One-Sided Derivative of Distance in Alexandrov Spaces

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Abstract

We give a complete, self-contained proof of a folklore theorem which says that in an Alexandrov space the distance between a point γ(t) on a geodesic γ and a compact set K is a right-differentiable function of t. Furthermore, the value of this right-derivative is given by the negative cosine of the minimal angle between the geodesic and any shortest path to the compact set (Theorem 5.3). Alexandrov spaces, of which Riemannian manifolds are examples, are metric spaces with a one-sided (upper or lower) curvature bound. Presumably, these are the most general spaces for which such a result holds. Our treatment also serves as a general introduction to metric geometry.

1 Introduction

Let (X, d) be a metric space. Given a compact set K ⊆ X and a unit-speed geodesic γ : [0, T] → X, the distance from γ to K at any given time is defined by the function ℓ(t) = d(γ(t), K). We consider the limit

\[ \lim_{t \to 0^+} \frac{\ell(t) - \ell(0)}{t} \]

(i.e. the one-sided derivative of ℓ at 0).

In an Alexandrov space (see Section 4 for definition), if we replace the compact set K with a point p, it is well known that the limit (1) exists and is equal to −cos(∠min), where ∠min is the infimum of angles between γ and any distance minimizing path connecting γ(0) to p. This result is commonly known as the First Variation Formula (after the similar result for Riemannian manifolds) and can be found in [ABN86] (Proposition 3.3), [BH91] (Corollary II.3.6), [BB01] (Corollary 4.5.7), and [Pla02] (Corollary 62).

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It is asserted, in publications such as [BBI01] (Exercise 4.5.11) and [BGP92] (Example 11.4), that for distance to a compact set, the limit exists and is again equal to \(-\cos(\angle_{\text{min}})\). However, we have been unable to find a published proof. We present here a complete, self-contained proof of this result. Our approach is largely based on techniques presented in [BBI01], with insights taken from [BH91], [Pla02], [Shi93], and others.

Our motivation is twofold. It is well-known ([Vee19] and references therein) that if \(K\) is a convex set in \(\mathbb{R}^n\), then the derivative of \(\ell(t)\) equals the negative cosine of the angle between \(\gamma\) and the distance minimizing path connecting \(\gamma(0)\) to \(K\). The result considered here is in fact the ultimate generalization of that fact. On the other hand, the result is also crucial for the study of mediatrices. Given two compact sets \(K_1\) and \(K_2\), the mediatrix is the sets of points \(x\) for which \(d(x,K_1) = d(x,K_2)\). In that context, a restricted version of our main theorem is used in [HPV17].

2 Length Spaces

Let \((X,d)\) be a metric space. A path (or curve) in \(X\) is a continuous injective function \(\gamma : [a,b] \to X\) where \([a,b]\) is an interval of \(\mathbb{R}\) (possibly degenerate). We define the length of any path \(\gamma\) as the supremum of the distance along finite partitions of the path:

\[
L(\gamma) = \sup \left\{ \sum_{k=1}^{n-1} d(\gamma(t_k),\gamma(t_{k+1})) : a = t_1 < t_2 < \cdots < t_n = b \right\}.
\]

The metric \(d\) is called intrinsic if for all \(x,y \in X\),

\[
d(x,y) = \inf \{ L(\gamma) : \gamma \text{ is a path connecting } x \text{ and } y \}.
\]

A path-connected metric space with an intrinsic metric is known as a length space.

In a length space, a shortest path is a path \(\gamma : [0,T] \to X\) such that the length of \(\gamma\) is precisely the distance between its endpoints; \(L(\gamma) = d(\gamma(0),\gamma(T))\). In a length space which is both complete and locally compact, we make use of the well known theorem of Hopf and Rinow, although we will not reference it directly.

**Theorem 2.1** (Hopf-Rinow). If \(X\) is a complete and locally compact length space, then every closed and bounded subset of \(X\) is compact; and any two points in \(X\) can be connected by a shortest path.

Note that shortest paths are not necessarily unique (consider antipodal points on a sphere). Any curve which is locally a shortest path is known as a geodesic.

Two paths, \(\gamma : [0,T] \to X\) and \(\eta : [0,S] \to X\), which have the same image but are not the same function are said to have different parameterizations. A path \(\gamma : [0,T] \to X\) is parameterized by arc-length (or unit-speed, for short) if for any \(t,t' \in [0,T]\),

\[
L(\gamma|_{[t,t']}) = |t' - t|.
\]
It follows that if $\gamma$ is a unit-speed shortest path, then for any $t \in [0, T]$, $d(\gamma(0), \gamma(t)) = t$. From now on, all parametrizations will be unit-speed unless otherwise mentioned.

A sequence of paths $\{\gamma_n\}_{n=1}^{\infty}$ is said to converge uniformly to a path $\gamma$ if each $\gamma_n$ admits a parameterization such that $\{\gamma_n\}_{n=1}^{\infty}$ converges uniformly to a parameterization of $\gamma$.

3 Angles

We denote by $M_k^2$ the 2-dimensional simply-connected space form\(^1\) of curvature $k$, equipped with intrinsic metric $d_k$ induced by the Riemannian metric. The diameter of the space $M_k^2$ is denoted $D_k$ and defined by

$$D_k = \sup\{d_k(x, y) : x, y \in M_k^2\} = \begin{cases} \pi/\sqrt{k} & \text{for } k > 0 \\ \infty & \text{for } k \leq 0. \end{cases}$$

Given any three points $x, y, z \in X$ with $d(x, y) + d(y, z) + d(x, z) < D_k$, we can fix three points $\bar{x}$, $\bar{y}$, and $\bar{z}$ in $M_k^2$ such that

$$d(x, y) = d_k(\bar{x}, \bar{y}), \quad d(x, z) = d_k(\bar{x}, \bar{z}), \quad \text{and} \quad d(y, z) = d_k(\bar{y}, \bar{z}). \quad (2)$$

The points $\bar{x}$, $\bar{y}$, and $\bar{z}$, together with the shortest paths joining them, form a geodesic triangle in $M_k^2$, which we call the comparison triangle and denote it $\overline{\Delta}(x, y, z)$. Such a comparison triangle is unique up to isometry. The interior angle in the geodesic triangle $\overline{\Delta}(x, y, z)$ (in $M_k^2$) with vertex $\bar{x}$ is denoted $\angle_k^e(y, z)$ and referred to as the $k$-comparison angle.

If $\gamma : [0, T] \to X$ and $\eta : [0, S] \to X$ are shortest paths in $X$ with $\gamma(0) = \eta(0)$, then for any sufficiently small $t \in (0, T]$ and $s \in (0, S]$, we can consider the comparison triangle $\overline{\Delta}(\gamma(0), \gamma(t), \eta(s))$. The upper angle between $\gamma$ and $\eta$ is defined as

$$\angle_k^+(\gamma, \eta) = \limsup_{t,s \to 0^+} \angle_k^{\gamma(0)}(\gamma(t), \eta(s)) = \limsup_{\varepsilon \to 0} \{\angle_k^{\gamma(0)}(\gamma(t), \eta(s)) : 0 < s, t \leq \varepsilon\}.$$

In the case that $\gamma$ connects the point $x$ to $y$, and $\eta$ connects the point $x$ to $z$, the upper angle above may also be denoted $\angle_k^+(y, z)$; although it is important to note that unless all shortest paths are unique, the angle always depends on the shortest paths connecting the points. Additionally, when it is understood that $\gamma(0)$ is the point at which we are measuring the angle, the subscript for the vertex is often omitted (i.e. $\angle_k^+(\gamma, \eta) = \angle_k^+(\gamma, \eta)$ and $\angle_k^{\gamma(0)}(\gamma(t), \eta(s)) = \angle_k^{\gamma(0)}(\gamma(t), \eta(s)))$).

Besides the upper angle, one may also consider the lower angle between two shortest paths, which is defined as $\angle_k^-(\gamma, \eta) = \liminf_{t,s \to 0^+} \angle_k^{\gamma(0)}(\gamma(t), \eta(s))$. If the upper angle and

\(^1\)A space form is a complete Riemannian manifold of constant sectional curvature.

\(^2\)Sufficiently small meaning the inequality $d(\gamma(0), \gamma(t)) + d(\gamma(0), \eta(s)) + d(\gamma(t), \eta(s)) < 2D_k$ is satisfied.
lower angle are equal, then we say the angle exists and denote it by $\angle(\gamma, \eta)$. We note that the upper and lower angles are indeed independent of the curvature of the space form chosen (as per [Pla02], all space forms are infinitesimally Euclidean; see also Appendix C.1).

The following proposition is commonly referred to as the triangle inequality for angles. The proof is similar to that found in [BH91].

**Proposition 3.1.** Let $X$ be a length space and let $\gamma$, $\eta$, and $\sigma$ be shortest paths in $X$ with $\gamma(0) = \eta(0) = \sigma(0)$. Then $\angle^{+}(\gamma, \eta) \leq \angle^{+}(\gamma, \sigma) + \angle^{+}(\sigma, \eta)$.

**Proof.** If $\angle^{+}(\gamma, \sigma) + \angle^{+}(\sigma, \eta) \geq \pi$, then the result is trivial, so we assume that $\angle^{+}(\gamma, \sigma) + \angle^{+}(\sigma, \eta) < \pi$. By way of contradiction, suppose that there is an $\varepsilon > 0$ such that

$$\angle^{+}(\gamma, \eta) > \angle^{+}(\gamma, \sigma) + \angle^{+}(\sigma, \eta) + \varepsilon. \quad (3)$$

By the definition of $\lim \sup$ there is a $\delta > 0$ such that

$$\angle^{k}(\gamma(t), \eta(r)) > \angle^{+}(\gamma, \eta) - \varepsilon/3 \quad \text{for some } t, r < \delta \quad (4)$$

$$\angle^{k}(\gamma(t), \sigma(s)) < \angle^{+}(\gamma, \sigma) + \varepsilon/3 \quad \text{for all } t, s < \delta \quad (5)$$

$$\angle^{k}(\sigma(s), \eta(r)) < \angle^{+}(\sigma, \eta) + \varepsilon/3 \quad \text{for all } s, r < \delta. \quad (6)$$

Fix $t$ and $r$ satisfying (4) and let $\bar{p}, \bar{t}, \bar{r} \in M_k^2$ be such that $t = d_k(\bar{t}, \bar{p})$, $r = d_k(\bar{r}, \bar{p})$, and

$$\angle^{k}(\gamma(t), \eta(r)) > \angle^{+}(\gamma, \eta) - \varepsilon/3$$

where $\theta_{\bar{t}, \bar{p}}$ is the angle between $\bar{p}\bar{t}$ and $\bar{p}\bar{r}$ in $M_k^2$. The left side of the above inequality tells us that $d(\gamma(t), \eta(r)) > d_k(\tilde{t}, \tilde{r})$. Combining the right side of the above inequality with (3), we have

$$\theta_{\bar{t}, \bar{p}} > \angle^{+}(\gamma, \sigma) + \angle^{+}(\sigma, \eta) + 2\varepsilon/3.$$ 

Therefore, we can fix $\bar{s} \in M_k^2$ along the path $\bar{t}\bar{r}$ such that

$$\theta_{\bar{t}, \bar{s}} > \angle^{+}(\gamma, \sigma) + \varepsilon/3 \quad \text{and} \quad \theta_{\bar{t}, \bar{s}} > \angle(\sigma, \eta) + \varepsilon/3.$$

Set $s = d_k(\bar{s}, \bar{p})$. Since $d_k(\bar{s}, \bar{p}) \leq \max\{d_k(\bar{t}, \bar{p}), d_k(\bar{r}, \bar{p})\} < \delta$, by (5) and (6) we have

$$\theta_{\bar{t}, \bar{s}} > \angle^{k}(\gamma(t), \sigma(s)) \quad \text{and} \quad \theta_{\bar{s}, \bar{p}} > \angle^{k}(\sigma(s), \eta(r)).$$

It follows that $d_k(\bar{t}, \bar{s}) > d(\gamma(t), \sigma(s))$ and $d_k(\bar{s}, \bar{r}) > d(\sigma(s), \eta(r))$. Thus, we have

$$d(\gamma(t), \eta(r)) > d_k(\bar{t}, \bar{s}) = d_k(\bar{t}, \tilde{s}) + d_k(\tilde{s}, \bar{r}) > d(\gamma(t), \sigma(s)) + d(\sigma(s), \eta(r))$$

which contradicts the triangle inequality in $X$. \hfill $\Box$

The next lemma is arguably the crux of this work. As observed above, it is clear that small triangles in space forms are essentially Euclidean. However, what we need here are the properties of **long, thin triangles**, that is: triangles with only one small side (and two long sides). The surprising — and perhaps counter-intuitive — fact is that these also behave as Euclidean triangles!
Lemma 3.2. Let $X$ be a length space and let $\gamma : [0,T] \to X$ and $\eta : [0,S] \to X$ be shortest paths such that $\gamma(0) = \eta(0)$. There is a $C > 0$ such that if $s, t < D_{|k|}$ and $s$ is held fixed as $t \to 0^+$ then

$$\left| \cos \left( \angle^k(\gamma(t), \eta(s)) \right) - \frac{s - d(\gamma(t), \eta(s))}{t} \right| \leq Ct/s. $$

Proof. We first look at $k = 0$ and summarize the proof found in [BBI01] as Lemma 4.5.5. For simplicity of notation, let $\theta = \angle^k(\gamma(t), \eta(s))$ and $d = d(\gamma(t), \eta(s))$. Recall that $t = d(\gamma(0), \gamma(t))$ and $s = d(\gamma(0), \eta(s))$. Employing the Euclidean law of cosines, we find

$$d^2 = s^2 + t^2 - 2st \cos \theta.$$

A trivial computation confirms that

$$\left| \cos \theta - \frac{s - d}{t} \right| = \left| \frac{s - d}{t} \cdot \frac{d - s}{2s} + \frac{t}{2s} \right|.$$

By the triangle inequality, $|s - d| = |d - s| \leq t$, which gives the desired result.

We next consider the case $k > 0$. If we radially project the triangle with sides of lengths $s$, $t$, and $d$ to the unit sphere, we can use the spherical law of cosines on the unit-sphere to derive

$$\cos(\theta) = \frac{\cos(d\sqrt{k}) - \cos(t\sqrt{k}) \cos(s\sqrt{k})}{\sin(t\sqrt{k}) \sin(s\sqrt{k})}.$$

Projecting the triangle back to the sphere of radius $1/\sqrt{k}$ does not affect the angle $\theta$. Thus

$$\cos \theta = \frac{\cos(d) - \cos(s)}{\sin(t) \sin(s)} + \frac{\cos(s)(1 - \cos(t))}{\sin(t) \sin(s)}.$$

Recall from the trigonometric relations that

$$\cos(d) - \cos(s) = 2 \sin \left( \frac{s + d}{2} \right) \sin \left( \frac{s - d}{2} \right);$$

$$1 - \cos(t) = 2 \sin^2 \left( \frac{t}{2} \right);$$

and

$$\sin(t) = 2 \sin \left( \frac{t}{2} \right) \cos \left( \frac{t}{2} \right).$$

Combining all of the above, we get

$$\cos(\theta) = \left( \frac{2 \sin \left( \frac{s + d}{2} \right)}{\sin(s)} \right) \left( \frac{\sin \left( \frac{s - d}{2} \right)}{\sin(t)} \right) + \frac{\cos(s) \sin \left( \frac{t}{2} \right)}{\sin(s) \cos \left( \frac{t}{2} \right)}.$$
Note that $d \to s$ as $t \to 0$. Using the limit of $\frac{\sin x}{x}$, we get as $t \to 0$

$$\cos(\theta) - \frac{s - d}{t} \to \frac{s \cos(s)}{2 \sin(s)} \left(\frac{t}{s}\right).$$

The bound given in the lemma follows as $\frac{s \cos(s)}{\sin s}$ is bounded with $0 < s < D|k|$.

The proof for $k < 0$ follows from the relationships $\cos(ix) = \cosh(x)$ and $\sin(ix) = is\sinh(x)$ and is very similar. It can be found in [Ale51] and [Shi93] (Lemma 4.1). The upshot is that

$$\cos(\theta) - \frac{s - d}{t} \to \frac{s \sqrt{|k|} \cosh(s \sqrt{|k|})}{2 \sinh(s \sqrt{|k|})} \left(\frac{t}{s}\right).$$

Again the bound follows immediately. \(\square\)

**Lemma 3.3.** If $X$ is a length space, then for all shortest paths $\gamma : [0,T] \to X$ and $\eta : [0,S] \to X$ with $\gamma(0) = \eta(0)$, for every fixed $s > 0$, we have

$$\limsup_{t \to 0^+} \angle^k(\gamma(t), \eta(s)) \leq \angle^+(\gamma, \eta).$$

**Proof.** If $s' < s$, then by the triangle inequality

$$s - s' \geq d(\gamma(t), \eta(s)) - d(\gamma(t), \eta(s')) \implies s - d(\gamma(t), \eta(s)) \geq s' - d(\gamma(t), \eta(s')).$$

Substituting this into Lemma 3.2, we see that

$$\liminf_{t \to 0^+} \cos \left(\angle^k(\gamma(t), \eta(s))\right) \geq \liminf_{t \to 0^+} \cos \left(\angle^k(\gamma(t), \eta(s'))\right).$$

As cosine is nonincreasing on $[0, \pi]$, we have

$$\cos \left(\limsup_{t \to 0^+} \angle^k(\gamma(t), \eta(s))\right) \geq \cos \left(\limsup_{s,t \to 0^+} \angle^k(\gamma(t), \eta(s))\right).$$

The right hand equals $\cos(\angle^+(\gamma, \eta))$. \(\square\)

**4 Alexandrov Spaces**

A length space $X$ is said to be of **curvature bounded above** (or **curvature $\leq k$**)
if there is a $k \in \mathbb{R}$ for which the following holds: At every point in $X$ there is a neighborhood $U$ such that for every geodesic triangle $\Delta \subseteq U$ with comparison triangle $\overline{\Delta} \subseteq \overline{M}_k^2$,

$$d(u, v) \leq d_k(\overline{u}, \overline{v})$$

(7)
Figure 1: A geodesic triangle with vertices $x$, $y$, and $z$ in a length space (left) and the respective comparison triangle $\Delta(x, y, z)$ in $M_0^2$ (right). The points $\bar{u}$ and $\bar{v}$ are chosen to satisfy $d(x, u) = d_k(\bar{x}, \bar{u})$ and $d(x, v) = d_k(\bar{x}, \bar{v})$.

for all $u, v \in \Delta$ and their comparison points $\bar{u}, \bar{v} \in \Delta$ (see Figure 1). Similarly, $X$ is said to be of curvature bounded below (or curvature $\geq k$) if $d(u, v) \geq d_k(\bar{u}, \bar{v})$. In either case, the neighborhood $U$ is referred to as a region of bounded curvature.

An Alexandrov space is a complete and locally compact length space with curvature bounded either above or below. It is well known that for Alexandrov spaces, the angle between two geodesics emanating from a common point always exists.

Lemma 4.1. Let $X$ be an Alexandrov space. If $\gamma : [0, T] \to X$ and $\eta : [0, S] \to X$ are shortest paths with $\gamma(0) = \eta(0)$ then the angle $\angle(\gamma, \eta)$ exists and

$$\angle(\gamma, \eta) = \lim_{t \to 0^+} \angle^k(\gamma(t), \eta(t)).$$

Proof. Suppose that $X$ is of curvature $\leq k$. Fix $s \in (0, S]$ and $a, b \in (0, T]$ such that $a < b$. We will consider two distinct comparison triangles in $M_k^2$. For simplicity of notation, we will denote them

$$\Delta(a) := \Delta(\gamma(a), \gamma(0), \eta(s)) \quad \text{and} \quad \Delta(b) := \Delta(\gamma(b), \gamma(0), \eta(s)).$$

From the definition of $\Delta(a)$ we have (see Figure 2)

$$d_k(\gamma(a), \eta(s)) = d(\gamma(a), \eta(s)).$$

Let $\bar{a}$ be the comparison point of $\gamma(a)$ in $\Delta(b)$ (as opposed to $\gamma(a)$, which is the comparison point in $\Delta(a)$). The upper bound $k$ for the curvature gives

$$d(\gamma(a), \eta(s)) \leq d_k(\bar{a}, \eta(s)).$$

Thus

$$d_k(\gamma(a), \eta(s)) \leq d_k(\bar{a}, \eta(s)),$$

which in turn implies

$$\angle^k(\gamma(a), \eta(s)) \leq \angle^k(\gamma(b), \eta(s)).$$
Thus, for any fixed \( s_0 \in (0, S] \), the map \( t \mapsto \angle^k(\gamma(t), \eta(s)) \) is monotonically nondecreasing. By the same reasoning the map \( s \mapsto \angle^k(\gamma(t_0), \eta(s)) \) is nondecreasing for any fixed \( t_0 \in (0, T] \). It follows from the monotonicity in both coordinates\(^4\) that

\[
\angle(\gamma, \eta) = \limsup_{s,t \to 0^+} \angle^k(\gamma(t), \eta(s)) = \liminf_{s,t \to 0^+} \angle^k(\gamma(t), \eta(s)) = \angle^-(\gamma, \eta).
\]

We conclude that the angle \( \angle(\gamma, \eta) \) exists and is equal to \( \lim_{t \to 0^+} \angle^k(\gamma(t), \eta(t)) \).

If \( X \) is of curvature \( \geq k \), the same method of proof applies, but the inequalities are reversed and the maps \( t \mapsto \angle^k(\gamma(t), \eta(s)) \) and \( s \mapsto \angle^k(\gamma(t_0), \eta(s)) \) are monotonically nonincreasing.

**Corollary 4.2.** Let \( X \) be an Alexandrov space of curvature \( \leq k \) (resp. \( \geq k \)). If the shortest paths \( \gamma : [0, T] \to X \) and \( \eta : [0, S] \to X \) (with \( \gamma(0) = \eta(0) \)) are contained in a region of bounded curvature, then

\[
\angle(\gamma, \eta) \leq \angle^k(\gamma(t), \eta(s)) \quad \text{(resp. } \angle(\gamma, \eta) \geq \angle^k(\gamma(t), \eta(s)) \text{)}
\]

for any \( s, t > 0 \).

*Proof.* By Lemma 4.1, if \( X \) is of curvature \( \leq k \) (resp. \( \geq k \)) the map \( t \mapsto \angle^k(\gamma(t), \eta(t)) \) is nondecreasing (resp. nonincreasing). It follows immediately that \( \angle(\gamma, \eta) \leq \angle^k(\gamma(t), \eta(s)) \) (resp. \( \angle(\gamma, \eta) \geq \angle^k(\gamma(t), \eta(s)) \)) for any \( t \in (0, T] \) and \( s \in (0, S] \).

While spaces of curvature bounded above and below share many properties, the following lemma gives an example of a property of spaces of curvature \( \geq k \) which is not

\(^3\)Note that the left distance is in \( \overline{\Delta}(a) \) while the right distance is in \( \overline{\Delta}(b) \).

\(^4\)For clarification on monotonicity in functions of two variables, see Proposition B.1 of the Appendix.
valid in spaces of curvature $\leq k$. This lemma also makes use of notation we shall need again, so we introduce it here. Let $\gamma : [0, T] \to X$ be a path and fix $t \in (0, T)$. The path $\gamma|_{[t,0]}$ is defined by $\gamma|_{[t,0]}(s) = \gamma(t-s)$ for $s \in [0, t]$. In other words, $\gamma|_{[t,0]}$ is the path that runs backwards along $\gamma$ from $\gamma(t)$ to $\gamma(0)$.

**Lemma 4.3.** If $X$ is an Alexandrov space of curvature bounded below, $\gamma : [0, T] \to X$ is a shortest path, $0 < t < T$, and $\sigma_t : [0, S] \to X$ is a shortest path with $\sigma_t(0) = \gamma(t)$ then

$$\angle_{\gamma(t)}(\gamma|_{[t,T]}, \sigma_t) + \angle_{\gamma(t)}(\gamma|_{[t,0]}, \sigma_t) = \pi.$$ 

In other words, adjacent angles along a shortest path sum to $\pi$.

**Proof.** By Proposition 3.1, we know that

$$\angle_{\gamma(t)}(\gamma|_{[t,T]}, \sigma_t) + \angle_{\gamma(t)}(\gamma|_{[t,0]}, \sigma_t) \geq \angle_{\gamma(t)}(\gamma|_{[t,T]}, \gamma|_{[0,t]}) = \pi$$

so it suffices to prove the reverse inequality.

Fix a small $\delta > 0$. We will consider a configuration of comparison points in $M_k^2$ for the points $\gamma(t-\delta)$, $\gamma(t)$, $\gamma(t+\delta)$, and $\sigma_t(\delta)$.

First, consider the comparison triangle $\overline{\Delta(\gamma(t-\delta), \sigma_t(\delta), \gamma(t+\delta)}$ with the comparison point $\overline{\gamma(t)}$. Second, consider the comparison triangle $\overline{\Delta(\gamma(t), \sigma_t(\delta), \gamma(t+\delta))}$ with the points $\gamma(t)$ and $\gamma(t+\delta)$ aligned as in Figure 3. Given that each triangle has a vertex representing $\sigma_t(\delta)$, we have labeled them $\sigma_t(\delta)$ and $\sigma_t(\delta)$ respectively, to distinguish them.

**Figure 3:** The comparison point construction of Lemma 4.3

By the definition of curvature $\geq k$, we know that

$$d_k(\sigma_t(\delta), \gamma(t)) \leq d_k(\sigma_t(\delta), \gamma(t)) = d(\sigma_t(\delta), \gamma(t)).$$

Considering that

$$d_k(\sigma_t(\delta), \gamma(t+\delta)) = d_k(\sigma_t(\delta), \gamma(t+\delta)) \quad \text{and} \quad d_k(\gamma(t), \gamma(t+\delta)) = d_k(\gamma(t), \gamma(t+\delta)),$$
we have an inequality between the interior angles at $\gamma(t)$:

$$\angle_{\gamma(t)}(\sigma_t(\delta), \gamma(t + \delta)) = \angle_{\gamma(t)}^k(\sigma_t(\delta), \gamma(t + \delta)) \leq \angle_{\gamma(t)}^k(\sigma_t(\delta), \gamma(t + \delta)).$$

Applying the analogous argument to the comparison triangle $\Delta(\gamma - \delta, \sigma_t(\delta), \gamma(t))$, we see that

$$\angle_{\gamma(t)}(\sigma_t(\delta), \gamma(t - \delta)) + \angle_{\gamma(t)}^k(\sigma_t(\delta), \gamma(t + \delta)) \leq \angle_{\gamma(t)}^k(\gamma(t - \delta), \gamma(t + \delta)).$$

Taking the limit as $\delta \to 0^+$ yields the result. \qed

**Proposition 4.4** (Semi-continuity of angles). Let $X$ be an Alexandrov space of curvature bounded above (resp. below). Suppose that the sequences of shortest paths $\{\gamma_n\}_{n=1}^\infty$ and $\{\sigma_n\}_{n=1}^\infty$, with $\gamma_n(0) = \sigma_n(0)$ for all $n$, converge uniformly to shortest paths $\gamma$ and $\sigma$ respectively. Then $\angle(\gamma, \sigma) \geq \lim \sup_{n \to \infty} \angle(\gamma_n, \sigma_n)$ (resp. $\angle(\gamma, \sigma) \leq \lim \inf_{n \to \infty} \angle(\gamma_n, \sigma_n)$).

**Proof.** First, suppose that $X$ is of curvature $\leq k$. For any $t \in [0, T]$, since $\gamma_n \to \gamma$ uniformly, $\gamma_n(t) \to \gamma(t)$; and the same can be said for the path $\sigma$. By Lemma 4.1 and Corollary 4.2

$$\angle(\gamma, \sigma) = \lim_{t \to 0^+} \angle_{\gamma(0)}^k(\gamma(t), \sigma(t))$$

$$= \lim_{t \to 0^+} \left( \lim_{n \to \infty} \angle_{\gamma_n(0)}^k(\gamma_n(t), \sigma_n(t)) \right)$$

$$\geq \lim_{t \to 0^+} \left( \lim \sup_{n \to \infty} \angle(\gamma_n, \sigma_n) \right).$$

As the final quantity above is independent of $t$, we have $\angle(\gamma, \sigma) \geq \lim \sup_{n \to \infty} \angle(\gamma_n, \sigma_n)$. \qed

Alternatively, if we suppose that $X$ is of curvature $\geq k$. Then (8) and (9) above still hold, but in (10) we make use of the other inequality of Corollary 4.2 to obtain $\angle(\gamma, \sigma) \leq \lim \inf_{n \to \infty} \angle(\gamma_n, \sigma_n)$. \qed

## 5 Distance to a Compact Set

**Lemma 5.1.** If $X$ is an Alexandrov space, $\gamma : [0, T] \to X$ is a unit-speed shortest path, and $p$ is an element of $X$ such that $\gamma(0) \neq p$, then

$$\lim \sup_{t \to 0^+} \frac{d(\gamma(t), p) - d(\gamma(0), p)}{t} \leq -\cos(\angle_{\min})$$

where $\angle_{\min}$ is the infimum of angles between $\gamma$ and shortest paths from $\gamma(0)$ to $p$. 

Lemma 5.2. Let $X$ be an Alexandrov space, $K$ a compact set in $X$, and $\gamma : [0, T] \to X$ a unit-speed shortest path such that $\gamma(0) \notin K$. For each $t \in [0, T]$ let $\sigma_t$ be a shortest path connecting $\gamma(t)$ to $K$. If there is a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \to 0$ and the sequence of shortest paths $\{\sigma_{t_n}\}_{n=1}^\infty$ converges to $\sigma_0$, then
\begin{equation*}
\limsup_{n \to \infty} \angle_{\gamma(t_n)}(\gamma(0), \sigma_{t_n}(s)) \leq \pi - \angle(\gamma, \sigma_0)
\end{equation*}
for all sufficiently small $s > 0$. (See Figure 4.)

Proof. Begin by fixing $N \in \mathbb{N}$ and $s' \in (0, S]$ such that $\gamma(0)$, $\gamma(t_n)$, and $\sigma_{t_n}(s)$ all lie in a region of bounded curvature whenever $n \geq N$ and $s \leq s'$. For simplicity of notation let $s_n = \sigma_{t_n}(s)$ for some fixed $s > 0$.

Suppose $X$ is of curvature $\geq k$. Then
\begin{align*}
\limsup_{n \to \infty} \angle_{\gamma(t_n)}^k(\gamma(0), s_n) &\leq \limsup_{n \to \infty} \angle_{\gamma(t_n)}(\gamma|_{[t_n, 0]}, \sigma_{t_n}) \\
&= \pi - \liminf_{n \to \infty} \angle_{\gamma(t_n)}(\gamma|_{[t_n, T]}, \sigma_{t_n}) \\
&\leq \pi - \angle(\gamma, \sigma_0)
\end{align*}
by Corollary 4.2, Lemma 4.3 and Proposition 4.4 respectively.

Next, suppose $X$ is of curvature $\leq k$. For each $n$, let $\eta_n$ be a shortest path connecting $\gamma(0)$ to $s_n$ (see Figure 5). By Corollary 4.2, $\angle(\eta_n, \sigma_0) \leq \angle_{\gamma(0)}^k(s_n, \sigma_0(s))$ for all $n$. Since $\sigma_{t_n} \to \sigma_0$, we have $\angle_{\gamma(0)}^k(s_n, \sigma_0(s)) \to 0$ and so $\angle(\eta_n, \sigma_0) \to 0$. By Proposition 3.7 twice,
\begin{align*}
\angle(\gamma, \sigma_0) \leq \angle(\gamma, \eta_n) + \angle(\eta_n, \sigma_0) \leq \angle(\gamma, \sigma_0) + 2\angle(\eta_n, \sigma_0).
\end{align*}
So, as $\angle(\eta_n, \sigma_0) \to 0$, we have $\angle(\gamma, \eta_n) \to \angle(\gamma, \sigma_0)$. Thus, using Corollary 4.2 again
\begin{align*}
\angle(\gamma, \sigma_0) &= \liminf_{n \to \infty} \angle(\gamma, \eta_n) \\
&\leq \liminf_{n \to \infty} \angle_{\gamma(t_n)}^k(\gamma(t_n), s_n).
\end{align*}
Let $X$ be an Alexandrov space, $\gamma : [0, T] \to X$ a unit-speed shortest path, and $K$ a compact set not containing $\gamma(0)$. If $\ell(t) = d(\gamma(t), K)$, then
\[
\lim_{t \to 0^+} \frac{\ell(t) - \ell(0)}{t} = -\cos(\angle_{\min})
\]
where $\angle_{\text{min}}$ is the infimum of angles between $\gamma$ and any shortest path of length $\ell(0)$ which connects $\gamma(0)$ to $K$.

**Proof.** First, let $\eta_0$ be a shortest path connecting $\gamma(0)$ to $K$ and let $a \in K$ be the endpoint of $\eta_0$. Note that for each $t > 0$, $\ell(t) \leq d(\gamma(t), a)$. Therefore, by Lemma 5.1

$$\limsup_{t \to 0^+} \frac{\ell(t) - \ell(0)}{t} \leq \limsup_{t \to 0^+} \frac{d(\gamma(t), a) - d(\gamma(0), a)}{t} \leq -\cos(\angle_{\text{min}}).$$

To get the reverse estimate, let $\{t_n\}_{n=1}^\infty$ be a sequence in $(0, T]$ such that $t_n \to 0$ and

$$\lim_{n \to \infty} \frac{\ell(t_n) - \ell(0)}{t_n} = \liminf_{t \to 0^+} \frac{\ell(t) - \ell(0)}{t}.$$

Similar to Lemma 5.2, for each $n$ let $\sigma_{t_n}$ be a shortest path connecting $\gamma(t_n)$ to $K$. Since $K$ is compact, the length of each path in the sequence $\{\sigma_{t_n}\}_{n=1}^\infty$ is uniformly bounded. Therefore, by the Arzela-Ascoli Theorem, $\{\sigma_{t_n}\}_{n=1}^\infty$ contains a subsequence which converges uniformly to a shortest path $\sigma_0$ connecting $\gamma(0)$ to $K$. Without loss of generality, we assume that the sequence $\{\sigma_{t_n}\}_{n=1}^\infty$ is this uniformly convergent subsequence.

Fix $s$ sufficiently small to satisfy the hypothesis of Lemma 5.2. For simplicity of notation let $p_n \in K$ be the endpoint of $\sigma_{t_n}$, let $p \in K$ be the endpoint of $\sigma_0$, and let $s_n = \sigma_{t_n}(s)$ (Figures 4 and 5). By Lemma 5.2, for some $C > 0$

$$\cos \left( \angle_{\gamma(t_n)}(\gamma(0), s_n) \right) \leq \frac{s - d(s_n, \gamma(0))}{t_n} + \frac{Ct_n}{s}. \quad (13)$$

Note that $\ell(t_n) = s + d(s_n, p_n)$ and

$$\ell(0) \leq d(\gamma(0), p_n) \leq d(\gamma(0), s_n) + d(s_n, p_n).$$

Combining these observations with (13), we get

$$\cos \left( \angle_{\gamma(t_n)}(\gamma(0), s_n) \right) \leq \frac{\ell(t_n) - \ell(0)}{t_n} + \frac{Ct_n}{s}.$$

Since $t_n \to 0$ while $s$ is held constant,

$$\liminf_{n \to \infty} \frac{\ell(t_n) - \ell(0)}{t_n} \geq \liminf_{n \to \infty} \cos \left( \angle_{\gamma(t_n)}(\gamma(0), s_n) \right) = \cos \left( \limsup_{n \to \infty} \angle_{\gamma(t_n)}(\gamma(0), s_n) \right).$$

Then by Lemma 5.2

$$\cos \left( \limsup_{n \to \infty} \angle_{\gamma(t_n)}(\gamma(0), s_n) \right) \geq \cos \left( \pi - \angle(\gamma, \sigma_0) \right) = -\cos(\angle(\gamma, \sigma_0)).$$

Thus,

$$\liminf_{t \to 0^+} \frac{\ell(t) - \ell(0)}{t} = \liminf_{n \to \infty} \frac{\ell(t_n) - \ell(0)}{t_n} \geq -\cos(\angle(\gamma, \sigma_0)) \geq -\cos(\angle_{\text{min}}),$$

which is the desired reverse estimate.

We note here that the one-sided derivative for distance to a point (Theorem 4.5.6 and Corollary 4.5.7 of [BB101]) follow as immediate corollaries of Theorem 5.3.

\[\textit{Note:} \text{see Appendix B.2 for clarification on how Arzela-Ascoli is used here.}\]
A Counterexamples

A.1 Upper Angle $\neq$ Lower Angle

Consider $\mathbb{R}^2$ with the metric $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, known as the $\ell^1$ metric, or the ‘taxicab’ metric. The space $(\mathbb{R}^2, d)$ is a complete and locally compact length space.

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be defined by $\gamma(t) = (t, t)$ and let $\eta : [0, 1] \to \mathbb{R}^2$ be defined by $\eta(s) = (s, 0)$. Note that these are both shortest paths with

$$L(\gamma) = 2 = d((0, 0), (1, 1)) \quad \text{and} \quad L(\eta) = 1 = d((0, 0), (1, 0))$$

Since $\gamma(0) = \eta(0)$, we may consider the upper and lower angle between them.

\[\begin{array}{c}
\gamma(0) = \eta(0) = (0, 0) \\
\eta(s) \quad \eta(1) = (1, 0) \\
\gamma(t) \\
\gamma(1) = (1, 1)
\end{array}\]

\textbf{Figure 6:} The paths $\gamma$ and $\eta$.

First, let $s = t$. Then (see Figure 6),

$$d(\gamma(0), \gamma(t)) = 2t \quad \text{and} \quad d(\eta(0), \gamma(t)) = d(\gamma(t), \eta(t)) = t.$$ 

Therefore, $\angle^0(\gamma(t), \eta(t)) = 0$ for all $t$. It follows that

$$\angle^-(\gamma, \eta) = \liminf_{s, t \to 0^+} \angle^0(\gamma(t), \eta(s)) \leq \lim_{t \to 0^+} \angle^0(\gamma(t), \eta(t)) = 0$$

Furthermore, as 0 is the minimum possible angle, we have $\angle^-(\gamma, \eta) = 0$.

Next, consider $t = s^2$. We have

$$d(\gamma(0), \gamma(s^2)) = 2s^2 \quad d(\eta(0), \eta(s)) = s \quad \text{and} \quad d(\gamma(s^2), \eta(s)) = s - s^2 + s^2 = s.$$ 

Note that the (Euclidean) comparison triangle $\Delta(\gamma(0), \gamma(t), \eta(s))$ is isosceles with a very small base. Using elementary plane geometry, one easily derives that $\cos(\angle^0(\gamma(s^2), \eta(s))) = s$. Since cosine is continuous and nonincreasing on the interval $[0, \pi]$, we see that

$$\cos(\angle^+(\gamma, \eta)) \leq \lim_{s \to 0^+} \cos(\angle^0(\gamma(s^2), \eta(s))) = \lim_{s \to 0^+} s = 0.$$ 

Thus we have $\angle^+(\gamma, \eta) \geq \pi/2$. In fact, we can show that the upper angle equals $\pi/2$. 

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If the upper angle were greater than $\pi/2$, then there would have to be points $s, t \in [0, 1]$ such that $d(\gamma(t), \eta(s))$ is greater than both $d(\gamma(0), \gamma(t))$ and $d(\eta(0), \eta(s))$. However, this is impossible since $d(\eta(0), \eta(s)) = s$, $d(\gamma(0), \gamma(t)) = 2t$, and

$$d(\gamma(t), \eta(s)) = |s - t| + |t - 0| = t + |s - t|$$

which cannot be simultaneously greater than $s$ and $2t$ for any $s, t \in [0, 1]$.

### A.2 Unbounded Curvature

Consider again the length space $(\mathbb{R}^2, d)$ from Appendix A.1. While we could use the result of the previous section combined with Lemma 4.1 to establish that $(\mathbb{R}^2, d)$ does not have bounded curvature, we instead provide here a direct proof using the definition of bounded curvature given in Section 4.

Given any neighborhood $U$ in $\mathbb{R}^2$, and any $k \in \mathbb{R}$, we can find three points $x, y, z \in U$ such that

$$d(x, y) = d(x, z) = d(y, z)$$

and $d(x, y) < D_k/2$, which forms an equilateral comparison triangle $\Delta(x, y, z) \subseteq M^2_k$.

![Figure 7: The geodesic triangles A (left) and B (right) with vertices x, y, and z in the 'taxicab' space $(\mathbb{R}^2, d)$.](image)

While there are many geodesic triangles in $(\mathbb{R}^2, d)$ with vertices $x, y, z$, we will only consider two. First, we choose the shortest paths which form a rectangle around the points and call this triangle $A$ (left side of Figure 7). We may fix points $u_a, v_a \in A$ on each side of the point $z$ such that $d(u_a, v_a) = d(z, u_a) + d(z, v_a)$. However, in our comparison triangle $\Delta(x, y, z)$, we have

$$d_k(\bar{u}_a, \bar{v}_a) < d_k(\bar{u}_a, \bar{z}) + d_k(\bar{z}, \bar{v}_a) = d(u_a, z) + d(z, v_a) = d(u_a, v_a).$$

Since $d(u_a, v_a) > d_k(\bar{u}_a, \bar{v}_a)$, the curvature of $(\mathbb{R}^2, d)$ cannot be $\leq k$.

Second, we consider the geodesic triangle $B$ consisting of three branching geodesics (right side of Figure 7). Let $u_b, v_b \in B$ be such that $d(u_b, z) = d(v_b, z) > 0$ and $d(u_b, v_b) = 0$. We know that the points $u_b$ and $v_b$ exist since the shortest paths connecting
z to x and z to y are branches from a common geodesic. Recalling that our comparison triangle $\Delta(x, y, z)$ is equilateral, we have

$$d_k(\bar{u}_b, \bar{v}_b) > 0 = d(u_b, v_b)$$

so the curvature of $(\mathbb{R}^2, d)$ cannot be $\geq k$. As no neighborhood $U$ can satisfy the definition of curvature bounded above or below by any $k$, the space $(\mathbb{R}^2, d)$ is not of bounded curvature.

### A.3 Supplementary Upper Angles May Not Sum to $\pi$

If on a geodesic $\gamma : [-T, T] \mapsto X$ where $X$ is Alexandrov with lower bound on the curvature, we choose 3 nearby points $b = \gamma(-t)$, $a = \gamma(0)$, and $c = \gamma(t)$, then the upper angle $\angle^+_{\gamma(t)}(\gamma|_{[0,t]}, \gamma|_{[0,-t]})$ equals $\pi$. This follows immediately from the definition of angles. See Figure 8.

However, spaces without lower bound on the curvature, such as the space $(\mathbb{R}^2, d)$ from Appendix [A.1](#), have the property that two geodesics that agree on a segment may bifurcate. Thus, suppose in Figure 8 $bad$ and $dac$ are geodesics. By the previous observation all three angles $\alpha$, $\beta$, and $\theta$ are equal to $\pi$, and we have a counter example to Lemma [4.3](#). Notice that it also follows that if there is a lower bound on the curvature, then geodesics cannot bifurcate.

### B Results in Analysis

#### B.1 Monotonicity in Functions of Two Real Variables

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. We say that $f$ is **component-wise monotonic** if for any constant $r \in \mathbb{R}$, the maps $x \mapsto f(x, r)$ and $x \mapsto f(r, x)$ are both nondecreasing or both nonincreasing. We further define the right-sided limit superior of $f$ at $a$ as

$$\limsup_{x, y \to a^+} f(x, y) = \lim_{\epsilon \to 0} \sup \{ f(x, y) : a < x, y \leq a + \epsilon \}.$$
Similarly, the right-sided limit inferior of $f$ at $a$ is given by
\[
\liminf_{x,y \to a^+} f(x,y) = \lim_{\varepsilon \to 0} \inf \{ f(x,y) : a < x, y \leq a + \varepsilon \}.
\]

**Lemma B.1.** If $f : \mathbb{R}^2 \to \mathbb{R}$ is component-wise monotonic, then for any $a \in \mathbb{R}$,
\[
\limsup_{x,y \to a^+} f(x,y) = \liminf_{x,y \to a^+} f(x,y).
\]

**Proof.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given and assume that $f$ is component-wise nondecreasing (the result for nonincreasing follows by symmetry). If we fix $x_1 < x_2$, then by our assumptions,
\[
f(x_1, x_1) \leq f(x_1, x_2) \leq f(x_2, x_2).
\]
Therefore, for any $a \in \mathbb{R}$, it follows from the monotonicity of $x \mapsto f(x, x)$ that
\[
\limsup_{x \to a^+} f(x, x) = \liminf_{x \to a^+} f(x, x) = \lim_{x \to a^+} f(x, x).
\]
Denote the limit by $f(a)^+$. To finish the proof, it suffices to show that
\[
\limsup_{x,y \to a^+} f(x,y) \leq f(a)^+ \leq \liminf_{x,y \to a^+} f(x,y).
\]

Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $x_n, y_n > a$ for all $n$ and $x_n, y_n \to a$. For each $n$, define $m_n = \min\{x_n, y_n\}$ and $M_n = \max\{x_n, y_n\}$. Then
\[
\limsup_{n \to \infty} f(x_n, y_n) \leq \lim_{n \to \infty} f(M_n, M_n) = f(a)^+ = \lim_{n \to \infty} f(m_n, m_n) \leq \liminf_{n \to \infty} f(x_n, y_n). \quad \Box
\]

### B.2 The Arzela-Ascoli Theorem for Paths

While there are many equivalent statements of the Arzela-Ascoli Theorem, the version which best fits our needs is that found in [BH91].

**Theorem B.2** (Arzela-Ascoli). If $X$ is a compact metric space and $Y$ is a separable metric space, then every sequence of equicontinuous maps $f_n : Y \to X$ contains a uniformly convergent subsequence.

**Corollary B.3.** If $X$ is a compact metric space and $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of paths in $X$ with uniformly bounded lengths, then the sequence $\{\gamma_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence.

**Proof.** Without loss of generality, we may assume that each $\gamma_n$ is constant-speed with domain $[0, 1]$. Since the length of the paths is uniformly bounded by, say $M \in \mathbb{R}$, for all $t, t' \leq [0, 1]$
\[
d(\gamma_n(t), \gamma_n(t')) \leq M |t - t'|
\]
so $\{\gamma_n\}_{n=1}^{\infty}$ is equicontinuous. As $[0, 1]$ is separable and $X$ is compact, by Theorem B.2, $\{\gamma_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence. \quad \Box
C A Geometric Observation

C.1 Space Forms are Infinitesimally Euclidean

Lemma C.1. \( \limsup_{t,s \to 0^+} \angle^k_0(\gamma(t), \eta(s)) = \limsup_{t,s \to 0^+} \angle^0(\gamma(t), \eta(s)) \)

Proof. There are many ways of proving this. The first is by using the geodesic equation to establish that the smaller the domain, the closer a geodesic crossing it resembles a "straight line". This is made rigorous in [BV07] (Proposition 1.10).

Another more informal way is to realize that instead of shrinking a triangle with sides of lengths \( a, b, \) and \( c \) in \( M^2_k \) by a factor of, say \( \varepsilon \), we might equally well define a new space \( M \) by changing coordinates in \( M^2_k \) from \( x \) to \( \bar{x} = x/\varepsilon \). Now consider a triangle with those same lengths \( a, b, \) and \( c \) again. Gauss’ original definition of (Gaussian) curvature is

\[
k = \lim_{A \to 0} \frac{A'}{A}
\]

where \( A \) is the area of a small disk in \( M^2_k \) and \( A' \) is the area in the unit sphere swept out by the unit normals in \( A \). Clearly, in our rescaled space \( A' \) has been shrunk by a factor \( \varepsilon^2 \) while \( A \) changes very little. \( \square \)

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