ONE MORE PATHOLOGY OF $C^*$-ALGEBRAIC TENSOR PRODUCTS

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ABSTRACT. We define a collection of tensor product norms for $C^*$-algebras and show that a symmetric tensor product functor on the category of separable $C^*$-algebras need not be associative.

1. INTRODUCTION

Following E. Kirchberg, [3], we call a bifunctor $(A, B) \rightarrow A \otimes B$ a $C^*$-algebraic tensor product functor if it is obtained by completing of the algebraic tensor product $A \otimes B$ of $C^*$-algebras in a functional way with respect to a suitable $C^*$-norm $\| \cdot \|_\alpha$. We call such a functor symmetric if the standard isomorphism $A \otimes B \cong B \otimes A$ extends to an isomorphism $A \otimes \alpha B \cong B \otimes \alpha A$. Similarly, we call it associative if the standard isomorphism $A \circ (B \circ C) \cong (A \circ B) \circ C$ extends to an isomorphism $A \otimes \alpha (B \otimes \alpha C) \cong (A \otimes \alpha B) \otimes \alpha C$ for any $C^*$-algebras $A, B, C$. It is well known that both the minimal tensor product functor $\otimes_{\text{min}}$ and the maximal tensor product functor $\otimes_{\text{max}}$ are symmetric and associative.

In this paper we construct a collection of symmetric $C^*$-algebraic tensor product functors related to asymptotic homomorphisms of $C^*$-algebras. For technical reasons we restrict ourselves to the category of separable $C^*$-algebras. Using $C^*$-algebras related to property T groups [9] we show that some of these tensor product functors are not associative.

Recall that asymptotic homomorphisms of $C^*$-algebras were first defined and studied in [2] in relation to topological properties of $C^*$-algebras. The most important and the best known case is the case of asymptotic homomorphisms from a suspended $C^*$-algebra $SA$ to the $C^*$-algebra $\mathbb{K}$ of compact operators, since the homotopy classes of those are the $K$-homology of $A$, the $E$-theory. Asymptotic homomorphisms to other $C^*$-algebras are less known. For example, it is known that any asymptotic homomorphism to the Calkin algebra is homotopic to a genuine homomorphism [4, 6]. Even less is known about asymptotic homomorphisms to $\mathbb{B}(H)$, where there is no topological obstruction (recall that the $K$-groups of $\mathbb{B}(H)$ are trivial). Such asymptotic homomorphisms are called asymptotic representations and were first studied in relation to the asymptotic tensor product $C^*$-algebras [7] and to semi-invertibility of $C^*$-algebra extensions [8].

2. DEFINITION OF ASYMPTOTIC $C^*$-TENSOR PRODUCTS

Recall [2] that an asymptotic homomorphism $\varphi$ from a $C^*$-algebra $A$ to a $C^*$-algebra $D$ is a family of maps $\varphi = (\varphi_t)_{t \in [0, \infty)} : A \rightarrow D$ satisfying the following properties:

1. the map $t \mapsto \varphi_t(a)$ is continuous for any $a \in A$;
2. $\lim_{t \rightarrow \infty} \varphi_t(a + \lambda b) - \varphi_t(a) - \lambda \varphi_t(b) = \lim_{t \rightarrow \infty} \varphi_t(a^*) - \varphi_t(a)^* = \lim_{t \rightarrow \infty} \varphi_t(ab) - \varphi_t(a) \varphi_t(b) = 0$ for any $a, b \in A$ and any $\lambda \in \mathbb{C}$.

Let $L(H)$ be the algebra of bounded operators on a separable Hilbert space $H$. Our point is that we would like to consider $D$ as a $C^*$-subalgebra of $L(H)$: $D \subseteq L(H)$. We also

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Lemma 1. The functor $\otimes_D$ is symmetric.
Proof. Obvious. \qed

3. Asymptotic representations taking values in \( \mathbb{K}^\infty \)

Let \( G \) be a residually finite infinite property T group, let \( \pi_n \) be the sequence of all non-equivalent irreducible unitary representations on finitedimensional Hilbert spaces \( H_n \) and let \( \mathcal{A} \) be the \( C^* \)-algebra generated by operators \( \oplus_{n=1}^{\infty} \pi_n(g), g \in G \). We denote by \( E \) the \( C^* \)-subalgebra in \( \mathbb{L}(\oplus_{n=1}^{\infty} H_n) \) generated by \( \mathcal{A} \) and by compact operators: \( E = \mathcal{A} + \mathbb{K} \). Put \( A = E/\mathbb{K} \). This \( C^* \)-algebra was first considered by S. Wassermann and we refer to his paper \[9\] for more details.

Lemma 2. Let \( \varphi = (\varphi_t)_{t \in [0, \infty)} : A \to \mathbb{K}^\infty \) be an asymptotic homomorphism. Then \( \varphi \) is asymptotically equivalent to zero, i.e. \( \lim_{t \to \infty} \varphi_t(a) = 0 \) for any \( a \in A \).

Proof. Let \( q_n : \prod_{n=1}^{\infty} \mathbb{K} \to \mathbb{K} \) is the projection onto the \( n \)-th copy. Then, for any \( \varepsilon > 0 \) there exists \( t_0 \) and continuous families \( p_n(t) \in \mathbb{K}_n \) of finitedimensional projections such that

\[
\|q_n \circ \varphi_t(1) - p_n(t)\| < \varepsilon
\]

for any \( t > t_0 \) and any \( n \in \mathbb{N} \). Let \( N_n \) denote the rank of \( p_n(t) \). Since all projections of the same finite rank are unitarily equivalent, without loss of generality (by changing \( \varphi \) by a unitarily equivalent asymptotic homomorphism) we may assume that all \( p_n(t) \) are \( t \)-independent, \( p_n(t) = p_n \). Then the formula \( \psi_t(a) = p_n(q_n \circ \varphi_t(a))p_n \) defines an asymptotic homomorphism from \( A \) to the matrix algebra. The group \( G \) with the stated properties is known to be finitely generated, so without loss of generality we may assume that \( \psi_t(g_i) \) are unitaries, where \( g_i \in G, i = 1, \ldots, k \), are generators for \( G \).

Since the direct product of \( k \) copies of the unitary group \( U_{N_n} \) is compact, so the set \( \{(\psi_t(g_1), \ldots, \psi_t(g_k)) : t \in [0, \infty)\} \) has an accumulation point \((u_1, \ldots, u_k) \in U_{N_n}^k \). If we put \( \sigma(g_i) = u_i \) then this map extends to a genuine representation of \( G \) of dimension \( N_n \). Indeed, \( G \) is a quotient of the free group \( \mathbb{F}_k \) generated by \( g_1, \ldots, g_k \) modulo some relations and each \( \psi_t \) and \( \sigma \) obviously define representations of \( \mathbb{F}_k \), which we denote by the same characters. If \( r \in \mathbb{F}_k \) is a relation then \( \lim_{t \to \infty} \|\psi_t(r) - p_n\| = 0 \). Therefore, \( \sigma(r) = p_n = \lim_{t \to \infty} \|\psi_t(r) - p_n\| = 0 \). Hence, \( \sigma \) factorizes through a representation of \( G \).

Suppose that \( p_n \neq 0 \) for some \( n \). This implies that the representation \( \sigma \) is non-zero, hence it equals one of \( \pi_j \). Since

\[
\|a\| \geq \lim \sup_{t \to \infty} \|\psi_t(a)\| \geq \|\sigma(a)\| = \|\pi_j(a)\|
\]

we have \( \lim \sup_{n \to \infty} \|\pi_n(a)\| \geq \|\pi_j(a)\| \) for any \( a \in A \). Then the identity map of \( G \) extends to a \( \ast \)-homomorphism \( i : A \to C_{\pi_j}^*(G) \), where \( C_{\pi_j}^*(G) \) denotes the \( C^* \)-algebra generated by the representation \( \pi_j \). Tensoring it by \( id_{C_{\pi_j}^*(G)} \), where \( \pi_j \) denotes the contragredient representation for a representation \( \pi \), we get a \( \ast \)-homomorphism

\[
i \otimes id_{C_{\pi_j}^*(G)} : A \otimes C_{\pi_j}^*(G) \to C_{\pi_j}^*(G) \otimes C_{\pi_j}^*(G).
\]

We do not specify the tensor product norm here because \( C_{\pi_j}^*(G) \) is finitedimensional, hence nuclear. It was shown in \[9\] that the norm on the left hand side of \(2\) is strictly smaller than the norm on the right hand side, so this \( \ast \)-homomorphism cannot exist. This contradiction shows that \( p_n = 0 \) for all \( n \), hence \((1)\) implies that \( \lim_{t \to \infty} \|q_n \circ \varphi_t(1)\| = 0 \) uniformly in \( n \), therefore, \( \lim_{t \to \infty} \|q_n \circ \varphi_t(a)\| = 0 \) uniformly in \( n \) for any \( a \in A \). \qed

Corollary 3. For \( A \) defined above, one has \( A \otimes_{\mathbb{K}^\infty} B = A \otimes_{\min} B \) for any \( C^* \)-algebra \( B \).
Proof. Since  
\[ \| \varphi \otimes \psi (a \otimes b) \| = \limsup_{t \to \infty} \| \varphi_t (a) \otimes \psi_t (b) \| = \limsup_{t \to \infty} \| \varphi_t (a) \| \cdot \| \psi_t (b) \| = 0 \]
for any \( a \in A, b \in B \) and for any asymptotic representations \( \varphi \) and \( \psi \), one has  
\[ \| a \otimes b \|_{\mathbb{K}^\infty} = \sup_{\varphi, \psi} \| \varphi \otimes \psi (a \otimes b) \| = 0, \]
hence \( \| c \|_{\mathbb{K}^\infty} = 0 \) for any \( c \in A \otimes B \), therefore,  
\[ \| c \|_{\mathbb{K}^\infty} = \max \{ \| c \|_{\min, 0} \} = \| c \|_{\min}. \]
\[ \square \]

4. An example of an asymptotic representation taking values in \( \mathbb{K}^\infty \)

Let \( C = C_0(0, 1] \). We are going to construct an asymptotic representation \( \varphi^0 \) of \( C \otimes A \) taking values in \( \mathbb{K}^\infty \). (We do not specify here the tensor norm since \( C \) is nuclear.)

Let \( \chi : A \to E \) be a continuous homogeneous selfadjoint selection map, cf. [1]. We denote by \( P_n \) the projection in \( \oplus_{n=1}^\infty H_n \) onto \( H_n \). For \( a \in A \) put \( \alpha(a) = \iota \circ \chi(a) \), where \( \iota : E \to \mathbb{L}(\oplus_{n=1}^\infty H_n) \) is the standard inclusion.

Let \( \{ \tau_n \}_{n \in \mathbb{N}} \) be a dense sequence of points in \( (0, 1) \). For \( t = k \in \mathbb{N} \) and for \( f \in C \) put  
\[ \beta_k(f) = \sum_{n=k+1}^\infty f(\tau_n)P_n, \]
(this sum also is convergent with respect to the \(*\)-strong topology). If \( k < t < k + 1 \) then put  
\[ \beta_t(f) = f((t - k)\tau_{k+1})P_{k+1} + \beta_{k+1}(f). \]

Let \( F \in C \otimes A \). One can consider \( F \) as a continuous function on \([0, 1]\) taking values in \( A \) such that \( F(0) = 0 \). Put  
\[ \phi_k(F) = \sum_{n=k+1}^\infty P_n \alpha(F(\tau_n))P_n, \]
where the sum is \(*\)-strongly convergent, and  
\[ \phi_t(F) = P_{k+1} \alpha(F((t - k)\tau_{k+1}))P_{k+1} + \phi_k(F) \]
for \( k < t < k + 1 \).

Lemma 4. The family of maps \( (\phi_t)_{t \in [0, \infty)} \) is an asymptotic representation of \( C \otimes A \) taking values in \( \prod_{n=1}^\infty \mathbb{L}(H_n) \subset \mathbb{K}^\infty \).

Proof. By the definition, the maps \( \phi_t, t \in [0, \infty) \), take values in \( \prod_{n=1}^\infty \mathbb{L}(H_n) \), so we only need to check that algebraic properties hold asymptotically. Let us check that for multiplication, as other properties can be checked in the same way. Let \( F_1, F_2 \in C \otimes A \), then the operators  
\[ K_n = \alpha(F_1(\tau_n)F_2(\tau_n)) - \alpha(F_1(\tau_n))\alpha(F_2(\tau_n)) \in \mathbb{K} \]
lie in a compact subset of \( \mathbb{K} \) (the image of \([0, 1]\) under the continuous map \( \tau \mapsto \alpha(F_1(\tau)F_2(\tau)) - \alpha(F_1(\tau))\alpha(F_2(\tau)) \)), hence  
\[ \lim_{k \to \infty} \phi_k(F_1F_2) - \phi_k(F_1)\phi_k(F_2) = \lim_{k \to \infty} \sum_{n=k+1}^\infty P_n K_n P_n = 0. \]
Finally, we easily pass to the continuous parameter: \( \lim_{t \to \infty} \phi_t(F_1F_2) - \phi_t(F_1)\phi_t(F_2) = 0. \)  \[ \square \]
Note that if $F = f \otimes a \in C \otimes A$ then
\[ \phi_t(f \otimes a) = \sum_{n=k}^{\infty} P_n \alpha(a) P_n \cdot \beta_t(f). \]

5. Comparing tensor norms

Let $B$ be the $C^*$-algebra generated by operators $\oplus_{n=1}^{\infty} \pi_n(g)$, $g \in G$, where $\pi$ denotes the contragredient representation for $\pi$.

Theorem 5. The tensor products $C \otimes_{K^\infty} (A \otimes_{K^\infty} B)$ and $(C \otimes_{K^\infty} A) \otimes_{K^\infty} B$ are not canonically isomorphic.

Proof. Let $f \in C$ be the identity function, $f(\tau) = \tau$, and let $\{g_1, \ldots, g_m\}$ be a symmetric set of generators of the group $G$ as above. We identify the group elements with the corresponding unitaries in $C^*$-algebras generated by representations of $G$ (like $B$) and in their quotients (like $A$). Let $d = \sum_{i=1}^{m} f \otimes g_i \otimes g_i \in C \otimes A \otimes B$. Denote by $\| \cdot \|_1$ and by $\| \cdot \|_2$ the norms on $C \otimes A \otimes B$ inherited from $C \otimes_{K^\infty} (A \otimes_{K^\infty} B)$ and $(C \otimes_{K^\infty} A) \otimes_{K^\infty} B$ respectively. Our aim is to show that $\|d\|_1 \neq \|d\|_2$.

It follows from Lemma 3 and from amenability of $C$ that $C \otimes_{K^\infty} (A \otimes_{K^\infty} B) = C \otimes_{\min} (A \otimes_{\min} B)$, so
\[ \|d\|_1 = \|f\| \cdot \left\| \sum_{i=1}^{m} g_i \otimes g_i \right\|_{\min} = \left\| \sum_{i=1}^{m} g_i \otimes g_i \right\|_{\min}. \]

It was shown in [9] that the latter norm is strictly smaller than $m$, so
\[ \|d\|_1 < m. \] (3)

When estimating the norm $\| \cdot \|_2$ from below, we may use two special asymptotic representations instead of taking the supremum over all of them. Let us take $\phi_t$ for $C \otimes A$ and the identity representation for $B$. Then
\[ \|d\|_2 \geq \limsup_{t \to \infty} \left\| \sum_{i=1}^{m} \phi_t(f \otimes g_i) \otimes \sum_{n=1}^{\infty} \pi_n(g_i) P_n \right\| = \limsup_{t \to \infty} \left\| \sum_{i=1}^{m} \sum_{n=1}^{\infty} P_n \beta_t(f) \alpha(g_i) P_n \otimes \sum_{n=1}^{\infty} \pi_n(g_i) P_n \right\| \geq \limsup_{n \to \infty} \left\| P_n \beta(f) \sum_{i=1}^{m} \alpha(g_i) P_n \otimes \pi_n(g_i) P_n \right\| \geq \limsup_{n \to \infty} \left\| P_n \alpha(g_i) P_n \otimes \pi_n(g_i) P_n \right\| \geq \left\| \sum_{i=1}^{m} P_n \alpha(g_i) P_n \otimes \pi_n(g_i) P_n \right\|, \]
where $\{n_j\}$ is any increasing subsequence of integers. Since the sequence $\{\tau_n\}_{n=1}^{\infty}$ is dense in $[0, 1]$, we can find a subsequence $\{n_j\}_{j=1}^{\infty}$ such that $\lim_{j \to \infty} \tau_{n_j} = 1$. Then
\[ \|d\|_2 \geq \limsup_{j \to \infty} \left\| f(\tau_{n_j}) \cdot \sum_{i=1}^{m} P_{n_j} \alpha(g_i) P_{n_j} \otimes \pi_{n_j}(g_i) \right\| = \limsup_{j \to \infty} \left\| \sum_{i=1}^{m} P_{n_j} \alpha(g_i) P_{n_j} \otimes \pi_{n_j}(g_i) \right\| = \left\| \sum_{i=1}^{m} \pi_{n_j}(g_i) \otimes \pi_{n_j}(g_i) \right\| = \sum_{i=1}^{m} 1 = m. \]
On the other hand, \( \|d\|_2 \leq \sum_{i=1}^{m} \|f \otimes g_i \otimes g_i\|_2 = m \), so we have
\[
\|d\|_2 = m.
\] (4)

Comparing (3) and (4), we conclude that these two norms are different. \( \square \)

References

[1] R. G. Bartle, L. M. Graves, Mappings between function spaces. Trans. Amer. Math. Soc. 72 (1952), 400–413.
[2] A. Connes, N. Higson, Déformations, morphismes asymptotiques et K-théorie bivariante. C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), 101–106.
[3] E. Kirchberg. Exact \( C^* \)-algebras, tensor products, and the classification of purely infinite algebras. Proc. Internat. Congress Math., Birkhäuser, 1995, 943–954.
[4] V. Manuilov, Asymptotic homomorphisms into the Calkin algebra. J. Reine Angew. Math. 557 (2003), 159–172.
[5] V. Manuilov, K. Thomsen, The Connes–Higson construction is an isomorphism. J. Funct. Anal. 213 (2004), 154–175.
[6] V. Manuilov, K. Thomsen, E-Theory is a Special Case of KK-Theory. Proc. London Math. Soc. 88 (2004), 455–478.
[7] V. Manuilov, K. Thomsen, On the asymptotic tensor \( C^* \)-norm. Arch. Math. 86 (2006), 138–144.
[8] V. Manuilov, K. Thomsen, On the lack of inverses to \( C^* \)-extensions related to property T groups. Canad. Math. Bull. 50 (2007), 268–283.
[9] S. Wassermann, \( C^* \)-algebras associated with groups with Kazhdan’s property T. Ann. Math. 134 (1991), 423–431.

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