Conformal invariance of the writhe of a knot

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Abstract
We give a new proof of the conformal invariance of the writhe of a knot from a conformal geometric viewpoint.

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1 Introduction
Suppose $K$ is a framed knot, i.e. there is a unit normal vector field $e_2$ along $K$. A 2-component link $K \cup K + \varepsilon e_2$ ($|\varepsilon| \ll 1$) can be considered a closed ribbon. Let $Lk$ and $Tw$ be the linking number and the total twist of $K \cup K + \varepsilon e_2$, and $Wr$ the writhe of $K$. Then we have

$$Lk = Wr + Tw.$$ (1)

When the knot $K$ has nowhere vanishing curvature and $e_2$ is the principal normal vector, $Lk$ is called the self-linking number of the knot and denoted by $Sl$. The equation (1) was proved in this case in [Ca1, Ca2, Ca3, Po], and in [Wh] in general. It plays an important role in the application of the knot theory to molecular biology ([Fu1, Fu2, Wr-Ba]).

When the knot $K$ is given by $K = f(S^1)$ the writhe is given by the Gauss integral:

$$Wr(f) = \frac{1}{4\pi} \iint_{S^1 \times S^1} \frac{\det(f'(s), f'(t), f(s) - f(t))}{|f(s) - f(t)|^3} \, ds \, dt.$$ (2)

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(the reader is referred to [AKT] for the details concerning writhe). Banchoff and White showed that the absolute value of the writhe is conformally invariant ([BW]). To be precise, they showed

**Theorem 1.1** ([BW]) Suppose $I$ is an inversion in a sphere. Then we have

$$\text{Wr}(I(K)) = -\text{Wr}(K).$$

It is a corollary of

**Theorem 1.2** ([BW]) Suppose $I$ is an inversion in a sphere. Then we have

$$\text{Tw}(I(K)) \equiv -\text{Tw}(K) \pmod{\mathbb{Z}}.$$  \hspace{1cm} (4)

In this paper, using techniques in conformal geometry, we give new proofs of Theorem 1.1 and a special case of Theorem 1.2 when $K$ has nowhere vanishing curvature and $e_2$ is the unit principal normal vector field.

### 2 Notations

Throughout the paper we use the following notations.

We assume that a knot $K = f(S^1)$ is oriented. We denote the positive tangent vector $\dot{f}$ by $v = v(x)$.

We denote the circle through $x, y,$ and $z$ by $\Gamma(x, y, z)$. When one of $x, y,$ and $z$, say $z$ is $\infty$, $\Gamma(x, y, \infty)$ means the line through $x$ and $y$. When $x$ is a point on a knot $K$, $\Gamma(x, x, y)$ denotes the circle (or line) which is tangent to $K$ at $x$ that passes through $y$. We assume that it is oriented by $v$ at $x$. Especially, when $y = x$ $\Gamma(x, x, x)$ denotes the oriented osculating circle. (We consider lines as circles through $\infty$.)

The union of osculating circles $\bigcup_{x \in K} \Gamma(x, x, x)$ is called the curvature tube ([BW]).

Suppose $x$ is a point on a knot $K$. We denote a sphere through the osculating circle $\Gamma(x, x, x)$ and $y$ ($y \neq x$) by $\Sigma(x, x, x, y)$. It is uniquely determined generically, i.e., unless $y \in \Gamma(x, x, x)$. When $y = \infty$ $\Sigma(x, x, x, \infty)$ means the plane through $\Gamma(x, x, x)$.

### 3 Proof of Theorem 1.2 in special case

In this section we assume that a knot $K = f(S^1)$ has nowhere vanishing curvature and the unit normal vector field $e_2$ along $K$ is given by the principal normal vectors. In this case the total twist is equal to $\frac{1}{2\pi}$ times the total torsion which we denote by $T\omega$:

$$T\omega = \frac{1}{2\pi} \int_K \tau dx,$$

where $\tau$ denotes the torsion of the knot. We will show in this section

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Proposition 3.1 If both $K$ and $I(K)$ have nowhere vanishing curvatures then
\[ T\omega(I(K)) \equiv -T\omega(K) \pmod{\mathbb{Z}}, \] (5)
where $I$ is an inversion in a sphere.

We remark that the above proposition can be proved by Theorem 6.3 of [CSW] which gives
\[ T\omega(K) = \frac{1}{2\pi} \int_K \tau dx \equiv \frac{1}{2\pi} \int_K Td\rho \pmod{\mathbb{Z}}, \]
where $T$ is the conformal torsion and $\rho$ is the conformal arc-length.

Lemma 3.2 Suppose a point $P$ does not belong to the osculating circle $\Gamma(x_0, x_0, x_0)$ of a knot $K$ at $x_0$. (We allow $P = \infty$.) Then, infinitesimally speaking, the sphere $\Sigma(x, x, x, P)$ rotates around the circle $\Gamma(x_0, x_0, P)$ at $x = x_0$ as $x$ travels in $K$. In other words,
\[ \lim_{x_1 \to x_0} (\Sigma(x_0, x_0, x_0, P) \cap \Sigma(x_1, x_1, x_1, P)) \supset \Gamma(x_0, x_0, P). \] (6)

Proof: Let $\Sigma(x, y, z, w)$ denote a sphere through $x, y, z,$ and $w$. Then
\[
\lim_{x_1 \to x_0} (\Sigma(x_0, x_0, x_0, P) \cap \Sigma(x_1, x_1, x_1, P)) \\
= \lim_{x_1 \to x_0} \left( \lim_{y_1 \to x_1} (\Sigma(y_0, x_0, x_1, P) \cap \Sigma(x_0, x_1, y_1, P)) \right) \\
\supset \lim_{x_1 \to x_0} \Gamma(x_0, x_1, P) = \Gamma(x_0, x_0, P).
\]

There is an alternative computational proof using the Lorentzian exterior product introduced in [LO].

□

We show that the total torsion of the image of $K$ by an inversion in a sphere with center $P$ is equal to the total angle variation of the rotation of the sphere $\Sigma(x, x, x, P)$ as $x$ goes around in $K$. In order to take into account the sign of the torsion, we have to consider the orientations.

Fix $P \notin \Gamma(x, x, x)$, where we allow $P = \infty$. The osculating circle $\Gamma(x, x, x)$ divide the sphere $\Sigma(x, x, x, P)$ into two domains. Let $D_1$ be one of the two that does not contain $P$ (gray disc of Figure 1 left). Assume $\Sigma(x, x, x, P)$ is given the orientation such that the restriction to $D_1$ induces the same orientation to the boundary $\partial D_1$ as that of $\Gamma(x, x, x)$ which is fixed in the previous section. Let $n(x)$ and $n_P(x)$ be the positive unit normal vectors to $\Sigma(x, x, x, P)$ at $x$ and $P$ respectively (Figure 1 left). Let $\Pi(x)$ and $\Pi_P(x)$ be the normal planes to $\Gamma(x, x, P)$ at $x$ and $P$ respectively. We assume that $\Pi(x)$ (or $\Pi_P(x)$) is oriented so that the algebraic intersection number of $\Gamma(x, x, P)$ and $\Pi(x)$ (or $\Pi_P(x)$) at $x$ (or respectively, at $P$) is equal to $+1$.

Since $\Pi(x) \perp \Gamma(x, x, P)$, Lemma 3.2 implies
**Corollary 3.3** Infinitesimally, the normal vector $n(x)$ to the sphere $\Sigma(x, x, x, P)$ rotates in the plane $\Pi(x)$ to $K$:

$$\frac{d}{dx} n(x) \in \Pi(x).$$  \hfill (7)

Let $I$ be an inversion in a sphere with center $P$. We denote $I(x)$ and $I(K)$ by $\tilde{x}$ and $\tilde{K}$. Then $I$ maps the osculating circle $\Gamma(x, x, x)$ to the osculating circle $\Gamma(\tilde{x}, \tilde{x}, \tilde{x})$ of $\tilde{K}$ at $\tilde{x}$, and the sphere $\Sigma(x, x, x, P)$ to the plane $\Sigma(\tilde{x}, \tilde{x}, \tilde{x}, \infty)$. Let $\tilde{n}(\tilde{x})$ be the positive unit normal vector to $\Sigma(\tilde{x}, \tilde{x}, \tilde{x}, \infty)$. Then

$$\tilde{n}(\tilde{x}) = -n_P(x)$$ \hfill (8)

(Figure 1).

![Diagram of geometric transformation](image)

**Figure 1:**

The above convention of orientation implies that $\tilde{n}(\tilde{x})$ is equal to the unit binormal vector of $\tilde{K}$. Lemma 3.2 implies that $\tilde{n}(\tilde{x})$ rotates in $\Pi(\tilde{x})$ at $\tilde{x}$. Our convention of the orientation of $\Pi(\tilde{x})$ implies

**Lemma 3.4** The torsion of $\tilde{K}$ is equal to the angle velocity of $\tilde{n}$ with respect to the arc-length $\tilde{s}$:

$$\tau(\tilde{K})(\tilde{x}) = \varepsilon(\tilde{x}) \left| \frac{d\tilde{n}}{d\tilde{s}}(\tilde{x}) \right| \quad (\varepsilon(\tilde{x}) \in \{+1, -1\}),$$ \hfill (9)

where $\varepsilon(\tilde{x})$ is the signature of the rotation of $\tilde{n}$ at $\tilde{x}$ with respect to the orientation of $\Pi(\tilde{x})$.

In other words, we have

$$\tau(\tilde{K})(\tilde{x}) = \tilde{v} \cdot \left( \tilde{n} \times \frac{d\tilde{n}}{d\tilde{s}} \right),$$

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where \( \tilde{v} \) denotes the positive unit tangent vector to \( \tilde{K} \).

Since the positive unit tangent vector to \( \Gamma(x, x, P) \) at \( P \) is equal to \(-\tilde{v}(\tilde{x})\), the orientations of \( \Pi_P(x) \) is opposite to that of \( \Pi(\tilde{x}) \). Therefore, (8) and Lemma 3.4 imply

\[
\tau(\tilde{K})(\tilde{x}) = -\varepsilon_P \left| \frac{ds}{ds} \left| \frac{dn_P}{ds} \right| \right. (\varepsilon_P \in \{+1, -1\}),
\]

where \( \varepsilon_P \) is the signature of the rotation of \( n_P \) with respect to the orientation of \( \Pi_P(x) \). Then Lemma 3.2 implies

\[
\left| \frac{ds}{ds} \right| = \left| \frac{dn}{ds} \right| \quad \text{(Figure 3)},
\]

therefore we have

**Lemma 3.5** *The torsion of \( \tilde{K} \) at \( \tilde{x} \) is equal to the negative of the angle velocity of \( n \) in \( \Pi(x) \) with respect to the arc-length \( s \) of \( K \) up to the multiplication by the Jacobian:*

\[
\tau(\tilde{K})(\tilde{x}) = -\varepsilon(x) \left| \frac{ds}{ds} \right| \left| \frac{dn}{ds} \right| (\varepsilon(x) \in \{+1, -1\}),
\]

where \( \varepsilon(x) \) is the signature of the rotation of \( n \) at \( x \) with respect to the orientation of \( \Pi(x) \).

In other words, we have

\[
\tau(\tilde{K})(\tilde{x}) = -\left| \frac{ds}{ds} \right| v \cdot \left( n \times \frac{dn}{ds} \right).
\]

Thus we are led to
Figure 3: $\angle n(x) \cdot n(x_1) = \angle n_P(x) \cdot n_P(x_1) = \angle \Sigma(x, x, x, P) \cdot \Sigma(x_1, x_1, x_1, P)$

**Proposition 3.6** Let $K$ be a knot with nowhere vanishing curvature, $I$ an inversion in a sphere with center $P$, and $\widetilde{K} = I(K)$. Assume that $\widetilde{K}$ has nowhere vanishing curvature, which happens if and only if $P$ is not contained in the curvature tube of $K$. Then the total torsion of $\widetilde{K}$ is the negative of the total angle variation of the positive unit normal vector $n(x)$ to the sphere $\Sigma(x, x, x, P)$ as $x$ goes around in $K$:

$$\int_I \tau(K)(\tilde{x}) d\tilde{x} = -\int_K \varepsilon |dn| = -\int_K v \cdot (n \times dn).$$

(11)

**Corollary 3.7** Let $K$ be a knot with nowhere vanishing curvature, and $I_j$ ($j = 1, 2$) an inversion in a sphere with center $P_j$ which is not contained in the curvature tube of $K$. Then the total torsion of $I_1(K)$ and $I_2(K)$ coincide modulo $2\pi \mathbb{Z}$:

$$\int_{I_1(K)} \tau d\tilde{x}_1 \equiv \int_{I_2(K)} \tau d\tilde{x}_2 \pmod{2\pi \mathbb{Z}}.$$

In other words,

$$T\omega(I_1(K)) - T\omega(I_2(K)) \in \mathbb{Z}.$$

**Proof:** Let $n_j$ ($j = 1, 2$) be the positive unit normal vector to $\Sigma(x, x, x, P_j)$ at $x$, and $\theta_{21}$ the angle from $n_1$ to $n_2$ in the oriented plane $\Pi(x)$. Then Corollary 3.4 and the above Proposition imply

$$\int_{I_1(K)} \tau d\tilde{x}_1 - \int_{I_2(K)} \tau d\tilde{x}_2 = \int_K \varepsilon \theta_{21},$$

which is equal to $2\pi k$ ($k \in \mathbb{Z}$) since $K$ is closed.

**Proof of Proposition 3.1** Put $P_2 = \infty$ in the above Corollary.
4 Proof of Theorem 1.1

We first prove the following fact in our context, which implies Theorem 3.1 in the case when the knot has nowhere vanishing curvature.

**Proposition 4.1** (Ch1, Pd) Suppose $K$ has nowhere vanishing curvature. Then

$$4\pi \text{Wr}(K) = -2\int_K \tau + 4\pi k$$

for some $k \in \mathbb{Z}$.

In other words, $\text{Wr} + T\omega$ is an integer.

**Proof:** Assume $K = f(S^1)$ is parametrized by the arc-length. Suppose $S^1 = [0, L]/\sim$, where $L$ is the length of $K$. Let $\varphi$ be a map from $S^1 \times S^1 \setminus \Delta$ to $S^2$ given by

$$\varphi(s, t) = \frac{f(s) - f(t)}{|f(s) - f(t)|} (s \neq t).$$

Then the integrand of the writhe is equal to the pull-back of the standard area element of $S^2$ by $\varphi$:

$$\det(f'(s), f'(t), f(s) - f(t)) |f(s) - f(t)|^3 = \varphi \cdot (\varphi_s \times \varphi_t),$$

where $\varphi_s = \frac{\partial \varphi}{\partial s}$ and $\varphi_t = \frac{\partial \varphi}{\partial t}$. Therefore, $4\pi$ times the writhe is equal to the signed area of the image of

$$D = \{(s, t) | 0 \leq s \leq L, s < t < s + L\}$$

by $\varphi$. Let $S$ be the closure of $\varphi(D)$. Then $S$ is an oriented “surface” (a continuous image of $S^1 \times [0, L]$) possibly with self-overlaps in $S^2$ whose “boundary” (the image of $S^1 \times \{0, L\}$) is given by

$$\partial S = C_+ \cup (-C_-),$$

where $C_+$ and $C_-$ denote the positive and negative tangential indicatrices:

$$C_+ = \varphi(\Delta_+) = \{v(s) = \hat{f}(s) | 0 \leq s \leq L\},$$

$$C_- = \varphi(\Delta_-) = \{-v(s) = -\hat{f}(s) | 0 \leq s \leq L\}.$$ 

We assume that the indicatrices are oriented so that $s$ increases in the positive direction.

Let $\Gamma(s)$ denote a great circle in $S^2$ which is tangent to $C_+$ at $v(s)$ (and to $C_-$ at $-v(s)$). We remark that $C_+$ has nowhere vanishing tangent vector since the knot has nowhere vanishing curvature. Note that $C_+ \cup C_-$ is the envelope of the family $\{\Gamma(s)\}_{0 \leq s \leq L}$. We assume that $\Gamma(s)$ has an orientation compatible with that of $C_+$ at $v(s)$. Let $\Gamma_+(s)$ be a semi-circle of $\Gamma(s)$ from $v(s) \in C_+$ to
\[ -v(s) \in C_- \] in the positive direction of \( \Gamma(s) \), and \( S' \) a region in \( S^2 \) swept by \( \Gamma_+(s) \) as \( s \) varies in \([0, L] \). Then \( S' \) is given by
\[
S' = \{ w(s, t) = (\cos t)v(s) + (\sin t)\frac{\dot{v}}{|\dot{v}|}(s) \mid 0 \leq s \leq L, 0 \leq t \leq \pi \}.
\]

It is an oriented “surface” (a continuous image of \( S^1 \times [0, \pi] \)) possibly with self-overlaps in \( S^2 \) whose “boundary” (the image of \( S^1 \times \{0, \pi\} \)) is \( C_+ \cup (-C_-) \).

The signed area of \( S' \) is given by
\[
\text{Area}(S') = \int_0^L \int_0^\pi w \cdot (w_s \times w_t)\, ds\, dt.
\]

Since \( \frac{\dot{v}}{|\dot{v}|} \) is equal to the principal normal vector \( e_2 \) of the knot \( K \), we have
\[
w \cdot (w_s \times w_t) = \det ((\cos t)e_1 + (\sin t)e_2, (\cos t)\kappa e_2 + (\sin t)e_1, -(\sin t)e_1 + (\cos t)e_2) = -\tau \sin t,
\]

which implies
\[
\text{Area}(S') = -2 \int_0^L \tau \, ds = -4\pi T \omega(K).
\]

Since both \( S \) and \( S' \) have the boundary \( C_+ \cup (-C_-) \), \( S \cup (-S') \) is a cycle of \( S^2 \), i.e. an oriented “surface” (a continuous image of a torus) possibly with self-overlaps without a boundary. Therefore, the signed area of \( S \cup (-S') \) is equal to \( 4\pi k \) for some integer \( k \). It follows that
\[
\text{Area}(S \cup (-S')) = \text{Area}(S) - \text{Area}(S') = 4\pi \text{Wr} + 4\pi T \omega = 4\pi k \ (k \in \mathbb{Z}),
\]

which completes the proof. \( \square \)

**Corollary 4.2** If both \( K \) and \( I(K) \) have nowhere vanishing curvatures then
\[
\text{Wr}(I(K)) = -\text{Wr}(K).
\]

**Proof:** Let \( I_j \ (j = 0, 1) \) be an inversion in a sphere with center \( P_j \) which is not contained in the curvature tube of \( K \). Then Proposition \ref{prop} implies that
\[
\text{Wr}(I_0(K)) + T\omega(I_0(K)) \in \mathbb{Z},
\]
\[
\text{Wr}(I_1(K)) + T\omega(I_1(K)) \in \mathbb{Z}.
\]

Then Corollary \ref{cor} implies
\[
\text{Wr}(I_0(K)) - \text{Wr}(I_1(K)) \in \mathbb{Z}.
\]

Join \( P_0 \) and \( P_1 \) by a smooth path \( P_t \). Let \( I_t \) be an inversion in a sphere with center \( P_t \). Then \( \text{Wr}(I_t(K)) \) is a continuous function of \( t \) (\ref{prop}), and hence
\[
\text{Wr}(I_0(K)) = \text{Wr}(I_1(K)).
\]
When \( P_1 \) goes to \( \infty \) and the radius of the sphere of the inversion also goes to \( +\infty \), \( I_1(K) \) approaches the mirror image of \( K \) and hence \( \text{Wr}(I_1(K)) \) approaches \( -\text{Wr}(K) \), which completes the proof. \( \square \)

**Proof of Theorem 1.1** Suppose the curvature of \( K \) vanishes somewhere. We have only to show that \( K \) can be approximated, with respect to the \( C^2 \)-topology, by a knot with non-vanishing curvature. This can be done as follows. The curvature tube of \( K \) is non-compact as it contains a line. But we can still find a point \( P \) with a very big distance from \( K \) which is not contained in the curvature tube. Let \( I \) be an inversion in a sphere with center \( P \) and radius approximately equal to the distance between \( P \) and \( K \). We can get a desired knot by taking the mirror image of \( I(K) \) thus constructed. \( \square \)

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