The homotopy principle in the existence level for maps with only singularities of types $A$, $D$ and $E$

YOSHIFUMI ANDO

Abstract

Let $N$ and $P$ be smooth manifolds of dimensions $n$ and $p$ ($n \geq p \geq 2$) respectively. Let $\Omega(N,P)$ denote an open subspace of $J^\infty(N,P)$ which consists of all regular jets and jets with prescribed singularities of types $A$, $D$ and $E$. An $\Omega$-regular map $f : N \to P$ refers to a smooth map having only singularities in $\Omega(N,P)$ and satisfying the transversality condition. We will prove what is called the homotopy principle for $\Omega$-regular maps in the existence level. Namely, a continuous section $s$ of $\Omega(N,P)$ over $N$ has an $\Omega$-regular map $f$ such that $s$ and $j^\infty f$ are homotopic as sections.

Introduction

Let $N$ and $P$ be smooth ($C^\infty$) manifolds of dimensions $n$ and $p$ respectively with $n \geq p \geq 2$. Let $A_k$, $D_k$ and $E_k$ denote the types of the singularities of function germs studied in [Ar]. We say that a $C^\infty$ stable map germ $f : (N,x_0) \to (P,y_0)$ has a singularity of type $A_k$, $D_k$ or $E_k$, when $f$ is $C^\infty$ equivalent to $g : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$, which is a versal unfolding of the function germ with respective singularities as follows. Here, we assume $n > p \geq 2$ only when we deal with the singularity of types $D_k$ and $E_k$. We take the coordinates

$$x = (x_1, \ldots, x_{p-k}, x_p, \ldots, x_{n-1}, t_0, \ldots, t_{k-2}, u) \quad \text{for } A_k,$$

$$x = (x_1, \ldots, x_{p-k}, x_p, \ldots, x_{n-2}, t_0, \ldots, t_{k-2}, u, \ell) \quad \text{for } D_k \text{ and } E_k,$$

and $(y_1, \ldots, y_p)$ of $\mathbb{R}^p$. Then $g$ is expressed by

$$y_i \circ g(x) = x_i \quad \text{for } 1 \leq i \leq p - k,$$

$$y_i \circ g(x) = t_{i-p+k-1} \quad \text{for } p - k < i \leq p - 1,$$

and by $y_p \circ g(x)$, which is written as

$$(A_k) \pm u^{k+1} + \sum_{i=1}^{k-1} t_{i-1} u^i \pm x_p^2 \pm \cdots \pm x_{n-1}^2 \quad (k \geq 1),$$

$$(D_k) \ u^2 \ell \pm \ell^{k-1} + t_0 u + t_1 \ell + \sum_{i=2}^{k-2} t_i \ell^i + Q \quad (k \geq 4),$$

$$(E_6) \ u^3 \pm \ell^4 + t_0 u + t_1 \ell + t_2 u \ell + t_3 \ell^2 + t_4 u \ell^2 + Q,$$

for $A_k$, $D_k$ and $E_k$ respectively.
\[ (E_7) \quad u^3 + ul^3 + t_0u + t_1l + t_2ul + t_3l^2 + t_4l^3 + t_5l^4 + Q, \]
\[ (E_8) \quad u^3 + l^6 + t_0u + t_1l + t_2ul + t_3l^2 + t_4ul^2 + t_5l^3 + t_6ul^3 + Q, \]
where \( Q = \pm x_0^2 \pm \cdots \pm x_n^2 \) ([An3], [Math1, (5.8)] and [Mo]).

In [B] there has been defined what is called the Boardman-Thom manifold \( \Sigma'(N, P) \) in \( J(\infty, N, P) \). For the symbol \( I = (n-p+1, 1, \ldots, 1, 0) \), a smooth map germ \( f : (N, x) \to (P, y) \) has \( x \) as a singularity of type \( A_k \) if and only if \( j^\infty f \in \Sigma'(N, P) \) and \( j^\infty f \) is transverse to \( \Sigma'(N, P) \) (see [Mo]). By using the method developed in [B], we have constructed in [An3] the submanifolds \( \Sigma D_k(N, P) (k \geq 4) \) and \( \Sigma E_k(N, P) (k = 6, 7 \text{ or } 8) \) in \( J(\infty, N, P) \), which play the similar role for the singularities \( D_k \) and \( E_k \) respectively as \( \Sigma'(N, P) \) does for \( A_k \). Let \( \Omega(N, P) \) denote an open subset of \( J(\infty, N, P) \), which consists of all regular jets and a number of prescribed submanifolds \( \Sigma A_i(N, P), \Sigma D_j(N, P) \) and \( \Sigma E_k(N, P) \). It is known when \( \Omega(N, P) \) becomes open by the adjacency relations of these singularities given in [Ar, Corollary 8.7] (Lemma 3.5). It is known that \( \Omega(N, P) \) is the subbundle of \( J(\infty, N, P) \) with the projection \( \pi^\infty_N \times \pi^\infty_P \), whose fiber is denoted by \( \Omega(n, p) \). A smooth map \( f : N \to P \) is called an \( \Omega \)-regular map if and only if (i) \( j^\infty f(N) \subset \Omega(N, P) \) and (ii) \( j^\infty f \) is transverse to all \( \Sigma A_i(N, P), \Sigma D_j(N, P) \) and \( \Sigma E_k(N, P) \) in \( \Omega(N, P) \).

We will study when a given continuous map is homotopic to an \( \Omega \)-regular map. Let \( C^\infty_\Omega(N, P) \) denote the space consisting of all \( \Omega \)-regular maps equipped with the \( C^\infty \)-topology. Let \( \Gamma_\Omega(N, P) \) (resp. \( \Gamma^\infty_\Omega(N, P) \)) denote the space consisting of all continuous sections (resp. sections transverse to all \( \Sigma X_i(N, P) \) for \( X_i = A_i, D_i \) and \( E_i \)) of the fiber bundle \( \pi^\infty_N \Omega(N, P) : \Omega(N, P) \to N \) equipped with the compact-open topology. Then there exists a continuous map
\[ j_\Omega : C^\infty_\Omega(N, P) \to \Gamma_\Omega(N, P) \]
defined by \( j_\Omega(f) = j^\infty f \). It follows from the well-known theorem due to Gromov[G1] that if \( N \) is a connected open manifold, then \( j_\Omega \) is a weak homotopy equivalence. This property is called the homotopy principle (the terminology used in [G2]). If \( N \) is a closed manifold, then it becomes a hard problem for us to prove the homotopy principle. As the investigation preceding [G1], we should refer to the Smale-Hirsch Immersion Theorem ([H1]), the \( k \)-mersion Theorem due to [F] and the Phillips Submersion Theorem for open manifolds ([P]). Du Plessis has introduced the extensibility condition under which the homotopy principle holds for maps with certain Thom-Boardman singularities in [duP1], or with prescribed \( C^\infty \) simple singularities in [duP2]. Éliassberg has proved in [E1] and [E2] the homotopy principle in the 1-jet level for fold-maps. In [An9] we have proved the homotopy principle in the existence level for maps with prescribed Thom-Boardman singularities in dimensions \( n \geq p \geq 2 \).

We prove the following theorem for closed manifolds.

**Theorem 0.1.** Let \( n \geq p \geq 2 \). Let \( N \) and \( P \) be connected manifolds of dimensions \( n \) and \( p \) respectively with \( \partial N = \emptyset \). Assume that \( \Omega(N, P) \) contains \( \Sigma A_1(N, P) \) at least. Let
C be a closed subset of N. Let s be a section of \( \Gamma_\Omega(N,P) \) which has an \( \Omega \)-regular map \( g \) defined on a neighborhood of \( C \) into \( P \), where \( j^\infty g = s \).

Then there exists an \( \Omega \)-regular map \( f : N \to P \) such that \( j^\infty f \) is homotopic to \( s \) relative to a neighborhood of \( C \) by a homotopy \( s_\lambda \) in \( \Gamma_\Omega(N,P) \) with \( s_0 = s \) and \( s_1 = j^\infty f \).

This theorem has been stated in [An2, Theorems 1.1 and 2.1] and has generalized [An1, Theorem 1]. However, the proof of Theorem 0.1 was not given, because a more general theorem was possibly supposed to appear. As far as the author knows, the proof of Theorem 0.1 has not been published until now. Recently, it turns out that this kind of the homotopy principle has many applications. Theorem 0.1 is very important even for fold-maps in [An6, Theorem 1] and [An7, Theorems 0.2 and 0.3] and the homotopy type of \( \Omega(n,p) \) associated to fold-maps, which consists of all regular jets and jets with \( A_1 \)-singularities, are determined in [An5] and [An7]. The famous theorem about the elimination of cusps in [L1] (see also [T]) is now a simple consequence of these theorems. In [Sady1] Sadykov has applied [An1, Theorem 1] to the elimination of higher \( A_k \) singularities \( (k \geq 3) \) for Morin maps between orientable manifolds when \( n - p \) is odd. We can describe the cobordism group of maps with prescribed singularities of types \( A, D \) and \( E \) in terms of certain stable homotopy groups by using the result in [An10, Theorem 4.2].

We refer the reader to [Saek] and [Sady2] as other applications of the homotopy principle.

We give the following application of Theorem 0.1.

**Theorem 0.2.** Let \( n > p \geq 2 \). If \( n - p \) is even, then we have

1. a smooth map \( f : N \to P \) admitting only singularities of types \( A_i \) \( (i \geq 1) \) and \( D_j \) \( (j \geq 4) \) is homotopic to a smooth map admitting only singularities of types \( A_i \) \( (i \geq 1) \), \( D_4 \) and \( D_5 \),

2. a smooth map \( f : N \to P \) admitting only singularities of types \( A_i \) \( (i \geq 1) \), \( D_j \) \( (j \geq 4) \) and \( E_k \) \( (8 \geq k \geq 6) \) is homotopic to a smooth map admitting only singularities of types \( A_i \) \( (i \geq 1) \), \( D_4 \), \( D_5 \) and \( E_6 \).

In Section 1 we explain notations which are used in this paper. In Sections 2 and 3 we review the definitions and the fundamental properties of the Boardman submanifolds, \( \Sigma D_k(N,P) \) and \( \Sigma E_k(N,P) \) respectively. In Section 4 we give Theorem 4.1 which is a simpler version of Theorem 0.1. We will prove Theorem 0.1 by the induction, and prepare a certain rotation of the tangent spaces defined around the singularities in \( N \) to deform \( s \). In Section 5 we prepare several lemmas which are necessary in the deformation of \( s \). In Section 6 we prove Theorem 4.1. In Section 7 we prove Theorem 0.2 by applying Theorem 0.1.

The author would like to thank the referee for his kind and helpful comments.

### 1 Notations

Throughout the paper all manifolds are Hausdorff, paracompact and smooth of class \( C^\infty \). Maps are basically continuous, but may be smooth (of class \( C^\infty \)) if necessary. Given a
fiber bundle \( \pi : E \rightarrow X \) and a subset \( C \) in \( X \), we denote \( \pi^{-1}(C) \) by \( E|_C \). Let \( \pi' : F \rightarrow Y \) be another fiber bundle. A map \( \tilde{b} : E \rightarrow F \) is called a fiber map over a map \( b : X \rightarrow Y \) if \( \pi' \circ \tilde{b} = b \circ \pi \). The restriction \( \tilde{b}(E|_C) : E|_C \rightarrow F \) (or \( F|_{\tilde{b}(C)} \)) is denoted by \( \tilde{b}|_C \). In particular, for a point \( x \in X \), \( E|_x \) and \( \tilde{b}|_x \) are simply denoted by \( E_x \) and \( \tilde{b}_x : E_x \rightarrow F_{b(x)} \) respectively. We denote, by \( b^f \), the induced fiber bundle \( b^f(F) \rightarrow F \) over \( b \). For a map \( j : W \rightarrow X \), let \( (b \circ j)^* \tilde{b} : j^*E \rightarrow (b \circ j)^*F \) be the fiber map canonically induced from \( b \) and \( j \). A fiberwise homomorphism \( E \rightarrow F \) is simply called a homomorphism. For a vector bundle \( E \) with metric and a positive function \( \delta \) on \( X \), let \( D_\delta(E) \) be the associated disk bundle of \( E \) with radius \( \delta \). If there is a canonical isomorphism between two vector bundles \( E \) and \( F \) over \( X = Y \), then we write \( E \cong F \).

When \( E \) and \( F \) are vector bundles over \( X = Y \), \( \text{Hom}(E, F) \) denotes the vector bundle over \( X \) with fiber \( \text{Hom}(E_x, F_x) \), \( x \in X \), which consists of all homomorphisms \( E_x \rightarrow F_x \).

Let \( J^k(N, P) \) denote the \( k \)-jet space of manifolds \( N \) and \( P \) (\( k \) may be \( \infty \)). Let \( \pi_N \) and \( \pi_P \) be the projections mapping a jet to its source and target respectively. The map \( \pi_N \times \pi_P : J^k(N, P) \rightarrow N \times P \) induces a structure of a fiber bundle with structure group \( L^k(p) \times L^k(n) \), where \( L^k(m) \) denotes the group of all \( k \)-jets of local diffeomorphisms of \((\mathbb{R}^m, 0)\). The fiber \( (\pi_N \times \pi_P)^{-1}(x, y) \) is denoted by \( J^k_{x,y}(N, P) \).

Let \( \pi_N \) and \( \pi_P \) be the projections of \( N \times P \) onto \( N \) and \( P \) respectively. We set

\[
J^k(T_N, TP) = \text{Hom}(\pi_N^*(TN) \oplus S^2(\pi_N^*(TN))) \oplus \cdots \oplus S^k(\pi_N^*(TN)), \pi_P^*(TP)) \tag{1.1}
\]

over \( N \times P \). Here, for a vector bundle \( E \) over \( X \), let \( S^i(E) \) be the vector bundle \( \cup_{x \in X} S^i(E_x) \) over \( X \), where \( S^i(E_x) \) denotes the \( i \)-fold symmetric product of \( E_x \). If we provide \( N \) and \( P \) with Riemannian metrics, then the Levi-Civita connections induce the exponential maps \( \exp_{N,x} : T_xN \rightarrow N \) and \( \exp_{P,y} : T_yP \rightarrow P \). In dealing with the exponential maps we always consider the convex neighborhoods \((K-N)\). We define the smooth bundle map

\[
J^k(N, P) \rightarrow J^k(TN, TP) \quad \text{over} \quad N \times P \tag{1.2}
\]

by sending \( z = j^k_{x,y} \in J^k_{x,y}(N, P) \) to the \( k \)-jet of \((\exp_{P,y})^{-1} \circ f \circ \exp_{N,x} \) at \( 0 \in T_xN \), which is regarded as an element of \( J^k(T_xN, T_yP)(= J^k_{x,y}(TN, TP)) \). The structure group of \( J^k(TN, TP) \) is reduced to \( O(p) \times O(n) \).

Recall that \( S^i(E) \) has the inclusion \( S^i(E) \rightarrow \otimes^i E \) and the canonical projection \( \otimes^i E \rightarrow S^i(E) \) (see [B, Section 4] and [Mats, Ch. III, Section 2]). Let \( E_j \) be subbundles of \( E \) \( (j = 1, \ldots, r) \). We define \( E_1 \bigcap \cdots \bigcap E_i = \bigcap_{j=1}^i E_j \) to be the image of \( E_1 \otimes \cdots \otimes E_i = \otimes_{j=1}^i E_j \rightarrow \otimes^i E \rightarrow S^i(E) \). When \( E_{j+1} = \cdots = E_{j+\ell} \), we often write \( E_1 \bigcap \cdots \bigcap E_j \bigcap E_{j+1} \bigcap \cdots \bigcap E_i \) in place of \( \bigcap_{j=1}^i E_j \).

## 2 Boardman manifolds

We review well-known results about Boardman manifolds in \( J^\infty(N, P) \) ([B] and [L2]). Let \( I = (i_1, i_2, \ldots, i_k, \ldots) \) be a Boardman symbol, which is a sequence of nonnegative integers with \( i_1 \geq i_2 \geq \cdots \geq i_k \geq \cdots \). Set \( I_k = (i_1, i_2, \ldots, i_k) \) and \( (I_k, 0) = (i_1, i_2, \ldots, i_k, 0) \). In the infinite jet space \( J^\infty(N, P) \), there have been defined a sequence of the submanifolds
\[ \Sigma^I_k(N,P) \supseteq \cdots \supseteq \Sigma^I_k(N,P) \supseteq \cdots \]

with the following properties. In this paper we often write \( \Sigma^l_k \) for \( \Sigma^I_k(N,P) \) if there is no confusion.

Let \( P = (\pi^N)^*(TP) \) and \( D \) be the total tangent bundle defined over \( J^\infty(N,P) \). We explain an important property of the total tangent bundle \( D \), which is often used in this paper. Let \( f : (N,x) \to (P,y) \) be a germ and \( F \) be a smooth function in the sense of [B, Definition 1.4] defined on a neighborhood of \( j_x^\infty f \). Given a vector field \( v \) defined on a neighborhood of \( x \) in \( N \), there is a total vector field \( D \) defined on a neighborhood of \( j_x^\infty f \) such that \( DF \circ j^\infty f = v(F \circ j^\infty f) \). It follows that \( d(j^\infty f)(v)(F) = DF(j^\infty f) \) for \( d(j^\infty f) : TN \to T(J^\infty(N,P)) \) around \( x \). This implies \( d(j^\infty f)(v) = D \). Hence, we have \( D \cong (\pi^N)^*(TN) \).

First we have the first derivative \( d_1 : D \to P \) over \( J^\infty(N,P) \). We define \( \Sigma^I_1(N,P) \) to be the submanifold of \( J^\infty(N,P) \) which consists of all jets \( z \) such that the kernel rank of \( d_{1,z} \) is 1. Since \( d_{1,|\Sigma^I_1(N,P)} \) is of constant rank \( n - 1 \), we set \( K_1 = \text{Ker}(d_1) \) and \( Q_1 = \text{Cok}(d_1) \), which are vector bundles over \( \Sigma^I_1(N,P) \). Set \( K_0 = D \), \( P_0 = P \) and \( \Sigma^I_0(N,P) = J^\infty(N,P) \). We can inductively define \( \Sigma^I_k(N,P) \) and the bundles \( K_k \) and \( P_k \) over \( \Sigma^I_k(N,P) \) \((k \geq 1)\) with the properties:

1. \( K_{k-1}|_{\Sigma^I_k(N,P)} \supseteq K_k \) over \( \Sigma^I_k(N,P) \).
2. \( K_k \) is an \( i_k \)-dimensional subbundle of \( T(\Sigma^I_{k-1}(N,P))|_{\Sigma^I_k(N,P)} \).
3. There exists the \((k+1)\)-th intrinsic derivative \( d_{k+1} : T(\Sigma^I_{k-1}(N,P))|_{\Sigma^I_k(N,P)} \to P_k \) over \( \Sigma^I_k(N,P) \), so that it induces the exact sequence

\[
0 \to T(\Sigma^I_k(N,P)) \overset{\text{inclusion}}{\to} T(\Sigma^I_{k-1}(N,P))|_{\Sigma^I_k(N,P)} \overset{d_{k+1}}{\to} P_k \to 0
\]

over \( \Sigma^I_k(N,P) \).

Namely, \( d_{k+1} \) induces the isomorphism of the normal bundle

\[
\nu(I_k \subset I_{k-1}) = (T(\Sigma^I_{k-1}(N,P))|_{\Sigma^I_k(N,P)})/T(\Sigma^I_k(N,P))
\]

of \( \Sigma^I_k(N,P) \) in \( \Sigma^I_{k-1}(N,P) \) onto \( P_k \).

4. \( \Sigma^I_{k+1}(N,P) \) is defined to be the submanifold of \( \Sigma^I_k(N,P) \), which consists of all jets \( z \) with \( \dim(\text{Ker}(d_{k+1,z}|K_{k+1})) = i_k+1 \).

5. Set \( K_{k+1} = \text{Ker}(d_{k+1}|K_k) \) and \( Q_{k+1} = \text{Cok}(d_{k+1}|K_k) \) over \( \Sigma^I_{k+1}(N,P) \). Then it follows that \( (K_{k+1}|_{\Sigma^I_{k+1}(N,P)}) \cap T(\Sigma^I_k(N,P))|_{\Sigma^I_{k+1}(N,P)} = K_{k+1} \).

6. The intrinsic derivative \( d(d_{k+1}|K_k) : T(\Sigma^I_k(N,P))|_{\Sigma^I_{k+1}(N,P)} \to \text{Hom}(K_{k+1}, Q_{k+1}) \) over \( \Sigma^I_{k+1}(N,P) \)

of \( d_{k+1}|K_k \) is of constant rank \( \dim(\Sigma^I_k(N,P)) - \dim(\Sigma^I_{k+1}(N,P)) \). We set \( P_{k+1} = \text{Im}(d(d_{k+1}|K_k)) \) and define \( d_{k+2} \) to be

\[
d_{k+2} = d(d_{k+1}|K_k) : T(\Sigma^I_k(N,P))|_{\Sigma^I_{k+1}(N,P)} \to P_{k+1}
\]

as the epimorphism.

7. The codimension of \( \Sigma^I_k(N,P) \) in \( J^k(N,P) \) is described in [B, Theorem 6.1].

8. We define \( \Sigma^k_{1,0}(N,P) = \Sigma^I_k(N,P) \setminus \Sigma^I_{k+1}(N,P) \).
(9) The submanifold $\Sigma^J(N, P)$ is actually defined so that it coincides with the inverse image of the submanifold in $J^k(N, P)$ by $\pi_k^\infty$.

We apply the above construction to the two symbols $J = (n-p+1, 1, \ldots, 1, \ldots)$ and $\mathfrak{J} = (n-p+1, 2, 1, \ldots, 1, \ldots)$. For both symbols, we write $K$ and $Q$ for $K_1$ and $Q_1$ respectively in the rest of the paper. We have $\dim K = n-p+1$, $\dim Q = 1$ and

$$d_2 : T(J^\infty(N, P))|_{\Sigma^{n-p+1}(N, P)} \to P_1 = \text{Hom}(K, Q) \quad \text{over } \Sigma^{n-p+1}(N, P). \quad (2.1)$$

When $k \geq 2$, we usually write $K^I_k$, $Q^I_k$, $P^I_k$ and $d^I_k$ in place of $K_k$, $Q_k$, $P_k$ and $d_k$ respectively for $I = J$ and $\mathfrak{J}$. We first deal with the symbol $J$. In this paper $\Sigma^J(N, P)$ and $\Sigma^J(N, P)$ are often denoted by $\Sigma A_k(N, P)$ and $\Sigma A_k(N, P)$ respectively.

(J-1) We have that $\dim K^2_1 = 1$, $Q^1_k = \text{Hom}(\bigodot^{k-1}K^J_2, Q)|_{\Sigma \mathcal{A}_k}$ and the exact sequence

$$0 \to T(\Sigma \mathcal{A}_k(N, P)) \cong T(\Sigma \mathcal{A}_{k-1}(N, P))|_{\Sigma \mathcal{A}_k} \xrightarrow{d^I_{k+1}} P^I_k \cong \text{Hom}(\bigodot^{k-1}K^J_2, Q)|_{\Sigma \mathcal{A}_k} \to 0. \quad (2.2)$$

Set $\nu(\mathcal{A}_k \subset \mathcal{A}_{k-1}) = \nu(J_k \subset J_{k-1})$. Then $d^I_{k+1}$ induces the canonical isomorphism $\nu(\mathcal{A}_k \subset \mathcal{A}_{k-1}) \to P^I_k$.

(J-2) $\Sigma^J(N, P)$ is of codimension $n-p+k$ in $J^\infty(N, P)$.

(J-3) A smooth map germ $f : (N, x) \to (P, y)$ has $x$ as a singularity of type $A_k$ if and only if $j^\infty f$ is transverse to $\Sigma^J(N, P)$ at $x$ and $j^\infty f \in \Sigma^J(N, P)$ (see [Mo]).

We next turn to the symbol $\mathfrak{J} = (n-p+1, 2, 1, \ldots, 1, \ldots)$. Then we have the following by using [B, Corollary 7.10].

(J-1) We have that $\dim K^2_2 = 2$, $Q^2_3 = \text{Hom}(K^3_2, Q)|_{\Sigma^2_3}$, and

$$d^3_3 : T(\Sigma^{n-p+1}(N, P))|_{\Sigma^2_3} \to P^3_2 \cong \text{Hom}(\bigodot^2K^2_2, Q)|_{\Sigma^2_3}, \quad (2.3)$$

and the exact sequence

$$0 \to K^3_3 \to K^2_2 \xrightarrow{d^3_3|K^3_3} \text{Hom}(\bigodot^2K^2_3, Q) \to \text{Hom}(K^3_2 \bigodot K^3_2, Q) \to 0 \quad \text{over } \Sigma^{3_3}(N, P). \quad (2.4)$$

Here, $K^3_3 = \text{Ker}(d^3_3|K^3_3)$ and $\text{Im}(d^3_3|K^3_3) \cong \text{Hom}(K^3_2 \bigodot K^3_2, Q)|_{\Sigma^3_3}$ are of dimension 1 and $Q^3_3 \cong \text{Hom}(K^3_3 \bigodot K^3_2, Q)|_{\Sigma^3_3}$ with $\dim Q^3_3 = 2$.

(3E-1) Since $P^3_3 = \text{Hom}(K^3_3, Q^3_3) \cong \text{Hom}(\bigodot^2K^3_3, Q)|_{\Sigma^3_3}$, we have the epimorphism

$$d^3_3 : T(\Sigma^{3_3}(N, P))|_{\Sigma^3_3} \to \text{Hom}(\bigodot^2K^3_3 \bigodot K^3_2, Q)|_{\Sigma^3_3}. \quad (2.5)$$

(3E-2) By $K^3_4 = K^3_3|_{\Sigma^3_4}$, $Q^3_4 \cong \text{Hom}(\bigodot^3K^3_4, Q)|_{\Sigma^3_4}$ and $P^3_4 = \text{Hom}(K^3_4, Q^3_4) \cong \text{Hom}(\bigodot^3K^3_4 \bigodot K^3_2, Q)|_{\Sigma^3_4}$, we have the epimorphism

$$d^3_3 : T(\Sigma^{3_3}(N, P))|_{\Sigma^3_4} \to \text{Hom}(\bigodot^3K^3_4 \bigodot K^3_2, Q)|_{\Sigma^3_4}. \quad (2.6)$$

(3E-3) The homomorphisms $d^3_3$, $d^4_3$ and $d^4_3$ induce the homomorphisms

$$\tilde{d}^3_3 : \bigodot^3K^3_2 \to Q \quad \text{over } \Sigma^{3_3}(N, P),$$

$$\tilde{d}^4_3 : \bigodot^4K^3_3 \bigodot K^3_2 \to Q \quad \text{over } \Sigma^{3_3}(N, P),$$

$$\tilde{d}^4_3 : \bigodot^4K^3_4 \bigodot K^3_2 \to Q \quad \text{over } \Sigma^{3_3}(N, P). \quad (2.7)$$

respectively.
3 Submanifolds $\Sigma D_k(N, P)$ and $\Sigma E_k(N, P)$

In this section we review the definition and the properties of the manifolds $\Sigma X_k(N, P)$ and $\Sigma X_k(N, P)$ for $X_k = D_k$ and $E_k$ in [An3]. We identify, as usual, $\text{Hom}(\mathcal{O}^3R^2, \mathcal{R})$ with the set of all cubic forms with variables $u$ and $v$ on $\mathbb{R}^2$. By [Ar, Lemma 5.1] it is decomposed into the five manifolds $S^1_4$, $S_5$, $S_5$ and $0$ which are orbit manifolds by $GL(2)$ through $u^2\ell \pm \ell^2$, $u^2\ell$, $u^3$ and 0 respectively. Let us recall the bundle $\text{Hom}(\mathcal{O}^3K_3^2, \mathcal{Q})$ over $\Sigma^{3z}(N, P)$. Then we obtain the subbundles $S^1_4(\mathcal{O}^3K_2^2, \mathcal{Q})$, $S_5(\mathcal{O}^3K_2^2, \mathcal{Q})$, $S_E(\mathcal{O}^3K_2^2, \mathcal{Q})$ of $\text{Hom}(\mathcal{O}^3K_3^2, \mathcal{Q})$ associated to $S^1_4$, $S_5$, $S_E$ respectively. Let $\nu(X_{k+1} \subset X_k)$ be the normal bundle $(T(\Sigma X_k(N, P))|_{\Sigma X_{k+1}})/T(\Sigma X_{k+1}(N, P))$ and set $\nu(X_{k+1} \subset X_k) = \nu(X_{k+1} \subset X_k)|_{\Sigma X_{k+1}}$.

Let us consider $\mathfrak{d}_3^3|_{\Sigma^{3z,0}_2} : \mathcal{O}^3K_2^3 \rightarrow \mathcal{Q}$ over $\Sigma^{3z,0}(N, P)$. We define the submanifolds

\[
\Sigma D_4(N, P) = \left(\mathfrak{d}_3^3|_{\Sigma^{3z,0}_2}\right)^{-1}(S^1_4(\mathcal{O}^3K_2^3, \mathcal{Q})),
\]

\[
\Sigma D_5(N, P) = \left(\mathfrak{d}_3^3|_{\Sigma^{3z,0}_2}\right)^{-1}(S_5(\mathcal{O}^3K_2^3, \mathcal{Q})),
\]

in $\Sigma^{3z,0}(N, P)$ respectively. Its transversality follows from [An3, Proposition 3.5].

(D-i) We set $\Sigma D_4(N, P) = \Sigma D^+(N, P) \cup \Sigma D^-(N, P)$, and $\Sigma D_4(N, P) = \Sigma^{3z,0}(N, P)$.

(D-ii) There exist a small neighborhood $U(\Sigma \mathcal{D}_5)$ of $\Sigma \mathcal{D}_5(N, P)$ in $\Sigma^{3z,0}(N, P)$ and the line bundle $L$ defined over $\Sigma \mathcal{D}_5(N, P)$, which is uniquely extended to the subbundle $\tilde{L}$ over $U(\Sigma \mathcal{D}_5)$ of $K_2^3$ such that for any $z \in \Sigma \mathcal{D}_5(N, P)$, $L_z$ coincides with $(\mathfrak{d}_3^3|_{\Sigma^{3z,0}_2})^{-1}(H_z)$, where $H_z$ is the subset of $\text{Hom}(\mathcal{O}^3K_2^3, \mathcal{Q}_z)$ of all quadratic forms of rank 1 or 0. Furthermore, we have $\text{Cok}(\mathfrak{d}_3^3|K_2^3) = \text{Hom}(\mathcal{O}^2L, \mathcal{Q})$ over $\Sigma \mathcal{D}_5(N, P)$.

(D-iii) Let $z \in U(\Sigma \mathcal{D}_5)$. Then $z$ lies in $\Sigma \mathcal{D}_5(N, P)$ if and only if the restriction $\mathfrak{d}_3^3|_{z_2} : \mathcal{O}^3L_z$ is a null homomorphism.

(D-iv) We successively construct the submanifolds $\Sigma \mathcal{D}_k(N, P)$ ($k \geq 5$) with $\Sigma \mathcal{D}_4(N, P) \supset U(\Sigma \mathcal{D}_5) \supset \Sigma \mathcal{D}_5(N, P) \supset \cdots \supset \Sigma \mathcal{D}_k(N, P)$ and the homomorphisms

\[
\mathbf{r}_3 : \mathcal{O}^3\tilde{L} \rightarrow \mathcal{Q} \quad \text{over} \quad U(\Sigma \mathcal{D}_5) \quad \text{and} \quad \mathbf{r}_{k-1} : \mathcal{O}^{k-1}L \rightarrow \mathcal{Q} \quad \text{over} \quad \Sigma \mathcal{D}_k(N, P) \quad (k \geq 5) \quad (3.1)
\]

with the following properties:

(D-iv-1) $\mathfrak{d}_3^3|_{\mathcal{O}^3\tilde{L}} = \mathbf{r}_3$.

(D-iv-2) $\Sigma \mathcal{D}_k(N, P)$ is of codimension $n - p + k$ in $J^\infty(N, P)$.

(D-iv-3) An element $z \in U(\Sigma \mathcal{D}_5)$ lies in $\Sigma \mathcal{D}_5(N, P)$ if and only if $\mathbf{r}_{3,z}$ vanishes. An element $z \in \Sigma \mathcal{D}_k(N, P)$ lies in $\Sigma \mathcal{D}_{k+1}(N, P)$ if and only if $\mathbf{r}_{k-1,z}$ vanishes.

(D-iv-4) The intrinsic derivative of $\mathbf{r}_{k-1}$

\[
d(\mathbf{r}_{k-1}) : T(\Sigma \mathcal{D}_k(N, P))|_{\Sigma \mathcal{D}_{k+1}} \rightarrow \text{Hom}(\mathcal{O}^{k-1}L, \mathcal{Q})|_{\Sigma \mathcal{D}_{k+1}} \quad (3.2)
\]
is surjective. In other words, \( r_{k-1} \) is, as a section, transverse to the zero-section of \( \text{Hom}(\mathcal{O}^{k-1}L, \mathcal{Q})|_{z \Sigma D_k} \) on \( \Sigma \overline{D}_{k+1}(N, P) \). Then, \( d(r_{k-1}) \) induces the isomorphism \( \nu: (\overline{D}_{k+1} \subset \mathcal{D}_k) \). 

(D-v) We define \( \Sigma D_k(N, P) = \Sigma \overline{D}_k(N, P) \). Then a smooth map germ \( f : (N, x) \to (P, y) \) has \( x \) as a singularity of type \( D_k \) if and only if \( j^\infty f \) is transverse to \( \Sigma D_k(N, P) \) at \( x \) and \( j^\infty f \in \Sigma D_k(N, P) \).

Next we turn to define \( \Sigma E_k(N, P) \). We note that the singularities \( E_6, E_7 \) and \( E_8 \) have the Boardman symbols \((\mathcal{J}_3, 0), (\mathcal{J}_3, 0) \) and \((\mathcal{J}_4, 0) \) respectively (the method for the calculation of Boardman symbols in [Math2] may be convenient).

(E-i) We define \( \Sigma E_6(N, P) \) to be the open submanifold of \( \Sigma^{3,0}(N, P) \) which consists of all jets \( z \) such that \( \tilde{d}_4^3|_z \bigcirc^4 K_3^3 \) does not vanish. We set \( \Sigma E_7(N, P) = \Sigma^{3,0}(N, P) \setminus \Sigma E_6(N, P) \). Namely, for \( z \in \Sigma E_7(N, P) \), \( \tilde{d}_4^3|_z \bigcirc^3 K_3^3 \bigcirc (K_2^3 / K_3^3) \) does not vanish.

(E-ii) Define \( \Sigma \overline{E}_7(N, P) \) in \( \Sigma^{3,0}(N, P) \) to be the inverse image of the zero-section of \( \tilde{d}_4|_z \bigcirc^4 K_3^3 \) over \( \Sigma^{3,0}(N, P) \), which is regarded as the section over \( \Sigma^{3,0}(N, P) \). By Lemma 3.1 (i) below, \( \Sigma \overline{E}_7(N, P) \) is a submanifold of \( \Sigma^{3,0}(N, P) \) and the intrinsic derivative \( d(\tilde{d}_4|_z \bigcirc^4 K_3^3) : T(\Sigma^{3,0}(N, P))|_{\Sigma \overline{E}_7} \to \text{Hom}(\bigcirc^4 K_3^3, \mathcal{Q})|_{\Sigma \overline{E}_7} \) of \( \tilde{d}_4|_z \bigcirc^4 K_3^3 \) is an epimorphism. Hence, we have the exact sequence

\[
0 \to T(\Sigma \overline{E}_7(N, P)) \to T(\Sigma^{3,0}(N, P))|_{\Sigma \overline{E}_7} \to \text{Hom}(\bigcirc^4 K_3^3, \mathcal{Q})|_{\Sigma \overline{E}_7} \to 0.
\]

Let \( z \in \Sigma^{3,0}(N, P) \). Since \( K_3^3 = K_3^3, \) we have that \( \tilde{d}_4|_z \bigcirc^4 K_3^3 \) vanishes. Namely, \( \tilde{d}_4|_z \bigcirc^4 K_3^3 \) vanishes. This implies that \( \Sigma^{3,0}(N, P) \subset \Sigma \overline{E}_7(N, P) \).

(E-iii) We define \( \Sigma E_8(N, P) \) to be the submanifold of \( \Sigma \overline{E}_7(N, P) \) which consists of all jets \( z \) such that

(i) \( \tilde{d}_4^3|_z \bigcirc^4 K_3^3 \bigcirc (K_2^3 / K_3^3) \) vanishes,

(ii) \( \tilde{d}_4|_z \bigcirc^5 K_3^3 \) does not vanish.

We have \( \Sigma E_8(N, P) \subset \Sigma^{3,0}(N, P) \) by (i), (3E-2) and (E-ii). By (5) in Section 2 for \( z \in \Sigma^{3,0}(N, P), \) \( K_3^3 \cap T_z(\Sigma^{3,0}(N, P)) = \{0\} \) if and only if \( z \in \Sigma^{3,0}(N, P) \). Hence, we have \( \Sigma E_8(N, P) \subset \Sigma^{3,0}(N, P) \) by (ii). By Lemma 3.1 (ii) below, the section \( \tilde{d}_4|_z \bigcirc^4 K_3^3 \bigcirc (K_2^3 / K_3^3) \) of \( \text{Hom}(\bigcirc^4 K_3^3 \bigcirc (K_2^3 / K_3^3), \mathcal{Q}) \) over \( \Sigma \overline{E}_7(N, P) \) is transverse to the zero-section, whose inverse image of this section coincides with \( \Sigma^{3,0}(N, P) \). Hence, \( \Sigma^{3,0}(N, P) \) is a submanifold of \( \Sigma \overline{E}_7(N, P) \) and the intrinsic derivative \( d(\tilde{d}_4|_z \bigcirc^4 K_3^3 \bigcirc (K_2^3 / K_3^3)) \) over \( \Sigma^{3,0}(N, P) \) is an epimorphism. Namely, we have the exact sequence

\[
0 \to T(\Sigma^{3,0}(N, P)) \to T(\Sigma \overline{E}_7(N, P))|_{\Sigma^{3,0}} \to \text{Hom}(\bigcirc^4 K_3^3 \bigcirc (K_2^3 / K_3^3), \mathcal{Q})|_{\Sigma^{3,0}} \to 0,
\]

which yields

\[
0 \to T(\Sigma E_8(N, P)) \to T(\Sigma \overline{E}_7(N, P))|_{\Sigma E_8} \to \text{Hom}(\bigcirc^4 K_3^3 \bigcirc (K_2^3 / K_3^3), \mathcal{Q})|_{\Sigma E_8} \to 0.
\]

(E-iii-a) By (E-i), we have that

\[
T(\Sigma^{3,0}(N, P))|_{\Sigma E_6} \ni K_3^3|_{\Sigma E_6} \neq \{0\} \quad \text{and} \quad K_3^3|_{\Sigma E_6 \cap T(\Sigma E_6(N, P))} = \{0\}.
\]
(E-iii-b) Since $K_3^3 \cap T(\Sigma^3,0)(N,P) = \{0\}$, we have $K_3^3|_{\Sigma E_7} \cap T(\Sigma^3,0)(N,P)|_{\Sigma E_7} = \{0\}.$

(E-iii-c) Let $r : \text{Hom}(\bigcirc^3 K_3^3 \cap K_3^2, Q)|_{\Sigma E_8} \to \text{Hom}(\bigcirc^4 K_3^3, Q)|_{\Sigma E_8}$ be the map induced from the restriction. From this definition, it follows that

$$r \circ d^3_4|_S (K_3^3|_{\Sigma E_8}) = d(d^3_1 \bigcirc^4 K_3^3)(K_3^3|_{\Sigma E_8}) : K_3^3|_{\Sigma E_8} \to \text{Hom}(\bigcirc^4 K_3^3, Q)|_{\Sigma E_8}$$

is an isomorphism. From (3.3), it follows that $K_3^3|_{\Sigma E_8} \cap T(\Sigma E_7(N,P))|_{\Sigma E_8} = \{0\}$, and the inclusion $K_3^3|_{\Sigma E_8} \to T(\Sigma E_7(N,P))|_{\Sigma E_8}$ induces the isomorphism $K_3^3|_{\Sigma E_8} \to \nu(E_7 \subset E_8)|_{\Sigma E_8}$.

(E-iv) $\Sigma E_k(N,P)$ is of codimension $n - p + k$ in $J^\infty(N,P)$.

(E-v) A smooth map germ $f : (N, x) \to (P, y)$ has $x$ as a singularity at $x$ of type $E_k$ if and only if $f^\infty$ is transverse to $\Sigma E_k(N,P)$ at $x$ and $j^\infty f \in \Sigma E_k(N,P)$.

When we deal with only $\Omega$-regular maps, it is convenient in the proof of Lemma 3.1 and also in the calculation of Thom polynomials of $\Sigma E_k(N,P)$ in [An4] to set $\Sigma E_6(N,P) = \Sigma^3(N,P)$ and $\Sigma E_7(N,P) = \Sigma^4(N,P)$, since we can readily deal with $\Sigma^3(N,P)$ and $\Sigma^4(N,P)$. This does not matter, since the subset of all jets $z \in \Sigma^3,0(N,P)$, such that $\widetilde{d}^3_4|_S \bigcirc^3 K_3^3|_{\Sigma E_8}$ vanishes and $\widetilde{d}^3_4|_S \bigcirc^4 K_3^3 \cap (K_3^2, K_3^3)$ does not vanish, has modality ([Ar, Lemma 6.1, 7]).

Namely, such a jet is not simple and does not appear in $\Omega(N,P)$.

**Lemma 3.1.** (i) The section $\widetilde{d}^3_4|_S \bigcirc^3 K_3^3$ of $\text{Hom}(\bigcirc^4 K_3^3, Q)$ over $\Sigma^3(N,P)$ is transverse to the zero-section on $\Sigma E_7(N,P)$.

(ii) The section $\widetilde{d}^3_4|_S \bigcirc^3 K_3^3 \cap (K_3^2, K_3^3)$ of $\text{Hom}(\bigcirc^3 K_3^3 \cap (K_3^2, K_3^3), Q)$ over $\Sigma E_7(N,P)$ is transverse to the zero-section on $\Sigma E_8(N,P)$.

**Proof.** The proof is very like the proof of [An3, Lemmas 3.4 and Proposition 3.5]. We only give the proof of (i), since the proof of (ii) is similar.

Take an element $z \in \Sigma E_7(N,P)$ over $(x_0, y_0)$. We choose local coordinates $x = (x_1, \ldots, x_{n-2}, u, \ell)$ around $x_0$ and $y = (y_1, \ldots, y_p)$ around $y_0$ such that $z$ is expressed as $j^\infty f^\infty h^z$ with

$$y_i \circ h^z(x) = x_i \quad (1 \leq i \leq p - 1),
$$

$$y_p \circ h^z(x) = \pm x_p^2 \pm \cdots \pm x_{n-2}^2 + \overline{h}(x_1, \ldots, x_{n-2}, u, \ell),$$

where $\overline{h} \in (x_1, \ldots, x_{n-2}, u, \ell)^2$ and $(\partial^4 \overline{h}/\partial \ell^4)|_{z=0} = 0$. Let $\delta_{x_1}, \ldots, \delta_{x_{n-2}}, \delta_u$ and $\delta_\ell$ express the vector fields of the total tangent bundle $D$, which correspond to the coordinates $x_1, \ldots, x_{n-2}, u, \ell$ respectively defined in [B, Definition 1.6] (we avoid to use the notation used there, which are used for the singularities $D_k$). Then $K_3^3$ is spanned by $\delta_{x_p}, \ldots, \delta_{x_{n-2}}, \delta_u$ and $\delta_\ell$, $K_3^3$ spanned by $\delta_u$ and $\delta_\ell$, $K_3^3$ spanned by $\delta_\ell$, and $Q_3^3$ spanned by the image of $\partial/\partial y_p$, say $e_p$. We consider the submanifold $S$ in $\Sigma^3(N,P) \cap J^\infty_{x_0,y_0}(N,P)$, which consists of all jets $j^\infty h$ such that $y_i \circ h(x) = x_i \quad (1 \leq i \leq p - 1)$ and that $y_p \circ h(x) = y_p \circ h^z(x) + \overline{h}(x_1, \ldots, x_{n-1}, u, \ell)$, where $\overline{h}$ is a polynomial of degree 5. Then the vector fields $\delta_u$ and $e_p$ determine the trivializations $K_3^3|_S \cong S \times R$ and $Q_3^3|_S \cong S \times R$ respectively. Under these trivializations, we calculate $\widetilde{d}^3_4|_S : \bigcirc^4 K_3^3|_S \to Q|_S$ as follows.
Let $d^j = a^j \delta \ell$ (1 ≤ $j$ ≤ 4) be the vector fields of $K_3^3|s$. Then we have
\[
\tilde{d}^3(d^1 \circ d^2 \circ d^3 \circ d^4) = (a^1 \partial / \partial \ell)(a^2 \partial / \partial \ell)(a^3 \partial / \partial \ell)(a^4 \partial / \partial \ell)y_p \circ h
= a^1a^2a^3a^4(4 \partial y_p \circ h / \partial \ell).
\]

This implies that $\tilde{d}^3|s$ is transverse to the zero-section of $\text{Hom}(\bigwedge^4 K_3^3, Q)$ at $z \in S \cap \Sigma E_7(N,P)$. Hence, (i) is proved. □

**Remark 3.2.** We can prove by the above definition of $\Sigma X_k(N,P)$ that the normal form for $X_k$ in Introduction has the origin as the singularity of the type $X_k$ respectively.

**Remark 3.3.** (i) The submanifold $\Sigma D_k(N,P)$ is actually defined so that it coincides with the inverse image of the submanifold in $J^{k-1}(N,P)$ by $\pi^\infty_{k-1}$.

(ii) The submanifold $\Sigma E_6(N,P)$ (resp. $\Sigma E_7(N,P)$ and $\Sigma E_8(N,P)$) is actually defined so that it coincides with the inverse image of the submanifold in $J^4(N,P)$ by $\pi^\infty_4$ (resp. in $J^5(N,P)$ by $\pi^\infty_5$).

**Remark 3.4.** We can entirely do the arguments in the definition of Boardman manifolds, $\Sigma D_k(N,P)$ and $\Sigma E_j(N,P)$ on $J^\ell(N,P)$ for a large $\ell$ in Sections 2 and 3. We provide $N$ and $P$ with Riemannian metrics, which enable us to consider the exponential maps $TN \to N$ and $TP \to P$ by the Levi-Civita connections. For points $x \in N$, $y \in P$ and orthonormal basis of $T_xN$ and $T_yP$, we take the associated convex neighborhoods $U \subset N$ around $x$ and $V \subset P$ around $y$ with the normal coordinate systems $(x_1, ..., x_n)$ and $(y_1, ..., y_p)$ so that $TU \approx U \times T_xN$ and $TV \approx V \times T_yP$ by the connections respectively (see [K-N]).

Let us define the canonical embedding $\mu^\ell: J^\ell(TN,TP) \to J^\infty(TN,TP)$ by putting the null homomorphisms of $\text{Hom}(S^i(\pi^\infty_N(TN)), \pi^\infty_P(TP))$ as the $i$-th component for $i > \ell$. We regard $\mu^\infty$ as the map to $J^\infty(N,P)$ under the identification (1.2). Any element $z \in \mu^\ell(J^\ell(TN,TP))$ is represented by a $C^\infty$ polynomial map germ $f : (N,x) \to (P,y)$ of degree $\ell$ under these coordinates. It is clear that $\pi^\infty_0 \circ \mu^\ell = \text{id}_{J^\ell(TN,TP)}$ and $\mu^\ell(\partial f / \partial \ell)|_{\mu^\ell(J^\ell(TN,TP))} = \partial \mu^\ell(J^\ell(TN,TP)) / \partial \ell$.

We can prove that $\text{D}|_{\mu^\ell(J^\ell(TN,TP))}$ is tangent to $\mu^\ell(\partial f / \partial \ell)$. Indeed, let $x \in U \subset N$ and $y \in V \subset P$ be as above. For any points $u \in U$ and $v \in V$, we take the normal coordinate systems $(u_1, ..., u_n)$ around $u$ in $U$ and $(v_1, ..., v_p)$ around $v$ in $V$ associated to the orthonormal basis of $T_uU$ and $T_vV$, which are induced from the above orthonormal basis of $T_xN$ and $T_yP$ by the parallel transformation along the geodesics of $x$ to $u$ and $y$ to $v$ respectively. We note that $u_1, ..., u_n$ are smooth functions of $x_1, ..., x_n$. Let $\sigma = (\sigma_1, ..., \sigma_n)$ with non-negative integers $\sigma_i$. We define the coordinate system $X_i$, $Y_j$ and $W_{j,\sigma}$ of $J^\infty(TU,TV)$ by
\[
X_i = x_i \circ \pi^\infty, \quad Y_j = y_j \circ \pi^\infty, \quad W_{j,\sigma}(j^\infty f) = \frac{\partial |\sigma|(v_j \circ f)}{\partial u^\sigma}(u),
\]
where $|\sigma| \geq 0$ and $Y_j = W_{j,0}$ for $|\sigma| = 0$. In the definition, we should note that the normal coordinate systems $(u_1, ..., u_n)$ and $(v_1, ..., v_p)$ vary depending on points $u$ and $v$. 

---

1. [K-N] Kenji-Neumann, Complex Analytic Varieties.
A smooth function $\Phi$ defined on an open subset of $\mu^\ell_\infty(J^f(TN,TP))$ is written as $\Phi \circ \mu^\ell_\infty \circ \pi^\ell_\infty$ on the same open subset. Hence, $\Phi$ is extended to $(\Phi \circ \mu^\ell_\infty) \circ \pi^\ell_\infty$ defined on an open subset of $J^\infty(TN,TP)$, which is a smooth function in the sense of [B, Definition (1.4)]. Let us consider the total tangent bundle $D$ defined on $J^\infty(TN,TP) = J^\infty(N,P)$ and its vector field $D_i$ corresponding to $\partial/\partial x_i$. By using [B, (1.8)], we have, for $z = j^\infty u \in \mu^\ell_\infty(J^f(TN,TP))$,

$$D_i(\Phi)(z) = \frac{\partial(\Phi \circ j^\infty f)}{\partial x_i}(u) = \frac{\partial \Phi}{\partial X_i}(z) \frac{\partial(X_i \circ j^\infty f)}{\partial x_i}(u) + \sum_{j,\sigma} \frac{\partial \Phi}{\partial W_{j,\sigma}}(z) \frac{\partial(W_{j,\sigma} \circ j^\infty f)}{\partial x_i}(u),$$

where

(i) if we define $g^h_i(u) = \frac{\partial u_h}{\partial x_i}(u)$, then they are smooth functions of $x_1, \ldots, x_n$,

(ii) $\frac{\partial W_{j,\sigma}}{\partial u_h} = W_{j,\sigma'}$ with $\sigma' = (\sigma_1, \ldots, \sigma_{h-1}, \sigma_h + 1, \sigma_{h+1}, \ldots, \sigma_n)$,

(iii) $|\sigma| \leq \ell - 1$, since $W_{j,\sigma}$ vanishes for $|\sigma| = \ell$.

Since $\Phi$ is a function of variables $X_i, Y_j$ and $W_{j,\sigma}$ with $|\sigma| \leq \ell$, so is $D_i(\Phi)$. Therefore, $D_i$ is tangent to $\mu^\ell_\infty(J^f(TN,TP))$ at $z$ for each $i$. Since $D$ is generated by $D_i$, $D_z$ is tangent to $\mu^\ell_\infty(J^f(TN,TP))$ at $z$.

For a symbol $I = (i_1, i_2, \ldots, i_r)$ we define $\Omega^I(n,p)$ to be a subset of $J^\infty(n,p)$ which consists of all Boardman manifolds $\Sigma^I(n,p)$ with symbols $J$ of length $r$ satisfying $J \leq I$ in the lexicographic order. The following lemma is well known.

**Lemma 3.5.** $\Omega^I(n,p)$ is an open subset of $J^\infty(n,p)$.

We next describe the adjacency relations between $\Sigma A_{k}(n,p), \Sigma D_{k}(n,p)$ and $\Sigma E_{k}(n,p)$ in $J^\infty(n,p)$. This adjacency relations are quite parallel to the result given in [Ar, Corollary 8.7]. However, we need to explain them, since $\Sigma X_{j}(n,p)$ is not necessarily an orbit through the jet of a germ with singularity $X_{j}$ given in Introduction by $L^\infty(p) \times L^\infty(n)$.

**Lemma 3.6.** Let $n \geq p \geq 2$. A subset $\Omega(n,p)$ consisting of all regular jets and a number of $\Sigma A_{k}(n,p), \Sigma D_{k}(n,p)$ and $\Sigma E_{k}(n,p)$ is an open subset of $J^\infty(n,p)$ if and only if the following three conditions are satisfied.

(i) If $\Sigma A_{k}(n,p) \subset \Omega(n,p)$, then $\Sigma A_{k}(n,p) \subset \Omega(n,p)$ for all $k$ with $1 \leq k < i$.

(ii) If $\Sigma D_{i}(n,p) \subset \Omega(n,p)$, then $\Sigma A_{k}(n,p)$ ($1 \leq k < i$) and $\Sigma D_{i}(n,p)$ ($4 \leq k < i$) are all contained in $\Omega(n,p)$.

(iii) If $\Sigma E_{i}(n,p) \subset \Omega(n,p)$, then $\Sigma A_{k}(n,p)$ ($1 \leq k < i$), $\Sigma D_{i}(n,p)$ ($4 \leq k < i$) and $\Sigma E_{i}(n,p)$ ($6 \leq k < i$) are all contained in $\Omega(n,p)$. 

11
Then there exist a jet 

\[ z \in \Omega(n, p) \] 

and a sequence \( \{z_k\} \) with 

\[ z_k = j_0^\infty f_k \] 

such that \( \lim_{k \to \infty} z_k = z \). If \( z \in \Sigma X_i(n, p) \) and \( j_0^\infty f_k \) is transverse to \( \Sigma X_i(n, p) \) at \( 0 \). Therefore, (i), (ii) and (iii) follow from the adjacency relation in [Ar, Corollary 8.7].

We next assume (i), (ii) and (iii). Suppose to the contrary that \( \Omega(n, p) \) is not open. Then there exist a jet \( z \in \Omega(n, p) \) and a sequence \( \{z_k\} \) with 

\[ z_k = j_0^\infty f_k \] 

such that \( \lim_{k \to \infty} z_k = z \) and \( z_k \notin \Omega(n, p) \) for all \( k \). If there exists infinite \( k \)'s with \( z_k \notin \Omega^{(3, 0)}(n, p) \), then we have 

\[ z \notin \Omega^{(3, 0)}(n, p) \] 

by Lemma 3.5. This is a contradiction. Note that 

\[ \Sigma \exists_{k=1}^\infty \Sigma A_i(n, p), \Sigma \exists_{i=4} \Sigma D_i(n, p) \] 

and 

\[ \Sigma \exists_{i=0}^\infty \Sigma E_i(n, p) \] 

are closed subsets of \( \Sigma \). We assume that there exists a number \( q \) and infinite \( k \)'s with \( z_k \in \Sigma X_q(n, p) \). Let \( q_0 \) be the smallest number among such number \( q \)'s with \( z_k \in \Sigma X_q(n, p) \). From the adjacency relation above there exists a number \( q \geq q_0 \) such that \( z \) lies in \( \Sigma A_q(n, p), \Sigma D_q(n, p) \) or \( \Sigma E_q(n, p) \). Consequently, it follows that 

(A) if \( z \in \Sigma A_q(n, p) \), then we have that 

\[ X_{q_0} = A_{q_0} \] 

and 

\[ z_k \in \Sigma A_{q_0}(n, p) \subset \Omega(n, p) \] 

by (i),

(D) if \( z \in \Sigma D_q(n, p) \), then we have that 

\[ X_{q_0} = A_{q_0} \] 

or 

\[ D_{q_0} \] 

and 

\[ z_k \in \Sigma X_{q_0}(n, p) \subset \Omega(n, p) \] 

by (ii),

(E) if \( z \in \Sigma E_q(n, p) \), then we have that 

\[ X_{q_0} = A_{q_0}, D_{q_0} \] 

or 

\[ E_{q_0} \] 

and 

\[ z_k \in \Sigma X_{q_0}(n, p) \subset \Omega(n, p) \] 

by (iii).

This is a contradiction. \( \square \)

4 Primary obstruction

In what follows we set \( \Sigma D_j(N, P) = \emptyset \) for \( 1 \leq j \leq 3 \) and \( \Sigma E_j(N, P) = \emptyset \) unless 

\[ 6 \leq j \leq 8 \]. For a section \( s \in \Gamma \Omega(N, P) \), we set 

\[ S_k(s) = s^{-1}(\Sigma X_k(N, P)), S_{X_k}(s) = s^{-1}(\Sigma X_k(N, P)) \] 

and 

\[ \Sigma X_k(N, P) \] 

where \( X_k \) refers to \( A_k, D_k \) or \( E_k \). We have 

\[ S_{X_k}(s) = \bigcup_{j=k}^\infty S_{X_j}(s) \]. \ We often write \( S_k, S_{X_k} \) and \( \Sigma X_k \) by omitting \( s \) if there is no confusion. Furthermore, we set 

\[ (s|S_{X_k})^* (K) = K(S_{X_k}(s)), (s|S_{X_k})^* Q = Q(S_{X_k}(s)) \] 

and 

\[ (s|S_{X_k})^*(K^j_l) = K^j_l(S_{X_k}(s)) \] 

We may assume that the section \( s \) given in Theorem 0.1 lies in \( \Gamma \Omega(N, P) \). Let \( C_{k+1} \) refer to the union of 

\[ C \cup S_{X_k+1}(s) \cup S_{D_k+1}(s) \cup S_{E_k+1}(s) \] 

\( k \geq 0 \). We assume that there exists an \( \Omega \)-regular map \( g_{k+1} \) defined on a neighborhood of \( C_{k+1} \), where \( j_0^\infty g_{k+1} = s \) holds.

**Theorem 4.1.** Let \( n \geq p \geq 2 \) and \( k \geq 1 \). Let \( N, P, \Omega(N, P) \) and \( s \in \Gamma \Omega(N, P) \) be given 

as in Theorem 0.1. Let \( C_{k+1} \) and \( g_{k+1} \) be given as above. Then there exists a homotopy 

\[ s_\lambda \in \Gamma \Omega(N, P) \] 

relative to a neighborhood of \( C_{k+1} \) with the following properties.

(4.1.1) \( s_0 = s \) and \( s_1 \in \Gamma \Omega(N, P) \).

(4.1.2) There exists an \( \Omega \)-regular map \( g_k \) which is defined on a neighborhood of \( C_k \) for \( k > 1 \) and on \( N \) for \( k = 1 \), where \( j_0^\infty g_k = s_1 \) holds.

(4.1.3) In the situation that \( \Omega(N, P) \) does not contain \( \Sigma D_j(N, P) \) or \( \Sigma E_k(N, P) \), we have that 

\[ S_{X_k+1}(S_{A_i}(N, P)) = s^{-1}(\Sigma A_i(N, P)) \] 

for \( k \geq i \).
Remark 4.2. In Theorem 0.1 we assume, in addition, that $N \cup S^{X_j}(s) \cup S^{X_k}(s) \cup S^{X_k}(s)$, $\Omega^{n-p+1,0}(N, P) = \Sigma^{n-p}(N, P) \cup \Sigma^{n-p+1,0}(N, P)$ and $U$, $U'$ be closed neighborhoods of $C_2$ with $U \subset \text{Int} U'$, where $g_2$ is defined. Since $s \in \Gamma^{n}_{t}(N, P)$, $s|N_0$ is a section of $\Gamma_{t}^{r}(N, P)$ and $g_2|U' \cap N_0$ is a fold-map. Hence, we obtain a homotopy $u_{\lambda} \in \Gamma_{t}^{r}(N, P)$ relative to a neighborhood of $U \cap N_0$ and a fold map $f_0 : N_0 \to P$ such that $s_0|N_0 = u_0$ and $u_1 = j_\infty f_0$. Then we obtain a homotopy $s_{\lambda}$ required in the case $k = 1$ of Theorem 4.1 by defining $s_{\lambda}|N_0 = u_{\lambda}$ and $s_{\lambda}|U = j_\infty g_2$.

We will prove Theorem 4.1 for $k \geq 2$ in Section 6.

Here we give a proof of Theorem 0.1 by using Theorem 4.1.

Proof of Theorem 0.1. We may assume without loss of generality that $s \in \Gamma^{n}_{0}(N, P)$. Then we may set $C_{n+1} = C$. By the assumption there exists the $\Omega$-regular map $g_{n+1} = g$ defined on a neighborhood of $C_{n+1}$. Then we can prove Theorem 0.1 by the downward induction on $k$, and by using the above argument for the final step $k = 1$. \hfill \Box

Remark 4.2. In Theorem 0.1 we assume, in addition, that $\Omega(N, P)$ does not contain any $\Sigma D_j(N, P)$ or $\Sigma E_k(N, P)$ and that $s \in \Gamma^{n}_{0}(N, P)$. Then we have that $s_{\lambda}^{-1}(\Sigma A_i(N, P)) = s^{-1}(\Sigma A_i(N, P)) = (j_\infty f)^{-1}(\Sigma A_i(N, P))$ for each $i$ with $i > 1$.

In what follows we take a number $k$ with $k > 1$. We begin by preparing several notions and results, which are necessary for the proof of Theorem 4.1. For the map $g_{k+1}$ and the closed subset $C_{k+1}$, we take an open neighborhood $U(C_{k+1})'$ of $C_{k+1}$, where $j_\infty g_{k+1} = s$. Without loss of generality we may assume that $N \setminus U(C_{k+1})'$ is nonempty. Take a smooth function $h_{C_{k+1}} : N \to [0, 1]$ such that

\[
\begin{align*}
\text{if } & \quad x \in C_{k+1}, \\
\text{if } & \quad x \in N \setminus U(C_{k+1})', \\
0 & < h_{C_{k+1}}(x) < 1 \quad \text{if } x \in U(C_{k+1})' \setminus C_{k+1}.
\end{align*}
\]

By the Sard Theorem ([H2]) there is a regular value $r$ of $h_{C_{k+1}}$ with $0 < r < 1$. Then $h_{C_{k+1}}^{-1}(r)$ is a submanifold and we set $U(C_{k+1}) = h_{C_{k+1}}^{-1}([r, 1])$. We decompose $N \setminus \text{Int} U(C_{k+1})$ into the connected components, say $L_1, \ldots, L_j, \ldots$. It suffices to prove Theorem 4.1 for each $L_j \cap \text{Int} U(C_{k+1})$. Since $\partial N = \emptyset$, we have that $N \setminus U(C_{k+1})$ has empty boundary. If $L_j$ is not compact, then Theorem 4.1 holds for $L_j \cap \text{Int} U(C_{k+1})$ by Gromov’s theorem ([G1, Theorem 4.1.1]). Therefore, it suffices to consider the special case where

\begin{enumerate}
\item[(C1)] $N \setminus \text{Int} U(C_{k+1})$ is compact, connected and nonempty,
\item[(C2)] $\partial U(C_{k+1})$ is a submanifold of dimension $n - 1$,
\item[(C3)] for the smooth function $h_{C_{k+1}} : N \to [0, 1]$ satisfying (4.1) there is a sufficiently small positive real number $\varepsilon$ with $r - 2\varepsilon > 0$ such that $r - t\varepsilon$ $(0 \leq t \leq 2)$ are all regular values of $h_{C_{k+1}}$. We have that $h_{C_{k+1}}^{-1}([r - 2\varepsilon, 1])$ is contained in $U(C_{k+1})'$. We set $U(C_{k+1})_t = h_{C_{k+1}}^{-1}([r - (2 - t)\varepsilon, 1])$. In particular, we have $U(C_{k+1})_2 = U(C_{k+1})$. Furthermore, we may assume that
\item[(C4)] $s \in \Gamma^{n}_{0}(N, P)$ and $S^{X_j}(s)$ $(j \leq k)$ are transverse to $\partial U(C_{k+1})_0$ and $\partial U(C_{k+1})_2$ on a neighborhood of $S^{X_k}(s)$.
\end{enumerate}

In what follows we choose and fix a Riemannian metric of $N$, which satisfies
Orthogonality Condition: Let $I = J$ or $\mathfrak{J}$. If $K^j_i(S^{X_k})/K^j_{i+1}(S^{X_k})$ is of positive dimension, then $K^j_i(S^{X_k})/K^j_{i+1}(S^{X_k})$ is orthogonal to $S^j_i(s)$ in $S^j_{i-1}(s)$ over $S^{X_k}(s)$ for $k \geq j \geq 1$ ($S^0(s) = N$).

Let $\nu(\overline{X}_k)$ be the normal bundle $(T(J^\infty(N,P))|_{\Sigma_\overline{X}_k})/T(\Sigma_\overline{X}_k(N,P))$ and $\nu(X_k) = \nu(\overline{X}_k)|_{\Sigma_\overline{X}_k}$. Let $j_K : K \to \nu(\overline{X}_k)$ over $\Sigma X_k(N,P)$ be the composition of the inclusion $K \to T(J^\infty(N,P))$ and the projection $T(J^\infty(N,P)) \to \nu(\overline{X}_k)$. We have the monomorphism

$$j_K \circ (s|S^{X_k})^K : K(S^{X_k}) \to K \to \nu(X_k).$$

For $s \in \Gamma^0_I(N,P)$, let $n(s,X_k)$ or simply $n(X_k)$ be the orthogonal normal bundle of $S^{X_k}(s)$ in $N$. Furthermore, we have the bundle map

$$ds|n(s,X_k) : n(s,X_k) \to \nu(X_k)$$

covering $s|S^{X_k}(s) : S^{X_k}(s) \to \Sigma X_k(N,P)$. Let $i_{n(s,X_k)} : n(s,X_k) \subset TN|_{S^{X_k}}$ denote the inclusion. We define $\Psi(s,X_k) : K(S^{X_k})|_{S^{X_k}} \to n(s,X_k) \subset TN|_{S^{X_k}}$ to be the composition

$$i_{n(s,X_k)} \circ (ds|n(s,X_k))^{-1} \circ j_K \circ (s|S^{X_k})^K \circ (K(S^{X_k})|_{S^{X_k}}) : K(S^{X_k})|_{S^{X_k}} \to \nu(X_k) \to n(s,X_k) \hookrightarrow TN|_{S^{X_k}}.$$

Let $i_{K(S^{X_k})} : K(S^{X_k}) \to TN$ be the inclusion.

**Remark 4.3.** If $f$ is an $\Omega$-regular map, then it follows from the definition of the total tangent bundle $D$ that $i_{K(S^{X_k}(j^\infty f))} = \Psi(j^\infty f, X_k)$.

In what follows let $M = S^{X_k}(s) \setminus \text{Int}(U(C_{k+1}))$. Let $\text{Mono}(K(S^{X_k})|_M, TN|_M)$ denote the subset of $\text{Hom}(K(S^{X_k})|_M, TN|_M)$, which consists of all monomorphisms $K(S^{X_k})_c \to T_c N$, $c \in M$. We denote the bundle of the local coefficients $B(\pi_j(\text{Mono}(K(S^{X_k})_c, T_c N)))$, $c \in M$, by $B(\pi_j)$, which is a covering space over $M$ with fiber $\pi_j(\text{Mono}(K(S^{X_k})_c, T_c N))$ defined in [Ste, 30.1]. From the obstruction theory due to [Ste, 36.3], it follows that the obstructions for $i_{K(S^{X_k})}|_M$ and $\Psi(s,X_k)|_M$ to be homotopic are the primary differences $d(i_{K(S^{X_k})}|_M, \Psi(s,X_k)|_M)$, which are defined in the cohomology groups with local coefficients $H^j(M, \partial M; B(\pi_j))$. We show that all of them vanish. In fact, we have that $\dim M = n - (n - p + k) = p - k < p - 1$. Since $\text{Mono}(\mathbb{R}^{n-p+1}, \mathbb{R}^n)$ is identified with $GL(n)/GL(p-1)$, it follows from [Ste, 25.6] that $\pi_j(\text{Mono}(\mathbb{R}^{n-p+1}, \mathbb{R}^n)) \cong \{0\}$ for $j < p - 1$. Hence, there exists a homotopy $\psi^M(s,X_k)_\lambda : K(S^{X_k})|_M \to TN|_M$ relative to $M \cap U(C_{k+1})$ in $\text{Mono}(K(S^{X_k})|_M, TN|_M)$ such that $\psi^M(s,X_k)_0 = i_{K(S^{X_k})}|_M$ and $\psi^M(s,X_k)_1 = \Psi(s,X_k)|_M$ by the definition of the primary difference. Let $\text{Iso}(TN|_M, TN|_M)$ denote the subspace of $\text{Hom}(TN|_M, TN|_M)$, which consists of all isomorphisms of $T_c N$, $c \in M$. The restriction map

$$r_M : \text{Iso}(TN|_M, TN|_M) \to \text{Mono}(K(S^{X_k})|_M, TN|_M)$$

defined by $r_M(h) = h|_{K(S^{X_k})_c}$, $h \in \text{Iso}(T_c N, T_c N)$, induces a structure of a fiber bundle with fiber $\text{Iso}(\mathbb{R}^{p-1}, \mathbb{R}^{p-1}) \times \text{Hom}(\mathbb{R}^{p-1}, \mathbb{R}^{n-p+1})$. By applying the covering homotopy property of the fiber bundle $r_M$ to the sections $id_{TN|_M}$ and the homotopy $\psi^M(s,X_k)_\lambda$,
we obtain a homotopy $\Psi^M(s, X_k)_\lambda : TN|_M \to TN|_M$ such that $\Psi^M(s, X_k)_0 = id_{TN|_M}$, $\Psi^M(s, X_k)|_c = id_{TN}$ for all $c \in M \cap U(C_{k+1})_1$ and $r_M \circ \Psi^M(s, X_k) = \psi^M(s, X_k)_\lambda$. We define $\Phi(s, X_k)_\lambda : TN|_M \to TN|_M$ by $\Phi(s, X_k)_\lambda = (\Psi^M(s, X_k)_\lambda)^{-1}$.

Let us fix a direct sum decomposition

\[
\begin{align*}
\nu(A_k) = \oplus_{j=1}^k \nu(J_j \subset J_{j-1}), \\
\nu(D_k) = (\oplus_{j=1}^3 \nu(3_j \subset 3_{j-1})) \oplus (\oplus_{j=5}^8 \nu(\overline{3}_j \subset \overline{3}_{j-1})), \\
\nu(E_k) = \oplus_{j=1}^7 \nu(3_j \subset 3_{j-1}) \oplus (\oplus_{j=7}^9 \nu(\overline{E}_j \subset \overline{E}_{j-1})),
\end{align*}
\]  

(4.2)

over $\Sigma X_k(N, P)$, which induces the direct sum decomposition $K = K_2 \oplus K_2' \oplus K_3 \oplus K_3'$.

Let $n(s, \overline{X}_j \subset \overline{X}_{j-1})$ be the orthogonal normal bundle of $S\overline{X}_j(s)$ in $S\overline{X}_{j-1}(s)$ and set $n(s, X_j \subset \overline{X}_{j-1}) = n(s, \overline{X}_j \subset \overline{X}_{j-1})|_{S^X_j}$ where $\overline{X}_j = I_j$, or $X_j = A_j, D_j, E_j$. Then we have the canonical direct sum decomposition such as

\[
\begin{align*}
n(s, A_k) = \oplus_{j=1}^k n(s, J_j \subset J_{j-1}), \\
n(s, D_k) = (\oplus_{j=1}^2 n(s, 3_j \subset 3_{j-1})) \oplus (\oplus_{j=5}^8 n(s, \overline{3}_j \subset \overline{3}_{j-1})), \\
n(s, E_k) = (\oplus_{j=1}^3 n(s, 3_j \subset 3_{j-1})) \oplus (\oplus_{j=7}^9 n(s, \overline{E}_j \subset \overline{E}_{j-1})).
\end{align*}
\]  

(4.3)

over $S^X_k(s)$. We can take the direct sum decompositions in (4.2) to be compatible with those in (4.3).

\section{Lemmas}

In the proof of the following lemma, $\Phi(s, X_k)_\lambda|_c$ ($c \in M$) is regarded as a linear isomorphism of $T_c N$. Let $r_0$ be a small positive real number with $r_0 < 1/10$.

\textbf{Lemma 5.1.} Let $k > 1$. Let $X_k$ be any of $A_k, D_k$ and $E_k$. Let $s \in \Gamma^r_\Omega(N, P)$ be a section satisfying the hypotheses of Theorem 4.1. Then there exists a homotopy $s_\lambda$ relative to $U(C_{k+1})_{2-3r_0}$ in $\Gamma^r_\Omega(N, P)$ with $s_0 = s$ satisfying

\begin{enumerate}
\item[(5.1.1)] for any $\lambda$, $S^{X_k}(s_\lambda) = S^{X_k}(s)$ and $\pi_p^\infty \circ s_\lambda|_{S^{X_k}(s_\lambda)} = \pi_p^\infty \circ s|_{S^{X_k}(s)}$,
\item[(5.1.2)] we have that $i_{K(S^{X_k}(s_1))}|_{S^{X_k}} = \Psi(s_1, X_k)$ and $K(S^{X_k}(s_1)) \subset n(s, X_k)_c$ for any point $c \in S^{X_k}(s_1)$.
\end{enumerate}

\textbf{Proof.} Recall the exponential map $\exp_{N,x} : T_x N \to N$ defined near $0 \in T_x N$. We write an element of $n(X_k)_c$ as $v_c$. There exists a small positive number $\delta$ such that the map

\[
e : D_{\delta}(n(X_k)_c)|_M \to N
\]

defined by $e(v_c) = \exp_{N,c}(v_c)$ is an embedding and that the images of $e$ for $X_k = A_k, D_k$ and $E_k$ do not mutually intersect, where $c \in M$ and $v_c \in D_{\delta}(n(X_k)_c)$ (note that $e|_M$ is the inclusion). Let $\rho : [0, \infty) \to \mathbb{R}$ be a decreasing smooth function such that $0 \leq \rho(t) \leq 1$, $\rho(t) = 1$ if $t \leq \delta/10$ and $\rho(t) = 0$ if $t \geq \delta$.

15
If we represent $s(x) \in \Omega(N, P)$ by a jet $j^\infty_x \sigma_x$ for a germ $\sigma_x : (N, x) \to (P, \pi_P^\infty \circ s(x))$, then we define the homotopy $s_\lambda$ of $\Gamma^r_\Omega(N, P)$ using $\Phi(s, X_k)_\lambda$ by

$$
\begin{aligned}
    s_\lambda(c(v_c)) &= j^\infty_{c(v_c)}(\sigma(c(v_c)) \circ \exp_{N,c} \circ \Phi(s, X_k)_\rho(||v_c||)_c \circ \exp_{N,c}^{-1}) & \text{if } c \in M \text{ and } ||v_c|| \leq \delta, \\
    s_\lambda(x) &= s(x) & \text{if } x \notin \text{Im}(e).
\end{aligned}
$$

Here, $\Phi(s, X_k)_\rho(||v_c||)_c$ refers to $\ell(v_c) \circ (\Phi(s, X_k)_\rho(||v_c||)_c) \circ \ell(-v_c)$, where $\ell(v)$ is the parallel translation defined by $\ell(v)(a) = a + v$. If $||v_c|| \geq \delta$, then $s_\lambda$ is well defined. Furthermore, we have that

1. $\pi_P^\infty \circ s_\lambda(x) = \pi_P^\infty \circ s(x)$,
2. $S^X_k(s_\lambda) = S^X_k(s)$,
3. if $c \in S^X_k(s)$, then we have that $n(s_1, X_k)_c \supset K(S^X_k(s_1))_c$,
4. $s_\lambda$ is transverse to $\Sigma X_k(N, P)$.

The property (5.1.2) is satisfied for $s_1$ by (5.1). □

In what follows we set $d_1(s, X_k) = (s|S^X_k)^*(d_1)$. We also choose and fix a Riemannian metric of $P$ and identify $Q(S^X_k)$ with the orthogonal complement of $\text{Im}(d_1(s, X_k))$ in $(\pi_P^\infty \circ s|S^X_k)^*(TP)$.

**Lemma 5.2.** Let $k \geq 1$. Let $X_k$ be any of $A_k$, $D_k$ and $E_k$. Let $s$ be a section of $\Gamma^r_\Omega(N, P)$ satisfying the property (5.1.2) for $s$ (in place of $s_1$) of Lemma 5.1. Then there exists a homotopy $s_\lambda$ relative to $U(C_{k+1})_{2-3r_0}$ in $\Gamma^r_\Omega(N, P)$ with $s_0 = s$ such that

1. $S^X_k(s_\lambda) = S^X_k(s)$ for any $\lambda$,
2. $\pi_P^\infty \circ s|S^X_k(s_\lambda)$ is an immersion into $P$ such that $d(\pi_P^\infty \circ s|S^X_k(s_\lambda)) : T(S^X_k(s_\lambda)) \to TP$ is equal to $(\pi_P^\infty \circ s|S^X_k(s_\lambda))^{TP} d_1(s, X_k) |T(S^X_k(s_\lambda))$, where $(\pi_P^\infty \circ s|S^X_k(s_\lambda))^{TP} : (\pi_P^\infty \circ s|S^X_k(s_\lambda))^{*(TP)} \to TP$ is the canonical induced bundle map,
3. we have that $i_{K(S^X_k(s_1))} |S^X_k = \Psi(s_1, X_k)$ and $K(S^X_k(s_1))_c \subset n(s, X_k)_c$ for any point $c \in S^X_k(s_1)$.

**Proof.** Since

$$
\begin{aligned}
    K \cap T(\Sigma^A_{k,0}(N, P)) &= \{0\} & \text{for } A_k \ (k \geq 1), \\
    K \cap T(\Sigma^D_{k,0}(N, P)) &= \{0\} & \text{for } D_k \ (k \geq 4), \\
    K \cap T(\Sigma^E_{k,0}(N, P)) &= \{0\} & \text{for } E_6 \text{ and } E_7, \\
    K \cap T(\Sigma^F_{k,0}(N, P)) &= \{0\} & \text{for } E_8,
\end{aligned}
$$

it follows that $(\pi_P^\infty \circ s)^{TP} \circ d_1(s, X_k) |T(S^X_k)$ is a monomorphism. By the Hirsch Immersion Theorem ([H1, Theorem 5.7]) there exists a homotopy of monomorphisms $m'_\lambda : T(S^X_k) \to TP$ covering a homotopy $m_\lambda : S^X_k \to P$ relative to $U(C_{k+1})_{2-4r_0}$ such that $m'_0 = (\pi_P^\infty \circ s)^{TP} \circ d_1(s, X_k)|T(S^X_k)$ and that $m_1$ is an immersion with $d(m_1) = m'_1$. Then we can extend $m'_\lambda$ to a homotopy $\widetilde{m'_\lambda} : TN|_{S^X_k} \to TP$ of homomorphisms of constant rank $p - 1$ relative to $U(C_{k+1})_{2-3r_0}$ so that $m'_0 = (\pi_P^\infty \circ s)^{TP} \circ d_1(s, X_k)$. In fact, let $m : S^X_k \times [0, 1] \to P \times [0, 1]$ and $m' : T(S^X_k) \times [0, 1] \to TP \times [0, 1]$ be the maps defined by $m(c, \lambda) = (m_\lambda(c), \lambda)$
and $m'(v, \lambda) = (m'_\lambda(v), \lambda)$ respectively. Let $m^*(m') : T(S^k_X) \times [0, 1] \to m^*(TP \times [0, 1])$ be the canonical monomorphism induced from $m'$ by $m$. Let $\mathcal{F}_1 = \text{Im}(m^*(m'))$ and $\mathcal{F}_2$ be the orthogonal complement of $\mathcal{F}_1$ in $m^*(TP \times [0, 1])$. Since $\mathcal{F}_2$ is isomorphic to $(\mathcal{F}_2|_{S^k_{x_0} \times 0}) \times [0, 1]$, we obtain a monomorphism of rank $k - 1$

$$j_{\mathcal{F}} : \text{Im}(d_1(s, X_k)|n(X_k)) \times [0, 1] \to \mathcal{F}_2 \quad \text{over} \quad S^k \times [0, 1].$$

Since $d_1(s, X_k)|(TN|_{S^k_k})$ is of constant rank $p - 1$ and induces the homomorphism of kernel rank $n - p + 1$

$$d : n(X_k) \times [0, 1] \to \text{Im}(d_1(s, X_k)|n(X_k)) \times [0, 1] \xrightarrow{\tilde{j}_{\mathcal{F}}} \mathcal{F}_2,$$

we define $\tilde{m}'$ to be the composition

$$TN|_{S^k_k} \times [0, 1] \cong (T(S^k_X) \oplus n(X_k)) \times [0, 1] \xrightarrow{m^*(m') \oplus d \mathcal{F}_1 \oplus \mathcal{F}_2} \text{Im}(m^*(m')) \oplus \text{Cok}(m^*(m')) \cong m^*(TP \times [0, 1]) \xrightarrow{m^{TP \times [0, 1]}} TP \times [0, 1],$$

where $m^{TP \times [0, 1]} : m^*(TP \times [0, 1]) \to TP \times [0, 1]$ is the canonical bundle map. We define $\tilde{m}'_\lambda$ to be $(\tilde{m}'_\lambda(v), \lambda) = \tilde{m}'(v, \lambda)$.

Next we construct a homotopy $s_\lambda : N \to \Omega(N, P)$ from $\tilde{m}'_\lambda$. We write, by $\Sigma^{n-p+1}(N, P)'$, the submanifold of $J^1(N, P) = J^1(TN, TP)$, which corresponds to $\Sigma^{n-p+1}(N, P)$ to distinguish them. Then $\pi'_\infty|\Sigma X_k(N, P) : \Sigma X_k(N, P) \to \Sigma^{n-p+1}(N, P)'$ becomes a fiber bundle. We regard $\tilde{m}'_\lambda$ as a homotopy $S^k \to \Sigma^{n-p+1}(N, P)'$. By the covering homotopy property to $s|S^k_X$ and $\tilde{m}'_\lambda$, we obtain a homotopy $s'_\lambda : S^k \to \Sigma X_k(N, P)$ covering $\tilde{m}'_\lambda$ relative to $U(C_{k+1})_{2-3r_0}$ such that $s'_0 = s|S^k_X$.

By the transversality of $s$, we regard small tubular neighborhoods of $S^k_X(s)$ and $\Sigma X_k(N, P)$ as vector bundles and that $s$ induces a bundle map between them when restricted. Applying the homotopy extension property to $s$ and $s'_\lambda$, we first extend $s'_\lambda$ to a homotopy defined on this tubular neighborhood of $S^k_X$ and then extend it to a required homotopy $s_\lambda \in \Gamma^T_{1, 1}((N, P)$, which satisfies $s_0 = s$, $s_\lambda|S^k_X = s'_\lambda$ and $s_\lambda|U(C_{k+1})_{2-3r_0} = s|U(C_{k+1})_{2-3r_0}$. This is a standard argument in topology and the details are left to the reader.

Here we give two lemmas necessary for the proof of Theorem 4.1. Let $\pi : E \to S$ be a smooth $(n - p + k)$-dimensional vector bundle with a metric over a $(p - k)$-dimensional manifold, where $S$ is identified with the zero-section. Then we can identify $\exp_E|D_\varepsilon(E)$ with $id_{D_\varepsilon(E)}$.

**Lemma 5.3.** Let $\pi : E \to S$ be given as above. Let $f_i : E \to P$ $(i = 1, 2)$ be $\Omega$-regular maps which have only singularities of types $A_j$, $D_j$ and $E_j$ with $j \leq k$ and those of type $X_k$, where $X_k$ is one of $A_k$, $D_k$ and $E_k$, exactly on $S$ such that, for every point $c \in S$,

1. $f_1|S = f_2|S$, which are immersions,
2. $S = S^k_{X_k}(j \circ f_1) = S^k_{X_k}(j \circ f_2)$,
3. $K(S^k_{X_k}(j \circ f_1))_c = K(S^k_{X_k}(j \circ f_2))_c$ are tangent to $\pi^{-1}(c)$,
(5.3.4) \( T_c(S_{I-j}^j(j^\infty f_1)) = T_c(S_{I-j}^j(j^\infty f_2)), \) \((j^\infty f_1|S)^*P_j)_c = ((j^\infty f_2|S)^*P_j)_c \) and
\[(j^\infty f_1|S)^*(d_{j+1}^I \circ d(j^\infty f_1))_c = (j^\infty f_2|S)^*(d_{j+1}^I \circ d(j^\infty f_2))_c \]
for each number \( j \) and \( I = J \) or \( \bar{J} \),

(5.3.5) if \( X_k = D_k \), then for each number \( j \geq 3 \),
\[(j^\infty f_1|S)^*(r_j)_c = (j^\infty f_2|S)^*(r_j)_c \quad \text{and} \quad (j^\infty f_1|S)^*(d(r_j))_c = (j^\infty f_2|S)^*(d(r_j))_c, \]

(5.3.6) if \( X_k = E_7 \), then
\[(j^\infty f_1|S)^*(d(\tilde{d}_3^I \bigcirc 4 K_3^3))_c = (j^\infty f_2|S)^*(d(\tilde{d}_3^I \bigcirc 4 K_3^3))_c, \]

(5.3.7) if \( X_k = E_8 \), then
\[(j^\infty f_1|S)^*(d(\tilde{d}_3^I \bigcirc 3 K_3^1 \bigcirc (K_2^3/K_3^3)))_c = (j^\infty f_2|S)^*(d(\tilde{d}_3^I \bigcirc 3 K_3^1 \bigcirc (K_2^3/K_3^3)))_c, \]
\[(j^\infty f_1|S)^*(\tilde{d}_3^I \bigcirc 5 K_3^1)_c = (j^\infty f_2|S)^*(\tilde{d}_3^I \bigcirc 5 K_3^1)_c. \]

Let \( \eta : S \rightarrow [0,1] \) be any smooth function. Let \( \varepsilon : S \rightarrow \mathbb{R} \) be a sufficiently small positive smooth function. For any \( c \in S \), \( v_c \in \pi^{-1}(c) \) with \( \|v_c\| \leq \varepsilon(c) \), let \( F^0(v_c) \) denote \( \exp_{P,f_1(c)}((1 - \eta(c)) \exp_{P,f_1(c)}(f_1(v_c)) + \eta(c) \exp_{P,f_2(c)}(f_2(v_c))) \).

Then the map \( F^0 : D_\varepsilon(E) \rightarrow P \) is a well-defined \( \Omega \)-regular map such that
\[
(1) \quad F^0|S = f_1|S = f_2|S,
\]
\[
(2) \quad S = S^{X_k}(j^\infty f^0),
\]
\[
(3) \quad K(S^{X_k}(j^\infty f^0)) = K(S^{X_k}(j^\infty f_1)) = K(S^{X_k}(j^\infty f_2)) \text{ are tangent to } \pi^{-1}(c),
\]
\[
(4) \quad T_c(S_{I-j}^j(j^\infty f^0)) = T_c(S_{I-j}^j(j^\infty f_1)), \quad ((j^\infty f^0|S)^*P_j)_c = ((j^\infty f_1|S)^*P_j)_c \text{ and}
\]
\[(j^\infty f^0)^*(d_{j+1}^I \circ d(j^\infty f^0))_c = (j^\infty f_1)^*(d_{j+1}^I \circ d(j^\infty f_1))_c = (j^\infty f_2)^*(d_{j+1}^I \circ d(j^\infty f_2))_c, \]
for each number \( j \) and \( I = J \) or \( \bar{J} \),

(5) if \( X_k = D_k \), then, for each number \( j \geq 3 \),
\[(j^\infty f^0)^*(r_j)_c = (j^\infty f_1)^*(r_j)_c = (j^\infty f_2)^*(r_j)_c, \quad (j^\infty f^0)^*(d(r_j))_c = (j^\infty f_1)^*(d(r_j))_c = (j^\infty f_2)^*(d(r_j))_c, \]

(6) if \( X_k = E_7 \), then
\[(j^\infty f^0)^*(d(\tilde{d}_3^I \bigcirc 4 K_3^3))_c = (j^\infty f_1)^*(d(\tilde{d}_3^I \bigcirc 4 K_3^3))_c = (j^\infty f_2)^*(d(\tilde{d}_3^I \bigcirc 4 K_3^3))_c, \]

(7) if \( X_k = E_8 \), then
\[
(j^\infty f^0)^*(d(\tilde{d}_3^I \bigcirc 3 K_3^1 \bigcirc (K_2^3/K_3^3)))_c = (j^\infty f_1)^*(d(\tilde{d}_3^I \bigcirc 3 K_3^1 \bigcirc (K_2^3/K_3^3)))_c = (j^\infty f_2)^*(d(\tilde{d}_3^I \bigcirc 3 K_3^1 \bigcirc (K_2^3/K_3^3)))_c, \]
\[
(j^\infty f^0)^*(\tilde{d}_3^I \bigcirc 5 K_3^1)_c = (j^\infty f_1)^*(\tilde{d}_3^I \bigcirc 5 K_3^1)_c = (j^\infty f_2)^*(\tilde{d}_3^I \bigcirc 5 K_3^1)_c. \]
Proof. Let us take a Riemannian metric on $E$ which is compatible with the metric of the vector bundle $E$ over $S$. In particular, $S$ is a Riemannian submanifold of $E$. Furthermore, take a Riemannian metric on $P$ such that $f(S) \cap P$ is a Riemannian submanifold of $P$ around $f(c)$. Then the local coordinates of $\exp_{N(c)}(K_c)$ and $\exp_{P,f(c)}(Q_c)$ are independent of the coordinates of $S$, where $Q_c$ is regarded as a line subspace of $T_{f(c)}P$.

Since $j^\infty f_1$ are transverse to $\Sigma X_k(E, P)$, we have $H(j^\infty f_1, I_j \subset I_{j-1}) = H(j^\infty f_2, I_j \subset I_{j-1})$ for $I = J$ or $\emptyset$. Furthermore, it follows similarly that $H(j^\infty f_1, \bar{D}_{j+1} \subset \bar{D}_j) = H(j^\infty f_2, \bar{D}_{j+1} \subset \bar{D}_j)$ for $X_k = D_k$ and $H(j^\infty f_1, E_{j+1} \subset E_j) = H(j^\infty f_2, E_{j+1} \subset E_j)$ for $X_k = E_k$.

We may consider $\eta(c)$ as a constant when dealing with higher intrinsic derivatives in the lemma by the identification (1.2) and the property of the total tangent bundle $\mathbf{D}$ given in the beginning of Section 2. Then the assertions (1)-(7) follow from the assumptions and the properties of $\Sigma X_k(N, P)$.

Let $\varepsilon$ be sufficiently small. Then since $\Omega(n, p)$ is open and $j^\infty f^0(S) \subset \Sigma X_k(N, P)$, $f^0$ is an $\Omega$-regular map. Since $f^0$ is transverse to $\Sigma X_k(N, P)$, we have $S^{X_k}(j^\infty f^0) = S$. 

The proof of the following lemma is elementary, and so is left to the reader.

Lemma 5.4. Let $\pi : E \to S$ be given as above, and let $(\Omega, \Sigma)$ be a pair of a smooth manifold and its submanifold of codimension $n - p + k$. Let $\varepsilon : S \to \mathbf{R}$ be a sufficiently small positive smooth function. Let $h : D_\varepsilon(E) \to (\Omega, \Sigma)$ be a smooth map such that $h : D_\varepsilon(E) \to (\Omega, \Sigma)$ is transverse to $\Sigma$. Then there exists a smooth homotopy $h_0 : (D_\varepsilon(E), S) \to (\Omega, \Sigma)$ between $h$ and $\exp_\Omega \circ dh|D_\varepsilon(E)$ such that

1. $h_0|S = h_0|S$, $S = h_\lambda^{-1}(\Sigma) = h_0^{-1}(\Sigma)$ for any $\lambda$,
2. $h_\lambda$ is smooth and is transverse to $\Sigma$ for any $\lambda$,
3. $h_0 = h$ and $h_1(v_c) = \exp_{\Omega,h(c)} \circ dh(v_c)$ for $c \in S$ and $v_c \in D_\varepsilon(E_c)$.

6 Proof of Theorem 4.1

For the normal bundles $n(X_k) (= n(s, X_k))$ and $Q (= Q(S^{X_k}))$ over $S^{X_k}(s)$, we recall that $\text{Hom}(\Sigma^\ell \cap \Sigma^j n(X_k), Q)|_{S^{X_k}(s)}$ is identified with the set of polynomials of degree $\leq \ell$ having the constant 0 with coefficients depending on a point of $S^{X_k}(s)$ (see [Mats, Ch. III, Section 2]). For a point $c \in S^{X_k}(s)$, take an open neighborhood $U$ around $c$ such that $n(X_k)|_U$ and $Q|_U$ are the trivial bundles, say $U \times \mathbf{R}^{n-p+k}$ and $U \times \mathbf{R}$ respectively, where $\mathbf{R}^{n-p+k}$ has coordinates $(x_1, \ldots, x_{n-p+k})$ and $\mathbf{R}$ has $y$. Then an element of $\text{Hom}(\mathbb{O}^j n(X_k), Q)|_U$ is identified with a polynomial $y(c) = \sum_{|\omega| = j} a^\omega(c) x_1^{\omega_1} x_2^{\omega_2} \cdots x_{n-p+k}^{\omega_{n-p+k}}$, $c \in U$, where $\omega = (\omega_1, \omega_2, \ldots, \omega_{n-p+k})$, $\omega_i \geq 0$ ($i = 1, \ldots, n-p+k$), $|\omega| = \omega_1 + \cdots + \omega_{n-p+k}$ and $a^\omega(c)$ is a real number. If $a^\omega(c)$ are smooth functions of $c$, then $\{a^\omega(c)\}$ defines a smooth section of $\text{Hom}(\mathbb{O}^j n(X_k), Q)|_{S^{X_k}(s)}$ over $U$.

We first introduce several homomorphisms between vector bundles over $S^{X_k}(s)$, which are used for the construction of the required $\Omega$-regular map in Theorem 4.1.

Let $s \in \Gamma_\Omega(N, P)$. By deforming $s$ if necessary, we may assume without loss of generality that $s$ satisfies (5.1.2) of Lemma 5.1 and (5.2.2) of Lemma 5.2, where $s_1$ is replaced by $s$. 

19
In the following, let \( K = K(S^X_k) \), \( Q = Q(S^X_k) \), \( K_j \) \((j \geq 2)\) refer to \( K^j(S^X_k) \) for \( X_k = A_k \) and \( K^3_j(S^X_k) \) for \( X_k = D_k \) or \( E_k \), and let \( L = (s|S^X_k)^*(L) \). We now describe the isomorphisms

\[
\begin{align*}
(s|S^{I_j})^*(d_{j+1} \circ ds|n(s, I_j \subset I_{j-1})) & \quad \text{for } I = J \text{ or } J,
(s|S^{D_{j+1}})^*(d(r_{j-1} \circ ds|n(s, D_{j+1} \subset D_j)),
(s|S^{E_7})^*(d(\bar{d}_{1|} \circ 4 \, K_3^3) \circ ds|n(s, E_7 \subset E_6)),
(s|S^{E_8})^*(d(\bar{d}_{1|} \circ 3 \, K_3^3 \circ (K_2^3/K_3^3)) \circ ds|n(s, E_8 \subset E_7)),
\end{align*}
\]

which are induced by \( s|S^{I_j} \), \( s|S^{D_{j+1}} \), \( s|S^{E_7} \) and \( s|S^{E_8} \) respectively. They yield the decompositions of the target bundles derived from \( K \supset K^j_2 \supset L \) and \( K^3_j \supset K^3_2 \), and hence the decomposition of the normal bundles as follows.

For \( A_k \),

\[
\begin{align*}
n(s, J_1 \subset J_0) &= K/K_2 \oplus T^A_2 \rightarrow \text{Hom}(K/K_2 \oplus K_2, Q) \text{ by (2.1),}
n(s, J_j \subset J_{j-1}) &= T^A_{j+1} \rightarrow \text{Hom}(O^j K_2, Q) \ (2 \leq j \leq k-1) \text{ by (2.2),}
n(s, J_k \subset J_{k-1}) &= K_2 \rightarrow \text{Hom}(O^k K_2, Q) \text{ by (5), (8) in Section 2 and (2.2),}
\end{align*}
\]

over \( S^A_k \), where (i) \( K/K_2 \) refers to the orthogonal complements of \( K_2 \) in \( K \), and (ii) \( T^A_2 \) refers to the 1-subbundle of \( n(s, J_1 \subset J_0) \), which corresponds to the direct summand in the right-hand side.

For \( D_k \),

\[
\begin{align*}
n(s, J_1 \subset J_0) &= K/K_2 \oplus T^D_2 \rightarrow \text{Hom}(K/K_2 \oplus K_2, Q) \ (k = 4) \text{ by (2.1),}
n(s, J_1 \subset J_0) &= K/K_2 \oplus T^2_{2,1} \oplus T^2_{2,2} \rightarrow \text{Hom}(K/K_2 \oplus K_2/L \oplus L, Q) \ (k \geq 5) \text{ by (2.1),}
n(s, J_1 \subset J_0) &= K_2 \oplus T^3_2 \rightarrow \text{Hom}(P^D_2, Q) \ (k = 4) \text{ by (2.3), (D-i), (D-iv-3),}
n(s, J_2 \subset J_1) &= K_2/L \oplus L \oplus T^2_{2k} \rightarrow \text{Hom}(P^D, Q) \ (k \geq 5) \text{ by (2.3), (D-ii), (D-iv-3),}
n(s, J_3 \subset J_2) &= T^2_{2k} \rightarrow \text{Hom}(O^{j-2} L, Q) \ (5 \leq j \leq k) \text{ by (3.2),}
\end{align*}
\]

over \( S^D_k \), where

(i) \( K/K_2 \) and \( K_2/L \) refer to the orthogonal complements of \( K_2 \) in \( K \), \( L \) in \( K_2 \) respectively,

(ii) \( T^D_2, T^D_{2,1}, T^D_{2,2} \) and \( T^D_3 \) refer to the subbundles of the normal bundles concerned which corresponds to the direct summands in the right-hand side respectively (dim\( T^D_2 = 2 \) and \( \dim \text{dim}^D_{T_3} = 1 \)),

(iii) \( P^D_2 = O^2 K_2 = V_1 \oplus V_2 \) and \( P^D = L \ominus K_2/L \oplus O^2 K_2/L \oplus O^2 L \). Here, \( V_2 \) is the 1-subbundle of \( O^2 K_2 \) which consists of all elements \( v \) with \((s|S^D_2)^*(\bar{d}_{3|})|(K_2 \ominus \{v\}) = 0\)

and \( V_1 \) is the orthogonal complement of \( V_2 \) in \( O^2 K_2 \).
For $E_k$,

$$\text{n}(s, 1_1 \subset J_0) = K/K_2 \oplus T_{E}^{E_1} \oplus T_{E}^{E_2} \to \text{Hom}(K/K_2 \oplus K_2/K_3 \oplus K_3, Q) \quad \text{by (2.1)},$$

$$\text{n}(s, 3_2 \subset J_1) = K_2/K_3 \oplus T_{E}^{E_1} \oplus T_{E}^{E_2} \to \text{Hom}(\bigcirc^2 K_2/K_3 \oplus \bigcirc^2 K_3 \oplus K_3 \bigcirc K_2/K_3, Q) \quad \text{by (2.3), (2.4)},$$

$$\text{n}(s, E_6 \subset J_2) = K_3 \oplus T_{E}^{E_6} \to \text{Hom}(\mathcal{P}^E, Q) \quad \text{for } E_6 \text{ by (2.5), (E-i), (E-iii-a)},$$

$$\text{n}(s, 3_3 \subset J_3) = T_{E}^{E_7} \oplus K_3 \to \text{Hom}(\mathcal{P}^E, Q) \quad \text{for } E_7 \text{ by (2.5), (E-i), (E-iii-b)},$$

$$\text{n}(s, E_7 \subset \mathcal{E}_6) = T_{E}^{E_7} \to \text{Hom}(\bigcirc^4 K_3, Q) \quad \text{for } E_7 \text{ by (3.3), (E-iii-b)},$$

$$\text{n}(s, E_7 \subset \mathcal{E}_6)|_{S^E} = K_4|_{S^E} \to \text{Hom}(\bigcirc^4 K_4, Q)|_{S^E} \quad \text{for } E_8 \text{ by (E-iii-c).},$$

$$\text{n}(s, E_8 \subset \mathcal{E}_7) = T_{E}^{E_8} \to \text{Hom}(\bigcirc^3 K_4 \oplus K_2/K_4, Q) \quad \text{for } E_8 \text{ by (3E-2), (E-iii)},$$

(6.4)

over $S^{E_k}$, where

(i) $\text{n}(s, E_6 \subset J_2) = \text{n}(s, 3_3 \subset J_2)|_{S^E}$ and $\mathcal{P}^{E} = \bigcirc^3 K_2 \oplus \bigcirc^2 K_3 \oplus K_2/K_3$,

(ii) $K/K_2$, $K_2/L$, and $K_2/K_3$ or $K_2/K_4$ refer to the orthogonal complements of $K_2$ in $K$, $L$ in $K_2$, and $K_3$ or $K_4$ in $K_2$ respectively,

(iii) $T_{j+1, j}^{E}, T_{j+2, j}^{E}, T_{j+3, j}^{E}, T_{j+4, j}^{E}, T_{j+5, j}^{E}$ and $T_{j+6, j}^{E}$ refer to 1-subbundle of the normal bundles concerned, which correspond to the direct summands in the right-hand side respectively.

The isomorphisms in (6.2), (6.3) and (6.4) canonically induce the following homomorphisms over $S^{E_k}(s)$ respectively.

For $A_k$,

$$\tilde{d}^A_k(s) : \bigcirc^2 K/K_2 \oplus K_2 \bigcirc T_{E}^{A} \to Q,$$

(iii) $\tilde{d}^A_k(s) : \bigcirc^2 K_2 \bigcirc T_{E}^{A} \to Q$ ($2 \leq j \leq k - 1$),

$$\tilde{d}^A_k(s) : \bigcirc^{k+1} K_2 \to Q.$$

For $D_k$,

$$\tilde{d}^{D_4}_k(s) : \bigcirc^2 K/K_2 \oplus K_2 \bigcirc T_{E}^{D_4} \to Q \quad (k = 4),$$

$$\tilde{d}^{D_5}_k(s) : \bigcirc^2 K/K_2 \bigcirc T_{E}^{D_5} \to Q \quad (k \geq 5),$$

$$\tilde{d}^{D_7}_k(s) : V_1 \bigcirc K_2 \bigcirc T_{E}^{D_7} \to Q \quad (k = 4),$$

$$\tilde{d}^{D_8}(s) : \bigcirc^5 K_4 \bigcirc T_{E}^{D_8} \to Q \quad (5 \leq j \leq k).$$

Furthermore, the homomorphism $r_{j-1}(s)$ induces the homomorphism by $s|S^{D_k}$

$$r_{j-1}(s) : \bigcirc^{j-1} L \to Q \quad (5 \leq j \leq k) \quad \text{over } S^{D_k}(s) \quad (k \geq 5).$$

For $E_k$,

$$\tilde{d}^{E}_2(s) : \bigcirc^2 K/K_2 \oplus K_2/K_3 \bigcirc T_{E}^{E_1} \oplus K_2 \bigcirc T_{E}^{E_2} \to Q,$$

$$\tilde{d}^{E}_3(s) : \bigcirc^3 K_3 \oplus \bigcirc^2 K_3 \bigcirc T_{E}^{E_3} \oplus K_2/K_3 \bigcirc T_{E}^{E_4} \to Q,$$

$$\tilde{d}^{E}_4(s) : \bigcirc^4 K_3 \oplus \bigcirc^2 K_3 \bigcirc T_{E}^{E_4} \oplus K_2/K_3 \bigcirc T_{E}^{E_5} \to Q \quad \text{for } E_6,$$

$$\tilde{d}^{E}_4(s) : \bigcirc^4 K_3 \bigcirc T_{E}^{E_4} \oplus \bigcirc^2 K_3 \bigcirc T_{E}^{E_5} \to Q \quad \text{for } E_7,$$

$$\tilde{d}^{E}_4(s) : \bigcirc^4 K_3 \bigcirc T_{E}^{E_4} \oplus \bigcirc^2 K_3 \bigcirc K_2/K_3 \bigcirc T_{E}^{E_5} \to Q \quad \text{for } E_8,$$

$$\tilde{d}^{E}_5(s) : \bigcirc^5 K_4 \bigcirc T_{E}^{E_5} \to Q \quad \text{for } E_6,$$

$$\tilde{d}^{E}_5(s) : \bigcirc^5 K_4 \bigcirc T_{E}^{E_5} \bigcirc K_2/K_4 \bigcirc T_{E}^{E_6} \to Q \quad \text{for } E_7,$$

$$\tilde{d}^{E}_5(s) : \bigcirc^5 K_4 \bigcirc T_{E}^{E_5} \bigcirc K_2/K_4 \bigcirc T_{E}^{E_6} \to Q \quad \text{for } E_8.$$
where \( \tilde{d}_5^E(s) \big| O^5 K_4 \) comes from \( (s|S^E)^*(\tilde{d}_5^3 \big| O^5 K_3^3) \).

We define the sections of \( \text{Hom}(\bigoplus_{j=1}^5 O^j n(X_k), Q) \):

\[
\begin{align*}
q^{A_k}(s) &= \tilde{d}_2^A(s) + \Sigma_{k=3}^{\infty} \tilde{d}_k^A(s) + \tilde{d}_{k+1}^A(s) & \text{over } S^{A_k}(s), \\
q^{D_4}(s) &= \tilde{d}_2^D(s) + \tilde{d}_3^D(s) + \tilde{d}_4^D(s) & \text{over } S^{D_4}(s), \\
q^{D_k}(s) &= \tilde{d}_2^D(s) + \tilde{d}_3^D(s) + \Sigma_{j=5}^{\infty} \tilde{d}(r_{j-2})(s) + r_{k-1}(s) \quad (k \geq 5) & \text{over } S^{D_k}(s), \\
q^{E_6}(s) &= \tilde{d}_2^E(s) + \tilde{d}_3^E(s) + \tilde{d}_4^E(s) & \text{over } S^{E_6}(s), \\
q^{E_7}(s) &= \tilde{d}_2^E(s) + \tilde{d}_3^E(s) + \tilde{d}_4^E(s) & \text{over } S^{E_7}(s), \\
q^{E_8}(s) &= \tilde{d}_2^E(s) + \tilde{d}_3^E(s) + \tilde{d}_4^E(s) + \tilde{d}_5^E(s) & \text{over } S^{E_8}(s).
\end{align*}
\]

Then we obtain the smooth fiber map

\[
(\pi_P^\infty \circ s|S^X_K)^TP \circ (d_1(s, X_k)|n(s, X_k) + q^{X_k}(s)) : n(s, X_k) \rightarrow (\pi_P^\infty \circ s|S^X_K)^*(TP) \rightarrow TP \tag{6.5}
\]
covering the immersion \( \pi_P^\infty \circ s|S^X_K : S^{X_k}(s) \rightarrow P \).

**Remark 6.1.** We explain what \( V_2 \) is and how the normal form for \( D_4 \) in Introduction is induced from \( \tilde{d}_3^{D_4}(s) \). For \( c \in S^{D_4}(s) \) we set \( \tilde{d}_3 = \tilde{d}_3^{D_4}(s) \big| O^3 K_{2,c} = a_0u^3 + a_1u^2\ell + a_2u\ell^2 + a_3\ell^3 \). A generator of \( \text{Hom}((V_2)_c, Q_c) \) is a nonsingular quadratic form of \( \text{Hom}(O^2 K_{2,c}, Q_c) \).

Suppose to the contrary. Then there are coordinates \( (u, \ell) \) around \( c \) such that \( (V_2)_c \) is generated by \( \partial/\partial u \bigcup \partial/\partial \ell = \partial^2 / \partial u^2 \). It follows from \( (s|S^{D_4})^*(\tilde{d}_3^3)(K_2 \bigcap V_2) = 0 \) that the coefficients of \( u^3 \) and \( u^2\ell \) in the polynomial \( \tilde{d}_3 \) vanish. Namely, \( \tilde{d}_3 \) is written as \( a_2u^2\ell + a_3\ell^3 = (a_2u + a_3\ell)\ell^2 \). This is of type \( S_5 \) or \( S_6 \). This is a contradiction.

Suppose that \( \partial^2 / \partial u^2 \pm \partial^2 / \partial \ell^2 \) is the generator of \( (V_2)_c \). Then \( (s|S^{D_4})^*(\tilde{d}_3^3)(K_2 \bigcap V_2) = 0 \) implies that

\[
\frac{\partial^3}{\partial u^3} \pm \frac{\partial^3}{\partial u\partial\ell^2} \tilde{d}_3 = 0 \quad \text{and} \quad \frac{\partial^3}{\partial u^2\partial\ell} \pm \frac{\partial^3}{\partial \ell^3} \tilde{d}_3 = 0.
\]

Then we have \( 3a_0 \pm a_2 = 0 \) and \( a_1 \pm 3a_3 = 0 \). Hence, if \( a_0 = 0 \), then we have \( \tilde{d}_3 = \mp 3a_3u^2\ell + a_3\ell^3 \) \( (a_3 \neq 0) \). If \( a_0 \neq 0 \), then we may assume \( a_0 = 1 \) and

\[
\tilde{d}_3 = u^3 \mp 3a_3u^2\ell \mp 3u\ell^2 + a_3\ell^3.
\]

Let \( k \) be a real number such that \( k^3 \mp 3a_3k^2 \mp 3k + a_3 = 0 \), and let \( u = v + k\ell \). Then we have

\[
u^3 \mp 3a_3u^2\ell \mp 3u\ell^2 + a_3\ell^3 = v^2 + 3(k \mp a_3)v\ell + 3(k^2 \mp 2a_3k \mp 1)(\ell^2).
\]

If we set \( A = (k^2 \mp 2a_3k \mp 1) \), then \( A \neq 0 \) and \( 4A \neq 3(k \mp a_3)^2 \). In fact, if \( A \neq 0 \), then we have

\[
v^2 + 3(k \mp a_3)v\ell + 3A\ell^2 = (1 - \frac{3(k \mp a_3)^2}{4A})v^2 + 3A(\ell + \frac{k \mp a_3}{2A}v)^2.
\]

If \( 4A = 3(k \mp a_3)^2 \), then \( \tilde{d}_3 \) is of type \( S_5 \). Otherwise this is of type \( S_5^\pm \). If \( A = 0 \), then \( a_3k^2 \mp 2k \mp a_3 = 0 \), and hence \( k = 0 \) or \( a_3^2 = 1 \). Since \( A = 0 \), the case \( k = 0 \) does not occur. If \( a_3^2 = 1 \), then \( \tilde{d}_3 = (u \mp \ell)^3 \), which is of type \( S_E \).
Proof of Theorem 4.1. By Lemmas 5.1 and 5.2 we may assume that γ for some small positive function δ.

\( \text{Im}(5.2.2), \) where δ is a sufficiently small positive smooth function defined on \( \Sigma X_k(N, P) \) for \( X_k = A_k, \) \( D_k \) and \( E_k \) such that

1. \( \delta \circ s \vert (S^{X_k} \setminus \text{Int}U(C_{k+1})_2) \) is constant,
2. \( \exp_N(D_{\delta \circ s}(n(X_k))) \setminus \text{Int}U(C_{k+1})_2 \) for \( X_k = A_k, \) \( D_k \) and \( E_k \) do not intersect mutually.

This is a tubular neighborhood of \( S^{X_k}'s. \)

It is enough for the proof of Theorem 4.1 except for (4.1.3) to prove the following assertion:

(A) There exists a homotopy \( H_\lambda \) relative to \( U(C_{k+1})_{2-r_0} \) in \( \Gamma^\omega_{2}(N, P) \) with \( H_0 = s \) satisfying the following for each \( X_k = A_k, D_k \) and \( E_k. \)

1. \( S^{X_k}(H_\lambda) = S^{X_k} \) for any \( \lambda. \)
2. We have an Ω-regular map \( G \) defined on a neighborhood of \( U(C_{k+1})_{2-r_0} \cup E(k) \) to \( P \) such that \( j^\infty G = H_1 \) on \( U(C_{k+1})_{2-r_0} \cup E(k). \)

By the Riemannian metric on \( P, \) we identify \( Q \) with the orthogonal line bundle of \( \text{Im}(d_1(s, X_k)) \) in \( (\pi^\infty_P \circ s)(S^{X_k})^*(TP). \) Then \( \exp_P \circ (\pi^\infty_P \circ s)(S^{X_k})^TP \vert D_\gamma(Q) \) is an immersion for some small positive function γ. In the proof we express a point of \( E(k) \) as \( v_c, \) where \( c \in S^{X_k}, \) \( v_c \in n(X_k)c \) and \( \|v_c\| \leq \delta(s(c)). \) In the proof we say that a smooth homotopy

\[
 k_\lambda : (E(k), \partial E(k)) \to (\Omega(N, P), \Omega(N, P) \setminus \Sigma X_k(N, P))
\]

has the property (C) if it satisfies that for any \( \lambda
\]

\[
 (C-1) k_\lambda^{-1}(\Sigma X_k(N, P)) = S^{X_k}, \text{ and } \pi^\infty_P \circ k_\lambda \vert S^{X_k} = \pi^\infty_P \circ k_0 \vert S^{X_k} \text{ and,}
\]

\[
 (C-2) k_\lambda \text{ is smooth and transverse to } \Sigma X_k(N, P).
\]

Recall the fiber map \( d_1(s, X_k)(n(X_k) + q^{X_k}(s)) \) over \( S^{X_k} \) in (6.5). If we choose δ sufficiently small compared with γ, then we can define the Ω-regular map \( g_0 : E(k) \to P \) by

\[
g_0(v_c) = \exp_P, \pi^\infty_P \circ s(c) \circ (\pi^\infty_P \circ s)(S^{X_k})^TP \circ (d_1(s, X_k)(n(X_k) + q^{X_k}(s)))c \circ \exp^{-1}_{N,c}(v_c) \tag{6.6}
\]

for each \( X_k = A_k, D_k \) and \( E_k. \) It follows from Remark 3.2 that \( g_0 \) has each point \( c \in S^{X_k} \) as the singularity of type \( X_k \) and vice versa. Now we need to modify \( g_0 \) by using Lemma 5.3 so that \( g_0 \) is compatible with \( g_{k+1}. \) Let \( \eta : S^{X_k} \to \mathbb{R} \) be a smooth function such that

1. \( 0 \leq \eta(c) \leq 1 \) for \( c \in S^{X_k}, \)
2. \( \eta(c) = 0 \) for \( c \in S^{X_k} \cap U(C_{k+1})_{2-3r_0}, \)
3. \( \eta(c) = 1 \) for \( c \in S^{X_k} \setminus U(C_{k+1})_{2-4r_0}. \)

Then consider the map \( G : U(C_{k+1})_{2-3r_0} \cup E(k) \to P \) defined by

\[
 \begin{cases} 
 G(x) = g_{k+1}(x) & \text{if } x \in U(C_{k+1})_{2-3r_0}, \\
 G(v_c) = (1 - \eta(c))g_{k+1}(v_c) + \eta(c)g_0(v_c) & \text{if } v_c \in E(k).
 \end{cases}
\]

It follows from Lemma 5.3 that \( G \) is an Ω-regular map defined on \( U(C_{k+1})_{2-3r_0} \cup E(k), \) that \( G \vert E(k) \) has the singularities of type \( X_k \) exactly on \( S^{X_k}, \) and that for any \( c \in
The properties (1)-(7) of Lemma 5.3 hold for $G$. Furthermore, we note that if $c \in E(k) \cap U(C_{k+1})_{2-3r_0}$, then $G(v_c) = g_{k+1}(v_c)$.

Set $\exp_{\Omega} = \exp_{\Omega(N,P)}$ for short. Let $h_1^c, h_0^c : (E(k), S^{X_k}) \to (\Omega(N, P), \Sigma X_k(N, P))$ be the maps defined by

$$h_1^c(v_c) = \exp_{\Omega,s(c)} \circ d_s \circ (\exp_N v_c)^{-1}(v_c),$$
$$h_0^c(v_c) = \exp_{\Omega, j^\infty G(c)} \circ d_c (j^\infty G) \circ (\exp_N v_c)^{-1}(v_c).$$

By applying Lemma 5.4 to the section $s$ and $h_1^c$, we first obtain a homotopy $h_1^\lambda \in \Gamma_{h_1}^\Omega(E(k), P)$ between $h_0^c = s$ and $h_1^c$ on $E(k)$ satisfying the properties (1), (2) and (3) of Lemma 5.4. Similarly, we obtain a homotopy $h_2^\lambda \in \Gamma_{h_2}^\Omega(E(k), P)$ between $h_0^c = s$ and $h_1^c = j^\infty G$ on $E(k)$ satisfying the properties (1), (2) and (3) of Lemma 5.4.

Next we construct a homotopy of bundle maps $n(X_k) \to \nu(X_k)$ covering a homotopy $S^{X_k} \to \Sigma X_k(N, P)$ between $d s n(X_k)$ and $d (j^\infty G) n(X_k)$. Recall the additive structure of $J^{\infty}(N, P)$ in (1.2). Then we have the homotopy $\kappa_\lambda : E(k) \to J^{\infty}(N, P)$ defined by

$$\kappa_\lambda(c) = (1 - \lambda)s(c) + \lambda j^\infty G(c)$$

covering $\pi_r^\infty \circ s|S^{X_k} : S^{X_k} \to P$, where $\pi_r^\infty \circ s|S^{X_k}$ is the immersion as in (5.2.2).

We show that $\kappa_\lambda|S^{X_k}$ is actually a homotopy of $S^{X_k}$ into $\Sigma X_k(N, P)$. Recall the identification $(s|S^{X_k})^*P \cong (\pi_r^\infty \circ s)*(TP)$ and $s^*D \cong TN$.

It follows from the decomposition of $n(X_k)$ in (4.3) that

$$\begin{align*}
(s|S^{I_2})^* (d_{I_2}^l + d_s n(s, I_2 \subset I_{l-1}), & \quad (\exp_N G)|S^{J_2} (d_{J_2}^l + d(j^\infty G)|n(s, I_2 \subset I_{l-1}), \\
(s|S^{D_{J_2+1}})^* (d(r_{J_2-1}) \circ d(j^\infty G)|n(s, D_{J_2+1} \subset D_{J_2}), & \quad (\exp_N G)|S^{D_{J_2+1}} (d(r_{J_2-1}) \circ d(j^\infty G)|n(s, D_{J_2+1} \subset D_{J_2}), \\
(s|S^{E_7})^* (d(\tilde{d}_3^l \circ \tilde{\kappa}_3^l) \circ d_s n(s, E_7 \subset E_6), & \quad (\exp_N G)|S^{E_7} (d(\tilde{d}_3^l \circ \tilde{\kappa}_3^l) \circ d(j^\infty G)|n(s, E_7 \subset E_6)), \\
(s|S^{E_8})^* (d(\tilde{d}_3^l \circ \tilde{\kappa}_3^l) \circ \tilde{\kappa}_3^l \circ (\tilde{\kappa}_3^l / \tilde{\kappa}_3^l) \circ d_s n(s, E_8 \subset E_7), & \quad (\exp_N G)|S^{E_8} (d(\tilde{d}_3^l \circ \tilde{\kappa}_3^l) \circ \tilde{\kappa}_3^l \circ (\tilde{\kappa}_3^l / \tilde{\kappa}_3^l) \circ d(j^\infty G)|n(s, E_8 \subset E_7)).
\end{align*}$$

(6.7)

over respective $S^{X_k}$, where $I = J$ or $\tilde{J}$. These formulas are the direct consequence of the construction of $d_1(s, X_k)n(X_k) + q^{X_k}(s)$ appearing in the definition of $G$ and the definitions of the intrinsic derivatives and $r_{k-1}$ in Sections 2 and 3. Hence, it follows from (6.7) that for any $c \in S^{X_k}$ we have $n(\kappa_\lambda, X_k)|c = n(X_k)|c$ and $Q(\kappa_\lambda)|c = Q_c$. Hence, the equalities of the homomorphisms in (6.7) also hold when $s$ is replaced by $\kappa_\lambda (0 \leq \lambda \leq 1)$. Therefore, $\kappa_\lambda|S^{X_k}$ gives a homotopy of $S^{X_k}$ into $\Sigma X_k(N, P)$.

Let us consider the commutative diagram

$$\begin{array}{cccccc}
n(X_k) & d(\kappa_\lambda)|n(X_k) & \nu(X_k) & d(X_k) & H(X_k) \\
\| & \uparrow(\kappa_\lambda|S^{X_k}) \nu(X_k) & & \uparrow(\kappa_\lambda|S^{X_k}) \nu(X_k) & \\
n(X_k) & \kappa_\lambda|S^{X_k} \nu(X_k) & (\kappa_\lambda|S^{X_k}) \nu(X_k) & (\kappa_\lambda|S^{X_k}) \nu(X_k) & H(X_k).
\end{array}$$

24
Here, \( H(X_k) \) (resp. \( d(X_k) \)) denotes the direct sum of the target bundles of the homomorphisms (resp. the sum of the homomorphisms) \( d^I_{j+1}|_\nu(I_j \subset I_{j-1}) \) for \( I = J \) or \( J \), \( d(r_{j-1})|_\nu(D_{j+1} \subset D_j) \), \( d(\tilde{d}I_{3j}^2 \cup \tilde{K}_3^3)|_\nu(e_7 \subset e_6) \) and \( d(\tilde{d}I_{3j}^2 \cup \tilde{K}_3^3 \cup (\tilde{K}_2^3/\tilde{K}_2^3))|_\nu(e_8 < e_7) \) such that \( \nu(I_j \subset I_{j-1}) \), \( \nu(D_{j+1} \subset D_j) \), \( \nu(e_7 \subset e_6) \) and \( \nu(e_8 < e_7) \), which appear in the direct sum decompositions of \( \nu(X_k) \) in (4.2), and let \( H(X_k) = (s|S^{X_k})^*(H(X_k)) \). Namely, \( H(X_k) \) is the direct sum of the target bundles of the isomorphisms in (6.2), (6.3) and (6.4) whose source normal bundles appear in the direct sum decomposition of \( n(X_k) \) in (4.3). Then it follows from (6.7) that

\[
(k_\lambda|S^{X_k})^*(d(X_k) \circ d(k_\lambda)|n(X_k)) = (s|S^{X_k})^*(d(X_k) \circ ds|n(X_k))
\]

for any \( \lambda \). This implies that \( k_\lambda \) is transverse to \( \Sigma X_k(N, P) \) for any \( \lambda \).

We define \( h^3_\lambda : (E(k), S^{X_k}) \rightarrow (\Omega(N, P), \Sigma X_k(N, P)) \) by

\[
h^3_\lambda(v_c) = \exp_{\Omega, k_\lambda(c)} \circ d_c(k_\lambda) \circ (\exp_{\Omega, v_c(c)})^{-1}(v_c).
\]

Then we have that \( h^3_0(v_c) = h^3_1(v_c) = s(v_c) \) for \( v \in E(k) \cap (U(C_{k+1})_{2-3r_0} \setminus \text{Int}(U(C_{k+1})) \) in general, we need to modify \( h_\lambda \) as follows.

Since \( h^3_0(v_c) = h^3_1(v_c) = s(v_c) \) for \( v \in E(k) \cap (U(C_{k+1})_{2-3r_0} \), we may assume in the construction of \( h^3_\lambda \), \( h^3_\lambda \) and \( h^3_\lambda \) that \( v \in E(k) \cap (U(C_{k+1})_{2-3r_0} \), then \( h^3_\lambda(v_c) = h^3_0(v_c) = h^3_1(v_c) \) and \( h^3_\lambda(v_c) = h^3_{1-\lambda}(v_c) \) for any \( \lambda \). By using these properties of \( h^3_\lambda \), \( h^3_\lambda \) and \( h^3_\lambda \), we can modify \( h_\lambda \) to a homotopy \( h_\lambda \in \Gamma_{\Omega}^r(E(k), P) \) satisfying the property (C) such that

1. \( h_\lambda(v_c) = h_0(v_c) = s(v_c) \) for any \( \lambda \) and any \( v \in E(k) \cap (U(C_{k+1})_{2-2r_0} \),
2. \( h_0(v_c) = s(v_c) \) for any \( v \in E(k) \),
3. \( h_1(v_c) = j^\infty G(v_c) \) for any \( v \in E(k) \).

By (1), we can extend \( h_\lambda \) to the homotopy \( H_\lambda : \Gamma_{\Omega}^r(E(k) \cup U(C_{k+1})_{2-2r_0} \) defined by \( H_\lambda(E(k)) = h_\lambda \) and \( H_\lambda|U(C_{k+1})_{2-2r_0} = s|U(C_{k+1})_{2-2r_0} \).

By applying the homotopy extension property to \( s \) and \( H_\lambda \) for each \( X_k = A_k, D_k \) and \( E_k \), we obtain an extended homotopy

\[
H_\lambda : (N, S^{X_k}) \rightarrow (\Omega(N, P), \Sigma X_k(N, P))
\]

relative to \( U(C_{k+1})_{2-2r_0} \) with \( H_0 = s \). Furthermore, we replace \( \delta \) and \( E(k) \) by smaller ones. Then \( H_\lambda \) is a required homotopy in \( \Gamma_{\Omega}^r(N, P) \) in the assertion (A).

We need further argument for (4.1.3) using the Thom’s first Isotopy Lemma (see [Math3, (8.1)]). We suppose that \( \Omega(N, P) \) does not contain \( \Sigma D_j(N, P) \) or \( \Sigma E_k(N, P) \). Let \( \delta \) and \( \delta' \) be smooth functions with \( 0 < \delta < \delta' < \delta \). Let \( U(A_k) = \text{Int}(E(k)) \cup U(C_{k+1})_{2-2r_0} \) and \( U(k) = \exp_N(D_{\cos(n(A_k))}) \cup U(C_{k+1})_{2-3/2r_0} \) for \( \zeta = \delta, \delta' \). Let \( p^j : U(A_k) \times I \rightarrow I \) be the projection onto the second factor. Then \( S^{A_j}(\{H_\lambda\}) \) is the subset of \( U(A_k) \times I \) which consists of all points \( (x, \lambda) \) such that \( H_\lambda(x) \in \Sigma A_j(N, P) \) for each \( j \leq k \). Since \( H_\lambda \) is transverse to \( \Sigma A_k(N, P) \), we may assume that \( H_\lambda \) is transverse to all \( \Sigma A_j(N, P) \) on
By the downward induction on \( j \), we obtain an extended smooth homotopy
\[
H_\lambda : (N, S^{A_j}(s)) \to (\Omega(N, P), \Sigma A_{j-1}(N, P))
\]
relative to \( U(C_{k+1})_{2-r_0} \) such that \( H_0 = s, \ S^{A_j}(H_\lambda) = S^{A_j}(s) \) for \( 1 \leq j \leq k \). This completes the proof.

The author does not know whether the Whitney condition (b) for the Thom’s first Isotopy Lemma holds or not among \( \Sigma A_i(N, P), \Sigma D_j(N, P) \) and \( \Sigma E_k(N, P) \).
7 Proof of Theorem 0.2

In this section we prove Theorem 0.2 by applying Theorem 0.1 in the elimination of higher $D_k$ and $E_k$ singularities.

Let $\Omega^{D}(N, P)$ (resp. $\Omega^{E}(N, P)$) denote the open subbundle of $J_{\infty}(N, P)$ which consists of all regular jets and $\Sigma A_i(N, P)$ ($i \geq 1$) and $\Sigma D_j(N, P)$ ($j \geq 4$) (resp. $\Sigma A_i(N, P)$ ($i \geq 1$), $\Sigma D_j(N, P)$ ($j = 4, 5$) and $\Sigma E_k(N, P)$ ($k \geq 6$)). Let $\Omega^{D_k}(N, P)$ (resp. $\Omega^{D_5E_6}(N, P)$) denote the open subbundle of $J_{\infty}(N, P)$ which consists of all regular jets and $\Sigma A_i(N, P)$ ($i \geq 1$) and $\Sigma D_j(N, P)$ ($j \geq 4$) (resp. $\Sigma A_i(N, P)$ ($i \geq 1$), $\Sigma D_j(N, P)$ ($j = 4, 5$) and $\Sigma E_6(N, P)$).

\textbf{Proof of Theorem 0.2.} By the assumption, $j^\infty f$ is a section $N \to \Omega^{D}(N, P)$ (resp. $N \to \Omega^{E}(N, P)$). By Proposition 7.1 below, we have the section $s^D : N \to \Omega^{D_k}(N, P)$ (resp. $s^E : N \to \Omega^{D_{5}E_{6}}(N, P)$) such that $\pi P^\infty \circ j^\infty f = \pi P^\infty \circ s^D$ (resp. $\pi P^\infty \circ j^\infty f = \pi P^\infty \circ s^E$). By Theorem 0.1 we obtain a required smooth map $g$ such that $j^\infty g$ and $s^D$ (resp. $s^E$) are homotopic. This proves the assertion. \hfill $\square$

\textbf{Proposition 7.1.} Let $n > p \geq 2$ and $n - p$ be even. Then we have the following.

1. Given a section $s : N \to \Omega^{D}(N, P)$, there exists a section $s^D : N \to \Omega^{D_k}(N, P)$ such that $\pi P^\infty \circ s = \pi P^\infty \circ s^D$.

2. Given a section $s : N \to \Omega^{E}(N, P)$, there exists a section $s^E : N \to \Omega^{D_{5}E_{6}}(N, P)$ such that $\pi P^\infty \circ s = \pi P^\infty \circ s^E$.

We need the following lemma for the proof of Proposition 7.1.

\textbf{Lemma 7.2.} Let $n > p \geq 2$ and $n - p$ be even. Then we have that, for $3, Q|_{\Sigma^{3}_{2}(N, P)}$, $L|_{\Sigma_{5}^{3}(N, P)}$ and $(K_{3}^{2}/K_{3}^{3})|_{\Sigma^{3}_{3}(N, P)}$ are trivial line bundles equipped with the canonical orientations respectively.

\textbf{Proof.} By Section 2 (5), $d_{2}|K : K \to \text{Hom}(K, Q)$ induces the isomorphism

$$K/K_{2}^{3} \cong \text{Hom}(K/K_{2}^{3}, Q) \quad \text{over } \Sigma^{3}_{2}(N, P),$$

which yields the nonsingular quadratic form $q : K/K_{2}^{3} \circ K/K_{2}^{3} \to Q$ over $\Sigma^{3}_{2}(N, P)$. Since $\dim K/K_{2}^{3} = n - p - 1$ is odd, we choose the unique orientation of $Q_{2}$, expressed by the unit vector $e_{p}$, so that the index (the number of the negative eigen values) of $q_{z}$, $z \in \Sigma^{3}_{2}(N, P)$ is less than $(n - p - 1)/2$.

By (3-1) and (D-ii) we have the following direct sum decompositions over $\Sigma_{5}^{3}(N, P)$:

$$\nu(\mathbb{J}_{2} \subset \mathbb{J}_{1})|_{\Sigma_{5}^{3}} = (K_{2}^{3}/L \oplus L \oplus T^{1})|_{\Sigma_{5}^{3}},$$

$$\text{Hom}(\bigodot_{2} K_{2}^{3}, Q)|_{\Sigma_{5}^{3}} \cong \text{Hom}(K_{2}^{3}/L \bigodot L \bigodot \bigodot (K_{2}^{3}/L)|_{\Sigma_{5}^{3}}),$$

where $T^{1}$ is the orthogonal complement of $K_{2}^{3}$ in $\nu(\mathbb{J}_{2} \subset \mathbb{J}_{1})$ over $\Sigma_{5}^{3}(N, P)$. 

27
By (2.3), (D-ii) and (D-iii), \((d^3_i|\nu(\mathfrak{J}_2 \subset \mathfrak{J}_1))|_{\Sigma D_5}\) induces the isomorphisms

\[
(K_3^3 / L)|_{\Sigma D_5} \to \text{Hom}(K_3^3 / L \circ L, Q)|_{\Sigma D_5}, \\
L|_{\Sigma D_5} \to \text{Hom}(\bigcirc^2 (K_3^3 / L), Q)|_{\Sigma D_5}, \\
T^1|_{\Sigma D_5} \to \text{Hom}(\bigcirc^2 L, Q)|_{\Sigma D_5}.
\]

Since \(\bigcirc^2 (K_3^3 / L)\) has the canonical orientation, \(L\) has the canonical orientation, expressed by the unit vector \(e(L)\) over \(\Sigma D_5(N, P)\), by the second isomorphism.

Let us provide \(K_3^3 / K_3^3\) with the orientation. By (3-1), \(d^3_3|K_3^3\) induces the isomorphism \(K_3^3 / K_3^3 \to \text{Hom}(\bigcirc^2 (K_3^3 / K_3^3), Q)\) over \(\Sigma^3(N, P)\). Then it comes from the orientation of \(\text{Hom}(\bigcirc^2 (K_3^3 / K_3^3), Q)\).

Lemma 7.3. Let \(n > p \geq 2\). Let \(z\) be a point of \(\Sigma E_6(N, P), \Sigma E_7(N, P)\) or \(\Sigma E_8(N, P)\). Let \(\{z_m\}\) be a sequence of \(\Sigma D_5(N, P)\) which converges to \(z\). Then we have

1. \(\{K_{2,z_m}\}\) converges to \(K_{2,z}\),
2. \(\{L_{z_m}\}\) converges to \(K_{3,z}\).

Proof. (1) Since \(\text{Ker}(d^3_2|K) = K_3^3\) over \(\Sigma^3(N, P)\) and since \(\Sigma D_5(N, P) \subset \Sigma^3(N, P)\) and \(\Sigma E_k(N, P) \subset \Sigma^3(N, P)\), the assertion (1) follows from the continuity of \(d^3_2|K\) on \(\Sigma^3(N, P)\).

(2) By (1), we take a subsequence of \(\{z_m\}\) for which \(\{L_{z_m}\}\) converges to a 1-dimensional subspace of \(K_{3,z}\), say \(V_z\). By (2.4) it is enough for the proof of the assertion (2) to show that \(d^3_3|V_z\) vanishes. Suppose that \(V_z \neq K_{3,z}\). Since \(\Sigma E_k(N, P) \subset \Sigma^3(N, P)\), it follows that \(\text{Ker}(d^3_3|K_{2,z}) = K_{3,z}\). Hence, \(d^3_3|V_z\) induces the isomorphism \(\bigcirc^3(K_{2,z}/K_{3,z}) \to Q_z\), which yields the isomorphism \(\bigcirc^3 V_z \to Q_z\) (on the contrary, we have that \(d^3_3|K_{3,z}\) and \(d^3_3|\bigcirc^3 K_{3,z}\) vanish). Since \(\lim_{m \to \infty} L_{z_m} = V_z\), there is a number \(m_0\) such that if \(m > m_0\), then \(d^3_3|\bigcirc^3 L_{z_m}\) does not vanish. This contradicts to the definition of \(\Sigma D_5(N, P)\) in (D-iii). Hence, we obtain the assertion (2).

Rem 7.4. Under the same assumption of Lemma 7.2

1. \(\tilde{L}\) has the canonical orientation if \(U(\Sigma D_5)\) is chosen as a tubular neighborhood,
2. the normal bundles for the respective inclusions \(U(\Sigma D_5) \supset \Sigma D_5(N, P) \supset \cdots \supset \Sigma D_5(N, P)\) are all trivial by (3.2).

Proof of Proposition 7.1. We give the proof only for (2), since the proof for (1) is parallel by setting \(S^{E_6}(s) = \emptyset\).

In the proof we identify \(J^k(N, P)\) with \(J^k(TN, TP)\) by (1.2). By Remark 3.4, there exists the open subbundle \(\Omega_{D_4}^N(N, P)'\) of \(J^3(N, P)\) (resp. \(\Omega_{D_5}^{E_6}(N, P)\) of \(J^4(N, P)\)) such that \((\pi_3^{-1})^{-1}(\Omega_{D_4}^N(N, P)' = \Omega_{D_4}^N(N, P)\) (resp. \((\pi_4^{-1})^{-1}(\Omega_{D_5}^{E_6}(N, P)') = \Omega_{D_5}^{E_6}(N, P)\)). We may assume \(s \in \Gamma_{D_5}(N, P)\). It follows that \((\pi_3^{-1} \circ s)(N \prod_{D_5}(s) \cup S^{E_6}(s)) \subset \Omega_{D_4}^N(N \times S^{E_6}(s) \cup S^{E_6}(s))\). Since \(s(S^{E_6}(s)) \subset \Sigma D_5(N, P)\) and \(s(S^{E_6}(s)) \subset \Sigma^3(N, P)\), we consider \(L = (s|S^{E_6}(s))^*L\) and \(K_3 = (s|S^{E_6}(s))^*K_3^3\).
We now construct a new section
\[ \tilde{u} : N \rightarrow \Omega^{D_{3}E_{6}}(N, P) \]
as follows.

We have that \( r_{3}(s) : \mathbb{O}^{3}L \rightarrow Q \) over \( S^{D_{3}}(s) \) is an isomorphism and is the nullhomomorphism over \( S^{T_{3}}(s) \). Furthermore, \( r_{1}(s) : \mathbb{O}^{4}L \rightarrow Q \) over \( S^{D_{3}}(s) \) is an isomorphism and is the null-homomorphism over \( S^{T_{3}}(s) \). Let \( e_{P}, e_{L} \) and \( e(K_{3} \circ K_{3}) \) be the unit vectors induced from \( e_{P}, e(L) \) and \( e(K_{3} \circ K_{3}) \), which represents the canonical orientation of \( K_{3} \circ K_{3} \), by \( s \) respectively. Then by using Lemma 7.3 we define the smooth isomorphisms \( \phi^{D} : \mathbb{O}^{4}L \rightarrow Q \) over \( S^{T_{3}}(s) \) and \( \phi^{E} : \mathbb{O}^{4}K_{3} \rightarrow Q \) over \( S^{E_{6}}(s) \) by \( \phi^{D}(\mathbb{O}^{4}e(L)) = e_{P} \) and \( \phi^{E}(\mathbb{O}^{3}e(K_{3} \circ K_{3})) = e_{P} \) respectively. Then we can find a section \( u_{\phi} : S^{T_{3}}(s) \cup S^{E_{6}}(s) \rightarrow \text{Hom}(S^{4}((\pi_{N}^{\infty} \circ s)^{*}(TN)), (\pi_{P}^{\infty} \circ s)^{*}(TP)) \) such that \( u_{\phi}(x)|\mathbb{O}^{4}L_{x} = \phi^{D}|_{x} \) for \( x \in S^{T_{3}}(s) \) and \( u_{\phi}(x)|\mathbb{O}^{4}K_{3,x} = \phi^{E}|_{x} \) for \( x \in S^{E_{6}}(s) \). Since \( S^{T_{3}}(s) \cup S^{E_{6}}(s) \) is a closed subset and since \( \text{Hom}(S^{4}((\pi_{N}^{\infty} \circ s)^{*}(TN)), (\pi_{P}^{\infty} \circ s)^{*}(TP)) \) is a vector bundle, we can extend \( u_{\phi} \) arbitrarily to the section \( \tilde{u} : N \rightarrow \text{Hom}(S^{4}((\pi_{N}^{\infty} \circ s)^{*}(TN)), (\pi_{P}^{\infty} \circ s)^{*}(TP)) \).

Then we define \( \tilde{u} \) by \( \tilde{u} = \pi_{N}^{3} \circ s \oplus \tilde{u}_{\phi} \) as the section of \( J^{4}(N, P) = J^{4}(TN, TP) \). We regard \( \tilde{u} \) as the section of \( J^{\infty}(N, P) \) over \( N \). We prove that \( \tilde{u} \in \Omega^{D_{3}E_{6}}(N, P) \). By the construction we have that \( r_{4}(\tilde{u})_{x} = u_{\phi}(x)|\mathbb{O}^{4}L_{x} = \phi^{D}|_{x} \) for \( x \in S^{T_{3}}(s) \) and \( d_{4,x}^{3}|\mathbb{O}^{4}K_{3,x} = u_{\phi}(x)|\mathbb{O}^{4}K_{3,x} = \phi^{E}|_{x} \) for \( x \in S^{E_{6}}(s) \). For any point \( x \in S^{T_{3}}(s) \cup S^{E_{6}}(s) \), let \( U_{x} \) be a convex neighborhood of \( x \) and let \( t \) and \( k \) be the coordinates of \( \exp_{N}(L_{x}) \) and \( \exp_{N}(K_{3,x}) \) respectively. As in the proof of Lemma 3.1, it follows from the definition of \( D \) that
\[
\begin{align*}
(\mathbb{O}^{4}d_{e})_{y_{l}}|_{\tilde{u}(x)} &= \partial^{4}y_{l}/\partial t^{4}(x) \neq 0 \quad \text{for } x \in S^{T_{3}}(s), \\
(\mathbb{O}^{4}d_{k})_{y_{l}}|_{\tilde{u}(x)} &= \partial^{4}y_{l}/\partial k^{4}(x) \neq 0 \quad \text{for } x \in S^{E_{6}}(s).
\end{align*}
\]
Hence, we have that \( \tilde{u}(S^{T_{3}}(s)) \subseteq \Sigma D_{3}(N, P) \) and \( \tilde{u}(S^{E_{6}}(s)) \subseteq \Sigma E_{6}(N, P) \). It is clear that \( \tilde{u}(N \setminus (S^{T_{3}}(s) \cup S^{E_{6}}(s))) \subseteq \Omega^{D_{4}}(N, P) \). This completes the proof.

Let \( X = D \) or \( E \) and \( f : N \rightarrow P \) be an \( \Omega^{\infty} \)-regular map. In this paper the Thom polynomial \( P(X_{k+1}, f) \) of \( S^{T_{k+1}}(j^{\infty}f) \) refers to the Poincaré dual in \( N \) of the fundamental class of \( S^{T_{k+1}}(j^{\infty}f) \). Let \( W = 1 + W_{1} + \cdots + W_{j} + \cdots \) be the total Stiefel-Whitney class and \( \overline{W} \) be the formal inverse of \( W \). Let \( [a] \) denote the greatest integer not greater than \( a \).

If \( n - p \) is odd, then Theorem 0.2 does not hold in general. We have the following theorem ([An4, Remark 4.3 and Section 8]).

**Theorem 7.5.** Let \( n = p + 1 \). Let \( W_{j} = W_{j}(TN - f^{*}(TP)) \). Then

1. if \( f : N \rightarrow P \) is an \( \Omega^{E} \)-regular map, then the Thom polynomial \( P(E_{k+1}, f) \) of \( S^{E_{k+1}}(j^{\infty}f) \) is equal to \( W_{k}W_{2} + W_{k-1}(W_{3} + W_{1}W_{2}) \).
2. if \( f : N \rightarrow P \) is an \( \Omega^{D} \)-regular map, then the Thom polynomial \( P(D_{k+1}, f) \) of \( S^{D_{k+1}}(j^{\infty}f) \) is equal to the part of degree \( k + 2 \) of the polynomial
\[
W(\overline{W}_{1} + W_{2}) \left\{ \sum_{j=0}^{\left[\frac{k}{2}\right]-2} \left( \begin{array}{c}
\left[\frac{k}{2}\right] - 2 \\
2j
\end{array} \right) \right\} W_{j} + W \left\{ \sum_{j=0}^{\left[\frac{k}{2}\right]-1} \left( \begin{array}{c}
\left[\frac{k}{2}\right] - 1 \\
2j
\end{array} \right) \right\} W_{j+1}.
\]
Recall that $\text{codim}\Sigma^2(N, P) = 9$ ([B, Theorem 6.1]) and $\text{codim}\Sigma^3(N, P) = 6$. By [H1, Theorem 7.1], there is an immersion $\mathbb{RP}^6 \to \mathbb{R}^7$. Hence, any map $f : \mathbb{RP}^6 \times \mathbb{RP}^2 \to \mathbb{R}^7$ is homotopic to an $\Omega_{E}$-regular map. We have that $W(\mathbb{RP}^6 \times \mathbb{RP}^2) = (1 + a)^7 \otimes (1 + b)^3$, where $a$ and $b$ are the generators of $H^1(\mathbb{RP}^6; \mathbb{Z}/(2))$ and $H^1(\mathbb{RP}^2; \mathbb{Z}/(2))$ respectively. By Theorem 7.5 (1), we have that $P(E_7, f) = a^6 \otimes b^2 \neq 0$ in this case. This implies that we cannot eliminate the singularity of type $E_7$ from $f : \mathbb{RP}^6 \times \mathbb{RP}^2 \to \mathbb{R}^7$.

For $\Omega_{\overline{D}}$-regular maps it is not easy to find this kind of examples. By Theorem 7.5 (2), we have that $P(D_4, f) = W_1 W_4$, $P(D_5, f) = P(D_6, f) = 0$ and

$$P(D_{k+1}, f) = W_k W_2 + W_{k-1}(W_3 + W_1 W_2) \text{ for } k = 6, 7.$$ 

Since an $\Omega_{\overline{D}}$-regular map does not have the singularity of type $E_k$, we have by Theorem 7.5 (1) that $P(E_{k+1}, f) = W_k W_2 + W_{k-1}(W_3 + W_1 W_2) = 0$ for $k = 5, 6, 7$. Hence, we have that $P(D_{k+1}, f) = 0$ for $k = 5, 6, 7$.

References

[An1] Y. Ando, On the elimination of Morin singularities, J. Math. Soc. Japan 37(1985), 471-487.

[An2] Y. Ando, An existence theorem of foliations with singularities $A_k$, $D_k$ and $E_k$, Hokkaido Math. J. 19(1991), 571-578.

[An3] Y. Ando, On local structures of the singularities $A_k$, $D_k$ and $E_k$ of smooth maps, Trans. Amer. Math. Soc. 331(1992), 639-651.

[An4] Y. Ando, On Thom polynomials of the singularities $D_k$ and $E_k$, J. Math. Soc. Japan 48(1996), 593-606.

[An5] Y. Ando, The homotopy type of the space consisting of regular jets and folding jets in $J^2(n, n)$, Japanese J. Math. 24(1998), 169-181.

[An6] Y. Ando, Fold-maps and the space of base point preserving maps of spheres, J. Math. Kyoto Univ. 41(2002), 691-735.

[An7] Y. Ando, Invariants of fold-maps via stable homotopy groups, Publ. RIMS, Kyoto Univ. 38(2002), 397-450.

[An8] Y. Ando, Existence theorems of fold-maps, Japanese J. Math. 30(2004), 29-73.

[An9] Y. Ando, A homotopy principle for maps with prescribed Thom-Boardman singularities, preprint, available online at http://front.math.ucdavis.edu/math.GT/0309204.

[An10] Y. Ando, Cobordisms of maps without prescribed singularities, submitted, 2004.
[Ar] V. I. Arnold, Normal forms for functions near degenerate critical points, the Weyl groups $A_k, D_k, E_k$ and Lagrangian singularities, Funct. Anal. Appl. 6(1972), 254-272.

[B] J. M. Boardman, Singularities of differentiable maps, IHES Publ. Math. 33(1967), 21-57.

[duP1] A. du Plessis, Maps without certain singularities, Comment. Math. Helv. 50(1975), 363-382.

[duP2] A. du Plessis, Contact invariant regularity conditions, Springer Lecture Notes 535(1976), 205-236.

[E1] J. M. ŘEliašberg, On singularities of folding type, Math. USSR. Izv. 4(1970), 1119-1134.

[E2] J. M. ŘEliašberg, Surgery of singularities of smooth mappings, Math. USSR. Izv. 6(1972), 1302-1326.

[F] S. Feit, $k$-mersions of manifolds, Acta Math. 122(1969), 173-195.

[G1] M. Gromov, Stable mappings of foliations into manifolds, Math. USSR. Izv. 3(1969), 671-694.

[G2] M. Gromov, Partial Differential Relations, Springer-Verlag, Berlin Heiderberg, 1986.

[H1] M. Hirsch, Immersions of manifolds, Trans. Amer. Math. Soc. 93(1959), 242-276.

[H2] M. Hirsch, Differential Topology, Springer-Verlag, 1976.

[K-N] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol.1, Interscience Publishers, New york, 1963.

[L1] H. I. Levine, Elimination of cusps, Topology 3(1965), 263-296.

[L2] H. I. Levine, Singularities of differentiable maps, Proc. Liverpool Singularities Symposium, Springer Lecture Notes 192(1971), 1-85.

[Math1] J. N. Mather, Stability of $C^\infty$ mappings, IV: Classification of stable germs by $\mathbb{R}$-algebra, Publ. Math. Inst. Hautes Étud. Sci. 37(1970), 223-248.

[Math2] J. N. Mather, On Thom-Boardman singularities, Dynamical Systems, Academic Press, 1973, 233-248.

[Math3] J. N. Mather, Stratifications and mappings, Dynamical Systems, Academic Press, 1973, 195-232.

[Mats] Y. Matsushima, Differentiable Manifolds, Marcel Dekker, New York, 1972.
[Mo] B. Morin, Formes canoniques des singularités d'une application différentiable, C. R. Acad. Sci. Paris 260(1960), 6503-6506.

[P] A. Phillips, Submersions of open manifolds, Topology 6(1967), 171-206.

[Sady1] R. Sadykov, The Chess conjecture, Algebr. Geom. Topol. 3(2003), 777-789.

[Sady2] R. Sadykov, Elimination of singularities of smooth mappings of 4-manifolds into 3-manifolds, Topology Appl. 144(2004), 173-199.

[Saek] O. Saeki, Fold maps on 4-manifolds, Comment. Math. Helv. 78(2003), 627-647.

[Ste] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, Princeton, 1951.

[T] R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier 6(1955-56), 43-87.

Department of Mathematical Sciences, Faculty of Science, Yamaguchi University, Yamaguchi 753-8512, Japan
e-mail: andoy@po.cc.yamaguchi-u.ac.jp