Improved Bounds for Track Numbers of Planar Graphs

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Abstract

A track layout of a graph consists of a vertex coloring and of a total order of each color class, such that the edges between each pair of colors form a non-crossing set. The track number of a graph is the minimum number of colors required by a track layout of the graph.

This paper improves lower and upper bounds on the track number of several families of planar graphs. We prove that every planar graph has track number at most 225 and every planar 3-tree has track number at most 25. Then we show that there exist outerplanar graphs whose track number is 5, which leads the best known lower bound of 8 for planar graphs. Finally, we investigate leveled planar graphs and tighten bounds on the track number of weakly leveled graphs, Halin graphs, and X-trees.

1 Introduction

A track layout of a graph is a partition of its vertices into sequences, called tracks, such that the vertices in each sequence form an independent set and the edges between every pair of tracks form a non-crossing set. The track number of a graph is the minimum number of tracks in a track layout. Track layouts are formally introduced by Dujmović, Morin, and Wood [8], although similar concepts are implicitly studied in several earlier works [4,11,12]. An original motivation for studying track layouts is their connection with the existence of low-volume three-dimensional graph drawings: A graph with \( n \) vertices has a three-dimensional straight-line drawing in a grid of size \( O(1) \times O(1) \times O(n) \) if and only if it has track number \( O(1) \) [8,9].

Track layouts are closely related to other models of linear graph layouts, specifically, stack and queue layouts. A stack (queue) layout of a graph consists of a linear order on the vertices and a partition of its edges so that no edges in a single part cross (nest) each other. The minimum number of parts needed in a stack (queue) layout of a graph is called its stack number (queue number) [12,13]. A major result in the field is that track number is tied to queue number in a sense that one is bounded by a function of the other [9]. In particular, every \( t \)-track graph has a \((t-1)\)-queue layout, and every \( q \)-queue graph has track number at most \( 4q \cdot 4q^{(2q-1)(4q-1)} \). The relationship between stack and track layouts is not that prominent but it is known that stack number is bounded by track number for bipartite graphs [9]: whether the reverse is true is an open question. Therefore, a study on track layouts may shed light on the relationship between the linear graph layouts.

In this paper we investigate lower and upper bounds on the track number of various families of planar graphs. Table 1 summarizes new and existing bounds described in the literature. A recent breakthrough result by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [7] implies that all planar graphs have bounded track number (in fact, the result extends to every proper minor-closed class of graphs). Their analysis leads to a very large constant, as it is based on a fairly generic technique. We provide an alternative proof resulting in the upper bound of 225 for the track number of planar graphs (Section 4.1).
### Table 1: Track numbers of various families of planar graphs

| Graph class          | Upper bound | Lower bound |
|----------------------|-------------|-------------|
|                      | Old Ref. New Ref. | Old Ref. New Ref. |
| tree                 | 3 [11]      | 3 [11]      |
| outerplanar          | 5 [9]       | 4 [9] 5 [Thm. 3] |
| series-parallel      | 15 [4]      | 6 [8]       |
| planar 3-tree        | 4,000 [2] 25 [Thm. 1] | 6 [8] 8 [Thm. 4] |
| planar               | 461,184,080 [7] 225 [Thm. 2] | 7 [9] 8 [Thm. 4] |
| X-tree               | 6 [3] 5 [Lm. 7] | 3 [4] 5 [Sect. 5.2] |
| Halin                | 6 [3]       | 3 [9] 5 [Sect. 5.2] |
| weakly leveled       | 6 [3]       | 3 [3] 6 [Sect. 5.2] |

An important ingredient of our construction is an improved upper bound of 25 for the track number of planar 3-trees (Section 3). For the lower bounds, we find an outerplanar graph that requires 5 tracks, which is worst-case optimal. This resolves an open question posed in [9] and provides the best lower bound of 8 on the track number of general planar graphs (Section 4.2).

Finally in Section 5, we study track layouts of (weakly) leveled planar graphs, which are the graphs with planar leveled drawings having no dummy vertices. We prove that the existing upper bound of 6 is worst-case optimal while certain subfamilies (e.g., X-trees) admit a layout on 5 tracks. The results close the gaps between upper and lower bounds on the track numbers for the subclasses of planar graphs. Our lower bounds in the section rely a new SAT formulation of the track layout problem described in Section 5.2. We conclude the paper in Section 6 with possible future directions and several open problems.

## 2 Preliminaries

In this section we introduce necessary definitions and recall some known results about track layouts. Throughout the paper, $G = (V(G), E(G))$ is a simple undirected graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges.

**Track Layouts** Let $\{V_i : 1 \leq i \leq t\}$ be a partition of $V$ such that for every edge $(u, v) \in E$, if $u \in V_i$ and $v \in V_j$, then $i \neq j$. Suppose that $<_i$ is a total ordering of the vertices in $V_i$. Then the ordered set $(V_i, <_i)$ is called a track and the partition is called a $t$-track assignment of $G$. An X-crossing in a track assignment consists of two edges, $(u, v)$ and $(x, y)$, such that $u$ and $x$ are on the same track $V_i$ with $u <_i x$, and $v$ and $y$ are on a different track $V_j$ with $y <_j v$. A track layout of graph $G$, denoted $T(G)$, is a track assignment with no X-crossings, and track number, $\text{tn}(G)$, is the minimum $t$ such that $G$ has a $t$-track layout. In particular, 1-track graphs are the independents sets, and 2-track graphs are the forests of caterpillars.

Some authors consider a relaxed definition of the concept, called improper track layouts, in which edges between consecutive vertices in a track are allowed [6,11]. It can be easily seen that the tracks of such a layout can be doubled to obtain a proper track layout [8]. Thus, every graph with improper track number $t$ has (proper) track number at most $2t$. In Section 5, we show that the upper bound can be smaller than $2t$ for some subclasses of graphs. In this paper we study only proper track layouts.
A basic result on track layouts is a “wrapping” lemma, which is due to Dujmović et al. [9] (which in turn is based on the ideas of Felsner et al. [11]). Consider a track assignment whose index set is two-dimensional. That is, let \( \{V_{i,j} : 1 \leq i \leq b, 1 \leq j \} \) be a track layout of a graph \( G \). Define the partial span of an edge \((u,v)\) in \( E \) with \( u \in V_{i_1,j_1} \) and \( v \in V_{i_2,j_2} \) to be \(|j_1 - j_2|\). The following lemma describes how to modify the track layout of \( G \) with possibly many tracks into a layout whose track number is bounded by a function of \( b \) and the maximum partial span.

**Lemma 1** (Dujmović et al. [9]). Let \( \{V_{i,j} : 1 \leq i \leq b, 1 \leq j \} \) be a track layout of a graph \( G \) with maximum partial span \( s \geq 1 \). Then \( \text{tn}(G) \leq b \cdot (2s + 1) \).

We stress that Lemma 1 can be applied for track layouts whose index set is one-dimensional (in that case, \( b = 1 \)) or multi-dimensional. In the latter case, one can wrap the layout by reducing the cardinality of a single dimension with a bounded partial span.

**Treewidth and Tree-Partitions** A tree-decomposition of \( G \) represents the vertices of \( G \) as subtrees of a tree, in such a way that the vertices are adjacent if and only if the corresponding subtrees intersect. The width of a tree-decomposition is one less than the maximum size of a set of mutually intersecting subtrees, and the treewidth of \( G \) is the minimum width of a tree-decomposition of \( G \). For a fixed integer \( k \geq 0 \), a \( k \)-tree is a maximal graph of treewidth \( k \), such that no more edges can be added without increasing its treewidth. In our proofs we do not directly use a tree-decomposition of a graph but utilize a related concept, a tree-partition, which is defined next.

Given a graph \( G \), a tree-partition of \( G \) is a pair \( (T, \{T_x : x \in V(T)\}) \) consisting of a tree \( T \) and a partition of \( V \) into sets \( \{T_x : x \in V(T)\} \), such that for every edge \((u,v)\) in \( E \) one of the following holds: (i) \( u,v \in T_x \) for some \( x \in V(T) \), or (ii) there is an edge \((x,y)\) of \( T \) with \( u \in T_x \) and \( v \in T_y \). The vertices of \( T \) are called the nodes and the sets \( T_x, x \in V(T) \) are called the bags of the tree-partition.

The following well-known result provides a tree-partition of a \( k \)-tree.

**Lemma 2** (Dujmović et al. [8]). There exists a rooted tree-partition \( (T, \{T_x : x \in V(T)\}) \) of a \( k \)-tree \( G \) such that

- for every node \( x \) of \( T \), the subgraph of \( G \) induced by the vertices of \( T_x \) is a connected \((k - 1)\)-tree;

- for every non-root node \( x \) of \( T \), if \( y \) is a parent node of \( x \) in \( T \), then the set of vertices in \( T_y \) having a neighbor in \( T_x \) form a clique of size \( k \) in \( G \).

**Layerings and \( H \)-Partitions** A generalization of a tree-partition is the notion of an \( H \)-partition. An \( H \)-partition of a graph \( G \) is a partition of \( V(G) \) into disjoint bags, \( \{A_x : x \in V(H)\} \) indexed by the vertices of \( H \), such that for every edge \((u,v)\) in \( E(G) \) one of the following holds: (i) \( u,v \in A_x \) for some \( x \in V(H) \), or (ii) there is an edge \((x,y)\) in \( E(H) \) with \( u \in A_x \) and \( v \in A_y \). In the former case, \((u,v)\) is an intra-bag edge and in the latter case, it is an inter-bag edge.

A layering of a graph \( G = (V,E) \) is an ordered partition \( (V_0,V_1,\ldots) \) of \( V \) such that for every edge \((v,w)\) in \( E \), if \( v \in V_i \) and \( w \in V_j \), then \(|i - j| \leq 1 \). If \( r \) is a vertex in a connected graph \( G \) and \( V_i = \{v \in V \mid \text{dist}(r,v) = i\} \) for all \( i > 0 \), then \( (V_0,V_1,\ldots) \) is called a BFS-layering of \( G \). The layered width of an \( H \)-partition of a graph \( G \) is the minimum integer \( \ell \) such that for some layering \( (V_0,V_1,\ldots) \) of \( G \), we have \(|A_x \cap V_i| \leq \ell \) for every bag \( A_x \) of the partition and every integer \( i \geq 0 \).
The following lemma is a key ingredient for our proof of the upper bound on the track number of planar graphs.

**Lemma 3** (Dujić et al. [7]). *Every planar graph $G$ has an $H$-partition of layered width 3 such that $H$ is planar and has treewidth at most 3. Moreover, there is such a partition for every BFS-layering of $G$."

### 3 Planar Graphs of Bounded Treewidth

In this section, we study track numbers of planar graphs having bounded treewidth. Our primary goal is improving the existing upper bound for planar 3-trees.

The following notion is introduced by Dujić et al. [8]; it is also implicit in the work of Di Giacomo et al. [4]. Let $\{V_i : 1 \leq i \leq t\}$ be a $t$-track layout of a $k$-tree, $G_k$. We say that a clique $C_1$ of $G_k$ proceeds a clique $C_2$ of $G_k$ with respect to the track layout if for all $u \in V_i \cap C_1$ and $w \in V_i \cap C_2$, it holds that $u \leq i w$; we denote the relation by $C_1 \prec C_2$. Let $S = \{C_1, \ldots, C_{|S|}\}$ be a set of maximal cliques of $G_k$. We say that $S$ is nicely ordered if $\prec$ is a total order on $S$, that is, $C_i \prec C_j$ for all $1 \leq i < j \leq |S|$. Finally, for a track layout of $G_k$, we say that a given set of maximal cliques is $c$-colorable if the set can be partitioned into $c$ nicely ordered subsets. Such a track layout is called $c$-clique-colorable.

**Lemma 4.** *The following holds:*

(a) *Every path admits a 1-clique-colorable 2-track layout.*

(b) *Every tree admits a 2-clique-colorable 3-track layout.*

(c) *Every outerplanar graph admits a 2-clique-colorable 5-track layout.*

**Proof.** Claim (a) of the lemma is straightforward, as every 2-track layout of a path is 1-clique-colorable. Now we prove (b). Consider a natural plane drawing of a given tree, $T$, such that the vertices having the same distance from the root, $r$, are drawn on the same horizontal line; see Figure 1a. Let $x(v)$ and $y(v)$ denote $x$ and $y$ coordinates of a vertex $v \in V(T)$. Clearly the drawing corresponds to a track layout with maximum span 1, which by Lemma 1 can be converted into a 3-track layout by assigning $\text{track}(v) = y(v) \pmod{3}$.
for every vertex $v$. The maximal cliques in the graph are edges of $T$. We partition the
dges into $S_1 = \{ (u, v) \in E(T) : \text{track}(u) = 1 \text{ or } \text{track}(v) = 1 \}$ and $S_2 = E(T) \setminus S_1$. It is
easy to verify that both sets of edges are nicely ordered with respect to the 3-track layout.

In order to prove (c), we utilize a 5-track layout of an outerplanar graph suggested by
Dujmović et al. [9]; see also [2]. They prove that every maximal outerplanar graph, $G$,
has a straight-line outerplanar drawing in which vertex coordinates are integers, and the
absolute value of the difference of the $y$-coordinates of the endvertices of each edge of $G$ is
either one or two; see Figure 1b. That is, $|y(v)−y(u)| \leq 2$ for all $(u, v) \in E$. Such a drawing
 corresponds to a track layout (with possibly many tracks) with maximum edge span of 2.

By Lemma 1, the layout is wrapped onto 5 tracks by assigning $\text{track}(v) = y(v) \pmod{5}$
and ordering vertices within a track lexicographically by $(\lfloor y/5 \rfloor, x)$.

The maximal cliques in $G$ are triangular faces. Denote a face containing vertices
$u, v, w$ by $(u, v, w)$ and the set of all faces in $G$ by $F$. We partition $F$ into two sets
$S_1 = \{ (u, v, w) \in F : \text{track}(u) = 2 \text{ or } \text{track}(v) = 2 \text{ or } \text{track}(w) = 2 \}$ and $S_2 = F \setminus S_1$; see Figure 1b where members of $S_2$ are shaded. All the faces of $S_1$ contain a vertex in
track 2; choose the leftmost such vertex $v^*$ with $\text{track}(v^*) = 2$. The faces containing $v^*$
are nicely ordered with respect to the track layout, as the drawing is planar. Removing
$v^*$ from the layout and applying the same argument for the remaining faces, yields a nice
order of $S_1$. An analogous procedure can be applied to construct a nice order of $S_2$, since
all those faces contain a vertex in track 0 and a vertex in track 4.

Notice that the claims of Lemma 4 are tight in terms of clique-colorability: There is
no constant $t$ such that every tree admits a 1-clique-colorable $t$-track layout. The next
lemma shows how to use clique-colorable track layouts for graphs of bounded treewidth.

**Lemma 5.** Assume that every $k$-tree admits a $c$-clique-colorable $t$-track layout. Then
every $(k + 1)$-tree admits a layout on $t \cdot (2c + 1)$ tracks.

**Proof.** Let $(T, \{T_x : x \in V(T)\})$ be a tree-partition of a $(k + 1)$-tree, $G_{k+1}$, given by
Lemma 2. In order to construct a track layout of $G_{k+1}$, consider a vertex $v \in V(G_{k+1})$
that belongs to a bag $x \in V(T)$ in the tree-partition. The track of $v$ is defined by two
indices, $(i_v, j_v)$, where $i_v$ is derived from a track layout of $T_x$ and $j_v$ is derived from a
certain track layout of the tree $T$. Next we define $i_v$ and $j_v$.

- By Lemma 2, vertices of every bag of the tree-partition form a connected $k$-tree.
  Thus, $T_x$ admits a $c$-clique-colorable $t$-track layout, which we denote by $T(T_x)$. The
  first index, $i_v$, is the track of $v$ in $T(T_x)$. Clearly the index ranges from 1 to $t$.

- Consider a parent node, $y \in V(T)$, of node $x$ in the tree-partition. By Lemma 2,
  vertices of $T_y$ adjacent to a vertex of $T_x$ form a maximal clique, which we call the
  parent clique of $x$. By our assumption, the clique has an assigned color, $c(x)$, ranging
  from 1 to $c$, and cliques with the same color are nicely ordered in $T(T_y)$.

We layout $T$ on (possibly many) tracks such that, for every parent node $y$, its child
nodes are on $c$ consecutive tracks. Formally, if $x$ is a child of $y$ and the corresponding
parent clique has color $1 \leq c(x) \leq c$, then $\text{track}(x) = \text{track}(y) + c(x)$. The order of
child nodes having the same color follows the nice order of the corresponding parent
cliques. The constructed track layout of $T$ is denoted $T(T)$.

We assign $j_v$ to be the track of node $x$ in $T(T)$. The index can be as large as
$\Omega(|V(T)|)$ but its maximum span is $c$. 

Now we define the order of the vertices of $G_{k+1}$ within the same track. If two vertices, $u$ and $v$, belong to the same bag $x \in V(T)$ in the tree-partition, their relative order is inherited from track layout $T(T_x)$. If the vertices are in different bags, that is, $u \in T_x, v \in T_y$ for some $x \in V(T), y \in V(T)$, then the order is dictated by the ordering of nodes $x$ and $y$ in track layout $T(T)$.

Next we show that the constructed track assignment has no X-crossings. Intra-bag edges do not form X-crossings by the assumption of the lemma. Consider two inter-bag edges, $(u_1, v_1) \in E(G_{k+1})$ and $(u_2, v_2) \in E(G_{k+1})$. Since there are no crossings in $T(T)$, the inter-bag edges mapped to edges of $T$ without a common parent are not in an X-crossing. Thus we may assume that $u_1 \in T_p, u_2 \in T_p, v_1 \in T_x, v_2 \in T_y$ for some bags $p, x, y \in V(T)$ such that $p$ is a parent of $x$ and $y$. If parent cliques of $x$ and $y$ are of different colors, then by construction of $T(T)$, vertices $v_1$ and $v_2$ are in different tracks. If parent cliques of $x$ and $y$ are of the same color, then the order between $v_1$ and $v_2$ is consistent with the order between $u_1$ and $u_2$, since the cliques are nicely ordered in $T(T_p)$. Therefore, an X-crossing between $(u_1, v_1)$ and $(u_2, v_2)$ is impossible.

Finally, we apply Lemma 1 for the constructed two-dimensional track layout with maximum partial span $c$ to get the desired claim.

Combining Lemma 5 with Lemma 4, we get an upper bound of 15 for planar 2-trees (series-parallel graphs), and an upper bound of 25 for planar 3-trees. While the former bound is already known [4], the latter upper bound is new.

**Theorem 1.** The track number of a planar 3-tree is at most 25.

### 4 General Planar Graphs

In this section we investigate upper and lower bounds on the track number of general planar graphs.

#### 4.1 An Upper Bound

Recently Dujmović et al. [7] used $H$-partitions of bounded layered width to prove that the queue number of a planar graph is a constant. By analogy with their result, we show that the track number of a graph is bounded by a function of the track number of $H$ and the layered width.

**Lemma 6.** If a graph $G$ has a layered $H$-partition of layered width $\ell$, then $G$ has track number at most $3\ell \cdot \tn(H)$.

**Proof.** Assume $G$ has a layered $H$-partition of width $\ell$, and suppose $T(H)$ is a layout of $H$ on $\tn(H)$ tracks. To define a track assignment of $G$, $T(G)$, consider a vertex $v \in V(G)$ that belongs to a bag $x \in V(H)$. The track of $v$ is defined by three indices, $(i_v, j_v, d_v)$.

- The first index, $i_v$, is the track of $x$ in track layout $T(H)$; it ranges from 1 to $\tn(H)$.
- The bag $x \in V(H)$ contains at most $\ell$ vertices in every layer. Label these vertices arbitrarily from 1 to $\ell$, and assign the second index, $j_v$, to the label. Thus, $1 \leq j_v \leq \ell$.
- The last index, $d_v$, represents the layer of $v$ in the given layered $H$-partition of $G$. Clearly, $d_v \geq 1$ and $d_v$ is at most the number of layers in $G$, which can be as large as $\Omega(|V(G)|)$.
In order to complete the track assignment, we define the ordering of vertices in the same track. Notice that for every bag $x \in V(H)$, the corresponding vertices $v \in A_x$ are on different tracks defined by the second and the third indices of the track assignment. Therefore, only the vertices of $G$ corresponding to different bags can belong to the same track. For those vertices, the ordering is inherited from the given track layout $T(H)$. That is, $v < u$ in $T(G)$ with $v \in A_x, u \in A_y$ if and only if $x < y$ in $T(H)$; see Figure 2.

Now we verify that $T(G)$ is a valid track layout, that is, it contains no X-crossings. For a contradiction suppose that $(u_1, v_1) \in E(G)$ and $(u_2, v_2) \in E(G)$ form an X-crossing, that is, $track(u_1) = track(u_2), track(v_1) = track(v_2)$ and $u_1 < u_2, v_2 < v_1$. Since all vertices of a bag of $H$ are on different tracks, it follows that $u_1$ and $u_2$ belong to different bags and that $v_1$ and $v_2$ belong to different bags. Therefore, the two edges correspond either to an X-crossing in $T(H)$ (if the two edges are intra-bag edges), or to an edge of $H$ with both endpoints on the same track of $T(H)$ (if one of the edges is inter-bag edge). Both of the options violate the definition of $T(H)$; hence, $T(G)$ contains no X-crossings.

Finally, observe that the partial span in $T(G)$ corresponding to the third dimension of the track assignment, $d_v$, is at most one, as it is based on a layering of $G$. By Lemma 1, the track layout can be wrapped onto $3\ell \cdot tn(H)$ tracks; Figure 2 illustrates the process.

Combining Lemma 6 with Lemma 3, we get the following result.

**Theorem 2.** The track number of a planar graph is at most 225.

### 4.2 Lower Bounds

Dujmović et al. [9] show an outerplanar graph that requires 4 tracks and prove that every outerplanar graph has a 5-track layout. Our next result closes the gap between the lower and the upper bounds answering the question posed in [9].

**Theorem 3.** The outerplanar graph in Figure 3a has track number 5.

Before proving the theorem, we introduce two configurations that we use in the proof. The first configuration, illustrated in Figure 3b, is defined on four vertices forming a cycle. If $track(a) = 1, track(b) = track(c) = 2, track(d) = 3, and b < c$, then for every vertex $v$ that is “inside” the quadrangle (that is, $track(v) = 2$ and $b < v < c$), all its neighbors are not on tracks 1 and 3. We call this a $Q(a,b < c,d)$-configuration for vertex $v$. The second construction, illustrated in Figure 3c, is defined on four vertices, $a, b, c, d$, with...
track(a) = track(b) = 1, track(c) = track(d) = 2 and a < b, c < d. If there exist two vertices u and v together with edges (u, a), (u, d), (v, b), (v, c), then track(u) ≠ track(v). We call this an W(a<b,c<d)-configuration for vertices u and v.

Proof of Theorem 3. Assume that the graph in Figure 3a has a 4-track layout. Without loss of generality, we may assume that vertex a is on track 1, vertex b is on track 2, and vertex c is on track 3. Next we consider tracks of vertices z1, z2, and z3, and distinguish four cases depending on how many of the vertices are on track 4.

- None of z1, z2, z3 are on track 4.

It is easy to see that track(z1) = 3, track(z2) = 1, and track(z3) = 2; see Figure 4a. Assume without loss of generality that b < z3, that is, vertex b precedes vertex z3 on track 2. Then in order for edge (c, z3) to avoid a crossing with edge (b, z1), vertex z1 should precede c on track 3, that is, z1 < c. Then a and z2 form an W(z1 < c, b < z3)-configuration, a contradiction.

- All of z1, z2, z3 are on track 4.

Assume without loss of generality that z1 < z2 < z3. We show that the graph in Figure 4b is not embeddable in four tracks.

Observe that z2’s neighbors cannot be on track 1; otherwise, the edge from z2 to the neighbor crosses one of the edges (a, z1) or (a, z3). Therefore, track(y2) = 3 and track(x2) = 2. Notice that y2 < c, as otherwise edges (y2, z2) and (c, z3) cross. Now x2 forms an W(z1 < z2, y2 < c)-configuration with vertex b.

- One of z1, z2, z3 is on track 4; suppose track(z1) = 4.

It is easy to see that track(z2) = 1 and track(z3) = 2. Assume without loss of generality that z3 < b; it follows that a < z2, as otherwise (b, z2) and (a, z3) cross. Next we prove that the graph in Figure 4c is not embeddable in four tracks. We distinguish two cases depending on the track of vertex y1.

First assume track(y1) = 3. It holds that y1 < c, as otherwise (c, z2) and (y1, a) cross. Now it is impossible to assign a track for y3: If track(y3) = 1, then y3 and a form an W(y1 < c, z3 < b)-configuration; if track(y3) = 4, then y3 and z1 form an W(y1 < c, z3 < b)-configuration.

Second assume track(y1) = 2. It holds that y1 < b, as otherwise (b, z2) and (a, y1) cross. If z3 < y1 < b, then y1 forms a Q(c, z3 < b, a)-configuration. If y1 < z3 < b, then z3 forms a Q(z1, y1 < b, a)-configuration, a contradiction.
Two of $z_1$, $z_2$, $z_3$ are on track 4. Suppose $track(z_1) = track(z_2) = 4$; thus, $track(z_3) = 2$.

Assume without loss of generality that $z_1 < z_2$. Next we prove that the graph in Figure 4d is not embeddable in four tracks. We distinguish two cases depending on the relative order of $b$ and $z_3$.

First assume $b < z_3$. Consider the track of $y_2$. If $track(y_2) = 1$, then $a$ and $y_2$ form an $W(z_1 < z_2, b < z_3)$-configuration; thus, $track(y_2) = 3$. To avoid a crossing between edges $(c, z_3)$ and $(b, y_2)$, we have $y_2 < c$. Now it is not possible to layout $x_2$. If $track(x_2) = 1$, then $a$ and $x_2$ form an $W(z_1 < z_2, y_2 < c)$-configuration. Otherwise if $track(x_2) = 2$, then $b$ and $x_2$ form an $W(z_1 < z_2, y_2 < c)$-configuration.

Second assume $z_3 < b$. Consider the track of $y_1$. If $track(y_1) = 3$, then $y_1 < c$, as otherwise $(y_1, z_1)$ and $(c, z_2)$ cross. Now it is impossible to layout $y_3$, which forms and an $W(y_1 < c, z_3 < b)$-configuration with vertex $a$ if $track(y_3) = 1$, and it forms an $W(y_1 < c, z_3 < b)$-configuration with vertex $z_1$ if $track(y_3) = 4$.

Thus, $track(y_1) = 2$. It follows that $y_1 < b$, as otherwise $(y_1, z_1)$ and $(b, z_2)$ cross. If $z_3 < y_1 < b$, then $y_1$ is in a $Q(a, z_3 < b, c)$-configuration and its neighbor, $x_1$, cannot be laid out. Otherwise if $y_1 < z_3 < b$, then $z_3$ is in a $Q(a, y_1 < b, z_1)$-configuration, and its neighbor, $y_3$, cannot be laid out.

Dujmović et al. (Lemma 22 of [9]) show that, for every outerplanar graph $H$ with track number $\tn(H)$, there exists a planar graph whose track number is $\tn(H) + 3$. We stress that the graph in their construction is a planar 3-tree. Therefore, we have the following improved bound for track number of planar 3-trees.

**Theorem 4.** There exists a planar 3-tree $G$ with track number $\tn(G) = 8$.

## 5 Other Subclasses of Planar Graphs

Track layouts of planar graphs are related to leveled planar graph drawings that were introduced by Heath and Rosenberg [12] in the context of queue layouts. A leveled planar drawing of a graph is a straight-line crossing-free drawing in the plane, such that the vertices are placed on a sequence of parallel lines (levels) and every edge joins vertices in two consecutive levels. A graph is **leveled planar** if it admits a leveled planar drawing. Bannister et al. [3] characterize the class of graphs by showing that a graph is leveled planar if and only if it is bipartite and admits a 3-track layout. In a relaxed definition of leveled
drawings, edges between consecutive vertices on the same level are allowed; this leads to weakly leveled planar graphs. For graphs that have a weakly leveled planar drawing, Bannister et al. [3] show the upper bound of 6 for the track number, while leaving the question of the lower bound open. Next we answer the question by providing an example of a weakly leveled planar graph whose track number is 6.

Certain families of planar graphs are known to admit weakly leveled planar graphs; for example, Halin graphs (an embedded tree with no vertices of degree 2 whose leaves are connected by a cycle) and X-trees (a complete binary tree with extra edges connecting vertices of the same level). Although track numbers of the graphs have been investigated in several earlier works [3, 4, 6], the gaps between lower and upper bounds remain open. In the following, we present a Halin graph and an X-tree that require 5 tracks. In addition we provide an algorithm that constructs a 5-track layout for every X-tree, thus, closing the gap between lower and upper bounds of the track number of such graphs.

Our lower bound examples in the section rely on computational experiments. To this end, we propose a SAT formulation of the track layout problem, and share the source code of our implementation [1]. The formulation is simple-to-implement but efficient enough to find optimal track layouts of graphs with up to a few hundred of vertices in a reasonable amount of time.

5.1 An Upper Bound for X-trees

An X-tree is a complete binary tree with extra edges connecting vertices of the same level. Formally, if \( v_1, v_2, \ldots, v_{2^d} \) are the vertices of level \( d \geq 0 \) in the tree in the left-to-right order, then the extra edges are \((v_i, v_{i+1})\) for \( 1 \leq i < 2^d \); see Figure 5a. Since X-trees admit a weakly leveled planar drawing, its track number is at most 6 [3, 4, 6]. Next we show that the right answer for the upper bound is 5.

**Lemma 7.** Every X-tree has a 5-track layout.

**Proof.** We call the edges between the vertices of the same level in the X-tree level edges, while the remaining edges are tree edges. In order to construct a 5-track layout, we build a planar straight-line drawing of the graph such that every vertex \( v \) is laid out on an integer grid with coordinates \( x(v) \in \mathbb{N} \) and \( y(v) \in \mathbb{N} \). In the drawing we maintain a property that \(|y(u) - y(v)| \leq 2\) for every edge \((u, v)\); see Figure 5b. It is easy to see that such a drawing corresponds to a track layout with span 2 in which the vertices having equal \( y \)-coordinates belong to the same track. By Lemma 1, the layout can be wrapped onto 5 tracks.

The drawing is built inductively on the level, \( d \geq 0 \), of a given X-tree using the following hypothesis: Every X-tree admits a planar straight-line drawing such that:

- \(|y(u) - y(v)| = 1\) for every level edge \((u, v)\), and \(|y(u) - y(v)| \leq 2\) for every tree edge;

- for the level-\( d \) vertices in the tree, \( v_1, v_2, \ldots, v_{2^d} \), it holds that \( x(v_1) < x(v_2) < \cdots < x(v_{2^d}) \); equivalently, the corresponding level edges form a strictly \( x \)-monotone polyline;

- every edge of the highest level, \((u, v)\), is on the boundary of the drawing; that is, it is visible from points \((x(u), +\infty)\) and \((x(v), +\infty)\).

The basis of the induction for \( d \leq 1 \) is trivial; see Figure 5b. In order to construct the drawing for an X-tree, \( G \), for \( d > 1 \), we start with an inductively constructed X-tree, \( G' \), of level \( d - 1 \). Denote the vertices of \( G \) of level \( d - 1 \) by \( w_1, \ldots, w_{2^{d-1}} \) and the vertices of level \( d \) by \( v_1, \ldots, v_{2^d} \). Observe that \( G \) can be constructed from \( G' \) by iteratively adding a
Every vertex is assigned to one track, which is ensured by the track assignment rules:

\[ \phi_1(v) \lor \phi_2(v) \lor \cdots \lor \phi_t(v) \; \forall v \in V \quad \text{and} \quad \neg \phi_i(v) \lor \neg \phi_j(v) \; \forall \; v \in V, 1 \leq i < j \leq t \]

To guarantee a valid track assignment, we forbid adjacent vertices on the same track:

\[ \neg \phi_i(v) \lor \neg \phi_i(u) \; \forall \; (u, v) \in E, 1 \leq i \leq t \]
For the relative encoding of vertices, we ensure asymmetry and transitivity:

$$\sigma(v, u) \leftrightarrow \neg \sigma(u, v) \quad \forall \text{ distinct } u, v \in V$$

$$\sigma(v, u) \land \sigma(u, w) \rightarrow \sigma(v, w) \quad \forall \text{ distinct } u, v, w \in V$$

To forbid X-crossings among edges, recall that an X-crossing between edges \((u_1, v_1) \in E, (u_2, v_2) \in E\) occurs when \(\text{track}(u_1) = \text{track}(u_2), \text{track}(v_1) = \text{track}(v_2)\) and \(u_1 < u_2, v_1 > v_2\). This can be expressed by the following rules:

$$\phi_i(u_1) \land \phi_i(u_2) \land \phi_j(v_1) \land \phi_j(v_2) \rightarrow (\sigma(u_1, u_2) \land \sigma(v_1, v_2)) \lor (\sigma(u_2, u_1) \land \sigma(v_2, v_1))$$

$$\forall (u_1, v_1) \in E, (u_2, v_2) \in E \text{ such that } u_1 \neq u_2, v_1 \neq v_2 \text{ and } \forall 1 \leq i, j \leq t$$

The resulting CNF formula contains \(\Theta(n^2)\) variables and \(\Theta(n^3 + m^2 t^2)\) clauses. Using a modern SAT solver, one can evaluate small and medium size instances (with up to a few hundred of vertices) within a reasonable time. For example, we computed optimal track layouts for all 977,526,957 maximal planar graphs having \(n = 18\) vertices. In total the computation took 5000 machine-hours, and all the graphs turned out to have a \(t\)-track layout for some \(4 \leq t \leq 7\). Larger graphs, such as one in Theorem 4, are solved within a few hours on a regular machine. Our implementation is available at [1].

Using the formulation, we identified examples of an X-tree and a Halin graph that require 5 tracks; see Figure 6a and 6b. In particular, X-trees of depth \(d \leq 5\) admit a 4-track layout but X-trees with \(d \geq 6\) have track number 5. Similarly, we found a weekly leveled planar graph with 14 vertices that has track number 6; see Figure 6c. Thus, the algorithm of Bannister et al. [3] for constructing 6-track layouts of weakly leveled planar graphs is worst-case optimal.

### 6 Conclusions and Open Problems

In this paper we improved upper and lower bounds on the track number of several families of planar graphs. A natural future direction is to close the remaining gaps for graphs listed in Table 1. Next we discuss several open questions related to track layouts.

Our approach for building track layouts of planar graphs relies on a construction for graphs of bounded treewidth. To the best of our knowledge, the upper bound on the track number of \(k\)-trees is \((k + 1)(2^{k+1} - 2)^k\) [14], while the lower bound is only quadratic in \(k\) [8]. It seems unlikely that the existing upper bound is the right answer, and finding a polynomial or even an exponential \(2^{O(k)}\) bound would already be an exciting improvement.
Open Problem 1. Improve the upper bound of the track number of $k$-trees.

One way of attacking the above problem is tightening a gap between track and queue
numbers of a graph, as the queue number of a $k$-tree is at most $2^k - 1$ [14]. As mentioned
earlier, every $t$-track graph has a $(t-1)$-queue layout. It is easy to see that this bound
is worst-case optimal, that is, there exist $t$-track graphs that require $t-1$ queues in
every layout. In the order direction, every $q$-queue graph has track number at most
$4q \cdot 4q^{(2q-1)(4q-1)}$ [9]. This bound can likely be improved. For example, 1-queue graphs
always admit 4-track layout but no better bound is known for the case $q \geq 2$.

Open Problem 2. What is the largest track number of a $q$-queue graph?

Finally, we would like to see some progress on upward track layouts [5,10]. An upward
track layout of a dag $G$ is a track layout of the underlying undirected graph of $G$, such
that the directed graph obtained from $G$ by adding arcs between consecutive vertices in
a track is acyclic. To the best of our knowledge, very little is known about the variant
of layouts. For example, the upward track number of (directed) paths and caterpillars is
known to be 3 [5], while the upward track number of (directed) trees is at most 5 [10].
Improving the bounds and investigating other classes of graphs is an interesting direction.

Open Problem 3. Investigate the upward track number of various families of graphs.

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References

[1] A SAT-based solver for constructing optimal linear layouts of graphs:
https://github.com/spupyrev/bob.

[2] J. M. Alam, M. A. Bekos, M. Gronemann, M. Kaufmann, and S. Pupyrev. Queue
layouts of planar 3-trees. In International Symposium on Graph Drawing and Network
Visualization, pages 213–226. Springer, 2018.

[3] M. J. Bannister, W. E. Devanny, V. Dujmović, D. Eppstein, and D. R. Wood. Track
layouts, layered path decompositions, and leveled planarity. Algorithmica, 81(4):1561–
1583, 2019.

[4] E. Di Giacomo, G. Liotta, and H. Meijer. Computing straight-line 3D grid drawings
of graphs in linear volume. Computational Geometry, 32(1):26–58, 2005.

[5] E. Di Giacomo, G. Liotta, H. Meijer, and S. K. Wismath. Volume requirements of
3D upward drawings. Discrete Mathematics, 309(7):1824–1837, 2009.

[6] E. Di Giacomo and H. Meijer. Track drawings of graphs with constant queue number.
In International Symposium on Graph Drawing, pages 214–225. Springer, 2003.

[7] V. Dujmović, G. Joret, P. Micek, P. Morin, T. Ueckerdt, and D. R. Wood. Planar
graphs have bounded queue-number. In Symposium on Foundations of Computer
Science, 2019.
[8] V. Dujmović, P. Morin, and D. R. Wood. Layout of graphs with bounded tree-width. *SIAM Journal on Computing*, 34(3):553–579, 2005.

[9] V. Dujmović, A. Pór, and D. R. Wood. Track layouts of graphs. *Discrete Mathematics & Theoretical Computer Science*, 6(2):497–522, 2004.

[10] V. Dujmović and D. R. Wood. Upward three-dimensional grid drawings of graphs. *Order*, 23(1):1–20, 2006.

[11] S. Felsmer, G. Liotta, and S. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. *Graph Algorithms and Applications*, 4:363–398, 2006.

[12] L. S. Heath and A. L. Rosenberg. Laying out graphs using queues. *SIAM J. Comput.*, 21(5):927–958, 1992.

[13] L. T. Ollmann. On the book thicknesses of various graphs. In *Southeastern Conference on Combinatorics, Graph Theory and Computing*, volume 8, page 459, 1973.

[14] V. Wiechert. On the queue-number of graphs with bounded tree-width. *Electr. J. Comb.*, 24(1):P1.65, 2017.