Two-loop and $n$-loop vertex corrections for eikonal diagrams with massive partons

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Abstract

We present the ultraviolet poles and finite terms of two-loop vertex corrections for diagrams with massive partons in the eikonal approximation. We discover and prove that the results for a set of the corrections generalize to $n$-loop eikonal diagrams. These results will enhance theoretical understanding and allow greater calculational accuracy for the many partonic processes where the eikonal approximation is applied.
Perturbation theory for partonic processes entails the calculation of increasingly complicated loop diagrams as one moves to higher orders. At present, many QCD processes and some beyond the Standard Model processes have been calculated to next-to-leading order (NLO) in the strong coupling $\alpha_S$, and they require the evaluation of one-loop diagrams. Higher-order calculations involve diagrams at two-loops and higher \[1, 2\].

Beyond the use of standard fixed-order calculations in perturbation theory, it is possible to evaluate higher-order contributions to partonic cross sections, that are dominant in certain regions of phase space, using a panoply of resummation methods and related techniques. The eikonal approximation has been an extremely useful tool in these other approaches to perturbative calculations. The eikonal approximation is valid for emission of soft gluons from partons in the hard scattering and leads to a simplified form of the Feynman rules. When the gluon momentum goes to zero, the usual Feynman rules for the quark propagator and quark-gluon vertex in the diagram in Figure 1 simplify as follows:

$$\bar{u}(p) \left(- i g_s T_F^c \gamma^\mu \right) \frac{i(p' + k' + m)}{(p + k)^2 - m^2 + i\epsilon} \rightarrow \bar{u}(p) g_s T_F^c \gamma^\mu \frac{p' + m}{2p \cdot k + i\epsilon} = \bar{u}(p) g_s T_F^c \frac{v^\mu}{v \cdot k + i\epsilon} \tag{1}$$

with $v$ a dimensionless vector, $p \propto v$, $g_s^2 = 4\pi \alpha_s$, and $T_F^c$ the generators of SU(3) in the fundamental representation.

The eikonal approximation is an essential element in calculations of QCD cross sections in certain kinematical regions. Apart from its use in deriving the soft part of the cross section in standard fixed-order QCD calculations, it has been applied to calculations of the high-energy behavior of elastic quark-quark scattering, (near) forward scattering amplitudes, and wide-angle elastic scattering \[3, 4, 5, 6\]. In addition, the eikonal approximation is invaluable in the development of transverse momentum and threshold resummations for a variety of partonic processes \[1, 2, 7-14\].

The ultraviolet (UV) pole structure of $n$-loop vertex correction diagrams involving eikonal lines is particularly worthy of attention, because, in addition to its intrinsic theoretical interest, it plays a direct role in renormalization group evolution equations that have been used in studies of hard elastic scattering \[3, 4\] and threshold resummations \[10-14\]. At present, results are known to one loop and have been used in state-of-the-art next-to-leading logarithm (NLL) resummations and their expansions to NNLO and higher orders for many QCD processes \[10-17\]. Threshold resummations will also play a major role in calculations of cross sections for Higgs production and supersymmetric particles since these particles are expected to be discovered close to threshold. The successes of NLL resummations, including dramatic reductions in scale dependence, have been quite remarkable; however there are still sizable uncertainties.
in the calculations due to formally subleading terms which in practice can have significant contributions. To go beyond the level of NLL accuracy and control subleading terms, two loop and even higher-order eikonal calculations are needed. These calculations are also useful for increasing theoretical accuracy in the other many applications of the eikonal approximation, especially when renormalization group evolution equations are used. The calculation of two-loop vertex corrections with eikonal lines representing massive partons will be the subject of this letter. The result for the two-loop vertex correction will suggest a generalization for the form of a set of the corrections at arbitrary n loops. We conjecture and then prove this n-loop result by induction. Results for massless partons will be presented in future work.

We begin by presenting one-loop vertex corrections in the eikonal approximation using the axial gauge. A representative one-loop vertex correction diagram is given in Figure 2. The general axial gauge gluon propagator is

\[ D^{\mu \nu}(k) = \frac{-i k^2}{k^2 + i \epsilon} N^{\mu \nu}(k), \quad N^{\mu \nu}(k) = g^{\mu \nu} - \frac{n^\mu k^\nu + k^\mu n^\nu}{n \cdot k} + n^2 \frac{k^\mu k^\nu}{(n \cdot k)^2}, \tag{2} \]

with \(n^\mu\) the axial gauge-fixing vector. The propagator for a quark, antiquark, or gluon eikonal line is \(i/(\delta v \cdot k + i \epsilon)\) with \(\delta = +1(-1)\) when the momentum \(k\) flows in the same (opposite) direction as the dimensionless vector \(v\). The interaction gluon-eikonal line vertex for a quark or antiquark eikonal line is \(-i g_s T^c_F v^\mu \Delta\) with \(\Delta = +1(-1)\) for a quark (antiquark).

![Figure 2: One-loop vertex correction diagram](image)

We denote the kinematic (i.e. color-independent) part of the n-loop correction to the vertex, with the virtual gluon linking lines \(i\) and \(j\), as \(\omega^{(n)}_{ij}(v_i, v_j)\). The eikonal lines represent massive partons with mass \(m\), for example a heavy quark-antiquark pair. The one-loop expression for \(\omega_{ij}\) is then

\[ \omega^{(1)}_{ij}(v_i, v_j) = g_s^2 \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2 + i \epsilon} N^{\mu \nu}(k) \frac{\Delta_i v_i^\mu}{\delta_i v_i \cdot k + i \epsilon} \frac{\Delta_j v_j^\nu}{\delta_j v_j \cdot k + i \epsilon}. \tag{3} \]

After isolating the UV poles of the integral in dimensional regularization (with \(\epsilon = 4 - D\)) using a variety of techniques, we can present the UV-pole part of \(\omega^{(1)}_{ij}\) as

\[ \omega^{(1) \text{UV}}_{ij}(v_i, v_j) = S^{(1)}_{ij} \frac{\alpha_s}{\pi \epsilon} [L_\beta + L_i + L_j - 1], \tag{4} \]

where \(S^{(1)}_{ij}\) is an overall sign, \(S^{(1)}_{ij} = \Delta_i \Delta_j \delta_i \delta_j\), \(L_\beta\) is the velocity-dependent eikonal function

\[ L_\beta = \frac{1 - 2m^2/s}{\beta} \left[ \ln \left( \frac{1 - \beta}{1 + \beta} \right) + \pi i \right], \tag{5} \]
with $\beta = \sqrt{1 - 4m^2/s}$, $s = (p_i + p_j)^2$, and the $L_i$ and $L_j$ are rather complicated functions of the gauge vector $n$. We note that these gauge-dependent functions are cancelled by the inclusion of contributions from one-loop self-energy corrections [10].

We now continue with a calculation of the two-loop diagram in Figure 3.

![Figure 3: Two-loop vertex correction diagram](image)

At two loops, the expression for $\omega_{ij}$ is

$$
\omega_{ij}^{(2)}(v_i, v_j) = g_s^4 \frac{d^Dk_1}{(2\pi)^D} \frac{(-i)}{k_1^2 + i\epsilon} N^{\mu\nu}(k_1) \int \frac{d^Dk_2}{(2\pi)^D} \frac{(-i)}{k_2^2 + i\epsilon} N^{\mu\nu}(k_2) \times \frac{\Delta_{1i} v_i^\mu}{\delta_{1i} v_i \cdot k_1 + i\epsilon} \frac{\Delta_{2i} v_i^\sigma}{\delta_{2i} v_i \cdot (k_1 + k_2) + i\epsilon} \frac{\Delta_{ij} v_j^\nu}{\delta_{ij} v_j \cdot k_1 + i\epsilon}. 
$$

(6)

As for the one-loop calculation, the result for $\omega_{ij}^{(2)}$ depends on whether the eikonal lines represent massless or massive partons. For massive eikonal lines, that we discuss here, the highest UV poles encountered are $1/\varepsilon^2$.

We may rewrite Eq. (6) as

$$
\omega_{ij}^{(2)}(v_i, v_j) = S_{ij}^{(2)} \sum_{k,l=1,2,3} I_{kl}^{(2)}(v_i, v_j),
$$

(7)

where $I_{kl}^{(2)}$ denotes the contribution of the $k$-th term in the gluon propagator $N^{\mu\nu}(k_1)$ in Eq. (2) with the $l$-th term in $N^{\mu\sigma}(k_2)$, and $S_{ij}^{(2)}$ is an overall sign, $S_{ij}^{(2)} = \Delta_{1i} \Delta_{1j} \Delta_{2i} \delta_{1i} \delta_{1j} \delta_{2i} \delta_{2j}$.

For example, the integral $I_{11}^{(2)}$ is defined as

$$
I_{11}^{(2)}(v_i, v_j) \equiv g_s^4 \frac{d^Dk_1}{(2\pi)^D} \frac{(-i)}{k_1^2 + i\epsilon} \frac{v_i \cdot v_j}{(v_i \cdot k_1 + i\epsilon)(v_j \cdot k_1 + i\epsilon)} \int \frac{d^Dk_2}{(2\pi)^D} \frac{(-i)}{k_2^2 + i\epsilon} \frac{v_i \cdot v_j}{[v_i \cdot (k_1 + k_2) + i\epsilon][v_j \cdot (k_1 + k_2) + i\epsilon]}
$$

$$
= g_s^4 \frac{d^Dk_1}{(2\pi)^D} \frac{i \pi^2/2^2 \Gamma(1 + \varepsilon/2)(1 - 2m^2/s)^2}{4\pi^2} \frac{1}{v_i \cdot (k_1 + i\epsilon)(v_j \cdot k_1 + i\epsilon)} \frac{1}{v_j \cdot (k_1 + k_2 + i\epsilon)} \left[ \frac{1}{\varepsilon} F(x) + \int_0^1 dx \int_0^1 dz \frac{f(x, z)}{(1 - z)_+} \right],
$$

(8)

where we used $v_i \cdot v_j = 1 - 2m^2/s$, with $m$ the mass of the heavy quark. The function $f$ is of the form

$$
f(x, z) = [A(x, z) + B_i(x, z) k_1 \cdot v_i + B_j(x, z) k_1 \cdot v_j]^{-1}
$$

(9)
and the $k_1$ integral over it is UV finite, while

$$F(x) = \int_0^1 dx \, f(x, 1) = -\frac{L_\beta}{1 - 2m^2/s}$$

(10)

with $L_\beta$ defined in Eq. (3).

After calculating all the integrals in $\omega_{ij}^{(2)}$, we can present the two-loop UV-pole terms:

$$\omega_{ij}^{(2)\text{UV}}(v_i, v_j) = S_{ij}^{(2)} \frac{\alpha_s^2}{\pi^2 \varepsilon^2} [L_\beta + L_i + L_j - 1]^2$$

$$+ S_{ij}^{(2)} \frac{\alpha_s^2}{\pi^2 \varepsilon} [L_\beta + L_i + L_j - 1] [[L_\beta + L_i + L_j - 1] \ln 2 + \ln(4\pi) - \gamma_E]$$

$$+ \ln \left( \frac{n^2}{2} \right) + v_i \cdot n \left( L_i - \frac{L''}{2} \right) + v_j \cdot n \left( L_j - \frac{L''}{2} \right) - M_\beta \right) ,$$

(11)

where $\gamma_E$ is the Euler constant, $L'$ and $L''$ are complicated functions of the gauge vector $n$, and

$$M_\beta = \left( \frac{1 - 2m^2/s}{\beta} \right) \left[ \ln 2 \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{2} \ln^2(1 - \beta) - \frac{1}{2} \ln^2(1 + \beta) + \text{Li}_2 \left( \frac{1 + \beta}{1 - \beta} \right)$$

$$- \text{Li}_2 \left( \frac{1 - \beta}{1 + \beta} \right) - \text{Li}_2 \left( \frac{1 + \beta}{2} \right) + \text{Li}_2 \left( \frac{1 - \beta}{2} \right) \right] .$$

(12)

We note that at two loops there are several more diagrams, in addition to Figure 3, that include two-loop self-energy corrections; one-loop self-energy corrections with one-loop vertex corrections as in Figure 2; diagrams with three-gluon vertices; and crossed diagrams. Some of these diagrams are zero, some are UV finite, and some contribute gauge-dependent terms that cancel against the gauge terms with $\ln(n^2)$, $L'$, $L''$ and $M_\beta$ in $\omega_{ij}^{(2)}$. Details will be given elsewhere.

For the discussion to follow, we define $\omega_{ij}^{(n)} \equiv \omega_{ij}^{(n)} / S_{ij}^{(n)}$ modulo UV-finite integrals encountered as in Eq. (8) and infrared terms [18]. We now note that the coefficient of the leading UV pole, i.e. the $1/\varepsilon^2$ pole, in $\omega_{ij}^{(2)}$ is simply the square of the coefficient of the $1/\varepsilon$ pole in $\omega_{ij}^{(1)}$. This fact suggests the possibility that the $1/\varepsilon$ pole in $\omega_{ij}^{(2)}$ might be derived by squaring $\omega_{ij}^{(1)}$ after we calculate the finite pieces in $\omega_{ij}^{(1)}$.

A calculation of the finite pieces of $\omega_{ij}^{(1)}$ gives

$$\omega_{ij}^{(1)\text{finite}}(v_i, v_j) = \frac{\alpha_s}{2\pi} \left[ [L_\beta + L_i + L_j - 1] \ln 2 + \ln(4\pi) - \gamma_E] + \ln \left( \frac{n^2}{2} \right)$$

$$+ v_i \cdot n \left( L_i - \frac{L''}{2} \right) + v_j \cdot n \left( L_j - \frac{L''}{2} \right) - M_\beta \right) .$$

(13)

Then

$$\omega_{ij}^{(1)} = \frac{\alpha_s}{\pi \varepsilon} [L_\beta + L_i + L_j - 1] + \omega_{ij}^{(1)\text{finite}} + \mathcal{O}(\varepsilon) .$$

(14)

We see then that indeed $\omega_{ij}^{(2)} = [\omega_{ij}^{(1)}]^2$. This holds not only for all the UV poles but also for the finite terms in $\omega_{ij}^{(2)}$, as an explicit calculation, keeping terms of $\mathcal{O}(\varepsilon)$ in $\omega_{ij}^{(1)}$, verifies.
We now propose a conjecture for the generalization of this finding to \( n \)-loops. The conjecture is that the UV poles and finite terms in the \( n \)-loop vertex correction in Figure 4 can be derived by raising the one-loop result to the \( n \)-th power. In other words, we want to prove that

\[
\omega'(n)_{ij}(v_i, v_j) = \left[ \omega'(1)_{ij}(v_i, v_j) \right]^n. \tag{15}
\]

We prove this by induction. As we have just seen, the above equation holds for \( n = 2 \). Next, we assume that it holds for arbitrary \( n \), and show that it must then also hold for \( n + 1 \). Now,

\[
\omega'(n+1)_{ij} = g_s^{2(n+1)} \int \frac{d^Dk_1}{(2\pi)^D} \cdots \frac{d^Dk_n}{(2\pi)^D} \frac{d^Dk_{n+1}}{(2\pi)^D} \\
\times \frac{(-i)N^{\mu_1\nu_1}(k_1)}{k_1^2 + i\epsilon} \cdots \frac{(-i)N^{\mu_n\nu_n}(k_n)}{k_n^2 + i\epsilon} \frac{(-i)N^{\mu_{n+1}\nu_{n+1}}(k_{n+1})}{k_{n+1}^2 + i\epsilon} \\
\times \frac{v_i^{\mu_1}}{v_i^{\mu_1}} \cdots \frac{v_i^{\mu_n}}{v_i^{\mu_n}} \frac{v_j^{\nu_1}}{v_j^{\nu_1}} \cdots \frac{v_j^{\nu_n}}{v_j^{\nu_n}} \frac{v_i^{\nu_{n+1}}}{v_i^{\nu_{n+1}}} \frac{v_j^{\nu_{n+1}}}{v_j^{\nu_{n+1}}}. \tag{16}
\]

The \( k_{n+1} \) integral gives the same UV poles and finite terms as \( \omega'(1)_{ij} \) plus an integral which is UV finite after integration over \( k_1 \cdots k_n \). Therefore, \( \omega'(n+1)_{ij} = \omega'(n)_{ij} \omega'(1)_{ij} = \left[ \omega'(1)_{ij} \right]^{n+1} \). So our formula also holds for \( n + 1 \). This concludes the proof.

Thus, we see that the vertex corrections of the form in Figure 4 can be readily derived for any number of loops \( n \), by calculating the one-loop result and keeping terms in it of \( \mathcal{O}(\epsilon^{n-1}) \). Of course, there is an increasing number of self-energy and other graphs as we move to higher \( n \), and these graphs are expected to provide contributions that will cancel the gauge dependence in \( \omega'(n)_{ij} \).

The theoretical loop calculations outlined in this letter should provide a powerful tool in the numerous applications of the eikonal approximation to QCD and beyond, and specifically
in more accurate resummations for a large variety of QCD and beyond the Standard Model partonic processes. Each process has a specific partonic content and color structure so the details have to be worked separately for each application. But the universal ingredient in all such studies is the $n$-loop result presented in this letter.

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