Agnostic Online Learning and Excellent Sets

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Abstract

We use algorithmic methods from online learning to revisit a key idea from the interaction of model theory and combinatorics, the existence of large “indivisible” sets, called “ε-excellent,” in k-edge stable graphs (equivalently, Littlestone classes). These sets arise in the Stable Regularity Lemma, a theorem characterizing the appearance of irregular pairs in Szemerédi’s celebrated Regularity Lemma. Translating to the language of probability, we find a quite different existence proof for ε-excellent sets in Littlestone classes, using regret bounds in online learning. This proof applies to any ε < 1/2, compared to < 1/2k or so in the original proof. We include a second proof using closure properties and the VC theorem, with other advantages but weaker bounds. As a simple corollary, the Littlestone dimension remains finite under some natural modifications to the definition. A theme in these proofs is the interaction of two abstract notions of majority, arising from measure, and from rank or dimension; we prove that these densely often coincide and that this is characteristic of Littlestone (stable) classes. The last section lists several open problems.

1 Overview

In this section we briefly present three complementary points of view (combinatorics §1.1, online learning §1.2, model theory §1.3) which inform this work, and state our main results in §1.4. The aim is to allow the paper to be readable by people in all three communities and hopefully to stimulate further interaction.

In the recent papers [4], [7] ideas from model theory played a role in the conjecture, and then the proof, that Littlestone classes are precisely those which can be PAC learned in a differentially private way (we direct the reader to the introductions of those papers for precise statements and further literature review). The present work may be seen as complementary to those papers in that it shows, perhaps even more surprisingly, that ideas and techniques can travel profitably in the other direction.

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1.1 Context from combinatorics

Szemerédi’s celebrated Regularity Lemma for finite graphs says essentially that any huge finite graph $G$ can be well approximated by a much smaller random graph. The lemma gives a partition of any such $G$ into pieces of essentially equal size so that edges are distributed uniformly between most pairs of pieces (“$\epsilon$-regularity”). Szemerédi’s original proof allowed for some pairs to be irregular, and he asked if this was necessary \[23\]. As described in \[12 \S1.8\], it was observed by several researchers including Alon, Duke, Leffman, Rödl and Yuster \[3\] and Lovász, Seymour, Trotter that irregular pairs are unavoidable due to the counterexample of half-graphs. A $k$-half graph has distinct vertices $a_1, \ldots, a_k, b_1, \ldots, b_k$ such that there is an edge between $(a_i, b_j)$ if and only if $i < j$.

Malliaris and Shelah showed that half-graphs characterize the existence of irregular pairs in Szemerédi’s lemma, by proving a stronger regularity lemma for $k$-edge stable graphs called the Stable Regularity Lemma \[16\]. (A graph is called $k$-edge stable if it contains no $k$-half graph.\) A central idea in the stable regularity lemma was that $k$-edge stability for small $k$ means it is possible to find large “indivisible” sets, so-called $\epsilon$-excellent sets. A partition into sets of this kind is quickly seen to have no irregular pairs (for a related $\epsilon$). For an exposition of this proof, and the model theoretic ideas behind it, see \[17\].

To recall the definition, let $0 < \epsilon < \frac{1}{2}$. Let $G = (V, E)$ be a finite graph. Following \[16\], say $B \subseteq V$ is $\epsilon$-good if for any $a \in V$, one of $\{b \in B : (a, b) \in E\}, \{b \in B : (a, b) \notin E\}$ has size $< \epsilon |B|$. If the first [most $b \in B$ connect to $a$], write $t(a, B) = 1$, and if the second [most $b \in B$ do not connect to $a$] write $t(a, B) = 0$. Say that $A \subseteq V$ is $\epsilon$-excellent if for any $B \subseteq V$ which is $\epsilon$-good, one of $\{a \in A : t(a, B) = 1\}, \{a \in A : t(a, B) = 0\}$ has size $< \epsilon |A|$. Informally, any $a \in A$ has a majority opinion about any $\epsilon$-good $B$ by definition of good, and excellence says that additionally, a majority of elements of $A$ have the same majority opinion. Observe that if $A$ is $\epsilon$-excellent it is $\epsilon$-good, because any set of size one is $\epsilon$-good.

Notice that while, e.g., $\frac{1}{4}$-good implies $\frac{1}{8}$-good, the same is a priori not true for $\epsilon$-excellent, because the definition of $\epsilon$-excellence quantifies over $\epsilon$-good sets. For the stable regularity lemma, it was sufficient to show that large $\epsilon$-excellent sets exist in $k$-edge stable graphs for $\epsilon < \frac{1}{2^k}$ or so. In this language, one contribution of the present paper is a new proof for existence of $\epsilon$-excellent sets in $k$-edge stable graphs, which works for any $\epsilon < \frac{1}{2^k}$, i.e. any $\epsilon$ for which excellence is well defined.

1.2 Context from online learning

The online learning setting shifts the basic context from graphs to hypothesis classes, i.e. pairs $(X, \mathcal{H})$ where $X$ is a finite or infinite set and $\mathcal{H} \subseteq \mathcal{P}(X)$ is a set of subsets of $X$, called hypotheses or predictors. We will identify elements $h \in \mathcal{H}$ with their characteristic functions, and write “$h(x) = 1$” for “$x \in h$” and $h(x) = 0$ otherwise. \[It\] is also usual to consider $h$ as having range $\{-1, 1\}$\. Any such hypothesis class can be naturally viewed as a bipartite graph on the disjoint sets of vertices $X$ and $\mathcal{H}$ with an edge between $x \in X$ and $h \in \mathcal{H}$ if and only if $h(x) = 1$. However, something which is a priori lost in this translation is a powerful understanding in the computer science community of the role of dynamic/adaptive/predictive arguments. This perspective is an important contribution to the proofs below, and seems to highlight some understanding currently

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1The Stable Regularity Lemma says that a finite $k$-edge stable graph can be equipartitioned into $\leq m$ pieces (where $m$ is polynomial in $\frac{1}{\epsilon^2}$) such that all pairs of pieces are regular, with densities close to 0 or 1. Note that two of these conditions, the improved size of the partition and the densities of regular pairs being close to 0 or 1, are already expected from finite VC dimension, see \[3, 15\], though here by a different proof.
missing in the other contexts.

A mistake tree is a binary decision tree whose internal nodes are labeled by elements of $X$. We can think of the process of traversing a root-to-leaf path in a mistake tree of height $d$ as being described by a sequence of pairs $(x_i, y_i) \in X \times \{0, 1\}$ (for $1 \leq i \leq d$) recording that at step $i$ the node we are at is labeled by $x_i$ and we then travel right (if $y_i = 1$) or left (if $y_i = 0$) to a node labeled by $x_{i+1}$, and so on. Say that $h \in H$ realizes a given branch (root-to-leaf path) $(x_1, y_1), \ldots, (x_d, y_d)$ if $h(x_i) = y_i$ for all $1 \leq i \leq d$, and say that a given mistake tree is shattered by $H$ if each branch is realized by some $h \in H$. The Littlestone dimension of $H$, denoted $\text{Ldim}(H)$, is the depth of the largest complete [binary] tree that is shattered by $H$. $H$ is called a Littlestone class if it has finite Littlestone dimension; for reasons explained in the next subsection, we may prefer to say that $(X, H)$ is a Littlestone pair. Littlestone [13] and Ben-David, Pál, and Shalev-Shwartz [3] proved that $\text{Ldim}$ characterizes online learnability of the class.

A basic setting within online learning is that of prediction with expert advice. Informally, we are given a set $X$ and a set of experts $H$; an expert may be oblivious in which case it is a function $h$ from $X$ to $\{0, 1\}$, or adaptive in which case it may change its prediction based on the past. (So, an oblivious expert always predicts according to the same rule $h : X \to \{0, 1\}$, whereas an adaptive experts may change its prediction rule based on past experience.)

The following repeated game is than played between two players called learner and adversary (denoted $L$ and $A$ respectively): in each round $t = 1, \ldots, T$ the adversary $A$ poses a query $x_t \in X$ to $L$; the learner $L$ then responds with a prediction $\hat{y}_t$, and the round ends with the adversary revealing the “true” label $y_t$. The goal of $L$ is to make as few mistakes as possible: specifically to minimize the regret with respect to the set of experts $H$, which is defined as the difference between the number of mistakes $L$ makes and the number of mistakes the best expert in $H$ makes.

A fundamental theorem in the area uses the Littlestone dimension to characterize the optimal (minimal) possible regret that can be achieved when the class $H$ contains only oblivious experts [13, 5, 20, 19, 18, 11]. This characterization hinges on an important property of Littlestone dimension which informally asserts that every (possibly infinite) Littlestone class can be covered by a finite number of adaptive experts. More formally: for every Littlestone class $H$ there exists a finite class of adaptive experts $E = E(H, T)$ such that every for every $x_1, \ldots, x_T \in X$ and every $h \in H$ there exists an expert in $E$ which given an input sequence $x_1, \ldots, x_T$, produces labels $y_1 = h(x_1), \ldots, y_T = h(x_T)$. This property essentially reduces any (possibly infinite) Littlestone class to a finite one. Below, in Theorem 3.1 we prove a mild extension of this important property.

1.3 Context from model theory

Consider again the case of a finite bipartite graph $G$ with vertex set $X \cup Y$ and edge relation $R$ (abstracting the study of a formula $\varphi(\bar{x}, \bar{y})$). Following logical notation we write $R(a, b)$ or $\neg R(a, b)$ to denote an edge or a non-edge. Define a [full] special tree of height $n$ to have internal nodes \{a_\eta : \eta \in {}^n \mathbb{2}\} from $X$ and indexed by binary sequences of length $\leq n$ and leaves \{b_\rho : \rho \in {}^n \mathbb{2}\} from $Y$ and indexed by binary sequences of length exactly $n$, which satisfy the following:\footnote{In general (in the fully agnostic setting), any optimal learner must be randomized and use random predictions $\hat{y}_t$, in this case the adversary only sees the expected value of $\hat{y}_t$. (I.e. the randomness of the learner is private and inaccessible by the adversary.)} For any $a_\eta$ and $b_\rho$, if $\eta$ is an initial segment of $\rho$ (notation: $\eta \leq \rho$), then $R(a_\eta, b_\rho)$ if $\eta^\frown(1) \leq \rho$ and $\neg R(a_\eta, b_\rho)$ if $\eta^\frown(0) \leq \rho$. A key ingredient in the proof of the Stable Regularity Lemma was the following special

\footnote{On this notation, see Convention 2.50}
case of Shelah’s Unstable Formula Theorem [22, II.2]. (The bounds are due to Hodges [11].) For a bipartite graph $G$ as above, if $G$ has a full special tree of height $n$, then it has a half-graph of size about $\log n$, with the $a$’s chosen from $X$ and the $b$’s chosen from $Y$ (i.e., it is not $k$-edge stable in the sense above). Moreover, if $G$ has a half-graph of size $k$, it has a full special tree of height about $\log k$.

It was noticed by Chase and Freitag [8] that the condition of model-theoretic stability (Shelah’s 2-rank; e.g., in this language, no full special tree of height $n$ for some finite $n$) corresponds to finite Littlestone dimension, and they used this to give natural examples of Littlestone classes using stable theories.

The following discussion reflects an understanding developed in the learning theoretic papers Alon-Livni-Malliaris-Moran [4], Bun-Livni-Moran [7], where model theoretic ideas had played a role in the proof that the Littlestone classes are precisely those which can be PAC-learned in a differentially private way. One contribution of the model theoretic point of view for online learning in general, and for our present argument in particular, is that a condition in online learning which appears inherently asymmetric, namely the Littlestone dimension (it treats elements and hypotheses as different kinds of objects; they play fairly different roles in the partitioning) is equivalent to a condition which is extremely symmetric, namely existence of half-graphs (when switching the roles of $X$ and $Y$ in a half-graph, it suffices to rotate the picture). Thus if $\mathcal{H}$ is a Littlestone class, the “dual” class obtained by setting $X' = \mathcal{H}$ and $\mathcal{H}' = \{ h \in \mathcal{H} : h(x) = 1 \} : x \in X$ is also. In online learning, $k$-edge stability also has a natural meaning: Threshold dimension $k$, that is, there do not exist elements $a_1, \ldots, a_k$ from $X$ and hypotheses $h_1, \ldots h_k$ from $\mathcal{H}$ such that $h_j(a_i) = 1$ if and only if $i < j$. In what follows, we sometimes refer to $(X, \mathcal{H})$ as a Littlestone pair, rather than simply saying that $\mathcal{H}$ is a Littlestone class, to emphasize this line of thought.

1.4 Results of the paper

The plan for the rest of the paper is as follows. In section two, we spell out the translation from [16] of good and excellent to Littlestone classes and explain why existence of nontrivial good sets characterizes such classes. For the rest of the paper we work primarily in the language of online learning with hypothesis classes $(X, \mathcal{H})$. In section three, we prove a dynamic Sauer-Shelah-Perles lemma for Littlestone classes in the form we will need (a mild extension of Lemma 12 in [5]). The theorem says that given any Littlestone class $\mathcal{H}$ of dimension $d$ and $T \in \mathbb{N}$ there exists a collection $A$ of $\binom{T}{d}$ algorithms such that for every binary tree $T$ with $X$-labeled internal nodes, every branch of $T$ which is realized by some $h \in \mathcal{H}$ is also realized by some algorithm from $A$.

Section four gives the existence proof for $\epsilon$-excellent sets in Littlestone classes for any $\epsilon < \frac{1}{2}$ using regret bounds. We first translate the definitions to the language of probability and define an $\epsilon$-good tree to be a binary tree whose nodes are labelled by $\epsilon$-good distributions over $X$. The section’s main theorem says that an $\epsilon$-good complete binary tree of finite depth $T$ which is shattered by $\mathcal{H}$ (naturally defined) witnesses a lower bound on the regret of any online learning algorithm. This is used to show that if $\mathcal{H}$ is a Littlestone class of dimension $d$ then the maximum possible depth of a complete $\epsilon$-good tree which is shattered by $\mathcal{H}$ is about $d/(\epsilon^2 - \epsilon^3)$ times a universal constant. In section five we give an alternate argument for bounding the maximal height of an $\epsilon$-good tree shattered by a Littlestone class $\mathcal{H}$ using some closure properties of Littlestone classes and the VC theorem. In section six, we analyze two a priori strengthenings of Littlestone dimension, which depend on a choice of $\epsilon$: the “virtual” Littlestone dimension considers $\epsilon$-good sets to be essentially virtual elements and asks that the maximal height of a tree labeled by actual or virtual elements be
finite, and the “approximate” Littlestone dimension asks that the maximal height of an \( X \)-labeled tree shattered by \( H \) be finite even when we allow for a small fraction of mistaken edges in the definition of “\( h \) realizes a branch.” As a corollary of earlier sections, these are both finite exactly when the original \( \text{Ldim} \) is finite (though we do not investigate the distance). In section seven, we compare the notion of majority arising from measure (such as good, excellent) and from rank or dimension (such as \( \text{Ldim} \)). We give some basic axioms for a notion of largeness and prove that any class admitting such a notion contains nontrivial sets on which a given notion of majority arising from measure and the notion of majority arising from largeness are both well defined and agree. Moreover, this characterizes Littlestone classes. Section eight contains several open questions suggested by these proofs.

2 Prior results and a characterization

The original facts about good and excellent sets in [16] were proved for \( k \)-edge stable graphs; the translation to Littlestone classes is immediate, but we record this here for completeness.

Definitions. Let \( 0 < \epsilon < \frac{1}{2} \) and let \((X, H)\) be a hypothesis class.

1 Say \( B \subseteq X \) is \( \epsilon \)-good if for any \( h \in H \), one of \( \{b \in B : h(b) = 1\} \), \( \{b \in B : h(x) = 0\} \) has size \( < \epsilon |B| \). Write \( t(h, B) = 1 \) in the first case, \( t(h, B) = 0 \) in the second.

2 Say that \( H \subseteq H \) is \( \epsilon \)-excellent if for any \( B \subseteq X \) which is \( \epsilon \)-good, one of \( \{h \in H : t(h, B) = 1\} \), \( \{h \in H : t(h, B) = 0\} \) has size \( < \epsilon |H| \). Write \( t(H, B) = 1 \) in the first case, \( t(H, B) = 0 \) in the second.

3 Define “\( H \) is an \( \epsilon \)-good subset of \( H \)” and “\( A \) is an \( \epsilon \)-excellent subset of \( X \)” in the parallel way switching the roles of \( X \) and \( H \).

Remark 2.1. The definition of \( \epsilon \)-good is monotonic in \( \epsilon \): it becomes weaker as \( \epsilon \) increases (below \( \frac{1}{2} \)). This is a priori not the case for excellence, since as \( \epsilon \) becomes larger the collection of \( \epsilon \)-good sets \( B \) quantified over may increase.

Fact 2.2 ([16] Claim 5.4 or [17] Claim 1.8, in our language). Suppose \((X, H)\) is a Littlestone class of dimension \( d \) and \( 0 < \epsilon < \frac{1}{2^d} \). Then for any finite \( H \subseteq H \) there exists \( A \subseteq H \), \( |A| \geq \epsilon^d |H| \) such that \( A \) is \( \epsilon \)-excellent.

The proof of fact 2.2 proceeds by noting that if \( H = H_\emptyset \) is not \( \epsilon \)-excellent, then there is some \( \epsilon \)-good \( A = A_\emptyset \) which witnesses this failure, splitting \( H_\emptyset \) naturally into \( H_{(0)} \) and \( H_{(1)} \) according to \( \mathbf{t} \). If either of these is excellent, we stop; if not, continue inductively to label the internal nodes and leaves of a full binary tree with \( A \)’s and \( H \)’s respectively. Suppose we arrive to height \( d \). To extract a Littlestone tree, or equivalently a full special tree (p. 3 above), choose a \( h_\rho \) from each \( H_\rho \) and then show it is possible to choose a suitable \( a_\eta \) from each \( A_\eta \) by using \( \epsilon < 2^{-d} \), the definition of good (really, of \( \mathbf{t} \)) and the union bound. Since this contradicts \( \text{Ldim}(H) = d \), at some earlier level some \( H_\rho \) must become excellent.

Note moreover that the same proof works to show existence of \( \epsilon \)-good sets, simply by taking the sets \( A \) to be singletons (note that any singleton is trivially \( \epsilon \)-good). In this case, the union bound is not needed and the proof works for any \( \epsilon < \frac{1}{2} \).
Fact 2.3. Suppose $(X, \mathcal{H})$ is a Littlestone class, $\text{Ldim}(\mathcal{H}) = d$ and $0 < \epsilon < \frac{1}{2}$. Then for every finite $A \subseteq \mathcal{H}$ there exists $B \subseteq A$, $|B| \geq \epsilon^d|A|$ such that $B$ is $\epsilon$-good.

We conclude this section by observing that existence of large good sets (both in $X$ and in $\mathcal{H}$) is characteristic of Littlestone classes. At this point, a similar result for excellence could also be stated, for some or for all sufficiently small $\epsilon$.

Claim 2.4. The following are equivalent for any hypothesis class $(X, \mathcal{H})$.

1. For every $\epsilon < \frac{1}{2}$ there is a constant $c = c(\epsilon) > 0$ such that for every finite $A \subseteq \mathcal{H}$ there exists $B \subseteq A$, $|B| \geq c|A|$ such that $B$ is $\epsilon$-good.

2. For some $\epsilon < \frac{1}{2}$ and some constant $c > 0$, for every finite $A \subseteq \mathcal{H}$ there exists $B \subseteq A$, $|B| \geq c|A|$ such that $B$ is $\epsilon$-good.

3. $\mathcal{H}$ is a Littlestone class.

4. For every $\epsilon < \frac{1}{2}$ there is a constant $c = c(\epsilon) > 0$ such that for every finite $A \subseteq X$ there exists $B \subseteq A$, $|B| \geq c|A|$ such that $B$ is $\epsilon$-good.

5. For some $\epsilon < \frac{1}{2}$ and some constant $c > 0$, for every finite $A \subseteq X$ there exists $B \subseteq A$, $|B| \geq c|A|$ such that $B$ is $\epsilon$-good.

Proof. By the remarks above (on the unstable formula theorem), $(X, \mathcal{H})$ is a Littlestone class if and only if the “dual class” $X' = \mathcal{H}$ and $\mathcal{H}' = \{\{h \in \mathcal{H} : h(x) = 1\} : x \in X\}$ is also. So it suffices to prove that (1), (2), (3) are equivalent, and equivalence of (3), (4), (5) will hold by a parallel argument.

(1) implies (2) is immediate, and (3) implies (1) is Fact 2.3. It remains to prove (2) implies (3).

Suppose we are given $\epsilon$ and $c$ from (2). Since any finite hypothesis class is necessarily Littlestone, we may assume $X$ is infinite. Choose $n$ large enough so that $\lfloor \epsilon cn \rfloor \geq 1$ and so that for any $k \geq \lceil cn \rceil$, we have $\min\{|\frac{k}{2}| - 1, \lceil \frac{k+1}{2} \rceil \} \geq ek$. If $\mathcal{H}$ is not a Littlestone class, then we know it has infinite (i.e. not finite) Threshold dimension and so for our chosen $n$, there are elements $\{x_i : i < n\}$ from $X$ and $H := \{h_j : j < n\}$ from $\mathcal{H}$ such that $x_i \in h_j$ if and only if $i < j$. But for any $H' \subseteq H$ of size $k \geq cn$, we can pick out all the cuts of $H'$ using the $x_i$'s. In particular, if $H' = \{h_{i_\ell} : \ell < k\}$, let $m = \lceil ek \rceil$ (by choice of $n$, this is not larger than whichever of $k/2$ or $(k-1)/2$ is an integer). Then $x_m$ partitions $H'$ into $\{h_{i_\ell} : 0 \leq \ell < m\}$ and $\{h_{i_\ell} : m \leq \ell < k\}$, both of which have size $\geq ek$, contradicting its being $\epsilon$-good.

\[ \Box \]

Convention 2.5. We clarify some notational points which hopefully will not cause confusion if explicitly pointed out.

- The word “label” in online learning usually refers to a value such as 0 or 1 attached to an element of $X$. Although we will generally follow this, we also use phrases such as “$X$-labeled tree” to mean a tree in which we associate to each node an element of $X$.

- In set theoretic notation, for each integer $n$, $n = \{0, \ldots, n - 1\}$. Also, $y^x$ denotes the set of functions from $x$ to $y$, as distinguished from $y^x$ which is the size of the set of functions from $x$ to $y$. In this notation, a tree of height $T$ has levels $0$ to $T - 1$, whereas in online learning the same tree would have levels $1$ to $T$.  

6
3 Dynamic Sauer-Shelah-Perles lemmas for Littlestone classes

We take the occasion to state and prove a variant of the celebrated Sauer-Shelah-Perles (SSP) lemma [21] which is adapted to Littlestone classes; this is a mild extension of versions known in the literature. The use of this lemma could be circumvented below by quoting prior work, but it seems to us possibly fruitful for future interactions of online learning, model theory, and combinatorics, and so worth spelling out. Let \( H \) be a class with Littlestone dimension \( d < \infty \).

Two results which could be considered variants of the SSP lemma are known for Littlestone classes: the first one, observed by Bhaskar [6] provides an upper bound of \( \binom{T}{\leq d} \) on the number of leaves in a binary-tree of height \( T \) with \( X \)-labeled nodes that are reachable by \( H \). The second, dynamic version is due to Ben-David, Pal, and Shalev-Shwartz [5]. This lemma is a key ingredient in the characterization of (agnostic) online-learnability by Littlestone dimension; it asserts the existence of \( \binom{T}{\leq d} \) online algorithms (or experts or dynamic-sets) such that for every sequence \( x_1, \ldots, x_T \) and for every \( h \in H \) there exists an algorithm among the \( \binom{T}{\leq d} \) algorithms which produces the labels \( h(x_1), \ldots, h(x_T) \) when given the sequence \( x_1, \ldots, x_T \) as input. The version we shall prove is a mild extension of the Ben-David, Pal, Shalev-Shwartz lemma to the case of trees.

After stating the result, we define the key terms and then prove the characterization.

**Theorem 3.1.** Let \((X, H)\) be a Littlestone class of dimension \( d \). For every \( T \in \mathbb{N} \) there exists a collection \( A \) of \( \binom{T}{\leq d} \) algorithms (dynamic sets) such that for every binary tree \( T \) of height \( T \) with \( X \)-labeled internal nodes, every branch in \( T \) which is realized by some \( h \in H \) is also realized by some algorithm from \( A \).

**Remark 3.2.** For simplicity, we define dynamic sets to be deterministic, but it is also reasonable for them to be random. In general, a randomized algorithm is simply a distribution over deterministic algorithms. When a randomized algorithm is a distribution over prefix-dependent deterministic algorithms, see below, then we may say it is prefix-dependent.

**Definition 3.3** (In the language of online learning). Fix \( T \in \mathbb{N} \) and a set \( X \). A dynamic set (or adaptive expert) \( A \) is a function which assigns to each internal node in each \( X \)-labeled binary tree of height \( \leq T \) a value in \( \{0, 1\} \) in a prefix-dependent way.

To explain, notice that \( A \) naturally defines a walk in any such tree: it starts at the root which is labeled by some \( a_0 =: a_0 \), it outputs \( A(\langle a_0 \rangle) =: t_0 \), then travels left (if its output was 0) or right (if its output was 1) to a node labeled by \( a_{\langle t_0 \rangle} =: a_1 \), where it outputs \( A(\langle a_0, a_1 \rangle) =: t_1 \) and so on. Prefix-dependence means that for any \( \ell \leq T \), if in two different trees the sequences of values \( a_0, \ldots, a_\ell \) produced in this way are the same, then also the output \( t_\ell \) of \( A \) in both cases is the same.

It should now be clear what it means for an algorithm \( A \) to realize a branch in a tree (the directions it gives instruct us to walk along this root-to-leaf path). Note that we can think of each \( h \in H \) as a very simple dynamic set in its guise as a characteristic function.

**Remark 3.4.** In online learning one distinguishes between adaptive and oblivious experts (or between experts with and without memory): an oblivious expert is simply an \( X \to \{0, 1\} \) function, whereas an adaptive expert has memory and can change its prediction based on previous observations. The above definition captures adaptive experts. The above definition slightly deviates from the standard definition of adaptive experts. In the standard definition, one usually only considers sequences (or oblivious trees), rather than general trees. Notice that the distinction between oblivious and general trees can be expressed analogously with respect to the adversary: the adversary,
who presents the examples to the online learner, can be oblivious – in which case it decided on the sequence of examples in advance, or it can be adaptive – in which case it decides which example to present at time \( t \) based on the predictions the online learning algorithm made up to time \( t \). In this language, our version of the dynamic SSP applies also to adaptive adversaries, whereas the previous version was restricted to oblivious adversaries.

**Definition 3.5** (In more set-theoretic language). Let \( T \in \mathbb{N} \) and \( \kappa = |X|^T \). Consider the set \( \mathcal{E} = \{e_i : i < \kappa\} \) of all \( T \)-element sequences of elements of \( X \). A dynamic set assigns to each enumeration \( e_i \) a function \( f_i : T \to \{0,1\} \), and the assignment must be coherent in the sense that if \( e_i \upharpoonright \beta = e_j \upharpoonright \beta \) then \( f_i \upharpoonright \beta = f_j \upharpoonright \beta \).

**Example 3.6.** Let \( X = \mathbb{N} \). Let \( A \) be the algorithm which receives \( a_t \) at time \( t \) and outputs \( 1 \) if \( a_t \) is the largest prime it has seen so far and \( 0 \) otherwise.

**Definition 3.7.** Given a possibly partial characteristic function \( g \) with \( \text{dom}(g) \subseteq X \) and \( \text{range}(g) \subseteq \{0,1\} \), define \( H_g = \{ h \in H : g \subseteq h \} \).

**Remark 3.8.** Observe that if \( H \) is a Littlestone class, \( H \subseteq H \) and \( f \) is a possibly partial, possibly empty characteristic function with \( \text{dom}(f) \subseteq X \) and \( a \in X \), then

\[
(*) \quad \text{min}\{\text{Ldim}(H_{f \cup \{(a,0)\}}), \text{Ldim}(H_{f \cup \{(a,1)\}})\} < \text{Ldim}(H_f)
\]

i.e., on one side of any partition by a half-space the dimension must drop, since \( \text{Ldim}(H_f) \leq \text{Ldim}(H) = d \) is defined and finite. (This property is what enables the notion of Littlestone majority vote.)

**Proof of Theorem 3.4.** To define our algorithms some notation will be useful. By “tree” in this proof we always mean an \( X \)-labeled binary tree of height \( T \). Given an algorithm \( A \) and a tree \( T \), let \( \sigma = \sigma(A,T) = \langle a_i : i < T \rangle \) and let \( \tau = \tau(A,T) = \langle t_i : i < T \rangle \) denote the sequence of elements of \( X \) associated to the nodes traversed, and the corresponding outputs of \( A \), respectively. Again, given \( A \) and \( T \), let \( \gamma = \gamma(A,T) = \langle g_i : i < T \rangle \) be the sequence of partial characteristic functions given by \( g_i = \{(a_j,t_j) : j < i\} \).

We define the \( \langle T\rangle_{d} \) algorithms as follows. Each algorithm \( A \) is parametrized by a set \( A \subseteq \{0,\ldots,T-1\} \) of size \( \leq d \) (and there is an algorithm for each such set). Given any tree \( T \), the algorithm proceeds as follows. Upon reaching a node at level \( i \) labeled by \( a_i \), it computes the values \( \text{Ldim}(H_{g_i \cup \{(a_i,0)\}}) \) and \( \text{Ldim}(H_{g_i \cup \{(a_i,1)\}}) \). Informally, it asks how the Littlestone dimension of the set \( H_{g_i} \) will change according to the decision on \( a_i \). It then makes its decision by cases. If \( i \in A \), then the algorithm chooses the value of \( t_i \) which will make \( \text{Ldim}(H_{g_i \cup \{(a_i,t_i)\}}) \) smaller, and in case of ties chooses \( 0 \). If \( i \notin A \), then the algorithm chooses the value of \( t_i \) which will make \( \text{Ldim}(H_{g_i \cup \{(a_i,t_i)\}}) \) larger, and in case of ties chooses \( 1 \). This finishes the definition of our class \( A \). Clearly the algorithms involved are all prefix dependent.

Let us verify that for any tree \( T \) and any \( h \in H \) there is an algorithm in \( A \) realizing the same branch as \( h \). Let \( (b_0,s_0),\ldots,(b_{t-1},s_{t-1}) \) denote the root-to-leaf path traversed by \( h \). For each \( i < T \), let \( f_i = \{(b_j,s_j) : j < i\} \) denote the partial characteristic function in play as we arrive to \( b_i \). (Notice that necessarily each \( f_i \subseteq h \).) Let us consider how we may use \( A \) to signal what to do. Let \( d_i^0 = \text{Ldim}(H_{f_i}) \), let \( d_i^0 = \text{Ldim}(H_{f_i \cup \{(a_i,0)\}}) \) and let \( d_i^1 = \text{Ldim}(H_{f_i \cup \{(a_i,1)\}}) \). There are several cases. If we know that at stage \( i \) the \( \text{Ldim} \) does not drop then by \( \text{3.5} \) the choice is determined. If we know that the \( \text{Ldim} \) drops and \( d_0^i \neq d_1^i \) then the choice is determined by
knowing whether we chose the larger or smaller. If we know that \( \text{Ldim} \) drops and \( d_i^0 = d_i^1 \) then the choice is determined by knowing whether or not we went left. With this in mind, define \( B \subseteq \{0, \ldots, T - 1\} \) to be \( B = \{i < T : (\text{Ldim}(H_{f_i}) \geq \text{Ldim}(H_{f_i \cup \{a_i,1-s_i\}})) \lor (\text{Ldim}(H_{f_i \cup \{a_i\}})) = \text{Ldim}(H_{f_i \cup \{a_i,1-s_i\}}) = \text{Ldim}(H_{f_i}) - 1 \text{ and } s_i = 1\} \). In English, \( B \) is the set of all \( i < T \) at which either there was only one way to make the dimension drop as much as possible, or both ways the dimension dropped by the same amount and we went left. Since at every \( i \in B \) the Littlestone dimension drops, necessarily \( |B| \leq d \).

Consider the algorithm \( A \in A \) parameterized by \( B \). We argue by induction on \( i < T \) that \( g_i = f_i \), that is, \( a_i = b_i \) and \( t_i = s_i \). To start, \( a_0 = b_0 \) is the label of the root. If \( i \notin B \), then at this stage along the path traversed by \( h \), either the Littlestone dimension did not drop as much as possible or \( s_i = 1 \). In the first case, there is only one value of \( t_i \) which will keep the dimension larger, and that is \( t_i = s_i \). If the dimension went down equally for both successors, \( A \) will choose \( t_i = 1 = s_i \). If \( i \in B \), then here the Littlestone dimension must drop as much as possible, so either there is only one way to achieve this and so \( t_i = s_i \), or both successors drop equally and \( t_i = 0 = s_i \). This completes the proof.

We now verify that this is a characterization.

**Lemma 3.9.** Suppose \((X, H)\) does not have finite Littlestone dimension. Then for every \( d \in \mathbb{N} \), for all sufficiently large \( T \in \mathbb{N} \) and every collection \( A \) of \( \binom{T}{d} \) dynamic sets, there is some binary tree \( T \) of height \( T \) with \( X \)-labeled internal nodes and some \( h \in H \) which realizes a branch in \( T \) not realized by any algorithm from \( A \).

**Proof.** Suppose \( d \) is given. Choose \( T \) large enough so that \( 2^T > T^d \). Since \( \text{Ldim} \) is not finite, we may construct a full binary tree of height \( T \) whose nodes are labeled by \( X \) and such that every branch is realized by some \( h \in H \). However, every algorithm \( A \in A \) realizes one and only one branch in \( T \), so there are not enough of them to cover all branches.

One final point. We have verified that the algorithms \( A \) built for \( H \) in \(|B| \) can realize any branch in any relevant tree which is realized by an element of \( H \). It is useful to observe that \( A \) simulates \( H \) in an even stronger way: its algorithms can continue to simulate the realization of branches by \( H \) even when we weaken the notion of realization to allow a certain number of mistakes.

**Corollary 3.10.** Let \( H \) be a Littlestone class of dimension \( d \), let \( T \in \mathbb{N} \), and let \( A \) be the family of \( \binom{T}{d} \) algorithms constructed for \( H \) in Theorem 3.1. Let \( T \) be any binary tree of height \( T \) with \( X \)-labeled internal nodes. Given any branch \((a_0, t_0), \ldots, (a_{T-1}, t_{T-1})\) and any \( h \in H \), let \( S = \{i < T : h(a_i) \neq t_i\} \) be the set of “mistakes” made by \( h \) for this branch. Then there is an algorithm \( A \in A \) which makes the same set of mistakes for this branch.

**Proof.** Consider a new tree \( T_* \) where all the nodes at level \( i \) have the same label \( a_i \) (the tree is “oblivious”). So branches through \( T_* \) amount to choosing subsets of \( \{a_i : i < T\} \). This special tree also falls under the jurisdiction of Theorem 3.1 and so gives our corollary.

### 4 Existence

In this section we give an existence proof for \( \epsilon \)-excellent sets via regret bounds. To begin we represent the definitions of good and excellent in the language of probability.
**ε-Good and ε-Excellent Distributions.** Let $\mathcal{H} \subseteq \{0, 1\}^X$. We say a distribution $P$ over $X$ is $\epsilon$-good w.r.t $\mathcal{H}$ if

$$\bigl(\forall h \in \mathcal{H}\bigr) : \Pr_{x \sim P}[h(x) = 1] \in [0, \epsilon] \cup [1 - \epsilon, 1].$$

Similarly, a distribution $Q$ over $\mathcal{H}$ is $\epsilon$-good if

$$\bigl(\forall x \in X\bigr) : \Pr_{h \sim Q}[h(x) = 1] \in [0, \epsilon] \cup [1 - \epsilon, 1].$$

Next, a distribution $P$ over $X$ is $\epsilon$-excellent if

$$\bigl(\forall \epsilon\text{-good } Q\bigr) : \Pr_{h \sim Q, x \sim P}[h(x) = 1] \in [0, \epsilon] \cup [1 - \epsilon, 1].$$

Finally, a distribution $Q$ over $\mathcal{H}$ is $\epsilon$-excellent if

$$\bigl(\forall \epsilon\text{-good } P\bigr) : \Pr_{h \sim P, x \sim Q}[h(x) = 1] \in [0, \epsilon] \cup [1 - \epsilon, 1].$$

We say that a subset of $\mathcal{H}$ or of $X$ is $\epsilon$-good ($\epsilon$-excellent) if the uniform distribution on it is $\epsilon$-good ($\epsilon$-excellent).

In the process of extracting large $\epsilon$-excellent subsets of $\mathcal{H}$ we use trees whose nodes are labelled by $\epsilon$-good distributions; let us refer here to such trees as $\epsilon$-good trees.

**Definition 4.1.** Note that each hypothesis $h \in \mathcal{H}$ naturally realizes a branch in an $\epsilon$-good tree $T$. The tree $T$ is said to be shattered by $\mathcal{H}$ if every branch is realized by some $h \in \mathcal{H}$.

**Discussion 4.2.** Note that for any $\mathcal{H}$ the definitions of $\epsilon$-good and excellent distributions are always trivially meaningful: consider a distribution concentrating on a single element. When $(X, \mathcal{H})$ is a Littlestone pair, the natural adaptation of Fact 2.2 to this setting will give many nontrivial examples.

**Theorem 4.3.** Let $\mathcal{H}$ be an hypothesis class, let $T$ be an $\epsilon$-good complete binary tree that is shattered by $\mathcal{H}$, and let $T$ denote the depth of $T$. Then, for every online learning algorithm $A$, the tree $T$ witnesses a lower bound on the regret of $A$ in the following sense. There exist distributions $D_1, \ldots, D_T$ over $X \times \{0, 1\}$ such that an independent sequence of random examples $(x_t, y_t) \sim D_t, t = 1, \ldots, T$ satisfies the following:

- The expected number of mistakes $A$ makes on the random sequence is at least $\frac{T}{2}$.
- $\exists h \in \mathcal{H}$ whose expected number of mistakes on the random sequence is at most $\epsilon \cdot T$.

Thus, the expected regret of $A$ w.r.t $\mathcal{H}$ on the random sequence is at least $(\frac{T}{2} - \epsilon) \cdot T$.

Before we prove this theorem, let us demonstrate how one can use it to bound the maximum depth of an $\epsilon$-good tree which is shattered by a Littlestone class $\mathcal{H}$: it is known that for every class $\mathcal{H}$ and for every $T \in \mathbb{N}$ there exists an algorithm $A$ whose expected regret w.r.t any sequence of examples $(x_1, y_1), \ldots, (x_T, y_T)$ is

$$O\left(\sqrt{d \cdot T}\right), \quad (1)$$

4 The algorithm $A$ is randomized.
where $d$ is the Littlestone dimension of $\mathcal{H}$, and the big oh notation conceals a fixed numerical constant. Thus, by Theorem 4.3, it follows that if there exists a (complete) $\epsilon$-good tree that is shattered by $\mathcal{H}$ of depth $T$ then $T$ must satisfy the following inequality:

$$
\left(\frac{1}{2} - \epsilon\right) \cdot T \leq O\left(\sqrt{d \cdot T}\right).
$$

Indeed, the LHS in the above inequality is a lower bound on the expected regret of $\mathcal{A}$, where as the RHS is an upper bound on it. A simple arithmetic manipulation yields that $T = O(d/(1/2 - \epsilon)^2)$. Thus, we get the following corollary:

**Corollary 4.4.** Let $\mathcal{H}$ be a class with Littlestone dimension $d < \infty$ and let $\epsilon \in \left[0, \frac{1}{2}\right]$. Denote by $d_\epsilon$ the maximum possible depth of a complete $\epsilon$-good tree which is shattered by $\mathcal{H}$. (Note that $d_0 = d$.) Then,

$$
d_\epsilon = O\left(\frac{d}{(\frac{1}{2} - \epsilon)^2}\right).
$$

**Proof of Theorem 4.3.** Let $\mathcal{T}$ be a tree and $\mathcal{A}$ be an online algorithm as in the premise of the theorem. We begin with defining the distributions $D_t$. We first note that the label in each distribution $D_t$ is deterministic; that is, there exist a distribution $D_t$ over $X$ and a label $y_t \in \{0, 1\}$ such that a random example $(x, y) \sim D_t$ satisfies that $y = y_t$ always (with probability = 1) and $x_t \sim D_t$. The distributions $D_t$ and labels $y_t$ correspond to a branch of $\mathcal{T}$ as follows:

- Initialize $t = 1$, set the “current” node $v_t$ to be the root of the tree.
- For $t = 1, \ldots, T$
  1. Let $D_t$ denote the $\epsilon$-good distribution $D_{v_t}$ which is associated with $v_t$.
  2. Define the label $y_t$ to be 1 if and only if

$$
\Pr_{(x_t)'_{t=1}^T \sim \prod_{t=1}^{T} D_t, \mathcal{A}}\left[A(x_t; (x_{t-1}, y_{t-1}), \ldots, (x_1, y_1)) = 1\right] \leq 1/2,
$$

where $\mathcal{A}$ is the considered online algorithm. (Note that the above probability is taken w.r.t the sampling of the $x_t$’s, as well as the randomness of $\mathcal{A}$ in case it is a randomized algorithm.) I.e. the adversary forces that the algorithm errs with probability at least 1/2 on $x_t$ when given an input sequence $(x_1, y_1), \ldots, (x_{t-1}, y_{t-1}), x_t$, where the $x_i$’s are sampled from the $D_t$’s.

3. Set $v_{t+1}$ to be the root of the subtree corresponding to the label $y_t$.

- Output the sequence $(D_1, y_1), \ldots, (D_T, y_T)$.

Let $(x_t)'_{t=1}^T \sim \prod_{t=1}^{T} D_t$ and fix $t \leq T$. Let $\hat{y}_t = A(x_t; (x_{t-1}, y_{t-1}), \ldots, (x_1, y_1))$ be the prediction of $\mathcal{A}$ on $x_t$. Thus, by construction $\hat{y}_t \neq y_t$ with probability at least 1/2, and therefore, by linearity of expectation:

$$
\mathbb{E}_{(x_t)'_{t=1}^T \sim \prod_{t=1}^{T} D_t, \mathcal{A}}\left[\sum_{t=1}^{T} 1[\hat{y}_t \neq y_t]\right] \geq \frac{T}{2},
$$

The derivation of the (optimal) bound of $O(\sqrt{d \cdot T})$ is somewhat involved [1], however a slightly weaker bound of $O(\sqrt{d \cdot T \log T})$ can be proven using elementary arguments [2].
It thus remains to show that there exists \( h \in \mathcal{H} \) whose expected number of mistakes is at most \( \epsilon \cdot T \). Indeed, this follows by considering an hypothesis \( h \in \mathcal{H} \) which realizes the branch corresponding to \( (D_t, y_t)^T_{t=1} \). Indeed, for each fixed distribution \( D_t \) on the branch \( \mathcal{B} \), the probability that \( h \) errs on \( x_t \sim D_t \) is at most \( \epsilon \). Thus, by linearity of expectation, the expected number of mistakes is at most \( \epsilon \cdot T \), and therefore there exists \( h \) as stated.

**Discussion 4.5.** To explain the use of Littlestone dimension hidden in this argument we emphasize that the above proof relies on deep ideas from online learning which are worth highlighting, although this is not formally necessary at this point. We also emphasize that this discussion surveys much prior work and not only what we do here.

There are two main ideas: the first one has to do with no-regret algorithms (such as the multiplicative-weights algorithm, see e.g. [14]). Consider a set of \( m < \infty \) experts (say weather forecasters), and every evening each of them tells us whether they think it will rain tomorrow or not. Then, we use this list of predictions to make a prediction of our own. Can we come up with a strategy that will guarantee that over a course of \( T \) days our prediction will not be much worse than that of the best expert in hindsight? No-regret algorithms address this problem and provide regret bounds of roughly \( \sqrt{T \cdot \log m} \): namely such algorithms make at most roughly \( \sqrt{T \cdot \log m} \) mistakes more than the best expert. We stress that the \( m \) experts can be arbitrary algorithms; in particular they may base their prediction at day \( t \) based on all information up to that point (i.e. prefix-dependence).

The second main idea, which is where the Littlestone dimension is used, is that any (possibly infinite) Littlestone class \( \mathcal{H} \) can be covered by a finite set of experts: that is, for every \( T < \infty \) there is a set of \( (\leq \text{Ldim}(\mathcal{H})) \) dynamic-sets which cover all hypotheses on \( \mathcal{H} \) with respect to sequences of length \( T \). Thus, by applying no-regret algorithms on the (finite!) set of experts ensure our regret is small relative to the best \( h \in \mathcal{H} \).

These two ideas imply that there are no-regret algorithms for any Littlestone class. Our bound on \( d_\epsilon \), the maximal height of an \( \epsilon \)-good complete shattered tree, exploits these connections in the new setting of \( \epsilon \)-good trees.

**Corollary 4.6.** Let \( \mathcal{H} \) be a Littlestone class of finite Littlestone dimension \( d_\epsilon \), \( \epsilon < \frac{1}{2} \), and \( d_\epsilon \) from 4.4. Then for any finite \( H \subseteq \mathcal{H} \) there is \( A \subseteq H \), \( |A| \leq e^{d_\epsilon}|H| \) such that \( A \) is \( \epsilon \)-excellent.

## 5 An Alternative Derivation Using Closure Properties

In this section we lay out an alternative argument for upper bounding \( d_\epsilon \), the maximal depth of an \( \epsilon \)-good tree which is shattered by \( \mathcal{H} \) when \( \epsilon < \frac{1}{2} \) and \( \mathcal{H} \) is a Littlestone class. The resulting bound is significantly weaker than the one stated in Corollary 4.4, but the reasoning may perhaps be more intuitive. In particular, it does not rely on the notion of regret from online learning.

In the rest of this section, let \( \epsilon < \frac{1}{2} \) be arbitrary but fixed.

The first idea is to exploit certain closure properties of Littlestone classes. Fix for awhile some \( k \in \mathbb{N} \) and some function \( B : \{0,1\}^k \rightarrow \{0,1\} \). Given \((X, \mathcal{H})\), let \((X, \mathcal{H}^{(B)})\) denote the class

\[
\mathcal{H}^{(B)} = \left\{ B(h_1, \ldots, h_k) : h_i \in \mathcal{H} \right\},
\]

where \( B(h_1, \ldots, h_k) \) denotes the function which takes \( x \in X \) to \( B(h_1(x), \ldots, h_k(x)) \in \{0,1\} \). Informally, we enrich \( \mathcal{H} \) by adding some additional hypotheses which come from applying \( B \) to \( k \)-tuples
of elements of $\mathcal{H}$. We stress that although $B$ can be arbitrary, it is fixed for any instance of this construction.

It has been shown that if $\mathcal{H}$ is a Littlestone class (i.e., $\text{Ldim}(X, \mathcal{H}) < \infty$) then also $\mathcal{H}^{(B)}$ is a Littlestone class\[^2\]\[^9\]. In particular,

$$\text{Ldim}(X, \mathcal{H}^{(B)}) = O\left(\text{Ldim}(X, \mathcal{H}) \cdot k \cdot \log k\right),$$

where the big oh notation conceals a universal numerical constant. Let us sketch a proof of this fact using the language of section three. If $\text{Ldim}(X, \mathcal{H}) = d$ then for any integer $T$ we have a set $\mathcal{E}_T$ of $(\frac{T}{\leq d})$ dynamic sets which simulate $\mathcal{H}$ on any $X$-labeled binary tree of height $T$. To see that $\mathcal{H}^{(B)}$ is also a Littlestone class it would suffice to show that the same is true for some $d'$ replacing $d$. For each $T$, and for each $k$-tuple of dynamic sets $E_1, \ldots, E_k$ from $\mathcal{E}_T$, let $B(E_1, \ldots, E_k)$ denote the dynamic set which operates by applying $B$ to the outputs of $E_1, \ldots, E_k$. Let $\mathcal{E}_T(B) = \{B(E_1, \ldots, E_k) : E_1, \ldots, E_k \in \mathcal{E}_T\}$. Observe that this collection of dynamic sets simulates $\mathcal{H}^{(B)}$ on any $X$-labeled binary tree of height $T$ and its size will remain polynomial in $T$ (at most roughly $T^{dk}$). On the other hand, had $\mathcal{H}^{(B)}$ not been a Littlestone then one would need $2^T >> T^{dk}$ dynamic sets to cover it.

One more step is needed here: we may also apply $B$ dually to $X$ rather than $\mathcal{H}$. To make sense of this, consider $(X, \mathcal{H})$ as a bipartite graph with an edge between $x \in X$ and $h \in \mathcal{H}$ if $h(x) = 1$. In this picture, $\mathcal{H}^{(B)}$ added some new points to the side of $X$, and defined a rule for putting an edge between any such new point and any given element of $X$. To apply $B$ dually, we carry out the parallel operation for $X$ instead. That is, let $(X^{(B)}, \mathcal{H})$ be the class where $X$ is enriched by new elements as follows: for any $x_1, \ldots, x_k \in X$ define an element $B(x_1, \ldots, x_k)$ and for any $h \in \mathcal{H}$, define $h(B(x_1, \ldots, x_k)) = 1$ if and only if $B(h(x_1), \ldots, h(x_k)) = 1$. Recalling section one, the dual of a Littlestone class is a Littlestone class, so $(X^{(B)}, \mathcal{H})$ is Littlestone, though the $\text{Ldim}$ may be quite a bit larger.\[^6\]

We may summarize this part by saying: for any $k \in \mathbb{N}$ and any function $B : \{0,1\}^k \to \{0,1\}$, if $(X, \mathcal{H})$ is a Littlestone class, then $(X^{(B)}, \mathcal{H})$ is a Littlestone class too.

How is this useful for deriving a bound on $d_\epsilon$? Suppose we are given an $\epsilon$-good tree $T$ which is shattered by $\mathcal{H}$. Choose $k$ large enough: $k = O(\text{VCdim}(X, \mathcal{H})/(\frac{1}{2} - \epsilon)^2)$ will suffice. Let $B : \{0,1\}^k \to \{0,1\}$ be the majority vote operation given by $(x_1, \ldots, x_k) \mapsto 0$ if $\{1 \leq i \leq k : a_i = 0\} \geq \frac{1}{2}k$ and $(x_1, \ldots, x_k) \mapsto 1$ otherwise. Suppose we independently sample $k$ elements $x_1, \ldots, x_k$ from one of the $\epsilon$-good distributions labeling our given tree. Then by our choice of $k$, the VC theorem tells us that, with positive probability, the trace of each $h \in \mathcal{H}$ on this sample is close enough to its true proportion. Here “close enough” means that the error is less than $\frac{1}{2} - \epsilon$. In particular, with positive probability, for every $h \in \mathcal{H}$ the majority vote $B(h(x_1), \ldots, h(x_k))$ on this sample agrees with the opinion of the $\epsilon$-good distribution on $h$. We can therefore sample $k$ elements from each of the distributions labeling the nodes of the tree and with positive probability, all samples will be correct in this way.

The crucial observation follows: any full binary $\epsilon$-good tree $T$ which is shattered by $\mathcal{H}$ can be transformed to a (standard) full binary tree $T'$ of the same height which is shattered by the class $(X^{(B)}, \mathcal{H})$. That is, there exists a choice of $k$ elements for each node in $T$ such that the corresponding tree $T'$ whose nodes are labelled by the $k$-wise majority votes of these elements (i.e. the appropriate $B(x_1, \ldots, x_k)$) is shattered by $(X^{(B)}, \mathcal{H})$. This shows that the length of $T'$ (and

\[^6\]It is known that $\text{Ldim}(X, \mathcal{H}) \leq 2^{\text{Ldim}(X, \mathcal{H})}$.
also of $T$) is bounded by $\text{Ldim}(X^{(B)}, \mathcal{H})$ which is finite. (Note that this argument implicitly gives an inequality between the approximate and virtual Littlestone dimension, which are defined in the next section.)

This completes the sketch of the proof.

This argument is closer in spirit to similar arguments in VC theory concerning the variability of the VC dimension under natural operations. The obtained bounds however are much weaker (at least double-exponentially weaker than the bound in Corollary 4.1).

6 Approximate and virtual Littlestone dimension

In this section we apply our earlier results to analyze two a priori strengthenings of Littlestone dimension. Both depend on a choice of $\epsilon < \frac{1}{2}$. The first comes from observing that $\epsilon$-good sets behave much like elements; we might call them “virtual elements.” The “virtual” Littlestone dimension asks about the maximal height of a tree where the nodes can be labeled by possibly virtual elements of $X$. A second definition comes from allowing a small fraction of mistaken edges in the definition of “$h$ realizes a branch.”

**Convention 6.1.** For this section, let $(X, \mathcal{H})$ be some hypothesis class, and for most of this section, let $\epsilon < \frac{1}{2}$ be arbitrary but fixed.

**Definition 6.2.** A virtual element is an $\epsilon$-good set.

Notice that all elements of $X$ are also virtual elements. In 6.2 “virtual element” really abbreviates “possibly virtual element;” we may use “strictly virtual element” otherwise.

**Definition 6.3.** Virtual dimensions:

1. The virtual Littlestone dimension of $\mathcal{H}$ is the depth $\ell$ of the largest complete binary tree whose nodes are labeled by virtual elements and such that every branch is realized by some element of $\mathcal{H}$, i.e. for every root-to-leaf path $(a_0, t_0),\ldots,(a_{\ell-1}, t_{\ell-1})$, where the $a_i$’s are virtual elements and the $t_i$’s are 0 or 1, there is $h \in \mathcal{H}$ such that $t(a_i, h) = t_i$ for $i = 0,\ldots, \ell - 1$.

2. The virtual Threshold dimension of $\mathcal{H}$ is the largest $k$ such that there exist $\bar{a} = (a_i : i < k)$, $\bar{h} = (h_i : i < k)$ where $\bar{a}$ is a sequence of distinct virtual elements of $X$ and $\bar{h}$ is a sequence of distinct elements of $\mathcal{H}$ and $t(a_i, h_j) = 1$ if and only if $i < j$.

**Remark 6.4.** In 6.3, it would also be reasonable to consider a case where the elements of $\mathcal{H}$ are allowed to be virtual; note that for the definitions to make sense, the $h$’s would then need to possess some degree of excellence (not necessarily faced with any good set, but at least towards those relevant to the present configuration).

**Remark 6.5.** Also, in 6.3 it may be interesting to consider the case where virtual elements are required to have some minimal size (say, $\geq \eta|\mathcal{H}|$ for some $\eta = \eta(\epsilon)$), since such elements are likely to be visible to random sampling; however, we do not follow this direction here.

**Definition 6.6.** Approximate dimensions:
1. The approximate Littlestone dimension of $\mathcal{H}$ is the depth $\ell$ of the largest complete binary tree whose nodes are labeled by elements and such that every branch is $\epsilon$-realized by some element of $\mathcal{H}$, meaning that for every root-to-leaf path $(a_0, t_0), \ldots, (a_{\ell-1}, t_{\ell-1})$, where the $a_i$'s are elements of $X$ and the $t_i$'s are 0 or 1, there is $h \in \mathcal{H}$ such that $|\{i < \ell : h(a_i) \neq t_i\}| < \epsilon \ell$.

2. The approximate Threshold dimension of $\mathcal{H}$ is the largest $k$ such that there exist $\bar{a} = \langle a_i : i < k \rangle$, $\bar{h} = \langle h_j : j < k \rangle$ where $\bar{a}$ is a sequence of distinct elements of $X$ and $\bar{h}$ is a sequence of distinct elements of $\mathcal{H}$ and at most $\epsilon \cdot k^2$ of the pairs $(i, j) \in [k] \times [k]$ satisfy $h_j(x_i) \neq 1[i < j]$. Equivalently, a random pair $(i, j)$ is connected according to the half-graph relation with probability at least $1 - \epsilon$.

In both cases, we may say “$\epsilon$-approximate” or “$\epsilon$-virtual” when the specific value of $\epsilon$ is important.

**Theorem 6.7.** Let $\epsilon < \frac{1}{2}$ be given. Then the following are equivalent:

1. The Littlestone dimension of $\mathcal{H}$ is finite.
2. The Threshold dimension of $\mathcal{H}$ is finite.
3. The virtual Littlestone dimension of $\mathcal{H}$ is finite.
4. The virtual Threshold dimension of $\mathcal{H}$ is finite.
5. The approximate Littlestone dimension of $\mathcal{H}$ is finite.

**Proof.** Clearly (3), (5) imply (1), and (4) implies (2).

We know (1) $\iff$ (2) by the unstable formula theorem (or “Hodges’ lemma”) and (3) $\iff$ (4) by an identical proof, working with the tree and graph given by considering each $a_i$ to be an element and putting an edge between $a_i$ and $h_j$ if and only if $t(a_i, h_j) = 1$.

(1) implies (3) is the special case of our theorem above (see Corollary 4.4 where the distributions assign zero measure to every element not in our given $\epsilon$-good set and use the counting measure otherwise.

(1) implies (5) also follows just as in the proof of Theorem 4.3 (in fact, it is simpler because we don’t need to deal with distributions). Given a full-binary $X$-labeled tree $T$ of height $T$ which is $\epsilon$-shattered by $\mathcal{H}$ and an online algorithm $A$, we can pick a branch on $T$ which forces $A$ to err in each step (or to err with probability $\geq 1/2$ if $A$ is randomized.) By $\epsilon$-shattering we know that there exists $h \in \mathcal{H}$ which errs on at most $\epsilon$-fraction of the nodes in this branch, and thus this branch witnesses that the regret of $A$ is at least $(1/2 - \epsilon) \cdot T$. This implies an upper bound on $T$ by picking $A$ to be an online learner which exhibits an optimal regret of $O(\sqrt{d \cdot T})$. (Just like in Corollary 4.4).

We have left approximate Threshold dimension to a separate claim, because its relation to $\epsilon$ is different. (We could also have given a definition in 6.6(2) which would have fit the equivalence, but this seemed perhaps more natural.)

**Claim 6.8.** The Threshold dimension of $\mathcal{H}$ is finite if and only if for some $\epsilon_s > 0$, for all $\epsilon_s > \epsilon > 0$ the $\epsilon$-approximate Threshold dimension is finite.

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Proof. If Threshold dimension is infinite, clearly \( \epsilon \)-approximate Threshold dimension is infinite for any \( \epsilon \). Suppose the \( \epsilon \)-approximate Threshold dimension is infinite. Begin with an approximate half-graph of size \( n \geq k = \lceil \sqrt{1/\epsilon} \rceil \). We will use the probabilistic method: pick uniformly at random \( k \) distinct indices \( \ell_1, \ldots, \ell_k \in [n] \) and consider \( \langle a_{\ell_i} \rangle_{i \leq k} \) and \( \langle h_{\ell_j} \rangle_{j \leq k} \). Note that for every \( i, j \leq k \) the probability that \( h_{\ell_i} (a_{\ell_i}) \neq 1 \) is at most \( \epsilon \). (Let us call a pair \( i, j \) for which this happens a bad pair.) Thus, with probability \( \leq k^2 \cdot \epsilon < 1 \) there exists a bad pair; equivalently, with probability \( > 0 \) no pair is bad and we get a half-graph of size \( k \). Since our assumption allows \( \epsilon \to 0 \), we find that the Threshold dimension is infinite. \( \square \)

7 Majorities in Littlestone classes

So far we have been guided by the thesis that Littlestone classes are characterized by frequent, large sets with well-defined notions of majority. However, there are at least two candidate notions of majority which are quite distinct: the majority arising from the counting measure, which we have been exploring via \( \epsilon \)-excellent and \( \epsilon \)-good, and the notion of majority arising from Littlestone rank.

In this section we prove that these two notions of majority “densely often agree” in Littlestone classes and indeed this is true of any simple axiomatic notion of majority, as defined below.

**Definition 7.1.** Say that \( A \subseteq H \) is Littlestone-opinionated if for any \( a \in X \), one and only one of

\[
\text{Ldim} (\{ h \in H : h(a) = 0 \}), \text{Ldim} (\{ h \in H : h(a) = 1 \})
\]

is strictly smaller than \( \text{Ldim} (A) \).

As a warm-up, we prove several claims. As above, a partition of \( H \) by a half-space means that for some element \( a \in X \) we separate \( H \) into \( \{ h \in H : h(a) = 0 \} \) and \( \{ h \in H : h(a) = 1 \} \).

**Claim 7.2.** Suppose \( \text{Ldim} (H) = d \) and \( 0 < \epsilon < \frac{1}{2} \). Then for any finite \( H \subseteq H \) there is \( A \subseteq H \) of size \( \geq \epsilon^d |H| \) such that \( A \) is both \( \epsilon \)-good and Littlestone-opinionated and these two notions of majority agree, i.e. for any \( a \in X \)

\[
\text{Ldim} (\{ h \in A : h(a) = t \}) = \text{Ldim} (H) \text{ iff } |\{ h \in A : h(a) = t \}| \geq (1 - \epsilon)|A|.
\]

**Proof.** It suffices to observe that given any finite \( H \subseteq H \) which \( (a) \) is not \( \epsilon \)-good, \( (b) \) is \( \epsilon \)-good but is not Littlestone-opinionated, or \( (c) \) is both \( \epsilon \)-good and Littlestone-opinionated but the two notions of majority do not always agree, we can find \( G \subseteq H \) (arising from a partition of \( H \) by a half-space) with \( |G| \geq \epsilon |H| \) and \( \text{Ldim} (G) < \text{Ldim} (H) \), because this initiates a recursion which cannot continue more than \( \text{Ldim} (H) \leq \text{Ldim} (H) = d \) steps.

Why? In case (a), there is a partition into two pieces of size \( \geq \epsilon |H| \); choose the one of smaller \( \text{Ldim} \). In case (b), there is a partition into two pieces each of \( \text{Ldim} \) strictly smaller than \( \text{Ldim} (H) \); choose the one of larger counting measure. In case (c), there is a partition where the majorities disagree, and we can choose the piece of larger counting measure and thus smaller \( \text{Ldim} \). This completes the proof. \( \square \)

**Definition 7.3.** Call \( P \) a good property (or: \( \epsilon \)-good property) for \( H \) if it is a property of finite subsets of \( H \) which implies \( \epsilon \)-good and which satisfies: for some constant \( c = c(P) > 0 \), for any finite \( H \subseteq H \) there is \( B \subseteq H \) of size \( \geq \epsilon^d |H| \) with property \( P \).
Lemma 7.5. Let \( H \) be a Littlestone class of dimension \( d \) and \( 0 < \varepsilon < \frac{1}{2} \). Let \( P \) be a good property for \( H \) and \( c = c(P) \). Then for any finite \( H \subseteq \mathcal{H} \) there is \( A \subseteq H \) of size \( \geq \varepsilon^{(c+1)d}|H| \) such that \( A \) has property \( P \) (so is also \( \varepsilon \)-good) and is Littlestone-opinionated, and for any \( a \in X \),

\[
\text{Ldim}(\{h \in A : h(a) = t\}) = \text{Ldim}(H) \iff |\{h \in A : h(a) = t\}| \geq (1 - \varepsilon)|A|
\]

i.e. the \( \varepsilon \)-good majority and the Littlestone majority agree.

Proof. Modify the recursion in the previous proof as follows. At a given step, if \( H \) does not have property \( P \), replace it by a subset \( C \) of size \( \geq c|H| \) which does. Since \( P \) implies \( \varepsilon \)-good, if we are not finished, then we are necessarily in case (b) or (c) and at the cost of an additional factor of \( \varepsilon \) we can find \( B \subseteq C \) where the \( \text{Ldim} \) drops. At the end of each such round, we have replaced \( H \) by \( B \subseteq H \) with \( |B| \geq \varepsilon^{c+1}|H| \) and \( \text{Ldim}(B) < \text{Ldim}(H) \).

Discussion 7.6. Note that this majority agreement deals with half-spaces, which is arguably the interesting case for “Littlestone-opinionated” as it relates to the SOA. In [7.5] it is a priori not asserted that every subset of \( A \) which is large in the sense of counting measure (but does not arise from a half-space) has large \( \text{Ldim} \).

Definition 7.7 (Axiomatic largeness). Define \( \mathcal{M} \) to be an axiomatic notion of relative largeness for the class \( \mathcal{H} \) if it satisfies the following properties:

1. \( \mathcal{M} \) is a subset of \( \mathcal{P} = \{(B, A) : B \subseteq A \subseteq \mathcal{H}\} \).
2. Define \( \mathcal{P}_{\text{half}} := \{(B, A) \in \mathcal{P} : B \text{ arises as the intersection of } A \text{ with a half-space}\} \).
3. If \((B, A) \in \mathcal{M}\), we say “\( B \) is a large subset of \( A \)” and we may write \( B \subseteq_{\mathcal{M}} A \).
4. The rules are:

   - (a) (monotonicity in the set) if \( C \subseteq B \subseteq A \) and \( C \subseteq_{\mathcal{M}} A \) then \( B \subseteq_{\mathcal{M}} A \).
   - (b) (monotonicity in the superset) if \( C \subseteq B \subseteq A \) and \( C \subseteq_{\mathcal{M}} A \) then \( C \subseteq_{\mathcal{M}} B \).
   - (c) (identity) \((A, A) \in \mathcal{M}\).
   - (d) (non-contradiction) If \((B, A) \in \mathcal{P}_{\text{half}} \) and \( C = A \setminus B \) [so also \((C, A) \in \mathcal{P}_{\text{half}} \)] then at most one of \((B, A) \) and \((C, A) \) belongs to \( \mathcal{M} \).
   - (e) (chain condition) There is \( n = n(\mathcal{M}) < \omega \) such that if \((A_i : i < m) \) is a set of subsets of \( \mathcal{H} \) and \((A_{i+1}, A_i) \notin \mathcal{M} \) for all \( i < m - 2 \) then \( m \leq n \). In other words, the length of any descending chain

\[
A_{m-1} \subseteq \cdots \subseteq A_0
\]

of non-large subsets is upper bounded by \( n \).

Example 7.8. Suppose \( \mathcal{H} \) is a Littlestone class. Then

\[
\mathcal{M} = \{(B, A) \in S : \text{Ldim}(A) = \text{Ldim}(B)\}
\]

satisfies Definition 7.7.\footnote{This captures relative majority or relative largeness since “being large” is a two-place relation.}
Proof. Conditions (3)(a),(b),(c) are immediate; (d) follows by the definition of $\text{Ldim}$. Condition (e) is clear because if $(A_t, A_{t+1}) \notin \mathcal{M}$ then $\text{Ldim}(A_{t+1}) < \text{Ldim}(A_t)$ so $n(\mathcal{M}) \leq d$. \qed

**Example 7.9.** A range of other examples are provided by various model theoretic notions of rank (using multiplicity one).

In the next result, we don’t need to assume a priori that $\mathcal{H}$ is a Littlestone class, though it will follow from the proof that it is.

**Lemma 7.10.** Suppose $\mathcal{H}$ admits a notion of relative largeness $\mathcal{M}$. Let $0 < \epsilon < \frac{1}{2}$ and let $P$ be an $\epsilon$-good property for $\mathcal{H}$. Let $c = c(P)$ and $n = n(\mathcal{M})$. Then for any nonempty finite $H \subseteq \mathcal{H}$ there is $A \subseteq H$ of size $\geq \epsilon^{(c+1)n}|H|$ such that:

1. $A$ has property $P$, and thus is $\epsilon$-good, so for any partition of $A$ by a half-space into $B \cup C$, at least one (so exactly one) of $B, C$ has size $< \epsilon|A|$.

2. For any partition of $A$ by a half-space into $B \cup C$, at least one (so exactly one) of $(B, A)$, $(C, A)$ belongs to $\mathcal{M}$.

3. The two notions agree, i.e. $(B, A) \in \mathcal{M}$ if and only if $|B| \geq (1 - \epsilon)|A|$.

**Proof.** Let $n = n(\mathcal{M})$ and set $A_0 = H$. By induction on $t \geq 0$ we shall prove that if $A_t$ does not satisfy conditions 1, 2, and 3 then either it contains a subset of size $\geq \epsilon^{|A_t|}$ which does, or there is $A_{t+1} \subseteq A_t$ such that $|A_{t+1}| \geq \epsilon^{c+1}|A_t|$ and $(A_{t+1}, A_t) \notin \mathcal{M}$. Our chain condition 7.7(4)(e) will then ensure $t$ is bounded above by $n$.

For each $t \geq 0$ proceed as follows. If $A_t$ has property $P$, define $A'_t = A_t$. If not, replace $A_t$ by a subset of size $\geq \epsilon^{|A_t|}$ which does, and set this to be $A'_t$. A priori, we have no information on whether $(A'_t, A_t) \in \mathcal{M}$. Since $A'_t$ has property $P$, if $A'_t$ does not already satisfy 1, 2, and 3, then condition 2 or 3 must fail; in either case, there must be some half-space which partitions $A'_t$ into two non-trivial sets at least one of which, call it $B$, has size at least $\epsilon|A'_t|$ and satisfies $(B, A'_t) \notin \mathcal{M}$. Set $A_{t+1} = B$. Then $|B| \geq \epsilon^{c+1}|A_t|$ and by condition 7.7(4)(b), $(A_{t+1}, A_t) \notin \mathcal{M}$. This completes the inductive step and the proof. \qed

**Theorem 7.11.** The following are equivalent for $(X, \mathcal{H})$.

1. $\mathcal{H}$ admits a notion of relative largeness $\mathcal{M}$.

2. $\mathcal{H}$ is a Littlestone class.

3. For every $\mathcal{M}$ and $0 < \epsilon < \frac{1}{2}$ there is $n = n(\epsilon, \mathcal{M})$ such that every finite nonempty $H \subseteq \mathcal{H}$ has a subset $A$ which satisfies:

   (a) $|A| \geq \epsilon^n|H|$, and

   (b) $A$ is $\epsilon$-good, and

   (c) for every partition of $A$ by a half-space into $B \cup C$, $(B, A) \in \mathcal{M}$ if and only if $|B| \geq \epsilon|A|$ if and only if $|B| \geq (1 - \epsilon)|A|$.

i.e. the counting majority and the $\mathcal{M}$-majority are well defined and agree.
4. In item (3) we may replace (b) by “A has property $P$” when $P$ is an $\epsilon$-good property for $\mathcal{H}$, at the cost of changing the exponent $n$ in (a) to $(c + 1)n$ for $c = c(P)$.

Proof. (2) implies (1) is Example 7.8. (1) implies (3) [or (4)] is Lemma 7.10. Clearly (4) implies (3). For (3) implies (2), note that (3) tells us a fortiori that we can always find large $\epsilon$-good subsets, so $\mathcal{H}$ must be a Littlestone class by 2.4.

Remark 7.12. Although we have formulated these largeness properties for subsets of $\mathcal{H}$, the symmetric results would hold for subsets of $X$.

It is interesting to inspect relative largeness from the perspective of online learning. Indeed, note that any such notion gives rise to an online learning strategy with a bounded mistake bound: the online learner maintains a version space $\mathcal{H}_i \subseteq \mathcal{H}$, starting with $\mathcal{H}_0 = \mathcal{H}$. For each input example $x_i$ received, the learner predicts $\hat{y}_i$ such that

$$(\{h \in \mathcal{H}_i : h(x_i) = \hat{y}_i\}, \mathcal{H}_i) \in \mathcal{M}$$

and note that there can be at most such $\hat{y}_i$, if no such $\hat{y}_i$ exists then the learner predicts $\hat{y}_i = 0$. Then, upon receiving the true label $y_i$, if $y_i = \hat{y}_i$ then the learner sets $\mathcal{H}_{i+1} = \mathcal{H}_i$ and else, when $y_i \neq \hat{y}_i$, the learner sets $\mathcal{H}_{i+1} = \{h \in \mathcal{H}_i : h(x_i) = y_i\}$. Observe that given any sequence $(x_1, y_1), \ldots, (x_T, y_T)$, this learner makes at most $n(\mathcal{M})$ mistakes: indeed, if the learner makes a mistake on $x_i$ then $(\mathcal{H}_{i+1}, \mathcal{H}_i) \notin \mathcal{M}$, and $\mathcal{H}_{i+1}$ is obtained by intersecting $\mathcal{H}$ with a halfspace.

This point of view offers an alternative explanation to the fact that only Littlestone classes admit notions of relative largeness. Moreover, it implies that for every notion of relative largeness $\mathcal{M}$, we have that $n(\mathcal{M})$ is at least the Littlestone dimension. This follows because the Littlestone dimension is equal to the optimal mistake-bound. Thus, the notion of relative largeness which arises from the Littlestone dimension is optimal in the sense that it minimizes $n(\mathcal{M})$.

8 Discussion and open problems

To conclude the paper we mention several natural open problems and directions for further work suggested by the proofs and interactions above.

For VC classes, recall that we have the usual Sauer-Shelah-Perles lemma, and Haussler’s covering lemma which says that every VC class of VC-dimension $d$ can be $\epsilon$-covered by roughly $\frac{1}{\epsilon^d}$ hypotheses [10] (The SSP lemma can be thought of as the special case of Haussler’s covering lemma where the domain has size $n$ and when $d = \frac{1}{n}$.) This is clearly useful for learning. It is natural to ask whether there is a dynamic version of this covering lemma for Littlestone classes. That is, is there a function $f = f(\epsilon, d)$ such that for any Littlestone class $\mathcal{H}$ of $\text{Ldim} d$ we can always find $\leq f(\epsilon, d)$ dynamic sets which approximately cover the whole class $\mathcal{H}$, meaning that for every sequence $x_1, \ldots, x_n$ from $X$ and every $h \in \mathcal{H}$ there is a dynamic set in our list which is $\epsilon$-close to it. This is also related to [1].

In the classical case, there is a fundamental relationship between the Littlestone dimension and the half-graph/Threshold dimension, as explained by Shelah’s unstable formula theorem: both are finite together, and bounds are known [22], [11]. Note, however, that tight quantitative bounds

\footnote{Informally, there is a list of approximately $\frac{1}{\epsilon^d}$ hypotheses such that every other hypothesis in our class is $\epsilon$-close to one of the hypotheses in our list.}
still remain open. To reiterate what we said in section one, a useful aspect of this relationship is connecting a symmetric or self-dual quantity with Littlestone dimension (see 1.3 above). In addition to sorting out the quantitative bounds in the classical case, it may be interesting to explore related questions for the approximate and virtual variants of Littlestone dimension directly.

It also may be worth while to explore further the significance of dynamic Sauer-Shelah-Perles lemmas for model theory.
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