CYLINDRICAL BLACK HOLE IN GENERAL RELATIVITY

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ABSTRACT

A black hole solution of Einstein’s field equations with cylindrical symmetry is found. Using the Hamiltonian formulation one is able to define mass and angular momentum for the cylindrical black hole through the corresponding and equivalent three dimensional theory. The causal structure is analyzed.

1. Introduction

In a classical framework black hole solutions are of great relevance since they might reflect the manner into which spacetime settles after complete gravitational collapse of some form (e.g., collapse of stars or clusters of stars) has occurred [1,2]. At the quantum level black holes are being used as theoretical laboratories in the sense that they can give clues for the underlying nature of the interaction between the geometry and the quantum world. Examples of this interaction are the Hawking radiation [3], scattering processes involving particles and black holes [4], and the statistics a black hole gas should obey [5].

In General Relativity the black hole solutions which have so far been found form a four parameter family called the generalized Kerr-Newman family of black holes. The four parameters are mass $M$, angular momentum $J$, charge $Q$, and the cosmological constant $\Lambda$ [6]. These are axisymmetric solutions and depending on the cosmological constant have different asymptotic behavior, they can be asymptotically flat ($\Lambda = 0$), de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$).
Axial symmetry has two important particular cases. One is spherical symmetry which has been extensively studied through the Schwarzschild solution and the Schwarzschild black hole. The other is cylindrical symmetry. Within the field of exact solutions, cylindrical symmetry has played an important role in the discussion of the internal consistency of General Relativity itself, through the static solutions of Levi-Civita [7,8] and Chazy-Curzon [9,10], and the stationary solutions of Lewis [11]. In an astrophysical context, cylindrical symmetry has been applied to the study of cosmic strings [12] which in turn offered a gain in understanding the role conical singularities and spacetime topological defects play in the gravitational field. However, up to now there is no black hole solutions with cylindrical symmetry. In this work we show that cylindrically symmetric rotating black hole solutions of Einstein’s field equations with a negative cosmological constant do indeed exist.

2. The equations and the solution

Einstein-Hilbert action in four dimensions is given by

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda),
\]

where \( g \) is the determinant of the metric and \( R \) the Ricci scalar. We assume that the spacetime is cylindrically symmetric and time-independent, i.e, there are three Killing vectors [13]: \( \frac{\partial}{\partial z} \) which corresponds to the translational symmetry along the axis, \( \frac{\partial}{\partial \phi} \) which has closed periodic trajectories around the axis and \( \frac{\partial}{\partial t} \) corresponding to the invariance under time translations. We then find that the following solution satisfies the equations of motion derived from (1),

\[
ds^2 = -\left( \alpha^2 r^2 - \frac{b}{ar} \right) dt^2 + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{ar}} + r^2 d\phi^2 + \alpha^2 r^2 dz^2,
\]

\[-\infty < t < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \phi < 2\pi, \quad -\infty < z < \infty.\]
Here $r$ is the radial circumferential coordinate, $\alpha^2 \equiv -\frac{1}{3}\Lambda > 0$ and $b$ is a constant which as we will see is related to the mass and we now assume positive. Equation (2) represents a static black hole with an event horizon at $ar = b^\frac{2}{3}$. Since the Kretschmann scalar is given by $R_{abcd}R^{abcd} = 24\alpha^4 \left(1 + \frac{b^2}{2\alpha^4 r^4}\right)$ there is a scalar polynomial singularity at $ar = 0$. To add angular momentum to the spacetime we perform the following coordinate transformation

$$\tilde{t} = \lambda t - \frac{\omega}{\alpha^2} \phi,$$  \hspace{1cm} (3)

where $\omega$ and $\lambda$ are constant parameters. In order to get rid of minor coordinate difficulties we still have to change to rotating axes by doing,

$$\bar{\phi} = \lambda \phi - \omega t.$$  \hspace{1cm} (4)

Substituting (3) and (4) into (2) we obtain

$$ds^2 = -\left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) \alpha^2 r^2 - \frac{b\lambda^2}{\alpha r}\right] dt^2 - \frac{\omega b}{\alpha^3 r} 2d\phi dt + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} + \left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) r^2 + \frac{\omega^2 b^2}{\alpha^3 r}\right] d\phi^2 + \alpha^2 r^2 dz^2,$$  \hspace{1cm} (5)

$$-\infty < t < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \phi < 2\pi, \quad -\infty < z < \infty.$$

This represents a stationary cylindrical black hole and, of course, also solves (1), as can be checked directly through the equations of motion generated by action (1). One can then be tempted to say that by inverting the coordinate transformations (3) and (4) one gets back the static spacetime (2). On a first glance this is indeed the case. However, transformation (3) is not a permitted goibal coordinate transformation. This is shown in a clear way in the work of Stachel [14,15]. Transformation (3) can be done locally, but not globally. Spacetime is not simply connected. This
means that the first Betti number of the manifold is one since closed curves encircling the horizon cannot be shrunk to zero. This happens in either spacetime, static given by (2) and stationary given by (5). Both are homeomorphic to each other and homeomorphic to $\{\mathbb{R}^3 - R\} \times \mathbb{R}$. Now, in both spacetimes there is a timelike Killing field $\xi = \frac{\partial}{\partial t}$. In the static spacetime this corresponds to an exact one-form $\nabla$ inverse to $\xi$ (i.e., $\nabla_{\mu} \equiv \frac{\xi_{\mu}}{|\xi|}$) given then by $\nabla = dt$ (see [14] for details). In the stationary spacetime the corresponding one form is $\nabla = dt + \frac{\omega}{a^2} d\phi$ which is a closed one-form but not exact. De Rham’s cohomology theorems then state that, since the first Betti number of the manifolds is one, there are global diffeomorphisms which map the $\xi$ of the two manifolds, but there is no such global diffeomorphisms mapping $\nabla$ and $V$. Since the metric maps vectors into one-forms it means that metrics (2) and (5) cannot be globally mapped into each other. In this case, the map is given by equation (3) which is immediately understood as a local map. This is because $\varphi$ is a periodic coordinate which in turn requires time to be also periodic. Thus, metrics (2) and (5) can be locally mapped into each other but not globally, and therefore they are distinct. As suggested by Stachel [14] this distinction could be tested by an Arahanov-Bohm type experiment. We have used the coordinate transformation trick (3) and (4) to convert (2) into (5). But once we realize it is a trick we cannot go back. Spacetime (5) gives a stationary spacetime, while (2) gives a static one. Note that in the Schwarzschild solution closed curves can always be shrunk to a point. So this type of coordinate transformation will not generate a rotating black hole, but a rotating infinite string superposed on a Schwarzschild black hole.

Linet [16] has found the general static solution of Einstein’s field equations with cylindrical symmetry and cosmological constant. However Linet’s solution uses a coordinate system which looses the black hole. Santos [17] has generalized Linet’s
solution for stationary fields. Since the coordinate system used is related to Linet’s it also looses the black hole.

3. Definition of mass and angular momentum

We now tackle the delicate issue of the mass and angular momentum of the black hole. Asymptotically, as \( r \to \infty \), the black hole spacetime is not Minkowski but anti-de Sitter. As shown by Henneaux and Teitelboim [18] one can give meaningful definitions for fields that approach at large spacelike distances the anti-de Sitter configuration, whose group of motions is \( \text{O}(3,2) \). However there are two problems here. Firstly, for large \( \pm z \) (keeping \( r \) fixed) the black hole does not approach the anti-de Sitter solution. Secondly, in a spacetime in which the singularity extends uniformly over the infinite \( z \)-line one expects that the total energy (i.e., the ADM mass) is infinite. We now show that one can deal with both difficulties simultaneously. Since the trouble lies in the infinity of the \( z \) direction we have to find a procedure to eliminate the \( z \) coordinate altogether. To this end we recur to an equivalent three-dimensional (3D) theory, i.e., a 3D theory which reproduces the equations of motion of cylindrically symmetric General Relativity.

The most general metric with one Killing vector, invariant under \( z \to -z \) can be written as [13],

\[
d s^2 = g_{ab} dx^a dx^b + e^{-4\phi} dz^2,
\]

(6)

Where \( a, b = t, r, \varphi \), \( g_{ab} = g_{ab}(t, r) \) is the 3D metric and \( \phi = \phi(t, r) \). In fact the most general metric includes another metric function \( A \) in which case the metric is not invariant under \( z \to -z \), see ref. [13]. For our purposes it is enough to put \( A = 0 \).
From standard dimensional reduction techniques on (6) and (1) we obtain the following 3D action,

$$ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} e^{-2\phi} (R - 2\Lambda). $$  \hspace{1cm} (7)

The $z$ dimension has disappeared but left its mark on the dilaton field $\phi$. The field $A$ commented above would appear as a gauge field in the dimensional reduction process. The theory with action (7) could be called $\Omega = 0$ 3D Brans-Dicke theory (where $\Omega$ is the Brans-Dicke parameter) or 3D Teitelboim-Jackiw theory (since these authors proposed (7) for the 2D case [19,20]). By varying action (7) with respect to $g_{ab}$ and $\phi$ one obtains equations of motion identical to cylindrically symmetric General Relativity as given by action (1). Of course the solutions of the equations of motion can be transferred from one theory to the other. In particular there is the 3D black hole solution obtained directly from (5) which now reads

$$ ds^2 = -\left[ \left( \lambda^2 - \frac{\omega^2}{\alpha^2} \right) \alpha^2 r^2 - \frac{b\lambda^2}{\alpha r} \right] dt^2 - \frac{\omega b}{\alpha^3 r^2} 2d\varphi dt + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} + \left[ \left( \lambda^2 - \frac{\omega^2}{\alpha^2} \right) r^2 + \frac{\omega^2 b}{\alpha^5 r} \right] d\varphi^2, \tag{8} $$

$$ e^{-2\phi} = c\alpha r, \tag{9} $$

where $c$ is a dimensionless constant, $c > 0$. The point is that the black hole solution (8)-(9) allows a meaningful definition of (ADM) mass and angular momentum. Spacetime is asymptotically anti-de Sitter, now with $O(2,2)$ as the group of motions. The calculations needed to find the mass and angular momentum are similar to those related to the 3D black hole of Bañados, Henneaux, Teitelboim and Zanelli [21]. We mention here that Horowitz and Welch [22] also arrived at the stationary black hole of 3D General Relativity through a coordinate transformation of the static black hole. With that black hole of 3D General Relativity one can apply almost
directly the formalism developed by Regge and Teitelboim [23,24]. Here, there is an extra dilaton field.

In order to find the Hamiltonian formulation of the black hole we write the metric in the canonical form

\[ ds^2 = -N_0^2 dt^2 + R^2 (N^\varphi dt + d\varphi)^2 + \frac{dR^2}{f^2}, \]

(10)

where

\[ R^2 = \left( \lambda^2 - \frac{\omega^2}{\alpha^2} \right) r^2 + \frac{\omega b}{\alpha^5 r}, \quad N_0^2 = \left( \alpha^2 r^2 - \frac{b}{\alpha r} \right) \left( \lambda^2 - \frac{\omega^2}{\alpha^2} \right) \frac{r^2}{R^2} \]

\[ N^\varphi = -\frac{\lambda \omega b}{\alpha^3 R^2} + \text{constant}, \quad f^2 = \left( \alpha^2 r^2 - \frac{b}{\alpha r} \right) \left( \frac{dR}{dr} \right)^2. \]

(11)

\( N^0 \) and \( N^\varphi \) are the lapse and shift functions. By the usual procedure [21,23] one can bring action (7) with the help of (8)-(10) into the Hamiltonian form, which reads

\[ S = -\Delta t \int \left\{ N \left[ \frac{1}{2} e^{-2\phi} R^3 \left( N^\varphi, R \right)^2 + e^{-2\phi} \left( f^2 \right)_R \left( 1 - 2R \frac{d\phi}{dR} \right) - \right. \right. \]

\[ -4f^2 \left( e^{-2\phi} R \frac{d\phi}{dR} \right)_R + 2e^{-2\phi} RA \left. \right] + N^\varphi \left[ e^{-2\phi} R^3 \left( N^\varphi, R \right) \left( \frac{N^\varphi}{N} \right)_R \right] \right\} dR + B. \]

(12)

\( N \equiv \frac{N^0}{f} \) and \( N^\varphi \) are Lagrange multipliers imposing constraints on the action, namely the terms inside the squared brackets should be zero. \( B \) is a surface term needed to ensure that Hamilton’s equations are satisfied. Our task is to find \( B \) and associate it with the mass and angular momentum. To obtain the equations of motion one has to vary (12) with respect to \( \phi, f^2 \) and the momentum conjugate to the metric \( \pi = R^3 \frac{(N^\varphi)_R}{N} \). Here, we are interested only in the surface terms that one acquires by the variation of the action [25]. That is, as \( R \to \infty \), one finds

\[ \delta S = -\Delta t \left[ N(\infty) e^{-2\phi} \left( 1 - 2R \frac{d\phi}{dR} \right) \delta f^2 \right]_{R=\infty} + \]
We have \( \delta f^2 = f^2_{BH} - f^2_{AdS} \), etc, (BH=black hole, AdS =anti-de Sitter). Then by carefully examining each term in (13) we arrive at

\[
\delta S = \Delta t \left[ N(\infty) \delta \left( \frac{bc}{2 \lambda^2 + \frac{\omega^2}{\alpha^2}} \right) - N^\phi(\infty) \delta \left( 3bc\lambda \frac{\omega}{\alpha^2} \right) \right] + \delta B. \tag{14}
\]

Thus \( \delta B \) has to be equal to minus the first term on the right hand side of (14). So the boundary term \( B \) is well defined and Hamilton’s equations follow. Since mass and angular momentum are defined as the terms conjugated to \( N \) and \( N^\phi \) we have,

\[
M = bc \left( 2\lambda^2 + \frac{\omega^2}{\alpha^2} \right), \tag{15}
\]

\[
J = 3bc\lambda \frac{\omega}{\alpha^2}. \tag{16}
\]

Note that \( M \) and \( J \) depend on the strength of the dilaton through the constant \( c \), as it happens with 2D dilaton gravity theories \([26,27,28]\). Solving for \( \lambda \) and \( \frac{\omega}{\alpha} \) yields

\[
\lambda^2 = \frac{1}{b} \frac{M + M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{4c}, \tag{17}
\]

\[
\frac{\omega^2}{\alpha^2} = \frac{1}{b} \frac{M - M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{2c}. \tag{18}
\]

In (17) we have taken the + sign in front of the descriminant (equation (18) has the corresponding – sign). Now, choosing \( b \) is choosing a scale for the coordinate \( r \).
In order to have the standard form of the anti-de Sitter spacetime at spatial infinity we have to set \( b = \frac{1}{4c} \left( -M + 3M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}} \right) \). Then the 3D black hole (8)-(9) is

\[
\begin{align*}
    ds^2 &= -\left( \alpha^2 r^2 - \frac{M + M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{4\alpha r} \right) dt^2 - \frac{J}{3\alpha r} 2 dtd\phi + \\
    &+ \frac{dr^2}{\alpha^2 r^2 - \frac{-M+3M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{4\alpha r}} + \left( r^2 + \frac{M - M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{2\alpha^3 r} \right) d\phi^2, \\
    e^{-2\phi} &= c\alpha r.
\end{align*}
\]

By a coordinate transformation one can put \( c = 1 \). The cylindrical black hole solution is then given by,

\[
\begin{align*}
    ds^2 &= -\left( \alpha^2 r^2 - \frac{M + M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{4\alpha r} \right) dt^2 - \frac{J}{3\alpha r} 2 dtd\phi + \\
    &+ \frac{dr^2}{\alpha^2 r^2 - \frac{-M+3M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{4\alpha r}} + \left( r^2 + \frac{M - M \sqrt{1 - \frac{8J^2\alpha^2}{9M^2}}}{2\alpha^3 r} \right) d\phi^2 + \\
    &+ \alpha^2 r^2 dz^2.
\end{align*}
\]

The mass-energy of this system is a mass-energy per unit length, which is the meaningful quantity in cylindrical systems. According to the literature it can be considered more appropriate to call (21) a (straight) black string instead of a black hole, although it seems to us that in 4D the names black string and cylindrical black hole are synonymous.
4. Causal Structure

Now we turn to the causal structure which has many interesting aspects. First assume that \( r \geq 0 \). Then there are three distinct cases with real solutions. (i) \( 0 \leq J_\alpha < M \): this is the black hole solution. There is an event horizon located at

\[
\begin{align*}
r_{\text{eh}}^3 &= \frac{1}{\alpha^3} \frac{-M+3M\sqrt{1-\frac{6J_\alpha^2}{9M^2}}}{4}.
\end{align*}
\]

The infinite redshift surface, always outside the horizon, is at

\[
\begin{align*}
r_{\text{rs}}^3 &= \frac{1}{\alpha^3} \frac{M+M\sqrt{1-\frac{6J_\alpha^2}{9M^2}}}{4}.
\end{align*}
\]

There is another important radius which gives the place at which upon decreasing \( r \) the perimeter starts to increase. This turning point is at

\[
\begin{align*}
r_{\text{tp}}^3 &= \frac{1}{\alpha^3} \frac{M-M\sqrt{1-\frac{6J_\alpha^2}{9M^2}}}{4}. \quad \text{When } 0 \leq J_\alpha \leq \frac{3\sqrt{3}}{4\sqrt{2}}M \text{ then } r_{\text{tp}} \leq r_{\text{eh}}.
\end{align*}
\]

For \( \frac{3\sqrt{3}}{4\sqrt{2}}M < J_\alpha < M \) one has \( r_{\text{eh}} < r_{\text{tp}} \). At \( r = 0 \) there is a spacelike singularity. For \( r \to \infty \) spacetime is anti-de Sitter. (ii) \( J_\alpha = M \): there is a null singularity at \( r = 0 \). This is the extremal limit of the black hole. (iii) \( M < J_\alpha \leq \sqrt{\frac{9}{8}}M \): there are no horizons. The singularity is timelike and naked. Like Kerr’s the singularity has a ring like structure. Unlike Kerr’s one cannot penetrate through the inside of the ring since the black hole is 3D (and in 4D the symmetry is cylindrical). There is an infinite redshift surface. The corresponding Penrose diagrams are very simple and we do not draw them here. For \( J_\alpha > \sqrt{\frac{9}{8}}M \) the solution turns complex.

There is another set of solutions when we do \( r \to -r \) in (14) or (16). In this set there are also three distinct cases all of them have closed timelike curves. Therefore, if one wants chronology protection these cases should be discarded.

5. Temperature

To include quantum field effects on the classical geometry one must compute the temperature, the entropy and other associated potentials. To display beyond
doubt what corresponds to the extreme black hole we find here the temperature \( T \) as a function of \( M \) and \( J \). By Euclideanizing metric (21) one can show that

\[
T = \frac{\alpha}{2\pi} \frac{3}{2} \left( \frac{3M}{\sqrt{1 - \frac{8J^2\alpha^2}{9M^2}} - M} \right)^{\frac{1}{4}} \left( \frac{3}{\sqrt{1 - \frac{8J^2\alpha^2}{9M^2}} + 1} \right)^{\frac{1}{2}}.
\] (22)

For \( J = 0 \) the temperature goes with \( M^{\frac{3}{2}} \). Thus it tends to zero as the horizon disappears. This is analogous to the black hole of 3D General Relativity [29] and in contrast to the Schwarzschild black hole. The extreme case is then \( J\alpha = M \), for which \( T = 0 \).

6. Conclusions

The cylindrical black hole has a rich structure which can be further explored at the quantum and the classical level. The extension to include electromagnetic fields is under study [25]. It is remarkable that the 3D version has given us insights into ill-defined quantities (such as mass and angular momentum) in the 4D spacetime.
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