Distribution Amplitudes of $K^*$ and $\phi$ at Physical Pion Mass from Lattice QCD
(Lattice Parton Collaboration (LPC))

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We present the first lattice QCD calculation of the distribution amplitudes of longitudinally and transversely polarized vector mesons $K^*$ and $\phi$ using large momentum effective theory. We use the clover fermion action on three ensembles with 2+1+1 flavors of highly improved staggered quarks fm lattice spacings, and choose three different hadron momenta $P_z = \{1.29, 1.72, 2.15\}$ GeV. The resulting lattice matrix elements are nonperturbatively renormalized in a hybrid scheme proposed recently. An extrapolation to the continuum and infinite momentum limit is carried out. We find that while the longitudinal distribution amplitudes tend to be close to the asymptotic form, the transverse ones deviate rather significantly from the asymptotic form. Our final results provide crucial ab initio theory inputs for analyzing pertinent exclusive processes.

Introduction. Searching for new physics beyond the standard model (SM) is a primary goal of particle physics nowadays. A unique possibility of doing so is to investigate flavor-changing neutral current processes which are highly suppressed in the SM. Some prominent examples of such processes include $B \to K^*\ell^+\ell^-$ and $B_s \to \phi\ell^+\ell^-$ decays. Recent experimental analyses by Belle and LHCb collaborations [1–5] have revealed notable tensions between the SM predictions of such processes and data, and attracted quite considerable theoretical interests (see Refs. [6–8] and many references therein). Various new physics interpretations have been proposed to resolve such tensions, but to firmly establish their existence requires an accurate and reliable theoretical understanding of the dynamics of weak decays.

In the low recoil region (high $q^2$), the $B \to K^*$ and $B_s \to \phi$ form factors can be directly calculated on the lattice (see for instance Refs. [9–11]), however these decays at large recoil are also of experimental interests, and for instance the $P_2^*$ anomaly has attracted many theoretical and experimental attentions [11, 12]. In the latter kinematics region, decay amplitudes are split into short-distance hard kernels and long-distance universal inputs. The universal inputs that enter include the light-cone distribution amplitudes (LCDAs) of the vector mesons $K^*, \phi$ which, to the leading-twist accuracy, specify the longitudinal momentum distribution amongst the valence quark and antiquark in the meson. While the hard scattering kernel is perturbatively calculable, the LCDAs can only be extracted from nonperturbative methods or from fits to relevant data. A reliable knowledge of LCDAs is essential in making predictions on physical observables, and in particular the transition form factors at large recoil can be typically affected by $\mathcal{O}(10\%)$ by the non-asymptotic terms of LCDAs in light-cone sum rules approach [13–14]. To date most of the available analyses have made use of estimates based on QCD sum rules [15] or Dyson-Schwinger equation [16], but a first-principle description of LCDAs for the vector $(K^*, \phi)$ meson is still missing.

Lattice QCD provides an ideal ab initio tool to access nonperturbative quantities in strong interaction. Though some lowest moments of the $\rho$ LCDA have been studied in Ref. [17], a direct calculation of the entire distribution has not been feasible until the proposal of large momentum effective theory (LaMET) [18, 19] recently. This is realized by simulating on the lattice appropriately chosen equal-time correlations, and then convert them to the lat-
ter through a perturbative matching. Since LaMET was proposed, a lot of progress has been achieved in calculating various parton distribution functions \cite{28, 29} and proposed, a lot of progress has been achieved in calculating various parton distribution functions \cite{20, 21} (and 

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\text{Ensemble} & $a$ (fm) & $L^3 	imes T$ & $c_{SW}$ & $m_{ud}$ & $m_s$ \\
\hline
a12m130 & 0.12 & 48x64 & 1.05088 & -0.0785 & -0.0191 \\
a09m130 & 0.09 & 64x96 & 1.04239 & -0.0580 & -0.0174 \\
a06m130 & 0.06 & 96x192 & 1.03493 & -0.0439 & -0.0191 \\
\hline
\end{tabular}
\caption{Information on the simulation setup. The light and strange quark mass (both valence and sea quark) of the clover action are tuned such that $m_s = 140$ MeV and $m_{ud} = 670$ MeV.}
\end{table}

According to LaMET, the above LCDAs can be obtained by first calculating the following bare equal-time correlations on the lattice

\begin{equation}
\langle 0 | \bar{\psi}_1(0) \gamma^a U(0, z^2) \psi_2(\zeta^2) | V \rangle = H_{V,L}(z) f_V^R \langle \zeta^2 \rangle \delta m(\mu) \zeta_5 / Z(z, \mu), \quad (3)
\end{equation}

where $H_{V,L}(z)$ is the renormalization factor computed from

\begin{equation}
Z(z, \mu) = \frac{1}{12} \text{Tr} \left[ \langle S(p) \rangle^{-1} \times \langle S(p) \rangle \gamma_5 \gamma_5 \right] x_{pz} \gamma_5, \quad \text{(5)}
\end{equation}

The $Z_{\text{hybrid}}$ denotes the endpoint renormalization constant which can be determined by imposing a continuity condition at $z = z_S$.

\begin{equation}
Z_{\text{hybrid}}(z, \mu) = e^{\delta m(\mu) \zeta_5} / Z(z, \mu). \quad \text{(6)}
\end{equation}

The mass counterterm $\delta m(\mu)$ can be extracted from the RI/MOM renormalization factor \cite{31}. The $z_{\mu}$ is chosen as 0.24 fm and 0.36 fm within perturbative region, and their difference is treated as a systematic error. By Fourier transforming $H_{V,L}^R$ to momentum space, we then obtain the quasi-DAs

\begin{equation}
\hat{f}_{V,L}(y, P_z) = \int d^4x e^{-i y \cdot x} e^{-i P_z x} \hat{A}(x), \quad \text{(7)}
\end{equation}
where the continuum limit has been taken. It can be factorized into the LCDAs through the factorization theorem [41]:

$$\tilde{\Phi}_{V,(L,T)}(y, P_z, \mu_R) = \int_0^1 dx C_{V,(L,T)}(x, y, P_z, \mu_R, \gamma) \Phi_{V,(L,T)}(x, \mu),$$

(8)

where the matching kernel $C_{V,(L,T)}$ was derived first in the transverse momentum cutoff scheme in Ref. [12] and then in the RI/MOM scheme in Ref. [33]. The $\mu$ and $\mu_R$ reflect the generic renormalization scale dependence of LCDAs and quasi-DAs. The matching formula and more details of the hybrid scheme can be found in the supplemental material [32].

**Numerical setup.** On the lattice, one directly calculates the two-point correlation function defined as:

$$C_{2}^{m}(z, \vec{P}, t) = \int d^3 y e^{-i\vec{P} \cdot \vec{y}} \times \langle 0 | \bar{\psi}_1(\vec{y}, t) \Gamma_1 U(\vec{y}, z, z \vec{z}) \psi_2(\vec{y} + z \vec{z}, t) \times \bar{\psi}_2(0, 0) \Gamma_2 \psi_1(0, 0) | 0 \rangle,$$

(9)

where the longitudinal polarization case $(m = L)$ has $\Gamma_1 = \gamma_t$ and $\Gamma_2 = \gamma_z$, and the transverse polarization case $(m = T)$ has $\Gamma_1 = \gamma_y$, and $\Gamma_2 = \gamma_x/\gamma_y$. Then the quasi-DAs can be extracted from the following parameterization:

$$\frac{C_{2}^{m}(z, \vec{P}, t)}{C_{2}^{m}(z = 0, \vec{P}, t)} = \frac{H_{V,m}^{b}(z)(1 + c_m(z)e^{-\Delta E t})}{(1 + c_m(0)e^{-\Delta E t})},$$

(10)

where $c_m(z)$ and $\Delta E$ are free parameters accounting for the excited state contaminations, and $H_{V,m}^{b}(z)$ is the bare matrix elements for quasi-DA. When $t$ is large enough, the excited state contaminations parameterized by $c_m(z)$ and $\Delta E$ are suppressed exponentially, and the ratio defined in Eq. (10) approaches the ground state matrix element $H_{V,m}^{b}(z)$. Based on the comparison between the joint two-state fit and constant fit shown in the supplemental material [32], we choose to use the constant fit in the range of $t \geq 0.54$ fm to provide a conservative error estimate in the following calculation.

The numerical simulation is based on three ensembles with $2+1+1$ flavors of HISQ [28] at physical pion mass with 0.06, 0.09 and 0.12 fm lattice spacings. The momentum smeared grid source [14] with the source positions $(x_0 + j_z L/2, y_0 + j_y L/2, z_0 + j_z L/2)$ are used in the calculation, where $(x_0, y_0, z_0)$ is a random position and $j_{x,y,z} = 0/1$. It allows us to obtain the even momenta in unit of $2\pi/L$ with $\sim 8$ times of the statistics. We also repeat the calculation at 8, 6, 4 time slices and fold the data in the normal and reversed time directions, which is equivalent to having $570 \times 8 \times 8 \times 2$, $730 \times 8 \times 6 \times 2$ and $970 \times 8 \times 4 \times 2$ measurements at three ensembles at $a = 0.06$, 0.09 and 0.12 fm, respectively. We have further reversed the $\vec{z}$ direction in Eq. (9) to double the statistics. Based on the numerical results, we confirmed that the dispersion relation can be satisfied for all the cases up to the $O(a^2 p^4)$ correction, and the continuum extrapolation in the coordinate space or momentum space provides consistent results [32].

**Results.** After renormalization in the hybrid scheme, we perform a phase rotation $e^{i2\pi/3}$ to the renormalized correlation, so that the imaginary part directly reflects the flavor asymmetry between the strange and up/down quarks. Taking the transversely-polarized $K^*$ as an example, we show in Fig. 1 the real (upper panel) and imaginary (lower panel) of the renormalized quasi-DA matrix elements $e^{i2\pi/3}H_{K^*,T}(z)$ with the momentum $P_z = 2\pi/L \times 10 = 2.15$ GeV. As shown in the upper panel, the matrix elements at different lattice spacings are consistent with each other, indicating that linear divergences arising from the gauge-link have been canceled up to the current numerical uncertainty. In the lower panel, we find a positive imaginary part at all the lattice spacings, which corresponds to a non-zero asymmetry with the peak at $x < 1/2$. This is consistent with expectations that lighter quarks carry less momentum of the parent meson.

As one can see from Fig. 1 the uncertainty of lattice data grows rapidly with the spatial separation of the non-local operator. Thus, to have a reasonable control of uncertainties in the final result we need to truncate the correlation at certain point. The missing long-range information can be supplemented by a physics-based extrapolation proposed in Ref. [31], which removes unphysical oscillations in a naive truncated Fourier transform with the price of altering the endpoint distribution (at $x \sim 0$ or 1), which cannot be reliably predicted by LaMET anyway due to increasingly important higher-twist contributions. Following Ref. [31], we adopt the following extrapolation form:

$$H_{V,(L,T)}(z, P_z) = \left[ \frac{c_1}{(-i\lambda)^a} + e^{i\lambda} \frac{c_2}{(i\lambda)^b} \right] e^{-\lambda/\lambda_0},$$

(11)

where the exponential term accounts for the finite correlation length for a hadron at finite momentum, and the two algebraic terms account for a power law behavior of the momentum distribution at $x$ close to 0 and 1, respectively. $\lambda = zP_z$, and the parameters $c_{1,2}, a, b, \lambda_0$ are determined by a fitting to the lattice data in the region where it exhibits an exponential decay behavior. To account for systematics from such an extrapolation, we have done two different extrapolations, one including the exponential term and the other not, and taken their difference as an estimate of systematics. This can be attributed as a source of the uncertainty from the inverse problem, and a more systematic strategy to handle the inverse problem of the Fourier transform is available in Ref. [45]. The detailed comparison of two extrapolations can be found in the supplemental material [32].
After renormalization and extrapolation, we can Fourier transform to momentum space and apply the corresponding matching. In Fig. 2, we show as an example the comparison of the quasi-DA and extracted LCDA for the transversely polarized $K^*$. The results correspond to the case with $P_z = 2.15$ GeV, and $a = 0.09$ fm. One notices that there is a non-vanishing tail for quasi-DA (yellow curve) in the unphysical region ($x > 1$ or $x < 0$), but it becomes much better for the LCDA (blue curve) after the perturbative matching is applied.

FIG. 1. The two-point correlation function for the transversely-polarized $K^*$ in coordinate space. We make a phase rotation by multiplying a factor $e^{ixP_z/2}$ with $P_z = 2.15$ GeV.

FIG. 2. Quasi-DA and LCDA extracted from it for the transversely-polarized $K^*$ using data at $a = 0.09$ fm, $P_z = 2.15$ GeV.

![Graph showing Re and Im of $e^{ixP_z/2}$](image)

![Graph showing results for $a\to0$, $P_z = 1.29$ GeV, $1.72$ GeV, and $2.15$ GeV](image)

We have performed a simple extrapolation to the continuum limit using the results at three different lattice spacings and the following formula

$$\psi(a) = \psi(a \to 0) + c_1 a + O(a^2),$$

with $O(a)$ correction being due to the mixed action effect from the clover valence fermion on HISQ sea. As an example, we show the extrapolated results for the transversely polarized $K^*$ in Fig. 3 for three different momenta, $P_z = \{1.29, 1.72, 2.15\}$ GeV. From this figure, one can see that the asymmetry slightly increases with $P_z$. Defining the asymmetry as $c_{asy} = \int_0^{x/2} dx\phi(x)/\int_{1/2}^{x} dx\phi(x)$, we find $c_{asy}$ is $1.090(15), 1.176(07), 1.227(08)$ for the three momenta. Since the strange quark is heavier than the up/down quark, a slight preference of $x < 1/2$ to $x > 1/2$ is expectable. It suggests that a large $P_z$ extrapolation is essential to suppress the power corrections and reproduce this correct preference behavior. Such a behavior is also observed in the study of Kaon LCDAs in Ref. [24].

After matching from Quasi-DA to LCDAs with $\mu = 2$ GeV, our final results for LCDAs of the $K^*$ and $\phi$ are given in Fig. 4 and 5, respectively, where the upper and lower panels correspond to the longitudinal and transverse polarization cases. In these figures, we have made...
FIG. 4. LCDAs for the longitudinally-polarized $K^*$ (upper panel) and transversely-polarized $K^*$ (lower panel). The results are extrapolated to the continuous limit ($a \to 0$) and the infinite momentum limit ($P_z \to \infty$). Regions with $x < 0.1, x > 0.9$ are shaded, as systematic errors in these regions are difficult to estimate.

We have chosen two renormalization scale $1.82\text{GeV}$ and $3.04\text{GeV}$ and treated their difference an estimate of systematic error from matching. It is worth emphasizing that the endpoint regions are difficult to access in LaMET. The endpoint can be roughly estimated from the largest attainable $\lambda$ (conjugate variable of $x$ in the Fourier transform) as $1/\lambda_{\text{max}}$. In the present calculation, we have $\lambda_{\text{max}} \approx 14$ (specifically $z_{P_z}^{P_z=1.29\text{GeV}} \approx 2.1f\text{m}$, $z_{P_z}^{P_z=1.72\text{GeV}} \approx 1.6f\text{m}$, $z_{P_z}^{P_z=2.15\text{GeV}} \approx 1.3f\text{m}$), thus we take a conservative estimate of the predictable region as $[0.1, 0.9]$. Beyond this region, we plot a shaded area with systematic errors difficult to estimate. As a comparison, we also show in Fig. 4 the asymptotic form $6x(1-x)$, the model results from earlier QCD sum rule calculations \cite{15} and the Dyson-Schwinger equations (DSE) results \cite{16} in Fig. 5. Our results indicate that while the longitudinal LCDAs tend to be close to the asymptotic form, the transverse LCDAs have relatively large deviations from the asymptotic form. These behaviors might have important implications on the study of semileptonic $B \to K^* \ell^+ \ell^-$ decay towards the search for new physics, and can be explored in the future.

Summary: We have presented the first lattice QCD calculation of LCDAs of longitudinally and transversely polarized vector mesons $K^*, \phi$ using LaMET. We did not consider the $\rho$ meson due to its large width which will introduce sizable systematic errors. The continuum and infinite momentum limits are taken based on calculations at physical light and strange quark mass with three lattice spacings and momenta. Our final results are then compared to the asymptotic form and QCD sum rule results. While the longitudinal LCDAs tend to be close to the asymptotic form, the transverse ones have relatively large deviations from the asymptotic form. Our final results provide crucial \textit{ab initio} theory inputs for analyzing pertinent exclusive processes.
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[1] J. T. Wei et al. [Belle], Phys. Rev. Lett. 103, 171801 (2009) doi:10.1103/PhysRevLett.103.171801 [arXiv:0904.0770 [hep-ex]].
[2] R. Aaij et al. [LHCb], JHEP 09, 179 (2015) doi:10.1007/JHEP09(2015)179 [arXiv:1506.08777 [hep-ex]].
[3] R. Aaij et al. [LHCb], JHEP 02, 104 (2016) doi:10.1007/JHEP02(2016)104 [arXiv:1512.04442 [hep-ex]].
[4] R. Aaij et al. [LHCb], JHEP 08, 055 (2017) doi:10.1007/JHEP08(2017)055 [arXiv:1705.05802 [hep-ex]].
[5] R. Aaij et al. [LHCb], Phys. Rev. Lett. 125, no.1, 011802 (2020) doi:10.1103/PhysRevLett.125.011802 [arXiv:2003.04831 [hep-ex]].
[6] B. Capdevila, A. Crivellin, S. Descotes-Genon, J. Matias and J. Virto, JHEP 01, 093 (2018) doi:10.1007/JHEP01(2018)093 [arXiv:1704.05340 [hep-ph]].
[7] D. Buttazzo, A. Greljo, G. Isidori and D. Marzocca, JHEP 11, 044 (2017) doi:10.1007/JHEP11(2017)044 [arXiv:1706.07808 [hep-ph]].
[8] A. Cerri, V. V. Gligorov, S. Malvezzi, J. Martin Camalich, J. Zupan, S. Akar, J. Allison, B. C. Allanach, W. Altmannshofer and L. Anderlini, et al. CERN Yellow Rep. Monogr. 7, 867-1158 (2019) doi:10.23731/CYRM-2019-007.867 [arXiv:1812.07638 [hep-ph]].
[9] R. R. Horgan, Z. Liu, S. Meinel and M. Wingate, Phys. Rev. D 89, no.9, 094501 (2014) doi:10.1103/PhysRevD.89.094501 [arXiv:1310.3722 [hep-lat]].
[10] R. R. Horgan, Z. Liu, S. Meinel and M. Wingate, Phys. Rev. Lett. 112, 212003 (2014) doi:10.1103/PhysRevLett.112.212003 [arXiv:1310.3887 [hep-ph]].
[11] S. Descotes-Genon, J. Matias, M. Ramon and J. Virto, JHEP 01, 048 (2013) doi:10.1007/JHEP01(2013)048 [arXiv:1207.2753 [hep-ph]].
[12] R. Aaij et al. [LHCb], JHEP 02, 104 (2016) doi:10.1007/JHEP02(2016)104 [arXiv:1512.04442 [hep-ex]].
[13] P. Ball and V. M. Braun, Phys. Rev. D 55, 5561-5576 (1997) doi:10.1103/PhysRevD.55.5561 [arXiv:hep-ph/9701238 [hep-ph]].
[14] P. Ball and R. Zwicky, Phys. Rev. D 71, 014029 (2005) doi:10.1103/PhysRevD.71.014029 [arXiv:hep-ph/0412079 [hep-ph]].
[15] P. Ball, V. M. Braun and A. Lenz, JHEP 08, 090 (2007) doi:10.1088/1126-6708/2007/08/090 [arXiv:0707.1201 [hep-ph]].
[16] F. Gao, L. Chang, Y. X. Liu, C. D. Roberts and S. M. Schmidt, Phys. Rev. D 90, no.1, 014014 (2014) doi:10.1103/PhysRevD.90.014014 [arXiv:1405.0289 [nucl-th]].
[17] V. M. Braun, P. C. Bruns, S. Collins, J. A. Gracey, M. Gruber, M. Göckeler, F. Hutzler, P. Pérez-Rubio, A. Schäfer and W. Söldner, et al. JHEP 04, 082 (2017) doi:10.1007/JHEP04(2017)082 [arXiv:1612.02955 [hep-lat]].
[18] X. Ji, Phys. Rev. Lett. 110, 262002 (2013) doi:10.1103/PhysRevLett.110.262002 [arXiv:1305.1539 [hep-ph]].
[19] X. Ji, Sci. China Phys. Mech. Astron. 57, 1407-1412 (2014) doi:10.1007/s11433-014-5492-3 [arXiv:1404.6680 [hep-ph]].
[20] X. Ji, Y. Liu, Y. S. Liu, J. H. Zhang and Y. Zhao, arXiv:2004.03543 [hep-ph].
[21] K. Cichy and M. Constantinou, Adv. High Energy Phys. 2019, 3036904 (2019) doi:10.1155/2019/3036904 [arXiv:1811.07248 [hep-lat]].
[22] J. H. Zhang, J. W. Chen, X. Ji, L. Jin and H. W. Lin, Phys. Rev. D 95, no.9, 094514 (2017) doi:10.1103/PhysRevD.95.094514 [arXiv:1702.00008 [hep-lat]].
[23] J. H. Zhang et al. [LP3], Nucl. Phys. B 939, 429-446 (2019) doi:10.1016/j.nuclphysb.2018.12.020 [arXiv:1712.10025 [hep-ph]].
[24] R. Zhang, C. Honkala, H. W. Lin and J. W. Chen, Phys. Rev. D 102, no.9, 094519 (2020) doi:10.1103/PhysRevD.102.094519 [arXiv:2005.13955 [hep-lat]].
[25] Y. Q. Ma and J. W. Qiu, Phys. Rev. D 98, no.7, 074021 (2018) doi:10.1103/PhysRevD.98.074021 [arXiv:1404.6860 [hep-ph]].
[26] Y. Q. Ma and J. W. Qiu, Phys. Rev. Lett. 120, no.2, 022003 (2018) doi:10.1103/PhysRevLett.120.022003 [arXiv:1709.03018 [hep-ph]].
[27] A. V. Radyushkin, Phys. Rev. D 96, no.3, 034025 (2017) doi:10.1103/PhysRevD.96.034025 [arXiv:1705.01488 [hep-ph]].
[28] E. Follana et al. [HPQCD and UKQCD], Phys. Rev. D 95, 034022 (2017) doi:10.1103/PhysRevD.95.034022 [arXiv:1608.08091 [hep-ph]].
See the supplementary material of this work for a collection of perturbative matching kernel, and more detailed results, which includes Ref. [33].

[34] A. Ali, V. M. Braun and H. Simma, Z. Phys. C 63, 437-454 (1994) doi:10.1007/BF01580324 [arXiv:hep-ph/9401277 [hep-ph]].

[35] T. Ishikawa, Y. Q. Ma, J. W. Qiu and S. Yoshida, arXiv:1609.0218 [hep-lat].

[36] J. W. Chen, X. Ji and J. H. Zhang, Nucl. Phys. B 915, 1-9 (2017) doi:10.1016/j.nuclphysb.2016.12.004 [arXiv:1609.08102 [hep-ph]].

[37] J. Green, K. Jansen and F. Steffens, Phys. Rev. Lett. 121, no.2, 022004 (2018) doi:10.1103/PhysRevLett.121.022004 [arXiv:1707.07152 [hep-lat]].

[38] I. W. Stewart and Y. Zhao, Phys. Rev. D 97, no.5, 054512 (2018) doi:10.1103/PhysRevD.97.054512 [arXiv:1709.04933 [hep-ph]].

[39] C. Alexandrou, K. Cichy, M. Constantinou, K. Hadjiyiannakou, K. Jansen, H. Panagopoulos and F. Steffens, Nucl. Phys. B 923, 394-415 (2017) doi:10.1016/j.nuclphysb.2017.08.012 [arXiv:1706.00265 [hep-lat]].

[40] V. M. Braun, A. Vladimirov and J. H. Zhang, Phys. Rev. D 99, no.1, 014013 (2019) doi:10.1103/PhysRevD.99.014013 [arXiv:1810.00048 [hep-ph]].

[41] X. Ji, A. Schäfer, X. Xiong and J. H. Zhang, Phys. Rev. D 92, 014039 (2015) doi:10.1103/PhysRevD.92.014039 [arXiv:1506.00248 [hep-ph]].

[42] J. Xu, Q. A. Zhang and S. Zhao, Phys. Rev. D 97, no.11, 114026 (2018) doi:10.1103/PhysRevD.97.114026 [arXiv:1804.01042 [hep-ph]].

[43] Y. S. Liu, W. Wang, J. Xu, Q. A. Zhang, S. Zhao and Y. Zhao, Phys. Rev. D 99, no.9, 094036 (2019) doi:10.1103/PhysRevD.99.094036 [arXiv:1810.10879 [hep-ph]].

[44] Y. B. Yang, A. Alexandru, T. Draper, M. Gong and K. F. Liu, Phys. Rev. D 93, no.3, 034503 (2016) doi:10.1103/PhysRevD.93.034503 [arXiv:1509.04616 [hep-lat]].

[45] J. Karpie, K. Orginos and S. Zafeiropoulos, JHEP 11, 178 (2018) doi:10.1007/JHEP11(2018)178 [arXiv:1807.10933 [hep-lat]].

[46] R. G. Edwards et al. [SciDAC, LHPC and UKQCD], Nucl. Phys. B Proc. Suppl. 140, 832 (2005) doi:10.1016/j.nuclphysbps.2004.11.254 [arXiv:hep-lat/0409003 [hep-lat]].

[47] M. A. Clark, R. Babich, K. Barros, R. C. Brower and C. Rebbi, Comput. Phys. Commun. 181, 1517-1528 (2010) doi:10.1016/j.cpc.2010.05.002 [arXiv:0911.3191 [hep-lat]].

[48] R. Babich, M. A. Clark, B. Joo, G. Shi, R. C. Brower and S. Gottlieb, doi:10.1145/2063384.2063478 [arXiv:1109.2935 [hep-lat]].

[49] M. A. Clark, B. Joo, A. Strelchenko, M. Cheng, A. Gambhir and R. Brower, arXiv:1612.07873 [hep-lat].

[50] Y. J. Bi, Y. Xiao, W. Y. Guo, M. Gong, P. Sun, S. Xu and Y. B. Yang, PoS LATTICE2019, 286 (2020) doi:10.22323/1.363.0286 [arXiv:2001.05706 [hep-lat]].