Analytical Approach to the Fuel Optimal Impulsive Transfer Problem Using Primer Vector Method

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Abstract. One of the objectives of mission design is selecting an optimum orbital transfer which often translated as a transfer which requires minimum propellant consumption. In order to assure the selected trajectory meets the requirement, the optimality of transfer should first be analyzed either by directly calculating the $\Delta V$ of the candidate trajectories and select the one that gives a minimum value or by evaluating the trajectory according to certain criteria of optimality. The second method is performed by analyzing the profile of the modulus of the thrust direction vector which is known as primer vector. Both methods come with their own advantages and disadvantages. However, it is possible to use the primer vector method to verify if the result from the direct method is truly optimal or if the $\Delta V$ can be reduced further by implementing correction maneuver to the reference trajectory. In addition to its capability to evaluate the transfer optimality without the need to calculate the transfer $\Delta V$, primer vector also enables us to identify the time and position to apply correction maneuver in order to optimize a non-optimum transfer. This paper will present the analytical approach to the fuel optimal impulsive transfer using primer vector method. The validity of the method is confirmed by comparing the result to those from the numerical method. The investigation of the optimality of direct transfer is used to give an example of the application of the method. The case under study is the prograde elliptic transfers from Earth to Mars. The study enables us to identify the optimality of all the possible transfers.

1. Introduction

In an interplanetary mission, one of the mission analysis objectives is designing a trajectory with minimum propellant to allow the spacecraft brings more useful mass [1]. The interplanetary flight requires the spacecraft to leave the orbit of one celestial body to enter the orbit of another planet or performing rendezvous with another planet or object in space.

During the preliminary design phase, the designed trajectory can be modeled as a simple conic arc. Alternatively, it is possible to design a more complex transfer involving other celestial bodies passed by the spacecraft during the transfer, commonly known as gravity assist trajectory. Although the latter is much more complicated than a conic arc, it is widely used by a mission designer due to its advantage of significantly reducing the amount of required propellant for the transfer.

Gravity assist is not the only way to reduce the required propellant for the interplanetary transfer. Mission designers often include corrective maneuver(s) to further reduce the amount of required propellant which is known as Deep Space Maneuver (DSM). Such trajectory, which combines gravity assist and deep space maneuver, is referred to as Multiple Gravity Assist - Deep Space Maneuver (MGA-DSM) trajectory [2].

One example of a recent MGA-DSM application is Juno, the interplanetary space mission to Jupiter under NASA's New Horizon program. Juno, which arrived at Jupiter's orbit in July 2016, has
successfully performed two DSMs along its way in addition to the earth gravity assists. It initially launched into a heliocentric orbit to obtain a short flight time to Jupiter [3]. On its way to Jupiter, Juno performed two DSMs in 2012, approximately one year after launch [4]. The two DSMs were performed to correct its orbit so that it entered the Earth's sphere of influence (SOI) and obtained 'free' energy from the Earth to travel to Jupiter. The second maneuver was performed on February 2016, correcting its path to prepare a rendezvous with Jupiter. Figure 1.1 below illustrates Juno's mission profile from its launch in 2011 until its arrival at Jupiter orbit in July 2016. Table 1.1 lists the application of DSM on some of the most recent missions.

![Juno interplanetary mission profile](image)

**Figure 1.** Juno interplanetary mission profile

| Mission  | Date of performing DSM | V (m/s) | Correction | Target  |
|----------|------------------------|--------|------------|---------|
| Exomars  | 28 July 2016           | 334.176| DSM-1      | Mars    |
|          | 11 August 2016         | 17.7   | DSM-2      |         |
| Juno     | 28 September 2012      | 345    | DSM-1      | Jupiter |
|          | 3 August 2016          | 385    | DSM-2      |         |
| Messenger| 12 December 2005       | 315.6  | DSM-1      | Mercury |
|          | 17 October 2007        | 227.4  | DSM-2      |         |
|          | 17 March 2008          | 72.2   | DSM-3      |         |
|          | 6 December 2008        | 222.1, 24.7 | DSM-4 (two segments) |         |
|          | 29 November 2009       | 117.75 | DSM-5      |         |
An optimal MGA-DSM transfer is solved using modern mathematical tools to obtain the global solution such as genetic algorithm [8] and evolutionary algorithm [9]. However, this approach suffers from the dimensionality of the search space. On the other hand, one can use an analytical approach to solve for an optimum trajectory by using the calculus of variation [10]. One of the most established work in trajectory optimization using an analytical method based on Calculus of Variation is Primer Vector Theory. The theory which was introduced by Lawden in the 1960s [11], providing a method to determine a transfer optimality using a set of Necessary Conditions (NC).

There are two categories of trajectory optimization methods; direct and indirect methods. Trajectory optimization using primer vector is classified as an indirect method because the optimum trajectory is not defined by calculating the cost, instead, it determines the optimality using the NC [12]. The NC allows a mission planner to analyze whether a transfer is optimum or non-optimum by evaluating the primer vector profile along the transfer trajectory and see if it satisfy the NC. Transfer trajectory refers to the trajectory of the spacecraft from orbit around the departure planet to the orbit of around the destination.

A non-optimum trajectory can be altered by modifying the orbital velocity vector as such to satisfy the NC of optimality [13]. The modification of the velocity vector is achieved through the implementation of DSM (corrective maneuver). A linear method to optimize a transfer based on the gradient of the cost functional was first proposed by Jezewski, which estimates the position and time to apply the DSM [14].

The propagation of primer vector profile requires solving the primer vector equation which is a second order differential equation. The primer profile is commonly obtained using a propagation of state transition matrix to solve the primer vector equation. This approach relies on the numerical integration of the STM and the primer vector hence the computation time becomes a disadvantage. The paper by Iorfida [15] presented a novel approach to solve the primer vector equation analytically, giving a focus on the out of plane component (normal to orbital plane). This paper presents the study based on Iorfida’s work to solve the in plane component of the primer vector.

2. Method

The primer vector method provides a tool to check the optimality of a transfer by using a set of necessary conditions for transfer optimality. The necessary conditions depend on the types of the transfer and for the impulsive transfer case, there are four necessary conditions for optimality [10]:

1. The primer vector and its derivative are continuous everywhere.
2. The magnitude of the primer vector is always below 1, aside from the instants where the impulse occurs, where \( \| \mathbf{p} \| = 1 \).
3. At the impulse times, the primer vector is a unit vector in the direction of the thrust, \( \mathbf{p} = \frac{\Delta \mathbf{v}}{\|\Delta \mathbf{v}\|} \).
4. As a consequence of the above conditions, the first derivative, \( \frac{dp}{dt} = \dot{p} = \mathbf{p}^T \mathbf{p} = 0 \) at an intermediate impulse.

In order to evaluate a transfer with regards to the necessary conditions above, the primer profile along the transfer needs to be propagated. The propagation requires solving the primer vector equation in the form:

\[
\dot{\mathbf{p}}(t) = \mathbf{G}(\mathbf{r})\mathbf{p}.
\]  

(1)

\( \mathbf{p} \) in equation 1 is the primer vector which belongs to the 3-dimensional space while \( \mathbf{G} \) is the gravity gradient matrix. The gravity gradient matrix in the inverse gravitational form is given by

\[
\mathbf{G} = \frac{-\mu}{r^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{3\mu}{r^5} \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}.
\]  

(2)
A new approach proposed by Iorfiда and Palmer to solve the primer vector equation without the use of state transition matrix [15]. The approach used a coordinate transformation from Cartesian coordinate (x, y, z) to local Polar coordinate (r, θ, h). The Polar coordinate system is a local coordinate, which means the coordinate system change with time. The local Polar coordinate is defined as in figure 2.

**Figure 2.** Polar coordinate system definition

This reference coordinate change leads to the decoupling of the in plane (r and θ) and out of plane (h) components of the primer derivative as expressed in equation 3 below:

\[
\dot{\mathbf{p}} = \left( \dot{p}_r - \frac{h}{r^2} p_\theta \right) \mathbf{e}_r + \left( \dot{p}_\theta + \frac{h}{r^2} p_r \right) \mathbf{e}_\theta + \dot{p}_h \mathbf{e}_h, \tag{3}
\]

hence, the in plane and the out of plane components can be analyzed separately. In equation 3, \( p_r, p_\theta, p_h \) are the radial, circumferential and normal components of the primer vector (see figure 2). For an in plane transfer case where the initial and final orbit lies in the same orbital plane, the primer vector is obtained by solving the in plane part of equation 3:

\[
\dot{\mathbf{p}} = \left( \dot{p}_r - \frac{h}{r^2} p_\theta \right) \mathbf{e}_r + \left( \dot{p}_\theta + \frac{h}{r^2} p_r \right) \mathbf{e}_\theta. \tag{4}
\]

The analytical solution of equation 4 is derived by introducing the relative energy (\( H_R \)) and relative angular momentum (\( L_R \)) into the equation. (More explanation about the concept of relative energy and relative angular momentum can be found in the paper by Imre and Palmer [16]). As a result, fully analytical solutions to the in plane primer vector component are obtained in the forms:

\[
p_r = A_1 \bar{x} + A_3 \left( \frac{1}{x} - 1 \right) - A_4 \left[ x - \frac{3}{2} \frac{(1 - e^2)^{-3/2}}{M} M \bar{x} \right] \tag{5}
\]

and

\[
p_\theta = \frac{A_1}{x} - A_2 x - A_3 \bar{x}(x + 1) + \frac{3}{2} A_4 \left[ \frac{1}{x} M (1 - e^2)^{-3/2} \right] \tag{6}
\]

where

\[
x = \frac{r}{l}, \tag{7}
\]

\[
\bar{x} = \frac{lr}{h}. \tag{8}
\]

M is the mean anomaly of the transfer orbit:
\[ M = h(1 - e^2)^{3/2}t / l^2 \]  

\( r \) is the orbital radius, \( l \) is the parameter semi latus rectum, \( e \) is the eccentricity and \( h \) is the true anomaly of the transfer orbit. In equations 5 and 6 above, there are four constants, \( A_1, A_2, A_3, A_4 \) which are solved from the boundary conditions of the transfer. In a fixed transfer case (fixed initial and final positions) with fixed transfer time, the boundary values; \( p_r, p_\theta_0, p_r f, p_\theta f \) are known.

3. Results and Discussions

In order to validate the analytical solution of the in-plane primer vector components expressed in equations 5 and 6, the propagation results are compared to the result from numerical integration using state transition matrix. The numerical solution of the primer vector is defined in the Cartesian coordinates \((x, y, z)\) with \(x\) and \(y\) axes lie in the orbital plane and \(z\) axis normal to the orbital plane as in figure 3.

![Figure 3. Transfer orbit in Cartesian coordinate frame](image)

In the Cartesian frame, the origin lies in the main body (Sun). The \(x\)-axis is in the direction of eccentricity vector. The angular momentum direction is the direction normal to the orbital plane which is defined as the \(z\)-axis. The \(y\)-axis follows the right-hand rule. The two-points boundary value problem is represented by specifying the initial and final true anomalies of the transfer. For a fixed transfer case, where the eccentricity and semi major axis are known, the radius and velocity vectors at the departure and arrival for a planar case, can be calculated using these formulas [17]:

\[ \mathbf{r} = r \cos \theta \mathbf{i}_x + r \sin \theta \mathbf{i}_y \]

\[ \mathbf{v} = -\mu / h \sin \theta \mathbf{i}_x + \mu / h (e + \cos \theta) \mathbf{i}_y . \]

The polar components of the velocity vector are calculated by multiplying \( \mathbf{v} \) with the rotation matrix from Cartesian to Polar coordinate.

The 3\(^{rd}\) axis in Cartesian coordinate (z-axis) is parallel to that of the Polar coordinate (h-axis) so there is no transformation required even if the case is 3-dimensional. Since the true anomaly at departure and arrival are known, the transfer time is solved using Kepler’s equation. The algorithm to solve the primer vector using numerical propagation is presented in the flowchart shown in figure 4.

The boundary value in the 4\(^{th}\) step of the algorithm in figure 4 came from Lawden’s 2\(^{nd}\) necessary condition, which states that the magnitude of primer vector at the time of impulse has to be equal to one. This is also used to verify the result from the final step where the modulus of the primer vector, \( p_f \) at \( t = t_f \) from propagating the primer vector using the initial condition has to be equal to one. It is important to note that for the initial and final conditions, any values of \( p_r \) and \( p_\theta \) can be chosen, as long as \( \| \mathbf{p} \| \) is equal to one. Since the transfer trajectory is fixed, different values of the boundary conditions determine the departure and arrival orbits.
Figure 4. Algorithm for primer vector numerical propagation

The algorithm to propagate the in-plane primer vector profile using the analytical solutions are presented in figure 5.

Figure 5. Algorithm for primer vector analytical propagation

The graphs in figure 6 are generated from the analytical and numerical approaches. In the examples presented in those graphs, the chosen transfer trajectory represents Earth to Mars transfer which has these following parameters: semi-major axis = 1.8875 x 10^8 km, eccentricity = 0.6, \( \theta_0 = 20 \) deg and \( \theta_f = 310 \) deg. The primer vector boundary conditions in Polar coordinate are given by

\[
\mathbf{p}_0 = \begin{bmatrix} p_{r0} \\ p_{\theta_0} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}; \quad \mathbf{p}_f = \begin{bmatrix} p_{r_f} \\ p_{\theta_f} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}.
\]
As can be seen from figure 6, the primer vector from both approaches looks similar. The integration method is validated since both results show that the primer vector magnitudes are equal to one at initial and final points which are the boundary conditions applied to the case under analysis. The results are validated further by comparing the primer profiles for each component (radial and tangential) as in figure 7 which also show similar profiles.

The numerical integration of the state transition matrix and the propagation of the primer vectors were done in Matlab using an ODE45 function with an error tolerance of $10^{-12}$. The ODE45 function in Matlab is an implementation of the 4th order Runge Kutta algorithm which is essentially Taylor expansion of the 4th order. It can be seen in figure 8, the absolute error and the relative errors between the numerical and the analytical solution are in the order of $10^{-11}$, which is closed to the predefined error tolerance.
Figure 8. The error between the analytical and the numerical solutions

For further verification, a graph is generated for the same transfer trajectory but with different final conditions of primer vector. Various final conditions of the primer vector are represented by various $\beta$ angles between the radial components, $p_r$ and the primer vector, $p$ at the end of transfer as defined in figure 9.

Figure 9. Definition of $\beta$ angle in the transfer orbit geometry

In figure 10, the primer vector boundary conditions at the end of transfer, $p_{r_f}$ and $p_{\theta_f}$ were varied using different $\beta$ angles. The transfer orbit parameters are kept fixed as well as the primer vector initial conditions.

Another case similar to the test case above is studied, with the difference in the eccentricity value of the transfer orbit which is changed into 0.1. The result is shown in figure 11. As expected, both results show that the errors are in the order of $10^{12}$. 
Figure 10. Absolute error for various \( \beta \) angle, \( e = 0.6 \)

Figure 11. Absolute error for various \( \beta \) angle, \( e = 0.1 \)

As mentioned before, the primer vector is used to identify the optimality of a transfer so that the mission designer can decide if it is necessary to include the corrective maneuver to reduce the fuel. The next example presents the application of primer vector method to investigate the optimality of direct transfer possibilities from Earth to Mars.

Figure 12 shows the typical \( \Delta V \) for various options of departure date (\( T_E \)) and transfer duration (\( T_{EM} \)). It can be seen from the graph that the fuel minimum transfers lie in the darkest blue region which also means that the optimal transfer exist in this region. The lowest \( \Delta V \) is approximately 5.7 km/s.

Figure 12. \( \Delta V \) for Earth – Mars direct transfer (adapted from [2])
Figure 13 presents the optimality map as a result of the primer vector simulation of the respective transfers. The blue region represents optimum transfers (the region where the modulus of \( p \) always less or equal to one) while the non-optimum transfers are represented by the yellow region (the region where the modulus of the primer vector reach a value bigger than one). It means that if the selected transfer lies in the yellow region in figure 13, the required \( \Delta V \) can be minimized further while if the transfer lies within the blue region it is not necessary to apply a corrective maneuver.

\[
\begin{align*}
\text{Figure 13. Earth-Mars transfer optimality analysis}
\end{align*}
\]

In case the selected transfer is not optimum, corrective maneuver can be included to minimize the fuel and the primer vector can provide a good initial guess to apply this maneuver, which is the time and position where its modulus reach its maximum value above one.

4. Conclusions
The analytical solution to the in-plane primer vector equations has been derived using an orbital relative energy and relative angular momentum. The validation of the solution is performed by comparing the results to the numerical solution. The errors between the two approaches are within the expected range of around \( 10^{-12} \). It has been verified that the source of error is the numerical integration process due to the fact that the order of error is roughly equal to the error tolerance specified for the ODE45.

The primer vector can be used to improve the optimization result of the direct method by evaluating if the candidate transfer can be optimize further by applying DSM. The study shows that the analytical solutions of the primer vector can identify the optimum region of the transfer. Having the analytical solution of the primer vector means that it can be derived further to obtain its first derivative. The first derivative is important for identifying the position and time to apply a correction maneuver without having to propagate the whole primer profile of the transfer.
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References
[1] Kemble, S., Interplanetary Mission Analysis and Design. Springer-Praxis Publishing Ltd, Chicester UK, 2016.
[2] Zazzera, F. B. et al. Global Trajectory Optimization: can we prune the solution space when considering Deep Space Maneuvers? Technical report, ESA, 2007.
[3] Juno mission and trajectory design. http://spaceflight101.com/juno/juno-mission-trajectory-design/.
[4] New frontiers program, mission juno. https://discoverynewfrontiers.nasa.gov/missions/missions_juno.cfm.
[5] Critical deep-space maneuver targets messenger for its first mercury encounter. http://www.spaceref.com.
[6] Exomars-2016 completes deep space maneuvers. http://www.russianspaceweb.com/exomars2016-mission-dsm.html.
[7] Deep space manoeuvre part 2 – the little burn. http://blogs.esa.int
[8] Gad, A. H., Space Trajectories Optimization Using Variable-Chromosome-Length Genetic Algorithms. PhD thesis, Michigan Technological University, 2011.
[9] Vasile, M., Pascale, P. D. Preliminary design of multiple gravity-assist trajectories. Journal of Spacecraft and Rockets (AIAA), Vol. 43(No. 4): pp. 794-805, 2006.
[10] Prussing, J. E., Primer Vector Theory and Applications. Cambridge University Press, 2010.
[11] Lawden, D. F., Optimal trajectory for space navigation, Butterworths Mathematical Text, University of Wisconsin – Madison, 1963.
[12] Conway, B. A., Spacecraft Trajectory Optimization. Cambridge University Press, 2010.
[13] Handelsman, M. and Lion, P. M., Primer vector on fixed-time impulsive trajectories. AIAA Journal, Vol. 6, No. 1: pp 127-132, 2007.
[14] Jezewski, D. J., Primer vector theory and applications. Technical report, Lyndon B. Johnson Space Center, NASA, 1975.
[15] Palmer, P. L., Iorfida, E., Roberts, M., Geometric approach to the perpendicular thrust case for trajectory optimization. Journal of Guidance, Control, and Dynamics, Vol. 39(No. 5): pp. 1059-1068, 2016.
[16] Palmer, P. L. and Imre, E., Relative motion between satellites on neighbouring keplerian orbits. Journal of Guidance, Control, and Dynamics, Vol. 30(No. 2):pp. 521-528, 2007.
[17] R. H. Battin. An Introduction to the Mathematics and Methods of Astrodynamics. American Institute of Aeronautics and Astronautics, Inc. Reston, Virginia, 1999.