CANAL HYPERSURFACES GENERATED BY PSEUDO NULL, PARTIALLY NULL AND NULL CURVES IN LORENTZ-MINKOWSKI 4-SPACE

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Abstract. In this paper, we obtain the parametric expressions of the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a pseudo null, partially null or null curves in $E_1^4$ and give their some geometric invariants such as unit normal vector fields, Gaussian curvatures and mean curvatures. Also, we construct some examples for these canal hypersurfaces and finally, we give some characterizations for tubular hypersurfaces in $E_1^4$.

1. GENERAL INFORMATION AND BASIC CONCEPTS

The class of surfaces formed by sweeping a sphere is called canal surfaces and they have been investigated by Monge in 1850. So, one can see a canal surface as the envelope of a moving sphere with varying radius, defined by the trajectory $\gamma(s)$ of its centers and a radius function $r(s)$. The canal surface is called a tubular surface or pipe surface if the radius function is constant. After investigating by Monge in 1850. So, one can see a canal surface as the envelope of a moving sphere.

A vector $\mathbf{u} \in E_1^4 - \{0\}$ is called spacelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$; timelike if $\langle \mathbf{u}, \mathbf{u} \rangle < 0$ and lightlike (null) if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. In particular, the vector $\mathbf{u} = 0$ is spacelike. Also, the norm of the vector $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$. A curve $\gamma(s)$ in $E_1^4$ is spacelike, timelike or lightlike (null), if all its velocity vectors $\gamma'(s)$ are spacelike, timelike or lightlike, respectively and a non-null (i.e. timelike or spacelike) curve has unit speed if $\langle \gamma'(s), \gamma'(s) \rangle = 1$. If $\langle \gamma''(s), \gamma''(s) \rangle = 1$, then the null curve $\gamma$ is parametrized by...
arclength function \( s \). Also, if the principal normal vector or the binormal vector of a spacelike curve \( \gamma(s) \) in \( E_1^4 \) is null, then the spacelike curve \( \gamma(s) \) is called a pseudo null curve or a partially null curve (for detail, one can see [9], [10], [30], [33], and etc.)

If \( \{F_1, F_2, F_3, F_4\} \) is the moving Frenet frame along a curve \( \gamma(s) \) in \( E_1^4 \) consisting of the unit tangent vector field, principal normal vector field, binormal vector field and trinormal vector field, then we have the following cases according to the causal character of \( \gamma(s) \) (see [9], [10], [30], [33], and etc):

\[
\begin{align*}
\text{i)} & \quad \text{If the curve } \gamma(s) \text{ is pseudo null, then the Frenet formulas are} \\
& \quad \left\{ \begin{array}{l}
F_1' = k_1 F_2, \\
F_2' = k_2 F_3, \\
F_3' = k_3 F_2 - k_2 F_4, \\
F_4' = -k_1 F_1 - k_3 F_3,
\end{array} \right. \\
& \quad \text{and we have} \\
& \quad \left\{ \begin{array}{l}
\langle F_1, F_1 \rangle = \langle F_3, F_3 \rangle = 1, \\
\langle F_2, F_2 \rangle = \langle F_4, F_4 \rangle = 0, \\
\langle F_3, F_4 \rangle = 1, \\
\langle F_2, F_3 \rangle = \langle F_2, F_4 \rangle = (F_2, F_3) = (F_3, F_4) = 0.
\end{array} \right. \\
& \quad \text{(1.3)}
\end{align*}
\]

\[
\begin{align*}
\text{ii)} & \quad \text{If the curve } \gamma(s) \text{ is partially null, then the Frenet formulas are} \\
& \quad \left\{ \begin{array}{l}
F_1' = k_1 F_2, \\
F_2' = -k_1 F_1 + k_2 F_3, \\
F_3' = k_3 F_2, \\
F_4' = -k_2 F_2 - k_3 F_4,
\end{array} \right. \\
& \quad \text{and we have} \\
& \quad \left\{ \begin{array}{l}
\langle F_1, F_1 \rangle = \langle F_2, F_2 \rangle = 1, \\
\langle F_3, F_3 \rangle = \langle F_4, F_4 \rangle = 0, \\
\langle F_3, F_4 \rangle = 1, \\
\langle F_2, F_3 \rangle = \langle F_2, F_4 \rangle = (F_2, F_3) = (F_3, F_4) = 0.
\end{array} \right. \\
& \quad \text{(1.5)}
\end{align*}
\]

\[
\begin{align*}
\text{iii)} & \quad \text{If the curve } \gamma(s) \text{ is null, then the Frenet formulas are} \\
& \quad \left\{ \begin{array}{l}
F_1' = k_1 F_2, \\
F_2' = k_2 F_1 - k_1 F_3, \\
F_3' = -k_2 F_2 + k_3 F_4, \\
F_4' = -k_3 F_1,
\end{array} \right. \\
& \quad \text{and we have} \\
& \quad \left\{ \begin{array}{l}
\langle F_2, F_2 \rangle = \langle F_4, F_4 \rangle = 1, \\
\langle F_1, F_1 \rangle = \langle F_3, F_3 \rangle = 0, \\
\langle F_1, F_3 \rangle = 1, \\
\langle F_1, F_2 \rangle = \langle F_3, F_4 \rangle = \langle F_2, F_4 \rangle = (F_2, F_3) = (F_3, F_4) = 0.
\end{array} \right. \\
& \quad \text{(1.7)}
\end{align*}
\]

Here, the first curvature \( k_1(s) = 0 \), if \( \gamma \) is a straight line and \( k_1(s) = 1 \) in all other cases for pseudo null and null curves. So, the pseudo null and null curves have two curvatures \( k_2(s) \) and \( k_3(s) \). Also, the third curvature \( k_3(s) = 0 \) for each \( s \) for a partially null curve and thus, a partially null curve has two curvatures \( k_1(s) \) and \( k_2(s) \).

Furthermore, if \( p \) is a fixed point in \( E_1^4 \) and \( r \) is a positive constant, then the pseudo-Riemannian hypersphere, pseudo-Riemannian hyperbolic space and pseudo-Riemannian null hypercone are defined by

\[
\begin{align*}
S^3_1(p, r) &= \{ u \in E_1^4 : \langle u - p, u - p \rangle = r^2 \}, \\
H^3_0(p, r) &= \{ u \in E_1^4 : \langle u - p, u - p \rangle = -r^2 \}, \\
Q^3_1 &= \{ u \in E_1^4 : \langle u - p, u - p \rangle = 0 \},
\end{align*}
\]

respectively.

In the present study, we construct the canal hypersurfaces in \( E_1^4 \) as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a pseudo null, partially null or null curve.
Furthermore, the differential geometry of different types of (hyper)surfaces in 4-dimensional spaces has been a popular topic for geometers, recently ([3], [4], [5], [6], [8], [23], [24], [25], [28], and etc). In this context, let $\Omega$ be a hypersurface in $E_4^1$ given by

$$
\Omega : U \subset E^3 \rightarrow E_4^1 \quad (u_1, u_2, u_3) \rightarrow \Omega(u_1, u_2, u_3) = (\Omega_1(u_1, u_2, u_3), \Omega_2(u_1, u_2, u_3), \Omega_3(u_1, u_2, u_3), \Omega_4(u_1, u_2, u_3)).
$$

Then the Gauss map (i.e., the unit normal vector field), the matrix forms of the first and second fundamental forms are

$$
N_\Omega = \frac{\Omega_{u_1} \times \Omega_{u_2} \times \Omega_{u_3}}{||\Omega_{u_1} \times \Omega_{u_2} \times \Omega_{u_3}||}, \quad [g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \quad \text{and} \quad [h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix},
$$

respectively. Here $g_{ij} = \langle \Omega_{u_i}, \Omega_{u_j} \rangle$, $h_{ij} = \langle \Omega_{u_i u_j}, N_\Omega \rangle$, $\Omega_{u_i} = \frac{\partial \Omega}{\partial u_i}$, $\Omega_{u_i u_j} = \frac{\partial^2 \Omega}{\partial u_i \partial u_j}$, $i, j \in \{1, 2, 3\}$.

Also, the matrix of shape operator of the hypersurface (1.10) is

$$
S = [a_{ij}] = [g^{ij}][h_{ij}],
$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. With the aid of (1.11) and (1.12), the Gaussian curvature and mean curvature of a hypersurface in $E_4^1$ are given by

$$
K = \varepsilon \frac{\det[h_{ij}]}{\det[g_{ij}]}, \quad 3\varepsilon H = \text{tr}(S),
$$

respectively. Here, $\varepsilon = \langle N_\Omega, N_\Omega \rangle$. We say that a hypersurface is flat or minimal, if it has zero Gaussian or zero mean curvature, respectively. For more details about hypersurfaces in $E_4^1$, we refer to [14], [31] and etc.

## 2. CANAL HYPERSURFACES GENERATED BY PSEUDO NULL, PARTIALLY NULL AND NULL CURVES IN $E_4^1$

In this section, firstly we construct the canal hypersurfaces that are formed as the envelope of a family of pseudo hyper-spheres or pseudo hyperbolic hyperspheres whose centers lie on a pseudo null curve, partially null curve or null curve in $E_4^1$. After that, we obtain some important geometric invariants such as unit normal vector fields, Gaussian curvatures and mean curvatures of these canal hypersurfaces, separately.

**Theorem 1.** The canal hypersurfaces that are formed as the envelope of a family of pseudo hyper-spheres whose centers lie on a pseudo null curve $\gamma(s)$ with Frenet vector fields $F_i$, $i \in \{1, 2, 3, 4\}$, in $E_4^1$ can be parametrized by

$$
\begin{align*}
C_1(s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{1 - r'(s)^2} \left( g(t, w) \sin(f(t, w))F_2(s) + \cos(f(t, w))F_3(s) + \frac{\sin(f(t, w))}{2g(t, w)}F_4(s) \right), \\
C_2(s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{1 - r'(s)^2} \left( g(t, w) \cos(f(t, w))F_2(s) + \sin(f(t, w))F_3(s) + \frac{\cos(f(t, w))}{2g(t, w)}F_4(s) \right), \\
C_3(s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{1 - r'(s)^2} \left( g(t, w) \sin(f(t, w))F_2(s) + \cosh(f(t, w))F_3(s) - \frac{\sinh(f(t, w))}{2g(t, w)}F_4(s) \right), \\
C_4(s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{r'(s)^2 - 1} \left( g(t, w) \cosh(f(t, w))F_2(s) + \sinh(f(t, w))F_3(s) - \frac{\cosh(f(t, w))}{2g(t, w)}F_4(s) \right),
\end{align*}
$$

(2.1)
where we suppose \( r'(s)^2 < 1 \) for canal hypersurfaces \( C_1, C_2, C_3 \) and \( r'(s)^2 > 1 \) for canal hypersurface \( C_4 \).

Furthermore, the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a pseudo null curve \( \gamma(s) \) with Frenet vector fields \( F_i, i \in \{1, 2, 3, 4\} \), in \( E_4^{\text{psd}} \) can be parametrized by

\[
Psd^{-n} C_5 (s, t, w) = \gamma(s) + r(s)r'(s)F_1(s)
\]

\[
\pm r(s) \sqrt{1 + r'(s)^2} \left( g(t, w) \cosh(f(t, w))F_2(s) + \sinh(f(t, w))F_3(s) - \frac{\cosh(f(t, w))}{2g(t, w)}F_4(s) \right).
\]

Here, the canal hypersurfaces \( C_1, C_2, C_3, C_4 \) are timelike and the canal hypersurface \( C_5 \) is spacelike.

**Proof.** Let the center curve \( \gamma : I \subseteq \mathbb{R} \to E_1^{\text{psd}} \) be a pseudo null curve with non-zero curvature with Frenet vector fields \( F_1(s), F_2(s), F_3(s), F_4(s) \) called unit tangent, principal normal, binormal and trinormal vectors of \( \gamma(s) \), respectively. Then, the parametrization of the envelope of pseudo hyperspheres (or pseudo hyperbolic hyperspheres) defining the canal hypersurfaces \( psd^{-n} C \) \((s, t, w)\) in \( E_4^{\text{psd}} \) can be given by

\[
psd^{-n} C \ (s, t, w) - \gamma(s) = a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) + a_3(s, t, w)F_3(s) + a_4(s, t, w)F_4(s),
\]

where \( a_i(s, t, w) \) are differentiable functions of \( s, t, w \) on the interval \( I \). Furthermore, since \( psd^{-n} C \) \((s, t, w)\) lies on the pseudo hyperspheres \((\lambda = 1)\) (or pseudo hyperbolic hyperspheres \((\lambda = -1)\)), we have

\[
g(\ psd^{-n} C \ (s, t, w) - \gamma(s), \ psd^{-n} C \ (s, t, w) - \gamma(s) ) = \lambda r^2(s)
\]

which leads to from \( (2.3) \) and \( (1.4) \) that

\[
a_1^2 + a_3^2 + 2a_2a_4 = \lambda r^2
\]

and

\[
a_1a_1s + a_3a_3s + a_4a_2s + a_2a_4s = \lambda rr_s,
\]

where \( r(s) \) is the radius function; \( r = r(s), r_s = \frac{dr(s)}{ds}, a_i = a_i(s, t, w), a_i_s = \frac{\partial a_i(s, t, w)}{\partial s} \).

So, differentiating \( (2.3) \) with respect to \( s \) and using the Frenet formula \( (1.3) \), we get

\[
\frac{psd^{-n} C}{ds} = (1 - a_4k_1 + a_1s) F_1 + (a_1k_1 + a_3k_3 + a_2s) F_2
\]

\[
+ (a_2k_2 - a_4k_3 + a_3s) F_3 + (-a_3k_2 + a_4s) F_4,
\]

where \( \frac{psd^{-n} C}{ds} = \frac{\partial (psd^{-n} C \ (s, t, w))}{\partial s} \). Furthermore, \( psd^{-n} C \ (s, t, w) - \gamma(s) \) is a normal vector to the canal hypersurfaces, which implies that

\[
g(\ psd^{-n} C \ (s, t, w) - \gamma(s), \ psd^{-n} C \ (s, t, w) - \gamma(s) ) = 0
\]

and so, from \( (2.3), (2.7) \) and \( (2.8) \) we have

\[
( a_1(1 - a_4k_1 + a_1s) + a_2(-a_3k_2 + a_4s)
+ a_3(a_2k_2 - a_4k_3 + a_3s) + a_4(a_1k_1 + a_3k_3 + a_2s) ) = 0.
\]

Using \( (2.6) \) in \( (2.9) \), we get

\[
a_1 = -\lambda rr_s.
\]

Hence, using \( (2.10) \) in \( (2.5) \), we reach that

\[
a_1^2 + 2a_2a_4 = \lambda r^2(1 - \lambda r_s^2).
\]
Therefore from (2.10) and (2.11), the canal hypersurfaces \( C (s, t, w) \) that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a pseudo null curve in \( E_1^4 \) can be parametrized by (2.1) or (2.2), respectively.

Here we must note that, from now on we will state \( r' = r_s(s), \, r'' = \frac{d^2r(s)}{ds^2}, \, f = f(t, w), \, g = g(t, w), \) \( \sin f = \sin(f(t, w)), \, f_t = \frac{\partial f(t, w)}{\partial t}, \, f_{tt} = \frac{\partial^2 f(t, w)}{\partial t^2}, \) and so on.

**Theorem 2.** The Gaussian and mean curvatures of the canal hypersurfaces \( C_{1i} \), \( i \in \{1, 2, 3, 4, 5\} \), given by (2.1) and (2.2) in \( E_1^4 \) are

\[
\begin{align*}
K_{p_{s-d-n}}^{C_{1i}} &= -r(1-r'^2)k_i^2 \sinh f + 4r''(1-r'^2-r''')g^2 + 2\sqrt{1-r'^2}(1-r'^2-r'')k_i g \sin f \\
&\quad r^2(r\sqrt{1-r'^2}k_i \sin f - 2(1-r'^2-r''')g)^2 \\
H_{p_{s-d-n}}^{C_{1i}} &= 2r(1-r'^2)k_i^2 \sinh f + 4r''(1-r'^2-r''')g^2 - 4\sqrt{1-r'^2}(1-r'^2-r'')k_i g \cos f \\
&\quad r^2(r\sqrt{1-r'^2}k_i \cos f - 2(1-r'^2-r''')g)^2 \\
K_{p_{s-d-n}}^{C_{2i}} &= -2r(1-r'^2)k_i^2 \cos^2 f - 4r''(1-r'^2-r''')g^2 - 2\sqrt{1-r'^2}(1-r'^2-r'')k_i \sinh f \\
&\quad r^2(r\sqrt{1-r'^2}k_i \sinh f - 2(1-r'^2-r''')g)^2 \\
H_{p_{s-d-n}}^{C_{2i}} &= -2r(1-r'^2)k_i^2 \cosh f + 4r''(1-r'^2-r''')g^2 + 2\sqrt{1-r'^2}(1-r'^2-r'')k_i \cosh f \\
&\quad r^2(r\sqrt{1-r'^2}k_i \cosh f - 2(1-r'^2-r''')g)^2 \\
K_{p_{s-d-n}}^{C_{3i}} &= -2r(1-r'^2)k_i^2 \sinh f + 4r''(1-r'^2-r''')g^2 + 2\sqrt{1-r'^2}(1-r'^2-r'')k_i g \sin f \\
&\quad r^2(r\sqrt{1-r'^2}k_i \sin f - 2(1-r'^2-r''')g)^2 \\
H_{p_{s-d-n}}^{C_{3i}} &= -2r(1-r'^2)k_i^2 \cosh f - 4r''(1-r'^2-r''')g^2 + 2\sqrt{1-r'^2}(1-r'^2-r'')k_i \cosh f \\
&\quad r^2(r\sqrt{1-r'^2}k_i \cosh f - 2(1-r'^2-r''')g)^2 \\
K_{p_{s-d-n}}^{C_{4i}} &= r(1+r'^2)k_i^2 \cosh^2 f - 4r''(1+r'^2+r'')g^2 + 2\sqrt{1-r'^2}(1+r'^2+r'')k_i \cosh f \\
&\quad r^2(r\sqrt{1+r'^2}k_i \cosh f - 2(1+r'^2+r''')g)^2 \\
H_{p_{s-d-n}}^{C_{4i}} &= -2r(1+r'^2)k_i^2 \cosh f + 4r''(1+r'^2+r'')g^2 + 2\sqrt{1-r'^2}(1+r'^2+r'')k_i \cosh f \\
&\quad r^2(r\sqrt{1+r'^2}k_i \cosh f - 2(1+r'^2+r''')g)^2 \\
K_{p_{s-d-n}}^{C_{5i}} &= r(1+r'^2)k_i^2 \cosh^2 f + 4r''(1+r'^2+r'')g^2 + 2\sqrt{1+r'^2}(1+r'^2+r'')k_i \cosh f \\
&\quad r^2(r\sqrt{1+r'^2}k_i \cosh f - 2(1+r'^2+r''')g)^2 \\
H_{p_{s-d-n}}^{C_{5i}} &= -2r(1+r'^2)k_i^2 \cosh f + 4r''(1+r'^2+r'')g^2 + 2\sqrt{1+r'^2}(1+r'^2+r'')k_i \cosh f \\
&\quad r^2(r\sqrt{1+r'^2}k_i \cosh f - 2(1+r'^2+r''')g)^2 \\
\end{align*}
\]

**Proof.** Here we will obtain the unit normal vector field, Gaussian and mean curvatures of the canal hypersurfaces \( C_{1i} \) \( (s, t, w) \) given by

\[
C_{1i} \quad (s, t, w) = \gamma(s) - r(s)r'(s)F_1(s) + r(s)\sqrt{1-r'(s)^2}\left( g(t, w) \sin(f(t, w))F_2(s) + \cos(f(t, w))F_3(s) + \frac{\sin(f(t, w))}{2g(t, w)}F_4(s) \right), \quad (2.13)
\]
in $E^4$. Firstly, from (1.13), the first derivatives of the canal hypersurface (2.13) are obtained as

\[
\begin{align*}
(C_1)_{s}^{\text{psd-n}} &= A_1 F_1 + A_2 F_2 + A_3 F_3 + A_4 F_4, \\
(C_1)_{t}^{\text{psd-n}} &= r \sqrt{1 - r''} \left( g f_t \cos f + g_t \sin f \right) F_2 - r \sqrt{1 - r''} f_t \sin f F_3 + \frac{r \sqrt{1 - r''}}{2g^2} \left( g f_t \cos f - g_t \sin f \right) F_4, \\
(C_1)_{w}^{\text{psd-n}} &= r \sqrt{1 - r''} \left( g f_w \cos f + g_w \sin f \right) F_2 - r \sqrt{1 - r''} f_w \sin f F_3 + \frac{r \sqrt{1 - r''}}{2g^2} \left( g f_w \cos f - g_w \sin f \right) F_4,
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= 1 - r'' - r \sqrt{1 - r''} k_1 \sin f, \\
A_2 &= \frac{1}{\sqrt{1 - r''}} \left( \frac{1}{2g^2} \left( r (1 - r'' - r) k_1 \cos f - r' \left( \frac{r}{2g^2} \cos f \right) \right) \right), \\
A_3 &= \frac{1}{\sqrt{1 - r''}} \left( \frac{r (1 - r'' - r) k_2 g_2 \cos f}{2g^2} \right), \\
A_4 &= -2r(1 - r')^2 k_2 g_2 \cos f - r' (1 - r'' - r) \sin f.
\end{align*}
\]

From (1.11) and 2.14-2.16, the unit normal vector field of $C_1$ in $E^4$ is

\[
N = -r' F_1 + \sqrt{1 - r''} \left( g \sin f F_2 + \cos f F_3 + \frac{\sin f}{2g} F_4 \right).
\]

and we get $\langle N, N \rangle = 1$. Also, the coefficients of the first fundamental form are given by

\[
\begin{align*}
g_{11} &= \frac{1}{4g^2 (1 - r'')^2} \left( (1 - r') \left( \frac{r}{2g^2} (1 - r'') k_1 \sin f - 2(1 - r' - r) g \right)^2 + (1 - r') \left( 2k_2 g^2 - k_3 \right) \sin f + 2r' \left( 1 - r' - r \right) \sin f \right)^2 + 4g^2 \left( r (1 - r') k_2 g \cos f \right), \\
g_{12} &= \frac{r^2}{2g^2} \left( 2r (1 - r') k_2 g \cos f + r' \left( 1 - r' - r \right) \sin f \right), \\
g_{13} &= \frac{r^2}{2g^2} \left( 2r (1 - r') k_2 g \cos f - r' \left( 1 - r' - r \right) \sin f \right), \\
g_{22} &= \frac{r^2}{g^2} \left( 1 - r' \right) \left( g^2 f_t^2 - g_t^2 \sin^2 f \right), \\
g_{23} &= \frac{r^2}{g^2} \left( 1 - r' \right) \left( g^2 f_t f_w - g_t g_w \sin^2 f \right), \\
g_{33} &= \frac{r^2}{g^2} \left( 1 - r' \right) \left( g^2 f_w^2 - g_w^2 \sin^2 f \right),
\end{align*}
\]

and it follows that

\[
\det[g_{ij}] = \frac{-r^4 (1 - r') \left( \frac{r}{2g^2} (1 - r'') k_1 \sin f - 2(1 - r' - r) g \right)^2 \left( g_w f_t - f_w g_t \right)^2 \sin^2 f}{4g^4}.
\]
Now, for obtaining the coefficients of the second fundamental form, let us give the second derivatives \((C)_{x,x_j}\) of the canal hypersurface (2.13):

\[
(C_1)_{ss} = B_1 F_1 + B_2 F_2 + B_3 F_3 + B_4 F_4, \tag{2.20}
\]

\[
(C_1)_{st} = (C_1)_{ts} = \frac{r}{2g^2} \left( \frac{r (1-r^2) k_3 f_i}{\sqrt{1-r^2}} \sin f - r' \left(1 - r^2 - rr' \right) (g f_t \cos f + g_t \sin f) \right) F_1 \tag{2.21}
\]

\[
(C_1)_{sw} = (C_1)_{ws} = \frac{r}{2g^2} \left( \frac{r (1-r^2) k_3 f_i}{\sqrt{1-r^2}} \sin f + r' \left(1 - r^2 - rr' \right) (g f_w \cos f + g_w \sin f) \right) F_1 + \frac{1}{\sqrt{1-r^2}} \left( \frac{r (1-r^2) k_3 f_i}{2g^2} \sin f \right) F_2 + \frac{1}{2\sqrt{1-r^2} g^2} \left( \frac{r (1-r^2) k_3 f_i}{2g^2} \sin f \right) F_3 + \frac{1}{2\sqrt{1-r^2} g^2} \left( \frac{r (1-r^2) k_3 f_i}{2g^2} \sin f \right) F_4 \tag{2.22}
\]

\[
(C_1)_{tt} = r \sqrt{1-r^2} \left( (2f_i g_t + g f_t) \cos f + (-g f_i^2 + g_t) \sin f \right) F_2 \tag{2.23}
\]

\[
- r \sqrt{1-r^2} \left( f_i^2 \cos f + f_t \sin f \right) F_3
\]

\[
- \frac{r}{2g^3} \left( g (2f_i g_t - g f_t) \cos f + (-2g_i^2 + g (g f_i^2 + g_t)) \sin f \right) F_4
\]

\[
(C_1)_{tw} = (C_1)_{wt} = r \sqrt{1-r^2} \left( (f_i g_w + f_w g_t + g f_tw) \cos f + (-g f_tw + g_t) \sin f \right) F_2 \tag{2.24}
\]

\[
- r \sqrt{1-r^2} \left( f_i f_w \cos f + f_tw \sin f \right) F_3
\]

\[
- \frac{r}{2g^3} \left( g (f_i g_w + f_w g_t - g f_tw) \cos f + (-2g_i g_w + g (g f_i f_w + g_tw)) \sin f \right) F_4
\]

and

\[
(C_1)_{ww} = r \sqrt{1-r^2} \left( (2f_w g_w + g f_{ww}) \cos f + (-g f_i^2 + g_{ww}) \sin f \right) F_2 \tag{2.25}
\]

\[
- r \sqrt{1-r^2} \left( f_w^2 \cos f + f_{ww} \sin f \right) F_3
\]

\[
- \frac{r}{2g^3} \left( g (2f_w g_w - g f_{ww}) \cos f + (-2g_i^2 + g (g f_i^2 + g_{ww})) \sin f \right) F_4.
\]
Thus, from (1.11), (2.17) and (2.20)-(2.25), the coefficient \( s \) of the second fundamental form are given
obtained and so using the shape operator, we get the mean curvature
\[
8 M. ALTIN, A. KAZAN AND D.W. YOON
\]
\[
B_1 = \frac{-r\sqrt{1-r'^2}k_1' \sin f}{2g} + \frac{k_1(r(1-r'^2)k_2g \cos f - r'(1-r'^2 - rr'') \sin f)}{g\sqrt{1-r'^2}} - 3r'r'' - rr'',
\]
\[
B_2 = \frac{-r\sqrt{1-r'^2}k_1' \sin f}{2g} + (1 - 2r'^2 - 2rr'') k_1
\[
- \frac{1}{2g(1-r'^2)^{3/2}} \begin{pmatrix}
(1 - r'^2) \right) k_3^2 \sin f + 2 (1 - r'^2) g \left( \frac{rr' \sqrt{1-r'^2} k_1'}{2rr' - 2rr''} \right) \cos f \\
-2g^2 \left( r(1-r'^2)^2 k_2 k_3 + r'' (1-4r'^2 + 3r'^4 - rr'') - rr'' (1-r'^2) \right) \sin f
\end{pmatrix},
\]
\[
B_3 = \frac{1}{2g(1-r'^2)^{3/2}} \begin{pmatrix}
2 (1-r'^2) (-1-r'^2) (rk_2' + 2r'k_2 + 2rr''k_2) g^2 \sin f \\
-2g^2 \left( -r(1-r'^2) k_2 \left( r' \sqrt{1-r'^2} k_1 - 2 (1-r'^2) k_3 \cos f \right) \right)
\end{pmatrix},
\]
\[
B_4 = \frac{1}{2g(1-r'^2)^{3/2}} \begin{pmatrix}
-2 (1-r'^2) (-1-r'^2) (rk_2' + 2r'k_2 + 2rr''k_2) g \cos f \\
+ 2r^2 (1-r'^2)^2 k_2 g^2 - r(1-r'^2)^2 k_3^2 k_1 k_3 - 2 (1-r'^2) + 3r^4 - rr'' (1-r'^2) \right) \sin f
\end{pmatrix}.
\]

Thus, from (1.11), (2.17) and (2.20)-(2.25), the coefficients of the second fundamental form are given by

\[
h_{11} = \frac{-1}{4g^2(1-r'^2)} \begin{pmatrix}
(1 - r'^2)^2 k_1^2 \sin f^2 + 4r (1 - r'^2)^2 k_2^2 g^4 \sin^2 f + r(1-r'^2)^2 k_3^2 \sin^2 f \\
-2g^2 (r(1-r'^2)^2 k_2 k_3 (3 + \cos(2f)) + 2r''(1-r'^2 - rr'')) \\
+2\sqrt{1-r'^2} k_1 g (4rr' - 1 - 2rr'') k_2 g \cos f - (1-r'^2 - 2rr'') \sin f
\end{pmatrix},
\]
\[
h_{12} = h_{21} = \frac{-1}{2g^2} \begin{pmatrix}
-2 (1-r'^2) k_2 g^2 f_1 - \left( r' \sqrt{1-r'^2} k_1 \cos f - (1-r'^2) k_3 \right) g_f f_t \\
- (1-r'^2) k_2 g^2 g_t \sin(2f) + \left( r' \sqrt{1-r'^2} k_1 - (1-r'^2) k_3 \cos f \right) g_t f_t \sin f
\end{pmatrix},
\]
\[
h_{13} = h_{31} = \frac{-1}{2g^2} \begin{pmatrix}
-2 (1-r'^2) k_2 g^2 f_w - \left( r' \sqrt{1-r'^2} k_1 \cos f - (1-r'^2) k_3 \right) g_f w \\
- (1-r'^2) k_2 g^2 g_w \sin(2f) + \left( r' \sqrt{1-r'^2} k_1 - (1-r'^2) k_3 \cos f \right) g_w f_w \sin f
\end{pmatrix},
\]
\[
h_{22} = \frac{r(1-r'^2)(g_f^2 \sin^2 f - g^2 f_t^2)}{g^2},
\]
\[
h_{23} = h_{32} = \frac{r(1-r'^2)(g_w g^2 \sin^2 f - g^2 f_t f_w)}{g^2},
\]
\[
h_{33} = \frac{r(1-r'^2)(g_w^2 \sin^2 f - g^2 f_w^2)}{g^2}
\]
and it implies
\[
\det[h_{ij}] = \frac{r^2 (1-r'^2)}{4g^4} \begin{pmatrix}
r(1-r'^2) k_1^2 \sin^2 f - 4r'^2 (1-r'^2 - rr'') g^2 \\
-2\sqrt{1-r'^2} (1-r'^2 - 2rr'') k_1 g \sin f
\end{pmatrix} (f_t g_w - f_w g_t)^2 \sin^2 f. \tag{2.27}
\]

So, from (1.13), (2.19) and (2.27), we obtain the Gaussian curvature \( K_{pad-n} \) as given in (2.12).

Also, using (2.18) and (2.26) in (1.12), the shape operator of the canal hypersurface (2.13) can be obtained and so using the shape operator, we get the mean curvature \( H_{pad-n} \) as given in (2.12).
Similarly, the Gaussian and mean curvatures $K_{psd-n} \ C_i$ and $H_{psd-n} \ C_i$ of the canal hypersurfaces $C_i$, $i \in \{2, 3, 4, 5\}$, given by (2.12) can be obtained.

Here, from (2.12), we can state the following theorem which gives an important relation between Gaussian and mean curvatures of the canal hypersurfaces:

**Proposition 1.** The Gaussian curvatures and the mean curvatures of the canal hypersurfaces $C_1$, $C_3$ and $C_4$ given by (2.1) in $E^4_1$ satisfy

$$3H_{psd-n} \ C_i - r^2K_{psd-n} \ C_i + \frac{2}{r} = 0, \ i = 1, 3, 4 \quad (2.28)$$

and the Gaussian curvature and the mean curvature of the canal hypersurfaces $C_2$ and $C_5$ given by (2.1) and (2.2) in $E^4_1$ satisfy

$$3H_{psd-n} \ C_i - r^2K_{psd-n} \ C_i - \frac{2}{r} = 0, \ i = 2, 5. \quad (2.29)$$

Now, we will give some results for the canal hypersurface $C_1$ given by (2.1) in $E^4_1$. Similarly, one can obtain similar results for the canal hypersurfaces $C_2$, $C_3$, $C_4$ and $C_5$ given by (2.1) and (2.2) in $E^4_1$, too.

**Proposition 2.** Let the pseudo null curve $\gamma(s)$, which generates the canal hypersurface $C_1$ given by (2.1) in $E^4_1$, be a straight line. Then, the canal hypersurface $C_1$ is flat, if $r(s) = as + b, a, b \in \mathbb{R}$, $a \neq \pm 1$.

*Proof.* If the pseudo null curve $\gamma(s)$ is a straight line, then we have $k_1(s) = 0$. So from (2.12), we have

$$K_{psd-n} \ C_1 = \frac{r''}{r^2(1 - r^2 - rr'')}. \quad (2.30)$$

If we use $r(s) = as + b, a, b \in \mathbb{R}$, $a \neq \pm 1$ in (2.30), then $K_{psd-n} \ C_1$ vanishes and this completes the proof.

**Proposition 3.** Let the pseudo null curve $\gamma(s)$, which generates the canal hypersurface $C_1$ given by (2.1) in $E^4_1$, not be a straight line. If $g(t, w) \neq \sin(f(t, w))$, then the canal hypersurface $C_1$ cannot be flat.

Also, if $g(t, w) = \sin(f(t, w))$, then the canal hypersurface $C_1$ is flat when the equation

$$2(1 - r^2) \left(\sqrt{1 - r^2} + 2r''\right) - r \left(1 - r^2 + 4r'' \left(\sqrt{1 - r^2} + r''\right)\right) = 0 \quad (2.31)$$

holds.

*Proof.* If the pseudo null curve $\gamma(s)$, which generates the canal hypersurface $C_1$ given by (2.1) in $E^4_1$, isn’t a straight line, then we have $k_1(s) = 1$. So, from (2.12) we get

$$K_{psd-n} \ C_1 = -r(1 - r^2) \sin^2 f + 4r''(1 - r^2 - rr'')g^2 + 2\sqrt{1 - r^2}(1 - r^2 - 2rr'')g \sin f. \quad (2.32)$$

From (2.32), if the canal hypersurface $C_1$ is flat, then we have

$$-r(1 - r^2) \sin^2 f + 4r''(1 - r^2 - rr'')g^2 + 2\sqrt{1 - r^2}(1 - r^2 - 2rr'')g \sin f = 0. \quad (2.33)$$
Firstly, let us suppose \( g(t, w) \neq \sin(f(t, w)) \). Since the set \( \{\sin^2 f, g^2, g \sin f\} \) is linearly independent, we get from (2.33) that
\[
-r(1 - r^2) = 4r''(1 - r^2 - 2rr'') = 2\sqrt{1 - r^2}(1 - r^2 - 2rr'') = 0 \tag{2.34}
\]
and this is a contradiction. Thus, \( C_1 \) cannot be flat in this situation and this proves the first part of this Proposition.

Secondly, if \( g(t, w) = \sin(f(t, w)) \) holds, then from (2.32) we have
\[
K_{psd-n}^{C_1} = \frac{2 (1 - r^2) \left( \sqrt{1 - r^2 + 2r''} - r \left( 1 - r^2 + 4r'' \left( \sqrt{1 - r^2 + r''} \right) \right) \right)}{r^2 \left( 2 - 2r^2 - r \left( \sqrt{1 - r^2 + 2r''} \right) \right)^2} \tag{2.35}
\]
and this completes the proof. \( \square \)

**Proposition 4.** Let the pseudo null curve \( \gamma(s) \), which generates the canal hypersurface \( C_1 \) given by (2.1) in \( E_1^4 \), be a straight line. Then, the canal hypersurface \( C_1 \) is minimal, if \( r(s) \) satisfies
\[
\int \frac{dr}{\sqrt{1 - (4r')^2}} = \pm s + b, \quad a, b \in \mathbb{R}.
\]

**Proof.** If the pseudo null curve \( \gamma(s) \) is a straight line, from (2.12), we have
\[
H_{psd-n}^{C_1} = -\frac{2 - 2r^2 - 3rr''}{3r \left( 1 - r^2 - rr'' \right)}. \tag{2.36}
\]
So, if the equation
\[
2 - 2r^2 - 3rr'' = 0 \tag{2.37}
\]
holds, then the canal hypersurface \( C_1 \) is minimal. The solution of the equation (2.37) can be found in [23] and this completes the proof. \( \square \)

**Proposition 5.** Let the pseudo null curve \( \gamma(s) \), which generates the canal hypersurface \( C_1 \) given by (2.1) in \( E_1^4 \), not be a straight line. If \( g(t, w) \neq \sin(f(t, w)) \), then the canal hypersurface \( C_1 \) cannot be minimal.

Also, if \( g(t, w) = \sin(f(t, w)) \), then the canal hypersurface \( C_1 \) is minimal when the equation
\[
8 \left( 1 - r^2 \right)^2 - 2r \left( 1 - r^2 \right) \left( \sqrt{1 - r^2 + 10r''} \right) - 3r^2 \left( 1 - r^2 - 4r'' \right) = 0 \tag{2.38}
\]
holds.

**Proof.** If the pseudo null curve \( \gamma(s) \), which generates the canal hypersurface \( C_1 \) given by (2.1) in \( E_1^4 \), isn’t a straight line, then from (2.12) we get
\[
H_{psd-n}^{C_1} = \frac{2r \left( 1 - r^2 \right)^{3/2} g \sin f + 3r^2 \left( 1 - r^2 \right) \sin^2 f - 4 \left( 1 - r^2 - rr'' \right) \left( 2 - 2r^2 - 3rr'' \right) g^2}{3 \left( -r^3 \left( 1 - r^2 \right) \sin^2 f + 4r \left( 1 - r^2 - rr'' \right) \right) ^2 g^2}. \tag{2.39}
\]
From (2.39), if the canal hypersurface \( C_1 \) is minimal, then we have
\[
2r \left( 1 - r^2 \right)^{3/2} g \sin f + 3r^2 \left( 1 - r^2 \right) \sin^2 f - 4 \left( 1 - r^2 - rr'' \right) \left( 2 - 2r^2 - 3rr'' \right) g^2 = 0. \tag{2.40}
\]
Firstly, let us suppose that \( g(t, w) \neq \sin(f(t, w)) \). Since the set \( \{g \sin f, \sin^2 f, g^2\} \) is linearly independent, we get from (2.40) that
\[
2r \left( 1 - r^2 \right)^{3/2} = 3r^2 \left( 1 - r^2 \right) = 4 \left( 1 - r^2 - rr'' \right) \left( 2 - 2r^2 - 3rr'' \right) = 0 \tag{2.41}
\]
and this is a contradiction. Thus, \( C_1 \) cannot be minimal in this situation and this proves the first part of this Proposition.

Secondly, if \( g(t, w) = \sin(f(t, w)) \) holds, then we have

\[
H_{\text{psd}^{-n}} = -\frac{8 \left(1 - r'^2\right)^2 - 2r \left(1 - r'^2\right) \left(\sqrt{1 - r'^2} + 10r''\right) - 3r^2 \left(1 - r'^2 - 4r''^2\right)}{3r \left(4 \left(1 - r'^2\right)^2 - 8r \left(1 - r'^2\right) r'' - r^2 \left(1 - r'^2 - 4r''^2\right)\right)}
\]  

(2.42)

and this completes the proof.

Here, let us construct an example for the canal hypersurface \( C_1 \) \((s,t,w)\) that is formed as the envelope of a family of pseudo hyperspheres whose center lie on a pseudo null curve in \( E^4_1 \). One can construct the canal hypersurfaces \( C_2, C_3, C_4 \) and \( C_5 \), similarly.

**Example 1.** Let us take the pseudo null curve (given in \( [18] \))

\[
\gamma(s) = \frac{1}{2\sqrt{2}} (\cosh(2s), \sinh(2s), \sin(2s), -\cos(2s))
\]

(2.43)

in \( E^4_1 \). The Frenet vectors and curvatures of the curve (2.43) are

\[
\begin{align*}
F_1 &= \frac{1}{\sqrt{2}} (\sinh(2s), \cosh(2s), \cos(2s), \sin(2s)), \\
F_2 &= \frac{1}{\sqrt{2}} (\cosh(2s), \sinh(2s), -\sin(2s), \cos(2s)), \\
F_3 &= \frac{1}{\sqrt{2}} (\sinh(2s), \cosh(2s), -\cos(2s), -\sin(2s)), \\
F_4 &= \frac{1}{2\sqrt{2}} (-\cosh(2s), -\sinh(2s), -\sin(2s), \cos(2s)), \\
k_1 &= 1, \ k_2 = 4, \ k_3 = 0.
\end{align*}
\]

(2.44)

If we assume that \( g(t, w) = t \) and \( f(t, w) = w \) in (2.1), the canal hypersurfaces \( C_1 \) \((s,t,w)\) can be parametrized by

\[
C_1^{\text{psd}^{-n}} (s,t,w) = \frac{1}{16\sqrt{2}t} \left(\begin{array}{l}
\cosh(2s) (8t + \sqrt{3}s (-1 + 8t^2) \sin w) + 4st (-1 + \sqrt{3} \cos w) \sinh(2s), \\
\sinh(2s) (8t + \sqrt{3}s (-1 + 8t^2) \sin w) + 4st (-1 + \sqrt{3} \cos w) \cosh(2s), \\
\sin(2s) (8t - \sqrt{3}s (1 + 8t^2) \sin w) + 4st (-1 - \sqrt{3} \cos w) \cos(2s), \\
\cos(2s) (-8t + \sqrt{3}s (1 + 8t^2) \sin w) + 4st (-1 - \sqrt{3} \cos w) \sin(2s)
\end{array}\right)
\]

(2.45)

where the radius function has been taken as \( r(s) = \frac{s}{2} \). From (2.12), the Gaussian and mean curvatures of the canal hypersurfaces \( C_1^{\text{psd}^{-n}} \) are obtained as

\[
\begin{align*}
K_{C_1^{\text{psd}^{-n}}} &= -\frac{8}{s^2 \left(s - 2\sqrt{3} \csc w\right)}, \\
H_{C_1^{\text{psd}^{-n}}} &= \frac{-4st^2 + 2s^2 \left(2\sqrt{3} + 3s \sin w\right) \sin w}{36st^2 - 3s^2 \sin^2 w}.
\end{align*}
\]

(2.46)

In Figure 1 (a), one can see the projection of the canal hypersurfaces (2.45) for \( w = \frac{\pi}{3} \) into \( x_1 x_3 x_4 \)-space.

**Theorem 3.** The canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a partially null curve \( \gamma(s) \) with Frenet vector fields \( F_i, i \in \{1,2,3,4\} \), in
$E_1^4$ can be parametrized by

$$
\begin{align}
\text{part-} n 
C_1 (s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{1 - r'(s)^2} \left( \sin(f(t, w))F_2(s) + g(t, w)\cos(f(t, w))F_3(s) + \frac{\cos(f(t, w))}{2g(t, w)}F_4(s) \right), \\
\text{part-} n 
C_2 (s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{1 - r'(s)^2} \left( \cos(f(t, w))F_2(s) + g(t, w)\sin(f(t, w))F_3(s) + \frac{\sin(f(t, w))}{2g(t, w)}F_4(s) \right), \\
\text{part-} n 
C_3 (s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{1 - r'(s)^2} \left( \cosh(f(t, w))F_2(s) + g(t, w)\sinh(f(t, w))F_3(s) + \frac{\sinh(f(t, w))}{2g(t, w)}F_4(s) \right), \\
\text{part-} n 
C_4 (s, t, w) &= \gamma(s) - r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{r'(s)^2 - 1} \left( \sinh(f(t, w))F_2(s) + g(t, w)\cosh(f(t, w))F_3(s) + \frac{\cosh(f(t, w))}{2g(t, w)}F_4(s) \right), \\
\end{align}
$$

where we suppose $r'(s)^2 < 1$ for canal hypersurfaces $C_1, C_2, C_3$ and $r'(s)^2 > 1$ for canal hypersurface $C_4$.

Furthermore, the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a partially null curve $\gamma(s)$ with Frenet vector fields $F_i$, $i \in \{1, 2, 3, 4\}$, in $E_1^4$ can be parametrized by

$$
\begin{align}
\text{part-} n 
C_5 (s, t, w) &= \gamma(s) + r(s)r'(s)F_1(s) \\
&\pm r(s)\sqrt{1 + r'(s)^2} \left( \sinh(f(t, w))F_2(s) + g(t, w)\cosh(f(t, w))F_3(s) - \frac{\cosh(f(t, w))}{2g(t, w)}F_4(s) \right). \\
\end{align}
$$

Here, the canal hypersurfaces $C_1, C_2, C_3, C_4$ are timelike and the canal hypersurface $C_5$ is spacelike.

**Proof.** Let the center curve $\gamma : I \subseteq \mathbb{R} \to E_1^4$ be a partially null curve with non-zero curvature with Frenet vector fields. Then, the parametrization of the envelope of pseudo hyperspheres (or pseudo hyperbolic hyperspheres) defining the canal hypersurfaces $X(s, t, w)$ in $E_1^4$ can be given by

$$
\text{part-} n 
C (s, t, w) - \gamma(s) = a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) + a_3(s, t, w)F_3(s) + a_4(s, t, w)F_4(s). \\
$$

Furthermore, since $C(s, t, w)$ lies on the pseudo hyperspheres ($\lambda = 1$) (or pseudo hyperbolic hyperspheres ($\lambda = -1$)), we have

$$
g(\text{part-} n \ C (s, t, w) - \gamma(s), \text{part-} n \ C (s, t, w) - \gamma(s)) = \lambda r^2(s)
$$

which leads to from (2.49) and (1.6) that

$$
a_1^2 + a_2^2 + 2a_3a_4 = \lambda r^2
$$

and

$$
a_1a_1 + a_2a_2 + a_4a_4 = \lambda rr.s.
$$

So, differentiating (2.49) with respect to $s$ and using the Frenet formula (1.5), we get

$$
(\text{part-} n \ C)_s = (1 - a_2k_1 + a_1s)F_1 + (a_1k_1 - a_4k_2 + a_2s)F_2 \\
+ (a_2k_2 + a_3k_3 + a_4s)F_3 + (-a_4k_3 + a_4s)F_4.
$$

Furthermore, $C(s, t, w) - \gamma(s)$ is a normal vector to the canal hypersurfaces, which implies that

$$
g(\text{part-} n \ C (s, t, w) - \gamma(s), (\text{part-} n \ C)_s(s, t, w)) = 0
$$
and so, from (2.49), (2.53) and (2.55) we have
\[
\left( a_1 (1 - a_2 k_1 + a_1 s) + a_2 (a_1 k_1 - a_4 k_2 + a_2 s) + a_3 (-a_4 k_3 + a_4 s) + a_4 (a_2 k_2 + a_3 k_3 + a_3 s) \right) = 0. \tag{2.55}
\]

Using (2.52) in (2.56), we get
\[
a_1 = -\lambda \nu r s. \tag{2.56}
\]

Hence, using (2.56) in (2.51), we reach that
\[
a_1^2 + a_2^2 + 2a_3 a_4 = \lambda r^2
\]
\[
a_1^2 + 2a_3 a_4 = \lambda r^2 (1 - \nu r_0^2). \tag{2.57}
\]

Therefore from (2.56) and (2.57), the canal hypersurfaces \( C_{part-n} \) \((s, t, w)\) that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a partially null curve in \( E^4 \) can be parametrized by (2.47) or (2.48), respectively.

Using the similar procedure with proof of the Theorem 2, we can obtain the following Theorem:

**Theorem 4.** The Gaussian and mean curvatures of the canal hypersurfaces \( C_{part-n} \), \( i \in \{1, 2, 3, 4\} \), given by (2.47) and (2.48) in \( E^4 \) are

\[
\begin{align*}
K_{part-n} & = -r \left(1-r^2\right) k_1^2 \sin^2 f + r^2 \left(1-r^2-r''\right) \sin f + \sqrt{1-r^2} \left(1-r^2-r''\right) k_1 \sin f \left(r^2 - r^2 - r'' - \sqrt{1-r^2} k_1 \sin f\right) \right) \right) \right) \right) \right) \right), \\
H_{part-n} & = r \left(1-r^2\right) k_1^2 \sin f + 3r^2 \left(1-r^2\right) k_1^2 \sin f \left(1-r^2-r''\right) \left(2-2r^2-3r''\right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right), \tag{2.58}
\end{align*}
\]

Here, from (2.58), we can state the following theorem which gives an important relation between Gaussian and mean curvatures of the canal hypersurfaces:

**Proposition 6.** The Gaussian curvatures and the mean curvatures of the canal hypersurfaces \( C_{part-n} \), \( C_3 \) and \( C_4 \) given by (2.47) in \( E^4 \) satisfy

\[
3H_{part-n} - r^2 K_{part-n} + \frac{2}{r} = 0, \quad i = 1, 3, 4 \tag{2.59}
\]
and the Gaussian curvature and the mean curvature of canal hypersurfaces $C_2$ and $C_5$ given by (2.47) and (2.48) in $E^4_1$ satisfy

$$3H_{part-n} - r^2 K_{part-n} - \frac{2}{r} = 0, \ i = 2, 5.$$  \hspace{1cm} (2.60)

Now, we will give some results for the canal hypersurface $C_5$ given by (2.48) in $E^4_1$. Similarly, one can obtain similar results for the canal hypersurfaces $C_1$, $C_2$, $C_3$ and $C_4$ given by (2.37) in $E^4_1$, too.

**Proposition 7.** The canal hypersurface $C_5$ that is formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a partially null curve $\gamma(s)$ in $E^4_1$ is flat, if $k_1(s) = 0$ and the radius function satisfies $r(s) = as + b$, $a, b \in \mathbb{R}$.

**Proof.** If we use $k_1(s) = 0$ in (2.58), then we have

$$K_{part-n} = \frac{r''}{r^2 (1 + r'^2 + rr'')} \hspace{1cm} (2.61)$$

and this completes the proof. \hfill \Box

**Proposition 8.** The canal hypersurface $C_5$ that is formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a partially null curve $\gamma(s)$ in $E^4_1$ cannot be flat when $k_1(s) \neq 0$.

**Proof.** Let the canal hypersurface $C_5$ be flat. Then from (2.58), we get

$$r \left(1 + r'^2\right) k_1^2 \sinh^2 f + r'' \left(1 + r'^2 + rr''\right) - \sqrt{1 + r'^2} \left(1 + r'^2 + 2rr''\right) k_1 \sinh f = 0. \hspace{1cm} (2.62)$$

Since the set $\{\sinh f, \sinh^2 f, 1\}$ is linearly independent, we get from (2.62) that

$$r \left(1 + r'^2\right) k_1^2 = r'' \left(1 + r'^2 + rr''\right) = \sqrt{1 + r'^2} \left(1 + r'^2 + 2rr''\right) k_1 = 0 \hspace{1cm} (2.63)$$

and this is a contradiction when $k_1(s) \neq 0$. So, this completes the proof. \hfill \Box

**Proposition 9.** The canal hypersurface $C_5$ that is formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a partially null curve $\gamma(s)$ in $E^4_1$ is minimal, if $k_1(s) = 0$ and the radius function $r(s)$ satisfies

$$\int \frac{dr}{\sqrt{\left(\frac{\dot{r}}{s}\right)^2 - 1}} = \pm s + b, \ a, b \in \mathbb{R}. \hspace{1cm} \text{(2.58)}$$

**Proof.** If we use $k_1(s) = 0$ in (2.58), then we have

$$H_{part-n} = \frac{2 + 2r'^2 + 3rr''}{3r(1 + r'^2 + rr'')} \hspace{1cm} (2.64)$$

So, if the equation

$$2 + 2r'^2 + 3rr'' = 0 \hspace{1cm} \text{(2.65)}$$

holds, then the canal hypersurface $C_5$ is minimal.

Now, let us solve the equation (2.65). If we take $r'(s) = h(s)$, we get

$$r'' = h' = \frac{dh}{dr} \frac{dr}{ds} = \frac{dh}{dr}. \hspace{1cm} (2.66)$$

Using (2.66) in (2.65), we have

$$3r \frac{dh}{dr} h + 2h^2 + 2 = 0. \hspace{1cm} (2.67)$$
From (2.65), \( r'(s) = h(s) \neq 0 \) and so we reach that
\[
\frac{-3h}{2(1+h^2)} dh = \frac{dr}{r}. \tag{2.68}
\]
By integrating (2.68), we have
\[
h = \pm \sqrt{\left( \frac{a}{r} \right)^{\frac{2}{3}} - 1}, \tag{2.69}
\]
where \( a \) is constant. Since \( r' = \frac{dr}{ds} = h \), from (2.69) we get
\[
\int \frac{dr}{\sqrt{\left( \frac{a}{r} \right)^{\frac{2}{3}} - 1}} = \pm \int ds \tag{2.70}
\]
and this completes the proof. \( \square \)

**Proposition 10.** The canal hypersurface \( C_5 \) that is formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a partially null curve \( \gamma(s) \) in \( E_1^4 \) cannot be minimal when \( k_1(s) \neq 0 \).

**Proof.** The proof can be done with similar method used in the proof of Proposition 8 \( \square \)

Here, let us construct an example for the canal hypersurface \( C_5 \) \((s,t,w)\) that is formed as the envelope of a family of pseudo hyperspheres whose center lie on a partially null curve in \( E_1^4 \). One can construct the canal hypersurfaces \( C_1, C_2, C_3 \) and \( C_4 \), similarly.

**Example 2.** Let us take the partially null curve (given in [8])
\[
\gamma(s) = \frac{1}{2} (2e^s, 2e^s, \cos(2s), \sin(2s)) \tag{2.71}
\]
in \( E_1^4 \). The Frenet vectors and curvatures of the curve (2.43) are
\[
F_1 = (e^s, e^s, -\sin(2s), \cos(2s)), \\
F_2 = \frac{1}{4} (e^s, e^s, -2\cos(2s), -2\sin(2s)), \\
F_3 = \frac{1}{5} (1, 1, 0, 0), \\
F_4 = \left( -\frac{e^{2s}}{4} - \frac{1}{5}, \frac{e^{2s}}{4} + \frac{1}{5}, \frac{e^s}{5}(\cos(2s) + 2\sin(2s)), \frac{e^s}{5}(\sin(2s) - 2\cos(2s)) \right), \\
k_1 = 2, k_2 = e^s, k_3 = 0. \tag{2.72}
\]
If we assume that \( g(t,w) = t \) and \( f(t,w) = w \) in (2.48), the canal hypersurfaces \( C_5 \) \((s,t,w)\) can be parametrized by
\[
C_5 \left( s, t, w \right) = \left( \begin{array}{c}
\frac{s(4+5e^{2s}+100t^2)}{32\sqrt{5t}} \cosh w + \frac{e^s}{8} \left( 8 + 2s + \sqrt{5s} \sinh w \right), \\
\frac{s(-4+5e^{2s}+100t^2)}{32\sqrt{5t}} \cosh w + \frac{e^s}{8} \left( 8 + 2s + \sqrt{5s} \sinh w \right), \\
\frac{1}{4} \left( 2 \cos(2s) - s \sin(2s) + \sqrt{5s} \left( \frac{e^s}{10t} \cosh w (\cos(2s) + 2\sin(2s)) - \cos(2s) \sinh w \right) \right), \\
\frac{1}{4} \left( 2 \sin(2s) + s \cos(2s) + \sqrt{5s} \left( \frac{e^s}{10t} \cosh w (\sin(2s) - 2\cos(2s)) - \sin(2s) \sinh w \right) \right),
\end{array} \right) \tag{2.73}
\]
where the radius function has been taken as \( \frac{r}{s} \). From (2.55), the Gaussian and mean curvatures of the canal hypersurfaces \( C_5 \) are obtained as
\[
\begin{align*}
K_{\text{part.-n}} &= -\frac{16 \sinh w}{s^2(\sqrt{5} - 2s \sinh w)}, \\
H_{\text{part.-n}} &= -\frac{20 - 4s(\sqrt{5} + 6s \sinh w) \sinh w}{3s(5 + 4s^2 \sinh^2 w)}. \tag{2.74}
\end{align*}
\]
Theorem 5. The canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a null curve $\gamma(s)$ with Frenet vector fields $F_i$, $i \in \{1, 2, 3, 4\}$, in $E_4^1$ can be parametrized by
\[
\begin{align*}
C_1(s, t, w) &= \gamma(s) + a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) - r(s)r'(s)F_3(s) + a_4(s, t, w)F_4(s), \tag{2.75}
\end{align*}
\]
where
\[
a_2(s, t, w)^2 + a_4(s, t, w)^2 = r(s) \left( r(s) + 2a_1(s, t, w)r'(s) \right).
\]
Furthermore, the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a null curve $\gamma(s)$ with Frenet vector fields $F_i$, $i \in \{1, 2, 3, 4\}$, in $E_4^1$ can be parametrized by
\[
\begin{align*}
C_2(s, t, w) &= \gamma(s) + a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) + r(s)r'(s)F_3(s) + a_4(s, t, w)F_4(s). \tag{2.76}
\end{align*}
\]
where
\[
a_2(s, t, w)^2 + a_4(s, t, w)^2 = -r(s) \left( r(s) + 2a_1(s, t, w)r'(s) \right).
\]
Proof. Let the center curve $\gamma : I \subseteq \mathbb{R} \to E_4^1$ be a null curve with non-zero curvature with Frenet vector fields. Then, the parametrization of the envelope of pseudo hyperspheres (or pseudo hyperbolic hyperspheres) defining the canal hypersurfaces $C(s, t, w)$ in $E_4^1$ can be given by
\[
\begin{align*}
C(s, t, w) - \gamma(s) &= a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) + a_3(s, t, w)F_3(s) + a_4(s, t, w)F_4(s). \tag{2.77}
\end{align*}
\]
Furthermore, since $X(s, t, w)$ lies on the pseudo hyperspheres ($\lambda = 1$) (or pseudo hyperbolic hyperspheres ($\lambda = -1$)), we have
\[
\begin{align*}
\begin{align*}
g(C(s, t, w) - \gamma(s), C(s, t, w) - \gamma(s)) &= \lambda r^2(s) \tag{2.78}
\end{align*}
\end{align*}
\]
which leads to from (2.77) and (1.8) that
\[
a_2^2 + a_4^2 + 2a_1a_3 = \lambda r^2 \tag{2.79}
\]
and
\[
a_2a_{2s} + a_4a_{4s} + a_1a_{3s} + a_3a_{1s} = \lambda rr_s. \tag{2.80}
\]
So, differentiating (2.77) with respect to $s$ and using the Frenet formula (1.7), we get
\[
\begin{align*}
(\dot{C})_s &= (1 + a_2k_2 - a_4k_3 + a_{1s})F_1 + (a_1k_1 - a_3k_2 + a_{2s})F_2 \tag{2.81}
\end{align*}
\]
\[
\begin{align*}
+ (-a_2k_1 + a_{3s})F_3 + (a_3k_3 + a_{4s})F_4.
\end{align*}
\]
Furthermore, $C(s, t, w) - \gamma(s)$ is a normal vector to the canal hypersurfaces, which implies that
\[
\begin{align*}
g(C(s, t, w) - \gamma(s), (\dot{C})_s(s, t, w)) &= 0 \tag{2.82}
\end{align*}
\]
and so, from (2.71), (2.81) and (2.82) we have
\[
\begin{align*}
\begin{align*}
\begin{pmatrix}
 a_1 (-a_2k_1 + a_{3s}) + a_2 (a_1k_1 - a_3k_2 + a_{2s}) + a_3 (1 + a_2k_2 - a_4k_3 + a_{1s}) + a_4 (a_3k_3 + a_{4s})
\end{pmatrix} &= 0. \tag{2.83}
\end{align*}
\end{align*}
\]
Using (2.80) in (2.83), we get
\[
a_3 = -\lambda rr_s. \tag{2.84}
\]
Hence, using (2.81) in (2.79), we reach that
\[
a_2^2 + a_4^2 = \lambda r(r + 2a_1r_s). \tag{2.85}
\]
Therefore from (2.84) and (2.85), the canal hypersurfaces \( C(s, t, w) \) that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a null curve in \( E^4_1 \) can be parametrized by (2.75) or (2.76), respectively. □

Here, let us construct an example for the canal hypersurface \( C_1(s, t, w) \) that is formed as the envelope of a family of pseudo hyperspheres whose center lie on a null curve in \( E^4_1 \). One can construct the canal hypersurface \( C_2 \), similarly.

**Example 3.** Let us take the null curve (given in [1])

\[
\gamma(s) = \frac{1}{\sqrt{2}} (\sinh s, \cosh s, \sin s, \cos s) \tag{2.86}
\]

in \( E^4_1 \). The Frenet vectors and curvatures of the curve (2.43) are

\[
\begin{align*}
F_1 &= \frac{1}{\sqrt{2}} (\cosh s, \sinh s, \cos s, -\sin s), \\
F_2 &= \frac{1}{\sqrt{2}} (\sinh s, \cosh s, -\sin s, -\cos s), \\
F_3 &= \frac{1}{\sqrt{2}} (-\cosh s, -\sinh s, \cos s, -\sin s), \\
F_4 &= \frac{1}{\sqrt{2}} (\sinh s, \cosh s, \sin s, \cos s), \\
k_1 &= 1, \quad k_2 = 0, \quad k_3 = -1.
\end{align*}
\tag{2.87}
\]

If we assume that \( g(t, w) = t \) and \( f(t, w) = w \) in (2.48), the canal hypersurfaces \( C_1(s, t, w) \) can be parametrized by

\[
C_1(s, t, w) = \frac{1}{8\sqrt{2}t}
\begin{pmatrix}
-2st (1 + \sqrt{3} \cos w) \cosh s + (8t + \sqrt{3}s (1 + 2t^2) \sin w) \sinh s, \\
-2st (1 + \sqrt{3} \cos w) \sinh s + (8t + \sqrt{3}s (1 + 2t^2) \sin w) \cosh s, \\
-2st (1 - \sqrt{3} \cos w) \cos s + (8t + \sqrt{3}s (1 - 2t^2) \sin w) \sin s, \\
-2st (1 - \sqrt{3} \cos w) \sin s + (8t + \sqrt{3}s (1 - 2t^2) \sin w) \cos s
\end{pmatrix}
\tag{2.88}
\]

where the radius function has been taken as \( r(s) = \frac{s}{2} \).

In Figure 1 (c), one can see the projection of the canal hypersurfaces (2.88) for \( w = \frac{\pi}{3} \) into \( x_1 x_3 x_4 \)-space.
3. TUBULAR HYPERSURFACES GENERATED BY PSEUDO-NULL, PARTIALLY NULL AND NULL CURVES IN $E_4$

In this section, by taking the radius function $r(s) = r$ is constant in the Section 2, firstly we give the parametric expressions of the tubular hypersurfaces which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a pseudo-null curve, partially null curve or null curve in $E_4$. After that, we obtain some results for Weingarten tubular hypersurfaces with the aid of the Gaussian curvatures and mean curvatures of these tubular hypersurfaces.

**Theorem 6.** The tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on pseudo null curves with Frenet vector fields $F_i$, $i \in \{1, 2, 3, 4\}$, in $E_4$ can be parametrized by

$$
\begin{align*}
T_{1}(s, t, w) &= \gamma(s) \pm r \left( g(t, w) \sin(f(t, w))F_2(s) + \cos(f(t, w))F_3(s) + \frac{\sin(f(t, w))}{2g(t, w)} F_4(s) \right), \\
T_{2}(s, t, w) &= \gamma(s) \pm r \left( g(t, w) \cos(f(t, w))F_2(s) + \sin(f(t, w))F_3(s) + \frac{\cos(f(t, w))}{2g(t, w)} F_4(s) \right), \\
T_{3}(s, t, w) &= \gamma(s) \pm r \left( g(t, w) \sinh(f(t, w))F_2(s) + \cosh(f(t, w))F_3(s) - \frac{\sinh(f(t, w))}{2g(t, w)} F_4(s) \right). \\
\end{align*}
$$

(3.1)

Furthermore, the tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on pseudo null curves with Frenet vector fields $F_i$, $i \in \{1, 2, 3, 4\}$, in $E_4$ can be parametrized by

$$
\begin{align*}
T_{4}(s, t, w) &= \gamma(s) \pm r \left( g(t, w) \cosh(f(t, w))F_2(s) + \sinh(f(t, w))F_3(s) - \frac{\cosh(f(t, w))}{2g(t, w)} F_4(s) \right).
\end{align*}
$$

(3.2)

**Theorem 7.** The Gaussian and mean curvatures of the tubular hypersurfaces $T_{i}$, $i \in \{1, 2, 3, 4\}$, given by (3.1) and (3.2) in $E_4$ are

$$
\begin{align*}
K_{psd-n}^{T_1} &= \frac{k_1}{r^2(2g \cos f - rk_1)}, & H_{psd-n}^{T_1} &= -\frac{1}{r^2} + \frac{4g}{r + 4g - 3rk_1} \cos f, \\
K_{psd-n}^{T_2} &= \frac{k_1}{r^2(rk_1 - 2g \sec f)}, & H_{psd-n}^{T_2} &= -\frac{1}{r^2} + \frac{4g}{r + 4g - 3rk_1} \sin f, \\
K_{psd-n}^{T_3} &= \frac{-k_1}{r^2(2g \csc f + rk_1)}, & H_{psd-n}^{T_3} &= -\frac{1}{r^2} + \frac{4g}{r + 4g - 3rk_1} \sin f, \\
K_{psd-n}^{T_4} &= \frac{k_1}{r^2(2g \sec f + rk_1)}, & H_{psd-n}^{T_4} &= -\frac{1}{r^2} + \frac{4g}{r + 4g - 3rk_1} \cos f.
\end{align*}
$$

(3.3)

So from (3.3), we can give the following results:

**Proposition 11.** i) If the pseudo null curve $\gamma(s)$, which generates the tubular hypersurface $T_1$ given by (3.1) in $E_4$, is a straight line, then the tubular hypersurface is flat.

ii) Let the pseudo null curve $\gamma(s)$, which generates the tubular hypersurface $T_1$, given by (3.1) in $E_4$, not be a straight line. If $g(t, w) = \sin(f(t, w))$, then the Gaussian curvature is constant with $-\frac{2}{r^2}$.

**Proposition 12.** i) The tubular hypersurface $T_1$ given by (3.1) in $E_4$ cannot be minimal.

ii) If the pseudo null curve $\gamma(s)$, which generates the tubular hypersurface $T_1$ given by (3.1) in $E_4$, is a straight line, then the mean curvature of the tubular hypersurface $T_1$ given by (3.1) in $E_4$ is constant with $-\frac{2}{r^2}$. 
iii) Let the pseudo null curve $\gamma(s)$, which generates the tubular hypersurface $T_1$ given by (3.1) in $E_1^4$, not be a straight line. If $g(t, w) = \sin(f(t, w))$, then the mean curvature of the tubular hypersurface $T_1$ given by (3.1) in $E_1^4$ is constant with $\frac{3r-4}{3(2-r)^2}$, $2 \neq r \neq \frac{4}{3}$.

Also, if

$$
\begin{align*}
\left( \frac{H_{psd-n}}{T_1} \right)_s (K_{psd-n})_t - \left( \frac{H_{psd-n}}{T_1} \right)_t (K_{psd-n})_s &= 0, \\
\left( \frac{H_{psd-n}}{T_1} \right)_s (K_{psd-n})_w - \left( \frac{H_{psd-n}}{T_1} \right)_w (K_{psd-n})_s &= 0, \\
\left( \frac{H_{psd-n}}{T_1} \right)_w (K_{psd-n})_w - \left( \frac{H_{psd-n}}{T_1} \right)_w (K_{psd-n})_t &= 0
\end{align*}
$$

(3.4)

hold on a hypersurface, then we call the hypersurface as $(H_{psd-n}, K_{psd-n})$ - Weingarten, $(H_{psd-n}, K_{psd-n})_{st}$ - Weingarten hypersurface, respectively, where

$$
\left( \frac{H_{psd-n}}{T_1} \right)_s = \frac{\partial (H_{psd-n})}{\partial s}
$$

and so on. Thus,

$$(H_{psd-n}, K_{psd-n})_{tw} - \text{Weingarten, } (H_{psd-n}, K_{psd-n})_{sw} - \text{Weingarten hypersurface}, \text{ respectively, where } (H_{psd-n})_s = \frac{\partial (H_{psd-n})}{\partial s}$$



Theorem 8. The tubular hypersurfaces $T_i$, $i \in \{1, 2, 3, 4\}$, given by (3.1) and (3.2) in $E_1^4$, are

$$(H_{psd-n}, K_{psd-n})_{st} - \text{Weingarten, } (H_{psd-n}, K_{psd-n})_{sw} - \text{Weingarten and } (H_{psd-n}, K_{psd-n})_{tw} - \text{Weingarten.}$$

Proof. Let us give the proof for $i = 1$. From (3.3), we have

$$
\begin{align*}
\left( \frac{H_{psd-n}}{T_1} \right)_s &= \frac{2k_1 g f}{3(-2g+rk_1 \sin f)}, \\
\left( \frac{H_{psd-n}}{T_1} \right)_t &= \frac{2k_1 (g f \cos f - g_1 \sin f)}{3(-2g+rk_1 \sin f)^2}, \\
\left( \frac{H_{psd-n}}{T_1} \right)_w &= \frac{2k_1 (g f_2 \cos f - g_2 \sin f)}{3(-2g+rk_1 \sin f)^2}
\end{align*}
$$

and

$$
\begin{align*}
\left( \frac{K_{psd-n}}{T_1} \right)_s &= \frac{2k_2 g f}{r^2(-2g+rk_1 \sin f)}, \\
\left( \frac{K_{psd-n}}{T_1} \right)_t &= \frac{2k_1 (g f \cos f - g_1 \sin f)}{r^2(-2g+rk_1 \sin f)^2}, \\
\left( \frac{K_{psd-n}}{T_1} \right)_w &= \frac{2k_1 (g f_2 \cos f - g_2 \sin f)}{r^2(-2g+rk_1 \sin f)^2}
\end{align*}
$$

(3.5)

where $f_t = \frac{\partial f(t, w)}{\partial t}$, $f_w = \frac{\partial f(t, w)}{\partial w}$ and so on. So, from (3.3) we get

$$
\begin{align*}
\left( \frac{H_{psd-n}}{T_1} \right)_s (K_{psd-n})_t - \left( \frac{H_{psd-n}}{T_1} \right)_t (K_{psd-n})_s &= 0, \\
\left( \frac{H_{psd-n}}{T_1} \right)_s (K_{psd-n})_w - \left( \frac{H_{psd-n}}{T_1} \right)_w (K_{psd-n})_s &= 0, \\
\left( \frac{H_{psd-n}}{T_1} \right)_w (K_{psd-n})_w - \left( \frac{H_{psd-n}}{T_1} \right)_w (K_{psd-n})_t &= 0
\end{align*}
$$

(3.6)

and this completes the proof. Similarly, one can obtain the results for $i = 2, 3, 4$. □

Theorem 9. The tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on partially null curves with Frenet vector fields $F_i$, $i \in \{1, 2, 3, 4\}$, in $E_1^4$ can be parametrized by

$$
\begin{align*}
p_{part-n}T_1 (s, t, w) &= \gamma(s) \mp r \left( \sin(f(t, w))F_2(s) + g(t, w) \cos(f(t, w))F_3(s) + \frac{\cos(f(t, w))}{2g(t, w)}F_4(s) \right), \\
p_{part-n}T_2 (s, t, w) &= \gamma(s) \mp r \left( \cos(f(t, w))F_2(s) + g(t, w) \sin(f(t, w))F_3(s) + \frac{\sin(f(t, w))}{2g(t, w)}F_4(s) \right), \\
p_{part-n}T_3 (s, t, w) &= \gamma(s) \mp r \left( \cosh(f(t, w))F_2(s) + g(t, w) \sinh(f(t, w))F_3(s) - \frac{\sinhf(t, w)}{2g(t, w)}F_4(s) \right)
\end{align*}
$$

(3.7)
Furthermore, the tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on partially null curves with Frenet vector fields $F_i, i \in \{1, 2, 3, 4\}$, in $E^4$ can be parametrized by
\[ T_{i}^{\text{part-}n}(s, t, w) = \gamma(s) \mp r \left( \sinh(f(t, w))F_2(s) + g(t, w)\cosh(f(t, w))F_3(s) - \frac{\cosh(f(t, w))}{2g(t, w)}F_4(s) \right). \] (3.8)

**Theorem 10.** The Gaussian and mean curvatures of the tubular hypersurfaces $T_{i}^{\text{part-}n}, i \in \{1, 2, 3, 4\}$, given by (3.7) and (3.8) in $E^4$ are
\[
\begin{align*}
K_{i}^{\text{part-}n} & = \frac{k_1}{r^2 (\sec f - rk_1)}, & H_{i}^{\text{part-}n} & = \frac{1}{-r + 2 + 3rk_1 \cos f}; \\
K_{i}^{\text{part-}n} & = \frac{-k_1}{r^2 (\sec f + rk_1)}, & H_{i}^{\text{part-}n} & = \frac{1}{-r - 2 + 3rk_1 \cos f}; \\
K_{i}^{\text{part-}n} & = \frac{k_3}{r^2 (\sec f - rk_1)}, & H_{i}^{\text{part-}n} & = \frac{1}{-r + 2 + 3rk_1 \cosh f}; \\
K_{i}^{\text{part-}n} & = \frac{-k_3}{r^2 (\sec f + rk_1)}, & H_{i}^{\text{part-}n} & = \frac{1}{-r - 2 + 3rk_1 \cosh f}.
\end{align*}
\] (3.9)

Thus,

**Proposition 13.** If $k_1(s) = 0$ for the partially null curve $\gamma(s)$, which generates the tubular hypersurfaces $T_{i}^{\text{part-}n}$ given by (3.7) and (3.8) in $E^4$, then these tubular hypersurfaces are flat and also, the tubular hypersurfaces $T_{i}^{\text{part-}n}$ have constant mean curvature with $\frac{2(1)^{i}}{3i}$.

**Theorem 11.** The tubular hypersurfaces $T_{i}^{\text{part-}n}, i \in \{1, 2, 3, 4\}$, given by (3.7) and (3.8) in $E^4$, are \( H_{i}^{\text{part-}n}, K_{i}^{\text{part-}n} \) - Weingarten, \( H_{i}^{\text{part-}n}, K_{i}^{\text{part-}n} \) - Weingarten and \( H_{i}^{\text{part-}n}, K_{i}^{\text{part-}n} \) - Weingarten.

**Proof.** Here we will give the proof for $i = 3$. From (3.9), we have
\[
\begin{align*}
H_{i}^{\text{part-}n} & = \frac{k_1 \cosh f}{3(-1+rk_1 \cosh f)}, & H_{i}^{\text{part-}n} & = \frac{k_1 \cosh f}{r^2(-1+rk_1 \cosh f)^2}; \\
K_{i}^{\text{part-}n} & = \frac{k_3}{3(-1+rk_1 \cosh f)^2}, & K_{i}^{\text{part-}n} & = \frac{k_3}{r^2(-1+rk_1 \cosh f)^2}.
\end{align*}
\] (3.10)

So, from (3.10) we get
\[
\begin{align*}
\left( H_{i}^{\text{part-}n} \right)_s & \left( K_{i}^{\text{part-}n} \right)_t - \left( H_{i}^{\text{part-}n} \right)_t \left( K_{i}^{\text{part-}n} \right)_s = 0, \\
\left( H_{i}^{\text{part-}n} \right)_s & \left( K_{i}^{\text{part-}n} \right)_w - \left( H_{i}^{\text{part-}n} \right)_w \left( K_{i}^{\text{part-}n} \right)_s = 0, \\
\left( H_{i}^{\text{part-}n} \right)_t & \left( K_{i}^{\text{part-}n} \right)_w - \left( H_{i}^{\text{part-}n} \right)_w \left( K_{i}^{\text{part-}n} \right)_t = 0
\end{align*}
\] (3.11)
and this completes the proof. Similarly, one can obtain the results for $i = 1, 2, 4$. \( \square \)

**Theorem 12.** The tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a null curve $\gamma(s)$ with Frenet vector fields $F_i, i \in \{1, 2, 3, 4\}$, in $E^4$ can be parametrized by
\[ T_{1}^{\text{null}}(s, t, w) = \gamma(s) + a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) + a_4(s, t, w)F_4(s), \] (3.12)
where
\[ a_2(s, t, w)^2 + a_4(s, t, w)^2 = r^2. \]
There is no tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperbolic hyperspheres whose centers lie on a null curve \( \gamma(s) \) with Frenet vector fields \( F_i, i \in \{1, 2, 3, 4\} \), in \( E_4^1 \).

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