Multiuser cognitive radio networks: an information-theoretic perspective

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Abstract Achievable rate regions and outer bounds are derived for three-user interference channels where the transmitters cooperate in a unidirectional manner via a noncausal message-sharing mechanism. The three-user channel facilitates different ways of message-sharing between the primary and secondary (or cognitive) transmitters. Three natural extensions of unidirectional message-sharing from two users to three users are introduced: (i) cumulative message sharing; (ii) primary-only message sharing; and (iii) cognitive-only message sharing. To emphasize the notion of interference management, channels are classified based on different rate-splitting strategies at the transmitters. The techniques of superposition coding and Gel’fand–Pinsker’s binning are employed to derive an achievable rate region for each of the cognitive interference channels. The results are specialized to the Gaussian channel, which enables a visual comparison of the achievable rate regions through simulations and helps us achieve some additional rate points under extreme assumptions. We also provide key insights into the role of rate-splitting at the transmitters as an aid to better interference management at the receivers.

Keywords Cognitive radio channels · Achievable regions · Outer bounds · Superposition coding

1 Introduction

Cognitive radios (CRs) [1] are devices that gather and exploit knowledge of their Radio Frequency (RF) environment to adjust their transmission and reception parameters, and improve the efficiency of spectral utilization. An overview of the potential benefits offered by the CRs in physical layer research is provided in [2]. In [3], three main CR paradigms are identified—underlay, overlay and interweave. In the underlay paradigm, CR users are allowed to operate only if their interference to noncognitive (or primary) users is below a certain threshold. While operating in the overlay paradigm, the CRs transmit their data simultaneously with the primary users but employ sophisticated techniques that maintain (or even improve) the performance of primary users. In the interweave paradigm, the CRs sense unused frequency bands called spectrum holes to communicate without disrupting primary transmissions. Of these, the information theoretic research has focused primarily on the overlay paradigm where CR transmitters cooperate using unidirectional message sharing in a noncausal manner. Here, the cognitive user gains access to messages and the corresponding codewords of the primary user before transmission. This is also referred to as noncausal or genie-aided message-sharing. Then, the primary and cognitive users simultaneously transmit their messages, but the encoding is performed in such a way that the primary user’s achievable rates do not suffer. We first present a short survey of recent information-theoretic work in this area, followed by a summary of our contributions.
1.1 Related work

Besides identifying the three CR network paradigms mentioned above, [3] explored some of the fundamental capacity limits and associated transmission strategies for CR wireless networks. In [4, 5], Devroye et al. defined the two-user discrete memoryless genie-aided CR channel and derived an achievable rate region by employing ratesplitting at both transmitters. An outer bound was proposed for the corresponding Gaussian channel by allowing bidirectional cooperation between the transmitters in a noncausal manner, resulting in a multiple antenna (MIMO) broadcast channel whose capacity region is well-known [6]. In [7], Wu et al. introduced terms like dumb and smart antennas to refer to primary and cognitive senders, respectively. Without resorting to rate-splitting, they derived inner and outer bounds for the general discrete memoryless channel, along with capacity results for some special cases. They also derived the capacity region for the Gaussian CR channel under weak interference assumption. In [8], Jovičić et al. presented the Gaussian CR channel and derived capacity results for the low interference regime where the primary receiver uses single-user decoding to achieve the capacity, while in the high interference regime joint code design at the two transmitters and multiuser decoding at the primary receiver is shown to be optimal to maximize the sum rate for the primary and cognitive users. In [9], an achievable rate region is derived for the two-user CR interference channel, where only the CR transmitter employs rate-splitting. In the high interference regime, the region presented in [9] subsumes the ones derived in [7, 8].

Capacity bounds for two-user interference channels with cognitive and partially cognitive transmitters were reported in [10–15], while [16–19] concerned with interference channels with common information. The sum-capacity of the Gaussian MIMO cognitive radio network was presented in [20], where the results applied to the single-antenna CR channel as well, while [21] presented capacity scaling laws for CR networks. Multiple access channels with cooperation have been considered in [22–23]. Broadcast channel with a cognitive relay has been addressed in [24]. Most recent results include [25–28], where a new achievable rate region for the two-user CR channel has been derived encompassing all the previous ones, with capacity results for a few classes of channels. The above mentioned references employ a combination of the coding scheme proposed by Han and Kobayashi for the interference channel [29], the binning technique proposed by Gel’fand and Pinsker (GP) for coding over channels with random parameters [30], superposition coding proposed first for the broadcast channel [31] and dirty-paper coding [32] to cancel out noncausal interference at the transmitter in the Gaussian channel case.

1.2 Our contribution

In this paper, we consider the case of three-user CR interference channels, where two (or one) CRs and one (or two) primary user communicate with three respective receivers. The following points summarize our main contributions:

1. Message-sharing mechanism: The first interesting observation we make is that there are multiple ways in which the two-user CR channel can be extended to the three-user CR channel, depending on the message sharing mechanism employed. We consider three intuitive schemes, namely (i) cumulative message sharing (CuMS); (ii) primary-only message sharing (PrMS); and (iii) cognitive-only message sharing (CoMS).

2. Interference management: Growing network-size presents issues related to interference management. To deal with interference in this three-user channel, we use rate-splitting, which was first reported in [29] to enlarge the achievable rate region for the classical two-user interference channel. To highlight the benefits and limitations of rate-splitting, we define five cognitive channel models, two each for CuMS and PrMS, and one for CoMS, which correspond to different ratesplitting strategies. The different types of message-sharing mechanisms and rate-splitting strategies will be made precise in the next section.

3. Achievable rate regions: We derive an achievable rate region for each of the five models by first considering the discrete memoryless version of the channel. To this end, we employ the technique of combining GP’s binning principle [30] with superposition coding [31]. As a result, we illustrate the generality of the techniques employed here, and provide useful and novel insights into the rate regions and their characterization.

4. Gaussian channel case: We specialize the achievable rate regions to the important special case of Gaussian
In Sect. 3, we consider the Gaussian channel model and derive outer bounds. Simulation results also state corollaries that enlarge the rate regions in the Gaussian case. We describe now the message-sharing mechanisms considered in this paper.

1. In the case of cumulative message-sharing (CuMS), sender \( S_2 \) has noncausal knowledge of the message \( m_1 \) and the corresponding codewords of the primary sender, \( S_1 \). Sender \( S_3 \) has noncausal knowledge of the message \( m_1 \) of the primary transmitter as well as the message \( m_2 \) of \( S_2 \), and their respective codewords. A schematic of CuMS is shown in Fig. 1.

2. In the case of primary-only message-sharing (PrMS), senders \( S_2 \) and \( S_3 \) have noncausal knowledge of the message \( m_1 \) and the corresponding codewords of the primary sender, \( S_1 \). There is no message-sharing mechanism between \( S_2 \) and \( S_3 \) themselves. See Fig. 2 for a channel schematic.

3. In the case of cognitive-only message-sharing (CoMS), sender \( S_1 \) has noncausal knowledge of messages \( m_1 \) and \( m_2 \), and the corresponding codewords of senders, \( S_1 \) and \( S_2 \). There is no message-sharing mechanism between \( S_1 \) and \( S_2 \). A channel schematic for CoMS is shown in Fig. 3.

An \((M_1, M_2, M_3, n, P^n)\) code exists for these channels, if there exists the following encoding functions:

\[
\begin{align*}
&f_1 : \mathcal{M}_1 \rightarrow \mathcal{X}_1^n, \quad f_1' : \mathcal{M}_1 \rightarrow \mathcal{X}_1^n, \quad f_1'' : \mathcal{M}_1 \rightarrow \mathcal{X}_1^n \\
f_2 : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{X}_2^n, \quad f_2' : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{X}_2^n, \quad f_2'' : \mathcal{M}_2 \rightarrow \mathcal{X}_2^n \\
f_3 : \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \rightarrow \mathcal{X}_3^n, \quad f_3' : \mathcal{M}_1 \times \mathcal{M}_3 \rightarrow \mathcal{X}_3^n, \\
f_3'' : \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \rightarrow \mathcal{X}_3^n
\end{align*}
\]

and the following decoding functions, for \( k = 1, 2, 3 \):

\[
\begin{align*}
g_k : \mathcal{Y}_k^n \rightarrow \mathcal{M}_k, \quad g_k' : \mathcal{Y}_k^n \rightarrow \mathcal{M}_k, \quad g_k'' : \mathcal{Y}_k^n \rightarrow \mathcal{M}_k
\end{align*}
\]

\[
\begin{align*}
&\begin{array}{c|c}
S_1 & m_1 \\
S_2 & (m_1, m_2) \\
S_3 & (m_1, m_2, m_3)
\end{array}
\end{align*}
\]

Fig. 1 Three-user cognitive channel with CuMS
such that the decoding error probability $\max \{p^{(n)}_{e,1}, p^{(n)}_{e,2}, p^{(n)}_{e,3}\}$ is $\leq p^{(n)}_{e,k}$. $p^{(n)}_{e,k}$ is the average probability of decoding error, which is computed as:

$$p^{(n)}_{e,k} = \frac{1}{M_1M_2M_3} \sum_{m_1,m_2,m_3} p[\hat{m}_k \neq m_k](m_1, m_2, m_3) \text{ sent};$$

$k = 1, 2, 3$.

$f_k$ (or $g_k$) correspond to the encoders (or decoders) used by channels with CuMS, $f^{(i)}_k$ (or $g^{(i)}_k$) correspond to the encoders (or decoders) used by channels with PrMS and $f^{(ii)}_k$ (or $g^{(ii)}_k$) correspond to the encoders (or decoders) used by channels with CoMS.

We define two channels denoted $C^t_{\text{CoMS}}$, two channels denoted $C^t_{\text{PrMS}}$ and one channel denoted $C_{\text{CoMS}}$; $t = 1, 2$. A non-negative rate triple $(R_1, R_2, R_3)$ is achievable for each of the channels, if there exists a sequence of $(2^{[nR_1]}, 2^{[nR_2]}, 2^{[nR_3]}, n, P^{(n)}_e)$ codes such that $P^{(n)}_e \rightarrow 0$ as $n \rightarrow \infty$. The capacity region for the channels is the closure of the set of all achievable rate triples $(R_1, R_2, R_3)$. A subset of the capacity region gives an achievable rate region.

### 2.2 Rate-splitting strategies

In [29], it has been shown that the achievable rate region for the classical two-user interference channel can be enlarged by rate-splitting. Specifically, each transmitter encodes part of the message at a possibly low rate and constructs its codewords using superposition coding. This results in the unintended or non-pairing receiver being able to decode and cancel out the low rate sub-message from the interfering transmitter using simultaneous decoding, thereby enlarging the achievable rate region. This forms the motivation for employing rate-splitting as an effective interference management mechanism. In the three-user scenario, however, many more rate-splitting strategies exist compared to the two-user case. For example, sender $S_1$ can perform rate-splitting in one of the following four ways: it can (i) encode a part of its message such that both unintended receivers, $R_2$ and $R_3$, can decode the sub-message; (ii) encode a part of the message such that $R_2$ can decode it but not $R_3$; (iii) encode a part of the message such that $R_3$ can decode it but not $R_2$; and finally, (iv) encode in a manner such that the sub-message is not decodable at either $R_2$ or $R_3$ (i.e., decodable only at the $R_1$). In this paper, we consider the following rate-splitting strategies:

1. In $C^1_{\text{CoMS}}$ and $C^1_{\text{PrMS}}$, the senders encode part of their respective messages at a rate such that it can be reliably decoded by all the receivers. The other part of the message is encoded at a rate such that only the intended or pairing receiver is required to be able to decode it.

2. In $C^2_{\text{CoMS}}$ and $C^2_{\text{PrMS}}$, one part of the message is encoded such that only the intended receiver can decode it, while the other part is encoded at a rate such that it can only be decoded at the intended receiver and the receiver $R_1$.

3. In $C^3_{\text{CoMS}}$, sender $S_3$ encodes one part of the message at a rate such that all receivers can decode it, while the other part is encoded at a rate such that it can only be decoded at its pairing receiver, $R_3$. There is no rate-splitting at $S_1$ and $S_2$.

Note that, regardless of the manner in which rate-splitting is performed, $R_i$ should always be able to reliably decode the codewords from $S_i$, $i = 1, 2, 3$. We consider the above described rate-splitting strategies for the following reasons:

1. To better understand the role of rate-splitting as a mechanism for interference management at the receivers, especially with growing network-size (for example, from two-user to three-user CR channels).
2. To demonstrate the increasing difficulty in characterizing theoretical limits with the number of rate-splits at the encoder. The number of probability-of-error terms at a decoder increases exponentially with the number of rate-splits at the encoder (both pairing and non-pairing encoders). Though the error analysis can be carried out, characterizing the rate/capacity region becomes cumbersome, leading to difficulties in the graphical representation of simulation results. This fact will become clearer when we try to plot the rate regions.

3. To demonstrate the effect of reduction in network-size on the rate region characterization. Specifically, we show that the achievable rate regions for the three-user CR channel reduces to that of the two-user case, corresponding to the rate-splitting strategies employed, when the network-size is scaled down from three users to two users.

The notation for describing the achievable rates of these sub-messages and their respective description is tabulated in Table 1. The decoding capabilities of receivers, resulting from rate-splitting at the transmitters, are summarized in Tables 2, 3 and 4. We also introduce auxiliary random variables defined on finite sets and tabulate them in Table 5. Depending on the rate-splitting strategy employed by the senders, only a subset of these sub-messages, their corresponding rates, and the corresponding auxiliary random variables will be used to derive an achievable rate region for each channel model. Note that, similar rate-splitting strategies in the context of the X-channel have been reported in [36–38].

2.3 Channel modification

Rate-splitting necessitates modification of the channels $C_{\text{CuMS}}^t, C_{\text{PMS}}^t$ and $C_{\text{CoMS}}^t$; $t = 1, 2$. Here, we explicitly show the modification for one channel ($C_{\text{CuMS}}$); the modification for the other channel models is similar. Referring to the rate-splitting strategy for the channel $C_{\text{CuMS}}^t$, the messages at the three senders in the modified channel can be written as:

Sender 1 $m_{11} \in M_{11} = \{1, \ldots, M_{11}\}$,
Sender 2 $m_{21} \in M_{21} = \{1, \ldots, M_{21}\}$,
$m_{22} \in M_{22} = \{1, \ldots, M_{22}\}$,
Sender 3 $m_{31} \in M_{31} = \{1, \ldots, M_{31}\}$,
$m_{33} \in M_{33} = \{1, \ldots, M_{33}\}$,

with all messages being defined on sets with finite number of elements. Note that, there is no rate-splitting at sender $S_1$, but for consistency in notation, we write $m_1$ as $m_{11}$.

We define an $\{M_{11}, M_{21}, M_{22}, M_{31}, M_{33}, n, P_{e_m}^n\}$ code for the modified channel as a set of $M_{11}$ codewords for $S_1$, $M_{11}M_{21}M_{22}$ codewords for $S_2$, and $M_{11}M_{21}M_{22}M_{31}M_{33}$ codewords for $S_3$, such that the average probability of decoding error is less than or equal to $P_{e_m}^n$. We call a tuple $(R_{11}, R_{21}, R_{22}, R_{31}, R_{33})$ achievable if there exists a sequence of $\{2^{[nR_{11}]}, 2^{[nR_{21}]}, 2^{[nR_{22}]}, 2^{[nR_{31}]}, 2^{[nR_{33}]}, n, P_{e_m}^n\}$ codes such that $P_{e_m}^n \to 0$ as $n \to \infty$. Here, $R_{11}$ corresponds to $R_1$. The capacity region for the modified channel is the closure of the set of all achievable rate tuples $(R_{11}, R_{21}, R_{22}, R_{31}, R_{33})$. It can be shown that if the rate tuple $(R_{11}, R_{21}, R_{22}, R_{31}, R_{33})$ is achievable for the modified channel, then the rate triple $(R_{11} + R_{21} + R_{22}, R_{31} + R_{33})$ is achievable for the channel $C_{\text{CuMS}}^2$ (see [29, Corollary 2.1]). In a similar fashion, the remaining channel models

### Table 1

| Sub-message | Rate | Description |
|-------------|------|-------------|
| $m_{10} \in \{1, \ldots, 2^{nR_{10}}\}$ | $R_{10}$ | Rate achieved: $S_1 \to (R_{11}, R_{21}, R_{31})$ |
| $m_{11} \in \{1, \ldots, 2^{nR_{11}}\}$ | $R_{11}$ | Rate achieved: $S_1 \to R_{11}$ |
| $m_{20} \in \{1, \ldots, 2^{nR_{20}}\}$ | $R_{20}$ | Rate achieved: $S_2 \to (R_{11}, R_{21}, R_{31})$ |
| $m_{21} \in \{1, \ldots, 2^{nR_{21}}\}$ | $R_{21}$ | Rate achieved: $S_2 \to (R_{11}, R_{21}, R_{31})$ |
| $m_{30} \in \{1, \ldots, 2^{nR_{30}}\}$ | $R_{30}$ | Rate achieved: $S_3 \to R_{33}$ |
| $m_{31} \in \{1, \ldots, 2^{nR_{31}}\}$ | $R_{31}$ | Rate achieved: $S_3 \to (R_{11}, R_{21}, R_{33})$ |
| $m_{33} \in \{1, \ldots, 2^{nR_{33}}\}$ | $R_{33}$ | Rate achieved: $S_3 \to (R_{11}, R_{21}, R_{33})$ |
| $m_{1} \in \{1, \ldots, 2^{nR_{1}}\}$ | $R_{1}$ | Rate achieved: $S_1 \to R_{1}$ |
| $m_{2} \in \{1, \ldots, 2^{nR_{2}}\}$ | $R_{2}$ | Rate achieved: $S_2 \to R_{2}$ |

For ex., $R_{11}$ is the rate achieved between $S_1$ and $R_{11}$, while $R_{21}$ is the rate achieved between $S_2$ and $R_{21}, R_{31}$, etc. The last two rows correspond to the channel $C_{\text{CoMS}}^1$, wherein the senders $S_1$ and $S_2$ do not employ rate-splitting.

### Table 2

| Receiver | Decoding capability |
|----------|---------------------|
| $R_1$ | $m_{10}, m_{11}, m_{20}, m_{30}$ |
| $R_2$ | $m_{10}, m_{20}, m_{22}, m_{30}$ |
| $R_3$ | $m_{10}, m_{20}, m_{30}, m_{33}$ |

For ex., receiver $R_2$ can decode messages $m_{10}, m_{20}, m_{22}, m_{30}$.

### Table 3

| Receiver | Decoding capability |
|----------|---------------------|
| $R_1$ | $m_{11}, m_{21}, m_{31}$ |
| $R_2$ | $m_{21}, m_{22}$ |
| $R_3$ | $m_{31}, m_{33}$ |

For ex., receiver $R_3$ can decode messages $m_{31}, m_{33}$.
can be appropriately modified; the details are omitted to avoid repetition.

2.4 Probability distributions

Here, we present the probability distribution functions which characterize the channels $C_{t}^{1}$, $C_{t}^{2}$, $C_{t}^{1}$, and $C_{t}^{2}$. Let $\mathcal{P}_{\text{CoMS}}$ denote the set of all joint probability distributions $p_{\text{CoMS}}(.)$; $t = 1,2$ respectively, that factor as follows:

$$
p_{\text{CoMS}}(q, w_{1}, w_{2}, x_{1}, u_{1}, u_{2}, x_{2}, v_{1}, v_{2}, y_{1}, y_{2}, y_{3}) = p(q)p(w_{1}, w_{2}, x_{1} | q)p(u_{1} | w_{1})p(u_{2} | w_{2})p(x_{2} | w_{1}, w_{2}, x_{1})p(v_{1} | w_{1}, u_{1}, u_{2}, v_{2})p(v_{2} | w_{2}, u_{2}, v_{1}, x_{2}, y_{1}, y_{2}, y_{3})
$$

(1)

$$
p_{\text{PrMS}}(q, w, x, u_{1}, u_{2}, x_{1}, y_{1}, y_{2}, y_{3}) = p(q)p(w, x_{1} | q)p(u_{1} | w)q(u_{2} | w)q(x_{2} | w, x_{1})p(v_{1} | w, u_{1}, u_{2}, v_{2})p(v_{2} | w, u_{2}, v_{1}, x_{2}, y_{1}, y_{2}, y_{3})
$$

(2)

Let $\mathcal{P}_{\text{PrMS}}$ denote the set of all joint probability distributions $p_{\text{PrMS}}(.)$; $t = 1,2$ respectively, that factor as follows:

$$
p_{\text{PrMS}}(q, w_{1}, w_{2}, x_{1}, u_{1}, u_{2}, x_{2}, v_{1}, v_{2}, y_{1}, y_{2}, y_{3}) = p(q)p(w_{1}, w_{2}, x_{1} | q)p(u_{1} | w_{1})p(u_{2} | w_{2})p(x_{2} | w_{1}, w_{2}, x_{1})p(v_{1} | w_{1}, u_{1}, u_{2}, v_{2})p(v_{2} | w_{2}, u_{2}, v_{1}, x_{2}, y_{1}, y_{2}, y_{3})
$$

(3)

$$
p_{\text{PrMS}}(q, w, x, u_{1}, u_{2}, x_{1}, y_{1}, y_{2}, y_{3}) = p(q)p(w, x_{1} | q)p(u_{1} | w)q(u_{2} | w)q(x_{2} | w, x_{1})p(v_{1} | w, u_{1}, u_{2}, v_{2})p(v_{2} | w, u_{2}, v_{1}, x_{2}, y_{1}, y_{2}, y_{3})
$$

(4)

Let $\mathcal{P}_{\text{CoMS}}$ denote the set of all joint probability distributions $p_{\text{CoMS}}(.)$ respectively, that factor as follows:

$$
p_{\text{CoMS}}(q, w_{1}, x_{1}, u_{2}, x_{2}, v_{1}, v_{2}, y_{1}, y_{2}, y_{3}) = p(q)p(w_{1}, x_{1} | q)p(u_{2} | x_{2})p(v_{1} | w_{1}, u_{2}, v_{2})p(v_{2} | w_{2}, u_{2}, v_{1}, y_{1}, y_{2}, y_{3})
$$

(5)

The lower case letters ($q, w, u, v, y, etc.$) are realizations of their corresponding random variables, and note that for notational simplicity, the same letter ($p$) is used to denote all the different probability distributions above. An achievable rate region for each channel is defined by a set of non-negative real numbers (referred to as rate tuples) that satisfy certain information-theoretic inequalities. An achievable rate region for each of the channels considered in this paper are given in Appendices 1–3.

2.5 Achievability theorem

**Theorem 2.1** Let $C_{t}^{1}$, $C_{t}^{2}$, $C_{t}^{1}$, and $C_{t}^{2}$ denote the capacity region of the channel $C_{t}^{1}$, $C_{t}^{2}$, and $C_{t}^{1}$, $C_{t}^{2}$; $t = 1,2$. Let

$$
\mathcal{R}_{\text{CoMS}} = \bigcup_{p_{\text{CoMS}}(.) \in \mathcal{P}_{\text{CoMS}}} \mathcal{R}_{\text{CoMS}}(p_{\text{CoMS}});
$$

(6)

$$
\mathcal{R}_{\text{PrMS}} = \bigcup_{p_{\text{PrMS}}(.) \in \mathcal{P}_{\text{PrMS}}} \mathcal{R}_{\text{PrMS}}(p_{\text{PrMS}});
$$

(7)

$$
\mathcal{R}_{\text{CoMS}} = \bigcup_{p_{\text{CoMS}}(.) \in \mathcal{P}_{\text{CoMS}}} \mathcal{R}_{\text{CoMS}}(p_{\text{CoMS}});
$$

(8)

In the above, $\mathcal{R}_{\text{CoMS}}(p_{\text{CoMS}})$ denotes a set of achievable rates when the channel is characterized by the joint probability distribution function $p_{\text{CoMS}}$, and similar definitions apply for the other notations used. The region $\mathcal{R}_{\text{CoMS}}$ or $\mathcal{R}_{\text{PrMS}}$ or $\mathcal{R}_{\text{CoMS}}$ is an achievable rate region for the channel $C_{t}^{1}$, $C_{t}^{2}$, $C_{t}^{1}$, and $C_{t}^{2}$. i.e., $\mathcal{R}_{\text{CoMS}}$ or $\mathcal{R}_{\text{PrMS}}$ or $\mathcal{R}_{\text{CoMS}} \subseteq \mathcal{R}_{\text{CoMS}}$ or $\mathcal{R}_{\text{PrMS}}$ or $\mathcal{R}_{\text{CoMS}}$.

We employ the technique of combining the GP binning principle [30] with superposition coding [31] to prove the coding theorem and derive a set of achievable rates for each of the channel models. We present the proof for the channel $C_{t}^{1}$ in Appendix 4. The proof for the remaining three channels ($C_{t}^{1}$, $C_{t}^{2}$, $C_{t}^{1}$, $C_{t}^{2}$, and $C_{t}^{2}$) are along similar lines and are omitted.

3 The Gaussian case

In this section, we introduce the Gaussian CR channel to (i) evaluate and plot the rate region for the different channel models considered in this paper, (ii) describe several extensions, in the form of corollaries, to the achievable rate regions described above, and (iii) derive some outer bounds to help us test the optimality of the coding techniques that we have employed to derive the achievable rate regions.

3.1 The Gaussian CR channel

The achievable rate regions described for the discrete memoryless channels can be extended to the Gaussian channels by quantizing the channel inputs and outputs [39]. Let $C_{t}^{G}$ denote the cognitive Gaussian channel with cumulative message sharing, $C_{t}^{G}$ the cognitive Gaussian channel with PrMS and $C_{t}^{G}$ the cognitive Gaussian channel with CoMS ($G$ for Gaussian, CuMS, PrMS and CoMS are the same as before); $t = 1,2$. We show the extension for only one of the channel models—from $C_{t}^{2, \text{CuMS}}$ to $C_{t}^{2, \text{CuMS}}$.
The cognitive Gaussian channel is described by a discrete-time input $X_k$, a corresponding output $Y_k$, and a random variable $Z_k$ denoting noise at the receiver; $k = 1, 2, 3$. Following the maximum-entropy theorem [40], the input random variable $X_k; k = 1, 2, 3$ is assumed to have a Gaussian distribution. The transmitted codeword $\tilde{X}_k = (\tilde{x}_{k1}, \ldots, \tilde{x}_{kN})$ satisfies the average power constraint given by $\mathbb{E}(\|\tilde{X}_k\|^2) \leq \tilde{P}_k; k = 1, 2, 3$, where $\mathbb{E}(\cdot)$ is the expectation operator. The zero-mean random variable $Z_k$ is drawn i.i.d from a Gaussian distribution with variance $\bar{N}_k; k = 1, 2, 3$, and is assumed to be independent of the signal $X_k$. The Gaussian CR channel can be converted to a standard from using invertible transformations [8, 41].

For the channel $C^0_{G,CuMS}$, we have $W, U_1, U_2, V_1$ and $V_3$ as the random variables (RV) which describe the sources at the transmitters. We also some consider additional RVs—$W, U_1, U_2, V_1$ and $V_3$—with the following statistics:

$\tilde{W} \sim N(0, P_1), \tilde{U}_1 \sim N(0, \tau P_2), \tilde{U}_2 \sim N(0, \tau P_2)$, with $\tau + \tilde{\tau} = 1, \tilde{V}_1 \sim N(0, \bar{N} P_3)$, $\tilde{V}_3 \sim N(0, \bar{N} P_3)$, with $\bar{\tau} + \tilde{\tau} = 1$. Further, $W = \tilde{W}, U_1 = \tilde{U}_1 + x_1 X_1, U_2 = \tilde{U}_2 + x_2 X_1, V_1 = \tilde{V}_1 + x_3 X_1 + \beta_1 X_2, V_3 = \tilde{V}_3 + x_4 X_1 + \beta_2 X_2$, where the input RV’s $X_1, X_2$ and $X_3$ are given by $X_1, X_2, X_3$ and $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$. Notice that $W, U_1, U_2, V_1$ and $V_3$ are mutually independent. Therefore, $X_1 \sim N(0, P_1), X_2 \sim N(0, P_2)$ and $X_3 \sim N(0, P_3)$. The values of $\tau$ and $\bar{\tau}$ are randomly selected from the interval $[0, 1]$. The values of $x_1, x_2, x_3, x_4, \beta_1$ and $\beta_2$ are repeatedly generated according to $\tilde{N}(0, 1)$. The channel outputs are

$Y_1 = X_1 + a_{12} X_2 + a_{13} X_3 + Z_1,$

$Y_2 = a_{21} X_1 + X_2 + a_{23} X_3 + Z_2,$

$Y_3 = a_{31} X_1 + a_{32} X_2 + X_3 + Z_3,$

where $Z_1 \sim N(0, Q_1), Z_2 \sim N(0, Q_2)$ and $Z_3 \sim N(0, Q_3)$ are independent additive noise, and $Q_1, Q_2$ and $Q_3$ are noise variances when the input-output relations are represented in the standard form. Substituting for $X_1, X_2$ and $X_3$, we get

$Y_1 = W + a_{12}(\tilde{U}_1 + \tilde{U}_2) + a_{13}(\tilde{V}_1 + \tilde{V}_3) + Z_1,$

$Y_2 = a_{21} \tilde{W} + (\tilde{U}_1 + \tilde{U}_2) + a_{23}(\tilde{V}_1 + \tilde{V}_3) + Z_2,$

$Y_3 = a_{31} \tilde{W} + a_{32}(\tilde{U}_1 + \tilde{U}_2) + \tilde{V}_1 + \tilde{V}_3 + Z_3,$

where the interference coefficients $a_{12}, a_{13}, a_{21}, a_{23}, a_{31}$ and $a_{32}$ are assumed to be real and globally known. The rate region $\mathcal{R}^2_{CuMS}$ for the channel $C^2_{CuMS}$ can be extended to its respective Gaussian channel model by evaluating the mutual information terms. To this end, we construct a covariance matrix and compute its entries. Let us define a vector $\Theta = (Y_1, Y_2, Y_3, W, U_1, U_2, V_1, V_3)$. The covariance matrix is given by $\text{COV}(Y_1, Y_2, Y_3, W, U_1, U_2, V_1, V_3) = \mathbb{E}(\Theta^\top \Theta)$. The entries of this covariance matrix are used to compute the differential entropy terms, which are further used to evaluate the mutual information.

**Theorem 3.1** Let $\mathcal{R} = (\tau, \bar{\tau}, x_1, x_2, x_3, a_{13}, \beta_1, \beta_2)$. For a fixed $\mathcal{R}$, let $\mathcal{R}^2_{CuMS}(\mathcal{R})$ be achievable. The rate region $\mathcal{R}^2_{CuMS}$ is achievable for the Gaussian channel $C^2_{G,CuMS}$ with

$\mathcal{R}^2_{CuMS} = \bigcup_{\mathcal{R}_t} \mathcal{R}^2_{CuMS}(\mathcal{R})$.

Since the computation procedure is cumbersome and lengthy albeit straightforward, we do not provide the proof here. The same procedure is followed to compute the mutual information terms for the remaining channel models—$C^1_{G,CuMS}, C^1_{G,PMS}; t = 1, 2,$ and $C^1_{G,CMS}$.

### 3.2 Extensions

We state some corollaries in this subsection which allow additional achievable rate tuples for the Gaussian channel to be identified by treating the cognitive transmitters as relays, depending on their knowledge of the other user’s message. Also note that the achievable rate points below are presented as separate corollaries for clarity of presentation; one could state them together as one single result as well. The proofs for some of the corollaries can be found in Appendix 5.

### 3.2.1 $C^1_{G,CMS}$

**Corollary 3.2** Let $\mathcal{R}^2_{CuMS}$ be the set of all points $(R_1, R_{21} + R_{22}, R_{31} + R_{33})$ where $(R_1, R_{21}, R_{22}, R_{31}, R_{33})$ is an achievable rate tuple of Theorem 3.1. Then, the convex hull of the region $\mathcal{R}^2_{CuMS}$ with the points $(R_1, 0, 0)$ and $(0, R_2, R_3)$ is achievable for the $C^1_{G,CMS}$ model, where

$R^1_1 = \frac{1}{2} \log_2 \left( 1 + \frac{(\sqrt{P_1} + |a_{12}| \sqrt{P_2} + |a_{13}| \sqrt{P_3})^2}{Q_1} \right),$  

$R^1_2 = \frac{1}{2} \log_2 \left( 1 + \frac{P_2}{Q_2 + |a_{23}|^2 P_3} \right),$  

$R^1_3 = \frac{1}{2} \log_2 \left( 1 + \frac{P_3}{Q_3} \right).$

**Corollary 3.3** Let $\mathcal{R}^2_{CuMS}$ be the set of all points $(R_1, R_{21} + R_{22}, R_{31} + R_{33})$, where $(R_1, R_{21}, R_{22}, R_{31}, R_{33})$ is an achievable rate tuple of Theorem 3.1. Then the convex hull of the region $\mathcal{R}^2_{CuMS}$ with the points $(R_1, 0, 0)$ and $(0, R_2, R_3)$ are achievable for the $C^0_{G,CMS}$ model, where

$R^0_1 = \frac{1}{2} \log_2 \left( 1 + \frac{(\sqrt{P_1} + |a_{12}| \sqrt{P_2} + |a_{13}| \sqrt{P_3})^2}{Q_1 + |a_{13}|^2 P_3} \right).$
\[
R_2^* = \frac{1}{2} \log_2 \left(1 + \frac{\left(\sqrt{P_2} + |a_{23}| \sqrt{P_3^S} \right)^2}{Q_2 + |a_{23}|^2 P_3^S} \right),
\]
(10)

\[
r = \frac{1}{2} \log_2 \left(1 + \frac{P_3^S}{Q_3} \right),
\]
(11)

where \( P_3^S = P_3^S = P_3 - P_3^S, \forall P_3^S \in [0, P_3]. \)

**Corollary 3.4** The convex hull of the region \( \mathcal{C}_{G,CAM}^2 \) with the points \((R_1^*, 0, r)\) and \((0, r, R_3^*)\) is achievable for the \( C_{G,CAM}^* \) model, where

\[
R_1^* = \frac{1}{2} \log_2 \left(1 + \frac{\left(\sqrt{P_1} + |a_{12}| \sqrt{P_2^S} + |a_{13}| \sqrt{P_3} \right)^2}{Q_1 + |a_{12}|^2 P_2^S} \right),
\]
(12)

\[
r = \frac{1}{2} \log_2 \left(1 + \frac{P_3^S}{Q_2 + |a_{23}|^2 P_3} \right),
\]
(13)

\[
R_3^* = \frac{1}{2} \log_2 \left(1 + \frac{P_3}{Q_3} \right),
\]
(14)

where \( P_3^S = (2^{2r} - 1)(Q_2 + |a_{23}|^2 P_3), P_2^S = P_2 - P_2^S \) and \( r \) is the minimum rate that \( S_2 \) is guaranteed to achieve.

**Corollary 3.5** The convex hull of the region \( \mathcal{C}_{G,CAM}^2 \) with the points \((0, R_2^*, 0)\) and \((0, 0, R_3^*)\) is achievable for the \( C_{G,CAM}^* \) model, where

\[
R_2^* = \frac{1}{2} \log_2 \left(1 + \frac{\left(\sqrt{P_2} + |a_{23} \sqrt{P_3} \right)^2}{Q_2} \right),
\]
(15)

\[
R_3^* = \frac{1}{2} \log_2 \left(1 + \frac{P_3}{Q_3} \right),
\]
(16)

The following theorem follows directly from standard time-sharing arguments.

**Theorem 3.6** The convex hull of the region \( \mathcal{C}_{G,CAM}^2 \) with the achievable points in the Corollaries 3.2–3.5 results in an achievable rate region of the \( C_{G,CAM}^* \) channel model.

### 3.2.2 \( C_{G,PMS}^* \)

**Corollary 3.7** Let \( \mathcal{C}_{PMS} \) be the set of all points \((R_1, R_2, R_3) \in \mathbb{R}^3 \) such that \((R_1, R_2, R_2, R_3, R_3) \) is an achievable rate tuple. Then the convex hull of the region \( \mathcal{C}_{PMS} \) with the points \((R_1^*, 0, r)\) and \((0, r, R_3^*)\) are achievable for the \( C_{G,PMS}^* \) model, where

\[
R_1^* = \frac{1}{2} \log_2 \left(1 + \frac{\left(\sqrt{P_1} + |a_{12}| \sqrt{P_2} + |a_{13}| \sqrt{P_3} \right)^2}{Q_1} \right),
\]
(17)

\[
R_2^* = \frac{1}{2} \log_2 \left(1 + \frac{P_2}{Q_2 + |a_{23}|^2 P_3} \right),
\]
(18)

\[
R_3^* = \frac{1}{2} \log_2 \left(1 + \frac{P_3}{Q_3 + |a_{32}|^2 P_2} \right),
\]
(19)

**Corollary 3.8** The convex hull of the region \( \mathcal{C}_{PMS}^2 \) with the points \((R_1^*, 0, r)\) and \((0, r, R_3^*)\) are achievable for the \( C_{G,PMS}^* \) model, where

\[
R_1^* = \frac{1}{2} \log_2 \left(1 + \frac{\left(\sqrt{P_1} + |a_{12}| \sqrt{P_2} + |a_{13}| \sqrt{P_3} \right)^2}{Q_1 + |a_{12}|^2 P_2^S} \right),
\]
(20)

\[
R_2^* = \frac{1}{2} \log_2 \left(1 + \frac{P_2}{Q_2 + |a_{23}|^2 P_3} \right),
\]
(21)

\[
r = \frac{1}{2} \log_2 \left(1 + \frac{P_3^S}{Q_3 + |a_{32}|^2 P_2} \right),
\]
(22)

where \( P_3^S = P_3 - P_3^S, \forall P_3^S \in [0, P_3]. \)

**Corollary 3.9** The convex hull of the region \( \mathcal{C}_{PMS}^2 \) with the points \((0, R_2^*, 0)\) and \((0, 0, R_3^*)\) are achievable for the \( C_{G,PMS}^* \) model, where

\[
R_2^* = \frac{1}{2} \log_2 \left(1 + \frac{\left(\sqrt{P_2} + |a_{23} \sqrt{P_3} \right)^2}{Q_2} \right),
\]
(23)

\[
r = \frac{1}{2} \log_2 \left(1 + \frac{P_3^S}{Q_3 + |a_{32}|^2 P_2} \right),
\]
(24)

\[
R_3^* = \frac{1}{2} \log_2 \left(1 + \frac{P_3}{Q_3 + |a_{32}|^2 P_2} \right),
\]
(25)

where \( P_3^S = P_2 - P_2^S, \forall P_2^S \in [0, P_2]. \)

The following theorem follows directly from standard time-sharing arguments.

**Theorem 3.10** The convex hull of the region \( \mathcal{C}_{PMS}^2 \) with the achievable points in the Corollaries 3.7–3.9 results in an achievable rate region of the \( C_{G,PMS}^* \) channel model.

### 3.2.3 \( C_{G,CAM}^* \)

**Corollary 3.11** Let \( \mathcal{C}_{CAM} \) be the set of all points \((R_1, R_2, R_3) \in \mathbb{R}^3 \) such that \((R_1, R_2, R_3, R_3) \) is an achievable rate tuple. Then the convex hull of the region

\[
R_1^* = \frac{1}{2} \log_2 \left(1 + \frac{\left(\sqrt{P_1} + |a_{12}| \sqrt{P_2} + |a_{13}| \sqrt{P_3} \right)^2}{Q_1} \right),
\]
(17)
The convex hull of the region \( \mathcal{G}_{\text{CoMS}} \) with the points \((R_1^*, 0, 0), (0, R_2^*, 0)\) and \((0, 0, R_3^*)\) are achievable for the \( \mathcal{C}_{G,\text{CoMS}} \) model, where

\[
R_1^* = \frac{1}{2} \log_2 \left( 1 + \frac{\left( \sqrt{P_1} + |a_{13}| \sqrt{P_3} \right)^2}{Q_1 + |a_{12}|^2 P_2} \right),
\]

(26)

\[
R_2^* = \frac{1}{2} \log_2 \left( 1 + \frac{\left( \sqrt{P_2} + |a_{23}| \sqrt{P_3} \right)^2}{Q_2 + |a_{21}|^2 P_1} \right),
\]

(27)

\[
R_3^* = \frac{1}{2} \log_2 \left( 1 + \frac{P_3}{Q_3} \right).
\]

(28)

**Corollary 3.12** The convex hull of the region \( \mathcal{G}_{\text{CoMS}} \) with the points \((R_1^*, 0, r), (0, R_2^*, r)\) and \((0, 0, r)\) are achievable for the \( \mathcal{C}_{G,\text{CoMS}} \) model, where

\[
R_1^* = \frac{1}{2} \log_2 \left( 1 + \frac{\left( \sqrt{P_1} + |a_{13}| \sqrt{P_3} \right)^2}{Q_1 + |a_{12}|^2 P_2 + |a_{13}|^2 P_3} \right),
\]

(29)

\[
R_2^* = \frac{1}{2} \log_2 \left( 1 + \frac{\left( \sqrt{P_2} + |a_{23}| \sqrt{P_3} \right)^2}{Q_2 + |a_{21}|^2 P_1 + |a_{23}|^2 P_3} \right),
\]

(30)

\[
r = \frac{1}{2} \log_2 \left( 1 + \frac{P_3^{S_1}}{Q_3} \right),
\]

(31)

where \( P_3^{S_1} = P_3^{S_2} = P_3 - P_3^{S_1}, \forall P_3^{S_1} \in [0, P_3]. \)

The following theorem follows directly from standard time-sharing arguments.

**Theorem 3.13** The convex hull of the region \( \mathcal{G}_{\text{CoMS}} \) with the achievable points in the Corollaries 3.11–3.12 results in an achievable rate region of the \( \mathcal{C}_{G,\text{CoMS}} \) channel model.

### 3.3 Outer bounds

For the channel models considered in this paper, we derive outer bounds by considering a scenario where the transmitters cooperate in a bidirectional manner, i.e., cooperating senders know each other’s message in a noncausal manner. Since bidirectional message sharing is tantamount to having additional message at the transmitters compared to the CR channels, it cannot hurt the capacity. Then, the channel models reduce to a multiple antenna broadcast channel (MIMO-BC) with one sender having three antennas and three receivers with one antenna each. Hence, the capacity region of the MIMO-BC (see [6]) is an outer bound on our achievable rate regions. We resort to duality results of the broadcast (BC) and the multiple access channels (MAC), reported first in [42] to calculate the capacity of the MIMO-BC. A similar approach was followed in [4] to derive an outer bound for the two-user CR channel.

Let \( P \) be the total power constraint for the MIMO-BC and \( P_1, P_2 \) and \( P_3 \) be the individual power constraint for the MAC. On the MAC channel, the rate achieved by user \( j \) is given by

\[
R_{\text{MAC},j} = \log_2 \left| \frac{I + \sum_{k=j}^{K} H_k^H P_j H_k}{I + \sum_{k=j+1}^{K} H_k^H P_j H_k} \right|,
\]

(32)

where \(|A|\) denotes the determinant of \( A \); and the channel matrices are \( H_1 = [a_{12} \ a_{13}], H_2 = [a_{23} \ a_{21}] \) and \( H_3 = [a_{31} \ a_{32}] \); and \( I + \sum_{k=j+1}^{K} H_k^H P_j H_k \) is the interference experienced by the \( j \)th user. The MIMO-BC capacity region with power constraint \( P \) is equal to the union of capacity regions of the dual MAC, where the union is taken over all individual power constraint, \( P_1, P_2 \) and \( P_3 \), such that \( P = P_1 + P_2 + P_3 \). Therefore,

\[
C_{\text{BC}}(P, H) = \bigcup_{P_1, P_2, P_3} C_{\text{MAC}}(P_1, P_2, P_3; H^T),
\]

(33)

where

\[
C_{\text{MAC}}(P_1, P_2, P_3; H^T) = \bigcup_{j\in\{1,2,3\}} R_{\text{MAC},j},
\]

(34)

where \( R_{\text{MAC},j} \) is given by (32). We thus obtain the capacity region of the MIMO-BC, which forms an outer bound for the channel models considered in this paper. Generally, this outer bound tends to be loose, since the MIMO-BC capacity region was obtained by allowing bidirectional (or complete) transmitter cooperation. Nonetheless, these outer bounds provide useful insights into the strengths and weaknesses of the proposed achievable rate regions, as will be shown in the simulation results section, Sect. 4. To the best of our knowledge, this is the first set of outer bounds that have been derived for the three-user Gaussian CR channel. The rates of individual users can be further bounded depending on the model (CuMS, PrMS or CoMS).

1. In the case of CuMS, senders \( S_2 \) and \( S_3 \) have complete knowledge of the \( S_1 \)’s message and \( S_3 \) has knowledge of \( S_2 \)’s message but not vice-versa. Note that, the rate of \( S_1 \) cannot be bounded by the interference-free case where \( a_{12} = 0 \) and \( a_{13} = 0 \). This is because unidirectional message sharing enables \( S_2 \) and \( S_3 \) to transmit the message of \( S_1 \), thereby increasing the rate of \( S_1 \) beyond what is achievable with the \( S_1 \) alone transmitting its message. Hence, the rate \( R_1 \) can upper bounded as follows.

\[
R_1 \leq \frac{1}{2} \log_2 \left( 1 + \frac{(\sqrt{P_1} + |a_{12}| \sqrt{P_2} + |a_{13}| \sqrt{P_3})^2}{Q_1} \right).
\]

(35)
Similarly, the rate of $S_2$ cannot be bounded by the interference free rate, as $S_1$ can use its knowledge of $S_2$’s message to enable $S_2$ increase its rate. Hence, the rate of $S_2$ can upper bounded as

$$R_2 \leq \frac{1}{2} \log_2 \left( 1 + \frac{\left( \sqrt{P_2} + |a_{23}| \sqrt{P_3} \right)^2}{Q_2} \right). \quad (36)$$

Finally, the rate of $S_3$ can be upper bounded by the interference free case.

$$R_3 \leq \frac{1}{2} \log_2 \left( 1 + \frac{P_3}{Q_3} \right). \quad (37)$$

2. In the case of PrMS, although $S_2$ and $S_3$ have complete knowledge of $S_1$’s message, they do not have each other’s message. Therefore, the bound on the $S_1$’s rate given by (35) remains valid, as the $S_2$ and $S_3$ can use their knowledge of $S_1$’s message to increase its rate. The bound on $S_3$’s rate is the same as in the case of CuMS and is given by (37). Lastly, the $S_2$’s rate can be upper bounded by the interference-free case, as follows:

$$R_2 \leq \frac{1}{2} \log_2 \left( 1 + \frac{P_3}{Q_2} \right). \quad (38)$$

3. We now upper bound the sum rate of $S_2$ and $S_3$ by allowing full cooperation between their transmitters and pairing receivers. This results in a point-to-point MIMO channel, whose capacity is expressed as follows:

$$C_{\text{MIMO}} = \max_{\sum p_i \leq P} \sum_{i=1}^{N} \log_2 \left( 1 + \frac{P_i \sigma_i^2}{Q} \right), \quad (39)$$

where $\frac{P_i \sigma_i^2}{Q}$ is the signal-to-noise ratio associated with the $i$th channel, $\sigma_i$’s are the singular values and $N$ represents the number of singular values of the MIMO channel. The optimum power allocation $P_i$ can be obtained by the water-filling algorithm [40].

4. In the case of CoMS, sender $S_3$ has noncausal knowledge of $S_1$ and $S_2$. Therefore, the rates of $S_1$ and $S_2$ cannot be bounded by the interference free scenario. The rate of $S_1$ can be upper bounded as follows:

$$R_1 \leq \frac{1}{2} \log_2 \left( 1 + \frac{\left( \sqrt{P_1} + |a_{13}| \sqrt{P_3} \right)^2}{Q_1} \right). \quad (40)$$

The rates of $S_2$ and $S_3$ can be upper bounded as in (36) and (37), respectively. To bound the sum rate of $S_1$ and $S_2$ we allow full cooperation between the transmitters and pairing receivers, resulting in a point-to-point MIMO channel. The capacity of this channel is given by (39).

Note that the outer bounds presented in this paper have been derived by providing additional side-information to the transmitters. Such genie-aided outer bounds can also be derived by giving side-information to the receivers, as can be found in [43].

4 Simulation results and discussion

4.1 Simulation Setup

We consider a 3-user Gaussian cognitive channel with CuMS, PrMS and CoMS for the simulations. We generate the source and channel symbols as described in Sect. 3. The direct channel gains are $a_{11} = a_{22} = a_{33} = 1$. The interference coefficients $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.55$. The values of $\tau$ and $\kappa$ are assumed to be randomly selected from the interval [0,1]. The values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1$ and $\beta_2$ are repeatedly generated according to $\mathcal{N}(0,1)$. The transmit powers $P_1 = P_2 = P_3 = 7.8$ dB or 10 dB, as specified.

4.2 Results and discussion

1. Two-user channels: We present first results for the two-user interference and CR channels. Though these results exist in the literature, we focus on the interplay between rate-splitting and message sharing on the rates achievable on the channels. Figure 4 shows rate regions for the two-user interference and CR channels with different rate-splitting strategies. In Fig. 4a, we show the rate region for the two-user interference channel where only one of the senders employ rate-splitting. The interference channel where both senders employ rate-splitting is shown in Fig. 4b. This is the Han-Kobayashi region [31]. The rate region for the CR channel where only one of the senders employ rate-splitting is plotted in Fig. 4c. This is the region presented in [9]. Finally, the CR channel where both senders employ rate-splitting is shown in Fig. 4d. This rate region is slightly larger than the one presented in [4], we further highlight this in Sect. 4.3. We also plot the capacity region of the two-user MIMO Gaussian broadcast channel [6] as an outer bound to achievable rate regions.

We can transition from Fig. 4a to b by rate-splitting. We use message sharing to transition from Fig. 4a to c, which is larger than Fig. 4b. This indicates that message sharing is more beneficial than rate splitting to
2. Comparison of the three-user CR channels: The rate region for the channel $C_{\text{CoMS}}$ is obtained following Theorem 3.1. Similar procedures are adopted for $C_{\text{PMS}}$ and $C_{\text{CoMS}}$. We also plot the rate region for the threeuser interference channels corresponding to $C_{\text{CoMS}}$ and $C_{\text{PMS}}$, by considering a simple extension of the Han-Kobayashi scheme [29] to the three-user case. Table 3 summarizes the rate-splitting strategy employed by the encoders and the resulting decoding ability of receivers for the interference channel.

(a) We first consider the achievable rate regions for the channels $C_{\text{CoMS}}$, $C_{\text{PMS}}$, and the three-user interference channel. In Fig. 5, we plot the rates of $S_1$ and $S_3$, when $S_2$ achieves a minimum rate of $R_2 = 0.8 \text{ bps/Hz}$. We notice that $C_{\text{CoMS}}$ has a bigger rate region than $C_{\text{PMS}}$. This follows directly from the fact that in $C_{\text{CoMS}}$, the cognitive transmitter $S_2$ benefits from $S_3$, while in $C_{\text{PMS}}$ there is no cooperation between $S_2$ and $S_3$. Further, the rate regions for the CR channels are bigger than the corresponding three-user interference channel, a well-established fact in the classical two user scenario.

(b) In Fig. 6, we plot the rates of $S_1$ and $S_2$, when $S_3$ achieves a minimum rate of $R_3 = 1 \text{ bps/Hz}$, for the channels $C_{\text{CoMS}}, C_{\text{PMS}}$, and $C_{\text{CoMS}}$. As in the previous scenario (Fig. 5), $C_{\text{CoMS}}$ has a bigger rate region than $C_{\text{PMS}}$. We make an interesting observation by comparing the rate region of $C_{\text{CoMS}}$ with those of $C_{\text{CoMS}}$ and $C_{\text{PMS}}$. We notice that, for $C_{\text{CoMS}}$, the maximum achievable $R_1$ is lesser than those of $C_{\text{CoMS}}$ and $C_{\text{PMS}}$. This is due to the message-sharing strategy adopted by $C_{\text{CoMS}}$, where only $S_3$ aids the communication of $(S_1, R_1)$. We also observe that the maximum achievable $R_3$ is greater than those of $C_{\text{CoMS}}$ and $C_{\text{PMS}}$, when one expects it to be the same as in $C_{\text{CoMS}}$, since in both $C_{\text{CoMS}}$ and $C_{\text{CoMS}}$, $S_3$ aids the communication of $(S_2, R_3)$. The reason can be attributed to the difference in the rate-splitting strategy employed by these channel models (compare Tables 3 and 4). In case of $C_{\text{CoMS}}, S_2$ performs rate-splitting, thereby reducing the effective maximum achievable $R_2$. But, in case of $C_{\text{CoMS}}, S_2$ does not employ rate-splitting. This suggests that, similar to the two-user scenario, rate-splitting seems to be less effective than message-sharing. It also suggests that one cannot comment on the superiority of a particular message-sharing scheme compared to another. For example, consider the following. When a transmitter $S_1$ shares its message with another transmitter $S_2$, message splitting by $S_1$ is not necessary, as it does not have a significant impact on the rates achievable by $S_1$ and $S_2$. On the other hand, message splitting by $S_2$ helps improve the rate achievable by $S_1$, but does not significantly impact the rate achievable by $S_2$. Several such instances are possible, and one could argue that performing both rate-splitting and message-sharing would enlarge the overall achievable region. Also note that, in Fig. 6, we have not plotted the rate region of the interference channel. This is because, it is not fair to compare the rate region of the three-user interference channel corresponding to the rate-splitting strategy in Table 3 with that of $C_{\text{CoMS}}$ which is based on the rate-splitting strategy in Table 4: these correspond, inherently, to completely different channel models.

In the following, we plot the rate regions, corollaries and outer bounds (see Sect. 3.2) to obtain interesting insights into the achievable rate regions of the different channel models considered in this paper.

3. Three user channels with CuMS (channel $C_{\text{CoMS}}$):

(a) In Fig. 7, we plot the rate of sender $S_1$ ($R_1$) versus the sum of the rates of $S_2$ and $S_3$ (i.e., $R_2 + R_3$) for the channel $C_{\text{CoMS}}$. In the figure, the outer bound is the intersection of (33), (35)–(37) and (39). The innermost region corresponds to the achievable region given in Theorem 3.1. The second largest region corresponds to Corollary 3.2. Note that our inner bound is for a specific rate-splitting strategy at the transmitters, which the outer bounds do not account for, due to which the outer bound may be loose in the examples considered in this paper. More insight on the $R_2$ and $R_3$ achievable via our scheme, and how it compares with the outer bound, can be obtained from the plots presented later in the discussion.

(b) Figure 8 shows plots of the rate of $S_2$ ($R_2$) versus the rate of $S_3$ ($R_3$) when $S_1$ achieves a minimum...
rate of 0, 1 and 1.5 bps/Hz. Although there is a gap between the inner bound and the outer bound, Corollary 3.3 coincides with the outer bound at the corner points. Note that, due to the noncausal knowledge of $S_1$’s message, by employing dirty paper coding, the interference from $S_1$ can be eliminated at $R_3$. Owing to this, with increase in rate $R_1$, the rate $R_2$ does not decrease much for relatively small values of $R_1$. On the other hand, as the rate $R_1$ increases, sender $S_1$ cannot achieve the required rate without senders $S_2$ and $S_3$ using their noncausal message knowledge to help $S_1$. Due to this, for higher values of $R_1$, the achievable rates of $R_2$ and $R_3$ decrease, as expected. Similar observations can also be made in the remaining cases presented below.

(c) In Fig. 9, we plot the rate of $S_1$ ($R_1$), versus that of $S_2$ ($R_2$), when $S_1$ achieves a minimum rate of $R_3 = 0, 1$ and 1.5 bps/Hz. The case with $R_3 = 0$ corresponds to a two-user CR channel with a cognitive relay, where the relay $S_3$ has noncausal knowledge of the messages from $S_1$ and $S_2$. We note that this is the first instance in the literature where an achievable region for the two-user CR channel with a cognitive relay is presented. The gap between the inner bound and the outer bound is relatively small. The rate of $S_2$ does not decrease much, as it employs dirty paper coding to eliminate interference when $S_1$ and $S_3$ achieve relatively smaller rates. It can be observed that as $S_3$ achieves higher rates, the achievable rate region of the $S_1$ and $S_2$ shrinks. Also, when $R_3 > 0$, the rates achievable using the extensions provided by the corollaries lies completely above the rates achievable by the coding scheme in Sect. 2.5, which is due to the suboptimality of that scheme with respect to the achievable rates of $S_1$ and $S_2$ for a fixed $R_3$. The rate of $S_1$ has a larger relative reduction compared to that of $S_2$, yet $S_1$ achieves a higher rate than $S_2$, as expected.

5. Three-user channel with CoMS: Figure 13 shows the plots of the rates of $S_1$ and $S_2$, when $S_1$ achieves a minimum rate of 0, 1 and 1.5 bps/Hz, along with the plot of Corollary 3.12. We see that the rates of $S_1$ and $S_2$ decrease with increasing rate of $S_3$. Also, the reduction in the size of the region is symmetric i.e., both $R_1$ and $R_2$ simultaneously decrease, and roughly speaking, by the same relative amount. When $R_3 = 0$, we get the achievable rate region for the two-user interference channel with a cognitive relay [44, 45].

Remark The inner bounds for the $C_{\text{CoMS}}^1$ and $C_{\text{PrMS}}^1$ have not been plotted here. This is mainly because applying the Fourier-Motzkin elimination procedure on the rate region is a formidable task, given the number of inequalities involved (see Appendices 1 and 2). This demonstrates practical difficulties involved with rate-splitting, especially with growing network size. Nevertheless, one can expect (i) the achievable rate regions for $C_{\text{CoMS}}^1$ and $C_{\text{PrMS}}^1$ to be larger than that for $C_{\text{CoMS}}^2$ and $C_{\text{PrMS}}^2$ and (ii) the gap between the achievable rate region and the outer bound for $C_{\text{CoMS}}^1$ and $C_{\text{PrMS}}^1$ to be smaller than that to $C_{\text{CoMS}}^2$ and $C_{\text{PrMS}}^2$ respectively, because $S_1$ also employs rate-splitting strategy in the former case.

As a concluding remark, note that, as mentioned above, there is a gap between the inner and outer bounds in all the cases plotted. There are a couple reasons for this.

1. In the case of $C_{\text{CoMS}}^1$ and $C_{\text{PrMS}}^1$, $S_1$ does not perform rate-splitting, thereby rendering the receivers of $S_2$ and $S_3$ vulnerable to interference caused due to $S_1$’s transmissions. In the case of $C_{\text{CoMS}}$, neither $S_1$ nor $S_2$ performs rate-splitting, leading to poor interference management at all the receivers. However, several
corollaries were derived based on the idea of allowing senders to dedicate (part of) their power for transmitting the primary sender’s message, which resulted in several additional rate points being achievable. And, it was shown that these rate tuples meet the outer bounds at corner points. A systematic way of expanding the rate region by including the different coding schemes is an open problem, which can be further explored in the future.

2. The outer bounds were derived by taking the intersection of the capacity region with bidirectional sharing and the individual user rates with unidirectional sharing, and hence have a natural advantage over the purely-unidirectional model assumed in deriving the rate regions. Furthermore, the duality result implicitly assumes that the receivers can successfully decode the interfering signals to a large extent.

Fig. 4 Two-user CR and interference channels with different ratesplitting strategies. The power at the transmitters are 7.8 dB

Fig. 5 Rate of $S_1 (R_1)$ versus the rate of $S_3 (R_3)$ when $S_2$ is guaranteed to achieve a minimum rate $R_2 = 0.8$ bps/Hz, for $C_{CuMS}^2$ and $C_{PrMS}^2$ along with the rate region of the corresponding interference channel. The power at the transmitters is 10 dB

Outer bounds can be made tighter by considering discrete memoryless channel models and introducing auxiliary RVs. Existing literature lacks results for tighter outer bounds, except for the most recent work in [28] which is for the two user CR channels, and is not directly applicable to our channel models. From the above discussion, we conclude that, though the three-user channel models considered in this paper are logical extensions of the classical two-user scenario, we are able to make interesting observations and draw several inferences on the effect of rate-splitting and message-sharing on larger networks. The techniques to analyze two-user networks

Table 4 Effect of rate-splitting on the decoding capability of receivers for the channel $C_{CuMS}$

| Receiver | Can decode |
|---------|------------|
| $R_1$   | $m_1, m_{31}$ |
| $R_2$   | $m_2, m_{31}$ |
| $R_3$   | $m_{31}, m_{33}$ |

Note that, there is no rate-splitting at the senders $S_1$ and $S_2$.

Table 5 Auxiliary random variables and their description

| Variable | Description |
|----------|-------------|
| $W_0 \in W_0$ | Public information: $S_1 \rightarrow (R_1, R_2, R_3)$ |
| $W_1 \in W_1$ | Private information: $S_1 \rightarrow R_1$ |
| $U_0 \in U_0$ | Public information: $S_2 \rightarrow (R_1, R_2, R_3)$ |
| $U_1 \in U_1$ | Public information: $S_2 \rightarrow (R_1, R_2)$ |
| $U_2 \in U_2$ | Private information: $S_2 \rightarrow R_2$ |
| $V_0 \in V_0$ | Public information: $S_3 \rightarrow (R_1, R_2, R_3)$ |
| $V_1 \in V_1$ | Public information: $S_3 \rightarrow (R_1, R_3)$ |
| $V_2 \in V_2$ | Private information: $S_3 \rightarrow R_3$ |

For ex., $U_1$ denotes public information from $S_2$ decodable at $R_1$ and $R_2$.
may carry over to these larger networks, but issues related to interference management via rate-splitting are nontrivial and need further investigation.

4.3 Effect of reduction in size of the network

Let us consider the case of removing a transmitter-receiver pair from the three-user CR channel model. In particular, let
us assume that \((S_3, R_3)\)-pair is removed, resulting in a two-user CR channel. We make the following observations:

1. The channels \(C^2_{\text{CuMS}}\) and \(C^2_{\text{PrMS}}\) will now reduce to the model employed in [9]. The achievability scheme results in a rate region which coincides with [9, Theorem 1] (see Fig. 4c), which includes the rate regions derived in [7] and [8]. Furthermore, the rate regions derived in [7, Theorem 3.5] and [8, Theorem 4.1] are in fact the capacity regions for the two-user CR channels in the low-interference regime. In our three-user channel models \(C^2_{\text{CuMS}}\) and \(C^2_{\text{PrMS}}\), low-interference regime can be considered by letting the auxiliary RVs \(U_1\) and \(V_1\) be constants. However, this does not yield the capacity region (unlike the two-user scenario) for \(C^2_{\text{CuMS}}\) and \(C^2_{\text{PrMS}}\).

2. The channels \(C^1_{\text{CuMS}}\) and \(C^1_{\text{PrMS}}\) will reduce to the model employed in [4]. However, our achievability scheme results in a slightly larger rate region compared to the one presented in [4, Theorem 1]. This is because of the fact that the rate region of [4, Theorem 1] takes into account noisy message-sharing (captured by Eq. (3)–(5) in [4, Theorem 1]), while our problem setup concerns degraded message sets.

3. For the channel \(C^1_{\text{CoMS}}\), let us consider removing \((S_2, R_2)\)-pair. This results in the model employed in [9], which has been addressed in the above discussion.

4.4 A recent result on two-user CR channel

Recently, in [28, Theorem V.1] a new inner bound has been derived for the two-user CR channel which encompasses all of the previously known achievable regions. The technique used to prove their main achievability theorem employs rate-splitting, superposition coding and a sequential binning procedure. However, we notice that their achievability scheme involves a rate-split at both encoders. This does not conform well to some of our channel models, specifically \(C^2_{\text{CuMS}}, C^2_{\text{PrMS}}\) and \(C^2_{\text{CoMS}}\), where one (or two) encoder(s) do not employ rate-splitting. This suggests that their technique may not be appropriate for our problem setup since, owing to the presence of three transmitters, more than one rate-splitting scheme can be considered leading to a large class of three-user channel models. Furthermore, it remains to be ascertained whether the achievable scheme of [28] generalizes to the three-user channel irrespective of the rate-splitting technique employed by the encoders.

A new outer bound has also been derived in [28, Theorem IV.1], which is looser than previously known outer bounds, but has the advantage of not involving auxiliary RVs. However, we have not investigated outer bounds for the discrete memoryless case, because of the complexity of our problem setup. Instead, we resorted to suggesting some outer bounds with the aid of the Gaussian channel model.

5 Conclusions

In this paper, we introduced multiuser channels with noncausal transmitter cooperation in the overlay CR network paradigm and presented three different ways of message sharing which we termed CuMS, PrMS and CoMS. We derived an achievable rate region for each of the channels by employing a combination of superposition and Gel’fand–Pinsker coding techniques. We considered the Gaussian channel model to plot the rate regions and presented some corollaries using which several achievable rate tuples for the Gaussian channel were readily identified. Later, we derived some outer bounds for the Gaussian case by considering bidirectional cooperation between the transmitters, and calculating the capacity region of the resulting Gaussian MIMO broadcast channel using BC-MAC duality results. Simulation results enabled us to compare rate-splitting and message-sharing as a mechanism to improve spectral efficiency. We observed that, while message-sharing is superior to rate-splitting in both two and three-user scenarios, it is not fully clear as to which type of message-sharing mechanism (CuMS, PrMS or CoMS) gives the largest rate region. Interesting open problems include (i) deriving tighter outer bounds and (ii) considering rate-constrained cooperation, wherein the cognitive radio estimates the message index transmitted by the primary user in a causal manner and is an important and practically relevant scenario to consider for future work.

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Appendix 1: Achievable rate region for $C_{\text{cAMS}}^t$: $t = 1, 2$

Here, we present rate inequalities which characterize the achievable rate region for the channels $C_{\text{cAMS}}^t$: $t = 1, 2$. The channel model $C_{\text{cAMS}}^t$ is symmetric, in the sense that all transmitters perform rate-splitting so that each receiver can decode and cancel out the interfering signals from the non-pairing senders. However, the receivers are not required to decode the public part of the non-pairing transmitter’s message correctly. Considering these, for the channel $C_{\text{cAMS}}^t$, we can derive a total of 36 inequalities, as given below.

$$
R_{10} \leq I(W_0; W_1, U_0, V_0, Y_1|Q), \\
R_{11} \leq I(W_1; W_0, U_0, V_0, Y_1|Q), \\
R_{10} + R_{11} \leq I(W_0, W_1; U_0, V_0, Y_1|Q) + I(W_0; W_1), \\
R_{10} + R_{20} \leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{30} \leq I(W_0, V_0; W_1, U_0, Y_1|Q) + I(W_0; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \\
R_{11} + R_{20} \leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1; U_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{11} + R_{30} \leq I(W_1, V_0; W_0, V_0, Y_1|Q) + I(W_1; V_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{11} + R_{20} \leq I(W_0, W_1, U_0, V_0, Y_1|Q) + I(W_0; W_1; U_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{11} + R_{30} \leq I(W_0, W_1, V_0; U_0, Y_1|Q) + I(W_0, W_1; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \\
R_{10} + R_{20} + R_{30} \leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0; U_0|Q) + I(W_0; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \\
R_{11} + R_{20} + R_{30} \leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1; U_0|Q) + I(W_1; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \\
R_{10} + R_{11} + R_{20} + R_{30} \leq I(W_0, W_1, U_0, V_0; U_2, Y_1|Q) + I(W_0, W_1; U_0, U_2; V_0|Q) + I(W_0, W_1; U_0|Q) + I(W_0, W_1; U_0, U_2; V_0|Q) \\
- I(W_0, W_1; U_0; V_0, V_1|Q) - I(W_0, W_1, U_0; U_2; V_0|Q), \\
R_{20} \leq I(U_0; W_0, U_2, V_0, Y_2|Q) - I(W_0, W_1; U_0|Q), \\
R_{22} \leq I(U_2; W_0, U_0, V_0, Y_2|Q) - I(W_0, W_1; U_2|Q), \\
R_{20} + R_{22} \leq I(U_0; U_2, W_0, V_0, Y_2|Q) + I(U_0; U_2|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q), \\
R_{10} + R_{20} \leq I(U_0; W_0, U_2, V_0, Y_2|Q) + I(U_0; W_0, U_2|Q) + I(U_0; W_0, U_2|Q), \\
R_{10} + R_{22} \leq I(U_0; W_0, U_2, V_0, Y_2|Q) + I(U_0; W_0, U_2|Q) - I(W_0, W_1; U_2|Q), \\
R_{20} + R_{30} \leq I(U_0; W_0, U_2, V_0, Y_2|Q) + I(U_0; W_0, U_2|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \\
R_{22} + R_{30} \leq I(U_2; W_0, U_0, V_0, Y_2|Q) + I(U_2; V_0|Q) - I(W_0, W_1; U_2|Q), \\
R_{10} + R_{20} + R_{22} \leq I(U_0, U_2, W_0, V_0, Y_2|Q) + I(U_0, U_2; V_0|Q) + I(U_0, U_2|Q), \\
R_{10} + R_{22} \leq I(U_0, W_0, U_2, V_0, Y_2|Q) + I(U_0, W_0, U_2; V_0|Q) + I(U_0, W_0, U_2; V_0|Q), \\
R_{20} + R_{30} + R_{30} \leq I(U_0, U_2, W_0, V_0, Y_2|Q) + I(U_0, U_2; V_0|Q) + I(U_0, U_2; V_0|Q) - I(W_0, W_1; U_0, U_2; V_0|Q) \\
- I(W_0, W_1; U_0, U_2; V_0|Q), \\
R_{20} + R_{30} \leq I(W_0, U_0, U_2, V_0, Y_2|Q) + I(W_0, U_0, U_2; V_0|Q) - I(W_0, W_1; U_0, U_2; V_0|Q), \\
R_{30} \leq I(V_0; W_0, U_0, V_0, Y_3|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \\
R_{33} \leq I(V_3; W_0, U_0, V_0, Y_3|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \\
R_{30} + R_{33} \leq I(V_0, V_5; W_0, U_0, Y_3|Q) + I(V_0; V_3|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \\
R_{10} + R_{30} \leq I(W_0, U_0, V_0, Y_3|Q) + I(W_0; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \\
R_{10} + R_{33} \leq I(W_0, V_0, V_3; U_0, Y_3|Q) + I(W_0; V_3|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \\
R_{20} + R_{30} \leq I(U_0, V_0, W_0, V_3; Y_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \\
20 + R_{33} \leq I(U_0, V_3; W_0, V_0, Y_3|Q) + I(U_0; V_3|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \\
R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, V_0, Y_3|Q) + I(W_0; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \\
R_{10} + R_{20} + R_{33} \leq I(W_0, U_0, V_0, Y_3|Q) + I(W_0; U_3|Q) + I(W_0; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q),
$$
The channel model $C_{cAMS}^2$ is not symmetric, in the sense that only senders $S_2$ and $S_3$ perform rate-splitting. This results in receiver $R_1$ being able to decode the public part of the non-pairing sender’s message, while receivers $R_2$ and $R_3$ can only decode the message from the pairing transmitter. We have a total of 10 inequalities which are given below.

\[
R_{11} \leq I(W; U_1, V_1, Y_1|Q), \\
R_{11} + R_{21} \leq I(W; U_1, V_1, Y_1|Q), \\
R_{11} + R_{31} \leq I(W; U_1, V_1, Y_1|Q) + I(W; V_1|Q) - I(W, U_1, U_2; V_1|Q), \\
R_{11} + R_{21} + R_{31} \leq I(W; U_1, V_1, Y_1|Q) I(W, U_1; V_1|Q) - I(W, U_1, U_2; V_1|Q), \\
R_{31} \leq I(U_1; U_2, Y_3|Q) - I(W, U_1|Q), \\
R_{22} \leq I(U_2; U_2, Y_3|Q) - I(W, U_2|Q), \\
R_{21} + R_{22} \leq I(U_1, U_2; Y_3|Q) + I(U_1; U_2|Q) - I(W; U_2|Q), \\
R_{31} \leq I(V_1; V_3, Y_3|Q) - I(W, U_1, U_2; V_1|Q), \\
R_{33} \leq I(V_1; V_3, Y_3|Q) - I(W, U_1, U_2; V_3|Q), \\
R_{31} + R_{33} \leq I(V_1; V_3; Y_3|Q) + I(V_1; V_3|Q) - I(W, U_1, U_2; V_1|Q).
\]

## Appendix 2: Achievable rate region for $C_{PMMS}^t$: $t = 1, 2$

In this appendix, we present the achievable rate region for the channels $C_{PMMS}^t$: $t = 1, 2$. An achievable rate region for the channel $C_{PMMS}^1$ is given by the following inequalities. The number of inequalities is the same as with the case of $C_{cAMS}^1$. Note that, the difference between the two channel models is that they have different message-sharing schemes.

\[
R_{10} \leq I(W_0; W_1, U_0, V_0, Y_1|Q), \\
R_{11} \leq I(W_1; W_0, U_0, V_0, Y_1|Q), \\
R_{10} + R_{11} \leq I(W_0, W_1; U_0, V_0, Y_1|Q) + I(W_0; W_1|Q), \\
R_{10} + R_{20} \leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{30} \leq I(W_0, V_0; W_1, U_0, Y_1|Q) + I(W_0; V_0|Q) - I(W_0, W_1; V_0|Q), \\
R_{11} + R_{20} \leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1; U_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{11} + R_{30} \leq I(W_1, V_0; W_0, U_0, Y_1|Q) + I(W_1; V_0|Q) - I(W_0, W_1; V_0|Q), \\
R_{10} + R_{11} + R_{20} + R_{30} \leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1; V_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{11} + R_{30} \leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1; V_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{20} + R_{30} \leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0, U_0; V_0|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{11} + R_{20} + R_{30} \leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1, U_0; V_0|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{11} + R_{20} + R_{30} \leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1; V_0|Q) + I(W_0; W_1|Q) + I(W_0, W_1; U_0|Q) + I(W_0, W_1; V_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{20} \leq I(U_0; U_2, V_0, Y_2|Q) - I(W_0, W_1; U_0|Q), \\
R_{22} \leq I(U_2; U_0, V_0, Y_2|Q) - I(W_0, W_1; U_2|Q), \\
R_{20} + R_{22} \leq I(U_0, U_2; V_0, W_0, Y_2|Q) + I(U_0; U_2|Q) - I(W_0, W_1; U_2|Q), \\
R_{10} + R_{20} \leq I(W_0, U_0; U_2, V_0, Y_2|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q), \\
R_{10} + R_{22} \leq I(U_0, U_2; V_0, W_0, Y_2|Q) + I(U_0; U_2|Q) - I(W_0, W_1; U_2|Q), \\
R_{20} + R_{30} \leq I(U_0, V_0; U_2, V_0, Y_2|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q), \\
R_{22} + R_{30} \leq I(U_2, V_0; U_0, Y_2|Q) + I(U_2; V_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q), \\
R_{21} + R_{22} \leq I(U_1, U_2; Y_2|Q) + I(U_1; U_2|Q) - I(W; U_2|Q), \\
R_{31} \leq I(V_1; V_3, Y_3|Q) - I(W, U_1, U_2; V_1|Q), \\
R_{33} \leq I(V_3; V_1, Y_3|Q) - I(W, U_1, U_2; V_3|Q), \\
R_{31} + R_{33} \leq I(V_1; V_3; Y_3|Q) + I(V_1; V_3|Q) - I(W, U_1, U_2; V_1|Q).
\]
\[
\begin{align*}
R_{10} + R_{20} + R_{22} &\leq I(W_0, U_0, U_2; V_0, Y_2|Q) + I(W_0, U_0; U_2|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) \\
R_{10} + R_{20} + R_{30} &\leq I(W_0, U_0, V_0; U_2, Y_2|Q) + I(W_0; U_0; U_2|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) \\
R_{10} + R_{22} + R_{30} &\leq I(W_0, U_2, V_0; U_2, Y_2|Q) + I(W_0; U_2; V_0|Q) + I(W_0; U_2|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q) \\
R_{20} + R_{22} + R_{30} &\leq I(U_0, U_2, V_0; W_2|Q) + I(U_0, U_2; V_0|Q) + I(U_0; U_2; V_0) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q) \\
&\quad - I(W_0; W_1; V_0|Q) \\
R_{10} + R_{20} + R_{22} + R_{30} &\leq I(W_0, U_0, U_2, V_0; Y_2|Q) + I(W_0, U_0, U_2; V_0|Q) + I(W_0, U_0; U_2; V_0) + I(W_0; U_0; U_2; V_0) - I(W_0; W_1; V_0|Q) \\
&\quad - I(W_0; W_1; V_0|Q) \\
R_{30} &\leq I(V_0; W_0, U_0, V_3, Y_3|Q) - I(W_0, W_1; V_0|Q) \\
R_{33} &\leq I(V_3; W_0, V_0, V_3, Y_3|Q) - I(W_0, W_1; V_3|Q) \\
R_{30} + R_{33} &\leq I(V_0; V_3; W_0, U_0, V_3, Y_3|Q) + I(V_0; V_3|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; V_3|Q) \\
R_{10} + R_{30} &\leq I(W_0, V_0; U_0, V_3, Y_3|Q) + I(W_0; V_0|Q) - I(W_0, W_1; V_0|Q) \\
R_{10} + R_{33} &\leq I(W_0, V_3; U_0, V_3, Y_3|Q) + I(W_0; V_3|Q) - I(W_0, W_1; V_3|Q) \\
R_{20} + R_{30} &\leq I(U_0, V_0; U_0, V_3, Y_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1; V_0|Q) \\
R_{20} + R_{33} &\leq I(U_0, V_3; U_0, V_3, Y_3|Q) + I(U_0; V_3|Q) - I(W_0, W_1; V_3|Q) \\
R_{10} + R_{30} + R_{33} &\leq I(W_0, U_0, V_0; V_3, Y_3|Q) + I(W_0; U_0; V_0; V_3|Q) + I(W_0; U_0; V_0; Y_3|Q) - I(W_0; W_1; V_0|Q) - I(W_0; W_1; V_3|Q) \\
R_{10} + R_{30} + R_{33} &\leq I(W_0, U_0, V_0, V_3; Y_3|Q) + I(W_0; U_0; V_0; V_3; Y_3|Q) + I(W_0; U_0; V_0; Y_3|Q) - I(W_0; W_1; V_0|Q) - I(W_0; W_1; V_3|Q) \\
R_{10} + R_{30} + R_{33} &\leq I(W_0, U_0, V_0, V_3, Y_3|Q) + I(W_0; U_0; V_0; V_3; Y_3|Q) + I(W_0; U_0; V_0; Y_3|Q) - I(W_0; W_1; V_0|Q) - I(W_0; W_1; V_3|Q) \\
&\quad - I(W_0; W_1; V_3|Q) \\
R_{10} + R_{30} + R_{33} &\leq I(W_0, V_0, V_3; Y_3|Q) + I(W_0; V_0; V_3|Q) + I(W_0; V_0; Y_3|Q) - I(W_0; W_1; V_0|Q) - I(W_0; W_1; V_3|Q) \\
&\quad - I(W_0; W_1; V_3|Q) \\
R_{31} + R_{33} &\leq I(V_1; V_3; Y_3|Q) + I(V_1; V_3|Q) - I(W_0; V_3|Q) - I(W_0; V_3|Q) \\
&\quad - I(W_0; V_1|Q).
\end{align*}
\]

An achievable rate region for the channel $c^2_{\text{PMS}}$ is given by the following inequalities. As before, the number of inequalities is same as that for the channel $c^2_{\text{CoMS}}$. One should also guard against direct comparison of the rate region equations of various channel models, since these channels are governed by joint distributions with different underlying factorizations.

\[
\begin{align*}
R_{11} &\leq I(W; U_1, V_1, Y_1|Q) \\
R_{31} + R_{21} &\leq I(W; U_1; V_1, Y_1|Q) \\
R_{11} + R_{31} &\leq I(W; U_1, V_1, Y_1|Q) - I(W; V_1|Q) \\
R_{11} + R_{21} + R_{31} &\leq I(W; U_1, V_1, Y_1|Q) + I(W; U_1; V_1|Q) \\
R_{21} &\leq I(U_1; U_2, Y_2|Q) - I(W; U_1|Q) \\
R_{22} &\leq I(U_2; U_1, Y_2|Q) - I(W; U_2|Q) \\
R_{21} + R_{22} &\leq I(U_1, U_2; Y_2|Q) + I(U_1; U_2|Q) \\
&\quad - I(W; U_1|Q) - I(W; U_2|Q), \\
R_{31} &\leq I(V_1; V_3, Y_3|Q) - I(W; V_1|Q) \\
R_{33} &\leq I(V_3; V_1, Y_3|Q) - I(W; V_3|Q) \\
R_{31} + R_{33} &\leq I(V_1; V_3; Y_3|Q) + I(V_1; V_3|Q) - I(W; V_3|Q) \\
&\quad - I(W; V_1|Q).
\end{align*}
\]

**Appendix 3: Achievable rate region for $C_{\text{CoMS}}$**

Here, we present the rate region for the channel $C_{\text{CoMS}}$. An achievable rate region for the channel $C_{\text{CoMS}}$ is given by the following inequalities. Here, sender $S_3$ has noncausal knowledge of the messages and codewords of $S_1$ and $S_2$, and performs rate-splitting. There is no rate-splitting at senders $S_1$ and $S_2$.

\[
\begin{align*}
R_1 &\leq I(W; V_0, Y_1|Q), \\
R_1 + R_{31} &\leq I(W, V_0; Y_1|Q) + I(W; V_0|Q) - I(W, U; V_0|Q), \\
R_2 &\leq I(U; V_0, Y_2|Q), \\
R_2 + R_{31} &\leq I(U, V_0; Y_2|Q) + I(U; V_0|Q) - I(W, U; V_0|Q), \\
R_{33} &\leq I(V_0; V_3, Y_3|Q) - I(W, U; V_3|Q), \\
R_{31} + R_{33} &\leq I(V_0; V_3; Y_3|Q) + I(V_0; V_3|Q) - I(W, U; V_3|Q) \\
&\quad - I(W, U; V_0|Q) - I(W, U; V_3|Q).
\end{align*}
\]

**Appendix 4: Proof of Theorem 2.1 for $C_{\text{CoMS}}$**

Here, we present the proof of achievability for the channel $C_{\text{CoMS}}$. The proof is presented in four parts, namely, codebook generation, encoding, decoding and analysis of
probabilities of decoding errors at the three receivers. We start with the codebook generation scheme.

Codebook generation

Let us fix \( p(.) \in \mathcal{P}_\mathcal{C}_{\text{CMS}} \). Generate a random time sharing codeword \( \mathbf{q} \), of length \( n \), according to the distribution \( \prod_{i=1}^n p(q_i) \). Generate \( 2^{nR_1} \) independent codewords \( \mathbf{W}(j) \), according to \( \prod_{i=1}^n p(w_i|q_i) \). For every \( w(j) \), generate one codeword \( \mathbf{X}(j) \) according to \( \prod_{i=1}^n p(x_i|w_i(j), q_i) \).

For \( \tau = 1, 2 \), generate \( 2^{n(R_\tau+W(U_i|Q)+4\epsilon)} \) independent codewords \( \mathbf{U}(l_i) \), according to \( \prod_{i=1}^n p(u_i|l_i, q_i) \). For every codeword triple \([\mathbf{u}_1(l_1), \mathbf{u}_2(l_2), \mathbf{w}(j)]\), generate one codeword \( \mathbf{X}(j) \) according to \( \prod_{i=1}^n p(x_i|u_i(l_i), u_2(l_2), w_i(j), q_i) \). Uniformly distribute the \( 2^{n(R_\tau+W(U_i|Q)+4\epsilon)} \) codewords \( \mathbf{U}(l_i) \) into \( 2^{nR_\tau} \) bins indexed by \( k_x \in \{1, \ldots, 2^{nR_\tau}\} \) such that each bin contains \( 2^{n(W(U_i|Q)+4\epsilon)} \) codewords.

For \( \rho = 1, 3 \), generate \( 2^{n(R_\rho+W(U_i|Q)+4\epsilon)} \) independent codewords \( \mathbf{V}(\rho, p) \), according to \( \prod_{i=1}^n p(v_i|p, q_i) \). For every codeword quadruple \([\mathbf{v}_1(t_1), \mathbf{v}_2(t_2), \mathbf{u}_1(l_1), \mathbf{u}_2(l_2), \mathbf{w}(j)]\), generate one codeword \( \mathbf{X}(j) \) according to \( \prod_{i=1}^n p(x_i|v_i(t_i), u_i(l_i), u_2(l_2), w_i(j), q_i) \). Distinctly distribute the \( 2^{n(R_\rho+W(U_i|Q)+4\epsilon)} \) codewords \( \mathbf{V}(\rho, p) \) uniformly into \( 2^{nR_\rho} \) bins indexed by \( p \in \{1, \ldots, 2^{nR_\rho}\} \) such that each bin contains \( 2^{n(W(U_i|Q)+4\epsilon)} \) codewords. The indices are given by \( j \in \{1, \ldots, 2^{nR_\rho}\}, l_x \in \{1, \ldots, 2^{nR_\rho}\} \) and \( \rho \in \{1, \ldots, 2^{nR_\rho}\} \).

Encoding & transmission

Let \( A^{(n)} \) be a typical set. We will be using the notation \( A^{(n)} \) to describe a typical set over many different random variables, but the definition will be clear from the context.

Let us suppose that the source message vector generated at the three senders is \((m_{11}, m_{21}, m_{22}, m_{31}, m_{32}) = (q, k_1, k_2, r_1, r_2, r_3)\). At the encoders, the first component is treated as the message index and the last four components are treated as the bin indices. \( S_2 \) looks for a codeword \( \mathbf{u}_1(l_1) \) in bin \( k_1 \) and a codeword \( \mathbf{u}_2(l_2) \) in bin \( k_2 \) such that \((\mathbf{u}_1(l_1), \mathbf{w}(j), q) \in A^{(n)}\) and \((\mathbf{u}_2(l_2), \mathbf{w}(j), q) \in A^{(n)}\), respectively. \( S_1 \) looks for a codeword \( \mathbf{v}_1(t_1) \) in bin \( r_1 \) and a codeword \( \mathbf{v}_2(t_2) \) in bin \( r_2 \) such that \((\mathbf{v}_1(t_1), \mathbf{u}_1(l_1), \mathbf{u}_2(l_2), \mathbf{w}(j), q) \in A^{(n)}\) and \((\mathbf{v}_3(t_3), \mathbf{u}_1(l_1), \mathbf{u}_2(l_2), \mathbf{w}(j), q) \in A^{(n)}\), respectively. \( S_1, S_2 \) and \( S_3 \) then transmit codewords \( \mathbf{x}(j), \mathbf{x}_2(l_1, l_2, j) \) and \( \mathbf{x}_3(t_1, t_2, t_3, l_1, l_2, j) \), respectively, through \( n \) channel uses. The transmissions are assumed to be synchronized.

Decoding

Recall that in \( \mathcal{C}_{\text{CMS}}^2 \), the primary receiver can decode the public parts of the non-pairing sender’s messages, while the secondary receivers can only decode the messages from their pairing transmitters. The three receivers accumulate an \( n \)-length channel output sequence: \( y_1 \) at \( R_1, y_2 \) at \( R_2 \) and \( y_3 \) at \( R_3 \). Decoders 1, 2 and 3 look for all indices \((j, l_1, t_1), (l_1, l_2)\) and \((t_1, t_2)\), respectively, such that \((\mathbf{w}(j), \mathbf{u}_1(l_1), \mathbf{v}_1(t_1), \mathbf{v}_3(t_3), \mathbf{y}_3, q) \in A^{(n)}\) and \((\mathbf{v}_1(t_1), \mathbf{v}_3(t_3), \mathbf{y}_3, q) \in A^{(n)}\). If \( j \) in all the index triples found are the same, \( R_1 \) declares \( m_{11} = j \), for some \( l_1 \) and \( t_1 \). If \( l_{11} \) in all the index pairs found are indices of codewords \( \mathbf{u}_1(l_1) \) from the same bin with index \( k_1 \), and \( l_{21} \) in all the index pairs found are indices of codewords \( \mathbf{u}_2(l_2) \) from the same bin with index \( k_2 \), then \( R_2 \) determines \( (m_{21}, m_{22}) = (k_1, k_2) \). Similarly, if \( t_1 \) in all the index pairs found are indices of codewords \( \mathbf{v}_1(t_1) \) from the same bin with index \( r_1 \), and \( t_2 \) in all the index pairs found are indices of codewords \( \mathbf{v}_3(t_3) \) from the same bin with index \( r_3 \), then \( R_3 \) determines \( (m_{31}, m_{32}) = (r_1, r_3) \). Otherwise, the receivers \( R_1, R_2 \) and \( R_3 \) declare an error.

Analysis of the probabilities of error

In this section we derive upperbounds on the probabilities of error events which could happen during encoding and decoding processes. We assume that a source message vector \((m_{11}, m_{21}, m_{22}, m_{31}, m_{32}) = (q, k_1, k_2, r_1, r_2, r_3)\) is encoded and transmitted. We consider the analysis of the probability of encoding error at senders \( S_2 \) and \( S_3 \), and the analysis of the probability of decoding error at each of the three receivers \( R_1, R_2 \), and \( R_3 \) separately.

First, let us define the following events:

(i) \( E_{ji} \triangleq \{ (\mathbf{W}(j), \mathbf{U}_1(l_1), q) \in A^{(n)} \} \)

(ii) \( E_{ji} \triangleq \{ (\mathbf{W}(j), \mathbf{U}_2(l_2), q) \in A^{(n)} \} \)

(iii) \( E_{ji} \triangleq \{ (\mathbf{W}(j), \mathbf{U}_1(l_1), \mathbf{U}_2(l_2), \mathbf{V}_1(t_1), q) \in A^{(n)} \} \)

(iv) \( E_{ji} \triangleq \{ (\mathbf{W}(j), \mathbf{U}_1(l_1), \mathbf{U}_2(l_2), \mathbf{V}_3(t_3), q) \in A^{(n)} \} \)

(v) \( E_{ji} \triangleq \{ (\mathbf{W}(j), \mathbf{U}_1(l_1), \mathbf{V}_1(t_1), \mathbf{Y}_3, \mathbf{q}) \in A^{(n)} \} \)

(vi) \( E_{ji} \triangleq \{ (\mathbf{U}_1(l_1), \mathbf{U}_2(l_2), \mathbf{Y}_2, \mathbf{q}) \in A^{(n)} \} \)

(vii) \( E_{ji} \triangleq \{ (\mathbf{V}_1(t_1), \mathbf{V}_3(t_3), \mathbf{Y}_3, \mathbf{q}) \in A^{(n)} \} \)

\( E_{ji} \triangleq \) complement of the event \( E_{ji} \). Events (i)–(vi) will be used in the analysis of probability of encoding error while events (v)–(vii) will be used in the analysis of probability of decoding error.

Probability of Error at the Encoder of \( S_2 \)

An error is made if (a) the encoder cannot find a \( \mathbf{u}_1(l_1) \) in the bin indexed by \( k_1 \) such that \((\mathbf{w}(j), \mathbf{u}_1(l_1), q) \in A^{(n)} \) or (b) it cannot find a \( \mathbf{u}_2(l_2) \) in the bin indexed by \( k_2 \) such that \((\mathbf{w}(j), \mathbf{u}_2(l_2), q) \in A^{(n)} \). The probability of encoding error at \( S_2 \) can be bounded as
\[ P_{e,S_2} \leq P \left( \bigcap_{U_1(l_1) \in \text{bin}(k_1)} (W(j), U_1(l_1), q) \notin A^n_{e} \right) \]
\[ + P \left( \bigcap_{U_2(l_2) \in \text{bin}(k_2)} (W(j), U_2(l_2), q) \notin A^n_{e} \right), \]
\[ \leq (1 - P(E_{j_1}))^{2^{n(H(W,U_1(l_1)|Q) + 3e)}} + (1 - P(E_{j_2}))^{2^{n(H(W,U_2(l_2)|Q) + 3e)}}, \]

where \( P(.) \) is the probability of an event. Since \( q \) is predetermined, and \( w \) and \( u_1 \) are independent given \( q \).

\[ P(E_{j_1}) = \sum_{(w,u_1,q) \in A^n_{e}} P(W(j) = w|q)P(U_1(l_1)) \]
\[ = u_1|q \geq 2^{n(I(W,U_1(l_1)|Q) - e)} - 2^{n(H(W|Q) + e)} 2^{n(H(U_1|Q) + e)} \]
\[ = 2^{-n(I(W,U_1(l_1)|Q) + 3e)}. \]

Similarly, \( P(E_{j_2}) \geq 2^{-n(I(W,U_2(l_2)|Q) + 3e)}. \)

Therefore,
\[ P_{e,S_2} \leq (1 - 2^{-n(I(W,U_1(l_1)|Q) + 3e)})^{2^{n(H(W,U_1(l_1)|Q) + 3e)}} \]
\[ + (1 - 2^{-n(I(W,U_2(l_2)|Q) + 3e)})^{2^{n(H(W,U_2(l_2)|Q) + 3e)}}. \]

Now,
\[ (1 - 2^{-n(I(W,U_1(l_1)|Q) + 3e)})^{2^{n(H(W,U_1(l_1)|Q) + 3e)}} \]
\[ = e^{-2^{n(I(W,U_1(l_1)|Q) + 3e)}} \]
\[ \leq e^{-2^{n(I(W,U_1(l_1)|Q) + 3e)}} \]
\[ = e^{-2^n}. \]

Clearly, \( P_{e,S_2} \to 0 \) as \( n \to \infty \).

### Probability of error at the encoder of \( S_3 \)

An error is made if (a) the encoder cannot find a \( v_1(t_1) \) in the bin indexed by \( r_1 \) such that \( (w(j), u_1(l_1), u_2(l_2), v_1(t_1), q) \in A^n_{e} \) or (b) it cannot find a \( v_3(t_3) \) in the bin indexed by \( r_3 \) such that \( (w(j), u_1(l_1), u_2(l_2), v_3(t_3), q) \in A^n_{e} \). The probability of encoding error at \( S_3 \) can be bounded as

\[ P_{e,S_3} \leq P \left( \bigcap_{V_1(t_1) \in \text{bin}(k_1)} (W(j), U_1(l_1), U_2(l_2), V_1(t_1), q) \notin A^n_{e} \right) \]
\[ + P \left( \bigcap_{V_3(t_3) \in \text{bin}(k_3)} (W(j), V_1(t_1), U_2(l_2), V_3(t_3), q) \notin A^n_{e} \right), \]
\[ \leq (1 - P(E_{j_1,t_1}))^{2^{n(H(W,U_1, V_1(t_1)|Q) + 3e)}} \]
\[ + (1 - P(E_{j_2,t_3}))^{2^{n(H(W,U_2, V_3(t_3)|Q) + 3e)}}. \]

With
\[ P(E_{j_1,t_1}) \geq 2^{-n(I(W,U_1, V_1(t_1)|Q) + 3e)}, \]
\[ P(E_{j_2,t_3}) \geq 2^{-n(I(W,U_2, V_3(t_3)|Q) + 3e)}, \]

we get
\[ P_{e,S_3} \leq \left( 1 - 2^{-n(I(W,U_1, V_1(t_1)|Q) + 3e)} \right)^{2^{n(I(W,U_1, V_1(t_1)|Q) + 3e)}} \]
\[ + \left( 1 - 2^{-n(I(W,U_2, V_3(t_3)|Q) + 3e)} \right)^{2^{n(I(W,U_2, V_3(t_3)|Q) + 3e)}}. \]

Proceeding in a way similar to the encoder error analysis at \( S_2 \), we can show that \( P_{e,S_3} \to 0 \) as \( n \to \infty \).

### Probability of Error at the Decoder of \( R_1 \)

There are two possible events which result in errors: (a) The codewords transmitted are not jointly typical i.e., \( E^n_{j_1,t_1} \) happens or (b) there exists some \( j \neq j \) such that \( E^n_{j_1,t_1} \) happens. Note that \( t_1 \) need not equal \( l_1 \), and \( t_1 \) need not equal \( t_1 \), since \( R_1 \) is not required to decode \( l_1 \) and \( l_1 \) correctly. The probability of decoding error can, therefore, be expressed as

\[ P_{e,R_1}^{(a)} = P \left( \bigcup_{j \neq j} E^n_{j_1,t_1} \right) \quad (41) \]

Applying union of events bound, (41) can be written as,

\[ P_{e,R_1}^{(a)} \leq P(E^n_{j_1,t_1}) + P \left( \bigcup_{j \neq j} E^n_{j_1,t_1} \right) \]
\[ = P(E^n_{j_1,t_1}) + \sum_{j \neq j} P(E^n_{j_1,t_1}) + \sum_{j \neq j} P(E^n_{j_1,t_1}) \]
\[ + \sum_{j \neq j} P(E^n_{j_1,t_1}) \]
\[ P(E^n_{j_1,t_1}), P(E^n_{j_1,t_1}), P(E^n_{j_1,t_1}), P(E^n_{j_1,t_1}) \]
\[ \text{can be upper bounded as follows.} \]

\[ P(E^n_{j_1,t_1}) \leq 2^{-n(I(W,U_1,V_1,Y_1|Q) - 3e)}, \]
\[ P(E^n_{j_1,t_1}) \leq 2^{-n(I(W,U_1,V_1,Y_1|Q) + I(W,U_1,Y_1|Q) - 4e)}, \]
\[ P(E^n_{j_1,t_1}) \leq 2^{-n(I(W,U_1,V_1,Y_1|Q) + I(W,V_1,Y_1|Q) - 4e)}, \]
\[ P(E^n_{j_1,t_1}) \leq 2^{-n(I(W,U_1,V_1,Y_1|Q) + I(W,U_1,Y_1|Q) + I(W,V_1,Y_1|Q) - 5e)}. \]

Substituting these in the probability of decoding error at \( R_1 \), we have,

\[ P_{e,R_1}^{(a)} = \epsilon + 2^{nR_1} 2^{-n(I(W,U_1,V_1,Y_1|Q) - 3e)} \]
\[ + 2^{nR_1} 2^{-n(I(W,U_1,V_1|Q) + 4e)} 2^{-n(I(W,U_1,Y_1|Q)} \]
\[ + I(W; U_1|Q) - 4e} \]
\[ + 2^{nR_1} 2^{-n(I(W,U_1,V_1|Q) + 4e)} 2^{-n(I(W,V_1,U_1,Y_1|Q)} \]
\[ + I(W; V_1|Q) - 4e} \]
\[ + 2^{nR_1} 2^{-n(I(W,U_1,V_1|Q) + 4e)} 2^{-n(I(W,U_1,V_1|Q) + 4e)} + R_{31} \]
\[ + I(W, U_1; U_1; V_1|Q) + 4e} \]
\[ + I(W, U_1; V_1|Q) + 4e} \]
\[ + I(W, U_1; V_1|Q) + 4e} \]
\[ + I(W; U_1; V_1|Q) + I(W; U_1|Q) - 5e}. \]

\[ P_{e,R_1}^{(a)} \to 0 \] as \( n \to \infty \) if \( R_{11}, R_{21} \) and \( R_{31} \) satisfy the following constraints:
Probability of error at the decoder of $\mathcal{R}_2$

The two possible error events are: (a) The codewords transmitted are not jointly typical i.e., $E_{i_1, i_2}^c$ happens or (b) there exists some $(l_1 \neq l_1, l_2 \neq l_2)$ such that $E_{i_1, l_2}^c$ happens. The probability of decoding error can be written as

$$P_{e,R_2}^{(n)} = P\left( E_{i_1, l_2}^c \cup \bigcup_{(i_1 \neq l_1, i_2 \neq l_2)} E_{i_1, i_2}^c \right)$$

Applying union of events bound, (46) can be written as,

$$P_{e,R_2}^{(n)} \leq P\left( E_{i_1, l_2}^c \right) + P\left( \bigcup_{(i_1 \neq l_1, i_2 \neq l_2)} E_{i_1, i_2}^c \right) = P\left( E_{i_1, l_2}^c \right) + \sum_{i_1 \neq l_1} P\left( E_{i_1, l_2}^c \right) + \sum_{i_2 \neq l_2} P\left( E_{i_1, l_2}^c \right) \leq P\left( E_{i_1, l_2}^c \right) + \sum_{i_1 \neq l_1} P\left( E_{i_1, i_2}^c \right) + \sum_{i_2 \neq l_2} P\left( E_{i_1, l_2}^c \right).$$

$P(\cdot)$ and $P(\cdot)$ can be upper bounded as follows.

$$P(\cdot) \leq 2^{-n(I(V_1; V_3, Y_3)) - 3\epsilon},$$

$$P(\cdot) \leq 2^{-n(I(W_1; U_2, V_3)) - 3\epsilon},$$

$$P(\cdot) \leq 2^{-n(I(U_1; U_2, V_3)) - 4\epsilon}.$$

Substituting these in the probability of decoding error at $\mathcal{R}_2$, we get $P_{e,R_2}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$ if $R_{31}$ and $R_{33}$ satisfy the following constraints:

$$R_{31} \leq I(V_1; V_3, Y_3) - I(W_1; U_1, U_2; V_1|Q),$$

$$R_{33} \leq I(V_3; V_1, Y_3) - I(W_1; U_1, U_2; V_3|Q),$$

$$R_{31} + R_{33} \leq I(V_1, V_3; Y_3) + I(V_1; V_3|Q) - I(W, U_1, U_2; V_3|Q) - I(W, U_1, U_2; V_1|Q).$$

The inequalities (42)–(45), (47)–(49) and (51)–(53) together constitute the achievable rate region for the channel $C_{CuMS}^2$.

Appendix 5: Proofs of some corollaries

Proof of Corollary 3.2

In the case of $C_{CuMS}^2$, when senders $S_2$ and $S_3$ do not have any message of their own to transmit, they can use their noncausal message knowledge to entirely help sender $S_1$. The rate tuple $(R_1', 0, 0)$ is therefore achievable, where $R_1'$ is the capacity of the vector channel $(S_1, S_2, S_3) \rightarrow R_1$, given by (6).

Next, when the rate achieved by sender $S_1$ is zero, $S_2$ can cancel the interference from $S_1$ completely by employing dirty paper coding. However, due to the message splitting model assumed here, $\mathcal{R}_2$ sees interference from $S_1$ regardless of the $R_3$ achieved, except in the case where $S_1$ helps $\mathcal{R}_2$ in receiving its message. This case is dealt with in Corollary 3.3. Hence, the rate achievable by $(S_2, R_2)$ is given by (7).

When $R_1 = 0$ and $R_2 = R_2^*$, due to the noncausal knowledge of $S_1$ and $S_2$’s messages, $S_3$ can completely mitigate the effect of interference and achieve the interference free rate, $R_3$, given by (8). Hence, the rate tuple $(0, R_2, R_3)$ is achievable. Finally, the convex hull of the rate region $\Omega_C$ with these points is achievable by standard time-sharing arguments.
Proof of Corollary 3.3

If $S_3$ achieves a rate less than the interference free rate, then it can use its remaining power to help either $S_1$ or $S_2$. The power required for $S_1$ to achieve a rate of $r$ ($r \leq R_3^*$) is $P_3^{S_1} = (2^{2r} - 1)Q_3$. The power that can be used to help $S_1$ or $S_2$ is $P_3^{S_1} = P_3 - P_3^{S_1}$. When $S_2$ achieves a rate of zero, then $S_2$ can completely help $S_1$. Further, $S_3$ can use the power of $P_3^{S_1^*}$ to help $S_1$. Therefore, $S_1$ can achieve a rate given by (9).

When $R_2^* = 0$ and $R_3 = r$, then $S_3$ can use the power of $P_3^{S_1^*}$ to transmit the message of $S_2$, and $S_2$ can cancel the interference from $S_1$ by employing dirty paper coding and achieve a rate given by (10). Hence, the rate tuples $(R_1^*, 0, r)$ and $(0, R_2^*, r)$ are achievable. The convex hull can be achieved by time-sharing arguments.

As $S_3$ has noncausal knowledge of $S_1$ and $S_2$, it can completely mitigate the effect of interference and achieve a rate given by (11).

Proof of Corollary 3.4

When $S_2$ achieves certain specific rate of $r$ ($r \leq R_3^*$), the power required is $P_2^{S_2} = (2^{2r} - 1)(Q_2 + |a_{23}|^2 P_3)$. The remaining power $P_3^{S_1} = P_3 - P_2^{S_2}$ can be used to help $S_1$. Therefore, $S_3$ can achieve a rate given by (12).

Due to the knowledge of $S_1$ message, $S_2$ can completely cancel the effect of interference from primary but it will always see the interference from $S_3$. Hence $S_2$ can achieve a rate given by (13).

When $R_1^* = 0$, the sender $S_1$ can mitigate the interference from $S_3$ and achieves the interference free rate given by (14).

Therefore, the rate tuples $(R_1^*, r, 0)$ and $(0, r, R_3^*)$ are achievable. The convex hull is achieved using time-sharing arguments.

Proof of Corollary 3.7

In case of $C_{GPM}^c$ channel model, the senders $S_2$ and $S_3$ can only help sender $S_1$ in its transmission. When $R_2 = 0$ and $R_3 = 0$, $S_1$ can achieve a rate given by (17).

When $R_1^* = 0$, both $S_2$ and $S_3$ can completely eliminate the effect of interference from $S_1$. However, they will experience the interference from each other. Hence, $S_2$ and $S_3$ can achieve a rate of $R_2^*$ and $R_3^*$ given by (18) and (19).

Therefore, it is clear that the rate tuples $(R_1^*, 0, 0)$ and $(0, R_2^*, R_3^*)$ are achievable. By standard time-sharing arguments, the convex hull is achievable.

Proof of Corollary 3.8

In order to achieve a rate $r$ ($r \leq R_3^*$), the power required by $S_3$ is $P_3^{S_1^*} = (1 + (2^{2r} - 1)|a_{23}|^2 P_3)$. The remaining power $P_3^{S_1} = P_3 - P_3^{S_1^*}$ can be used to help $S_1$. When $R_2 = 0$ and $R_3 = r$, $S_2$ and $S_3$ can use the power of $P_2$ and $P_3^{S_1^*}$ respectively, to help the primary. The rate achieved by $S_1$ is given by (20), where the $|a_{23}|^2 P_3^{S_1^*}$ in the denominator arises because of sender $S_1$ transmitting the message to its pairing receiver.

When $R_1 = 0$ and $R_3 = r, R_2$ can achieve a rate given by (21). Therefore, the rate tuples $(R_1^*, 0, r)$ and $(0, R_2^*, r)$ are achievable. The convex hull can be achieved by time-sharing.

As $S_3$ has noncausal knowledge of primary message, it can employ dirty paper coding to completely mitigate the effect of interference from $S_1$. However it sees interference from $S_2$ due to the rate splitting. Hence, $S_3$ can achieve a rate given by (22).

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