Regularity criterion for 3D nematic liquid crystal flows in terms of finite frequency parts in $\dot{B}_{\infty,\infty}^{-1}$

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Abstract

In this paper, we establish the regularity criterion for the weak solution of nematic liquid crystal flows in three dimensions when the $L^\infty(0,T;\dot{B}_{\infty,\infty}^{-1})$-norm of a suitable low frequency part of $(u, \nabla d)$ is bounded by a scaling invariant constant and the initial data $(u_0, \nabla d_0)$. Our result refines the corresponding one in (Liu and Zhao in J. Math. Anal. Appl. 407:557-566, 2013) and that in (Ri in Nonlinear Anal. TMA 190:111619, 2020).

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Keywords: Liquid crystal flow; Regularity criterion; Weak solution; $\dot{B}_{\infty,\infty}^{-1}$

1 Introduction

This note focuses on the regularity criteria for the following 3D nematic liquid crystal fluid flow:

\[
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla \pi &= -\lambda \nabla \cdot (\nabla d \otimes \nabla d), & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\
\partial_t d + (u \cdot \nabla) d &= \gamma (\Delta d + |\nabla d|^2 d), & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\
\text{div} u &= 0, & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\
(u, d)|_{t=0} &= (u_0, d_0), & x \in \mathbb{R}^3,
\end{align*}
\]

where $u(x,t)$ is the unknown velocity field, $d(x,t) : \mathbb{R}^3 \times (0, +\infty) \to S^2$, the unit sphere in $\mathbb{R}^3$, is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow and $\pi$ is the scalar pressure. $\nu$, $\lambda$, $\gamma$ are positive constants that represent viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relaxation time for the molecular orientation field. The notation $\nabla d \otimes \nabla d$ denotes the $3 \times 3$ matrix whose $(i,j)$ entry is given by $\partial_i d \cdot \partial_j d$ ($1 \leq i, j \leq 3$).

It is well-known that Ericksen and Leslie ([3–5, 8]) established the hydrodynamic theory of liquid crystals in 1960s. Lin [9] first introduced the above liquid crystal flow (1.1). Later Lin and Liu [11] obtained the global existence theorem for a weak solution and the local existence for the strong solution to the system (1.1).

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We first introduce the definition of Morrey spaces.

**Definition 1.1** For $1 \leq p \leq q \leq \infty$, we call $\dot{M}_{p,q}(\mathbb{R}^3)$ a Morrey space, if and only if

$$
\|f\|_{\dot{M}_{p,q}(\mathbb{R}^3)} = \sup_{x \in \mathbb{R}^3, 0 < R < \infty} R^{\frac{3}{p} - \frac{3}{q}} \left( \int_{B(x, R)} |f(y)|^q \, dy \right)^{\frac{1}{q}} < +\infty,
$$

here $B(x, R)$ denotes the ball in $\mathbb{R}^3$ with center $x$ and radius $R$.

In 2008, Fan and Guo [4] showed that, if $u$ satisfies one of the following conditions:

$$
\begin{align*}
&u \in L^s(0, T; \dot{M}_{p,q}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{s} + \frac{3}{p} = 1, p \geq 3, p \geq q \geq 1, \\
&\nabla u \in L^s(0, T; \dot{M}_{p,q}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{s} + \frac{3}{p} = 2, p \geq \frac{3}{2}, p \geq q \geq 1,
\end{align*}
$$

then $(u, d)$ is extended beyond $t = T$. Later Liu, Zhao and Cui [12] obtained the regularity criterion to the system (1.1) under the assumption that $\partial_3 u \in L^\beta(0, T; L^\alpha)$ with $\frac{2}{\beta} + \frac{3}{\alpha} \leq 1, \alpha > 3$. Recently, Wei, Li and Yao [16] proved that, if the weak solution $(u, d)$ satisfies

$$
\begin{align*}
&u_3, \nabla d \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{\alpha}, \alpha > \frac{10}{3},
\end{align*}
$$

then $(u, b)$ can be extended beyond $t = T$. Liu and Zhao [13] proved that the solution $(u, d)$ to (1.1) is smooth up to time $T$ provided that

$$
\| (u, \nabla d) \|_{L^\infty(0, T; B_{\infty,\infty}^{-1}(\mathbb{R}^3))} \leq \varepsilon_0.
$$

When $d = 0$, the system (1.1) becomes an incompressible Navier–Stokes equation. There is a large literature on the regularity criteria on the Navier–Stokes equation; see [1, 6, 7, 15].

By traditional turbulence theory, viscous incompressible flows develop in such a way that energy is transferred from large scales to neighboring smaller scales. Hence, it is important to study regularity for the Navier–Stokes equation based on various wave-number band parts of weak solutions is important since it reveals in a way the relationship between regularity of weak solutions and turbulent flows. Cheskidov and Shvydkoy [2] proved that a Leray–Hopf weak solution $u$ to the Navier–Stokes equation is regular in $(0, T]$ if

$$
\| u^k \|_{B_{\infty,\infty}^{-1}(\mathbb{R}^3)} < C_V,
$$

where $u^k$ is high frequency part of $u$ with Fourier models $|\xi| \geq k$. Kim, Kwak and Yoo [5] proved that, if sufficiently high frequency parts of a weak solution to the Navier–Stokes equation on a torus belong to Serrin’s class, then the weak solution is regular. Very recently, Ri [14] proved that a Leray–Hopf weak solution $u$ to 3D Navier–Stokes equations is regular if the $L^\infty(0, T; B_{\infty,\infty}^{-1}(\mathbb{R}^3))$-norm of a suitable low frequency part of $u$ is bounded by a scaling invariant constant depending on the kinematic viscosity $\nu$ and initial value $u_0$. Motivated by [2, 5, 13] and [14], we will investigate the regularity criteria for the weak solution $(u, d)$ to the liquid crystal fluid flows (1.1) in the critical function space $L^\infty(0, T; \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3))$. 

based on low and medium frequency parts, respectively. Before stating our result, we shall present some symbols and notations.

Let
\[
\begin{align*}
  u_k &= \int_0^k u_s \, ds, \\
  u^k &= \int_k^\infty u_s \, ds, \\
  u_{h,k} &= u_k - u_h, 0 < h < k < \infty.
\end{align*}
\] (1.2)

Here
\[
  u_k(t, x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|\xi| = k} \hat{u}(t, \xi) e^{ix \cdot \xi} \, d\sigma_\xi,
\]

and \( \hat{u} \) denotes Fourier transform of \( u \). Our result is stated as follows.

**Theorem 1.2** Let \((u, d)\) be a weak solution to (1.1) with \((u_0, d_0) \in H^3(\mathbb{R}^3) \times H^4(\mathbb{R}^3), \text{div} u_0 = 0\). Assume that, for \( 0 < T < \infty \), there exists \( \delta \in (0, T) \) such that if \((u, d)\) is regular in \((0, T)\) the inequalities

\[
\left\| (u_k, \nabla d_k) \right\|_{L^\infty(T-\delta, T; \dot{B}^{-1}_\infty, \infty)} < C_1
\]

and

\[
\left\| (u_{k/2}, \nabla d_{k/2}) \right\|_{L^\infty(T-\delta, T; \dot{B}_{\infty, \infty}^\infty)} < C_2 \left( \| u_0 \|_{L^2} \right) \left( \| \nabla d_0 \|_{L^2} \right)^{-1} \left( \| \nabla u_0 \|_{L^2} + \| \Delta d_0 \|_{L^2} \right)^{-1}
\]

hold. Then \((u, d)\) is regular on \((0, T]\), where \( \tilde{k} > 0 \) is defined by

\[
\tilde{k} = C_3 \left( \| \nabla u_0 \|_{L^2} + \| \Delta d_0 \|_{L^2} \right)^2,
\]

and the \( C_i, i = 1, 2, 3, \) are absolute constants.

**Remark 1.1** Theorem 1.2 can be regarded as the generalization of Theorem 1.1 in [13] and Theorem 1.1 in [14].

The rest of this paper is organized as follows. Some useful facts are presented in Sect. 2. The proof of Theorem 1.2 is given in Sect. 3.

### 2 Preliminaries and some basic facts

In order to define Besov spaces, we first introduce the Littlewood–Paley decomposition theory. Let \( \mathcal{S}(\mathbb{R}^n) \) be the Schwartz class of rapidly decreasing functions. Let \( \mathcal{S}(\mathbb{R}^n) \) be the Schwartz class of rapidly decreasing functions.

For given \( f \in \mathcal{S}(\mathbb{R}^n) \), its Fourier transform \( \mathcal{F}(f) = \hat{f} \) and its inverse Fourier transform \( \mathcal{F}^{-1}(f) = \check{f} \) are given by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx
\]

and

\[
\check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(x) \, d\xi,
\]
respectively. Let us choose two nonnegative radial functions \( \chi, \varphi \in \mathcal{S}(\mathbb{R}^n) \) satisfying 
\[
supp \chi \subset B = \{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3} \} \quad \text{and} \quad supp \varphi \subset C = \{ \xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \text{ such that }
\]
\[
\sum_{j \in \mathbb{Z}} \varphi(2^j \xi) = 1, \quad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\}
\]
and
\[
\chi(\xi) + \sum_{j \geq 0} \varphi(2^j \xi) = 1, \quad \text{for any } \xi \in \mathbb{R}^n.
\]

For \( j \in \mathbb{Z} \), the homogeneous Littlewood–Paley projection operators \( S_j \) and \( \hat{\Delta}_j \) are defined by
\[
\hat{S}_j f = \chi(2^j D)f = 2^{nj} \int_{\mathbb{R}^n} \tilde{h}(2^j y) f(x - y) \, dy, \quad \text{where } \tilde{h} = \mathcal{F}^{-1} \chi,
\]
and
\[
\hat{\Delta}_j f = \varphi(2^j D)f = 2^{nj} \int_{\mathbb{R}^n} h(2^j y) f(x - y) \, dy, \quad \text{where } h = \mathcal{F}^{-1} \varphi.
\]
\( \hat{\Delta}_j \) is a frequency projection to the annulus \( \{ |\xi| \sim 2^j \} \), and \( \hat{S}_j \) is a frequency projection to the ball \( \{ |\xi| \leq 2^j \} \). Let \( s \in \mathbb{R} \), \( p, q \in [1, \infty) \). The homogeneous Besov space \( \dot{B}^s_{p,q}(\mathbb{R}^n) \) is presented by the distributions \( f \in \mathcal{S}'_h \) such that
\[
\left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \hat{\Delta}_j f \|_{L^p}^q \right)^{\frac{1}{q}} < \infty,
\]
with the norm
\[
\| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \begin{cases} 
(\sum_{j \in \mathbb{Z}} 2^{jsq} \| \hat{\Delta}_j f \|_{L^p}^q)^{\frac{1}{q}}, & 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} \{ 2^j \| \hat{\Delta}_j f \|_{L^p} \}, & q = \infty.
\end{cases}
\label{eq:2.1}
\]

On the other hand, we recall some facts that can be found in \([14]\). If \( u \in L^2(\mathbb{R}^3) \), then it follows from the definition of \( u_k \) and \( \dot{u}^k \) that
\[
(u_k, \dot{u}^k) = 0, \quad \forall k > 0.
\label{eq:2.2}
\]
Moreover, for \( 0 \leq r < s \), by Plancherel’s theorem,
\[
\begin{align*}
\| u_k \|_{\dot{H}^r} &= \| 2^r \xi^s \hat{u}_k \|_{L^2} \leq 2^r \| \hat{\Delta}_r \hat{u}_k \|_{L^2} = 2^r \| \Delta u_k \|_{\dot{H}^r}, \\
\| \dot{u}^k \|_{\dot{H}^r} &= \| 2^r \xi^s \hat{u}_k \|_{L^2} \geq 2^r \| \hat{\Delta}_r \hat{u}_k \|_{L^2} = 2^r \| \Delta u_k \|_{\dot{H}^r}.
\end{align*}
\label{eq:2.3}
\]
Since \( \| \Delta u \|_{L^2} \sim \| \nabla^2 u \|_{L^2}, \forall u \in \dot{H}^2(\mathbb{R}^3) \), we have
\[
k \| \nabla \dot{u}^k \|_{L^2} \leq \| \nabla^2 \dot{u}^k \|_{L^2} \leq c \| \Delta \dot{u}^k \|_{L^2}, \quad \forall u \in H^2(\mathbb{R}^3),
\label{eq:2.4}
\]
with some $c > 0$. Moreover, it can be easily seen that

$$(u_4v_l)^m = 0, \quad \forall u, v \in L^2(\mathbb{R}^3), \forall k, l > 0, \forall m > k + l,$$

(2.5)

because the Fourier transform of $u_4v_l$ is supported in $\{\xi \in \mathbb{R}^3 : |\xi| \leq k + l\}$.

3 Proof of Theorem 1.2

For convenience, we assume $\mu = \lambda = 1$ throughout the proof of Theorem 1.2.

Proof Assume that a weak solution $(u, d)$ of (1.1) is regular in $[0, T)$, but not in $[0, T]$. Then $\lim_{t \rightarrow T^-} \|\nabla u(t)\|_{L^2} + \|\Delta d(t)\|_{L^2} = \infty$. Notice that, for all smooth solutions to system (1.1), one has the following basic energy law (see [10]):

$$
\|u(t, t)\|_{L^2}^2 + \|\Delta d(t)\|_{L^2}^2 + \int_0^t (\|\nabla u(\tau, t)\|_{L^2}^2 + \|\Delta d + \Delta d^2(\tau, t)\|_{L^2}^2) d\tau
\leq \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2,
$$

(3.1)

for all $0 < t < \infty$. By (1.2), one has

$$
\|\nabla u(t)\|_{L^2}^2 + \|\Delta d(t)\|_{L^2}^2 \leq k^2 \|u_0\|_{L^2}^2 + k^2 \|\nabla d_0\|_{L^2}^2 + \|\nabla u^k(t)\|_{L^2}^2 + \|\Delta d^k(t)\|_{L^2}^2.
$$

(3.2)

Thus,

$$
\lim_{t \rightarrow T^-} \|\nabla u^k(t)\|_{L^2}^2 + \|\Delta d^k(t)\|_{L^2}^2 = \infty.
$$

(3.3)

We can see from [13] that, if there exists a positive constant $\varepsilon_0 > 0$ such that

$$
\|(u, \nabla d)\|_{L^\infty(0, T; \dot{H}^1_{\lambda} L^2)} \leq \varepsilon_0,
$$

then the solution $(u, d)$ is smooth up to time $T$.

Now we multiply the first equation of (1.1) with $-\Delta u^k$ and integrate over $\mathbb{R}^3$ to get by (2.2)

$$
\frac{1}{2} \frac{d}{dt} \|\nabla u^k\|_{L^2}^2 + \|\Delta u^k\|_{L^2}^2 = (u \cdot \nabla u, \Delta u^k) + \left(\Delta d \cdot \nabla d + \frac{1}{2} \nabla |\nabla d|^2, \Delta u^k\right).
$$

(3.4)

Applying $\nabla$ to the second equation of (1.1) and making an $L^2$ inner product with respect to $\nabla \Delta d^k$, we can verify

$$
\frac{1}{2} \frac{d}{dt} \|\Delta d^k\|_{L^2}^2 + \|\nabla \Delta d^k\|_{L^2}^2 = (\nabla (u \cdot \nabla d), \nabla \Delta d^k) + (\nabla (|\nabla d|^2 d), \nabla \Delta d^k).
$$

(3.5)

Adding (3.3) and (3.4) gives rise to

$$
\frac{1}{2} \frac{d}{dt} \left(\|\nabla u^k\|_{L^2}^2 + \|\Delta d^k\|_{L^2}^2 + \|\Delta u^k\|_{L^2}^2 + \|\nabla \Delta d^k\|_{L^2}^2\right)
\leq (u \cdot \nabla u, \Delta u^k) + (\Delta d \cdot \nabla d, \Delta u^k) + \frac{1}{2} (\nabla |\nabla d|^2, \Delta u^k)
$$

$$
+ (\nabla (u \cdot \nabla d), \nabla \Delta d^k) + (\nabla (|\nabla d|^2 d), \nabla \Delta d^k)
:= I_1 + I_2 + I_3 + I_4 + I_5.
$$
Next we estimate $I_1 - I_5$, respectively. From [14], we have

$$|I_1| = |(u \cdot \nabla, \Delta u^k)|$$

$$\leq C k^2 \|u_0\|_{L^2}^2 \|u_{2, k}\|_{L^\infty}^2 + C \|u_k\|_{B^1_{2, \infty}} \|\Delta u^k\|_{L^2}^2$$

$$+ C k^{-\frac{1}{2}} \|\nabla u^k\|_{L^2} \|\Delta u^k\|_{L^2}^2 + \frac{1}{4} \|\Delta u_{4, 2k}\|_{L^2}^2. \quad (3.6)$$

Since $d = d_k + d^k$, we write

$$(\Delta d \cdot \nabla)d = (\Delta d_k \cdot \nabla)d_k + (\Delta d^k \cdot \nabla)d_k + (\Delta d_k \cdot \nabla)d^k + (\Delta d^k \cdot \nabla)d^k.$$

Then

$$I_2 = (\Delta d \cdot \nabla, \Delta u^k)$$

$$= ((\Delta d_k \cdot \nabla)d_k, \Delta u^k) + ((\Delta d^k \cdot \nabla)d_k, \Delta u^k) + ((\Delta d_k \cdot \nabla)d^k, \Delta u^k)$$

$$+ ((\Delta d^k \cdot \nabla)d^k, \Delta u^k)$$

$$:= I_{21} + I_{22} + I_{23} + I_{24}. \quad (3.7)$$

Note that $d_k = d_{\frac{1}{2}} + d_{\frac{3}{2}, k}$ and the Fourier transform of $(\Delta d_k \cdot \nabla)d_k$ is supported in $|\xi| < 2k$, thus we deduce

$$I_{21} = ((\Delta d_k \cdot \nabla)d_k, \Delta u^k)$$

$$= (([\Delta d_k \cdot \nabla]d_{\frac{1}{2}} + (\Delta d_{\frac{3}{2}, k} \cdot \nabla)d_{\frac{3}{2}, k}, \Delta u_{4, 2k})$$

$$= ([\Delta d_{\frac{1}{2}} \cdot \nabla]d_{\frac{1}{2}} + (\Delta d_{\frac{3}{2}, k} \cdot \nabla)d_{\frac{3}{2}, k} + (\Delta d_{\frac{1}{2}, k} \cdot \nabla)d_{\frac{1}{2}, k}, \Delta u_{4, 2k})$$

$$= ([\Delta d_{\frac{1}{2}} \cdot \nabla]d_{\frac{1}{2}} + (\Delta d_{\frac{3}{2}, k} \cdot \nabla)d_{\frac{3}{2}, k} + (\Delta d_{\frac{1}{2}, k} \cdot \nabla)d_{\frac{1}{2}, k}, \Delta u_{4, 2k})$$

$$:= I_{211} + I_{212}, \quad (3.8)$$

where we used the fact $([\Delta d_{\frac{1}{2}} \cdot \nabla]d_{\frac{3}{2}, k})_{k, 2k} = 0$. Thanks to the Hölder inequality, the Young inequality and (3.1), we get

$$|I_{211}| \leq \|\Delta d_k\|_{L^2} \|\nabla d_{\frac{1}{2}, k}\|_{L^\infty} \|\Delta u_{4, 2k}\|_{L^2}$$

$$\leq C k \|\nabla d_k\|_{L^2} \|\nabla d_{\frac{1}{2}, k}\|_{L^\infty} \|\Delta u_{4, 2k}\|_{L^2}$$

$$\leq C k^2 \|\nabla d_k\|_{L^2}^2 \|\nabla d_{\frac{1}{2}, k}\|_{L^\infty}^2 + \frac{1}{8} \|\Delta u_{4, 2k}\|_{L^2}^2$$

$$\leq C k^2 \|\nabla d_{\frac{1}{2}, k}\|_{L^\infty}^2 \|u_0\|_{L^2}^2 + \|\nabla d_{\frac{3}{2}, k}\|_{L^2}^2 + \frac{1}{8} \|\Delta u_{4, 2k}\|_{L^2}^2. \quad (3.9)$$
With the help of Hölder’s inequality, (2.4), Gagliardo–Nirenberg’s inequality, Sobolev’s embedding and Young’s inequality, one has

\[ |I_{212}| \leq \| \Delta d_{\frac{1}{2}} k \|_{L^\infty} \| \nabla d_{\frac{1}{2}} \|_{L^2} \| \Delta u_{k,2k} \|_{L^2} \]

\[ \leq CK \| \nabla d_{\frac{1}{2}} k \|_{L^\infty} \| \nabla d_{\frac{1}{2}} \|_{L^2} \| \Delta u_{k,2k} \|_{L^2} \]

\[ \leq CK^2 \| \nabla d_{\frac{1}{2}} k \|_{L^\infty} \| \nabla d_{\frac{1}{2}} \|_{L^2}^2 + \frac{1}{8} \| \Delta u_{k,2k} \|_{L^2}^2 \]

(3.10)

which along with (3.9) implies

\[ |I_{21}| \leq CK^2 \| \nabla d_{\frac{1}{2}} k \|_{L^\infty} \| \Delta d_{\frac{1}{2}} \|_{L^2} \]

\[ \leq C \| \Delta d_{\frac{1}{2}} \|_{L^\infty} \| \nabla \Delta d_{\frac{1}{2}} \|_{L^2} \| \nabla^2 d_{\frac{1}{2}} \|_{L^2} \| \Delta u_{k} \|_{L^2} \]

\[ \leq CK^{-\frac{1}{2}} \| \nabla \Delta d_{\frac{1}{2}} \|_{L^2} \| \Delta d_{\frac{1}{2}} \|_{L^2} \| \Delta u_{k} \|_{L^2} \]

\[ \leq CK^{-\frac{1}{2}} \| \Delta d_{\frac{1}{2}} \|_{L^2} \left( \| \Delta u_{k} \|_{L^2}^2 + \| \nabla \Delta d_{\frac{1}{2}} \|_{L^2}^2 \right). \]

By the definition of the \( \dot{B}_{\infty,\infty} \) norm, we have

\[ \| u_k \|_{L^\infty} \leq CK \| u_k \|_{\dot{B}_{\infty,\infty}^{\frac{1}{2}+}}, \quad \forall k > 0. \]

(3.13)

From the Hölder inequality, (3.13) and the Young inequality, we can conclude that

\[ |I_{23}| \leq \| \Delta d_{k} \|_{L^\infty} \| \nabla d_{k} \|_{L^2} \| \Delta u_{k} \|_{L^2} \]

\[ \leq CK \| \nabla d_{k} \|_{L^\infty} \| \nabla d_{k} \|_{L^2} \| \Delta u_{k} \|_{L^2} \]

\[ \leq CK^2 \| \nabla d_{k} \|_{\dot{B}_{\infty,\infty}^{\frac{1}{2}+}} \| \nabla d_{k} \|_{L^2} \| \Delta u_{k} \|_{L^2} \]

\[ \leq C \| \nabla d_{k} \|_{\dot{B}_{\infty,\infty}^{\frac{1}{2}+}} \left( \| \Delta u_{k} \|_{L^2}^2 + \| \nabla \Delta d_{k} \|_{L^2}^2 \right). \]

(3.14)

Similarly,

\[ |I_{24}| \leq \| \Delta d_{k} \|_{L^2} \| \nabla d_{k} \|_{L^\infty} \| \Delta u_{k} \|_{L^2} \]

\[ \leq CK \| \nabla d_{k} \|_{\dot{B}_{\infty,\infty}^{\frac{1}{2}+}} \| \Delta d_{k} \|_{L^2} \| \Delta u_{k} \|_{L^2} \]

\[ \leq C \| \nabla d_{k} \|_{\dot{B}_{\infty,\infty}^{\frac{1}{2}+}} \| \nabla \Delta d_{k} \|_{L^2} \| \Delta u_{k} \|_{L^2} \]

\[ \leq C \| \nabla d_{k} \|_{\dot{B}_{\infty,\infty}^{\frac{1}{2}+}} \left( \| \Delta u_{k} \|_{L^2}^2 + \| \nabla \Delta d_{k} \|_{L^2}^2 \right). \]

(3.15)
Combining (3.7), (3.11), (3.12), (3.14) and (3.15), one arrives at

\[ |I_2| \leq Ck^{-\frac{1}{2}} \| \Delta d^k \|_{L^2} (\| \Delta u^k \|_{L^2}^2 + \| \nabla \Delta d^k \|_{L^2}^2) \]
\[ + C \| \nabla d_k \|_{\dot{H}^{-\frac{1}{2}}_{\infty}} (\| \Delta u^k \|_{L^2}^2 + \| \nabla \Delta d^k \|_{L^2}^2) \]
\[ + Ck^2 \| \nabla d_{2,k} \|_{L^\infty}^2 (\| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2) + \frac{1}{4} \| \Delta u_{k,2k} \|_{L^2}^2. \]

To estimate \( I_3 \), we make the following decomposition:

\[ \frac{1}{2} \nabla |\nabla d|^2 = \frac{1}{2} |\nabla d^k + \nabla d_k|^2 \leq |\nabla d^k|^2 + |\nabla d_k|^2 \]
\[ = 2 \nabla d^k \cdot \nabla^2 d^k + 2 \nabla d_k \cdot \nabla^2 d_k. \]

Then

\[ |I_3| \leq 2 \left| (\nabla d^k \cdot \nabla^2 d^k, \Delta u^k) \right| + 2 \left| (\nabla d_k \cdot \nabla^2 d_k, \Delta u^k) \right| := I_{31} + I_{32}. \]

Applying the same method to the bound (3.12) gives rise to

\[ I_{31} \leq \| \nabla d^k \|_{L^2} \| \nabla^2 d^k \|_{L^2} \| \Delta u^k \|_{L^2} \]
\[ \leq C \| \nabla d^k \|_{H^{\frac{1}{2}}} \| \Delta d^k \|_{\dot{H}^{\frac{1}{2}}} \| \Delta u^k \|_{L^2} \]
\[ \leq C \| \nabla d^k \|_{L^2}^2 \| \nabla d^k \|_{L^2} \| \Delta d^k \|_{\dot{H}^{\frac{1}{2}}} \| \Delta u^k \|_{L^2} \]
\[ \leq C \| \nabla d^k \|_{L^2}^2 \| \Delta d^k \|_{L^2} \| \nabla \Delta d^k \|_{L^2} \| \Delta u^k \|_{L^2} \]
\[ \leq Ck^{-\frac{1}{2}} \| \Delta d^k \|_{L^2} \| \nabla \Delta d^k \|_{L^2} \| \Delta u^k \|_{L^2} \]
\[ \leq Ck^{-\frac{1}{2}} \| \Delta d^k \|_{L^2} (\| \nabla \Delta d^k \|_{L^2}^2 + \| \Delta u^k \|_{L^2}^2). \]

Similarly to (3.8), we have

\[ I_{32} = 2 \left| (\nabla d_k \cdot \nabla^2 d_k, \Delta u^k) \right| \]
\[ = 2 \left| (\nabla d_k \cdot \nabla^2 d_{2,k}, \Delta u_{k,2k}) + (\nabla d_{2,k} \cdot \nabla^2 d_k, \Delta u_{k,2k}) \right| \]
\[ \leq 2 \left| (\nabla d_k \cdot \nabla^2 d_{2,k}, \Delta u_{k,2k}) \right| + 2 \left| (\nabla d_{2,k} \cdot \nabla^2 d_k, \Delta u_{k,2k}) \right| \]
\[ := I_{321} + I_{322}. \]

Using Hölder’s inequality, (2.4), Young’s inequality and (3.1), one can verify

\[ I_{321} \leq 2 \| \nabla d_k \|_{L^2} \| \Delta d_{2,k} \|_{L^\infty} \| \Delta u_{k,2k} \|_{L^2} \]
\[ \leq C k \| \nabla d_{2,k} \|_{L^\infty} \| \nabla d_k \|_{L^2} \| \Delta u_{k,2k} \|_{L^2} \]
\[ \leq Ck^2 \| \nabla d_{2,k} \|_{L^\infty}^2 (\| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2) + \frac{1}{8} \| \Delta u_{k,2k} \|_{L^2}^2. \]
Similarly,

\[ I_{322} \leq 2 \| \Delta d_2 \|_{L^2} \| \nabla d_{\frac{1}{2}} \|_{L^\infty} \| \Delta u_{k,2k} \|_{L^2} \]

\[ \leq CK \| \nabla d_{\frac{1}{2}} \|_{L^2} \| \nabla d_{\frac{1}{2}} \|_{L^\infty} \| \Delta u_{k,2k} \|_{L^2} \]

\[ \leq CK^2 \| \nabla d_{\frac{1}{2}} \|_{L^\infty}^2 \left( \| u_0 \|_{L^2}^2 + \| d_0 \|_{L^2}^2 \right) + \frac{1}{4} \| \Delta u_{k,2k} \|_{L^2}^2, \]  

which along with (3.20) implies

\[ I_{32} \leq CK^2 \| \nabla d_{\frac{1}{2}} \|_{L^\infty}^2 \left( \| u_0 \|_{L^2}^2 + \| d_0 \|_{L^2}^2 \right) + \frac{1}{4} \| \Delta u_{k,2k} \|_{L^2}^2. \]  

From (3.17), (3.18) and (3.22), we can deduce

\[ |I_3| \leq CK^{-\frac{1}{2}} \| \Delta d^k \|_{L^2} \left( \| u^k \|_{L^2}^2 + \| \nabla \Delta d^k \|_{L^2}^2 \right) \]

\[ + CK^2 \| \nabla d_{\frac{1}{2}} \|_{L^\infty}^2 \left( \| u_0 \|_{L^2}^2 + \| d_0 \|_{L^2}^2 \right) + \frac{1}{4} \| \Delta u_{k,2k} \|_{L^2}^2. \]

We now address the term \( I_4 \). We decompose \( I_4 \) into the following form:

\[ I_4 = \left( (\nabla u \cdot \nabla) d, \nabla \Delta d^k \right) + \left( (u \cdot \nabla) \nabla d, \nabla \Delta d^k \right) := I_{41} + I_{42}. \]

Since

\[ (\nabla u \cdot \nabla) d = (\nabla u^k \cdot \nabla) d^k + (\nabla u^k \cdot \nabla) d_k + (\nabla u_k \cdot \nabla) d^k + (\nabla u_k \cdot \nabla) d_k, \]

we can get

\[ I_{41} = \left( (\nabla u^k \cdot \nabla) d^k, \nabla \Delta d^k \right) + \left( (\nabla u^k \cdot \nabla) d_k, \nabla \Delta d^k \right) + \left( (\nabla u_k \cdot \nabla) d^k, \nabla \Delta d^k \right) \]

\[ + \left( (\nabla u_k \cdot \nabla) d_k, \nabla \Delta d^k \right) := I_{411} + I_{412} + I_{413} + I_{414}. \]

Similar to the estimate (3.12), one has

\[ |I_{411}| \leq \| \nabla u^k \|_{L^6} \| \nabla d^k \|_{L^3} \| \nabla \Delta d^k \|_{L^2} \]

\[ \leq C \| \nabla u^k \|_{H^1} \| \nabla d^k \|_{L^2}^{\frac{1}{2}} \| \nabla d^k \|_{H^1}^{\frac{1}{2}} \| \nabla \Delta d^k \|_{L^2} \]

\[ \leq CK^{-\frac{1}{2}} \| \Delta d^k \|_{L^2} \| \Delta u^k \|_{L^2} \| \nabla \Delta d^k \|_{L^2} \]

\[ \leq CK^{-\frac{1}{2}} \| \Delta d^k \|_{L^2} \left( \| \Delta u^k \|_{L^2}^2 + \| \nabla \Delta d^k \|_{L^2}^2 \right). \]

The Hölder inequality, the Young inequality and (3.13) imply

\[ I_{412} \leq \| \nabla u^k \|_{L^2} \| \nabla d_k \|_{L^\infty} \| \nabla \Delta d^k \|_{L^2} \]

\[ \leq CK \| \nabla d_k \|_{L^\infty} \| \nabla u^k \|_{L^2} \| \nabla \Delta d^k \|_{L^2} \]

\[ \leq C \| \nabla d_k \|_{L^\infty} \| \Delta u^k \|_{L^2} \| \nabla \Delta d^k \|_{L^2} \]

\[ \leq C \| \nabla d_k \|_{L^\infty} \left( \| \Delta u^k \|_{L^2}^2 + \| \nabla \Delta d^k \|_{L^2}^2 \right). \]
By Hölder’s inequality, (2.4) and Young’s inequality, we get

\[ |I_{414}| \leq \| \nabla u_k \|_{L^\infty} \| \nabla d_{\frac{1}{2},k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq CK \| u_k \|_{L^\infty} \| \nabla d_{\frac{1}{2},k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq CK^2 \| u_k \|_{B^1_{\infty,\infty}} \| \nabla d_{\frac{1}{2},k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq C \| u_k \|_{B^1_{\infty,\infty}} \| \nabla \Delta d_{k,2k} \|_{L^2}^2. \]  

(3.29)

Similarly,

\[ |I_{414}| \leq \| \nabla u_{\frac{1}{2},k} \|_{L^\infty} \| \nabla d_{\frac{1}{2},k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq CK \| u_{\frac{1}{2},k} \|_{L^\infty} \| \nabla d_{\frac{1}{2},k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq CK^2 \| u_{\frac{1}{2},k} \|_{B^1_{\infty,\infty}} \| \nabla d_{\frac{1}{2},k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq CK^2 \| u_{\frac{1}{2},k} \|_{B^1_{\infty,\infty}} \| \nabla \Delta d_{k,2k} \|_{L^2}^2. \]  

(3.30)

which together with (3.29) reads

\[ |I_{414}| \leq CK^2 \| u_{\frac{1}{2},k} \|_{B^1_{\infty,\infty}} \| \nabla d_{\frac{1}{2},k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2}^2 + \frac{1}{8} \| \nabla \Delta d_{k,2k} \|_{L^2}^2. \]  

(3.31)

Combining (3.26)–(3.28) and (3.31) yields

\[ |I_{41}| \leq \frac{1}{8} \| \nabla \Delta d_{k,2k} \|_{L^2}^2 + CK^{-\frac{1}{2}} \| \Delta u^k \|_{L^2} \| \Delta u^k \|_{L^2} + \| \Delta d^k \|_{L^2} \]
\[ + C \| u_k \|_{B^1_{\infty,\infty}} \| \nabla d_{\frac{1}{2},k} \|_{B^1_{\infty,\infty}} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ + CK^2 \| u_{\frac{1}{2},k} \|_{B^1_{\infty,\infty}} \| \nabla d_{\frac{1}{2},k} \|_{B^1_{\infty,\infty}} \| \nabla \Delta d_{k,2k} \|_{L^2} \| \nabla d_{k,2k} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2}^2. \]  

(3.32)
To handle $I_{42}$, we split $I_{42}$ into

$$I_{42} = \left( (u_k \cdot \nabla) \nabla d_k, \nabla \Delta d_k \right) + \left( (u_k \cdot \nabla) \nabla d_k, \nabla \Delta d_k \right) + \left( (u_k \cdot \nabla) \nabla d_k, \nabla \Delta d_k \right)$$

$$+ \left( (u_k \cdot \nabla) \nabla d_k, \nabla \Delta d_k \right) \quad \text{(3.33)}$$

By Hölder’s inequality and (2.4), we get

$$|I_{421}| \leq \|u_k\|_{L^\infty} \left\| \nabla^2 d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2}$$

$$\leq \|u_k\|_{L^\infty} \left\| \Delta d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2} \quad \text{(3.34)}$$

Similarly to (3.12), one has

$$|I_{422}| \leq \|u_k\|_{L^2} \left\| \nabla^2 d_k \right\|_{L^1} \left\| \nabla \Delta d_k \right\|_{L^2}$$

$$\leq C \|u_k\|_{L^2} \left\| \Delta d_k \right\|_{L^3} \left\| \nabla \Delta d_k \right\|_{L^2}$$

$$\leq C \left\| \nabla u_k \right\|_{L^3} \left\| \Delta d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2} \quad \text{(3.35)}$$

Hölder’s inequality, (2.4), and Young’s inequality guarantee

$$|I_{423}| \leq \|u_k\|_{L^2} \left\| \nabla^2 d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2}$$

$$\leq \|u_k\|_{L^2} \left\| \Delta d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2}$$

$$\leq c \|u_k\|_{L^2} \left\| \nabla d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2} \quad \text{(3.36)}$$

Similarly to (3.8), we write

$$I_{424} = \left( (u_{2,k} \cdot \nabla) \nabla d_{2,k}, \nabla \Delta d_{2,k} \right) + \left( (u_{2,k} \cdot \nabla) \nabla d_{2,k}, \nabla \Delta d_{2,k} \right) := I_{4241} + I_{4242}.$$ 

From the Hölder inequality and the Young inequality, we conclude

$$|I_{4241}| \leq \|u_{2,k}\|_{L^2} \left\| \nabla^2 d_{2,k} \right\|_{L^\infty} \left\| \nabla \Delta d_{2,k} \right\|_{L^2}$$

$$\leq C \|u_{2,k}\|_{L^2} \left\| \Delta d_{2,k} \right\|_{L^\infty} \left\| \nabla \Delta d_{2,k} \right\|_{L^2}$$

$$\leq C \|u_{2,k}\|_{L^2} \left\| \nabla d_{2,k} \right\|_{L^\infty} \left\| \nabla \Delta d_{2,k} \right\|_{L^2} \quad \text{(3.37)}$$
and

\[ |I_{4242}| \leq \|u_{2,k}\|_{L^\infty} \| \nabla^2 d_{\frac{k}{2}} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq C \|u_{2,k}\|_{L^\infty} \| \Delta d_{\frac{k}{2}} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq c_k \|u_{2,k}\|_{L^\infty} \| \nabla d_{\frac{k}{2}} \|_{L^2} \| \nabla \Delta d_{k,2k} \|_{L^2} \]
\[ \leq Ck^2 \|u_{2,k}\|_{L^\infty}^2 \left( \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \right) + \frac{1}{16} \|\nabla \Delta d^k\|_{L^2}^2. \quad (3.38) \]

Therefore, by (3.24)–(3.38), we have

\[ |I_4| \leq C k^{-\frac{1}{2}} \left( \|\nabla u^k\|_{L^2} + \|\Delta d^k\|_{L^2} \right) \left( \|\nabla u^k\|_{L^2}^2 + \|\nabla \Delta d^k\|_{L^2}^2 \right) \]
\[ + C \|\nabla d_k\|_{B^{-1}_{\infty,\infty}} \left( \|\nabla u^k\|_{L^2}^2 + \|\nabla \Delta d^k\|_{L^2}^2 \right) \]
\[ + C \|u_k\|_{B^{-3}_{\infty,\infty}} \|\nabla \Delta d^k\|_{L^2}^2 + C k^2 \left( \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \right) \]
\[ \times \left( \|u_{2,k}\|_{L^\infty}^2 + \|\nabla d_{2,k}\|_{L^\infty}^2 \right) + \frac{1}{4} \|\nabla \Delta d_{k,2k}\|_{L^2}^2. \quad (3.39) \]

It is left to deal with the last term, $I_5$. Using the fact that

\[ \nabla (|\nabla d^2 d|) = 2\nabla^2 d \nabla dd + |\nabla d|^2 \nabla d, \]

we can rewrite $I_5$ as follows:

\[ I_5 = 2 \langle \nabla^2 d \nabla d, \nabla \Delta d^k \rangle + \langle |\nabla d|^2 \nabla d, \nabla \Delta d^k \rangle := I_{51} + I_{52}. \quad (3.40) \]

Since

\[ 2\nabla^2 d \nabla dd = \left( 2\nabla^2 d_3 \nabla d_3 + 2\nabla^2 d_3 \nabla d_3 + 2\nabla^2 d_3 \nabla d_3 + 2\nabla^2 d_3 \nabla d_3 \right) d, \]

we have

\[ I_{51} = 2 \langle \nabla^2 d_3 \nabla d_3, \nabla \Delta d^k \rangle + 2 \langle \nabla^2 d_3 \nabla d_3, \nabla \Delta d^k \rangle + 2 \langle \nabla^2 d_3 \nabla d_3, \nabla \Delta d^k \rangle \]
\[ + 2 \langle \nabla^2 d_3 \nabla d_3, \nabla \Delta d^k \rangle \]
\[ := I_{511} + I_{512} + I_{513} + I_{514}. \quad (3.41) \]

Reasoning as (3.8), one has

\[ I_{511} = \langle \nabla^2 d_3 \nabla d_{\frac{3}{2},k} d, \nabla \Delta d_{k,2k} \rangle + \langle \nabla^2 d_{\frac{3}{2},k} \nabla d_{\frac{3}{2},k} d, \nabla \Delta d_{k,2k} \rangle \]
\[ := I_{5111} + I_{5112}. \quad (3.42) \]
Using $|d| = 1$, Hölder’s inequality, inequality (2.4) and Young’s inequality, we have

\[
|I_{511}| \leq 2 \left\| \nabla^2 d_k \right\|_{L^1} \left\| \nabla d_{\frac{3}{2},k} \right\|_{L^\infty} \left\| \nabla \Delta d_{\frac{1}{2},k} \right\|_{L^2} \\
\leq C \left\| \Delta d_k \right\|_{L^1} \left\| \nabla d_{\frac{3}{2},k} \right\|_{L^\infty} \left\| \nabla \Delta d_{\frac{1}{2},k} \right\|_{L^2} \\
\leq C k \left\| \nabla d_k \right\|_{L^1} \left\| \nabla d_{\frac{3}{2},k} \right\|_{L^\infty} \left\| \nabla \Delta d_{\frac{1}{2},k} \right\|_{L^2} \\
\leq C k^2 \left\| \nabla d_{\frac{3}{2},k} \right\|_{L^\infty}^2 \left( \| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2 \right) + \frac{1}{8} \left\| \nabla \Delta d_{\frac{1}{2},k} \right\|_{L^2}^2.
\]

Similarly,

\[
|I_{512}| \leq \left\| \nabla^2 d_{\frac{3}{2},k} \right\|_{L^\infty} \left\| \nabla d_{\frac{7}{2},k} \right\|_{L^2} \left\| \nabla \Delta d_{\frac{11}{2},k} \right\|_{L^2} \\
\leq C k \left\| \nabla d_{\frac{3}{2},k} \right\|_{L^\infty} \left\| \nabla d_{\frac{7}{2},k} \right\|_{L^2} \left\| \nabla \Delta d_{\frac{11}{2},k} \right\|_{L^2} \\
\leq C k^2 \left\| \nabla d_{\frac{3}{2},k} \right\|_{L^\infty}^2 \left( \| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2 \right) + \frac{1}{8} \left\| \nabla \Delta d_{\frac{11}{2},k} \right\|_{L^2}^2.
\]

which all taken together implies

\[
|I_{511}| \leq C k^2 \left\| \nabla d_{\frac{3}{2},k} \right\|_{L^\infty}^2 \left( \| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2 \right) + \frac{1}{4} \left\| \nabla \Delta d_{\frac{11}{2},k} \right\|_{L^2}^2.
\]

By the fact $|d| = 1$, the Hölder inequality, (2.4) and (3.13), we can get

\[
|I_{512}| \leq 2 \left\| \nabla^2 d_k \right\|_{L^\infty} \left\| \nabla d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2} \\
\leq C \left\| \Delta d_k \right\|_{L^\infty} \left\| \nabla d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2} \\
\leq C k \left\| \nabla d_k \right\|_{L^\infty} \left\| \nabla d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2} \\
\leq C k^2 \left\| \nabla d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2}^2. \\
\leq C \left\| \Delta d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2}^2.
\]

Similarly,

\[
|I_{513}| \leq 2 \left\| \nabla^2 d_k \right\|_{L^2} \left\| \nabla d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2} \\
\leq C \left\| \nabla d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2}^2.
\]

Reasoning as (3.12) again, one has

\[
|I_{514}| \leq 2 \left\| \nabla^2 d_k \right\|_{L^2} \left\| \nabla d_k \right\|_{L^\infty} \left\| \nabla \Delta d_k \right\|_{L^2} \\
\leq C \left\| \nabla^2 d_k \right\|_{L^2} \left\| \nabla^2 d_k \right\|_{H^1} \left\| \nabla d_k \right\|_{H^1} \left\| \nabla \Delta d_k \right\|_{L^2} \\
\leq C \left\| \Delta d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2} \left\| \Delta d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2} \\
\leq C k^{-\frac{1}{2}} \left\| \Delta d_k \right\|_{L^2} \left\| \nabla \Delta d_k \right\|_{L^2}^2.
\]
Therefore, inequalities (3.45)–(3.48) yield
\[
|I_{51}| \leq CK^2 \| \nabla d_{k,k} \|^2_{L^2} + (\| u_0 \|^2_{L^2} + \| \nabla d_0 \|^2_{L^2}) + C \| \nabla d_k \|_{L^\infty}^2 \| \nabla \Delta d^k \|^2_{L^2} + C \| \Delta d^k \|^2_{L^2} \| \nabla \Delta d^k \|^2_{L^2} + \frac{1}{4} \| \nabla \Delta d_{k,2k} \|^2_{L^2}.
\] (3.49)

It is easy to get \( \Delta d \cdot d = -|\nabla d|^2 \) due to \( |d| = 1 \). Then \( |\nabla d|^2 \nabla d = -\Delta d \cdot d \nabla d \). Hence we decompose \( I_{52} \) in the following way:
\[
I_{52} = -I_{521} + I_{522} + I_{523} + I_{524}.
\]

Repeating the methods to prove (3.12), we obtain
\[
|I_{521}| \leq C \| \Delta d^k \|^2_{L^2} \| \nabla d^k \|^2_{L^2} \| \nabla \Delta d^k \|^2_{L^2} \leq C \| \Delta d^k \|^2_{L^2} \| \nabla d^k \|^2_{H^1} \| \nabla \Delta d^k \|^2_{L^2} \leq C \| \Delta d^k \|^2_{L^2} \| \nabla \Delta d^k \|^2_{L^2} \leq CK^{-\frac{1}{2}} \| \Delta d^k \|^2_{L^2} \| \nabla \Delta d^k \|^2_{L^2}.
\] (3.51)

Similarly to (3.46), we have
\[
|I_{522}| + |I_{523}| \leq C \| \nabla d_k \|_{L^\infty} \| \nabla \Delta d^k \|^2_{L^2}.
\] (3.52)

Similarly to (3.45), one has
\[
|I_{524}| \leq CK^2 \| \nabla d_{k,k} \|^2_{L^\infty} (\| u_0 \|^2_{L^2} + \| \nabla d_0 \|^2_{L^2}) + \frac{1}{4} \| \nabla \Delta d_{k,2k} \|^2_{L^2}.
\] (3.53)

Thus
\[
|I_{52}| \leq CK^{-\frac{1}{2}} \| \Delta d^k \|^2_{L^2} \| \nabla \Delta d^k \|^2_{L^2} + C \| \nabla d_k \|_{L^\infty} \| \nabla \Delta d^k \|^2_{L^2} + C \| \nabla d_k \|_{L^\infty} \| \nabla \Delta d^k \|^2_{L^2} + \frac{1}{4} \| \nabla \Delta d_{k,2k} \|^2_{L^2}.
\] (3.54)

From (3.49) and (3.54), we deduce
\[
|I_5| \leq CK^2 \| \nabla d_{k,k} \|^2_{L^\infty} (\| u_0 \|^2_{L^2} + \| \nabla d_0 \|^2_{L^2}) + C \| \nabla d_k \|_{L^\infty} \| \nabla \Delta d^k \|^2_{L^2} + CK^{-\frac{1}{2}} \| \Delta d^k \|^2_{L^2} \| \nabla \Delta d^k \|^2_{L^2} + \frac{1}{2} \| \nabla \Delta d_{k,2k} \|^2_{L^2}.
\] (3.55)
Combining (3.6), (3.16), (3.23), (3.39) and (3.55), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u^k \|_{L^2}^2 + \| \Delta d^k \|_{L^2}^2 \right) \\
\leq C_1 k^2 (\| u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2) (\| u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2) \\
+ C_2 (\| u_k \|_{L^2}^2 + \| \nabla u_k \|_{L^2}^2 \| \Delta u_k \|_{L^2}^2 + \| \nabla d_k \|_{L^2}^2 \| \Delta d_k \|_{L^2}^2) \\
+ C_3 k^{-2} (\| \nabla u^k \|_{L^2} \| \Delta d^k \|_{L^2}) (\| \nabla u^k \|_{L^2}^2 + \| \Delta d^k \|_{L^2}^2) + \frac{3}{4} \| \Delta u_{k,2k} \|_{L^2}^2 \\
+ \frac{3}{4} \| \nabla d_{k,2k} \|_{L^2}^2
\]

and

\[
\frac{d}{dt} \left( \| \nabla u^k \|_{L^2}^2 + \| \Delta d^k \|_{L^2}^2 \right) \\
\leq \left[ c_1 k^2 (\| u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2) (\| u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2) \\
- \frac{1}{8} (\| \Delta u^k \|_{L^2}^2 + \| \nabla d^k \|_{L^2}^2) \right] + \left[ c_2 (\| u_k \|_{L^2}^2 + \| \nabla u_k \|_{L^2}^2 \| \Delta u_k \|_{L^2}^2 + \| \nabla d_k \|_{L^2}^2 \| \Delta d_k \|_{L^2}^2) \\
+ \left( c_3 k^{-1/2} (\| \nabla u^k \|_{L^2} \| \Delta d^k \|_{L^2}) - \frac{1}{8} (\| \Delta u^k \|_{L^2}^2 + \| \nabla d^k \|_{L^2}^2) \right). \right.
\]

Let

\[
\tilde{k} = 128 \times 4 c_2^2 (\| \nabla u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2)^2.
\]

Then

\[
\| \nabla u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2 < \frac{\tilde{k}^2}{16c_3}.
\]

Since \( \lim_{\tau \to 0} \| \nabla u^k(\tau) \|_{L^2} + \| \Delta d^k(\tau) \|_{L^2} = \infty \), there is some \( \delta \in (0, T) \) such that

\[
\| \nabla u^k(T - \delta) \|_{L^2} + \| \Delta d^k(T - \delta) \|_{L^2} = \frac{\tilde{k}^2}{16c_3},
\]

\[
\| \nabla u^k(t) \|_{L^2} + \| \Delta d^k(t) \|_{L^2} > \frac{\tilde{k}^2}{16c_3}.
\]

From (3.60), we get for any \( t \in (T - \delta, T) \)

\[
c_1 \tilde{k}^2 (\| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2) (\| u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2) - \frac{1}{8} (\| \Delta u_0 \|_{L^2}^2 + \| \nabla d_0 \|_{L^2}^2) \\
\leq \tilde{k}^2 \left[ c_1 \left( \| u_0 \|_{L^2} \| \nabla d_0 \|_{L^2} \| \Delta u_0 \|_{L^2} + \| \nabla d_0 \|_{L^2} \| \Delta d_0 \|_{L^2} \right) - \frac{1}{8} \left( \| \nabla u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2 \right) \right] \\
\leq \tilde{k}^2 \left[ c_1 \left( \| u_0 \|_{L^2} \| \nabla d_0 \|_{L^2} \| \Delta u_0 \|_{L^2} + \| \nabla d_0 \|_{L^2} \| \Delta d_0 \|_{L^2} \right) - \frac{1}{8} \left( \| \nabla u_0 \|_{L^2}^2 + \| \Delta d_0 \|_{L^2}^2 \right) \right]
\[ \leq 0, \]

provided

\[ \| u_{2, \xi} (t) \|_{L^\infty} + \| \nabla d_{2, \xi} (t) \|_{L^\infty} \leq \frac{\tilde{k}}{32c_3 \sqrt{c_1} (\| u_0 \|_{L^2} + \| \nabla d_0 \|_{L^2})}, \quad \forall t \in (T - \delta, T). \]  \hfill (3.61)

In view of (3.58), the inequality (3.61) is equivalent to

\[
\| u_{2, \xi} (t) \|_{L^\infty \{T - \delta, T; \Omega \}}, \| \nabla d_{2, \xi} (t) \|_{L^\infty \{T - \delta, T; \Omega \}} < \frac{1}{c} \frac{1}{k^{\frac{1}{2}} 32c_3 \sqrt{c_1} (\| u_0 \|_{L^2} + \| \nabla d_0 \|_{L^2})},
\]

\[
< \frac{1}{16 \sqrt{2} c_3 (\| u_0 \|_{L^2} + \| \Delta d_0 \|_{L^2}) \times 32c_3 \sqrt{c_1} (\| u_0 \|_{L^2} + \| \nabla d_0 \|_{L^2})},
\]

\[
< \frac{1}{512 \sqrt{2} c_3 \sqrt{c_1} (\| u_0 \|_{L^2} + \| \nabla d_0 \|_{L^2}) (\| \nabla u_0 \|_{L^2} + \| \Delta d_0 \|_{L^2})}.
\]  \hfill (3.62)

Thus, if (3.62) and

\[ c_2 (\| u_\xi \|_{L^\infty \{T - \delta, T; \Omega \}}, \| \nabla d_\xi \|_{L^\infty \{T - \delta, T; \Omega \}}) \leq \frac{1}{4} \]  \hfill (3.63)

hold, we can infer from (3.57) that

\[
\frac{d}{dt} \left( \| \nabla u_\xi \|_{L^2}^2 + \| \Delta d_\xi \|_{L^2}^2 \right) 
\leq \left( c_3 k^{\frac{1}{2}} (\| \nabla u_\xi \|_{L^2} + \| \Delta d_\xi \|_{L^2}) - \frac{1}{8} \right) (\| \nabla u_\xi \|_{L^2}^2 + \| \Delta d_\xi \|_{L^2}^2).
\]  \hfill (3.64)

Since \( c_3 k^{\frac{1}{2}} (\| \nabla u_\xi (T - \delta) \|_{L^2} + \| \Delta d_\xi (T - \delta) \|_{L^2}) - \frac{1}{8} = c_3 k^{\frac{1}{2}} \frac{1}{16c_3} - \frac{1}{8} < 0 \), there is a right neighborhood \( I \) of \( t = T - \delta \) such that

\[ c_3 k^{\frac{1}{2}} (\| \nabla u_\xi (t) \|_{L^2} + \| \Delta d_\xi (t) \|_{L^2}) - \frac{1}{8} < 0, \quad \forall t \in I. \]

Hence, it follows by (3.64) that the function \( t \rightarrow \| \nabla u_\xi \|_{L^2} + \| \Delta d_\xi \|_{L^2} \) decreases in \( I \), which contradicts (3.59) and (3.60). Thus, when (3.62) and (3.63) are satisfied, \( u \) and \( \nabla d \) cannot blow up at \( t = T \), and \( u \) and \( \nabla d \) are regular in \( (0, T] \). The proof of the theorem is completed.

\[ \square \]

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### Availability of data and materials

Not applicable.

### Competing interests

The authors declare that they have no competing interests.
Authors’ contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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