ON SOME INTEGRAL TRANSFORMS OF COULOMB FUNCTIONS RELATED TO THREE-DIMENSIONAL PROPER LORENTZ GROUP

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ABSTRACT. Considering the relationship between two bases in representation space of the three-dimensional proper Lorentz group, we derive some formulas with integrals involving Coulomb wave functions, which can be considered as Fourier, Mellin, K-Bessel, Hankel and Mehler-Fock transforms of these functions.

1. Introduction

As usually, let \( \mathbb{R} \) and \( \mathbb{C} \) be the sets of real and complex numbers, respectively. In addition, throughout this paper we use the definition \( \mathbb{R}_a \) for the ray \( (a; +\infty) \).

Let us recall that the Coulomb (wave) functions \( F_{\sigma}(\rho, \lambda) \) and \( H_{\sigma}^{\pm}(\rho, \lambda) \) are functions belonging to the kernel of the Coulomb differential operator

\[
\mathfrak{d} := \frac{d^2}{d\lambda^2} + 1 - \frac{2\rho}{\lambda} - \frac{\sigma(\sigma + 1)}{\lambda^2},
\]

where \( \lambda \in \mathbb{R}^0, \rho \in \mathbb{R} \) (Sommerfeld parameter), and \( \sigma \) is non-negative integer (angular momentum quantum number). These functions are defined by formulas

\[
F_{\sigma}(\rho, \lambda) = 2^{-\sigma-1}C_{\sigma}(\rho)(\mp i)^{\sigma+1}M_{\pm i\rho, \sigma + \frac{1}{2}}(\pm 2i\lambda),
\]

\[
H_{\sigma}^{\pm}(\rho, \lambda) = (\mp i)^{\sigma}\exp \left(\frac{\pi \rho}{2} \pm i c_{\sigma}(\rho)\right)W_{\mp i\rho, \sigma + \frac{1}{2}}(\mp 2i\lambda),
\]

where \( M_{\mu,\nu}(z) \) and \( W_{\mu,\nu}(z) \) are Whittaker functions of the first and second kind, respectively, and the normalizing constant (Gamow factor) \( C_{\sigma}(\rho) \) and Coulomb phase shift \( c_{\sigma}(\rho) \) are defined as follow:

\[
C_{\sigma}(\rho) = 2^\sigma \exp \left(\frac{-\pi \rho}{2}\right) [\Gamma(2\sigma + 2)]^{-1} |\Gamma(\sigma + 1 + i\rho)|,
\]

\[
c_{\sigma}(\rho) = \arg \Gamma(\sigma + 1 + i\rho).
\]

They can be considered for complex values of \( \lambda(\neq 0), \rho \) and \( \sigma \). The definitions (1.1) and (1.2) are correct since the choice of upper or lower signs in (1.1) and (1.2) isn’t important in view of identity \( _1F_1(a; b; z) = \exp(z) \ _1F_1(b - a; b; -z) \) (known as Kummer’s transformation) for confluent hypergeometric function \( _1F_1 \) which determines the both Whittaker functions.

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We recall also that \( F_\sigma(\rho, \lambda) \) and \( H^\pm_\sigma(\rho, \lambda) \) are regular and irregular (respectively) solutions of the Coulomb wave equation \( \mathcal{D}[y] = 0 \) and connected by the identity
\[
F_\sigma(\rho, \lambda) = \pm \text{ imaginary part of } H^\pm_\sigma(\rho, \lambda).
\] (1.3)

Below we also use another function belonging to \( \text{Ker} \mathcal{D} \):
\[
G_\sigma(\rho, \lambda) = \text{ real part of } H^\pm_\sigma(\rho, \lambda).
\]

Since defect of \( \mathcal{D} \) is equal to 2, the linearly independent functions \( F_\sigma(\rho, \lambda) \) and \( G_\sigma(\rho, \lambda) \) form a basis in \( \text{Ker} \mathcal{D} \). Another basis consists of \( H^+_\sigma(\rho, \lambda) \) and \( H^-_\sigma(\rho, \lambda) \). In [1] the author considered also two other bases in \( \text{Ker} \mathcal{D} \) consisting of functions, also named Coulomb functions and introduced by Hartree in [5] and H. and B. Jeffreys in [6].

2. Representation space, its bases, and functionals \( F_1 \) and \( F_2 \)

We recall that the three-dimensional Lorentz group is the subgroup of matrices \( (g_{ij}) \) in \( GL(3, \mathbb{R}) \) satisfying the equalities \( g_{11}^2 - g_{22}^2 - g_{33}^2 = (-1)^E(\frac{1}{2}) \) for \( i \in \{1, 2, 3\} \), where \( E(n) \) denotes the entire part of an integer \( n \). In this paper we consider its intersection \( G \) with \( SL(3, \mathbb{R}) \), calling \( G \) the proper Lorentz group.

Let \( \sigma \in \mathbb{C} \) and \( T \) be the representation of \( G \) in the linear space \( \mathcal{D} \) consisting of \( \sigma \)-homogeneous and infinitely differentiable functions defined on the cone \( \Lambda : x_1^2 - x_2^2 - x_3^2 = 0 \) acting according to rule \( T(g)[f(x)] = f(g^{-1}x) \). We recall that the functions \( x^\mu_{\pm} \) on \( \mathbb{R} \), which generate the generalized functions \( (x^\mu_{\pm}, f) \) [6], are defined as follow: \( x^\mu_{\pm} \) is equal to \( |x|^\mu \) for \( x \in \mathbb{R}^\pm \) and coincides with zero function otherwise. In this paper we deal with the bases [17]
\[
B_1 = \left\{ f_{\lambda}(x) = (x_1 + x_2)^\sigma \exp \frac{\lambda x_3}{x_1 + x_2} \mid \lambda \in \mathbb{R} \right\}
\]
and
\[
B_2 = \left\{ f_{\rho,\pm}(x) = (x_2)^{\sigma-1}\rho (x_1 + x_3)^{\rho} \mid \rho \in \mathbb{R} \right\}.
\]

Below we use two bilinear functionals defined on pairs of representation spaces in the same way as in [14]. In order to introduce them, we define the following subsets on \( \Lambda \): parabola \( \gamma_1 : x_1 + x_2 = 1 \) and hyperbola \( \gamma_2 = \gamma_{2,+} \cup \gamma_{2,-} \), where \( \gamma_{2,\pm} : x_2 = \pm 1 \). Let \( H_i \) be a subgroup of \( G \), which acts transitively on \( \gamma_i \). We define \( F_1 \) and \( F_2 \) as
\[
F_i : (\mathcal{D}, \mathcal{D}) \rightarrow \mathbb{C}, \ (f, g) \rightarrow \int_{\gamma_i} f(x)g(x) \, d\gamma_i,
\]
where \( d\gamma_i \) is a \( H_i \)-invariant measure on \( \gamma_i \). Let us parameterize \( \gamma_1 \) and \( \gamma_2 \) as follow:
\[
\gamma_1 : \begin{cases}
x_1 = \frac{1}{2} \left( 1 + \alpha_1^2 \right), \\
x_2 = \frac{i}{2} \left( 1 - \alpha_1^2 \right), \\
x_3 = \alpha_1,
\end{cases} \quad \gamma_{2,\pm} : \begin{cases}
x_1 = \cosh \alpha_2, \\
x_2 = \pm 1, \\
x_3 = \sinh \alpha_2,
\end{cases}
\]
where \( \alpha_1, \alpha_2 \in \mathbb{R} \). Since the subgroups \( H_1 \) and \( H_2 \) consist of matrices
\[
h_1(\theta_1) = \frac{1}{2} \begin{pmatrix}
2 + \theta_1^2 & \theta_1^2 & 2\theta_1 \\
-\theta_1^2 & 2 - \theta_1^2 & -2\theta_1 \\
2\theta_1 & -2\theta_1 & 2
\end{pmatrix},
\]
we can define the functionals \( F_1 \) and \( F_2 \) as
\[
F_1 : (\mathcal{D}, \mathcal{D}) \rightarrow \mathbb{C}, \ (f, g) \rightarrow \int_{\gamma_1} f(x)g(x) \, d\gamma_1,
\]
and
\[
F_2 : (\mathcal{D}, \mathcal{D}) \rightarrow \mathbb{C}, \ (f, g) \rightarrow \int_{\gamma_2} f(x)g(x) \, d\gamma_2.
\]
and

\[
h_2(\theta_2) = \begin{pmatrix} \cosh \theta_2 & 0 & \sinh \theta_2 \\ 0 & 1 & 0 \\ \sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix},
\]

respectively, where \( \theta_1 \in [0; 2\pi) \) and \( \theta_2 \in \mathbb{R} \), and

\[
T(h_1(\theta_1))[f_\lambda(\alpha_1)] = f_\lambda(\alpha_1 - \theta_1),
\]
\[
T(h_2(\theta_2))[f_{\rho,\pm}(\alpha_2)] = f_{\rho,\pm}(\alpha_2 - \theta_2),
\]

we have \( d\gamma_i = d\alpha_i \). It has been showed in [14] that \( F_1 \) and \( F_2 \) coincide on pairs \( (D, D^*) \) such that degree of homogeneity of \( D^* \) is equal to \(-\sigma - 1\).

3. Matrix elements of \( B_1 \hookrightarrow B_2 \) and \( B_1^* \hookrightarrow B_2^* \) transformations in terms of Coulomb functions

Let us express a function \( f_\lambda \in B_1^* \) as a linear combination of vectors belonging to \( B_2^* \):

\[
f^*_\lambda(x) = \int \left[ c^*_{\lambda,\rho,\pm}f^*_\rho,\pm(x) + c^*_{\lambda,\rho,-}f^*_\rho,-(x) \right] d\rho. \tag{3.1}
\]

Since

\[
f_{\rho,\pm}|_{\gamma_{2,\pm}} = f^*_{\rho,\pm}|_{\gamma_{2,\pm}} = \exp(i\rho\alpha_2) \quad \text{and} \quad f_{\rho,\pm}|_{\gamma_{2,\mp}} = f^*_{\rho,\pm}|_{\gamma_{2,\mp}} = 0, \tag{3.2}
\]

we have

\[
F_i(f^*_{\lambda}; f_{\rho,\pm}) = \int \left[ c^*_{\lambda,\rho,\pm}F_2(f^*_\rho,\pm, f_{\rho,\pm}) \right] d\rho
\]

\[
= \int c^*_{\lambda,\rho,\pm} d\rho \int \exp(i(\rho + \hat{\rho})\alpha_2) d\alpha_2 = 2\pi \int c^*_{\lambda,\rho,\pm} \delta(\rho + \hat{\rho}) d\rho = 2\pi c^*_{\lambda,-\rho,\pm},
\]

where \( \delta(\rho + \hat{\rho}) \) is the \( \hat{\rho} \)-delayed Dirac delta function, therefore,

\[
c^*_{\lambda,\rho,\pm} = \frac{1}{2\pi} F_i(f^*_{\lambda}; f_{-\rho,\pm}).
\]

In the same way, if

\[
f^*_{\rho,\pm}(x) = \int \left[ c^*_{\rho,\pm,\lambda} f^*_\lambda(x) \right] d\lambda, \tag{3.3}
\]

then

\[
c^*_{\rho,\pm,\lambda} = \frac{1}{2\pi} F_i(f^*_{\rho,\pm}; f_{-\lambda}) = c_{-\lambda,-\rho,\pm}. \tag{3.4}
\]

Considering that \( \sigma \) is the third argument (after \( \rho \) and \( \lambda \)) of \( c^*_{\rho,\pm,\lambda} \), we derive from (3.1) that \( c^*_{\rho,\pm,\lambda}(\sigma) = c_{-\lambda,-\rho,\pm}(-\sigma - 1) \).

**Theorem 1.** Let \( \sigma \in \mathbb{R}_{-1} \) and \( \lambda \neq 0 \),

\[
c^*_{\lambda,\rho,+} = \frac{\Gamma(\sigma + 1 + i\rho)}{\pi \lambda^{\sigma+1}} \exp \left( \frac{\pi \rho}{2} \right) F_\sigma(\rho, \lambda).
\]
Proof. Let us use the known formula [12, see, e.g., Entry 2.3.6(1)]
\[ \int_0^a (a - x)^{\alpha - 1} \beta^{-1} \exp(-px) dx = B(\alpha, \beta) a^{\alpha + \beta - 1} \, _1F_1(\alpha; \alpha + \beta; -ap), \] (3.5)
which holds true for \( \Re(\alpha), \Re(\beta) \in \mathbb{R}_0 \), to computing of \( c_{\lambda, \rho, +} \):

\[ c_{\lambda, \rho, +} = \frac{1}{2\pi} \, _1F_1(f^*_\lambda, f_{-\rho, +}) \]
\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{1 - \alpha_1^2}{2} \right)^\sigma \left( \frac{1 + \alpha_1^2}{2} + \alpha_1 \right)^{-i\rho} \exp(i\lambda_1) \, d\alpha_1 \]
\[ = \frac{2^{-\sigma - 1}}{\pi} \int_{-1}^1 (1 - \alpha_1)^\sigma (1 + \alpha_1)^{-i\rho} \exp(i\lambda_1) \, d\alpha_1 \]
\[ = \frac{2^{-\sigma - 1}}{\pi} \exp(-i\lambda) \int_0^2 t^{-i\rho} (2 - t)^\sigma \exp(i\lambda t) \, dt \]
\[ = \frac{2^\sigma}{\pi} \exp(-i\lambda) B(\sigma + 1 + i\rho, \sigma + 1 - i\rho) \, _1F_1(\sigma + 1 - i\rho; 2\sigma + 2; 2i\lambda). \]

Using here the relation (see, e.g., [10, p. 290])
\[ M_{\mu, \nu}(z) = z^{\nu + \frac{1}{2}} \exp\left( -\frac{z}{2} \right) \, _1F_1 \left( \nu - \mu + \frac{1}{2}; 2\nu + 1; z \right), \]
we obtain

\[ c_{\lambda, \rho, +} = \frac{2^{-2\sigma - 2} (i\lambda)^{-\sigma - 1}}{\pi} B(\sigma + 1 + i\rho, \sigma + 1 - i\rho) M_{\nu, \sigma + \frac{1}{2}}(2i\rho). \]

Using here (1.1) and considering the equalities \( B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \) and \( \Gamma(\bar{z}) = \overline{\Gamma(z)} \), where \( \bar{z} \) is the complex conjugate of \( z \), we complete the proof. \( \square \)

Let us note that

\[ c_{\lambda, \rho, -} = \frac{1}{2\pi} \, _1F_1(f^*_\lambda, f_{-\rho, -}) \]
\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{1 - \alpha_1^2}{2} \right)^\sigma \left( \frac{1 + \alpha_1^2}{2} + \alpha_1 \right)^{-i\alpha_1} \exp(i\lambda_1) \, d\alpha_1 \]
\[ = \frac{2^{-\sigma - 1}}{\pi} \exp(-i\lambda) \left\{ \int_{\mathbb{R}_0} t^{-i\rho} (t + 2)^\sigma \exp(i\lambda t) \, dt \right. \]
\[ + \exp(i\lambda) \left. \int_{\mathbb{R}_0} t^{-i\rho} (t + 2)^{\sigma - i\rho} \exp(-i\lambda t) \, dt \right\}. \] (3.6)
Theorem 2. Let $\sigma$ of in terms of Coulomb functions, we consider the following theorem for one particular value for $\sigma$. However, in order to represent the matrix elements $c_{\lambda,\rho,-}$ in terms of Coulomb functions, we consider the following theorem for one particular value of $\sigma$ not belonging to the above domain.

**Theorem 2.** Let $\sigma = \frac{1}{4}$, $\lambda \neq 0$, and

$$A = \Re \left(1^\frac{1}{4} \exp \left(i \frac{1}{4} \rho \right)\right), \quad B = \Im \left(1^\frac{1}{4} \exp \left(i \frac{1}{4} \rho \right)\right).$$

Then

$$c_{\lambda,\rho,-} = \frac{2 \left(AG_{\frac{3}{4}}(\rho, \lambda) - BF_{\frac{3}{4}}(\rho, \lambda)\right)}{\lambda^\frac{7}{4} \sqrt{\cosh(2\pi \rho)}}.$$  

**Proof.** In view of (3.6), we have

$$c_{\lambda,\rho,-} = \frac{1}{2^n \pi} \sum_{j=0}^{\infty} \exp((-1)^j i \lambda) \int_{\mathbb{R}_0} t^{-\frac{1}{2} + (-1)^j i \rho} (t + 2) - \frac{1}{2} (-1)^j i \rho \exp((-1)^j i \lambda t) dt.$$  

Using here the formula (see, e.g., [7, Entry 3.383(6)])

$$\int_{\mathbb{R}_0} x^{\nu - 1} (x + \beta)^{\frac{1}{2} - \nu} \exp(-\mu x) dx = 2^{\nu - \frac{1}{2}} \frac{\Gamma(\nu)}{\mu^\frac{1}{2}} \exp \left(\frac{\beta \mu}{2}\right) D_{1 - 2 \nu} \left(\sqrt{2\beta \mu}\right), \quad (3.7)$$

where $|\arg \beta| < \pi$, $\Re(\nu) \in \mathbb{R}_0$, $\Re(\mu) \geq 0$, and $D_r$ is the parabolic cylinder function, and considering that [7, Entry 9.240]

$$D_r(z) = 2^{\frac{r+1}{4}} W_{\frac{r+1}{4}} \left(-\frac{z^2}{2}\right),$$

we obtain

$$c_{\lambda,\rho,-} = \frac{1}{(2\lambda)^{\frac{7}{2}} \pi} \sum_{j=0}^{\infty} \left((-1)^j i \right)^\frac{7}{4} \Gamma \left(\frac{3}{4} + (-1)^j i \rho\right) W_{(-)j+1,\rho-\frac{1}{4}} \left((-1)^j i + 2i \lambda\right).$$

Using here (1.2) and considering that

$$\left|\Gamma \left(\frac{1}{4} \pm i \rho\right)\right| = \Gamma \left(\frac{1}{4} \pm i \rho\right) \exp \left(i \frac{\lambda}{4} \rho\right),$$

we have

$$c_{\lambda,\rho,-} = \frac{1^\frac{7}{4}}{(2\lambda)^{\frac{7}{2}} \pi} \left[i^\frac{7}{4} \Gamma \left(\frac{3}{4} + i \rho\right) H^+_\frac{3}{4}(\rho, \lambda) + (-i)^\frac{7}{4} \Gamma \left(\frac{3}{4} - i \rho\right) H^-_{\frac{3}{4}}(\rho, \lambda)\right].$$
Theorem 3. Let correctly. In view of (3.2),

\[
\lambda \left( \frac{1}{4} + i \rho \right) \Gamma \left( \frac{3}{4} - i \rho \right) = \frac{\pi \sqrt{2}}{\cosh(\pi \rho) + i \sinh(\pi \rho)},
\]

we obtain

\[
c_{\lambda, \rho, -} = \frac{1}{\lambda^\frac{1}{2} \cosh(2\pi \rho)} \left[ i^\frac{3}{2} \exp \left( i c_{-\frac{3}{4}}(\rho) \right) H_{-\frac{3}{4}}(\rho, \lambda) + (-i)^\frac{3}{2} \exp \left( -i c_{-\frac{3}{4}}(\rho) \right) H_{-\frac{3}{4}}(\rho, \lambda) \right].
\]

Therefore,

\[
c_{\lambda, \rho, -} = \frac{2}{\lambda^\frac{1}{2} \sqrt{\cosh(2\pi \rho)}} \Re \left( i^\frac{3}{2} \exp \left( i c_{-\frac{3}{4}}(\rho) \right) H_{-\frac{3}{4}}(\rho, \lambda) \right).
\]

\[\square\]

4. Integrals involving products of Coulomb and modified Bessel functions, converging to Legendre functions and related to expression of \( f_{\rho, \pm} \) with respect to basis \( B_1^* \)

Let \( \xi \in \mathbb{R}^3 \). It is clear that the function \( F_\xi(x) = (\xi_1 x_1 - \xi_2 x_2 - \xi_3 x_3)^\sigma \) belongs to \( \mathcal{D} \).

Theorem 3. Let

\[
|\xi_2| < \sqrt{\xi_1^2 - \xi_3^2}, \quad \xi_1 > \xi_3, \quad (4.1)
\]

and \(-1 < \sigma < 0\). Then

\[
\int_{\mathbb{R}_0} \lambda^{-\frac{1}{2}} K_{\sigma+\frac{1}{2}} \left( \lambda \sqrt{\xi_1^2 - \xi_2^2 - \xi_3^2} \right) \left[ \exp \left( \frac{i \xi_3 \lambda}{\xi_1 + \xi_2} \right) F_{-\sigma-1}(-\rho, -\lambda) + (-1)^\sigma \exp \left( -\frac{i \xi_3 \lambda}{\xi_1 + \xi_2} \right) F_{-\sigma-1}(-\rho, \lambda) \right] d\lambda = \frac{\pi}{2 (\xi_1 + \xi_2)^\sigma} \]

\[
\cdot \left( \frac{\xi_1 - \xi_3}{\xi_1 + \xi_3} \right)^{\sigma+\frac{1}{2}} \left( \frac{|\xi_1 + \xi_2|}{\sqrt{\xi_1^2 - \xi_2^2}} \right)^{\sigma+\frac{1}{2}} \exp \left( \frac{\pi \rho}{2} \right) \Gamma(-2\sigma) \]

\[
\cdot B(-\sigma - i \rho, -\sigma + i \rho, |\Gamma(-\sigma + i \rho)| \rho^\sigma + i \frac{1}{2} \left( \frac{|\xi_2|}{\sqrt{\xi_1^2 - \xi_3^2}} \right).
\]

Proof. Let us not that (4.1) excepts the case \( \xi = (\xi_1, -\xi_1, \xi_1) \), thus numbers \( \xi_1 + \xi_2, \xi_1 - \xi_3, \xi_1 + \xi_3 \) are not equal to zero and, in particular, the current theorem is formulated correctly. In view of (3.2),

\[
\mathcal{F}_1(f_{\rho, \pm}, F_\xi) = \mathcal{F}_2(f_{\rho, \pm}^*, F_\xi) = \int_{\gamma_2, +} f_{\rho, \pm}^* (x) F_\xi(x) \, d\gamma_2
\]

\[
= \int_{\mathbb{R}_0} \exp(i \rho \alpha_2) (\xi_1 \cosh \alpha_2 - \xi_3 \sinh \alpha_2 - \xi_2)^\sigma \, d\alpha_2
\]

\[
= 2^{-\sigma} \int_{\mathbb{R}_0} t^{-\sigma - 1 + i \rho} \left[ (\xi_1 - \xi_3) t^2 + 2 \xi_2 t + \xi_1 + \xi_3 \right]^\sigma \, dt,
\]
where the polynomial \((\xi_1 - \xi_3)t^2 + 2\xi_2t + \xi_1 + \xi_3\) doesn’t have real roots in view of (4.1).

In order to evaluate this integral, we use the formula \[12, \text{Entry 2.2.9.}(7)\]

\[
\int_{\mathbb{R}_0} \frac{x^{m-1} \, dx}{(ax^2 + 2bx + c)^\nu} = a^{-\frac{m}{2}} c^{-\nu} B(\mu, 2\nu - \mu) \, _2F_1 \left( \frac{\mu}{2}, \nu - \frac{\mu}{2}; \nu + \frac{1}{2}; 1 - \frac{b^2}{ac} \right), \tag{4.2}
\]

where \(a \in \mathbb{R}_0\), \(b^2 < ac\), \(0 < \Re(\mu) < 2\Re(\nu)\), and \(_2F_1\) is the Gaussian hypergeometric function. The condition (4.1) means that the argument of this function in (4.2) belongs to the interval \((0; 1)\), thus we use the formula \[13, \text{Entry 7.3.1.}(41)\]

\[2F_1(a, b; c; x) = 2^{a+b-x} x^{\frac{1-2a-2b}{4}} \Gamma(a + b + \frac{1}{2}) P_{\frac{1}{2} - a - b}^{\frac{1}{2} - \frac{1}{2}} (\sqrt{1-x}). \tag{4.3}\]

On the other hand, in view of (4.3),

\[F_i(f_j^{\bullet}, F_{\xi}) = \int_{\mathbb{R}} c_{\rho, +, \lambda} F_i(f_j^{\bullet}, F_{\xi}) \, d\lambda, \tag{4.4}\]

where

\[F_i(f_j^{\bullet}, F_{\xi}) = F_1(f_j^{\bullet}, F_{\xi}) = \left( \frac{\xi_1 + \xi_2}{2} \right)^\sigma \int_{\mathbb{R}} \exp(i\lambda \sigma) \left[ \alpha^2 - \frac{2\xi_3 \alpha_1}{\xi_1 + \xi_2} + \frac{\xi_1 - \xi_2}{\xi_1 + \xi_2} \right]^\sigma d\alpha. \]

In order to evaluate this integral, we use the formula \[11, \text{p. 202}\]

\[\int_{\mathbb{R}} [a^2 + (x + b)^2]^{-\nu} \exp(ixy) \, dx = 2\sqrt{\pi} \frac{\exp(\pm iby)}{\Gamma(\nu)} \left( \frac{|y|}{2a} \right)^{-\frac{\nu}{2}} K_{\nu - \frac{1}{2}}(a|y|), \tag{4.5}\]

where \(\Re(\nu) \in \mathbb{R}_0\).

Considering (4.3) and \(\Gamma(-\sigma) \Gamma\left( \frac{1}{2} - \sigma \right) = 2^{2\sigma + 1} \sqrt{\pi} \Gamma(-2\sigma)\) (in view of Legendre Duplication Formula), we complete the proof. \(\Box\)

Let us note that under condition of Theorem 3, the real parts of numbers \(\sigma \pm i\rho\) and \(\sigma + 1 \pm i\rho\) are positive. It means that function \(F_{\sigma - 1}\) in this theorem can be expressed via \(F_\sigma\) and \(G_\sigma\): Dziecol, Yngve and Froman derived in \[2\] the reflection formula (for complex \(\sigma, \rho\) and \(\lambda\))

\[F_{-\sigma - 1}(\rho, \lambda) = \cos \theta F_\sigma(\rho, \lambda) + \sin \theta G_\sigma(\rho, \lambda), \]

where

\[\theta = \left( \sigma + \frac{1}{2} \right) \pi + c_{-\sigma - 1}(\rho) - c_\sigma(\rho), \]

which holds for \(z, w \neq 0, -\pi < \arg z, \arg w < \pi, \) and \(\ln \Gamma(z), \ln \Gamma(w) \in \mathbb{R}\) for \(z, w \in \mathbb{R}_0\), where \(z = -\sigma \pm i\rho\) and \(w = \sigma + 1 \pm i\rho\). In this way, it is possible to represent \(c_{\lambda, \rho, -}\) in Theorem 2 as a linear combinations of functions \(F_{\frac{1}{2}}(\rho, \lambda)\) and \(G_{\frac{1}{2}}(\rho, \lambda)\).
Theorem 4. For
\[ |\xi_2| > \sqrt{\xi_1^2 - \xi_3^2} > 0, \quad \xi_1 > \xi_3, \]  
(4.5)
and \(-1 < \sigma < 0\),
\[
\int_{\mathbb{R}} \lambda^{-\frac{1}{2}} \exp \left( \frac{i \xi_3 \lambda}{\xi_1 + \xi_2} \right) F_{-\sigma-1}(-\rho, -\lambda) \\
\cdot \left[ \left(1 - \sec(\pi \sigma)\right) J_{\sigma + \frac{1}{2}}(2|k|\lambda) + \tan(\pi \sigma) J_{-\sigma - \frac{1}{2}}(2|k|\lambda) \right] d\lambda \\
= \frac{(-1)^{\sigma+1} (\xi_1 + \xi_2)^{\sigma} |\xi_1 + \xi_2|^{\sigma + \frac{1}{2}} (\xi_1 + |\xi_3|)^i \sin(\pi \sigma) \exp\left(\frac{3\pi \rho}{2}\right)}{2^{\sigma} \sqrt{\pi} \sqrt{\xi_2^2 + \xi_3^2 - \xi_1^2} (\xi_2^2 - \xi_3^2) \frac{\lambda}{\sigma} \Gamma(\sigma + 1) \Gamma(-\sigma)} \\
\cdot Q_{-\sigma-1}^i \left( -\frac{\xi_2}{\sqrt{\xi_2^2 + \xi_3^2 - \xi_1^2}} \right),
\]
where \( k = \sqrt{\frac{\xi_1^2 + \xi_2^2 - \xi_3^2}{\xi_1 + \xi_2}} \).

Proof. Let us note that [15] yields the inequality \(|\xi_1| \neq |\xi_3|\).

In view of (3.2),
\[
F_i(f_{\rho^+}, F_\xi) = F_2(f_{\rho^+}, F_\xi) = \int_{\gamma_2, \pm} f_{\rho^+}(x) F_\xi(x) d\gamma_2 \\
= \int_{\mathbb{R}} \exp(i\rho \alpha_2) (\xi_1 \cosh \alpha_2 - \xi_3 \sinh \alpha_2 + \xi_2)^\sigma d\alpha_2 \\
= (\xi_1^2 - \xi_3^2)^{\frac{-i \rho \alpha_2}{2}} (\xi_1 + |\xi_3|)^i \int_{\mathbb{R}} \exp(i\rho u) \left[ \cosh u - \frac{\xi_2}{\sqrt{\xi_1^2 - \xi_3^2}} \right]^\sigma du,
\]
where \( \frac{|\xi_2|}{\sqrt{\xi_1^2 - \xi_3^2}} \in \mathbb{R}_1 \) according to (4.5). Meaning here the last integral as its principal value and using formula [12] Entry 2.5.48(6)]
\[
\int_0^{\infty} \frac{\cos bx \, dx}{(a + \cosh cx)^\nu} = \exp\left(\frac{b \pi}{c} \right) \frac{\Gamma\left(\nu - \frac{1}{2}\right)}{\Gamma(\nu) \sqrt{\pi}} Q_{-\nu-1}^i \frac{a}{\sqrt{a^2 - 1}},
\]
which is valid for \( b, \Re(c \nu) \in \mathbb{R}_0 \) and \( a \notin [-1; 1] \), we have
\[
F_i(f_{\rho^+}, F_\xi) = 2 \exp(\rho \pi) (\xi_1^2 - \xi_3^2)^{\frac{-i \rho}{2}} (|\xi_1| + |\xi_3|)^i \rho \\
\cdot (\xi_2^2 - \xi_3^2 - \xi_1^2)^{\frac{\sigma}{2}} \frac{\Gamma(-\sigma - i \rho \sigma)}{\Gamma(-\sigma)} Q_{-\sigma-1}^i \left( \frac{\xi_2}{\sqrt{\xi_2^2 + \xi_3^2 - \xi_1^2}} \right). 
\]
(4.6)

On the other hand, (3.3) yields (4.3), where the polynomial \( \alpha_1^2 - \frac{2\xi_3 \alpha_1 + \xi_2}{\xi_1 + \xi_2} \) has two different real roots \( \frac{\xi_4}{\xi_1 + \xi_2} \pm \pm k \). Using the substitution \( \alpha_1 = t + \frac{\xi_4}{\xi_1 + \xi_2} \), we have
\[
F_i(f_{\rho^+}, F_\xi) = F_1(f_{\rho^+}, F_\xi) = \left( \frac{\xi_1 + \xi_2}{2} \right)^\sigma \exp\left(\frac{i \xi_3 \lambda}{\xi_1 + \xi_2} \right) \int_{\mathbb{R}} \exp(i\lambda t) (t^2 - 4k^2)^\sigma dt,
\]
where (according to [12, Entry 2.3.5.3])
\[
\int_{-2|k|}^{2|k|} \exp(i\lambda t)(4k^2 - t^2)^\sigma \, dt = \left(\frac{4|k|}{\lambda}\right)^{\sigma + \frac{1}{2}} \sqrt{\pi} \Gamma(\sigma + 1) J_{\sigma + \frac{1}{2}}(2\lambda|k|)
\]
and (according to [12, Entry 2.3.5.5])
\[
\sum_{j=0}^{1} \int_{\mathbb{R}_{2|k|}} \exp((-1)^j i\lambda t)(t^2 - 4k^2)^\sigma \, dt = 4^\sigma \sqrt{\pi} i \left(\frac{\lambda}{|k|}\right)^{-\sigma - \frac{1}{2}} \Gamma(\sigma + 1) \left(H_{-\sigma - \frac{1}{2}}^{(1)}(2\lambda|k|) - H_{-\sigma - \frac{1}{2}}^{(2)}(2\lambda|k|)\right),
\]
where \(H_{-\sigma - \frac{1}{2}}^{(1)}\) and \(H_{-\sigma - \frac{1}{2}}^{(2)}\) are Hankel functions of the first and second kind, respectively.

To finish the proof we use the identity
\[
H_{-\sigma - \frac{1}{2}}^{(1)}(2\lambda|k|) - H_{-\sigma - \frac{1}{2}}^{(2)}(2\lambda|k|) = 2i \left(\sec(\pi\sigma) J_{\sigma + \frac{1}{2}}(2\lambda|k|) - \tanh(\pi\sigma) J_{-\sigma - \frac{1}{2}}(2\lambda|k|)\right).
\]

5. Integrals involving products of Coulomb and Legendre functions, converging to modified Bessel functions and related to expression of \(f^*_\lambda\) with respect to basis \(B^*_2\)

The results obtained in Theorems 3 and 4 may be characterised as formulae, on the one hand, for exponential Fourier and, for the second hand, for Mellin transform of Coulomb functions. In addition, Theorem 3 is being formula for K-transform and Theorem 4 is being formula for sum of Hankel transforms \([4]\) of Coulomb functions. All these formulas have been derived from the expression of the function belonging to 'hyperbolic' basis with respect to 'parabolic' basis. Choosing the opposite direction, in this section we derive one formula for generalized index Mehler–Fock transform \([18]\).

**Theorem 5.** For \((4.1)\) and \(\lambda \in \mathbb{R}_0\),
\[
\int_\mathbb{R} \left(\frac{\lambda}{\pi}\right)^{\frac{3}{4}} \left(\frac{3}{4} + i\rho\right) \exp\left(\frac{\pi\rho}{2}\right) F_{\lambda \rho}(\rho, \lambda) + \frac{2}{\sqrt{\cosh(2\pi\rho)}} \left(AG_{\lambda \rho}(\rho, \lambda) - BF_{\lambda \rho}(\rho, \lambda)\right) \left(\xi_1 - \xi_3\right)^{-\frac{1}{8} - i\rho} \left(\xi_1 + \xi_3\right)^{-\frac{1}{8} + i\rho} B \left(\frac{1}{4} + i\rho, \frac{1}{4} - i\rho\right) P_{\frac{3}{4}}^{1/4} \left(\frac{|\xi_1|}{\sqrt{\xi_1^2 - \xi_3^2}}\right) \, d\rho
\]
\[
= 2^3 \left(\frac{\lambda}{\pi}\right)^{\frac{1}{4}} (\xi_1^2 - \xi_3^2)^{\frac{1}{8}} K_{\frac{3}{4}} \left(\frac{\lambda \sqrt{\xi_1^2 - \xi_3^2}}{|\xi_1 + \xi_2|}\right).
\]

**Proof.** From \((3.1)\) we have
\[
F_i(f_{\lambda}, F_{\xi}) = \int_\mathbb{R} \left[c_{\lambda, \rho, +}^* F_j(f_{\rho, +}, F_{\xi}) + c_{\lambda, \rho, -}^* F_k(f_{\rho, -}, F_{\xi})\right] \, d\rho,
\]
where \( i, j, k \in \{1, 2\} \). Choosing here \( i = 1 \) and \( j = k = 2 \), we calculate \( F_1(f_1^*, F_2) \) and \( F_2(f_{p, \pm}, F_\xi) \) according to formulas (4.4) and (4.2), respectively, and use Theorems 1 and 2.

By using the formula (see, for example, \([13]\))

\[
P_\nu^\mu(z) = \sqrt{\frac{2}{\pi}} \frac{\exp(i\nu\pi)}{(\nu^2 - 1)^{\frac{1}{4}}} Q^{-\frac{1}{2}} - \frac{1}{2} \left( \frac{z}{\sqrt{z^2 - 1}} \right),
\]

which is valid for \( \Re(z) \in \mathbb{R}_0 \), it is possible to obtain the result, which is similar to Theorem 5, for (4.5).

6. The relationship with the Poisson transform in \( \mathfrak{D}^* \)

Let us note that the integrals \( F_i(f_1^*, F_\xi) \) and \( F_i(f_{p, \pm}, F_\xi) \), which do not depend on the choice of integration contour \( \gamma_i \), are the particular cases of the so-called Poisson transform \([16, \text{Section 10.3.1}]\)

\[
\mathcal{P}[f](y) = \int_{\gamma_i} f(x) (x_1 y_1 - x_2 y_2 - x_3 y_3)^\sigma \, d\gamma_i,
\]

where \( f \in \mathfrak{D}^* \) and the point \( y \) belongs to the hyperboloid \( \Upsilon : y_1^2 - y_2^2 - y_3^2 = 1 \). The image of this integral transform consists of \( \sigma \)-homogeneous functions which are defined on \( \Upsilon \) and belong to the kernel of the following \('(1, 2)\)-Laplace operator':

\[
\Box = \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_3^2}.
\]

The Poisson transform intertwines the representation \( T^* \) and the representation defined by the shifts \( \mathcal{P}[f](y) \rightarrow \mathcal{P}[f](y^{-1} y) \). Thus, computing \( F_i(f_1^*, F_\xi) \) and \( F_i(f_{p, \pm}, F_\xi) \) for the case

\[
\xi = y = \left( \cosh \alpha_3, \sinh \alpha_3 \cos \beta_3, \sinh \alpha_3 \sin \beta_3 \right),
\]

where \( \alpha_3 \in \mathbb{R} \) and \( \beta_3 \in (-\pi, \pi) \), which satisfies the condition \([4.1]\), we obtain the values of the Poisson transform with the kernel \( F_y \) of the basis functions \( f_\lambda^* \) and \( f_{p, \pm}^* \). For example, we have

\[
\mathcal{P}[f_\lambda^*](y) = \frac{2\pi}{\cosh \alpha_3 + \sinh \alpha_3 \cos \beta_3} \left[ \lambda^{\sigma - \frac{1}{2}} \Gamma(-\sigma) \right] \exp \left( \frac{i\lambda \sinh \alpha_3 \sin \beta_3}{\cosh \alpha_3 + \sinh \alpha_3 \cos \beta_3} \right) K_{\sigma + \frac{1}{2}} \left( \frac{\lambda}{\cosh \alpha_3 + \sinh \alpha_3 \cos \beta_3} \right).
\]

The function \( F_y \) as a function \( F_y(y) \) defined on \( \Upsilon \) is also being the kernel of the so-called Gelfand–Graev integral transform. The applications of this transform to generalizations of Funk–Hecke theorem had been considered in \([16]\) and \([15]\).
We note also that choosing $\beta = 0$ in (6.1), we obtain from Theorem 3 the following integral representation of Legendre function:

$$
P_{\sigma - \frac{1}{2}}^{\frac{1}{2}}(\vert \tanh \alpha_3 \vert) = \frac{2 \exp \left( -\frac{\alpha_3^2 \mu}{2} \right)}{\pi \Gamma(-2\sigma) B(-\sigma - i\rho, -\sigma + i\rho) |\Gamma(-\sigma + i\rho)|} \int_{\mathbb{R}_0} \frac{F_{-\sigma-1}(-\rho, \lambda) + (-1)^\sigma F_{-\sigma-1}(-\rho, \lambda)}{\sqrt{\lambda}} K_{\sigma+1} \left( \frac{\lambda}{\exp \alpha_3} \right) d\lambda.
$$

A more general representation for $P_{\sigma - \frac{1}{2}}^{\frac{1}{2}}(\vert \tanh \alpha_3 \vert)$ can be obtained by using the following parametrization of $\Upsilon$:

$$
y = (\cosh \alpha_3 \cosh \beta_3, \sinh \alpha_3, \cosh \alpha_3 \sinh \beta_3).
$$

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