On limiting characteristics for a non-stationary two-processor heterogeneous system

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1 Introduction

In this paper we study a non-stationary Markovian queueing model of a two-processor heterogeneous system with time-varying arrival and service rates which was firstly investigated in [21], see also time-dependent analysis of this model in the recent paper [20]. In general, non-stationary queueing models have been actively studied during some decades, see, for instance [3, 5, 6, 11, 16, 18] and the references therein.

In the paper [20] the authors deal with the so-called “time-dependent analysis”, in other words, they try to find the state probabilities on a finite interval under some initial conditions (as a rule, initially, the number of customers in the queue is zero), see for instance [2]. Another approach is connected with the determination of the limiting mode, see [1].

Essentially more information about queue-length process can be obtained using ergodicity and the corresponding estimates of the rate of convergence. A general approach to obtaining sharp bounds on the rate of convergence via the notion of the logarithmic norm of an operator function was recently discussed in details in our papers [23, 24, 25]. The first studies in this direction were published since 1980-s for birth-death models, see [14, 15]. In [23] we proved that there are four classes of Markovian queueing models for which the reduced forward Kolmogorov system can be transformed to

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the system with essentially nonnegative matrix. Although the model under consideration does not belong to one of these classes, we can apply the same approach and obtain some useful bounds on the rate of convergence for it. Moreover, we can compute the limiting characteristics of the model using bounds on the rate of convergence and truncations technique introduced in [17, 22].

Note an interesting fact: exact estimates of the rate of convergence yield exact estimates of stability (perturbation bounds), see [7, 8, 9, 10, 13, 19] and references therein.

An important feature of multiprocessor queueing systems is the presence of risks related to the overload of the system. In the present paper it is demonstrated that under natural conditions on the arrival/service rates these risks vanish and the system rather easily approaches the ergodic mode.

2 Description of the model

Here we consider a multiprocessor system consisting of two types of processors, which for convenience will be referred to as the “main” and “backup” processors [20]. Each job requires exactly one processor for its execution. When both processors are idle, the main processor is scheduled for service before the backup processor. A computer system consists of two processors, a main processor, and a backup processor. A description of the model is as follows:

(i) jobs arrive at the system according to the Poisson process with an arrival rate $\lambda(t)$. Service is exponentially distributed, and two servers provide heterogeneous service rates $\mu_1(t)$, $\mu_2(t)$ such that $\mu_2(t) \leq \mu_1(t)$.

(ii) each job needs only one server to be served and the jobs select the servers on the basis of fastest server first (FSF).

![Figure 1: Transitions for a two-processor heterogeneous model](image-url)
The probabilistic dynamics of the process is represented by the forward Kolmogorov system of differential equations:

$$\frac{dp}{dt} = A(t)p, \quad (1)$$

where \( p = (p_{00}, p_{01}, p_{10}, p_{11}, \ldots, p_{1n}, \ldots)^T, \)

\[
A(t) = \begin{pmatrix}
  -\lambda & \mu_1 & \mu_2 & 0 & 0 & \cdots \\
  \lambda & -(\lambda + \mu_1) & 0 & \mu_2 & 0 & \cdots \\
  0 & 0 & -(\lambda + \mu_2) & \mu_1 & 0 & \cdots \\
  0 & \lambda & \lambda & -(\lambda + \mu) & \mu & \cdots \\
  0 & 0 & 0 & \lambda & -(\lambda + \mu) & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}, \quad (2)
\]

where \( \mu(t) = \mu_1(t) + \mu_2(t), \quad A(t) = Q^T(t), \) and \( Q(t) \) - the intensity matrix.

### 3 Bounds on the rate of convergence

Since \( p_{00}(t) = 1 - p_{01}(t) - \sum_{j=0}^{\infty} p_{1j}(t) \) due to the normalization condition, the system \((1)\) can be rewritten as

$$\frac{dz}{dt} = B(t)z + f(t), \quad (3)$$

where

\[
f(t) = (\lambda, 0, \ldots, 0, \ldots)^T, \quad z(t) = (p_{10}, p_{01}, p_{11}, \ldots, p_{1n}, \ldots)^T,
\]

and \( B(t) = (b_{ij}(t))_{i,j=1}^{\infty} =

\[
= \begin{pmatrix}
  -(2\lambda + \mu_1) & -\lambda & \mu_2 - \lambda & -\lambda & -\lambda & \cdots \\
  0 & -(\lambda + \mu_2) & \mu_1 & 0 & 0 & \cdots \\
  \lambda & \lambda & -(\lambda + \mu) & \mu & 0 & \cdots \\
  0 & 0 & \lambda & -(\lambda + \mu) & \mu & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}. \quad (4)
\]
Denote by $T$ the upper triangular matrix
\[ T = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}. \tag{5} \]

Consider the matrix $TB(t)T^{-1} =$
\[ = \begin{pmatrix}
-(\lambda + \mu_1) & \mu_1 - \mu_2 & \mu_2 & 0 & 0 & \cdots \\
\lambda & -(\lambda + \mu_2) & 0 & \mu_2 & 0 & \cdots \\
\lambda & 0 & -(\lambda + \mu) & \mu & 0 & \cdots \\
0 & 0 & \lambda & -(\lambda + \mu) & \mu & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \tag{6} \]

Let $\{d_i\}, i \geq 1$, be a sequence of positive numbers such that
\[ d_1 = 1, \ d_2 = \epsilon, \ d_3 = 1, \ d_4 = \delta_1 > 1, \ \frac{d_5}{d_4} = \frac{d_6}{d_5} = \ldots = \delta > 1. \]

Let $D = \text{diag}(d_1, d_2, \ldots)$ be the corresponding diagonal matrix and $l_{1D}$ be a space of vectors $l_{1D} = \{x = (x_1, x_2, \ldots)/\|x\|_{1D} = \|Dx\|_1 < \infty\}$, where $D = DT$.

Consider the matrix $DTB(t)T^{-1}D^{-1} = DB(t)D^{-1} =$
\[ = \begin{pmatrix}
-(\lambda + \mu_1) & \frac{d_1}{d_2}(\mu_1 - \mu_2) & \frac{d_2}{d_3}\mu_2 & 0 & 0 & 0 & \cdots \\
\frac{d_2}{d_1}\lambda & -(\lambda + \mu_2) & 0 & \frac{d_2}{d_3}\mu_2 & 0 & 0 & \cdots \\
\frac{d_3}{d_1}\lambda & 0 & -(\lambda + \mu) & \frac{d_2}{d_4}\mu & 0 & 0 & \cdots \\
0 & 0 & \frac{d_4}{d_3}\lambda & -(\lambda + \mu) & \frac{d_3}{d_4}\mu & 0 & \cdots \\
0 & 0 & 0 & \frac{d_5}{d_4}\lambda & -(\lambda + \mu) & \frac{d_3}{d_4}\mu & \cdots \\
0 & 0 & 0 & 0 & \frac{d_6}{d_5}\lambda & -(\lambda + \mu) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \tag{7} \]
The approach used in this paper is based on the notion of the logarithmic norm of a linear operator function and the corresponding bounds of the Cauchy operator, see the detailed discussion, for instance, in [4]. Namely, if \( B(t), \ t \geq 0, \) is a one-parameter family of bounded linear operators on a Banach space \( B, \) then

\[
\gamma(B(t))_B = \lim_{h \to +0} \frac{\| I + hB(t) \| - 1}{h}
\]

(8)

is called the logarithmic norm of the operator \( B(t). \)

If \( B = l_1, \) then the operator \( B(t) \) is given by the matrix \( B(t) = (b_{ij}(t))_{i,j=0}^{\infty}, \ t \geq 0, \) and the logarithmic norm of \( B(t) \) can be found explicitly:

\[
\gamma(B(t))_{1D} = \gamma(DB(t)D^{-1})_1 = \sup_b \left( b_{jj}(t) + \sum_{i \neq j} |b_{ij}(t)| \right), \ t \geq 0.
\]

(9)

Hence the following bound on the rate of convergence holds:

\[
\| x(t) \| \leq e^{\int_0^t \gamma(B(\tau)) \, d\tau} \| x(0) \|,
\]

where \( x = z^* - z^{**} \) and \( x \) is the solution of the differential equation

\[
\frac{dx}{dt} = B(t)x,
\]

which we obtain instead of the system (3).

Let \( \alpha_i(t) \) be negative sums of the elements of corresponding columns for the matrix (7), such as:

\[
\alpha_1 = (\lambda + \mu_1) - \frac{d_2}{d_1} \lambda - \frac{d_4}{d_1} \lambda,
\]

\[
\alpha_2 = (\lambda + \mu_2) - \frac{d_1}{d_2} (\mu_1 - \mu_2),
\]

\[
\alpha_3 = (\lambda + \mu) - \frac{d_1}{d_3} \mu_2 - \frac{d_4}{d_3} \lambda,
\]

\[
\alpha_4 = (\lambda + \mu) - \frac{d_2}{d_4} \mu_2 - \frac{d_3}{d_4} \mu - \frac{d_5}{d_4} \lambda,
\]

\[
\alpha_5 = (\lambda + \mu) - \frac{d_4}{d_5} \mu - \frac{d_6}{d_5} \lambda, \quad \alpha_6 = (\lambda + \mu) - \frac{d_5}{d_6} \mu - \frac{d_7}{d_6} \lambda, \quad \ldots,
\]

where \( \alpha_5 = \alpha_6 = \ldots, \) since \( \frac{dx}{dt} = \frac{dx}{dt} = \frac{dx}{dt} = \ldots = \delta. \)

Then we obtain the logarithmic norm:

\[
\gamma(B(t))_{1D} = \gamma(DB(t)D^{-1})_1 = - \inf_{i \geq 1} (\alpha_i(t)) = - \min_{i \leq 5} (\alpha_i(t)).
\]

(10)
4 The case \( \mu_1 = \mu_2 \).

First, let \( \lambda, \mu_1 = \mu_2 \) be constant, \( 0 < \lambda < \mu = \mu_1 + \mu_2 \). Then the exact value of the decay parameter (or the spectral gap) for a simple birth-death process with intensities \( \lambda \) and \( \mu \) is well-known, namely, it equals \( \beta^* = (\sqrt{\lambda} - \sqrt{\mu})^2 \), see, e. g., [12], and the corresponding \( \delta = \sqrt{\frac{\mu}{\lambda}} \). Hence, we consider the same \( \delta \) and put \( d_2 = \epsilon << 1, \delta_1 = \delta \).

Then
\[
\alpha_1 = \frac{\mu}{2} - \epsilon \lambda, \\
\alpha_2 = \lambda + \frac{\mu}{2}, \\
\alpha_3 = \frac{\mu}{2} + \lambda - \sqrt{\lambda \mu}, \\
\alpha_4 = (\sqrt{\lambda} - \sqrt{\mu})^2 - \frac{\epsilon}{\sqrt{\lambda \mu}}, \\
\alpha_k = (\sqrt{\lambda} - \sqrt{\mu})^2, \quad k \geq 5,
\]
Put \( \beta^* = \min(\alpha_1, \alpha_3, \alpha_4) \). Then we have
\[
\gamma(B(t))_{1D} = -\inf_i \alpha_i(t) = -\beta^*(t). \tag{11}
\]

Hence the following bound holds:
\[
||x(t)||_{1D} \leq e^{-\beta^* t}||x(0)||_{1D}. \tag{12}
\]

Let now the intensities \( \lambda(t), \mu_1(t) = \mu_2(t) \) be 1-periodic. Put
\[
\mu_* = \int_0^1 \mu(t) dt, \quad \lambda_* = \int_0^1 \lambda(t) dt.
\]
Then the best possible bound for a pure birth-death process is attained, if we take \( \delta = \sqrt{\frac{\mu_*}{\lambda_*}} \).

Then for these \( \delta, d_2 = \epsilon << 1, \delta_1 = \delta \) we have
\[
\alpha_1(t) = \frac{\mu(t)}{2} - \epsilon \lambda(t), \quad \alpha_2(t) = \lambda(t) + \frac{\mu(t)}{2}, \tag{13}
\]
\[
\alpha_3(t) = \frac{\mu(t)}{2} + \lambda(t) - \sqrt{\lambda(t)\mu(t)}, \tag{14}
\]
\[
\alpha_4(t) = (\sqrt{\lambda(t)} - \sqrt{\mu(t)})^2 - \frac{\epsilon}{\sqrt{\lambda(t)\mu(t)}}, \tag{15}
\]
\[
\alpha_k(t) = (\sqrt{\lambda(t)} - \sqrt{\mu(t)})^2, \quad k \geq 5. \tag{16}
\]
Put $\beta_*(t) = \min(\alpha_1(t), \alpha_3(t), \alpha_4(t))$. We have
\[
\gamma(B(t))_{1D} = -\inf \alpha_i(t) = -\beta_*(t).
\] (17)
Hence the following bound holds:
\[
\|x(t)\|_{1D} \leq e^{-\int_0^t \beta_*(\tau) d\tau}\|x(0)\|_{1D}.
\] (18)

**Remark 1.** It should be noted that in [20] there are some misprints in the plots, namely, the intensities must have a multiplier $\pi$, say $1 + \sin 2\pi t$. Moreover, on Fig 3 of that paper the sum of all probabilities evidently is greater than 1.

**Remark 2.** It can be seen that actually the periodic terms in the intensities do not affect the rate of convergence, see the plots related to the examples. Hence it is essentially easier to find the parameter $\beta_0$ for the corresponding homogeneous model. Namely, if we put
\[
\alpha_{10} = \frac{\mu_1}{2} - \epsilon \lambda_s, \quad \alpha_{20} = \lambda_s + \frac{\mu_1}{2}, \quad \alpha_{30} = \frac{\mu_1}{2} + \lambda_s - \sqrt{\lambda_s \mu_1},
\]
\[
\alpha_{40} = (\sqrt{\lambda_s} - \sqrt{\mu_1})^2 - \frac{\epsilon}{2} \sqrt{\lambda_s \mu_1}, \quad \alpha_{k0} = (\sqrt{\lambda_s} - \sqrt{\mu_1})^2, \quad k \geq 5,
\] (19)
instead of (13)-(16), then we obtain $\beta_{0*} = \min(\alpha_{10}, \alpha_{30}, \alpha_{40})$, and the bound on the rate of convergence in the form
\[
\|x(t)\|_{1D} \leq N e^{-\beta_{0*} t}\|x(0)\|_{1D},
\] (21)
for some positive $N$.

**Remark 3.** In the following examples we consider the behavior of the 'first' state probabilities $P_{00}(t)$, $P_{01}(t)$, $P_{10}(t)$, $P_{11}(t)$, $P_{21}(t)$, $P_{31}(t)$ and the mathematical expectation (the mean) of the queue-length process $E(t) = p01 + p10 + 2*p11 + 3*p21 + 4*p31 + \ldots$. One can see that for all examples the rates of convergence for original model with 1-periodic intensities and for the corresponding homogeneous model are the same, as it was noted in Remark 2.

5 Examples

**Example 1.** Let $\mu_1 = \mu_2 = 2, \lambda = 1 + \sin 2\pi t, \lambda_s = 1$ (Example 2 from [20]). Put $\delta = \sqrt{\frac{\mu}{\lambda_s}} = 2$. Then we obtain
\[
\alpha_1(t) = 2 - \epsilon(1 + \sin 2\pi t) \geq 2 - 2\epsilon,
\]
\[ \alpha_3 = 3 + \sin 2\pi t - 2\sqrt{1 + \sin 2\pi t} = 1 + \left(1 - \sqrt{1 + \sin 2\pi t}\right)^2 \geq 1, \]
\[ \alpha_4 = \left(2 - \sqrt{1 + \sin 2\pi t}\right)^2 - \epsilon \sqrt{1 + \sin 2\pi t} \geq \left(2 - \sqrt{2}\right)^2 - \epsilon \sqrt{2} \geq 0.3, \]
for sufficiently small \( \epsilon \). Therefore, we have \( \beta_* = 0.3 \). Thus, we can obtain the following bound
\[ \|p^*(t) - p^{**}(t)\|_1 \leq 2\|z^*(t) - z^{**}(t)\|_1 \leq 4\|z^*(t) - z^{**}(t)\|_{1D} \leq 4e^{-0.3t}\|z^*(0) - z^{**}(0)\|_{1D}. \]

On the other hand, we can obtain a simpler bound by applying Remark 2. Namely, \( \alpha_{10} = 2 - \epsilon, \alpha_{30} = 1, \alpha_{40} = 1 - \epsilon, \beta_{*0} = 1 - \epsilon, \) and
\[ \|p^*(t) - p^{**}(t)\|_1 \leq 4Ne^{-(1-\epsilon)t}\|z^*(0) - z^{**}(0)\|_{1D}. \]

Now, applying our standard truncations technique, see the detailed discussion and bounds in [17, 22], we can find the limiting characteristics of the queue-length process, the respective plots are shown in pictures X-Y.

**Example 2.** Let \( \mu_1 = \mu_2 = 2, \lambda = 3(1 + \sin 2\pi t), \lambda_* = 3 \) (Example 3 from [20]). Put \( \delta = \sqrt{\frac{\mu_1}{\lambda_*}} = \frac{2}{\sqrt{3}} \). Then we obtain \( \alpha_1 = 2 - 3\epsilon(1 + \sin 2\pi t), \alpha_3 = 5 + 3\sin 2\pi t - 2\sqrt{3(1 + \sin 2\pi t)}, \alpha_4 = 7 + 3\sin 2\pi t + (4 - \epsilon)\sqrt{3(1 + \sin 2\pi t)}. \)

On the other hand, using Remark 2 we have simple corresponding bounds: \( \alpha_{10} = 2 - 3\epsilon, \alpha_{30} = 5 - 2\sqrt{3}, \alpha_{40} = 7 - (4 - \epsilon)\sqrt{3} \geq 7 - 4\sqrt{3} \geq 0.07. \) Hence \( \beta_{*0} = 0.07. \)

Thus we can obtain the following bound
\[ \|p^*(t) - p^{**}(t)\|_1 \leq 4Ne^{-0.07t}\|z^*(0) - z^{**}(0)\|_{1D}. \]

6 The case \( \mu_1 > \mu_2. \)

First, let the intensities be constant. Put \( \mu_1(t) = (1 + \chi)\mu_2(t), \) where for \( \chi > 0. \) Then we have
\[ \alpha_1 = (1 + \chi)\mu_2 - \epsilon\lambda, \]
\[ \alpha_2 = \lambda + \mu_2 \left(1 - \frac{\chi}{\epsilon}\right), \]
\[ \alpha_3 = \lambda(1 - \delta_1) + (1 + \chi)\mu_2, \]
\[ \alpha_4 = \lambda(1 - \delta) + \mu_2 \left(2 + \chi - \frac{2 + \epsilon + \chi}{\delta_1}\right), \]
and
\[ \alpha_k = \lambda(1 - \delta) + \mu_2 \left(1 - \frac{1}{\delta}\right)(2 + \chi), \quad k \geq 5. \]
Put $\beta_* = \min_{i \leq 4}(\alpha_i)$. Then we have
\[ \gamma(B(t))_{1D} = -\min(\alpha_i(t)) = -\beta_. \] 
Hence, the following bound on the rate of convergence holds:
\[ \|p^*(t) - p^{**}(t)\|_1 \leq 4e^{-\beta_* t}\|z^*(0) - z^{**}(0)\|_{1D}. \] 
(22)

Let now the intensities $\lambda(t), \mu_1(t) = (1 + \chi)\mu_2(t)$ be 1-periodic. Put
\[ \mu_2* = \int_0^1 \mu_2(t) \, dt, \quad \lambda_* = \int_0^1 \lambda(t) \, dt. \]

Then, in accordance with Remark 2, we can find the corresponding parameter $\beta_{0*}$ for the respective homogeneous model. Namely, we have $\beta_{0*} = \min_{i \leq 4}(\alpha_{0i})$, and the bound on the rate of convergence (21) for some positive $N$.

**Example 3.** Let $\mu_1(t) = 6(1 + \cos 2\pi t), \mu_2(t) = 5(1 + \cos 2\pi t), \lambda(t) = 8(1 + \sin 2\pi)$. (Example 1 from [20]). Then $\chi = 0.2, \mu_{2*} = 5, \mu_{1*} = 6, \lambda_* = 8$. Hence, we have
\[ \alpha_{10} = 6 - 8\epsilon, \quad \alpha_{20} = 13 - \frac{1}{\epsilon}, \]
\[ \alpha_{30} = 14 - 8\delta_1, \quad \alpha_{40} = 19 - 8\delta - \frac{11 + 5\epsilon}{\delta_1}, \]
and
\[ \alpha_{k0} = 8(1 - \delta) + 11 \left(1 - \frac{1}{\delta}\right), \quad k \geq 5. \]

As we have already noted, the best value of the bound is attained, when $\delta = \sqrt{\frac{7}{5}} = \sqrt{\frac{14}{8}}$, then
\[ \alpha_{k0} = \left(\sqrt{11} - \sqrt{8}\right)^2 \approx 0.2, \quad k \geq 5. \]

Now put $\epsilon = \frac{1}{12}$ and $\delta_1 = \frac{13}{8}$. Then $\alpha_{10} > 1, \alpha_{20} = 1, \alpha_{30} = 1$ and $\alpha_{40} = 19 - 11/14 > 1$.

Then we obtain
\[ \beta_{0*} = \inf_{i \geq 1} \alpha_{i0} = \alpha_{50} = \left(\sqrt{11} - \sqrt{8}\right)^2 > 0.2, \]
and the following bound on the rate of convergence holds:
\[ \|p^*(t) - p^{**}(t)\|_1 \leq 4Ne^{-0.2t}\|z^*(0) - z^{**}(0)\|_{1D}. \]
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Figure 2: Example 1. Approximation of the mean $E(t, k)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$ for original and homogeneous situations.
Figure 3: Example 1. Approximation of the mean $E(t, k)$ for $t \in [50, 51]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 4: Example 2. Approximation of the mean $E(t, k)$ for $t \in [0, 200]$ with initial conditions $X(0) = 0$ and $X(0) = 100$ for original and homogeneous situations.
Figure 5: Example 2. Approximation of the mean $E(t, k)$ for $t \in [200, 201]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 6: Example 3. Approximation of the mean $E(t, k)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$ for original and homogeneous situations.
Figure 7: Example 3. Approximation of the mean $E(t, k)$ for $t \in [50, 51]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 8: Example 1. Approximation of the probability $P_{00}(t)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 9: Example 1. Approximation of the probability $P_{01}(t)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 10: Example 1. Approximation of the probability $P_{10}(t)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 11: Example 1. Approximation of the probability $P_{11}(t)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 12: Example 2. Approximation of the probability $P_{00}(t)$ for $t \in [0, 200]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 13: Example 2. Approximation of the probability $P_{01}(t)$ for $t \in [0, 200]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
**Figure 14:** Example 2. Approximation of the probability $P_{10}(t)$ for $t \in [0, 200]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 15: Example 2. Approximation of the probability $P_{11}(t)$ for $t \in [0,200]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 

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Figure 16: Example 3. Approximation of the probability $P_{00}(t)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 17: Example 3. Approximation of the probability $P_{01}(t)$ for $t \in [0,50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 18: Example 3. Approximation of the probability $P_{10}(t)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 
Figure 19: Example 3. Approximation of the probability $P_{11}(t)$ for $t \in [0, 50]$ with initial conditions $X(0) = 0$ and $X(0) = 100$. 