POLYNOMIAL TIME RELATIVELY COMPUTABLE TRIANGULAR ARRAYS IN A MULTINOMIAL SETTING

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Abstract. We extend the methods and results of [3] to the setting of multinomial distributions satisfying certain properties. These include all the multinomial distributions arising from the direct proof of the Central Limit Theorem given in [2], which, by results of that paper, constitutes essentially full generality for the situations in which the Central Limit Theorem holds.

1. Introduction

The two papers, [2] and [3], provide the context for our work here. The first of these papers provides a direct proof of the Central Limit Theorem that proceeds by introducing a sequence of approximating multinomial distributions. The second deals more narrowly with the sequence \( \{S_n\} \) based on Rademacher random variables, so that the basic setting is that of the binomial distributions with \( p = \frac{1}{2} \). It focuses on the notion of strong, trim triangular array representations of the sequence of quantiles of the sequence \( \{S_n\} \) in this restricted setting, classifying them by sequences of admissible permutations and analyzing the complexity of the simplest of these classifying sequences.

We extend the methods and many (but not all) of the results of the latter to the setting of multinomial distributions provided by the former. These multinomial distributions essentially capture (via sufficiently good approximation) the full generality of situations where the Central Limit Theorem holds: this is one of the main results of [2]. Thus, the results of this paper represent significant progress on the program laid out in the first paragraph of [3]: obtaining (and analyzing the complexity of) strong trim triangular representations for the sequences of quantiles of weakly convergent sequences.

As set forth in [3], a triangular array representation of a sequence \( \{X_n\} \) of random variables is strong if, for each \( n \), \( X_n \) is equal pointwise, as a function, not just in distribution, to the sum of the random variables in the \( n \)th row of the triangular array. Thus (as was already noted in [3]), there is very clear analogy: strong triangular array representations are to triangular array representations as almost sure convergence is to weak convergence. Therefore, it is very natural to seek strong triangular array representations for sequences that converge almost surely, as do the sequences of quantiles of weakly convergent sequences. It is reasonable to expect that a more detailed understanding of the sequence of quantiles can be obtained via analysis in terms of a (particularly nice) strong triangular array representation thereof.

The issue of “trimness” will be discussed in more detail below, in the second paragraph of (1.1); for now, it will suffice to say that it addresses, for each \( n \), the issue of how many bits in the binary expansion of \( x \in (0, 1) \) are required to determine the values of the random variables in the \( n \)th row of the triangular array. In the setting of [3], it was shown (Proposition 2 of that paper) that it is impossible that each of the random variables in the \( n \)th row depends only on a single bit (even if the bit in question is allowed to vary with the random variable involved); as

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Garmirian and Stanley dedicate this paper to the memory of our dear departed mentor and friend, Vladimir Dobrić.
noted there, this can also be seen via other routes. The analogue of Proposition 2 of [3] holds in the setting of this paper, but we do not prove it here. In the setting of [3], the trimness notion was that each of the random variables in the $n^{th}$ row was to depend only on the first $n$ bits; this was presented there as the natural “next best hope” as to how much information about $x$ was needed. The representations we obtain in this paper will be “trim” in a sense that is the clear analogue of the trimness notion of [3]: the random variables in the $n^{th}$ row will depend only on the first $n(M + 1)$ bits, where $M$ is a parameter associated with the multinomial distribution in question. It was shown in [3] (Proposition 1) that “trimness does not come for free”: there are strong triangular array representations that are not trim; while this result goes over to the setting of this paper, via similar arguments, we do not prove it here.

As was noted in (1.3.1) of [3], there is another view of trimness, suggested by A. Nerode (private communication); trimness is naturally associated with a Lipschitz continuity condition with $\delta = \epsilon$ for a transformation from $(0,1)$ to the the product of the $\{-1,1\}^n$ (with the product of the discrete topologies on the $\{-1,1\}^n$). This view of trimness carries over to the setting of this paper, by replacing each $\{-1,1\}^n$ by $\{-1,1\}^{n(M + 1)}$.

The content of Corollary 1 of [3] and the analogous result, Corollary 1 of this paper, is that strong trim triangular array representations correspond exactly to sequences of admissible permutations. Any admissible permutation can be thought of as providing a “rearrangement” of $(0,1)$ (the domain of the random variables) that transforms the disorderly $S_n$ into the orderly step function, $S_n^*$. In the setting of [3], the rearrangement is literally a permutation of the dyadic intervals (of the depth appropriate for the permutation). The situation is somewhat more complicated in this paper and is addressed in (2.1.1) and especially (2.1.2) where the sets $E_{n,\ell}$ are introduced. On each of these sets, $S_n$ is constant and each of these sets is the union of dyadic intervals of a fixed depth still greater than $n(M + 1)$. By permuting the second index ($\ell$), the rearrangement corresponding to a level $n$ admissible permutation maps the sets $E_{n,\ell}$ to depth $n(M + 1)$ dyadic intervals, $D_{n,\ell}$.

Further, the translation into the language of admissible permutations makes it possible to pose and answer questions about the complexity of strong trim triangular array representations in terms of the complexity of the corresponding sequence of admissible permutations. These measure the complexity of the rearrangements needed to transform the sequence $\{S_n\}$ into the almost surely convergent sequence of its quantiles.

We will say more, below, about the connections and differences between this paper, and the papers [2] and [3], but both of these end up working closely with a sequence, $(R_n|n \in \mathbb{Z}^+)$, of i.i.d. random variables defined on $(0,1)$; both consider $S_n := \sum_{i=1}^n R_n$. The main concern of [3] was, for a very specific choice of $R_n$, to provide strong trim triangular array representations, $(R^*_n,i|n \in \mathbb{Z}^+ \text{ and } 1 \leq i \leq n)$ of the sequence, $(S^*_n|n \in \mathbb{Z}^+)$ of quantiles of the sequence $(S_n|n \in \mathbb{Z}^+)$, and to analyze the complexity of the simplest of these representations. We carry out this program here, starting from the more complicated and general sequences $(R_n|n \in \mathbb{Z}^+)$ whose partial sums, $S_n$, are the multinomial distributions of [2].

Echoing the above discussion, what made the triangular array representations of [3] strong is that for each $n$, $\sum_{i=1}^n R^*_n,i$ is not only equal in distribution to $S^*_n$, but it is equal to it pointwise, as a function on $(0,1)$. The representations $(R^*_n,i|n \in \mathbb{Z}^+ \text{ and } 1 \leq i \leq n)$ we obtain in this paper will be strong in exactly the same way. The issue of “trimness” was also discussed above, and, as noted there, will be discussed in more detail below, in the second paragraph of (1.1).

The first main result of [3] was (for each $n$) to classify by admissible permutations of $\{0, \ldots, 2^n - 1\}$ all of the sequences $(R^*_n,i|1 \leq i \leq n)$ that sum to $S^*_n$ and satisfy the “trimness” condition along with some additional properties. The analogue of this result is Theorem 1 of this paper, proved in (2.2). Now the classifying permutations are permutations of $\{0, \ldots, 2^{n(M + 1)} - 1\}$, where $M$ is a parameter associated with one of the multinomial distributions arising from the setting of [2], and admissibility is defined in an analogous fashion appropriate to the multinomial distribution
under consideration. An immediate consequence is that sequences, \((\pi_n|n \in \mathbb{Z}^+)\), of admissible permutations classify strong trim triangular array representations \((R^*_n|n \in \mathbb{Z}^+ \text{ and } 1 \leq i \leq n)\) of \((S^*_n|n \in \mathbb{Z}^+)\); this is Corollary 1, also (2.2).

As was the case in [3], analysis of the admissibility condition leads readily to the conclusion that there are “many” admissible permutations of \(\{0, \ldots, 2^n(M+1) - 1\}\). This is item (3) of Lemma 1, proved in (3.1). Combining Theorem 1, Corollary 1 and Lemma 1 immediately gives Corollary 2, also in (3.1): the existence of (continuum many) strong trim triangular array representations of \((S^*_n|n \in \mathbb{Z}^+)\). The second main result of [3] was the polynomial time computability of the sequence \((F_n|n \in \mathbb{Z}^+)\), of admissible permutations in which each \(F_n\) is, in a suitable sense, the simplest possible permutation of \(\{0, \ldots, 2^n-1\}\); by extension, the corresponding strong trim triangular array representation is also polynomial time computable. Here too, we obtain the analogous results: these are Theorems 2, 3 in (3.2). It would be too much to hope for polynomial time space rather than in \(\mathcal{O}\). Thus, our assumption is that \(O\) were in Cantor space rather than in \((0,1)\), this would be completely accurate; we will make it precise in the next paragraph. First, however, we note that while it is natural to think that the \(J_i\) should form a partition of \(\mathbb{Z}^+\), in our implementation, this will not be the case, and this will offer definite advantages.

For each positive integer, \(i\), let \((j_{i,k}|k \in \mathbb{Z}^+)\) be the increasing enumeration of \(J_i\) and let:

\[
\mathcal{J}_i := \{j_{i,1}, \ldots, j_{i,M+1}\} \text{ and } \mathcal{J}^n := \mathcal{J}_1 \cup \ldots \cup \mathcal{J}_n.
\]
We then take $P_i(x)$ to be that member of $(0,1)$, satisfying:

\[
\text{for all } k, \, \varepsilon_k(P_i(x)) = \varepsilon_{j_{i,k}}(x).
\]

In \cite{2}, $J_i$ is chosen to be “the $i^{th}$ column of a triangular array of positive integers” (not to be confused with the triangular arrays of random variables which are the main objects of study in this paper) in which the “rows” do partition $\mathbb{Z}^+$, where each row is a finite set of consecutive integers, and where the sequence of row lengths, $(\ell g_{\rho})_{\rho \in \mathbb{Z}^+}$, is monotone non-decreasing and unbounded. Note that with the stipulation that the rows form a partition into finite sets of consecutive integers, the triangular array of positive integers is completely specified by specifying the sequence of row lengths $(\ell g_{\rho})_{\rho \in \mathbb{Z}^+}$.

Dobrić and Garmirian adopt a specific choice of $(\ell g_{\rho})_{\rho \in \mathbb{Z}^+}$ but note that, for the results of \cite{2}, any reasonable choice suffices. For the purposes of this paper, and, in particular, for the proof of Lemma 2, we make an apparently somewhat odd specific choice of $(\ell g_{\rho})_{\rho \in \mathbb{Z}^+}$. For positive integers, $\rho$, we let:

\[
\ell g_{\rho} \text{ be the unique positive integer } b \text{ such that } (b - 1)(M + 1) < \rho \leq b(M + 1).
\]

Thus, the rows come in “blocks” of size $(M + 1)$; for rows in the $b^{th}$ block, the row length is $b$. We also decompose a row index, $\rho$, as:

\[
(\ell g(\rho) - 1)(M + 1) + r(\rho), \text{ with } 1 \leq r \leq M + 1.
\]

Thus, $\ell g(\rho)$ is the block index of $\rho$ while $r(\rho)$ is the index of $\rho$ within its block.

Via the $R_i$, for all positive integers, $n$, $O$ determines a multinomial distribution, $\text{MN}_n^O$ with domain $(0,1)$. The values, $\text{MN}_n^O(x)$, of the multinomial distributions are the inner products, $k_1(x)a_1 + \ldots + k_m(x)a_m$, where the $k_s(x)$ are the frequencies with which $o_s$ occurs in $(R_1(x),\ldots,R_n(x))$. Thus, the $k_s(x)$ are non-negative integers that sum to $n$ and so there are exactly $\binom{n+m-1}{m-1}$ vectors $(k_1(x),\ldots,k_m(x))$ and at most this many values of $\text{MN}_n^O$.

With all of this in place, we now "zoom in a bit more closely on" our main objects of study. For $x \in (0,1)$, and positive integers, $i, n$, we take:

\[
(1) \quad R_i(x) := O(P_i(x)) \quad \text{and} \quad S_n(x) := \sum_{i=1}^{n} O(P_i(x)) = \sum_{i=1}^{n} R_i(x).
\]

As will be discussed in the next subsection, it follows from the results of \cite{2} that $S_n/\sqrt{n}$ converges (weakly) to the standard normal with domain $(0,1)$. Note that in view of our assumption on $O$, we have that:

\[
(2) \quad \text{each } R_i \text{ depends only on the } j \in J_i \text{ and so each } S_n \text{ depends only on the } j \in J''.
\]

As in \cite{3}, we are also interested in the quantiles, $S_n^*$, of $S_n$. The significance of the quantiles comes, in large measure, from the work of Skorokhod. He showed, \cite{3}, that if $\{X_n\}$ is a sequence of random variables (on any probability space) converging weakly to $X$, then the sequence of quantiles, $X_n^*$ (which are defined on $(0,1)$, regardless of the domain of the original $X_n$), converges almost surely to $X^*$, the quantile of $X$ (also defined on $(0,1)$).

Letting $C$ be the cumulative distribution of $X_n$:

\[
X_n^*(x) := \inf \{ t \in \mathbb{R} | C(t) \geq x \}.
\]

It is well-known that $X_n$ and $X_n^*$ are equal in distribution. It then follows from the results of \cite{2}, that:

The sequence $\{S_n^*/\sqrt{n}\}$ converges almost surely to the standard normal with domain $(0,1)$.

We will focus on representations:

\[
S_n^* = \sum_{i=1}^{n} R_{n,i}^* \text{ where each } R_{n,i}^* \text{ is } n - \text{trim}
\]

(and satisfies a number of other properties laid out in the statement of Theorem 1). The admissibility condition for permutations of $\{0, \ldots, 2^n(M+1) - 1\}$ will be given in (2.1.4), below.
For $n > 0$ and non-negative integers $\ell < 2^{n(M+1)}$, we let $D_{n,\ell}$ be the $\ell$th depth $n(M + 1)$ dyadic interval, i.e.:

$$D_{n,\ell} = \begin{cases} 
(0, 2^{-n(M+1)}) & \text{if } \ell = 0, \\
(2^{-n(M+1)}, (\ell + 1)2^{-n(M+1)}) & \text{otherwise}.
\end{cases}$$

We argue, in (2.1.1.) below, that each $S_n^*$ is constant on each $D_{n,\ell}$. As in [3], this will allow us to define “integer versions”, $IS_n^*$, of the $S_n^*$:

$$IS_n^*(\ell) = \text{the constant value of } S_n^* \text{ on } D_{n,\ell}, \text{ for } 0 \leq \ell < 2^{n(M+1)}.$$  

As in [3], this serves as a paradigm case for introducing integer versions of notions (usually functions, but sometimes relations) naturally defined on $(0,1)$, when the notion is invariant on a system of subsets of $(0,1)$, indexed by integers. The indexing will typically also depend on $n$ as a parameter, and the subsets in question will typically be either the $D_{n,\ell}$ or the $E_{n,\ell}$ (the analogues, for $S_n$, of the $D_{n,\ell}$ which we introduce in (2.1.2), below).

Our approach to complexity issues will follow that of [3], especially as laid out in the final two paragraphs of subsection (1.2) of that paper. We also include the references [1], [4] from that paper for general background on computational complexity.

### 1.2. Connections with [2] and [3]

In this subsection we elaborate further on the connections between this paper and the work of [2] (in (1.2.1)) and that of [3] (in (1.2.2)).

#### 1.2.1. The Context Provided by the Results of [2]

In [2], Dobrić and Garmirian give an explicit proof of the CLT directly from the definition of weak convergence, which states that (with $S$ a Polish space equipped with the $\sigma$ – algebra $\mathcal{B}(S)$ of Borel subsets of $S$) a sequence of measures $\mu_n$ (on $S$) converges to $\mu$ weakly provided that for each bounded, continuous function $f : S \to \mathbb{R}$,

$$\lim_{n \to \infty} \int f(x) \, d\mu_n(x) = \int f(x) \, d\mu(x).$$

In the setting of [2], $S$ is $\mathbb{R}$ and $\{\mu_n\}$ is the sequence of measures induced by a specified i.i.d. sequence of random variables.

The starting point in [2] is not the random variable $O$ of (1.1), but a much more general random variable, $Q$, on which the only assumptions are that $Q$ has mean 0 and variance 1. The random variable $O$ of (1.1) will be obtained as a “truncated version” of the expansion of $Q$ with respect to the Haar basis. Such an expansion is available since $Q \in L^2(0,1)$.

Using that paper’s versions of the sort of $P_i$ described in (1.1), an i.i.d. sequence $(Q_i | i \in \mathbb{Z}^+)$ of copies of $Q$ is obtained by taking $Q_i(x)$ to be $Q(P_i(x))$. Expanding $Q$, respectively the $Q_i$, with respect to the Haar basis yields:

$$Q(x) = \sum_{k=0}^{\infty} 2^k c_{k,|2^k x|} (1)^{x_{k+1}}(x),$$

$$Q_i(x) = \sum_{k=0}^{\infty} 2^k c_{k,|2^k P_i(x)|} (1)^{x_{k+1}}(P_i(x)).$$

The next step in [2] is to consider the following sums of $n$ i.i.d. random variables:

$$\Sigma_n(x) := \sum_{i=1}^{n} Q_i(x).$$

In [2], these $\Sigma_n$ were denoted by $S_n$. The following truncated versions of $Q$, the $Q_i$ and $\Sigma_n$ were also considered in [2]; the last was denoted there by $S_{n,M}$. For $M \in \mathbb{N}$, define:

$$O_M(x) := \sum_{k=0}^{M} 2^k c_{k,|2^k x|} (1)^{x_{k+1}}(x).$$
It follows from properties of the outcomes that $O_M$ will also have mean 0 and that
\[ Var(O_M) = \theta_M^2 = \sum_{j=0}^{M} \sum_{k=0}^{2^j-1} c_{j,k}. \]

In [2], $\sigma_M^2$ is used in place of $\theta_M^2$.

For positive integers, $i$, define:
\[ R_{i,M}(x) := O_M(P_i(x)) \left( \sum_{k=0}^{M} 2^k c_{k,\lfloor 2^k P_i(x) \rfloor} (-1)^{\lfloor 2^k P_i(x) \rfloor} \right) \]
and:
\[ \Sigma_{n,M}(x) := \sum_{i=1}^{n} R_{i,M}(x). \]

Dobrinić and Garmirian showed that for each bounded and continuous $f : \mathbb{R} \to \mathbb{R}$ and for each $\epsilon > 0$, there exists a natural number $M_0$ such that for all $M \geq M_0$, with $\theta_M$ as above,
\[ \left| \int_0^1 f \left( \frac{\Sigma_n(x)}{\sqrt{n}} \right) d\lambda(x) - \int_0^1 f \left( \frac{\Sigma_{n,M}(x)}{\theta_M \sqrt{n}} \right) d\lambda(x) \right| < \epsilon. \]
Thus, they showed that these Haar expansions are in fact “sufficiently close” to their truncated versions. They then further approximated these truncated versions by the standard Gaussian measure on $\mathbb{R}$, completing their proof of the CLT.

It is then clear that for each $i$, each $M$ and for all $x$:
\[ (3) \quad R_{i,M}(x) \text{ depends only on } \{\varepsilon_{j,k}(x) | 1 \leq k \leq M+1 \} \quad \text{and so} \]
\[ (4) \quad \text{for all } x, \Sigma_{n,M}(x) \text{ depends only on } \{\varepsilon_{j,i}(x) | 1 \leq k \leq M+1, 1 \leq i \leq n \}. \]

The context for this paper is then provided by fixing a suitable $M$, letting $O$ be $O_M$, and (referring back to Equation (1)) letting $R_i = O \circ P_i = R_{i,M}$ and $S_n$ be $\Sigma_{n,M}$, but with the specific choice of the $P_i$ described in (1.1).

1.2.2. The Special Case of [3]: $M = 0$. Here, we indicate how the work of [3] can and should be understood as the analysis of the special case $M = 0$, and also discuss similarities and some differences in the approach taken in this paper. The [3] analogue of the random variable $O$ should be one of two Rademacher random variables defined on $(0,1)$, taking on values -1 and 1, each with probability 1/2, and depending only on $\varepsilon_1$. This brings us to a minor discordance between the treatment in [3] and that of [2] in [3], implicitly, $O(x)$ is taken to be $(-1)^{1+\varepsilon_1(x)}$, so that the natural order on $(0,1)$ coincides with the lexicographic order on $\{-1,1\}^{2^i}$, whereas the true analogue of the treatment in [2] would take $O(x)$ to be $(-1)^{\varepsilon_1(x)}$. Here, we have opted for the latter choice, so that the analogy with [3] is imperfect in this minor respect. The role of $O$ in [3] is left in the background, however, mainly because the notions developed above depend so much more transparently on $x$ in the setting of that paper. This will be a recurring theme, and we shall comment on its influence in various places in what follows.

Continuing with the system of analogies, the [3] analogue of the triangular array of integers is simply the diagonal: each “row” has length 1 and $i$ is the unique entry in the $i^{th}$ row. Thus, (in view of the observations in the previous paragraph) the [3] analogue of $R_i$ is $R_i(x) = (-1)^{1+\varepsilon_1(x)}$. The analogue of $P_i$ is denoted by $X_i$ in [3], and $X_i(x)$ is simply $\varepsilon_i(x)$. In [3], the $R_i$ are simply taken to be given, as in the previous sentence, rather than “created” as copies of $O$ via the various $P_i$.

In view of the preceding, note that the [3] analogue of $J_i$ is simply $\{i\}$. One motivation for the choice we have made, above, of the triangular array of integers, is to try to recapture as much as possible of the simplicity of this situation; our $J_i$ also are not only pairwise disjoint, but come in consecutive blocks: if $i_1 < i_2$ then all members of $J_{i_1}$ are less than all members of $J_{i_2}$. In fact, specifying this property along with the specification that the cardinality of each $J_i$ is $M+1$ almost completely determines the choice of the $P_i$, we have made in this paper and described
in (1.1), in that it requires that each row length is repeated at least, but not necessarily exactly \( M + 1 \) times. After the proof of Lemma 2, in (3.2), we will indicate exactly how this facilitates the proof of this Lemma.

2. Theorem 1 and Corollary 1

2.1. Preliminaries for Theorem 1.

2.1.1. Level Sets for \( S_n^* \). As in [3], the level sets for \( S_n^* \) are unions of dyadic intervals, but here they are of depth \( n(M + 1) \), the \( D_{n,\ell} \) introduced in the penultimate paragraph of (1.1). This is because the intervals on which \( S_n^* \) is constant have lengths which are integer multiples of \( 2^{-n(M+1)} \). This, in turn, follows from the fact that \( S_n^* \) is the inverse of the cdf of \( S_n \).

2.1.2. Level Sets for \( S_n \). Here, as a result of the multinomial distribution and the related “shuffle” involved in the definition of the \( P_i(x) \), the situation is rather different. The level sets for \( S_n \) will not be unions of depth \( n(M + 1) \) dyadic intervals; greater depth is necessarily involved once we leave behind the simple \( M = 0 \) case of [3]. The correct picture emerges from a more detailed analysis which we now undertake.

Recall the \( J_i \) from (1.1) and that their increasing enumerations are the \( (j_{i,k} | k \in \mathbb{Z}^+ ) \). Recall also from there the \( \mathcal{J}_i \) and the \( \mathcal{T}^n \). Also, as noted in Equations (2) and (3), above, the sequence \( (\epsilon_{j_{i,k}}(x)| 1 \leq k \leq M + 1 ) \) completely determines the value of \( R_i(x) \).

Now, fix \( n \geq 1 \). The sets \( \mathcal{J}_i, \) for \( 1 \leq i \leq n, \) are (of course) pairwise disjoint and therefore, their union, \( \mathcal{T}^n \), has cardinality \( n(M + 1) \). So, as noted in Equations (3) and (4), above, taken together, \( (\epsilon_{j_{i,k}}(x)| 1 \leq i \leq n, \ 1 \leq k \leq M ) \) completely determines the sequence \( (R_i(x)| 1 \leq i \leq n ), \) and therefore determines \( S_n(x), \) the sum of this sequence. Let \( (\tilde{\epsilon}_\ell | 0 \leq \ell \leq 2^{n(M+1)} ) \) be the increasing enumeration of \( \{0,1\}^n \) with respect to lexicographic order; thus, each \( \tilde{\epsilon}_\ell \) is a bitstring with domain \( \mathcal{T}^n \); we’ll denote its value at a coordinate \( j \in \mathcal{T}^n \) by \( (\tilde{\epsilon}_\ell)_j \). Finally, define

\[
E_{n,\ell} := \left\{ x \in (0,1)^n | \epsilon_{j_{i,k}}(x) = (\tilde{\epsilon}_\ell)_j \text{ for all } j \in \mathcal{T}^n \right\}.
\]

Then, each \( E_{n,\ell} \) has measure \( 2^{-n(M+1)} \) and the level sets of \( S_n \) are unions of these \( E_{n,\ell} \).

2.1.3. Sequences of binary digits and outcomes. For \( x \in (0,1) \), we set:

\[
\bar{b}_n(x) := (\epsilon_1(x), \ldots, \epsilon_{n(M+1)}(x)),
\]

\[
\tilde{b}_n(x) := (R_1(x), \ldots, R_n(x)).
\]

For \( 1 \leq \ell \leq n, \) \( \bar{b}_n(x) \) is constant on \( D_{n,\ell} \) and \( \tilde{b}_n(x) \) is constant on \( E_{n,\ell} \). Therefore, following the convention in the final paragraph of (1.1), we denote these constant values by \( \mathcal{I}_n(\ell), \mathcal{O}_n(\ell), \) respectively. Of course, the same also applies to each individual \( R_i \) and so we also denote by \( (|R_i|)_n(\ell) \) the constant value of \( R_i(x) \) on \( E_{n,\ell} \). As in [3], \( \mathcal{O}_n(\ell) \) is the reverse of the binary representation of \( \ell \):

\[
\ell = \sum_{i=1}^{n(M+1)} 2^{n(M+1)-i} \left( \mathcal{O}_n(\ell) \right)_i,
\]

and \( \left( \mathcal{I}_n(\ell) | \ell < 2^{n(M+1)} \right) \) enumerates \( \{0,1\}^{n(M+1)} \) in increasing order with respect to the lexicographic ordering. For \( \bar{b} \in \{0,1\}^{n(M+1)} \), we also let:

\[
D_{\bar{b}} := \left\{ x \in (0,1) | \bar{b}_n(x) = \bar{b} \right\}.
\]

Letting \( \ell \) be such that \( \bar{b} = \mathcal{I}_n(\ell) \), we note that \( D_{\bar{b}} = D_{n,\ell} = \left\{ x \in (0,1) | \bar{b}_n(x) = \bar{b} \right\} \).

Note that \( \mathcal{I}_n(\ell) \) does NOT denote the \( \ell \)-th component of a vector, \( \bar{b}_n \), rather it denotes the vector (a length \( n(M + 1) \) bitstring) itself. We will denote the \( i \)-th component of this vector by \( \left( \mathcal{I}_n(\ell) \right)_i \). Note that this is just \( \epsilon_i(x) \) for any \( x \in D_{n,\ell} \). Similar observations hold with \( \mathcal{O} \) in place of \( \bar{b} \) (and the set of outcomes of \( O \) replacing \( \{0,1\} \)).
Let $O_n$ denote the set of vectors $\{I\bar{\sigma}_n(\ell)\mid 0 \leq \ell < 2^{n(M+1)}\}$ and note that this gives an enumeration without repetitions of $O_n$ (this is essentially because there is a bijection between outcomes of $O$ and subsets of $\{1, \ldots, M + 1\}$). Thus $\bar{b}_n(\ell) \mapsto \bar{\sigma}_n(\ell)$ is a bijection between $\{0,1\}^{n(M+1)}$ and $O_n$. We use $\nu_n$ to denote this bijection. Note that for $\bar{b} \in \{0,1\}^{n(M+1)}$, $x \in D_\bar{b}$ and $1 \leq i \leq n$, $(\nu_n(\bar{b}))_i = R_i(x)$.

2.1.4. Admissible Permutations. We are now ready to define this paper’s version of the notion of admissible permutation (of $\{0, \ldots, 2^{n(M+1)} - 1\}$) and to prove Theorem 1, which is best understood in the following way. As already noted, each $E_{n,\ell}$ tells us much more than a value of $S_n$: it tells us how this value arises as the sum of values of the $R_i$ for $1 \leq i \leq n$. The content of Theorem 1 amounts to unravelling the analogous additional information about how a value of $S_n^*$ arises: as the sum of values of the members of an i.i.d. family of $n$ random variables, each of which is $n$-trim, i.e., as the sum of the $R_{n,i}$ for $1 \leq i \leq n$.

Definition 1. A permutation, $\pi$, of $\{0, \ldots, 2^{n(M+1)} - 1\}$ is admissible iff for all $0 \leq \ell < 2^{n(M+1)}$: $S_n^*(x) = S_n(y)$ for all $x \in D_{\pi,\ell}$ and all $y \in E_{n,\pi(\ell)}$.

2.2. Theorem 1 and Corollary 1.

Theorem 1. For each $n$, there is a canonical bijection between admissible permutations of $\{0, \ldots, 2^{n(M+1)} - 1\}$ and representations of $S_n^*$ as a sum $S_n^* = \sum_{i=1}^{n} R_{n,i}^*$, where $(R_{n,i}^*)_{1 \leq i \leq n}$ is an i.i.d. family of $n$-trim random variables each of which has mean 0, variance 1 and has outcomes of $O$ as values, with equal probability.

Proof. Fix $n > 0$. We first construct the $R_{n,i}^*$; given $\pi$, and then construct $\pi$ given the $R_{n,i}^*$. We then carry out the necessary verifications in each direction.

Let $\pi$ be an admissible permutation of $\{0, \ldots, 2^{n(M+1)} - 1\}$. For $x \in (0,1)$, let $\ell$ be such that $x \in D_{\pi,\ell}$, and let $1 \leq i \leq n$. Then:

(5) \hspace{1cm} \text{let } y \text{ be any member of } E_{n,\pi(\ell)} \text{ and define: } R_{n,i}^*(x) := R_i(y).

Conversely, given an independent family, $(R_{n,i}^*)_{1 \leq i \leq n}$, such that $S_n^* = \sum_{i=1}^{n} R_{n,i}^*$, where each $R_{n,i}^*$ is $n$-trim, has mean 0, variance 1 and has outcomes of $O$ as values, with equal probability, we obtain $\pi$ as follows. Given $0 \leq \ell < 2^{n(M+1)}$, let $x \in D_{\pi,\ell}$, let $\bar{\sigma} = (R_{n,i}^*(x))_{1 \leq i \leq n}$ and define:

(6) \hspace{1cm} \pi(\ell) = \text{ that } \ell' \text{ with } 0 \leq \ell' < 2^{n(M+1)} \text{ such that } \bar{\sigma} = I\bar{\sigma}_n(\ell').

Clearly these constructions yield a bijection, so we turn to the necessary verifications.

First suppose $\pi$ is admissible and that the $R_{n,i}^*$ are defined by Equation (5). Clearly these $R_{n,i}^*$ are $n$-trim, have mean 0 and variance 1. In order to see that they sum to $S_n^*$, note that:

for all $\ell < 2^{n(M+1)}$ and all $x \in D_{\pi,\ell}$, $S_n^*(x) = S_n(y)$, for any $y \in E_{n,\pi(\ell)}$,

i.e. $S_n^*(x) = \sum_{i=1}^{n} R_i(y)$, for any such $y$, i.e. $S_n^*(x) = \sum_{i=1}^{n} R_{n,i}^*(x)$; this suffices.

We next show that for each $1 \leq i \leq n$ and each of the $m$ outcomes, $o$ of $O$, $P(R_{n,i}^* = o) = \frac{1}{m} = 2^{(n-1)(M+1)} \times 2^{-n(M+1)}$ by showing that the event “$R_{n,i}^* = o$” is the union of $2^{(n-1)(M+1)}$ dyadic intervals $D_{n,\ell}$. For this, note that the event “$R_i = o$” does have probability $\frac{1}{m}$ and therefore is the union of $2^{(n-1)(M+1)}$ of the sets $E_{n,\ell}$. It is routine to see that a set $E_{n,\ell}$ is included in the event “$R_i = o$” iff the set $D_{n,i-1}(\ell)$ is included in the event “$R_{n,i} = o$” and so this suffices.
In order to see that these \( R_{n,i}^* \) are independent, it suffices to show that:

\[
\text{for all } \vec{d} = (o_1, \ldots, o_n) \in \mathcal{O}_n, \quad p((o_1, \ldots, o_n) = p_1(o_1) \cdot \ldots \cdot p_n(o_n),
\]

where \( p \) is the joint pmf of the \( R_{n,i}^* \) and each \( p_i \) is the pmf of \( R_{n,i}^* \) alone. We have already seen that \( p_1(o_1) \cdot \ldots \cdot p_n(o_n) = 2^{-n(M+1)} \), so, again viewing \( \pi \) as a permutation of \( \{0,1\}^{n(M+1)} \), let \( \vec{b} := \pi^{-1} \circ (\nu_n)^{-1}(\vec{d}) \) and note that:

\[
P(R_{n,i}^* = o_1, \ldots, R_{n,n}^* = o_n) = \lambda \left( \left\{ \left(x | \pi \left( \vec{b}_n(x) \right) = \nu_n^{-1}(\vec{d}) \right) \right\} \right) = \lambda(D_{\vec{b}}) = 2^{-n(M+1)}.
\]

For the opposite direction, suppose that \( R_{n,i}^*|1 \leq i \leq n \) is given with the stated properties. Let \( \pi \) be defined by Equation (6). We first show that \( \pi \) is one-to-one. For this, let \( x \in D_{n,\vec{d}}, \vec{d} = (o_1, \ldots, o_n) = (R_{n,i}^*(x)|1 \leq i \leq n) \) and note that if \( \vec{u} \in \{0,1\}^{n(M+1)} \) is such that

\[
\text{for } y \in D_{\vec{u}}, \quad (R_{n,i}^*(y)|1 \leq i \leq n) = \vec{d}, \text{ then } \vec{u} = \vec{b}_n(\ell).
\]

If this were to fail we would have that

\[
P(R_{n,1}^* = o_1, \ldots, R_{n,n}^* = o_n) \geq \lambda(D_{\vec{d}}) + \lambda(D_{n,\vec{d}}) = 2^{-n(M+1)+1},
\]

which contradicts our hypotheses on the \( R_{n,i}^* \). Thus, \( \pi \) is one-to-one. Admissibility then follows, because now, by hypothesis, if \( x \in D_{n,k} \) and \( m = \pi(k) \), then:

\[
S_n^*(x) = \sum_{i=1}^{n} R_{n,i}^*(x) = \sum_{i=1}^{n} o_i, \text{ but also for any } y \in D_{n,m}, \quad S_n(y) = \sum_{i=1}^{n} o_i,
\]

as required.

\[ \square \]

**Corollary 1.** There is a canonical bijection between sequences, \( \{\pi_n\} \), of admissible permutations of \( \{0, \ldots, 2^{n(M+1)} - 1\} \) and trim, strong triangular arrays for \( \{S_n^*\} \).

\[ \square \]

Given \( n \) and \( (R_{n,i}^*|1 \leq i \leq n) \), Equation (6) is best seen as a finer version of the displayed formula of Definition 1 (the definition of admissible permutation). Incorporating the additional information in the representations \( (R_{n,i}|1 \leq i \leq n) \) and \( (R_{n,i}^*|1 \leq i \leq n) \) singles out a specific admissible permutation, whereas the displayed formula of Definition 1 defines the set of all of them. Equation (5) reverses this, taking as given the canonical representation, \( (R_{n,i}|1 \leq i \leq n) \), together with a specific admissible permutation and singling out a specific representation, \( (R_{n,i}^*|1 \leq i \leq n) \), of \( S_n^* \).

### 3. Theorems 2 and 3

We follow the general outline of §3 of [3], omitting certain items, and in particular, those specifically targeted at §4 of [3], as well as Propositions 1 - 3 of that paper. As part of our concluding remarks in (3.3), we will discuss in more detail the status of the omitted items.

In (3.1.1), we introduce a number of notions related to the correspondence between values of the \( \text{MN}^O_n \) and multinomial vectors \( \vec{k} \); the latter arise as vectors (of length \( m \)) of frequencies of the outcomes, \( o_n \) of \( O \). We also introduce the functions \( \tau^O_1 \) and \( \tau^O_2 \). This is the substance of Definitions 2 - 5. The role of \( \tau^O_1 \) is essential and has been mentioned in the Introduction, just prior to the beginning of (1.1). On the other hand, despite its formal dependence on \( O \), the role of the function \( \tau^O_2 \) is merely a convenience for the proof of the important Lemma 2. This is discussed in somewhat more detail below.

In (3.1.2), Definition 6 introduces the function \( \text{SMC}^O \) and the \( \gamma^O_{n,\ell} \), which are this paper’s analogues of the function \( \text{SBC} \) of [3] and the binomial coefficients, respectively. In Definition 7, we introduce this paper’s versions of the functions \( \text{Step}_{\ell,n} \), \( \text{Weight}_{\ell,n} \). Incorporating Remark 1 paves the way for introducing their “integer versions” \( I_{\text{Step}}_{\ell,n} \), \( I_{\text{Weight}}_{\ell,n} \), \( IS \) and \( IL^* \), respectively, following the paradigm laid out in the penultimate paragraph of (1.1). We also introduce the functions \( I_{\text{Step}}, I_{\text{Weight}} \) which are the natural encodings (as functions of two variables) of the family of the \( I_{\text{Step}}_{\ell,n} \), \( I_{\text{Weight}}_{\ell,n} \), respectively. Some obvious properties of these “integer versions” are stated in Remark 2.
In Definition 8, we introduce the sets $A_{n,t}$ and $B_{n,t}$ (this paper’s version of the sets $A_{n,i}$ and $B_{n,i}$ of [3]), their “integer versions” $IA_{n,t}$ and $IB_{n,t}$, the three-place relations $RIA$ and $RIB$, which encode the families $IA_{n,i}$ and $IB_{n,i}$ respectively, and, finally, their cardinality functions, $\alpha(n,t,\xi)$ and $\beta(n,t,\xi)$, respectively, and their enumerating functions $\alpha_{n,t,s}$ and $\beta_{n,t,s}$ respectively. The cardinality and enumerating functions play an important role in the proofs of Lemma 2 and Theorem 2. The notion of “tameness” which was introduced in [3], remains in the background here, but plays an important role implicitly. This will be discussed in connection with the summary of (3.2).

The material of Definition 8 also sets the stage for the concluding items, Lemma 1 and Corollary 2, of (3.1.2). In part (a) of Lemma 1, we show that, in analogy with the situation in [3], the $\gamma_{n,t}$ are the common cardinalities, $\alpha_{n,t}$ of $IA_{n,t}$ and $\beta_{n,t}$ of $IB_{n,t}$. In part (b), we give an alternate characterization of admissibility which enables part (c), where we finally tie up some loose ends from §2 by proving the existence of many admissible permutations of $\{0, \ldots, 2^{n(M+1)} - 1\}$. In Corollary 2, we invoke Theorem 1 and Corollary 1 to extend this to the existence of representations (with the needed properties) of each $S^\alpha_n$ and to strong, trim triangular representations of the sequence $\{S^\alpha_n\}$. We conclude (3.1) by developing, in (3.1.3), some notation related to the $\mathcal{T}_1$. This will be used in the proof of Lemma 2.

We begin to address complexity issues in earnest in (3.2). In Definition 9 ((3.2.1)), we introduce the sequence $\{F_n\}$ of admissible permutations and note, in Remark 3, its connection to the view of admissibility developed in the final paragraph of the proof of Lemma 1. This Remark justifies the somewhat imprecise claim that each $F_n$ is the simplest admissible permutation of $\{0, \ldots, 2^{n(M+1)} - 1\}$.

In (3.2.2) we prove a sequence of results that culminates in the proof of Theorem 2: that the two-place function $F$ which naturally encodes the sequence $\{F_n\}$ is P-TIME relative to $\tau^O_1$. Before laying out in detail the route to this result, it will be helpful to comment briefly on the role of “tameness” and some of the related results of [3]. The context is that we are dealing with a $d+1$-place relation, $R$ on $\mathbb{N}$ and $R$ has the property that for all $\vec{u} = (u_1, \ldots, u_d) \in \mathbb{N}^d$,

$$R(\vec{u}) = \{w|R(u_1, \ldots, u_d, w)\} \text{ is finite.}$$

The cardinality function for $R$ was then defined to be the function, which, to $d+1$-tuples, $(\vec{u}, \xi) \in \mathbb{N}^{d+1}$, assigns the cardinality of $R[\vec{u}] \cap \{1, \ldots, \xi\}$. Then, in [3], such a relation, $R$, was defined to be “tame” iff its cardinality function is P-TIME. The important contribution of tameness was provided by item (6) of Remark 3, and by Lemma 2 of [3], and both of these required a mild additional assumption: that it is P-TIME decidable whether, given $\vec{u} \in \mathbb{N}^d$, there is some $w$ such that $R(\vec{u}, w)$. Given this additional assumption on $R$, item (6) of Remark 3 of [3] notes that if $R$ is tame, then $R$ itself is P-TIME decidable. This is simply because, for $\vec{u} \in \mathbb{N}^d$ and $w \in \mathbb{Z}^+$, letting $cd$ denote the cardinality function for $R$, $R(\vec{u}, w)$ holds iff $cd(\vec{u}, w) = cd(\vec{u}, w - 1) + 1$.

The content of Lemma 2 of [3] involves the “enumerating function” for a relation $R$, as above. This is the function, $E$, whose domain consists of $d + 1$-tuples, $(\vec{u}, s)$ such that $1 \leq s \leq |R[\vec{u}]|$, and to such a $d + 1$-tuple assigns the $s$th member of $R[\vec{u}]$ (in the increasing enumeration of $R[\vec{u}]$). The result is that if $R$ is tame, then $E$ is P-TIME.

Returning to the context of this paper, of course the relations we have in mind are $RIA$ and $RIB$. We relativize all of the notions in the last two paragraphs to the function $\tau^O_1$. Then, relative to $\tau^O_1$, both of these relations satisfy the mild additional hypothesis: their domains are P-TIME decidable relative to $\tau^O_1$. The proofs of item (6) of Remark 3, and of Lemma 2, both of [3] then relativize in a completely straightforward way to $\tau^O_1$.

We can now continue with our account of the sequence of results in (3.2.2). We begin with Proposition 1 which establishes that the following functions are P-TIME relative to $\tau^O_1$: $SMC^O$, the $\gamma^O_{n,t}$, the function $IStep$ and the function $\alpha$ of Equation (7c) (Definition 8). As already noted, above, $\alpha$ is the “cardinality function” for the relation $RIA$. Therefore, in virtue of the preceding discussion, it follows (and this is Remark 4) that $RIA$ is tame relative to $\tau^O_1$, and so it is also P-TIME decidable relative to $\tau^O_1$. This is part of Corollary 3 which also notes that for the same
Definition 4. The main work is done in Lemma 2, where we show that the function $\beta$ of Equation (7c) is also P-TIME relative to $\tau_1^O$. It follows (as above for RIA and the function $a_{n,t,s}$) that RIB is tame relative to $\tau_2^O$ and therefore P-TIME decidable relative to $\tau_1^O$, and that the function $b_{n,t,s}$ of Equation (7di) (Definition 8) is also P-TIME relative to $\tau_2^O$ (since it is the enumerating function of RIB). This is Corollary 4, the analogue “on the $B$ – side” of Corollary 3.

This completes the preparation for the proof of Theorem 2. We conclude (3.2) with the obvious translation of the result of Theorem 2 into the language of strong trim triangular representations of the sequence $\{S_n\}$. This is Theorem 3 and it follows immediately from Theorem 2 and Corollary 1. We conclude the paper in (3.3) with a retrospective comparison with [3], focusing for reasons and since the function $a_{n,t,s}$ of Equation (7i) (Definition 8) is the enumerating function for RIA, it follows that, as a function of $(n, t, s)$, $a_{n,t,s}$ is P-TIME relative to $\tau_1^O$.

In Proposition 2 we show that $\tau_2^O$ is outright P-TIME, despite its formal dependence on $O$. The main work is done in Lemma 2, where we show that the function $\beta$ of Equation (7c) is also P-TIME relative to $\tau_1^O$. It follows (as above for RIA and the function $a_{n,t,s}$) that RIB is tame relative to $\tau_2^O$ and therefore P-TIME decidable relative to $\tau_1^O$, and that the function $b_{n,t,s}$ of Equation (7di) (Definition 8) is also P-TIME relative to $\tau_2^O$ (since it is the enumerating function of RIB). This is Corollary 4, the analogue “on the $B$ – side” of Corollary 3.

3.1. Preliminaries for Theorems 2 and 3. Our first task will be to take a closer look at the multinomial distributions $MN_n^O$.

3.1.1. Multinomial vectors, values and frequencies: the functions $\tau_1^O$ and $\tau_2^O$.

Definition 2. We denote by $K_n$ the set of level $n$ multinomial vectors, $\vec{k} = (k_1, \ldots, k_m)$, where each $k_i$ is a non-negative integer and $\sum_i k_i = n$. As usual, $\binom{n}{k}$ denotes the multinomial coefficient corresponding to $n$ and $\vec{k}$, i.e. \( \binom{n}{\vec{k}} = \frac{n!}{k_1! \cdots k_m!} \). We denote by $C_n$ the cumulative distribution function of $MN_n^O$.

As already noted in (1.1), the values of $MN_n^O$ are the inner products, $\vec{k} \cdot \vec{\alpha} = k_1 \alpha_1 + \ldots + k_m \alpha_m$, of the $\vec{k} \in K_n$ with the fixed vector, $(\alpha_1, \ldots, \alpha_m)$, of outcomes of $O$. This leads naturally to the following equivalence relations, $\equiv_n$ on the $K_n$.

Definition 3. For $\vec{k}, \vec{r} \in K_n$, we set $\vec{k} \equiv_n \vec{r}$ iff $\vec{k} \cdot \vec{\alpha} = \vec{r} \cdot \vec{\alpha}$.

Thus, it is the equivalence classes of $\equiv_n$ that correspond bijectively to the values of $MN_n^O$. The structure of the $\equiv_n$ depends on $O$, and therefore, ultimately, on $Q$ (recall (1.2)) and thus we cannot control this structure aside from the minimal requirements imposed by the hypotheses that $Q$ has been normalized. In particular, we cannot control the number of values/equivalence classes beyond the obvious bound of $\binom{n+m-1}{m-1}$ mentioned in (1.1) (just prior to Equation (1)).

Definition 4. We let $v^t_n \mid 0 \leq t \leq T_n$ be the increasing enumeration of the values of $MN_n^O$. For $0 \leq t \leq T_n$, we let $K_{n,t}$ be the equivalence class of $\equiv_n$ corresponding to the value $v^t_n$, i.e., $K_{n,t} = \{ \vec{k} \in K_n | \vec{k} \cdot \vec{\alpha} = v^t_n \}$ and we let $K_{n,t} = \{ \vec{k} \in K_n | \vec{k} \cdot \vec{\alpha} < v^t_n \}$. We let $\tau_1^O$ be that function whose domain consists of all $m + 2$-tuples, $(n, k_1, \ldots, k_m, t)$ such that $n > 0$, $0 \leq t \leq T_n$, $(k_1, \ldots, k_m) \in K_n$, and such that:

$\tau_1^O(n, k_1, \ldots, k_m, t) = 1$ iff $(k_1, \ldots, k_m) \in K_{n,t}$.

We will also denote elements of the domain of $\tau_1^O$ by $(n, \vec{k}, t)$.

The $t$ – indices of these values will serve as the analogues of the notions of Step and Weight from [3]. The analogues in [3] of the $v^t_n$ are the $-n + 2i$, for $0 \leq i \leq n$, while the analogue of $\tau_1^O$ in [3] is simply the (obviously P-TIME) function which (for $n, k$, and $t$, with $0 \leq t \leq n$ and $-n \leq t \leq n$ such that $n \equiv t (\text{mod} 2)$) takes on value $1$ iff $t = -n + 2i$. Thus, it checks whether or not $t$ is the value of $S_n$ that corresponds to the Hamming weight, $k$. Our $\tau_1^O$ furnishes the analogous check. As usual, in our setting, the correspondence between patterns of binary digits and values of $S_n$ is no longer at the surface: it depends on $O$ in an essential way.
Definition 5. For \( x \in (0, 1) \), \( 0 \leq b \leq n \) and \( 1 \leq s \leq m \), let \( k_{n,s}(x,b) := \) the frequency of \( o_s \) in \((\bar{o}_{n}(x))_i \) \( i > b \) and let \( \bar{k}_{n}(x,b) := (k_{n,1}(x,b), \ldots, k_{n,m}(x,b)) \). Since these are constant on the \( E_{n,t} \), we have also defined the integer versions, \( 1\bar{k}_{n,s}(\ell,b) \) and \( \bar{l}_{n}(\ell,b) \), for \( 0 \leq \ell < 2^{n(M+1)} \). We let \( \tau^O_1 \) be the function defined on four-tuples \((n,s,\ell,b)\) with \( 0 \leq b \leq n \), \( 1 \leq s \leq m \) and \( 0 \leq \ell < 2^{n(M+1)} \), and such that for such \((n,s,\ell,b)\):

\[
\tau^O_1(n,s,\ell,b) := 1\bar{k}_{n,s}(\ell,b).
\]

3.1.2. Multinomial analogues of Step, Weight and related notions. We now develop the analogues of the notions introduced in (3.1) of [3].

Definition 6. For \( n > 0 \) and \( 0 \leq t \leq T_n + 1 \), set

\[
SMC^O(n,t) := \sum_{\bar{k} \in \mathfrak{K}_{n,t}} \left( \frac{n}{\bar{k}} \right) \text{ and } \gamma^O_{n,t} := SMC^O(n,t+1) - SMC^O(n,t).
\]

Thus, \( \gamma^O_{n,t} = \sum_{\bar{k} \in \mathfrak{K}_{n,t}} \left( \frac{n}{\bar{k}} \right) \). As we shall soon see (and as may already be obvious), \( \gamma^O_{n,t} \) is the combinatorial counterpart of the binomial coefficient \( \binom{n}{\bar{k}} \) in [3], while \( SMC^O \) is the combinatorial counterpart of the function \( SBC \) of [3]. This said, even these “aggregated” functions fall short of providing all of the information needed for the proof of the crucial Lemma 2, below, in (3.2).

For \( \bar{k} \in \mathfrak{K}_n \) and \( 0 \leq t \leq T_n \), \( \tau^O_1 \) tells whether or not \( \bar{k} \in \mathfrak{K}_{n,t} \). Item (2) of Remark 1, and the related Proposition 1, below, show that the information encoded by \( \tau^O_1 \) allows us to recover the \( \gamma^O_{n,t} \) and \( SMC^O \) in a simple fashion. The function \( \tau^O_2 \) provides information about the frequencies of the outcomes \( o_s \) that is “stratified” by \( b \), i.e., about the frequencies among the \((1\bar{o}_{n}(\ell))_i \) for \( i > b \).

Both \( \tau^O_1 \) and \( \tau^O_2 \) play prominent roles in the proof of Lemma 2, but the role of \( \tau^O_2 \) will turn out to be a convenience, since, despite its formal dependence on \( O \), we will nevertheless be able to show, in Proposition 2, of (3.2), that it is outright P-TIME. The finer information encoded in \( \tau^O_2 \) is indispensable, however, and thus, in (3.2), our upper complexity bounds will be obtained relative to the function \( \tau^O_1 \).

Definition 7. For \( n > 0 \), and \( x \in (0,1) \), \( \text{Step}_n(x) \) (respectively \( \text{Weight}_n(x) \)) is the unique \( t \) with \( 0 \leq t \leq T_n \) such that \( S^*_n(x) = v^n_t \) (respectively \( S^+_n(x) = v^n_t \)).

Remark 1. For \( x \in (0,1) \), and \( n > 0 \), the following observations are obvious:

\begin{enumerate}
  \item \( \text{SMC}^O(n,t) = 2^{n(M+1)} \text{C}_n(v^n_t) \).
  \item \( \gamma^O_{n,t} = \sum_{\bar{k} \in \mathfrak{K}_n} \tau^O_1(n,\bar{k},t) \) so \( \text{SMC}^O(n,t) = \sum_{\ell \leq n} \sum_{\bar{k} \in \mathfrak{K}_n} \tau^O_1(n,\bar{k},t) \).
  \item \( \text{Step}_n(x) \) is the unique \( t \) such that \( \text{SMC}^O(n,t) \leq x2^{n(M+1)} < \text{SMC}^O(n,t+1) \).
  \item \( \text{Step}_n(x), \text{Weight}_n(x) \) depend at most on \( \tilde{b}_n(x), \tilde{o}_n(x) \), respectively.
\end{enumerate}

In view of the penultimate paragraph of (1.1) and 3. of Remark 1, we can introduce the notions \( \text{Step}^n(x), \text{Weight}^n(x) \) for \( 0 \leq \ell < 2^{n(M+1)} \). We already knew that \( S_n \) and \( S^*_n \) also depend at most on \( \tilde{b}_n(x), \tilde{o}_n(x) \), respectively, and the \( \text{IS}^n_\ell \), \( \text{IS}^n_\ell \) for \( 0 \leq \ell < 2^{n(M+1)} \) were already introduced at the end of (1.1). Via the usual identification, we have also introduced the notions \( \text{IS}^n_\ell(\bar{b}) \), \( \text{IS}^n_\ell(\bar{o}) \) for \( \bar{b} \in \{0,1\}^{n(M+1)} \) and \( \bar{o} \in \mathfrak{O}_n \). Unlike the situation in [3], here, due to the role of \( O \), \( \text{Weight}^n_\ell \) will typically not be independent of \( n \) and purely intrinsic to \( \ell \); there is greater symmetry between Step and Weight in the present context. In what follows, we shall use the notations \( \text{ISStep}^n(\ell), \text{IWeight}^n(\ell) \) rather than \( \text{ISStep}^n_\ell \), \( \text{IWeight}^n_\ell \), respectively. The following is then also obvious.

Remark 2. For \( n > 0 \) and \( k < 2^n \):

\begin{enumerate}
  \item \( \text{ISStep}(n,0) = 0 \) and for \( 0 \leq \ell < 2^{n(M+1)} \), \( \text{ISStep}(n,\ell) \) is the least positive \( t \leq T_n \) such that \( \ell < \text{SMC}^O(n,t+1) \).
(2) For all $0 \leq t \leq T_n$, $\text{SMCO}^O(n, t)$ is the unique (and so least) $\ell$ such that $\text{IStep}(n, \ell) = t$. 

\textbf{Definition 8.} For $0 \leq t \leq T_n$, we define $A_{n,t}$, $B_{n,t}$ by:

\[(7) \quad A_{n,t} := \{x \in (0, 1) | \text{Step}_n(x) = t \} \text{ and } B_{n,t} := \{x \in (0, 1) | \text{Weight}_n(x) = t \}.
\]

In view of Remark 2, for fixed $n > 0$, each $A_{n,t}$ is a union of $D_{n,t}$’s and each $B_{n,t}$ is the union of $E_{n,t}$’s, and therefore, in view of the last paragraph of (1.1), we have introduced the notions $I A_{n,t}$, $IB_{n,t}$ for the corresponding subsets of $\{0, \ldots, 2^{n(M+1)} - 1 \}$ (or of $\{0, 1\}^{n(M+1)}$ via the usual identification):

\[(7a) \quad IA_{n,t} := \left\{ \ell < 2^{n(M+1)} | D_{n,t} \subseteq A_{n,t} \right\} \text{ and } IB_{n,t} := \left\{ \ell < 2^{n(M+1)} | E_{n,t} \subseteq B_{n,t} \right\}.
\]

We also let:

\[(7b) \quad a_{n,t} := |IA_{n,t}| \text{ and } b_{n,t} := |IB_{n,t}|.
\]

and for positive integers, $\xi < 2^{n(M+1)}$, we let

\[(7c) \quad \alpha(n, t, \xi) = |IA_{n,t} \cap \{1, \ldots, \xi\}|, \text{ and } \beta(n, t, \xi) = |IB_{n,t} \cap \{1, \ldots, \xi\}|.
\]

For $n > 0$ and $1 \leq t \leq T_n$ we let:

\[(7di) \quad (a_{n,t,s} | 1 \leq s \leq a_{n,t}) \text{ be the increasing enumeration of } IA_{n,t},
\]

\[(7dii) \quad (b_{n,t,s} | 1 \leq s \leq b_{n,t}) \text{ be the increasing enumeration of } IB_{n,t}.
\]

Finally, we let:

\[(7e) \quad \text{RI}A(n, t, \ell) \text{ iff } \ell \in IA_{n,t} \text{ and } \text{RIB}(n, t, \ell) \text{ iff } \ell \in IB_{n,t}.
\]

In the terminology of [3], the functions of Equation (7c) are the cardinality functions for the relations of Equation (7e), and the functions of Equations (7di) and (7dii) are the enumerating functions for these relations.

**Lemma 1.** Let $n > 0$. Then:

1. for all $0 \leq t \leq T_n$, $\alpha_{n,t} = \gamma^O_{n,t} = \beta_{n,t}$.
2. If $\pi$ is a permutation of $\{0, \ldots, 2^{n(M+1)} - 1 \}$, the admissibility of $\pi$ is equivalent to each of the following conditions:
   a. for all $\ell < 2^{n(M+1)}$, $\text{IWeight}(n, \pi(\ell)) = \text{IStep}(n, \ell)$,
   b. for all $0 \leq t \leq T_n$, $\pi |IA_{n,t}| = IB_{n,t}$,
   c. $IS_n^* = IS_n \circ \pi$.
3. There are $\prod_{i=0}^{T_n} (\gamma^O_{n,t})$ admissible permutations of $\{0, \ldots, 2^{n(M+1)} - 1 \}$.

**Proof.** Since $\alpha_{n,t} = 2^{n(M+1)} P(\text{Step}_n(x) = v^t_n)$ and $\beta_{n,t} = 2^{n(M+1)} P(\text{Weight}_n(x) = v^t_n)$,

1. follows. For 2., it is clear that (c) is equivalent to (a), and from (1) it then follows that (b) is also equivalent to (a), so we argue that the admissibility of $\pi$ is equivalent to (a). Let $\pi$ be any permutation of $\{0, \ldots, 2^n - 1 \}$, let $\ell < 2^{n(M+1)}$ and let $x \in D_{n,t}$, $y \in E_{n,\pi(t)}$. Let $s = \text{IStep}(n, \ell)$ and $t = \text{IWeight}(n, \pi(\ell))$. Then $s = t$ iff $v^t_n = v^s_n$ iff $S^*_n(x) = S_n(y)$ and (a) is equivalent to the admissibility of $\pi$.

For 3., note that an admissible permutation $\pi$ decomposes into the system of its restrictions to the $IA_{n,t}$. Complete information about $\pi | IA_{n,t}$ is encoded by the permutation, $\pi_{n,t}$ of $\{1, \ldots, \gamma^O_{n,t}\}$ defined by:

\[(8) \quad \text{if } 1 \leq s \leq \gamma^O_{n,t} \text{, then } \pi(a_{n,t,s}) = b_{n,t,\pi_{n,t}(s)}.
\]

Further, the $\pi_{n,t}$ are arbitrary in the sense that if, for $0 \leq t \leq T_n$, $\phi_{n,t}$ is any permutation of $\{1, \ldots, \gamma^O_{n,t}\}$, then for each $n$ there is a (unique) admissible permutation $\pi_n$ of $\{0, \ldots, 2^{n(M+1)} - 1 \}$ such that for each $0 \leq t \leq T_n$, $\pi_{n,t} = \phi_{n,t}$. Finally, for fixed $n$, the product in 3. counts the number of such systems $(\phi_{n,t}|0 \leq t \leq T_n)$, and so 3. follows. 

\[\square\]
The next Corollary is an immediate consequence of Theorem 1, Corollary 1 and item 3. of Lemma 1; it gives the existence of strong trim triangular arrays for \( \{S_n^k\} \).

**Corollary 2.** For each \( n \), there are \( \prod_{i=0}^{T_n} (S^O_{i,n,t}) \) representations of \( S^*_n \) as a sum, \( S^*_n = \sum_{i=1}^{n} R^*_n,i \),

where \( (R^*_n,i|1 \leq i \leq n) \) is an i.i.d. family of random variables each of which has outcomes of \( O \) as values, with equal probability, has mean \( 0 \), variance \( 1 \) and depends only on \( b_n \). Therefore, there exist (continuum many) strong trim triangular array representations of the sequence \( \{S^*_n\} \). \( \square \)

### 3.1.3. A Closer Look at the \( J_i \).

We develop here some additional properties of the the \( J_i \) needed for the proof of Theorem 2, and especially of the crucial Lemma 2, both in the next subsection. It is there that we will see the point of the particular choice of the triangular array of positive integers presented in (1.1) and its apparently peculiar “repetition” \( (M+1) \) times of each row length.

With \( \binom{1}{1} = 0 \), as usual, it is easy to see that the integer entries in the \( b^\text{th} \) block of rows are precisely those \( \eta \) satisfying \( \binom{b}{2}(M+1) < \eta \leq \binom{b+1}{2}(M+1) \), and therefore, that the integer entries in the \( r^\text{th} \) row of the \( b^\text{th} \) block are precisely those \( \eta \) satisfying \( \binom{b}{2} + (r-1)(M+1) < \eta \leq \binom{b}{2} + r(M+1) \). For \( 1 \leq i \leq b, 1 \leq r \leq M+1 \), we let \( \eta_{b,r,i} \) be the entry in row \( r \) and column \( i \) of block \( b \). It is then immediate that \( \eta_{b,r,i} = \left( \binom{b}{2} + (r-1)(M+1) + i \right. \).

For fixed \( b \) and fixed \( 1 \leq i \leq b \), the entries in the \( i^\text{th} \) column of the \( b^\text{th} \) block are precisely the \( \eta_{b,r,i} \), with \( 1 \leq r \leq M+1 \), i.e., the \( \left( \binom{b}{2} + (r-1)(M+1) + i \right. \). These are therefore exactly the members of the intersection of \( J_i \) with the \( b^\text{th} \) block of rows. Finally, we have that the elements of the \( J_i \) are the entries in the \( i^\text{th} \) (and last) column in the \( i^\text{th} \) block of rows; i.e., \( J_i \) is the intersection of the \( i^\text{th} \) column with the \( i^\text{th} \) block of rows, and that for \( 1 \leq r \leq M+1 \), \( j_{i,r} = \eta_{b,r,i} = \left( \binom{b}{2} + (r-1)(M+1) + i \right. \).

### 3.2. Complexity Estimates: Theorems 2 and 3.

#### 3.2.1. The sequence \( \{F_n\} \), its natural encoding, \( F \), and Theorem 2.

For the next Definition, recall Definition 8 in (3.1), where we defined the \( b_{n,t,s} \) and \( \alpha(n,t,x) \).

**Definition 9.** For all \( n > 0 \), \( F_n \) is the permutation of \( \{0, \ldots, 2^n(M+1) - 1\} \) defined as follows. If \( 0 \leq \ell < 2^n(M+1) \), let \( t = \text{IStep}(n,\ell) \). Then

\[
F_n(\ell) := b_{n,t,s}, \text{ where } s = \alpha(n,t,\ell).
\]

Then, take \( F : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) to be the natural encoding of the sequence \( \{F_n\} \).

In view of 2. of Lemma 1, these \( F_n \) are (obviously) very natural admissible permutations of the \( \{0, \ldots, 2^n(M+1) - 1\} \).

**Remark 3.** Note that our definition of \( F_n \) is equivalent to stipulating that, in terms of the notation of Equation (8), for all \( 1 \leq t \leq T_n \), \( (F_n)_t \) is the identity permutation of \( \{1, \ldots, \gamma^O_{n,t}\} \). This is one aspect of what, in our eyes, justifies the claim that \( F_n \) is the simplest admissible permutation of \( \{0, \ldots, 2^n(M+1) - 1\} \). Note, also, that with \( t = \text{IStep}(n,\ell) \) and \( s = \alpha(n,t,\ell) \), then, in fact, \( s = 1 + \ell - \text{SCMO}(n,t) \); further, for these \( t, s \), we also have that \( s = \beta(n, t, F(n,\ell)) \). \( \square \)

#### 3.2.2. \( F \) is P-TIME relative to \( \tau^O_1 \).

It was noted in Proposition 4 of [3] that, as functions of \( (n,i) \), with \( i \leq n \), the binomial coefficients \( \binom{n}{i} \) and SBC are computable in time polynomial in \( n \). Building on this, we have:

**Proposition 1.** As functions of \( (n,t) \), with \( 1 \leq t \leq T_n \), the function \( \text{SCMO}^O \) and the function \( \gamma^O_{n,t} \) are computable in time polynomial in \( n \), relative to the function \( \tau^O_1 \). The function \( \text{IStep} \) and the function \( \alpha \) of Equation (7c) (Definition 8) are P-TIME relative to \( \tau^O_1 \).
Proof. First note that, as a function of $\vec{k} \in K_n$ and of $n$, $\binom{n}{\vec{k}}$ is computable in time polynomial in $n$, by Proposition 4 of [3] and the standard expression for the multinomial coefficient as a $m$–fold product of binomial coefficients. The first sentence of the Proposition is then immediate from item (2) of Remark 1.

That IStep is P-TIME relative to $\tau_1^O$ then follows immediately from item (1) of Remark 2, and that $\alpha$ is P-TIME relative to $\tau_1^O$ follows immediately from (the first part of) the final sentence of Remark 3. $\square$

Remark 4. In (a generalization of) the terminology of [3], we have shown that the relation RIA is tame relative to $\tau_1^O$. This is because, as noted, immediately after Definition 8, in the terminology of [3], $\alpha$ is the cardinality function for RIA.

The next Corollary then follows immediately from Proposition 1 and from Lemma 2 of [3].

Corollary 3. The relation RIA is P-TIME decidable relative to $\tau_1^O$. As a function of $(n, t, s)$ with $1 \leq s \leq \alpha_{n,t}$, the function $a_{n,t,s}$ of Equation (7di) (Definition 8) is P-TIME relative to $\tau_1^O$.

Proof. This is simply because, as noted following Definition 8, the function $a_{n,t,s}$ is the enumerating function for RIA. $\square$

We next show, as promised, that the dependence of $\tau_2^O$ on $O$ is really only formal. It is now that we make use of the enumeration, $(\zeta_s | 1 \leq s \leq m)$ of $\{0, 1\}^m$ introduced in the third paragraph of (1.1).

Proposition 2. As a function of $(n, s, \ell, b)$ with $n > 0$, $1 \leq s \leq m$, $0 \leq \ell \leq 2^{n(M+1)}$, $0 \leq b \leq n$, $\tau_2^O$ is computable in time polynomial in $n$.

Proof. $\tau_2^O(n, s, \ell, b)$ is the sum over $i$ such that $b < i < n$ of the characteristic (indicator) function of the relation, $\Theta(n, s, \ell, i)$ which holds iff $(\vec{I}_n(\ell))_i = \alpha_s$. This relation is P-TIME decidable since it holds iff for all $j$ with $1 \leq j \leq M + 1$, $(\vec{B}_n(\ell))_j = (\zeta_s)_j$. $\square$

We now have the analogue of Lemma 3 of [3]. The argument here is considerably more involved and delicate, and requires the information encoded in $\tau_1^O$. As already noted, appeals to $\tau_2^O$ are just a convenience, in view of Proposition 2. Nevertheless, as with its prototype in [8], a key ingredient is S. Buss’s suggestion of (something like) a binary search.

Lemma 2. The function $\beta(n, t, \xi)$ of Definition 8 is P-TIME relative to $\tau_1^O$.

Proof. In analogy with the proof of Lemma 3 of [3], given $n, t$, and $0 \leq x < 2^{n(M+1)}$, we “walk down the 1’s of the binary representation of $\xi$”; here we use the multinomial coefficients to count as we go. More precisely, with

$$\xi = \sum_{\zeta=1}^{n(M+1)} 2^{n(M+1)\zeta} \left( \vec{B}_n(\xi) \right)_\zeta,$$

as in (2.1.3) with $\xi$ in place of $x$,

if $\left( \vec{B}_n(\xi) \right)_\zeta = 0$, we let $\vec{I}_B(n, t, \xi, \zeta) := \emptyset$, while if $\left( \vec{B}_n(\xi) \right)_\zeta = 1$, we let

$$\vec{I}_B(n, t, \xi, \zeta) := \left\{ \ell \in I_B(n, t) \left| \left( \vec{B}_n(\ell) \right)_\zeta = 0 \right. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left.
union is \( IB_{n,t} \cap \{1, \ldots, \xi\} \), we’ll have that:

\[
\beta(n, t, \xi) = \sum_{\zeta = 1}^{n(M+1)} \beta^*(n, t, \xi, \zeta).
\]

Thus, it suffices to show that \( \beta^* \) is \( \text{P-TIME} \) relative to \( \tau_1^D \), which we now undertake. When \( \left( I\bar{b}_n(\xi) \right)_\zeta = 1 \), we consider the superset, \( \overline{C}(n, \xi, \zeta) \), of \( I\bar{b}_{n,t,\xi,\zeta} \) consisting of those \( \ell \) with \( 1 \leq \ell < \xi \) such that \( \left( I\bar{b}_n(\ell) \right)_\zeta = 0 \) and for all \( j < \zeta \), \( \left( I\bar{b}_n(\ell) \right)_j = \left( \overline{I\bar{b}}_n(\xi) \right)_j \). If \( \xi \in IB_{n,t,\xi,\zeta} \), we let \( C(n, \xi, \zeta) := \overline{C}(n, \xi, \zeta) \cup \{\xi\} \); otherwise, we let \( C(n, \xi, \zeta) := \overline{C}(n, \xi, \zeta) \).

For \( \ell \in C(n, \xi, \zeta) \), \( \ell \in IB_{n,t,\xi,\zeta} \) iff \( I\bar{b}_n(\ell) \in K_{n,t} \) iff \( \tau_2^D(n, I\bar{b}_{n,1}(\ell), \ldots, I\bar{b}_{n,m}(\ell), t) = 1 \) (note that if \( \ell = \xi \), then this will automatically be true, by the definition of \( C(n, \xi, \zeta) \)).

For \( n, \xi, \zeta \) as above, and for \( \bar{k} \in K_n \), let:

\[
\tau(n, \xi, \zeta, \bar{k}) := \text{the frequency of } \bar{k} \text{ in } \left( I\bar{b}_n(\ell) | \ell \in \overline{C}(n, \xi, \zeta) \right), \quad \text{and}
\]

\[
c(n, \xi, \zeta, \bar{k}) := \text{the frequency of } \bar{k} \text{ in } \left( I\bar{b}_n(\ell) | \ell \in C(n, \xi, \zeta) \right).
\]

It is then immediate that:

\[
(9) \quad \beta^*(n, t, \xi, \zeta) = \sum_{\bar{k} \in K_n} c(n, \xi, \zeta, \bar{k}) \tau_1^D(n, \bar{k}, t).
\]

It is also clear that \( c(n, \xi, \zeta, \bar{k}) = \tau(n, \xi, \zeta, \bar{k}) \), unless \( \xi \in C(n, \xi, \zeta) \) and \( I\bar{b}_n(\xi) = \bar{k} \); in this case \( c(n, \xi, \zeta, \bar{k}) = \tau(n, \xi, \zeta, \bar{k}) + 1 \). It will, therefore, suffice to show that the function \( \tau \) is \( \text{P-TIME} \), which is the burden of what follows.

Let \( b(\zeta), r(\zeta), i(\zeta) \) be those \( b, r, i \), respectively, such that, as in (3.1.3), \( \zeta = \eta_{b,r,i} \). Clearly, these are \( \text{P-TIME} \) computable from \( \zeta \). Note that if \( \ell \in \overline{C}(n, \xi, \zeta) \) and \( i > b(\zeta) \), then \( (I\bar{b}_n(\ell))_i = (I\bar{b}_n(\xi))_i \). Let \( s(n, \xi, \zeta, \ell) \) be that \( s \) such that \( (I\bar{b}_n(\ell))_\zeta = s \), and note that \( s(n, \xi, \zeta, \ell) \) is also that \( s \) such that \( I\bar{b}_{n,s}(\ell, b(\zeta)) < I\bar{b}_{n,s}(\ell, b(\zeta) - 1) \), i.e., such that \( \tau_2^D(n, s, \ell, b(\zeta)) < \tau_2^D(n, s, \ell, b(\zeta) - 1) \); thus, \( s(n, \xi, \zeta, \ell) \) is polynomial time computable.

Let \( D := \overline{J}_{\overline{b}(\zeta)} \cap \{1, \ldots, \zeta - 1\} \), and let \( d = |D| \). For \( \ell \in \overline{C}(n, \xi, \zeta) \), let \( \sigma_\ell \) be that function \( \sigma \) from \( D \) to \( \{0, 1\} \) defined by \( \sigma(\ell) = \left( I\bar{b}_n(\ell) \right)_\ell \). Clearly \( s(n, \xi, \zeta, \ell) \) depends only on \( \sigma_\ell \), and equally clearly, every \( \sigma \in \{0, 1\}^D \) arises as \( \sigma_\ell \), for some \( \ell \in \overline{C}(n, \xi, \zeta) \). A key step in what follows will be to compute the frequency with which this occurs, but for now, this justifies defining \( s^* : \{0, 1\}^D \to \{1, \ldots, m\} \) by

\[
(10) \quad s^*(\sigma) = s(n, \zeta, \ell) \text{ for any } \ell \in \overline{C}(n, \xi, \zeta) \text{ such that } \sigma = \sigma_\ell.
\]

We are now in a position to prove:

\[
(11) \quad |\overline{C}(n, \xi, \zeta)| = 2^{(b(\zeta)-1)(M+1)+d}.
\]

**Proof. (of Equation (11))** We give an explicit bijection \( h \) from \( \overline{C}(n, \xi, \zeta) \) to the set \( C^* \) of pairs, \( (\ell^*, \sigma) \) with \( 0 \leq \ell^* < 2^{(b(\zeta)-1)(M+1)} \) and \( \sigma \in \{0, 1\}^D \). Given \( \ell \in \overline{C}(n, \xi, \zeta) \), we let \( h_1(\ell) \) be that \( \ell^* \) such that \( E_{n,\ell^*} \subseteq E_{b(\zeta)-1,\ell^*} \), and let \( h_2(\ell) = \sigma_\ell \). Let \( h(\ell) := (h_1(\ell), h_2(\ell)) \). Then, \( h \) is one-to-one, since if \( \ell, \ell' \in \overline{C}(n, \xi, \zeta) \) and \( \ell \neq \ell' \), choosing \( y \in E_{n,\ell^*} \), then for some \( j \in D \cup \{\ell | 1 \leq i < b(\zeta) \} \) and all \( y' \in E_{n,\ell^*}, \ell_2(\zeta) \neq \ell_2(y) \). Fixing such a \( j \), if \( j \in D \), then \( h_2(\ell') \neq h_2(\ell) \), while if \( j \notin D \), then \( h_1(\ell') \neq h_1(\ell) \). But \( h \) is also onto, since if \( 0 \leq \ell^* < 2^{(b(\zeta)-1)(M+1)} \) and \( \sigma : D \to \{0, 1\} \), let \( y^* \in E_{b(\zeta)-1,\ell^*} \), let \( y \in E_{n,\xi} \). We may assume, WLOG, that \( y^* \) has the following properties, since, if necessary, it can be modified to have them, without affecting membership in \( E_{b(\zeta)-1,\ell^*} \):

- \( \varepsilon_j(y^*) = \varepsilon_j(y) \), for all \( j > \zeta \),
- \( \varepsilon_\zeta(y^*) = 0 \),
- \( \varepsilon_j(y^*) = \sigma(j) \), for all \( j \in D \).
Let \( \ell \) be such that \( y^* \in E_{n,\ell} \). By construction, \( \ell \in \overline{C}(n, \xi, \zeta) \) and \( h(\ell) = (\ell^*, \sigma) \).

We will continue to work with the bijection \( h \) constructed in the proof of Equation (11). Let \( \ell \in \overline{C}(n, \xi, \zeta) \), and let \( h(\ell) = (\ell^*, \sigma) \). It is then clear from the construction of \( h \) that for all \( 1 \leq s \leq m \):

\[
I_{k_{b(\zeta)-1,s}}(\ell^*, 0) = \begin{cases}
\tau_2^O(n, s, \ell, 0) - \tau_2^O(n, s, x, b(\zeta)) - 1, & \text{if } s = s(\sigma), \\
\tau_2^O(n, s, \ell, 0) - \tau_2^O(n, s, x, b(\zeta)), & \text{otherwise}.
\end{cases}
\]

It follows from Equation (12) (and the proof of Equation (11)) that whenever \((k_1^{s}, \ldots, k_m^{s}) \in K_{b(\zeta)-1}\), there is some \( \ell \in \overline{C}(n, \xi, \zeta) \) such that (with \( \sigma = \sigma_\ell \)) for all \( 1 \leq s \leq m \), the analogue of Equation (12) holds, with \( k_s^* \) in place of \( I_{k_{b(\zeta)-1,s}}(\ell^*, 0) \). Also for any such \( \bar{k}^* = (k_1^{s}, \ldots, k_m^{s}) \), the frequency of \( \bar{k}^* \) in \( \left( I_{\bar{k}^*_{b(\zeta)-1}}(\ell^*) | 0 \leq \ell^* < 2^{b(\zeta)-1}(M+1) \right) \) is simply \( \beta^*(\bar{k}^*) \).

For \((n, \xi, \zeta, \bar{k}) \in \text{dom} \; \tau, \sigma \in \{0, 1\}^D \), and \( 1 \leq s \leq m \), let:

\[
k_s^*(n, \xi, \zeta, \bar{k}, \sigma) = \begin{cases}
\tau_2^O(n, s, \xi, \zeta, \bar{k}, \sigma) - 1, & \text{if } s = s(\sigma), \\
\tau_2^O(n, s, \xi, \zeta, \bar{k}, \sigma), & \text{otherwise}.
\end{cases}
\]

We say that \((n, \xi, \zeta, \bar{k}, \sigma) \) is \emph{bad} if \((k_1^{s}, \ldots, k_m^{s}) \notin K_{b(\zeta)-1}, \) where \( k_s^* = k_s^*(n, \xi, \zeta, \bar{k}, \sigma) \), for \( s = 1, \ldots, m \); otherwise, \((n, \xi, \zeta, \bar{k}, \sigma) \) is \emph{good}. Note that it is \( \text{P-TIME} \)-decidable whether \((n, \xi, \zeta, \bar{k}, \sigma) \) is bad, and that it is good iff for some \( \ell \in \overline{C}(n, \xi, \zeta) \), \( \bar{k} = I_{\bar{k}^*}(\ell) \) and \( \sigma = \sigma_\ell \). When \((n, \xi, \zeta, \bar{k}, \sigma) \) is good, we also let \( \bar{k}^*(n, \xi, \zeta, \bar{k}, \sigma) = (k_1^{s}, \ldots, k_m^{s}) \), with each \( k_s^* = k_s^*(n, \xi, \zeta, \bar{k}, \sigma) \).

Then, we define:

\[
c^*(n, \xi, \zeta, \bar{k}, \sigma) = \begin{cases}
0, & \text{if } (n, \xi, \zeta, \bar{k}, \sigma) \text{ is bad}, \\
\beta^*(\bar{k}^*) - 1, & \text{otherwise}.
\end{cases}
\]

It is then clear that \( c^* \) is \( P \)-\( \text{TIME} \) and that:

\[
\tau(n, \xi, \zeta, \bar{k}) = \sum_{\sigma \in \{0, 1\}^D} c^*(n, \xi, \zeta, \bar{k}, \sigma).
\]

Thus, as required, \( \tau(n, \xi, \zeta, \bar{k}) \) is also \( \text{P-TIME} \).

It is now, in the light of the proof of Lemma 2, that the “raison d’être” for our choice of the \( P_i \) finally appears clearly. For \( \ell \in \overline{C}(n, \xi, \zeta) \), for \( i > b(\zeta) \), the \( (I0_i(\ell))_i \) are completely determined (by \( \xi \)) , while for \( i < b(\zeta) \), the \( (I0_i(\ell))_i \) are completely arbitrary. It is only \( (I0_{\ell}(\ell))_{\ell(\zeta)} \) which is “partially determined”, and the partial determination (by \( \xi \)) is completed by the \( \sigma_\ell \). This is the key to the proof of Equation (11), from which the rest of the proof follows easily.

Omitting the analogue of Remark 4, we proceed directly to the analogue of Corollary 3.

**Corollary 4.** The relation \( RIB \) is \( \text{P-TIME} \)-decidable relative to \( \tau_1^D \). As a function of \((n, t, s)\) with \( 1 \leq s \leq \beta_{n,t} \), the function \( b_{n,t,s} \) of Equation (7dii) (Definition 8) is \( \text{P-TIME} \)-relative to \( \tau_1^D \).

**Proof.** This follows from Lemma 2 and Lemma 2 of [3], because \( b \) is an enumerating function of \( RIB \) while \( \beta \) is its cardinality function, so Lemma 2, above, establishes that \( RIB \) is tame. Since it is tame, it is \( \text{P-TIME} \)-decidable, by Lemma 2 of [3].

**Theorem 2.** \( F \) is \( \text{P-TIME} \)-relative to \( \tau_1^D \).

**Proof.** Let \( \chi \) denote the characteristic function of the relation \( RIB \). Let \( \Gamma \) be the three place relation on \( N \) such that \( \Gamma(n, \ell, \ell') \) holds iff \( n > 0 \), \( \ell, \ell' < 2^{n(M+1)} \) and

\[
\beta(n, I\text{Step}(n, \ell), \ell') \cdot \chi(n, I\text{Step}(n, \ell), \ell') = \alpha(n, I\text{Step}(n, \ell), \ell).
\]

Much as in [3] (where the analogue of Equation (16) is Equation (7)), we show, first, that \( \Gamma \) is \( \text{P-TIME} \)-decidable relative to \( \tau_1^D \) and, given \( n > 0 \) and \( \ell < 2^{n(M+1)} \) there is unique \( \ell' \) such that
\(\Gamma(n, \ell, \ell')\) holds. We next argue that, as a function of \((n, \ell)\), the unique solution, \(\ell'\), is computable relative to \(\tau_1^O\) in time polynomial in \(n\). Finally, we show that \(\Gamma(n, \ell, \ell')\) holds iff \(\ell' = F(n, \ell)\). Of course this will suffice to show that.

By Proposition 1, the functions \(\alpha\) and \(\text{IStep}\) are \(P\)-TIME relative to \(\tau_1^O\), and by Lemma 2, the function \(\beta\) is, as well. Since \(\text{RIB}\) is \(P\)-TIME decidable, the function \(\chi\) is \(P\)-TIME. It is then clear that \(\Gamma\) is \(P\)-TIME decidable.

Now suppose \(n > 0\) and \(\ell < 2^{n(M+1)}\). Let \(t = \text{IStep}(n, k)\) and, as in Remark 3, let \(s = 1 + \ell - \text{SMC}^O(n, t)\). Note that \(\ell = a_{n, t, s}\) (and that \(s = \alpha(n, t, \ell)\), so, in particular, \(\alpha(n, t, \ell) > 0\)). By part (2) of Lemma 1, \(s \leq \beta_{n, t}\), so \(b_{n, t, s} \in I B_{n,t}\). Of course we have that \(s = \beta(b_{n, t, s})\). It is then clear that \(\Gamma(n, t, b_{n, t, s})\) and so there is a solution \(\ell'\) to \(\Gamma(n, t, \ell')\). As was the case in [3], it is the fact of multiplying by \(\chi(n, t, \ell)\) that makes \(\ell' = b_{n, t, s}\) the unique solution, since this guarantees that if \(\Gamma(n, t, \ell'')\) holds then \(\ell'' \in IB_{n,t}\). Given this, and that \(s = \alpha(n, t, \ell) = \beta(n, t, \ell')\) it is then clear that \(\ell'' = \ell' = b_{n, t, s}\). This also makes it clear that, as a function of \((n, \ell)\), the unique solution, \(\ell'\), is computable relative to \(\tau_1^O\) in time polynomial in \(n\): this is simply because the function \(b\) is \(P\)-TIME relative to \(\tau_1^O\) (by Corollary 4), while \(t\) and \(s\) are also computable in time polynomial in \(n\), relative to \(\tau_1^O\), because \(t = \text{IStep}(n, \ell)\) and \(s = 1 + \ell - \text{SMC}^O(n, t)\).

Combining Corollary 1 and Theorem 2 immediately gives:

**Theorem 3.** The trim strong triangular array representation of \(\{S_n^*\}\) corresponding to the sequence of admissible permutations encoded by \(F\) is \(P\)-TIME.

3.3. **Concluding Remarks: Retrospective Comparisons with [3].** The list of results of [3] for which we have presented no clear analogues in this paper is as follows:

1. Propositions 1 and 2 in (3.2) of [3].
2. All of the numbered items of §4 of [3].
3. Corollary 4 and Remark 5 of (3.5) of [3].
4. Proposition 3 of (3.3) of [3], Equation (8) of (3.4) of [3].

Of these, the results listed in items 1., 2., above, do have clear analogues. Further, the proofs from [3] go over in a straightforward fashion. Nevertheless, we have chosen to omit them on the grounds that the important point had already been made in the setting of [3]. For the sake of completeness, we recall that Proposition 1 of [3] shows that “trimness does not come for free” by constructing a non-trim strong triangular array representations starting from a trim strong one, while Proposition 2 of [3] is a “non-persistence” result, showing that in any sequence, \((\pi_n|n \in \mathbb{Z}^+)\), of admissible permutations and any \(n\), it is never true that \(\pi_{n+1}\) extends \(\pi_n\) and that in any trim strong triangular array representation of \(\{S_n^*\}\), it is never true that all of the random variables \(R_{n,i}^*\) in the “\(n^{th}\) row reappear among the \(R_{n+1,i}^*\) in the \(n + 1^{th}\) row”. The material of §4 of [3] dealt with the question of constructing variants, \((G_n|n \in \mathbb{Z}^+)\) and \((H_n|n \in \mathbb{Z}^+)\) of (that paper’s version of) the sequence \((F_n|n \in \mathbb{Z}^+)\). The motivation for the variants is to build some additional desirable properties into the individual \(G_n\) and \(H_n\); for example, the \(G_n\) have the property that if (in the notation of [3]) \(\ell \in A_{n,i} \cap B_{n,i}\), then \(G_n(\ell) = \ell\).

The situation is different, unfortunately, regarding the results listed in item 4. above. These results involve obtaining a simple expression for the base-2 exponential function in terms of any sequence \((\pi_n|n \in \mathbb{Z}^+)\), of admissible permutations (Proposition 3 of [3]) and an even simpler one in terms of the sequence \((F_n|n \in \mathbb{Z}^+)\) (Equation (8) of [3] for the [3] version of this sequence). There are no clear analogues of these results, although there is a clear analogue of the method by which they were obtained in [3]. The basis of that method was the equivalence between being a power of 2 and having (Hamming) weight 1 (and so being a member of \(B_{n,1}\)). The analogue of the approach taken in [3] for a general sequence, \((\pi_n|n \in \mathbb{Z}^+)\), of admissible permutations is to consider the increment by 1 of the sum of all of the weight 1 elements, i.e., of all of the members of \(B_{n,1}\) (note: \(\gamma_{n,1}^O\) is still equal to \(n\) as is easy to see). The problem is that this may no longer be
the sum of the pure 2-powers: \( \sum_{s=1}^{n} \pi_s(n) \) is still the natural thing to consider, but it will depend on \( O \) in ways that make it difficult to identify what function this gives (though it seems quite plausible that it is at least as complex as the exponential function with base 2). Here it is, once again, the absence of a transparent connection between the notion of Weight and the patterns of 1’s in binary expansions.

Finally, concerning the results listed in item (3), above, the content of Corollary 4 of [3] is that the function SBC of that paper can be simply computed from \( F \) and Inv\( F \), the (natural encoding of the) sequence of inverses, \( \{ F_{n}^{-1} \} \). The natural analogue, here would have the function \( \tau^O_1 \) in place of SBC, and we conjecture that this is true, but we have yet to work this out. Whether or not SBC can be obtained from \( F \) alone (or, somewhat more plausibly, from Inv\( F \) alone is the subject of the speculative Remark 5 of [3]; similar questions arise naturally here, but, here too, we have yet to work this out.

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