Comparison results for highly degenerate parabolic equations with univariate convex data and optimal strategies for options on trading accounts

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Abstract

For linear multivariate purely second order highly degenerated parabolic equations with univariate convex data, monotonicity of coefficient matrices implies monotonicity of the related value functions. For multivariate data, comparison holds only for trivial coefficients, where we recover and extend existing results for uniformly elliptic equations to highly degenerate equations by a different method of proof based on Green’s identity. The results extend to multivariate parabolic equations with first order terms and monotonically increasing univariate data. Related convexity criteria are derived from this new perspective. The univariate data are assumed to have some upper bound on exponential growth. Extensions of the argument to lognormal coordinates allow for applications beyond power options. Representation formulas of Greeks are implied. Multivariate comparison with univariate data is used in order to determine optimal strategies for multivariate passport options. These optimal strategies reveal new features of multivariate passport options compared to univariate passport options due to correlation effects. Passport option values are determined by HJB-Cauchy problems with control spaces of measurable bounded functions. Especially the values of optimal strategy functions of passport options, where the control space of strategy values forms a hypercube, are located on the inverse images of the vertices of that cube under the rotation matrix mapping which defines the diagonalization of the correlation matrix of the underlying assets. Especially, multivariate passport options cannot be reduced to lookback options as in the univariate case. However multivariate passport options inherit from univariate passport options the feature that optimal strategies prescribe switching between long and short limit positions on a high frequency basis. This corresponds to control spaces of measurable functions, where more than Hölder regularity cannot be expected. As this is often not feasible, it is interesting to introduce a new product of symmetric passport options with positive trading position constraints. The comparison result is applied to this new product in the case of one underlying share (next to a money account), where optimal strategies prescribe a maximal limit position in the lower asset at each time. Hence, coefficients related to optimal strategies are not continuous in general, but in a more regular class which is more interesting from the trading perspective and from the point of view of regularity theory than the case of classical passport options.
1 Introduction

Natural sufficient conditions for the existence of smooth densities for parabolic equations of second order were established by Hörmander in [8]. Later in the 1980’s Gaussian priori estimates of these densities and of multivariate spatial, time and mixed derivatives of arbitrary order were established from the point of view of Malliavin calculus in [12]. Our interest in this paper is to combine these results with Green’s identity and related properties of the adjoint of fundamental solutions in order to obtain comparison results for multivariate pure diffusions with univariate convex data. This extends generalizations of Hajek’s univariate comparison results obtained in [11]. The Hörmander condition seems natural in this context, because

a) stronger conditions of degeneracy may lead to regularity constraints or even non-existence of densities,

b) comparison of jump diffusions does not hold in general since it does not hold for simple Poisson processes as is shown in [19].

Furthermore, the restriction of univariate data is essential as the result cannot be extended to multivariate data. Here we give a different proof of a result obtained in [14]. We also extend convexity criteria to the class of highly degenerate parabolic equations of pure second order. No-go results of comparison for multivariate data results are strengthened also in the sense that exponential upper bounds for the univariate data are allowed. This allows for applications to power options, where the payoff has polynomial growth conditions in lognormal coordinates which transfers to exponential growth conditions for normal coordinates. Even these growth conditions can be weakened, as we show in section 4. Fortunately, comparison results with univariate data are sufficient in order to determine optimal strategies for multivariate forms of classical passport options, and of a new class of symmetric passport options which are considered in the essentially univariate case of one share and one money account. A review of the literature on passport options and an introduction to this new product is given in the last section of this paper. Additional applications concerning the representations of Greeks are considered in section 5.

2 Comparison for univariate data and the adjoint of the fundamental solution

First, let us consider pure diffusions of second order on the domain \( D = [0, T] \times \mathbb{R}^n \), where \( T > 0 \) is arbitrarily large and and \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space, i.e., we consider Cauchy problems of the form

\[
\left\{ \begin{array}{l}
Lv \equiv v_t - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = 0 \\
v(0, x) = f(x),
\end{array} \right.
\]

(1)

on the domain \( D \) with spatially dependent coefficients \( a_{ij} : \mathbb{R}^n \rightarrow \mathbb{R} \) which form a nonnegative coefficient matrix \( (a_{ij}) \geq 0 \) at each \( x \in \mathbb{R}^n \). Since we have local convexity violation for nontrivial diffusions with multivariate payoffs
(see below), and since some optimal control problems of practical interest can be formulated with restricted forms of payoffs, we are interested especially in univariate data, i.e., data of the form
\[ f(x) = h(x_1) \]  
(2)

after appropriate renumeration of the components of \( x = (x_1, \ldots, x_n)^T \). Note that convexity criteria and comparison transfer to stochastic sums. Uniform ellipticity simplifies the argument a bit, but our methods apply to Hörmander diffusions as well, and we will prove our results in the latter case. We denote the fundamental solution of (1) by \( p \). The adjoint equation is
\[ L^*u \equiv u_t + \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}u) = 0, \]  
(3)

where we denote the fundamental solution of (3) by \( p^* \). For any regular functions \( u, v \in C^{1,2} \), we have Green’s identity for variable coefficients
\[ vL^*u - uLv = (uv)_t - \sum_{i=1}^n \left( \sum_{j=1}^n \left( u_{x_j} \frac{\partial v}{\partial x_i} - v_{x_j} \frac{\partial u}{\partial x_i} - uv \frac{\partial a_{ij}}{\partial x_j} \right) \right) dSd\sigma, \]  
(4)

Now let \( u, v \) be the fundamental solutions of \( Lv = 0 \) and \( L^*u = 0 \) respectively. We consider
\[ v(\sigma, z) = p(\sigma, z; s, y), \quad u(\sigma, z) = p^*(\sigma, z; t, x), \quad t > s. \]  
(5)

Let \( \epsilon > 0 \) be small enough such that \( s + \epsilon < t - \epsilon \), and let \( B_R = \{ z \in \mathbb{R}^n \mid |z| \leq R \} \), where \( R \) is much larger than \( |x|, |y| \). Integrating equation (4) over \( [s+\epsilon, t-\epsilon] \times B_R \), we get
\[ \int_{B_R} u(t-\epsilon, z)v(t-\epsilon, z) - u(s+\epsilon, z)v(s+\epsilon, z) dz = \int_{s+\epsilon}^{t-\epsilon} \int_{\partial B_R} \sum_{i,j=1}^n \left( u_{x_j} \frac{\partial v}{\partial x_i} - v_{x_j} \frac{\partial u}{\partial x_i} - uv \frac{\partial a_{ij}}{\partial x_j} \right) dSd\sigma, \]  
(6)

where \( \partial B_R \) denotes the boundary of \( B_R \), and where the right side of (6) is a surface integral over the boundary \( \partial B_R \) of the ball \( B_R \). We are interested in conditions, where the right side of (6) converges to zero as \( R \uparrow \infty \). We then have
\[ \int_{B_R} u(t-\epsilon, z)v(t-\epsilon, z) dz = \int_{B_R} u(s+\epsilon, z)v(s+\epsilon, z) dz, \]  
(7)

or
\[ \int_{B_R} v(t-\epsilon, z)p^*(t-\epsilon, z; t, x) dz = \int_{B_R} u(s+\epsilon, z)p(s+\epsilon; s, y) dz. \]  

In the limit \( \epsilon \downarrow 0 \) we get
\[ v(t, x) = u(s, y), \]  
(8)

and, similarly for any \( h \in \mathbb{R}^n \), we have
\[ v(t, x+h) = u(s, y+h). \]  
(9)
Next, for directions $h$, consider finite difference quotients

$$D^+_h u(s, w) = \frac{u(s, w + h) - u(s, w)}{h}, \quad D^-_h u(s, w) = \frac{u(s, w) - u(s, w - h)}{h},$$

and

$$D^2_h u(s, w) = \frac{D^+_h u(s, w) - D^-_h u(s, w)}{h},$$

and for multiindices $\alpha$, let $D^\alpha_h$ denote the coordinate versions of these difference equations. Then

$$D^\alpha_h v(t, x) = D^\alpha_h u(s, y)$$

holds for all $0 \leq |\alpha| \leq 2$, and if the finite difference of the right side of (6) goes to zero as $R \uparrow \infty$ and $h \downarrow 0$, then we get indeed

$$D^\alpha_x v(t, x) = D^\alpha_y u(s, y)$$

Such a formula can then be used in partial integration in order to obtain comparison results, but as convex functions usually have some growth at spatial infinity, such partial integrations are useful only for approximating functions, i.e., for data which live in a suitable functions space such as $H^2 \cap C^2$, where $H^2$ denotes the standard Sobolev space of order 2, i.e., the space where the derivatives up to second order are in $L^2$.

Before we introduce the Hörmander conditions, we consider a simple set of conditions such that comparison holds. Here, we mention the more specific condition of uniform ellipticity, since this is a typical condition used in the literature. Indeed, for some applications, such as passport options, global regular existence results for the associated Hamilton-Jacobi-Bellmann equations are unknown or may not exist if the control space is not regular (as is the case especially for classical multivariate passport options). Next, we state the uniform ellipticity condition.

$(C)$ A uniform ellipticity condition holds, i.e., there exist $0 < \lambda < \Lambda < \infty$ such that for all $s_i^1, 1 \leq i \leq n$ and $\sigma \in [0, T]$

$$\lambda |z|^2 \leq \sum_{ij} a_{ij} z_i z_j \leq \Lambda |z|^2.$$  

The second order coefficients $a_{ij}$ themselves and their partial derivatives up to second order are bounded and Hölder continuous.

Next, we consider the univariate data condition.

$(D)$ For a finite constant $c > 0$, the convex univariate data $f$ satisfy for some small $\epsilon > 0$ and all $x \in \mathbb{R}$

$$|f(x_1)| \leq c \exp \left( c |x_1|^{2-\epsilon} \right).$$

Remark 2.1. Note the constant $c$ in the exponent in (15). For lognormal coordinates $x_1 = \ln(S_1)$ a power option payoff $f(S_1) = S_1^m = \exp(mx_1)$ satisfies the condition D. The argument for a comparison considered here can be adapted to this situation.
Next, let us reconsider the assumption C above. Hölder continuity and boundedness of second derivatives of $a_{ij}$ implies that a) we have Lipschitz continuous matrix entries $\sigma = (\sigma_{ij})$ such that $\sigma \sigma^T = (a_{ij})$, and that b) the adjoint density exists and is positive (as the density itself). Lipschitz continuity of $\sigma$ implies that ordinary stochastic ODE-theory is available and implies the existence of strong solutions of associated stochastic ODEs without reference to ellipticity conditions. Furthermore, the comparison arguments below depend essentially on an integrated Green’s identity which leads to relation of finite differences of a density and its adjoint, and a priori estimates such that the right side in (6) goes to zero as the radius $R$ goes to infinity. Hence this argument does not depend on uniform ellipticity of the operator. Indeed, what is essentially needed is the existence of a smooth density and appropriate a priori upper bounds such that the right side of the Green’s identity in (6) goes to zero as the radius $R$ goes to infinity. Hence the Hörmander estimates which are strengthened a bit by Kusuoka and Stroock in the context of Malliavin calculus (cf. [12]) are the appropriate estimates. We next introduce this class of highly degenerate parabolic equations defined by Hörmander.

For positive natural numbers $m, n$, consider a matrix-valued function

$$x \rightarrow (v_{ji})(x), \quad 1 \leq j \leq n, \quad 0 \leq i \leq m, \quad x \in \mathbb{R}^n$$

on $\mathbb{R}^n$, and $m$ smooth vector fields

$$V_i = \sum_{j=1}^n v_{ji}(x) \frac{\partial}{\partial x_j}$$

where $0 \leq i \leq m$. These vector fields define a Cauchy problem on $[0, \infty) \times \mathbb{R}^n$ for the distribution $p$ of the form

$$\left\{ \begin{array}{l}
\frac{\partial p}{\partial t} = \frac{1}{2} \sum_{i=1}^m V_i^2 p + V_0 p \\
p(0, x; y) = \delta_y(x),
\end{array} \right.$$  

(18)

where $\delta_y(x) = \delta(x-y)$ is the Dirac delta distribution with an argument shifted by the vector $y \in \mathbb{R}^n$. The density with arguments $(t, x)$ is smooth on $(0, \infty) \times \mathbb{R}^n$ for all parameters $s < t$ and $y \in \mathbb{R}^n$ if for all $x \in \mathbb{R}^n$ we have

$$H_x = \mathbb{R}^n.$$  

(19)

Here, for each $x \in \mathbb{R}^n$ the set $H_x = \bigcup_{n=0}^\infty H_x^n$ is defined inductively as follows. For $n = 0$ let

$$H_x^0 := \text{span}\{V_i(x) | 1 \leq i \leq m\},$$  

(20)

and given $H_x^n$ for $n \geq 0$ define

$$H_x^{n+1} := H_x^n \cup \text{span}\{[V_j, V_k](x), | 0 \leq j, k \leq m\},$$  

(21)

and where $[.,.]$ are the Lie bracket of vector fields. We say that the Hörmander condition (H) for (15) is satisfied, if

$$\forall x \in \mathbb{R}^n \quad H_x := \bigcup_{n=0}^\infty H_x^n = \mathbb{R}^n.$$  

(22)
Usually this goes with the assumption that the coefficients of the vector fields are smooth (i.e., $C^\infty$) and bounded with bounded derivatives, i.e., $v_{ji} \in C^\infty_b(R^n)$. Linear growth for the functions $v_{ji}$ themselves is allowed such that we use the weaker assumption of coefficients with linear growth and bounded derivatives of arbitrary order or $v_{ji} \in C^\infty_{b,l}(R^n)$ in symbols. If the equation in (18) equals a pure diffusion without drift as in (1), then we speak of a pure Hörmander diffusion. Note that in this case

$$\frac{1}{2} \sum_{i=1}^m V_i^2 + V_0 = \sum_{ij} a^h_{jk} \frac{\partial^2}{\partial x_j \partial x_k}$$  \hspace{1cm} (23)$$

for some matrix-valued function $x \to (a^h_{jk}(x))$ which is elliptic at each $x \in R^n$. Here the upper script $h$ just reminds us that the coefficient matrix $(a^h_{ij})$ represents the second order coefficient matrix of a pure Hörmander diffusion.

The relation to diffusion processes is via $(a_{jk}) = \sigma \sigma^T$, where the condition H ensures that the latter condition exists. The main result in [12] extending the analysis in [8] is

**Theorem 2.2.** Consider a $d$-dimensional diffusion process of the form

$$dX_t = \sum_{i=1}^d \sigma_{0i}(X_t)dt + \sum_{j=1}^d \sigma_{ij}(X_t)dW^j_t$$  \hspace{1cm} (24)$$

with $X(0) = x \in R^d$ with values in $R^d$ and on a time interval $[0,T]$. Assume that $\sigma_{0i}, \sigma_{ij} \in C^\infty$. Then the law of the process $X$ is absolutely continuous with respect to the Lebesgue measure, and the density $p$ exists and is smooth, i.e.

$$p : (0, T] \times R^d \times R^d \to R \in C^\infty ((0, T] \times R^d \times R^d).$$  \hspace{1cm} (25)$$

Moreover, for each nonnegative natural number $j$, and multiindices $\alpha$, $\beta$ there are increasing functions of time

$$A_{j,\alpha,\beta}, B_{j,\alpha,\beta} : [0, T] \to R,$$  \hspace{1cm} (26)$$

and functions

$$n_{j,\alpha,\beta}, m_{j,\alpha,\beta} : N \times N^d \times N^d \to N,$$  \hspace{1cm} (27)$$

such that

$$\left| \frac{\partial^{|\alpha|} \sigma_{ij}^{|\beta|}}{\partial x_j} \right| p(t, x, y) \leq A_{j,\alpha,\beta}(t)(1 + x)^{m_{j,\alpha,\beta}} \exp \left( -B_{j,\alpha,\beta}(t) \frac{(x-y)^2}{t} \right).$$  \hspace{1cm} (28)$$

Moreover, all functions $A_{j,\alpha,\beta}$ and $B_{j,\alpha,\beta}$ depend on the level of iteration of Lie-bracket iteration at which the Hörmander condition becomes true.

Theorem 2.2 is also sometimes formulated in a probabilistic manner. We note

**Corollary 2.3.** In the situation of Theorem 2.2 above, solution $X^x_t$ starting at $x$ is in the standard Malliavin space $D^\infty$, and there are constants $C_{i,q}$ depending
on the derivatives of the drift and dispersion coefficients such that for some constant \( \gamma_{l,q} \)

\[
|X^r_t|_{l,q} \leq C_{l,q}(1 + |x|)^{\gamma_l,q}.
\]

(29)

Here \(|\cdot|_{l,q}\) denotes the norm where derivatives up to order \( l \) are in \( L^q \) (in the Malliavin sense).

However, the upper bounds of spatial derivatives obtained in [12] are not finite on the whole space, an effect which limits existence of regular global solutions of Hörmander diffusions. Since these upper bounds are natural in general, we have no comparison without existence of global regular solutions.

This has to be investigated on a case by case basis. For our method we need Hölder continuity for spatial derivative up to second order for the solution of the the pure Hörmander diffusion problem. Therefore we include existence of global regular solutions in the following assumption.

(HE) We assume that the Hörmander condition in (22) is satisfied and that we have a pure Hörmander diffusion, i.e., that the condition in (23) is satisfied for the coefficient matrix \((a_{ij})\) in (30) below. Furthermore we assume that the Cauchy problem in (30) has a global classical solution which is in \( C^{2,\alpha}_X \), i.e., a classical solution in \( C^{2,1}_X \) with finite spatial Hölder norms for spatial derivatives up to second order on the whole domain of \( \mathbb{R}^n \).

We remark that there is no loss of generality if we assume the coefficient matrices to be symmetric. The main comparison results is

**Theorem 2.4.** Let \((a_{ij}), (a'_{ij})\) be two matrices of component functions which equal \( n \times n\)-matrices \( \sigma \sigma^T \) and \( \sigma' \sigma'^T \) respectively, and such that the conditions in Theorem 2.3 hold for \( \sigma_{ij} \). Assume that \( f(x) = h_1(x_1) \) satisfies condition D. Assume that condition(HE is satisfied for \((a_{ij}), (a'_{ij})\). On the domain \([0, T] \times \mathbb{R}^n\) consider a Cauchy problem of the form

\[
\begin{cases}
Lv' \equiv v_t - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v'}{\partial x_i \partial x_j} = 0, \\
v'(0, x) = h_1(x_1),
\end{cases}
\]

(30)

and an analogous Cauchy problem with coefficient functions \((a'_{ij})\) respectively. If \((a_{ij}) \leq (a'_{ij})\) in the sense that \((a'_{ij}) - (a_{ij})\) is a nonnegative matrix, i.e., has nonnegative eigenvalues, and \(a_{11} < a'_{11}\), then \( v < v' \) on the domain \([0, T] \times \mathbb{R}^n\).

The existence assumption in (HE) can be eliminated if we know existence for other reasons, e.g., if the assumptions D and C hold, because uniform ellipticity and regularity of coefficients together with the growth and data condition in D implies existence. Therefore, we have

**Corollary 2.5.** Let \((a_{ij}), (a'_{ij})\) be two matrices of component functions which satisfy the condition C, and assume that \( f(x) = h_1(x_1) \) satisfies condition (D).

We compare the solution of (7) with data (8) with the solution of the Cauchy problem

\[
\begin{cases}
Lv' \equiv v'_t - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v'}{\partial x_i \partial x_j} = 0, \\
v'(0, x) = h_1(x_1),
\end{cases}
\]

(31)

If \((a_{ij}) \leq (a'_{ij})\) in the sense that \((a'_{ij}) - (a_{ij})\) is a nonnegative matrix, i.e., has nonnegative eigenvalues, and \(a_{11} < a'_{11}\), then \( v < v' \) on the domain \([0, T] \times \mathbb{R}^n\).
Proof. (Martingale theorem.) For a small \( \epsilon \), consider a convolution \( h_{1,\epsilon} = h_1 \ast \epsilon \) of \( G_\epsilon \in C^\infty \) such that \( h_{1,\epsilon} \) remains convex and \( \lim_{\epsilon \downarrow 0} h_{1,\epsilon} = h_1 \) (pointwise). For example, \( G_\epsilon \) may be chosen to be a heat kernel, i.e., the fundamental solution of \( p_t - \epsilon p_{x_1 x_1} = 0 \). Next, for small \( \delta > 0 \) and large \( R \) define

\[
h_{1,\epsilon,\delta}^R := \begin{cases} h_1(x) \text{ if } |x| \leq R \\ h_1(x) \exp(-\delta|x - R|^2) \text{ if } |x| > R,
\end{cases}
\]  

(32)

and let \( h_{1,\epsilon,\delta}^R = h_{1,\epsilon,\delta}^R \ast \epsilon \) be the fundamental solutions of \( p_t - \sum \alpha_i \epsilon p_{x_i x_i} = 0 \) and of \( p_t' - \sum \alpha_i \epsilon p_{x_i x_i}' = 0 \). The value functions \( v, v' \) have the representations

\[
v(t, x) = \int_{\mathbb{R}^n} h_1(y_1)p(t, x; 0, y)dy, \]

(33)

and

\[
v'(t, x) = \int_{\mathbb{R}^n} h_1(y_1)p'(t, x; 0, y)dy, \]

(34)

and the approximative value functions have the representations

\[
v_{\epsilon,\delta}^R(t, x) = \int_{\mathbb{R}^n} h_{1,\epsilon,\delta}^R(y_1)p(t, x; 0, y)dy \]

(35)

and

\[
v_{\epsilon,\delta}^{R'}(t, x) = \int_{\mathbb{R}^n} h_{1,\epsilon,\delta}^R(y_1)p'(t, x; 0, y)dy. \]

(36)

We have \( p, p' > 0 \) by assumption \( C \) and there exists an adjoint fundamental solution, i.e., \( \langle Lv, u \rangle = \langle v, L^*u \rangle \), where \( L \) is the operator of (31), and \( L^* \) is the adjoint operator. In order to use partial integration and the adjoint, we introduce rotated coordinates \( \tilde{x} \) such that \( \tilde{x} \rightarrow h_{1,\epsilon,\delta}(\tilde{x}) \) becomes a function on the whole domain of \( \mathbb{R}^n \) which is in \( H^2 \cap C^2 \) with respect to this multidimensional domain. Since \( x_1 \rightarrow h_{1,\epsilon,\delta}(x_1) \) is univariate, we can always consider small rotations such that \( \tilde{x} = \sum_{j=1}^n \lambda_j x_j \), where \( \lambda_j > 0 \), \( 1 \leq j \leq n \) and such that \( \lambda_1 \) is close to 1 and \( \lambda_j, j \neq 0 \) are small. Let \( \tilde{p} \) and \( \tilde{p}' \) denote the corresponding fundamental solution in rotated coordinates, i.e., corresponding to \( p, p' \) via the rotation transformation outlined. Next, we use the relation (36) which leads to (37), or for multiindices \( 0 \leq |\alpha| \leq 2 \) we have

\[
D_\tilde{x}^\alpha \tilde{p}(t, \tilde{x}; s, \tilde{y}) = D_\tilde{y}^\alpha \tilde{p}^*(s, \tilde{y}; t, \tilde{x}), \quad t > s,
\]

(37)

where \( \tilde{p}^* \) is the adjoint of \( \tilde{p} \). Let \( \tilde{y} \rightarrow h_{1,\epsilon,\delta}^{R,\lambda}(\tilde{y}) = h_{1,\epsilon,\delta}^R(y_1) \) denote the rotational transform of the data. Then we have

\[
D_\tilde{x}^\alpha \tilde{v}_{\epsilon,\delta}^{R,\lambda}(t, \tilde{x}) = \int_{\mathbb{R}^n} h_{1,\epsilon,\delta}^{R,\lambda}(\tilde{y})D_\tilde{y}^\alpha \tilde{p}(t, \tilde{x}; 0, \tilde{y})d\tilde{y} \\
= \int_{\mathbb{R}^n} h_{1,\epsilon,\delta}^{R,\lambda}(\tilde{y})D_\tilde{y}^\alpha \tilde{p}^*(0, \tilde{y}; t, \tilde{x})d\tilde{y} \\
= \int_{\mathbb{R}^n} \left(D_\tilde{y}^\alpha h_{1,\epsilon,\delta}^{R,\lambda}(\tilde{y})\right) \tilde{p}^*(0, \tilde{y}; t, \tilde{x})d\tilde{y},
\]

(38)
Note that $\tilde{p}^* > 0$ by $\text{(37)}$ and assumption (C). Now let $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{A}' = (\tilde{a}'_{ij})$ denote the coefficient matrix in transformed rotated coordinates such that for all $1 \leq i, j \leq n$ and all $\tilde{x}, \tilde{x}$, we have $\tilde{a}_{ij}(\tilde{x}) = a_{ij}(x)$ and $\tilde{a}'_{ij}(\tilde{x}) = a'_{ij}(x)$. Assume that $\tilde{p}$ is fundamental solution of

\[ \tilde{u}_{t}^\epsilon,\delta,R - \text{Tr} \left( \tilde{A}D^2\tilde{u}_{t}^\epsilon,\delta,R \right) = 0, \tag{39} \]

where we consider this problem along with some data $\tilde{u}(0, \tilde{x}) = h^{R,A}_{\epsilon,\delta}(\tilde{x})$, and where

\[ \text{Tr} \left( \tilde{A}D^2\tilde{u}_{t}^\epsilon,\delta,R \right) = \sum_{j=0}^{n} \tilde{A}_{ij}(D^2_{jk}\tilde{u}_{t}^\epsilon,\delta,R)\delta_{ik}, \tag{40} \]

and compare this to a solution

\[ \tilde{u}_{t}'^\epsilon,\delta,R - \text{Tr} \left( \tilde{A}'D^2\tilde{u}_{t}'^\epsilon,\delta,R \right) = 0 \tag{41} \]

with the same data. For $\delta\tilde{u} = \tilde{u}' - \tilde{u}$, we get

\[ \delta\tilde{u}_{t}^\epsilon,\delta,R - \text{Tr} \left( \tilde{A}'D^2\delta\tilde{u}_{t}^\epsilon,\delta,R \right) = \text{Tr} \left( (\tilde{A}' - \tilde{A})D^2\tilde{u}_{t}^\epsilon,\delta,R \right). \tag{42} \]

As we have zero data for the difference, we have

\[ \delta\tilde{u}_{t}^\epsilon,\delta,R(t, \tilde{x}) = \int_{0}^{t} \int_{\mathbb{R}^n} \text{Tr} \left( (A' - A)D^2\tilde{u}_{t}^\epsilon,\delta,R \right) (s, \tilde{y})\tilde{p}'(t, \tilde{x}; s, \tilde{y})d\tilde{y}ds. \tag{43} \]

Now let $(t, \tilde{x})$ be given. For $\frac{R}{2} \gg |\tilde{x}|$ and $\delta, \epsilon > 0$ small, we observe from $\text{(38)}$ that for given $T$ we may choose $R$ large enough

\[ \frac{\partial^2 \tilde{u}_{t}^\epsilon,\delta,R}{\partial \tilde{x}_i^2}(s, \tilde{y}) > 0, \quad |\tilde{y}| \leq \frac{R}{2}, \quad s \in [0, T] \tag{44} \]

where for $\lambda_1$ close to 1 and $\lambda_i$, $2 \leq i \leq n$ small this term dominates the other Greeks. We conclude that

\[ \left( (\tilde{a}_{11} - \tilde{a}_{11}) \frac{\partial^2 \tilde{u}_{t}^\epsilon,\delta,R}{\partial \tilde{x}_1^2} \right) (s, \tilde{y}) > 0, \quad |\tilde{y}| \leq \frac{R}{2}, \quad s \in [0, T] \tag{45} \]

is the dominating term in $\text{Tr} \left( (A' - A)D^2\tilde{u}_{t}^\epsilon,\delta,R \right)$ in a ball of radius $\frac{R}{2}$. Outside that ball (since $|\tilde{x}| \ll \frac{R}{2}$) $\tilde{p}'(t, \tilde{x}; s, y)$ becomes small such that from $\text{(43)}$ we get

\[ \delta\tilde{u}_{t}^\epsilon,\delta,R(\tau, \tilde{x}) > 0, \quad R \text{ large, } \epsilon, \delta \text{ small}, \tag{46} \]

which relation holds also in the limit $\delta, \epsilon \downarrow 0$ and $R \uparrow \infty$. \hfill \square

Next, we give consider variations of the argument above for comparison and derive convexity criteria for pure Hörmander diffusions as further corollaries. A further simple conclusion is that for pure Hörmander diffusions with spatially nonconstant coefficients there are multivariate convex data such that convexity and comparison are locally violated.
For smooth convex data \( f : \mathbb{R}^n \to \mathbb{R} \), consider the function \( v^f := v - f \), where \( v \) satisfies the equation (1). Since \( f \) is independent of time, we obviously have

\[
\begin{aligned}
\frac{\partial v^f}{\partial t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v^f}{\partial x_i \partial x_j} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} &= 0 \\
v^f(0, x) &= 0
\end{aligned}
\]

Hence, we have the representation

\[
v^f(t, x) = \int_{t}^{0} \int_{\mathbb{R}^n} \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right)(s, y) p(t, x; s, y) \, dy \, ds,
\]

where \( p \) is the fundamental solution of \( \frac{\partial q}{\partial t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 q}{\partial x_i \partial x_j} = 0 \).

For data \( f : \mathbb{R}^\epsilon,\delta =: \begin{cases} f^\epsilon(x) & \text{if } |x| = \sqrt{\sum_{i=1}^{n} x_i^2} \leq R \\
\exp(-\delta|x - R|^2) & \text{if } |x| = \sqrt{\sum_{i=1}^{n} x_i^2} > R,
\end{cases} \)

we get the representation

\[
\frac{\partial^2 v^f_{R, \epsilon, \delta}}{\partial x_i \partial x_j}(t, x) = \int_{t}^{0} \int_{\mathbb{R}^n} \left( D_{y_1}^2 \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right)(s, y) \right) p^*(t, x; s, y) \, dy \, ds.
\]

In the case of univariate data this simplifies considerably to

\[
\frac{\partial^2 v^f_{1, \epsilon}}{\partial x_i \partial x_j}(t, x) = \int_{t}^{0} \int_{\mathbb{R}^n} \left( D_{y_1}^2 \left( a_{11} \frac{\partial^2 f}{\partial x_1} \right)(s, y) \right) p^*(t, x; s, y) \, dy \, ds.
\]

From (50) we get convexity criteria and failure of convexity of the value function for all nontrivial regular bounded coefficient matrices and some convex data. From (51) we get partial convexity and comparison for regular bounded coefficients and univariate convex data. In order to state the convexity criterion, we need assumptions for more general data. We assume

\[(\text{D'})\] For a finite constant \( c > 0 \) the convex data \( f : \mathbb{R}^n \to \mathbb{R} \) satisfy for some small \( \epsilon > 0 \) and all \( x \in \mathbb{R}^n \)

\[|f(x)| \leq c \exp\left(\epsilon |x|^2 - \epsilon\right).
\]

We may abbreviate the coefficient matrix by \( A = (a_{ij}) \) and the Hessian of the data \( f \) by \( D^2 f \). Alexandrov’s result (cf. [1]) tells us that \( D^2 f \) exists almost everywhere for convex functions, but in order to have a pointwise well-defined Hessian everywhere, we may convolute with a heat kernel of small dispersion, i.e., with fundamental solutions of \( q_\epsilon^t = \epsilon \Delta q_\epsilon \) in order to have smooth convex approximations at hand. We denote \( f_\epsilon = f * q_\epsilon \), where ‘*’ denotes convolution. In the following, we say that a regular function is convex if its Hessian is nonnegative. We say that a function is strictly convex, if its Hessian is positive at all arguments of the domain \( \mathbb{R}^n \). In the following we assume that the assumptions D’ and HE hold. Alternatively, we may assume that D’ and C hold of course. Now we have
The behavior of regular convex functions at critical points is also made in [17].

C we restrict the sufficient criterion to the critical set

the sufficient criterion of if criterion of Theorem 2.6 becomes an iff-criterion if

We can combine this observations with our observations so far. This means that

solution function

Corollary 2.7. Assume D' and HE hold for data and coefficients. Then the

Assume D' and HE hold for data and coefficients. Then the

and

D

for any partial second order derivative of

Proof. If the assumptions C and D' hold, then the limit \( \delta \downarrow 0 \) and \( R \uparrow \infty \) of

exists on both sides of the equation for arbitrary small \( \epsilon > 0 \). Since \( p^* > 0 \)

by assumptions C and D' and \( \mathrm{Tr} (AD^2 f_\epsilon) \) is convex for \( \epsilon > 0 \) small, this limit of

shows that the value function \( v^\epsilon_1 \) is convex for small \( \epsilon > 0 \). Finally, if \( v^\epsilon \)

is convex, then \( v \) is convex. \( \square \)

The latter criterion is sufficient and rather strong in the sense that it essentially says that the time derivative of the value functions has a nonnegative Hessian. Let us assume that \( f \in C^2 \) for a moment. In order to have an iff-criterion, we need to consider the critical set \( C_r \), where \( C_r := \{ x \mid D^2 f(x) = 0 \} \). If \( x \in C_r \) and \( f \) is convex then \( D^2 f(x) \geq 0 \) such that this \( x \in C \) is a minimum for any partial second order derivative of \( f \). It follows that \( D^3 f(x) = 0 \) and \( D^4 f(x) \geq 0 \). The latter very simple but effective observation about the behavior of regular convex functions at critical points is also made in [17]. We can combine this observations with our observations so far. This means that the sufficient criterion of if criterion of Theorem 2.6 becomes an 'iff'-criterion if we restrict the sufficient criterion to the critical set \( C_r \). We have

Corollary 2.7. Assume D' and HE hold for data and coefficients. Then the

solution function \( v \) of (4) is convex iff

\[ \forall x \in C_r \quad \mathrm{Tr} (AD^2 f) \text{ is convex in } U, \quad (54) \]

where \( U \) is a local neighborhood of \( x \), or if for all \( x \in C_r \) the Hessian is a nonnegative

\[ \left( D^2 f(x) \right) \geq 0 \text{ for any small } \epsilon > 0. \]

This criterion is sufficient and necessary, and we can use it in order to obtain a partially different proof of comparison for univariate data. Note, however, that the main idea of connecting Green's identity for variable coefficients with properties of the relations between (derivatives) the fundamental solution and (derivatives) of its adjoint. We have

Corollary 2.8. Assume D' and HE hold for data and coefficients. Then the

local convexity criterion of Corollary 2.7 holds.

Proof. We assume that \( h_1 \in C^2 \) and that the assumptions D' and HE hold. Then

the limit with \( \delta, \epsilon \downarrow 0 \) and \( R \uparrow \infty \) of (51) holds and we have the representation

\[ \frac{\partial^2 v_1}{\partial x \partial y} (t, x) = \int_0^t \int_{\mathbb{R}^n} \left( D^2 g_{y, y} \left( a_{11} \frac{\partial h_1}{\partial t} \right) (s, y) \right) p^* (t, x; s, y) dy ds, \quad (55) \]

Consider the critical set \( C_{h_1} := \{ x \mid D^2 h_1(x_1) = 0 \} \). If \( x \in C_{h_1} \) and \( h_1 \in C^2 \)

is convex, then \( D^2 h_1(x_1) \geq 0 \) such that this \( x \in C_{h_1} \) is a minimum for any partial second order derivative of \( h_1 \). It follows that \( D^3 h_1(x_1) = 0 \) and
\[ D^2_{x_1,x_1} h_1(x_1) \geq 0, \text{ where only } D^2_{x_1,x_1,x_1,x_1} h_1(x_1) \geq 0 \text{ may be different from zero for } x \in C_{h_1}. \] We get
\[
\forall x \in C_{h_1} \left( D^2_{x_1,x_1} \left( a_{11} \frac{d^2 h_1}{dx_1^2} \right)(s,x) \right) = a_{11}(s,x)\delta_{i_1}\delta_{j_1} \frac{\partial^2}{\partial x_1 \partial x_1} \frac{d^2 h_1}{dx_1^2} \geq 0 \quad (56)
\]
and the local convexity criterion is satisfied. Comparison follows.

Essentially, in the latter corollary we have derived a generalization of the locally convexity preserving condition in [17] where a uniform ellipticity condition in [17] is generalized to the condition (HE), and the polynomial upper bound in [17] is generalized to an exponential upper bound in D'. Recall this condition. Let \( L := \frac{\partial}{\partial t} - L_{sp} \) with \( L \) as in (1). Assume that HE and D'. For regular functions \( f: \mathbb{R}^n \to \mathbb{R} \) and arbitrary 'directions' \( u \in \mathbb{R}^n \) define as usual
\[
D_u f(x) = \lim_{h \downarrow 0} \frac{f(x + hu) - f(x)}{h}, \quad D_{uu} f(x) = \lim_{h \downarrow 0} \frac{D_u f(x + hu) - D_u f(x)}{h}.
\]
Then the condition in Corollary 2.7 can be rephrased by saying that under the condition (D') and (HE) the operator \( L \) is called locally convexity preserving at \( x \in \mathbb{R}^n \) if there exists a neighborhood \( U = U(x) \) of \( x \) in \( \mathbb{R}^n \) with respect to the standard topology such that
\[
\forall u \in \mathbb{R}^n \forall \text{convex } f \in C^2(U) \quad (D_{uu} f(x) = 0) \Rightarrow D_{uu} (L_{sp} f)(x) \geq 0 \quad (58)
\]
This is a reformulation of the condition in Corollary 2.7 is stated in [17] under the restricted condition C, and where a more restrictive condition than D or D' was imposed on the data, i.e., in [17] it is assumed that the data have a polynomial upper bound. Under these restricted conditions equivalent conditions for convexity preservation can be obtained (cf. [17]). We mention that these criteria can be generalized to the condition (HE), but the data condition of a polynomially upper bound is essentially used. However this is enough in order to assert that the local convexity condition is violated for multivariate data \( f \) in general a fortiori.

**Corollary 2.9.** Assume D' and HE hold for data and coefficients. Then for all nonconstant coefficients there are data such that convexity of the value function is locally violated.

**Proof.** For polynomially bounded data and the condition C the proof in [17] applies. If the conditions HE and D hold, then a solution can be represented by limits of solution functions of parabolic problems which satisfy C and have polynomial bounded data, where the convexity is locally violated. The violation of convexity is then preserved in the limit.

**Remark 2.10.** Note that we consider the essential case of time-homogeneous models in this paper. Convexity criteria are satisfied for purely time dependent coefficients for analogous extensions of condition C, of course.
3 Applications to finance I: Representations of Greeks and beyond power options

Consider functions $u, v$ with $Lv = 0$ and $L^*u = 0$ respectively, where for for fixed $x + h, y + h$

$$v(\sigma, z) = p(\sigma, z; s, y + h), \quad u(\sigma, z) = p^*(\sigma, z; t, x + h), \quad t > s. \quad (59)$$

Integrating equation (4) over $[s + \epsilon, t - \epsilon] \times B_R$ and using $Lv = 0$ and $L^*u = 0$ we observed that the left side of (4) becomes zero, and that the right side of the resulting equation

$$\int_{B_R} u(t - \epsilon, z)v(t - \epsilon, z) - u(s + \epsilon, z)v(s + \epsilon, z)dz$$

$$= \int_{s + \epsilon}^{t - \epsilon} \int_{\partial B_R} \left( \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} - va_{ij} \frac{\partial u}{\partial x_j} - uv a_{ij} \right) \right) (\sigma, z) dS d\sigma, \quad (60)$$

converges to zero as $R \uparrow \infty$. We then have

$$\int_{B_R} u(t - \epsilon, z)v(t - \epsilon, z)dz = \int_{B_R} u(s + \epsilon, z)v(s + \epsilon, z)dz,$$

which is

$$\int_{B_R} v(t - \epsilon, z)p^*(t - \epsilon, z; t, x + h)dz = \int_{B_R} u(s + \epsilon, z)p(s + \epsilon, z; s, y + h)dz$$

or

$$\int_{B_R} p(t - \epsilon, z; s, y)p^*(t - \epsilon, z; t, x + h)dz$$

$$= \int_{B_R} p^*(s + \epsilon, z; t, x)p(s + \epsilon, z; s, y + h)dz \quad (61)$$

In the limit $\epsilon \downarrow 0$ the integrand on the left side contributes only for $z = x + h$ and the integrand on the right side contributes only for $z = y + h$ such that indeed

$$v(t, x + h) = u(s, y + h), \quad (62)$$

where $h \in \mathbb{R}^n$ was free fixed choice.

Hence, the relation holds for finite difference quotients

$$D^+_h u(s, y) = \frac{u(s, y + h) - u(s, y)}{h}, \quad D^-_h u(s, y) = \frac{u(s, y) - u(s, y - h)}{h}. \quad (63)$$

Similarly for higher order finite differences. Hence for $h \downarrow 0$ and any multiindex $\alpha$ we get indeed

$$D^\alpha_x v(t, x) = D^\alpha_y u(s, y). \quad (64)$$

In the notation above with $h = 0$ we have

$$v(t, x) = p(t, x; s, y), \quad u(s, y) = p^*(\sigma, z; t, x), \quad t > s. \quad (65)$$

Such a formula can then be used in partial integration and for the representations of Greeks. For a Cauchy problem of a second order linear equation with fundamental solution $p$ and initial data $f$ at time $t_0$ we have the representation

$$v^f(t, x) = \int_{\mathbb{R}^n} f(y)p(t, x; t_0; y)dy = \int_{\mathbb{R}^n} f(y)p^*(t_0; y; t, x)dy \quad (66)$$
For spatial derivatives we get the representation

\[ v^f(t,x) = \int_{\mathbb{R}^n} f(y) D^2_y p(t,x,y) dy = \int_{\mathbb{R}^n} f(y) D^2_y p^*(t_0, y, t, x) dy \quad (67) \]

If \( f \in H^2 \cap C^2 \), then the derivatives can be shifted to the payoff functions. Next we consider a application concerning growth conditions of initial data for problems written in lognormal coordinates, where we observe that we can improve on the data assumption (D) if some boundary conditions are satisfied. In finance, diffusions of second order are usually considered in lognormal coordinates, where we observe that we can interchange the variable coefficients

\[ \partial_s v \equiv v_t - \sum_{i,j} a_{ij} \partial_{s_i s_j} v = 0, \]

and the Cauchy problem in \( \Pi \) becomes a Cauchy problem on the domain \( \Omega^* \) of the form

\[ \left\{ \begin{array}{l}
L^* v \equiv v_t - \sum_{i,j=1}^n a_{ij}^* s_i s_j \frac{\partial^2 v^*}{\partial s_i \partial s_j} = 0, \\
v^*(0, s) = f^*(s),
\end{array} \right. \quad (69) \]

where we assume univariate data

\[ f^*(s) := f(x) = h^*(s_1) := h(x_1), \quad (70) \]

and where

\[ a_{ij}^*(s) := a_{ij}(x), \quad (71) \]

again after appropriate renumeration of the components of \( s = (s_1, \cdots, s_n)^T \) and \( x = (x_1, \cdots, x_n)^T \) respectively. Here we mention that option value problems are often given in the form of a final value problem which can be obtained from \( (69) \) by a time transformation \( \tau = T - t \). Here, we stick with Cauchy problem formulation for convenience. Note that the constant \( c > 0 \) in the assumption D above is arbitrary such that power options, i.e., options with payoff \( h^*(s_1) = s_1^m \) for some integer \( m \), are subsumed. However, if we have natural zero boundary conditions of the value functions at \( s_i = 0 \) for all \( i \) then we can weaken the data condition to

\[ (D^*) \quad \text{For a finite constant } c > 0 \text{ the convex univariate data } f \text{ satisfy for some small } \epsilon > 0 \text{ and all } s_1 \in \mathbb{R}_+ \]

\[ |f^*(s_1)| \leq c \exp \left( c|s_1|^{2-\epsilon} \right). \quad (72) \]

In order to observe that this generalization is possible note first that the adjoint equation is

\[ L^*_x u \equiv u_t + \sum_{i,j=1}^n s_i s_j \frac{\partial^2 u}{\partial s_i \partial s_j}(a_{ij}^* u^*) = 0. \quad (73) \]

We may denote the fundamental solution of \( (3) \) by \( p^*_x \). Then for regular functions \( u^x, v^x \in C^{1,2} \) on \([0,T] \times \mathbb{R}_+^n \) we have Green’s identity for variable coefficients

\[ v^x L^*_x u^x - u^x L^*_x v^x = (u^x v^x)_t \]

\[ - \sum_{i=1}^n s_i \frac{\partial}{\partial s_i} \left( \sum_{j=1}^n \left( u^x a_{ij}^* s_j \frac{\partial v^x}{\partial s_j} - v^x a_{ij}^* s_j \frac{\partial u^x}{\partial s_j} - u^x v^x s_j \frac{\partial a_{ij}^*}{\partial s_j} \right) \right), \quad (74) \]
and for fundamental solutions $u^s, v^s$ to $L_s v^s = 0$ and $L_s^* u^s = 0$ we may integrate this identity over $s + \epsilon < t - \epsilon$ with $\epsilon > 0$ as before, and over $B^+_R := \{ s \in \mathbb{R}^n | ||s|| \leq R \}$ for large $R$.

4 Applications to finance II: Passport options, symmetric passport options, and optimal strategies

Options on a traded account have been widely studied in the previous literature. In the simplest setup, the client is free to trade in two underlying assets subject to specific contractual limits. At the time of the maturity, he can keep the profits from this trading strategy while his losses are forgiven. Some of the contracts within this family, like passport options, have been even actively traded. However, the popularity of such contracts has been rather small. For univariate passport options passport options can be subsumed by lookback options. However, this does not hold for multivariate passport options, as optimal strategies have a much more complicated structure, a consequence, which we draw below from the comparison theorem above. Secondly, mathematical optimal strategies, e.g. for a passport call written on one share imply high frequency trading between short and long limit positions. For multivariate passport options this statement has to be modified according to the correlation structure of the underlying assets, as we observe below. High frequency trading between short and long limit positions may be related may be also related to the opinion that passport options are expensive. But up to possible costs for handling the transactions passport options, the costs are just replication costs. Extreme short positions may also be unpopular to such an extent that they may be restricted by law from time to time. Another reason may be the fact that the previously considered contracts treat the two underlying assets asymmetrically. In this traditional setup, the restriction of the trading position is set to only one asset and the residual wealth is invested in the second asset. For the passport option, the restriction on the position in the first asset is $[-1,1]$, meaning that the agent can take any position between long and short. The position in the second asset is given by the residual wealth.

Passport options were introduced in [9]. The authors derived the optimal strategy in the geometric Brownian motion model, which is achieved by a short position when the traded account is negative and a long position when the traded account is positive. They also found the corresponding option value by solving the corresponding pricing partial differential equation. Henderson and Hobson (2000) showed that the same strategy remains optimal in the presence of stochastic volatility. Shreve and Vecer (2000) considered more general trading limits on the first asset. The optimality of the solution was proved using the probabilistic arguments based on a comparison theorem of Hajek (1985). Vecer (2001) later showed that Asian options are special cases of options on a traded account when the restriction on the first asset has a specific deterministic form and found a novel pricing partial differential equation. Delbaen and Yor (2002) showed that the strategy for the passport option remains optimal when the portfolio rebalancing is restricted to a discrete time. Kampen (2008)
considered multivariate passport options, where the traded account consists of more than one asset, and observed that optimal strategies may depend on the sign of the correlations between assets. However a full determination of optimal strategies for multivariate passport options was not given in this note. We shall do this below. Then in a second step we consider a basic example of a symmetric passport option with just one share and one money account. We shall discuss the advantages of this idea. We also find interesting optimal strategies very different from optimal strategies of classical passport options which illustrate the practicability of this new product.

First let us extend the results known so far for multivariate passport options. First recall the structure of the product itself. Given a trading account

$$\Pi = \Pi_\Delta = \sum_{i=1}^{n} \Delta_i S_i, \text{ where } dS_i = \sigma_i S_i dW_i, \ S(0) = x \in \mathbb{R}^n \quad (75)$$

with \( n \) lognormal processes \((S_i)_{1 \leq i \leq n} \) where correlations of Brownian motions \( W_i \) are encoded in \((\rho_{ij})_{1 \leq i,j \leq n}\), and \( q_i \in [-1, 1] \) are bounded trading positions, the price of a classical passport option is given by the the solution of an optimal control problem

$$\sup_{-1 \leq \Delta_i \leq 1, \ 1 \leq i \leq n} E^{x,p}(f(\Pi_\Delta)), \ f \text{ convex, exponentially bounded,} \quad (76)$$

for the trading positions \( \Delta_i \in [-1, 1] \), and where \( p \) indicates the initial value of the portfolio variable. Actually, we may assume that the volatilities are functions of the assets as long as the regularity assumptions on the coefficients above in (C) are satisfied. As we shall observe below, the comparison result above then implies that an optimal strategy maximizes the basket volatility, i.e.,

$$\sup_{-1 \leq \Delta_i \leq 1, \ 1 \leq i \leq n} \sqrt{\frac{\sum_{i,j=1}^{n} \rho_{ij} \Delta_i \sigma_i \sigma_j S_i S_j}{\sum_{i=1}^{n} S_i}}. \quad (77)$$

Hence, signs of correlations (and space-time dependence of the signs of correlations) can change an optimal strategy essentially. This indicates also that multivariate mean comparison results are significant extensions of univariate results. However, the mathematically determined optimal strategies are strategies which switch between maximal short and long positions with high frequency such that standard passport option are considered to be expensive. Let us go deeper into this result and draw some new consequences. Note that the strategy processes \( \Delta \) define a family of value functions

$$v^\delta(t, s, p) := \mathbb{E}[\Pi_\Delta(T)]^+ | S(t) = s, \Pi(t) = p]. \quad (78)$$

which satisfy a pure diffusion equations. These value functions can be compared for regular volatility matrices, i.e. regular strategies \( \delta = (\delta_1, \cdots, \delta_n) \) especially, according to the comparison result above. For simplicity of notation, we rewrite the volatility matrix of the underlying assets in the form

$$\sigma \sigma^T(\delta) := (\delta_i \sigma_i S_i \rho_{ij} \delta_j \sigma_j S_j). \quad (79)$$

Components of this matrix may be denoted by \( (\sigma \sigma^T)_{ij}(\delta) \). Let \( \Sigma^T(\delta) \) the volatility matrix where \( \sigma \sigma^T(\delta) \) is augmented by the basket volatility term on the
diagonal (corresponding to the quadratic variation of $\Pi$) and by the correlation term related to the correlations of the portfolio variable and the underlyings. For time to expiration $\tau = T - t$ the passport option value function $v^p$ satisfies the HJB-equation

$$\frac{\partial v^p}{\partial \tau} - \sup_{-1 \leq \delta_i \leq 1, 1 \leq i \leq n} \text{Tr} \left( \Sigma \Sigma^T (\delta) D^2 v^p \right) = 0,$$

(80)

which has to be solved along with the initial condition $v^p(0, s, p) = (p - K)^+$. Here in the expression $-1 \leq \delta_i \leq 1$, $\delta_i$ refers to the value of a strategy function also denoted by $\delta_i$ for simplicity of notation. Here, $D^2 v^p$ is the Hessian with respect to the variables $(p, s)$. The supremum of the volatility matrix over the set of regular strategies is outside the regular control space required (say $C^3_b$) by the comparison result in general, but comparison over regular control spaces leads to the monotonicity condition

$$\delta, \delta' \in C^3_b, \quad (\sigma \sigma^T)(\delta) < (\sigma \sigma^T)(\delta') \Rightarrow v^\delta < v^{\delta'} \text{ on } (0, T] \times \mathbb{R}^n_+ \quad (81)$$

Here, note that we have expiry at $\tau = 0$ which corresponds to $t = T$. The limit or optimal strategy is a function which is just measurable in general, and the order of matrices means that the difference $(\sigma \sigma^T)(\delta') - (\sigma \sigma^T)(\delta)$ is positive definite.

Next let us go deeper into the question of existence of global solutions to the HJB-equations. First we have to remark that we cannot expect the existence of classical solutions, i.e., solutions which exist in $C^{1,2}$. Standard estimates for classical solutions of second order HJB-Cauchy problems even require data in $C^3$ (cf. [13] and [14]) in contrast to the Lipschitz-continuous data usually given in finance theory. However, as we have only measurable coefficients in the case of classical passport options, there are deeper reasons that we can only expect continuous or Hölder continuous solutions. Therefore it seems natural to consider such problems in the context of viscosity solutions. We do not need to reconsider this theory here, and refer to the classical reference in [2] and to [5] for the special case of HJB- equations.

The difficulties of regularity of solution related to lower regularity of coefficients may be a reason for considering discrete time control spaces (as Delbaen did). In this case the semi-group property of the operators ensures that we can reduce the HJB-problem to Cauchy problems with regular coefficients at each time step where the control remains constant. Superficially, this seems flexible enough as we may have variable time step sizes and arbitrarily small time step sizes. We may consider a partition of the time horizon interval $[0, T]$. However, continuous time control spaces have an interest in their own since we are interested in limit behavior. We are interested in limit behavior because it reveals new features which are not apparent in discrete models. This is true in many areas of scientific modelling. The control space has to be just large enough in order to construct the optimal strategy as a limit from strategies of this control space, where the limit has not to be a part of that control space if we know that it is well-defined for other reasons.

For each natural number $N \geq 1$ consider a discretization of a finite time horizon $[0, T]$, i.e., a set of adjacent intervals $\Delta^N_T := \{\Delta^i_T | 0 \leq i \leq 2^N - 1\}$,
where $\Delta_T^i = [T_{i-1}, T_i]$ The comparison result then implies that an optimal strategy function $\delta = (\delta_1, \cdots, \delta_n) : \Delta_N^S \times \mathbb{R}^n_+ \rightarrow [-1, 1]^n$ for a classical multivariate passport option on $n$ uncorrelated assets with a discrete time set $\{t_i = \frac{i}{N} | 0 \leq i \leq 2^N - 1\}$ has its values at the vertices of the hypercube $[-1, 1]^n$. Indeed such a strategy is independent from the asset values (a further difference to symmetric passport options introduced below, which is an interesting feature of the latter new type of product). For the former product we may define the strategy function as a piecewise constant step function on the time intervals $\Delta_T^i$ which is Lebesgue measure for each $N$ sure. Such a construction lead to natural limit control spaces of measurable functions as $N \uparrow \infty$. Such control spaces may be considered for classical passport options. We call them natural Lebesgue-measurable control spaces. The strategy function members of such natural Lebesgue-measurable control spaces can also be constructed from spaces of smooth functions with bounded derivatives, and we call the limit HJB-Cauchy problems with natural Lebesgue-measurable control spaces the associated HJB-Cauchy problems.

Next, what can we expect about solutions of HJB-Cauchy equations with measurable coefficients (related to optimal strategies)? Well, the best estimates for parabolic equations in this case are based on the works of Nash and de Giorgi. Especially de Giorgi’s estimates were adapted and extended to parabolic equations and quasilinear parabolic equations in [15]. However, these estimates require an uniform ellipticity condition as in C, and they provide no more than H"older continuous solutions. We cannot expect more in our framework of highly degenerate parabolic equations with measurable coefficients. In this generality a regularity and existence theory is not available, but viscosity solutions can be established in specific cases.

Therefore for the purpose of this paper, it is natural to impose the following extended assumption for HJB-equations related to multivariate classical passport options.

**HEHJB** For a HJB Cauchy problems in (80) we assume that the condition HE holds for smooth strategy functions $\delta_i$ with bounded derivatives. For a natural Lebesgue-measurable Limit control space of strategy functions we consider the associated HJB-Cauchy problem. Then this HJB-Cauchy problem is assumed to have a unique viscosity solution on the time interval $[0, T]$.

Now the matrix order in (81) corresponds to the basket volatility order

$$\sqrt{\sum_{i,j=1}^n (\sigma \sigma^T)_{ij}(\delta)} \leq \sqrt{\sum_{i,j=1}^n (\sigma \sigma^T)_{ij}(\delta')} \sum_{i=1}^n S_i$$

such that we can indeed consider the order of basket volatilities in order to determine optimal strategies. This is worth noting in our context since the stochastic sum $\Pi = \sum_i S_i^\Delta := \Delta_i S_i$ with increment $d\Pi = d(\sum_{i=1}^n dS_i^\Delta) = \sum_{i=1}^n dS_i^\Delta$ have the representation (with some standard Brownian motion $\tilde{W}$)

$$d\Pi = \sum_{ij} (\sigma \sigma^T)_{ij}(\Delta, S) d\tilde{W}.$$
Note that the HJB-equation in (80) depends on \( n + 1 \) variables corresponding to the processes \( \Pi, S_1, \cdots, S_n \) with a univariate payoff which depends only on the variable \( p \) corresponding to the portfolio process \( \Pi \). The comparison result above can be applied then directly. We do not need the representation in (83) in order to obtain optimal strategies, but this representation shows that optimal strategies maximize the volatility of the corresponding stochastic sum process, which is remarkable. With this preparations we get the following result.

**Theorem 4.1.** For a passport call written on \( n \) assets as above assume that the assumption HEHJB is satisfied. Then any optimal strategy maximizes the basket volatility function

\[
\sigma_B(\delta, s) := \sqrt{\sum_{ij} (\sigma \sigma^T)_{ij}(\delta)} = \sqrt{\langle Qs, \Lambda Qs \rangle}\]

(84)

where \( s = (\delta s_1, \cdots, \delta s_n)^T \) are the weighted asset values and \( Q^T \Lambda Q = (\sigma_i \rho_{ij} \sigma_j) \) with eigenvalue matrix \( \Lambda = \text{diag}(\lambda_i, 1 \leq i \leq n) \), i.e. \( Q \) denotes the diagonalization matrix for the volatility matrix \( (\sigma_i \rho_{ij} \sigma_j) \). As a consequence any optimal strategy satisfies at any time

\[
\delta_{opt} \in \{Q^T \delta_{vc}^l | \delta_{vc}^l \in V_C \}
\]

(85)

where \( V_C \) denotes the set of strategy functions with values in the set of vertices of the cube \([-1, 1] \). i.e.,

\[
V_C := \{\delta | \delta : [0, T] \times \mathbb{R}^n_s \times \mathbb{R} \to \{-1, 1\}^n \text{ is a measurable function}\}.
\]

(86)

Here, \( \delta_{vc}^l(t, s, p) = (\delta_{vc1}^l(t, s, p), \cdots, \delta_{vcn}^l(t, s, p)) \) and for a matrix \( Q \) and a vector function \( \delta \) the expression \( Q \delta = \left( \sum_{j} q_{1j} \delta_j, \cdots, \sum_{j} q_{nj} \delta_j \right)^T \) is understood pointwise. Optimal strategies are then determined as follows. Start with \( (t, s, p) \to \delta_{vc}^l(t, s, p) = (\delta_{vc1}^l(t, s, p), \cdots, \delta_{vcn}^l(t, s, p)) \in V_C \) where each component is determined by the corresponding one-dimensional marginal problem (for each \( 1 \leq i \leq n \) passport option written on \( S_i \) where all \( S_j \) for \( j \neq i \) are set to zero). Then \( Q^T \delta_{vc}^{opt} \) is an optimal strategy of the multivariate passport option.

**Proof.** Follows from the assumption HEHJB, the construction of a natural control space of measurable functions and the comparison theorem.

The result above has the interesting consequence that correlations can have the effect that in situations of special correlations an investor who follows an optimal strategy may switch between long and short limit positions of one asset or some assets and does not invest in some or all other assets at all. Nevertheless the high frequency switching strategy between long and short positions which is known for marginal univariate problems is somehow preserved in an optimal strategy of the multivariate problem. From a financial point of view multivariate passport options cannot be subsumed by lookback options as in the univariate case, because the optimal strategies are much more complex. Nevertheless the high frequency shifting of huge amount of sums between large long and short positions survives essentially at least for some of the underlying in an optimal strategy. This may be considered as impractical.
Furthermore, the fact that the trading constraints on the assets are asymmetric for classical options on a traded account limits the applicability of such contracts. If we consider a foreign exchange type option with the underlying currencies dollar and euro, the contract with asymmetric constraints would be different from the perspective of the dollar and the euro investor. This is not the case for the plain vanilla options, where a call option from the perspective of the first currency is a put option from the perspective of the second currency. This leads to a natural question, whether we can formulate an option on a traded account with the symmetrical treatment of the two underlying assets.

The approach that treats both assets symmetrically is rather straightforward. Instead of imposing an absolute restriction on the position in the first asset, one can simply require a relative restriction in terms of the fraction of the current wealth. The most natural restriction is to allow the client to invest any proportion of his wealth to the first asset, so the $\alpha$ fraction of the invested wealth in that asset is in the interval $[0, 1]$. Obviously, this is symmetric with respect to the second asset as the residual wealth proportion $1 - \alpha$ invested in the second asset is restricted to the same interval $[0, 1]$. Moreover, this approach generalizes to any number of assets, so we can formulate the symmetric problem for an arbitrary number of assets $N$. This is a very natural approach as the investors are typically free to invest any portion of their wealth to assets of their choice, corresponding to $[0, 1]$ fraction of their total wealth. The problem of finding the optimal strategy that maximizes the option value is rather complex for any $N > 2$, and thus we limit ourselves only to $N = 2$ in this application section.

Imposing symmetric trading restrictions is only the first necessary step for the symmetric treatment of the underlying assets. We also need to use the reference asset that treats the individual assets symmetrically. A reference asset candidate here is an index consisting of 50% of both assets. In the following text, we mathematically formalize the definition of the option on a traded account that treats both assets symmetrically. This is model independent. Next, we assume geometric Brownian motion dynamics and first derive the evolution of the asset prices with respect to the index. In the following step, we find the evolution of the actively traded account with respect to the index. In order to find the optimal strategy, we need the above generalization of Hajek's comparison theorem (which extends a comparison result for stochastic sums published in Kampen (2016)) and adapt it to our problem. It is still true that the optimal strategy has the largest volatility with respect to the index, which is interesting in itself as the resulting portfolio has the largest price variance and thus it determines the maximal possible distributional departure in the sense of $L^2$ norm from the index that can be achieved by an active trading. This strategy is well known stop-loss strategy, which invests all the wealth in the weaker asset.

So let us consider just two underlying assets, let us call them $S$ and $M$. For instance, they can represent a stock market and a money market for the stock market type contracts, or two currencies in the foreign exchange type contracts. These are the names of the assets with no numerical value rather than the prices. The price $S_M(t)$ of the asset $S$ with respect to the asset $M$ is defined as how many units of $M$ are needed at time $t$ to acquire a single unit of the asset $S$. As the price is a number representing a relationship of two assets, we systematically...
use the above two asset notation in the following text to reflect it. The asset appearing in the subscript is traditionally referred to as a reference asset or a numeraire. We will use different reference assets in our analysis. For simplicity, we consider only assets that have its own martingale measure, such as the stocks that reinvests dividends or the money markets. For instance, the prices expressed with respect to a reference asset \( M \) are \( P^M \) martingales, in particular \( S_M(t) \) is a \( P^M \) martingale. Assets that do not have its own martingale measure, such as the currencies, can be linked to the corresponding money markets using the proper discounting.

We can further simplify the setup and introduce the following scaling

\[ S_M(0) = 1. \]

The investor creates a self-financing portfolio \( X \) by starting at \( X(0) = S(0) = M(0) \) and at time \( t \):

\[ X(t) = \Delta^S(t)S(t) + \Delta^M(t)M(t). \tag{87} \]

A natural restriction we consider in this paper is

\[ \Delta^S(t) \geq 0, \quad \Delta^M(t) \geq 0, \tag{88} \]

so he is not allowed to be short in any of the funds. The constraint in Equation (88) means that the investor is free to invest any fraction between \([0, 1]\) of his wealth \( X \) into the stock market \( S \) with the remaining fraction of his wealth going to the money market \( M \). This follows from Equation (87) by using \( X \) as a numeraire:

\[ 1 = \Delta^S(t)S_X(t) + \Delta^M(t)M_X(t). \tag{89} \]

The lower bound condition in one of the markets imposes an upper bound condition in the second market, so the positions are constrained by

\[ X_S(t) \geq \Delta^S(t) \geq 0, \quad X_M(t) \geq \Delta^M(t) \geq 0. \tag{90} \]

This is a very natural condition. Moreover, it treats both assets equivalently, imposing the same restriction. Note that the upper bounds are random and depend on the current value of the investor’s wealth \( X \). A trivial observation is that \( X \) must be always non-negative with zero wealth being an absorbing boundary.

**Remark 4.2 (Relationship to passport options).** The constraint for the univariate classical passport option is on the position in the stock market only

\[ a \leq \Delta^S(t) \leq b, \]

the position in the second market \( M \) follows from

\[ \Delta^M(t) = X_M(t) - \Delta^S(t)S_M(t). \]

In particular, it can be negative even in the situation when we constrain the investor to have a positive position in \( S(t) \) by requiring \( \Delta^S(t) \geq a \geq 0 \). The condition is not symmetric for both assets, the imposed restriction does not treat them equivalently. This is arguably one of the main reasons why such contract is not appealing to the investors. Moreover, the traded account can become negative in contrast to the situation that treats both assets symmetrically.
For preservation of the symmetry of the contract, it is necessary that the reference asset also treats both assets equally. One obvious choice is to use
\[ N(t) = \frac{1}{2}(S(t) + M(t)). \] (91)
The asset \( N \) can be regarded as an index consisting of the two assets, or equivalently, a basket of the two assets.

The contract on the actively traded account can be then defined by a payoff at the terminal time \( T \)
\[ (X_N(T) - K)^+ \text{ units of } N(T) \] (92)
for some contractually defined strike \( K \). As \( X_N(0) = 1 \), the strike that corresponds to the at the money option is equal to \( K = 1 \). In order to preserve the symmetry, the contract has to be settled in the index \( N \) rather than a single asset \( S \) or \( N \). For instance, if the contract is written on two currencies, say dollar and euro, the contract seen from the position of the investor or the euro investor is identical. Next we exemplify the previous discussion in the context of a GBM model. We note that the following considerations can be generalized rather straightforwardly to the case of variable volatilities. The seller of the option must be ready to cover any trading strategy used by the holder of the contract. In order to indicate the dependence of the portfolio on the strategy we may sometimes write \( X^\Delta := X \) in the following. The fair price of the contract corresponds to the trading strategy \( \Delta^S(t) \) that maximizes the expectation of the
\[ \mathbb{E}^N(X^\Delta_N(T) - 1)^+. \] (93)
Let us assume geometric Brownian motion model for the stock price \( S_M(t) \), so
\[ dS_M(t) = \sigma S_M(t) dW^M(t). \] (94)
Any discounting is already incorporated in the money market \( M \) and the price \( S_M(t) \) is \( P^M \) martingale. Similarly, the inverse price
\[ dM_S(t) = \sigma M_S(t) dW^S(t) \] (95)
is a \( P^S \) martingale. The relationship between \( W^M(t) \) and \( W^S(t) \) is
\[ dW^S(t) = -dW^M(t) + \sigma dt. \] (96)
From the self-financing trading assumption, the evolution of the trading portfolio \( X \) is
\[ dX_M(t) = \Delta^S(t) dS_M(t) \] (97)
and
\[ dX_S(t) = \Delta^M(t) dM_S(t). \] (98)
In order to find the optimal strategy, we need to find price evolutions with respect to the index \( N \).

**Lemma 4.3.** The evolution of the price \( M_N(t) \) under the probability measure \( P^N \) is given by
\[ dM_N(t) = \frac{1}{2}\sigma M_N(t)(2 - M_N(t))dW^N(t). \] (99)
Proof. Note that
\[ M_N(t) = \frac{M(t)}{\frac{1}{2}(M(t) + S(t))} = \frac{2}{1 + S_M(t)} \] (100)
and thus
\[
dM_N(t) = d \left( \frac{2}{1 + S_M(t)} \right)
= - \left( \frac{2}{(1 + S_M(t))^2} \right) \sigma S_M(t) dW^M(t) + \left( \frac{2}{(1 + S_M(t))^2} \right) \sigma^2 S^2_M(t) dt \\
= \left( \frac{2}{(1 + S_M(t))^2} \right) \sigma S_M(t) \left[ -dW^M(t) + \frac{\sigma S_M(t)}{(1 + S_M(t))} dt \right] \\
= \left( \frac{2}{(1 + S_M(t))^2} \right) \sigma S_M(t) dW^N(t) \\
= \sigma M_N(t) \left( \frac{S_M(t)}{1 + S_M(t)} \right) dW^N(t) \\
= \frac{1}{2} \sigma M_N(t) S_N(t) dW^N(t) \\
= \frac{1}{2} \sigma M_N(t) (2 - M_N(t)) dW^N(t).
\]
The process \( M_N(t) \) must be \( P^N \) martingale, which determines \( W^N(t) \) as
\[
dW^N(t) = -dW^M(t) + \frac{\sigma S_M(t)}{(1 + S_M(t))} dt \] (101)
\[
= -dW^M(t) + \frac{1}{2} \sigma S_N(t) dt.
\]

The SDE in Equation (101) is interesting on its own as it represents the evolution of the asset with respect to the index. From the definition of \( N \) in Equation (91), we have
\[ 2 = M_N(t) + S_N(t), \] (102)
constraining the \( M_N(t) \) process between 0 and 2:
\[ 0 \leq M_N(t) \leq 2. \]
One can think about \( M_N(t) \) as the scaled proportion of the money market \( M \) in the index \( N \). The price \( M_N(t) \) has the largest volatility when \( M_N(t) = 1 \), or in other words, when \( M(t) = S(t) \). The process \( M_N(t) \) loses volatility in two extreme cases, when \( M_N(t) = 0 \) and when \( M_N(t) = 2 \). The first case corresponds to \( S_M(t) = \infty \), so the asset \( M \) is worthless in comparison with the asset \( S \), the second case corresponds to \( S_M(t) = 0 \) when the asset \( S \) is worthless in comparison with the asset \( M \).

Note that from the symmetry of the problem, we have immediately
\[
dS_N(t) = -\frac{1}{2} \sigma S_N(t) (2 - S_N(t)) dW^N(t). \] (103)
It also follows from Equation \(102\).

Now we are ready to compute the evolution of \(X_N(t)\).  

**Lemma 4.4.** The evolution of the actively traded portfolio \(X\) with respect to the index \(N\) follows:

\[
dX_N(t) = \frac{1}{2} \left( X_N(t) - 2\Delta^S(t) \right) \sigma S_N(t) dW^N(t). \tag{104}
\]

*Proof.* We have

\[
dX_N(t) = d(X_M(t) \cdot M_N(t))
\]

\[
= X_M(t) dM_N(t) + M_N(t) dX_M(t) + dX_M(t) dM_N(t)
\]

\[
= X_M(t) \frac{1}{2} \sigma M_N(t) S_N(t) dW^N(t) + M_N(t) \Delta^S(t) \sigma S_M(t) dW^M(t)
\]

\[
- \Delta^S(t) \sigma S_M(t) \frac{1}{2} \sigma M_N(t) S_N(t) dt
\]

\[
= \frac{1}{2} \sigma X_N(t) S_N(t) dW^N(t) - \Delta^S(t) \sigma S_N(t) \left[ -dW^M(t) + \frac{1}{2} \sigma S_N(t) dt \right]
\]

\[
= \frac{1}{2} \left( X_N(t) - 2\Delta^S(t) \right) \sigma S_N(t) dW^N(t).
\]

\[
\square
\]

From \(\Delta^S(t) = X_S(t) - \Delta^M(t) M_S(t)\), we also have an alternative representation

\[
dX_N(t) = -\frac{1}{2} \left( X_N(t) - 2\Delta^M(t) \right) \sigma M_N(t) dW^N(t). \tag{105}
\]

For a given convex payoff function \(f\) the related symmetric passport option price function \(v^{op}\) has the representation

\[
v^{\delta}(t, x, y) = \sup_{0 \leq \delta, 0 \leq \Delta \leq X_N} \mathbb{E}^{(t, x, y)} \left( f(X_N) \right).
\]

(106)

where \(S_N(t) = x, X_N^2(t) = y\) are initial values of the respective processes at time \(t\).

For a given stochastic strategy \(\Delta\) we may define a value function

\[
v^{\delta}(t, x, y) := \mathbb{E}^N[(X_N^2(T) - 1)^+] | S_N(t) = x, X_N^2(t) = y]. \tag{107}
\]

Let us consider the transformation to normal coordinates \(u^{\delta}(\tau, z_1, z_2) := v^{\delta}(t, z_1, z_2)\), where \(\tau = T - t, z_1 = \ln(x)\) and \(z_2 = \ln(y)\). The stochastic strategy \(\Delta\) corresponds to a strategy \(\delta\) in value space which is a function of the underlyings. The function \(u^{\delta}\) satisfies the initial- boundary value problem

\[
u^{\delta}_\tau - \frac{1}{8} \sigma^2 (2 - \exp(z_1))^2 u^{\delta}_{z_1 z_1} + \frac{1}{4} \sigma^2 (2 - \exp(z_1))(\exp(z_1) - 2\delta) u^{\delta}_{z_2 z_2}
\]

\[
= -\frac{1}{8} \sigma^2 (\exp(z_1) - 2\delta)^2 u^{\delta}_{z_2 z_2} = 0. \tag{108}
\]

with initial condition

\[
u(0, z_1, z_2) = (\exp(z_2) - 1)^+ \tag{109}
\]

24
We impose natural boundary conditions at spatial infinity, and have an additional finite boundary condition at \( z_1 = \log(2) \). We get
\[
    u_\tau^\delta - \frac{1}{8} \sigma^2 (2 - 2\delta)^2 u_{z_2 z_2} = 0, \quad \text{at} \quad z_1 = \log(2).
\] (110)

This equation corresponds to the process
\[
    dX_N(t) = \frac{1}{2} (X_N(t) - 2 \Delta^S(t)) \sigma S_N(t) dW^N(t)
\] (111)

such that we can apply Hajek’s result at the boundary where \( z_1 = \log(2) \). Hence we know \( \delta = 0 \) at \( \{(\tau, z_1, z_2) | z_1 = \log(2)\} \) a priori. We may say that \( \delta \) lives in reduced control space if \( \delta \in C_c := \{\delta \in C^3 | \delta_{|z_1=\ln(2)} = 0\} \). The boundary condition reduces to
\[
    u_\tau^\delta - \frac{1}{2} \sigma^2 u_{z_2 z_2} = 0, \quad \text{at} \quad z_1 = \log(2),
\] (112)

and such a boundary condition can be considered if the volatilities are regular functions. In case of constant volatilities the latter condition simplifies to
\[
    u_\tau^\delta(t, \log(2), z_2) = \exp(z_2) \cdot N(d_+) - N(d_-),
\] (113)

where \( d_{\pm} = \frac{z_2 \pm \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \). The problem may be considered on the domain \( D = [0, T] \times (-\infty, \log(2)] \times \mathbb{R} \). There are three further issues here concerning comparison: a) in which space does the strategy function \( \delta \) live?; b) the problem has a boundary in finite space, and comparison has to be adapted to this situation, and c) the spatial part of the operator is not strictly elliptic. We formulate the comparison theorem in regular strategy spaces and for a regularized problem. More precisely, we modify the asset dynamics, where for small \( \epsilon > 0 \) we define
\[
    dS_N^\epsilon(t) = -\frac{1}{2} \sigma S_N(t) (2 - S_N(t)) dW^{N,\epsilon}(t)
\] (114)

where \( W^{N,\epsilon}(t) \) is constructed by adding a small perpendicular process, i.e.,
\[
    dW^{N,\epsilon}(t) = dW^N + \epsilon dW^{\perp,N}, \quad \langle dW^N, dW^{\perp,N} \rangle = 0
\] (115)

The corresponding equation for \( u_\tau^{\delta,\epsilon} \) gets an additional factor \((1 + \epsilon)^2\) in the second term of the equation (108) and becomes strictly elliptic. Concerning issue a) we compare \( C^3 \) strategies in order to prove an identity for derivatives of the density and its adjoint up to second order. The issue in b) is addressed in the proof of the following theorem.

**Theorem 4.5 (Comparison Theorem).** Let \( \delta, \delta' \in C^3_c \) and \( \epsilon > 0 \) be strategies of the value functions \( u^{\delta,\epsilon}, u^{\delta',\epsilon} \) defined on the domain \( D \). Then the order of these value functions is induced by the order of the volatility of the portfolio term alone, i.e., for \( \tau \in (0, T] \)
\[
    \frac{1}{8} \sigma^2 (x - 2\delta)^2 < \frac{1}{8} \sigma^2 (x - 2\delta')^2 \Rightarrow u^{\delta,\epsilon}(\tau, .) < u^{\delta',\epsilon}(\tau, .).
\] (116)

**Proof.** For small positive angle \( \theta \) consider the transformed coordinates
\[
    \begin{pmatrix}
    \tilde{z}_1 \\
    \tilde{z}_2
    \end{pmatrix} =
    \begin{pmatrix}
    \cos(\theta) & \sin(\theta) \\
    -\sin(\theta) & \cos(\theta)
    \end{pmatrix}
    \begin{pmatrix}
    z_1 \\
    z_2
    \end{pmatrix}
\] (117)
Multiplying with the inverse (rotation by $-\theta$), we observe that $-\sin(-\theta)\tilde{z}_1 + \cos(-\theta)\tilde{z}_2 = \sin(\theta)\tilde{z}_1 + \cos(\theta)\tilde{z}_2 = \tilde{z}_2$ such that both coefficients $\sin(\theta), \cos(\theta)$ in the sum representation of $\tilde{z}_2$ are positive for $\theta$ positive and small. Let $u^{\delta,\theta}$ and $u^{\delta',\theta}$ denote the value functions in rotated coordinates. Recall that we have a reduced boundary condition which does not depend on $\delta$ (due to the application of the classical Hajek result on the boundary). For the sake of comparison of both functions we can reduce to zero boundary conditions and extend both functions trivially to the whole space. These trivial extensions to the whole space may be denoted still by $u^{\delta,\theta}$ and $u^{\delta',\theta}$ for simplicity. The latter reduction simplifies classical representations of the value functions $u^{\delta}$ and $u^{\delta'}$ and classical representations of their derivatives up to second order. Especially, we can avoid the boundary terms in these representations (boundary layer terms). Here, by classical representations we mean the classical representations of solutions of initial boundary value problems in terms of the fundamental solution.

Remark 4.6. Such reductions to zero boundary conditions seem to be not familiar to all readers in the probabilistic community. We shall give a more detailed description of this step in the more interesting case of multivariate symmetric passport options in a subsequent paper.

In this context we remark that for almost all regular functions $\delta$ the fundamental solution is well-defined. Hence in these transformed extended coordinates, the problem is defined on the whole space where initial data are defined as a payoff of a weighted sum $(\exp(\tilde{z}_2) - 1)^+ = (\exp(\sin(\theta)\tilde{z}_1 + \cos(\theta)\tilde{z}_2) - 1)^+$. For a payoff $f$ define for small $\delta_0 > 0$ and large $R$ an approximation of the payoff function

$$f^R_{\delta_0}(w) = \begin{cases} f(w) & \text{if } |w| \leq R, \\
 f(w) \exp(-\delta_0 |w - R|^2) & \text{if } |w| > R, \end{cases} \quad (118)$$

and let $f^R_{\delta,\delta_0}$ be a smoothed version of $f^R_{\delta,0}$ (smoothing close to identity). Note that the function $f^R_{\delta,\delta_0}$ is in $H^2 \cap C^2$. Let $u^{\delta,\theta,\epsilon,\delta_0, R}$ be a value function of the regularized (i.e., strictly elliptic approximation) form of the equation (108) in rotated coordinates with data $f^R_{\delta,\delta_0}(\sin(\theta)\tilde{z}_1 + \cos(\theta)\tilde{z}_2)$, let $p^{\delta,\theta,\epsilon,\delta_0, R}$ be the corresponding fundamental solution, and let $p^{\theta,\epsilon,\delta_0, R}$ be its adjoint (backward and forward equation density in probabilistic terms). The approximative value function itself and the multivariate spatial derivatives of order $|\alpha| \leq 2$ have essentially the representation

$$D^\alpha u^{\delta,\theta,\epsilon,\delta_0, R}(\tau, \tilde{z}_1, \tilde{z}_2) = \int_0^\tau \int_{\mathbb{R}^n} f^R_{\delta,\delta_0}(\xi)d\tau \left( D^\alpha \right) p^{\delta,\theta,\epsilon,\delta_0, R}(\tau, \tilde{z}_1, \tilde{z}_2; \sigma, \xi_1, \xi_2)d\sigma,$$

where we can suppress the identical boundary term due to the reduction indicated above. Next, for $w = (w_1, w_2)$ and $\sigma < s < \tau$ define

$$\overline{\pi}(s, w) = p^{\delta,\theta,\epsilon,\delta_0, R}(s, w; \sigma, \xi_1, \xi_2)$$

$$\overline{\pi}(s, w) = p^{\Theta,\epsilon,\delta_0, R}(s, w; \tau, \tilde{z}_1, \tilde{z}_2). \quad (120)$$

Let $L\overline{\pi} = 0$ and $L^*\overline{\pi} = 0$ abbreviate the equations for $\overline{\pi}, \overline{\pi}$ (approximative equations for (108)). We may assume that $\epsilon > 0$ is small enough such that
Theorem 4.7. Let \( \delta \) be monotonically increasing as \( \delta \) increases. The sequence of functions \( (\delta^\alpha) \) converges to \( \delta \). Hence we can do partial integration and obtain comparison for arbitrary small \( \theta \) which is preserved in the limit \( \theta \downarrow 0 \).

We conclude that

\[
(D^\alpha_\xi \overline{\pi})(\tau, z) = (D^\alpha_\xi \overline{\pi})(\sigma, \xi).
\]

Hence we can do partial integration and obtain comparison for arbitrary small \( \theta \) which is preserved in the limit \( \theta \downarrow 0 \).

The regular control space \( C^3 \) does not contain the volatility-maximizing function \( I(z_1 \leq 0) = I \exp(z_1 \leq 1) \) or \( I(x \leq 1) \) of the portfolio term. Define the sequence of functions \( h^\epsilon \), where

\[
h^\epsilon(z_1) = \begin{cases} 
1 & \text{if } z_1 \leq -\epsilon, \\
\exp\left(-1 - \frac{z_1}{\epsilon}\right) & \text{if } -\epsilon \leq z_1 \leq 0, \\
0 & \text{else.}
\end{cases}
\]

Let \( \delta^\epsilon \in C^\infty \) be defined by a convolution of \( h^\epsilon \) with a smoothing Gaussian kernel which is close to identity. Then this a sequence of functions \( \delta^\epsilon \) which is monotonically increasing as \( \epsilon \) decreases and \( \lim_{\epsilon \downarrow 0} \delta^\epsilon = \delta^\text{opt} = I(x \leq 1) \). According to the Comparison Theorem we have

\[
v^\delta(t, x, y) = \lim_{\epsilon \downarrow 0} \mathbb{E}^N[(X^N_\delta(T) - 1^+) | S_N(t) = x, X^N_\delta(t) = y]
\]

and stochastic ODE theory shows that the limit \( v^\text{opt}(t, x, y) \) exists as \( \epsilon \downarrow 0 \).

**Theorem 4.7 (Optimal strategy).** The optimal strategy maximizing \( \mathbb{E}^N[X_N(T) - K]^+ \) is given by

\[
dx_N(t) = \frac{1}{2} (S_N(t) - 2 \cdot I(S_N(t) \leq 1)) \sigma X_N(t) dW_N(t).
\]

27
Proof. According to the Comparison theorem, the optimal strategy maximizes the absolute value of the $dW^N(t)$ term. The optimal position $\Delta^S(t)$ is attained at one of the ends of the interval for its possible range. When $\Delta^S(t) = 0$, the absolute value reduces to $X_N(t)$.

When $\Delta^S(t) = X_S(t)$, the absolute value is equal to $-X_N(t) + 2X_S(t)$.

Thus $\Delta^S(t) = 0$ is optimal when

$$X_N(t) \geq -X_N(t) + 2X_S(t),$$

which is equivalent to

$$S(t) \geq M(t).$$

$$\Delta^S(t) = \begin{cases} X_S(t), & S(t) \leq M(t), \\ 0, & S(t) \geq M(t), \end{cases}$$

and

$$\Delta^M(t) = \begin{cases} 0, & S(t) \leq M(t), \\ X_M(t), & S(t) \geq M(t), \end{cases}$$

More succinctly,

$$\Delta^S(t) = X_S(t) \cdot I(S_N(t) \leq 1), \quad \Delta^M(t) = X_M(t) \cdot I(S_N(t) \geq 1).$$

Thus it is optimal to be fully invested in the weaker asset. The evolution of the optimal portfolio is given by

$$d\bar{X}_N = \frac{1}{2} (\bar{X}_N - 2\Delta^S(t)) \sigma S(t) dW^N(t)$$

$$d\bar{X}_M = \frac{1}{2} (\bar{X}_M - 2\Delta^M(t)) \sigma M(t) dW^M(t)$$

\[\square\]
Appendix A: Value function representation, the adjoint, and derivatives

We consider some observations of Section 3 in more detail. For $x, y, h \in \mathbb{R}^n$ fixed we consider

$$v(\sigma, z) = p(\sigma, z; s, y + h), \ u(\sigma, z) = p^*(\sigma, z; t, x + h), \ t > s. \quad (132)$$

We have $Lv = 0$ and $L^*u = 0$, and integrating equation $\ref{130}$ over $[s + \epsilon, t - \epsilon] \times B_R$ we have

$$\int_{B_R} u(t - \epsilon, z)v(t - \epsilon, z) - u(s + \epsilon, z)v(s + \epsilon, z)dz = \int_{B_R} \left( \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{n} \left( u \partial a_{ij} - v \partial a_{ij} - u \partial a_{ij} - v \partial a_{ij} \right) \right) (\sigma, z) dSd\sigma. \quad (133)$$

$a_{ij}$ and $\frac{\partial a_{ij}}{\partial x_j}$ are usually assumed to be bounded. However for the purpose of estimating the right side of $\ref{133}$ it is sufficient that $a_{ij}$ is of linear growth and that all the other assumption in $\ref{12}$ are matched. In the case we are of estimating the right side of $\ref{133}$ it is sufficient that $a_{ij}$ and any multiindex $\alpha$ is of linear growth factor time a Gaussian where we have an exponential decay as the Euclidean distances $|\sigma, z - (s, y + h)|$ and $|(\sigma, z) - (s, x + h)|$ become large. Hence, the right side of $\ref{133}$ goes to zero as $R \uparrow \infty$, and we have

$$\int_{\mathbb{R}^n} u(t - \epsilon, z)v(t - \epsilon, z)dz = \int_{\mathbb{R}^n} u(s + \epsilon, z)v(s + \epsilon, z)dz,$$

which is

$$\int_{\mathbb{R}^n} v(t - \epsilon, z)p^*(t - \epsilon, z; t, x + h)dz = \int_{\mathbb{R}^n} u(s + \epsilon, z)p(s + \epsilon, z; s, y + h)dz \quad (134)$$

In the limit $\epsilon \downarrow 0$ in the integrand on the left side we have $p^*(t - \epsilon, z; t, x + h) \to \delta(z - (x + h))$ and in the integrand on the right side we have $p(s + \epsilon, z; t, x + h) \to \delta(z - (y + h))$ such that indeed

$$v(t, x + h) = u(s, y + h), \quad (135)$$

where $h \in \mathbb{R}^n$ was free fixed choice.

Hence, the relation holds for finite difference quotients

$$D_h^+ u(s, y) = \frac{u(s, y + h) - u(s, y)}{h}, \ D_h^- u(s, y) = \frac{u(s, y) - u(s, y - h)}{h}, \quad (136)$$

and

$$D_{h_i, h_j}^2 u(s, y) = \frac{D_{h_i}^+ u(s, y) - D_{h_i}^- u(s, y)}{h_j} = \frac{u(s, y + h_i) - 2u(s, y) + u(s, y - h_i)}{h_i h_j}. \quad (137)$$

for second order finite differences. Applying regularity of $v$ and $u$, for $h \downarrow 0$ (or $h_i, h_j \downarrow 0$ and any multiindex $\alpha$ we get indeed

$$D_\alpha^+ v(t, x) = D_\alpha^0 u(s, y). \quad (138)$$

For the understanding of Greek representations it is useful to reconsider for
any fixed $h \in \mathbb{R}^n$ we first observe
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} u(t - \epsilon, z)v(t - \epsilon, z)dz = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} u(s + \epsilon, z)v(s + \epsilon, z)dz,
\]
iff
\[
p(t, x + h; s, y + h) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} p(t - \epsilon, z; s, y + h)p^*(t - \epsilon, z; t, x + h)dz
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} p^*(s + \epsilon, z; t, x + h)p(s + \epsilon, z; s, y + h)dz = p^*(s, y + h; t, x + h)
\]
for $t > s$.

Let $\{e_i\}_{1 \leq i \leq n}$ be the Euclidean basis of $\mathbb{R}^n$. For any given $1 \leq i \leq n$ and $h_i$ the mean value theorem implies that for some $y_{i(h_1)}^*, y_{i(h_2)}^* \in [y, y + h_ie_i]$ we have
\[
\lim_{h_i \to 0} p(t + h_ie_i, s; y + h_ie_i) - p(t, s; y + h_ie_i) = \lim_{h_i \to 0} p(t, s; y + h_ie_i) - p(t, s; y)
+ \lim_{h_i \to 0}(p_{y_i}(t + h_ie_i, s, y_{i(h_1)}^*) - p(t, s, y_{i(h_2)}^*))
= p_{x_i}(t, x; s, y)
\]
Similarly
\[
\lim_{h_i \to 0} p^*(s, y + h_ie_i; t + h_ie_i) - p^*(s, y; t + h_ie_i) = \lim_{h_i \to 0} p^*(s, y; t + h_ie_i) - p^*(s, y; t)
+ \lim_{h_i \to 0}(p_{y_i}^*(s, y + h_ie_i, t, x_{i(h_1)}^*) - p(s, y; t, x_{i(h_2)}^*))
= p_{x_i}^*(s, y; t, x).
\]
Invoking (139) the terms in (140) and (141) are equal, i.e., for all $t > s, x, y \in \mathbb{R}^n$
\[
p_{x_i}(t, x; s, y) = p_{x_i}^*(s, y; t, x).
\]
Then we may start with these expressions and shifted versions
\[
p_{x_i}(t, x + h; s, y + h) = p_{x_i}^*(s, y + h; t, x + h),
\]
and repeat the argument above which leads to identities as in (138). The finite difference difference approximations above can be used in order to develop computation schemes for the Greeks (with derivative shifts for $f \in C^{[\alpha]} \cap H^{[\alpha]}$
\[
D_{x}^{\alpha}v^f(t, x) = \int_{\mathbb{R}^n} f(y)D_{y}^{\alpha}p(t, x; t_0, y)dy = \int_{\mathbb{R}^n} f(y)D_{y}^{\alpha}p^*(t_0, y; t, x)dy.
\]
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