On the spectral theory of linear differential-algebraic equations with periodic coefficients

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Received: 4 November 2022 / Revised: 30 August 2023 / Accepted: 28 October 2023 / Published online: 24 November 2023 © The Author(s), under exclusive licence to Springer Nature Switzerland AG 2023

Abstract
In this paper, we consider the spectral theory of linear differential-algebraic equations (DAEs) for periodic DAEs in canonical form, i.e.,

\[ J \frac{df}{dt} + Hf = \lambda Wf, \]

where \( J \) is a constant skew-Hermitian \( n \times n \) matrix that is not invertible, both \( H = H(t) \) and \( W = W(t) \) are \( d \)-periodic Hermitian \( n \times n \)-matrices with Lebesgue measurable functions as entries, and \( W(t) \) is positive semidefinite and invertible for a.e. \( t \in \mathbb{R} \) (i.e., Lebesgue almost everywhere). Under some additional hypotheses on \( H \) and \( W \), called the local index-1 hypotheses, we study the maximal and the minimal operators \( L \) and \( L'_{0} \), respectively, associated with the differential-algebraic operator \( L = W^{-1}(J \frac{d}{dt} + H) \), both treated as an unbounded operators in a Hilbert space \( L^2(\mathbb{R}; W) \) of weighted square-integrable vector-valued functions. We prove the following: (i) the minimal operator \( L'_{0} \) is a densely defined and closable operator; (ii) the maximal operator \( L \) is the closure of \( L'_{0} \); (iii) \( L \) is a self-adjoint operator on \( L^2(\mathbb{R}; W) \) with no eigenvalues of finite multiplicity, but may have eigenvalues of infinite multiplicity. Finally, we show that for 1D photonic crystals with passive lossless media, Maxwell’s equations for the electromagnetic fields become, under separation of variables, periodic DAEs in canonical form satisfying our hypotheses so that our spectral theory applies to them.

This work is based in part on the Ph.D. dissertation of the first author Bader Alshammari.

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Keywords  Linear differential-algebraic equations · Periodic coefficients · Operator theory · Spectral theory · Electromagnetics · 1D photonic crystals

Mathematics Subject Classification 34A09 · 34L05 · 47B25 · 47A75 · 78M22 · 46N20 · 47N20 · 47B38 · 47B93 · 47N50

1 Introduction

The spectral theory of linear differential-algebraic equations (DAEs) with periodic coefficients is of significant interest from both a theoretical point of view and in applications (see, for instance, [1–3] and [4, Sec. 6.6.7]) and this is especially true for electromagnetic problems involving 1D photonic crystals (see, for instance, [5–9]). For linear ordinary differential equations (ODEs), the spectral theory is well-developed (see, for instance, [10, 11]). But this is not true for DAEs when the interval under consideration is unbounded or when the spectral problem is posed on a Hilbert space of weighted $L^2$ functions. Yet, such DAEs arise naturally when considering electromagnetic phenomena in periodic layered media (1D photonic crystals) involving passive lossless media (especially when the materials are anisotropic, biisotropic, or bianisotropic instead of isotropic). In this case, solving for the time-harmonic electromagnetic fields using separation of variables, Maxwell’s equations are reduced to exactly such a spectral problem on the whole real line for DAEs in canonical form.

Motivated by such applications, we consider the spectral theory for periodic DAEs in the following implicit canonical (cf. [12]) or Hamiltonian form:

$$ J \frac{df}{dt} + Hf = \lambda Wf, \quad (1) $$

where $J$ is a constant skew-Hermitian $n \times n$ matrix that is not invertible, both $H = H(t)$ and $W = W(t)$ are $d$-periodic Hermitian $n \times n$ matrices with Lebesgue measurable functions as entries, and $W(t)$ is positive semidefinite and invertible for a.e. $t \in \mathbb{R}$ (i.e., Lebesgue almost everywhere).

In this paper we study the maximal and the minimal operators $L$ and $L'_0$ (see Definition 4), respectively, associated with the differential-algebraic (DA) operator $\mathcal{L} = W^{-1}(J \frac{d}{dt} + H)$ (see Definition 1), both treated as an unbounded operators in a Hilbert space $L^2(\mathbb{R}; W)$ [as defined by (12) with inner product (13); see also Remark 3] of weighted square-integrable vector-valued functions. Our main result is Theorem 30 [that requires the local index-1 hypotheses (see Hyp. 11) are true for $H - z_0 W$, $W$ with respect to $J$ on the interval $\mathbb{R}$ for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$] which can be summarized as follows: (i) the minimal operator $L'_0$ is a densely defined and closable operator; (ii) the maximal operator $L$ is the closure of $L'_0$; (iii) $L$ is a self-adjoint operator on $L^2(\mathbb{R}; W)$ with no eigenvalues of finite multiplicity, but may have eigenvalues of infinite multiplicity. We also give two examples which show the latter can occur (see Examples 31 and 35).

An important result of this paper is Theorem 34 which gives simple hypotheses on the coefficients $H, W$ in terms of $J$ that are sufficient for Theorem 30 to be true.
[i.e., for (i)-(iii) to be true] and they can be stated as follows: Let \( v_1, \ldots, v_{n_1} \) and 
\( v_{n_1+1}, \ldots, v_n \) be an orthonormal basis of \( n \times 1 \) column vectors for the range and 
kernel of \( J \), respectively, and define the \( n \times n \) unitary matrix \( V \) to be the column 
matrix 
\[
\begin{bmatrix}
v_1 & \cdots & v_{n_1} & v_{n_1+1} & \cdots & v_n 
\end{bmatrix}
\]. We next define \( n_2 = n - n_1 \) and the \( n_i \times n_j \) matrix-valued functions \( H_{ij} \) and 
\( W_{ij} \) (for \( i, j = 1, 2 \)) by forming a \( 2 \times 2 \) block matrix partitioning of 
\( V^{-1}HV \) and \( V^{-1}WV \) as
\[
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}, 
\begin{bmatrix}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{bmatrix}.
\]

Then Theorem 34 tells us that if all the entries of \( H_{11}, W_{11}, \) and \( H_{12}W_{22}^{-1}H_{21} \) are in 
\( L^1([0, d]) \) (i.e., integrable on the interval \([0, d]\)) then Theorem 30 is true.

Let us compare all this to the spectral theory for periodic ODEs in the form (1) when \( J \) is invertible, 
also known as canonical or Hamiltonian (system of) ODEs (see, for instance, [10, 11, 13–20]). In [10], 
V. I. Derguzov proves, under the hypotheses \( H(t) \) and \( W(t) \) are also smooth symmetric matrix-valued functions, 
that the maximal operator \( L \) is self-adjoint on \( L^2(\mathbb{R}; W) \) by giving an explicit spectral representation of
\( L \) expanded in terms of the Bloch solutions of the periodic ODEs (1) which proves \( L \) has purely absolutely continuous (AC) spectrum and is unitarily equivalent to multiplication by \( \lambda \) in a special Hilbert space of square-integrable functions. Furthermore, 
he proves that the spectral multiplicity at each \( \lambda \) is equal to the number of linearly independent 
bounded solutions (corresponding to the dimension of the space of those Bloch solutions) of (1). In [11], J. Weidmann has shown using different methods 
that you can weaken the hypotheses that Derguzov had to \( H \) and \( W \) are Hermitian with entries in \( L^1([0, d]) \) and still prove the maximal operator \( L \) is self-adjoint on 
\( L^2(\mathbb{R}; W) \) [11, Theorem 12.3] with purely AC spectrum and \( L \) is the closure of the 
minimal operator \( L_0 \) [11, Theorem 12.4]. In particular, the self-adjoint operator \( L \) has 
no eigenvalues. It should be pointed out that we are not aware of any weaker hypothe-
ses that allow such a broad class of coefficients \( H, W \) which yield these results and 
in this sense, [11] is the “state-of-the-art” for the spectral theory of canonical ODEs 
with periodic coefficients.

Our goal here is to develop such a spectral theory for DAEs in the form (1) that is 
comparable to those results above for ODEs, under the weakest hypotheses on \( H, W \) we can get. Based on techniques from [11] adapted to DAEs together with the spectral 
theory for unbounded operators [21, 22] and the theory of Schur complements [23], we 
are able to prove the results (i)–(iii) in Theorem 30 which represent significant progress 
toward this goal. Furthermore, as the local index-1 hypotheses in that theorem may 
not be easy to verify in applications, we also provide Proposition 33 and Theorem 34 
to help simplify those hypotheses. Finally, given the difficulties of proving our main 
results (i)–(iii), we do not try to prove the self-adjoint operator \( L \) has no singular 
continuous spectrum (as done for ODEs in [10] and [11]) nor do we try to develop 
an analog spectral representation of \( L \) (as was done in [10] for ODEs) in terms of an 
expansion in Bloch solutions. Nevertheless, for a complete spectral theory on \( L \), it 
would be desirable to do so (if possible) for the applications mentioned above, but this 
is left for future work.
As a final remark, the literature on DAEs using functional analysis and operator theory that is closest in spirit to our approach, especially Sect. 3, is [24–27]. They are mainly interested in closedness and normal solvability of the maximal operator $L$ on bounded or compact intervals $I$ for Hilbert spaces of $L^2$ functions and its adjoint relationship to the minimal operator $L_0'$. However, they do not treat the self-adjoint spectral theory of DAEs nor do they treat unbounded intervals nor weighted Hilbert spaces of $L^2$ functions as we need to in this paper. In addition, if we were to use the approach that they do then the weakest hypotheses on $H, W$ that we could use would require its entries to be in $L^\infty$ (i.e., essentially bounded functions) among other things. Nevertheless, these works are complementary to our paper.

The rest of this paper is organized as follows. In Sect. 2, we motivate our studies on the spectral theory of periodic DAEs by considering electromagnetism and Maxwell’s equations associated with 1D photonic crystals and passive lossless media. In Sect. 3, we study the differential operators $L_0', L, L$ associated with the DAEs on an arbitrary interval $I \subseteq \mathbb{R}$ (with nonempty interior). In Sect. 4, we study the periodic case of these operators (with $I = \mathbb{R}$) and prove our main results (i)–(iii) above. In Subsec. 4.1, we show how to simplify the hypotheses needed to prove those results. Finally, Appendix 1 provides the basic notations and auxiliary results that are needed.

2 One-dimensional photonic crystals

Here we motivate our studies on periodic linear DAEs by showing how they arise in electromagnetism (EM) involving 1D photonic crystals. By starting from Maxwell’s equations and consider time-harmonic electromagnetic fields in a periodic layered media with passive lossless materials, it will be shown how to reduce the problem to periodic linear DAEs in the canonical form (1).

Consider Maxwell’s equations (in Gaussian units) without sources [28–30]:

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0,
\]

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{D}$ and $\mathbf{B}$ are the electric displacement and magnetic induction fields, and $c$ is the speed of light (in a vacuum). For passive lossless (linear) homogeneous media, the constitutive relations are of the form (see, for instance, [29, Sec. 80], [31, 32])

\[
\begin{bmatrix}
\mathbf{D} \\
\mathbf{B}
\end{bmatrix} = W
\begin{bmatrix}
\mathbf{E} \\
\mathbf{H}
\end{bmatrix}, \quad W = \begin{bmatrix}
\varepsilon & \xi \\
\xi & \mu
\end{bmatrix},
\]

where the $3 \times 3$ matrices $\varepsilon, \mu, \xi, \mu \in M_3(\mathbb{C})$ are the dielectric permittivity, magnetic permeability, and magnetoelectric coupling tensors, respectively, and $W \in M_6(\mathbb{C})$ is a Hermitian and positive definite matrix, i.e., $W^* = W > 0$.

Next, consider a 1D photonic crystal made of such materials. Specifically, a $d$-periodic layered medium whose layers are normal to $z$-axis, homogeneous in the $xy$-plane, with matrix-valued function $W = W(z) \in M_6(\mathcal{M}(\mathbb{R}))$ satisfying
\[ W(z + d) = W(z), \quad W(z)^* = W(z) > 0 \text{ for a.e. } z \in \mathbb{R}. \]

Using separation of variables, one finds that Maxwell’s equations admit time-harmonic electromagnetic fields of the form

\[ E = E(z)e^{i(k_1x + k_2y - \omega t)}, \quad H = H(z)e^{i(k_1x + k_2y - \omega t)} \]

[with frequency \( \omega \in \mathbb{C} \), \( (k_1, k_2) \in \mathbb{R}^2 \), and \( i = \sqrt{-1} \)], provided \( E(z), H(z) \) are solutions to the periodic Maxwell's DAEs:

\[
\begin{align*}
J\frac{d}{dz}f + Hf &= \lambda Wf, \quad (2) \\
f(z) &= \begin{bmatrix} E(z) \\ H(z) \end{bmatrix}, \quad \lambda = \frac{\omega}{c}, \quad W = W(z) = \begin{bmatrix} \varepsilon(z) & \xi(z) \\ \zeta(z) & \mu(z) \end{bmatrix}, \\
J &= i^{-1} \begin{bmatrix} 0 & -e_3 \times \\ e_3 \times & 0 \end{bmatrix}, \quad e_3 \times = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
H &= \begin{bmatrix} 0 & k_{\perp} \times \\ -k_{\perp} \times & 0 \end{bmatrix}, \quad k_{\perp} \times = \begin{bmatrix} 0 & 0 & k_2 \\ 0 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}. \\
\end{align*}
\]

These are periodic DAEs in canonical form (1). Indeed, \( J \) is a constant skew-Hermitian 6 \( \times \) 6 matrix that is not invertible, both \( H \) and \( W = W(z) \) are \( d \)-periodic Hermitian 6 \( \times \) 6 matrices with Lebesgue measurable functions as entries, and \( W(z) \) is positive semidefinite and invertible for a.e. \( z \in \mathbb{R} \).

Our general approach to reducing the DAEs to ODEs (see Proposition 14 below) for this example can start with the following [cf. (14)]:

\[
V = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad V^* = V^{-1}, \quad V^{-1}J V = \begin{bmatrix} J_{11} & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
J_{11} = i^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]

where \( V \) acts by block partitioning the tangential and normal components of the electromagnetic fields as

\[
E = \begin{bmatrix} E_1 & E_2 & E_3 \end{bmatrix}^T, \quad H = \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}^T,
\]
\[ V^{-1} \begin{bmatrix} E \\ H \end{bmatrix} = V^{-1} f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad f_1(z) = \begin{bmatrix} E_1(z) \\ E_2(z) \\ H_1(z) \\ H_2(z) \end{bmatrix}, \quad f_2(z) = \begin{bmatrix} E_3(z) \\ H_3(z) \end{bmatrix}. \]

The tangential components \( f_1 \) give rise to the differential part of the periodic Maxwell’s DAEs (2) (see also Example 2), whereas the normal components \( f_2 \) are derived from them algebraically. This statement can be made rigorous using Proposition 14 [by taking the interval \( I = \mathbb{R} \) and the coefficients \( J, H, W \) to be as in (3)–(5)], on solvability of linear DAEs.

In conclusion, the methods we use in this paper to reduced linear DAEs to linear ODEs is implicit in a standard and well known approach in studying the electrodynamics of layered media [5, 33–38] (as opposed to other standard methods such as those in [39–41]) and has been used effectively by the second author of this paper, for instance, in [6, 8, 9, 42]. But using DA operators and spectral theory in this context is completely new and deserves to be further studied from this perspective, especially for 1D photonic crystals. More broadly, it is known that linear DAEs arise in computational electromagnetics [43] and in circuit theory [44], and as such it would be interesting and worth investigating how our methods and results would be useful in these contexts.

### 3 The minimal and maximal operators associated with linear differential-algebraic equations

In this section we will consider the linear operators associated to the linear DAEs in the following form:

\[ J \frac{d}{dt} f + H f = W g \quad (6) \]

on an interval \( I \subseteq \mathbb{R} \) (with nonempty interior) and with coefficients \( J, H, W \) satisfying the following (see Appendix 1 for notation):

\[ J \in M_n(\mathbb{C}), \quad J \neq 0, \quad J^* = -J, \quad (7) \]

\[ H, W : I \rightarrow M_n(\mathbb{C}), \quad H, W \in M_n(M(I)), \quad (8) \]

\[ W(t)^* = W(t) \geq 0, \quad \det W(t) \neq 0, \quad \text{for a.e.} \ t \in I. \quad (9) \]

**Definition 1** The linear operator \( \mathcal{L} : D(\mathcal{L}) \rightarrow [M(I)]^n \) defined by

\[ D(\mathcal{L}) = \left\{ f \in [M(I)]^n : Jf \in [W_{loc}^{1,1}(I)]^n \right\}, \]

\[ \mathcal{L} f = W^{-1} \left( \frac{d}{dt} Jf + Hf \right), \quad f \in D(\mathcal{L}) \]
is called the differential-algebraic (DA) operator associated with \( I, J, H, W \).

Alternatively, since \( J \) is a skew-Hermitian matrix (which implies its range is orthogonal to its kernel), \( D(\mathcal{L}) \) and \( \mathcal{L} \) are given by

\[
D(\mathcal{L}) = \left\{ f \in [\mathcal{M}(I)]^n : Pf \in [W_{loc}^{1,1}(I)]^n \right\},
\]

\[
\mathcal{L} f = W^{-1} \left[ J \frac{d}{dt} (Pf) + Hf \right], \quad f \in D(\mathcal{L}),
\]

where \( P \in M_n(\mathbb{C}) \) is the orthogonal projection of \( \mathbb{C}^n \) onto \( \text{ran} \, J \).

**Example 2** Consider the periodic Maxwell’s DAEs (2). The corresponding DA operator \( \mathcal{L} \), with \( f \in D(\mathcal{L}) \), is associated with the interval \( I = \mathbb{R} \) and \( J, H, W \) defined by (3)–(5), and the orthogonal projection \( P \in M_6(\mathbb{C}) \) of \( \mathbb{C}^6 \) onto \( \text{ran} \, J \) is

\[
P = J^* J = \begin{bmatrix} \Pi & 0 \\ 0 & \Pi \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where}
\]

\[
[W_{loc}^{1,1}(\mathbb{R})]^6 \ni Pf(z) = P \begin{bmatrix} \mathcal{E}(z) \\ \mathcal{H}(z) \end{bmatrix} = \begin{bmatrix} E_1(z) & E_2(z) & 0 & H_1(z) & H_2(z) & 0 \end{bmatrix}^T.
\]

Next, define the Hilbert space \( L^2(I; W) \) with inner product \( \langle \cdot, \cdot \rangle_W \) by

\[
L^2(I; W) = \left\{ f \in [\mathcal{M}(I)]^n : \int_I \langle W(t) f(t), f(t) \rangle dt < \infty \right\},
\]

\[
\langle f, g \rangle_W = \int_I \langle W(t) f(t), g(t) \rangle dt, \quad f, g \in L^2(I; W),
\]

and the subspace \( L^2_{loc}(I; W) \) of \( \mathcal{M}(I) \) by

\[
L^2_{loc}(I; W) = \left\{ f \in [\mathcal{M}(I)]^n : f \in L^2([a, b]; W), \text{ for every compact interval } [a, b] \subseteq I \right\}.
\]

**Remark 3** It easy to verify that left multiplication by \( W^{-1/2} \) is an isometric isomorphism between the Hilbert spaces \( [L^2(I)]^n \) and \( L^2(I; W) \) implying that

\[
L^2(I; W) = W^{-1/2}[L^2(I)]^n, \quad L^2_{loc}(I; W) = W^{-1/2}[L^2_{loc}(I)]^n.
\]

**Definition 4** The maximal operator \( L : D(L) \to L^2(I; W) \) generated by \( \mathcal{L} \) and the minimal operator \( L'_0 : D(L'_0) \to L^2(I; W) \) generated by \( \mathcal{L} \) are defined by

\[
D(L) = \{ f \in L^2(I; W) : f \in D(\mathcal{L}), \mathcal{L} f \in L^2(I; W) \},
\]

\[
L f = \mathcal{L} f, \quad \text{for } f \in D(L),
\]

\[
D(L'_0) = \{ f \in D(L) : Jf \text{ has compact support contained in the interior of } I \},
\]

\[
L'_0 f = \mathcal{L} f, \quad \text{for } f \in D(L'_0).
\]
Remark 5 Note that the sets $D(L_0')$ and $D(L)$ are subspaces of the Hilbert space $L^2(I; W)$ and both $L_0' : D(L_0') \to L^2(I; W)$, $L : D(L) \to L^2(I; W)$ are linear operators. This only required mild regularity on the coefficients, but closedness of these operators and completeness of their domains will be proven under stronger regularity conditions that are described by the local index-1 hypotheses (Hyp. 11) below.

Notation 6 When we need to be explicit about the dependence of $\mathcal{L}$, $L$ or $L_0'$ on $H$ and/or $J$ we will use the subscript $(\cdot)_H$ or $(\cdot)_{J,H}$ with these operators, e.g., $\mathcal{L}_H$, $L_H$ or $(L_0')_H$ and for the latter, we will just write $L_{H,0}'$ instead. Similarly, $(L_0')_{J,H}$ will be written instead as $L_{J,H,0}'$.

The following theorem is fundamental in developing the spectral theory associated with the operator $L$ on the Hilbert space $L^2(I; W)$.

Theorem 7 (a) For $f \in D(L_0')$ and $g \in D(L_{H^*})$ we have

$$\langle L_0' f, g \rangle_W = \langle f, L_{H^*} g \rangle_W.$$

(b) For $f \in D(L_0')$ and $g \in D(L_{H^*,0}')$ we have

$$\langle L_0' f, g \rangle_W = \langle f, L_{H^*,0}' g \rangle_W.$$

Proof (a) Let $f \in D(L_0')$. Then there exists a compact interval $[a, b] \subseteq I$ such that $(Jf)(t) = 0$ for every $t \in I \setminus (a, b)$ and, in particular, $(Jf)(a) = (Jf)(b) = 0$. Hence, for any $g \in D(L_{H^*})$ and using (10) and (11) (with $H^*$ replacing $H$), we have

$$\langle L_0' f, g \rangle_W = \left\langle \int \left[ \frac{d}{dt} Jf(t) + H(t)f(t) \right], g(t) \right\rangle dt$$

$$= \int \frac{d}{dt} \langle Jf(t), Pg(t) \rangle - \langle Jf(t), \frac{d}{dt} Pg(t) \rangle + \langle f(t), H(t)^*g(t) \rangle dt$$

$$= \int \frac{d}{dt} \langle Jf(t), Pg(t) \rangle + \langle W(t)f(t), W(t)^{-1} \frac{d}{dt} Jg(t) + H(t)^*g(t) \rangle dt$$

$$= \langle (Jf)(b), (Pg)(b) \rangle - \langle (Jf)(a), (Pg)(a) \rangle + \langle f, L_{H^*} g \rangle_W$$

$$= \langle f, L_{H^*} g \rangle_W.$$

(b) This follows immediately from part a). □

In the case $\det(J) = 0$, since $J^* = -J$, there exists a matrix $V$ such that

$$V \in M_n(\mathbb{C}), \ V^* = V^{-1}, \ V^{-1} J V = \begin{bmatrix} J_{11} & 0 \\ 0 & 0 \end{bmatrix}, \ \det(J_{11}) \neq 0, \ (14)$$

and, in particular, it follows that $J_{11}^* = -J_{11}$. Next, block partition the matrices $V^{-1}HV, V^{-1}WV \in M_n(\mathcal{M}(I))$ conformal to the block structure of $V^{-1}JV$ in (14),
Similarly, the positive square roots \( W^{±} \) of \( W \) i.e.,

\[
V^{-1}HV = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad V^{-1}WV = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \quad H_{ij}, \ W_{ij} \in M_{n_i \times n_j}(\mathcal{M}(I)),
\]

\( n_1 := \text{rank} \ J = \text{rank} \ J_{11}, \quad n_2 := n - n_1 = \text{nullity}(J). \) (15)

If \( H_{22}^{-1} \in M_{n_2}(\mathcal{M}(I)) \) [equivalently, \( \det H_{22}(t) \neq 0 \) for a.e. \( t \in I \)] then for the Schur complement \( H/H_{22} \) of \([H_{ij}]_{i,j=1,2}\) with respect to \( H_{22} \) (see [23]), we have

\[
H/H_{22} := H_{11} - H_{21}H_{22}^{-1}H_{12} \in M_{n_1}(\mathcal{M}(I)).
\]

The next set of lemmas lead to a useful alternative characterization of \( L^2(I; W) \) and \( L^2_{1 loc}(I; W) \).

**Lemma 8** If \( W \) satisfies the hypotheses (8) and (9) then the inverse \( W^{-1}(t) := W(t)^{-1} \) of \( W(t) \) exists for a.e. \( t \in I \) (and setting it to say the identity matrix \( I_n \) when it doesn’t exist) and defines a function \( W^{-1} : I \to M_n(\mathbb{C}) \) which satisfies

\[
W^{-1} \in M_n(\mathcal{M}(I)), \quad W^{-1}(t)^* = W^{-1}(t)^* \geq 0, \quad \det W^{-1}(t) \neq 0, \quad \text{for a.e. } t \in I.
\]

Similarly, the positive square roots \( W^{±1/2}(t) := W(t)^{±1/2} \) of \( W^{±1}(t) \) exists for a.e. \( t \in I \) (and setting it to say the identity matrix \( I_n \) when it doesn’t exist) and defines functions \( W^{±1/2} : I \to M_n(\mathbb{C}) \) which satisfy

\[
W^{±1/2} \in M_n(\mathcal{M}(I)), \quad W^{±1/2}(t)^* = W^{±1/2}(t)^* \geq 0, \quad \det W^{±1/2}(t) \neq 0, \quad \text{for a.e. } t \in I.
\]

**Proof** The proof is obvious from the elementary theory of matrices and that for positive definite matrices. \( \square \)

**Lemma 9** If \( W \) satisfy the hypotheses (8) and (9) and \( V \) satisfy the hypotheses (14) then, with respect to the \( 2 \times 2 \) block matrix partitioning \( V^{-1}WV = [W_{i,j}]_{i,j=1,2} \) in (15), we have \( W_{11}^{-1} \in M_{n_1}(\mathcal{M}(I)) \) and \( W_{22}^{-1} \in M_{n_2}(\mathcal{M}(I)) \), and the Schur complements of \([W_{i,j}]_{i,j=1,2}\) with respect to \( W_{11} \) and \( W_{22} \), i.e.,

\[
W/W_{11} := W_{22} - W_{21}W_{11}^{-1}W_{12}, \quad W/W_{22} := W_{11} - W_{12}W_{22}^{-1}W_{21},
\]

respectively, have the following properties for \( i = 1, 2 \):

\[
W/W_{ii} \in M_{n_i}(\mathcal{M}(I)),
\]

\[
(W/W_{ii})(t)^* = (W/W_{ii})(t)^* \geq 0, \quad \det(W/W_{ii})(t) \neq 0, \quad \text{for a.e. } t \in I,
\]

\[
0 \leq (W/W_{ii})(t) \leq W_{jj}(t) \quad \text{for a.e. } t \in I, \quad \text{for each } j = 1, 2, \quad j \neq i.
\]

**Proof** The proof is immediate from elementary properties of \( 2 \times 2 \) block partitioned positive definite matrices and their Schur complements (see, for instance, [23]) since \( W(t)^* = W(t) \geq 0 \) and \( \det W(t) \neq 0 \) for a.e. \( t \in I \) implies \( W(t) \) is a positive definite
matrix for a.e. $t \in I$ which implies so is $V^{-1}W(t)V = [W_{i,j}(t)]_{i,j=1,2}$ and hence so are the blocks $W_{11}(t)$ and $W_{22}(t)$ with $W_{12}(t)^* = W_{21}(t)$.

\[\square\]

**Lemma 10** If $W$ satisfies the hypotheses (8) and (9), $V$ satisfies the hypotheses (14), and $V^{-1}WV = [W_{i,j}]_{i,j=1,2}$ block partitioned as in (15) then

\[
f \in (\mathcal{M}(I))^{n} \iff V^{-1}f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad f_i \in (\mathcal{M}(I))^{n_i}, \quad i = 1, 2.
\]

Furthermore,

\[
f \in L^2(I; W) \iff \begin{array}{c}
(W/W_{11})^{1/2}f_2 \in (L^2(I))^{n_2}, \\
W_{11}^{-1/2}(W_{11}f_1 + W_{12}f_2) \in (L^2(I))^{n_1}
\end{array}
\]

\[
\iff \begin{array}{c}
(W/W_{22})^{1/2}f_1 \in (L^2(I))^{n_1}, \\
W_{22}^{-1/2}(W_{21}f_1 + W_{22}f_2) \in (L^2(I))^{n_2}.
\end{array}
\]

Similarly, the statement remains true if we replace $L^2(I; W), (L^2(I))^{n_1}, (L^2(I))^{n_2}$ in (17), (18), (19) by $L^2_{loc}(I; W), (L^2_{loc}(I))^{n_1}, (L^2_{loc}(I))^{n_2}$, respectively.

**Proof** The proof of the statement (16) is obvious and so will be omitted. Also, once we prove the equivalence of the statements (17), (18), and (19) then the equivalence of these statements in the local, i.e., “loc,” case follows immediately. Now we prove that (17) holds iff (19) is true, but we will omit the proof of the statement that (17) holds iff (18) is true as it is similar. By Lemma 8, we have the block factorization

\[
V^{-1}WV = [W_{ij}]_{i,j=1,2} = \begin{bmatrix} I_{n_1} & W_{12}W_{22}^{-1} \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} W/W_{22} & 0 \\ W_{22}^{-1}W_{21} & I_{n_2} \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix}.
\]

Next, it follows from this and Lemma 9 together with Lemma 8 applied to $W/W_{22}$ that for any $f \in (\mathcal{M}(I))^n$ we have

\[
\int_I \langle W(t)f(t), f(t) \rangle dt = \int_I \langle V^{-1}W(t)V[V^{-1}f(t)], [V^{-1}f(t)] \rangle dt
\]

\[
= \int_I \left\{ \begin{bmatrix} W/W_{22} & 0 \\ 0 & W_{22} \end{bmatrix} \begin{bmatrix} I_{n_1} & f_1 \\ 0 & f_2 \end{bmatrix}, \begin{bmatrix} I_{n_1} & 0 \\ 0 & f_2 \end{bmatrix} \end{bmatrix} \right\}(t) dt
\]

\[
= \int_I \langle (W/W_{22})(t)f_1(t), f_1(t) \rangle + (W_{22}(t)[W_{22}^{-1}(t)W_{21}(t)f_1(t) + f_2].
\]

\[
W_{22}^{-1}(t)W_{21}(t)f_1(t) + f_2(t) \rangle dt
\]

\[
= \int_I \|[(W/W_{22})^{1/2}f_1](t)\|^{2}dt + \int_I \|W_{22}^{-1/2}(W_{21}f_1 + W_{22}f_2)(t)\|^{2}dt,
\]

from which the proof follows. \[\square\]

We now give the main hypotheses used throughout the paper.
**Hypothesis 11** The following set of hypotheses are called the *local index-1 hypotheses* for $H, W$ with respect to $J$ on the interval $I$:

If $\det J \neq 0$ then $H, W \in M_n(L_{loc}^1(I))$.  \hspace{1cm} (20)

If $\det J = 0$ then

$$
H_{22} \text{ is invertible, i.e., } H_{22}^{-1} \in M_{n_2}(\mathcal{M}(I)),
$$

$$
H_{12}H_{22}^{-1}W_{22}^{1/2} \in M_{n_1 \times n_2}(L_{loc}^2(I)),
$$

$$
H/W_{22}, W_{11} \in M_{n_1}(L_{loc}^1(I)),
$$

$$
W_{22}^{1/2}H_{22}^{-1}W_{22}^{1/2} \in M_{n_2}(L_{loc}^\infty(I)),
$$

$$
W_{22}^{1/2}(W_{22}^{-1}W_{21} - H_{22}^{-1}H_{21}) \in M_{n_2 \times n_1}(L_{loc}^2(I)),
$$

$$
W/W_{22} \in M_{n_1}(L_{loc}^1(I)).
$$

Similarly, if we drop “loc” in the above then these are called the *index-1 hypotheses*.

The next lemma simplifies these hypotheses in a more symmetric form which is useful when considering adjoints as in the proof of the next corollary.

**Lemma 12** Suppose $\det J = 0$. Then the local index-1 hypotheses for $H, W$ with respect to $J$ on the interval $I$ are true if and only if $H_{22}$ is invertible,

$$
H_{22}^{-1} \in M_{n_2}(\mathcal{M}(I)), \hspace{1cm} W_{22}^{1/2}H_{22}^{-1}W_{22}^{1/2} \in M_{n_2}(L_{loc}^\infty(I)),
$$

$$
H/W_{22}, W_{11} \in M_{n_1}(L_{loc}^1(I)),
$$

$$
H_{12}H_{22}^{-1}W_{22}^{1/2} \in M_{n_1 \times n_2}(L_{loc}^2(I)), \hspace{1cm} W_{22}^{1/2}H_{22}^{-1}H_{21} \in M_{n_2 \times n_1}(L_{loc}^2(I)).
$$

Similarly, the statement with the “local” and “loc” dropped is true.

**Proof** By hypotheses on $W$, we have $W_{11} - W/W_{22} = W_{12}W_{22}^{-1}W_{21} = (W_{22}^{-1/2}W_{21})^*(W_{22}^{-1/2}W_{21})$ and $0 \leq W/W_{22} \leq W_{11}$. Assume that $W_{11} \in M_{n_1}(L_{loc}^1(I))$. Then this implies $W/W_{22} \in M_{n_1}(L_{loc}^1(I))$ and $W_{22}^{1/2}W_{21} \in M_{n_2 \times n_1}(L_{loc}^2(I))$. And from this it follows that $W_{22}^{1/2}(W_{22}^{-1}W_{21} - H_{22}^{-1}H_{21}) \in M_{n_2 \times n_1}(L_{loc}^2(I)) \iff W_{22}^{1/2}H_{22}^{-1}H_{21} \in M_{n_2 \times n_1}(L_{loc}^2(I))$. The proof of the lemma now follows immediate from this. \hfill $\square$

**Corollary 13** The (local) index-1 hypotheses for $H, W$ with respect to $J$ on the interval $I$ are true if and only if the (local) index-1 hypotheses for $H^*, W$ with respect to $J$ on the interval $I$ are true.

**Proof** The proof follows immediately from considering matrix adjoints of the conditions in Lemma 12 if $\det J = 0$ or those in Hyp. 11 if $\det J \neq 0$. \hfill $\square$

The reason for such hypotheses will become clear as we move forward in this section, but the next proposition gives the main reason why.
Proposition 14 Suppose $J$, $H$, $W$ satisfy the hypotheses (7)–(9) and, in the case $\det J \neq 0$, that $V$ satisfies the hypotheses (14), and $V^{-1} HV = [H_{i,j}]_{i,j=1,2}$, $V^{-1} WV = [W_{i,j}]_{i,j=1,2}$ are block partitioned as in (15). Then the following statements are true:

(i) If (21) holds then

\[ f \in D(L), \; Lf = g, \]  
\[ \iff \]  
\[ g \in [M(I)]^n, \]  
\[ f_1 \in [W_{loc}^{1,1}(I)]^{n_1}, \]  
\[ J_{11} \frac{df_1}{dt} + H/H_{22} f_1 = F, \]  
\[ f_2 \in [M(I)]^{n_2}, \; f_2 = H^{-1}_{22}(W_{21}g_1 + W_{22}g_2) - H^{-1}_{22}H_{21}f_1, \]

where $F \in [M(I)]^{n_2}$ is defined by

\[ F = (W_{11}g_1 + W_{12}g_2) - H_{12}H^{-1}_{22}(W_{21}g_1 + W_{22}g_2), \]

and $f$, $g$, $f_1$, $f_2$, $g_1$, $g_2$ are related by

\[ V^{-1} f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \; V^{-1} g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \]

(ii) If (20) or (21)–(23) holds then for any $g \in L^2_{loc}(I; W)$, $t_0 \in I$, and $f_0 \in \text{ran} J$ there is a unique solution $f \in D(L)$ to the IVP

\[ Lf = g, \; (Jf)(t_0) = f_0. \]

In addition, in the case $\det J \neq 0$ or if (24)–(26) holds in the case $\det J = 0$, then

\[ f \in L^2_{loc}(I; W). \]

Proof (i),($\Rightarrow$): Suppose (27) is true. Then $Jf \in [W_{loc}^{1,1}(I)]^n$ and this implies that $f_1 \in [W_{loc}^{1,1}(I)]^{n_1}$, $f_2 \in [M(I)]^{n_2}$, where $f_1$, $f_2$ are related to $f$ by (33). Next, since $Lf = g$ then $g \in [M(I)]^n$ and $\frac{d}{dt}(Jf) + Hf = WLf = Wg$ (with equality in $[M(I)]^n$) which yields the system of equations:

\[ J_{11} \frac{df_1}{dt} + H_{11} f_1 + H_{12} f_2 = W_{11}g_1 + W_{12}g_2, \]  
\[ H_{21} f_1 + H_{22} f_2 = W_{21}g_1 + W_{22}g_2, \]

where $g_1$, $g_2$ are related to $f$ by (33). Solving these equations for $f_2$ in terms of $f_1$, $g_1$, $g_2$ yields the equivalent system of equations:

\[ J_{11} \frac{df_1}{dt} + H/H_{22} f_1 = W_{11}g_1 + W_{12}g_2 - H_{12}H^{-1}_{22}(W_{21}g_1 + W_{22}g_2). \]
\[ f_2 = H_{22}^{-1}(W_{21}g_1 + W_{22}g_2) - H_{22}^{-1}H_{21}f_1, \]

which are equivalent to the system of equations (30) and (31), where \( F \) is defined by (32) implying \( F \in [\mathcal{M}(I)]^n \). Thus, we have proven that (27) implies (28)–(32), with \( f, g \) and \( f_1, f_2, g_1, g_2 \) are related by (33). (i.) Conversely, suppose \( f, g \) satisfy (28)–(32), where \( f, g \) and \( f_1, f_2, g_1, g_2 \) are related by (33). Then one can verify that \( f \in D(\mathcal{L}) \) and \( \frac{d}{dt}(Jf) + Hf = Wg \) (with equality in \([\mathcal{M}(I)]^n\)), which implies \( \mathcal{L}f = g \). This completes the proof of (i). (ii): Assume (20) holds, in the case \( \det J \neq 0 \), or, in the case \( \det J = 0 \), that (21)–(23) holds. Let \( g \in L_{loc}^2(I; W), t_0 \in I, f_0 \in \text{ran } J \).

Consider the first case in which \( \det J \neq 0 \). Then \( Wg \in [L_{loc}^1(I)]^n \) since, for any compact interval \([a, b] \subseteq I\), we have \( g \in L^2([a, b]; W) \) and \( W \in M_n(L^1([a, b])) \) by hypothesis so it follows by Holder’s inequality that

\[
\int_{[a, b]} \|W(t)g(t)\|dt = \int_{[a, b]} \langle W(t)g(t), W(t)g(t)\rangle^{1/2}dt \\
\leq \int_{[a, b]} \|W(t)\|\|W(t)^{1/2}g(t)\|dt \\
\leq \int_{[a, b]} (\|W(t)\|^{1/2})^2dt^{1/2} \int_{[a, b]} \|W(t)^{1/2}g(t)\|^2 dt^{1/2} \\
= \left[ \int_{[a, b]} \|W(t)\|dt \int_{[a, b]} \langle W(t)g(t), g(t)\rangle dt \right]^{1/2} < \infty.
\]

It also follows from this that \( J^{-1}Wg \in [L_{loc}^1(I)]^n \) and by hypotheses that \( J^{-1}H \in M_n(L_{loc}^1(I)) \). By Theorem 36, there exists a unique solution \( f \in [W_{1,1}^1(I)]^n = D(\mathcal{L}) \) to the ODE IVP

\[
\frac{d}{dt}f + J^{-1}Hf = J^{-1}Wg, \quad f(t_0) = J^{-1}f_0
\]

on \( I \). The proof of the statement (ii) in the case \( \det J \neq 0 \) now follows immediately from this with the exception that we still need to prove \( f \in L_{loc}^2(I; W) \). But this follows from the fact that \( f \in [W_{1,1}^1(I)]^n \subseteq [L_{loc}^\infty(I)]^n \) so by Holder’s inequality we have for any compact interval \([a, b] \subseteq I\),

\[
\int_{[a, b]} \langle W(t)f(t), f(t)\rangle dt \leq \int_{[a, b]} \|W(t)\|\|f(t)\|^2 dt \\
\leq (\text{ess sup}_{t \in [a, b]}\|f(t)\|)^2 \int_{[a, b]} \|W(t)\|dt < \infty,
\]

implying that \( f \in L_{loc}^2(I; W) \).

Consider now, the second case in which \( \det J = 0 \). First, it follows from the block structure of \( V^{-1}JV \) in (14) that

\[
V^{-1}f_0 = \begin{bmatrix} (f_1)_0 \\ 0 \end{bmatrix}
\]
for a unique \((f_1)_0 \in \mathbb{C}^n\). Next, is the hypothesis (23) is true we know that \(W_{11} \in M_{n_1}(L^1_{\text{loc}}(I))\) from which it follows that \(W_{11}^{-1/2} \in M_{n_1}(L^2_{\text{loc}}(I))\). This together with \(g \in L^2_{\text{loc}}(I; W)\) and Lemma 10 implies by Holder’s inequality together with the assumption that the hypothesis (22) holds, that \(F\) defined in (22) satisfies

\[
F = W_{11}^{1/2}[W_{11}^{-1/2}(W_{11}g_1 + W_{12}g_2)] - H_{12}H_{22}^{-1}W_{22}^{1/2}[W_{22}^{-1/2}(W_{21}g_1 + W_{22}g_2)],
\]

\(F \in [L^1_{\text{loc}}(I)]^{n_1}\).

This and the assumption that the hypothesis (23) holds implies that \(J_{11}^{-1}F \in [L^1_{\text{loc}}(I)]^{n_1}\) and \(J_{11}^{-1}H/H_{22} \in M_{n_1}(L^1_{\text{loc}}(I))\). By Theorem 36, there exists a unique solution \(f_1 \in [W^1_{11}(I)]^{n_1}\) to the ODE IVP

\[
\frac{df_1}{dt} + J_{11}^{-1}H/H_{22}f_1 = J_{11}^{-1}F, \quad f_1(t_0) = J_{11}^{-1}(f_1)_0.
\]

on \(I\). Thus, if we take \(f_2\) to be defined by (31) then \(f, g\) satisfy (28)–(32), where \(f, g\) and \(f_1, f_2, g_1, g_2\) are related by (33). It follows from this and part (i) of this proposition, that \(f \in D(L), Lf = g\), and we have

\[
(Jf)(t_0) = (JVV^{-1}f)(t_0) = V \left[ \begin{array}{c} J_{11}f_1(t_0) \\ 0 \end{array} \right] = V \left[ \begin{array}{c} (f_1)_0 \\ 0 \end{array} \right] = f_0.
\]

The uniqueness portion of statement (ii) in this case follows immediately by the uniqueness of the solution to the ODE IVP (34) on \(I\). This proves statement (ii) in the case \(\det J = 0\) with the exception that we still need to prove \(f \in L^2_{\text{loc}}(I; W)\) if (24)–(26) holds. We do this next.

Suppose, in addition, that hypotheses (24)–(26) are true. To prove \(f \in L^2_{\text{loc}}(I; W)\) it suffices by Lemma 10 to prove \(f_1, f_2\) satisfy (19) in the “loc” case, i.e., when in (19) we replace \([L^2(I)]^{n_1}, [L^2(I)]^{n_1} \) with \([L^2_{\text{loc}}(I)]^{n_1}, [L^2_{\text{loc}}(I)]^{n_1}\), respectively. Let \([a, b] \subseteq I\) be a compact interval. First, by the hypothesis (26) we have \(W/W_{22} \in M_{n_1}(L^1([a, b]))\), and so by Lemma 9 and then Lemma 8 applied to \(W/W_{22}\) (instead of \(W\)) it follows that \((W/W_{22})^{1/2} \in M_{n_1}(L^2([a, b]))\). Hence, since \(f_1 \in [W^{1,1}([a, b])]^{n_1} \subseteq [L^\infty([a, b])]^{n_1}\), this implies by Holder’s inequality that \((W/W_{22})^{1/2}f_1 \in [L^2([a, b])]^{n_1}\). Also, as \(f_1 \in [L^\infty(I)]^{n_1}\) and by hypothesis (25), it follows by Holder’s inequality that \((W/W_{22})^{1/2}(W_{22}^{-1}H_{22}^{-1}H_{21})f_1 \in (L^2([a, b]))^{n_2}\).

Next, as \(g \in L^2([a, b]; W)\), it follows by Lemma 10 that \((W_{22}^{-1/2}(W_{21}g_1 + W_{22}g_2) \in (L^2([a, b]))^{n_2}\). Hence, from this and hypothesis (24), it follows by Holder’s inequality that \((W_{22}^{-1/2}H_{22}^{-1}W_{22}^{-1/2}([W_{22}^{-1/2}(W_{21}g_1 + W_{22}g_2)]) \in (L^2([a, b]))^{n_2}\). Finally, it follows from these facts, the formula (31) for \(f_2\), and Minkowski’s inequality that

\[
W_{22}^{-1/2}(W_{21}f_1 + W_{22}f_2)
\]

\[
= W_{22}^{-1/2}(W_{21}f_1 + W_{22}(H_{22}^{-1}(W_{21}g_1 + W_{22}g_2) - H_{22}^{-1}H_{21}f_1)))
\]

\[
= W_{22}^{-1/2}(W_{21}f_1 - W_{22}H_{22}^{-1}H_{21}f_1) + W_{22}^{-1/2}H_{22}^{-1}(W_{21}g_1 + W_{22}g_2)
\]
This proves that \( f \in L^2([a, b]; W) \). As \([a, b]\) was an arbitrary compact interval in \( I \),
this proves \( f \in L^2_{loc}(I; W) \) which completes the proof. \( \square \)

**Corollary 15** If the index-1 hypotheses (20) or (21)–(26) are true on a compact interval \( I \) then

\[
\text{ran } L = L^2(I; W), \quad \ker L = \ker \mathcal{L}, \quad \dim \ker L = \text{rank } J < \infty.
\]

**Proof** Let \( g \in L^2(I; W) \), where \( I \) is a compact interval. Fix any \( t_0 \in I \) and \( f_0 \in \text{ran } J \).
Then by Proposition 14, the unique solution \( f \in D(\mathcal{L}) \) to the IVP \( \mathcal{L}f = g \) with
\( (Jf)(t_0) = f_0 \), is an element of \( L^2_{loc}(I; W) = L^2(I; W) \). Hence, \( f \in D(\mathcal{L}) \) and
\( Lf = \mathcal{L}f = g \) which implies \( f \in \text{ran } L \). This proves that \( L^2(I; W) \subseteq \text{ran } L \) and
since \( \text{ran } L \subseteq L^2(I; W) \) (by definition of \( L \)), we conclude that \( \text{ran } L = L^2(I; W) \).
Next, as we can take \( g = 0 \) in this proof, it follows immediately that \( \ker L = \ker \mathcal{L} \).
It remains to prove \( \text{dim } \ker L = \text{rank } J < \infty \). First, as \( J \in \mathbb{C}^n \), \( J \neq 0 \), then \( 1 \leq r := \text{rank } J = \text{dim } \ker J \leq n < \infty \). Next, let \( \beta_1, \ldots, \beta_r \) be a basis for \( \text{ran } J \).
Then in our proof with \( g = 0 \) and \( t_0 \in I \) fixed, there exists a unique solution \( f_j \in D(\mathcal{L}) \)
\( Lf_j = 0 \) with \( (Jf_j)(t_0) = \beta_j \), for each \( j = 1, \ldots, r \). We claim that \( f_1, \ldots, f_r \)
is a basis for \( \ker L \). Obviously, \( f_1, \ldots, f_r \in \ker L \). Next, let \( c_1, \ldots, c_r \in \mathbb{C} \) be such
that \( c_1 f_1 + \cdots + c_r f_r = 0 \). Then \( 0 = (J0)(t_0) = [J(c_1 f_1 + \cdots + c_r f_r)](t_0) =
\( c_1 \beta_1 + \cdots + c_r \beta_r \) implying \( c_1 = \cdots = c_r \). This proves the vectors \( f_1, \ldots, f_r \)
are linearly independent. Finally, let \( h \in \ker L \). Then there exists scalars \( c_1, \ldots, c_r \in \mathbb{C} \)
such that \( (Jh)(t_0) = c_1 \beta_1 + \cdots + c_r \beta_r = [J(c_1 f_1 + \cdots + c_r f_r)](t_0) \). By the uniqueness
of the solution \( f \in D(\mathcal{L}) \) to \( Lf = 0 \), \( (Jf)(t_0) = (Jh)(t_0) \), it follows that \( h =
c_1 f_1 + \cdots + c_r f_r \). This proves that the vectors \( f_1, \ldots, f_r \) span \( \ker L \). Therefore,
the vectors \( f_1, \ldots, f_r \) are a basis for \( \ker L \) which proves our claim and proves that
\( \dim \ker L = r = \text{rank } J < \infty \). \( \square \)

**Definition 16** For a compact interval \( I = [a, b] \), the **closed minimal operator** \( L_0 : D(L_0) \rightarrow L^2(I; W) \) generated by \( \mathcal{L} \) is the linear operator defined by

\[
D(L_0) = \{ f \in D(\mathcal{L}) : (Jf)(a) = (Jf)(b) = 0 \}, \quad L_0 f = \mathcal{L} f, \text{ for } f \in D(L_0).
\]

**Notation 17** When we need to be explicit about the dependence of \( L_0 \) on \( H \) and/or \( J \)
we will use the subscript \( (\cdot)_H \) or \( (\cdot)_{J,H} \) with these operators, e.g., \( (L_0)_H \) and for
the latter, we will write \( L_{H,0} \). Similarly, \( (L_0)_{J,H} \) will be written instead as \( L_{J,H,0} \).

The importance of the next two theorems becomes clearer by comparing it to
Theorem 37 with \( A = L_0, B = L_{H*} \) as it is used to prove Theorem 20 below.

**Theorem 18** Suppose the index-1 hypotheses are true for \( H, W \) with respect to \( J \) on
a compact interval \( I \). Then the following are true:

(a) For \( f \in D(L_0) \) and \( g \in D(L_{H^*}) \) we have
\[
\langle L_0 f, g \rangle_W = \langle f, L_{H^*} g \rangle_W.
\]
(b) For \( f \in D(L_0) \) and \( g \in D(L_{H^*,0}) \) we have
\[
\langle L_0 f, g \rangle_W = \langle f, L_{H^*,0} g \rangle_W.
\]

**Proof** The proof is similar to the proof of Theorem 7 and so it is omitted. \( \square \)

**Theorem 19** If the index-1 hypotheses are true for \( H \), \( W \) with respect to \( J \) on a compact interval \( I \) then

\[
\text{ran } L_0 = (\ker L_{H^*})^\perp, \quad (\text{ran } L_0)^\perp = \ker L_{H^*}, \tag{35}
\]
\[
\ker L_0 = \{0\}, \quad \text{ran } L_{H^*} = (\ker L_0)^\perp = L^2(I; W), \tag{36}
\]
\[
\ker L_{H^*} + \text{ran } L_0 = \text{ran } L_{H^*} + \ker L_0 = L^2(I; W), \tag{37}
\]

and, in particular, \( \text{ran } L_0 \) and \( \ker L_{H^*} \) are closed subspaces of \( L^2(I; W) \).

**Proof** First, by Corollary 13 the index-1 hypotheses are also true for \( H^*, W \) with respect to \( J \) on the compact interval \( I = [a, b] \). Second, (37) follows immediately from (35) and (36). Third, from Corollary 15 (applied to \( L_{H^*} \)) it follows that \( \text{ran } L_{H^*} = L^2(I; W) \) and, by Corollary 15 (applied to \( L \) and Proposition 14.(ii), we have \( \ker L_0 = \{0\} \), which proves the equalities in (36). Next, the second identity of (35) follows from the first identity since \( \ker L_{H^*} \) is finite-dimensional by Corollary 15 (applied to \( L_{H^*} \)) and hence \( (\ker L_{H^*})^\perp = \ker L_{H^*} = \ker L_{H^*} \). Thus, it remains to prove that \( \text{ran } L_0 = (\ker L_{H^*})^\perp \).

Let \( f \in L^2(I; W) \). Then, by Proposition 14.(ii) and the hypotheses, there is a unique solution \( u \in D(L) \) to the IVP \( Lu = f, (Ju)(a) = 0 \), and, in addition, \( u \in L^2_{loc}(I; W) = L^2(I; W) \) so that \( u \in D(L) \) with \( Lu = L_u = f \). By applying Corollary 15 to \( L_{H^*} \) we know that \( \ker L_{H^*} = \ker L_{H^*} \) and \( \infty > \dim \ker L_{H^*} = \text{rank } J := r > 0 \). Furthermore, using a similar proof as in Corollary 15, we get a basis \( \{z_i : 1 \leq i \leq r\} \) of \( \ker L_{H^*} \) satisfying \( (Jz_i)(b) = JVe_i, i = 1, \ldots, r \), where \( e_1, \ldots, e_n \) denote the standard basis vectors for \( \mathbb{C}^n \). Then it follows that

\[
\langle f, z_i \rangle_W = \langle Lu, z_i \rangle_W = \langle Lu, z_i \rangle_W - \langle u, L_{H^*} z_i \rangle_W
\]
\[
= \langle (Ju)(b), J^+(Jz_i)(b) \rangle - \langle (Ju)(a), J^+(Jz_i)(a) \rangle = \langle (Ju)(b), J^+(Jz_i)(b) \rangle
\]
\[
= \langle (Ju)(b), J^+ JV e_i \rangle = \langle JJ^+(Ju)(b), Ve_i \rangle = \langle JJ^+(Ju)(b), Ve_i \rangle
\]
\[
= \langle Ju(b), Ve_i \rangle = \langle V^*(Ju)(b), e_i \rangle = \langle V^{-1}(Ju)(b), e_i \rangle
\]
\[
= \langle V^{-1} JVV^{-1} u(b), e_i \rangle = \langle V^{-1} JVV^{-1} u(b), e_i \rangle = \langle J_{11} u_1(b), e_i \rangle.
\]

for each \( i = 1, \ldots, r \), where \( V = I_n, J_{11} = J \) if \( \det J \neq 0 \), otherwise,

\[
V^{-1} u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad V^{-1} JVV^{-1} u = \begin{bmatrix} J_{11} u_1 \\ 0 \end{bmatrix}.
\]
Suppose \( f \in \text{ran } L_0 \). Then we must have \( u \in D(L_0) \), \( L_0 u = Lu = f \), and \((Ju)(a) = (Ju)(b) = 0\) so that by the above calculation it follows that \( \langle f, z_i \rangle = 0 \) for all \( i = 1, \ldots, r \). As \( z_1, \ldots, z_r \) is a basis of solutions to \( \ker L_{H^*} \) then this implies that \( \langle f, z \rangle_W = 0 \) for all \( z \in \ker L_{H^*} \) and hence \( f \in (\ker L_{H^*})^\perp \). This proves that \( \text{ran } L_0 \subseteq (\ker L_{H^*})^\perp \). Conversely, suppose \( f \in (\ker L_{H^*})^\perp \). Then \( \langle f, z_i \rangle_W = 0 \) for every \( z \in \ker L_{H^*} \). By the above calculation for this \( f \) and the corresponding \( u \), we have \( 0 = \langle f, z_i \rangle = (J_{11}u_1)(b) \) for each \( i = 1, \ldots, r \) implying \((J_{11}u_1)(b) = 0\). As this implies \((Ju)(b) = 0\) then it follows that \( u \in D(L_0) \) [as \( u \in D(L) \) with \((Ju)(a) = (Ju)(b) = 0\) and so \( L_0 u = Lu = f \) which proves \( f \in \text{ran } L_0 \). This proves \((\ker L_{H^*})^\perp \subseteq \text{ran } L_0 \). Therefore, \( \text{ran } L_0 = (\ker L_{H^*})^\perp \), which completes the proof.

\[ \square \]

**Theorem 20** Suppose the index-1 hypotheses are true for \( H, W \) with respect to \( J \) on a compact interval \( I \). Then the following statements hold:

- **(a)** The subspaces \( D(L_0), D(L_{H^*},0), D(L), \) and \( D(L_{H^*}) \) are all dense in \( L^2(I; W) \).
- **(b)** The operators \( L_0 : D(L_0) \to L^2(I; W) \) and \( L_{H^*} : D(L_{H^*}) \to L^2(I; W) \) are densely defined closed operators with closed ranges and are adjoints of each other. In particular,
  \[
  (L_0)^* = L_{H^*} = \overline{L_{H^*}} = (L_{H^*})^{**}, \quad L_{H^*}^* = L_0 = \overline{L_0} = (L_0)^{**}.
  \]
- **(c)** The operators \( L_{H^*},0 : D(L_{H^*},0) \to L^2(I; W) \) and \( L : D(L) \to L^2(I; W) \) are densely defined closed operators with closed ranges and are adjoints of each other. In particular,
  \[
  (L_{H^*},0)^* = L = \overline{L} = L^{**}, \quad L^* = L_{H^*},0 = \overline{L_{H^*},0} = (L_{H^*},0)^{**}.
  \]

**Proof** First, by Corollary 13, the index-1 hypotheses are also true for \( H^*, W \) with respect to \( J \) on the compact interval \( I \). Second, by duality (i.e., \( H^{**} = H \)), if we prove the statements for \( L_0 \) and \( L_{H^*} \) then the same results for \( L_{H^*},0 \) and \( L_H \) also follow. Third, once we have proven that \( L_0 \) and \( L_{H^*} \) are densely defined closed operators then it follows from general results on adjoints [22, Theorem VIII.1.(b)] that the closure of \( L_0 \) and \( L_{H^*} \) (i.e., \( \overline{L_0} \) and \( \overline{L_{H^*}} \)) satisfies the relationships \( L_{H^*} = \overline{L_{H^*}} = (L_{H^*})^{**} \) and \( L_0 = \overline{L_0} = (L_0)^{**} \). Fourth, once we have proven that \( L_0 \) is a densely defined it will follow that \( L \) is densely defined [since \( D(L_0) \subseteq D(L) \)] and then by duality (i.e., \( H^{**} = H \)) it follows that \( L_{H^*},0, L_{H^*} \) are also densely defined. Fifth, if we can then prove that \( (L_0)^* = L_{H^*} \), it will follow from general results on adjoints [22, Theorem VIII.1] that \( L_{H^*} \) is a closed operator (so that \( L_{H^*} = \overline{L_{H^*}} \)), \( L_0 \) is a closable operator, and \( L_{H^*}^* = \overline{L_0} \). Finally, if we can then prove \( L_0 \) is a closed operator, i.e., \( \overline{L_0} = L_0 \), we will have completed the proof of the theorem. In summary, to prove the theorem we need only show: (i) \( L_0 \) is densely defined; (ii) \((L_0)^* = L_{H^*} \); (iii) \( L_0 \) is a closed operator.

(i) Let \( h \in D(L_0)^\perp \). Then for any solution \( g \in D(L_{H^*}) \) of \( L_{H^*} g = h \) we have by Proposition 14(ii) (applied to \( H^* \) instead of \( H \)) that \( g \in L^2_{loc}(I; W) \) and since \( I \) is compact we have \( L^2_{loc}(I; W) = L^2(I; W) \). It now follows that \( g \in D(L_{H^*}) \).
and $L_{H^*} g = h$. Hence, from this and by Theorem 18.a), it follows that for every $f \in D(L_0)$, we have $\langle L_0 f, g \rangle_W = \langle f, L_{H^*} g \rangle_W = \langle f, h \rangle_W = 0$. This implies that $g \in \text{ran}(L_0)^\perp$. By Theorem 19 we know that $\ker L_{H^*} = (\text{ran} L_0)^\perp$ so that $h = L_{H^*} g = 0$. From which we conclude that $D(L_0)^\perp = \{0\}$ and thus $D(L_0) = D(L_0)^\perp = \{0\}^\perp = L^2(I; W)$. Therefore, $L_0$ is densely defined, which proves (i).

(ii) From part (i) we know that the linear operator $L_0 : D(L_0) \to L^2(I; W)$ is densely defined and so now we consider its adjoint $(L_0)^*$. By Theorem 18.a), it follows that $D((L_0)^*) \subseteq \text{ran}(L_0)^\perp$ and $(L_0)^* g = L_{H^*} g$ for all $g \in D((L_0)^*)$. Thus, to prove that $(L_0)^* = L_{H^*}$, it suffices to prove that $D((L_0)^*) \subseteq D(L_{H^*})$. Let $g \in D((L_0)^*)$ and set $h = (L_0)^* g$. Then, since we know by Theorem 19 that $\text{ran} L_{H^*} = L^2(I; W)$, there exists an $f \in D(L_{H^*})$ such that $L_{H^*} f = h$. Hence, from this and Theorem 18.a), it follows that for every $u \in D(L_0)$, we have $\langle L_0 u, g \rangle_W = \langle u, (L_0)^* g \rangle_W = \langle u, L_{H^*} f \rangle_W = \langle L_0 u, f \rangle_W$ which implies that $g - f \in (\text{ran} L_0)^\perp$. By Theorem 19 we know that $(\text{ran} L_0)^\perp = \ker L_{H^*} \subseteq D(L_{H^*})$ and hence $g - f \in D(L_{H^*})$. As $f \in D(L_{H^*})$ and $D(L_{H^*})$ is a subspace of $L^2(I; W)$, it follows that $g \in D(L_{H^*})$. Therefore, $D((L_0)^*) \subseteq D(L_{H^*})$.

(iii) From part (ii) we know that $D(L_0)$ is dense in $L^2(I; W)$ and since $D(L_0) \subseteq D(L)$ then $D(L)$ is dense in $L^2(I; W)$. Hence, by the hypotheses using $H^*$ instead of $H$, it follows that $D(L_{H^*})$ is also dense in $L^2(I; W)$. Thus, as we proved $(L_0)^* = L_{H^*}$, it follows from general results on adjoints [22, Theorem VIII.1] that $L_{H^*}$ is a closed operator (so that $L_{H^*} = \overline{L_{H^*}}$), $L_0$ is a closable operator, and $L_{H^*} ^\perp = \overline{L_0}$. We will now prove $\overline{L_0} = L_0$, i.e., $L_0$ is closed. Let $\{u_m\}_{m \in \mathbb{N}} \subseteq D(L_0)$ be a sequence converging in $L^2(I; W)$ converging to $u$ such that $\{L_0 u_m\}_{m \in \mathbb{N}}$ converges in $L^2(I; W)$ to $f$. If we can prove that $u \in D(L_0)$ and $L_0 u = f$ then this will prove that $L_0$ is a closed operator. First, since $L_0$ is closable and $L_{H^*} ^\perp = \overline{L_0}$, it follows by general results on closable operators [22, Proposition, p. 250] that $u \in D(L_{H^*} ^\perp)$ and $L_{H^*} ^\perp u = f$. Next, let $g \in \ker L_{H^*}$. Then by Theorem 18.a) it follows that $\langle f, g \rangle_W = \lim_{m \to \infty} \langle L_0 u_m, g \rangle_W = \lim_{m \to \infty} \langle u_m, L_{H^*} g \rangle_W = 0$ which implies $f \in (\ker L_{H^*}) ^\perp = \text{ran} L_0$, where the latter equality follows from Theorem 19. Thus, there exists $v \in D(L_0)$ such that $L_0 v = f = L_{H^*} u$. Hence, by Theorem 18.a) it follows that for any $h \in D(L_{H^*})$, we have $\langle v, L_{H^*} h \rangle_W = \langle L_0 v, h \rangle_W = \langle L_{H^*} u, h \rangle_W = \langle u, L_{H^*} h \rangle_W$ implying $u - v \in (\text{ran} L_{H^*}) ^\perp = \ker L_0 = \{0\}$, where the latter two equalities follow from Theorem 19, and so $u = v \in D(L_0)$ with $L_0 u = L_0 v = f$. This proves $L_0$ is a closed operator.

We now extend Theorem 20 to allow for an arbitrary interval $I$.

**Theorem 21** Suppose the local index-1 hypotheses are true for $H$, $W$ with respect to $J$ on an interval $I$. Then the following statements hold:

(i) The subspaces $D(L_0')$, $D(L_{H^*})', D(L)$, and $D(L_{H^*})$ are all dense in $L^2(I; W)$.

(ii) The operators $L_0' : D(L_0') \to L^2(I; W)$ and $L_{H^*} : D(L_{H^*}) \to L^2(I; W)$ are densely defined. Furthermore, $L_{H^*}$ is a closed operator and $L_0'$ is a closable operator such that its closure, i.e., $\overline{L_0'} = (L_0')^\ast$, and $L_{H^*}$ are adjoints of each
other. In particular,
\((L'_0)^* = (L_0')^* = L_H^* = L_{H^*} = (L_{H^*})^*, \quad (L_{H^*})^* = L''_0.\)

(iii) The operators \(L'_{H^*,0} : D(L'_{H^*,0}) \to L^2(I; W)\) and \(L : D(L) \to L^2(I; W)\) are densely defined. Furthermore, \(L\) is a closed operator and \(L'_{H^*,0}\) is a closable operator such that its closure, i.e., \((L'_{H^*,0})^* = (L'_{H^*,0})^*\), and \(L\) are adjoints of each other. In particular,
\((L'_{H^*,0})^* = (L'_{H^*,0})^* = L = L^* = L^* = L''_0.\)

**Proof** First, as the hypotheses are valid for both \(H\) and \(H^*\) (by Corollary 13), and \((H^*)^* = H\), then (by duality) we need only prove the statement for \(L'_0\) and \(L_{H^*}\).

Second, once we have proven that \(L'_0\) and \(L_{H^*}\) are densely defined operators with \(L'_0\) and \(L_{H^*}\) a closable and closed operator, respectively, then it follows from general results that the closure of \(L'_0\) and \(L_{H^*}\) satisfy the relationships \(L_{H^*} = L_{H^*} = (L_{H^*})^*\)

and \((L'_0)^* = (L_0')^* = L = L^* = L^* = L''_0.\) On the other hand, once we know that \(L'_0\) is dense for any arbitrary \(H\) satisfying the local index-1 hypotheses then (by duality) \(D(L'_{H^*,0})\) is also dense so that since \(D(L'_{H^*,0}) \subseteq D(L_{H^*})\), it will follow that \(D(L_{H^*})\) is dense which implies then by Theorem 7.a) that the adjoint \((L'_0)^*\) of \(L'_0\) is an extension of \(L_{H^*}\), i.e., \(D(L_{H^*}) \subseteq D((L'_0)^*)\) with \(L_{H^*} f = (L'_0)^* f\) for every \(f \in D(L_{H^*})\) [which we denote by \(L_{H^*} \subseteq (L'_0)^*\)], so that \((L'_0)^*\) is densely defined from which it follows from general results on adjoints that \(L'_0\) is closable, and finally since \(L_{H^*} \subseteq (L'_0)^*\) then to prove \((L'_0)^* = L_{H^*}\) (which implies from this by general results that \(L_{H^*}\) is closed), we need only prove that \((L'_0)^* \subseteq L_{H^*}\). Thus, to recap, to complete the proof we need only prove that \(D(L'_0)\) is dense for any arbitrary \(H\) satisfying the local index-1 hypotheses and that \((L'_0)^* \subseteq L_{H^*}\), i.e., \(D((L'_0)^*) \subseteq D(L_{H^*})\) and \(L_{H^*} f = (L'_0)^* f\) for every \(f \in D((L'_0)^*)\). We will do this next.

For any arbitrary \(\alpha, \beta \in \mathbb{R}\) with \(\alpha < \beta\) such that \(\Delta = [\alpha, \beta] \subseteq \text{int } I\), we consider \(L\) on the compact interval \(\Delta\). Denote by \(L_{\Delta,0}, L_{\Delta}\) the closed minimal and maximal operators generated by \(L\) in \(L^2(\Delta; W)\). With abuse of notation, we can treat any element of \(D(L_{\Delta,0})\) as an element of \(D(L'_0)\) by zero extension, i.e., setting it equal to zero on \(I \setminus \Delta\). It then follows that \(\bigcup_{\Delta \subseteq I} D(L_{\Delta,0}) = D(L'_0)\). As each \(D(L_{\Delta,0})\) is dense in \(L^2(\Delta; W)\) it then follows that \(D(L'_0)\) is dense in \(L^2(I; W)\). Hence as \(D(L'_0) \subseteq D(L)\) with \(L_0 f = L f\) for all \(f \in D(L'_0)\), it follows that both the linear operators \(L'_0 : D(L'_0) \to L^2(I; W)\) and \(L : D(L) \to L^2(I; W)\) are densely defined with \(L'_0 \subseteq L\). It now follows by duality, using the same argument but with \(H^*\) instead of \(H\), that both the linear operators \(L'_{H^*,0} : D(L'_{H^*,0}) \to L^2(I; W)\) and \(L : D(L_{H^*}) \to L^2(I; W)\) are densely defined. Hence, to complete the proof we need only prove that \((L'_0)^* \subseteq L_{H^*}\). We now write \((\cdot, \cdot)_\Delta\) for the inner product in \(L^2(\Delta; W)\); by \(h_{\Delta}\) we denote the restriction of a function \(h\) from \(L^2(I; W)\) to \(L^2(\Delta; W)\) (in the natural way). We already know that \(L_{\Delta,0}^* = L_{H^*,\Delta}\) and \(L_{\Delta,0} \subseteq L'_0\), hence \(L_{\Delta,0} \subseteq L'_0 \subseteq L\). We also know that for every \(h \in D(L)\) we have \(h_{\Delta} \in D(L_\Delta)\). Let
If the hypotheses of Theorem 21 are true then this theorem tells us \( L'_0 \) is a densely defined closable operator. As such, we know that its closure \( \overline{L'_0} = (L'_0)^{**} \) is the smallest closed operator extension of \( L'_0 \) (see [22, Sec. VIII.2]). Because of this, we extend the Definition 16 to unbounded intervals as follows.

**Definition 22** If the local index-1 hypotheses are true for \( H, W \) with respect to \( J \) on an interval \( I \), then **closed minimal operator** \( L_0 \) generated by \( \mathcal{L} \) is defined by

\[
L_0 := \overline{L'_0} = (L'_0)^{**}.
\]

The next theorem shows in what sense this definition is an extension of the definition of \( L_0 \) in Definition 16.

**Theorem 23** If the index-1 hypotheses are true for \( H, W \) with respect to \( J \) on a compact interval \( I \) then

\[
L_0 = \overline{L'_0} = (L'_0)^{**}.
\]

**Proof** First, by Corollary 13, the local index-1 hypotheses are also true for \( H^*, W \) with respect to \( J \) on the interval \( I \). Thus, by Theorem 21 we know that \( L'_0 : D(L'_0) \to L^2(I; W) \) is a densely defined closable operator and hence it follows that \( \overline{L'_0} = (L'_0)^{**} \) (see [22, Sec. VIII.2]). Now, by Theorem 20 we have \( L^*_{H^*} = L_0 \) and by Theorem 21 we know that \( (L'_0)^* = L_{H^*} \). Therefore, \( L_0 = L^*_{H^*} = (L'_0)^{**} = \overline{L'_0} \).

One of the main problems in the spectral theory of DAEs is to answer the following question: If \( H^* = H \), what additional hypotheses imply the maximal operator \( L : D(L) \to L^2(I; W) \) generated by \( \mathcal{L} \) is a self-adjoint operator? The goal of the next section is to consider this question under the additional assumption that \( H \) and \( W \) are periodic.

### 4 On the spectral problem for periodic DAEs

In this section we will study the linear differential-algebraic equations with periodic coefficients [say, \( d \)-periodic for some fixed period \( d \in (0, \infty) \)], i.e., \( d \)-periodic linear DAEs (6), in terms of the spectral problem (1) and the corresponding spectral theory of their associated DA operators \( \mathcal{L}, L^*_0, L_0, L \) (Definition 1, Definition 4, and Definition 22) on the Hilbert space \( L^2(\mathbb{R}; W) \) [as defined by (12) with inner product (13)]. Thus,
we continue to assume that (7)-(9) are true for $J, H, W$ with the interval $I = \mathbb{R}$, but now we have the addition periodicity hypothesis:

\[ H(t + d) = H(t), \; W(t + d) = W(t), \; \text{for a.e. } t \in \mathbb{R}. \]  

(38)

The next operators and their basic properties will be needed.

**Definition 24** For any $m, n \in \mathbb{N}$, the left $(\pm)$ and right $(-)$ translation (or shift) operators $U_{\pm} : M_{m,n}(\mathcal{M}(\mathbb{R})) \to M_{m,n}(\mathcal{M}(\mathbb{R}))$, are the linear operators defined by

\[ f \in M_{m,n}(\mathcal{M}(\mathbb{R})), \; (U_{\pm}f)(t) = f(t \pm d), \; \forall t \in \mathbb{R}. \]

**Lemma 25** The operators $U_{\pm} : M_{m,n}(\mathcal{M}(\mathbb{R})) \to M_{m,n}(\mathcal{M}(\mathbb{R}))$ are invertible,

\[ U_{\pm}^{-1} = U_{-\pm}, \; U_{\pm}(L^2(\mathbb{R}; W)) = L^2(\mathbb{R}; W), \]

\[ U_{\pm}(D(A)) = D(A), \; AU_{\pm} = U_{\pm}A, \; U_{\pm}(\ker A) = \ker A, \; \forall A \in \{L, L'_0, \mathcal{L}\}. \]

and the restrictions $U_{\pm} : L^2(\mathbb{R}; W) \to L^2(\mathbb{R}; W)$ are unitary operators with no eigenvalues.

**Proof** We omit the proof of this statement as it is trivial following immediately from our definitions, the assumptions (7)-(9), and the periodicity hypothesis (38). \qed

The main goal of this section is to prove the maximal operator $L$ is self-adjoint under the additional hypotheses:

\[ H(t)^* = H(t), \; \text{for a.e. } t \in (0, d). \]  

(39)

\[ D(L'_0) \text{ is dense in } L^2(\mathbb{R}; W). \]  

(40)

\[ (L'_0)^* = L. \]  

(41)

**Remark 26** To prove $L$ is self-adjoint we need only consider the spectral problem (1) and show that $\ker (L - z I) = \ker (L - \overline{z} I) = \{0\}$ for some $z \in \mathbb{C}\setminus\mathbb{R}$. Indeed, by Lemma 5 we know that $D(L'_0), D(L)$ are subspaces of $L^2(\mathbb{R}; W)$ with $D(L'_0) \subseteq D(L)$ and $L'_0 : D(L'_0) \to L^2(\mathbb{R}; W)$ and $L : D(L) \to L^2(\mathbb{R}; W)$ defined by $Lg = \mathcal{L}g$ for $g \in D(L)$ and $L'_0 f = \mathcal{L} f = Lf$ for $f \in D(L'_0)$ are well-defined linear operators. Second, if (40) is true, i.e., $L'_0$ is densely defined, then $L$ is also densely defined since $L'_0 \subseteq L$, and also by Theorem 7 we know $L'_0$ is symmetric, hence closable, with $L \subset (L'_0)^* = (\overline{L'_0})^*$ which implies from this and the fact $(L'_0)^*$ is closed that $L$ must also be closable with $L \subset \overline{T} \subset (L'_0)^* = (\overline{L'_0})^*$. Third, it follows from this that if (40) is true then (41) is equivalent to $D((L'_0)^*) \subseteq D(L)$, in which case $(\overline{L'_0})^* = (L'_0)^* = L$. Fourth, if (40) is true then [22, Corollary, p. 257] and [21, Theorem X.1] imply the following two conditions are equivalent: a) $L'_0$ is essentially self-adjoint [i.e., its closure $\overline{L'_0} = (L'_0)^{**}$ is self-adjoint]; b) $\ker ((L'_0)^* - z I) = \ker ((L'_0)^* - \overline{z} I) = \{0\}$ for some $z \in \mathbb{C}\setminus\mathbb{R}$. Hence, if (40) and (41) are true then the following two conditions
are equivalent: a) \( L \) is self-adjoint; b) \( \ker(L - zI) = \ker(L - \overline{z}I) = \{0\} \) for some \( z \in \mathbb{C} \setminus \mathbb{R} \).

The next theorem is our main result on the eigenvalues of finite multiplicities for the operators \( \mathcal{L}, L \) and \( L_0' \).

**Theorem 27** Let \( z \in \mathbb{C} \). Then following statements are true:

(a) \( \ker(L_0' - zI) \subseteq \ker(L - zI) \subseteq \ker(\mathcal{L} - zI) \).

(b) If \( \dim \ker(\mathcal{L} - zI) < \infty \) then either \( \ker(\mathcal{L} - zI) = \{0\} \) or there exists \( f \in \ker(\mathcal{L} - zI), \lambda \in \mathbb{C} \) such that

\[
\mathcal{L}f = zf, \quad U_+ f = \lambda f, \quad f \neq 0, \quad \lambda \neq 0, \tag{42}
\]

for some \( \lambda \in \mathbb{C} \).

(c) If \( \dim \ker(L - zI) < \infty \) then \( \ker(L - zI) = \{0\} \).

(d) If \( \dim \ker(L_0' - zI) < \infty \) then \( \ker(L_0' - zI) = \{0\} \).

**Proof** As the proof for \( z \neq 0 \) is the same for the proof with \( z = 0 \) (as we can just replace \( H \) with \( H - zW \) and use the fact that \( \mathcal{L}_H - zI = \mathcal{L}_{H - zW}, L_H - zI = L_{H - zW}, L_{H,0}' - zI = L_{H - zW,0}' \)), we may assume \( z = 0 \) without loss of generality.

(a) This follows immediately since \( D(L_0') \subseteq D(L) \subseteq D(\mathcal{L}), Lf = \mathcal{L}f \) for every \( f \in D(L) \), and \( L_0' f = \mathcal{L} f \) for every \( f \in D(L_0') \).

(b) By Lemma 25 we know that the restrictions \( U_\pm : \ker \mathcal{L} \to \ker \mathcal{L} \) are linear operators which are inverses of each other. Hence, if \( \dim \ker \mathcal{L} < \infty \) and \( \ker \mathcal{L} \neq \{0\} \) then this implies by elementary linear algebra that there exists an eigenvector \( f \in \ker \mathcal{L} \) and corresponding eigenvalue \( \lambda \in \mathbb{C} \) of \( U_+ \) and since \( U_+ \) is invertible then \( \lambda \neq 0 \). This proves statement (b).

(c) By Lemma 25 we know that the restriction \( U_+ : \ker L \to \ker L \) is a linear operator which commutes with \( L \), and is the restriction of the operator \( U_+ : L^2(\mathbb{R}; W) \to L^2(\mathbb{R}; W) \) to \( \ker L \). Hence, if \( \dim \ker L < \infty \) and \( \ker L \neq \{0\} \) then this implies by elementary linear algebra of the existence of an eigenvalue of \( U_+ \) of \( L \) and hence of \( U_+ : L^2(\mathbb{R}; W) \to L^2(\mathbb{R}; W) \), a contradiction that \( U_+ \) has no eigenvalues. This proves statement (c).

(d) The proof of (d) is similar to the proof of (c) and so is omitted.

\[ \square \]

**Remark 28** Any function \( f \) and scalar \( \lambda = e^{ikd} \) satisfying the eigenvalue problem (42) with \( z \in \mathbb{C} \) is called a Bloch solution and Floquet multiplier, respectively, of the periodic DAEs \( \mathcal{L} \) in which case the scalar \( k \) is called a wavenumber. The multivalued function \( z = z(k) \) of \( k \) is then referred to as the (complex) dispersion relation.

The next two theorems are our main results on the self-adjoint spectral theory of periodic DAEs associated with the maximal operator \( L \).

**Theorem 29** Suppose (39), (40), and (41) are true. If there exists \( z_0 \in \mathbb{C} \setminus \mathbb{R} \) such that

\[ \dim \ker(L - z_0 I) < \infty \text{ and } \dim \ker(L - \overline{z_0} I) < \infty \]
then \( \ker(L - z_0 I) = \ker(L - \overline{z}_0 I) = \{0\} \) and \( L : D(L) \to L^2(\mathbb{R}; W) \) is a self-adjoint operator on \( L^2(\mathbb{R}; W) \) with no eigenvalues of finite multiplicity.

**Proof**  The proof follows immediately from Remark 26 and Theorem 27. \( \square \)

**Theorem 30**  Suppose (39) and that the local index-1 hypotheses (see Hyp. 11) are true for \( H - z_0 W \), \( W \) with respect to \( J \) on the interval \( \mathbb{R} \) for some \( z_0 \in \mathbb{C} \setminus \mathbb{R} \). Then (40) and (41) are true, \( L : D(L) \to L^2(\mathbb{R}; W) \) is a self-adjoint operator on \( L^2(\mathbb{R}; W) \), \( L'_0 : D(L'_0) \to L^2(\mathbb{R}; W) \) is essentially self-adjoint with closure \( L \), and \( L \) has no eigenvalues of finite multiplicity.

**Proof**  First, we have that \( (H - z_0 W)^* = H - \overline{z}_0 W \). Thus, the local index-1 hypotheses are true for both \( H - z_0 W \), \( W \) and \( (H - z_0 W)^* \), \( W \) with respect to \( J \) on the interval \( \mathbb{R} \) (by Corollary 13). Second,

\[
\mathcal{L} - z_0 I = L_{H - z_0 W}, \quad L - z_0 I = L_{H - z_0 W}, \quad L'_0 - z_0 I = L'_{H - z_0 W,0}, \\
\mathcal{L} - \overline{z}_0 I = L_{H - \overline{z}_0 W}, \quad L - \overline{z}_0 I = L_{H - \overline{z}_0 W}, \quad L'_0 - \overline{z}_0 I = L'_{H - \overline{z}_0 W,0}. 
\]

Thus, by Theorem 21 it follows that the subspaces \( D(L'_{H - z_0 W,0}), D(L'_{(H - z_0 W)^*,0}), D(L_{H - z_0 W}) \), and \( D(L_{(H - z_0 W)^*}) \) are all dense in \( L^2(\mathbb{R}; W) \), and

\[
(L'_{H - z_0 W,0})^* = (L'_{H - z_0 W,0})^* = L_{(H - z_0 W)^*} = L_{H - \overline{z}_0 W}, \\
L'_{(H - z_0 W)^*} = L_{(H - z_0 W)^*} = (L_{H - z_0 W})^* = L'_{H - z_0 W,0}, \\
(L'_{H - \overline{z}_0 W,0})^* = (L'_{(H - \overline{z}_0 W)^*,0})^* = (L'_{(H - z_0 W)^*,0})^* = L_{H - z_0 W}, \\
L_{H - z_0 W} = L_{H - z_0 W} = (L_{H - z_0 W})^*, (L_{H - z_0 W})^* = L'_{(H - z_0 W)^*,0}. 
\]

As we have

\[
D(L'_{H - z_0 W,0}) = D(L_0 - z_0 I), \quad D(L'_{(H - z_0 W)^*,0}) = D(L_0 - \overline{z}_0 I), \\
D(L_{H - z_0 W}) = D(L - z_0 I), \quad D(L'_{(H - z_0 W)^*,0}) = D(L - \overline{z}_0 I), \\
D(L'_0 - z_0 I) = D(L'_0), \quad D(L - z_0 I) = D(L),
\]

it follows from Theorem 21 that (40) and (41) are true. Thus, by Theorems 27 and 29, we need only prove the claim: \( \dim \ker(\mathcal{L} - z_0 I) < \infty \) and \( \dim \ker(\mathcal{L} - \overline{z}_0 I) < \infty \). If this was not true then either \( \dim \ker(\mathcal{L} - z_0 I) = \infty \) or \( \dim \ker(\mathcal{L} - \overline{z}_0 I) = \infty \) and this would imply that \( \dim \ker(\mathcal{L}_{[0,d]} - z_0 I) = \infty \) or \( \dim \ker(\mathcal{L}_{[0,d]} - \overline{z}_0 I) = \infty \) (where \( \mathcal{L}_{[0,d]} \) denotes the DA operator associated with \([0, d], J, H_{[0,d]}, W_{[0,d]}\) ), a contradiction of Corollary 15. This proves the claim. \( \square \)

Below is an example of a self-adjoint maximal operator \( L \) with an eigenvalue of infinite multiplicity.
Example 31 Consider the $d$-periodic DAEs (for any $d > 0$) in canonical form:

$$
J \frac{d}{dt} f + H f = \lambda W f,
$$

where

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},
$$

that is,

$$i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \lambda \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.
$$

Then the DA operator $\mathcal{L} : D(\mathcal{L}) \to [\mathcal{M}(\mathbb{R})]^2$ associated with $\mathbb{R}, J, H, W$ is

$$D(\mathcal{L}) = \left\{ f \in [\mathcal{M}(\mathbb{R})]^2 : Jf \in [W_{loc}^{1,1}(\mathbb{R})]^2 \right\} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : f_1 \in W_{loc}^{1,1}(\mathbb{R}), f_2 \in \mathcal{M}(\mathbb{R}) \right\},$$

$$\mathcal{L} f = W^{-1} \left( \frac{d}{dt} Jf + Hf \right) = \left[ \begin{bmatrix} i \frac{df_1}{dt} \\ 0 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right] \in D(\mathcal{L}).$$

Next, the maximal and minimal operators, $L : D(L) \to L^2(\mathbb{R}; W)$ and $L_0 : D(L_0) \to L^2(\mathbb{R}; W)$ generated by $\mathcal{L}$ on the Hilbert space $L^2(\mathbb{R}; W) = [L^2(\mathbb{R})]^2$ are

$$A \in \{ L_0, L \}, \quadAf = \mathcal{L} f = \left[ \begin{bmatrix} i \frac{df_1}{dt} \\ 0 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right] \in D(A),$$

$$D(L) = \{ f \in L^2(\mathbb{R}; W) : f \in D(\mathcal{L}), \mathcal{L} f \in L^2(\mathbb{R}; W) \} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : f_1 \in W_{loc}^{1,1}(\mathbb{R}), f_1, \frac{df_1}{dt}, f_2 \in L^2(\mathbb{R}) \right\},$$

$$D(L_0) = \{ f \in D(L) : Jf \text{ has compact support contained in the interior of } \mathbb{R} \} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : f_2 \in L^2(\mathbb{R}), f_1 \in W_{loc}^{1,1}(\mathbb{R}), f_1 \text{ has compact support} \right\}.$$

Let us now prove that $L$ has only one eigenvalue, namely, $\lambda = 0$, and it is an eigenvalue of infinite multiplicity. If $\lambda \in \mathbb{C}, \lambda \neq 0$ then

$$\mathcal{L} f = \lambda f \iff f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, f_2 = 0, f_1(t) = ce^{-i\lambda t}, c \in \mathbb{C}$$

implying $Lf = \lambda f \iff f = 0$, which proves $\lambda$ is not an eigenvalue of $L$. Next,

$$\ker \mathcal{L} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : f_1, f_2 \in \mathcal{M}(\mathbb{R}), f_1 = c \in \mathbb{C} \right\}, \quad \ker L = \left\{ \begin{bmatrix} 0 \\ f_2 \end{bmatrix} : f_2 \in L^2(\mathbb{R}) \right\},$$

which proves that $\lambda = 0$ is an eigenvalue of $L$ with infinite multiplicity.
Next, we prove $L^* = L, \overline{L}_0 = L$, and $L$ has no eigenvalues of finite multiplicity by showing the hypotheses of Theorem 30 are satisfied. Let $z_0 \in \mathbb{C}$. Now we consider the local index-1 hypotheses (Hyp. 11) for $H - z_0 W = -z_0 I_2$, $W$ with respect to $J$ on the interval $\mathbb{R}$. First, the unitary matrix $V = I_2 \in M_2(\mathbb{C})$, has the desired block form $J = V^{-1} J V = [J_{ij}]_{i,j=1,2}$ in (14) with $J_{11} = i I_1 = [i], n_1 = 1, n_2 = 1$. Furthermore, $H = V^{-1} H V = [H_{ij}]_{i,j=1,2}, W = V^{-1} W V = [W_{ij}]_{i,j=1,2} \in M_2(\mathbb{M}(\mathbb{R}))$ are already conformal to the block structure of $V^{-1} J V$, where

$$H_{ij} = 0, W_{ij} \in M_{n_i \times n_j}(\mathbb{M}(\mathbb{R})), \ i, j = 1, 2; \ W_{11} = W_{22} = I_1, \ W_{12} = W_{21} = 0.$$  

It follows from this that the local index-1 hypotheses for $H - z_0 W, W$ with respect to $J$ on the interval $\mathbb{R}$ are not satisfied if $z_0 = 0$ since $H_{22} = 0$ is not invertible, but are satisfied if $z_0 \neq 0$ since

$$\frac{-1}{z_0} I_1 = (H - z_0 W)^{-1}_{22} \in M_{n_2}(\mathbb{M}(\mathbb{R})), \quad 0 = (H - z_0 W)_{12}(H - z_0 W)_{22}^{-1} \in M_{n_1 \times n_2}(L^2_{loc}(\mathbb{R})), \quad -z_0 I_1 = (H - z_0 W)/(H - z_0 W)_{22}, \ W_{11} = I_1 \in M_{n_1}(L^1_{loc}(\mathbb{R})), \quad \frac{-1}{z_0} I_1 = W_{22}^{1/2} (H - z_0 W)^{-1}_{22} W_{22}^{1/2} \in M_{n_2}(L^\infty_{loc}(\mathbb{R})), \quad 0 = W_{22}^{1/2} (W_{22}^{-1} W_{21} - H_{22}^{-1} H_{21}) \in M_{n_2 \times n_1}(L^2_{loc}(\mathbb{R})), \quad I_1 = W/W_{22} \in M_{n_1}(L^1_{loc}(\mathbb{R})).$$

We conclude the local index-1 hypotheses (Hyp. 11) for $H - z_0 W, W$ with respect to $J$ on the interval $\mathbb{R}$ are true in this example if and only if $z_0 \in \mathbb{C} \setminus \{0\}$. In particular, we can take $z_0 = i$ from which it follows by Theorem 30 that $L^* = L, \overline{L}_0 = L$, and $L$ has no eigenvalues of finite multiplicity.

### 4.1 Simplifying the local index-1 hypotheses

It is desirable to have simpler hypotheses that are sufficient for Theorem 30 to be true. The next proposition and the theorem that follows are our main results in this regard. We conclude with an example to illustrate the usefulness of these two results. First, we begin with a lemma.

**Lemma 32** If (39) is satisfied and $z_0 \in \mathbb{C} \setminus \mathbb{R}$, then $(W^{-1/2} H W^{-1/2} - z_0 I_n)^{-1}$ and $H - z_0 W$ are invertible for a.e. $t \in \mathbb{R}$ and

$$W^{1/2}(H - z_0 W)^{-1} W^{1/2} = (W^{-1/2} H W^{-1/2} - z_0 I_n)^{-1} \in M_n(L^\infty_{loc}(\mathbb{R})), \quad (H - z_0 W)^{-1} = W^{-1/2} (W^{-1/2} H W^{-1/2} - z_0 I_n)^{-1} W^{-1/2} \in M_n(\mathbb{M}(\mathbb{R})).$$
Furthermore, if \( \det J = 0 \) then \( W_{22}^{-1/2} H_{22} W_{22}^{-1/2} - z_0 I_{n_2} \) and \( (H - z_0 W)_{22} \) are also invertible for a.e. \( t \in \mathbb{R} \) and

\[
W_{22}^{1/2} (H - z_0 W)_{22}^{-1} W_{22}^{1/2} = (W_{22}^{-1/2} H_{22} W_{22}^{-1/2} - z_0 I_{n_2})^{-1} \in M_{n_2}(L^{\infty}_{loc}(\mathbb{R})),
\]

so that

\[
(H - z_0 W)_{22}^{-1} = (H_{22} - z_0 W_{22})^{-1} - W_{22}^{-1/2} (W_{22}^{-1/2} H_{22} W_{22}^{-1/2} - z_0 I_{n_2})^{-1} W_{22}^{-1/2} \in M_{n_2}(\mathcal{M}(\mathbb{R})).
\]

**Proof** Let \( z_0 \in \mathbb{C} \setminus \mathbb{R} \). Now recall, for any \( A \in M_m(\mathbb{C}) \) such that \( A^* = A \), it follows that for any \( x \in \mathbb{C}^m \),

\[
\| (A - z_0 I_m) x \| \| x \| \geq | \langle x, (A - z_0 I_m) x \rangle | \geq | \text{Im} \langle x, (A - z_0 I_m) x \rangle | = | \text{Im}(z_0) | \| x \|^2
\]

which simultaneously proves \( A - z_0 I_m \) is invertible and \( \| (A - z_0 I_m)^{-1} \| \leq | \text{Im}(z_0) |^{-1} \).

Next, if \( C, B \in M_m(\mathbb{C}) \) such that \( C^* = C \) and \( B \) is positive semidefinite and invertible then \( A := B^{-1/2} C B^{-1/2} \) satisfies \( A^* = A \) and

\[
C - z_0 B = B^{1/2} (B^{-1/2} C B^{-1/2} - z_0 I_m) B^{1/2} = B^{1/2} (A - z_0 I_m) B^{1/2}
\]

so that \( C - z_0 B \) is invertible with

\[
(C - z_0 B)^{-1} = B^{-1/2} (A - z_0 I_m)^{-1} B^{-1/2}, \quad (A - z_0 I_m)^{-1} = B^{1/2} (C - z_0 B)^{-1} B^{1/2}.
\]

Finally, recall that if \( D \in M_m(\mathbb{C}) \) is any invertible matrix then, in terms of its determinant \( \det D \) and its adjugate \( \text{adj} D \) (i.e., the transpose of the cofactor matrix of \( D \)), the inverse formula holds: \( D^{-1} = \frac{\text{adj} D}{\det D} \). The proof of this lemma now follows immediately from these elementary facts and the hypotheses on \( H, W \) along with the equality \( (H - z_0 W)_{22} = H_{22} - z_0 W_{22} \) in \( M_{n_2}(\mathcal{M}(\mathbb{R})) \) in the case \( \det J = 0 \).

**Proposition 33** Suppose (39) is satisfied. Then the following statements are true:

(a) If \( \det J \neq 0 \) then \( H, W \in M_n(L^1_{loc}(\mathbb{R})) \) if and only if the local index-1 hypotheses are true for \( H - z_0 W, W \) with respect to \( J \) on the interval \( \mathbb{R} \), for some \( z_0 \in \mathbb{C} \setminus \mathbb{R} \) (in which case, its true for every \( z_0 \in \mathbb{C} \)).

(b) If \( \det J = 0 \) and \( z_0 \in \mathbb{C} \setminus \mathbb{R} \) then the local index-1 hypotheses are true for \( H - z_0 W, W \) with respect to \( J \) on the interval \( \mathbb{R} \) if and only if

\[
W_{11} \in M_{n_1}(L^1_{loc}(\mathbb{R})), \quad H_{11} - H_{12} (H_{22} - z_0 W_{22})^{-1} H_{21} \in M_{n_1}(L^1_{loc}(\mathbb{R})).
\]

Moreover, the statements (a) and (b) are still true if we replace \( L^1_{loc}(\mathbb{R}) \) by \( \mathcal{L}^1([0, d]) \).

**Proof** (a) First, suppose \( H, W \in M_n(L^1_{loc}(\mathbb{R})) \). Let \( z_0 \in \mathbb{C} \). Then we have \( H - z_0 W \in M_n(L^1_{loc}(\mathbb{R})) \). Hence, if \( \det J \neq 0 \) then it follows immediately that the local index-1 hypotheses are true for \( H - z_0 W, W \) with respect to \( J \) on the interval
\( \mathbb{R} \). Conversely, suppose \( \det J \neq 0 \) and the local index-1 hypotheses are true for \( H - z_0 W, W \) with respect to \( J \) on the interval \( \mathbb{R} \), for some \( z_0 \in \mathbb{C} \setminus \mathbb{R} \). Then \( \text{Im} \, z_0 \neq 0 \) and \( H - z_0 W, H - \overline{z_0} W = (H - z_0 W)^* \in M_n(L^{1}_{loc}(\mathbb{R})) \). Hence,

\[
W = \frac{1}{\text{Im} \, z_0} \frac{1}{2i} \{ (H - \overline{z_0} W) - (H - z_0 W) \} \in M_n(L^{1}_{loc}(\mathbb{R}))
\]

which implies \( H = (H - z_0 W) + z_0 W \in M_n(L^{1}_{loc}(\mathbb{R})) \). This proves statement (a).

(b) Assume \( \det J = 0 \) and let \( z_0 \in \mathbb{C} \setminus \mathbb{R} \). Then the local index-1 hypotheses for \( H - z_0 W, W \) with respect to \( J \) on the interval \( \mathbb{R} \) are equivalent, by Lemma 12 and Lemma 32, to the following:

\[
\begin{align*}
(H - z_0 W)/(H - z_0 W)_{22}, & 
W_{11} \in M_{n_1}(L^{1}_{loc}(\mathbb{R})), \\
(H - z_0 W)_{12} (H - z_0 W)^{-1}_{22} W_{22}^{1/2} & 
\in M_{n_1 \times n_2}(L^{2}_{loc}(\mathbb{R})),
\end{align*}
\]

\[
W_{22}^{1/2} (H - z_0 W)_{22}^{-1} H_{21} \in M_{n_2 \times n_1}(L^{2}_{loc}(\mathbb{R})).
\]

Considering this even further, using Holder’s inequality and the facts that \( (H - z_0 W)_{12} = H_{12} - z_0 W_{12} \), \( (H - z_0 W)_{21} = H_{21} - z_0 W_{21} \), and \( W_{22}^{1/2} (H - z_0 W)_{22}^{-1} W_{22}^{1/2} \in M_{n_2}(L^{\infty}_{loc}(\mathbb{R})) \) (by Lemma 32), and \( W_{22}^{-1/2} W_{21} \in M_{n_2 \times n_1}(L^{2}_{loc}(I)) \) (see the proof of Lemma 12), we see that those conditions are equivalent to the following:

\[
\begin{align*}
(H - z_0 W)/(H - z_0 W)_{22}, & 
W_{11} \in M_{n_1}(L^{1}_{loc}(\mathbb{R})), \\
H_{12} (H - z_0 W)_{22}^{-1} W_{22}^{1/2} & 
\in M_{n_1 \times n_2}(L^{2}_{loc}(\mathbb{R})),
\end{align*}
\]

\[
W_{22}^{1/2} (H - z_0 W)_{22}^{-1} H_{21} \in M_{n_2 \times n_1}(L^{2}_{loc}(\mathbb{R})).
\]

Therefore, by expanding

\[
(H - z_0 W)/(H - z_0 W)_{22} = (H - z_0 W)_{11} - (H - z_0 W)_{12} (H - z_0 W)_{22}^{-1} (H - z_0 W)_{21}
\]

\[
= H_{11} - z_0 W_{11} - H_{12} (H - z_0 W)_{22}^{-1} H_{21} + z_0 H_{12} (H - z_0 W)_{22}^{-1} W_{22}^{1/2} W_{22}^{-1/2} W_{21}
\]

\[
+ z_0 W_{12} W_{22}^{-1/2} W_{22}^{1/2} (H - z_0 W)_{22}^{-1} H_{21} - z_0^2 W_{12} W_{22}^{-1/2} W_{22}^{1/2} (H - z_0 W)_{22}^{-1} W_{22}^{1/2} W_{22}^{-1/2} W_{21},
\]

we see that those conditions are equivalent to the following:

\[
W_{11} \in M_{n_1}(L^{1}_{loc}(\mathbb{R})),
\]

\[
H_{12} (H_{22} - z_0 W_{22})^{-1} W_{22}^{1/2} \in M_{n_1 \times n_2}(L^{2}_{loc}(\mathbb{R})),
\]

\[
W_{22}^{1/2} (H_{22} - z_0 W_{22})^{-1} H_{21} \in M_{n_2 \times n_1}(L^{2}_{loc}(\mathbb{R})),
\]

\[
H_{11} - H_{12} (H_{22} - z_0 W_{22})^{-1} H_{21} \in M_{n_1}(L^{1}_{loc}(\mathbb{R})).
\]
Now considering the imaginary part of the last matrix function [recall, for a matrix \( A \in M_m(\mathbb{C}) \), the imaginary part of \( A \) is defined by \( \text{Im} \ A = \frac{1}{2i}(A - A^*) \) and if \( A \) is invertible then \( \text{Im}(A^{-1}) = \text{Im}(A^{-1}A^*(A^{-1})^*) = -A^{-1}\text{Im}(A)(A^{-1})^* \) and \( \text{Im}(A^{-1}) = \text{Im}((A^{-1})^*A^*A^{-1}) = -(A^{-1})^*(\text{Im} \ A)A^{-1} \) we find that

\[
\text{Im}[H_{11} - H_{12}(H_{22} - z_0 W_{22})^{-1}H_{21}]
\]

\[
= -(\text{Im} z_0) H_{12}(H_{22} - z_0 W_{22})^{-1} W_{22}[(H_{22} - z_0 W_{22})^{-1}]^* H_{21}
\]

\[
= -(\text{Im} z_0) H_{12}(H_{22} - z_0 W_{22})^{-1} W_{22}^{1/2} [H_{12}(H_{22} - z_0 W_{22})^{-1} W_{22}^{1/2}]^*
\]

and

\[
\text{Im}[H_{11} - H_{12}(H_{22} - z_0 W_{22})^{-1}H_{21}]
\]

\[
= -(\text{Im} z_0) H_{12}(H_{22} - z_0 W_{22})^{-1} W_{22}[(H_{22} - z_0 W_{22})^{-1}]^* H_{21}
\]

\[
= -(\text{Im} z_0)[W_{22}^{1/2} (H_{22} - z_0 W_{22})^{-1} H_{21}]^* W_{22}^{1/2} (H_{22} - z_0 W_{22})^{-1} H_{21},
\]

and we see from this that those conditions are equivalent to the conditions (43) and (44). This proves (b).

Finally, the statements (a) and (b) are still true if we replace \( L^1_{loc}(\mathbb{R}) \) by \( L^1([0,d]) \). The reason is that since \( H, W \) are \( d \)-periodic functions, then \( H, W \in M_n(L^1_{loc}(\mathbb{R})) \) if and only if, for their restrictions, \( H, W \in M_n(L^1([0,d])) \). Similarly, due to the \( d \)-periodicity, (43) and (44) are true if and only if \( W_{11}, H_{11} - H_{12}(H_{22} - z_0 W_{22})^{-1}H_{21} \in M_n(L^1([0,d])) \).

**Theorem 34** If (39) is satisfied and \( \det J = 0 \) then

\[
H_{11}, W_{11}, H_{12} W_{22}^{-1} H_{21} \in M_n(L^1_{loc}(\mathbb{R}))
\]

(45)

then the local index-1 hypotheses are true for \( H - z_0 W, W \) with respect to \( J \) on the interval \( \mathbb{R} \), for every \( z_0 \in \mathbb{C} \setminus \mathbb{R} \). Moreover, the theorem is still true if we replace \( L^1_{loc}(\mathbb{R}) \) by \( L^1([0,d]) \) in (45).

**Proof** Let \( z_0 \in \mathbb{C} \setminus \mathbb{R} \). Then by hypotheses (43) is true so to prove the theorem, it suffices by Proposition 33.(b), to show (44) is true. First, by our hypotheses

\[
(W_{22}^{-1/2} H_{21})^* W_{22}^{-1/2} H_{21} = H_{12} W_{22}^{-1/2} (H_{12} W_{22}^{-1})^* = H_{12} W_{22}^{-1} H_{21} \in M_n(L^1_{loc}(\mathbb{R}))
\]

implying \( H_{12} W_{22}^{-1/2} \in M_{n_1 \times n_2}(L^2_{loc}(\mathbb{R})), W_{22}^{-1/2} H_{21} \in M_{n_2 \times n_1}(L^2_{loc}(\mathbb{R})) \). Second, it follows from this, Holder’s inequality, and Lemma 32 that

\[
H_{11} - H_{12}(H_{22} - z_0 W_{22})^{-1}H_{21}
\]

\[
= H_{11} - H_{12} W_{22}^{-1/2} [W_{22}^{1/2} (H_{22} - z_0 W_{22})^{-1} W_{22}^{1/2} H_{21}] W_{22}^{-1/2} H_{21} \in M_n(L^1_{loc}(\mathbb{R})).
\]

This proves the theorem except for the “Moreover,...” part. But this follows from the fact that since \( H_{11}, W_{11}, H_{12}, W_{22}, H_{21} \) are \( d \)-periodic functions, then (45) is true if and only if, for their restrictions, \( H_{11}, W_{11}, H_{12} W_{22}^{-1} H_{21} \in M_n(L^1([0,d])) \).
**Example 35** Let \( n \in \mathbb{N} \) and \( J \in M_n(\mathbb{C}) \) with \( J^* = -J \neq 0 \). Consider the \( d \)-periodic DAEs (for any \( d > 0 \)) in canonical form with \( H = 0 \), \( W = I_n \), i.e.,

\[
J \frac{d}{dt} f = \lambda f.
\]

The DA operator \( \mathcal{L} : D(\mathcal{L}) \to [\mathcal{M}(\mathbb{R})]^n \) associated with \( \mathbb{R}, J, H, W \) is

\[
D(\mathcal{L}) = \left\{ f \in [\mathcal{M}(\mathbb{R})]^n : Jf \in [\mathcal{W}_{1,1}(\mathbb{R})]^n \right\}, \quad \mathcal{L}f = Jf = \frac{d}{dt}f.
\]

Next, the maximal and minimal operators, \( L : D(L) \to L^2(\mathbb{R}; W) \) and \( L_0 : D(L_0) \to L^2(\mathbb{R}; W) \), generated by \( \mathcal{L} \) on the Hilbert space \( L^2(\mathbb{R}; W) = [L^2(\mathbb{R})]^n \) are

\[
L \mathcal{L}f = \frac{d}{dt}Jf, \quad \text{for } f \in D(L); \quad L_0 \mathcal{L}f = \frac{d}{dt}Jf, \quad \text{for } f \in D(L_0),
\]

\[
D(L) = \left\{ f \in L^2(\mathbb{R}; W) : f \in D(\mathcal{L}), \mathcal{L}f \in L^2(\mathbb{R}; W) \right\} = \left\{ f \in [L^2(\mathbb{R})]^n : Jf \in [\mathcal{W}_{1,1}(\mathbb{R})]^n, \frac{d}{dt}Jf \in [L^2(\mathbb{R})]^n \right\},
\]

\[
D(L_0) = \left\{ f \in D(L) : Jf \text{ has compact support contained in the interior of } \mathbb{R} \right\} = \left\{ f \in D(L) : (Jf)(t) = 0 \text{ for all } t \text{ sufficiently large} \right\}.
\]

Let us now compare and contrast the spectral theory for \( L \) in the two different possible cases: (i) \( \det J \neq 0 \); (ii) \( \det J = 0 \). First, in either case (i) or (ii), Proposition 33 and Theorem 34 imply that the local index-1 hypotheses are true for \( H - z_0 W, W \) with respect to \( J \) on the interval \( \mathbb{R} \) for every \( z_0 \in \mathbb{C} \setminus \mathbb{R} \) and so Theorem 30 implies \( L^* = L, L_0^* = L_0 \), and \( L \) has no eigenvalues of finite multiplicity.

As we will show in the case (i) \( \det J \neq 0 \), \( L \) has no eigenvalues, whereas in the case (ii) \( \det J = 0 \), \( L \) has one eigenvalue (of infinite multiplicity). Suppose (i) \( \det J \neq 0 \). Then

\[
D(L) = \left\{ f \in [L^2(\mathbb{R})]^n : f \in [\mathcal{W}_{1,1}(\mathbb{R})]^n, \frac{d}{dt}f \in [L^2(\mathbb{R})]^n \right\}.
\]

Hence if \( f \) is an eigenvector of \( L \) with corresponding eigenvalue \( \lambda \) then \( f \in D(L) \) and \( Lf = \lambda f \) implying \( \frac{df}{dt} = \lambda f \) and hence \( f(t) = e^{\lambda t}f(0), \forall t \in \mathbb{R} \). As \( (ij)^* = iJ \), this implies that \( iJ \) has an orthonormal basis of eigenvectors \( v_1, \ldots, v_n \in \mathbb{C}^n \) with corresponding eigenvalues \( \omega_1, \ldots, \omega_n \in \mathbb{R} \) and there exists \( a_1, \ldots, a_n \in \mathbb{C} \) such that

\[
f(0) = \sum_{j=1}^n a_j v_j \text{ implying } f(t) = \sum_{j=1}^n a_j e^{-i\lambda \omega_j} v_j, \forall t \in \mathbb{R} \text{ as } f \in D(L) \text{ and is an eigenvector of } L \text{ then } 0 < \langle f, f \rangle < \infty \text{ but}
\]

\[
\langle f, f \rangle = \int_{\mathbb{R}} f(t)^* f(t) dt = \sum_{j=1}^n \sum_{k=1}^n \int_{\mathbb{R}} |a_j e^{-i\lambda \omega_j} v_j|^2 |a_k e^{-i\lambda \omega_k} v_k| dt
\]

\[
= \sum_{a_j \neq 0} |a_j|^2 \int_{\mathbb{R}} (e^{-i\lambda \omega_j}) e^{-i\lambda \omega_j} dt = \sum_{a_j \neq 0} |a_j|^2 \int_{\mathbb{R}} e^{-i2\omega_j} \text{ Im } \lambda dt
\]
and since the integral
\[
\int_{-\infty}^{\infty} e^{-i2t\omega} \Im \lambda \, dt = \lim_{r \to -\infty, R \to +\infty} \int_{r}^{R} e^{-i2t\omega} \Im \lambda \, dt
\]
doesn’t exist as a finite limit for any \( \omega \in \mathbb{R} \), this yields a contradiction that \( L \) had an eigenvector. This proves that \( L \) has no eigenvalues if \( \det J \neq 0 \). Now suppose (ii) \( \det J = 0 \). Then there is a unitary matrix \( V \in M_n(\mathbb{C}) \), such that \( V^{-1} J V \) has the block form (14) with \( J_{11} \in M_{n_1}(\mathbb{C}) \), \( \det(J_{11}) \neq 0 \), \( n_1 = \text{rank} \ J \) and setting \( n_2 = \text{nullity}(J) \), then matrices \( H = V^{-1} H V = [H_{ij}]_{i,j=1,2} = 0 \), \( W = V^{-1} W V = [W_{ij}]_{i,j=1,2} = I_n \in M_n(\mathcal{M}(\mathbb{R})) \) are already partitioned conformal to the block structure of \( V^{-1} J V \) with

\[
H_{ij} = 0, \ W_{ij} \in M_{n_i \times n_j}(\mathcal{M}(\mathbb{R})), \ i, j = 1, 2; \ W_{11} = I_{n_1}, \ W_{22} = I_{n_2}, \ W_{12} = W_{21} = 0.
\]

Hence,
\[
D(L) = \left\{ V \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : f_1 \in [W^{1,1}_{loc}(\mathbb{R})]^{n_1}, \ f_1, \frac{df_1}{dt} \in [L^2(\mathbb{R})]^{n_1}, \ f_2 \in [L^2(\mathbb{R})]^{n_2} \right\},
\]

\[
L \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = V \begin{bmatrix} J_{11} \frac{df_1}{dt} \\ 0 \end{bmatrix}, \ \text{for all} \ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in D(L).
\]

Thus, \( f \) is an eigenvector of \( L \) with corresponding eigenvalue \( \lambda \) if and only if

\[
f = V \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ J_{11} \frac{df_1}{dt} = \lambda f_1, \ f_1 \in [W^{1,1}_{loc}(\mathbb{R})]^{n_1}, \ f_1, \frac{df_1}{dt} \in [L^2(\mathbb{R})]^{n_1}, \ \lambda f_2 = 0, \ f_2 \in [L^2(\mathbb{R})]^{n_2}.
\]

As \( J_{11}^* = -J_{11} \in M_{n_1}(\mathbb{C}) \) and \( \det J_{11} \neq 0 \), it follows from the case (i) we just considered, \( \lambda = 0 \) is the only eigenvalue of \( L \) and has infinite dimensional eigenspace

\[
\ker L = \left\{ V \begin{bmatrix} 0 \\ f_2 \end{bmatrix} : f_2 \in [L^2(\mathbb{R})]^{n_2} \right\}.
\]

**Acknowledgements**  The authors would like to thank Fritz Gesztesy, Anthony Stefan, and the anonymous reviewers for their suggestions and feedback on our original manuscript that helped improve it.

**Author Contributions**  The authors contributed equally on this paper.

**Funding**  No funds, grants, or other support was received.

**Data availability**  Not applicable.

**Declarations**

**Conflict of interest**  The authors have no relevant financial or non-financial interests to disclose.

**Ethics approval**  Not applicable.
Consent to participate  Not applicable.
Consent for publication  Not applicable.
Code availability  Not applicable.

Appendix A: Notation and auxiliary theorems

For any $z \in \mathbb{C}$, its complex conjugate and norm are $\overline{z}$ and $|z| = (\overline{z}z)^{1/2}$, respectively. For any interval $I \subseteq \mathbb{R}$, the complex vector space of Lebesgue measurable functions (with equality in the sense of equal a.e. on $I$) is denoted by $\mathcal{M}(I)$. For each $p \in [1, \infty]$, the subspace $L^p(I)$ of $\mathcal{M}(I)$, defined by

$$L^p(I) = \left\{ f \in \mathcal{M}(I) : \int_I |f(t)|^p dt < \infty \right\}, \text{ if } p \neq \infty,$$

$$L^\infty(I) = \left\{ f \in \mathcal{M}(I) : \text{ess sup}_{t \in I} |f(t)| < \infty \right\}$$

is a Banach space with norm

$$\| f \|_p = \begin{cases} \left( \int_I |f(t)|^p dt \right)^{1/p}, & \text{if } p \neq \infty, \\ \text{ess sup}_{t \in I} |f(t)|, & \text{if } p = \infty \end{cases}$$

and, in the case $p = 2$, is a Hilbert space with inner product

$$\langle f, g \rangle_2 = \int_I \overline{f(t)} g(t) dt, \quad f, g \in L^2(I).$$

For any compact interval $I = [a, b]$, we denote the Banach space of all complex-valued absolutely continuous functions on the interval $I$ by $AC(I)$ with norm

$$\| f \|_{AC(I)} = |f(a)| + \int_a^b \left| \frac{df}{dt}(\tau) \right| d\tau, \quad f \in AC(I).$$

For any interval $I$, the subspace $L^p_{loc}(I)$ of $\mathcal{M}(I)$ and the complex vector space $AC_{loc}(I)$ are defined by

$$L^p_{loc}(I) = \{ f \in \mathcal{M}(I) : f \in L^p([a, b]), \text{ for every compact interval } [a, b] \subseteq I \},$$

$$AC_{loc}(I) = \{ f : I \to \mathbb{C} \mid f \in AC([a, b]), \text{ for every compact interval } [a, b] \subseteq I \},$$

respectively. The subspace $W^{1,p}(I)$ of $L^p(I)$, defined by

$$W^{1,p}(I) = \left\{ f \in L^p(I) : f \in AC_{loc}(I), \frac{df}{dt} \in L^p(I) \right\},$$
is a Banach space with norm
\[\|f\|_{1,p} = \|f\|_p + \left\| \frac{df}{dt} \right\|_p, \quad f \in W^{1,p}(I).\]

The subspace \(W^{1,p}_{loc}(I)\) of \(L^p_{loc}(I)\) is defined by
\[W^{1,p}_{loc}(I) = \{ f \in \mathcal{M}(I) : f \in W^{1,p}([a,b]), \text{ for every compact interval } [a,b] \subseteq I \}.\]

If \(f \in W^{1,p}_{loc}(I)\) then there is a unique \(g \in AC_{loc}(I)\) such that \(f(t) = g(t)\) for a.e. \(t \in I\), and as such, we will always use this representative of \(f\) when we evaluate \(f\) at a point, i.e., for each \(t_0 \in I\) will define \(f(t_0) := g(t_0)\). Similarly for \(f \in W^{1,p}(I)\).

Let \(m, n \in \mathbb{N}\), \(I\) an interval, \(p \in [1, \infty]\), and \(V \in \{\mathbb{C}, \mathcal{M}(I), L^p_{loc}(I), W^{1,p}_{loc}(I), AC_{loc}(I)\}\).

We denote the set of all \(n \times m\) matrices with entries in \(V\) by \(M_{n,m}(V)\), and define
\[\mathcal{V}^n = M_{n,1}(\mathcal{V}), \quad M_n(\mathcal{V}) = M_{n,n}(\mathcal{V})\]
and identify \(\mathcal{V}\) with \(\mathcal{V}^1\). If \(\mathcal{V}\) is a Banach space (Hilbert space) with norm \(\| \cdot \|_{\mathcal{V}}\) then \(M_{n,m}(\mathcal{V})\) will denote the Banach space (Hilbert space) with norm
\[\|[a_{ij}]\| = \left( \sum_{i=1}^{n} \sum_{j=1}^{m} \|a_{ij}\|^2_{\mathcal{V}} \right)^{1/2}, \quad [a_{ij}] \in M_{n,m}(\mathcal{V}).\]

In particular, for the Hilbert spaces \(\mathbb{C}^n\) and \((L^2(I))^n\), their inner products \(\langle \cdot, \cdot \rangle\) and \(\langle \cdot, \cdot \rangle_2\), respectively, are defined as
\[\langle x, y \rangle = x^* y, \quad x, y \in \mathbb{C}^n,\]
\[\langle f, g \rangle_2 = \int_I \langle f(t), g(t) \rangle dt, \quad f, g \in (L^2(I))^n,\]
where \(\ast\) denotes conjugate-transpose.

If \(\mathcal{V}\) is a functional space in which integration \(\int_U \langle \cdot \rangle \ dt\) over an interval \(U \subseteq I\) or differentiation \(\frac{d}{dt}\) (either in classical or weak sense) is well-defined then for any \([a_{ij}] \in M_{n,m}(\mathcal{V})\) we define, respectively,
\[\int_U [a_{ij}] (t) dt = \left[ \int_U a_{ij}(t) dt \right], \quad \frac{d[a_{ij}]}{dt} = \left[ \frac{da_{ij}}{dt} \right].\]

We need the following from [45, Theorem 1.2.1], [46, Theorem 3.2] (resp.):
Theorem 36  Let $I$ be any interval and $m, n \in \mathbb{N}$. If

$$ A \in M_n(L^1_{loc}(I)), \quad F \in M_{n,m}(L^1_{loc}(I)) \quad (A1) $$

then every initial-value problem (IVP)

$$ \frac{dX}{dt} + AX = F, \quad X(t_0) = C, \quad t_0 \in I, \quad C \in M_{n,m}(\mathbb{C}) $$

on $I$, has a unique solution

$$ X \in M_{n,m}(W^1_{loc}(I)). \quad (A2) $$

Similarly, in the case in which the interval $I$ is bounded, the statement is true if the “loc” is dropped in the hypotheses (A1) and in the conclusion (A2).

Theorem 37  If $H, K$ are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle_K$, respectively, $D(A)$ and $D(B)$ are subspaces of $H$ and $K$, respectively, and $A : D(A) \to K$, $B : D(B) \to H$ are linear operators satisfying

$$ \langle Ax, y \rangle_K = \langle x, By \rangle_H, \quad \text{for all } x \in D(A), y \in D(B), $$

$$ \ker A + \operatorname{ran} B = H, \quad \ker B + \operatorname{ran} A = K, $$

then $A$ and $B$ are densely defined closed operators with closed ranges and are adjoints of each other, i.e., $A^* = B, \quad B^* = A$.

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