A tale of two dyons

Gérard Clément\textsuperscript{a,*}, Dmitri Gal’tsov\textsuperscript{b,c†}

\textsuperscript{a} LAPTh, Université Savoie Mont Blanc, CNRS, 9 chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux cedex, France
\textsuperscript{b} Department of Theoretical Physics, Faculty of Physics, Moscow State University, 119899, Moscow, Russia
\textsuperscript{c} Kazan Federal University, 420008 Kazan, Russia

Abstract

We present a one-parameter family of stationary, asymptotically flat solutions of the Einstein-Maxwell equations with only a mild singularity, which are endowed with mass, angular momentum, a dipole magnetic moment and a quadrupole electric moment. We briefly analyze the structure of this solution, which we interpret as a system of two extreme co-rotating black holes with equal masses and electric charges, and opposite magnetic and gravimagnetic charges, held apart by an electrically charged, magnetized string which also acts as a Dirac-Misner string.

\textsuperscript{*}Email: gclement@lapth.cnrs.fr
\textsuperscript{†}Email: galtsov@phys.msu.ru
1 Introduction

Stationary asymptotically flat black holes with regular connected event horizons are known to be strongly limited by uniqueness theorems and given in the vacuum case by the Kerr metric, and in the electrovacuum case by the Kerr-Newman solution. In the search for alternatives to this standard black hole scenario, prompted by the coming perspectives to receive high accuracy data from the center of the Galaxy, one is led to relax some basic assumptions, among which one could most safely sacrifice horizon connectedness. Many such solutions of the Einstein-Maxwell equations known in the Weyl-Papapetrou form are nicely reviewed in the Griffiths and Podolsky book [1]. Leaving aside the well-known Israel-Wilson-Perjéès and Weyl linear superpositions, these solutions can be roughly classified in two families. One was obtained via soliton generating techniques and consists of non-linear superpositions of aligned black holes possibly endowed with charges and NUT parameters. The other contains solutions generated by other methods, such as the static magnetized Bonnor solution [2], or the one-parameter class of static Zipoy-Voorhees (ZV) [3, 4] vacuum solutions, also known as \( \gamma \)-metrics, and their rotating Tomimatsu-Sato (TS) [5] cousins with integer \( \gamma \), the \( \gamma = 1 \) TS solution coinciding with the Kerr metric. The physical interpretation of these last solutions is far from trivial, and it was only relatively recently recognized that the Bonnor solution [6] or the TS solution with \( \gamma = 2 \) (TS2) [7] actually describe black-hole pairs. Both families generically contain naked ring-type curvature singularities, and/or milder conical line singularities (strings), which however can be avoided by imposing external fields [6], at the expense of asymptotic flatness. Here we adopt a tolerant attitude with respect to mild naked singularities and do not reject solutions endowed with novel and very intriguing features for the sake of cosmic censorship. It is worth noting that the possibility of violation of cosmic censorship in gravitational collapse has received more attention recently [8].

In this short note (a more detailed version will be published elsewhere) we wish to draw attention to a new electrovacuum solution obtained some time ago by one of us [9] via an original generating technique which produces a one-parameter family of rotating electrovacuum solutions from a given static vacuum one. Application of this procedure to the static ZV metric generates a rotating solution with a magnetic dipole moment, which is the unique so far known rotating generalization of the ZV metric with non-integer \( \gamma \), and has the advantage to be free from a naked ring singularity. For \( \gamma = 1 \) it is again the Kerr metric, while for other integer values of \( \gamma \) it is not the vacuum TS metric, but some new electrovacuum metric. Postponing the case of generic real \( \gamma \) for future publication, we concentrate here on the \( \gamma = 2 \) version which looks the most interesting physically.

2 The solution

This one-parameter family of rotating solutions was generated from the static \( \gamma = 2 \) Zipoy-Voorhees (ZV2) vacuum solution in [9]. The Ernst potentials of the rotating solution are \( E = (U - W)/(U + W) \), \( \psi = V/(U + W) \), with the Kinnersley potentials:

\[
\begin{align*}
U &= \frac{p^2 + 1}{2x} + iqy, \\
V &= \varepsilon(W - 1), \\
W &= 1 + \frac{q^2}{2} \frac{1 - y^2}{x^2 - 1} - i \frac{pq y}{2x}, \quad (\varepsilon^2 = 1)
\end{align*}
\]

(2.1)

where the prolate spheroidal coordinates \( x \geq 1, \ y \in [-1, +1] \) are related to the Weyl cylindrical coordinates \( \rho, \ z \) by

\[
\rho = \kappa \sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \kappa xy,
\]

(2.2)

(the positive constant \( \kappa \) setting the length scale), and the real parameters \( p \) and \( q \) are related by \( p = \sqrt{1 - q^2} \). The potentials of the ZV2 solutions are recovered for \( q = 0 \).
The form (2.1) of the solution is only implicit. Dualization of the imaginary part of the scalar Ernst potentials to vector potentials leads to the explicit metric and electromagnetic fields

\[
\begin{align*}
 ds^2 &= -f (dt - \omega d\varphi)^2 + \kappa^2 \varpi \left[ e^{2\nu} \left( \frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} \right) 
+ f^{-1}(x^2-1)(1-y^2) d\varphi^2 \right], \\
 A &= \frac{\epsilon}{\varpi} dt + \kappa \Theta d\varphi,
\end{align*}
\]  

(2.3)

(\varphi \text{ periodic with period } 2\pi), \text{ the various functions of } x \text{ and } y \text{ occurring in (2.3) being}

\[
\begin{align*}
f &= \frac{p^2(x^2-1)^2 - q^2 x^2(1-y^2)}{4x^2}, \\
\Sigma &= \left[ \frac{p x^2 + 2x + p}{2x} + \frac{q^2(1-y^2)}{2(x^2-1)} \right] + q^2 \left( 1 - \frac{p}{2x} \right)^2 y^2, \\
e^{2\nu} &= \frac{4x^2(x^2-1)^2}{p^2 x^2 - y^2)^3}, \quad \omega = \kappa \frac{f}{p}, \\
\Pi &= \Pi_1(x)(1-y^2) + \Pi_2(x)(1-y^2)^2, \\
\Pi_1 &= -\frac{q}{2p} \left\{ \frac{(px + 2)[4x^2 + p^2(x^2-1)] + 4p(1+p^2)x + 8 + 2p^2 - p^4}{x^2-1} \right\}, \\
\Pi_2 &= -\frac{q^3}{2} \left[ \frac{p}{4x^2} + \frac{2x - p}{x^2 - 1} \right], \\
\varpi &= \frac{q^2}{4} \left\{ -\frac{2(px - p)}{x^2} + \left[ \frac{p(2x - p)}{x^2} + \frac{px^2 + 2x + p}{x(x^2-1)} \right] (1-y^2) + \frac{q^2(1-y^2)^2}{(x^2-1)^2} \right\}, \\
\Theta &= \Theta_1(x)(1-y^2) + \Theta_2(x)(1-y^2)^2 + \Theta_3(x)(1-y^2)^3, \\
\Theta_1 &= -\frac{q}{4p} \left\{ \frac{p}{x^2} \left[ p^2 x^3 + 5px^2 - (8 - 4p^2 + p^4)x + 2p - 3p^3 \right] + \frac{16 - p^2 + p^4}{x^2-1} \right\}, \\
\Theta_2 &= -\frac{q^3}{8p} \left[ \frac{p(4x^3 - 3px^2 + 2p^2 x + 5p)}{x^2(x^2 - 1)} + \frac{2(4x + 3p + p^3)}{x(x^2 - 1)^2} \right], \\
\Theta_3 &= -\frac{q^5}{8x(x^2 - 1)^2}.
\end{align*}
\]  

(2.4)

The metric (2.3),(2.4) is asymptotically (for \( x \to \infty \)) Minkowskian in spherical coordinates, with \( r = \kappa x, \cos \theta = y \). The associated conserved charges are the mass \( M \), angular momentum \( J \), dipole magnetic moment \( \mu \), and quadrupole electric moment \( Q_2 \):

\[
\begin{align*}
 M &= \frac{2\kappa}{p}, \quad J = \frac{\kappa^2 q(4 + p^2)}{p^2}, \quad \mu = \epsilon \kappa^2 q, \quad Q_2 = -\epsilon \kappa^3 q^2 / p.
\end{align*}
\]  

(2.5)

The ratio \(|\mu/J|\) is bounded above by 1/5, in agreement with the Barrow-Gibbons bound \([10]\) \(|\mu/J| \leq 1\), while the ratio \(|J/M^2|\) satisfies the Kerr-like bound

\[
|J|/M^2 \leq 1,
\]  

(2.6)

the upper bound being attained in the limit \( p \to 0 \) with \( M \) fixed, in which case the solution reduces to the extreme Kerr vacuum solution.

To the difference of the TS2 vacuum solution, which is another rotating generalization of the ZV2 solution, the present solution is free from a naked ring singularity. Such a singularity would arise as a zero of the function \( \Sigma(x, y) \), however it is clear from (2.4) that \( \Sigma \) admits the lower bound
$\Sigma(x, y) > (p+1)^2$. Another possible locus for singularities is the symmetry axis $\rho = 0$ which from (2.2) can be divided in three pieces: $z > \kappa$ ($y = +1$, $x > 1$), $-\kappa < z < \kappa$ ($x = 1$, $-1 < y < 1$), and $z < -\kappa$ ($y = -1$, $x > 1$). By construction the solution is regular for $y = \pm 1$ if $x \neq 1$. However it presents coordinate singularities on the segment $S$ ($-\kappa < z < \kappa$) as well as at its two ends $H_\pm$ ($z = \pm \kappa$). Other coordinate singularities are the ergosurface $F \equiv f/\Sigma = 0$, which consists of two disjoint components: the surface $f(x, y) = 0$, which contains $S$ ($f < 0$ for $x^2 \rightarrow 1$), and $S$ itself, on which $\Sigma \rightarrow \infty$; the causal boundary, where $g_{\varphi \varphi} \equiv F^{-1} \rho^2 - F\omega^2 = 0$, which contains $S$ ($F^{-1} \rho^2 < 0$ and $\omega$ finite) and is contained within the ergosurface component $f = 0$; and the horizons, where $N^2 \equiv \rho^2/g_{\varphi \varphi} = 0$ with $g_{\varphi \varphi} > 0$, obvious candidates being the the end points $H_\pm$ of the segment $S$.

A first integral of the geodesic equation in the metric (2.3) is $ds^2/d\lambda^2 = \epsilon$ (with $\lambda$ an affine parameter, and $\epsilon = -1, 0, 1$), which can be written as

$$T + U = \epsilon, \quad (2.7)$$

with

$$T = \kappa^2 \Sigma \epsilon^{2\nu} \left( \frac{\dot{x}^2}{x^2 - 1} + \frac{\dot{y}^2}{1 - y^2} \right) > 0,$$

$$U = \frac{(L - E\omega)^2 F}{\rho^2} - \frac{E^2}{F}, \quad (2.8)$$

where $\dot{} = d/d\lambda$, and $E$ and $L$ are the conserved energy and orbital angular momentum. Near the segment $S$, where $\xi^2 \equiv x^2 - 1 \rightarrow 0$, $-F$ and $\rho^2$ both scale as $\xi^2$, so that for $E \neq 0$ geodesics turn back before reaching $S$, while geodesics with $E = 0$ terminate on $S$, but cannot originate from $\infty$ if they are timelike or null ($\epsilon = -1$ or $0$), so that $S$ is a “harmless” naked singularity. Near the two endpoints $H_\pm$, geodesics such that, near $x = 1$, $1 - y^2 \sim X^2(x^2 - 1)$ with $X$ fixed can be continued through $x = \pm y = 1$ to a region with $x < 1$ and $y^2 > 1$, suggesting that these are actually two double horizons. We thus arrive at a possible interpretation of this solution as describing a system of two black holes $H_\pm$ held apart by a string $S$, which shall be validated in the following.

3 The horizons

When $x$ decreases from infinity to $1$, an ergosurface $f(x, y) = 0$ appears. As $x$ goes to $1$, $f$ goes to $-\infty$, unless $y$ goes simultaneously to $1$ with the ratio

$$X^2 = \frac{1 - y^2}{x^2 - 1} \quad (3.1)$$

held fixed. Then,

$$f \rightarrow -q^2 X^2, \quad \Pi \rightarrow -q X^2 \lambda(p),$$

$$\Sigma \rightarrow \frac{p\lambda(p)}{2} + q^2 (1 + p) X^2 + \frac{q^4}{4} X^4, \quad (3.2)$$

with

$$\lambda(p) = \frac{(1 + p)(8 - 4p + 5p^2 - p^3)}{2p} \geq 8 \quad (3.3)$$

(the lower bound being attained in the limit $q \rightarrow 0$). Rewriting the metric in the ADM form as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (3.4)$$

we then see that

$$g_{\varphi \varphi} = \kappa^2 \left[ \frac{\Sigma}{f}(x^2 - 1)(1 - y^2) - \frac{\Pi^2}{\Sigma f} \right] \quad (3.5)$$
goes over to a positive function of $X$, while the lapse function

$$N^2 = \frac{\kappa^2(x^2 - 1)(1 - y^2)}{g_{\varphi \varphi}}$$

(3.6)
develops a double zero at $x = 1$, $y = \pm 1$, corresponding to two double horizons, co-rotating at the angular velocity

$$\Omega_H = -N\varphi|_H = -\frac{\kappa \Pi}{\Sigma g_{\varphi \varphi}}|_H = \frac{f}{\kappa \Pi}|_H = \frac{q}{\kappa \lambda(p)}$$

(3.7)

To investigate the geometry of the horizons, we follow [7] and transform from the prolate spheroidal coordinates to the coordinates $X$ (the positive square root of (3.1)) and $Y = y/x$. On the horizons $Y = \pm 1$, the metric degenerates to

$$ds^2 = \frac{4\kappa^2 \Sigma(X) dX^2}{p^2(X^2 + 1)^4} + \frac{\kappa^2 \lambda^2(p) X^2 d\hat{\varphi}^2}{\Sigma(X)},$$

(3.8)
in the co-rotating near-horizon frame ($\hat{t}, X, Y, \hat{\varphi}$) defined by $\hat{t} = t$, $\hat{\varphi} = \varphi - \Omega_H t$. Introducing then a new angular coordinate $\eta$ by

$$X = \tan(\eta/2) \quad (0 \leq \eta \leq \pi),$$

(3.9)

(3.8) can be rewritten as

$$ds^2 = \frac{\kappa^2 \lambda(p)}{2 l(\eta)} (d\eta^2 + l^2(\eta) \sin^2 \eta d\hat{\varphi}^2),$$

(3.10)
where

$$l(\eta) = \frac{p \lambda(p)}{2} \frac{(X^2 + 1)^2}{\Sigma(X)}$$

(3.11)
is everywhere positive and finite, and such that $l(0) = 1$. It follows that each horizon is homeomorphic to $S^2$, with an area

$$A = 4\pi \kappa^2 \frac{\lambda(p)}{2},$$

(3.12)
The corresponding areal radius is of the order of the total mass $M$. At $\eta = \pi$ ($X \to \infty$), the metric (3.10) presents a conical singularity with deficit angle $2\pi(1 - \alpha)$, where

$$\alpha = l(\pi) = \frac{2 p \lambda(p)}{q^4} > \frac{8}{q^4} > 8.$$  

(3.13)

The evaluation of the electromagnetic potential on the horizon leads to

$$\hat{A}_H = -\varepsilon \left( \frac{q^2(2 - p)}{2 \lambda(p)} dt + \frac{\kappa q}{4} \frac{\delta(p) X^2 + q^2 \gamma(p) X^4}{\Sigma(X)} d\hat{\varphi} \right)$$

(3.14)
in the co-rotating frame, with

$$\gamma(p) = \frac{(1 + p)(4 - p + p^2)}{p}, \quad \delta(p) = \frac{(1 + p)^2(8 - p^2 + p^3)}{p}.$$  

(3.15)
Using the Tomimatsu formula [11]

$$Q_H = -\frac{1}{4\pi} \oint_H \omega d\text{Im}\psi d\varphi,$$

(3.16)
with $\omega_H = 1/\Omega_H$, we find that the horizons carry electric charges

$$Q_+ = Q_- = -\frac{\varepsilon \kappa(1 + p)}{2},$$

(3.17)
which also means that, to ensure global electric neutrality, the string must be also charged, which we
shall check in the next section. The vector potential (3.14) generates a magnetic field perpendicular
to the horizon. Because the normals to the two horizons \( Y = 1 \) and \( Y = -1 \) are oppositely oriented
and the net magnetic charge is zero, the magnetic lines of force must emerge from one horizon and
flow into the other horizon, so that the two horizons can be considered as carrying exactly opposite
magnetic charges

\[
P_+ = -P_- = \frac{1}{4\pi} \oint_{H_+} dA_\varphi d\varphi = \frac{\varepsilon \kappa (p)}{2q}.
\]

(3.18)

Using the Ostrogradsky theorem and the Einstein-Maxwell equations, the Komar mass and angular
momentum at infinity

\[
M = \frac{1}{4\pi} \oint_{\infty} D^\mu k^\nu d\Sigma_{\mu\nu}, \quad J = -(1/8\pi) \oint_{\infty} D^\mu m^\nu d\Sigma_{\mu\nu}
\]

(3.19)

\((k^\mu = \delta^\mu_\tau, m^\mu = \delta^\mu_\varphi)\) can be transformed \([11]\) into the sums over the boundary surfaces (here, the two
horizons and the string) \( M = \sum_n M_n, J = \sum_n J_n, \) with

\[
M_n = \frac{1}{8\pi} \oint_{\Sigma_n} \left[ g^{ij} g^a \partial_j g_{a\mu} + 2(A_t F^{it} - A_\varphi F^{i\varphi}) \right] d\Sigma_i.
\]

\[
J_n = -\frac{1}{16\pi} \oint_{\Sigma_n} \left[ g^{ij} g^a \partial_j g_{a\mu} + 4A_\varphi F^{it} \right] d\Sigma_i.
\]

(3.20)

As shown by Tomimatsu, these reduce on the horizons to

\[
M_H = \frac{1}{8\pi} \oint_H \left[ \omega d \text{Im}\mathcal{E} + 2d(A_\varphi \text{Im}\mathcal{\psi}) \right] d\varphi,
\]

\[
J_H = \frac{1}{8\pi} \oint_H \omega \left[ \frac{1}{2} \omega d \text{Im}\mathcal{E} + d(A_\varphi \text{Im}\mathcal{\psi}) + \omega A_t \text{dIm}\mathcal{\psi} \right] d\varphi,
\]

(3.21)

which yield

\[
M_+ = M_- = \frac{\kappa p}{p} + \frac{\kappa p}{2},
\]

(3.22)

\[
J_+ = J_- = \frac{\kappa^2}{8qp} \left[ 2\lambda(p)(2 + p^2) + q^2 p(1 + p)(2 - p) \right].
\]

(3.23)

The horizon mass (3.22) is larger than half of the global mass \( 2\kappa/p \), so the string must have negative
mass. Finally, we will show in the next section that the two horizons carry opposite NUT charges

\[
-N_+ = N_- = \frac{\kappa \lambda(p)}{4q}.
\]

(3.24)

4 The string

For \( \xi^2 \equiv x^2 - 1 \to 0 \) (with \( y^2 < 1 \)), the solution (2.3) reduces to:

\[
ds^2 \sim -\frac{\kappa^2 q^2}{4} (1 - y^2)^2 d\varphi^2 + \frac{\kappa^2 q^4}{p^2 (1 - y^2)} \left[ \frac{dy^2}{1 - y^2} \right.
\]

\[
+ d\xi^2 + \alpha^2 \Omega_H^2 \xi^2 \left( dt - \kappa \left( \frac{\lambda(p)}{q} + q(1 - p/2)(1 - y^2) \right) d\varphi \right)^2 \right]
\]

\[
A \sim \varepsilon \left[ 1 - \frac{2(1 + p) \xi^2}{q^2 (1 - y^2)} \right] dt - \kappa \left( \frac{\gamma(p)}{q} + \frac{q(1 - y^2)}{2} \right) d\varphi,
\]

(4.1)

(4.2)
where we have neglected irrelevant terms of order \( \xi^2 \) and higher. The singularity at \( \xi = 0 \) is obviously a conical one, with finite Ricci square scalar

\[
R^{\mu \nu} R_{\mu \nu} \sim \frac{64 p^4}{\kappa^4 q^2} \left( 1 + p \right)^2 + q^2 y^2 \right)^2.
\]

Transforming to the horizon co-rotating frame by \( d\phi = d\hat{\phi} + \Omega_H dt \), we find that the near-string metric (4.1) transforms to that of a spinning cosmic string [12, 13] in a curved spacetime,

\[
ds^2 \sim q^4 \left[ \frac{1 - y^2}{4\lambda^2(p)} \left( dt + \Omega_H^{-1} d\hat{\phi} \right)^2 + \frac{\kappa^2}{p^2(1 - y^2)} \left( \frac{dy^2}{1 - y^2} + d\xi^2 + \alpha^2 \xi^2 d\varphi^2 \right) \right],
\]

with (negative) tension per unit length \((1 - \alpha)/4\) where \( \alpha \) is given in (3.13), and “spin” \( \Omega_H^{-1}/4 \) where \( \Omega_H \) is given in (3.7).

In view of the fact that the finite-length string connects two black holes, this spin should actually be interpreted as a gravimagnetic flow along the Misner string connecting two opposite NUT sources at \( \rho = 0, z = \pm \kappa \), with the gravimagnetic potential \( \omega/2 = N_+ \cos \theta_+ + N_- \cos \theta_- \) where \( \cos \theta_{\pm} = \mp 1 \) along the string, with \( N_{\pm} \) given by (3.24). Similarly, the constant contribution \(-\varepsilon \kappa \gamma(p)/q\) to \( \mathcal{A}_\varphi \) in (4.2) should be interpreted as the magnetic flow along a Dirac string connecting two opposite monopoles at \( z = \pm \kappa \) with magnetic charges given by (3.18). The non-constant contribution gives rise to a magnetic field density \( \sqrt{|g|}B^\xi = F_{y\varphi} = \varepsilon \kappa q y \), which leads to an intrinsic string magnetic moment

\[
\mu_S = \frac{1}{4\pi} \int_{-1}^{+1} \sqrt{|g|} B^\xi z 2\pi dy = \varepsilon \kappa^2 q \frac{3}{3} = \frac{\mu}{3}.
\]

(to obtain the total magnetic moment \( \mu \), the magnetic dipole contribution \( 2\kappa P_H \) and the sum of the horizon magnetic moments must be added to this).

Although it is not immediately obvious from (4.2), the string also carries an electric charge. The near-string covariant component \( F_{t\xi} \) of the radial electric field vanishes to order \( O(\xi) \), but on account of \( g_{tt} = O(\xi^2) \) and \( \sqrt{|g|} = O(\xi) \), the radial electric field density \( \sqrt{|g|} F_{t\xi} \) is finite and constant along the string, leading to the electric charge

\[
Q_S = \frac{1}{4\pi} \int_{-1}^{+1} \sqrt{|g|} F_{t\xi} 2\pi dy = \varepsilon \kappa (1 + p).
\]

This string electric charge together with the horizon electric charges lead to a vanishing total electric charge

\[
Q_+ + Q_- + Q_S = 0,
\]

a vanishing electric dipole moment, and a contribution to the total electric quadrupole moment, to which must be added that of the two opposite horizon electric dipole moments generated by the rotation of the horizon magnetic charges, and the sum of the horizon electric quadrupole moments.

The string mass and angular momentum can be evaluated from (3.20) integrated over the string \( x = 1, -1 < y < 1 \), and are the sum of gravitational and electromagnetic contributions. Although in the co-rotating frame the string is a spinning cosmic string with negative tension, and thus presumably negative gravitational mass, in the global frame the gravitational contribution to the string mass is – surprisingly – positive. However it is overwhelmed by the negative electromagnetic contribution \(-Q_S A_t(\xi = 0)\), resulting in a net negative string mass

\[
M_S = \kappa - \kappa(1 + p) = -\kappa p,
\]
which represents the binding energy between the two black holes of mass \((\kappa/p + \kappa p/2)\), leading to the total mass

\[
M = M_+ + M_- + M_S = \frac{2\kappa}{p}.
\]  

(4.9)

The fact that the string mass is negative explains the repulsion experienced by test particles in geodesic motion near the string (antigravity).

Similarly, the string angular momentum is the sum of gravitational and electromagnetic contributions

\[
J_S = \kappa^2 \left[ \frac{\lambda(p)}{2q} + \frac{q}{3} \left( 1 - \frac{p}{2} \right) \right] - \kappa^2 (1 + p) \left[ \frac{\gamma(p)}{q} + \frac{q}{3} \right] 
\]

\[
= \frac{\kappa^2}{2q} \left[ \lambda(p) - 2(1 + p)\gamma(p) - pq^2 \right].
\]  

(4.10)

The first term \(\kappa^2 \lambda(p)/2q\) is the NUT dipole \(2\kappa N_H\), the remainder corresponding to the intrinsic string angular momentum. It can be checked that the horizon angular momenta (3.23) and the total string angular momentum (4.10) add up to the net angular momentum (2.5):

\[
J = J_+ + J_- + J_S = \frac{\kappa^2 q(4 + p^2)}{p^2}.
\]  

(4.11)

## 5 Summary and outlook

Summarizing the features of the solution, we can present it schematically as shown in Fig.1. It is a system of two extreme co-rotating black holes endowed with masses, NUT charges, electric and magnetic charges held apart by a magnetized, electrically charged string of negative tension, which is also a Dirac and Misner string. The whole system lies inside the ergosphere. As the Misner string does not extend to infinity, the solution is asymptotically flat. There are no strong curvature singularities, while the string, which can formally be considered as a mild naked singularity, is inaccessible from infinity. The charges compensate each other so that the asymptotic parameters are the mass, the angular momentum and the gravitational and electromagnetic multipole moments. The family interpolates between the static vacuum ZV2 solution for \(q = 0\) and extreme Kerr for \(q = 1\). As usual, the Misner string is surrounded by a region containing CTCs. We have argued elsewhere [14, 15] that NUT-related CTCs do not necessarily lead to observable violations of causality.

As will be discussed elsewhere [16], this solution can be analytically continued beyond the horizons. The most economical maximal analytical extension contains two interior regions between an outer and an inner horizon (both degenerate), and beyond the inner horizons a third region extending to spacelike infinity and containing a timelike ring singularity.

Comparing with other known stationary solutions describing two-black hole systems, we think that this one has a minimal number of physically undesirable features and can be considered as “almost” physical. This surprising property is presumably related to the specific generating technique of [9] which endows the static seed solution with rotation and charges, all depending on a single parameter. Applying this transformation to the general-\(\gamma\) ZV solution generically gives singular space-times with novel features [16], which may be interesting in view of recent discussions of alternatives to Kerr in astrophysical modeling [8]. It is worth noting that our solution belongs to a subclass of the nine-parameter family of solutions of the Einstein-Maxwell equations constructed in [17], whose physical features are still unexplored.

### Acknowledgments

DG thanks LAPTh Annecy-le-Vieux for hospitality at different stages of this work. DG also acknowledges the support of the Russian Foundation of Fundamental Research under the project 17-02-01299a
Figure 1: Schematic picture of the solution.

and the Russian Government Program of Competitive Growth of the Kazan Federal University.

References

[1] J.B. Griffiths and J. Podolsky, Exact space-times in Einstein’s General Relativity, CUP, 2009.
[2] W.B. Bonnor, Z. Phys. 190, 444 (1966).
[3] D.M. Zipoy, Journ. Math. Phys. 7, 1137 (1966).
[4] B.H. Voorhees, Phys. Rev. D2, 2119 (1970).
[5] A. Tomimatsu and H. Sato, Phys. Rev. Lett. 29, 1344 (1972); Progr. Theor. Phys. 50, 95 (1973).
[6] R. Emparan, Phys. Rev. D61, 104009 (2000) [arXiv:hep-th/9906160].
[7] H. Kodama and W. Hikida, Class. Quantum Grav. 20, 5121 (2003) [arXiv: gr-qc/0304064].
[8] K. Yagi and L. C. Stein, Class. Quant. Grav. 33, 054001 (2016) [arXiv:1602.02413 [gr-qc]].
[9] G. Clément, Phys. Rev. D 37, 4885 (1998) [arXiv:gr-qc/9710109].
[10] J.D. Barrow and G.W. Gibbons, Phys. Rev. D 95, 064040 (2017) [arXiv:1701.06343].
[11] A. Tomimatsu, Progr. Theor. Phys. 72, 73 (1984).
[12] S. Deser, R. Jackiw and G. ’t Hooft, Annals Phys. 152, 220 (1984).
[13] G. Clément, Int. J. Theor. Phys 24, 267 (1985).
[14] G. Clément, D. Gal’tsov and M. Guenouche, Phys. Lett. B 750, 591 (2015) [arXiv:1508.07622 [hep-th]].

[15] G. Clément, D. Gal’tsov and M. Guenouche, Phys. Rev. D 93, 024048 (2016) [arXiv:1509.07854 [hep-th]].

[16] G. Clément and D. Gal’tsov, in preparation.

[17] V. S. Manko, J. D. Sanabria-Gomez and O. V. Manko, Phys. Rev. D 62, 044048 (2000).