Generating minimally transitive permutation groups

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Abstract

We improve the upper bounds (in terms of $n$) in [9] and [13] on the minimal number of elements required to generate a minimally transitive permutation group of degree $n$.

1 Introduction

A transitive permutation group $G \leq S_n$ is called minimally transitive if every proper subgroup of $G$ is intransitive. In this paper, we consider the minimal number of elements $d(G)$ required to generate such a group $G$, in terms of its degree $n$. For a prime factorisation $n = \prod_{p \text{ prime}} p^{n(p)}$ of $n$, we will write $\omega(n) := \sum_p n(p)$ and $\mu(n) := \max\{n(p) : p \text{ prime}\}$.

The question of bounding $d(G)$ in terms of $n$ was first considered by Shepperd and Wiegold in [13]; there, they prove that every minimally transitive group of degree $n$ can be generated by $\omega(n)$ elements. It was then suggested by Pyber (see [12]) to investigate whether or not $\mu(n) + 1$ elements would always suffice. A. Lucchini gave a partial answer to this question in [9], proving that: if $G$ is a minimally transitive group of degree $n$, and $\mu(n) + 1$ elements are not sufficient to generate $G$, then $\omega(n) \geq 2$ and $d(G) \leq \lceil \log_2(\omega(n) - 1) + 3 \rceil$.

In this note, we offer a complete solution to the problem, proving

**Theorem 1.1.** Let $G$ be a minimally transitive permutation group of degree $n$. Then $d(G) \leq \mu(n) + 1$.

Our approach follows along the same lines as Lucchini’s proof of the main theorem in [9]. Indeed, his methods suffice to prove Theorem 1.1 in the case when a minimal normal subgroup of $G$ is abelian. Thus, our main efforts will be concerned with the case when a minimal normal subgroup of $G$ is a direct product of isomorphic nonabelian simple groups. The key step in this direction is Lemma 3.1 which we prove in Section 3. We use Section 2 to outline the method of crown-based powers due to Lucchini and F. Dalla Volta; this will serve as the basis for our arguments. Finally, we prove Theorem 1.1 in Section 4.

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2 Crown-based powers

In this section, we outline an approach to study the minimal generation of finite groups, which is due to F. Dalla Volta and A. Lucchini. So let $G$ be a finite group, with $d(G) = d > 2$, and let $M$ be a normal subgroup of $G$, maximal with the property that $d(G/M) = d$. Then $G/M$ needs more generators than any proper quotient of $G/M$, and hence, as we shall see below, $G/M$ takes on a very particular structure.

We describe this structure as follows: let $L$ be a finite group, with a unique minimal normal subgroup $N$. If $N$ is abelian, then assume further that $N$ is complemented in $L$. Now, for a positive integer $k$, set $L_k$ to be the subgroup of the direct product $L_k$ defined as follows

$$L_k := \{(x_1, x_2, \ldots, x_k) : x_i \in L, Nx_i = Nx_j \text{ for all } i, j\}$$

Equivalently, $L_k := \text{diag}(L^k)^N k$, where $\text{diag}(L^k)$ denotes the diagonal subgroup of $L^k$. The group $L_k$ is called the crown-based power of $L$ of size $k$.

We can now state the theorem of Dalla Volta and Lucchini.

**Theorem 2.1 ([2], Theorem 1.4).** Let $G$ be a finite group, with $d(G) \geq 3$, which requires more generators than any of its proper quotients. Then there exists a finite group $L$, with a unique minimal normal subgroup $N$, which is either nonabelian or complemented in $L$, and a positive integer $k \geq 2$, such that $G \cong L_k$.

It is clear that, for fixed $L$, $d(L_k)$ increases with $k$. To use this result, however, we will need a bound on $d(L_k)$, in terms of $k$. This is provided by the next two theorems. Before giving the statements, we require some additional notation: for a group $G$ and a normal subgroup $M$ of $G$, let $P_{G,M}(d)$ denote the conditional probability that $d$ randomly chosen elements of $G$ generate $G$, given that their images modulo $M$ generate $G/M$.

**Theorem 2.2 ([9], Theorem 2.1 and [2], Theorem 2.7).** Let $L$ be a finite group with a unique minimal normal subgroup $N$ which is either nonabelian or complemented in $L$, and let $k$ be a positive integer. Assume also that $d(L) \leq d$. Then

(i) If $N$ is abelian, then $d(L_k) \leq \max\{d(L), k + 1\}$;

(ii) If $N$ is nonabelian, then $d(L_k) \leq d$ if and only if $k \leq P_{L,N}(d)|N|^{d}/|C_{\text{Aut}(N)}(L/N)|$.

We will also need an estimate for $P_{L,N}(d)$.

**Theorem 2.3 ([4], Theorem 1.1).** Let $L$ be a finite group, with a unique minimal normal subgroup $N$, which is nonabelian, and suppose that $d \geq d(L)$. Then $P_{L,N}(d) \geq 53/90$.

3 Indices of proper subgroups in finite simple groups

Before stating and proving the main result of this section, we need some standard notation: for a positive integer $m$, $\pi(m)$ denotes the set of prime divisors of $m$. Our lemma can now be stated as follows.
Lemma 3.1. Let $S$ be a nonabelian simple group. Then there exists a set of primes $\Gamma = \Gamma(S)$ with the following properties:

(i) $|\Gamma| \leq f(S)$, where $f(S) := r/2 + 1$ if $S$ is an alternating group of degree $r$, and $f(S) := 4$ otherwise;

(ii) $\pi(|S : H|)$ intersects $\Gamma$ nontrivially for every proper subgroup $H$ of $S$.

Proof. If $S = L_2(p)$, for some prime $p$, then since every maximal subgroup $M$ of $S$ has index divisible by either $p$ or $p + 1$ (see [5], for example), the result is clear. If $S = L_2(8), L_3(3), U_3(3)$ or $Sp_4(8)$, then direct computation using MAGMA (or Tables 8.1 to 8.6 and Table 8.14 in [1]), implies that each maximal subgroup of $S$ has index divisible by at least one of the primes in $\{2, 3\}$, $\{2, 13\}$, $\{3, 7\}$, and $\{2, 3\}$, respectively.

Next, assume that $S = A_r$ is an alternating group of degree $r$, and let $p$ and $q$ be the two largest primes not exceeding $r$, where $p > q$. If $r = p$, then we can take $\Gamma := \{r, q\}$, by Theorem 4 of [7]. So assume that $p < r$, and for each $k$ in $p \leq k \leq r - 1$, choose a prime divisor $q_k$ of $\binom{r}{k}$. Then set $\Gamma := \Gamma(A_r) = \{q_p, \ldots, q_r-1\} \cup \{p, q\}$. We claim that $\Gamma$ satisfies (i) and (ii). To see this, note that $|\Gamma| \leq r - p + 2$, which is less than $r/2 + 2$ by Bertrand’s postulate. This proves (i). To see that (ii) holds, let $H$ be a proper subgroup of $A_r$. If $p$ or $q$ does not divide $|H|$ then we are done, so assume that $pq$ divides $|H|$. Then $A_k \leq H \leq S_k \times S_{r-k}$, for some $k$ with $p \leq k \leq r - 1$, by Theorem 4 of [7]. Hence, $H$ has index divisible by $\binom{r}{k}$, and (ii) follows.

So assume that $S$ is not one of the simple groups considered in the first two paragraphs above, and let $\Pi = \Pi(S)$ be the set of prime divisors of $|S|$ discussed in Corollary 6 of [7], so that $|\Pi| \leq 3$. If $S$ does not occur in the left hand column of Table 10.7 in [7], then $\Gamma := \Pi$ satisfies the conclusion of the lemma, by Corollary 6 of [7], so assume otherwise.

Then $S$ is one of the simple groups in the left hand column of Table 10.7 in [7]; we need to prove that there exists a set $\Gamma$ as in the statement of the lemma. If $H < S$ is not one of the exceptions listed in the middle column of Table 10.7, then $|S : H|$ intersects $\Pi$ non-trivially. Thus, all we need to prove is that there exists a prime $p$ such that, whenever $H$ is one of these exceptional subgroups, then $p$ divides $|S : H|$. Indeed, in this case, $\Gamma := \Pi \cup \{p\}$ gives us what we need.

So let $H$ be one of these subgroups. We consider each of the possibilities from Table 10.7 of [7]:

1. $S = PSp_{2m}(q)$ ($m$, $q$ even) or $P\Omega_{2m+1}(q)$ ($m$ even, $q$ odd), and $\Omega_{2m}^{-}(q) \leq H$. Then $H \leq N_S(\Omega_{2m}(q))$, so $|S : N_S(\Omega_{2m}(q))|$ divides $|S : H|$. But $|N_S(\Omega_{2m}(q)) : \Omega_{2m}(q)| \leq 2$ using Corollary 2.10.4 part (i) and Table 2.1.D of [6] and, for each of the two choices of $S$, we have $|S : \Omega_{2m}(q)| = q^m(q^m - 1)$. Choosing $p$ so that $q = p^f$ now works.

2. $S = P\Omega_{2m+1}^{+}(q)$ ($m$, $q$ odd), and $\Omega_{2m-1}(q) \leq H$. As above, $H \leq N_S(\Omega_{2m-1}(q))$, and we use Corollary 2.10.4 part (i) and Table 2.1.D of [6] to conclude that $|N_S(\Omega_{2m-1}(q)) : \Omega_{2m-1}(q)| \leq 2$. It follows that $\frac{1}{2}q^{m-1}(q^m - 1) = |S : \Omega_{2m-1}(q)|$ divides $2|S : H|$. Since $m \geq 4$, choosing $p$ so that $q = p^f$ again works.
3. $S = \text{PSp}_4(q)$ and $\text{PSp}_2(q^2) \leq H$. Then $H \leq N_S(\Omega_{2m-1}(q))$, and Corollary 2.10.4 part (i) and Table 2.1.D of [6] gives $|N_S(\text{PSp}_2(q^2)) : \text{PSp}_2(q^2)| \leq 2$. It follows that $q^2(q^2 - 1) = |S : \text{PSp}_2(q^2)|$ divides $2|S : H|$. Again, the prime $p$ satisfying $q = p^f$, for some $f$, gives us what we need.

4. In each of the remaining cases (see Table 10.7 in [6]), we are given a tuple $(S, Y_1, \ldots, Y_{t(S)})$, where $t(S) \leq 4$, $S$ is one of $L_2(4)$, $L_3(3)$, $L_6(2)$, $U_3(3)$, $U_3(5)$, $U_4(2)$, $U_4(3)$, $U_5(2)$, $U_6(2)$, $\text{PSp}_4(7)$, $\text{PSp}_4(8)$, $\text{PSp}_6(2)$, $\text{PΩ}^+_8(2)$, $G_2(3)$, $2F_4(2)'$, $M_{11}$, $M_{12}$, $M_{24}$, $\text{HS}$, $M_2L$, $C_{02}$ or $C_{03}$, $Y_i < S$ for each $1 \leq i \leq t(S)$, and $H$ is contained in at least one of the groups $Y_i$. In each case, we can easily see that there is a prime $p$, with $p$ dividing $|S : Y_i|$ for each $i$ in $1 \leq i \leq t(S)$.

This completes the proof.\[\square\]

4 The proof of Theorem 1.1

Before proceeding to the proof of Theorem 1.1 we need three lemmas.

**Lemma 4.1.** Let $G$ be a transitive subgroup of $S_n$ $(n \geq 1)$, let $1 \neq M$ be a normal subgroup of $G$, and let $\Omega$ be the set of $M$-orbits. Then

(i) Either $M$ is transitive, or $\Omega$ forms a system of blocks for $G$. In particular, the size of an $M$-orbit divides $n$.

(ii) $|\Omega| = |G : AM|$, where $A$ is a point stabiliser in $G$.

(iii) If $G$ is minimally transitive, then $G^\Omega$ acts minimally transitively on $\Omega$.

**Proof.** Part (i) is clear, so we prove (ii): if $M$ is transitive, then $AM = G$, so $|\Omega| = 1 = |G : AM|$. Otherwise, part (i) implies that the size of each $M$-orbit is $|M : M \cap A| = |AM : A|$, so the number of $M$-orbits is $n/|AM : A| = |G : AM|$. Part (ii) follows. Finally, part (iii) is Theorem 2.4 in [3].\[\square\]

**Lemma 4.2 (III, Proof of Lemma 3).** Let $L$ be a finite group with a unique minimal normal subgroup $N$, which is nonabelian, and write $N \cong S^t$, where $S$ is a nonabelian simple group. Then $|C_{\text{Aut}(N)}(L/N)| \leq t|S|^t|\text{Out}(S)|$.

**Lemma 4.3 (III, Proposition 4.4).** Let $S$ be a nonabelian finite simple group. Then $|\text{Out}(S)| \leq |S|^{1/4}$.

The preparations are now complete.

**Proof of Theorem 1.1.** Assume that the theorem is false, and let $G$ be a counterexample of minimal degree. Also, let $A$ be the stabiliser in $G$ of a point $\alpha$, and let $m := \mu(n) + 1$.

First, we claim that $G$ needs more generators than any proper quotient of $G$. To this end, let $M$ be a normal subgroup of $G$, and let $K$ be the kernel of the action of $G$ on the set of
$M$-orbits. Then $G/K$ is minimally transitive of degree $s := |G : AM|$, by Lemma 4.1 and hence, since $s$ divides $n$, the minimality of $G$ implies that there exists elements $x_1, x_2, \ldots, x_m$ in $G$ such that $G = \langle x_1, x_2, \ldots, x_m, K \rangle$. But then $H := \langle x_1, x_2, \ldots, x_m \rangle$ acts transitively on the set of $M$-orbits, so $HM = G$ by minimal transitivity of $G$. Hence $d(G/M) \leq m$, which proves the claim.

Hence, by Theorem 2.1, $G \cong L_k$, for some $k \geq 2$, and some group $L$ with a unique minimal normal subgroup $N$, which is either nonabelian, or complemented in $L$. We now fix some notation: write $\text{Soc}(G) = N_1 \times N_2 \times \ldots \times N_k$, where each $N_i \cong N \cong S^t$, for some simple group $S$, and $t \geq 1$, and set $X_1 := N_1 \times N_2 \times \ldots \times N_t$. We will also write $X_0 := 1$, $H_{i+1} = N_{i+1} \cap X_i A$, and we denote by $\Delta_i$ the $X_i$-orbit containing $\alpha$, for $0 \leq i \leq k$. Then $|\Delta_i| = n |X_i A|/|G|$ by Lemma 4.1 part (ii), and hence

$$\frac{|\Delta_{i+1}|}{|\Delta_i|} = \frac{|X_{i+1} A|}{|X_i A|} = \frac{|N_{i+1} X_i A|}{|X_i A|} = |N_{i+1} : H_{i+1}|$$

Furthermore, it is shown in the proof of the main theorem in [9], that $|\Delta_{i+1}|/|\Delta_i| = |N_{i+1} : H_{i+1}|$ is greater than 1 for $0 \leq i \leq k - 2$, and also for $i = k - 1$ if $N$ is abelian. Note also that $G/\text{Soc}(G) \cong L/M$ is $m$-generated, by the previous paragraph; thus, $L$ is $m$-generated (see [10]).

We now separate the cases of $N$ being abelian or nonabelian. If $N$ is abelian, then $N \cong C^t_p$, for some prime $p$, so by the previous paragraph, $p$ divides $|N_{i+1} : H_{i+1}| = |\Delta_{i+1}|/|\Delta_i|$ for each $0 \leq i \leq k - 1$. Thus, $p^k$ divides $|\Delta_k|$, and hence divides $n$, by Lemma 4.1 part (i). It follows that $k \leq \mu(n)$, which, by Theorem 2.2 part (i), contradicts our assumption that $d(G) > \mu(n) + 1$.

Thus, $N$ is nonabelian. Hence, by the third paragraph, for each $i$ in $0 \leq i \leq k - 2$, $N_{i+1}$ has a direct factor $S_{i+1} (S_{i+1} \cong S)$, with $|S_{i+1} : S_{i+1} \cap H_{i+1}| > 1$. Let $\Gamma = \Gamma(S)$ be the set of primes in Lemma 3.1, so that $|\Gamma| \leq f(S)$, where $f(S)$ is as defined in Lemma 3.1. Then Lemma 3.1 implies that for each $0 \leq i \leq k - 2$, the index $|S_{i+1} : S_{i+1} \cap H_{i+1}|$, and hence $|\Delta_{i+1}|/|\Delta_i| = |N_{i+1} : H_{i+1}|$, is divisible by some prime $p_{i+1}$ in $\Gamma$.

So we now have a list of primes $p_1, p_2, \ldots, p_{k-1}$, with each $p_i$ in $\Gamma$, such that the product $\prod_{i=1}^{k-1} p_i$ divides $|\Delta_{k-1}|$. For each prime $p$ in $\Gamma$, let $a(p)$ be the number of times that $p$ occurs in this product. Then, since $|\Delta_{k-1}|$ divides $n$ by Lemma 4.1 (i), $\prod_{p \in \Gamma} p^{a(p)}$ divides $n$. Since $|\Gamma| \leq f(S)$, and $\sum_{p \in \Gamma} a(p) = k - 1$, we have $a(p) \geq (k - 1)/f(S)$ for at least one prime $p$ in $\Gamma$. Hence, $(k - 1)/f(S) \leq \mu(n)$, and it follows that

$$k \leq f(S) \mu(n) + 1 \leq \frac{53|S|^{\mu(n)} 90|\text{Out}(S)|}{53|N|^{m}}$$

(4.1)

$$\leq \frac{|\text{Aut}((N/L)/N)|}{90|C_{\text{Aut}(N)}(L/N)|} \quad (\text{by Lemma 4.2})$$

(4.2)

$$\leq \frac{P_{L,N}(m)|N|^m}{|C_{\text{Aut}(N)}(L/N)|} \quad (\text{by Theorem 2.3})$$

(4.3)

The inequality at (4.1) above follows easily when $S$ is an alternating group of degree $r$, since $|S| = r!/2$, and $|\text{Out}(S)| \leq 4$ in this case (also, $|\text{Out}(S)| \leq 2$ if $r \neq 6$). It also follows easily
when $S$ is not an alternating group, using Lemma 4.3. Now, by Theorem 2.2 part (ii), the inequality at (4.3) contradicts our assumption that $d(G) > m$. This completes the proof. \[\Box\]

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