Periodic Strategies II: Generalizations and Extensions

V. K. Oikonomou1, J. Jost1,2
1Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, 04103 Leipzig, Germany
2Santa Fe Institute, New Mexico, USA

May 27, 2020

Abstract

At a mixed Nash equilibrium, the payoff of a player does not depend on her own action, as long as her opponent sticks to his. In a periodic strategy, a concept developed in a previous paper [3], in contrast, the own payoff does not depend on the opponent’s action. Here, we generalize this to multi-player simultaneous perfect information strategic form games. We show that also in this class of games, there always exists at least one periodic strategy, and we investigate the mathematical properties of such periodic strategies. In addition, we demonstrate that periodic strategies may exist in games with incomplete information; we shall focus on Bayesian games. Moreover we discuss the differences between the periodic strategies formalism and cooperative game theory. In fact, the periodic strategies are obtained in a purely non-cooperative way, and periodic strategies are as cooperative as the Nash equilibria are. Finally, we incorporate the periodic strategies in an epistemic game theory framework, and discuss several features of this approach.

1 Introduction

John Nash [1] showed that every strategic form game possesses at least one Nash equilibrium (for an alternative proof, that avoids the use of Brouwer’s fixed point theorem and only needs simple topological facts about bifurcations, see [2]). Here, it is assumed that players act rationally in the sense that they try to maximize their payoffs, and this rationality of all players is common knowledge, as are the possible actions and payoffs of each player. The Nash equilibrium then is consistent in the sense that when everybody plays it, no single player could gain an advantage from a unilateral deviation. Such a Nash equilibrium can be pure, that is, each player plays some definite strategy, or mixed, where some players choose among their actions with certain probabilities. For instance, the matching pennies game has only one Nash equilibrium, and this is mixed, as each player plays either strategy randomly with probability 1/2. Such a mixed Nash
equilibrium has a curious property. To see this, for simplicity, we consider a game with two players $i = A, B$ who have two possible actions 1, 2 each. When $A$ and $B$ play actions $\alpha$ and $\beta$ with respective probabilities $p_\alpha$ and $q_\beta$ (with $p_1 + p_2 = 1 = q_1 + q_2$), then the (expected) utility of $A$ is (in obvious notation)

$$U_A = \sum_{\alpha, \beta} U_A(\alpha, \beta) p_\alpha q_\beta.$$  \hfill (1)

When now $A$ wants to maximize her payoff, she adjusts her probabilities $p_1$ and applies calculus to get as a first order necessary condition at a mixed value $0 < p_1 < 1$

$$0 = \sum_\beta U_A(1, \beta) q_\beta - \sum_\beta U_A(2, \beta) q_\beta.$$ \hfill (2)

This then is a condition about the probabilities $q_\beta$ of her opponent which is independent of her own probabilities $p_\alpha$. That is, when the opponent plays according to those values, it is irrelevant for $A$ what she plays. She will always get the same payoff. Thus, at a mixed Nash equilibrium, when every player has a mixed strategy, no single player can change her outcome by changing her strategy, as long as all others stick to their probabilities.

Of course, this is well known. The phenomenon is simply a consequence of the fact that the utility $\mathcal{U}$ depends linearly on the probabilities of the individual players. Therefore, taking the derivative w.r.t. them makes the resulting condition independent of them.

In [3], we have investigated what happens when $A$ seeks a critical point of (1) not with respect to her own probability $p_1$, but with respect to the opponent probability $q_1$. We then get the condition

$$0 = \sum_\alpha U_A(\alpha, 1) p_\alpha - \sum_\alpha U_A(\alpha, 2) p_\alpha.$$  \hfill (3)

This is now independent of the opponent’s probabilities $q_\beta$. That is, when $A$ plays according to (3), her payoff is unaffected by the choice of strategy of her opponent.

Let us consider a simple example where the payoff table is given by,

|   | 1   | 2   |
|---|-----|-----|
| 1 | 2.1 | 0.0 |
| 2 | 0.0 | 1.1 |

with $A$ being the row player and $B$ the column player. (2) for $A$ yields $q_1 = 1/3, q_2 = 2/3$, and the analogous computation for $B$ gives $p_1 = 1/2 = p_2$. The expected payoffs for at this mixed Nash equilibrium are

$$\bar{U}_A = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{3}, \quad \bar{U}_B = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}.$$ \hfill (4)

In contrast, (3), when applied for both players, yields $p_1 = \frac{1}{3}, q_1 = \frac{1}{2}$. The expected utilities remain the same. As investigated in [3], the latter property does not always
hold, that is, the payoffs at a mixed Nash and at an equilibrium computed according to (3) need not always be the same, and depending on the game, either of them could be larger than the other. But an equilibrium according to (3), called periodic for reasons to be discussed in a moment, exists in the same generality as a mixed Nash equilibrium and to show this is the main purpose of this paper.

In order to explore the consistency of such an equilibrium, it is useful to recall the concept of rationalizability of Bernheim [37] and Pearce [38]. Here, a sequence of alternating strategy choices of A and B is called rationalizable if each of them is a best response to the previous strategy of the opponent. A Nash equilibrium is rationalizable, but in general, there are other sequences of rationalizability strategies. For instance, in the matching pennies game, there is a sequence where each player alternates between her/his two options. In such a sequence, each strategy periodically repeats itself. A similar phenomenon exists also for our type of equilibrium, and this is the reason why it was called periodic. The interpretation is somewhat curious, however. Since a player can of course not directly choose the opponent’s probabilities to maximize her payoff, which was the assumption underlying (3), the logic has to become somewhat different. While A cannot choose $q_1$, her opponent can choose his $q_1$ so as to maximize A’s payoff, and conversely, A could then choose her $p_1$ to maximize B’s payoff. Again, this can be done as an iterative response as in the rationalizability paradigm, and when both players act that way, this is perfectly self-consistent. That is, when everybody believes that everybody else operates in that way and acts accordingly, a periodic cycle exists that confirms everybody’s belief.

In this paper, we first generalize the theorems related to periodic strategies to simultaneous multi-player perfect information strategic form games. Several examples will illustrate the new features brought by the presence of three or more players. Periodic strategies for non-trivial perfect information simultaneous strategic form games are related (or in some cases are identical) to all the existing rationalizable strategies [1, 20], as demonstrated in [3]. We shall then turn to the question whether such periodic strategies also exist in strategic form games with incomplete information. We shall mainly focus on Bayesian games [6, 21, 31], in which case the presence of rationalizable strategies (ex ante and interim) suggests that periodic strategies should also exist in this type of games. In fact, games where the players are uncertain about the setting and only know that certain scenarios occur with certain probabilities can sometimes be modelled as a games with an additional nature player who chooses among the scenarios with those probabilities. This will also be useful for our reasoning. Also with regard to incomplete information and cooperative game theory, in Ref. [32] an interesting approach was used in order to study reentrant phase transitions and defensive alliances in social dilemmas with informed strategies. Furthermore a review on co-evolutionary games was provided in Ref. [33].

One of the most important features of the periodic strategy algorithm is that periodic strategies do apply to non-cooperative game theory [34, 35]. As explained, however, by construction, the periodic strategies are based on maximizing the payoffs of a game for a player by using the probability distribution of the opponents. We should point out, however, that this is different from the setting of cooperative game theory, which
is about coalitions and distributions of payoffs inside them. In contrast, the procedure for obtaining the periodic strategies involves maximization of a player’s own utility function, without any cooperation with the opponent, or any apparent agreement. Finally, we shall attempt to incorporate the periodicity concept into an epistemic game theory [24,36] theoretical framework.

The outline of the paper is as follows: In section 2, we generalize the periodic solution concept to multi-player finite, perfect information simultaneous strategic form games. In section 3 we study the periodic solution concept for games with incomplete information, quantified in terms of Bayesian games. The non-cooperativity argument on which periodicity is based is discussed in section 4, while in section 5, we incorporate the periodicity concept in a very simple epistemic game theory framework by connecting types to the periodicity number, without getting into much details however. The conclusions along with future perspectives of the periodicity concept follow at the end of the paper.

2 Generalization of the Periodicity Concept to Multi-player Games – The Perfect Information Case

In this section we generalize the concept of periodicity to multi-player strategic form games with perfect information. We start with a concrete example. Consider a three player game with

- The set of players: \( I = A, B, C \)
- Their strategy spaces \( \mathcal{M}(A), \mathcal{M}(B), \mathcal{M}(C) \) and the total strategy space \( \bar{G} = \mathcal{M}(A) \times \mathcal{M}(B) \times \mathcal{M}(C) \)
- The payoff functions \( \mathcal{U}(i) : \bar{G} \to \mathbb{R}, i = A, B, C \)

We define six continuous maps between the strategy spaces \( \mathcal{M}(i) \) and \( \mathcal{M}(j) \),

\[
\varphi_{ij} : \mathcal{M}(i) \to \mathcal{M}(j)
\]  

(5)

We usually write \( \varphi_{ij} \circ \varphi_{km} \) for \( \varphi_{ij} \circ \varphi_{km} \). The maps \( \varphi_{ij} \) and \( \varphi_{ji} \), act in such a way that when we start with an action \( x_k \) of player \( i \), the following inequality holds:

\[
\mathcal{U}(i)(x_k, \varphi_{ij}(x_k), \varphi_{im}(x_k)) > \mathcal{U}(i)(x_k, x, y) \quad \forall (x, y) \in \mathcal{M}(j) \cup \mathcal{M}(m) \setminus \{\varphi_{ij}(x_k), \varphi_{im}(x_k)\}
\]

(6)

In the example of GAME 1 in Fig. 1 each player has two actions available. In Fig. 2 we can see the periodicity chains for the action \( a_1 \) of player A, recalling the periodicity concept we gave in the 2-player game case in Ref. [3]. Let us give a verbal description of the periodicity diagram. The letters \( ABC \) on the arrows indicate the player whose action is considered.

Player A will play \( a_1 \) if player B plays \( b_2 \) and player C plays \( c_2 \) simultaneously. In the map notation, this becomes \( \varphi_{AB}(a_1), \varphi_{AC}(a_1) \), as indicated in the figure. By following
Figure 1: A 3-Player Game payoff matrix. The game is a simultaneous action game of three players A, B and C. The actions of the players A, B and C are denoted as \((a_1, a_2)\), \((b_1, b_2)\) and \((c_1, c_2)\) respectively.

the B arrow, B will play \(b_2\) if player A plays \(a_2\) and player C plays \(c_2\). Following C in node "1", C would play \(c_2\) if player B plays \(b_2\) and A plays \(a_1\) (we have reached a periodic cycle at this point but we continue in order to show the new structures). Back in node 2, following the C arrow, C will play \(c_2\) if B plays \(b_2\) and A plays \(a_1\). Back in node 2 following the arrow A, A will play \(a_2\) if B plays \(b_2\) and C plays \(c_2\). Accordingly, in node 3, following arrow B, B will play \(b_2\) if A plays \(a_2\) and C plays \(c_2\) (we have reached a set stable cycle of \(a_2\) as we will see) and so on. Back at node 3, following

Figure 2: The periodicity of the strategy \(a_1\) for the 3-player game of Fig. 1. In the figure it is shown how the periodicity concept is realized for the strategy \(a_1\) of player A in detail. The graph returns to the original strategy \(a_1\).

A, A will play \(a_1\), if B plays \(b_2\) and C plays \(c_2\) and so on. Thus, we have the periodic
The most striking new feature of the multi-player game case is the fact that in the periodicity algorithm, the utility functions appear in a rather different order as we shall see. Let us take the first type, $\varphi_{CA\varphi_{AC}(a_1)} = a_1$. The periodic algorithm in terms of the utility functions is

$$U_A(a_1, \varphi_{AB}(a_1), \varphi_{AC}(a_1)) > U_A(a_1, x, y) \quad \forall (x, y) \in \mathcal{M}(B) \cup \mathcal{M}(C) \{\varphi_{AB}(a_1), \varphi_{AC}(a_1)\}$$

(8)

$$U_C(\varphi_{AC}(a_1), \varphi_{CA\varphi_{AC}(a_1)}, \varphi_{CB\varphi_{AC}(a_1)}) > U_C(\varphi_{AC}(a_1), x, y) \quad \forall (x, y) \in \mathcal{M}(A) \cup \mathcal{M}(B) \{\varphi_{CA\varphi_{AC}(a_1)}, \varphi_{CB\varphi_{AC}(a_1)}\}$$

$$U_A(\varphi_{CA\varphi_{AC}(a_1)}, \varphi_{AB}\varphi_{CA\varphi_{AC}(a_1)}, \varphi_{AC}\varphi_{CA\varphi_{AC}(a_1)}) > U_A(\varphi_{CA\varphi_{AC}(a_1), x, y}) \quad \forall (x, y) \in \mathcal{M}(C) \cup \mathcal{M}(B) \{\varphi_{AB}\varphi_{CA\varphi_{AC}(a_1)}, \varphi_{AC}\varphi_{CA\varphi_{AC}(a_1)}\}$$

For the other type, $\varphi_{CA\varphi_{BC\varphi_{AB}(a_1)} = a_1}$, the periodic algorithm becomes

$$U_A(a_1, \varphi_{AB}(a_1), \varphi_{AC}(a_1)) > U_A(a_1, x, y) \quad \forall (x, y) \in \mathcal{M}(B) \cup \mathcal{M}(C) \{\varphi_{AB}(a_1), \varphi_{AC}(a_1)\}$$

(9)

$$U_B(\varphi_{AB}(a_1), \varphi_{BA\varphi_{AB}(a_1), \varphi_{BC\varphi_{AB}(a_1)}) > U_C(\varphi_{AB}(a_1), x, y) \quad \forall (x, y) \in \mathcal{M}(A) \cup \mathcal{M}(B) \{\varphi_{BA\varphi_{AB}(a_1), \varphi_{BC\varphi_{AB}(a_1)}\}$$

$$U_C(\varphi_{BC\varphi_{AB}(a_1)}, \varphi_{CA\varphi_{BC\varphi_{AB}(a_1)}), \varphi_{CB\varphi_{BC\varphi_{AB}(a_1)}) > U_C(\varphi_{BC\varphi_{AB}(a_1), x, y}) \quad \forall (x, y) \in \mathcal{M}(C) \cup \mathcal{M}(B) \{\varphi_{CA\varphi_{BC\varphi_{AB}(a_1)}, \varphi_{CB\varphi_{BC\varphi_{AB}(a_1)}\}$$

In terms of utility functions, this looks like

$$U_A \xrightarrow{P} U_C \xrightarrow{P} U_A$$

(10)

$$U_A \xrightarrow{P} U_C \xrightarrow{P} U_A \xrightarrow{P} U_C \xrightarrow{P} U_A$$

$$U_A \xrightarrow{P} U_B \xrightarrow{P} U_B \xrightarrow{P} U_B$$

$$U_A \xrightarrow{P} U_B \xrightarrow{P} U_A \xrightarrow{P} U_C$$

$$U_A \xrightarrow{P} U_C \xrightarrow{P} U_B \xrightarrow{P} U_C$$

If we include all the periodic points we found in the graph, we have the following new...
types of periodicity (some of which belong to set stable cycles):

\[
\begin{align*}
\varphi_{CA}\varphi_{AC}(a_1) &= a_1 \\
\varphi_{BC}\varphi_{AB}(a_1) &= a_1 \\
\varphi_{BC}\varphi_{CB}\varphi_{AC}(a_1) &= a_1 \\
\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1) &= a_1 \\
\varphi_{BC}\varphi_{AB}\varphi_{CA}\varphi_{AC}(a_1) &= a_1 \\
\varphi_{BC}\varphi_{CB}\varphi_{AC}(a_1) &= a_1
\end{align*}
\]

(11)

The periodicity corresponding to the \(a_2\) action is shown in Fig. 3. In general, there can

be various types of periodicity, with their number, type and form not directly depending on the numbers of players and actions. As the number of players increases, depending on the payoffs, the complexity of the periodic strategies significantly increases. But as will become obvious, the complexity of the algorithm depends strongly on the payoffs.

Now we generalize this type of games and we proceed to a 4-player game with each player having again two available actions, as shown in Fig. 5. In Fig. 4 and 6, we see the periodic structure for the actions \(a_1\) and \(a_2\), resp. We now look at the periodic

Figure 3: Periodicity of strategy \(a_2\) for the 3-player game of Fig. 1.
strategies for \( a_1 \) in Fig. 4. These are the following:

\[
\begin{align*}
\varphi_{DA}\varphi_{AD}(a_1) &= a_1 \\
\varphi_{DA}\varphi_{CD}\varphi_{AC}(a_1)(a_1) &= a_1 \\
\varphi_{DA}\varphi_{AD}\varphi_{CA}(a_1)\varphi_{AC}(a_1) &= a_1 \\
\varphi_{DA}\varphi_{BD}\varphi_{AB}(a_1)\varphi_{BA}(a_1)\varphi_{AB}(a_1) &= a_1 \\
\varphi_{DA}\varphi_{AD}\varphi_{BA}\varphi_{AB}(a_1) &= a_1
\end{align*}
\]

In terms of utility functions, this looks as follows, with Type \( j \) referring to line \( j \) in Table 2.

**Type 1**

\[
\begin{align*}
U_A(a_1, \varphi_{AD}(a_1), \varphi_{AC}(a_1), \varphi_{AB}(a_1)) &> U_A(a_1, x, y, z) \\
&\forall (x, y, z) \in M(B) \cup M(C) \cup M(D) \setminus \{\varphi_{AD}(a_1), \varphi_{AC}(a_1), \varphi_{AB}(a_1)\}
\end{align*}
\]

\[
\begin{align*}
U_D(\varphi_{AD}(a_1), \varphi_{DA}\varphi_{AD}(a_1), \varphi_{DC}\varphi_{AD}(a_1), \varphi_{DB}\varphi_{AD}(a_1)) &> U_D(\varphi_{AD}(a_1), x, y, z) \\
&\forall (x, y, z) \in M(A) \cup M(B) \cup M(C) \setminus \{\varphi_{BA}\varphi_{AB}(a_1), \varphi_{BC}\varphi_{AB}(a_1)\}
\end{align*}
\]

**Type 2**
Figure 5: A 4-player game payoff matrix. The game is a simultaneous action game of four players A, B, C and D.

\[ U_A(a_1, \varphi_{AC}(a_1), \varphi_{AD}(a_1), \varphi_{AB}(a_1)) > U_A(a_1, x, y, z) \] (15)
\[ \forall (x, y, z) \in M(B) \cup M(C) \cup M(D) \setminus \{ \varphi_{AD}(a_1), \varphi_{AC}(a_1), \varphi_{AB}(a_1) \} \]
\[ U_C(\varphi_{AC}(a_1), \varphi_{CA}\varphi_{AC}(a_1), \varphi_{CB}\varphi_{AC}(a_1), \varphi_{CD}\varphi_{AC}(a_1)) > U_C(\varphi_{AC}(a_1), x, y, z) \]
\[ \forall (x, y, z) \in M(A) \cup M(B) \cup M(D) \setminus \{ \varphi_{CA}\varphi_{AC}(a_1), \varphi_{CB}\varphi_{AC}(a_1), \varphi_{CD}\varphi_{AC}(a_1) \} \]
\[ U_D(\varphi_{CD}\varphi_{AC}(a_1), \varphi_{DA}\varphi_{CD}\varphi_{AC}(a_1), \varphi_{DB}\varphi_{CD}\varphi_{AC}(a_1), \varphi_{DC}\varphi_{CD}\varphi_{AC}(a_1)) > U_C(\varphi_{CD}\varphi_{AC}(a_1), x, y, z) \]
\[ \forall (x, y, z) \in M(C) \cup M(B) \cup M(A) \setminus \{ \varphi_{DA}\varphi_{CD}\varphi_{AC}(a_1), \varphi_{DB}\varphi_{CD}\varphi_{AC}(a_1), \varphi_{DC}\varphi_{CD}\varphi_{AC}(a_1) \} \]

Type 3
\[\mathcal{U}_A(a_1, \varphi_{AC}(a_1), \varphi_{AD}(a_1), \varphi_{AB}(a_1)) > \mathcal{U}_A(a_1, x, y, z)\] (16)

\[\forall (x, y, z) \in M(B) \cup M(C) \cup M(D) \setminus \{\varphi_{AD}(a_1), \varphi_{AC}(a_1), \varphi_{AB}(a_1)\}\]

\[\mathcal{U}_C(\varphi_{AC}(a_1), \varphi_{CA}\varphi_{AC}(a_1), \varphi_{CB}\varphi_{AC}(a_1), \varphi_{CD}\varphi_{AC}(a_1)) > \mathcal{U}_C(\varphi_{AC}(a_1), x, y, z) \forall (x, y, z)\]

\[\forall (x, y, z) \in M(A) \cup M(B) \cup M(D) \setminus \{\varphi_{CA}\varphi_{AC}(a_1), \varphi_{CB}\varphi_{AC}(a_1), \varphi_{CD}\varphi_{AC}(a_1)\}\]

\[\mathcal{U}_A(\varphi_{CA}\varphi_{AC}(a_1), \varphi_{AC}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{AB}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1)) > \]

\[\mathcal{U}_A(\varphi_{CA}\varphi_{AC}(a_1), x, y, z)\]

\[\forall (x, y, z) \in M(C) \cup M(B) \cup M(A) \setminus \{\varphi_{DA}\varphi_{CD}\varphi_{AC}(a_1), \varphi_{DB}\varphi_{CD}\varphi_{AC}(a_1), \varphi_{DC}\varphi_{CD}\varphi_{AC}(a_1)\}\]

\[\mathcal{U}_D(\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{DA}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{DA}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{DB}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1)) > \]

\[\mathcal{U}_D(\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), x, y, z)\]

\[\forall (x, y, z) \in M(C) \cup M(B) \cup M(A) \setminus \{\varphi_{DA}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{DA}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{DB}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1)\}\]

**Type 4**
\[ U_A(a_1, \varphi_{AC}(a_1), \varphi_{AD}(a_1), \varphi_{AB}(a_1)) > U_A(a_1, x, y, z) \] ∀(x, y, z)

\[ \in M(B) \cup M(C) \cup M(D) \{ \{ \varphi_{AD}(a_1), \varphi_{AC}(a_1), \varphi_{AB}(a_1) \} \}
\]

\[ U_B(\varphi_{AB}(a_1), \varphi_{BA}\varphi_{AB}(a_1), \varphi_{BC}\varphi_{AB}(a_1), \varphi_{BD}\varphi_{AB}(a_1)) > U_B(\varphi_{AB}(a_1), x, y, z) \]

\[ \in M(A) \cup M(C) \cup M(D) \{ \{ \varphi_{BA}\varphi_{AB}(a_1), \varphi_{BC}\varphi_{AB}(a_1), \varphi_{BD}\varphi_{AB}(a_1) \} \}
\]

\[ U_A(\varphi_{BA}\varphi_{AB}(a_1), \varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{AB}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{AD}\varphi_{BA}\varphi_{AB}(a_1)) > \]

\[ U_A(\varphi_{BA}\varphi_{AB}(a_1), x, y, z) \quad \forall(x, y, z) \in M(C) \cup M(B) \cup M(D) \{ \{ \varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{AB}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{AD}\varphi_{BA}\varphi_{AB}(a_1) \} \}
\]

\[ U_C(\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{CA}\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{CD}\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{CB}\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{DC}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1)) > U_C(\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), x, y, z) \]

\[ \in \quad M(D) \cup M(B) \cup M(A) \{ \{ \varphi_{DA}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{DA}\varphi_{AD}\varphi_{CA}\varphi_{AC}(a_1), \varphi_{DB}\varphi_{CD}\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{DB}\varphi_{CD}\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1), \varphi_{DB}\varphi_{CD}\varphi_{AC}\varphi_{BA}\varphi_{AB}(a_1) \} \}
\]

Player A would play \( a_1 \) if players D, B and C play \( d_2, b_2 \) and \( c_2 \). Following arrow C, player C would play \( c_2 \) if players A, B and D play simultaneously \( a_2, b_1 \) and \( d_2 \). Following arrow B at node 2, player B would play \( b_1 \) if players A, C and D play \( a_2, c_2 \) and \( d_1 \). Following arrow A at node 2, player A would play \( a_2 \) if players B, C and D play \( b_2, c_2 \) and \( d_2 \). Following arrow D at node 4, player D would play \( d_2 \) if players A, B and C play \( a_1, b_2 \) and \( c_1 \).

We have reached the first periodic point. Following arrow b at node 4, player B would play \( b_2 \) if players A, C and D play \( a_2, c_2 \) and \( d_2 \). Following arrow C at node 4, player C would play \( c_2 \) if players A, B and D play \( a_2, b_1 \) and \( d_2 \). Going back to node 2, following arrow D, player D would play \( d_2 \) if players A, B and C play \( a_1, b_2 \) and \( c_1 \). Going back to node 1, following arrow B at node 1, player B would play \( b_2 \) if players A, C and D play \( a_2, c_2 \) and \( d_1 \). Following arrow A at node 4, player A would play \( a_2 \) if players B, C and D play \( b_2, c_2 \) and \( d_2 \). Following arrow B at node 5, player B would play \( b_2 \) if players A, C and D play \( a_2, c_2 \) and \( d_2 \). Player D would then play \( d_2 \), if players A, B and C play \( a_1, b_2 \) and \( c_1 \). Following arrow D at node 5, player D would play \( d_2 \) if players A, B and C play \( a_1, b_2 \) and \( c_1 \). Following arrow C at node 5, player C would play \( c_2 \) if players A, B and D play \( a_2, b_1 \) and \( d_2 \). Finally, D would play \( d_2 \) if players A and C play \( a_1, b_2 \) and \( c_1 \).
2.1 The Periodicity Concept for Simultaneous Perfect Information Multi-Player Games

After these examples, we shall now generalize the concept of periodicity to general multi-player simultaneous perfect information strategic form games. Consider a finite player, finite action, perfect information, simultaneous, strategic form game, with

- The set \( I = 1, \ldots, N \) of players
- Their strategy spaces \( \mathcal{M}(i), \ldots \) and the total strategy space \( \mathcal{G} = \mathcal{M}(1) \times \cdots \times \mathcal{M}(N) \)
- Their payoff functions \( \mathcal{U}(\mathcal{G}) : \mathcal{G} \to \mathbb{R} \).

We define \( 2N \) continuous maps between the strategy spaces \( \mathcal{M}(i) \) and \( \mathcal{M}(j) \),

\[
\varphi_{ij} : \mathcal{M}(i) \to \mathcal{M}(j)
\]

The maps act in such a way that, when starting with an action \( x_i \) of player \( i \), the following inequalities hold:

\[
\mathcal{U}_k(\varphi_{ik}(x_i), \varphi_{kj}(x_i), \ldots, \varphi_{kl}(x_i)) > \mathcal{U}_l(x_i, y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{ij}(x_i), \varphi_{ik}(x_i), \ldots, \varphi_{il}(x_i))\}
\]

\[
\mathcal{U}_k(\varphi_{ik}(x_i), \varphi_{kj}(x_i), \varphi_{kl}(x_i), \ldots, \varphi_{kl}(x_i)) > \mathcal{U}_l(\varphi_{ik}(x_i), y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{ik}(x_i), \varphi_{kj}(x_i), \varphi_{kl}(x_i), \ldots, \varphi_{kl}(x_i))\}
\]

\[
\vdots
\]

\[
\mathcal{U}_m(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i) \ldots \varphi_{mk1} \varphi_{mk2} \ldots \varphi_{ik}(x_i)) >
\]

\[
\mathcal{U}_k(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i), y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i))\}
\]

\[
\mathcal{U}_m(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i) \ldots \varphi_{mk1} \varphi_{mk2} \ldots \varphi_{ik}(x_i)) >
\]

\[
\mathcal{U}_k(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i), y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i))\}
\]

\[
\mathcal{U}_m(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i) \ldots \varphi_{mk1} \varphi_{mk2} \ldots \varphi_{ik}(x_i)) >
\]

\[
\mathcal{U}_k(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i), y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i))\}
\]

\[
\mathcal{U}_m(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i) \ldots \varphi_{mk1} \varphi_{mk2} \ldots \varphi_{ik}(x_i)) >
\]

\[
\mathcal{U}_k(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i), y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i))\}
\]

\[
\mathcal{U}_m(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i) \ldots \varphi_{mk1} \varphi_{mk2} \ldots \varphi_{ik}(x_i)) >
\]

\[
\mathcal{U}_k(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i), y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i))\}
\]

\[
\mathcal{U}_m(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i) \ldots \varphi_{mk1} \varphi_{mk2} \ldots \varphi_{ik}(x_i)) >
\]

\[
\mathcal{U}_k(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i), y_1, y_2, \ldots, y_l)
\]

\[
\forall (y_1, y_2, \ldots, y_l) \in \mathcal{M}(j) \cup \mathcal{M}(k) \cup \cdots \cup \mathcal{M}(l) \setminus \{(\varphi_{mm1} \varphi_{mm2} \ldots \varphi_{ik}(x_i))\}
\]
We call the action $x_i$ periodic if at some step of the periodicity algorithm [3], we have

$$x_i = \varphi_{m1} \circ \ldots \circ \varphi_{ik}(x_i)$$

(20)

Let us explain the meaning of each step of the algorithm. Start with the first step, when $i$ plays $x_i$, his payoff is maximized when his opponents play a combination of actions (simultaneously), namely the actions $(\varphi_{i1}(x_i), \varphi_{i2}(x_i), \ldots, \varphi_{iN}(x_i))$. This procedure is repeated at every step.

**Definition 1** (Periodicity). In an $N$-player simultaneous move strategic form game with finite actions, we define periodic strategies for player A to be the subset $P_A$ of his available strategies $M_A$ for which there exists an operator $Q: M_A \rightarrow M_A$, with $Q = \varphi_{ij}(x_i), \varphi_{ik}(x_i), \varphi_{il}(x_i)$ for which $Qx_i = x_i$ such that the inequalities of relation (19) are fulfilled at each step.

Periodic strategies are structures inherent to every non-trivial finite action $N$-player strategic form game.

**Theorem 1.** Every finite action simultaneous $N$-player strategic form game contains at least one periodic action.

**Proof.** The proof of the theorem is very easy, since the inequalities (19) hold. Let us consider player $i$ and start from an action $x_*$ which is assumed to be non-periodic. If we apply the maps $\varphi_{ij}$ to $x_*$, so that the inequalities (19) are satisfied at every step, then, since the game contains a finite number $n$ of actions, there will be an action $x_a$ for which there exists an operator constructed from a finite number of maps $Q = \varphi_{ij}(x_i), \varphi_{ik}(x_i), \varphi_{il}(x_i)$, so that $Qx_a = x_a$. If the above is not true for any other action apart from $x_a$, then since the game contains a finite number of actions, this would imply that $x_a$ is periodic. So every finite action game contains at least one periodic action. A more detailed proof goes as follows. Suppose we start with the non-periodic action $x_i$ of player $i$. Then

$$U_i(x_i, \varphi_{ij}(x_i), \varphi_{ik}(x_i), \ldots, \varphi_{il}(x_i)) > U_i(x_i, y_1, y_2, \ldots, y_l)$$

(21)

The algorithm will continue for some player $k$,

$$U_k(\varphi_{ik}(x_i), \varphi_{ikij}(x_i), \varphi_{ikij}(x_i), \ldots, \varphi_{ikij}(x_i)) > U_k(\varphi_{ikij}(x_i), y_1, y_2, \ldots, y_l)$$

(22)

After this step, the algorithm will continue for some of the actions $\varphi_{ik}\varphi_{ikij} \circ \ldots \circ \varphi_{ikij}\varphi_{ikij}$, if none of the actions is repeated. Suppose the algorithm continues and it is the turn of player $m$, with

$$U_m(\varphi_{km}(x_i), \varphi_{kmij}(x_i), \varphi_{kmij}(x_i), \ldots, \varphi_{kmij}(x_i)) > U_m(\varphi_{kmij}(x_i), y_1, y_2, \ldots, y_l)$$

(23)

Since this is deterministic and there are only finitely many players and actions, it eventually has to become periodic. \qed
The above reasoning reveals another property of the set of periodic actions in finite multi-player simultaneous strategic form games. Recall the definition of set stable strategies from Bernheim [37]. We modify this definition of set stability as follows:

**Definition 2** (Set Stability). Let $\mathcal{Q}$ be an automorphism $\mathcal{Q} : \mathcal{M}(A) \to \mathcal{M}(A)$. In addition, let $A \subseteq A \cup B \subseteq \mathcal{M}(A)$, with $A \cap B = \emptyset$. The set $A$ is set stable under the action of the map $\mathcal{Q}$ if, for any initial $x_0 \in A \cup B$ and any sequence $x_k$ formed by taking $x_{k+1} \in \mathcal{Q}(x_k)$, there exists $x_K \in A \cup B$ such that $d(x_K, x^1) < \epsilon$, with $x^1 \in A$. For finite sets, this implies that any sequence formed by applying the operator $\mathcal{Q}$ produces an $x_k$ for any initial $x_0$, with $x_k$ belonging to the set stable set $A$.

**Theorem 2.** Let $\mathcal{P}(i)$ denote the set of periodic strategies for player $i$. The set $\mathcal{P}(i)$ is set stable, under the action of the maps $\varphi_{ij}$.

Thus, the periodicity diagram of any non-periodic action $x_0$ results in the periodicity cycle of some action $x_K$.

**Proof.** The proof of this theorem is contained in the proof of Theorem 1.

2.2 New Features; Remarks

There is one difference between the 2- and the multi-player periodicity. In the two-player case, the utility functions chain is

\[
U_A \xrightarrow{P} U_B \xrightarrow{P} U_A \xrightarrow{P} U_B \ldots
\]

(24)

and the periodicity occurs for $U_A$, if we start with a periodic action of player A. In the multi-player case, although we may start with an action $x_i$ of player $i$ and the utility $U_i$, the periodicity might occur at the utility function of another player, say $U_m$. Let us further explain this version of periodicity. At the end of the algorithm, player $m$ will play one of his actions, when his opponents play some actions, one of which is $x_i$, corresponding to player $i$. However, this does not exclude the fact that we might return to the utility function of player $i$ again. One example of this kind is Type 2 periodicity of player C, for the three player game we studied previously in this section, or the periodicity of $a_2$ corresponding to the same game. Having studied the perfect information case, we now generalize our framework to include non-perfect information games.

3 Non Perfect Information Games – Bayesian Games

In this section we address the issue of periodicity in the case of finite games with incomplete information. Our analysis on incomplete information games is based mainly on references [8,21,31] and references therein. In Bayesian strategic form games and more generally in strategic form games with incomplete information, we can always associate some related complete information strategic form games to the game in question. The corresponding strategic form games are called ex-ante and interim strategic form games.
Exploiting these two, we will define and study the ex-ante and interim rationalizable strategies and through these, the periodicity in the case of non-perfect information games. With respect to the latter, the interim rationalizability has two versions, the interim independent and interim correlated rationalizability. Both can be found by constructing the interim independent and interim correlated strategic form game from the initial Bayesian game. Since the Bayesian games can be represented in terms of strategic form games, all the periodicity concepts that we developed in the 2-player and multi-player cases hold true. For simplicity, we shall only present the case with two players and two actions for each player. The findings can be easily be generalized to the multi-player case. The interim independent strategic form game with the Bayesian game having initially two players corresponds to a three player game. Let us start with the ex-ante game. A Bayesian game is a list \((N, A, \Theta, T, u, p)\), with

- \(N\), the number of players
- \(A = (A_i)_{i \in N}\), the set of action profiles with generic member \(a = (a_i)_{i \in N}\)
- \(\Theta\), the set of all possible parameters \(\theta_i\) (in our case usually two different matrices for one of the two players)
- \(T = (T_i)_{i \in N}\) the set of types with generic member \(t = (t_i)_{i \in N}\)
- \(u_i : \Theta \times A \rightarrow R\), the payoff function of player \(i\)
- \(p_i = p_i(\cdot | t_i) \in \Delta(\Theta \times T_{-i})\) is the belief of the type \(t_i\) about \(\theta, t_{-i}\)

Each player \(i\) knows his own type \(t_i\) but does not necessarily know \(\theta\), or the other players’ types, about which he has a belief \(p_i(\cdot | t_i)\). The game is defined in terms of players interim beliefs \(p_i(\cdot | t_i)\), which they obtain after they observe their own type, but before taking their action. The game can also be defined by ex-ante beliefs \(p_i\) \(\in \Delta(\Theta \times T)\) for some belief \(\pi\). The game has a common prior, if there exists \(\pi \in \Delta(\Theta \times T)\) such that:

\[
p_i(\cdot | t_i) = \pi(\cdot | t_i), \forall t_i \in T_i, \forall i \in N
\]

In that case, the game is denoted by \(\langle N, A, \Theta, u, \pi \rangle\). When modelling incomplete information, there is often no ex-ante stage or an explicit information structure in which players observe values of some signals. In the modelling stage, each player \(i\) has the following hierarchical belief system:

- Some belief \(\tau_i^1 \in \Delta(\Theta)\), about the payoffs (and the other aspects of the physical world), a belief that is often referred to as the first order belief of \(i\)
- Some belief \(\tau_i^2 \in \Delta(\Theta \times \Delta(\Theta_i))\) about the payoffs and the other players’ first order beliefs \((\theta, \tau_{-i}^1)\)
- Iteratively, for each \(n\), some belief \(\tau_i^n\) about the payoffs and the other players’ beliefs of all orders \(< n\), \((\theta, \tau_{-i}^1, \tau_{-i}^2, \ldots \tau_{-i}^{n-1})\)
In the Harsanyi type space formalism \cite{21,23}, the infinite belief hierarchies are modelled using a type space \((\Theta, T, P)\) and also using a type \(t_i \in T_i\) in the following way: Given a type \(t_i\) and a type space \((\Theta, T, P)\), one can compute the first order belief of a type \(t_i\), by

\[
h^1_i(\cdot \mid t_i) = \text{marg}_\theta p(\cdot \mid t_i) \tag{26}
\]

so that

\[
h^1_i(\theta \mid t_i) = \sum_{t_{-i}} \text{marg}_\theta p(\theta, t_{-i} \mid t_i) \tag{27}
\]

and the second order by

\[
h^2_i(\theta, \hat{h}^1_{-i}) = \sum_{t_{-i} \mid h^1_{-i}(\cdot \mid t_{-i} = \hat{h}^1_{-i})} p(\theta, t_{-i} \mid t_i) \tag{28}
\]

A type space \((\Theta, t, p)\), and a type \(t_i \in T_i\) model a belief hierarchy \((\tau^1_i, \tau^2_i, \ldots)\) if

\[
h^k_i(\cdot \mid t_i = \tau^k_i), \forall k \tag{29}
\]

Given any Bayesian game \((N, A, \Theta, u, \pi)\), with common prior \(\pi\), one can define the ex-ante game, which we denote by \(G_{\text{ex}} = (N, S, U)\), where \(S_i = A_{T_i}^T\) and

\[
U_i = E_{\pi}[u_i(\theta, s(t))] \tag{30}
\]

for each \(i \in N\) and \(s \in S\). For any Bayesian game \((N, A, \Theta, T, u, p)\) one can also define the interim game, which we denote by \(G_{\text{int}} = (\hat{N}, \hat{S}, \hat{U})\), where \(\hat{N} = \cup_{t \in T}\) and also \(\hat{S}_i = A_i\) for each \(t_i \in \hat{N}\) and

\[
U_{t_i}(\hat{s}) = E[u_i(\theta, \hat{s}_{t_{-i}} \mid p_i(\cdot \mid t_i))] = \sum_{(\theta, t_{-i})} u_i(\theta, \hat{s}_{t_{-i}})p(\theta, t_{-i} \mid t_i) \tag{31}
\]

for each \(i \in N\) and \(s \in S\).

### 3.1 Ex-ante game and Ex-ante Rationalizability

Given any Bayesian game \((N, A, \Theta, T, u, p)\) and a player \(i \in N\), a strategy \(s_i : T_i \rightarrow A_i\) is said to be ex-ante rationalizable iff \(s_i\) is rationalizable in the corresponding ex-ante strategic form game \(G_{\text{ant}} \cite{5,6,36}. Ex-ante rationalizability makes sense if there is an ex-ante stage in the game. In that case, ex-ante rationalizability captures precisely the implications of common knowledge of rationality as perceived in the ex-ante planning stage of the game \cite{5,6}. It does impose unnecessary restrictions on players’ beliefs from an interim perspective however. Let us look at the following example \cite{5,6,36}: Consider a Bayesian game with the following characteristics:

- \(N = (1, 2)\)
- \(\Theta = (\theta, \theta')\)
- \(T = (t_1, t'_1) \times t_2\)
- \(p(\theta, t_1, t_2) = p(\theta', t'_1, t_2) = \frac{1}{2}\)
The action space and the payoff functions are given by

\[
\begin{array}{c|cc}
\theta & L & R \\
\hline
U & 1,\epsilon & -2,0 \\
D & 0,0 & 0,1 \\
\end{array}
\quad \begin{array}{c|cc}
\theta' & L & R \\
\hline
U & -2,\epsilon & 1,0 \\
D & 0,0 & 5,1 \\
\end{array}
\]

Here, player A has two types corresponding to two different payoff actions. Player B has only one payoff table and one type. The ex-ante representation of this game is equal to

\[
\begin{array}{c|cc}
  & L & R \\
\hline
UU & -1/2,\epsilon & -1/2,\epsilon \\
UD & 1/2,\epsilon & -1,1/2 \\
DU & -1,\epsilon/2 & 1/2,1/2 \\
DD & 0,0 & 0,1 \\
\end{array}
\]

To every Bayesian game corresponds an ex-ante perfect information strategic form game. The actions that are rationalizable in the ex-ante strategic form game are called ex-ante rationalizable actions. The rationalizable strategy profile in the case at hand is \(S^\infty(G_{ant} = (DU, R))\). The periodicity cycle of this strategy is

\[
DU \xrightarrow{p} R \xrightarrow{p} DU
\]

In addition, we can see that the theorem which relates types to periodicity number holds true, since there are two types needed to describe this periodic cycle. In this case, the types are the ones that correspond to the perfect information ex-ante strategic form game, so these are seen in a perfect information perspective. Of course all the theorems holding true for finite simultaneous strategic form games, hold also true for Bayesian games since the latter are equivalent to perfect information strategic form games. We now proceed to interim rationalizability related periodic equilibria.

### 3.2 Interim Rationalizability

There are conflicting notions of interim rationalizability in incomplete information games in the literature. One straightforward notion of interim rationalizability is to apply rationalizability to the interim game \(G_{int}\). An embedded assumption of the interim game is that it is common knowledge that the belief of a player \(i\) about \(\theta_{-i}\), which is given by \(p_t(\cdot \mid t_i)\), is independent of his belief about the other players’ actions. In particular, his belief about \((\theta, t_{-i}, a_i)\) is derived from some belief \(p_t(\cdot \mid t_i) \times \mu_{t_i}\) for some \(\mu_{t_i} \in \Delta(A_{-i})\). This is because we have taken the expectations with respect to \(p_t(\cdot \mid t_i)\), in defining the interim game \(G_{int}\), before considering his beliefs about the other players’ actions. Because of this independence assumption, such a rationalizability notion is called interim independent rationalizability. Through the interim rationalizability we will make contact with the periodicity concept in this case as well.
3.2.1 Interim Independent Rationalizability

Given any Bayesian game $B = (N, A, \Theta, T, u, p)$ and any type $t_i$ of player $i \in N$, an action $a_i \in A_i$ is said to be interim independent rationalizable for $t_i$, iff $a_i$ is rationalizable for $t_i$ in the interim game $G_{int}$. The interim independent Rationalizability is the most complex type of rationalizability among all the rationalizability types for Bayesian games. Consider the Bayesian game we used in the previous example of the ex-ante game. The corresponding interim independent game is actually a 3-player game with player-type set $N = (t_1, t_1', t_2)$, and with the following payoff table:

|       | $\theta$ | $L$   | $R$   |
|-------|----------|-------|-------|
| $U_1$ | 1, $\epsilon$, -2 | -2, 0, 1 |
| $D_1$ | $0, \epsilon/2$, -2 | 0, 1/2, 1 |

The first player $t_1$ chooses the rows, the player $t_2$ the columns and finally type $t_1'$ chooses the matrices. All actions are rationalizable as can be easily checked. Let us see the periodicity graphs for the above game. For instance for $U$, the corresponding periodicity graph appears in Fig. 7.

This example is somewhat degenerate, but the periodicity study is identical to the study of periodicity in a 3-player strategic form game. This also proves that indirectly, using the interim rationalizability strategies, we relate the non-perfect information game to a multi-player, perfect information, simultaneous, strategic form game and therefore all the periodicity theorems hold true in this case as well. We further proceed in the same fashion and relate periodicity to the Interim Correlated Rationalizability concept.

**Bayesian Periodic Interim Independent Rationalizability**

![Figure 7: Periodicity for a 3-player Bayesian Game. The first player with type $t_1$ chooses the rows with actions $U$ and $D$, the second player with type $t_2$ the columns, with actions $L$ and $R$ and finally the third player with type $t_1'$ chooses the two matrices of the game.](image-url)
3.2.2 Interim Correlated Rationalizability

Consider a Bayesian game \( B = (N, A, \Theta, T, u, p) \). Interim correlated rationalizability \([5, 6]\) allows more beliefs than interim independent rationalizability, and it is a weaker concept in reference to the latter. When all types have positive probability, ex ante rationalizability is stronger than the other two interim rationalizabilities. So all ex-ante rationalizable actions are interim independent and all interim independent rationalizable actions are interim correlated rationalizable actions. The converse is not true. Thus the following holds true \([5, 6]\):

\[
\text{ex – ante} \subset \text{Interim – independent} \subset \text{interim – correlated} \tag{33}
\]

Interim correlated rationalizability captures the implications of common knowledge of rationality precisely \([5, 6]\). In addition, interim independent rationalizability depends on the way the hierarchies are modelled, in that there can be multiple representations of the same hierarchy, with distinct sets of interim independent rationalizable actions. Moreover, one cannot have any extra robust prediction from refining interim correlated rationalizability. Any prediction that does not follow from interim correlated rationalizability alone relies on the assumptions about the infinite hierarchy of beliefs. A researcher cannot verify such a prediction in the modelling stage without the knowledge of the infinite hierarchy of beliefs. Now, the interim correlated rationalizable actions are the ones that are rationalizable in the interim correlated game. Let us see how this game is found, by using a Bayesian game \([5, 6]\). Take \( \Theta = (-1, 1) \), \( N = (1, 2) \) and the payoff matrices are:

\[
\begin{array}{c|ccc|c|ccc}
\theta = 1 & b_1 & b_2 & b_3 & \theta = -1 & b_1 & b_2 & b_3 \\
a_1 & 1,1 & -10,10 & -10,0 & a_1 & -10,-10 & 1,1 & -10,0 \\
a_2 & -10,-10 & 1,1 & -10,0 & a_2 & 1,1 & -10,-10 & -10,0 \\
a_3 & 0,-10 & 0,-10 & 0,0 & a_3 & 0,-10 & 0,-10 & 0,0 \\
\end{array}
\]

Table 1: Game 1B

We consider the type space \( T = (t_1, t_2) \), with \( p(\theta = 1, t) = p(\theta = -1, t) = 1/2 \). The interim game is the following complete information game:

\[
\begin{array}{c|ccc|c|ccc}
\theta = 1 & b_1 & b_2 & b_3 & \theta = -1 & b_1 & b_2 & b_3 \\
a_1 & -9/2,-9/2 & -9/2,-9/2 & -10,0 & a_1 & -9/2,-9/2 & -9/2,-9/2 & -10,0 \\
a_2 & -9/2,-9/2 & -9/2,-9/2 & -10,0 & a_2 & -9/2,-9/2 & -9/2,-9/2 & -10,0 \\
a_3 & 0,-10 & 0,-10 & 0,0 & a_3 & 0,-10 & 0,-10 & 0,0 \\
\end{array}
\]

Table 2: Game 1B

It is easy to show that even in this Bayesian framework we can find a periodic action and specifically in the interim reduced game. Thereby, we indirectly demonstrated that by using the various imperfect information rationalizability concepts, we relate periodicity with Bayesian games in general. Therefore we may formalize the periodicity concept in Bayesian games.
3.2.3 Periodicity and Bayesian Games

We can easily understand that since every Bayesian game corresponds to some perfect information, finite player, finite action, strategic form game, the following theorem holds.

**Theorem 3.** Every finite action simultaneous \( N \)-player Bayesian strategic form game contains at least one periodic action.

**Proof.** Every finite player finite action strategic form game corresponds to an interim game or an ex-ante game, which are finite action finite player games. Therefore since every finite action, finite player strategic form game has at least a periodic action, it follows that this is also true for every finite action, finite player, Bayesian strategic form game.

Moreover, all the arguments that hold for perfect information games also hold for the ex-ante and interim representations of a strategic form game. So we can generalize these arguments to Bayesian games. For the ex-ante and interim correlated representations of a Bayesian game, the following theorem holds.

**Theorem 4.** In a two player perfect information ex-ante and interim correlated representation of a two-player Bayesian strategic form game, the number of types \( N_{t_i} \) corresponding to the periodic cycle of an ex-ante or interim correlated rationalizable periodic action is

\[
N_{t_i} = 2^n
\]  

(34)

The types are those corresponding to the perfect information representation of the Bayesian game and not those corresponding to the incomplete information game.

**Proof.** We shall call rationalizable strategies those which are rationalizable for the corresponding ex-ante or interim correlated strategic form game, without specifying to which we refer \([5, 6]\). The results hold for either case. Having this in mind, for every such action, if the periodicity number is \( n \), it is possible to construct a periodic chain with exactly \( 2n \) rationalizable actions appearing in that chain. Therefore, we need to prove that for each action appearing in the rationalizability chain there exists at least one type, so the minimum number of types corresponding to all the actions of the rationalizability chain is \( 2n \). As is proved in \([36]\), in a static game with finitely many choices for every player, it is always possible to construct an epistemic model in which,

- Every type expresses common belief in rationality
- Every type assigns for every opponent probability 1 to one specific choice and one specific type for that opponent.

Thus, for two player games, each type for player A, for example, assigns probability 1 to one of his opponent’s actions and one specific type for that action, such that this action is optimal for his opponent. In addition, in two player games, rationalizable actions and choices that can be made under common belief in rationality coincide. Hence,
we can associate to every rationalizable action of player \( A \) exactly one type which in turn assigns probability 1 to one specific rationalizable action and one specific type of his opponents’ type’s and actions. Moreover, as proved in [36], the actions that can rationally be made under common belief in rationality are rationalizable. To state this more formally, in a static game with finitely many actions for every player, the choices that can rationally be made under common belief in rationality, are exactly those choices that survive iterated elimination of strictly dominated strategies. Hence, for two player games, we conclude that strategies which express common belief in rationality and rationalizable strategies coincide. This is because all beliefs in two-player games are independent. (This is not always true in games with more than two players, however.) Therefore, when periodic rationalizable strategies are considered, the total number of types needed for a rationalizability cycle is equal to \( 2^n \). This concludes the proof.

4 Periodicity and Cooperatvity

While our concept of a periodic solution seems to involve some form of cooperativity, this is of course different from what is called cooperative game theory. The latter is about binding commitments, coalitions and the distribution of payoffs inside such coalitions. All these features are absent in our setting. For further illustration, we shall now discuss one of the most refined cooperative game theory concepts, that of a cooperative-competitive (CO-CO) solution [39] (see also [40]) and we shall compare the results of this solution concept with those that result from the periodic strategies algorithm.

4.1 Cooperative-Competitive Equilibrium

Consider a general, two player non-zero sum game with players \( A \) and \( B \), described by the payoff functions \( \Phi^A \) and \( \Phi^B \), with:

\[
\Phi^A : \mathcal{M}(A) \times \mathcal{M}(B) \rightarrow \mathbb{R}, \quad \Phi^B : \mathcal{M}(A) \times \mathcal{M}(B) \rightarrow \mathbb{R}
\]  

with the strategy spaces \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \) being compact metric spaces, and the payoff functions being continuous functions from \( \mathcal{M}(A) \times \mathcal{M}(B) \) into \( \mathbb{R} \). If cooperativity and communication between players is allowed, the players \( A \) and \( B \) can adopt a set of strategies \( (a^\sharp, b^\sharp) \) that maximizes their combined payoffs,

\[
V^\sharp = \Phi^A(a^\sharp, b^\sharp) + \Phi^B(a^\sharp, b^\sharp) = \max_{a,b \in \mathcal{M}(A) \times \mathcal{M}(B)} \left[ \Phi^A(a, b) + \Phi^B(a, b) \right]
\]

The choice of the strategy \( (a^\sharp, b^\sharp) \) may favor one player more than the other. In such a case, the player that is better off must provide some incentive to the other player, in order that he complies with the strategy \( (a^\sharp, b^\sharp) \). This incentive is actually a side payment. Splitting the total payoff, \( V^\sharp \) into two equal parts will not be acceptable, because this does not reflect the relative strength of the players and their personal
contributions to their cooperativity outcomes [10]. A more realistic approach was introduced by [39] which we shall now describe. Define the following game:

$$
\Phi^\sharp(a, b) = \Phi^A(a, b) + \Phi^B(a, b)
$$

(37)

$$
\Phi^S(a, b) = \Phi^A(a, b) - \Phi^B(a, b)
$$

(38)

These relations actually imply that the original game is split into two games, a purely cooperative one, with payoff $\Phi^\sharp(a, b)$, and a competitive one (which is a zero sum game), with payoff $\Phi^S(a, b)$. In the cooperative game, the players have equal payoffs, that is, they both receive $\Phi^\sharp(a, b)$, while in the purely competitive part, the players have opposite payoffs, namely $\Phi^S(a, b)$ and $-\Phi^S(a, b)$.

Denote the value of the zero-sum game by $V^S$, with utility function $\Phi^S(a, b)$. Having found the value of the game, the cooperative-competitive value of the game is defined as the payoff pair

$$
\left(\frac{V^\sharp}{2} + V^S, \frac{V^\sharp}{2} - V^S\right)
$$

(39)

The cooperative-competitive solution of the game is defined as the pair of strategies $(a^\sharp, b^\sharp)$, together with a side payment $P_S$ from player B to player A, such that:

$$
\Phi^A(a^\sharp, b^\sharp) + P_S = \frac{V^\sharp}{2} + V^S
$$

$$
\Phi^B(a^\sharp, b^\sharp) - P_S = \frac{V^\sharp}{2} - V^S
$$

Obviously, the side payment can be negative, in which case player A pays player B the amount $P_S$.

Conceptually, the cooperative-competitive solution is opposite to the algorithm that yields periodic strategies, owing to the fact that the cooperative-competitive solution, namely the strategy pair $(a^\sharp, b^\sharp)$, is determined by maximizing the sum of the player’s and his opponent’s utility. The periodic strategies on the other hand are computed by maximizing each player’s own payoff, with respect to the opponent’s actions. We shall now present some characteristic examples and compare the cooperative-competitive solution and the periodic algorithm solution.

### 4.2 Cooperative-Competitive Solution and Periodicity Algorithm–Some Examples

Consider the Battle of Sexes game that appears in Table 3.

|   | $b_1$ | $b_2$ |
|---|-------|-------|
| $a_1$ | 2,1   | 0,0   |
| $a_2$ | 0,0   | 1,2   |

Table 3: Battle of Sexes

As we demonstrated in Ref. [3], for this game both the pure strategy pairs $(a_1, b_1)$ and $(a_2, b_2)$ are periodic strategies. Moreover, when we apply the periodic strategies
algorithm to mixed strategies, we obtain a mixed strategy that yields the same payoffs as the mixed Nash equilibrium, with the difference that each player’s payoff does not depend on his opponent’s actions. Let us recall the results:

The mixed Nash equilibrium for this game is \((p^*_N = \frac{2}{3}, q^*_N = \frac{1}{3})\) and moreover, the application of the periodic strategies algorithm yields the strategy, \((p^*_p = 1/3, q^*_p = 2/3)\).

The expected utilities of the players are:

\[
\begin{align*}
U_{1,p,q}(p^*_p = 1/3, q) &= \frac{2}{3}, \\
U_{2,p,q}(p, q^*_p = 2/3) &= \frac{2}{3}, \\
U_{1,p,q}(p, q^*_N = 1/3) &= \frac{2}{3}, \\
U_{1,p,q}(p^*_N = 2/3, q) &= \frac{2}{3}
\end{align*}
\]

Hence, the payoff corresponding to the mixed Nash equilibrium is \((U_{1,N}, U_{2,N}) = (2/3, 2/3)\) and the algorithm of periodic strategies yields the payoffs \((U_{1,P}, U_{2,P}) = (2/3, 2/3)\). Let us now turn to the cooperative-competitive solution of the Battle of Sexes game. By the procedure described in the previous subsection, the zero-sum game of the Battle of Sexes game is given in table 4.

| b₁   | b₂   |
|------|------|
| a₁   | 1/2  | 0    |
| a₂   | 0    | -1/2 |

Table 4: Battle of Sexes

We compute \(V^t = 3\) and \(V^S = 0\). It is obvious that the cooperative-competitive strategy is constituted from any of the two strategy sets \((a_1, b_1)\) or \((a_2, b_2)\). Within the cooperative-competitive solution, player B must make a side payment \(P_S = 1\) to player A. Hence, in the cooperative-competitive solution the final utilities are \((U_{1,CC}, U_{2,CC}) = (2, 2)\). As we can see, when players cooperate, they receive a higher payoff than in all other non-cooperative payoffs we presented for this game. Consequently, the strategies that are obtained from the periodic strategies algorithm are, in expected utility terms, as non-cooperative as the mixed Nash equilibrium.

Let us give another example of the non-cooperativity of the mixed and non-mixed periodic strategies. Consider the game that appears in Table 5.

The payoffs corresponding to the mixed Nash equilibrium \((p^*_N = \frac{5}{6}, q^*_N = \frac{48}{49})\) and the
ones corresponding to the periodic strategies algorithm \((p_p^* = 1/49, q_p^* = 48/49)\) are

\[
\begin{align*}
U_1P(p_p^* = 1/49, q) &= \frac{146}{49} \\
U_2P(p, q_p^* = 1/6) &= \frac{35}{6} \\
U_1N(p, q_N^* = 48/49) &= \frac{146}{49} \\
U_1N(p_N^* = 5/6, q) &= \frac{35}{6}
\end{align*}
\]

The strategy \((a_1, b_2)\) corresponds to the cooperative-competitive strategy. The values \(V^z\) and \(V^S\) are equal to \(V^z = 56\) and \(V^S = -\frac{3}{2}\), and hence the side payment of player A to player B is \(P_S = -\frac{47}{2}\). The cooperative-competitive value of the game (the final payoffs of the two players) is \((U_{1CC}, U_{2CC}) = (\frac{53}{2}, \frac{59}{2})\). By comparing the cooperative payoffs with the non-cooperative ones, appearing in equation (41), it is obvious that the non-cooperative ones are smaller than the cooperative ones. Thus, the strategies that result from applying the periodic strategies algorithm are again non-cooperative.

Nevertheless, for some games, the cooperative-competitive strategies payoff value (in the terminology of cooperative-competitive equilibria) may coincide with the periodic mixed or pure strategies payoff. But this occurs only for a rather particular class of games, like the Prisoner-Dilemma. For example, for the game in Table 4.2

\[
\begin{array}{c|c|c}
   & B_1 & B_2 \\
\hline
A_1 & 4.4 & -1.6 \\
A_2 & 6.1 & 0.0 \\
\end{array}
\]

Table 6: Prisoners Dilemma

the application of the periodic strategies results to the strategy pair \((A_1, B_1)\), with payoffs \((U_{1P}, U_{2P}) = (4, 4)\). For this game the values \(V^z\) and \(V^S\) are equal to \(V^z = 8\) and \(V^S = 0\), and the side payment of player A to player B is \(P_S = 0\). Consequently, the cooperative-competitive value of the game is \((U_{1CC}, U_{2CC}) = (4, 4)\), which is the same as the periodic one. However, this is accidental and an artifact of the details of the payoff matrix.
5 Epistemic Game Theory Framework and Periodic Strategies

In this section, we shall connect the periodicity number $n$ appearing in the automorphism $Q^n$ defined earlier to the number of types needed to describe a two player simultaneous strategic form game within an epistemic framework. We shall assume a perfect information context. The epistemic game theory formalism was introduced by Harsanyi, in order to describe incomplete information games [21–23] and thereafter adopted by other authors (see for example [4–6] and references therein). Our approach mimics the one used in [24] and also the one adopted from Perea in [36]. For completeness, we shall briefly present the appropriate formalism and reasoning.

5.1 Belief Hierarchies in Complete Information Games and Types and Common Belief in Rationality

Consider a two player game with a set of finite actions for each player, A and B. A belief hierarchy for player A of the game is constructed from a chain of increasing order beliefs in terms of objective probabilities as follows [36]:

- A first order belief is the belief that player A holds for player B’s actions
- Iteratively, a $k$-th order belief represents the belief that player A holds for the $(k-1)$-th order belief of player B.

The belief hierarchy expresses in general rational choices of the players under common belief in rationality, that is, every player believes in his opponent’s rationality and believes that his opponent believes that he acts rationally and so on. Since belief hierarchies are not so easy to use in practice, the concept of a type is introduced, which encompasses all the information that a belief hierarchy contains, but is a more compact way to describe such a hierarchy.

Before doing that, let us quantify the belief hierarchies in a more formal way, in terms of spaces of probability distributions. With a suitable topology and metric, the space of probability distributions on a compact metric space is again a compact metric space, and therefore, the construction can be iterated, that is, we can consider probability distributions on spaces of probability distributions.

The first order belief hierarchy is given by all the probabilities distributions over the space of actions that player $i$ considers possible for his opponents. By assumption, this set $X_i^1$ is finite, hence in particular compact, and we may also equip it with a metric. The space of first order beliefs then is the space of probability distributions on that space,

$$B_i^1 = \Delta(X_i^1) \quad (42)$$

Iteratively, we obtain the $k$-th order of uncertainty,

$$X_i^k = X_i^{k-1} \times (\times_{j \neq i} B_j^{k-1}) \quad (43)$$

25
which embodies the \((k-1)\)-th order space of uncertainty and also the \((k-1)\)-th order of the opponent’s beliefs. Thus, the space of \(k\)-th order beliefs is the set \(\Delta(X_i^k)\). A belief hierarchy \(b_i\) for the player \(i\) is an infinite chain of beliefs \(b_i^k \in B_i^k, \forall k\), that is:

\[
b_i = (b_i^1, b_i^2, ..., b_i^k)
\]  \(\text{(44)}\)

Relation \(\text{(44)}\) encodes what was said above. The belief hierarchy is assumed to be coherent, which means that the various beliefs in the belief hierarchy do not contradict each other, that is, for \(m > k\)

\[
mrg(b_i^m | X_i^{k-1}) = b_i^{k-1}  \tag{45}\]

Having defined coherent belief hierarchies, the epistemic framework is constructed using the definition of an epistemic type which is simply a coherent belief hierarchy for a player \(i\). A type corresponds to some epistemic model constructed for the game, so let \(T_i\) be the total number of types needed to describe player \(i\). In addition, for every player \(i\) and for every \(t_i \in T_i\), the epistemic model specifies a probability distribution \(b_i(t_i)\) over the set \(C_{-i} \times T_{-i}\), which represents the set of choice-types of player \(i\)’s opponent \(-i\). The probability distribution \(b_i(t_i)\) stands for the belief that a player \(i\)’s type \(t_i\) holds about player’s \(-i\) actions and types, so

\[
b_i : T_i \rightarrow \Delta(T_{-i} \times C_{-i})  \tag{46}\]

for a two player game. The type of a player \(i\) is the complete belief hierarchy. Now a choice \(c_i\) of player \(i\) is optimal for his type \(t_i\) if it is optimal for the first order beliefs that \(t_i\) holds about the opponent’s choices. Within the epistemic game theoretic framework, one can easily define common belief in rationality. Indeed, we say that the type \(t_i\) believes in the opponent’s rationality if \(t_i\) assigns positive probability to his opponents \(-i\) choice types \((c_{-i}, t_{-i})\), in which case \(c_{-i}\) is optimal for type \(t_{-i}\). Having defined the belief in opponent’s rationality, we define the \(k\)-fold belief in rationality \([36]\):

- Type \(t_i\) expresses 1-fold belief in rationality if \(t_i\) believes in the opponent’s rationality
- Iteratively, type \(t_i\) expresses \(k\)-fold belief in rationality if \(t_i\) assigns positive probability to opponent types that express \((k-1)\)-fold belief in rationality.
- Type \(t_i\) corresponding to player \(i\) expresses common belief in rationality, if it expresses \(k\)-fold belief in rationality for every \(k\).

In addition, we can formally define a rational choice, when common belief in rationality is assumed in the game, as follows: A choice \(c_i\) of player \(i\) is rational under common belief in rationality, if there is some type \(t_i\) such that:

- Type \(t_i\), expresses common belief in rationality
- Choice \(c_i\) is optimal for this type \(t_i\)

Our aim is to connect the periodicity number \(n\) defined earlier to the number of types that are necessary to describe a simultaneous two player finite action game. This connection will use the point rationalizable strategies.
5.2 The Connection of the Periodicity Number to the total Number of Types of the Epistemic Model

As demonstrated in Ref. [3] the rationalizable actions that are also periodic are particularly interesting, since for these we can connect the total periodicity number \( n \) to the numbers of types needed to describe the game with an epistemic model. This relation can be described by the following theorem:

**Theorem 5.** In a two player perfect information strategic form game, the number of types \( N_{t_i} \) corresponding to the periodic cycle of a rationalizable periodic action is

\[
N_{t_i} = 2^n
\]  

(47)

**Proof.** For every such action if the periodicity number is \( n \), it is possible to construct a periodicity chain with exactly \( 2n \) rationalizable actions appearing in that chain. Therefore what is necessary to prove is that for each action appearing in the rationalizability chain, there exist at least one type, so the minimum number of types corresponding to all the actions of the rationalizability chain is \( 2n \). As proved in [36], in a static game with finitely many choices for every player, it is always possible to construct an epistemic model in which,

- Every type expresses common belief in rationality
- Every type assigns for every opponent probability 1 to one specific choice and one specific type for that opponent.

Therefore, for two player games, each type for player A for example, assigns probability 1 to one of his opponents actions and one specific type for that action, such that this action is optimal for his opponent. In addition, in two player games, rationalizable actions and choices that can be made under common belief in rationality coincide. Hence, we can associate to every rationalizable action of player A exactly one type which in turn assigns probability 1 to one specific rationalizable action and one specific type of his opponent’s types and actions. Moreover, as proved in [36], the actions that can rationally be made under common belief in rationality are rationalizable. To state this more formally, in a static game with finitely many actions for every player, the choices that can rationally be made under common belief in rationality are exactly those that survive iterated elimination of strictly dominated strategies. Hence, for two player games, we conclude that strategies which express common belief in rationality and rationalizable strategies coincide. This is because all beliefs in two-player games are independent, something that is not always true in games with more than two players. Therefore, when periodic rationalizable strategies are considered, the total number of types needed for a rationalizability cycle is equal to \( 2n \).

5.2.1 A Comment on Simple Belief Hierarchies and Nash Equilibria

Within an epistemic game theory context, a type \( t_i \) is said to have a simple belief hierarchy, if \( t_i \)’s belief hierarchy is generated by some combination \( \sigma_i \) of probabilistic
beliefs about the players choices. Thus, a type has a simple belief hierarchy if it is believed that his opponents are correct about his beliefs. As proved in [36], a simple belief hierarchy, given by probabilistic beliefs $\sigma_i$ about players’ choices, expresses common belief in rationality, if the combination $\sigma_i$ of beliefs is itself a Nash equilibrium. The converse is not always true. Hence, using the theorem above, the number of types needed to describe a simple belief hierarchy for a Nash equilibrium is 2. Obviously, if a Nash action is periodic, then $n = 1$ and applying relation (47), we find that the types needed in the periodic Nash case are two.

There is an interesting point regarding simple belief hierarchies. When considering two player games, it is proved (see [36], theorem 4.4.3) that a type $t_i$ has a simple belief hierarchy iff $t_i$ believes that his opponent holds correct beliefs and believes that his opponent believes that he holds correct beliefs himself. Thus, he believes that he does not err in his prediction about his opponent’s beliefs, and he believes that for his opponent too. In higher order beliefs this is no longer true, and therefore we could argue that the total number of wrong beliefs of all the two players about each other’s beliefs is equal to $2n - 1$. Thus, the total number of errors of the two players is $2n - 1$. Errors here are the beliefs $\sigma_i$ due to which the higher order belief hierarchy fails to be a simple belief hierarchy.

Concluding Remarks

In this work we have studied extensions and generalizations of the periodicity concept introduced in [3]. In particular, we have shown the existence of periodic strategies in multi-player perfect information simultaneous strategic form games. We also proved that the set of periodic strategies is set-stable under the periodicity map. In addition, we discussed the presence of periodic strategies in games with incomplete information, focusing on Bayesian games. In that case we made extensive use of various generalizations of Bernheim’s rationalizability concept. The issue of cooperativity and periodicity was formally addressed as well. The periodic strategies are simply as cooperative as the mixed Nash equilibrium. In an epistemic framework, the number of types needed to describe the rationalizability cycle of a rationalizable periodic strategy equals twice the periodicity number of that action. The next step would be the inclusion of mixed strategies in multi-player games. Actually, the cooperativity issue in games with more than two players becomes more complex, because the players are free to form coalitions. Periodicity then has to be reconsidered under this perspective.

Clearly, the periodicity feature for finitely many actions of strategic form games can be very useful. Indeed, all the periodic actions can be found using some simple program. This result is actually a common feature of every non-degenerate finite action game, that is, every non-Nash rationalizable action is usually periodic. This can be very useful for games that have, as we mentioned, finitely many actions, since the potential non-Nash rationalizable actions can be determined by finding the periodic strategies. Furthermore, an interesting future study would be to consider 3-player mixed strategies and their relation to periodic strategies. One should carefully examine whether there is
any exceptional class of games with the special attributes of the two player games that we presented in the present article. In particular, we should check whether the algorithm of periodic strategies leads to strategies for which the expected utility of players is higher than the corresponding Nash one, and in addition if the periodic strategies for a player are independent of the other player’s action, as in the two player case. In addition, the multi-player cooperativity issue should also be formally addressed. The question whether the periodic strategies imply any sort of cooperativity has to be re-addressed in a multi-player context. This is because, in cases with $N \geq 3$ players, two or more players may form coalitions in order to cooperate against the rest. Moreover, one can investigate the case of continuum utility functions. Finally, in the case of Bayesian games, one might look for a connection between the types of the imperfect information case and the corresponding Ex-ante or interim game, or a connection between periodicity imperfect information types spaces.

An important feature of periodic strategies as examined in this paper is that they make a player robust against the way that the opponent-rival decides to play the game. In contrast to the Nash strategies, where each player relies on his opponent’s rationality and on the fact that the opponent will actually play the Nash strategy too, the payoff of a player that uses a periodic strategy is not affected by the opponent’s actual actions. This is valuable in non-trivial games, like the prisoner’s dilemma. It is remarkable that although we used a non-trivial non-cooperative context, we ended up that the optimal equilibrium of the game is the socially optimal solution. In this work we demonstrated how periodic strategies can be realized in multi-player simultaneous perfect information games and also in games with imperfect information. Hence this shows that the periodicity concept seems to be an inherent feature of every non-trivial game. The advantage of the periodic strategies over the Nash strategies is that the periodic strategies players do not depend on the rationality of the opponent. Although rationality is considered a prerequisite in most games, there exist many modern politics and economics related examples where rationality is questioned. More importantly, in many cases the opponents may have hidden information, so although a player might think that the payoff are given and the game is played with perfect information about the payoffs of the game, the opponent might act non-rationally with respect to the perfect information game, but rationally with respect to the hidden information game. the periodic strategies then are safe strategies in the sense that the possibility of loosing is minimized or controlled in a formal way.

References

[1] J.Nash, Equilibrium points in $n$-person games, Proc.Nat.Ac.Sc. 36 (1950), 48 – 49

[2] J. Jost, N. Bertschinger, E. Olbrich, and D. Wolpert. Information geometry and game theory. In Nihat Ay, Paolo Gibilisco, and Frantiek Mat, editors, Information geometry and its applications : on the occasion of Shun-ichi Amaris 80th Birthday, IGAIA IV Liblice, Vol. 252 of Proceedings in Mathematics and Statistics, pages 19–46. Springer, Cham, 2018.
[3] V.K. Oikonomou, J. Jost, Periodic Strategies: A New Solution Concept and an Algorithm for NonTrivial Strategic Form Games, Advances in Complex Systems, Vol. 20, No. 5 (2017) 1750009

[4] Battigalli, P. (1997), On rationalizability in extensive games, Journal of Economic Theory 74, 40-61

[5] Pierpaolo Battigalli, Rationalizability in infinite, dynamic games with incomplete information, Research in Economics 57, 1-38.

[6] Battigalli, P. (1996), Strategic independence and perfect Bayesian equilibria, Journal of Economic Theory 70, 201-234.

[7] Battigalli, P. and M. Siniscalchi (1999), Hierarchies of conditional beliefs, and interactive epistemology in dynamic games, Journal of Economic Theory 88, 188-230.

[8] Battigalli, P. and M. Siniscalchi (2002), Strong belief and forward induction reasoning, Journal of Economic Theory 106, 356-39

[9] Blume, L.E., Brandenburger, A. and E. Dekel (1991a), Lexicographic probabilities and choice under uncertainty, Econometrica 59, 61-79.

[10] Blume, L.E., Brandenburger, A. and E. Dekel (1991b), Lexicographic probabilities and equilibrium refinements, Econometrica 59, 81-98.

[11] Dekel, E. and D. Fudenberg (1990), Rational behavior and payoff uncertainty, Journal of Economic Theory 52, 243-267.

[12] Perea, A. (2003), Rationalizability and minimal complexity in dynamic games, Maastricht University.

[13] Rubinstein, A. (1991), Comments on the interpretation of game theory, Econometrica 59, 909-924

[14] Schuhmacher, F. (1999), Proper rationalizability and backward induction, International Journal of Game Theory 28, 599-615.

[15] Asheim, G.B. (2001), Proper rationalizability in lexicographic beliefs, International Journal of Game Theory 30, 453-478.

[16] Epstein, L. and T. Wang (1996), ”Beliefs about beliefs” without probabilities, Econometrica 64 1343-

[17] Reny, P.J. (1992), Rationality in extensive-form games, Journal of Economic Perspectives 6, 103-

[18] Stalnaker, R. (1998), Belief revision in games: forward and backward induction, Mathematical Social Sciences 36, 31-56
van Damme, E. (1984), A relation between perfect equilibria in extensive form games and proper equilibria in normal form games, International Journal of Game Theory 13, 1-13.

Srihari Govindan, Robert Wilson, 2009. "On Forward Induction," Econometrica, Econometric Society, vol. 77(1), pages 1-28

J.C. Harsanyi, Games with incomplete information played by bayesian play- ers, I, Management Science, 14, 159-182, (1967)

J.C. Harsanyi, Games with incomplete information played by bayesian play- ers, II, Management Science, 14, 320-334, (1967)

J.C. Harsanyi, Games with incomplete information played by bayesian play- ers, III, Management Science, 14, 486-502, (1968)

T. Tan and S.R.C. Werlang, The bayesian foundations of solution concepts of games, Journal of Economic Theory, 45 (1988), 370-391

V. Zandt, Interim Bayesian Equilibrium on Universal type space for Supermodular Games, Journal of Economic Theory, 145, 249-263

Jonathan Levin, Solution Concept, Notes

Stephen Morris, Satoru Takahashi, Games in Preference Form and Preference Rationalizability, Economic Theory Center Working Paper No. 43-2012

Geoffroy de Clippel, Values for cooperative games with incomplete information: An eloquent example, Center for Operations Research and Econometrics (CORE) in its series CORE Discussion Papers with number 2002014

Branislav L. Slantchev, Static and Dynamic Games of Incomplete Information, Department of Political Science, University of California- San Diego

F. Forges, R. Serrano, Cooperative games with incomplete information: some open problems, working paper, Brown University, Department of Economics in its series Working Papers with number 2011-15

Erik J. Balder, Nicholas C. Yannelis, Bayesian, Walrasian equilibria: beyond the rational expectations equilibrium, Economic Theory 38 (2009) 385-397

Attila Szolnoki, Matjaz Perc, Reentrant phase transitions and defensive alliances in social dilemmas with informed strategies, EPL 110, 38003 (2015)

Matjaz Perc, Attila Szolnoki, Coevolutionary games - a mini review, BioSystems 99 (2010) 109-125

Fudenberg, Drew; Tirole, Jean (1991), Game theory, MIT Press
[35] Osborne, Martin J. (2004), An introduction to game theory, Oxford University Press

[36] A. Perea, Epistemic Game Theory (2012), Cambridge University Press

[37] Bernheim, B.D. (1984), Rationalizable strategic behavior, Econometrica 52, 1007

[38] D. Pearce, Rationalizable strategic behavior and the problem of perfection, Econometrica 52, 1029-1050, 1984

[39] Adam Kalai, Ehud Kalai, Cooperation in Strategic Games Revisited, The Quarterly Journal of Economics (2012) doi: 10.1093/qje/qjs074

[40] Alberto Bressan, Noncooperative Differential Games, Milan Journal of Mathematics, Volume 79, Issue 2, pp 357, December 2011