A possible mechanism of generation of extreme waves in shallow water

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Abstract

We consider a mechanism of generation of huge waves by multi-soliton resonant interactions. A non-stationary wave amplification phenomenon is found in some exact solutions of the Kadomtsev-Petviashvili (KP) equation. The mechanism proposed here explains the character of extreme waves and of those in Tsunami.

1 Introduction

The phenomenon of particularly high and steep waves on the sea surface is one of the most dangerous events. The large waves such as tsunami and freak (rogue) wave have a significant impact on the safety of people and infrastructure and are responsible for the erosion of coastlines and sea bottoms and the biological environment. For example, the
Indian ocean tsunami of 26 December 2004 caused an estimated 250,000 or more deaths and extensive damage due to run-up, landward inundation, and wave-structure interactions. The prediction of extreme waves is an important task for human being. Understanding the physics of large extreme waves may save lives.

Freak waves occur much more frequently than it might be expected from surface wave statistics whereas they are particularly steep [1]. Freak waves may happen in both deep water and shallow water. Although several physical mechanisms of generation of freak waves in deep water have been proposed, the mechanism in shallow water is few [2].

Recently, it has been established that a moving pressure disturbance can generate solitary waves also in open sea areas. This mechanism may occur in relatively shallow areas with heavy fast ferry traffic where soliton-like disturbances frequently occur. In a fatal accident which occurred in Harwich, a port on England’s east coast, in July 1999, one surviving victim reported that the wave looked like 'the white cliffs of Dover'. Research carried out in Europe shows that the soliton produced by a fast ferry was probably responsible for the disaster. This extreme wave is so-called 'solitary killer' wave [3, 4, 5].

Solitary waves in shallow water have been studied since the 19th century. The theories of asymptotic expansion and integrable system have brought the remarkable progress to this research field. With the assumption of weak nonlinearity the dynamics of solitary waves in shallow water is well described by the Korteweg-de Vries (KdV) equation. Moreover, for weakly two-dimensional cases, that is, for the cases that the scale of variation in the direction normal to the propagation direction is much longer than that in the propagation direction, the fundamental equations may be reduced to the Kadomtsev-Petviashvili (KP) equation. Both of the equations are integrable and it makes the precise analysis of the wave motion possible.
Peterson et al. studied the amplitude of two soliton solutions of the KP equation. It was pointed out that the interaction of two solitary waves may be one of mechanism of generation of extreme waves \([6, 7, 8]\). In the case of resonant Y-shape soliton which was found by Miles, the maximum amplitude of a solitary wave can reach four times the amplitude of an incoming solitary wave \([9, 10]\). The weak point of this theory is that the interaction pattern of 2-soliton is stationary and does not describe non-stationary phenomena. If a high wave is generated by soliton resonance once, the high wave can not disappear within finite time. There is a gap to describe the generation of extreme wave using 2-soliton solutions of the KP equation \([11]\).

Recently, Biondini and Kodama studied a class of exact solutions of the KP equation and found that the interaction patterns of solitons in the class have non-stationary web-like structures \([12]\). Such solutions are resonant-type soliton solutions which are the generalization of the Y-shape resonant soliton solution which was found by Miles. These general resonant solutions are simply expressed by a Wronskian. Furthermore, Kodama gave a general formulation of the classification of interaction patterns of the \(N\)-soliton solutions of the KP equation \([13]\) (See also \([14]\)).

In this paper, we propose a mechanism of generation of huge waves by multi-soliton resonant interactions. The key of analysis is the Wronskian solution of the KP equation.

2 Physical derivation of the Kadomtsev-Petviashvili equation

Let us consider an object region of the ocean as an inviscid liquid layer of uniform undisturbed depth \(h\) while the influence of the open air is assumed to be negligibly small. Assume
the plane \( z^* = 0 \) of the Cartesian coordinates \((x^*, y^*, z^*)\) coincides with the flat bottom of the region and \( z^* = h + \zeta^*(x^*, y^*, t^*) \) denotes the free surface. Here, \( t^* \) is time. Let us denote the velocity components along the axes \( x^*, y^*, z^* \) by \( u^*(x^*, y^*, z^*, t^*) \), \( v^*(x^*, y^*, z^*, t^*) \) and \( w^*(x^*, y^*, z^*, t^*) \), respectively. Assume the fluid motion is irrotational, then the fluid velocity components are written as \( u^* = \Phi^* x^* \), \( v^* = \Phi^* y^* \), and \( w^* = \Phi^* z^* \) in terms of the velocity potential \( \Phi^*(x^*, y^*, z^*, t^*) \). Here we consider the long waves of small but finite amplitude. Hence, we make the nondimensionalization as follows:

\[
(x^*, y^*) = \ell (x, y), \quad z^* = h \zeta, \quad t^* = \frac{\ell}{c_0} t, \quad \zeta^* = a \zeta, \quad \Phi^* = \frac{a \ell c_0}{h} \Phi,
\]

(1)

where \( \ell \) is the characteristic horizontal length, \( a \) the characteristic amplitude, \( g \) the gravitational acceleration, and \( c_0 \equiv \sqrt{gh} \) the linear long-wave phase speed. Then the governing equations and the boundary conditions are:

\[
\beta \Delta \Phi + \Phi_{zz} = 0, \quad (0 < z < 1 + \alpha \zeta), \quad (2)
\]

\[
\Phi_z = 0, \quad (z = 0), \quad (3)
\]

\[
\Phi_x + \frac{\alpha}{2} (\nabla \Phi)^2 + \frac{\alpha}{2 \beta} \Phi^2_x + \zeta = 0, \quad (z = 1 + \alpha \zeta), \quad (4)
\]

\[
\zeta + \alpha \nabla \Phi \cdot \nabla \zeta - \frac{1}{\beta} \Phi_z = 0, \quad (z = 1 + \alpha \zeta), \quad (5)
\]

where the subscript variables denote the partial differentiations with respect to the variables and \( \nabla \equiv (\partial / \partial x, \partial / \partial y) \), \( \Delta \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \), \( \alpha = a / h \), and \( \beta = (h / \ell)^2 \). Here we consider weakly nonlinear long waves where weak nonlinearity and weak dispersion balance out, thus we assume \( \alpha \ll 1, \beta \ll 1 \) and \( \beta = O(\alpha) \) (hereafter we assume \( \varepsilon = \alpha = \beta \), that is, \( \varepsilon = a / h \) and \( \ell = h / \sqrt{\varepsilon} \)). Then, from (2) and (3), the velocity potential \( \Phi(x, y, z, t) \) is represented as follows:

\[
\Phi(x, y, z, t) = \sum_{m=0}^{\infty} (-\varepsilon \Delta)^m \phi(x, y, t) \frac{z^{2m}}{(2m)!},
\]

(6)
where $\phi(x,y,t)$ is the value of the velocity potential at the bottom. Substituting (6) into the boundary conditions (4) and (5), we obtain the coupled equations for the functions $\phi$ and $\zeta$

\[
\phi_t + \zeta + \frac{1}{2} \varepsilon \nabla \phi \cdot \nabla \phi - \frac{1}{2} \varepsilon \Delta \phi = O(\varepsilon^2),
\]

(7)

\[
\zeta_t + \Delta \phi + \varepsilon \nabla \cdot (\zeta \nabla \phi) - \frac{1}{6} \varepsilon \Delta^2 \phi = O(\varepsilon^2).
\]

(8)

For unidirectional propagation (in the positive $x$ direction) of waves, if we introduce the new variables $\xi = x - t$, $\sigma = \varepsilon t$ and assume the expansions

\[
\zeta = \zeta_0(\xi, \sigma) + \varepsilon \zeta_1(\xi, \sigma) + \cdots, \quad \phi = \phi_0(\xi, \sigma) + \varepsilon \phi_1(\xi, \sigma) + \cdots,
\]

(9)

we obtain the KdV equation for $\zeta_0$:

\[
2 \frac{\partial \zeta_0}{\partial \sigma} + 3 \zeta_0 \frac{\partial \zeta_0}{\partial \xi} + \frac{1}{3} \frac{\partial^3 \zeta_0}{\partial \xi^3} = 0, \quad \frac{\partial \phi_0}{\partial \xi} = \zeta_0.
\]

(10)

As is well known, this equation describes the dynamics of one dimensional soliton (line soliton). Miles formulated the interaction of two solitons propagating in different directions $\mathbf{n}_1$ and $\mathbf{n}_2$ by means of a perturbation method [9]. Miles showed the followings: If the parameter $\kappa = \frac{1}{2} (1 - \mathbf{n}_1 \cdot \mathbf{n}_2) = \sin^2 \psi$ ($2\psi$ is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$) is not small, in the lowest order of approximation $\zeta$ can be represented by the superposition of two solitons propagating in the directions $\mathbf{n}_1$ and $\mathbf{n}_2$ which are governed by the respective KdV equations. But, if $\kappa = O(\varepsilon)$, the perturbation expansion breaks down. Miles called the former the weak interaction and the latter the strong interaction.

In the case of $\kappa = O(\varepsilon)$, the directions of propagation of two solitons are almost the same. If we choose the $x$ axis as the main propagation direction, the scale of variation of the wave field in the $y$ direction would be $O(\varepsilon^{-1/2})$. Thus, introducing the variables

\[
\xi = x - t, \quad \eta = \varepsilon^{1/2} y, \quad \sigma = \varepsilon t,
\]

(11)
and expanding $\zeta$ and $\phi$ as

$$\zeta = \zeta_0(\xi, \eta, \sigma) + \varepsilon \zeta_1(\xi, \eta, \sigma) + \cdots, \quad \phi = \phi_0(\xi, \eta, \sigma) + \varepsilon \phi_1(\xi, \eta, \sigma) + \cdots, \quad (12)$$

we obtain the KP equation [15, 16]

$$2 \frac{\partial \zeta_0}{\partial \sigma} + 3 \zeta_0 \frac{\partial \zeta_0}{\partial \xi} + \frac{1}{3} \frac{\partial^3 \zeta_0}{\partial \xi^3} + \frac{\partial^2 \phi_0}{\partial \eta^2} = 0, \quad \zeta_0 = \frac{\partial \phi_0}{\partial \xi} \quad (13)$$

or

$$\frac{\partial}{\partial \zeta} \left( 2 \frac{\partial \zeta_0}{\partial \sigma} + 3 \zeta_0 \frac{\partial \zeta_0}{\partial \xi} + \frac{1}{3} \frac{\partial^3 \zeta_0}{\partial \xi^3} \right) + \frac{\partial^2 \zeta_0}{\partial \eta^2} = 0. \quad (14)$$

In terms of the dimensional variables, this is written as

$$\frac{\partial}{\partial x^*} \left( \frac{\partial \zeta_0^*}{\partial t^*} + c_0 \frac{\partial \zeta_0^*}{\partial x^*} + \frac{3}{2} c_0 h \frac{\partial \zeta_0^*}{\partial x^*} + \frac{1}{6} c_0 h^2 \frac{\partial^3 \zeta_0^*}{\partial x^*} \right) + \frac{c_0}{2} \frac{\partial^2 \zeta_0^*}{\partial y^{2*}} = 0, \quad (15)$$

where $\zeta_0^* = a \zeta_0$.

Here, if we introduce the following dimensionless variables:

$$u = \frac{3 \zeta_0^*}{2h}, \quad X = \frac{1}{h} (x^* - c_0 t^*), \quad Y = \frac{y^*}{h}, \quad T = -\frac{2c_0}{3h} t^*, \quad (16)$$

then (15) becomes

$$(-4u_T + uu_{XXX} + 6uu_X)_X + 3u_{YY} = 0. \quad (17)$$

Further, introducing $\tau$-function through

$$u = \frac{3 \zeta_0^*}{2h} = 2(\log \tau)_{XX}, \quad (18)$$

we obtain the following bilinear form from (17):

$$(D_X(-4D_T + D_X^3) + 3D_T^2) \tau \cdot \tau = 0, \quad (19)$$

where $D_X$, $D_T$ and $D_T$ are the Hirota bilinear operators defined, for example, by

$$D_x^n D_y^m f \cdot g = \left( \partial_x - \partial_{x'} \right)^n \left( \partial_y - \partial_{y'} \right)^m f(x,x')g(y,y')|_{x'=x,y'=y}.$$
Using the Hirota bilinear method [17], the \( \tau \)-function of 1-soliton solution is given by
\[
\tau = 1 + \exp(2\Theta), \quad \Theta = K \cdot X + \Omega T + \Theta^0 = KX + LY + \Omega T + \Theta^0,
\] (20)
\[
\Omega = K \left( K^2 + \frac{3L^2}{4K^2} \right),
\] (21)
and then the dimensionless elevation \( \tilde{\zeta} = \zeta^*/h \) is given by
\[
\tilde{\zeta} = \frac{4}{3} (\log \tau)_{XX} = \frac{4K^2}{3} \text{sech}^2 \Theta.
\]
The \( \tau \)-function of 2-soliton solution is given by
\[
\tau = 1 + \exp(2\Theta_1) + \exp(2\Theta_2) + A \exp(2\Theta_1 + 2\Theta_2),
\] (22)
\[
\Theta_i = K_i \cdot X + \Omega_i T + \Theta_i^0 = K_iX + L_iY + \Omega_i T + \Theta_i^0,
\] (23)
\[
\Omega_i = K_i \left( K_i^2 + \frac{3L_i^2}{4K_i^2} \right), \quad (i = 1, 2),
\] (24)
\[
A = \frac{4(K_1 - K_2)^2 - (\tan \psi_1 - \tan \psi_2)^2}{4(K_1 + K_2)^2 - (\tan \psi_1 - \tan \psi_2)^2},
\] (25)
\[
\tan \psi_i = L_i/K_i, \quad (i = 1, 2).
\] (26)
which is the most well-known 2-soliton solution of the KP equation [18]. In the next section, we study more general soliton solutions which are expressed by the Wronskian form.

3 Wronskian solutions, resonant-soliton and extreme elevations

If the functions \( f_i(X,Y,T) \) satisfy the relations
\[
\frac{\partial f_i}{\partial Y} = \frac{\partial^2 f_i}{\partial X^2}, \quad \frac{\partial f_i}{\partial T} = \frac{\partial^2 f_i}{\partial X^3}, \quad (i = 1, \ldots, N),
\] (27)
then the Wronskians

$$\tau_N = \begin{vmatrix} f_1^{(0)} & \cdots & f_N^{(0)} \\ \vdots & \ddots & \vdots \\ f_1^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix}, \text{ with } f_i^{(n)} = \frac{\partial^n f_i}{\partial X^n}, \quad (i = 1, \cdots, N)$$

(28)

are solutions to the bilinear equation \([19]\). For example, we can obtain the ordinary \(N\)-soliton solutions by setting \(f_i\) as

$$f_i = e^{\theta_{i-1}} + e^{\theta_i}, \quad (i = 1, \cdots, N),$$

(29)

$$\theta_j = -k_j X + k_j^2 Y - k_j^3 T + \theta_j^0, \quad (j = 1, \cdots, 2N),$$

(30)

where \(k_j\) and \(\theta_j^0\) are constants.

Let us choose the following \(f_i\) as \(f_i\) satisfying (27)

$$f_1 = \sum_{i=1}^{M} e^{\theta_i} = f, \quad f_i = f^{(i-1)}, \quad 1 < i \leq N \leq M,$$

(31)

where \(\theta_i, (i = 1, 2, \cdots, M)\) are given in the form (30) and the ordering \(k_1 < k_2 < \cdots < k_M\) is assumed. Then \(\tau\)-function in (28) takes the form

$$\tau_N = \begin{vmatrix} f^{(0)} & \cdots & f^{(N-1)} \\ \vdots & \ddots & \vdots \\ f^{(N-1)} & \cdots & f^{(2N-2)} \end{vmatrix}.$$  

(32)

Using the Binet-Cauchy theorem, the \(\tau\)-function (32) with (31) can be expanded as a sum of exponential functions,

$$\tau_{M_+} = \sum_{1 \leq i_1 < \cdots < i_{M_+} \leq M} \Delta(i_1, \ldots, i_{M_+}) \exp \left( \sum_{j=1}^{M_+} \theta_{i_j} \right),$$

(33)

where \(M = M_+ + M_-, \quad N = M_+, \quad (1 \leq M_+ \leq M - 1)\). \(\Delta(i_1, \ldots, i_{M_+})\) is the square of Vandermonde’s determinant, \(\Delta(i_1, \ldots, i_{M_+}) = \prod_{1 \leq j < l \leq M_+} (k_{i_j} - k_{i_l})^2 \) \([12]\). This is called the fully resonant \((M_-, M_+)-\)soliton solution. This solution has \(M_-\) solitons in asymptotics for
Figure 1: Fully resonant soliton solutions

$y \to -\infty$, $M_+$ solitons in asymptotics for $y \to \infty$, in the intermediate region these solitons interact resonantly. The $\tau$-function (33) is a general form of fully resonant-type soliton solutions.

It is also possible to set $f_i$ as

$$f_i = \sum_{j=1}^{M} a_{ij} e^{\theta_j}, \quad \text{for} \quad i = 1, \ldots, N, \quad \text{and} \quad M > N,$$

(34)

using the phases $\theta_j$ given in the form (30) and a $N \times M$-matrix $A_{(N,M)} := (a_{ij})$. Then the $\tau$-function (28) with (34) is expanded as a sum of exponential functions,

$$\tau = \sum_{1 \leq i_1 < \cdots < i_N \leq M} \xi (i_1, \ldots, i_N) \prod_{1 \leq j < l \leq N} (k_{ij} - k_{il}) \exp \left( \sum_{j=1}^{N} \theta_{ij} \right),$$

(35)

where $\xi (i_1, \ldots, i_N)$ is the $N \times N$-minor whose $j$-th columns ($j = 1, \cdots, N$) are given by the $i_j$-th columns in the matrix $A_{(N,M)} = (a_{ij})$, respectively,

$$\xi (i_1, \ldots, i_N) := \begin{vmatrix} a_{1,i_1} & \cdots & a_{1,i_N} \\ \vdots & \ddots & \vdots \\ a_{N,i_1} & \cdots & a_{N,i_N} \end{vmatrix}.$$

This is a general form of solutions in which the line solitons interact resonantly and non-resonantly [13]. The $\tau$-function (33) of the fully resonant line soliton solutions is a special case of the general $\tau$-function (35). In this general form of line soliton solutions, 2-soliton solutions in the sense that the solutions have the same sets of two line solitons in both asymptotics for $y \to \pm \infty$ are classified into three types, O-type, P-type and T-type. The $A$-matrices $A_{(2,4)}$ corresponding these three types have the following row reduced echelon
Figure 2: The interaction pattern of the fully resonant $(2,3)$-soliton ($k_1 = -2, k_2 = -1, k_3 = -1/100, k_4 = 1/100, k_5 = 2$)

forms

\[
A^{(O)}_{(2,4)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad A^{(P)}_{(2,4)} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad A^{(T)}_{(2,4)} = \begin{pmatrix} 1 & 0 & - & - \\ 0 & 1 & + & + \end{pmatrix},
\]

respectively, where ‘+, −’ shows the signs of the entries (also nonzero). T-type soliton is resonant two soliton solution, and O-type soliton is corresponding to two soliton solution obtained by the Hirota direct method [17]. Note that all $N$-soliton solutions in the sense that the solutions have the same sets of $N$ line solitons in both asymptotics for $y \to \pm \infty$ are classified by using Young diagrams, i.e. we can construct the $A_{(N,2N)}$-matrices from the Young diagrams [13].

Biondini and Kodama [12] and Kodama [13] considered the interaction patterns but did not the amplitudes of line solitons. Here, we consider the amplitudes generated by soliton interactions. First, we study fully resonant soliton solutions given by the $\tau$-function [33]. Figure 1 shows three examples of resonant soliton solutions. Figure 1(a) shows the Y-shape resonant soliton solution (resonant $(2,1)$-soliton). The two incoming solitons interact resonantly and generate a high amplitude soliton. The amplitude of the high amplitude soliton generated by soliton resonance is four times as large as the amplitude of the incoming soliton. We see the amplitude of soliton is amplified by soliton resonance. Figures 1(b) and 1(c) show the resonant $(2,2)$-soliton (T-type 2 soliton). In general, T-type 2-soliton has a hole as seen in Fig.1(b). However, in Fig.1(c) we see no hole but H-shape soliton. This is a degenerate case of the T-type 2-soliton which occurs due to the closeness of two parameters $k_2$ and $k_3$. The amplitudes of the intermediate interaction region in Fig.1(b) are
Figure 3: The interaction pattern of the fully resonant $(3,3)$-soliton $(k_1 = -5/2, k_2 = -5/4, k_3 = -1/2, k_4 = 1/2, k_5 = 3/2, k_6 = 5/2)$

not so high, i.e. smaller than the double of the amplitude of the incoming soliton. On the other hand, in Fig.1(c) we see the amplitude of the interaction region of the H-shape soliton is very near to four times the amplitude of the incoming soliton, i.e. higher than the sum of the amplitudes of two incoming solitons. In Fig.1(b) the interaction region makes a hole, and then the amplitude spreads out, then the amplitude becomes higher. In contrast, Fig.1(c) has no hole and the amplitude concentrates on small region. Physically, the existence of the H-shape soliton is predicted by Miles by using symmetry argument of the Y-shape resonant soliton [10]. The H-shape soliton is not a degenerate solution of 2-soliton solution. The resonant 2-solitons having a hole have been found in numerical simulations of the (2+1)-dimensional soliton equations [20, 21]. It is natural that the H-shape solitons appear in physical situations such as ocean surface waves. The Y-shape soliton is a limit of the H-shape soliton.

Figure 2 shows the interaction pattern of fully resonant $(2,3)$-soliton. In contrast to ordinary nonresonant soliton solutions, the interaction pattern of fully resonant $(2,3)$-soliton shows complex nonstationary behavior.

Figure 3 shows the interaction pattern of fully resonant $(3,3)$-soliton. We see there exist 4 holes. In general, resonant $(M_-,M_+)$-soliton solutions given by the $\tau$-function generate web-like structure having $(M_- - 1)(M_+ - 1)$ holes [12].

The existence of web structure makes us expect the existence of nonstationary solutions having maximum amplitude for finite time. Thus, we have a following question: are there
Figure 4: Time evolution of the maximum amplitude of fully resonant \((3,3)\)-soliton. Exact solution.

Figure 5: The time evolution of the partial resonant soliton solution given by the \(\tau\)-function \((35)\) with the \(A\)-matrix \(A_{(2,5)}\) \((36)\), i.e., \(2 \times 2\)-Wronskian, \(f_1 = \sum_{k=1}^{3} e^{\theta_k}, f_2 = \sum_{k=4}^{5} e^{\theta_k} (k_1 = -2, k_2 = 0, k_3 = 2, k_4 = 2.01, k_5 = 4)\)

Figure 6: Plots of figure 5 in the plane \(y =\) constant at \(t = 1\): (a) \(y=5.5\), (b) \(y=4\), (c) \(y=-2\)

any exact solutions having characters of extreme waves? In Fig. 2 and Fig. 3, we see that the maximum amplitude of fully resonant multi-soliton solutions decrease at certain time by contraries. Figure 4 shows the time evolution of the maximum amplitude of Fig. 3. Fully resonant soliton solutions having web structure do not have characters of the extreme waves.

Next, we consider the \(\tau\)-function \((35)\) which is the general form of \(N\)-line soliton solutions of the KP equation. This solution includes rich variety of solutions and there are nonstationary solutions which are similar to extreme wave. Figure 5 shows an example. The solution of Fig. 5 was made by the \(2 \times 2\)-Wronskian solutions, the \(A\)-matrix is

\[
A_{(2,5)} = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}.
\] (36)

We see that the stem of Y-shape soliton and the other soliton interact near-resonantly. When 1-soliton cross with the stem of Y-shape soliton, the interaction region is amplified into high amplitude (See Fig. 5, Fig. 6, and Fig. 7). Thus, if there are several resonant solitons and each interaction parts of resonant solitons interact resonantly, huge waves are generated. Using
this principle, we can easily understand that there is a mechanism of generation of the extreme waves.

In Fig.8 we show the interaction of inelastic 2-soliton given by the $A_{(2,4)}$-matrix

$$A_{(2,4)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \end{pmatrix}.$$  \hspace{1cm} (37)

We see that the interaction pattern is a combination of two Y-shape resonant solitons at $t < 0$, i.e. there are two resonant stems. At $t = 0$, two resonant stems have a head-on collision, then a big amplitude wave is generated at $t > 0$ (see Fig.8 and 10).

In Fig.11 we show the contour plots of solitons corresponding to Fig.5 and Fig.8. Based on the asymptotic analysis which was discussed in [12], each solitons are labeled by the index $[i,j]$ which represents a soliton made by two dominant exponentials $e^{\theta_i}$ and $e^{\theta_j}$. Taking the limit $k_4 \to k_3$, $k_5 \to k_4'$ in the soliton solution of the case of $A_{(2,5)}$ in Fig.8 we can recover the interaction pattern of the soliton solution of the case of $A_{(2,4)}$ in Fig.8. Because a new soliton resonance is made by the limit, the arrangement of solitons is changed.

The mechanism of generation of extreme waves is as follows: (i) Several solitons are generated by some reasons, e.g. fast ferry traffic, tsunami, current and topography. (ii) Several generated solitons interact resonantly. (iii) When the interaction region of a resonant soliton interacts with one of the other resonant soliton, the amplitude will be 16 times of the amplitude of a soliton. (iv) After the interaction of the stem of resonant solitons, the amplitude decreases.

This shows that the soliton resonance can be a mechanism of generation of extreme waves. The resonant interaction of several solitary waves may cause extreme waves higher than 4 times of amplitude of a solitary wave. This mechanism may happen in ocean; for instance, tsunami, freak wave and solitary killer wave may be amplified by this mechanism.
Figure 7: The change of maximum amplitude of figure 5.

Figure 8: The time evolution of the inelastic 2-soliton solution given by the $\tau$-function (35) with the A-matrix $A_{(2,4)}$ (37) ($k_1 = -2, k_2 = 0, k_3 = 2, k_4 = 4$)

Figure 9: Plots of figure 8 in the plane $y =$ constant at $t = 1$: (a) $y=5.5$, (b) $y=3.5$, (c) $y=-0.5$

4 Conclusion

The KP equation has rich mathematical structure and many interesting solutions. Finding physically interesting solutions of the KP equation is an important problem. In this paper, we have searched solutions having characters of extreme waves and found such solutions which are related resonant soliton solutions expressed by the Wronskian form. We have found there exists the solutions similar to properties of the extreme waves on the sea surface. The mechanism of generation of extreme waves is, (i) several solitons are generated by some reasons, (ii) several generated solitons interact resonantly, (iii) when the interaction region of a resonant soliton interacts with one of the other resonant soliton, the amplitude will be 16 times of the amplitude of a soliton, (iv) after the interaction of the stem of resonant solitons, the amplitude decreases.

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Figure 10: The change of maximum amplitude of figure 8.

Figure 11: The contour lines of interaction patterns in Fig.5 and Fig.8. Each solitons are labeled by the index $[i, j]$ which represents a soliton made by two dominant exponentials $e^{\theta_i}$ and $e^{\theta_j}$. The upper graphs are corresponding to Fig.5, the lower graphs are corresponding to Fig.8.

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