Composite branch-point twist fields in the Ising model and their expectation values

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Abstract
We investigate a particular two-point function of the $n$-copy Ising model. That is, the correlation function $\langle \varepsilon(r) T(0) \rangle$ involving the energy field and the branch-point twist field. The latter is associated with the symmetry of the theory under cyclic permutations of its copies. We use a form factor expansion to obtain an exact integral representation of $\langle \varepsilon(r) T(0) \rangle$ and find its complete short-distance expansion. This allows us to identify all the fields contributing in the short-distance massive OPE of the correlation function under examination, and fix their expectation values, conformal structure constants and massive corrections thereof. Most contributions are given by the composite field $\varepsilon T$ and its derivatives. We find all non-vanishing form factors of this latter operator.

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1. Introduction

The idea of associating a local field with a branch point in a CFT on a Riemann surface was first introduced in [1], and deepened in [2]. It became then natural in the context of the evaluation of bi-partite entanglement entropy for two-dimensional integrable QFTs (see i.e. [3] for a review), and for non-integrable ones [4]. These models are usually quantized on a line, and then evolved in Euclidean time $\tau = it$. The idea is to divide the system at $\tau = 0$ into two regions, and then consider $n$ copies of the original model. These ‘replicas’ are connected cyclically along a branch cut which covers one of the two regions (the choice is indeed arbitrary). The boundary points of the cut are branch points of degree $n$, and a field $T$ is associated with them, related to the $\mathbb{Z}_n$ symmetry of the global model. These fields are called twist fields, and are local with respect to the Hamiltonian density of the $n$-copy model $h_n(x) = \sum_{j=1}^n h_j(x)$, where $h_j(x)$ is evaluated in the $j$th sheet of the $n$-root map which connects the Riemann surface with $\mathbb{R}^2$. Consider for simplicity $\mathcal{M}_0$ the manifold with a single branch point in $x = 0$ and $\mathbb{R}^2$ the disconnected $n$ replicas, then the action of the twist field on the expectation values of the theory is defined as

$$\langle \cdots \rangle_{\mathcal{M}_0} = \frac{\langle T(0) \cdots \rangle_{\mathbb{R}^2_n}}{\langle T \rangle_{\mathbb{R}^2_n}}. \quad (1)$$
The counterpart of the twist field in the ultraviolet conformal theory is a spinless operator of conformal dimension $[2]$
$$\Delta_\tau = \bar{\Delta}_\tau = \frac{c}{24} \left( n - \frac{1}{n} \right).$$

In a recent paper [5] involving the author, it was first observed that a function that generalizes $\Delta_\tau$ to massive theories, known in the literature as $\Delta$-function [6], has all the right properties to be a $c$-function of the theory. That is, it has the same qualitative features of Zamolodchikov’s $c$-function [7], even if it is different from it. This observation was put on a more solid basis in [8], where the correlation function $\langle \phi(t)T(0) \rangle$ between the perturbing field and the twist field in general $1 + 1$-dimensional integrable theories was considered. In particular, it was noted that it can be obtained exactly for the $n$-copy thermally perturbed Ising model, where $\phi = \varepsilon$, that is, the energy field of conformal dimension $\Delta = \bar{\Delta} = 1/2$. This gives us the opportunity of investigating the massive OPE for the operator product $\varepsilon(r)T(0)$ in great detail. This is the main issue we want to address in this paper.

This work is organized as follows. Section 2 is devoted to describing the model and setting the notation. The analysis of the two-point function $\langle \varepsilon(r)T(0) \rangle$ and the identification of the operator : $\varepsilon T :$ are carried out in section 3. The two-particle form factor of this composite operator is given in section 4 and higher particle form factors are given in section 5. In section 6, we collect all our main conclusions and discuss open problems.

2. The Ising model and twist field form factors

The two-dimensional thermally perturbed Ising model can be described by a relativistic quantum field theory on a line with action
$$S_E = i \int d^2x (\bar{\psi} (\partial_t - \partial_x) \psi - \psi (\partial_t + \partial_x) \bar{\psi} - m \bar{\psi} \psi),$$
which can be obtained by perturbing the critical Ising model with the energy operator and introducing a mass scale $m$ which is related to the temperature. While in the conformal theory this field can be taken to be the product of two fermionic fields, in the off-critical model, due to the presence of $m$, there is an additional mixing with the identity operator, so that
$$\varepsilon = a \bar{\psi} \psi + bm \| \quad \text{with} \quad a, b \in \mathbb{R} \setminus \{0\}. \quad (4)$$

In (3), $\psi$ and $\bar{\psi}$ are the two real components of a Majorana spinor, and they can be represented as
$$\psi(\tau, x) = \sqrt{\frac{m}{4\pi}} \int d\theta e^{\frac{i}{2} \{ a(\theta) e^{m(\tau \sinh \theta - \tau \cosh \theta)} + a^\dagger(\theta) e^{-m(\tau \sinh \theta + \tau \cosh \theta)} \}},$$
$$\bar{\psi}(\tau, x) = -i \sqrt{\frac{m}{4\pi}} \int d\theta e^{-\frac{i}{2} \{ a(\theta) e^{m(\tau \sinh \theta - \tau \cosh \theta)} - a^\dagger(\theta) e^{-m(\tau \sinh \theta + \tau \cosh \theta)} \}}, \quad (5)$$
where $\theta$ is the rapidity, and the mode operators $a(\theta)$ and $a^\dagger(\theta)$ satisfy the canonical commutation relations
$$\{ a(\theta), a^\dagger(\theta') \} = \delta(\theta - \theta') \{ a(\theta), a(\theta') \} = \{ a(\theta), a^\dagger(\theta') \} = 0. \quad (6)$$

The Hilbert space of this model is the Fock space built over the postulated vacuum $|0\rangle$ and the algebra (6). The action of the operators $a$ and $a^\dagger$ is
$$a(\theta) |0\rangle = |\theta\rangle a^\dagger(\theta) |0\rangle = 0 \quad a^\dagger(\theta) |\theta\rangle = |\theta\rangle \quad 0 |a(\theta)\rangle = |\theta\rangle; \quad (7)$$
and a general vector is defined by multiple applications of (7)
$$|\theta_1, \theta_2, \ldots, \theta_n > = a^\dagger(\theta_1) a^\dagger(\theta_2) \cdots a^\dagger(\theta_n) |0\rangle, \quad (8)$$
where a basis is chosen setting $\theta_1 > \theta_2 > \cdots > \theta_n$. 

The fields of the theory act as linear operators on the Hilbert space. For an operator $O$, the general matrix element

$$F^O_k(\theta_1, \ldots, \theta_k) = \langle 0|O(0)|\theta_1, \ldots, \theta_k \rangle_{\text{in}}$$  \hspace{1cm} (9)

is known in the literature as a form factor\(^1\). These quantities are particularly useful in the context of integrable theories, and they provide a non-perturbative expansion for two-point functions. Indeed by the introduction of the resolution of the identity over the basis (8), it is possible to express the correlator between two operators $O_1$ and $O_2$ as

$$\langle O_1(x)O_2(0) \rangle = \sum_{k=1}^{\infty} \int_{\theta_1 > \theta_2 > \cdots > \theta_k} \frac{d\theta_1 \cdots d\theta_k}{(2\pi)^k} F^{O_1}_k(\theta_1, \ldots, \theta_k)F^{O_2}_k(\theta_1, \ldots, \theta_k) e^{-m \sum_i \cosh \theta_i}. \hspace{1cm} (10)$$

The convergence of this series is remarkably fast for most integrable models and leads to a good approximation just considering the first few terms. The aforementioned constants $a$ and $b$ in (4) are fixed by the normalization

$$\epsilon(x) = 2\pi(m+1) : \bar{\psi}(x)\psi(x) :,$$  \hspace{1cm} (11)

where $aa^\dagger = \eta = +1$ is the definition of normal ordering in this case. The two-particle form factor can be easily extracted using (5):

$$F^2_k(\theta_1, \theta_2) = \langle 0|\epsilon(0)\theta_1\theta_2 \rangle_{\text{in}} = -\frac{i m}{2} \int d\phi d\eta \, e^{\frac{\epsilon^2}{2}} \langle 0|a(\eta)aa^\dagger(\theta_1)aa^\dagger(\theta_2)|0 \rangle$$

$$= -im \sinh \frac{\theta_1 - \theta_2}{2}, \hspace{1cm} (12)$$

where the Wick theorem for fermion algebra is used to define contractions. The normalization (11) is chosen in light of the relation between $\epsilon$ and the trace of the stress energy tensor $\Theta(x) = 2\pi m \epsilon(x)$ [9], to give $F^2_k(i\pi) = 2\pi m^2$.

2.1. Form factors construction for $T$

The twist field is uniquely defined by its scaling dimension (2), its exchange relations with the other fields of the theory, and by the requirement that it is invariant under all those symmetries of the $n$-copy model which commute with $\mathbb{Z}_n$. In particular, its action on the fermion fields is

$$\psi_i(\tau, y)T(\tau, x)\psi_{i+1}(\tau, y) = T(\tau, x)\psi_{i+1}(\tau, y) \hspace{1cm} x > y,$$

$$\psi_i(\tau, y)T(\tau, x)\psi_{i-1}(\tau, y) = T(\tau, x)\psi_{i-1}(\tau, y) \hspace{1cm} x < y,$$  \hspace{1cm} (13)

where $\psi_i$ is the fermionic field of the $i$th copy, and the same relations hold for $\bar{\psi}_i$. These non-trivial relations modify Watson’s equations

$$F^T_{k,n_1,\ldots,n_{i+1}}(\ldots, \theta_1, \theta_{i+1}, \ldots) = S_{n_1,\ldots,n_{i+1}}^{(n)}(\theta_{i+1})F^T_{k,n_1,\ldots,n_{i+1}}(\ldots, \theta_{i+1}, \theta_i, \ldots), \hspace{1cm} (14)$$

$$F^T_{k,n_1,\ldots,n_i}(\theta_1 + 2\pi i, \ldots, \theta_k) = F^T_{k,n_1,\ldots,n_{i+1}}(\theta_2, \ldots, \theta_k, \theta_i), \hspace{1cm} (15)$$

where particles’ indices $n_i$ label the copy they belong to, $\theta_{i+1} = \theta_i - \theta_{i+1}$ and, for the Ising model, $S_{n_1,\ldots,n_{i+1}}^{(n)}(\theta_{i+1}) = (-1)^{n_{i+1}/2}$. The pole structure is also modified by (13) which

\(^1\) In general, form factors depend on a complete set of quantum numbers. The one-copy Ising model though has only one kind of ‘particle’ in its spectrum, so that a form factor needs no other specifications than rapidities of incoming particles, whereas for the $n$-copy theory there is an additional number that has to be specified that labels the copy the ‘particles’ belong to.\n
\(^2\) Throughout this paper, the following notation will be adopted: $O(\tau = 0, x) = O(x)$.\n
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introduces an extra kinematic equation as follows:

$$\text{Res}_{\theta_0 = \theta_\text{lab}} F_{k+2}^{T|n_1 n_2 \ldots n_k} (\theta_0 + i\pi, \theta_0, \theta_1, \ldots, \theta_k) = i F_{k}^{T|n_1 n_2 \ldots n_k} (\theta_1, \ldots, \theta_k), \quad (16)$$

$$\text{Res}_{\theta_0 = \theta_\text{lab}} F_{k+2}^{T|n_1(n_1+1)n_2 \ldots n_k} (\theta_0 + i\pi, \theta_0, \theta_1, \ldots, \theta_k) = -i \prod_{j=1}^k S_{n_j n_j} (\theta_{0j}) F_{k}^{T|n_1 \ldots n_k} (\theta_1, \ldots, \theta_k). \quad (17)$$

These equations connect form factors of different orders, and can be used to check the validity of any form factors associated with the twist field. The two-particle form factor for the Ising model was found in [10], and can be expressed as

$$F_{2}^{T|11} (\theta) = \frac{i}{n} \frac{\sinh \left( \frac{\theta}{2} \right)}{\sinh \left( \frac{n\theta}{2} \right) \sinh \left( \frac{2n\theta}{n^2} \right)}, \quad (18)$$

and the exact expression of $\langle \langle T \rangle \rangle$ was also given in that paper. Moreover, higher particle form factors were also extracted, and they showed a Pfaffian structure. As $T$ is an even operator under the $\mathbb{Z}_2$ symmetry of the Ising model only even particle form factors are non-vanishing, and they have the form

$$F_{2k}^{T|11 \ldots 1} (\theta_1, \ldots, \theta_{2k}) = \langle \langle T \rangle \rangle \text{Pf}(K), \quad (19)$$

where $\text{Pf}(K)^2 = \det(K)$ is a Pfaffian, and $K$ is an anti-symmetric $2k \times 2k$ matrix, with entries

$$K_{ij} = \frac{F_{2}^{T|11} (\theta_j)}{\langle \langle T \rangle \rangle}. \quad (20)$$

### 3. An important OPE and composite twist fields

In [8], the authors were interested in the computation of the delta-function of the twist field

$$\Delta (x) = -\frac{n}{2} \int_x^\infty d s \left[ \frac{\langle \Theta (s) T (0) \rangle_{\mathbb{R}_n^2}}{\langle \langle T \rangle \rangle_{\mathbb{R}_n^2}} - \langle \Theta (s) \rangle_{\mathbb{R}_n^2} \right], \quad (21)$$

where $\Theta$ is the trace of the energy–momentum tensor. In this and the following paragraphs instead of dealing with fields on a single copy (as we did in the introduction), we will rather be dealing with the sum of all the branches of the $n$-root map which connects $M_0$ to $\mathbb{R}^2$, such that $\varepsilon (x) = \sum_{\eta=1}^n \epsilon_\eta (x)$. There, evidence was given that this function has all the right properties to be a $c$-function. In light of the relation between $\Theta$ and $\varepsilon$ mentioned in section 2, we will henceforth refer directly to this second operator. Indeed for $\varepsilon$ only the two-particle form factor is non-vanishing and it is possible to find an exact integral representation for the correlation function

$$\langle \langle \varepsilon (r) T (0) \rangle \rangle = \langle \langle T \rangle \rangle - \frac{\langle \langle T \rangle \rangle m}{2\pi^2} \cos \frac{\pi}{2n} \int_\infty^{-\infty} d \phi \frac{K_0 (2m r \sinh \frac{\phi}{2} \cosh \frac{\phi}{2}) \sin \frac{\phi}{2} \sinh \frac{\phi}{2}}{\cosh \frac{\phi}{n} - \cos \frac{\phi}{n}}, \quad (22)$$

where $K_0 (\phi)$ is the modified Bessel function of the second kind of argument $\phi$. In the CFT, where the correlation length tends to infinity, one is allowed to use the OPE:

$$\varepsilon (r) T (0) = \sum_k C_k^\varepsilon T^{\Delta_k - \frac{1}{2} - \Delta_T} \mathcal{O}_k (0), \quad (23)$$

$\varepsilon$ From now on the subscript labelling the manifold VEVs are evaluated on will be omitted. It is implicit that $\langle \langle T \rangle \rangle$ is always evaluated on $n$ disconnected copies of $\mathbb{R}^2$, while other VEVs are on the real plane.

$^4$ This OPE is as usual to be taken in the ‘weak’ sense; indeed it has a meaning only once plugged into a matrix element of the CFT on $\mathbb{R}^2_n$. 

$^4$
where \( \mathcal{O}_k \) is a basis of fields and \( C_{k,T}^0 \) are the dimensionless constants of the expansion. The most relevant operator appearing in (23) is the composite twist field \( \mathcal{O}_0 \equiv: \varepsilon T : \), and it is defined implicitly as the twist operator which corresponds to the leading term. Its conformal weight and structure constant with \( \varepsilon \) and \( T \) are known [8]:

\[
\Delta_{\varepsilon T} = \frac{1}{2n} + \Delta_T \quad \text{and} \quad C_{\varepsilon T}^0 = \frac{1}{n}. \tag{24}
\]

We know that, due to the arbitrariness in the choice of the argument of\(^5\): \( \varepsilon T : \), also derivatives of this field play a role in the OPE (23). We want to fix the weights and structure constants for these corrections. In order to address this issue, we use the same methodology applied in [8] to the leading term: we consider the transformation law of the field \( \varepsilon \) under the conformal map which connects \( M_0 \) to the plane, and we Taylor expand about the origin in the variables \( z(\bar{z}) = x + (-)i\tau \) on the manifold. This allows us to identify \( \mathcal{O}_\alpha \equiv: \partial^{2\alpha} \varepsilon T : \) with \( \alpha \in \mathbb{N} \)

\[
\Delta_{\partial^{2\alpha} \varepsilon T} = \frac{1 + 2\alpha}{2n} + \Delta_T \quad \text{and} \quad C_{\partial^{2\alpha} \varepsilon T} = \frac{1}{n(\alpha!)}^2, \tag{25}
\]

where \( \partial^{2\alpha} \equiv (\partial^2)^\alpha \).

Note that in (25), we considered only the case in which the number of derivatives on \( z \) matches the one of those in \( z \), even though in (23) also non-matching terms could appear in principle. This is due to the fact that we are eventually interested in the correlation function \( \langle \varepsilon(r)T(0) \rangle \) in the massive theory, where vacuum expectation values (VEVs) \( \langle \partial^k \partial^l \varepsilon T \rangle \) for \( k \neq l \) are null. From now on we will refer directly to the operators : \( \partial^{2\alpha} \varepsilon T : \), denoting : \( \varepsilon T : \) as the case \( \alpha = 0 \).

In the massive theory an analogous but different OPE exists [11] which can be extracted perturbatively from (23), and can be used to express the lhs of (22). Here and throughout this paper, the assumption that there be the same basis of operators in the massless and massive theories is pushed forward, such that we will refer with the same symbols (with an abuse of notation) to the CFT and perturbed fields. This expansion is in a sense richer than the conformal notation) to the CFT and perturbed fields. This expansion is in a sense richer than the conformal

5 Indeed the choice to take as argument 0 is arbitrary, and one could have chosen any point in the interval \([0, r]\).

The difference between these choices is represented by a Taylor expansion about 0; therefore; the OPE (23) should include all derivatives of the field : \( \varepsilon T : \).
to determine all \( C_{j}^{a_{2}eT} \): in a systematic way. Therefore, we expect the OPE in the massive theory to take the form

\[
\varepsilon(r) T(0) = \sum_{a=0}^{\infty} C_{j}^{a_{2}eT} \cdot (r) + \partial^{2a} \varepsilon T : (0);
\]

although we see later that further corrections to (28), that are not predictable from CFT arguments, will also appear.

### 3.1. Computation of \( \langle \partial^{2a} \varepsilon T \rangle \) and \( C_{j}^{a_{2}eT} \)

Let us now proceed by expanding the integral on the rhs of (22) in a small \( r \) region, and then compare the result with the massive OPE (28), that is we compare terms with the same dimensions. In order to extract all the information needed, it is convenient first to reexpress the integral in terms of the quantity \( t = mr \varepsilon \).

\[
- \frac{m}{\pi^{2}} \cos \frac{\pi}{2n} (mr)^{\frac{1}{2}-1} \int_{mr}^{\infty} dt \frac{1}{t} K_{0} \left( t + \frac{(mr)^{2}}{t} \right) \left[ 1 - \left( \frac{t}{mr} \right)^{2} \right] \left[ 1 - \left( \frac{mr}{t} \right)^{2} \right] \]

\[
\left( \frac{t}{mr} \right)^{\frac{2}{2}} - 2 \cos \frac{\pi}{2n} \left( \frac{t}{mr} \right)^{\frac{2}{2}} + 1.
\]

One can note that leading contributions to this integral are given for large \( t/(mr) \). This provides a natural parameter over which it is possible to expand the fraction in (29) as

\[
\left[ 1 - \left( \frac{t}{mr} \right)^{2} \right] \left[ 1 - \left( \frac{mr}{t} \right)^{2} \right] = \sum_{a=0}^{\infty} \Omega_{a} \left( \frac{t}{mr} \right)^{-\frac{2}{2}}
\]

where the coefficients \( \Omega_{a} \) are real numbers, evaluated in appendix A. Once (30) is plugged into (29) one obtains

\[
- \frac{m}{\pi^{2}} \cos \frac{\pi}{2n} \sum_{a=0}^{\infty} \Omega_{a} \left( mr \right)^{\frac{1}{2}+a-1} \int_{mr}^{\infty} dt \frac{1}{t} \frac{1}{2} K_{0} \left( t + \frac{(mr)^{2}}{t} \right).
\]

The dependence on \( mr \) of the Bessel function can be extracted by expanding it for small \( r \) as

\[
K_{0} \left( t + \frac{(mr)^{2}}{t} \right) = K_{0}(t) - K_{1}(t) (mr)^{2} + \frac{K_{0}(t) + K_{1}(t)}{4t} (mr)^{4} + O(mr)^{6}.
\]

The resulting integrals can be thought of as a special case of a known integral of the Maijer-G-function, as explained in appendix B. Substituting (32) into (31) and carrying out the integrals gives

\[
\langle \varepsilon(r) T(0) \rangle = - \frac{(T)m}{\pi^{2}} \cos \frac{\pi}{2n} \sum_{a=0}^{\infty} \Omega_{a} \left[ \frac{\Gamma \left( \frac{n-2a}{2} \right)}{2^{1+\frac{2n}{2}}} (mr)^{\frac{1+2a}{2}} \right. 
\]

\[
+ \frac{n}{n+1+2a} \frac{\Gamma \left( \frac{n-2a}{2} \right)}{2^{1+\frac{2n}{2}}} (mr)^{\frac{1+2a}{2}+1} \left( \frac{n}{n+1+2a} \right)^{2} \frac{\Gamma \left( \frac{n-2a}{2} \right)}{2^{2+\frac{2n}{2}}} (mr)^{\frac{1+2a}{2}+3} 
\]

\[- \left. \frac{n^{3}}{(n+1+2a)^{2}(3n+1+2a)} \frac{\Gamma \left( \frac{n-2a}{2} \right)}{2^{2+\frac{2n}{2}}} (mr)^{\frac{1+2a}{2}+3} + \ldots \right] ,
\]

where we reported only the first contributions related to the composite twist field and its descendants, as we are mainly interested in those. After a bit of manipulation, we can compare term by term this expansion with (28), and by matching the terms with the same perturbative
order we are able to extract the following VEVs:

\[ \langle \partial^{2\alpha} \epsilon T \rangle = -\frac{\cos \left( \frac{(1+2\alpha)\pi}{2n} \right)}{2^{1+\frac{1}{2n}}\pi^2} - 1 . \tag{34} \]

This is the main result of this section. In the same way, we can fix the constants \( C_{\alpha}^{\partial^{2\alpha} T} \) to every order. A major challenge when doing this is to be able, for terms proportional to the same power of \( r \), to distinguish between contributions to expectation values and to structure constants. It turns out that this ambiguity can be resolved by requiring that the expectation values (34) are continuous functions of \( n \) for each fixed value of \( \alpha \). This requirement is natural because of the special relation between \( T \) and the entanglement entropy, in which context it is necessary to analytically continue all physical quantities to \( n \in [1, \infty) \). The first few non-vanishing coefficients are

\[
C_2^{\partial^{2\alpha} T} = \frac{n}{n + 1 + 2\alpha} + \frac{n^2}{(n + 1 + 2\alpha)^2} \tan \left( \frac{(1+2\alpha)\pi}{2n} \right) \tan \frac{\pi}{2n} \\
C_4^{\partial^{2\alpha} T} = \frac{n^2}{2(n + 1 + 2\alpha)(3n + 1 + 2\alpha)} + \frac{n^3(1 + 2\alpha + 2n)}{(1 + 2\alpha + n)^2(1 + 2\alpha + 3n)^2} \frac{\tan \left( \frac{(1+2\alpha)\pi}{2n} \right)}{\tan \frac{\pi}{2n}} \\
C_6^{\partial^{2\alpha} T} = \frac{n^3(4n + 1 + 2\alpha)}{6(n + 1 + 2\alpha)^2(3n + 1 + 2\alpha)(5n + 1 + 2\alpha)} + \frac{n^4(4n + 1 + 2\alpha)}{\sin \left( \frac{(1+2\alpha)\pi}{2n} \right)} - \frac{n^4(4n + 1 + 2\alpha)}{\sin \left( \frac{\pi}{2n} \right)^2(3n + 1 + 2\alpha)^2(5n + 1 + 2\alpha)} \left( 1 - 2n - \frac{2n^2}{1 + 3n + 2\alpha} \right). \tag{35} \]

We find that \( C_{2\alpha+1}^{\partial^{2\alpha} T} = 0 \), for \( j \in \mathbb{N}_0 \).

It is worth noting that VEVs of these composite operators are singular whenever the argument of the Gamma-function is either zero or a negative integer. Analysing (34), one can see that when \( n = 1 + 2\alpha \) the VEV of the 2\( \alpha \)th derivative is divergent. In other words, such singularities can only occur for odd values of \( n \). Therefore, it follows that the analysis above is only really consistent when restricting \( n \) to be even, in which case no singularities for special values of \( n \) and \( \alpha \) arise. In the \( n \) odd case, the singularities that occur at various orders of the expansion actually cancel each order, generating a well-defined short-distance expansion. This stark contrast between the \( n \) even and \( n \) odd cases is a priori rather surprising. At present, we do not have a clear understanding of why it happens. It is a matter which we might come back to in future research. The leading term in (28) is well defined for all values of \( n > 1 \), for both \( n \) even and odd so that our identification of the expectation value of \( : \epsilon T : \), already performed in [8], does still hold for general values of \( n \).

### 3.2. Logarithmic corrections to the massive OPE

Analysing carefully the contributions reported in appendix B, one can note that further corrections involving terms of the form \( (mr)^{2\alpha} \) and \( (mr)^{2\alpha} \log(mr) \), with \( \alpha = 0, 1, \ldots \), occur (note for example the logarithmic terms in (B.7) and (B.8)), such that (28) is modified to

\[ \epsilon(r)T(0) = \sum_{\alpha=0}^{\infty} [\gamma_{\epsilon T}^{\partial^{2\alpha} T}(r) : \partial^{2\alpha} \epsilon T : (0) + m \gamma_{\epsilon T}^{\partial^{2\alpha} T}(r) \partial^{2\alpha} T(0)]. \tag{36} \]

where the new terms mentioned above are contained in the coefficients \( \gamma_{\epsilon T}^{\partial^{2\alpha} T}(r) \). It is clear that they are features of the massive theory as they have no counterpart in the CFT. To understand
their presence one has to consider once more (4). The term proportional to $\bar{\psi}\psi$ is responsible for all the composite fields: $\partial^2 \alpha T$: in the OPE, whereas the term proportional to the identity generates contributions proportional to $mT$ and its derivatives $m\partial^2 \alpha T$. It is interesting that this paves the way for the evaluation of $\langle \partial^2 \alpha T \rangle$, although in this case, due to the structure of the expansion, this would need a resummation of infinitely many terms. The presence of logarithmic terms is imputable to the freedom in choosing $b$ in (4). Indeed in the expansions carried out in appendix B we have general terms of the type

$$(mr)^{2\alpha} \sum_{\beta=0}^{\infty} \left( \Sigma(\alpha, \beta, n) + \Lambda(\alpha, \beta, n) \log m r \right),$$

where $\Sigma$ and $\Lambda$ are two rational functions of $\alpha$, $\beta$ and $n$. When a correction of the kind of (37) is plugged into (33), it gives for fixed $\alpha$

$$-\langle T \rangle m \pi \frac{\cos \pi}{2n} (mr)^{2\alpha} \sum_{\beta=0}^{\infty} \Omega_\beta(n) (\Sigma(\alpha, \beta, n) + \Lambda(\alpha, \beta, n) \log m r).$$

The term containing $\Sigma$ contributes to $\langle \psi \rangle \langle \partial^2 \alpha T \rangle$. The presence of the logarithmic term allows us to rewrite (38) as

$$-\langle T \rangle m \pi \frac{\cos \pi}{2n} (mr)^{2\alpha} \sum_{\beta=0}^{\infty} \Omega_\beta(n) \left( \Sigma(\alpha, \beta, n) + \Lambda(\alpha, \beta, n) \log m r + \Sigma(\alpha, \beta, n) \log m \right).$$

where $\delta \in \mathbb{R}^+ \setminus \{0\}$. This corresponds to a redefinition of $\langle \psi \rangle$, and one can note that a logarithmic correction is the only functional form that allows this to happen. This is not the first time a logarithmic correction to two-point functions of the Ising model has been observed (see for example the spin–spin correlation function in [15]). In both cases, their presence is fully explained by the ambiguity in the definition of $\langle \psi \rangle$.

4. Two-particle form factors of composite operators

This section is dedicated to the evaluation of form factors for composite twist fields of the type introduced in section 3. These operators are formally defined as the regularized limit of an operator $O$ approaching the twist field in the original CFT. The regularization defines the meaning of the ordered product, which can be taken to be a point splitting

$$O\alpha T : (x) \sim \lim_{\epsilon \to 0} O(x+\epsilon) T(x).$$

One can start by considering $\psi T :$ as a benchmark. Since $T$ is even and $\psi$ is odd under the $\mathbb{Z}_2$ symmetry of the Ising model, only odd particle form factors will be non-vanishing. Considering then the matrix element $\langle 0 | \psi(x) T(0) | \theta \rangle$, the one-particle form factor can be extracted by looking at the leading contribution as $x$ approaches 0. It is useful to introduce the resolution of the identity for the $n$-copy system

$$I = \sum_{k=1}^{\infty} \sum_{\mu_1, \ldots, \mu_k} \int_{\theta_1, \ldots, \theta_k} (2\pi)^k \delta(\theta_1, \ldots, \theta_k) \mu_1, \ldots, \mu_k, \mu, \ldots, \mu, \theta_1, \ldots, \theta_1),$$

where $\mu_i$ label the number of the copy and all quantum numbers of the $i$th particle. In this way one can write

$$\langle 0 | \psi(x) T(0) | \theta \rangle = \frac{r}{2\pi} \int_{-\infty}^{\infty} d\phi \langle 0 | \psi(x) | \phi \rangle = \langle \phi | T(0) | \theta \rangle.$$
The matrix element $\langle 0 | \psi (x) | \phi \rangle$ can be easily extracted from (5)
\[
\langle 0 | \psi (x) | \phi \rangle = \sqrt{\frac{m}{4\pi}} e^{i \pi p x},
\]
while $\langle \phi | T(0) | \theta \rangle$ is linked to the two-particle form factor by crossing [12]
\[
\langle \phi | T(0) | \theta \rangle = \langle 0 | T(0) | \theta, \phi + i \pi - i \varepsilon^+ \rangle + (T) \delta (\theta - \phi),
\]
where the introduction of $i \varepsilon^+$ has to be thought of in the distributions sense, and describes how to avoid the pole at $i \pi$.

Plugging (43) and (44) into (42) yields
\[
\langle 0 | \psi (x) | T(0) | \theta \rangle = \langle T \rangle \sqrt{\frac{m}{4\pi}} e^{i \pi p x} + \frac{i}{2\pi} \sqrt{\frac{m}{4\pi}} (T) \cos \frac{\pi}{2n} \times \int d\phi e^{i \pi p x} \frac{\sinh (\theta - \phi - i \pi/2)}{\sinh (\theta - \phi + i \pi/2)}.
\]

The leading term when $x$ approaches 0 is given by the integral part that is divergent, but can be made convergent by shifting the domain of integration from the real axes $\mathbb{R}$ to $\mathbb{R} + i \pi/2$. Such a change of variable does not affect the result of the integration, as the integrand has no poles in the region between the two axes.\(^6\)

The integral in (45) becomes
\[
e^{-i\frac{\pi}{2}} \int d\phi e^{\frac{i}{2} - i E_\theta} \frac{\sinh (\theta - \phi - i \pi/2)}{\sinh (\theta - \phi + i \pi/2)}.
\]

In this integral the main contributions come from large $|\phi|$, so that, after splitting the integration path into positive and negative regions, one can find two series expansions for the fraction in (46). The leading term for small $x$ is given by the positive $\psi$ part, that is
\[
-2 e^{-i \frac{\pi}{2}} (1 + \frac{1}{2}) e^{\frac{\pi}{2}} \int_0^{\infty} d\phi e^{\frac{\pi}{2} (1 - \frac{1}{2}) - i E_\theta} + \ldots
\]
\[
= -2 e^{-i \frac{\pi}{2}} (1 + \frac{1}{2}) e^{\frac{\pi}{2}} \int_{-\infty}^{0} d\phi e^{-m \cosh \phi} \cos \left[ \frac{\phi}{2} \left( 1 - \frac{1}{n} \right) \right] + \ldots
\]
\[
= -2 e^{-i \frac{\pi}{2}} (1 + \frac{1}{2}) e^{\frac{\pi}{2}} 2^{1/2} (1 - \frac{1}{2}) \Gamma \left( \frac{n}{2n} \right) (m \pi)^{-\frac{1}{2}} (1 - \frac{1}{2}) + \ldots.
\]

The one-particle form factor for $\psi T$ can be read out from (45) and (47):
\[
F_1^{\psi T | 1} (\theta) = -\frac{e^{-i \frac{\pi}{2}} (1 + \frac{1}{2})}{\pi} 2^{1/2} (1 - \frac{1}{2}) \Gamma \left( \frac{n}{2n} \right) \cos \left( \frac{\pi}{2n} \right) \frac{m \pi}{\sqrt{4\pi}} e^{\pi \theta}.
\]

The procedure used to obtain this result was introduced for descendants of twist fields in [13]. In that case the authors were dealing with the twist field associated with the global $U(1)$ symmetry of the Dirac Lagrangian. In this and next sections extensive use of this procedure is made, demonstrating its consistency for a different kind of twist fields.

4.1. Two-particle form factor of $\epsilon T$

One of the main focuses of this paper is the computation of form factors of composite operators defined as in (40) with $O = \epsilon$. The matrix element $\langle 0 | \theta(x) T(0) | \theta_1, \theta_2 \rangle$ can be written by means of (41) as
\[
\langle 0 | \epsilon(x) T(0) | \theta_1, \theta_2 \rangle = \frac{1}{4\pi^2} \sum_{j_1, j_2} \int d\phi_1 d\phi_2 \langle 0 | \epsilon(x) | \phi_1, \phi_2 \rangle \langle j_1, j_2 | \phi_2, \phi_1 | T(0) | \theta_1, \theta_2 \rangle.
\]

\(^6\) Note that it is the term $i \varepsilon^+$ which allows us to perform this shift.
where $f$ labels the copy number, and is repeated because $\epsilon$ connects only particles on the same copy.

The matrix element $j,j|\phi_2, \phi_1|T(0)|\theta_1, \theta_2\rangle$ is connected to the four-particle form factor of the twist field by the crossing relation (44), which used repeatedly gives

$$j,j|\phi_2, \phi_1|T(0)|\theta_1, \theta_2\rangle = \langle 0|T(0)|\theta_1, \theta_2, \phi_1 + i\sigma - i\epsilon^+, \phi_2 + i\sigma - i\epsilon^+\rangle_{1,1,j}$$

$$+ \delta(\theta_1 - \phi_1)\delta_{1,j}(0|T(0)|\theta_2, \phi_2 + i\sigma - i\epsilon^+)_{1,j}$$

$$+ \delta(\theta_2 - \phi_2)\delta_{1,j}(0|T(0)|\theta_1, \phi_1 - i\sigma - i\epsilon^+)_{1,j}$$

$$- \delta(\theta_1 - \phi_2)\delta_{1,j}(0|T(0)|\theta_2, \phi_2 + i\sigma - i\epsilon^+)_{1,j}$$

$$- \delta(\theta_2 - \phi_1)\delta_{1,j}(0|T(0)|\theta_1, \phi_1 - i\sigma - i\epsilon^+)_{1,j}$$

$$+ \delta(\theta_1 - \phi_1)\delta(\theta_2 - \phi_2)\delta_{1,j} - \delta(\theta_1 - \phi_2)\delta(\theta_2 - \phi_1)\delta_{1,j}.$$  \[(50)\]

Even if (50) is quite cumbersome one can note that many terms can be extracted from the others with the exchange $\theta_1 \leftrightarrow \theta_2$, which leaves the integration untouched. The leading contribution for small $x$ is given by the first term (the one without deltas). This four-particle form factor considers two particles on the first copy and two on copy $j$, but can be reduced to one where particles are considered on the same copy (say 1) with multiple applications of equations (14) and (15):

$$\langle 0|T(0)|\theta_1, \theta_2, \phi_1 + i\sigma - i\epsilon^+, \phi_2 + i\sigma - i\epsilon^+\rangle_{1,1,j}$$

$$= \mathcal{F}_{2}^{T|11}(\theta_1, \theta_2, \phi_1 + i\sigma - i\epsilon^+, \phi_2 + i\sigma - i\epsilon^+)$$

$$= \mathcal{F}_{2}^{T|1111}(\theta_1, \theta_2, \phi_1 + (2j - 1)i\sigma, \phi_2 + (2j - 1)i\sigma)_{+},$$  \[(51)\]

and this allows us to use the Pfaffian structure (19) to re-express it. In the last step of (51) the notation $(\cdots)_{+}$ is introduced to indicate that any pole on the real axes of what is in the brackets is avoided with the $i\epsilon^+$ prescription. With the help of (19), (20) and (12), one can rewrite the leading term of (49) as

$$\frac{\text{im}}{\langle T \rangle} \frac{1}{4\pi^2} \sum_{j=1}^{n} \int_{\phi_1,\phi_2} d\phi_1 d\phi_2 \sinh \left(\frac{\phi_1 - \phi_2}{2}\right) e^{-i\epsilon(p_{\phi_1} + p_{\phi_2})}$$

$$\left[\mathcal{F}_{2}^{T|11}(\theta_1, \theta_2)\mathcal{F}_{2}^{T|11}(\phi_1, \phi_2) - \mathcal{F}_{2}^{T|11}(\theta_1, \phi_1 + (2j - 1)i\sigma)\mathcal{F}_{2}^{T|11}(\theta_2, \phi_2 + (2j - 1)i\sigma)\right]_{+}$$

$$+ \mathcal{F}_{2}^{T|11}(\theta_2, \phi_1 + (2j - 1)i\sigma)\mathcal{F}_{2}^{T|11}(\theta_1, \phi_2 + (2j - 1)i\sigma)_{+}$$  \[(52)\]

and proceed by evaluating the leading contribution for the three terms in (52). The first one is

$$- \frac{m\cos \frac{T}{\pi}}{8\pi^2} \mathcal{F}_{2}^{T|11}(\theta_1, \theta_2) \int d\phi_1 d\phi_2 \sinh \left(\frac{\phi_1 - \phi_2}{2}\right) \sinh \left(\frac{\phi_1 - \phi_2}{2}\right) \sinh \left(\frac{\phi_1 - \phi_2}{2}\right) \sinh \left(\frac{\phi_1 - \phi_2}{2}\right)$$

$$\sinh \left(\frac{\phi_1 - \phi_2}{2}\right) e^{-i\epsilon(p_{\phi_1} + p_{\phi_2})},$$  \[(53)\]

on which we can perform the change of variables $t = (\phi_1 - \phi_2)/2$ and $s = (\phi_1 + \phi_2)/2$ and carry out the $s$ integration to obtain

$$- \frac{m\cos \frac{T}{\pi}}{2\pi^2} \mathcal{F}_{2}^{T|11}(\theta_1, \theta_2) \int dt \frac{\sinh t}{\sinh \left(t \frac{\zeta}{n} + \frac{\zeta}{2}\right) \sinh \left(t \frac{\zeta}{n} + \frac{\zeta}{2}\right)} K_0(2mx \cosh t).$$  \[(54)\]

We note again that the leading contribution is given for large $t$ so that we can use the parity of the integrand to reduce the region of integration to $(0, \infty)$, and expand the fraction in (54) in a convergent way on this domain. The resulting integral is

$$- \frac{m\cos \frac{T}{\pi}}{\pi^2} \mathcal{F}_{2}^{T|11}(\theta_1, \theta_2) \int_0^\infty dt e^{\left(\frac{t}{2}\right)} K_0(mx e^t) + \cdots,$$  \[(55)\]

and with the change of variable $u = mx e^t$ we can extract the leading order for small $x$, that is

$$- \frac{m\cos \frac{T}{\pi}}{\pi^2} \mathcal{F}_{2}^{T|11}(\theta_1, \theta_2) \int_0^\infty du e^{-\frac{t}{2}} K_0(u) \left(mx e^t\right)^{-\frac{1}{2}} + \cdots.$$  \[(56)\]
Solving the integral we finally obtain
\[
- \frac{m \cos \frac{n}{2}}{2^{1+\frac{n}{2}} \pi^{1+\frac{n}{2}}} \left( \frac{n-1}{2n} \right)^2 F_2^{(11)}(\theta_1, \theta_2)(m)\sqrt{1-\frac{1}{n}} + \ldots.
\] (57)

The second and third terms in (52) are more involved than the first, due to their explicit dependence on the parameter \(j\). One can start by noting that the value of the third term can be extracted from the value of the second one by changing sign, and performing the exchange \(\theta_1 \leftrightarrow \theta_2\). Hence, we focus on the first term. As, in order to have a convergent integral, the integration axes have to be risen by \(i\pi/2\), we need to study the pole structure of this term. The only kinematic poles which lie on the real axes arise from the cases \(j = 1, n\) and \(\phi_1 = \theta_1\) and \(\phi_2 = \theta_2\); but they are avoided with the \(i\pi/2\) prescription. In general \(F_2^{(11)}(\theta, \phi + (2j - 1)i\pi)\) has kinematic poles for \(\phi = \theta + 2(n - j + 1)i\pi\) and \(\phi = \theta + 2(n - j)i\pi\). Considering that \(j\) runs from 1 to \(n\) one can see that all poles group in even multiples of \(i\pi\), so that the first group above the real axes is in \(\theta + 2i\pi\), and correspond to \(j = n, n - 1\). Hence, the needed shift can be safely performed.

The integral we want to evaluate is then
\[
\frac{\text{im}(T) \cos^2 \frac{\pi}{2}}{8\pi^2 n^2} \sum_{j=1}^{n} \int d\phi_1 d\phi_2 e^{-i\pi(\phi_1 + \phi_2)} \sinh \left( \frac{\phi_1 - \phi_2}{2} \right) \sinh \left( \frac{\theta_1 - (2j - 1)i\pi}{2n} \right) \sinh \left( \frac{\theta_2 - (2j - 1)i\pi}{2n} \right) \sinh \left( \frac{\phi_1 - (2j - 1)i\pi}{2n} \right) \sinh \left( \frac{\phi_2 - (2j - 1)i\pi}{2n} \right).
\] (58)

Although the fraction in (58) is rather complicated and mixes integration variables with parameters it can be dramatically simplified by means of the following identity:
\[
\sinh (\alpha_1 - \beta_1 \pm \gamma) \sinh (\alpha_2 - \beta_2 \pm \gamma) = \frac{1}{2} \left[ \cosh (\alpha_1 + \alpha_2 - (\beta_1 + \beta_2) \pm 2\gamma) - \cosh (\alpha_1 - \alpha_2 - (\beta_1 - \beta_2)) \right],
\] (59)

leading to
\[
\frac{\text{im}(T) \cos^2 \frac{\pi}{2}}{4\pi^2 n^2} \sum_{j=1}^{n} \int d\phi_1 d\phi_2 e^{-i\pi(\phi_1 + \phi_2)} \sinh \left( \frac{\phi_1 - \phi_2}{2} \right) \cosh \left( \frac{\theta_1 + \phi_1 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right) - \cosh \left( \frac{\theta_1 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right)
\]
\[
\times \cosh \left( \frac{\theta_2 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right) \cosh \left( \frac{\phi_1 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right) - \cosh \left( \frac{\phi_2 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right).
\] (60)

Now performing the same change of variable as in (54) gives
\[
\frac{\text{im}(T) \cos^2 \frac{\pi}{2}}{2\pi^2 n^2} \sum_{j=1}^{n} \int dt \, ds \, e^{-2m\pi t \sin \phi_1 t} \sinh t \cosh \left( \frac{\theta_1 + \phi_1 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right) - \cosh \left( \frac{\theta_1 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right)
\]
\[
\times \cosh \left( \frac{\theta_2 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right) \cosh \left( \frac{\phi_1 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right) - \cosh \left( \frac{\phi_2 - (\phi_1 + \phi_2) - 2(2j - 1)i\pi}{2n} \right),
\] (61)

and to make it convergent the shift \(s \rightarrow s - i\pi/2\) has been performed.

As \(x\) approaches 0 the main contribution is given by large \(t\), and \(s\) peaks around 0. It is then natural to expand the fraction in (61) in powers of \(t\). As before there is no such expansion on the whole real axes, but splitting it into the positive and negative parts allows us to consider two different series which converge respectively on the two regions. We can start considering
For the Ising model, the result of the third part of the integral in (52) can be obtained from (66) with a minus sign in front, which doubles the result. Putting (57), (63) and (65) together, we finally obtain the two-particle form factor for the field : $\epsilon T:$, that is

$$F_{\epsilon T : \vert \alpha} (\theta_1, \theta_2) = -\frac{\cos \frac{\pi}{\kappa n}}{2^{1+\frac{n}{2} \pi}} \Gamma \left( \frac{n-1}{2n} \right)^2 \left[ F_{T : \vert \alpha} (\theta_1, \theta_2) + \frac{4i \cos \frac{\pi}{\kappa n} (T) \sinh \frac{\theta_1 - \theta_2}{2n}}{2n} \right].$$

(66)

This result was the aim of this section. To check its validity one can employ (16). Indeed, since : $\epsilon T:$ is still a twist field, the same type of residue equations as for $T$ must be satisfied. Then, one can easily check using (34) and (66) that

$$\lim_{\theta \to \theta} (\theta - \theta) F_{\epsilon T : \vert \alpha} (\theta + i \pi, \theta) = i \epsilon T :,$$

(67)

which confirms the compatibility of the two results of these sections.

Note that this result satisfies all form factors equations for the twist field and has a structure of the type

$$F_{\epsilon T : \vert \alpha} (\theta_1, \theta_2) = \alpha \left[ Q_{\epsilon T : \vert \alpha} (\theta_1, \theta_2) + \beta \kappa (\theta_1, \theta_2) \right] F_{\text{min}} (\theta_1, \theta_2),$$

(68)

where $\alpha$ and $\beta$ are two-dimensional constants, $F_{\text{min}}$ is the minimal form factor of the theory, and $\kappa$ is a kernel solution of the form factor equations (see e.g. [5] for a discussion). Even if we could have understood that (66) should have had the form (68) to fulfill all twist properties it has to, we would have never been able to fix $\alpha$ and $\beta$ without the methods used in this section, and section 3.

5. Higher particle form factors of composite operators

In this section, we deal with the computation of higher particle form factors for : $\psi T :$ and : $\epsilon T :$. They can be extracted with the same methods for both operators, employing higher
particle form factors of the twist field (19). Indeed, focusing on : $\psi^T :$, when looking for the leading term for the $(2k-1)$-particle form factor, one has to deal with

$$\langle 0\vert \langle \psi(x)\vert T(0)\vert 0\rangle \vert \theta_1, \theta_2, \ldots, \theta_{2k-1} \rangle \sim \frac{n}{2\pi} \int d\phi \langle 0\vert \langle \psi(x)\vert \phi \rangle \langle 0\vert T(0)\vert 0\rangle \vert \theta_1, \theta_2, \ldots, \theta_{2k-1}, \phi + i\pi \rangle \langle \phi \rangle.$$

(69)

One is then able to isolate term by term the higher particle part, and reduce it to the same evaluation carried out in (42)–(48), obtaining finally

$$F_{2k-1}^{\psi^T|11\ldots 1} = \langle T \rangle \text{Pf}(K_{\psi^T}),$$

(70)

where $K_{\psi^T}$ is the $2k \times 2k$ matrix defined as

$$K_{\psi^T} = \begin{pmatrix} 0 & F_1^{\psi^T|1} (\theta_1) & \cdots & F_1^{\psi^T|1} (\theta_{2k-1}) \\ -F_1^{\psi^T|1} (\theta_1) & 0 & \cdots & F_1^{\psi^T|1} (\theta_{2k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ -F_1^{\psi^T|1} (\theta_{2k-1}) & -F_2^{\psi^T|1} (\theta_1, \theta_{2k-1}) / \langle T \rangle & \cdots & 0 \end{pmatrix}.$$

(71)

Higher particles form factors for : $\varepsilon^T :$ can be evaluated with the same logic, although they show a more complicated pattern, and cannot be reduced to a Pfaffian form. This is due to the presence of a kernel part in the two-particle form factor. The $(2k+2)$-particle form factor is

$$F_{2k+2}^{\varepsilon^T|11\ldots 1} (\theta_1, \theta_2, \ldots, \theta_{2k+2}) = \sum_{i<j} (-1)^{\sigma(i,j)} / \langle T \rangle 2k F_{2k}^{\varepsilon^T|11\ldots 1} (\theta_1, \theta_2, \ldots, \theta_{2k+2})_{ij} F_{2}^{\varepsilon^T|11} (\theta_i, \theta_j).$$

(72)

where $\sigma(i, j)$ is the permutation that brings $\theta_i$ and $\theta_j$ to the right of all other rapidities, while with $F(\cdots)_{ij}$ we mean a form factor of all rapidities but those two.

### 6. Conclusions

In this paper, we have investigated the short-distance behaviour of the correlation function $\langle \varepsilon (r) T(0) \rangle$ for the two-dimensional Ising model in the vicinity of the critical point beyond the leading contribution. This led us to the identification of the VEVs of a new class of twist fields, including

$$: \varepsilon^T : (x) \sim \lim_{\delta \to 0} \varepsilon (x + \delta) T(x),$$

(73)

and its derivatives. Furthermore, we managed to compute massive corrections to the structure constants up to $(mr)^6$ for all these fields. The very fact that we are able to compute VEVs of such a large set of local operators is remarkable, as the computation of VEVs for general theories and fields is known to be a very hard task and there is no general procedure to tackle it. In addition, to the contributions that would be naturally expected from the CFT theory we have found logarithmic corrections, a phenomenon that has already been observed for other correlation functions in the off-critical Ising model (see e.g. [15]). These logarithmic terms are due to the arbitrariness in the definition of $\langle \varepsilon \rangle$. In addition, we have computed all higher particle form factors of : $\varepsilon^T :$ and $\psi^T :$. By exploiting (40), we have been able to fully determine the normalization of all form factors and to provide new solutions to the form factor equations for twist fields. A byproduct of our investigation is the fact that all the expectation values $\langle : \partial^{2m} \varepsilon^T : \rangle$ are negative. Although we did not put much emphasis on this feature in this paper, this provides further evidence for the negativity of the connected correlator $\langle \varepsilon (r) T(0) \rangle$ claimed in [8].

There are a number of open problems related to this work which we would like to address in future. First, the short-distance expansion of $\langle \varepsilon (r) T(0) \rangle$ that we considered here is only well
defined for even \( n \). It would be interesting to have a deeper understanding of what happens when we consider \( n \) odd, and why these two cases are distinct. Second, this paper introduces an OPE which involves the twist field in a replica theory. This type of OPEs have never been studied, neither in the massive nor in the critical theories. A better understanding of the operator content and OPEs in replica theories is desirable. Finally, it would be interesting to understand if these composite twist fields are in any way related to the entanglement entropy of particular states in the Ising model.

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Appendix A. Coefficients \( \Omega_a(n) \)

This section is devoted to the computation of coefficients in (30). As explained in section 3, the aim is to expand the fraction on the lhs of that equation for large \( t/mr \). First of all, let us introduce the more convenient variable \( u = (t/mr)^{-2/n} \). The denominator can then be treated, as long as \( t > mr \), as the generating function of the Chebishev polynomials of the second kind, that is

\[
\frac{1}{u^2 - 2xu + 1} = \sum_{\alpha=0}^{\infty} U_\alpha(x) u^\alpha, \quad (A.1)
\]

for \(-1 < x < 1\) and \( |x| < 1 \). The first condition is satisfied for every \( n \geq 2 \) as \( x = \cos(\pi/n) \), while the second is satisfied in the whole integration domain of (29) except for the lower limit \( t = mr \). This divergence is ‘cured’ by integrating over the domain \([mr + \epsilon, \infty)\), where \( \epsilon \) is a small parameter. Once this expansion is plugged into (29) the sum and the integration can be safely exchanged. After the integration is performed one then has to be sure that the result does not depend on \( \epsilon \), and finally set it to zero. We have performed these steps showing that indeed (A.1) in this case can be taken as valid also at the point \( t = mr \). The details are technical and cumbersome, and they will not be reported here. From now on, and throughout the calculation in section 3 we will take (A.1) as series representation and the whole integration path. The polynomials \( U_\alpha(x) \) in our case are formally defined as follows:

\[
U_\alpha \left( \cos \frac{\pi}{n} \right) = \frac{\sin \left( \frac{1 + 2\alpha}{2}\pi \right)}{\sin \frac{\pi}{n}}. \quad (A.2)
\]

The lhs in (30) can then be expanded as shown on the rhs with

\[
\Omega_a(n) = \begin{cases} 
\cos \frac{(1 + 2\alpha)\pi}{2n} & \text{if } \alpha < n \\
\cos \frac{2\alpha\pi}{2n} & \\
\cos \frac{(1 + 2\alpha)\pi}{2n} + \frac{\sin \frac{(1 + 2\alpha)\pi}{2n}}{\cos \frac{2\alpha\pi}{2n}} & \text{if } \alpha \geq n
\end{cases}. \quad (A.3)
\]
Appendix B. Definite integrals of Bessel functions and powers

In this appendix, we present a solution to integrals of the kind

\[ \int_{mr}^{\infty} dt \, t^{-\mu} K_\nu(t), \]  

where both \( \mu \) and \( mr \) are positive real numbers. In section 3, in particular, an expansion for small values of \( mr \) was needed, so this will be the aim of this appendix. First of all let us introduce the function

\[ G_{m,n}^{p,q}(t | a_1, \ldots, a_p | b_1, \ldots, b_q) = \frac{1}{2\pi i} \int L \prod_{j=1}^{p} \Gamma(b_j - s) \prod_{j=0}^{n} \Gamma(1 - a_j + s) \prod_{j=p+1}^{p+q+1} \Gamma(t_j - s) \, ds. \]  

This is a representation of the Meijer G-function, in the formalism adopted by [14], and the details and properties about this function will not be reported here. A useful identity is

\[ K_\nu(t) = \frac{1}{2} \, G_{0,2}^{2,0}(\frac{1}{4} | -1 \nu, 1 \nu) \],

which holds for \( | \arg(t) | \leq \pi/2 \), and the empty sets of Gamma-functions’ poles are omitted. In the light of (B.3) the integral in (B.1) can be rewritten as follows:

\[ \frac{(mr)^{1-\mu}}{4} \int_{1}^{\infty} dt \, t^{-\mu} G_{0,2}^{2,0}(\frac{m^2}{4} | \nu, -\nu) \].

This is a special case of a known integral of the G-function, which in the most general form is

\[ \int_{1}^{\infty} dt \, t^{-\mu} (t - 1)^{\sigma-1} G_{p,q}^{m,n}(a | a_1, \ldots, a_p | b_1, \ldots, b_q) = \Gamma(\sigma) G_{p+1,q+1}^{m+1,n}(\alpha | a_1, \ldots, a_p, \rho | \rho - \sigma, b_1, \ldots, b_q), \]

which holds for real \( | \arg(t) | \leq (m+q)/2 - (q/2) \pi \), \( p + q \leq 2(m + n) \), \( \Re(\sigma) > 0 \) and \( \Re(\rho - \sigma - a_j) > -1 \forall j \in [1, n] \). These conditions are all clearly satisfied by (B.4), so that the result is

\[ \int_{mr}^{\infty} dt \, t^{-\mu} K_\nu(t) = \frac{(mr)^{1-\mu}}{4} \, G_{1,3}^{3,0}(\frac{m^2}{4} | \mu - 1 \nu, -\nu, -\nu) \].

Now this result has to be restricted to the cases (32) to be useful for the OPE that was considered in section 3. The calculations are tedious and the results cumbersome; hence, only the first few terms of the expansion of the first two contributions are given,

\[ \int_{mr}^{\infty} dt \, t^{-\frac{\mu+1}{2}} K_0(t) = 2^{-1-\frac{1+2\alpha}{2}} \Gamma \left( \frac{n - 1 - 2\alpha}{2} \right)^2 + \frac{n(n - 1 - 2\alpha)(\eta - \log 2) - n}{(n - 1 - 2\alpha)^2} (mr)^{1-\frac{1+2\alpha}{2}} \log(mr) + \frac{n}{4(3n - 1 - 2\alpha)^2} (mr)^{3-\frac{1+2\alpha}{2}} \log(mr) + O(mr)^{5-\frac{1+2\alpha}{2}} \]  

\[ \int_{mr}^{\infty} dt \, t^{-1-\frac{\mu+1}{2}} K_1(t) = \frac{(mr)^{1-\frac{1+2\alpha}{2}} n}{1 + 2\alpha + n} + 2^{-2-\frac{1+2\alpha}{2}} \Gamma \left( \frac{n - 1 - 2\alpha}{2n} \right) \Gamma \left( \frac{-1 + 2\alpha + n}{2n} \right) \]  

\[ + \frac{n(n - 1 - 2\alpha)(1 + 2\log 2 - 2\eta - 2n)}{4(n - 1 - 2\alpha)^2} (mr)^{1-\frac{1+2\alpha}{2}} \]
\[- \frac{n}{2(n-1-2\alpha)} (mr)^{1-\frac{1+2\alpha}{\alpha}} \log(mr) + \frac{n}{64(3n-1-2\alpha)^2} (mr)^{3-\frac{1+2\alpha}{\alpha}} + \frac{n}{16(3n-1-2\alpha)} (mr)^{3-\frac{1+2\alpha}{\alpha}} \log(mr) + \text{O}(mr)^{5-\frac{1+2\alpha}{\alpha}} \] 

(B.8)

where \( \gamma = 0.577216 \) is the Euler–Mascheroni constant.

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