Neutrino Emission From Direct Urca Processes in Pion Condensed Quark Matter

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We study neutrino emission from direct Urca processes in pion condensed quark matter. In compact stars with high baryon density, the emission is governed by the gapless modes of the pion condensation which leads to an enhanced emissivity. While for massless quarks the enhancement is not remarkable, the emissivity is significantly larger and the cooling of the condensed matter is considerably faster than that in normal quark matter when the mass difference between $u$- and $d$-quarks is sizable.

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I. INTRODUCTION

Compact stars are born at temperatures $T \gtrsim 10^{11}$ K (about tens of MeV) and rapidly cool down to $10^{8}$ K (about a few MeV) within minutes, and then the cooling process is dominated by neutrino emission for about $10^{4}$ years during which the temperature drops down to $10^{6}$ K (about a few KeV) \cite{1, 2}. Since the typical density in a compact star could be several times the normal nuclear density, matter with strong interaction at such high density and low temperature must be highly degenerate and may be in a superfluidity/superconductivity or hybrid state \cite{3, 4, 5}. Neutrino cooling processes in these phases may behave differently from that in the normal phase, and the observations of cooling rate can provide constraints on the phase structure of dense matter.

Many cooling mechanisms have been investigated in literatures. If a compact star is in the state of nuclear matter, the cooling is governed by the modified Urca processes and the temperature dependence of the neutrino emissivity is $\epsilon \sim T^{8}$ \cite{6, 7, 8, 9, 10}. When the proton abundance is large enough ($\gtrsim 11\%$), the direct Urca processes can take place and lead to a larger neutrino emissivity $\epsilon \sim T^{6}$ \cite{11, 12}. When pion or kaon condensation is present, such a fast cooling can be realized as well \cite{13, 14, 15}. In the nuclear superfluid, the quasi-particle spectrum of nucleons is gapped and the cooling mechanism is dramatically changed. At low temperatures $T \ll T_{c}$ being the critical temperature of nuclear superfluid, the Cooper pairs can rarely be broken by thermal fluctuations, the neutrino emissivity is suppressed by a Boltzman factor $e^{-\Delta/T}$ with $\Delta$ being the energy gap \cite{16, 17}. When the temperature approaches $T_{c}$, the effect of pair breaking and recombination results in an emissivity of $\epsilon \sim T^{7}$ for neutrino pair radiation \cite{18}.

Compact stars can be in the state of quark matter \cite{19, 20}, when the baryon density is high enough. The neutrino processes in normal quark matter are similar to that in nuclear matter. For example, the direct Urca processes including the $\beta$-decay ($d \rightarrow u + e^{-} + \bar{\nu}_{e}$) and the electron capture ($u + e^{-} \rightarrow d + \nu_{e}$) in normal quark matter lead to an emissivity of $\epsilon \sim T^{6}$ \cite{21, 22, 23}, while the modified Urca processes give $\epsilon \sim T^{8}$ \cite{24}. Quark matter at asymptotically high density is in the color-flavor locked (CFL) phase \cite{25} of color superconductivity (CSC). The CSC phases at moderate density are still unclear due to the complicated non-perturbative effect. Various candidates have been proposed, such as two flavor CSC (2SC) \cite{26}, gapless 2SC \cite{27}, gapless CFL (gCFL) \cite{28}, crystalline CSC \cite{29}, and spin-1 CSC \cite{30, 31, 32}. Neutrino emission from CSC quark matter is exponentially suppressed at low temperature, if the quasi-particle spectra are fully gapped \cite{33, 34, 35, 36, 37, 38}. Similar to nuclear matter, the pair breaking and recombination effect results in an emissivity of $\epsilon \sim T^{5}$ at temperature close to $T_{c}$ \cite{39}. The emissivity behaves as $\epsilon \sim T^{5-5}$ for gCFL \cite{40} and $\epsilon \sim T^{8}$ for gSC, see, e.g. Table 1 of Ref. \cite{28}.

At moderate baryon density, the state with spontaneous isospin symmetry breaking is a possible ground state of quark matter, if the isospin chemical potential (or equivalently electron chemical potential) is larger than the pion mass \cite{41, 42}. In this state, the nonzero quark-antiquark condensate $\langle \bar{d}i\gamma_{5}u \rangle$ or $\langle \bar{u}i\gamma_{5}d \rangle$ can be considered as pion condensate when the coupling between the quark and antiquark is strong enough. When the temperature of the system increases, the condensate will melt and there will be, respectively, BCS and BEC phase transitions in weak and strong coupling regions. Without making confusion, we call in the following the quark-antiquark condensate of light flavors as pion condensate. We emphasize that the pion condensation in the current paper is very different from the Goldstone boson one in the CFL phase \cite{42, 43}, where the Goldstone bosons are diquark-anti-diquark excitations. In particular the color symmetry is spontaneously broken in the CFL phase while it is kept in our case. For example, the kaons $K^{0}$ and $K^{+}$ with quantum numbers $d\bar{s}$ and $u\bar{s}$ in the CFL phase are actually the lowest
excitations and made of \((\bar{u}s)(ud)\) and \((\bar{d}s)(ud)\) respectively, where all diquarks (anti-diquarks) are in anti-triplet (triplet) in color and flavor space. In our case the pions are really made of \(ud\), \(d\bar{u}\) and \(u\bar{d} - d\bar{d}\), the same as in vacuum. For neutrino emission, the role of pion condensation in quark matter is very different from that in nuclear matter. In nuclear matter, the pion degrees of freedom can open a new channel for nucleon interaction and result in an enhanced neutrino emissivity \([8, 13]\). In quark matter, however, the pion condensate will suppress, on one hand, the neutrino emission of pairing quarks due to the cost of breaking the pairs via thermal excitations, and, on the other hand, more phase space for direct Urca processes of those unpaired quarks. The competition between the two effects determines whether the neutrino emissivity is suppressed or enhanced.

At moderate baryon density, the system is controlled by light quarks and the strange quark excitation is not yet important. In this paper, we will calculate the neutrino emissivity and cooling rate through direct Urca processes in pion condensed quark matter with only two flavors. Since the typical baryon chemical potential \(\mu_B\) is of the order of 1 GeV and the isospin chemical potential \(\mu_I\) is about \(100 - 250\) MeV in compact stars, the mismatch between the two Fermi surfaces of the pairing \(u(\bar{u})\) and \(d(\bar{d})\) quarks is large enough and the pion superfluid will be in gapless state when the condensate \(\Delta\) satisfy the condition \(\mu_B/3 > \Delta\). We will focus on the gapless pion condensation.

The paper is organized as follows. In Section II we derive the general formulation for the emissivity in pion condensed quark matter. The calculations of the emissivity and cooling rate for positive energy excitations which are dominant at low temperature are presented in Sections III and IV. The contribution from negative energy excitations to the emissivity is discussed in Section V. The summary is given in the final section. Our units are \(\hbar = k_B = c = 1\) except particular specification. As a convention, we denote a 4-momentum as \(K^\mu = (k_0, \mathbf{k})\), and its 3-momentum magnitude as \(k = |\mathbf{k}|\).

II. GENERAL FORMULATION FOR NEUTRINO EMMISSIVITY

At a typical temperature for a relatively aging neutron star, the neutrino mean free path is larger than the star radius, and there is no accumulation of neutrinos inside the star. In this case, the chemical potential for neutrinos can be taken to be zero. We assume \(\beta\)-equilibrium, so the chemical potentials satisfy \(\mu_\ell = \mu_u + \mu_e\) for the quark chemical potentials \(\mu_u\) and \(\mu_d\) and the electron chemical potential \(\mu_e\). Since the pion condensation favors a high isospin or equivalently electron chemical potential, we assume that \(\mu_e\) is of the same order of \(\mu_u\) and \(\mu_d\).

![FIG. 1: The one-loop diagram of W-boson polarization tensor.](image)

It is convenient to use the Fermi effective model of weak interactions to describe the Urca processes where the characteristic energy scale is about a few hundred MeV and much less than the mass of W-bosons. The interaction Lagrangian of the model is

\[
\mathcal{L}_{\text{int}} = \frac{G}{\sqrt{2}} J^\mu J^\mu_{\mu} \tag{1}
\]

with the weak current

\[
J^\mu(x) = \bar{\nu} \gamma^\mu (1 - \gamma_5) \nu + \bar{u} \gamma^\mu (1 - \gamma_5) u, \tag{2}
\]

where \(u, d, e, \nu\) denote the fields for \(u\) and \(d\) quarks, electrons and neutrinos, respectively, and \(G = G_F \cos \theta_C \approx 1.13488 \times 10^{-11} \text{ MeV}^{-2}\) is the four-fermion coupling constant with the Cabibbo angle \(\theta_C\).

The neutrino emissivity, defined as the total energy per unit time and per unit volume carried away by neutrinos and anti-neutrinos, can be written as

\[
\epsilon_\nu = G^2 \int \frac{d^3 p_e}{(2\pi)^3 2E_e} \int \frac{d^3 p_{\nu}}{(2\pi)^3 2E_{\nu}} \left( E_e n_B(- E_e + \mu_e + E_{\nu}) \right) n_F(E_e - \mu_e) \times \mathcal{L}^{\mu\nu}(P_e, P_{\nu}) \text{Im} \Pi_{\lambda_3}^{\nu}(E_e - \mu_e - E_{\nu}, P_e - P_{\nu}), \tag{3a}
\]

\[
\epsilon_{\bar{\nu}} = -G^2 \int \frac{d^3 p_e}{(2\pi)^3 2E_e} \int \frac{d^3 p_{\nu}}{(2\pi)^3 2E_{\nu}} \left( E_e n_B(E_e - \mu_e + E_{\nu}) \right) n_F(\mu_e - E_e) \times \mathcal{L}^{\mu\nu}(P_e, P_{\nu}) \text{Im} \Pi_{\lambda_3}^{\nu}(E_e - \mu_e + E_{\nu}, P_e + P_{\nu}), \tag{3b}
\]

where \(P_i = (E_i, \mathbf{p}_i)\) for \(i = u, d, e, \nu\) are on-shell 4-momenta for quarks and leptons. The energies are \(E_i = \sqrt{p_i^2 + m_i^2}\) with the masses \(m_i\) for \(i\) and with \(m_\nu = 0\) and \(m_e \approx 0\). Here \(n_B(x) = (e^{x/T} - 1)^{-1}\) and
\[ n_F(x) = (e^{x/T} + 1)^{-1} \] are the Bose-Einstein and Fermi-Dirac distribution functions, respectively. \( L^{\lambda \sigma}(P_x, P_y) \) denotes the leptonic tensor and \( \Pi^{R \lambda \sigma}_M(Q) \) the retarded polarization tensor for W-bosons. The leptonic tensor reads

\[
L^{\lambda \sigma}(P_x, P_y) = \text{Tr} [\gamma^{\lambda} (1 - \gamma_5) \gamma^\sigma (1 - \gamma_5) \cdot P_y]
= 8 [P_x^\lambda P_y^\sigma + P_y^\lambda P_x^\sigma - P_x \cdot P_y g^{\lambda \sigma} + i e^{\lambda \sigma \beta} P_{ea} P_{b\beta}].
\] (4)

As illustrated in Fig. 1.1 the W-boson polarization tensor can be written as

\[
\Pi^{\lambda \sigma}(q_0, q) = T \sum_n \int \frac{d^3 p_a}{(2\pi)^3} \text{Tr} [\Gamma^{\lambda} S(p_{a0}, p_a) \Gamma^{\sigma} S(p_{d0}, p_d)],
\] (5)

where \( p_{d0} = p_{a0} + q_0, p_a = p_{a0} + q_a \), and \( \Gamma^{\lambda}_a = \gamma^{\lambda} (1 - \gamma_5) \tau_a \) with the ladder operators in flavor space \( \tau_a = (\tau_1 \pm i \tau_2)/2 \).

The bosonic and fermionic Matsubara frequencies are given by \( q_0 = 2\pi T + p_{ab} = i(2n + 1)\pi T \) \((l, n \text{ are integers})\) respectively. The retarded polarization tensor \( \Pi^R \) is obtained from the above after taking the analytic extension for the Matsubara frequency \( q_0 = 2\pi T \rightarrow q_0 + i0^+ \), where the second \( q_0 \) is real. We can then extract the imaginary part \( \text{Im} \Pi^R \) from \( \Pi^R \). The full propagator for quarks

\[
S(K) = \begin{bmatrix}
S_{uu}(K) & S_{ud}(K) \\
S_{du}(K) & S_{dd}(K)
\end{bmatrix}
\] (6)

in flavor space can be derived from its inverse \([11]\),

\[
S^{-1}(K) = \begin{pmatrix}
\gamma^\nu K_{\nu} + \mu_a \gamma_0 - m & i\gamma_5 \Delta^-
\gamma^\nu K_{\nu} + \mu_d \gamma_0 - m & \end{pmatrix},
\] (7)

where \( m \) is the average effective quark mass, and \( \Delta^+ \sim \langle \bar{d}i\gamma_5 d \rangle \) and \( \Delta^- \sim \langle \bar{u}i\gamma_5 u \rangle \) are the charged pion condensates.

Without loss of generality, we assume real condensates and let \( \Delta^+ = \Delta^- = \Delta \). In this case, the four matrix elements of \( S \) are explicitly expressed as

\[
S_{uu} = \sum_{a,r=\pm} B^a_r(k) \frac{\Lambda^a_k \gamma_0}{k_0 + \mu_B/3 + r E^a_k},
\] (8)

\[
S_{dd} = \sum_{a,r=\pm} B^a_r(k) \frac{\Lambda^a_k \gamma_0}{k_0 + \mu_B/3 + r E^a_k},
\] (9)

\[
S_{ud} = -i \sum_{a,r=\pm} r \sqrt{B^a_+(k) B^a_-(k)} \frac{\Lambda^a_k \gamma_5}{k_0 + \mu_B/3 + r E^a_k},
\] (10)

\[
S_{du} = -i \sum_{a,r=\pm} r \sqrt{B^a_+(k) B^a_-(k)} \frac{\Lambda^a_k \gamma_5}{k_0 + \mu_B/3 + r E^a_k},
\] (11)

where the summation over \( a \) (and \( b \) in the following) is for positive and negative energy excitations for quarks, and the that over \( r \) (and \( s \) in the following) is for quarks and anti-quarks. The Bogoliubov coefficients \( B^a_r(k) \), energy projectors \( \Lambda^a_k \) and energies \( E^a_k \) are defined by

\[
B^a_r(k) = \frac{1}{2} \left[ 1 - a r \frac{E_k + a \delta \mu}{E^a_k} \right],
\]

\[
\Lambda^a_k = \frac{1}{2} \left[ 1 + a \frac{\gamma_0 (\gamma \cdot k + m)}{E_k} \right],
\]

\[
E^a_k = \sqrt{(E_k + a \delta \mu)^2 + \Delta^2}.
\] (12)

We use \( \delta \mu = (\mu_d - \mu_u)/2 = -\mu_t/2 = \mu_c/2 \) and \( \mu \equiv \mu_B/3 \) to replace \( \mu_B \) and \( \mu_c \), where \( \mu_B, B, c \) are chemical potentials for the isospin, baryons and electrons respectively. From the poles of the quark propagators, the quasi-quark energy is then given by \( E^a_k + r \mu \). The chemical potentials for \( u \) and \( d \) quarks can be expressed in terms of \( \delta \mu \) and \( \mu \) as \( \mu_u = \mu - \delta \mu \) and \( \mu_d = \mu + \delta \mu \).

Substituting the quark propagator \([10]\) into the W-boson polarization tensor \([12]\) and performing the trace in color and flavor space, we obtain

\[
\Pi^{\lambda \sigma}(q_0, q) = N_c T \sum_n \int \frac{d^3 p_a}{(2\pi)^3} \text{Tr} [\gamma^{\lambda} (1 - \gamma_5) S_{uu}(p_{a0}, p_a) \gamma^\sigma (1 - \gamma_5) S_{dd}(p_{d0}, p_d)]
\]

\[
= N_c \sum_{a,b,r,s=\pm} \int \frac{d^3 p_a}{(2\pi)^3} H^{ab}_{rs} (P_x, P_y) B^a_r(p_a) B^b_s(p_d) \frac{n_F(-r E^a_u - \mu) - n_F(-s E^b_d - \mu)}{q_0 - r E^a_u + s E^b_d - \mu},
\] (13)
with the quark tensor \( H_{ab}^{\lambda\sigma}(P_u, P_d) \) given by

\[
H_{ab}^{\lambda\sigma}(P_u, P_d) = 4E_u E_d \Im \Gamma(\gamma_\lambda(1 - \gamma_5) \Sigma_{\mu} \gamma_\mu \gamma_\gamma(1 - \gamma_5) \Sigma_{\nu} \gamma_\nu) \\
= 8 \left[ P_{uu} \delta_{bd} + P_{ud} \delta_{ab} - P_{uu} \cdot P_{ud} \gamma^{\lambda\sigma} + \Im \epsilon_{\alpha\beta}\gamma_{\alpha} P_{uu}^\beta P_{bd}^\gamma \right].
\]

(14)

where \( P_{uu} = (E_u, a^u_\mu) \) with \( a = +, - \) and \( P_{bd} = (E_d, b^d_\mu) \) with \( b = +, - \) for positive and negative energy excitations respectively. We have also used the number of colors \( N_c \) which is finally set to 3 in our calculation. Using \( \Im \Pi_{R}^{\lambda\sigma}(q_0, q) = \Im \Pi_{L}^{\lambda\sigma}(q_0 + i0^+, q) \), the imaginary part of the retarded polarization tensor of W-bosons can be read out directly,

\[
\Im \Pi_{R}^{\lambda\sigma}(q_0, q) = \frac{\pi N_c}{(2\pi)^3 4 E_u E_d} \int \frac{d^3 p_u}{(2\pi)^3 4 E_u E_d} \frac{d^3 p_d}{(2\pi)^3 4 E_u E_d} \rho^{\lambda\sigma}(P_u, P_d) B^a_r(p_u) B^b_{-s}(p_d)n_\nu^{-1}(q_0) \\
\times n_F((-rE_u^a - \mu)n_F(sE_d^b - \mu)\delta(q_0 - rE_u^a + sE_d^b)) \\
= -\frac{\pi N_c}{(2\pi)^3 4 E_u E_d} \int \frac{d^3 p_u}{(2\pi)^3 4 E_u E_d} \frac{d^3 p_d}{(2\pi)^3 4 E_u E_d} \rho^{\lambda\sigma}(P_u, P_d) B^a_r(p_u) B^b_{-s}(p_d)n_\nu^{-1}(q_0) \\
\times n_F(rE_u^a + \mu)n_F(-sE_d^b - \mu)\delta(q_0 - rE_u^a + sE_d^b),
\]

(15)

where the first equality is for the neutrino emissivity with \( q_0 = E_e - \mu_e - E_\nu \) and the second for the anti-neutrino one with \( q_0 = E_e - \mu_e + E_\nu \). Here we have used the identity \( n_F(x) - n_F(y) = n_F(-x)n_F(y)/n_B(-x + y) = -n_F(x)n_F(-y)/n_B(x - y) \).

Substituting the imaginary polarization \( \Im \Pi^{\lambda\sigma}_{R} \) into the neutrino emissivity \( \Im \Pi^{\lambda\sigma}_{L} \), we arrive at

\[
\epsilon_{\nu} = 2N_c \int \frac{d^3 p_u}{(2\pi)^3 2E_u} \int \frac{d^3 p_d}{(2\pi)^3 2E_d} \rho^{\lambda\sigma}(P_u, P_d) B^a_r(p_u) B^b_{-s}(p_d)n_F(-rE_u^a - \mu)n_F(sE_d^b - \mu)|M_{ab}|^2, \\
\epsilon_{\bar{\nu}} = 2N_c \int \frac{d^3 p_u}{(2\pi)^3 2E_u} \int \frac{d^3 p_d}{(2\pi)^3 2E_d} \rho^{\lambda\sigma}(P_u, P_d) B^a_r(p_u) B^b_{-s}(p_d)n_F(-rE_u^a - \mu)n_F(sE_d^b - \mu)|M_{ab}|^2,
\]

(16)

where the shorthand notation \( |M_{ab}|^2 \) is the spin-averaged matrix element for the \( \beta \)-decay or the electron capture processes \( 21 \).

\[
|M_{ab}|^2 = \frac{G^2}{4} L_{\alpha\beta}(P_e, P_\nu) H_{ab}^{\lambda\sigma}(P_u, P_d) = 64G^2(P_e \cdot P_{uu})(P_\nu \cdot P_{bd}).
\]

(17)

Here we have suppressed energy projection indices \( a, b \) in quark momenta as before. When \( a = + \) and \( b = + \) for \( u \) and \( d \) quarks respectively, \( |M_{u+}|^2 \) is just the matrix element of the Urca processes in vacuum. For other combinations of \( a \) and \( b \), it represents the matrix elements with at least one negative energy excitation for \( u, d \) quasi-quarks.

### III. NEUTRINO EMISSION WITH POSITIVE ENERGY EXCITATIONS

It is well known that the negative energy excitations are normally neglected at low temperature due to their strong suppression relative to the positive excitations, see, e.g. Eqs. (21-24) for fully gapped phases in Ref. [35]. This can be understood from the analysis of the parts involving Bogoliubov coefficients and quark number distributions in Eq. \( 10 \).

\[
B^a_r(p_u) B^b_{-s}(p_d)n_F(\mp rE_u^a \pm \mu)n_F(\pm sE_d^b \pm \mu),
\]

(18)

where the upper and lower signs correspond to neutrinos and anti-neutrinos respectively. Let us consider the part related to the negative energy excitations for \( u \) quarks, \( B^a_r(p_u)n_F(-rE_u^a - \mu) \) and \( B^a_r(p_u)n_F(rE_u^a + \mu) \).

Note that the dominant contribution to the emissivity comes from the phase space near gapless nodes which correspond to the vanishing arguments of quark distribution functions by selecting the branch with \( r = - \), see Appendix A. The gapless node for \( u \) quarks in this case is located at \( E_u^0 = \sqrt{\mu^2 - \Delta^2} + \delta \mu \), then we obtain \( B^a_r(p_u) = \left(1 - \sqrt{\mu^2 - \Delta^2}/\mu \right)/2 \) which approaches zero for large \( \mu \) or small \( \Delta \). The same conclusion is also true for \( d \) quarks. The above arguments are valid only at low temperatures \( T \ll \mu_{u,d} \) where the dominant contribution of phase space integrals comes from the region near gapless nodes.

In this section we calculate the neutrino emissivity by dropping out the negative energy excitations. We will briefly discuss the contribution from the negative energy excitations in Section IV.
When quarks are treated as free and massless constituents in normal quark matter, phase space for direct Urca processes vanishes, since quarks and electrons are collinear in momenta on their Fermi surfaces [21]. To get non-vanishing phase space which requires that the Fermi momenta of $u$ and $d$ quarks and electrons should satisfy a triangular relation, one must take into account nonzero quark masses and/or modified quark dispersion relations due to interactions and/or pairings between (anti-)quarks. We will consider these effects together. When the baryon chemical potential is high enough, the dispersion relations of quasi-quarks are gapless, the neutrino emissivity is dominated by the quasi-quarks around the gapless momenta. For $u$ and $d$ quarks, the gapless momenta are $p_{u}^0 = (1 - \kappa)\sqrt{(E_u^0)^2 - m_u^2}$ and $p_{d}^0 = (1 - \kappa)\sqrt{(E_d^0)^2 - m_d^2}$ with the gapless energies $E_u^0 = \sqrt{\mu^2 - \Delta^2 - \delta \mu}$ and $E_d^0 = \sqrt{\mu^2 - \Delta^2 + \delta \mu}$. The dispersion relations of quasi-quarks are schematically shown in Fig. 2. We have included the interaction among quarks by introducing a phenomenological constant $\kappa$ which reflects the reduction of the Fermi momenta and can be derived in Landau's Fermi liquid theory or in field theory. In perturbative QCD we have $\kappa = 2\alpha_s/(3\pi)$ with $\alpha_s$ being the strong coupling constant [44], and in the NJL model it can be expressed as $\kappa = 4g^2\mu_B^2/(3\pi^2)$ with the four-fermion coupling constant $g$ [38]. In the normal phase with vanishing pion condensate, $E_{u,d}^0$ and $p_{u,d}^0$ are just the Fermi energies and momenta, i.e. $E_{u,d}^0 = \mu_{u,d}$ and $p_{u,d}^0 = (1 - \kappa)\mu_{u,d}$.

At the gapless momenta, the angles between $u$-quark and electron momenta and between $u$- and $d$-quark momenta can be written as

$$
\cos \theta_{uc}^0 = \frac{(p_u^0)^2 - (p_e^0)^2 - \mu_e^2}{2\mu_e p_u^0},
$$

$$
\cos \theta_{ud}^0 = \frac{(p_u^0)^2 + (p_d^0)^2 - \mu_d^2}{2\mu_d p_u^0}.
$$

Substituting them into Eq. (17), one obtains approximately

$$
|M_{++}|^2 \approx 64G^2 E_u E_d (1 - \frac{p_u^0}{E_u^0} \cos \theta_{uc}^0) \left(1 - \frac{p_d^0}{E_d^0} \cos \theta_{ud}^0\right),
$$

where $\theta_{ud}$ is the angle between neutrino and $d$-quark momenta. In the brackets the $u$- and $d$-quark energies and momenta and the angle between $u$-quark and electron momenta are replaced by the corresponding values at the gapless nodes. Taking into account the relation $\mu_d = \mu_{u} + \mu_e$, the $\delta$-functions in Eq. (16) can be simplified as $\mu_{e}/(p_u p_d) \delta(\cos \theta_{ud} - \cos \theta_{ud}^0)$ to the leading order of $p_u/\mu_e$, since the higher order contribution is small in the kinetic range of the processes. Using the above approximation, the neutrino and anti-neutrino emissivity can be expressed in the following form

$$
\epsilon_\nu \approx 16\pi N_c G^2 \mu_e E_u^0 E_d^0 \left(1 - \frac{p_u^0}{E_u^0} \cos \theta_{uc}^0\right) J \sum_{r,s=\pm} \int_0^\infty dE_\nu E_\nu^3 \int_m^\infty dE_u \int_m^\infty dE_d \times B_s^+(p_u) B_{-s}^-(p_d) n_F(E_\nu + rE_u^+ - sE_d^-) n_F(-rE_u^- - \mu) n_F(sE_d^- + \mu),
$$

$$
\epsilon_\bar{\nu} \approx 16\pi N_c G^2 \mu_e E_u^0 E_d^0 \left(1 - \frac{p_u^0}{E_u^0} \cos \theta_{uc}^0\right) J \sum_{r,s=\pm} \int_0^\infty dE_\nu E_\nu^3 \int_m^\infty dE_u \int_m^\infty dE_d \times B_s^+(p_u) B_{-s}^-(p_d) n_F(E_\nu - rE_u^+ + sE_d^-) n_F(rE_u^- + \mu) n_F(-sE_d^- - \mu),
$$

where we have used $d\mu_d d\sigma d\nu dE_u dE_d/(p_u p_d)$, and the angular integration $J$ has been evaluated as

$$
J = \int \frac{d\Omega_\nu}{(2\pi)^3} \int \frac{d\Omega_\nu}{(2\pi)^3} \int \frac{d\Omega_\nu}{(2\pi)^3} (1 - \frac{p_u^0}{E_u^0} \cos \theta_{ud}) \delta(\cos \theta_{ud} - \cos \theta_{ud}^0) = \frac{1}{16\pi^6}.
$$

To further simplify the emissivity, it is convenient to introduce the dimensionless variables

$$
x = \frac{E_u + \delta \mu}{T}, \ y = \frac{E_d - \delta \mu}{T}, \ u = \frac{\mu}{T}, \ v = \frac{E_\nu}{T}, \ w = \frac{\Delta}{T},
$$

FIG. 2: The schematic dispersion relations for gapped and gapless quasi-quarks.
and relax the integration ranges of $x, y$ to $(-\infty, \infty)$, since the integrations are dominated by the gapless nodes. Dropping out odd terms of $x$ and $y$ from the integrations, and noting that the summation over $r$ and $s$ is symmetric, $\epsilon_\nu$ and $\epsilon_\bar{\nu}$ in (24) are approximately the same. Therefore, we have

$$
\epsilon = \epsilon_\nu + \epsilon_\bar{\nu} \approx \frac{6}{\pi^2} G^2 \mu_e E_u E_d \left(1 - \frac{p^0_u}{E_u} \cos \theta^0_u \right) T^6 F(u, w),
$$

with

$$
F(u, w) = \sum_{r,s=\pm} F_{rs}(u, w),
$$

$$
F_{rs}(u, w) = \int_0^\infty dx dy dv^3 \frac{1}{e^{-x\sqrt{x^2 + w^2 - u}} + 1} \frac{1}{e^{s\sqrt{y^2 + w^2 + u}} + 1} \frac{1}{e^{-s\sqrt{y^2 + w^2 + r\sqrt{x^2 + w^2 + v}} + 1}}.
$$

While the two integrals in (24) for neutrinos and anti-neutrinos are different in general case, see Appendix [13] they approach to the same value at low temperature, because the gapless momenta scaled by temperature are far from zero providing high effective Fermi surfaces. In this case, the integrals perfectly center at the gapless momenta and other contributions quickly damp out. However this is not the case at $T_c$. The difference and coincidence in the neutrino and anti-neutrino emissivities also occur in fully gapped superfluid/superconducting states.

Eq. (24) is our main result, which presents the neutrino emissivity $\epsilon$ as a function of temperature $T$, chemical potentials $\mu_B$ and $\mu_c$, the effective quark mass $m_{u,d}$, and the pion condensate $\Delta$.

To clearly see the effects of the pion condensate, we consider two limit cases. One is the high temperature limit with $T$ close to $T_c$ where the condensate is small, $\Delta \ll \mu_u, \mu_d, \mu_c, T$, and its temperature dependence can be modeled as $\Delta = \Delta_0 \sqrt{1 - (T/T_c)^2}$ with $\Delta_0$ being the pion condensate at $T = 0$. The other is the low temperature limit with $T \ll T_c$. Since the chiral symmetry is largely restored in the phase of pion condensation, the effective quark mass is small compared with the quark chemical potential (actually, $m/\mu \propto \sqrt{T}$), we can assume $m \ll \mu_u, \mu_d, \mu_c$.

In the limit of high temperature, we can expand $E^0_u, p^0_u$, and $\cos \theta^0_u$ in terms of $\kappa, m_{u,d}^2/\mu^2$ and $\Delta^2/\mu^2$, see Appendix [C]. By keeping only the leading order and considering the limit $\lim_{\mu \to 0} F(u, w) = 457 \pi^6/5040$, we obtain the emissivity near $T_c$.

$$
\epsilon \approx \left(1 - \frac{\Delta^2}{\mu_u \mu_d} \right) \frac{1371}{2520} \pi G^2 \mu_e \mu_d \left(1 + \frac{\mu_u}{\mu_d} \kappa + \frac{m_u^2 - m_d^2}{2 \mu_u \mu_d} \right) T^6.
$$

Comparing it with the result in normal quark matter with massless quarks [21],

$$
\epsilon_0 \approx \left(1 - \frac{\Delta_0^2}{\mu_u \mu_d} \right) \frac{1371}{2520} \pi G^2 \mu_e \mu_d \left(1 - \frac{p^0_u}{E_u} \cos \theta^0_u \right) T^6
$$

$$
\approx \left(1 - \frac{\Delta_0^2}{\mu_u \mu_d} \right) \frac{1371}{2520} \pi G^2 \mu_e \mu_d \left(1 + \frac{\mu_d}{\mu_u} \kappa T^6, \right)
$$

we conclude that, near $T_c$, a small pion condensate reduces the emissivity slightly but the correction due to the quark mass splitting may enhance it significantly.

In the low temperature limit, at $\mu_B = 0$ ($u = 0$) we can recover the fully gapped phase and the emissivity has an exponential suppression factor $e^{-\Delta/T}$ (see, e.g. [33]). For non-vanishing $\mu_B$ or $u$, such a suppression is absent due to the gapless modes appeared in quark number distribution functions. In the case with $u > w \gg 1$, we derived in Appendix [A] the asymptotic behavior of $F_{rs}(u, w)$ which can be summarized as

$$
F_{++} \sim w e^{-u - w},
$$

$$
F_{+-} \sim w^{1/2} (u^2 - w^2)^{1/2} e^{-u - w},
$$

$$
F_{-+} \sim w^{1/2} (u^2 - w^2)^{1/2} e^{-u - w},
$$

$$
F_{--} \sim O(1).
$$

It is easy to see that the term $F_{--}(u, w)$ dominates $F(u, w)$ due to the gapless modes in both $u$ and $d$ quark number distribution functions. Therefore, at low temperature the emissivity is of the same order as that in normal quark matter.

To do numerical calculation, we choose the parameters $\mu_B = 900$ MeV, $\mu_f = 150$ MeV, $\kappa = 2/(3\pi)$ and $\Delta_0 = 100$ MeV. These are typical values for the possible pion condensed quark matter in compact stars. The numerical result for the ratio $\epsilon/\epsilon_0$ as functions of $T/T_c$ is illustrated in Fig. [3] where the critical temperature is given by $T_c = 0.57 \Delta_0$ according to the BCS theory. When the chiral symmetry is fully restored, the quarks are massless, and the difference between pion condensed and normal quark matter is small at low temperature and becomes considerable at intermediate temperature. At high temperature $T \to T_c$, the emissivity in pion condensed matter is a little lower than that in normal matter, just as we expected above. However, in general case with
nonzero baryon and isospin chemical potentials, the effective quark masses \( m_u \) and \( m_d \) for \( u \)- and \( d \)-quarks are not the same, there should exist a quark mass splitting induced by the isospin symmetry breaking. In the massive case with \( m_u = 75 \) MeV and \( m_d = 125 \) MeV, the enhancement factor for the emissivity becomes 1.2 in the whole temperature range. A similar behavior was found in a two-flavor gapless color superconductor \(^{36}\).

\section{IV. COOLING RATE}

In this section, we compute the cooling rate due to neutrino emission. To this end, we integrate the energy equation \( \epsilon(T) = -c_V(T) \frac{dT}{dt} \) and obtain

\[
t - t_0 = - \int_{t_0}^{T} c_V(T') \frac{dT'}{\epsilon(T')},
\]

where \( T_0 \) is the initial temperature at time \( t_0 \). From the entropy density

\[
s = -2N_c \sum_{a,r=\pm} \int \frac{d^3k}{(2\pi)^3} \left\{ n_F(E^a_k + r\mu) \ln n_F(E^a_k + r\mu) + [1 - n_F(E^a_k + r\mu)] \ln [1 - n_F(E^a_k + r\mu)] \right\},
\]

the specific heat \( c_V \) is given by

\[
c_V(T) = T \frac{\partial s}{\partial T} = 2N_c \sum_{a,r=\pm} \int \frac{d^3k}{(2\pi)^3} n_F(E^a_k + r\mu)n_F(-E^a_k - r\mu)
\times \left[ \frac{1}{T^2}(E^a_k + r\mu)^2 - \frac{1}{T}(E^a_k + r\mu) \right] \frac{\partial E^a_k}{\partial \Delta} \frac{\partial \Delta}{\partial T}
\approx T \frac{N_c}{\pi^2} \sum_{a,u,d} \rho_a^0 \rho_a^0 K(u, w) + \frac{\Delta^2}{T^2} G(u, w),
\]

where we have used the relation \( \Delta(T) = \Delta_0 \sqrt{1 - (T/T_c)^2} \) and made the approximation that the integral is dominated by the gapless nodes, and the functions \( K(u, w) \) and \( G(u, w) \) are defined by

\[
K(u, w) = \sum_{r=\pm} \int_0^{\infty} dx \left( \sqrt{x^2 + w^2 + ru} \right)^2 \frac{e^{\sqrt{x^2 + w^2 + ru}}}{(e^{\sqrt{x^2 + w^2 + ru}} + 1)^2},
\]

\[
G(u, w) = \sum_{r=\pm} \int_0^{\infty} dx \left[ 1 + \frac{ru}{\sqrt{x^2 + w^2}} \right] \frac{e^{\sqrt{x^2 + w^2 + ru}}}{(e^{\sqrt{x^2 + w^2 + ru}} + 1)^2}.
\]

Note that the contribution from the excitation branch with \( a = + \) is suppressed by a factor \( e^{-\mu/T} \) at low temperature, it is negligible except near the critical temperature \( T_c \). At \( T_c \) with \( w = 0 \), we recover \( K(u, 0) \simeq \pi^2/3 \) in the normal phase. The first term of \( G(u, 0) \) is approximately unit and leads to the standard jump of specific heat at the critical temperature for a symmetric superfluid without Fermi surface mismatch between the two species. The second term of \( G(u, 0) \) reflects the gapless nature of the asymmetric pairing. If one turns off the interaction between quarks (\( \kappa = 0 \)), the \( K \)-term reproduces the low temperature specific heat of two-flavor quark matter

\[\text{FIG. 3: The scaled neutrino emissivity } \epsilon/\epsilon_0 \text{ as functions of the scaled temperature } T/T_c \text{ for massless and massive quarks.}\]
$c_{V0} = \gamma T$ with $\gamma = E_u^F p_u^F + E_d^F p_d^F$ with the $u$- and $d$-quark Fermi energies and Fermi momenta, while the $G$-term gives the jump of the specific heat at the normal-superfluid transition point. The numerical result of $c_V(T)$ is shown in Fig. 4. The quark mass dependence of $c_V$ gives the jump of the specific heat at the normal-superfluid transition point. The numerical result of Eq. (16) reads

$\Delta = 100\text{MeV}$

FIG. 4: The scaled specific heat $c_V/(\gamma T_c)$ as functions of scaled temperature $T/T_c$. The straight dashed line is the result in normal quark matter, and the solid line is the result in the pion condensation. There is a jump at the phase transition point.

Substituting the neutrino emissivity (24) and the specific heat (31) into Eq. (29), we obtain

$$t - t_0 = -\frac{\pi^3}{2G^2\mu_e} \int_{T_0}^{T'} dT' \frac{p_0^0 E_0^u + p_0^0 E_0^d}{E_d^0(E_d^0 - p_0^0 \cos \theta_{uc})} K(u, w) + \Delta_0^2 T_c^{-2} G(u, w) F(u, w),$$

where $p_{0, d}^0$ and $E_{u, d}^0$ depend on $T$ through $\Delta(T)$. Fig. 5 shows the time evolution of the pion condensed quark matter with initial temperature $T_0 = 1$ MeV at $t_0 = 1$ yr. For comparison we also show the cooling evolution in normal quark matter. In the case with mass difference between $u$- and $d$-quarks, the cooling of the condensed phase is faster. This is due to the larger neutrino emissivity and smaller specific heat in the pion condensed quark matter. For massless quarks, the two cooling curves for condensed and normal quark matter are almost the same due to the fact that the emissivity difference between the two cases is small at low temperature, see Fig. 5.

FIG. 5: The cooling curves in pion condensed (solid lines) and normal (dashed lines) quark matter for massive (left panel) and massless (right panel) quarks.

V. CONTRIBUTIONS FROM NEGATIVE ENERGY EXCITATIONS

As we have argued above that the positive energy excitations dominate the neutrino emissivity at $\mu_B \gg \Delta$ and at low temperature $T \ll \mu_{u, d}$. In principle, the contribution from negative energy excitations must be considered if we follow the exact calculation [16] at high temperature. In order to better understand the role of negative energy excitations, we discuss the emissivity at $T \rightarrow T_c$. In this case, the $j, k, b, c$ related part of the integrands in Eq. (30) reads

$$I_\nu = \frac{B_j^u(p_u) B_{-j}^d(p_d) n_F(E_{\nu} + r E_d^a - s E_d^b) n_F(-r E_d^a - \mu) n_F(s E_d^b + \mu) |M_{ab}|^2}{\gamma \Theta[-r a \text{Sgn}(E_u + a \delta \mu) \Theta[-s b \text{Sgn}(E_d - b \delta \mu)] n_F(E_{\nu} + r |E_d + a \delta \mu| - s |E_d - b \delta \mu|) \times n_F(-r |E_u + a \delta \mu| - \mu) n_F(s |E_d - b \delta \mu| + \mu) |M_{ab}|^2}$$

$$I_{\bar{\nu}} \simeq n_F(E_{\bar{\nu}} - a E_u + b E_d - \mu_c) n_F(a E_u - \mu_u) n_F(-b E_d + \mu_d) |M_{ab}|^2,$$

$$(34)$$
where we have used the approximation $B_r^u(p_u) \simeq \Theta[-ra\text{Sgn}(E_u + a\delta\mu)]$ at $T \to T_c$ which means that only the terms with $r\text{Sgn}(E_u + a\delta\mu) = -a$ are relevant in the calculation of neutrino emission, the step and sign functions are defined as $\Theta(x) = 1$ for $x > 0$ and 0 for $x < 0$, and $\text{Sgn}(x) = 1$ for $x > 0$ and $-1$ for $x < 0$.

The above expressions with $a = +$ and $b = +$ reproduces the conventional result for normal quark matter,

$$I_\nu \sim n_F(E_\nu - E_u + E_d - \mu_e)n_F(E_u - \mu_u)n_F(-E_d + \mu_d)|M_+|^2,$$

$$I_{\bar{\nu}} \sim n_F(E_\nu + E_u - E_d + \mu_e)n_F(-E_u + \mu_u)n_F(E_d - \mu_d)|M_+|^2.$$  \hspace{1cm} (35)

At low temperature $T \ll \mu_{u,d}$, this contribution dominates the neutrino emission, since the integrals in have sharp peaks at $E_u \sim \mu_u$ and $E_d \sim \mu_d$ and the contribution from negative energy excitations is suppressed at least by a factor of $e^{-\mu_{u,d}/T}$. However, at temperature $T \sim \mu_{u,d}$, such a suppression disappears, and those processes which are forbidden by energy conservations at low temperature can happen. For example, for $ab = -+$ the processes with energy conservations $E_\nu + E_u + E_d = \mu_e$ and $E_u + E_d = E_\nu + \mu_e$ for neutrinos and ant-neutrinos are forbidden at low temperature, but allowed at $T \sim \mu_{u,d}$ since the quark energies could be as large as the chemical potentials.

VI. SUMMARY

Quark matter may exist in the core of a compact star. Pion condensed phase is one possible ground state of the quark matter due to the non-zero isospin density as a result of the $\beta$ equilibrium between $u$ and $d$ quarks. For the neutrino emission, the role of the pion condensation in quark matter is different from that in nuclear matter. In nuclear matter, the presence of pion degrees of freedom effectively opens a new channel for the nucleon interaction and results in an enhancement of the neutrino emissivity. In this paper we considered the gapless pion excitations which can be realized in compact stars. We calculated the neutrino emissivity and cooling rate through direct Urca processes in pion condensed quark matter with two flavors. We derived general expressions for neutrino and anti-neutrino emissivity. At temperature much lower than quark chemical potentials, we demonstrated that the positive energy excitation dominates the emissivity if the baryon chemical potential is much larger than the magnitude of the condensate. We also discussed the role of negative energy excitations at temperature comparable to the chemical potential.

We showed the numerical results for the neutrino emissivity and cooling rate at low temperature. The total neutrino emissivity is of the same order as that in the normal phase, which is a result of gapless excitations. While the results in the two phases are quite close to each other when quarks are massless, the mass difference between $u$- and $d$-quarks due to isospin symmetry breaking leads to a remarkable enhancement of the emissivity. For typical values of the initial temperature, the chemical potential and the pion condensate corresponding to the compact star interior, the cooling of the condensed phase is found to be faster than that of the normal one, when the quark mass splitting is sizable.

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In the limit of \( u > w \gg 1 \) at low temperature, the four components of \( F(u, w) \) can be simplified as

\[
F_{++}(u, w) \simeq \int_0^\infty dv dx dy v^3 \left[ e^{\sqrt{v^2 + w^2 + u}} + e^{\sqrt{x^2 + w^2 + v}} \right]^{-1} \\
\simeq e^{-u-w} \int_0^\infty dv dx dy v^3 \left[ e^{9/2w} + e^{x^2/(2w) + v} \right]^{-1} \\
\simeq \frac{w}{e^{u-w}},
\]

\[
F_{+-}(u, w) \simeq \int_0^\infty dv dx dy v^3 \left[ e^{\sqrt{v^2 + w^2 - v}} + e^{\sqrt{x^2 + w^2 - u}} \right]^{-1} \\
\simeq e^{w-u} \int_0^\infty dv dx dy v^3 \left[ e^{\sqrt{w^2 - u} + v} - e^{x^2/(2w) - u} \right]^{-1} \\
\simeq 3\sqrt{2\pi} e^{w-u} \int_0^{\infty} \frac{dy}{\sqrt{x^2+w^2}} \\
\simeq w^{1/2}(u^2 - w^2)^{1/2} e^{-u-w},
\]

\[
F_{-+}(u, w) \simeq \frac{1}{4} \int_0^{\infty} dx \int_0^\infty dy e^{-\sqrt{y^2+w^2} - u} \left[ \sqrt{x^2 + w^2} + \sqrt{v^2 + w^2} \right]^4 \\
\simeq \frac{1}{4} \int_0^{\infty} dx \int_0^\infty dy e^{-\sqrt{y^2+w^2} - u} \left[ \sqrt{x^2 + w^2} + \sqrt{y^2 + w^2} \right]^4 \\
\simeq w^{1/2}(u^2 - w^2)^{9/2} e^{-u-w},
\]

\[
F_{--}(u, w) \simeq \int_0^\infty dx \int_0^\infty dy \int_0^\infty dv \frac{v^3}{e^{-\sqrt{x^2+w^2} + \sqrt{y^2+w^2} + v} + 1} \\
\simeq O(1),
\]

where we have used

\[
\text{Lim}_{T \to 0} \frac{1}{e^{x/T} + 1} \simeq \Theta(-x).
\]

The dominant contribution to the integral over \( x \) in \( F_{--} \) comes from the regions near the upper limit \( x = \sqrt{u^2 - w^2} + 0^+ \) and lower limit \( x = \sqrt{u^2 - w^2} + 0^+ \).

**APPENDIX B: EXACT CALCULATION OF EMISSIVITY FOR NEUTRINOS AND ANTI-NEUTRINOS**

In this appendix, we calculate exactly the neutrino and anti-neutrino emissivity listed in \( [21] \), in order to see how good is the approximation we made in deriving \( [24] \). From \( [21] \), the pre-factors in \( \epsilon_\nu \) and \( \epsilon_\bar{\nu} \) are exactly the same, and the difference is from the summations and integrals. We denote the summation and integration in \( \epsilon_\nu \) by \( F_\nu \) and that in \( \epsilon_\bar{\nu} \) by \( F_\bar{\nu} \). It is easy to see the relation \( F_\nu(-\delta \mu) = F_\bar{\nu}(\delta \mu) \). In fact, it is true even for the original \( \epsilon_\nu \) and \( \epsilon_\bar{\nu} \) in \( [19] \). For \( \mu \gg \delta \mu \), the two integrals are dominated by the phase space near gapless momenta at low temperature, the \( \delta \mu \) dependence of \( F_\nu \) and \( F_\bar{\nu} \) is negligible, and we have therefore \( F_\nu = F_\bar{\nu} \). However, if the temperature is not very low or \( \delta \mu \) approaches to \( \mu \), the gapless regions might not dominate the integrals, and the explicit dependence on \( \delta \mu \) must enter \( F_\nu \) and \( F_\bar{\nu} \). In Fig. 6 the exact \( F_\nu \) and \( F_\bar{\nu} \) scaled by \( F_0 \equiv F_\nu(\Delta T = 0, T = 0) \) are presented as functions of \( T/T_c \). While at low temperature \( F_\nu \) and \( F_\bar{\nu} \) coincide very well, indicating that the approximation we used is good, the difference between \( F_\nu \) and \( F_\bar{\nu} \) increases as the temperature grows and reaches 10% at \( T \to T_c \).
APPENDIX C: PARAMETER EXPANSION NEAR $T_c$

At $T \to T_c$ in the pion condensation, to the leading order of $\kappa, m_u^2/\mu_u^2, \Delta^2/\mu^2$, the gapless energies and momenta of quarks and the corresponding angle between $u$-quark and electron momenta can be written as

$$E_0^u \approx \mu_u \left( 1 - \frac{\Delta^2}{2\mu_u^2} \right),$$
$$E_0^d \approx \mu_d \left( 1 - \frac{\Delta^2}{2\mu_d^2} \right),$$
$$p_0^u \approx \mu_u \left[ 1 - \kappa - \frac{m_u^2}{2\mu_u^2} - \frac{\Delta^2}{2\mu_u^2} \right],$$
$$p_0^d \approx \mu_d \left[ 1 - \kappa - \frac{m_d^2}{2\mu_d^2} - \frac{\Delta^2}{2\mu_d^2} \right],$$
$$\cos \theta_{ue}^0 \approx 1 - \frac{\mu_d}{\mu_u} \frac{m_u^2 - m_d^2}{2\mu_u^2}.\tag{C1}$$

which lead to

$$E_0^u E_0^d \left( 1 - \frac{p_0^u}{E_0^u} \cos \theta_{ue}^0 \right) \approx \mu_u \mu_d \left( 1 - \frac{\Delta^2}{2\mu_u^2} \right) \left( 1 - \frac{\Delta^2}{2\mu_d^2} \right) \times \left[ 1 - \left( 1 - \frac{m_u^2}{2\mu_u^2} \right) \left( 1 - \frac{\mu_d}{\mu_u} \frac{m_u^2 - m_d^2}{2\mu_u^2} \right) \right]$$

$$\approx \mu_u \mu_d \left( 1 - \frac{\Delta^2}{\mu_u \mu_d} \right) \left( 1 + \frac{\mu_d}{\mu_u} \kappa + \frac{m_u^2 - m_d^2}{2\mu_u \mu_e} \right). \tag{C2}$$

Substituting this relation into (21), we obtain (26).

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