NON-AUTONOMOUS ORNSTEIN-UHLENBECK EQUATIONS IN EXTERIOR DOMAINS

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Abstract. In this paper, we consider non-autonomous Ornstein-Uhlenbeck operators in smooth exterior domains \( \Omega \subset \mathbb{R}^d \) subject to Dirichlet boundary conditions. Under suitable assumptions on the coefficients, the solution of the corresponding non-autonomous parabolic Cauchy problem is governed by an evolution system \( \{ P_{\Omega}(t, s) \}_{0 \leq s \leq t} \) on \( L^p(\Omega) \) for \( 1 < p < \infty \). Furthermore, \( L^p \)-estimates for spatial derivatives and \( L^p-L^q \) smoothing properties of \( P_{\Omega}(t, s) \), \( 0 \leq s \leq t \), are obtained.

1. Introduction

In recent years, parabolic equations with unbounded and time-independent coefficients were investigated intensively in various function spaces over the whole space \( \mathbb{R}^d \) or exterior domains; we refer e.g. to [6,8,9,13,15] and the monograph [5]. However, it is also interesting to consider parabolic equations with unbounded coefficients in the non-autonomous case. In particular, analytically there is a great interest in the prototype situation of time-dependent Ornstein-Uhlenbeck operators in exterior domains, as operators of this type arise e.g. in the study of the Navier-Stokes flow in the exterior of a rotating obstacle; see e.g. [12,16].

Therefore, in this paper we consider non-autonomous Cauchy problems with Dirichlet boundary condition of the type

\[
\begin{aligned}
    u_t(t, x) - L_{\Omega}(t)u(t, x) &= 0, \quad t \in (s, \infty), \ x \in \Omega, \\
    u(t, x) &= 0, \quad t \in (s, \infty), \ x \in \partial \Omega, \\
    u(s, x) &= f(x), \ x \in \Omega,
\end{aligned}
\]  

(1.1)

where \( s \geq 0 \) is fixed, \( \Omega \subset \mathbb{R}^d \) is a domain and \( \{ L_{\Omega}(t) \}_{t \geq 0} \) is a family of time-dependent Ornstein-Uhlenbeck operators formally defined by

\[
L_{\Omega}(t)\varphi(x) = \frac{1}{2} \text{Tr} (Q(t)Q^*(t)D_x^2\varphi(x)) + \langle M(t)x + c(t), D_x\varphi(x) \rangle, \quad x \in \Omega, \quad t \geq 0.
\]  

(1.2)

Throughout the paper we assume that \( Q, M \in C_\text{loc}^\alpha(\mathbb{R}_+, \mathbb{R}^{d \times d}), c \in C_\text{loc}^\alpha(\mathbb{R}_+, \mathbb{R}^d) \) for some \( \alpha \in (0, 1) \) and there is \( \mu > 0 \) such that

\[
|Q(t)x| \geq \mu|x|, \quad t \geq 0, \ x \in \mathbb{R}^d.
\]

The above assumption guarantees that the operators \( L_{\Omega}(t) \) are uniformly elliptic.

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The main purpose of this paper is to consider problem (1.1) in the $L^p$-setting for the case of smooth exterior domains $\Omega$. However, in the course of this paper we also consider the situation where $\Omega$ is $\mathbb{R}^d$ and a smooth bounded domain.

In the following the $L^p$-realization of $L_\Omega(t)$ will be denoted by $L_\Omega(t)$ with an appropriate domain $\mathcal{D}(L_\Omega(t)) \subset L^p(\Omega)$, specified later. Then we can rewrite equation (1.1) as an abstract non-autonomous Cauchy problem
\[(nACP) \quad \begin{cases} u'(t) = L_\Omega(t)u(t), & 0 \leq s < t, \\ u(s) = f, \end{cases} \tag{1.3}\]
where $f \in L^p(\Omega)$.

**Definition 1.1.** A continuous function $u : [s, \infty) \to L^p(\Omega)$ is called a (classical) solution of (nACP) if $u \in C^1([s, \infty), L^p(\Omega))$, $u(s) = f$, and $u'(t) = L_\Omega(t)u(t)$ for $0 \leq s < t$.

**Definition 1.2** (Well-posedness). We say that the Cauchy problem (nACP) is well-posed (on regularity spaces $\{Y_s\}_{s \geq 0}$) if the following statements are true.

(i) **(Existence and uniqueness)** There are dense subspaces $Y_s \subset \mathcal{D}(L_\Omega(s))$ of $L^p(\Omega)$ such that for $f \in Y_s$ there is a unique solution $t \mapsto u(t; s, f) \in Y_t$ of (nACP).

(ii) **(Continuous dependence)** The solution depends continuously on the data; i.e., for $s_n \to s$ and $Y_{s_n} \ni f_n \to f \in Y_s$, we have $\tilde{u}(t; s_n, f_n) \to \tilde{u}(t; s, f)$ uniformly for $t$ in compact subsets of $[0, \infty)$, where we set $\tilde{u}(t; s, f) := u(t; s, f)$ for $t \geq s$ and $\tilde{u}(t; s, f) := f$ for $t < s$.

In order to discuss well-posedness of (nACP) we introduce the concept of strongly continuous evolution systems.

**Definition system 1.3** (Evolution system). A two parameter family of linear, bounded operators $\{P_\Omega(t, s)\}_{0 \leq s \leq t}$ on $L^p(\Omega)$ is called a (strongly continuous) evolution system if

(i) $P_\Omega(s, s) = \text{Id}$ and $P_\Omega(t, s) = P_\Omega(t, r)P_\Omega(r, s)$ for $0 \leq s \leq r \leq t$,

(ii) for each $f \in L^p(\Omega)$, $(t, s) \mapsto P_\Omega(t, s)f$ is continuous on $0 \leq s \leq t$.

We say $\{P_\Omega(t, s)\}_{0 \leq s \leq t}$ solves the Cauchy problem (nACP) (on spaces $\{Y_s\}_{s \geq 0}$) if there are dense subspaces $Y_s$ of $L^p(\Omega)$ such that $P_\Omega(t, s)Y_s \subset Y_t \subset \mathcal{D}(L_\Omega(t))$ for $0 \leq s \leq t$ and the function $u(t) := P_\Omega(t, s)f$ is a solution of (nACP) for $f \in Y_s$.

It is well-known that the Cauchy problem (nACP) is well-posed on $\{Y_s\}_{s \geq 0}$ if and only if there is an evolution system solving (nACP) on $\{Y_s\}_{s \geq 0}$ (see e.g. [20, Sect. 3.2]).

The main result of this paper (see Theorem 3.1) is to show that for smooth exterior domains $\Omega \subset \mathbb{R}^d$ problem (nACP) is solved by a strongly continuous evolution system $\{P_\Omega(t, s)\}_{0 \leq s \leq t}$ on $L^p(\Omega)$ and thus, is well-posed. Since in unbounded domains the operators $L_\Omega(t)$ have unbounded drift coefficients, the present situation does not fit into the well-studied framework of evolution systems of parabolic type (see e.g. the monograph by Lunardi [17, Chapter 6] or the fundamental papers by Tanabe [22, 24] and Acquistapace, Terreni [1, 3]). Therefore the well-posedness of (nACP) and regularity properties of the solution do not follow from abstract arguments. Here lies the major difficulty. In order to prove our result we proceed as follows: In Section 2 we consider (nACP) in the case
that $\Omega$ is the whole space $\mathbb{R}^d$ or a smooth bounded domain. For the whole space case we use a representation formula for the evolution system as done in [10]. In the case of bounded domains we can apply the standard results for non-autonomous Cauchy problems of parabolic type. These auxiliary results are then applied in Section 3 to construct an evolution system $\{P_{\Omega}(t, s)\}_{0 \leq s \leq t}$ on $L^p(\Omega)$ for smooth exterior domains $\Omega \subset \mathbb{R}^d$, by some cut-off techniques. Moreover, our method allows us to prove $L^p$-$L^q$ estimates and estimates for spatial derivatives of $\{P_{\Omega}(t, s)\}_{0 \leq s \leq t}$.

**Notations.** The euclidian norm of $x \in \mathbb{R}^d$ will be denoted by $|x|$. By $B(R)$ we denote the open ball in $\mathbb{R}^d$ with centre at the origin and radius $R$. For $T > 0$ we use the notations:

$$\Lambda_T := \{(t, s) : 0 \leq s \leq t \leq T\}$$
$$\tilde{\Lambda}_T := \{(t, s) : 0 \leq s < t \leq T\}$$
$$\Lambda := \{(t, s) : 0 \leq s \leq t\}$$
$$\tilde{\Lambda} := \{(t, s) : 0 \leq s < t\}.$$ 

If $u : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^d$ is a domain, we use the following notation:

$$D_i u = \frac{\partial u}{\partial x_i}, \quad D_{ij} u = D_i D_j u, \quad D_x u = (D_1 u, \ldots, D_d u), \quad D_2 u = (D_{ij} u).$$ 

Let us come to notation for function spaces. For $1 \leq p < \infty$, $j \in \mathbb{N}$, $W^{j,p}(\Omega)$ denotes the classical Sobolev space of all $L^p(\Omega)$-functions having weak derivatives in $L^p(\Omega)$ up to the order $j$. Its usual norm is denoted by $\| \cdot \|_{j,p}$ and by $\| \cdot \|_p$ when $j = 0$. By $W_0^{1,p}(\mathbb{R}^d)$ we denote the closure of the space of test functions $C_c^\infty(\mathbb{R}^d)$ with respect to the norm of $W^{1,p}(\mathbb{R}^d)$. For $0 < \alpha < 1$ we denote by $C_0^{\alpha}(\mathbb{R}_+, \mathbb{R}^{d \times d})$ the space of all $\alpha$-Hölder continuous functions in $[0, T]$ for all $T > 0$. The space of all bounded continuous functions $u : \Omega \to \mathbb{R}$ is denoted by $C_b(\Omega)$. For $k \in \mathbb{N}$, $C_b^k(\Omega)$ is the subspace of $C_b(\Omega)$ consisting of all functions which are differentiable up to the order $k$ in $\Omega$ such that the derivatives are bounded.

Finally, we denote by $C^{1,2}(I \times \Omega)$ the space of all functions $u : I \times \Omega \to \mathbb{R}$ which are continuously differentiable with respect to $t \in I$ and $C^2$ with respect to the space variable $x \in \Omega$, where $I \subseteq [0, \infty)$ is an interval.

### 2. Auxiliary results: whole space and bounded domains

In this section we prove some auxiliary results concerning the evolution systems in the case of the whole space $\mathbb{R}^d$ and smooth bounded domains. These results are needed in Section 3 for the construction of the evolution system in the case of exterior domains.

#### 2.1. The evolution system in the whole space.

The realizations of $\{L_{\mathbb{R}^d}(t)\}_{t \geq 0}$ are defined by

$$\mathcal{D}(L_{\mathbb{R}^d}(t)) := \{u \in W^{2,p}(\mathbb{R}^d) : (M(t) x, D_x u(x)) \in L^p(\mathbb{R}^d)\},$$

$$L_{\mathbb{R}^d}(t) u := L_{\mathbb{R}^d}(t) u. \quad (2.1)$$
Here the domain of $L_{\Omega}(t)$ depends on the time parameter $t$. However, note that the subspace
\[
Y_{\mathbb{R}^d} := \{ u \in W^{2,p}(\mathbb{R}^d) : |x| \cdot D_j u(x) \in L^p(\mathbb{R}^d) \text{ for all } j = 1, \ldots, d \}
\]
is contained in $\mathcal{D}(L_{\Omega}(t))$ for all $t \geq 0$ and is dense in $L^p(\mathbb{R}^d)$. The space $Y_{\mathbb{R}^d}$ will serve as a regularity space in order to discuss well-posedness of (nACP).

It follows directly from [19] (see also [18]) that in the autonomous case (i.e. for fixed $s \geq 0$) the operator $(L_{\mathbb{R}^d}(s), \mathcal{D}(L_{\mathbb{R}^d}(s)))$ generates a strongly continuous semigroup, which is however not analytic. Second order elliptic operators in $\mathbb{R}^d$ with more general unbounded and time-independent coefficients were considered e.g. in [21, 14].

In the following we denote by $\{U(t,s)\}_{t,s \geq 0}$ the evolution system in $\mathbb{R}^d$ that satisfies
\[
\begin{cases}
\frac{\partial}{\partial t} U(t,s) = -M(t)U(t,s), \\
U(s,s) = \text{Id}.
\end{cases}
\]
The existence of $\{U(t,s)\}_{t,s \geq 0}$ follows directly from the Picard-Lindelöf theorem. Now for $f \in L^p(\mathbb{R}^d)$ and $s \geq 0$ we set $P_{\mathbb{R}^d}(s,s) = \text{Id}$ and for $(t,s) \in \tilde{\Lambda}$ we define
\[
P_{\mathbb{R}^d}(t,s)f(x) = (k(t,s,\cdot) \ast f)(U(t,s)x + g(t,s)), \quad x \in \mathbb{R}^d,
\]
where
\[
k(t,s,x) := \frac{1}{(2\pi)^{d/2}(\det Q_{t,s})^{\frac{d}{2}}} e^{-\frac{1}{2} \langle Q_{t,s}^{-1}x,x \rangle}, \quad x \in \mathbb{R}^d,
\]
\[
g(t,s) = \int_s^t U(s,r)c(r)dr \quad \text{and} \quad Q_{t,s} = \int_s^t U(s,r)Q(r)Q^*(r)U^*(s,r)dr.
\]
As in [7 Proposition 2.1] (see also [12 Proposition 2.1]) it can be shown that for initial value $f \in C_0^\infty(\mathbb{R}^d)$, the function $u(t,x) := P_{\mathbb{R}^d}(t,s)f(x)$ is a classical solution to
\[
\begin{cases}
\partial_t u(t,x) - L_{\mathbb{R}^d}(t)u(t,x) = 0, \quad (t,s) \in \tilde{\Lambda}, \ x \in \mathbb{R}^d, \\
u(s,x) = f(x), \quad x \in \mathbb{R}^d,
\end{cases}
\]
and $u \in C^{1,2}((s,\infty) \times \Omega)$ and $u$ solves (2.7). Further, the two parameter family of operators $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s) \in \Lambda}$ is a strongly continuous evolution system on $L^p(\mathbb{R}^d)$.

**Proposition 2.1.** Let $1 < p < \infty$. Then the family of operators $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s) \in \Lambda}$ defined in (2.2) is a strongly continuous evolution system on $L^p(\mathbb{R}^d)$ with the following properties.

(a) For $(t,s) \in \Lambda$, the operator $P_{\mathbb{R}^d}(t,s)$ maps $Y_{\mathbb{R}^d}$ into $Y_{\mathbb{R}^d}$.

(b) For every $f \in Y_{\mathbb{R}^d}$ and every $s \in [0, \infty)$, the map $t \mapsto P_{\mathbb{R}^d}(t,s)f$ is differentiable in $(s,\infty)$ and
\[
\frac{\partial}{\partial t} P_{\mathbb{R}^d}(t,s)f = L_{\mathbb{R}^d}(t)P_{\mathbb{R}^d}(t,s)f.
\]

(c) For every $f \in Y_{\mathbb{R}^d}$ and $t \in (0,\infty)$, the map $s \mapsto P_{\mathbb{R}^d}(t,s)f$ is differentiable in $[0,t)$ and
\[
\frac{\partial}{\partial s} P_{\mathbb{R}^d}(t,s)f = -P_{\mathbb{R}^d}(t,s)L_{\mathbb{R}^d}(s)f.
\]
Definition 1.3) holds for every \( f \in \mathcal{C}_c^\infty(\mathbb{R}^d) \). Since \( \mathcal{C}_c^\infty(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d) \) the law of evolution holds even for all \( f \in L^p(\mathbb{R}^d) \). The strong continuity of the map \( \Lambda \ni (t, s) \mapsto P_{\mathbb{R}^d}(t, s) \) can be shown as in \([12\text{, Proposition 2.3}]\). Equalities \((2.6)\) and \((2.7)\) follow by differentiating the kernel \( k(t, s, x) \) with respect to \( t \) and \( s \), respectively.

Let us now show that the evolution system \( \{P_{\mathbb{R}^d}(t, s)\}_{(t, s)\in \Lambda} \) leaves the regularity space \( Y_{\mathbb{R}^d} \) invariant. Since \( k(t, s, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^d) \) it follows that \( P_{\mathbb{R}^d}(t, s) f \in \mathcal{C}^\infty(\mathbb{R}^d) \) for all \( f \in L^p(\mathbb{R}^d) \) and \((t, s)\in \Lambda\). Moreover, we note that

\[
D_x P_{\mathbb{R}^d}(t, s) f = U^*(s, t) (k(t, s, \cdot) * D_x f) (U(s, t) x + g(t, s))
\]

holds for all \( f \in W^{1,p}(\mathbb{R}^d) \). Thus, it suffices to show that for all \( j = 1, \ldots, d \) we have

\[
|D_j f| (k(t, s, \cdot) * D_j f)(x) \in L^p(\mathbb{R}^d).
\]

So let \( h \in L^q(\mathbb{R}^d) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Then we obtain

\[
\int_{\mathbb{R}^d} \left| (|x| \cdot (k(t, s, \cdot) * D_j f)(x)) h(x) \right| \, dx
\]

\[
\leq C \int_{\mathbb{R}^d} |x| |h(x)| \int_{\mathbb{R}^d} |D_j f(x - y)e^{-\frac{i}{2}Q^{-1}_{\xi}(y, y)}| \, dy \, dx
\]

\[
\leq C \left[ \int_{\mathbb{R}^d} e^{-\frac{i}{2}Q^{-1}_{\xi}(y, y)} \int_{\mathbb{R}^d} \left| (|x| \cdot D_j f(x - y)) h(x) \right| \, dx \, dy + \int_{\mathbb{R}^d} |y| e^{-\frac{i}{2}Q^{-1}_{\xi}(y, y)} \int_{\mathbb{R}^d} |D_j f(x - y)| |h(x)| \, dx \, dy \right]
\]

\[
\leq C \left[ \||x| D_j f\|_p \|h\|_q + \|D_j f\|_p \|h\|_q \right].
\]

Here the constant \( C \) may change from line to line. Thus

\[
\int_{\mathbb{R}^d} \left| (|x| \cdot (k(t, s, \cdot) * D_j f)(x)) h(x) \right| \, dx < \infty
\]

holds for all \( h \in L^q(\mathbb{R}^d) \) and this proves the assertion. \(\square\)

As a consequence of Proposition \([2.1]\) Cauchy problem (nACP) is well-posed in the case of \( \mathbb{R}^d \) with regularity space \( Y_{\mathbb{R}^d} \). Now we prove \( L^p - L^q \) estimates and estimates for higher order spatial derivatives of \( \{P_{\mathbb{R}^d}(t, s)\}_{(t, s)\in \Lambda} \). For this purpose we need the following estimates for the matrices \( Q_{t, s} \). For a proof we refer to \([10\text{, Lemma 3.2}]\) and \([12\text{, Lemma 2.4}]\).

**Lemma 2.2.** Let \( T > 0 \). Then there exists a constant \( C := C(T) > 0 \) such that

\[
\|Q_{t, s} \| \leq C(t - s)^{-\frac{1}{q}}, \quad (t, s) \in \bar{\Lambda}_T,
\]

\[
(\det Q_{t, s})^{\frac{1}{q}} \geq C(t - s)^{-\frac{1}{q}}, \quad (t, s) \in \Lambda_T. \tag{2.8}
\]

**Proposition 2.3.** Let \( T > 0 \), \( 1 < p \leq q < \infty \) and \( \beta \in \mathbb{N}^d_0 \) be a multi-index. Then there exists a constant \( C := C(T) > 0 \) such that for every \( f \in L^p(\mathbb{R}^d) \)

\[
(a) \quad \|P_{\mathbb{R}^d}(t, s) f\|_q \leq C(t - s)^{-\frac{\beta}{q} \left( \frac{1}{p} - \frac{1}{q} \right)} \|f\|_p, \quad (t, s) \in \bar{\Lambda}_T,
\]

\[
(b) \quad \|D_x^\beta P_{\mathbb{R}^d}(t, s) f\|_p \leq C(t - s)^{-\frac{\beta}{p}} \|f\|_p, \quad (t, s) \in \bar{\Lambda}_T.
\]
Moreover,
\[ \| P_{\mathbb{R}^d}(t, s)f \|_{k,p} \leq C \| f \|_{k,p}, \quad (t, s) \in \Lambda_T, \]
for all \( f \in W^{k,p}(\mathbb{R}^d) \), \( k = 1, 2, \) and
\[ \| P_{\mathbb{R}^d}(t, s)f \|_{2,p} \leq C(t - s)^{-\frac{1}{2}} \| f \|_{1,p}, \quad (t, s) \in \tilde{\Lambda}_T, \]
for all \( f \in W^{1,p}(\mathbb{R}^d) \).

**Proof.** Let \( T > 0 \). By the change of variables \( \xi = U(s, t)x \) and by Young’s inequality we obtain
\[ \| P_{\mathbb{R}^d}(t, s)f \|_q \leq | \det U(s, t) |^{\frac{1}{2}} \| k(t, s, \cdot) \|_r \| f \|_p, \]
where \( 1 < r < \infty \) with \( \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q} \). Moreover, by the change of variables \( y = Q_{t,s}^{1/2}z \) we obtain
\[ \| k(t, s, \cdot) \|_r = \frac{(\det Q_{t,s})^{\frac{1}{2}(1-r)}}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}} \, dz \leq C(\det Q_{t,s})^{\frac{1}{2}(1-r)}. \]
Now Lemma 2.2 yields (a).

To prove (b) we first note that
\[ | D_x^2 P_{\mathbb{R}^d}(t, s)f(x) | \leq | U^*(s, t)|^{[\beta]} |(D_x^2 k(t, s, \cdot) * f)(U(s, t)x + g(t, s)) | \]
holds. Thus, we have to estimate the norm of \( D_x^2 k(t, s, \cdot) \). Since
\[ D_x k(t, s, x) = -k(t, s, x) \left( Q_{t,s}^{-1} x \right)^* \]
holds, we obtain by differentiating further
\[ | D_x^2 k(t, s, x) | \leq C k(t, s, x) | Q_{t,s}^{-1} x |^{[\beta]} \]
for some constant \( C > 0 \). As above, by the change of variables \( y = Q_{t,s}^{1/2}z \), we obtain
\[ \| D_x^2 k(t, s, \cdot) \|_1 \leq \frac{\| Q_{t,s}^{-1} \|^{[\beta]}}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}^d} |z|^{[\beta]} e^{-\frac{|z|^2}{2}} \, dz \leq C \| Q_{t,s}^{-\frac{1}{2}} \|^{[\beta]}. \]
Now Lemma 2.2 yields assertion (b). The last assertions follow by a direct computation. \( \square \)

**Remark 2.4.** If \( \{ U(t, s) \}_{t,s \geq 0} \) is uniformly bounded, i.e. \( \| U(t, s) \| \leq M \) for some constant \( M > 0 \) and all \( t, s \geq 0 \), then the estimates in Lemma 2.2 and Proposition 2.3 hold in \( \Lambda \) and \( \tilde{\Lambda} \) respectively. In particular, in this case the evolution system \( \{ P(t, s) \}_{(t,s) \in \Lambda} \) is uniformly bounded.

2.2. **The evolution system in bounded domains.** In this subsection we assume that \( D \subset \mathbb{R}^d \) is a bounded domain with \( C^{1,1} \)-boundary. For \( t \geq 0 \) we set
\[ \mathcal{D}(L_D(t)) := \mathcal{D}(L_D) := W^{2,p}(D) \cap W_0^{1,p}(D), \]
\[ L_D(t)u := L_D(t)u. \quad (2.9) \]
Note that in this situation the domain is independent of the time parameter \( t \), i.e. all the operators \( L_D(t) \) are defined on the same domain \( \mathcal{D}(L_D) \).
Lemma 2.5. Let $D \subset \mathbb{R}^d$ be a bounded domain with $C^{1,1}$-boundary and $1 < p < \infty$.

(a) For fixed $s \in [0, \infty)$, the operator $(L_D(s), \mathcal{D}(L_D))$ generates an analytic semigroup on $L^p(D)$.

(b) The map $t \mapsto L_D(t)$ belongs to $C_\text{loc}^0(\mathbb{R}_+, \mathcal{L}(\mathcal{D}(L_D), L^p(D)))$.

Proof. Assertion (a) follows from the classical theory of elliptic second order operators in bounded domains (see also [17, Lemma 2.4]). Assertion (b) follows from the assumptions on the coefficients of $L_D(\cdot)$.

The following proposition now follows directly from the theory of evolution systems of parabolic type; see [17, Chapter 6] and [11, Sect. 2.3]. See also [4, Sect. 7] for bounded domains of class $C^2$.

Proposition 2.6. Let $D \subset \mathbb{R}^d$ be a bounded domain with $C^{1,1}$-boundary and $1 < p < \infty$. Then there is a unique evolution system $\{P_D(t,s)\}_{(t,s) \in \Lambda}$ on $L^p(D)$ with the following properties.

(a) For $(t,s) \in \tilde{\Lambda}$, the operator $P_D(t,s)$ maps $L^p(D)$ into $\mathcal{D}(L_D)$.

(b) The map $t \mapsto P_D(t,s)$ is differentiable in $(s, \infty)$ with values in $\mathcal{L}(L^p(D))$ and

$$\frac{\partial}{\partial t} P_D(t,s) = L_D(t) P_D(t,s). \quad (2.10)$$

(c) For every $f \in \mathcal{D}(L_D)$ and $t \in (0, \infty)$, the map $s \mapsto P_D(t,s)f$ is differentiable in $[0, t)$ and

$$\frac{\partial}{\partial s} P_D(t,s)f = -P_D(t,s)L_D(s)f. \quad (2.11)$$

(d) Let $T > 0$. Then there exists a constant $C := C(T) > 0$ such that

$$\| P_D(t,s)f \|_p \leq C \| f \|_p, \quad (2.12)$$

and

$$\| P_D(t,s)f \|_{2p} \leq C(t-s)^{-1} \| f \|_p. \quad (2.13)$$

for all $f \in L^p(D)$ and all $(t,s) \in \tilde{\Lambda}_T$.

The following estimates follow directly from the proposition above and simple interpolation.

Corollary 2.7. Let $T > 0$, $1 < p < \infty$ and $p \leq q < \infty$. Then there exists a constant $C := C(T) > 0$ such that for every $f \in L^p(D)$

(a) $\| P_D(t,s)f \|_q \leq C(t-s)^{-\frac{q}{2} \left(\frac{1}{p} - \frac{1}{2}\right)} \| f \|_p$, \quad $(t,s) \in \tilde{\Lambda}_T$,

(b) $\| \partial_s P_D(t,s)f \|_p \leq C(t-s)^{-\frac{1}{2}} \| f \|_p$, \quad $(t,s) \in \tilde{\Lambda}_T$.

Moreover,

$$\| P_D(t,s)f \|_{k,p} \leq C \| f \|_{k,p}, \quad (t,s) \in \Lambda_T,$$

for all $f \in W^{k,p}(D)$, $k = 1, 2$, and

$$\| P_D(t,s)f \|_{2,p} \leq C(t-s)^{-\frac{1}{2}} \| f \|_{1,p}, \quad (t,s) \in \tilde{\Lambda}_T,$$

for all $f \in W^{1,p}(D)$. 

Proof. Let us start with the case \( q \geq p \geq d/2 \). Then, by the Gagliardo-Nierenberg inequality (cf. [25, Theorem 3.3]) and Proposition 2.6 (d), we immediately obtain
\[
\| P_D(t,s)f \|_q \leq C \| D_x^2 P_D(t,s)f \|_{p}^a \| P_D(t,s)f \|_{p}^{1-a} \leq C(t-s)^{-a} \| f \|_p, \quad (t,s) \in \tilde{\Lambda}_T,
\]
where \( a = \frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \). The case \( 1 < p < \frac{d}{2} \) follows by iteration. Assertion (b) is also proved by the Gagliardo-Nierenberg inequality. By setting \( a = \frac{1}{2} \) and \( p = q \) we obtain
\[
\| D_x P_D(t,s)f \|_p \leq C \| D_x^2 P_D(t,s)f \|_{p}^\frac{1}{2} \| P_D(t,s)f \|_{p}^{\frac{1}{2}} \leq C(t-s)^{-\frac{d}{2}} \| f \|_p, \quad (t,s) \in \tilde{\Lambda}_T.
\]
For the last assertions we refer, for example, to [17, Corollary 6.1.8].

\[ \square \]

3. The Evolution system in exterior domains

In this section we come to the main part of this paper. In the sequel we always assume that \( \Omega \subset \mathbb{R}^d \) is an exterior domain with \( C^{1,1} \)-boundary, i.e., \( \Omega = \mathbb{R}^d \setminus K \), where \( K \subset \mathbb{R}^d \) is a compact set with \( C^{1,1} \)-boundary. For \( t \geq 0 \) we set
\[
\mathcal{D}(L_\Omega(t)) := \{ u \in W^{2,p}(\mathbb{R}^d) \cap W^{1,p}_0(\Omega) : (M(t)x, D_x u(x)) \in L^p(\Omega) \},
\]
\[
L_\Omega(t)u := L_\Omega(t)u.
\]
(3.1)
Here the domain of \( L_\Omega(t) \) depends on the time parameter \( t \), however the subspace
\[
Y_\Omega := \{ u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) : |x| \cdot D_j u(x) \in L^p(\Omega) \text{ for } j = 1, \ldots, d \}
\]
is contained in \( \mathcal{D}(L_\Omega(t)) \) for all \( t \geq 0 \) and is dense in \( L^p(\Omega) \). It follows from [9] that in the autonomous case (i.e. for fixed \( s \geq 0 \)) the operator \( (L_\Omega(s), \mathcal{D}(L_\Omega(s))) \) generates a strongly continuous semigroup on \( L^p(\Omega) \). For more general second order elliptic operators with unbounded and time-independent coefficients in exterior domains we refer to [13]. Our main result is the existence of an evolution system in \( L^p(\Omega) \), \( 1 < p < \infty \), associated to the operators \( L_\Omega(\cdot) \).

Theorem 3.1. Let \( \Omega \subset \mathbb{R}^d \) be an exterior domain with \( C^{1,1} \)-boundary and \( 1 < p < \infty \). Then there exists a unique evolution system \( \{ P_\Omega(t,s) \}_{(t,s) \in \Lambda} \) on \( L^p(\Omega) \) with the following properties.

(a) For \( (t,s) \in \Lambda \), the operator \( P_\Omega(t,s) \) maps \( Y_\Omega \) into \( Y_\Omega \).
(b) For every \( f \in Y_\Omega \) and \( s \geq 0 \), the map \( t \mapsto P_\Omega(t,s)f \) is differentiable in \( (s, \infty) \) and
\[
\frac{\partial}{\partial t} P_\Omega(t,s)f = L_\Omega(t)P_\Omega(t,s)f.
\]
(3.2)
(c) For every \( f \in Y_\Omega \) and \( t > 0 \), the map \( s \mapsto P_\Omega(t,s)f \) is differentiable in \( [0, t) \) and
\[
\frac{\partial}{\partial s} P_\Omega(t,s)f = -P_\Omega(t,s)L_\Omega(s)f.
\]
(3.3)

As a direct consequence we obtain well-posedness of the abstract non-autonomous Cauchy problem (nACP) on the regularity space \( Y_\Omega \).

Corollary 3.2. Let \( \Omega \) be an exterior \( C^{1,1} \)-domain. Then the Cauchy problem (nACP) is well-posed on \( Y_\Omega \).
In the following, we describe the construction of the evolution system \( \{ P_\Omega(t, s) \}_{(t, s) \in \Lambda} \) in detail. The general idea is to derive the result for exterior domains from the corresponding results in the case of \( \mathbb{R}^d \) and bounded domains. For this purpose let \( R > 0 \) be such that \( K \subset B(R) \). We set \( D := \Omega \cap B(R + 3) \). We denote by \( \{ P_\mathbb{R}^d(t, s) \}_{(t, s) \in \Lambda} \) the evolution system in \( L^p(\mathbb{R}^d) \) and by \( \{ P_D(t, s) \}_{(t, s) \in \Lambda} \) the evolution system in \( L^p(D) \) for the bounded domain \( D \). Next we choose cut-off functions \( \varphi, \eta \in C^\infty(\Omega) \) such that \( 0 \leq \varphi, \eta \leq 1 \) and
\[
\varphi(x) := \begin{cases} 
1, & |x| \geq R + 2, \\
0, & |x| \leq R + 1,
\end{cases}
\]
and
\[
\eta(x) := \begin{cases} 
1, & |x| \leq R + 2, \\
0, & |x| \geq R + \frac{\delta}{2}.
\end{cases}
\]
For \( f \in L^p(\Omega) \) we define \( f_0 \in L^p(\mathbb{R}^d) \) and \( f_D \in L^p(D) \), respectively, by
\[
f_0(x) := \begin{cases} 
f(x), & x \in \Omega, \\
0, & x \not\in \Omega,
\end{cases}
\quad\text{and}\quad f_D(x) = \eta(x) f(x).
\]
These definitions ensure that for every function \( f \in \mathcal{D}(L_0(t)) \) we have \( f_0 \in \mathcal{D}(L_{\mathbb{R}^d}(t)) \) and \( f_D \in \mathcal{D}(L_D(t)) \). Now for \( (t, s) \in \Lambda \) and \( f \in L^p(\Omega) \) we set
\[
W(t, s) f = \varphi P_{\mathbb{R}^d}(t, s) f_0 + (1 - \varphi) P_D(t, s) f_D.
\] (3.4)

A short calculation yields
\[
D_x W(t, s) f = \varphi D_x P_{\mathbb{R}^d}(t, s) f_0 + (1 - \varphi) D_x P_D(t, s) f_D + D_x \varphi (P_{\mathbb{R}^d}(t, s) f_0 - P_D(t, s) f_D),
\]
and
\[
D_x^2 W(t, s) f = \varphi D_x^2 P_{\mathbb{R}^d}(t, s) f_0 + (1 - \varphi) D_x^2 P_D(t, s) f_D + 2 (D_x \varphi)^* \cdot (D_x P_{\mathbb{R}^d}(t, s) f_0 - D_x P_D(t, s) f_D) + D_x^2 \varphi (P_{\mathbb{R}^d}(t, s) f_0 - P_D(t, s) f_D).
\]

Thus, for \( f \in Y_\Omega \), we obtain
\[
\begin{cases} 
\frac{\partial}{\partial t} W(t, s) f = L_\Omega(t) W(t, s) f - F(t, s) f, & (t, s) \in \Lambda, \\
W(s, s) f = f,
\end{cases}
\] (3.5)

with
\[
F(t, s) f = \text{Tr} [Q(t) Q^*(t) (D_x \varphi)^* \cdot (D_x P_{\mathbb{R}^d}(t, s) f_0 - D_x P_D(t, s) f_D)] + L_\Omega(t) \varphi (P_{\mathbb{R}^d}(t, s) f_0 - P_D(t, s) f_D).
\] (3.6)

From the properties of the evolution systems \( \{ P_{\mathbb{R}^d}(t, s) \}_{(t, s) \in \Lambda} \) and \( \{ P_D(t, s) \}_{(t, s) \in \Lambda} \) it follows that the function \( F(t, s) f \) in \( L^p(\Omega) \) is well-defined for every \( f \in L^p(\Omega) \) and \( (t, s) \in \Lambda \).
Moreover, for every \( f \in L^p(\Omega) \), \( F(\cdot, \cdot) f \) is continuous in \( \tilde{\Lambda} \) with values in \( L^p(\Omega) \). By using Proposition 2.3 and Corollary 2.7 we obtain the estimate
\[
\| F(t, s) f \|_p \leq C \left( 1 + (t - s)^{-\frac{1}{2}} \right) \| f \|_p, \quad (t, s) \in \tilde{\Lambda}_T, \tag{3.7}
\]
for any \( T > 0 \) and a suitable constant \( C := C(T) > 0 \).

It is clear, that if an evolution system \( \{ \Pi(t, s) \}_{(t, s) \in \Lambda} \) exists on \( L^p(\Omega) \), then the solution \( u(t) \) to the inhomogeneous equation (3.5) is given by the variation of constant formula
\[
u(t) = \Pi(t, s) f - \int_s^t \Pi(t, r) F(r, s) f dr.
\]

This consideration suggests to consider the integral equation
\[
\Pi(t, s) f = W(t, s) f + \int_s^t \Pi(t, r) F(r, s) f dr, \quad (t, s) \in \Lambda, \ f \in L^p(\Omega). \tag{3.8}
\]

Let us state a lemma which will be very useful. Its proof is analogous to the proof in the case of one-parameter families (see [8, Lemma 4.6]). But for the sake of completeness we give here the details of the proof.

**Lemma 3.3.** Let \( X_1 \) and \( X_2 \) be two Banach spaces, \( T > 0 \) and let \( R : \tilde{\Lambda}_T \to \mathcal{L}(X_2, X_1) \) and \( S : \tilde{\Lambda}_T \to \mathcal{L}(X_2) \) be strongly continuous functions. Assume that
\[
\| R(t, s) \|_{\mathcal{L}(X_2, X_1)} \leq C_0 (t - s)^{\alpha}, \quad \| S(t, s) \|_{\mathcal{L}(X_2)} \leq C_0 (t - s)^{\beta}, \quad (t, s) \in \tilde{\Lambda}_T,
\]
holds for some \( C_0 := C_0(T) > 0 \) and \( \alpha, \beta > -1 \). For \( f \in X_2 \) and \( (t, s) \in \tilde{\Lambda}_T \), set \( T_0(t, s) f := R(t, s) f \) and
\[
T_n(t, s) f := \int_s^t T_{n-1}(t, r) S(r, s) f dr, \quad n \in \mathbb{N}, \ (t, s) \in \tilde{\Lambda}_T.
\]

Then there exists a constant \( C > 0 \) such that
\[
\sum_{n=0}^{\infty} \| T_n(t, s) f \|_{X_1} \leq C (t - s)^{\alpha} \| f \|_{X_2}, \quad (t, s) \in \tilde{\Lambda}_T. \tag{3.9}
\]

Moreover, if \( \alpha \geq 0 \), the convergence of the series in (3.9) is uniform on \( \Lambda_T \).

**Proof.** For \( f \in X_2 \) and \( (t, s) \in \tilde{\Lambda}_T \) we have
\[
\| T_1(t, s) f \|_{X_1} \leq C_0^2 \int_s^t (t - r)^{\alpha} (r - s)^{\beta} dr = C_0^2 (t - s)^{\alpha+\beta+1} B(\beta + 1, \alpha + 1) \| f \|_{X_2},
\]
where \( B(\cdot, \cdot) \) denotes the Beta function. So, by induction, we obtain
\[
\| T_n(t, s) f \|_{X_1}
\leq C_0^{n+1} (t - s)^{\alpha+n(\beta+1)} B(\beta + 1, \alpha + 1) \cdots B(\beta + 1, \alpha + 1 + (n - 1)(\beta + 1)) \| f \|_{X_2}
= C_0^{n+1} (t - s)^{\alpha+n(\beta+1)} \Gamma(\beta + 1)^n \Gamma(\alpha + 1 + n(\beta + 1)) \| f \|_{X_2}, \ n \in \mathbb{N}, \ (t, s) \in \tilde{\Lambda}_T,
\]
where \( \Gamma(\cdot) \) is the Gamma function.
where \( \Gamma \) denotes the Gamma function. Let us recall now the identity \( \Gamma(x+1) = x\Gamma(x) \), \( x \geq -1 \), and denotes by \([\cdot]\) the Gaussian brackets. Then, it follows that
\[
\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+n(\beta+1))} \leq \frac{C_\alpha}{[n(\beta+1)]!}, \quad n \in \mathbb{N}
\]
for some \( C_\alpha > 0 \). Hence,
\[
\|T_n(t,s)f\|_{X_1} \leq C_\alpha C_0(t-s)^{\alpha} \Gamma(\beta+1)^n \frac{(t-s)^{n(\beta+1)}}{[n(\beta+1)]!} \|f\|_{X_2}
\leq C_\alpha C_0(t-s)^{\alpha} e^{t-s} \Gamma(\beta+1)^n \frac{(t-s)^{[n(\beta+1)]}}{[n(\beta+1)]!} \|f\|_{X_2}, \; n \in \mathbb{N}, \; (t,s) \in \tilde{\Lambda}_T.
\]

Since
\[
\sum_{n=0}^{\infty} (C_0 \Gamma(\beta+1))^n \frac{(t-s)^{[n(\beta+1)]}}{[n(\beta+1)]!} \leq C_\beta e^{\beta(t-s)} \leq C_\beta e^{\beta T} =: C_T, \quad (t,s) \in \Lambda_T
\]
for some constants \( C_\beta, c_\beta > 0 \), it follows that
\[
\sum_{n=0}^{\infty} \|T_n(t,s)f\|_{X_1} \leq C_T C_\alpha e^{T(t-s)} \|f\|_{X_2}, \quad (t,s) \in \tilde{\Lambda}_T.
\]

It is clear that if \( \alpha \geq 0 \) then the convergence of the above series is uniform on \( \Lambda_T \).

**Proof of Theorem 3.1.** Let \( T > 0 \). By using Proposition 2.3 and Corollary 2.7 we have
\[
\|W(t,s)f\|_p \leq C\|f\|_p, \quad \text{for } f \in L^p(\Omega), \; (t,s) \in \Lambda_T.
\]

So, by (3.7), we can apply Lemma 3.3 with \( R = W, \; S = F, \; \alpha = 0, \; \beta = -\frac{1}{2} \) and \( X_1 = X_2 = L^p(\Omega) \). Thus, for any \( f \in L^p(\Omega) \), the series \( \sum_{k=0}^{\infty} P_k(t,s)f \) converges uniformly in \( \Lambda_T \), where \( P_0(t,s)f = W(t,s)f \) and
\[
P_{k+1}(t,s)f = \int_s^t P_k(t,r)F(r,s)f \; dr, \; \quad (t,s) \in \Lambda_T, \; f \in L^p(\mathbb{R}^d). \quad (3.10)
\]

Since \( T > 0 \) is arbitrary,
\[
P_\Omega(t,s) := \sum_{k=0}^{\infty} P_k(t,s), \quad (t,s) \in \Lambda \quad (3.11)
\]
is well-defined. It is easy to check that \( P_\Omega(t,s) \) satisfies the integral equation \( \Box \). Moreover, from the strong continuity of \( W(\cdot, \cdot) \) and (3.7) we deduce inductively that \( P_k(\cdot, \cdot) \) is strongly continuous and hence, by the uniform convergence of the series we get the strong continuity of \( P_\Omega(\cdot, \cdot) \).

In order to show that \( \{P_\Omega(t,s)\}_{(t,s) \in \Lambda} \) leaves \( Y_\Omega \) invariant, we consider the Banach space \( X_1 := \{f \in W_0^{1,p}(\Omega) : |x| \cdot D_x f(x) \in L^p(\Omega) \; \text{for} \; j = 1, \ldots, d\} \) endowed with the norm
\[
\|f\|_{X_1} := \|f\|_{1,p} + \|x| \cdot D_x f\|_p, \quad f \in X_1.
\]
Proposition [2.3, Corollary 2.7] and the last part of the proof of Proposition [2.1] permit us to apply Lemma 3.3 with Proposition 2.3, Corollary 2.7 and the last part of the proof of Proposition 2.1. Moreover, by taking \( X_1 = W^{2,p}(\Omega) \), \( X_2 = W^{1,p}(\Omega) \), \( R = W \), \( S = F \), \( \alpha = 0 \) and \( \beta = -\frac{1}{2} \). So, we obtain that \( \Lambda(\Omega) \) leaves \( Y_\Omega \) invariant and

\[
\sum_{n=0}^\infty [\|P_k(t,s)f\|_{2,p} + \|x|D_x P(t,s)f\|_p] < C_T(1 + (t-s)^{-\frac{1}{2}})(\|f\|_{1,p} + \|x| \cdot D_x f\|_p), \quad (t,s) \in \Lambda_T, \ f \in Y_\Omega. \tag{3.12}
\]

Let us now prove Equation (3.2). For \( f \in Y_\Omega \) we compute

\[
\begin{align*}
\frac{\partial}{\partial t} P_0(t,s)f &= L_\Omega(t) P_0(t,s)f - F(t,s)f \\
\frac{\partial}{\partial t} P_1(t,s)f &= L_\Omega(t) P_1(t,s)f + F(t,s)f - \int_s^t F(t,r) F(r,s) fdr \\
\frac{\partial}{\partial t} P_2(t,s)f &= L_\Omega(t) P_2(t,s)f + \int_s^t F(t,r) F(r,s) fdr \\
&\quad - \int_s^t \int_{r_1}^t F(t,r_2) F(r_2, r_1) F(r_1, s) fdr_2 dr_1.
\end{align*}
\]

Inductively we see that

\[
\frac{\partial}{\partial t} \sum_{k=0}^n P_k(t,s)f = L_\Omega(t) \sum_{k=0}^n P_k(t,s)f - R_n(t,s)f \tag{3.13}
\]

holds for \( n \in \mathbb{N} \), where

\[
R_n(t,s)f := \int_s^t \int_{r_1}^t \ldots \int_{r_{n-1}}^t F(t,r_n) F(r_n, r_{n-1}) \ldots F(r_1, s) fdr_n \ldots dr_2 dr_1.
\]

Now, we estimate the norm of \( R_n(t,s)f \). Estimate (3.6) yields

\[
\begin{align*}
\|R_1(t,s)f\|_p &\leq C^2 \int_s^t (t-r)^{-\frac{1}{2}} (r-s)^{-\frac{1}{2}} dr \|f\|_p = C^2 B(1/2, 1/2) \|f\|_p, \\
\|R_2(t,s)f\|_p &\leq C^3 B(1/2, 1/2) \int_s^t (r-s)^{-\frac{1}{2}} dr \|f\|_p \\
&= C^3 B(1/2, 1/2) B(1/2, 1/2) (t-s)^{\frac{1}{2}} \|f\|_p.
\end{align*}
\]

Inductively, we see that

\[
\|R_n(t,s)f\|_p \leq C^{n+1} B(1/2, 1/2) B(1/2, 1) \ldots B(1/2, n/2) (t-s)^{\frac{n-1}{2}} \|f\|_p \leq \frac{C^{n+1} \Gamma(1/2)^n}{[\Gamma(n-1)/2]} (t-s)^{\frac{n-1}{2}} \|f\|_p \tag{3.14}
\]
holds for \(n \in \mathbb{N}\). Here the constant \(C\) may change from line to line. From estimate (3.14), it follows that \(\|R_n\|_p\) tends to zero as \(n \to \infty\). So, by (3.12) and the closedness of \(L_\Omega(t)\), we can conclude that
\[
\frac{\partial}{\partial t} P_\Omega(t, s) f = L_\Omega(t) \sum_{k=0}^{\infty} P_k(t, s) f, \quad t > s, \ f \in Y_\Omega,
\]
holds and this proves (3.2).

Let us now show Equation (3.3). For \(f \in Y_\Omega\) we have
\[
L_D(s)(\eta f) = \eta L_\Omega(s)f + \text{Tr}[Q(t)Q^*(t)(D_x\eta)^* \cdot D_x f] + (L_\Omega(s)\eta)f
\]
holds. Thus,
\[
W(t, s)L_\Omega(s)f = \varphi P_{\mathbb{R}^d}(t, s)(L_\Omega(s)f)_0 + (1 - \varphi) P_D(t, s)(L_\Omega(s)f)_D
\]
\[
= \varphi P_{\mathbb{R}^d}(t, s)L_{\mathbb{R}^d}(s)f_0 + (1 - \varphi) P_D(t, s)L_D(s)f_D - G(t, s)f,
\]
where
\[
G(t, s)f := (1 - \varphi) P_D(t, s) \left( \text{Tr}[Q(t)Q^*(t)(D_x\eta)^* \cdot D_x f] + (L_\Omega(s)\eta)f \right)
\]
and \(f \in Y_\Omega\). This yields
\[
\frac{\partial}{\partial s} W(t, s)f = -W(t, s)L_\Omega(s)f - G(t, s)f
\]
for \((t, s) \in \Lambda\) and \(f \in Y_\Omega\).

Now, let \(T > 0\) be arbitrary but fixed. Then, from the definition of \(G\) and Corollary 2.7, it follows that we can apply Lemma 3.3 with \(X_1 = X_2 = W^{1, p}(\Omega), R = S = G\) and \(\alpha = \beta = -\frac{1}{2}\). So, the series
\[
T(t, s)f := \sum_{k=0}^{\infty} T_k(t, s)f, \quad (t, s) \in \bar{\Lambda}_T,
\]
is well-defined and
\[
\|T(t, s)f\|_{1, p} \leq C(t - s)^{-\frac{1}{2}}\|f\|_{1, p}, \quad (t, s) \in \bar{\Lambda}_T,
\]
for \(f \in W^{1, p}(\Omega)\). On the other hand, \(T(\cdot, \cdot)\) satisfies the integral equation
\[
T(t, s)f = G(t, s)f + \int_s^t T(t, r)G(r, s)f \, dr, \quad (t, s) \in \Lambda_T, \ f \in W^{1, p}(\Omega).
\]
In particular \(T(t, \cdot)f\) is continuous on \([0, t]\) with respect to the \(L^p\)-norm for any \(f \in W^{1, p}(\Omega)\) and \(t \geq 0\). Now, for \(f \in L^p(\Omega)\) and \((t, s) \in \Lambda_T\) we set
\[
S(t, s)f := W(t, s)f + \int_s^t T(t, r)W(r, s)f \, dr.
\]
It follows from the continuity of \(T(t, \cdot)W(\cdot, s)f\) on \([s, t]\), Proposition 2.3 and Corollary 2.7 that the above integral is well-defined for any \(f \in L^p(\Omega)\). Computing the derivative with
respect to $s$ yields
\[
\frac{\partial}{\partial s} S(t,s)f = -W(t,s)L_\Omega(s)f - G(t,s)f + T(t,s)f - \int_s^t T(t,r)W(r,s)L_\Omega(s)f\,dr \\
- \int_s^t T(t,r)G(r,s)f\,dr
\]
for all $f \in Y_\Omega$, due to (3.10). From this equality together with (3.2) and since $P_\Omega(t,s)Y_\Omega \subset Y_\Omega$, $(t,s) \in \Lambda$, we can conclude that
\[
\frac{\partial}{\partial t}(S(t,r)P_\Omega(r,s)f) = 0
\]
holds for all $f \in Y_\Omega$. This yields that for $f \in Y_\Omega$, the function $S(t,r)P_\Omega(r,s)f$ is constant on $\Lambda_T$ and thus, by the density of $Y_\Omega$ in $L^p(\Omega)$ and by the fact that $T > 0$ was arbitrary, it follows that $S(t,s)f = P_\Omega(t,s)f$ holds for all $f \in L^p(\Omega)$ and all $(t,s) \in \Lambda$. This proves (3.3).

Let us now show the uniqueness of the solution $P_\Omega(t,s)f$ of (nACP) for initial value $f \in Y_\Omega$. For this purpose we assume that there exists another solution $t \mapsto u(t) \in Y_\Omega$. Since $u(r) \in Y_\Omega$ for all $r \in [s, \infty)$ it follows from equality (3.3) that the map $r \mapsto P_\Omega(t,r)u(r)$ is differentiable for $0 \leq s < r < t$ and
\[
\frac{\partial}{\partial r}(P_\Omega(t,r)u(r)) = -P_\Omega(t,r)L_\Omega(r)u(r) + P_\Omega(t,r)L_\Omega(r)u(r) = 0.
\]
Therefore $P_\Omega(t,r)u(r)$ is constant on $0 \leq s < r < t$. Thus, by letting $r \to s$ and $r \to t$ we obtain $P_\Omega(t,s)f = u(t)$. The uniqueness now directly implies that the law of evolution (Property (i) of Definition 1.3) holds.

To conclude this section we prove $L^p$-$L^q$ smoothing properties of the evolution system \{P_\Omega(t,s)\}_{(t,s) \in \Lambda}$ and $L^p$-estimates for its spatial derivatives. The following estimates follow basically directly via the representation (3.11) from Lemma 3.3, Proposition 2.3 and Corollary 2.7.

**Proposition 3.4.** Let $T > 0$, $1 < p < \infty$ and $p \leq q < \infty$. Then there exists a constant $C := C(T) > 0$ such that
\[
(i) \quad \|P_\Omega(t,s)f\|_q \leq C(t-s)^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}\|f\|_p, \\
(ii) \quad \|D_x P_\Omega(t,s)f\|_p \leq C(t-s)^{-\frac{d}{2}}\|f\|_p
\]
for $(t,s) \in \tilde{\Lambda}_T$ and $f \in L^p(\Omega)$. Moreover, for $1 < p < q < \infty$ and $f \in L^p(\Omega)$
\[
\lim_{t \to s}\|((t-s)^{\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}P_\Omega(t,s)f\|_q + \|(t-s)^{\frac{d}{2}}D_x P_\Omega(t,s)f\|_p = 0.
\]

**Proof.** To obtain (i) we apply Lemma 3.3 with $X_1 = L^q(\Omega)$, $X_2 = L^p(\Omega)$, $R = W$, $S = F$, $\alpha = -\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, $\beta = -\frac{1}{2}$, Proposition 2.3 and Corollary 2.7 in the case where $q \geq p \geq \frac{d}{2}$. By iteration (i) holds also for $1 < p < \frac{d}{2}$. 
The second assertion follows by applying Lemma 3.3 with $X_1 = W^{1,p}(\Omega)$, $X_2 = L^p(\Omega)$, $R = W$, $S = F$, $\alpha = \beta = -\frac{1}{2}$, Proposition 2.3 and Corollary 2.7. Finally, the last assertion can be obtained as in [15, Proposition 3.4]. □

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