2D quantum gravity from quantum entanglement

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In quantum systems with many degrees of freedom the replica method is a useful tool to study the entanglement of arbitrary spatial regions. We apply it in a way which allows them to back-react. As a consequence, they become dynamical subsystems whose position, form and extension is determined by their interaction with the whole system. We analyze in particular quantum spin chains described at criticality by a conformal field theory (CFT). Its coupling to the Gibbs’ ensemble of all possible subsystems is relevant and drives the system into a new fixed point which is argued to be that of the 2D quantum gravity coupled to this system. Numerical experiments on the critical Ising model show that the new critical exponents agree with those predicted by the formula of Knizhnik, Polyakov and Zamolodchikov.

PACS numbers: 03.65.Ud, 05.50.+q, 04.60.Kz

Understanding the effects of quantum entanglement of systems with many degrees of freedom such as quantum spin chains or quantum field theories is a challenging problem which connects statistical mechanics to quantum information science. If the system under study is taken to be in a pure state \( |\Psi\rangle \), a complete description of the information available to an observer who has access only to a subsystem \( A \) will be given by the reduced density matrix

\[
\rho_A = \text{tr}_B |\Psi\rangle \langle \Psi| \quad (1)
\]

obtained by tracing over the degrees of freedom of the remainder \( B \), inaccessible to the observer.

There are various functions of \( \rho_A \) which could be used as useful probes to measure how closely entangled, or how ‘quantum’, a given state is (see [1] for reviews). Most of them are expressed in terms of \( \text{tr} \rho_A^n \). For instance, the R\"enyi entropy is \( R_A(n) = \log \text{tr} \rho_A^n / (1 - n) \) and the entanglement entropy is the limit \( S_A = \lim_{n \to 1} R_A(n) = -\text{tr} \rho_A \log \rho_A \).

In quantum field theory the quantity \( \text{tr} \rho_A^n \) for integer \( n \) can be computed without explicit knowledge of the ground state through the so-called replica method \([2,3]\). Following this procedure the partition function \( Z \) of a \( d \) dimensional quantum system is computed in the standard way by doing the functional integration on \( n \) copies (or replicas) of the corresponding Euclidean classical system in \( d + 1 \) dimensions. These copies interact among themselves through the inaccessible subsystems \( B \) as explained below. In this set up the quantity \( \text{tr} \rho_A^n \) is proportional to the canonical partition function \( Z_n(A) \) of the coupled system of \( n \) replicas. More precisely we have \( \text{tr} \rho_A^n = Z_n(A) / Z^n \).

In most previous studies the accessible subsystem \( A \) is chosen to be fixed and a major goal is to investigate how \( \text{tr} \rho_A^n \) and the entanglement entropy depend on \( A \). The point of view which is taken in this paper is different. We treat \( A \) (or equivalently \( B \)) as a back-reacting, dynamical, subsystem whose position, form and extension is determined by its interaction with the whole system. We implement it by ‘summing over histories’, i.e. by putting the system in equilibrium with the Gibbs’ ensemble \( \{ A \} \) of all possible subsystems.

As we will see, when the system is put on a lattice the sum over the ensemble \( \{ A \} \) is unambiguously defined. An interesting consequence is that the sum over \( \{ A \} \) defines a non-trivial, translation invariant, modification of the system uniquely generated by the quantum entanglement of the set of all the accessible subsystems. This is particularly interesting when the system undergoes a critical transition.

One important question to address is whether the interaction with the ensemble \( \{ A \} \) changes the universality class of the critical system i.e., in renormalization group language, whether the coupling to \( \{ A \} \) is relevant when the critical system is in a pure state. In this paper we answer this question for the class of \( 1 + 1 \) dimensional quantum systems described at criticality by a relativistic, massless, field theory, i.e. a conformal field theory (CFT) with central charge \( c \). It turns out that any fixed point with \( c > 0 \) becomes unstable when coupled to \( \{ A \} \) and that the system flows to a fixed point with \( c = 0 \). A series of numerical experiments on the Ising model indicates that for a suitable choice of the coupling parameters the new fixed point is critical. The scaling dimensions of the primary fields turn out to be those expected in the coupling of CFT to two dimensional quantum gravity.

The partition function of many quantum systems at inverse temperature \( \beta \) may be rewritten as a Feynman path integral in imaginary time \( 0 \leq \tau \leq \beta \). In the case of a \( d \) dimensional quantum lattice system we may regard the quantum partition function \( Z = \text{tr} e^{-\hat{H}} \) as the canonical partition function of a classical system in \( d + 1 \) dimensions in a slab geometry with thickness \( \beta \). The boundary conditions in the imaginary time direction are periodic for bosonic degrees of freedom. The case when the classical system is infinite in all \( d + 1 \) directions corresponds to zero temperature in the quantum system. For sake of
simplicity we assume that the classical system is defined on a \( d + 1 \) dimensional hyper-cubic lattice \( \Lambda = \{ \bar{x}, \tau \} \) \( (x_i, \tau \in \mathbb{Z}) \). Its partition function can be computed by doing the Euclidean functional integral \( Z = \int \mathcal{D}[\phi] e^{-S[\phi]} \) over fields \( \phi_x = \phi(\bar{x}, \tau) \) periodic under \( \tau \to \tau + \beta \). The Euclidean action \( S[\phi] \) is assumed to be decomposable as the sum \( S = \sum_{xy} S[\phi_x, \phi_y] \) of the contributions of the links \( \langle xy \rangle \) of \( \Lambda \).

Working in the framework of replica method, we consider a stack of \( n \) copies of the original system, range them in a cyclical order and couple them together in the following way. We pick all the lattice nodes belonging to the inaccessible subsystem \( B \) and replace their links in the imaginary time direction with links connecting two consecutive copies

\[
S^{(k)}_{\langle xy \rangle} = \begin{cases} 
S[\phi^{(k)}_x, \phi^{(k+1)}_y] & x \in B \\
S[\phi^{(k)}_x, \phi^{(k)}_y] & x \notin B
\end{cases} 
\]

where \( \phi^{(k)} \) denotes the field in the \( k \)-th copy. The subsystem \( B \) lies in a slice of the lattice at a given value of \( \tau \). Since the imaginary time and space directions enter into the problem on very different footings, one might expect the corresponding classical system to exhibit intrinsically anisotropic scaling. Here we restrict our analysis to systems which are sufficiently isotropic, in such a way that their critical behavior is described by a relativistic \( d + 1 \) dimensional field theory. In this context it is convenient to define a slightly generalized coupling among the \( n \) replicas, getting rid of the constraint of the \( B \) subsystem to lie in a constant slice and treating spatial and temporal links in the same way. Each stack of \( n \) links associated to the nodes \( x \) and \( y \) in the \( n \) replicas is set in two possible states. In the state ‘\( A \)’ each link of the stack connects points of the same replica while in the state ‘\( B \)’ it connects them cyclically, like in \( (2) \); the links in the state ‘\( B \)’ single out the subsystem \( B \). An advantage of this more general setting is that it is easy to show that the coupled system of \( n \) replicas is endowed with an important local symmetry: flipping from ‘\( A \)’ to ‘\( B \)’ or vice versa the state of all links intersecting an arbitrary closed \( d \) dimensional manifold keeps invariant the partition function \( [5] \). A direct consequence of such a symmetry is that not only the entropy but all the thermodynamic functions depend only on the boundary of \( B \) \( [6] \).

In order to promote accessible subsystems to dynamical variables one has simply to sum over all possible assignments of the states ‘\( A \)’ and ‘\( B \)’ to the lattice links, so the partition function of our coupled system of \( n \) replicas can be written as

\[
Z_n = \sum_{\{G\}} \prod_{k=1}^{n} \mathcal{D}[\phi^{(k)}] e^{-\sum_{k=1}^{n} \sum_{xy} S^{(k)}_{\langle xy \rangle}} \tag{3}
\]

where \( G \) is the subgraph of links which are set in the state ‘\( B \)’ and the summation is over all subgraphs.

In the spirit of replica method, the evaluation of the entanglement entropy would require taking the limit \( n \to 1 \), even if there are indications that the analytic continuation from positive integer \( n \) to real values could be rather difficult \( [6] \). Fortunately, we do not need to do that, because the new phenomenon we want to describe can be observed for any integer \( n > 1 \).

To make the discussion concrete and explicit, we specialize now to the case where the system in question is a quantum spin chain described at criticality by a CFT. Its two dimensional lattice description \( [3] \) is a discretized version of a \( n \)-sheeted covering of the plane, where the dual \( \tilde{G} \) of the subgraph \( G \) is formed by the set of cuts connecting these sheets. The local symmetry mentioned above turns out to express the invariance of the system under the addition (or the removal) of closed cuts or under continuous deformations of open cuts with fixed ends. It is worth noting that the only elements of the graph \( \tilde{G} \) having an intrinsic geometrical -and physical- meaning are the end points of the cuts, i. e. the branch points of the Riemann surface. They correspond to conical singularities with deficit angle \( 2\pi(n-1) \). On the contrary, the cuts joining different branch points are in no way distinguished lines on the surface: their introduction has a similar role as the choice of a reference frame on the surface. Thus, the dynamical effects of the back-reaction of the accessible subsystems of a critical quantum spin chain are intimately related to a 2D CFT in statistical equilibrium with a gas of conical singularities.

Note that the \( n \)-sheeted covering of the plane with \( N \) branch points is a Riemann surface of genus \( g = (n-1)(N-2)/2 \), therefore summing over all accessi-
ple subsystems corresponds, in the replica approach, to a double sum over genera and moduli of these Riemann surfaces. This is the first indication that this issue is related to 2D quantum gravity.

A stronger indication comes from an observation made by Knizhnik long time ago \cite{9,10}, namely that branch points correspond to primary fields $\Phi_n(z,\bar{z})$ of scaling dimensions

$$\Delta_n = \Delta_n = \frac{c}{24} \left( 1 - \frac{1}{n^2} \right),$$

where $c$ is the central charge of the system at the critical point. We see that for $c$ not too large the conical singularities are associated to relevant operators. Therefore the effect of considering the accessible subsystems as dynamical quantities is equivalent, in the underlying field theory, to perturbing the CFT by the relevant operator $\Phi_n(z,\bar{z})$, so the action is

$$S = S^* + \mu \int \Phi_n(z,\bar{z}) \, d^2z$$

where $S^*$ is the CFT action and $\mu$ is the chemical potential which controls the appearance of conical singularities. Such a perturbation drives the system away from the critical point. According to the $c$-theorem \cite{9}, a generic relevant operator generates renormalization group flows into fixed points with a value of $c$ that cannot exceed its initial value. However, there is more information to be gained. According to \cite{9} any CFT with $c > 0$ is unstable whenever perturbed with $\Phi_n$. This restricts the possible fixed points to which any CFT with $c > 0$ may flow to those with $c = 0$. For generic values of $\mu$ these are trivial fixed points corresponding to massive theories. There is however one value of $\mu$, at least, at which the system is critical. Thus the total central charge of the CFT coupled to the ensemble of the accessible subsystems is zero, which is precisely what it happens in CFT’s coupled to quantum gravity.

The strongest indication that this new universality class corresponds to 2D quantum gravity comes from numerical experiments, where one may accurately evaluate how change the scaling dimensions of local operators when the coupling to accessible subsystems is switched on. We use as a guidance the formula of Knizhnik, Polyakov and Zamolodchikov (KPZ) \cite{9,10}

$$\Delta^o = \Delta + \frac{\gamma^2}{4} \Delta (\Delta - 1), \quad \gamma = \sqrt{\frac{25 - c}{6}} - \sqrt{\frac{1 - c}{6}},$$

which relates the scaling dimensions $\Delta^o$ of a primary field of a CFT to the scaling dimensions $\Delta$ of the same operator when the theory is coupled to quantum gravity.

Let us consider a spin-$\frac{1}{2}$ quantum chain coupled to a transverse magnetic field $h$. The quantum Hamiltonian is $\hat{H} = -J \sum_i \sigma_i^x \sigma_{i+1}^x - H \sum_i \sigma_i^z$, where $\sigma^x$ and $\sigma^z$ are the usual Pauli matrices. As is well known, this system exhibits a quantum phase transition for $J = h$. This manifests itself as a power law decay of the correlators which lies in the universality class of the 2D critical Ising model, described by the CFT with $c = \frac{1}{2}$.

We simulated this system at the self-dual point with $n = 2, \ldots, 5$ replicas of a square lattice enclosed in a square box with a side of $L$ lattice spacings with toroidal boundary conditions. In a first set of Monte Carlo calculations the ensemble $\{A\}$ was taken on a 1D slice at a fixed value of $\tau$. On the same temporal slice we measured the (difference of) correlators

$$C(L, s) = \langle \sigma_i^z \sigma_{i+L/2}^z - \sigma_i^x \sigma_{i+L/2}^x \rangle,$$

where $s$ is a (proper) factor of the integer $L$. In our simulations we chose $s = 8$ and $s = 4$. In order to control the coupling with the ensemble $\{A\}$, we modified the Ising model by allowing a non-zero fugacity $z = e^{-\mu}$ counting the number of branch points.

Since $\{A\}$ affected only a one dimensional boundary of our system, this remained critical even at $z \neq 0$. We simulated the Ising part of the model with a standard non-local cluster algorithm, while the update of the subset $A$, consisting of an arbitrary number $N$ of disjoint intervals with $N = 0, 1, \ldots, L/2$, was performed with a heat bath method. Although the specific form of $C(L, s)$ is not known, we have, in the thermodynamic limit,

$$C(\lambda L, s) = \lambda^{-\tau} C(L, s)$$

where $\lambda$ is any positive rescaling factor. As $z$ varies from 0 to 1, $x$ drops from the expected value of the pure Ising model $x = 4\Delta^o$ to a new scaling dimension (see figure 1).
According to the previous discussion, the expected value is the one suggested by KPZ formula. As sometimes it happens in critical systems, the observed value of $x$ differs by an integer with respect the expected value. Precisely we found $x = 4\Delta_x + 1$ for $n = 2$ and $x = 4\Delta_x + 2$ for $n \geq 3$ (see figures 1 and 2). $\Delta_x = \frac{1}{6}$ is the KPZ value associated to $\Delta_o = \frac{1}{17}$ for $c = \frac{5}{2}$.

Let us emphasize that in the first set of numerical experiments the simulated system is one dimensional, in the sense that the correlator and the subsystems are taken on a fixed 1D slice at a given value of $\tau$. In the second set of numerical experiments we simulated instead a truly two dimensional system, with no limitations on the location of cuts representing the accessible subsystems. In this new setting the gas of conical singularities is spread in the bulk and drives the system away from the critical point of the pure system. There is however a critical value $z_c$ of the fugacity at which the whole interacting system undergoes a second order phase transition which is presumably in the same universality class of the 1D quantum system described above. The system composed of two replicas at the self-dual point of the pure Ising system exhibits a critical behavior for $z_c = 0.01127(1)$. The scaling dimension of the spin operator turns out to be, within the numerical accuracy, the one of the 2D quantum gravity (see figure 2).

Likewise, the scaling dimensions of the energy operator turn out to be the KPZ value $\Delta_e = \frac{7}{4}$, corresponding to $\Delta_o = \frac{1}{2}$. In order to extract this critical exponent, we measured the vacuum expectation value of the link operator $\langle \sigma_x^z \sigma_y^z \rangle$, where $x$ and $y$ represent two nearest neighbors of the lattice. At the critical point this quantity is expected to have the following functional form

$$\langle \sigma_x^z \sigma_y^z \rangle = c_0 + c_1/L^{2\Delta_e} + c_2/L^{2\Delta_e+1} + \ldots$$

where we wrote explicitly only the terms necessary to accurately fit the data, as shown in figure 3.

From a geometrical point of view, the two-sheeted system we described is defined on an hyperelliptic Riemann surface and the sum $\sum_{(G)}$ over all possible subsets of links made in eq.(3) corresponds, in the continuum limit, to a double sum over the space of moduli and the genera of these surfaces. This has some similarity with the double scaling limit of matrix models \cite{12}. A promising aspect of the present approach is that it may be extended in a straightforward way to higher dimensions.

Let us conclude with a general remark. In the past years, 't Hooft \cite{12} and several other authors \cite{2, 13–15} have suggested that the quantum entanglement and its ensuing entropy might be related to the Bekenstein-Hawking entropy of black holes. More recently, in the light of the AdS/CFT correspondence, a comprehensive gravitational interpretation of the entanglement entropy has been proposed \cite{16}. The present study captures a different facet of the same fascinating relationship between gravity and quantum entanglement, in which the back-reaction of the accessible subsystems of a 1D quantum system has the same effect as the coupling to 2D quantum gravity.

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