High-speed collapse of a hollow sphere of type I matter

Zahid Ahmad, Tomohiro Harada, Ken-ichi Nakao and M Sharif

1 Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan
2 Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan
3 Department of Mathematics and Physics, Graduate School of Science, Osaka City University, Osaka 558-8585, Japan

E-mail: zahid_rp@yahoo.com, harada@rikkyo.ac.jp, knakao@sci.osaka-cu.ac.jp and msharif@math.pu.edu.pk

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Abstract
In this paper, we study the dynamics of a hollow spherical matter collapsing with a very large initial velocity. The spacetime is initially very similar to the Vaidya solution, and the deviations from this background are treated perturbatively. The equations of state for radial pressure \( p_R = k \rho \) and tangential one \( p_T = w \rho \) with constants \( k \) and \( w \) are assumed. We find for the case of equations of state \( k < 1 \) and \( 0 < w \leq 1 \) that the initial velocity, which is nearly the speed of light, is strongly decelerated. This result implies that the pressure is essential to the property of singularity formation in gravitational collapse even for initially nearly light-speed collapse. By contrast, in cases with the negative tangential pressure, the present result implies that the central naked singularity similar to that of the Vaidya spacetime can be formed, even though the radial pressure is positive, and the weak, strong and dominant energy conditions hold. Especially, in the case of \( w < - (1 - k)/4 \), the high-speed collapse will produce the spacetime structure very similar to that of the Vaidya spacetime.

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1. Introduction
Einstein presented the theory of general relativity which describes the gravitational force in terms of the spacetime curvature. He formulated the field equations which relate the geometry of the spacetime to matter fields. The earliest exact solutions of these equations were the Schwarzschild metric representing the exterior of a spherically symmetric star and the Friedmann cosmological models. These solutions have spacetime singularities where the energy density or spacetime curvature diverges and the usual description of the spacetime is impossible there [1].
It is a well-known phenomenon that gravitational collapse of massive objects results in the formation of spacetime singularities in our universe. There are two kinds of spacetime singularities from a point of view of their visibilities. A spacetime singularity is said to be \textit{naked} when it is observable to local or distant observers, and the remaining one is said to be \textit{covered}. The singularity theorems of Hawking and Penrose show that the formation of spacetime singularities is not rare in our universe if the general relativity is correct and the matter or radiation fields satisfy physically reasonable energy conditions [2]. However, these theorems do not provide information about the visibility of the spacetime singularity.

About the visibility of the singularity, Penrose proposed a conjecture called the cosmic censorship conjecture which has two versions [3]. The weak version states that the spacetime singularities produced by gravitational collapse of physically reasonable matter fields, which develops from generic non-singular initial data, are always covered by horizon, whereas the strong version claims that there is no singularity visible to any observers. Some rather serious counterexamples have been found for the strong version [4, 5]. As for the weak version, any precise theoretical or mathematical proof has not yet been given, although it has many physical applications in black hole and other areas in astrophysics. This motivates that a detailed study of dynamically developing gravitational collapse models is necessary to obtain a correct form of the cosmic censorship. Several examples ([6–14] and references therein) have been studied so far which admit both black hole and naked singularity solutions depending on the choice of the initial data. Most of the work on gravitational collapse has been done by considering dust fluid due to the existence of an exact solution. However, the assumption of dust fluid might be too restricted as the effects of pressure cannot always be neglected in the formation processes of the spacetime singularities. Thus it is important to discuss this issue by including pressure.

Ori and Piran [15–17] investigated self-similar spherically symmetric perfect fluid collapse by assuming the equation of state \( p = k \rho \). They found that a naked singularity is formed for \( 0 < k \lesssim 0.0105 \). They have also shown that there exist naked-singular solutions with oscillations in the velocity field for \( 0 < k \lesssim 0.4 \). Later, these results were extended for \( 0 < k \lesssim 0.5625 \) by Foglizzo and Henriksen [18]. The same results were also provided without self-similarity assumption by one of the present authors TH and Maeda [19–21]. Giambo et al have investigated naked singularity formation in perfect fluid collapse without self-similarity assumption analytically [22]. Goswami and Joshi [23] have also investigated analytically the local geometry near the central shell focusing singularity formed by a spherical collapse of a perfect fluid with the equation of state in the form \( p = k \rho \) and have shown that the initial condition is crucial for whether this singularity is naked. However, it is difficult to construct a general global solution analytically. Toward the progress of analytical studies, it is important to develop a new approximation scheme (analytical procedure) to discuss gravitational collapse which leads to more definite results.

One of the present authors KN and Morisawa studied the cylindrically symmetric gravitational collapse of a thick shell composed of dust by introducing a high-speed approximation scheme [24]. The same authors generalized this work for the perfect fluid case [25]. In a recent paper [26], two of the present authors MS and ZA have extended this work by considering two perfect fluids. These investigations have provided interesting results about the gravitational collapse. It would be worthwhile to explore whether these results hold for spherical collapse or not. This motivated us to develop a high-speed approximation scheme for a spherically symmetric system with the type I matter [27]. There are many studies about the naked singularity formation by the same type of matter in the spherically symmetric system. Dwivedi and Joshi showed by the local analysis that the initial data are crucial for whether the naked singularity forms at the symmetric center by the gravitational collapse of general type I matter [28]. Recently, this work extended to the higher dimensional spacetime...
by Goswami and Joshi [29]. The spherically symmetric matter with vanishing radial pressure and non-vanishing tangential pressure has been studied by various authors [30–35]. In this paper, we consider the case treated by [28] but we construct global analytic solutions by using high-speed approximation.

This paper is organized as follows. In section 2, we write the Einstein equations for the spherically symmetric spacetime with a type I matter in the single null coordinate system. The null dust solution is investigated in section 3. Section 4 is devoted to discussing the high-speed approximation scheme for the general type I matter. The effects of pressure on the high-speed gravitational collapse are discussed in section 5. Finally, the summary of the results is given in section 6.

In this paper, we adopt the geometrized unit, i.e., \( c = 1 = G \) and follow the convention of the Riemann and metric tensors and the abstract index notation adopted in the textbook by Wald [36]; the Latin indices denote the type of a tensor, whereas the Greek indices denote the components of a tensor.

2. Spherically symmetric matter with anisotropic pressure

We focus on the spacetime with spherical symmetry. For later convenience, we adopt the single null coordinate system in which the line element is given by

\[
ds^2 = -A(v, r) \, dv^2 + 2B(v, r) \, dv \, dr + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

where we assume that \( B \) is positive, and this assumption implies that the coordinate \( v \) is the advanced time, i.e., a radial curve of constant \( v \) is the future directed ingoing null. Then the non-trivial components of the Einstein equations are given by

\[
\frac{A}{r B^2} \left( \frac{A'}{A} - \frac{2B'}{B} \right) - \frac{1}{r^2} \left( 1 - \frac{A}{B^2} \right) = 8\pi T^v_v, \tag{2}
\]

\[
\frac{2B'}{r B^2} = 8\pi T^v_r, \tag{3}
\]

\[
\frac{A}{r B^2} \left( \frac{2B}{B} - \frac{A}{A} \right) = 8\pi T^r_v, \tag{4}
\]

\[
\frac{A'}{r B^2} - \frac{1}{r^2} \left( 1 - \frac{A}{B^2} \right) = 8\pi T^r_r, \tag{5}
\]

\[
\frac{1}{B^2} \left( B' \left( B + \frac{A'}{2} \right) - \frac{B'}{B^2} \left( B + \frac{A'}{2} \right) + \frac{A}{r B^2} \left( \frac{A'}{A} - \frac{B'}{B} \right) \right) = 8\pi T^\theta_\theta, \tag{6}
\]

where the dot denotes the derivative with respect to the advanced time \( v \), while the prime denotes the derivative with respect to the radial coordinate \( r \).

We study the dynamics of the spherically symmetric type I matter whose stress–energy tensor is [27]

\[
T^a_b = \rho u^a u_b + p_R s^a s_b + p_T \Omega^a_b,
\]

where \( u^a \) is 4-velocity of a constituent particle, \( s^a \) is the unit radial vector normal to \( u^a \) and \( \Omega^a_b \) is defined by

\[
\Omega^a_b = \delta^a_b + u^a u_b - s^a s_b,
\]

and thus \( \rho, p_R \) and \( p_T \) are the energy density, radial pressure and the tangential pressure, respectively.
We write the components of $\textcolor{red}{u}\mu$ and $\textcolor{red}{s}\mu$ in the forms

$$u_\mu = N(V, -1 + V, 0, 0),$$

$$s_\mu = NB(1 - V, V, 0, 0),$$

where

$$N = \frac{1}{\sqrt{V[2B + V(A - 2B)]}}.$$  

We define new variables $D$, $P_R$ and $P_T$ as

$$D := \frac{N^2 \sqrt{g}(\rho + p_R)}{\sin \theta} = \frac{r^2 B(\rho + p_R)}{V[2B + V(A - 2B)]},$$

$$P_R := \frac{N^2 \sqrt{g}p_R}{\sin \theta} = \frac{r^2 Bp_R}{V[2B + V(A - 2B)]},$$

$$P_T := \frac{N^2 \sqrt{g}p_T}{\sin \theta} = \frac{r^2 Bp_T}{V[2B + V(A - 2B)]},$$

where $g$ is the determinant of the metric tensor. Then the stress–energy tensor is written in the form

$$T^a_b = \frac{1}{r^2 B} \left[ Dk^ak_b + V\{2B + V(A - 2B)\} \left\{ P_R \delta^a_b + (P_T - P_R)/\Omega_b \right\} \right],$$

where the components of the vector field $k^a$ are

$$k^\mu = \frac{u^\mu}{N} = (V, -1 + V, 0, 0).$$

3. Null dust limit

3.1. Metric

It is easy to see that in the limit of $V \to 0$ with $D$, $P_R$ and $P_T$ fixed, the stress–energy tensor (15) becomes that of the null dust which belongs to the type II matter [27],

$$T^a_b = \frac{D}{r^2 B}k^a_k_b,$$

where in this limit we have

$$k^\mu = (0, -1, 0, 0).$$

We can easily check that $B^{-1}k^\mu$ is the tangent of the ingoing null geodesic, i.e.,

$$k^\mu \nabla_\mu (B^{-1}k_b) = 0.$$  

Thus the equation of motion $\nabla_\mu T^a_b = 0$ leads

$$\nabla_\mu (r^{-2}Dk^\mu) = 0.$$  

The above equation reduces to

$$(BD)' = 0,$$
and thus we find that BD is a function of the advanced time \( v \) only. Further since equation (3) reduces to \( B' = 0 \), we have \( B = B(v) \). Therefore, we have \( D = D(v) \). If we introduce a new advanced time \( \bar{v} \) defined by

\[
\bar{v} = \int B(v) \, dv,
\]

(22)

we have a new line element

\[
d\bar{s}^2 = -\bar{A}(\bar{v}, r) \, d\bar{v}^2 + 2 \, d\bar{v} \, dr + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

(23)

where \( \bar{A} = A/B^2 \). Thus without loss of generality, we can assume \( B = 1 \) and will do so below in this section.

The remaining non-trivial components of the Einstein equations become

\[
\frac{A'}{r} - \frac{1}{r^2} (1 - A) = 0,
\]

(24)

\[
\dot{A} = -\frac{8\pi D}{r},
\]

(25)

\[
\frac{A''}{2} + \frac{A'}{r} = 0.
\]

(26)

From equation (24), we obtain

\[
A = 1 - \frac{2M(v)}{r}.
\]

(27)

From equation (25) and the above equation, we have

\[
\frac{dM}{dv} = 4\pi D(v).
\]

(28)

If we fix the dependence of \( D \) on \( v \), the solution is completely determined. This solution is known as the Vaidya solution [37].

3.2. Central singularity

The precursory singularity in spherical gravitational collapse formed at the symmetry center can be naked singularity [38]. The central singularity in Vaidya spacetime can be naked if the mass function \( M(v) \) satisfies some condition [39, 40, 42]. Here we give a brief review about this issue by following the analysis in [40].

The future directed ingoing null condition is \( v = \text{const} \), whereas the future directed outgoing null condition is given by

\[
\frac{dr}{dv} = \frac{1}{2} \left( 1 - \frac{2M}{r} \right).
\]

(29)

We assume that \( D(v) \) has a compact support so that we have \( M(v) = 0 \) for \( v \leq 0 \) and \( M(v) \neq 0 \) for \( v > 0 \). This assumption guarantees that the symmetry center \( r = 0 \) is regular initially, i.e., for \( v \leq 0 \). The central singularity forms at \( v = 0 = r \). To know whether the central singularity is naked, we investigate the existence of the outgoing null geodesics from \( v = 0 = r \).

\[\text{For the case of dust, Lemos and Hellaby showed that the Vaidya solution is obtained by taking a limit of infinite specific energy in the different coordinate system from equation (1).}\]
Suppose that the $r = r_0(v)$ is a solution for equation (29) which emanates from the central singularity $v = 0 = r$. The mass function is written by using this solution as

$$M(v) = \frac{1}{2} r_0(v) [1 - 2r_0(v)].$$

(30)

The above equation is just the condition on the mass function for which the central singularity becomes naked. It should be noted that there is only one ingoing null geodesic $v = 0 = r$ that hits the central singularity. Hence, if there is a one-parameter family of null geodesics emanating from the central singularity, the central naked singularity is null, while, if there is only one, the central singularity is instantaneous, i.e., an event in a conformally extended spacetime manifold. Solutions in such a one-parameter family are written in the form

$$r(v) = r_0(v) + z(v; \sigma),$$

(31)

where $\sigma$ parametrizes the solutions. Substituting the above form into equation (29), we have

$$\frac{dz}{dv} = \frac{z(2r_0 - 1)}{2(z + r_0)}.$$  

(32)

We search for the solutions which behave as $z \to 0$ for $v \to 0$.

Suppose $r_0 \sim \beta v^\alpha$ near the central singularity, where both of $\alpha$ and $\beta$ are positive. Then we find the following three cases:

(i) $M(v) \sim \beta v^\alpha / 2$ for $\alpha > 1$;
(ii) $M(v) \sim \beta (1/2 - \beta) v$ for $\alpha = 1$;
(iii) $M(v) \sim -\alpha \beta^2 v^{2\alpha - 1}$ for $\alpha < 1$.

For case (i), the solution for equation (32) is given by

$$z \sim \sigma v^{-\alpha} \exp\left(-\frac{1}{2 \beta (\alpha - 1) v^{\alpha - 1}}\right) \quad \text{and} \quad z \sim \frac{v}{2}.$$  

(33)

For case (ii), we have, for $\beta \neq 1/4$,

$$v \sim \sigma |z|^\frac{1}{\beta} + \frac{2z}{1 - 4\beta},$$

(34)

while we have, for $\beta = 1/4$,

$$v \sim \sigma z + 4z \ln|z|.$$  

(35)

We see from equation (34) that $\beta$ should be less than $1/2$ so that $z \to 0$ for $v \to 0$. This condition guarantees that the mass function $M(v)$ is non-negative near $v = 0$. For case (iii), the mass function $M(v)$ is negative near $v = 0$. Thus in this case, the spacetime singularity of $v > 0$ and $r = 0$ is necessarily naked but it is non-trivial whether the central singularity at $v = 0 = r$ is null. We can see that the solution for equation (32) is only $z \sim -2\beta v^\alpha$, and from equation (31), we have

$$r \sim -\beta v^\alpha < 0.$$  

(36)

Since $r$ should be non-negative, this is an unacceptable solution. Thus for case (iii), the solution $r = r_0(v)$ is the unique outgoing radial null geodesic which emanates from the central singularity. Further the radial curve $v = 0$ is the unique future directed ingoing radial null which hits the central singularity. Thus we may conclude that the central singularity of negative mass function is instantaneous.

We see from the above results that if the mass function $M$ behaves as $M \sim \mu v^\gamma$ near $v = 0$ with $\mu > 1/16$ and $\gamma = 1$, then the central singularity is not naked.
Figure 1. The conformal diagram of the Vaidya spacetime in cases (i) and (ii). The central naked singularity is null and thus there is a family of the future directed outgoing radial null geodesics.

Figure 2. The conformal diagram of the Vaidya spacetime in case (iii). The central naked singularity is instantaneous. The final product of the gravitational collapse is the Schwarzschild spacetime with negative mass.

We depict the conformal diagrams for cases (i) and (ii) in figure 1, while for case (iii) in figure 2. The covered case is depicted in figure 3.
4. High-speed approximation scheme

The high-speed approximation is a kind of linear perturbation scheme in which $V$ is a small variable of order $\epsilon$. Then we write

\begin{align*}
A &= 1 - \frac{2M_B(v)}{r} + \delta_A(v, r), \\
B &= 1 + \delta_B(v, r), \\
D &= D_B(v)[1 + \delta_D(v, r)],
\end{align*}

where $\delta_A, \delta_B$ and $\delta_D$ are also assumed to be $O(\epsilon)$, and

\[ \dot{M}_B = 4\pi D_B. \] (40)

We take into account the terms up to the first order of $\epsilon$. Hereafter, we assume that $V$ is non-negative so that $u^\mu$ is a causal vector. As in the previous section, we also assume here that $D_B(v)$ has a compact support so that we have $M_B(v) = 0$ for $v \leq 0$ and $M_B(v) \neq 0$ for $v > 0$.

From equation (15), we have

\[ T^v_r = \frac{D}{r^2} V^2. \] (41)

This equation and equation (3) lead $B' = O(\epsilon^2)$, and thus we have

\[ B = 1 + \delta_B(v) + O(\epsilon^2). \] (42)

The above result means that $B$ is the function of only $v$ up to the first order of $\epsilon$, and hence, without loss of generality, we can assume $B = 1$ up to this order by the same reason as in the case of the Vaidya solution. Then equations (2) and (5) agree with each other up to the first order of $\epsilon$. 

\[ \text{Figure 3. The conformal diagram of the Vaidya spacetime in the case that the central singularity is not naked. The mass function $M$ behaves as $M \sim v^{\gamma}$ near $v = 0$ with $\mu > 1/16$ and $\gamma = 1$.} \]
The remaining apparently independent components of the Einstein equations of \( O(\epsilon) \) are

\[
(r\delta_A)' = -8\pi (D_B - 2P_K)V, \tag{43}
\]

\[
 r\dot{\delta}_A = 8\pi D_B \left( 1 + \frac{2M_B}{r} \right) V - \delta_D, \tag{44}
\]

\[
(r\delta_A)'' = \frac{32\pi}{r} P_T V. \tag{45}
\]

The equations of motion for the matter \( \nabla_a T^a_b = 0 \) of order \( O(\epsilon) \) are given by

\[
[(D_B - 2P_K)V]' + D_B \left( 1 + \frac{2M_B}{r} \right) V - \delta_D)' = 0, \tag{46}
\]

\[
[(D_B - 2P_K)V]' + \frac{4}{r} P_T V = 0. \tag{47}
\]

Equation (45) is not necessary, since this equation is derived from equations (43) and (47). Differentiating equation (43) with respect to \( r \), we have

\[
(r\delta_A)'' = -8\pi [(D_B - 2P_K)V]'. \tag{48}
\]

Substituting equation (47) into the right-hand side of the above equation, we have equation (45). Equation (46) is also not necessary, since this equation is derived from equations (43) and (44). Differentiating equation (43) with respect to \( v \), we have

\[
(r\dot{\delta}_A)' = -8\pi [(D_B - 2P_K)V]. \tag{49}
\]

Differentiating equation (44) with respect to \( r \), we have

\[
(r\dot{\delta}_A)' = 8\pi D_B \left( 1 + \frac{2M_B}{r} \right) V - \delta_D'. \tag{50}
\]

From equations (49) and (50), we have equation (46). Thus independent equations are only equations (43), (44) and (47).

5. The analysis of the high-speed collapse

In order to write formal solutions for the perturbation variables, we introduce new variables \( k(v, r) \) and \( w(v, r) \), defined by

\[
p_R = k(v, r)\rho \quad \text{and} \quad p_T = w(v, r)\rho. \tag{51}
\]

Here, we briefly review the so-called weak, strong and dominant energy conditions, which might be satisfied by physically reasonable matter. These conditions are rewritten in the form of the conditions on the values of \( k \) and \( w \). The weak energy condition leads to the following conditions on \( \rho, p_R \) and \( p_T \),

\[
\rho \geq 0, \quad \rho + p_R \geq 0 \quad \text{and} \quad \rho + p_T \geq 0, \tag{52}
\]

the strong energy condition adds one more condition

\[
\rho + p_R + 2p_T \geq 0, \tag{53}
\]

the dominant energy condition leads further to two conditions,

\[
\rho \geq |p_R| \quad \text{and} \quad \rho \geq |p_T|. \tag{54}
\]
Assuming that $\rho$ is non-negative, all the three energy conditions are guaranteed, if and only if the following conditions for $k$ and $w$ are satisfied:

$$-1 \leq k \leq 1, \quad -1 \leq w \leq 1 \quad \text{and} \quad 1 + k + 2w \geq 0.$$

(55)

Hereafter we assume the non-negativity of $\rho$. It is worthy of note that both of $k$ and $w$ are bounded above and below by virtue of the energy conditions.

From equation (51), we have

$$\frac{P_R}{\rho + p_R} = \frac{k}{k + 1} \quad \text{and} \quad \frac{P_T}{\rho + p_T} = \frac{w}{k + 1}.$$

(56)

Using the above equation, equation (47) becomes

$$\left(\frac{1 - k}{1 + k} V\right)^{\prime} + \frac{4w}{r(1 + k)} V = 0,$$

(57)

where we have used the fact $D_B = D_B(v)$. From the above equation, we have

$$\left[\ln \left|\frac{1 - k}{1 + k} V\right|\right]^{\prime} = -\frac{4w}{r(1 - k)}.$$

(58)

Clearly, the high-speed approximation scheme is not applicable to the case of $k = \pm 1$: the upper sign corresponds to the stiff matter and the lower sign corresponds to the cosmological-constant-like matter. Hereafter we assume $k \neq \pm 1$.

The formal solution of equation (58) is given by

$$\frac{1 - k}{1 + k} V = C(v) \exp \left(-\int_{r}^{r^\prime} \frac{4w(v, x)}{1 - k(v, x)} \frac{dx}{x}\right),$$

(59)

where $C(v)$ is an arbitrary function of $v$ but it is set so that $V$ is positive at least initially. Substituting the above equation into equation (43) and integrating it, we have

$$\delta_A = -\frac{8\pi CD_B}{r} \int r^\prime \exp \left(-\int_{r}^{r^\prime} \frac{4w(v, x)}{1 - k(v, x)} \frac{dx}{x}\right) dy,$$

(60)

where we set the integration constant so that $\delta_A$ is finite in the limit of $r \to 0$ if possible. Then substituting equations (59) and (60) into equation (44), we have

$$D_B\delta_D = CD_B \left(\frac{1 + k}{1 - k}\right) \left(1 + \frac{2M_B}{r}\right) \exp \left(-\int_{r}^{r^\prime} \frac{4w(v, x)}{1 - k(v, x)} \frac{dx}{x}\right)

+ \int r^\prime \left[CD_B - CD_B \int_{r}^{r^\prime} \left(\frac{4w(v, z)}{1 - k(v, z)} + \frac{4w(v, z)k(v, z)}{[1 - k(v, z)]^2}\right) dz\right]

\times \exp \left(-\int_{r}^{r^\prime} \frac{4w(v, x)}{1 - k(v, x)} \frac{dx}{x}\right) dy.$$

(61)

Once $k$ and $w$ are determined, we can know the behavior of the first-order perturbations by performing the integrations in the formal solutions (59), (60) and (61).

The behavior of the solutions (59)–(61) near the spacetime singularity will be determined by the asymptotic values of $k(v, r)$ and $w(v, r)$ in the limit that the hollow sphere shrinks to its symmetry center. Thus the solutions obtained by assuming the constancy of $k$ and $w$ will give us sufficient information about what we would like to know. Hereafter, we assume that $k$ and $w$ are constants. Integration in equation (59) is then easily performed and we obtain

$$V = \frac{1 + k}{1 - k} C(v) r^{-\frac{4w}{5}}.$$

(62)
Using the above result, integrations in equations (60) and (61) are also easily performed, and we obtain

\[
\delta A = -8\pi C DB F(r; k),
\]

(63)

\[
DB\delta D = CD B \left( \frac{1 + k}{1 - k} \right) \left( 1 + \frac{2MB}{r} \right) r^{-\frac{4w}{1-k}} + (DB C + DB \dot{C}) F(r; k),
\]

(64)

where

\[
F(r; k) = \begin{cases} 
\frac{1}{1-k} - 4w - 1 & \text{for } 1-k - 4w \neq 0, \\
\frac{1}{1-k} \ln r & \text{for } 1-k - 4w = 0.
\end{cases}
\]

(65)

From the causality requirement [41], we assume \( k < 1 \) and \( w \leq 1 \). Then we study the following three cases, \( 0 < w \leq 1 \), \( -(1 - k)/4 < w \leq 0 \) and \( w \leq -(1-k)/4 \), separately, below.

5.1. \( 0 < w \leq 1 \)

It can be seen from equation (62) that \( V \) diverges in the limit of \( r \to 0 \). This implies that, in this case, the high-speed collapse is necessarily decelerated by the pressure effect so significantly that the high-speed approximation breaks down before the singularity formation. This is a somewhat unexpected result, since, at first glance, one might infer that the pressure effect would be negligible in the situation with the strong gravity near the spacetime singularity.

5.2. \( -(1 - k)/4 < w \leq 0 \)

It is easily seen from equations (62) and (63) that both \( V \) and \( \delta A \) are finite in the limit of \( r \to 0 \). Equation (64) leads

\[
DB\delta D \sim \frac{2MB}{r} CD B r^{-\frac{4w}{1-k}} \quad \text{near } r = 0.
\]

(66)

Here note that by assumption, we have

\[
0 \leq -\frac{4w}{1-k} < 1.
\]

(67)

\( MB/r \) diverges in the limit of \( r \to 0 \) with \( v \) fixed in the domain with non-vanishing \( MB \). Thus, due to equation (67), the density perturbation \( DB\delta D \) also diverges in the limit of \( r \to 0 \) within this domain, if it initially does not vanish there. The high-speed approximation in this domain breaks down when the matter particles approach the symmetry center \( r = 0 \).

Since \( MB/v \) vanishes at \( v = 0 \), the behavior of the density perturbation \( DB\delta D \) in the neighborhood of the central singularity might be different from the above case. The behavior of \( MB/r \) near the central singularity has already been shown in section 3. Our special interests are in cases (i) and (ii), since the background central singularity has the extent in the future directed ingoing null direction, while in case (iii) and in the covered case, the central singularity is instantaneous (see figures 1–3). Thus we focus on cases (i) and (ii) in which the mass function \( MB \) behaves as \( MB \sim \mu v^\alpha \) with \( \alpha \geq 1 \). In these cases, \( MB/r \) is finite at the central singularity in the background Vaidya spacetime. Thus the central singularity will be very similar to the central naked singularity of the Vaidya solution.

Here it should be noted that this case includes a case of dust \( k = w = 0 \). The dust is described by the Lemaître–Tolman–Bondi solution and it is well known that the central singularity can also be naked [43–46]. The present result implies that the central singularity
formed by the gravitational collapse of a hollow dust sphere is very similar to that formed by the null dust, whereas the non-central singularity is not so. The latter is in contrast to the cylindrically symmetric case in which the cylindrical hollow dust collapsing with very high speed is well described by the null dust solution even at the spacetime singularity \[24\].

5.3. \( w \leq -(1 - k)/4 \)

In contrast to the case B, the perturbation variables \( V \) and \( \delta_A \) vanish in the limit of \( r \to 0 \) with \( v \) fixed. In the case of \( w < -(1 - k)/4 \), \( D_B \delta_D \) also vanishes in the same limit, whereas it is finite in the case of \( w = -(1 - k)/4 \). Thus the spacetime structure in the neighborhood of the singularity at \( r = 0 \) is very similar to the Vaidya solution. Even if the radial pressure is positive, the high-speed approximation is consistent until the spacetime singularity forms. The consistency of the high-speed approximation depends not only on the radial pressure but also on the tangential one. Here note that all of the energy conditions, equation (55), hold only if

\[
k \geq -\frac{1}{3}
\]

holds.

Recent observations imply the acceleration of cosmic volume expansion [47–49], and this means the existence of unknown matter components with violation of the strong energy condition. Thus it might be important to consider cases with the violation of the energy conditions as special examples in the case of \( w \leq -(1 - k)/4 \). A case of \( k \) smaller than \(-1/3\) corresponds to the so-called dark energy, and thus the result obtained here implies that the spacetime singularity formed by spherically symmetric high-speed collapse of the dark energy is similar to that of the Vaidya solution. Further, it is worthy of note that in the phantom energy case \( k < -1 \) which is the special case of the dark energy [50], \( D_B \) is negative and thus the mass function \( M_B \) is also negative by equation (40). Hence the first-order solution of \( k < -1 \) implies that the high-speed collapse of the phantom energy forms the timelike singularity similar to that in the Schwarzschild spacetime with a negative mass (see figure 2).

6. Summary and discussion

Gravitational collapse is one of the most important topics in gravitational physics. The cosmic censorship conjecture provides major motivation to study this issue. Since there is no theorem proving or disproving this conjecture or no theorem stating the generic feature of physical spacetime singularities, it is interesting to investigate this issue in the situation different from previously studied ones.

This paper continues to study this issue and provides an extension of the previous work on high-speed cylindrical collapse of perfect fluid [25] to spherically symmetric spacetime with type I matter. To see the pressure effects on the high-speed approximation scheme, assuming that the energy density \( \rho \) is non-negative, we have studied a linear equation of state for the radial and tangential pressures, i.e., \( p_R = k \rho \) and \( p_T = w \rho \) with constants \( k \) and \( w \), in detail. By the causality requirement, we have restricted our attention to the case of \( k < 1 \) and \( w \leq 1 \). (The causality requirement implies \( k \leq 1 \). The reason why \( k = 1 \) is excluded is that the high-speed approximation is not applicable to the cases of \( k = \pm 1 \).)

One might think that the collapsing speed becomes very large due to the strong gravity just before the formation of the spacetime singularity. However, we have found that, in the case of positive tangential pressure \( 0 < w \leq 1 \), the large initial imploding velocity is necessarily decelerated by the pressure effect and thus the high-speed approximation scheme becomes invalid before the singularity formation.
In the case of the negative or vanishing tangential pressure \( w \leq 0 \), the behavior of the perturbation variables is different from the case of \( 0 < w \leq 1 \). In the case of \(-1-k)/4 < w \leq 0\), all of the perturbations are finite at the central singularity, if the background central singularity is null and naked, although the density perturbations blows up at the non-central singularity of the background Vaidya spacetime. This result implies that the central singularity will be null and naked like that of the Vaidya spacetime, for the case of \(-(1-k)/4 < w \leq 0\). In the cases of \( w \leq -(1-k)/4 \), all of the perturbation variables are finite everywhere. This result implies that, in this case, the geometrical structure near not only the central singularity but also the non-central singularity is very similar to that of the Vaidya solution. Here it is worthy to note that as long as \( k > -1/3 \), this case satisfies all of the physically reasonable energy conditions, i.e., the weak, strong and dominant energy conditions. The result obtained here strongly suggests that the spacetime structure realized by the spherical matter with large enough tangential tension is well described by the Vaidya solution even if the physically reasonable energy conditions are satisfied.

It is a remarkable result that the consistency of the high-speed approximation depends not on the radial pressure \( p_\text{R} \) but on the tangential one \( p_\text{T} \). In the case of \( k \leq 0 \), the gradient of radial pressure will not stop the high-speed collapse. Thus it is reasonable that the consistency of the high-speed approximation depends on the only tangential pressure. It is non-trivial that even if \( k > 0 \), the consistency of the high-speed approximation also depends on the only tangential pressure. At first glance, the gradient of the radial pressure with \( k > 0 \) seems to affect the gravitational collapse, but this is not true. The reason is that if the sound speed in the radial direction is less than the speed of light \((0 < k < 1)\), the spacetime singularity formation by the nearly light-speed collapse can be completed before the effect of radial pressure gradient spreads out, since the sound cone is significantly narrowed down in the frame in which the matter moves with nearly the speed of light. This is also the reason why the high-speed approximation is not applicable to the case of \( k = 1 \). In this case, the sound speed in the radial direction is equal to the speed of light and thus the sound cone in the radial direction is equivalent to the light cone which is Lorentz invariant. As a result, the effect of the radial pressure gradient can spread out before the singularity formation by the nearly light-speed collapse is completed. The perturbative construction of approximate solutions on the background null dust solution is impossible in the case of \( k = 1 \).

The dark energy case, \( k \leq -1/3 \) and \( w < -(1-k)/4 \), might be important in connection to the issue of the accelerated cosmic expansion [47–49], although the strong energy is not satisfied in this case. These are included in the case of \( w < -(1-k)/4 \). The repulsive gravity due to the dark energy cannot decelerate the high-speed collapse, and the formed central singularity can be null, and the non-central part of the spacetime singularity will be spacelike. As mentioned in the above, the pressure gradient force of the dark energy will also not stop the collapse but rather accelerate it. This property of the dark energy will be the reason why the high-speed collapse becomes a good approximation at the singularity formation in the present case and also why black holes may grow self-similarly due to the accretion of dark energy in the accelerated universe [51, 52].

Finally, it should be noted that the present results are valid up to the only first order and could be modified by the higher order effects. Thus we need to investigate the higher order, but this is a future work.

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