A Commutative Family of Integral Transformations and Basic Hypergeometric Series. I. Eigenfunctions

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Abstract

It is conjectured that a class of \( n \)-fold integral transformations \( \{ I(\alpha) | \alpha \in \mathbb{C} \} \) forms a mutually commutative family, namely, we have \( I(\alpha)I(\beta) = I(\beta)I(\alpha) \) for \( \forall \alpha, \forall \beta \in \mathbb{C} \). The commutativity of \( I(\alpha) \) for the two-fold integral case is proved by using several summation and transformation formulas for the basic hypergeometric series. An explicit formula for the complete system of the eigenfunctions for \( n = 3 \) is conjectured. In this formula and in a partial result for \( n = 4 \), it is observed that all the eigenfunctions do not depend on the spectral parameter \( \alpha \) of \( I(\alpha) \).

1 Introduction

It was pointed out in [1] that a certain class of \( n \)-fold integral transformations plays an essential role for a study of the vertex operator \( \Phi(\zeta) \) for Baxter’s eight-vertex model [2 3 4]. (As for the definition of the intertwiner \( \Phi(\zeta) \), see [5 6] and [1].) More precisely, in [1] the matrix elements \( \langle \Phi(\zeta_1)\Phi(\zeta_2)\cdots\Phi(\zeta_n) \rangle \) were represented by applying the \( n \)-fold integral transformation to a basic hypergeometric series. (See Eq. (47) of [1].) The aim of the present paper is to investigate the structure of the integral transformation to obtain a better understanding of the eight-vertex vertex operator \( \Phi(\zeta) \).
Let us first recall the notations for the integral representation given in [1]. Let $h(\zeta)$ and $g(\zeta)$ be the functions defined by

$$h(\zeta) = (1 - \zeta) \frac{(qt^{-1}\zeta; q)_\infty}{(t\zeta; q)_\infty}, \quad (1)$$

$$g(\zeta) = \frac{(q^{\frac{1}{2}}t^{-\frac{1}{2}}\zeta; q)_\infty}{(q^{\frac{1}{2}}t^{-\frac{1}{2}}\zeta; q)_\infty}. \quad (2)$$

Here we have used the standard notation for the $q$-shifted factorial [9]. Note that we have slightly modified the definition of $h(\zeta)$ from the one given by Eq.(4) in [1]. One finds this change in $h(\zeta)$ will simplify our discussion given in what follows.

Let us introduce a space of formal power series of degree zero in variables $\zeta_i$ ($i = 1, 2, \cdots, n$) corresponding to the positive cone of the $A_{n-1}$ type root lattice

$$F_n = \left\{ \sum_{i_1, i_2, \cdots, i_{n-1} \geq 0} c_{i_1, i_2, \cdots, i_{n-1}} \left( \frac{\zeta_2}{\zeta_1} \right)^{i_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{i_2} \cdots \left( \frac{\zeta_n}{\zeta_{n-1}} \right)^{i_{n-1}} \right\}. \quad (3)$$

Note that the matrix elements of the eight-vertex vertex operators belong to this space, namely $\langle \Phi(\zeta_1)\Phi(\zeta_2)\cdots\Phi(\zeta_n)\rangle \in F_n$. Our task in this paper is to propose an integral operator which acts on $F_n$, and study some basic properties of it. Applications will be considered in the continuations of the present paper [7,8].

The central object in the present article is given as an integral transformation acting on $F_n$.

**Definition 1.1** Let $(s_1, s_2, \cdots, s_n) \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$ be parameters. Let $q, t \in \mathbb{C}$ be parameters satisfying the condition $|qt^{-1}| < 1$. We will assume that all these parameters are generic unless otherwise stated. The $n$-fold integral transformation $I(\alpha) = I(\alpha; s_1, s_2, \cdots s_n, q, t)$ is defined by

$$I(\alpha)f(\zeta_1, \cdots, \zeta_n) = \prod_{i=1}^{n} \left( \frac{(qt^{-1}; q)_\infty}{(\alpha s_i^{-1}qt^{-1}; q)_\infty} \frac{(q; q)_\infty}{(\alpha^{-1}s_i q; q)_\infty} \right)$$

$$\times \prod_{i<j} h(\zeta_j/\zeta_i) \oint_{C_1} \cdots \oint_{C_n} \frac{d\xi_1}{2\pi i \xi_1} \cdots \frac{d\xi_n}{2\pi i \xi_n} \prod_{i=1}^{n} \frac{\Theta_q(\alpha s_i^{-1}q^{\frac{1}{2}}t^{-\frac{1}{2}}\zeta_i/\xi_i)}{\Theta_q(\alpha q^{\frac{1}{2}}t^{-\frac{1}{2}}\zeta_i/\xi_i)}$$

$$\times \prod_{k=1}^{n} \left[ \prod_{j<k} g(\zeta_k/\xi_j) \prod_{j\geq k} g(\zeta_j/\zeta_k) \right] f(\xi_1, \cdots, \xi_n),$$

(4)
where the integration contours $C_i$ are given by the conditions $|ζ_i/ξ_i| = 1$, and the theta function $Θ_q(ζ_i)$ is given by (12).

Then our main statement in this article is

**Conjecture 1.2** For $∀ α, ∀ β ∈ C$, the integral transformations $I(α), I(β)$ acting on the space $F_n$ are mutually commutative

$$I(α)I(β) = I(β)I(α).$$

Here the parameters $(s_1, s_2, ⋯, s_n), q$ and $t$ should be fixed, namely Eq.(5) means

$$I(α; s_1, ⋯, s_n, q, t)I(β; s_1, ⋯, s_n, q, t) = I(β; s_1, ⋯, s_n, q, t)I(α; s_1, ⋯, s_n, q, t).$$

In Section 2, we will prove the commutativity of the integral transformation $I(α)$ for the case of $n = 2$ by using some summation and transformation formulas for the basic hypergeometric series.

**Theorem 1.3** For $n = 2$, we have

$$I(α)I(β) = I(β)I(α),$$

on $F_2$.

In Section 3, the complete system of the eigenfunctions for $n = 3$ will be conjectured. This gives us a strong support for Conjecture 1.2 for $n = 3$, since no dependence on the spectral parameter $α$ is observed in all the eigenfunctions. Some evidence for the commutativity for $n = 4$ will also be given.

Conjecture 1.2 means that there exists a quantum mechanical integrable system whose Hamiltonian is given by the integral operator $I(α)$. It is an interesting problem to relate this quantum mechanical system with already known one. This will be considered in the next paper [7]. A Macdonald-type difference operator will be introduced and its commutativity with the action of $I(α)$ will be discussed.

The meaning of the commutativity $[I(α), I(β)] = 0$, however, still remains unclear from the lattice model point of view. Firstly, we do not understand the role of Eq.(5) for the characterization of the correlation functions of
the eight-vertex model, since the integral operator $I(\alpha)$ was heuristically introduced through the investigation based on the free field representation of $\Phi(\zeta)$. Next, we lack an explanation for Eq.(5) based on the commutative family generated by the row-to-row transfer matrix of the eight-vertex model.

The plan of the paper is as follows. In Section 2, a proof of Theorem 1.3 is given. Several summation and transformation formulas for the basic hypergeometric series will be used there. In Section 3, basic properties of the eigenfunctions of $I(\alpha)$ are described. Explicit formulas for the eigenfunctions for $n = 3, 4$ will be conjectured and shown that these are independent of the spectral parameter $\alpha$. For $n = 3$, it gives us the complete system of the eigenfunctions. For $n = 4$, however, only several lower terms will be obtained. Concluding remarks are given in Section 4.

Throughout the paper, we use the standard notations for the $q$-shifted factorials

$$(a; q)_n = \begin{cases} 1 & (n = 0), \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & (n = 1, 2, \cdots), \end{cases} \quad (7)$$

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n(a_2; q)_n\cdots(a_m; q)_n, \quad (8)$$

$$(a; q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad (9)$$

$$(a; q)_{-n} = \frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n})} \quad (n = 1, 2, \cdots), \quad (10)$$

and the basic hypergeometric series

$$r+1\phi_r \left( \begin{array}{c} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{r+1}; q)_n}{(b_1, b_2, \cdots, b_r, q; q)_n} z^n. \quad (11)$$

We also use the notation for the elliptic theta function as

$$\Theta_q(z) = (z; q)_\infty(q/z; q)_\infty(q; q)_\infty. \quad (12)$$
2 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. For the two-fold integral case, the integral transformation reads

\[ I(\alpha)f(\zeta_1, \zeta_2) = \prod_{i=1}^{2} \left( \frac{(qt^{-1}; q)_{\infty}}{(q^{-1}t^{1-1}; q)_{\infty}} \right) h(\zeta_2/\zeta_1) \]
\[ \times \oint \oint \frac{d\xi_1}{2\pi i \xi_1} \frac{d\xi_2}{2\pi i \xi_2} \Theta_q(\alpha s_1^{-1} q^{t^{-1}} \zeta_1/\xi_1) \Theta_q(\alpha s_2^{-1} q^{t^{-1}} \zeta_2/\xi_2) \]
\[ \times g(\xi_1/\zeta_1) g(\xi_2/\zeta_1) g(\zeta_2/\zeta_1) g(\zeta_2/\zeta_2) f(\xi_1, \xi_2). \]

Let us prove the following theorem which is equivalent to Theorem 1.3

**Theorem 2.1** For the two-fold integral case, the operator \( I(\alpha) \) has the eigenvalues and eigenvectors:

\[ I(\alpha)f_j(\zeta_1, \zeta_2) = \lambda_j(\alpha)f_j(\zeta_1, \zeta_2), \]
for \( j = 0, 1, 2, \ldots \), where

\[ \lambda_j(\alpha) = \frac{(\alpha s_1^{-1}; q)_{-j}}{(\alpha s_1^{-1}qt^{-1}; q)_{-j}} \frac{(\alpha s_2^{-1}; q)_{j}}{(\alpha s_2^{-1}qt^{-1}; q)_{j}} \]
\[ = \frac{(\alpha^{-1}s_1t; q)_{j}}{(\alpha^{-1}s_1; q)_{j}} \frac{(\alpha s_2^{-1}; q)_{j}}{(\alpha s_2^{-1}qt^{-1}; q)_{j}} (qt^{-1})_j, \]

and

\[ f_j(\zeta_1, \zeta_2) = \zeta^j \times \Phi_3 \left( \frac{sq^2j+q^{i+1}t^{1-1}}{s^{1/2}q^{i+1/2}, -s^{1/2}q^{i+1/2}}, qt^{1-1}, sq^{2j+1} : q, t \zeta \right) \]
\[ = \zeta^j (1 - \zeta) \times \Phi_1 \left( \frac{qt^{-1}, sq^{2j+1}t^{1-1}}{sq^{2j+1}, : q, t \zeta} \right). \]

Here we have denoted \( \zeta = \zeta_2/\zeta_1 \) and \( s = s_1/s_2 \) for short.

Since the eigenfunctions \( f_j(\zeta_1, \zeta_2) \) do not depend on the parameter \( \alpha \) and our space of series \( \mathcal{F}_2 \) is spanned by \( \{ f_j(\zeta_1, \zeta_2) : j = 0, 1, 2 \ldots \} \), we immediately see that Theorem 2.1 is equivalent to Theorem 1.3.

Note that we have expressed the eigenfunctions \( f_j(\zeta_1, \zeta_2) \) in two ways by using
Lemma 2.2 We have
\[(1 - z)_{2\phi_1} \left( \frac{a, b}{aqb^{-1}} ; q, zqb^{-1} \right) = \psi_3 \left( \frac{a + \frac{1}{2}q^2, -a + \frac{1}{2}q^2, aq^{-1}, bq^{-1}}{aq^{-\frac{1}{2}}, -aq^{-\frac{1}{2}}, aq^{-1}, bq^{-1}} ; q, zqb^{-1} \right). \] (19)

Proof.
\[\text{LHS} = 1 + \sum_{n=1}^{\infty} \frac{(a, b; q)_n}{(aqb^{-1}, q)_n} (q^{-1})^n z^n = 1 + \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(aqb^{-1}, q)_n} (q^{-1})^n z^{n+1} = 1 + \sum_{n=1}^{\infty} \frac{(a, b; q)_n}{(aqb^{-1}, q)_n} \frac{(1 - bq^{-1})(1 - aq^{-2n-1})}{(1 - bq^{-1})(1 - aq^{-n-1})} (zqb^{-1})^n = \text{RHS}.\]

We will see this identity will play an important role in what follows.

2.1 monomial basis representation of $I(\alpha)$
We represent the action of the integral transformation $I(\alpha)$ in terms of the monomial basis \{$(\zeta_i/\zeta_1)^i | i = 0, 1, 2, \cdots$\} of $F_2$. To this end, first we need the Laurent series expansion of the function
\[g(\xi/\zeta) \frac{\Theta_q(\alpha q^{1/2} t^{-1/2} \zeta/\xi)}{\Theta_q(q^{1/2} t^{-1/2} \zeta/\xi)}.\] (20)

This is given by Ramanujan’s summation formula for the $1\psi_1$ series. (Eq. (5.2.1) of Gasper and Rahman [9], hereafter referred to as GR).

Lemma 2.3 We have
\[g(\xi/\zeta) \frac{\Theta_q(\alpha q^{1/2} t^{-1/2} \zeta/\xi)}{\Theta_q(q^{1/2} t^{-1/2} \zeta/\xi)} = \frac{(\alpha q^{-1}; q)_\infty (q^{-1}; q)_\infty}{(q^{-1}; q)_\infty} \sum_{m \in \mathbb{Z}} \frac{(\alpha; q)_m}{(\alpha q t^{1/2}; q)_m} (q^{1/2} t^{-1/2} \zeta/\xi)^m,\] (21)
for $|q^{-1/2} t| < |\zeta/\xi| < |q^{-1/2} t|$. Let us represent the action of $I(\alpha)$ on a formal power series of the form
\[f(\zeta_1, \zeta_2) = \sum_{j=0}^{\infty} f_j (\zeta_2/\zeta_1)^j,\] (22)
in terms of the basic hypergeometric series $2\phi_1$.
Proposition 2.4 Let \( f(\zeta_1, \zeta_2) \) be as above. Then we have

\[
I(\alpha)f(\zeta_1, \zeta_2) = (1 - \zeta) \frac{(q t^{-1} \zeta; q)_\infty}{(t \zeta; q)_\infty} \times \sum_{k=0}^\infty 2^k \phi_1 \left( \frac{\alpha^{-1} s_1 q^k t, t}{\alpha^{-1} s_1 q^{k+1}; q, q t^{-1} \zeta} \right) h \left( \frac{\alpha s_{2}^{-1} q^k, t}{\alpha s_{2}^{-1} q^{k+1}; q, q t^{-1} \zeta} \right) \lambda_k(\alpha) f_k \zeta^k,
\]

where \( \zeta = \zeta_2/\zeta_1 \).

Proof. By the \( q \)-binomial theorem (Eq. (1.3.2) of GR [2]), we have

\[
g(\zeta) = \sum_{j=0}^\infty g_j \zeta^j, \quad g_j = \frac{(t; q)_j}{(q; q)_j} (q^{1/2} t^{-1/2})^j.
\]

Thus

LHS

\[
= h(\zeta_2/\zeta_1) \int \frac{d\xi_1}{2\pi i \xi_1} \int \frac{d\xi_2}{2\pi i \xi_2} \sum_{m,n \in \mathbb{Z}} \frac{(\alpha s_{1}^{-1}; q)_m}{(\alpha s_{1}^{-1} q^{-1}; q)_m} (q^{1/2} t^{-1/2} \zeta_2/\zeta_1)^m \times \sum_{i,j,k \geq 0} g_i g_j f_k (\xi_2/\xi_1)^i (\xi_2/\xi_1)^j (\xi_2/\xi_1)^k
\]

\[
= h(\zeta_2/\zeta_1) \sum_{i,j,k \geq 0} \frac{(\alpha s_{1}^{-1}; q)_{-j-k}}{(\alpha s_{1}^{-1} q^{-1}; q)_{-j-k}} g_j (q^{1/2} t^{-1/2})^{-j} (\zeta_2/\zeta_1)^j \times \frac{(\alpha s_{2}^{-1}; q)_{i+k}}{(\alpha s_{2}^{-1} q^{-1}; q)_{i+k}} g_i (q^{1/2} t^{-1/2})^i (\zeta_2/\zeta_1)^i \times f_k (\zeta_2/\zeta_1)^k
\]

\[
= h(\zeta_2/\zeta_1) \sum_{i,j,k \geq 0} \frac{(\alpha^{-1} s_{1} q^k t, t; q)_j}{(\alpha^{-1} s_{1} q^{k+1}; q)_j} (q t^{-1} \zeta_2/\zeta_1)^j \times \frac{(\alpha s_{2}^{-1} q^k t; q)_i}{(\alpha s_{2}^{-1} q^{k+1+t^{-1}}; q)_i} (q t^{-1} \zeta_2/\zeta_1)^i \times \frac{(\alpha s_{1}^{-1} t, \alpha s_{2}^{-1}; q)_k}{(\alpha s_{1}^{-1} t, \alpha s_{2}^{-1} q^{-1}; q)_k} (q t^{-1} \zeta_2/\zeta_1)^k f_k
\]

= RHS.
To obtain a more useful series expression for $I(\alpha)$ from Proposition 2.4, we need a lemma.

**Lemma 2.5** We have

$$\frac{(zq/b; q)_{\infty}}{(b z; q)_{\infty}} 2\phi_1 \left( \frac{a, b}{aq/b ; q, zq/b} \right) 2\phi_1 \left( \frac{c, b}{cq/b ; q, zq/b} \right) = \sum_{k=0}^{\infty} \frac{(cq/b^2; q)_k(q/b; q)_k b^k z^k}{(cq/b; q)_k(q; q)_k} \phi_3 \left( \frac{q^{-k}, b, bq^{-k}/c, a}{bq^{-k}, b^2q^{-k}/c, aq/b ; q} \right).$$

**Proof.** By using Heine’s transformation (Eq. (1.4.3) of GR [9])

$$\text{LHS} = 2\phi_1 \left( \frac{a, b}{aq/b ; q, zq/b} \right) 2\phi_1 \left( \frac{cq/b^2, q/b}{cq/b ; q, bz} \right) = \sum_{m,n=0}^{\infty} \frac{(a; q)_m(b; q)_m (cq/b^2; q)_n(q/b; q)_n}{(cq/b; q)_m(q; q)_n} (zq/b)^m(bz)^n$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(a; q)_m(b; q)_m (cq/b^2; q)_k(q/b; q)_k}{(cq/b; q)_k(q; q)_k} (zq/b)^m(bz)^k$$

$$\times \sum_{m=0}^{k} \frac{(bq^{-k}/c; q)_k(q^{-k}; q)_k}{(aq/b; q)_m(q; q)_m} (b^2q^{-k}/c; q)_k(bq^{-k}; q)_k q^m$$

$$= \text{RHS}.$$ 

**□**

Using Lemma 2.5, we can express the action of the operator $(1-\zeta_2/\zeta_1)^{-1}I(\alpha)$ by using a terminating and balanced $4\phi_3$ series.

**Proposition 2.6**

$$\frac{1}{1-\zeta_2/\zeta_1}I(\alpha)f(\zeta_1, \zeta_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} (\zeta_2/\zeta_1)^i e_{ij} f_j,$$

(26)
where

\[ e_{ij} = \frac{(qt^{-1}; q)_{i-j}(\alpha^{-1}s_1q^{j+1}t^{-1}; q)_{i-j}}{(q; q)_{i-j}(\alpha^{-1}s_1q^{j+1}; q)_{i-j}} t^{i-j} \lambda_j(\alpha) \times 4\phi_3 \left( \begin{array}{c} q^{-i+j}, t, \alpha s_1^{-1}q^{-i}, \alpha s_2^{-1}q^j \\ q^{-i+j}t, \alpha s_1^{-1}q^{-i}t, \alpha s_2^{-1}q^{j+1}t^{-1}; q, q \end{array} \right). \]  

(27)

Note that, if we work with the monomial basis \{ (\zeta_2/\zeta_1)^i | i = 0, 1, 2, \ldots \} and identifying \((\zeta_2/\zeta_1)^k\) with the column unit vector \(i(0, \ldots, 0, 1, 0, \ldots)\) having 1 at the \(k\)-th place, we have an infinite lower triangular matrix representation of the integral operator \(I(\alpha)\),

\[ \frac{1}{1 - \zeta_2/\zeta_1} I(\alpha) = \begin{pmatrix} 1 & e_{10} & 1 \\ e_{20} & e_{21} & 1 \\ e_{30} & e_{31} & e_{32} & 1 \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \]  

(28)

Because of this lower triangular nature, the action of \(I(\alpha)\) on \(\mathcal{F}_n\) is well defined.

**Proof.**

\[ \text{LHS of Eq.(26)} \]

\[ = \frac{(qt^{-1}; q)_{\infty}}{(t; q)_{\infty}} \sum_{j=0}^{\infty} 2\phi_1 \left( \alpha^{-1}s_1q^j t, t \right) \frac{(\alpha^{-1}s_1q^{j+1}; q)_{\infty}}{} \times 2\phi_1 \left( \alpha s_2^{-1}q^j, t \right) \frac{(\alpha s_2^{-1}q^{j+1}t^{-1}; q)_{\infty}}{\left( q, q \right)_{k}\left( \alpha^{-1}s_1q^{j+1}; q \right)_{k}} \right) \lambda_j(\alpha)f_j \zeta^j \]

\[ = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(qt^{-1}; q)_k(\alpha^{-1}s_1q^{j+1}t^{-1}; q)_k}{(q; q)_k(\alpha^{-1}s_1q^{j+1}; q)_k} (t\zeta)^k \lambda_j(\alpha)f_j \zeta^j \times 4\phi_3 \left( \begin{array}{c} q^{-k}, t, \alpha s_1^{-1}q^{-j+k}, \alpha s_2^{-1}q^j \\ q^{-k}t, \alpha s_1^{-1}q^{-j+k}t, \alpha s_2^{-1}q^{j+1}t^{-1}; q, q \end{array} \right) \]

\[ = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(qt^{-1}; q)_{i-j}(\alpha^{-1}s_1q^{j+1}t^{-1}; q)_{i-j}}{(q; q)_{i-j}(\alpha^{-1}s_1q^{j+1}; q)_{i-j}} t^{i-j} \lambda_j(\alpha)f_j \zeta^i \times 4\phi_3 \left( \begin{array}{c} q^{-i+j}, t, \alpha s_1^{-1}q^{-i}, \alpha s_2^{-1}q^j \\ q^{-i+j}t, \alpha s_1^{-1}q^{-i}t, \alpha s_2^{-1}q^{j+1}t^{-1}; q, q \end{array} \right) \]

\[ = \text{RHS of Eq.(26)}. \]
2.2 properties of the coefficients of the functions $f_j(\zeta_1, \zeta_2)$

Nextly, we need to study some properties of the coefficients of the functions $f_j(\zeta_1, \zeta_2)$ defined by Eq. (17) in some detail. The coefficients of $f_j(\zeta_1, \zeta_2)$

$$f_j(\zeta_1, \zeta_2) = \sum_{i=0}^{\infty} c_{ij}(\zeta_2/\zeta_1)^i,$$

are given by

$$c_{ij} = \begin{cases} \frac{(sq^{2j}t^{-1}, s^{\frac{1}{2}}q^{i+1}t^{-\frac{i}{2}}, -s^{\frac{1}{2}}q^{i+1}t^{-\frac{i}{2}}, t^{-1}; q)_{i-j}}{(sq^{2j+1}, s^{\frac{1}{2}}q^{i}t^{-\frac{i}{2}}, -s^{\frac{1}{2}}q^{i}t^{-\frac{i}{2}}, q; q)_{i-j}} & (i \geq j), \\ 0 & (i < j) \end{cases}$$

(30)

where we have denoted $s = s_1/s_2$ for short. If we identify the functions $f_j(\zeta_1, \zeta_2)$'s with the column vectors $^t(c_{0j}, c_{1j}, c_{2j}, \cdots)$ and use $\{f_j(\zeta_1, \zeta_2)|j = 0, 1, 2 \cdots\}$ as our basis of $F_2$, we may have another matrix representation of $I(\alpha)$. With this basis, Theorem 2.1 is recast as

$$I(\alpha)C = CA(\alpha),$$

(31)

where

$$C = \begin{pmatrix} 1 & 1 \\ c_{10} & c_{21} & 1 \\ c_{20} & c_{31} & c_{32} & 1 \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

(32)

$$A(\alpha) = \text{diag}(\lambda_0(\alpha), \lambda_1(\alpha), \lambda_2(\alpha), \cdots).$$

(33)

We find that all the entries of $C^{-1}$ can be factorized.

Proposition 2.7 The inverse of $C$ is written as

$$C^{-1} = \begin{pmatrix} 1 \\ d_{10} & 1 \\ d_{20} & d_{21} & 1 \\ d_{30} & d_{31} & d_{32} & 1 \end{pmatrix},$$

(34)
where

\[
d_{ij} = \begin{cases} 
  \frac{(sq^{i+j+1}t^{-1}, t; q)_{i-j}}{(sq^{i+j}, q; q)_{i-j}} & (i \geq j), \\
  0 & (i < j).
\end{cases} \tag{35}
\]

**Proof.** It suffices to show

\[
\sum_{k=j}^{i} d_{ik} c_{kj} = \delta_{i,j},
\]

for \( i \geq j \). Rewrite the left hand side as

\[
d_{i,j} \sum_{k=j}^{i-j} d_{i,k+j} c_{k+j,j},
\]

and we realize this summation is a terminating very-well-poised \( 6\phi_5 \) series, since

\[
d_{i,k+j} \frac{d_{i,k+j}}{d_{i,j}} = \frac{(q^{-i+j}, sq^{i+j}; q)_k}{(q^{-i+j+1}t^{-1}, sq^{i+j+1}t^{-1}; q)_k} (qt^{-1})^k,
\]

\[
c_{k+j,j} = \frac{(sq^{2j}t^{-1}, s^{\frac{1}{2}}q^{j+1}t^{-\frac{1}{2}}, -s^{\frac{1}{2}}q^{i+j+1}t^{-\frac{1}{2}}, -t^{-1}; q)_k t^k}{(sq^{2j+1}, s^{\frac{3}{2}}q^j t^{-\frac{3}{2}}, -s^{\frac{3}{2}}q^{j+1}t^{-\frac{3}{2}}, q; q)_k}.
\]

One then finds that this \( 6\phi_5 \) series can be summed by using Eq. (2.4.2) of GR [9]. Thus we have

\[
d_{i,j} \sum_{k=0}^{i-j} d_{i,k+j} c_{k+j,j} = d_{i,j} 6\phi_5 \left( \frac{sq^{2j}t^{-1}, s^{\frac{1}{2}}q^{j+1}t^{-\frac{1}{2}}, -s^{\frac{1}{2}}q^{i+j+1}t^{-\frac{1}{2}}, -t^{-1}, sq^{i+j}, q^{-i+j}}{s^{\frac{1}{2}}q^j t^{-\frac{1}{2}}, -s^{\frac{1}{2}}q^j t^{-\frac{1}{2}}, sq^{2j+1}, q^{-i+j+1}t^{-1}, sq^{i+j+1}t^{-1}; q, q} \right)
\]

\[
= d_{i,j} \frac{(sq^{2j+1}t^{-1}, q^{-i+j+1}, q)_{i-j}}{(sq^{2j+1}, q^{-i+j+1}t^{-1}; q)_{i-j}}
\]

\[
= \delta_{i,j}.
\]

\[ \square \]

Our next task is to study the matrix \( C \Lambda(\alpha) C^{-1} \) and compare it with the coefficients given in Proposition 2.6. Since we have realized \( (1-\zeta_2/\zeta_1)^{-1} I(\alpha) \) instead of \( I(\alpha) \) itself, we need to modify the function \( f_j(\zeta_1, \zeta_2) \) accordingly. Let us set

\[
f_j(\zeta_1, \zeta_2) = (1-\zeta_2/\zeta_1) \tilde{f}_j(\zeta_1, \zeta_2), \tag{36}
\]

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where
\[
\tilde{f}_j(\zeta_1, \zeta_2) = (\zeta_2/\zeta_1)^j \times 2\phi_1 \left( \frac{q^{2j+1}t^{-1}, qt^{-1}}{q^{2j+1}}; q, qt\zeta_2/\zeta_1 \right).
\] (37)

Then Theorem 2.1 can be written as
\[
\frac{1}{1 - \zeta_2/\zeta_1} I(\alpha) f_j(\zeta_1, \zeta_2) = \lambda_j(\alpha) \tilde{f}_j(\zeta_1, \zeta_2) \quad (j = 0, 1, 2, \ldots).
\] (38)

Setting
\[
\tilde{f}_j(\zeta_1, \zeta_2) = \sum_{i=0}^{\infty} \tilde{c}_{ij}(\zeta_2/\zeta_1)^i,
\] (39)
we have a matrix equation
\[
\frac{1}{1 - \zeta_2/\zeta_1} I(\alpha) = \tilde{C} \Lambda(\alpha) C^{-1},
\] (40)

where
\[
\tilde{C} = (\tilde{c}_{ij})_{0 \leq i,j \leq \infty},
\] (41)
\[
\tilde{c}_{ij} = \begin{cases} 
(s q^{2j+1} t^{-1}, q t^{-1} ; q)_{i-j} t^{i-j} & (i \geq j), \\
(s q^{2j+1}, q ; q)_{i-j} & (i < j),
\end{cases}
\] (42)

and $s = s_1/s_2$.

Let us examine the matrix $\tilde{C} C^{-1}$ before we start to calculate $\tilde{C} \Lambda(\alpha) C^{-1}$. It is not necessary, but this way of calculating the matrix multiplication may reduce our task a little.

**Proposition 2.8** We have
\[
\tilde{C} C^{-1} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}.
\] (43)
Proof. By using
\[
\frac{(t; q)_{i-j-1}}{(q; q)_{i-j-1}} = \frac{(t; q)_{i-j}}{(q; q)_{i-j}} \frac{(q^{-i+j}; q)_i}{(q^{-i+j+1}; q)_i} (qt^{-1})^l, \\
\]
\[
(sq^{2i-2l+1}t^{-1}; q)_l (sq^{i+j+l+1}t^{-1}; q)_{i-j-1} = (sq^{i+j+l+1}t^{-1}; q)_{i-j} \\
= (sq^{i+j+1}t^{-1}; q)_{i-j} \frac{(s-1q^{-i-j}; q)_l}{(s-1q^{-2i}t; q)_l} q^{-l(i-j)}, \\
\]
\[
(sq^{2i-2l+1}; q)_l (sq^{i+j-l}; q)_{i-j-1} = \frac{1 - sq^{2i-l}}{1 - sq^{2i-2l}} (sq^{i+j-l}; q)_{i-j} \\
= \frac{1 - sq^{2i-l}}{1 - sq^{2i-2l}} (sq^{i+j-l}; q)_{i-j} \frac{(s-1q^{-i-j+1}; q)_l}{(s-1q^{-2i+1}; q)_l} q^{-l(i-j)}, \\
\]
and
\[
1 - sq^{2i-2l} \frac{1 - sq^{2i-2l}}{1 - sq^{2i-2l}} \frac{1 - sq^{-2i}}{1 - sq^{-2i+1}} q^{-l} \\
= \frac{(s^{-\frac{i}{2}}q^{-i+1}, -s^{-\frac{i}{2}}q^{-i+1}, s^{-1}q^{-2i}; q)_l}{(s^{-\frac{i}{2}}q^{-i}, -s^{-\frac{i}{2}}q^{-i}, s^{-1}q^{-2i+1}; q)_l} q^{-l}, \\
\]
we have
\[
\sum_{k=j}^{i} \bar{c}_{ik} d_{kj} = \sum_{l=0}^{i-j} \bar{c}_{i,i-l} d_{i-l,j} \\
= \sum_{l=0}^{i-j} (sq^{2i-2l+1}; q)_l (sq^{i+j-l+1}; q)_l (s^{-1}q^{-2i}; q)_l (s^{-1}q^{-i-j}; q)_l \phi_5 \left( \begin{array}{c} s^{-1}q^{-2i}, s^{-\frac{i}{2}}q^{-i+1}, -s^{-\frac{i}{2}}q^{-i+1}, qt^{-1}, s^{-1}q^{-i-j}, q^{-i+j} \\
+ s^{-\frac{i}{2}}q^{-i}, -s^{-\frac{i}{2}}q^{-i}, s^{-1}q^{-2i}; q^{-i+j+1}, t^{-1}, s^{-1}q^{-i-j+1}; q, 1 \end{array} \right) \\
= \frac{d_{ij}}{(s^{-1}q^{-2i+1}; q^{-i+j}; q)_{i-j}} \\
= 1.
\]
Here, we have used Eq. (2.4.2) of GR [9].

Now we proceed to studying the matrix elements of $\bar{C}\Lambda(\alpha)C^{-1}$. 

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Proposition 2.9  The nonzero entries for $\tilde{\mathcal{C}}\Lambda(\alpha)C^{-1}$ are are given by

\[
(\tilde{\mathcal{C}}\Lambda(\alpha)C^{-1})_{ij} = \lambda_i(\alpha)d_{ij} \times 8W_7\left(s^{-1}q^{-2i}; qt^{-1}, s^{-1}q^{-i-j}t, \alpha s_1^{-1}q^{-i}, \alpha^{-1}s_2q^{-i}t, q^{-i+j}; q, qt^{-1}\right),
\]

for $i \geq j$, and $(\tilde{\mathcal{C}}\Lambda(\alpha)C^{-1})_{ij} = 0$ if $i < j$.

Note that here we have used the compact notation

\[
8W_7(a_1; a_4, a_5, \cdots, a_{r+1}; q, z)
\]

\[
= r+1\phi_r\left(\frac{a_1, qa_1^{1/2}, -qa_1^{1/2}, a_4, \cdots, a_{r+1}}{a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \cdots, qa_1/a_{r+1}}; q, z\right),
\]

for the very-well-poised $r+1\phi_r$ series.

Proof. From the identity

\[
\lambda_{i-l}(\alpha) = \frac{(\alpha s_2^{-1}, \alpha^{-1}s_1t; q)_{i-l}}{(\alpha s_2^{-1}q^{-1}, \alpha^{-1}s_1q; q)_{i-l}}(qt^{-1})^{i-l}
\]

\[
= \frac{(\alpha s_2^{-1}, \alpha^{-1}s_1t; q)_{i}}{(\alpha s_2^{-1}q^{-1}, \alpha^{-1}s_1q; q)_{i}}(qt^{-1})^{i} \frac{(\alpha^{-1}s_2q^{-i}t, \alpha s_1^{-1}q^{-i}; q)_l}{(\alpha^{-1}s_2q^{-i+1}, \alpha s_1^{-1}q^{-i+1}t^{-1}; q)_l}(qt^{-1})^{l}
\]

\[
= \lambda_i(\alpha) \frac{(\alpha^{-1}s_2q^{-i}t, \alpha s_1^{-1}q^{-i}; q)_l}{(\alpha^{-1}s_2q^{-i+1}, \alpha s_1^{-1}q^{-i+1}t^{-1}; q)_l}(qt^{-1})^{l},
\]

and the calculation given in the proof of Proposition 2.8, we have

\[
\text{LHS of Eq.(44)} = \sum_{k=j}^{i} \tilde{c}_{ik}\lambda_k(\alpha)d_{kj} = \sum_{l=0}^{i-j} \tilde{c}_{i,i-l}\lambda_{i-l}(\alpha)d_{i-l,j}
\]

\[
= \text{RHS of Eq.(44)}.
\]

Then, one can apply Watson's formula (Eq. (2.5.1) of GR [9]) to transform the above terminating $8W_7$ series into a terminating balanced $4\phi_3$ series.
Proposition 2.10 We have

\[
(C\Lambda(\alpha)C^{-1})_{ij} = \lambda_i(\alpha)d_{ij} \frac{(s^{-1}q^{-2i+1}, qt^{-1}; q)_{i-j}}{(\alpha^{-1}s_2q^{-i+1}, \alpha s_1^{-1}q^{-i+1}; q)_{i-j}} \times 4\phi_3 \left( q^{-i+j}, \alpha s_1^{-1}q^{-i}, \alpha^{-1}s_2q^{-it}, q^{-i+j}q^{-i+j+1t-1}, s^{-1}q^{-2it}; q, q \right),
\]

for \( i \geq j \).

Finally, by using Sears’ transformation (Eq. (2.10.4) of GR [9]), we see that the coefficients \((C\Lambda(\alpha)C^{-1})_{ij}\) and \(e_{ij}\) are the same.

Proposition 2.11 We have

\[
(C\Lambda(\alpha)C^{-1})_{ij} = e_{ij}
\]

for \( i \geq j \).

Proof. By using

\[
\frac{(t; q)_{i-j}}{(q^{-i+j+1t-1}; q)_{i-j}} = (-1)^{i-j}t^{i-j}q^{\frac{(i-j)(i-j-1)}{2}},
\]

\[
\frac{(s^{-1}q^{-2i+1}; q)_{i-j}}{(sq^{i+j}; q)_{i-j}} = (-1)^{i-j}s^{-i+j}q^{-i+j(i-j)}q^{\frac{(i-j)(i-j-1)}{2}},
\]

\[
\frac{(sq^{i+j+1t-1}; q)_{i-j}}{(s^{-1}q^{-2i}; q)_{i-j}} = (-1)^{i-j}s^{i-j}t^{-i+j}q^{-i+j+1(i-j)}q^{\frac{(i-j)(i-j-1)}{2}},
\]

\[
\frac{(\alpha^{-1}s_2q^{-it}; q)_{i-j}}{(\alpha^{-1}s_2q^{-i+1}; q)_{i-j}} = \frac{(\alpha s_2^{-1}qt^{-1}; q)_{i-j}}{(\alpha s_2^{-1}qt^{-1}; q)_{j}} \frac{(\alpha^{-1}s_2q^{-1}qt^{-1}; q)_{j}}{(\alpha^{-1}s_2q^{-1}qt^{-1}; q)_{j}} \frac{(\alpha^{-1}s_2qt^{-1}; q)_{j}}{(\alpha^{-1}s_2qt^{-1}; q)_{j}} \times (-1)^{-i+j}(\alpha^{-1}s_1q; q)_{i-j} \frac{(\alpha^{-1}s_1q; q)_{i-j}}{(\alpha^{-1}s_1q; q)_{i-j}},
\]

we have

\[
(C\Lambda(\alpha)C^{-1})_{ij} = \lambda_i(\alpha)d_{ij} \frac{(s^{-1}q^{-2i+1}, qt^{-1}; q)_{i-j}}{(\alpha^{-1}s_2q^{-i+1}, \alpha s_1^{-1}q^{-i+1}; q)_{i-j}}
\]
\[ \times (\alpha^{-1} s_2 q^{-i} t, \alpha^{-1} s_1 q^j t^{-1}; q)_{i-j} (\alpha s_1 q^{-i})^{i-j} \\
\times 4 \phi_3 \left( q^{-i+j}, \alpha s_1^{-1} q^{-i}, \alpha s_2^{-1} q^j, t \right) (q^{-i+j} t, \alpha s_1^{-1} q^{-i} t, \alpha s_2^{-1} q^j t^{-1}; q, q) \]
\[ = \frac{(q t^{-1}; q)_{i-j} (\alpha^{-1} s_1 q^j t^{-1}; q)_{i-j} (q^{-i+j} t, \alpha s_1^{-1} q^{-i} t, \alpha s_2^{-1} q^j t^{-1}; q, q)}{(q; q)_{i-j} (\alpha^{-1} s_1 q^j t^{-1}; q)_{i-j} (q^{-i+j} t, \alpha s_1^{-1} q^{-i} t, \alpha s_2^{-1} q^j t^{-1}; q, q)} \lambda_j(\alpha) \times 4 \phi_3 \left( q^{-i+j} t, \alpha s_1^{-1} q^{-i} t, \alpha s_2^{-1} q^j t^{-1}; q, q \right) = e_{ij}. \]

We obtain Eq. (38) from Proposition 2.6 and Proposition 2.11. Hence we have completed the proof of Theorem 2.1.

### 3 Properties of the Eigenfunctions

We examine some basic properties of the eigenfunctions of \( I(\alpha) \) for general \( n \).

An explicit formula for all the eigenfunctions of \( I(\alpha) \) for \( n = 3 \) is conjectured, and a partial result is given for the case of \( n = 4 \). In these explicit formulas, no dependence on the spectral parameter \( \alpha \) is observed. This supports our Conjecture 1.2.

#### 3.1 existence of the eigenfunctions

Let us study the existence of the eigenfunctions which form a basis of \( \mathcal{F}_n \).

**Proposition 3.1** Let the parameters \((s_1, s_2, \cdots, s_n), \alpha, q \) and \( t \) be generic. Let \( j_1, j_2, \cdots, j_{n-1} \) be nonnegative integers. In the space \( \mathcal{F}_n \), there exist a unique solution to the equation

\[ \begin{align*}
I(\alpha) f_{j_1, j_2, \cdots, j_{n-1}}(\zeta_1, \cdots, \zeta_n) &= \lambda_{j_1, j_2, \cdots, j_{n-1}}(\alpha) f_{j_1, j_2, \cdots, j_{n-1}}(\zeta_1, \cdots, \zeta_n), \\
\end{align*} \]

with the conditions

\[ f_{j_1, j_2, \cdots, j_{n-1}}(\zeta_1, \cdots, \zeta_n) = \sum_{i_1 \geq j_1, \cdots, i_{n-1} \geq j_{n-1}} c_{i_1, i_2, \cdots, i_{n-1}} \left( \frac{\zeta_2}{\zeta_1} \right)^{i_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{i_2} \cdots \left( \frac{\zeta_n}{\zeta_{n-1}} \right)^{i_{n-1}}, \]
and \( c_{j_1,j_2,\ldots,j_{n-1}} = 1 \), if and only if

\[
\lambda_{j_1,j_2,\ldots,j_{n-1}}(\alpha) = \prod_{i=1}^{n} \frac{(\alpha s_{i}^{-1}; q)_{j_{i-1} - j_{i}}}{(\alpha s_{i}^{-1} q t^{-1}; q)_{j_{i-1} - j_{i}}},
\]

(50)
is satisfied. Here \( j_0 = 0 \) and \( j_n = 0 \) are assumed.

**Proof.** Working with the dominance order, or the lexicographic order in the monomial basis

\[
1 < \frac{\zeta_2}{\zeta_1} < \left( \frac{\zeta_2}{\zeta_1} \right)^2 < \cdots < \frac{\zeta_3}{\zeta_2} < \frac{\zeta_2 \zeta_3}{\zeta_1 \zeta_2} < \left( \frac{\zeta_2}{\zeta_1} \right)^2 \frac{\zeta_3}{\zeta_2} < \cdots,
\]

we see the representation of the integral operation \( I(\alpha) \) becomes infinite lower triangular matrix. The explicit form of the action of \( I(\alpha) \) on a monomial is obtained by Lemma 2.3. It reads

\[
I(\alpha) \left( \frac{\zeta_2}{\zeta_1} \right)^{j_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{j_2} \cdots \left( \frac{\zeta_n}{\zeta_{n-1}} \right)^{j_{n-1}} = \left( \frac{\zeta_2}{\zeta_1} \right)^{j_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{j_2} \cdots \left( \frac{\zeta_n}{\zeta_{n-1}} \right)^{j_{n-1}} \prod_{i<j} h(\zeta_j/\zeta_i)
\]

\[
\times \sum_{k_{l,m} \geq 0} \prod_{r=1}^{n} \mu(\alpha s_{r}^{-1}; k_{1,r} + \cdots + k_{r-1,r} - k_{r+1,r} - \cdots - k_{n,r} + j_{r-1} - j_{r})
\]

\[
\times \prod_{l<m} g_{k_{l,m}} \left( \frac{\zeta_m}{\zeta_l} \right)^{k_{l,m}} \prod_{l>m} g_{k_{l,m}} \left( \frac{\zeta_l}{\zeta_m} \right)^{k_{l,m}},
\]

where we have used the notations Eq. (24) and

\[
\mu(\alpha; k) = \frac{(\alpha; q)_k}{(\alpha q t^{-1}; q)_k} (q^{\frac{1}{2}} t^{-\frac{1}{2}})^k.
\]

Thus the lower triangularity is explicitly seen. It is easy to see that the diagonal elements are given by Eq. (50), by setting \( k_{l,m} = 0 \) and forgetting the factor \( \prod_{l<j} h(\zeta_j/\zeta_i) \) in the above expression. Since all the diagonal entries are distinct if the parameters are generic, there is no obstruction for the construction of the eigenfunction.
By noting that

\[ \mu(\alpha; k + l) = \mu(\alpha; k)\mu(\alpha q^k, l), \]

it can be easily seen that all the eigenfunctions are related by shifting the parameters \( s_i \) suitably.

**Proposition 3.2** The eigenfunctions of \( I(\alpha) \) satisfy

\[ f_{j_1, j_2, \ldots, j_{n-1}}(\zeta_1, \ldots, \zeta_n) = \prod_{i=1}^{n} \zeta_i^{j_i - j_i - j_i} f_{0,0,\ldots,0}(\zeta_1, \ldots, \zeta_n). \]

Here, \( j_0 = 0, j_n = 0 \) are assumed, and the shift operator \( T_{q,s_i} \) is defined by

\[ T_{q,s_i} g(s_1, \ldots, s_n) = g(s_1, \ldots, qs_i, \ldots, s_n). \]

We have constructed the set of eigenfunctions \( \{f_{j_1, j_2, \ldots, j_{n-1}}(\zeta_1, \ldots, \zeta_n)\} \) which forms a basis of \( \mathcal{F}_n \). Note that, we have a good structure in the eigenfunctions associated to the \( A_{n-1} \) root system. Namely, the relation between the leading term and the shifts in the parameters \( s_i \) are nicely organized by the \( A_{n-1} \) roots \( \alpha_1 = (-1, 1, 0, 0, \ldots) \), \( \alpha_2 = (0, -1, 1, 0, \ldots) \) and so on.

**3.2 conjecture for \( n = 3 \)**

Let us examine the eigenfunctions for \( n = 3 \) and argue that our Conjecture 1.2 seems true for \( n = 3 \). We have not obtained a good enough understanding of \( I(\alpha) \) to be able to derive the eigenfunctions for \( n \geq 3 \) in a rigorous manner. Fortunately, however, a conjectural form of the eigenfunctions can be obtained for \( n = 3 \) by a brute force calculation up to certain degrees in \( \zeta \)-variables.

**Conjecture 3.3** The first eigenfunction of the integral transformation \( I(\alpha) \) for \( n = 3 \) is given by

\[
\begin{aligned}
&= f_{0,0}(\zeta_1, \zeta_2, \zeta_3)
&= \sum_{k=0}^{\infty} \frac{(qt^{-1}, qt^{-1}, t, t; q)_k (qs_1/s_3)^k (\zeta_3/\zeta_1)^k (1 - \zeta_j/\zeta_i) \cdot 2\phi_1 \left( q^{k+1}t^{-1}, qt^{-1}s_i/s_j; q, t\zeta_j/\zeta_i \right)}{(q, qs_1/s_2, qs_2/s_3, qs_1/s_3; q)_k}.
\end{aligned}
\]
Proposition 3.4

The first eigenfunction \( 1.2 \) seems true for \( n \) several terms of the eigenfunctions to state our observation that Conjecture \( \ref{conjecture1.2} \) we have not completely understood the structure of them yet. So we give first Note that all the other eigenfunctions can be obtained by Proposition 3.2.

3.3 partial result for \( n \) parameter \( \alpha \n \)

For \( n = 4 \), the study of the eigenfunctions becomes much more difficult, and we have not completely understood the structure of them yet. So we give first several terms of the eigenfunctions to state our observation that Conjecture \( \ref{conjecture1.2} \) seems true for \( n = 4 \).

**Proposition 3.4** The first eigenfunction \( f_{0,0,0}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \) up to the powers \[
\left( \frac{\zeta_2}{\zeta_1} \right)^{i_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{i_2} \left( \frac{\zeta_4}{\zeta_3} \right)^{i_3} \quad (0 \leq i_1 \leq 2, 0 \leq i_2 \leq 2, 0 \leq i_3 \leq 2),
\] (54) is given by the following expression.

\[
f_{0,0,0}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = y_{0,0,0} + y_{1,1,0} + y_{0,1,1} + y_{1,1,1} + y_{2,2,0} + y_{1,2,1} + y_{0,2,2} + y_{2,2,1} + y_{1,2,2} + y_{2,2,2} + \cdots,
\] (55)

where

\[
y_{0,0,0} = \varphi(0, 0, 0, 0, 0), \quad \tag{56}
\]

\[
y_{1,1,0} = \frac{\zeta_4}{\zeta_1} \left( q \frac{s_1}{s_3} \right) \frac{(qt^{-1})_1(qt^{-1})_1(t)_1(t)_1}{(q)_1(qs_{12})_1(qs_{23})_1(qs_{34})_1(qs_{14})_1} \varphi(1, 1, 0, 1, 0, 0), \quad \tag{57}
\]

\[
y_{0,1,1} = \frac{\zeta_4}{\zeta_2} \left( q \frac{s_2}{s_4} \right) \frac{(qt^{-1})_1(qt^{-1})_1(t)_1(t)_1}{(q)_1(qs_{23})_1(qs_{34})_1(qs_{24})_1} \varphi(0, 1, 1, 0, 1, 0), \quad \tag{58}
\]

\[
y_{1,1,1} = \frac{\zeta_4}{\zeta_1} \left( q \frac{s_1}{s_4} \right) \frac{(qt^{-1})_1(qt^{-1})_1(t)_1(t)_1}{(q)_1(qs_{13})_1(qs_{34})_1(qs_{14})_1} \varphi(1, 0, 0, 1, 0, 1) + \frac{\zeta_4}{\zeta_1} \left( q \frac{s_1}{s_4} \right) \frac{(qt^{-1})_1(qt^{-1})_1(t)_1(t)_1}{(q)_1(qs_{13})_1(qs_{34})_1(qs_{14})_1} \varphi(0, 0, 1, 1, 0, 1) + \frac{\zeta_4}{\zeta_1} \left( q \frac{s_1}{s_4} \right) \frac{(qt^{-1})_1(qt^{-1})_1(t)_1(t)_1}{(q)_1(qs_{13})_1(qs_{34})_1(qs_{14})_1} \varphi(1, 0, 0, 1, 0, 1) \times \varphi(1, 1, 1, 1, 1, 1), \quad \tag{59}
\]
\begin{align}
y_{2,0} &= \frac{\zeta_3 \zeta_3}{\zeta_1 \zeta_1} \left( q^2 s_1 s_1 \right) \left( q \right) (q - 1) \left( q - 1 \right)^2 \left( 2 t_1 \right) \left( 2 t_2 \right) \varphi(2, 2, 0, 0, 0), \quad (60)
y_{0,2} &= \frac{\zeta_4 \zeta_4}{\zeta_2 \zeta_2} \left( q^2 s_2 s_2 \right) \left( q \right) (q - 1) \left( q - 1 \right)^2 \left( 2 t_1 \right) \left( 2 t_2 \right) \varphi(0, 2, 0, 0, 0), \quad (61)
y_{1,1} &= \frac{\zeta_3 \zeta_4}{\zeta_1 \zeta_2} \left( q^2 s_1 s_2 \right) \left( q \right) (q - 1) \left( q - 1 \right)^2 \left( 1 \right) \left( 1 \right) \left( 2 \right) \left( 3 \right) \varphi(1, 1, 1, 1, 1) \times \varphi(1, 1, 1, 1, 1) \times \varphi(1, 1, 1, 1, 1), \quad (62)
y_{2,1} &= \frac{\zeta_3 \zeta_4}{\zeta_1 \zeta_1} \left( q^2 s_1 s_1 \right) \left( q \right) (q - 1) \left( q - 1 \right)^2 \left( 1 \right) \left( 1 \right) \left( 2 \right) \left( 3 \right) \varphi(1, 1, 1, 1, 1) \times \varphi(1, 1, 1, 1, 1) \times \varphi(1, 1, 1, 1, 1), \quad (63)
y_{1,2} &= \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_2} \left( q^2 s_1 s_2 \right) \left( q \right) (q - 1) \left( q - 1 \right)^2 \left( 1 \right) \left( 2 \right) \left( 3 \right) \varphi(1, 1, 1, 1, 1) \times \varphi(1, 1, 1, 1, 1) \times \varphi(1, 1, 1, 1, 1), \quad (64)
\end{align}
\[ \times \varphi(1,1,1,1,2,1) \]
\[ + \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( -q \frac{s_1 s_2}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_2(q^{-1})_1(t)_2(t)_2(t)_1(q^3 s_1 s_2 / s_3 s_4)_1}{(q)_1(q)_1(q s_1)_2(q s_3)_2(q s_{13})_1(q s_{24})_2(q s_{14})_1} \times \varphi(1,2,2,1,1,2) \]

\[ \text{and} \]

\[ y_{2,2,2} = \]
\[ - \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( q^2 \frac{s_1 s_1}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_2(q^{-1})_1(t)_2(t)_2(t)_1(q^2 s_{24} / s_1 s_3)_1}{(q)_1(q s_1)_2(q s_3)_2(q s_{13})_1(q s_{24})_1(q s_{14})_1} \times \varphi(1,1,1,1,1,1) \]
\[ + \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( q \frac{s_1 s_1}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_1(q^{-1})_1(t)_1(t)_1(q^2 s_{23})_1}{(q)_1(q s_1)_1(q s_{23})_1(q s_{34})_1(q s_{13})_1(q s_{24})_1(q s_{14})_1} \times \varphi(1,1,1,1,1,2) \]
\[ + \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( q \frac{s_1 s_1}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_1(q^{-1})_1(t)_1(t)_1(q^2 s_{21})_1}{(q)_1(q s_1)_1(q s_{23})_1(q s_{34})_2(q s_{13})_1(q s_{24})_1(q s_{14})_1} \times \varphi(1,1,2,1,1,1) \]
\[ + \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( q \frac{s_1 s_1}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_1(q^{-1})_1(t)_2(t)_1(t)_1(q^2 s_{43})_1}{(q)_1(q s_1)_1(q s_{12})_1(q s_{23})_1(q s_{34})_1(q s_{13})_1(q s_{24})_1(q s_{14})_1} \times \varphi(2,1,1,1,1,1) \]
\[ + \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( q \frac{s_1 s_1}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_2(q^{-1})_1(t)_2(t)_1(t)_1(q^3 s_1 s_2 / s_3 s_4)_1}{(q)_1(q s_1)_1(q s_{12})_1(q s_{23})_1(q s_{34})_2(q s_{13})_2(q s_{24})_1(q s_{14})_1} \times \varphi(1,1,2,2,1,2) \]
\[ + \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( q \frac{s_1 s_1}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_2(q^{-1})_1(t)_2(t)_1(t)_1(q^3 s_1 s_2 / s_3 s_4)_1}{(q)_1(q s_1)_1(q s_{12})_1(q s_{23})_1(q s_{34})_1(q s_{13})_1(q s_{24})_2(q s_{14})_1} \times \varphi(2,1,1,1,2,2) \]
\[ + \frac{\zeta_4 \zeta_4}{\zeta_1 \zeta_1} \left( q \frac{s_1 s_1}{s_4 s_4} \right) \frac{(q^{-1})_2(q^{-1})_2(q^{-1})_1(t)_2(t)_1(t)_1(q^3 s_1 s_2 / s_3 s_4)_1}{(q)_1(q s_1)_1(q s_{12})_1(q s_{23})_1(q s_{34})_1(q s_{13})_1(q s_{24})_2(q s_{14})_1} \times \varphi(2,1,1,1,2,2) \]
Here, we have used the notations $s_{ij} = s_i/s_j$, $(a)_k = (a;q)_k$ and

\[
\varphi(k_{12}, k_{23}, k_{34}, k_{13}, k_{24}, k_{14}) = \prod_{1 \leq i < j \leq 4} (1 - \zeta_j/\zeta_i) \cdot {}_2\phi_1 \left( {q^{k_{ij}+1} t^{-1}, q^{k_{ij}+1} s_i/s_j; q, t \zeta_j/\zeta_i} \right).
\]

We may convince ourselves that the above expression gives us an efficient way to produce correct coefficients for the eigenfunction. Furthermore, even the following can be observed.

**Conjecture 3.5** The expression for the first eigenfunction given in Proposition 3.4 gives correct coefficients for the powers $(\zeta_2/\zeta_1)^{i_1} (\zeta_3/\zeta_2)^{i_2} (\zeta_4/\zeta_3)^{i_3}$ satisfying the conditions:

\[
0 \leq i_1 < \infty, \quad 0 \leq i_2 \leq 2, \quad 0 \leq i_3 < \infty, \quad (67)
\]

or

\[
0 \leq i_1 \leq 2, \quad 0 \leq i_2 < \infty, \quad 0 \leq i_3 \leq 2. \quad (68)
\]

The author has extended the formula given in Proposition 3.4 for a little higher powers in $\zeta_i$'s, and observed that there still exists a nice hypergeometric-like structure. At this moment, however, it is not clear how to organize the general terms, and we omit writing.

In the first several terms of the eigenfunctions given in Proposition 3.4, we do not see any dependence on the parameter $\alpha$. Thus, Conjecture 1.2 is very likely true for $n = 4$.

## 4 Concluding Remarks

Let us summarize the results obtained in this paper. In Section 1, we have introduced the integral transformation $I(\alpha)$, and it was conjectured that $I(\alpha)$ generates a commutative family of operators acting on the space of formal series $\mathcal{F}_n$ (Conjecture 1.2). In Section 2, a proof of the commutativity for the case $n = 2$ was given (Theorem 2.1). In Section 3, the existence of the eigenfunctions for general $n$ was studied, when all the parameters are generic.
Proposition 3.1. Explicit formulas for the eigenfunctions were examined for $n = 3, 4$ (Conjecture 3.3 and Proposition 3.4). Since these eigenfunctions do not depend on the spectral parameter $\alpha$ of the integral transformation $I(\alpha)$, it is expected that Conjecture 1.2 is true for $n = 3, 4$.

Finally, let us make some comments to show the directions toward the next papers [7, 8]. When we specialize the parameters as $s_1 = s_2 = \cdots = s_n$, there exist several phenomena which are interesting both from the hypergeometric series and the lattice model viewpoints. In [7], a class of hypergeometric-type functions will be introduced, which is characterized by a certain covariant transformation property with respect to the action of $I(\alpha)$ together with an initial condition given by an infinite product expression. To be more precise, by setting $s_1 = s_2 = \cdots = s_n = 1$, we will introduce a series $F(\alpha) = F(\zeta_1, \zeta_2, \zeta_3, \alpha, q, t) \in F_n$ characterized by the conditions

(I) $I(\alpha^{-1}t) \cdot F(\alpha) = F(\alpha^{-1}t)$,  
(II) $F(t^{1/2}) = \prod_{1 \leq i < j \leq n} (1 - \zeta_j/\zeta_i) (qt^{-1/2}\zeta_j/\zeta_i; q)_\infty \frac{(qt^{-1/2}\zeta_j/\zeta_i; q)_\infty}{t^{1/2}\zeta_j/\zeta_i}$.  

This function $F(\alpha)$ will be called ‘quasi-eigenfunction’ for short. We are interested in the function $F(\alpha)$ because of the following observations:

1. The explicit formula of $F(\alpha)$ for $n = 2$ can be easily derived [7]. Further, we are able to have a conjectural expression of $F(\alpha)$ for $n = 3$:

$$F(\zeta_1, \zeta_2, \zeta_3, \alpha, q, t) = \sum_{k=0}^{\infty} \frac{(\alpha^{-2}t, qt^{-1}, t^{-1}; q)_k}{(q, \alpha^{-1}q, \alpha^{-1}q; q)_k} (q\zeta_3/\zeta_1)^k \frac{(\alpha^{-1}, q^{-k}}{\alpha q^{-k+1}; q, \alpha t}) \times \prod_{1 \leq i < j \leq 3} (1 - \zeta_j/\zeta_i) \frac{\phi_1(q^{k+1}t^{-1}, \alpha qt^{-1}, \alpha^{-1}q^{k+1}; q, \alpha^{-1}t\zeta_j/\zeta_i)}{\phi_1(q^{k+1}t^{-1}, \alpha qt^{-1}, \alpha^{-1}q^{k+1}; q, \alpha^{-1}t\zeta_j/\zeta_i)}.$$  

2. We may find a variety of infinite product expressions for $F(\alpha)$ when we suitably specialize the parameter $\alpha$. Simplest examples among these are:

$$F(-t^{1/2}) = \prod_{1 \leq i < j \leq n} (1 - \zeta_j/\zeta_i) \frac{(-qt^{-1/2}\zeta_j/\zeta_i; q)_\infty}{(-t^{1/2}\zeta_j/\zeta_i; q)_\infty},$$  
$$F(t) = \prod_{1 \leq i < j \leq n} (1 - \zeta_j/\zeta_i) \frac{gt^{-1}\zeta_j/\zeta_i; q)_\infty}{(t\zeta_j/\zeta_i; q)_\infty}. $$
Here, we have used the notation $\prod_{1 \leq i < j \leq n} f_{ij} = f_{13} f_{15} \cdots f_{24} f_{26} \cdots$.

3. The highest-to-highest matrix elements of the eight-vertex vertex operators can be realized as

\begin{equation}
\langle \Phi(\zeta_1) \Phi(\zeta_2) \cdots \Phi(\zeta_n) \rangle = \prod_{1 \leq i < j \leq n} \frac{\xi(\zeta_j^2/\zeta_i^2; p, q)}{1 - \zeta_j/\zeta_i} \cdot F(\zeta_1, \cdots, \zeta_n; -1, p^{1/2}, q),
\end{equation}

where

\begin{equation}
\xi(z; p, q) = \frac{(q^2 z; p, q^4)_{\infty} (pq^2 z; p, q^4)_{\infty}}{(q^4 z; p, q^4)_{\infty} (p z; p, q^4)_{\infty}}.
\end{equation}

One finds that the integral formulas of the matrix elements of the vertex operators for the cases $p^{1/2} = q^{3/2}$ and $p^{1/2} = -q^2$ given in \[\text{[1]}\] can be easily recovered from the above properties of $F(\alpha)$. Note, however, the case $p^{1/2} = q^3$ corresponds to another type of product formula for $F(\alpha)$ which is not given here.

In the continuation of the present work \[\text{[7, 8]}\], basic properties of the quasi-eigenfunction $F(\alpha)$ will be discussed.

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