On Lebesgue measure of integral self-affine sets

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Abstract

Let $A$ be an expanding integer $n \times n$ matrix and $D$ be a finite subset of $\mathbb{Z}^n$. The self-affine set $T = T(A, D)$ is the unique compact set satisfying the equality $A(T) = \bigcup_{d \in D} (T + d)$. We present an effective algorithm to compute the Lebesgue measure of the self-affine set $T$, the measure of the intersection $T \cap (T + u)$ for $u \in \mathbb{Z}^n$, and the measure of the intersection of self-affine sets $T(A, D_1) \cap T(A, D_2)$ for different sets $D_1, D_2 \subset \mathbb{Z}^n$.

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Let $A$ be an expanding integer $n \times n$ matrix, where expanding means that every eigenvalue has modulus greater than 1, and let $D$ be a finite subset of $\mathbb{Z}^n$. There exists a unique nonempty compact set $T = T(A, D) \subset \mathbb{R}^n$, called (integral) self-affine set, satisfying $A(T) = \bigcup_{d \in D} (T + d)$. It can be given explicitly by

$$T = \left\{ \sum_{k=1}^{\infty} A^{-k} d_k : d_k \in D \right\}.$$ 

The self-affine set $T$ with $|D| = |\text{det} A|$ and of positive Lebesgue measure is called a self-affine tile. Self-affine tiles were intensively studied for the last two decades in the context of self-replicating tilings, radix systems, Haar-type wavelets, etc.

The question of how to find the Lebesgue measure $\lambda(T)$ of the self-affine set $T$ was considered by Lagarias and Wang in [7], where some partial cases were studied. In particular, it was shown that self-affine tiles have integer Lebesgue measure. He, Lau and Rao [4] reduced the problem of finding $\lambda(T)$ to the case when $D$ is a coset transversal for $\mathbb{Z}^n/A(\mathbb{Z}^n)$. The last case was treated by Gabardo and Yu [3] and in more general settings

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by Bondarenko and Kravchenko [1]. The positivity of the Lebesgue measure of self-affine sets was also studied in [3, 4].

In this note, we present a simple method to compute the Lebesgue measure \( \lambda(T) \) of the self-affine set \( T \). We construct a finite labeled graph (automaton) and show that \( \lambda(T) \) is equal to the uniform Bernoulli measure of the left-infinite sequences which can be read along paths in this graph. Similar graphs when \( D \) is a coset transversal were constructed in [3, 10] and other papers. In addition this method allows to find the measure of the intersection \( T \cap (T + u) \) for \( u \in \mathbb{Z}^n \), and the measure of the intersection of self-affine sets \( T(A, D_1) \cap T(A, D_2) \) for different sets \( D_1, D_2 \subseteq \mathbb{Z}^n \). Our construction seems to be very natural and actually works for any contracting self-similar group action (here the self-affine sets correspond to the self-similar actions of \( \mathbb{Z}^n \), see [9, Section 6.2] and [1]).

We proceed as follows. If the set \( D \) does not contain all coset representatives of \( \mathbb{Z}^n / A(\mathbb{Z}^n) \), we extend it to the set \( K \supset D \) which does, and choose a coset transversal \( C \subseteq K \).

Construct a directed labeled graph (automaton) \( \Gamma = \Gamma(A, K) \) with the set of vertices \( \mathbb{Z}^n \), and we put a directed edge from \( u \) to \( v \) for \( u, v \in \mathbb{Z}^n \) labeled by the pair \( (x, y) \) for \( x, y \in K \) if \( u + x = y + Av \). The nucleus of the graph \( \Gamma \) is the subgraph (subautomaton) \( \mathcal{N} \) spanned by all cycles of \( \Gamma \) and all vertices that can be reached following directed paths from the cycles. Since the matrix \( A \) is expanding the nucleus \( \mathcal{N} \) is a finite graph and it can be algorithmically computed. Indeed, if \( u + x = y + Av \) then

\[
\|v\| < \|u\| \quad \text{whenever} \quad \|u\| > (1 - \|A^{-1}\|)^{-1} \max_{x,y \in K} \|A^{-1}(x - y)\|,
\]

and the nucleus \( \mathcal{N} \) is contained in the ball centered at the origin of radius given by the right-hand side above. Remove every edge in \( \mathcal{N} \) whose label is not in \( C \times D \), and replace every label \( (a, b) \) by \( a \). We get some finite graph \( \mathcal{N}_D \) whose edges are labeled by elements of the set \( C \).

Let \( C^{-\omega} \) be the space of all left-infinite sequences \( \ldots x_2x_1, x_i \in C \), with the product topology of discrete sets. Let \( \mu \) be the uniform Bernoulli measure on \( C^{-\omega} \), i.e. the product measure with \( \mu(x) = 1/|C| \) for every \( x \in C \). For every vertex \( v \) of the graph \( \mathcal{N}_D \) denote by \( F_v \) the set of all left-infinite sequences which can be read along left-infinite paths in \( \mathcal{N}_D \) that end in \( v \). The sets \( F_v \) are closed in \( C^{-\omega} \), thus compact and measurable.

**Theorem 1.** The Lebesgue measure of the self-affine set \( T \) is equal

\[
\lambda(T) = \sum_{v \in \mathcal{N}_D} \mu(F_v).
\]

**Proof.** Consider the map \( \Phi : K^{-\omega} \times \mathbb{Z}^n \rightarrow \mathbb{R}^n \) given by the rule

\[
\Phi(\ldots x_2x_1, v) = v + A^{-1}x_1 + A^{-2}x_2 + \ldots,
\]

where \( x_i \in K \) and \( v \in \mathbb{Z}^n \). Since \( \mathbb{Z}^n = K + A(\mathbb{Z}^n) \), the map \( \Phi \) is onto (see [3] or [1, Section 6.2]). Two elements \( \xi = (\ldots x_2x_1, v) \) and \( \zeta = (\ldots y_2y_1, u) \) for \( x_i, y_i \in K \) and
\( v, u \in \mathbb{Z}^n \) represent the same point \( \Phi(\xi) = \Phi(\zeta) \) in \( \mathbb{R}^n \) if and only if there is a finite subset \( B \subset \mathbb{Z}^n \) and a sequence \( \{v_m\}_{m \geq 1} \in B \) such that there exists the path

\[
v_m \xrightarrow{(x_m, y_m)} v_{m-1} \xrightarrow{(x_{m-1}, y_{m-1})} \ldots \xrightarrow{(x_2, y_2)} v_1 \xrightarrow{(x_1, y_1)} u \quad \text{for every } m \geq 1. \tag{1}
\]

in the graph \( \Gamma \) for every \( m \geq 1 \). Indeed, this path implies that

\[
v_m + x_m + Ax_{m-1} + \ldots + A^{m-1}x_1 + A^mv = y_m + Ay_{m-1} + \ldots + A^{m-1}y_1 + A^mu. \tag{2}
\]

Applying \( A^{-m} \) and using the facts that \( A^{-1} \) is contracting and the sequence \( \{v_m\}_{m \geq 1} \) attains a finite number of values, we get the equality \( \Phi(\xi) = \Phi(\zeta) \). For the converse, we choose \( v_m \) such that (2) holds, and using the equality \( \Phi(\xi) = \Phi(\zeta) \) we get that \( \{v_m\}_{m \geq 1} \) attains a finite number of values. Notice that since the set \( B \) is assumed to be finite, every element \( v_m \) lies either on a cycle or there is a directed path from a cycle to \( v_m \). In particular, all elements \( v_m \) should belong to the nucleus \( \mathcal{N} \), and we have that the elements \( \xi \) and \( \zeta \) represent the same point in \( \mathbb{R}^n \) if and only if there exists a left-infinite path in \( \mathcal{N} \) labeled by \( (\ldots x_2x_1, \ldots y_2y_1) \) and ending in \( u - v \).

Take the restriction \( \Phi_C : C^{-\omega} \times \mathbb{Z}^n \to \mathbb{R}^n \) of the map \( \Phi \) on the subset \( C^{-\omega} \times \mathbb{Z}^n \). Since \( \mathbb{Z}^n = C + A(\mathbb{Z}^n) \), the map \( \Phi_C \) is also onto, and this gives an encoding of points in \( \mathbb{R}^n \) by elements of \( C^{-\omega} \times \mathbb{Z}^n \). Consider the uniform Bernoulli measure \( \mu \) on the space \( C^{-\omega} \) and the counting measure on the group \( \mathbb{Z}^n \), and put the product measure on the space \( C^{-\omega} \times \mathbb{Z}^n \).

Since the set \( C \) is a coset transversal, the push-forward of this measure under \( \Phi_C \) is the Lebesgue measure on \( \mathbb{R}^n \) (see [1, Proposition 25]). Hence to find the Lebesgue measure of the self-affine set \( T \) it is sufficient to find the measure of its preimage in \( C^{-\omega} \times \mathbb{Z}^n \). However, \( T \) is equal to \( \Phi(D^{-\omega} \times \{0\}) \), and hence the sequence \( (\ldots x_2x_1, v) \) for \( x_i \in C \) and \( v \in \mathbb{Z}^n \) represents a point in \( T \) if and only if there exists a left-infinite path in the nucleus \( \mathcal{N} \), which ends in \( -v \) and is labeled by \( (\ldots x_2x_1, \ldots y_2y_1) \) for some \( y_i \in D \). Hence

\[
\Phi_{C^{-1}}^{-1}(\Phi(D^{-\omega} \times \{0\})) = \bigcup_{v \in \mathcal{N}D} F_v \times \{-v\}, \tag{3}
\]

and the statement follows.

The Bernoulli measure of the sets \( F_v \) for any finite graph \( \Gamma = (V, E) \) can be effectively computed (see [1, Section 2]). First, we can assume that the graph is left-resolving, i.e. for every vertex \( v \in V \) the incoming edges to \( v \) have different labels. Indeed, for any finite graph \( \Gamma = (V, E) \) there exists a left-resolving graph \( \Gamma' = (V', E') \) with the property that for every \( v \in V \) there exists \( v' \in V' \) such that \( F_v = F_{v'} \), and this graph can be easily constructed (here every vertex \( v' \) corresponds to some subset of \( V \), see [3, Section 2.3]). For a left-resolving graph the vector \( (\mu(F_v))_{v \in V} \) (if it is nonzero) is the left eigenvector of the adjacency matrix of the graph for the eigenvalue \( |C| = |\det A| \). This eigenvector is uniquely defined if we know its entries \( \mu(F_v) \) for vertices \( v \) in the strongly connected components without incoming edges. For every such a component \( \Gamma' \), we have \( F_v = C^{-\omega} \) and \( \mu(F_v) = 1 \) for every vertex \( v \) in \( \Gamma' \) if inside this component every vertex has incoming edges labeled by every element of the set \( C \), and \( \mu(F_v) = 0 \) otherwise. In particular, the entries \( \mu(F_v) \) are rational numbers, and we recover the following result of [4].
Corollary 2. Every self-affine set has rational Lebesgue measure.

It is also easy to check when the measure of $T$ is non-zero without calculating its precise value but just looking at the left-resolving graph (not the graph $N_D$) constructed above. The measure $\lambda(T)$ will be positive if and only if there exists a strongly connected component such that inside this component every vertex has incoming edges labeled by every letter of the alphabet.

Example 1. Let $A = (3)$ and $D = \{0, 1, 5, 6\}$. The self-affine set $T$ is $[0, \frac{4}{3}] \cup [\frac{5}{3}, 3]$, and $\lambda(T) = 8/3$. Choose $K = D$ and the coset transversal $C = \{0, 1, 5\}$. The associated automaton $N_D$ is shown in Figure 1. Here $\mu(F_0) = 1$, $\mu(F_1) = 1/3$, $\mu(F_2) = 1/8$, $\mu(F_{-1}) = 7/12$, $\mu(F_{-2}) = 5/8$, and $\mu(F_{-3}) = \mu(F_3) = 0$.

The above method can be used to find $\lambda(T \cap (T + u))$ for $u \in \mathbb{Z}^n$. The set $T + u$ is the image of the set $D^{-\omega} \times u$, and its preimage under $\Phi_C$ can be described as in (3). In particular

$$\lambda(T \cap (T + u)) = \sum_{v_1, v_2 \in N_D} \mu(F_{v_1} \cap F_{v_2}).$$

Similarly, one can find the measure of the intersection of self-affine sets $T_1 = T(A, D_1)$ and $T_2 = T(A, D_2)$ for different sets $D_1, D_2 \subset \mathbb{Z}^n$. We take a set $E$ which contains $D_1, D_2$, and some coset transversal $C$, and as above we construct the nucleus $N$ and its subgraphs $N_{D_1}$ and $N_{D_2}$. Then

$$\lambda(T_1 \cap T_2) = \sum_{v \in N} \mu(F_v^{(1)} \cap F_v^{(2)}).$$
where $F_{v}^{(i)}$ is calculated in the graph $N_{D_{i}}$. Hence these two problems are reduced to the question of how to find the measure of the intersection $F_{v_{1}}^{(1)} \cap F_{v_{2}}^{(2)}$, where each set $F_{v_{i}}^{(i)}$ is defined in some finite graph $\Gamma^{(i)} = (V^{(i)}, E^{(i)})$ with its vertex $v_{i}$. One can construct a new finite graph $\Gamma$ (sometimes called the labeled product of graphs $\Gamma^{(i)}$) with the set of vertices $V^{(1)} \times V^{(2)}$, where we put an edge $(u_{1}, u_{2}) \xrightarrow{\tau} (w_{1}, w_{2})$ for every edges $u_{1} \xrightarrow{\tau} w_{1}$ in $\Gamma^{(1)}$ and $u_{2} \xrightarrow{\tau} w_{2}$ in $\Gamma^{(2)}$. Then $F_{(v_{1}, v_{2})} = F_{v_{1}}^{(1)} \cap F_{v_{2}}^{(2)}$ (see [3, Section 3.2]).

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