COMPLEX STRUCTURES AND SLICE-REGULAR FUNCTIONS ON REAL ASSOCIATIVE ALGEBRAS

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Abstract. In this paper, we study the (complex) geometry of the set $S$ of the square roots of $-1$ in a real associative algebra $A$, showing that $S$ carries a natural complex structure, given by an embedding into the Grassmannian of $\mathbb{C} \otimes A$. With this complex structure, slice-regular functions on $A$ can be lifted to holomorphic maps from $\mathbb{C} \times S$ to $\mathbb{C} \otimes A \times S$ and the values of the original slice-regular functions are recovered by looking at how the image of such holomorphic map intersects the leaves of a particular foliation on $\mathbb{C} \otimes A \times S$, constructed in terms of incidence varieties. In this setting, the quadratic cone defined by Ghiloni and Perotti is obtained by considering some particular (compact) subvarieties of $S$, defined in terms of some inner product on $A$.

Moreover, by defining an analogue of the stereographic projection, we extend the construction of the twistor transform, introduced by Gentili, Salamon and Stoppato, to the case of an associative algebra, under the hypothesis of the existence of sections for a given projective bundle.

Finally, we introduce some more general classes of ”slice-regular” functions to which the present theory applies in all qualitative aspects.

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1. Introduction

Slice-regular quaternionic functions of a quaternionic variable were introduced by Gentili and Struppa in [7]; since then, their theory has been extensively studied by many authors (for reference, see [2,3,6]) and extended to the context of Clifford algebras (see [1]) and of general real alternative algebras (see [9]).

These regular functions share an impressive number of properties with holomorphic functions: they can be written as power series, they enjoy a maximum property, a Cauchy formula holds, the structure of the zeros is rigid and so on.

In [9], Ghiloni and Perotti built on an idea that dates back to Fueter [4], for the quaternionic case, and to Sce [15] and Rinehart [14], for the case of an algebra, exploring the link between slice-regular functions and intrinsic complex functions (i.e. functions that satisfy $F(\overline{z}) = \overline{F(z)}$); they proved that every slice-regular quaternionic function $f$ is induced, in a suitable sense, by a holomorphic function $F$ from an open set of $\mathbb{C}$ to $\mathbb{C}^4$. They generalized this point of view to define slice-regular functions on a real associative algebra.

In [5], Gentili, Salamon and Stoppato, with the aim of producing examples of interesting complex structures in 4 real dimensions, found another way to associate to every slice-regular function a holomorphic map, this time from $U \times \mathbb{CP}^1$ to $\mathbb{CP}^3$, where $U$ is an open domain in $\mathbb{C}$.

In [13], the author proved that, in the setting established by Ghiloni and Perotti, the values of $f$ are linked to the values of $F$ in a “holomorphic” fashion, i.e. for each $q \in \mathbb{H}$ there exists a complex hypersurface $Z(q)$ of $\mathbb{C}^4$ such that the solutions of $f(x) = q$ can be deduced from the intersections of the image of $F$ with $Z(q)$; this shows that many properties of slice-regular functions can actually be deduced by the corresponding properties of holomorphic functions.

The starting point of all the explorations of holomorphicity of slice-regular functions cited above is the generalization of the writing of a complex number in terms of its real and imaginary part, which is related to the choice of an imaginary unit; in terms of the algebra structure of the complex numbers, this is equivalent to the choice of an element of the algebra which induces a complex structure by (left) multiplication, i.e. a square root of $-1$. In the complex field, we obviously have only two square roots of $-1$, which, in terms of induced complex structures, are quite similar, as they have the same eigenspaces in $\mathbb{C} \otimes \mathbb{C}$ (they just swap signs); however, in a generic real algebra, the set of square roots of $-1$ is larger and the corresponding set of complex structures more varied.

We consider the set $S$ of square roots of $-1$ in a real associative algebra and we define a natural almost complex structure $J_S$ on it, which turns out to be integrable when $A$ is associative (see Proposition 3.6 with this complex structure, a slice-regular function $f$ (defined on the quadratic cone $Q_A$, following the definition of Ghiloni and Perotti in [9]).
induces a holomorphic map $\mathcal{F}$ from (an open domain in) $\mathbb{C} \times S$ to $\mathbb{C} \otimes A \times S$. In this way, we extend the domain of definition of a slice-regular function, as the quadratic cone is, in general, a subset of $S \times \mathbb{C}$ (see Proposition 3.9).

This complex structure on $S$ also makes the inclusion of $S$ into the space of linear complex structures on $A$ a holomorphic map, therefore the set $S$ is identified with a subvariety of the Grassmannian of $\mathbb{C} \otimes A$. This allows us to define an incidence variety in $\mathbb{C} \otimes A \times S$ and, by means of it, a real-analytic foliation of $\mathbb{C} \otimes A \times S$ with complex leaves; the space of parameters of this foliation is $A$ and the set of points where $f$ assumes a given value $a \in A$ is uniquely determined by the intersection of the image of $\mathcal{F}$ with the leaf $\mathcal{I}(a)$ associated to $a$ (see Theorem 4.2 and Proposition 4.4).

Once we fix $u \in S$, we can use it to give $A$ the structure of a complex vector space; moreover, every invertible element of $A$ acts on $S$ by conjugation (i.e. by sending $s$ to $a^{-1}sa$) and the map that sends an element of $A$ to $a^{-1}ua$ is holomorphic with respect to the complex structure given by $u$ on $A$. Extending this action also in the case of zerodivisors, we consider all the elements of $A$ that associate $u$ with a given $s \in S$, thus defining a fiber bundle on $S$, which is actually a vector bundle (see Proposition 5.3). The existence of a never vanishing section of such a bundle, on (an affine chart of) $S$ produces a way of parametrizing $S$ with elements of $A$, exactly as the stereographic projection does for the unit sphere; with such a parametrization, we can define a general version of the twistor transform (see Proposition 5.4).

We also show how to recover the quadratic cone defined by Ghiloni and Perotti in [9] and the associated set of imaginary units $\mathbb{S}_A$, by considering the subset $S_0$ containing the elements of $S$ that induce complex structures which are orthogonal with respect to a given inner product on $A$; as the inner product varies, the complex subvarieties $S_0$ cover all of $S$. An advantage of this viewpoint is the possibility to find global holomorphic equations for the associated zero variety $Z_0$ inside $A \otimes \mathbb{C}$ (see Theorem 6.9).

From such a description, it is clear that many of the qualitative properties of slice-regular functions come from the fact that the map $\mathcal{F}$ (or some equivalent map) is holomorphic; however, the theory developed up to know assumes a stronger hypothesis, namely that such map is of the form $\mathcal{F}(z, s) = (F(z), s)$, with $F$ holomorphic; if one drops this restrictions and considers just any holomorphic map (with the right symmetries), a larger family of functions is defined, which shares many behaviours in common with holomorphic functions (and, obviously, slice-regular functions).

The content is organized as follows. In Section 2, we present the various definitions of slice-regular function, the results of [9], and we work out the explicit correspondence with
the twistor transform given by [5]; we also remark how the latter interpretation is linked to the results in [13].

Section 3 deals with the natural complex structure carried by $S$ and its role in the definition of slice-regular functions; in Section 4, we describe the incidence variety and the zero variety that generalize the discussion carried out in [13].

Section 5 is devoted to the construction of a parametrization of an affine chart of the set $S$ and the consequent definition of a twistor transform in the general setting of an associative algebra; in Section 4, we restrict the admissible set of square roots of $-1$ to those inducing an orthogonal transformation of $A$ (via left multiplication), with respect to some inner product, and we study the consequences of this. In particular, the setting of Ghiloni and Perotti can be obtained by choosing a particular inner product.

Finally, Section 7 proposes a generalization of slice-regular functions, for which many of the properties that come from holomorphic functions still hold.

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## 2. Holomorphicity in the quaternionic case

This section is intended as a brief introduction to the different ways of defining slice-regular functions (on the quaternions or on a real alternative algebra with an involution) and of linking them to actual holomorphic maps; we want to emphasize the connections between the various definition and work out the explicit correspondence between the holomorphic stem function and the twistor transform, both associated to a slice-regular function on quaternions.

Let $\mathbb{H}$ be the algebra of quaternions and let $S$ be the 2-sphere of square roots of $-1$, i.e.

$$S = \{ q \in \mathbb{H} : q^2 = -1 \}.$$  

Consider the map $\pi : \mathbb{C} \times S \to \mathbb{H}$ given by $\pi((x + iy), u) = x + uy$. On $\mathbb{C} \times S$, the map $\sigma : \mathbb{C} \times S \to \mathbb{C} \times S$ given by $\sigma(z, u) = (\overline{z}, -u)$ is an involution with no fixed points and $\pi \circ \sigma = \pi$; let $U \subseteq \mathbb{C}$ be an open set, symmetric with respect to the real axis (i.e. invariant under complex conjugation), then $U \times S$ is invariant under $\sigma$.

The open sets $V \subseteq \mathbb{H}$ that can be obtained as $V = \pi(U \times S)$, with $U$ as above, are called *axially symmetric*. We note that $\pi \circ \sigma(\cdot) = \pi(\cdot)^c$, where $q \mapsto q^c$ is the quaternionic conjugation.

Given $V \subseteq \mathbb{H}$, a function $f : V \to \mathbb{H}$ is said to be a *left slice function* if we can find $\alpha, \beta : \mathbb{C} \to \mathbb{H}$ such that
• $\alpha(\overline{z}) = \alpha(z)$
• $\beta(\overline{z}) = -\beta(z)$
• $f \circ \pi(z, u)) = \alpha(z) + u\beta(z)$ for all $z \in U$, $u \in S$.

Let $C \otimes H$ be the tensor product of $C$ and $H$ over $R$; we define an involution on $C \otimes H$ by requiring that $z \otimes q = \overline{z} \otimes q$ and taking the linear extension. Let $\tau : C \otimes H \times S \to C \otimes H \times S$ the involution given by $\tau(z \otimes q, u) = (\overline{z} \otimes q, -u)$ and linear on the first component. The space $C \otimes H$ has a natural complex structure given by $J_0 \otimes I$, where $J_0$ is the natural complex structure of $C$ induced by the multiplication by $i$ and $I$ is the identity on $H$.

Moreover, we extend the map $\pi$ to $C \otimes H \times S$ as follows $\pi(z \otimes q, u) = \pi(z, u)q$, where the product on the left hand side is in $H$, and then extend it linearly on $C \otimes H \times S$. We denote this extension again by $\pi$.

In [9], Ghiloni and Perotti prove the following.

**Proposition 2.1.** Let $V \subseteq H$ an axially symmetric domain, $V = \pi(U \times S)$, and let $f : V \to H$ be a left slice function, then there exists a unique $F : U \to C \otimes H$ such that

- $F \circ \sigma = \overline{F}$
- $f \circ \pi(z, u) = \pi(F(z), u)$ for all $(z, u) \in U \times S$.

We say that $f$ is induced by $F$ and we write $f = \mathcal{I}(F)$; every left slice function can be obtained this way and every function obtained as $\mathcal{I}(F)$ is left slice. Moreover, $f$ is left slice-regular if and only if $F$ is holomorphic from $(U, J_0)$ to $(C \otimes H, J_0 \otimes I)$.

In other words, they prove that there exists a holomorphic function $F : U \to C \otimes H$ such that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{F} & C \otimes H \\
\pi(\cdot, u) \downarrow & & \downarrow \pi(\cdot, u) \\
V & \xrightarrow{f} & H
\end{array}
\]

commutes for every $u \in S$.

In [13], it was shown that, for every $q \in H$, there exists a complex hypersurface $V(q) \subset C \otimes H$ such that $F(z) \in V(q)$ if and only if there exists $u \in S$ such that $f(\pi(z, u)) = q$. As $q$ varies in $H$, the hypersurfaces $V(q)$ change by a translation of a real vector.

Also in [13], we constructed a diffeomorphism between $S$ and a complex submanifold of the Grassmannian of 2-planes in $C^4$, interpreting $u \in S$ as a linear complex structure on $R^4$ and associating it to its ($-i$)-eigenspace. This gives a complex structure on $S$ (which can only be the standard one on $CP^1$), naturally associated to slice-regular functions; we define the (almost) complex structure $\mathcal{J}$ on $H \setminus R$ as follows: given $q \in H \setminus R$, there exist unique $z \in C_+, u \in S$ such that $\pi(z, u) = q$, then we consider on $T_q(H \setminus R) \cong H$ the linear involution
\[ L_u(x) = ux, \text{ with } x \in \mathbb{H}, \text{ so, we set} \]
\[ J_q(x) = L_u(x) \quad \forall q \in \mathbb{H} \setminus \mathbb{R}, \quad \forall x \in T_q(\mathbb{H} \setminus \mathbb{R}) \cong \mathbb{H}. \]

It is easy to show that \( J \) is integrable and that \((\mathbb{H} \setminus \mathbb{R}, J)\) is biholomorphic to \( \mathbb{C}_+ \times \mathbb{S} \), where \( \mathbb{S} \) has the standard complex structure of \( \mathbb{CP}^1 \). These considerations are carried out in detail in [5].

The following result is an easy consequence of the remarks we just made.

**Lemma 2.2.** Let \( V = \pi(U \times \mathbb{S}) \subseteq \mathbb{H} \) an axially symmetric domain, \( f : V \to \mathbb{H} \) a slice-regular function. Then there exists a unique function
\[ \mathcal{F} : U \times \mathbb{S} \to \mathbb{C} \otimes \mathbb{H} \times \mathbb{S} \]
such that
\begin{itemize}
  \item \( \mathcal{F} \circ \sigma = \tau \circ \mathcal{F} \)
  \item \( \mathcal{F} \) is holomorphic
  \item \( \mathcal{F} \) is the identity on the second component
  \item \( f \circ \pi = \pi \circ \mathcal{F} \).
\end{itemize}

In particular, \( \mathcal{F} \) can be interpreted as a holomorphic function from \((V \setminus \mathbb{R}, J)\) to \( \mathbb{C} \otimes \mathbb{H} \times \mathbb{CP}^1 \).

The situation is encoded in the following commutative diagram:
\[
\begin{array}{ccc}
U \times \mathbb{S} & \xrightarrow{\mathcal{F}} & \mathbb{C} \otimes \mathbb{H} \times \mathbb{S} \\
\pi \downarrow & & \downarrow \pi \\
V & \xrightarrow{f} & \mathbb{H}
\end{array}
\]

2.1. **Fixing a basis.** Let \( \{1, I, J, K\} \) be an orthonormal basis for \( \mathbb{H} \) such that \( IJ = K \); in fact, we will just need that \( IJ +JI = 0 \) and that \( IJ = K \), the orthonormality is a consequence of these two requests. Given \( z \in \mathbb{C} \), we denote \( \pi(z,u) \) by \( z_u \) for any \( u \in \mathbb{S} \). Moreover, we have that
\[ \mathbb{H} \cong \mathbb{C}_I \oplus (\mathbb{C}_I J), \]
i.e. we have an isomorphism of additive groups \( \gamma : \mathbb{C}^2 \to \mathbb{H} \) given by \( \gamma(z,w) = z_I + w_I J \).

**Remark 2.1.** It is obvious that \( \gamma(\lambda z, \lambda w) = \lambda_I \gamma(z, w) \) for all \( \lambda \in \mathbb{C} \).

Now, let us consider two elements \( u, v \in \mathbb{S} \) and \( z \in \mathbb{C} \); the quaternions \( z_u \) and \( z_v \) differ by a rotation of \( \mathbb{H} \) that fixes the real axis, i.e. by a transformation of the form
\[ Q_p(q) = p^{-1}qp \]
for some \( p \in \mathbb{H} \setminus \{0\} \). If we write \( p = \gamma(z, w), p' = \pi(z', w') \), it is quite clear that \( Q_p = Q_{p'} \) if and only if \( (z, w) = \lambda(z', w') \) with \( \lambda \in \mathbb{C}^* \); therefore the set of maps \( Q_p \) is identified with \( \mathbb{C}P^1 \), via \( \gamma \). Chosing \( I \) as a special imaginary unit, we define a map
\[
\rho : \mathbb{C} \times \mathbb{C}P^1 \to \mathbb{H}
\]
as follows
\[
\rho(z, [u_0 : u_1]) = Q_{\gamma(u_0, u_1)}(z) = (\gamma(u_0, u_1))^{-1}z\gamma(u_0, u_1).
\]
By Remark 2.1,
\[
\rho(z, [u_0 : u_1]) = (\gamma(u_0, u_1))^{-1}\gamma(zu_0, zu_1)
\]
is actually a function of the homogeneous 4-tuple \([u_0 : u_1 : z_0 : z_1]\). Let
\[
S_1 : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3
\]
be the Segre embedding
\[
S_1([x : y], [a : b]) = [xa : xb : ya : yb]
\]
and let \( \rho_1 : \mathbb{C}P^3 \to \mathbb{H} \) be given by
\[
\rho_1([w_0 : w_1 : w_2 : w_3]) = (\gamma(w_0, w_1))^{-1}\gamma(w_2, w_3).
\]
Then \( \rho(z, [u_0 : u_1]) = \rho_1 \circ S_1([1 : z], [u_0 : u_1]) \). We have thus factorized the leftmost vertical arrow in (2) as
\[
\mathbb{C} \times \mathbb{S} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}P^1 \xrightarrow{S_1} \mathbb{C}P^3 \xrightarrow{\rho_1} \mathbb{H}.
\]
On the other hand, considering the map \( \pi : \mathbb{C} \otimes \mathbb{H} \times \mathbb{S} \to \mathbb{H} \), we obtain a map
\[
\delta : \mathbb{C} \otimes \mathbb{H} \times \mathbb{C}P^1 \to \mathbb{H}
\]
which, on the elements \((z \otimes q, [u_0 : u_1])\) acts as follows
\[
\delta(z \otimes q, [u_0 : u_1]) = \rho(z, [u_0 : u_1])q = \gamma(u_0, u_1)^{-1}zI\gamma(u_0, u_1)q.
\]

**Remark 2.2.** For every \((z, w) \in \mathbb{C}^2\), we have
\[
\gamma(z, w)I = \gamma(iz, -iw) \quad \gamma(z, w)J = \gamma(-w, z) \quad \gamma(z, w)K = \gamma(iw, iz).
\]
Thanks to Remarks 2.1, 2.2 and to the additivity of \( \gamma \), we obtain that \( \delta(z \otimes q, [u_0 : u_1]) \) is
\[
\gamma(u_0, u_1)^{-1}\gamma(q_0zu_0 + iq_1zu_1 - q_2zu_1 + iq_3zu_1, q_0zu_1 - iq_1zu_1 + q_2zu_0 + iq_3zu_0)
\]
where \( q = q_0 + q_1I + q_2J + q_3K \). If we use the basis \( \{1, I, J, K\} \) to identify \( \mathbb{C} \otimes \mathbb{H} \) with \( \mathbb{C}^4 \), then \( \delta([z_0, z_1, z_2, z_3], [u_0 : u_1]) = \rho_1 \circ \delta_1 \circ S_2([1 : z_0 : z_1 : z_2 : z_3], [u_0 : u_1]) \), where
\[
S_2 : \mathbb{C}P^4 \times \mathbb{C}P^1 \to \mathbb{C}P^9
\]
is the Segre embedding and
\[
\delta_1 : \mathbb{C}P^9 \to \mathbb{C}P^3
\]
is given by
\[ \delta_1([w_0 : \ldots : w_9]) = [w_0 : w_1 : w_2 + iw_4 - w_6 + iw_8 : w_3 - iw_5 + w_7 + iw_9]. \]

Therefore, we factored the rightmost vertical arrow of the diagram (2) as
\[ C \otimes H \times S \xrightarrow{\cong} C^4 \times \mathbb{CP}^1 \xrightarrow{S_2} \mathbb{CP}^9 \xrightarrow{\delta_1} \mathbb{CP}^3 \xrightarrow{\rho_1} H. \]

**Remark 2.3.** It could surprise that the map \( \delta_1 \) does not depend on the choice of the basis \( \{1, I, J, K\} \); however, for \( q \in \mathbb{H} \), let us consider the \( \mathbb{C} \)-linear map \( F_q : \mathbb{C}^2 \to \mathbb{C}^2 \) defined by
\[ F_q(z, w) = \gamma^{-1}(\gamma(z, w)q). \]

Then,
\[ \delta_1([w_0 : \ldots : w_9]) = \begin{bmatrix} F_1 & 0 & 0 & 0 & 0 \\ 0 & F_1 & F_I & F_J & F_K \end{bmatrix} \begin{bmatrix} [w_0 : w_1] \\ \vdots \\ [w_8 : w_9] \end{bmatrix}. \]

Moreover, the fact that \( I \gamma(z, w) = \gamma(iz, iw) \) and that \( IJ = K = -JI \) uniquely identifies all the linear maps involved.

If we expand the diagram (2) incorporating the two factorizations found above, we obtain the following diagram.

The next result, contained in [5], is now easy to prove, where \( \tilde{F} \) is defined by the commutative diagram above and can be expressed in terms of the function \( \mathcal{F} \), or (which is equivalent) in terms of the function \( F \) given by Ghiloni and Perotti.
Lemma 2.3. Let $V = \pi(U \times S) \subseteq \mathbb{H}$ an axially symmetric domain, $f : V \to \mathbb{H}$ a slice-regular function. Then there exists a unique function
\[
\tilde{F} : \mathcal{U} \times \mathbb{C}P^1 \to \mathbb{C}P^3
\]
such that
- $\tilde{F}$ is holomorphic
- $\tilde{F}$ sends the complex spheres $\{z\} \times \mathbb{C}P^1$ into lines of $\mathbb{C}P^3$
- $f \circ \rho = \rho_1 \circ \tilde{F}$.

In particular, $\tilde{F}$ can be interpreted as a holomorphic function from $(V \setminus \mathbb{R}, J)$ to $\mathbb{C}P^3$.

By the Segre embedding $S_1$, we can identify $\mathcal{U} \times \mathbb{C}P^1$ with an open set of the Klein quadric in $\mathbb{C}P^3$; moreover, we have the following corollary, also present in [5].

Corollary 2.4. Given $V, \mathcal{U}$ as before, every slice-regular function $f : V \to \mathbb{H}$ induces a holomorphic function
\[
\mathcal{F} : \mathcal{U} \to \text{Gr}_{\mathbb{C}}(2, \mathbb{C}^4).
\]

Both in Lemma 2.3 and Corollary 2.4 we could also impose some symmetry conditions on the holomorphic function, obtaining a bijective correspondence between slice-regular functions and this class of holomorphic functions. Those symmetry conditions can be written in the form of equivariance with respect to a real structure on $\mathbb{C}P^3$, or equivalently on the Grassmannian $\text{Gr}_{\mathbb{C}}(2, \mathbb{C}^4)$. The detailed description can be found in [5].

Remark 2.4. We want to stress that, whereas the functions $\mathcal{F}$ and $\mathcal{F}$ are defined only in terms of $f$, the functions $\tilde{F}$ and $\tilde{F}$ depend on the choice of the orthonormal basis $\{1, I, J, K\}$.

From this detailed analysis, we resume the result from [13] about the complex-analyticity of $V(q)$: consider $V(0)$, which is the projection on the factor $\mathbb{C}P^4$ of the set $(\delta_1 \circ S_2)^{-1}(L)$ where $L \subseteq \mathbb{C}P^3$ is the line $\{[w_0 : w_1 : 0 : 0] : [w_0 : w_1] \in \mathbb{C}P^1\}$. Indeed, from this description one recovers completely [13, Theorem 3.3] just working out explicitly the coordinate expressions of the maps involved.

3. A natural complex structure

The aim of this section is to extend the construction of the complex structure $J$ in [11] to a more general setting.

Let $A$ be an associative real algebra with unity; as a real vector space, $A$ is isomorphic to $\mathbb{R}^N$. We consider the set
\[
S = \{a \in A : a^2 = -1\}.
\]

We define some families of linear operators on $A$. 
**Definition 3.1.** Given \( a \in A \), we define \( L_a, R_a \in \text{End}(A) \) as
\[
L_a(x) = ax, \quad R_a(x) = xa.
\]
We denote their sum by \( F_a \), i.e. \( F_a(x) = ax + xa \), and their composition by \( T_a \), i.e. \( T_a(x) = axa \).

The following properties are easy to prove.

**Lemma 3.2.** For \( a \in A \), we have
1. \([R_a, L_a] = 0\),
2. \( F_a^2 = L_a^2 + R_a^2 + 2T_a \).

For \( s \in S \), we have
3. \( R_s^2 = I_s^2 = -I \), therefore \( R_s \) and \( L_s \) are invertible,
4. \( R_sF_s = T_s - I \),
5. \( F_s^2 = 2T_s - 2I \),
6. \( T_s^2 = I \).

We start by exploring the geometry of the set \( S \).

**Lemma 3.3.** The set \( S \) is an algebraic submanifold and \( T_sS = \{ h \in T_sA : sh + hs = 0 \} \), where we identified \( T_sA \) with \( A \) in the usual way.

**Proof.** Let us consider the map \( f : A \to A \) given by \( f(a) = a^2 + 1 \); this is a polynomial map. Moreover, given \( a \in A \) and \( h \in T_aA \cong A \), we have
\[
Df_a[h] = \lim_{t \to 0} \frac{f(a + th) - f(a)}{t} = \frac{a^2 + t^2h^2 + t(ah + ha) + 1 - a^2 - 1}{t} = ah + ha.
\]
Now, obviously \( S = f^{-1}(0) \), therefore it is an algebraic subvariety; if \( s \in S \), i.e. if \( s^2 + 1 = 0 \), \( Df_s[h] = 0 \) if and only if \( F_s(h) = 0 \) if and only if, by Lemma 3.2, \( T_s(h) = h \). As \( T_s^2 = I \), \( T_s \) is then diagonalizable with eigenspaces \( E_1(s) = \ker(T_s - I) \) and \( E_{-1}(s) = \ker(T_s + I) \). The map \( s \mapsto T_s \) is continuous from \( S \) to the set
\[
B = \{ T \in \text{End}(A) : T^2 = \text{Id}_A \},
\]
moreover, the function \( T \mapsto \text{tr}T \) is continuous from \( \text{End}(A) \) to \( \mathbb{R} \). However, if \( T \in B \), then \( \text{tr}T = \dim E_1 - \dim E_{-1} \in \mathbb{Z} \), so we can partition \( B \) into connected components
\[
B_j = \{ T \in B : \text{tr}T = j \} \quad j \in \mathbb{Z}.
\]
Therefore, if \( S' \subseteq S \) is a connected component, then \( \text{tr}T_s \) is constant for \( s \in S' \), so it is \( \dim E_1(s) \). This means that \( \dim \ker Df_s \) is constant for \( s \in S' \), for every connected component \( S' \) of \( S \).
Now, consider the set $Y \subset A$ of invertible elements; its complement is real analytic and with empty interior, so $Y$ is an open manifold of dimension $\dim A$. Given an element $s_0 \in S$, we define the map
\[
\Phi : Y \to S \quad \Phi(y) = ys_0y^{-1}
\]
which is obviously well defined; we note that $\Phi(y_1) = \Phi(y_2)$ if and only if $y_1^{-1}y_2 \in E^{-1}(s_0) \cap Y$. Moreover,
\[
D\Phi_y[h] = \lim_{t \to 0} \frac{\Phi(y + th) - \Phi(y)}{t} = \lim_{t \to 0} \frac{ys_0y^{-1} - tys_0y^{-1}hy^{-1} + ths_0y^{-1} - ys_0y^{-1} + O(t^2)}{t}
\]
and clearly $D\Phi_y[h] = 0$ if and only if $hs_0y^{-1} = ys_0y^{-1}hy^{-1}$ if and only if $hs_0 = ys_0y^{-1}h$ if and only if $y^{-1}h \in E^{-1}(s_0)$.

If $S'$ is the connected component containing $s_0$ and $Y_0$ is the connected component of $Y$ containing 1, the $\Phi(Y_0) \subseteq S'$. However, $\Phi(Y_0)$ is a connected component of the orbit of the point $s_0$ under the action of the inner automorphisms of $A$, therefore it is an immersed manifold; on the other hand, as $\Phi(y_1) = \Phi(y_2)$ if and only if $y_1^{-1}y_2 \in E^{-1}(s_0) \cap Y$, then $\Phi(Y_0)$ has always a unique and well defined tangent, of dimension $\dim A - \dim E^{-1}(s_0) = \dim E_1(s_0) = \dim \ker Df_{s_0}$, which is constant on $S'$. So, $\Phi(Y_0) = S'$ and it is actually an embedded submanifold of $A$.

Therefore, each connected component is a manifold and for every $s \in S$
\[
T_sS = \{h \in A : sh + hs = 0\}
\]
as we claimed. \qed

**Remark 3.1.** The set $B_j$ is empty if $j$ is odd or $|j| > N$, moreover $B_N = \{I\}$ and $B_{-N} = \{-I\}$. In general, if $k = (N + j)/2$ is a positive integer not bigger than $N$, then $\dim B_j = 2k(N - k)$. So, the top dimensional component is $B_0$, with dimension $N^2/2$, which corresponds to a union $S_0$ of connected components of $S$ of dimension $N/2$.

**Lemma 3.4.** For $s \in S$, the operators $L_s$, $R_s$, $F_s$ have zero trace.

**Proof.** By Lemma 3.2, we have that $L_s^2 = R_s^2 = -I$, therefore $L_s$ and $R_s$ have eigenvalues $\pm i$; as the trace has to be real, the only possibility is $\tr L_s = \tr R_s = 0$.

Similarly, $F_s^2 = 2T_s - 2I$ and $F_s^2 + 4I = 2T_s + 2I$, therefore
\[
F_s^2(F_s^2 + 4I) = 4(T_s + I)(T_s - I) = 0
\]
so $F_s$’s eigenvalues can be 0, $\pm 2i$; again, as the trace is a real number, the only possibility is $\tr F_s = 0$. \qed
Remark 3.2. From Lemma 3.4, $S$ is contained in the kernel of the linear map $A \ni a \mapsto \text{tr}(F_a) \in \mathbb{R}$, whereas $\text{tr}(F_1) = 2N$, so $S$ is contained in a hyperplane that does not contain 1, hence the map $(x + iy, s) \mapsto x + sy$ is injective when $y > 0$.

In the case of a Clifford algebra, the previous remark follows also from a more precise computation of the trace of the operator $L_s$.

Corollary 3.5. If $A$ is a Clifford algebra, $\text{tr} L_s = 0$ for all $s \in S$, i.e. $S$ is contained in a hyperplane which does not contain 1. In particular, this implies that the map $(x + iy, s) \mapsto x \cdot 1 + y \cdot s$ is injective from $\mathbb{C} \times S$ into $A$.

Proof. Let $e_1, \ldots, e_n$ be generators of $A$; a basis of $A$ as a real vector space is
\[ \{ e_I \mid I \subseteq \{1, \ldots, n\} \} \]
where, for $I = (i_1, \ldots, i_p)$,
\[ e_I = e_{i_1} \cdots e_{i_p}, \quad e_0 = e_{\emptyset} = 1, \]
so, the dimension $N$ of $A$ is $2^n$.

For $a \in A$, we have that $a = \sum a_I e_I$, then, the $e_J$-component of the element
\[ L_a(e_J) = \sum a_I e_I e_J \]
is $a_0$, because the only $I$ such that $e_I e_J = e_J$ is $I = \emptyset$, as all the elements of the basis are invertible. Therefore
\[ \text{tr} L_a = 2^n a_0. \]

In the same way, we prove that $\text{tr} R_a = 2^n a_0$. As, for $s \in S$, $0 = \text{tr} F_s = \text{tr} L_s + \text{tr} R_s = 2^{n+1} s_0$, then $S$ is contained in the hyperplane $s_0 = 0$.

Given that 1 is contained in the hyperplane $s_0 = 1$, we have that two elements of $A$ of the form $x + sy$ and $u + sv$ with $x, u \in \mathbb{R}$ and $y, v \in \mathbb{R}_+$ coincide if and only $x = u$ and $y = v$. \hfill $\square$

Remark 3.3. The case of $j = 0$ is, in some sense, generic. If $h \in T_s S$, then the map $L_h : A \to A$, given by $L_h(x) = hx$, is such that $L_h(E_1(s)) \subseteq E_1(s)$ and $L_h(E_{-1}(s)) \subseteq E_{-1}(s)$; if $h$ is invertible, this implies that $\dim E_1(s) = \dim E_{-1}(s)$. Therefore, if $T_s S$ contains at least one invertible element, then $s$ belongs to $S_0$.

For every $s \in S$, the map $L_s : A \to A$ can be identified with an element of $\text{End}(T_s A)$; moreover, if $h \in T_s S$, then $sh + hs = 0$ and
\[ sL_s(h) + L_s(h)s = ssh + shs = s(h + hs) = 0 \]
so $L_s$ restricts to an endomorphism of $T_s S$. We define the almost complex structure $J_S : TS \to TS$ as
\[ J_S(s, h) = (s, L_s(h)). \]
Proposition 3.6. The structure $J_S$ is integrable.

Proof. Let $X$, $Y$ be vector-fields on $S$. The Nijenhuis tensor is then

$$N(X,Y) = [X,Y] + J_S([J_SX,Y]+[X,J_SY]) - [J_SX,J_SY].$$

As it is well known, $[X,Y] = dX(Y) - dY(X)$, where the differential is computed as the differential of the maps $X, Y : S \to A$; so we need to compute the quantities

$$d(J_SX)(J_SY) = J_SY \cdot X + J_S(dX(J_SY))$$
$$J_S(d(J_SX)(Y)) = J_S(Y \cdot X) - dX(Y),$$

where \cdot denotes the product in $A$, and substitute them in the formula for the Nijenhuis tensor, obtaining

$$N(X,Y) = J_S(Y \cdot X) - J_S(X \cdot Y) + J_SY \cdot X - J_SX \cdot Y.$$ 

As $N(\cdot, \cdot)$ is a tensor, the value of $N(X,Y)$ at a point $s$ is determined by the values of $X(s)$ and $Y(s)$; so, if $X(s) = x \in A$ and $Y(s) = y \in A$, then

$$N(X,Y)(s) = s(xy) - s(xy) + (sy)x - (sx)y = 0$$

because $A$ is associative. Therefore the almost complex structure $J_S$ is integrable. \hfill \Box

The following corollary is now trivial.

Corollary 3.7. Every connected component of $S$ is a complex manifold.

Remark 3.4. Let $S_j = \{ s \in S : \text{tr}T_s = j \}$, then $S_j = \emptyset$ if $j$ is odd or $|j| > N$. We note that each $S_j$ is a complex manifold (maybe disconnected), which is invariant under all internal automorphisms of $A$; moreover dim$_\mathbb{R} S_j = (N + j)/2$ if it is non-empty.

Remark 3.5. In case $A = \mathbb{H}$, then $N = 4$ and $S$ is the unit sphere of $\mathbb{R}^3$; the complex structure $J_S$ is the standard one, induced on $S^2 \subset \mathbb{R}^3$ by the vector product. We could also carry on the same computations in the case of the algebra of octonions, but then the almost complex structure is not integrable.

Corollary 3.8. If $A$ is a Clifford algebra of signature $(p,q)$ with $p - q \not\equiv 3 \mod 8$, all the connected components of $S$ are of real dimension $N/2$.

Proof. It is enough to show that $\text{tr}T_s = 0$ for all $s \in S$. This follows because

$$\text{tr}T_s = 2^ns_0^2,$$ (3)

so, by Corollary 3.5 $\text{tr}T_s = 0$. 


We now prove Equation (3). We consider the basis defined in Corollary 3.5 we fix \( s = \sum s_I e_I \) and we compute

\[
T_s(e_J) = \sum_{I,K} s_I s_K e_I e_J e_K^* .
\]

The \( e_J \)-component of such a sum is clearly given by

\[
\sum_I s_I^2 e_I e_J e_I ,
\]

therefore

\[
\text{tr} T_s = \sum_J \sum_I s_I^2 e_I^2 k(I,J)
\]

where \( k(I,J) \in \{1, -1\} \) is such that

\[
e_J e_I = k(I,J) e_I e_J .
\]

We note that

\[
k(I,J) = (-1)^{|I||J|} (-1)^{|I \cap J|}
\]

so, we define \( i = |I|, \) \( k = |I \cap J|, \) \( j + k = |J| \). We rewrite the previous sum as

\[
\text{tr} T_s = \sum_I s_I^2 e_I^2 \sum_{k=0}^{n-i} \binom{i}{k} \binom{n-i}{j} (-1)^{i(j+k)} (-1)^k
\]

\[
= \sum_I s_I^2 e_I^2 \sum_{k=0}^{n-i} \binom{i}{k} (-1)^{(i+1)k} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^{ij} = \sum_I s_I^2 e_I^2 ((-1)^{i+1} + 1)^i ((-1)^i + 1)^{n-i}
\]

\[
= \begin{cases} 2^n s_0^2 & \text{if } 2 \mid n \\ 2^n s_0^2 + 2^n \omega^2 s_\omega^2 & \text{if } 2 \nmid n \end{cases},
\]

where \( \omega = e_1 \cdots e_n \) and \( s_\omega \) is the corresponding coefficient. Suppose that the Clifford algebra has \( p \) generators that square to 1 and \( q \) that square to \(-1\), so that \( n = p + q \). It is an easy computation to show that \( \omega^2 = -1 \) if \( p - q \equiv 3 \mod 4 \) and \( \omega^2 = +1 \) if \( p - q \equiv 1 \mod 4 \). So

\[
\text{tr} T_s = \begin{cases} 2^n s_0^2 & \text{if } 2 \mid p - q \\ 2^n (s_0^2 - s_\omega^2) & \text{if } p - q \equiv 3 \mod 4 \\ 2^n (s_0^2 + s_\omega^2) & \text{if } p - q \equiv 1 \mod 4 \end{cases} .
\]

Moreover, if \( n \) is odd, \( \omega \) is in the center of \( A \), so \( (\omega s)^2 = \omega^2 s^2 \); therefore, if \( p - q \equiv 1 \mod 4 \), the map \( s \mapsto \omega s \) sends \( S \) to itself, but \( \text{tr} L_{\omega s} = 2^n s_\omega \), so \( s_\omega = 0 \) when \( p - q \equiv 1 \mod 4 \). □
Remark 3.6. In the case \( p - q \equiv 3 \mod 8 \), the trace of \( T_s \) is \(-2n s^2\) and, as it is given by \( \dim E_1(s) - \dim E_{-1}(s) \), we know it is an integer; therefore, there are only a finite number of possibilities for \( s_\omega \), giving a partition of \( S \) into families of connected components with different dimensions.

Remark 3.7. Corollaries 3.5, 3.8 imply that, for a Clifford algebra \( A \) with signature \((p, q)\) with \( p - q \not\equiv 3 \mod 8 \), the map
\[
\mathbb{C}_+ \times S \ni (x + iy, s) \to x + sy \in A \setminus \mathbb{R}
\]
is a proper smooth embedding, whose image is a \((N + 4)/2\)-dimensional real manifold on which the complex structure \( J_S \) can be extended as an integrable complex structure, because to every point we can associate a unique element of \( S \). We denote by \( Q \) the closure in \( A \) of the image of this embedding.

Given \( a \in Q \setminus \mathbb{R} \), obtained from \((z, s)\) through the map described above, we define
\[
\text{tr}(a) = \text{Re}(z) \quad |a| = |z| .
\]
Then, every such \( a \) satisfies a real quadratic equation, namely
\[
x^2 - 2\text{tr}(a)x + |a|^2
\]
(\( \text{compare with [9, Proposition 1, (3)]} \)). These two definitions can be extended to \( \mathbb{R} \) in a trivial way, but, in general, they do not come from a trace and a norm on \( A \), which are induced by a \(*\)-involution.

3.1. Slice-regular functions on a real associative algebra. In this section, we want to obtain an analogue of Lemma 2.2 for a general real associative algebra, employing the complex structure \( J_S \) defined above; we start by recalling some relevant definitions.

Let \( \mathbb{R}_n \) be the Clifford algebra over \( n \) units, with the notation used above; we identify \( \mathbb{R}_{n+1}^+ \) with \( \text{Span}\{e_0, \ldots, e_n\} \) in \( \mathbb{R}_n \) and we note by \( \mathbb{S}^{n-1} \) the unit sphere of \( \text{Span}\{e_1, \ldots, e_n\} \subseteq \mathbb{R}_n \); quite obviously \( \mathbb{S}^{n-1} \) is a subset of the square roots of \(-1\) in \( \mathbb{R}_n \).

The first attempt at extending the theory of slice-regular functions from quaternions (and octonions) to other algebras was made in [8].

In [1], Colombo, Sabadini and Struppa defined the concept of \textit{slice monogenic function}: let \( U \subset \mathbb{R}^{n+1} \) be an open domain and \( f : U \to \mathbb{R}_n \) a real differentiable function, the function \( f \) is called left slice monogenic if, for every \( I \in \mathbb{S}^{n-1} \), we have
\[
(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}) f(e_0 x + I y) = 0 \quad \forall x, y \in \mathbb{R}^2 \text{ s.t. } e_0 x + I y \in U .
\]

In [9], Ghiloni and Perotti proved that, when \( U \) intersects the line generated by \( e_0 \in \mathbb{R}_n \), every such a function can be extended to a larger set inside \( \mathbb{R}_n \). Indeed, for any real
alternative algebra $A$, they define a set $S_A$ of units, which contains, in the Clifford case, the sphere $S^{n-1}$, and consider a map $\pi : \mathbb{C} \times S_A \to A$ given by $\pi(x + iy, u) = x + uy$, then, any slice monogenic function on $U$ extends to a function $f : V \to A$, where $V = U \times S_A$ is a saturated set for $\pi$. Such a function is induced, in the sense of Proposition 2.1, by a holomorphic function $F : U \to \mathbb{C} \otimes A$, hence we call it a (left) slice regular function.

Such extension still satisfies that
\[
\left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f(x + Iy) = 0 \quad \forall x + iy \in U
\]
for every $I \in S^{n-1}$ and, in fact, for every $I \in S_A$. Moreover, we can find $\alpha, \beta : U \to A$ such that
\begin{enumerate}
  \item $\alpha(\bar{z}) = \alpha(z)$
  \item $\beta(\bar{z}) = -\beta(z)$
  \item $f(\pi(z, u)) = \alpha(z) + u\beta(z)$
  \item $\partial_x \alpha = \partial_y \beta$ and $\partial_y \alpha = -\partial_x \beta$.
\end{enumerate}

**Remark 3.8.** When $A$ is an associative algebra, the set $S_A$ is properly contained in the set $S$ of square roots of $-1$ in $A$.

Let $A$ be an associative algebra and $S$ the set of square roots of $-1$ in $A$; as in Section 2 we define the map
\[
\pi : \mathbb{C} \times S \to A
\]
by $\pi(x + iy, s) = x + sy$ and its extension to the tensor product $\mathbb{C} \otimes A$,
\[
\pi : \mathbb{C} \otimes A \times S \to A
\]
by asking that $\pi(z \otimes a, s) = \pi(z, s)a$ and imposing linearity in the first component.

We define the involutions
\[
\sigma : \mathbb{C} \times S \to \mathbb{C} \times S \quad \tau : \mathbb{C} \otimes A \times S \to \mathbb{C} \otimes A \times S
\]
by asking
\[
\sigma(z, s) = (\bar{z}, -s) \quad \tau(z \otimes a, s) = (\bar{z} \otimes a, -s)
\]
and imposing, for $\tau$, linearity in the first component.

Thanks to what we proved in the previous pages, we obtain a generalization of Lemma 2.2.

**Proposition 3.9.** Let $A$ be a real associative algebra, $U \subseteq \mathbb{C}$ an open set and set $V = \pi(U \times S_A)$; then for every slice regular function $f : V \to A$ there exists a function
\[
\mathcal{F} : U \times S \to \mathbb{C} \otimes A \times S
\]
such that
\[ F \circ \sigma = \tau \circ F \]

(2) \( F \) is holomorphic

(3) \( F \) is the identity on the second component

(4) \( f \circ \pi(z, s) = \pi \circ F(z, s) \) for all \((z, s) \in U \times \mathbb{S}_A\).

In particular, \( f \) can be extended as a slice regular function to the set \( \pi(U \times S) \).

**Proof.** By definition, we have a holomorphic function \( F : U \to \mathbb{C} \otimes A \) such that \( \pi(F(z), s) = f(\pi(z, s)) \) for all \((z, s) \in U \times \mathbb{S}_A\). We define

\[ F(z, s) = (F(z), s) \]

and we immediately see that the requests of the Proposition are fullfilled. To prove that \( f \) can be extended as a function on \( \pi(U \times S) \), we just need to check that the relation

\[ f \circ \pi = \pi \circ F \]

gives a well-posed extension.

This is true because the map \( \pi \) is injective from \( \mathbb{C}_+ \times S \) to \( A \), as follows from Remark 3.2. \( \square \)

4. The incidence variety

We recall some facts about the space of complex structures on a real vector space; we deem these facts well known, however we were not able to find an appropriate reference, so we present also a sketch of proof. Let \( V \) be a real vector space of dimension \( 2p \).

**Theorem 4.1.** The set

\[ \mathcal{E}_V = \{ J \in \text{End}(V) : J^2 = -\text{Id}_V \} \]

is a smooth manifold and the map \( \mathcal{L}_J : T_J \mathcal{E}_V \to T_J \mathcal{E}_V \) given by \( \mathcal{L}_J(A) = JA \) defines a complex structure on it. The complex manifold \( \mathcal{E}_V \) is then biholomorphic to

\[ \mathcal{C}_V = \{ W \in \text{Gr}_{\mathbb{C}}(p, \mathbb{C} \otimes V) : W \cap \overline{W} = \{0\} \} \]

via the map

\[ \mathcal{E}_V \ni J \mapsto W(J) = \{1 \otimes v + i \otimes Jv : v \in V \} \in \mathcal{C}_V \]

which sends \( J \) to the \((-i)\)-eigenspace of its extension to \( \mathbb{C} \otimes V \).

**Sketch of the proof.** We define the map \( W^{-1} : \mathcal{C}_V \to \mathcal{E}_V \). Consider the following commutative diagram

\[
\begin{array}{c}
0 \rightarrow W \rightarrow \mathbb{C} \otimes V \rightarrow \overline{W} \rightarrow 0 \\
\uparrow \phi \\
V \end{array}
\]
which is exact on the first line and where the vertical arrow is the map sending \( v \) to \( 1 \otimes v \). As the image of \( V \) through this map intersects \( W \) and \( W \) only in \( \{0\} \), the map \( \omega \) is a real isomorphism between \( V \) and \( W \).

Let \( J_0 \) be the standard complex structure on \( \mathbb{C} \otimes V \), i.e. defined by \( J_0(z \otimes v) = (iz) \otimes v \), then

\[
W^{-1}(W) = \omega^{-1} \circ J_0 \circ \omega.
\]

It is easy to check that \( W^{-1} \) is indeed the inverse of \( W \) and that these two maps are continuous. Therefore \( E \) is an algebraic subvariety of \( \text{End}(V) \) which is homeomorphic to an open set of \( \text{Gr}_C(n, \mathbb{C} \otimes V) \), so \( E \) is a smooth manifold.

The complex structure \( J_0 \) lifts to a complex structure \( \tilde{J}_0 \) on \( \text{Gr}_C(n, \mathbb{C} \otimes V) \) and one can check that, for \((J, A) \in T_{E} \) one has

\[
W^* \tilde{J}_0 (J, A) = L_J(A).
\]

□

It is a simple matter of checking definitions to note that \((S, J_S)\) is then a complex submanifold of \( E \) upon identifying \( s \) with \( L_s \in \text{End}(A) \); the complex structure \( J_S \) coincides, on \( T_s S \), with the restriction of the map \( L_{L_s} \) and, by associativity, \( L_s L_h = L_{sh} \) for \( s \in S, h \in T_s S \).

Hence, through the map \( W \), we identify \( S \) with a complex submanifold \( S = \{W(L_s) : s \in S\} \) of an open domain of the complex Grassmannian \( \text{Gr}_C(n, \mathbb{C} \otimes A) \).

We have all the ingredients to generalize the results of Section 3 in [13] (see, in particular, Theorem 3.3).

**Theorem 4.2.** Let \( A \) be an associative algebra and consider

\[
Z = \{w \in \mathbb{C} \otimes A : \pi(w, s) = 0 \text{ for some } s \in S\}
\]

\[
\mathfrak{Z} = \{(w, s) \in \mathbb{C} \otimes A \times S : \pi(w, s) = 0\}.
\]

Then the incidence variety \( \mathfrak{Z} \) is a complex submanifold of \( \mathbb{C} \otimes A \times S \) and the zero variety \( Z \) is a complex subspace of \( \mathbb{C} \otimes A \); moreover, there exists a proper surjective holomorphic map from \( \mathfrak{Z} \) to \( Z \).

**Proof.** Write \( w \in \mathbb{C} \otimes A \) as \( 1 \otimes a + i \otimes b \), with \( a, b \in A \), then \( \pi(w, s) = a + sb \). We note that if \( \pi(w, s) = 0 \), then \( -a = sb \), i.e. \( sa = b \), so \( w = 1 \otimes a + i \otimes sa \in W(L_s) \); on the other hand, if \( w \in W(L_s) \), then \( w = 1 \otimes a + i \otimes L_s(a) \) so \( \pi(w, s) = a + s^2a = a - a = 0 \).

Therefore, the holomorphic embedding \( \mathbb{C} \otimes A \times S \ni (w, s) \to (w, W(L_s)) \in \mathbb{C} \otimes A \times \text{Gr}_C(n, \mathbb{C} \otimes A) \) sends \( \mathfrak{Z} \) to

\[
\{(w, W) \in \mathbb{C} \otimes A \times \text{Gr}_C(n, \mathbb{C} \otimes V) : w \in W, W \in S\}
\]
i.e the total space of the restriction to $S$ of the tautological bundle of the Grassmannian, which is clearly a complex submanifold of the trivial bundle $\mathbb{C} \otimes A \times S$.

Now, it is easy to see that the projection $\mathbb{C} \otimes A \times S \ni (w, s) \mapsto w \in \mathbb{C} \otimes A$ is proper and holomorphic and that its restriction to $\mathcal{F}$ is surjective on $Z$, which is then a complex subvariety of $\mathbb{C} \otimes A$, by the Proper Mapping Theorem by Remmert (see [12]).

As in [13], we can define the sets

$$Z(a) = \left\{ w \in \mathbb{C} \otimes A : \pi(w, s) = a \text{ for some } s \in S \right\}$$

$$\mathcal{F}(a) = \left\{ (w, s) \in \mathbb{C} \otimes A \times S : \pi(w, s) = a \right\}$$

and we have the following.

**Corollary 4.3.** The set $Z(a)$ is a complex subvariety of $\mathbb{C} \otimes A$, obtained from $Z$ by a translation of a real vector.

The complex space $\mathbb{C} \otimes A \times S$ is foliated by the sets $a$; such a foliation is real-analytic, with complex leaves, and $A$ is the space of parameters.

Unlike the case of the quaternions, $Z$ is not a hypersurface in $\mathbb{C} \otimes A$, so it is not the zero of a single holomorphic function. We can, nonetheless, use these results to obtain some information on the behaviour of slice-regular functions, for example, on the structure of their zeroes. For a more detailed study of the zeroes of a slice-regular function on a real algebra, the interested reader could see the recent works of Ghiloni, Perotti and Stoppato [10, 11].

**Proposition 4.4.** With the previous notation, consider $V = \pi(U \times S)$, with $U$ intersecting the real axis, and a slice-regular function $f : V \to A$, then the connected components of $f^{-1}(0)$ are either isolated real points or complex subvarieties of $\pi(U_+ \times S)$, where $U_+ = U \cap \mathbb{C}_+$, biholomorphic to subvarieties of $S$.

**Proof.** We have that $\pi(z, s) \in f^{-1}(0)$ if and only if $\mathcal{F}(z, s) \in \mathcal{F}$. It is obvious that, for $a \in A$, $(1 \otimes a, s) \in \mathcal{F}$ if and only if $a = 0$, therefore, as $\mathcal{F}(U \times S)$ contains $\mathcal{F}(U \cap \mathbb{R}) \times S)$, which contains only points of the form $(1 \otimes a, s)$, by property (1) in Proposition 3.9 we have that $\mathcal{F}(U \times S) \subseteq Z$ if and only if $\mathcal{F}(U \times S) = \{0\} \times S$.

This means that the set $\{z \in U : \mathcal{F}(\{z\} \times S) \cap Z \neq \emptyset\}$ is discrete in $U$; therefore, its connected components are of the form $\{z_j\} \times Y$, with $Y \subseteq S$ a complex subvariety; if $z_j \not\in \mathbb{R}$, then $\pi$ embeds $\{z_j\} \times Y$ as a complex subvariety in $\pi(U_+ \times S)$, otherwise, it is sent to the real point $z_j$.

**Remark 4.1.** If $U$ does not intersect the real axis, it could also happen that $F(U) \subseteq Z(0)$. In that case, for each $z \in U$, we define the set

$$S_z = \{s \in S : \mathcal{F}(z, s) \in \mathcal{F}(0)\}$$
and we know that it is not empty; in general, if $F(z)$ is invertible, $S_z$ is a singleton, hence the set

$$\{(z,s) \in U \times S : s \in S_z\}$$

is, outside a discrete union of subvarieties of $S$, a Riemann surface in $U \times S$.

We close this section by showing one interesting property of the zero variety: the set $Z$ "absorbs the products", from the right and, if $S$ is compact, from the left. Given its connection with the zero set of slice-regular functions, this sounds like a very reasonable property, however the proof, at least for the left side, is not trivial; we don't know if the compactness assumption could be dispensed with.

It is worth noticing that the proofs proposed do not use the relationship with slice-regular functions.

**Proposition 4.5.** If $w \in Z$ and $w' \in \mathbb{C} \otimes A$, then $ww' \in Z$.

*Proof.* Write $w = 1 \otimes x + i \otimes y$, as $z \in V_0$, then there exists $a \in S$ such that $x + ay = 0$, that is $x = -ay$. Let $w' = 1 \otimes s + i \otimes t$, then

$$ww' = (1 \otimes x + i \otimes y)(1 \otimes s + i \otimes t) = 1 \otimes (xs - yt) + i \otimes (xt + ys) = -1 \otimes (ays + yt) + i \otimes (-ayt + ys)$$

so, $\pi(ww', s) = -ays - yt - a^2 yt + ays = 0$, i.e. $ww' \in Z$. \square

**Proposition 4.6.** Suppose $S$ is compact. If $w \in Z$ and $w' \in \mathbb{C} \otimes A$, then $w'w \in Z$.

*Proof.* Write $w = 1 \otimes x + i \otimes y$, as $z \in Z$, then there exists $a \in S$ such that $x + ay = 0$, that is $x = -ay$. Let $w' = 1 \otimes s + i \otimes t$, then

$$w'w = (1 \otimes s + i \otimes t)(1 \otimes x + i \otimes y) = 1 \otimes (sx - ty) + i \otimes (sy + tx)$$

$$= -1 \otimes (say + ty) + i \otimes (sy - tay) = (-1 \otimes (sa + t) + i \otimes (ta - s))1 \otimes y .$$

We note that $(sa + t)a = ta - s$, so $sa + t$ is invertible if and only if $ta - s$ is so.

If $ta - s$ is invertible, then we set

$$b = (sa + t)(ta - s)^{-1}$$

and we note that

$$b^2 = (sa + t)(ta - s)^{-1}(sa + t)(ta - s)^{-1} =$$

$$= (s - ta)a(ta - s)^{-1}(ta - s)(-a)(ta - s)^{-1} =$$

$$= -(ta - s)a(-a)(ta - s)^{-1} = -1 .$$

Moreover, $\pi(w'w, b) = 0$, hence $w'w \in Z$.

If $ta - s$ is not invertible, consider the set

$$X_a = \{(s,t,b) \in A \times A \times A : sa + t = b(ta - s) \text{ and } b^2 = -1\} .$$
It is easy to show that $X_a$ is a real analytic space in $\mathbb{R}^{3N}$, hence closed; we consider also the map $h : A^3 \to \mathbb{C} \otimes A$ given by $h(s, t, b) = 1 \otimes s + i \otimes t$.

We note that the set $Y$ of invertible elements in $A$ is an open dense subset and its complement is a real-analytic subset, hence

$$W = \{ \omega \in \mathbb{C} \otimes A : \pi(\omega, a) \in Y \}$$

is an open dense subset of $\mathbb{C} \otimes A$; for $(s + it) \in W$, we know that $h^{-1}(s + it) \cap X_a \neq \emptyset$. Therefore, $h(X_a)$ contains $W$. Now, let $w_0 \in \mathbb{C} \otimes A \setminus W$ and take $\{w_j\} \subset W$ such that $w_j \to w_0$; let $b_j \in S$ be such that $(w_j, b_j) \in X_a$. By compactness of $S$, up to taking a subsequence, we have that $b_j \to b_0 \in S$ and, as $X_a$ is closed, $(w_0, b_0) \in X_a$.

This implies that $h(X_a) = \mathbb{C} \otimes A$; therefore we can always find $b \in S$ such that $sa + t = b(ta - s)$. Hence $\pi(w'w, b) = 0$, so $w'w \in Z$. □

5. A generalized stereographic map

Given $u \in S$, we define on $A$ the structure of a complex vector space by setting, for $\lambda \in \mathbb{C}$, $\lambda = x + iy$,

$$\lambda \cdot a = (x + uy) a .$$

Let us denote by $E$ the complex vector space $(A, L_u)$ and by $\gamma : E \to A$ the real isomorphism between $E$ and $A$; let $D$ be the subset of zerodivisors of $A$ (and hence of $E$).

We define the space

$$V = \{(z, -s) \in E \times S : \gamma(z) s = w\gamma(z)\}$$

and the projection $p : V \to S$.

**Lemma 5.1.** For every $s \in S$, $p^{-1}(-s) = V_s \times \{-s\}$, where $V_s$ is a (complex) linear subspace of $E$. If $u$ does not belong to the center of $A$, then $V_s$ is of positive dimension for every $s \in S$.

**Proof.** Let us fix $s \in S$, then

$$V_s = \{ z \in E : R_s L_u \gamma(z) = -\gamma(z) \}$$

where $R_s \in \text{End}(A)$ is defined by $R_s(q) = qs$. We note that $R_s L_u = L_u R_s$, hence

$$(R_s L_u)^2 = R_s^2 L_u^2 = (-I)(-I) = I .$$

Therefore, $R_s L_u$ is diagonalizable with eigenvalues $\pm 1$ and we are looking for its eigenspace relative to the eigenvalue $-1$; this eigenspace is trivial if and only if $R_s L_u = I$.

If $R_s L_u = I$, then $R_s = -L_u$ and also $su = 1$, i.e. $s = -u$. Therefore $R_u = -L_u$, so $qu + uq = 0$ for every $q \in A$, i.e. $u$ belongs to the center; if $u$ is not in the center, $V_s$ is never trivial.

Finally, we check that, if $z \in V_s$, then $\gamma(iz) = L_u \gamma(z)$, so

$$R_s L_u \gamma(iz) R_s L_u L_u \gamma(z) = L_u R_s L_u \gamma(z) = L_u (-\gamma(z)) = -L_u \gamma(z) = -\gamma(iz) .$$
which means $iz \in \mathcal{V}_s$. \qedhere

**Remark 5.1.** The map $E \setminus D \ni z \mapsto \gamma(z)^{-1} u \gamma(z) \in S$ is holomorphic.

As usual, if $A$ is a Clifford algebra, we have more precise information.

**Corollary 5.2.** If $A$ is a Clifford algebra, $\dim_{\mathbb{C}} \mathcal{V}_s = N/4$ for all $s \in S$.

**Proof.** We note that $\text{tr}(R_s L_u) = 0$ for all $s \in S$: this follows from a computation very similar to the one carried out in the proof of Corollary 3.8. So the dimension of the $(1)$-eigenspace and the dimension of the $(-1)$-eigenspace coincide, hence the desired result. \qed

The construction of the complex structure $J_S$ employed the embedding of $A$ into $\text{End}(A)$ by sending an element $a$ to the endomorphism $L_a$; however, we could as well consider the embedding $a \mapsto R_a$. This map turns out to be antiholomorphic, from $S$ to $\mathcal{V}_A$ (with the notation of Theorem 4.1): if $h \in T_s S$, then $sh = -hs$ and

$$
\lim_{t \to 0} \frac{R_{s+th} - R_s}{t} = R_h,
$$

so $L_sh = sh = -hs$ is sent by the differential of the map to $R_{-hs} = -R_s R_h \in T_{R_s} \mathcal{V}_A$ and $-R_s R_h = -L_{R_s}(R_h)$.

**Proposition 5.3.** The space $\mathcal{V}$ is a complex submanifold of $E \times S$ and, in particular, it has a structure of a fiber bundle over $S$.

**Proof.** The complex vector space $E$ is isomorphic to the $i$-eigenspace of $L_u$ in $\mathbb{C} \otimes A$ via the map

$$v \mapsto 1 \otimes v - i \otimes L_u v .$$

Under this map, the $(-1)$-eigenspace of $R_s L_u$ is sent to the intersection of the $i$-eigenspaces of $L_u$ and $R_s$. We denote by $E_c(T)$ the $c$-eigenspace of the map $T$ in $\mathbb{C} \otimes A$, then

$$\mathcal{V}_s = E_i(L_u) \cap E_{-i}(R_{-s}) .$$

As we noted before, the map $a \mapsto R_a \mapsto E_{-i}(R_a)$ is antiholomorphic; therefore $s \mapsto E_i(L_u) \cap E_{-i}(R_{-s})$ is a holomorphic map from $S$ to $\text{Gr}_C(E)$ and $\mathcal{V}$ is obtained by pulling back the tautological bundle, hence it is a complex subvariety of the trivial bundle $E \times S$. \qed

By a standard procedure, $\text{Gr}_C(E)$ can be embedded as a closed subvariety of a projective space $\mathbb{CP}^M$.

Suppose that, possibly by removing a closed subvariety $Y \subset S$, we obtain a section $\sigma' : S \setminus Y \to \mathcal{V}$ which never vanishes. Let $D \subset A$ be the set of zerodivisors of $A$, it is easy to show that $L_u(D) \subset D$; suppose that $\sigma(s)$ is contained in $\mathbb{P}(\mathcal{V}_s \setminus D)$ for every $s \in S \setminus Y$. We call $\sigma$ a parametrization of $S$. 
Remark 5.2. If $S \setminus Y$ embeds as a contractible affine subvariety in some affine space, then every holomorphic bundle is holomorphically trivial, hence a never vanishing section $\sigma'$ exists for $V$ restricted to $S \setminus Y$. Taking the projective closure of the graph, we obtain a map $\sigma : S \to \mathbb{P}(E)$; as long as $p(D)$ is nowhere Zariski-dense in $S$, we can find $\sigma$ that avoids $D$.

We fix a basis $\{v_1, \ldots, v_N\}$ of $A$ as a real vector space, with $v_1 = 1, v_2 = u$; we define the maps $T_j : E \to E$ by setting

$$T_j(z) = \gamma^{-1}(\gamma(z)v_j) .$$

It is easy to check that the $T_j$'s are $\mathbb{C}$-linear maps. Then, $\pi : \mathbb{C} \otimes A \times S \to A$ is given by

$$\pi((z \otimes q, s)) = \gamma(\sigma(s))^{-1}z_0 \gamma(\sigma(s))q = \gamma(\sigma(s))^{-1}\gamma(zT_q(\sigma'(s)))$$

where $T_q = \sum q_jT_j$ and $q = \sum q_jv_j$.

Therefore, we have a generalization of Lemma 2.3.

Proposition 5.4. Let $\sigma$ be a parametrization of $S$. If $V = \pi(U \times S)$ an axially symmetric domain and $f : V \to A$ a slice-regular function, then there exists a unique holomorphic function

$$\tilde{F} : U \times S \to \mathbb{P}(E \oplus E)$$

which induces $f$.

Proof. By [9], we have a holomorphic function $F : U \to \mathbb{C} \otimes A$ inducing $f$; we write

$$F = F_0v_1 + F_1v_2 + F_2v_3 + \ldots F_Nv_N .$$

Given $(z, s)$, we set

$$\tilde{F}(z, s) = [(\gamma(\sigma(s)), \gamma(T_{F(z)}(\sigma(s))))]$$

where

$$T_{F(z)}(w) = \sum_j F_j(z)T_j(w) .$$

We define the map $\rho_1 : E \oplus E \to A$ by $\rho_1(z, w) = \gamma(z)^{-1}\gamma(w)$ and we note that such map passes to the quotient a map $\rho_1 : \mathbb{P}(E \oplus E) \to A$ such that $f \circ \pi = \rho_1 \circ \tilde{F}$. \hfill $\square$

6. Orthogonal complex structures

In this section, we study the elements of $A$ which induce on $A$, by left multiplication, a complex structure which is orthogonal with respect to some given inner product; let $\langle \cdot, \cdot \rangle$ be an inner product on $A$, i.e. a bilinear symmetric $\mathbb{R}$-valued positive form. We remark that this is a completely generic inner product, with no links with the algebra structure of $A$; in particular, given $a \in A$, the adjoint endomorphism $L_a^t$ may very well not be of the form $L_b$ for any $b \in A$. 
We define 
\[ S_0 = \{ s \in S : L_s L_s^t = I \} , \]
i.e. the set of elements of \( A \) which induce, by left multiplication, a complex structure on \( A \) which is orthogonal with respect to \( \langle \cdot, \cdot \rangle \).

**Remark 6.1.** We note that, when \( A = \mathbb{H} \), there is no difference between \( S \) and \( S_0 \), when we consider the inner product that makes the basis \( \{1, i, j, k\} \) orthonormal and positively oriented.

We have the following alternative description of \( S_0 \).

**Lemma 6.1.** Given \( a \in A \), any two of the following conditions imply the third one:

1. \( s \in S \)
2. \( L_s L_s^t = I \)
3. \( L_s + L_s^t = 0 \).

**Proof.** First of all, \( s \in S \) if and only if \( L_s^2 = -I \), i.e. if and only if \( L_s^{-1} = -L_s \). If (1) and (2) hold, then \( L_s^t = L_s^{-1} = -L_s \), so (3) holds; if (1) and (3) hold, then \( L_s^t = -L_s = L_s^{-1} \), so (2) holds; if (2) and (3) hold, \( -L_s = L_s^t = L_s^{-1} \), so (1) holds. \( \square \)

Contrary to \( S \), the set \( S_0 \) is always compact.

**Corollary 6.2.** The set \( S_0 \) is compact.

**Proof.** The map \( a \mapsto L_a \) is a linear injective map from \( A \) to \( \text{End}(A) \), hence proper, and the set of orthogonal antisymmetric endomorphisms of \( A \) is compact, therefore also its preimage is compact. By the previous lemma, such preimage is \( S_0 \). \( \square \)

It is easy to see that \( S_0 \) is defined by a finite number of algebraic equations in \( A \), so it is an algebraic subvariety of \( A \) (and of \( S \)); by Lemma 6.1, \( S_0 \) can be described as the intersection of \( S \) with the linear subspace \( A_0 = \{ a \in A : L_a + L_a^t = 0 \} \), therefore, at any regular point \( s \in S_0 \), we have that \( T_s S_0 = T_s S \cap A_0 \).

**Lemma 6.3.** For any \( s \in S_0 \), the subspace \( T_s S \cap A_0 \) of \( T_s A \) is \( J_S \)-invariant.

**Proof.** If \( h \in T_s S \cap A_0 \), then \( sh + hs = 0 \) and \( L_h + L_h^t = 0 \). By definition
\[ J_S(s, h) = (s, sh) \]
and we already know that \( sh \in T_s S \). On the other hand, \( L_{sh} = L_s L_h \) and \( L_{sh}^t = L_h^t L_s = L_h L_s \) (because \( s, h \in A_0 \)), so
\[ L_{sh} + L_{sh}^t = L_{sh} + L_{hs} = 0 \]
as \( h \in T_s S \). Therefore the subspace \( T_s S \cap A_0 \) is \( J_S \)-invariant. \( \square \)
Therefore, the regular part of $S_0$ is a complex submanifold of $S$; moreover, $S_0$ is of finite volume around its singular points, being algebraic and compact, therefore, by King’s extension theorem, $S_0$ is a complex subvariety of $S$, possibly singular.

**Theorem 6.4.** The set $S_0$ is a complex subvariety of $S$, with respect to the complex structure $J_S$.

If we let the inner product vary, we obtain a family of complex subvarieties of $S$, whose union is clearly $S$. Another description of the set $S_0$ is the following.

**Proposition 6.5.** We have
\[ S_0 = \{ s \in A : \langle sa, sa \rangle = \langle a, a \rangle, \langle sa, a \rangle = 0 \ \forall a \in A \}. \]

**Proof.** Let $a_1, \ldots, a_N$ be an orthonormal basis for $A$ as a real vector space; then $L_s$ is orthogonal if and only if $L_s^2 L_s = I$, i.e. if and only if
\[ \langle L_s^2 L_s a_j, a_j \rangle = 1 \quad \langle L_s^2 L_s a_j, a_k \rangle = 0 \text{ if } j \neq k \]
and, as $L_s^2 L_s$ is symmetric, we have that $\langle L_s^2 L_s a_j, a_k \rangle = \langle L_s^2 L_s a_k, a_j \rangle$, which gives
\[ \langle L_s^2 L_s (a_j + a_k), (a_j + a_k) \rangle = \langle L_s^2 L_s a_j, a_j \rangle + \langle L_s^2 L_s a_k, a_k \rangle + 2 \langle L_s^2 L_s a_j, a_k \rangle. \]
Therefore,
\[ \langle L_s^2 L_s a_j, a_j \rangle = 1 \quad \langle L_s^2 L_s a_j, a_k \rangle = 0 \text{ if } j \neq k \]
is equivalent to
\[ \langle L_s^2 L_s a_j, a_j \rangle = 1 \quad \langle L_s^2 L_s (a_j + a_k), (a_j + a_k) \rangle = 2 \text{ if } j \neq k. \]
By bilinearity and the symmetry of $L_s^2 L_s$, this is equivalent to $\langle L_s^2 L_s a, a \rangle = \langle a, a \rangle$, i.e. to $\langle sa, sa \rangle = \langle a, a \rangle$ for all $a \in A$.

In the same way, $L_s^2$ is antisymmetric if and only if
\[ \langle (L_s + L_s^2) a, a \rangle = 0 \quad \forall a \in A, \]
+ as before, because $L_s + L_s^2$ is symmetric. We note that
\[ \langle (L_s + L_s^2) a, a \rangle = 0 \iff \langle L_s a, a \rangle + \langle L_s^2 a, a \rangle = 0 \iff \langle L_s a, a \rangle + \langle a, L_s a \rangle = 0 \]
which happens if and only if $\langle sa, a \rangle = 0$ for all $a \in A$. \qed

We consider the set $Q_0 = \pi(\mathbb{C} \times S_0) \subseteq A$, where $\pi : \mathbb{C} \times S \to A$ is as defined above; we can characterize the set $Q_0$ as follows.

**Proposition 6.6.** The set $Q_0$ is the largest subset of $A$ which is invariant under real translations and such that every element $x$ satisfies $L_x^2 L_x = \|x\|^2 I$.  

Proof. Obviously, $Q_0$ is invariant under real translations: if $r \in \mathbb{R}$ and $x \in A$ is of the form $x_0 + sx_1$ for some $s \in S_0$, then $x + r = (x_0 + r) + sx_1$ is again in $Q_0$.

Moreover, if $x = x_0 + sx_1$, then $L_x = x_0 I + x_1 L_s$ and $L_x^t = x_0 I - x_1 L_s$, as $s \in S_0$; so

$$L_x^t L_x = (x_0^2 + x_1^2) I .$$

On the other hand,

$$\|x\|^2 = (x_0 + sx_1, x_0 + sx_1) = |x_0|^2 + |x_1|^2 \|s\|^2 + 2x_0 x_1 \langle 1, s \rangle$$

and, as $u \in S_0$, we know that $\|s\| = 1$ and $\langle s, 1 \rangle = 0$.

For the converse implication, take $x \in A$, with $\|x\| = 1$, such that $L_x^t L_x = I$ and suppose that also $(L_x^t + I)(L_x + I) = \alpha^2 I$. Then

$$\alpha^2 I = L_x^t L_x + I + (L_x + L_x^t) = 2I + L_x + L_x^t$$

so

$$L_x + L_x^t = 2 \beta I .$$

Therefore, $L_x - \beta I = L_{x-\beta}$ is antisymmetric and, by hypothesis,

$$L_{x-\beta} L_{x-\beta} = -L_{x-\beta}^t L_{x-\beta} = -\gamma^2 I .$$

This means that $\gamma^{-1} L_{x-\beta} = L_s$ for $s \in S_0$, that is, $x - \beta = \gamma s$, i.e. $x = \beta + \gamma s$, i.e. $x \in Q_0$. \hfill \qed

The following alternative description is easily deduced.

**Corollary 6.7.** We have that $x \in Q_0$ if and only if there exist real numbers $\alpha, \beta$ with $\alpha^2 \geq \beta^2$, such that

$$L_x + L_x^t = 2 \beta I \quad L_x^t L_x = \alpha^2 I .$$

**Proof.** If $x \in Q_0$, then $x = x_0 + sx_1$, so $L_x = x_0 I + x_1 L_s$. Therefore

$$L_x^t + L_x = 2x_0 I \quad L_x^t L_x = (x_0^2 + x_1^2) I ,$$

so we set $\beta = x_0$, $\alpha = x_0^2 + x_1^2$ and we notice that $\alpha^2 \geq \beta^2$.

For the converse implication, if $x \in A$ is such that

$$L_x + L_x^t = 2 \beta I \quad L_x^t L_x = \alpha^2 I ,$$

with $\alpha^2 \geq \beta^2$, then $L_x - \beta I$ is antisymmetric and

$$(L_x - \beta I)^t (L_x - \beta I) = L_x^t L_x - \beta (L_x + L_x^t) + \beta^2 I = (\alpha^2 - \beta^2) I .$$

If $\alpha^2 = \beta^2$, then $L_x - \beta I = 0$, i.e. $x = \beta \in \mathbb{R}$.

If $\alpha^2 - \beta^2 > 0$, then we can take $\gamma = \sqrt{\alpha^2 - \beta^2}$ and conclude as in the proof of Proposition 6.6. \hfill \qed
The two previous results establish a clear correspondence between our set $Q_0$ and the quadratic cone $Q_A$ defined by Ghiloni and Perotti in [9]; therefore, for the appropriate choice of the inner product $\langle \cdot, \cdot \rangle$, the set $S_0$ is the set $S_A$ defined in [9].

6.1. Equations for the zero variety. The set $Z$ defined in Section 4 is a complex subvariety of the affine complex space $A$, so it has global equations; sadly, we have no general method to compute them.

On the other hand, if we consider $S_0$ as a set of square roots of $-1$ and we define

$$Z_0 = \{ w \in \mathbb{C} \otimes A : \pi(w, s) = 0 \text{ for some } s \in S_0 \},$$

then we can obtain some results. Let $Q : \mathbb{C} \otimes A \times \mathbb{C} \otimes A \rightarrow \mathbb{C}$ be the $\mathbb{C}$-bilinear extension of the inner product on $A$ and $\Phi : \mathbb{C} \otimes A \rightarrow \mathbb{C}$ be the quadratic form $\Phi(w) = Q(w, w)$.

Proposition 6.8. If $w \in \mathbb{C} \otimes A$ belongs to $Z_0$ then $\Phi(ww') = 0$ for all $w' \in \mathbb{C} \otimes A$.

Proof. If $w \in Z_0$, then there exists $s \in S_0$ such that $\pi(w, s) = 0$. As in the proof of Theorem 1.2 we note that $\pi(w, s) = 0$ if and only if $w \in W(L_s)$. Hence

$$w = 1 \otimes v + i \otimes L_v$$

for some $v \in A$. Therefore, writing $I_1$ for the identity map in $\text{End}(\mathbb{C})$ and $I$ for the identity map in $\text{End}(A)$,

$$ww' = (I_1 \otimes L_v)w' + i(I_1 \otimes L_s L_v)w'.$$

We note that

$$((I_1 \otimes L_v) + i(I_1 \otimes L_s L_v))^t = (I_1 \otimes L_v^t) - i(I_1 \otimes L_v^t L_s)$$

so

$$((I_1 \otimes L_v) + i(I_1 \otimes L_s L_v))^t((I_1 \otimes L_v) + i(I_1 \otimes L_s L_v)) = ((I_1 \otimes L_v^t) - i(I_1 \otimes L_v^t L_s))((I_1 \otimes L_v) + i(I_1 \otimes L_s L_v)) =

= (I_1 \otimes L_v^t)(I_1 \otimes I + I_1 \otimes L_s^2)(I_1 \otimes L_v) = 0$$

Therefore

$$\Phi(ww') = Q((I_1 \otimes L_v)w' + i(I_1 \otimes L_s L_v)w', (I_1 \otimes L_v)w' + i(I_1 \otimes L_s L_v)w') = Q(w', 0) = 0,$$

as we have $Q(T \cdot, \cdot) = Q(\cdot, T^t \cdot)$ for every endomorphism $T$ of $\mathbb{C} \otimes A$.

We note that the set

$$Z = \{ w \in \mathbb{C} \otimes A : \Phi(ww') = 0 \text{ for every } w' \in \mathbb{C} \otimes A \}$$

is given by a finite number of quadratic equations in $\mathbb{C} \otimes A$, namely, by the $N(N + 1)/2$ equations $L_v^t L_v = 0$, therefore it is a complex subvariety of $\mathbb{C} \otimes A$.

Theorem 6.9. Suppose that the space $Z$, described by (5), is irreducible as a complex subvariety of $\mathbb{C} \otimes A$, then it coincides with $Z_0$. 

□
Proof. By the previous Proposition, we already know that
\[ Z_0 \subseteq \{ w \in \mathbb{C} \otimes A : \Phi(ww') = 0 \text{ for every } w' \in \mathbb{C} \otimes A \} = \{ w \in \mathbb{C} \otimes A : L^t_w L_w = 0 \} . \]
On the other hand, if we write \( w \in \mathbb{C} \otimes A \) as \( w = 1 \otimes a + i \otimes b \), then
\[ L^t_w L_w = (I_1 \otimes (L^t_a L_a - L^t_b L_b)) + i(I_1 \otimes (L^t_b L_a + L^t_a L_b)) ; \]
so, \( L^t_w L_w = 0 \) if and only if \( L^t_a L_a = L^t_b L_b \) and \( L^t_b L_a + L^t_a L_b = 0 \).

If \( L^t_w L_w = 0 \) and \( a \) is invertible, then also \( b \) is invertible, and vice versa; in such case, we set \( u = yx^{-1} \) and it is easy to check that
\begin{align*}
(1) & \quad L^t_b L_u = I \\
(2) & \quad L^t_a + L_u = 0 \\
(3) & \quad L_b = L_a L_u .
\end{align*}
Therefore, \( w \in Z_0 \). Now, being invertible is an open condition, so there is an open subset of the space describe by (5) which is contained in \( Z_0 \); by irreducibility, the two complex subvarieties coincide. \( \square \)

7. Generalized slice-regular functions

Consider the following definition.

**Definition 7.1.** Let \( A \) be a real associative algebra, \( U \subseteq \mathbb{C} \) an open domain and \( V = \pi(U \times S) \); a function \( f : V \to A \) is called a *generalized slice-regular function* if there exists a holomorphic map \( \Phi : U \times S \to \mathbb{C} \otimes A \times S \) such that \( f \circ \pi = \pi \circ \Phi \) and \( \Phi \circ \sigma = \tau \circ \Phi \).

**Lemma 7.2.** Let \( A \) be an associative algebra and suppose that \( S \) is compact and connected. If \( f : V \to A \) is a generalized slice-regular function, there exist a holomorphic function \( F : \mathbb{C} \otimes A \to S \) and a holomorphic map \( \Phi : U \times S \to S \) such that
\[ f(\pi(z, s)) = \pi(F(z), \Phi(z, s)) . \]

**Proof.** We write \( \Phi = (F, \Phi) \) with \( F : U \times S \to \mathbb{C} \otimes A \) and \( \Phi : U \times S \to S \) holomorphic. However, as \( S \) is compact and \( \mathbb{C} \otimes A \) is Stein, \( F(z \times S) \) is a single point, as \( S \) is connected, therefore we can interpret \( F \) as a holomorphic function \( F : U \to \mathbb{C} \otimes A \). \( \square \)

The classical definition of slice-regular function is obtained with \( \Phi(z, s) = s \) for all \( (z, s) \in U \times S \). A number of properties, that are true for holomorphic functions, hold also for generalized slice-regular functions, for example:

- \( f(\pi(z, s)) = \alpha(z) + \Phi(z, s)\beta(z) \), with \( \alpha, \beta \) as in Section 3.1
- the maximum property holds
- zeros are a discrete union of subvarieties of \( S \)
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• when $A = \mathbb{H}$, a Rouchè formula, an argument principle and a Hurwitz theorem hold.

If one is interested in more quantitative properties, like representation formulas, the form of the function $\Phi$ comes into play; we can put some restrictions on the form of $\Phi$ and obtain some subclasses of generalized slice-regular functions.
1. If we ask that for every $z \in U$ the map $s \mapsto \Phi(z, s)$ is injective (or even an automorphism of $S$), we obtain functions which have generically isolated zeros.
2. If we fix a holomorphic map $z \mapsto \phi_s \in \text{Aut}(S)$, we can consider all the functions $f$ for which $\Phi(z, s) = \phi_s(s)$.
3. If we fix $\phi \in \text{Aut}(S)$, we can consider all the functions $f$ for which $\Phi(z, s) = \phi(s)$.

In particular, for the third class of functions a representation result similar to the Cauchy formula should hold.

Suppose now that we have two associative algebras $A_1$ and $A_2$, with $S_1$, $S_2$ associated sets of square roots of $-1$; we define the functions $\pi_1 : \mathbb{C} \times S_1 \to A_1$ and $\pi_2 : \mathbb{C} \otimes A_2 \times S_2 \to A_2$ as above. Consider also the involutions $\sigma_1 : \mathbb{C} \times S_1 \to \mathbb{C} \times S_1$ and $\tau_2 : \mathbb{C} \otimes A_2 \times S_2 \to \mathbb{C} \otimes A_2 \times S_2$.

**Definition 7.3.** Let $U \subseteq \mathbb{C}$ an open domain and $V_i = \pi_i(U \times S_i)$; a function $f : V \to A_2$ is called a generalized slice-regular function if there exists a holomorphic map $\mathfrak{F} : U \times S_1 \to \mathbb{C} \otimes A_2 \times S_2$ such that $f \circ \pi_1 = \pi_2 \circ \mathfrak{F}$ and $\mathfrak{F} \circ \sigma_1 = \tau_2 \circ \mathfrak{F}$.

As before, such a function is of the form $\mathfrak{F}(z, s) = (F(z), \Phi(z, s))$ and we can define analogues of the three classes above.

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