Abstract

Dual form of 3+1 dimensional Yang-Mills theory is obtained as another SO(3) gauge theory. Duality transformation is realized as a canonical transformation. The non-Abelian Gauss law implies the corresponding Gauss law for the dual theory. The dual theory is non-local. There is a non-local version of Yang-Mills theory which is self-dual.

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I. INTRODUCTION

Duality transformation plays an important role in many contexts in quantum field theory and statistical physics. It relates a model at a strong coupling or high temperature to another at weak coupling or low temperature. Therefore it provides a valuable tool in understanding some strongly interacting theories. In some cases, there is invariance under duality transformation. The standard example is the Ising model in two dimensions. In such a situation it provides further valuable information regarding properties of the system. Another mysterious aspect of duality transformations is that it often exposes topological degrees of freedom which play crucial roles in determining properties of the system. A classic example is the Berezinsky-Kosterlitz-Thouless transition in two dimensional x-y model.

In this article we consider duality transformation of 3+1 dimensional Yang-Mills theory. Such transformations have already played crucial roles for understanding many aspects of gauge theories. Indeed the first examples of lattice gauge theories appeared as dual theories of certain Ising models [1].

Duality transformation is especially important for understanding the confinement aspects of gauge theories [2]. It is expected, and in some cases checked, that monopoles play a crucial role for this property. Three dimensional compact $U(1)$ gauge theory is a well understood example [3].

Duality transformation of an Abelian gauge theory gives the dual potential [4] i.e. one which couples minimally to magnetic matter. Therefore it exposes the monopole degrees of freedom. This is brought out in a powerful way in four dimensional super symmetric gauge theories [5].

Deser and Teitelboim [6] analyzed the possibility of duality invariance of 3+1 dimensional Yang-Mills theory in close analogy to Maxwell theory and concluded that invariance is not realized. The first work to address duality transformation of 3+1 dimensional Yang-Mills theory retaining all the non-Abelian features was by Halpern [7]. Using complete axial gauge fixing, he brought out the crucial role played by the Bianchi identity. The dual theory was a gauge theory with a new gauge potential, though the action was non-local.

Another issue closely related to duality transformation is reformulation of the gauge theory dynamics using gauge invariant degrees of freedom. Several authors [8] consider rewriting the functional integral using a gauge covariant second rank tensor. Anishetty, Cheluvara, Sharatchandra and Mathur [9] pointed out that $SO(3)$ lattice gauge theory in 2+1 dimensions is closely related to gravity. This can be used to formulate the dynamics using local gauge invariant degrees of freedom [10]. Similar situation is true in 3+1 dimensions also [11].

In this article we bring in new techniques which are useful for duality transformation of non-Abelian gauge theories. Though we use the language of functional integrals, our procedure can be stated directly for classical Yang-Mills theory. We adopt the Hamiltonian formalism. This is the most direct method for duality transformation in Maxwell’s theory as reviewed in the first section. This also brings out the crucial role played by the Gauss law and the Hodge decomposition in duality transformation which we developed in [12]. This approach automatically gives the dual theory as a $SO(3)$ gauge theory, with a non-Abelian dual gauge field.
We also use generating functions of canonical transformations to perform duality transformation (section III.B). We find that it is an extremely powerful technique for handling non-Abelian theories. It is very helpful for obtaining the implication of the non-Abelian Gauss law for the dual theory. It turns out that it is natural to treat the dual gauge field as a background gauge field of the Yang-Mills theory and vice-versa. (We use rescaled fields such that the gauge transformations do not involve the coupling constants.) Choosing the generating function to be invariant under a common gauge transformation, the Gauss law constraint simply goes over to a similar constraint in the dual theory (section III.C). Another important issue is the gauge copy problem [14,16], i.e. gauge inequivalent potentials which give the same non-Abelian magnetic field. In analogy to the Abelian case, we would like to replace $\vec{E}_i$, the non-Abelian electric field by $\vec{B}_i[C]$, the non-Abelian magnetic field of the dual gauge potential $C$. But if gauge copies are present, then this naive replacement runs into problems. We have argued in [1] that there is only boundary degrees of freedom for the gauge field copies. As a consequence the number of degrees of freedom provided by $\vec{B}_i[C]$ are sufficient.

We explore the possibility of self duality of 3+1-dimensional Yang-Mills theory in section IV and conclude that it is absent. All the canonical transformations that we consider lead to a dual theory which is non-local.

We summarize our results in sec V.

II. GAUSS LAW AND DUALITY TRANSFORMATION IN MAXWELL’S THEORY

Consider the free Maxwell theory. The extended phase space has the canonical variables, the vector potential $A_i$ and the electric field $E_i$, $i = 1, 2, 3$ with the Poisson bracket

$$[A_i(x), E_j(y)]_{PB} = \delta_{ij} \delta(x - y).$$

(1)

The Hamiltonian density is,

$$H(x) = \frac{1}{2}(E_i^2(x) + B_i^2[A](x))$$

(2)

where the magnetic field $B_i[A] = \epsilon_{ijk}\partial_j A_k$. $A_i$ and $A_i + \partial_i \Lambda$ give rise to same $B_i[A]$. The physical phase space is the subspace given by the Gauss law constraint,

$$\partial_i E_i = 0.$$  

(3)

A very easy way of obtaining the dual theory is to solve the Gauss law constraint. We have the general solution,

$$E_i = \epsilon_{ijk}\partial_j C_k$$

(4)

We can compute the Poisson bracket of the new variable $C$ with the old variables as follows. We have the Poisson bracket

$$[B_i(x), E_j(y)]_{PB} = -\epsilon_{ijk}\partial_k \delta(x - y).$$

(5)
Substituting the above ansatz for $E$ we get as a consistent solution the non-zero Poisson bracket

$$[B_i(x), C_j(y)]_{PB} = \delta_{ij} \delta(x - y).$$

(6)

Thus we have the new canonical pair $(C, E = B[A])$ in contrast to the old set $(A, E)$. In terms of this new pair the Hamiltonian takes the form

$$H(x) = \frac{1}{2}(E^2(x) + B^2[C](x)).$$

(7)

Thus we have made a canonical transformation from the pair $(A, E)$ to $(C, B)$ and the Hamiltonian has the same form in terms the new variables. The analogy is complete since $C$ is also a gauge field (the dual gauge field), with $C_i(x)$ and $C_i(x) + \partial_i \lambda(x)$ giving rise to the same $B[C]$. This is the dual local gauge transformation. Also the new extended phase space has the dual Gauss law constraint

$$\partial_i \mathcal{E}_i = 0.$$  

(8)

The old vector potential $A$ couples minimally to the electric currents. In contrast the new vector potential couples minimally to the magnetic current as can be verified by introducing sources. Thus the dual symmetry is complete.

The duality transformation can be viewed as a canonical transformation induced by the generating function

$$S(A, C) \equiv \langle C|B[A]\rangle = \int \epsilon_{ijk} C_i \partial_j A_k$$

(9)

of the old and the new coordinates $A$ and $C$ respectively. We have the symmetry

$$\langle C|B[A]\rangle = -\langle A|B[C]\rangle.$$  

(10)

This is a very convenient technique for obtaining the new momentum and for computing the Poisson brackets of the old and the the new variables. We get the old and new momenta to be,

$$E_i = \frac{\delta S}{\delta A_i} = \epsilon_{ijk} \partial_j C_k = B_i[C],$$

(11)

and

$$\mathcal{E}_i = -\frac{\delta S}{\delta C_i} = -B_i[A];$$

(12)

respectively. The generating function is invariant under the old gauge transformation. This gives the identity, that for any $\lambda$

$$\int \partial_i \lambda \frac{\delta S}{\delta A_i} = 0.$$ 

(13)

As $\lambda$ is arbitrary, it follows

$$\partial_i \frac{\delta S}{\delta A_i} = 0,$$  

(14)

which is the Gauss law constraint. This is a very convenient way of making the duality transformation preserving the Gauss law constraints. The generating function is also invariant under the new gauge transformation which implies the new Gauss law $\partial_i \mathcal{E}_i = 0$.

We extend and generalize these techniques for non-Abelian gauge theories.
III. TECHNIQUES FOR DUALITY TRANSFORMATION

In this section we introduce various techniques useful for the duality transformation of non-Abelian gauge theories.

A. Functional integral with phase space variables

The Euclidean functional integral for 3+1-dimensional Yang-Mills theory is formally

$$Z = \int \mathcal{D}A^a_i \exp\{-\frac{1}{4g^2} \int \tilde{F}_{\mu\nu} \cdot \tilde{F}_{\mu\nu}\}$$

(15)

where

$$\tilde{F}_{\mu\nu} = \partial_{\mu} \tilde{A}_{\nu} - \partial_{\nu} \tilde{A}_{\mu} + \tilde{A}_{\mu} \times \tilde{A}_{\nu}$$

(16)

With this choice the gauge transformation does not involve the coupling constant. We could as well have started with the Minkowski space functional integral. However the Euclidean version makes the role of the non-Abelian Gauss law even more transparent.

Introducing an auxiliary field $E_a^i$, (15) becomes

$$Z = \int \mathcal{D}A^a_i \mathcal{D}A^a_0 \mathcal{D}E^a_i \exp \{ \int (\frac{-g^2}{2} \tilde{E}_i \cdot \tilde{E}_i - \frac{1}{2g^2} \tilde{B}_i[A] \cdot \tilde{B}_i[A])

+i \tilde{E}_i \cdot (\partial_0 \tilde{A}_i - D_i[A] \tilde{A}_0)\}$$

(17)

where

$$D_i[A] = \partial_i + \tilde{A}_i \times$$

(18)

is the covariant derivative and

$$\tilde{B}_i[A] = \frac{1}{2} \epsilon_{ijk}(\partial_j \tilde{A}_k - \partial_k \tilde{A}_j + \tilde{A}_j \times \tilde{A}_k)$$

(19)

is the non-Abelian magnetic field. Integration over $A_0$ gives

$$Z = \int \mathcal{D}A^a_i \mathcal{D}E^a_i \delta(D_i[A]E_i) \exp\{ \int (-\mathcal{H} + i\tilde{E}_i \cdot \partial_0 \tilde{A}_i)\}.$$  

(20)

Using the Feynman time slicing procedure, it is clear that $A_i, E_i$ are the conjugate variables of the phase space and

$$\mathcal{H} = \frac{1}{2}(g^2 E^2 + \frac{1}{g^2} B^2)$$

(21)

is the hamiltonian density. There are also three first class constraints, the non-Abelian Gauss law:

$$D_i[A] \tilde{E}_i = 0.$$  

(22)
B. Duality transformation via a canonical transformation

In close analogy to the Abelian case, we consider a change of variable $s$ from $E$ to $\tilde{C}$.

$$\vec{E}_i = \epsilon_{ijk} D_j [A] \tilde{C}_k$$  \hspace{1cm} (23)

where $\tilde{C}$ transforms homogeneously under gauge transformations. Naively $\tilde{C}_i^a$ is the canonical conjugate of the non-Abelian magnetic field $B_i^a$. This can be checked directly. Note that

$$[E_m^d(x), B_i^a(y)]_{PB} = \epsilon_{ijm} (\delta^{da} \partial_j + \epsilon^{dab} A^b_j) \delta(x - y).$$  \hspace{1cm} (24)

Using (23), the left hand side is

$$\epsilon_{ijm} (\delta^{de} \partial_j + \epsilon^{dab} A^b_j) [\tilde{C}_m^e(x), B_i^a(y)]_{PB}. \hspace{1cm} (25)$$

This is consistent with

$$[\tilde{C}_m^e(x), B_i^a(y)]_{PB} = \delta^{ea} \delta_{mi} \delta(x - y). \hspace{1cm} (26)$$

An easy way to see this is by using the generator of canonical transformations

$$S(A, \tilde{C}) = \int \tilde{C}_i^a B_i^a [A]$$  \hspace{1cm} (27)

Then $E_i^a = \frac{\delta S}{\delta A_i^a} = \epsilon_{ijk} (D_j [A] \tilde{C}_k)^a$ and the new momentum conjugate to the new variable $\tilde{C}_i^a$ is

$$\mathcal{E}_i^a = - \frac{\delta S}{\delta \tilde{C}_i^a} = -B_i^a [A]. \hspace{1cm} (28)$$

The great advantage of realizing duality transformation via a canonical transformation is that the phase space measure in the functional integral is invariant.

$$\mathcal{D} A D \mathcal{E} = \mathcal{D} C D \mathcal{E} \hspace{1cm} (29)$$

Also

$$\sum p_i \dot{q}_i = \sum P_i \dot{Q}_i \hspace{1cm} (30)$$

and

$$H'(P, Q) = H(p(P, Q), q(P, Q))$$  \hspace{1cm} (31)

under a canonical transformation $(q, p) \rightarrow (Q, P)$. Therefore it is easy to express the exponent in equation (20) also in terms of the new variables.
C. New Gauss law from the old Gauss law

In order to satisfy the Gauss law constraint (22), we need
\[ \vec{B}_i[A] \times \vec{C}_i = 0, \] (32)
where sum over \( i \) is implied. Here we have used
\[ \epsilon_{ijk} D_j[A] D_k[A] \vec{C}_i = \vec{B}_i[A] \times \vec{C}_i. \] (33)

It is of interest to have the dual field also a gauge field. With that in mind we introduce a new gauge field \( C_i \). The covariant derivative with respect to the gauge field \( C \) can be written as
\[ D_i[C] = D_i[A] + (\vec{C} - \vec{A})_i \times . \] (34)

We also have the Bianchi identity
\[ D_i[A] B_i[A] = 0. \] (35)

If we could preserve relation (28) in terms of the new field \( C \), then we would get
\[ D_i[C] \vec{E}_i = - (\vec{C} - \vec{A})_i \times \vec{B}_i[A] \] (36)
(30) together with (32) immediately indicates that to get the new Gauss law, it is better to rewrite \( \vec{C} \) as \( \vec{C} - \vec{A} \). This changes our ansatz (23) to
\[ \vec{E}_i = \epsilon_{ijk} D_j[A](\vec{C} - \vec{A})_k. \] (37)

This corresponds to the generating function
\[ S(A, C) = \int (\vec{C} - \vec{A})_i \vec{B}_i[A]. \] (38)

With this choice the old Gauss law (22) simply goes over to the new Gauss law
\[ D_i[C] \vec{E}_i = 0. \] (39)

Such a feature is very useful for the duality transformation. It can be easily realized in general as shown below. In ansatz (23), \( C \) transforms homogeneously (as an isotriplet vector field) under the \( A \)-gauge transformation, whereas \( A \) transforms inhomogeneously.
\[ \delta A_i = D_i[A] \Lambda \] (40)
In contrast, in ansatz (37) \( C \) transforms as a gauge field under \( A \)-gauge transformations.

Note that if \( C \) and \( A \) both transform as gauge fields, \( \alpha C + (1 - \alpha)A \) also transforms like a gauge field for any choice of a real parameter \( \alpha \). However \( (C - A) \) transforms homogeneously, i.e. as a matter field in the adjoint representation. Consider a canonical transformation
$S(A, C)$ which is gauge invariant under the common gauge transformations as in equation (B8). Some choices of terms in $S(A, C)$ are

(a) $\epsilon_{ijk} \vec{A}_i \cdot \partial_j \vec{A}_k + \frac{1}{3} \vec{A}_i \cdot \vec{A}_j \times \vec{A}_k \equiv CS[A]
(b) \epsilon_{ijk} (\vec{C}_i \cdot \partial_j \vec{C}_k + \frac{1}{3} \vec{C}_i \cdot \vec{C}_j \times \vec{C}_k) \equiv CS[C]
(c) (\vec{C} - \vec{A})_i \cdot \vec{B}_i[A] \equiv det(C - A).
(d) \epsilon_{ijk} \frac{1}{3!} (\vec{C} - \vec{A})_i \cdot (\vec{C} - \vec{A})_j \times (\vec{C} - \vec{A})_k \equiv det(C - A).

Here $CS$ is the Chern-Simons density. Since

$$\frac{\delta CS[A]}{\delta A_i} = B_i[A],$$

it contributes a piece which is independent of $C$ to $E_i$. Note that the functional integral (20) is insensitive to shifts

$$E_i \rightarrow E_i + \alpha B_i[A]$$

where $\alpha$ is an arbitrary real parameter. First of all, the Gauss law condition

$$D_i[A] \vec{E}_i = 0$$

does not change as a consequence of the Bianchi identity (35). Next, the term $E_i \dot{A}_i$ changes by

$$\alpha B_i[A] \dot{A}_i = \alpha \frac{\partial}{\partial t} CS[A].$$

This being a total derivative, does not matter. (This conclusion is not correct when instanton number [17] is non-zero.) This invariance is reflected in the possible addition of $CS[A]$ (41a) to the generating function $S[A, C]$

Invariance of $S(A, C)$ under simultaneous gauge transformation of $A$ (40) and $C$, where,

$$\delta \vec{C}_i = D_i[C] \vec{A}$$

implies

$$\int \left\{ (D_i[A] \Lambda)^a \frac{\delta S}{\delta A_i^a} + (D_i[C] \Lambda)^a \frac{\delta S}{\delta C_i^a} \right\} = 0$$

As this is true for any arbitrary choice of $\Lambda$, we get,

$$D_i[A] \vec{E}_i = D_i[C] \vec{E}_i$$

so that the old Gauss law constraint implies the new Gauss law constraint. Another advantage of such a choice of $S(A, C)$ is that the dual field $C$ appears as a background gauge field for $A$ and vice-versa.
The new gauss law may be realized through an auxiliary field $C_0$ which would play the role played by $A_0$ in (17). This naturally leads to the action functional formulation of the dual theory, once we integrate over $E_i$:

$$Z = \int D C_0 D C_i D E_i \exp \left\{ -H'[C, E] + i(\partial_0 \vec{C} - D_i[C] \vec{C}_0) \cdot \vec{E}_i \right\}$$

$$= \int D C_0 D C_i \exp (-S[C_0, C_i])$$

(49)

where $S[C_0, C_i]$ is gauge invariant under the full gauge transformation, $\delta \vec{C}_\mu = D_\mu [C] \vec{A}$.

**D. Degrees of freedom**

The constraint equation (32) can be handled in a different way. In the generic case where $\text{det} \; B \equiv |B|$, the determinant of the $3 \times 3$ matrix $B^a_i (i, a = 1, 2, 3)$ is non-zero, it is easy to solve this constraint on $C$ [12]. Use $B^a_i$ to “lower” the color index in $C_i^a$.

$$C_i^a = C_{ij} B_j^a.$$  

(50)

Equation (32) is satisfied if and only if $C_{ij}$ is a symmetric tensor. This corresponds to the choice

$$S(A, C) = \int C_{ij} b_{ij}$$

(51)

where $C_{ij}$ would be the new coordinates and $b_{ij} = \vec{B}_i^a[A] \cdot \vec{B}_j^a[A]$, the new conjugate momenta.

Thus the “physical” phase space of Yang-Mills theory may be described in terms of the conjugate pair $C_{ij}, b_{ij}$ which are gauge invariant symmetric second rank tensors. Each of these have six degrees of freedom at each $x$ which appears to match the required degrees of freedom. The situation could have been more involved because of the Wu-Yang ambiguities [14]. But as was analyzed in [16] this is not a generic phenomenon. The equation

$$\epsilon_{ijk} D_j [A] e_k = 0.$$  

(52)

does not have a continuous family of solutions. Therefore we can write

$$\vec{E}_i = \epsilon_{ijk} D_j [A] (\vec{C}_k - \vec{A}_k)$$

(53)

Alternately we can use the decomposition of the form [12]

$$\vec{E}_i = \vec{B}_i [C]$$

(54)

This seems to be closest to the choice in the Abelian case which had duality invariance. Note that

$$\vec{B}_i [C] = \vec{B}_i [A] + \epsilon_{ijk} D_j [A] (\vec{C}_k - \vec{A}_k) + \frac{1}{2} \epsilon_{ijk} (\vec{C} - \vec{A})_j \times (\vec{C} - \vec{A})_k$$

(55)

which corresponds to an expansion of $B_i [C]$ about a “background gauge field” $A$ with $(\vec{C} - \vec{A})$ as the quantum fluctuation. If $E_i$ satisfies the Gauss law (22), so does $E_i - B_i [A]$. Therefore
the ansatz (37) and (54) essentially differ through the last term on the right hand side of (55). This is obtained by including the term \( \det(C_A) \) (41d) in the generating functional of the canonical transformation.

The choice (54) is appealing for many reasons. We have,

\[
\int \frac{1}{2} E_i^2 = \int \left( \frac{1}{2} B_i^2[C] \right)
\]

also

\[
\frac{\delta S}{\delta A_i} \partial_0 \tilde{A}_i + \frac{\delta S}{\delta C_i} \partial_0 \tilde{C}_i = \partial_0 S,
\]

a total derivative, so that,

\[
\int \tilde{E}_i \partial_0 \tilde{A}_i = \int \tilde{E}_i \cdot \partial_0 \tilde{C}_i
\]

Therefore the exponent in (17) can be expressed easily in terms of the new variables as before.

**IV. DUALITY TRANSFORMATION**

In Maxwell theory we had duality invariance because \( E_i = B_i[C] \) and \( \mathcal{E}_i = -B_i[A] \). Such a simple interchange does not work for the non-Abelian case as seen from equations (28) and (37). Note that if we add \( CS[A] \), equation (41) to the generating function (38), we can make

\[
\tilde{E}_i = \tilde{B}_i[A] + \epsilon_{ijk} D_j[A] (\tilde{C} - \tilde{A})_k.
\]

As seen from (55) the quadratic term in \( (C - A) \) is missing.

We now weaken our requirement. It is sufficient if,

\[
g^2 E^2 + \frac{1}{g^2} B^2[A] = g^2 B^2[C] + \frac{1}{g^2} \mathcal{E}^2
\]

If we use a generating function \( S(A, C) \), we require

\[
-g^2 \left( \frac{\delta S}{\delta A_i} \right)^2 + \frac{1}{g^2} \left( \frac{\delta S}{\delta C_i} \right)^2 = -g^2 B^2[C] + \frac{1}{g^2} B^2[A].
\]

Consider the \( g = 1 \) case. Now equation (51) can be rewritten as

\[
\frac{\delta S}{\delta \left( \frac{A + C}{2} \right)_i} \frac{\delta S}{\delta \left( \frac{A - C}{2} \right)_i} = \epsilon_{ijk} D_j \left[ \frac{A + C}{2} \right] \left( \frac{A - C}{2} \right)_k \cdot \left\{ \tilde{B}_i \left[ \frac{A + C}{2} \right] \right.
\]

\[
+ \frac{1}{2} \epsilon_{ijk} \left( \frac{A - \tilde{C}}{2} \right)_j \times \left( \frac{A - \tilde{C}}{2} \right)_k \right\}
\]
using equation (53) for the background gauge field \( \left( \frac{A + C}{2} \right) \). It is amusing to note that the generating function

\[
S \left( \frac{A + C}{2}, \frac{A - C}{2} \right) = \left( \frac{\bar{A} - \bar{C}}{2} \right)_i \cdot \bar{B}_i \left[ \frac{A + C}{2} \right] + \text{det} \left( \frac{A - C}{2} \right)
\]  

(63)

gives the right hand side of the above equation, but with the opposite sign. Self duality is achieved in the Abelian case by using

\[
S = CS \left( \frac{C + A}{2} \right) - CS \left( \frac{C - A}{2} \right).
\]  

(64)

The non-Abelian case should have something similar and not (63). Unfortunately there is no \( S \) satisfying (62). As a consequence self duality is ruled out.

We consider generating functions

\[
S(A, C) = \alpha_1 CS(A) + \alpha_2 CS(C) + \alpha_3 (\bar{A} - \bar{C})_i \cdot \bar{B}_i[A] \\
+ \frac{\alpha_4}{2} \epsilon_{ijk} (\bar{A} - \bar{C})_i \cdot D_j[A] (\bar{A} - \bar{C})_k + \alpha_5 \text{det}(A - C).
\]  

(65)

where \( \alpha_1, \ldots, \alpha_5 \) are arbitrary real parameters for the present. Now we get

\[
\bar{E}_i = \beta_1 \bar{B}_i[A] + \beta_2 \epsilon_{ijk} D_j[A] (\bar{A} - \bar{C})_k \\
+ \frac{\beta_3}{2} \epsilon_{ijk} (\bar{A} - \bar{C})_j \times (\bar{A} - \bar{C})_k
\]  

(66)

\[
\bar{E}_i = \gamma_1 \bar{B}_i[A] + \gamma_2 \epsilon_{ijk} D_j[A] (\bar{A} - \bar{C})_k \\
+ \frac{\gamma_3}{2} \epsilon_{ijk} (\bar{A} - \bar{C})_j \times (\bar{A} - \bar{C})_k
\]  

(67)

where \( \beta_1 = \alpha_1 + \alpha_3; \beta_2 = \alpha_3 + \alpha_4; \beta_3 = \alpha_4 + \alpha_5; \) and \( \gamma_1 = -\alpha_2 + \alpha_3; \gamma_2 = \alpha_2 + \alpha_4; \gamma_3 = -\alpha_2 + \alpha_5. \) For no choice of the parameters \( \alpha_1, \ldots, \alpha_5 \) do we get a local Hamiltonian in the dual variables. We illustrate this for a specific choice, \( \alpha_2 = \alpha_4 = \alpha_5 = 0, \alpha_1 = -2 \) and \( \alpha_3 = 1. \) We get \( \bar{E}_i = \bar{B}_i[A] \) but \( \bar{E}_i = -\bar{B}_i[C] - \frac{1}{2} \epsilon_{ijk} (\bar{A} - \bar{C})_j \times (\bar{A} - \bar{C})_k. \) Therefore the dual action becomes

\[
g^2 \{ B_i[C] + \frac{1}{2} \epsilon_{ijk} (\bar{A} - \bar{C})_j \times (\bar{A} - \bar{C})_k \}^2 + \frac{1}{g^2} \bar{E}^2.
\]  

(68)

\((A - C)\) may be regarded as a non-local functional of the dual variables \((C, \bar{E});\) solution of

\[
\epsilon_{ijk} D_j[C] (\bar{A} - \bar{C})_k + \frac{1}{2} \epsilon_{ijk} (\bar{A} - \bar{C})_j \times (\bar{A} - \bar{C})_k = \bar{E}_i - \bar{B}_i[C]
\]  

(69)

Consider a modified Yang-Mills Hamiltonian

\[
H = \int \left( \frac{1}{2} g^2 \bar{E}_i^2 + \frac{1}{2g^2} \bar{E}_i^2 \right)
\]  

(70)

where it is presumed that \( \bar{E}_i \) is expressed in terms of \((A, E)\). This theory would be self dual, if the generating function \( S(A, C) \) is symmetric under the interchange \( A \leftrightarrow C \). A simple way of realizing this is to have \( S \) (regarded as a functional of \((A + C)\) and \((A - C)\)), even in \((A - C)\). For all choices of \( S \) we have considered, the theory is non-local.
V. CONCLUSION

In this article we have constructed a dual form of the 3+1 Yang-Mills theory. We have argued that the functional integral using phase space variables is best suited for the purpose. Now the duality transformation can be realized as a canonical transformation. This provides a powerful tool, because the action and the measure in the dual variables as also the implications of the Gauss law constraint for the dual theory are easily written. The dual theory is also a SO(3) gauge theory. The dual theory, though a SO(3) gauge theory, is a non-local theory. However Yang-Mills theory with a non-local action is self dual. Our techniques for obtaining the dual theory may provide a firm basis for the computations of the confining properties in the dual QCD approach of Baker, Ball and Zachariasen [18].
REFERENCES

[1] F. Wegner, J. Math. Phys. 12 (1971), 2259.
[2] John B. Kogut, Rev. Mod. Phys. 51 (1979), 659.
[3] A. Polyakov, Phys. Lett. B59 (1975), 82; A. Polyakov, Nucl. Phys. B120 (1977), 429.
[4] T. Banks, R. Myerson and J. Kogut, Nucl. Phys. B129 (1977), 493; J. L. Cardy, Nucl. Phys. B205 (1982), 1; Manu Mathur and H. S. Sharatchandra, Phys. Rev. Lett. 66 (1991), 3097.
[5] W. Seiberg and E. Witten, Nucl. Phys. B426 (1994), 19.
[6] S. Deser and C. Teitelboim, Phys. Rev. D13 (1976), 1592.
[7] M. B. Halpern, Phys. Rev. D16 (1977), 1798; M. B. Halpern, Phys. Rev. D19 (1979), 517.
[8] Y. Kazama and R. Savit, Phys. Rev. D21 (1980), 2916; G. Chamlers and W. Siegel, hep-th/9712191.
[9] R. Anishetty and H. S. Sharatchandra, Phys. Rev. Lett. 65 (1990), 813; R. Anishetty, S. Cheluveraja, H. S. Sharatchandra and M. Mathur, Phys. Lett. B314 (1993), 387.
[10] B. Gnanapragasam and H. S. Sharatchandra, Phys. Rev. D45 (1992), R1010; R. Anishetty, Pushan Majumdar and H. S. Sharatchandra, Phys. Lett. B478 (2000), 373.
[11] R. Anishetty, Phys. Rev. D44 (1991), 1895; F. A. Lunev, Phys. Lett. B295 (1992), 99; F. A. Lunev, Mod. Phys. Lett. A9 (1994), 2281; F. A. Lunev, J. Math. Phys. 37 (1996), 5351; M. Bauer, D. Z. Freedman and P. E. Haagensen, Nucl. Phys. B428 (1994), 147; P. E. Haagensen and K. Johnson, Nucl. Phys. B439 (1995), 597.
[12] Pushan Majumdar and H. S. Sharatchandra, Phys.Rev. D58 (1998), 067702.
[13] E. T. Newman and C. Rovelli, Phys. Rev. Lett. 69 (1992), 1300.
[14] T. T. Wu and C. N. Yang, Phys. Rev. D12 (1975), 3845.
[15] D. Z. Freedman and R. R. Khuri, Phys. Lett. B329 (1994), 263.
[16] Pushan Majumdar and H. S. Sharatchandra, imsc/98/04/14, hep-th/9804091 (1998).
[17] A. A. Belavin, A. M. Polyakov, A. S. Shvarts, Yu. S. Tyupkin Phys. Lett. 59B (1975), 85.
[18] M. Baker, J. S. Ball and F. Zachariasen, Phys. Rep. 209 (1991), 73.