Consistency Checking and Improving for Interval-Valued Hesitant Preference Relations

Yuling Zhai 1,2 and Zeshui Xu 2,3,*

1 School of Mathematics and information science, Weifang University, Weifang 261061, China; zxr1223@126.com
2 Business School, State Key Laboratory of Hydraulics and Mountain River Engineering, Sichuan University, Chengdu 610065, China
3 School of Computer and Software, Nanjing University of Information Science and Technology, Nanjing 210044, China
* Correspondence: xuzeshui@263.net

Received: 1 March 2019; Accepted: 28 March 2019; Published: 2 April 2019

Abstract: Group decision making (GDM), which aims to obtain a sensible decision result with several decision makers, is a common occurrence in daily life. Since the uncertainty of the objects is a thorny issue in the process of GDM, it is important to eliminate uncertainty in order to achieve an optimal decision result. Considerations of some types of preference relations based on various fuzzy sets have been presented and investigated in previous studies; in this paper, we define the interval-valued hesitant multiplicative preference relation (IVHMPR) and the multiplicative consistency of IVHMPR. Based on these, we provide a detailed discussion on the connections between the interval-valued hesitant fuzzy preference relation (IVHFPR) and the IVHMPR. Then, we give a method to check the for unacceptable consistency of IVHFPR and IVHMPR, and improve them to make the consistency acceptable. Finally, an illustrative example of selecting the optimal treatment for a lung cancer patient is given to demonstrate our work in detail.

Keywords: interval-valued hesitant fuzzy preference relation; interval-valued hesitant multiplicative preference relation; transforming function; expectation additive consistency; multiplicative consistency

1. Introduction

The interval-valued hesitant fuzzy set (IVHFS) was presented by Chen et al. [1] as a capable tool to describe the uncertainty in real practical application, especially in group decision making (GDM) circumstances. As can be seen from its acronym, IVHFS, it is related to the interval-valued fuzzy set (IVFS) [2] and the hesitant fuzzy set (HFS) [3]. In comparison to IVFS and HFS, IVHFS has a superior advantage with respect to characterizing uncertainty in GDM contexts. For instance, it may be not easy for a decision maker to exactly quantify his/her opinion with a crisp number. One often creates fluctuations in the neighborhood of a crisp value when he/she makes a decision. Then, an interval-neighborhood of this explicit value is an advisable expression of the decision maker’s comments in this practical situation. Therefore, interval description increases the imitativeness of people’s estimation when considering indeterminacy. In other words, interval-valued expression is more appropriate than crisp numbers to model cognitive behavior, due to its the better flexibility. Additionally, while brittleness is a drawback of HFS, IVHFS can overcome it to a certain extent. For example, it is common in GDM processes that there are several distinct opinions from different decision makers, whereby they cannot persuade the other to reach a unanimous decision. The HFS shows better performance in characterizing these discrete opinions than some other traditional fuzzy sets. Although the envelopment of a HFS can incorporate all the opinions, the deviation between the
truth result and the calculated one will be increased, since some impossible opinions will be included. Therefore, HFS has a unique superiority in describing this hesitancy with regard to an interval fuzzy number. This superiority is retained in IVHFS, although it is a flaw of IVF. In summary, IVHFS is generated on the basis of the advantages of both IVFS and HFS. It is arguably superior for describing uncertain information based on individual indetermination (relatively microscopic) and group inconsistency (relatively macroscopic). Even so, because IVHFS has been developed in just a short time, the investigations and applications of IVHFS are still in their infancy, and include the properties of operations [4], the similarity/distance measures [5–8], the entropy [8], the aggregation operator [9–12], the correlation coefficient [13], the preference relation [1], the approaches to obtaining the weight vectors about alternatives [14–21], etc. Thus, we select IVHFS as the basis to commence our work in this paper.

The preference relation is a common way to present people’s preferences through pairwise comparisons between some alternatives. It may not be easy for people to make an exact judgement with an independent alternative because of their inevitable lack of knowledge and cognitive behaviors. However, it is not difficult for them to conveniently rank all the alternatives with a preference relation. Hence, the preference relation has been explored in depth, and it has been widely applied to solve various GDM problems. In terms of categories, the preference relation can be classified into the fuzzy preference relation (FPR) and the multiplicative preference relation (MPR). Recently, Chen et al. [1] defined the concept of interval-valued hesitant fuzzy preference relation (IVHFR) and applied it to the GDM situation based on some distance measurements and aggregation operations. In this paper, it is shown both theoretically and practically that IVHFR can handle GDM problems in a flexible and objective manner under an interval-valued hesitant fuzzy environment. Zhang [19] extended the QUALIFLEX method to IVHFPR, which is able to directly deal with original INHFS information without performing aggregation operations. Tang and Meng [20] developed an approach based on IVHFPR to rank alternatives in GDM that does not require the addition of extra values to each element in IVHFPR and doesn’t disregard any offered information. Zhang et al. [21] built several programming models to estimate missing values in incomplete IVHFPR, which also don’t require extra values for each element in IVHFPR or disregard any offered information. Through carefully collating the researches of IVHFSs and IVHFPR, we find that the procedure of using it to resolve a GDM problem can be roughly divided into five steps (see Figure 1). Each step focuses on resolving a special issue of IVHFPR in the decision-making process.

As shown in Figure 1, because there is no concept of interval-valued hesitant multiplicative preference relation (IVHMPR), in the existing results, the authors studied the decision-making process with IVHFSs based only on IVHFPRs, without using IVHMPRs, and studied the relation between IVHFPR and IVHMPR. This is very important for the development of the theory of interval-valued hesitant preference relations and forms the basis for their applications in practical decision-making problems. In the following, we point out two interesting issues worthy of study: (1) What is the relationship between IVHFPR and IVHMPR? Can it be theoretically explained? (2) If the property of connection between IVHFPR and IVHMPR results in a good performance, what is the utility of this connection for consistency checking and the improvement of the preference relation with IVHFSs? In this paper, we will address the above two issues.

It is clearly seen from Figure 1 that the consistency level is vital for applying the preference relation in the decision-making process. A poor decision will be obtained when we use a preference relation with poor consistency because of the contradiction and conflict within it. Although the aggregation manner can also disturb the final ranking of all alternatives, a high quality of consistency is without doubt the necessary prerequisite for making a good decision by using a preference relation. In the existing results on FPR and MPR, the scholars have proposed some consistency concepts and constructed some checking and improving methods based on them. For example, Liao et al. [22] developed a method to obtain the approximate multiplicatively consistent interval-valued intuitionistic FPR, which only needs to know the off-diagonal elements in the preference relation. Liao et al. [23]
proposed a method to construct the acceptably multiplicatively consistent hesitant FPR, which does not need the participation of the decision maker. Liu et al. [24] introduced the density function to determine the multiplicative consistency index of hesitant FPR to solve the problem of the existing method that there is no any theoretical evidence to give a consistency threshold. Viewing this research comprehensively, there are two flaws in the existing methods of checking and improving consistency for preference relations:

- Some specific methods of checking and repairing consistency for IVFPR and HFPR have been presented based on the multiplicative transitivity \( r_{ij} r_{jk} = r_{ij} r_{jk} \) [22,23], which is explicitly interpreted in Definition 9. This formula is the necessary condition of multiplicative transitivity \( r_{ik} r_{kj} = r_{ij} \), which is described in detail in Definition 8, but not the sufficient condition of it. That is to say, \( r_{ij} r_{jk} r_{ki} = r_{ij} r_{jk} r_{ki} \) holds when \( r_{ik} r_{kj} = r_{ij} \) is satisfied. However, \( r_{ik} r_{kj} = r_{ij} \) may be not true when \( r_{ij} r_{jk} r_{ki} = r_{ij} r_{jk} r_{ki} \). Therefore, it is important to find a way which is sufficient and necessary when \( r_{ik} r_{kj} = r_{ij} \).

Motivated by all of the above analysis, in this paper, we will do the following work:

- Develop the concepts of expectation additive consistency for IVHFPR, IVHMPR, and the multiplicative consistency for IVHMPR. Construct a connection between the IVHFPR with expectation additive consistency and the IVHMPR with multiplicative consistency.
- Give a checking and improving method for the multiplicative consistency of IVHMPR that is directly deduced from the original preference data and is also applied to IVHFPR.

To do so, the rest of this paper is organized as follows: Section 2 briefly reviews some related concepts and basic operators of the preference relation that we will use later. Section 3 defines IVHMPR and constructs the connection between it and IVHFPR. Thereafter, Section 4 establishes a method to check and improve the multiplicative consistency of IVHMPR, which is also applied to examine and repair the expectation additive consistency of IVHFPR. Then, an illustration of determining the optimal treatment for a lung cancer patient is demonstrated in Section 5. Section 6 compares the new proposed consistency checking and improvement method to the related existing results. Finally, Section 7 concludes the paper.

![Figure 1. The steps of handling GDM problem with IVHFPR.](image-url)
2. Preliminaries

To facilitate the later presentations, in this section, we briefly review some related concepts and operators.

2.1. The Concepts of IVHFS and IVHFPR

Definition 1. [1] Let \( X = \{x_1, x_2, \cdots, x_n\} \) be a reference set, and \([0,1]^2\) be the set of all closed subintervals of \([0,1]\). Then, an IVHFS associated with \( X \) is \( A = \{(x_i, h(x_i))|x_i \in X, i = 1,2,\cdots,n\} \), where \( h(x_i) : X \to [0,1]^2 \) contains all of the possible interval-valued membership degrees on the element \( x_i \) to a set. For convenience, Chen and Xu [1] denoted \( h(x_i) = \begin{pmatrix} \mu^1(x_i), \mu^2(x_i) \\ \mu^3(x_i), \mu^4(x_i) \end{pmatrix} \) as an interval-valued hesitant fuzzy element (IVHFE).

Definition 2. [1] Let \( X = \{x_1, x_2, \cdots, x_n\} \) be a reference set. An interval-valued hesitant preference relation on \( X \) can be denoted by a matrix \( R = (r_{ij})_{n \times n} \subseteq X \times X \), where \( r_{ij} \) is an IVHFE, each of which indicates all the possible degrees to which \( x_i \) is preferred to \( x_j \), and \( \|r_{ij}\| \) represents the number of \( r_{ij} \). Moreover, each element \( r_{ij} \) in the preference relation matrix \( R \) should satisfy the condition that \( \inf r_{ij} = \sup r_{ij} = 1 \), \( r_{ij} = [0.5, 0.5], i, j = 1, 2, \cdots, n \). Additionally, all of the elements in \( r_{ij} \) are arranged in an increasing order, and \( r_{ij}^{(\tau)} \) is the \( \tau \)-th element in \( r_{ij} \). Similarly, we know that \( r_{ij}^{(\tau)} \) is the \( \sigma(L(r_{ij}) - \tau + 1) \)-th element in \( r_{ij} \).

2.2. Interval Number and IVMPR

Definition 3. [25] An interval number can be defined as: \( I = [u_l, u_r] \), where \( u_l, u_r \in R \), and \( u_l \leq u_r \).

Definition 4. [25] Let \( I_1 = [u_l, u_r] \) and \( I_2 = [v_l, v_r] \) be two interval numbers, where \( u_l, u_r, v_l, v_r \geq 0 \), then their addition, multiplication, and logarithm operations are shown as follows:

Addition operation: \( I_1 + I_2 = [u_l + v_l, u_r + v_r] \);
Multiplication operation: \( I_1 \times I_2 = [u_l \times v_l, u_r \times v_r] \);
Logarithm operation: \( \log_a I_1 = \left[ \log_a u_l, \log_a u_r \right] \).

Definition 5. [26] Let \( I_1 = [u_l, u_r] \) and \( I_2 = [v_l, v_r] \) be two interval numbers, where \( u_l, u_r, v_l, v_r \geq 0 \), then \( p(I_1 \geq I_2) = \max \{1 - \max \left( \frac{u_l - v_l}{v_r - u_r}, \frac{u_r - v_r}{u_r - u_l}, \frac{u_l - v_r}{v_r - u_l}, \frac{u_r - v_l}{v_r - u_l} \right) \} \), where \( p(I_1 \geq I_2) \) is the probability of \( I_1 \geq I_2 \).

Definition 6. [27] An IVMPR can be given as follows:

\[
U = (u_{ij})_{n \times n} = \begin{pmatrix}
[1,1] & u_{12}^- & u_{12}^+ & \cdots & u_{1n}^- & u_{1n}^+ \\
\bar{u}_{21}^- & [1,1] & \cdots & \cdots & \cdots \\
\bar{u}_{2n}^- & \bar{u}_{2n}^+ & [1,1] & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\bar{u}_{n1}^- & \bar{u}_{n1}^+ & \bar{u}_{n2}^- & \bar{u}_{n2}^+ & [1,1]
\end{pmatrix}
\]

where \( u_{ij} \) indicates that \( x_i \) is between \( u_{ij}^- \) and \( u_{ij}^+ \) times superior to \( x_j \), and \( [u_{ij}^-, u_{ij}^+] \) is an interval number, which satisfies the conditions that \( \frac{1}{9} \leq u_{ij}^- \leq u_{ij}^+ \leq 9 \), and \( u_{ij} = \frac{1}{u_{ij}^-} \).
2.3. The Related Concepts on Consistency for Preference Relation

Definition 7. [28]. The pair-wise comparison matrix can pass the consistency test, if the consistency ratio C.R. = \( \frac{C}{R} \) < 0.1, where the consistency index C.I. = \( \frac{\lambda_{\text{max}} - n}{n-1} \), R.I. is the average random index, \( \lambda_{\text{max}} \) and \( n \) are the maximum eigenvalue and the order of the matrix \( A \) respectively.

Definition 8. [28]. A reciprocal matrix is perfectly consistent if \( \mu_{ik} \mu_{kj} = \mu_{ij} \) for all \( i, k \), \( j = 1, 2, \ldots, n \).

Definition 9. [29]. Let \( R = (r_{ij})_{n \times n} \) be a FPR, it is multiplicatively consistent if it satisfies \( r_{ij} = \frac{r_{ik} \cdot r_{kj}}{r_{ik} \cdot r_{kj}} \) for all \( i, k, j = 1, 2, \ldots, n \).

Definition 10. [29]. Let \( R = (r_{ij})_{n \times n} \) be a FPR, then \( R \) is called an additive transitive FPR if the following additive transitivity is satisfied: \( r_{ij} = r_{ik} + r_{kj} - 0.5 \), for all \( i, k, j = 1, 2, \ldots, n \).

3. The IVHMPR and Its Connection with IVHFPR

In this section, we first define the concept of the interval-valued hesitant multiplicative set (IVHMS), whose basic elements are the components of an IVHMPR, and then construct the IVHMPR and discuss its relation with the IVHFPR.

3.1. The IVHMPR

Definition 11. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a fixed set. Then, an IVHMS pertaining to \( X \) can be defined as \( M = \{(x, m(x)) | x \in X\} \), where \( m(x) = \left\{ m^{e}, m^{o} \right\} = \left\{ \overline{m}^{e}, \overline{m}^{o} \right\} \), \( \sigma = 1, 2, \ldots, L(m) \) denotes all of the possible interval membership degrees of the element \( x \in X \) with the condition that \( \frac{1}{N} \leq m_{ij}^{e} \leq \overline{m}_{ij}^{e} \leq N \), and \( N \in \mathbb{Z}^{+} \), where \( N \) is a positive integer and \( \mathbb{Z}^{+} \) is the set of all positive integers. For convenience, \( m(x) \) is called an interval-valued hesitant multiplicative number (IVHMN).

The IVHMS is a particular set to describe uncertain preference information. Each element of it is an interval that indicates the possible times range of an alternative being superior to another one. Due to the randomness and complexity of determination, we consider only that the values of \( m_{ij}^{e} \) and \( \overline{m}_{ij}^{e} \) fall into a suitable interval \( \left[ \frac{1}{N}, N \right] \), which is not too extensive or narrow. In general, we take \( N = 9 \).

In what follows, we first define some basic operations of IVHMs:

Definition 12. Let \( m(x) = \left\{ m^{e}, m^{o} \right\} = \left\{ \overline{m}^{e}, \overline{m}^{o} \right\} \), \( \sigma = 1, 2, \ldots, L(m) \) be an IVHMN, then the mean value of \( m(x) \) can be captured by the formula below:

\[
E(m(x)) = \left( \prod_{\sigma=1}^{L(m)} m^{o}\right)^{1/L(m)} = \left( \prod_{\sigma=1}^{L(m)} m^{e}\right)^{1/L(m)}, \left( \prod_{\sigma=1}^{L(m)} \overline{m}^{o}\right)^{1/L(m)}, \left( \prod_{\sigma=1}^{L(m)} \overline{m}^{e}\right)^{1/L(m)}
\]  

(1)

Lemma 1. (Infill element criterion). We should infill a few elements into an IVHMN \( m \) when its number is not big enough in the practical calculations. However, the infilled element is just the mean value of \( m \). Furthermore, the reformed \( m \), namely \#m, is also an IVHMN.

Proof. It is not difficult to prove that this conclusion is true by using Equation (1). We do not address this in detail here. □

For example, suppose that there is an IVHMN \( m = ([5.6, 6], [7.8, 8.2], [8, 9.68]) \); its mean value can be calculated by Equation (1) as \( E(m) = [7.0435, 7.8093] \). If the target number of an IVHMN needs to
be a positive integer \( L \), where \( L > 3 \), then we infill \( L - 3 \) numbers of \([7.0435, 7.8093]\) into \( m \). In this way, we obtain a new IVHMN that satisfies the number requirement, namely:

\[
#m = \left[ [5.6, 6], [7.0435, 7.8093], \ldots, [7.0435, 7.8093], [7.8, 8.2], [8, 9.68] \right]_{L-3}
\]

Additionally, we should note that there are other methods to make the number of an IVHMN into an ideal value, such as the method based on the optimism and pessimism principles. Based on Lemma 1, we give the operations of IVHMNs below:

**Definition 13.** Let \( M = \left\{ m_i(x) = \left[ m_i^0, \sigma_i = 1, 2, \ldots, L(m_i) \right], i = 1, 2, \ldots, L(M) \right\} \) be an IVHMS, where \( L(m_i) \) and \( L(M) \) are the positive integers, respectively. Then we denote the corresponding infilled IVHMS as:

\[
#M = \left\{ #m_i(x) = \left[ #m_i^0, \sigma = 1, 2, \ldots, L \right], i = 1, 2, \ldots, L(M) \right\}
\]

where \( L = \max(L(m_1), L(m_2), \ldots, L(m_N)) \), and all the elements in \( #m_i \) are sorted in an increasing order. Thus, three basic operations can be defined as follows:

1. **The first kind of multiplication operation:**
   \[
   \bigotimes_{i=1}^{N} m_i = \left[ \left( \prod_{\alpha=1}^{i} m^\alpha \right)^\frac{1}{L(m_i)}, \left( \prod_{\alpha=1}^{i} \overline{m}^\alpha \right)^\frac{1}{L(m_i)} \right]
   \]

2. **The second kind of multiplication operation:**
   \[
   \bigotimes_{i=1}^{N} m_i = \left[ \frac{1}{L(m_i)} \prod_{\alpha=1}^{i} m^\alpha, \frac{1}{L(m_i)} \prod_{\alpha=1}^{i} \overline{m}^\alpha \right]
   \]

3. **The mean value of \( M \):**
   \[
   E(M) = \left( \prod_{i=1}^{L(M)} E(m_i(x)) \right)^\frac{1}{L(M)}
   \]

4. **The variance value of \( M \):**
   \[
   \text{Var}(M) = E(M - E(M))^2
   \]

From the viewpoint of mapping, the fist kind of multiplication is defined from a set of IVHMSs into an IVHMS. If we call \((S(M), \bigodot)\) an IVHM space based on the multiplication \( \bigodot \), where \( S(M) \) is the set of all of the IVHMNs, then \((S(M), \bigodot)\) is closed on \( \bigodot \) or the set \( S(M) \) is closed on \( \bigodot \). For the second kind of multiplication, it is defined from a set of IVHMSs into an interval number set. Logically, the set \( S(M) \) is not closed on \( \bigodot \).

**Definition 14.** Let \( X = [x_1, x_2, \ldots, x_n] \) be a reference set. An IVHMPR on \( X \) can be denoted by a matrix:

\[
M = (m_{ij})_{n \times n} = \begin{pmatrix}
    m_{11} & m_{12} & \cdots & m_{1n} \\
    m_{21} & m_{22} & \cdots & m_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{n1} & m_{n2} & \cdots & m_{nn}
\end{pmatrix}_{n \times n}
\]

where \( M : X \times X \to IVHMS \). Each \( m_{ij} \) contains all the possible intervals, which indicates that \( x_i \) is between \( m^\sigma_{ij} \) and \( \overline{m}^\sigma_{ij} \) times superior to \( x_j \). \( m^\sigma_{ij}, \overline{m}^\sigma_{ij} (\sigma = 1, 2, \ldots, L(m_{ij})) \) is the \( \sigma \) - th interval preference value in \( m_{ij} \). Moreover, the following conditions should be satisfied when the matrix \( M \) is an IVHMPR:

1. \( m_{ij} = \left\{ [1, 1], \ldots, [1, 1] \right\}, N \in \mathbb{Z}^+, \text{if } i = j; \)
2. \( L(m_{ij}) = L(m_{ji}), \forall i, j = 1, 2, \ldots, n; \)
3. All the elements in \( m_{ij} \) are in an ascending order when \( i < j \) (the criterion of comparing any two elements in an IVHMN follows Definition 5);
Theorem 1. Let $R$ be an IVHFPR pertaining to a reference set $X$

3.2. The Connection between IVHFPR and IVHMPR

Now we investigate the connection between IVHFPR and IVHMPR. Let’s begin with a transforming function, which can reflect the attitude of decision makers in some way:

Definition 15. Let $R = \left( r_{ij} \right)_{n \times n}$, where $r_{ij} = \left\{ \left( r_{ij}^\sigma, \overline{r}_{ij}^\sigma \right) \mid \sigma = 1, 2, \cdots, L(r_{ij}) \right\}$ is an IVHFPR associated with a reference set $X = \{x_1, x_2, \cdots, x_n\}$; a transforming function can be defined as:

$$V\left( \left( r_{ij}^\sigma, \overline{r}_{ij}^\sigma \right) \right) = \left[ a^{2r_{ij}^{\sigma-1}}, a^{2\overline{r}_{ij}^{\sigma-1}} \right]$$

where $a^{(\cdot)} : [-1, 1] \to \mathbb{R}$ is an exponential function whose base number is always bigger than 1. Generally, the base number of $a^{(\cdot)}$ should be bigger than 1, but not bigger than 9, according to Saaty’s scale [28].

It is important to note that when the transforming function affects $r_{ij}$, it means that the function can cope with all of the elements in $r_{ij}$. Similarly, it can be used to deal with each element in the IVHFPR $R$.

For example, let $R = \left( r_{ij} \right)_{n \times n}$ be an IVHFPR of $X$, and $a^{(t)} = e$. Then

$$V(R) = \left( \left[ a^{2r_{ij}^{\sigma-1}}, a^{2\overline{r}_{ij}^{\sigma-1}} \right]_{\sigma = 1, 2, \cdots, L(r_{ij})} \right)_{n \times n}$$

Based on Equation (2), we can transform any IVHFPR $R$ into $V(R)$.

Furthermore, the lower and upper indices of $a^{(\cdot)}$ in Equation (2) are $2r_{ij}^{\sigma-1}$ and $2\overline{r}_{ij}^{\sigma-1}$, respectively, which are equal to $r_{ij}^\sigma - (1 - r_{ij}^\sigma)$ and $\overline{r}_{ij}^\sigma - (1 - \overline{r}_{ij}^\sigma)$, respectively. We can interpret the same framework of them as the superior degree subtracts the inferior degree, which expresses the decision maker’s attitude. While the base number of $a^{(\cdot)}$ can theoretically be arbitrarily selected, and only needs to satisfy the requirements of Definition 15, for convenience of calculation, it would be much better to determine a proper value, which is not too big or too small, as the base value of $a^{(\cdot)}$, because changing its value will not alter the preference ranking.

Theorem 1. Let $R$ be an IVHFPR pertaining to a reference set $X = \{x_1, x_2, \cdots, x_n\}$; $V(\cdot)$ is a transforming function with the condition that the base number is a $\left( \frac{1}{N} \leq a \leq N \right)$. Then, $V(R)$ is an IVHMPR.

Proof. Let $R = \left( r_{ij} \right)_{n \times n}$ be an IVHFPR on $X$. Then, it is very easy to prove that $V(R)$ meets all the requirements of Definition 14 except the Axiom (4), according to the concepts of $R$ and $V(\cdot)$. Therefore, here we only prove Axiom (4). Since $\forall i, j = 1, 2, \cdots, n$, $\sigma = 1, 2, \cdots, L(r_{ij})$, $V(r_{ij}) = \left( \left[ a^{2r_{ij}^{\sigma-1}}, a^{2\overline{r}_{ij}^{\sigma-1}} \right]_{\sigma = 1, 2, \cdots, L(r_{ij})} \right)_{n \times n}$, then, $V\left( r_{ij}^\sigma \right) \times V\left( \frac{L(r_{ij})}{\mu}^{\sigma+1} \right) = a^{2r_{ij}^{\sigma-1}} \times a^{2(\frac{L(r_{ij})}{\mu})^{(\sigma+1)-1}} = a^{2(r_{ij}^\sigma + \frac{L(r_{ij})}{\mu})^{(\sigma+1)-1}} = a^0 = 1$. This completes the proof of Theorem 1. \(\Box\)
Theorem 2. Let \( X = \{x_1, x_2, \cdots, x_n\} \) be a reference set, \( R \) and \( M \) be the IVHFPR and IVHMPR on \( X \), respectively, and the function \( V(\cdot) : [R] \to [M] \) be any one of the transforming functions. Then, the mapping \( V(\cdot) : V(R) = M \) is a one-to-one correspondence.

Proof. Let \( R_1 = (r_{ij}) = \{[r^i_{ij}, \bar{r}^i_{ij}] \mid \sigma = 1, 2, \cdots, L(m_{ij})\} \) and \( R_2 = (r_{ij}) = \{[r^i_{ij}, \bar{r}^i_{ij}] \mid \sigma = 1, 2, \cdots, L(m_{ij})\} \) be any two IVHFPRs in \([R]\). According to Theorem 1, we know that \( M = V(R) \) is an IVHMPR. Consequently, we denote the set of all \( V(R) \) as \([M]\). Now, we first prove \( \forall R_1, R_2 \in [R], \) if \( R_1 \neq R_2 \), then \( V(R_1) \neq V(R_2) \).

Suppose that \( M_1 = M_2 \) when \( R_1 \neq R_2 \). It is logical to get the following two results according to the equivalence definition of matrices:

(a) \( L(V(r_{ij})) = L(V(r_{ij})), \forall i, j = 1, 2, \cdots, n; \)
(b) \( V(r_{ij}) = V(r_{ij}), \forall i, j = 1, 2, \cdots, n. \)

Since the transforming function \( V(\cdot) \) is strictly increasing, \( V^{-1}(\cdot) \) is also strictly increasing. Thus, there is an equivalence relation \( L(r_{ij}) = L(r_{ij}) = L(V(r_{ij})) = L(V(r_{ij})), \) where the value is denoted as \( L \). Synthesizing this and the Axiom (a), we know that \( \forall \sigma = 1, 2, \cdots, L, \) \( V^{-1}(V^l(r_{ij})) = V^{-1}(V^l(r_{ij})). \) Then \( r^l_{ij} = r^l_{ij} \) can be deduced. Furthermore, we can get \( r_{ij} = r_{ij}. \) Based on the above analysis, we can conclude that \( R_1 = R_2. \) Clearly, this result is in conflict with the premise \( R_1 \neq R_2. \) Therefore, our assumption, i.e., \( M_1 = M_2 \) when \( R_1 \neq R_2 \) is wrong. Thus, \( R_1 \neq R_2 \Rightarrow M_1 \neq M_2. \)

Similarly, we can prove that \( M_1 \neq M_2 \Rightarrow R_1 \neq R_2 \) \( (\forall M_1, M_2 \in [M]). \) In addition, we suppose that \( R_1 = R_2 \) when \( M_1 \neq M_2. \) It is logical to get the following two results in accordance with the equivalence definition of the matrices:

(c) \( L(r_{ij}) = L(r_{ij}), \forall i, j = 1, 2, \cdots, n; \)
(d) \( r_{ij} = r_{ij}, \forall i, j = 1, 2, \cdots, n. \)

Since the transforming function \( V(\cdot) \) is strictly increasing, we know that \( V^{-1}(\cdot) \) is also strictly increasing. Taking all of these inferred results into account, we get an equation relation \( L(r_{ij}) = L(m_{ij}) = L(m_{ij}), \) and denote the result as \( L. \) Then, \( \forall \sigma = 1, 2, \cdots, L, \) we get \( V(r_{ij}) = V(r_{ij}). \)

That is, \( m^l_{ij} = m^l_{ij}, \) i.e., \( M_1 = M_2. \) Thus, the consequence conflicts with the premise \( M_1 \neq M_2, \) which means that our assumption, \( R_1 = R_2 \) when \( M_1 \neq M_2, \) is wrong. Therefore, \( R_1 \neq R_2 \Rightarrow M_1 \neq M_2. \) This completes the proof of Theorem 2. \( \square \)

Corollary 1. Let \( M \) be an IVHMPR associated with a reference set \( X = \{x_1, x_2, \cdots, x_n\}, \) and \( V(\cdot) \) be a transforming function with the base number \( a (a > 1). \) Then, \( R = V^{-1}(M) \) is an IVHFPR, where \( V^{-1}(\cdot) \) is the inverse function of \( V(\cdot), \) which can be described in detail by the following formula:

\[
V^{-1}\left[\left[\begin{array}{c} m^l_{ij} \\ \bar{m}^l_{ij} \end{array}\right]\right] = \left[\begin{array}{c} \frac{1 + \log_a m^l_{ij}}{2} \\ \frac{1 + \log_a \bar{m}^l_{ij}}{2} \end{array}\right]\left\{a \geq \max\left(\bar{m}^l_{ij}, \frac{1}{m^l_{ij}}\right), a > 1\right\}
\]

(3)

Proof. Let \( M = (m_{ij})_{n \times n}, \) where \( m_{ij} = \{[m^l_{ij}, \bar{m}^l_{ij}] \mid \sigma = 1, 2, \cdots, L(m_{ij})\} \) is an IVHMPR over \( X. \) Then

\[
r^l_{ij} + \sigma^l_{ij} = V^{-1}(m^l_{ij}) + V^{-1}(m^l_{ij}) = 2 + \log_a \left(\frac{m^l_{ij} + \bar{m}^l_{ij}}{2}\right) = 2 + \frac{1}{2} = 1
\]

(4)
Additionally, we can verify that $V^{-1}(M)$ satisfies all the other conditions of Definition 2, according to the concepts of IVHMPR and the transforming function. Thus, we complete the proof of Corollary 1.

\textbf{Example 2.} Let $R$ be an IVHFPR, where

$$
R = \begin{pmatrix}
(0.5, 0.5) & (0.2, 0.4), (0.3, 0.5) & (0.5, 0.6), (0.6, 0.7), (0.65, 0.8) \\
(0.5, 0.7), (0.6, 0.8) & (0.5, 0.5) & (0.78, 0.89) \\
(0.2, 0.35), (0.3, 0.4), (0.4, 0.5) & (0.11, 0.22) & (0.5, 0.5)
\end{pmatrix}
$$

If we use the transforming function $V(\cdot)$ with the base number 7, then we can obtain an IVHMPR, which is denoted as $M$:

$$
M = \begin{pmatrix}
(1, 1) & (0.7^{-0.6}, 7^{-0.2}) & (0.7^{-0.4}, 1) & (1, 7^{0.2}) & (7^{0.2}, 7^{0.4}) & (7^{0.3}, 7^{0.6}) \\
(1, 7^{0.4}) & (7^{0.2}, 7^{0.6}) & (1, 1) & (7^{0.56}, 7^{0.78}) \\
(7^{-0.6}, 7^{-0.3}) & (7^{-0.4}, 7^{-0.2}) & (7^{-0.2}, 1) & (7^{-0.78}, 7^{-0.56}) & (1, 1)
\end{pmatrix}
$$

It is easy to validate that $M = V(R)$ satisfies all of the requirements of Definition 14. Moreover, we can easily obtain the IVHFPR $R$ by $M$ and $V^{-1}\left(\left[m^a_{ij}, m^a_{ji}\right]\right) = \frac{1+\log m^a_{ij}}{2}, \frac{1+\log m^a_{ji}}{2}$.

All in all, we have found a connection between IVHFPR and IVHMPR, based on the transforming function. This provides a very good basis for the later analysis of the consistency of IVHFPR and IVHMPR. In the next section, we will aim at establishing a method to check and improve the consistency for IVHFPR and IVHMPR.

4. Analysis of the Consistency for IVHFPR and IVHMPR

Let’s first take a look at the IVHFPR $R$ and the IVHMPR $M$ described in Example 2. We naturally find a ubiquitous phenomenon that the numbers of the elements of IVHFNs in $R$ may not be same. For instance, in Example 2, $L(r_{12}) = 2$, but $L(r_{13}) = 3$, and so do the IVHMPR $M$. Under this situation, this difficulty may be brought into the decision-making process. Hence, before the discussion of the consistency for IVHFPR and IVHMPR, we have to normalize the IVHFPR $R$ and the IVHMPR $M$ under the principle of minimal modification.

4.1. The Normalizations of IVHFPR and IVHMPR

The minimal modification principle refers to the idea that every normalized element should be mostly identical to the original version. Furthermore, the being “mostly identical” between the normalized element and its corresponding original one means that the smaller the influence of editing on the original one, the better the normalized result obtained. This small influence of editing on the corresponding original one means that the difference in expectation between the edited and original elements should be as small as possible, and the same for their variance.

In the following, we present a normalizing method for the IVHFPR:

\textbf{Definition 16.} Let $R = \left(r_{ij}\right)_{n \times m}$ be an IVHFPR on a reference set $X$. Then the corresponding normalized matrix of $X$, namely $N_R(R) = \left(\#r_{ij}\right)_{n \times m}$, should fulfils the following conditions:

(1) $\forall i, j = 1, 2, \cdots, n, L(\#r_{ij}) = L_R$, where $L_R = \max(L(r_{ij})|i, j = 1, 2, \cdots, n)$

(2) If $i = j$, then $\#r_{ij} = \left\{ \begin{array}{ll}
[1, 1], \cdots, [1, 1] & \\
L_R
\end{array} \right.$
(3) If \( L_R > L(r_{ij}) \), where \( i < j \), then we should infill \( L_R - L(r_{ij}) \) number of \( \#r^i_{ij}, \#\overline{r}^i_{ij} \) \( \notin r_{ij} \) into \( r_{ij} \), where
\[
\left[ \#r^i_{ij}, \#\overline{r}^i_{ij} \right] = \left[ \frac{L(r_{ij})}{\sum_{i=1}^{r^i_{ij}} \sum_{j=1}^{\overline{r}^i_{ij}} L(r_{ij})} \right].
\]
Furthermore, all the elements in \( \#m_{ij} \) (\( i < j \)) are in a non-descending order.

(4) If \( i > j \), then \( \#r_{ij} = \left\{ \left[ \#r^i_{ij}, \#\overline{r}^i_{ij} \right] | \sigma = 1, 2, \cdots, L_R \right\} \), where
\[
\left[ \#r^i_{ij}, \#\overline{r}^i_{ij} \right] = \left[ 1 - \#r_i^{L(r_{ij})-\sigma+1}, 1 - \#r_i^{L(r_{ij})-\sigma+1} \right]
\]

**Theorem 3.** The normalized matrix \( N_R = (\#r_{ij})_{n \times n} \) calculated by Definition 16 has the minimum impact on \( R \) through packing other interval-valued fuzzy number(s) in \( r_{ij} \) or removing some interval-valued fuzzy number(s) from \( r_{ij} \).

**Proof.** Let \( R = (r_{ij}) = \left\{ \left[ r^i_{ij}, \overline{r}^i_{ij} \right] | \sigma = 1, 2, \cdots, L(r_{ij}) \right\} \) be any one of IVHFPRs over \( X \). Then, its corresponding normalized form is \( \#R = (\#r_{ij}) = \left\{ \left[ \#r^i_{ij}, \#\overline{r}^i_{ij} \right] | \sigma = 1, 2, \cdots, L(r_{ij}) \right\} \). Next, we prove Theorem 3 from the two cases: \( i \leq j \) and \( i > j \).

(1) If \( i \leq j \), then we calculate the mean value of each \( r_{ij} \), and denote it as \( E(r_{ij}) = \left[ \frac{L(r_{ij})}{\sum_{i=1}^{r^i_{ij}} \sum_{j=1}^{\overline{r}^i_{ij}} L(r_{ij})} \right] \).

Thus,
\[
E(\#r_{ij}) = \left[ \frac{L(r_{ij})}{\sum_{i=1}^{r^i_{ij}} \sum_{j=1}^{\overline{r}^i_{ij}} L(r_{ij})} \right] \left[ \frac{L(r_{ij})}{\sum_{i=1}^{r^i_{ij}} \sum_{j=1}^{\overline{r}^i_{ij}} L(r_{ij})} \right] = \left[ \#r^i_{ij}, \#\overline{r}^i_{ij} \right] = E(r_{ij})
\]

Therefore, no matter how many \( \left[ \#r^i_{ij}, \#\overline{r}^i_{ij} \right] \notin r_{ij} \), we infill in \( r_{ij} \), and thus, the edited \( r_{ij} \), i.e., \( \#r_{ij} \), have the same expectation with \( r_{ij} \). Their variance can be proven in a similar way, so we omit it here.

(2) If \( i > j \), then we can prove that the theorem is true in the same way based on Axiom (4) of Definition 15. This completes the proof of Theorem 3.

**Corollary 2.** Let \( R \) be an IVHFPR over a fixed set \( X \), and \( N_R(R) \) be the corresponding normalized form of \( R \). Then \( N_R(R) = \#R \) is also an IVHFPR.

**Proof.** The proof process of Corollary 2 is easy and simple according to Definition 2 and Definition 16, so we omit it here.

**Example 3.** Let \( R \) be an IVHFPR as the same as in Example 2. Then, its normalization form can be calculated as:
\[
\#R = \left\{ \begin{array}{c}
(0.5,0.5), (0.5,0.5), (0.5,0.5) \\
(0.2,0.4), (0.25,0.45), (0.3,0.5) \\
(0.5,0.6), (0.6,0.7), (0.65,0.8) \\
(0.5,0.7), (0.55,0.75), (0.6,0.8) \\
(0.5,0.5), (0.5,0.5), (0.5,0.5) \\
(0.78,0.89), (0.78,0.89), (0.78,0.89) \\
(0.2,0.35), (0.3,0.4), (0.4,0.5) \\
(0.11,0.22), (0.11,0.22), (0.11,0.22) \\
(0.5,0.5), (0.5,0.5), (0.5,0.5)
\end{array} \right\}
\]
It is easy to verify that \( \#R \) is also an IVHFPR. So far, we have proposed the normalized method for IVHFPR and investigated the effect of the normalizing method over IVHFPR. However, is there also a similar situation for IVHMRR? We now discuss it as follows:

**Definition 17.** Let \( M = (\overline{m}_{ij})_{n \times n} \) be an IVHMRR associated with a reference set \( X \). Then its corresponding normalized matrix \( N(M) = (\overline{n}_{ij})_{n \times n} \) should fulfill the following conditions:

1. \( \forall i, j = 1, 2, \ldots, n, L(\overline{m}_{ij}) = L_M, \) where \( L_M = \max(L(m_{ij}), i, j = 1, 2, \ldots, n) \)

2. If \( i = j, \overline{m}_{ij} = \left\{ \left[ \overline{m}_{11}, \ldots, \overline{m}_{ij}, \ldots, \overline{m}_{nn} \right] \right\} \), where \( L_M \)

3. If \( L_M > L(m_{ij}), \) where \( i < j, \) then we should add \( \sum L_M - L(m_{ij}) \) number of \( \left[ \overline{m}_{ij} \right] \) into \( m_{ij}, \)

and calculate it by the formula \( \left[ \overline{m}_{ij}, \overline{m}_{ij} \right] \).

4. All the elements in \( \overline{m}_{ij} \) are in a non-descending order.

5. If \( i > j, \) then \( \overline{m}_{ij} = \left\{ \left[ \overline{m}_{ij} \right] \right\} \).

**Theorem 4.** Let \( M = (m_{ij})_{n \times n} \) be an IVHMRR on a reference set \( X \). Then, the corresponding normalized matrix \( N(M) = (\overline{n}_{ij})_{n \times n} \) has a more minimal impact on \( M \) than through packing other interval-valued multiplicative number(s) from \( m_{ij} \) or removing interval-valued multiplicative number(s) from \( m_{ij}. \)

**Proof.** Let \( M = (m_{ij}) = \left\{ \left[ m_{ij}, \overline{m}_{ij} \right] \right\} \) be any one of IVHMRRs over \( X \). Then its corresponding normalized form is \( \overline{M} = (\overline{m}_{ij}) = \left\{ \left[ \overline{m}_{ij}, \overline{m}_{ij} \right] \right\} \). Next, we prove Theorem 4 based on two cases (\( i \leq j \) and \( i > j \):

If \( i \leq j, \) then we calculate the mean value of each \( m_{ij} \) by Equation (1) and denote it as \( E(m_{ij}) \). Then

\[
E(m_{ij}) = \left[ E(m_{ij}), E(\overline{m}_{ij}) \right] = \left[ \sum_{a=1}^{L(m_{ij})} \frac{\overline{m}_{ij}}{a}, \sum_{a=1}^{L(\overline{m}_{ij})} \frac{\overline{m}_{ij}}{a} \right] = \left[ \sum_{a=1}^{L(m_{ij})} \frac{m_{ij}}{a}, \sum_{a=1}^{L(\overline{m}_{ij})} \frac{\overline{m}_{ij}}{a} \right] = \left[ \sum_{a=1}^{L(m_{ij})} \frac{m_{ij}}{a}, \sum_{a=1}^{L(\overline{m}_{ij})} \frac{\overline{m}_{ij}}{a} \right] = \left[ \sum_{a=1}^{L(m_{ij})} \frac{m_{ij}}{a}, \sum_{a=1}^{L(\overline{m}_{ij})} \frac{\overline{m}_{ij}}{a} \right]
\]

which indicates that the infilled element \( \left[ \overline{m}_{ij}, \overline{m}_{ij} \right] \) is essentially the mean value of \( m_{ij}. \)

Now we prove:
Let \( R \) be an IVHFPR over a reference set \( X \). This completes the proof of Theorem 4.

Firstly, \( \forall \) \( \#m_{ij} \) is an IVHMPR over \( X \). Therefore, \( \#m_{ij} \) normalized form can be denoted as \( \#m_{ij} = \frac{\#m_{ij}}{\sum_{i=1}^{n} \#m_{ij}} \). Then the corresponding normalized matrix of \( M \) is also an IVHMPR over \( X \).

\[
E(\#m_{ij}) = \left[ \frac{\#m_{ij}}{\sum_{j=1}^{n} \#m_{ij}} \right] = \left[ \frac{\#m_{ij}}{\sum_{j=1}^{n} \#m_{ij}} \right]
\]

Therefore, no matter how many \( \#m_{ij}, \#m_{ij} \notin m_{ij} \) we infill in \( m_{ij} \), the edited \( m_{ij} \), namely \( \#m_{ij} \), has the same expectation as \( m_{ij} \). Their variance can be similarly proven, so we omit it here.

If \( i > j \), then we can prove Theorem 4 in the same way according to Axiom (5) of Definition 17. This completes the proof of Theorem 4. \( \square \)

**Corollary 3.** Let \( M \) be an IVHMPR over a reference set \( X = \{x_1, x_2, \ldots, x_n\} \). Then the corresponding normalized matrix of \( M \) is also an IVHMPR over \( X \).

**Proof.** Let \( M = (m_{ij})_{n \times n} \) where \( m_{ij} = \left\{ \#m_{ij}, \#m_{ij} \right\}_{\sigma = 1, 2, \ldots} \) is an IVHMPR over \( X \). Then its normalized form can be denoted as \( N_M(M) = (\#m_{ij})_{n \times n} = \left\{ \left( \#m_{ij}, \#m_{ij} \right) \right\}_{\sigma = 1, 2, \ldots} \). Among all the axioms of Definition 14, Axioms (1)–(3) are very easy to prove based on Axioms (1), (2), and (4) in Definition 17. We only need to prove Axiom (4) of Definition 14: Firstly, if \( \#m_{ij} \in m_{ij} \), \( \sigma = 1, 2, \ldots, \#m_{ij} \), then it surely meets Axiom (4) of Definition 14, due to the fact that \( M \) itself is an IVHMPR. Secondly, if \( \#m_{ij} \notin m_{ij} \), \( \sigma = 1, 2, \ldots, \#m_{ij} \), then \( \#m_{ij}^{(m_{ij})} = \frac{1}{\#m_{ij}} \), according Axioms (3) and (5) in Definition 17. Therefore, \( \#m_{ij} \times \#m_{ij}^{(m_{ij})} = 1 \) holds. This completes the proof of Corollary 3. \( \square \)

**Theorem 5.** Let \( R \) be an IVHFPR over a reference set \( X = \{x_1, x_2, \ldots, x_n\} \). Let \( M \) be the corresponding IVHMPR transformed from \( R \) by a transforming function \( V(\cdot) \). Then \( V(N_R(R)) = N_M(V(R)) \), where \( N_R(\cdot) \) and \( N_M(\cdot) \) are the normalizations over IVHFPR and IVHMPR, respectively.

**Proof.** Let \( R = (r_{ij}) = \left\{ \#r_{ij}, \#r_{ij} \right\}_{\sigma = 1, 2, \ldots} \) be any one of IVHFPRs over \( X \). Then we denote \( N_R(R) = \left\{ \#r_{ij}, \#r_{ij} \right\}_{\sigma = 1, 2, \ldots} \) as the corresponding normalized matrix of \( R \). Firstly, \( \forall r_{ij} \in N_R(R), i, j = 1, 2, \ldots, n \) according Definition 16 and Equation (2), we know

\[
V(N_R(r_{ij})) = \left\{ \begin{array}{ll}
\left\{ \#r_{ij}, \#r_{ij} \right\}_{\sigma = 1, 2, \ldots, \max(L(r_{ij})}, & \text{if } i < j;
\left\{ \frac{L(r_{ij})}{2}, \#r_{ij} \right\}_{\sigma = 1, 2, \ldots, \max(L(r_{ij})}, & \text{if } i > j.
\end{array} \right.
\]
which completes the proof of Theorem 5.

Corollary 4. Let M be an IVHMPR over a reference set X = \{x_1, x_2, \ldots, x_n\}. Let R be the corresponding IVHFPR transformed from M by a reverse transforming function V^{-1}(-). Then V^{-1} (N_M(M)) = N_R(V^{-1}(M)), where N_M(\cdot) and N_R(\cdot) are the normalization for IVHMPR and IVHFPR, respectively.

Proof. The proof process of Corollary 4 is similar to that of Corollary 3, so we omit it here.

Example 4. Let R be an IVHFPR on the fixed set X = \{x_1, x_2, \ldots, x_n\}, as shown below:

\[
R = \begin{pmatrix} \(0.5, 0.5\) & (0.28, 0.5), [0.65, 0.75], [0.72, 0.8) \) & (0.5, 0.68), [0.6, 0.7]\) \\
(0.2, 0.28), [0.25, 0.35], [0.5, 0.72) & (0.5, 0.5) & (0.8, 0.9) \\
(0.3, 0.4), [0.32, 0.5) & (0.1, 0.2) & (0.5, 0.5) \end{pmatrix}
\]  

(4)

Suppose that the transforming function is \(V(\{r^i_{ij}, \bar{r}^i_{ij}\}) = [6^{2r^i_{ij}-1}, 6^{2\bar{r}^i_{ij}-1}]\), then the normalization form of R can be obtained as:

\[
V(N_R(R)) = \begin{pmatrix} \[6^{-0.4}, 6^{-0.4}, 1\], [1, 1, 1] \) & \[6^{-0.44}, 1\], [0.8, 0.9], [0.94, 0.94] \) & \[6^{-0.4}, 6^{-0.4}, 1\], [1, 1, 1] \) & \[6^{-0.44}, 1\], [0.8, 0.9], [0.94, 0.94] \) \\
\[6^{-0.4}, 6^{-0.4}, 1\], [1, 1, 1] \) & \[6^{-0.44}, 1\], [0.8, 0.9], [0.94, 0.94] \) & \[6^{-0.4}, 6^{-0.4}, 1\], [1, 1, 1] \) & \[6^{-0.44}, 1\], [0.8, 0.9], [0.94, 0.94] \) \\
\[6^{-0.4}, 6^{-0.4}, 1\], [1, 1, 1] \) & \[6^{-0.44}, 1\], [0.8, 0.9], [0.94, 0.94] \) & \[6^{-0.4}, 6^{-0.4}, 1\], [1, 1, 1] \) & \[6^{-0.44}, 1\], [0.8, 0.9], [0.94, 0.94] \) \end{pmatrix}
\]  

(5)

which can also be calculated by the transforming and normalizing functions as \(N_M(V(R))\) (refer to Corollary 4).

By now, we have completed the discussions on the normalizations for IVHFPR and IVHMPR. In the next subsection, we will present a method for checking the consistency of IVHFPR and IVHMPR, and establish a method for improving the unacceptable consistency for IVHFPR and IVHMPR.

4.2. Checking and Improving the Consistency of IVHFPR and IVHMPR

In this section, we first present the concepts of expectation additive consistency of IVHFPR and expectation multiplicative consistency of IVHMPR. Then, we investigate the relationship between these two kinds of consistency of IVHFPR and IVHMPR. Based on which, we finally establish a method to synchronously check and improve the expectation additive consistency of IVHFPR and the expectation multiplicative consistency of IVHMPR.

Definition 18. Let \(N_R(R) = \{r^i_{ij}\}_{i \neq j}\) be a normalized IVHFPR on a fixed set \(X = \{x_1, x_2, \ldots, x_n\}\). Then, we denote \(R\) as being expectation additively consistent if the equivalence relation \(E(b_{ik}) + E(b_{ij}) - 0.5 = E(b_{ij})\)
Let $M$ be an IVHMPR on a fixed set $X$. Theorem 6.

**Symmetry 2019**

Let $M$ be an IVHMPR on a fixed set $X$. Theorem 6.

Improving method to address this issue: IVHFPR to be of expectation additive consistency, in the following, we propose a specific checking and not a su

any other orders of $i$, $j$.

Let $M$ be an IVHMPR on a fixed set $X$. Theorem 6.

**Proof.** Let $M$ be an IVHMPR on a fixed set $X = \{x_1, x_2, \ldots, x_n\}$, and the matrix $R$ be the corresponding IVHFPR of $M$ transformed by Equation (3). Then $R$ is expectation additively consistent if $M$ is multiplicatively consistent, not vice versa.

By algebra, we can easily prove that the expectation additive consistency of IVHFPR is weaker than the additive consistency defined in [20].

**Definition 19.** Let $N_M(M) = \{m_{ij}\}_{i,j}$ be a normalized IVHMPR on a fixed set $X = \{x_1, x_2, \ldots, x_n\}$. Then, we call $M$ to be multiplicatively consistent if the equation $b_{ik} \times b_{kj} = b_{ij}$ holds. Additionally, $b_{ik} \times b_{kj} = b_{ij}$ can be described as: $b_{ij} = \left( b_{ik} \times b_{kj} \right)$ and $\left( b_{ik} \times b_{kj} \right) = b_{ij}$ for all $i, j, k \in \{1, 2, \ldots, n\}$, where $\sigma = 1, 2, \ldots, L \{m_{ij}\}$, and

$B(\#m_{ij}) = b_{ij} = \left\{ \left[ b_{ij}^\sigma, b_{ij}^\tau \right]_{\sigma = 1, 2, \ldots, L \{m_{ij}\}} \right\}$ is an extraction function, where $b_{ij}^\sigma = \left\{ \#m_{ij}^{\sigma}, \#m_{ij}^{\tau}, \#m_{ij}^\sigma, \#m_{ij}^\tau \right\},$ if $i < j$;

and $\overline{b}_{ij} = \left\{ \#m_{ij}^{\sigma}, \#m_{ij}^{\tau}, \#m_{ij}^\sigma, \#m_{ij}^\tau \right\}$ if $i > j$.

Through Equation (3) and Definition 19, we know that there is a one-to-one mapping between IVHMPR with multiplicative consistency and IVHFPR with additive consistency based on number $a (a > 1)$, where the additive consistency is defined in [20]. For the relation between IVHMPR with multiplicative consistency and IVHFPR with expectation additive consistency, we give the following theorem to explain it.

**Theorem 6.** Let $M$ be an IVHMPR on a fixed set $X = \{x_1, x_2, \ldots, x_n\}$, and the matrix $R$ be the corresponding IVHFPR of $M$ transformed by Equation (3). Then $R$ is expectation additively consistent if $M$ is multiplicatively consistent, not vice versa.

**Proof.** Let $M$ be an IVHMPR on a fixed set $X = \{x_1, x_2, \ldots, x_n\}$, and the matrix $R$ be the corresponding IVHFPR of $M$ transformed by Equation (3). Additionally, let $N_M(M) = \{m_{ij}\}_{i,j}$ be the normalized form $M$, and $N_R(R) = \{r_{ij}\}_{i,j}$ be the normalized form of $R$. Then, if $M$ is multiplicatively consistent, we can know that $b_{ik}^\sigma \times b_{kj}^\sigma = b_{ij}^\sigma$ holds.

If $i < k < j$, then

$$\log_a \left( b_{ik}^\sigma \times b_{kj}^\tau \right) = \log_a \left( b_{ij}^\sigma \right) \Leftrightarrow \log_a \left( m_{ik}^\sigma \times m_{kj}^\tau \right) = \log_a \left( m_{ij}^\sigma \right)$$

holds obviously. Thus, we can deduce the equivalence

$$\log_a \left( a^{2b_{ik}^\sigma - 1} \times a^{2b_{kj}^\tau - 1} \right) = \log_a a^{2b_{ij}^\sigma - 1}$$

obviously. Thus, we can deduce the equivalence $\log_a \left( a^{2b_{ik}^\sigma - 1} \times a^{2b_{kj}^\tau - 1} \right) = \log_a a^{2b_{ij}^\sigma - 1}$

$\Leftrightarrow \#m_{ik}^\sigma + \#m_{kj}^\tau - 0.5 = \#m_{ij}^\sigma \Leftrightarrow E\left( \#m_{ik}^\sigma \right) + E\left( \#m_{kj}^\tau \right) - 0.5 = E\left( \#m_{ij}^\sigma \right)$. Similarly, we can also prove the cases for any other orders of $i, k, j$. This completes the proof of Theorem 6. \(\square\)

From the proof of Theorem 6, we know that the expectation consistency of IVHFPR is a necessary but not a sufficient condition for additive consistency as defined in [20]. Considering that there are no methods to check the expectation inconsistency of IVHFPR and improve the expectation inconsistent IVHFPR to be of expectation additive consistency, in the following, we propose a specific checking and improving method to address this issue:
Theorem 7. Let $R$ and $M$ be as before. Then, $\text{Var}(B^2(N_M(M)) - nB(N_M(M))) = O$ and $E(B^2(N_M(M)) - nB(N_M(M))) = O$.

Proof. The sufficiency is obvious, and the necessity is proven below.

Because $\text{Var}(B^2(N_M(M)) - nB(N_M(M))) = O$ and $E(B^2(N_M(M)) - nB(N_M(M))) = O$, we know that if $\text{Var}(B^2(N_M(M)) - nB(N_M(M))) = O$, then $B^2(N_M(M)) - nB(N_M(M))$ is equal to $O$ almost everywhere.

Let $N_M(M) = \{\#m_{ij}\}_{n \times n}$ and $B(N_M(M))$ be as before. According to the matrix product, we obtain the following result:

$$B^2(N_M(M)) = (B(N_M(M)) \circ B(N_M(M))) = \left( \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right) = A_{\text{cons}}$$

where $a_{ij} = \sum_{k=1}^{n} b_{ik} \cdot b_{kj}$. If $i < k < j$, $\forall \sigma = 1, 2, \cdots, L_M$, then $a_{ij} = \sum_{k=1}^{n} \left\{ b_{ik} \times b_{kj}, \sigma_{ik} \times \sigma_{kj} \right\}$.

According to Theorem 6, we know that if $\text{Var}(B^2(N_M(M)) - nB(N_M(M))) = O$ and $E(B^2(N_M(M)) - nB(N_M(M))) = O$, then the IVHFP $R$ is expectation additively consistent almost everywhere. Therefore, Theorem 7 is proven. $\square$

In real practical situations, the expectation additively consistent IVHFPR [20] and the multiplicatively consistent IVHMPR are in frequent and difficult to obtain. Therefore, we use a consistency with an acceptable level to substitute for the strict consistency. That is to say, if an IVHFPR/IVHMPR does not satisfy the expectation additive/multiplicative consistency requirements, but it meets a high consistency level (such as $C.I. < 0.1$), then we call it an acceptable expectation additively consistent IVHFPR/acceptably multiplicatively consistent IVHMPR. The acceptable expectation additively consistent IVHFPR and the acceptably multiplicatively consistent IVHMPR are very useful in group decision making under uncertainty.

Definition 20. Let $R$ and $M$ be as before, and let $N_R(R)$ and $N_M(M)$ be the corresponding normalized IVHFPR and IVHMPR over a reference set $X = \{x_1, x_2, \cdots, x_n\}$, respectively. $B(\cdot)$ is the extracting function presented in Definition 18 and Definition 19. If the consistency index $C.I.$ in Definition 7 is smaller than or equal to $\lambda$ ($0 \leq \lambda \leq 1$) for all $B(N_M(M))$, then we denote IVHMPR $M$ as being acceptably multiplicatively consistent on level $\lambda$ and denote IVHFPR $R$ as being acceptably additively consistent on level $\lambda$. Otherwise, we denote $M$ as being unacceptably multiplicatively consistent IVHMPR on level $\lambda$ and denote $R$ as being unacceptably additively consistent on level $\lambda$.

According to Definition 18 and Theorem 6, it is obvious that if the IVHMPR $N_M(M)$ is acceptably multiplicatively consistent on the level $\lambda$, then the corresponding IVHFPR $N_R(R)$ is acceptably expectation additively consistent on the level $\lambda$. In the following, motivated by Theorems 6 and 7 and Ref. [30], we give a specific algorithm to improve unacceptably consistent IVHMPR with multiplicative consistency:
The prevalence of lung cancer is rather high in China, and it has been one of the major causes of cancer. As fourth in the risk factors influencing disease. More importantly, air pollution is responsible for about 20% of cases that lead to lung cancer. The results indicated that outdoor particulate air pollution (PM 2.5) had reached as high as fourth in the risk factors influencing disease. More importantly, air pollution is responsible for about 20% of cases that lead to lung cancer.

In recent decades, China’s rapid overall economic growth and development has been truly impressive. However, unfortunately, China is experiencing air pollution, while people enjoy the benefits of economic expansion. In particular, since 2013, the urbanization of rural areas has aggravated the frequent extreme weather, such as haze, with dust pollution from the demolition and construction of buildings, which is directly reflected in the exceeding of the allowable limit for outdoor particulate air pollution (PM 2.5) by up to 95.5%. In 2010, the prestigious medical journal the Lancet published a series of studies on the global burden of diseases, especially mentioning respiratory infections and lung cancer. The results indicated that outdoor particulate air pollution (PM 2.5) had reached as high as fourth in the risk factors influencing disease. More importantly, air pollution is responsible for about 20% of cases that lead to lung cancer.

Lung cancer is the most common cancer in the world, particularly in some developing countries [31]. The prevalence of lung cancer is rather high in China, and it has been one of the major causes of cancer deaths. There are 47.5 and 22.2 women suffering from lung cancer per 100,000 people, according

Algorithm 1.

**Input:** The IVHFPR \( R \) or the IVHMPR \( M \).

**Output:** the acceptably expectation additively consistent IVHFPR \( \overline{R} \) and the acceptably multiplicatively consistent IVHMPR \( \overline{M} \).

**Step 1.** If the preference information provided by the decision maker is expressed as an IVHFPR, then we should transform it into an IVHMPR and normalize it through the methods presented in Sections 3 and 4.1. When the preferences are directly expressed as an IVHMPR, we only need to normalize it by the approach defined in Definition 17. Let \( N_M(M) \) be the final normalized IVHMPR, go to Step 2.

**Step 2.** Choose the value of the threshold \( \lambda \). Based on Definitions 7 and 20, check whether the multiplicative consistency of \( N_M(M) \) is up to level \( \lambda \). Denote the consistency index of \( N_M(M) \) by \( C.I.(N_M(M)) \). If \( C.I.(N_M(M)) < \lambda \), then go to Step 5. Otherwise, go to Step 3.

**Step 3.** Construct an induced matrix \( C = B^2(N_M(M)) - nB(N_M(M)) \), where \( B^2(N_M(M)) = B(N_M(M)) \otimes B(N_M(M)) \).

**Step 4.** Identify all the elements in \( |E(C)| + |Var(C)| \); there should be one of the following situations:

- **Case 1.** All of the elements in \( |E(C)| + |Var(C)| \) are zero elements \([0,0] \), which means that the present normalized IVHMPR is multiplicatively consistent. In this case, go to Step 5.

- **Case 2.** There are some zero elements in \( |E(C)| + |Var(C)| \). Thus, we should adjust the corresponding original element(s) in the normalized IVHMPR \( N_M(M) \) according to the following sub-steps:

  **Sub-step 1.** Find the maximal value within \( |E(C)| + |Var(C)| \) and denote its location by \((\xi, \eta)\). Go to the next sub-step.

  **Sub-step 2.** Denote the \( \xi \)th row in \( N_M(M) \) by \( #m_x = (#m_{x1} #m_{x2} \cdots #m_{xL_N(M)}) \) and denote the \( \eta \)th column in \( N_M(M) \) by \( #m_\eta = (#m_{1\eta} #m_{2\eta} \cdots #m_{NM(M)\eta}) \). Compute \( #m_x - \#m_\eta = (#m_{x1\eta} #m_{x2\eta} \cdots #m_{xL_N(M)\eta}) \).

  **Sub-step 3.** Adjust \( #m_{x1\eta} \) as \( #m_{x1} \) and adjust \( #m_{x2} \) according to \( #m_{x1\eta} \) based on Definition 17.

Denote the adjusted matrix as a new IVHMPR \( M \), go to Step 2.

**Step 5.** Denote the IVHMPR as \( \overline{M} \) and compute the corresponding IVHFPR \( \overline{R} \) using Equation (3). Output the results and end Algorithm 1.

This approach for checking and improving the consistency using Algorithm 1 is presented based on the multiplicatively consistent IVHMPR in order to obtain an acceptably multiplicatively consistent IVHMPR and an acceptably expectation additively consistent IVHFPR. Considering that there is a good connection between IVHFPR with additive consistency and IVHMPR with multiplicative consistency, the proposed method can also be applied to check and improve the additive consistency of IVHFPR.

5. Illustrative Example
to the Chinese cancer registration annual report of 2013 [32]. Furthermore, this situation is exhibiting an upward trend, which presents a great challenge to limited medical resources. Simultaneously, the prevention and treatment of lung cancer is becoming one of the most important current health care problems in our country.

Currently, treatments for lung cancer can be separated into three categories: surgically removing the tumor; blasting the cancer cells with radiation; and Chinese traditional medicine. Treatment may be changed to accommodate different patients based on physical capacity, age, seriousness of the illness, etc. As a result, selecting the best treatment is important, and is crucial for the patient’s recovery and the effective use of medical resources. The preference relation is a powerful tool to help make rational decisions. Thus, we illustrate the consistency checking and improving process in the context of selecting the most optimal and sensible treatment for lung cancer.

Suppose that four indices (\(I_1\): age and physiological condition, \(I_2\): the severity of illness, \(I_3\): the desired effect, and \(I_4\): expense) need to be considered, when a doctor measures the three treatments: \(A_1\): surgery, \(A_2\): chemotherapy, and \(A_3\): traditional Chinese medicine. Then, we can expediently get an IVHPR based on the doctor’s evaluation as follows:

\[
R = \begin{bmatrix}
(0.5, 0.5) & (0.58, 0.6) & (0.6, 0.68) & (0.65, 0.8) \\
(0.3, 0.4) & (0.4, 0.42) & (0.5, 0.5) & (0.6, 0.79) \\
(0.2, 0.35) & (0.3, 0.4) & (0.21, 0.4) & (0.5, 0.5)
\end{bmatrix}
\]

**Step 1.** Normalize \(R\) and translate it into an IVHPR as \(#M\) according to Theorem 5 with the base number \(\alpha = 6\):

\[
#M = \begin{bmatrix}
(1, 1, 1, 1) & (1.33, 1.43, 1.38, 1.71, 1.43, 2.05) & (1.43, 1.91, 1.43, 2.05, 1.71, 2.93) \\
(0.49, 0.70, 0.56, 0.72, 0.70, 0.75) & (1, 1, 1, 1, 1, 1) & (1.43, 2.83, 1.43, 2.83, 1.43, 2.83) \\
(0.34, 0.58, 0.49, 0.70, 0.52, 0.70) & (0.35, 0.70, 0.35, 0.70, 0.35, 0.70) & (1, 1, 1, 1, 1, 1)
\end{bmatrix}
\]

**Step 2.** Let \(\lambda = 0.02\) be the threshold of consistency index. According to Definitions 7 and 20, we know that the multiplicatively consistent indexes of \(B(#M)\) are 0.00, 0.03, 0.01, 0.05, 0.00, and 0.02. Therefore, the multiplicative consistency of \(#M\) should be improved.

**Step 3.** Construct an induced matrix \(C = B^2(N_M(M)) - nB(N_M(M))\) as follows:

\[
\begin{align*}
C &= (c_{ij})_{i,j=1,13} = B^2(N_M(M)) - 3B(N_M(M)) \\
&= \begin{bmatrix}
(1.43, 2.83, 1.43, 2.83, 1.43, 2.83) & (1.43, 2.83, 1.43, 2.83, 1.43, 2.83) & (1, 1, 1, 1, 1, 1) \\
(1.69, 1.65, 1.69, 1.23, 1.28, 1.491) & (1.69, 1.65, 1.69, 1.23, 1.28, 1.491) & (1.69, 1.65, 1.69, 1.23, 1.28, 1.491) \\
(1.72, 1.63, 1.61, 1.00, -1.42, -1.513) & (1.72, 1.63, 1.61, 1.00, -1.42, -1.513) & (1.72, 1.63, 1.61, 1.00, -1.42, -1.513)
\end{bmatrix}
\]

**Step 4.** Identify all the elements in \(|E(C)| + |Var(C)|\), where

\[
|E(C)| + |Var(C)| = \begin{bmatrix}
4.37 & 5.29 & 26.93 \\
3.11 & 4.16 & 7.50 \\
3.35 & 2.67 & 4.23
\end{bmatrix}
\]

Obviously, this is in line with Case 2 in Step 4 of Algorithm 1. In the following, we adjust elements in \(#M\):

**Substep 1.** From the above \(|E(C)| + |Var(C)|\), we find the maximal value to be 26.93 and denote its location as \((\xi, \eta) = (1, 3)\). Go to the next sub-step.

**Substep 2.** Denote the 1st row in \(#M\) as \(#m_1\), and denote the 3rd column in \(N_M(M)\) as \(#m_3\). Compute \(#m_1 \cdot #m_3 - #m_{13} = (2.34, 46, 21, 2.65, 148.67, 1.83, 167.67)\) and find the absolute minimal nonzero value within it as 1.83, which is denoted by \(#m \cdot #m - #m\). Go to the next sub-step.

**Substep 3.** Adjust all elements in \(#m_{13}\) as \(#m_{13} = 1.71\). Denote the adjusted matrix as a new IVHPR \(M\), go to Step 2.
Repeating Steps 2–4 in Algorithm 1 twice, we obtain the results as follows:

\[
\#M^2 = \begin{bmatrix}
1 & 1.38 & 2.05 \\
0.72 & 1 & [1.43, 2.83] \\
0.49 & [0.35, 0.7] & 1
\end{bmatrix}
\quad \text{and} \quad
\#R^2 = \begin{bmatrix}
0.5 & 0.59 & 0.7 \\
0.41 & 0.5 & [0.6, 0.79] \\
0.3 & [0.21, 0.4] & 0.5
\end{bmatrix}
\]

where \#M^2 and \#R^2 are the process results that have been revised twice. Please note that we show the simplest expression of the revised results based on the understanding that a number is the special case of an interval.

From Step 2, we get the multiplicatively consistent indexes of \#M^2 are 0.00, 0.02, 0.00, 0.02, 0.00, and 0.02. Therefore, the multiplicative consistency of \#M^2 does not need to be further improved.

**Step 5.** Output \#M^2 as the acceptably multiplicatively consistent IVHMPR \(\overline{M}\) and output \#R^2 as the acceptably additively consistent IVHFPR \(\overline{R}\). Obviously, \(\overline{R}\) also is an acceptably expectation additively consistent IVHMPR. End Algorithm 1.

By now, we have numerically illustrated the checking and improving process for an IVHFPR \(R\) with the additive consistency or an IVHMPR \(M\) with multiplicative consistency. After that, we move into the stage of obtaining a ranking of these three treatments and making a decision by integrating all of the preference information described by \(\overline{M}\) or \(\overline{R}\). If we use the interval-valued hesitant fuzzy weighted geometric (IVHFWG) operator [1], we can get the final ranking of these three treatments as \(A_1 > A_2 > A_3\). Here we do not give the detailed calculation process due to the space limitations of this paper.

6. Comparisons

In this section, we compare the new proposed method in this paper to several related results, such as the consistency checking and improving method in [30] and the 0–1 programming model in [20].

6.1. Comparing Algorithm 1 to the Method with \(D^2 - nD = O\)

Ergu et al. [30] proposed an equivalent condition for checking the multiplicative consistency for the preference relation in AHP [27,28]. Let \(D\) be the preference relation in AHP, the consistency improving process of [30] is applied based on the equation \(D^2 - nD = O\). If the multiplicative consistency of \(D\) does not meet requirements, \(D^2 - nD = O\) can guide us to improve it, because \(D^2 - nD = O\) is a sufficient and necessary condition for getting a multiplicatively consistent preference relation in AHP [27,28]. However, the method in Ref. [30] cannot be extended into decision-making situations with IVHMPR. Algorithm 1, as proposed in this paper, is a multiplicative consistency checking and improving method for IVHMPR. Since IVHFS is a general expression of a real number, Algorithm 1 can be seen as an extended application of the method in Ref. [30] to the decision-making situations with IVHMPR.

Furthermore, from the input and output of Algorithm 1, it is easy to find that Algorithm 1 can be applied to the decision-making process with IVHFPR. However, the method in Ref. [30] cannot be applied in such cases, because the equation \(D^2 - nD = O\) can only evaluate the multiplicative consistency for MPR. It cannot judge the additive consistency for FPR.

Furthermore, from Step 4 in Algorithm 1, we know that the consistency checking and improving method proposed in this paper is constructed based on Theorem 7, which is related to the expectation additive consistency of IVHFPR. If the MPR in Ref. [30] were transformed into the FPR by Equation (2), the method in Ref. [30] could be applied for checking and improving the additive consistency of FPR based on Equation (3). However, the additive consistency of FPR is different from the expectation additive consistency, where additive consistency is stricter than expectation additive consistency. Thus, Algorithm 1 and the method in Ref. [30] are in line with different adjusting principles, where the adjusting principle in Algorithm 1 is more relaxed. In particular, due to the fact that the expectation of
any real number is the number and the variance of the real number is zero, Algorithm 1 can be applied to the examples in Ref. [30] and will obtain the same results.

6.2. Comparing Algorithm 1 to the Method with 0–1 Programming Models

In Ref. [20], the authors proposed several 0–1 programming models to get additively consistent IVHFPR. These are automatic ways of adjusting the elements in IVHFPR, and cannot be applied into IVHMPR. However, using Algorithm 1, we can know the details of the adjustment process on IVHFPR, which can then be applied into IVHMPR. Furthermore, the programming models in Ref. [20] are constructed based on the additive consistency of IVHFPR, which is stricter than the expectation additive consistency used in Algorithm 1. Thus, using Algorithm 1, we can get more acceptably additively consistent IVHFPRs from the consistency improving process. Moreover, from the numerical illustration in Section 5, we can find that the elements in the resulting IVHFPR obtained using Algorithm 1 may be included within the resulting IVHFPR obtained using the method in Ref. [20].

For example, applying Algorithm 1 to the IVHFPR in the example in Ref. [20], we can first obtain an IVHFPR whose element in the first row and the forth column is 0.8, where the corresponding element in the under triangle of this IVHFPR is 0.2. This verifies the above statement about the difference between Algorithm 1 and the method in Ref. [20].

7. Conclusions

In this paper, we first proposed the concepts of IVHMPR, the expectation additive consistency of IVHFPR, and the multiplicative consistency of IVHMPR. This work enriches the theory of IVHFSs and lays the foundation for the application of IVHFSs in practical GDM problems with FPR and MPR. Then, we investigated the relationship between IVHFPR and IVHMPR, and established an interesting link and built a bridge between IVHFPR with expectation additive consistency and IVHMPR with multiplicative consistency. Thus, we can optionally convert and use IVHFPR and IVHMPR according to the reality and people’s habits.

Furthermore, we constructed a checking and improving algorithm to deal with IVHMPR with unacceptable multiplicative consistency, which is also able to cope with IVHFPR with unacceptable expectation additive consistency. Several numerical examples illustrated that the new proposed algorithm can check and improve the multiplicative consistency of IVHMPR, and can also check and improve the expectation additive consistency of IVHFPR. Some comparisons between Algorithm 1 and related results showed the effectiveness of the new method proposed in this paper.

Because it is a hard work to obtain real data for diseases from hospitals, due to issues of patient privacy, we only illustrate Algorithm 1 using a numerical example with a randomly generated IVHFPR. This is one of the drawbacks of our work in this paper: we have not applied the new results in real applications. Additionally, for Algorithm 1, when the orders of IVHFPR and IVHMPR are high, too much calculation is required to get the induced matrix \( C = B^2 - nB \) (refer to Step 3 in Algorithm 1). Because the expression of IVHFS is more complex than HFS, IVFS or traditional FS, the calculation of preference information expressed by IVHFS is more difficult and complicated than for preference information expressed using HFS, IVFS or traditional FS. Finally, according the expectation and variance, adjusting the elements in IVHFPR/IVHMPR may be too slow to obtain a FPR/MPR with a high level of consistency in emergency situations.

In the future, we will focus on applying IVHFPR and IVHMPR to other decision-making methods in practical applications.

Author Contributions: Conceptualization, Z.X.; Formal analysis, Y.Z.

Funding: The work was supported by the National Natural Science Foundation of China (Nos. 71571123, 71532007, 71771155, 71801174).

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper. This research does not involve any human or animal participation.
References

1. Chen, N.; Xu, Z.S.; Xia, M.M. Interval-valued hesitant preference relations and their applications to group decision making. *Knowl.-Based Syst.* **2013**, *37*, 528–540. [CrossRef]
2. Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]
3. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [CrossRef]
4. Chen, N.; Xu, Z.S. Properties of interval-valued hesitant fuzzy sets. *J. Intel. Fuzzy Syst.* **2014**, *27*, 143–158.
5. Wei, G.W.; Lin, R.; Wang, H.J. Distance and similarity measures for hesitant interval-valued fuzzy sets. *J. Intel. Fuzzy Syst.* **2014**, *27*, 19–36.
6. Farhadinia, B. Study on division and subtraction operations for hesitant fuzzy sets, interval-valued hesitant fuzzy sets and typical dual hesitant fuzzy sets. *J. Intell. Fuzzy Syst.* **2015**, *28*, 1393–1402.
7. Farhadinia, B. Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. *Inf. Sci.* **2013**, *240*, 129–144. [CrossRef]
8. Zhai, Y.L.; Xu, Z.S.; Liao, H.C. Measures of probabilistic interval-valued intuitionistic hesitant fuzzy sets and their application in reducing excessive medical examinations. *IEEE Trans. Fuzzy Syst.* **2017**, *26*, 1651–1670. [CrossRef]
9. Zhu, B.; Xu, Z.S.; Xu, J.P. Deriving a ranking from hesitant fuzzy preference relations under group decision making. *IEEE Trans. Cybern.* **2014**, *44*, 1328–1337. [CrossRef]
10. Li, L.G.; Peng, D.H. Interval-valued hesitant fuzzy Hamacher synergetic weighted aggregation operators and their application to shale gas areas selection. *Math. Probil. Eng.* **2014**, *1*, 759–765. [CrossRef]
11. Wei, G.W.; Zhao, X.F.; Lin, R. Some hesitant interval-valued fuzzy aggregation operators and their applications to multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2013**, *46*, 43–53. [CrossRef]
12. Jun, Y. Interval-valued hesitant fuzzy prioritized weighted aggregation operators for multiple attribute decision making. *J. Algorithms Comput. Technol.* **2014**, *8*, 179–192.
13. Meng, F.Y.; Wang, C.; Chen, X.H.; Zhang, Q. Correlation coefficients of interval-valued hesitant fuzzy sets and their application based on the Shapley function. *Int. J. Intell. Syst.* **2016**, *31*, 17–43. [CrossRef]
14. Meng, F.Y.; Chen, X.H. An approach to interval-valued hesitant fuzzy multi-attribute decision making with incomplete weight information based on hybrid Shapley operators. *Informatica* **2014**, *25*, 617–642. [CrossRef]
15. Tavakkoli-Moghaddam, R.; Gitinavard, H.; Mousavi, S.M.; Siadat, A. An interval-valued hesitant fuzzy TOPSIS method to determine the criteria weights. *Lect. Notes Bus. Inf. Process.* **2015**, *218*, 157–169.
16. Wei, G.W.; Zhao, X.F.; Lin, R.; Wang, H.J. Models for hesitant interval-valued fuzzy multiple attribute decision making based on the correlation coefficient with incomplete weight information. *J. Intell. Fuzzy Syst.* **2014**, *26*, 1631–1644.
17. Pérez-Fernández Raúl Alonso, P.; Bustince, H.; Díazd, I.; Montes, S. Applications of finite interval-valued hesitant fuzzy preference relations in group decision making. *Inf. Sci.* **2016**, *326*, 89–101. [CrossRef]
18. Khalid, A.; Beg, I. Incomplete interval-valued hesitant fuzzy preference relations in decision making. *Iran. J. Fuzzy Syst.* **2015**, *18*, 6. [CrossRef]
19. Zhang, Z.M. Multi-criteria decision-making using interval-valued hesitant fuzzy QUALIFLEX methods based on a likelihood-based comparison approach. *Neural Comput. Appl.* **2017**, *28*, 1835–18354. [CrossRef]
20. Tang, J.; Meng, F.Y. Ranking objects from group decision making with interval-valued hesitant fuzzy preference relations in view of additive consistency and consensus. *Knowledge-Based Systems* **2018**, *162*, 46–61. [CrossRef]
21. Zhang, Y.N.; Tang, J.; Meng, F.Y. Programming model-based method for ranking objects from group decision making with interval-valued hesitant fuzzy preference relations. *Appl. Intell.* **2018**, *49*, 2969–2985.
22. Liao, H.C.; Xu, Z.S.; Xia, M.M. Multiplicative consistency of interval-valued intuitionistic fuzzy preference relation. *J. Intel. Fuzzy Syst.* **2014**, *27*, 1985–1995.
23. Liao, H.C.; Xu, Z.S.; Xia, M.M. Multiplicative consistency of hesitant fuzzy preference relation and ITS application in group decision making. *Int. J. Inf. Technol. Decis. Mak.* **2014**, *13*, 47–76. [CrossRef]
24. Liu, H.F.; Xu, Z.S.; Liao, H.C. The multiplicative consistency index of hesitant fuzzy preference relation. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 82–93. [CrossRef]
25. Moore, R.E. *Interval analysis*; Prentice-Hall INC.: Englewood Cliffs, NJ, USA, 1966.
26. Xu, Z.S.; Da, Q.L. Research on method for ranking interval numbers. *Syst. Eng.* **2001**, *19*, 94–96.
27. Saaty, T.L.; Vargas, L.G. Uncertainty and rank order in the analytic hierarchy process. *Eur. J. Oper. Res.* **1987**, *32*, 107–117. [CrossRef]

28. Saaty, T.L. Modeling unstructured decision problems—The theory of analytical hierarchies. *Math. Comput. Simul.* **1978**, *20*, 147–158. [CrossRef]

29. Tanino, T. Fuzzy preference orderings in group decision making. *Fuzzy Sets and Systems*. *Fuzzy Sets Syst.* **1984**, *12*, 117–131. [CrossRef]

30. Ergu, D.; Kou, G.; Peng, Y.; Shi, Y. A simple method to improve the consistency ratio of the pair-wise comparison matrix in ANP. *Eur. J. Oper. Res.* **2011**, *213*, 246–259. [CrossRef]

31. Jemal, A.; Bray, F.; Center, M.M.; Ferlay, J.; Ward, E.; Forman, D. Global cancer statistics. *CA Cancer J. Clin.* **1999**, *49*, 33–64. [CrossRef]

32. Chen, W.; Zheng, R.; Zhang, S.; Zhao, P.; Li, G.; Wu, L.; He, J. Report of incidence and mortality in China cancer registries 2009. *Chin. J. Cancer Res.* **2013**, *24*, 171–180. [CrossRef]

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).