An eXtended HDG method for Darcy-Stokes-Brinkman interface problems

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Abstract

This paper proposes an interface/boundary-unfitted eXtended hybridizable discontinuous Galerkin (X-HDG) method for Darcy-Stokes-Brinkman interface problems in two and three dimensions. The method uses piecewise linear polynomials for the velocity approximation and piecewise constants for both the velocity gradient and pressure approximations in the interior of elements inside the subdomains separated by the interface, uses piecewise constants for the numerical traces of velocity on the inter-element boundaries inside the subdomains, and uses piecewise constants or linear polynomials for the numerical traces of velocity on the interface. Optimal error estimates are derived for the interface-unfitted X-HDG scheme. Numerical experiments are provided to verify the theoretical results and the robustness of the proposed method.

Key Words: eXtended HDG method, Darcy-Stokes-Brinkman interface problem, interface/boundary-unfitted mesh, error estimate.

1 Introduction

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain divided into two subdomains, \( \Omega_i (i = 1, 2) \), by a piecewise smooth interface \( \Gamma \) (cf. Figure 1). We consider the following Darcy-Stokes-Brinkman interface problem: find the velocity \( u \) and the pressure \( p \) such that

\[
\begin{aligned}
  \nabla \cdot (\nu \nabla u) + \nabla p + \alpha u &= f \quad &\text{in} \quad \Omega_1 \cup \Omega_2, \\
  \nabla \cdot u &= 0 \quad &\text{in} \quad \Omega_1 \cup \Omega_2, \\
  u &= g_D \quad &\text{on} \quad \partial \Omega, \\
  J u K &= 0, \quad (\nu \nabla u - p I) n &= g_N \quad &\text{on} \quad \Gamma.
\end{aligned}
\]

(1.1)

Here the viscosity coefficient \( \nu \) and the zeroth-order term coefficient \( \alpha \) are piecewise constants with

\[
\nu|_{\Omega_i} = \nu_i > 0, \quad \alpha|_{\Omega_i} = \alpha_i \geq 0, \quad i = 1, 2.
\]

(1.2)

The jump of a function \( w \) across the interface \( \Gamma \) is defined by \( [w] := (w|_{\Omega_1})|_{\Gamma} - (w|_{\Omega_2})|_{\Gamma} \), \( I \) the \( d \times d \) identity matrix, and \( n \) denotes the unit normal vector along \( \Gamma \), pointing to \( \Omega_2 \). \( f \) denotes the body force, \( g_N \) the interface traction, and \( g_D \) the source term satisfying

\[
\int_{\partial \Omega} g_D \cdot n = 0,
\]

(1.3)

where \( n \) is the outward unit normal vector along \( \partial \Omega \). The Darcy-Stokes-Brinkman model (1.1) is usually used to describe porous media flow coupled with viscous fluid flow in a single form of equation (cf. [21, 23, 35, 43, 49]).

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For an elliptic interface problem, the low regularity of the solution due to the coefficient discontinuity may result in reduced accuracy of finite element discretization [4,50]. One strategy for this situation is to use interface(or body)-fitted meshes (cf. Figure 2) so as to dominate the approximation error caused by the non-smoothness of solution [7,9,11,14,32,44]. However, the generation of interface-fitted meshes is usually expensive, especially when the interface is of complicated geometry or moving with time or iteration.

Another strategy avoiding the loss of numerical accuracy is to use certain modification of the finite element approximation around the interface. The resultant finite element methods do not need interface-fitted meshes (cf. Figure 2). One representative of such interface-unfitted methods is the eXtended/Generalized Finite Element Method (XFEM or GFEM), where additional basis functions characterizing the singularity of the solution around the interface are enriched into the corresponding approximation space. We refer to [45] for an overview work and [5,6,8,10,28,48,51] for some developments of XFEM/GFEM. In particular, we refer to [12,15,29,30,34,47] for several XFEMs using additional cut basis functions for Stokes or Darcy interface problems. It should be pointed out that the immersed finite element method (IFEM) is another type of interface-unfitted method, where special finite element basis functions are constructed to satisfy the interface jump conditions (cf. [1,2,31,39–41,52] and the references therein).

The hybridizable discontinuous Galerkin (HDG) framework [16] provides a unifying strategy for hybridization of finite element methods for second order elliptic problems. By the local elimination of the unknowns defined in the interior of elements, the HDG method leads to a system where the unknowns are only the globally coupled degrees of freedom describing the introduced Lagrange multiplier. We refer to [3,17–20,22,24,25,33,36–38,42,46] for some developments and applications of the HDG method. We
also mention that arbitrary order interface-unfitted eXtended HDG methods with optimal convergence were analyzed in [26,27] for elliptic and elasticity interface problems, respectively.

In this paper we aim to propose a low order eXtended HDG (X-HDG) method for the Darcy-Stokes-Brinkman interface problem (1.1). The main features of the method are as follows:

- The method is a low order scheme, which uses piecewise linear polynomials for the velocity approximation and piecewise constants for both the velocity gradient and pressure approximations in the interior of elements inside the subdomains separated by the interface, and uses piecewise constants for the numerical traces of velocity on the inter-element boundaries inside the subdomains.
- To deal with the interface conditions, the interface is approximated by a fold line/plane, on which the numerical traces of velocity adopt piecewise constants or piecewise linear polynomials.
- The method is parametric-friendly in the sense that optimal error estimates are obtained without requiring “sufficiently large” stabilization parameters in the scheme.
- The method uses interface-unfitted polygonal/polyhedral meshes, and applies to curved domains with boundary-unfitted meshes.

The rest of the paper is organized as follows. Section 2 introduces the X-HDG scheme for the interface problem with a polygonal/polyhedral domain. Section 3 is devoted to the error estimation. Section 4 applies the X-HDG method to a curved domain problem. Numerical examples are provided in Section 5 to verify the theoretical results. Finally, Section 6 gives some concluding remarks.

2 X-HDG scheme for interface problem

2.1 Notation and XFE spaces

For any bounded domain $D \subset \mathbb{R}^s$ ($s = d, d - 1$) and nonnegative integer $m$, let $H^m(D)$ and $H^m_0(D)$ be the usual $m$-th order Sobolev spaces on $D$, with norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. In particular, $L^2(D) := H^0_0(D)$ is the space of square integrable functions, with inner product $(\cdot, \cdot)_D$. When $D \subset \mathbb{R}^{d-1}$, we use $(\cdot, \cdot)_D$ to replace $(\cdot, \cdot)_{D}$. For $m = 1, 2$, we set

$$H^m(\Omega_1 \cup \Omega_2) := \{ v \in L^2(\Omega), v|_{\Omega_1} \in H^m(\Omega_1), \text{ and } v|_{\Omega_2} \in H^m(\Omega_2) \},$$

$$\|\cdot\|_m := \|\cdot\|_{m,\Omega_1\cup\Omega_2} = \sum_{i=1}^{2} \|\cdot\|_{m,\Omega_i}, \quad |\cdot|_m := |\cdot|_{m,\Omega_1\cup\Omega_2} = \sum_{i=1}^{2} |\cdot|_{m,\Omega_i}. $$

For any integer $k \geq 0$, $P_k(D)$ denotes the set of all polynomials on $D$ with degree at most $k$.

Assume that $\Omega$ is a polygonal/polyhedral domain. Let $T_h = \cup\{K\}$, consisting of arbitrary open polygons/polyhedrons, be a shape-regular partition of the domain $\Omega$ in the sense that the following two assumptions hold (cf. [13]):

(M1). There exists a positive constant $\theta_\ast$ such that the following holds: for each element $K \in T_h$, there exists a point $M_K \in K$ such that $K$ is star-shaped with respect to every point in the circle (or sphere) of center $M_K$ and radius $\theta_\ast h_K$.

(M2). There exists a positive constant $l_\ast$ such that for every element $K \in T_h$, the distance between any two vertexes is no less than $l_\ast h_K$.

We define the set of all elements intersected by the interface $\Gamma$ as

$$T^\Gamma_h := \{K \in T_h : K \cap \Gamma \neq \emptyset \}. $$

For any $K \in T^\Gamma_h$, called an interface element, let $\Gamma_K := K \cap \Gamma$ be the part of $\Gamma$ in $K$, and $\Gamma_{K,h}$ be the straight line/plane segment connecting the intersection between $\Gamma_K$ and $\partial K$ (Figure 4).

To ensure that $\Gamma$ is reasonably resolved by $T_h$, we make the following standard assumptions on $T_h$ and $\Gamma$( cf. Figure 3 for two cases violating the assumptions):

(A1). For $K \in T^\Gamma_h$ and any edge/face $F \subset \partial K$ which intersects $\Gamma_h$, $F_\Gamma := \Gamma \cap F$ is simply connected with either $F_\Gamma = F$ or $\text{meas}(F_\Gamma) = 0$. 

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(A2). For $K \in T^T_h$, $\Gamma_K$ is sufficiently smooth such that for any two different points $x, y \in \Gamma_K$, the unit normal vectors $n(x)$ and $n(y)$, pointing to $\Omega_2$, at $x$ and $y$ satisfy
\[
|n(x) - n(y)| \leq \gamma h_K,
\]
with $\gamma \geq 0$(cf. [14,50]). Note that $\gamma = 0$ when $\Gamma_K = \Gamma_{K,h}$, i.e. $\Gamma_K$ is a straight line/plane segment.

Figure 3: Two cases violating assumptions: violating (A1)(left) and violating (A2)(right).

Figure 4: An example (a triangulation with circular interface): $T^\Gamma_h$ (green part), $\varepsilon^\Gamma_h$ (collection of blue segments inside the elements of $T^\Gamma_h$), and $\varepsilon^*_h$ (collection of all triangle edges not in $\varepsilon^\Gamma_h$).

Let $\varepsilon_h$ be the set of all edges(faces) of all elements in $T_h$. Denote by $\varepsilon^\Gamma_h$ the partition of the fold line/plane approximation of $\Gamma$ with respect to $T_h$, i.e.
\[
\varepsilon^\Gamma_h := \{F : F = \Gamma_{K,h}, \text{ or } F = \Gamma \cap \partial K \text{ if } \Gamma \cap \partial K \text{ is an edge/face of } K, \ K \in T_h\}.
\]
We set $\varepsilon_h^\Gamma := \varepsilon_h \setminus \varepsilon^\Gamma_h$. For any $K \in T_h$ and $F \in \varepsilon^\Gamma_h$ or $\varepsilon_h$, let $h_K$ and $h_F$ be respectively the diameters of $K$ and $F$, and let $n_K$ be the unit outward normal vector along $\partial K$. Denote by $h := \max_{K \in T_h} h_K$ the mesh size of $T_h$, and by $\nabla_h$ and $\nabla^\Gamma_h$ the piecewise-defined gradient and divergence operators with respect to $T_h$, respectively.

Since the rest of the paper deals with the discrete problem, in what follows and without ambiguity, let $\Omega_1$ and $\Omega_2$ denote the two sides of $\Gamma_h$ rather than of $\Gamma$, and set $K_i = K \cap \Omega_i$ for $i = 1, 2$.

Throughout the paper, we use $a \lesssim b$ ($a \geq b$) to denote $a \leq Cb$ $(a \geq Cb)$, where $C$ is a generic positive constant independent of mesh parameters $h, h_K, h_e$, the coefficients $\nu_i, \alpha_i$ $(i = 1, 2)$ and the location of the interface relative to the mesh.
2.2 X-HDG scheme

The X-HDG method is based on the following first-order formulations of Darcy-Stokes-Brinkman interface problem (1.1):

\[
L = \nu \nabla u \quad \text{in} \quad \Omega_1 \cup \Omega_2, \tag{2.2a}
\]

\[
-\nabla \cdot L + \nabla p + \alpha u = f \quad \text{in} \quad \Omega_1 \cup \Omega_2, \tag{2.2b}
\]

\[
\nabla \cdot u = 0 \quad \text{in} \quad \Omega_1 \cup \Omega_2, \tag{2.2c}
\]

\[
u^{-1} L_h, w \rangle_{\Gamma_h} - \langle \hat{u}_h, w n \rangle_{\partial \Omega \setminus \Gamma} - \langle \hat{u}_h, w n \rangle_{o, \Gamma_h} = 0, \tag{2.3a}
\]

\[
(\alpha u_h, v)_{\Gamma_h} + \langle \tau(Q_h^b u_h - \tilde{u}_h), v \rangle_{\partial \Omega \setminus \Gamma} + \langle \tau(Q_m^b u_h - \tilde{u}_h), v \rangle_{o, \Gamma_h} = (f, v), \tag{2.3b}
\]

\[
\langle \hat{u}_h \cdot n, q \rangle_{\partial \Omega \setminus \Gamma} + \langle \hat{u}_h \cdot n, q \rangle_{o, \Gamma_h} = 0, \tag{2.3c}
\]

\[
\langle L_h n, \mu \rangle_{\partial \Omega \setminus \Gamma} - \langle p_h n, \mu \rangle_{\partial \Omega \setminus \Gamma} - \langle \tau(Q_h^b u_h - \tilde{u}_h), \mu \rangle_{\partial \Omega \setminus \Gamma} = 0, \tag{2.3d}
\]

\[
\langle L_h n, \tilde{\mu} \rangle_{o, \Gamma_h} - \langle p_h n, \tilde{\mu} \rangle_{o, \Gamma_h} - \langle \tau(Q_m^b u_h - \tilde{u}_h), \tilde{\mu} \rangle_{o, \Gamma_h} = (g_N, \tilde{\mu})_{o, \Gamma_h}, \tag{2.3e}
\]

Let \( \chi_i \) be the characteristic function on \( \Omega_i \) for \( i = 1, 2 \). For any integer \( r \geq 0 \), \( F \in \mathcal{E}_h \cup \mathcal{E}_h^T \) and \( K \in \mathcal{T}_h \), let \( Q_h^b : L^2(D) \to P_r(D) \) and \( Q_r : L^2(D) \to P_r(D) \) be the standard \( L^2 \) orthogonal projection operators for \( D = F \cap \Omega_i \) and \( D = K \cap \Omega_i \), respectively. Vector or tensor analogues of \( Q_h^b \) and \( Q_r \) are denoted by \( Q_h^b \) and \( Q_r \), respectively. Set

\[
\oplus \chi_i P_r(K) := \chi_1 P_r(K) + \chi_2 P_r(K), \quad r = 0, 1.
\]

We introduce the following X-HDG finite element spaces:

\[
W_h = \{ w \in L^2(\Omega)^{d \times d} : \forall K \in \mathcal{T}_h, w|_K \in P_0(K)^{d \times d} \text{ if } K \cap \Gamma = \emptyset; w|_K \in (\oplus \chi_i P_0(K))^{d \times d} \text{ if } K \cap \Gamma \neq \emptyset \},
\]

\[
V_h = \{ v \in L^2(\Omega)^d : \forall K \in \mathcal{T}_h, v|_K \in P_0(K)^d \text{ if } K \cap \Gamma = \emptyset; v|_K \in (\oplus \chi_i P_0(K))^d \text{ if } K \cap \Gamma \neq \emptyset \},
\]

\[
Q_h = \{ q \in L^2(\Omega) : \forall K \in \mathcal{T}_h, q|_K \in P_0(K) \text{ if } K \cap \Gamma = \emptyset; q|_K \in (\oplus \chi_i P_0(K)) \text{ if } K \cap \Gamma \neq \emptyset \},
\]

\[
M_h = \{ m \in L^2(\varepsilon_h)^d : \forall F \in \varepsilon_h^*, m|_F \in P_0(F)^d \text{ if } F \cap \Gamma = \emptyset; m|_F \in (\oplus \chi_i P_0(F))^d \text{ if } F \cap \Gamma \neq \emptyset \},
\]

\[
M_h(g_D) = \{ m \in M_h : m|_F = Q_h^b g_D, \forall F \in \varepsilon_h \text{ with } \alpha \leq 0 \}.
\]

To describe the X-HDG scheme, we also define

\[
(\cdot, \cdot)_{\Gamma_h} := \sum_{K \in \mathcal{T}_h} (\cdot, \cdot)_K, \quad (\cdot, \cdot)_{\partial \Omega \setminus \Gamma} := \sum_{K \in \mathcal{T}_h} (\cdot, \cdot)_{\partial K \setminus \varepsilon_h}^K,
\]

and, for scalars \( q \), vector \( u, v \) and tensor \( w \) with \( q_i = q|_{\Omega_i \setminus \Gamma}, u_i = u|_{\Omega_i \setminus \Gamma}, v_i = v|_{\Omega_i \setminus \Gamma} \) and \( w_i = w|_{\Omega_i \setminus \Gamma} \),

\[
\langle q, v \cdot n \rangle_{o, \varepsilon_h^T} := \sum_{F \in \varepsilon_h^T} \langle q_1, v_1 \cdot n_1 \rangle_{\Omega_1 \setminus \Gamma} + \langle q_2, v_2 \cdot n_2 \rangle_{\Omega_2 \setminus \Gamma},
\]

\[
\langle u, v \rangle_{o, \varepsilon_h^T} := \sum_{F \in \varepsilon_h^T} \langle u_1, v_1 \rangle_{\Omega_1 \setminus \Gamma} + \langle u_2, v_2 \rangle_{\Omega_2 \setminus \Gamma},
\]

\[
\langle w n, v \rangle_{o, \varepsilon_h^T} := \sum_{F \in \varepsilon_h^T} \langle w_1 n_1, v_1 \rangle_{\Omega_1 \setminus \Gamma} + \langle w_2 n_2, v_2 \rangle_{\Omega_2 \setminus \Gamma},
\]

where \( n_i \) denotes the unit normal vector along \( \Gamma_h \) pointing from \( \Omega_i \) to \( \Omega_j \) with \( i, j = 1, 2 \) and \( i \neq j \).
for all \((w, v, q, μ, ϕ) \in W_h \times V_h \times Q_h \times M_h(0) \times \tilde{M}_h\). Here the stabilization function \(τ\) is defined as following: for any \(K \in T_h\) and \(i = 1, 2\),

\[
τ|_{F∩Ω_h} = ν_i h^{-1}_K, \quad \text{for} \ F ⊂ ∂K \setminus ε^Γ_h \text{ or } F \in ε^Γ_h. \tag{2.4}
\]

When \(F \in ε^Γ_h\) is a line segment/straight plane, we take \(g^Γ_N|_F := g^Γ_N\), and when \(F = Γ_{K,h} \neq Γ_K\) for some \(K \in T^Γ_h\), we set \(g^Γ_N|_F\) to be some linear interpolation of \(g^Γ_N\) using data of \(g^Γ_N\) at two (2D case) / three (3D case) intersection points of \(Γ_K\) and \(Γ_{K,h}\).

**Remark 2.1.** From the first-order system (2.2), there should be some terms like \((u_h, ∇_h \cdot w)_{τ_h}, −(∇_h \cdot L_h, v)_{τ_h}, (∇_h p_h, v)_{τ_h}\) and \(−(u_h, ∇ q)_{τ_h}\) in the scheme (2.3). In fact, they are all vanish since \(W_h\) and \(Q_h\) are piecewise constant tensor/scalar spaces.

**Remark 2.2.** We note that in the implementation, we can locally eliminate the interior of elements, and the reduced system only involves the unknowns of \(p_h, \hat{u}_h\) and \(u_h\).

**Theorem 2.1.** The X-HDG scheme (2.3) (with \(m = 0\) or \(1\)) admits a unique solution.

**Proof.** Since the (2.3) is a linear square system, it suffices to show that if all of the given data vanish, i.e. \(f = g_D = g^Γ_N = 0\), then we get the zero solution. By taking the \((w, v, q, μ, ϕ) = (L_h, u_h, p_h, \hat{u}_h, u_h)\) in (2.3) and adding these equations together, we have

\[
(ν^{-1}L_h, L_h)_{τ_h} + (α u_h, u_h)_{τ_h} + \langle τ(Q^b_m u_h - \hat{u}_h), (u_h - \hat{u}_h)\rangle_{∂τ_h \setminus ε^Γ_h} + \langle τ(Q^b_m u_h - \hat{u}_h), (u_h - \hat{u}_h)\rangle_{*,ε^Γ_h} = 0,
\]

which indicates \(L_h = 0\),

\[
Q^b_m u_h - \hat{u}_h = 0, \quad \text{on} \ ∂K \setminus ε^Γ_h, ∀K \in T_h, \tag{2.5}
\]

\[
\{Q^b_m u_h - \hat{u}_h\} = 0, \quad \text{on} \ ε^Γ_h. \tag{2.6}
\]

where \(\{\cdot\}\) is defined by \(\{w\} = \frac{1}{2}(w_1 + w_2)\) with \(w_i = w_{i|Ω_i \cup Γ}\) for \(i = 1, 2\). These relations, together with the equation (2.3a), the definition of projection and integration by parts, yield

\[
0 = (ν^{-1}L_h, w)_{τ_h} + (∇_h u_h, w)_{τ_h} - ⟨Q^b_m u_h - \hat{u}_h, wn⟩_{∂τ_h \setminus ε^Γ_h} - ⟨Q^b_m u_h - \hat{u}_h, wn⟩_{*,ε^Γ_h} = (∇_h u_h, w)_{τ_h}.
\]

Taking the \(w_h = ∇_h u_h\) in this relation gives \(∇_h u_h = 0\). Then \(u_h\) is piecewise constant, which, together with (2.5), (2.6) and the fact that \(\hat{u}_h = 0\) on \(∂Ω\), implies

\[
u_h = \hat{u}_h = \tilde{u}_h = 0.
\]

The thing left is to show \(p_h = 0\). In view of (2.3d) and (2.3e), we have

\[
[p_h] = 0, \quad \text{on} \ ∂K \text{ and } ε^Γ_h, ∀K \in T_h.
\]

Thus, \(p_h\) is a constant in \(Ω\), and the fact \(p_h \in L^2_Ω(Ω)\) means \(p_h = 0\). This completes the proof.

### 3 A priori error estimation: a case of fold line/plane interface

This section is devoted to an error analysis of the X-HDG scheme (2.3) with a fold line/plane interface \(Γ\). We note that in this case \(Γ_K = Γ_{K,h}\) is a line segment/quadrilateral for any \(K \in T^Γ_h\), and \(g^Γ_N = g^Γ_N\) in the equation (2.3e).

#### 3.1 Optimal error estimation for velocity gradient and pressure

Firstly we introduce the following standard estimates for the \(L^2\) orthogonal projection operators \(Q_r\) and \(Q^Γ_r\) (cf. [13, 26]).
Lemma 3.1. Let $s$ be an integer with $1 \leq s \leq r+1$. For any $K \in \mathcal{T}_h$ and $v \in H^s ((K \cap \Omega_1) \cup (K \cap \Omega_2))$, we have
\[
\begin{align*}
\|v - Q_r v\|_{0,K} + h \|v - Q_r v\|_{1,K} & \lesssim h^{s-1} \|v\|_{s,K}, \\
\|v - Q_r v\|_{0,\partial K} + \|v - Q_r v\|_{0,\Gamma_K} & \lesssim h^{s-1/2} \|v\|_{s,K}, \\
\|v - Q_r^b v\|_{0,\partial K} + \|v - Q_r^b v\|_{0,\Gamma_K} & \lesssim h^{s-1/2} \|v\|_{s,K},
\end{align*}
\]
where the notations $\|\cdot\|_{s,K}$ and $\|\cdot\|_{0,\partial K}$ are understood respectively as $\|\cdot\|_{s,K} = \sum_{i=1}^{2} \|\cdot\|_{s,\partial K \cap \Omega_i}$ and $\|\cdot\|_{s,\partial K} = \sum_{i=1}^{2} \|\cdot\|_{s,\partial K \cap \Omega_i}$ when $K \in \mathcal{T}_h^i$.

For simplicity of presentation, denote
\[
e_h^L := L_h - Q_0 L, \quad e_h^u := u_h - Q_1 u, \quad e_h^p := p_h - Q_0 p, \quad e_h^\alpha := \alpha_h - Q_0^h u, \quad e_h^i := \alpha_h - Q_m^h u, \quad (3.1)
\]
where, for $i = 1, 2$ and $m = 0, 1$,
\[
\begin{align*}
(Q_0 L)|_{K \cap \Omega_i} & := Q_0 (L|_{K \cap \Omega_i}), \quad (Q_1 u)|_{K \cap \Omega_i} := Q_1 (u|_{K \cap \Omega_i}), \quad (Q_0 p)|_{K \cap \Omega_i} := Q_0 (p|_{K \cap \Omega_i}), \quad \forall K \in \mathcal{T}_h, \\
(Q_0^h u)|_{F \cap \Omega_i} & := Q_0^h (u|_{F \cap \Omega_i}), \quad \forall F \in \mathcal{E}_h, \\
(Q_m^h u)|_{F} & := Q_m^h (u|_F), \quad \forall F \in \mathcal{E}_h^i.
\end{align*}
\]

Then we have the following lemma for error equations.

Lemma 3.2. For all $(w, v, q, \mu, \bar{\mu}) \in W_h \times V_h \times Q_h \times M_h(0) \times \bar{M}_h$, it holds
\[
\begin{align*}
& (v - e_h^L \cdot w)_{\mathcal{E}_h} - \langle e_h^u \cdot w, n \rangle_{\partial \mathcal{E}_h} - \langle e_h^\alpha \cdot w, n \rangle_{s,\mathcal{E}_h} = 0, \quad (3.2a) \\
& (\alpha e_h^u \cdot v)_{\mathcal{E}_h} + \langle \tau(Q_0 e_h^u - e_h^\alpha), v \rangle_{\partial \mathcal{E}_h} + \langle \tau(Q_m e_h^u - e_h^\alpha), v \rangle_{s,\mathcal{E}_h} = \sum_{i=1}^{2} L_i(v), \quad (3.2b) \\
& \langle e_h^\alpha \cdot n, \mu \rangle_{\partial \mathcal{E}_h} - \langle e_h^\alpha \cdot n, \mu \rangle_{s,\mathcal{E}_h} = 0, \quad (3.2c) \\
& \langle e_h^\alpha \cdot \bar{\mu}, \bar{\mu} \rangle_{s,\mathcal{E}_h} - \langle e_h^\alpha \cdot \bar{\mu}, \bar{\mu} \rangle_{s,\mathcal{E}_h} = -L_2(\bar{\mu}), \quad (3.2d)
\end{align*}
\]
where for any $\psi \in H^1(\Omega_1 \cup \Omega_2) \cup W_h \cup V_h \cup M_h \cup \bar{M}_h$,
\[
\begin{align*}
L_1(\psi) & := \langle (Q_0 L - L)n, \psi \rangle_{\partial \mathcal{E}_h} + \langle \tau(Q_0^h \cdot u - Q_1 \cdot u), \psi \rangle_{\partial \mathcal{E}_h} + \langle (Q_0 p - p)n, \psi \rangle_{\partial \mathcal{E}_h}, \\
L_2(\psi) & := \langle (Q_0 L - L)n, \psi \rangle_{s,\mathcal{E}_h} + \langle \tau(Q_m^h \cdot u - Q_1 \cdot u), \psi \rangle_{s,\mathcal{E}_h} + \langle (Q_0 p - p)n, \psi \rangle_{s,\mathcal{E}_h}.
\end{align*}
\]

Proof. Let $(L, u, p)$ be the solution of (2.2). From the definitions of the projection operators we obtain
\[
\begin{align*}
& (v - e_h^L \cdot w)_{\mathcal{E}_h} - \langle Q_0 u, w \rangle_{\partial \mathcal{E}_h} - \langle Q_m^h u, w \rangle_{s,\mathcal{E}_h} = 0, \\
& \langle (Q_0 L - L)n, \psi \rangle_{\partial \mathcal{E}_h} + \langle (Q_0 L - L)n, v \rangle_{s,\mathcal{E}_h} + (\alpha Q_1 u, v)_{\mathcal{E}_h} \\
& - \langle (Q_0 p - p)n, \psi \rangle_{s,\mathcal{E}_h} - \langle (Q_0 p - p)n, v \rangle_{s,\mathcal{E}_h} = (f, v), \\
& \langle Q_0^h u \cdot n, q \rangle_{\partial \mathcal{E}_h} + \langle Q_m^h u \cdot n, q \rangle_{s,\mathcal{E}_h} = 0,
\end{align*}
\]
for any $(w, v, q) \in W_h \times V_h \times Q_h$. Then, subtracting (2.3a), (2.3b) and (2.3c) respectively from the above three equations yields (3.2a), (3.2b) and (3.2c). Finally, (3.2d), (3.2e) follows from (2.3d), (2.3e) and the relations
\[
\langle (L - pI)n, \mu \rangle_{s,\mathcal{E}_h} = \langle g, \mu \rangle_{s,\mathcal{E}_h}
\]
for $\mu \in M_h(0)$, $\bar{\mu} \in \bar{M}_h$. 

Define a seminorm $\| \cdot \|: w, v, \mu, \tilde{\mu} \in L^2(\Omega)^d \times L^2(\Omega)^d \times L^2(\varepsilon_h^*) \times L^2(\varepsilon_h^*)$ by
\[
\|(w, v, \mu, \tilde{\mu})\|^2 := \|\nu^{-\frac{1}{2}}w\|_{0,\mathcal{H}^2}^2 + \|\alpha^{-\frac{1}{2}}v\|_{0,\mathcal{H}^2}^2 + \|\nabla (Q_0^b v - \mu)\|_{\partial T_h \setminus \mathcal{H}_h}^2 + \|\nabla (Q_m^b v - \tilde{\mu})\|_{\mathcal{H}_h}^2,
\]
(3.3) where
\[
\|\cdot\|_{0,\mathcal{H}^2}^2 := \sum_{K \in \mathcal{T}_h} \|\cdot\|_{0,K}^2, \quad \|\cdot\|_{\partial T_h \setminus \mathcal{H}_h}^2 := \sum_{K \in \mathcal{T}_h} \langle\cdot,\cdot\rangle_{\partial K \setminus \mathcal{H}_h}, \quad \|\cdot\|_{\mathcal{H}_h}^2 := \langle\cdot,\cdot\rangle_{\mathcal{H}_h}.
\]

**Lemma 3.3.** Let $(L, u, p) \in H^1(\Omega_1 \cup \Omega_2)^d \times H^2(\Omega_1 \cup \Omega_2)^d \times H^1(\Omega_1 \cup \Omega_2)$ and $(L_h, u_h, p_h, \tilde{u}_h) \in W_h \times V_h \times Q_h \times M_h (g) \times M_h$ be the solutions of the problem (2.2) and the X-HDG scheme (2.3), respectively. Then it holds
\[
\|\nu^{\frac{1}{2}}\nabla e_h^n\|_{0,\mathcal{T}_h} \lesssim \|(e_h^L, e_h^m_n, e_h^u_n, e_h^\mu_n)\| \lesssim h\|\nu^{\frac{1}{2}}u\|_{2,\Omega_1 \cup \Omega_2} + \|\nu^{\frac{1}{2}}p\|_{1,\Omega_1 \cup \Omega_2}.
\]
(3.4)

**Proof.** We first show
\[
\|\nu^{\frac{1}{2}}\nabla e_h^n\|_{0,\mathcal{T}_h} \lesssim \|(e_h^L, e_h^m_n, e_h^u_n, e_h^\mu_n)\|.
\]
(3.5)

In fact, taking $w = \nu \nabla e_h^n$ in (3.2a) and applying integration by parts yield
\[
(e_h^L, \nabla e_h^n)_{\mathcal{T}_h} - (\nu(e_h^L - e_h^n), \nabla e_h^n)_{\partial T_h \setminus \mathcal{H}_h} - (\nu(e_h^L - e_h^n), \nabla e_h^n)_{\mathcal{H}_h} = 0,
\]
which together with the property of projection, implies
\[
\|\nu^{\frac{1}{2}}\nabla e_h^n\|_{0,\mathcal{T}_h} = \|(e_h^L, \nabla e_h^n)_{\mathcal{T}_h} - (\nu(Q_0^b e_h^n - e_h^n), \nabla e_h^n)_{\partial T_h \setminus \mathcal{H}_h} - (\nu(Q_m^b e_h^n - e_h^n), \nabla e_h^n)_{\mathcal{H}_h}\|_{\mathcal{H}_h},
\]
for $m = 0, 1$. In view of the Cauchy-Schwarz inequality, the trace inequality and the definition of $\| \cdot \|$, we then have
\[
\|\nu^{\frac{1}{2}}\nabla e_h^n\|_{0,\mathcal{T}_h} \leq \|\nu^{\frac{1}{2}}e_h^n\|_{0,\mathcal{T}_h} + \|\nabla (Q_0^b e_h^n - e_h^n)\|_{\partial T_h \setminus \mathcal{H}_h} + \|\nabla (Q_m^b e_h^n - e_h^n)\|_{\mathcal{H}_h} \leq \|(e_h^L, e_h^m_n, e_h^u_n, e_h^\mu_n)\|,
\]
where we recall that $\tau$ is given by (2.4).

The thing left is to estimate the term $\|(e_h^L, e_h^m_n, e_h^u_n, e_h^\mu_n)\|$. Taking $(w, v, q, \mu, \tilde{\mu}) = (e_h^L, e_h^m_n, e_h^u_n, e_h^\mu_n)$ in (3.2) and adding up the five equations, we obtain
\[
\|\nu^{\frac{1}{2}}\nabla e_h^n\|_{0,\mathcal{T}_h} = \sum_{i=1}^{2} E_i,
\]
where
\[
E_1 = ((Q_0 L - L)n, e_h^n - e_h^m_n)_{\partial T_h \setminus \mathcal{H}_h} + ((Q_0 L - L)n, e_h^n - e_h^m_n)_{\mathcal{H}_h}
\]
\[
+ (Q_0 p - p) n, e_h^n - e_h^m_n)_{\partial T_h \setminus \mathcal{H}_h} + (Q_0 p - p) n, e_h^n - e_h^m_n)_{\mathcal{H}_h},
\]
\[
E_2 = (\tau Q_0^b (u - Q_1 u), e_h^n - e_h^m_n)_{\partial T_h \setminus \mathcal{H}_h} + (\tau Q_0^b (u - Q_1 u), e_h^n - e_h^m_n)_{\mathcal{H}_h}.
\]
We just need to estimate $E_i$ ($i = 1, 2$) term by term. From Lemma 3.1, the Cauchy-Schwarz inequality and (3.5) it follows
\[
E_1 \lesssim h\|\nu^{\frac{1}{2}}L\|_{1,\Omega_1 \cup \Omega_2} + \|\nu^{\frac{1}{2}}p\|_{1,\Omega_1 \cup \Omega_2} \left(\|\nabla (e_h^L - e_h^m_n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (e_h^L - e_h^m_n)\|_{\mathcal{H}_h}\right)
\]
\[
\lesssim h\|\nu^{\frac{1}{2}}L\|_{1,\Omega_1 \cup \Omega_2} + \|\nu^{\frac{1}{2}}p\|_{1,\Omega_1 \cup \Omega_2} \left(\|\nabla (Q_0^b e_h^n - e_h^n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (Q_0^b e_h^n - e_h^n)\|_{\mathcal{H}_h}\right)
\]
\[
+ \|\nabla (e_h^L - Q_0^b e_h^n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (e_h^L - Q_0^b e_h^n)\|_{\mathcal{H}_h}\right)
\]
\[
\lesssim h\|\nu^{\frac{1}{2}}L\|_{1,\Omega_1 \cup \Omega_2} + \|\nu^{\frac{1}{2}}p\|_{1,\Omega_1 \cup \Omega_2} \left(\|\nabla (e_h^L - e_h^m_n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (e_h^L - e_h^m_n)\|_{\mathcal{H}_h}\right)
\]
\[
\lesssim h\|\nu^{\frac{1}{2}}u\|_{2,\Omega_1 \cup \Omega_2} + \|\nu^{\frac{1}{2}}p\|_{1,\Omega_1 \cup \Omega_2} \left(\|\nabla (e_h^L - e_h^m_n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (e_h^L - e_h^m_n)\|_{\mathcal{H}_h}\right)
\]
\[
\lesssim h\|\nu^{\frac{1}{2}}u\|_{2,\Omega_1 \cup \Omega_2} \left(\|\nabla (e_h^L - e_h^m_n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (e_h^L - e_h^m_n)\|_{\mathcal{H}_h}\right).
\]
Similarly, we have
\[
E_2 \leq \|\nabla (u - Q_1 u)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (u - Q_1 u)\|_{0,\mathcal{H}_h} \left(\|\nabla (e_h^L - e_h^m_n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (e_h^L - e_h^m_n)\|_{\mathcal{H}_h}\right)
\]
\[
\lesssim h\|\nu^{\frac{1}{2}}u\|_{2,\Omega_1 \cup \Omega_2} \left(\|\nabla (e_h^L - e_h^m_n)\|_{0,\partial T_h \setminus \mathcal{H}_h} + \|\nabla (e_h^L - e_h^m_n)\|_{\mathcal{H}_h}\right).
\]
As a result, the desired conclusion follows.
Lemma 3.4. Under the same conditions as in Lemma 3.3, it holds
\[
\|e^p_h\|_{0,\tau_h} \lesssim h(\nu_{\text{max}}^{\frac{1}{2}} + \alpha_{\text{max}}^{\frac{1}{2}}) \left( \|\nu^{\frac{1}{2}} u\|_{2,\Omega_1 \cup \Omega_2} + \|\nu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right). \tag{3.6}
\]
Here \(\nu_{\text{max}} = \max_{i=1,2} \nu_i\) and \(\alpha_{\text{max}} = \max_{i=1,2} \alpha_i\).

Proof. Since \(e^p_h \in L^2_h(\Omega)\), there exists \(v^* \in H^1_0(\Omega)\) such that
\[
\|e^p_h\|_{0,\tau_h} \lesssim \|\nu^{\frac{1}{2}} v^*\|_{1,\Omega_1 \cup \Omega_2}. \tag{3.7}
\]

In view of integration by parts, the properties of projections, and (3.2b), (3.2d) and (3.2e) with \((v, \mu, \tilde{\mu}) = (Q_1 v^*, Q_0 v^*, Q_m^h v^*)\), we have
\[
(\nabla \cdot v^*, e^p_h)_{\tau_h} = (Q_0^h v^*, e^p_h n)_{\partial \tau_h \setminus c^h} + (Q_m^h v^*, e^p_h n)_{c^h} = T_1 + T_2 + T_3,
\]
where
\[
T_1 = (e^p_h, \nabla_h Q_1 v^*)_{\tau_h} + (e^p_h n, Q_1 v^* - Q_0^h v^*)_{\partial \tau_h \setminus c^h} + (e^p_h n, Q_1 v^* - Q_m^h v^*)_{c^h} + (\alpha e^p_h, Q_1 v^*)_{\tau_h},
\]
\[
T_2 = (\tau Q_0^h e^p_h - e^p_h, Q_1 v^* - Q_0^h v^*)_{\partial \tau_h \setminus c^h} + (\tau (Q_m^h e^p_h - e^p_h), Q_1 v^* - Q_m^h v^*)_{c^h},
\]
\[
T_3 = -L_1(\hat{T}_1 v^* - Q_0^h v^*) - L_2(\hat{T}_2 v^* - Q_m^h v^*).
\]

From integration by parts, the Cauchy-Schwarz inequality and the properties of projections it follows
\[
T_1 \lesssim \left( \nu_{\text{max}}^{\frac{1}{2}} \|\nu^{\frac{1}{2}} e^p_h\|_{0,\tau_h} + \alpha_{\text{max}}^{\frac{1}{2}} \|e^p_h\|_{0,\tau_h} \right) \|v^*\|_{1,\Omega} \lesssim \left( \nu_{\text{max}}^{\frac{1}{2}} + \alpha_{\text{max}}^{\frac{1}{2}} \right) \|v^*\|_{1,\Omega},
\]
\[
T_2 \lesssim \nu_{\text{max}}^{\frac{1}{2}} \|\nabla_h (e^p_h, e^p_h, e^p_h, e^p_h)\| \|v^*\|_{1,\Omega},
\]
\[
T_3 \lesssim \nu_{\text{max}}^{\frac{1}{2}} h \left( \|\nu^{\frac{1}{2}} L\|_{1,\Omega_1 \cup \Omega_2} + \|\nu^{\frac{1}{2}} u\|_{2,\Omega_1 \cup \Omega_2} + \|\nu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right) \|v^*\|_{1,\Omega}.
\]

So by (3.7) and the relation (2.2a) we have
\[
\|e^p_h\|_{0,\tau_h} \lesssim \left( \nu_{\text{max}}^{\frac{1}{2}} + \alpha_{\text{max}}^{\frac{1}{2}} \right) \left( \|e^p_h\|_{0,\tau_h} + h \left( \|\nu^{\frac{1}{2}} u\|_{2,\Omega_1 \cup \Omega_2} + \|\nu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right) \right),
\]
which, together with Lemma 3.3, yields the desired result (3.6). \(\blacksquare\)

Based on Lemmas 3.1, 3.3, 3.4, and the triangle inequality, we can easily obtain the following main result.

Theorem 3.1. Let \((L, u, p) \in H^1(\Omega_1 \cup \Omega_2)^{d \times d} \times H^2(\Omega_1 \cup \Omega_2)^d \times H^1(\Omega_1 \cup \Omega_2)\) and \((L_h, u_h, p_h, \tilde{u}_h, \tilde{u}_h) \in W_h \times V_h \times Q_h \times M_h(g) \times M_h\) be the solutions of the problem (2.2) and the X-HDG scheme (2.3), respectively. Then it holds
\[
\|\nu^{\frac{1}{2}} (L - L_h)\|_{0,\tau_h} + \|\nu^{\frac{1}{2}} (\nabla u - \nabla_h u_h)\|_{0,\tau_h} + (\nu_{\text{max}}^{\frac{1}{2}} + \alpha_{\text{max}}^{\frac{1}{2}})^{-1} \|p - p_h\|_{0,\tau_h} \lesssim h \left( \|\nu^{\frac{1}{2}} u\|_{2,\Omega_1 \cup \Omega_2} + \|\nu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right), \tag{3.8}
\]
\[
\|\alpha^{\frac{1}{2}} (u - u_h)\|_{0,\tau_h} \lesssim h \left( \|\nu^{\frac{1}{2}} u\|_{2,\Omega_1 \cup \Omega_2} + \|\alpha^{\frac{1}{2}} u\|_{1,\Omega_1 \cup \Omega_2} + \|\nu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right). \tag{3.9}
\]
### 3.2 $L^2$ error estimation for velocity

In this subsection, we shall derive an $L^2$ error estimate for the velocity approximation by the Aubin-Nitsche’s technique of duality argument. To this end, we introduce the auxiliary problem

\[
\Phi = \nu \nabla \phi \quad \text{in} \quad \Omega_1 \cup \Omega_2, \quad (3.10a)
\]

\[-\nabla \cdot \Phi + \nabla \psi + \alpha \phi = e_h^u \quad \text{in} \quad \Omega_1 \cup \Omega_2, \quad (3.10b)
\]

\[\nabla \cdot \phi = 0 \quad \text{in} \quad \Omega_1 \cup \Omega_2, \quad (3.10c)
\]

\[\phi = 0 \quad \text{on} \quad \partial \Omega, \quad (3.10d)
\]

\[\|\phi\| = 0, \|\Phi - \psi I\|n = 0 \quad \text{on} \quad \Gamma, \quad (3.10e)
\]

and assume the following regularity estimate:

\[
\|\Phi\|_{1,\Omega_1 \cup \Omega_2} + \|\nu \Phi\|_{2,\Omega_1 \cup \Omega_2} + \|\alpha \phi\|_{2,\Omega_1 \cup \Omega_2} + \|\psi\|_{1,\Omega_1 \cup \Omega_2} \lesssim \|e_h^u\|_{0,\tau_h}. \quad (3.11)
\]

Here we recall that $e_h^u = u_h - Q_1 u$.

**Theorem 3.2.** Let $(L, u, p) \in H^1(\Omega_1 \cup \Omega_2)^d \times H^2(\Omega_1 \cup \Omega_2)^d \times H^1(\Omega_1 \cup \Omega_2)$ and $(L_h, u_h, p_h, \bar{u}_h, \bar{u}_h) \in W_h \times V_h \times M_h(g) \times M_h$ be the solutions of the problem (3.2b) and the X-HDG scheme (3.2c), respectively. Then, under the assumption (3.11) it holds

\[
\|u - u_h\|_{0,\tau_h} \lesssim h^{2-\frac{1}{2}} \nu_{\min} \left(\|\nu \frac{1}{2} u\|_{2,\Omega_1 \cup \Omega_2} + \|\nu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2}\right) \quad (3.12)
\]

for (i) $m = 1$ or (ii) $m = 0$ and $g_N$ is a constant on any $F \in e_{h}^\Gamma$. Here $\nu_{\min} := \min_{i=1,2} \nu_i$.

**Proof.** Testing the equations (3.10b) and (3.10c) by $e_h^u$ and $e_h^p$, respectively, adding them up, and using integration by parts and the properties of projections, we obtain

\[
\|e_h^u\|^2_{0,\tau_h} = - \langle \nabla_h \cdot \Phi, e_h^u \rangle_{\tau_h} + \langle \nabla_h \psi, e_h^u \rangle_{\tau_h} + \langle \alpha \phi, e_h^u \rangle_{\tau_h} - \langle \nabla_h \cdot \phi, e_h^u \rangle_{\tau_h}
\]

\[
= (\langle Q_0 \Phi - \Phi \rangle n, e_h^u)_{\partial \tau_h \setminus e_{h}^\Gamma} + (\langle Q_0 \Phi - \Phi \rangle n, e_h^u)_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle \psi, e_h^u \cdot n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + (\langle Q_0 \Phi - \Phi \rangle n, e_h^u)_{\partial \tau_h \setminus e_{h}^\Gamma}
\]

\[
+ \langle \psi, e_h^u \cdot n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle \alpha Q_1 \phi, e_h^u \rangle_{\tau_h} - \langle e_h^p n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - \langle e_h^p n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma}.
\]

Due to the error equations (3.2a) and (3.2c), we have

\[
\langle \nu^{-1} e_h^u, Q_0 \Phi \rangle_{\tau_h} - \langle e_h^u, Q_0 \Phi n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - \langle e_h^u, Q_0 \Phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} = 0,
\]

\[
\langle e_h^u, Q_0 \Phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle e_h^u, Q_0 \psi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} = 0.
\]

which, together with the fact $\|\Phi - \psi I\|n = 0$, imply

\[
\|e_h^u\|^2_{0,\tau_h} = \langle (Q_0 \Phi - \Phi) n, e_h^u \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle (Q_0 \Phi - \Phi) n, e_h^u \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle \psi, e_h^u \cdot n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle \psi, e_h^u \cdot n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle \psi, e_h^u \cdot n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma}
\]

\[
+ (\psi Q_1, \phi, e_h^u)_{\tau_h} - \langle e_h^p n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - \langle e_h^p n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle Q_0 \Phi, \nu^{-1} e_h^u \rangle_{\tau_h}.
\]

Notice that by (3.10a), the properties of projections and integration by parts it holds

\[
\langle Q_0 \Phi, \nu^{-1} e_h^u \rangle_{\tau_h} = \langle \Phi, \nu^{-1} e_h^u \rangle_{\tau_h} = \langle \Phi, \nu^{-1} e_h^u \rangle_{\tau_h}
\]

\[
= \langle \phi, e_h^u n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle \phi, e_h^u n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle Q_0 \Phi, e_h^u n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \langle Q_m \phi, e_h^u n \rangle_{\partial \tau_h \setminus e_{h}^\Gamma},
\]

and that taking $(v, \mu, \bar{\mu}) = (Q_1 \phi, Q_m \phi, Q_m \phi)$ in (3.2b),(3.2d)-(3.2e) shows

\[
\langle \nu e_h^u, Q_1 \phi \rangle_{\tau_h} = - \langle \tau(Q_m e_h^u - e_h^u), Q_1 \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - \langle \tau(Q_m e_h^u - e_h^u), Q_1 \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} + \sum_{i=1}^2 L_i(Q_1 \phi),
\]

\[\langle e_h^u n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - \langle e_h^u n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} = \langle \tau(Q_m e_h^u - e_h^u), Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - L_1(Q_1 \phi),
\]

\[\langle e_h^u n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - \langle e_h^u n, Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} = \langle \tau(Q_m e_h^u - e_h^u), Q_m \phi \rangle_{\partial \tau_h \setminus e_{h}^\Gamma} - L_2(Q_m \phi).
\]
The four equations above mean that

$$
\begin{align*}
\langle \alpha Q_1 \phi, e_h^m \rangle_{\Omega} & = -\langle e_h^m n, Q_b^0 \phi \rangle_{\partial \Omega} - \langle e_h^m n, Q_b^0 \phi \rangle_{\partial \Omega} + (Q_0 \Phi, \nu^{-1} e_h^m)_{\Omega} \\
= & \langle \tau(Q_b^0 e_h^m - e_h^m), Q_b^0 \phi - Q_1 \phi \rangle_{\partial \Omega} + \langle \tau(Q_b^0 e_h^m - e_h^m), Q_b^0 \phi - Q_1 \phi \rangle_{\partial \Omega} + L_1(Q_1 \phi - Q_b^0 \phi) + L_2(Q_1 \phi - Q_b^0 \phi).
\end{align*}
$$

As a result, we obtain

$$
\| e_h^m \|_{0, \Omega}^2 = \sum_{j=1}^4 I_j \tag{3.13}
$$

with

$$
\begin{align*}
I_1 & := \langle (Q_0 \Phi - \Phi) n, e_h^m - e_h^m \rangle_{\partial \Omega} + \langle (Q_0 \Phi - \Phi) n, e_h^m - e_h^m \rangle_{\partial \Omega} \\
I_2 & := \langle \psi - Q_0 \psi, (e_h^m - e_h^m) \cdot n \rangle_{\partial \Omega} + \langle \psi - Q_0 \psi, (e_h^m - e_h^m) \cdot n \rangle_{\partial \Omega} \\
I_3 & := \langle \tau(Q_b^0 e_h^m - e_h^m), Q_b^0 \phi - Q_1 \phi \rangle_{\partial \Omega} + \langle \tau(Q_b^0 e_h^m - e_h^m), Q_b^0 \phi - Q_1 \phi \rangle_{\partial \Omega} \\
I_4 & := L_1(Q_1 \phi - Q_b^0 \phi) + L_2(Q_1 \phi - Q_b^0 \phi).
\end{align*}
$$

From the Cauchy-Schwarz inequality, Lemmas 3.1 and 3.3, and the regularity assumption (3.11) it follows

$$
\begin{align*}
I_1 + I_2 + I_3 & \lesssim h^{n-\frac{1}{2}} \| \Phi \|_{1, \Omega} \| u \|_{2, \Omega} + \| \psi \|_{1, \Omega} \| e_h^m, e_h^m, e_h^m \| \\
& \lesssim h^{n-\frac{1}{2}} \| e_h^m \|_{0, \Omega} \left( \| \frac{1}{2} u \|_{2, \Omega} + \| \frac{1}{2} p \|_{1, \Omega} \right). \tag{3.14}
\end{align*}
$$

In light of the property of projection, it holds

$$
\begin{align*}
I_4 & = \left( \langle (Q_0 L - L) n, Q_1 \phi - Q_1 \phi \rangle_{\partial \Omega} + \langle \tau(Q_b^0 (u - Q_1 u), Q_1 \phi - \phi \rangle_{\partial \Omega} + \langle (Q_0 p - p) n, Q_1 \phi - \phi \rangle_{\partial \Omega} \right) \\
& \quad + \langle (Q_b^0 (u - Q_1 u), Q_1 \phi - \phi \rangle_{\partial \Omega} + \langle (Q_0 p - p) n, Q_1 \phi - \phi \rangle_{\partial \Omega} \\
& \quad + \langle (Q_b^0 (u - Q_1 u), Q_1 \phi - \phi \rangle_{\partial \Omega} + \langle (Q_0 p - p) n, Q_1 \phi - \phi \rangle_{\partial \Omega} \right) \\
& =: \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3. \tag{3.15}
\end{align*}
$$

Again by Lemma 3.1 and (3.11) we get

$$
\begin{align*}
\tilde{I}_1 & \lesssim h^{n-\frac{1}{2}} \| \nu \|_{2, \Omega} \left( \| \frac{1}{2} L \|_{1, \Omega} + \| \frac{1}{2} p \|_{1, \Omega} \right) \\
& \lesssim h^{n-\frac{1}{2}} \| e_h^m \|_{0, \Omega} \left( \| \frac{1}{2} u \|_{2, \Omega} + \| \frac{1}{2} p \|_{1, \Omega} \right), \tag{3.16}
\end{align*}
$$

and, for case (i) with $m = 1$,

$$
\begin{align*}
\tilde{I}_3 & = \langle (Q_0 L - L) n, \phi - Q_b^0 \phi \rangle_{\partial \Omega} + \langle (Q_0 p - p) n, \phi - Q_b^0 \phi \rangle_{\partial \Omega} \\
& \lesssim h^{n-\frac{1}{2}} \| \nu \|_{2, \Omega} \left( \| \frac{1}{2} L \|_{1, \Omega} + \| \frac{1}{2} p \|_{1, \Omega} \right) \\
& \lesssim h^{n-\frac{1}{2}} \| e_h^m \|_{0, \Omega} \left( \| \frac{1}{2} u \|_{2, \Omega} + \| \frac{1}{2} p \|_{1, \Omega} \right). \tag{3.17}
\end{align*}
$$

Since $\| (L - p I) n \|_F = 0$ for $F \in \varepsilon_h \setminus \varepsilon_h^F$, we have

$$
\begin{align*}
\tilde{I}_2 & = \langle (Q_0 L - L) n, \phi - Q_b^0 \phi \rangle_{\partial \Omega} + \langle (Q_0 p - p) n, \phi - Q_b^0 \phi \rangle_{\partial \Omega} \\
& = \langle (Q_0 L, \phi - Q_b^0 \phi \rangle_{\partial \Omega} + \langle (Q_0 p, \phi - Q_b^0 \phi \rangle_{\partial \Omega} \rangle \\
& = 0. \tag{3.18}
\end{align*}
$$
Similarly, for case (ii) with \( m = 0 \) and \( \|(L - pI)n\| = g_N^I = \text{const.} \) on any \( F \in \tilde{c}_h \), it holds
\[
\tilde{I}_h = ((Q_0L - L)n, \phi - Q_0^b\phi)_{\tilde{c}_h} + ((Q_0p - p)n, \phi - Q_0^b\phi)_{\tilde{c}_h} = 0. \tag{3.19}
\]
Finally, combining (3.13)-(3.19) completes the proof.

Remark 3.1. For the Stokes equation with \( \nu_1 = \nu_2 = \nu \) and \( \alpha = 0 \), from Theorems 3.1 and 3.2 we easily obtain
\[
\|u - u_h\|_{0, \mathcal{T}_h} + h\|\nabla u - \nabla_h u_h\|_{0, \mathcal{T}_h} + \nu^{-1}h\|p - p_h\|_{0, \mathcal{T}_h} \lesssim h^2\left(\|u\|_{2, \Omega_1 \cup \Omega_2} + \nu^{-1}\|p\|_{1, \Omega_1 \cup \Omega_2}\right).
\]

4 Application of X-HDG method to curved domains

Let \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) be a curved domain with piecewise smooth boundary. Consider the following problem:
\[
\begin{cases}
-\nu\Delta u + \nabla p + \alpha u = f & \text{in } \Omega, \\
\nabla \cdot u = 0 & \text{in } \Omega, \\
u \equiv 0, \; p \equiv 0 & \text{in } \Omega^c.
\end{cases}
\tag{4.1}
\]
Here \( \nu > 0 \) and \( \alpha \geq 0 \) are two constants, and \( g_D \) satisfies the compatibility condition (1.3).

Figure 5: The geometry of a curved domain (left) and a boundary unfitted mesh (right).

Let \( \mathbb{B} \supset \Omega \) be a simpler domain than \( \Omega \) (cf. Figure 5), and denote \( \Omega^c := \mathbb{B} \setminus \bar{\Omega} \). Then we can rewrite problem (4.1) as an interface problem:
\[
\begin{cases}
-\nu\Delta u + \nabla p + \alpha u = \chi_{\Omega}f & \text{in } \Omega \cup \Omega^c, \\
\nabla \cdot u = 0 & \text{in } \Omega \cup \Omega^c, \\
u \equiv 0, \; p \equiv 0 & \text{in } \Omega^c.
\end{cases}
\tag{4.2a}
\]
Here \( \chi_{\Omega} \) is the characteristic function on \( \Omega \), which satisfying \( \chi_{\Omega} = 1 \) in \( \Omega \) and \( \chi_{\Omega} = 0 \) in \( \Omega^c \). We note that the problem (4.2) is a special interface problem with \( \partial \Omega \) being the interface, for which we only need to approximate the solution in \( \Omega \) due to (4.2d).

Let \( \mathcal{T}_h = \cup\{K\} \) be a shape-regular partition of the domain \( \Omega \) consisting of arbitrary open polygons/polyhedrons. For any \( K \) satisfying \( K \cap \Gamma \neq \emptyset \), called an boundary element, let \( \Gamma_K := K \cap \Gamma \) be the part of \( \Gamma \) in \( K \), and \( \Gamma_{K, h} \) be the straight line/plane segment connecting the intersection between \( \Gamma_K \) and \( \partial K \).
Define the following sets of elements or edges/faces:

\[ T_h := \{ K \in T_h : K \cap \Omega = K \} , \]
\[ T_h^\Gamma := \{ K \cap \Omega : \forall K \in T_h \text{ with } K \cap \partial \Omega \neq \emptyset \} , \]
\[ T_h^\varepsilon := T_h \cup T_h^\Gamma , \]
\[ \varepsilon_h := \{ F \cap \Omega : \forall \text{ edge/face } F \text{ of all elements in } T_h \} , \]
\[ \varepsilon_h^\Gamma := \{ F : F = \Gamma_{K,h}, \forall K \in T_h^\Gamma , \text{ or } F \text{ is an edge/face of some } K \in T_h^\varepsilon \text{ with } F \subset \bar{K} \cap \partial \Omega \} , \]
\[ \varepsilon_h := \varepsilon_h^\varepsilon \cup \varepsilon_h^\Gamma . \]

We also introduce the following X-HDG finite element spaces:

\[ W_h := \{ w \in L^2(\Omega)^{d \times d} : w|_K \in P_0(K)^{d \times d} \forall K \in T_h^* \} , \]
\[ V_h := \{ v \in L^2(\Omega)^d : v|_K \in P_1(K)^d \forall K \in T_h^* \} , \]
\[ Q_h := \{ q \in L^2(\Omega)^d : q|_K \in P_0(K) \forall K \in T_h^* \} , \]
\[ M_h := \{ \mu \in L^2(\varepsilon_h^\varepsilon)^d : \mu|_F \in P_0(F)^d \forall F \in \varepsilon_h^\varepsilon \} , \]
\[ \tilde{M}_h := \{ \tilde{\mu} \in L^2(\varepsilon_h^\varepsilon)^d : \tilde{\mu}|_F \in P_m(F)^d , \forall F \in \varepsilon_h^\varepsilon \} \text{ with } m = 0, 1 . \]

Then the X-HDG scheme for (4.2) is given as follows: find \((l_h, u_h, p_h, \tilde{u}_h, \tilde{u}_h) \in W_h \times V_h \times Q_h \times M_h \times \tilde{M}_h\) such that

\[
\begin{align*}
\langle (\nu^{-1}L_h(w), w)_{T_h^*} - \langle \tilde{u}_h, wn \rangle_{\partial \varepsilon_h^\varepsilon} - \langle \hat{u}_h, wn \rangle_{\varepsilon_h^\varepsilon} = 0, \\
(\alpha u_h, v)_{T_h^\varepsilon} + \langle \tau(Q_h^b u_h - \tilde{u}_h), v \rangle_{\partial \varepsilon_h^\varepsilon} + \langle \tau(Q_m^b u_h - \tilde{u}_h), v \rangle_{\varepsilon_h^\varepsilon} = (f, v), \\
\langle \hat{u}_h \cdot n, q \rangle_{\partial \varepsilon_h^\varepsilon} + \langle \tilde{u}_h \cdot n, q \rangle_{\varepsilon_h^\varepsilon} = 0, \\
\langle L_h n, \mu \rangle_{\partial \varepsilon_h^\varepsilon} - \langle p_h n, \mu \rangle_{\partial \varepsilon_h^\varepsilon} - \langle \tau(Q_h^b u_h - \tilde{u}_h), \mu \rangle_{\partial \varepsilon_h^\varepsilon} = 0, \\
\langle \hat{u}_h, \mu^* \rangle_{\varepsilon_h^\varepsilon} = (g_D, \mu^*)_{\varepsilon_h^\varepsilon},
\end{align*}
\]

for all \((w, v, q, \mu, \tilde{\mu}) \in W_h \times V_h \times Q_h \times M_h \times \tilde{M}_h\). Here

\[
\langle w, v \rangle_{\partial \varepsilon_h^\varepsilon} := \sum_{K \in T_h^*} \langle w, v \rangle_{\partial K \setminus \varepsilon_h^\varepsilon} , \quad \langle w, v \rangle_{\varepsilon_h^\varepsilon} := \sum_{F \in \varepsilon_h^\varepsilon} \langle w, v \rangle_F
\]

for any scalars/vectors \(w, v, q, \mu, \tilde{\mu}\) in \(W_h \times V_h \times Q_h \times M_h \times \tilde{M}_h\). Here

\[
\tau|_F = \nu h^{-1}_K , \forall F \subset \partial K \text{ with } K \in T_h^* .
\]

When \(F \in \varepsilon_h^\varepsilon\) is a line segment/straight plane, we take \(g_D|_F := g_D\), and when \(F = \Gamma_{K,h} \neq \Gamma_K\) for some \(K \in T_h^\varepsilon\), we set \(g_D|_F\) to be some linear interpolation of \(g_D\) using data of \(g_D\) at two (2D case)/three (3D case) intersection points of \(\Gamma_K\) and \(\Gamma_{K,h}\).

By following the same line as in the proof of Theorem 2.1, we can obtain the existence and uniqueness of the solution to (4.3).

**Theorem 4.1.** The X-HDG scheme (4.3) admits a unique solution.

## 5 Numerical experiments

In this section, we provide five 2-dimensional numerical examples to verify the performance of the proposed X-HDG method.

**Example 5.1.** Square domain with circular interface [1]: \(g_D^\Gamma = 0\)
Consider the problem (1.1) with \( \Omega = [-1,1]^2 \), \( \Omega_1 = \{(x,y) \in \Omega : r = \sqrt{x^2+y^2} > r_0 = \sqrt{3/10} \} \), \( \Omega_2 = \Omega \setminus \Omega_1 \), and \( \alpha_1 = \alpha_2 = 0 \). The exact solution \((u = (u_1, u_2), p)\) is given by
\[
\begin{align*}
  u_1(x,y) &= \frac{1}{\nu_i} y(x^2+y^2 - 0.3) & \text{in } \Omega_i, & i = 1, 2, \\
  u_2(x,y) &= -\frac{1}{\nu_i} x(x^2+y^2 - 0.3) & \text{in } \Omega_i, & i = 1, 2, \\
  p(x,y) &= \frac{1}{10} (x^3 - y^3) & \text{in } \Omega_1 \cup \Omega_2.
\end{align*}
\]

The force term, boundary conditions and interface conditions can be derived explicitly.
We take \((\nu_1, \nu_2) = (10^{-3}, 1), (1, 10^{-3}), (10^{-3}, 10^{-3})\), and use \(N \times N\) uniform triangular/rectangular meshes (cf. Figure 6) in the X-HDG scheme (2.3). Error results of the numerical solutions are listed in Table 1, and the solutions \(u_h\) and \(p_h\) at \((\nu_1, \nu_2) = (10^{-3}, 1)\) and \(128 \times 128\) triangular mesh are shown in Figure 7.

From Table 1 we can see that in all cases the X-HDG method yields optimal convergence orders, i.e. first order for \(\|L - L_h\|_0\), \(\|
abla u - \nabla h u_h\|_0\) and \(\|p - p_h\|_0\), and second order for \(\|u - u_h\|_0\). These results are conformable to Theorems 3.1 and 3.2.

Figure 6: The square domain with circular interface at 16 x 16 meshes: triangular mesh(left) and rectangular mesh(right).

Figure 7: The X-HDG solutions \(u_{1h}\) (left), \(u_{2h}\) (middle) and \(p_h\) (right) at \((\nu_1, \nu_2) = (10^{-3}, 1)\) and \(128 \times 128\) triangular mesh: Example 5.1.
Table 1: History of convergence for the X-HDG scheme (2.3): Example 5.1

(a) Triangular meshes

| ν1 | ν2 | mesh | \|u_i-u_h\|_h | \|L_i-L_h\|_h | \|V_i-V_h\|_h | \|p-p_h\|_h |
|----|----|------|------------|------------|------------|------------|
| 1  | 10^{-3} | 16 × 16 | 1.4633E-01 | 8.8492E-02 | 3.0103E-01 | 3.3596E-01 |
|    |        | 32 × 32 | 3.7945E-02 | 4.4647E-02 | 1.4818E-01 | 1.7004E-01 |
|    |        | 64 × 64 | 9.4540E-03 | 2.2413E-02 | 7.4780E-02 | 8.8473E-02 |
|    |        | 128 × 128 | 2.3843E-03 | 1.1227E-02 | 3.7645E-02 | 4.2430E-02 |

(b) Rectangular meshes

| ν1 | ν2 | mesh | \|u_i-u_h\|_h | \|L_i-L_h\|_h | \|V_i-V_h\|_h | \|p-p_h\|_h |
|----|----|------|------------|------------|------------|------------|
| 1  | 10^{-3} | 16 × 16 | 2.5011E-02 | 8.8606E-02 | 9.8525E-02 | 3.4740E-01 |
|    |        | 32 × 32 | 6.1643E-03 | 4.4647E-02 | 4.9169E-02 | 1.7214E-01 |
|    |        | 64 × 64 | 1.5228E-03 | 2.2415E-02 | 2.4596E-02 | 8.5033E-02 |
|    |        | 128 × 128 | 3.7796E-04 | 1.1227E-02 | 1.2301E-02 | 4.2462E-02 |

Example 5.2. Square domain with circular interface [1]: \( g_N^x \neq 0 \)

Consider the same domain and interface as in Example 5.1. The exact solution \((u_1, u_2, p)\) is given by

\[
u_i(x, y) = 1 + \frac{1}{\nu_i} y \sin(x^2 + y^2 - 0.3) \quad \text{in} \quad \Omega, \quad i = 1, 2,
\]

\[
u_2(x, y) = 2 - \frac{1}{\nu_i} x \sin(x^2 + y^2 - 0.3) \quad \text{in} \quad \Omega, \quad i = 1, 2,
\]

\[
p(x, y) = \left\{ \begin{array}{ll}
\frac{e^{x+y} - 1.3798535909816816}{\sqrt{1 + x^2 + y^2}} & \text{in} \quad \Omega_1, \\
\frac{1.3798535909816816}{\sqrt{1 + x^2 + y^2}} & \text{in} \quad \Omega_2,
\end{array} \right.
\]

and the coefficients \(\nu\) and \(\alpha\) are taken as: (i) \((\nu_1, \nu_2) = (1, 10^{-3}), (\alpha_1, \alpha_2) = (0, 0)\); (ii) \((\nu_1, \nu_2) = (10^{-2}, 1), (\alpha_1, \alpha_2) = (1, 0)\).

From Table 2 we can observe that the convergence rates of \(\|u - u_h\|_0, \|L - L_h\|_0, \|V - V_h\|_0\) and \(\|p - p_h\|_0\) are all optimal at triangular and rectangular meshes. We also plot in Figure 8 the numerical solutions \(u_h\) and \(p_h\) at 128 × 128 triangular mesh with \((\nu_1, \nu_2) = (1, 10^{-3}), (\alpha_1, \alpha_2) = (0, 0)\).
Table 2: History of convergence for the X-HDG scheme (2.3): Example 5.2

(a) Triangular meshes

| $\alpha_1$ | $\alpha_2$ | $\nu_1$ | $\nu_2$ | mesh       | $\|\mathbf{e}_h\|_0$ | $\|\mathbf{L}_{\mathbf{h}}\|_0$ | $\|\nabla \mathbf{u} - \nabla \mathbf{u}_h\|_0$ | $\|p - p_h\|_0$ |
|-----------|-----------|--------|--------|------------|----------------|----------------|----------------|----------------|
| 0         | 0         | 1      | $10^{-3}$ | 16 x 16   | 1.4780E-01   | 9.1176E-02 | 3.0346E-01 | 4.9323E-02 |
|           |           |        |        | 32 x 32   | 3.8392E-02   | 4.6059E-02 | 1.4955E-01 | 2.3874E-02 |
|           |           |        |        | 64 x 64   | 9.5775E-03   | 2.3134E-02 | 7.5525E-02 | 1.1738E-02 |
|           |           |        |        | 128 x 128 | 2.4162E-03   | 1.1590E-02 | 3.8060E-02 | 5.8293E-03 |

(b) Rectangular meshes

| $\alpha_1$ | $\alpha_2$ | $\nu_1$ | $\nu_2$ | mesh       | $\|\mathbf{e}_h\|_0$ | $\|\mathbf{L}_{\mathbf{h}}\|_0$ | $\|\nabla \mathbf{u} - \nabla \mathbf{u}_h\|_0$ | $\|p - p_h\|_0$ |
|-----------|-----------|--------|--------|------------|----------------|----------------|----------------|----------------|
| 0         | 0         | 1      | $10^{-3}$ | 16 x 16   | 1.4245E-02   | 9.5200E-02 | 9.3735E-02 | 1.8235E-02 |
|           |           |        |        | 32 x 32   | 4.3235E-03   | 4.6587E-02 | 5.0761E-02 | 5.9483E-02 |
|           |           |        |        | 64 x 64   | 1.1439E-03   | 2.3291E-02 | 2.6161E-02 | 1.8745E-02 |
|           |           |        |        | 128 x 128 | 2.8998E-04   | 1.1598E-02 | 1.3196E-02 | 6.9168E-03 |

Figure 8: The X-HDG solutions $u_{1h}(\text{left}), u_{2h}(\text{middle})$ and $p_h(\text{right})$ at $(\nu_1, \nu_2) = (1, 10^{-3}), \alpha_1 = \alpha_2 = 0$ and 128 x 128 triangular mesh: Example 5.2.

Example 5.3. A laminar flow test in a square domain with straight line interface; $g_N^\Gamma \neq 0$

Take $\Omega = [0, 1]^2$ (Figure 9) in (1.1). Two kinds of fluids with different viscosity flow in the subdomains $\Omega_1 = [0, 1] \times [b_0, 1]$ and $\Omega_2 = [0, 1] \times [0, b_0]$, respectively, with $b_0 = 0.4031$. The exact solution is given by

\[
\begin{align*}
    u_1(x,y) &= 1 - e^{\lambda_i} \sin\left(\frac{\pi}{b_0} y\right) & \text{in } \Omega_1, & i = 1, 2, \\
    u_2(x,y) &= 0 & \text{in } \Omega_1 \cup \Omega_2, \\
    p(x,y) &= \frac{1}{2} e^{2\lambda_i} x - b_0 \left(\frac{1}{\nu_i} e^{2\lambda_2} - 1\right) - (1 - b_0) \left(\frac{1}{\nu_i} e^{2\lambda_1} - 1\right) & \text{in } \Omega_1, & i = 1, 2,
\end{align*}
\]

where $\lambda_i = \frac{1}{2\nu_i} - \sqrt{\frac{1}{4\nu_i^2} + 4\pi^2}$, $i = 1, 2$, and $(\nu_1, \nu_2) = (1, 10^{-2})$. We take $(\alpha_1, \alpha_2) = (0, 0), (0, 1)$. 

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From Table 3, we can see that the X-HDG method yields optimal convergence rates for the numerical solutions at both triangular and rectangular meshes. We also show in Figure 10 the numerical solutions $u_{1h}$ and $p_h$ at $(\nu_1, \nu_2) = (1, 10^{-2}), (\alpha_1, \alpha_2) = (0, 0)$ and $128 \times 128$ triangular mesh.

![Figure 9: The domain with straight line interface at 8 x 8 meshes: triangular mesh(left) and rectangular mesh(right).](image)

Table 3: History of convergence for the X-HDG scheme (2.3): Example 5.3

| $\alpha_1$ | $\alpha_2$ | $\nu_1$ | $\nu_2$ | mesh | $\|u_{1h} - u_0\|_{L^2}$ | $\|u_{1h} - u_0\|_{\mathbb{H}(0)}$ | $\|\nabla u_{1h} - \nabla u_0\|_{L^2}$ | $\|\nabla u_{1h} - \nabla u_0\|_{\mathbb{H}(0)}$ | $\|p_h - p_0\|_0$ | $\|p_h - p_0\|_{L^2}$ |
|----------|----------|--------|--------|---------|-----------------|-----------------|-----------------|-----------------|----------------|----------------|
|          |          |        |        | $8 \times 8$ | 1.044E-01 | 2.574E-01 | 3.513E-01 | 1.590E-01 | 1.06 | 0.92 |
| 0        | 0        | 1      | $10^{-2}$ | $16 \times 16$ | 2.687E-02 | 1.338E-01 | 1.680E-01 | 8.382E-02 | 1.06 | 1.00 |
| 0        | 0        | 1      | $10^{-2}$ | $32 \times 32$ | 6.871E-03 | 6.833E-02 | 8.361E-02 | 4.191E-02 | 1.01 | 1.00 |
| 0        | 0        | 1      | $10^{-2}$ | $64 \times 64$ | 1.724E-03 | 3.439E-02 | 4.189E-02 | 2.064E-02 | 1.00 | 1.00 |
| 0        | 0        | 1      | $10^{-2}$ | $128 \times 128$ | 4.319E-04 | 1.725E-02 | 2.101E-02 | 1.02 | 1.02 | 1.02 |

|          |          |        |        | $8 \times 8$ | 7.521E-02 | 2.142E-01 | 2.842E-01 | 1.944E-01 | 1.07 | 1.07 |
| 0        | 1        | 1      | $10^{-2}$ | $16 \times 16$ | 2.376E-02 | 1.279E-01 | 1.554E-01 | 9.250E-02 | 1.07 | 1.00 |
| 0        | 1        | 1      | $10^{-2}$ | $32 \times 32$ | 6.524E-03 | 6.725E-02 | 8.171E-02 | 4.336E-02 | 1.09 | 1.09 |
| 0        | 1        | 1      | $10^{-2}$ | $64 \times 64$ | 1.671E-03 | 3.424E-02 | 4.163E-02 | 2.085E-02 | 1.06 | 1.06 |
| 0        | 1        | 1      | $10^{-2}$ | $128 \times 128$ | 4.208E-04 | 1.724E-02 | 2.098E-02 | 1.03 | 1.03 | 1.03 |

|          |          |        |        | $8 \times 8$ | 1.566E-01 | 7.307E-01 | 3.411E-01 | 1.826E-01 | 1.01 | 0.99 |
| 0        | 0        | 1      | $10^{-2}$ | $16 \times 16$ | 4.151E-02 | 2.850E-01 | 1.535E-01 | 9.905E-02 | 0.88 | 0.88 |
| 0        | 0        | 1      | $10^{-2}$ | $32 \times 32$ | 1.066E-02 | 1.116E-01 | 7.284E-02 | 5.572E-02 | 0.97 | 0.97 |
| 0        | 0        | 1      | $10^{-2}$ | $64 \times 64$ | 2.698E-03 | 4.577E-02 | 3.562E-02 | 2.520E-02 | 1.00 | 1.00 |
| 0        | 0        | 1      | $10^{-2}$ | $128 \times 128$ | 6.774E-04 | 1.765E-02 | 1.265E-02 | 1.265E-02 | 1.01 | 1.01 |

|          |          |        |        | $8 \times 8$ | 3.371E-02 | 3.191E-01 | 1.457E-01 | 1.284E-01 | 0.99 | 0.99 |
| 0        | 1        | 1      | $10^{-2}$ | $16 \times 16$ | 3.371E-02 | 3.191E-01 | 1.457E-01 | 1.284E-01 | 0.99 | 0.99 |
| 0        | 1        | 1      | $10^{-2}$ | $32 \times 32$ | 9.802E-03 | 7.224E-02 | 5.337E-02 | 5.337E-02 | 1.08 | 1.08 |
| 0        | 1        | 1      | $10^{-2}$ | $64 \times 64$ | 2.579E-03 | 4.764E-02 | 3.558E-02 | 2.563E-02 | 1.06 | 1.06 |
| 0        | 1        | 1      | $10^{-2}$ | $128 \times 128$ | 6.546E-04 | 1.765E-02 | 1.265E-02 | 1.265E-02 | 1.01 | 1.01 |

Example 5.4. Curved domain test 1: circular boundary

Set $\Omega = \{(x - x_0)^2 + (y - y_0)^2 \leq r^2\}$ in the model problem (4.1) with a homogeneous boundary
condition, where $x_0 = y_0 = \frac{1}{2}$ and $r = \frac{\sqrt{3}}{4}$ (Figure 11). The exact solution is given by

$$u_1(x, y) = y(x^2 + y^2 - r^2),$$
$$u_2(x, y) = -x(x^2 + y^2 - r^2),$$
$$p(x, y) = \frac{1}{10}(x^3 - y^3),$$

and the coefficients $(\nu, \alpha) = (1, 0), (1, 1), (0.01, 1)$.

We take $B = [0, 1]^2$ in the X-HDG scheme (4.3). Table 4 shows that the boundary-unfitted X-HDG method is of optimal convergence rates for the numerical solutions. Figure 12 plots the numerical solutions $u_h$ and $p_h$ at $128 \times 128$ triangular mesh.
### Table 4: History of convergence for the X-HDG scheme (4.3): Example 5.4

| α  | ν   | mesh     | \[\| u_1 - u_h \|_0 \] | error | order | \[\| u_2 - u_h \|_0 \] | error | order | \[\| p - p_h \|_0 \] | error | order |
|----|-----|----------|-----------------|-------|-------|-----------------|-------|-------|-----------------|-------|-------|
| 0  | 1   | 8 × 8    | 2.0363E-02      | 7.5156E-02 | 1.1122E-01 | 8.1187E-01      | 1.1122E-01 | 1.1122E-01 | 1.1122E-01 | 1.1122E-01 | 1.1122E-01 |
|    |     | 16 × 16  | 5.1184E-03      | 3.8783E-02 | 0.95   | 7.5156E-02      | 1.00  | 1.1122E-01 | 8.1187E-01 | 1.1122E-01 | 1.1122E-01 |
|    |     | 32 × 32  | 1.2773E-03      | 1.9648E-02 | 0.98   | 3.8783E-02      | 0.99  | 1.1122E-01 | 8.1187E-01 | 1.1122E-01 | 1.1122E-01 |
|    |     | 64 × 64  | 3.1972E-04      | 9.8908E-03 | 0.99   | 1.9648E-02      | 0.99  | 1.1122E-01 | 8.1187E-01 | 1.1122E-01 | 1.1122E-01 |
|    |     | 128 × 128| 8.0032E-05     | 4.9607E-03 | 0.99   | 9.8908E-03      | 0.99  | 1.1122E-01 | 8.1187E-01 | 1.1122E-01 | 1.1122E-01 |

### Example 5.5. Curved domain test 2: five-star shaped boundary

Let Ω be a five-star shaped domain with boundary \( \Gamma = \{(r, \theta) : \rho(r, \theta) = 0, \ 0 \leq \theta < 2\pi\} \) in (4.1), where \( \rho(r, \theta) = r - \frac{x^3}{4} - \frac{1}{10} \sin(5\theta + \frac{\pi}{2}), \ r = \sqrt{x^2 + y^2}. \) The exact solution of (4.1) is given by

\[
\begin{align*}
    u_1(x, y) &= x^2y, \\
    u_2(x, y) &= -xy^2, \\
    p(x, y) &= \frac{1}{3}(x^3 - y^3),
\end{align*}
\]

and the coefficients \((\nu, \alpha) = (1, 0), (1, 1), (0.01, 1).\)

We take \( B = [-1, 1]^2 \) in the X-HDG scheme (4.3). Table 5 shows that the boundary-unfitted X-HDG method is of optimal convergence rates for the numerical solutions. Figure 14 plots the numerical solutions \( u_h \) and \( p_h \) at 128 × 128 triangular mesh.
Table 5: History of convergence for the X-HDG scheme (4.3); Example 5.5

| $\alpha$ | $\nu$ | mesh         | $\|u-u_{h}\|_{0}$ | $\|L-L_{h}\|_{0}$ | $\|\nabla u-\nabla u_{h}\|_{0}$ | $\|p-p_{h}\|_{0}$ |
|----------|-------|--------------|-------------------|-------------------|---------------------------------|-----------------|
| 0        | 1     | $16 \times 16$ | $2.0455E-02$     | $1.0942E-01$     | $5.2046E-01$                    |                  |
|          |       | $32 \times 32$ | $5.9306E-03$     | $5.5978E-02$     | $1.0942E-01$                    | $5.2046E-01$    |
|          |       | $64 \times 64$ | $1.5742E-03$     | $2.8320E-02$     | $2.5209E-01$                    | $2.5200E-01$    |
|          |       | $128 \times 128$ | $3.9680E-04$    | $1.4260E-02$     | $1.2400E-01$                    | $1.2398E-01$    |
| 1        | 1     | $16 \times 16$ | $2.0401E-02$     | $1.0941E-01$     | $5.2097E-01$                    |                  |
|          |       | $32 \times 32$ | $5.9174E-03$     | $5.5976E-02$     | $1.0941E-01$                    | $5.2097E-01$    |
|          |       | $64 \times 64$ | $1.5714E-03$     | $2.8319E-02$     | $2.5209E-01$                    | $2.5200E-01$    |
|          |       | $128 \times 128$ | $3.9615E-04$    | $1.4259E-02$     | $1.2400E-01$                    | $1.2398E-01$    |
| 1        | 0.01  | $16 \times 16$ | $4.3717E-01$     | $9.8775E-01$     | $1.9356E-01$                    |                  |
|          |       | $32 \times 32$ | $1.3610E-01$     | $5.8835E-01$     | $7.5767E-01$                    | $1.35$          |
|          |       | $64 \times 64$ | $3.6656E-02$     | $3.1398E-01$     | $3.2769E-02$                    | $1.21$          |
|          |       | $128 \times 128$ | $9.3985E-03$   | $1.6092E-01$     | $1.5592E-02$                    | $1.07$          |

Figure 14: The X-HDG solutions $u_{1h}$ (left), $u_{2h}$ (middle) and $p_{h}$ (right) at $\nu = 1, \alpha = 0$ and $128 \times 128$ triangular mesh: Example 5.5.

6 Conclusions

For the Darcy-Stokes-Brinkman interface problems, the proposed low order interface-unfitted X-HDG method is of optimal convergence and applies to curved domains with boundary-unfitted meshes. Numerical experiments have demonstrated the performance of the method.
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