ABSOLUTE CONNECTEDNESS AND CLASSICAL GROUPS

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ABSTRACT. We prove for groups of rational points of split semisimple linear groups (Chevalley groups) over arbitrary infinite fields, connected perfect linear algebraic groups, some infinite permutation groups and infinite dimensional general linear groups, a model theoretical property called absolute connectedness, which is related to diameters of the Lascar strong types.

INTRODUCTION

In this article we examine the notion of an absolutely connected group. By that one means the following. Suppose $G$ is an arbitrary infinite group. Consider an expansion of $G$ by some first order structure $\mathcal{G} = (G, \cdot, \ldots)$, where $\cdot$ is usual group multiplication and $\ldots$ represents some additional structure i.e. relations, functions and constants. Let $\mathcal{G}^* = (G^*, \cdot, \ldots)$ be a sufficiently saturated elementary extension of $\mathcal{G}$. In model theory we consider the following notions of invariant normal subgroups of bounded index, $G^*_\emptyset \subseteq G^*_{00} \subseteq G^*_0 \subseteq G^*$, called model theoretic connected components (see [18, 12, 22]). We mainly concentrate on the $\infty$-component $G^*_\emptyset$, that is the smallest subgroup of $G$ of bounded index which is invariant under $\text{Aut}(\mathcal{G})$ (see Section 1.4 below). One says that a group $G$ (without any logic structure) is absolutely connected if for all possible first order expansions $\mathcal{G}$ of $G$, and elementary extensions $\mathcal{G}^*$ of $\mathcal{G}$, the $\infty$-component coincide with the whole group:

$$G^*_\infty = G^*.$$

In general, the quotient $G^*/G^*_\emptyset$ with “the logic topology” is a “quasi-compact” topological group (see for example [21 Proposition 3.5(1)] where we used the notation $G^*_L$ for $G^*_\emptyset$), which is an invariant of the structure $\mathcal{G}$. That is, does not depend on the choice of saturated extension $\mathcal{G}^*$ (see [21 Proposition 3.3(3)]). Hence, for an absolutely connected group this invariant is always trivial, no matter what the ambient logic structure is.

In our previous work [21 Section 3] we make a link between model theoretic connected components of $\mathcal{G}$ and strong types of the structure $(\mathcal{G}, \circ, X)$, where $\circ$ is a regular action of $G$ on $X$ (see Section 4). Strong types are fundamental objects in model theory. They
correspond to orbits on $G^*$ of some canonical subgroups of Aut($G^*$). In [29] Newelski considers the diameters of Lascar strong types. Motivating by his idea we introduce the notion of $N$-(definable) absolutely connected group (Definition 2.6), for a natural number $N$. Then $G$ is $N$-definable absolute connected if and only if the corresponding Lascar strong type has diameter $N$ (Remark 4.2).

One of the main aim of the paper is the following structural result about the group of rational points of split semisimple linear groups, in the context of absolute connectedness. We recall that a linear group defined over field $k$ is called split over $k$ or $k$-split if some maximal torus $T$ in $G$ is split over $k$, that is $T$ is isomorphic over $k$ to a direct product of copies of the multiplicative groups $\mathbb{G}_m$ [4, 18.6] (over an algebraically closed field, every semisimple group is split).

**Theorem.** 3.14 Let $k$ be an arbitrary infinite field and $G$ be a $k$-split, semisimple linear algebraic group $G$ defined over $k$. The derived subgroup $[G(k), G(k)]$ of $G(k)$ is 12-absolutely connected. Moreover $[G(k), G(k)]$ is a definable over $\emptyset$ subgroup of $G(k)$ in the pure group language.

We first prove absolute connectedness of simply connected semisimple split groups, where $G(k) = [G(k), G(k)]$ (Theorem 3.11) and then, using universal $k$-covering, establish the general case. If $G$ is not simply connected, then the group $[G(k), G(k)]$ might be a proper Zariski dense subgroup of $G(k)$. For example if $G = \text{PGL}_n$, then $[G(k), G(k)] = \text{PSL}_n(k)$ is a proper subgroup for some fields. Likewise the result of Platonov [32, Main Theorem] and Lemma 3.13 imply, that if $G$ is a non simply connected semisimple $k$-split group and $k$ is a finitely generated field, then $[G(k), G(k)]$ is a proper subgroup of $G(k)$.

The proof of the above theorems goes through the theory of Borel-Tits about rationality issues of reductive groups [6, 7] and the Gauss decomposition with prescribed semisimple part from [11].

In an early version of the present paper we stated Theorem 3.14 for the class of Chevalley groups, where by a Chevalley group we mean a group generated by the root subgroups [35]. By [34, 16.3.2] and the construction of Chevalley groups, if $G$ is connected, split over $k$, simply connected and absolutely almost simple $k$-group, then $G(k)$ is a Chevalley group. In general (that is when $G$ is non necessary simply connected) it can be shown that Chevalley groups are derived subgroups $[G(k), G(k)]$, where $G$ is semisimple and split over prime subfield of $k$ [5, Section 3.3].

We establish in Section 3 absolute connectedness of some infinite permutation groups, infinite-dimensional general linear groups and non-reductive perfect algebraic groups.

- For an infinite set $\Omega$, the group
  $$\text{Alt}(\Omega) = \{ \sigma \in \text{Sym}(\Omega) : \text{supp}(\sigma) \text{ is finite and } \sigma \text{ is even} \}$$
  is 8-absolutely connected (Proposition 3.1).
- For an uncountable cardinal $\kappa$, the group $\text{Sym}^\kappa(\Omega) = \{ \sigma \in \text{Sym}(\Omega) : |\text{supp}(\sigma)| < \kappa \}$ is 16-absolutely connected (Proposition 3.2).
- Suppose $V$ is an infinite-dimensional vector space over a division ring $D$. The group $\text{GL}(V)$ of all linear automorphisms of $V$ is 128-weakly simple (Proposition 3.3).
- Let $G$ be a connected linear algebraic group defined over an algebraically closed field $K$. If $G(K)$ is prefect of the commutator width $R$ and the solvable radical
\( R(G(K)) \) is of derived length \( M \), then \( G(K) \) is \( 12(4R) \) derived length \( M \)-absolutely connected (Proposition 3.18). In particular the group \( K^n \rtimes \text{SL}_n(K) \) is \( 48 \)-absolutely connected (Example 3.19).

In Section 2 we examine several quantitative measures of simplicity of groups in the context of absolute connectedness. Namely, \( N \)-bounded simplicity (Definition 2.1) and a more general notion of \( N \)-weak simplicity (Definition 2.11). Weak simplicity implies absolute connectedness (Proposition 2.2, Theorem 2.15). We prove that classes of absolute connected and weakly simple groups consist of perfect groups (Theorem 2.16) and are closed under several standard group theoretic operation (extensions, products, homomorphic images, Proposition 2.10, Lemma 2.12). We highlight the following proposition.

**Proposition. 2.13** Suppose \( H \) is \( N \)-absolutely connected \( [N \cdot \text{weakly simple}] \), and for some prime \( p \)

\[
1 \to \mathbb{Z}/p\mathbb{Z} \to G \xrightarrow{f} H \to 1
\]

is an exact sequence, which does not split (that is \( G \) is not a product \( \ker(f) \times H \)), where \( \mathbb{Z}/p\mathbb{Z} \cong \ker(f) < Z(G) \). Let \( h \) be the corresponding 2-cocycle. Then

1. \( G \) is \( N \cdot (3p - 2) \)-absolutely connected \( [N \cdot (3p - 2)-\text{ws respectively}] \),
2. if the image \( \text{Im}(h) \subset \mathbb{Z}/p\mathbb{Z} \) of \( h \) does not generate \( \mathbb{Z}/p\mathbb{Z} \) in \( M \) steps, then \( G \) is not \( \frac{1}{2}M \)-absolutely connected.

As an application of this result, we find absolutely connected groups with strictly growing coefficient of absolute connectedness (see the end of Section 4). Using this we construct (Proposition 4.4) a structure \( G \) of the form \( (F^\omega, \cdot, \mathcal{P}_N) \), where \( F^\omega \) is a free group on countably many generators and each \( \mathcal{P}_N \) is a predicate, with \( G^{00} \neq G^\infty \). The first example of a group with \( G^{00} \neq G^\infty \) has been found recently by Conversano and Pillay in [12].

In Section 3.3 we deal with linear groups over an arbitrary infinite field \( S \). In Proposition 3.6 we prove, using a standard compactness argument, bounded simplicity of the class of \( W \)-trivial groups. The rest this section is focused mainly on the split semisimple groups over arbitrary infinite fields. A more general case of isotropic groups, within this context, will be considered in [19].

Some results of Sections 2, 3, 3.1, 3.2 and 4 of this paper appeared in author’s Ph.D. thesis.

### 1. Basic notation and prerequisites

In this section we repeat the notation and some basic facts, mainly from [18].

#### 1.1. Group theory

For a group \( G \), elements \( a, b \in G \) and \( A, B \subseteq G \) we use the following notation: \( A \cdot B = \{ab : a \in A, b \in B \} \), \( a^b = b^{-1}ab \), \( A^B = \bigcup_{a \in A, b \in B} a^b \), \( A^n = A \cdot \ldots \cdot A \), \( A^{\leq n} = \bigcup_{i \leq n} A^i \) and \( [a, b] = a^{-1}b^{-1}ab \). The subset \( X \subseteq G \) is called normal if \( X^g = X \) for every \( g \in G \). A group \( G \) is called perfect if \( G \) equals its derived subgroup \( [G, G] \). The commutator length of an element \( g \in [G, G] \) is the minimal number of commutators sufficient to express \( g \) as their product. The commutator width \( \text{cw}(G) \) of a perfect group \( G \) is the maximum of the commutator lengths of elements from derived subgroup or the sign \( \infty \) if the maximum does not exists. By \( Z(G) \) we denote the center of the group \( G \).
1.2. Model theory. We assume that the reader is familiar with basic notions of model theory. The model-theoretic background can be found in [27]. By abuse of notation we write \( \bar{b} \in X \) to express the fact that each component of the tuple \( \bar{b} \) belongs to \( X \).

For every cardinal \( \kappa \), every infinite model can be elementarily extended to a \( \kappa \)-saturated and \( \kappa \)-strongly homogeneous model \( M \). We shall work in a \( \kappa \)-saturated and \( \kappa \)-strongly homogeneous model \( M \). Such models are called monster models or sufficiently saturated models. By \( A \) we always denote some small set of parameters from \( M \), that is \( |A| < \kappa \). Usually we will deal with groups with some first order structure, but sometimes we consider group only in a group language without any extra structure. In the latter case we say that the group is pure.

1.3. Boundedness and thickness. A formula \( \varphi(\bar{x}, \bar{y}) \), with free variables \( \bar{x}, \bar{y} \) and over parameters, is called thick if \( \varphi \) is symmetric and for some natural \( n \), for every \( n \)-sequence \( (\bar{a}_i)_{i<n} \) from \( M \) (we do not require \( \bar{a}_0, \ldots, \bar{a}_{n-1} \) to be pairwise distinct), there exist \( i < j < n \) such that \( \varphi(\bar{a}_i, \bar{a}_j) \) holds in \( M \) ([10] Section 3, [10] Definition 1.10).

By \( \Theta_A(\bar{x}, \bar{y}) \) we denote the conjunction of all thick formulas over \( A \). Then \( \Theta_A(\bar{x}, \bar{y}) \) gives a relation on \( M \) which is \( \bigwedge \)-definable over \( A \), but is not necessarily transitive. By \( \Theta_A^2 \) we mean the following relation: \( \Theta_A^2(\bar{a}, \bar{b}) \) holds if and only if there is \( \bar{a} \) such that \( \Theta_A(\bar{a}, \bar{a}) \wedge \Theta_A(\bar{a}, \bar{b}) \) holds. Also, \( \Theta_A \) has the following properties ([10] Lemmas 6.7, [10] Facts 1.11, 1.12): for \( \bar{a}, \bar{b} \in M \)

- if \( \Theta_A(\bar{a}, \bar{b}) \), then there is a small model \( M' \prec M \) containing \( A \), such that \( \bar{a} \equiv_{M'} \bar{b} \),
- if for some small model \( M' \prec M \) containing \( A \), \( \bar{a} \equiv_{M'} \bar{b} \) holds, then \( \Theta_A^2(\bar{a}, \bar{b}) \),
- \( \Theta_A(\bar{a}_0, \bar{a}_1) \) holds if and only if \( (\bar{a}_0, \bar{a}_1) \) can be extended to an order \( A \)-indiscernible sequence \( (\bar{a}_i)_{i<\kappa} \), that is for an arbitrary finite sequences \( i_1 < \ldots < i_m \) and \( j_1 < \ldots < j_m \) from \( \kappa \), \( \text{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_k}/A) = \text{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_k}/A) \).

We provide here, for the completeness of the exposition, the proof of the following well known property of sufficiently saturated models.

**Lemma 1.1** (Cardinality gap property). Suppose \( E \) is an equivalence relation on \( M^n \).

1. If \( E \) is definable over \( A \), then either \( E \) has finitely many equivalence classes or at least \( \kappa \) many classes.
2. If \( E \) is \( \bigwedge \)-definable over \( A \) or \( A \)-invariant, then either the number of classes of \( E \) is at most \( 2^{|L(A)|} \) or at least \( \kappa \).

**Proof.** (1) follows from compactness.

(2) We may assume that \( E \) is \( A \)-invariant, because every \( \bigwedge \)-definable over \( A \) relation is \( A \)-invariant. By downward Löwenheim-Skolem theorem, there is an elementary submodel \( M' \prec M \) containing \( A \), and of cardinality at most \( |A| + \aleph_0 \). Assume \( E \) has less than \( \kappa \) classes. It is enough to show that if \( \bar{x} \) and \( \bar{b} \) have the same type over \( M' \), then \( E(\bar{x}, \bar{b}) \) (because there are at most \( 2^{|L(A)|} \) many complete types over \( M' \)). If \( \text{tp}(\bar{x}/M') = \text{tp}(\bar{b}/M') \), then \( \Theta_A^2(\bar{x}, \bar{b}) \) holds, and by homogeneity and \( A \)-invariance of \( E \), \( E(\bar{x}, \bar{b}) \) is true. \(\square\)

An equivalence relation \( E \) on a \( \kappa \)-saturated and \( \kappa \)-strongly homogeneous model is called bounded if \( E \) has less than \( \kappa \) many equivalence classes.
1.4. Model theoretical connected components. Unless otherwise stated, we assume in this subsection that $G$ is a group equipped with extra first order structure which is moreover a sufficiently saturated model. Every $A$-invariant subgroup $H$ of $G$ induces a natural equivalence relation (equality of cosets). By the cardinality gap property, then the index $[G : H]$ is either bounded (at most $2^{|L(A)|}$) or unbounded (at least $\geq \kappa$). The following subgroups are called the model theoretic connected components of $G$.

- $G^0 = \bigcap \{H < G : H$ is $A$-definable and $[G : H]$ is finite $\}$
- $G^0_A = \bigcap \{H < G : H$ is $A$-definable over $A$ and $[G : H]$ is bounded $\}$
- $G^\infty = \bigcap \{H < G : H$ is $A$-invariant and $[G : H]$ is bounded $\}$

We say that $G^\infty$ exists and define it as $G^\infty_0$, if for every small set of parameters $A \subseteq G$, $G^\infty_A = G^\infty_0$. Similarly we define existence of $G^0_0$ and $G^0_A$. By [18], the groups $G^\infty_A \subseteq G^0_0 \subseteq G^0_A$ are normal subgroups of $G$ of bounded index. We will use another description of $G^\infty_A$.

Following the notation from [18] Section 1, define $X_{\Theta_A} = \{a^{-1}b : a, b \in G, \Theta_0(a, b)\}$.

Lemma 1.2 ([18] Lemma 2.2(2)). The group $G^\infty_A$ is generated by $X_{\Theta_A}$.

In [18] Lemma 3.3 we gave another description of $X_{\Theta_A}$. The key idea is the notion of a thick subset of a group (based on the definition of thick formula): a subset $P$ of an arbitrary group $G$ is $N$-thick if it is symmetric (that is $P = P^{-1}$) and for every $N$-sequence $g_1, \ldots, g_N$ from $G$, there are $1 \leq i < j \leq N$ such that

$$g_i^{-1}g_j \in P.$$ 

We say $P$ is thick if it is $N$-thick for some $N \in \mathbb{N}$. An obvious example of thick subset of the group is a subgroup of finite index.

For the properties of thick sets see [18] Section 3.

We denote by $R(n, m)$ the corresponding Ramsey number, that is, the least $N$ such that for any complete graph on $N$ vertices, whose edges are coloured black or white, there exists either a complete subgraph on $n$ vertices which is entirely white, or a complete subgraph on $m$ vertices which is entirely red.

Suppose $H_1$ and $H_2$ are groups (not necessarily sufficiently saturated), $f : H_1 \rightarrow H_2$ is an epimorphism, $X, Y \subseteq H_1$, $Z \subseteq H_2$ and $n \in \mathbb{N}$.

Lemma 1.3. (1) If $X$ is $n$-thick and $Y$ is $m$-thick, then $X \cap Y$ is $R(n, m)$-thick.

(2) The set $f^{-1}[X]$ is $n$-thick if and only if $X$ $n$-thick.

(3) If $Z$ is $n$-thick, then $f[Z]$ is $n$-thick.

Proof. (1) Assume that $X \cap Y$ is not $R(n, m)$-thick and take the sequence $(a_i)_{i < R(n, m)}$ witnessing this. Consider the complete graph on $R(n, m)$ vertices with the following coloring: $\{i, j\}$ is black if and only if $a_i^{-1}a_j \notin X$, otherwise $\{i, j\}$ is white. By the Ramsey theorem either there is a clique of size $n$ with all edges being black, then $X$ is not $n$-thick, or a clique of size $m$ with white edges, then $Y$ is not $m$-thick. (2) and (3) are immediate.

Lemma 1.4. The group $G^\infty_A$ is generated, by the intersection of all $A$-definable thick subsets of $G$. More general, suppose $n \in \mathbb{N}$, then

(1) $X^A_n = \bigcap \{P^n : P \subseteq G$ is $A$-definable and thick $\}$,

(2) $X^A_G = \bigcap \{P^G : P \subseteq G$ is $A$-definable and thick $\}$.
Recall that for $A, B \subseteq G$ and natural $n$, $A^B = \bigcup_{a \in A, b \in B} b^{-1}ab$ and $A^n = A \cdot \ldots \cdot A$.

**Proof.** (1) is just [KS Lemma 3.3]. (2) follows from compactness. Note that if $P$ is $A$-definable, then $P^G$ is also $A$-definable. $\subseteq$ is clear. For $\supseteq$, take $a$ in the intersection. By (1) for $n = 1$, it is enough to observe that the type $p(x) = \{a^x \in P : P \subseteq G$ is $A$-definable and thick} is finitely satisfiable (thick sets are closed under intersection, Lemma [13](1)).

2. Absolutely connected and weakly simple groups

The aim of present section is to introduce the notion of (definable) absolute connectedness (Definition 2.6), and give the basic results on it. In short, $G$ is **absolutely connected** if $G^\infty_A = G$, for every small $A$ (working in a sufficiently saturated expansion of an arbitrary expansion of $G$). We relate also this notion to the known concepts of bounded simplicity and quasi-simplicity (see the definition below). We introduce also an auxiliary class of weakly simple groups (Definition 2.11).

The symbol $G$ denotes a group possibly equipped with extra first order structure, but we no longer assume $G$ is sufficiently saturated. We denote by $G^*$ a sufficiently saturated elementary extension of $G$.

**Definition 2.1.** (1) We say that a group $G$ is $N$-boundedly simple if

$$\left(g^G \cup g^{-1}G\right)^{\leq N} = G,$$

for every $g \in G \setminus Z(G)$. A group is **boundedly simple** if it is $N$-boundedly simple for some natural number $N$. Note that a boundedly simple group need not be simple as an abstract group.

(2) The group $G$ is called **quasi-simple** if it is perfect and $G/Z(G)$ is simple as an abstract group.

**Proposition 2.2.** If $G^*$ is quasi-simple, then $G^{*\infty}$ exists and is equal to $G^*$.

**Proof.** For every small set $A$ of parameters, the component $G^{*\infty}_A$ is a normal subgroup of $G^*$. [KS Lemma 2.2(3)]. Note that in any uncountable quasi-simple group $G$, the center $Z(G)$ has infinite index. Otherwise, the set of all commutators $\{[x, y] : x, y \in G\}$ would be finite, so $G = [G, G]$ would be countable. Therefore $[G^* : Z(G^*)]$ is infinite, so by Lemma [11](1), index $Z(G^*)$ is unbounded. Thus $G^* = G^{*\infty}_A \cdot Z(G^*)$ (because $G^*/Z(G^*)$ is simple), but $G^*$ is perfect, so $G^* = G^{*\infty}_A$.

The following question arises: When is a sufficiently saturated group $G^*$ quasi-simple? The next proposition says in the non-abelian case that it can only happen if $G$ itself is boundedly simple. Note that $N$-bounded simplicity, for a fixed $N$, is a first order property and quasi-simplicity in general is not a first order condition.

**Proposition 2.3.** Suppose that $G$ is an infinite and nonabelian group.

(1) If $G$ is boundedly simple, then $G$ is quasi-simple.

(2) Assume additionally that $G$ realizes every type from $S_2(\emptyset)$ in the pure group language.

(a) If $G$ is perfect, then $G$ has finite commutator width.

(b) If $G$ is quasi-simple, then $G$ is boundedly simple.
Proof. (1) It is clear that $G/Z(G)$ is simple. Bounded simplicity is preserved under taking homomorphic images. Hence $G/[G,G]$ is abelian and boundedly simple. Since $G$ is nonabelian, $G$ is perfect.

(2) The proof is an application of the compactness argument.

(2a) If $G$ would have infinite commutator width, then the following type is consistent: $p(x) = \{ x \text{ is not a product of } n \text{ commutators} : n \in \mathbb{N} \}$. However by the assumption $G$ cannot realize $p$.

(2b) Consider a quasi-simple $G$. By compactness one can find a natural number $M$, such that

$$(\Delta) \quad \left(g^G \cup g^{-1}G\right)^{\leq M} \cdot Z(G) = G,$$

for every $g \in G \setminus Z(G)$. Otherwise the following type

$$p(x, y) = \left\{ x \not\in Z(G), y \not\in \left(x^G \cup x^{-1}G\right)^{\leq M} \cdot Z(G) : M \in \mathbb{N} \right\}$$

would be consistent, so if $(a, b) \in G$ satisfy $p$, then $H/Z(G)$, where $H$ is the normal closure of $a$, would be a proper and nontrivial normal subgroup of $G/Z(G)$.

Fix $g \in G \setminus Z(G)$. By $(\Delta)$, every commutator $[x, y]$ is in $\left(g^G \cup g^{-1}G\right)^{\leq 4M}$. By (1), $\text{cw}(G)$ is finite. Thus $G$ is $4M$ $\text{cw}(G)$-boundedly simple. \qed

The weak saturation assumption in Proposition 2.3 is essential. Consider for example the group $G = \text{SO}_3(\mathbb{R})$ with the inherited structure from $\mathbb{R}$. It is well known that $G$ is simple. However $G$ is not boundedly simple (in a sufficiently saturated extension $\mathbb{R}^*$ of $\mathbb{R}$ the group $G^* = \text{SO}_3(\mathbb{R}^*)$ has a nontrivial proper normal subgroup consisting of rotations by infinitesimal angles).

By Propositions 2.2 and 2.3 if $G^*$ is a boundedly simple sufficiently saturated group, then $G^{\ast\infty}$ exists and is equal to $G^*$. We characterize groups satisfying this property.

**Proposition 2.4.** Suppose $G$ is an arbitrary infinite group possibly with some extra structure and $G^*$ is a sufficiently saturated expansion of $G$. Then the following are equivalent

1. $G^{\ast\infty}$ exists and $G^* = G^{\ast\infty}$
2. there exists a natural number $N$ such that for every definable (with parameters) thick subset $P \subseteq G$, $P^N = G$.

Note that, since every thick set $P$ contains the neutral element, $P^N = P^{\leq N}$.

Proof. (2) $\Rightarrow$ (1) It is enough to show that for a small set $A \subseteq G^*$, $G^* = X_{\emptyset A}^N$. Let $P^* = \varphi(G^*; \overline{a})$ be $k$-thick and $\overline{a} \subseteq A$. Since in $G$ it is true that for every $\varphi$-definable (with parameters) and $k$-thick $P \subseteq G$, $P^N = G$, the same is true in $G^*$, that is $\varphi(G^*; \overline{a})^N = G^*$. Hence by Lemma 1.4(1), $G^* = X_{\emptyset A}^N = G^{\ast\infty}$.

(1) $\Rightarrow$ (2) Suppose, on the contrary, that for every natural $N$ there is $P_N \subseteq G^*$, definable over some finite $A_N \subseteq G^*$ and thick, with $P_N^N \neq G^*$. Let $A = \bigcup_{N \in \mathbb{N}} A_N$. Then clearly, for every natural $N$, we have $X_{\emptyset A}^N \subseteq P_N^N \neq G^*$. By compactness $G^{\ast\infty}_A \neq G^*$, which is impossible. \qed

In the next proposition we consider groups having the property $G^* = G^{\ast\infty}$ in all first order expansions.
Proposition 2.5. Let $G$ be a non-trivial pure group (that is without any additional structure). The following conditions are equivalent.

1. There exists a natural number $N$ such that for every thick subset $P \subseteq G$ (not necessarily definable), $P^N = G$.
2. $G$ is infinite and if $G^*$ is a sufficiently saturated extension of an arbitrary first order expansion of $G$, then $G^{*\infty}$ exists and $G^* = G^{*\infty}$.

Furthermore, when (1) holds, $G^{*\infty} = \Theta_G^N$, for every small $A \subseteq G^*$ and an arbitrary first order structure on $G$.

Proof. $(1) \Rightarrow (2)$ is because otherwise the thick subset $\{e\}$ of $G$ does not fulfill the condition from (1). The rest of (2) follows easily from Proposition 2.4.

$(2) \Rightarrow (1)$ Suppose contrary to our claim, that for every natural $N$ there is a thick subset $P_N$ of $G$ with $P_N^N \neq G$. Expand the structure of $G$ by the predicates for all $P_N$. If $G^*$ is a sufficiently saturated extension of $G$, then clearly for every natural $N$ we have $X_{\Theta}^N \subseteq P_N^N \neq G^*$, so $G^{*\infty}_G \neq G^*$, contrary to (2). $\square$

Definition 2.6. (1) A group $G$ is called $N$-absolutely connected or $N$-ac if it satisfies the condition (1) from Proposition 2.5 that is for every thick subset $P \subseteq G$, we have $P^N = G$.

(2) We say that a group $(G, \cdot, \ldots)$ with some first order structure, is $N$-definably absolutely connected, if for every definable (with parameters) thick $P \subseteq G$, we have $P^N = G$.

(3) A group is [definably] absolutely connected, if it is $N$-[definably] absolutely connected, for some natural $N$.

Problem 2.7. Characterize the class of groups $G$ having the property “$G^{*00}$ exists and is equal to $G^*$ in all first order expansions”. Is it an elementary class in the pure group language?

The component $G^{*0}$ exists and $G^* = G^{*0}$ if and only if $G$ has no parameter-definable subgroups of finite index. Therefore if we start just with a group $G$ with no structure, then $G^{*0}$ exists and $G^* = G^{*0}$ in an arbitrary monster model $G^*$ of an arbitrary first order expansion of $G$, if and only if $G$ has no proper subgroup of finite index. Denote by NFQ the class of groups without proper subgroups of finite index. In [20] we proved that NFQ is not an elementary class: the free product $\mathbb{Q} * \mathbb{Q}$ of rationals is in NFQ, but the ultrapower of $\mathbb{Q} * \mathbb{Q}$ has a normal subgroup of index 2.

In Proposition 2.10 below we collect basic properties of absolute connected groups.

Lemma 2.8. [13] Lemma 3.6] If $P$ is an $m$-generic subset of a group $G$, $1 \in P$ and $P = P^{-1}$, then $P^{3m-2}$ is a subgroup of $G$ of finite index at most $m$.

The next lemma is a constructive version of the well know Iwasawa Lemma from the classical proof of simplicity of groups of Lie type.

Lemma 2.9. Let $G$ be a perfect group with a finite commutator width $\text{cw}(G) = N$, and $A \subseteq G$ be a normal and symmetric subset. If $B < G$ is a solvable subgroup of derived length $M$ and $G = A \cdot B$, then $G = A^{(4N)^M}$.
Proof. Let $C$ be the set of all commutators. By the assumption $G = C^N$. We use the following commutator identity

$$[a_1b_1, a_2b_2] = (a_1^{-1})^{b_1} (a_2^{-1}a_1)^{b_1b_2} a_2b_1 \cdot [b_1, b_2].$$

It is enough to prove $G = A^{(4N)^n} \cdot B^{(n)}$ for every natural $n$. The case $n = 0$ follows from the assumption. For the induction step, note that (♣) implies that $C \subseteq A^{(4N)^n} \cdot [B^{(n)}, B^{(n)}]$ and then $G = A^{(4N)^{n+1}} \cdot B^{(n+1)}$, since $G = C^N$ and $A$ is a normal subset of $G$. \qed

Absolutely connectedness is preserved under homomorphic images and under taking certain finite or soluble extensions.

**Proposition 2.10.** Let $f : G \to H$ be an epimorphism of groups.

1. If $G$ is $N$-ac, then $H$ is also $N$-ac.
2. If $H$ is $N_1$-ac and $\ker(f)$ is $N_2$-ac, then $G$ is $(N_1 + N_2)$-ac.
3. If $G = \bigcup_{i \in I} G_i$ for $N$-ac subgroups of $G_i \subseteq G$, $i \in I$, then $G$ is also $N$-ac.
4. If $H$ is $N$-ac, $G$ has no subgroups of finite index and $\ker(f)$ is finite of cardinality $M$, then $G$ is $N \cdot (3M - 2)$-ac.
5. If $H$ is $N$-ac, $G$ is perfect with $\text{cw}(G) = R < \infty$ and $\ker(f)$ is solvable of derived length $M$, then $G$ is $N \cdot (4R)^M$-ac.

**Proof.** (1) This easily follows from Lemma 1.3(3).
(2) Let $P \subseteq G$ be thick. Then $f[P] \subseteq H$ is also thick, so $f[P]^{N_1} = H$. Thus

$$G = f^{-1}[f[P]^{N_1}] = P^{N_1} \cdot \ker(f).$$

The set $P \cap \ker(f)$ is thick in $\ker(f)$, therefore $\ker(f) \subseteq P^{N_2}$ and $G = P^{N_1 + N_2}$.
(3) If $P \subseteq G = \bigcup_{i \in I} G_i$ is thick, then $P \cap G_i$ is thick in $G_i$ for every $i \in I$. By the assumption $G_i \subseteq P^N$, so $G = P^N$.
(4) To prove (4), let $P \subseteq G$ be thick. Then $f[P]^N = H$, so $G = P^N \cdot \ker(f)$ and $P^N$ is $M$-generic. By the Lemma 2.8, $G = P^{N(3M - 2)}$.
(5) Take a thick subset $P \subseteq G$. Then as in (4), $G = P^N \cdot \ker(f)$. By Lemma 2.9, $G = P^{N(4R)^M}$. \qed

In order to give examples of absolutely connected groups, we introduce the class of weakly simple groups. For a natural number $N$ define

$$\mathcal{G}_N(G) = \left\{ g \in G : \left( g^G \cup g^{-1}G \right)^{\leq N} = G \right\}.$$

If $G$ is $N$-boundedly simple, then $\mathcal{G}_N(G) = G \setminus Z(G)$ and $G$ is absolutely connected. In Definition 2.11 below we require the set $\mathcal{G}_N(G)$ to be big in a weaker sense.

**Definition 2.11.** We say that a group $G$ is $N$-weakly simple or $N$-ws if $G \setminus \mathcal{G}_N(G)$ is not thick, that is for every $n$, there is a sequence $(g_i)_{i < n}$ in $G$ such that $g_i^{-1}g_j \in \mathcal{G}_N(G)$ for all $i < j < n$. A group is weakly simple, if it is $N$-weakly simple for some natural number $N$.

Our goal is prove that weakly simple groups are absolutely connected (Theorem 2.15) and that the latter are perfect (Theorem 2.16).

Likewise in Proposition 2.10, we can prove that the class of weakly simple groups is closed under certain operations.
Lemma 2.12.  

(1) The class of all $N$-weakly simple groups is elementary in the pure language of groups and is closed under taking homomorphic images, direct sums and arbitrary products.

(2) Let $f : G \to H$ be an epimorphism of groups.

(a) If $H$ is $N$-ws, $G$ has no subgroups of finite index and $\ker(f)$ is finite of cardinality $M$, then $G$ is $N \cdot (3M - 2)$-ws.

(b) If $H$ is $N$-ws, $G$ is perfect with $\text{cw}(G) = R < \infty$ and $\ker(f)$ is solvable of derived length $M$, then $G$ is $N \cdot (4R)^M$-ws.

Proof. (1) This follows from easy remarks: if $f : G \to H$ is an epimorphism, then $f \left[ G_N(G) \right] \subseteq G_N(H)$. $G_N \left( \bigoplus_{i \in I} G_i \right) = \bigoplus_{i \in I} G_N \left( G_i \right)$ and $G_N \left( \prod_{i \in I} G_i \right) = \prod_{i \in I} G_N \left( G_i \right)$.

(2) The proof is identical with the proof of Proposition 2.10(4) and (5). □

There is an interesting special case of Proposition 2.10(4) and Lemma 2.12(2a). Namely, in the next proposition we prove that an extension of an absolutely connected group by $\mathbb{Z}/p\mathbb{Z}$, where $p$ is prime, is also absolutely connected unless the extension is split. This observation will be important in the Section 4.

We give some preliminaries. Suppose $1 \to \mathbb{Z}/p\mathbb{Z} \to G \xrightarrow{f} H \to 1$ is an exact sequence of groups, where $p$ is a prime number and $\ker(f) \subset \mathbb{Z}(G)$. Then there exists a 2-cocycle $h : H \times H \to \mathbb{Z}/p\mathbb{Z}$, such that the group $G$ is isomorphic to the product $(\mathbb{Z}/p\mathbb{Z}, H, h)$, where the multiplication is defined by the rule $(a_1, x_1) \cdot (a_2, x_2) = (a_1 + a_2 + h(x_1, x_2), x_1x_2)$.

Proposition 2.13. Suppose $H$ is $N$-absolutely connected [$N$-weakly simple], and for some prime $p$

$$1 \to \mathbb{Z}/p\mathbb{Z} \to G \xrightarrow{f} H \to 1$$

is an exact sequence, which does not split (that is $G$ is not a product $\ker(f) \times H$), where $\mathbb{Z}/p\mathbb{Z} \cong \ker(f) \subset \mathbb{Z}(G)$. Let $h$ be the corresponding 2-cocycle. Then

(1) $G$ is $N \cdot (3p - 2)$-absolutely connected $[N \cdot (3p - 2)$-ws respectively],

(2) if the image $\text{Im}(h) \subset \mathbb{Z}/p\mathbb{Z}$ of $h$ does not generate $\mathbb{Z}/p\mathbb{Z}$ in $M$ steps, then $G$ is not $\frac{1}{2}M$-absolutely connected.

Proof. (1) By Proposition 2.10(4) and Lemma 2.12(2a) it is enough to prove that $G$ has no subgroups of finite index. Suppose $G_1 < G$ is such a subgroup. By considering the intersection of all the conjugates of $G_1$ we may assume that $G_1 \subset G$. Therefore $f[G_1] = H$, so $G = G_1 \cdot \ker(f)$. However $\ker(f)$ is simple, so $G_1 \cap \ker(f) = \{1\}$ (otherwise $G = G_1$).

Since $\ker(f) \subseteq \mathbb{Z}(G)$, group $G$ is the product $\ker(f) \times G_1$.

(2) We may assume that $M$ is even. In order to prove that $G$ is not $\frac{1}{2}M$-absolutely connected we need to find a thick subset $P \subseteq G$ such that $P^M \neq G$. Take $P = \{1\} \times H$.

Since $\mathbb{Z}/p\mathbb{Z} \cdot P' = G$, the set $P'$ is a p-generic subset of $G$ (see Lemma 2.8, so by [18, Lemma 3.2(4)] $P = P'P'^{-1}$ is thick. Suppose that 1 is the identity element of $H$. Using the description of $G$ in terms of cocycle $h$ we have that: the identity element of $G$ is $(-h(1, 1), 1)$; the inverse element of $(a, x)$ is $(-a - h(x, x^{-1}) - h(1, 1), x^{-1})$. Hence we have $P \subseteq (\text{Im}(h) - \text{Im}(h) - h(1, 1)) \times H$. More general, one can prove by induction that

$$P^n \subseteq ((2n - 1) \text{Im}(h) - n \text{Im}(h) - nh(1, 1)) \times H \subseteq 2n(\text{Im}(h) - \text{Im}(h)) \times H,$$

so (3) implies that $P^M \neq G$. □

The next lemma is crucial in the proof of Theorem 2.15.
Lemma 2.14. If $P \subseteq G$ is thick, then there exists a thick subset $Q \subseteq P^4$ which is normal, (that is $Q^g = Q$ for all $g \in G$) and $\emptyset$-definable in the structure $(G, \cdot, P)$, where $P$ is a predicate.

Proof. Consider the structure $\mathcal{G} = (G, \cdot, P)$ and let $\mathcal{G}^* = (G^*, \cdot, P^*)$ be a sufficiently saturated extension. Recall that for a small model $M \prec G^*$, the set $X_{\mathcal{M}}$ is defined as $\{a^{-1}b : a, b \in G^*, a \equiv_M b\}$. By the properties of $\Theta_A$ from Section 1.3 we have

$$X_{\Theta_\emptyset} \subseteq \bigcup_{M \prec G^* \text{ small}} X_{\mathcal{M}} \subseteq X_{\emptyset}^2.$$

Moreover $(X_{\mathcal{M}})^{G^*} \subseteq X_{\mathcal{M}}^{G^*}$ holds. Indeed, if $y \in (X_{\mathcal{M}})^{G^*}$, then there are $a, x \in G^*$ and $h \in \text{Aut}(G^*/M)$, such that $y = (a^{-1}h(a))^x = (ax)^{-1}h(a)x = ((ax)^{-1}h(ax))$ $(h(x)^{-1}x) \in X_{\mathcal{M}}^{G^*}$. Hence

$$(X_{\mathcal{M}})^{G^*} \subseteq \bigcup_{M \prec G^* \text{ small}} X_{\mathcal{M}}^{G^*} \subseteq \bigcup_{M \prec G^* \text{ small}} X_{\mathcal{M}}^{2}.$$

By (1), (2) and Lemma 1.3.1 we have $X_{\Theta_\emptyset}^{G^*} \subseteq X_{\emptyset}^{G^*} \subseteq P^4$. By compactness and Lemma 1.4(2) we can find a thick and $\emptyset$-definable $Q \subseteq G^*$, with $Q^{G^*} \subseteq P^4$. \qed

Theorem 2.15.  

1. If $G$ is $N$-boundedly simple and $Z(G)$ has infinite index, then $G$ is $N$-weakly simple.

2. Every $N$-weakly simple group is $4N$-absolutely connected.

Proof. (1) follows by definition. We prove (2). Let $P \subseteq G$ be a thick subset. We will prove that $P^{4N} = G$. By Lemma 2.14 there is a thick and normal subset $Q \subseteq P^4$. Weak simplicity of $G$ implies that $Q \cap G_N(G) \neq \emptyset$ (because otherwise $Q \subseteq G \setminus G_N(G)$ and the last set would be thick). Take $g \in Q \cap G_N(G)$. Since $Q$ is symmetric, we have that $g^Q \cup g^{-1}Q^g \subseteq Q^g = Q \subseteq P^4$. Now is it enough to use the definition of $G_N(G)$. \qed

If a group is abelian, then clearly it is not weakly simple. The next theorem extends this remark to absolutely connected and non-perfect groups.

Theorem 2.16. Absolutely connected groups are perfect.

Proof. First we verify that the abelian groups

- $(\mathbb{Z}/p\mathbb{Z}, +)$, $(\mathbb{Q}/\mathbb{Z}, +)$ and $\mathbb{Z}(p^\infty) = \left(\mathbb{Z} \left[\frac{1}{p}\right] / \mathbb{Z}, +\right)$ are not absolutely connected,

where $p$ is a prime number and $\mathbb{Z}\left[\frac{1}{p}\right]$ is the subgroup of $(\mathbb{Q}, +)$ generated by the set $\left\{\frac{1}{p^n} : n \in \mathbb{N}\right\}$.

The group $\mathbb{Z}/p\mathbb{Z}$ is not absolutely connected, because it is finite. Let $H = \mathbb{Q}/\mathbb{Z}$ or $H = \mathbb{Z}(p^\infty)$. Then $H$ can be viewed as a dense subgroup of the circle group $S^1$. Small connected neighborhoods of $e$ in $S^1$ form a collection of thick subsets contradicting the condition from Definition 2.6. Since $H$ is dense in $S^1$, intersections of these neighborhoods with $H$ also work for $H$. Thus $H$ is not absolutely connected.
Suppose, for a contradiction, that there is a non-perfect absolutely connected group $G$. Then by Proposition 2.10(1) a nontrivial abelian group $G/[G, G]$ is absolutely connected. The next lemma shows that our group $G/[G, G]$ may be mapped homomorphically onto one of the groups $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Z}(p^\infty)$, contradicting (♠) and Proposition 2.10(1). □

The next lemma is presumably already known, however for the convenience of the reader we give here the proof.

**Lemma 2.17.** Every abelian group can be homomorphically mapped onto one of the following groups:

$$\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}/\mathbb{Z} \text{ or } \mathbb{Z}(p^\infty)$$

for some prime number $p$.

**Proof.** Let $G$ be an abelian group. We may assume that $G$ is infinite, because otherwise $G$ can be mapped onto $\mathbb{Z}/p\mathbb{Z}$ (for some prime $p$). We may further assume that $G$ is a torsion group. To see this suppose there is $g \in G$ of infinite order. Let $H < G$ be a maximal subgroup of $G$ disjoint from the set $\{g^n : n \in \mathbb{Z} \setminus \{0\}\}$. When $G/H \cong \mathbb{Z}$, $G$ can be mapped onto $\mathbb{Z}/2\mathbb{Z}$. If $G/H \not\cong \mathbb{Z}$, then $G/(H, g)$ is a nontrivial torsion group (for every $a \in G \setminus H$, there is $n$, such that $g^n \in (H, a)$, so $a^m \in (H, g)$ for some $m$).

Every torsion abelian group $G$ splits as a direct sum of Sylow $p$-subgroups

$$G = \bigoplus_{p \in \mathbb{P}} G_p,$$

where each element of $G_p$ has order $p^n$ for some natural $n$. Hence we may assume that $G = G_p$ is a $p$-group. If $G \neq p \cdot G$, then $G/pG$ is a vector space over finite field $\mathbb{F}_p$, so there is a mapping from $G$ onto $\mathbb{Z}/p\mathbb{Z}$. If $G = pG$, then we claim that $G$ is divisible. Indeed, let $g \in G$ and $s = p^k \cdot m$, where $p \nmid m$. We want to find a solution in $G$ of the equation $g = s \cdot x$. Since $G = pG$, there is $g' \in G$ such that $g = p^k \cdot g'$. Let $p^n$ be the order of $g'$ in $G$. The numbers $p^n$ and $m$ are coprime, therefore there are $d_1, d_2 \in \mathbb{Z}$ such that $1 = d_1 \cdot p^n + d_2 \cdot m$. The element $d_2g'$ is a solution of our equation, because $g = p^k \cdot g' = p^k(d_1p^n + d_2m) \cdot g' = (p^k m) \cdot (d_2g') = s \cdot (d_2g')$. We proved that $G$ is divisible abelian group. By a well known fact (see for example [24, Theorem 9.1.6]), every divisible abelian group splits as a direct sum of groups isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}[p^\infty]$. This finishes the proof. □

**Example 2.18.** Theorem 2.16 cannot be generalized to definably absolutely connected groups. Let $\mathbb{R}^*$ be a sufficiently saturated extension of the real field $\mathbb{R}$. By Proposition 2.4, $(\mathbb{R}^*, +)$ is definably absolutely connected group and non-perfect (because by [33, Theorem 1.10] or [18, Theorem 5.3], $(\mathbb{R}^*, +)^\infty$ exists and $(\mathbb{R}^*, +)^\infty = \mathbb{R}^*$, since $(\mathbb{R}^*, +)^\infty$ is an ideal in $\mathbb{R}^*$). On the other hand, consider the following structure $G = (\mathbb{R}, +, \cdot, 0, 1, Q_n)_{n \in \mathbb{N}}$, where $Q_n = (\frac{-1}{n}, \frac{1}{n}) + \mathbb{Z}$. Since all $Q_n$’s are thick, in the saturated extension $G^* : G^* \subseteq \bigcap_{n \in \mathbb{N}} Q_n \not\subseteq G^*$. 

Since every absolutely connected group is perfect (Theorem 2.16), the natural question arises about the commutator width of these groups. Proposition 2.19 answers this question for weakly simple groups.

**Proposition 2.19.** For every natural $N$ there is a constant $k_N$ such that every $N$-weakly simple group has commutator width $\leq k_N$.

**Proof.** Suppose that for every natural $k$ there is an $N$-weakly simple group $G_k$ with commutator width $\geq k$ and take $g_k \in G_k$ of commutator length $\geq k$. Consider the
product $G = \prod_{k \in \mathbb{N}} G_k$. Group $G$ is $N$-weakly simple (Lemma 2.12(1)). Hence by Theorem 2.16, $G$ is perfect. However $G \neq [G, G]$, because the element $g = (g_k)_{k \in \mathbb{N}} \in G$ has infinite commutator length, so is not in $[G, G]$. 

We deal with connected components of the quotient group. Suppose $G$ is a sufficiently saturated group and let $H < G$ be a $\emptyset$-definable subgroup with infinite index. Let $j: G \to G/H$ be the quotient map. Consider on $G/H$ the natural quotient structure (for example consisting of unary predicates for all subsets of the form $j[P]$, where $P \subseteq G$ is $\emptyset$-definable and thick). $G/H$ with this structure is sufficiently saturated.

**Proposition 2.20.** Let $x \in \{\infty, 00, 0\}$. Then

1. $j\left[\bigcap_{P \subseteq G, \emptyset-\text{def. thick}} P\right] = \bigcap_{P \subseteq G, \emptyset-\text{def. thick}} j[P]$.
2. If $H$ is definably absolutely connected, then $G^0_\Theta = j^{-1}\left[(G/H)^0_\Theta\right]$.

**Proof.** (1) When $x = \infty$, it is enough to note the following equalities in $G/H$:

$$j\left[\bigcap_{P \subseteq G, \emptyset-\text{def. thick}} P\right] = \bigcap_{P \subseteq G, \emptyset-\text{def. thick}} j[P] = \bigcap_{Q \subseteq G/H, \emptyset-\text{def. thick}} Q.$$ 

The first equality follows from compactness and the fact that the intersection of finitely many thick sets is thick. For the second: if $P \subseteq G$ is $\emptyset$-definable and thick, then (by Lemma 1.3(3)) $j[P]$ is also thick and $\emptyset$-definable in $G/H$, if $Q \subseteq G/H$ is $\emptyset$-definable and thick, then $Q = j\left[j^{-1}[Q]\right]$, where $j^{-1}[Q]$ is thick and $\emptyset$-definable. From (1) we conclude $j[X^G_{\Theta_0}] = X^{G/H}_{\Theta_0}$, hence $j\left[\bigcap_{P \subseteq G, \emptyset-\text{def. thick}} P\right] = \bigcap_{Q \subseteq G/H, \emptyset-\text{def. thick}} Q$.

Now assume that $x = 00$. The inclusion $\supseteq$ is clear, because the image of $\bigwedge$-definable subgroup over $\emptyset$ of bounded index under $j$ is also $\bigwedge$-definable over $\emptyset$ ($G^0_0$ is an intersection of some $\emptyset$-definable thick subsets) and of bounded index. For $\subseteq$ note that $j^{-1}\left[\bigcap_{P \subseteq G, \emptyset-\text{def. thick}} P\right]$ is $\bigwedge$-definable over $\emptyset$ of bounded index. Therefore $G^0_0 \subseteq j^{-1}\left[(G/H)^0_0\right]$.

When $x = 0$, the proof is similar and is left to the reader.

(2) By (1) we have $j^{-1}\left[(G/H)^0_0\right] = G^0_0 \cap H$. Therefore it suffices to prove that $G^\infty_0 \supseteq H$. If $P \subseteq G$ is $\emptyset$-definable and thick, then $P \cap H$ is definable and thick in $H$. Since $H$ is $N$-definable absolutely connected, we have $P^N \supseteq H$. Thus $G^\infty_0 \supseteq X^N_{\Theta_0} = \bigcap P^N \supseteq H$. 

### 3. Examples

The aim of the present section is to give the structural result (Theorem 3.14) about the groups of rational points of split semisimple linear groups over arbitrary infinite fields. However first we consider other examples of absolutely connected and weakly (boundedly) simple groups.

At the beginning, we would like to recall a very elegant example of Daniel Lascar from [25]. Let $K < L$ be an extension of algebraically closed fields with $\deg_K(L) > \aleph_0$. Consider the automorphism group $G = \text{Aut}(L/K)$. By [25] Proposition 3, $G$ is 4-boundedly simple; that is $G_4(G) = G \setminus \{e\}$ (the proof of [25] Proposition 3) is for the case $L = \mathbb{C}$ and $K = \hat{\mathbb{Q}}$, but only the assumption $\deg_K(L) > \aleph_0$ is used).

#### 3.1. Infinite permutation groups.

Let $\Omega$ be an infinite set. Consider the alternating group of permutations of $\Omega$:

$$\text{Alt}(\Omega) = \{\sigma \in \text{Sym}(\Omega) : \text{supp}(\sigma) \text{ is finite and } \sigma \text{ is even}\}.$$
The converse of Theorem 2.15 is not true.

**Proposition 3.1.** $\text{Alt}(\Omega)$ is 8-absolutely connected but not weakly simple.

**Proof.** $\text{Alt}(\Omega)$ is not weakly simple because $\mathcal{G}_N(\text{Alt}(\Omega)) = \emptyset$, for every natural $N$. To prove the absolute connectedness, take an arbitrary $n$-thick subset $P \subseteq \text{Alt}(\Omega)$. By considering $P^4$ and using Lemma 2.14, we may assume that $P$ is normal. Now it is enough to prove that $P^2 = \text{Alt}(\Omega)$.

We claim that $P$ contains any finite product of disjoint even cycles. Take $n$th-thick subset $P \subseteq \text{Alt}(\Omega)$. By considering $P^4$ and using Lemma 2.14, we may assume that $P$ is normal. Now it is enough to prove that $P^2 = \text{Alt}(\Omega)$.

We claim that $P$ contains any finite product of disjoint even cycles. Take $n$th-thick subset $P \subseteq \text{Alt}(\Omega)$. By considering $P^4$ and using Lemma 2.14, we may assume that $P$ is normal. Now it is enough to prove that $P^2 = \text{Alt}(\Omega)$.

The following two cycles $(x, a_1, \ldots, a_{2p+1})$ and $(y, b_1, \ldots, b_{2q+1})$ are in $P$, so

\[(x, a_1, \ldots, a_{2p+1}) \circ (y, b_1, \ldots, b_{2q+1}) = (x, a_1, \ldots, a_{2p+1}) \circ (y, b_1, \ldots, b_{2q+1})\]

is in $P^2$. Thus $P^2$ contains every product of two disjoint odd cycles. Similarly to the above, we can prove that every even permutation (which is a disjoint product of even cycles and even number of odd cycles) is in $P^2$. \hfill \square

Now we concentrate on the group $\text{Sym}(\Omega)$ and its normal subgroups. For a cardinal $\kappa$, define $\text{Sym}^\kappa(\Omega) = \{\sigma \in \text{Sym}(\Omega) : |\text{supp}(\sigma)| < \kappa\}$. By the previous proposition, $\text{Sym}^{\aleph_0}(\Omega)$ has absolutely connected subgroup $\text{Alt}(\Omega)$ of index 2.

**Proposition 3.2.** Let $\kappa$ be an uncountable cardinal.

1. If $\kappa = \lambda^+$ is a successor cardinal, then $\text{Sym}^\kappa(\Omega)$ is 4-weakly simple, so it is 16-absolutely connected.

2. If $\kappa$ is a limit cardinal, then $\text{Sym}^\kappa(\Omega)$ is 16-absolutely connected, but not weakly simple.

**Proof.** We use the following result of Bertram, Moran, Droste and Göbel (3, 28, 13).

**Theorem** (13 Theorem p. 282]). If $\sigma, \tau \in \text{Sym}(\Omega)$, $|\text{supp}(\tau)| \leq |\text{supp}(\sigma)|$ and $|\text{supp}(\sigma)|$ is infinite, then $\tau \in (\text{Sym}(\Omega))^4$.

The group $\text{Sym}^\kappa(\Omega)$ is a normal subgroup of $\text{Sym}(\Omega)$ and two elements from $\text{Sym}^\kappa(\Omega)$ are conjugate in $\text{Sym}(\Omega)$ if and only if they are conjugate in $\text{Sym}^\kappa(\Omega)$. Therefore in the case when $\kappa = \lambda^+$ is a successor cardinal, from the above theorem we conclude that the group $\text{Sym}^\kappa(\Omega)$ is 4-weakly simple (and non-simple), that is

$$\mathcal{G}_4(\text{Sym}^\kappa(\Omega)) = \text{Sym}^\kappa(\Omega) \setminus \text{Sym}^\lambda(\Omega).$$

When $\kappa$ is a limit cardinal, $\text{Sym}^\kappa(\Omega) = \bigcup_{\lambda < \kappa} \text{Sym}^\lambda(\Omega)$. In this case $\text{Sym}^\kappa(\Omega)$ is not weakly simple ($\mathcal{G}_N(\text{Sym}^\kappa(\Omega)) = \emptyset$). However for a successor cardinal $\lambda < \kappa$, the group $\text{Sym}^\lambda(\Omega)$ is 4-weakly simple. By Theorem 2.15 $\text{Sym}^\lambda(\Omega)$ is 16-absolutely connected and by Proposition 2.10(3) the same is true for $\text{Sym}^\kappa(\Omega)$. \hfill \square
3.2. **Infinite-dimensional general linear group.** We consider groups of infinite matrices. The following proposition is derived from the result of Tolstyk (39).

**Proposition 3.3.** Assume that \( V \) is an infinite-dimensional vector space over a division ring \( D \). Then the group \( \text{GL}(V) \) of all linear automorphisms of \( V \) is 32-weakly simple and 128-absolutely connected.

**Proof.** Call a subspace \( U \) of \( V \) moietous if \( \dim U = \dim V = \text{codim} U \). We say that \( \pi \in G \) a moietous involution if there exists a decomposition \( V = U \oplus U' \oplus W \) into a direct sum of moietous subspaces, such that \( \pi \) on \( W \) is identity and exchanges \( U \) and \( U' \). Proposition 1.1 from [39] says that every moietous involution is in \( G_{32}(\text{GL}(V)) \).

In order to prove that \( \text{GL}(V) \) is 32-weakly simple it is enough to find in \( \text{GL}(V) \) an infinite sequence \( (g_i)_{i \in \mathbb{N}} \), such that for all \( i < j \in \mathbb{N} \) the element \( g_i^{-1}g_j \) is a moietous involution (so \( g_i^{-1}g_j \in G_{32}(\text{GL}(V)) \)). Take an infinite decomposition \( V = \bigoplus_{i \in \mathbb{N}} V_i \) of \( V \) into moietous subspaces. Let \( g_i \) be the moietous involution of \( V \) with respect to the following decomposition:

\[
V = V_{2i} \oplus V_{2i+1} \oplus \bigoplus_{2i, 2i+1 \neq j \in \mathbb{N}} V_j,
\]

that is \( g_i[V_{2i}] = V_{2i+1} \). Then clearly \( g_i^{-1}g_j \) for \( i \neq j \) is also a moietous involution. \( \square \)

3.3. **Linear groups.** In this section we study absolute connectedness and weak simplicity of linear groups. By a linear algebraic group we mean an affine algebraic group (see [4, 31]). We assume throughout that \( k \) is an arbitrary infinite field. All linear groups are assumed to be connected.

We say that a linear algebraic group \( G \) is a \( k \)-group or \( G \) is defined over \( k \), if the ideal of polynomials vanishing on \( G \) is generated by polynomials over \( k \); see [4, AG §11]. By \( G(k) \) we denote the group of \( k \)-rational points of a \( k \)-group \( G \). A Borel subgroup of \( G \) is a maximal connected Zariski closed solvable subgroup of \( G \); a subgroup of \( G \) is called parabolic if it contains some Borel subgroup.

We recall the basic types of algebraic groups.

**Definition 3.4.** Let \( G \) be a connected linear algebraic group defined over \( k \). The group \( G \) is called

1. reductive (respectively semisimple), if the unipotent radical \( R_u(G) \) (respectively the solvable radical \( R(G) \)) of \( G \) is trivial,
2. isotropic over \( k \) or \( k \)-isotropic, if there is a proper parabolic subgroup of \( G \) which is defined over \( k \) [4 §20],
3. split over \( k \) or \( k \)-split if some maximal torus \( T \) in \( G \) is split over \( k \) [4, 18.6],
4. absolutely almost simple (respectively almost simple over \( k \) or almost \( k \)-simple) if \( G \) has no proper normal connected closed subgroup (respectively defined over \( k \)) [6 §0.7], [37, 1.1.1, 3.1.2],
5. simply connected if \( G \) does not admit any nontrivial central isogeny \( \pi: \tilde{G} \to G \) ([31, 2.1.13], [34, 8.1.11], [37, 1.5.4, 2.6.2]).

Let \( G \) be a \( k \)-group. If \( G \) is absolutely almost simple, then of course \( G \) is almost \( k \)-simple, but the converse is not true in general. However, when \( G \) is \( k \)-split, semisimple and simply connected, then these two notions are equivalent. This observation seems to be well known and we use it in the proof of Theorem 3.11. We could not find relevant reference in the literature, so we prove this fact in the next lemma.
For every separable field extension $k' \supseteq k$ there is a functor $R_{k'/k}$ [6 6.17] which assigns, by restriction of scalars, to each affine $k'$-group $H'$ an affine $k$-group $H = R_{k'/k}(H')$ such that $H'(k') \cong H(k)$.

**Lemma 3.5.** Every almost $k$-simple $k$-split simply connected semisimple $k$-group $G$ is absolutely almost simple.

*Proof.* By [6 6.21 (II)], [37 3.1.2], there exists a finite separable field extension $k'$ of $k$ and absolutely almost simple and simply connected $k'$-group $G'$ such that $G = R_{k'/k}(G')$.

It is enough to prove that $k' = k$. Let $T'$ be a maximal $k'$-split torus of $G'$. Since $G$ is $k$-split, by [34 16.2.7], $T = R_{k'/k}(T')$ is a maximal $k$-split torus of $G$ and $\dim(T) = \dim(T')$. However, by [6 6.17], $\dim(T) = \dim(T')[k' : k]$, so $k' = k$. \qed

We review below some facts about bounded simplicity of linear groups.

Over any algebraically closed field $K$ we have the following well known fact: if $G$ is absolutely almost simple, then $G(K)$ is $2\dim(G)$-boundedly simple [1 2.2]. For an arbitrary infinite field $k$, bounded simplicity of $G(k)$ for certain classes of groups, that is $W$-trivial groups, can be derived from Proposition 2.3 We recall the classical result of Tits [38 Main theorem]: if $G$ is reductive, isotropic over $k$ and almost $k$-simple, then $G(k)$ contains a certain Zariski dense normal subgroup, denoted by $G(k)^+$ (see Definition 3.12 below), which is quasi-simple. Following Gille [17 1.1] we say that the group $G$ is $W$-trivial over $k$ if for every field extension $F$ of $k$,

$$G(F) = G(F)^+.$$  

In other words, the group $G(F)$ is quasi-simple. Proposition 2.3 implies the following fact, which has been also observed by Keller and Rapinchuk some time ago.

**Proposition 3.6.** If $G$ is $W$-trivial over $k$, then $G(F)$ is boundedly simple for every field extension $F$ of $k$.

*Proof.* If $F^*$ is a sufficiently saturated elementary extension of $F$, then $G(F^*)$ is also sufficiently saturated and quasi-simple. Also $G(F^*)$ is elementary equivalent to $G(F)$. By Proposition 2.3 the group $G(F^*)$ is boundedly simple and hence $G(F)$ is boundedly simple. \qed

We refer the reader to [17 Theorem 5.9, Theorem 6.1] for some examples of $W$-trivial groups: almost simple, simply connected, isotropic over $k$ groups of type $B_n$, $C_n$ ($\text{SL}_n(F)$, $\text{Sp}_{2n}(F)$) and quasi-split groups (having Borel subgroup defined over $k$).

The concept of bounded simplicity is closely related to the extended covering number of a group [2]. Namely for a simple nonabelian group $G$, the covering number $\text{cn}(G)$ of $G$ is the smallest natural number $n$, such that $C^n = G$, whenever $C$ is noncentral conjugacy classes of $G$. If no such $N$ exists, we put $\text{cn}(G) = \infty$. If $\text{cn}(G) < \infty$, then clearly $G$ is $\text{cn}(G)$-boundedly simple. Unfortunately, Proposition 2.3 does not give any estimation for bounded simplicity. Nevertheless, for a class of Chevalley groups we have the following result.

**Theorem 3.7.** [16] There is a constant $d$ such that for an arbitrary quasi-simple Chevalley group $H$ the covering number $\text{cn}(H)$ is at most $d \cdot n$, where $n$ is the rank of $H$.

In the theorem above by a Chevalley group the authors mean a group generated by the root subgroups [35] (see also explanations in the Introduction). Thus, every Chevalley group is $dn$-boundedly simple, but one can also prove that there is no a
universal constant $d$ such that all Chevalley groups are $d$-boudedly simple. In Theorem 3.14 below we prove that the derived subgroup $[G(k), G(k)]$ of semisimple and $k$-split group is definable and 12-absolute connected, hence also Chevalley groups have this property.

Throughout this section unless otherwise is stated we use the following notation ([4, 18.6, §20, §21]):

- $G$ is a connected reductive $k$-split $k$-group,
- $T$ is a maximal $k$-split torus in $G$,
- $\Sigma = \Sigma(T, G)$ is the root system of relative $k$-roots of $G$ with respect to $T$ (each $\alpha \in \Sigma$ is a homomorphism $\alpha : T \to k^\times$); in fact $\Sigma$ can be regarded as a root system in $\mathbb{R}^n$, satisfying the crystallographic condition: $<\alpha, \beta> = 2\frac{[\alpha, \beta]}{[\beta, \beta]} \in \mathbb{Z}$, for all $\alpha, \beta \in \Sigma$, where $(,)$ is the usual scalar product on $\mathbb{R}^n$ [4, 14.6, 14.7],
- $\Pi \subset \Sigma$ is the simple root system generating $\Sigma$; that is every root $\alpha \in \Sigma$ can be written as a linear combination of roots from $\Pi$, where all non-zero coefficients are positive integers or all are negative integers, $U_\alpha$ for $\alpha \in \Sigma$, is the $k$-root group of $G$ corresponding to $\alpha \in \Sigma$,
- $\Sigma^+$ and $\Sigma^-$ are the sets of all positive and negative $k$-roots from $\Sigma$,
- $U$ (respectively $U^-$) is the group generated by all $U_\alpha$ for $\alpha \in \Sigma^+$ (respectively $\alpha \in \Sigma^-$).
- $B$ is the product of $T$ and $U^+$.

The group $B$ is a Borel subgroup of $G$ and $U$ is the unipotent radical of $B$.

If $\alpha = \sum_{\beta \in \Pi} k_\beta \beta$ is the representation of $\alpha \in \Sigma$ with respect to $\Pi$, then the height of $\alpha$ is $ht(\alpha) = \sum_{\beta \in \Pi} k_\beta$.

For each $\alpha \in \Sigma$ the group $U_\alpha(k)$ is a connected and unipotent subgroup, normalised by $T(k)$. In particular [4, 18.6], there is an isomorphism $x_\alpha : k \to U_\alpha(k)$ such that for $s \in k$ and $t \in T(k)$,

$$(*) \quad tx_\alpha(s)t^{-1} = x_\alpha(\alpha(t)s).$$

More precisely, for $u \in k^\times$, $\alpha \in \Sigma$, define

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u) \quad \text{and} \quad t_\alpha(u) = w_\alpha(u)w_\alpha(1)^{-1}.$$ 

Furthermore, if $G$ is simply connected, then [33] Lemma 20(c), p. 29] the torus $T(k) = \langle t_\alpha(u) : u \in k^\times, \alpha \in \Pi \rangle$ and the action of $t_\alpha(u)$ on the $k$-root subgroup $U_\beta$ is given by the formula

$$(**) \quad t_\alpha(u)x_\beta(s)t_\alpha(u)^{-1} = x_\beta(u^{<\beta, \alpha> s}),$$

where $< \beta, \alpha> = 2\frac{[\beta, \alpha]}{[\alpha, \alpha]} \in \mathbb{Z}$.

If we fix an ordering on $\Sigma^+$, then for every element $x \in U$ there is a unique tuple $(s_\alpha)_{\alpha \in \Sigma^+} \in k^{|\Sigma^+|}$, such that $x$ can be uniquely written in the form

$$(***) \quad x = \prod_{\alpha \in \Sigma^+} x_\alpha(s_\alpha),$$

where the product is taken in a fixed order. The analogous fact is true for negative roots $\Sigma^-$ and $U^-$. The next definition is well known for linear algebraic groups [4, 12.2].
Definition 3.8. An element \( t \in T(k) \) is called regular if
\[
C_{G(k)}(t) \cap U(k) = \{ e \},
\]
where \( C_{G(k)}(t) \) is the centralizer of \( t \) in \( G(k) \). Equivalently (by \( \textsc{iii} \) and \( \textsc{iii} \)) \( t \) is regular if and only if for every root \( \beta \in \Sigma^+ \), \( \beta(t) \neq 1 \).

Proposition 3.10 below is a variant of [14, Proposition]. In the proof we use the Gauss decomposition in Chevalley groups.

Theorem 3.9. [11] Gauss decomposition with prescribed semisimple part, Theorem 2.1.]
Suppose \( k \) is an arbitrary infinite field and \( G \) is an absolutely almost simple and simply connected \( k \)-split \( k \)-group. Then for every noncentral \( g \in G(k) \) and \( t \in T(k) \) there exist \( v \in U^-(k), u \in U(k) \) and \( x \in G(k) \) such that \( g^x = v \cdot t \cdot u \).

Proposition 3.10. Suppose that \( G \) is an absolutely almost simple and simply connected \( k \)-split \( k \)-group. Then for every regular element \( t \in T(k) \), we have \( G(k) = (t^{G(k)})^3 \).

Proof. Let \( C = t^{G(k)} \). Consider the following functions
- \( \varphi : U(k) \to U(k), \varphi(u) = [t, u] = t^{-1}u^{-1}tu \),
- \( \psi : U^-(k) \to U^-(k), \psi(v) = [v, t^{-1}] = v^{-1}tvt^{-1} \).

The functions \( \varphi \) and \( \psi \) are well defined, because \( U(k) \) and \( U^-(k) \) are normal subgroups of \( B(k) \). Since \( t \) is regular, \( \varphi \) and \( \psi \) are injective. In fact, \( \varphi \) and \( \psi \) are bijections. We prove that \( \varphi \) is surjective — the argument for \( \psi \) is similar. Consider the central series \( U(k) = U_1(k) \supset U_2(k) \supset \ldots \supset U_m(k) = \{ e \} \) for \( U(k) \), where \( U_i(k) = \langle x_\alpha(s) : \alpha \in \Sigma^+, \mathrm{ht}(\alpha) \geq i, s \in k \rangle \). By the Chevalley commutator formula [11, 14.5, Remark(2)], we have
\[
U_i(k)/U_{i+1}(k) \subseteq Z(U(k)/U_{i+1}(k)).
\]
Each factor \( U_i(k)/U_{i+1}(k) \) is not only an abelian group, but even a finite dimensional vector space over \( k \). Namely, by \( \textsc{iii} \),
\[
U_i(k)/U_{i+1}(k) \cong \left\{ \sum_{\mathrm{ht}(\alpha) = i} x_\alpha(s_\alpha) : s_\alpha \in k \right\}.
\]
By \( \textsc{ii} \), \( \varphi[U_i(k)] \subseteq U_i(k) \); hence \( \varphi \) induces a \( k \)-linear transformation
\[
\varphi_i : U_i(k)/U_{i+1}(k) \to U_i(k)/U_{i+1}(k),
\]
with the matrix \( \text{diag}(1 - \alpha(t)) : \alpha \in \Sigma, \mathrm{ht}(\alpha) = i \). Since \( t \) is regular, each \( \varphi_i \) is a bijection. Now, using \( \textit{(i)} \) one can easily prove, by induction on \( 1 \leq i \leq m-1 \), that for every \( u \in U(k) \), there is \( u_i \) such that \( u \equiv [t, u_i] \mod U_{i+1} \). Thus \( \varphi \) is surjective.

Claim. \( C^2 \supseteq G(k) \setminus Z(G(k)) \)

Proof. Take an arbitrary \( g \in G(k) \setminus Z(G(k)) \). By [11, Theorem 2.1]
\[
g^G \cap U^-(k) \cdot t^2 \cdot U(k) \neq \emptyset.
\]
Since \( \varphi \) and \( \psi \) are surjective, for some \( g' \in g^G \) there are \( v \in U^-(k), u \in U(k) \) satisfying \( g' = \psi(v)t^2\varphi(u) = v^{-1}tvu^{-1}tu \in C^2 \).

Our conclusion follows from the claim, because if \( x \notin C^2 \cdot C \), then \( xC^{-1} \subseteq G(k) \setminus C^2 \subseteq Z(G(k)) \); this is impossible since \( Z(G(k)) \) is finite and \( C \) is infinite.
Simply connected split groups are absolutely connected.

**Theorem 3.11.** Let $k$ be an arbitrary infinite field and $G$ be a $k$-split, semisimple, simply connected, $k$-group. Then $G(k)$ is 3-weakly simple and 12-absolutely connected.

**Proof.** Without loss of generality we may assume that $G$ is absolutely almost simple. Indeed, by [4, 22.10], [37, 3.1.2], the group $G$ is a direct product over $k$

$$G_1 \times \ldots \times G_n,$$

where each $G_i$ is $k$-split, semisimple, simply connected and almost $k$-simple. Thus $G(k) \cong \prod_{1 \leq i \leq n} G_i(k)$. By Lemma 3.5 every $G_i$ is absolutely almost simple, so by Lemma 2.12(1), it is enough to prove that $G_i(k)$ is 3-weakly simple.

By Proposition 3.10 and Theorem 2.15 in order to prove the conclusion of the theorem, one needs to show that the set of non-regular elements in $T(k)$ is non-thick (Section 1.4). By Definition 3.8, it is enough to find, for each natural $m$, a sequence $(t_i)_{1 \leq i \leq m}$ in $T(k)$ such that $\beta(t_i^{-1} t_j) \neq 1$ for $i < j < m$ and $\beta \in \Sigma$.

By (1) and (2), $\beta(t_a(s)) = s^{<\beta, \alpha>}$. Recall that $\Pi \subset \Sigma$ is a simple root system. Fix some sequence $(\lambda_a)_{a \in \Pi} \subset \mathbb{Z}$ and for $s \in k^\times$, define $a(s) = \prod_{a \in \Pi} t_a(s^{\lambda_a})$. Then for $i < j < m$,

$$\beta\left(a(s^i)^{-1}a(s^j)\right) = s^{(j-i)\sum_{a \in \Pi} \lambda_a <\beta, \alpha>}.$$

Fix a natural number $m$. Since the Cartan matrix $(<\alpha, \beta>)_{a, \beta \in \Pi}$ of the irreducible root system $\Sigma$ is nondegenerate (see [9, Section 3.6]), one can easily find a suitable sequence $(\lambda_a)_{a \in \Pi}$ of integers such that, for every simple root $\beta \in \Pi$, the sum $\sum_{a \in \Pi} \lambda_a <\beta, \alpha>$ is positive. Every root $\beta$ is a $\mathbb{Z}$-linear combination of simple roots with all coefficients positive, or all negative. Moreover $<\cdot, \cdot>$ is additive in the first coordinate. Hence for all $\beta \in \Sigma$,

$$\sum_{a \in \Pi} \lambda_a <\beta, \alpha> \neq 0.$$ 

The field $k$ is infinite, so there is $t \in k^\times$ such that $\beta\left(a(s^i)^{-1}a(s^j)\right) \neq 1$ for every $i < j < m$. Thus $(t_i)_{1 \leq i \leq m} = (a(s^i))_{1 \leq i \leq m}$ satisfies our requirements.

Now we consider the case when $G$ is $k$-split but not necessarily simply connected. Our goal is Theorem 3.14 below, which is a more general version of Theorem 3.11.

Every semisimple $k$-group $G$ has a universal $k$-covering $\pi: \tilde{G} \to G$ defined over $k$ ([31, Proposition 2.10], [37, 2.6.1]); that is $\tilde{G}$ is a simply connected $k$-group and $\pi$ is a central isogeny defined over $k$.

**Definition 3.12.** Suppose $G$ is a $k$-isotropic $k$-group. By $G(k)^+$ we denote the canonical normal subgroup of $G(k)$ (denoted in [36] as $G_0(k)$) generated by the $k$-rational points of the unipotent radicals of parabolic subgroups of $G$ defined over $k$

$$G(k)^+ = \langle R_u(P)(k) : P \text{ is a parabolic subgroup defined over } k \rangle.$$

Suppose that $G$ is $k$-isotropic and almost $k$-simple reductive $k$-group. Tits in [36] proved that every noncentral subgroup of $G(k)$ normalized by $G(k)^+$ contains $G(k)^+$. For a simply connected $G$ there is the Kneser-Tits conjecture which asks whether $G(k)^+ = G(k)$ [38, 17].

**Lemma 3.13.** [7, 6.5, 6.6] Let $G$ be a semisimple $k$-split $k$-groups and $\pi: \tilde{G} \to G$ a universal $k$-covering of $G$. Then
\((1)\) \(\tilde{G}(k)^+ = \tilde{G}(k)\), \\
\((2)\) \(\pi\left(\tilde{G}(k)\right) = G(k)^+\) and \(G(k)^+\) is the derived subgroup \([G(k), G(k)]\) of \(G(k)\).

**Proof.** (1) is precisely the Kneser-Tits conjecture for split groups. The validity of this conjecture (even for a more general class of quasi-split groups) is established, for example, in [35, Lemma 64, p. 183]. 

(2) follows by (1) and [7, 6.5]. \(\square\)

**Theorem 3.14.** Let \(k\) be an arbitrary infinite field and \(G\) be a \(k\)-split, semisimple \(k\)-group. Then the derived subgroup \([G(k), G(k)]\) is 3-weakly simple and 12-absolutely connected. Moreover \([G(k), G(k)]\) is a definable over \(\emptyset\) subgroup of \(G(k)\) in the pure group language.

**Proof.** Let \(\pi: \tilde{G} \to G\) be a universal \(k\)-covering of \(G\). The group \(\tilde{G}\) is simply connected, so by Theorem 3.11 \(\tilde{G}(k)\) is 3-weakly simple. By Lemmas 3.13 and 2.12 (1) \(\pi\left(\tilde{G}(k)\right) = G(k)^+ = [G(k), G(k)]\) is 3-weakly simple and 12-absolutely connected. By 3-weak simplicity of \((G(k))^+\) one can find \(g \in G_3(G(k)^+\) (see Definition 2.11). Therefore, since \((G(k))^+\) is a normal subgroup of \(G(k)\), it is definable over \(g\). However, \([G(k), G(k)]\) is \(\emptyset\)-invariant, so is definable over \(\emptyset\). \(\square\)

Theorem 3.14 and Proposition 2.20 imply the following corollary.

**Corollary 3.15.** Suppose \(G\) is a \(k\)-split, semisimple \(k\)-group and \(G(k)\) is equipped with some first order structure. Denote by \(A\) the quotient 
\[G(k)/[G(k), G(k)]\]
with the natural quotient structure (see Proposition 2.20) and by \(G^*\) and \(A^*\) — sufficiently saturated extensions of \(G(k)\) and \(A\). Then 
\[G^*/G^*\emptyset_0 \cong A^*/A^*\emptyset_0, \text{ for } x \in \{\infty, 00, 0\} .\]

In particular \(G^*/G^*\emptyset_0\) is abelian.

For semisimple split groups model theoretic components can be described in terms of components of the abelianization. Not much is known in general about components of abelian groups. In a forthcoming paper [8] we consider the case of finitely generated abelian groups.

**Remark 3.16.** If \(G\) is a reductive \(k\)-split \(k\)-group, then \([G, G]\) is a \(k\)-split semisimple \(k\)-group [4, 2.3, 14.2]. Hence the group \([[G, G] (k)\), [G, G] (k)]\) is a 3-weakly simple subgroup of \(G(k)\). We have the following sequence of subgroups 
\[G_0 = [[G, G] (k), [G, G] (k)] \subseteq [G, G] (k) \subseteq G(k) .\]
If \([G, G](k)\) is definable in \(G(k)\), then by Theorem 3.14 the group \(G_0\) is 3-weakly simple and a definable subgroup of \(G(k)\). Therefore the conclusion of Corollary 3.15 in this case is true, where as \(A\) we take the group \(G(k)/[[G, G] (k), [G, G] (k)]\). However \(A\) is not necessarily abelian.

Let \(\text{SL}_\infty(k)\) be the union (the direct limit) of groups \(\text{SL}_n(k)\), where for \(n < m\), \(\text{SL}_n(k)\) is embedded in \(\text{SL}_m(k)\) in a natural way, that is \(A \mapsto \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\). By Proposition 2.10(3), \(\text{SL}_\infty(k)\) is 12-absolutely connected. Since a union of weakly simple groups is not
necessarily weakly simple (for example $\text{Sym}^\kappa(\Omega)$, for limit $\kappa$), we cannot easily deduce that $\text{SL}_\infty(k)$ is weakly simple.

**Problem 3.17.** Is $\text{SL}_\infty(k)$ weakly simple?

Theorem 3.14 settles the issue of absolute connectedness of groups of rational points of split semisimple groups. The problem of determining which linear $k$-groups $G(k)$ are absolutely connected, where $G$ is an arbitrary connected group and $k$ is an arbitrary infinite field, seems to be difficult. Therefore, consider the case when $k = K$ is an algebraically closed field and $G$ is a connected $K$-group. Every absolutely connected group is perfect (Theorem 2.16), so if $G(K)$ is absolute connected, then $G(K)$ must be perfect. This is also a sufficient condition for absolute connectedness.

**Proposition 3.18.** Let $G$ be a connected linear algebraic group defined over an algebraically closed field $K$. Then the following conditions are equivalent:

(a) $G(K)$ is weakly simple,
(b) $G(K)$ is absolutely connected,
(c) $G(K)$ is perfect.

More precisely, if $\text{cw}(G(K)) = R$ and the solvable radical $\mathcal{R}(G(K))$ is of derived length $M$, then $G(K)$ is $3(4R)^M$-weakly simple.

**Proof.** (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) follow from Theorem 2.15 and Theorem 2.16. For (c) $\Rightarrow$ (a), note that $G/\mathcal{R}(G)$ is a semi-simple linear algebraic group. Since $G(K)$ is perfect, group $G(K)$ has a commutator width $R$ at most $2\dim(G)$ [4, 2.2]. By Lemma 2.12(2b), $G$ is $3(4R)^M$-weakly simple. $\square$

Since perfect reductive group is semisimple, the content of the proposition is in the non-reductive case. We do not know if there is a universal bound for weak simplicity of such groups, as there is for semisimple groups.

**Example 3.19.** Consider a semidirect product of unipotent and perfect group $G = K^n \rtimes \text{SL}_n(K)$ with the multiplication given by $(\overline{v}, f) \ast (\overline{w}, g) = (\overline{v} + f(\overline{w}), fg)$. The group $G$ is non-reductive, because its unipotent radical $\mathcal{R}_u(G)$ contains $K^n$. By [15], $\text{cw}(\text{SL}_n(K)) = 1$, so one can show that $G$ is perfect with $\text{cw}(G) = 1$. Indeed

$$[(\overline{v}, f), (\overline{w}, g)] = \left(f^{-1}(g^{-1}(\overline{v} + f(\overline{w}) - \overline{v}) - \overline{v}), f^{-1}g^{-1}fg \right).$$

Every element of $\text{SL}_n(K)$ can be written as a commutator $[f, g]$, where $f - \text{id}$ is invertible, that is $1$ is not an eigenvalue of $f$. Therefore every element of $G$ is a commutator. By Lemma 2.12(2), $G$ is 12-weakly simple.

4. **Diameters of Lascar strong types, $G^{00}$ and $G^\infty$**

In this section we make a link between the notion of (definable) absolute connectedness, diameters of Lascar strong types, and $G$-compactness. Examples of groups $G$ where $G^\infty \neq G^{00}$ are also considered.

We use the notation from Section 1.3. Suppose $M$ is a sufficiently saturated model (that is, $\kappa$-saturated and $\kappa$-strongly homogeneous) and $A \subset M$ is a small subset. The group of Lascar strong automorphisms over $A$ is defined by [25]:

$$\text{Aut}_L(M/A) = \langle \{\text{Aut}(M/M') : M' \prec M \text{ is a small submodel containing } A \} \rangle.$$
Let $k < \kappa$ be some ordinal. We say that (possibly infinite) tuples $\bar{\pi}, \bar{b} \in M^k$ have the same Lascar strong type, and write $E_{L/A}(\bar{\pi}, \bar{b})$, if there exists $f \in Aut_{L}(M/A)$ such that $\bar{\pi} = f(\bar{b})$. The relation $E_{L/A}$ is the transitive closure of $\Theta_{A}(\bar{\pi}, \bar{\nu})$ (Section 1.3), so is an $A$-invariant and bounded equivalence relation on $M^k$. If $E_{L/A}(\bar{\pi}, \bar{b})$ holds, we say that $\bar{\pi}$ and $\bar{b}$ are at distance $n$ [29, Section 1.] and write $d(\bar{\pi}, \bar{b}) = n$, if $n$ is the minimal natural number such that for some $\bar{\pi}_0 = \bar{\pi}, \bar{\pi}_1, \ldots, \bar{\pi}_n = \bar{b}$, we have $\Theta_{A}(\bar{\pi}_i, \bar{\pi}_{i+1})$ for $0 \leq i \leq n - 1$. By the Lascar strong type of $\bar{\pi}$ over $A$, denoted $stp_{L}(\bar{\pi}/A)$, we mean the orbit of $\bar{\pi}$ under $Aut_{L}(M/A)$. The diameter $diam(stp_{L}(\bar{\pi}/A))$ of $stp_{L}(\bar{\pi}/A)$ is defined to be the supremum of $d(\bar{\pi}, \bar{b})$, for $E_{L/A}(\bar{\pi}, \bar{b})$, or the sign $\infty$.

The relation $E_{L/A}$ is the finest bounded $A$-invariant equivalence relation on $M^k$. There exists also the finest bounded $\bigwedge$-definable over $A$ equivalence relation on $M^k$, denoted by $E_{KP/A}$ and known as equality of Kim-Pillay strong types over $A$. Namely, let $E_{KP/A}$ be the intersection of all bounded $\bigwedge$-definable over $A$ equivalence relations on $M^k$. Every $\bigwedge$-definable over $A$ equivalence relation is $A$-invariant, so $E_{L/A} \subseteq E_{KP/A}$. There exists an appropriate group of automorphisms $Aut_{KP}(\mathcal{C}/A) \triangleleft Aut(\mathcal{C}/A)$ such that $E_{KP/A}(\bar{\pi}, \bar{b})$ holds if and only if for some $f \in Aut_{KP}(M/A)$, $\bar{\pi} = f(\bar{b})$. By the Kim-Pillay strong type or KP-type of $\bar{\pi}$ over $A$ we mean $stp_{KP}(\bar{\pi}/A)$, the orbit of $\bar{\pi}$ under $Aut_{KP}(\mathcal{C}/A)$. Every KP-type over $A$ is of course type definable over $A$. If the diameter $diam(stp_{L}(\bar{\pi}/A))$ is finite, then obviously $stp_{L}(\bar{\pi}/A) = stp_{KP}(\bar{\pi}/A)$. Newelski in [29, Corollary 1.8, 1.9] proved the following.

**Proposition 4.1.**

(1) If $stp_{L}(\bar{\pi}/A) = stp_{KP}(\bar{\pi}/A)$ then $diam(stp_{L}(\bar{\pi}/A))$ is finite.

(2) If for each natural $n$ there exists in $M$ a Lascar strong type of diameter at least $n$, then there exists in $M$ a Lascar strong type of infinite diameter, which is not $\bigwedge$-definable.

**Remark 4.2.** The notion of $N$-absolute connectedness has a close relationship with the diameters of the Lascar strong types (see [29, 21]). Namely, for an infinite group $G$ with extra structure, consider the following 2-sorted structure $\mathcal{G} = (G, X, \cdot)$, where $\cdot : G \times X \to X$ is a regular action of $G$ on $X$, and $X$ is a predicate (on $G$ we take its original structure) [21, Section 3]. In other words $X$ is affine copy of the group $G$. Suppose $G^*$ is a sufficiently saturated extension of $\mathcal{G}$. The group $G^*$ acts by automorphisms on $X^*$ [21, Section 3.]. By [21, Lemma 3.7] we have the following correspondence between strong types on sort $X^*$ and components of $G^*$. That is for $x, y \in X^*$

- $E_{KP/\emptyset}(x, y)$ holds if and only if $x = g(y)$ for some $g \in G^{*00}$,
- $E_{L/\emptyset}(x, y)$ holds if and only if $x = g(y)$ for some $g \in G^{*0}$,
- $\Theta_{\emptyset}(x, y)$ holds if and only if $x = g(y)$ for some $g \in X_{\Theta_{\emptyset}}$,

where $X_{\Theta_{\emptyset}} = \{a^{-1}b : a, b \in G^*, \Theta_{\emptyset}(a, b)\}$ (see Section 1.4). Therefore, the group $G$ is $N$-definably absolutely connected if and only if the following two conditions hold:

- every two elements of $X^*$ have the same Lascar strong type,
- the diameter of $X^*$ is at most $N$.

By Lemma 1.2 Proposition 4.1(1) and the last remark we have the following observation.

**Remark 4.3.** Suppose $G$ is a sufficiently saturated group equipped with some first order structure, and $A \subseteq G$ is small. Then $G^{00}_{A} = G_A^{\infty}$ if and only if there exists $N$ such that $G^{00}_{A} = X_{\Theta_{A}}^{N}$.
By the downward Löwenheim–Skolem Theorem we may assume that $A = \emptyset$ and that $G$ is sufficiently saturated. If $G^{00}_A = G^\infty_A$, then $X$ is simultaneously the Lascar strong type and the Kim–Pillay strong type. Thus by Proposition 4.11, $\text{diam}(X) = N$ is finite, so $G^{00}_0 = X^N_{\emptyset}$.

Hrushovski and Pillay proved $G^{00} = G^\infty$ for any definably amenable group $G$ whose ambient structure has NIP ([23], see also [22], with notation $G^{000}$ for $G^\infty$). For certain groups the coefficient $N$ from Remark 4.3 is known. For example, for groups with simple theory ([21, Proposition 4.2]) and for definably amenable groups with NIP ([23] and [18, Proposition 5.4]), $X^N_{\emptyset}$ is already a group. For all groups in Section 3 (possibly except some connected perfect linear groups from Proposition 3.18), $X^{128}_{\emptyset}$ is a group.

We prove an analogue of Proposition 4.1(2) in a group setting.

**Proposition 4.4.**  (1) Let $N$ be a natural number. Suppose that for a sufficiently saturated group $G$ equipped with some first order structure, the set $X^N_{\emptyset}$ is not a group. Then

(a) there exists a definable and thick $P \subseteq G$ such that in the reduct $(G, \cdot, P)$ of $G$ (where $P$ is a predicate), the set $X^N_{\emptyset}$ is not a group,

(b) there exists a thick subset $P'$ of the free group on countably many generators $\mathbb{F}_\omega$, such that in the structure $(\mathbb{F}_\omega, \cdot, P')$, the set $X^N_{\emptyset}$ is not a group.

(2) Assume that for every natural $N$ there exists a sufficiently saturated group $G$ such that $X^N_{\emptyset}$ is not a group. Then, there exist a family of countably many thick subsets $(P_N)_{N \in \mathbb{N}}$ of the group $\mathbb{F}_\omega$, such that in a sufficiently saturated extension $G^*$ of the structure $G = (\mathbb{F}_\omega, \cdot, P_N)_{N \in \mathbb{N}}$, the component $G^*_0$ is not $\wedge$-definable. Thus $G^{00}_0 \neq G^*_0$.

**Proof.** (1) Note that $X^N_{\emptyset}$ is a group if and only if $X^{N+1}_{\emptyset} = X^N_{\emptyset}$. Hence, from the assumption we have $(g, i)_{i \leq N}$ in $X_{\emptyset}$ such that $g = g_0 \cdot \ldots \cdot g_N \notin X^N_{\emptyset}$.

(a) By Lemma 1.4(1) we have $g \notin P^N$ for some $\emptyset$-definable thick subset $P$ of $G$. In the reduct $(G, \cdot, P)$ of $G$, the set $X_{\emptyset}$ contains $(g, i)_{i \leq N}$ too, but also $g \notin P^N \supseteq X^N_{\emptyset}$. Therefore the set $X^N_{\emptyset}$ is not a group.

(b) The fact that $X^N_{\emptyset}$ is not a group is witnessed by the existence of suitable indiscernible sequences. Namely, there are order indiscernible sequences $(g_{0, i})_{i \in \mathbb{N}}, (g_{1, i})_{i \in \mathbb{N}}, \ldots, (g_{N, i})_{i \in \mathbb{N}}$ in $G$ such that (Section 1.3)

$$g_k = g_{k, 0}^{-1} \cdot g_{k, 1}, \text{ for } k = 0, \ldots, N \text{ and } g_0 \cdot \ldots \cdot g_N \notin X^N_{\emptyset}.$$

By the downward Löwenheim–Skolem Theorem we may assume that $G$ is countable. Also, $G$ contains the aforementioned indiscernible sequences, and there are automorphisms of $G$ witnessing order indiscernibility of these sequences. Consider the free group $\mathbb{F}_\omega = \{ (x_g : g \in G), \cdot \}$ and the natural projection $\pi : \mathbb{F}_\omega \to G$, $\pi(x_g) = g$. Define $P' = \pi^{-1}[P]$, where $P$ is from (a). Every $f \in \text{Aut}(G, \cdot, P)$ lifts to an $F \in \text{Aut}(F_0)$, by the rule $F(x_g) = x_{f(g)}$; we have $\pi \circ F = f \circ \pi$, so $F$ preserves $P'$. Whence in the structure

$$F_0 = (\mathbb{F}_\omega, \cdot, P')$$
the set $P'$ is thick (by Lemma 4.3(2)) and the sequences $(x_{g,i})_{i \in \mathbb{N}},$ $j = 0, \ldots, N$ are indiscernible too. Thus each $x_{g,0}^{-1} x_{g,1}$ belongs to $X_{\Theta}$ of the structure $F_0$. Since

$$\prod_{k \leq N} x_{g,0}^{-1} x_{g,1} \in X_{\Theta} \setminus P_N \subseteq X_{\Theta} \setminus X_{\Theta},$$

the set $X_{\Theta}$ is not a group.

(2) By (1), for each natural number $N$ we have

(1) a thick subset $P_N' \subseteq F_{\omega}$,

(2) the collection of indiscernible sequences $(g_{N,0,i})_{i \in \mathbb{N}}, (g_{N,1,i})_{i \in \mathbb{N}}, \ldots, (g_{N,N,i})_{i \in \mathbb{N}}$ of $(F_{\omega}, \cdots, P_N')$, such that $\prod_{0 \leq k \leq N} g_{N,k,0}^{-1} g_{N,k,1} \not\in P_N'$.

Let $(F_{\omega}, P_N)_{N \in \mathbb{N}}$ be the product of structures $(F_{\omega}, \cdots, P_N')$, that is, define $P_N = \prod_{i \in \mathbb{N}} P_{i,N} \subseteq F_{\omega}$, where $P_{i,N} = F_{\omega}$ if $i = N$ and $P_{i,N} = F_{\omega}$ otherwise. It is easy to see that the $P_N$'s are thick in the product $F_{\omega}$. Consider the structure $(F_{\omega}, \cdots, P_N)_{N \in \mathbb{N}}$ and its sufficiently saturated extension $G'$. Note that the sequences from (2) are still indiscernible after embedding into $F_{\omega}$ (by the natural embedding), hence they are indiscernible in $G'$. Therefore, we have that, for each $N$, the set $X_{\Theta}$ of $G'$ is not a group. Hence, by Remark 4.3, $G'' \not\cong G''$.

If $G$ is an absolutely connected but not $N$-absolutely connected group, then $G$ satisfies the condition from (1) of the previous proposition. Therefore, in order to obtain a group $G$ with $G'' \not\cong G''$ it is enough to find a sequence of absolutely connected groups $G_n$, $n \in \mathbb{N}$, such that $G_n$ is not $n$-absolutely connected. The existence of such a sequence, using finite covers of $SL_2(\mathbb{R})$, has been recently observed by Krupiński and Pillay (unpublished). Using Proposition 2.13 we also construct such a sequence below. Fix a natural number $n$. By Proposition 2.13 it is enough to find an absolutely connected group $H$ with some nontrivial cocycle $h: H \times H \to \mathbb{Z}$ with finite image $\text{Im}(\tilde{h})$. Indeed, let $h$ be such a cocycle. For a prime number $p$ take the composition

$$h_p = \text{mod } p \circ h: G \times G \to \mathbb{Z}/p\mathbb{Z}.$$

If $p$ is big enough, then the central extension $G' = \mathbb{Z}/p\mathbb{Z} \times_{h_p} G$ of $G$ corresponding to the cocycle $h_p$ is absolutely connected but not $n$-absolutely connected.

A suitable candidate for $G$ might be a Lie group with torsion-free fundamental group $\pi_1(G)$. For example take $G = SL_2(\mathbb{R})$ with the Petersson’s cocycle of the standard section [1] [30]

$$h: SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \to \{ -1, 0, 1 \} \subseteq \mathbb{Z}$$

that gives the topological universal cover of $SL_2(\mathbb{R})$ (since $\pi_1(SL_2(\mathbb{R})) = \mathbb{Z}$, $h$ has values in $\mathbb{Z}$). Note that by [12], the topological universal cover $\tilde{G}$ of $SL_2(\mathbb{R})$ satisfies $\tilde{G}'' \not\cong \tilde{G}''$.

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References

[1] Asai, T.: The reciprocity of Dedekind sums and the factor set for the universal covering group of $SL(2, \mathbb{R})$. Nagoya Math. J. 37, 67 — 80 (1970)
[2] Arad, Z., Herzog M.: Products of conjugacy classes in groups. Lecture Notes in Mathematics, 1112. Springer-Verlag, Berlin (1985)
[3] Bertram, E. A.: On a theorem of Schreier and Ulam for countable permutations. J. Algebra 24, 316 -- 322 (1973)
[4] Borel, A.: Linear algebraic groups. Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, (1991)
[5] Borel, A.: Properties and linear representations of Chevalley groups. In: Borel A., Carter R. W., Curtis C. W., Iwahori N., Springer T. A., Steinberg R. (eds.) Seminar on Algebraic Groups and Related Finite Groups The Institute for Advanced Study, Princeton, N.J., 1968/69, pp. 1 — 55. Lecture Notes in Mathematics, Vol. 131 Springer, Berlin
[6] Borel, A., Tits, J.: Groupes réductifs. Inst. Hautes Études Sci. Publ. Math. No. 27, 55 – 150 (1965)
[7] Borel, A., Tits, J.: Homomorphismes “abstraits” de groupes algébriques simples. Ann. of Math. 97, 499 -- 571 (1973)
[8] Bowler, N., Chen, C., Gismatullin, J.: Model theoretic connected components of finitely generated nilpotent groups., preprint 2011
[9] Carter, R.: Simple groups of Lie type. Reprint of the 1972 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, (1989)
[10] Casanovas, E., Lascar, D., Pillay, A., Ziegler M.: Galois groups of first order theories. J. Math. Log. 1, 305 — 319 (2001)
[11] Chernousov, V., Ellers, E., Gordeev, N.: Gauss decomposition with prescribed semisimple part: short proof. J. Algebra 229, 314 — 332 (2000)
[12] Conversano, A., Pillay, A.: Connected components of definable groups and o-minimality I., preprint 2011
[13] Droste, M., Göbel, R.: On a theorem of Baer, Schreier and Ulam for permutations. J. Algebra 58, 282 — 290 (1979)
[14] Ellers, E., Gordeev, N.: Gauss decomposition with prescribed semisimple part in classical Chevalley groups. Comm. Algebra 22, 5935 — 5950 (1994)
[15] Ellers, E., Gordeev, N.: On the conjectures of J. Thompson and O. Ore. Trans. Amer. Math. Soc. 350, 3657 — 3671 (1998)
[16] Ellers, E., Gordeev, N., Herzog, M.: Covering numbers for Chevalley groups. Israel J. Math. 111, 339 — 372 (1999)
[17] Gille, P.: Le problème de Kneser-Tits. Séminaire Bourbaki. Vol. 2007/2008. Astérisque No. 326 (2009), Exp. No. 983, vii, 39 -- 81 (2010)
[18] Gismatullin, J.: Model theoretic connected components of groups., Israel J. Math. 184, 251 — 274 (2011)
[19] Gismatullin, J.: Bounded simplicity of isotropic groups. preprint 2011
[20] Gismatullin, J., Muranov, A.: A remark on groups without finite images. note 2009, arXiv:0903.2536
[21] Gismatullin, J., Newelski, L.: G-compactness and groups. Arch. Math. Logic. 47, 479 — 501 (2008)
[22] Hrushovski, E., Peterzil, Y., Pillay, A.: Groups, measures, and the NIP. J. Amer. Math. Soc. 21, 563 — 596 (2008)
[23] Hrushovski, E., Pillay, A.: On NIP and invariant measures. J. of the European Mathematical Society, 13, 1005 – 1061 (2011)
[24] Kargapolov, M. I., Merzljakov, Ju. I.: Fundamentals of group theory. Third edition. “Nauka”, Moscow (1982)
[25] Lascar, D.: The group of automorphisms of the field of complex numbers leaving fixed the algebraic numbers is simple. In: Evans D. M. (ed.) Model theory of groups and automorphsim groups. Blaubeuren, 1995, London Math. Soc. Lecture Note Ser., 244 pp 110 — 114. Cambridge Univ. Press, Cambridge (1997)
[26] Lascar, D.: Automorphism groups of saturated structures; a review Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 25 — 33, Higher Ed. Press, Beijing (2002)
[27] Marker, D.: *Model theory*, Graduate Texts in Mathematics, 217. Springer-Verlag, New York (2002)

[28] Moran, G.: *The product of two reflection classes of the symmetric group*. Discrete Math. 15, 63 – 77 (1976)

[29] Newelski, L.: *The diameter of a Lascar strong type*. Fundam. Math. 176, 157 — 170 (2003)

[30] Petersson, H.: *Zur analytischen Theorie der Grenzkreisgruppen I*. Math. Ann., 115, 23 — 67 (1938)

[31] Platonov, V. P., Rapinchuk A. S.: *Algebraic groups and number theory*. Pure and Applied Mathematics, 139. Academic Press, Inc., Boston, MA, (1994)

[32] Platonov, V. P.: *The Dieudonné conjecture, and the non-surjectivity of coverings of algebraic groups at k-points*. Dokl. Akad. Nauk SSSR 216 (1974), 986 – 989. English translation: Soviet Math. Dokl. 15 (1974), 927 — 931

[33] Shelah, S.: *Definable groups and 2-dependent theories*. preprint

[34] Springer, T. A.: *Linear algebraic groups*. Second edition. Progress in Mathematics, 9. Birkhäuser Boston, Inc., Boston, MA, (1998)

[35] Steinberg, R.: *Lectures on Chevalley groups*. Notes prepared by John Faulkner and Robert Wilson. Yale University, New Haven, Conn., (1968)

[36] Tits, J.: *Algebraic and abstract simple groups*. Ann. of Math. 80, 313 – 329 (1964)

[37] Tits, J.: *Classification of algebraic semisimple groups*. In: Borel, A., Mostow, G.D. (eds.) *Algebraic Groups and Discontinuous Subgroups*. Proc. Sympos. Pure Math., Boulder, Colo., 1965, pp. 33 – 62. Amer. Math. Soc., Providence, R.I., (1966)

[38] Tits, J.: *Groupes de Whitehead de groupes algébriques simples sur un corps (d’après V. P. Platonov et al.*)*. Séminaire Bourbaki, 29e année (1976/77), Exp. No. 505, pp. 218 – 236; Lecture Notes in Math., 677, Springer, Berlin, (1978)

[39] Tolstykh, V. A.: *Infinite-dimensional general linear groups are groups of finite width*. Siberian Math. J. 47, 950 — 954 (2006)

[40] Ziegler, M., *Introduction to the Lascar group*. In: Tent, K. (ed.) *Tits Buildings and the Model Theory of Groups*. London Math. Soc. Lecture Note Ser., 291 pp. 279 — 298. Cambridge Univ. Press, Cambridge (2002)

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