MONOMIAL AND RODRIGUES ORTHOGONAL POLYNOMIALS ON THE CONE

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Abstract. We study two families of orthogonal polynomials with respect to the weight function
\[ w(t)(t^2 - \|x\|^2)^{\mu - \frac{1}{2}}, \quad \mu > -\frac{1}{2}, \]
on the cone \( \{(x,t) : \|x\| \leq t, x \in \mathbb{R}^d, t > 0\} \) in \( \mathbb{R}^{d+1} \). The first family consists of monomial polynomials
\[ V_{k,n}(x,t) = t^{n-|k|}x^k + \cdots \]for \( k \in \mathbb{N}_0^d \) with \( |k| \leq n \), which has the least \( L^2 \) norm among all polynomials of the form \( t^{n-|k|}x^k + P \) with \( \deg P \leq n - 1 \), and we will provide an explicit construction for \( V_{k,n} \). The second family consists of orthogonal polynomials defined by the Rodrigues type formulas when \( w \) is either the Laguerre weight or the Jacobi weight, which satisfies a generating function in both cases. The two families of polynomials are partially biorthogonal.

1. Introduction

One of the fundamental differences between orthogonal polynomials in one variable and several variables lies in the multiplicity of orthogonal bases. Let \( L^2(\Omega, W) \) be a weighted \( L^2 \) space on the domain \( \Omega \) in \( \mathbb{R}^d \). For \( n = 0, 1, 2, \ldots \), let \( V_n^d \) denote the space of orthogonal polynomials of degree \( n \) in \( L^2(\Omega, W) \), which consists of polynomials of degree \( n \) that are orthogonal to all polynomials of lower degrees for the inner product of \( L^2(\Omega, W) \). For \( d = 1 \), the space \( V_n^d \) is one dimension, so the orthogonal polynomial of degree \( n \) is unique up to a constant multiple. For \( d > 1 \), the space \( V_n^d \) has the dimension \( \binom{n + d - 1}{n} \) and it has infinitely many distinct bases. Different bases are often needed for different occasions or applications, and each basis may catch a specific perspective of the orthogonal structure. One could choose an orthonormal basis and there are infinitely many of those. There are also outstanding bases that may not be mutually orthogonal.

Among the plethora of choices, we single out two that are not orthogonal but are biorthogonal with each other. The first basis consists of monomial polynomials
\[ V_k(x) = x^k + R_k(x), \quad k \in \mathbb{N}_0^d, \quad R_k \in \Pi_{n-1}^d. \]

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where $\Pi^d_n$ denotes the space of polynomials of degree $n$ in $d$ variables, which is uniquely determined by the requirement that $\{V_k : |k| = n\}$ is a basis of $\mathcal{V}^d_n$. This basis is not necessarily an orthogonal basis of $\mathcal{V}^d_n$. One of the important features of this basis is that $V_k$ has the least $L^2$ norm among all polynomials of the form $x^k - P$, $P \in \Pi^d_{n-1}$; more precisely,

$$\inf_{P \in \Pi^d_{n-1}} \|x^k - P\|_{L^2(\Omega,W)} = \|V_k\|_{L^2(\Omega,W)}. \tag{1.1}$$

The second basis consists of polynomials $U_k$, $|k| = n$, defined by the Rodrigues type formulas that involve repeated differentiation, which can only be defined for special weight functions and domains. The study of such bases was initiated for classical orthogonal polynomials on the unit disk and the unit ball $\mathbb{B}^d$ by Hermite and his contemporaries (see, for example, [3, 7, 8]), where the orthogonality is defined for the weight function $(1 - \|x\|^2)^{\mu - \frac{d}{2}}$, $\mu > -\frac{d}{2}$, on $\mathbb{B}^d$. For the unit ball, these two families of polynomials can be given explicitly, satisfy elegant generating functions, and turn out to be mutually biorthogonal [3, 7] (see Section 2.2 below).

In the present paper, we study orthogonal polynomials on the conic domain

$$\mathcal{V}^{d+1} = \{(x,t) \in \mathbb{R}^{d+1} : |x| \leq t, x \in \mathbb{R}^d, t \geq 0\},$$

on $\mathbb{R}^{d+1}$ and the orthogonality is defined with respect to the weight function

$$W(x,t) = w(t)(t^2 - |x|^2)^{\mu - \frac{d}{2}}, \quad \mu > -\frac{1}{2},$$

where $w$ is a weight function defined on $\mathbb{R}_+ = [0, \infty)$ and the cone could be regarded as finite if $w$ is supported on the finite interval, say, $[0,1]$. Orthogonal polynomials on the cone are studied only recently. The two most important cases are the Laguerre polynomials on the cone, with $w(t) = e^{-t}$, and the Jacobi polynomials on the cone, with $w(t) = t^\alpha(1 - t)^\beta$. In these cases, an orthonormal basis is defined in [15] and used to show that orthogonal polynomials in $\mathcal{V}^{d+1}$ are eigenfunctions of a second-order differential operator and they satisfy an addition formula. The latter gives a closed-form formula for the reproducing kernel of the space $\mathcal{V}^{d+1}$, which provides essential tools for an extensive study on approximation theory and computational harmonic analysis over the cone [16, 17]. The study of the orthogonal structure over the cone is still in its beginning, the present paper aims to explore the monomial basis and the basis defined by the Rodrigues formulas.

Our main work lies in understanding the monomial basis. In the variable $(x,t) \in \mathbb{R}^{d+1}$, such a basis takes of the form

$$V_{k,n}(x,t) = t^{-|k|}x^k + P_{k,n} \quad \text{with} \quad P_{k,n} \in \Pi^{d+1}_{n-1}.$$  

Given the nature of the weight function, one may expect that the polynomial $V_{k,n}$ can be constructed simply in terms of the monomial polynomials $V_k$ on the unit ball and orthogonal polynomial of one variable in the $t$ variable. The problem, however, turns out to be more subtle, since $V_k$ can only appear in the form of $t^{|k|}V_k\left(\frac{x}{t}\right)$, which however is a homogeneous polynomial, far from a monomial. Even the case $d = 1$, or $\mathcal{V}^2 \subset \mathbb{R}^2$, turns out to be non-trivial. One of our main results provides an explicit formula for the polynomials $V_{k,n}$ and its $L^2$ norm. For the Laguerre and the Jacobi cases, we also define the second family of polynomials, denoted by $U_{k,n}$, via the Rodrigues type formulas. The polynomials $U_{k,n}$ for $|k| = n$ is also a basis of $\mathcal{V}^{d+1}$ and they satisfy a generating function. The two families, $\{V_{k,n}\}$ and $\{U_{k,n}\}$, are, however, only partially biorthogonal, in contrast to the case of the unit ball.
The paper is organized as follows. The next section is preliminary, where we recall the results of classical orthogonal polynomials on the unit ball and develop orthogonal polynomials on the cone. The monomial polynomials on the cone, called the V-family, are defined and studied in the third section, where we first consider the case \( d = 1 \) to illustrate the idea. The second family of polynomials, the U-family, is defined via Rodrigues formulas in the fourth section and shown to be partially biorthogonal to the V-family and, in \( d = 1 \) case, a basis that is biorthogonal to the V-family is explicitly constructed in terms of the U-family. Finally, we discuss generating functions for these polynomials in the fifth section.

2. Preliminary

In the first subsection, we discuss orthogonal polynomials, especially monomial orthogonal polynomials, in general. In the second section, we illustrate the general setup by recalling results from classical orthogonal polynomials on the unit ball, which will also be needed in our study. In the last subsection, we lay down the basics for orthogonal polynomials on the cone.

Throughout this paper, we use multi-index notation: for \( \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \) and \( \mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d \), we shall write, for example,

\[
\mathbf{k}! = k_1! \cdots k_d!, \quad (\mathbf{a})_\mathbf{k} = (a_1)^{k_1} \cdots (a_d)^{k_d}, \quad \sum_{j \leq k} A_j = \sum_{j_1=0}^{k_1} \cdots \sum_{j_d=0}^{k_d} A_j,
\]

where \( (\mathbf{a})_\mathbf{k} = (a + 1) \cdots (a + k - 1) \) denotes the Pochhammer symbol.

2.1. Monomial orthogonal polynomials. Let \( w \) be a weight function on a domain \( \Omega \subset \mathbb{R}^d \). Let \( \langle \cdot, \cdot \rangle_w \) be the inner product defined by

\[
\langle f, g \rangle_w = \int_\Omega f(x)g(x)w(x)dx.
\]

Let \( \Pi_d^n \) denote the space of polynomials of degree at most \( n \) in \( d \) variables. A polynomial \( P \) of degree \( n \) is an orthogonal polynomial with respect to the inner product if

\[
\langle P, Q \rangle_w = 0, \quad \forall Q \in \Pi_d^{n-1}.
\]

Assume that \( w \) is regular so that orthogonal polynomials with respect to the inner product are well-defined. Let \( \mathcal{V}_d^n(\Omega, w) \) be the space of orthogonal polynomials of degree \( n \) in \( d \) variables with respect to the inner product. It is known that

\[
r_d^n := \dim \mathcal{V}_d^n(\Omega, w) = \binom{n + d - 1}{n}, \quad n = 0, 1, 2, \ldots.
\]

For \( d > 1 \), the space \( \mathcal{V}_d^n(\Omega, w) \) contains infinitely many bases. Moreover, since orthogonality is defined as orthogonal to all polynomials of lower degrees, elements of a basis may not be mutually orthogonal. A basis \( \{P_j^n : 1 \leq j \leq r_d^n\} \) of \( \mathcal{V}_d^n(\Omega, w) \) is called an orthogonal basis if \( \langle P_j^n, P_k^n \rangle_w = 0 \) whenever \( j \neq k \) and it is called an orthonormal basis if, in addition, \( \langle P_j^n, P_j^n \rangle_w = 1 \) for all \( j \). It is often convenient to consider an orthonormal basis for many applications, such as dealing with the Fourier orthogonal series. Notice, however, that if \( \mathbb{P}_n = \{P_j^n : 1 \leq j \leq r_d^n\} \) is an orthonormal basis, then so is \( \mathbb{Q}\mathbb{P}_n \) if \( \mathbb{P}_n \) is regarded as a column vector and \( \mathbb{Q} \) is an orthogonal matrix of size \( \binom{n + d - 1}{n} \). Hence, there are infinitely many distinct orthonormal bases. Our goal in this
paper is to consider the monomial basis \(\{V_k : |k| = n, k \in \mathbb{N}^d\}\), where \(V_k \in \mathcal{V}_n^d(\Omega, w)\) is characterized by
\[
V_k(x) = x^k + P_k, \quad P_k \in \Pi_{n-1}, \quad n = |k|;
\]
in other words, the polynomial \(V_k\) of degree \(n\) contains a single monomial, \(x^k\), of degree \(n\). The monomial orthogonal polynomial \(V_k\) is the error of the least square approximation to the monomial \(x^k\) from \(\Pi_{|k|-1}^d\); more precisely, it satisfies (1.1) since \(V_k = x^k + P_k\) is orthogonal to all polynomials in \(\Pi_{n-1}^d\). Moreover, in the case of several families of classical orthogonal polynomials, such as those on the unit ball or the standard simplex, the monomial orthogonal polynomials possess fine structures and are biorthogonal to another family of orthogonal polynomials defined by the Rodrigues type formulas. We illustrate this in the next subsection.

2.2. Orthogonal polynomials on the unit ball. For \(\mu > -\frac{1}{2}\), the classical weight function on the unit ball \(\mathbb{B}^d = \{x \in \mathbb{R}^d : ||x|| \leq 1\}\) of \(\mathbb{R}\) is
\[
w_\mu(x) = (1 - ||x||^2)^{\mu - \frac{1}{2}}, \quad \mu > -\frac{1}{2}, \quad x \in \mathbb{B}^d,
\]
the normalization constant \(b_\mu^b\), so that \(b_\mu^b w_\mu(x)\) has the unit integral, is given by
\[
b_\mu^b = \frac{1}{\int_{\mathbb{B}^d} w_\mu(x)dx} = \frac{\Gamma(\mu + \frac{d+1}{2})}{\pi^{\frac{d}{2}} \Gamma(\mu + \frac{1}{2})}.
\]

When \(d = 1\), the ball is \([-1, 1]\) and the associated orthogonal polynomials are the classical Gegenbauer polynomials \(C_n^\mu\), which satisfy the orthogonal relation
\[
c_\mu \int_{-1}^{1} C_n^\mu(t)C_m^\mu(t)(1 - t^2)^{\mu - \frac{1}{2}} dt = h_n^\mu \delta_{n,m}, \quad h_n^\mu = \frac{\mu(2\mu)_n}{(n + \mu)n!},
\]
where \(c_\mu = \frac{\Gamma(n+1)}{\sqrt{\pi(n+\frac{1}{2})}}\) is the same as \(b_\mu^b\) with \(d = 1\) and \(\mu \neq 0\). If \(\mu = 0\), the polynomial \(C_n^0\) becomes the Chebyshev polynomial \(T_n\) under the limit
\[
\lim_{\mu \rightarrow 0} \frac{1}{c_\mu} C_n^\mu(x) = 2 \frac{n}{n} T_n(x), \quad n \geq 1.
\]

For \(d > 1\), the orthogonal polynomials with respect to \(w_\mu\) on \(\mathbb{B}^d\) are well studied; see [7] Section 5.2]. Let \(\mathcal{V}_n(\mathbb{B}^d, w_\mu)\) be the space of OPs for this weight function. There are many distinct bases for this space and two orthonormal bases can be given explicitly in either Cartesian or polar coordinates. We are mostly interested in the monomial basis and the basis defined by the Rodrigues formula. While the former has been discussed before, the latter is an extension of the Rodrigues formula for classical orthogonal polynomials in one variable to several variables; see, for instance, [3, 7, 8, 10, 14], for some examples. For the unit ball, both bases can be traced back to the work of Hermite [3]. Following the classical presentation (cf. [8, Chapter 12] and [7, Chapter 5]), we denote the elements of these two bases by \(V_k\) and \(U_k\) with \(k \in \mathbb{N}^d\), respectively.

We consider the \(U\)-basis given by the Rodrigues formula first. For each \(k \in \mathbb{N}_0^d\), the polynomials \(U_k\) is defined by
\[
U_k(x) = \frac{(-1)^{|k|}(2\mu)_{|k|}}{2^{|k|}(\mu + \frac{1}{2})_{|k|} |k|!} \frac{1}{(1 - ||x||^2)^{\mu - \frac{1}{2}}} \frac{\partial^{|k|}}{\partial_{x_1} |k|} \cdots \partial_{x_d} |k| (1 - ||x||^2)^{\mu - \frac{1}{2}}.
\]
Then \( \{ U_k : |k| = n \} \) is a basis of \( \mathcal{V}_n(\mathbb{B}^d, w_\mu) \). These polynomials also satisfy a generating function given by
\[
(2.3) \quad \frac{1}{((1 - \langle a, x \rangle)^2 + \|a\|^2(1 - \|x\|^2))^{\mu}} = \sum_{k \in \mathbb{N}_0^d} U_k(x)a^k, \quad a \in \mathbb{B}^d.
\]
The \( V \)-basis is the monomial basis, for each \( k \in \mathbb{N}_0^d \), the polynomial \( V_k \) is defined by
\[
(2.4) \quad V_k(x) = x^k F_B \left( -\frac{k}{2}, \frac{1}{2} - |k| - \mu - \frac{d - 3}{2}; \frac{1}{x_1^2}, \ldots, \frac{1}{x_d^2} \right),
\]
where \( 1 = (1, \ldots, 1) \in \mathbb{N}^d \) and \( F_B \) is the Lauricella hypergeometric series of \( d \) variables defined by
\[
F_B(a, b; c; x) = \sum_{m \in \mathbb{N}_0^d} \frac{(a)_m(b)_m}{(c)_m m!} x^m, \quad \max_{1 \leq j \leq d} |x_j| < 1.
\]
It is easy to see that \( V_k(x) = x^k + Q_k \) with \( Q_k \in \Pi_{n-1}^d \), so that \( V_k \) is monomial for \( k \in \mathbb{N}_0^d \). Moreover, \( \{ V_k : |k| = n \} \) is a basis for \( \mathcal{V}_n(\mathbb{B}^d, w_\mu) \). These polynomials satisfy a generating function given by
\[
(2.5) \quad \frac{1}{(1 - 2\langle a, x \rangle + \|a\|^2)^{\mu + \frac{d - 1}{2}}} = \sum_{k \in \mathbb{N}_0^d} \frac{2|k|}{k!} \left( \mu + \frac{d - 1}{2} \right)^{|k|} V_k(x)a^k, \quad a \in \mathbb{B}^d.
\]
As we mentioned before, the polynomial \( V_k \) has the least \( L^2 \) norm among all polynomials of the form \( x^k - P(x) \), \( P \in \Pi_{n-1}^d \). Its norm is computed in [13 Corollary 4.8], which shows in particular that, for \( k \in \mathbb{N}_0^d \) and \( |k| = n \),
\[
(2.6) \quad \inf_{P \in \Pi_{n-1}^d} \| x^k - P \|_{L^2(\mathbb{B}^d, w_\mu)}^2 = \frac{\lambda k!}{2^{n-1}(\lambda)^n} \int_0^1 \prod_{i=1}^d P_{\lambda_i}(t) t^{|n + 2\lambda - 1|} dt,
\]
where \( \lambda = \mu + \frac{d - 1}{2} > 0 \) and \( P_n \) is the Legendre polynomial of degree \( n \), which implies, in particular, that the integral on the right-hand side is positive, a fact that is not obvious. For \( d = 2 \), a different expression for this quantity is given in [5]. For small \( n \), the least square of some symmetric monomials are given in [2], which can be deduced from the norm of \( V_k \), see [13]. Orthogonal polynomials are also used in deriving the exact constant in the Bernstein inequality on the ball in the recent work of [9].

Neither \( U \)-basis nor \( V \)-basis are orthogonal bases for \( \mathcal{V}_n(\mathbb{B}^d, w_\mu) \). They are, however, biorthogonal with respect to each other. More precisely, they satisfy
\[
(2.7) \quad b_{ij}^\mu \int_{\mathbb{B}^d} U_k(x)V_j(x)w_\mu(x)dx = \frac{(2\mu)|k|}{2^{|j|}\mu + \frac{d - 1}{2}|k|} \delta_{j,k},
\]
We will need the explicit expression of \( V_k(x) \) in monomials, which is derived by rewriting (2.4) in the following form
\[
V_k(x) = \sum_{2j \leq k} b_{kj}^\mu x^{k - 2j}, \quad \text{where} \quad b_{kj}^\mu := \frac{(-k/2)_j(-k+1/2)_j}{(-|k| - \lambda + 1)|j|!},
\]
and we shall denote, throughout the rest of the paper,
\[
\lambda = \mu + \frac{d - 1}{2}.
\]
The relation (2.7) can be reversed, as seen in [13, Corollary 6.2], as

\[ x^k = \sum_{2i \leq k} c_{k,i} V_{k-2i}(x), \]

where the coefficients are given by

\[ c_{k,i} = \frac{(-1)^{\mu}(-\frac{k}{2})^{i}(-\frac{k+1}{2})^{i}}{(2(-|k| - \lambda + 1)_{2i})!} (2(-|k| - \lambda)_{2i} - (\lambda)_{2i}). \]

Putting the two identities together, we obtain

\[
\begin{align*}
    x^k &= \sum_{2i \leq k} c_{k,i} V_{k-2i}(x) &= \sum_{2i \leq k} c_{k,i} \sum_{2j \leq k} b_{k-2i,j} x^{k-2j} \\
    &= \sum_{2i \leq k} c_{k,i} b_{k-2i,j} x^{k-2j} = \sum_{2j \leq k} c_{k,i} b_{k-2i,j} x^{k-2j} \\
    &= \sum_{i \leq j} x^{k-2j} \sum_{i \leq j} c_{k,i} b_{k-2i,j} x^{2i-2j}
\end{align*}
\]

Consequently, it follows readily that

\[ \sum_{i \leq j} b_{k,i} b_{k-2i,j-1} = \delta_{j,0}, \quad j \in \mathbb{N}_0. \]

This identity will be used in the sequel.

2.3. **Orthogonal polynomials on the cone.** Let \( w \) be a weight function on an interval in \( \mathbb{R} \). We can assume that the interval is either \([0, 1]\) or \([0, \infty)\) without losing generality. On the cone \( \mathbb{R}^{d+1}_+ \), we define

\[ W_{\mu}(x, t) = w(t)(t^2 - \|x\|^2)^{\mu - \frac{1}{2}}, \quad \mu > -\frac{1}{2}, \]

and define the inner product

\[ (f, g)_\mu := b_\mu \int_{\mathbb{R}^{d+1}} f(x, t)g(x, t)W_{\mu}(x, t)dxdt, \]

where \( b_\mu \) is a normalization constant so that \( (1, 1)_\mu = 1 \) and

\[ b_\mu = b_\mu^w \times b_\mu^w \quad \text{with} \quad b_\mu^w = \frac{1}{\int_0^\infty \|x\|^2 + 2\mu - 1 w(t)dt}. \]

This can be easily verified by using the separation of variables

\[ \int_{\mathbb{R}^{d+1}} f(x, t)W_{\mu}(x, t)dxdt = \int_0^\infty \int_{\mathbb{R}^d} f(ty, t)(1 - \|y\|^2)^{\mu - \frac{1}{2}} dy t^{d+2\mu - 1} w(t)dt. \]

Let \( \mathcal{V}_n(\mathbb{R}^{d+1}, W_{\mu}) \) be the space of orthogonal polynomials of degree \( n \) in \( d+1 \) variables with respect to the inner product \((f, g)_\mu\). A basis of this space can be given in terms of orthogonal polynomials on the unit ball and a family of orthogonal polynomials in one variable.

**Proposition 2.1.** Let \( \mathbb{P}_m = \{P_k : |k| = m \} \) be a basis of \( \mathcal{V}_m(\mathbb{B}^d, w_\mu) \) for \( m \leq n \) and let \( q_{\mu}^n \) be an orthogonal polynomial in one variable with respect to the weight function \( t^\mu w(t) \) on \( \mathbb{R}_+ \). Define

\[ Q_k,n(x, t) = q_{\alpha-m}^n(t) e^m P_k \left( \frac{x}{t} \right), \quad |k| = m, \quad 0 \leq m \leq n, \]

\[ ^\text{1} \text{Throughout this paper, we will adopt the convention that all orthogonal polynomials on the cone will be denoted by letters in the sans serif font, such as } Q_k, S_k, \text{ and } V_k. \]
where \( \alpha_m = d + 2m + 2\mu - 1 \). Then \( Q_n = \{ Q_{k,n} : |k| = m, 0 \leq m \leq n \} \) is a basis of \( \mathcal{V}_n(\mathbb{V}^{d+1}, W_\mu) \). In particular, if \( \mathbb{P}_m \) is an orthogonal basis for \( \mathcal{V}_m(\mathbb{B}^d, w_\mu) \) for \( 0 \leq m \leq n \), then \( Q_n \) is an orthogonal basis for \( \mathcal{V}_n(\mathbb{V}^{d+1}, W_\mu) \).

**Proof.** This can be easily seen from

\[
\langle Q_{k,n}, Q_{k',n'} \rangle_\mu = b_\mu \int_0^\infty t^d + m + m' + 2\mu - 1 \, q_{n-m}^\alpha(t) q_{n'-m'}^{\alpha'}(t) \int_{\mathbb{B}^d} P_k^m(y) P_{k'}^{m'}(y) w_\mu(y) dy w(t) dt
\]

\[
= b_\mu \int_0^\infty q_{n-m}^\alpha(t) q_{n'-m'}^{\alpha'}(t) t^\alpha w(t) dt \int_{\mathbb{B}^d} P_k^m(y) P_{k'}^{m'}(y) w_\mu(y) dy \delta_{m,m'}
\]

\[
= \| q_{n-m}^\alpha \|^2 b_\mu \int_{\mathbb{B}^d} P_k^m(y) P_{k'}^{m'}(y) w_\mu(y) dy \delta_{m,m'} \delta_{n,n'},
\]

where we have used the orthogonality of \( P_k^m \). Moreover, if \( \mathbb{P}_m \) is an orthogonal basis, then the last integral can be replaced by a constant multiple of \( \delta_{k,k'} \) so that the basis \( Q_n \) is an orthogonal basis on the cone. \( \square \)

What we are interested in is the monomial basis for \( \mathcal{V}_n(\mathbb{V}^{d+1}, W_\mu) \). Monomials of degree \( n \) in \((x,t)\) variables are \( t^{n-|k|} x^k \) for \( k \in \mathbb{N}_0^d \) and \( 0 \leq |k| \leq n \). We assume that the monomial orthogonal polynomial \( V_{k,n} \) is of the form

\[
V_{k,n}(x,t) = t^{n-|k|} x^k + R_k(x,t), \quad R_k \in \Pi_{n-1}^d.
\]

Given the Proposition 2.1, it is tempting to consider orthogonal polynomials

\[
S_{k,n}(x,t) := q_{n-m}^\alpha(t) t^m V_k \left( \frac{t}{x} \right), \quad |k| = m, \ k \in \mathbb{N}_0^d, \ 0 \leq m \leq n,
\]

where \( V_k \) denotes the monomial orthogonal polynomials on the unit ball, defined in (2.4), and \( q_{n-m}^\alpha \) is the monic orthogonal polynomial in the \( t \) variable. However, although both \( q_{n-m}^\alpha \) and \( V_k \) are monomial polynomials, the polynomials \( S_{k,n} \) are monomial only when \( k = 0 \) since the polynomial \( t^{|k|} V_k \left( \frac{t}{x} \right) \) is a homogeneous polynomial of degree \( m \) in the \((x,t)\) variables when \( |k| > 0 \), hence not monomial. The polynomials \( S_{k,n} \) nevertheless play an important role in our construction of the monomial basis in the next section.

For our study, the most interesting cases are when \( w \) is a classical weight function, for which \( q_{n-m}^\alpha \) can be given explicitly in classical orthogonal polynomials. This comes down to two families of classical weight functions, which lead to the Laguerre polynomials on the cone and the Jacobi polynomials on the cone studied in [15]. We review these two families of polynomials below.

### 2.3.1. Laguerre polynomials on the cone

In this case, the cone is unbounded,

\[
\mathbb{V}^{d+1} = \{ (x,t) : \|x\| \leq t, \ x \in \mathbb{R}^d, \ 0 \leq t < \infty \},
\]

and the weight function \( w(t) = t^\beta e^{-t} \), so that \( W_\mu \) in (2.11) becomes

\[
W_{\beta,\mu}(x,t) = (t^2 - \|x\|^2)^\mu t^\beta e^{-t}, \quad \mu > -\frac{1}{2}, \ \beta > -d,
\]

and its normalization constant becomes \( b_{\beta,\mu}^L \) given by

\[
b_{\beta,\mu}^L = \frac{1}{\Gamma(2\mu + \beta + d)} \beta^\mu.
\]
Recall that the Laguerre polynomial $L_n^α$ is defined by, for $α > -1$,

$$L_n^α(t) = \frac{(α + 1)n}{n!} \frac{1}{I_n} F_1(-n; α + 1; t) = \frac{(α + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(α + 1)_k k!} t^k,$$

and it satisfies the orthogonal relation

$$\frac{1}{\Gamma(α + 1)} \int_{0}^{∞} L_n^α(t) L_m^α(t) t^α e^{-t} dt = \frac{(α + 1)_n}{n!} δ_{m,n}.$$

The orthogonal polynomials $Q_{k,n}$ in (2.12) on the cone are now given in terms of the Laguerre polynomials, which we denote by $L_{k,n}$ and they are

$$L_{k,n}(x, t) = L^{2m+2µ+β+d-1}_{n-m}(t) t^m P^m_k \left( \frac{x}{t} \right), \quad |k| = m, \ 0 \leq m \leq n.$$

2.3.2. Jacobi polynomials on the cone. In this case, the cone is bounded,

$$∀^{d+1} = \{(x, t) : ||x|| \leq t, \ x \in \mathbb{B}^d, \ 0 \leq t \leq 1\},$$

and the weight function $w(t) = t^β (1 - t)^γ$ with $β > -d$ and $γ > -1$. Now $W_µ$ in (2.11) becomes

$$W_{β,γ,µ}(x, t) = (t^2 - ||x||^2)^{µ-\frac{1}{2}} t^β (1 - t)^γ, \quad µ > -\frac{1}{2}, \ γ > -1, \ β > -d,$$

and its normalization constant becomes $b_{β,γ,µ}$ given by

$$b_{β,γ,µ} = \frac{2t^β}{c_{α,β}} = \frac{c_{α,β}}{c_{α,γ}} = \frac{\Gamma(α + γ + 2)}{\Gamma(α + 1)\Gamma(γ + 1)}.$$

Recall that the Jacobi polynomial $P^α_β(t)$ is given by, for $α, β > -1$,

$$P^α_β(t) = \frac{(α + 1)_n}{n!} \frac{1}{I_n} F_1 \left( -n, n + α + β + 1; \frac{1 - t}{α + 1}; \frac{1}{2} \right),$$

in terms of the hypergeometric function and it satisfies the orthogonal relation

$$\frac{c_{α,β}}{2^{α+β+1}} \int_{-1}^{1} P^α_β(t) P^α_β(t) (1 - t)^α (1 + t)^β dt = h^α_β δ_{n,m},$$

where the norm square is given by

$$h^α_β = \frac{(α + 1)_n(β + 1)_n(α + β + n + 1)}{n!(α + β + 2)_n(α + β + 2n + 1)}.$$

The orthogonal polynomials $Q_{k,n}$ in (2.12) on the cone are given in terms of the Jacobi polynomials, which we denote by $J_{k,n}$ and they are

$$J_{k,n}(x, t) = P^{2m+2µ+β+d-1,γ}_{n-m}(1 - 2t) t^m P^m_k \left( \frac{x}{t} \right), \quad |k| = m, \ 0 \leq m \leq n.$$

3. Monomial orthogonal polynomials on the cone

This section aims to construct monomial orthogonal polynomials on the cone. To illustrate our construction, we consider the case $d = 1$ in the first subsection and present our construction for $d > 1$ in the second subsection for the setting of $W_µ$ with a generic weight $w$. We specialize the result to the Laguerre and the Jacobi polynomials on the cone in the third subsection.
3.1. **Monomial orthogonal polynomials for** $d = 1$. Our construction is based on the orthogonal polynomials $S_{k,n}$ defined in (2.14). To illustrate our approach, we consider the case $d = 1$ first, for which

$$W_\mu(x, t) = w(t)(t^2 - x^2)^{\mu - \frac{1}{2}}, \quad |x| \leq t, \quad \mu > -\frac{1}{2},$$

and we assume the polynomial $S_{k,n}$ takes the form

$$S_{k,n}(x, t) := q^{2k+2\mu}_n(t) t^k C^\mu_k \left( \frac{x}{t} \right), \quad 0 \leq k \leq n,$$

where $q^{2k+2\mu}_n(t) = t^{n-k} + \cdots$ is a monic polynomial of degree $n - k$ and it is orthogonal with respect to $t^{2k+2\mu} w(t)$ on $\mathbb{R}$, and we retain $C^\mu_k$ instead of using the monic Gegenbauer polynomial for convenience.

3.1.1. The case of a generic weight $w$. Here we construct monomial orthogonal polynomials for $W_\mu$ with a generic function $w$. The polynomial $S_{0,n}(x, t) = q^2_n(t) = t^n + \cdots$ is a monomial orthogonal polynomial of degree $n$ in the $(x, t)$ variable. For $k > 0$, $S_{k,n}$ is not monomial since the polynomial $t^k C^\mu_k \left( \frac{x}{t} \right)$ is a homogeneous polynomial of degree $k$ in the $(x, t)$ variables. For $k > 0$, we use the expansion of the Gegenbauer polynomials given by [6] (18.5.10)

$$C^\mu_n(x) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} b^\mu_{n,i} x^{n-2i} \text{ with } b^\mu_{n,i} = (-1)^i 2^{n-2i} \frac{(\mu)_{n-i}}{i!(n-2i)!},$$

which can also be reversed to give [6] (18.18.17)

$$x^k = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} c^\mu_{k,j} C^\mu_{k-2j}(x) \text{ with } c^\mu_{k,j} = \frac{k!}{2^k} \frac{k-2j+\mu}{(\mu)_{k-j+1} j!}.$$ 

The above formulas hold for $\mu \neq 0$ and $\mu > -\frac{1}{2}$. If $\mu = 0$, then $C^0_k$ becomes the Chebyshev polynomial $T_k$, for which the above relations become

$$x^k = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} c^0_{k,j} T_{k-2j}(x) \text{ with } c^0_{k,j} = \frac{1}{2^k} \frac{k!}{(k-j)!} \begin{cases} 2 & k \neq 2j, \\ 1 & k = 2j, \end{cases}$$

$$T_n(x) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} b^0_{n,i} x^{n-2i} \text{ with } b^0_{n,i} = (-1)^i 2^{n-2i-1} \frac{(n-i-1)!}{i!(n-2i)!}, \quad n \in \mathbb{N}.$$ 

**Theorem 3.1.** For $0 \leq k \leq n$, define

$$V_{k,n}(x, t) = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} c^\mu_{k,j} S_{k-2j,n}(x, t).$$

Then $V_{k,n}(x, t) = t^{n-k} x^k + \cdots$ is a monomial polynomial of degree $n$ and $\{V_{k,n} : 0 \leq k \leq n\}$ is the monomial basis of $V_n(\mathbb{R}^2, W_\mu)$.

**Proof.** Since $q^{2k+2\mu}_{n-k+2j}(t) - t^{n-k+2j} = R_{n-k+2j-1}(t)$, which is a polynomial of degree $n - k + 2j - 1$, it follows from the definition of $c_{k,j}$ that

$$V_{k,n}(x, t) = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} c^\mu_{k,j} q^{2(k-2j)+2\mu}_{n-k+2j}(t) t^{k-2j} C^\mu_{k-2j} \left( \frac{x}{t} \right).$$
\[ t^n \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c_{k,j}^\mu C_{k-2j}^\mu \left( \frac{x}{t} \right) + \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c_{k,j}^\mu R_{n-k+2j-1}(t) t^{k-2j} C_{k-2j}^\mu \left( \frac{x}{t} \right) \]
\[ = t^{n-k} x^k + \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c_{k,j}^\mu R_{n-k+2j-1}(t) t^{k-2j} C_{k-2j}^\mu \left( \frac{x}{t} \right). \]

Since the second term is evidently of degree at most \( n - 1 \), \( V_{k,n} \) is monomial, and the set \( \{ V_{k,n} : 0 \leq k \leq n \} \) is clearly independent. Moreover, as a linear combination of \( S_{j,n} \), \( V_{k,n} \) is an element of \( \mathbb{V}_n(\mathbb{V}^2, W_\mu) \). This completes the proof. \( \square \)

**Proposition 3.2.** For \( 0 \leq k \leq n \), the norm of \( V_{k,n} \) is given by

\[ b_\mu \int_{\mathbb{V}^2} |V_{k,n}(x,t)|^2 W_\mu(x,t)dxdt = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} |c_{i,k,i}^\mu|^2 h_{k-2i}^\mu \| q_{n-k+2i}^{2(k-2i)+2\mu} \|^2, \]

where \( h_{n-2i}^\mu \) is given by (2.1) and it can be rewritten as

\[ h_{k-2i}^\mu = \frac{(2\mu)_{k-2i} \alpha^{k-2i}}{k!(k-2i+\mu)(k-2i+\mu)} \]

and \( \| q_{n-j}^{2j+2\mu} \| \) is defined by

\[ ||q_{n-j}^{2j+2\mu}||^2 = b_\mu \int_0^\infty \left| q_{n-j}^{2j+2\mu}(t) \right|^2 t^{2j+2\mu} w(t)dt. \]

**Proof.** For \( d = 1 \), the polynomials \( S_{k,n} \) are mutually orthogonal and, more precisely,

\[ b_\mu \int_{\mathbb{V}^2} S_{k,n}(x,t)S_{j,n}(x,t)W_\mu(x,t)dxdt = b_\mu \int_0^\infty q_{n-k}^{2k+2\mu}(t) \alpha^{2k+2\mu} C_k^\mu(x)(1-x^2)^\mu dx \]

Finally, we use \( (a)_{k-2i} = (a)/(1-a)_{2i} \) and \( (x)_{2j} = 2^j (\frac{x}{2})_j \) to rewrite \( h_{k-2i}^\mu \) into the given form.

**3.1.2. The Laguerre polynomials on the cone.** In this case, the weight function is \( w(t) = t^\beta e^{-t} \) and the polynomial \( q_{n-k}^{2k+2\mu} \) is given by the monic Laguerre polynomial, denoted by \( \tilde{L}_{n-k}^\alpha \); more precisely,

\[ q_{n-k}^{2k+2\mu}(t) = \tilde{L}_{n-k}^{2k+2\mu+\beta}(t) = (-1)^n k! I_{n-k}^{2k+2\mu+\beta}(t), \]
so that the monomial polynomial \( V_{k,n}^\beta(x,t) \) is given by

\[
V_{k,n}^\beta(x,t) = \sum_{j=0}^{\lfloor k/2 \rfloor} c_{k,j}^{\mu} S_{k-2j,n}(x,t) = \sum_{j=0}^{\lfloor k/2 \rfloor} c_{k,j}^{\mu} L_{n-k+2j}^{2(k-2j)+2\mu+\beta}(t) t^k C_k^\mu \left( \frac{x}{t} \right).
\]

It is now easy to verify that

\[
\left\| 4^{2k+2\mu} \right\|^2 = \frac{1}{\Gamma(2\mu + \beta + 1)} \int_0^\infty \left| L_{n-k}^{2k+2\mu+\beta}(t) \right|^2 t^{2k+2\mu+\beta} e^{-t} dt = (n-k)! (2\mu + \beta + 1)_{n+k}.
\]

Hence, the norm of the monomial polynomial given in Proposition 3.2 can be specified. Thus, we obtain

\[
(3.1) \quad b_{\beta,\mu}^L \int_{\mathbb{R}^2} |V_{k,n}^\beta(x,t)|^2 W_{\beta,\mu}(x,t) dx dt
= \sum_{i=0}^{\lfloor k/2 \rfloor} |c_{k,i}^{\mu}|^2 h_{k-2i}^{\mu}(n-k+2i)!(2\mu + \beta + 1)_{n+k-2i}.
\]

The sum can be written as a sum of two hypergeometric functions \( _2F_1 \) evaluated at 1, which however do not have a closed-form formula. It remains to be seen if the error can be expressed by a formula similar to that of (2.6) on the cone. By (1.1), the right-hand side of the identity gives the explicit formula for the error of the least square approximation to the monomial \( t^{n-|k|} \) by lower degree polynomials. Let

\[
E_n(f)_{\beta,\mu} = \inf_{P \in H_{n+1}^d} \| f - P \|_{L^2(\mathbb{R}^2, W_{\beta,\mu})},
\]

where the norm is defined by normalized weight function. Using (3.1), we can compute this quantity when \( f \) is a monomial. Below we list the first few for small \( k \):

\[
\begin{align*}
[E_n(t^n)_{\beta,\mu}]^2 & = n! (\beta + 2\mu + 1)_n; \\
[E_n(t^{n-1}x)_{\beta,\mu}]^2 & = \frac{(n-1)! (\beta + 2\mu + 1)_{n+1}}{2(\mu + 1)_2}; \\
[E_n(t^{n-2}x^2)_{\beta,\mu}]^2 & = \frac{n! (\beta + 2\mu + 1)_n}{4(\mu + 1)^2} + \frac{(2\mu + 1)(n-2)! (\beta + 2\mu + 1)_{n+2}}{4(\mu + 1)^2(\mu + 2)}; \\
[E_n(t^{n-3}x^3)_{\beta,\mu}]^2 & = \frac{9(n-1)! (\beta + 2\mu + 1)_{n+1}}{8(\mu + 1)(\mu + 2)^2} + \frac{3(2\mu + 1)(n-3)! (\beta + 2\mu + 1)_{n+3}}{8(\mu + 1)(\mu + 2)^2(\mu + 3)}.
\end{align*}
\]

It should be noted that, for \( d = 1 \), a change of variables \((t,x) \mapsto (u,v)\) given by

\[
u = \frac{t + x}{2} \quad \text{and} \quad v = \frac{t - x}{2}
\]

maps \( \mathbb{R}^2 \) to \( \mathbb{R}^2_+ \) and the weight function \( t^\beta e^{-t} \) to

\[
W_{\beta}^L = (u + v)^\beta e^{-u-v}, \quad (u,v) \in \mathbb{R}^2_+.
\]

In particular, when \( \beta = 0 \), the orthogonal polynomials become the product of classical Laguerre polynomials. For \( \beta > 0 \), orthogonal polynomials with respect to \( W_{\beta}^L \) are studied in [10]. We notice, however, that the monomial polynomial \( V_{k,n} \) is no longer monomial under the mapping since

\[
V_{k,n}(x,t) = V_{k,n}(u-v, u+v) = (u+v)^{n-k}(u-v)^k + \cdots,
\]

so that our result remains new even when \( \beta = 0 \).
The Jacobi polynomials on the cone. Here the weight function is \( w(t) = t^\beta (1-t)^\gamma \) and the polynomial \( q_{n-k}^{2+2\mu} \) is given in terms of the monic Jacobi polynomial, denoted by \( \tilde{P}_{n-k}^{(\alpha,\beta)} \); more precisely,

\[
q_{n-k}^{2+2\mu}(t) = \frac{1}{(-2)^{n-k}} \tilde{P}_{n-k}^{(2k+2\mu+\beta,\gamma)}(1-2t)
\]

\[
= \frac{(-1)^{n-k}(n-k)!}{(n+k+2\mu+\beta+\gamma+1)_{n-k}} \tilde{P}_{n-k}^{(2k+2\mu+\beta,\gamma)}(1-2t),
\]

so that the monomial polynomial \( V_k^l \) is given by

\[
V_k^l(x, t) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} c_{k,j}^l S_{k-2j,n}(x, t)
\]

\[
= \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{c_{k,j}^l}{(-2)^{n-k-2j}} \tilde{P}_{n-k+2j}^{(2k-2j)+2\mu+\beta,\gamma}(1-2t)t^k C_k^\mu \left( \frac{x}{t} \right).
\]

In this case, it is easy to verify that

\[
\left\| q_{n-k}^{2+2\mu} \right\|^2 = \frac{c_{2k+2\mu,\beta,\gamma}}{(-2)^{2n-2k}} \int_0^1 \frac{\tilde{P}_{n-k}^{(2k+2\mu+\beta,\gamma)}(1-2t)^2 t^{2k+2\mu+\beta}(1-t)^\gamma dt}{(n-k)! (2\mu+\beta+1)_{n+k}(\gamma+1)_{n-k}}
\]

Hence, the norm of the monomial polynomial is given by the sum

\[
b_{\beta,\gamma,\mu}^l \int_{\mathbb{R}^2} \left| V_k^l(x, t) \right|^2 W_{\beta,\gamma,\mu}(x, t) dx dt
\]

\[
= \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left| c_{k,i}^l \right|^2 h_k^l \frac{(n-k+2i)! (2\mu+\beta+1)_{n+k-2i}(\gamma+1)_{n-k+2i}}{(2\mu+\beta+\gamma+2)_{2n}(n+k+2\mu+\beta+\gamma+1)_{n-k+2i}}.
\]

The sum can be written as a sum of three hypergeometric functions \( sF_7 \) evaluated at 1 but no closed-form formula. It remains to be seen if it can be expressed by a formula similar to that of (2.6). Let

\[
E_n(f)_{\beta,\gamma,\mu} = \inf_{P \in \mathfrak{P}_{n-1}} \left\| f - P \right\|_{L^2(\mathbb{R}^2, W_{\beta,\gamma,\mu})},
\]

where the norm is defined by normalized weight function. We can use (3.3) to compute this quantity for \( f \) is a monomial. Below are the first few cases for \( f(x, y) = t^n |x|^k \), in which \( \alpha = \beta + \gamma + 2\mu + 1 \),

\[
\left[ E_n(t^n)_{\beta,\gamma,\mu} \right]^2 = \frac{n!(\gamma+1)_n(\alpha)_n(\beta+2\mu+1)_n}{n!(\alpha+1)_{2n}};
\]

\[
\left[ E_n(t^n x)_{\beta,\gamma,\mu} \right]^2 = \frac{(n-1)!(\gamma+1)_{n-1}(\alpha+1)_n(\beta+2\mu+1)_{n+1}}{2(\mu+1)(\alpha+1)_{2n}(\alpha+1)_{2n-1}};
\]

\[
\left[ E_n(t^n x^2)_{\beta,\gamma,\mu} \right]^2 = \frac{n!(\gamma+1)_n(\alpha)_{2n}(\beta+2\mu+1)_n}{4(\mu+1)^2(\alpha+1)_{2n}} + \frac{(2\mu+1)(n-2)!(\gamma+1)_{n-2}(\alpha+1)_n(\alpha+2)_n(\beta+2\mu+1)_{n+2}}{4(\mu+1)^2(\mu+2)(\alpha+1)_{2n}(\alpha+2)_{2n-2}}.
\]
In particular, \( E_n(t^n)_{\beta,\gamma,\mu} \) is the \( L^2 \) norm of \( \| \hat{P}_n^{(2\mu+\beta,\gamma)} \|_{L^2([-1,1],w_{2\mu+\beta,\gamma})} \), as can be easily verified.

We note that, for \( d = 1 \), the change of variables \((t,x) \mapsto (u,v)\) in (3.6) sends
\[
(x,t) \in \mathbb{V}^2 \mapsto (u,v) \in \mathbb{T}^2 = \{(u,v) : u \geq 0, v \geq 0, u + v \leq 1\},
\]
where \( \mathbb{T}^2 \) is the standard triangle; and sends the weight function \( t^\beta(1-t)^\gamma \) to
\[
W_{\beta}^l = (u+v)^\beta(1-u-v)^\gamma, \quad (u,v) \in \mathbb{T}^2.
\]
In particular, when \( \beta = 0 \), the orthogonal polynomials on the cone become a special case of the classical Jacobi polynomials on the triangle. For \( \beta \neq -1 \), the orthogonal polynomials are not the classical Jacobi polynomials but are special cases studied in [11], which has also been extended to simplexes in [1]. Nevertheless, as in the Laguerre case, the monomial polynomial \( V_{k,n} \) is no longer monomial under the mapping, and our result remains new even when \( \beta = 0 \).

### 3.2. Monomial orthogonal polynomials for \( d \geq 1 \)

Following the idea for \( d = 1 \), we now construct monomial orthogonal polynomials on the cone for \( d > 1 \). Again we deal with the weight function \( W_\mu \) with a generic weight function \( w \) first.

#### 3.2.1. The case of a generic weight function \( w \)

We consider \( W_\mu \) defined in (2.11) and use \( S_{k,n} \) defined in (2.14) to construct monomial orthogonal polynomial \( V_{k,n} \) that satisfies (2.13).

**Theorem 3.3.** Let \( d \geq 2 \) and let \( c_{k,j}^\mu \) be the coefficients defined in (2.9). For \( k \in \mathbb{N}_0^d \) and \(|k| \leq n\), define
\[
V_{k,n}(x,t) = \sum_{2j \leq k} c_{k,j}^\mu S_{k-2j,n}(x,t).
\]
Then \( V_{k,n}(x,t) = t^{n-|k|}x^{|k|} \cdots \) is a monomial polynomial of degree \( n \) and the set \( \{V_{k,n} : 0 \leq |k| \leq n, k \in \mathbb{N}_0^d\} \) is the monomial basis of \( \mathcal{V}_n(\mathbb{V}^{d+1}, W_\mu) \). Furthermore, \( V_{k,n} \) satisfies
\[
V_{k,n}(x,t) = \sum_{2i \leq k} x^{k-2i} \sum_{m=0}^{\frac{|i|}{2}} q_{n-|k|+2m}^2(t) t^{2|j|-2m} B_{i,m},
\]
where \( B_{i,m} \) depends only on \(|i|\) and is defined by, with \( \lambda = \mu + \frac{d-1}{2} \),
\[
B_{i,m} = \frac{(-1)^m(-\frac{1}{2})^i(-\frac{k+1}{2})^i}{(-|k| - \lambda + 1)|i+m|!} \binom{m}{i} (2(-|k| - \lambda + 1)m - (-|k| - \lambda)m).
\]

**Proof.** Using that \( q_{n-m}^{a_m}(t) - t^{n-m} = R_{n-m-1}(t) \) is a polynomial of degree \( n - m - 1 \), \( a_m = 2m + 2\mu \), we obtain, by (2.13),
\[
V_{k,n}(x,t) = \sum_{2j \leq k} c_{k,j}^\mu q_{n-|k|+2j|}^{2(|k|-2j)+2\mu}(t) x^{k-2j} V_{k-2j} \left( \frac{x}{t} \right)
\]
\[
= t^n \sum_{2j \leq k} c_{k,j}^\mu V_{k-2j} \left( \frac{x}{t} \right) + \sum_{2j \leq k} c_{k,j}^\mu R_{n-|k|+2j|}^{k-2j}(t) x^{k-2j} V_{k-2j} \left( \frac{x}{t} \right)
\]
\[
= t^{n-|k|}x^{|k|} + \sum_{2j \leq k} c_{k,j}^\mu R_{n-|k|+2j|}^{k-2j}(t) x^{k-2j} V_{k-2j} \left( \frac{x}{t} \right),
\]
Proposition 3.4. For \( k \in \mathbb{N}_0^d \), \( k = |k| \), and \( 0 \leq k \leq n \), the norm of \( V_{k,n} \) is given by

\[
b_{\mu} \int_{\mathbb{R}^{d+1}} |V_{k,n}(x,t)|^2 W_{\mu}(x,t)dxdt
\]
For the last integral, let

\[ T = \left( \frac{1}{2} \right)_{k} \frac{\sum |i|}{(\mu + \frac{d+1}{2})_{k}} \sum_{2i \leq k} \sum_{m=0}^{\left\lfloor \frac{k}{4} \right\rfloor} B_{i,m} \left\| q_{n-k+2m} \right\|^{2 (k-2m)+2\mu} \left( \frac{k}{2} - \mu - \frac{d-1}{2} \right)_{i} \left( -k + \frac{1}{2} \right)_{i}, \]

where \( \left\| q_{n-j} \right\| \) is defined by

\[ \left\| q_{n-j} \right\|^{2} = b_{\mu} \int_{0}^{\infty} \left| q_{n-j}(t) \right|^{2} t^{2j+2\mu} w(t) dt. \]

**Proof.** Since \( V_{k,n} \) is monomial orthogonal polynomial, we obtain

\[ \text{LH} := b_{\mu} \int_{\mathbb{R}^{d+1}} \left[ V_{k,n}(x, t) \right]^{2} w_{\mu}(x, t) dx \]

\[ = b_{\mu} \int_{\mathbb{R}^{d+1}} V_{k,n}(x, t) t^{n-k} x^{k} w_{\mu}(x, t) dx dt. \]

By \( \mathbb{R}^{d} \) and setting \( x = ty \), it follows that

\[ \text{LH} := \sum_{2i \leq k} b_{\mu} \int_{\mathbb{R}^{d+1}} y^{2i} \sum_{m=0}^{\left\lfloor \frac{k}{4} \right\rfloor} \left( \frac{1}{2} \right)_{k} B_{i,m} \left\| q_{n-k+2m} \right\|^{2 (k-2m)+2\mu} \left( \frac{k}{2} - \mu - \frac{d-1}{2} \right)_{i} \left( -k + \frac{1}{2} \right)_{i} \int_{\mathbb{R}^{d}} y^{2k-2i} (1 - \left\| y \right\|^{2})^{\mu - \frac{1}{2}} dy \]

\[ = \sum_{2i \leq k} \sum_{m=0}^{\left\lfloor \frac{k}{4} \right\rfloor} B_{i,m} \left\| q_{n-k+2m} \right\|^{2 (k-2m)+2\mu} \left( \frac{1}{2} \right)_{k} \int_{\mathbb{R}^{d}} \left( \frac{1}{2} \right)_{k} (1 - \left\| y \right\|^{2})^{\mu - \frac{1}{2}} dy \]

For the last integral, let \( T^{d} = \{ y \in \mathbb{R}^{d} : y_{i} \geq 0, 1 \leq i \leq d, 1 - \left\| y \right\| \geq 0 \} \), where \( \left\| y \right\| = y_{1} + \ldots + y_{d} \), be the simplex in \( \mathbb{R}^{d} \). Using Lemma 4.4.1 in [7], we obtain that

\[ b_{\mu} \int_{\mathbb{R}^{d}} y^{2k-2i} (1 - \left\| y \right\|^{2})^{\mu - \frac{1}{2}} dy = b_{\mu} \int_{T^{d}} y^{k-i} (1 - \left\| y \right\|^{2})^{\mu - \frac{1}{2}} dy \]

\[ = \frac{\left( \frac{1}{2} \right)_{k-i}}{\left( \frac{1}{2} \right)_{k-i}} \frac{\left( \frac{1}{2} \right)_{k-i} (-\mu - \frac{d-1}{2} - k)_{i}}{\left( -k + \frac{1}{2} \right)_{i} (\mu + \frac{d+1}{2})_{i}}. \]

Putting together, this completes the proof. \( \square \)

Our next proposition shows that \( \mathbb{R}^{d} \) can also be reversed.

**Proposition 3.5.** Let \( b_{k,j}^{\mu} \) be the coefficients in (2.7). Then, for \( k \in \mathbb{N}_{0}^{d} \) and \( |k| \leq n \),

\[ S_{k,n}(x, t) = \sum_{2i \leq k} b_{k,j}^{\mu} V_{k-2i,n}(x, t). \]

This follows from a simple computation:

\[ \sum_{2i \leq k} b_{k,j}^{\mu} V_{k-2i,n}(x, t) = \sum_{2i \leq k} b_{k,j}^{\mu} \sum_{2i \leq k-2j} c_{k-2j,i}^{\mu} S_{k-2i,n}(x, t) \]

\[ = \sum_{2i \leq k} b_{k,j}^{\mu} \sum_{2i \leq k} c_{k-2j,i}^{\mu} S_{k-2i,n}(x, t) \]

\[ = \sum_{2i \leq k} S_{k-2i,n}(x, t) \sum_{j \leq i} b_{k,j}^{\mu} c_{k-2j,i}^{\mu}. \]
Hence, for $k \in \mathbb{N}_0^d$, $k = |k|$, and $0 \leq k \leq n$, the norm of the monic Jacobi polynomial is given by

$$
\left\| \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) \right\|_2^2 = (n - |k| + 2m)! (2\mu + d + \beta)_{n+|k|-2m}.
$$

Hence, for $k \in \mathbb{N}_0^d$, $k = |k|$, and $0 \leq k \leq n$, the norm of $\mathcal{V}^J_{k,n}$ given in Proposition 3.6 is specified to

$$
b_{\mu} \int_{\mathbb{Y}_{d+1}} \left| \mathcal{V}^J_{k,n}(x,t) \right|^2 W_{\beta,\mu}(x,t) dx dt
$$

$$
= \sum_{2j \leq k} x^{k-2j} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} B_{j,m} \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) t^{2j-2m},
$$

where $B_{j,m}$ is given by (3.6). As in the case of $d = 1$, the norm of the monic Jacobi polynomial is given by

$$
\left\| \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) \right\|_2^2 = (n - |k| + 2m)! (2\mu + d + \beta)_{n+|k|-2m}.
$$

Hence, for $k \in \mathbb{N}_0^d$, $k = |k|$, and $0 \leq k \leq n$, the norm of $\mathcal{V}^J_{k,n}$ given in Proposition 3.6 is specified to

$$
b_{\mu} \int_{\mathbb{Y}_{d+1}} \left| \mathcal{V}^J_{k,n}(x,t) \right|^2 W_{\beta,\gamma,\mu}(x,t) dx dt
$$

$$
= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\left(\frac{|k|}{2}\right)_{k-i}^2}{(n-|k|+2i)!} \frac{(n-k+2i)! (2\mu + \beta + d - 1)_{n-k+2i} (\gamma + 1)_{n-k+2i}}{(2\mu + \beta + \gamma + d + 1)_{2n} (n-k-2i + 2\mu + \beta + \gamma + d)_{n-k+2i}}.
$$

3.2.3. Monomial Jacobi polynomials. For the Jacobi weight function $w(t) = t^\beta (1-t)^\gamma$, we use monic Jacobi polynomials $\mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t)$, so that the monomial polynomial, denoted by $\mathcal{V}^J_{k,n}$, becomes by (3.5)

$$
\mathcal{V}^J_{k,n}(x,t) = \sum_{2j \leq k} x^{k-2j} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} B_{j,m} \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) t^{2j-2m},
$$

where $B_{j,m}$ is given by (3.6). As in the case of $d = 1$, the norm of the monic Jacobi polynomial is given by

$$
\left\| \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) \right\|_2^2 = \left\| \mathcal{P}^{(\alpha, \beta)}_{n-k}(1-2t) \right\|_2^2 = (n - |k| + 2m)! (2\mu + d + \beta)_{n+|k|-2m}.
$$

Hence, for $k \in \mathbb{N}_0^d$, $k = |k|$, and $0 \leq k \leq n$, the norm of $\mathcal{V}^J_{k,n}$ given in Proposition 3.6 is specified to

$$
b_{\beta,\gamma,\mu} \int_{\mathbb{Y}_{d+1}} \left| \mathcal{V}^J_{k,n}(x,t) \right|^2 W_{\beta,\gamma,\mu}(x,t) dx dt
$$

$$
= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\left(\frac{|k|}{2}\right)_{k-i}^2}{(n-|k|+2i)!} \frac{(n-k+2i)! (2\mu + \beta + d - 1)_{n-k+2i} (\gamma + 1)_{n-k+2i}}{(2\mu + \beta + \gamma + d + 1)_{2n} (n-k-2i + 2\mu + \beta + \gamma + d)_{n-k+2i}}.
$$
4. ORTHOGONAL POLYNOMIALS VIA RODRIGUES FORMULAS

The Laguerre and Jacobi polynomials can be defined by the Rodrigues formulas, which we use to define $U$-family of orthogonal polynomials on the cone.

4.1. Laguerre polynomials on the cone. Recall that on the infinite cone $\mathbb{V}^{d+1}$, the Laguerre weight function is defined by

$$W_{\beta,\mu}(x,t) = (t^2 - \|x\|^2)^{\mu - \frac{1}{2}} t^\beta e^{-t}, \quad \mu > -\frac{1}{2}, \quad \beta > -d.$$ 

We define a family of polynomials $U_{k,n}^I$ by the Rodrigues type formula.

**Definition 4.1.** Let $\mu > -\frac{1}{2}$ and $\beta > -d$. For $k \in \mathbb{N}^d_0$ and $|k| = k$, $0 \leq k \leq n$, define

$$U_{k,n}^I(x,t) = \frac{1}{W_{\beta,\mu}(x,t)} \frac{\partial^k}{\partial x^k} (t^2 - \|x\|^2)^{k+\mu - \frac{1}{2}} \partial_{\mu-k}^n t^{n+k+\beta+2\mu+d-1} e^{-t}.$$ 

We note that the order of the derivatives in the above definition cannot be exchanged since $(t^2 - \|x\|^2)^{k+\mu - \frac{1}{2}}$ depends on both $t$ and $x$.

**Theorem 4.2.** The family of polynomials $\{U_{k,n}^I : |k| \leq n\}$ is a basis of $V_n(\mathbb{V}^{d+1}, W_{\beta,\mu})$.

**Proof.** The definition of $U_{k,n}^I$ can be rewritten as

$$U_{k,n}^I(x,t) = \frac{1}{(t^2 - \|x\|^2)^{\mu - \frac{1}{2}}} \frac{\partial^k}{\partial x^k} (t^2 - \|x\|^2)^{k+\mu - \frac{1}{2}} \frac{1}{t^{2k+2\mu+d-1}} \partial_{\mu-k}^n t^{n+k+\beta+2\mu+d-1} e^{-t},$$

from which it follows immediately that $U_{k,n}^I$ is indeed a polynomial of degree $n$. We now prove that it is an orthogonal polynomial.

Let $P$ be a polynomial of total degree at most $n-1$ in the $(x,t)$ variables. By the definition of $U_{k,n}^I$,

$$I := \int_{\mathbb{V}^{d+1}} U_{k,n}^I(x,t) P(x,t) W_{\beta,\mu}(x,t) dx dt$$

$$= \int_{\mathbb{V}^{d+1}} P(x,t) \frac{\partial^k}{\partial x^k} (t^2 - \|x\|^2)^{k+\mu - \frac{1}{2}} dx \frac{1}{t^{2k+2\mu+d-1}} \partial_{\mu-k}^n t^{n+k+\beta+2\mu+d-1} dt,$$

which becomes, after integration by parts $k$ times with respect to $x$,

$$I = (-1)^k \int_{\mathbb{V}^{d+1}} \frac{1}{t^{2k+2\mu+d-1}} (t^2 - \|x\|^2)^{k+\mu - \frac{1}{2}} \frac{\partial_{\mu-k}^n}{\partial t^n} \left( e^{-t} t^{n+k+2\mu+\beta+d-1} \right) Q(x,t) dx dt,$$

where $Q = \frac{\partial^k}{\partial x^k} P$ is a polynomial of total degree at most $n-k-1$ in the $(x,t)$ variables. Under the substitution $x = ty$, it follows

$$I = (-1)^k \int_0^\infty \int_{\mathbb{S}^d} (1 - \|y\|^2)^{k+\mu - \frac{1}{2}} \frac{\partial_{\mu-k}^n}{\partial t^n} \left( e^{-ty} t^{n+k+2\mu+\beta+d-1} \right) Q(ty,t) dy dt.$$

Since $Q$ is a polynomial of total degree $n-k-1$ in the $(x,t)$ variables, $Q(ty,t)$ is a polynomial of degree at most $n-k-1$ in the $t$ variable. In particular, $\frac{\partial_{\mu-k}^n}{\partial t^n} Q = 0$. Hence, applying integration by parts $n-k$ times with respect to $t$, we conclude that $I = 0$. This completes the proof. \qed
Unlike orthogonal polynomials on the unit ball, the two bases of \( \{ U_{k,n}^L \} \) and \( \{ V_{k,n}^L \} \) are not mutually biorthogonal. They satisfy the following partial biorthogonality.

**Proposition 4.3.** The polynomials \( U_{k,n}^L(x,t) \) and \( V_{k,n}^L(x,t) \) are partial biorthogonal in the sense that, for \( k, j \in \mathbb{N}_0^n \) with \( k = |k| \leq n \) and \( j = |j| \leq n \), the relation

\[
b_{k,\mu}^L \int_{\mathbb{R}^{d+1}} U_{k,n}^L(x,t) V_{j,n}^L(x,t) W_{\beta,\mu}(x,t) dx dt = k!(n-k)!(-1)^n (2\mu + \beta + d)_{n+k} \frac{(\mu + \frac{k}{2})_k}{(\mu + \frac{d+k}{2})_k} \delta_{kj},
\]

holds if \( k \geq j \) or \( j > k \) and \( j - k \) has an odd component.

**Proof.** Since \( U_{k,n}^L \) is an orthogonal polynomial of degree \( n \) and \( V_{k,n}^L \) is monomial, we obtain immediately that

\[
\langle U_{k,n}^L, V_{j,n}^L \rangle = b_{k,\mu}^L \int_{\mathbb{R}^{d+1}} U_{k,n}^L(x,t) V_{j,n}^L(x,t) W_{\beta,\mu}(x,t) dx dt
\]

\[
= b_{k,\mu}^L \int_{\mathbb{R}^{d+1}} U_{k,n}^L(x,t) t^{n-|j|} x^j W_{\beta,\mu}(x,t) dx dt.
\]

Integrating by parts \( k \) times with respect to \( x \) as in the previous theorem, we obtain

\[
\langle U_{k,n}^L, V_{j,n}^L \rangle = (-1)^k b_{k,\mu}^L \int_{\mathbb{R}^{d+1}} \frac{t^{n-k}}{2^{k+2\mu+d-1}} (t^2 - \|x\|^2)^{k+\mu-\frac{d}{2}}
\]

\[
\times \frac{\partial^{n-k}}{\partial^k x^j} (e^{-t(n+k+2\mu+\beta+d-1)} \frac{\partial^k}{\partial^k x^j} dx dt,
\]

which is zero if \( k > j \). If, however, \( j > k \) and there is an \( i \) such that \( j_i - k_i \) is odd, then \( \frac{\partial^k}{\partial x^j} x^j = \frac{\partial^k}{\partial (j-k)^i} x^j-k \) has a component \( x_i^{j_i-k_i} \) of odd power, so that the last integral is zero if we make a change of variable \( x_i \mapsto -x_i \). Finally, for \( j = k \), we use substitution \( x = ty \) and integrate by parts with respect to \( t \), as in the previous proof, to obtain

\[
\langle U_{k,n}^L, V_{j,n}^L \rangle = (-1)^k k! \frac{\partial^k}{\partial^k (j-k)} \int_{\mathbb{R}^d} (1 - \|y\|^2)^{k+\mu-\frac{d}{2}} dy
\]

\[
\times \frac{1}{\Gamma(2\mu + \beta + d)} \int_0^\infty e^{-t} \frac{\partial^{n-k}}{\partial^k t^{n+k+2\mu+\beta+d-1}} e^{-t(n+k+2\mu+\beta+d-1)} dt
\]

\[
= (-1)^n k! \frac{\partial^k}{\partial^k (j-k)} \frac{(n-k)!}{\Gamma(2\mu + \beta + d)} \int_0^\infty e^{-t} e^{-t(n+k+2\mu+\beta+d-1)} dt
\]

\[
= (-1)^n k! (n-k)! \frac{(\mu + \frac{k}{2})_k}{(\mu + \frac{d+k}{2})_k},
\]

which completes the proof. \( \square \)

We can find a basis in terms of \( U_{k,n}^L \) that is biorthogonal with respect to the basis of \( V_{k,n}^L \). This is illustrated in the following for the case \( d = 1 \).

**Proposition 4.4.** For \( d = 1 \), define the polynomials \( \{ Y_{n-k,n}^L : 0 \leq k \leq n \} \) recursively as follows: for \( k = 0,1 \), \( Y_{n-k,n}^L = U_{n-k,n}^L \), and for \( k = 2,3,\ldots,n \),

\[
Y_{n-k,n}^L(x,t) = U_{n-k,n}^L - \sum_{1 \leq i \leq \lfloor k/2 \rfloor} c_{i,k} Y_{n-k+2i,n}^L,
\]

where \( c_{i,k} \) are constants to be determined.
where the coefficients are
\[ c_{i,k} = \frac{k!(\mu + i + n - k + 1)}{2^i(k - 2i)i!(\mu + n - k + 1/2)_{2i}(2\mu + \beta + 2n - k + 1)_{2i}}. \]

Then \( \{\mathcal{Y}_{n-k,n}^k : 0 \leq k \leq n\} \) is a basis of \( V_n(\mathcal{V}^2, W_{\beta,\mu}) \) and it is mutually biorthogonal to the basis \( \{\mathcal{V}_{k,n}^k : 0 \leq k \leq n\} \).

**Proof.** We define \( \mathcal{Y}_{n-k,n}^k \) as in (4.1) and determine the coefficients \( c_{i,k} \) by requiring
\[ I_{n-k,n-j} = \langle \mathcal{Y}_{n-k,n}^k, \mathcal{V}_{n-j,n}^l \rangle = h_{n-k}\delta_{k,j}, \]
where, by (4.1) and the required biorthogonality,
\[ h_{n-k} = \langle \mathcal{U}_{n-k,n}^k, \mathcal{V}_{n-k,n}^k \rangle. \]
The case \( k = 0 \) and 1 holds trivially by Proposition 4.3. Assume that \( \mathcal{Y}_{n-k+2i,n}^k \) has been determined for \( i = 1, \ldots, \lfloor k/2 \rfloor \) for \( 2 \leq k \leq n \). We now define \( \mathcal{Y}_{n-k,n}^k \) as given in (4.1). Then
\[ I_{n-k,n-j} = \langle \mathcal{U}_{n-k,n}^k, \mathcal{V}_{n-j,n}^l \rangle - h_{n-j} \sum_{1 \leq l \leq \lfloor k/2 \rfloor} c_{i,k}h_{n-k+2l,n-j}, \]
which is zero if \( j > k \) or \( k \) and \( j \) have different parity when \( j < k \). Thus, we only need to consider \( j = k - 2i \), for which \( I_{n-k,n-j} = 0 \) becomes
\[ 0 = \langle \mathcal{U}_{n-k,n}^k, \mathcal{V}_{n-k+2i,n}^k \rangle - c_{i,k}h_{n-k+2i}, \quad 1 \leq i \leq \lfloor k/2 \rfloor, \]
which gives immediately
\[ c_{i,k} = \langle \mathcal{U}_{n-k,n}^k, \mathcal{V}_{n-k+2i,n}^k \rangle / h_{n-k+2i}. \]
The value of \( h_{n-k+2i} \) is already given in Proposition 4.3. Furthermore, following the proof there, we also obtain
\[
\langle \mathcal{U}_{n-k,n}^k, \mathcal{V}_{n+2i,n}^k \rangle = b_{\beta,\mu}^k(-1)^k(2i)! \int_{-1}^1 y^{2i}(1 - y^2)^{k+\mu - \frac{1}{2}} dy \\
\times \int_0^\infty t^{n-k} \frac{\partial^{n-k}}{\partial t^{n-k}} (t^{n+k+\beta+2\mu} e^{-t}) dt \\
= \frac{(-1)^n(k + 2i)!(n-k)!(2\mu + \beta + 1)n+k(\mu + \frac{1}{2})_{k}k(\frac{1}{2})_{k}}{(2i)!\mu(k+i)}. 
\]
where the last step follows from integrating by parts \( n-k \) times in the \( t \) variable and evaluating the two integrals of one variable. Using this identity with \( k \) replaced by \( n-k \), we obtain the formula of the coefficient \( c_{i,k} \) after rewriting the Pochhammer symbols.

**4.2. Jacobi polynomials on the cone.** Recall that on the finite cone \( \mathcal{V}^{d+1} \), the Jacobi weight function is defined by
\[ W_{\beta,\gamma,d}(x,t) = (t^2 - ||x||^2)^{\mu - \frac{d}{2}} t^\beta (1-t)^\gamma, \quad \mu > -\frac{1}{2}, \quad \beta > -d, \quad \gamma > -1. \]

We define a family of polynomials \( \mathcal{U}_{n,k}^k \) by the Rodrigues type formula.
Definition 4.5. Let $\mu > -\frac{1}{2}$, $\beta > -d$ and $\gamma > -1$. For $k \in \mathbb{N}_0^d$ and $|k| = k$, $0 \leq k \leq n$, define
\[
U_{k,n}^j(x,t) = \frac{1}{W_{\beta,\gamma,\mu}(x,t)} \frac{1}{t^{2k+2\mu+d-1}} \frac{\partial^k}{\partial x^k} \left( t^2 - \|x\|^2 \right)^{k+\mu - \frac{d}{2}} \times \frac{1}{t^{n-k}} \left[ t^n + k + 2\mu + d - 1 (1-t)^{\gamma+n-k} \right].
\]

Theorem 4.6. The family of polynomials $\{U_{k,n}^j : |k| \leq n\}$ is a basis of $V_n(W_{\beta,\gamma,\mu})$.

Proof. The definition of $U_{k,n}^j$ can be rewritten as
\[
U_{k,n}^j(x,t) = \frac{1}{(t^2 - \|x\|^2)^{\mu - \frac{d}{2}}} \frac{\partial^k}{\partial x^k} \left( t^2 - \|x\|^2 \right)^{k+\mu - \frac{d}{2}} \times \frac{1}{t^{2k+\beta+2\mu+d-1}} \left[ t^{n-k} + 2k + 2\mu + d - 1 (1-t)^{\gamma+n-k} \right],
\]
from which it follows immediately that $U_{k,n}^j$ is indeed a polynomial of degree $n$. We can prove that it is an orthogonal polynomial similar to the case of the Laguerre polynomials. Let $P$ be a polynomial of degree $n-1$ in $(x,t)$ variable. Integrating by parts $k$ times with respect to $x$ gives
\[
\int_{\mathbb{R}^{d+1}} U_{k,n}^j(x,t) P(x,t) W_{\beta,\gamma,\mu}(x,t) dx dt = (-1)^k \int_{\mathbb{R}^{d+1}} \frac{1}{t^{2k+\beta+2\mu+d-1}} \left( t^2 - \|x\|^2 \right)^{k+\mu - \frac{d}{2}} \times \frac{\partial^{n-k}}{\partial x^{n-k}} \left[ (1-t)^{\gamma+n-k} t^n + 2k + 2\mu + d - 1 \right] Q(x,t) dx dt,
\]
which becomes zero upon integrating by parts $n-k$ times with respect to $t$. □

Theorem 4.7. The polynomials $U_{k,n}^j(x,t)$ and $V_{j,n}^j(x,t)$ are partially biorthogonal in the sense that, for $k, j \in \mathbb{N}_0^d$ and $k = |k| \leq n$ and $j = |j| \leq n$,
\[
b_{\beta,\gamma,\mu}^j \int_{\mathbb{R}^{d+1}} U_{k,n}^j(x,t) V_{j,n}^j(x,t) W_{\beta,\gamma,\mu}(x,t) dx dt = \delta_{k,j} \frac{k!(n-k)!(\mu + \frac{1}{2})_k (2\mu + \beta + d)_n (\gamma + 1)_n t^{-n-k}}{(\mu + \frac{d+1}{2})_k (2\mu + \beta + \gamma + d + 1)_n},
\]
holds if $k \geq j$ or $j > k$ and $j - k$ has an odd component.

Proof. Since $U_{k,n}^j$ is orthogonal to polynomials of lower degree and $V_{j,n}^j$ is monomial, we obtain
\[
\langle U_{k,n}^j, V_{j,n}^j \rangle = b_{\beta,\gamma,\mu}^j \int_{\mathbb{R}^{d+1}} U_{k,n}^j(x,t) V_{j,n}^j(x,t) W_{\beta,\gamma,\mu}(x,t) dx dt = \delta_{k,j} \int_{\mathbb{R}^{d+1}} U_{k,n}^j(x,t) V_{j,n}^j(x,t) W_{\beta,\gamma,\mu}(x,t) dx dt.
\]
Integrating by parts $k$ times with respect to $x$, we obtain
\[
\langle U_{k,n}^j, V_{j,n}^j \rangle = (-1)^k b_{\beta,\gamma,\mu}^j \int_{\mathbb{R}^{d+1}} \frac{1}{t^{2k+2\mu+d-1}} \left( t^2 - \|x\|^2 \right)^{k+\mu - \frac{d}{2}} \times \frac{\partial^{n-k}}{\partial x^{n-k}} \left[ (1-t)^{\gamma+n-k} t^n + 2k + 2\mu + d - 1 \right] \partial^k x^j dx dt,
\]
from which the partially biorthogonal follows as in the Laguerre case. For \( j = k \), we use substitution \( x = ty \) and integrate by parts with respect to \( t \) to obtain

\[
\langle U_{k,n}^j, V_{k,n}^j \rangle = (-1)^k k! b_{\mu + \frac{d}{2}} \int_{B^d} (1 - \|y\|^2)^{k + \mu - \frac{d}{2}} dy 
\times c_{2\mu + \beta + d - 1, \gamma} \int_0^1 t^{n-k} \frac{\partial^{n-k}}{\partial t^{n-k}} [(1-t)^{\gamma+n-k} t^{n+k+2\mu+\beta+d-1}] dt 
= (-1)^k k! \frac{b_{\mu + \frac{d}{2}}}{b_{\mu + k}} (n-k) c_{2\mu + \beta + d - 1, \gamma} \int_0^1 (1-t)^{\gamma+n-k} t^{n+k+2\mu+\beta+d-1} dt 
= (-1)^k k! (n-k)! \frac{(\mu + \frac{1}{2})k}{(\mu + \frac{d+1}{2})k} \frac{c_{2\mu + \beta + d - 1, \gamma}}{c_{n+k+2\mu + d - 1, n-k+\gamma}},
\]

where \( c_{\alpha, \gamma} \) is defined in (2.13), from which (4.2) follows. \( \square \)

We can also define a basis in terms of \( U_{k,n}^j \) so that it is biorthogonal with respect to the basis of \( V_{k,n}^j \). We state the result for the case \( d = 1 \) as in the Laguerre case.

**Proposition 4.8.** For \( d = 1 \), define the polynomials \( \{Y_{n-k,n}^j : 0 \leq k \leq n\} \) recursively as follows: for \( k = 0, 1 \), \( Y_{n-k,n}^j = U_{n-k,n}^j \), and for \( k = 2, 3, \ldots, n \),

\[
Y_{n-k,n}^j(x,t) = U_{n-k,n}^j - \sum_{1 \leq i \leq \lfloor k/2 \rfloor} d_{i,k} Y_{n-k+2i,n}^j
\]

recursively, where the coefficients are given by

\[
d_{i,k} = \frac{k!(\mu + i + n - k + 1)!(\gamma + k - 2i + 1)_{2i}}{2^{2i}(k-2i)!i!(\mu + n - k + 1/2)_{2i}(2\mu + \beta + 2n - k + 1)}.
\]

Then \( \{Y_{n-k,n}^j : 0 \leq k \leq n\} \) is a basis of \( V_{n} \) and it is mutually biorthogonal to the basis \( \{U_{k,n}^j : 0 \leq k \leq n\} \).

The proof is parallel to the Jacobi case, we shall omit the details.

5. Generating function for \( U \)-family and \( V \)-family

Like the case of the unit ball, the \( U \)-family of orthogonal polynomials on the cone satisfies a generating function. For the \( V \)-family, however, we do not have a generating function unless we use the relation in Proposition 4.8 to relate \( V_{k,n} \) to \( S_{k,n} \) first, since then we can obtain a generating function for \( V_{k,n} \) in terms of a generating function for \( S_{k,n} \). We shall give the generating function for \( V_{k,n} \) in this section as well. In both cases, the generating functions are given for the Laguerre polynomials and the Jacobi polynomials on the cone.

5.1. Laguerre polynomials on the cone. The polynomials \( U_{k,n}^L \) are indexed by \((k,n)\) with \( k \in \mathbb{N}_0^d \), \( |k| \leq n \) and \( n \in \mathbb{N}_0 \). Its generating function is given below.

**Proposition 5.1.** For \( b \in \mathbb{R}^d \), \( r > 0 \) and \( r < 1 \), the generating function for \( U_{k,n}^L \) is given by

\[
\sum_{n=0}^{\infty} \sum_{|k| \leq n} \frac{(-1)^k (2\mu)|k|}{2^{|k|}(\mu + \frac{1}{2})|k|! |(n - |k|)|!} U_{k,n}^L(x,t)b^k r^n = (1 - r)^{2\mu} e^{-\frac{t}{1-r}} (1 - r)^{\beta + d} \left( \frac{((1-r)^2 - r\langle b, x \rangle)^2 + r^2 \|b\|^2(t^2 - \|x\|^2)}{2} \right)^{\beta}.
\]
Proof. Changing variable \(x = ty\), we can write
\[
\frac{1}{(t^2 - ||x||^2)^{\frac{\mu}{2}}} \frac{\partial^k}{\partial x^k} (t^2 - ||x||^2)^{\frac{k + \frac{\mu}{2}}{2}} = \frac{t^k}{(1 - ||y||^2)^{\frac{k + \frac{\mu}{2}}{2}}} \frac{\partial^k}{\partial y^k} (1 - ||y||^2)^{\frac{k + \frac{\mu}{2}}{2}}.
\]
Consequently, using the Rodrigues formulas for \(U_k\) in (5.2) and the Laguerre polynomial, we see that the polynomial \(U_{k,n}^\alpha\) can be written as
\[
U_{k,n}^\alpha(x,t) = \left(\frac{2^k (\mu + \frac{1}{2})^k |k|! (n - |k|)!}{(-1)^k (2\mu)^{|k|}}\right) t^k U_k \left(\frac{x}{t}\right) L_{2k+\alpha}(t),
\]
where \(\alpha = 2\mu + \beta + d - 1\). Hence, using the generating formula for the Laguerre polynomials first, we see that the left-hand side of (5.1) is equal to
\[
\text{LHS} = \sum_{n=0}^\infty \sum_{k=0}^n t^k U_k \left(\frac{x}{t}\right) L_{2k+\alpha}(t) b^k r^n
\]
\[
= \left(\sum_{k=0}^\infty t^k U_k \left(\frac{x}{t}\right) b^k\right) \left(\sum_{n=0}^\infty L_{2n}(t) r^n\right)
\]
\[
= \left(\sum_{k=0}^\infty t^k U_k \left(\frac{x}{t}\right) b^k\right) \frac{1}{(1 - r)^{2\alpha+1}} e^{\frac{tr}{1-r}}.
\]
Now, applying the generating formula for \(U_k\) in (2.3) with \(a = tr|b|/(1-r)^2\), we obtain
\[
\text{LHS} = \frac{1}{(1 - \frac{r}{1-r})^2 (b, x)^2 + r^2 \frac{|b|^2}{(1-r)^4} (t^2 - ||x||^2)^\mu} \frac{1}{(1 - r)^{2\alpha+1}} e^{\frac{tr}{1-r}},
\]
which becomes (5.1) after rearranging the terms. \(\square\)

We now give a generating function for \(S_{k,n}^\alpha\) given by
\[
S_{k,n}^\alpha(x,t) := \tilde{L}_{2k+\alpha,\beta+\gamma}(t) t^k V_k \left(\frac{x}{t}\right), \quad |k| = k, \quad k \in \mathbb{N}_0, \quad 0 \leq k \leq n.
\]
Using the generating function (2.5) in place of (2.3), the generating function can be derived exactly as in the proof for (5.1). The result is the following.

**Proposition 5.2.** For \(b \in \mathbb{R}^d\), \(r > 0\) and \(r < 1\), the generating function for \(S_{k,n}^\alpha\) is given by
\[
\sum_{n=0}^\infty \sum_{|k| \leq n} \frac{(-1)^{n-|k|} 2^{|k|} (\mu + \frac{d-1}{2}) |k|! (n - |k|)!}{(n - |k|)! |k|!} S_{k,n}^\alpha(x,t) b^k r^n
\]
\[
= \frac{1}{(1 - r)^{2\alpha+d} e^{\frac{tr}{1-r}}}.
\]

### 5.2. Jacobi polynomials on the cone.
Similarly to the Laguerre case, the generating function for the \(U\)-family of the Jacobi polynomials on the cone is given as follows.

**Proposition 5.3.** For \(b \in \mathbb{R}^d\), \(r > 0\) and \(r < 1\), the generating function for \(U_{k,n}^\alpha\) is given by
\[
\sum_{n=0}^\infty \sum_{|k| \leq n} \frac{(-1)^k (2\mu)^{|k|}}{2^{|k|} (\mu + \frac{d}{2}) |k|! (n - |k|)!} U_{k,n}^\alpha(x,t) b^k r^n
\]
\[ R = \left[ (1 - r)^2 + 4rt \right]^\frac{1}{2}. \]

Proof. Changing variable \( x = ty \), we can rewrite the polynomial \( U^J_{k,n} \) as

\[ U^J_{k,n}(x, t) = \frac{2^{|k|}(\mu + \frac{1}{2})|k|!(n - |k|)!}{(-1)^{|k|}(2\mu)|k|} t^k U_k \left( \frac{x}{t} \right) P_{n-k}^{(2k+\alpha, \gamma)}(1 - 2t), \]

where \( \alpha = 2\mu + \beta + d - 1 \). We now use the generating formula for the Jacobi polynomials, given by

\[ \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(1 - 2t)r^n = 2^{\alpha + \beta} R^{-1} (1 - r + R)^{-\alpha} (1 + r + R)^{-\beta}, \]

with \( R \) defined as in the statement, which shows that the left-hand side of (5.2) is equal to

\[
\text{LHS} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{|k|=k} t^k U_k \left( \frac{x}{t} \right) P_{n-k}^{(2k+\alpha, \gamma)}(1 - 2t)b^k r^n
\]

\[ = \sum_{k=0}^{\infty} \sum_{|k|=k} t^k U_k \left( \frac{x}{t} \right) b^k \sum_{n=k}^{\infty} P_{n-k}^{(2k+\alpha, \gamma)}(1 - 2t)r^n
\]

\[ = \sum_{k=0}^{\infty} \sum_{|k|=k} t^k U_k \left( \frac{x}{t} \right) b^k r^k 2^{2k+\alpha+\gamma} R^{-1} (1 - r + R)^{-2k-\alpha} (1 + r + R)^{-\gamma}. \]

Now, we apply the generating formula for \( U_k \) in (2.3) with \( a = 4tr|b|/(1 - r + R)^2 \) and simplify the result to complete the proof. \( \square \)

Similarly, as in the Laguerre case, we can give a generating function for \( S^J_{k,n} \),

\[
S^J_{k,n}(x, t) := \sum_{n=0}^{\infty} \sum_{|k|=k} \frac{(-1)^{n-k}2^{|k|}(\mu + \frac{d-1}{2})|k|(n + k + 2\mu + \beta + \gamma + 1)_{n-k}}{(n - |k|)!} S^J_{k,n}(x, t)b^k r^n,
\]

for \( |k| = k, k \in \mathbb{N}_0^d \) and \( 0 \leq k \leq n \). Using the generating function (2.5) in place of (2.3), the proof follows exactly as that of (5.1). The result is the following.

**Proposition 5.4.** For \( b \in \mathbb{B}^d, r > 0 \) and \( r < 1 \), the generating function for \( S^J_{k,n} \) is given by

\[
\sum \frac{(-1)^{n-k}2^{|k|}(\mu + \frac{d-1}{2})|k|(n + k + 2\mu + \beta + \gamma + 1)_{n-k}}{(n - |k|)!} \sum \frac{(-1)^{n-k}2^{|k|}(\mu + \frac{d-1}{2})|k|(n + k + 2\mu + \beta + \gamma + 1)_{n-k}}{(n - |k|)!} S^J_{k,n}(x, t)b^k r^n,
\]

where \( R \) is the same as in (5.2).
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