GLOBAL POINTWISE ESTIMATES OF POSITIVE SOLUTIONS TO SUBLINEAR EQUATIONS

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Abstract. We give bilateral pointwise estimates for positive solutions $u$ to the sublinear integral equation

$$u = G(\sigma u^q) + f \text{ in } \Omega,$$

for $0 < q < 1$, where $\sigma \geq 0$ is a measurable function, or a Radon measure, $f \geq 0$, and $G$ is the integral operator associated with a positive kernel $G$ on $\Omega \times \Omega$. Our main results, which include the existence criteria and uniqueness of solutions, hold for quasimetric, or quasi-metrically modifiable kernels $G$.

As a consequence, we obtain bilateral estimates, along with the existence and uniqueness, for positive solutions $u$, possibly unbounded, to sublinear elliptic equations involving the fractional Laplacian,

$$(-\Delta)^{\frac{\alpha}{2}} u = \sigma u^q + \mu \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c,$$

where $0 < q < 1$, and $\mu, \sigma \geq 0$ are measurable functions, or Radon measures, on a bounded uniform domain $\Omega \subset \mathbb{R}^n$ for $0 < \alpha \leq 2$, or on the entire space $\mathbb{R}^n$, a ball or half-space, for $0 < \alpha < n$.

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1. Introduction

Let $\Omega$ be a locally compact, Hausdorff space with countable base. We denote by $\mathcal{M}^+(\Omega)$ the class of nonnegative Radon measures (locally finite) in $\Omega$. Let $G: \Omega \times \Omega \to (0, +\infty]$ be a lower semicontinuous function (kernel) on $\Omega \times \Omega$. The $G$-potential of $\sigma \in \mathcal{M}^+(\Omega)$ is defined by

$$G\sigma(x) = \int_{\Omega} G(x, y) \, d\sigma(y), \quad x \in \Omega.$$ 

If $d\nu = f \, d\sigma$, where $f \in L^1_{\text{loc}}(\Omega, \sigma)$, we write

$$G\nu = G(f \, d\sigma) = G^\sigma f.$$ 

For $\sigma \in \mathcal{M}^+(\Omega) (\sigma \neq 0)$, we study pointwise behavior of solutions to the sublinear integral equation

$$(1.1) \quad u = G(u^q d\sigma) + f, \quad u \geq 0 \quad \text{in } \Omega,$$ 

where $0 < q < 1$ and $f \geq 0$ is a Borel measurable function, under certain assumptions on $G$.

Our main goal is to obtain bilateral pointwise estimates of all solutions $u$ to (1.1) for quasi-metric kernels $G$ (see the definition below).

The linear case $q = 1$ was treated earlier in [10] for non-homogeneous equations ($f \neq 0$) and quasi-metric kernels $G$. The “smallness” of the operator norm,

$$\|G\sigma\|_{L^2(\Omega, \sigma) \to L^2(\Omega, \sigma)} < 1,$$ 

plays a crucial role. Under this assumption, bilateral pointwise estimates of the minimal solution $u = (I - G\sigma)^{-1} f$ were obtained in [10].

The sublinear case $0 < q < 1$ differs from the linear one in many ways. Related weighted norm inequalities are of $(1, q)$-type, instead of the $(2, 2)$-type used in [10]. In contrast to the case $q = 1$, there is no “smallness” assumption on $\sigma$. For $0 < q < 1$, we first obtain bilateral estimates of nontrivial solutions to the homogeneous equation ($f = 0$), which are then invoked in the study of non-homogeneous equations. To estimate all solutions pointwise, we use nonlinear potentials $K\sigma$ intrinsic to sublinear problems, along with linear potentials $G\sigma$.

For the superlinear case $q > 1$, we refer to [22], where quasi-metric kernels were introduced for the first time in the framework of nonlinear equations. Bilateral pointwise estimates of the (minimal) solution were obtained for $f \neq 0$ under the “smallness” assumption

$$\|G\sigma\|_{L^{q'}(\Omega, \sigma) \to L^{q'}(\Omega, \nu)} \leq c,$$ 

where $\frac{1}{q} + \frac{1}{q'} = 1$, $d\nu = f^q d\sigma$, and $c$ depends on $q, G$. In [18], more precise, but one-sided estimates of solutions were given for all $q \neq 0$. 


In this paper, we make use of some elements of potential theory outlined in [5], [14], along with a sublinear version of Schur’s lemma obtained in [28] for kernels $G$ which are quasi-symmetric (QS) and satisfy a weak maximum principle (WMP); see Sec. 2 below. In particular, the kernels $G$ considered below are assumed to be lower semicontinuous on $\Omega \times \Omega$. This makes it possible to invoke the notion of capacity $\text{cap}(\cdot)$ associated with $G$.

The restriction $G(x, y) > 0$ for all $x, y \in \Omega$, rather than $G(x, y) \geq 0$ used in [14], [28], is introduced mainly for the sake of simplicity. It can be relaxed in many instances, and often replaced with the condition $G(x, x) > 0$ for all $x \in \Omega$.

**Definition.** A kernel $G$ on $\Omega \times \Omega$ is said to be quasi-metric if $G$ is symmetric, i.e., $G(x, y) = G(y, x)$ for all $x, y \in \Omega$, and $d(x, y) := \frac{1}{G(x, y)}$ satisfies the quasi-triangle inequality

$$d(x, y) \leq \kappa [d(x, z) + d(z, y)], \quad \forall x, y, z \in \Omega,$$

with quasi-metric constant $\kappa > 0$.

**Remark.** Without loss of generality, we assume in the definition above that $d$ is nontrivial: $d(x, y) \neq 0$ for some $x, y \in \Omega$; in this case, $\kappa \geq \frac{1}{2}$.

Quasi-metric kernels have numerous applications in Analysis and PDE, including weighted norm inequalities, Schrödinger operators and 3-G inequalities, spectral theory, semilinear elliptic problems on $\mathbb{R}^n$ and complete, non-compact Riemannian manifolds, etc. (see, for instance, [3], [10], [17], [19], [20], [22], [27], [28]).

More generally, we consider equations (1.1) with quasi-metrically modifiable kernels $G$ discussed below (see also [11], [12], [18], [28]). Such results are applicable to sublinear elliptic problems in a domain $\Omega \subseteq \mathbb{R}^n$ (a non-empty connected open set) under certain restrictions on $\Omega$. We consider equations involving the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \sigma u^q + \mu, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{ in } \Omega^c, \end{cases}$$

where $0 < q < 1$, $0 < \alpha < n$, and $\mu, \sigma \in \mathcal{M}^+(\Omega)$.

Sublinear problems of this type have been extensively studied, especially in the classical case $\alpha = 2$. In bounded smooth domains $\Omega$, existence and uniqueness were established originally by Krasnoselskii [23, Sec. 7.2.6], Brezis and Oswald [7], and others, for more general concave nonlinearities, but under heavy restrictions on classes of solutions, coefficients and data. (See [4], [8], [29], and the literature cited there.)
On the entire space $\mathbb{R}^n$, sharp existence and uniqueness results, along with certain global pointwise estimates, were obtained by Brezis and Kamin [6] for bounded solutions $u > 0$ to the equation $-\Delta u = \sigma u^q$. The proof of the uniqueness property given in [6] for bounded solutions $u$ such that $\liminf_{x \to \infty} u(x) = 0$ is especially subtle; it is based on an analysis of a related parabolic porous medium equation.

In this paper, we obtain bilateral pointwise estimates for all solutions to (1.5) using nonlinear potential theory. As a result, we solve the uniqueness problem and give sharp existence criteria, for arbitrary $\mu$ and $\sigma$, in bounded uniform domains $\Omega$ for $0 < \alpha \leq 2$, and on the entire space $\mathbb{R}^n$ for $0 < \alpha < n$, as well as on complete, non-compact Riemannian manifolds $M$ with nonnegative Ricci curvature (see the Example below).

If $(-\Delta)^{\frac{\alpha}{2}}$ has a positive Green’s function $G$ in $\Omega$, then applying Green’s operator $G$ to both sides, we obtain an equivalent problem where solutions $u$ satisfy the integral equation (1.1) with $f = G\mu$. In the case $\alpha = 2$, such solutions $u$ to (1.5) in bounded $C^2$-domains $\Omega$ are usually called very weak solutions (see [11], [24]).

We obtain sharp lower estimates of solutions to (1.1) for quasi-metric kernels $G$, and more generally, for quasi-symmetric (QS) kernels $G$ which satisfy the weak maximum principle (WMP). Many examples of elliptic differential operators whose Green’s kernels have these properties are given in [3].

We also obtain matching upper estimates of solutions for quasi-metric, or quasi-metrically modifiable kernels $G$. In particular, Green’s kernels for $(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha \leq 2$, in uniform domains $\Omega$ are quasi-metrically modifiable. Hence, our general results yield bilateral pointwise estimates of all solutions to (1.5) in this case.

When $2 < \alpha < n$, we can treat Green’s kernels for nice domains $\Omega \subseteq \mathbb{R}^n$, such as the balls or half-spaces, where the Green kernel is known to be quasi-metrically modifiable (see [10], [12], [13]).

On the entire space $\Omega = \mathbb{R}^n$, the Green kernel, i.e., the Newtonian kernel if $\alpha = 2$, $n \geq 3$, and the Riesz kernel of order $\alpha$ if $0 < \alpha < n$, are quasi-metric. Sublinear equations (1.5) in this case were treated earlier in [8], [31].

For $\sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$, we consider weighted norm inequalities of $(1,q)$-type,

\begin{equation}
\|G\nu\|_{L^q(\Omega,\sigma)} \leq C \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega),
\end{equation}

where we use the notation $\|\nu\| = \nu(\Omega)$ for $\nu \in \mathcal{M}^+(\Omega)$. We denote by $\kappa = \kappa(\Omega, \sigma)$ the least constant $C$ in (1.6).
It is more convenient to use here $\mathcal{M}^+(\Omega)$ in place of $L^1(\Omega, \sigma)$. The latter corresponds to the $(1, q)$-type inequality
\begin{equation}
\|G^\sigma f\|_{L^q(\Omega, \sigma)} \leq C \|f\|_{L^1(\Omega, \sigma)}, \quad \forall f \in L^1(\Omega, \sigma).
\end{equation}

**Remark.** It follows from [16, Lemma 3.I] and [28, Theorem 1.1] that \eqref{1.4} $\iff$ \eqref{1.5} for (QS)&(WMP) kernels $G$. On the other hand, simple examples show that \eqref{1.5} $\implies$ \eqref{1.4} may fail even for symmetric kernels without the (WMP) restriction.

We observe that inequality \eqref{1.5} is the end-point case $p = 1$ of the $(p, q)$-type weighted norm inequality
\begin{equation}
\|G^\sigma f\|_{L^q(\Omega, \sigma)} \leq C \|f\|_{L^p(\Omega, \sigma)}, \quad \forall f \in L^p(\Omega, \sigma),
\end{equation}
when $0 < q < p$ and $p \geq 1$.

For $p > 1$, $0 < q < p$, inequality \eqref{1.6} was characterized recently in [30], in the context of studying solutions $u \in L^r(\Omega, \sigma)$ with $r > q$ to the homogeneous equation \eqref{1.1} with $f = 0$. The case $p = 1$, or $r = q$, is substantially more complicated.

In [28, Theorem 1.1], we proved that \eqref{1.4} is equivalent to the existence of a nontrivial supersolution $u \in L^q(\Omega, \sigma)$ to the homogeneous equation, so that
\begin{equation}
G(u^q d\sigma) \leq u < +\infty \quad d\sigma\text{-a.e. in } \Omega,
\end{equation}
for kernels $G \geq 0$ that satisfy (QS)&(WMP) conditions. This can be viewed as a sublinear version of Schur’s lemma (see [16]).

For a kernel $G$, we set
\begin{equation}
B(x, r) := \{ y \in \Omega : G(x, y) > \frac{1}{r} \}, \quad x \in \Omega, \; r > 0.
\end{equation}

If $G$ is a quasi-metric kernel, then $B(x, r)$ is a quasi-metric ball.

By Fubini’s theorem we have
\begin{equation}
G^\sigma(x) = \int_0^\infty \frac{\sigma(B(x, r))}{r^2} dr, \quad x \in \Omega.
\end{equation}

Let $E \subset \Omega$ be a Borel set. By $d\sigma_E = \chi_E d\sigma$ we denote the restriction of $\sigma$ to $E$. We consider a localized version of inequality \eqref{1.4},
\begin{equation}
\|G^\nu\|_{L^q(\Omega, \sigma_E)} \leq C \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega),
\end{equation}

By $\varkappa(E) = \varkappa(E, \sigma)$ we denote the least constant $C$ in \eqref{1.7}. In most cases, it suffices to consider the constants $\varkappa(B) = \varkappa(B, \sigma)$ for “balls” $B = B(x, r)$ defined by \eqref{1.9}. The dependence on $\sigma$ will often be dropped, particularly when $\sigma$ is the measure in \eqref{1.8}. Notice that, obviously,
\begin{equation}
\varkappa(B \cap E, \sigma) = \varkappa(B, \sigma_E).
\end{equation}
We remark that, for (QS) & (WMP) kernels, there are estimates of \( \kappa(B) \) in terms of the norms of potentials \( G_\sigma B \) in Lorentz spaces ([28], Theorem 1.2):

\[
C_1 \| G_\sigma B \|_{L^{\frac{q}{1-q}}(\Omega, \sigma_B)} \leq \kappa(B) \leq C_2 \| G_\sigma B \|_{L^{\frac{q}{1-q}}(\Omega, \sigma_B)},
\]

where \( C_1 \) depends only on \( q \) and \( b \), and \( C_2 \) on \( q, a \) and \( b \). Here \( a \) and \( b \) are the constants in the conditions (QS) and (WMP), respectively.

Using the constants \( \kappa(B) \), we construct the nonlinear potential \( K_\sigma \), intrinsic to sublinear problems, by

\[
K_\sigma(x) = \int_0^\infty \left[ \kappa(B(x, r)) \right]^{\frac{q}{1-q}} \frac{dr}{r^2}, \quad x \in \Omega.
\]

We remark that nonlinear potentials of this type for Riesz kernels \( G(x, y) = |x - y|^{n-\alpha} \) on \( \mathbb{R}^n \) were introduced for the first time in [8]. They are related to nonlinear potentials of Havin–Maz’ya–Wolff type, which appeared originally in the paper of Havin and Maz’ya [26], and were used extensively by Hedberg and Wolff [21] (see also [1], [25], and the literature cited there).

Let \( \mu, \sigma \in \mathcal{M}^+(\Omega) \) \( (\sigma \neq 0) \) and \( 0 < q < 1 \). A Borel measurable function \( u: \Omega \to [0, +\infty] \) is called a nontrivial supersolution associated with the equation

\[
u = G(u^q d\sigma) + G\mu \quad d\sigma\text{-a.e. in } \Omega,
\]

if \( u > 0 \) \( d\sigma\text{-a.e.}, \) and

\[
G(u^q d\sigma) + G\mu \leq u < +\infty \quad d\sigma\text{-a.e. in } \Omega.
\]

A subsolution is defined similarly as a Borel measurable function \( u: \Omega \to [0, +\infty] \) such that

\[
u \leq G(u^q d\sigma) + G\mu < +\infty \quad d\sigma\text{-a.e. in } \Omega.
\]

A nontrivial solution to (1.15) is both a subsolution and a nontrivial supersolution. If \( u \) is a (super) solution, then \( u \in L^q_{\text{loc}}(\Omega, \sigma) \) (see Lemma 2.2 below).

We now state our main theorem for quasi-metric kernels.

**Theorem 1.1.** Let \( \mu, \sigma \in \mathcal{M}^+(\Omega) \) \( (\sigma \neq 0) \) and \( 0 < q < 1 \). Suppose \( G \) is a quasi-metric kernel. Then the following statements hold.

(i) Any nontrivial solution \( u \) to equation (1.15) satisfies the bilateral pointwise estimates

\[
\begin{align*}
c \left[ (G\sigma(x))^{\frac{1}{1-q}} + K_\sigma(x) \right] + G\mu(x) & \leq u(x) \\
u(x) & \leq C \left[ (G\sigma(x))^{\frac{1}{1-q}} + K_\sigma(x) + G\mu(x) \right],
\end{align*}
\]

and
$d\sigma$-a.e. in $\Omega$, where $c, C$ are positive constants which depend only on $q$ and the quasi-metric constant $\kappa$ of the kernel $G$. Moreover, such a solution $u$ is unique.

(ii) Estimate (1.18) holds for any nontrivial supersolution $u$ at all $x \in \Omega$ such that

$$u(x) \geq G(u^q d\sigma)(x) + G\mu(x).$$

Similarly, (1.19) holds for any subsolution $u$ at all $x \in \Omega$ such that

$$u(x) \leq G(u^q d\sigma)(x) + G\mu(x).$$

(iii) A nontrivial (super) solution $u$ to (1.15) exists if and only if the following three conditions hold:

$$\int_a^\infty \frac{\sigma(B(x_0, r))}{r^2} dr < \infty,$$

$$\int_a^\infty \frac{[2(K(B(x_0, r)))^{\frac{1}{q-1}}]}{r^2} dr < \infty,$$

$$\int_a^\infty \frac{\mu(B(x_0, r))}{r^2} dr < \infty,$$

for some (or, equivalently, all) $x_0 \in \Omega$ and $a > 0$. Any nontrivial solution $u$ satisfies (1.18), (1.19) at all $x \in \Omega$ such that

$$u(x) = G(u^q d\sigma)(x) + G\mu(x).$$

Remarks. 1. If $u$ is a nontrivial solution to (1.15), understood $d\sigma$-a.e., then

$$\tilde{u}(x) := G(u^q d\sigma)(x) + G\mu(x)$$

is a nontrivial solution to (1.25) for all $x \in \Omega$. Notice that $\tilde{u} = u$ $d\sigma$-a.e., and consequently

$$\tilde{u}(x) = G(\tilde{u}^q d\sigma)(x) + G\mu(x), \quad \forall x \in \Omega.$$  

Such representatives $\tilde{u}$ can be used to obtain estimates of solutions that hold everywhere in $\Omega$. (See [12] in the linear case $q = 1$.)

2. Under the assumptions of Theorem 1.1 conditions (1.22)–(1.24) hold if and only if $G\sigma < +\infty$, $K\sigma < +\infty$, and $G\mu < +\infty$ $d\sigma$-a.e., or equivalently $G\sigma \not\equiv +\infty$, $K\sigma \not\equiv +\infty$, and $G\mu \not\equiv +\infty$. Another existence criterion is given below (see Lemma 4.6 and Corollary 5.5).

3. An analogue of Theorem 1.1 holds for equation (1.1) with arbitrary Borel measurable function $f \geq 0$ (see Theorem 6.1 below). One only needs to replace $G\mu$ with $G(f^q d\sigma) + f$ in (1.18), (1.19), and the corresponding estimates for sub/super solutions. Notice that in the special case $f = G\mu$ the extra term $G(f^q d\sigma)$ may be dropped.
Example. Let $M$ be a complete, non-compact Riemannian manifold with the volume doubling condition. If the minimal Green’s function $G$ satisfies the Li–Yau estimates, then $G$ is known to be a quasi-metric kernel [17, Lemma 6.1]. In particular, this is true on manifolds $M$ with nonnegative Ricci curvature, and in many other circumstances (see [17]). Under these assumptions, Theorem 1.1 gives existence, uniqueness, and bilateral estimates of positive solutions to the sublinear elliptic equation $-\Delta u = \sigma u^q + \mu$, where $\Delta$ is the Laplace–Beltrami operator on $M$.

As was mentioned above, in many applications the kernel $G$ can be modified using a positive function $m \in C(\Omega)$, which is called a modifier. If the modified kernel

$$\tilde{G}(x, y) = \frac{G(x, y)}{m(x) m(y)}, \quad x, y \in \Omega,$$

satisfies the (QS)&(WMP), then our lower estimates of solutions are applicable (see Sec. 5). Upper estimates require stronger assumptions.

Definition. A kernel $G$ on $\Omega \times \Omega$ is said to be quasi-metrically modifiable, with modifier $m$, if the kernel $\tilde{G}$ defined by (1.26) is a quasi-metric kernel with quasi-metric constant $\tilde{\kappa}$.

For quasi-metrically modifiable kernels $G$, Theorem 1.1 is applicable with $\tilde{G}$ in place of $G$, which leads to matching lower and upper global estimates of solutions up to the boundary of $\Omega$.

A typical modifier is given by

$$g(x) = \min\{1, G(x, x_0)\}, \quad x \in \Omega,$$

where $x_0$ is a fixed pole in $\Omega$, provided $g \in C(\Omega)$. (Sec. 5 below; see also [20], Sec. 8.)

In particular, this procedure is applicable to Green’s kernels $G$ for $(-\Delta)^{\frac{\alpha}{2}}$ in certain domains $\Omega \subset \mathbb{R}^n$, for instance, balls or half-spaces, if $0 < \alpha < n$, or uniform domains discussed below if $0 < \alpha \leq 2$. For bounded $C^{1,1}$-domains $\Omega$ if $0 < \alpha \leq 2$, as well as balls or half-spaces if $0 < \alpha < n$, it is well known that $g(x) \approx [\text{dist}(x, \Omega^c)]^{\frac{\alpha}{2}}$.

This approach works also for Green’s functions of uniformly elliptic, symmetric operators $L$ in divergence form,

$$Lu = \text{div}(A \nabla u), \quad A = (a_{ij}(x))_{i,j=1}^n, \quad a_{ij} = a_{ji} \in L^\infty(\Omega),$$

with real-valued coefficients $a_{ij}$ in place of the Laplacian (see [12], [19], and the literature cited there).

Suppose $G$ is a quasi-metrically modifiable kernel, with modifier $m$, associated with the quasi-metric $\tilde{d} = 1/\tilde{G}$. We denote by $\tilde{B}(x, r)$ a
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quasi-metric ball

\begin{equation}
\tilde{B}(x, r) := \left\{ y \in \Omega : \tilde{G}(x, y) > \frac{1}{r} \right\}, \quad x \in \Omega, \quad r > 0.
\end{equation}

Let \( d\tilde{\sigma} = m^{1+q}d\sigma \). For a Borel set \( E \subseteq \Omega \), by \( \tilde{\nu}(E) = \tilde{\nu}(E, \tilde{\sigma}) \) we denote the least constant in the inequality

\begin{equation}
\| \tilde{G}\nu \|_{L^q(\tilde{\sigma}^\nu)} \leq \tilde{\nu}(E) \| \nu \|, \quad \forall \nu \in \mathcal{M}^+(\Omega).
\end{equation}

Using the constants \( \tilde{\nu}(\tilde{B}(x, r)) \), we construct the modified intrinsic potential \( \tilde{K}\sigma \) defined by

\begin{equation}
\tilde{K}\sigma(x) := \int_0^\infty \frac{[\tilde{\nu}(\tilde{B}(x, r))]^{\frac{1}{1-q}}}{r^2} dr, \quad x \in \Omega.
\end{equation}

**Theorem 1.2.** Let \( \mu, \sigma \in \mathcal{M}^+(\Omega) \) (\( \sigma \neq 0 \)) and \( 0 < q < 1 \). Suppose \( G \) is a quasi-metrically modifiable kernel with modifier \( m \). Then any nontrivial solution \( u \) to equation (1.15) is unique and satisfies the bilateral pointwise estimates

\begin{equation}
cm \left( \left[ \frac{G(m^q d\sigma)}{m} \right]^{\frac{1}{1-q}} + \tilde{K}\sigma \right) + G\mu \leq u
\end{equation}

and

\begin{equation}
u \leq Cm \left( \left[ \frac{G(m^q d\sigma)}{m} \right]^{\frac{1}{1-q}} + \tilde{K}\sigma \right) + C G\mu,
\end{equation}

d\sigma\text{-a.e. in } \Omega, \text{ where } c, C \text{ are positive constants which depend only on } q \text{ and the quasi-metric constant } \tilde{\kappa} \text{ of the modified kernel } G.

The lower bound (1.32) holds for any nontrivial supersolution \( u \), whereas the upper bound (1.33) holds for any subsolution \( u \).

**Remarks.** 1. Under the assumptions of Theorem 1.2 a nontrivial (super) solution to (1.15) exists if and only if \( G(m^q d\sigma) < +\infty, \tilde{K}\sigma < +\infty \), and \( G\mu < +\infty \) \( d\sigma\)-a.e., or equivalently (see Sec. 3)

\begin{equation}G(m^q d\sigma) \neq +\infty, \quad \tilde{K}\sigma \neq +\infty, \quad G\mu \neq +\infty.
\end{equation}

2. If \( G \) is quasi-metrically modifiable with modifier \( m = g \) given by (1.27), a nontrivial (super) solution to (1.15) exists if and only if

\begin{equation}\tilde{\nu}(\Omega) < \infty \quad \text{and} \quad \int_{\Omega} g \, d\mu < \infty.
\end{equation}

3. The lower bound (1.32) holds for any nontrivial supersolution \( u \) if \( \tilde{G} \) is a (QS) kernel which satisfies the (WMP); see Sec. 3 below.

In the following definition of a uniform domain (or, equivalently, an interior NTA domain), we rely on the notions of the interior corkscrew
condition and the Harnack chain condition. We refer to [19] for related definitions, in metric spaces, along with a discussion of quasi-metric properties, 3-G inequalities, and the uniform boundary Harnack principle (see also [2]).

**Definition.** A uniform domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain which satisfies the interior corkscrew condition and the Harnack chain condition.

Notice that uniform domains are not necessarily regular in the sense of Wiener. Bounded Lipschitz and non-tangentially accessible (NTA) domains are examples of regular uniform domains.

The next corollary is a direct consequence of Theorem [1.2] and the fact that Green’s function $G$ of $(-\Delta)^{\frac{\alpha}{2}}$ in a uniform domain $\Omega$ for $0 < \alpha \leq 2$ is quasi-metrically modifiable, with $m = g$ and quasi-metric constant $\tilde{\kappa}$ which does not depend on the choice of $x_0 \in \Omega$ (see [3], [19]).

**Corollary 1.3.** Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a uniform domain. Suppose $G$ is Green’s kernel of $(-\Delta)^{\frac{\alpha}{2}}$ in $\Omega$, where $0 < \alpha \leq 2$, $\alpha < n$. Define the modifier $m = g$ by (1.27) with pole $x_0 \in \Omega$.

Let $0 < q < 1$, and let $\mu, \sigma \in M^+(\Omega)$, and $d\tilde{\sigma} = g^{1+q}d\sigma$. Then the following statements hold.

(i) Any nontrivial solution $u$ to equation (1.5) is unique and satisfies estimates (1.32), (1.33) $d\tilde{\sigma}$-a.e., and at all $x \in \Omega$ where (1.25) holds.

(ii) Any nontrivial supersolution $u$ satisfies the lower bound (1.32), and any subsolution $u$ satisfies the upper bound (1.33).

(iii) A nontrivial (super) solution to (1.5) exists if and only if (1.35) holds, for some (or, equivalently, all) $x_0 \in \Omega$.

**Remarks.** 1. In the case $0 < \alpha < 2$, $n \geq 2$, Corollary 1.3 holds in any bounded domain $\Omega \subset \mathbb{R}^n$ with the interior corkscrew condition, without requiring that the Harnack chain condition holds. When $n = \alpha = 2$, Corollary 1.3 holds in any finitely connected domain $\Omega \subset \mathbb{R}^2$ with positive Green’s function, in particular, in a bounded Lipschitz domain. In all of these cases, Green’s function $G$ of $(-\Delta)^{\frac{\alpha}{2}}$ is known to be quasi-metrically modifiable with modifier $g$ (see [19] Sections 3 and 4).

2. Corollary 1.3 holds for uniformly elliptic operators $L$ in divergence form given by (1.28), in place of the Laplacian $\Delta$, in NTA domains, as well as uniform domains with Ahlfors regular boundary (see [12], [19] and the references given there).
3. Uniqueness of solutions to sublinear problems of the type (1.5) was previously known only under various restrictions on solutions, coefficients, and data, for instance, for bounded solutions [6], [7], or finite energy solutions [29].

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2. Kernels and potential theory

Let $\Omega$ be a locally compact Hausdorff space with countable base. In this section, we consider nonnegative lower semicontinuous kernels $G: \Omega \times \Omega \to (0, +\infty]$ (see [3], [14], [15]).

By $S_\mu$ we denote the closed support of $\mu \in \mathcal{M}^+(\Omega)$. We set $\|\mu\| := \mu(\Omega)$. For $\mu \in \mathcal{M}^+(\Omega)$, the potential $G\mu$, and the adjoint potential $G^*\mu$, are defined, respectively, by

$$G\mu(x) := \int_\Omega G(x, y)\,d\mu(y), \quad \forall x \in \Omega,$$

$$G^*\mu(x) := \int_\Omega G(y, x)\,d\mu(y), \quad \forall x \in \Omega.$$

**Definition.** The kernel $G$ on $\Omega \times \Omega$ satisfies the *Weak Maximum Principle* (WMP), with constant $b \geq 1$, if

$$(2.1) \quad G\mu(x) \leq 1, \quad \forall x \in S_\mu \implies G\mu(x) \leq b, \quad \forall x \in \Omega,$$

for any $\mu \in \mathcal{M}^+(\Omega)$.

When $b = 1$, we say that $G$ satisfies the (Frostman) *Maximum Principle*.

The following lemma was stated in [28, Lemma 3.5]. However, the proof given there is valid only under the usual assumptions of potential theory, namely, that the kernel $G$ is finite off the diagonal and continuous in the extended sense in $\Omega \times \Omega$. A complete proof for general quasi-metric kernels is given below.

**Lemma 2.1.** Let $G$ be a quasi-metric kernel with quasi-metric constant $\kappa$. Then $G$ satisfies the (WMP) with constant $b = 2\kappa$.

**Proof.** Without loss of generality we may assume that the measure $\mu \in \mathcal{M}^+(\Omega)$ in (2.1) is compactly supported. Suppose that $G\mu \leq 1$ on $K := S_\mu$. Let us fix $x \in \Omega \setminus K$, and set

$$\rho(x) := \inf\{d(x, z): \ z \in K\},$$

where $d = 1/G$. 

We first consider the case $\rho(x) > 0$. As in the proof of \cite[Lemma 3.5]{28}, for any $\epsilon > 0$ we choose $y' \in K$ for which $d(x, y') < (1 + \epsilon) \rho(x)$. It follows that $d(x, y') < (1 + \epsilon) d(x, y)$ for any $y \in K$, since $\rho(x) \leq d(x, y)$. Then

$$
d(y', y) \leq \kappa [d(x, y) + d(x, y')] < \kappa (2 + \epsilon) d(x, y).
$$

Hence, $G(x, y) < \kappa (2 + \epsilon) G(y', y)$ for all $y \in K$, which yields

$$
G\mu(x) \leq \kappa (2 + \epsilon) G\mu(y') \leq \kappa (2 + \epsilon).
$$

Letting $\epsilon \to 0$ proves (2.11) with $b = 2\kappa$ if $\rho(x) > 0$.

In the case $\rho(x) = 0$, we set $E_0(x) := \{z \in K : d(x, z) = 0\}$. Clearly, $E_0(x)$ is a Borel subset of $K$.

If $E_0(x) \neq \emptyset$, then there exists $y' \in K$ such that $d(x, y') = 0$, so that

$$
d(y', y) \leq \kappa [d(x, y) + d(x, y')] \leq \kappa d(x, y),
$$

for all $y \in K$. Hence, $G(x, y) \leq \kappa G(y', y)$, and consequently

$$
G\mu(x) \leq \kappa G\mu(y') \leq \kappa.
$$

In the case $E_0(x) = \emptyset$, there exists a sequence of points $y_m \in K$ such that $d(x, y_m) > 0$ and $d(x, y_m) \downarrow 0$ as $m \to \infty$. Set $a_m := 1/d(x, y_m)$. For all $y \in K$, we have

$$
d(y_m, y) \leq \kappa [d(x, y) + d(x, y_m)] \leq 2\kappa \max\{d(x, y), d(x, y_m)\}.
$$

Hence, $\min\{G(x, y), a_m\} \leq 2\kappa G(y_m, y)$, and consequently

$$
\int_{\{y \in K : G(x, y) \leq a_m\}} G(x, y) d\mu(y) \leq 2\kappa G\mu(y_m) \leq 2\kappa.
$$

Since $a_m \uparrow \infty$ and $E_0(x) = \emptyset$, the monotone convergence theorem gives

$$
G\mu(x) \leq 2\kappa.
$$

This completes the proof in the case $\rho(x) = 0$. \hfill \Box

Let $0 < q < 1$ and $\sigma \in \mathcal{M}^+(\Omega)$. Suppose $G$ is a kernel in $\Omega \times \Omega$. We consider nontrivial solutions $u > 0$ $d\sigma$-a.e. to the \textit{homogeneous} integral equation

$$
(2.2) \quad u = G(u^q d\sigma) < \infty \quad d\sigma\text{-a.e. in } \Omega,
$$

We also study nontrivial \textit{supersolutions} $u > 0$ $d\sigma$-a.e. to the corresponding integral inequality

$$
(2.3) \quad G(u^q d\sigma) \leq u < \infty \quad d\sigma\text{-a.e. in } \Omega,
$$

and \textit{subsolutions} $u$ such that

$$
(2.4) \quad 0 \leq u \leq G(u^q d\sigma) < \infty \quad d\sigma\text{-a.e. in } \Omega.
The following lemma proved in [28, Lemma 2.2] shows that, if there exists a (super) solution \( u < +\infty \) \( d\sigma \)-a.e., then actually \( u \in L^q_{\text{loc}}(\Omega, \sigma) \).

**Lemma 2.2.** Let \( G \) be a kernel on \( \Omega \times \Omega \). Suppose \( u \geq 0 \) is a super-solution such that (2.3) holds. Then \( u \in L^q_{\text{loc}}(\Omega, \sigma) \).

**Definition.** A kernel \( G \) on \( \Omega \times \Omega \) is quasi-symmetric \((QS)\) if, for a positive constant \( a \),

\[
\begin{align*}
    a^{-1}G(y, x) &\leq G(x, y) \leq a G(y, x), &\forall x, y \in \Omega.
\end{align*}
\]

A symmetrized kernel \( G^s \) is defined by

\[
G^s(x, y) := G(x, y) + G(y, x).
\]

Clearly, \( G^s \) is symmetric. For a \((QS)\) kernel \( G \), \( G^s \) is comparable to \( G \):

\[
\left(1 + \frac{1}{a}\right) G(x, y) \leq G^s(x, y) \leq (1 + a) G(x, y), \quad x, y \in \Omega.
\]

We denote the integral operator with kernel \( G^s \) by \( G^s \). For a \((QS)\) kernel \( G \), the least constants \( \kappa \) in the inequality

\[
\|G\nu\|_{L^q(\Omega, \sigma)} \leq \kappa \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega),
\]

and \( \kappa_s \) in the inequality

\[
\|G^s\nu\|_{L^q(\Omega, \sigma)} \leq \kappa_s \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega),
\]

are equivalent:

\[
\left(1 + \frac{1}{a}\right) \kappa \leq \kappa_s \leq (1 + a) \kappa.
\]

If \( G \) is a \((QS)\) kernel, then there is a nontrivial supersolution \( u > 0 \) \( d\sigma \) a.e. such that \( G(u^q d\sigma) \leq u < \infty \) \( d\sigma \) a.e. if and only if there is a nontrivial supersolution \( u_s \geq 0 \) \( d\sigma \) a.e. to the symmetrized version, \( G^s(u_s^q d\sigma) \leq u_s < \infty \) \( d\sigma \) a.e. This is easy to see using a scaled version \( u_s = c_s u \) with an appropriate positive constant \( c_s \) which depends only on \( a \) and \( q \).

There are numerous notions of capacity in analysis. For a discussion of \( L^p \)-capacities \((1 < p < \infty)\) and the corresponding nonlinear potential theory, as well as capacities associated with Sobolev spaces and other function spaces we refer to [1], [25], and the literature cited there. Capacity associated with Green’s function of the Laplacian in a domain \( \Omega \subseteq \mathbb{R}^n \) (whenever \( \Omega \) admits a nontrivial Green’s function) is fundamental to classical potential theory (see, e.g., [2]).

We use capacities studied by Choquet in the framework of linear potential theory \((p = 1)\) for potentials \( G\mu \), with kernel \( G : \Omega \times \Omega \rightarrow [0, +\infty] \) and \( \mu \in \mathcal{M}^+(\Omega) \) (see [3], [14], [15]).
For a compact set $K \subset \Omega$, the capacity $\capa_0(K)$ can be defined as follows (see [5, 15]),

$$
\capa_0(K) := \sup\{\mu(K) : \mu \in \mathcal{M}^+(K), \quad G^*\mu(y) \leq 1, \forall y \in \Omega\}.
$$

We will mostly use the following version of capacity.

**Definition.** The *Wiener capacity* $\capa(K)$ of a compact set $K \subset \Omega$ is defined by

$$
\capa(K) := \sup\{\mu(K) : \mu \in \mathcal{M}^+(K), \quad G^*\mu(y) \leq 1, \forall y \in S_{\mu}\}.
$$

(2.5)

Obviously, $\capa_0(K) \leq \capa(K)$, and for (WMP) kernels,

$$
\capa_0(K) \leq \capa(K) \leq b \capa_0(K).
$$

It is known that, for a kernel $G > 0$, we have $\capa(K) < +\infty$ for every compact $K \subset \Omega$ [14, Sec. 2.5].

The capacity $\capa$ can be extended as an “exterior” set function, first to open sets $B \subset \Omega$, and then to arbitrary sets $A \subset \Omega$. In particular,

$$
capa(B) := \sup\{\capa(K) : \text{for all compact sets } K, \ K \subset B\},
$$

$$
capa(A) := \inf\{\capa(B) : \text{for all open sets } B, \ A \subset B\}.
$$

A measure $\mu \in \mathcal{M}^+(\Omega)$ is *absolutely continuous with respect to capacity* if

$$
\capa(K) = 0 \implies \mu(K) = 0, \quad \text{for every compact set } K.
$$

The notions of capacity and *equilibrium measure*, i.e., an extremal measure in (2.5), were an essential feature of our approach in [28], which is developed further in this paper. A proof of the following lemma can be found in [28, Lemma 4.2].

**Lemma 2.3.** Let $0 < q < 1$ and $\sigma \in \mathcal{M}^+(\Omega)$. Let $G$ be a kernel on $\Omega \times \Omega$. Suppose $G^*(u^q\sigma) \leq u \, d\sigma$-a.e., where $u \geq 0$, $u \in L^q_{\text{loc}}(\Omega, \sigma)$. Then $d\omega := u^q\sigma$ is absolutely continuous with respect to capacity.

If in addition $u > 0$ $d\sigma$-a.e., then $\sigma$ is absolutely continuous with respect to capacity.

3. **Lower bounds for supersolutions**

We will need the following lower bound for supersolutions obtained in [18, Theorem 1.3].

**Lemma 3.1.** Let $\sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$. Suppose $G$ is a kernel on $\Omega \times \Omega$ which satisfies the (WMP) with constant $b$. Then any nontrivial
supersolution $u > 0$ $d\sigma$-a.e. such that $G(u^q d\sigma) \leq u < +\infty$ $d\sigma$-a.e. satisfies the estimate
\begin{equation}
(3.1) \quad u(x) \geq c \left[ G\sigma(x) \right]^{\frac{1}{1-q}},
\end{equation}
where $c = (1 - q)^{\frac{1}{1-q}} b^{-\frac{1}{1-q}}$, for all $x \in \Omega$ such that $G(u^q d\sigma)(x) \leq u(x)$.

Another lower estimate for supersolutions $u$, deduced in the next lemma, complements (3.1) in a crucial way. It holds for kernels $G$ which satisfy both the (WMP) and (QS) conditions. Using a symmetrized kernel, we may assume without loss of generality that $G$ is symmetric; for (QS) kernels, the constant $C$ in (3.2) will depend on $q$, $b$, and the quasi-symmetry constant $a$.

**Lemma 3.2.** Let $\sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$. Suppose $G$ is a symmetric kernel on $\Omega \times \Omega$ which satisfies the (WMP) with constant $b$. Then any nontrivial supersolution $u > 0$ $d\sigma$-a.e. such that $G(u^q d\sigma)(x) \leq u(x)$ satisfies the estimate
\begin{equation}
(3.2) \quad u(x) \geq c K\sigma(x),
\end{equation}
where $c = (1 - q)^{\frac{1}{1-q}} b^{-\frac{1}{1-q}}$, for all $x \in \Omega$ such that $G(u^q d\sigma)(x) \leq u(x)$.

**Proof.** The proof of (3.2) makes use of an idea employed in the proof of [25, Lemma 5.11]. Let $u$ be a nontrivial supersolution. Then $u \in L^q_{\text{loc}}(\Omega, \sigma)$ by Lemma 2.2. We set $d\omega := u^q d\sigma$, so that $\omega \in \mathcal{M}^+(\Omega)$.

For $x \in \Omega$ and $t > 0$, let $B(x, t)$ be a “ball” defined by (1.10). We first prove the estimate
\begin{equation}
(3.3) \quad \kappa(B(x, t)) \leq \frac{b}{(1 - q)^{\frac{1}{q}}} \|u\|_{L^q(\Omega, \sigma \delta(x,t))}^{1-q}, \quad \forall x \in \Omega, \ t > 0,
\end{equation}
where without loss of generality we assume that $\sigma(B(x, t)) > 0$ and $\|u\|_{L^q(\Omega, \sigma \delta(x,t))} < \infty$. We set $d\mu = d\omega_{B(x,t)} = u^q d\sigma_{B(x,t)}$.

Suppose $\nu \in \mathcal{M}^+(\Omega)$ is a probability measure. Since $u \geq G\omega \geq G\mu$, it follows that $G\mu < \infty$ $d\mu$-a.e., and
\begin{align*}
\int_{\Omega} (G\nu)^q d\sigma_{B(x,t)} &= \int_{\Omega} \left( \frac{G\nu}{u} \right)^q u^q d\sigma_{B(x,t)} \\
&\leq \int_{\Omega} \left( \frac{G\nu}{G\mu} \right)^q d\mu.
\end{align*}

For $\lambda > 0$, we set $E_\lambda := \left\{ y \in B(x, t) : \frac{G\nu(y)}{G\mu(y)} > \lambda \right\}$. Then, for any $\beta > 0$, we clearly have
\begin{align*}
\int_{\Omega} \left( \frac{G\nu}{G\mu} \right)^q d\mu &= \int_{\Omega}^{\infty} \mu(E_\lambda) \lambda^{q-1} d\lambda.
\end{align*}
\[ q \int_{0}^{\beta} \mu(E_{\lambda}) \lambda^{q-1} d\lambda + q \int_{\beta}^{\infty} \mu(E_{\lambda}) \lambda^{q-1} d\lambda \]
\[ := I + II. \]

Clearly,
\[ I \leq q \| \mu \| \int_{0}^{\beta} \lambda^{q-1} d\lambda = \beta^{q} \| \mu \|. \]

To estimate \( II \), we use the \((1,1)\) weak-type bound [28, Lemma 5.10],
\[ \mu(E_{\lambda}) \leq \frac{b \| \nu \|}{\lambda} = \frac{b}{\lambda}. \]

Notice that by Lemma [2.2] and Lemma [2.3], \( \omega \) is absolutely continuous with respect to capacity. Hence, the same is true for \( \mu \).

It follows,
\[ II \leq q b \int_{\beta}^{\infty} \lambda^{q-2} d\lambda = \frac{q}{1-q} b \beta^{q-1}. \]

Choosing \( \beta = \frac{b}{\| \mu \|} \), we deduce
\[ \int_{\Omega} (G\nu)^{q} d\sigma_{B(x,t)} \leq \frac{b^{q}}{1-q} \| \mu \|^{1-q} = \frac{b^{q}}{1-q} \left( \int_{B(x,t)} u^{q} d\sigma \right)^{1-q}. \]

For a general (finite, nonzero) measure \( \nu \in \mathcal{M}^{+}(\Omega) \), by homogeneity we obtain the inequality
\[ \int_{\Omega} (G\nu)^{q} d\sigma_{B(x,t)} \leq \frac{b^{q}}{1-q} \left( \int_{B(x,t)} u^{q} d\sigma \right)^{1-q} \| \nu \|^{q}, \]

which proves [3.3]. Therefore,
\[
\begin{align*}
K\sigma(x) := \int_{0}^{\infty} \frac{[x(B(x,t))]^{\frac{q}{1-q}}}{t^{2}} dt & \leq \frac{b^{\frac{q}{1-q}}}{(1-q)^{\frac{1}{1-q}}} \int_{0}^{\infty} \frac{\int_{B(x,t)} u^{q} d\sigma}{t^{2}} dt \\
& = \frac{b^{\frac{q}{1-q}}}{(1-q)^{\frac{1}{1-q}}} G(u^{q}d\sigma)(x) \leq \frac{b^{\frac{q}{1-q}}}{(1-q)^{\frac{1}{1-q}}} u(x).
\end{align*}
\]

\[ \square \]

**Corollary 3.3.** Let \( \mu, \sigma \in \mathcal{M}^{+}(\Omega) \) and \( 0 < q < 1 \). Suppose \( G \) is a quasi-symmetric kernel on \( \Omega \times \Omega \) with quasi-symmetry constant \( a \), which satisfies the (WMP) with constant \( b \). Then any nontrivial supersolution \( u \) to \((1.15)\) satisfies the estimate
\[ u(x) \geq c \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right] + G\mu(x), \]
where \( c = c(q, a, b) \), for all \( x \in \Omega \) such that
\[ G(u^{q}d\sigma)(x) + G\mu(x) \leq u(x). \]
In particular, (3.4) holds $d\sigma$-a.e.

Proof. Suppose $u$ is a nontrivial supersolution to (1.15). Let us set $v(x) := G(u^q d\sigma)(x), x \in \Omega$. Then, obviously, $0 < v \leq u < \infty$ $d\sigma$-a.e. Hence, $v$ is a nontrivial supersolution such that

$$G(v^q d\sigma)(x) \leq G(u^q d\sigma)(x) = v(x), \quad \forall x \in \Omega.$$ 

Then by Lemma 3.1 and Lemma 3.2 we have

$$v(x) \geq c \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right], \quad \forall x \in \Omega,$$

where $c = c(q, a, b)$. Notice that Lemma 3.2 is stated for symmetric kernels with $c = c(q, b)$, but for quasi-symmetric kernels, we use a symmetrized form of the kernel $G$, which yields the same estimate with a constant $c$ that additionally depends on $a$ (see Sec. 2).

If $u(x) \geq v(x) + G\mu(x)$ for $x \in \Omega$, then (3.4) is an immediate consequence of (3.6). In particular, (3.4) holds $d\sigma$-a.e. □

4. QUASI-METRIC KERNELS

In this section, our main goal is to deduce the upper estimates (1.19) of subsolutions associated with equation (1.15), for quasi-metric kernels $G$. They match the lower estimates of supersolutions (1.18) obtained in Sec. 3.

We start with our main lemma.

Lemma 4.1. Let $G$ be a quasi-metric kernel on $\Omega \times \Omega$ with quasi-metric constant $\kappa$. Let $0 < q < 1$ and $\nu, \sigma \in M^+(\Omega)$. Then, for all $x \in \Omega,$

$$G[(G\nu)^q d\sigma](x) \leq C (G\nu(x))^q \left[ G\sigma(x) + (K\sigma(x))^{1-q} \right],$$

where $C = (2\kappa)^q$.

Proof. Let $d\omega := (G\nu)^q d\sigma$. Then

$$G\omega(x) = \int_0^\infty \frac{\omega(B(x, t))}{t^2} dt.$$ 

Clearly,

$$\omega(B(x, t)) = \int_{B(x,t)} (G\nu)^q d\sigma$$

$$\leq \int_{B(x,t)} (G\nu_{B(x,2\kappa t)})^q d\sigma + \int_{B(x,t)} (G\nu_{B(x,2\kappa t)^c})^q d\sigma$$

$$:= I + II.$$
We estimate the first term,
\[ I = \int_{B(x,t)} (G\nu_{B(x,2\kappa t)})^q d\sigma \leq \left( \mathcal{K}(B(x,t)) \right)^q \left( \nu(B(x,2\kappa t)) \right)^q. \]

To estimate the second term, notice that
\[ G\nu_{B(x,2\kappa t)}(y) = \int_0^\infty \frac{\nu(B(y,r) \cap B(x,2\kappa t))}{r^2} dr. \]

For all \( y \in B(x,t) \) and \( z \in B(y,r) \), we have
\[ d(x,z) \leq \kappa \left[ d(x,y) + d(y,z) \right] \leq \kappa (t + r). \]
Consequently, \( B(y,r) \subset B(x,2\kappa t) \) if \( 0 < r \leq t \), so that \( B(y,r) \cap B(x,2\kappa t)^c = \emptyset \). Moreover, \( B(y,r) \subset B(x,2\kappa r) \) if \( r > t \). Hence, for all \( y \in B(x,t) \),
\[ G\nu_{B(x,2\kappa t)}(y) = \int_0^\infty \frac{\nu(B(y,r) \cap B(x,2\kappa t))}{r^2} dr \leq \int_t^\infty \frac{\nu(B(x,2\kappa r))}{r^2} dr \leq 2\kappa \int_0^\infty \frac{\nu(B(x,s))}{s^2} ds = 2\kappa G\nu(x). \]

It follows that
\[ II = \int_{B(x,t)} (G\nu_{B(x,2\kappa t)}^c(y))^q d\sigma \leq (2\kappa)^q (G\nu(x))^q \sigma(B(x,t)). \]

Combining the preceding estimates, we obtain
\[ \omega(B(x,t)) \leq \left( \mathcal{K}(B(x,t)) \right)^q \left( \nu(B(x,2\kappa t)) \right)^q + (2\kappa)^q (G\nu(x))^q \sigma(B(x,t)). \]

Hence,
\[ G\omega(x) = \int_0^\infty \frac{\omega(B(x,t))}{t^2} dt \leq \int_0^\infty \frac{\left( \mathcal{K}(B(x,t)) \right)^q \left( \nu(B(x,2\kappa t)) \right)^q}{t^2} dt + (2\kappa)^q (G\nu(x))^q \int_0^\infty \frac{\sigma(B(x,t))}{t^2} dt. \]

Using Hölder’s inequality in the first integral, we deduce
\[ G\omega(x) \leq \left( \int_0^\infty \frac{\left( \mathcal{K}(B(x,t)) \right)^q}{t^2} dt \right)^{1-q} \left( \int_0^\infty \frac{\left( \nu(B(x,2\kappa t)) \right)^q}{t^2} dt \right)^q + (2\kappa)^q (G\nu(x))^q \int_0^\infty \frac{\sigma(B(x,t))}{t^2} dt. \]
Lemma 4.2. Let \( G \) be a quasi-metric kernel on \( \Omega \times \Omega \) with quasi-metric constant \( \kappa \). Let \( 0 < q < 1 \) and \( \mu, \sigma \in M^+(\Omega) \). Then any subsolution \( u \geq 0 \) such that \( u \leq G(u^q d\sigma) + G\mu < +\infty \) d\( \sigma \)-a.e., satisfies the estimate
\[
(4.2) \quad u(x) \leq C \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) + G\mu(x) \right],
\]
for all \( x \in \Omega \) such that \( u(x) \leq G(u^q d\sigma)(x) + G\mu(x) < +\infty \), where \( C = (8\kappa)^{\frac{1}{1-q}} \). In particular, (4.2) holds d\( \sigma \)-a.e.

Proof. Let \( dv := u^q d\sigma + d\mu \), so that \( u \leq Gv \) d\( \sigma \)-a.e., and consequently \( dv \leq (Gv)^q d\sigma + d\mu \). Then
\[
Gv(x) \leq G[(Gv)^q d\sigma](x) + G\mu(x), \quad \forall x \in \Omega.
\]
By Lemma 4.1,
\[
G[(Gv)^q d\sigma](x) \leq (2\kappa)^q (Gv(x))^q \left[ G\sigma(x) + (K\sigma(x))^{1-q} \right].
\]
Hence,
\[
Gv(x) \leq G[(Gv)^q d\sigma] + G\mu(x)
\]
\[
\leq (2\kappa)^q (Gv(x))^q \left[ G\sigma(x) + (K\sigma(x))^{1-q} \right] + G\mu(x)
\]
\[
\leq (4\kappa)^q (Gv(x))^q \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right]^{1-q} + G\mu(x).
\]
By Young’s inequality,
\[
(4\kappa)^q (Gv(x))^q \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right]^{1-q}
\]
\[
\leq q Gv(x) + (1-q) (4\kappa)^{\frac{q}{1-q}} \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right].
\]
Hence,
\[
G\nu(x) \leq q G\nu(x) + (1-q) (4\kappa)^{\frac{q}{1-q}} \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right] + G\mu(x).
\]
For \( x \in \Omega \) such that \( G\nu(x) < +\infty \), we can move \( q G\nu(x) \) to the left-hand side. Then, dividing both sides by \( 1 - q \), we obtain
\[
G\nu(x) \leq (4\kappa)^{\frac{q}{1-q}} \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right] + \frac{1}{1-q} G\mu(x).
\]
Using the inequality \( x + 1 \leq e^x \) with \( x = \frac{q}{1-q} \) we estimate roughly
\[
\frac{1}{1-q} \leq e^{\frac{q}{1-q}} \leq (8\kappa)^{\frac{q}{1-q}}, \quad \text{since} \quad 2\kappa \geq 1 > \frac{e}{4}. \text{ Hence,}
\]
\[
G\nu(x) \leq (8\kappa)^{\frac{q}{1-q}} \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) + G\mu(x) \right].
\]
Thus,

\[ u(x) \leq G\nu(x) \leq (8\kappa)^{\frac{q}{2-q}} \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) + G\mu(x) \right], \]

for all \( x \in \Omega \) such that \( G\nu(x) = G(u^qd\sigma)(x) + G\mu(x) < +\infty \). This completes the proof of Lemma 4.2. \( \square \)

In the remaining part of this section, we rely on the fact that a quasi-metric kernel with quasi-metric constant \( \kappa \) obeys the (WMP) with constant \( b = 2\kappa \) by Lemma 2.1 above.

From the next lemma, we will deduce estimate (4.2) for all \( x \in \Omega \), provided \( u(x) \leq G(u^qd\sigma)(x) + G\mu(x) \), including the case \( u(x) = +\infty \).

**Lemma 4.3.** Let \( G \) be a quasi-metric kernel on \( \Omega \times \Omega \) with quasi-metric constant \( \kappa \). Let \( 0 < q < 1 \) and \( \mu, \sigma \in \mathcal{M}^+(\Omega) \). Then the function

\[ h(x) := (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) + G\mu(x), \quad x \in \Omega, \]

satisfies the estimate

\[ G(h^qd\sigma)(x) \leq C h(x), \quad \forall x \in \Omega, \]

where \( C \) is a constant which depends only on \( q \) and \( \kappa \).

**Proof.** Notice that by Lemma 4.1 with \( \nu = \mu \), we have

\[ G[(G\mu)^qd\sigma](x) \leq (2\kappa)^q (G\mu(x))^q[G\sigma(x) + (K\sigma(x))^{1-q}], \]

\[ \leq (4\kappa)^q h(x), \quad \forall x \in \Omega. \]

Therefore, it remains to prove (4.3) for \( \mu = 0 \). The proof is similar to that of Lemma 4.1 but with nonlinear potentials \((G\sigma)^{\frac{1}{1-q}}\) and \( K\sigma \) in place of the linear potential \( G\nu \). Fix \( x \in \Omega \), where without loss of generality we may assume that \( h(x) < \infty \). In particular, \( K\sigma(x) < \infty \), and hence \( \kappa(B(x, s)) < \infty \) for every \( s > 0 \).

We have

\[ G(h^qd\sigma)(x) = \int_0^\infty \frac{\int_{B(x,t)} (h(y))^q d\sigma(y)}{t^2} dt. \]

Notice that, for a constant \( c(q) > 0 \) which depends only on \( q \), we have \( h(y) \leq c(q)(h_1(y) + h_2(y)) \), where

\[ h_1(y) = [G\sigma_{B(x,2\kappa t)}(y)]^{\frac{1}{1-q}} + K\sigma_{B(x,2\kappa t)}(y) \]

\[ h_2(y) = [G\sigma_{B(x,2\kappa t)^c}(y)]^{\frac{1}{1-q}} + K\sigma_{B(x,2\kappa t)^c}(y). \]

Clearly,

\[ \int_{B(x,t)} (h(y))^q d\sigma(y) \leq c(q)^q \int_{B(x,t)} (h_1(y))^q d\sigma(y) \]
+ c(q)^q \int_{B(x,t)} (h_2(y))^q d\sigma(y) := I + II.

To estimate term $I$, notice that $\mathcal{K}(B(x,2\kappa t)) < \infty$. Hence, by [28, Theorem 1.1 and Corollary 5.9] there exists a nontrivial solution $u_{B(x,2\kappa t)} \in L^q(\Omega, \sigma_{B(x,2\kappa t)})$ to the equation $u = G(u^q d\sigma_{B(x,2\kappa t)})$, and

$$\int_{B(x,2\kappa t)} [u_{B(x,2\kappa t)}(y)]^q d\sigma(y) \leq [\mathcal{K}(B(x,2\kappa t))]^{\frac{q}{1-q}}.$$ 

We use the lower estimate

$$u_{B(x,2\kappa t)} \geq c(q, \kappa) h_1 \quad \text{d}\sigma\text{-a.e. in } B(x,2\kappa t),$$

which follows by combining (3.1) and (3.2). We estimate,

$$I = \int_{B(x,t)} (h_1(y))^q d\sigma(y) \leq C_1(q, \kappa) [\mathcal{K}(B(x,2\kappa t))]^{\frac{q}{1-q}},$$

for some $C_1(q, \kappa)$ depending only on $q, \kappa$. Integrating both sides over $(0, +\infty)$ with respect to $dt/t^2$, we deduce

$$G(h_1^q d\sigma)(x) \leq C_1(q, \kappa) \int_0^\infty \frac{[\mathcal{K}(B(x,2\kappa t))]^{\frac{q}{1-q}}}{t^2} dt$$

$$= 2\kappa C_1(q, \kappa) K \sigma(x) \leq 2\kappa C_1(q, \kappa) h(x).$$

We next estimate term $II$ as in the proof of Lemma 4.1. If $y \in B(x,t)$, then by the quasi-triangle inequality we have

$$B(y, s) \subset B(x,2\kappa t) \text{ if } 0 < s \leq t, \quad B(y, s) \subset B(x,2\kappa s) \text{ if } s > t.$$ 

In particular, $B(y, s) \cap B(x,2\kappa t)^c = \emptyset$ for $0 < s \leq t$.

Hence, using (4.13) with $E = B(x,2\kappa t)^c$, we obtain, for all $y \in B(x,t)$,

$$h_2(y) = \left[ \int_0^\infty \frac{\sigma(B(y, s) \cap B(x,2\kappa t)^c)}{s^2} ds \right]^{\frac{1}{1-q}}$$

$$+ \int_0^\infty \frac{[\mathcal{K}(B(y, s) \cap B(x,2\kappa t)^c)]^{\frac{q}{1-q}}}{s^2} ds$$

$$\leq \left[ \int_t^\infty \frac{\sigma(B(x,2\kappa s))}{s^2} ds \right]^{\frac{1}{1-q}} + \int_t^\infty \frac{[\mathcal{K}(B(x,2\kappa s))]^{\frac{q}{1-q}}}{s^2} ds$$

$$\leq c(q, \kappa) h(x).$$

This estimate yields

$$II \leq C_2(q, \kappa) (h(x))^q \sigma(B(x,t)).$$
where \( C_2(q, \kappa) \) depends only on \( q \) and \( \kappa \). Integrating again both sides of the preceding inequality over \((0, +\infty)\) with respect to \( dt/t^2 \), we see that

\[
G(h^2d\sigma)(x) \leq C_2(q, \kappa) (h(x))^q G\sigma(x) \leq C_2(q, \kappa) h(x),
\]

since obviously \( G\sigma(x) \leq (h(x))^{1-q} \). Combining the above estimates completes the proof of (4.4) for all \( x \in \Omega \) in the remaining case \( \mu = 0 \). \( \square \)

**Corollary 4.4.** Let \( G \) be a quasi-metric kernel on \( \Omega \times \Omega \) with quasi-metric constant \( \kappa \). Let \( 0 < q < 1 \) and \( \mu, \sigma \in \mathcal{M}^+(\Omega) \). Then every subsolution \( u \) for which \( u \leq G(u^q d\sigma) + G\mu < +\infty \) \( d\sigma\)-a.e. satisfies the estimate

\[
(4.5) \quad u(x) \leq (8\kappa)^{\frac{2}{1-q}} \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) + G\mu(x) \right],
\]

for all \( x \in \Omega \) such that \( u(x) \leq G(u^q d\sigma)(x) + G\mu(x) \). In particular, (4.5) holds \( d\sigma\)-a.e.

**Proof.** Let \( h := (G\sigma)^{\frac{1}{1-q}} + K\sigma + G\mu \). Fix \( x \in \Omega \). In view of Lemma 4.2, \( u(x) \leq (8\kappa)^{\frac{2}{1-q}} h(x) \) provided \( u(x) \leq G(u^q d\sigma)(x) + G\mu(x) < +\infty \). Therefore, it only remains to show that \( h(x) = +\infty \) whenever \( G(u^q d\sigma)(x) + G\mu(x) = +\infty \). This is obvious if \( G\mu(x) = +\infty \).

Suppose \( G\mu(x) < +\infty \), but \( G(u^q d\sigma)(x) = +\infty \). Since \( u \leq (8\kappa)^{\frac{2}{1-q}} h \) \( d\sigma\)-a.e., we have

\[
G(u^q d\sigma)(x) \leq (8\kappa)^{\frac{2}{1-q}} G(h^2d\sigma)(x).
\]

By Lemma 4.3, \( G(h^2d\sigma)(x) \leq C(q, \kappa) h(x) \) for all \( x \in \Omega \). Hence, \( G(u^q d\sigma)(x) = +\infty \) \( \Rightarrow h(x) = +\infty \). \( \square \)

**Lemma 4.5.** Let \( \mu, \sigma \in \mathcal{M}^+(\Omega) \) (\( \sigma \neq 0 \)) and \( 0 < q < 1 \). Suppose \( G \) is a quasi-metric kernel on \( \Omega \times \Omega \). Then a nontrivial solution \( u \) to (1.13) exists if and only if \( G\sigma < +\infty \), \( K\sigma < +\infty \), and \( G\mu < +\infty \) \( d\sigma\)-a.e., and satisfies the bilateral pointwise estimates

\[
(4.6) \quad c \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) \right] + G\mu(x) \leq u(x),
\]

and

\[
(4.7) \quad u(x) \leq C \left[ (G\sigma(x))^{\frac{1}{1-q}} + K\sigma(x) + G\mu(x) \right],
\]

d\( \sigma\)-a.e. in \( \Omega \), where \( c, C \) are positive constants which depend only on \( q \) and the quasi-metric constant \( \kappa \) of the kernel \( G \).
Proof. Let \( u_0 = c_0 \left[ (G\sigma)^{\frac{1}{1-q}} + G\mu \right] \), where a small constant \( 0 < c_0 \leq 1 \) is to be determined later. We claim that \( u_0 \) is a positive subsolution to \([2.3]\), i.e., \( u_0 \leq G(u_0^q d\sigma) + G\mu \). It suffices to verify the inequality

\[
(4.8) \quad c_0 \left[ (G\sigma)^{\frac{1}{1-q}} + G\mu \right] \leq c_0^q G \left[ (G\sigma)^{\frac{q}{1-q}} d\sigma \right] + G\mu.
\]

By \([18, Lemma 2.5 \text{ and Remark 2.6}]\) with \( r = \frac{1}{1-q} \), we have

\[
(4.9) \quad (G\sigma)^{\frac{1}{1-q}} \leq \frac{1}{1-q} b^{\frac{q}{1-q}} G \left[ (G\sigma)^{\frac{q}{1-q}} d\sigma \right],
\]

where \( b \) is the constant in the \((\text{WMP})\) for \( G \). Hence, \((4.8)\) holds if we pick \( c_0 \) small enough so that \( c_0^q \geq \frac{1}{1-q} b^{\frac{q}{1-q}}, \) which proves the claim.

We next define the sequence \( \{u_j\}_{j=0}^\infty \) by

\[
u_{j+1} := G(u_j^q d\sigma) + G\mu, \quad j = 0, 1, 2, \ldots.
\]

Since \( u_0 \) is a subsolution, we have \( u_1 \geq u_0 \). By induction, we see that \( \{u_j\} \) is non-decreasing, so that \( u_j \uparrow u \), and

\[
u_j \leq u_{j+1} := G(u_j^q d\sigma) + G\mu, \quad j = 0, 1, 2, \ldots.
\]

To show that each \( u_j \) is a subsolution, arguing by induction and applying Lemma 4.3, we estimate

\[
u_j \leq C \left[ (G\sigma)^{\frac{1}{1-q}} + K\sigma + G\mu \right] < +\infty \quad d\sigma\text{-a.e.}
\]

where \( C = C(j, q, \kappa) \) may depend on \( j \). Then by Corollary 4.4, the preceding inequality holds with a constant \( C = C(q, \kappa) \) which does not depend on \( j \). By the monotone convergence theorem, \( u < +\infty d\sigma\)-a.e. is a nontrivial solution to \([2.3]\), which satisfies \((4.7)\). The lower estimate \((4.6)\) holds for any (super) solution \( u \) by Corollary 3.3, since \( G \) is a quasi-metric kernel which satisfies the \((\text{WMP})\) by Lemma 2.1. \(\square\)

**Lemma 4.6.** Let \( \mu, \sigma \in \mathcal{M}^+(\Omega) \ (\sigma \neq 0) \) and \( 0 < q < 1 \). Suppose \( G \) is a quasi-metric kernel on \( \Omega \times \Omega \). Then the following conditions are equivalent:

(i) \( G\sigma < +\infty, K\sigma < +\infty, \) and \( G\mu < +\infty d\sigma\)-a.e.

(ii) \( G\sigma \neq +\infty, K\sigma \neq +\infty, \) and \( G\mu \neq +\infty. \)

(iii) Conditions \((1.22)-(1.24)\) hold for some (or, equivalently, all) \( x_0 \in \Omega \) and \( a > 0 \).

**Proof.** We first show that, if any one of conditions \((1.22)-(1.24)\) holds for some \( x_0 \in \Omega \) and \( a > 0 \), then it holds for all \( x_0 \in \Omega \) and \( a > 0 \). Let \( x \in \Omega \). Since \( G(x, x_0) > 0 \), it follows that \( x \in B(x_0, R) \) for \( R \) large, where we may assume \( R > a \). Then, by the quasi-triangle inequality,
\[ B(x, t) \subset B(x_0, 2\kappa t), \text{ for all } t \geq R. \] Consequently, for all \( t \geq R \), we have
\[
\begin{align*}
\sigma(B(x, t)) &\leq \sigma(B(x_0, 2\kappa t)), \\
\kappa(B(x, t)) &\leq \kappa(B(x_0, 2\kappa t)), \\
\mu(B(x, t)) &\leq \mu(B(x_0, 2\kappa t)).
\end{align*}
\]

It follows that
\[
\begin{align*}
\int_{R}^{\infty} \frac{\sigma(B(x, t))}{t^2} dt &\leq \int_{R}^{\infty} \frac{\sigma(B(x_0, 2\kappa t))}{t^2} dt < \infty, \\
\int_{R}^{\infty} \left( \frac{\kappa(B(x, t))}{t^2} \right)^{\frac{q-1}{q}} dt &\leq \int_{R}^{\infty} \left( \frac{\kappa(B(x_0, 2\kappa t))}{t^2} \right)^{\frac{q-1}{q}} dt < \infty, \\
\int_{R}^{\infty} \frac{\mu(B(x, t))}{t^2} dt &\leq \int_{R}^{\infty} \frac{\mu(B(x_0, 2\kappa t))}{t^2} dt < \infty,
\end{align*}
\]
respectively, since the substitution \( s = 2\kappa t \) in the integrals on the right-hand side shows that \( s > 2\kappa R \geq R > a \).

It follows from (4.10)–(4.12) that \( \sigma(B(x, t)) < \infty, \kappa(B(x, t)) < \infty \) and \( \mu(B(x, t)) < \infty \) for all \( t > R \), and hence for all \( t > 0 \). We deduce that, for all \( x \in \Omega \) and \( a > 0 \), we have (independently for each of the following integrals)
\[
\begin{align*}
\int_{a}^{\infty} \frac{\sigma(B(x, t))}{t^2} dt &< \infty, \\
\int_{a}^{\infty} \left( \frac{\kappa(B(x, t))}{t^2} \right)^{\frac{q-1}{q}} dt &< \infty, \\
\int_{a}^{\infty} \frac{\mu(B(x, t))}{t^2} dt &< \infty.
\end{align*}
\]

Thus, if condition (iii) of Lemma 4.6 holds for some \( x_0 \) and \( a \), then it holds for all \( x_0 \in \Omega \) and \( a > 0 \).

We notice that, obviously, (i)\( \Rightarrow \) (ii)\( \Rightarrow \) (iii). Let us show (iii)\( \Rightarrow \) (i). As was noticed above, it follows from (4.14) that \( \kappa(B(x, t)) < \infty \) for all \( x \in \Omega \) and \( t > 0 \). Hence, for any finite measure \( \nu \in \mathcal{M}^+(\Omega) \), we have
\[
\int_{B(x, t)} (G\nu)^q d\sigma \leq \left[ \kappa(B(x, t)) \right]^q \| \nu \|^q < \infty.
\]

Applying the preceding inequality to \( \nu = \sigma_{B(x, 2\kappa t)} \) and \( \nu = \mu_{B(x, 2\kappa t)} \), we deduce, for all \( x \in \Omega \) and \( t > 0 \),
\[
\int_{B(x, t)} (G\sigma_{B(x, 2\kappa t)})^q d\sigma < \infty, \quad \int_{B(x, t)} (G\mu_{B(x, 2\kappa t)})^q d\sigma < \infty.
\]
It follows that $G \sigma_{B(x, 2\kappa t)} < \infty$ and $G \mu_{B(x, 2\kappa t)} < \infty$ $d\sigma$-a.e. in $B(x, t)$. As was shown above in the proof of Lemma 4.3 (the estimate of term II), we have, for all $y \in B(x, t)$,

\[
G \sigma_{B(x, 2\kappa t)}(y) \leq C \int_t^\infty \frac{\sigma(B(x, 2\kappa s))}{s^2} ds < \infty,
\]

\[
G \mu_{B(x, 2\kappa t)}(y) \leq C \int_t^\infty \frac{\mu(B(x, 2\kappa s))}{s^2} ds < \infty.
\]

Therefore, $G \sigma < \infty$ and $G \mu < \infty$ $d\sigma$-a.e. in all quasi-metric balls $B(x, t)$, and consequently in $\Omega = \cup_{t>0} B(x, t)$.

To verify that $K \sigma < \infty$ $d\sigma$-a.e. in all quasi-metric balls $B(x, t)$, notice that $\kappa(B(x, 2\kappa s)) < \infty$, and hence by Gagliardo’s lemma (see [16, Lemma 3.1] or [28, Theorem 1.1]), there exists a nontrivial solution $u_{B(x, 2\kappa t)} \in L^q(\Omega, \sigma_{B(x, 2\kappa t)})$ to the equation $u = G(u^q \sigma_{B(x, 2\kappa t)})$ understood $d\sigma_{B(x, 2\kappa t)}$-a.e. By Lemma 3.2

\[K \sigma_{B(x, 2\kappa t)} \leq C u_{B(x, 2\kappa t)} < \infty \quad d\sigma$-a.e. in $B(x, t).\]

Moreover, as in the proof of Lemma 4.3 for any $y \in B(x, t)$, we have

\[K \sigma_{B(x, 2\kappa t)}(y) \leq C \int_t^\infty \frac{\kappa(B(x, 2\kappa s))}{s^2} ds < \infty.
\]

Thus, $K \sigma < \infty$ $d\sigma$-a.e. in all quasi-metric balls $B(x, t)$, and consequently in $\Omega$. This completes the proof of (iii) $\Rightarrow$ (i). □

5. Quasi-metrically modifiable kernels

In this section, we give bilateral pointwise estimates of solutions to sublinear integral equations (1.15) for quasi-metrically modifiable kernels $G$ with modifier $m$. We recall that the kernel $G$ is quasi-metrically modifiable, with constant $\kappa$, if for a positive function $m \in C(\Omega)$ called a modifier,

\[G(x, y) := \frac{G(x, y)}{m(x) m(y)}, \quad x, y \in \Omega,
\]

is a quasi-metric kernel with quasi-metric constant $\kappa$.

Examples of quasi-metrically modifiable kernels can be found in [3, 10, 12, 19]. In particular, for bounded domains $\Omega \subset \mathbb{R}^n$ satisfying the boundary Harnack principle, such as bounded Lipschitz, NTA or uniform domains, Green’s kernels $G$ for the Laplacian and fractional Laplacian (in the case $0 < \alpha \leq 2$, $\alpha < n$) are quasi-metrically modifiable.
Let $0 < q < 1$ and $\mu, \sigma \in \mathcal{M}^+(\Omega)$. We discuss relations between solutions (as well as sub- and super-solutions) to the equations
\begin{align}
(5.2) \quad u &= G(u^q d\sigma) + G\mu, \quad 0 < u < \infty \ d\sigma\text{-a.e. in } \Omega, \\
(5.3) \quad v &= \tilde{G}(v^q d\tilde{\sigma}) + \tilde{G}\tilde{\mu}, \quad 0 < v < \infty \ d\tilde{\sigma}\text{-a.e. in } \Omega,
\end{align}
where $\tilde{G}$ is the modified kernel $\tilde{G}(x, y) = \frac{1}{1 + \tilde{d}(x, y)}$, with modifier $m$, and
\begin{equation}
(5.4) \quad v := \frac{u}{m}, \quad d\tilde{\sigma} := m^{1+q} d\sigma, \quad d\tilde{\mu} = m d\mu.
\end{equation}
Clearly, equations (5.2) and (5.3) are equivalent.

We next consider a pair of $(1, q)$-weighted norm inequalities with the same least constant $\tilde{\kappa} = \tilde{\kappa}(\Omega)$,
\begin{align}
(5.5) \quad \|G\nu\|_{L^q(m \, d\sigma)} &\leq \tilde{\kappa} \int_{\Omega} m \, d\nu, \quad \forall \nu \in \mathcal{M}^+(\Omega), \\
(5.6) \quad \|\tilde{G}\nu\|_{L^q(\Omega, \tilde{\sigma})} &\leq \tilde{\kappa} \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega).
\end{align}
Here again (5.6) is simply an equivalent restatement of (5.5) in terms of $\tilde{G}$, $\tilde{\sigma}$ instead of $G$, $\sigma$.

For a Borel set $E \subset \Omega$, we denote by $\tilde{\kappa}(E)$ the least constant in the localized versions of inequalities (5.5), (5.6) with $\sigma_E$ in place of $\sigma$.

Let $\tilde{B}(x, r)$ be a quasi-metric ball associated with the quasi-metric $\tilde{d} = 1/\tilde{G}$,
\begin{equation}
(5.7) \quad \tilde{B}(x, r) := \left\{ y \in \Omega: \quad \tilde{G}(x, y) > \frac{1}{r} \right\}, \quad x \in \Omega, \ r > 0.
\end{equation}

We recall that the modified intrinsic potential $\tilde{K}\sigma$ is defined by
\begin{equation}
(5.8) \quad \tilde{K}\sigma(x) := \int_{0}^{\infty} \left[ \tilde{\kappa}(\tilde{B}(x, r)) \right]^{\frac{1}{1+q}} \frac{r^2}{r^2} dr, \quad x \in \Omega.
\end{equation}

**Lemma 5.1.** Let $G$ be a $(QS)$ kernel on $\Omega \times \Omega$ such that the modified kernel $\tilde{G}$, defined by (5.1) with modifier $m$, satisfies the (WMP). Then any nontrivial supersolution $u$ to (5.2) satisfies the estimate
\begin{equation}
(5.9) \quad u \geq c \, m \left( \frac{G(m^q d\sigma)}{m} \right)^{\frac{1}{1+q}} + \tilde{K}\sigma + G\mu \quad d\sigma\text{-a.e.},
\end{equation}
where $c$ is a positive constant which depends only on $q$, the quasi-symmetry constant $a$, and the constant $b$ in the (WMP) for $G$.

**Proof.** Let $v$, $\tilde{\sigma}$, and $\tilde{\mu}$ be defined by (5.4). Then $v > 0$ $d\tilde{\sigma}$-a.e. is a nontrivial supersolution, i.e., $\tilde{G}(v^q d\tilde{\sigma}) + \tilde{G}\tilde{\mu} \leq v < \infty \ d\tilde{\sigma}$-a.e. Since $\tilde{G}$
is a quasi-metric kernel, by Corollary 3.3 applied to \(v, \tilde{\sigma}, \tilde{\mu}\) and \(\tilde{G}\), we obtain
\[v \geq c \left( (\tilde{G}\tilde{\sigma})^{\frac{1}{1-q}} + \tilde{K}\sigma + \tilde{G}\tilde{\mu} \right) \ d\tilde{\sigma}\text{-a.e.}
\]
This gives estimate (5.9) for the supersolution \(u\). □

The next lemma provides a matching upper pointwise bound for subsolutions to sublinear integral equations, for quasi-metrically modifiable kernels \(G\).

**Lemma 5.2.** Let \(G\) be a quasi-metrically modifiable kernel on \(\Omega \times \Omega\) with modifier \(m\). Then any subsolution \(u\) to \((5.2)\) satisfies the estimate
\[(5.10) \quad u \leq C \left( \left[ \frac{G(m^q d\sigma)}{m} \right]^{\frac{1}{1-q}} + \tilde{K}\sigma + \tilde{G}\tilde{\mu} \right) \ d\sigma\text{-a.e.},\]
where \(C = C(q, \kappa)\), and \(\kappa\) is the quasi-metric constant for the modified kernel \(\tilde{G}\).

**Proof.** Let \(v, \tilde{\sigma}, \tilde{\mu}\) be defined by \((5.4)\). Then \(v \geq 0\) is a subsolution, i.e.,
\[v \leq \tilde{G}(v^q d\tilde{\sigma}) + \tilde{G}\tilde{\mu}.
\]
Since \(\tilde{G}\) is a quasi-metric kernel, by Lemma 4.2 applied to \(v, \tilde{\sigma}, \tilde{\mu}\) and \(\tilde{G}\), we obtain
\[v \leq C \left( (\tilde{G}\tilde{\sigma})^{\frac{1}{1-q}} + \tilde{K}\sigma + \tilde{G}\tilde{\mu} \right).
\]
This translates to estimate \((5.10)\) for the subsolution \(u\). □

In the remaining part of this section we consider the modifiers \(m = g\) given by
\[(5.11) \quad g(x) = g^a(x) := \min\{1, G(x, x_0)\}, \quad x \in \Omega,
\]
where \(x_0 \in \Omega\) is a fixed pole.

The following lemma was proved in \([28, \text{Lemma } 3.4]\) in a slightly weaker form, namely for the set \(\Omega \setminus \{x: G(x, x_0) = +\infty\}\) in place of \(\Omega\) (see also \([20, \text{Sec. } 8]\)). Its proof is based on the so-called Ptolemy’s inequality for quasi-metric spaces \([10, \text{Sec. } 3]\), which states that, if \(d\) is a quasi-metric with constant \(\kappa\) on \(\Omega\), then
\[(5.12) \quad d(x, y)d(x_0, z) \leq 4\kappa^2 \left[ d(x, z)d(y, x_0) + d(x_0, x)d(z, y) \right],
\]
for all \(x, y, z \in \Omega\).

**Lemma 5.3.** Let \(G\) be a quasi-metric kernel on \(\Omega \times \Omega\) with quasi-metric constant \(\kappa\). Let \(x_0 \in \Omega\), and let \(g(x) = \min\{1, G(x, x_0)\}\). Then
\[(5.13) \quad \tilde{G}(x, y) = \frac{G(x, y)}{g(x)g(y)}
\]
is a quasi-metric kernel on $\Omega \times \Omega$ with quasi-metric constant $4\kappa^2$. In particular, $\tilde{G}$ satisfies the (WMP) with constant $b = 8\kappa^2$.

**Proof.** Fix a pole $x_0 \in \Omega$. We need to prove the inequality

$$(5.14) \quad \frac{g(x)g(y)}{G(x,y)} \leq 4\kappa^2 \left[ \frac{g(x)g(z)}{G(x,z)} + \frac{g(y)g(z)}{G(z,y)} \right],$$

for all $x, y, z \in \Omega$.

Since $G$ is a quasi-metric kernel, multiplying both sides of $(1.4)$ by $g(x)g(y)$, we deduce

$$(5.15) \quad \frac{g(x)g(y)}{G(x,y)} \leq \kappa \left[ \frac{g(x)g(y)}{G(x,z)} + \frac{g(x)g(y)}{G(z,y)} \right],$$

for all $x, y, z \in \Omega$.

In the case $1 \leq G(z, x_0) \leq +\infty$, which ensures that $g(z) = 1$, we see that $(5.14)$ is immediate from $(5.15)$, since obviously $g(y) \leq g(z)$ and $g(x) \leq g(z)$, and $\kappa \geq \frac{1}{2}$.

In the remaining case $G(z, x_0) < 1$, we have $g(z) = G(z, x_0)$. Obviously, $g(y) \leq G(y, x_0)$ and $g(x) \leq G(x, x_0)$. Hence, letting $d := 1/G$ in $(5.12)$ and multiplying both of its sides by $g(x)g(y) G(z, x_0)$, we obtain

$$(5.16) \quad \frac{g(x)g(y)}{G(x,y)} \leq 4\kappa^2 \left[ \frac{g(x)}{G(x,z)} + \frac{g(y)}{G(z,y)} \right] G(z, x_0),$$

which coincides with $(5.14)$ in this case. Thus, $(5.14)$ follows from $(5.16)$. This proves that $\tilde{G}$ is a quasi-metric kernel with quasi-metric constant $\tilde{\kappa} = 4\kappa^2$. Then $\tilde{G}$ satisfies the (WMP) with constant $b = 2\tilde{\kappa} = 8\kappa^2$ by Lemma 2.1. \hfill $\Box$

In the next lemma, we give a criterion for the existence of (super) solutions in the case of quasi-metrically modified kernels $G$.

**Lemma 5.4.** Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \neq 0$) and $0 < q < 1$. Suppose $G$ is a quasi-metrically modifiable kernel on $\Omega \times \Omega$ with modifier $m = g \in C(\Omega)$ defined by $(5.11)$. Then there exists a nontrivial (super) solution to equation $(5.2)$ if and only if conditions $(1.35)$ hold, i.e.,

$$(5.17) \quad \int_{\Omega} g \, d\mu < \infty \quad \text{and} \quad \tilde{\kappa}(\Omega) < \infty,$$

where $\tilde{\kappa}(\Omega)$ denotes the least constant in the weighted norm inequality

$$(5.18) \quad \|G\nu\|_{L^q(\Omega, g\sigma)} \leq \tilde{\kappa}(\Omega) \int_{\Omega} g \, d\nu, \quad \forall \nu \in \mathcal{M}^+(\Omega).$$
Proof. Let $\tilde{G}$ be the modified kernel defined by (5.13) and $\tilde{B}(x,t)$ the corresponding quasi-metric ball. Let $m = g$ with a fixed pole $x_0 \in \Omega$. Obviously, $g(x) \leq G(x,x_0)$ and $g(x_0) \leq 1$. It follows that

\[
\tilde{G}(x,x_0) = \frac{G(x,x_0)}{g(x)g(x_0)} \geq 1, \quad \forall x \in \Omega.
\]

Hence,

\[
\tilde{g}(x) := \min\{1, \tilde{G}(x,x_0)\} \equiv 1, \quad \forall x \in \Omega.
\]

We set $d\tilde{\sigma} = g^{1+q} d\sigma$ and $d\tilde{\mu} = g d\mu$. By Lemma 4.5 and Lemma 4.6 with $\tilde{G}$, $\tilde{\sigma}$ and $\tilde{\mu}$ in place of $G$, $\sigma$ and $\mu$, respectively, we see that there exists a nontrivial solution $v$ to equation (5.3) if and only if

\[
\int_1^\infty \tilde{\sigma}(\tilde{B}(x_0,t)) \frac{dt}{t^2} + \int_1^\infty \frac{\tilde{\varpi}(\tilde{B}(x_0,t))^{1\over q} dt}{t^2} + \int_1^\infty \frac{\tilde{\mu}(\tilde{B}(x_0,t)) dt}{t^2} < \infty.
\]

It follows from (5.19) that $\tilde{B}(x_0,t) = \Omega$ if $t > 1$. Hence, the preceding condition is equivalent to

\[
\tilde{\sigma}(\Omega) < \infty, \quad \tilde{\varpi}(\Omega) < \infty, \quad \tilde{\mu}(\Omega) < \infty,
\]

where $\tilde{\varpi}(\Omega)$ is the least constant in the inequality

\[
\|\tilde{G}\tilde{\nu}\|_{L^q(\Omega,\tilde{\sigma}d\tilde{\mu})} \leq \tilde{\varpi}(\Omega) \int_\Omega \tilde{g} d\tilde{\nu}, \quad \forall \tilde{\nu} \in M^+(\Omega).
\]

But $\tilde{g} \equiv 1$ in $\Omega$ by (5.20). Hence, $\tilde{\varpi}(\Omega)$ coincides with the least constant in the inequality

\[
\|\tilde{G}\tilde{\nu}\|_{L^q(\Omega,\tilde{\sigma}d\tilde{\mu})} \leq \tilde{\varpi}(\Omega) \|\tilde{\nu}\|, \quad \forall \tilde{\nu} \in M^+(\Omega),
\]

or, equivalently, (5.18). Letting $\tilde{\nu} = \delta_{x_0}$ in (5.22), and noticing that $\tilde{G}\tilde{\nu}(x) = \tilde{G}(x,x_0) \geq 1$ in $\Omega$ by (5.20), we obtain

\[
[\tilde{\sigma}(\Omega)]^{1\over q} \leq \tilde{\varpi}(\Omega).
\]

Hence, the condition $\tilde{\sigma}(\Omega) < \infty$ in (5.21) is redundant. Since $\tilde{\mu}(\Omega) = \int_\Omega g d\mu$, it follows that (5.21) is equivalent to (5.17). Thus, a nontrivial (super) solution $v$ to (5.3) exists if and only if (5.17) holds. Using the relation (5.4) which translates (5.3) to (5.2), we see that condition (5.17) is necessary and sufficient for the existence of a nontrivial (super) solution $u$ to (5.2).

In the following corollary, we give an alternative criterion for the existence of (super) solutions to (5.2) in the case of quasi-metric kernels $G$, which complements Lemma 4.6 above.
**Corollary 5.5.** Let $\mu, \sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \neq 0$) and $0 < q < 1$. Suppose $G$ is a quasi-metric kernel on $\Omega \times \Omega$ such that $g \in C(\Omega)$, where $g$ is defined by (5.11). Then there exists a nontrivial (super) solution to equation (5.2) if and only if (5.17) holds.

**Proof.** By Lemma 5.3 the kernel $G$ is quasi-metrically modifiable with modifier $m = g$. Hence, by Lemma 5.4 there exists a nontrivial (super) solution to equation (5.2) if and only if (5.17) holds. $\square$

6. **Proofs of Theorem 1.1, Theorem 1.2, and Theorem 6.1**

**Proof of Theorem 1.1.** The lower bound (1.18) for nontrivial supersolutions follows from Corollary 3.3, for all (QS)&(WMP) kernels $G$ with $c = c(q, a, b)$. In particular, it holds with $c = c(q, \kappa)$ for quasi-metric kernels, which satisfy the (WMP) with constant $b = 2\kappa$ by Lemma 2.1.

The upper bound for subsolutions (1.19) is a consequence of Corollary 4.4 for quasi-metric kernels $G$ with $C = (8\kappa)^{1/q}$. This proves statement (ii) of Theorem 1.1.

Combining the upper and lower bounds for sub and supersolutions, respectively, yields bilateral estimates for solutions. The uniqueness of solutions is proved in Theorem 6.1 below for more general Borel measurable data $f \geq 0$ in place of $G\mu$. This yields statement (i).

Finally, the existence criterion in statement (iii) is a consequence of Lemma 4.5 and Lemma 4.6. $\square$

**Proof of Theorem 1.2.** Suppose that the kernel $G$ is quasi-metrically modifiable. Then the modified kernel $\tilde{G}$ is quasi-metric, and obeys the (WMP) by Lemma 2.1. Hence, the lower bound for supersolutions follows from Lemma 5.1. The upper bound for subsolutions is a consequence of Lemma 5.2. Combining these estimates gives the bilateral estimates of solutions. The uniqueness property of solutions is a consequence of the bilateral estimates, as shown below in the proof of Theorem 6.1. This completes the proof of Theorem 1.2. $\square$

The existence criteria mentioned in Remarks 1 and 2 after Theorem 1.2 follow from Lemma 4.5 and Lemma 4.6 (applied to $G$, $d\tilde{\sigma} = m^{1+q}d\sigma$ and $d\tilde{\mu} = md\mu$ in place of $G$, $\sigma$ and $\mu$, respectively) and Lemma 5.4.

In conclusion, we extend our results to equation (1.1) considered at the beginning of the Introduction, with an arbitrary Borel measurable function $f \geq 0$ in place of $G\mu$.

**Theorem 6.1.** Suppose $G$ is a quasi-metric kernel in $\Omega \times \Omega$. Suppose $0 < q < 1$, $\sigma \in \mathcal{M}^+(\Omega)$ ($\sigma \neq 0$) and $f \geq 0$ is a Borel measurable function in $\Omega$. Then the following statements hold.
(i) Any nontrivial solution \( u \) to equation (1.1) is unique and satisfies the bilateral pointwise estimates

\[
c \left[ (G\sigma)^{\frac{1}{1-q}} + K\sigma + G(f^q d\sigma) \right] + f \leq u
\]

and

\[
u \leq C \left[ (G\sigma)^{\frac{1}{1-q}} + K\sigma + G(f^q d\sigma) \right] + f,
\]

d\(\sigma\)-a.e. in \( \Omega \), where \( c, C \) are positive constants which depend only on \( q \) and the quasi-metric constant \( \kappa \) of the kernel \( G \).

(ii) Estimate (6.1) holds for any nontrivial supersolution \( u > 0 \) such that

\[
G(u^q d\sigma) + f \leq u < \infty \quad d\sigma\text{-a.e. in } \Omega,
\]

whereas estimate (6.2) holds for any subsolution \( u \) such that

\[
u \leq G(u^q d\sigma) + f < \infty \quad d\sigma\text{-a.e. in } \Omega.
\]

(iii) A nontrivial solution \( u \) to (1.1) exists if and only if

\[
G\sigma < \infty, \quad K\sigma < \infty, \quad f < \infty, \quad G(f^q d\sigma) < \infty, \quad d\sigma\text{-a.e. in } \Omega.
\]

**Proof.** We first prove statement (ii). Suppose that \( u \) is a nontrivial supersolution to (1.1). Let \( v := G(u^q d\sigma) \). Then obviously \( v > 0 \) and \( v + f \leq u < \infty \) \( d\sigma\)-a.e. By Lemma 2.3, \( v + f \in L^q_{\text{loc}}(\Omega) \), and in particular \( d\mu := f^q d\sigma \) is a measure in \( \mathcal{M}^+(\Omega) \). Since \( G[(v + f)^q d\sigma] \leq v \) and \( 2^{q-1}(v^q + f^q) \leq (v + f)^q \), we estimate

\[
2^{q-1} \left[ G(v^q d\sigma) + G\mu \right] \leq v < \infty \quad d\sigma\text{-a.e.}
\]

The constant \( 2^{q-1} \) is easily incorporated into \( \sigma \) by using \( \tilde{\sigma} := 2^{q-1}\sigma \). Hence, by Lemma 2.1 and Corollary 3.3 with \( \tilde{\sigma} \) in place of \( \sigma \), we deduce

\[
v \geq c \left[ (G\sigma)^{\frac{1}{1-q}} + K\sigma + G\mu \right] \quad d\sigma\text{-a.e.},
\]

where \( c = c(q, \kappa) \). Since \( u \geq v + f \), we obtain (6.1).

To prove the upper bound (6.2), suppose that \( u \geq 0 \) is a subsolution to (1.1). Without loss of generality we may assume that

\[
(G\sigma)^{\frac{1}{1-q}} + K\sigma + G(f^q d\sigma) \neq \infty,
\]

since otherwise (6.2) is trivial. From this it follows, as was shown above, that \( \int_{B(x,r)} f^q d\sigma < \infty \) for all \( x \in \Omega, \, r > 0 \), and consequently \( d\mu := f^q d\sigma \) is a measure in \( \mathcal{M}^+(\Omega) \). By Lemma 4.6

\[
(G\sigma)^{\frac{1}{1-q}} + K\sigma + G\mu < \infty \quad d\sigma\text{-a.e.}
\]
Let \( v := G(u^q d\sigma) \). Then \( u \leq v + f < \infty \) \( d\sigma \)-a.e. by (6.4). Hence,

\[
v \leq G((v + f)^q d\sigma) \leq G(v^q d\sigma) + G(f^q d\sigma)
\]

\[
= G(v^q d\sigma) + G\mu \quad d\sigma\text{-a.e.}
\]

We next show that \( G(v^q d\sigma) < \infty \) \( d\sigma \)-a.e. Using Lemma 4.1 with \( d\nu := u^q d\sigma \), so that \( v = G\nu \), we estimate

\[
G(v^q d\sigma) = G[(G\nu)^q d\sigma] \leq (2\kappa)^q (G\nu)^q [G\sigma + (K\sigma)^{1-q}]
\]

\[
= (2\kappa)^q v^q [G\sigma + (K\sigma)^{1-q}] < \infty \quad d\sigma\text{-a.e.}
\]

by (6.4) and (6.8). It follows that \( v \) is a subsolution satisfying (6.4) with \( f = G\mu \), i.e.,

\[
v \leq G(v^q d\sigma) + G\mu < \infty \quad d\sigma\text{-a.e.}
\]

Hence, by Corollary 4.4,

\[
v \leq (8\kappa)^{\frac{q}{1-q}} \left[ (G\sigma)^{\frac{1}{1-q}} + K\sigma + G\mu \right] d\sigma\text{-a.e.}
\]

Thus,

\[
u \leq v + f \leq (8\kappa)^{\frac{q}{1-q}} \left[ (G\sigma)^{\frac{1}{1-q}} + K\sigma + G\mu \right] + f.
\]

This completes the proof of statement (ii).

We next prove statement (i). Notice that the bilateral estimates (6.1) and (6.2) for any solution \( u \) to (1.1) follow from statement (ii). It remains to prove the uniqueness property, which is a consequence of these estimates. Indeed, suppose \( u_1 \) and \( u_2 \) are solutions to (1.1). Applying (6.1) and (6.2) with \( u_1 \) and \( u_2 \) in place of \( u \), we deduce

\[
a u_1 \leq u_2 \leq a^{-1} u_1 \quad d\sigma\text{-a.e.,}
\]

where \( a = cC^{-1} \leq 1 \) is a positive constant. Raising to the power \( q \) and applying the operator \( G^\sigma \), we deduce

\[
a^q G(u_1^q d\sigma) \leq G(u_2^q d\sigma) \leq a^{-q} G(u_1^q d\sigma).
\]

It follows that

\[
a^q [G(u_1^q d\sigma) + f] \leq G(u_2^q d\sigma) + f \leq a^{-q} [G(u_1^q d\sigma) + f].
\]

Hence,

\[
a^q u_1 \leq u_2 \leq a^{-q} u_1 \quad d\sigma\text{-a.e.}
\]

Iterating this procedure, we obtain, for any \( j = 1, 2, \ldots \),

\[
a^{jq} u_1 \leq u_2 \leq a^{-jq} u_1 \quad d\sigma\text{-a.e.}
\]

Passing to the limit as \( j \to \infty \) yields \( u_1 = u_2 \) \( d\sigma \)-a.e. This completes the proof of statement (i).
The proof of statement (iii) is similar to that of Lemma 4.5 in the special case $f = G\mu$; we omit the details. □

Remark. An analogue of Theorem 6.1 for quasi-metrically modifiable kernels $G$ is deduced in a similar way. This gives an extension of Theorem 1.2 for solutions of equation (1.1), with $f$ in place of $G\mu$. The corresponding estimates of solutions remain valid once we replace $G\mu$ with $c G(f^q d\sigma) + f$ in (1.32), and $C G(f^q d\sigma) + f$ in (1.33), respectively.

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