Conditional Infimum and Hidden Convexity in Optimization

Jean-Philippe Chancelier and Michel De Lara,
CERMICS, Ecole des Ponts, Marne-la-Vallée, France

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Abstract

Detecting hidden convexity is one of the tools to address nonconvex minimization problems. After giving a formal definition of hidden convexity, we introduce the notion of conditional infimum, as it will prove instrumental in detecting hidden convexity. We develop the theory of the conditional infimum, and we establish a tower property, relevant for minimization problems. Thus equipped, we provide a sufficient condition for hidden convexity in nonconvex minimization problems. We illustrate our result on nonconvex quadratic minimization problems. We conclude with perspectives for using the conditional infimum in relation to the so-called S-procedure, to couplings and conjugacies, and to lower bound convex programs.

1 Introduction

Convex minimization problems display well-known features that make their numerical resolution appealing. In particular, convex minimization algorithms are known to be simpler and less computationally intensive, in comparison with nonconvex ones. Thus, it is tempting to “convexify” a problem in order to solve it, rather than to use nonconvex optimization. More generally, it has long been searched how to relate a nonconvex minimization problem to a convex one. If the original nonconvex minimization problem is formulated on a convex set, then the convex lower envelope of the objective function has the same minimum and a solution (argmin) of the original nonconvex problem is solution of the convex lower envelope problem [8, Proposition 11]. Needless to say that computing the lower envelope can be at least as difficult as solving the original nonconvex problem. This is why other approaches have been developed, like convexification by domain or range transformation as exposed in [8] which provides a survey. The vocable of “hidden convexity” covers different approaches: duality and biduality analysis like in [6]; identifying classes of nonconvex optimization problems whose convex relaxations have optimal solutions which at the same time are global optimal solutions of the original nonconvex problems [5]. A survey of hidden convex optimization can be found in [15], with its focus on three widely used ways to reveal the hidden
convex structure for different classes of nonconvex optimization problems. In this paper, we propose a definition of “hidden convexity” and a new way to reveal it by means of what we call the “conditional infimum”. We discuss these two points now.

Regarding hidden convexity, we consider a set $W$, a function $h : W \rightarrow \mathbb{R}$ and a subset $W \subset W$. We say that the minimization problem $\min_{w \in W} h(w)$ displays hidden convexity if there exists a vector space $X$, a convex function $f : X \rightarrow \mathbb{R}$ and a convex subset $C \subset X$ such that $\min_{w \in W} h(w) = \min_{x \in C} f(x)$. So, in the definition we propose, the original minimization problem is not formulated on a vector space, even less on a convex domain.

Now for the conditional infimum. The operation of marginalization (that is, partial minimization as in [13, Theorem 5.3]) is widely used in optimization, especially in the context of studying how optimal values and optimal solutions depend on the parameters in a given problem. Another operation through which new functions are constructed by minimization is the so-called epi-composition, developed by Rockafellar (see [12, p. 27] and the historical note in [12, p. 36]); epi-composition is called infimal postcomposition in [4, p. 214]. Notice that the vocable of marginalization hinges at a corresponding operation (of partial integration) in probability theory. A nice parallelism between optimization and probability theories has been pointed out by several authors [9, 2]. Following this approach, we have relabelled epi-composition as conditional infimum in [7, Definition 2.4], with the notation $\inf[f \mid \theta]$. The expression “conditional infimum” appears in the conclusion part of [14], where it is defined with respect to a partition field, that is, a subset of the power set which is closed w.r.t. (with respect to) union and intersection, countable or not; however, the theory is not developed. Related notions can be found — but defined on a measurable space equipped with a unitary Maslov measure — in the following works: in [9], the theory of performance is sketched and the “conditional performance” is defined; in [2], the “conditional cost excess” is defined; in [3], the “conditional essential supremum” is defined. In this paper, we define the conditional infimum with respect to a correspondence between two sets, without requiring measurable structures, and we study its properties in the perspective of applications to optimization.

The paper is organized as follows. In Sect. 2, we provide a definition of the conditional infimum (and supremum) of a function with respect to a correspondence (between two sets), followed by examples and main properties. In Sect. 3 we develop applications of the conditional infimum to minimization problems. In Sect. 4 we provide a sufficient condition for hidden convexity in nonconvex minimization problems, and we illustrate our result on nonconvex quadratic minimization problems. In the concluding Sect. 5 we point out perspectives for using the conditional infimum in other contexts, namely in relation to the so-called S-procedure, to couplings and conjugacies, and to lower bound convex programs.

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1In this paper, we address hidden convexity in optimisation. In [7], we dealt with the stronger notion of hidden convexity in a function, that we characterized by means of one-sided linear conjugacies. Let $\theta : W \rightarrow X$ be a mapping. We say that the function $h : W \rightarrow \mathbb{R}$ displays hidden convexity with respect to the mapping $\theta$ if there exists a vector space $X$ and a convex function $f : X \rightarrow \mathbb{R}$ such that $h = f \circ \theta$. Thus, hidden convexity in a function resorts to a property of convex factorization.
2 Conditional infimum with respect to a correspondence

In §2.1, we provide a definition of the conditional infimum (and supremum) of a function with respect to a correspondence between two sets, followed by examples in §2.2. Then, we expose properties of the conditional infimum and supremum in §2.3.

As we manipulate functions with values in $\mathbb{R} = [-\infty, +\infty]$, we adopt the Moreau lower ($\underline{\cdot}$) and upper ($\overline{\cdot}$) additions [10], which extend the usual addition ($+$) with $(+\infty) + (-\infty) = -\infty$ and $(+\infty) + (-\infty) = (+\infty) = +\infty$.

2.1 Definitions of the conditional infimum and supremum

We now give formal definitions of the conditional infimum and supremum with respect to a correspondence between two sets. In optimization, one is more familiar with set-valued mappings [12, Chapter 5] than with correspondences, though the two notions are essentially equivalent. We favor the notion of correspondence because, regarding conditional infimum, we obtain a nicer formula with the composition of correspondences than with the composition of set-valued mappings (see Footnote 7).

Recalls on correspondences. We recall that a correspondence $\mathcal{R}$ between two sets $\mathbb{U}$ and $\mathbb{V}$ is a subset $\mathcal{R} \subset \mathbb{U} \times \mathbb{V}$. We denote $u \mathcal{R} v \iff (u, v) \in \mathcal{R}$. A foreset of a correspondence $\mathcal{R}$ is any set of the form $\mathcal{R} v = \{ u \in \mathbb{U} \mid \exists v \in \mathbb{V}, u \mathcal{R} v \}$, where $v \in \mathbb{V}$, or, by extension, of the form $\mathcal{R} V = \{ u \in \mathbb{U} \mid \exists v \in \mathbb{V}, u \mathcal{R} v \}$, where $V \subset \mathbb{V}$. An afterset of a correspondence $\mathcal{R}$ is any set of the form $u \mathcal{R} = \{ v \in \mathbb{V} \mid u \mathcal{R} v \}$, where $u \in \mathbb{U}$, or, by extension, of the form $U \mathcal{R} = \{ v \in \mathbb{V} \mid \exists u \in \mathbb{U}, u \mathcal{R} v \}$, where $U \subset \mathbb{U}$. The domain and the range of the correspondence $\mathcal{R}$ are given respectively by $\text{dom} \mathcal{R} = \{ u \in \mathbb{U} \mid \text{u} \mathcal{R} \neq \emptyset \}$ and $\text{range} \mathcal{R} = \{ v \in \mathbb{V} \mid \mathcal{R} v \neq \emptyset \}$. We denote by $\mathcal{R}^{-1} \subset \mathbb{V} \times \mathbb{U}$ the correspondence between the two sets $\mathbb{V}$ and $\mathbb{U}$ given by $v \mathcal{R}^{-1} u \iff u \mathcal{R} v$. For any pair of correspondences $\mathcal{R}$ on $\mathbb{U} \times \mathbb{V}$ and $\mathcal{S}$ on $\mathbb{V} \times \mathbb{W}$, the composition $\mathcal{R} \mathcal{S}$ denotes the correspondence between the two sets $\mathbb{U}$ and $\mathbb{W}$ given by, for any $(u, w) \in \mathbb{U} \times \mathbb{W}$, $u \mathcal{R} \mathcal{S} w \iff \exists v \in \mathbb{V}$ such that $u \mathcal{R} v$ and $v \mathcal{S} w$.

Optimization over a subset. Let $f : \mathbb{U} \to \mathbb{R}$ be a function. Like in Probability theory, where one starts by defining the conditional probability w.r.t. a subset of the sample space, we define

$$\inf \left[ f \mid U \right] = \inf_{u \in U} f(u), \quad \sup \left[ f \mid U \right] = \sup_{u \in U} f(u), \quad \forall U \subset \mathbb{U}. \quad (1)$$

Even if we draw parallels between optimization and probability theories, we do not develop the parallelism to its potential full extent as, for instance, we do not consider the equivalent of a generic probability distribution. Compared to [9, 2, 1, 3] which consider Maslov measures and densities — that is, an analog of probability measures — we could say that, in this paper, we only focus on the theory of the conditional infimum/supremum for the analog of the uniform probability (see the introduction of [1]).
To complete the link with optimization under constraint, we also define
\[
\arg\min_{u \in U} [f | U] = \arg\min_{u \in U} f(u) , \quad \arg\max_{u \in U} [f | U] = \arg\max_{u \in U} f(u) , \quad \forall U \subset \mathbb{U} .
\] (2)

**Definition of conditional infimum w.r.t. a correspondence.** In the existing definitions of the conditional infimum of a function in the literature \[9\] \[2\] \[3\], both the original function and its conditional infimum are defined on a measurable space equipped with a unitary Maslov measure. By contrast, our new definition (below) of the conditional infimum of a function does not require a measurable space (nor a Maslov measure) but a correspondence between two sets, a source set and a target set; what is more, for a function whose domain is the source set, its conditional infimum is defined on the target set.

**Definition 1** Let \( f : U \to \mathbb{R} \) be a function and \( \mathcal{R} \) be a correspondence between the sets \( U \) and \( \mathcal{V} \). We define the conditional infimum of the function \( f \) with respect to the correspondence \( \mathcal{R} \) (and resp. the conditional supremum) as the functions \( \inf [f | \mathcal{R}] : \mathcal{V} \to \mathbb{R} \) (and resp. \( \sup [f | \mathcal{R}] : \mathcal{V} \to \mathbb{R} \)) given by

\[
\begin{align*}
\inf [f | \mathcal{R}] : \mathcal{V} \to \mathbb{R} , & \quad \inf [f | \mathcal{R}] (v) = \inf \left[ f \right| \mathcal{R} \{ v \} , \quad \forall v \in \mathcal{V} , \\
\sup [f | \mathcal{R}] : \mathcal{V} \to \mathbb{R} , & \quad \sup [f | \mathcal{R}] (v) = \sup \left[ f \right| \mathcal{R} \{ v \} , \quad \forall v \in \mathcal{V} ,
\end{align*}
\] (3a)

where we have used the notation \[\|\].

We adopt the conventions\[3\] that \[12\] p. 1
\[
\inf_{\emptyset} f = \inf_{u \in \emptyset} f(u) = +\infty \quad \text{and} \quad \sup_{\emptyset} f = \sup_{u \in \emptyset} f(u) = -\infty .
\] (4)

As a consequence of (3a) and (4), the conditional infimum takes the value \(+\infty\) (and the conditional supremum takes the value \(-\infty\)) outside \( \text{range} \mathcal{R} \), that is,

\[
\begin{align*}
\inf [f | \mathcal{R}] (v) = +\infty , & \quad \forall v \notin \text{range} \mathcal{R} , \\
\sup [f | \mathcal{R}] (v) = -\infty , & \quad \forall v \notin \text{range} \mathcal{R} .
\end{align*}
\] (5a)

Recall that the **effective domain** of a function \( g : \mathcal{V} \to \mathbb{R} \) is \( \text{dom} g = \{ v \in \mathcal{V} | g(v) < +\infty \} \).

Therefore, regarding the effective domain, we have the inclusion\[4\]
\[
\text{dom} (\inf [f | \mathcal{R}]) \subset \text{range} \mathcal{R} = \text{dom} \mathcal{R}^{-1} .
\] (6)

All properties about the conditional infimum are easily carried to the conditional supremum (and conversely) because
\[
- \sup [f | \mathcal{R}] = \inf [-f | \mathcal{R}] , \quad - \inf [f | \mathcal{R}] = \sup [-f | \mathcal{R}] .
\] (7)

In the sequel, we will favour the conditional infimum as we are interested in applications to minimization problems.

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\[3\] Such conventions arise naturally as the mapping \( U \in 2^U \mapsto \inf_U f = \inf_{u \in U} f(u) \) is nonincreasing and as the mapping \( U \in 2^U \mapsto \sup_U f = \inf_{u \in U} f(u) \) is nondecreasing. However, one has to be careful because \( \inf_U f \leq \sup_U f \) if \( U \neq \emptyset \), but \( +\infty = \inf_{\emptyset} f > \sup_{\emptyset} f = -\infty \).

\[4\] To the left hand side of the inclusion, the notation \( \text{dom} \) refers to the effective domain of a function, whereas to the right hand side, the notation \( \text{dom} \) refers to the domain of a correspondence.
Example. Let $\mathcal{D} \subset \mathbb{R}^d \times \mathbb{R}^d$ be the binary relation (hence, correspondence) given by $x \mathcal{D} x' \iff \exists \lambda \in \mathbb{R} \setminus \{0\}, \ x = \lambda x'$. Thus, $\mathcal{D}$ is the equivalence relation on $\mathbb{R}^d$ whose classes are $\{0\}$ and the (unoriented) directions of $\mathbb{R}^d$. Let $\| \cdot \|$ be the Euclidian norm on $\mathbb{R}^d$, $A$ be a matrix with $d$ rows and $p$ columns, and $b \in \mathbb{R}^p$ be a vector. If we set $f(x) = \|Ax - b\|^2$, for $x \in \mathbb{R}^d$, an easy computation leads to

$$
(\forall x \in \mathbb{R}^d) \quad \inf \left[ f \big| \mathcal{D} \right](x) = \begin{cases} \|b\|^2 & \text{if } Ax = 0, \\ \|b\|^2 - \frac{\langle Ax, b \rangle^2}{\|Ax\|^2} & \text{if } Ax \neq 0. \end{cases}
$$

Conditional infimum w.r.t. a correspondence induced by a set-valued mapping $\Theta : U \rightrightarrows V$. Let $\Theta : U \rightrightarrows V$ be a set-valued mapping, that is, $\Theta : U \rightarrow 2^V$. We define the graph of $\Theta$ by

$$
\mathcal{G}_\Theta = \{(u, v) \in U \times V \mid v \in \Theta(u)\} \subset U \times V.
$$

As the graph $\mathcal{G}_\Theta$ defines a correspondence between the two sets $U$ and $V$, we introduce specific definitions and notations. For any function $f : U \rightarrow \mathbb{R}$, we define the conditional infimum $\inf \left[ f \big| \Theta \right] : V \rightarrow \mathbb{R}$ of the function $f$ with respect to the set-valued mapping $\Theta$ (and the conditional supremum $\sup \left[ f \big| \Theta \right] : V \rightarrow \mathbb{R}$) by

$$
\begin{align*}
\inf \left[ f \big| \Theta \right](v) &= \inf \left[ f \big| \mathcal{G}_\Theta \right](v), \ \forall v \in V, \\
\sup \left[ f \big| \Theta \right](v) &= \sup \left[ f \big| \mathcal{G}_\Theta \right](v), \ \forall v \in V.
\end{align*}
$$

Conditional infimum w.r.t. a correspondence induced by a set-valued mapping $\Phi : V \rightrightarrows U$ (the other way round). Let $\Phi : V \rightrightarrows U$ be a set-valued mapping. Beware that, to the difference of the set-valued mapping $\Theta : U \rightrightarrows V$ above, the source set is $V$ and the target set is $U$. We consider such set-valued mappings to make the connection with how they are used in optimization to handle constraints (see Footnote 5). Regarding the graph of $\Phi$ in (8), we have

$$
\mathcal{G}_\Phi = \{(v, u) \in V \times U \mid u \in \Phi(v)\} \subset V \times U,
$$

and, defining $\Phi^{-1} : U \rightrightarrows V$ by $\Phi^{-1}(u) = \{v \in V \mid u \in \Phi(v)\}$, we get that

$$
(\mathcal{G}_\Phi)^{-1} = \{(u, v) \in U \times V \mid u \in \Phi(v)\} = \mathcal{G}_{\Phi^{-1}} \subset U \times V.
$$

For any function $f : U \rightarrow \mathbb{R}$, we have the properties\footnote{Set-valued mappings are used in optimization because they offer a handy way to denote constraints as in the left hand side expressions in (11). We will explain in Footnote 4 why we have chosen to favor correspondences rather than set-valued mappings.}

\begin{align*}
\inf_{u \in \Phi(v)} f(u) &= \inf \left[ f \big| \mathcal{G}_{\Phi^{-1}} \right](v) = \inf \left[ f \big| \Phi^{-1} \right](v), \ \forall v \in V, \\
\sup_{u \in \Phi(v)} f(u) &= \sup \left[ f \big| \mathcal{G}_{\Phi^{-1}} \right](v) = \sup \left[ f \big| \Phi^{-1} \right](v), \ \forall v \in V,
\end{align*}

where we have used the notations in (9).
Conditional infimum w.r.t. a correspondence induced by a mapping $\theta : U \to \mathbb{V}$. Let $\theta : U \to \mathbb{V}$ be a mapping. The graph of $\theta$ in (8) is now

$$\mathcal{G}_\theta = \{ (u, v) \in U \times \mathbb{V} \mid \theta(u) = v \} \subset U \times \mathbb{V} \ .$$

For any function $f : U \to \mathbb{R}$, Equations (9) give (with the mapping $\theta$ identified with the set-valued mapping $u \mapsto \{ \theta(u) \}$)

$$\begin{align*}
\inf [f \mid \theta](v) &= \inf [f \mid \mathcal{G}_\theta](v) = \inf_{u \in \theta^{-1}(v)} f(u) \ , \ \forall v \in \mathbb{V} \ , \quad (13a) \\
\sup [f \mid \theta](v) &= \sup [f \mid \mathcal{G}_\theta](v) = \sup_{u \in \theta^{-1}(v)} f(u) \ , \ \forall v \in \mathbb{V} \ . \quad (13b)
\end{align*}$$

As $\theta : U \to \mathbb{V}$ is a mapping, it induces a set-valued mapping $\theta^{-1} : \mathbb{V} \rightrightarrows U$. Using Equations (10), for any function $g : \mathbb{V} \to \mathbb{R}$, we have the following properties:

$$\begin{align*}
\inf [g \mid \theta^{-1}](u) &= \inf [g \mid \mathcal{G}_{\theta^{-1}}](u) = \inf_{v = \theta(u)} g(v) = (g \circ \theta)(u) \ , \ \forall u \in U \ , \quad (14a) \\
\sup [g \mid \theta^{-1}](u) &= \sup [g \mid \mathcal{G}_{\theta^{-1}}](u) = \sup_{v = \theta(u)} g(v) = (g \circ \theta)(u) \ , \ \forall u \in U \ . \quad (14b)
\end{align*}$$

2.2 Examples

Examples with characteristic functions. For any subset $W \subset \mathbb{W}$ of a set $\mathbb{W}$, $\delta_W : \mathbb{W} \to \mathbb{R}$ denotes the characteristic function of the set $W$: $\delta_W(w) = 0$ if $w \in W$, and $\delta_W(w) = +\infty$ if $w \notin W$.

For any correspondence $\mathfrak{R}$ between the sets $U$ and $\mathbb{V}$, and for any $u \in U$ and any subset $V \subset \mathbb{V}$, we have that,

$$\begin{align*}
\inf [\delta_{\{u\}} \mid \mathfrak{R}] &= \delta_{\{u\}} \ , \\
\sup [-\delta_{\{u\}} \mid \mathfrak{R}] &= -\delta_{\{u\}} \\
\inf [\delta_U \mid \mathfrak{R}] &= \delta_{U \mathfrak{R}} \ , \\
\sup [-\delta_U \mid \mathfrak{R}] &= -\delta_{U \mathfrak{R}} .
\end{align*}$$

As a consequence, the conditional infimum and supremum with respect to a correspondence characterize this latter, as the mappings $\mathfrak{R} \mapsto \inf [\cdot \mid \mathfrak{R}]$ and $\mathfrak{R} \mapsto \sup [\cdot \mid \mathfrak{R}]$ are injective.

Examples with rectangular correspondences. For any two subsets $U \subset \mathbb{U}$ and $V \subset \mathbb{V}$, we define the rectangle correspondence $U \times V$. Then, for any function $f : U \to \mathbb{R}$, we have that

$$\begin{align*}
\inf [f \mid U \times V](v) &= \begin{cases} 
\inf_{u \in U} f(u) & \text{if } v \in V \ , \\
+\infty & \text{if } v \notin V \ ;
\end{cases} \\
\sup [f \mid U \times V](v) &= \begin{cases} 
\sup_{u \in U} f(u) & \text{if } v \in V \ , \\
-\infty & \text{if } v \notin V \ ;
\end{cases}
\end{align*}$$

That is, $\inf [f \mid U \times V] = \inf [f \mid U] + \delta_V$, and $\sup [f \mid U \times V] = \sup [f \mid U] + (-\delta_V)$.
Marginalization operations. Let $U$ and $V$ be two sets, $\Delta_V$ be the diagonal of $V^2$, and $U \times \Delta_V$ be the correspondence between the sets $U \times V$ and $V$ given by $(u, v) \mapsto (u \times \Delta_V) v' \iff v = v'$. The foresets of the correspondence $U \times \Delta_V$ satisfy $(U \times \Delta_V) v = U \times \{v\}$, for any $v \in V$. The following conditional infimum and supremum of a function $h : U \times V \to \mathbb{R}$ with respect to the correspondence $U \times \Delta_V$ provide the marginalization operations:

\[
\begin{align*}
\inf [h \mid U \times \Delta_V](v) &= \inf_{u \in U} h(u, v), \quad \forall v \in V, \quad (17a) \\
\sup [h \mid U \times \Delta_V](v) &= \sup_{u \in U} h(u, v), \quad \forall v \in V. \quad (17b)
\end{align*}
\]

2.3 Properties of the conditional infimum and supremum

We expose properties of the conditional infimum and supremum. We recall that the strict epigraph of a function $f : U \to \mathbb{R}$ is defined by the subset

\[
\text{epi}^+ f = \{ (u, t) \in U \times \mathbb{R} \mid f(u) < t \} \subset U \times \mathbb{R},
\]

hence $\text{epi}^+ f$ can be understood as a correspondence between $U$ and $\mathbb{R}$.

Proposition 2

1. **Strict epigraph:**
   for any function $f : U \to \mathbb{R}$ and for any correspondence $\mathcal{R}$ on $U \times V$, we have that
   \[
   \text{epi}^+ \inf [f \mid \mathcal{R}] = \mathcal{R}^{-1}(\text{epi}^+ f),
   \]
   where, on the right hand side, $\text{epi}^+ f$ is understood as a correspondence between $U$ and $\mathbb{R}$, and $\mathcal{R}^{-1}(\text{epi}^+ f)$ as a composition of two correspondences, hence as a subset of $V \times \mathbb{R}$.

2. **Linearity and sublinearity w.r.t. min-plus, $\land$ and $\lor$ operations:** for any correspondence $\mathcal{R}$ on $U \times V$, we have that
   \[
   \begin{align*}
   \text{• for any family } (f_i)_{i \in I} \text{ of functions } f_i : U \to \mathbb{R}, \\
   \inf_{i \in I} [\bigwedge_{i \in I} f_i \mid \mathcal{R}] &= \bigwedge_{i \in I} \inf [f_i \mid \mathcal{R}], \quad (20a) \\
   \bigvee_{i \in I} \inf [f_i \mid \mathcal{R}] &\leq \inf \left[ \bigvee_{i \in I} f_i \mid \mathcal{R} \right], \quad (20b) \\
   \text{• for any function } f : U \to \mathbb{R} \text{ and } r \in \mathbb{R}, \\
   \inf [f + r \mid \mathcal{R}] &= \inf [f \mid \mathcal{R}] + r, \quad (20c) \\
   \text{• for any functions } f : U \to \mathbb{R} \text{ and } h : U \to \mathbb{R}, \\
   \inf [f \mid \mathcal{R}] + \inf [h \mid \mathcal{R}] &\leq \inf [f + h \mid \mathcal{R}]. \quad (20d)
   \end{align*}
   \]
3. Monotony with respect to functions:
For any correspondence $R$ on $U \times V$, we have that

- for any functions $f : U \to \mathbb{R}$ and $h : U \to \mathbb{R}$,
  \[ f \leq h \implies \inf [f \mid R] \leq \inf [h \mid R], \quad (21a) \]
  \[ \inf [f \mid U] \leq \inf [f \mid RV] \leq \inf [f \mid R](v), \quad \forall v \in V, \quad (21b) \]
- for any function $f : U \to \mathbb{R}$ and for any nondecreasing function $\varphi : \mathbb{R} \to \mathbb{R}$,
  \[ \varphi \circ \inf [f \mid R] \leq \inf [\varphi \circ f \mid R]. \quad (21c) \]

4. Two correspondences on $U \times V$:
For any pair $R, S$ of correspondences on $U \times V$ and for any function $f : U \to \mathbb{R}$, we have that

\[ \inf [f \mid R \cup S] = \inf [f \mid R] \land \inf [f \mid S], \quad (22a) \]
\[ \inf [f \mid R \cap S] \geq \inf [f \mid R] \lor \inf [f \mid S], \quad (22b) \]
\[ R \subset S \implies \inf [f \mid R] \geq \inf [f \mid S]. \quad (22c) \]

5. Pushforward property\(^6\)
For any correspondence $R$ on $U \times V$, for any subset $V \subset V$ and for any function $f : U \to \mathbb{R}$, we have that

\[ \inf [\inf [f \mid R] \mid V] = \inf [f \mid RV]. \quad (23) \]

6. Tower property\(^7\)
For any pair of correspondences $R$ on $U \times V$ and $\mathcal{S}$ on $V \times W$, and for any function $f : U \to \mathbb{R}$, we have that

\[ \inf [\inf [f \mid R] \mid \mathcal{S}] = \inf [f \mid RS]. \quad (24) \]

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\(^6\)This formula for the conditional infimum has the flavour of the change of variable formula under push-forward probability.

\(^7\)This formula for conditional infima has the flavour of the tower property for conditional expectations. Had we defined the conditional infimum not w.r.t. a correspondence, but w.r.t. a set-valued mapping, the tower property would write, in a reverse way, as $\inf [\inf [f \mid \Phi^{-1}] \mid \Psi^{-1}] = \inf [f \mid (\Psi \circ \Phi)^{-1}]$, making appear the composition $\Psi \circ \Phi$ of two set-valued mappings $\Psi : W \rightrightarrows V$ and $\Phi : V \rightrightarrows U$ as in \([12\), p. 151]. Indeed, the composition $\Psi \circ \Phi$ satisfies $\mathcal{S}_{\Psi \circ \Phi} = \mathcal{S}_{\Phi} \mathcal{S}_{\Psi}$, hence $\mathcal{S}_{(\Psi \circ \Phi)^{-1}} = (\mathcal{S}_{\Psi \circ \Phi})^{-1} = (\mathcal{S}_{\Psi} \mathcal{S}_{\Phi})^{-1} = (\mathcal{S}_{\Psi})^{-1} (\mathcal{S}_{\Phi})^{-1}$. We prefer the formula $\inf [\inf [f \mid R] \mid \mathcal{S}] = \inf [f \mid RS]$ to the formula $\inf [\inf [f \mid \Phi^{-1}] \mid \Psi^{-1}] = \inf [f \mid (\Psi \circ \Phi)^{-1}]$.
7. Right composition with mappings:
   for any correspondence $\mathcal{R}$ on $U \times V$, we have that
   
   • for any function $h : W \to \mathbb{R}$ and for any mapping $\theta : U \to W$,
     \[
     \inf [h \circ \theta | \mathcal{R}] = \inf [h | \mathcal{G}_\theta^{-1} \mathcal{R}], \tag{25a}
     \]
   
   • for any function $f : U \to \mathbb{R}$ and for any mapping $\theta : W \to V$,
     \[
     \inf [f | \mathcal{R}] \circ \theta = \inf [f | \mathcal{R} \mathcal{G}_\theta^{-1}] . \tag{25b}
     \]

8. Joint conditional infimum and supremum:
   for any correspondence $\mathcal{R}$ on $U \times V$, and for any functions $f : U \to \mathbb{R}$ and $h : U \to \mathbb{R}$,
   we have that
   
   \[
   \inf \left[ f + h \mid \mathcal{R} \right] \leq \inf \left[ f \mid \mathcal{R} \right] + \sup \left[ h \mid \mathcal{R} \right] , \tag{26a}
   \]
   
   \[
   \sup \left[ f + h \mid \mathcal{R} \right] \geq \sup \left[ f \mid \mathcal{R} \right] + \inf \left[ h \mid \mathcal{R} \right] . \tag{26b}
   \]

Proof. Most of the claims are straightforward consequences of the Definition 1 of the conditional infimum, and are left to the reader.

• We prove (19):
   
   \[
   (v, t) \in \text{epi}^+ \inf \left[ f \mid \mathcal{R} \right] \iff \inf \left[ f \mid \mathcal{R} \right](v) < t \quad \text{(by definition (18) of the strict epigraph)}
   \iff \exists u \in \mathcal{R}v, \ f(u) < t \quad \text{(by definition (3a) of the conditional infimum $\inf [f \mid \mathcal{R}]$)}
   \iff \exists u \in U, \ v \mathcal{R}^{-1}u \text{ and } (u, t) \in \text{epi}^f
   \iff \exists u \in U, \ v \mathcal{R}^{-1}u \text{ and } u \text{ (epi}^+ f) \ t
   \iff (v, t) \in \mathcal{R}^{-1}(\text{epi}^+ f) .
   \quad \text{(by definition of the composition of correspondences)}
   \]

• We prove (25a) as follows. For any correspondence $\mathcal{R}$ on $U \times V$, any function $h : W \to \mathbb{R}$, any mapping $\theta : U \to W$ and any $v \in V$, we have that
   
   \[
   \inf [h \circ \theta | \mathcal{R}] = \inf [\inf [h | \mathcal{G}_\theta^{-1}] | \mathcal{R}] \quad \text{(as } h \circ \theta = \inf [h | \mathcal{G}_\theta^{-1}] \text{ by (14a))}
   \]
   
   \[
   = \inf [h | \mathcal{G}_\theta^{-1} \mathcal{R}] . \quad \text{(by the tower property (24))}
   \]

• We prove (25b) as follows. For any correspondence $\mathcal{R}$ on $U \times V$, any function $f : U \to \mathbb{R}$, any mapping $\theta : W \to V$ and any $w \in W$, we have that
   
   \[
   \inf [f \mid \mathcal{R}] \circ \theta = \inf [\inf [f | \mathcal{R}] | \mathcal{G}_\theta^{-1}] \quad \text{(by (14a))}
   \]
   
   \[
   = \inf [f | \mathcal{R} \mathcal{G}_\theta^{-1}] . \quad \text{(by the tower property (24))}
   \]
• We prove (24) as follows. For any pair of correspondences \( R \) on \( U \times V \) and \( S \) on \( V \times W \), any function \( f : U \to \mathbb{R} \) and any \( w \in W \), we have that

\[
\inf \left[ \inf_{v \in S} f \mid R \right] (w) = \inf_{v \in R} \inf_{u \in S} f(u) \quad \text{(by definition (3a) of the conditional infimum)}
\]

This ends the proof.

\[ \square \]

3 Applications of the conditional infimum to minimization problems

With the conditional infimum, we now establish equalities and inequalities between two minimization problems, an original problem on the set \( W \) and another one on the set \( X \), where the sets \( W \) and \( X \) are possibly different (in particular, \( X \) might be a vector space, whereas \( W \) is not). As we deal with optimization problems, we will often resort to the more telling usage \( \inf_{w \in W} h(w) \) or \( \min_{w \in W} h(w) \), rather than \( \inf \left[ h \mid W \right] \) as in (1).

**Proposition 3** We consider two sets \( W \) and \( X \), a correspondence \( R \) on \( W \times X \), and a function \( h : W \to \mathbb{R} \). For any subset \( X \subset X \), we have the equality

\[
\inf_{w \in R} h(w) = \inf_{x \in X} \left( \inf_{w \in R} [h \mid R](x) \right) . \tag{27}
\]

For any subset \( W \subset W \), we have the implications

\[
W \subset \text{dom} R \implies \inf_{w \in W} h(w) \geq \inf_{x \in W} \left( \inf_{w \in R} [h \mid R](x) \right) , \tag{28a}
\]

\[
\mathcal{R}^{-1} W \subset W \subset \text{dom} R \implies \inf_{w \in W} h(w) = \inf_{x \in \mathcal{R}^{-1} W} \left( \inf_{w \in R} [h \mid R](x) \right) . \tag{28b}
\]

**Proof.** The equality (27) is proved as follows:

\[
\inf_{w \in R} h(w) = \inf_{w \in W} [h \mid R] \quad \text{(by definition (1))}
\]

\[
= \inf_{x \in X} \left( \inf_{w \in R} [h \mid R](x) \right) \quad \text{(by the pushforward property (23))}
\]

\[
= \inf_{x \in X} \left( \inf_{w \in R} [h \mid R](x) \right) \quad \text{(by definition (1))}
\]

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We suppose that \( W \subset \text{dom} R \) and we prove the right hand side inequality in (28a). First, we prove that \( W \subset R^{-1}W \). Indeed, if \( w \in W \) we have that \( w \in \text{dom} R \) as \( W \subset \text{dom} R \). Therefore, there exists \( x \in X \) such that \( wRx \) or, equivalently, that \( xR^{-1}w \). Now, \( wRx \) and \( xR^{-1}w \) imply that \( wR^{-1}w \) and thus \( w \in R^{-1}W \). Second, we obtain that

\[
\inf_{w \in W} h(w) \geq \inf_{w \in R^{-1}W} h(w) \quad \text{(since } W \subset R^{-1}W) \]

\[
= \inf_{x \in R^{-1}W} \left( \inf \left[ h \mid R \right](x) \right) . \quad \text{(by Equation (27) with } X = R^{-1}W) \]

When \( R^{-1}W \subset W \subset \text{dom} R \), the right hand side equality in (28b) comes from the fact that the inequality above is an equality using that \( R^{-1}W \subset W \).

This ends the proof. \( \square \)

With the conditional infimum, we now state sufficient conditions to relate the optimal solutions of two minimization problems.

**Proposition 4** We consider a function \( h : W \to \mathbb{R} \), a subset \( W \subset W \) and the minimization problem

\[
\min_{w \in W} h(w) . \quad (29)
\]

Assume that there exists

1. a set \( X \), a correspondence \( R \) on \( W \times X \), and a function \( f : X \to \mathbb{R} \) such that

\[
f(x) \leq \inf \left[ h \mid R \right](x) , \ \forall x \in X , \quad (30a)
\]

a subset \( X \subset X \) such that

\[
W \subset RX , \quad (30b)
\]

and an optimal solution \( x^* \in X \) to the auxiliary minimization problem \( \min_{x \in X} f(x) \), that is,

\[
x^* \in \arg \min_{x \in X} f(x) , \quad (30c)
\]

2. an element \( w^* \in W \) such that

\[
h(w^*) = f(x^*) , \quad (30d)
\]

\[
w^* \in W . \quad (30e)
\]

Then, \( w^* \) is an optimal solution to the original minimization problem (29), that is,

\[
w^* \in \arg \min_{w \in W} h(w) . \quad (31)
\]
Proof. The equality (31) between solutions of minimization problems follows from
\[
\begin{align*}
  h(w^*) &= f(x^*) \quad \text{(by assumption (30a))} \\
  &= \min_{x \in X} f(x) \quad \text{(by assumption (30c))} \\
  &\leq \inf_{x \in X} [h \, | \, R](x) \quad \text{(because } f \leq \inf [h \, | \, R] \text{ by assumption (30a))} \\
  &= \inf_{w \in R_X} h(w) \quad \text{(by the equality (27))} \\
  &\leq \inf_{w \in W} h(w) \quad \text{(because } R_X \supset W \text{ by assumption (30b))}
\end{align*}
\]
As \( w^* \in W \) by assumption (30e), this ends the proof. \( \Box \)

4 Detecting hidden convexity using the conditional infimum

In §4.1 we provide a sufficient condition for hidden convexity in minimization problems. Then, in §4.2 we show how our result applies to quadratic optimization problems.

4.1 A sufficient condition for hidden convexity in minimization problems

We propose a formal definition of “hidden convexity” in minimization problems, using the notation (1)–(2).

Definition 5 We consider a set \( \mathbb{W} \), a function \( h : \mathbb{W} \to \mathbb{R} \) and a subset \( W \subset \mathbb{W} \). We say that the minimization problem \( \inf_{w \in W} h(w) = \inf [h \, | \, W] \) displays hidden convexity if there exists a vector space \( \mathbb{X} \), a convex function \( f : \mathbb{X} \to \mathbb{R} \) and a convex subset \( C \subset \mathbb{X} \) such that
\[
\inf_{w \in W} h(w) = \inf [h \, | \, W] = \inf [f \, | \, C] = \inf_{x \in C} f(x) . \quad (32a)
\]
Moreover, the minimization problem \( \min_{w \in W} h(w) \) is said to display strong hidden convexity if, in addition to (32a), \( \arg \min [f \, | \, C] \neq \emptyset \) and there exists a set-valued mapping \( \gamma : C \rightrightarrows W \) such that
\[
\gamma(\arg \min [f \, | \, C]) \subset \arg \min [h \, | \, W] . \quad (32b)
\]

We state a sufficient condition for hidden convexity in minimization problems, using the conditional infimum.

Proposition 6 Let \( \mathbb{X} \) be a vector space, \( \mathbb{W} \) be a set and \( R \subset \mathbb{W} \times \mathbb{X} \) be a correspondence between the sets \( \mathbb{W} \) and \( \mathbb{X} \). Let \( h : \mathbb{W} \to \mathbb{R} \) be a function, and \( C \subset \mathbb{X} \) be a convex subset
such that the function $\inf_{\mathbb{R}} [h | \mathbb{R}] : \mathbb{X} \to \mathbb{R}$ is convex on $C$. Then, the minimization problem $\inf_{w \in \mathbb{R}C} h(w)$ displays hidden convexity as in (32a), with $f = \inf_{\mathbb{R}} [h | \mathbb{R}]$:

$$\inf_{w \in \mathbb{R}C} h(w) = \inf_{x \in C} \left( \inf_{\mathbb{R}} [h | \mathbb{R}] \right)(x).$$

(33)

Moreover, if there exists an optimal solution $x^* \in \mathbb{X}$ to the auxiliary convex minimization problem $\min_{x \in C} \left( \inf_{\mathbb{R}} [h | \mathbb{R}] \right)(x)$, that is, if

$$x^* \in \arg \min_{x \in C} \left( \inf_{\mathbb{R}} [h | \mathbb{R}] \right)(x),$$

(34a)

and if there exists an optimal solution $w^*$ to the minimization problem $\min_{w \in \mathbb{R}x^*} h(w)$ — which is the original minimization problem but with stronger constraint $w \in \mathbb{R}x^*$ instead of $w \in \mathbb{R}C$ — that is, if

$$w^* \in \arg \min_{w \in \mathbb{R}x^*} h(w),$$

(34b)

then $w^*$ is an optimal solution to the original minimization problem $\min_{w \in \mathbb{R}C} h(w)$, that is,

$$w^* \in \arg \min_{w \in \mathbb{R}C} h(w).$$

(34c)

**Proof.** The equality (33) is a straightforward application of the equality (27) with $X = C$, in Proposition 3.

The second part regarding the arg min is an application of Proposition 4 whose assumptions (30a), (30b), (30c), (30d), (30e) are satisfied as follows.

Equation (30a) is satisfied by taking the function $f = \inf_{\mathbb{R}} [h | \mathbb{R}]$. Equation (30b) is satisfied by taking the subsets $X = C$ and $W = \mathbb{R}C$. Equation (30c) is exactly Equation (34a) in the assumptions as $f = \inf_{\mathbb{R}} [h | \mathbb{R}]$ and $X = C$. Equation (30d) holds true because

$$h(w^*) = \min_{w \in \mathbb{R}x^*} h(w) \quad \text{(by the assumption (34b))}$$

$$= \inf_{\mathbb{R}} [h | \mathbb{R}](x^*) \quad \text{(by definition (3a) of the conditional infimum $\inf_{\mathbb{R}} [h | \mathbb{R}]$)}$$

$$= f(x^*) \quad \text{(because $f = \inf_{\mathbb{R}} [h | \mathbb{R}]$)}$$

Equation (30e) is satisfied because $w^* \in \mathbb{R}x^*$ by (34b), where $x^* \in C$ by (34a), so that $w^* \in \mathbb{R}C = W$. □

In the next §12 we will use the following version of Proposition 6 where the correspondence $\mathbb{R}$ is induced by a mapping.

**Corollary 7** Let $\mathbb{X}$ be a vector space, $\mathbb{W}$ be a set and $\theta : \mathbb{W} \to \mathbb{X}$ be a mapping. Let $h : \mathbb{W} \to \mathbb{R}$ be a function, and $C \subset \mathbb{X}$ be a convex subset such that the function $\inf_{\theta} [h | \theta] : \mathbb{X} \to \mathbb{R}$ is
convex on \( C \). Then, the minimization problem \( \inf_{\theta(w) \in C} h(w) \) displays hidden convexity as in (32a), with \( f = \inf [h \mid \theta] : \)

\[
\inf_{\theta(w) \in C} h(w) = \inf_{x \in C} (\inf [h \mid \theta])(x) .
\]

Moreover, if there exists an optimal solution \( x^* \in X \) to the auxiliary convex minimization problem \( \min_{x \in C} (\inf [h \mid \theta])(x) \), that is, if

\[
x^* \in \arg \min_{x \in C} (\inf [h \mid \theta])(x) ,
\]

and if there exists an optimal solution \( w^* \) to the minimization problem \( \min_{\theta(w) \in C} h(w) \) — which is the original minimization problem but with the stronger constraint \( \theta(w) = x^* \) instead of \( \theta(w) \in C \) — that is, if

\[
w^* \in \arg \min_{\theta(w) = x^*} h(w) ,
\]

then \( w^* \) is an optimal solution to the original minimization problem \( \min_{\theta(w) \in C} h(w) \), that is,

\[
w^* \in \arg \min_{\theta(w) \in C} h(w) .
\]

### 4.2 Hidden convexity in the quadratic case

We study hidden convexity both for functions and for minimization problems in the quadratic case. Let \( d \in \mathbb{N}^* \) be a positive integer. We define the square mapping \( s : \mathbb{R}^d \to \mathbb{R}^d \) by

\[
s(w) = s(w_1, \ldots, w_d) = (w_2^1, \ldots, w_2^d) , \ \forall w \in \mathbb{R}^d .
\]

#### 4.2.1 Hidden convexity in linear-quadratic functions

We provide necessary and sufficient conditions under which the conditional infimum of a linear-quadratic function, w.r.t. the square mapping (37), is convex. We deduce a sufficient condition for hidden convexity of a linear-quadratic function w.r.t. to the square mapping.

**Proposition 8** Let \( d \in \mathbb{N}^* \) be a positive integer, \( b \in \mathbb{R}^d \) be a vector, and \( A \) be a \( d \times d \) symmetric matrix. Let the linear-quadratic function \( q : \mathbb{R}^d \to \mathbb{R} \) be given by (where \( ' \) denotes transposition)

\[
q(w) = w' Aw + b' w , \ \forall w \in \mathbb{R}^d .
\]

Then, the function \( f = \inf [q \mid s] : \mathbb{R}^d \to \mathbb{R} \), defined in (13a) by

\[
f(x) = \inf \{ w' Aw + b' w \mid w_1^2 = x_1, \ldots, w_d^2 = x_d \} , \ \forall x = (x_1, \ldots, x_d) \in \mathbb{R}^d ,
\]
is convex if and only if

$$\exists \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d \text{ such that } \begin{cases} \varepsilon_i b_i \leq 0, & \forall i = 1, \ldots, d, \\ \varepsilon_i \varepsilon_j A_{ij} \leq 0, & \forall i, j = 1, \ldots, d, \ i \neq j. \end{cases} \quad (38c)$$

In that case, the function $f = \inf [q | s]$ in (38c) is proper convex lsc with effective domain $\text{dom} f = \mathbb{R}^d_+$, and has the expression

$$f(x_1, \ldots, x_d) = \begin{cases} +\infty, & \text{if } (x_1, \ldots, x_d) \not\in \mathbb{R}^d_+, \\ \sum_{i=1}^d A_{ii} x_i - \sum_{i \neq j} A_{ij} |\sqrt{x_i x_j}| - \sum_{i=1}^d |b_i| \sqrt{x_i} & \text{if } (x_1, \ldots, x_d) \in \mathbb{R}^d_. \end{cases} \quad (38d)$$

As a consequence, if (38c) holds true, the linear-quadratic function $q : \mathbb{R}^d \to \mathbb{R}$ displays hidden convexity (see Footnote 7) with respect to the square mapping $s$ as we have that $q = f \circ s$, where the function $f$ is convex.

**Proof.** The proof is in three steps.

- First, we obtain different expressions of the function $f = \inf [q | s]$ in (38c). As $s(\mathbb{R}^d) = \mathbb{R}^d_+$ by definition (37) of the square mapping, we have $(\inf [q | s])(x) = +\infty$ for any $x \not\in \mathbb{R}^d_+$, by (6). Then, we have, for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+$,

$$f(x_1, \ldots, x_d) = (\inf [q | s])(x_1, \ldots, x_d)$$

$$= \inf \{w'Aw + b'w \mid w \in \mathbb{R}^d, \ (w_1^2, \ldots, w_d^2) = (x_1, \ldots, x_d)\}$$

by definition (38a) of the conditional infimum w.r.t. a mapping, and by definition (37) of the square mapping

$$= \inf \left\{ \sum_{i=1}^d A_{ii} w_i^2 + \sum_{i \neq j} A_{ij} w_i w_j + \sum_{i=1}^d b_i w_i \mid w_1^2 = x_1, \ldots, w_d^2 = x_d \right\}$$

$$= \sum_{i=1}^d A_{ii} x_i + \inf \left\{ \sum_{i \neq j} A_{ij} w_i w_j + \sum_{i=1}^d b_i w_i \mid w_1 = \pm \sqrt{x_1}, \ldots, w_d = \pm \sqrt{x_d} \right\}$$

$$= \sum_{i=1}^d A_{ii} x_i + \min \left\{ \sum_{i \neq j} A_{ij} \varepsilon_i' \varepsilon_j' \sqrt{x_i x_j} + \sum_{i=1}^d b_i \varepsilon_i' \sqrt{x_i} \mid \varepsilon' \in \{-1, 1\}^d \right\} \quad (39a)$$

$$\geq \sum_{i=1}^d A_{ii} x_i + \sum_{i \neq j} |A_{ij}|(-\sqrt{x_i x_j}) + \sum_{i=1}^d |b_i|(-\sqrt{x_i}) \quad (39b)$$

where we recognize, in this last expression (39b), the expression (38c) of the function $f$.

- Second, we suppose that (38c) holds true. Then, it is easy to check that $\varepsilon \in \{-1, 1\}^d$ given by (38c) provides an equality in the inequality between (39a) and (39b). Thus, we get that the
function \( \inf [g \mid s] \) is the function given by (38a). Now, it is easily checked (by computing the Hessian) that the functions \((w_i, w_j) \in \mathbb{R}_+^2 \mapsto (-\sqrt{x_i x_j})\) are convex, for all \(i \neq j\). Therefore, it is easily deduced that the function \( f : \mathbb{R}^d \to \mathbb{R} \) in (38d) is convex lsc with effective domain \( \text{dom } f = \mathbb{R}_+^d \), hence is proper convex lsc.

- Third, we suppose that the function \( f = \inf [g \mid s] : \mathbb{R}^d \to \overline{\mathbb{R}} \) is convex.

  For any \( \varepsilon \in \{-1, 1\}^d \), the following subset \( X_\varepsilon \) of \([0, +\infty[^d \) is closed (as easily follows from its second expression below)

\[
X_\varepsilon = \left\{ x \in [0, +\infty[^d \mid \varepsilon \in \arg \min \{ \sum_{i \neq j} A_{ij} \varepsilon_i \varepsilon_j \sqrt{x_i x_j} + \sum_{i=1}^d b_i \varepsilon_i \sqrt{x_i} \mid \varepsilon' \in \{-1, 1\}^d \} \} \right\}
\]

\[
= \left\{ x \in [0, +\infty[^d \mid \sum_{i \neq j} A_{ij} \varepsilon_i \varepsilon_j \sqrt{x_i x_j} + \sum_{i=1}^d b_i \varepsilon_i \sqrt{x_i} \leq \sum_{i \neq j} A_{ij} \varepsilon_i' \varepsilon_j' \sqrt{x_i x_j} + \sum_{i=1}^d b_i \varepsilon_i' \sqrt{x_i}, \ \forall \varepsilon' \in \{-1, 1\}^d \right\}.
\]

We are going to show that one of the subsets \( X_\varepsilon \), when \( \varepsilon \in \{-1, 1\}^d \), has nonempty interior. As \( \bigcup_{\varepsilon' \in \{-1, 1\}^d} X_{\varepsilon'} = [0, +\infty[^d \), there is at least one subset \( L \subset \{-1, 1\}^d \) such that \( \bigcup_{\varepsilon' \in L} X_{\varepsilon'} = [0, +\infty[^d \) and the subset \( L \) has the smallest possible cardinal. If \( |L| = 1 \), then there is one \( \varepsilon \in \{-1, 1\}^d \) such that \( X_\varepsilon = [0, +\infty[^d \), and this \( X_\varepsilon \) obviously has nonempty interior. If \( |L| \geq 2 \), then \( \bigcup_{\varepsilon' \in \{-1, 1\}^d} X_{\varepsilon'} = [0, +\infty[^d \) implies that, for any \( \varepsilon \in L \), we have that \( \emptyset \varsubsetneq \left( \bigcup_{\varepsilon' \in L \setminus \{\varepsilon\}} X_{\varepsilon'} \right)^\varepsilon \subset X_\varepsilon \). Therefore, the subset \( X_\varepsilon \) has nonempty interior since it contains the nonempty set \( \left( \bigcup_{\varepsilon' \in L \setminus \{\varepsilon\}} X_{\varepsilon'} \right)^\varepsilon \), which is open as the complementary set of a finite union of closed subsets.

As a consequence, there is one \( \varepsilon \in \{-1, 1\}^d \) and there is a ball \( B \) in \([0, +\infty[^d \) such that

\[
f(x_1, \ldots, x_d) = \sum_{i \neq j} A_{ij} \varepsilon_i \varepsilon_j \sqrt{x_i x_j} + \sum_{i=1}^d b_i \varepsilon_i \sqrt{x_i}, \ \forall x \in B.
\]

As the function \( f \) is convex, so is the function \( k : B \ni x \mapsto \sum_{i \neq j} A_{ij} \varepsilon_i \varepsilon_j \sqrt{x_i x_j} + \sum_{i=1}^d b_i \varepsilon_i \sqrt{x_i} \), and so are the restrictions \( x_i \mapsto k(0, \ldots, 0, x_i, 0, \ldots, 0) \) and \( (x_i, x_j) \mapsto k(0, \ldots, 0, x_i, 0, \ldots, 0, x_j, 0, \ldots, 0) \) for any \( i \neq j \). We conclude readily that \( \varepsilon \in \{-1, 1\}^d \) satisfies (38c).

This ends the proof. \( \square \)

4.2.2 Hidden convexity in linear-quadratic minimization problems

Now, we provide sufficient conditions under which the minimization of a linear-quadratic function, under constraints given by the square mapping (37), displays hidden convexity.

**Proposition 9** Let \( d \in \mathbb{N}^* \) be a positive integer and \( C \subset \mathbb{R}_+^d \) be a convex subset. Let \( b \in \mathbb{R}^d \) be a vector, and \( A \) be a \( d \times d \) symmetric matrix such that (38c) holds true. Then,
the minimization problem \( \min_{w \in \mathbb{R}^d} w'Aw + b'w \), under the constraint that \((w_1^2, \ldots, w_d^2) \in C\), displays strong hidden convexity, as in Definition 5.

Indeed, we have that
\[
\inf \left\{ w'Aw + b'w \mid w \in \mathbb{R}^d, \ (w_1^2, \ldots, w_d^2) \in C \right\} = \inf_{x \in C} f(x) ,
\]
where the function \( f : \mathbb{R}^d \to \mathbb{R} \) is proper convex lsc with effective domain \( \text{dom} f = \mathbb{R}^d_+ \), and is given by (38c).

Moreover, regarding \( \text{argmin} \), we have the following implication
\[
x^* \in \arg \inf_{w \in C} f(w) \implies \varepsilon \cdot \sqrt{x^*} \in \arg \inf \left\{ w'Aw + b'w \mid w \in \mathbb{R}^d, \ (w_1^2, \ldots, w_d^2) \in C \right\} ,
\]
where the vector \( \varepsilon \cdot \sqrt{x^*} \in \mathbb{R}^d \) has components \( \varepsilon_i \sqrt{x^*_i} \), for \( i = 1, \ldots, d \) and \( \varepsilon \) is given by (38c).

**Proof.** Equation (40a) is a straightforward application of Corollary 7. Indeed, Equation (40a) follows from Equation (33) with function \( h = q \) given by (38a), correspondence \( R = G_\varepsilon \) given by the graph of the square mapping (37), and convex subset \( C \subset \mathbb{R}_+^d \). The correspondence (40b) between argmins also follows from Corollary 7, by using the vector \( w^* = \varepsilon \cdot \sqrt{x^*} \in \mathbb{R}^d \) where \( \varepsilon \) is given by (38c).

This ends the proof. 

Our result covers (and extends) the following two cases.

**Corollary 10** For any convex subset \( C \subset \mathbb{R}_+^d \) and symmetric matrix \( M \) such that \( M_{ij} \geq 0 \) for all \( i \neq j \), the maximization problem
\[
\max \left\{ w'Mw \mid w \in \mathbb{R}^d, \ (w_1^2, \ldots, w_d^2) \in C \right\}
\]
is equivalent to the convex minimization problem
\[
\min \left\{ -\sum_{i=1}^d M_{ii}x_i - \sum_{i \neq j} M_{ij} \sqrt{x_ix_j} \mid (x_1, \ldots, x_d) \in C \right\} .
\]

**Proof.** It suffices to apply Proposition 9 with \( A = -M, b = 0 \) and \( \varepsilon = (1, \ldots, 1) \).

The following Corollary extends the result in [6, Theorem 7], as we do not require the simultaneous diagonalization property\(^8\).

**Corollary 11** For any scalars \( l \leq u \) and any diagonal matrices \( A \) and \( S \), the minimization problem
\[
\min \left\{ w'Aw + b'w \mid w \in \mathbb{R}^d, \ l \leq w'Sw \leq u \right\}
\]
\(\text{footnote}\)\(^8\)The simultaneous diagonalization property [6, Equation (3)] reads as: \( \exists \eta \in \mathbb{R} \) such that \( A + \eta S > 0 \).
is equivalent to the convex minimization problem

$$\min \left\{ \sum_{i=1}^{d} A_{ii} x_i - \sum_{i=1}^{d} |b_i| \sqrt{x_i} \mid (x_1, \ldots, x_d) \in \mathbb{R}_+^d, \ l \leq \sum_{i=1}^{d} S_{ii} x_i \leq u \right\}.$$  \hspace{1cm} (42b)

**Proof.** It suffices to apply Proposition 9 with $C = \{ (x_1, \ldots, x_d) \in \mathbb{R}_+^d \mid l \leq \sum_{i=1}^{d} S_{ii} x_i \leq u \}$ and $\varepsilon = -\text{sign}(b)$. □

5 Conclusion

Detecting hidden convexity is one of the tools to address nonconvex minimization problems. In this paper, we have contributed to this research program by giving a formal definition of hidden convexity in a minimization problem, and by putting forward the notion of conditional infimum. Building upon a well-known parallelism between optimization and probability theories, we have established a list of properties of the conditional infimum, among which a tower formula, relevant for minimization problems. Thus equipped, we have provided sufficient conditions for hidden convexity in nonconvex optimization problems, and we have illustrated our results on nonconvex quadratic minimization problems. We finish this conclusion by pointing out perspectives for using the conditional infimum in other contexts, namely in relation to the so-called S-procedure, to couplings and conjugacies, and to lower bound convex programs.

The conditional infimum appears in the so-called S-procedure (see the survey paper [11]), itself related to hidden convexity (see [5]) as follows. Let $f_0, f_1, \ldots, f_p : U \to \mathbb{R}$ be functions, and consider the statements

$$(I) \quad (f_i(u) \geq 0, \ \forall i = 1, \ldots, p) \implies f_0(u) \geq 0,$$  \hspace{1cm} (43a)

$$(C) \quad \exists \alpha_1 \geq 0, \ldots, \alpha_p \geq 0 \text{ such that } f_0 - \sum_{i=1}^{p} \alpha_i f_i \geq 0.$$  \hspace{1cm} (43b)

It is obvious that $(C) \implies (I)$. The S-procedure consists in finding sufficient conditions to ensure that $(I) \implies (C)$, that is, conditions such that $(I) + \text{conditions} \implies (C)$. We easily show that we can write statement $(I)$ in term of conditional infimum as

$$(I) \iff \inf [f_0 \mid f_1, \ldots, f_p] (v) \geq 0, \ \forall v \in \mathbb{R}_+^p.$$  \hspace{1cm} (44)

The conditional infimum is related to couplings and conjugacies. One-sided linear couplings are introduced in [7] as follows: letting $\mathcal{W}$ be a set and $\theta : \mathcal{W} \to \mathbb{R}^d$ be a mapping, we define the coupling $\star_\theta : \mathcal{W} \times \mathbb{R}^d \to \mathbb{R}$ by $\star_\theta (w, y) = \langle \theta(w), y \rangle$, for any $(w, y) \in \mathcal{W} \times \mathbb{R}^d$. Then, we show in [7, Proposition 2.5] that the $\star_\theta$-Fenchel-Moreau conjugate $h^{\star_\theta}$ of a function $h : \mathcal{W} \to \mathbb{R}$ can be expressed as the Fenchel conjugate of the conditional infimum $\inf [h \mid \theta]$:

$$h^{\star_\theta} = (\inf [h \mid \theta])^*.$$  \hspace{1cm} (45)
It appears that one-sided linear couplings are also related to hidden convexity: in [7, Proposition 2.6], we show that a function is \( \theta \)-convex if and only if it is the composition of a closed convex function on \( \mathbb{R}^d \) with the mapping \( \theta \) (see Footnote 1). More generally, letting \( W \) be a set and \( \mathcal{R} \subset W \times \mathbb{R}^d \) be a correspondence between \( W \) and \( \mathbb{R}^d \), we define the coupling \( \star \mathcal{R} : W \times \mathbb{R}^d \to \mathbb{R} \) by \( \star \mathcal{R}(w, y) = \sup_{x \in \mathcal{R}} \langle x, y \rangle \), for any \( (w, y) \in W \times \mathbb{R}^d \). Then, an easy computation shows that the \( \star \mathcal{R} \)-Fenchel-Moreau conjugate \( h^{\star \mathcal{R}} \) of a function \( h : W \to \mathbb{R} \) can be expressed as the Fenchel conjugate of the conditional infimum \( \inf [h | \mathcal{R}] \):

\[
\begin{align*}
\star \mathcal{R} &= (\inf [h | \mathcal{R}])^*.
\end{align*}
\]

(46)

We can use the conditional infimum in Proposition 3 to obtain lower bound convex programs for nonconvex problems as follows. Consider, on the one hand, a function \( h : W \to \mathbb{R} \) and a subset \( W \subset W \), and, on the other hand, a correspondence \( \mathcal{R} \) on \( W \times \mathbb{R}^d \), and a convex subset \( X \subset \mathbb{R}^d \). If \( \mathcal{R}X \subset W \), we have the inequality

\[
\inf_{w \in W} h(w) \geq \inf_{x \in X} (\inf [h | \mathcal{R}])^{* *}(x).
\]

(47)

This may be interesting when \( h^{* *}' \) is trivial, but \( (\inf [h | \mathcal{R}])^{* *}' \) is not, like when \( h \) is the \( \ell_0 \) pseudonorm on \( \mathbb{R}^d \) and \( \mathcal{R} \) is given by the normalization mapping onto the Euclidean sphere [7].

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