On the maximum number of cliques in a graph embedded in a surface

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Abstract
This paper studies the following question: given a surface \( \Sigma \) and an integer \( n \), what is the maximum number of cliques in an \( n \)-vertex graph embeddable in \( \Sigma \)? We characterise the extremal graphs for this question, and prove that the answer is between \( 8(n - \omega) + 2^{\omega} \) and \( 8n + \frac{3}{2} 2^{\omega} + o(2^{\omega}) \), where \( \omega \) is the maximum integer such that the complete graph \( K_\omega \) embeds in \( \Sigma \). For the surfaces \( S_0, S_1, S_2, N_1, N_2, N_3, \) and \( N_4 \) we establish an exact answer.

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1. Introduction

A clique in a graph is a set of pairwise adjacent vertices. Let \( c(G) \) be the number of cliques in a graph \( G \). For example, every set of vertices in the complete graph \( K_n \) is a clique, and \( c(K_n) = 2^n \). This paper studies the following question at the intersection of topological and extremal graph theory:

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5 We consider simple, finite, undirected graphs \( G \) with vertex set \( V(G) \) and edge set \( E(G) \). A \( K_3 \) subgraph of \( G \) is called a triangle of \( G \). For background graph theory, see [4].

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given a surface $\Sigma$ and an integer $n$, what is the maximum number of cliques in an $n$-vertex graph embeddable in $\Sigma$?

For previous bounds on the maximum number of cliques in certain graph families, see [5,6,13, 14,22,23] for example. For background on graphs embedded in surfaces, see [11,21]. Every surface is homeomorphic to $S_g$, the orientable surface with $g$ handles, or to $N_h$, the non-orientable surface with $h$ crosscaps. The Euler characteristic of $S_g$ is $2 - 2g$. The Euler characteristic of $N_h$ is $2 - h$. The orientable genus of a graph $H$ is the minimum integer $g$ such that $H$ embeds in $S_g$. The non-orientable genus of a graph $G$ is the minimum integer $h$ such that $G$ embeds in $N_h$. The orientable genus of $K_n$ ($n \geq 3$) is $\left\lceil \frac{1}{2}(n - 3)(n - 4) \right\rceil$, and its non-orientable genus is $\left\lceil \frac{1}{6}(n - 3)(n - 4) \right\rceil$, except that the non-orientable genus of $K_3$ is 3.

Throughout the paper, fix a surface $\Sigma$ with Euler characteristic $\chi$. If $\Sigma = S_0$ then let $\omega = 3$, otherwise let $\omega$ be the maximum integer such that $K_\omega$ embeds in $\Sigma$. Thus $\omega = \left\lfloor \frac{1}{2}(7 + \sqrt{49 - 24\chi}) \right\rfloor$ except for $\Sigma = S_0$ and $\Sigma = N_2$, in which case $\omega = 3$ and $\omega = 6$, respectively.

To avoid trivial exceptions, we implicitly assume that $|V(G)| \geq 3$ whenever $\Sigma = S_0$.

Our first main result is to characterise the $n$-vertex graphs embeddable in $\Sigma$ with the maximum number of cliques; see Theorem 1 in Section 2. Using this result we determine an exact formula for the maximum number of cliques in an $n$-vertex graph embeddable in each of the sphere $S_0$, the torus $S_1$, the double torus $S_2$, the projective plane $N_1$, the Klein bottle $N_2$, as well as $N_3$ and $N_4$; see Section 3. Our third main result estimates the maximum number of cliques in terms of $\omega$. We prove that the maximum number of cliques in an $n$-vertex graph embeddable in $\Sigma$ is between $8(n - \omega) + 2^\omega$ and $8n + \frac{3}{2}2^\omega + o(2^\omega)$; see Theorem 2 in Section 4.

2. Characterisation of extremal graphs

The upper bounds proved in this paper are of the form: every graph $G$ embeddable in $\Sigma$ satisfies $c(G) \leq 8|V(G)| + f(\Sigma)$ for some function $f$. Define the excess of $G$ to be $c(G) - 8|V(G)|$. Thus the excess of $G$ is at most $Q$ if and only if $c(G) \leq 8|V(G)| + Q$. Theorem 2 proves that the maximum excess of a graph embeddable in $\Sigma$ is finite.

In this section, we characterise the graphs embeddable in $\Sigma$ with maximum excess. A triangulation of $\Sigma$ is an embedding of a graph in $\Sigma$ in which each facial walk has three vertices and three edges with no repetitions. (We assume that every face of a graph embedding is homeomorphic to a disc.)

Lemma 1. Every graph $G$ embeddable in $\Sigma$ with maximum excess is a triangulation of $\Sigma$.

Proof. Since adding edges within a face increases the number of cliques, the vertices on the boundary of each face of $G$ form a clique.

Suppose that some face $f$ of $G$ has at least four distinct vertices in its boundary. Let $G'$ be the graph obtained from $G$ by adding one new vertex adjacent to four distinct vertices of $f$. Thus $G'$ is embeddable in $\Sigma$, has $|V(G)| + 1$ vertices, and has $c(G) + 16$ cliques, which contradicts the choice of $G$. Now assume that every face of $G$ has at most three distinct vertices.

Suppose that some face $f$ of $G$ has repeated vertices. Thus the facial walk of $f$ contains vertices $u, v, w, v$ in this order (where $v$ is repeated in $f$). Let $G'$ be the graph obtained from $G$ by adding two new vertices $p$ and $q$, where $p$ is adjacent to $\{u, v, w, q\}$, and $q$ is adjacent to $\{u, v, w, p\}$. So $G'$ is embeddable in $\Sigma$ and has $|V(G)| + 2$ vertices. If $S \subseteq \{p, q\}$ and $S \neq \emptyset$ and $T \subseteq \{u, v, w\}$, then $S \cup T$ is a clique of $G'$ but not of $G$. It follows that $G'$ has $c(G) + 24$ cliques, which contradicts the choice of $G$. Hence no face of $G$ has repeated vertices, and $G$ is a triangulation of $\Sigma$. \qed

Let $G$ be a triangulation of $\Sigma$. An edge $vw$ of $G$ is reducible if $vw$ is in exactly two triangles in $G$. We say $G$ is irreducible if no edge of $G$ is reducible [2,3,7,9,10,12,17,19,20]. Note that $K_3$ is a triangulation of $S_0$, and by the above definition, $K_3$ is irreducible. In fact, it is the only irreducible triangulation of $S_0$. We take this somewhat non-standard approach so that Theorem 1 holds for all surfaces.

Let $vw$ be a reducible edge of a triangulation $G$ of $\Sigma$. Let $vwx$ and $wyv$ be the two faces incident to $vw$ in $G$. As illustrated in Fig. 1, let $G/\overline{vw}$ be the graph obtained from $G$ by contracting $vw$; that is, delete the edges $vw, wy, wx$, and identify $v$ and $w$ into $v$. $G/\overline{vw}$ is a simple graph since $x$ and $y$ are the
only common neighbours of \(v\) and \(w\). Indeed, \(G/\{v, w\}\) is a triangulation of \(\Sigma\). Conversely, we say that \(G\) is obtained from \(G/\{v, w\}\) by splitting the path \(xy\) at \(v\). If, in addition, \(xy \in E(G)\), then we say that \(G\) is obtained from \(G/\{v, w\}\) by splitting the triangle \(xuv\) at \(v\). Note that \(xuv\) need not be a face of \(G/\{v, w\}\). In the case that \(xuv\) is a face, splitting \(xuv\) is equivalent to adding a new vertex adjacent to each of \(x, v, y\).

Graphs embeddable in \(\Sigma\) with maximum excess are characterised in terms of irreducible triangulations as follows.

**Theorem 1.** Let \(Q\) be the maximum excess of an irreducible triangulation of \(\Sigma\). Let \(X\) be the set of irreducible triangulations of \(\Sigma\) with excess \(Q\). Then the excess of every graph \(G\) embeddable in \(\Sigma\) is at most \(Q\), with equality if and only if \(G\) is obtained from some graph in \(X\) by repeatedly splitting triangles.

**Proof.** We proceed by induction on \(|V(G)|\). By Lemma 1, we may assume that \(G\) is a triangulation of \(\Sigma\). If \(G\) is irreducible, then the claim follows from the definition of \(X\) and \(Q\). Otherwise, some edge \(vw\) of \(G\) is in exactly two triangles \(vwx\) and \(vwy\). By induction, the excess of \(G/\{v, w\}\) is at most \(Q\), with equality if and only if \(G/\{v, w\}\) is obtained from some \(H \in X\) by repeatedly splitting triangles. Hence \(c(G/\{v, w\}) \leq 8|V(G/\{v, w\})| + Q\).

Observe that every clique of \(G\) that is not in \(G/\{v, w\}\) is in \(\{A \cup \{w\} : A \subseteq \{x, v, y\}\}\). Thus \(c(G) \leq c(G/\{v, w\}) + 8\), with equality if and only if \(xuv\) is a triangle. Hence \(c(G) \leq 8|V(G)| + Q\); that is, the excess of \(G\) is at most \(Q\).

Now suppose that the excess of \(G\) equals \(Q\). Then the excess of \(G/\{v, w\}\) equals \(Q\), and \(c(G) = c(G/\{v, w\}) + 8\) (implying \(xuv\) is a triangle). By induction, \(G/\{v, w\}\) is obtained from \(H\) by repeatedly splitting triangles. Therefore \(G\) is obtained from \(H\) by repeatedly splitting triangles.

Conversely, suppose that \(G\) is obtained from some \(H \in X\) by repeatedly splitting triangles. Then \(xuv\) is a triangle and \(G/\{v, w\}\) is obtained from \(H\) by repeatedly splitting triangles. By induction, the excess of \(G/\{v, w\}\) equals \(Q\), implying the excess of \(G\) equals \(Q\).  

\( \square \)

### 3. Low-genus surfaces

To prove an upper bound on the number of cliques in a graph embedded in \(\Sigma\), by Theorem 1, it suffices to consider irreducible triangulations of \(\Sigma\) with maximum excess. The complete list of irreducible triangulations is known for \(S_0, S_1, S_2, N_1, N_2, N_3\) and \(N_4\). In particular, Steinitz and Rademacher [16] proved that \(K_3\) is the only irreducible triangulation of \(S_0\) (under our definition of irreducible). Lavrenchenko [9] proved that there are 21 irreducible triangulations of \(S_1\), each with between 7 and 10 vertices. Sulanke [17] proved that there are 396,784 irreducible triangulations of \(S_2\), each with between 10 and 17 vertices. Barnette [1] proved that the embeddings of \(K_6\) and \(K_7 - K_3\) in \(N_1\) are the only irreducible triangulations of \(N_1\). Sulanke [20] proved that there are 29 irreducible triangulations of \(N_2\), each with between 8 and 11 vertices (correcting an earlier result by Lavrenchenko and Negami [10]). Sulanke [17] proved that there are 9708 irreducible triangulations of \(N_3\), each with between 9 and 16 vertices. Sulanke [17] proved that there are 6,297,982 irreducible triangulations of \(N_4\), each with between 9 and 22 vertices. Using the lists of all irreducible triangulations due to Sulanke [18] and a naive algorithm for counting cliques,\(^6\) we have computed the set \(X\) in Theorem 1 for each of the above surfaces; see Table 1. This data with Theorem 1 implies the following results.

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\( ^6 \) The code is available from the authors upon request.
Fig. 2. $K_7$ embedded in the torus, and $K_6$ embedded in the projective plane.

Table 1
The maximum excess of an $n$-vertex irreducible triangulation of $\Sigma$.

| $\Sigma$ | $\chi$ | $\omega$ | $n=3$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | Max |
|----------|--------|----------|-------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_0$    | 2      | 3        | -16   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   | -16 |
| $S_1$    | 0      | 7        | 72    | 48| 40| 32 |   |   |   |   |   |   |   |   |   |   |   |   |   |   | 72  |
| $S_2$    | -2     | 8        | 208   |160|136|128|120|96 |88 |80 |   |   |   |   |   |   |   |   |   |   | 208 |
| $N_1$    | 1      | 6        | 16    | 8 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   | 16  |
| $N_2$    | 0      | 6        | 48    | 48| 40| 32 |   |   |   |   |   |   |   |   |   |   |   |   |   |   | 48  |
| $N_3$    | -1     | 7        | 104   | 96|80 |80 |72 |64 |56 |   |   |   |   |   |   |   |   |   |   | 104 |
| $N_4$    | -2     | 8        | 216   |152|136|136|128|120|112|99 |83 |75 |   |   |   |   |   |   |   |   | 216 |

**Proposition 1.** Every planar graph $G$ with $|V(G)| \geq 3$ has at most $8|V(G)| - 16$ cliques, as proved by Wood [22]. Moreover, a planar graph $G$ has $8|V(G)| - 16$ cliques if and only if $G$ is obtained from the embedding of $K_3$ in $S_0$ by repeatedly splitting triangles.

**Proposition 2.** Every toroidal graph $G$ has at most $8|V(G)| + 72$ cliques. Moreover, a toroidal graph $G$ has $8|V(G)| + 72$ cliques if and only if $G$ is obtained from the embedding of $K_7$ in $S_1$ by repeatedly splitting triangles (see Fig. 2).

**Proposition 3.** Every graph $G$ embeddable in $S_2$ has at most $8|V(G)| + 208$ cliques. Moreover, a graph $G$ embeddable in $S_2$ has $8|V(G)| + 208$ cliques if and only if $G$ is obtained from one of the following two graph embeddings in $S_2$ by repeatedly splitting triangles$^7$:

- graph #1: bcdef,afghdef,abehfgd,acgdfhi,adgcfjib,bcdfebhdh,bfjgec,fghdj
- graph #6: bcdef,afghdef,abehfgd,acgdfhi,adgcfjib,bcdfebhdh,bfjgec,fghdj
- graph #26: bcdef,afghdef,abehfgd,acgdfhi,adgcfjib,bcdfebhdh,bfjgec,fghdj

**Proposition 4.** Every projective planar graph $G$ has at most $8|V(G)| + 16$ cliques. Moreover, a projective planar graph $G$ has $8|V(G)| + 16$ cliques if and only if $G$ is obtained from the embedding of $K_6$ in $N_1$ by repeatedly splitting triangles (see Fig. 2).

**Proposition 5.** Every graph $G$ embeddable in the Klein bottle $N_2$ has at most $8|V(G)| + 48$ cliques. Moreover, a graph $G$ embeddable in $N_2$ has $8|V(G)| + 48$ cliques if and only if $G$ is obtained from one of the following three graph embeddings in $N_2$ by repeatedly splitting triangles (see Fig. 3):

- graph #3: bcddef,afghdef,abehfgd,acgdfhi,adgcfjib,behdh,bfjgec
- graph #6: bcdef,afghdef,abehfgd,acgdfhi,adgcfjib,behdh,bfjgec,fghdj
- graph #26: bcddef,afghdef,abehfgd,acgdfhi,adgcfjib,behdh,bfjgec,fghdj.

$^7$ This representation describes a graph with vertex set $\{a, b, c, \ldots\}$ by adjacency lists of the vertices in order $a, b, c, \ldots$. The graph # refers to the position in Sulanke’s file [18].
Proposition 6. Every graph $G$ embeddable in $\mathbb{N}_3$ has at most $8|V(G)| + 104$ cliques. Moreover, a graph $G$ embeddable in $\mathbb{N}_3$ has $8|V(G)| + 104$ cliques if and only if $G$ is obtained from one of the following 15 graph embeddings in $\mathbb{N}_3$ by repeatedly splitting triangles:

- Graph #1: $bcde, aefgdhic, abiegfd, acfbgie, adichgfbl, bdfeicb, beghdb, bgejfh, bhgfdec$
- Graph #2: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #3: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #4: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #5: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #6: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #7: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #8: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #9: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #10: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #11: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #12: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #13: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #14: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #15: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #16: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #17: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #18: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #19: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #20: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #21: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #22: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #23: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #24: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$
- Graph #25: $bcde, aefgdhic, abiehd, achfjobg, adichgbf, begihdb, bdfeicb, beghdb, bhgfdec$

Proposition 7. Every graph $G$ embeddable in $\mathbb{N}_4$ has at most $8|V(G)| + 216$ cliques. Moreover, a graph $G$ embeddable in $\mathbb{N}_4$ has $8|V(G)| + 216$ cliques if and only if $G$ is obtained from one of the following three graph embeddings in $\mathbb{N}_4$ by repeatedly splitting triangles:

- Graph #1: $bcde, afgdheic, abiehd, acgfbhie, adgfbihc, bdechfj, bdfeicb, beghdb, bhgfdec$
- Graph #2: $bcde, afgdheic, abiehd, acgfbhie, adgfbihc, bdechfj, bdfeicb, beghdb, bhgfdec$
- Graph #3: $bcde, afgdheic, abiehd, acgfbhie, adgfbihc, bdechfj, bdfeicb, beghdb, bhgfdec$

Note that the three embeddings in Proposition 7 are of the same graph.

4. A bound for all surfaces

Recall that $\Sigma$ is a surface with Euler characteristic $\chi$, and if $\Sigma = S_0$ then $\omega = 3$, otherwise $\omega$ is the maximum integer such that $K_\omega$ embeds in $\Sigma$. We start with the following upper bound on the minimum degree of a graph.

Lemma 2. Assume $\Sigma \neq S_0$. Then every graph $G$ embeddable in $\Sigma$ has minimum degree at most

$$6 + \frac{\omega^2 - 5\omega - 7}{|V(G)|}.$$
Proof. By the definition of $\omega$, the complete graph $K_{\omega+1}$ cannot be embedded in $\Sigma$. Thus if $\Sigma = S_g$ then $g = \frac{1}{2}(2 - \chi) \leq \left\lceil \frac{1}{12}(\omega - 2)(\omega - 3) \right\rceil - 1$, and if $\Sigma = N_h$ then $h = 2 - \chi \leq \left\lceil \frac{1}{6}(\omega - 2)(\omega - 3) \right\rceil - 1$. In each case, it follows that $2 - \chi \leq \frac{1}{6}(\omega - 2)(\omega - 3) - \frac{1}{6}$. That is,
\[ -6\chi \leq \omega^2 - 5\omega - 7. \] (1)

Say $G$ has minimum degree $d$. It follows from Euler’s Formula that $|E(G)| \leq 3|V(G)| - 3\chi$. By (1),
\[ d \leq \frac{2|E(G)|}{|V(G)|} \leq 6 \frac{|V(G)| - 6\chi}{|V(G)|} \leq 6 + \omega^2 - 5\omega - 7 = 6 + \frac{\omega^2 - 5\omega - 7}{|V(G)|}. \]

For graphs in which the number of vertices is slightly more than $\omega$, Lemma 2 can be reinterpreted as follows.

Lemma 3. Assume $\Sigma \neq S_0$. Let $s := \left\lceil \sqrt{\omega + 11} - 3 \right\rceil \geq 1$. Let $G$ be a graph embeddable in $\Sigma$. If $G$ has at most $\omega + 1$ vertices, then $G$ has minimum degree at most $\omega - 1$. If $G$ has at least $\omega + j$ vertices, where $j \in [2, s]$, then $G$ has minimum degree at most $\omega - j + 1$.

Proof. Say $G$ has minimum degree $d$. If $|V(G)| \leq \omega$, then trivially $d \leq \omega - 1$. If $|V(G)| = \omega + 1$, then $G$ is not complete (by the definition of $\omega$), again implying that $d \leq \omega - 1$. Now assume $|V(G)| \geq \omega + j$ for some $j \in [2, s]$. By Lemma 2,
\[ d \leq \frac{\omega^2 - 5\omega - 7}{\omega + j} = \omega - j + 1 + \frac{j^2 + 5j - 7}{\omega + j}. \]

Since $j \leq s < \sqrt{\omega + 11} - 2$, we have $j^2 + 5j - 7 \leq s^2 + 4s - 7 + j < \omega + j$. It follows that $d \leq \omega - j + 1$. □

Now we prove our first upper bound on the number of cliques.

Lemma 4. Assume $\Sigma \neq S_0$. Let $s := \left\lceil \sqrt{\omega + 11} - 3 \right\rceil \geq 1$. Let $G$ be an $n$-vertex graph embeddable in $\Sigma$. Then
\[ c(G) \leq \begin{cases} 5 \omega^2 & \text{if } n \leq \omega + s, \\ 5 \omega^2 + (n - \omega - s)2^{\omega-s+1} & \text{otherwise}. \end{cases} \]

Proof. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of $G$ such that $v_i$ has minimum degree in the subgraph $G_i := G - \{v_1, \ldots, v_{i-1}\}$. Let $d_i$ be the degree of $v_i$ in $G_i$ (which equals the minimum degree of $G_i$). Charge each non-empty clique $C$ in $G$ to the vertex $v_i \in C$ with minimum degree. Charge the clique $\emptyset$ to $v_n$.

We distinguish three types of vertices. Vertex $v_i$ is type-1 if $i \in [1, n - \omega - s]$. Vertex $v_i$ is type-2 if $i \in [n - \omega - s + 1, n - \omega]$. Vertex $v_i$ is type-3 if $i \in [n - \omega + 1, n]$.

Each clique charged to a type-3 vertex is contained in $\{v_{n-\omega+1}, \ldots, v_n\}$, and there are at most $2^\omega$ such cliques.

Say $C$ is a clique charged to a type-1 or type-2 vertex $v_i$. Then $C - \{v_i\}$ is contained in $N_{G_i}(v_i)$, which consists of $d_i$ vertices. Thus the number of cliques charged to $v_i$ is at most $2^{d_i}$. Recall that $d_i$ equals the minimum degree of $G_i$, which has $n - i + 1$ vertices.

If $v_i$ is type-2 then, by Lemma 3 with $j = n - \omega - i + 1 \in [1, s]$, we have $d_i \leq \omega - j + 1$, and $d_i \leq \omega - j$ if $j = 1$. Thus the number of cliques charged to type-2 vertices is at most
\[ 2^{\omega-1} + \sum_{j=2}^{s} 2^{\omega-j+1} \leq 2^{\omega-1} + \sum_{j=1}^{\omega-1} 2^j \leq \frac{3}{2} 2^\omega. \]

If $v_i$ is type-1 then $G_i$ has more than $\omega + s$ vertices, and thus $d_i \leq \omega - s + 1$ by Lemma 3 with $j = \omega$. Thus the number of cliques charged to type-1 vertices is at most $(n - \omega - s)2^{\omega-s+1}$. □
We now prove the main result of this section; it provides lower and upper bounds on the maximum number of cliques in a graph embeddable in $\Sigma$.

**Theorem 2.** Every $n$-vertex graph embeddable in $\Sigma$ contains at most $8n + \frac{5}{2}2^{\omega} + o(2^{\omega})$ cliques. Moreover, for each $n \geq \omega$, there is an $n$-vertex graph embeddable in $\Sigma$ with $8(n - \omega) + 2^{\omega}$ cliques.

**Proof.** To prove the upper bound, we may assume that $\Sigma \neq S_0$, and by Theorem 1, we need only consider $n$-vertex irreducible triangulations of $\Sigma$. Joret and Wood [7] proved that, in this case, $n \leq 2^2 - 13\chi$. By Eq. (1),

$$n \leq 2^2 - 13\chi \leq 2^2 + \frac{13}{6}(\omega^2 - 5\omega - 7) < 3\omega^2.$$

If $n \leq \omega + s$ then $c(G) \leq \frac{5}{2}2^{\omega}$ by Lemma 4. If $n > \omega + s$ then by the same lemma,

$$c(G) \leq \frac{5}{2}2^{\omega} + (3\omega^2 - \omega - s)2^{\omega-s+1} < \frac{5}{2}2^{\omega} + 3\omega^22^{\omega-s+1} < \frac{5}{2}2^{\omega} + 2^{\omega-s+2\log\omega + 3}.$$

Since $s \in \Theta(\sqrt{\omega})$, we have $c(G) \leq \frac{5}{2}2^{\omega} + o(2^{\omega})$.

To prove the lower bound, start with $K_\omega$ embedded in $\Sigma$ (which has $2^{\omega}$ cliques). Now, while there are less than $n$ vertices, insert a new vertex adjacent to each vertex of a single face. Each new vertex adds at least 8 new cliques. Thus we obtain an $n$-vertex graph embedded in $\Sigma$ with at least $8(n - \omega) + 2^{\omega}$ cliques. \qed

## 5. Concluding conjectures

We conjecture that the upper bound in Theorem 2 can be improved to more closely match the lower bound.

**Conjecture 1.** Every graph $G$ embeddable in $\Sigma$ has at most $8|V(G)| + 2^{\omega} + o(2^{\omega})$ cliques.

If $K_\omega$ triangulates $\Sigma$, then we conjecture the following exact answer.

**Conjecture 2.** Suppose that $K_\omega$ triangulates $\Sigma$. Then every graph $G$ embeddable in $\Sigma$ has at most $8(|V(G)| - \omega) + 2^{\omega}$ cliques, with equality if and only if $G$ is obtained from $K_\omega$ by repeatedly splitting triangles.

By Theorem 1, this conjecture is equivalent to the following.

**Conjecture 3.** Suppose that $K_\omega$ triangulates $\Sigma$. Then $K_\omega$ is the only irreducible triangulation of $\Sigma$ with maximum excess.

The results in Section 3 confirm Conjectures 2 and 3 for $S_0$, $S_1$ and $N_1$.

Now consider surfaces possibly with complete graph triangulation. Then the bound $c(G) \leq 8(|V(G)| - \omega) + 2^{\omega}$ (in Conjecture 2) is false for $S_2$, $N_2$, $N_3$ and $N_4$. Loosely speaking, this is because these surfaces have ‘small’ $\omega$ compared to $\chi$. In particular, $\omega = \left\lceil \frac{1}{2}(7 + \sqrt{49 - 24\chi}) \right\rceil$ except for $S_0$ and $N_2$, and $\omega = \frac{1}{2}(7 + \sqrt{49 - 24\chi})$ if and only if $K_\omega$ triangulates $\Sigma \neq S_0$. This phenomenon motivates the following conjecture.

**Conjecture 4.** Every graph $G$ embeddable in $\Sigma$ has at most

$$8|V(G)| - 4(7 + \sqrt{49 - 24\chi}) + 2^{(7 + \sqrt{49 - 24\chi})/2}$$

cliques, with equality if and only if $K_\omega$ triangulates $\Sigma$ and $G$ is obtained from $K_\omega$ by repeatedly splitting triangles.
There are two irreducible triangulations of $S_2$ with maximum excess, there are three irreducible triangulations of $N_2$ with maximum excess, there are 15 irreducible triangulations of $N_3$ with maximum excess, and there are three irreducible triangulations of $N_4$ with maximum excess. This suggests that for surfaces with no complete graph triangulation, a succinct characterisation of the extremal examples (as in Conjecture 3) might be difficult. Nevertheless, we conjecture the following strengthening of Conjecture 3 for all surfaces.

**Conjecture 5.** Every irreducible triangulation of $\Sigma$ with maximum excess contains $K_\omega$ as a subgraph.

A triangulation of a surface $\Sigma$ is vertex-minimal if it has the minimum number of vertices in a triangulation of $\Sigma$. Of course, every vertex-minimal triangulation is irreducible. Ringel [15] and Jungerman and Ringel [8] together proved that the order of a vertex-minimal triangulation is $\omega$ if $K_\omega$ triangulates $\Sigma$, is $\omega + 2$ if $\Sigma \in \{S_2, N_2, N_3\}$, and is $\omega + 1$ for every other surface.

Triangulations #26 of $N_2$ and #2464 of $N_3$ are the only triangulations in Propositions 1–7 that are not vertex-minimal. Triangulation #26 of $N_2$ is obtained from two embeddings of $K_6$ in $N_1$ joined at the face $bdf$ (see Fig. 3). Triangulation #2464 of $N_3$ is obtained by joining an embedding of $K_6$ in $N_1$ and an embedding of $K_7$ in $S_1$ at the face $bdf$ (see Fig. 4).

Every other triangulation in Propositions 1–7 is obtained from an embedding of $K_\omega$ by adding (at most two) vertices and edges until a vertex-minimal triangulation is obtained. This provides some evidence for our final conjecture.

**Conjecture 6.** For every surface $\Sigma$, the maximum excess is attained by some vertex-minimal triangulation of $\Sigma$ that contains $K_\omega$ as a subgraph. Moreover, if $\Sigma \not\in \{N_2, N_3\}$ then every irreducible triangulation with maximum excess is vertex-minimal and contains $K_\omega$ as a subgraph.

We have verified Conjectures 4–6 for $S_0, S_1, S_2, N_1, N_2, N_3$ and $N_4$.

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