Analysis of Linear Stability and Bifurcations of Central Configurations in the Planar Restricted Circular Four-Body Problem

B S Bardin¹,² and E V Volkov¹,²

¹Moscow Aviation Institute (National Research University), Volokolamskoe shosse 4, 125993 Moscow, Russia
²Mechanical Engineering Research Institute of the Russian Academy of Sciences, 4, M. Kharitonyevskiy Pereulok, 101990 Moscow, Russia

E-mail: bsbardin@yandex.ru, evvolkov94@mail.ru

Abstract. The planar restricted four-body problem is considered. That is, motion of an infinitesimal body \( P \) in the gravitational field of three attracting bodies is studied. It is supposed, that the above three body form a stable equilateral Lagrange triangle and all four bodies move in a plane. In this case there are eight central configurations formed by the bodies. An analysis of stability and bifurcation of the central configurations is performed. In particular, it is shown that the bifurcation is only possible in cases of degeneracy, when mass of an attracting body vanishes. It is also established that in non-degenerate cases five central configurations are unstable and three central configurations can be both stable and unstable. Domains of stability in linear approximation are constructed in the plane of parameters.

1. Introduction

The restricted four-body problem is a simplest generalization of the classical restricted three-body problem. It is of great interest for celestial mechanics and dynamics of satellites. In this problem, the motion of four bodies is investigated, one of which has an infinitely small mass, three other bodies have finite masses and move under the influence of their mutual attraction according to Newton’s law of universal gravitation. The infinitesimal body moves under the gravitational attraction of other three bodies and does not affect their motion. There exists a partial type of motion of the bodies, when they form the so-called central configuration. It means that the resultant attracting force acting on a body is a central force directed to the system mass center.

A numerous papers are devoted to the study of the central configuration in the restricted planar four-body problem. The existence and bifurcation of central configurations were studied in [1-5]. In particular, it has been established [1] that there can be eight, nine, or ten configurations. A linear analysis of the stability of central configurations has been performed in [6-9]. In [10-11], conclusions on the Lyapunov stability of central configurations were obtained in the case of two equal masses of attracting bodies.

In what follows, we denote by \( m_i \) masses of attracting bodies \( M_i \) \( (i = 1, 2, 3) \). We assume that the attracting bodies form an equilateral triangle (Lagrangian triangle) and move in circular orbits. The infinitesimal body \( P \) moves in the plane of the Lagrangian triangle. For any values of the masses of
attracting bodies, we carry out a linear stability analysis and study possible scenarios of bifurcation for central configuration formed by small body $P$ with attracting bodies $M_i$.

2. Formulation of the problem

Let us introduce rotating coordinate system $Oxyz$, which is rigidly connected with the Lagrangian triangle. The origin of this system is located in the middle of the segment $M_2M_3$. The axis $Ox$ passes through this segment. The axis $Oy$ is perpendicular to the $Ox$ and passes through the attracting body $M_1$. The axis $Oz$ complements the coordinate system to the right, orthogonal triple.

![Figure 1. Coordinate system.](image)

The equations of motion of the infinitesimal body $P$ can be written in the form of Hamilton's equations

\[
\frac{dξ}{dτ} = \frac{∂H}{∂p_ξ}, \quad \frac{dn}{dτ} = \frac{∂H}{∂p_η}, \quad \frac{dp_ξ}{dτ} = -\frac{∂H}{∂ξ}, \quad \frac{dp_η}{dτ} = -\frac{∂H}{∂η}.
\]  

(1)

The dimensionless coordinates $ξ, η$ are introduced by means of formulae $x = rξ, y = rη$, where $x$ and $y$ are coordinates of the infinitesimal body in system $Oxyz$ and $r = const$ is the distance between the primaries. The dimensionless time $τ$ is introduced by means of formula $τ = ωt$, where $ω^2 = f(m_1 + m_2 + m_3) r^{-3}$ and $f$ is the gravitational constant. Let us note that $ω$ is the angular velocity of Lagrange triangular configuration.

The Hamiltonian function reads

\[
H = \frac{1}{2} (p_ξ^2 + p_η^2) + p_ξ η − p_η ξ = \frac{1}{2} \left(1 - \frac{μ_2 + μ_3}{2}\right)p_ξ - \frac{μ_2 - μ_3}{2}p_η - \frac{1 - μ_2 - μ_3}{ρ_1} - \frac{μ_2 - μ_3}{ρ_2} - \frac{μ_2 - μ_3}{ρ_3},
\]  

(2)

where

\[
ρ_1 = \left(ξ^2 + \left(η - \frac{\sqrt{3}}{2}\right)^2\right)^{1/2}, \quad ρ_2 = \left(\left(ξ + \frac{1}{2}\right)^2 + \eta^2\right)^{1/2}, \quad ρ_3 = \left(\left(ξ - \frac{1}{2}\right)^2 + \eta^2\right)^{1/2}.
\]  

(3)

\[
μ_2 = \frac{m_2}{m_1 + m_2 + m_3}, \quad μ_3 = \frac{m_3}{m_1 + m_2 + m_3}.
\]  

(4)

It is worth noting that the coordinates of the system mass center in dimensionless variables $ξ, η$ read

\[
ξ_c = -\frac{μ_2 - μ_3}{2}, \quad η_c = \frac{\sqrt{3}(1 - μ_2 - μ_3)}{2}.
\]  

(5)

In terms of the stability of the central configurations, we are only interested in values of the parameters $μ_2$ and $μ_3$, for which the Lagrangian triangular configuration of three bodies is stable. The necessary condition for the stability of such a configuration is the well-known Routh condition [12], which in terms of the parameters $μ_2$ and $μ_3$ reads
\[ 1 + 27(\mu_2^2 + \mu_2\mu_3 + \mu_3^2 - \mu_2 - \mu_3) > 0 \]  

Without loss of any generality, we assume that the mass of the body \( M_1 \) is greater than the masses of the bodies \( M_2 \) and \( M_3 \). It follows from inequality (6) that the quantities \( \mu_2 \) and \( \mu_3 \) can take values from the interval

\[ 0 < \mu_i < \frac{1}{2} - \frac{\sqrt{69}}{18} \quad (i = 2, 3). \]

If the Routh condition is not satisfied, then the central configuration of bodies \( M_i \) \((i = 1, 2, 3)\) is unstable. It yields the instability of the corresponding central configuration of four bodies. In what follows, we assume that the Routh condition is satisfied.

The equations of motion (1) have the following stationary solution describing the relative equilibria of infinitesimal body \( P \) in the rotating coordinate system \( Oxyz \).

\[ \xi = \xi_*, \quad \eta = \eta_*, \quad p_\xi = -\eta_* + \frac{\sqrt{3}}{2} (1 - \mu_2 - \mu_3), \quad p_\eta = \xi_* + \frac{1}{2} (\mu_2 - \mu_3), \]

where \( \xi_* \) and \( \eta_* \) satisfy the following equations

\[ \frac{\xi + \frac{\mu_2}{2} - \frac{\mu_3}{2}}{\rho_1^3} - \frac{(1 - \mu_2 - \mu_3)}{\rho_2^3} \xi - \frac{\mu_2}{\rho_2} \left( \xi + \frac{1}{2} \right) - \frac{\mu_3}{\rho_3^3} \left( \xi - \frac{1}{2} \right) = 0, \]

\[ \eta - \frac{\sqrt{3}(1 - \mu_2 - \mu_3)}{2} - \frac{(1 - \mu_2 - \mu_3)}{\rho_1^3} \left( \eta - \frac{\sqrt{3}}{2} \right) - \frac{\mu_2}{\rho_2} \eta - \frac{\mu_3}{\rho_3} \eta = 0. \]

The equilibrium of the body \( P \) in the rotating coordinate system \( Oxyz \) corresponds to its absolute motion in a circular orbit. In this motion the body \( P \) forms a constant central configuration with the bodies \( M_i \).

In the next sections for any parameters values we investigate linear stability of the corresponding central configuration and perform analysis of their bifurcations.

### 3. Bifurcation of relative equilibria

Let us first consider the limiting cases \( m_2 = 0 \) and \( m_3 = 0 \), when our problem degenerates into the restricted three-body problem. In this cases, the central configurations degenerate into Euler and Lagrangian libration points. The central configurations for the limiting cases are shown in figures 2 and 3. The positions of the body \( P \) is indicated by blue points. To designate the relative equilibrium \( P \) in limiting cases, we used the standard notation for libration points \( L_j^{(k)} \), where the superscript \( k \) corresponds to the case \( \mu_k = 0 \).

![Figure 2](image-url). Central configurations in the limiting case \( \mu_2 = 0 (m_2 = 0) \).
Figure 3. Central configurations in the limiting case $\mu_3 = 0$ ($m_3 = 0$).

For parameter values satisfying the Routh condition, the regions of relative equilibria for the infinitesimal body $P$ were numerically constructed (see figure 4). It turned out that for parameter values $\mu_2 \neq 0$ and $\mu_3 \neq 0$ there are exactly eight relative equilibria, which are located in narrow regions. These regions are located either along a circle of unit radius, whose center is in the body $M_1$ or along the straight lines passing through $M_1$, $M_3$ and $M_1$, $M_2$ respectively (see figure 4). The libration points $L_j^{(k)}$ are the limit points of the indicated regions, so that the relative equilibria pass into one of the libration points at $\mu_2 = 0$ or $\mu_3 = 0$.

Figure 4. Regions of relative equilibria.

By $P_{ij}$ we denote the relative equilibrium that degenerates into a libration point $L_i^{(2)}$ as $\mu_2 \to 0$ and degenerates into a libration point $L_j^{(3)}$ as $\mu_3 \to 0$. The relative equilibria $P_{ij}$ and corresponding libration points are shown in table 1.
Table 1. Relative equilibria and their limiting positions.

| $\mu_2 = 0$  | $\mu_2 \neq 0$, $\mu_3 \neq 0$ | $\mu_3 = 0$  |
|--------------|---------------------------------|--------------|
| $L_1^{(2)}$  | $P_{15}$                         | $L_5^{(3)}$  |
| $L_2^{(2)}$  | $P_{25}$                         | $L_5^{(3)}$  |
| $L_3^{(2)}$  | $P_{33}$                         | $L_3^{(3)}$  |
| $L_4^{(2)}$  | $P_{45}$                         | $L_3^{(3)}$  |
| $L_5^{(2)}$  | $P_{51}$                         | $L_1^{(3)}$  |
| $L_5^{(2)}$  | $P_{52}$                         | $L_2^{(3)}$  |
| $L_5^{(2)}$  | $P_{54}$                         | $L_4^{(3)}$  |
| $L_5^{(2)}$  | $P_{55}$                         | $L_5^{(3)}$  |

The regions of relative equilibria $P_{ij}$ have no common points except for the limiting positions $L_5^{(2)}$ and $L_5^{(3)}$ corresponding to the cases $\mu_2 = 0$ and $\mu_3 = 0$. At $\mu_k = 0$ ($k = 2, 3$) bifurcations of relative equilibria can occur. In fact, the following scenarios of bifurcations take place. The relative equilibria $P_{51}, P_{52}, P_{54}$ and $P_{55}$ approach $L_5^{(2)}$ as $\mu_2 \to 0$ and merge into the point $L_5^{(2)}$ at $\mu_2 = 0$. Similarly, the relative equilibria $P_{15}, P_{25}, P_{45}$ and $P_{55}$ approach $L_5^{(3)}$ as $\mu_3 \to 0$ and merge into the point $L_5^{(3)}$ at $\mu_3 = 0$. Regions of relative equilibria near $L_5^{(2)}$ and $L_5^{(3)}$ are shown in figures 5, 6.

![Figure 5](image1.png) **Figure 5.** The fragments of regions for existence of relative equilibria near the point $L_5^{(2)}$.

![Figure 6](image2.png) **Figure 6.** The fragments of regions for existence of relative equilibria near the point $L_5^{(3)}$. 
From the above numerical analysis, it is clear that bifurcations of relative equilibria are possible only in the limiting cases mentioned.

Note also that the regions of existence of relative equilibria were constructed using the method of numerical continuation in the parameters. In the vicinity of the points \( L_5^{(2)} \) and \( L_5^{(3)} \), the numerical analysis becomes more complicated and requires higher accuracy of calculations. Let us note that in the vicinity of the points \( L_5^{(2)} \) and \( L_5^{(3)} \) the study of the relative equilibria can be carried out analytically by using the small parameter method, which in this case is more efficient than the numerical analysis.

4. Stability of central configuration

In this section, we present the results of studying the stability of central configurations in linear approximation. If the Routh condition is satisfied, then the problem of stability of the central configuration is reduced to studying the stability of relative equilibria of the body \( P \) in the coordinate system \( O \xi \eta \) rotating together with the attracting centers.

In order to study the stability of equilibria \( P_{ij} \) we introduce local canonical variables \( q_1, q_2, p_1 \) and \( p_2 \) in a neighborhood of the above equilibria

\[
\xi = \xi_* + q_1, \quad \eta = \eta_* + q_2,
\]

(10)

and expand the Hamiltonian (2) in a series with respect to powers of the new canonical variables

\[
H = H_2 + H_3 + H_4 + \cdots.
\]

(11)

The quadratic part \( H_2 \) of the Hamiltonian reads

\[
H_2 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(aq_1^2 + b q_2^2) + c q_1 q_2 + p_1 q_2 - p_2 q_1,
\]

(12)

where the coefficients \( a, b, c \) have the following explicit form

\[
a = -\frac{(1 - \mu_2 - \mu_3)}{\rho_1^2}(3 \xi_*^2 - \rho_2^2) + \frac{\mu_2}{\rho_2^2} \left( \rho_2^2 - 3 \left( \xi_* + \frac{1}{2} \right)^2 \right) + \frac{\mu_3}{\rho_3^2} \left( \rho_3^2 - 3 \left( \xi_* - \frac{1}{2} \right)^2 \right),
\]

(13)

\[
b = -\frac{(1 - \mu_2 - \mu_3)}{\rho_1^2} \left( 3 \left( \eta_* - \frac{\sqrt{3}}{2} \right)^2 - \rho_1^2 \right) + \frac{\mu_2}{\rho_2^2} \left( \rho_2^2 - 3\eta_*^2 \right) + \frac{\mu_3}{\rho_3^2} \left( \rho_3^2 - 3\eta_*^2 \right),
\]

\[
c = -\frac{3(1 - \mu_2 - \mu_3)}{\rho_1^2} \xi_* \left( \eta_* - \frac{\sqrt{3}}{2} \right) - \frac{3\mu_2}{\rho_2^2} \left( \xi_* + \frac{1}{2} \right) \eta_* - \frac{3\mu_3}{\rho_3^2} \left( \xi_* - \frac{1}{2} \right) \eta_*.
\]

The question of the stability for this linear system with the Hamiltonian \( H_2 \) can be solved by analysing the roots of its characteristic equation

\[
\lambda^4 + (a + b + 2) \lambda^2 + ab - c^2 - a - b + 1 = 0.
\]

(14)

In accordance with Lyapunov's theorem [13], the linear system is stable if all roots of the characteristic equation are simple and purely imaginary. This condition satisfied if the coefficients of equation (14) satisfy the following inequalities

\[
a + b + 2 > 0, \quad ab - c^2 - a - b + 1 > 0,
\]

(15)

\[
a^2 + b^2 + 4c^2 - 2ab + 8(a + b) > 0.
\]

Conditions (15) were verified numerically for all possible values of the parameters \( \mu_2, \mu_3 \) and the following conclusions on stability were obtained. The relative equilibrium \( P_{33} \) is unstable for all values of the parameters \( \mu_2 \) and \( \mu_3 \). The relative equilibria \( P_{15} \) and \( P_{25} \) are unstable when \( \mu_3 \neq 0 \). In the limiting case \( \mu_3 = 0 \) the equilibria \( P_{15} \) and \( P_{25} \) degenerate into the libration point \( L_5^{(3)} \), which is stable in linear approximation for all values of the parameter \( \mu_2 \). Similarly, the relative equilibria \( P_{51} \)
and $P_{52}$ are unstable when $\mu_2 \neq 0$. If $\mu_2 = 0$, then $P_{51}$ and $P_{52}$ degenerate into the libration point $L_5^{(2)}$, which is stable in linear approximation for all values of the parameter $\mu_3$.

The most interesting situation takes place for the relative equilibria $P_{45}$, $P_{54}$ and $P_{55}$. These relative equilibria, depending on the values of the parameters, can be both stable and unstable. For these relative equilibria stability diagrams were constructed in the plane of the parameters (see figures 7, 8, 9). The instability domains are indicated by red color and domains of stability the linear approximation are indicated by blue color.

Note that in the limiting case $\mu_2 = 0$ the relative equilibria $P_{45}$, $P_{54}$ and $P_{55}$ degenerate into the libration point $L_5^{(2)}$, which is stable in linear approximation for all values of the parameter $\mu_3$. Similar, in the limiting case $\mu_3 = 0$ they degenerate into the point of libration $L_5^{(3)}$, which is stable in linear approximation for all values of the parameter $\mu_2$.

**Figure 7.** Domains of stability of relative equilibria $P_{45}$.

**Figure 8.** Domains of stability of relative equilibria $P_{54}$.

**Figure 9.** Domains of stability of relative equilibria $P_{55}$. 
Conclusions
Let us briefly formulate the results of the study. It has been numerically established that if Routh conditions are satisfied, then there exist exactly eight central configurations, which correspond to the relative equilibria of the infinitesimal body \( P \) in the coordinate system rotating together with the attracting centers \( M_1, M_2 \) and \( M_3 \). In the limiting cases, when an attracting center has zero mass, the relative equilibria degenerate into triangular (Lagrangian) or rectilinear (Euler) libration points. In this case, bifurcation is possible. It occurs according to the following scenarios. Four relative equilibria \( P_{51}, P_{52}, P_{54} \) and \( P_{55} \) pass into the libration point \( L_{5}^{(2)} \) at \( \mu_2 = 0 \), and four other relative equilibria \( P_{15}, P_{25}, P_{45} \) and \( P_{55} \) pass into the libration point \( L_{5}^{(3)} \) at \( \mu_3 = 0 \). There are no other bifurcations.

The regions of existence of possible relative equilibria of the infinitesimal body are numerically constructed for all values of the parameters that satisfy the Routh conditions. By using the numerical analysis, it was established that the position of relative equilibrium \( P_{33} \) is unstable for all values of the parameters. The relative equilibria \( P_{15} \) and \( P_{25} \) are linearly stable only in the limiting case \( \mu_3 = 0 \), when they pass into the libration point \( L_{5}^{(3)} \), and \( P_{51} \) and \( P_{52} \) are linearly stable only in the limiting case \( \mu_2 = 0 \), when they pass into the libration point \( L_{5}^{(2)} \). The relative equilibria \( P_{45}, P_{54} \) and \( P_{55} \) can be both stable and unstable depending on value of parameters.

Acknowledgments
This work was supported by the grant of the Russian Scientific Foundation (project No. 19-11-00116) at the Moscow Aviation Institute (National Research University).

References
[1] Pedersen P 1944 Dan. Mat.-Fys. Medd 21 1-80
[2] Pedersen P 1952 Dan. Mat.-Fys. Medd 26 1-37
[3] Budzko D A and Prokopenya A N 2010 Programming and Computer Software 36 68-74
[4] Leandro E S G 2006 J. Differential Equations 226 323–351
[5] Howell K C and Spencer D B 1986 Acta Astronomica 13 473-479
[6] Brumberg V A 1957 Soviet Astronomy 1 57–79
[7] Budzko D A 2009 Computer algebra systems in teaching and research 1 28-36
[8] Grebenikov E A, Gadomski L and Prokopenya A N 2007 Nonlinear Oscillations 10 62–77
[9] Budzko D A 2011 Bull. of the National Academy of Sciences of Belarus 4 55-59
[10] Bardin B S and Volkov E V 2020 Stability Study of a Relative Equilibrium in the Planar Circular Restricted Four-Body Problem IOP Conf. Ser.: Materials Science and Engineering 927 012012
[11] Bardin B S and Esipov P A 2018 Investigation of Lyapunov stability of a central configuration in the restricted four-body problem AIP Conf. Proc. 1959 040004
[12] Routh E J 1875 Proc. London Math. Soc. 6 86–97
[13] Lyapunov A M 1992 The General Problem of the Stability of Motion (London: Taylor & Francis)