

Existence for Time-Fractional Semilinear Diffusion Equation on the Sphere

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Fractional diffusion on the sphere plays a large role in the study of physical phenomena customs and meteorology and geophysics. In this paper, we examine two types of the sphere problem: the initial value problem and the end value problem. We are interested in focus on the solution existence in a local or global form. In order to overcome difficult evaluations when evaluating, we need some new techniques. The main analytical tool is the use of the Banach fixed point theorem.

1. Introduction

When examining many physical and geophysical phenomena, one encounters problems directly or indirectly related to the sphere. To describe and explain quantitatively these models, they will be simulated with mathematical equations on the sphere. That is also the reason why the spherical equations have attracted many scientists interested and studied them. We can give some examples as follows. The weather forecasting models and the currents of groundwater in the ocean bed were simulated by equations on the sphere. For the readers’ convenience, we have given a number of typical works that have had a great influence on the development of the analysis of PDEs on the sphere.

According to the development of the mathematics disciplines, especially calculus, in the last few decades, fractional analysis has been one of the most influential disciplines in mathematics. Most of the problems related to it often have applications in modeling real-world problems. Fractional analysis has many applications in mechanics, physics, engineering science, etc. We would like to share many published works on these issues such as Karapinar et al. [7–14] and Inc and his group [15–19]. The main goal of this paper is to investigate the existence of the following equation

\[
\frac{\partial}{\partial t} u(x, t) - \frac{\partial^{1-q}}{\partial t} \Delta^* u(x, t) = \theta(t) F(u(x, t)), \quad (x, t) \in S^2 \times [0, T],
\]

with the initial Cauchy condition

\[
u(x, 0) = f(x),
\]

or the terminal value condition

\[
u(x, T) = g(x),
\]
where $\partial^\alpha_s/\partial t^\alpha$ called the Riemann-Liouville fractional derivative of order $\alpha$, $0 < \alpha \leq 1$. Here, $S^2$ is a sphere on the $R^3$. It is given by $D^\alpha_s/\partial t^\alpha$ is the Riemann-Liouville fractional derivative of order $1 - \alpha$ given by

$$\frac{d^{1-\alpha}}{dt^{1-\alpha}} v(t) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \left( \int_0^t s^{\alpha-1} v(t-s) ds \right), \quad 0 < \alpha < 1,$$

and $D^{\alpha,\beta}_{0+} v(t) = (d/dt)^{\alpha,\beta} v(t)$ if $\alpha = 1$. The functions $\psi, F$ in (1) are defined later. The operator $\Delta^*$ is called Laplace-Beltrami which is introduced in more detail in Section 2. To the best of our knowledge, there are not any results on problem (1). Our main goal in this paper is to study two goals. Our first goal is to consider Cauchy initial problem (1)–(2). We get the global solution in a suitable space. Our second goal is to consider the Cauchy terminal problem (1)–(2). In this case, we get only local solutions in space $L^{15}$. For both of the above purposes, we use Banach fixed point theorem together with the evaluation of the sphere. In addition, to overcome the difficulties of proving, we also cleverly make use of spherical harmonics of degree $j$ on $S^2\,$, so it is expressed by the expansion of spherical harmonics

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} \hat{f}_{jk} W_{jk} \hat{f}_{jk} = \int_{S^2} f W_j dS,$$

where $dS$ is the surface measure of the unit sphere. The Sobolev space $H^s(S^2)$ is defined by

$$H^s(S^2) = \left\{ g \in L^2(S^2) : \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} |(j^2 + j + 1)^{s/2} |g\wedge_{jk}|^2 < \infty \right\},$$

with the following norm

$$\|g\|_{H^s(S^2)} = \left\{ \sum_{j=0}^{\infty} \sum_{k=1}^{2j+1} (j^2 + j + 1)^{s} |g\wedge_{jk}|^2 \right\}^{1/2}. $$

The space $L^\infty(0, T; H^s(S^2))$ is defined by

$$L^\infty(0, T; H^s(S^2)) = \{ f \in L^\infty(0, T; H^s(S^2)) \,| \, e^{-\alpha t} \|f(\cdot, t)\|_{H^s(S^2)} < \infty \},$$

with corresponding norm as follows

$$\|f\|_{L^\infty(0, T; H^s(S^2))} = \sup_{0 \leq t \leq T} e^{-\alpha t} \|f(\cdot, t)\|_{H^s(S^2)}. $$

2. Preliminaries

Mittag-Leffler is a function represented as the following form

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad (z \in \mathbb{C}), \text{ for } \alpha > 0 \text{ and } \beta \in \mathbb{R}. \text{ We call to mind the following lemma (see for example [20]). We have the following lemma which is useful for next proof.}

**Lemma 1.** Let $0 < \alpha < 1$. Then, the function $E_{\alpha,1}(-z)$ satisfies the following property

$$\frac{C_1}{1 + z} \leq E_{\alpha,1}(-z) \leq \frac{C_2}{1 + z}, \quad z > 0,$$

where $C_1$ and $C_2$ are the two positive constants.

Spherical harmonics are polynomials which satisfy $\Delta Y(x) = 0$ (where $\Delta$ is the Laplacian operator in $R^3$) and are restricted to the surface of the Euclidean sphere $S^2$. The eigenvalues for $-\Delta^*$ in $R^3$ are

$$\lambda_j = j^2 + j, \quad j = 0, 1, 2, \cdots,$$

and the eigenfunctions corresponding to $\lambda_j$ are the spherical harmonics $W_j(x)$ of order $l$, i.e.,

$$\Delta^* W_j(x) = -\lambda_j W_j(x).$$

The space of all spherical harmonics of degree $j$ on $S^2$, denoted by $V_j$, has an orthonormal basis $\{W_{j\nu}(x) : k = 1, 2, 3, \cdots, N(j, 2)\}$ where

$$N(2, j) = \frac{2j + 1}{\Gamma(2)}, \quad j \geq 1.$$ 

3. Global Existence for Mild Solution to Initial Value Problem

**Theorem 2.** Assume that $\theta : (0, T) \rightarrow \mathbb{R}$ such that

$$|\theta(z)| \leq C_0 z^m, \quad -\frac{1}{2} < m.$$
Let $F$ satisfies the condition

$$\|Fv_1 - Fv_2\|_{H^r(s')} \leq K_f \|v_1 - v_2\|_{H^r(s')},$$  \hspace{1cm} (16)$$

where $K_f$ is a positive constant and $\nu$ and $\nu'$ satisfy that

$$0 < \nu - \nu' < \min \left( 2, \frac{1}{\alpha} \right).$$  \hspace{1cm} (17)$$

Then, problem (1)–(2) has a global existence in $L^\infty_\omega(0, T; H^r(S^2))$ for a enough large.

Proof. As we know from [2] that $\Delta^*$ is the Laplace-Beltrami on the sphere $S^2$. Any function $u \in L^2(S^2)$ can be described by the terms of spherical harmonics

$$u(x, t) = \sum_{j=0}^{\infty} \sum_{k=1}^{C_0} \tilde{u}_{jk}(t) W_{jk}(x),$$

where $dS$ is the surface measure of the unit sphere. Let us first give an expression of the mild solution.

$$\tilde{u}_{jk}(t) = E_{a,1}(-j^2 + j \tau^a)\tilde{f}_{jk},$$

$$+ \int_0^t E_{a,1}(-j^2 + j)(t - r)^\alpha \theta(r) \bar{f}_{jk}(u)(r) dr.$$  \hspace{1cm} (19)$$

So, we get that the following equality

$$u(x, t) = \sum_{j=0}^{\infty} \sum_{k=1}^{C_0} \left( E_{a,1}(-j^2 + j \tau^a)\tilde{f}_{jk} \right) W_{jk}(x) + \sum_{j=0}^{\infty} \sum_{k=1}^{C_0} \left( \int_0^t E_{a,1}(-j^2 + j)(t - r)^\alpha \theta(r) \bar{f}_{jk}(u)(r) dr \right) W_{jk}(x).$$  \hspace{1cm} (20)$$

Setting the function

$$\mathcal{M}_1(x, t) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \left( E_{a,1}(-(j^2 + j \tau^a)\tilde{f}_{jk} \right) W_{jk}(x).$$  \hspace{1cm} (22)$$

Noting that $E_{a,1}(-z) \leq C_2$ for any $z > 0$, and $f \in H^r(S^2)$, the first term is bounded by

$$\|\mathcal{M}_1(\cdot, t)\|_{H^r(s')} = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \left( j^2 + j + 1 \right)^\nu \left( E_{a,1}(-j^2 + j \tau^a)\tilde{f}_{jk} \right)^2, \leq [C_2^2] \|f\|_{H^r(s')}^2,$$

which allows us to get that

$$\|\mathcal{M}_1(\cdot, t)\|_{H^r(s')} \leq C_2 \|f\|_{H^r(s')},$$  \hspace{1cm} (24)$$

for any $0 \leq t \leq T$. This gives immediately that if $\omega = 0$ then $\mathcal{F}\omega$ belongs to the space $L^\infty(0, T; H^r(S^2))$. Let us take two functions $\omega$ and $\tilde{\omega}$ belong to the space $L^\infty_\omega(0, T; H^r(S^2))$. Then, from (21), we have

$$\mathcal{F}\omega(t) - \mathcal{F}\tilde{\omega}(t) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \left( \int_0^t E_{a,1}(-(j^2 + j)(t - r)^\alpha \theta(r) \bar{f}_{jk}(u)(r) dr \right) W_{jk}(x).$$  \hspace{1cm} (25)$$

Using Parseval’s equality and Hölder inequality, one has

$$\|\mathcal{F}\omega(t) - \mathcal{F}\tilde{\omega}(t)\|_{H^r(s')}^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \left( j^2 + j + 1 \right)^\nu \left( \int_0^t E_{a,1}(-(j^2 + j)(t - r)^\alpha \theta(r) \bar{f}_{jk}(u)(r) dr \right)^2 \leq \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \left( j^2 + j + 1 \right)^\nu \left( \int_0^t E_{a,1}(-(j^2 + j)(t - r)^\alpha \theta(r) \bar{f}_{jk}(u)(r) dr \right)^2 \hspace{1cm} (26)$$

Let us review that $E_{a,1}(-z) \leq (C_2/(1 + z))$, we note that the following inequality

$$\|\mathcal{F}\omega(t) - \mathcal{F}\tilde{\omega}(t)\|_{H^r(s')}^2 \leq C_2 \|f\|_{H^r(s')},$$
Due to the condition \( \int_0^1 e^{-\alpha(t-r)} \, dr \leq (1/2a) \), we know that the following estimate

\[
e^{-\alpha t} \| \mathcal{J} w(t) - \mathcal{J} \tilde{w}(t) \|_{H^2(s)}^2 \\
\leq \frac{1}{2a} \| w - \tilde{w} \|_{L^2_w(0, T; H^2(S))}^2.
\]  

This implies immediately that

\[
e^{-\alpha t} \| \mathcal{J} w(t) - \mathcal{J} \tilde{w}(t) \|_{H^2(s)} \leq \sqrt{\frac{1}{2a} \| w - \tilde{w} \|_{L^2_w(0, T; H^2(S))}^2}.
\]

The right hand side of (34) is independent of \( t \), so we can deduce that the following estimate

\[
\| \mathcal{J} w - \mathcal{J} \tilde{w} \|_{L^2_w(0, T; H^2(S))} \leq \sqrt{\frac{1}{2a} \| w - \tilde{w} \|_{L^2_w(0, T; H^2(S))}^2}.
\]

By choose \( a \) enough large such that

\( \sqrt{\frac{1}{2a} \| w - \tilde{w} \|_{L^2_w(0, T; H^2(S))}^2} < 1 \), we find that \( \mathcal{J} \) is a contraction in \( L^2_w(0, T; H^2(S)) \). Based on the Banach fixed point theorem, we have immediately concluded that problem (1)–(2) has a global existence in \( L^2_w(0, T; H^2(S^2)) \).

\[\Box\]

4. Terminal Value Problem: Local Existence

In this section, we devoted the following problem with terminal condition

\[
\begin{cases}
\frac{\partial}{\partial t} u(x, t) - \frac{\partial^{1-a}}{\partial t^{1-a}} \Delta^\alpha u(x, t) = \theta(t) F(u(x, t)), & (x, t) \in S^2 \times (0, T), \\
w(x, T) = g(x), & x \in S^2,
\end{cases}
\]

where \( g \) is defined later. The purpose of this section is to study the existence and uniqueness of solution of problem (1)–(3).

**Theorem 3.** Let us assume that \( \psi : (0, T) \rightarrow \mathbb{R} \) such that

\[
|\theta(z)| \leq C_{\delta} e^{\delta z}, \delta > -1.
\]

Let the function \( g \in H^2(S^2) \). Let \( F \) satisfies the condition

\[
\| F v_1 - F v_2 \|_{L^2(S^2)} \leq K_f \| v_1 - v_2 \|_{L^2(S^2)},
\]

where \( K_f \) is a constant.
for any $K_f > 0$. Then, problem (1)–(3) has a local existence in $L^\infty(0, T; L^2(S^2))$.

Proof. Let us assume that $u(x, 0) = f(x)$. Then, we have

$$\tilde{w}_{jk}(t) = E_{a, 1}(-\langle j^2 + j \rangle t^\alpha)^j f_{jk}$$

$$+ \int_0^t E_{a, 1}(-\langle j^2 + j \rangle (t - r)^\alpha) \theta(r) \tilde{F}_{jk}(w(r))dr.$$  \hspace{1cm} (39)

Set $t = T$ into the above equation, we get that

$$\tilde{w}_{jk}(T) = E_{a, 1}(-\langle j^2 + j \rangle T^\alpha)^j f_{jk}$$

$$+ \int_0^T E_{a, 1}(-\langle j^2 + j \rangle (T - r)^\alpha) \theta(r) \tilde{F}_{jk}(w(r))dr.$$  \hspace{1cm} (40)

This implies that

$$\tilde{f}_{jk} = \frac{1}{E_{a, 1}(-\langle j^2 + j \rangle T^\alpha)}$$

$$\times \left(\tilde{g}_{jk} - \int_0^T E_{a, 1}(-\langle j^2 + j \rangle (T - r)^\alpha) \theta(r) \tilde{F}_{jk}(w(r))dr\right).$$  \hspace{1cm} (41)

After some simple calculation, we have that

$$\tilde{w}_{jk}(t) = E_{a, 1}(-\langle j^2 + j \rangle t^\alpha)^j f_{jk}$$

$$+ \int_0^t E_{a, 1}(-\langle j^2 + j \rangle (t - r)^\alpha) \theta(r) \tilde{F}_{jk}(w(r))dr$$

$$- E_{a, 1}(-\langle j^2 + j \rangle T^\alpha)^j f_{jk}$$

$$\cdot (-\langle j^2 + j \rangle (T - r)^\alpha) \theta(r) \tilde{F}_{jk}(w(r))dr.$$  \hspace{1cm} (42)

So, we get that the following equality

$$w(x, t) = \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)/T^2} \left( E_{a, 1}(-\langle j^2 + j \rangle t^\alpha)^j \tilde{g}_{jk} \right) W_{jk}(x)$$

$$+ \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)/T^2} \left( \int_0^t E_{a, 1}(-\langle j^2 + j \rangle (t - r)^\alpha) \theta(r) \tilde{F}_{jk}(w(r))dr \right) W_{jk}(x)$$

$$- \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)/T^2} \left( E_{a, 1}(-\langle j^2 + j \rangle T^\alpha)^j \tilde{F}_{jk}(w(r))dr \right) W_{jk}(x).$$  \hspace{1cm} (43)

In order to apply Banach fixed point theorem, we need to set the following function

$$\mathcal{F} \theta(t) = \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)/T^2} \left( E_{a, 1}(-\langle j^2 + j \rangle t^\alpha)^j \tilde{g}_{jk} \right) W_{jk}$$

$$+ \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)/T^2} \left( \int_0^t E_{a, 1}(-\langle j^2 + j \rangle (t - r)^\alpha) \theta(r) \tilde{F}_{jk}(\theta(r))dr \right) W_{jk}$$

$$- \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)/T^2} \left( E_{a, 1}(-\langle j^2 + j \rangle t^\alpha)^j \tilde{F}_{jk}(\theta(r))dr \right) W_{jk}. $$  \hspace{1cm} (44)

First, let us look at the expression as above and give an evaluation for $||\mathcal{F} \theta_1(t) - \mathcal{F} \theta_2(t)||$ for any $\theta_1, \theta_2 \in L^\infty(0, T; L^2(S^2))$. First, noting that

$$E_{a, 1}(-\langle j^2 + j \rangle T^\alpha)^j \geq \frac{C_1}{1 + \langle j^2 + j \rangle T^\alpha},$$

$$E_{a, 1}(-\langle j^2 + j \rangle (T - r)^\alpha) \leq \frac{C_2}{1 + \langle j^2 + j \rangle (T - r)^\alpha}. $$  \hspace{1cm} (45)
we get that
\[
\frac{E_{a,1}(-j^2 + j)(T - r)^a}{E_{a,1}(-j^2 + j)T^a} \leq \frac{C_2}{1 + (j^2 + j)(T - r)^a} \leq C_3 T^a(T - r)^{-a}.
\] (46)

This implies immediately that
\[
\begin{align*}
\|\mathcal{T}_2 \theta_1(t) - \mathcal{T}_2 \theta_2(t)\|_{L^2(S')}^2
\leq & \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)T(2)} \left( \frac{E_{a,1}(-j^2 + j)T^a}{E_{a,1}(-j^2 + j)T^a} \right) \\
& \cdot E_{a,1}(-j^2 + j)(T - r)^a \theta(r) \\
& \cdot \left( (F \wedge \delta_\rho(\theta_1)(r) - F \wedge \delta_\rho(\theta_2)(r))^2 \right) \\
\leq & \sum_{j=0}^{\infty} \sum_{k=1}^{(2j+1)T(2)} \left( \int_0^T E_{a,1}(-j^2 + j)(T - r)^a \right) \\
& \cdot \left( \int_0^T E_{a,1}(-j^2 + j)(T - r)^a \right) \\
& \cdot \left( \int_0^T E_{a,1}(-j^2 + j)(T - r)^a \right) \\
& \cdot \left( \int_0^T E_{a,1}(-j^2 + j)(T - r)^a \right) \\
& \cdot \left( \int_0^T E_{a,1}(-j^2 + j)(T - r)^a \right)
\end{align*}
\] (47)

where we have used Hölder inequality. Using (46), we have that
\[
\int_0^T E_{a,1}(-j^2 + j)(T - r)^a \theta(r) dr
\leq C_3 T^a C_0 \int_0^T (T - r)^{-a} \theta(r) dr
\leq C_3 C_1 C_0 T^{a-1} \theta(1 - \alpha, 1 + \delta)
\]

\[
= C_3 C_1 C_0 T^{a-1} \theta_B(1 - \alpha, 1 + \delta),
\]

\[
\int_0^T E_{a,1}(-j^2 + j)(T - r)^a \theta(r) dr
\leq C_3 C_1 C_0 T^{a-1} \theta_B(1 - \alpha, 1 + \delta),
\] (48)

Hence, we can deduce that
\[
\begin{align*}
\|\mathcal{T}_2 \theta_1(t) - \mathcal{T}_2 \theta_2(t)\|_{L^2(S')}^2
\leq & \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right) \\
& \cdot \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right) \\
& \cdot \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right) \\
& \cdot \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right)
\end{align*}
\] (49)

Lipschitz property of $F$ as in (38) gives that
\[
\begin{align*}
\int_0^T (T - r)^{-a} \|F(\theta_1)(r) - F(\theta_2)(r)\|_{L^2(S')}^2 dr
\leq & K_f \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \|\theta_1 - \theta_2\|_{L^2(S')}^2 dr
\leq & K_f \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \|\theta_1 - \theta_2\|_{L^2(S')}^2 \right)
= K_f T^{a-1} \theta_B(1 - \alpha, 1 + \delta) \|\theta_1 - \theta_2\|_{L^2(S')}^2,
\end{align*}
\] (50)

Combining (49) and (50), we find that
\[
\begin{align*}
\|\mathcal{T}_2 \theta_1(t) - \mathcal{T}_2 \theta_2(t)\|_{L^2(S')}^2
\leq & \left( C_3 C_1 C_0 \right)^2 \theta_B(1 - \alpha, 1 + \delta) \|\theta_1 - \theta_2\|_{L^2(S')}^2,
\end{align*}
\] (51)

which allows us to get that
\[
\begin{align*}
\|\mathcal{T}_2 \theta_1(t) - \mathcal{T}_2 \theta_2(t)\|_{L^2(S')}^2
\leq & C_3 C_1 C_0 T^{a-1} \theta_B(1 - \alpha, 1 + \delta) \|\theta_1 - \theta_2\|_{L^2(S')}^2.
\end{align*}
\] (52)

The term to the right of above expression is independent of $t$, so we get that
\[
\|\mathcal{T}_2 \theta_1(t) - \mathcal{T}_2 \theta_2(t)\|_{L^2(S')}^2
\leq C_3 C_1 C_0 T^{a-1} \theta_B(1 - \alpha, 1 + \delta) \|\theta_1 - \theta_2\|_{L^2(S')}^2.
\] (53)

By a similar argument as above, we deduce that
\[
\begin{align*}
\|\mathcal{T}_1 \theta_1(t) - \mathcal{T}_1 \theta_2(t)\|_{L^2(S')}^2
\leq & \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right) \\
& \cdot \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right) \\
& \cdot \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right) \\
& \cdot \left( \int_0^T (T - r)^{-a} \theta_B(1 - \alpha, 1 + \delta) \right)
\end{align*}
\] (54)

where in the last inequality, we have used Hölder inequality. Let us repeat that $E_{a,1}(-j^2 + j)(T - r)^a \leq C_2 (14)\theta(r) \leq C_0 T^a$, and thanks to that, we immediately have the following two estimations.
Using Banach fixed point theorem, we can conclude that $\mathcal{T}$ has a fixed point $w$ in $L^\infty(0, T; L^2(\mathbb{S}^2))$. Hence, we get the desired result.

\section{5. Conclusion}

In this paper, this is one of our first results about fractional diffusion on the sphere. In this article, we are interested in the existing existence in local form and global format with the main tool is the Banach fixed point theorem. In the future, we study the ill-posedness of this problem and show the convergent rate between the sought solution and the regularized solution.

\section{Data Availability}

No data were used to support this study.

\section{Conflicts of Interest}

The authors declare that they have no competing interests.

\section{Authors’ Contributions}

Both authors contributed equally and significantly in writing this paper. Four authors read and approved the final manuscript.

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