Explicit Form of the Evolution Operator of Tavis–Cummings Model: Three and Four Atoms Cases

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Abstract

In this letter the explicit form of evolution operator of the Tavis–Cummings model with three and four atoms is given. This is an important progress in quantum optics or mathematical physics.

The purpose of this paper is to give an explicit form to the evolution operator of Tavis–Cummings model ([1]) with some atoms. This model is a very important one in Quantum Optics and has been studied widely, see [2] as general textbooks in quantum optics.

We are studying a quantum computation and therefore want to study the model from this point of view, namely a quantum computation based on atoms of laser–cooled and trapped linearly in a cavity. We must in this model construct the controlled NOT gate or other controlled unitary gates to perform a quantum computation, see [3] as a general introduction to this subject.

For that aim we need the explicit form of evolution operator of the models with one, two, three and four atoms (at least). As to the model of one atom or two atoms it is more or less known (see [4]), while as to the case of three and four atoms it has not been given as far as we know. Since we succeeded in finding the explicit form for three and four atoms cases we report it ([5]).

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The Tavis–Cummings model (with \( n \)-atoms) that we treat in this paper can be written as follows (we set \( \hbar = 1 \) for simplicity).

\[
H = \omega_1 \mathbf{1} \otimes a \dagger a + \frac{\Delta}{2} \sum_{i=1}^{n} \sigma_i^{(3)} \otimes \mathbf{1} + g \sum_{i=1}^{n} \left( \sigma_i^{(+)} \otimes a + \sigma_i^{(-)} \otimes a \dagger \right),
\]

where \( \omega \) is the frequency of radiation field, \( \Delta \) the energy difference of two level atoms, \( a \) and \( a \dagger \) are annihilation and creation operators of the field, and \( g \) a coupling constant, and \( L = 2^n \). Here \( \sigma_i^{(+)}, \sigma_i^{(-)} \) and \( \sigma_i^{(3)} \) are given as

\[
\sigma_i^{(s)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma_s \otimes 1_2 \otimes \cdots \otimes 1_2 \quad (i \text{ position}) \in M(L, \mathbb{C})
\]

where \( s \) is +, – and 3 respectively and

\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Here let us rewrite the Hamiltonian \( \text{(1)} \). If we set \( S_+ = \sum_{i=1}^{n} \sigma_i^{(+)} \), \( S_- = \sum_{i=1}^{n} \sigma_i^{(-)} \), \( S_3 = \frac{1}{2} \sum_{i=1}^{n} \sigma_i^{(3)} \),

then \( \text{(1)} \) can be written as

\[
H = \omega_1 \mathbf{1} \otimes a \dagger a + \Delta S_3 \otimes \mathbf{1} + g \left( S_+ \otimes a + S_- \otimes a \dagger \right) \equiv H_0 + V,
\]

which is very clear. We note that \( \{S_+, S_-, S_3\} \) satisfy the \( su(2) \)-relation

\[
[S_3, S_+] = S_+, \quad [S_3, S_-] = -S_, \quad [S_+, S_-] = 2S_3.
\]

However, the representation \( \rho \) defined by \( \rho(\sigma_+) = S_+, \quad \rho(\sigma_-) = S_-, \quad \rho(\sigma_3/2) = S_3 \) is a reducible representation of \( su(2) \).

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} U = HU = (H_0 + V) U,
\]

where \( U \) is a unitary operator (called the evolution operator). We can solve this equation by using the \textbf{method of constant variation}. The result is well–known to be

\[
U(t) = \left( e^{-i \omega S_3 \otimes \mathbf{1}} e^{-it \omega a \dagger} \right) e^{-itg( S_+ \otimes a + S_- \otimes a \dagger)}
\]

under the resonance condition \( \Delta = \omega \), where we have dropped the constant unitary operator for simplicity. Therefore we have only to calculate the term \( \text{(8)} \) explicitly, which is however a very hard task \(^1\). In the following we set

\[
A_n = S_+ \otimes a + S_- \otimes a \dagger
\]

\(^1\)The situation is very similar to that of the paper \text{quant-ph}/0312060 in [7].
for simplicity. We can determine $e^{-itgA}$ for $n = 1$ (one atom case), $n = 2$ (two atoms case), $n = 3$ (three atoms case) and $n = 4$ (four atoms case) completely.

**One Atom Case** In this case $A$ in (9) is written as

$$A_1 = \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} \equiv B_{1/2}. $$ (10)

By making use of the simple relation

$$A_1^2 = \begin{pmatrix} aa^\dagger & 0 \\ 0 & a^\dagger a \end{pmatrix} = \begin{pmatrix} N + 1 & 0 \\ 0 & N \end{pmatrix}$$ (11)

with the number operator $N$ we have

$$e^{-itgA_1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (tg)^{2n} A_1^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} (tg)^{2n+1} A_1^{2n+1}$$

$$= \begin{pmatrix} \cos (tg\sqrt{N+1}) & -i \sin (tg\sqrt{N+1}) a \\ i \sin (tg\sqrt{N+1}) / \sqrt{N+1} a^\dagger & \cos (tg\sqrt{N}) \end{pmatrix}. $$ (12)

We obtained the explicit form of solution. However, this form is more or less well–known, see for example the second book in [2].

**Two Atoms Case** In this case $A$ in (9) is written as

$$A_2 = \begin{pmatrix} 0 & a & a & 0 \\ a^\dagger & 0 & 0 & a \\ a^\dagger & 0 & 0 & a \\ 0 & a^\dagger & a^\dagger & 0 \end{pmatrix}. $$ (13)

Our method is to reduce the $4 \times 4$–matrix $A_2$ in (13) to a $3 \times 3$–matrix $B_1$ in the following to make our calculation easier. For that aim we prepare the following matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then it is easy to see

$$T^\dagger A_2 T = \begin{pmatrix} 0 & 0 & \sqrt{2}a & 0 \\ 0 & 0 & 0 & \sqrt{2}a \\ \sqrt{2}a^\dagger & 0 & 0 & \sqrt{2}a \\ 0 & \sqrt{2}a^\dagger & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & B_1 \end{pmatrix},$$

where $B_1 = J_+ \otimes a + J_- \otimes a^\dagger$ and $\{ J_+, J_- \}$ are just generators of (spin one) irreducible representation of (3). We note that this means a well–known decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. 

3
Therefore to calculate $e^{-itgA}$ we have only to do $e^{-itgB_1}$. Noting the relation

$$B_1^2 = \begin{pmatrix} 2(N+1) & 0 & 2a^2 \\ 0 & 2(2N+1) & 0 \\ 2(a\dagger)^2 & 0 & 2N \end{pmatrix},$$

$$B_1^3 = \begin{pmatrix} 2(2N+3) & 2(2N+1) \\ 2(2N+1) & 2(2N-1) \end{pmatrix} B_1 \equiv DB_1,$$  \hspace{1cm} (14)

and so

$$B_1^{2n} = D^{n-1}B_1^2 \text{ for } n \geq 1, \quad B_1^{2n+1} = D^n B_1 \text{ for } n \geq 0$$

we obtain by making use of the Taylor expansion

$$e^{-itgB_1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (tg)^{2n} B_1^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (tg)^{2n+1} B_1^{2n+1}$$

$$= \begin{pmatrix} 1 + \frac{2N+2}{2N+3} f(N+1) & -ih(N+1) a & \frac{2}{2N+3} f(N+1) a^2 \\ -ih(N)a\dagger & 1 + 2f(N) & -ih(N) a \\ \frac{2}{2N-1} f(N-1)(a\dagger)^2 & -ih(N-1) a\dagger & 1 + \frac{2N}{2N-1} f(N-1) \end{pmatrix}$$  \hspace{1cm} (15)

where

$$f(N) = \frac{-1 + \cos \left( \frac{tg\sqrt{2(2N+1)}}{2} \right)}{\sqrt{2N+1}}, \quad h(N) = \frac{\sin \left( \frac{tg\sqrt{2(2N+1)}}{2} \right)}{\sqrt{2N+1}}.$$  \hspace{1cm} (16)

**Three Atoms Case** In this case $A$ in (9) is written as

$$A_3 = \begin{pmatrix} 0 & a & a & 0 & a & 0 & 0 & 0 \\ a\dagger & 0 & 0 & a & 0 & a & 0 & 0 \\ a\dagger & 0 & 0 & a & 0 & 0 & a & 0 \\ 0 & a\dagger & 0 & 0 & 0 & 0 & 0 & a \\ a\dagger & 0 & 0 & 0 & a & a & 0 & 0 \\ 0 & a\dagger & 0 & 0 & a\dagger & 0 & 0 & a \\ 0 & 0 & a\dagger & 0 & a\dagger & 0 & 0 & a \\ 0 & 0 & a\dagger & 0 & a\dagger & a\dagger & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (16)

We would like to look for the explicit form of solution like (12) or (15). If we set

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{\sqrt{6}} & 0 & \frac{\sqrt{2}}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{\sqrt{6}} & 0 & -\frac{\sqrt{2}}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
then it is not difficult to see

\[
T^\dagger A_3 T = \begin{pmatrix}
0 & a & 0 & 0 \\
0 & a & \sqrt{3}a & 0 \\
a^\dagger & 0 & 0 & 2a \\
a^\dagger & 0 & \sqrt{3}a & 0 \\
\end{pmatrix} = \begin{pmatrix}
B_{1/2} & B_{1/2} \\
B_{3/2} & B_{3/2} \\
\end{pmatrix}.
\]

This means a decomposition of spin \( \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \). Therefore we have only to calculate \( e^{-igB_{3/2}} \), which is however not easy. In this case there is no simple relation like \([\text{I}]\) or \([\text{II}]\), so we must find another one.

Let us state the key lemma for that. Noting

\[
B_{3/2}^2 = \begin{pmatrix}
3N + 3 & 2\sqrt{3}a^2 & 0 \\
0 & 7N + 4 & 2\sqrt{3}a^2 \\
2\sqrt{3}(a^\dagger)^2 & 0 & 3N \\
\end{pmatrix},
\]

\[
B_{3/2}^3 = \begin{pmatrix}
\sqrt{3}(7N + 4)a^\dagger & 0 & 6a^3 \\
0 & 20(N + 1)a^\dagger & 0 \\
6(a^\dagger)^3 & 0 & 3\sqrt{3}(7N + 3)a^\dagger \\
\end{pmatrix},
\]

and the relations

\[
B_{3/2}^{2n+1} = B_{3/2}B_{3/2}^{2n}, \quad B_{3/2}^{2n+2} = B_{3/2}^2B_{3/2}^{2n},
\]

we can obtain \( B_{3/2}^{2n} \) and \( B_{3/2}^{2n+1} \) like

\[
B_{3/2}^{2n} = \begin{pmatrix}
\alpha_n(N + 2) & 0 & 2\sqrt{3}\xi_n(N + 2)a^2 & 0 \\
0 & \beta_n(N + 1) & \gamma_n(N) & 0 \\
2\sqrt{3}\xi_n(N)(a^\dagger)^2 & 0 & \delta_n(N - 1) \\
0 & 2\sqrt{3}\xi_n(N - 1)(a^\dagger)^2 & \delta_n(N - 1) & 0 \\
\end{pmatrix},
\]

\[
B_{3/2}^{2n+1} = \begin{pmatrix}
0 & \sqrt{3}\beta_n(N + 2)a^\dagger & 0 & 6\xi_n(N + 2)a^3 \\
\sqrt{3}\beta_n(N + 1)a^\dagger & 0 & 2\xi_{n+1}(N + 1)a^\dagger & 0 \\
0 & 2\xi_{n+1}(N)a^\dagger & \sqrt{3}\gamma_n(N - 1)a^\dagger & 0 \\
6\xi_n(N - 1)(a^\dagger)^3 & 0 & \sqrt{3}\gamma_n(N - 1)a^\dagger & 0 \\
\end{pmatrix},
\]

where

\[
\alpha_n(N) = (v_+\lambda_+^n - v_-\lambda_-^n)/(2\sqrt{d}), \quad \beta_n(N) = (w_+\lambda_+^n - w_-\lambda_-^n)/(2\sqrt{d}),
\]

\[
\gamma_n(N) = (v_+\lambda_+^n - v_-\lambda_-^n)/(2\sqrt{d}), \quad \delta_n(N) = (w_+\lambda_+^n - w_-\lambda_-^n)/(2\sqrt{d}),
\]

\[
\xi_n(N) = (\lambda_+^n - \lambda_-^n)/(2\sqrt{d}),
\]
and $\lambda_{\pm} \equiv \lambda_{\pm}(N), \: v_{\pm} \equiv v_{\pm}(N), \: w_{\pm} \equiv w_{\pm}(N), \: d \equiv d(N)$ defined by

\[
\lambda_{\pm}(N) = 5N \pm \sqrt{d(N)}, \: v_{\pm}(N) = -2N - 3 \pm \sqrt{d(N)}, \: w_{\pm}(N) = 2N - 3 \pm \sqrt{d(N)},
\]

\[
d(N) = 16N^2 + 9.
\]

Then by making use of (17) and (18) we have

\[
e^{-itgB_{3/2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (tg)^{2n} B_{3/2}^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} (tg)^{2n+1} B_{3/2}^{2n+1}
\]

\[
= \begin{pmatrix}
    f_2(N + 2) & -\sqrt{3}iF_1(N + 2)a & 2\sqrt{3}h_1(N + 2)a^2 & -6iH_0(N + 2)a^3 \\
    -\sqrt{3}iF_1(N + 1)a^\dagger & f_1(N + 1) & -2iH_1(N + 1)a & 2\sqrt{3}h_1(N + 1)a^2 \\
    2\sqrt{3}h_1(N)(a^\dagger)^2 & -2iH_1(N)a^\dagger & f_0(N) & -\sqrt{3}iF_0(N)a \\
    -6iH_0(N - 1)(a^\dagger)^3 & 2\sqrt{3}h_1(N - 1)(a^\dagger)^2 & -\sqrt{3}iF_0(N - 1)a^\dagger & f_{-1}(N - 1)
\end{pmatrix}
\]

(19)

where

\[
f_2(N) = \left\{ v_{\pm}(N)\cos(tg\sqrt{\lambda_{\pm}(N)}) - v_{-}(N)\cos(tg\sqrt{\lambda_{-}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
f_1(N) = \left\{ w_{\pm}(N)\cos(tg\sqrt{\lambda_{\pm}(N)}) - w_{-}(N)\cos(tg\sqrt{\lambda_{-}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
f_0(N) = \left\{ v_{\pm}(N)\cos(tg\sqrt{\lambda_{-}(N)}) - v_{-}(N)\cos(tg\sqrt{\lambda_{+}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
f_{-1}(N) = \left\{ w_{\pm}(N)\cos(tg\sqrt{\lambda_{-}(N)}) - w_{-}(N)\cos(tg\sqrt{\lambda_{+}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
h_1(N) = \left\{ \cos(tg\sqrt{\lambda_{+}(N)}) - \cos(tg\sqrt{\lambda_{-}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
F_1(N) = \left\{ \frac{w_{\pm}(N)}{\sqrt{\lambda_{\pm}(N)}}\sin(tg\sqrt{\lambda_{\pm}(N)}) - \frac{w_{-}(N)}{\sqrt{\lambda_{-}(N)}}\sin(tg\sqrt{\lambda_{-}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
F_0(N) = \left\{ \frac{v_{\pm}(N)}{\sqrt{\lambda_{-}(N)}}\sin(tg\sqrt{\lambda_{-}(N)}) - \frac{v_{-}(N)}{\sqrt{\lambda_{+}(N)}}\sin(tg\sqrt{\lambda_{+}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
H_1(N) = \left\{ \sqrt{\lambda_{+}(N)}\sin(tg\sqrt{\lambda_{+}(N)}) - \sqrt{\lambda_{-}(N)}\sin(tg\sqrt{\lambda_{-}(N)}) \right\} / (2\sqrt{d(N)}),
\]

\[
H_0(N) = \left\{ \frac{1}{\sqrt{\lambda_{+}(N)}}\sin(tg\sqrt{\lambda_{+}(N)}) - \frac{1}{\sqrt{\lambda_{-}(N)}}\sin(tg\sqrt{\lambda_{-}(N)}) \right\} / (2\sqrt{d(N)}).\]
Four Atoms Case  In this case $A_4$ in (20) is written as

$$A_4 = \begin{pmatrix}
0 & a & a & 0 & a & 0 & 0 & a \\
0 & a^\dagger & 0 & 0 & a & 0 & a & 0 \\
0 & a^\dagger & 0 & 0 & a & 0 & a & a \\
0 & a^\dagger & a & 0 & 0 & 0 & 0 & a \\
0 & a^\dagger & 0 & 0 & a & 0 & a & a \\
0 & a^\dagger & 0 & 0 & a & a & 0 & a \\
0 & a^\dagger & 0 & 0 & a & a & 0 & a \\
0 & a^\dagger & 0 & 0 & a & a & 0 & a \\
0 & a & a & a & a & a & a & a
\end{pmatrix} \cdot (20)
$$

If we set $T$ as

$$T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

then it is not difficult to see

$$T^\dagger A_4 T =$$
This means a well–known decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \oplus 0 \oplus 1 \oplus 1 \oplus 2$. Since we have calculated $e^{-itgB_1}$ in (15) (15) we have only to do $e^{-itgB_2}$, which is of course hard. The proof is similar to that of the three atoms case, so we state only the result.

$$\exp(-itgB_2) =$$

$$\begin{pmatrix}
0 & \sqrt{2a} & 0 & 0 & 0 & 0 \\
\sqrt{2a}^\dagger & 0 & \sqrt{2a} & 0 & 0 & 0 \\
0 & \sqrt{2a} & 0 & \sqrt{2a} & 0 & 0 \\
\sqrt{2a}^\dagger & 0 & \sqrt{2a} & 0 & \sqrt{2a} & 0 \\
0 & \sqrt{2a} & 0 & \sqrt{2a} & 0 & \sqrt{2a} \\
0 & \sqrt{2a} & 0 & \sqrt{2a} & 0 & \sqrt{2a} \\
\end{pmatrix}
\begin{pmatrix}
0 & 2a & 0 & 0 & 0 & 0 \\
2a^\dagger & 0 & \sqrt{6a} & 0 & 0 & 0 \\
0 & \sqrt{6a}^\dagger & 0 & \sqrt{6a} & 0 & 0 \\
0 & \sqrt{6a} & 0 & 2a & 0 & 0 \\
0 & 0 & \sqrt{6a}^\dagger & 0 & 2a & 0 \\
0 & 0 & 0 & 2a^\dagger & 0 & 0 \\
\end{pmatrix}
\equiv 0 \oplus B_1 \oplus 0 \oplus B_1 \oplus B_1 \oplus B_2.$$

where

\begin{align*}
f_2(N) &= 1 + 4(N - 1)\{(u_+ / \lambda_+)(\cos tg \sqrt{\lambda_+} - 1) - (u_- / \lambda_-)(\cos tg \sqrt{\lambda_-} - 1)\} / \sqrt{d} \\
f_1(N) &= (u_+ \cos tg \sqrt{\lambda_+} - u_- \cos tg \sqrt{\lambda_-}) / \sqrt{d}, \\
f_0(N) &= 1 + 2\{(v_+ w_+ / \lambda_+)(\cos tg \sqrt{\lambda_+} - 1) - (v_- w_- / \lambda_-)(\cos tg \sqrt{\lambda_-} - 1)\} / \sqrt{d}, \\
f_{-1}(N) &= (u_+ \cos tg \sqrt{\lambda_-} - u_- \cos tg \sqrt{\lambda_+}) / \sqrt{d},
\end{align*}
We conclude this paper by making a comment. The Tavis–Cummings model is based
on (only) two energy levels of atoms. However, an atom has in general infinitely
many energy levels, so it is natural to use this possibility. We are also studying a quantum
computation based on multi–level systems of atoms (a qudit theory) \[7\]. Therefore we
would like to extend the Tavis–Cummings model based on two–levels to a model based
on multi–levels. This is a very challenging task.

\[ f_{-2}(N) = 1 + 4(N + 2)\{(u_+/\lambda_+)(\cos tg\sqrt{\lambda_+ - 1}) - (u_-/\lambda_-)(\cos tg\sqrt{\lambda_- - 1})\}/\sqrt{d}, \]
\[ h_1(N) = 2\{(v_+/\lambda_+)(\cos tg\sqrt{\lambda_+ - 1}) - (v_-/\lambda_-)(\cos tg\sqrt{\lambda_- - 1})\}/\sqrt{d}, \]
\[ h_0(N) = (\cos tg\sqrt{\lambda_+} - \cos tg\sqrt{\lambda_-})/\sqrt{d}, \]
\[ h_{-1}(N) = 2\{(w_+/\lambda_+)(\cos tg\sqrt{\lambda_+ - 1}) - (w_-/\lambda_-)(\cos tg\sqrt{\lambda_- - 1})\}/\sqrt{d}, \]
\[ k_0(N) = 4\{(1/\lambda_+)(\cos tg\sqrt{\lambda_+ - 1}) - (1/\lambda_-)(\cos tg\sqrt{\lambda_- - 1})\}/\sqrt{d}, \]
and
\[ F_1(N) = \{(u_+/\sqrt{\lambda_+})\sin tg\sqrt{\lambda_+} - (u_-/\sqrt{\lambda_-})\sin tg\sqrt{\lambda_-}\}/\sqrt{d}, \]
\[ F_{-1}(N) = \{(u_+/\sqrt{\lambda_+})\sin tg\sqrt{\lambda_+} - (u_-/\sqrt{\lambda_-})\sin tg\sqrt{\lambda_-}\}/\sqrt{d}, \]
\[ H_1(N) = 2\{(v_+/\sqrt{\lambda_+})\sin tg\sqrt{\lambda_+} - (v_-/\sqrt{\lambda_-})\sin tg\sqrt{\lambda_-}\}/\sqrt{d}, \]
\[ H_0(N) = \{(1/\sqrt{\lambda_+})\sin tg\sqrt{\lambda_+} - (1/\sqrt{\lambda_-})\sin tg\sqrt{\lambda_-}\}/\sqrt{d}, \]
\[ H_{-1}(N) = 2\{(w_+/\sqrt{\lambda_+})\sin tg\sqrt{\lambda_+} - (w_-/\sqrt{\lambda_-})\sin tg\sqrt{\lambda_-}\}/\sqrt{d} \]
, and \( d \equiv d(N), \lambda_\pm \equiv \lambda_\pm(N), u_\pm \equiv u_\pm(N), v_\pm \equiv v_\pm(N) \) and \( w_\pm \equiv w_\pm(N) \) defined by
\[ \lambda_\pm(N) = 10N + 5 \pm 3\sqrt{d(N)}, \quad u_\pm(N) = \frac{1}{2}(-3 \pm \sqrt{d(N)}), \]
\[ v_\pm(N) = \frac{\sqrt{3}}{\sqrt{2}}(2N - 1 \pm \sqrt{d(N)}), \quad w_\pm(N) = \frac{\sqrt{3}}{\sqrt{2}}(2N + 3 \pm \sqrt{d(N)}), \]
\[ d(N) = 4N^2 + 4N + 9. \]

A comment is in order. We note that in the process of calculation we used Mathematica
to the fullest (a calculation by hand might be “painful”). We would like to
generalize the results in this paper to the cases of more than four atoms. However, it is
not easy to perform a calculation due to some severe technical reasons. There is a (big ?)
gap between the four atoms and the five ones.

We obtained the explicit form of evolution operator of the Tavis–Cummings model
for three and four atoms cases. This is a big progress in quantum optics or mathematical
physics \[2\]. Therefore, many applications to quantum physics or mathematical physics
will be expected, see for example papers in \[4\].

We can also apply the result(s) to a quantum computation based on atoms of laser–
cooled and trapped linearly in a cavity \[6\].

We conclude this paper by making a comment. The Tavis–Cummings model is based
on (only) two energy levels of atoms. However, an atom has in general infinitely
many energy levels, so it is natural to use this possibility. We are also studying a quantum
computation based on multi–level systems of atoms (a qudit theory) \[7\]. Therefore we
would like to extend the Tavis–Cummings model based on two–levels to a model based
on multi–levels. This is a very challenging task.

\[ ^2 \text{To obtain an explicit solution for some (intersting) model is still important} \]
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