Almost periodic divisors, holomorphic functions, and holomorphic mappings *

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Abstract

We prove that to each almost periodic, in the sense of distributions, divisor \( d \) in a tube \( T_\Omega \subset \mathbb{C}^m \) one can assign a cohomology class from \( H^2(K, \mathbb{Z}) \) (actually, the first Chern class of a special line bundle over \( K \) generated by \( d \)) such that the trivial cohomology class represents the divisors of all almost periodic holomorphic functions on \( T_\Omega \); here \( K \) is the Bohr compactification of \( \mathbb{R}^m \). This description yields various geometric conditions for an almost periodic divisor to be the divisor of a holomorphic almost periodic function. We also give a complete description for the divisors of homogeneous coordinates for holomorphic almost periodic curves; in particular, we obtain a description for the divisors of meromorphic almost periodic functions.

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The classical theory of almost periodic functions has found a lot of applications in various branches of mathematics, from differential equations (cf. [37], [7], [25]) to number theory (cf. [3], [38]). In spite of the fact that the whole theory was originally motivated by problems in complex analysis (H.Bohr [5], p.3), analytic aspects of the theory are less known. Holomorphic almost periodic functions have certain specific properties, mainly because almost periodicity of a holomorphic function causes strong restrictions on the distribution of its values (for example, of its zeroes). The main contributions to the classical theory of holomorphic almost periodic functions of one variable are due to B. Jessen, H.Tornehave, B. Ja. Levin and M. G. Krejn; for a detailed presentation of the subject, see [20] and [24].

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The case of several complex variables is much more difficult. First results in this di-
rection concerned zero sets of exponential sums (O. Gelfond [15], [16], B. Ja. Kazarnovskii
[21], [22]). A fruitful approach developed by L. I. Ronkin has allowed to investigate gen-
eral holomorphic almost periodic functions and mappings (cf.[26]–[33]). This resulted,
in particular, in the notions of almost periodic divisors and holomorphic chains (cf. [35],
[36], [13], [14]; also see the survey [11]). Note that the main accent in those papers was
made to the asymptotic behavior and asymptotic characteristics.

The problem of inner description for the divisors of holomorphic almost periodic
functions has been raised by M. G. Krejn and B. Ja. Levin in [23], where it was solved
for entire almost periodic functions of exponential type on the plane with zeroes in a
strip of finite width. Note that the divisors of holomorphic almost periodic functions
inherit, in a sense, the property of being almost periodic, nevertheless there exist almost
periodic divisors that are not the divisors of any almost periodic holomorphic functions
(cf. [40] and [36]). For holomorphic almost periodic functions on a strip with spectrum
in a free group the problem was solved in [39]. A complete description of the divisors
of holomorphic and meromorphic almost functions on a strip was given in [8] and [10].

Notice that almost periodic divisors in a strip on the plane were classified in [39]
with the help of certain integer valued matrix, while in [8], [10] this was done in terms
of cohomology classes from $H^2(K, \mathbb{Z})$, where $K$ is the so-called Bohr's compactification
of the real axis (see Section 1).

The multi-dimensional problem of realizability of almost periodic divisors as the
divisors of almost periodic holomorphic functions was studied by L. I. Ronkin for periodic
divisors (cf.[34]) and for their restrictions to certain family of planes (cf.[36]) with the
help of an integer valued matrix, too. But for arbitrary almost periodic divisors the
problem has never been raised before.

Our approach to the realizability problem is very close to the classical method of
investigation of the Second Cousin Problem on a domain $D \subset \mathbb{C}^m$ (see [17], Ch.5).
Namely, in the classical case a line bundle over $D$ corresponds to each data for the
Cousin Problem on $D$ such that the problem has a solution if and only if this bundle
is trivial. Therefore, the first Chern class (i.e., the corresponding element of the group
$H^2(D, \mathbb{Z})$) of the bundle is assigned to each data such that the problem is solvable if
and only if this class is trivial. In particular, if $H^2(D, \mathbb{Z}) = \{0\}$, then every Second
Cousin Problem on $D$ has a solution. In our investigation, a line bundle over Bohr's
compact set $K$ is assigned to each almost periodic divisor on a tube domain $T_\Omega$ (the
case $T_\Omega = \mathbb{C}^m$ is not excluded), such that the bundle is trivial just for the divisors of
holomorphic almost periodic functions on the domain; therefore, the first Chern class
of the bundle corresponds to each almost periodic divisor in the domain such that it is the
divisor of a holomorphic almost periodic function if and only if this class is trivial.
Note that $H^2(K, \mathbb{Z}) \neq \{0\}$, therefore some problems of realizability for almost periodic
divisors have no solutions.

As for the case of the Second Cousin Problem, we need to solve the appropriate
$\bar{\partial}$-problem; we use a technique from [4] to get a required integral representation for a
solution with specific properties of the $\bar{\partial}$-problem in a tube domain.
The paper is organized as follows.
In Section 1, we give the main definitions, necessary notations, and some information about almost periodic functions and bundles.
In Section 2, we introduce the notion of holomorphic function on $K \times \Omega$ and establish a correspondence between such functions and holomorphic almost periodic functions in $T_\Omega$. Then we construct a line bundle over $K \times \Omega$, corresponding to an almost periodic divisor in $T_\Omega$.
In Section 3, we solve the appropriate $\bar{\partial}$-problem on $T_\Omega$ and prove that if the constructed bundle is trivial, then the divisor is the divisor of a holomorphic almost periodic function in $T_\Omega$.
In Section 4, we show that the Chern class of the bundle is actually an element of the group $H^2(K, \mathbb{Z})$ and obtain simple geometric conditions sufficient for realizability of divisors as the divisors of holomorphic almost periodic functions. For the case of divisors with spectrum in a finitely-generated additive subgroup of $\mathbb{R}^m$, we establish a correspondence between Chern classes of the divisors and skew-symmetric matrices with integer entries.
In Section 5, we investigate the Chern classes of periodic divisors and some classes of almost periodic divisors. We also find a structure formula for Chern classes of almost periodic divisors and prove that an arbitrary Chern class is a finite sum of the Chern classes for periodic divisors.
In Section 6, we obtain a complete classification of the divisors of homogeneous coordinates of almost periodic holomorphic mappings from tube domains into projective spaces.

1 Definitions, notations, and some preliminary information

A continuous function $f(x)$ on $\mathbb{R}^m$ is called almost periodic if the collection $\{S_t f\}_{t \in \mathbb{R}^m}$ is a relatively compact set with respect to the topology of uniform convergence on $\mathbb{R}^m$; here $S_t f(x) = f(x + t)$ is the shift along $t$.

The class of such functions coincides with the closure, with respect to the topology of the uniform convergence on $\mathbb{R}^m$, of the set of all finite exponential sums

$$\sum a_n e^{i<x, \lambda_n>}, \quad \lambda_n \in \mathbb{R}^m,$$

where $<x, \lambda_n> = x_1 \lambda_{n,1} + \ldots + x_m \lambda_{n,m}, \quad a_n \in \mathbb{C}$. In the case $m = 1$, this is one of the main results of the classical theory of almost periodic functions.

A continuous function $f(z)$ on a tube domain $T_\Omega = \{z = x + iy : x \in \mathbb{R}^m, y \in \Omega \subset \mathbb{R}^m\}$ is called almost periodic on $T_\Omega$ if the collection $\{S_t f\}_{t \in \mathbb{R}^m}$ is a relatively compact set with respect to the topology of uniform convergence on every tube domain $T_{\Omega'}$, $\Omega' \subset \subset \Omega$. 
The class of such functions coincides with the closure of the set of all finite exponential sums (1), where $a_n$ are continuous functions on $\Omega$.

Throughout the paper, $AP(\mathbb{R}^m)$ is the class of all almost periodic functions on $\mathbb{R}^m$, $AP(T_\Omega)$ is the class of all almost periodic functions on $T_\Omega$, $H(D)$ is the class of all holomorphic functions on $D$, $APH(T_\Omega) = AP(T_\Omega) \cap H(T_\Omega)$. We always consider tube domains with convex base $\Omega$.

Note that every function $f \in APH(T_\Omega)$ belongs to the closure of the set of all finite sums

$$\sum a_ne^{i(x,\lambda_n)}, \quad \lambda_n \in \mathbb{R}^m, \quad a_n \in \mathbb{C}. \quad (2)$$

The notion of spectrum for functions $f$ from $AP(\mathbb{R}^m)$ or $AP(T_\Omega)$ is introduced as

$$\text{sp} f = \{ \lambda \in \mathbb{R}^m : \lim_{T \to \infty} (2T)^{-m} \int_{[-T,T]^m} f(x + iy)e^{-i(x,\lambda)}dx \neq 0 \}; \quad (3)$$

the spectrum is at most countable, and all the exponents $\lambda$ in sums (1) or (2) belong to the spectrum of $f$.

For the case of functions on the axis or on a strip, all these facts are basic results of the theory of almost periodic function (see, for example, [6]); for the general case the proves are similar, see [33], [35].

By $C_{p,q}^\infty(T_\Omega)$ we denote the space of $(p,q)$-forms

$$\sum a_{i_1,\ldots,i_p,j_1,\ldots,j_q}dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q} \quad (4)$$

with infinitely smooth coefficients $a_{i_1,\ldots,i_p,j_1,\ldots,j_q}(z)$; we assume that the coefficients and all their derivatives are bounded on any domain $T_{\Omega'}$ for $\Omega' \subset \subset \Omega$. Also, by $C_{p,q}^0(T_\Omega)$ we denote the space of $(p,q)$-forms such that all the coefficients are continuous and have compact supports. Finally, by $M_{p,q}(T_\Omega)$ we denote the space of $(p,q)$-currents of order 0 on $T_\Omega$, i.e., forms (4) whose coefficients are complex measures on $T_\Omega$. This space is equipped with the weak topology of convergence on elements of $C_{m-p,m-q}^0(T_\Omega)$.

A typical example of a current from $M_{1,1}(T_\Omega)$ is the current of integration over the divisor $d_F$ of a function $F \in H(T_\Omega)$, i.e., the current $(i/2\pi)\partial\bar{\partial} \log |F(z)|$. We will identify occasionally a divisor $d$ with the corresponding current of integration and write $d \in M_{1,1}(T_\Omega)$.

A current $\theta \in M_{p,q}(T_\Omega)$ is almost periodic if for any $\phi \in C_{m-p,m-q}^0(T_\Omega)$ the function $(\theta, S_t\phi)$, as a function of $t$, belongs to $AP(\mathbb{R}^m)$; the space of such currents is denoted by $APM_{p,q}(T_\Omega)$. A divisor $d$ is almost periodic if $d \in APM_{1,1}(T_\Omega)$; all the divisors of functions $f \in APH(T_\Omega)$ are almost periodic (cf. [35]); if the measure $\sum_k \partial^2 \log |F|/\partial z_k \partial z_k$ for $F \in T_\Omega$ is almost periodic, then the divisor $d_F$ is almost periodic (see [14]).

We will make use of the following result (cf.[35], [11]).

**Proposition 1.1** If a sequence $\{f_n\} \subset H(D)$ converges uniformly on every compact subset of $D$, then the sequence $\{\log |f_n|\}$ converges on $D$ in the sense of distributions.

It was proved in [33] that the coefficients of a current $\theta \in APM_{p,q}(T_\Omega)$ can be approximated in the weak topology on sets $\{S_t\phi\}_{t \in \mathbb{R}^m}$, $\phi \in C_{0,0}^0(T_\Omega)$, uniformly in $t \in \mathbb{R}^m$, by finite sums (1); in this case $a_n$ are complex measures on $\Omega$.  

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The spectrum of $\theta \in APM_{p,q}(T_\Omega)$ is the union of the spectra of the functions $(\theta, S_\lambda \phi)$ over all $\phi \in C^0_{m-p,m-q}(T_\Omega)$. This set is countable as well, and all the exponents in sum (1) belong to the spectrum of $\theta$ (see [35] or [11]).

For any additive subgroup $\Gamma \subset \mathbb{R}^m$, we denote by $AP(\mathbb{R}^m, \Gamma)$, $AP(T_\Omega, \Gamma)$, $APH(T_\Omega, \Gamma)$, and $APM_{p,q}(T_\Omega, \Gamma)$ the corresponding classes of functions with spectra in $\Gamma$.

Bohr's compactification $\hat{\mathbb{R}}^m$ is the closure of the image of $\mathbb{R}^m$ under the map $j$ into Tikhonov's product of the circles $\mathbb{T}_\lambda$,

$$j : x \mapsto \{e^{i(x, \lambda)}\} \in \prod_{\lambda \in \mathbb{R}^m} \mathbb{T}_\lambda. \quad (5)$$

The preimage of Tikhonov's topology on $\mathbb{R}^m$ with respect to the map $j$ is called Bohr's topology and is denoted by $\beta$. It is clear that this topology is the weakest one where all the sums (1) are continuous. If we replace the product of the circles over all $\lambda \in \mathbb{R}^m$ by the product over $\lambda \in \Gamma$, $\Gamma$ being a subgroup of $\mathbb{R}^m$, then the corresponding Bohr's compactification is denoted by $\hat{\Gamma}$ and the corresponding topology by $\beta(\Gamma)$. Evidently, one can take the product of the circles only over all generators of $\Gamma$. In particular, for $\Gamma = \text{Lin}_\mathbb{Z}\{\lambda_1, \ldots, \lambda_N\}$ with vectors $\lambda_1, \ldots, \lambda_N \in \mathbb{R}^m$ linearly independent over $\mathbb{Z}$, the set $\hat{\Gamma}$ coincides with the torus $\mathbb{T}^N$. Then it follows from the approximation property that functions from $AP(\mathbb{R}^m)$ are uniformly continuous with respect to topology $\beta$ and functions from $AP(\mathbb{R}^m, \Gamma)$ are uniformly continuous with respect to topology $\beta(\Gamma)$. Hence functions from $AP(\mathbb{R}^m)$ are just the restrictions to $j\mathbb{R}^m$ of continuous functions on $\hat{\mathbb{R}}^m$, and functions from $AP(\mathbb{R}^m, \Gamma)$ are just the restrictions to $j\mathbb{R}^m$ of continuous functions on $\hat{\Gamma}$ (see [1] or [2]).

Following [17] and [18], we will consider line bundles $\mathcal{F}$ with the fiber $\mathbb{C}$ over a paracompact base $X$; in this paper $X = \hat{\mathbb{R}}^m$ or $X = \hat{\mathbb{R}}^m \times \Omega$, where $\Omega$ is an open convex subset of $\mathbb{R}^m$. We say that a section $\varphi$ of the bundle over an open set $\omega \subset \hat{\mathbb{R}}^m \times \Omega$ is holomorphic if $\varphi(\tau \cdot jx, y)$ is a smooth function of $x, y \in \mathbb{R}^m$ for small $x$, and $\partial_x \varphi(\tau \cdot jx, y) \big|_{x=0} = 0$ for all $(\tau, y) \in \omega$, $j$ being defined in (5). To each line bundle $\mathcal{F}$ over $X$ assign the (first) Chern class $c(\mathcal{F}) \in H^2(X, \mathbb{Z})$. Namely, let $\omega_\alpha$ be a sufficiently fine covering of $X$ and $g_{\alpha, \beta}(x), \ x \in X$, be the corresponding transition functions. The integer

$$c_{\alpha, \beta, \gamma} = \frac{1}{2\pi i} \log g_{\alpha, \beta}(x) + \frac{1}{2\pi i} \log g_{\beta, \gamma}(x) + \frac{1}{2\pi i} \log g_{\gamma, \alpha}(x) \quad (6)$$

corresponds to every triple of indices such that $\omega_\alpha \cap \omega_\beta \cap \omega_\gamma \neq \emptyset$ (if $\omega_\alpha \cap \omega_\beta \cap \omega_\gamma = \emptyset$, take $c_{\alpha, \beta, \gamma} = 0$). The collection $\{c_{\alpha, \beta, \gamma}\}$ forms a 2-cocycle; its cohomology class $[\{c_{\alpha, \beta, \gamma}\}] \in H^2(\{\omega_\alpha\}, \mathbb{Z})$ does not depend on the choice of branch of logarithm in (6). Moreover, if we multiply each function $g_{\alpha, \beta}(x)$ by $h_\alpha(x)/h_\beta(x)$, $h_\alpha(x) \neq 0$ for all $\alpha$ and $x \in \omega_\alpha$, then the cohomology class remains the same. Passing to the inductive limit with respect to refinement of covering, we obtain the first Chern class $c(\mathcal{F}) \in H^2(X, \mathbb{Z})$. If there exists a global section $\{\varphi_\alpha\}$ vanishing nowhere, then we can take $\log g_{\alpha, \beta} = \log \varphi_\alpha - \log \varphi_\beta$. This yields that $c_{\alpha, \beta, \gamma} = 0$, therefore $c(\mathcal{F}) = 0$.

On the other hand, to each $c \in H^2(X, \mathbb{Z})$ one can assign an element of the group $H^1(X, \mathbb{Z}^*)$, where $\mathbb{Z}^*$ is the multiplicative bundle of germs of continuous functions van-
ishing nowhere on $X$. Namely, the 1-cocycle $\{g_{\alpha,\beta}\}$ of the bundle $\Xi^*$ corresponds to $c_{\alpha,\beta,\gamma} \in H^2(\{\omega_n\}, \mathbb{Z})$ such that (6) is fulfilled. Note that the 1-cocycles $\{g_{\alpha,\beta}\}$ satisfy the conditions

$$g_{\alpha,\beta}g_{\beta,\alpha} = 1 \quad \text{and} \quad g_{\alpha,\beta}g_{\beta,\gamma}g_{\gamma,\alpha} = 1,$$

i.e., $\{g_{\alpha,\beta}\}$ are transition functions for some line bundle over $X$.

It is clear that the line bundle with the transition functions $g_{\alpha,\beta}/\tilde{g}_{\alpha,\beta}$ corresponds to the difference of cohomology classes $c - \tilde{c}$. Further, if $c = 0$, then $c_{\alpha,\beta,\gamma} = 0$ for sufficiently fine covering. Now it follows in the standard way that $g_{\alpha,\beta}$ is a continuous function on $\tilde{\Omega}$ for all $\tilde{\Omega}$ extends to $\hat{\tilde{\Omega}}$ on $\tilde{\Omega}$, and $\tilde{\Omega}$ is dense in $\tilde{\Omega}$. It is obvious that every function $f$ on $\tilde{\Omega}$ extends to the set $\hat{\tilde{\Omega}}$ as a continuous function $\hat{f}(\tau, y)$ such that $\hat{f}(j\tau, y) = f(z)$. Conversely, each continuous function $\tilde{f}(\tau, y)$ on $\tilde{\Omega}$ generates the function $f(z) = \hat{f}(j\tau, y) \in AP(T_\Omega)$. In addition, $f \in APH(T_\Omega)$ iff

$$\bar{\partial}_z \tilde{f}(\tau \cdot jx, y) \mid_{x=0} = 0$$

for all $(\tau, y) \in \tilde{\Omega}$.

\[ \Box \]

It is obvious that every function $e^{i(x, \lambda)}$ extends to $\hat{\Omega}$ as a continuous function $\tilde{e}_\lambda(\tau)$, therefore every exponential sum (1) with coefficients $a_n(y)$ continuous on $\Omega$ extends to $\hat{\Omega}$ as a continuous function, too. The same assertion is true for every $f \in AP(T_\Omega)$ as a uniform limit of such sums on every $T_{\Omega'}, \Omega' \subset \subset \Omega$. Since $f(z) = \hat{f}(j\tau, y)$ is valid for the functions $e^{i(x, \lambda)}$, the equality holds for all $f \in AP(T_\Omega)$. On the other hand, if $\tilde{f}(\tau, y)$ is continuous on $\hat{\Omega}$, then $\tilde{f}(\tau, y)$ is uniformly continuous on $\hat{\Omega}$ for each compact set $P \subset \Omega$, therefore the family of shifts $\{S_tf\}_{t \in \mathbb{R}}$ for functions $f(z) = \hat{f}(j\tau, y)$ is relatively compact set with respect to the topology of uniform convergence on every tube domain $T_{\Omega'}, \Omega' \subset \subset \Omega$, and $f \in AP(T_\Omega)$. Finally, since $j\mathbb{R}$ is dense in $\hat{\Omega}$ and $\tilde{f}(j(t + x), y) = S_tf(z)$, the function $\tilde{f}(\tau \cdot jx, y)$ is holomorphic, too.

\[ \Box \]

Proposition 2.2 Each divisor $d \in AP\text{M}_{1,1}(T_\Omega)$ defines a unique continuous mapping $\tilde{d} : \hat{\tilde{\Omega}} \mapsto M_{1,1}(T_\Omega)$ with the properties

$$\tilde{d}(\tau \cdot jt) = S_t\tilde{d}(\tau),$$

$$\tilde{d}(1) = d.$$
Conversely, each continuous mapping \( \bar{d} \) from \( \hat{R}^m \) to \( M_{1,1}(T_\Omega) \) with the property (8) generates a divisor \( d \in APM_{1,1}(T_\Omega) \) by equality (9); if \( d \) is the divisor of a function \( f \in AP\tilde{H}(T_\Omega) \), then \( \bar{d}(\tau) \) is the divisor of the holomorphic function \( \tilde{f}(\tau \cdot jx, y) \).

\( \Box \) Let \( \bar{e}_\lambda(\tau) \) be the function from the proof of the previous proposition. Then for every \( \lambda \in \hat{R}^m \) the map \( \tau \mapsto \bar{e}_\lambda(\tau \cdot jx) \) is a continuous map from \( \hat{R}^m \) to \( AP(\hat{R}^m) \) with the property (8). Therefore every sum (1) with complex measures \( a_n(y) \) on \( \Omega \) generates a continuous map from \( \hat{R}^m \) to \( M_{1,1}(T_\Omega) \). Every divisor \( d \in APM_{1,1} \) can be approximated by such sums, hence it generates a continuous map \( \bar{d} : \hat{R}^m \mapsto AP(T_\Omega) \) with the property (8). Conversely, each continuous map \( d \) from \( \hat{R}^m \) to \( AP(\hat{R}^m) \) with the properties (8) and each form \( \alpha \in C^\infty_{1,1}(T_\Omega) \) induce the continuous function \( (\bar{d}(\tau), \alpha) \) on \( \hat{R}^m \). Hence the function \( (\bar{d}(1), S_\alpha) = (\bar{d}(jt), \alpha) \) belongs to \( AP(R^m) \). This means that \( d \in APM_{1,1}(T_\Omega) \). Using the density of \( jR^m \) in \( \hat{R}^m \) and the equality \( \bar{d}(jt) = S\bar{d} \), we get the uniqueness of \( \bar{d} \). Finally, if \( d = d_jf \) for \( f \in AP\tilde{H}(T_\Omega) \), then the map \( \bar{d} : \tau \mapsto (i/2\pi)\partial \tilde{d} \log |f(\tau \cdot jx, y)| \) satisfies (8) and (9); this map is continuous because the map \( \tau \mapsto \tilde{f}(\tau \cdot jx, y) \) is continuous as a map from \( \hat{R}^m \) to \( AP(T_\Omega) \) and, by Proposition 1.1, the map \( f \mapsto \tilde{d} \log |f| \) is continuous as a map from \( H(T_\Omega) \) to \( M_{0,0}(T_\Omega) \). Thus \( \bar{d} = \bar{d} \).

**Remark.** In the previous propositions, the classes \( AP(T_\Omega), APM_{1,1}(T_\Omega) \) and the compact set \( \hat{R}^m \) can be replaced by \( AP(T_\Omega, \Gamma), APM_{1,1}(T_\Omega, \Gamma) \) and the compact set \( \hat{\Gamma} \), respectively. Really, we can use for approximation of the functions and currents with spectrum in \( \Gamma \) only the sums of exponents \( e^{i(x, \lambda)} \), \( \lambda \in \Gamma \). Conversely, the restriction to the set \( j\hat{R}^m \times \Omega \) of a continuous on \( \hat{\Gamma} \times \Omega \) function \( \bar{f}(\tau, y) \) is a continuous function \( f(z) \) on \( x \) with respect to the topology \( \beta(\Gamma) \), therefore \( \text{sp} f(x + iy_0) \subset \Gamma \) for each fixed \( y_0 \in \Omega \), and we obtain \( \text{sp} f \subset \Gamma \). In the same way, we get \( \text{sp} (\bar{d}(jt)) \subset \Gamma \) in Proposition 2.2.

We say that a function \( \varphi(\tau, y) \), continuous on an open set \( \omega \subset \hat{R}^m \times \Omega \), is holomorphic on \( \omega \) if the function \( \varphi(\tau \cdot jx, y) \) is smooth for small \( x \) and satisfies equation (7) for all \((\tau, y) \in \omega \). Further, \( \varphi(\tau, y) \) defines the divisor \( d \in APM_{1,1}(T_\Omega) \) on \( \omega \) if for each \( \tau, y_0 \in \omega \) there exists \( \delta > 0 \) such that the restriction of the divisor \( \bar{d}(\tau) \) to the ball \( B(iy_0, \delta) \subset \mathbb{C}^m \) coincides with the restriction of the divisor of the function \( \varphi(\tau \cdot jx, y) \).

It can be shown that a holomorphic function defining a divisor is essentially unique. Namely, we have

**Proposition 2.3** Suppose \( \bar{f}(\tau, y), \bar{g}(\tau, y) \) are holomorphic functions on \( \omega \subset \hat{R}^m \times \Omega \). If for each point \( z_0 = x_0 + iy_0 \), \( (jx_0, y_0) \in \omega \), the quotient \( \bar{f}(jx, y) / \bar{g}(jx, y) \) is a holomorphic function on some neighborhood of \( z_0 \), then \( \bar{f}(\tau, y) = \bar{g}(\tau, y) \bar{h}(\tau, y) \) with holomorphic on \( \omega \) function \( \bar{h}(\tau, y) \). Moreover, if the functions \( \bar{f}(jx, y) \) and \( \bar{g}(jx, y) \) have the same divisor, then \( \bar{h}(\tau, y) \neq 0 \).

\( \Box \) Fix \((\tau_0, y_0) \in \omega \). If \( \bar{g}(\tau_0, y_0) \neq 0 \), then we take \( \bar{h} = \bar{f} / \bar{g} \) on some neighborhood of \((\tau_0, y_0)\). Suppose \( \bar{g}(\tau_0, y_0) = 0 \). Let \( z = x + iy = (x^1, x) + iy^1(y^1, y) \), \( x, y, x^1, y^1 \in \mathbb{R} \), \( x', y' \in \mathbb{R}^{-1} \). Without loss of generality it can be assumed that \( a(x^1) = \bar{g}(\tau_0 \cdot j(x', 0), (y^1, y_0)) \neq 0 \). We have \(|a(z^1)| \geq \gamma > 0 \) whenever \(|z^1 - iy^1| = \varepsilon \) and \( \varepsilon \) is
sufficiently small. Let \( \tau \) be near \( \tau_0, 'y \) be near \( 'y_0, \zeta = \xi + i\eta \in \mathbb{C} \). Take

\[
\bar{h}(\tau, y) = \frac{1}{2\pi i} \int_{|\zeta-i'y_0|=\varepsilon} \frac{\bar{f}(\tau \cdot j(\xi,0), (\eta, y))}{\bar{g}(\tau \cdot j(\xi,0), (\eta, y))} \frac{d\zeta}{\zeta - iy^1}. 
\]

This function is continuous in some neighborhood of \((\tau_0, y_0)\). Take \( \tau = jt, \ t \in \mathbb{R}^m \). Using Cauchy’s integral formula, we obtain \( \bar{h}(jt, y)\bar{g}(jt, y) = \bar{f}(jt, y) \). Now the first statement of the proposition follows from the density of \( \mathbb{R}^m \) in \( \mathbb{R}^m \). To prove the last statement we interchange \( \bar{f} \) and \( \bar{g} \).

\[\square\]

**Proposition 2.4** For any divisor \( d \in APM_{1,1}(T_\Omega) \) and any point \((\tau_0, y_0) \in \mathbb{R}^m \times \Omega \) there exists a neighborhood \( \omega \) and a holomorphic function \( r(\tau, y) \) defining the divisor \( d \) on \( \omega \).

\[\square\]

Let \( \bar{d}(\tau) \) be the continuous map from proposition 2.2. Since \( T_\Omega \) is a convex domain, we see that for each \( \tau \in \mathbb{R}^m \) there exists a function \( f_\tau(z) \in H(T_\Omega) \) such that \((i/2\pi)\bar{d}(\log |f_\tau(z)|)\) is the current of integration over the divisor \( \bar{d}(\tau) \). Let \( z = (z^1, 'z)^1 = (x^1, x) + iy(1, y), x^1, y \in \mathbb{R}, 'x, 'y \in \mathbb{R}^{m-1}, \zeta = z^1 = x^1 + iy^1 \). Using a linear transformation of coordinates with real coefficients, we may assume that \( f_{\tau_0}(\zeta, iy_0) \neq 0 \).

Fix \( \delta > 0 \) such that \( f_{\tau_0}(\zeta, iy') \neq 0 \) for \( 0 < |\zeta - iy_0^1| \leq \delta \). Therefore we have \( \sup(\tau_0 \cap \{ z = (\zeta, iy_0) \in T_\Omega : \delta/2 \leq |\zeta - iy_0^1| \leq \delta \} = \emptyset \). Hence there exists a neighborhood \( U_{\tau_0} \) of \( \tau_0 \) and \( \varepsilon > 0 \) such that for \( \tau \in U_{\tau_0}, |'z - iy_0^1| < \varepsilon \) the functions \( f_{\tau}(\zeta, 'z) \) as functions of \( \zeta \) have no zeros on the set \( \delta/2 \leq |\zeta - iy_0^1| \leq \delta \).

Let \( \zeta_1(\tau', z), \ldots, \zeta_k(\tau', z), k = k(\tau', z) \), be all zeros of the function \( f_{\tau}(\zeta, 'z) \) on the set \( |\zeta - iy_0^1| < \delta/2 \) as a function of \( \zeta \). Let \( \varphi(\zeta) \) be a nonnegative infinitely smooth function such that \( \varphi(\zeta) = 1 \) for \( |\zeta - iy_0^1| \leq \delta/2 \) and \( \varphi(\zeta) = 0 \) for \( |\zeta - iy_0^1| \geq \delta \). \( \Delta_\zeta \) be Laplace operator on the plane \( C_\zeta \). We have

\[
k(\tau', z) = 2\pi(\Delta_\zeta \log |f_{\tau}(\zeta, 'z)|, \varphi(\zeta)) = \pi/i \int \log |f_{\tau}(\zeta, 'z)| \Delta_\zeta \varphi(\zeta) d\zeta \land d\bar{\zeta}.
\]

It follows from Hurwitz’ Theorem that for any fixed \( \tau \in U_{\tau_0} \) the number \( k(\tau', z) \) does not depend on \( 'z \) from the disc \( |'z - iy_0^1| < \varepsilon \).

Let \( \psi('z) \) be an infinitely smooth function with support in the set \( \{ 'z : |'z - iy_0^1| < \varepsilon \} \) such that \((2i)^{1-m} \int \psi('z) d'z \land d\bar{z} = 1 \). Then

\[
k(\tau', z) = 2\pi(2i)^{-m} \int \log |f_{\tau}(\zeta, 'z)| \psi('z) \Delta_\zeta \varphi(\zeta) d\zeta \land d\bar{\zeta} \land d'z \land d\bar{z}.
\]

The integral is the action of the current \((i/2\pi)\bar{d}(\log |f_{\tau}|)\) on the form \( \psi('z) \varphi(\zeta) d'z \land d\bar{z} \in \mathfrak{c}^m_{-1,m-1}(T_\Omega) \), hence it is continuous as a function of \( \tau \), and \( k(\tau', z) = \text{const} \) on \( U_{\tau_0} \times \{ 'z : |'z - iy_0^1| < \varepsilon \} \). Since \( \delta \) is arbitrary small, the functions \( \zeta_1(\tau', z), \ldots, \zeta_k(\tau', z) \) are continuous at the point \( (\tau_0, iy_0^1) \). By the same arguments, these functions are continuous at each point \( (\tau', z) \) with the property \( f_{\tau}(\zeta, 'z) \neq 0 \).

Now we define a function \( r(\tau, y) \). If \( iy \notin \sup \bar{d}(\tau_0) \), we can take \( r(\tau, y) = \equiv 1 \) and \( \omega \) is small enough. Otherwise we take \( \omega = \{ (\tau, y) : \tau \in U_{\tau_0}, |y^1 - y_0^1| < \delta, |'y - iy_0^1| < \delta \} \).
defines some divisor $d$ divisor of a function $f$. Every holomorphic section vanishing nowhere. such that some global holomorphic section of this bundle defines the divisor $d$. Theorem 2.2 analog of this theorem for divisors with spectrum in a subgroup $\Gamma$: 

$$
\varepsilon, r(\tau, y) = P(\tau, iy^l, i'y^l) \text{ with } P(\tau, \zeta^l, z) = (\zeta - \zeta_1(\tau, z)) \cdots (\zeta - \zeta_k(\tau, z)). \text{ Represent this polynomial in the form } P(\tau, \zeta^l, z) = \zeta^k + b_{k-1}(\tau, z)\zeta^{k-1} + \ldots + b_0(\tau, z). \text{ Since }
$$

$$
\sum_{1 \leq j \leq k} (\zeta_j(\tau, z))^s = \frac{1}{2\pi i} \int_{|z-iy| = \delta} \frac{\partial f_\tau(\zeta, z)}{\partial \zeta} \frac{\zeta^s d\zeta}{f_\tau(\zeta, z)},
$$

all the functions $b_l(\tau, z)$, $l = 1, \ldots, k-1$, are holomorphic on the set $\{z : |z - i'y| < \varepsilon\}$. Since the polynomial $P(\tau, z^l, z)$ has the same zeros as $f_\tau(z)$ on the set $\{z : |z^1 - i'y| < \delta, |z - y| < \varepsilon\}$, we get that the restriction of the divisor of $P(\tau, z)$ to this set is $d(\tau)$. It follows from (8) that $P(\tau \cdot j, z) = P(\tau, t + z)$ for small $t$ and $z$, hence $r(\tau, y)$ is a holomorphic function on $\omega$. 

Now we introduce the main result of this section.

**Theorem 2.1** To each $d \in AP_{1,1}(T_{\Omega})$ one can assign a line bundle $\mathbb{F}_d$ over $\hat{\mathbb{R}}^m \times \Omega$ and a global holomorphic section of this bundle defining the divisor $d$; $d$ is the divisor of a function $f \in AP(H(T\Omega))$ iff $\mathbb{F}_d$ has a global holomorphic section vanishing nowhere.

Take a sufficiently fine covering $\{\omega_\alpha\}$ of $\hat{\mathbb{R}}^m \times \Omega$ such that the divisor $d$ is defined by a holomorphic function $r_\alpha(\tau, y)$ on each $\omega_\alpha$. Taking into account Proposition 2.3, we get that the functions $g_{\alpha, \beta}(\tau, y) = r_\alpha(\tau, y)/r_\beta(\tau, y)$ vanish nowhere on $\omega_\alpha \cap \omega_\beta$. It is obvious that these functions satisfy all the conditions for being transition functions of some line bundle $\mathbb{F}_d$ over $\hat{\mathbb{R}}^m \times \Omega$. The equality

$$
r_\beta(\tau, y) = r_\alpha(\tau, y)g_{\alpha, \beta}(\tau, y)
$$

means that $\{r_\alpha\}$ is the section of $\mathbb{F}_d$. If $d = df$ for some $f \in AP(H(T\Omega))$, then the function $f(\tau, y)$ from Proposition 2.1 also defines $d$, hence $\{r_\alpha(\tau, y)/f(\tau, y)\}$ is a global holomorphic section vanishing nowhere.

Conversely, if $\{R_\alpha(\tau, y)\}$ is a global holomorphic section of $\mathbb{F}_d$ vanishing nowhere, then the holomorphic function $F(\tau, y) = r_\alpha(\tau, y)/R_\alpha(\tau, y)$ is well defined on $\hat{\mathbb{R}}^m \times \Omega$ and defines the same divisor as $r_\alpha(\tau, y)$.

Using Remark to Propositions 2.1 and 2.2 and arguing as above, we also get an analog of this theorem for divisors with spectrum in a subgroup $\Gamma$:

**Theorem 2.2** To each $d \in AP_{1,1}(T_{\Omega}, \Gamma)$ one can assign a line bundle over $\hat{\mathbb{R}} \times \Omega$ such that some global holomorphic section of this bundle defines the divisor $d$; $d$ is the divisor of a function $f \in AP(H(T\Omega), \Gamma)$ iff the bundle has a global holomorphic section vanishing nowhere.

In the end of this section we prove the converse statement to Theorem 2.1:

**Proposition 2.5** Every holomorphic section $\{r_\alpha(\tau, y)\}$ of a line bundle $\mathbb{F}$ over $\hat{\mathbb{R}}^m \times \Omega$ defines some divisor $d \in AP_{1,1}(T_{\Omega})$. 

9
Fix $\tau \in \mathbb{R}^m$. The equality
\[ \bar{d}(\tau) = (i/2\pi)\partial\overline{\partial} \log |r_{\alpha}(\tau \cdot jx, y)| \] (11)
for $\alpha$ such that $(\tau \cdot jx_0, y_0) \in \omega_0$ defines the divisor $\bar{d}(\tau)$ on the set $\{z = x + iy : |z - z_0| < \varepsilon\}$, $\varepsilon$ is small enough. It follows from (10) that the current (11) is well-defined and belongs to $M_{1,1}(T_{\Omega})$. Since $(\tau \cdot jt) \cdot jx_0 = \tau \cdot j(x_0 + t)$, we obtain (8). Furthermore, the function $r_{\alpha}(\tau \cdot jx, y)$ is continuous with respect to $\tau$; using Proposition 1.1, we obtain that the function $\log |r_{\alpha}(\tau \cdot jx, y)|$, as a mapping from $\mathbb{R}^m$ to $M_{0,0}(|\{z : |z - z_0| < \varepsilon\}$, is continuous. Therefore the mapping $\bar{d}(\tau)$ of $\mathbb{R}^m$ to $M_{0,0}(T_{\Omega})$ is continuous, and we can use Proposition 2.2.

3 Existence of a global holomorphic section vanishing nowhere

Here we get a sufficient condition for the existence of a global holomorphic section vanishing nowhere. First we need the following result.

Proposition 3.1 For each pair of convex domains $\Omega' \subset \subset \Omega$ the equation $\bar{\partial} u = a$ with a $\bar{\partial}$-closed $(0,1)$-differential form $a(z) = \sum_k a_k(z)dz^k$ with bounded and $C^1$-smooth on $T_{\Omega}$ coefficients has a solution $Aa \in C_{0,0}^\infty(T_{\Omega'})$ with the following properties:

- $A$ is a linear operator,
- $A\Sigma^0 = \Sigma^1 A \forall \tau \in \mathbb{R}^m$,
- $\sup_{z \in T_{\Omega'}} |Aa(z)| \leq c(\Omega, \Omega') \sup_{z \in T_{\Omega}} (\sum_k |a_k(z)|^2)^{1/2}$.

At first, assume that $\Omega$ is bounded and $\Omega = \{y : \kappa(y) < 0\}$ for some convex and $C^2$-smooth function $\kappa(y)$ in a neighborhood of $\Omega$, $\text{grad}_y \kappa(y) \neq 0$ on $\partial\Omega$, and $a(z)$ is any $\bar{\partial}$-closed $(0,1)$-differential form with bounded and $C^1$-smooth on $T_{\Omega}$ coefficients. Let $\beta(t)$ be a convex, $C^2$-smooth function on $\mathbb{R}$ such that $\beta(t) = 0$ for $t < 0$, $\beta(t) > 0$ for $t > 0$, and $\beta(1) > -\min_\Omega \kappa(y)$. For $z = x + iy \in \mathbb{C}^m$ and $r > 0$, put $\rho(z) = \kappa(y)$, $\rho_r(z) = \rho(z) + \beta(|x| - r)$, $G_r = \{z : \rho_r(z) < 0\}$. It is clear that $\{z \in T_{\Omega} : |z| < r\} \subset G_r \subset \{z \in T_{\Omega} : |z| < r + 1\}$.

Let $\zeta = \xi + i\eta \in \mathbb{C}^m$, $\langle z, \zeta \rangle = \sum_{k=1}^m z^k \zeta^k$. Fix a compact set $K \subset T_{\Omega}$. We claim that the inequality
\[ |\langle \text{grad}_\xi \rho_r(\zeta), \zeta - z \rangle| \geq \varepsilon \] (12)
is true with some $\varepsilon > 0$ for all $z \in K$, $\zeta \in T_{\Omega}$ such that $\text{dist}(\zeta, \partial G_r) < \delta$, $\delta = \delta(K)$, for all $r > r(K)$. It is sufficient to estimate from below the sum
\[ \langle \text{grad}_\xi \beta(|\xi| - r), \xi - x \rangle + \langle \text{grad}_\eta \kappa(y), \eta - y \rangle. \] (13)
Here the first term is nonnegative for $|x| < r$ and all $\xi \in \mathbb{R}^m$. Since
\[ \langle \text{grad}_\eta \kappa(y), \eta - y \rangle \geq 2\varepsilon > 0 \] (14)
for all \( \eta \in \partial \Omega \), \( z \in K \), and some \( \varepsilon > 0 \), we see that the value (13) is at least \( \varepsilon \) whenever \( \text{dist}(\eta, \partial \Omega) \leq 2\delta \) for some \( \delta > 0 \). On the other hand, if \( \text{dist}(\zeta, \partial G_r) < \delta \) and \( \text{dist}(\eta, \partial \Omega) > 2\delta \), then \( |\xi| - r \geq \gamma \) for some \( \gamma > 0 \), and we have

\[
\langle \text{grad}_z \beta(|\xi| - r), \xi - x \rangle = \beta'(|\xi| - r)\langle \xi, \xi - x \rangle / |\xi| \geq \beta'(|\xi| - |x|).
\]

Since the last term is greater than \( \beta'(|\xi| - r + \gamma - |x|) \), and since \( |\langle \text{grad}_\eta \kappa(\eta), \eta - y \rangle| \leq c < \infty \) for all \( \eta, y \in \Omega \), we get the desired estimate for a sufficiently large \( r \).

Furthermore, put

\[
u_r(z) = C_m \int_{G_r} \exp(z - \zeta, \zeta - z) \frac{a \land (s_r, d\zeta) \land (d\zeta(s_r, d\zeta))^{m-1}}{(s_r, \zeta - z)^m}
\]

with \( s_r = \langle \text{grad}_z \rho_r(\zeta), \zeta - z \rangle \text{grad}_z \rho_r(\zeta) + |\rho_r(\zeta)|(|\zeta - \bar{z}) \rangle \). Note that the kernel in (15) is, up to a constant, just the component of bidegree \( (0, 0) \) in \( z \) of the kernel given by equality (14) of [4] for \( Q = z - \zeta \). Moreover, it follows from (13) that \( s_r \) satisfies the necessary conditions of Theorem 2 of [4] (see also item ii) on p. 102 there), hence for the appropriate choice of the constant \( C_m \) in (15) we have \( \partial u_r(z) = a(z), z \in G_r \).

It follows from (12) that \( |\langle s_r, \zeta - z \rangle| \) are uniformly bounded from below by some positive value if \( z \in K, \zeta \in G_r, |\xi| \geq r \) and \( r \geq 2 \sup \{|x| : x + iy \in K \} \). Hence the integral

\[
C_m \int_{G_r \cap \{\zeta : |\zeta| > r\}} \exp(z - \zeta, \zeta - z) \frac{a \land (s_r, d\zeta) \land (d\zeta(s_r, d\zeta))^{m-1}}{(s_r, \zeta - z)^m}
\]

and all its derivatives in \( z \) tend to zero as \( r \to \infty \). For \( |\xi| < r \) we have \( \rho_r(\zeta) = \rho(\zeta) \) and the function \( s_r \) coincides with \( s = \langle \text{grad}_z \rho(\zeta), \zeta - z \rangle \text{grad}_z \rho(\zeta) + |\rho(\zeta)|(|\zeta - \bar{z}) \rangle \). Therefore the functions \( u_r \) tend to the integral

\[
Aa(z) = C_m \int_{T_1} \exp(z - \zeta, \zeta - z) \frac{a \land (s, d\zeta) \land (d\zeta(s, d\zeta))^{m-1}}{(s, \zeta - z)^m}
\]

as \( r \to \infty \) uniformly in \( z \in K \), and the function \( Aa(z) \) belongs to the class \( C^1 \) and \( \bar{\partial}Aa(z) = a(z) \) for all \( z \in \Omega \).

It is clear that the assertion a) is fulfilled. Since the function \( \rho(\zeta) \) does not depend on \( \xi \), the kernel in the integral (16) depends only on the variables \( y, \eta, x - \xi \), and the substitution \( \zeta \mapsto \zeta - t \) for all \( t \in \mathbb{R}^m \) gives the assertion b). Finally, it follows from (14) that

\[
|\langle s(z, \zeta), \zeta - z \rangle| \geq |\langle \text{grad}_\eta \kappa(\eta), \eta - y \rangle| \geq C_1 > 0
\]

for \( z \in T_{1'}, \zeta \in \Omega, \text{dist}(\eta, \partial \Omega) < \delta \), and

\[
|\langle s(z, \zeta), \zeta - z \rangle| \geq |\kappa(\eta)||\zeta - z| \geq C_2 |\zeta - z|
\]

for \( z \in T_{1'}, \zeta \in \Omega, \text{dist}(\eta, \partial \Omega) \geq \delta \). Moreover, for \( z, \zeta \in \Omega \) all the coefficients of the form \( \langle s(z, \zeta) d\zeta \rangle \) have the upper bound \( C_3 |\zeta - z| \), and the coefficients of the form
$d_\zeta \langle s(z, \zeta), \zeta - z \rangle$ have the upper bound $C_4(1 + |\zeta - z|)$. Hence the integrand in (16) estimates from above as

$$C_5 \exp(-|x - \xi|^2) \sup_{\zeta \in T_{\Omega}} |a(\zeta)| \frac{|\zeta - z|(1 + |\zeta - z|)^{m-1}}{\min\{1, |\zeta - z|^m\}}$$

for all $z \in T_{\Omega'}$, and we obtain the assertion c).

In the general case, it is sufficient to take an arbitrary convex domain $\tilde{\Omega}, \Omega' \subset \subset \tilde{\Omega} \subset \subset \Omega$, such that the operator $A$ with the desired properties for the pair $(\Omega', \tilde{\Omega})$ has been already constructed.

Our main result of this section is as follows.

**Theorem 3.1** Let $\mathcal{F}$ be a line bundle over $\mathbb{R}^m \times \Omega$ with holomorphic transition functions $\{g_{\alpha, \beta}\}$. If for some $y_0 \in \Omega$ there exists a section of $\mathcal{F}$ over $\mathbb{R}^m \times \{y_0\}$ which does not vanish on $\mathbb{R}^m \times y_0$, then $\mathcal{F}$ has a global holomorphic section vanishing nowhere.

As an immediate consequence of theorems 2.1 and 3.1 we get

**Corollary 3.1** (for $m = 1$ see [12]). If the restriction of a divisor $d \in APM_{1,1}(T_{\Omega})$ to the domain $T_{\Omega'} \subset T_{\Omega}$ is the divisor of a function $g \in APH(T_{\Omega})$ (for example, the projection of supp $d$ on $\Omega$ is not dense in $\Omega$), then $d$ is the divisor of a function $f \in APH(T_{\Omega})$.

Since the domain $\Omega$ is contractible to a point $\{y_0\}$, the bundle $\mathcal{F}$ is isomorphic to the restriction of $\mathcal{F}$ to the base $\mathbb{R}^m \times \{y_0\}$ (see [19], Chapter 3, Theorem 4.7). Therefore there exists some global section of $\mathcal{F}$ vanishing nowhere, and the Chern class $c(\mathcal{F}) \in H^2(\mathbb{R}^m \times \Omega, \mathbb{Z})$ is zero. Hence the cocycle $\{c_{\alpha, \beta, \gamma}\}$ from (6) is a coboudary for a sufficiently fine covering $\{\omega_\alpha\}$ of $\mathbb{R}^m \times \Omega$, i.e., there exists a 1-cochain $\{b_{\alpha, \beta}\}$ with the property

$$c_{\alpha, \beta, \gamma} = b_{\alpha, \beta} + b_{\beta, \gamma} + b_{\gamma, \alpha}.$$ 

Moreover,

$$b_{\alpha, \beta} = -b_{\beta, \alpha}.$$ 

Taking into account (6), we see that the functions

$$F_{\alpha, \beta}(\tau, y) = (1/2\pi i) \log g_{\alpha, \beta}(\tau, y)$$

satisfy the equalities

$$F_{\alpha, \beta}(\tau, y) = -F_{\beta, \alpha}(\tau, y), \quad (\tau, y) \in \omega_\alpha \cap \omega_\beta,$$

and

$$F_{\alpha, \beta}(\tau, y) + F_{\beta, \gamma}(\tau, y) + F_{\gamma, \alpha}(\tau, y) = 0 \quad (\tau, y) \in \omega_\alpha \cap \omega_\beta \cap \omega_\gamma,$$ 

for a suitable choice of the branches of the logarithm.
Fix a convex domain $\Omega' \subset \subset \Omega$. We can assume that the covering $\{\omega_n\}$ of the set $\mathbb{R}^m \times \Omega'$ has the form $\omega_n = U_n \times B_k$, where $\{U_n\}_{n=1}^N$ is an open covering of $\mathbb{R}^m$, and $\{B_k\}_{k=1}^K$ is an open covering of $\Omega'$. We can also assume that the oscillations of all the functions $F_{\alpha,\beta}(\tau, y)$ on each $\omega_n$ to be less than $1/4$.

Let $\{\chi_n(\tau)\}_{n=1}^N$ be a partition of unity corresponding to the covering $\{U_n\}_{n=1}^N$, $\{\xi_k(y)\}_{k=0}^K$ be a partition of unity corresponding to the covering $\{B_k\}_{k=0}^K$. Further, suppose $\nu(s) \geq 0$ is a $C^\infty$-smooth function with the support in the ball $|s| < \varepsilon$ and such that $\int_{\mathbb{R}^m} \nu(s) ds = 1$. Since the covering $\{U_n\}$ is finite, we see that for $\varepsilon$ sufficiently small the functions

$$\tilde{\chi}_n(\tau) = \int_{\mathbb{R}^m} \chi_n(\tau \cdot js) \nu(s) ds$$

form a partition of unity subjected to the covering $\{U_n\}$, too. Put for $(\tau, y) \in \omega_n$

$$F_{\alpha}(\tau, y) = \sum_{l,k} \tilde{\chi}_l(\tau)\xi_k(y)F_{(l,k),\alpha}(\tau, y).$$

(20)

It follows from (18) and (19) that for $(\tau, y) \in \omega_\alpha \cap \omega_\beta$,

$$F_{\alpha}(\tau, y) - F_{\beta}(\tau, y) = F_{\alpha,\beta}(\tau, y).$$

(21)

In addition, all the functions $F_{\alpha}(\tau \cdot jx, y)$ are $C^\infty$-smooth on their domains of definition. Put

$$a(\tau, x, y) = \partial F_{\alpha}(\tau \cdot jx, y).$$

for $(\tau \cdot jx, y) \in \omega_\alpha$ and fixed $\tau$. Since the function $F_{\alpha,\beta}(\tau, y)$ is holomorphic, relation (21) implies that all the coefficients of $a(\tau, x, y)$ are well-defined and uniformly bounded on $\mathbb{R}^m \times \mathbb{R}^m \times \Omega'$. Besides, $a(\tau \cdot jx, 0, y) = a(\tau, x, y)$ and $\partial_z a = 0$. Therefore it follows from Propositions 3.1 that for each convex domain $\Omega'' \subset \subset \Omega'$ there exists a function $b(\tau, x, y)$, uniformly continuous on $\mathbb{R}^m \times \mathbb{R}^m \times \Omega''$ and such that $\partial_z b = a$ and $b(\tau, x, y) = S_\beta b(\tau, 0, y) = b(\tau \cdot jx, 0, y)$. In view of (21), the functions

$$\varphi_{\alpha}(\tau, y) = \exp 2\pi i [F_{\alpha}(\tau, y) - b(\tau, 0, y)], \ (\tau, y) \in \omega_\alpha,$$

satisfy the equations

$$\varphi_{\alpha}(\tau, y) = g_{\alpha,\beta}(\tau, y)\varphi_{\beta}(\tau, y), \ (\tau, y) \in \omega_\alpha \cap \omega_\beta,$$

hence they form a holomorphic, vanishing nowhere section of $F$ over $\mathbb{R}^m \times \Omega''$.

Let $\Omega_n$, $n = 1, 2, \ldots$, be convex domains such that $\Omega_n \subset \subset \Omega_{n+1}$, $\cup_n \Omega_n = \Omega$, and $\{\omega_n\}$ be a sufficiently fine covering of $\mathbb{R}^m \times \Omega$. We have just proved that for each $n$ there exists a holomorphic, vanishing nowhere section $\{\varphi_{\alpha}(\tau, y)\}$ over $\mathbb{R}^m \times \Omega_n$. It is obvious that the functions

$$\tilde{h}_n(\tau, y) = \frac{\varphi_{\alpha}^{n+1}(\tau, y)}{\varphi_{\alpha}^n(\tau, y)}, \ (\tau, y) \in \omega_n,$$

are well-defined, holomorphic, and vanishing nowhere on the set $\tilde{\mathbb{R}}^m \times \Omega_n$. It follows from Proposition 2.1 that $h_n(z) = \tilde{h}_n(jx, y) \in APH(T_{\Omega_n})$ and $h_n(z) \neq 0$ on $T_{\Omega_n}$. Therefore (for $m = 1$ see [6], for $m > 1$ see [9])

$$h_n(z) = \exp [i(z, c_n) + \kappa_n(z)]$$

(22)
with \( c_n \in \mathbb{R}^m \), \( \kappa_n \in APH(T_{\Omega_n}) \). Take an exponential sum \( P_n(z) \) of the type (2) such that \( |\kappa_n(z) - P_n(z)| < 2^{-n} \) on \( T_{\Omega_{n-1}} \). It is easy to prove that the function

\[
f_n(z) = \prod_{k=1}^{n-1} \exp[-i\langle z, c_k \rangle - P_k(z)] \prod_{k=n}^{\infty} h_k(z) \exp[-i\langle z, c_k \rangle - P_k(z)]
\]

belongs to the class \( APH(T_{\Omega_{n-1}}) \). Besides, \( f_n(z)/f_{n+1}(z) = h_n(z) \). Therefore the function \( \Psi_\alpha(\tau, y) = \varphi_\alpha^n(\tau, y) f_\alpha(\tau, y) \) is well-defined on \( \omega_\alpha \) whenever \( \omega_\alpha \subset \Omega_{n-1} \). Hence the section \( \{ \Psi_\alpha(\tau, y) \} \) is well-defined, holomorphic, and vanishing nowhere on the set \( \hat{\mathbb{R}}^m \times \Omega \).

\[\text{Remark 1.} \quad \text{Instead of the existence of a section over } \hat{\mathbb{R}}^m \times y_0 \text{ vanishing nowhere we may assume that (18) and (19) is true for } y = y_0.\]

\[\text{Remark 2.} \quad \text{The condition } h_n \in APH(T_{\Omega_n}, \Gamma) \text{ implies } \kappa_n \in APH(T_{\Omega_n}, \Gamma) \text{ and } c_n \in \Gamma \text{ (see [9]). Therefore, since the class } APH(T_{\Omega}, \Gamma) \text{ is closed with respect to the uniform convergence, the previous theorem is true for line bundles over } \hat{\Gamma} \times \Omega.\]

4 Chern class of almost periodic divisors

As above, take a covering \( \{ U_\alpha \times B_s \} \) of the set \( \hat{\mathbb{R}}^m \times \Omega \) such that \( \{ U_\alpha \}_{\alpha \in A} \) is an open covering of \( \hat{\mathbb{R}}^m \) and \( \{ B_s \}_{0 \leq s < \infty} \) is an open covering of \( \Omega \). Suppose the functions \( r_{(\alpha, s)}(\tau, y) \) define a divisor \( d \in APM_{1,1}(T_\Omega) \) on the sets \( \{ U_\alpha \times B_s \} \); then the functions \( g_{(\alpha, s), (\beta, n)}(\tau, y) = r_{(\alpha, s)}(\tau, y)/r_{(\beta, n)}(\tau, y) \) are transition functions for the line bundle \( \mathbb{F}_d \) over \( \hat{\mathbb{R}}^m \times \Omega \). Fix \( y_0 \in B_0 \); the function \( g_{(\alpha, 0), (\beta, 0)}(\tau, y_0) \) defines the line bundle \( \mathbb{F}_{d, y_0} \) over \( \hat{\mathbb{R}}^m \). If we take another functions \( \tilde{r}_{(\alpha, s)}(\tau, y) \) defining the divisor \( d \) over \( U_\alpha \times B_s \), then \( \tilde{r}_{(\alpha, s)}(\tau, y) = h_{(\alpha, s)}(\tau, y) r_{(\alpha, s)}(\tau, y) \) with some functions \( h_{(\alpha, \tau, y)} \) vanishing nowhere. Therefore a unique Chern class is assigned to the divisor \( d \).

We will say that the Chern class \( c(\mathbb{F}_{d, y_0}) \) is the Chern class \( c(d) \) of the divisor \( d \).

Note that the sum of divisors \( d_1, d_2 \) is defined by the product of the functions defining the divisors \( d_1, d_2 \); hence, \( c(d_1 + d_2) = c(d_1) + c(d_2) \).

Also, if \( d \) is the divisor of a function \( f \in APH(T_\Omega) \), then by Theorem 2.5 the bundle \( \mathbb{F}_{d, y_0} \) has a section \( \varphi_\alpha(\tau) \) vanishing nowhere. Take in (6) \( \log g_{\alpha, \beta}(\tau) = \log \varphi_\alpha(\tau) - \log \varphi_\beta(\tau) \) for all \( \alpha, \beta \), then we obtain \( c(d) = 0 \). On the other hand, if \( c(d) = 0 \), then for a sufficiently fine covering of \( \hat{\mathbb{R}}^m \) and for a suitable branches of logarithm, the left-hand side in (6) is zero for all \( \alpha, \beta, \gamma \). Now, using Theorem 2.1 and Remark 1 to Theorem 3.1, we come to the following result.

**Theorem 4.1** \( d \in APM_{1,1}(T_\Omega) \) is the divisor of a function \( f \in APH(T_\Omega) \) iff \( c(d) = 0 \).

Suppose that a domain \( \Omega \subset \mathbb{R}^m \) is stable under the map \( L : y \mapsto 2y_0 - y \). If \( \tilde{d} \) is the image of a divisor \( d \in APM_{1,1}(T_\Omega) \) under this map, then the holomorphic functions \( \tilde{r}_\alpha(\tau, y) = r_\alpha(\tau, 2y_0 - y) \) define this divisor on the set \( U_\alpha \times B_0 \) (we may assume that
If the restriction of a divisor \( d \in APM_{1,1}(T_\Omega) \) to the ball \( B \subset \Omega \) with center in \( y_0 \) is stable under the map \( y \mapsto 2y_0 - y \), then \( d \) is the divisor of a function \( f \in APH(T_\Omega) \).

Suppose \( sp \, d \subset \Gamma \), where \( \Gamma \) is a subgroup of \( \mathbb{R}^m \). As above, we can introduce the Chern class \( c_\Gamma(d) \in H^2(\hat{\Gamma}, \mathbb{Z}) \) and prove the following statement.

**Theorem 4.3** \( d \in APM_{1,1}(T_\Omega, \Gamma) \) is the divisor of a function \( f \in APH(T_\Omega, \Gamma) \) iff \( c_\Gamma(d) = 0 \).

Consider a relation between \( c_\Gamma(d) \) and \( c(d) \). Let \( \iota \) be the identity mapping of \( \mathbb{R}^m \) with the topology \( \beta \) to \( \mathbb{R}^m \) with the topology \( \beta(\Gamma) \). Extend \( \iota \) to a continuous mapping \( \tilde{\iota} \) of \( \mathbb{R}^m \) to \( \hat{\Gamma} \). It is easy to prove that \( \tilde{\iota} \) is an open surjection. Since \( \tilde{\iota} \) is an epimorphism, \( \iota^* \) is a monomorphism. Using Theorem 4.1, Theorem 4.3 and (23), we derive the following result.

**Theorem 4.4** If \( d \) is the divisor of a function \( f \in APH(T_\Omega) \) and \( sp \, d \subset \Gamma \), then \( d \) is the divisor of a function \( f_1 \in APH(T_\Omega, \Gamma) \).

It follows from the next theorem that the case \( \Gamma = Lin_\mathbb{Z}\{\lambda_1, \ldots, \lambda_N\} \) with the vectors \( \lambda_1, \ldots, \lambda_N \in \mathbb{R}^m \) linearly independent over \( \mathbb{Z} \) is most important.

**Theorem 4.5** For each \( d \in APM_{1,1}(T_\Omega) \) there exists a divisor \( d' \in APM_{1,1}(T_\Omega) \) with the spectrum in the group \( Lin_\mathbb{Z}\{\lambda_1, \ldots, \lambda_n\} \) such that the both divisors have the same Chern class.

Let \( \{U_\alpha\} \) be any sufficiently fine covering of \( \mathbb{R}^m \). We may assume that

\[
U_\alpha = \{\tau \in \mathbb{R}^m : (\tau_{\lambda_1}, \ldots, \tau_{\lambda_n}) \in \tilde{U}_\alpha\},
\]

where \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}^m \) are linearly independent over \( \mathbb{Z} \), the number \( n \) and the vectors \( \lambda_1, \ldots, \lambda_n \) are the same for all \( \alpha \). The coordinates of \( \tau \in \mathbb{R}^m \subset \prod_{\lambda \in \mathbb{R}^m} \mathbb{T}_\lambda \), \( \tilde{U}_\alpha \) is an open set in the torus \( \mathbb{T}^m \). Clearly, the group \( H^2(\{U_\alpha\}, \mathbb{Z}) \) is isomorphic to the group \( H^2(\{\tilde{U}_\alpha\}, \mathbb{Z}) \). Passing to the inductive limit with respect to the refinement of coverings, we obtain a monomorphism

\[
\iota^* : H^2(\hat{\Gamma}, \mathbb{Z}) \mapsto H^2(\mathbb{R}^m, \mathbb{Z})
\]
exists a divisor $d$ on the branches of $\log \Psi_Z$ with $\Gamma = \text{Lin}_Z$. It will be proved in the next section that for any element $c \in H^2(\hat{\Gamma}, Z)$ with $\Gamma = \text{Lin}_Z\{\lambda_1, \ldots, \lambda_n\}$ there exists a divisor $d' \in APM_{1,1}(T_\Omega)$ with the Chern class $c$.

Let $d \in APM_{1,1}(T_\Omega, \Gamma)$ with $\Gamma = \text{Lin}_Z\{\lambda_1, \ldots, \lambda_n\}$, the vectors $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^m$ being linearly independent over $\mathbb{Z}$. Here $\hat{\Gamma} = \mathbb{T}^N = \{\zeta \in \mathbb{C}^N : \zeta^l = e^{2\pi i u^l}, l = 1, \ldots, N\}$. By definition, put

$$P : u \in \mathbb{R}^N \mapsto (e^{2\pi i u^1}, \ldots, e^{2\pi i u^N}).$$

Let $\{U_\alpha\}$ be any sufficiently fine open covering of $\mathbb{T}^N$ with connected $U_\alpha$, $\mathbb{F}$ be a line bundle over $\mathbb{T}^N$ with transition functions $\{g_{\alpha,\beta}(\zeta)\}$. Take an arbitrary connected component $\omega_{\alpha,0}$ of the set $P^{-1}(U_\alpha)$ and put

$$\omega_{\alpha,k} = \omega_{\alpha,0} + k, \quad k \in \mathbb{Z}^N.$$

It can be assumed that $\omega_{\alpha,k} \cap \omega_{\alpha,k'} = \emptyset$ for all $\alpha$ and $k \neq k'$. The functions $G_{(\alpha,k),(\beta,n)}(u) = g_{\alpha,\beta}(P(u)), \quad u \in \omega_{\alpha,k} \cap \omega_{\beta,n}$, are transition functions for some line bundle $\mathbb{F}$ over $\mathbb{R}^N$. Any bundle over $\mathbb{R}^N$ has a global section vanishing nowhere, and let $\{\Phi_{\alpha,k}(u)\}$ be just the same section for $\mathbb{F}$. Then we have

$$\Phi_{\alpha,k}(u) = g_{\alpha,\beta}(P(u))\Phi_{\beta,n}(u), \quad u \in \omega_{\alpha,k} \cap \omega_{\beta,n}. \quad (25)$$

Further, by $e_1, \ldots, e_N$ denote the basis vectors in $\mathbb{R}^N$. Since $g_{\alpha,\beta}(P(u+e_l)) = g_{\alpha,\beta}(P(u))$, we see that the functions

$$\Psi_l(u) = \Phi_{\alpha,k+e_l}(u+e_l)/\Phi_{\alpha,k}(u), \quad l = 1, \ldots, N, \quad u \in \omega_{\alpha,k}, \quad (26)$$

are well-defined on $\mathbb{R}^N$. Then the numbers

$$m_{p,q} = \frac{1}{2\pi i} [\log \Psi_p(u+e_q) - \log \Psi_p(u) - \log \Psi_q(u+e_p) + \log \Psi_q(u)], \quad 1 \leq p, q \leq N, \quad (27)$$

form a skew-symmetric matrix $M$ with integral entries. This matrix does not depend on the branches of $\log \Psi_l(u)$, $1 \leq l \leq N$. Moreover, since any other section $\{\tilde{\Phi}_{\alpha,k}(u)\}$ vanishing nowhere on $\mathbb{R}^N$ has the form $\tilde{\Phi}_{\alpha,k}(u) = \Xi(u)\Phi_{\alpha,k}(u)$ with $\Xi(u) \neq 0$ on $\mathbb{R}^N$, we see that $M$ is defined only by the bundle $\mathbb{F}$. It is easy to prove that if matrices $M$ and $\tilde{M}$ correspond to the bundles $\mathbb{F}$ and $\tilde{\mathbb{F}}$ over $\mathbb{T}^N$ with transition functions $\{g_{\alpha,\beta}\}$ and $\{\tilde{g}_{\alpha,\beta}\}$, then the matrix $M - \tilde{M}$ corresponds to the bundle with the transition functions $\{g_{\alpha,\beta}/\tilde{g}_{\alpha,\beta}\}$. If the bundle $\mathbb{F}$ has a global section $\{\varphi_\alpha(\tau)\}$ vanishing nowhere, then the functions $\Phi_{\alpha,k}(u) = \varphi_\alpha(P(u)), u \in \omega_{\alpha,k}$ form a global section of $\tilde{\mathbb{F}}$ over $\mathbb{R}^N$. Since this section satisfies the conditions

$$\Phi_{\alpha,k+e_l}(u+e_l) = \Phi_{\alpha,k}(u), \quad u \in \omega_{\alpha,k} \quad (28)$$

with $\Gamma = \text{Lin}_Z\{\lambda_1, \ldots, \lambda_n\}$, and the equality

$$H^2(\mathbb{R}^m, \mathbb{Z}) = \cup_{\Gamma}^* H^2(\hat{\Gamma}, \mathbb{Z}),$$

where $\Gamma$ runs over all the subgroups of the type $\text{Lin}_Z\{\lambda_1, \ldots, \lambda_n\}$. It will be proved in the next section that for any element $c \in H^2(\hat{\Gamma}, \mathbb{Z})$ with $\Gamma = \text{Lin}_Z\{\lambda_1, \ldots, \lambda_n\}$ there exists a divisor $d' \in APM_{1,1}(T_\Omega)$ with the Chern class $c$. 

\[\Box\]
for all $\alpha, k \in \mathbb{Z}^N, l = 1, \ldots, N$, we see that $\Psi_l(u) \equiv 1$, $l = 1, \ldots, N$, and $M = 0$. In particular, if the bundles $\mathbb{F}$ and $\tilde{\mathbb{F}}$ over $\mathbb{T}^N$ have the same Chern class, then the same matrix $M$ corresponds to these bundles. Thus the mapping $c \mapsto M$ is a well-defined homomorphism of $H^2(\mathbb{T}^N, \mathbb{Z})$ to the additive group of all skew-symmetric matrices with integral entries.

Let us check that this mapping is injective.

Suppose that the matrix $M = 0$ corresponds to the bundle $\tilde{\mathbb{F}}$ over $\mathbb{R}^N$ and $\{\Phi_{\alpha,k}(u)\}$ is a global section of $\tilde{\mathbb{F}}$ vanishing nowhere. Put

$$H_1(u) = \begin{cases} -\sum_{n=1}^{[u^1]} \log \Psi_1(u - ne_1) - (u^1 - [u^1]) \log \Psi_1(0^1, u) & \text{for } u^1 \geq 1, \\ -u^1 \log \Psi_1(0^1, u) & \text{for } 0 \leq u^1 < 1, \\ \sum_{n=0}^{-[u^1]} \log \Psi_1(u - ne_1) - (u^1 - [u^1]) \log \Psi_1(0^1, u) & \text{for } u^1 < 0, \end{cases}$$

and for all $\alpha, k$

$$\tilde{\Phi}_{\alpha,k}(u) = \Phi_{\alpha,k}(u) \exp H_1(u),$$

where $u = (u^1, u^i), u^1 \in \mathbb{R}, u^i \in \mathbb{R}^{N-1},$ and $[u^1]$ is the integral part of $u^1$. It is easy to see that the function $H_1(u)$ is continuous and satisfies the equality $H_1(u + e_1) - H_1(u) = -\log \Psi_1(u)$. It follows from (26) that the section $\{\tilde{\Phi}_{\alpha,k}(u)\}$ satisfies (28) for $l = 1$.

Further, suppose that $\{\Phi_{\alpha,k}\}$ satisfies (28) for some $l' \neq 1$. By (26), we can take $\log \Psi_{l'}(u) \equiv 0$. Using the equality $M_{l,l'} = 0$, we have $\log \Psi_1(u + e_{l'}) - \log \Psi_1(u) = \log \Psi_{l'}(u + e_1) - \log \Psi_{l'}(u) = 0$ and $H_1(u + e_{l'}) - H_1(u) = 0$. Hence the section $\Phi_{\alpha,k} \exp H_1(u)$ satisfies (28) with $l = 1, l = l'$. Therefore arguing as above, we can “improve” the section in each coordinate sequentially and obtain a section $\{\tilde{\Phi}_{\alpha,k}\}$ satisfying (28) for all $l$. Now it follows that the functions $\varphi_{\alpha}(\zeta) = \Phi_{\alpha,k}(P^{-1}(\zeta))$ are well-defined on $U_{\alpha} \subset \mathbb{T}^N$ for all $\alpha$ and form a global section vanishing nowhere.

Below we will prove that each skew-symmetric matrix with integer entries corresponds to some Chern class of a purely periodic divisor, therefore the constructed mapping $c \mapsto M$ is an isomorphism. Thus we will identify a Chern class $c \in H^2(\mathbb{T}^N, \mathbb{Z})$ with the matrix $M$.

5 Periodic divisors and Completing Theorem

Let $D$ be a divisor in the domain $T_G = \mathbb{R}^N + iG$, where $G \subset \mathbb{R}^N$ is a convex domain. The divisor is called $N$-periodic if there exist $N$ vectors $u_l \in \mathbb{R}^N$, linearly independent over $\mathbb{R}$ and such that $D(w + u_l) = D(w), l = 1, \ldots, N$. We can assume without loss of generality that these vectors are $e_1, \ldots, e_N$. Then $D \subset 2\pi \mathbb{Z}^N$. Remark that we do not exclude the case of divisors depending on $k < N$ coordinates.

Let $F(w) \in H(T_G)$ be an arbitrary function with the divisor $D$. The mapping $\tilde{D}(\zeta) = (i/2\pi) \partial \bar{\partial} \log |F(w + \log \zeta)|$, where $\log \zeta = (\log \zeta^1, \ldots, \log \zeta^N)$, is well-defined; by Proposition 1.1, it continuously maps from $\mathbb{T}^N$ to $M(T_G)$.

Let $\Lambda$ be the matrix with rows $\lambda_1, \ldots, \lambda_N \in \mathbb{R}^m$ linearly independent over $\mathbb{Z}$, $\Omega$ be the set $\{y \in \mathbb{R}^m : \Lambda y \in G\}$. If $\Gamma = \text{Lin}_\mathbb{Z}\{\lambda_1, \ldots, \lambda_N\}$, then the map $j : \mathbb{R}^m \mapsto \hat{\Gamma}(= \mathbb{T}^N)$
is $P \circ \Lambda$, where $P$ is defined in (24). The mapping $\ddot{d}(\zeta) = (i/2\pi) \partial \bar{\partial} \log |F(\Lambda z + \frac{\log \zeta}{2\pi i})|$ continuously takes $\mathbb{T}^N$ to $M_{1,1}(T_\Omega)$ too, and

$$\ddot{d}(\zeta \cdot jt) = \tilde{d}(\zeta \cdot P(\Lambda t)) = (i/2\pi) \partial \bar{\partial} \log |F(\Lambda z + \Lambda t + \frac{\log \zeta}{2\pi i})| = S_t \ddot{d}(\zeta)$$

for all $t \in \mathbb{R}^m$. Using Proposition 2.2, we see that the divisor $d = \ddot{d}(1)$ of the function $F(\Lambda z)$ is almost periodic with spectrum in $\Gamma$.

Take a sufficiently fine covering $\{U_\alpha\}_{\alpha \in A}$ of $\mathbb{T}^N$. As in the previous section, define open sets $\omega_{\alpha,k} \subset \mathbb{R}^N$, $k \in \mathbb{Z}^N$. The functions $r_\alpha(\zeta, y) = F(u + i\Lambda y)$ for $u \in \omega_{\alpha,k}, P(u) = \zeta, k = k(\alpha)$, define the divisor $d$ on $U_\alpha \times \Omega$ for any mapping $k(\alpha)$ from $A$ to $\mathbb{Z}^N$. Then the functions $g_{\alpha,\beta}(\zeta, y) = r_\alpha(\zeta, y)/r_\beta(\zeta, y)$ are transition functions for the line bundle $\mathbb{F}$ over $\mathbb{T}^N \times \Omega$.

Fix $y_0 \in \Omega$. Then the functions $\Phi_{\alpha, k}(u) = r_\alpha(P(u), y_0)/F(u + i\Lambda y_0)$ form a global holomorphic section of some line bundle over $\mathbb{R}^N$, vanishing nowhere. Now the equalities (26) and (27) define the functions $\Psi_l(u), l = 1, \ldots, N$ and the matrix $M$. Since $P(u + e_l) = P(u)$ for all $l = 1, \ldots, N$, we have

$$\Psi_l(u) = F(u + i\Lambda y)/F(u + e_l + i\Lambda y), \quad l = 1, \ldots, N.$$  

The entries of the matrix $M$ are integer and depend continuously on $\Psi$, therefore we can change $\Psi_l(u)$ to the functions

$$\tilde{\Psi}_l(w) = F(w)/F(w + e_l)$$

for all fixed $w \in G$.

Since all projections of the set $\{w : F(w) = 0\}$ to the hyperplanes $\Im w^l = 0, l = 1, \ldots, N$, have zero Lebesgue measure, we see that the rectangle $\Pi_{p,q}(w)$ with vertices $w, w + e_p, w + e_p + e_q, w + e_q$ does not intersect $\text{supp} \mathcal{D}$ for a.a. $w \in T_G$. Therefore for a.a. $w'$,

$$- \log \tilde{\Psi}_p(w') = \log F(w' + e_p) - \log F(w') = \Delta_{[w', w'^n + e_p]} \log F(w),$$
$$- \log \tilde{\Psi}_p(w' + e_q) = \log F(w' + e_p + e_q) - \log F(w' + e_q) = \Delta_{[w' + e_q, w'^n + e_p + e_q]} \log F(w),$$

where $\Delta_{[a,b]} f$ means the increment of the function $f$ on the segment $[a,b]$. Similarly,

$$- \log \tilde{\Psi}_q(w') = \Delta_{[w', w'^q + e_q]} \log F(w),$$
$$- \log \tilde{\Psi}_q(w' + e_p) = \Delta_{[w' + e_p, w'^q + e_p + e_q]} \log F(w),$$

Hence

$$m_{p,q} = (1/2\pi) \Delta_{\Pi_{p,q}(w)} \text{Arg} F(w).$$

**Remark.** In [34] L.Ronkin proves that a necessary and sufficient condition for a periodic divisor to be the divisor of a holomorphic periodic function is $M = 0$; in [36] he shows that a section of a periodic divisor $D$ (we denote this section by $d$) is the divisor of a holomorphic almost periodic function iff $D$ is the divisor of a holomorphic
periodic function. Here we have shown that the Chern classes of $d$ and $D$ coincide, hence Ronkin’s result follows from Theorem 4.1.

Example. Let $\phi(\zeta)$ be an entire function in the plane $C$ with simple zeroes at all points with integer coordinates. Put for $w = (w^1, \ldots, w^N) \in C^N$ and fixed $p, q \in \mathbb{N}$, $1 \leq p < q \leq N$,

$$F_{p,q}(w) = \phi(w^p + iw^q).$$

(32)

By $D_{p,q}$ denote the divisor of $F$. Since $\tilde{\Psi}_l(i) \equiv 1$ for $l \neq s, t$, we see that all the entries of the corresponding matrix $M_{p,q}$ (except for $m_{p,q}$ and $m_{q,p} = -m_{p,q}$) vanish. Take $w_0 = \{-1/2, \ldots, -1/2\}$. Since $F_{p,q}(w) \neq 0$ for $w \in \Pi_{p,q}(w_0)$, we have

$$m_{p,q} = \frac{1}{2} \Delta_{\Pi_{p,q}(w_0)} \text{Arg} F_{p,q}(w) = \frac{1}{2} \Delta_L \text{Arg} \phi(w) = 1,$$

(33)

here $L$ is the rectangle in $C$ with vertices $-1/2 - i/2, 1/2 - i/2, 1/2 + i/2, -1/2 + i/2$. If we change $\lambda$ to $\mu$, then we get (36) for all rational $n/k$.

The divisor $d_{\lambda, \mu}$ of the function

$$F_{p,q}(\Lambda z) = \phi(\langle z, \lambda \rangle + i \langle z, \lambda \rangle), \quad z \in C^m,$$

has the same Chern class (as above, $\Lambda$ is the $N \times m$-matrix with rows $\lambda_1, \ldots, \lambda_N \in \mathbb{R}^m$ linearly independent over $\mathbb{Z}$). It follows easily that if $\lambda_1, \lambda_2$ are linearly independent over $\mathbb{R}$, then $d_{\lambda_1, \lambda_2}$ is $N$-periodic. Otherwise, the support of $d_{\lambda_1, \lambda_2}$ is a union of parallel complex hyperplanes.

Now we describe Chern classes of divisors as finite sums of special type.

For vectors $\lambda, \mu \in \mathbb{R}^m$, denote by $\lambda \wedge_{\mathbb{Z}} \mu$ the Chern class of the divisor $d_{\lambda, \mu}$ of the function $\phi(\langle z, \lambda \rangle + i \langle z, \mu \rangle)$. First let $\lambda, \mu$ be linearly independent over $\mathbb{Z}$. The permutation of $\lambda$ and $\mu$ corresponds to the rearrangement of $p$ and $q$ in (32), therefore we get $-1$ in (33). Thus

$$\lambda \wedge_{\mathbb{Z}} \lambda = -\lambda \wedge_{\mathbb{Z}} \mu.$$

(34)

Taking $\lambda \mapsto -\lambda$ corresponds to $w^p \mapsto -w^p$ in (32), therefore we get $-1$ in (33) again. Hence

$$(-\lambda) \wedge_{\mathbb{Z}} \mu = -\lambda \wedge_{\mathbb{Z}} \mu.$$

(35)

Changing $\lambda$ to $k\lambda$, $k \in \mathbb{N}$, corresponds to changing $w^p$ to $kw^p$, so we obtain $m_{p,q} = k$ in (33). Since other coefficients of $M$ are zero, we get

$$(k\lambda) \wedge_{\mathbb{Z}} \mu = k(\lambda \wedge_{\mathbb{Z}} \mu);$$

(36)

if we change $\lambda$ to $(n/k)\lambda$ here, then we get (36) for all rational $n/k$.

Further, let a matrix $\tilde{M}$ represent the Chern class of the divisor of the function $\tilde{F}(w) = \phi(w^p + iw^q)$. Arguing as above, we get $\tilde{m}_{p,q} = \tilde{m}_{r,q} = 1$, $\tilde{m}_{p,r} = 0$. Therefore the Chern class of this divisor equals the sum of the Chern classes of the divisors of the functions $\phi(w^p + iw^q)$ and $\phi(w^r + iw^q)$. Let $\lambda', \lambda'', \mu \in \mathbb{R}^m$ be linearly independent over $\mathbb{Z}$. If we take a matrix $\Lambda$ with the rows $\lambda_1 = \lambda', \lambda_2 = \lambda''$, $\lambda_3 = \mu$, we see that the Chern class of the divisor of the function $\phi(\langle z, \lambda' + \lambda'' \rangle + i \langle z, \mu \rangle)$ equals the sum of the Chern classes of divisors $d_{\lambda', \mu}$ and $d_{\lambda'', \mu}$. We have

$$(\lambda' + \lambda'') \wedge_{\mathbb{Z}} \mu = \lambda' \wedge_{\mathbb{Z}} \mu + \lambda'' \wedge_{\mathbb{Z}} \mu.$$
Consider the divisor of the function $\phi(kw^p + iw^p)$. Since this function depends only on one variable, we see that the zero matrix $M$ corresponds to the divisor $d_{k\lambda,\lambda}$. Therefore we have

$$\lambda \wedge_{\mathbb{Z}} \mu = 0,$$

if the vectors $\lambda$, $\mu$ are linearly dependent over $\mathbb{Z}$. Now, arguing as in (33), we obtain that the divisor of the function $\phi(w^p + w^q + iw^q)$ has the same matrix $M$ as the divisor of the function $\phi(w^p + iw^q)$, hence

$$(\lambda + \mu) \wedge_{\mathbb{Z}} \mu = \lambda \wedge_{\mathbb{Z}} \mu.$$  

Therefore if $\lambda'' = r\lambda' + s\mu$ for rational $r$ and $s$, then

$$(\lambda'' + \lambda') \wedge_{\mathbb{Z}} \mu = s((r + 1)/s\lambda' + \mu) \wedge_{\mathbb{Z}} \mu = r\lambda' \wedge_{\mathbb{Z}} \mu + \lambda' \wedge_{\mathbb{Z}} \mu$$

$$= s(r/s\lambda' + \mu) \wedge_{\mathbb{Z}} \mu + \lambda' \wedge_{\mathbb{Z}} \mu = \lambda'' \wedge_{\mathbb{Z}} \mu + \lambda' \wedge_{\mathbb{Z}} \mu.$$  

Thus (37) is true for all $\lambda'$, $\lambda''$, $\mu$.

Since $M = \sum_{p<q} m_{p,q} M_{p,q}$, we get that the Chern class of each divisor with spectrum in $\text{Lin}_{\mathbb{Z}}\{\lambda_1, \ldots, \lambda_N\}$ has the form

$$\sum_{1 \leq p < q \leq N} m_{p,q} \lambda_p \wedge_{\mathbb{Z}} \lambda_q.$$  

Actually, this statement realizes a well-known isomorphism between $H^2(\hat{\mathbb{R}}^m, \mathbb{Z})$ and $\mathbb{R}^m / \mathbb{Z} \mathbb{R}^m$ (see [18]).

Now we can prove that every almost periodic divisor is complemented to the divisor of some holomorphic almost periodic function.

**Theorem 5.1** The Chern class of each divisor $d \in APM_{1,1}(T_\Omega)$ can be represented as a finite sum $\sum \lambda_s \wedge_{\mathbb{Z}} \mu_s$; the divisor $d + \sum d_{\mu_s,\lambda_s}$ is the divisor of some $f \in APH(T_\Omega)$. If $m > 1$, then we can choose $\mu_s$, $\lambda_s$ in such a way that all the divisors $d_{\mu_s,\lambda_s}$ are periodic.

\[\square\] The first statement follows from (39), (36), and Theorem 4.5. Using (34) and Theorem 4.1, we get the second statement of our theorem. Finally, note that for $m > 1$ one can take a basis in $\Gamma' \supset \Gamma = \text{Lin}_{\mathbb{Z}}\{\lambda_1, \ldots, \lambda_N\}$ such that any two elements of the basis are linearly independent over $\mathbb{R}$. Namely, choose

$$\lambda_0 \not\in \bigcup_{1 \leq j,k \leq N} \{t \lambda_j + (1 - t) \lambda_k : t \in \mathbb{R}\} \cup \bigcup_{1 \leq j \leq N} \{t \lambda_j : t \in \mathbb{R}\}$$

and take vectors $\lambda_0$, $\lambda_1 - \lambda_0$, $\ldots$, $\lambda_N - \lambda_0$ as a basis of the group $\Gamma' = \text{Lin}_{\mathbb{Z}}\{\lambda_0, \lambda_1, \ldots, \lambda_N\}$.
6 Almost periodic mappings into projective space

Let \( F(z) \) be a holomorphic mappings of \( T_\Omega \) into the projective space \( \mathbb{C}P^k \). Using the homogeneous coordinates, it can be written in the form \([f^0(z) : f^1(z) : \ldots : f^k(z)]\) with holomorphic functions \( f^l \) on \( T_\Omega \) without common zeroes. These functions is well-defined up to common holomorphic factor vanishing nowhere on \( T_\Omega \). We will say that divisors \( d_i \) of the functions \( f^l, l = 0, \ldots, k \), are coordinate divisors for \( F \) (we suppose that the mapping \( F \) is not degenerate, i.e., each coordinate \( f^l \) is not identically zero). We assume that \( \mathbb{C}P^k \) is equipped with the Fubini-Study metric.

In the case \( k = 1 \) we can interpret \( F(z) \) as a meromorphic function on \( T_\Omega \) with disjoint sets of zeroes and poles and the spherical metric on the set of values of mapping \( F \). Form homogeneous coordinates, it can be written in the form \( f \) that the values of the mapping \( \bar{\omega} \) up to common holomorphic factor vanishing nowhere on \( T \) for each \( \omega \). It follows from the inequality

\[
|\kappa(t, y) - \kappa(t', y)| \leq \sup_{x \in \mathbb{R}^m} \rho(F(x + iy + t), F(x + iy + t'))
\]

that \( \kappa(x, y) \in AP(T_\Omega) \), hence \( \kappa \) extends to a continuous function on \( \hat{\mathbb{R}}^m \times \Omega \). This means that both \( \kappa \) and \( F \) are uniformly continuous in the topology \( \beta \). Therefore \( F(z) \) extends to a continuous mapping \( \bar{F} : \hat{\Omega} \to \mathbb{C}P^k \). Fix a point \( (\tau_0, y_0) \in \hat{\mathbb{R}}^m \times \Omega \). Then the values of the mapping \( \bar{F} \) for \( \tau, y \) in some neighborhood \( \omega \) of this point belong to one of the local charts, for example to \( W^0 = 1 \). Hence \( \bar{F}(\tau, y) = [1 : \varphi^1(\tau, y) : \ldots : \varphi^k(\tau, y)] \) for \( (\tau, y) \in \omega \) with continuous functions \( \varphi^1, \ldots, \varphi^k \). Moreover, since \( \bar{F}(jx, y) = F(z) = [1 : f^1(z) : \ldots : f^k(z)] \) for \( (jx, y) \in \omega \), the functions \( \varphi^l(\tau, y), l = 1, \ldots, k \), are holomorphic on \( \omega \).

Let \( \{\omega_\alpha\} \) be a sufficiently fine open covering of \( \hat{\mathbb{R}}^m \times \Omega \). It can be assumed that for each \( \omega_\alpha \) at least one of the homogeneous coordinates of \( \bar{F} = [\varphi^0_\alpha : \varphi^1_\alpha : \ldots : \varphi^k_\alpha] \)
equals identically 1 on $\omega_\alpha$. Taking into account Proposition 2.3, we obtain that the functions $g_{\alpha,\beta} = \varphi^0_\alpha/\varphi^0_\beta = \ldots = \varphi^k_\alpha/\varphi^k_\beta$ have no zeroes on $\omega_\alpha \cap \omega_\beta$. It is clear that these functions define a line bundle $F$ over $\mathbb{R}^m \times \Omega$, and functions $\{\varphi^l_{\alpha}(\tau, y)\}$ for all fixed $l = 0, \ldots, k$, define global holomorphic sections of $F$. Proposition 2.5 shows that the coordinate divisors of $F(z)$ are defined by just these sections.

Conversely, any holomorphic global sections $\varphi^l = \{\varphi^l_{\alpha}(\tau, y)\}$, $l = 0, \ldots, k$, of the line bundle $F$ over $\mathbb{R}^m \times \Omega$ without common zeroes are homogeneous coordinates for the continuous holomorphic mapping $F = [\varphi^0_{\alpha}, \ldots, \varphi^k_{\alpha}] : \omega_\alpha \mapsto \mathbb{C}P^k$ for each $\alpha$. Since

$$\varphi^0_\alpha/\varphi^0_\beta = \ldots = \varphi^k_\alpha/\varphi^k_\beta,$$

the mapping $F$ is well-defined on $\mathbb{R}^m \times \Omega$; the continuity of $F(\tau, y)$ implies almost periodicity of $F(z) = F(jx, y)$. ■

The following theorem is the main result of this section.

**Theorem 6.1** In order that divisors $d_0, \ldots, d_k$ be the coordinate divisors of some holomorphic almost periodic mapping $F : T_\Omega \mapsto \mathbb{C}P^k$, it is sufficient and necessary that the following conditions are fulfilled:

a) $d_0, \ldots, d_k$ are almost periodic divisors,

b) $d_0, \ldots, d_k$ have the same Chern class,

c) for each $\Omega' \subset \subset \Omega$ there exists $\delta > 0$ such that every ball $B(z_0, \delta)$, $z_0 \in \Omega'$, intersects at most $k$ supports of the divisors $d_0, \ldots, d_k$.

Let $F : T_\Omega \mapsto \mathbb{C}P^k$ be a holomorphic almost periodic mapping. It follows from Proposition 6.1 that there exist holomorphic global sections $\varphi^0, \ldots, \varphi^k$ of the line bundle $F$ corresponding to the coordinate divisors of $F$. The continuity of $F(\tau, y)$ implies almost periodicity of $F(z) = F(jx, y)$. ■

Conversely, let conditions a), b), c) be fulfilled. It follows from Theorem 2.1 that global holomorphic sections $\varphi^0, \ldots, \varphi^k$ of line bundles $F_0, \ldots, F_k$ over $\mathbb{R}^m \times \Omega$ correspond to the divisors $d_0, \ldots, d_k$, respectively. Note that if holomorphic sections $\{\varphi_\alpha\}, \{\tilde{\varphi}_\alpha\}$ of line bundles $F, \tilde{F}$ with transition functions $\{g_{\alpha,\beta}\}, \{\tilde{g}_{\alpha,\beta}\}$ correspond to divisors $d, \tilde{d}$ with a same Chern class, then the line bundle with the transition functions $\{g_{\alpha,\beta}/\tilde{g}_{\alpha,\beta}\}$ has a global holomorphic section $\{\varphi_{\alpha}\}$ vanishing nowhere, therefore the sections $\{\varphi_{\alpha}\}, \{\tilde{\varphi}_{\alpha}\}$ of the same line bundle $F$ are assigned to the divisors $d, \tilde{d}$. Therefore we may suppose that $F_0 = F_1 = \ldots = F_k = F$.

Assume that all the sections $\varphi^l_\alpha$ have a common zero $(\tau_0, y_0)$. Then in some neighborhood of this point the sections $\varphi^l_\alpha$ are defined by the functions $\varphi^l_\alpha$, $l = 0, \ldots, k$. Choose a sequence $x_n \in \mathbb{R}^m$ such that $jx_n \to \tau_0$ as $n \to \infty$. It is clear that the holomorphic in $x + iy$ functions $\varphi^l_\alpha(j(x_n + x), y)$ converge as $n \to \infty$, uniformly in some neighborhood of the point $(0, y_0)$, to the functions $\varphi^l_{\alpha}(\tau_0 + jx, y)$, $l = 0, \ldots, k$, respectively. Hence
Hurvitz’ Theorem implies that for arbitrary small $\delta > 0$ and sufficiently large $n$ all the functions $\varphi^l_\alpha(jx, y)$ have zeroes in the ball $B(x_n + iy_0, \delta)$. This contradicts condition c). Thus all the sections $\varphi^l$ have no common zeroes, and Proposition 6.1 yields that there exists an almost periodic mapping with the coordinate divisors $d_0, \ldots, d_k$.

Now we prove a theorem which gives a method for constructing holomorphic almost periodic mappings.

**Theorem 6.2** Let $F(z) = [f^0(z) : f^1(z) : \ldots : f^k(z)]$, $\tilde{F}(z) = [\tilde{f}^0(z) : \tilde{f}^1(z) : \ldots : \tilde{f}^k(z)]$ be holomorphic almost periodic mappings from $T_\Omega$ to $\mathbb{C}P^k$, a function $h \in H(T_\Omega)$, and let the functions $f^l(z)\tilde{f}^l(z)/h(z)$, $l = 0, \ldots, k$, be holomorphic on $T_\Omega$ and their divisors satisfy condition c) of Theorem 6.1. Then these functions are homogeneous coordinates for some holomorphic almost periodic mapping.

It follows from Proposition 6.1 that there exist line bundles $\mathbb{F}$ and $\tilde{\mathbb{F}}$ over $\mathbb{R}^m \times \Omega$ with transition functions $\{g_{\alpha\beta}\}$ and $\{\tilde{g}_{\alpha\beta}\}$, whose sections $\{\varphi^l_\alpha\}$, $\{\tilde{\varphi}^l_\alpha\}$, $l = 0, \ldots, k$, correspond to the coordinate divisors of the mappings $F$, $\tilde{F}$. Then $\{\varphi^l_\alpha\tilde{\varphi}^l_\alpha\}$, $l = 0, \ldots, k$, are global holomorphic sections of the line bundle with the transition functions $\{g_{\alpha\beta}\tilde{g}_{\alpha\beta}\}$.

Fix a point $(\tau_0, y_0) \in \mathbb{R}^m \times \Omega$ and its neighborhood $\omega$ such that the sections $\{\varphi^l_\alpha\tilde{\varphi}^l_\alpha\}$ are defined by holomorphic functions $q^l(\tau, y)$, $l = 0, \ldots, k$, on $\omega$. Choose a sequence $x_n \in \mathbb{R}^m$ such that $jx_n \to \tau_0$ as $n \to \infty$. There exists $\delta > 0$ such that for $n$ sufficiently large the image of every ball $B(x_n + iy_0, \delta)$ under the map $j$ lies in $\omega$ and at least one of the holomorphic functions $q^l(jx, y)/h(x + iy)$ has no zeroes in each ball. For the sake of being definite, suppose that for an infinite sequence of numbers $n$ the functions $q^0(jx, y)$ and $h(x + iy)$ have the same zeroes in $B(x_n + iy_0, \delta)$. Since $h(x + iy)$ divides all the functions $q^l(jx, y)$, so does the function $q^0(jx, y)$. Using Proposition 2.3, we get that in a sufficiently small neighborhood $\omega'$ of the point $(\tau_0, y_0)$,

$$q^l(\tau, y) = q^0(\tau, y)r^l(\tau, y), \quad l = 1, \ldots, k,$$

with some functions $r^l(\tau, y)$ holomorphic in $\omega'$.

Choose an open covering $\{\omega_\alpha\}$ of the set $\mathbb{R}^m \times \Omega$ such that for each $\alpha$ there exists a number $l(\alpha)$ and functions $q^l_\alpha$, $r^l_\alpha$, holomorphic in $\omega_\alpha$ such that $q^l_\alpha$ define the sections $\{\varphi^l_\alpha\tilde{\varphi}^l_\alpha\}$, $l = 0, \ldots, k$. We have

$$q^l_\alpha(\tau, y) = q^{l(\alpha)}_\alpha(\tau, y)r^l_\alpha(\tau, y), \quad l = 0, \ldots, k.$$

It is clear that the functions $q^{l(\beta)}_\beta/q^{l(\alpha)}_\alpha$ are holomorphic and have no zeroes on $\omega_\alpha \cap \omega_\beta$, therefore the functions $\{g_{\alpha\beta}\tilde{g}_{\alpha\beta}q^{l(\beta)}_\beta/q^{l(\alpha)}_\alpha\}$ are transition functions for some line bundle $\mathbb{F}'$. Then the functions $\{r^l_\alpha\}$ form $k + 1$ holomorphic sections of this bundle and $r^0_\alpha \equiv 1$ on $\omega_\alpha$. Using Proposition 6.1, we obtain that the line bundle $\mathbb{F}'$ generates a holomorphic almost periodic mapping $H$ with the homogeneous coordinates $f^l(z)\tilde{f}^l(z)/h(z)$, $l = 0, \ldots, k$.

**Corollary 6.1** (for $m = 1$, see [10]). The product of two meromorphic almost periodic functions is almost periodic if and only if the zeroes and poles of the product are uniformly separated in every $T_\Omega$ with $\Omega' \subset \subset \Omega$. 

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Note that there exists a holomorphic almost periodic mapping $F$ such that for every homogeneous holomorphic representation $F(z) = [f^0(z) : f^1(z) : \ldots : f^k(z)]$, none of the functions $f^l$ belongs to $APH(T_\Omega)$.

Indeed, let $\phi(z)$ be a holomorphic function on the plane with zeroes at all points with integer coordinates. Fix points $\zeta_1, \ldots, \zeta_k$ in the plane such that neither the points, nor their differences have integer coordinates. Then the divisors $d_0, \ldots, d_k$ of the functions $h_0(z^1, z^2) = \phi(z^1 - iz^2), h_1(z^1, z^2) = \phi(z^1 - iz^2 + \zeta_1), \ldots, h_k(z^1, z^2) = \phi(z^1 - iz^2 + \zeta_k)$ are periodic, have the same nonzero Chern class, and satisfy the conditions c) of Theorem 6.1. Therefore there exists a holomorphic almost periodic mapping with the coordinate divisors $d_0, \ldots, d_k$, but there exist no functions from $APH(T_\Omega)$ with these divisors.

Nevertheless, the following theorem is true.

**Theorem 6.3** Let $F(z)$ be a holomorphic almost periodic mapping of $T_\Omega$ into the projective space $\mathbb{CP}^k$. Then there exists an almost periodic divisor $d$ in $T_\Omega$ and $g^0(z), \ldots, g^k(z) \in APH(T_\Omega)$ such that their common zeros are contained in the support of $d$, and

$$F(z) = [g^0(z) : \ldots : g^k(z)]$$

on $T_\Omega \setminus \text{supp} d$; the representation (40) with holomorphic almost periodic functions without common zeroes exists if and only if the coordinate divisors of $F$ have zero Chern class.

□ If all the functions $g^j$ in (40) have no common zeroes, then their divisors are the coordinate divisors of $F$. It follows from Theorem 4.1 that all these divisors have zero Chern class.

Further, let $[f^0(z) : \ldots : f^k(z)]$ be a homogeneous representation of $F$ with functions $f^j$ holomorphic in $T_\Omega$ without common zeroes. Let $d_0$ be the divisor of $f^0(z)$. Using Theorem 5.1, take an almost periodic divisor $d$ such that $d_0 + d$ is the divisor of some $g^0 \in APH(T_\Omega)$; we take $d = 0$ in the case $c(d_0) = 0$. Since the mapping $\tilde{F}(z) = [1 : g^0(z) : \ldots : g^k(z)]$ is almost periodic, Theorem 6.2 with $h(z) = f^0(z)$ implies that the mapping $[1 : g^j(z) : \ldots : g^k(z)]$ with $g^j(z) = f^j(z)g^0(z)/f^0(z)$, $j = 1, \ldots, k$, is almost periodic, too. Hence the functions $g^j(z)$, $j = 0, 1, \ldots, k$, are almost periodic and have common zeroes only in $\text{supp} d$. It is clear that the representation (40) is valid on the set $T_\Omega \setminus \text{supp} d$.

7 Further remarks

Our method of determining of holomorphic function by almost periodic divisor does not allow to control the growth of the function (even in the case $T_\Omega = \mathbb{C}$). Perhaps, we need a more precise integral representation for the $\bar{\partial}$-problem.

Note also that it looks natural to replace the group of mappings $\{z \mapsto z + t\}_{t \in \mathbb{R}^m}$ in the definition of almost periodic functions, to any group of automorphisms; apparently, then we will need another compactification instead of Bohr’s one and another integral representation for the $\bar{\partial}$-problem.
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