Measures of imaginarity and quantum state order

Qiang Chen\textsuperscript{1}, Ting Gao\textsuperscript{1*}, and Fengli Yan\textsuperscript{2*}

\textsuperscript{1}School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China;\textsuperscript{2}College of Physics, Hebei Key Laboratory of Photophysics Research and Application, Hebei Normal University, Shijiazhuang 050024, China

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Complex numbers are widely used in classical and quantum physics. Further, they play an important role in describing quantum systems and their dynamical behaviors. Herein, we propose several measures of the imaginarity of quantum states based on $l_1$ norm and convex functions in the framework of resource theory. Further, we investigate the quantum state order after a quantum system passes through a real channel. Rigorous proof shows that these proposed measures possess all the desirable properties for a measure of imaginarity. The connection between the measure of imaginarity based on the $l_1$ norm and the measure of imaginarity based on relative entropy is derived. Moreover, we demonstrate that the $l_1$ norm-based and the relative entropy-based measures of imaginarity are of the same order for qubit quantum states. Further we discuss the influences of the bit flip channel, phase damping channel, and amplitude flip channel on single qubit state order.

\textbf{measure, imaginarity, quantum state order}

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\textbf{1 Introduction}

Quantum resource theory provides a method for exploring the properties of quantum systems [1, 2]. In this theory, the resource of the quantum system is quantified by an operational method and the information processing tasks that can be realized are determined by the resource consumed. For example, in the resource theory of entanglement, the quantization of entanglement [3-7] and a series of applications of entanglement, such as quantum key distribution [8-14], quantum teleportation [15, 16], quantum direct communication [17-21], and quantum secret sharing [22, 23], have been provided. Recently, researchers have proposed many resource theories, such as resource theories of coherence [24-28], asymmetry [29], quantum thermodynamics [30], nonlocality [31], superposition [32], and incompatibility [33]. In addition, they have developed applicable quantities in the mathematical framework of resource theory [34].

One feature of quantum mechanics is the use of imaginary numbers. While imaginary numbers are used to describe the oscillatory motion in classical physics, they play a crucial role in quantum mechanics as well because the wave functions of the quantum system involve complex numbers [35]. For example, the polarization density matrix of a single photon in the $\{|H\rangle, |V\rangle\}$ basis, where $|H\rangle$ and $|V\rangle$ express the horizontal polarization, and vertical polarization, respectively. Further, the imaginary numbers in the density matrix...
cause the rotation of the electric-field vectors. Based on this phenomenon, Hickey and Gour [36] proposed the imaginariness resource theory. In this theory, the density matrix with imaginary elements is defined as a resource state and otherwise as a free state. Hickey and Gour [36] also defined the largest class of free operations. For the special physical constraints, some free operations are obtained, and then the theoretical framework of imaginarity resource is established. In this framework, several measures of imaginarity are given, and a state conversion condition for the pure states of a single qubit is discussed. Furthermore, in 2021, Wu et al. [37, 38] proposed a robustness measure of imaginarity and gave the transformation condition of states of a single qubit under free operation.

Herein, we present several new measures of imaginarity in the framework of resource theory and investigate the influence of quantum channels on the quantum state order for a single-qubit. The remainder of this paper is organized as follows. In sect. 2, we review some concepts, including the real states, free operations, and measures of imaginarity. In sect. 3, we mainly investigate whether the newly proposed measures of imaginarity based on $l_p$ norm, $p$-norm, and convex roof extended are good measures in the framework of the resource theory. The relation between these measures is also studied. The influence of quantum channels on the quantum state order for a single qubit is discussed in sect. 4.

3 Measures of imaginarity

Let us begin to discuss the quantization of imaginarity, which plays a very important role in determining the resources of a given quantum state.

Two measures of the imaginarity of the quantum state $\rho$ have been proposed in ref. [36]. They are the measure of imaginarity based on the 1-norm,

$$M(\rho) = \min_{\sigma \in \mathcal{F}} \|\rho - \sigma\|_1 = \frac{1}{2} \|\rho - \rho^T\|_1,$$

where $\rho^T$ denotes the transposition of the density matrix $\rho$, $\|A\|_1 = \text{Tr}(A^\dagger A)^{1/2}$ is the 1-norm of matrix $A$ [24], and the robustness of imaginarity

$$R(\rho) = \min_{\sigma \in \mathcal{D}(\mathcal{H})} \left\{ s \geq 0 : \frac{s \sigma + \rho}{1 + s} \in \mathcal{F} \right\}. $$

The geometric measure of imaginarity for pure states $|\psi\rangle$ is [37]

$$M_g(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \mathcal{F}} |\langle \phi |\psi \rangle|^2. $$

Next, we discuss several important distance-based imaginarity functions. Consider the function constructed based on the $l_p$ norm. The $l_p$ norm of a matrix $A$ [24] is defined as:

$$\|A\|_{l_p} = \left( \sum_{ij} |A_{ij}|^p \right)^{1/p}. $$

Specially we can define the function based on the $l_1$ norm as:

$$M_{l_1}(\rho) = \min_{\sigma \in \mathcal{F}} \|\rho - \sigma\|_{l_1}, $$

where $\rho$ is an arbitrary quantum state, $\sigma \in \mathcal{F}$ is the real quantum state. Then, one can obtain the following result.

**Theorem 1** $M_{l_1}(\rho) = \sum_{ij} |\text{Im}(\rho_{ij})|$, and $M_{l_1}(\rho)$ is a measure of imaginarity for free operations being all real operations within complete positivity trace-preserving (CPTP)
quantum operations, where \(\text{Im}(\rho_{ij})\) represents the imaginary part of the matrix element \(\rho_{ij}\).

**Proof** First, we prove \(M_1(\rho) = \sum_{i\neq j} |\text{Im}(\rho_{ij})|\). Obviously, each quantum state \(\rho = (\rho_{ij})\) in a \(d\)-dimensional Hilbert space can be written as: \(\rho = (\rho_{ij}) = (a_{ij} + ib_{ij})\), where \(a_{ij}, b_{ij}\) are real numbers, and when \(i = j\), then \(b_{ij} = 0\) holds. The real state \(\sigma = (\sigma_{ij}) = (c_{ij})\) with \(c_{ij}\) being real number. Hence,

\[
\|\rho - \sigma\|_1 = |(a_{11} - c_{11})| + |(a_{22} - c_{22})| + \cdots + |(a_{dd} - c_{dd})| \\
+ 2|a_{12} - c_{12}| + b_{12}| + 2|a_{13} - c_{13}| + b_{13}| + \cdots + 2|a_{dd} - c_{dd}| + b_{dd}|
\]

\[
= \sum_{ij} \sqrt{(a_{ij} - c_{ij})^2 + b_{ij}^2}.
\] (8)

Clearly, the minimum of \(\|\rho - \sigma\|_1\) occurs at \(c_{ij} = a_{ij}\). That is, when \(\sigma = \text{Re}(\rho)\), one gets

\[
M_1(\rho) = \sum_{i\neq j} |\text{Im}(\rho_{ij})|.
\] (9)

Here \(\text{Re}(\rho)\) stands for the real part of \(\rho\). This means that \(\text{Re}(\rho)\) is the closest real state of \(\rho\).

Now, we demonstrate that the function \(M_1(\rho)\) is a measure of the imaginarity of the quantum state \(\rho\).

Obviously, for an arbitrary quantum state \(\rho\), we have

\[
M_1(\rho) = \sum_{i\neq j} |\text{Im}(\rho_{ij})| \geq 0.
\] (10)

For a real quantum state \(\rho\) we can easily derive \(M_1(\rho) = 0\) by eq. (10).

When the function \(M_1(\rho) = 0\), one has

\[
|\text{Im}(\rho_{ij})| = 0.
\] (11)

This implies that the matrix elements of the quantum state \(\rho\) are real numbers. Hence, the quantum state \(\rho\) is real.

Further, \(M_1(\rho)\) is monotonic under an arbitrary real operation within the CPTP.

Assume \(\epsilon\) is the real operation within the CPTP, \(\rho\) and \(\sigma\) are two density operators, according to the definition of \(l_1\) norm [24], we have

\[
\|\epsilon(\rho) - \epsilon(\sigma)\|_1 \leq \|\rho - \sigma\|_1.
\] (12)

Evidently, a quantum state \(\rho\) can be written as \(\rho = \rho_R + i\rho_I\), where \(\rho_R = \frac{1}{2}(\rho + \rho^T), \rho_I = \frac{1}{2}(\rho - \rho^T)\). It is not difficult to observe that \(\rho_R\) is real symmetric, \(\rho_I\) is real antisymmetric, and

\[
\text{Tr} \rho_R = \text{Tr} \left[ \frac{1}{2}(\rho + \rho^T) \right] = \frac{1}{2} \left[ \text{Tr}(\rho) + \text{Tr}(\rho^T) \right] = 1,
\]

\[
\langle \psi | \rho_R | \psi \rangle = \frac{1}{2} \langle \psi | \rho | \psi \rangle + \frac{1}{2} | \langle \psi | \rho^T | \psi \rangle | \geq 0.
\] (14)

Therefore \(\rho_R\) is the real density matrix. According to the \(l_1\) norm of the matrix is contracted under CPTP, one can obtain

\[
M_1(\epsilon(\rho)) = \inf_{\epsilon \in \mathcal{F}} \|\epsilon(\rho) - \sigma\|_1
\]

\[
\leq \|\epsilon(\rho) - \epsilon(\rho_R)\|_1
\]

\[
= \|\epsilon(\rho_R + i\rho_I) - \epsilon(\rho_R)\|_1
\]

\[
\leq \|\rho - \rho_R\|_1
\]

\[
= \|i\rho_I\|_1 = M_1(\rho_R).
\] (15)

Thus, we conclude that the function \(M_1(\rho)\) is a measure of imaginarity for free operations, which are all real operations within CPTP quantum operations. The proof of Theorem 1 has been completed.

However, for the functions induced by the \(l_p\) norm or \(p\)-norm [25], we have the following conclusion.

**Theorem 2** For any quantum state \(\rho\) in a \(d\)-dimensional Hilbert space, when \(p > 1\), both the function \(M_{1/p}(\rho \otimes \frac{1}{d})\) and function \(M_{p}(\rho \otimes \frac{1}{d})\) induced by the \(l_p\) norm and \(p\)-norm, respectively, do not satisfy monotonicity under all real operations within CPTP mappings.

**Proof** It is not difficult to observe that for a particular real state

\[
\rho_1 = |0\rangle\langle 0|,
\]

there exists a real operation \(\Lambda\) which transforms the quantum state

\[
\rho_2 = \frac{I}{d}
\]

to the quantum state \(\rho_1\). Here \(I\) is the \(d\)-dimensional identity operator, and the Kraus operators of the real operation \(\Lambda\) are \(\{K_i = |i\rangle\langle i - 1|\}\), and \(\{K_i\}\) satisfy \(\sum_{i=1}^d K_i^\dagger K_i = I\).

We choose the real operation \(\Lambda\), whose Kraus operators are \(\{\bar{K}_i = I \otimes K_i\}\). Clearly, \(\{\bar{K}_i\}\) satisfy \(\sum_{i=1}^d \bar{K}_i^\dagger \bar{K}_i = I',\) where \(I'\) is the identity operator of the direct product space. Then, we have

\[
M_{1/p}(\Lambda(\rho \otimes \frac{1}{d})) = M_{1/p}(\rho \otimes |0\rangle\langle 0|) = \\
\left( \sum_{ij} |\text{Im}(\rho \otimes |0\rangle\langle 0|)_{ij}|^p \right)^{1/p}
\]

\[
= M_{1/p}(\rho) > M_{1/p}(\rho \otimes \frac{1}{d}).
\] (18)
Here $\text{Im}(\rho \otimes |0\rangle\langle 0|)_M$ represents the imaginary part of the matrix element $(\rho \otimes |0\rangle\langle 0|)_M$. The above inequality takes advantage of the following results:

$$M_p(\rho \otimes \frac{I}{d}) = \left\{ \sum_{ij} \text{Im}(\rho \otimes \frac{I}{d})_{ij}^p \right\}^{1/p}$$

$$= d^{1/p} M_p(\rho) < M_p(\rho).$$

(19)

Obviously, eq. (18) indicates that when $p > 1$, function $M_p$ does not satisfy the condition $M_p(\varepsilon(\rho)) \leq M_p(\rho)$ for arbitrary free operation $\varepsilon$ and quantum state $\rho$. That is, when $p > 1$, the function $M_p$ cannot be regarded as a measure of imaginarity.

For a matrix $A$, its $p$-norm $\|A\|_p$ is defined as $\{\text{Tr}(A^*A)^{\frac{1}{2}}\}^\frac{1}{p}$. When $p > 1$, for $p$-norm induced function

$$M_p(\rho) = \min_{\sigma \in F} \|\rho - \sigma\|_p,$$

we have

$$M_p(\tilde{A}(\rho \otimes \frac{I}{d})) = M_p(\rho \otimes |0\rangle\langle 0|)$$

$$= M_p(\rho) > M_p\left( \rho \otimes \frac{I}{d} \right).$$

(21)

The inequality above can be derived from

$$M_p(\rho \otimes \frac{I}{d}) \leq \min_{\sigma \in F} \|\rho \otimes \frac{I}{d} - \sigma \otimes \frac{I}{d}\|_p$$

$$= \min_{\sigma \in F} \|(\rho - \sigma) \otimes \frac{I}{d}\|_p$$

$$= \min_{\sigma \in F} \|\rho - \sigma\|_p \frac{1}{d}$$

$$= M_p(\rho) \frac{1}{d}$$

$$< M_p(\rho).$$

(22)

Thus we have demonstrated that when $p > 1$, the function $M_p$ violates monotonicity under all real operations within CPTP mappings. Hence Theorem 2 is true.

Let us discuss the measure of imaginarity based on relative entropy. In the resource theory of coherence, coherence measure $C(\rho)$ based on relative entropy satisfies the axiomatic condition of the coherence measure [2], and its expression being similar to coherence distillation [26] is

$$C(\rho) = S(\Delta'(\rho)) - S(\rho),$$

(23)

where $\Delta'$ is the decoherence operation and $S(\rho)$ stands for Von Neumann entropy of the quantum state $\rho$.

Similar to the resource theory of coherence, here we need an operator $\Delta$.

**Definition 1** The mathematical operator $\Delta$ is defined by

$$\Delta(\rho) = \frac{1}{2}(\rho + \rho^T),$$

(24)

where $\rho$ is any quantum state.

Evidently, $\Delta$ is just a simple mathematical operator, rather than a free operation. The relationship between $\Delta(\rho)$ and quantum real operation satisfying the physically consistent condition [36] can be stated as the following Theorem 3.

Each of the possible physical constraints corresponds to the same set of free operations, forming an equivalence class of operations that we call physically consistent [36]. The physically realizable operation is a physical constraint. A resource non-generating operation $E$ is said to be physically realizable [36] if it admits a free dilation. That is, there exists a Hilbert space $\mathcal{H}_E$, a free state $|0\rangle\langle 0|_E$, and a free unitary $U_{AE}$ acting on the joint state space $\mathcal{H}_A \otimes \mathcal{H}_E$ such that $E(\rho) = U_{AE}[U_{AE}(\rho \otimes |0\rangle\langle 0|_E)U_{AE}^T]$ for any density operator $\rho \in \mathcal{H}$. A quantum channel $E$ is physically consistent [36] if and only if it commutes with the transpose map. That is, $E(\rho)^T = E(\rho^T)$ for all $\mathcal{D}(\mathcal{H})$.

**Theorem 3** Let $\varepsilon$ be a real operation within the CPTP. If $\varepsilon$ satisfies the condition of physical consistency. Then, for any quantum state $\rho$, we have

$$\varepsilon(\Delta(\rho)) = \Delta(\varepsilon(\rho)).$$

(25)

**Proof** For any quantum state $\rho$, because the real operation $\varepsilon$ is linear, hence one has

$$\varepsilon[\Delta(\rho)] = \varepsilon\left[\frac{1}{2}(\rho + \rho^T)\right] = \frac{1}{2}[\varepsilon(\rho) + \varepsilon(\rho^T)]$$

$$\frac{1}{2}[\varepsilon(\rho) + \varepsilon(\rho^T)] = \Delta(\varepsilon(\rho)),$$

(26)

where the third equality of the above equation is obtained from the condition of physical consistency [36]. Therefore Theorem 3 holds.

The quantum relative entropy between the quantum states $\rho$ and $\sigma$ is usually taken as [39]:

$$S(\rho||\sigma) = \text{Tr}[\rho \log_2 \rho] - \text{Tr}[\rho \log_2 \sigma].$$

(27)

The relative entropy of the imaginarity of a quantum state $\rho$ is defined as [40]:

$$M_l(\rho) = \min_{\sigma \in F} S(\rho||\sigma).$$

(28)

Then, the relative entropy function $M_l(\rho)$ can be reexpressed as [40]:

$$M_l(\rho) = S(\Delta(\rho)) - S(\rho).$$

(29)

**Theorem 4** For any qubit pure state $|\psi\rangle$, the measure of imaginarity based on the relative entropy satisfies

$$M_l(|\psi\rangle) \leq M_l(|\psi\rangle),$$

(30)
the equality holds if \( M_I(|\psi\rangle) = 1 \).

**Proof** Choose a qubit pure state \( |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \), where \( \alpha, \beta \) are complex numbers and satisfy \( |\alpha|^2 + |\beta|^2 = 1 \). Assume that \( |\alpha|^2 > |\beta|^2 \). Then \( |\alpha|^2 = 1 - |\beta|^2 > x (1 - x) \log_2 (1 - x) \). It is not difficult to obtain

\[
M_I(|\psi\rangle) = H(\lambda_I),
\]

where

\[
\lambda_I = \frac{1 + \sqrt{1 - 4(cf - de)^2}}{2}.
\]

According to \( H(x) \leq 2\sqrt{x(1-x)} \) [41], we have

\[
M_I(|\psi\rangle) = H(\lambda_I)
\]

\[
\leq 2\sqrt{\lambda_I(1 - \lambda_I)}
\]

\[
= 2\sqrt{\frac{1 + \sqrt{1 - 4(cf - de)^2}}{2} \times \frac{1 - \sqrt{1 - 4(cf - de)^2}}{2}}
\]

\[
= 2\sqrt{\frac{1 - (1 - 4(cf - de)^2)}{4}}
\]

\[
= 2\sqrt{(cf - de)^2} = 2|cf - de| = M_I(|\psi\rangle).
\]

Thus, we have proved that for a qubit pure state \( |\psi\rangle \), \( M_I(|\psi\rangle) \leq M_I(|\psi\rangle) \) is true.

Clearly, when \( M_I(|\psi\rangle) = 1 \), one has \( |cf - de| = \frac{1}{2} \). Therefore, \( \lambda_I = \frac{1}{2} \), which induces \( 1 = H(\lambda_I) = M_I(|\psi\rangle) \). This fact shows that \( M_I(|\psi\rangle) = M_I(|\psi\rangle) \), if \( M_I(|\psi\rangle) = 1 \). So, Theorem 4 holds.

In addition to the above measures of imaginarity, there exist other measures. Next, based on the measure of the imaginarity of pure states, we give a measure of the imaginarity of mixed quantum states by convex roof extended [42].

**Proposition 1** If \( M(|\psi\rangle) \) is a measure of the imaginarity of pure state \( |\psi\rangle \), then the convex roof extended

\[
M(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i M(|\psi_i\rangle)
\]

is a measure of the imaginarity of mixed state \( \rho \) if \( M(\rho) \) is a convex function. Here \( \{p_i, |\psi_i\rangle\} \) is the decomposition of the quantum state \( \rho \), and \( \{p_i\} \) is a probability distribution, namely, \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \).

**Proof** According to the definition of the function \( M(\rho) \), when \( M(\rho) = 0 \), obviously we can obtain that the quantum state \( \rho \) is a real one. Conversely, if \( \rho \) is a real state, there is a real decomposition \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \) such that \( M(|\psi_i\rangle) = 0 \). So, \( M(\rho) = 0 \).

Now we prove that the function \( M(\rho) \) is monotonic.

For any quantum state \( \rho \), we take the best decomposition of the quantum state \( \rho \), expressed as: \( \rho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \), then one has

\[
M(\rho) = \sum_k p_k M(|\psi_k\rangle).
\]

Assume \( \{K_j\} \) is the set of Kraus operators of a real operation, \( c_{jk} = \langle \psi_k| K_j^T K_j \psi_k \rangle \), then \( \rho = \sum_j q_j M(\rho) = \sum_j q_j M \left( \frac{K_j \rho K_j^T}{q_j} \right) \)

\[
\leq \sum_j q_j M \left( \frac{p_k c_{jk}}{q_j} \frac{K_j |\psi_k\rangle \langle \psi_k| K_j^T}{c_{jk}} \right)
\]

\[
\leq \sum_j q_j M(|\psi_k\rangle) = M(\rho),
\]

where the first inequality is true because \( M(\rho) \) is a convex function. Thus, we demonstrate that the function \( M(\rho) \) is monotonic. The proof is complete.

### 4 Influence of quantum channel on quantum state order

In this section, we mainly investigate the ordering of quantum states based on the measure of imaginarity after passing through a real channel. The main real channels involved are the amplitude damping channel, phase flip channel, and bit flip channel. We restate the definition of the ordering of quantum states as follows [43-45].

**Definition 2** Let \( M_A \) and \( M_B \) be two measures of imaginarity. For two arbitrary quantum states \( \rho_1 \) and \( \rho_2 \), if the following relationship

\[
M_A(\rho_1) \leq M_A(\rho_2) \iff M_B(\rho_1) \leq M_B(\rho_2)
\]

is true, then the measures \( M_A \) and \( M_B \) are said to be of the same order; if the above relation is not satisfied, the measures \( M_A \) and \( M_B \) are considered to be of different order.

We only discuss the ordering of quantum states in the case of a single qubit. On a fixed reference basis, the state of a single qubit can always be written as:

\[
\rho = \frac{1}{2} \begin{pmatrix}
1 + r \cdot \sigma & \frac{t(n_x - i n_y)}{2} \\
\frac{t(n_x + i n_y)}{2} & 1 - t \cdot \sigma
\end{pmatrix},
\]

where \( \sigma \) is the Pauli vector, \( t = ||r|| \leq 1 \), \( n = (n_x, n_y, n_z) = \frac{1}{t} r \) is a unitary vector.
It is easy to obtain the measures of the imaginarity of the quantum state $\rho$
\[ M_i(\rho) = t|n_i|, \]  
\[ M_i(\rho) = H \left( \frac{1}{2} + \frac{t \sqrt{1 - n^2_i}}{2} \right) - H \left( \frac{1}{2} \right). \]  
\[ \text{(39)} \]
\[ \text{(40)} \]

Now let’s consider the monotonicity of these functions. One can easily obtain
\[ \frac{\partial M_i(\rho)}{\partial |n_i|} = \frac{t}{2} \left( \frac{1 - t \sqrt{1 - n^2_i}}{1 + t \sqrt{1 - n^2_i}} \right) \geq 0, \]
\[ \text{(41)} \]
\[ \frac{\partial M_i(\rho)}{\partial t} = \frac{1}{2} \log_2 \frac{1 + t}{1 - t} + \frac{\sqrt{1 - n^2_i}}{2} \log_2 \frac{1 - t \sqrt{1 - n^2_i}}{1 + t \sqrt{1 - n^2_i}} \geq \frac{1}{2} \log_2 \frac{1 + t}{1 - t} + \frac{1}{2} \log_2 \frac{1 - t}{1 + t} \geq 0. \]
\[ \text{(42)} \]
\[ \text{(43)} \]

Therefore, $M_i(\rho)$ is monotonically increasing about the independent variables $|n_i|$ and $t$. Evidently, $M_i(\rho)$ is also monotonically increasing about the independent variables $|n_i|$ and $t$. Thus, we reach the following conclusion.

**Proposition 2** The measure $M_i(\rho)$ and the measure $M_i(\rho)$ are of the same order for qubit quantum states.

A quantum channel can change the quantum state. Furthermore, it can affect the quantum state order. For a measure of quantum states, we define the influence of the quantum channel on the quantum state order as follows.

**Definition 3** Let $M$ be a measure of imaginarity and $\varepsilon$ be a quantum channel. For two arbitrary quantum states $\rho_1$ and $\rho_2$, if
\[ M(\rho_1) \leq M(\rho_2) \Leftrightarrow M(\varepsilon(\rho_1)) \leq M(\varepsilon(\rho_2)) \]
\[ \text{(44)} \]
holds, then we say that the quantum channel $\varepsilon$ does not change the quantum state order; otherwise, we say the quantum state order is changed by the quantum channel $\varepsilon$.

Next, we discuss the influence of quantum channels on the ordering of qubit states when one chooses a measure of imaginarity. First, we study the case of the bit flip channel $\varepsilon$ and the imaginarity measure $M_i(\rho)$. Here, the quantum state of the qubit is stated as eq. (38), the bit flip channel $\varepsilon$ is expressed by the real Kraus operators $[K_0 = \sqrt{p}I, K_1 = \sqrt{1 - p}\sigma_z]$, where $p \in [0, 1]$, $\sigma_z$ is the Pauli operator.

**Proposition 3** Suppose one chooses $M_i(\rho)$ as the measure of imaginarity, the quantum state order does not change after a single qubit goes through a bit flip channel.

**Proof** The state of the qubit system after passing through the bit flip channel $\varepsilon$ is
\[ \varepsilon(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger \]
\[ = \left( \frac{1 + m_x(2p - 1)}{2} \right) m_x - i m_y(2p - 1) \left( \frac{1 - m_x(2p - 1)}{2} \right), \]
\[ \text{(45)} \]
where $\rho$ is expressed by eq. (38).

It is easy to derive that
\[ M_i(\varepsilon(\rho)) = H \left( \frac{1 + t \sqrt{n^2_i + (2p - 1)^2 n^2_z}}{2} \right) - H \left( \frac{1 + t \sqrt{n^2_i + (2p - 1)^2 (1 - n^2_z)}}{2} \right). \]
\[ \text{(46)} \]

Obviously, $M_i(\varepsilon(\rho))$ contains four parameters $t, p, n_x, n_z$. We can easily get
\[ \frac{\partial M_i(\varepsilon(\rho))}{\partial |n_i|} = \frac{t}{2} \left( \frac{2p - 1}{n^2_i + (2p - 1)^2 n^2_z} \right) \log_2 \frac{1 - t \sqrt{n^2_i + (2p - 1)^2 n^2_z}}{1 + t \sqrt{n^2_i + (2p - 1)^2 n^2_z}} \leq 0. \]
\[ \text{(47)} \]

Using the monotonically increasing properties of
\[ f(x) = \frac{1}{x} \log_2 \frac{1 + tx}{1 - tx}, \quad (0 \leq x \leq 1), \]
\[ \text{(48)} \]
we have
\[ \frac{\partial M_i(\varepsilon(\rho))}{\partial n_i} = \frac{t}{2} \left( \frac{|n_i|}{\sqrt{n^2_i + (2p - 1)^2 n^2_z}} \right) \cdot \log_2 \frac{1 - t \sqrt{n^2_i + (2p - 1)^2 n^2_z}}{1 + t \sqrt{n^2_i + (2p - 1)^2 n^2_z}} - \frac{t}{2} \left( \frac{|n_i|[1 - (2p - 1)^2]}{\sqrt{n^2_i + (2p - 1)^2 n^2_z} + (2p - 1)^2 n^2_z} \right) \cdot \log_2 \frac{1 - t \sqrt{n^2_i + (2p - 1)^2 n^2_z} + (2p - 1)^2 n^2_z}{1 + t \sqrt{n^2_i + (2p - 1)^2 n^2_z} + (2p - 1)^2 n^2_z} \]
\[ \text{(49)} \]
\[ \geq \frac{t}{2} \cdot \frac{|n_1(2p-1)^2|}{\sqrt{n_1^2 + (2p-1)^2n_2^2}} \cdot \log_2 \left( 1 - t \sqrt{\frac{n_1^2 + (2p-1)^2n_2^2}{1 + t \sqrt{n_1^2 + (2p-1)^2n_2^2}}} \right). \quad (49) \]

So when \( n_1 \leq 0 \), we have \( \frac{\partial M_i(\varepsilon(\rho))}{\partial |n_1|} \geq 0 \). Because \( M_i(\varepsilon(\rho)) \) is an even function of the variable \( n_1 \), we can conclude that \( M_i(\varepsilon(\rho)) \) is a monotonic decreasing function of variable \( |n_1| \), i.e.,

\[ \frac{\partial M_i(\varepsilon(\rho))}{\partial |n_1|} \leq 0. \quad (50) \]

The partial derivative of \( M_i(\varepsilon(\rho)) \) with respect to \( t \) is

\[ \frac{\partial M_i(\varepsilon(\rho))}{\partial t} = \frac{\sqrt{n_1^2 + (2p-1)^2n_2^2}}{2} \cdot \log_2 \left( \frac{1 - t \sqrt{n_1^2 + (2p-1)^2n_2^2}}{1 + t \sqrt{n_1^2 + (2p-1)^2n_2^2}} \right) \]

\[ + \frac{\sqrt{n_1^2 + (2p-1)^2n_2^2}}{2} \cdot \log_2 \left( \frac{1 + t \sqrt{n_1^2 + (2p-1)^2n_2^2}}{1 - t \sqrt{n_1^2 + (2p-1)^2n_2^2}} \right) \geq 0. \quad (51) \]

Therefore, the measure \( M_i(\varepsilon(\rho)) \) is a monotonically decreasing function with respect to the variable \( |n_1|,|n_2| \), and a monotonically increasing function with respect to the variable \( i \).

On the other hand, we can obtain that

\[ \frac{\partial M_i(\rho)}{\partial |n_1|} = \frac{\partial M_i(\rho)}{\partial |n_2|} = \frac{\partial M_i(\rho)}{\partial |n_3|} = \frac{\partial M_i(\rho)}{\partial |n_4|} = \frac{\partial M_i(\rho)}{\partial |n_5|} = \frac{\partial M_i(\rho)}{\partial |n_6|} = \frac{\partial M_i(\rho)}{\partial |n_7|} = \frac{\partial M_i(\rho)}{\partial |n_8|}. \]

By using eq. (41), one gets

\[ \frac{\partial M_i(\rho)}{\partial |n_1|} \leq 0. \quad (53) \]

Similarly, we have

\[ \frac{\partial M_i(\rho)}{\partial |n_2|} \leq 0. \quad (54) \]

Combining eqs. (43), (47), (50), (51), (53), and (54), one arrives at that the quantum state order does not change after a single qubit goes through a bit flip channel. Thus, Proposition 3 is true.

**Proposition 4** Assume we choose \( M_i(\rho) \) as the measure of imaginarity, the quantum state order does not change after a single qubit goes through a bit flip channel.

**Proof** By using eq. (45) we have

\[ M_i(\varepsilon(\rho)) = t(2p-1)n_1. \quad (55) \]

Considering the above eqs. (55) and (39), it is not difficult to obtain that when we choose \( M_i(\rho) \) as the measure of imaginarity, the quantum state order does not change after a single qubit goes through a bit flip channel. This implies that Proposition 4 holds.

Now let us investigate the case when the imaginarity measure \( M_i(\rho) \) has been chosen, and the quantum channel is the phase flip channel \( \Lambda \). Here, the quantum state of the qubit is determined by the state eq. (38), the phase flip channel \( \Lambda \) is expressed by the real Kraus operators \( K_0 = \sqrt{p}1, K_1 = \sqrt{1-p}00, K_2 = \sqrt{1-p10}10, 0 \leq p \leq 1 \).

For this case, we can prove the following proposition.

**Proposition 5** Suppose we choose \( M_i(\rho) \) as the measure of imaginarity, the quantum state order does not change after a single qubit goes through a phase flip channel.

**Proof** After a qubit passes through a phase flip channel, the quantum state can be written as:

\[ \Lambda(\rho) = K_0\rho K_0^\dagger + K_1\rho K_1^\dagger + K_2\rho K_2^\dagger \]

\[ = \begin{pmatrix} 1 + m_z & i p(n_x - n_y) \\ t p(n_x + n_y) & 2 - m_z \end{pmatrix}. \quad (56) \]

One can easily deduce

\[ M_i(\Lambda(\rho)) = H \left( \frac{1 + t \sqrt{n_1^2 + p^2n_2^2}}{2} \right) \]

\[ - H \left( \frac{1 + t \sqrt{n_1^2 + p^2(1 - n_2^2)}}{2} \right). \quad (57) \]

The partial derivatives are

\[ \frac{\partial M_i(\Lambda(\rho))}{\partial t} = \frac{\sqrt{p^2n_1^2 + n_2^2}}{2} \cdot \log_2 \left( \frac{1 - t \sqrt{n_1^2 + p^2n_2^2}}{1 + t \sqrt{n_1^2 + p^2n_2^2}} \right) \]

\[ + \frac{\sqrt{n_1^2 + p^2(1 - n_2^2)}}{2} \cdot \log_2 \left( \frac{1 + t \sqrt{n_1^2 + p^2(1 - n_2^2)}}{1 - t \sqrt{n_1^2 + p^2(1 - n_2^2)}} \right) \geq 0; \quad (58) \]
\[
\frac{\partial M_t(\langle \rho \rangle)}{\partial |n_z|} = t \cdot \frac{n_z |1-p^2|}{\sqrt{n_z^2 + p^2 n_z^2}} \log_2 \frac{1 - t \sqrt{n_z^2 + p^2 n_z^2}}{1 + t \sqrt{n_z^2 + p^2 n_z^2}} \\
\leq 0; \tag{59}
\]

\[
\frac{\partial M_t(\langle \rho \rangle)}{\partial |n_z|} \\
= \frac{t}{2} \cdot \frac{n_z}{\sqrt{n_z^2 + p^2 n_z^2}} \log_2 \frac{1 - t \sqrt{n_z^2 + p^2 n_z^2}}{1 + t \sqrt{n_z^2 + p^2 n_z^2}} \\
\leq \frac{t}{2} \cdot \frac{n_z |1-p^2|}{\sqrt{n_z^2 + p^2 n_z^2}} \log_2 \frac{1 - t \sqrt{n_z^2 + p^2 n_z^2}}{1 + t \sqrt{n_z^2 + p^2 n_z^2}} \\
= 0. \tag{60}
\]

Here the first inequality in eq. (60) comes from the fact that when \( t \) is fixed, \( \frac{1}{2} \log_2 \frac{1-x}{1+x} \) is monotonically decreasing function with respect to \( x \) and \( n_z^2 + n_z^2 \leq 1 \); the second inequality in eq. (60) is based on that when \( n_z, t \) are fixed, \( -\frac{1}{2} \cdot \frac{n_z |1-p^2|}{\sqrt{n_z^2 + p^2 n_z^2}} \log_2 \frac{1-t \sqrt{n_z^2 + p^2(n_z - n_z)}^2}{1+t \sqrt{n_z^2 + p^2(n_z - n_z)}^2} \) is monotonically decreasing function with respect to \( p^2 \).

By eqs. (43), (53), (54), (58)-(60), we obtain that if we choose \( M_t(\langle \rho \rangle) \) as the measure of imaginarity, then the quantum state order does not change after a single qubit goes through a phase flip channel. Thus, Proposition 5 is true.

**Proposition 6** Suppose we choose \( M_t(\langle \rho \rangle) \) as the measure of imaginarity, then the quantum state order does not change after a single qubit goes through a phase flip channel.

**Proof** Using eq. (56), we have

\[
M_t(\langle \rho \rangle) = t |p| |n_z|. \tag{61}
\]

Proposition 7 When qubit state \( \rho \) satisfies \( n_z \leq 0 \) if one chooses \( M_t(\langle \rho \rangle) \) as the measure of imaginarity, then the quantum state order does not change after a single qubit goes through an amplitude damping channel.

**Proof** For a qubit state stated by eq. (38), the amplitude damping channel leads it to

\[
\Gamma(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger
\]

\[
= \left( \frac{1 + t n_z + p(1 - n_z)}{2} \frac{\sqrt{1 - p(n_z - i n_z)}}{2} \right) \left( \frac{1}{2} \frac{\sqrt{1 - pt(n_z + i n_z)}}{2} \right).
\]

One can easily obtain a measure of imaginarity based on relative entropy:

\[
M_t(\Gamma(\rho))
= H \left( \frac{1 + \sqrt{[p + m_z(1 - p)]^2 + (1 - p)^2 n_z^2}}{2} \right) - H \left( \frac{1 + \sqrt{[p + m_z(1 - p)]^2 + (1 - p)^2 (1 - n_z^2)}}{2} \right).
\]

Therefore, we get the partial derivatives

\[
\frac{\partial M_t(\Gamma(\rho))}{\partial t} \nonumber
= \frac{(p + m_z(1 - p)) n_z(1 - p) + t(1 - p) n_z^2}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2} \\
\times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}}{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}} \\
+ \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 (1 - n_z^2)}}{2} \times \log_2 \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 (1 - n_z^2)}}{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 (1 - n_z^2)}},
\]

\[\geq 0; \tag{64}\]

\[
\frac{\partial M_t(\Gamma(\rho))}{\partial |n_z|} = \frac{|p + m_z(1 - p)| |n_z|}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}} \times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}}{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}} \leq 0; \tag{65}\]

\[
\frac{\partial M_t(\Gamma(\rho))}{\partial n_z} = \frac{(p + m_z(1 - p)) |1 - p|}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}} \times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}}{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)^2 n_z^2}}.
\]
\[
\begin{align*}
&+ \frac{[p + m_z(1 - p)](1 - p) - (1 - p)\sqrt{n_z}}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
&\times \log_2 \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}.
\end{align*}
\tag{66}
\]

Using the monotonically increasing properties of
\[
f(x) = \frac{1}{x} \log_2 \frac{1 + x}{1 - x}, \quad (0 \leq x \leq 1),
\]
and \(0 \leq n_z^2 \leq 1 - n_z^2\), then we have
\[
\frac{\partial M_I(\Gamma(\rho))}{\partial n_z} \geq \text{Min} \left\{ \frac{[p + m_z(1 - p)](1 - p)}{2 \sqrt{(p + m_z(1 - p))^2}} \times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2}}{1 + \sqrt{(p + m_z(1 - p))^2}} + \frac{[p + m_z(1 - p)](1 - p) - (1 - p)\sqrt{n_z}}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
\times \log_2 \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}},
\right. \tag{67}
\]
\[
\frac{\partial M_I(\Gamma(\rho))}{\partial n_z}
\]
\[
\geq A = \frac{-t(1 - p)}{2} \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2}}{1 + \sqrt{(p + m_z(1 - p))^2}} + \frac{t(1 - p)}{2} \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{\sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
\times \log_2 \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}},
\tag{68}
\]
\[
\frac{\partial M_I(\Gamma(\rho))}{\partial n_z}
\]
\[
\geq B = \frac{(1 - p)^2n_z}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
\times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}.
\tag{69}
\]

Let
\[
A = \frac{[p + m_z(1 - p)](1 - p)}{2 \sqrt{(p + m_z(1 - p))^2}} \\
\times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2}}{1 + \sqrt{(p + m_z(1 - p))^2}} + \frac{[p + m_z(1 - p)](1 - p) - (1 - p)\sqrt{n_z}}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
\times \log_2 \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}.
\tag{70}
\]
\[
B = \frac{(1 - p)^2n_z}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
\times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}.
\tag{71}
\]

Therefore, when \(p + m_z(1 - p) \geq 0\), we have \(A \geq B\); when \(p + m_z(1 - p) \leq 0\), we have \(A \leq B\).

In the situation \(p + m_z(1 - p) \leq 0\), one gets
\[
\frac{\partial M_I(\Gamma(\rho))}{\partial n_z}
\]
\[
\geq A = \frac{-t(1 - p)}{2} \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2}}{1 + \sqrt{(p + m_z(1 - p))^2}} + \frac{t(1 - p)}{2} \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{\sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
\times \log_2 \frac{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}},
\tag{72}
\]
\[
\frac{\partial M_I(\Gamma(\rho))}{\partial n_z}
\]
\[
\geq B = \frac{(1 - p)^2n_z}{2 \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}} \\
\times \log_2 \frac{1 - \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}{1 + \sqrt{(p + m_z(1 - p))^2 + (1 - p)\sqrt{2(1 - n_z^2)}}}.
\tag{73}
\]

When \(n_z \leq 0\) and \(p + m_z(1 - p) \geq 0\), we have \(\frac{\partial M_I(\Gamma(\rho))}{\partial n_z} \geq 0\), that is, if \(n_z, t, p\) are fixed and satisfy \(n_z \leq 0\) and \(p + m_z(1 - p) \geq 0\), then the function \(M_I(\Gamma(\rho))\) monotonically increases with respect to the variables \(n_z\).

Combining eqs. (43), (53), (54), (64), (65), (72), and (73), we determine that when qubit state \(\rho\) satisfies \(n_z \leq 0\) if one chooses \(M_I(\rho)\) as the measure of imaginarity, then the quantum state order does not change after a single qubit goes through an amplitude damping channel. Thus, we have validated Proposition 7.

**Proposition 8** When we take \(M_I(\rho)\) as the measure of imaginarity, the quantum state order does not change after a single qubit goes through an amplitude damping channel.

**Proof** Using eq. (62), we deduce the measure of imaginarity
\[
M_I(\Gamma(\rho)) = t \sqrt{1 - p|n_z|},
\tag{74}
\]
Using eq. (39) and above eq. (74) we obtain that Proposition 8 is true.

5 Conclusion

We investigate the measures of imaginarity in the framework of resource theory and the quantum state order after a quantum system passes through a real channel. We define functions based on the $l_1$ norm and the convex roof extended and show that they are the measures of imaginarity. The relations between the relative entropy of imaginarity $M_l(\rho)$ and the imaginarity measure $M_1(\rho)$ based on the $l_1$ norm for the single qubit pure state $\rho$ are obtained. We also prove that the functions based on the $l_p$ ($p \geq 2$) norm and $p$-norm ($p \geq 2$) are not measures of imaginarity. Moreover, we demonstrate that the measure $M_l(\rho)$ and the measure $M_1(\rho)$ are of the same order for qubit quantum states and discuss the influences of the bit flip channel, phase damping channel and amplitude flip channel on single qubit state order.

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