The sine kernel, two corresponding operator identities, and random matrices

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Abstract

In the present paper, we consider the integral operator, which acts in Hilbert space and has sine kernel. This operator generates two operator identities and two corresponding canonical differential systems. We find the asymptotics of the corresponding resolvent and Hamiltonians. We use both the method of operator identities and the theory of random matrices.

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1 Introduction

Let us consider the operator

\[ S_\zeta f = f(x) - \int_0^\zeta k(x-y)f(y)dy, \quad \zeta > 0, \quad (1.1) \]

where

\[ k(x) = \frac{\sin(x\pi)}{x\pi}. \quad (1.2) \]

The operator \( S_\zeta \) is invertible (see [2, p. 167]). Hence, we have

\[ S_\zeta^{-1} f = f(x) + \int_0^\zeta R_\zeta(x,y)f(y)dy, \quad f(y) \in L^2(0,\zeta), \quad (1.3) \]
where the function $R_{\zeta}(x, y)$ is continuous with respect to the variables $x, y, \zeta$. The operator $S_{\zeta}$ plays an important role in a number of theoretical and applied problems (e.g., in random matrix theory [2,9,10] and in optical problems [3]). The operator $S_{\zeta}$ satisfies simultaneously two operator identities and generates two canonical differential systems [5–9]. In the present paper, we investigate the asymptotics of the kernel $R_{\zeta}(x, y)$ and of the Hamiltonians of the corresponding canonical systems when $\zeta \to \infty$ (see Theorem 3.3). In this case, we use both the method of operator identities and the results from theory of random matrices (see [10] and section 4 of the present paper). Let us introduce the operators:

$$K_{\pm}(\zeta) f = \int_{-1}^{1} k_{\pm}(x, t, \zeta) f(t) dt, \quad f(x) \in L^2(-1, 1), \quad (1.4)$$

and

$$K(\zeta) f = \int_{-1}^{1} k(x, t, \zeta) f(t) dt, \quad f(x) \in L^2(-1, 1), \quad (1.5)$$

where

$$k(x, t, \zeta) = \frac{\sin \zeta \pi(x - t)}{\pi(x - t)}, \quad (1.6)$$

$$k_{\pm}(x, t, \zeta) = \left[ \frac{\sin \zeta \pi(x - t)}{\pi(x - t)} \pm \frac{\sin \zeta \pi(x + t)}{\pi(x + t)} \right] / 2 \quad (1.7)$$

In section 4, we investigate the operators

$$S(\zeta, \lambda) = I - \lambda K(\zeta) \quad \text{and} \quad S_{\pm}(\zeta, \lambda) = I - \lambda K_{\pm}(\zeta), \quad 0 < \lambda \leq 1. \quad (1.8)$$

We note that Fredholm determinants

$$P_{\pm}(\zeta, \lambda) = \det(I - \lambda K_{\pm}(\zeta)), \quad P(\zeta, \lambda) = \det(I - \lambda K(\zeta)) \quad (1.9)$$

play an important role in the random matrix theory [10]. We found special integral representations for $P_{\pm}(\zeta, \lambda)$ and for $P(\zeta, \lambda)$. These results may be applied to a number of problems (see Remark 4.8).
2 Two operator identities and two canonical differential systems

The operator $S_\zeta$, which is defined by formulas (1.1) and (1.2), satisfies the following operator identity (see [5]-[9]):

$$(QS_\zeta - S_\zeta Q)f = -\frac{1}{2i\pi} \int_0^\zeta [e^{i(x-y)\pi} - e^{-i(x-y)\pi}]f(y)dy,$$

where

$$ Qf = xf(x).$$

The second operator identity has the form [8]:

$$(AS_\zeta - S_\zeta A)f = i \int_0^\zeta [M(x) + M(y)]f(y)dy;$$

where

$$ Af = i \int_0^x f(x)dx, \quad M(x) = \frac{1}{2} - \int_0^x k(x)dx. $$

The operator $S_\zeta$ and operator identities (2.1) and (2.3) generate two canonical differential systems. The first system is connected with identity (2.1) and has the form [5]:

$$ \frac{d}{dx}W_1(x, z) = -i J_1 H_1(x) \frac{W_1(x, z)}{x-z}, \quad W_1(0, z) = I_2, $$

where $J_1$ and $H_1(x)$ are defined by the relations

$$ H_1(x) = \frac{1}{2\pi} \begin{pmatrix} |q(x)|^2 & q^2(x) \\ q^2(x) & |q(x)|^2 \end{pmatrix}; \quad J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q(x) = S_{-1}^{-1} e^{ix\pi}. $$

The second system is connected with identity (2.3) and has the form [5]

$$ \frac{d}{dx}W_2(x, z) = iz J_2 H_2(x) W_2(x, z), \quad W_2(0, z) = I_2, $$

where $J_2$ and $H_2(x)$ are defined by the relations

$$ H_2(x) = \frac{1}{2\pi} \begin{pmatrix} q_1^2(x) & 1/2 \\ 1/2 & q_2^2(x) \end{pmatrix}; \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, $$

where $q_1(x)$ and $q_2(x)$ are defined by the relations

$$ q_1(x) = e^{-ix\pi}, \quad q_2(x) = S_{-1}^{-1} e^{ix\pi}. $$
\[ q_1(x) = S^{-1}1, \quad q_1(x)q_2(x) = 1/2. \] (2.10)

The operator \( S^{-1} \) is defined by the Krein formula (see [1]Ch.IV and (1.3)):
\[ S^{-1}f = f(x) + \int_0^x R_x(x,y)f(y)dy. \] (2.11)

We note that the operator \( S_\zeta \) admits the triangular factorization (see [1]Ch.IV):
\[ S_\zeta = S_-S^*. \] (2.12)

### 3 Asymptotic behaviour of Hamiltonians \( H_1(\zeta), H_2(\zeta) \) and resolvent kernel \( R_\zeta(x,y) \) as \( \zeta \to \infty \)

Along with the operator \( S_\zeta \) we consider the operator
\[ C_\zeta f = f(x) - \int_{-\zeta}^\zeta k(x-y)f(y)dy, \quad \zeta > 0, \] (3.1)
where the kernel \( k(x) \) is defined by (1.2). The operator \( U_\zeta f(x) = f(x-\zeta) \) maps unitary the space \( L^2(-\zeta,\zeta) \) onto \( L^2(0,2\zeta) \). It is easily to see that
\[ C_\zeta = U_\zeta^{-1}S_2U_\zeta. \] (3.2)

By (1.3) and (3.2) we have
\[ C_\zeta^{-1}f = f(x) + \int_{-\zeta}^\zeta Q_\zeta(x,y)f(y)dy, \quad f(y) \in L^2(-\zeta,\zeta) \] (3.3)
where
\[ R_{2\zeta}(x,y) = Q_\zeta(x-\zeta,y-\zeta). \] (3.4)

Relation (3.3) implies that
\[ R_{2\zeta}(2\zeta,2\zeta) = Q_\zeta(\zeta,\zeta), \quad R_{2\zeta}(2\zeta,0) = Q_\zeta(\zeta,-\zeta). \] (3.5)

Further we need the following two lemmas.

**Lemma 3.1** Let relations (3.1) and (3.3) be fulfilled. Then we have
\[ Q_\zeta(-x,-y) = Q_\zeta(x,y). \] (3.6)
Proof. From (3.1) and (3.3) we obtain

$$- k(x, y) + Q_\zeta(x, y) - \int_{-\zeta}^{\zeta} Q_\zeta(x, s)k(s, y)\,ds = 0. \quad (3.7)$$

Relation (3.7) implies that

$$- k(-x, -y) + Q_\zeta(-x, -y) - \int_{-\zeta}^{\zeta} Q_\zeta(-x, -s)k(-s, -y)\,ds = 0. \quad (3.8)$$

Using equality

$$k(-x, -y) = k(x, y). \quad (3.9)$$

and relation (3.8) we obtain (3.6). Lemma is proved.

Lemma 3.2 Let relations (3.1) and (3.3) be fulfilled. Then we have

$$q(2\zeta) = e^{i\zeta\pi}r(\zeta), \quad (3.10)$$

where $r(\zeta)$ is defined by the equality

$$r(\zeta) = e^{i\zeta\pi} + \int_{-\zeta}^{\zeta} Q_\zeta(-\zeta, s)e^{-is\pi}\,ds. \quad (3.11)$$

Proof. It follows from (3.11) that

$$e^{i\zeta\pi}r(\zeta) = e^{2i\zeta\pi} + \int_{-\zeta}^{\zeta} Q_\zeta(-\zeta, -s)e^{i(\zeta+s)\pi}\,ds. \quad (3.12)$$

Formula (3.12) can be written in the form:

$$e^{i\zeta\pi}r(\zeta) = e^{2i\zeta\pi} + \int_{0}^{2\zeta} Q_\zeta(-\zeta, \zeta - t)e^{it\pi}\,dt. \quad (3.13)$$

Taking into account (3.4) and (3.6), we have

$$e^{i\zeta\pi}r(\zeta) = e^{2i\zeta\pi} + \int_{0}^{2\zeta} R_{2\zeta}(2\zeta, t)e^{it\pi}\,dt. \quad (3.14)$$

The assertion of the lemma follows directly from (2.7), (2.11) and (3.14).

Let us formulate the main result of the present section:
Theorem 3.3 Let relation (1.1), (1.2) be fulfilled. Then the following asymptotic equalities are valid:

1) \[ R_{2\zeta}(2\zeta, 2\zeta) \sim \pi \left( \frac{1}{4}u + \frac{1}{4}u - \sum_{n=1}^{\infty} \frac{c_{2n}}{u^{2n+1}} \right), \quad u \to \infty, \quad (u = 2\pi \zeta), \quad (3.15) \]

where \( c_2 = -\frac{1}{4}, c_4 = -\frac{5}{2}, \ldots \)

2) \[ R_{\zeta}(\zeta, 0) \sim \pi \sum_{n=0}^{\infty} \frac{a_{2n}}{u^{2n}}, \quad u \to \infty, \quad (3.16) \]

where \( a_0^2 = 1/4, 2a_0a_2 = -1/4, a_4 + a_2^2 = 3c_2, a_6 + 2a_2a_4 = 5c_4 \)

3) \[ |q(2\zeta)|^2 \sim \pi \left( \frac{1}{2}u + \sum_{n=1}^{\infty} \frac{2nc_{2n}}{u^{2n+1}} \right), \quad u \to \infty. \quad (3.17) \]

4) \[ q^2(2\zeta) = e^{iu\pi} \left[ \sum_{n=0}^{\infty} \frac{a_{2n}(1-2n)}{u^{2n}} + i \sum_{n=0}^{\infty} \frac{a_{2n}}{u^{2n-1}} \right], \quad u \to \infty. \quad (3.18) \]

Proof. We use the well-known system [4], [10]:

\[ \frac{d}{d\zeta} [\zeta Q_\zeta(\zeta, \zeta)] = |r^2(\zeta)|, \quad 2\pi \zeta Q_\zeta(-\zeta, \zeta) = 3|r^2(\zeta)| \quad (3.19) \]

\[ \frac{d}{d\zeta} [Q_\zeta(\zeta, \zeta)] = 2Q_\zeta^2(-\zeta, \zeta) \quad \frac{d}{d\zeta} [\zeta Q_\zeta(-\zeta, \zeta)] = \Re[r^2(\zeta)]. \quad (3.20) \]

We need also the asymptotic relation (see [10], formula (89)):

\[ -2\zeta Q_\zeta(\zeta, \zeta) \sim -\frac{1}{4}u^2 - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{c_{2n}}{u^{2n}}, \quad u \to \infty, \quad (u = 2\pi \zeta). \quad (3.21) \]
where \( c_2 = -\frac{1}{4}, \ c_4 = -\frac{5}{2} \). Formulas (3.19) and (3.21) imply that

\[
|r(\zeta)|^2 \sim \pi \left( \frac{1}{2} u + \sum_{n=1}^{\infty} \frac{2nc_{2n}}{u^{2n+1}} \right), \quad u \to \infty.
\]

(3.22)

According to (3.21) we have

\[
Q_\zeta(\zeta, \zeta) \sim \pi \left( \frac{1}{4} u + 1 - \sum_{n=1}^{\infty} \frac{c_{2n}}{u^{2n+1}} \right), \quad u \to \infty.
\]

(3.23)

Using relations (3.20) and (3.23) we derive that

\[
Q_\zeta(\zeta, \zeta) \sim \pi^{2} \left( \frac{1}{4} u - \sum_{n=1}^{\infty} \frac{c_{2n}(2n+1)}{u^{2n+2}} \right), \quad u \to \infty.
\]

(3.24)

Consequently,

\[
Q_\zeta(-\zeta, \zeta) \sim \pi \sum_{n=0}^{\infty} \frac{a_{2n}}{u^{2n}}, \quad u \to \infty,
\]

(3.25)

where in view of (3.24) we get

\[
a_0^2 = 1/4, \ 2a_0a_2 = -1/4, \ 2a_0a_4 + a_2^2 = 3c_2, \ 2a_0a_6 + 2a_2a_4 = 5c_4, \ldots
\]

(3.26)

Taking into account (3.19) and (3.20) we have

\[
r^2(\zeta) = \frac{d}{d\zeta} \left[ \zeta Q_\zeta(-\zeta, \zeta) \right] + i2\pi \zeta Q_\zeta(-\zeta, \zeta).
\]

(3.27)

According to (3.25) and (3.27) the equality

\[
r^2(\zeta) \sim \pi \sum_{n=0}^{\infty} \frac{a_{2n}(1-2n)}{u^{2n}} + i\pi \sum_{n=0}^{\infty} \frac{a_{2n}}{u^{2n-1}}, \quad u \to \infty.
\]

(3.28)

holds. Comparing formulas (3.5), (3.10) with (3.22), (3.23) and (3.25), (3.28) we obtain the assertion of the theorem.

**Remark 3.4** In the next section we shall prove that

\[
a_0 = 1/2.
\]

(3.29)

Hence, using relations (3.20) we can find all coefficients \( a_{2n} \), \( n = 1, 2, \ldots \).
In order to derive the asymptotic of $H_2(\zeta)$ we use the well-known Krein’s formula (see [1], Ch.IV):

\[ q_1^2(\zeta) = \exp[2 \int_0^\zeta R_t(t,0)dt]. \] (3.30)

From (3.25), (3.27) and (3.30) we deduce that

\[ \log[q_1^2(\zeta)] = \pi \zeta + \beta + o(1), \quad \zeta \to \infty, \] (3.31)

where

\[ \beta = \int_0^\zeta [2R_t(t,0) - \pi]dt. \] (3.32)

It follows from (2.10) and (3.31) that

\[ \log[q_2^2(\zeta)] = -\pi \zeta - \beta - 2 \log 2 + o(1), \quad \zeta \to \infty. \] (3.33)

**Remark 3.5** Taking into account (2.6) and (3.17), (3.18), (3.29) we obtain the asymptotic of $H_1(\zeta)$.

**Remark 3.6** Taking into account (2.9) and (3.31) - (3.33) we obtain the asymptotic of $H_2(\zeta)$.

*Let us investigate the expressions: $(S^{-1}_\zeta e^{it\pi}, e^{it\pi})$ and $(S^{-1}_\zeta 1, 1)$, which are important in the theory of the integral operators with difference kernels (see [8]).

According to (2.7) and (2.10) we have

\[ (S^{-1}_\zeta e^{it\pi}, e^{it\pi}) = \int_0^\zeta |q^2(t)|dt, \quad (S^{-1}_\zeta 1, 1) = \int_0^\zeta |q_1^2(t)|dt \] (3.34)

Relations (3.17) and (3.34) imply that

\[ (S^{-1}_\zeta e^{it\pi}, e^{it\pi}) = \frac{\pi^2 \zeta^2}{4} + O(1), \quad \zeta \to \infty. \] (3.35)

Relations (3.31) and (3.34) imply that

\[ (S^{-1}_\zeta 1, 1) = \frac{1}{\pi} e^{\pi \zeta + \beta} [1 + o(1)], \quad \zeta \to \infty. \] (3.36)
4 Fredholm determinants, integral representations

Let us introduce the operators:

\[ K_\pm(\zeta)f = \int_{-1}^{1} k_\pm(x,t,\zeta)f(t)dt, \quad f(x) \in L^2(-1,1), \quad (4.1) \]

and

\[ K(\zeta)f = \int_{-1}^{1} k(x,t,\zeta)f(t)dt, \quad f(x) \in L^2(-1,1), \quad (4.2) \]

where

\[ k(x,t,\zeta) = \frac{\sin \zeta \pi (x-t)}{\pi (x-t)}, \quad (4.3) \]

\[ k_\pm(x,t,\zeta) = \left[ \frac{\sin \zeta \pi (x-t)}{\pi (x-t)} \pm \frac{\sin \zeta \pi (x+t)}{\pi (x+t)} \right] / 2 \quad (4.4) \]

In this section we shall investigate the operators

\[ S(\zeta,\lambda) = I - \lambda K(\zeta) \quad \text{and} \quad S_\pm(\zeta,\lambda) = I - \lambda K_\pm(\zeta), \quad 0 < \lambda \leq 1. \quad (4.5) \]

It is easy to see that

\[ K_\pm = \frac{I + J}{2} K = K \frac{I + J}{2}, \quad (4.6) \]

where \( Jf(x) = f(-x) \).

**Lemma 4.1** The following equality holds (see [17]):

\[ (I - \lambda K_\pm)^{-1} K_\pm = \frac{I + J}{2} (I - \lambda K)^{-1} K \quad (4.7) \]

**Proof.** It is easy to see that

\[ \left( \frac{I + J}{2} \right)^n = \frac{I + J}{2}, \quad n = 1, 2, \ldots \quad (4.8) \]

Using relations (4.6), (4.8) and inequality \( \|K\| < 1 \) we obtain

\[ (I - \lambda K_\pm)^{-1} - I = \frac{I + J}{2} \sum_{n=1}^{\infty} K^n \lambda^n = \frac{I + J}{2} [(I - \lambda K)^{-1} - I] \quad (4.9) \]
The assertion of Lemma 4.1 follows from equality (4.9).

We consider the Fredholm determinants

\[ P_\pm(\zeta, \lambda) = \det(I - \lambda K_\pm(\zeta)), \quad P(\zeta, \lambda) = \det(I - \lambda K(\zeta)) \]  

(4.10)

**Lemma 4.2** The following relation is valid

\[ \frac{d}{d\zeta} \log P_\pm(\zeta, \lambda) = -\lambda \{ ((I - \lambda K(\zeta))^{-1} \phi_1, \phi_1) \pm \Re[((I - \lambda K(\zeta))^{-1} \phi_1, \phi_1)] \}, \]

where \( \phi_1(x, \zeta) = e^{ix\pi\zeta/\sqrt{2}} \).

**Proof.** Let us write the equality

\[ \frac{d}{d\zeta} \log P_\pm(\zeta, \lambda) = -\lambda \text{tr}[(I - \lambda K_\pm(\zeta))^{-1} \frac{d}{d\zeta} K_\pm(\zeta)]. \]  

(4.11)

It follows from (4.4), (4.8) and (4.11) that

\[ \frac{d}{d\zeta} \log P_\pm(\zeta, \lambda) = -\lambda \text{tr}[(I - \lambda K(\zeta))^{-1} \frac{1}{2} \frac{d}{d\zeta} K(\zeta)]. \]  

(4.12)

According to (4.2) we have

\[ \frac{d}{d\zeta} K(\zeta)f = \frac{1}{2} \int_{-1}^{1} [e^{i\pi\zeta(x-t)} + e^{-i\pi\zeta(x-t)}] f(t) dt. \]

(4.13)

We introduce the one-dimensional operators

\[ T_\pm(\zeta, \lambda)f = \frac{\lambda}{4}(I - \lambda K(\zeta))^{-1} \int_{-1}^{1} e^{\pm i\pi\zeta(x-t)} f(t) dt, \]

\[ V_\pm(\zeta, \lambda)f = \frac{\lambda}{4}(I - \lambda K(\zeta))^{-1} \int_{-1}^{1} e^{\pm i\pi\zeta(-x-t)} f(t) dt, \]

(4.14)

(4.15)

Let us consider a complete orthonormal system functions \( \phi_n(x, \zeta), \quad (n = 1, 2, \ldots) \) in the space \( L^2(-1, 1) \) such, that \( \phi_1(x, \zeta) = e^{ix\pi\zeta/\sqrt{2}} \). Then we have

\[ \text{tr} T_+(\zeta, \lambda) = \frac{\lambda}{4} \sum_{n=1}^{\infty} ((I - \lambda K(\zeta))^{-1} \phi_n)(\phi_n)(\phi_n), \quad \phi = \sqrt{2} \phi_1. \]

(4.16)
It follows from (4.16), that

\[ trT_+(\zeta, \lambda) = \frac{\lambda}{2} ((I - \lambda K(\zeta))^{-1} \phi_1, \phi_1) \]  

(4.17)

In the same way we obtain the relations

\[ trT_-(\zeta, \lambda) = \frac{\lambda}{2} ((I - \lambda K(\zeta))^{-1} \phi_1, \phi_1), \]  

(4.18)

\[ trV_+(\zeta, \lambda) = \frac{\lambda}{2} ((I - \lambda K(\zeta))^{-1} \phi_1, \phi_1), \]  

(4.19)

\[ trV_-(\zeta, \lambda) = \frac{\lambda}{2} ((I - \lambda K(\zeta))^{-1} \phi_1, \phi_1), \]  

(4.20)

The kernel of the operator \( K \) is real. Hence, the relations (4.17)-(4.20) imply, that

\[ trT_-(\zeta, \lambda) = trT_+(\zeta, \lambda), \quad trV_-(\zeta, \lambda) = \overline{trV_+(\zeta, \lambda)}. \]  

(4.21)

Taking into account (4.22) we deduce

\[ \frac{d}{d\zeta} \log P_\pm(\zeta, \lambda) = -\lambda \{ trT_+(\zeta, \lambda) \pm \Re[trV_-(\zeta, \lambda)] \}. \]  

(4.22)

The assertion of the lemma follows directly from relations (4.17), (4.19) and (4.22).

Further we need the operator \( Uf(x) = g(s) = f(s/\zeta)/\sqrt{\zeta} \), which maps isometrically \( L^2(-1,1) \) onto \( L^2(-\zeta, \zeta) \). It is easy to see that \( U^{-1}g(s) = f(x) = g(x\zeta)/\sqrt{\zeta} \) and \( U^* = U^{-1} \). It follows from (4.2) and (4.3) that

\[ K(\zeta) = U^{-1}C_\zeta U, \quad K_\pm(\zeta) = U^{-1}C_\pm(\zeta)U \]  

(4.23)

where

\[ C_\zeta g(s) = \int_{-\zeta}^{\zeta} \frac{\sin \pi(s-t)}{\pi(s-t)} g(s) ds \]  

(4.24)

\[ C_\pm(\zeta) g(s) = \frac{1}{2} \int_{-\zeta}^{\zeta} \left[ \frac{\sin \pi(s-t)}{\pi(s-t)} \pm \frac{\sin \pi(s+t)}{\pi(s+t)} \right] g(s) ds \]  

(4.25)

Using relations (4.10) and (4.23) we obtain, that

\[ P_\pm(\zeta, \lambda) = \det(I - \lambda C_\pm(\zeta))^{-1}, \quad P(\zeta, \lambda) = \det(I - \lambda C_\zeta)^{-1} \]  

(4.26)

Lemma 4.2 and relations (4.23), (4.26) imply the assertion.
Lemma 4.3 The following relation is valid

$$2\zeta \frac{d}{d\zeta} \log P_\pm(\zeta, \lambda) = -\lambda\{(I - \lambda C\zeta)^{-1}\psi, \psi\} \pm \Re\{(I - \lambda C\zeta)^{-1}\psi, \bar{\psi}\},$$

where $\psi(x) = e^{ix\pi}$.

Using relations (3.2) and Lemma 4.3 we get

Lemma 4.4 If $\lambda = 1$, then

$$2\zeta \frac{d}{d\zeta} \log P_\pm(\zeta, 1) = -\{(S_{2\zeta}^{-1}\psi_1, \psi_1) \pm \Re\{(S_{2\zeta}^{-1}\psi_1, \bar{\psi}_1)\}\},$$

(4.27)

where $\psi_1(x) = U_\zeta \psi(x) = \psi(x - \zeta)$.

It follows from (2.7) and (4.27) that

$$2\zeta \frac{d}{d\zeta} \log P_\pm(\zeta, 1) = -\left\{\int_0^{2\zeta} q^2(s) |ds\pm\Re\left[\int_0^{2\zeta} e^{2is\pi} q^2(s) ds\right]\right\}. \quad (4.28)$$

Taking into account (3.10), (3.11) and (4.28) we obtain

Theorem 4.5 If $\lambda = 1$, then

$$\zeta \frac{d}{d\zeta} \log P_\pm(\zeta, 1) = -\left\{\int_0^{\zeta} |r^2(s)| ds \pm \Re\left[\int_0^{\zeta} r^2(s) ds\right]\right\}. \quad (4.29)$$

Now we show that Remark 3.4 is valid.

Corollary 4.6 The equality

$$a_0 = 1/2$$

(4.30)

holds.

Proof. According to (3.28) we have

$$\Re[r^2(\zeta)] = \pi a_0 + o(1), \quad \zeta \to \infty. \quad (4.31)$$

Taking into account (4.29) and (4.31) we obtain

$$\frac{d}{d\zeta} [\log P_+(\zeta, 1) - \log P_-(\zeta, 1)] = -2\pi[a_0 + o(1)]. \quad (4.32)$$
Hence, the following relation is valid
\[ \log P_+(\zeta, 1) - \log P_-(\zeta, 1) = -2\pi \zeta [a_0 + o(1)]. \] (4.33)

Using asymptotic formula (90) from the paper [10] we have
\[ \log P_+(\zeta, 1) - \log P_-(\zeta, 1) = -\pi \zeta [1 + o(1)]. \] (4.34)

Relations (4.33) and (4.34) imply relation (4.30). The Corollary is proved.

Let us introduce the notations
\[ \sigma_+(\zeta) = \zeta \frac{d}{d\zeta} \log P_+(\zeta, 1), \quad \sigma_-(\zeta) = \zeta \frac{d}{d\zeta} \log P_-(\zeta, 1). \] (4.35)

In view of (4.7) we get
\[ \sigma_+(\zeta) + \sigma_-(\zeta) = \sigma(\zeta). \] (4.36)

Theorem 4.5 and relation (4.36) imply the assertion:

**Corollary 4.7** If \( \lambda = 1 \) then
\[ \zeta \frac{d}{d\zeta} \log P(\zeta, 1) = -\int_0^{\zeta} |r(s)| ds. \] (4.37)

We note that relations (4.36) and (4.37) are well-known [10].

**Remark 4.8** Theorem 4.5 (see formula (4.29)) can be used by solving the following problems:
1. Find the asymptotics of \( \zeta \frac{d}{d\zeta} \log P_+(\zeta, 1) \) when \( \zeta \to \infty \) with the help of formulas (3.22) and (3.28).
2. Find the asymptotics of \( \zeta \frac{d}{d\zeta} \log P_-(\zeta, 1) \) when \( \zeta \to 0 \) with the help of formula (3.11).
3. Find the estimation of \( \zeta \frac{d}{d\zeta} \log P_+(\zeta, 1) \) with the help of formula (3.11).

**References**

[1] Gohberg I. and Krein M.G., *Theory and applications of Volterra operators in Hilbert space*, Amer.Math. Soc., 1970.
[2] Deift P.A., Its A.R., Zhou X., *A Riemann-Hilbert approach to asymptotic arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics*, Annals of Mathematics, 146, 149–235, 1997.

[3] Levin B.R., *Theoretical foundations of statistical radio engineering*, Moscow, 1968 (Russian).

[4] Mehta M.L., *A non-linear differential equation and Fredholm determinant*, J. Physique I 2, 1721–1729, 1992.

[5] Sakhnovich L.A., *Spectral Theory of Canonical Differential Systems. Method of Operator Identities*, Operator Theory: Advances and Applications, 107, 1999.

[6] Sakhnovich L.A., *Integrable operators and canonical differential systems*, Math. Nachr. 280, no.1-2, 205–220, 2007.

[7] Sakhnovich L.A., *Operators similar to unitary operators with absolutely continuous spectrum*, Funkcional. Anal i Prilozen. 2:1, 51–63, 1968 (Russian).

[8] Sakhnovich L.A., *Integral equations with difference kernels on finite intervals*, Operator Theory: Advances and Applications, 84, 2015, second edition.

[9] Sakhnovich L.A., *The Krein differential system and integral operators of random matrix theory*, St. Petersburg Math. J., 22, no. 5, 835–846, 2011.

[10] Tracy C.A. and Widom H., *Introduction to random matrices*, Springer Lecture Notes, 424, 103–130, 1993.