A remark on primitive cycles and Fourier-Radon transform

A. Beilinson

The aim of this note is to point out that Brylinski’s Radon transform \([B]\) is a natural instrument for the Green-Griffiths approach to Hodge conjecture \([GG]\), \([BFNP]\). In particular, some principal results of \([BFNP]\) follow from the general fact that Radon transform preserves primitive cohomology (while reversing its grading). As was noticed by Drinfeld, this assertion is immediate from the basic Fourier transform functoriality \([L]\).\(^1\)

This note originates from a talk given at a student Hodge theory seminar. I am grateful to V. Drinfeld for his enlightening comment, to D. Kazhdan for a discussion, and to M. Kerr and G. Pearlstein for an exchange of letters.

1. A reformulation of the Hodge conjecture. For a compact complex algebraic variety \(X\) let \(N_1 H^\cdot (X, \mathbb{Q})\) be the niveau filtration on its cohomology (it is Poincaré dual to more commonly used coniveau filtration; conjecturally, the two filtrations are complementary). Thus \(N_1 H^\cdot (X, \mathbb{Q})\) is the intersection of kernels of all restriction maps \(H^\cdot (X, \mathbb{Q}) \to H^\cdot (Y, \mathbb{Q})\), where \(Y \neq X\) is a closed algebraic subvariety of \(X\). According to Totaro and Thomas, see \([BFNP]\) th. 6.5, the Hodge conjecture amounts to the next assertion: For every projective smooth \(X\) of dimension \(2n\) the subspace of Hodge \((n, n)\)-classes in \(H^{2n} (X, \mathbb{Q})\) has zero intersection with \(N_1 H^{2n} (X, \mathbb{Q})\). Of course, it suffices to consider the subspace of primitive Hodge classes. Thus every description of \(N_1 H^{2n} (X, \mathbb{Q})^{\text{prim}}\) provides a reformulation of the Hodge conjecture. The articles \([GG]\) and \([BFNP]\) provide one such description; we present it in the last line of the note.

Remark. As was pointed out by the referee, Kerr and Pearlstein can treat similarly Grothendieck’s generalized Hodge conjecture.

Question. For \(\gamma\) in a given term of coniveau filtration, what can one say about simplest possible singularities of \(Y\) with \(\gamma|_Y \neq 0\)? (E.g., by Thomas, for algebraic \(\gamma\), i.e., for \(\gamma\) in the deepest term of coniveau filtration, the singularities are ODP.)

2. Radon transform \((B)\). We play with complex algebraic varieties and \(\mathbb{Q}\)-sheaves. An arbitrary ground field and \(\mathbb{Q}\ell\)-sheaves will do as well.
For an algebraic variety $Z$, we denote by $D(Z)$ the derived category of bounded constructible $\mathbb{Q}$-complexes on $Z$; let $\mathcal{M}(Z) \subset D(Z)$ be the category of perverse sheaves on $Z$, $\mathcal{P}: D(Z) \to \mathcal{M}(Z)$ the cohomology functor ([BBD]). For smooth $Z$ let $\mathcal{M}^{sm}(Z) \subset \mathcal{M}(Z)$ be the Serre subcategory of smooth perverse sheaves (i.e., local systems); it generates the thick subcategory $D^{sm}(Z) \subset D(Z)$ of complexes with smooth cohomology. The Verdier quotient $\tilde{D}(Z) := D(Z)/D^{sm}(Z)$ is a t-category with heart $\mathcal{M}(Z) := \mathcal{M}(Z)/\mathcal{M}^{sm}(Z)$. The latter is an Artinian $\mathbb{Q}$-category; the projection $\mathcal{M}(Z) \to \mathcal{M}(Z)$ identifies the subcategory of non-smooth irreducible perverse sheaves on $Z$ with that of irreducible objects in $\mathcal{M}(Z)$.

Let $V$ be a vector space of dimension $n \geq 2$, $V^\vee$ its dual. Let $P$, $P^\vee$ be the corresponding projective spaces, $i : T \to P \times P^\vee$ be the incidence correspondence. Let $p$, $p^\vee$ be the projections $P \times P^\vee \Rightarrow P$, $P^\vee$, and $p_i(T)$, $p_i^\vee(T)$ be their restrictions to $T$. The Radon transform functor $\mathcal{R} : D(P) \to D(P^\vee)$ is $\mathcal{R}(M) := p_i^\vee(T)^*p_i^*M[n - 2]$. Interchanging $P$ and $P^\vee$, we get $\mathcal{R}^\vee : D(P^\vee) \to D(P)$, etc. Notice that $\mathcal{R}$ sends $D^{sm}(P)$ to $D^{sm}(P^\vee)$, so we have $\mathcal{R} : D(P) \to D(P^\vee)$.

**Theorem ([B] 3.1).** The compositions $\tilde{D} R\tilde{D}^\vee$, $\tilde{D}^\vee \tilde{D}$ are Tate twist functors $M \mapsto M(2 - n)$. The functors $\tilde{D}$, $\tilde{D}^\vee$ are t-exact, hence they yield equivalences of the abelian categories $\mathcal{M}(P) \cong \mathcal{M}(P^\vee)$.

**3. Fourier transform ([B], [L]).** The formalism of constructible sheaves extends to algebraic stacks of finite type ([MB], [LO]). The group $G_m$ acts on any vector space by homotheties. Consider the quotient stacks $V := V/G_m$, $V^\vee := V^\vee/G_m$. The open embedding $j : V^\vee := V \setminus \{0\} \hookrightarrow V$ yields one $j : V \hookrightarrow V^\vee$, etc. The canonical pairing map $\mu : V \times V^\vee \to A^1$ yields $\mu : V \times V^\vee \to A^1$. Let $p_r$, $p_i^\vee : V \times V^\vee \Rightarrow V$, $V^\vee$ be the projections. One has the (homogenous) Fourier transform $F : D(V) \to D(V^\vee)$, $F(N) := pr_*^\vee(pr^*N \otimes \mu^*j_{A^1, *}(\mathbb{Q})[n - 1])$, see [L] 1.5, 1.9. Interchanging $V$ and $V^\vee$, we get $F^\vee : D(V^\vee) \to D(V)$.

**Theorem ([L] 3.1, 4.2).** The compositions $FF^\vee$, $F^\vee F$ are Tate twist functors $N \mapsto N(-n)$. The functors $F$, $F^\vee$ are t-exact, hence they yield equivalences of the abelian categories $\mathcal{M}(V) \cong \mathcal{M}(V^\vee)$.

Consider the closed embeddings $i_V : \{0\} \hookrightarrow V$, $i_V : B_{G_m} = \{0\}/G_m \hookrightarrow V$, etc. The projection $j_{A^1, *}(\mathbb{Q}) \to i_{A^1, *}(\mathbb{Q}(-1)[1])$ yields a natural morphism

\begin{equation}
 j^*_{i_V^V}F_{j_{i_V^V}} \to \mathcal{R}(-1),
\end{equation}

which becomes an isomorphism $j^*_{i_V^V}F_{j_{i_V^V}} \sim \mathcal{R}(-1)$ in $\tilde{D}(P^\vee)$ (see [L] 1.6). By [L] 1.8, one has a natural identification

\begin{equation}
 F_{i_V^V} \sim \pi_{B_{G_m}}[n],
\end{equation}

where $\pi_{B_{G_m}}$ is the projection $V^\vee \to B_{G_m}$. Notice that $\pi_{B_{G_m}} \sim i_{i_V}^*$, so $\pi_{B_{G_m}}$ is left adjoint to $i_{i_V}^*$. Passing in (2) to the right adjoint functors, we get

\begin{equation}
 i_{i_V}^*[n] \sim i_{i_V}^*F.
\end{equation}

**Remark.** Other settings for Fourier transform of constructible sheaves can be also used towards our aim (these are monodromic Fourier transform that identifies
the subcategories of complexes with monodromic cohomology in \(D(V)\) and \(D(V^\vee)\), and, for \(D\)-modules or for \(\ell\)-adic sheaves in finite characteristic, the full Fourier transform that identifies \(D(V)\) with \(D(V^\vee))\).

4. Primitive cycles. Let \(M\) be a non-constant irreducible perverse sheaf on \(\mathbb{P}\). By the theorem in 2, \(\mathcal{R}(M)\) is an irreducible object of \(\mathcal{M}(\mathbb{P}^\vee)\); let \(M^\vee\) be the corresponding non-constant irreducible perverse sheaf on \(\mathbb{P}^\vee\). Let \(c \in H^2(\mathbb{P}, \mathbb{Q}(1))\) be the class of a hyperplane section. We have the primitive decomposition\(^3\)

\[
\bigoplus_{j \geq \max\{a/2, 0\}} H^{a-2j}(\mathbb{P}, M(-j))_{\text{prim}} \cong H^a(\mathbb{P}, M),
\]

where \(H^{-i}(\mathbb{P}, M)_{\text{prim}} := \ker(c^i: H^{-i}(\mathbb{P}, M) \to H^i(\mathbb{P}, M)(i)), i \geq 0,\) the \(j\)-component of \(\cong\) is multiplication by \(c^j\). Set \(H^a(\mathbb{P}, M)_{\text{coprim}} := \ker(c: H^a(\mathbb{P}, M) \to H^{a+2}(\mathbb{P}, M)(1)), a \geq 0,\) which equals component \(j = 2a\) of (4). Ditto for \(M^\vee\).

**Theorem.** One has canonical identifications

\[
H^a(\mathbb{P}, M)_{\text{coprim}} \cong H^{a+2-n}(\mathbb{P}^\vee, M^\vee)_{\text{prim}}.
\]

**Proof.** The intermediate extension functor \(j_{V!*}: \mathcal{M}(\mathbb{P}) \to \mathcal{M}(V), j_{V!*}(M) := \text{Im}(\mu V^0 j_{V!*}(M) \to \mu V^0 j_{V!*}(M))\) identifies the category of irreducible perverse sheaves on \(\mathbb{P}\) with that of those irreducible perverse sheaves on \(\mathbb{V}\) which are not supported on \(\mathbb{V} \setminus \mathbb{P} = \{0\}/\mathbb{G}_m\). Since \(\mathcal{F}\) sends sheaves supported on \(\mathbb{V} \setminus \mathbb{P}\) to constant sheaves and \(\mu V^0 j_{V!*}(M) = j_{V!*}(M)\), we see that (1) yields \(j_{V!*} j_{V!*}(M) \cong M^\vee(1),\) hence \(j_{V!*}(M^\vee) = j_{V!*}(M)(1).\) Applying (3), we get \(i_{V!*} j_{V!*}(M)(1) = i_{V!*} j_{V!*}(M^\vee)[-n].\) Pulling it back by the smooth projections \(\pi_\mathbb{V} : V \to \mathbb{V},\) \(\pi_\mathbb{P} : V^\vee \to \mathbb{P}\) of relative dimension one, we get a canonical isomorphism

\[
i_{V!*} j_{V!*}(M^\vee)(1) \cong i_{V!*} j_{V!*}(M^\vee)[-n],
\]

where \(M^\vee := \pi_\mathbb{V}^* M^\vee[1], M^\vee_b := \pi_\mathbb{V}^* M^\vee[1]\) are irreducible perverse sheaves on \(V^\vee, \) \(V^\vee_b\). Since \(i_{V!*}\) is right t-exact and \(j_{V!*}(M^\vee)\) is irreducible, the complex \(i_{V!*} j_{V!*}(M^\vee)\) is acyclic in degrees \(\leq 0;\) dually, \(i_{V!*} j_{V!*}(M^\vee)\) is acyclic in degrees \(\leq 0.\) We get (5) combining (6) with the next (well-known) lemma:

**Lemma.** There are canonical identifications \(H^a i_{V!*} j_{V!*}(M^\vee)[1] \cong H^{a+1}(\mathbb{P}, M)_{\text{prim}},\)

\(H^a i_{V!*} j_{V!*}(M) \cong H^{a-1}(\mathbb{P}, M(1))_{\text{coprim}}.\)

**Proof of Lemma.** The canonical exact triangle \(i_{V!*} j_{V!*}(M^\vee) \to i_{V!*} j_{V!*}(M^\vee) \to i_{V!*} j_{V!*}(M^\vee)\) and the above acyclicity remark imply that

\[
i_{V!*} j_{V!*}(M^\vee)(1) = \tau_{\geq 0} i_{V!*} j_{V!*}(M^\vee), \quad i_{V!*} j_{V!*}(M^\vee) = \tau_{<0} i_{V!*} j_{V!*}(M^\vee).
\]

Now \(i_{V!*} j_{V!*}(M^\vee) \cong R\Gamma(V^\vee, M^\vee) \cong R\Gamma(\mathbb{P}, M(1))[1]\), the first isomorphism comes since \(M^\vee\) is \(\mathbb{G}_m\)-equivariant, the second is the projection formula. Thus the evident exact triangle \(\mathbb{Q}_\mathbb{P} \to \pi_\mathbb{V}^* \mathbb{Q}_\mathbb{V} \to \mathbb{Q}[-1]\), its boundary map is \(c\), yields isomorphism \(i_{V!*} j_{V!*}(M^\vee) \cong \text{Cone}(R\Gamma(\mathbb{P}, M(1))[-1] \to R\Gamma(\mathbb{P}, M)[1]).\) By (7) and (4), it provides the identifications of the lemma, q.e.d.

\(^2\)Monodromic Fourier transform is the functor \(N \mapsto \text{holim}_a \text{pr}_\mathbb{P}^*(\mu^* N \otimes \mu^* j_{\mathbb{P}}^\circ \mathcal{L}_a)[n + 1],\)

where \(\cdots \to \mathcal{L}_2 \to \mathcal{L}_1\) are local systems on \(\mathbb{A}^1 \setminus \{0\}\) with unipotent Jordan block monodromy; \(\text{rk} \mathcal{L}_a = a.\) For the analytic version, see [B] §6.

\(^3\)For an arbitrary irreducible \(F\) this was proven in [D] (via [BK] or [G]) and [M].
Remark. Only the case $a = 0$ is needed for the aims of [BFNP].

5. A description of $N_1$ ([GG], [BFNP]). Let $X$ be an irreducible projective variety, $F$ be an irreducible perverse sheaf on $X$ whose support equals $X$ (the case we need is $F = \mathbb{Q}_X[\dim X]$); $L$ be a very ample sheaf on $X$. Let us describe the subspace $N_1H^0(X, F)^{\text{prim}}$ of $H^0(X, F)^{\text{prim}}$ (which is the intersection of kernels of all restriction maps to $H^0(Y, F|_Y)$, $Y$ is a closed proper subspace of $X$).

We have the embedding $i_L : X \hookrightarrow \mathbb{P} = \mathbb{P}_X$, $n = \dim H^0(X, L)$, that corresponds to $L$; we assume that $X \neq \mathbb{P}$, so $M := i_L^*F$ is non-constant. Consider identification $\alpha : H^0(X, F)^{\text{prim}} \cong H^{2-n}(\mathbb{P}^\vee, M^\vee)$ defined as the composition $H^0(\mathbb{P}, M)^{\text{prim}} \to H^0(\mathbb{P}, M)^{\text{coprim}} \cong H^{2-n}(\mathbb{P}^\vee, M^\vee)^{\text{prim}} = H^{2-n}(\mathbb{P}^\vee, M^\vee)$ where $\cong$ is the isomorphism from (5).

For a constructible complex $G$ denote by $\mathcal{H}G$, $\tau_\geq$. its usual (not perverse) cohomology sheaves and the canonical truncation. The projection $M^\vee \to \tau_{\geq 2-n}M^\vee$ yields the map $H^{2-n}(\mathbb{P}^\vee, M^\vee) \to H^0(\mathbb{P}^\vee, H^{2-n}M^\vee)$. Let $K_L \subset H^0(X, F)^{\text{prim}}$ be the $\alpha$-preimage of its kernel. It coincides with the kernel of the composition $H^0(X, F)^{\text{prim}} \hookrightarrow H^0(X, F)^{\text{prim}} \overset{\hat{\nu} \tau_{\geq}}{\longrightarrow} H^{2-n}(\mathbb{P}^\vee, \mathcal{R}(M)) \to H^0(\mathbb{P}^\vee, H^{2-n}\mathcal{R}(M))$, which assigns to a primitive cycle the display of its images in $H^0(Y, F|_Y)$ for all hyperplane sections $Y$. (Indeed, by the decomposition theorem, $\mathcal{R}(M)$ is the direct sum of $M^\vee$ and constant sheaves, so the kernel of a projection $\mathcal{H}^{2-n}\mathcal{R}(M) \to \mathcal{H}^{2-n}(M^\vee)$ is a constant sheaf, and primitive cycles restrict to 0 on a general hyperplane section.) Therefore $K_L$ consists of all primitive cycles whose restriction to each hyperplane section is 0.

Clearly $K_L \subset K_L^{\otimes 2} \subset \ldots$. Since every closed subscheme $Y \subset X$, $Y \neq X$, lies on a hypersurface of sufficiently high degree, we see that $N_1H^0(X, F)^{\text{prim}}$ equals $K_L^{\otimes n}$ for $n \gg 0$.

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