On the largest-eigenvalue process
for generalized Wishart random matrices

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Abstract. Using a change-of-measure argument, we prove an equality in law be-
tween the process of largest eigenvalues in a generalized Wishart random-matrix
process and a last-passage percolation process. This equality in law was conjectured
by Borodin and Pêché (2008).

1. Introduction

The past decade has witnessed a surge of interest in connections between random
matrices on the one hand and applications to growth models, queueing systems,
and last-passage percolation models on the other hand; standard references are
Baryshnikov (2001) and Johansson (2000). In this note we prove a result of this
kind: an equality in law between a process of largest eigenvalues for a family of
Wishart random matrices and a process of directed last-passage percolation times.

To formulate the main result, we construct two infinite arrays of random vari-
ables on an underlying measurable space, along with a family \( \{ P^{\pi,\hat{\pi}} \} \) of prob-
ability measures parametrized by a positive \( N \)-vector \( \pi \) and a nonnegative sequence
\( \{ \hat{\pi}_n : n \geq 1 \} \). The elements of the first array \( \{ A_{ij} : 1 \leq i \leq N, j \geq 1 \} \) are in-
dependent and \( A_{ij} \) has a complex zero-mean Gaussian distribution with varian-
tce \( 1/(\pi_i + \hat{\pi}_j) \) under \( P^{\pi,\hat{\pi}} \). That is, both the real and complex part of \( A_{ij} \) have zero
mean and variance \( 1/(2\pi_i + 2\pi_j) \). Write \( A(n) \) for the \( N \times n \) matrix formed by the first
n columns of \( A \), and define the matrix-valued stochastic process \( \{ M(n) : n \geq 0 \} \) by setting
\( M(n) = A(n)A(n)^* \) for \( n \geq 1 \) and by letting \( M(0) \) be the \( N \times N \) zero
matrix. We call \( \{ M(n) : n \geq 0 \} \) a generalized Wishart random-matrix process,
since the marginals have a Wishart distribution if \( \pi \) and \( \hat{\pi} \) are identically one and
zero, respectively.

The elements of the second array \( \{ W_{ij} : 1 \leq i \leq N, j \geq 1 \} \) are independent and
\( W_{ij} \) is exponentially distributed with parameter \( \pi_i + \hat{\pi}_j \) under \( P^{\pi,\hat{\pi}} \). We define

\[
Y(N, n) = \max_{P \in \Pi(N,n)} \sum_{(ij) \in P} W_{ij},
\]
where \( \Pi(N,n) \) is the set of up-right paths from \((1,1)\) to \((N,n)\). The quantity 
\[ Y(N,n) \] 
arises in last-passage percolation models as well as in series Jackson networks in queueing theory, see for instance Dieker and Warren (2008) or Johansson (2009).

The following theorem, a process-level equality in law between the largest eigenvalue of \( M(n) \) and \( Y(N,n) \), is the main result of this note. Given a matrix \( C \), we write \( sp(C) \) for its vector of eigenvalues, ordered decreasingly.

**Theorem 1.1.** For any strictly positive vector \( \pi \) and any nonnegative sequence \( \hat{\pi} \), the processes \( \{sp(M(n))_1 : n \geq 1\} \) and \( \{Y(N,n) : n \geq 1\} \) have the same distribution under \( P^{\pi,\hat{\pi}} \).

It is known from Defossez (2008); Forrester and Rains (2006) that this holds in the 'standard' case, i.e., under the measure \( P := P^{(1,\ldots,1),(0,0,\ldots)} \). In its stated generality, the theorem was conjectured by Borodin and Péché (2008), who prove that the laws of \( Y(N,n) \) and the largest eigenvalue of \( M(n) \) coincide for fixed \( n \geq 1 \). Our proof is based on a change-of-measure argument, which is potentially useful to prove related equalities in law.

Throughout, we use the following notation. We let \( H_{N,N} \) be the space of all \( N \times N \) Hermitian matrices, and \( W^N \) the set \( \{x \in \mathbb{R}^N : x_1 \geq \ldots \geq x_N\} \). For \( x, x' \in W^N \), we write \( x \prec x' \) to mean that \( x \) and \( x' \) interlace in the sense that 
\[ x'_1 \geq x_1 \geq x'_2 \geq x_2 \geq \ldots \geq x'_N \geq x_N. \]

### 2. Preliminaries

This section provides some background on generalized Wishart random matrices, and introduces a Markov chain which plays an important role in the proof of Theorem 1.1.

#### 2.1. The generalized Wishart random-matrix process

Under \( P^{\pi,\hat{\pi}} \), the generalized Wishart process \( \{M(n) : n \geq 0\} \) from the introduction has independent increments since, for \( m \geq 1 \),
\[ M(m) = M(m-1) + (A_{im}A_{jm})_{1 \leq i,j \leq N}, \]

where \( \bar{A}_{jm} \) is the complex conjugate of \( A_{jm} \). In particular, the matrix-valued increment has unit rank. The matrix \( M(m) - M(m-1) \) can be parameterized by its diagonal elements together with the complex arguments of \( A_{im} \) for \( 1 \leq i \leq N \); under \( P^{\pi,\hat{\pi}} \), these are independent and the former have exponential distributions while the latter have uniform distributions on \([0,2\pi]\). (This fact is widely used in the Box-Muller method for computer generation of random variables with a normal distribution.) Since the \( i \)-th diagonal element has an exponential distribution under \( P^{\pi,\hat{\pi}} \) with parameter \( \pi_i + \hat{\pi}_m \), we obtain the following proposition.

**Proposition 2.1.** For any \( m \geq 1 \), the \( P^{\pi,\hat{\pi}} \)-law of \( M(m) - M(m-1) \) is absolutely continuous with respect to the \( P \)-law of \( M(m) - M(m-1) \), and the Radon-Nikodym derivative is
\[
\prod_{i=1}^{N} (\pi_i + \hat{\pi}_m) \exp \left( - \sum_{i=1}^{N} (\pi_i + \hat{\pi}_m - 1)(M_{ii}(m) - M_{ii}(m-1)) \right).
\]
2.2. A Markov transition kernel. We next introduce a time-inhomogeneous Markov transition kernel on $W^N$. We shall prove in Section 3 that this kernel describes the eigenvalue-process of the generalized Wishart random-matrix process of the previous subsection.

In the standard case ($\pi \equiv 1$, $\hat{\pi} \equiv 0$), it follows from unitary invariance (see Defosseux (2008, Sec. 5) or Forrester and Rains (2006)) that the process $\{\text{sp}(M(n) : n \geq 0)\}$ is a homogeneous Markov chain. Its one-step transition kernel $Q(z, \cdot)$ is the law of $\text{sp}(\text{diag}(z) + G)$, where $G = \{g_{ij} : 1 \leq i, j \leq N\}$ is a rank one matrix determined by an $N$-vector $g$ of standard complex Gaussian random variables. For $z$ in the interior of $W^N$, $Q(z, \cdot)$ is absolutely continuous with respect to Lebesgue measure on $W^N$ and can be written explicitly as in Defosseux (2008, Prop. 4.8):

$$Q(z, dz') = \frac{\Delta(z')}{\Delta(z)}e^{-\sum_{i < j}(z'_i - z_j)}1_{\{z < z'\}}dz',$$

where $\Delta(z) := \prod_{1 \leq i < j \leq N}(z_i - z_j)$ is the Vandermonde determinant.

We use the Markov kernel $Q$ to define the aforementioned time-inhomogeneous Markov kernels, which arise from the generalized Wishart random-matrix process. For general $\pi$ and $\hat{\pi}$, we define the inhomogeneous transition probabilities $Q_{n-1,n}^{\pi,\hat{\pi}}$ via

$$Q_{n-1,n}^{\pi,\hat{\pi}}(z, dz') = \prod_{i=1}^{N}(\pi_i + \hat{\pi}_n) \frac{h_{\pi}(z')}{h_{\hat{\pi}}(z)}e^{-\sum_{i=1}^{n}(z'_i - z_i)}Q(z, dz'),$$

with

$$h_{\pi}(z) = \frac{\det\{e^{-\pi_i z_j}\}}{\Delta(\pi)\Delta(z)}. \quad (2.2)$$

Note that $h_{\pi}(z)$ extends to a continuous function on $(0, \infty)^N \times W^N$ (this can immediately be seen as a consequence of the Harish-Chandra-Itzykson-Zuber formula, see (3.2) below).

One can verify that the $Q_{n-1,n}^{\pi,\hat{\pi}}$ are true Markov kernels by writing $1_{\{z < z'\}} = \det\{1_{\{z_i < z'_j\}}\}$ and applying the Cauchy-Binet formula

$$\int_{W^N} \det\{\xi_i(z_j)\} \det\{\psi_j(z_i)\}dz = \det\left\{\int_{\mathbb{R}}\xi_i(z)\psi_j(z)dz\right\}.$$

3. The generalized Wishart eigenvalue-process

In this section, we determine the law of the eigenvalue-process of generalized Wishart random-matrix process. Although it is not essential to the proof of Theorem 1.1, we formulate our results in a setting where $\text{sp}(M(0))$ is allowed to be nonzero.

Write $m_\mu$ for the ‘uniform distribution’ on the set $\{M \in \mathbb{H}_{N,N} : \text{sp}(M) = \mu\}$. That is, $m_\mu$ is the unique probability measure invariant under conjugation by unitary matrices, or equivalently $m_\mu$ is the law of $U\text{diag}(\mu)U^*$ where $U$ is unitary and distributed according to (normalized) Haar measure. We define measures $P_{n}^{\mu,\pi}$ by letting the $P_{n}^{\mu,\pi}$-law of $\{M(n) - M(0) : n \geq 0\}$ be equal to the $P_{n}^{\mu,\pi}$-law of $\{M(n) : n \geq 0\}$, and letting the $P_{n}^{\mu,\pi}$-distribution of $M(0)$ be independent of $\{M(n) - M(0) : n \geq 0\}$ and absolutely continuous with respect to $m_\mu$ with Radon-Nikodym derivative

$$\frac{CN}{h_{\pi}(\mu)}e^{-\sum_{i=1}^{N}\mu_i\exp(-\text{tr}((\text{diag}(\pi) - I)M(0)))}, \quad (3.1)$$
where $c_N$ is a constant depending only on the dimension $N$ and $I$ is the identity matrix. Recall that $h_\pi(\mu)$ is defined in (2.2). That this defines the density of a probability measure for all $\pi$ and $\mu$ follows immediately from the Harish-Chandra-Itzykson-Zuber formula (e.g., Mehta (2004, App. A.5))

$$
\int_U \exp \left( - \operatorname{tr}(\operatorname{diag}(\pi) U \operatorname{diag}(\mu) U^*) \right) dU = c_N^{-1} h_\pi(\mu), \quad (3.2)
$$

writing $dU$ for normalized Haar measure on the unitary group. Throughout, we abbreviate $P^{(1,\ldots,1),(0,0,\ldots)}_\mu$ by $P_\mu$. Note that the $P_\mu^{\pi,\hat{\pi}}$-law and the $P_\mu^{(i)}$-law of $\{M(n) : n \geq 0\}$ coincide if $\mu = 0$.

The following theorem specifies the $P_\mu^{\pi,\hat{\pi}}$-law of $\{\text{sp}(M(n)) : n \geq 0\}$.

**Theorem 3.1.** For any $\mu \in W^N$, $\{\text{sp}(M(n)) : n \geq 0\}$ is an inhomogeneous Markov chain on $W^N$ under $P_\mu^{\pi,\hat{\pi}}$, and it has the $Q^{\pi,\hat{\pi}}_{n-1,n}$ of Section 2.2 for its one-step transition kernels.

**Proof.** Fix some $\mu \in W^N$. The key ingredient in the proof is a change of measure argument. We know from Defosseux (2008) or Forrester and Rains (2006) that Theorem 3.1 holds for the ‘standard’ case $\pi = (1,\ldots,1), \hat{\pi} \equiv 0$.

Writing $P_\mu^{\pi,\hat{\pi}}$ and $P_\mu$ for the distribution of $(M(0),\ldots,M(n))$ under $P_\mu^{\pi,\hat{\pi}}$ and $P_\mu$ respectively, we obtain from Section 2.1 that for $n \geq 0$,

$$
\frac{dP_\mu^{\pi,\hat{\pi}}}{dP_\mu}(M(0),\ldots,M(n)) = C_{\pi,\hat{\pi}}(n,N) \frac{c_N}{h_\pi(\mu)} e^{-\sum_{i=1}^N \mu_i} \times \exp \left( - \operatorname{tr}((\operatorname{diag}(\pi) - I)M(n)) - \sum_{m=1}^n \hat{\pi}_m \operatorname{tr}(M(m) - M(m - 1)) \right),
$$

where $C_{\pi,\hat{\pi}}(n,N) = \prod_{i=1}^N \prod_{j=1}^n (\pi_i + \hat{\pi}_j)$. Let the measure $P_\mu^{\pi,\hat{\pi}}$ (and $P_\mu$) be the restriction of $P_\mu^{\pi,\hat{\pi}}$ (and $P_\mu$) to the $\sigma$-field generated by $(\text{sp}(M(0)),\ldots,\text{sp}(M(n)))$. Then we obtain for $n \geq 0$,

$$
\frac{dp_\mu^{\pi,\hat{\pi}}}{dp_\mu}(\text{sp}(M(0)),\ldots,\text{sp}(M(n))) = \mathbb{E}_{P_\mu} \left[ \frac{dP_\mu^{\pi,\hat{\pi}}}{dP_\mu}(M(0),\ldots,M(n)) \bigg| \text{sp}(M(0)),\ldots,\text{sp}(M(n)) \right],
$$

where $\mathbb{E}_{P_\mu}$ denotes the expectation operator with respect to $P_\mu$. Since the $P_\mu$-distribution of $(M(0),\ldots,M(n))$ given the spectra is invariant under componentwise conjugation by a unitary matrix $U$, we have for $\mu \equiv \mu^{(0)} \prec \cdots \prec \mu^{(n)}$,

$$
\mathbb{E}_{P_\mu} \left[ \exp \left( - \operatorname{tr}(\operatorname{diag}(\pi)M(n)) \right) \bigg| \text{sp}(M(0)) = \mu^{(0)},\ldots,\text{sp}(M(n)) = \mu^{(n)} \right]
$$

$$
= \int_U \exp \left( - \operatorname{tr}(\operatorname{diag}(\pi) U \operatorname{diag}(\mu^{(n)}) U^*) \right) dU
$$

$$
= c_N^{-1} h_\pi(\mu^{(n)}),
$$

where the second equality is the Harish-Chandra-Itzykson-Zuber formula. From the preceding three displays in conjunction with $\operatorname{tr}(M) = \sum_i \text{sp}(M)_i$, we conclude
that
\[ \frac{dp_n^{\pi,\hat{x}}}{dp_n}(\mu, \mu(1), \ldots, \mu(n)) \]
\[ = C_{\pi,\hat{x}}(n, N) \frac{\h_{\pi}(\mu(n))}{\h_{\pi}(\mu)} \exp \left( -\sum_{i=1}^{N} \sum_{r=1}^{n} \hat{x}_r \left[ \mu_i^{(r)} - \mu_i^{(r-1)} \right] + \sum_{i=1}^{N} [\mu_i^{(n)} - \mu_i] \right). \]

Since \( \text{sp}(M(\cdot)) \) is a Markov chain with transition kernel \( Q \) under \( P_\mu \), we have
\[ P_\mu^{\pi,\hat{x}}(\text{sp}(M(1)) \in dp_1^{(1)}, \ldots, \text{sp}(M(n)) \in dp_n^{(n)}) \]
\[ = \frac{dp_n^{\pi,\hat{x}}}{dp_n}(\mu, \mu(1), \ldots, \mu(n))P_\mu(\text{sp}(M(1)) \in dp_1^{(1)}, \ldots, \text{sp}(M(n)) \in dp_n^{(n)}) \]
\[ = \frac{dp_n^{\pi,\hat{x}}}{dp_n}(\mu, \mu(1), \ldots, \mu(n))Q(\mu, dp_1^{(1)}) \cdots Q(\mu(n-1), dp_{n-1}^{(n-1)}) \]
\[ = Q_{0,1,2}^{\pi,\hat{x}}(\mu, dp_1^{(1)}), Q_{1,2}^{\pi,\hat{x}}(\mu^{(2)}, dp_2^{(2)}) \cdots Q_{n-1,n}^{\pi,\hat{x}}(\mu(n-1), dp_{n-1}^{(n-1)}), \]
the last equality being a consequence of the definition of \( Q_{k-1}^{\pi,\hat{x}} \) and the expression for \( dp_n^{\pi,\hat{x}}/dp_n \). \( \square \)

### 4. Robinson-Schensted-Knuth and the proof of Theorem 1.1

This section explains the connection between the infinite array \( \{W_{ij}\} \) of the introduction and the Markov kernels \( Q_{n-1,n}^{\pi,\hat{x}} \). In conjunction with Theorem 3.1, these connections allow us to prove Theorem 1.1.

**The RSK algorithm.** The results in this section rely on a combinatorial mechanism known as the Robinson-Schensted-Knuth (RSK) algorithm. This algorithm generates from a \( p \times q \) matrix with nonnegative entries a triangular array \( x = \{x_{ij} : 1 \leq j \leq p, 1 \leq i \leq j\} \) called a Gelfand-Tsetlin (GT) pattern. A GT pattern with \( p \) levels \( x^1, \ldots, x^p \) is an array for which the coordinates satisfy the inequalities
\[ x^k_i \leq x^k_{i-1} \leq x^k_{i-1} \leq \ldots \leq x^k_2 \leq x^k_1 \leq x^k_i \]
for \( k = 2, \ldots, p \). If the elements of the matrix are integers, then a GT pattern can be identified with a so-called semistandard Young tableau, and the bottom row \( x^p = \{x^p_i : 1 \leq i \leq p\} \) of the GT pattern corresponds to the shape of the Young tableau. We write \( \mathcal{K}_p \) for the space of all GT patterns \( x \) with \( p \) levels.

By applying the RSK algorithm with row insertion to an infinite array \( \{\xi_{ij} : 1 \leq i \leq N, 1 \leq j \leq n\} \) for \( n = 1, 2, \ldots \), we obtain a sequence of GT patterns \( x(1), x(2), \ldots \). It follows from properties of RSK that
\[ x^N_i(n) = \max_{P \in \Pi(N, n)} \sum_{(ij) \in P} \xi_{ij}, \tag{4.1} \]
where \( \Pi(N, n) \) is the set of up-right paths from \( (1, 1) \) to \( (N, n) \) as before. Details can be found in, e.g., Johansson (2000) or Dieker and Warren (2008, case A).

Greene’s theorem generalizes (4.1), and gives similar expressions for each component of the pattern \( x^N_i(n) \), see for instance Chapter 3 of Fulton (1997) or Equation (16) in Doumerc (2003). As a consequence of these, we can consider the RSK algorithm for real-valued \( \xi_{ij} \) and each \( x(n) \) is then a continuous function of the input data.
We remark that the RSK algorithm can also be started from a given initial GT pattern $x(0)$. If RSK is started from the null pattern, it reduces to the standard algorithm and we set $x(0) = 0$.

**The bijective property of RSK.** RSK has a bijective property which has important probabilistic consequences for the sequence of GT patterns constructed from specially chosen random infinite arrays. Indeed, suppose that $\{\xi_{ij} : 1 \leq i \leq N, j \geq 1\}$ is a family of independent random variables with $\xi_{ij}$ having a geometric distribution on $\mathbb{Z}_+$ with parameter $a_i b_j$, where $\{a_i : 1 \leq i \leq N\}$ and $\{b_j : j \geq 1\}$ are two sequences taking values in $(0, 1]$. Write $\{X(n) : n \geq 0\}$ for the sequence of GT patterns constructed from $\xi$.

Using the bijective property of RSK it can be verified that the bottom rows $\{X_N(n) : n \geq 0\}$ of the GT patterns evolve as an inhomogeneous Markov chain with transition probabilities

$$P_{n-1,n}(x, x') = \prod_{i=1}^{N} \left(1 - a_i b_n \right) \frac{s_{x'}(a)}{s_x(a)} \prod_{i=1}^{N} \left(x'_i - x_i\right) 1_{\{0 \leq x_i < x'_i\}},$$

where $s_{\lambda}(a)$ is the Schur polynomial corresponding to a partition $\lambda$:

$$s_{\lambda}(a) = \sum_{x \in \mathbb{K}_{N; x^N = \lambda}} a^x,$$

with the weight $a^x$ of a GT pattern $x$ being defined as

$$a^x = a_1^{x_1} \prod_{k=2}^{N} a_k^{\sum_{i=1}^{k} x_i - \sum_{i=1}^{k-1} x_i}.$$

This is proved in O’Connell (2003) in the special case with $b_j = 1$ for all $j$, and the argument extends straightforwardly; see also Forrester and Nagao (2008).

Non-null initial GT patterns generally do not give rise to Markovian bottom-row processes. Still, the inhomogeneous Markov chain of bottom rows can be constructed starting from a given initial partition $\lambda$ with at most $N$ parts by choosing $X(0)$ suitably from the space of a GT patterns with bottom row $\lambda$: $X(0)$ should be independent of the family $\{\xi_{ij}\}$ with probability mass function

$$p(x) = \frac{a^x}{s_{\lambda}(a)}.$$

**Exponentially distributed input data.** We now consider the sequence of GT patterns $\{X_L(n) : n \geq 0\}$ arising from setting $a_i = 1 - \pi_i / L$ and $b_j = 1 - \hat{\pi}_j / L$ in the above setup, and we study the regime $L \to \infty$ after rescaling suitably. In the regime $L \to \infty$, the input variables $\{\xi_{ij} / L\}$ (jointly) converge in distribution to independent exponential random variables, the variable corresponding to $\xi_{ij} / L$ having parameter $\pi_i + \hat{\pi}_j$. Thus, the law of the input array $\xi$ converges weakly to the $P_{\pi, \hat{\pi}}$-law of the array $\{W_{ij} : 1 \leq i \leq N, j \geq 1\}$ from the introduction. Refer to Doumerc (2003) and Johansson (2000) for related results on this regime.

By the aforementioned continuity of the RSK algorithm and the continuous-mapping theorem, $\{X_L(n) / L : n \geq 0\}$ converges in distribution to a process $\{Z(n) :
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\[ Z^N_1(n) = \max_{P \in \Pi(N,n)} \sum_{(ij) \in P} W_{ij}. \]

Moreover, the process of bottom rows \( \{ Z^N(n) : n \geq 0 \} \) is an inhomogeneous Markov chain for which its transition mechanism can be found by letting \( L \to \infty \) in (4.2):

Lemma 4.1. Under \( P^{\pi,\hat{\pi}} \), the process \( \{ Z^N(n) : n \geq 0 \} \) is an inhomogeneous Markov chain on \( W^N \), and it has the \( Q^{\pi,\hat{\pi}}_{n-1,n} \) of Section 2.2 for its one-step transition kernels.

A similar result can be obtained given a non-null initial bottom row \( \mu \in W^N \). In case the components of \( \mu \) are distinct, the distribution of the initial pattern \( Z(0) \) should then be absolutely continuous with respect to Lebesgue measure on \( \{ z \in K^N : z^N = \mu \} \) with density

\[ \frac{\Delta(\pi)}{\det(e^{-\pi_{ij}\mu_j})} c^z, \]

where \( c = (e^{-\pi_1}, \ldots, e^{-\pi_N}) \).

Proof of Theorem 1.1. We now have all ingredients to prove Theorem 1.1. We already noted that \( Z^N_1(n) \) equals \( Y(N,n) \). Thus, for any strictly positive vector \( \pi \) and any nonnegative sequence \( \hat{\pi} \), \( \{ Y(N,n) : n \geq 1 \} \) has the same \( P^{\pi,\hat{\pi}} \)-distribution as \( \{ Z_1(n) : n \geq 1 \} \). In view of Theorem 3.1 and Lemma 4.1, in turn this has the same \( P^{\pi,\hat{\pi}} \)-distribution as the largest-eigenvalue process \( \{ \text{sp}(M(n))_1 : n \geq 1 \} \). This proves Theorem 1.1.

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