Anisotropic cosmological models with non-minimally coupled magnetic field

Alexander B. Balakin∗
Department of General Relativity and Gravitation
Kazan State University, 420008 Kazan, Russia
and
Winfried Zimdahl†
Institut für Theoretische Physik, Universität zu Köln
D-50937 Köln, Germany

July 25, 2018

Abstract

Motivated by the structure of one-loop vacuum polarization effects in curved spacetime we discuss a non-minimal extension of the Einstein-Maxwell equations. This formalism is applied to Bianchi I models with magnetic field. We obtain several exact solutions of the non-minimal system including those which describe an isotropization process. We show that there are inflationary solutions in which the cosmological constant is determined by the non-minimal coupling parameters. Furthermore, we find an isotropic de Sitter solution characterized by a “screening” of the magnetic field as a consequence of the non-minimal coupling.

PACS numbers: 04.40.Nr, 98.80.Jk

1 Introduction

The Einstein-Maxwell theory has been a subject of investigations since long [1]. As far as cosmological implications are concerned, the possible role of a primordial magnetic field has attracted particular interest. Studying the impact of a magnetic field on the dynamical evaluation requires anisotropic cosmological

∗Electronic address: Alexander.Balakin@ksu.ru
†Electronic address: zimdahl@thp.uni-koeln.de
models. A general discussion of this type of models with magnetic field and references to early activities along this line may be found in [2]. By now there are strong limits on the current magnitude of such a field [3] which seems to render magnetic fields on cosmological scales unimportant at the present stage of the cosmic evolution. This does not imply, however, that a magnetic field did not influence the dynamics of the early Universe. Moreover, quantum effects are expected to become relevant at early cosmic stages. Quantum electrodynamical consideration show that vacuum polarization effects in curved spacetime give rise to non-minimal modifications of the (minimal) Einstein-Maxwell Lagrangian [4]. The investigation of a non-minimal coupling of gravity with electromagnetic fields was initiated by Prasanna [5]. Prasanna introduced the additional invariant $R^{ikmn}F_{ik}F_{mn}$ ($R^{ikmn}$ is the Riemann tensor, $F_{ik}$ is the Maxwell tensor) into the Lagrangian for the gravito-electromagnetic system and obtained a non-minimal one-parameter modification of the Einstein-Maxwell equations [6]. Novello and Salim [7] included the (gauge-dependent) terms $RA^kA_k$ and $R^{ik}A_iA_k$ in the Lagrangian ($A_k$ is the electromagnetic potential four-vector, $R^{ik}$ is the Ricci tensor and $R$ is the curvature scalar). A qualitatively new step has then been made by Drummond and Hathrell [4] by calculating quantum-electrodynamical one-loop corrections in curved spacetime. The Lagrangian of such a theory contains the three U(1) gauge-invariant scalars $R^{ikmn}F_{ik}F_{mn}$, $R^{ik}g^{mn}F_{im}F_{kn}$ and $RF_{mn}F^{mn}$ with coefficients proportional to the square of the Compton wavelength of the electron. Subsequently, a non-minimal coupling of gravity and electromagnetism has been discussed by a number of authors in different settings [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Non-minimally extended theories were used as a framework to discuss potential limitations of the equivalence principle [19, 20, 21]. A further quantum electrodynamical motivation of the use of the generalized Maxwell equations can be found in [22, 23, 24]. The effect of birefringence induced by curvature, first discussed in [4], and some of its consequences for electrodynamic systems have been investigated for the pp-wave background in [25, 26, 27, 28, 29]. A curvature force has been introduced to describe the accelerated expansion of the universe [30, 31]. Non-minimal interactions in which torsion is coupled to the electromagnetic field were studied in [32, 33] (see also [34] for a review). Finally, we mention a mediated non-minimal coupling in which the scalar Higgs field $\phi$ is coupled to gravity via a $\xi\phi^2 R$ term and to a Yang-Mills potential $A_k$ by $\phi^2 A_k A^k$ [35].

Most of the investigations so far were devoted to the analysis of the non-minimally modified Maxwell equations on a given background. With the exceptions [8] and [15] the impact of the non-minimal coupling on the gravitational field has been outside the focus of interest. However, one may expect that the rich structure of such type of theories gives rise to novel features in the gravitational dynamics as well. Our purpose here is to demonstrate this aspect for a specific class of non-minimal interactions in Bianchi I cosmological models. These interactions are modelled according to the already mentioned general structure obtained in [4] as the result of quantum electrodynamical calculations. While the coupling constants are fixed in [4], they will be considered as arbitrary, constant parameters in our analysis (cf. [13]). On this basis we shall first obtain
the general set of equations for a non-minimally extended Einstein-Maxwell system with linear electrodynamics (For a non-linear, non-minimal extension of the Einstein-Maxwell theory see [36]). Then we specify this set to the homogenous but anisotropic case of Bianchi I cosmological models with magnetic field and a matter component with generally anisotropic equations of state. We obtain simple exact solutions for several choices of the non-minimal coupling parameters. Among them are solutions with axial symmetry which isotropize in the long time limit. As a specific feature of inflationary solutions of the non-minimal theory we find direct relations between a cosmological constant and the non-minimal coupling parameters. There exists an isotropic de Sitter solution with this property as well for which the corresponding set of parameters makes the gravitational dynamics independent of the (non-vanishing) magnetic field.

The paper is organized as follows. In section 2 we present the general formalism of the non-minimally extended set of gravito-electromagnetic field equations, based on [4]. The (linear) electrodynamical field equations are obtained and discussed in section 3. Section 4 is devoted to the gravitational field equations in the general case. In section 5 the material of the previous sections is applied to the Bianchi I geometry. Several particular models are then studied in section 6. Section 7 provides a summary of the paper.

2 Non-minimal coupling of gravity and electromagnetism

2.1 General formalism

A non-minimal extension of the Einstein-Maxwell theory can be derived from the action functional

\[ S[g, A] = \int d^4x \sqrt{-g} L \]

with the Lagrangian

\[ L = \frac{R + 2\Lambda}{\kappa} \mathcal{L}_{\text{matter}} + \frac{1}{2} F_{mn} F^{mn} + \frac{1}{2} R^{ikmn} F_{ik} F_{mn}. \]

Here, \( \Lambda \) is a cosmological constant, \( g \) is the determinant of the metric tensor \( g_{ik} \), the constant \( \kappa \) is equal to \( \kappa = \frac{8\pi G}{c^4} \) where \( G \) is Newton's gravitational constant. The quantity \( \mathcal{L}_{\text{matter}} \) is the Lagrangian of neutral matter and \( F_{ik} \) is the Maxwell tensor \( F_{ik} = \nabla_i A_k - \nabla_k A_i \), where \( \nabla_i \) denotes the covariant derivative and \( A_i \) is the four potential. The last term describes a \( U(1) \) gauge invariant non-minimal coupling between gravity and electromagnetism, mediated through the tensor

\[ R_{ikmn} = \frac{q_1}{2} (g^{im} g^{kn} - g^{in} g^{km}) R + \frac{d_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn}, \]
where \( q_1, q_2 \) and \( q_3 \) are phenomenological coupling constants with the dimension [length]\(^2\). This structure of the coupling is motivated by quantum electrodynamical calculations of vacuum polarization effects in curved spacetime by Drummond and Hathrell [4]. While \( q_1, q_2 \) and \( q_3 \) have definite values in [4], we assume them to be arbitrary constant parameters in our analysis. The choice [2], [3] is a generalization of previous non-minimal modifications of Maxwell’s theory. The case \( q_1 = q_2 = 0 \) was investigated in [5, 6]. For \( q_2 = q_3 = 0 \) one obtains a model considered in [15].

The tensor \( R^{ikmn} \) has the same symmetry properties as the Riemann tensor \( R_{ikmn} \). Contraction yields

\[
\begin{align*}
g_{kn} R^{ikmn} &= R^{im}(q_2 + q_3) + \frac{1}{2} Rg^{im}(3q_1 + q_2), \\
g_{kn}g_{im} R^{ikmn} &= R(6q_1 + 3q_2 + q_3).
\end{align*}
\] (4)

The case of a vanishing trace which will be of interest in later applications is characterized by

\[
g_{kn}g_{im} R^{ikmn} = 0 \implies 6q_1 + 3q_2 + q_3 = 0.
\] (5)

### 3 Electrodynomic equations

The equations of non-minimal electrodynamics are obtained by varying the action functional with the Lagrangian [2] with respect to the four-potential \( A_i \) of the electromagnetic field. They are of the standard form

\[
\nabla_k H^{ik} = 0, \quad \nabla_k F^{*ik} = 0,
\] (6)

where the induction tensor \( H^{ik} \) is given by

\[
H^{ik} = F^{ik} + R^{ikmn} F_{mn}.
\] (7)

This relation has the structure of a constitutive law in which \( R^{ikmn} \) plays the role of a susceptibility tensor.

#### 3.1 Non-minimal constitutive equations

The linear constitutive equation [4] has the standard form \( H^{ik} = C^{ikmn} F_{mn} \) with a “material” tensor

\[
C^{ikmn} = \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) + \mathcal{R}^{ikmn}.
\] (8)
This tensor describes the linear electromagnetic response of the system, which may also be characterized by the dielectric and magnetic permeabilities, as well as by possible magneto-electric effects \[37, 38, 39\]. \(C_{ikmn}\) can uniquely be decomposed with respect to the four velocity \(U^i\) (normalized by \(U^iU_i = 1\)) of the medium:

\[
C_{ikmn} = \frac{1}{2} \left[ \varepsilon_{im} U^k U^n - \varepsilon_{in} U^k U^m + \varepsilon_{kn} U^i U^m - \varepsilon_{km} U^i U^n \right] \\
- \frac{1}{2} \eta^{ikl} (\mu^{-1})_{ls} \eta_{mns} \\
- \frac{1}{2} [\eta^{ikl} (U^n \nu_{l}^m - U^m \nu_{l}^n) + \eta^{lmn} (U^i \nu_{k}^l - U^k \nu_{i}^l)] .
\]

Here \(\varepsilon_{im}\) and \((\mu^{-1})_{pq}\) are the dielectric and magnetic permeability tensors, respectively, and \(\nu_{m}^p\) is a tensor of magneto-electric coefficients. These quantities are defined as

\[
\varepsilon_{im} = 2C_{ikmn} U_k U_n, \quad (\mu^{-1})_{pq} = -\frac{1}{2} \eta^{ikl} C_{ikmn} \eta_{mnp} .
\]

The dot denotes the position of the second index when lowered. The tensors \(\eta_{mnl}\) and \(\eta^{ikl}\) are anti-symmetric and orthogonal to \(U^i\),

\[
\eta_{mnl} \equiv \varepsilon_{mnl} U^s, \quad \eta^{ikl} \equiv \varepsilon^{ikls} U_s .
\]

They satisfy the identity

\[
-\eta^{ikp} \eta_{mnp} = \delta^{ikl} U_l U^s = \Delta^i_n \Delta^k_m - \Delta^i_m \Delta^k_n ,
\]

where \(\delta^{ikl}_{mns}\) is the generalized 6-indices \(\delta\) -- Kronecker tensor. The spatial projection tensor \(\Delta^{ik}\) is defined by

\[
\Delta^{ik} = g^{ik} - U^i U^k .
\]

Contracting equation \(13\) yields

\[
\frac{1}{2} \eta^{ikl} \eta_{klm} = -\delta^{il}_{mns} U_l U^s = -\Delta^i_m .
\]

The tensors \(\varepsilon_{ik}\) and \((\mu^{-1})_{ik}\) are symmetric, but \(\nu_{ik}\) is generally non-symmetric. All these tensors are orthogonal to \(U^i\),

\[
\varepsilon_{ik} U^k = 0, \quad (\mu^{-1})_{ik} U^k = 0, \quad \nu_{l}^k U^l = 0 = \nu_{l}^k U_k .
\]

Use of \(8\) in \(10\) and \(11\) provides us with

\[
\varepsilon_{im} = \Delta^{im} + 2R^{ikmn} U_k U_n ,
\]

\[
(\mu^{-1})_{pq} = \Delta_{pq} - \frac{1}{2} \eta^{ikl} R^{ikmn} \eta_{mnp} .
\]
\begin{equation}
\nu_p \cdot \nu^m = \eta_{pik} R^{ikmn} U_n. \tag{19}
\end{equation}

The non-minimal coupling of gravitational and electromagnetic fields may be interpreted as a change of the dielectric and magnetic properties of the vacuum, including a specific magnetoelectric interaction. The vacuum acquires properties of a quasi-medium under the influence of a non-vanishing tensor $R^{ikmn}$. The analogy between non-minimally extended electrodynamics and macroscopic media was pointed out, e.g., in [22]. This analogy may be completed by introducing the electric induction $D^i$, the magnetic field $H^i$, the electric field $E^i$ and the magnetic induction $B^i$, [37]:

\begin{equation}
D^i = \varepsilon^{im} E_m - B^i \nu^i_m, \quad H_i = \nu_i^m E_m + (\mu^{-1})_im B^m. \tag{20}
\end{equation}

The vectors $D^i$, $H^i$, $E^i$ and $B^i$ are defined by [40]:

\begin{equation}
D^i = H^{ik} U_k, \quad H^i = H^{*ik} U_k, \quad E^i = F^{ik} U_k, \quad B^i = F^{*ik} U_k. \tag{21}
\end{equation}

They are orthogonal to the velocity four-vector $U^i$: $D^i U_i = 0 = E^i U_i$, $H^i U_i = 0 = B^i U_i$, and form the basis for the decomposition of the tensors $F_{mn}$ and $H_{mn}$:

\begin{equation}
F_{mn} = E_m U_n - E_n U_m - \eta_{mln} B^l, \quad H_{mn} = D_m U_n - D_n U_m - \eta_{mln} H^l. \tag{23}
\end{equation}

4 Gravitational field equations

The gravitational field equations are obtained by varying the action (1) with the Lagrangian (2) with respect to the metric tensor. They can be written in the standard form

\begin{equation}
R_{ik} - \frac{1}{2} g_{ik} R = \Lambda g_{ik} + \kappa T^{(\text{eff})}_{ik}, \tag{24}
\end{equation}

where

\begin{equation}
T^{(\text{eff})}_{ik} = T^{(\text{matter})}_{ik} + T^{(0)}_{ik} + q_1 T^{(1)}_{ik} + q_2 T^{(2)}_{ik} + q_3 T^{(3)}_{ik}. \tag{25}
\end{equation}

The stress-energy tensor of the matter $T^{(\text{matter})}_{ik}$ may be decomposed according to

\begin{equation}
T^{(\text{matter})}_{ik} = W U_i U_k + I_{ik}^{(q)} U_k + I_{ik}^{(p)} U_i + P_{ik}, \tag{26}
\end{equation}

where $W$ is the matter energy density scalar, $P_{ik}$ is the symmetric (generally anisotropic) pressure tensor, orthogonal to the velocity four-vector ($P_{ik} U^k = 0$), and $I_{ik}^{(q)}$ is the energy-flux four-vector, orthogonal to the four velocity ($I_{ik}^{(q)} U^i = 0$). By $T^{(0)}_{ik}$ we denote the usual stress-energy tensor of the electromagnetic field,

\begin{equation}
T^{(0)}_{ik} = \frac{1}{4} g_{ik} F_{mn} F^{mn} - F_{in} F^i_n. \tag{27}
\end{equation}
The contributions from the non-minimal interaction are

\[ T_{ik}^{(1)} = R T_{ik}^{(0)} - \frac{1}{2} R_{ik} F_{mn} F^{nm} \]

\[ \quad - \frac{1}{2} g_{ik} \nabla^{l} \nabla_{l} \left( F_{mn} F^{mn} \right) + \frac{1}{2} \nabla_{i} \nabla_{k} \left( F_{mn} F^{mn} \right), \quad (28) \]

\[ T_{ik}^{(2)} = - \frac{1}{2} g_{ik} \left[ \nabla_{m} \nabla_{l} \left( F_{mn} F^{kl} \right) - R_{lm} F_{mn} F^{kl} \right] - F^{ln} \left( R_{il} F_{kn} + R_{kl} F_{ln} \right) \]

\[ \quad - R_{mn} F_{kn} F_{k} F_{l} - \frac{1}{2} \nabla^{l} \nabla_{l} \left( F_{mn} F^{mn} \right) \]

\[ \quad + \frac{1}{2} \nabla_{l} \left[ \nabla_{i} \left( F_{kn} F^{ln} \right) + \nabla_{k} \left( F_{in} F^{ln} \right) \right], \quad (29) \]

and

\[ T_{ik}^{(3)} = \frac{1}{4} g_{ik} R^{mnl} F_{mn} F_{ls} - \frac{3}{4} F^{ls} \left( F_{i}^{n} R_{knls} + F_{k}^{n} R_{lnls} \right) \]

\[ - \frac{1}{2} g_{ik} \nabla_{m} \nabla_{n} \left( F_{i}^{n} F_{k}^{m} + F_{k}^{n} F_{i}^{m} \right). \quad (30) \]

The tensor \( T_{ik}^{(1)} \) is proportional to the corresponding term in [15], the part \( T_{ik}^{(3)} \) reproduces the stress-energy tensor of [16]. The tensor \( T_{ik}^{(2)} \) is new. All three terms are supposed to contribute to the total stress-energy tensor in the following. In contrast to the traceless electromagnetic stress-energy tensor \( T_{ik}^{(0)} \) the tensors \( T_{ik}^{(1)}, T_{ik}^{(2)} \) and \( T_{ik}^{(3)} \) have non-vanishing traces:

\[ g^{ik} T_{ik}^{(1)} = - q_{1} \left[ \frac{1}{2} R F_{mn} F^{mn} + \frac{3}{2} \nabla^{k} \nabla_{k} \left( F_{mn} F^{mn} \right) \right], \quad (31) \]

\[ g^{ik} T_{ik}^{(2)} = - q_{2} \left[ R_{mn} F_{k} F_{l} + \frac{1}{2} \nabla^{k} \nabla_{k} \left( F_{mn} F^{mn} \right) \right], \quad (32) \]

\[ g^{ik} T_{ik}^{(3)} = - q_{3} \frac{1}{2} R^{mnl} F_{mn} F_{ls} + \nabla^{m} \nabla_{n} \left( F_{kn} F_{km} \right), \quad (33) \]

Non-vanishing traces of effective stress energy tensors are also features of non-linear electrodynamic models (see, e.g., [41]).

The effective stress-energy tensor \( T^{(\text{eff})}_{ik} \) in eq (24) has to be divergence-free, i.e.

\[ \nabla^{k} T^{(\text{eff})}_{ik} = 0. \quad (34) \]

We assume the stress-energy tensor of the matter \( T_{ik}^{(\text{matter})} \) to be conserved separately, i.e., \( \nabla^{k} T_{ik}^{(\text{matter})} = 0 \). The remaining part of the effective stress-energy tensor is then automatically conserved if \( F_{ik} \) is a solution of Maxwell’s equations. In order to check this fact directly, one has to use the Maxwell equations [6] with [17], the Bianchi identities, the symmetry properties of the Riemann tensor and the commutation rules for the covariant derivatives. This procedure is analogous to the one described in [16] and we omit it.
5 Anisotropic cosmological models

5.1 Metric structure

We consider now the Bianchi I cosmological model with the line element \[ ds^2 = dt^2 - a^2(t) \, (dx^1)^2 - b^2(t) \, (dx^2)^2 - c^2(t) \, (dx^3)^2. \] (35)

Due to the symmetry of the metric only six components of the Riemann tensor are different from zero:

\[ R_{01}^{01} = -\ddot{a}/a, \quad R_{02}^{02} = -\ddot{b}/b, \quad R_{03}^{03} = -\ddot{c}/c, \]

\[ R_{12}^{12} = -\dot{a}/a \dot{b}/b, \quad R_{13}^{13} = -\dot{a}/a \dot{c}/c, \quad R_{23}^{23} = -\dot{b}/b \dot{c}/c. \] (36)

5.2 Exact solution of Maxwell’s equations

Since the susceptibility tensor $R^{k\rho\sigma\gamma} = 0$, it follows from relation (19) that all magneto-electric coefficients vanish. The gravitational field (35) does not mix pure electric and pure magnetic fields. From relations (17) and (18) we find the dielectric and magnetic permeability tensors

\[ \varepsilon^{\alpha\beta} = \delta^{\alpha\beta} + 2R_{00\rho}^{\alpha}, \quad (\mu^{-1})^{\alpha\beta} = \delta^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\gamma\sigma} R_{\gamma\sigma}^{\rho\mu} \eta_{\mu\nu\beta}. \] (37)

Let us consider now a magnetic field directed along the 0$^z$ axis (which is also the direction of the shear eigenvector [42]). The symmetry of the problem then fixes $F_{ik}$ and $H^{ik}$ to be of the following structure:

\[ F_{ik} = (\delta^1_i \delta^2_k - \delta^2_i \delta^1_k) F_{12}, \] (38)

\[ H^{ik} = (g^{i1} g^{k2} - g^{i2} g^{k1}) [1 + q_1 R + q_2 (R_1^1 + R_2^2) + q_3 R_{12}^{12}] F_{12}. \] (39)

The second set of the equations (6) yields $F_{12} = \text{const}$. The first set of eqs. (6) is identically satisfied since all the components $H^{00}$ are equal to zero. Introducing the scalar value of the magnetic field $B(t)$ by

\[ B^2(t) = \frac{1}{2} F_{ik} F^{ik} = F_{12} F^{12}, \] (40)

we reproduce the result [cf. [2]]

\[ B^2(t) = \frac{\text{const}^2}{a^2(t) b^2(t)}, \] (41)

i.e., $B(t) a(t) b(t) = \text{const}$. 

8
5.3 Einstein’s equations

In a co-moving frame with $U^i = \delta^i_0$, the field equations reduce to the following system:

$$\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{b}}{b} \frac{\ddot{c}}{c} = \Lambda + \kappa W + \frac{1}{2} \kappa B^2(t)$$

$$+ \kappa B^2(t) \left\{ q_1 \left[ 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{b}}{b} \right]^2 + 3 \frac{\dot{a}}{a} \right] + q_2 \left( \frac{\dot{a}}{a} \right)^2 \right\}, \tag{42}$$

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}}{b} \frac{\ddot{c}}{c} = \Lambda - \kappa P(1) - \frac{1}{2} \kappa B^2(t)$$

$$+ \kappa B^2(t) \left\{ q_1 \left[ 4 \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{b}}{b} \right] - 6 \left( \frac{\dot{a}}{a} \right)^2 - 4 \left( \frac{\dot{b}}{b} \right)^2 + 4 \frac{\ddot{b}}{b} + 4 \frac{\ddot{c}}{c} + 3 \frac{\ddot{c}}{c} \right]$$

$$+ q_2 \left[ 2 \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - 3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{b}}{b} \right] - 2 \frac{\ddot{b}}{b} + 2 \frac{\ddot{c}}{c} \right\} \right\}, \tag{43}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a} \frac{\ddot{c}}{c} = \Lambda - \kappa P(2) - \frac{1}{2} \kappa B^2(t)$$

$$+ \kappa B^2(t) \left\{ q_1 \left[ 4 \left( \frac{\dot{b}}{b} \right)^2 + \frac{\dot{a}}{a} + \frac{\ddot{c}}{c} - 6 \left( \frac{\dot{b}}{b} \right)^2 - 4 \left( \frac{\dot{a}}{a} \right)^2 - 4 \frac{\ddot{b}}{b} + 4 \frac{\ddot{c}}{c} + 3 \frac{\ddot{c}}{c} \right]$$

$$+ q_2 \left[ 2 \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - 3 \left( \frac{\dot{b}}{b} \right)^2 + \frac{\dot{c}}{c} \right] - 2 \frac{\ddot{b}}{b} + 2 \frac{\ddot{c}}{c} \right\} \right\}, \tag{44}$$
\[
\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{a} \dot{b}}{ab} = \Lambda - \kappa P_{(3)} + \frac{1}{2} \kappa B^2(t)
\]

\[
+ \kappa B^2(t) \left\{ q_1 \left[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} - 4 \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\dot{b}}{b} \right)^2 - \frac{\ddot{a} \dot{b}}{ab} \right] \right.

- q_2 \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right)^2 - q_3 \frac{\dot{a} \dot{b}}{ab} \right\}. \quad (45)
\]

Here, \( P_{(1)}, P_{(2)} \) and \( P_{(3)} \) are the eigenvalues of the anisotropic pressure tensor \( P_{ik} \). Summing up eqs. (42) - (45) we obtain the trace equation:

\[
2 \left( \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\ddot{a} \dot{b}}{ab} + \frac{\dot{a} \ddot{c}}{ac} + \frac{\dot{b} \ddot{c}}{bc} \right) = 4\Lambda + \kappa (W - P_{(1)} - P_{(2)} - P_{(3)})
\]

\[
+ \kappa B^2(t) \left\{ (8q_1 + 4q_2 + q_3) \left( \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} \right) + 2q_1 \frac{\ddot{c}}{c}

- 2(6q_1 + 3q_2 + q_3) \left( \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\dot{b}}{b} \right)^2 \right) - 2(5q_1 + 2q_2) \frac{\dot{a} \dot{b}}{ab}

+ (8q_1 + 4q_2 + q_3) \frac{\ddot{c}}{c} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) \right\}. \quad (46)
\]

Differentiating (42) and using (43) - (45) leads to the conservation law for the matter:

\[
W + \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) W + \frac{\ddot{a}}{a} P_{(1)} + \frac{\dot{b}}{b} P_{(2)} + \frac{\ddot{c}}{c} P_{(3)} = 0. \quad (47)
\]

It follows that \( T_{ik}^{(0)} + q_1 T_{ik}^{(1)} + q_2 T_{ik}^{(2)} + q_3 T_{ik}^{(3)} \) is conserved as well. The equations (42) - (45) represent a modified (compared to minimal coupling) dynamical system, since the non-minimal terms contribute to the coefficients before the second order derivatives. Its complete analysis should be the subject of future investigations along the lines described in [43].

6 Particular models

6.1 Quasi-one-dimensional solutions with pure magnetic field

As the first application we consider a model without matter \( (W = 0, P_{ik} = 0) \) and with constant values of two of the metric functions: \( a(t) = a_0 \) and \( b(t) = b_0 \).
Only $c(t)$ is assumed to vary. Then $B^2(t) = \text{const}$ (see, (41)) and the equations (42) reduce to

$$0 = \Lambda + \frac{1}{2} \kappa B^2(t_0), \quad \frac{\ddot{c}}{c} \left( 1 - q_1 \kappa B^2(t_0) \right) = \Lambda - \frac{1}{2} \kappa B^2(t_0). \quad (48)$$

Only one of the non-minimal parameters enters the dynamics. According to the first equation this model requires a negative cosmological constant which exactly compensates the magnetic field term, i.e., $\kappa B^2 = -2\Lambda$. The equation for $c(t)$ is

$$\ddot{c} + c(t) \left( \frac{\kappa B^2(t_0)}{1 - q_1 \kappa B^2(t_0)} \right) = 0. \quad (49)$$

For $B^2 = \Lambda = 0$ we obtain the vacuum solution $c \propto t$ which is the degenerate case of a Kasner solution, equivalent to the flat spacetime Milne universe (cf. [2]). For non-vanishing $\kappa B^2 = -2\Lambda$ we have three cases.

**First case**: $q_1 \kappa B^2(t_0) < 1$.

This condition includes the case $q_1 = 0$. The solution of eq. (49) is oscillatory:

$$c(t) = c(t_0) \cos \nu(t - t_0) + \frac{\dot{c}(t_0)}{\nu} \sin \nu(t - t_0), \quad (50)$$

where

$$\nu^2 \equiv \left[ \frac{\kappa B^2(t_0)}{1 - q_1 \kappa B^2(t_0)} \right]. \quad (51)$$

The zeros of $c(t)$ denote singularities of the model.

**Second case**: $q_1 \kappa B^2(t_0) = 1$.

The equations (49) contradict each other. This case is incompatible with $B(t_0) \neq 0$.

**Third case**: $q_1 \kappa B^2(t_0) > 1$.

This condition requires $q_1$ to be positive. The solution of (49) is

$$c(t) = c(t_0) \cosh \mu(t - t_0) + \frac{\dot{c}(t_0)}{\mu} \sinh \mu(t - t_0), \quad (52)$$

where

$$\mu^2 \equiv \left[ \frac{\kappa B^2(t_0)}{q_1 \kappa B^2(t_0) - 1} \right]. \quad (53)$$

For $t >> t_0$ the function $c(t)$ behaves as $c(t) \propto e^{\mu t}$. The model is non-singular when $|\frac{c(t)}{c(t_0)}| \geq 1$. If $|\frac{c(t)}{c(t_0)}| < 1$, then there exists a singularity at $t^*$, given by $\tanh \mu(t^* - t_0) = -\frac{\dot{c}(t_0)}{c(t_0)}\mu$ and $c(t^*) = 0$. 

11
6.2 Quasi-two-dimensional solutions with magnetic field and matter

A second model with $B(t) = \text{const}$ is obtained for $a(t)b(t) = \text{const}$. We may write $a(t) = a(t_0)E(t)$ and $b(t) = b(t_0)E^{-1}(t)$ with $E(t_0) = 1$ where $t_0$ is some reference time. If we additionally assume $c(t) = \text{const}$, the dynamics is restricted to the $x^1Ox^2$ plane. The equations (42)-(45) reduce to the system:

$$
-L \left( \frac{\dot{E}}{E} \right)^2 = \Lambda + \kappa W + \frac{1}{2}\kappa B^2, \quad -L \frac{\ddot{E}}{E} + 2 \left( \frac{\dot{E}}{E} \right)^2 = \Lambda - \kappa P_{(1)} - \frac{1}{2}\kappa B^2,
$$

(54)

$$
L \left( \frac{\dot{E}}{E} \right)^2 = \Lambda - \kappa P_{(3)} + \frac{1}{2}\kappa B^2, \quad L \frac{\ddot{E}}{E} + 2(1-L) \left( \frac{\dot{E}}{E} \right)^2 = \Lambda - \kappa P_{(2)} - \frac{1}{2}\kappa B^2,
$$

(55)

where

$$
L \equiv 1 + (q_1 - q_3)\kappa B^2 = \text{const}.
$$

(56)

For $L \neq 0$ the system (54), (55) is equivalent to

$$
2L \left( \frac{\dot{E}}{E} \right)^2 = -\kappa (W + P_{(3)}), \quad 2L \left( \frac{\dot{E}}{E} \right)^2 = \kappa (P_{(1)} - P_{(2)}),
$$

(57)

$$
P_{(3)} = W + B^2 + \frac{2\Lambda}{\kappa}, \quad P_{(1)} + P_{(2)} = 2W + \frac{4\Lambda}{\kappa}.
$$

(58)

The minimally coupled case $q_1 = q_2 = q_3 = 0$ corresponds to $L = 1$. It is dynamically indistinguishable from a non-minimal configuration with $q_1 = q_3$ and arbitrary $q_2$. The case $L = 1$ requires $W + P_{(3)} < 0$. This can only be achieved by a sufficiently negative cosmological constant, $2W + B^2 + \frac{2\Lambda}{\kappa} < 0$ which also implies that $P_{(1)} + P_{(2)} < 0$.

6.2.1 Case $L = 0$

For $q_3 - q_1 = \frac{1}{\kappa B^2}$ the quantity $E(t)$ is constant, i.e., the universe is static. The matter distribution is characterized by constant quantities as well:

$$
P_{(1)} = P_{(2)} = \frac{\Lambda}{\kappa} - \frac{1}{2}B^2, \quad W = -P_{(3)} = -\frac{\Lambda}{\kappa} - \frac{1}{2}B^2.
$$

(59)

The energy density $W$ is positive if $\Lambda < -\frac{1}{2}\kappa B^2$. All the pressure eigenvalues become negative in this case.
6.2.2 Case $L \neq 0$, $P(1) = P(2)$

Here we obtain

$$E(t) = e^{H_0(t-t_0)}, \quad W = -LH_0^2 - \frac{\Lambda}{\kappa} - \frac{1}{2}B^2,$$

(60)

$$P(3) = -LH_0^2 + \frac{\Lambda}{\kappa} - \frac{1}{2}B^2, \quad P(1) = P(2) = -LH_0^2 - \frac{\Lambda}{\kappa} - \frac{1}{2}B^2,$$

(61)

where $H_0$ is an arbitrary integration constant. For any $L > 0$ a sufficiently negative cosmological constant is required for the energy density $W$ to be positive. After re-parametrization of the coordinates and the time the metric takes a form

$$ds^2 = dt^2 - (e^{H_0t}dx^2 + e^{-H_0t}dy^2) - dz^2.$$  

(62)

6.2.3 Ultrarelativistic matter with $L \neq 0$, $P(1) = P(2)$

If the matter is ultrarelativistic, i.e., $T_{(\text{matter})}^{k} = 0$ and, consequently, $W = P(1) + P(2) + P(3)$, one obtains

$$H_0^2 = \frac{2\Lambda}{\kappa L}, \quad W = -\frac{3}{2}LH_0^2 - \frac{1}{2}B^2,$$

(63)

$$P(3) = -\frac{1}{2}LH_0^2 + \frac{1}{2}B^2, \quad P(1) = P(2) = -\frac{1}{2}LH_0^2 - \frac{1}{2}B^2.$$

(64)

The energy density $W$ is positive for $LH_0^2 < -\frac{1}{2}B^2$, which requires the constant $L$ to be negative, i.e., $q_3 - q_1 > \frac{1}{\kappa L}$. In this case the longitudinal pressure $P(3)$ is positive as well. The transversal pressure $P(1) = P(2)$ is positive for $LH_0^2 < -B^2$. The condition $L < 0$ necessarily implies a negative cosmological constant again.

6.3 Axial symmetry: $a(t) = b(t)$, $P(1) = P(2) \equiv P(3)$

All the metric functions are assumed to be time dependent now. Einstein’s equations reduce to the following system of three equations for two unknown functions:

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} = \Lambda + \kappa W + \kappa B^2(t) \left\{ \frac{1}{2} + (7q_1 + 4q_2 + q_3) \left(\frac{\dot{a}}{a}\right)^2 + 2q_1 \frac{\dot{a}}{a}\frac{\dot{c}}{c} \right\},$$

(65)

$$\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} = \Lambda - \kappa P(3) + \kappa B^2(t) \left\{ -\frac{1}{2} + q_1 \frac{\dot{c}}{c} + (7q_1 + 4q_2 + q_3) \left[ \frac{\dot{a}}{a} - 2 \left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{\dot{c}}{c} \right] \right\},$$

(66)

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \kappa P(3) + \kappa B^2(t) \left\{ \frac{1}{2} + 2q_1 \frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 (13q_1 + 4q_2 + q_3) \right\}.  

(67)
The matter conservation law takes the form

\[ \dot{W} + \frac{\dot{a}}{a} (W + P_{(tr)}) + \frac{\dot{c}}{c} (W + P_{(3)}) = 0. \] (68)

Formally, the function \( c(t) \) does not appear in equation (67). However, in general there is a coupling to \( c(t) \) via the pressure \( P_{(3)} \) through the conservation law (68). There are two simple cases for which eq. (67) decouples from eqs. (65) and (66). The first one is \( P_{(3)} = 0 \), i.e., a vanishing longitudinal pressure, the second one is \( W + P_{(3)} = 0 \), i.e., a vacuum type behavior in longitudinal direction together with a transversal equation of state \( P_{(tr)} = P_{(tr)}(W) \). In the following we consider both cases separately.

6.3.1 "Longitudinal dust": \( P_{(3)} = 0 \)

With the substitution

\[ \dot{a}(t) = a^\sigma \sqrt{Z(a)}, \quad \sigma = \frac{13q_1 + 4q_2 + q_3}{2q_1}, \quad B^2(t) = \frac{M^2}{a^4}, \] (69)

where \( M^2 = \text{const} \) and \( q_1 \neq 0 \), equation (67) is transformed into the first order equation

\[ \left( 1 - \frac{\kappa q_1 M^2}{a^4} \right) \frac{dZ(a)}{da} + (2\sigma + 1) \frac{Z}{a} = \left( \Lambda + \frac{\kappa M^2}{2a^4} \right) a^{1-2\sigma}. \] (70)

The solution of eq. (70) can be represented as quadrature:

\[
Z(a) = H_a^2(a_0)a_0^{2-2\sigma} \left( \frac{a_0^4 - \kappa q_1 M^2}{a^4 - \kappa q_1 M^2} \right)^{\frac{2\sigma + 1}{2\sigma - 3}} \\
+ \left( a^4 - \kappa q_1 M^2 \right)^{-\frac{2\sigma + 1}{2\sigma - 3}} \int_{a_0}^{a} dx x^{1-2\sigma} \left( x^4 - \kappa q_1 M^2 \right)^{\frac{2\sigma - 3}{2\sigma - 1}} \left( \Lambda x^4 + \frac{1}{2} \kappa M^2 \right),
\] (71)

where \( H_a(a_0) \equiv a^{\sigma - 1} \sqrt{Z(a_0)} \) with \( a_0 \equiv a(t_0) \) and \( H_a(t) \equiv \frac{\dot{a}}{a} = H_b(t) \equiv \frac{\dot{b}}{b} \) is the expansion rate in the \( x^3 \) and \( x^2 \) directions. Due to the expressions \( (x^4 - \kappa q_1 M^2)^{\frac{2\sigma - 3}{2\sigma - 1}} \) and \( (a^4 - \kappa q_1 M^2)^{\frac{2\sigma - 3}{2\sigma - 1}} \) in Eq. (71), the result of the integration is sensitive to the sign of \( q_1 \). For \( q_1 > 0 \) the term \( (x^4 - \kappa q_1 M^2)^{\frac{2\sigma - 3}{2\sigma - 1}} \) has two real zeros \( x_{1,2} = \pm (\kappa q_1 M^2)^{\frac{1}{2\sigma - 1}} \). If at least one zero belongs to the interval \( (a_0, a(t)) \), the integral in eq. (71) diverges for \( \frac{2\sigma - 3}{2\sigma - 1} \leq -1 \), i.e., for \( 2\sigma + 1 \leq 0 \). If \( q_1 < 0 \), such a singularity does not appear. The asymptotic behavior of the function \( Z(a) \) for \( a \to \infty \) is

\[ Z(a \to \infty) = \frac{\Lambda}{3} a^{2-2\sigma}. \] (72)
Thus, asymptotically all the models yield
\[ \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} = H_a = \text{const}, \quad a(t) = a(t_0)e^{H_a(t-t_0)}, \] (73)
i.e., a de Sitter type expansion for \( a(t) \), independent of the parameter \( \sigma \). Moreover, one can check directly that the solution (73) is an exact solution of the equation (67) with
\[ H_a^{-2} = 2(11q_1 + 4q_2 + q_3), \] (74)
where \( 11q_1 + 4q_2 + q_3 \neq 0 \). It is remarkable, that the constant expansion rate \( H_a \) is determined both by \( \Lambda \) via eq. (73) and by the non-minimal coupling parameters via eq. (74). The relation (74) has no counterpart in the minimal theory. Combining (73) and (74) allows us to establish the following relation between \( \Lambda \) and the coupling parameters:
\[ \Lambda = \frac{3}{2(11q_1 + 4q_2 + q_3)}. \] (75)

The cosmological constant is expressed in terms of quantities which are supposed to be the result of quantum field theoretical calculations. In quantum electrodynamics the parameters \( q_1, q_2, \) and \( q_3 \) are
\[ q_1 = -\frac{\alpha \lambda^2}{120 \pi}, \quad q_2 = -13q_1, \quad q_3 = 2q_1, \]
where \( \alpha \) is the fine structure constant, i.e., they are proportional to the square of the Compton wavelength \( \lambda_e \) of the electron. This would give rise to a value \( \Lambda_{\text{QED}} = \frac{90}{13\pi \alpha \lambda^2} \) of the cosmological constant. While one does not expect a quantum electrodynamical length to set the scale for an early de Sitter stage (recall that we assume \( q_1, q_2, \) and \( q_3 \) to be free parameters) this result may nevertheless indicate a potential relevance of non-minimal interactions for an early inflationary dynamics.

With a transversal equation of state \( P_{(\gamma)}(t) = (\gamma - 1)W(t) \) and with (68) the conservation law (68) yields
\[ W(t) = W(t_0) \frac{c(t_0)}{c(t)} e^{-2H_a(t-t_0)}. \] (76)
The equation (68) for \( c(t) \) can be rewritten in the form
\[ \frac{d}{dz} \left\{ [1 - \alpha z]^{-\xi} Y(z) \right\} = -[1 - \alpha z]^{-(\xi+1)} \frac{\kappa W(t_0)}{8H_a^2} z^{2-2\gamma}, \] (77)
where
\[ z = \left( \frac{a_0}{a(t)} \right)^4, \quad \alpha = \frac{q_1\kappa M^2}{a_0^4}, \quad \xi = \frac{1 - 2q_1H_a^2}{8q_1H_a^2}, \quad Y = \frac{c(t)a(t_0)}{a(t)c(t_0)}. \] (78)
For \( \alpha \neq 1 \) the solution for \( c(t) \) with the initial value \( c(t_0) \) has the following explicit form
\[ c(t) = c(t_0)e^{H_a(t-t_0)} \left[ 1 - \alpha e^{-4H_a(t-t_0)} \right]^{\xi} \cdot \left\{ (1 - \alpha)^{-\xi} \frac{\kappa W(t_0)}{8H_a^2} \int_1 e^{-4H_a(t-t_0)} dx x^{2-2\gamma} [1 - \alpha x]^{-(\xi+1)} \right\}. \] (79)
At $\alpha x = 1$ the integral in eq. (79) is non-singular for $\xi < 0$. In the asymptotic regime $t \to \infty$ one obtains from (79)
\[ c(t) \approx c(t_0) e^{H_a(t-t_0)} \Gamma, \] (80)
where the constant value $\Gamma$ is equal to
\[ \Gamma \equiv (1 - \alpha)^{-\xi} - \frac{\kappa W(t_0)}{8 H_a^2} \int_1^0 dx x^{2+3\gamma} \left[ 1 - \alpha x \right]^{-3(\xi+1)}. \] (81)
For $x \to 0$ the constant $\Gamma$ remains finite for $2\gamma - 3 > -1$, i.e., $2\gamma + 1 > 0$.

The expression (80) shows that the expansion rate in $0x^3$ direction tends asymptotically to the expansion rate in the orthogonal direction, i.e., the universe becomes isotropic. The isotropization rate is characterized by the function $K(t)$,
\[ K(t) \equiv \log Y, \quad \dot{K}(t) = \frac{\dot{c}(t)}{c(t)} - \frac{\dot{a}(t)}{a(t)}, \quad \dot{K}(t \to \infty) \to 0. \] (82)

Since $11q_1 + 4q_2 + q_3 \neq 0$ was assumed in Eq. (74), the minimal limit $q_1 = q_2 = q_3 = 0$ cannot be taken here, since it implies $H_a \to \infty$. To check whether or not a corresponding isotropization takes place in the minimally coupled theory as well one has to solve the system (65) - (67) with $q_1 = q_2 = q_3 = 0$. It is straightforward to realize that a solution $3H_a^2 = \Lambda$ of the minimally coupled system requires $P_{(3)} = \frac{1}{3}B^2$. Consequently, a dust equation of state $P_{(3)} = 0$ means the absence of the magnetic field. While the non-minimal theory in our example admits an isotropization in the presence of a magnetic field, the minimally coupled theory does not.

For $\alpha = 1$ the initial value problem for $c(t)$ with the initial value $c(t_0)$ degenerates. This requires a special treatment and we do not consider this case here.

6.3.2 “Longitudinal quasi-vacuum”: $W + P_{(3)} = 0$

When $P_{(tr)}(t) = (\gamma - 1)W(t)$ and $W + P_{(3)} = 0$, the conservation law yields
\[ W(t) = W(t_0) \left( \frac{a(t_0)}{a(t)} \right)^{2\gamma}, \] (83)
and $a(t)$ can be found in quadratures from the equation (77). The solution $a(t) = a(t_0) e^{H_a(t-t_0)}$ holds in this case as well if the magnetic field term with $B^2(t) \propto a^{-4}(t)$ is compensated by the longitudinal pressure $P_{(3)}(t) = -W(t) \propto a^{-2\gamma}(t)$. This requires a transversal stiff matter equation of state, i.e., $\gamma = 2$. As in the previous subsection (see the discussion following eq. (79)) we obtain two expressions for $H_a$,
\[ H_a = \sqrt{\frac{\Lambda}{3}} \quad \text{and} \quad H_a^2 = \frac{2W(t_0)a_0^3 + M^2}{2M^2 (11q_1 + 4q_2 + q_3)}, \] (84)
which again imply a relation between $\Lambda$ and the parameters of the non-minimal interaction. From equation (65) we find

$$c(t) = c(t_0)e^{H_a(t-t_0)} \left[1 - \alpha e^{-4H_a(t-t_0)}\right]^{\zeta} \left[1 - \alpha\right]^{-\zeta}, \quad (85)$$

where

$$\zeta \equiv \frac{2W(t_0)a_0^4 + M^2(1 - 2q_1 H_a^2)}{8q_1 M^2 H_a^2}. \quad (86)$$

Again, the solution for $c(t)$ has the same asymptotical behavior as $a(t)$, i.e., the universe becomes isotropic. This solution is non-singular if $q_1$ is negative. $c(t)$ increases monotonically for $4\zeta < 1$. If $4\zeta > 1$ the function $c(t)$ increases after it passed a minimum value. If $q_1$ is positive, there exists a time $t^*$ with $c(t^*) = 0$, where $t^*$ is given by

$$t^* = t_0 + \frac{1}{4H_a} \log \alpha, \quad \alpha = \frac{\kappa q_1 M^2}{a_0^4}. \quad (87)$$

The model is then applicable for $t > t^*$.

Also in this case the isotropization process is a property of the non-minimal theory only. The corresponding solution $H_a = \text{const}$ of the minimal theory is necessarily isotropic for $W + P_{(3)} = 0$.

### 6.4 Isotropic universe model with “hidden” magnetic field

Let us consider now the conditions, under which an isotropic model with $a(t) = b(t) = c(t)$ is compatible with the existence of a magnetic field. The Einstein equations reduce to

$$3 \left(\dot{\frac{a}{a}}\right)^2 = \Lambda + \kappa W + \kappa B^2(t) \left\{\frac{1}{2} + (9q_1 + 4q_2 + q_3) \left(\dot{\frac{a}{a}}\right)^2\right\}, \quad (88)$$

$$2\ddot{\frac{a}{a}} + \left(\dot{\frac{a}{a}}\right)^2 = \Lambda - \kappa P_{(tr)} + \kappa B^2(t) \left\{-\frac{1}{2} + (8q_1 + 4q_2 + q_3) \ddot{\frac{a}{a}} - \left(\dot{\frac{a}{a}}\right)^2 \left(7q_1 + 4q_2 + q_3\right)\right\} \quad (89)$$

and

$$2\ddot{\frac{a}{a}} + \left(\dot{\frac{a}{a}}\right)^2 = \Lambda - \kappa P_{(3)} + \kappa B^2(t) \left\{\frac{1}{2} + 2q_1 \ddot{\frac{a}{a}} - \left(\dot{\frac{a}{a}}\right)^2 (13q_1 + 4q_2 + q_3)\right\}. \quad (90)$$

Furthermore, the trace equation becomes

$$6 \left[\dot{\frac{a}{a}} + \left(\dot{\frac{a}{a}}\right)^2\right] = 4\Lambda + \kappa(W - 2P_{(tr)} - P_{(3)}) + 2\kappa B^2(t)(9q_1 + 4q_2 + q_3) \left[\frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right]. \quad (91)$$
and the conservation law simplifies to
\[ \dot{W} + \frac{\dot{a}}{a} \left( 3W + 2P_{(\tau)} + P_{(3)} \right) = 0. \] (92)

Equations (89) and (90) are compatible when
\[ \ddot{a} \left( 6q_1 + 4q_2 + q_3 \right) + 6q_1 \left( \frac{\dot{a}}{a} \right)^2 = 1 - \frac{P_{(3)} - P_{(\tau)}}{B^2(t)}. \] (93)

In the minimally coupled case compatibility requires \( P_{(3)} = P_{(\tau)} + B^2 \). This excludes an isotropic matter configuration \( P_{(3)} = P_{(\tau)} \) together with a non-vanishing magnetic field. For the present non-minimal coupling the situation is different. Here, the compatibility condition (93) takes the form
\[ \dot{H}(t)Q_1 + H(t)Q_2 = 1, \quad H(t) = \frac{\dot{a}(t)}{a(t)}. \] (94)

where
\[ Q_1 = 6q_1 + 4q_2 + q_3, \quad Q_2 = 12q_1 + 4q_2 + q_3. \] (95)

(In the present isotropic case we have \( H_a = \frac{\dot{a}}{a} = H_b = \frac{\dot{b}}{b} = H_c = \frac{\dot{c}}{c} = H \)). The solutions of this compatibility condition can be classified as follows.

Case \( Q_1 = 0 \)
The Hubble parameter is constant and satisfies the condition \( 6H^2q_1 = 1 \).

Case \( Q_1 \neq 0 \)
For positive \( Q_2 \) the function \( H(t) \) satisfies the relation
\[ \frac{1 - \sqrt{Q_2}H(t)}{1 + \sqrt{Q_2}H(t)} = \frac{1 - \sqrt{Q_2}H(t_0)}{1 + \sqrt{Q_2}H(t_0)} e^{-\frac{2\sqrt{|Q_2|}(t-t_0)}{Q_1}}. \] (96)

For the subcase \( H(t_0) = \frac{1}{\sqrt{Q_2}} \), there exists a special constant solution
\[ H^2(t) = H^2(t_0) = \frac{1}{(12q_1 + 4q_2 + q_3)}. \] (97)

For negative \( Q_2 \) the Hubble function is
\[ H(t) = \frac{1}{\sqrt{|Q_2|}} \tan \left\{ \arctan \sqrt{|Q_2|H(t_0)} + \frac{\sqrt{|Q_2|}(t-t_0)}{Q_1} \right\}. \] (98)

For \( Q_2 = 0 \) the Hubble parameter is linear in time
\[ H(t) = H(t_0) - \frac{t-t_0}{6q_1}. \] (99)

Furthermore, for non-minimal coupling the richer structure of the field equations admits a configuration for which a non-zero magnetic field does not appear in
Einstein’s equations. This happens if all the multipliers of the terms $\kappa B^2(t)$ in (88)-(90) vanish simultaneously. The corresponding conditions are

$$H(t) \equiv \dot{a} / a = \text{const} \equiv H_0, \quad q_1 = \frac{1}{2H_0^2}, \quad 4q_2 + q_3 = -\frac{5}{H_0^2}. \quad (100)$$

It is interesting to realize that this solution coincides with (97). The third relation admits the particular case

$$q_2 = -\frac{2}{H_0^2}, \quad q_3 = \frac{3}{H_0^2}, \quad (101)$$

for which $6q_1 + 3q_2 + q_3 = 0$ and the trace of the susceptibility tensor vanishes (see, eq. (5)). As a consequence, there exists a stationary cosmological solution with

$$P_{(3)} = P_{(tr)} \equiv P, \quad W(t) = \text{const} \equiv W_0, \quad P = \text{const} = -W_0,$$

$$3H_0^2 = \kappa W_0 + \Lambda, \quad H(t) = H_0, \quad a(t) = a(t_0)e^{H_0(t-t_0)}. \quad (102)$$

The non-vanishing magnetic field is hidden as far as the space-time evolution is concerned. The additional coupling terms give rise to a non-minimal screening of the magnetic field.

Also the solution (102) with (100) and (101) is characterized by a Hubble expansion that is directly determined by the non-minimal coupling strength. While this solution does not constitute a real inflationary model since there is no exit from the de Sitter phase, we hope that it may provide the starting point for a more general approach in which the parameters $q_1$, $q_2$ and $q_3$ are no longer constants but dynamical degrees of freedom, e.g., a multiplet of scalar fields. In such a context more “realistic” solutions might well occur. The circumstance that the impact of the non-minimal coupling weakens in the long time limit becomes obvious if we try to solve, say eq. (90), with a power law ansatz $a \propto t^\nu$. All the terms in the braces on the right hand side, except for the first one which is independent of the non-minimal coupling, decay as $t^{-2}$. Consequently, these terms play a role at early times but they become irrelevant in the long time limit.

7 Discussion

Inspired by a well motivated non-minimal coupling between gravity and electromagnetism we have explicitly demonstrated that the richer structure of the corresponding theory gives rise to novel features of the cosmological dynamics. We have obtained a number of simple exact solutions for Bianchi I models with magnetic field. For axially symmetric configurations we found inflationary type solutions with magnetic field which describe an isotropization process as a result of the non-minimal coupling, i.e., without a counterpart in the minimally coupled theory. Furthermore, some solutions of the non-minimal theory
establish a direct relation between a cosmological constant and the coupling parameters of the non-minimal interaction. Finally, we have shown that there exists an isotropic de Sitter solution for which the magnetic field is screened by the non-minimal coupling.

Acknowledgments

This work was supported by the Deutsche Forschungsgemeinschaft.

References

[1] Stephani H., Kramer D., MacCallum M., Hoenselaers C., Herlt E. Exact solutions of Einstein’s field equations. Cambridge University Press, 2003.
[2] Zel’dovich Ya.B. and Novikov, I.D. The structure and evolution of the universe. University of Chicago Press, 1983.
[3] Barrow J.D. and Ferreira P., and Silk, J. Phys. Rev. Lett. 78 (1997) 3610.
[4] Drummond I.T. and Hathrell S.J. Phys. Rev. D 22 (1980) 343.
[5] Prasanna A.R. Phys. Lett. A 37 (1971) 331.
[6] Prasanna A.R. Lett. Nuovo Cim. 6 (1973) 420.
[7] Novello M. and Salim J.M. Phys. Rev. D 20 (1979) 377.
[8] Accioly A.J., Vaidya A.N., and Som M.M. Phys. Rev. D 28 (1983) 1853.
[9] Novello M. and Heintzmann H. Gen. Rel. Grav. 16 (1984) 527.
[10] Souza J.G., Bedran M.L., and Lesche B. Rev. Bras. Fis. 14 (1984) 488.
[11] Goenner H. Found. Phys. 14 (1984) 865.
[12] Accioly A.J. and Pereira da Silva N.L.P. Prog. Theor. Phys. Lett. 76 (1986) 1179.
[13] Turner M.S. and Widrow L.M. Phys. Rev. D 37 (1988) 2743.
[14] Müller-Hoissen, F. Phys. Lett. B 201 (1988) 325.
[15] Novello M., Oliveira L.A.R., and Salim J.M. Class. Quantum Grav. 7 (1990) 51.
[16] Accioly A., Azeredo A.D., De Aragão C.M.L., and Mukai H. Class. Quantum Grav. 14 (1997) 1163.
[17] Mohanty S. and Prasanna A.R. Nucl. Phys. B. 526 (1998) 501.
[18] Teyssandier P. [gr-qc/0303081]
[19] Lafrance R. and Myers R.C. Phys. Rev. D 51 (1995) 2584.
[20] Prasanna A.R. and Mohanty S. Class. Quantum Grav. 20 (2003) 3023.
[21] Solanki S.K., Preuss O., Haugan M.P., Gandorfer A., Povel H.P., Steiner P., Stucki K., Bernasconi P.N., and Soltan D. Phys. Rev. D 69 (2004) 062001.
[22] Colladay D. and Kostelecký V.A. Phys.Rev.D 58 (1998) 116002.
[23] Kostelecký V.A. and Mewes M. Phys.Rev.Lett. 87 (2001) 251304.
[24] Kostelecký V.A. and Mewes M. Phys.Rev. D 66 (2002) 056005.
[25] Balakin A.B. Class. Quantum Grav. 14 (1997) 2881.
[26] Balakin A.B. Ann. Phys.(Leipzig) 9 (2000) SI-21-24.
[27] Balakin A.B. and Lemos J.P.S. Class. Quantum Grav. 18 (2001) 941.
[28] Balakin A.B., Kerner R., and Lemos J.P.S. Class. Quantum Grav. 18 (2001) 2217.
[29] Balakin A.B. and Lemos J.P.S. Class.Quant.Grav. 19 (2002) 4897.
[30] Zimdahl W., Schwarz D.J., Balakin A.B., and Pavón D. Phys.Rev. D 64 (2001) 063501.
[31] Balakin A.B., Pavón D., Schwarz D.J., and Zimdahl W. NJP 5 (2003) 85.1 -85.14.
[32] Rubilar G.F., Obukhov Yu.N., and Hehl F.W. Class. Quantum Grav. 20 (2003)L185.
[33] Itin Ya. and Hehl F.W. Phys. Rev. D 68 (2003) 127701.
[34] Hehl F.W. and Obukhov Yu.N. Lect. Notes Phys. 562 (2001) 479.
[35] van der Bij J.J. and Radu E. (hep-th/0003073).
[36] Balakin A.B. and Lemos J.P.S. (gr-qc/0503076).
[37] Eringen A.C. and Maugin G. A. Electrodynamics of continua( Springer-Verlag, New York, 1989).
[38] Landau L. D., Lifchitz E. M., and Pitaevskii L. P. Electrodynamics of Continuous Media (Butterworth Heinemann, Oxford 1996).
[39] Hehl F.W. and Obukhov Yu.N. Foundations of Classical Electrodynamics: Charge, Flux, and Metric. Birkhäuser. Boston. 2003.
[40] Lichnerowicz A. (1967). Relativistic Hydrodynamics and Magnetohydrodynamics, Benjamin, New York, p. 84 ff.
[41] Lemos J.P.S. and Kerner R. *Grav.Cosmol.* **6** (2000) 49-58.

[42] Tsagas C.G. and Maartens R. *Class. Quantum Grav.* **17** (2000) 2215.

[43] Wainwright J. and Ellis G.F.R. (eds.) *Dynamical systems in cosmology.* (Cambridge University Press, 1997).