THE LARGE-TIME BEHAVIOR OF SOLUTIONS IN THE CRITICAL 
$L^p$ FRAMEWORK FOR COMPRESSIBLE VISCOUS AND 
HEAT-CONDUCTIVE GAS FLOWS

WEIXUAN SHI AND JIANG XU

Abstract. The $L^p$ theory for non-isentropic Navier-Stokes equations governing compressible viscous and heat-conductive gases is not yet proved completely so far, because the critical regularity cannot control all nonlinear coupling terms. In this paper, we pose an additional regularity assumption of low frequencies in $\mathbb{R}^d (d \geq 3)$, and then the sharp time-weighted inequality can be established, which leads to the time-decay estimates of global strong solutions in the $L^p$ critical Besov spaces. Precisely, we show that if the initial data belong to some Besov space $\dot{B}^{-s_1}_{2,\infty}$ with $s_1 \in (1 - \frac{d}{2}, s_0)$ ($s_0 \equiv \frac{2d}{p} - \frac{d}{2}$), then the $L^p$ norm of the critical global solutions admits the time decay $t^{-s_1 - \frac{d}{2} \left(\frac{1}{2} - \frac{1}{p}\right)}$ (in particular, $t^{-\frac{d}{2}}$ if $s_1 = s_0$), which coincides with that of heat kernel in the $L^p$ framework. In comparison with [15], the low-frequency regularity $s_1$ can be improved to be the whole range.

1. Introduction

The compressible viscous and heat conductive gases reads as

$$\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla P &= \text{div} \tau, \\
\partial_t \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \text{div} \left[ u \left( \rho \left( \frac{|u|^2}{2} + e \right) + P \right) \right] &= \text{div} (\tau \cdot u - q)
\end{aligned}$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Here, $\rho = \rho(t, x) \in \mathbb{R}_+$ denotes the density, $u = u(t, x) \in \mathbb{R}^d$, the velocity field and $e = e(t, x) \in \mathbb{R}_+$, the internal energy per unit mass. We restrict ourselves to the case of a Newtonian fluid: the viscous stress tensor is $\tau = \lambda \text{div} u \text{Id} + 2\mu D(u)$, where $D(u) \equiv \frac{1}{2} (\nabla u + \nabla u^T)$ stands for the deformation tensor. The notations $\text{div}$ and $\nabla$ are the divergence operator and gradient operator with respect to the spatial variable $x$, respectively. The Lamé coefficients $\lambda$ and $\mu$ (the bulk and shear viscosities) are density-dependent functions, which are supposed to be smooth enough and to satisfy

$$\mu > 0 \quad \text{and} \quad \nu \triangleq \lambda + 2\mu > 0. \quad (1.2)$$

The heat conduction $q$ is given by $q = -\kappa \nabla T$, where $T$ stands for the temperature. The heat conduction coefficient $\kappa$ is assumed to be density-dependent smooth function satisfying $\kappa > 0$.

It follows from the second and third equations of (1.1) that

$$\partial_t (\rho e) + \text{div} (\rho u e) + P \text{div} u = \text{div} (\kappa (\rho \nabla T)) + 2\mu D(u) : D(u) + \lambda (\text{div} u)^2.$$
In order to reformulate (1.1) in light of ϱ, u and 𝕋 only, we make the additional assumption that the internal energy \( e = e(ϱ, 𝕋) \) satisfies Joule law:

\[
\partial_.mybatis = C_v \quad \text{for some positive constant} \ C_v
\]

and that the pressure function \( P = P(ϱ, 𝕋) \) is of the form

\[
P(ϱ, 𝕋) = \pi_0(ϱ) + 𝕋 \pi_1(ϱ),
\]

where \( \pi_0 \) and \( \pi_1 \) are given smooth functions. Such pressure laws cover the cases of ideal fluids (for which \( \pi_0(ϱ) = 0 \) and \( \pi_1(ϱ) = R\rho \) for a universal constant \( R > 0 \)), of barotropic fluids (\( \pi_1(ϱ) = 0 \)), and of Van der Waals fluids (\( \pi_0 = -\alpha \rho^2, \pi_1 = \beta \rho / (\delta - \rho) \) with \( \alpha, \beta, \delta > 0 \)). With the aid of the Gibbs relations for the internal energy and the Helmholtz free energy, we have the Maxwell relation

\[
\varrho^2 \partial_ombres e(ϱ, 𝕋) = P(ϱ, 𝕋) - 𝕋 \partial_ pornos P(ϱ, 𝕋) = \pi_0(ϱ),
\]

and end up with the following temperature equation:

\[
\varrho C_v (\partial_t 𝕋 + u \cdot \nabla 𝕋) + 𝕋 \pi_1(ϱ) \div u = 2\mu \div u + \lambda (\div u)^2 + \div (\kappa (ϱ) \nabla 𝕋).
\]

We focus on solutions that are close to some constant equilibrium \((ϱ_\infty, 0, 𝕋_\infty)\) with \( ϱ_\infty > 0 \) and \( 𝕋_\infty > 0 \) fulfilling the linear stability condition:

\[
\partial_ pornos P(ϱ_\infty, 𝕋_\infty) > 0 \quad \text{and} \quad \partial_ pornos P(ϱ_\infty, 𝕋_\infty) > 0.
\]

If System (1.1) is written in terms of \((ϱ, u, 𝕋)\), then it is not difficult to see that (1.1) is scaling invariant (neglecting the lower order pressure term) under the following transformation.

\[
ϱ(t, x) \sim ϱ(l^d t, lx), \quad u(t, x) \sim lu(l^d t, lx), \quad 𝕋(t, x) \sim l^2 𝕋(l^d t, lx), \quad l > 0.
\]

Consequently, some so-called critical spaces was employed to solve (1.1), whose norms are invariant with respect to the scaling. To the best of our knowledge, the point of view of scaling invariance is now classical and stems from the study of incompressible Navier-Stokes equations, see \([2, 16, 25]\) and references therein. In comparison with isentropic case (see \([4, 7, 9, 11, 12, 14, 17, 30, 32]\)), the \( L^p \) theory of (1.1) is not completely proved yet. Danchin \([10]\) first used general \( L^p \) Besov space (chain of spaces \( \dot{B}^{d/p}_p \times \dot{B}^{d/p-1}_p \) in fact) and established the local existence and uniqueness of solutions of (1.1). Later, Chikami and Danchin \([5]\) performed Lagrangian approach and Banach fixed point theorem to improve those results as in \([10]\) such that \( 1 < p < d \) and \( d \geq 3 \).

The exponent \( p \) seems to be optimal since the ill-posedness of (1.1) in dimension three in the sense that the continuity of data-solution map fails at the origin, was established by Chen, Miao and Zhang \([8]\) if \( p > 3 \). Danchin \([11]\) constructed the global existence and uniqueness of strong solutions to (1.1) in the \( L^2 \) critical hybrid Besov spaces (in space dimension \( d \geq 3 \)). Recently, Danchin & He \([13]\) gave the \( L^p \) extension of (1.1). For simplicity, those physical coefficients \( \lambda, \mu \) and \( \kappa \) are assumed to be constant. In fact, their results still hold true in case that \( \lambda, \mu \) and \( \kappa \) depend smoothly on the density.

A natural question is what is the large time asymptotic description of the constructed solution in \([13]\). For that issue, recall that in the framework of high Sobolev regularity, Matsumura and Nishida \([28]\) obtained the fundamental \( L^1-L^2 \) decay estimate, by assuming the initial data are the small perturbation in \( H^3(\mathbb{R}^3) \times L^1(\mathbb{R}^3) \) of \((ϱ_\infty, 0, 𝕋_\infty)\):

\[
\|(ϱ - ϱ_\infty, u, 𝕋 - 𝕋_\infty)(t)\|_{L^2(\mathbb{R}^3)} \lesssim \langle t \rangle^{-\frac{3}{4}} \quad \text{with} \quad \langle t \rangle \triangleq \sqrt{1 + t^2}.
\]
Shortly after Matsumura and Nishida, still for with high Sobolev regularity, there are a number of results on the large-time behavior of solutions to the compressible Navier-Stokes system (also including the present full case), see \cite{19,21,24,26,27,29,31,34} and references therein. Precisely, the result of \cite{25} was generalized to more physical situations where the fluid domain is not $\mathbb{R}^d$: for instance, the exterior domains were studied by Kobayashi \cite{23} and Kobayashi & Shibata \cite{24}, and the half spaces were investigated by Kagei & Kobayashi \cite{21,22}. On the other hand, there are some results available which are connected to the wave aspect of the solutions. In one dimension space, Zeng \cite{34} presented the $L^1$ convergence to the nonlinear Burgers’ diffusive wave. Hoff and Zimbrun \cite{19} performed the detailed analysis of the Green function for the multi-dimensional case and established the $L^\infty$ decay rates of diffusion waves. In \cite{27}, Liu and Wang gave pointwise convergence of solution to diffusion waves with the optimal time-decay estimate in odd dimension, where the phenomena of the weaker Huygens’ principle was also shown. This was generalized later to \cite{13} in \cite{20}. In the critical regularity framework however, there are few results concerning the time-decay estimates of (strong) global solutions to the Cauchy problem of (1.1). Very recently, Danchin and the second author \cite{15} made an attempt, where the initial data are additionally assumed to in $\dot{B}^{-s_1}_{2,\infty}$ with $[\max(0, 2 - \frac{d}{2}), s_0]$ ($s_0 \triangleq \frac{2d}{p} - \frac{d}{2}$). Consequently, the $L^p$ norm of solutions (the slightly stronger $\dot{B}^0_{p,1}$ norm in fact) decays as fast as $t^{-\frac{d}{2}}$. In particular, the rate is of $O(t^{-d(\frac{1}{p} - \frac{1}{2})})$ in case of $s_1 = s_0$. However, that is not optimal in sense of the decay rate of heat kernel (see Remark \cite{13} below).

1.1. Main results. To simplify the statement, let us assume that the density and the temperature tend to some positive constants $\varrho_\infty$ and $T_\infty$, at infinity. Setting $A \triangleq \mu_\infty \Delta + (\lambda_\infty - \mu_\infty) \nabla \text{div}$, $\varrho = \varrho_\infty (1 + b)$ and $T = T_\infty + \mathcal{E}$, we see from (1.1) and (1.5) that, whenever $b > -1$, the triplet $(b, u, \mathcal{E})$ fulfills

$$\begin{cases}
\partial_t b + u \cdot \nabla b + (1 + b) \text{div} u = 0, \\
\partial_t u + u \cdot \nabla u + \frac{\partial \varrho}{\varrho_\infty} \nabla \text{div} u = \nabla \mathcal{E} + \frac{\pi_1(\varrho_\infty(1 + b))}{\varrho_\infty(1 + b)} \nabla \mathcal{E} b \\
\partial_t \mathcal{E} + u \cdot \nabla \mathcal{E} + (T_\infty + \mathcal{E}) \frac{\pi_1(\varrho_\infty(1 + b))}{\varrho_\infty(1 + b)} \text{div} u = \frac{\kappa(\varrho_\infty(1 + b))}{\varrho_\infty(1 + b)} (\text{div} (\kappa(\varrho_\infty(1 + b)) \text{div} u)^2).
\end{cases}$$

Then, setting $\nu \triangleq 2\mu_\infty + \lambda_\infty$ ($\mu_\infty = \mu(\varrho_\infty)$ and $\lambda_\infty = \lambda(\varrho_\infty)$), $\nu_\infty \triangleq \frac{\nu}{\varrho_\infty}$, $\chi_0 \triangleq \partial_\theta P(\varrho_\infty, T_\infty)^{-\frac{1}{2}}$, and performing the change of unknowns

$$a(t, x) = b(\nu_\infty \chi_0^2 t, \nu_\infty \chi_0 x), \quad v(t, x) = \chi_0 u(\nu_\infty \chi_0^2 t, \nu_\infty \chi_0 x),$$

$$\theta(t, x) = \chi_0 \sqrt{\frac{C_\nu}{T_\infty}} \mathcal{E}(\nu_\infty \chi_0^2 t, \nu_\infty \chi_0 x),$$

we finally get

$$\begin{cases}
\partial_t a + \text{div} v = f, \\
\partial_t v - \mathcal{A} v + \nabla a + \gamma \nabla \theta = g, \\
\partial_t \theta - \beta \Delta \theta + \gamma \text{div} v = k,
\end{cases}$$

(1.9)
with
\[ \tilde{A} \triangleq \frac{A}{\nu}, \quad \beta \triangleq \frac{\kappa_\infty}{\nu C_v} \left( \kappa_\infty = \kappa(\varrho_\infty) \right), \quad \gamma \triangleq \frac{\chi_0}{\varrho_\infty} \sqrt{\frac{T_\infty}{C_v}} \pi_1(\varrho_\infty), \]
and where the nonlinear terms \( f, g \) and \( k \) are given by
\[
\begin{align*}
f & \triangleq -\text{div}(av), \\
g & \triangleq -v \cdot \nabla v - I(a)\tilde{A}v - K_1(a)\nabla a - K_2(a)\nabla \theta - \theta \nabla K_3(a) \\
& \quad + \frac{1}{(1 + a)\nu} \text{div} \left( 2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\text{div} v \text{Id} \right), \\
k & \triangleq -v \cdot \nabla \theta - \beta I(a)\Delta \theta - \left( \bar{K}_1(a) + \bar{K}_2(a)\theta \right) \text{div} v + \frac{1}{\nu(1 + a)} \text{div} (\bar{\kappa}(a)\nabla \theta) \\
& \quad + \frac{1}{\nu \chi_0} \sqrt{\frac{1}{T_\infty C_v}} \left( \frac{2\mu(\varrho_\infty(1 + a))}{1 + a} \text{div} v : D(v) + \frac{\lambda(\varrho_\infty(1 + a))}{1 + a}(\text{div} v)^2 \right)
\end{align*}
\]
with
\[
\begin{align*}
I(a) & \triangleq \frac{a}{1 + a}, \quad \tilde{\mu}(a) \triangleq \mu(\varrho_\infty(1 + a)) - \mu(\varrho_\infty), \quad \tilde{\lambda}(a) \triangleq \lambda(\varrho_\infty(1 + a)) - \lambda(\varrho_\infty), \\
K_1(a) & \triangleq \partial_\varrho P(\varrho_\infty(1 + a), T_\infty) - 1, \quad K_2(a) \triangleq \chi_0 \frac{\varrho_\infty}{\varrho_\infty} \sqrt{\frac{T_\infty}{C_v}} \left( \frac{\pi_1(\varrho_\infty(1 + a))}{1 + a} - \pi_1(\varrho_\infty) \right), \\
K_3(a) & \triangleq \chi_0 \sqrt{\frac{T_\infty}{C_v}} \int_0^a \frac{\pi_1(\varrho_\infty(1 + z))}{1 + z} dz, \quad \bar{K}_2(a) \triangleq \frac{\pi_1(\varrho_\infty(1 + a))}{C_v \varrho_\infty(1 + a)}, \\
\bar{K}_1(a) & \triangleq \frac{\chi_0}{\varrho_\infty} \sqrt{\frac{T_\infty}{C_v}} \left( \frac{\pi_1(\varrho_\infty(1 + a))}{1 + a} - \pi_1(\varrho_\infty) \right), \quad \bar{\kappa}(a) \triangleq \kappa(\varrho_\infty(1 + a)) - \kappa(\varrho_\infty).
\end{align*}
\]

Note that \( K_1, K_2, K_3, \bar{K}_1, \bar{K}_2, \bar{\mu}, \bar{\lambda} \) and \( \bar{\kappa} \) are smooth functions satisfying \( K_1(0) = K_2(0) = K_3(0) = \bar{K}_1(0) = \bar{\mu}(0) = \bar{\lambda}(0) = \bar{\kappa}(0) = 0 \).

The main result of the paper is stated as follows.

**Theorem 1.1.** Let \( \varrho_\infty > 0 \) and \( T_\infty \) be two constant such that \[ (1.6) \] is fulfilled. Suppose that \( d \geq 3 \), and that \( p \) satisfies
\[
2 \leq p < d \quad \text{and} \quad p \leq \frac{2d}{d - 2}.
\]

Let \((a, v, \theta)\) be the corresponding global solution to \[ (1.9) \] with the initial data \((a_0, v_0, \theta_0)\) such that was constructed in \[ [13] \]. Let
\[
1 - \frac{d}{2} < s_1 \leq s_0 \left( s_0 \triangleq \frac{2d}{p} - \frac{d}{2} \right).
\]

There exists a positive constant \( c = c(p, d, \lambda, \mu, P, \kappa, C_v, \varrho_\infty, T_\infty) \) such that if
\[
D_{p, 0} \triangleq \| (a_0, v_0, \theta_0) \|_{B_{2, \varrho_\infty}^{s_1}} \leq c,
\]
then it holds that
\[
D_p(t) \lesssim (D_{p, 0} + \| (\nabla a_0, v_0) \|_{B_{p, 1}^{s_1}}^h + \| \theta_0 \|_{B_{p, 1}^{s_1}}^h)^{\ell} \quad \text{for all} \quad t \geq 0,
\]
where the functional $\mathcal{D}_p(t)$ is defined by
\[
\mathcal{D}_p(t) \triangleq \sup_{s \in [e^{-s_1}, s_1]} \left\| (\tau)^{\frac{s_1}{2} + \frac{s}{2}} (a, v, \theta) \right\|_{L_t^\infty(B^{\frac{s}{2}}_{2,1})}^\ell + \left\| (\tau)^{\alpha} (\nabla a, v) \right\|_{L_t^\infty(B^{\frac{s}{2}}_{p,1})}^h
\]
(1.14) \[+ \left\| (\tau)^{\alpha} \theta \right\|_{L_t^\infty(B^{\frac{s}{2}}_{p,1})}^h + \left\| \tau^\alpha (\nabla v, \theta) \right\|_{L_t^\infty(B^{\frac{s}{2}}_{p,1})}^h \]

with $\alpha \triangleq s_1 + \frac{d}{2} + \frac{1}{2} - \varepsilon$ for sufficiently small $\varepsilon > 0$.

**Remark 1.1.** Theorem 1.1 investigates the case of $s_1$ belonging to the whole range $(1 - \frac{d}{4}, s_0]$, which is open left in (1.1). The sharp lower bound stems from the elementary time-decay inequality. More precisely,
\[
\int_0^t (t - \tau)^{-\frac{s_1}{2} + \frac{s}{2}} (\tau)^{-\delta} d\tau \leq (t)^{-\frac{s_1 + s}{2}}, \quad 0 \leq \frac{s_1 + s}{2} \leq \delta, \quad \delta > 1.
\]
In subsequent low-frequency analysis, the minimum value of $\delta$ is $s_1/2 + d/4 + 1/2$, owing to $s \leq d/2 + 1$. Consequently, $s_1/2 + d/4 + 1/2 > 1$ yields the desired lower bound. In addition, Theorem 1.1 holds in case that $\mu, \lambda$ and $\kappa$ depend smoothly on the density.

**Remark 1.2.** If (1.12) is replaced by the slightly stronger hypothesis:
\[
\left\| (a_0, v_0, \theta_0) \right\|_{B^{\frac{s_1 - 1}{2}, 1}_2}^\ell \leq c \ll 1,
\]
then one can take $\varepsilon = 0$ in both $\alpha$ and $\mathcal{D}_p(t)$.

As a consequence of Theorem 1.1, the time decay estimates of the $L^p$ norm (the slightly stronger $\dot{B}^0_{p,1}$ norm in fact) of solutions.

**Corollary 1.1.** Under the additional assumption (1.11)-(1.12), the global solution satisfies
\[
\| \Lambda^s a \|_{L^r} \lesssim A_0 \left( t \right)^{-\frac{s_1}{2} + \frac{s}{2} \left( \frac{1}{2} - \frac{1}{r} \right)} \text{ if } -\tilde{s}_1 < s + d \left( \frac{1}{p} - \frac{1}{r} \right) \leq \frac{d}{p},
\]
\[
\| \Lambda^s v \|_{L^r} \lesssim A_0 \left( t \right)^{-\frac{s_1}{2} + \frac{s}{2} \left( \frac{1}{2} - \frac{1}{r} \right)} \text{ if } -\tilde{s}_1 < s + d \left( \frac{1}{p} - \frac{1}{r} \right) \leq \frac{d}{p} + 1,
\]
\[
\| \Lambda^s \theta \|_{L^r} \lesssim A_0 \left( t \right)^{-\frac{s_1}{2} + \frac{s}{2} \left( \frac{1}{2} - \frac{1}{r} \right)} \text{ if } -\tilde{s}_1 < s + d \left( \frac{1}{p} - \frac{1}{r} \right) \leq \frac{d}{p},
\]
for $t \geq 0$ and $p \leq r \leq \infty$, where $-\tilde{s}_1 \triangleq s_1 + d \left( \frac{1}{p} - \frac{1}{r} \right), A_0 \triangleq (\mathcal{D}_{p,0} + \| (\nabla a_0, v_0) \|_{B^{\frac{s}{2}}_{p,1}}^h + \| \theta_0 \|_{B^{\frac{s}{2}}_{p,1}}^h)$ and the operator $\Lambda^s$ is defined by $\Lambda^s f \triangleq \mathcal{F}^{-1} (|\xi|^s \mathcal{F} f)$ for $s \in \mathbb{R}$.

**Remark 1.3.** For convenience of reader, let us show the decay rates of heat kernel $E(t)U_0 \triangleq e^{-t\Delta} U_0$ first. In Fourier variable, we have
\[
\mathcal{F}[E(t)U_0](\xi) = e^{-|\xi|^2} \mathcal{F}U_0(\xi).
\]
It follows from Hausdorff-Young and Hölder inequalities that
\[
\| E(t)U_0 \|_{L^p} \lesssim \| \mathcal{F}[E(t)U_0](\xi) \|_{L^p} \lesssim \| \mathcal{F}U_0(\xi) \|_{L^{p'}} \| e^{-|\xi|^2} \|_{L^m} \lesssim \| U_0 \|_{L^q} t^{-\frac{d}{2m}},
\]
where $1/p + 1/p' = 1/q + 1/q' = 1, 1/p' = 1/q' = 1/m$ and $p \geq 2$. Hence, one can get $m = p$ if choosing $q = p/2$, which is, the heat kernel enjoys the time-decay rate of $O(t^{-d/p})$ in $L^p$ norm if $U_0 \in L^{p/2}$. Noticing the embedding $L^{p/2} \hookrightarrow B^{s_0}_{2,\infty}$, we see that
the global solution of (1.9) decays to constant equilibrium with the same rate if taking the endpoint regularity $s_1 = s_0$. Those decay rates in Corollary 1.1 are thus optimal and satisfactory.

Prompted by the recent work dedicated to the compressible barotropic flow (see [14]), we here aim at proving Theorem 1.1. The additional unknown $\theta$ cannot contribute more regularities in term of (1.7), so those nonlinear terms between density, velocity field and temperature need to be treated carefully. Up to now, the global-in-time existence and large-time behavior of solutions of (1.1) remains open in dimension two, which is left for future consideration. In contrast to [15], the low-frequency analysis for $1 - d/2 < s_1 < \max(0, 2 - d/2)$ is much more technical. Owing to the heat smoothing effect, it is possible to adapt the standard Duhamel principle treating the nonlinear right-hand side $(f, g, k)$ of (1.9). Precisely, we split the nonlinear term $(f, g, k)$ into $(f^e, g^e, k^e)$ and $(f^h, g^h, k^h)$ (see the context below). In order to handle $(f^e, g^e, k^e)$ in the time-weighted integral, some new and non-standard Besov product estimates are well developed, see [3.0]-[3.3]. Secondly, bounding the term $(f^h, g^h, k^h)$ for example, $k_1(a, \theta^h)$ is more elaborate due to the less regularity of $\theta$, where different Sobolev embeddings are mainly employed. See Lemmas 3.2-3.3 for more details.

On the other hand, we proceed differently for the analysis of the high frequencies decay of the solution, since there is no smooth effect for $a$. Indeed, the idea is to work with a so-called “effective velocity” $w$ (which was initiated by Hoff [18] and first used in the context of critical regularity by Haspot [17]) such that, up to low order terms, the divergence-free part of $v$, the temperature $\theta$ and $w$ satisfy a parabolic system while $a$ fulfills a damped transport equation. Then, by employing $L^p$ energy argument directly on these equations after localization, one can eventually obtain optimal decay exponents for high frequencies.

The rest of the paper unfolds as follows. In Section 2, we recall Littlewood-Paley decomposition, Besov spaces and related analysis tools. Section 3 is devoted to the proofs of Theorem 1.1 and Corollary 1.1.

2. Preliminary

Throughout the paper, $C > 0$ stands for a generic “constant”. For brevity, we write $f \lesssim g$ instead of $f \leq Cg$. The notation $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$. For any Banach space $X$ and $f, g \in X$, we agree that $\|f\|_X = \|f\|_X + \|g\|_X$. For all $T > 0$ and $\rho \in [1, +\infty]$, we denote by $L^\rho_T(X) = L^\rho([0, T]; X)$ the set of measurable functions $f : [0, T] \to X$ such that $t \mapsto \|f(t)\|_X$ is in $L^\rho(0, T)$.

Let us next briefly recall Littlewood-Paley decomposition, Besov spaces and analysis tools. The interested reader is referred to Chap. 2 and Chap. 3 of [1] for more details. We begin with the homogeneous Littlewood-Paley decomposition. To this end, we fix some smooth radial non increasing function $\chi$ with $\operatorname{Supp} \chi \subset B(0, \frac{4}{3})$ and $\chi \equiv 1$ on $B(0, \frac{3}{4})$, then set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ so that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1 \text{ in } \mathbb{R}^d \setminus \{0\} \text{ and } \operatorname{Supp} \varphi \subset \left\{ \xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3 \right\}.$$ 

The homogeneous dyadic blocks $\hat{\Delta}_j$ are defined by

$$\hat{\Delta}_j f \triangleq \varphi(2^{-j} D) f = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} f) = 2^{jd} h(2^j \cdot) \ast f \text{ with } h \triangleq \mathcal{F}^{-1} \varphi.$$
Formally, we have the homogeneous decomposition as follows

\[(2.1) \quad f = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f,\]

for any tempered distribution \(f \in S'({\mathbb{R}^d})\). As it holds only modulo polynomials, it is convenient to consider the subspace of those tempered distributions \(f\) such that

\[(2.2) \quad \lim_{j \to -\infty} \|\hat{S}_j f\|_{L^\infty} = 0,\]

where \(\hat{S}_j f\) stands for the low frequency cut-off \(\hat{S}_j f \triangleq \chi(2^{-j}D)f\). As a matter of fact, if \((2.2)\) is fulfilled, then \((2.1)\) holds in \(S'(\mathbb{R}^d)\). For convenience, we denote by \(S'_0(\mathbb{R}^d)\) the subspace of tempered distributions satisfying \((2.2)\).

With the aid of the Littlewood-Paley decomposition, the homogeneous Besov space is defined as follows.

**Definition 2.1.** For \(\sigma \in \mathbb{R}\) and \(1 \leq p, r \leq \infty\), the homogeneous Besov spaces \(\dot{B}_{p,r}^\sigma\) is defined by

\[\dot{B}_{p,r}^\sigma \triangleq \left\{ f \in S'_0 : \|f\|_{\dot{B}_{p,r}^\sigma} < +\infty \right\},\]

where

\[(2.3) \quad \|f\|_{\dot{B}_{p,r}^\sigma} \triangleq \|(2^{j\sigma}\|\Delta_j f\|_{L^p})\|_{\ell^r(\mathbb{Z})}.\]

In many parts of this paper, we use the following classical properties (see [1]):

- **Scaling invariance:** For any \(\sigma \in \mathbb{R}\) and \((p, r) \in [1, \infty]^2\), there exists a constant \(C = C(\sigma, p, r, d)\) such that for all \(\lambda > 0\) and \(f \in \dot{B}_{p,r}^\sigma\), we have

  \[C^{-1}\lambda^{\sigma - \frac{d}{p}} \|f\|_{\dot{B}_{p,r}^\sigma} \leq \|f(\lambda \cdot)\|_{\dot{B}_{p,r}^\sigma} \leq C\lambda^{\sigma - \frac{d}{p}} \|f\|_{\dot{B}_{p,r}^\sigma}.\]

- **Completeness:** \(\dot{B}_{p,r}^\sigma\) is a Banach space whenever \(\sigma < \frac{d}{p}\) or \(\sigma \leq \frac{d}{p}\) and \(r = 1\).
- **Interpolation:** The following inequality is satisfied for \(1 \leq p_1, r_1, r_2, r \leq \infty\), \(\sigma_1 \neq \sigma_2\) and \(\theta_1 \in (0, 1)\):

  \[\|f\|_{\dot{B}^{\theta_1 \sigma_1 + (1 - \theta_1) \sigma_2}_{p_1, r_1, r} \lesssim \|f\|_{\dot{B}^{\sigma_1}_{p_1, r_1}}^\theta \|f\|_{\dot{B}^{\sigma_2}_{p_2, r_2}}^{1 - \theta}\]

  with \(\frac{1}{r} = \frac{\theta_1}{r_1} + \frac{1 - \theta_1}{r_2}\).
- **Action of Fourier multipliers:** If \(F\) is a smooth homogeneous of degree \(m\) function on \(\mathbb{R}^d \setminus \{0\}\) then

  \[F(D) : \dot{B}^{\sigma}_{p,r} \to \dot{B}^{-m}_{p,r}.\]

The following embedding properties are used several times in this paper.

**Proposition 2.1.** *(Embedding for Besov spaces on \(\mathbb{R}^d\))*

- For any \(p \in [1, \infty]\) we have the continuous embedding \(\dot{B}^{0}_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^{0}_{p,\infty}\).
- If \(\sigma \in \mathbb{R}\), \(1 \leq p_1, p_2 \leq \infty\) and \(1 \leq r_1, r_2 \leq \infty\), then \(\dot{B}^{\sigma}_{p_1, r_1} \hookrightarrow \dot{B}^{\sigma - \frac{d}{p_1} - \frac{1}{r_1}}_{p_2, r_2}\).
- The space \(\dot{B}^{\sigma}_{p,1}\) is continuously embedded in the set of bounded continuous functions (going to zero at infinity if, additionally, \(p < \infty\)).

Let us mention the following product estimate in the Besov spaces, which plays a fundamental role in bounding bilinear terms of \((1.9)\) (see [11,14]).
Proposition 2.2. Let $\sigma > 0$ and $1 \leq p, r \leq \infty$. Then $\dot{B}^\sigma_{p, r} \cap L^\infty$ is an algebra and
$$\|fg\|_{\dot{B}^\sigma_{p, r}} \lesssim \|f\|_{L^\infty} \|g\|_{\dot{B}^\sigma_{p, r}} + \|g\|_{L^\infty} \|f\|_{\dot{B}^\sigma_{p, r}}.$$

Let the real numbers $\sigma_1, \sigma_2, p_1$ and $p_2$ fulfill
$$\sigma_1 + \sigma_2 > 0, \quad \sigma_1 \leq \frac{d}{p_1}, \quad \sigma_2 \leq \frac{d}{p_2}, \quad \sigma_1 \geq \sigma_2, \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1.$$

Then we have
$$\|fg\|_{\dot{B}^\sigma_{p, r}} \lesssim \|f\|_{\dot{B}^\sigma_{p_1, 1}} \|g\|_{\dot{B}^\sigma_{p_2, 1}}$$
with $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma_1}{d}$.

Additionally, for exponents $\sigma > 0$ and $1 \leq p_1, p_2, q \leq \infty$ satisfying
$$\frac{d}{p_1} + \frac{d}{p_2} - d \leq \sigma \leq \min \left( \frac{d}{p_1}, \frac{d}{p_2} \right) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\sigma}{d},$$
we have
$$\|fg\|_{\dot{B}^{\sigma}_{p, r}} \lesssim \|f\|_{\dot{B}^{\sigma}_{p_1, 1}} \|g\|_{\dot{B}^{\sigma}_{p_2, \infty}}.$$

Proposition 2.3 are not enough to bound all nonlinear terms in the proof of Theorem 1.1 so we need to the following non standard product estimates (see [14][33]).

Proposition 2.3. Let the real numbers $\sigma_1, \sigma_2, p_1$ and $p_2$ be such that
$$\sigma_1 + \sigma_2 \geq 0, \quad \sigma_1 \leq \frac{d}{p_1}, \quad \sigma_2 < \min \left( \frac{d}{p_1}, \frac{d}{p_2} \right) \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1.$$

Then it holds that
$$\|fg\|_{\dot{B}^{\sigma_1+\sigma_2}_{p_1, p_2, \infty}} \lesssim \|f\|_{\dot{B}^{\sigma_1}_{p_1, 1}} \|g\|_{\dot{B}^{\sigma_2}_{p_2, \infty}}.$$

Proposition 2.4. Let $j_0 \in \mathbb{Z}$, and denote $z^\ell \triangleq \dot{S}_{j_0}z$, $z^h \triangleq z - z^\ell$ and, for any $\sigma \in \mathbb{R}$,
$$\|z\|_{\tilde{B}^\ell_{2, \infty}} \triangleq \sup_{j \leq j_0} 2^{j\sigma}\|\Delta_j z\|_{L^2}.$$

There exists a universal integer $N_0$ such that for any $2 \leq p \leq 4$ and $\sigma > 0$, we have

$$\|fg^h\|_{\tilde{B}^\ell_{2, \infty}} \leq C \left( \|f\|_{\dot{B}^\sigma_{p_1, 1}} + \|\dot{S}_{j_0+N_0}f\|_{L^{p_1}} \right) \|g^h\|_{\tilde{B}^{-\sigma}_{p, \infty}};$$

(2.4)

$$\|f^h g^h\|_{\tilde{B}^\ell_{2, \infty}} \leq C \left( \|f^h\|_{\dot{B}^\sigma_{p_1, 1}} + \|\dot{S}_{j_0+N_0}f^h\|_{L^{p_1}} \right) \|g^h\|_{\tilde{B}^{-\sigma}_{p, \infty}};$$

(2.5)

with $s_0 \triangleq \frac{2d}{p} - \frac{d}{2}$ and $\frac{1}{p} \triangleq \frac{1}{2} - \frac{1}{p}$, and $C$ depending only on $j_0$, $d$ and $\sigma$.

System (1.3) also involves compositions of functions (through $K_1(a), K_2(a), K_3(a)$ $\tilde{K}_1(a), \tilde{K}_2(a), \tilde{K}_3(a)$) that are handled due to the following proposition.

Proposition 2.5. Let $F : \mathbb{R} \to \mathbb{R}$ be smooth with $F(0) = 0$. For all $1 \leq p, r \leq \infty$ and $\sigma > 0$, we have $F(f) \in \dot{B}^\sigma_{p, r} \cap L^\infty$ for $f \in \dot{B}^\sigma_{p, r} \cap L^\infty$, and
$$\|F(f)\|_{\dot{B}^\sigma_{p, r}} \leq C(f) \|f\|_{\dot{B}^\sigma_{p, r}}$$
with $C$ depending only on $\|f\|_{L^\infty}$, $F'$ (and higher derivatives), $\sigma$, $p$ and $d$.

In the case $\sigma > - \min \left( \frac{d}{p}, \frac{d}{p_1} \right)$ then $f \in \dot{B}^\sigma_{p, r} \cap \dot{B}^{\frac{d}{p}}_{p, 1}$ implies that $F(f) \in \dot{B}^\sigma_{p, r} \cap \dot{B}^{\frac{d}{p}}_{p, 1}$, and we have
$$\|F(f)\|_{\dot{B}^\sigma_{p, r}} \leq C(1 + \|f\|_{\dot{B}^{\frac{d}{p}}_{p, 1}}) \|f\|_{\dot{B}^\sigma_{p, r}}.$$
In addition, we also notice the classical Bernstein inequality:
\begin{equation}
\|D^k f\|_{L^p} \leq C^{1+k}\lambda^{k+d\left(\frac{1}{a} - \frac{1}{b}\right)}\|f\|_{L^a}
\end{equation}
that holds for all function \(f\) such that \(\text{Supp} F f \subset \{\xi \in \mathbb{R}^d : |\xi| \leq R\lambda\}\) for some \(R > 0\) and \(\lambda > 0\), if \(k \in \mathbb{N}\) and \(1 \leq a \leq b \leq \infty\).

More generally, if we suppose \(f\) to satisfy \(\text{Supp} F f \subset \{\xi \in \mathbb{R}^d : R_1 \lambda \leq |\xi| \leq R_2 \lambda\}\) for some \(0 < R_1 < R_2\) and \(\lambda > 0\), then for any smooth homogeneous of degree \(m\) function \(A\) on \(\mathbb{R}^d \setminus \{0\}\) and \(1 \leq a \leq \infty\), we get (see e.g. Lemma 2.2 in [1]):
\begin{equation}
\|A(D)f\|_{L^a} \approx \lambda^m\|f\|_{L^a}.
\end{equation}

An obvious consequence of (2.6) and (2.7) is that
\begin{equation}
(2.8)
\end{equation}
Furthermore, Minkowskis inequality allows us to compare standard Lebesgue-Besov semi-norms of \(L^p\) space and \(\lambda > 0\) such that holds for all function \(f\) for some \(0 < R < R_1\) and \(\lambda > 0\).

In order to state optimal regularity estimates for the heat equation, a class of mixed space-time Besov spaces are also used, which was initiated by J.-Y. Chemin and N. Lerner [6] (see also [3] for the particular case of Sobolev spaces).

**Definition 2.2.** For \(T > 0, \sigma \in \mathbb{R}, 1 \leq r, \rho \leq \infty\), the homogeneous Chemin-Lerner space \(\tilde{L}_T^\rho(B^{\sigma}_{p,r})\) is defined by
\[
\tilde{L}_T^\rho(B^{\sigma}_{p,r}) \triangleq \left\{ f \in L^\rho(0,T; S_0^\rho) : \|f\|_{\tilde{L}_T^\rho(B^{\sigma}_{p,r})} < +\infty \right\},
\]
where
\begin{equation}
(2.8)
\end{equation}
For notational simplicity, index \(T\) is omitted if \(T = +\infty\). We denote
\[
\tilde{C}_b(\mathbb{R}^+; B^{\sigma}_{p,r}) \triangleq \left\{ f \in \mathcal{C}(\mathbb{R}^+; B^{\sigma}_{p,r}) \text{ s.t } \|f\|_{\tilde{L}_T^\infty(B^{\sigma}_{p,r})} < +\infty \right\}.
\]
Furthermore, Minkowskis inequality allows us to compare \(\| \cdot \|_{\tilde{L}_T^\rho(B^{\sigma}_{p,r})}\) with the more standard Lebesgue-Besov semi-norms of \(L_T^\rho(B^{\sigma}_{p,r})\) as follows.

**Remark 2.1.** It holds that
\[
\|f\|_{\tilde{L}_T^\rho(B^{\sigma}_{p,r})} \leq \|f\|_{L_T^\rho(B^{\sigma}_{p,r})} \text{ if } r \geq \rho, \quad \|f\|_{\tilde{L}_T^\rho(B^{\sigma}_{p,r})} \geq \|f\|_{L_T^\rho(B^{\sigma}_{p,r})} \text{ if } r \leq \rho.
\]

Restricting the above norms (2.3) and (2.8) to the low or high frequencies parts of distributions will be fundamental in our method. For that purpose, we shall often use the following notation for some suitable index \(j_0\) \footnote{Note that for technical reasons, we need a small overlap between low and high frequencies.}
\[
\|f\|_{\tilde{L}_T^{\rho}_{p,1}(B^{\sigma}_{p,1})} \triangleq \sum_{j \leq j_0} 2^{j\sigma}\|\hat{\Delta}_j f\|_{L^p} \quad \text{and} \quad \|f\|_{\tilde{L}_T^{\rho}_{p,1}(B^{\sigma}_{p,1})} \triangleq \sum_{j \geq j_0} 2^{j\sigma}\|\hat{\Delta}_j f\|_{L^p},
\]
\[
\|f\|_{\tilde{L}_T^{\rho}_{p,1}(B^{\sigma}_{p,1})} \triangleq \sum_{j \leq j_0} 2^{j\sigma}\|\hat{\Delta}_j f\|_{L^p(\tilde{L}^\infty)} \quad \text{and} \quad \|f\|_{\tilde{L}_T^{\rho}_{p,1}(B^{\sigma}_{p,1})} \triangleq \sum_{j \geq j_0} 2^{j\sigma}\|\hat{\Delta}_j f\|_{\tilde{L}^\rho(\tilde{L}^\infty)}.
\]

Finally, we end this section with the parabolic regularity estimates for the heat equation.

**Proposition 2.6.** Let \(\sigma \in \mathbb{R}, (p, r) \in [1, \infty]^2\) and \(1 \leq \rho_2 \leq \rho_1 \leq \infty\). Let \(u\) satisfy
\[
\left\{ \begin{array}{l}
\partial_t u - \mu \Delta u = f, \\
u|_{t=0} = u_0.
\end{array} \right.
\]
Then for all $T > 0$ the following a priori estimate is fulfilled:
\begin{equation}
\frac{1}{\mu^\sigma} \|u\|_{L_T^p(B_{p, r}^{\sigma, 2})} \lesssim \|u_0\|_{\dot{B}_{p, r}^\sigma} + \mu^{\sigma_2 - 1} \|f\|_{L_T^p(B_{p, r}^{\sigma - \sigma_2, 2})}.
\end{equation}

**Remark 2.2.** The solutions to the following Lamé system
\begin{equation}
\begin{cases}
\partial_t u - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u = f, \\
u|_{t=0} = u_0,
\end{cases}
\end{equation}
where $\lambda$ and $\mu$ are constant coefficients such that $\mu > 0$ and $\lambda + 2\mu > 0$, also fulfill (2.9) (up to the dependence w.r.t. the viscosity). Indeed, if we denote by $P \triangleq \text{Id} - \nabla(-\Delta)^{-1}\text{div}$ and $Q \triangleq \text{Id} - P$ the orthogonal projectors over divergence-free and potential vector fields, then we see both $Pf$ and $Qf$ satisfy the heat equation, as it can easily be observed by applying $P$ and $Q$ to (2.10).

### 3. The proof of Time-decay estimates

This section is devoted to the proof of Theorem 1.1 taking for granted the global existence result in [13]. We denote by $\mathcal{X}_p(t)$ the energy norm:
\begin{equation}
\mathcal{X}_p(t) \triangleq \|a(\tau, \theta)\|_{L_t^p(B_{p, 1}^{\sigma, -1})}^\ell + \|a(\tau, \theta)\|_{L_t^1(B_{p, 1}^{\sigma, 1})}^\ell + \|\nabla a(\tau, \theta)\|_{L_t^1(B_{p, 1}^{\sigma, 1})}^h + \|\nabla \text{div} u(\tau, \theta)\|_{L_t^1(B_{p, 1}^{\sigma, 1})}^h.
\end{equation}

In what follows, we shall use repeatedly the following obvious inequality that is satisfied whenever $0 \leq \sigma_1 \leq \sigma_2$ and $\sigma_2 > 1$:
\begin{equation}
\int_0^t \langle t - \tau \rangle^{-\sigma_1} \langle \tau \rangle^{-\sigma_2} d\tau \lesssim (t)^{-\sigma_1}.
\end{equation}

Let us keep in mind that the global solution $(a, v, \theta)$ satisfies
\begin{equation}
\|a\|_{L_t^\infty(B_{p, 1}^{\sigma, 1})} \leq c \ll 1 \text{ for all } t \geq 0.
\end{equation}

#### 3.1. First step: Bounds for the low frequencies

Let $(E(t))_{t \geq 0}$ be the semi-group associated with the left-hand side of (1.9). The standard Duhamel principle yields
\begin{equation}
\begin{pmatrix}
a(t) \\
v(t) \\
\theta(t)
\end{pmatrix} = E(t) \begin{pmatrix}
a_0 \\
v_0 \\
\theta_0
\end{pmatrix} + \int_0^t E(t - \tau) \begin{pmatrix}
f(\tau) \\
g(\tau) \\
k(\tau)
\end{pmatrix} d\tau.
\end{equation}

First of all, we state smoothing estimate of the linearized solution $(a_L, v_L, \theta_L) \triangleq E(t)(a_0, v_0, \theta_0)$, which behaves like that of heat kernel.

**Lemma 3.1.** Let $(a_L, v_L, \theta_L)$ be the solution to the following system
\begin{equation}
\begin{cases}
\partial_t a_L + \text{div} v_L = 0, \\
\partial_t v_L - \tilde{A} v_L + \nabla a_L + \gamma \nabla \theta_L = 0, \\
\partial_t \theta_L - \beta \Delta \theta_L + \gamma \text{div} v_L = 0
\end{cases}
\end{equation}
with the initial data $(a_L, v_L, \theta_L)|_{t=0} = (a_0, v_0, \theta_0)$. 

Then, for any \( j_0 \in \mathbb{Z} \), there exists a positive constant \( c_0 = c_0(\lambda_\infty, \mu_\infty, \beta, \gamma, j_0) \) such that
\[
\|(a_{ij}, v_{ij}, \theta_{ij})(t)\|_{L^2} \lesssim e^{-c_0t} \|(a_{ij}, v_{ij}, \theta_{ij})(0)\|_{L^2}
\]
for \( t \geq 0 \) and \( j \leq j_0 \), where we set \( z_j = \hat{\Delta}z_j \) for any \( z \in \mathcal{S}'(\mathbb{R}^d) \).

The interested reader is referred to [15] for the proof of Lemma 3.1. Set \( U \triangleq (a, v, \theta) \) and \( U_0 \triangleq (a_0, v_0, \theta_0) \). From Lemma 3.1, we perform the same procedure as in [14, 15] to obtain for \( s + s_1 > 0 \),
\[
\sup_{t \geq 0} \|E(t)U_0\|_{\tilde{H}^s_{g,1}} \lesssim \|U_0\|_{\tilde{H}^{s_1+s}_{g,1}}.
\]
Additionally, it is clear that for \( s + s_1 > 0 \),
\[
\|E(t)U_0\|_{\tilde{H}^s_{g,1}} \lesssim \|U_0\|_{\tilde{H}^{s_1+s}_{g,1}}.
\]
Then it follows that
\[
\sup_{t \geq 0} (t)^{-\frac{s_1+s}{2}} \|E(t)U_0\|_{\tilde{H}^s_{g,1}} \lesssim \|U_0\|_{\tilde{H}^{s_1+s}_{g,1}} \quad \text{with} \quad \langle t \rangle \triangleq \sqrt{1 + t^2}.
\]
Consequently, with the aid of Duhamel formula, we end up with
\[
\langle a, v, \theta \rangle \|_{\tilde{H}^s_{g,1}} \lesssim \langle \tau \rangle^{-\frac{s_1+s}{2}} \|a_0, v_0, \theta_0\|_{\tilde{H}^{s_1+s}_{g,1}} + \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|(f, g, k)(\tau)\|_{\tilde{H}^{s_1+s}_{g,1}} d\tau.
\]

Bounding the time-weighted integral on the right side of (3.4) is included in the following proposition.

**Proposition 3.1.** Let \( p \) fulfills (1.10), then it holds that for all \( t \geq 0 \),
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|(f, g, k)(\tau)\|_{\tilde{H}^{s_1+s}_{g,1}} d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( D^2_p(t) + \chi^2_p(t) \right),
\]
provided that \( -s_1 < s \leq \frac{d}{2} + 1 \), where \( \chi_p(t) \) and \( D_p(t) \) have been defined in (3.1) and (1.14), respectively.

Indeed, we decompose the nonlinear term \( g = \sum_{i=1}^6 G_i \) with
\[
G_1 = -v \cdot \nabla v, \quad G_2 = -K_1(a) \nabla a, \\
G_3 = \frac{1}{(1 + a)\nu} \left( 2\tilde{\mu}(a) \text{div} D(u) + \tilde{\chi}(a) \nabla \text{div} v \right) - I(a)\tilde{\nabla}v, \\
G_4 = \frac{1}{(1 + a)\nu} \left( \tilde{\mu}'(a) D(u) \cdot \nabla a + \tilde{\chi}'(a) \text{div} v \nabla a \right), \\
G_5 = -K_2(a) \nabla \theta, \quad G_6 = -\theta \nabla K_3(a).
\]
As shown by [32], we can get the following inequality
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \|(f, G_1, G_2, G_3, G_4)(\tau)\|_{\tilde{H}^{s_1+s}_{g,1}} d\tau \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} \left( D^2_p(t) + \chi^2_p(t) \right).
\]

In order to finish the proof of Proposition 3.1 it suffices to bound those “new” nonlinear terms, which are not available in the barotropic compressible Navier-Stokes system. For that end, let us decompose \( G_5, G_6 \) and \( k \) in terms of low-frequency and high frequency as follows:
\[
G_5 = G_5^f + G_5^b, \quad G_6 = G_6^f + G_6^b
\]
Hence, (3.8) directly stems from the embedding
\[ k = k^\ell + k^h \]
with
\[ k^\ell \triangleq -v \cdot \nabla \theta^\ell - (\tilde{K}_1(a) + \tilde{K}_2(a) \theta) \div v^\ell + \frac{\tilde{\kappa}(a)}{(1 + a) \nu} \nabla a \cdot \nabla \theta^\ell + k_1(a, \theta^\ell) + k_2(a, \nabla v, \nabla v^\ell) \]
\[ k^h \triangleq -v \cdot \nabla \theta^h - (\tilde{K}_1(a) + \tilde{K}_2(a) \theta) \div v^h + \frac{\tilde{\kappa}(a)}{(1 + a) \nu} \nabla a \cdot \nabla \theta^h + k_1(a, \theta^h) + k_2(a, \nabla v, \nabla v^h), \]
where
\[ k_1(a, \Theta) \triangleq \frac{\tilde{\kappa}(a)}{\nu(1 + a)} \Delta \Theta - \beta I(a) \Delta \Theta, \]
\[ k_2(a, V_1, V_2) \triangleq \frac{1}{\nu \chi_0} \sqrt{\frac{1}{\mathcal{T}_\infty C_v}} \left( \frac{\mu(\rho_\infty(1 + a))}{1 + a} (V_1 : V_2 + V_1 : V_2) + \frac{\lambda(\rho_\infty(1 + a))}{1 + a} \Tr V_1 \Tr V_2 \right) \]
and
\[ z^\ell \triangleq \sum_{j < j_0} \hat{\Delta}_j z, \quad z^h \triangleq z - z^\ell \quad \text{for } z = v, \theta. \]

Let us split the proof of Proposition 3.1 into two lemmas.

**Lemma 3.2.** If \( p \) satisfies (1.10), then it holds that for all \( t \geq 0, \)
\[ \int_0^t (t - \tau)^{-\frac{s_1 + s}{2}} \| (G_0^\ell, G_0^h, k^\ell)(\tau) \|_{B_{2, \infty}^{s_1}}^p d\tau \lesssim \langle t \rangle^{-\frac{s_1 + s}{2}} (D_p^2(t) + \lambda_p^2(t)), \]
provided that \(-s_1 < s \leq \frac{d}{2} + 1\).

**Proof.** Let us first claim that the following three non classical product inequalities
\[ \| FG \|_{B_{2, \infty}^{s_1}} \lesssim \| F \|_{B_p^{\sigma_1}} \| G \|_{B_{2, \infty}^{s_1}}, \]
\[ \| FG \|_{B_p^{\sigma_1}} \lesssim \| F \|_{B_p^{\sigma_1}} \| G \|_{B_p^{\sigma_2}}, \]
\[ \| FG \|_{B_p^{\sigma_1}} \lesssim \| F \|_{B_p^{\sigma_1}} \| G \|_{B_{2, \infty}^{s_1 + 1}} \]
for \( 1 - \frac{d}{2} < s_1 \leq s_0 \) and \( p \) satisfying (1.10). Indeed, the interested reader is referred to [31,32] for the proofs of (3.6)-(3.7). It follows from Proposition 2.3 with \( \sigma_1 = \frac{d}{p} - 1, \)
\( \sigma_2 = \frac{d}{p} - s_1 + 1, \) \( p_1 = p \) and \( p_2 = 2 \) that
\[ \| FG \|_{B_{2, \infty}^{s_1}} \lesssim \| F \|_{B_{p_1}^{\sigma_1}} \| G \|_{B_{2, \infty}^{s_1 + 1}}. \]
Hence, (3.8) directly stems from the embedding \( B_{2, 1}^{s_1 + 1} \hookrightarrow B_{2, \infty}^{s_1 + 1}. \)
On the other hand, due to Proposition 2.5 and the relations $-s_1 < \frac{d}{2} - 1 < \frac{d}{2} < \frac{d}{2} + 1$ and $\alpha \triangleq s_1 + \frac{d}{2} + \frac{1}{2} - \varepsilon > \frac{3d}{4} + \frac{d}{4}$ for small enough $\varepsilon > 0$, we infer that

$$\begin{align*}
(3.9) \quad & \sup_{\tau \in [0,t]} \langle \tau \rangle^{\frac{d}{2} + \frac{d}{2}} \| (a, v, \theta^\ell) (\tau) \|_{B^\frac{d}{2}} + \sup_{\tau \in [0,t]} \langle \tau \rangle^{\frac{d}{2} + \frac{d}{2} - \frac{1}{2}} \| (a, v, \theta^\ell) (\tau) \|_{B^\frac{d}{2} - 1} \lesssim D_p(t),
\end{align*}$$

and also that, thanks to $-s_1 < 1 - s_1 < 2 - s_1 < \frac{d}{2} + 1$,

$$\begin{align*}
(3.10) \quad & \| \nabla (a^\ell, v^\ell, \theta^\ell) (\tau) \|_{B^{-s_1}_{2,1}} \lesssim \langle \tau \rangle^{-\frac{1}{2}} D_p(\tau), \quad \| \nabla^2 \theta(\tau) \|_{B^{-s_1}_{2,1}} \lesssim \langle \tau \rangle^{-1} D_p(\tau) \quad \text{for all} \quad \tau \geq 0.
\end{align*}$$

Observe that (1.14) and the relations $-s_1 < \frac{d}{2} - 1 \leq \frac{d}{2} < \frac{d}{2} + 1 \leq \frac{d}{2} + 1$ and $\alpha \triangleq s_1 + \frac{d}{2} + \frac{1}{2} - \varepsilon > \frac{3d}{4} + \frac{d}{4}$ for small enough $\varepsilon > 0$, we obviously have

$$\begin{align*}
(3.11) \quad & \| \theta^\ell (\tau) \|_{B^\frac{d}{2} + 1} \lesssim \langle \tau \rangle^{-\frac{d}{2} + \frac{d}{2}} D_p(\tau), \quad \| \nabla (a^\ell, v^\ell, \theta^\ell) (\tau) \|_{B^\frac{d}{2} - s_1 + 1} \lesssim \langle \tau \rangle^{-\frac{d}{2} + \frac{d}{2} - 1} D_p(\tau),
\end{align*}$$

and

$$\begin{align*}
(3.12) \quad & \| (\nabla a^h, v^h) (\tau) \|_{B^\frac{d}{2} - 1} + \| \theta^h (\tau) \|_{B^\frac{d}{2} + 2} \lesssim \langle \tau \rangle^{-\alpha} D_p(\tau) \quad \text{for all} \quad \tau \geq 0.
\end{align*}$$

Now, let us begin with proving Lemma 3.2. To handle the term with $G^\ell = -K_2(a) \nabla \theta^\ell$, we write that, thanks to Proposition 2.5 together with (3.6), (3.9) and (3.11),

$$\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\frac{s_1 + s}{2}} \| K_2(a) \nabla \theta^\ell \|_{B^{s_1 - s}_{2,\infty}} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_1 + s}{2}} \| a \|_{B^\frac{d}{2}} \| \nabla \theta^\ell \|_{B^{s_1 - s}_{2,1}} d\tau \lesssim D^2_p(t) \int_0^t \langle t - \tau \rangle^{-\frac{3s_1 + s}{2}} \langle \tau \rangle^{-\frac{d}{2} + \frac{d}{2} - \frac{1}{2}} d\tau.
\end{align*}$$

According to $\frac{d}{2} + \frac{d}{2} > 1$ and $\frac{s_1}{2} + \frac{d}{2} + \frac{1}{2} \geq \frac{s_1 + s}{2}$ for $s_1$ satisfying (1.11) and $s \leq \frac{d}{2} + 1$, inequality (3.2) ensures that

$$\begin{align*}
\int_0^t \langle t - \tau \rangle^{-\frac{s_1 + s}{2}} \| K_2(a) \nabla \theta^\ell \|_{B^{s_1 - s}_{2,\infty}} d\tau \lesssim \langle t \rangle^{-\frac{s_1 + s}{2}} D^2_p(t).
\end{align*}$$

The terms $\tilde{K}_1(a) \div v^\ell$ and $k_1(a, \theta^\ell)$ (that is, the term $k_1(a, \theta)$ is of the type $K(a) \Delta \theta$ with $K(0) = 0$) may be treated at a similar way (use (3.6), (3.9), (3.10), (3.2) and Proposition 2.5), so we feel free to skip them for brevity. Let us decompose

$$\begin{align*}
\theta^\ell \nabla K_3(a) = \theta^\ell K'_3(a) \nabla a^\ell + \theta^\ell K'_3(a) \nabla a^h \quad \text{with} \quad K'_3(a) = \chi_0 \sqrt{\frac{1}{C_v} \sum_1^{\infty} \frac{\pi_1'(q_\infty(1 + a))}{1 + a}},
\end{align*}$$

$$\begin{align*}
v \cdot \nabla \theta^\ell = v^\ell \cdot \nabla \theta^\ell + v^h \cdot \nabla \theta^\ell \quad \text{and} \quad \tilde{K}_2(a) \theta \div v^\ell = \tilde{K}_2(a) \theta^\ell \div v^\ell + \tilde{K}_2(a) \theta^h \div v^\ell.
\end{align*}$$
Regarding the term with $\theta^\ell K_3'(a) \nabla a^\ell$, it follows from Propositions 2.2, 2.5, (3.6), (3.9), (3.10), (3.3) and (3.2) that

$$
\int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| \theta^\ell K_3'(a) \nabla a^\ell \|_{B_{2,1}^{s-1}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} (1 + \|a\|_{B^4_{p,1}}) \| \theta^\ell \|_{B^4_{p,1}} \| \nabla a^\ell \|_{B_{2,1}^{s-1}} d\tau \\
\lesssim \mathcal{D}_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \langle \tau \rangle^{-\frac{2p}{4} - \frac{d}{2}} d\tau \lesssim \langle t \rangle^{-\frac{3s+\varepsilon}{2}} \mathcal{D}_p^2(t).
$$

Bounding $v^\ell \cdot \nabla \theta^\ell$ and $\tilde{K}_2(a) \theta^\ell \div v^\ell$ essentially follows from the same procedure as $\theta^\ell K_3'(a) \nabla a^\ell$, we thus omit them. To handle the term with $\theta^\ell K_3'(a) \nabla a^h$, we note that, owing to [3.7], (3.3) and Propositions 2.2, 2.5,

$$
\int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| \theta^\ell K_3'(a) \nabla a^h \|_{B_{2,1}^{s-1}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| \theta^\ell K_3'(a) \nabla a^h \|_{B_{2,1}^{s-1}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| \theta^\ell \|_{B^4_{p,1}} \| \nabla a^h \|_{B_{2,1}^{s-1}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| \theta^\ell \|_{B^4_{p,1}} \| \nabla a^h \|_{B_{2,1}^{s-1}} d\tau,
$$

where we used the relations $s_1 \leq s_1 + \frac{d}{2} - \frac{d}{p} (p \geq 2)$ and $\frac{d}{p} - \frac{d}{2} - s_1 < \frac{d}{p} - 1$ for $s_1$ fulfilling [1.11]. According to $\frac{d}{2} + \frac{d}{2p} + \alpha > \frac{d}{2} + \frac{d}{2} + \frac{1}{2} > 1$ (as $\alpha > \frac{d}{2} + 1$ for sufficiently small $\varepsilon > 0$) and $\frac{d}{2} + \frac{d}{2p} + \alpha > \frac{3s+\varepsilon}{2}$ for all $s \leq \frac{d}{2} + 1$ as well as (3.11), (3.13) and (3.2), we arrive at

$$
\int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| \theta^\ell K_3'(a) \nabla a^h \|_{B_{2,1}^{s-1}} d\tau \\
\lesssim \mathcal{D}_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \langle \tau \rangle^{-\frac{2p}{4} - \frac{d}{2}} d\tau \\
\lesssim \langle t \rangle^{-\frac{3s+\varepsilon}{2}} \mathcal{D}_p^2(t).
$$

For the term $v^h \cdot \nabla \theta^\ell$, we take advantage of (3.7), (3.11), (3.13), (3.2) and the relations $s_1 \leq s_1 + \frac{d}{2} - \frac{d}{p} (p \geq 2)$ and $\frac{d}{p} - \frac{d}{2} - s_1 < \frac{d}{p} - 1$ to conclude that we still have

$$
\int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| v^h \cdot \nabla \theta^\ell \|_{B_{2,1}^{s-1}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| v^h \|_{B^4_{p,1}} \| \nabla \theta^\ell \|_{B^4_{p,1}} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \| v^h \|_{B^4_{p,1}} \| \nabla \theta^\ell \|_{B^4_{p,1}} d\tau \\
\lesssim \mathcal{D}_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{3s+\varepsilon}{2}} \langle \tau \rangle^{-\frac{2p}{4} - \frac{d}{2}} d\tau \\
\lesssim \langle t \rangle^{-\frac{3s+\varepsilon}{2}} \mathcal{D}_p^2(t).
$$

To bound the term corresponding to $\tilde{K}_2(a) \theta^h \div v^\ell$, we observe that applying [3.7] with $s_1 = s_0 = \frac{2d}{p} - \frac{d}{2}$ yields

$$
\| FG \|_{B_{2,\frac{d}{p}}^{s-1}} \lesssim \| F \|_{B_{p,1}^{s-1}} \| G \|_{B_{2,\frac{d}{p}}^{s-1}}.
$$
Note that $s_1 \leq s_0 \leq \frac{d}{p}$ and $-\frac{d}{p} < \frac{d}{p} - 2$ ($p < d$), we get from \((3.14), (3.3)\) and Propositions \(2.2, 2.5\)

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| \tilde{K}_2(a) \theta^h \text{div} \nu \|_{B_{2,2}^{-s_1}} \, d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| \tilde{K}_2(a) \theta^h \text{div} \nu \|_{B_{2,2}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| \theta^h \|_{B_{p,1}} \| \text{div} \nu \|_{B_{2,1}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| \theta^h \|_{B_{p,1}} \| \text{div} \nu \|_{B_{2,1}} \, d\tau.
\]

In light of \((3.11), (3.13)\) and \((3.2)\), we arrive at

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| \tilde{K}_2(a) \theta^h \text{div} \nu \|_{B_{2,2}^{-s_1}} \, d\tau \lesssim \mathcal{D}_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \langle \tau \rangle^{-\frac{\alpha}{2} - \frac{d}{2p} - \frac{1}{2}} \, d\tau \\
\lesssim \langle t \rangle^{-\frac{\alpha + 1}{2}} \mathcal{D}_p^2(t).
\]

Let us next look at the term with $\frac{\mathcal{K}(a)}{(1+a)^\nu} \nabla a \cdot \nabla \theta^\ell$. Denote by $H(a)$ the smooth function fulfilling $H'(a) = \frac{\mathcal{K}(a)}{(1+a)^\nu} \nabla a$. So it suffices to estimate the term $\nabla H(a) \cdot \nabla \theta^\ell$. With the aid of the fact $s_1 \leq s_1 + \frac{d}{2} - \frac{d}{p}$, \((3.8), (3.9), (3.12), (3.2)\) and Proposition \(2.5\) we end up with

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| \nabla H(a) \cdot \nabla \theta^\ell \|_{B_{2,2}^{-s_1}} \, d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| a \|_{B_{p,1}} \| \nabla \theta^\ell \|_{B_{2,1}} \, d\tau \\
\lesssim \mathcal{D}_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \langle \tau \rangle^{-\frac{\alpha}{2} - \frac{d}{2p} - \frac{1}{2}} \, d\tau \\
\lesssim \langle t \rangle^{-\frac{\alpha + 1}{2}} \mathcal{D}_p^2(t),
\]

where the relation $p \leq \frac{2d}{d - 2}$ ensures $\frac{\alpha}{2} + \frac{d}{2p} + 1 \geq \frac{\alpha}{2} + \frac{d}{2} + 1 > 1$ and $\frac{\alpha}{2} + \frac{d}{2p} + 1 \geq \frac{\alpha + 1}{2}$ for all $s \leq \frac{d}{2} + 1$. We finally decompose $k_2(a, \nabla v^\ell, \nabla v^\ell) = k_2(a, \nabla v^\ell, \nabla v^\ell) + k_2(a, \nabla v^b, \nabla v^\ell)$.

For the term with $k_2(a, \nabla v^\ell, \nabla v^\ell)$, it follows from \((3.8), (3.9), (3.12), (3.3), (3.2)\), Propositions \(2.2, 2.5\) and the relation $s_1 \leq s_1 + \frac{d}{2} - \frac{d}{p}$ that

\[
\int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| k_2(a, \nabla v^\ell, \nabla v^\ell) \|_{B_{2,2}^{-s_1}} \, d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \| \nu^\ell \|_{B_{p,1}} \| \nabla v^\ell \|_{B_{2,1}} \, d\tau \\
\lesssim \mathcal{D}_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{\alpha + 1}{2}} \langle \tau \rangle^{-\frac{\alpha}{2} - \frac{d}{2p} - \frac{1}{2}} \, d\tau \\
\lesssim \langle t \rangle^{-\frac{\alpha + 1}{2}} \mathcal{D}_p^2(t).
\]
Proposition 2.5, of (3.17). To bound the term involving $K$ for $t > 0$, we claim that

\begin{equation}
(3.18)
\end{equation}

It follows from (3.1) that

\begin{equation}
(3.16)
\end{equation}

Proof. Provided that

\begin{equation}
(3.1)
\end{equation}

Hence, putting all estimates together leads to Lemma 3.2.

Lemma 3.3. If $p$ satisfies (1.10), then it holds that for all $t \geq 0$,

\begin{equation}
(3.16)
\end{equation}

provided that $-s_1 < s \leq \frac{p}{2} + 1$.

Proof. In order to prove (3.16), we shall present the following inequality

\begin{equation}
(3.17)
\end{equation}

for $1 - \frac{d}{2} < s_1 \leq s_0$ and $p$ satisfying (1.10). The reader is referred to [31] for the proof of (3.17). To bound the term involving $K_2(a)\nabla \theta h$, we see that, thanks to (3.17) and Proposition 2.5,

\begin{equation}
(3.18)
\end{equation}

It follows from (3.1) that

\begin{equation}
I_1 \lesssim \langle t \rangle^{-\frac{s_1+s}{2}} X_p^2(1)
\end{equation}

Keeping in mind that the relations $s_1 \leq s_0 \leq \frac{d}{2}$ and $-\frac{d}{2} < \frac{p}{2} - 2$ (and $p < d$) and using (3.11), (3.13), (3.14), (3.3), (3.2) and Propositions 2.2, 2.5, we conclude that

\begin{equation}
\int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \| k_2(a, \nabla v^h, \nabla v^f) \|_{B_{2,\infty}^s} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \| \nabla v^h \|_{B_{2,1}^{\frac{p}{2}}} \| \nabla v^f \|_{B_{2,1}^{\frac{p}{2}}} d\tau 
\end{equation}

\begin{equation}
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \| \nabla v^h \|_{B_{2,1}^{\frac{p}{2}}} \| \nabla v^f \|_{B_{2,1}^{\frac{p}{2}}} d\tau 
\end{equation}

\begin{equation}
\lesssim D_p^2(t) \int_0^t \langle t - \tau \rangle^{-\frac{s_1+s}{2}} \| \nabla v^h \|_{B_{2,1}^{\frac{p}{2}}} \| \nabla v^f \|_{B_{2,1}^{\frac{p}{2}}} d\tau 
\end{equation}

\begin{equation}
\lesssim \langle t \rangle^{-\frac{s_1+s}{2}} D_p^2(t).
\end{equation}
and that, owing to (3.9), (3.15) and (3.2), if \( t \geq 1 \),
\[
I_2 \lesssim \int_1^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} (\tau)^{-\frac{d}{2} + \frac{1}{2} - \alpha} \left( \langle \tau \rangle^{\frac{d}{2} + \frac{1}{2}} \| a \|_{B_{p,1}^{\frac{d}{p}}} \right) (\tau^\alpha \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}}) \ d\tau
\]
\[
\lesssim \left( \sup_{\tau \in [1,t]} \langle \tau \rangle^{\frac{d}{2} + \frac{1}{2}} \| a \|_{B_{p,1}^{\frac{d}{p}}} \right) \left( \sup_{\tau \in [1,t]} \tau^\alpha \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}} \right) \int_1^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} (\tau)^{-\frac{d}{2} + \frac{1}{2} - \alpha} \ d\tau
\]
(3.19)
\[
\times \int_1^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} (\tau)^{-\frac{d}{2} + \frac{1}{2} - \alpha} \ d\tau \lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} D^2_p(t),
\]
where the fact \( \alpha > 1 \) for small enough \( \varepsilon > 0 \) implies \( \frac{d}{2} + \frac{1}{2} + \alpha > 1 \) and \( \frac{d}{2} + \frac{1}{2} + \alpha \geq \frac{d}{2} + \frac{1}{2} + \frac{1}{2} + \alpha \geq \frac{d}{2} + \frac{1}{2} + \alpha \geq \frac{s+\alpha}{2} \) for \( s_1 \) satisfying (1.11) and \( s \leq \frac{d}{2} + 1 \). Bounding \( v \cdot \nabla \partial_h \) and \( \tilde{K}_2(a) \partial_h \) essentially follow from the same procedure as \( K_2(a) \nabla \partial_h \), we thus omit them. For the term with \( \partial_h \nabla K_3(a) \), applying (3.17) and Proposition 2.5 yields
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \| \partial_h \nabla K_3(a) \|_{B_{2,\infty}} \ d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \| a \|_{B_{p,1}^{\frac{d}{p}}} \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}} \ d\tau
\]
\[
= \left( \int_0^1 + \int_1^t \right) \cdots \ d\tau \lesssim J_1 + J_2.
\]
It is clear that \( J_1 \lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} \mathcal{A}_p(1) \) and that, due to (3.9), (3.15) and (3.2), if \( t \geq 1 \),
\[
J_2 \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} (\tau)^{-\frac{d}{2} + \frac{1}{2} - \alpha} \left( \langle \tau \rangle^{\frac{d}{2} + \frac{1}{2}} \| a \|_{B_{p,1}^{\frac{d}{p}}} \right) (\tau^\alpha \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}}) \ d\tau
\]
\[
\lesssim \left( \sup_{\tau \in [1,t]} \langle \tau \rangle^{\frac{d}{2} + \frac{1}{2}} \| a \|_{B_{p,1}^{\frac{d}{p}}} \right) \left( \sup_{\tau \in [1,t]} \tau^\alpha \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}} \right) \int_1^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} (\tau)^{-\frac{d}{2} + \frac{1}{2} - \alpha} \ d\tau
\]
\[
\lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} D^2_p(t).
\]
The term \( \frac{\epsilon^\alpha}{(1+\alpha)^p} \nabla a \cdot \nabla \partial_h \) may be treated at a similar way, so we omit it. Let us look at the term with \( \tilde{K}_2(a) \partial_h \div v^h \). With the aid of (3.17), (3.3) and Propositions 2.2, 2.5, we arrive at
\[
\int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \| \tilde{K}_2(a) \partial_h \div v^h \|_{B_{2,\infty}} \ d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}} \| v^h \|_{B_{p,1}^{\frac{d}{p}}} \ d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} \left( \| \theta \|_{B_{p,1}^{\frac{d}{p}}} + \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}} \right) \| v^h \|_{B_{p,1}^{\frac{d}{p}}} \ d\tau
\]
\[
= \left( \int_0^1 + \int_1^t \right) \cdots \ d\tau \lesssim H_1 + H_2.
\]
It follows from (3.1) and the interpolation that \( H_1 \lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} \mathcal{A}_p(1) \). By using (3.9), (3.15) and (3.2), we get, if \( t \geq 1 \),
\[
H_2 \lesssim \left( \sup_{\tau \in [1,t]} \langle \tau \rangle^{\frac{d}{2} + \frac{1}{2} - \alpha} \| \theta \|_{B_{p,1}^{\frac{d}{p}}} + \sup_{\tau \in [1,t]} \tau^\alpha \| \partial_h \|_{B_{p,1}^{\frac{d}{p}}} \right) \left( \sup_{\tau \in [1,t]} \tau^\alpha \| v^h \|_{B_{p,1}^{\frac{d}{p}}} \right)
\]
\[
\times \int_1^t \langle t - \tau \rangle^{-\frac{s+\alpha}{2}} (\langle \tau \rangle^{-\frac{d}{2} + \frac{1}{2} + \alpha} + \langle \tau \rangle^{-\alpha}) \langle \tau \rangle^{-\alpha} \ d\tau \lesssim \langle t \rangle^{-\frac{s+\alpha}{2}} D^2_p(t),
\]
where we noticed the fact \( \alpha > \frac{d}{2} + \frac{1}{2} \) for small enough \( \varepsilon > 0 \). To bound the term corresponding to \( k_1(a, \partial_h) \), it suffices to handle \( K(a) \Delta \partial_h \) with \( K(0) = 0 \). To this end,
one has to consider the cases $2 \leq p \leq \frac{2d}{3}$ and $p > \frac{2d}{3}$ separately. If $2 \leq p \leq \frac{2d}{3}$, then we have, using (3.17) and Proposition 2.5,

$$
\|K(a)\Delta \theta^h\|_{B^2_{p,1}} \lesssim \|a\|_{B^\frac{d}{p-1}} \|\nabla^2 \theta^h\|_{B^{1-\frac{4}{p}}_{p,1}} \lesssim \|a\|_{B^\frac{d}{p-1}} \|\theta^h\|_{B^\frac{d}{p-1}}
$$

where the fact $2 \leq p \leq \frac{2d}{3}$ ensures $1 - \frac{4}{p} \leq \frac{d}{p} - 2$. By repeating the procedure leading to (3.18)-(3.19), we get

$$
\int_0^t \langle t - \tau \rangle^{-\frac{d+2}{2}} \|K(a)\Delta \theta^h\|_{B^2_{p,1}} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{4d+8}{2}} \|a\|_{B^\frac{d}{p-1}} \|\theta^h\|_{B^\frac{d}{p-1}} d\tau
$$

$$
\lesssim \langle t \rangle^{-\frac{4d+8}{2}} \big( \mathcal{X}^2_p(t) + D^2(t) \big).
$$

Notice that if $p > \frac{2d}{3}$, then applying (2.1) with $\sigma = 2 - \frac{d}{p} > \frac{d}{2}$ > yields

$$(3.20) \quad \|FG^h\|_{B^{s_1}_{p,1}} \lesssim \|FG^h\|_{B^{s_0}_{p,1}} \lesssim \big( \|F\|_{B^{2-s_0}_{p,1}} + \|F^\ell\|_{L^p} \big) \|G^h\|_{B^{\frac{4-s}{2}}_{p,1}}$$

since $s_1 \leq s_0$. Furthermore, using the composition inequality in Lebesgue spaces and the embeddings $\dot{B}^2_{p,1} \hookrightarrow L^p$ and $\dot{B}^{s_0}_{p,1} \hookrightarrow L^p$ gives

$$
\|K(a)\|_{L^p} \lesssim \|a\|_{L^p} \lesssim \|a\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} + \|a^h\|_{B^{s_0}_{p,1}} \lesssim \|a\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} + \|a^h\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}}
$$

where we noticed the relations $\frac{d}{2} - 1 \leq \frac{d}{p} \leq \frac{2d}{d-2}$ and $s_0 \leq \frac{d}{p}$. It follows from Proposition 2.5, the embedding $\dot{B}^{2-s_0}_{p,1} \hookrightarrow \dot{B}^\frac{d}{p-1}_{p,1}$ and the relations $2 - s_0 > 2 - 1$ ($p > \frac{2d}{3}$) and $2 - \frac{d}{p} < \frac{d}{p} < d$ ($p < d$) that

$$
\|K(a)\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \lesssim \|a\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \lesssim \|a\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} + \|a^h\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \lesssim \|a\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} + \|a^h\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}}
$$

According to (3.20), we deduce that

$$
\int_0^t \langle t - \tau \rangle^{-\frac{4d+8}{2}} \|k_1(a, \theta^h)\|_{B^{2-s_1}_{p,1}} d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{4d+8}{2}} \big( \|a\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} + \|a^h\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \big) \|\theta^h\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} d\tau
$$

$$
= \left( \int_0^1 + \int_1^t \right) \langle \tau \rangle^{\frac{d}{2} - \frac{4}{p} - \frac{1}{2}} d\tau \lesssim \tilde{I}_1 + \tilde{I}_2.
$$

With the aid of (3.1), we have $\tilde{I}_1 \lesssim \langle \tau \rangle^{\frac{d}{2} + \frac{4}{p} - \frac{1}{2}} D_p(\tau)$. Using the fact

$$
\|a^{\ell}(\tau)\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \lesssim \langle \tau \rangle^{\frac{d}{2} + \frac{4}{p} - \frac{1}{2}} D_p(\tau)
$$

together with (3.13), (3.15) and (3.2), we thus get, if $t \geq 1$,

$$
\tilde{I}_2 = \int_1^t \langle t - \tau \rangle^{-\frac{4d+8}{2}} \big( \|a\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} + \|a^h\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \big) \|\theta^h\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} d\tau
$$

$$
\lesssim \left( \sup_{\tau \in [1, t]} \langle \tau \rangle^{\frac{d}{2} + \frac{4}{p} - \frac{1}{2}} \|a^{\ell}(\tau)\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} + \sup_{\tau \in [1, t]} \langle \tau \rangle^{\alpha} \|a^h(\tau)\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \right) \left( \sup_{\tau \in [1, t]} \langle \tau \rangle^{\alpha} \|\theta^h(\tau)\|_{B^{\frac{d}{p} - \frac{4}{p}}_{p,1}} \right)
$$

$$
\times \int_1^t \langle t - \tau \rangle^{-\frac{4d+8}{2}} \big( \langle \tau \rangle^{-\frac{d}{2} - \frac{4}{p} + \frac{1}{2}} + \langle \tau \rangle^{-\alpha} \big) \langle \tau \rangle^{-\alpha} d\tau \lesssim \langle t \rangle^{-\frac{4d+8}{2}} D_p(t).
$$
Let us finally estimate the term with \( k_2(a, \nabla v, \nabla v^h) \). We observe that, owing to (3.17), (3.3), Propositions 2.2, 2.6 and the interpolation,

\[
\int_0^t (t - \tau) \frac{d^t + 2}{2} \| k_3(a, \nabla v, \nabla v^h) \|^\ell_{B_2, H^\infty} \, d\tau \lesssim \int_0^t (t - \tau) \frac{d^t + 2}{2} \| v \|^\ell_{B_2, \frac{d}{4}} \, \| v^h \|^\ell_{B_2, \frac{d}{4}} \, d\tau
\]

\[
\lesssim \int_0^t (t - \tau) \frac{d^t + 2}{2} (\| v \|^\ell_{B_2, \frac{d}{4}} + \| v^h \|^\ell_{B_2, \frac{d}{4}}) \| \nabla v^h \|^\ell_{B_2, \frac{d}{4}} \, d\tau
\]

\[
= \left( \int_0^1 + \int_1^t \right) \ldots \, d\tau \equiv \tilde{J}_1 + \tilde{J}_2.
\]

For \( \tilde{J}_1 \), it is clear that \( \tilde{J}_1 \lesssim \langle t \rangle^{-\frac{d+2}{4}} X_p^2(t) \) and that, owing to the fact \( \alpha > \frac{d+2}{4} \) for sufficiently small \( \varepsilon > 0 \) as well as (3.9), (3.13), (3.15) and (3.2), if \( t \geq 1 \),

\[
\tilde{J}_2 = \int_1^t (t - \tau)^{-\frac{d+2}{4}} (\| v \|^\ell_{B_2, \frac{d}{4}} + \| v^h \|^\ell_{B_2, \frac{d}{4}}) \| \nabla v^h \|^\ell_{B_2, \frac{d}{4}} \, d\tau
\]

\[
\lesssim \left( \sup_{\tau \in [1, t]} \langle \tau \rangle^\alpha \langle \tau \rangle^{\alpha + \frac{d}{4}} \| v \|^\ell_{B_2, \frac{d}{4}} + \sup_{\tau \in [1, t]} \langle \tau \rangle^{\alpha} \| v^h \|^\ell_{B_2, \frac{d}{4}} \right) \left( \sup_{\tau \in [1, t]} \tau^\alpha \| \nabla v^h \|^\ell_{B_2, \frac{d}{4}} \right)
\]

\[
\times \int_1^t (t - \tau)^{-\frac{d+2}{4}} \left( \langle \tau \rangle^{-\frac{d+2}{4}} + \langle t \rangle^{-\alpha} \right) \langle t \rangle^{-\alpha} \, d\tau \lesssim \langle t \rangle^{-\frac{d+2}{4}} D_p^2(t).
\]

Hence, the proof of Lemma 3.3 is finished.

Combining (3.5) and those estimates in Lemmas 3.2, 3.3 we get Proposition 3.1 eventually. Furthermore, with the aid of (3.2), we conclude that

\[
\langle t \rangle^\frac{d+2}{4} \| (a, v, \theta)(t) \|^\ell_{B_2, \frac{d}{4}} \lesssim D_{p, 0} + D_p^2(t) + X_p^2(t) \quad \text{for all} \quad t \geq 0,
\]

provided that \(-s_1 < s \leq \frac{d}{2} + 1\).

3.2. Second step: decay estimates for the high frequencies of \((\nabla a, v, \theta)\). In this section, we shall apply the energy method of \( L^p \) type in terms of the effective velocity. Let \( \mathcal{P} \equiv \text{Id} + \nabla (-\Delta)^{-1} \text{div} \) be the Leray projector onto divergence-free vector-fields. It follows from (1.9) that \( \mathcal{P} v \) satisfies

\[
\partial_t \mathcal{P} v - \bar{\mu}_\infty \Delta \mathcal{P} v = \mathcal{P} g \quad \text{with} \quad \bar{\mu}_\infty = \frac{\mu_\infty}{\nu}.
\]

Let us introduce the effective velocity \( w \):

\[
w \equiv \nabla (-\Delta)^{-1} (a - \text{div} v),
\]

which was initiated by Hoff [18] and first used in the context of critical regularity by Haspot [17] as well as developed by Danchin, the first author and the second author [14, 15, 31, 32]. We observe that \((a, w, \theta)\) satisfies

\[
\begin{align*}
\partial_t a + a &= f - \text{div} w, \\
\partial_t w - \Delta w &= \nabla (-\Delta)^{-1} (f - \text{div} g) - \gamma \nabla \theta + w - (-\Delta)^{-1} \text{div} a, \\
\partial_t \theta - \beta \Delta \theta &= k - \gamma \text{div} w - a.
\end{align*}
\]

**Proposition 3.2.** If \( p \) satisfies (1.10), then it holds that for all \( T \geq 0 \),

\[
\| \langle t \rangle^{\alpha} (\nabla a, v) \|_{L_p^\infty(B_{p, 1})}^h + \| \langle t \rangle^{\alpha} \theta \|_{L_p^\infty(B_{p, 1})}^h \lesssim \| (\nabla a_0, v_0) \|_{B_{p, 1}}^h + \| \theta_0 \|_{B_{p, 1}}^h + D_p^2(T) + X_p^2(T),
\]

(3.22)
with \( \alpha = s_1 + \frac{d}{p} + \frac{1}{3} - \varepsilon \) for sufficiently small \( \varepsilon > 0 \), where \( X_p(T) \) and \( D_p(T) \) have been defined by \((3.1)\) and \((1.14)\), respectively.

**Proof.** By performing the \( L^p \) energy method, we end up with (see [14, 15, 31, 32] for details)

\[
\begin{align*}
\| \langle t \rangle^\alpha (\nabla a, v) \|_{L^{2H}_T(B_{p,1}^{\frac{d}{p}-1})}^h + \| \langle t \rangle^{\alpha} \theta \|_{L^{2H}_T(B_{p,1}^{\frac{d}{p}-2})}^h & \lesssim \| (\nabla a_0, v_0) \|_{L^{2H}_T(B_{p,1}^{\frac{d}{p}-1})}^h + \| \theta_0 \|_{L^{2H}_T(B_{p,1}^{\frac{d}{p}-2})}^h \\
& + \sum_{j \geq j_{0-1}} \sup_{t \in [0,T]} \left( \langle t \rangle^\alpha \int_0^t e^{-c_0(t-\tau)} 2^{j(\frac{d}{p}-1)} Z_j(\tau) d\tau \right).
\end{align*}
\]

(3.23)

with \( Z_j = Z_j^1 + \cdots + Z_j^6 \) and

\[
\begin{align*}
Z_j^1 & \triangleq \| \widetilde{\Delta}^\delta_j (a v) \|_{L^p}, & Z_j^2 & \triangleq \| g_j \|_{L^p}, & Z_j^3 & \triangleq 2^{-j} \| k_j \|_{L^p}, \\
Z_j^4 & \triangleq \| \nabla \Delta^\delta_j (a \div v) \|_{L^p}, & Z_j^5 & \triangleq \| R_j \|_{L^p}, & Z_j^6 & \triangleq \| \div v \|_{L^{\infty}} \| \nabla a_j \|_{L^p},
\end{align*}
\]

where \( a_j \triangleq \Delta^\delta_j a, g_j \triangleq \Delta^\delta_j g, k_j \triangleq \Delta^\delta_j k \) and \( R_j \triangleq [u \cdot \nabla, \nabla \Delta^\delta_j]a \).

Firstly, we observe that

\[
\sum_{j \geq j_{0-1}} \sup_{t \in [0,2]} \left( \langle t \rangle^\alpha \int_0^t e^{-c_0(t-\tau)} 2^{j(\frac{d}{p}-1)} Z_j(\tau) d\tau \right) \lesssim \int_0^2 \sum_{j \geq j_{0-1}} 2^{j(\frac{d}{p}-1)} Z_j(\tau) d\tau.
\]

The terms in \( Z_j^1, Z_j^2, Z_j^5 \) and \( Z_j^6 \) as well as those in \( Z_j^3 \) corresponding to \( G_1, G_2, G_3 \) and \( G_4 \) may be estimated exactly as in [32]. Consequently, it is only a matter of handling those “new” nonlinear terms in \( Z_j^2 \) and \( Z_j^3 \). Precisely,

\[
\begin{align*}
K_2(a) \nabla \theta, & \quad \theta \nabla K_3(a), & v \cdot \nabla \theta, & \quad \widetilde{K}_2(a) \div v, \\
G_2(a) \theta \div v, & \quad \frac{\nabla^\delta(a)}{(1+|\nabla^\delta(a)|)^p} \nabla a \cdot \nabla \theta, & k_1(a, \theta), & \quad k_2(a, \nabla v, \nabla v).
\end{align*}
\]

To do this, we shall use frequently that, owing to \((3.1)\), interpolation and embeddings (recall that \( p \geq 2 \)),

\[
(3.24) \quad \| (a, v) \|_{L^2_t(B_{p,1}^{\frac{d}{p}})} \lesssim \| \nabla \theta^\ell, \theta^h \|_{L^1_t(B_{p,1}^{\frac{d}{p}})} \lesssim \| \nabla \theta \|_{L^1_t(B_{p,1}^{\frac{d}{p}})}
\]

and also that

\[
(3.25) \quad \| (\nabla \theta^\ell, \theta^h) \|_{L^1_t(B_{p,1}^{\frac{d}{p}-1})} \lesssim \| \nabla a, \nabla \theta^\ell \|_{L^\infty_t(B_{p,1}^{\frac{d}{p}-1})} \lesssim \| \nabla \theta \|_{L^1_t(B_{p,1}^{\frac{d}{p}})}
\]

For the terms with \( K_2(a) \nabla \theta \) and \( \theta \nabla K_3(a) \), we decompose

\[
K_2(a) \nabla \theta = K_2(a) \nabla \theta^\ell + K_2(a) \nabla \theta^h \quad \text{and} \quad \theta \nabla K_3(a) = \theta \nabla K_3(a)^\ell + \theta \nabla K_3(a)^h.
\]

With the aid of Propositions 2.2 and 2.5 the Hölder inequality, \((3.24)\) and \((3.25)\), we deduce that

\[
\begin{align*}
\| K_2(a) \nabla \theta \|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^h & \lesssim \| a \|_{L^2_t(B_{p,1}^{\frac{d}{p}})} \| \nabla \theta^\ell \|_{L^2_t(B_{p,1}^{\frac{d}{p}-1})} + \| a \|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})} \| \nabla \theta^h \|_{L^1_t(B_{p,1}^{\frac{d}{p}})} \\
& \lesssim \| a \|_{L^2_t(B_{p,1}^{\frac{d}{p}})} \| \nabla \theta \|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^2,
\end{align*}
\]

\[
\begin{align*}
\| \theta \nabla K_3(a) \|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^h & \lesssim \| \theta \|_{L^2_t(B_{p,1}^{\frac{d}{p}})} \| a \|_{L^2_t(B_{p,1}^{\frac{d}{p}})} + \| \theta \|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})} \| a \|_{L^1_t(B_{p,1}^{\frac{d}{p}})} \\
& \lesssim \| a \|_{L^2_t(B_{p,1}^{\frac{d}{p}})} \| \nabla \theta \|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^2.
\end{align*}
\]
Keep in mind that the term \( \tilde{\nu} (a) \frac{\nabla a \cdot \nabla \theta}{(1 + a)^{\nu}} \) of \( k \) is of the type \( \nabla H(a) \cdot \nabla \theta \) with \( H(0) = 0 \), and the term \( k_1(a, \theta) \) of \( k \) is of the type \( K(a) \Delta \theta \) with \( K(0) = 0 \). For the terms with \( v \cdot \nabla \theta, \tilde{K}_2(a) \theta \text{div} v, \tilde{\nu} (a) \frac{\nabla a \cdot \nabla \theta}{(1 + a)^{\nu}} \) and \( k_1(a, \theta) \), we decompose them as follows:

\[
 v \cdot \nabla \theta = v \cdot \nabla \theta^\ell + v \cdot \nabla \theta^h, \quad \tilde{K}_2(a) \theta \text{div} v = \tilde{K}_2(a) \theta^\ell \text{div} v + \tilde{K}_2(a) \theta^h \text{div} v,
\]

\[
 \nabla H(a) \cdot \nabla \theta = \nabla H(a) \cdot \nabla \theta^\ell + \nabla H(a) \cdot \nabla \theta^h, \quad k_1(a, \theta) = k_1(a, \theta^\ell) + k_1(a, \theta^h).
\]

Furthermore, we observe that, thanks to Propositions 2.2 and 2.5 and (3.24), (3.25), (3.3) as well as the relations \( p < d \) and \( d \geq 3 \),

\[
 \| v \cdot \nabla \theta \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| v \cdot \nabla \theta^\ell \|^h_{L^1(B_{p,1}^{\frac{d}{p}-1})} + \| v \cdot \nabla \theta^h \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| v \|^d_{L^2(B_{p,1}^{\frac{d}{p}})} \left( \| \nabla \theta^\ell \|^d_{L^2(B_{p,1}^{\frac{d}{p}-1})} + \| \nabla \theta^h \|^d_{L^2(B_{p,1}^{\frac{d}{p}-2})} \right) \lesssim \mathcal{X}^2_p(t),
\]

\[
 \| \tilde{K}_2(a) \theta \text{div} v \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| \tilde{K}_2(a) \theta^\ell \text{div} v \|^h_{L^1(B_{p,1}^{\frac{d}{p}-1})} + \| \tilde{K}_2(a) \theta^h \text{div} v \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| \theta^\ell \|^d_{L^2(B_{p,1}^{\frac{d}{p}})} \| v \|^d_{L^2(B_{p,1}^{\frac{d}{p}})} + \| \theta^h \|^d_{L^2(B_{p,1}^{\frac{d}{p}})} \| v \|^d_{L^\infty(B_{p,1}^{\frac{d}{p}-1})} \lesssim \mathcal{X}^2_p(t),
\]

\[
 \left\| \tilde{\nu} (a) \frac{\nabla a \cdot \nabla \theta}{(1 + a)^{\nu}} \right\|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| \nabla H(a) \cdot \nabla \theta^\ell \|^h_{L^1(B_{p,1}^{\frac{d}{p}-1})} + \| \nabla H(a) \cdot \nabla \theta^h \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| a \|^d_{L^\infty(B_{p,1}^{\frac{d}{p}})} \left( \| \nabla \theta^\ell \|^d_{L^1(B_{p,1}^{\frac{d}{p}-1})} + \| \theta^h \|^d_{L^1(B_{p,1}^{\frac{d}{p}-2})} \right) \lesssim \mathcal{X}^2_p(t),
\]

\[
 \| k_1(a, \theta) \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| k_1(a, \theta^\ell) \|^h_{L^1(B_{p,1}^{\frac{d}{p}-1})} + \| k_1(a, \theta^h) \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| a \|^d_{L^\infty(B_{p,1}^{\frac{d}{p}})} \left( \| \nabla \theta^\ell \|^d_{L^1(B_{p,1}^{\frac{d}{p}-1})} + \| \theta^h \|^d_{L^1(B_{p,1}^{\frac{d}{p}-2})} \right) \lesssim \mathcal{X}^2_p(t).
\]

It follows from Propositions 2.2 and 2.5 (3.3) and (3.24) that

\[
 \| \tilde{K}_1(a) \text{div} v \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| a \|^d_{L^1(B_{p,1}^{\frac{d}{p}})} \| v \|^d_{L^2(B_{p,1}^{\frac{d}{p}})} \lesssim \mathcal{X}^2_p(t),
\]

\[
 \| k_2(a, \nabla v, \nabla v) \|^h_{L^1(B_{p,1}^{\frac{d}{p}-2})} \lesssim \| v \|^2_{L^2(B_{p,1}^{\frac{d}{p}})} \lesssim \mathcal{X}^2_p(t).
\]

Therefore, putting together all the above estimates, we conclude that

\[
 (3.26) \quad \sum_{j \geq 30-1} \sup_{t \in [0,2]} \left( \langle t \rangle \alpha \int_0^t e^{-c_0(t-\tau)} 2^{j(\frac{d}{p}-1)} Z_j(\tau) d\tau \right) \lesssim C \mathcal{X}^2_p(2).
\]

Secondly, let us bound the supremum for \( 2 \leq t \leq T \) in the last term of (3.23). To this end, one can split the integral on \([0, t]\) into integrals \([0, 1]\) and \([1, t]\). The \([0, 1]\) part can be bounded exactly as the supremum on \([0, 2]\) handled before. In order to deal with
the $[1,t]$ part of the integral for $2 \leq t \leq T$, we start from (3.27)
\[
\sum_{j \geq j_0-1} \sup_{t \in [2,T]} \left( \langle t, e^{-c_0(t-\tau)} \right) 2^{j} \left( \frac{d}{2} - 1 \right) Z_j(\tau) d\tau \right) \lesssim \sum_{j \geq j_0-1} 2^{j} \left( \frac{d}{2} - 1 \right) \sup_{t \in [1,T]} t^\alpha Z_j(t).
\]

In what follows, we claim the following two inequalities
\[
\begin{align*}
\| \tau^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} a(\tau) \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} & \lesssim D_p(t), \\
\| \tau^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} \nabla^m(a^\ell, v^\ell, \theta^\ell)(\tau) \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} & \lesssim D_p(t) \quad \text{for } m = 0, 1.
\end{align*}
\]

Indeed, it follows from Proposition 2.1, the fact $\alpha > \frac{s}{d} + \frac{d}{2} - \frac{s}{2}$ for small enough $\varepsilon$, (1.14) and tilde norms that
\[
\begin{align*}
\| \tau^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} a(\tau) \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} & \lesssim \| \tau^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} a^\ell(\tau) \|_{L^\infty_t(B_{\frac{d}{2},1}^{d})} + \| \tau^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} a^h(\tau) \|_{L^\infty_t(B_{\frac{d}{2},1}^{d})} \\
& \lesssim \| \langle \tau \rangle^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} a(\tau) \|_{L^\infty_t(B_{\frac{d}{2},1}^{d})} + \| \langle \tau \rangle^\alpha a(\tau) \|_{L^\infty_t(B_{\frac{d}{2},1}^{d})} \\
& \lesssim D_p(t),
\end{align*}
\]
\[
\begin{align*}
\| \tau^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} \nabla^m(a^\ell, v^\ell, \theta^\ell)(\tau) \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} & \lesssim \| \langle \tau \rangle^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} \nabla^m(a^\ell, v^\ell, \theta^\ell)(\tau) \|_{L^\infty_t(B_{\frac{d}{2},1}^{d})} \\
& \lesssim \| \langle \tau \rangle^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} \nabla^m(a, v, \theta)(\tau) \|_{L^\infty_t(B_{\frac{d}{2},1}^{d})} \\
& \lesssim D_p(t) \quad \text{for } m = 0, 1.
\end{align*}
\]

To bound the right-hand side of (3.27), it only need to estimate the “new” nonlinear terms (say, $G_5, G_6$ and $k$), which are not available in the isentropic compressible Navier-Stokes system, see [32] for more details. Regarding the term with $K_2(a) \nabla \theta$, we still use the decomposition $K_2(a) \nabla \theta = K_2(a) \nabla \theta^h + K_2(a) \nabla \theta^\ell$. According to Propositions 2.2 and 2.5, (1.14), (3.25), (3.28), (3.29) and tilde norms, we get
\[
\begin{align*}
\| t^\alpha K_2(a) \nabla \theta^h \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} & \lesssim \| a \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} \| t^\alpha \nabla \theta^h \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} \lesssim \chi_p(T) D_p(T), \\
\| t^\alpha K_2(a) \nabla \theta^\ell \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} & \lesssim \| t^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} a \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} \| t^{\frac{s}{d} + \frac{d}{2} - \frac{s}{2}} \nabla \theta^\ell \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} \\
& \lesssim D^2_p(T).
\end{align*}
\]

For the term with $A \nabla K_3(a)$, we decompose $A \nabla K_3(a) = A^h \nabla K_3(a) + A^\ell \nabla K_3(a)$. To handle the term with $A^h \nabla K_3(a)$, we note that, due to Propositions 2.2 and 2.5, (1.14), (3.25) and tilde norms,
\[
\| t^\alpha A^h \nabla K_3(a) \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} \lesssim \| t^\alpha A^h \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} \| A \|_{L^\infty_t(B_{p,1}^{\frac{d}{2}})} \lesssim \chi_p(T) D_p(T).
\]

For the term containing $A^\ell \nabla K_3(a)$, we may write
\[
A^\ell \nabla K_3(a) = A^\ell K'_3(a) \nabla a \quad \text{with } K'_3(a) = \chi_0 \sqrt{\frac{T_\infty}{C_v} \frac{\pi_1}{\theta_\infty} \frac{(1 + a)}{1 + a}}.
\]
Now, we have thanks to Propositions 2.2 and 2.5 (1.14), (3.25) and (3.29),
\[
\|t^\alpha \theta^h K_1^2(a) \nabla a^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|\theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|t^\alpha \nabla a^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha \theta^h K_2^2(a) \nabla a^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|\theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|t^\alpha \nabla a^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha \theta^h \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|\theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha \theta^h \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|\theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
Now, let us keep in mind that the relations \( p < d \) and \( d \geq 3 \). For the term with \( v \cdot \nabla \theta \), it follows from (1.14), (3.25), (3.29) and Proposition 2.2 adapted to tilde spaces that
\[
\|t^\alpha v \cdot \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|v\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|t^\alpha \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha v^h \cdot \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|t^\alpha v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha v^h \cdot \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|t^\alpha v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
With the aid of Propositions 2.2 and 2.5 (1.14), (3.25), (3.28) and (3.29), we arrive at
\[
\|t^\alpha \tilde{K}_1(a) \nabla v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|\tilde{a}\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|t^\alpha \nabla v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha \tilde{K}_1(a) \nabla v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|t^\alpha v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
To deal with the term containing \( \tilde{K}_2(a) \theta \nabla v \), we observe that, thanks to Propositions 2.2 and 2.5 (1.14), (3.3), (3.25) and (3.29) that
\[
\|t^\alpha \tilde{K}_2(a) \theta^h \nabla v\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|t^\alpha \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla v\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha \tilde{K}_2(a) \theta^h \nabla v\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|\theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|t^\alpha v^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
Recall that the term \( \frac{\tilde{a}^2(a)}{(1+|\alpha|)^a} \nabla a \cdot \nabla \theta \) of \( k \) is of the type \( \nabla H(a) \cdot \nabla \theta \) with \( H(0) = 0 \), and the term \( k_1(a, \theta) \) of \( k \) is of the type \( K(a) \Delta \theta \) with \( K(0) = 0 \). Consequently, from Propositions 2.2 and 2.5 (1.14), (3.25), (3.28) and (3.29), we infer that
\[
\|t^\alpha \nabla H(a) \cdot \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|t^\alpha \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
\[
\|t^\alpha \nabla H(a) \cdot \nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-2}})} \lesssim \|t^\alpha \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \|\nabla \theta^h\|_{L_t^\infty(B_{p,1}^{d \frac{4}{p-1}})} \lesssim \mathcal{X}_p(T) D_p(T),
\]
To bound the term containing \( k_2(a, \nabla v, \nabla v) \), we take advantage of Propositions 2.2 and 2.3, (1.14), (3.3), (3.25) and (3.29), and get

\[
\| t^\alpha k_2(a, \nabla v, \nabla v^h) \|_{B^d_p(B^d_{p,1})}^h \lesssim \| \nabla v \|_{L^p(B^d_{p,1})^d}^h \| t^\alpha \nabla v^h \|_{L^q(B^d_{q,1})}^h \lesssim X_p(T)D_p(T),
\]

\[
\| t^\alpha k_2(a, \nabla v^h, \nabla v) \|_{B^d_p(B^d_{p,1})}^h \lesssim \| t^\alpha \nabla v^h \|_{L^p(B^d_{p,1})^d}^h \| \nabla v \|_{L^q(B^d_{q,1})}^h \lesssim X_p(T)D_p(T),
\]

\[
\| t^\alpha k_2(a, \nabla v^\ell, \nabla v^\ell) \|_{B^d_p(B^d_{p,1})}^h \lesssim \| t^\alpha \nabla v^\ell \|_{L^p(B^d_{p,1})^d}^h \| \nabla v^\ell \|_{L^q(B^d_{q,1})}^h \lesssim X_p(T)D_p(T),
\]

Putting all the above estimates together, we discover that

\[
\sum_{j \geq j_0-1} 2^{j(d-1)} \sup_{t \in [1,T]} t^\alpha Z_j(t) \lesssim X_p(T)D_p(T) + D_p^2(T).
\]

Plugging (3.30) in (3.27), and remembering (3.26) and (3.23), we end up with (3.22). This completes the proof of Proposition 3.2.

### 3.3. Third step: Decay and gain of regularity for the high frequencies of \((\nabla v, \theta)\)

Let us prove that the parabolic smoothing effect provided by the last two equations of (1.9) allows us to get gain of regularity and decay altogether for \(v\) and \(\theta\). Precisely, one has

**Proposition 3.3.** If \( p \) satisfies (1.10), then it holds that for all \( t \geq 0 \),

\[
\| t\alpha (\nabla v, \theta) \|_{B^d_p(B^d_{p,1})}^h \lesssim \| (\nabla v_0, v_0) \|_{B^d_{p,1}^d}^h + \| \theta_0 \|_{B^d_{p,1}^d}^h + \lambda^2_p(t) + D^2_p(t)
\]

with \( \alpha = s_1 + \frac{d}{2} + \frac{1}{2} - \epsilon \) for sufficiently small \( \epsilon > 0 \), where \( X_p(t) \) and \( D_p(t) \) have been defined by (3.1) and (1.14), respectively.

**Proof.** It follows from the second and third equations in (1.9) that

\[
\begin{cases}
\partial_t v - \tilde{A}v = g - \nabla a - \gamma \nabla \theta, \\
\partial_t \theta - \beta \Delta \theta = k - \gamma \text{div} v.
\end{cases}
\]

In order to prove (3.31), we reformulate (3.32) as follows

\[
\begin{cases}
\partial_t (t^\alpha \tilde{A}v) - \tilde{A}(t^\alpha \tilde{A}v) = t^\alpha \tilde{A}g + \alpha t^{\alpha-1} \tilde{A}v - t^\alpha \tilde{A} \nabla a - t^\alpha \tilde{A} \nabla \theta, \\
\partial_t (t^\alpha \Delta \theta) - \beta (t^\alpha \Delta \theta) = t^\alpha \Delta k + \alpha t^{\alpha-1} \Delta \theta - t^\alpha \Delta \text{div} v,
\end{cases}
\]

\[ (t^\alpha \tilde{A}v, t^\alpha \Delta \theta)|_{t=0} = (0,0). \]

Taking advantage of Proposition 2.6, Remark 2.2, and Bernstein inequality, we have for \( j \geq j_0 - 1 \),

\[
\| t^\alpha \nabla v \|_{L^p(B^d_{p,1})}^h \lesssim \| t^\alpha g \|_{L^p(B^d_{p,1})}^h + \| t^{\alpha-1} v \|_{L^p(B^d_{p,1})}^h + \| t^\alpha \nabla a \|_{L^p(B^d_{p,1})}^h + \| t^{\alpha-1} \theta \|_{L^p(B^d_{p,1})}^h + \| t^\alpha v \|_{L^p(B^d_{p,1})}^h
\]

\[
\| t^\alpha \theta \|_{L^p(B^d_{p,1})}^h \lesssim \| t^\alpha k \|_{L^p(B^d_{p,1})}^h + \| t^{\alpha-1} \theta \|_{L^p(B^d_{p,1})}^h + \| t^\alpha v \|_{L^p(B^d_{p,1})}^h.
\]
As \( \alpha > 1 \) for small enough \( \varepsilon > 0 \), we see that
\[
\| \tau^{\alpha-1}(\nabla v, \theta) \|^h_{L^\infty_t(B^{s-\frac{d}{p}}_{p,1})} \lesssim \| \tau^\alpha(\nabla v, \theta) \|^h_{L^\infty_t(B^{s}_{p,1})},
\]
\[
\| \tau^\alpha(\nabla a, v) \|^h_{L^\infty_t(B^{s-\frac{d}{p}}_{p,1})} \lesssim \| \tau^\alpha(\nabla a, v) \|^h_{L^\infty_t(B^{s}_{p,1})}.
\]
Furthermore, we deduce that
\[
\| \tau^\alpha(\nabla v, \theta) \|^h_{L^\infty_t(B^{s}_{p,1})} \lesssim \| \tau^\alpha g \|^h_{L^\infty_t(B^{s}_{p,1})} + \| \tau^\alpha k \|^h_{L^\infty_t(B^{s}_{p,1})} + \| \tau^\alpha(\nabla a, v) \|^h_{L^\infty_t(B^{s}_{p,1})} + \| \tau^\alpha(\nabla a, v) \|^h_{L^\infty_t(B^{s-\frac{d}{p}}_{p,1})}.
\]

(3.33)

With the aid of (3.22), the last two norms of the r.h.s of (3.33) can be bounded by
\[
\| (\nabla a_0, v_0) \|^h_{B^{s}_{p,1}} + \| \theta_0 \|^h_{B^{s}_{p,1}} + K_p(t) + X_p^2(t).
\]

Bounding the norms \( \| \tau^\alpha g \|^h_{L^\infty_t(B^{s}_{p,1})} \) and \( \| \tau^\alpha k \|^h_{L^\infty_t(B^{s}_{p,1})} \) are exactly same as the second step and those work of [32], one can conclude that (3.31) readily.

Finally, adding up (3.31) to (3.22) and (3.21) yields for all \( T \geq 0 \),
\[
D_p(T) \lesssim D_{p,0} + \| (\nabla a_0, v_0) \|^h_{B^{s}_{p,1}} + \| \theta_0 \|^h_{B^{s}_{p,1}} + X_p^2(T) + D_p^2(T).
\]

The global existence result (see for example Theorem 1.1 in [15]) ensures that \( X_p(t) \lesssim X_{p,0} \ll 1 \) and as
\[
\| (a_0, v_0, \theta_0) \|^\ell_{B^{s-\frac{d}{p}}_{2,\infty}} \lesssim \| (a_0, v_0, \theta_0) \|^\ell_{B^{s}_{2,\infty}},
\]
on one can conclude that (1.13) is satisfied for all time if \( D_{p,0} \) and \( X_{p,0} \) are small enough. This completes the proof of Theorem 1.1.

3.4. The proof of Corollary 1.1

Proof. It is suffices to show the decay estimate for \( \theta \). With the aid of the embedding \( B^{s+d(\frac{1}{p} - \frac{1}{2})}_{p,1} \hookrightarrow B^s_{r,1} \) for \( p \leq r \leq \infty \), we arrive at
\[
\sup_{t \in [0,T]} \| t^{-s_1 + \frac{s}{2} - d(\frac{1}{2} - \frac{1}{r})} \| B^s_{r,1} \| \| \tau^\alpha \theta \|_{B^s_{r,1}} \lesssim \sup_{t \in [0,T]} \| t^{-s_1 + \frac{s}{2} - d(\frac{1}{2} - \frac{1}{r})} \| \theta \|_{B^s_{r,1}}
\]
\[
\lesssim \| t^{-s_1 + \frac{s}{2} - d(\frac{1}{2} - \frac{1}{r})} \| \theta \|_{B^{s+d(\frac{1}{p} - \frac{1}{2})}_{p,1}}
\]
\[
\lesssim \| t^{-s_1 + \frac{s}{2} - d(\frac{1}{2} - \frac{1}{r})} \| \theta \|_{B^{s+d(\frac{1}{p} - \frac{1}{2})}_{p,1}}
\]

Thanks to (1.13) and (1.14), we discover that
\[
\| t^{-s_1 + \frac{s}{2} - d(\frac{1}{2} - \frac{1}{r})} \| \theta \|_{L^\infty_t(B^{s+d(\frac{1}{p} - \frac{1}{2})}_{2,1})} \lesssim \| \theta \|^\ell_{L^\infty_t(B^{s+d(\frac{1}{p} - \frac{1}{2})}_{2,1})}
\]
\[
\lesssim (D_{p,0} + \| (\nabla a_0, v_0) \|^h_{B^{s}_{p,1}} + \| \theta_0 \|^h_{B^{s}_{p,1}}),
\]

where we used the fact \(-s_1 < s + d(\frac{1}{2} - \frac{1}{r}) \leq \frac{d}{p} + 1 \) for \(-s_1 < s + d(\frac{1}{p} - \frac{1}{r}) \leq \frac{d}{p} + 1 \).

On the other hand, if \( \varepsilon > 0 \) is small enough, then we have \( \frac{d}{2} + \frac{1}{2} - \varepsilon > 1 + \frac{d}{4} - \varepsilon \geq \)
\[
\frac{d}{s} \geq \frac{s}{d} + \frac{d}{2} \left( \frac{1}{r} - \frac{1}{p} \right) \quad \text{for} \quad -s_1 < s + d \left( \frac{1}{r} - \frac{1}{p} \right) \leq \frac{d}{2}, \quad \text{which ensures that} \quad \alpha \geq \frac{s_1 + \frac{s}{d} + \frac{d}{2} \left( \frac{1}{r} - \frac{1}{p} \right)}{2}. 
\]

Consequently, we deduce that

\[
\left\| \frac{t^{\frac{s_1 + s}{2}} - \frac{d}{2} \left( \frac{1}{r} - \frac{1}{p} \right)}{\theta} \right\|^{h}_{L^{\infty}(B_{p,1}^{s+d} \left( \frac{1}{r} - \frac{1}{p} \right))} \lesssim \left( D_{p,0} + \| \nabla a_0, v_0 \|^{\frac{d}{2}}_{B_{p,1}^{s+d}} + \| \theta_0 \|^{\frac{d}{2}}_{B_{p,1}^{s+d}} \right).
\]

Hence, using \( \hat{B}_{p,1}^{0} \rightarrow L^r \) yields the desired result for \( \theta \). Proving the inequalities for \( a \) and \( v \) is similar. The proof of Corollary 1.1 is complete.

\[\square\]

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Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, P.R.China

Department of Mathematics and Statistics, McGill University, Montreal, Quebec H3A 2K6, Canada

E-mail address: wxshi168@163.com

Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, P.R.China

E-mail address: jiangxu_79math@yahoo.com