ON KNOT FLOER HOMOLOGY AND LENS SPACE SURGERIES

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Abstract. In an earlier paper, we used the absolute grading on Heegaard Floer homology $HF^+$ to give restrictions on knots in $S^3$ which admit lens space surgeries. The aim of the present article is to exhibit stronger restrictions on such knots, arising from knot Floer homology. One consequence is that the non-zero coefficients of the Alexander polynomial of such a knot are $\pm 1$. This information can in turn be used to prove that certain lens spaces are not obtained as integral surgeries on knots. In fact, combining our results with constructions of Berge, we classify lens spaces $L(p, q)$ which arise as integral surgeries on knots in $S^3$ with $|p| \leq 1500$. Other applications include bounds on the four-ball genera of knots admitting lens space surgeries (which are sharp for Berge’s knots), and a constraint on three-manifolds obtained as integer surgeries on alternating knots, which is closely related to a theorem of Delman and Roberts.

1. Introduction

Let $K \subset S^3$ be a knot for which some integral surgery gives a lens space $L(p, q)$. The surgery long exact sequence for Heegaard Floer homology $HF^+$, together with the absolute grading on the latter group, can be combined to give a number of restrictions on $K$, see [15].

Consequently, for each fixed lens space $L(p, q)$, there is an explicitly determined, finite list of symmetric polynomials which might arise as the Alexander polynomials of such knots. The indeterminacy can be clarified from the following dual point of view. A knot in $S^3$ whose surgery gives $L(p, q)$ induces a knot $K'$ in $L(p, q)$ on which some surgery gives $S^3$. The stated indeterminacy, then, corresponds to the possible homology classes for $[K'] \in H_1(L(p, q); \mathbb{Z})$. Indeed, there are straightforward homological obstructions to realizing a given homology class in $H_1(L(p, q); \mathbb{Z})$ in this way from a knot in $S^3$ (or indeed from any integer homology three-sphere).

The results of [15] go beyond these homological considerations to give additional constraints on the Alexander polynomials of the knots $K$. These further constraints are specific to $S^3$: they can be used to rule out lens space surgeries even in cases where the lens space is realized as a surgery on a knot in some other integral homology three-sphere.

The aim of the present article is to strengthen considerably these constraints, with the help of the Floer homology of knots, see [19] and also [23]. In terms of the Alexander polynomial, our results here show that if $K$ is a knot with the above properties, then
all the coefficients of its Alexander polynomial are \( \pm 1 \), and the non-zero coefficients alternate in sign. Actually, since our results apply to a wider class of three-manifolds than lens spaces, before stating the theorems precisely, we discuss the class of three-manifolds we study.

1.1. **Knot Floer homology and L-space surgeries.** The appropriate generalization of the notion of lens spaces, for our purposes, is given in the following definition. Note that \( \hat{H}F(Y) \) is the three-manifold invariant defined in [17].

**Definition 1.1.** A closed three-manifold \( Y \) is called an L-space if \( H_1(Y; \mathbb{Q}) = 0 \), and \( \hat{H}F(Y) \) is a free Abelian group whose rank coincides with the number of elements in \( H_1(Y; \mathbb{Z}) \), which we write as \( |H_1(Y; \mathbb{Z})| \).

The set of L-spaces includes all lens spaces \( L(p, q) \) and, indeed, all spaces with “elliptic geometry,” i.e. all the finite, free quotients of \( S^3 \) by groups of isometries (c.f. Proposition 2.3 below). It also includes a class of plumbing manifolds which are obtained as plumbings specified by trees, for which the surgery coefficient associated to each vertex is no smaller than the number of edges meeting at that vertex (according to Theorem 7.1 of [22]). The set of L-spaces is closed under connected sums, and the following additional operation: fix an L-space \( Y \), and a knot \( K \subset Y \) with a choice of framing \( \lambda \) for which

\[
|H_1(Y_{\lambda+\mu}(K))| = |H_1(Y)| + |H_1(Y_{\lambda}(K))|,
\]

where \( \mu \) denotes the meridian for the knot, and \( Y_{\lambda}(K) \) denotes the three-manifold obtained from \( Y \) by performing surgery on \( Y \) along \( K \) with framing \( \lambda \). Then, if both \( Y \) and \( Y_{\lambda}(K) \) are L-spaces, then so is \( Y_{\lambda+\mu}(K) \). This construction gives infinitely many hyperbolic L-spaces. For instance, let \( P(a, b, c) \) denote the three-stranded pretzel knot with \( a \), \( b \), and \( c \) twists respectively. As observed by Fintushel and Stern [7], \(+18\) surgery on \( P(-2, 3, 7) \) is a lens space. Thus, applying the above principle to the knot \( K \) in the L-space \( S^3 \), and induction, we see that for all integers \( p \geq 18 \), \( S^3_p(P(-2, 3, 7)) \) is an L-space. These are hyperbolic for all sufficiently large \( p \), according to a theorem of Thurston [27], [28] (in fact, the fundamental group is infinite for all \( p > 19 \), c.f. [10]). A more in-depth discussion of L-spaces with more examples is given in Section 2.

The results of this paper are built on the following theorem about the Floer homology of a knot which admits an L-space surgery. To state the result, recall that there is a knot Floer homology group associated to a knot \( K \) in \( S^3 \) and an integer \( i \), which is a graded Abelian group, denoted \( \hat{H}FK(K, i) \), c.f. [19], see also [23].

**Theorem 1.2.** Suppose that \( K \subset S^3 \) is a knot for which there is a positive integer \( p \) for which \( S^3_p(K) \) is an L-space. Then, there is an increasing sequence of non-negative integers

\[
n_{-k} < \ldots < n_k
\]
with the property that \( n_i = -n_{-i} \), with the following significance. If for \(-k \leq i \leq k\) we let
\[
\delta_i = \begin{cases} 
0 & \text{if } i = k \\
\delta_{i+1} - 2(n_{i+1} - n_i) + 1 & \text{if } k - i \text{ is odd} \\
\delta_{i+1} - 1 & \text{if } k - i > 0 \text{ is even},
\end{cases}
\]
then \( \widehat{HF}(K, j) = 0 \) unless \( j = n_i \) for some \( i \), in which case \( \widehat{HF}(K, j) \cong \mathbb{Z} \) and it is supported entirely in dimension \( \delta_i \).

Since
\[
\sum_i \chi(\widehat{HF}(K, i)) \cdot T^i = \Delta_K(T)
\]
is the symmetrized Alexander polynomial (c.f. Proposition 4.2 of [19]), the above theorem says that \( \widehat{HF} \) is determined explicitly from the Alexander polynomial of \( K \). Conversely, the above theorem gives strong restrictions on the Alexander polynomials of knots which admit \( L \)-space surgeries:

**Corollary 1.3.** Let \( K \subset S^3 \) be a knot for which there is an integer \( p \) for which \( S^3_p(K) \) is an \( L \)-space. Then the Alexander polynomial of \( K \) has the form
\[
\Delta_K(T) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(T^{n_j} + T^{-n_j}),
\]
for some increasing sequence of positive integers \( 0 < n_1 < n_2 < \ldots < n_k \).

For a fixed \( L \)-space \( Y \), the possible polynomials which could occur as the Alexander polynomials of knots \( K \subset S^3 \) for which \( S^3_p(K) \cong Y \) is determined up to a finite indeterminacy by the absolute grading on \( \widehat{HF}(Y) \) (c.f. [15], but observe that this result also follows from the methods of the present paper, see Section 3). Thus, the above corollary can be used to give new restrictions on which \( L \)-spaces arise as \(+p\) surgeries on knots in \( S^3 \).

1.2. **Alexander polynomials and lens space surgeries.** As an illustration, let \( d(L(p, q), i) \) denote the absolute grading of the generator of \( \widehat{HF}(L(p, q), i) \). (Here, we use the orientation convention that \( L(p, q) \) is obtained by \( p/q \) surgery on the unknot in \( S^3 \).) We showed in [15] (c.f. Proposition 4.8; compare also [12] and [29]), that this quantity is determined by the recursive formula
\[
d(-L(1, 1), 0) = 0
\]
\[
d(-L(p, q), i) = \left( \frac{pq - (2i + 1 - p - q)^2}{4pq} \right) - d(-L(q, r), j),
\]
where \( r \) and \( j \) are the reductions modulo \( q \) of \( p \) and \( i \) respectively. Note that we are implicitly using here a specific identification \( \mathbb{Z}/p\mathbb{Z} \cong \text{Spin}^c(L(p, q)) \) (defined explicitly
in Subsection 4.1 of [15], but not crucial for our purposes here). We have the following consequence of Corollary 1.3:

**Corollary 1.4.** The lens space $L(p,q)$ is obtained as surgery on a knot $K \subset S^3$ only if there is a one-to-one correspondence

$$\sigma : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Spin}^c(L(p,q))$$

with the following symmetries:

- $\sigma(-[i]) = \sigma([i])$
- there is an isomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ with the property that

$$\sigma([i]) - \sigma([j]) = \phi([i - j]),$$

with the following properties. For $i \in \mathbb{Z}$, let $[i]$ denote its reduction modulo $p$, and define

$$t_i = \begin{cases} 
-d(L(p,q), \sigma([i])) + d(L(p,1), [i]) & \text{if } 2|i| \leq p \\
0 & \text{otherwise,}
\end{cases}$$

then the Laurent polynomial

$$1 + \sum_i \left( \frac{t_{i-1}}{2} - t_i + \frac{t_{i+1}}{2} \right) T^i = \sum_i a_i \cdot T^i$$

has integral coefficients, all of which satisfy $|a_i| \leq 1$, and all of its non-zero coefficients alternate in sign.

For instance, a straightforward if tedious calculation using the above result shows that $L(17, 2)$ does not occur as integral surgery on any knot in $S^3$, even though it passes all the criteria from [15] (in particular, it is realizable as $+17$ surgery on a knot in some other integral homology three-sphere). Similar remarks hold for $L(19, 17)$ and $L(26, 23)$ (compare the list at the end of [15]).

In fact, these obstructions are particularly powerful when one combines them with Berge’s construction of knots which admit lens space surgeries, see [1]. Indeed, there is evidence suggesting that the conditions on $L(p,q)$ in Corollary 1.4 necessary for it to be realized as integral surgery on a knot in $S^3$ are also sufficient. We return to this point at the end of the present introduction, after describing Berge’s constructions. But first, we turn to some other immediate applications of Theorem 1.2.

1.3. **Alternating knots and L-space surgeries.** In another direction, we obtain the following consequence of Corollary 1.3 (together with properties of the Alexander polynomials of alternating knots, c.f. Section 4), which is rather similar in spirit to a theorem of Delman and Roberts [6] obtained using the theory of laminations:

**Theorem 1.5.** If $K \subset S^3$ is an alternating knot with the property that some integral surgery along $K$ is an $L$-space, then $K$ is a $(2, 2n + 1)$ torus knot, for some integer $n$. 
1.4. Bounding the four-ball genus. To go beyond the Alexander polynomial, recall that the knot Floer homology \( \bigoplus_m \widehat{HF}(K, m) \) is the homology of the graded complex associated to a filtration

\[ \ldots \subseteq \mathcal{F}(K, m) \subseteq \mathcal{F}(K, m + 1) \subseteq \ldots \]

of the chain complex \( \widehat{CF}(S^3) \) (whose homology is \( \mathbb{Z} \), in a single dimension), induced by the knot \( K \). This filtration gives an integer \( \tau(K) \) which is defined to be the smallest integer \( m \) for which the induced map on homology

\[ \iota^m_K: H_*(\mathcal{F}(K, m)) \to \widehat{HF}(S^3) \cong \mathbb{Z} \]

is non-trivial. In [20] (c.f. Corollary 1.3 of [20]) and also [23], it is shown that if \( g^*(K) \) denotes the four-ball genus, then

\[ |\tau(K)| \leq g^*(K). \tag{1} \]

Note that \( g^*(K) \) gives a lower bound on the unknotting number of \( K \). Combining Inequality (1) with Theorem 1.2, we obtain the following:

**Corollary 1.6.** Suppose that \( K \subset S^3 \) is a knot which admits an integral \( L \)-space surgery, then \( |\tau(K)| \) coincides with the degree of the symmetrized Alexander polynomial of \( K \). In particular, the four-ball genus \( g^*(K) \) is bounded below by this degree.

**Proof.** The first claim follows immediately from the description of the knot Floer homology given in Theorem 1.2, while the second follows from the first, together with Inequality (1).

Corollary 1.6 also gives an illustration of how Theorem 1.2 goes beyond the Alexander polynomial. As an amusing application, consider the knot \( K = 10_{132} \) pictured in Figure 1, the ten-crossing knot whose Alexander polynomial is

\[ \Delta_K(T) = T^{-2} - T^{-1} + 1 - T + T^2. \]

This Alexander polynomial satisfies the criteria of Corollary 1.3. However, the knot clearly has unknotting number one, and hence according to Corollary 1.6, this knot admits no \( L \)-space surgeries.

Let \( T_{p,q} \) denote the \((p, q)\) torus knot. Since \( pq \pm 1 \)-surgery on the \((p, q)\) torus knot is a lens space, the above corollary shows that \( \tau(T_{p,q}) = \frac{(p-1)(q-1)}{2} \), and hence (after a careful choice of unknotting) that the unknotting number of \( T_{p,q} \) is given by this quantity. This result was first proved by Kronheimer and Mrowka [9] (and conjectured by Milnor [13]).

In fact, Corollary 1.6 gives a calculation of the four-ball genera of all knots coming from Berge's constructions, see [1]. Specifically, recall that Berge’s constructions have a particularly nice description from the point of view of knots in lens spaces.
**Definition 1.7.** Consider the standard genus one Heegaard diagram for \( L(p, q) \), where the two attaching circles \( \alpha \) and \( \beta \) meet in exactly \( p \) points. A lens space Berge knot \( K' \subset L(p, q) \) is one which is formed from a pair of arcs, one of which is supported in the attaching disk for \( \alpha \) and the other is supported in the attaching disk for \( \beta \), with the additional property that \([K'] \in H_1(L(p, q); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}\) is a generator.

The following result is verified in Section 5, using results of Stallings [26] and Brown [3]:

**Proposition 1.8.** All lens space Berge knots are fibered.

**Definition 1.9.** When integral surgery of \( L(p, q) \) along some lens space Berge knot \( K' \) gives \( S^3 \), there is a naturally induced knot \( K \subset S^3 \) for which some integral surgery gives \( L(p, q) \). We call this induced knot a classical Berge knot.

**Corollary 1.10.** Let \( K \) be a classical Berge knot. Then, the degree of the Alexander polynomial agrees with both the Seifert and four-ball genera of \( K \).

**Proof.** Since \( K \) is fibered, its Seifert genus agrees with the degree of its Alexander polynomial. The statement about the four-ball genus now follows from Corollary 1.6. \( \square \)

**Figure 1.** The knot \( 10_{132} \). This knot has Alexander polynomial \( 1 - (T + T^{-1}) + (T^2 + T^{-2}) \), and unknotting number one.
1.5. Realizing lens spaces. In [1], Berge proves the following theorem:

**Theorem 1.11. (Berge)** The lens space $L(p, q)$ arises as integral surgery on a knot in $S^3$ if we can find integers $A, a, B, b$ so that $p = |Aa + Bb|$, $a^2q \equiv \pm b^{\pm 2} \pmod{p}$, which satisfy at least one of the following additional constraints

1. $A = 1, a = \pm 1, (B, b) = 1, B \geq 2$,
2. $A = 1, a = \pm 1, (B, b) = 2, B \geq 4$,
3. $A > 1, a = \pm 1, and there is an integer $\epsilon = \pm 1$ so that $(B + \epsilon)/A$ is an odd integer and $b \equiv -2\epsilon Aa \pmod{B}$,
4. $A > 3, a = \pm 1$ and there is an $\epsilon = \pm 1$ so that $(2B + \epsilon)/A$ is integral, and $b \equiv -\epsilon Aa \pmod{B}$,
5. $A > 1, A$ is odd, $a = \pm 1$, and there is an $\epsilon = \pm 1$ such that $(B - \epsilon)/A$ is an integer, and $b \equiv -\epsilon Aa \pmod{B}$,
6. $A > 2, A$ is even, $a = \pm 1$ $B = 2A + 1$, and $b \equiv -a(A - 1) \pmod{B}$,
7. $a = -(A + B), b = -B$,
8. $a = -(A + B), b = B$,
9. $(A, B, a, b) = (4J + 1, 2J + 1, 6J + 1, -J)$ for some integer $J$,
10. $(A, B, a, b) = (6J + 2, 2J + 1, 4J + 1, -J)$ for some integer $J$,
11. $(A, B, a, b) = (6J + 4, 2J + 1, -4J - 3, J + 1)$ for some integer $J$,
12. $(A, B, a, b) = (4J + 3, 2J + 1, -6J - 5, J + 1)$ for some integer $J$.

Berge proves the above theorem by explicitly constructing the corresponding knots in $S^3$. For instance, lens spaces of Type (1) are realized by surgeries on torus knots, Type (2) by surgeries on cables of torus knots, Types (3)-(6) by other knots supported in a solid torus (for which some surgery gives another solid torus), Type (7) by surgeries on knots supported in the Seifert surface of a trefoil, Type (8) by surgeries on knots supported in the Seifert surface of the figure eight knot, and Types (9)-(12) are some other “sporadic” examples. Experimental verification suggests the following purely combinatorial conjecture (which we have verified for $|p| \leq 1500$ using a program written in Mathematica [30]):

**Conjecture 1.12.** A lens space $L(p, q)$ appears on Berge’s list above if and only if it passes the conditions of Corollary 1.4.

A proof of the above conjecture, of course, would prove that Berge’s conditions on a lens space are necessary and sufficient for it to be realized as integral surgery on a knot in $S^3$. Thus, our computer verification can be alternately phrased as follows:

**Proposition 1.13.** For all $p \leq 1500$, the lens spaces which are realized as integer surgeries on knots in $S^3$ are precisely those lens spaces which are realized on Berge’s list.

Note that Berge conjectures a stronger statement: he conjectures that his scheme [1] enumerates all knots which admit lens space surgeries. The above proposition could be viewed as partial evidence supporting his conjecture.
1.6. Further remarks. Recall that in [18], we proved that if \( K \) is a fibered knot with genus \( g \), then \( \widehat{HFK}(K, g) \cong \mathbb{Z} \), generalizing the standard fact that a fibered knot has monic Alexander polynomial. Thus Theorem 1.2 could be seen as evidence supporting the conjecture that all knots with lens space surgeries are fibered. (Note that at the time of the writing of this paper, the authors know of no non-fibered knots for which \( \widehat{HFK}(K, d) \cong \mathbb{Z} \), where \( d \) denotes the degree of the Alexander polynomial of \( K \).)

1.7. Organization. We discuss \( L \)-spaces in Section 2. In Section 3 we prove Theorem 1.2 and its immediate corollaries, Corollaries 1.3 and 1.4, and also Proposition 1.13. In Section 4, we prove Theorem 1.5 from Theorem 1.2, and a result on the Alexander polynomials of alternating knots. Finally, in Section 5, we verify Proposition 1.8.

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2. L-spaces

The aim of the present section is to collect some of the key properties of L-spaces, and to give some constructions. In fact, much of the material here is not new, but can be found sprinkled throughout most of our papers on Heegaard Floer homology. It is for this reason that we feel that it might be useful to collect the properties in one place.

Recall that $\widehat{HF}(Y)$ is a finitely generated, $\mathbb{Z}/2\mathbb{Z}$-graded Abelian group which, for rational homology three-spheres, satisfies the relation that

$$\chi(\widehat{HF}(Y)) = |H_1(Y;\mathbb{Z})|$$

(c.f. Proposition 5.1 of [16]) and, in particular, for any rational homology three-sphere, $|H_1(Y;\mathbb{Z})| \leq \text{rk}\widehat{HF}(Y)$. When $Y$ is an L-space, this inequality is an equality.

In general, $\widehat{HF}(Y)$ depends on the orientation used for $Y$, but its total rank does not (c.f. Proposition 2.5 of [16]). Using coefficients in an arbitrary field, we see that the condition of being an L-space is also independent of the orientation of $Y$. Note that L-spaces have an alternate characterization in terms of other elements of the Heegaard Floer homology package defined in [17]: a rational homology three-sphere is an L-space if and only if $HF^+_{red}(Y) = 0$. We will have no further use for this characterization in the present paper, but point it out as it appears to be the characterization which generalizes more neatly to the case where $b_1(Y) > 0$.

The fact that lens spaces are L-spaces follows immediately from their standard genus one Heegaard diagrams (c.f. Proposition 3.1 of [16]).

The functor $\widehat{HF}(Y)$ enjoys a Künneth principle for connected sums (Proposition 6.1 of [16]), from which it follows readily that the set of L-spaces is closed under connected sums.

Suppose that $K \subset Y$ is a knot in a rational homology three-sphere, and let $\mu$ be the meridian for $K$ and let $\lambda$ be any choice of longitude (i.e. simple, closed curve in the torus $\partial K$ which meets $\mu$ in a single transverse point of intersection). Indeed, suppose that $\lambda$ is chosen so that the three-manifolds $Y_{\lambda}(K)$ and $Y_{\lambda+\mu}(K)$ are both rational homology three-spheres. We have the following result (compare Lemma 7.12 of [15]):

**Proposition 2.1.** Let $K \subset Y$ be a knot in a rational homology three-sphere, and let $\lambda$ be a choice of longitude for the knot, so that $Y_{\lambda}(K)$, $Y_{\lambda+\mu}(K)$ are also rational homology three-spheres, and

$$|H_1(Y_{\lambda+\mu}(K))| = |H_1(Y)| + |H_1(Y_{\lambda}(K))|.$$  

If $Y$ and $Y_{\lambda}(K)$ are L-spaces, then so is $Y_{\lambda+\mu}(K)$.

**Proof.** This follows readily from the long exact surgery sequence for $\widehat{HF}$ (c.f. Theorem 9.16 of [16]), which in the present case reads:

$$\cdots \longrightarrow \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y_{\lambda}(K)) \longrightarrow \widehat{HF}(Y_{\lambda+\mu}(K)) \longrightarrow \cdots$$
In particular, we see that
\[ \text{rk}\hat{HF}(Y_{\lambda+\mu}(K)) \leq \text{rk}\hat{HF}(Y) + \text{rk}\hat{HF}(Y_{\lambda}(K)). \]

It follows immediately that if \( Y \) and \( Y_{\lambda}(K) \) are \( L \)-spaces, then
\[ \text{rk}\hat{HF}(Y_{\lambda+\mu}(K)) \leq |H_1(Y_{\lambda+\mu}(K))|, \]
while the opposite inequality is provided by Equation (2), forcing equality to hold. Moreover, repeating the above argument with coefficients in any finite field, one verifies that \( \hat{HF}(Y_{\lambda+\mu}(K)) \) has no torsion.

The above proposition guarantees that if \( K \subset S^3 \) is a knot with the property that \( S^3_n(K) \) is an \( L \)-space, for some positive integer \( p \), then so is \( S^3_n(K) \) for all integers \( n \geq p \).

In [21], we give a characterization of Seifert fibered \( L \)-space which we recall presently. Recall that a Seifert fibered rational homology three-sphere is specified by a collection of integers \( b, \{\alpha_i\}_{i=1}^n, \) and \( \{\beta_i\}_{i=1}^n \) (sometimes abbreviated \( (b; \beta_1/\alpha_1, ..., \beta_n/\alpha_n) \)), where here all \( \alpha_i > 2 \) and the \( 0 < \beta_i < \alpha_i \), and \( (\alpha_i, \beta_i) = 1 \) (see [25], see also [24] for a modern treatment). The \( \{\alpha_i\} \) specify the base orbifold (which in the present case must have genus zero). The number \( n \) is the number of singular orbits for the circle action on \( Y \).

A Seifert fibered space has an orbifold degree given by the formula
\[ b + \sum_{i} \frac{\beta_i}{\alpha_i} \]
When the base has genus zero, the orbifold degree is non-zero precisely when \( Y \) is a rational homology three-sphere. Note that the orbifold degree changes sign under orientation reversal of \( Y \).

A Seifert space \( (b; \beta_1/\alpha_1, ..., \beta_n/\alpha_n) \) can be realized as the boundary of a plumbing of spheres, where the spheres are arranged in a star-like pattern, so that the central node has self-intersection number \( b \), and the multiplicities of the chains of spheres is given by the Hirzebruch-Jung fractional expansion of \( \alpha_i/\beta_i \). Let \( G \) denote this labeled graph: i.e. this is a tree equipped with a function \( m \) from the vertices of \( G \) to the integer, which gives rise in the usual manner to an inner product on the vector space \( V \) generated by the vertices. In topological terms, \( V \) is \( H_2(W(G)) \) where \( W(G) \) is the four-manifold constructed from the plumbing diagram specified by \( G \), and the induced inner product corresponds to the intersection form. Note that the induced intersection form on \( V \) is negative-definite if and only if the orbifold degree is negative.

A characteristic vector \( K \) for \( H_2(W(G)) \) is a vector \( K \) in the dual space for \( V \) with the property that \( \langle K, v \rangle \equiv m(v) \) (mod 2) for each vertex \( v \). A sequence \( \{K_i\}_{i=1}^\ell \) of characteristic vectors is called a full path if
- for each \( i \) and each vertex \( v \) for \( G \),
  \[ |\langle K_i, v \rangle| \leq -m(v) \]
for each vertex $v$ for $G$,
\[ m(v) + 2 \leq \langle K_1, v \rangle \leq -m(v) \]
and
\[ m(v) \leq \langle K_\ell, v \rangle \leq -m(v) - 2 \]

- for each $i < \ell$, there is a vertex $v$ with the property that $\langle K_i, v \rangle = -m(v)$, and $K_{i+1} = K_i + 2\text{PD}[v]$.

We call two full paths equivalent if they start with the same initial vector. As proved in [21], equivalent full paths also have the same final vector.

Full paths are related to the Heegaard Floer homology of Seifert fibered spaces, according to the following result from [21]:

**Theorem 2.2.** Let $Y$ be a Seifert fibered rational homology sphere with Seifert invariants $(b; \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n)$, and let $G$ denote its corresponding negative-definite plumbing graph. If $b \leq -n$, then $Y$ is an L-space. More generally, $Y$ is an L-space if and only if the number of equivalence classes of full paths for $G$ agrees with the number of elements $|H_1(Y; \mathbb{Z})|$.

**Proof.** The first statement is a consequence of Proposition 2.1 and indeed, a proof is spelled out in [22]. The second is an application of the main result in [21].

**Proposition 2.3.** Every three-manifold with elliptic geometry is an L-space.

**Proof.** Spaces with elliptic geometry are those Seifert fibered fibered spaces over a base orbifold $\Sigma$ with positive orbifold Euler characteristic $\chi^\text{orb} (\Sigma)$, which have non-zero orbifold degree over their base (see [24] for a discussion of these notions).

Now, positivity of the Euler characteristic of the base forces it to have genus zero and at most three singular fibers. If the number of singular fibers is less than three, the total space is a lens space, and hence covered by our earlier discussion. If there are three singular fibers, with integral multiplicities $\alpha_1$, $\alpha_2$, and $\alpha_3$ (all $> 1$), then the formula for the orbifold Euler characteristic is
\[ \chi^\text{orb} (\Sigma) = -1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}. \]

The positivity criterion here forces $\{\alpha_1, \alpha_2, \alpha_3\}$ to be either $\{2, 3, n\}$ with $n \leq 5$ or $\{2, 2, n\}$ with $n$ arbitrary.

By reversing the orientation of $Y$, we can arrange for the orbifold degree to be negative. For these values of the $\alpha_i$, negativity of the orbifold degree now clearly forces $b \leq -2$. Indeed, by Theorem 2.2, it suffices to consider the case where $b = -2$.

We consider first the case where $\alpha_2 = 2$. In this case, we apply first an induction on the length of the third chain of spheres, and a subinduction on the multiplicity on the
last leaf. For the basic case, (where the graph has four vertices with multiplicities $-2$), we again appeal to the second criterion from Theorem 2.2. For the inductive step, we consider the case where we add one more vertex with multiplicity $-2$. It is easy to see that this is realized as $Y_{\lambda+\mu}(K)$, where $Y$ is the Seifert fibered space obtained by the graph where this last vertex is deleted, and $Y_{\lambda}(K)$ is obtained as a graph with fewer vertices (gotten by exchanging the multiplicity with $-1$, and then successively blowing down $-1$ spheres). Indeed, our hypotheses force

$$|H_1(Y_{\lambda+\mu}(K))| = |H_1(Y)| + |H_1(Y_{\lambda}(K))|,$$

so the inductive hypotheses and Proposition 2.1 applies to show that $Y_{\lambda+\mu}(K)$ is an $L$-space. The induction required to reduce the multiplicity of this last vertex by one works in the same way.

Indeed, the case where $\alpha_2 = 3$ and $\beta_2 = 1$ follows from the case where $\alpha_2 = 2$ by another application of Proposition 2.1.

The finitely many (seven) cases where $\beta_2 = 2$ and $\alpha_2 = 3$ which are not covered above all follow from calculations using the second criterion from Theorem 2.2.

There are examples of non-elliptic Seifert fibered $L$-spaces. In fact, recall (see [2], see also [7] and [10]) that if we consider the pretzel knot $P(-2, 3, n)$, where $n$ is an odd integer, then $S^3_{2n+4}(K)$ is $\pm Y$ where $Y$ is the Seifert fibered rational homology three-sphere with invariants $(-2; 1/2, 1/4, (n-8)/(n-6))$. It is easy to see that when $n \geq 9$, $S^3_{2n+4}(K)$ is an $L$-space (though when $n > 9$, it is not elliptic). As we mentioned in the introduction, in the case where $n = 7$, similar considerations show that $S^3_{18}(K)$ is a lens space (c.f. [7] and [2]). In view of Proposition 2.1, if $n$ is any odd integer with $n \geq 7$, and $p$ be any integer with $p \geq 2n+4$, then the three-manifold $S^3_{p}(P(-2, 3, n))$ is an $L$-space.
3. Proof of Theorem 1.2.

Theorem 1.2 follows from the relationship between the knot Floer homology associated to $K \subset S^3$ and the Heegaard Floer homology of $\widehat{HF}(S^3_p(K))$ for all large enough $p$. This relationship is established in Section 4 of [19] (see especially Theorem 4.4). We recall these constructions briefly here.

Recall that a knot $K$ induces a filtration on $\widehat{CF}(S^3)$. More precisely, the (finitely many) generators for $\widehat{CF}(S^3)$ have a filtration level taking values in $(0, \mathbb{Z})$ (the relevance of the first coordinate will become apparent in a moment), and the differential is non-increasing in this filtration. For $m \in \mathbb{Z}$, we let $C\{0, m\} \subset \widehat{CF}(S^3)$ denote the subgroup generated by elements with filtration level $(0, m)$. More generally, we can let $C\{(\ell, m)\}$ denote the set of generators of $C\{(0, m-\ell)\}$, now shifted by a group isomorphism $U_\ell: C\{(\ell, m)\} \rightarrow C\{(0, m-\ell)\}$.

In [19], we equip $C = \bigoplus_{(i,j) \in \mathbb{Z}} C\{(i, j)\}$ with a differential $D$ which commutes with the maps $U_\ell$, and which is compatible with the initial differential on $\widehat{CF}(S^3) = \bigoplus_{m \in \mathbb{Z}} C\{(0, m)\}$.

Indeed, the differential $D$ respects the $\mathbb{Z} \oplus \mathbb{Z}$ filtration which sends the summand $C\{(i, j)\} \subset C$ to $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$. This means that if $\xi$ is supported in this summand $C\{(\ell, m)\}$, then $\partial \xi$ is supported in the group

$$
\bigoplus_{\{(i,j)\} | i \leq \ell, \text{ and } j \leq m} C\{(i, j)\} \subset C.
$$

The complex $C$ is graded by the convention that $H_*(C\{i = 0\}) \cong H_*(\widehat{CF}(S^3)) \cong \mathbb{Z}$ is supported in dimension zero, the differential $D$ lowers degree by one, and the map $U_\ell$ lowers it by $2\ell$. The chain complex referred to here as $C$ is the $\mathbb{Z} \oplus \mathbb{Z}$-filtered complex $CFK^\infty(S^3, K)$ from [19].

A $\mathbb{Z} \oplus \mathbb{Z}$-filtered complex $C$ induces many other chain complexes. We introduce the following notational shorthand. If $R$ is a region in the $(i, j)$ plane, then let $C(R)$ denotes the naturally induced complex on the set of generators of $C$ whose filtration level $(i, j)$ lies in the region $R$. Of course, this does not make sense for any region $R$ but there are three cases of interest to us (here, we write $(i_1, j_1) \leq (i_2, j_2)$ if $i_1 \leq i_2$ and $i_2 \leq j_2$):

- suppose $R$ has the property that $(i_1, j_1) \in R$ and $(i_2, j_2) \leq (i_1, j_1) \Rightarrow (i_2, j_2) \in R$, then $C(R)$ is naturally a subcomplex;
• suppose \( R \) has the property that
\[
(i_1, j_1) \in R \text{ and } (i_2, j_2) \geq (i_1, j_1) \Rightarrow (i_2, j_2) \in R,
\]
then \( C(R) \) is naturally a quotient complex;

• suppose \( R \) has the property that
\[
(i_1, j_1) \leq (i_2, j_2) \leq (i_3, j_3) \text{ and } (i_1, j_1), (i_3, j_3) \in R \Rightarrow (i_2, j_2) \in R,
\]
then \( C(R) \) is naturally the subcomplex of a quotient complex of \( C \).

If \( n \) is any integer, there is a natural affine identification \( \text{Spin}^c(S^3_n(K)) \cong \mathbb{Z}/n\mathbb{Z} \) made explicit in [19] (but not crucial for our present applications). If \([m] \in \mathbb{Z}/n\mathbb{Z}\), we let \( \tilde{HF}(S^3, [m]) \) denote the summand of the Floer homology in the \( \text{Spin}^c \) structure corresponding to \([m]\). Theorem 4.4 of [19] states that given any knot \( K \subset S^3 \), there is an integer \( N \) so that for all \( n \geq N \), there is an isomorphism of chain complexes
\[
(3) \quad \tilde{CF}(S^3_n(K), [m]) \cong C\{\max(i, j - m) = 0\}.
\]

Theorem 1.2 is now an algebraic consequence of the above theorems. This algebra is encoded in the following two lemmas.

**Lemma 3.1.** Let \( C \) be a \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered chain complex over a field \( \mathbb{F} \), and let \( m \) be an integer with the property that
\[
(4) \quad H_*(C\{\max(i, j - m) = 0\}) \cong \mathbb{F} \cong H_*(C\{\max(i, j - (m - 1)) = 0\})
\]
(ignoring the grading).

Suppose also that \( H_*(C\{i < 0, j = m\}) = 0 \).

Then, either \( H_*(C\{(0, m)\}) = 0 \), in which case \( H_*(C\{i < 0, j = m - 1\}) = 0 \) as well; or \( H_*(C\{(0, m)\}) \cong \mathbb{F} \), in which case \( H_*(C\{i < 0, j = m - 1\}) \cong \mathbb{F} \) and \( H_*(C\{i = 0, j \leq m - 1\}) = 0 \).

**Proof.** Let
\[
X = \{i \leq 0, j = m\} \text{ and } Y = \{i = 0, j \leq m - 1\},
\]
so that \( UX = \{i \leq -1, j = m - 1\} \). In this notation, Equation (4) says that:
\[
H_*(C\{UX \cup Y\}) \cong H_*(C\{X \cup Y\}) \cong \mathbb{F}.
\]

By the long exact sequence associated to the short exact sequence
\[
0 \longrightarrow C\{i < 0, j = m\} \longrightarrow C\{X\} \longrightarrow C\{i = 0, j = m\} \longrightarrow 0,
\]
combined with our hypothesis that \( H_*(C\{i < 0, j = m\}) \) is trivial, we see that
\[
H_*(C\{X\}) \cong H_*(C\{(0, m)\}).
\]
We have the following pair of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C\{UX\} & \longrightarrow & C\{UX \cup Y\} & \xrightarrow{B} & C\{Y\} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & C\{X \cup Y\} & \longrightarrow & C\{X\} & \longrightarrow & 0
\end{array}
\]

Let \(n\) denote the rank of \(H_*(C\{(0, m)\}) \cong H_*(C\{X\})\). There are two cases according to whether or not the map induced by \(B\) on homology

\[
b: H_*(C\{UX \cup Y\}) \cong \mathbb{F} \longrightarrow H_*(C\{Y\})
\]

is trivial.

If \(b\) is trivial, then it is easy to see that \(H_*(C\{Y\})\) has rank \(n - 1\), and that the coboundary associated to the horizontal short exact sequence

\[
\delta_h: H_*(C\{Y\}) \cong \mathbb{F}^{n-1} \longrightarrow H_*(C\{UX\}) \cong \mathbb{F}^n
\]

is injective. Moreover, another count of ranks then ensures that the coboundary map associated to the vertical short exact sequence

\[
\delta_v: H_*(C\{X\}) \longrightarrow H_*(C\{Y\})
\]

is surjective. In particular, the image of the composite

\[
\delta_h \circ \delta_v: H_*(C\{X\}) \longrightarrow H_*(C\{UX\})
\]

has rank \(n - 1\). On the other hand, the relation that \(D^2 = 0\) (on the induced complex \(C\{X \cup UX \cup Y\}\)) ensures that the composite is trivial (indeed, the relation gives a null-homotopy of the composite on the chain level). Thus, we have established that \(n = 1\) and \(H_*(C\{Y\})\) is trivial.

On the other hand, if \(b\) is non-trivial, then \(H_*(C\{Y\})\) has rank \(n + 1\), and indeed the map

\[
\delta_h: H_*(C\{Y\}) \cong \mathbb{F}^{n+1} \longrightarrow H_*(C\{UX\}) \cong \mathbb{F}^n
\]

is surjective, while the map

\[
\delta_v: H_*(C\{X\}) \cong \mathbb{F}^n \longrightarrow H_*(C\{Y\}) \cong \mathbb{F}^{n+1}
\]
is injective. On the one hand, this implies that the image of \((U\delta_v) \circ \delta_h\) is \(n\)-dimensional; on the other hand, the composite is trivial (which follows from the fact that \(D^2 = 0\) on the complex \(C\{UX \cup Y \cup UY\}\)), and hence \(n = 0\), and \(H_*(C\{Y\})\) is one-dimensional. These two cases cover the two cases in the conclusion of the lemma.

Lemma 3.2. Suppose once again that \(C\) is a bigraded complex, with the property that

\[ H_*(C\{\max(i, j - m + 1) = 0\}) \cong \mathbb{F} \]

for some integer \(m\).

Suppose furthermore that \(H_*(C\{i < 0, j = m\}) \cong \mathbb{F}\) and \(H_*(C\{i = 0, j \leq m\}) = 0\).

Then either \(H_*(C\{(0, m)\}) = 0\), in which case \(H_*(C\{X \cup Y\}) \cong \mathbb{F}\), or \(H_*(C\{i = 0, j = m\}) \cong \mathbb{F}\), in which case \(H_*(C\{i < 0, m - 1\}) = 0\).

Proof. We continue using the notation for the proof of Lemma 3.1, with

\[ X = \{i \leq 0, j = m\} \quad \text{and} \quad Y = \{i = 0, j \leq m - 1\}. \]

In this case, the hypothesis that

\[ 0 = H_*(C\{i = 0, j \leq m\}) = H_*(C\{Y \cup (0, m)\}) \]

ensures that

\[ H_*(C\{(0, m)\}) \cong H_*(C\{Y\}). \]

Let \(n\) denote the rank of \(H_*(C\{(0, m)\})\).

Again, we have the two exact sequences illustrated in the Diagram (5). Since \(H_*(UX \cup Y) \cong \mathbb{F}\) (and our hypotheses also ensure the \(H_*(X \cup Y) \cong \mathbb{F}\)), we have two cases according to whether or not the map on homology \(b\) trivial as before.

If \(b\) is trivial, a diagram chase shows that

\[ \delta_h : H_*(C\{Y\}) \cong \mathbb{F}^n \longrightarrow H_*(C\{UX\}) \cong \mathbb{F}^{n+1} \]

is injective and also that

\[ \delta_v : H_*(C\{X\}) \longrightarrow H_*(C\{Y\}) \]

is surjective. Again, this implies that the image of \(\delta_h \circ \delta_v\) is \(n\)-dimensional, but since the composite is trivial, \(n = 0\), and hence \(H_*(C\{UX\})\) is one-dimensional.

If \(b\) is non-trivial,

\[ \delta_h : H_*(C\{Y\}) \cong \mathbb{F}^n \longrightarrow H_*(C\{UX\}) \cong \mathbb{F}^{n-1} \]

is surjective and also

\[ \delta_v : H_*(C\{X\}) \longrightarrow H_*(C\{Y\}) \]

is injective. Thus, the image of \((U\delta_v) \circ \delta_h\) is \((n - 1)\)-dimensional; but again this is trivial, forcing \(n = 1\), and \(H_*(C\{UX\})\) to be zero-dimensional.

These two cases cover the two cases in the conclusion of the lemma. \(\square\)
Proof of Theorem 1.2. By the universal coefficients theorem, it suffices to establish Theorem 1.2 over an arbitrary field $\mathbb{F}$. First, note that according to Equation (3) (and our hypothesis on $K$), $H_*(C\{\max(i,j-m)\}) = \mathbb{F}$ for all $m$.

Since $C\{i=0\}$ is finitely generated, for all sufficiently large $m$, $C\{i<0,j=m\} = 0$. Moreover, $C\{i=0,j\leq m\} = C\{i=0\}$, so that $H_*(C\{i=0,j\leq m\}) \cong \mathbb{F}$ is supported in even (zero) degree. In particular, the hypotheses of Lemma 3.1 apply. Indeed, by descending induction on $m$, and using Lemma 3.1 and 3.2, we have that for all $m$, the rank of $\widetilde{HFK}(K,m)$ is at most one, and for each integer $m$, exactly one of the following two possibilities holds:

1. either $H_*(C\{i<0,j=m\}) = 0$, in the case where there either is no $\ell > m$ with $\widetilde{HFK}(K,\ell) \neq 0$ or the smallest such $\ell$ has the corresponding $\widetilde{HFK}(K,\ell)$ supported in odd degree;
2. or $H_*(C\{i<0,j=m\}) \cong \mathbb{F}$ and $H_*(C\{i=0,j\leq m\}) = 0$ and the smallest $\ell > m$ with $\widetilde{HFK}(K,\ell) \neq 0$ has the corresponding $\widetilde{HFK}(K,\ell)$ supported in even degree.

Indeed, let $\ell > m$ be a pair of integers for which $\widetilde{HFK}(K,\ell) \cong \mathbb{F} \cong \widetilde{HFK}(K,m)$ (ignoring gradings), and for all intermediate $m < j < \ell$, $\widetilde{HFK}(K,j) = 0$. Let $d$ denote the dimension in which $\widetilde{HFK}(K,\ell)$ is supported.

If $d$ is even, then it is easy to see (once again, by one application of Lemma 3.1 followed by repeated applications of Lemma 3.2) that $H_*(C\{i<0,j=m\}) \cong \mathbb{F}$ is supported in dimension $d-2(\ell-m)$. It follows now from Lemma 3.2, that the coboundary map for the short exact sequence

$$0 \longrightarrow C\{i<0,j=m\} \longrightarrow C\{i\leq0,j=m\} \longrightarrow \widetilde{CFK}(K,m) \longrightarrow 0,$$

which drops dimension by one, induces an isomorphism in homology

$$\delta_h: \widetilde{HFK}(K,m) \longrightarrow H_*(C\{i<0,j=m\});$$

thus $\widetilde{HFK}(K,m)$ is supported in dimension $d-2(\ell-m)+1$.

If $d$ is odd, then it follows that the coboundary map for the short exact sequence

$$0 \longrightarrow C\{i=0,j<\ell\} \longrightarrow C\{i=0,j\leq\ell\} \longrightarrow \widetilde{CFK}(K,\ell) \longrightarrow 0$$

induces an isomorphism on homology. Thus, $H_*(C\{i=0,j<\ell\}) \cong \mathbb{F}$ is supported in dimension $d-1$. Indeed, it is easy to see that the natural inclusion $C\{i=0,j\leq m\} \subset C\{i=0,j<\ell\}$ induces an isomorphism in homology. In fact since $H_*(C\{i=0,j=m-1\}) = 0$ (c.f. Lemma 3.1), the projection $C\{i=0,j\leq m\} \longrightarrow C\{i=0,j=m\}$ also induces a (degree-preserving) isomorphism in homology: i.e. $\widetilde{HFK}(K,m)$ is supported in dimension $d-1$.

Together, these claims establish the theorem. \qed
It is worth pointing out that the above lemmas hold even in the case where \( S^3_p(K) \) is not an \( L \)-space. In particular, we have the following:

**Proposition 3.3.** Let \( K \subset S^3 \) be a knot with the properties that \( \hat{\mathit{HF}}(K, m) = 0 \) for all \( m > d \), \( \hat{\mathit{HF}}(K, d) \neq 0 \). Then, if for all sufficiently large \( n \), \( \hat{HF}(S^3_n(K), [d]) \cong \hat{HF}(S^3_n(K), [d-1]) \cong \mathbb{Q} \), we have that \( \tau(K) = d \).

**Proof.** Follows immediately from Lemma 3.1.

We turn now to some of the consequences of Theorem 1.2 which were described in the introduction.

**Proof of Corollary 1.3.** This is an immediate consequence of Theorem 1.2 and the relationship between the knot Floer homology and the Alexander polynomial, Proposition 4.2 of [19].

**Proof of Corollary 1.4.** Note that the above proof could be modified to give the relationship between the absolute gradings on \( \hat{HF}(L) \) with the Alexander polynomial of \( K \). We have not spelled this out, as it was already determined in [15], c.f. Theorem 7.2 and especially Corollary 7.5, both in [15].

**Proof of Proposition 1.13.** Proposition 1.13 is verified by first calculating \( d(-L(p, q), i) \) for \( p \) in some range, then enumerating all possible correspondences \( \sigma \), keeping only those \( q \) for which one of the correspondences \( \sigma \) satisfies the conditions of Corollary 1.4, and then verifying that this list of allowed lens spaces is covered by Berge’s list. This verification is algorithmic, if tedious. We used code written in Mathematica [30].
4. Alternating knots and \(L\)-space surgeries

For the proof of Theorem 1.5, we use the following characterization of the \((2,n)\) torus knots, which follows easily from standard properties of the Alexander polynomial for alternating knots, compare [14], [4], [8], [11]:

**Proposition 4.1.** If \(K\) is an alternating knot with the property that all the coefficients \(a_i\) of its Alexander polynomial \(\Delta_K\) have \(|a_i| \leq 1\), then \(K\) is the \((2,2n+1)\) torus knot.

**Proof.** According to a theorem of Menasco [11], a non-prime alternating knot factors as a sum of (non-trivial) alternating knots. According to a theorem of Crowell and Murasugi [4] and [14], the Alexander polynomials of these factors are non-trivial polynomials whose coefficients alternate in sign. It follows at once that the Alexander polynomial of our original knot has coefficients greater than one.

Thus, it suffices to consider the case where \(K\) is prime. Consider an alternating projection, and let \(w\) resp. \(b\) denote the number of white resp. black regions in the checkerboard coloring. Form the “black graph” \(B\) of the knot projection, whose vertices correspond to the black regions and whose edges correspond to double-points in the knot projection. Recall that the Alexander polynomial of a knot can be interpreted as a suitable count of spanning trees of \(B\), where each tree is weighted by some \(T\)-power, and a sign. Moreover, the number of distinct \(T\)-powers appearing this polynomial is bounded above by the number double-points in the knot projection plus one, which in turn is given by \(w + b - 1\). Recall that the main step in the Crowell-Murasugi theorem shows that for an alternating projection, the trees contributing a fixed \(T\)-power contribute with the same sign.

According to a result of Crowell [5], the total number of such trees for a prime, alternating knot is bounded below by \(1 + (w - 1)(b - 1)\). Thus, according to our hypothesis that \(|a_i| \leq 1\), no two trees can contribute to the same \(T\)-power, and hence

\[
1 + (w - 1)(b - 1) \leq w + b - 1.
\]

This inequality immediately forces either \(w = b = 3\) or at least one of \(w\) or \(b\) = 2. In the case where \(w = b = 3\), it is easy to see that the knot in question is the figure eight knot, whose Alexander polynomial does not satisfy the hypotheses of the theorem. In the case where \(w\) or \(b\) = 2, it is easy to see that \(K\) is the \((2,2n+1)\) torus knot.

**Proof of Theorem 1.5.** Put together Corollary 1.3 and Proposition 4.1.
5. Berge knots are fibered

The aim of this section is to verify Proposition 1.8. This result seems to be known to the experts, but we include a proof here for completeness.

According to a theorem of Stallings [26], a connected three-manifold $Y$ with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$ is fibered if and only if the kernel of the Abelianization map

$$\pi_1(Y) \to H_1(Y; \mathbb{Z})$$

is a finitely generated group. We use this characterization to show that all knots coming from Berge’s construction (c.f. Definition 1.7) are fibered. Specifically, the knot complements arising from Berge’s constructions have Heegaard genus two, and thus their fundamental group admits a presentation with two generators and one relator

$$G = \langle X, Y \rangle / R(X, Y).$$

A theorem of Brown [3] concerns conditions under which a homomorphism

$$\chi: G \to \mathbb{R}$$

has finitely generated kernel. Specifically, write the word

$$R(X, Y) = A_1 \cdots A_m,$$

where $A_i \in \{X, Y, X^{-1}, Y^{-1}\}$, and then consider the sequence of real numbers

$$S = \left\{ \chi \left( \prod_{i=1}^{n} A_i \right) \right\}_{n=1}^{m},$$

then Brown’s theorem states that the kernel of $\chi$ is finitely generated if the sequence $S$ achieves its maximum and minimum only once. We will apply this condition to the Abelianization map in the following proof of Proposition 1.8:

**Proof of Proposition 1.8.** Berge’s construction gives a knot for each generator for the homology of $H_1(L(p, q); \mathbb{Z})$, also giving rise to a genus two Heegaard diagram for the knot complement $L(p, q) - K$. Explicitly, we start with a genus one Heegaard diagram for $L(p, q)$: the Heegaard surface is given as a square torus, and $\alpha$ is a straight line with slope $p/q$, while $\beta_1$ is given as a line with slope zero. Now, fix an integer $0 < k < p$ which is relatively prime to $p$, and draw a segment with slope zero which is disjoint from $\beta_1$ and which intersects $\alpha$ in $k$ points. After attaching a one-handle to the torus at near the endpoints of the arc, we can close up the arc to give a closed circle $\beta_2$ in the surface of genus two which continues to meet $\alpha$ in $k$ points and is disjoint from $\beta_1$. The associated Heegaard diagram is easily seen to represent a knot complement $L(p, q) - K$, where $K$ is a lens space Berge knot in the sense of Definition 1.7 representing a homology class which is $k$ times a generator for $H_1(L(p, q), \mathbb{Z})$.

This description can be used to give a presentation of the fundamental group of $Y = L(p, q) - K$. Of course, the description gives $Y$ as a genus two handlebody and an attached disk, and hence $\pi_1(Y)$ as a group with two generators and one relation. This
description can be given explicitly: let $X$ and $Y$ be the curves dual to the attaching disks for $\beta_1$ and $\beta_2$. The relation arising from the attaching disk $\alpha$ is found by following the curve $\alpha$, and recording in order which of the curves $\beta_1$ and $\beta_2$ are encountered – with $\beta_1$ contributing a factor of $X^\pm$ and $\beta_2$ contributing a factor of $Y^\pm$, where here the exponent is given by the local intersection number of $\alpha$ with the corresponding $\beta$-curve. In fact, for the curves coming from Berge’s construction, the obtained relator has the following simple form. Let

$$E(i) = E(i, p, q, k) = \begin{cases} 1 & \text{if there is some integer } 0 \leq j < k \text{ with } j \equiv i \cdot q \pmod{p} \\ 0 & \text{otherwise,} \end{cases}$$

then the presentation of $G = \pi_1(L(p, q) - K)$ given by the above procedure is

$$G \cong \langle X, Y \rangle / \prod_{i=1}^{p} (XY^{E(i, p, q, k)}).$$

See Figure 2 for an example.

It follows that $G/[G, G]$ is the lattice spanned by $[X]$ and $[Y]$, modulo the relation $p[X] + k[Y] = 0$. Thus, Abelianization can be viewed as a map

$$\chi: G \rightarrow \mathbb{Z}$$

which sends $[X]$ to $-k$ and $[Y]$ to $p$.

Now, our relator $R(X, Y) = \prod_{i=1}^{p}(XY^{E(i, p, q, k)})$ contains $k$ instances of $Y$; explicitly, writing $R(X, Y) = \prod_{i=1}^{m} A_i$, there is a sequence of distinct integers $\{n_i\}_{i=1}^{k}$ with the property that $A_{n_i} = Y$. It is easy to see that the maxima of the sequence $S$ described above are achieved amongst the $k$ words of the form $\{w_i = A_1 \cdot ... \cdot A_{n_i}\}_{i=1}^{k}$. Moreover, it is straightforward to see that

$$\chi(w_i) \equiv i \cdot p \pmod{k},$$

and hence, since $(k, p) = 1$, these $k$ values are distinct, showing uniqueness of the maximum. Similarly, the minima are achieved on the $k$ words $\{u_i = A_1 \cdot ... \cdot A_{n_i-1}\}$. Once again, these $k$ values $\chi(u_i) = \chi(w_i) - p$ are distinct modulo $k$, and hence the minimum is uniquely achieved. It follows now from Brown’s theorem [3] that the kernel of the Abelianization map is finitely generated, and hence according to Stallings’ theorem [26] that the knot complement is fibered. □
Figure 2. The (3, 4) torus knot. We draw here the Heegaard diagram for a Berge knot in $L(11, 2)$. This drawing takes place in a square torus (i.e. make the usual identifications on this square), with an additional one-handle added along the two hollow circles. The $\alpha$-curve is the diagonal curve with slope $11/2$, $\beta_1$ is the long horizontal dashed line, and $\beta_2$ has an arc indicated by the other dashed line, which then closes up inside the attached handle. Tracing along $\alpha$, we see that the fundamental group of $L(11, 2) - K$ is generated by elements $X$ and $Y$ satisfying the relation $XYXYX^5YXYX^3 = e$. Indeed, the Heegaard diagram we obtain in this manner describes the complement of the (3, 4) torus knot $T_{3,4}$ in $S^3$ (corresponding to the fact that $+11$ surgery on $T_{3,4}$ gives $-L(11, 2)$).
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