On a family of Schreier graphs of intermediate growth associated with a self-similar group

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Abstract

For every infinite sequence \( \omega = x_1x_2\ldots \), with \( x_i \in \{0,1\} \), we construct an infinite 4-regular graph \( X_\omega \). These graphs are precisely the Schreier graphs of the action of a certain self-similar group on the space \( \{0,1\}^\infty \). We solve the isomorphism and local isomorphism problems for these graphs, and determine their automorphism groups. Finally, we prove that all graphs \( X_\omega \) have intermediate growth.

Keywords: self-similar group, Schreier graph, intermediate growth, local isomorphism.

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1 Introduction

In [CFS04], F. Fiorenzi, F. Scarabotti and the second named author of the present paper gave an exposition of some results of M. Gromov from [Gro99] on symbolic dynamics on infinite graphs. Let \( X = (V,E) \) be a graph of uniformly bounded valence. Let \( r \geq 0 \), \( u \in V \) and denote by \( B_r(u) = \{ w \in V : d(u,w) \leq r \} \) the ball of radius \( r \) centered at \( u \), where \( d : V \times V \to \mathbb{R}^+ \) is the geodesic distance on \( X \). Given two vertices \( u,v \in V \), one says that \( u \sim_r v \) if there exists a graph isomorphism \( \psi_r(u,v) : B_r(u) \to B_r(v) \) such that \( \psi_r(u) = v \). It is immediate that \( \sim_r \) is an equivalence relation on \( V \); the corresponding equivalence classes are called \( r \)-classes. Since \( X \) has uniformly bounded valence, for each integer \( r \geq 0 \) there exist only finitely many \( r \)-classes for the vertices of \( X \). The collection \( P(X) \) of all graph isomorphisms of the form \( \psi_r(u,v) \), where \( u \sim_r v \), \( r \geq 0 \), constitutes the pseudogroup of partial isometries of the graph \( X \). Gromov called such a pseudogroup of dense holonomy provided that for every \( r > 0 \) there exists a \( D_r > 0 \) such that for every \( u,v \in V \) there exists \( w \in V \) such that \( d(u,w) < D_r \) and \( u \sim_r w \). For example, if \( G \) is a finitely generated group, \( S \subset G \) is a finite symmetric generating subset, then the Cayley graph \( X = C(G,S) \) of \( G \) with respect to \( S \) is a regular graph of degree \( |S| \) (and thus of uniformly bounded valence). Moreover, for all \( u,v \in V \), the left multiplication by \( h = vu^{-1} \) yields a graph isomorphism \( B_r(u) \to B_r(v) \) such that \( u \leftrightarrow v \). It follows that for each \( r \geq 0 \) there exists a unique \( r \)-class in \( X \). In fact \( G \hookrightarrow \text{Aut}(X) \) and this constitutes a trivial example of such dense holonomy pseudogroups of isometries of a (bounded valence) graph. A. Žuk asked for non-trivial examples of such graphs \( X \), possibly with a trivial automorphism group \( \text{Aut}(X) \), and in [CFS04, Example 3.23] an explicit example is provided. Consider the Cayley graph \( X = C(\mathbb{Z},\{\pm 1\}) = (V,E) \) and add new edges according to the following recursive rule. We first connect all pairs of vertices of
the form \((2n, 2n + 2)\), where \(n \in \mathbb{Z}\). After this step, all vertices corresponding to even integers have degree 4. Consider now the remaining vertices of degree 2: these are the odd vertices. Note that 1 and \(-1\) are the two vertices of degree 2 which are closest to 0. We choose the vertex 1 and we then connect all pairs of vertices of the form \((2n + 1, 2n + 5)\), where \(n \in \mathbb{Z}\). This way, also the odd vertices which are congruent to 1 mod 4 now have degree 4. Observe that the vertices that still have degree 2 are the odd vertices which are congruent to 3 mod 4: in particular, the vertex of degree 2 which is closest to 0 is \(-1\). We then connect all pairs of vertices of the form \((2n - 1, 2n + 7)\), where \(n \in \mathbb{Z}\). This way, also the odd vertices which are congruent to \(-1\) mod 8 now have degree 4. And so on (see Section 2 for more details). The resulting graph \(X'\) (which in the present paper is denoted by \(X_{(10)^\infty}\)) is regular of degree 4, its pseudogroup of partial isometries has dense holonomy and, moreover, \(\text{Aut}(X')\) is trivial. This last result is easily deduced from the following fact. As one easily checks, the graph \(X'\) is generic, in the sense that for all \(u, v \in V(X')\) there exists \(r > 0\) such that \(u \nsim_r v\). As genericity is equivalent to the triviality of the automorphism group of the graph, this gives our claim. In [CFS04], it is also shown that \(X'\) is an amenable graph and it is observed, after a remark of the last named author of the present paper, that \(X'\) is the Schreier graph associated with the action of a self-similar group on the boundary of the rooted binary tree.

In [BH05], I. Benjamini and C. Hoffman considered a family of amenable graphs, called \(\omega\)-periodic graphs, whose construction is similar to that of \(X'\). In particular, their “basic example” corresponds to the graph that in the present paper is denoted by \(X_{0^\infty}\). They proved that this graph has intermediate growth, that is, it is superpolynomial and subexponential and they also proved, after a remark of L. Bartholdi, that this graph is an example of Schreier graph. They also provided examples within the family of \(\omega\)-periodic graphs having polynomial (resp. exponential) growth.

In the present paper, with each right-infinite sequence \(\omega = x_1x_2 \ldots \in \{0, 1\}^\infty\), we associate an infinite 4-regular graph \(X_\omega\). Moreover, after saying that two sequences \(\omega = x_1x_2 \ldots\) and \(\omega' = y_1y_2 \ldots\) are cofinal (resp. anticofoodinal) provided that there exists \(i_0\) such that \(x_i = y_i\) (resp. \(x_i = 1 - y_i\)), for every \(i \geq i_0\), we prove the following results:

- \(X_\omega \cong X_{\omega'}\) if and only if \(\omega\) and \(\omega'\) are either cofinal or anticofoodinal (Theorem 1);
- \(\text{Aut}(X_\omega)\) is trivial if \(\omega\) is neither cofinal nor anticofoodinal to \(0^\infty\), and \(\text{Aut}(X_\omega) = \mathbb{Z}/2\mathbb{Z}\) otherwise (Corollary 2);
- \(X_\omega\) and \(X_{\omega'}\) are locally isomorphic if and only if either both \(\omega\) and \(\omega'\) are cofinal or anticofoodinal with \(0^\infty\), or both \(\omega\) and \(\omega'\) are neither cofinal nor anticofoodinal with \(0^\infty\) (Theorem 3);
- \(X_\omega\) has dense holonomy and is generic if and only if \(\omega\) is neither cofinal nor anticofoodinal with \(0^\infty\) (Theorem 4);
- for each \(\omega\), the graph \(X_\omega\) is isomorphic to the orbital Schreier graph \(\Gamma_\omega\) of the word \(\omega\) under the action of a self-similar group \(G\) (Theorem 5);
- \(X_\omega\) has intermediate growth, and therefore is amenable, for all \(\omega\) (Theorem 6).

In the Appendix, a detailed study of finite Gelfand pairs associated with the action of the group \(G\) on each level of the rooted binary tree is presented. This leads in particular to a description of the decomposition of the corresponding permutations representation into
irreducible submodules and to an explicit expression for the associated spherical functions. The key step is to prove that the action of $G$ on each level of the tree is 2-point homogeneous.

Incidentally, this automatically gives the symmetry of the Gelfand pairs.

2 Definition of the graphs $X_\omega$ and the isomorphism problem

Consider the binary alphabet $\{0,1\}$, and let $\{0,1\}^\infty$ be the set of all (right-)infinite sequences $x_1x_2\ldots$ with $x_i \in \{0,1\}$. We associate an infinite 4-regular graph $X_\omega = (V_\omega, E_\omega)$ with each sequence $\omega \in \{0,1\}^\infty$. The vertex set $V_\omega$ of every graph $X_\omega$ is the set $\mathbb{Z}$ of integer numbers. The edge set $E_\omega$ depends on the sequence $\omega = x_1x_2\ldots$ and is defined as follows. For every $n \geq 1$, we set

$$a_n^\omega = x_1 + x_22 + x_32^2 + \cdots + x_{n-1}2^{n-2} - x_n2^{n-1} = \sum_{i=1}^{n} x_i2^{i-1} - 2^{n-1},$$

where $x = 1 - x$, for each $x \in \{0,1\}$. Notice that if $\omega = 0^\infty$ (resp. $\omega = 1^\infty$) one has $a_n^\omega = -2^{n-1}$ (resp. $a_n^\omega = 2^{n-1} - 1$) for all $n \geq 1$. In the general case, we get the inequalities

$$-2^{n-1} \leq a_n^\omega \leq 2^{n-1} - 1,$$

for every $n \geq 1$. Put $E_0^\omega = \{(z, z + 1) : z \in \mathbb{Z}\}$ and, for every $n \geq 1$, define

$$E_n^\omega = \{(2^n z - a_n^\omega, 2^n(z + 1) - a_n^\omega) : z \in \mathbb{Z}\}.$$

Then the edge set $E_\omega$ of the graph $X_\omega$ is given by the disjoint union $\bigsqcup_{n=0}^\infty E_n^\omega$, with possibly a loop rooted at the (unique) vertex which is not incident to any edge of $\bigsqcup_{n=1}^\infty E_n^\omega$.

The graph $X_\omega$ can be constructed step by step by reading consequently the letters of the binary sequence $\omega$ and adding the edges from the set $E_n^\omega$. Denote by $X_\omega^0$ the graph with the vertex set $\mathbb{Z}$ and the edge set $E_0^\omega$ and, for each $n \geq 1$, let $X_\omega^n$ be the graph with the vertex set $\mathbb{Z}$ and the edge set $\bigsqcup_{i=0}^{n-1} E_i^\omega$. Note that all vertices of the graph $X_\omega^0$ have degree 2, while the graph $X_\omega^n$, for $n \geq 1$, contains also vertices of degree 4. Suppose we have constructed the graph $X_\omega^{n-1}$ and we read the $n$-th letter $x_n$ of the sequence $\omega$. If $x_n = 0$, then we find the smallest positive integer which has degree 2, viewed as a vertex of the graph $X_\omega^{n-1}$ (the first vertex of degree 2 to the right from zero). If $x_n = 1$, then we find the largest nonpositive integer which has degree 2 as a vertex of $X_\omega^{n-1}$ (the first vertex of degree 2 to the left from zero). In both cases this integer number is precisely $-a_n^\omega$, see Figure 1. Then the graph $X_\omega^n$ is obtained from the graph $X_\omega^{n-1}$ by connecting all second consecutive vertices which are congruent to $-a_n^\omega$ modulo $2^n$ (they are of degree 2 in $X_\omega^{n-1}$). These vertices become the new vertices of degree 4 in the graph $X_\omega^n$ and the corresponding new edges constitute the set $E_n^\omega$.

In particular, the vertex of $X_\omega$ corresponding to the integer $-a_n^\omega$ is the closest vertex to 0 which is incident to an edge in $E_n^\omega$, in formulæ,

$$|a_n^\omega| = \min \left\{ |z| : z \in \mathbb{Z} \setminus \bigcup_{i=1}^{n-1} (2^i\mathbb{Z} - a_i^\omega) \right\}.$$ (2)

If the sequence $\omega$ contains both infinitely many 0’s and 1’s, then each vertex of the graph $X_\omega$ is incident to an edge of $\bigsqcup_{n=1}^\infty E_n^\omega$ and so there is no loop in $X_\omega$. Indeed, if there exists a vertex $v$ which is not incident to any edge of $\bigsqcup_{n=1}^\infty E_n^\omega$ and this vertex corresponds to a
Let us consider the isomorphism problem for the family of graphs $X_\omega$, with $\omega \in \{0,1\}^\infty$.

**Lemma 1.** Let $\omega, \omega' \in \{0,1\}^\infty$ and suppose that $\omega$ and $\omega'$ differ only at the $n$-th letter,

nonpositive (resp. positive) integer, then there exists $n_0 \geq 1$ such that $x_n = 0$ (resp. $x_n = 1$) for all $n \geq n_0$. On the other hand, if $\omega = 0^\infty$, then 0 is the unique vertex of $X_{0^\infty}$ which is not incident to any edge of $\coprod_{n=1}^\infty E_n^\omega$ and so there is a loop at 0. Similarly, if $\omega = 1^\infty$, then 1 is the unique vertex of $X_{1^\infty}$ which is not incident to any edge of $\coprod_{n=1}^\infty E_n^\omega$ and so there is a loop at 1. In the general case, if $\omega = x_1x_2\ldots x_n0^\infty$ then the graph $X_\omega$ has a loop at the vertex $-\sum_{i=1}^n x_i2^{i-1}$; similarly, if $\omega = x_1x_2\ldots x_n1^\infty$ then the graph has a loop at $1 - \sum_{i=1}^n (x_i - 1)2^{i-1}$.
Figure 3: The graph $X_{(10)\infty}$, with $a_{(10)\infty} = \frac{-1+(-2)^{n-1}}{3}$.

namely

$$\omega = x_1x_2\ldots x_{n-1}x_nx_{n+1}\ldots$$
$$\omega' = x_1x_2\ldots x_{n-1}\overline{x}_nx_{n+1}\ldots$$

Then the map $\Phi_n : V_\omega \rightarrow V_{\omega'}$ defined by the rule

$$\Phi_n(z) = z - (\overline{x}_n - x_n)2^{n-1},$$

for all $z \in V_\omega = \mathbb{Z}$, is a graph isomorphism.

Proof. It suffices to observe that, under our assumptions,

$$a_{i}^{\omega'} = a_{i}^{\omega} \quad \text{for each } 1 \leq i \leq n - 1$$

and

$$a_{i}^{\omega'} = a_{i}^{\omega} + (\overline{x}_n - x_n)2^{n-1} \quad \text{for each } i \geq n,$$

so that $\Phi_n$ preserves the adjacency relation between vertices. \qed

Lemma 2. Let $\omega = x_1x_2\ldots \in \{0,1\}^\infty$ and put $\overline{\omega} = \overline{x}_1\overline{x}_2\ldots$. Then the map $\Psi : V_\omega \rightarrow V_{\overline{\omega}}$ defined by the rule

$$\Psi(z) = -z + 1,$$

for all $z \in V_\omega = \mathbb{Z}$, is a graph isomorphism.

Proof. Since $x_i + \overline{x}_i = 1$ for each $i \geq 1$, it follows from [1] that

$$a_{i}^{\omega} + a_{i}^{\overline{\omega}} = 1 + 2 + 2^2 + \cdots + 2^{i-2} - 2^{i-1} = -1$$

for each $i \geq 1$ and so $\Psi$ preserves the adjacency relation between vertices. \qed

Two sequences $\omega = x_1x_2\ldots$ and $\omega' = y_1y_2\ldots$ over the alphabet $\{0,1\}$ are said to be cofinal (resp. anticofinal) if there exists $i_0$ such that $x_i = y_i$ (resp. $x_i = \overline{y}_i$), for every $i \geq i_0$. It is clear that cofinality is an equivalence relation. The corresponding equivalence classes are called the cofinality classes and we denote by $Cof(\omega)$ the cofinality class of $\omega \in \{0,1\}^\infty$.  

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Theorem 1. Two graphs $X_\omega$ and $X_{\omega'}$, with $\omega, \omega' \in \{0, 1\}^\infty$, are isomorphic if and only if the sequences $\omega$ and $\omega'$ are either cofinal or antico-final.

Proof. Let $\omega = x_1x_2\ldots$ and $\omega' = y_1y_2\ldots$ with $x_i, y_i \in \{0, 1\}$. First, consider the case when the sequences $\omega$ and $\omega'$ are cofinal. Then $\omega$ and $\omega'$ differ only at finitely many indices, say $i_1, i_2, \ldots, i_k$, i.e., $y_{i_j} = \pi_{i_j}$ for $j = 1, \ldots, k$, and $y_{i_l} = x_l$ for $l \not\in \{i_1, \ldots, i_k\}$. Then, the composition $\Phi = \Phi_{i_k} \circ \cdots \circ \Phi_{i_1}$ of the isomorphisms defined in (3) yields an isomorphism between the graphs $X_\omega$ and $X_{\omega'}$. If $\omega$ and $\omega'$ are antico-final, then $\omega$ and $\omega'$ are cofinal, and hence the graphs $X_\omega$ and $X_{\omega'}$ are isomorphic. Then the graphs $X_\omega$ and $X_{\omega'}$ are isomorphic by Lemma 2.

For the converse, suppose that the graphs $X_\omega$ and $X_{\omega'}$ are isomorphic. If the sequence $\omega$ is cofinal or antico-final with $0^\infty$, then there exists a vertex with a loop in the graph $X_\omega$. Hence the graph $X_{\omega'}$ must contain a vertex with a loop and thus $\omega'$ is cofinal or antico-final with $0^\infty$. So, in the sequel, we assume that both $\omega$ and $\omega'$ have infinitely many 0's and 1's.

The following remark is crucial. The edges in the set $E_\omega^0$ have a unique property, namely that for two consecutive edges $(z, z+1)$ and $(z+1, z+2)$, the vertices $z$ and $z+2$ are adjacent in the graph $X_\omega$ (by an edge in $E_\omega^1$) if either $z$ is even and $\omega = 1x_2x_3\ldots$, or if $z$ is odd and $\omega = 0x_2x_3\ldots$. This property does not hold for the edges in $E_\omega^i$ with $i \geq 1$, because the graph is 4-regular and each vertex is incident to exactly two edges in $E_\omega^0$ and to only other two edges in $E_\omega^0$. Then it is easy to see that the edge set $E_\omega^0$ must be preserved under graph isomorphisms. This implies that a graph isomorphism $\varphi : V_\omega \rightarrow V_{\omega'}$ is either of the form $z \mapsto z + t$ or of the form $z \mapsto -z + t$, for a fixed $t \in \mathbb{Z}$.

Suppose that $\varphi : V_\omega \rightarrow V_{\omega'}$ is an isomorphism of the form $\varphi(z) = z + t$, for all $z \in \mathbb{Z}$. We will show that the sequence $\omega'$ can be eventually recovered by using the property (3). This property implies that the vertex $\varphi(-a_n^\omega) = -a_n^\omega + t$ of the graph $X_{\omega'}$ is the closest one to the vertex $\varphi(0) = t$ that is incident to an edge of $E_{\omega'}^0$. We set

$$I_t = \{ n \in \mathbb{N} : 1 - 2^{n-1} \leq -a_n^\omega + t \leq 2^{n-1} \}.$$

We claim that the set $I_t$ satisfies the following properties:

1. the set $I_t$ is nonempty;
2. if $n \in I_t$, then $n + k \in I_t$ for all $k \geq 1$.

Proof of 1. Case $t > 0$. There exists a unique integer $n_1 \geq 1$ such that $2^{n_1-1} \leq t < 2^{n_1}$. We choose the smallest $i \geq 1$ such that $x_{n_1+i} = 1$ (the index $i$ is well defined since $\omega$ contains infinitely many 1's). With this choice, one has

$$a_{n_1+i+1}^\omega \leq 1 + 2 + \cdots + 2^{n_1+i-1} = 2^{n_1+i} - 1$$

and so $-a_{n_1+i+1}^\omega \geq 1 - 2^{n_1+i}$. Then a fortiori $-a_{n_1+i+1}^\omega + t \geq -2^{n_1+i} + 1$, because $t$ is positive. On the other hand, we also have $a_{n_1+i+1}^\omega \geq 2^{n_1+i-1} - 2^{n_1+i}$ by construction and so $-a_{n_1+i+1}^\omega \leq 2^{n_1+i} - 2^{n_1+i-1}$. Hence,

$$-a_{n_1+i+1}^\omega + t \leq 2^{n_1+i} - 2^{n_1+i-1} + 2^{n_1} \leq 2^{n_1+i},$$

because $-2^{n_1+i-1} + 2^{n_1} \leq 0$. This implies that $n_1 + i + 1 \in I_t$. 


Case \( t < 0 \). There exists a unique integer \( n_1 \geq 1 \) such that \(-2^{n_1} < t \leq -2^{n_1-1}\). We choose the smallest \( i \geq 1 \) such that \( x_{n_1+i} = 0 \) and \( x_{n_1+i+1} = 1 \) (the index \( i \) is well defined since \( \omega \) has infinitely many 0’s and 1’s). With this choice, one has

\[
a_{n_1+i+1}^\omega \leq 1 + 2 + \cdots + 2^{n_1+i-2} = 2^{n_1+i-1} - 1
\]

and so \(-a_{n_1+i+1}^\omega \geq -2^{n_1+i-1} + 1\). This implies

\[
-a_{n_1+i+1}^\omega + t \geq -2^{n_1+i-1} + 1 - 2^{n_1} \geq -2^{n_1+i} + 1,
\]

because \(-2^{n_1} \geq -2^{n_1+i-1}\). Moreover, we also have \( a_{n_1+i+1}^\omega \geq -2^{n_1+i} \) by construction and so \(-a_{n_1+i+1}^\omega \leq 2^{n_1+i}\). Then a fortiori \(-a_{n_1+i+1}^\omega + t \leq 2^{n_1+i} - 1\), because \( t \leq -1 \). Hence, \( n_1 + i + 1 \in I_t \).

**Proof of 2.** Suppose that \( n \in I_t \), i.e. \( 1 - 2^{n-1} \leq -a_n^\omega + t \leq 2^{n-1} \). It follows from (i) that

\[
a_{n+k}^\omega - a_n^\omega = 2^{n-1} + x_{n+1}2^n + \cdots + x_{n+k-1}2^{n+k-2} - x_{n+k}2^{n+k-1}
\]

and so

\[
a_{n+k}^\omega - a_n^\omega \leq 2^{n-1} + 2^n + \cdots + 2^{n+k-2} = 2^{n-1}(2^k - 1) = 2^{n+k-1} - 2^{n-1}.
\]

This gives

\[
-a_{n+k}^\omega + t \geq -a_n^\omega - 2^{n+k-1} + 2^{n-1} + t \geq -2^{n+k-1} + 1.
\]

On the other hand, it follows from (ii) that \( a_{n+k}^\omega - a_n^\omega \geq 2^{n-1} - 2^{n+k-1} \) and so

\[
-a_{n+k}^\omega + t \leq -a_n^\omega + t - 2^{n-1} + 2^{n+k-1} \leq 2^{n+k-1}.
\]

The properties 1 and 2 are proved.

Let \( n_0 = n_0(t) \) be the least element of the set \( I_t \). Then it follows from the definition of \( I_t \) and the property 2 that

\[
-a_n^\omega = -a_n^\omega' = -a_n^\omega + t \quad \text{for all } n \geq n_0.
\]

(5)

In other words, there exists \( n_0 \) depending on the translation \( t \) such that the closest integer to \( \varphi(0) \) which is incident to an edge of \( E_\omega^n \) coincides with the closest integer to 0 which is incident to an edge of \( E_{\omega'}^n \), for all \( n \geq n_0 \). We will rewrite (ii) in the form

\[
a_n^\omega = a_{n-1}^\omega + 2^{n-2}(2x_n - 1), \quad a_n^\omega' = a_{n-1}^\omega' + 2^{n-2}(2y_n - 1).
\]

Then (ii) implies

\[
0 = -a_n^\omega + t + a_n^\omega' = (-a_{n-1}^\omega + t + a_{n-1}^\omega') + 2^{n-1}(y_n - x_n)
\]

\[
= 2^{n-1}(y_n - x_n),
\]

for all \( n > n_0 \). Hence \( x_n = y_n \) for all \( n > n_0 \), and therefore the sequences \( \omega \) and \( \omega' \) are cofinal.

Suppose now that \( \varphi : V_\omega \rightarrow V_{\omega'} \) is an isomorphism of the form \( \varphi(z) = -z + t \). Consider the isomorphism \( \Psi : V_{\omega'} \rightarrow V_{\overline{\omega}} \) defined in Lemma 2. Then the composition

\[
\Psi \circ \varphi : V_\omega \rightarrow V_{\overline{\omega}}
\]

is an isomorphism of the form \( z \mapsto z + t \) between the graphs \( X_\omega \) and \( X_{\overline{\omega}} \). Then it follows from the first part of the proof that \( \omega \) and \( \overline{\omega} \) are cofinal, and this implies that \( \omega \) and \( \omega' \) are anticofinal. \( \Box \)
Corollary 2. If \( \omega \) is either cofinal or anticofinal with \( 0^\infty \) then \( \text{Aut}(X_\omega) \cong \mathbb{Z}/2\mathbb{Z} \), otherwise the automorphism group \( \text{Aut}(X_\omega) \) is trivial.

Proof. Suppose first that the sequence \( \omega = x_1x_2\ldots \in \{0,1\}^\infty \) contains infinitely many 0’s and 1’s. Then an automorphism \( \varphi \) of the graph \( X_\omega \) is necessarily of the form \( \varphi(z) = z + t \). It follows from [4], with \( \omega' = \omega \), that \( t = 0 \) and therefore \( \varphi = \text{id}_{X_\omega} \). Hence the group \( \text{Aut}(X_\omega) \) is trivial.

Now suppose that \( \omega \) contains either finitely many 0’s or 1’s. In both cases, the sequence \( \omega \) is either cofinal or anticofinal with the infinite word \( 0^\infty \). Hence the graph \( X_\omega \) is isomorphic to the graph \( X_{0^\infty} \) and so we are only left to show that \( \text{Aut}(X_{0^\infty}) \cong \mathbb{Z}/2\mathbb{Z} \). Take an automorphism \( \varphi \in \text{Aut}(X_{0^\infty}) \). Since the vertex 0 is the unique vertex with a loop, we get \( \varphi(0) = 0 \). As in the proof of Theorem [4], one can show that \( \varphi \) preserves the edge set \( E_0^\omega \). It follows that \( \varphi(1) \) is either either 1 or \( -1 \) and so \( \varphi \) is either the identity map or the inversion \( z \mapsto -z \). Hence \( \text{Aut}(X_\omega) \cong \text{Aut}(X_{0^\infty}) \cong \mathbb{Z}/2\mathbb{Z} \).

Recall that we denoted by \( X_\omega^n \) the graph with the vertex set \( \mathbb{Z} \) and the edge set \( \prod_{k=0}^{n} E_\omega^k \).

The following proposition solves the isomorphism problem for these graphs.

Proposition 3. The graphs \( X_\omega^n \) and \( X_{\omega'}^n \) are isomorphic for all \( \omega, \omega' \in \{0,1\}^\infty \) and \( n \geq 1 \). The automorphism group \( \text{Aut}(X_\omega^n) \) is isomorphic to the infinite dihedral group \( D_\infty \) for every \( n \geq 1 \).

Proof. As we mentioned when constructing these graphs, the graph \( X_\omega^n \) only depends on the first \( n \) letters of the sequence \( \omega \). Let \( \omega \) and \( \omega' \) start with letters \( x_1x_2\ldots x_n \) and \( y_1y_2\ldots y_n \), respectively. Then the translation map \( \Phi : V(X_\omega^n) \to V(X_{\omega'}^n) \) defined by

\[
\Phi(z) = z + (x_1 - y_1) + (x_2 - y_2)2 + \cdots + (x_n - y_n)2^{n-1},
\]

for all \( z \in V(X_\omega^n) = \mathbb{Z} \), yields a graph isomorphism between \( X_\omega^n \) and \( X_{\omega'}^n \) (see Lemma [4].

As for the corresponding automorphism groups, notice that the translation \( \alpha : z \mapsto z + 2^n \) and the inversion \( \beta : z \mapsto -z \) are automorphisms of the graph \( X_{0^\infty}^n \). As in the proof of Theorem [4], one can show that every automorphism preserves the edge set \( E_\omega^0 \) and hence \( \text{Aut}(X_{0^\infty}^n) \) is a subgroup of \( \text{Aut}(\mathbb{Z}) \cong D_\infty \). Since \( \alpha \) and \( \beta \) generate \( D_\infty \), we deduce that \( \text{Aut}(X_{0^\infty}^n) \) is also isomorphic to the group \( D_\infty \). □

3 Dense holonomy and the local isomorphism problem for the graphs \( X_\omega \)

In this section we consider the local structure of the graphs \( X_\omega \), with \( \omega \in \{0,1\}^\infty \). We construct a sequence of finite 4-regular graphs \( X_n = (V_n, E_n) \), for \( n \geq 1 \), which will be used to approximate the graphs \( X_\omega \). For each \( n \), the vertices of the graph \( X_n \) are the residues \( \mathbb{Z}/2^n\mathbb{Z} = \{0,1,\ldots,2^n - 1\} \) modulo \( 2^n \), and the edges are \( (z \mod 2^n, z + 1 \mod 2^n) \) and

\[
(2^k z + 2^{k-1} \mod 2^n, 2^k(z + 1) + 2^{k-1} \mod 2^n),
\]

for every \( z \in \mathbb{Z} \) and \( k \geq 1 \). Every graph \( X_n \) is 4-regular with two loops at the vertices 0 and \( 2^{n-1} \). The graphs \( X_1, X_2, \) and \( X_3 \) are shown in Figure [4].

Take a sequence \( \omega \in \{0,1\}^\infty \) and consider the graph \( X_\omega \). There is a natural sequence of quotients \( X_\omega \mod 2^n \) of the graph \( X_\omega \) when we factorize the vertex set \( \mathbb{Z} \) modulo \( 2^n \). The
vertices of the graph \(X_\omega \mod 2^n\) are the residues modulo \(2^n\) and two vertices \(z_1 + 2^n\mathbb{Z}\) and \(z_2 + 2^n\mathbb{Z}\) are adjacent if the integers \(z_1 + 2^n t_1\) and \(z_2 + 2^n t_2\) are adjacent in the graph \(X_\omega\), for some \(t_1, t_2 \in \mathbb{Z}\). For example, the graph \(X_{(10)^\infty} \mod 8\) is shown in Figure 5.

**Proposition 4.** For every \(\omega \in \{0, 1\}^\infty\), the quotient graph \(X_\omega \mod 2^n\) is isomorphic to the graph \(X_n\).

**Proof.** Let \(\omega\) start with the letters \(x_1 x_2 \ldots x_n\), with \(x_i \in \{0, 1\}\). Then, as in the proof of Proposition 3, we have that the map \(\Phi : V(X_\omega \mod 2^n) \to V_n\) defined by

\[
\Phi(z) = z + x_1 + x_2 2 + \cdots + x_n 2^{n-1} \mod 2^n,
\]

for all \(z \in V(X_\omega \mod 2^n) = \mathbb{Z}/2^n\mathbb{Z}\), yields a graph isomorphism between the graphs \(X_\omega \mod 2^n\) and \(X_n\). \(\square\)
Moreover, the graphs $X_\omega$ can be recovered as limits of suitable sequence of pointed graphs $(X_n,v_n)$ in the local topology. To be more precise, recall that a sequence of pointed graphs $(Y_n,v_n)$ converges to a pointed graph $(Y,v)$ in the Gromov-Hausdorff metric if, for every ball $B_Y(v,r)$ in the graph $Y$ with center $v$ and radius $r$, there exists an isomorphism between $B_Y(v,r)$ and $B_{Y_n}(v_n,r)$ that maps $v$ to $v_n$, for all $n$ large enough.

**Theorem 5.** For every sequence $\omega = x_1x_2\ldots \in \{0,1\}^\infty$, one has the convergence

$$
(X_\omega,0) = \lim_{n \to \infty} (X_n,x_1 + x_2 2 + \cdots + x_n 2^{n-1}).
$$

Moreover, every limit of a convergent sequence of pointed graphs $(X_n,v_n)$ is isomorphic to a graph $X_\omega$, for some $\omega \in \{0,1\}^\infty$.

**Proof.** Let $I_m$ be the subgraph of the graph $X_\omega$ induced by the vertices in the interval $(-2^m,2^m)$. Choose $n$ such that, if a vertex $v$ of $I_m$ is adjacent to some vertex $u$ of $X_\omega$, then their difference $|v-u|$ is less than $2^n$. Then the natural projection $I_m \to I_n \mod 2^n$ is an isomorphism. Composing this projection with the isomorphism from the proof of Proposition [4] we get the isomorphism between $I_m$ and the induced subgraph of $X_n$ that maps $0$ to $x_1 + x_2 2 + \cdots + x_n 2^{n-1}$. Since the subgraphs $\{I_m\}_{m \geq 1}$ cover the graph $X_\omega$, we deduce (6).

For the second statement, consider a convergent sequence of pointed graphs $(X_n,v_n)$. Using the diagonal argument construct a sequence $\omega = x_1x_2\ldots \in \{0,1\}$ such that for every $m$ the equality

$$
x_1 + x_2 2 + \cdots + x_m 2^{m-1} \equiv v_n \mod 2^m
$$

holds for infinitely many $n$. By passing to a subsequence, if necessary, we can assume that this equality holds for all $n$. Then the sequence $(X_n,v_n)$ converges to the graph $(X_\omega,0)$ by the first statement.

In particular, locally at every point the graph $X_\omega$ looks like a part of some graph $X_n$. This can be used to classify the graphs $X_\omega$ up to local isomorphisms. Two graphs $X$ and $Y$ are said to be **locally isomorphic** if, for every ball in one graph, there exists an isomorphic ball in the other graph.

**Theorem 6.** Let $\omega,\omega' \in \{0,1\}^\infty$. The graphs $X_\omega$ and $X_{\omega'}$ are locally isomorphic if and only if either both $\omega$ and $\omega'$ are cofinal or anticoinal with $0^\infty$, or both $\omega$ and $\omega'$ are neither cofinal nor anticoinal with $0^\infty$. In particular, the family of graphs $X_\omega$, for $\omega \in \{0,1\}^\infty$, contains precisely two graphs up to local isomorphisms, for example, $X_{0^\infty}$ and $X_{1(10)\infty}$.

**Proof.** If $\omega$ and $\omega'$ are cofinal or anticoinal with $0^\infty$, then the graphs $X_\omega$ and $X_{\omega'}$ are even isomorphic by Theorem [4].

If $\omega$ contains both infinitely many $0$'s and $1$'s, while $\omega'$ is either cofinal or anticoinal with $0^\infty$, then the graph $X_{\omega'}$ contains a vertex with a loop in contrast to the graph $X_\omega$. Hence, these graphs are not locally isomorphic.

We are left to consider the case when both $\omega$ and $\omega'$ contain infinitely many $0$'s and $1$'s. Let $\omega = x_1x_2\ldots$ and $\omega' = y_1y_2\ldots$. Consider the induced subgraph $I_n$ of the graph $X_\omega$, whose vertices are the integers in the interval $(-2^{n-1},2^{n-1})$. Then the map $\Phi$ from the proof of Proposition [4] is an isomorphism between $I_n$ and the induced subgraph of $X_{\omega'}$, whose vertices are the integers in the interval $(z_0-2^{n-1},z_0+2^{n-1})$ with $z_0 = (x_1-y_1) + (x_2-y_2)2 + \cdots + (x_n-y_n)2^{n-1}$. Since the subgraphs $\{I_n\}_{n \geq 1}$ eventually cover any ball, the graphs $X_\omega$ and $X_{\omega'}$ are locally isomorphic. 

\[\square\]
Let $X$ be a graph of uniformly bounded valence, and consider all balls $B(v, r)$ of radius $r$ in $X$. We consider the isomorphism relation on the balls as on pointed graphs, where $B(v, r)$ and $B(u, r)$ are isomorphic if there exists a graph isomorphism $B(v, r) \rightarrow B(u, r)$ mapping $v$ to $u$. Note that, since $X$ has uniformly bounded valence, the set $T_X(r)$ of isomorphism classes of pointed $r$-balls is finite. We then define the $r$-type of a vertex $v$ of $X$ as the element $\alpha(v, r) \in T_X(r)$ representing the ball $B(v, r)$. Following Gromov \cite{Gro99}, we say that the graph $X$ has dense holonomy if for any radius $r$ there exists $R = R(r)$ such that every $R$-ball in $X$ contains vertices of each $r$-type. Equivalently, for any radius $r$ there exists $R = R(r)$ such that for every $r$-type $\alpha \in T_X(r)$ the balls $B(v, R)$ at the vertices $v$ of type $\alpha(v, r) = \alpha$ cover the whole of the graph $X$

**Theorem 7.** The graph $X_\omega$, with $\omega \in \{0, 1\}^\infty$, has dense holonomy if and only if the sequence $\omega$ has both infinitely many 0’s and 1’s.

**Proof.** If $\omega$ is cofinal or anticoinal with $0^\infty$, then the graph $X_\omega$ contains a unique vertex with loop, and hence it cannot have dense holonomy.

Suppose that $\omega$ has infinitely many 0’s and 1’s. Given $r \geq 0$, consider the finite set $T_{X_\omega}(r)$ and let $z_1, z_2, \ldots, z_m$ be the representatives for the corresponding $r$-types. Choose $n$ large enough so that every ball $B(z_i, r + 1)$ is contained in the interval $(-2^n - 1, 2^n - 1)$. Then the $r$-type of each vertex $z_i$ coincides with its $r$-type as a vertex of the graph $X_\omega \mod 2^{n+1}$. It follows that $\alpha(z_i + k2^{n+1}, r) = \alpha(z_i, r)$ for all $k \in \mathbb{Z}$, and the property in the definition of dense holonomy holds with $R = 2^n$.

Finally, we say that a graph $X$ is generic if distinct vertices $u, v$ of $X$ have distinct $r$-types $\alpha(u, r) \neq \alpha(v, r)$ for some $r$. We observe that genericity is equivalent to the triviality of the automorphism group. Indeed, if the balls $B(u, r)$ and $B(v, r)$ are not isomorphic as pointed graphs, then there is no automorphism $\varphi \in Aut(X)$ that maps $v$ to $u$. Therefore, if the graph $X$ is generic, then the automorphism group $Aut(X)$ is trivial. Conversely, suppose that two distinct vertices $v$ and $u$ have the same $r$-types, for all $r \geq 0$. Then, for every $r$, there exists a graph isomorphism $\varphi_r : B(v, r) \rightarrow B(u, r)$ that maps $v$ to $u$. Note that the restrictions $\varphi_r|_{B(v, r')}, B(v, r') \rightarrow B(u, r')$, with $r' \leq r$, are also isomorphisms and that there are only finitely many graph isomorphisms $B(v, r') \rightarrow B(u, r')$. By a compactness argument, there exists an isomorphism $\varphi \in Aut(X)$ such that $\varphi(v) = u$, and therefore $Aut(X)$ is nontrivial. Hence, Corollary \ref{cor:genericity} implies that the graph $X_\omega$ for $\omega \in \{0, 1\}^\infty$ is generic if and only if the sequence $\omega$ has both infinitely many 0’s and 1’s.

### 4 The graphs $X_\omega$ as Schreier graphs of a self-similar group

A faithful action of a group $G$ on the space $\{0, 1\}^\infty$ is called self-similar if, for every $g \in G$ and every $x \in \{0, 1\}$, there exists $h \in G$ and $y \in \{0, 1\}$ such that

$$g(x\omega) = yh(\omega),$$

for every sequence $\omega \in \{0, 1\}^\infty$. In this case, the element $h$ is called the restriction of $g$ at $x$ and is denoted by $h = g|_x$. We also get the action of $G$ on $\{0, 1\}$, where $y$ is the image of $x$ under $g$. Then, every element $g \in G$ can be uniquely given by the tuple $(g|_0, g|_1)\pi$, where $\pi \in \text{Sym}(\{0, 1\})$ is the permutation induced by $g$ on $\{0, 1\}$. Inductively, we can define the
action of $G$ on the set $\{0,1\}^n$, and the restriction $g|_{x_1x_2...x_n} = (((g|_{x_1}|_{x_2})...)|_{x_n}$, for any $x_i \in \{0,1\}$.

If a group $G$ acts self-similarly on the space $\{0,1\}^\infty$, it can also be regarded as an automorphism group of the rooted binary tree $T_2$ (see Figure 6). In fact, the $2^n$ vertices of the $n$-th level of the tree can be identified with the words in $\{0,1\}^n$, for each $n \geq 1$ (the root of the tree is identified with the empty word $\emptyset$). The elements of the boundary $\partial T_2$ of the tree can be identified with the (right-)infinite binary words, i.e., the elements of $\{0,1\}^\infty$. In particular, for every automorphism $g \in G$ whose self-similar representation is $g = (g|_0, g|_1)\pi$, the permutation $\pi \in Sym(\{0,1\})$ describes the action of $g$ on the first level of the tree, and $g|_i$ is its restriction on the subtree rooted at the vertex $i$ of the first level, with $i \in \{0,1\}$. More generally, $g|_{x_1...x_n}$ is the restriction of the action of $g$ to the subtree rooted at the vertex $x_1...x_n$ of the $n$-th level of $T_2$. Observe that such a subtree is isomorphic to the whole tree $T_2$. Then, the property of self-similarity means that these restrictions are elements of $G$.

Consider the self-similar group $G$ generated by the transformations $a$ and $b$ of the set $\{0,1\}^\infty$, whose actions satisfy the following recursive rules

$$
\begin{align*}
a(0x_2x_3...) &= 1x_2x_3..., & b(0x_2x_3...) &= 0b(x_2x_3...), \\
a(1x_2x_3...) &= 0a(x_2x_3...), & b(1x_2x_3...) &= 1a(x_2x_3...),
\end{align*}
$$

for all $x_i \in \{0,1\}$. Using restrictions, the generators $a$ and $b$ can be written recursively as

$$
\begin{align*}
a &= (e, a)\sigma, & b &= (b, a),
\end{align*}
$$

where $\sigma$ is the transposition $(0,1)$, and $e$ is the identity transformation.

The group $G$ is the simplest example of a group generated by a polynomial but not bounded automaton (see definition in [Sid00] or [Bon07, Chapter IV], the generating automaton of $G$ is shown in Figure 7).

Moreover, all the groups generated by polynomial and not bounded automata minimizing the sum of the number of states and the number of letters are isomorphic to the group $G$. 

---

**Figure 6: The rooted binary tree $T_2$.**
Indeed, all such generating automata have 3 states over an alphabet with 2 letters, and one state defines the trivial automorphism. Up to passing to inverses of generators, permuting the states of the automaton, permuting letters of the alphabet, there are only 3 such automata, whose states satisfy the following recursions:

\[
\begin{align*}
  a &= (e,a)\sigma \\
  b &= (b,a) \\
  a_1 &= (e,a_1)\sigma \\
  b_1 &= (b_1,a_1)\sigma \\
  a_2 &= (e,a_2)\sigma \\
  b_2 &= (a_1,b_2)\sigma.
\end{align*}
\]

Then \(a = a_1 = a_2, b_1a = (b_1a,a), a^{-1}b_2 = (a^{-1}b_2,a)\). The transformations \(b, b_1, a\), and \(a^{-1}b_2\) satisfy the same recursion and thus are all equal. Hence \(\{a, b\}, \{a_1, b_1\}\), and \(\{a_2, b_2\}\) are just different generating sets of the group \(G\).

The action of the transformation \(a = (e,a)\sigma\) on the space \(\{0,1\}^\infty\) corresponds to the addition of 1 to dyadic integers \(\mathbb{Z}_2\), when the sequence \(x_1x_2x_3\ldots\) is identified with the binary integer \(x_1 + x_22 + x_32^2 + \ldots\); for this reason, \(a\) is called the (binary) adding machine. Indeed, \(a(x_1x_2\ldots) = y_1y_2\ldots\), with \(x_i, y_i \in \{0,1\}\), if and only if

\[
1 + x_1 + x_22 + \ldots + x_n2^{n-1} + \ldots = y_1 + y_22 + \ldots + y_n2^{n-1} + \ldots.
\]

In particular, the action of \(a\) is transitive on every set \(\{0,1\}^n\), and hence the group \(G\) also acts transitively on \(\{0,1\}^n\), for every \(n \geq 1\). This is expressed by saying that the action of \(G\) is level-transitive on the tree. In the Appendix, it is also shown that \(G\) is self-replicating (recurrent) and regular weakly branch over its commutator subgroup.

**Proposition 8.** The union of the cofinality classes \(\text{Cof}(0^\infty) \cup \text{Cof}(1^\infty)\) forms one orbit of the action of \(G\) on \(\{0,1\}^\infty\). Any other orbit consists of precisely one cofinality class.

**Proof.** The orbits of the action of the adding machine \(a\) on the ring \(\mathbb{Z}_2\) of dyadic integers precisely correspond to the cofinality classes in the statement. Hence, it is sufficient to prove that the generator \(b\) preserves these orbits. To achieve this, notice that if \(b(x_1x_2\ldots) = y_1y_2\ldots\) and we take the first position \(n\) with \(x_n \neq y_n\), then \(b|_{x_1x_2\ldots x_{n-1}} = a\) and \(a(x_nx_{n+1}\ldots) = y_ny_{n+1}\ldots\).

Every action of a finitely generated group can be described by the associated Schreier graph. The Schreier graph \(\Gamma_n\) of the action of the group \(G\) on the set \(\{0,1\}^n\) is the graph with the set of vertices \(\{0,1\}^n\) and there is an edge between \(v\) and \(s(v)\) labeled by \(s\), for every \(v \in \{0,1\}^n\) and \(s \in \{a, b\}\). For a point \(\omega \in \{0,1\}^\infty\) the (orbital) Schreier graph \(\Gamma_\omega\) of \(\omega\) under the action of \(G\) is the graph, whose vertex set is the orbit \(G(\omega)\) of \(\omega\) under the action of \(G\), 

![Figure 7: The generating automaton for the group G.](image-url)
and there is an edge between every two vertices $v$ and $s(v)$ labeled by $s$ for every $s \in \{a, b\}$. It follows from Proposition 8 that the vertex set of the graph $\Gamma_{0^\infty}$ is the union $\text{Cof}(0^\infty) \cup \text{Cof}(1^\infty)$; every other graph $\Gamma_\omega$ has the set of vertices $\text{Cof}(\omega)$. All the Schreier graphs $\Gamma_n$ and $\Gamma_\omega$ are connected. For every $\omega = x_1x_2 \ldots x_n \in \{0, 1\}^\infty$, the sequence of the pointed Schreier graphs $(\Gamma_n, x_1x_2 \ldots x_n)$ converges in the local topology on pointed graphs to the pointed graph $(\Gamma_\omega, \omega)$ (see [BGN03, Proposition 7.2] and [GZ93]). Schreier graphs of self-similar actions of groups have largely studied from the viewpoint of spectral computations, growth, amenability, topology of Julia sets [BG00, Bon07, Bon11, GN05, DDMN10, GS06, GS08].

It is shown in [BH05] that the graph $X_{0^\infty}$ is isomorphic to the Schreier graph of the infinite word $0^\infty$ under the action of the group $G$. We generalize this analysis by the following statement.

**Theorem 9.** The map \( \varphi : V(\Gamma_\omega) \to V_\omega \) defined by the rule

\[ \varphi(a^m(\omega)) = m \quad \text{for} \quad m \in \mathbb{Z} \]

is a graph isomorphism for every $\omega \in \{0, 1\}^\infty$.

The map $\varphi_n : V(\Gamma_n) \to V_n$ defined by the rule

\[ \varphi_n(a^m(0^n)) = m \quad \text{for} \quad m = 0, 1, \ldots, 2^n - 1, \]

is a graph isomorphism for every $n \geq 1$.

**Proof.** We will only prove the first statement, the second is similar.

The map $\varphi$ is well defined and bijective because the action of $a$ is faithful and transitive on every orbit. We need to show that $\varphi$ preserves the adjacency relation. First, if two vertices of the graph $\Gamma_\omega$ are joined by an edge labeled by $a$, then the corresponding binary words differ by $\pm 1$ and so they are mapped to vertices of the graph $X_\omega$ which are connected by an edge of $E_\omega$. We are left to consider edges labeled by $b$. The actions of $a$ and $b$ have the following properties:

\[ b(0^{n-1}1\omega) = 0^{n-1}1a(\omega) \quad \text{and} \quad a^{2^n}(v\omega) = va(\omega), \]

for all $v \in \{0, 1\}^n$ and $\omega \in \{0, 1\}^\infty$. Let $\omega = x_1x_2 \ldots x_n\omega'$, with $x_i \in \{0, 1\}$. Using the correspondence with binary numbers and the definition of $a_n^\omega$, we get

\[ x_1 + x_22 + \cdots + x_n2^{n-1} + \omega'2^n - a_n^\omega = 2^{n-1} + \omega'2^n \quad \Rightarrow \quad a^{-a_n^\omega}(\omega) = 0^{n-1}1\omega'. \]

Hence

\[ b(a^{2^n-k-a_n^\omega}(\omega)) = b(0^{n-1}1a^k(\omega')) = 0^{n-1}1a^{k+1}(\omega') = a^{2^n(k+1)-a_n^\omega}(\omega), \]

which corresponds to the edge $(2^n k - a_n^\omega, 2^n (k + 1) - a_n^\omega)$ of the graph $X_\omega$.

We can apply the results from the previous section to the graphs $\Gamma_\omega$.

**Corollary 10.** Let $\omega, \omega' \in \{0, 1\}^\infty$. The orbital Schreier graphs $\Gamma_\omega$ and $\Gamma_{\omega'}$ are isomorphic if and only if $\omega$ and $\omega'$ are cofinal or anticofinal.

If $\omega$ is eventually constant (and so cofinal or anticofinal with $0^\infty$), then $\Gamma_\omega$ and $\Gamma_{\omega'}$ are isomorphic if and only if also $\omega'$ is eventually constant and in this case $\Gamma_\omega = \Gamma_{\omega'}$, because $\omega$ and $\omega'$ belong to the same orbit. If $\omega$ contains infinitely many 0’s and infinitely many 1’s, then $\Gamma_\omega = \Gamma_{\omega'}$ if $\omega'$ is cofinal with $\omega$ (they are in the same orbit), and $\Gamma_\omega \cong \Gamma_{\omega'}$ if $\omega'$ is anticofinal with $\omega$ (they are in different orbits).
Corollary 11. The set \( \{ \Gamma_\omega \}_{\omega \in \{0,1\}^\infty} \) contains uncountably many isomorphism classes of graphs. The class of \( \Gamma_0^\infty \) contains only one graph; any other class contains two isomorphic graphs (corresponding to two different orbits).

Moreover, all graphs \( \Gamma_\omega \) except \( \Gamma_0^\infty \) are locally isomorphic.

Let us list at the end of this section known properties of the group \( G \). First of all, we observe that the monoid generated by \( a \) and \( b \) is free, and hence \( G \) has exponential growth (see [BGK+08] for the proof of these properties, where the group \( G \) appears under the number 929).

It is proved by S. Sidki in [Sid04] that groups generated by polynomially growing automata do not have free non-abelian subgroups, which implies that \( G \) has no free non-abelian subgroups. For a shorter proof of S. Sidki’s result (for the case of locally finite trees) see [Nek10].

In [AAV] it is shown that the class of linear-activity automata groups is contained in the class of amenable groups. In particular, the group \( G \), which is discussed there in the example following (the statement of) Theorem 1 (where the generator \( a \) (resp. \( b \)) corresponds to our generator \( b \) (resp. \( a^{-1} \))) and which is called the long-range group, is amenable. This answered a question posed by the fourth named author in the Kourovka notebook [Kou06], Question 16.74, and also in Guido’s book of conjectures [Gui08], Conjecture 35.9.

Note that the amenability of \( G \) provides another proof of the fact that \( G \) contains no free nonabelian subgroups.

5 Growth of the graphs \( X_\omega \)

Let \( X \) be a connected graph of uniformly bounded valence. The growth function of \( X \) with respect to one of its vertices \( v \in V(X) \) is defined by \( \gamma_v(n) = |B(v,n)| \). In order to avoid dependence on a vertex, one introduces the equivalence relation on the growth functions. Given two functions \( f, g : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\} \) we say that \( f \preceq g \) if there exists a constant \( C > 0 \) and an integer \( n_0 \) such that \( f(n) \leq g(Cn) \) for all \( n \geq n_0 \), and then \( f \) and \( g \) are called equivalent \( f \sim g \) if \( f \preceq g \) and \( g \preceq f \). The equivalence class of a function is called its growth.

Then, for any two vertices of the graph \( X \), the respective growth functions are equivalent, and one can talk about the growth \( \gamma \) of \( X \). We say that the graph \( X \) has intermediate growth if its growth \( \gamma \) satisfies \( P \preceq \gamma \preceq E \) for any polynomial \( P \) and every exponential function \( E(n) = a^n \), with \( a > 1 \). Recall that the diameter of a finite graph \( \Gamma = (V,E) \) is defined as \( \max_{u,v \in V} d(u,v) \).

Theorem 12. The diameters of the Schreier graphs \( \Gamma_n \) have intermediate growth. Indeed, there exist constants \( c, d > 0 \) such that

\[
c\sqrt{n}2^{\sqrt{2n}} \leq \text{Diam}(\Gamma_n) \leq d\sqrt{n}2^{\sqrt{2n}}
\]

for all \( n \geq 1 \).

Proof. The statement basically follows from Lemmas 3 and 4 in [BH05]. Using the symmetries of the graph \( \Gamma_n \) one can check that the diameter \( \text{Diam}(\Gamma_n) \) is bounded from above by the distance \( 2d(0^n,0^{n-1}) \), and of course \( d(0^n,0^{n-1}) \) is the lower bound. It is shown in Lemma 3 that

\[
d \left( 0^{\frac{n^2+3n+2}{2}}, 0^{\frac{n^2+3n}{2}} \right) = n2^n + 1,
\]
for all \( n \geq 1 \). Hence, for \( m \approx \frac{n^2}{2} \), we get that \( \text{Diam}(\Gamma_m) \) is equal to \( \sqrt{m} 2^{\sqrt{2m}} \) up to a bounded multiplicative constant dependent on \( m \).

It is proved in [BH05] that the graph \( \Gamma^0_\infty \) has intermediate growth. We generalize this result in the next theorem.

**Theorem 13.** All orbital Schreier graphs \( \Gamma_\omega \) for \( \omega \in \{0,1\}^\infty \) of the group \( G \) have intermediate growth.

**Proof.** The lower bound follows from Theorem 12. Indeed, every ball \( B(\omega, r) \) in the graph \( \Gamma_\omega \) of radius \( r \geq 2 \text{Diam}(\Gamma_n) \) contains at least \( 2^n \) vertices, which correspond to the integers in the interval \([0, 2^n]\) when we use the identification \( \Gamma_\omega = X_\omega \). Hence, the ball \( B(\omega, n) \) contains \( \geq n^{\frac{1}{2} \log_2 n} \) vertices, which gives the super-polynomial lower bound.

Let us prove the bound from above. Let \( l(g) \) for \( g \in G \) be the length of the element \( g \) in the generators \( a, b \), i.e., \( l(g) \) is equal to the minimal number \( n \) such that \( g \) can be expressed as a product \( g = s_1 s_2 \ldots s_n \), with \( s_i \in \{a^\pm 1, b^\pm 1\} \). Notice that \( l(g|_v) \leq l(g) \) for all words \( v \) over the alphabet \( \{0, 1\} \).

Fix a sequence \( \omega = x_1 x_2 \ldots \in \{0,1\}^\infty \) and consider the ball \( B(\omega, n) \) in the graph \( \Gamma_\omega \), centered at the vertex \( \omega \) and of radius \( n \). If \( g|v_1 \omega_1 = v_2 \omega_2 \) for \( v_1, v_2 \in \{0,1\}^k \) and \( \omega_1, \omega_2 \in \{0,1\}^\infty \) then \( g|v_1 (\omega_1) = \omega_2 \). Hence, for every fixed \( k \), each sequence in the ball \( B(\omega, n) \) is of the form \( v \omega_1 \) for some \( v \in \{0,1\}^k \) and \( \omega_1 = h(x_{k+1} x_{k+2} \ldots) \) for some \( h \in \mathcal{N}(n,k) \), where

\[ \mathcal{N}(n,k) = \{ g|_{x_1 x_2 \ldots x_k} : g \in G \text{ and } l(g) \leq n \}. \]

It follows that

\[ |B(\omega, n)| \leq 2^k \cdot |\mathcal{N}(n,k)|. \tag{7} \]

Let us show that, for every \( n \), we can find \( k \) such that the values \( 2^k \) and \( |\mathcal{N}(n,k)| \) are small enough.

Consider an element \( g \in G \) of length \( \leq n \) written as a word in generators

\[ g = a^{\alpha_0} b^{\beta_1} a^{\alpha_1} b^{\beta_2} \ldots b^{\beta_m} a^{\alpha_m}, \tag{8} \]

where the interior powers are non-zero. Notice that \( a^i|_v \in \{1, a, a^{-1}\} \) for all words \( v \) of length \( \geq \log_2 l \), and \( b^j|_v \) is equal to \( b^j \) or \( a^j \) with \( |i| \leq |l| \) for all finite words \( v \). Hence the restriction \( g|_v \) for words \( v \) of length \( \geq \log_2 n \) can be written in the form

\[ a^{\varepsilon_0} \cup a^{\varepsilon_1} \cup \ldots \cup a^{\varepsilon_m}, \tag{9} \]

where \( \varepsilon_i \in \{-1, 0, 1\} \) and on every position \( \cup \) we get a power of \( a \) or of \( b \). If one of the places \( \cup \) in the expression \( a^{\varepsilon_i} \cup \) from (8) is filled with a power of \( a \), then this expression contains at most one position with a power of \( b \). The same holds if \( \varepsilon_i = 0 \). If \( \varepsilon_i \in \{-1, 1\} \), then

\[
\begin{align*}
  b^{\beta_i} a b^{\beta_{i+1}}|_0 &= b^{\beta_i} a^{\beta_{i+1}} \\
  b^{\beta_i} a^{-1} b^{\beta_{i+1}}|_0 &= b^{\beta_i} a^{-1} a^{\beta_{i+1}}
\end{align*}
\]

and in all cases there is only one position with a power of \( b \). Hence, the restriction \( g|_v \) for words \( v \) of length \( \geq (\log_2 n + 1) \) can be expressed in the form (8) with \( \leq m/2 \) positions with
a power of $b$. By applying the same procedure $\log_2 m$ times, we get an element with at most one position with a power of $b$. It follows that

$$g|_v \in \mathcal{N} = \{a^{\varepsilon_1} b^k a^{\varepsilon_2}, a^k : k \in \mathbb{Z} \text{ and } \varepsilon_i \in \{-1, 0, 1\}\}$$

for words $v$ of length $\geq (\log_2 n)(\log_2 n + 1)$ (here we use $m \leq n/2$). Notice that the set $\mathcal{N}$ contains $\leq 20n$ elements of length $\leq n$. We can apply estimate (7) with $k = (\log_2 n)(\log_2 n + 1)$ and get

$$|B(\omega, n)| \leq 20n^{\log_2 n + 2} \sim n^{\log_2 n}.$$ 

\[\square\]

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