Finite groups whose commuting graphs are integral

Jutirekha Dutta and Rajat Kanti Nath

Department of Mathematical Sciences,
Tezpur University, Napaam-784028, Sonitpur, Assam, India.
Emails: jutirekhadutta@yahoo.com, rajatkantinath@yahoo.com

Abstract: A finite non-abelian group $G$ is called commuting integral if the commuting graph of $G$ is integral. In this paper, we show that a finite group is commuting integral if its central factor is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or $D_{2m}$, where $p$ is any prime integer and $D_{2m}$ is the dihedral group of order $2m$.

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1 Introduction

Let $G$ be a non-abelian group with center $Z(G)$. The commuting graph of $G$, denoted by $\Gamma_G$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two vertices $x$ and $y$ are adjacent if and only if $xy = yx$. In recent years, many mathematicians have considered commuting graph of different finite groups and studied various graph theoretic aspects (see [4, 6, 11, 12, 13, 14]). A finite non-abelian group $G$ is called commuting integral if the commuting graph of $G$ is integral. It is natural to ask which finite groups are commuting integral. In this paper, we compute the spectrum of the commuting graphs of finite groups whose central factors are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, for any prime integer $p$, or $D_{2m}$, the dihedral group of order $2m$. Our computation reveals that those groups are commuting integral.

Recall that the spectrum of a graph $\mathcal{G}$ denoted by $\text{Spec}(\mathcal{G})$ is the set $\{\lambda^{k_1}_1, \lambda^{k_2}_2, \ldots, \lambda^{k_n}_n\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $\mathcal{G}$ with multiplicities $k_1, k_2, \ldots, k_n$ respectively. A graph $\mathcal{G}$ is called integral if $\text{Spec}(\mathcal{G})$ contains only integers. It is well known that the complete graph $K_n$ on $n$ vertices is integral and $\text{Spec}(K_n) = \{(-1)^{n-1}, (n-1)^1\}$. Further, if $\mathcal{G} = K_{m_1} \sqcup K_{m_2} \sqcup \cdots \sqcup K_{m_l}$, where $K_{m_i}$ are complete graphs on $m_i$ vertices for $1 \leq i \leq l$, then

$$\text{Spec}(\mathcal{G}) = \{((-1)^{m_1-1}, (m_1-1)^1, (m_2-1)^1, \ldots, (m_l-1)^1\}.$$ 

The notion of integral graph was introduced by Harary and Schwenk [9] in the year 1974. Since then many mathematicians have considered integral graphs, see for example [2, 10, 15]. A very impressive survey on integral graphs can be found in [8].

*Corresponding author
Ahmadi et. al noted that integral graphs have some interest for designing the network topology of perfect state transfer networks, see [3] and the references there in.

For any element $x$ of a group $G$, the set $C_G(x) = \{ y \in G : xy = yx \}$ is called the centralizer of $x$ in $G$. Let $|\text{Cent}(G)| = |\{C_G(x) : x \in G\}|$, that is the number of distinct centralizers in $G$. A group $G$ is called an $n$-centralizer group if $|\text{Cent}(G)| = n$. In [7], Belcastro and Sherman characterized finite $n$-centralizer groups for $n = 4, 5$. As a consequence of our results, we show that 4, 5-centralizer finite groups are commuting integral. Further, we show that a finite $(p + 2)$-centralizer $p$-group is commuting integral for any prime $p$.

## 2 Main results and consequences

We begin this section with the following theorem.

**Theorem 2.1.** Let $G$ be a finite group such that $\frac{\mathbb{Z}}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime integer. Then

$$\text{Spec}(\Gamma_G) = \{(-1)^{(p^2-1)|Z(G)|-p-1}, ((p-1)|Z(G)|-1)^{p+1}\}.$$  

*Proof.* Let $|Z(G)| = n$ then since $\frac{\mathbb{Z}}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ we have $\frac{\mathbb{Z}}{Z(G)} = \langle aZ(G), bZ(G) : a^p, b^2, aba^{-1}b^{-1} \in Z(G) \rangle$, where $a, b \in G$ with $ab \neq ba$. Then for any $z \in Z(G)$, we have

$$C_G(a) = C_G(a^i z) = Z(G) \cup aZ(G) \cup \ldots \cup a^{p-1}Z(G) \text{ for } 1 \leq i \leq p-1,$$

$$C_G(a^i b) = C_G(a^i b z) = Z(G) \cup a^i bZ(G) \cup \ldots \cup a^{(p-1)i}b^{p-1}Z(G) \text{ for } 1 \leq j \leq p.$$  

These are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$. Therefore

$$\Gamma_G = K_{C_G(a) \setminus Z(G)} \sqcup \left( \bigsqcup_{j=1}^{p+1} K_{C_G(a^j b) \setminus Z(G)} \right).$$  

Thus $\Gamma_G = K_{(p-1)n} \sqcup \left( \bigsqcup_{j=1}^{p+1} K_{(p-1)n} \right)$, since $|C_G(a)| = pn$ and $|C_G(a^i b)| = pn$ for $1 \leq j \leq p$ where as usual $K_m$ denotes the complete graph with $m$ vertices. That is, $\Gamma_G \cong \biguplus_{j=1}^{p+1} K_{(p-1)n}$. Hence the result follows. \qed

Above theorem shows that $G$ is commuting integral if the central factor of $G$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime integer $p$. Some consequences of Theorem 2.1 are given below.

**Corollary 2.2.** Let $G$ be a non-abelian group of order $p^3$, for any prime $p$, then

$$\text{Spec}(\Gamma_G) = \{(-1)^{p^3-2p-1}, (p^2 - p - 1)^{p+1}\}.$$  

Hence, $G$ is commuting integral.

*Proof.* Note that $|Z(G)| = p$ and $\frac{\mathbb{Z}}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.1. \qed

**Corollary 2.3.** If $G$ is a finite 4-centralizer group then $G$ is commuting integral.
Let $G$ be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 2.4

$$\text{Spec}(\Gamma_G) = \{((-1)^3|Z(G)|^{-1}, (|Z(G)| - 1)^3 \}.$$

This shows that $G$ is commuting integral. $\square$

Further, we have the following result.

**Corollary 2.4.** If $G$ is a finite $(p + 2)$-centralizer $p$-group, for any prime $p$, then

$$\text{Spec}(\Gamma_G) = \{((-1)^3|Z(G)|^{-1}, (|Z(G)| - 1)^3 \}.$$  

Hence, $G$ is commuting integral.

**Proof.** If $G$ is a finite $(p + 2)$-centralizer $p$-group then by Lemma 2.7 of [3] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Now the result follows from Theorem 2.4. $\square$

The following theorem shows that $G$ is commuting integral if the central factor of $G$ is isomorphic to the dihedral group $D_{2m} = \{a, b : a^m = b^2 = 1, bab^{-1} = a^{-1}\}$.

**Theorem 2.5.** Let $G$ be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geq 2$. Then

$$\text{Spec}(\Gamma_G) = \{((-1)^{(2m-1)|Z(G)|-m-1}, (|Z(G)| - 1)^m, ((m - 1)|Z(G)| - 1)^3 \}.$$  

**Proof.** Since $\frac{G}{Z(G)} \cong D_{2m}$ we have $\frac{G}{Z(G)} = \langle xZ(G), yZ(G) : x^2, y^m, xyx^{-1}y \in Z(G) \rangle$, where $x, y \in G$ with $xy \neq yx$. It is not difficult to see that for any $z \in Z(G)$,

$$C_G(y) = C_G(y'z) = Z(G) \cup yZ(G) \cup \cdots \cup y^{m-1}Z(G), 1 \leq i \leq m - 1$$

and

$$C_G(xy^j) = C_G(xy^jz) = Z(G) \cup xy^jZ(G), 1 \leq j \leq m$$

are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$. Therefore

$$\Gamma_G = K_{\{C_G(y) : Z(G) \cup \cdots \cup y^{m-1}Z(G)\} \cup \bigcup_{j=1}^{m} K_{\{C_G(xy^j) : Z(G) \cup \cdots \cup y^{m-1}Z(G)\}}.$$

Thus $\Gamma_G = K_{(m-1)n} \cup \bigcup_{j=1}^{m} K_n$, since $|C_G(y)| = mn$ and $|C_G(xy^j)| = 2n$ for $1 \leq j \leq m$, where $|Z(G)| = n$. Hence the result follows. $\square$

**Corollary 2.6.** If $G$ is a finite $5$-centralizer group then $G$ is commuting integral.

**Proof.** If $G$ is a finite $5$-centralizer group then by Theorem 4 of [2] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $D_6$. Now, if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then by Theorem 2.4 we have

$$\text{Spec}(\Gamma_G) = \{((-1)^8|Z(G)|^{-4}, (2|Z(G)| - 1)^4 \}.$$  

Again, if $\frac{G}{Z(G)} \cong D_6$ then by Theorem 2.4 we have

$$\text{Spec}(\Gamma_G) = \{((-1)^5|Z(G)|^{-4}, (|Z(G)| - 1)^3, (2|Z(G)| - 1)^4 \}.$$  

In both the cases $\Gamma_G$ is integral. Hence $G$ is commuting integral. $\square$
We also have the following result.

**Corollary 2.7.** Let $G$ be a finite non-abelian group and \{$x_1, x_2, \ldots, x_r$\} be a set of pairwise non-commuting elements of $G$ having maximal size. Then $G$ is commuting integral if $r = 3, 4$.

**Proof.** By Lemma 2.4 of [1], we have that $G$ is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4. Hence the result follows from Corollary 2.3 and Corollary 2.6.

We now compute the spectrum of the commuting graphs of some well-known groups, using Theorem 2.5.

**Proposition 2.8.** Let $M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1} \rangle$ be a metacyclic group, where $m > 2$. Then

$$\text{Spec}(\Gamma_{M_{2mn}}) = \begin{cases} \{(-1)^{2mn-m-n-1}, (n-1)^{m}, (mn-n-1)^{1}\} & \text{if } m \text{ is odd} \\ \{(-1)^{2mn-2n-3}, (2n-1)^{3}, (mn-2n-1)^{1}\} & \text{if } m \text{ is even} \end{cases}$$

**Proof.** Observe that $Z(M_{2mn}) = \langle b^{2} \rangle$ or $\langle a \rangle \cup \langle b^{2} \rangle$ according as $m$ is odd or even. Also, it is easy to see that $M_{2mn} Z(M_{2mn}) \cong D_{2m}$ or $D_{m}$ according as $m$ is odd or even. Hence, the result follows from Theorem 2.5.

The above Proposition 2.8 also gives the spectrum of the commuting graph of the dihedral group $D_{2m}$, where $m > 2$, as given below:

$$\text{Spec}(\Gamma_{D_{2m}}) = \begin{cases} \{(-1)^{m-2}, 0^{m}, (m-2)^{1}\} & \text{if } m \text{ is odd} \\ \{(-1)^{3m-3}, 1^{3}, (m-3)^{1}\} & \text{if } m \text{ is even} \end{cases}$$

**Proposition 2.9.** The spectrum of the commuting graph of the dihedral group or the generalized quaternion group $Q_{4m} = \langle a, b : a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} \rangle$, where $m \geq 2$, is given by

$$\text{Spec}(\Gamma_{Q_{4m}}) = \{(-1)^{3m-3}, 1^{m}, (2m-3)^{1}\}.$$ 

**Proof.** The result follows from Theorem 2.8 noting that $Z(Q_{4m}) = \{1, a^{m}\}$ and $Q_{4m} Z(Q_{4m}) \cong D_{2m}$. 

**Proposition 2.10.** Consider the group $U_{6n} = \{a, b : a^{2n} = b^{3} = 1, a^{-1} ba = b^{-1} \}$. Then $\text{Spec}(\Gamma_{U_{6n}}) = \{(-1)^{3n-4}, (n-1)^{3}, (2n-1)^{1}\}$.

**Proof.** Note that $Z(U_{6n}) = \langle a^{3} \rangle$ and $U_{6n} Z(U_{6n}) \cong D_{6}$. Hence the result follows from Theorem 2.8.

We conclude the paper by noting that the groups $M_{2mn}, D_{2m}, Q_{4m}$ and $U_{6n}$ are commuting integral.
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