A tight Cramér-Rao bound for joint parameter estimation with a pure two-mode squeezed probe

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Abstract

We calculate the Holevo Cramér-Rao bound for estimation of the displacement experienced by one mode of an two-mode squeezed vacuum state with squeezing $r$ and find that it is equal to $4 \exp(-2r)$. This equals the sum of the mean squared error obtained from a dual homodyne measurement, indicating that the bound is tight and that the dual homodyne measurement is optimal.

Keywords. Quantum physics, Quantum information, Quantum optics, Parameter estimation, Cramér-Rao bound
I. INTRODUCTION

In quantum mechanics, there is a limit to the precision to which we can simultaneously measure two observables that do not commute. If the observables are complementary variables, such as position and momentum, this limit is described by the Heisenberg uncertainty principle.

In continuous variable quantum optics, the bosonic quadrature operators $Q$ and $P$ are another pair of complementary variables, and obey the canonical commutation relation $[Q, P] = 2i$, where throughout the paper we use units where $\hbar = 2$. The bosonic field is also described by the annihilation operator $a$ and creation operator $a^\dagger$, which are related to the quadrature operators by $Q = a + a^\dagger$ and $P = i(a^\dagger - a)$. The displacement operator is given by

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a),$$

where $\alpha = (q + ip)/2$ is the complex amplitude.

In this paper we address this question: Given the probe state $\rho_0$ which undergoes a displacement of $D(\alpha)$ resulting in $\rho_\theta = D(\alpha)\rho_0 D^\dagger(\alpha)$, how well can we estimate the two parameters $\theta_1 := q = 2\text{Re}(\alpha)$ and $\theta_2 := p = 2\text{Im}(\alpha)$? Our figure of merit is sum of the mean square error (MSE), $V := E[(\hat{\theta}_1 - \theta_1)^2] + E[(\hat{\theta}_2 - \theta_2)^2]$ where $E$ is the expectation value, and $\hat{\theta}_1$ and $\hat{\theta}_2$ are the estimates of $\theta_1$ and $\theta_2$ respectively. We aim to find a lower bound to $V$. These bounds are called Cramér-Rao bounds (CR bounds) and will only depend on the state $\rho_\theta$ and independent of the measurement performed on it. We calculate the CR bound based on the work of Holevo [1, 2], which we call Holevo CR bound. First, we calculate the Holevo CR bound when the probe state $\rho_0$ is a single mode squeezed state, for which tight bounds are already known. Next, we calculate the Holevo CR bound when the probe is a two mode squeezed state. We find that it is superior to bounds calculated by previous authors [3, 4]; and that the bound can be reached by a simple measurement.

Our paper is divided up into sections as follows. In section III we briefly describe the Gaussian quantum optics used in our results. In section III we summarize parameter estimation theory including CR bounds. In section IV we summarize the bounds found by other authors and the MSE from a dual homodyne measurement. In section V we calculate the Holevo CR bound for one- and two-mode squeezed probes and discuss our results.
II. GAUSSIAN QUANTUM OPTICS

Consider a state consisting of \( m \) bosonic modes. Let the annihilation and creation operators of the \( k \)th mode be \( a_k \) and \( a_k^\dagger \), respectively, and the quadrature operators be \( Q_k \) and \( P_k \). Define a vector \( Z \) to contain all the quadrature operators:

\[
\vec{Z} = (Q_1, P_1, ..., Q_m, P_m)
\]  

(2)

The mean of the quadrature operators of a state \( \rho \), otherwise known as the displacement vector of the state, is given by

\[
M = [\langle Z_j \rangle]_j,
\]  

(3)

where \( \langle A \rangle = \text{tr}(\rho A) \) is the expectation value of operator \( A \). Define the covariance matrix \( V \), which contains the variances of the quadrature operators, by

\[
V = \left[ \frac{1}{2} \langle Z_j Z_k - Z_k Z_j \rangle - \langle Z_j \rangle \langle Z_k \rangle \right]_{jk}.
\]  

(4)

A thermal state is given by

\[
\rho_{\text{th}}(N) = \frac{1}{N + 1} \sum_{n=0}^{\infty} \left( \frac{N}{1 + N} \right)^n |n\rangle \langle n|
\]  

(5)

where \( |n\rangle \) are the Fock states, and \( N \) is the mean number of photons in the bosonic mode. A thermal state has a zero displacement vector and a covariance matrix of \( V_{\text{th}} = (2N + 1)I_2 \) where \( I_2 \) is the \( 2 \times 2 \) identity matrix.

The single-mode squeezing operator is given by

\[
S(r) = \exp \left( \frac{1}{2} (ra^2 - ra'^2) \right),
\]  

(6)

where \( r \) is the squeezing parameter. When acting on the vacuum state, this gives the squeezed vacuum state \( |S(r)\rangle = S(r) |0\rangle \). The squeezed vacuum state has a zero displacement vector and a covariance matrix of

\[
V_{\text{sq}} = \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}.
\]  

(7)

The two-mode squeezing operator is given by

\[
S_2(r) = \exp \left( ra_1 a_2 - ra_1^\dagger a_2^\dagger \right).
\]  

(8)
When acting on two vacuum states it gives the two-mode squeezed vacuum state, also known as the Einstein-Podolski-Rosen (EPR) state. The two-mode squeezed vacuum state has zero displacement vector and covariance matrix of

\[
V_{\text{EPR}} = \begin{pmatrix}
\cosh(2r) & 0 & \sinh(2r) & 0 \\
0 & \cosh(2r) & 0 & -\sinh(2r) \\
\sinh(2r) & 0 & \cosh(2r) & 0 \\
0 & -\sinh(2r) & 0 & \cosh(2r)
\end{pmatrix}.
\] (9)

A beam splitter is used to mix two modes. It is described by the unitary transformation

\[
B(\phi) = \exp\left(\phi(a_1^\dagger a_2 - a_1 a_2^\dagger)\right)
\] (10)

where \(\tau = \cos^2 \phi\) is the transmissivity of the beam splitter.

### III. PARAMETER ESTIMATION THEORY

Let \(\rho_\theta\) be a family of states parametrized by \(d\) parameters \(\theta = (\theta_1, \theta_2, \ldots, \theta_d)\). The goal of parameter estimation is to estimate the value of \(\theta\) based on the outcome of a measurement on \(\rho_\theta\). In quantum mechanics, a measurement is described by a positive operator-valued measure (POVM) \(\Pi = \{\Pi_x\}\). Each measurement outcome \(x\) has a corresponding non-negative hermitian operator \(\Pi_x\) associated with it, where the probability of measuring \(x\) on a state \(\rho_\theta\) is \(p_\theta(x) = \text{tr}(\Pi_x \rho_\theta)\), and the POVM elements sum to Identity: \(\sum_x \Pi_x = I\). We then need an estimator \(\hat{\theta}(x)\), which maps the observed outcome \(x\) to an estimate for \(\theta\). An estimator is called locally unbiased at \(\theta\) if \(E[\hat{\theta}(x)] = \theta\) at the point \(\theta\). An estimator is called unbiased if and only if it is locally unbiased at every \(\theta\).

The MSE matrix \(V_\theta[\hat{\theta}]\) of the estimator \(\hat{\theta}\) is given by

\[
V_\theta[\hat{\theta}] = \left[ \sum_x p_\theta(x)(\hat{\theta}_j(x) - \theta_j)(\hat{\theta}_k(x) - \theta_k) \right]_{jk}.
\] (11)

The sum of the MSE \(\nu\) is the trace of the MSE matrix:

\[
\nu = \text{Tr}\{V_\theta[\hat{\theta}]\}.
\] (12)

Here, \(\text{Tr}\{\cdot\}\) denotes the trace of an estimator matrix. The Cramér-Rao bound provides a lower bound to the MSE matrix for a classical probability distribution \(p_\theta(x)\):

\[
V_\theta[\hat{\theta}] \geq (J_\theta[p_\theta])^{-1}
\] (13)
where \( A \geq B \) for matrices \( A \) and \( B \) means \( A - B \) is positive semi-definite. Taking the trace we get a bound for the MSE,

\[
\mathcal{V} \geq \text{Tr}\{(J_\theta[p_\theta])^{-1}\}. \tag{14}
\]

\( J_\theta[p_\theta] \) is the classical Fisher information matrix is given by

\[
J_\theta[p_\theta] = \left[ \sum_x p_\theta(x) \frac{\partial \log p_\theta(x)}{\partial \theta_j} \frac{\partial \log p_\theta(x)}{\partial \theta_k} \right]_{jk}. \tag{15}
\]

This provides a bound to the MSE matrix for a fixed measurement \( \Pi \). Next we define the most informative quantum Cramér-Rao bound, by minimizing over all POVMs.

\[
C^{MI}_\theta = \min_{\Pi} \text{Tr}\{J_\theta[\Pi]^{-1}\} \tag{16}
\]

In practice, this minimization is difficult to perform, so lower bounds are used instead. The first one is base of the symmetric log derivative (SLD) operator \( L_{\theta,j} \) which is defined by

\[
\frac{\partial \rho_\theta}{\partial \theta_j} = \frac{1}{2} (\rho_\theta L_{\theta,j} + L_{\theta,j} \rho_\theta). \tag{17}
\]

This is used to calculate the SLD quantum fisher information matrix defined by

\[
G_\theta = [\langle L_{\theta,j}, L_{\theta,k} \rangle_{\rho_\theta}]_{jk}, \tag{18}
\]

where we use an inner product defined by

\[
\langle X, Y \rangle_{\rho_\theta} = \text{tr}\left(\rho_\theta \frac{1}{2} (Y X^\dagger + X^\dagger Y)\right), \tag{19}
\]

where \( \text{tr}(\cdot) \) denotes trace of a density matrix. This leads to a bound on the sum of the MSE, which we call the SLD CR bound \( C^S_\theta \).

\[
\mathcal{V} \geq C^S_\theta = \text{Tr}(G_\theta^{-1}). \tag{20}
\]

The next quantum CR bound we consider is based on the right log derivative (RLD) operator \( \tilde{L}_{\theta,j} \) defined by

\[
\frac{\partial \rho_\theta}{\partial \theta_j} = \rho_\theta \tilde{L}_{\theta,j}. \tag{21}
\]

This is used to calculate the RLD quantum fisher information matrix

\[
\tilde{G}_\theta = [\langle \tilde{L}_{\theta,j}, \tilde{L}_{\theta,k} \rangle_{\rho_\theta}]_{jk}. \tag{22}
\]
where we use an inner product defined by

\[ \langle X, Y \rangle^+_\rho = \text{tr}(\rho Y X^\dagger). \]  

This leads to a bound on the sum of the MSE \[ V \], which we call the RLD CR bound \[ C^R_\theta \], given by

\[ V \geq C^R_\theta = \text{Tr}\{\text{Re} \tilde{G}^{-1}_\theta\} + \text{TrAbs}\{\text{Im} \tilde{G}^{-1}_\theta\}, \]  

where \( \text{TrAbs}\{\cdot\} \) denotes the sum of the absolute values of the eigenvalues of a matrix.

While the RLD and SLD CR bounds are easy to compute \[ 8, 9 \], in general they are not always achievable. The SLD CR bound corresponds to performing the optimal measurements for the estimation of each parameter ignoring the other. But for non-commuting observables, it might not be possible to perform the two optimal measurements simultaneously. Similarly, the RLD CR bound is in general not obtainable by a valid measurement.

Holevo derived another bound for the MSE \[ 1, 2 \], which we shall call the Holevo CR bound. The Holevo CR bound is defined through the following minimisation

\[ C^H_\theta := \min_{\vec{X} \in \mathcal{X}} h_\theta[\vec{X}]. \]  

and \( \mathcal{X} := \{(X_1, X_2, \ldots, X_d)\} \) where \( X_j \) are Hermitian operators satisfying the locally unbiased conditions

\[ \text{tr}(\rho_\theta X_j) = 0 \]  

\[ \text{tr}\left(\frac{\partial \rho_\theta}{\partial \theta_j} X_k\right) = \delta_{jk} \]  

and \( h_\theta \) is the function

\[ h_\theta[\vec{X}] := \text{Tr}\left\{\text{Re} Z_\theta[\vec{X}]\right\} + \text{TrAbs}\left\{\text{Im} Z_\theta[\vec{X}]\right\}. \]  

\( Z_\theta[\vec{X}] \) is a \( d \times d \) matrix

\[ Z_\theta[\vec{X}] := [\text{tr}(\rho_\theta X_j X_k)]_{j,k}. \]  

For any \( \vec{X} \) satisfying the condition Eq. \( 27 \), \( h_\theta[\vec{X}] \geq C^S_\theta \) and \( h_\theta[\vec{X}] \geq C^R_\theta \). At the minimum of \( h_\theta \), defined above as \( C^H_\theta \), \( V \geq C^H_\theta \). So, the Holevo CR bound is always greater than or equal to the RLD and SLD bounds. See for example \[ 10 \] for proof of the above statements.

The Holevo CR bound involves a minimisation over the measurement space, is in general hard to compute, and can be attained by a collective measurement \[ 11 \]. When the probe state has rank one, an individual measurement is sufficient to attain the Holevo CR bound \[ 12 \].
FIG. 1: Bounds for $V$ as a function of squeezing parameter $r$ when the number of thermal photons $N = 0.1$ are shown in green (SLD) and red (RLD). The black line is the most informative bound from these two bounds. The sum of the mean squared error we get from a dual homodyne measurement is plotted in blue. We see that the bounds are not achieved by the dual homodyne measurement for some values of $r$.

IV. SLD AND RLD CR BOUNDS FOR TWO-MODE SQUEEZED PROBE

Let the probe be a two-mode squeezed thermal state given by

$$\rho_0 = S_2(r)(\rho_{th}(N) \otimes \rho_{th}(N))S_2^\dagger(r),$$

where if $N = 0$ we get the two-mode squeezed vacuum. The first mode of the probe state undergoes a unitary displacement operation $D(\theta)$ and ends up in the state $\rho_\theta$. The SLD CR bound and RLD CR bounds are given by \[3, 4\]:

$$C^{S}_{\theta} := \frac{2 + 4N}{\cosh 2r}, \quad (30)$$

$$C^{R}_{\theta} := \frac{8N(1 + N)}{(1 + 2N) \cosh 2r - 1}. \quad (31)$$

Now let us consider a measurement which we call the dual homodyne measurement. The measurement consists of interfering the two modes on a beam splitter with transmissivity $\tau = 1/2$, followed by homodyne measurement of the $Q$ quadrature of the first mode and a homodyne measurement of the $P$ quadrature of the second mode. The dual homodyne measurement gives $V = (8N + 4) \exp(-2r) := V_{DH}$.

We plot $V_{DH}$ and the two bounds in Fig. 1. The dual homodyne measurement MSE does
not reach the bounds for most values of $r$. This means that either the measurement is not optimal or the bounds are not tight. To help determine which is the case, we will calculate the Holevo CR bound and compare it to $V_{DH}$.

V. RESULTS

Here, we calculate the Holevo CR bound for two cases: a pure single-mode squeezed probe and a pure two-mode squeezed probe.

A. Calculation of Holevo bound for pure single-mode squeezed probe

Consider a single mode squeezed probe $\rho_0 = S_1(r)\rho_{th}(N)S_1(r)\dagger$. Applying the displacement operator $D(\theta)$, we end up with the state $\rho_\theta = D(\theta)\rho_0D(\theta)\dagger$. For pure states when $N = 0$, this case is shown in [12–14] to be coherent and the bound from the RLD is known to be tight [13–15]. One optimal joint estimation strategy is to alternatively apply the optimal strategy for each parameter. In general, when $\rho$ is a mixed state, the Cramér-Rao bounds are [3, 4]

\begin{align}
C_\theta^S := (2 + 4N) \cosh 2r \\
C_\theta^R := 2 + (2 + 4N) \cosh 2r .
\end{align}

The RLD bound is always greater than the SLD bound and is hence a more informative lower bound. In fact, the dual homodyne measurement gives $V = 2 + (2 + 4N) \cosh 2r$ which saturates the RLD bound [3]. As the bound increases with $r$, squeezed state probes perform worse than a coherent state probe ($r = 0$).

Although we already know what the result will be, as an exercise we compute the Holevo bound for the pure single-mode squeezed probe. In this case, $\rho_\theta = |\psi \rangle \langle \psi |$ where $|\psi \rangle = D(\theta) |S(r)\rangle$.

The derivatives of the displacement operator $D(\theta) = \exp \left( \frac{i}{2} \theta_2 Q - \frac{i}{4} \theta_1 P \right)$ with respect to $\theta_1$ and $\theta_2$ are

\begin{align}
\frac{\partial}{\partial \theta_1} D(\theta) &= \left( -\frac{i}{2} P + \frac{i}{4} \theta_2 \right) D(\theta) \\
\frac{\partial}{\partial \theta_2} D(\theta) &= \left( \frac{i}{2} Q - \frac{i}{4} \theta_1 \right) D(\theta) .
\end{align}
See appendix A.1 for the derivation.

To simplify the calculation, we calculate the Holevo CR bound when \( \theta \) is small, and hence evaluate at \( \theta = 0 \), and assert that the bound will be the same for all \( \theta \). The reason we can do this is because the Holevo CR bound is asymptotically attainable with an adaptive measurement scheme, given a set of \( n \) identical states \( \rho_0^\otimes n \) with \( n \to \infty \). A rough estimate for \( \theta \) can be obtained for a small number of measurements using \( \sqrt{n} \) states, then the remaining \( n - \sqrt{n} \) states can be displaced by \( D(-\tilde{\theta}) \) where \( \tilde{\theta} \) is the rough estimate for \( \theta \), resulting in states with a small \( \theta \).

We compute \( |\psi\rangle \) and the derivatives of \( |\psi\rangle \) with respect to \( \theta_1 \) and \( \theta_2 \) evaluated at \( \theta = 0 \).

\[
|\psi_0\rangle = |\psi\rangle \bigg|_{\theta=0} = |S(r)\rangle \tag{36}
\]

\[
|\psi_1\rangle = \frac{\partial}{\partial \theta_1} |\psi\rangle \bigg|_{\theta=0} = -P |S(r)\rangle \frac{i}{2} \tag{37}
\]

\[
|\psi_2\rangle = \frac{\partial}{\partial \theta_2} |\psi\rangle \bigg|_{\theta=0} = Q |S(r)\rangle \frac{i}{2}. \tag{38}
\]

The inner products are

\[
\langle \psi_0 | \psi_0 \rangle = 1 \tag{39}
\]

\[
\langle \psi_0 | \psi_1 \rangle = -\frac{i}{2} \langle S(r) | P | S(r) \rangle \tag{40}
\]

\[
\langle \psi_0 | \psi_2 \rangle = \frac{i}{2} \langle S(r) | Q | S(r) \rangle \tag{41}
\]

\[
\langle \psi_1 | \psi_1 \rangle = \frac{1}{4} \langle S(r) | P^2 | S(r) \rangle \tag{42}
\]

\[
\langle \psi_1 | \psi_2 \rangle = e^{2r} \frac{1}{4} \langle S(r) | Q^2 | S(r) \rangle \tag{43}
\]

\[
= e^{-2r},
\]
where we have used that the displacement vector of \(|S(r)\rangle\) is zero, and the covariance matrix given by Eq. (7).

\[ \langle \psi_1 | \psi_2 \rangle = -\frac{1}{4} \langle S(r) | PQ | S(r) \rangle. \]  

(44)

From the commutation relation \([Q, P] = 2i\), we get \(\text{Im} \langle PQ \rangle = -1\). The covariance of \(Q\) and \(P\) is given by

\[ V_{QP} = \frac{1}{2} \langle PQ + PQ \rangle - \langle Q \rangle \langle P \rangle \]  

(45)

\[ = \text{Re} \langle PQ \rangle - \langle Q \rangle \langle P \rangle. \]  

(46)

From Eq. (7) this should equal zero, and since the displacement vector is also zero, \(\text{Re}\{\langle S(r) | PQ | S(r) \rangle\} = 0\), so we have that

\[ \langle \psi_1 | \psi_2 \rangle = \frac{i}{4}. \]  

(47)

We introduce a set of orthonormal vectors \(\{|e_0\rangle, |e_1\rangle\}\) such that

\[ |\psi_0\rangle = |e_0\rangle \]  

(48)

\[ |\psi_1\rangle = |e_1\rangle \frac{e^r}{2} \]  

(49)

\[ |\psi_2\rangle = |e_1\rangle \frac{ie^{-r}}{2}, \]  

(50)

which satisfies the inner products. With this, the constraint Eq. (26) becomes

\[ \langle e_0 | X_1 | e_0 \rangle = 0 \]  

(51)

\[ \langle e_0 | X_2 | e_0 \rangle = 0, \]  

(52)

The density matrix for \(|\psi\rangle\) and its derivatives at the point \(\theta = 0\) are

\[ \rho_0 = \rho \bigg|_{\theta=0} \]  

\[ = |\psi_0\rangle \langle \psi_0 | \]  

(53)

\[ \rho_1 = \frac{\partial}{\partial \theta_1} \rho \bigg|_{\theta=0} \]  

\[ = |\psi_0\rangle \langle \psi_1 | + |\psi_1\rangle \langle \psi_0 | \]  

(54)

\[ \rho_2 = \frac{\partial}{\partial \theta_2} \rho \bigg|_{\theta=0} \]  

\[ = |\psi_0\rangle \langle \psi_2 | + |\psi_2\rangle \langle \psi_0 |. \]  

(55)
The constraint Eq. (27) is therefore

\[ \langle e_0 | X_1 | e_1 \rangle = e^{-r} \]  \hspace{1cm} (56)

\[ \langle e_0 | X_2 | e_1 \rangle = -ie^r. \]  \hspace{1cm} (57)

Because we are interested in the minimization of Eq. (25), we can set to zero all components of \( X_1 \) and \( X_2 \) not involved in the constraints or not complex conjugates of components involved in constraints. The minimization is trivial, and the solution occurs when

\[ Z_\theta = \begin{pmatrix} e^{-2r} & i \\ -i & e^{2r} \end{pmatrix}. \]  \hspace{1cm} (58)

The Holevo CR bound for a pure single-mode squeezed state probe is therefore

\[ C_H^\theta = 2 + 2 \cosh 2r, \]  \hspace{1cm} (59)

which equals the RLD bound and the variance from a dual homodyne measurement when \( N = 0 \) as expected.

B. Calculation of Holevo CR bound for pure two-mode squeezed probe

To calculate the Holevo CR bound for a pure two-mode squeezed probe, we follow a similar procedure as the single-mode case. The two-mode probe state can be transformed into a product state of two single-mode squeezed probes by a beam splitter with transmissivity \( \frac{1}{2} \). The beam splitter is a unitary transformation, which does not affect the Holevo CR bound. Furthermore, when \( N = 0 \), \( \rho \) has rank one, and this transformed version of \( \rho \) can be written as \( U \rho U^\dagger = |\psi\rangle \langle \psi| \) where

\[ |\psi\rangle = D(\theta/\sqrt{2}) |S(r)\rangle \otimes D(-\theta/\sqrt{2}) |S(-r)\rangle, \]  \hspace{1cm} (60)

We compute \( |\psi\rangle \) and the derivatives of \( |\psi\rangle \) with respect to \( \theta_1 \) and \( \theta_2 \) evaluated at \( \theta = 0 \).

\[ |\psi_0\rangle = |\psi\rangle \bigg|_{\theta=0} = |S(r)\rangle |S(-r)\rangle \]  \hspace{1cm} (61)

\[ |\psi_1\rangle = \frac{\partial}{\partial \theta_1} |\psi\rangle \bigg|_{\theta=0} \]
\begin{align*}
- P |S(r)\rangle |S(-r)\rangle \frac{i}{2\sqrt{2}} + |S(r)\rangle P |S(-r)\rangle \frac{i}{2\sqrt{2}}
\end{align*}

\begin{align*}
|\psi_2\rangle &= \frac{\partial}{\partial \theta_2} |\psi\rangle \bigg|_{\theta=0} \\
&= Q |S(r)\rangle |S(-r)\rangle \frac{i}{2\sqrt{2}} - |S(r)\rangle Q |S(-r)\rangle \frac{i}{2\sqrt{2}}.
\end{align*}

Of interest are the inner products involving the states $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$, so we calculate them now.

\begin{align*}
\langle \psi_0 | \psi_0 \rangle &= 1 \\
\langle \psi_0 | \psi_1 \rangle &= -\frac{i}{2\sqrt{2}} \langle S(r) | P |S(r)\rangle + \frac{i}{2\sqrt{2}} \langle S(-r) | P |S(-r)\rangle \\
&= 0 \\
\langle \psi_0 | \psi_2 \rangle &= \frac{i}{2\sqrt{2}} \langle S(r) | Q |S(r)\rangle - \frac{i}{2\sqrt{2}} \langle S(-r) | Q |S(-r)\rangle \\
&= 0 \\
\langle \psi_1 | \psi_1 \rangle &= \frac{1}{8} \langle S(r) | P^2 |S(r)\rangle + \frac{1}{8} \langle S(-r) | P^2 |S(-r)\rangle \\
&= \frac{1}{8} e^{2r} + \frac{1}{8} e^{-2r} = \frac{\cosh 2r}{4} \\
\langle \psi_2 | \psi_2 \rangle &= \frac{1}{8} \langle S(r) | Q^2 |S(r)\rangle + \frac{1}{8} \langle S(-r) | Q^2 |S(-r)\rangle \\
&= \frac{1}{8} e^{-2r} + \frac{1}{8} e^{2r} = \frac{\cosh 2r}{4} \\
\langle \psi_1 | \psi_2 \rangle &= -\frac{1}{8} \langle S(r) | PQ |S(r)\rangle - \frac{1}{8} \langle S(-r) | PQ |S(-r)\rangle \\
&= \frac{i}{4}.
\end{align*}

To satisfy the inner products, we introduce an orthonormal set of states \{\ket{e_0}, \ket{e_1}, \ket{e_2}\} such that

\begin{align*}
\ket{\psi_0} &= \ket{e_0} \\
\ket{\psi_1} &= \ket{e_1} \cosh \frac{r}{2} + \ket{e_2} \frac{\sinh r}{2} \\
\ket{\psi_2} &= \ket{e_1} \frac{i \cosh r}{2} - \ket{e_2} \frac{i \sinh r}{2}
\end{align*}

Using the construction in Eq. (71) the constraint Eq. (26) becomes

\begin{align*}
\langle e_0 | X_1 | e_0 \rangle &= 0 \\
\langle e_0 | X_2 | e_0 \rangle &= 0,
\end{align*}
and the constraint Eq. (27) becomes

\[
\begin{align*}
\text{Re}(\cosh r \langle e_0 | X_1 | e_1 \rangle + \sinh r \langle e_0 | X_1 | e_2 \rangle) &= 1 \\
\text{Re}(\cosh r \langle e_0 | X_2 | e_1 \rangle + \sinh r \langle e_0 | X_2 | e_2 \rangle) &= 0 \\
\text{Re}(i \cosh r \langle e_0 | X_1 | e_1 \rangle - i \sinh r \langle e_0 | X_1 | e_2 \rangle) &= 0 \\
\text{Re}(i \cosh r \langle e_0 | X_2 | e_1 \rangle - i \sinh r \langle e_0 | X_2 | e_2 \rangle) &= 1.
\end{align*}
\]

(74)\hspace{2cm} (75)\hspace{2cm} (76)\hspace{2cm} (77)

The matrix \( Z_\theta \) in Eq. (29) is given by

\[
Z_\theta = \begin{pmatrix}
\text{tr}(\rho_0 X_1 X_1) & \text{tr}(\rho_0 X_1 X_2) \\
\text{tr}(\rho_0 X_2 X_1) & \text{tr}(\rho_0 X_2 X_2)
\end{pmatrix}.
\]

(78)

Because we are interested in the minimization of Eq. (25), we can set to zero all components of \( X_1 \) and \( X_2 \) not involved in the constraints Eq. (72–77) or not complex conjugates of components involved in constraints. Define the components in terms of their real and imaginary parts:

\[
\begin{align*}
\langle e_0 | X_1 | e_1 \rangle &= t_1 + ij_1 \\
\langle e_0 | X_1 | e_2 \rangle &= s_1 + ik_1 \\
\langle e_0 | X_2 | e_1 \rangle &= t_2 + ij_2 \\
\langle e_0 | X_2 | e_2 \rangle &= s_2 + ik_2.
\end{align*}
\]

(79)

And so

\[
Z_\theta = \begin{pmatrix}
t_1^2 + j_1^2 + i_1^2 + k_1^2 & t_1 t_2 + j_1 j_2 + s_1 s_2 + k_1 k_2 + ((j_1 t_2 - j_2 t_1 + k_1 s_2 - k_2 s_1) \\
t_2^2 + j_2^2 + s_2^2 + k_2^2 & t_2 t_1 + j_2 j_1 + s_2 s_1 + k_2 k_1 - ((j_1 t_2 - j_2 t_1 + k_1 s_2 - k_2 s_1))
\end{pmatrix},
\]

(80)

The Holevo function from Eq. (28) becomes

\[
h = t_1^2 + t_2^2 + s_1^2 + s_2^2 + j_1^2 + j_2^2 + k_1^2 + k_2^2 + 2\abs{j_1 t_2 - j_2 t_1 + k_1 s_2 - k_2 s_1}.
\]

(81)

Using the constraints Eq. (74–77), we can eliminate four variables by making the substitutions

\[
\begin{align*}
t_1 &= \text{sech } r - s_1 \tanh r \\
j_1 &= k_1 \tanh r \\
t_2 &= -s_2 \tanh r
\end{align*}
\]
\[ j_2 = - \text{sech} \, r + k_2 \tanh r . \quad (82) \]

The problem now is to minimize \( h \) over the four remaining variables. This is accomplished in appendix A.2. We find that the minimum of \( h \) and the Holevo CR bound is \( C_H = 4 \exp(-2r) \). This is equal to the sum of the MSE obtained from the dual homodyne measurement. Hence, the dual homodyne measurement is the optimal measurement and the Holevo CR bound is tight.

C. Conclusion

We calculated the Holevo CR bound for a pure single-mode squeezed state probe experiencing a unknown displacement of \( D(\theta) \). As expected, this equals the RLD CR bound which is known to be tight.

We calculated the Holevo CR bound for a pure two-mode squeezed state probe to be \( C_H = 4 \exp(-2r) \). This bound is superior to the SLD and RLD CR bound found by \[3, 4\]. The dual homodyne measurement obtains our bound indicating that the bound is tight.

Our calculation relied on the probe state being pure, so a natural extension to our work is to find the Holevo CR bound for \( N > 0 \), i.e. when the probe is a two mode squeezed thermal state, and to determine whether the dual homodyne measurement is also optimal is this case.

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Appendix A: Calculations required for results

1. Derivatives of displacement operator

To calculate the derivatives of the displacement operator, we use the Baker-Campbell-Hausdorff identity

\[ e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} , \quad (A1) \]
where $A$ and $B$ are operators that do not commute, but commute with $[A, B]$.

$$\frac{\partial}{\partial \theta_1} D(\theta) = \frac{\partial}{\partial \theta_1} \exp \left( i \frac{\theta_2 Q - \theta_1 P}{2} \right)$$

$$= \frac{\partial}{\partial \theta_1} \exp \left( -i \theta_1 P \right) \exp \left( \frac{i}{2} \theta_2 Q \right) \exp \left( -\frac{1}{8} \theta_1 \theta_2 [P, Q] \right)$$

$$= \left( -\frac{i}{2} P - \frac{1}{8} \theta_2 [P, Q] \right) D(\theta)$$

$$= \left( -\frac{i}{2} P + \frac{i}{4} \theta_2 \right) D(\theta). \quad \text{(A2)}$$

where we have used $[Q, P] = 2i$. Similarly,

$$\frac{\partial}{\partial \theta_2} D(\theta) = \frac{\partial}{\partial \theta_2} \exp \left( i \frac{\theta_2 Q - \theta_1 P}{2} \right)$$

$$= \frac{\partial}{\partial \theta_2} \exp \left( i \theta_2 Q \right) \exp \left( -i \theta_1 P \right) \exp \left( -\frac{1}{8} \theta_1 \theta_2 [Q, P] \right)$$

$$= \left( \frac{i}{2} Q - \frac{1}{8} \theta_1 [Q, P] \right) D(\theta)$$

$$= \left( \frac{i}{2} Q - \frac{i}{4} \theta_1 \right) D(\theta). \quad \text{(A7)}$$

2. Performing the minimization for the two-mode probe

In order to contend with the absolute value in Eq. (81), we consider two cases, case 1: $g$ is greater or equal to zero, and case 2: $g$ is less than zero. We consider the case when $r \neq 0$. When $r = 0$, the problem reduces to a single-mode probe and is discussed in section \S A.

**case 1: $g \geq 0$**

Minimize

$$h = f + 2g \quad \text{(A12)}$$

subject to

$$g \geq 0 \quad \text{(A13)}$$
From the Karush-Kuhn-Tucker conditions, necessary (but not sufficient) conditions for the minimum are

\[-\nabla(f + 2g) = -\lambda \nabla g \quad (A14)\]

\[g \geq 0 \quad (A15)\]

\[\lambda \geq 0 \quad (A16)\]

\[\lambda g = 0 \quad (A17)\]

where \(\nabla = (\frac{\partial}{\partial s_1}, \frac{\partial}{\partial k_2}, \frac{\partial}{\partial k_1}, \frac{\partial}{\partial s_2})\). Equation \((A14)\) becomes

\[
\begin{pmatrix}
-2 \cosh 2r & \lambda - 2 & 0 & 0 \\
\lambda - 2 & -2 \cosh 2r & 0 & 0 \\
0 & 0 & 2 \cosh 2r & \lambda - 2 \\
0 & 0 & \lambda - 2 & 2 \cosh 2r
\end{pmatrix}
\begin{pmatrix}
s_1 \\
k_2 \\
k_1 \\
s_2
\end{pmatrix}
= \begin{pmatrix}
-\lambda \sinh r \\
-\lambda \sinh r \\
0 \\
0
\end{pmatrix}. \quad (A18)
\]

For \(r \neq 0\), this set of equations has no solutions when \(\lambda = 4 \cosh^2 r\). We thus consider \(\lambda \neq 4 \cosh^2 r\), and find

\[s_1 = k_2 = \frac{\lambda \sinh r}{4 \cosh^2 r - \lambda} \quad \text{and} \quad k_1 = s_2 = 0. \quad (A19)\]

From Eq. \((A17)\) we have that either \(\lambda = 0\) or \(g = 0\). Let us consider each case separately.

**case 1a: \(\lambda = 0\)**

When \(\lambda = 0\) the solutions \((A19)\) become

\[s_1 = k_2 = k_1 = s_2 = 0, \quad (A20)\]

which gives \(g = -\text{sech}^2 r < 0\) and violates condition \((A15)\). Hence this is not a valid solution.

**case 1b: \(g = 0\)**

When \(g = 0\), we solve for \(\lambda\) to get \(\lambda = 4e^{\pm r} \cosh r\). Both are valid solutions and give \(h = 4e^{\pm 2r}\). Although \(h = 4e^{2r}\) satisfies the Karush-Kuhn-Tucker conditions, is not the minimum so we can ignore this solution.
**case 2:** $g < 0$

Minimize

$$h = f - 2g$$ \hspace{1cm} \text{(A21)}

subject to

$$g < 0$$ \hspace{1cm} \text{(A22)}

The Karush-Kuhn-Tucker conditions now become

$$-\nabla (f - 2g) = \lambda \nabla g$$ \hspace{1cm} \text{(A23)}

$$g \leq 0$$ \hspace{1cm} \text{(A24)}

$$\lambda \geq 0$$ \hspace{1cm} \text{(A25)}

$$\lambda g = 0.$$ \hspace{1cm} \text{(A26)}

Since we require $g < 0$, condition (A26) implies $\lambda = 0$ for which condition (A23) becomes

$$\begin{pmatrix}
-\cosh 2r & 1 & 0 & 0 \\
1 & -\cosh 2r & 0 & 0 \\
0 & 0 & \cosh 2r & 1 \\
0 & 0 & 1 & -\cosh 2r
\end{pmatrix}
\begin{pmatrix}
s_1 \\
k_2 \\
k_1 \\
s_2
\end{pmatrix}
= \begin{pmatrix}
-2 \sinh r \\
-2 \sinh r \\
k_2 \\
0
\end{pmatrix}.$$ \hspace{1cm} \text{(A27)}

For $r \neq 0$, this has the solution $s_2 = k_1 = 0$, $s_1 = k_2 = \text{csch } r$ which gives $g = \text{csch}^2 r > 0$, hence is not a valid solution.

**solution**

Putting it all together, the smallest solution satisfying the Karush-Kuhn-Tucker conditions, and hence the minimum of $h$ and the Holevo bound is $C_H = 4 \exp(-2r)$.

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