Scaling limits for the block counting process and the fixation line of a class of Λ-coalescents

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Abstract

We provide scaling limits for the block counting process and the fixation line of Λ-coalescents as the initial state $n$ tends to infinity. The main result states that the block counting process, properly logarithmically scaled, converges in the Skorohod space to an Ornstein–Uhlenbeck type process as $n$ tends to infinity.

The reader is referred to [3] for a survey of Λ-coalescents. Unless Λ(\{1\}) > 0, either Πₜ has infinitely many blocks for all $t > 0$ almost surely or finitely many blocks for all $t > 0$ almost surely. The Λ-coalescent is said to stay infinite in the first case and to come down from infinity in the second. An atom of Λ at 1 corresponds to the rate of jumping to the trivial (and absorbing) partition consisting only of the block $\mathbb{N}$. For $t ≥ 0$ let $N^{(n)}_t$ denote the number of blocks of the restriction $\Pi^{(n)}_t := \{B \cap [n]| B \in \Pi_t, B \cap [n] \neq \emptyset\}$ of $\Pi_t$ to $[n] := \{1, \ldots, n\}$. The block counting process $N^{(n)} := (N^{(n)}_t)_{t≥0}$ is a $[n]$-valued conservative Markov process with càdlàg paths that jumps from $k ≥ 2$ to $j \in \{1, \ldots, k-1\}$ at the rate

$$q_{k,j} = \binom{k}{j-1} \int_{[0,1]} u^{k-j-1}(1-u)^{j-1}\Lambda(du).$$

The process $N^{(n)}$ starts in $n$ at time $t = 0$, has decreasing paths and eventually reaches the absorbing state 1.

1 Introduction

The Λ-coalescent, independently introduced by Pitman [16] and Sagitov [18], is a Markov process $\Pi = (\Pi_t)_{t≥0}$ with càdlàg paths, values in the space of partitions of $\mathbb{N} := \{1, 2, \ldots\}$, starting at time $t = 0$ from the partition $\{\{1\}, \{2\}, \ldots\}$ of $\mathbb{N}$ into singletons, whose behavior is fully determined by a finite measure $\Lambda$ on the Borel subsets of $[0, 1]$. If the process is in a state with $k ≥ 2$ blocks, any particular $j \in \{2, \ldots, k\}$ blocks merge at the rate

$$\lambda_{k,j} = \int_{[0,1]} u^{j-2}(1-u)^{k-j}\Lambda(du).$$

Clearly, $N^{(n)}$ starts in $n$ at time $t = 0$, has decreasing paths and eventually reaches the absorbing state 1. Our main objective is to analyze the limiting behavior of the block counting process of Λ-coalescents that stay infinite as the initial state $n$ tends to infinity by determining suitable scaling constants. For Λ-coalescents with dust the scaling is $n$. A block $B \in \Pi_t$ of size $|B| = 1$ is called a singleton. The number of singletons in $[n]$ divided by $n$ converges to the frequency of singletons as $n$ tends to infinity, and, if $\int_{[0,1]} u^{-1}\Lambda(du) < \infty$ and $\Lambda(\{0\}) = 0$, then the frequency of singletons is strictly positive or the Λ-coalescent is said to have dust. It is known that $N^{(n)}/n$ converges in the Skorohod space $D_{[0,1]}[0, \infty)$ to the frequency of singletons process as $n$ tends to infinity [9, Theorem 2.13].

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The Bolthausen–Sznitman coalescent, where \( \Lambda \) is the uniform distribution on \([0,1]\), is an example of a dust-free \( \Lambda \)-coalescent that stays infinite \([5]\). In this case it was shown in \([13]\) via moment calculations that \( (N_t^{(n)}/n^{\exp(-t)})_{t \geq 0} \) converges in \( D_{[0,\infty)}[0,\infty) \) to the Mittag–Leffler process as \( n \) tends to infinity.

We provide unified proofs for both limit theorems, stated later as Corollaries \([3]\) and \([4]\) and extend the convergence to \( \Lambda \)-coalescents, where the measure \( \Lambda \) is the sum of the uniform distribution on \([0,1]\) multiplied by a constant \( b \geq 0 \) and a measure that corresponds to a coalescent with dust. This includes the \( \Lambda \)-coalescent where \( \Lambda = \beta(1, b) \) is the beta distribution with parameters 1 and \( b > 0 \). Theorem \([1]\) below states that \( (\log N_t^{(n)} - \exp(-bt) \log n)_{t \geq 0} \) converges in \( D_R[0,\infty) \) as \( n \) tends to infinity. The logarithmic version of the convergence result has the advantage of putting the limiting process in Theorem 1 to a class of processes, which has been studied in the literature. The limiting process can be represented as the solution of the Langevin equation with Lévy noise instead of a Brownian motion and is sometimes called Ornstein–Uhlenbeck type process \([19]\) or generalized Ornstein–Uhlenbeck process. First Wolfe \([25]\) studied Ornstein–Uhlenbeck type processes on \( \mathbb{R} \), later Jurek and Vervaat \([12]\) on Banach spaces and Sato and Yamazato \([19]\) on \( \mathbb{R}^d \), then Applebaum \([2]\) on Hilbert spaces. Under the logarithmic moment condition \([17]\) the marginal distributions of the Ornstein–Uhlenbeck type process converge in distribution as \( n \) tends to infinity to the unique stationary distribution. The stationary distribution is self-decomposable. A real-valued random variable \( S \) is self-decomposable if for every \( \alpha \in [0, 1] \) there exists a random variable \( S_\alpha \) independent of \( S \) such that \( S \) has the same distribution as \( \alpha S + S_\alpha \). If \( \phi \) is the characteristic function of \( S \) then \( S \) is self-decomposable if and only if \( x \mapsto \phi(x)/\phi(\alpha x) \), \( x \in \mathbb{R} \), is the characteristic function of a real-valued random variable for every \( \alpha \in [0, 1] \). A distribution \( \mu \) on \( \mathbb{R} \) is self-decomposable if there exists a self-decomposable random variable with distribution \( \mu \). Conversely, every self-decomposable distribution can be obtained in this way. For the \( \beta(1, b) \)-coalescent \([17]\) is satisfied.

The fixation line \( L = (L_t)_{t \geq 0} \) is a \( \mathbb{N} \)-valued Markov process that jumps from \( k \in \mathbb{N} \) to \( j \in \{k + 1, \ldots\} \) at the rate

\[
\gamma_{k,j} = \left( \frac{j}{j-k+1} \right) \int_{[0,1]} u^{-k-1}(1-u)^k \Lambda(du).
\]

The fixation line is the ‘time-reversal’ of the block counting process, in the sense that the hitting times \( \inf\{t \geq 0|N_t^{(n)} \leq m\} \) and \( \inf\{t \geq 0|L_t^{(m)} \geq n\} \) share the same distribution \([11]\), Lemma 2.1. Equivalently, the process \( L \) is Siegmund-dual \([22]\) to the block counting process, i.e., if \( L_t^{(m)} = (L_t^{(m)})_{t \geq 0} \) denotes the fixation line starting in \( L_0^{(m)} = m \), \( m \in \mathbb{N} \), at time \( t = 0 \) then (see \([13]\))

\[
P(L_t^{(m)} \geq n) = P(N_t^{(n)} \leq m), \quad m, n \in \mathbb{N}, t \geq 0.
\]  

(1)

For a thorough definition of the fixation line see \([11]\) and the references therein. Theorem \([5]\) states that \( (\log L_t^{(n)} - \exp(bt) \log n)_{t \geq 0} \) converges in \( D_R[0,\infty) \) as the initial value \( L_0^{(n)} = n \) tends to infinity.

The article is organized as follows. In Section 2 the main convergence result (Theorem \([1]\)) is stated and applied to the \( \beta(1, b) \)-coalescent with parameter \( b > 0 \). The limiting process is an Ornstein–Uhlenbeck type process. Well-known results are applied to our setting in Section 3. In particular, the generator of the limiting process is determined. The line of proof is in some sense reversed. First we prove Corollaries \([3]\) and \([4]\) in Sections 4 and 5 by showing the convergence of the generators of the (logarithm of the) scaled block counting processes. The decomposition of \( \Lambda \) into the uniform distribution multiplied by a constant and a measure that corresponds to a coalescent with dust is transferred to the generators. This enables us to use relations obtained in Sections 4 and 5 to prove Theorem \([1]\) in Section 6. The proof of Theorem \([5]\) is conducted in Section 7 with similar methods.

**Notation.** Let \( E \) be a complete separable metric space. The Banach space \( B(E) \) of bounded measurable functions \( f : E \to \mathbb{R} \) is equipped with the usual supremum norm \( \|f\| := \sup_{x \in E} |f(x)| \) and the Banach subspace \( \hat{C}(E) \subset B(E) \) consists of continuous functions vanishing at infinity.
If $E \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$ then $C_k(E)$ denotes the space of $k$-times continuously differentiable functions. The Borel-$\sigma$-field on $\mathbb{R}$ is denoted by $\mathcal{B}$, $\Lambda$ is a (non-zero) finite measure on $B\cap[0,1]$ with $\Lambda(\{0\}) = \Lambda(\{1\}) = 0$ and $\lambda$ denotes the Lebesgue measure on $([0,1], B \cap [0,1])$. The generators, usually denoted by $A$, are understood to be defined on (a subspace of) $\hat{C}(E)$. For a measure space $(\Omega, \mathcal{F}, \mu)$ and $p > 0$ the space of measurable functions $f : \Omega \to \mathbb{R}$ with $\int |f|^p d\mu < \infty$ is denoted by $L^p(\mu)$ or in short, $L^p$.

2 Results

Let $\Lambda$ be a finite measure on $([0,1], B\cap[0,1])$ with no mass at 0 and 1. For $b \geq 0$ the map $B \mapsto (\Lambda - b\lambda)(B) = \Lambda(B) - b\lambda(B)$, $B \in B\cap[0,1]$, might possibly be a signed measure. Hahn’s decomposition theorem states the existence of some $A \in B\cap[0,1]$ such that $(\Lambda - b\lambda)^+(B) := (\Lambda - b\lambda)(B \cap A)$, $B \in B\cap[0,1]$, and $(\Lambda - b\lambda)^-(B) := -(\Lambda - b\lambda)(B \cap A^c)$, $B \in B\cap[0,1]$, define nonnegative measures. The two nonnegative measures $(\Lambda - b\lambda)^+$ and $(\Lambda - b\lambda)^-$ constitute the Jordan decomposition of $\Lambda - b\lambda$. Using this decomposition one can integrate with respect to a signed measure by defining $\int f d(\Lambda - b\lambda) := \int f d(\Lambda - b\lambda)^+ - \int f d(\Lambda - b\lambda)^-$ for $f \in L^1((\Lambda - b\lambda)^+) \cap L^1((\Lambda - b\lambda)^-)$. The assumption of Theorem 1 below is the following.

Assumption A. There exists $b \geq 0$ such that $\int_{[0,1]} u^{-1}(\Lambda - b\lambda)^+(du) < \infty$ and $\int_{[0,1]} u^{-1}(\Lambda - b\lambda)^-(du) < \infty$.

Note that the constant $b \geq 0$ is uniquely determined by $\Lambda$, if it exists. Schweinsberg’s criterion [21] shows that the $\Lambda$-coalescent does not come down from infinity under Assumption A, see Lemma 8 in the appendix. Moreover, the $\Lambda$-coalescent is dust-free if $b > 0$. Assumption A is for example satisfied, if $\Lambda$ has density $f \in C_1([0,1])$ with respect to $\lambda$ for which $\lim_{u \searrow 0} f'(u)$ exists and is finite. In this case $b = \lim_{u \searrow 0} f(u)$.

Suppose that $\Lambda$ satisfies Assumption A. Let $\Gamma(z) := \int_{[0,\infty]} u^{a-1}e^{-u}du$, $\Re(z) > 0$, denote the gamma function and $\Psi(z) := (\log \Gamma)'(z) = \Gamma'(z)/\Gamma(z)$, $\Re(z) > 0$, the digamma function. Define $a : = b(1 + \Psi(1)) - \int_{[0,1]} u^{-1}(\Lambda - b\lambda)(du)$ (2)

and $\psi : \mathbb{R} \to \mathbb{C}$ via $\psi(x) := ix\lambda + \int_{[0,1]} (e^{ix\lambda}u^{-1} - 1 + ixu)u^{-2}\lambda(du)$, $x \in \mathbb{R}$. (3)

Substituting $g : (0,1) \to \mathbb{R}$, $g(u) := \log(1 - u)$, $u \in (0,1)$, shows that

$\psi(x) = iax + \int_{(-\infty,0)} (e^{ixu} - 1 + ix(1 - e^u))g(du)$, $x \in \mathbb{R}$,

where the measure $g$, defined via

$\varrho(A) := \int_{g^{-1}(A)} u^{-2}\lambda(du) = \int_A (1 - e^u)^{-2}\varrho(du)$, $A \in B$, (4)

satisfies $\int \varrho(u^2 \wedge 1)g(du) < \infty$ and $g(\{0\}) = 0$. Hence $\varrho$ is a Lévy measure and $e^{\psi(x)}$, $x \in \mathbb{R}$, is the characteristic function of an infinitely divisible distribution.

Theorem 1 Suppose that $\Lambda$ satisfies Assumption A. Then the possibly time-inhomogeneous Markov process $X^{(n)} := (X^{(n)}_t)_{t \geq 0} := (\log N^{(n)}_t - e^{-bt}\log n)_{t \geq 0}$ converges in $D_0[0,\infty)$ as $n \to \infty$ to the time-homogeneous Markov process $X = (X_t)_{t \geq 0}$ with initial value $X_0 = 0$ and semigroup $(T_t)_{t \geq 0}$ given by

$T_t f(x) := \mathbb{E}(f(X_{s+t})|X_s = x) = \mathbb{E}(f(e^{-bt}x + X_t))$, $x \in \mathbb{R}, f \in B(\mathbb{R}), s, t \geq 0$, (5)

where $X_t$ has characteristic function $\phi_t$ given by

$\phi_t(x) = \exp \left( \int_0^t \psi(e^{-bs}x)ds \right)$, $x \in \mathbb{R}, t \geq 0$, (6)

and $\psi$ is given by (3).
Distributions with characteristic functions of the form \( \Psi \), where \( \psi \) is more generally the characteristic exponent of an infinitely divisible distribution, have been studied in [25] on \( \mathbb{R} \) and on more general state spaces in [2], [12] and [19]. Since \( \phi_{t+s}(x) = \phi_t(e^{-bx})\phi_s(x), x \in \mathbb{R} \), for \( s, t \geq 0 \), the semigroup \( (T_t)_{t \geq 0} \) belongs to the class of generalized Mehler semigroups [4]. In particular, the semigroup \( (T_t)_{t \geq 0} \) is Feller, i.e., \( T_t(\mathcal{C}(\mathbb{R})) \subseteq \mathcal{C}(\mathbb{R}) \) for each \( t \geq 0 \) and \( (T_t)_{t \geq 0} \) is strongly continuous on \( \mathcal{C}(\mathbb{R}) \). Hence the Markov process \( X \) in Theorem 1 exists and has paths in \( D_{\mathbb{R}}[0, \infty) \).

The beta distribution \( \beta(a, b) \) with parameters \( a, b > 0 \) has density \( u \mapsto \Gamma(a + b)/(\Gamma(a)\Gamma(b)) \) \( u^{a-1}(1-u)^{b-1}, u \in (0, 1) \), with respect to Lebesgue measure on \( (0, 1) \). The class of beta coalescents are the processes for which \( \Lambda = \beta(a, b) \) for some \( a, b > 0 \). They have been extensively studied in the literature due to the easy computability of the jump rates

\[
q_{k,j} = \frac{(a+b)\Gamma(k+1)\Gamma(j-1+b)\Gamma(k-j+1+a)}{(a\Gamma(k-2+a+b)\Gamma(j)\Gamma(k-j+2))}, \quad j \in \{1, \ldots, k-1\}, k \geq 2.
\]

The beta coalescent comes down from infinity if and only if \( 0 < a < 1 \) [21 Example 15]. If \( a > 1 \) then the beta coalescent is dust-free and does not come down from infinity.

**Example 2** Suppose that \( \Lambda = \beta(1, b) \) with \( b > 0 \). From the observation stated below Assumption A we conclude that Assumption A is satisfied with the same constant \( b \). The ’dust-part’ \( \Lambda - b\lambda \) has possibly negative density \( u \mapsto b((1-u)^{b-1} - 1), u \in (0, 1), \) with respect to Lebesgue measure on \( (0, 1) \). The underlying Lévy measure \( \varrho \) has density \( f \) with respect to Lebesgue measure on \( \mathbb{R} \setminus \{0\} \) given by \( f(u) := be^{bu}(1-e^u)^{-2} \) for \( u < 0 \) and \( f(u) := 0 \) for \( u > 0 \). Calculations involving Gauss’ representation [24, p. 247] for the digamma function \( \Psi \) (see Proposition 7 in the appendix) show that \( a = b(1 + \Psi(b)) \) and

\[
\psi(x) = b((1-b)\Psi(b) - (1-b - ix)\Psi(b + ix)), \quad x \in \mathbb{R}.
\]

According to Theorem 1 the process \( (\log N_t(n) - e^{-bt}\log n)_{t \geq 0} \) converges in \( D_{\mathbb{R}}[0, \infty) \) as \( n \to \infty \) to a Markov process \( X = (X_t)_{t \geq 0} \) with initial value \( X_0 = 0 \) and semigroup \( (T_t)_{t \geq 0} \) given by

\[
T_t f(x) := \mathbb{E}(f(X_{s+t}) | X_s = x) = \mathbb{E}(f(e^{-bt}x + X_t)), \quad x \in \mathbb{R}, f \in \mathcal{B}(\mathbb{R}), s, t \geq 0,
\]

where \( X_t \) has characteristic function \( \phi_t \) given by (2). Since \( \int_{[1-e^{-1}, 1]} \log(1-u)^{-1}\Lambda(du) = \int_{[1-e^{-1}, \infty]} \log(1-u)^{-1}b(1-u)^{b-1}du < \infty \), the logarithmic moment condition of Lemma 2 is satisfied and \( X_t \) converges in distribution as \( t \to \infty \) to the unique stationary distribution \( \mu \) of \( X \). The distribution \( \mu \) is self-decomposable with characteristic function \( \phi \) given by

\[
\phi(x) = \exp \left( \int_{[1-e^{-1}, \infty]} \psi(e^{-bx}) dx \right) = \exp \left( (1-b) \int_{[1-e^{-1}, \infty]} \frac{\Psi(b) - \Psi(b + iu)}{u} \frac{\Gamma(b + ix)}{\Gamma(b)} du \right), \quad x \in \mathbb{R}.
\]

In the last step equation (3) and the fact that \( \Psi(z) = (\log \Gamma(z))^\prime \), \( \text{Re}(z) > 0 \), has been used. The characteristic function \( \phi_t \) of \( X_t \) is hence given by

\[
\phi_t(x) = \frac{\phi(x)}{\phi(e^{-bt}x)} = \exp \left( (1-b) \int_{[e^{-bx}, \infty]} \frac{\Psi(b) - \Psi(b + iu)}{u} \frac{\Gamma(b + ix)}{\Gamma(b + i e^{-bt}x)} du \right), \quad x \in \mathbb{R}, t \geq 0.
\]

If \( Z \) has a gamma distribution with parameter \( b \) and \( 1, i.e., Z \) has density \( u \mapsto u^{b-1}e^{-u}(\Gamma(b))^{-1} \), \( u > 0 \), with respect to Lebesgue measure on \( (0, \infty) \) then \( \log Z \) has a self-decomposable distribution and characteristic function \( \Gamma(b + ix)/\Gamma(b), x \in \mathbb{R} \), see [23, V, Example 9.18]. If \( b < 1 \) then the first factor on the right-hand side of (4) is the characteristic function of a self-decomposable distribution (see [23, V, Theorem 6.7] and the proof of Proposition 7). The underlying characteristic exponent \( (1-b)(\Psi(b) - \Psi(b + iu)), u \in \mathbb{R} \), corresponds to the negative of a drift-free subordinator. Similarly to the convergence above, \( (N_t^{(n)}/n^{e^{-\lambda t}})_{t \geq 0} \) converges in \( D_{\mathbb{R}}[0, \infty) \) to \( (\exp(X_t))_{t \geq 0} \) as \( n \to \infty \).
The two cases mentioned in the introduction arise from Assumption A as follows. If \( \int_{[0,1]} u^{-1}\Lambda(du) < \infty \), then the \( \Lambda \)-coalescent has dust and Assumption A is satisfied with \( b = 0 \). Corollary 3 below has been proven in [9] and [13]. In both articles the blocks of the coalescent are allowed to merge simultaneously. In [13] the convergence of the generators has been proven and even a rate of convergence has been determined. In this article the uniform convergence of the generators is going to be proven as well, but with different techniques. In [9] the convergence of the corresponding semigroups has been shown, which is equivalent to the convergence of the generators on a core. We carry out the proof since parts are used to verify Theorem 1.

**Corollary 3 (dust case)** Suppose \( \int_{[0,1]} u^{-1}\Lambda(du) < \infty \). Then the time-homogeneous Markov process \( X^{(n)} := (X_t^{(n)})_{t \geq 0} := (N_t^{(n)} - \log n)_{t \geq 0} \) converges in \( D_\mathbb{R}[0, \infty) \) as \( n \to \infty \) to a limiting process \( X = (X_t)_{t \geq 0} \) with initial value \( X_0 = 0 \) and semigroup \( (T_t)_{t \geq 0} \) given by

\[
T_tf(x) := \mathbb{E}(f(X_{s+t}) \mid X_s = x) = \mathbb{E}(f(x + X_t)), \quad x \in \mathbb{R}, f \in B(\mathbb{R}), s, t \geq 0,
\]

where \( X_t \) has characteristic function \( \mathbb{E}(\exp(izX_t)) = \exp(t\psi(x)) \), \( x \in \mathbb{R}, t \geq 0 \), with

\[
\psi(x) = \int_{[0,1]} (e^{ix\log(1-u)} - 1)u^{-2}\Lambda(du), \quad x \in \mathbb{R}.
\]

Observe that \( -X \) is a pure-jump subordinator with characteristic exponent \( x \mapsto \psi(-x), x \in \mathbb{R} \).

Note that here \( b = 0, a = \int_{[0,1]} u^{-1}\Lambda(du) \) and the definitions [3] and (11) for \( \psi \) coincide. Hence Theorem 1 and Corollary 3 describe the same limiting process.

For \( \Lambda = \lambda \) Assumption A is satisfied with \( b = 1 \). The block counting process of the Bolthausen–Sznitman coalescent has been treated in [13] and [14]. [13] and [14] have proven that the semigroup of \( (N_t^{(n)} - \log n)_{t \geq 0} \) converges on a dense subset of \( B([0, \infty)) \) to the semigroup of a Feller process as \( n \) tends to infinity, hence the processes converge in \( D_\mathbb{R}[0, \infty) \). Taking logarithms does not spoil the convergence. If \( f \in \hat{C}(\mathbb{R}) \) then \( g := f \circ \log \in \hat{C}([0, \infty)) \), and the semigroup and hence the generator \( A^{(n)} \) of the logarithm of the scaled block counting process \( X^{(n)} := (X_t^{(n)})_{t \geq 0} = (\log N_t^{(n)} - e^{-t}\log n)_{t \geq 0} \) converge as well. We prove the convergence of \( A^{(n)} \) in Section 5. Since the scaling depends on \( t \), the process \( X^{(n)} \) is time-inhomogeneous, and [13] introduces the time-space process in order to transfer the question of convergence to time-homogeneous Markov processes. The time-space process is revisited in Section 5. Since \( \lambda = \beta(1, 1) \), the following result is the particular case \( b = 1 \) of Example 2.

**Corollary 4 (Bolthausen–Sznitman case)** Suppose \( \Lambda = \lambda \). Then the time-inhomogeneous Markov process \( X^{(n)} := (X_t^{(n)})_{t \geq 0} := (N_t^{(n)} - e^{-t}\log n)_{t \geq 0} \) converges in \( D_\mathbb{R}[0, \infty) \) as \( n \to \infty \) to the time-homogeneous Markov process \( X = (X_t)_{t \geq 0} \) with initial value \( X_0 = 0 \) and semigroup \( (T_t)_{t \geq 0} \) given by

\[
T_tf(x) := \mathbb{E}(f(X_{s+t}) \mid X_s = x) = \mathbb{E}(f(e^{-t}x + X_t)), \quad x \in \mathbb{R}, f \in B(\mathbb{R}), s, t \geq 0,
\]

where \( X_t \) has characteristic function \( \phi_t(x) := \mathbb{E}(\exp(izX_t)) = \Gamma(1+ix)/\Gamma(1+ie^{-t}x), x \in \mathbb{R}, t \geq 0 \).

Note that here \( b = 1, a = 1 + \Psi(1) \) and the underlying Lévy measure \( \mu \) has density \( f \) with respect to the Lebesgue measure on \( \mathbb{R} \setminus \{0\} \) given by \( f(u) := e^u(1 - e^u)^{-2} \) for \( u < 0 \) and \( f(u) := 0 \) for \( u > 0 \). Example 2 with \( b = 1 \) states that \( \psi(x) = ix\Psi(1+ix), x \in \mathbb{R} \), and that \( X_t \) converges in distribution as \( t \to \infty \) to the unique stationary distribution \( \mu \) of \( X \) with characteristic function \( \phi(x) = \Gamma(1+ix), x \in \mathbb{R} \). Let \( Z \) have an exponential distribution with parameter 1. Then (see e.g. [23], V, Example 9.15) \( \log Z \) is the negative of a Gumbel distributed random variable and \( \mathbb{E}(e^{ix\log Z}) = \Gamma(1+ix), x \in \mathbb{R} \). Hence \( -X_t \) converges in distribution as \( t \to \infty \) to the Gumbel distribution.

A convergence result for the fixation line can be stated analogously to Theorem 1.
Theorem 5 Suppose that \( \Lambda \) satisfies Assumption A. Then the possibly time-inhomogeneous Markov process \( Y^{(n)} := (Y^{(n)}_t)_{t \geq 0} := (\log L^{(n)}_t - e^{bt} \log n)_{t \geq 0} \) converges in \( D_N[0, \infty) \) as \( n \to \infty \) to the time-homogeneous Markov process \( Y := (Y_t)_{t \geq 0} \) with initial value \( Y_0 = 0 \) and semigroup \((T_t)_{t \geq 0}\) given by

\[
T_tf(y) := \mathbb{E}(f(Y_{s+t})|Y_s = y) = \mathbb{E}(f(e^{bt}y + Y_t)), \quad y \in \mathbb{R}, f \in B(\mathbb{R}), s, t \geq 0,
\]

where \( Y_t \) has characteristic function \( \chi_t \) given by

\[
\chi_t(y) = \exp \left( \int_0^t \psi(-e^{bs}y)ds \right), \quad y \in \mathbb{R}, t \geq 0,
\]

and \( \psi \) is given by \( \psi \).

Remark.

1. The process defined by (12) and (13) is an Ornstein–Uhlenbeck type process with underlying characteristic exponent \( y \mapsto \psi(-y), y \in \mathbb{R} \). The semigroup defined by (12) belongs to the class of generalized Mehler semigroups, since \( \chi_t(y) = \chi(e^{bt}y) \chi_s(y), y \in \mathbb{R}, \) for \( s, t \geq 0 \).

2. Let the random variable \( S_t \) have characteristic function \( \phi_t \), given by (6), for \( t \geq 0 \). Conditional on \( X_s = x \), \( X_{s+t} \) is distributed as \( e^{-bt}x + S_t \) for all \( x \in \mathbb{R} \). Note that \( Y_t = e^{-bt}X_t \) and \( Y_{t+s} = e^{bt}y - e^{bt}S_t \) such that

\[
\mathbb{P}(e^{Y_{t+s}} \geq x|e^{Y_s} = y) = \mathbb{P}(e^{bt}e^{-e^{bt}S_t} \geq x) = \mathbb{P}(e^{-e^{bt}S_t} \leq y) = \mathbb{P}(e^{X_{t+s}} \leq y|e^{X_s} = x)
\]

for all \( x, y, s, t \geq 0 \), i.e., \( e^Y \) is Siegmund-dual to \( e^X \) (see [22]).

3. The Bolthausen–Sznitman case \( \Lambda = \lambda \) is stated in [13, Theorem 3.1 b)] in non-logarithmic form. The fixation line in the Bolthausen–Sznitman coalescent is a continuous-time discrete state space branching process in which the offspring distribution has probability generating function \( f(s) = s + (1 - s)\log(1 - s), s \in [0, 1] \). The limiting process described in Theorem 5 is the logarithm of Neveu’s continuous-state branching process. By Corollary 5 the characteristic functions \( \chi_t \) of the marginal distributions are given by (see [13, Eq. (19)])

\[
\chi_t(y) = \phi_t(-e^ty) = \Gamma(1 - ie^{bt}y)/\Gamma(1 - iy), \quad y \in \mathbb{R}, t \geq 0.
\]

3 The limiting process

Standard computations (see [20, Lemma 17.1]) show that \( \phi_t \), given by (6), is the characteristic function of an infinitely divisible distribution for each \( t \geq 0 \) without Gaussian component and Lévy measure \( \varrho_t \) given by

\[
\varrho_t(A) = \int_{\mathbb{R} \setminus \{0\}} \int_0^t 1_A(e^{-bs}u)dsd\varrho(u), \quad A \in \mathcal{B}, t \geq 0.
\]

Sato and Yamazato [19, Theorem 3.1] provide a formula for the generator corresponding to the semigroup \((T_t)_{t \geq 0}\) given by (5).

Lemma 6 Suppose that \( \Lambda \) satisfies Assumption A. (Let \( \psi \) be given by (3), \( \phi_t \) be defined by (6) and let the random variable \( X_t \) have characteristic function \( \phi_t \) for each \( t \geq 0 \).) The family of operators \((T_t)_{t \geq 0}\) defined by (5) is a Feller semigroup. Let \( D \) denote the space of twice differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f, f', f'' \in C_b(\mathbb{R}) \) and such that the map \( x \mapsto xf'(x) \), \( x \in \mathbb{R} \), belongs to \( C_b(\mathbb{R}) \). Then \( D \) is a core for the generator \( A \) corresponding to \((T_t)_{t \geq 0}\) and

\[
Af(x) = f'(x)(a - bx) + \int_{[0,1]} (f(x + \log(1 - u)) - f(x) + u f'(x))u^{-2}\Lambda(du)
\]

for \( x \in \mathbb{R} \) and \( f \in D \), where \( a \) is given by (4).
The process has no stationary distribution.

Proof. Substituting \( g : (0,1) \to \mathbb{R}, g(u) := \log(1-u), u \in (0,1) \) shows that (14) is an integro-differential operator of the form (1.1) of Sato and Yamazato [19] with dimension \( d = 1 \). In [19], operators of this form are initially considered as acting on the space \( C^2_{\text{c}} \) of twice differentiable functions with compact support (see the explanations after Eq. (1.2) in [19]), but Step 3 of the proof of [19, Theorem 3.1] shows that (14) even holds for functions \( f \in D (\supset C^2_{\text{c}}) \). Note that the space \( D \) is denoted by \( F_1 \) in [19]. The fact that \( D \) is a core for \( A \) is only a different phrasing of the claim in Step 5 of the proof of [19, Theorem 3.1].

Remark. The limiting process in Theorem 1 arises as the solution of a certain stochastic differential equation. Let the Lévy process \( L = \{L_t\}_{t \geq 0} \) with characteristic functions \( \mathbb{E}(e^{ixL_t}) = e^{t\psi(x)}, x \in \mathbb{R}, t \geq 0, \) be adapted to the filtration \( (\mathcal{F}_t)_{t \geq 0} \) which satisfies the usual hypotheses such that \( L_{t+s} - L_s \) is independent of \( \mathcal{F}_s \) for all \( s, t \geq 0 \). In this remark \( \psi \) is allowed to be the characteristic exponent of an arbitrary infinitely divisible distribution on \( \mathbb{R} \) and \( b > 0 \) is fixed. The Langevin equation with Lévy noise instead of a Brownian motion

\[
dX_t = -bX_t \, dt + \, dL_t, \quad t \geq 0, \tag{15}
\]

with initial value \( X_0 = 0 \) has an unique \( (\mathcal{F}_t)_{t \geq 0} \)-adapted solution \( X = \{X_t\}_{t \geq 0} \) with càdlàg paths. The solution of (15) or the corresponding semigroup are hence sometimes called Ornstein–Uhlenbeck type or generalized Ornstein–Uhlenbeck process or semigroup. It holds that

\[
X_t = \int_0^t e^{-bt-s} \, dL_s, \quad t \geq 0. \tag{16}
\]

Various constructions for the integral (15) are possible. In Applebaum [1] Sections 6.3 and 6.2 the stochastic integral is the Itô-integral with respect to semimartingales. Wolfe [25] constructed the integral as a random Bochner integral, which exists in the sense of convergence in probability. Jurek and Vervaat [12] constructed the stochastic integral as a pathwise Laplace-Stieltjes integral (and using integration by parts). The process \( X \) is a stochastically continuous Markov process and the corresponding semigroup is given by (4), where the characteristic functions of \( X_t \) are given by (13) with underlying infinitely divisible characteristic exponent \( \psi \) for \( t \geq 0 \).

Suppose that \( b > 0 \) and that the Lévy measure \( \varrho \) of the characteristic exponent \( \psi \) satisfies

\[
\int_{\{|x| > 1\}} \log(1 + |u|) \varrho(du) < \infty. \tag{17}
\]

According to [19] Theorems 4.1 and 4.2, \( X_t \) converges in distribution as \( t \to \infty \) to the unique stationary distribution \( \mu \) of \( X \). The distribution \( \mu \) is self-decomposable and conversely every self-decomposable distribution can be obtained as the stationary distribution of an Ornstein–Uhlenbeck type process. If (17) does not hold, then there exists no stationary distribution. The following Lemma is an application to our setting.

Lemma 7 Suppose that \( \Lambda \) satisfies Assumption A. Let \( X = \{X_t\}_{t \geq 0} \) be as in Theorem 1. If further \( \int_{(0,1)} \log(1-u)^{-1} \Lambda(du) < \infty \) for some \( 1 - e^{-1} < \varepsilon < 1 \) then \( X_t \) converges in distribution as \( t \to \infty \) to the unique stationary distribution \( \mu \) of \( X \). The distribution \( \mu \) is self-decomposable with characteristic function \( \phi \) given by

\[
\phi(x) = \exp \left( \int_0^\infty \psi(e^{-bs}x) \, ds \right), \quad x \in \mathbb{R}. \tag{18}
\]

The characteristic function \( \phi_t \) of \( X_t \) satisfies \( \phi_t(x) = \phi(x)/\phi(e^{-bt}x), x \in \mathbb{R} \).

If \( \int_{(0,1)} \log(1-u)^{-1} (u^{-2}) \Lambda(du) = \infty \) for \( 0 < \varepsilon < 1 \), then, for every \( l \),

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \sup_{y \in \mathbb{R}} \mathbb{P}(|e^{-bt}x + X_t - y| \leq l) = 0. \tag{19}
\]

The process has no stationary distribution.
4 Proof of Corollary 3

In this section Λ satisfies the dust condition \( \int_{[0,1]} u^{-1} \Lambda(du) < \infty \). Let \( E_n := \{ x \in \mathbb{R} | e^{x} n \in [n] \} \) denote the state space of \( X^{(n)} = (X_t^{(n)}, t \geq 0) = (\log N_t^{(n)} - \log n)_{t \geq 0} \) for each \( n \in \mathbb{N} \). Define \( k := k(n, x) := e^{x} n \in [n] \) for \( x \in E_n \) and \( n \in \mathbb{N} \) such that the generator \( A^{(n)} \) of \( X^{(n)} \) can be represented as

\[
A^{(n)}f(x) = \sum_{j=1}^{k-1} (f(x + \log \frac{j}{k}) - f(x)) q_{k,j}, \quad x \in E_n, f \in \tilde{C}^{1}(\mathbb{R}), n \in \mathbb{N}. \tag{20}
\]

The process \( X = (X_t)_{t \geq 0} \) defined by \( \ref{20} \) and \( \ref{21} \) is a Feller process in \( \tilde{C}^{1}(\mathbb{R}) \). Let \( A \) denote the generator. From \( \ref{20} \) Theorem 31.5 it follows that the space \( \tilde{C}^{2}(\mathbb{R}) \) of twice differentiable functions \( f \in C^{2}(\mathbb{R}) \) with \( f, f', f'' \in \tilde{C}(\mathbb{R}) \) is a core for \( A \) and

\[
Af(x) = \int_{[0,1]} (f(x + \log(1-u)) - f(x)) u^{-2} \Lambda(du), \quad x \in \mathbb{R}, f \in \tilde{C}^{2}(\mathbb{R}). \tag{21}
\]

The idea to prove the uniform convergence of the generators is the following: write the jump rates as values of a distribution depending on \( k \) (with some minor rectifications) whose limiting behavior as \( k \to \infty \) can be determined. The generator can then be written as the mean of a random variable and classical weak convergence results can be applied.

**Proof.** [of Corollary 3] Let \( f \in \tilde{C}^{2}(\mathbb{R}) \). Define \( h : [0, 1] \times \mathbb{R} \to \mathbb{R} \) via \( h(u, x) := u^{-1}(f(x + \log(1-u)) - f(x)), u \in (0, 1), h(0, x) := \lim_{u \to 0} h(u, x) = -f'(x) \) and \( h(1, x) := \lim_{u \to 1} h(u, x) = -f(x) \) for \( x \in \mathbb{R} \). Differentiating \( s \mapsto f(x + \log(1-us)) \), \( s \in (0, 1) \), leads to

\[
\int_{0}^{1} \frac{f'(x + \log(1-us))}{1-us} ds, \quad u \in [0, 1), x \in \mathbb{R},
\]

such that

\[
h(u, x) = -\int_{0}^{1} \frac{f'(x + \log(1-us))}{1-us} ds, \quad u \in [0, 1), x \in \mathbb{R},
\]

and \( h \) stays bounded even as \( u \) tends to 0. Define

\[
S(k, x) := \sum_{j=1}^{k-1} (f(x + \log \frac{j}{k}) - f(x)) q_{k,j}, \quad I(x) := \int_{[0,1]} h(u, x) u^{-1} \Lambda(du), \quad k \in \mathbb{N}, x \in \mathbb{R}, \tag{22}
\]

such that \( A^{(n)}f(x) = S(k, x) \) for \( x \in E_n \) and \( n \in \mathbb{N} \) and \( I(x) = Af(x) \) for \( x \in \mathbb{R} \). Substituting \( k-j \) for \( j \) and the definition of \( h \) yield

\[
S(k, x) = \sum_{j=1}^{k-1} (f(x + \log(1-\frac{j}{k})) - f(x)) q_{k,k-j}
\]

\[
= \sum_{j=1}^{k-1} h(\frac{j}{k}, x) j \left( \frac{k}{k-j+1} \right) \int_{[0,1]} u^{j-1}(1-u)^{k-j-1} \Lambda(du)
\]

\[
= \sum_{j=0}^{k-1} h(\frac{j}{k}, x) j \left( \frac{k-1}{j+1} \right) \int_{[0,1]} u^{j-1}(1-u)^{k-j-1} \Lambda(du).
\]

Set \( c := \int_{[0,1]} u^{-1} \Lambda(du) > 0 \) and define the probability measure \( Q \) on \( ([0,1], \mathcal{B} \cap [0,1]) \) via \( Q(A) := c^{-1} \int_{A} u^{-1} \Lambda(du) \), \( A \in \mathcal{B} \cap [0,1] \). Let the random variables \( Z_{k}, k \in \mathbb{N}, \) have distribution given by

\[
\mathbb{P}(Z_{k} = j) = \left( \frac{k-1}{j} \right) \int_{[0,1]} u^{j}(1-u)^{k-1-j} Q(du), \quad j \in \{0, \ldots, k-1\},
\]

8
Remark.

1. The generator $A(n)$ converges even if $\Lambda(\{1\}) > 0$. In this case the atom at 1 can be split off from $\Lambda$ such that $q_{k,j} = (\frac{k}{j+1}) \int_{0,1]} u^{j-1} (1-u)^21_{[0,1]}(du) + \Lambda(\{1\})1_{\{1\}}(j)$, $j \in \{1, \ldots, k-1\}, k \geq 2$, where the first summand are the jump rates of the block counting process corresponding to the restriction $\Lambda_{[0,1]}$ of $\Lambda$ to $[0,1]$, i.e., a measure with no atom at 1. Thus,

$$A(n)f(x) = S(k,x) + (f(\log n^{-1}) - f(x))\Lambda(\{1\}), \quad x \in E_n, f \in \hat{C}(\mathbb{R}), n \in \mathbb{N},$$

where the jump rates in $S(k,x)$ correspond to $\Lambda_{[0,1]}$, and

$$Af(x) = I(x) + h(1,x)\Lambda(\{1\}) = I(x) - f(x)\Lambda(\{1\}), \quad x \in (-\infty, 0],$$

where $I(x) = \int h(u,x)\Lambda_{[0,1]}(du), x \in \mathbb{R}$. The additional term corresponds to the killing of the subordinator $-X$ at the rate $\Lambda(\{1\})$. Since $f \in \hat{C}(\mathbb{R}), \lim_{n \to \infty} \sup_{x \in E_n} |(f(\log n^{-1}) - f(x))\Lambda(\{1\}) + f(x)\Lambda(\{1\})| = \Lambda(\{1\}) \lim_{n \to \infty} |f(\log n^{-1})| = 0$, i.e., the additional term converges, and again (26) holds true.

2. The approach to the convergence of the generators is related to Bernstein polynomials. The $(k-1)$-th Bernstein polynomial

$$\sum_{j=0}^{k-1} h\left(\frac{1}{k-1}, x\right) \binom{k-1}{j} u^j (1-u)^{k-1-j}$$

of $h(\cdot, x)$ converges uniformly in $u \in [0,1]$ to $h(u,x)$ as $k \to \infty$, if $x \in \mathbb{R}$ is fixed.
5 Proofs concerning the Bolthausen–Sznitman coalescent

In this section $\Lambda = \lambda$ is the Lebesgue measure on $[0,1]$. Define $\alpha := \alpha(t) := e^{t}$, $t \geq 0$. The process $X^{(n)} = (X_{t}^{(n)})_{t \geq 0} = (\log N_{t}^{(n)} - \alpha \log n)_{t \geq 0}$ is a time-inhomogeneous Markov process. In order to prove convergence in $D\tilde{\mathbb{R}}[0, \infty)$ to $X$ we want to show the uniform convergence of the generators. Typical convergence results are stated for time-homogeneous Markov processes and in order to use these we are going to introduce the time-space process.

5.1 Time-space process: semigroup and generator

Define the time-space processes $\tilde{X}^{(n)} := (t,X_{t}^{(n)})_{t \geq 0}$, $n \in \mathbb{N}$, and $\tilde{X} := (t,X_{t})_{t \geq 0}$. It has been proven in [6] that $\tilde{X}^{(n)}$ and $\tilde{X}$ are time-homogeneous Markov processes (and exist on a new probability space). In the following the tilde symbol indicates the time-space setting. Let $\tilde{\mathbb{E}}_{n} := \{(s,x) \in [0,\infty) \times \mathbb{R} | e^{x}n^{\alpha(s)} \in [n]\}$ denote the state space of $\tilde{X}^{(n)}$, $\tilde{\mathbb{E}} := [0,\infty) \times \mathbb{R}$ denote the state space of $\tilde{X}$ and define $k := k(s,x,n) := e^{x}n^{\alpha(s)} \in \mathbb{N}$ for $(s,x) \in \tilde{\mathbb{E}}_{n}$ and $n \in \mathbb{N}$. Given $f \in B(\tilde{\mathbb{E}})$ and $s \geq 0$, denote the function $x \mapsto f(s,x)$, $x \in \mathbb{R}$, by $\pi f(s,x)$. The limiting process $X$ already is time-homogeneous. Recall that $\mathcal{D}$, the space of twice differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f, f', f'' \in C(\mathbb{R})$ and such that the map $x \mapsto xf'(x)$, $x \in \mathbb{R}$, belongs to $C(\mathbb{R})$, is a core for the generator $A$ of the semigroup $(T_{t})_{t \geq 0}$ corresponding to $X$. The semigroup $(\tilde{T}_{t})_{t \geq 0}$ of $\tilde{X}$, given by

$$\tilde{T}_{t}f(s,x) := E(f(s+t,X_{s+t}),X_{s} = x) = E(f(s+t,\alpha(t)x + X_{t}), \ (s,x) \in \tilde{\mathbb{E}}, f \in B(\tilde{\mathbb{E}}), t \geq 0,$$

is a Feller semigroup. Let $\tilde{\mathcal{D}}$ denote the space of functions $f \in \tilde{C}(\tilde{\mathbb{E}})$ of the form $f(s,x) = \sum_{i=1}^{l} g_{i}(s)h_{i}(x)$ with $l \in \mathbb{N}, h_{i} \in \mathcal{D}$ and $g_{i} \in C_{1}([0,\infty))$ such that $g_{i}, g'_{i} \in \tilde{C}([0,\infty))$ for $i = 1, \ldots, l$. Proposition 10 states that $\tilde{D}$ is a core for the generator $\tilde{A}$ of $(\tilde{T}_{t})_{t \geq 0}$ and

$$\tilde{A}f(s,x) = \frac{\partial}{\partial s}f(s,x) + A \pi f(s,x), \ (s,x) \in \tilde{\mathbb{E}}, f \in \tilde{\mathcal{D}}. \quad (27)$$

The 'semigroup' $(\tilde{T}_{s,t})_{s,t \geq 0}$ of $X^{(n)}$ is given by

$$\tilde{T}_{s,t}^{(n)}f(x) := E(f(X_{s+t}^{(n)}),X_{s}^{(n)} = x) = E(f(\log N_{t}^{(n)} - \alpha(s+t) \log n | X_{s}^{(n)} = k)$$

$$= E(f(\log N_{t}^{(n)} - \alpha(s+t) \log n)), \ (s,x) \in \tilde{\mathbb{E}}_{n}, f \in B(\mathbb{R}), t \geq 0.$$

The 'generator' $(A_{s}^{(n)})_{s \geq 0}$ of $(\tilde{T}_{s,t}^{(n)})_{s,t \geq 0}$ is given by

$$A_{s}^{(n)}f(x) := \lim_{t \rightarrow 0} t^{-1}(\tilde{T}_{s,t}^{(n)}f(x) - f(x))$$

$$= \lim_{t \rightarrow 0} t^{-1}(E(f(\log N_{t}^{(n)} - \alpha(s+t) \log n)) - f(x))$$

$$= -f'(x)\alpha'(s)\log n + \sum_{j=1}^{k-1}(f(x + \log \frac{j}{k}) - f(x))q_{k,j}, \ (s,x) \in \tilde{\mathbb{E}}_{n}. \quad (28)$$

Here $f \in C_{1}(\mathbb{R})$ such that $f, f' \in \tilde{C}(\mathbb{R})$. The semigroup $(\tilde{T}_{t}^{(n)})_{t \geq 0}$ of $\tilde{X}^{(n)}$, given by

$$\tilde{T}_{t}^{(n)}f(s,x) := E(f(s+t,X_{s+t}),X_{s}^{(n)} = x)$$

$$= E(f(s+t,\log N_{t}^{(n)} - \alpha(s+t) \log n)), \ (s,x) \in \tilde{\mathbb{E}}_{n}, f \in B(\tilde{\mathbb{E}}_{n}), t \geq 0, n \in \mathbb{N},$$

is a Feller semigroup on $\tilde{C}(\tilde{\mathbb{E}}_{n})$ for every $n \in \mathbb{N}$. On $\tilde{D}$, or more precisely, for the restriction of $f \in \tilde{D}$ to $\tilde{\mathbb{E}}_{n}$ the generator $A^{(n)}$ of $\tilde{T}^{(n)}$ is given by

$$A^{(n)}f(s,x) = \frac{\partial}{\partial s}f(s,x) + A_{s}^{(n)}\pi f(s,x), \ (s,x) \in \tilde{\mathbb{E}}_{n}, n \in \mathbb{N}. \quad (29)$$
5.2 Proof of Corollary \[4\]

Proof. [of Corollary \[4\]] Let \( f \in D \). The approach to the proof is the same as in Section \[4\] but the function \( u \mapsto f(x + \log(1 - u)) \), \( u \in [0, 1] \), demands second order approximation like in the integral part of the limiting generator (\[11\]). Define \( h : [0, 1] \times \mathbb{R} \to \mathbb{R} \) via \( h(u, x) := u^{-2}(f(x + \log(1 - u)) - f(x) + uf'(x)), \) \( u \in (0, 1), \) \( h(0, x) := \lim_{u \to 0} h(u, x) = 2^{-1}(f''(x) - f'(x)) \) and, since \( f \in \tilde{C}(\mathbb{R}), h(1, x) := \lim_{u \to 1} h(u, x) = f'(x) - f(x) \) for \( x \in \mathbb{R} \). Taylor's theorem applied to \( u \mapsto f(x + \log(1 - u)) \), \( u < 1 \), with evaluation point \( u = 0 \) and exact integral remainder yields

\[
h(u, x) = u^{-2} \int_0^u \frac{u - s}{(1 - s)^2} f''(x + \log(1 - s)) - f'(x + \log(1 - s)) ds
\]

or

\[
= \int_0^1 \frac{1 - s}{(1 - us)^2} f''(x + \log(1 - us)) - f'(x + \log(1 - us)) ds, \quad u \in [0, 1), x \in \mathbb{R}.
\]

The latter formula of \( h(u, x) \) shows that \( h \) is bounded even as \( u \) tends to zero. Putting \( k = k(s, x, n) = e^\alpha n^{\alpha(s)} \) in (\[28\]) yields

\[
A^{(n)}_s f(x) = f'(x) R(k, x) + S(k, x), \quad (s, x) \in \tilde{E}_n, n \in \mathbb{N},
\]

where

\[
R(k, x) := \log k - \sum_{j=1}^{k-1} \frac{k-j}{k} q_{k,j} - x, \quad k \in \mathbb{N}, x \in \mathbb{R},
\]

and

\[
S(k, x) := \sum_{j=1}^{k-1} (f(x + \log \frac{j}{k}) - f(x) + \frac{j}{k} f'(x)) q_{k,j}, \quad k \in \mathbb{N}, x \in \mathbb{R}.
\]

Further define \( I(x) := \int_{[0,1]} h(u, x) \Lambda(du), x \in \mathbb{R} \). By Eq. (\[17\]) with \( a = b = 1, \frac{k-j}{k} q_{k,j} = (k - j + 1)^{-1}, j \in \{1, \ldots, k - 1\}, k \geq 2 \), such that \( \sum_{j=1}^{k-1} \frac{k-j}{k} q_{k,j} = \sum_{j=2}^{k} j^{-1} \) for \( k \geq 2 \). As \( n \to \infty, k = k(s, x, n) = e^\alpha n^{\alpha(s)} \to \infty \) or \( x \to -\infty \). Fix \( T > 0 \). E.g., if \( s \in [0, T] \) then either \( k \geq n^{\alpha(T + \delta)} \) or \( x < -\alpha(T)(1 - \alpha(T)) \log n \), where \( \delta > 0 \) is a constant. The well-known asymptotics of the harmonic numbers states that sup \( x \in [0, T] \mid R(k, x) - (1 + \Psi(1) - x) \rceil = |\log k - \sum_{j=1}^{k} j^{-1} - \Psi(1)| \to 0 \) as \( k \to \infty \). Clearly, \( \lim_{x \to -\infty} |f'(x)| = 0 \). Dividing the state space as above therefore implies

\[
\lim_{n \to \infty} \sup_{(s, x) \in \tilde{E}_n, s \in [0, T]} |f'(x)| |R(k, x) - (1 + \Psi(1) - x)| = 0.
\]

Substituting \( k - j - 1 \) for \( j \) in (\[31\]) yields

\[
S(k, x) = \sum_{j=0}^{k-2} (f(x + \log(1 - \frac{j+1}{k})) - f(x) + \frac{j+1}{k} f'(x)) q_{k,k-j-1}
\]

or

\[
= \sum_{j=0}^{k-2} h(\frac{j+1}{k}, x) \frac{(j+1)^2}{k^2} \left( \frac{k}{j+2} \right) \int_{[0,1]} u^j (1-u)^{k-2-j} \Lambda(du)
\]

or

\[
= \frac{k-1}{k} \sum_{j=0}^{k-2} h(\frac{j+1}{k}, x) \frac{j+1}{j+2} \left( \frac{k-2}{j} \right) \int_{[0,1]} u^j (1-u)^{k-2-j} \Lambda(du), \quad k \in \mathbb{N}, x \in \mathbb{R}.
\]

Set \( c := \Lambda([0, 1]) > 0 \) and define the probability measure \( Q \) on \(([0, 1], \mathcal{B} \cap [0, 1])\) as \( Q := c^{-1} \Lambda \). Let the random variables \( Z_k, k \in \mathbb{N} \), have distribution given by

\[
\mathbb{P}(Z_k = j) = \binom{k-2}{j} \int_{[0,1]} u^j (1-u)^{k-2-j} Q(du), \quad j \in \{0, \ldots, k-2\},
\]
i.e., \( Z_k \) has a mixed binomial distribution with sample size \( k - 2 \) and random success rate \( Q \). Let \( Z \) have distribution \( Q \). Then

\[
S(k, x) = c(1 - k^{-1}) E((1 - (Z_k + 2)^{-1}) h((Z_k + 1)/k, x)), \quad k \in \mathbb{N}, x \in \mathbb{R},
\]

and \( I(x) = c E(h(Z, x)), x \in \mathbb{R} \). It is easy to check that \((Z_k + 1)/k \to Z\) in distribution as \( k \to \infty \).

The family of functions \( \{h(\cdot, x)|x \in \mathbb{R}\} \) is equicontinuous on \([\delta, 1 - \delta]\) for every \( 0 < \delta < 1/2 \) and uniformly bounded on \([0, 1]\). Due to \( Q(\{0\}) = c^{-1} \Lambda(\{0\}) = 0 \), \( Z_k \to \infty \) a.s. as \( k \to \infty \) and thus \( \lim_{k \to \infty} E\left(1/(Z_k + 2)^{-1}\right) = 0 \) such that the additional factor \( 1 - (Z_k + 2)^{-1} \) in the mean above can be omitted when considering the limit of \( S(k, x) \) as \( k \to \infty \). From Lemma \( 5 \) it follows that

\[
\lim_{k \to \infty} \sup_{x \in \mathbb{R}} |S(k, x) - I(x)| = 0. \tag{33}
\]

From \( \lim_{x \to -\infty} h(Z, x) = 0 \) a.s., the fact that the functions \( h(\cdot, x), x \in \mathbb{R} \), are uniformly bounded and the dominated convergence theorem it follows that

\[
\lim_{x \to -\infty} |I(x)| = c \lim_{x \to -\infty} |E(h(Z, x))| = 0. \tag{34}
\]

Since \( f, f' \in \tilde{C}(\mathbb{R}), \lim_{x \to -\infty} S(k, x) = 0 \) for any \( k \in \mathbb{N} \) and, in view of \( 33 \) and \( 34 \),

\[
\lim_{x \to -\infty} \sup_{k \in \mathbb{N}} |S(k, x)| = 0. \tag{35}
\]

As seen in the proof of Corollary \( 8 \), Eqs. \( 33 \)–\( 35 \) imply

\[
\lim_{n \to \infty} \sup_{(s, x) \in \tilde{E}_n, s \in [0, T]} |S(k, x) - I(x)| = 0. \tag{36}
\]

By \( 32 \), \( \lim_{n \to \infty} \sup_{(s, x) \in \tilde{E}_n, s \in [0, T]} |A^{(n)}_s f(x) - Af(x)| = 0. \) Due to \( 27 \) and \( 29 \),

\[
\lim_{n \to \infty} \sup_{(s, x) \in \tilde{E}_n, s \in [0, T]} |A^{(n)}_s f(s, x) - Af(s, x)| = 0 \quad \text{for every function } f \text{ belonging to the core } \tilde{D} \text{ and each } T > 0. \quad \text{From } 8, \text{ IV, Corollary 8.7 it follows that } \tilde{X}^{(n)} \to \tilde{X} \text{ in } D_{\tilde{E}}[0, \infty), \text{ hence } X^{(n)} \to X \text{ in } D_E[0, \infty) \text{ as } n \to \infty. \quad \Box
\]

**Remark.**

1. Note that if \( \Lambda = \lambda \) then \( Z_k \) has a discrete uniform distribution on \( \{0, \ldots, k - 2\} \) and \( Z \) has a continuous uniform distribution on \( (0, 1) \).

2. Put \( \gamma(k) := \sum_{j=1}^{k-1} \frac{k-j}{k} q_{k,j} = \sum_{j=2}^{k} (j-1) \frac{k}{j} \lambda_{k,j} \) for \( k \geq 2 \). Among dust-free \( \Lambda \)-coalescents that stay infinite the proof works for the Bolthausen–Sznitman coalescent, because the precise asymptotics of \( \gamma(k)/k = \log k - 1 \) as \( k \to \infty \) is known. Observe that in the proof of Corollary \( 1 \), the fact that \( \Lambda = \lambda \) is only used to verify \( 32 \).

### 6 Proof of Theorem \( 1 \)

In this section \( \Lambda \) satisfies Assumption A. We continue to use the time-space setting and the notation of Subsection \( 5.1 \) with \( \alpha \) replaced by \( \alpha := \alpha(t) := e^{-bt}, t \geq 0 \). Define \( \Lambda_D := \Lambda - b \lambda \) and let \( \Lambda^*_D, \Lambda_D \) denote the nonnegative measures constituting the Jordan decomposition \( \Lambda_D = \Lambda^*_D - \Lambda_D \) of \( \Lambda_D \). The decomposition of \( \Lambda \) into a 'Bolthausen–Sznitman part' \( \Lambda^*_D \) and a 'dust part' \( \Lambda_D \) is transferred to the jump rates and the generator. Proving Theorem \( 1 \) now only requires to suitable arrange equations already obtained in Sections \( 4 \) and \( 5 \). To be precise, the results of Section \( 4 \) are applied to the summands \( \Lambda^*_D \) of \( \Lambda_D \), but we omit this detail in the following.

**Proof.** [of Theorem \( 1 \)] Let \( q_{k,j}^{D^+}, q_{k,j}^{D^-} \) and \( q_{k,j}^{D^+} \) denote the rates of the block counting process corresponding to \( \lambda, \Lambda^*_D \) and \( \Lambda_D \), respectively, and define \( q_{k,j}^D := q_{k,j}^{D^+} - q_{k,j}^{D^-} \) for \( j \in \{1, \ldots, k\} \) and \( k \in \mathbb{N} \). Obviously, \( q_{k,j} = b q_{k,j}^{D^+} + q_{k,j}^{D^-} \). Recall that \( k = k(s, x, n) = e^n \alpha(x) \in \mathbb{N} \) for \( (s, x) \in \tilde{E}_n \).
and $n \in \mathbb{N}$. From (23) it follows that the 'generator' $A^{(n)}_s$ of $X^{(n)} = (X^{(n)}_t)_{t \geq 0} = (\log N^{(n)}_t - \alpha(t) \log n)_{t \geq 0}$ is given by

$$A^{(n)}_s f(x) = bR(k, x)f'(x) + bS_{BS}(k, x) + S_D(k, x), \quad (s, x) \in \bar{E}_n, n \in \mathbb{N},$$

where

$$R(k, x) := \log k - \sum_{j=1}^{k-1} \frac{k-1-j}{k} q^{\lambda}_{k,j} - x, \quad k \in \mathbb{N}, x \in \mathbb{R},$$

$$S_{BS}(k, x) := \sum_{j=1}^{k-1} (f(x + \log \frac{1}{j}) - f(x) + \frac{k-1-j}{k} f'(x)) q^{\lambda}_{k,j}, \quad k \in \mathbb{N}, x \in \mathbb{R},$$

$$S_D(k, x) := \sum_{j=1}^{k-1} (f(x + \log \frac{1}{j}) - f(x)) q^{D}_{k,j}, \quad k \in \mathbb{N}, x \in \mathbb{R},$$

are defined as in (30), (31) and (22), and $f \in C_1(\mathbb{R})$ such that $f, f' \in \hat{C}(\mathbb{R})$. By Lemma 6 and (2), the generator $\hat{A}$ of $X = (X_t)_{t \geq 0}$ can be written as

$$Af(x) = b(1 + \Psi(1) - x)f'(x) + b \int_{[0,1]} f(x + \log(1-u)) - f(x) + uf'(x) \lambda(du)$$

$$+ \int_{[0,1]} f(x + \log(1-u)) - f(x) \Lambda_D(du), \quad x \in \mathbb{R}, f \in D.$$

From Eqs. (32), (30) and (23)-(25) it follows that $\lim_{n \to \infty} \sup_{(s, x) \in \bar{E}_n, s \in [0, T]} |A^{(n)}_s f(x) - Af(x)| = 0$ for $f \in D$. Due to (27) and (29),

$$\lim_{n \to \infty} \sup_{(s, x) \in \bar{E}_n, s \in [0, T]} |\bar{A}^{(n)} f(s, x) - \bar{A} f(s, x)| = 0 \quad (37)$$

for every $f \in \bar{D}$ and $T > 0$. By Proposition 10 the space $\bar{D}$ is a core for $\bar{A}$. Thus, it follows from [2, IV, Corollary 8.7] that $X^{(n)} \to X$ in $D_{\bar{E}}[0, \infty)$, hence $X^{(n)} \to X$ in $D_{\bar{E}}[0, \infty)$ as $n \to \infty$. \hfill $\Box$

## 7 Proof of Theorem 5

In this section $A$ satisfies Assumption A. The process $Y^{(n)} = (Y^{(n)}_t)_{t \geq 0} = (\log L^{(n)}_t - e^{bt} \log n)_{t \geq 0}$ is a possibly time-inhomogeneous Markov process, hence we set up the time-space framework. We provide two proofs. Using Theorem 1 and Siegmund-duality, in the first proof the convergence of the one-dimensional distributions and subsequently the uniform convergence of the semigroups is shown. The second proof, in which the uniform convergence of generators is shown, resembles previous ones.

**Proof.** [First proof of Theorem 5] For $x \in \mathbb{R}$ and $t \geq 0$ define $m := [e^y e^x] \in \mathbb{N}$. If $\theta_t((\infty, 0)) = \int_{[0,1]} u^{-2} \Lambda(du) = \infty$ then $X_t$ has a continuous distribution for every $t > 0$. Eq. (1) and Theorem 1 imply that

$$\mathbb{P}(Y^{(n)}_t \geq y) = \mathbb{P}(L^{(n)}_t \geq m) = \mathbb{P}(N^{(m)}_t \leq n) = \mathbb{P}(X^{(m)}_t \leq \log n - e^{-bt} \log m)$$

$$+ \mathbb{P}(X_t \leq -e^{-bt} y) = \mathbb{P}(-e^{bt} X_t \geq y), \quad y \in \mathbb{R}, t \geq 0, \quad (38)$$

as $n \to \infty$. If $\int_{[0,1]} u^{-2} \Lambda(du) < \infty$ then the dust condition is satisfied such that $b = 0$ and (38) holds true for $-y$ in the set $C_{X_t}$ of continuity points of $X_t$. Since $Y_t \overset{d}{=} -e^{bt} X_t$, $\lim_{n \to \infty} \mathbb{P}(Y^{(n)}_t \leq -y) = \mathbb{P}(-Y_t \leq -y)$ for every $-y \in C_{X_t} = C_{-Y_t}$. Thus, $Y^{(n)}_t$ converges in distribution to $Y_t$ as $n \to \infty$ for every $t \geq 0$.

Define the time-space processes $\bar{Y}^{(n)} := (t, Y^{(n)}_t)_{t \geq 0}$, $n \in \mathbb{N}$, and $\bar{Y} := (t, Y_t)_{t \geq 0}$. The processes $\bar{Y}^{(n)}$ and $\bar{Y}$ are time-homogeneous Markov processes with state spaces $\bar{E}_n = \{(s, y) | s \geq$
$0, e^{yn^{b_{ns}}} \in \{n, n + 1, \ldots\}$ and $\widetilde{E} = [0, \infty) \times \mathbb{R}$ and semigroups $(\widetilde{T}_t^{(n)})_{t \geq 0}$ and $(\widetilde{T}_t)_{t \geq 0}$. Define $k := k(s, y, n) := e^{yn^{b_{ns}}} \in \{n, n + 1, \ldots\}$ for $(s, y) \in \widetilde{E}_n$ and $n \in \mathbb{N}$. Then

$$
\widetilde{T}_t^{(n)} f(s, y) = \mathbb{E}(f(s + t, Y_s^{(n)})|Y_s^{(n)} = y) = \mathbb{E}(f(s + t, \log f_{t-s}(k) - e^{-t+s} \log n))
$$

$$
= \mathbb{E}(f(s + t, e^{bt}y + Y_t^{(k)})), \quad (s, y) \in \widetilde{E}_n, f \in B(\widetilde{E}), t \geq 0, n \in \mathbb{N}.
$$

Fix $t > 0$ and first let $f \in B(\widetilde{E})$ be of the form $f(s, y) = g(s)h(y)$, $(s, y) \in \widetilde{E}$, where $g \in B([0, \infty))$ and $h \in \mathcal{C}(\mathbb{R})$. Clearly, $\widetilde{T}_t^{(n)} f(s, y) = g(s + t) \mathbb{E}(h(e^{bt}y + Y_t^{(k)})), (s, y) \in \widetilde{E}_n, n \in \mathbb{N}$, and $T_t f(s, y) = \mathbb{E}(f(s + t, Y_{s+t})|Y_s = y) = g(s + t)T_th(y) = g(s + t) \mathbb{E}(h(e^{bt}y + Y_t)), (s, y) \in \widetilde{E}$, where the distribution of $Y_t$ is defined by its characteristic function $\chi_t$, given by $[19]$. Note that $h$ is uniformly continuous and bounded. For $y \in \mathbb{R}$ define the function $h_y : \mathbb{R} \to \mathbb{R}$ via $h_y(x) := h(e^{bt}y + x), x \in \mathbb{R}$. The family of functions $\{h_y|y \in \mathbb{R}\}$ is equicontinuous and uniformly bounded. From the weak convergence of $Y_t^{(k)}$ to $Y_t$ as $k \to \infty$ and $\mathbb{R}$ Theorem 3.1 it follows that $\lim_{k \to \infty} \sup_{y \in \mathbb{R}} |\mathbb{E}(h(e^{bt}y + Y_t^{(k)})) - \mathbb{E}(h(e^{bt}y + Y_t))| = 0$. Since $k = e^{yn^{b_{ns}}} \geq n$ for $(s, y) \in \widetilde{E}_n$ and $n \in \mathbb{N}$, $\lim_{n \to \infty} \sup_{(s,y)\in \widetilde{E}_n} |\mathbb{E}(h(e^{bt}y + Y_t^{(k)})) - \mathbb{E}(h(e^{bt}y + Y_t))| = 0$. Thus,

$$
\lim_{n \to \infty} \sup_{(s, y) \in \widetilde{E}_n} \left|\widetilde{T}_t^{(n)} f(s, y) - \widetilde{T}_t f(s, y)\right| = 0. \quad (39)
$$

The algebra of functions $f \in B(\widetilde{E})$ of the form $f(s, y) = \sum_{j=1}^{l} g_j(s)h_j(y), (s, y) \in \widetilde{E}$, where $l \in \mathbb{N}, g_j \in B([0, \infty))$ and $h_j \in \mathcal{C}(\mathbb{R})$, separates points and vanishes nowhere. According to the Stone–Weierstrass theorem for locally compact spaces (see e.g. [7]) it is a dense subset of $B(\widetilde{E})$ such that (39) holds true for $f \in B(\widetilde{E})$. [8] IV, Theorem 2.11] states that $Y_t^{(n)} \to Y$ in $D_{\mathbb{R}}[0, \infty)$, hence $Y^{(n)} \to Y$ in $D_{\mathbb{R}}[0, \infty)$ as $n \to \infty$.

The process $Y$ defined by [12] and [13] is an Ornstein–Uhlenbeck type process (with nonnegative linear drift) as in [19]. The underlying infinitely divisible distribution has characteristic exponent $u \mapsto \psi(-u), y \in \mathbb{R}$. According to [19] Theorem 3.1, $D$ is a core for the corresponding generator $A$ and

$$
Af(y) = f'(y)(-a + by) + \int_{[0,1]} (f(y - \log(1 - u)) - f(y) - uf'(y))u^{-2} \Lambda(du) \quad (40)
$$

for $y \in \mathbb{R}$ and $f \in D$; comparatively see Lemma 6 and its proof.

**Proof.** [Second proof of Theorem 7] The ‘generator’ $(A_{s}^{(n)})_{s \geq 0}$ of $Y^{(n)}$ is given by

$$
A_{s}^{(n)} f(y) = -f'(y)be^{bs} \log n + \sum_{j \geq e^{yn^{b_{ns}}}} (f(\log j - e^{bs} \log n) - f(y))\gamma_{\nu, n^{b_{ns}}, j}, \quad (s, y) \in \widetilde{E}_n, n \in \mathbb{N}.
$$

Here $f \in \mathcal{C}_1(\mathbb{R})$ such that $f, f' \in \mathcal{C}(\mathbb{R})$. Putting $k := k(s, y, n) := e^{yn^{b_{ns}}}$ for $(s, y) \in \widetilde{E}_n$ and $n \in \mathbb{N}$ yields

$$
A_{s}^{(n)} f(y) = b f'(y)(- \log k + y) + \sum_{j=1}^{\infty} (f(y + \log(1 + \frac{1}{k})) - f(y))\gamma_{k,k+j}, \quad (s, y) \in \widetilde{E}_n, n \in \mathbb{N}.
$$

Define $\Lambda_{D} := \Lambda - b\lambda$ and let $\Lambda_{D}^{+}, \Lambda_{D}^{-}$ denote the nonnegative measures constituting the Jordan decomposition $\Lambda_{D} = \Lambda_{D}^{+} - \Lambda_{D}^{-}$ of $\Lambda_{D}$. Let $\gamma_{k,j}^{D,\pm}$ and $\gamma_{k,j}^{D,-}$ denote the jump rates of the fixation line corresponding to $\lambda, \Lambda_{D}^{+}$ and $\Lambda_{D}^{-}$, respectively, and define $\gamma_{k,j}^{D,-} := \gamma_{k,j}^{D,\pm} - \gamma_{k,j}^{D,-}$ for $j \in \{k, k + 1, \ldots\}$ and $k \in \mathbb{N}$. Then $\gamma_{k,k+j} = b^{k}_{k,k+j} + \gamma_{k,k+j}^{D,\pm}, k \in \mathbb{N}, j \in \mathbb{N}_{0},$ and

$$
A_{s}^{(n)} f(y) = b f'(y)R(k,y) + bSBS(k,y) + SD(k,y), \quad (s, y) \in \widetilde{E}_n, n \in \mathbb{N}, \quad (41)
$$
where

\[ R(k, y) := -\log k + y + \sum_{j=1}^{k} \frac{k}{j} \gamma_{k,k+j}^{\lambda}, \quad k \in \mathbb{N}, y \in \mathbb{R}, \]

\[ S_{BS}(k, y) := \sum_{j=1}^{\infty} (f(y + \log(1 + \frac{j}{k})) - f(y) - \frac{j}{k}1_{[0,1]}(\frac{j}{k})f'(y))\gamma_{k,k+j}^{\lambda}, \quad k \in \mathbb{N}, y \in \mathbb{R}, \]

\[ S_{D}(k, y) := \sum_{j=1}^{\infty} (f(y + \log(1 + \frac{j}{k})) - f(y))\gamma_{k,k+j}^{D}, \quad k \in \mathbb{N}, y \in \mathbb{R}, \]

and \( f \in C_{1}(\mathbb{R}) \) such that \( f, f' \in \mathcal{C}(\mathbb{R}) \). Using the decomposition of \( \Lambda \) on Eq. (40) yields

\[ Af(y) = bf'(y)(-1 - \Psi(1) + y) + bI_{BS}(y) + I_{D}(y), \quad y \in \mathbb{R}, f \in D, \quad (42) \]

where

\[ I_{BS}(y) := \int_{[0,1]} (f(y - \log(1 - u)) - f(y) - uf'(y))u^{-2}\lambda(du), \quad y \in \mathbb{R}, \]

\[ I_{D}(y) := \int_{[0,1]} (f(y - \log(1 - u)) - f(y))u^{-2}\Lambda_{D}(du), \quad y \in \mathbb{R}. \]

Let \( f \in D \). In the Bolthausen–Sznitman coalescent \( \gamma_{k,j+1}^{\lambda} = k/(j(j + 1)) \) for \( k, j \in \mathbb{N} \) such that \( \sum_{j=1}^{k} \frac{k}{j} \gamma_{k,k+j}^{\lambda} = \sum_{j=1}^{k}(j + 1)^{-1} = H_{k+1} - 1 = \log k - \Psi(1) + o(1) \) as \( k \to \infty \). Here \( H_{k} \) denotes the \( k \)-th harmonic number for \( k \in \mathbb{N} \). Thus,

\[ \lim_{k \to \infty} \sup_{y \in \mathbb{R}} |R(k, y) - (-1 - \Psi(1) + y)| = 0. \quad (43) \]

The function \( h_{BS} : [0,1] \times \mathbb{R} \to \mathbb{R} \), defined via \( h_{BS}(u, y) := u^{-2}(f(y - \log(1 - u)) - f(y) - \frac{y}{1-u}1_{[0,1]}(u)f'(y)), u \in [0,1], y \in \mathbb{R}, \) is bounded. Let the random variables \( Z_{k}, k \in \mathbb{N}, \) have distribution given by

\[ \mathbb{P}(Z_{k} = j) = \binom{k + j - 2}{j - 1} \int_{[0,1]} w^{j-1}(1 - u)^{k} \lambda(du), \quad j, k \in \mathbb{N}, \]

i.e., \( Z_{k} - 1 \) has a mixed negative binomial distribution. Observe that \( h_{BS}(1 - (1 + \frac{j}{k})^{-1}, y) = (\frac{k-j}{k})^{-2}(f(y + \log(1 + \frac{j}{k})) - f(y) - \frac{j}{k}1_{[0,1]}(\frac{j}{k})f'(y)), y \in \mathbb{R}, \) and \( \gamma_{k,k+j}^{\lambda} = (\frac{k}{k+j})^{-2}(1 + (k+j)^{-1})(1 - (j+1)^{-1})\mathbb{P}(Z_{k} = j) \) for \( j, k \in \mathbb{N} \). Hence,

\[ S_{BS}(k, y) = \mathbb{E}(h_{BS}(1 - (1 + Z_{k}/k)^{-1}, y)(1 + (k + Z_{k})^{-1})(1 - (Z_{k} + 1)^{-1})). \]

Let \( Z \) have uniform distribution on \( (0, 1) \) such that \( I_{BS}(y) = \mathbb{E}(h_{BS}(Z, y)) \) for \( y \in \mathbb{R} \). Here it is used that \( \int_{[0,1]} u^{-2}(u - \frac{u}{1-u}1_{[0,1]}(u))\lambda(du) = \int_{0}^{1/2} -u^{-1}\lambda(du) + \int_{1/2}^{1} u^{-1} \lambda(du) = 0 \).

The function \( g : (0, \infty) \to (0, 1), \) defined via \( g(u) := 1 - (1 + u)^{-1}, u \in (0, \infty), \) is bounded and continuous. Since \( Z_{k}/k \to Z/(1 - Z) \) in distribution as \( k \to \infty, 1 - (1 + Z_{k}/k)^{-1} = g(Z_{k}/k) \to g(Z/(1 - Z)) = Z \) in distribution as \( k \to \infty \). In particular, the random variables have values in \( [0,1] \). When considering the limit \( k \to \infty \), the factor \((1 + (k + Z_{k})^{-1})(1 - (Z_{k} + 1)^{-1})\) has no influence on \( S_{BS}(k, y) \). From Lemma [9] it follows that

\[ \lim_{k \to \infty} \sup_{y \in \mathbb{R}} |S_{BS}(k, y) - I_{BS}(y)| = 0. \quad (44) \]

The measure \( \Lambda_{D} \) is real-valued. Eq. (45) below can be proved when \( \Lambda_{D} \) is replaced by \( \Lambda_{D}^{\gamma} \) and \( \Lambda_{D}^{\Psi} \) in this paragraph, and then holds true for \( \Lambda_{D} \) by linearity. The function \( h_{D} : [0,1] \times \mathbb{R} \to \mathbb{R} \), defined via \( h_{D}(u, y) := u^{-1}(f(y - \log(1 - u)) - f(y)), u \in [0,1], y \in \mathbb{R}, \) is bounded. By assumption \( c := \int_{[0,1]} u^{-1}\Lambda_{D}(du) < \infty \). Define the probability measure \( Q \) on \( ([0,1], \mathcal{B} \cap [0,1]) \) via \( Q(A) := \int_{A} d \Lambda_{D} \to \infty \), for \( A \in \mathcal{B} \cap [0,1] \).
\[ c^{-1} \int_A u^{-1} \Lambda_D(du), \; A \in \mathcal{B} \cap [0,1]. \]

Let the random variables \( Z_k, k \in \mathbb{N}, \) have distribution given by

\[ \mathbb{P}(Z_k = j) = \binom{k + j - 1}{j} \int_{[0,1]} u^j (1 - u)^k Q(du), \quad j \in \mathbb{N}_0, k \in \mathbb{N}, \]

i.e., \( Z_k \) has a mixed negative binomial distribution. Observe that \( h_D(1 - (1 + \frac{k}{2})^{-1}, y) = (f(y + \log(1 + \frac{k}{2})) - f(y)) \gamma_{k,k+j}^{D}, \; y \in \mathbb{R}, \) and \( \gamma_{k,k+j}^{D} = \frac{c \kappa_{k,j}}{k} (1 - (1 + j)^{-1}) \mathbb{P}(Z_k = j) \) for \( j, k \in \mathbb{N}. \) Hence,

\[ S_D(k,y) = \sum_{j=0}^{\infty} (f(y + \log(1 + \frac{k}{2}))- f(y)) \gamma_{k,k+j}^{D}, \quad k \in \mathbb{N}, y \in \mathbb{R}. \]

Let the random variable \( Z \) have distribution \( Q. \) In particular, \( I_D(y) = cE(h_D(Z,y)), \; y \in \mathbb{R}. \) By Lemma 9 and since \( 1 - (1 + \frac{Z_k}{k})^{-1} \) converges in distribution to \( Z \) as \( k \to \infty, \)

\[ \lim_{k \to \infty} \sup_{y \in \mathbb{R}} \mathbb{E} \left( h_D(1 - (1 + Z_k/k)^{-1}, y)(1 - (1 + Z_k)^{-1}) \right) = 0. \]

Thus,

\[ \lim_{k \to \infty} \sup_{y \in \mathbb{R}} |S_D(k,y) - I_D(y)| = 0. \]

Taking into account that \( k = e^y n^b \geq n \) for \( (s, y) \in \tilde{E}_n \) and \( n \in \mathbb{N}, \) Eqs. (11)-(15) imply

\[ \lim_{n \to \infty} \sup_{(s,y) \in \tilde{E}_n} |A^{(n)} f(y) - A f(y)| = 0. \]

The time-space variant of [8, IV, Corollary 8.7] as implemented in the proof of Theorem 11 yields the desired convergence of \( Y^{(n)} \to Y \) in \( D_{\mathbb{R}}[0, \infty) \) as \( n \to \infty. \)

\section{Appendix}

\textbf{Lemma 8} The \( \Lambda \)-coalescent does not come down from infinity under Assumption A.

\textbf{Proof.} Suppose that \( \Lambda \) satisfies Assumption A. Define \( \Lambda_D := \Lambda - b \lambda, \) let \( \Lambda_D^+ \) and \( \Lambda_D^- \) denote the nonnegative measures constituting the Jordan decomposition \( \Lambda_D = \Lambda_D^+ - \Lambda_D^- \) of \( \Lambda_D \) and let \( |\Lambda_D| := \Lambda_D^+ + \Lambda_D^- \) denote the (total) variation of \( \Lambda_D. \) Define \( \eta_k^b := k \sum_{j=0}^{k-1} \int_{[0,1]} (1 - u)^j \Lambda(du) \)

and \( \eta_k^{b \lambda} \) and \( \eta_k^{\lambda |\Lambda_D|} \) similarly with \( b \lambda \) and \( |\Lambda_D| \) in place of \( \Lambda \) for \( k \geq 2. \) By assumption,

\[ \lim_{k \to \infty} k^{-1} \eta_k^{\lambda |\Lambda_D|} = \int_{[0,1]} u^{-1} |\Lambda_D|(du) < \infty. \]

Since

\[ (k \log k)^{-1} \eta_k^{b \lambda} = b \log k \sum_{j=0}^{k-2} \int_{0}^{1} (1 - u)^j du = b \log k \sum_{j=0}^{k-2} (j + 1)^{-1} \to b, \quad k \to \infty, \]

it follows that \( \eta_k^{b \lambda} - \eta_k^{\lambda |\Lambda_D|} \sim bk \log k \) as \( k \to \infty. \) Due to \( \Lambda \leq b\lambda + |\Lambda_D|, \) it holds that \( \eta_k^{\lambda} \leq \eta_k^{b \lambda} + \eta_k^{\lambda |\Lambda_D|} \) for \( k \geq 2. \) Hence,

\[ \sum_{k=2}^{\infty} (\eta_k^\lambda)^{-1} \geq \sum_{k=2}^{\infty} \left( \eta_k^{b \lambda} + \eta_k^{\lambda |\Lambda_D|} \right)^{-1} \to \infty. \]

The claim then follows from Schweinsberg’s criterion [21 Corollary 2].

The following lemma is a generalization of the integral criterion of convergence in distribution and is used in Sections 4-7 to prove the uniform convergence of generators.
Lemma 9 Let $X, X_1, X_2, \ldots$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $[0, 1]$ such that $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 0$ and $X_n \to X$ in distribution as $n \to \infty$. Suppose that the family $F$ of functions $f : [0, 1] \to \mathbb{R}$ is uniformly bounded on $[0, 1]$ and equicontinuous on $[\delta, 1 - \delta]$ for every $0 < \delta < 1/2$. In particular, $f \in F$ is bounded and continuous on $(0, 1)$. Then

$$\lim_{n \to \infty} \sup_{f \in F} |\mathbb{E}(f(X_n)) - \mathbb{E}(f(X))| = 0.$$  

Proof. Define $M := \sup_{f \in F} \|f\| < \infty$ and let $\varepsilon > 0$ be arbitrary. The assumption $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 0$ and the convergence of $X_n$ to $X$ in distribution as $n \to \infty$ provide the existence of $0 < \delta < 1/2$ and $n_0 \in \mathbb{N}$ such that $\mathbb{P}(X_n \notin [\delta, 1 - \delta]) < \varepsilon/(4M)$ for $n \geq n_0$ and $\mathbb{P}(X \notin [\delta, 1 - \delta]) < \varepsilon/(4M)$. For $f \in F$ define $\tilde{f} : [0, 1] \to \mathbb{R}$ via $\tilde{f}(u) := f(\delta)$, $0 \leq u \leq \delta$, $\tilde{f}(u) := f(u)$, $\delta \leq u \leq 1 - \delta$, and $\tilde{f}(u) := f(1 - \delta)$, $1 - \delta \leq u \leq 1$. Then $\{\tilde{f} \in F\}$ is bounded (by $M$) and equicontinuous on $[0, 1]$. [17] Theorem 3.1 yields

$$\lim_{n \to \infty} \sup_{f \in F} |\mathbb{E}(\tilde{f}(X_n)) - \mathbb{E}(\tilde{f}(X))| = 0.$$  

From

$$|\mathbb{E}(f(X_n)) - \mathbb{E}(f(X))| \leq |\mathbb{E}(f(X_n)) - \mathbb{E}(\tilde{f}(X_n))| + |\mathbb{E}(\tilde{f}(X_n)) - \mathbb{E}(\tilde{f}(X))| + |\mathbb{E}(\tilde{f}(X)) - \mathbb{E}(f(X))|,$$

it follows that $\lim_{n \to \infty} \sup_{f \in F} |\mathbb{E}(f(X_n)) - \mathbb{E}(f(X))| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary the proof is complete. \qed

Remark. In [17] Theorem 3.1] the state space is more generally a separable metric space, but equicontinuity of $F$ is required to hold on the whole state space.

Let $E$ be a complete separable metric space and equip $\tilde{E} := [0, \infty) \times E$ with the product metric. The following proposition treats the generator of time-space processes of time-homogeneous Feller processes.

Proposition 10 Suppose that $(T_t)_{t \geq 0}$ is a Feller semigroup on $\hat{C}(E)$ with generator $A$ and that $D$ is a core for $A$. For $f \in \hat{C}(\tilde{E})$ and $s \in [0, \infty)$ let $\pi_f(s, x)$ denote the function $x \mapsto f(s, x)$, $x \in E$. The semigroup $(\bar{T}_t)_{t \geq 0}$, defined via

$$\bar{T}_t f(s, x) := T_t \pi_f(s + t, x), \quad (s, x) \in \tilde{E}, f \in B(\tilde{E}), t \geq 0,$$

is a Feller semigroup on $\hat{C}(\tilde{E})$. Let $\bar{D}$ denote the space of functions $f \in \hat{C}(\tilde{E})$ of the form $f(s, x) = \sum_{i=1}^l g_i(s)h_i(x)$, $(s, x) \in \tilde{E}$, where $l \in \mathbb{N}, h_i \in D$ and $g_i \in C_c([0, \infty))$ such that $g_i, g'_i \in C([0, \infty))$ for $i = 1, \ldots, l$. Then $\bar{D}$ is a core for the generator $\bar{A}$ of $(\bar{T}_t)_{t \geq 0}$ and

$$\bar{A} f(s, x) = \frac{\partial}{\partial s} f(s, x) + A \pi f(s, x), \quad (s, x) \in \tilde{E}, f \in \bar{D}.$$  

(46)

Proof. Observe that all functions involved in the proof are bounded and uniformly continuous. Clearly, the right-hand side of (46) lies in $\hat{C}(\tilde{E})$. The core $D$ is a dense subset of $\hat{C}(E)$. Hence $\bar{D}$ is a dense subset of the space $D_0$ of functions $f \in \hat{C}(\tilde{E})$ of the form $f(s, x) = \sum_{i=1}^l g_i(s)h_i(x)$, $(s, x) \in \tilde{E}$, where $l \in \mathbb{N}, h_i \in \hat{C}(E)$ and $g_i \in \hat{C}([0, \infty))$ for $i = 1, \ldots, l$. The algebra $D_0$ separates points and vanishes nowhere. The Stone–Weierstrass theorem for locally compact spaces (e.g. [2]) ensures that $D_0$ is a dense subset of $\hat{C}(E)$. In [2] the theorem is stated for complex-valued functions, but it remains true for real-valued functions. To see this, let $f \in \hat{C}(E) \subseteq \hat{C}(E, \mathbb{C})$ be
arbitrary. By the theorem there exist a sequence \( (k_n)_{n \in \mathbb{N}} \subseteq \hat{C}(E, \mathbb{C}) \) such that \( \lim_{n \to \infty} \|k_n - f\| = 0 \). Then \( f_n := \text{Re}(k_n) \in \hat{C}(E), \ n \in \mathbb{N}, \) and \( \|f_n - f\| \leq \|k_n - f\| \to 0 \) as \( n \to \infty \). Thus \( \hat{D} \) is a dense subset of \( \hat{C}(E) \) as well. If \( h \in D \) and \( g \in C_1([0, \infty)) \) such that \( g, g' \in \hat{C}([0, \infty)) \), then
\[
 t^{-1}(\tilde{T}_t g(s) h(x) - g(s) h(x)) = t^{-1}(g(s + t) - g(s))h(x) + g(h_s + t) - 1(\tilde{T}_t h(x) - h(x))
\]
converges uniformly in \( (s, x) \in \hat{E} \) to \( g'(s) h(x) + g(s) Ah(x) \) as \( t \to 0 \), thus \( \hat{D} \) lies in the domain of \( \hat{A} \) and \([10]\) holds true. By the same argument as above, the space \( D_1 \) of functions \( f \in \hat{C}(E) \) of the form \( f(s, x) = \sum_{i=1}^l g_i(s) h_i(x), (s, x) \in \hat{E} \), where \( g_i(s) = c_i \exp(-a_is), s \in [0, \infty) \) with \( c_i \in \mathbb{R} \) and \( a_i > 0 \) and \( h_i \in D \) for \( i = 1, \ldots, l \), is a dense subset of \( \hat{C}(E) \). By Hille–Yosida theory (see e.g. [8, Proposition 3.1]) it now suffices to show that the image of \( \lambda I - \hat{A}|_{D_1} \) is a dense subspace of \( \hat{C}(E) \) for some \( \lambda > 0 \) in order to prove that \( \hat{D} \) is a core for \( \hat{A} \). Here \( I \) denotes the identity map on \( \hat{C}(E) \) or \( \hat{C}(\hat{E}) \). Let \( \varepsilon > 0 \) and \( f \in \hat{C}(\hat{E}) \) be arbitrary. By density of \( D_1 \in \hat{C}(\hat{E}) \), there exists \( f_1 \in D_1 \) of the form \( f_1(s, x) = \sum_{i=1}^l g_i(s) h_i(x), (s, x) \in \hat{E}, \) such that \( \|f_1 - f\| < \varepsilon/2 \). Since \( D \) is a core for \( A \), the image of \( \lambda I - A|_{D} \) is a dense subset of \( \hat{C}(E) \) for every \( \lambda > 0 \), in particular for \( \lambda + a_i \) in place of \( \lambda \). Hence there exists \( r_i \in D \) such that \( \|(\lambda + a_i)r_i - Ar_i - h_i\| < \varepsilon/(2l\|g_i\|) \) for \( i = 1, \ldots, l \). Clearly, the function \( (s, x) \mapsto \sum_{i=1}^l g_i(s)r_i(x), (s, x) \in \hat{E} \), belongs to \( \hat{D} \) and, by \([10]\),
\[
\|(\lambda I - \hat{A})\sum_{i=1}^l g_i(s) r_i(x) - f(s, x)\| \leq \|(\lambda I - \hat{A})\sum_{i=1}^l g_i(s) r_i(x) - \sum_{i=1}^l g_i(s) h_i(x)\| + \|f_1 - f\|
\]
\[
\leq \sum_{i=1}^l \|g_i((\lambda + a_i)r_i - Ar_i - h_i)\| + \varepsilon/2 \leq \varepsilon.
\]
In the second last step it is used that \( g_i'(-s) = -a_ig_i(s), s \in [0, \infty) \) for \( i = 1, \ldots, l \). Since \( \varepsilon > 0 \) has been arbitrary the proof is complete. \( \square \)

**Remark.** The last part of the proof of Proposition \([10]\) can be simplified under the additional assumption that \( T_tD \subseteq D \) for every \( t > 0 \). Then \( T_tD \subseteq D \) for every \( t \geq 0 \) and the claim follows by applying the core theorem [8, Proposition 3.3].

The computations in the proof of the following proposition are based on Gaussian’ representation [24, p. 247] for the digamma function
\[
\Psi(z) = \int_0^\infty \left( \frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}} \right) du, \quad \text{Re}(z) > 0.
\]

**Proposition 11** Suppose that \( \Lambda = \beta(1, b) \) with \( b > 0 \). Then the measure \( \varrho \), defined by \([3]\), has density \( f \) with respect to Lebesgue measure on \( \mathbb{R} \setminus \{0\} \) given by \( f(u) := be^{bu}(1 - e^{-u})^{-2} \) for \( u < 0 \) and \( f(u) := 0 \) for \( u > 0 \). Let \( a \) and \( \psi \) be given by \([7]\) and \([3]\). Then
\[
a = b(1 + \Psi(b)) \quad \text{(47)}
\]
and
\[
\psi(x) = b((1 - b)\Psi(b) - (1 - b - ix)\Psi(b + ix)), \quad x \in \mathbb{R}.
\]

**Proof.** It is easily verified that \( \varrho \) has density as stated in the proposition. Eq. \([47]\) follows from
\[
\int_{[0, 1]} u^{-1}(\Lambda - b\lambda)du = b \int_0^1 u^{-1}((1 - u)^{b - 1} - 1)du
\]
\[
= b \int_0^\infty \left( \frac{e^{-bu}}{1 - e^{-u}} - \frac{e^{-u}}{1 - e^{-u}} \right) du = b(\Psi(1) - \Psi(b)).
\]
Next, note that

\[ \Psi(b) - \Psi(b + ix) = \int_0^\infty (e^{-ixu} - 1) \frac{e^{-bu}}{1 - e^{-u}} du, \quad x \in \mathbb{R}, \]

is the characteristic exponent of the negative of a drift-free subordinator, whose Lévy measure has density \( u \mapsto e^{bu}(1 - e^{u})^{-1}, \ u < 0, \) with respect to Lebesgue measure on \((-\infty, 0)\). If \( b < 1 \) and \( Z \) has characteristic function \( \exp((1 - b)(\Psi(b) - \Psi(b + ix))), \ x \in \mathbb{R}, \) then \( \mathbb{E}(\log(1 + |Z|)) < \infty. \) This fact is required in order to use [23, V, Theorem 6.7] in Example 2. Integration by parts yields

\[
i x(\Psi(b + ix) - \Psi(b)) = \int_0^\infty (ix - ix e^{-ixu}) \frac{e^{-bu}}{1 - e^{-u}} du
\]

\[
= (ixu + e^{-ixu} - 1) \frac{e^{-bu}}{1 - e^{-u}} \bigg|_{u=0}^{u=\infty}
\]

\[
- \int_0^\infty (ixu + e^{-ixu} - 1) \left( \frac{-be^{-bu}}{1 - e^{-u}} - \frac{e^{-bu}}{(1 - e^{-u})^2} e^{-u} \right) du
\]

\[
= \int_0^\infty (e^{-ixu} - 1 + ixu) \frac{e^{-bu}}{(1 - e^{-u})^2} (1 - (1 - b)(1 - e^{-u})) du, \quad x \in \mathbb{R}.\]

Hence,

\[
(1 - b)\Psi(b) - (1 - b - ix)\Psi(b + ix)
\]

\[
= ix\Psi(b) + (1 - b)(\Psi(b) - \Psi(b + ix)) + ix(\Psi(b + ix) - \Psi(b))
\]

\[
= ix\Psi(b) + (1 - b) \int_0^\infty (e^{-ixu} - 1) \frac{e^{-bu}}{1 - e^{-u}} du
\]

\[
+ \int_0^\infty (e^{-ixu} - 1 + ixu) \frac{e^{-bu}}{(1 - e^{-u})^2} (1 - (1 - b)(1 - e^{-u})) du
\]

\[
= ix\Psi(b) + \int_0^\infty (e^{-ixu} - 1 + ixu) \frac{e^{-bu}}{(1 - e^{-u})^2} du - ix(1 - b) \int_0^\infty u \frac{e^{-bu}}{(1 - e^{-u})} du
\]

\[
= ix(\Psi(b) - (1 - b)\Psi'(b)) + b^{-1} \int_{\mathbb{R}\setminus\{0\}} (e^{ixu} - 1 - ixu) g(du)
\]

\[
= ix \left( (1 - b)\Psi(b) - (1 - b)\Psi'(b) + b^{-1} \int_{\mathbb{R}\setminus\{0\}} (e^{u} - 1 - u) g(du) \right)
\]

\[
+ b^{-1} \int_{\mathbb{R}\setminus\{0\}} (e^{ixu} - 1 + ix(1 - e^{-u})) g(du).
\]

The calculation

\[-(1 - b)\Psi'(b) + b^{-1} \int_{\mathbb{R}\setminus\{0\}} (e^{u} - 1 - u) g(du)
\]

\[
= \int_0^\infty \left( -(1 - b)u(1 - e^{-u}) + e^{-u} - 1 + u \right) \frac{e^{-bu}}{(1 - e^{-u})^2} du = \left. \frac{e^{-bu}}{1 - e^{-u}} \right|_{u=\infty}^{u=0} = 1
\]

and multiplication with \( b \) complete the proof. \( \square \)
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