\textit{OSp}(1|2) and \textit{Sl}(2) reductions in generalised super-Toda models and factorization of spin 1/2 fields.}

E. Ragoucy

\textit{NORDITA}

\textit{Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark}

Abstract

I show that the classical Toda models built on superalgebras, and obtained from a reduction with respect to an \textit{Sl}(2) algebra, are "linearly supersymmetrizable" (by adding spin 1/2 fields) if and only if the \textit{Sl}(2) is the bosonic part of an \textit{OSp}(1|2) algebra. In that case, the model is equivalent to the one constructed from a reduction with respect to the \textit{OSp}(1|2) algebra, up to spin 1/2 fields. The corresponding \textit{W} algebras are related through a factorization of spin 1/2 fields (bosons and fermions). I illustrate this factorization on an example: the superconformal algebra built on \textit{Sl}(n + 1|2).
1 Introduction

In the search of classifying all the $W$ (super)algebras \cite{1}, the generalised (super) Toda models play a central role \cite{2, 3}. Indeed, for the study of $W$ algebras, two important ways have been distinguished: the direct computation with the use of Jacobi identities \cite{4, 5}, or the construction of the $W$ algebras as symmetry of the Toda models. In these models, the $W$ algebra appears as a symmetry of the Toda action, and the formalism of hamiltonian reduction provides a natural framework to study their Operator Product Expansion (OPE), as well as their free fields realization. Although it has been shown \cite{4, 5} that (at least at the quantum level) there are some $W$ algebras that do not seem to be obtained from Toda models, the classification of the ”Toda-$W$ algebras” is nevertheless a first step in the classification of all the $W$ algebras. Progress have been made in that direction, specially in the classical case, where all the different Hamiltonian reductions leading to $W$ algebras have been classified \cite{7}, as well as the (super)spin contents of the corresponding $W$ algebra \cite{8}. The Toda-$W$ algebras are classified by the embedding of $SL(2)$ in Lie (super)algebras $A$, and the superconformal Toda-$W$ algebras by the $OSp(1|2)$ subalgebras of Lie superalgebras. Due to the characterization of $OSp(1|2)$ and $SL(2)$ subalgebras, it means that one can can exhaustively label the $W$ algebras as $W(A, S)$ where $S$ is a sub(super)algebra of $A$ that defines the $SL(2)$ or $OSp(1|2)$ embeddings (see for instance \cite{8} for details). However, if this classification of Toda-$W$ algebras is exhaustive, it is still unknown whether the $W$ algebras are all distinct, and if they are some inclusions (or weaker relations) between them. In this paper, I will show how to relate some of the $W$ algebras obtained from $SL(2)$ reduction to the ones we get from $OSp(1|2)$ reductions. I will also show that the $OSp(1|2)$ reduction is the only way that provides a superconformal (i.e. linearly supersymmetric) $W$ algebra in the Toda models framework.

This paper is organised as follow. In section 2, I recall the basic tools that are needed to build a generalised superconformal Toda model from a Wess-Zumino-Witten (WZW) action. In particular, the role of the $OSp(1|2)$ subalgebra (of the Lie superalgebra the model is built on) is emphasized. Then, the section 3 presents the case of generalized Toda models associated to $SL(2)$ reduction of Lie superalgebras and show that the models which can be linearly supersymmetrized are equivalent to the models presented in section 2. In particular, it is shown that there exists a gauge fixing where the actions differ only by kinetic terms in spin 1/2 fields. As a consequence, the corresponding $W$ algebras (symmetries of the actions) are equivalent through a factorization of the spin 1/2 fields. These points are enlightened by an example in the section 4: the quadratic superconformal algebra obtained from the $SL(n+1|2)$ superalgebra. I compute the OPEs and the free fields realization in an $N = 1$ formalism, then factorize the fermions of the algebra and show that we recover the quadratic superconformal algebra obtained from the reduction with respect to $SL(2)$, that was already studied in \cite{9}. Finally, I list in the conclusion the questions that are still opened.

\footnote{The spin contents for $W$ algebras built on $SL(2)$ reduction of superalgebras was not explicitly given in \cite{8}. There was nevertheless all the tools needed for such a classification, and I develop the technic in this paper.}
2 $OSp(1|2)$ reductions

Let us first recall the basic ingredients that one needs to make a constrained super-WZW model [10]. One starts with a supersymmetric WZW model based on a Lie superalgebra $\mathcal{A}$. Its action reads

$$S_0(G) = \frac{\kappa}{2} \int d^2xd^2\theta < G^{-1}D_+ \hat{G}, \hat{G}^{-1}D_- G > +$$

$$+ \int d^2xd^2\theta dt < G^{-1}\partial_t G, (G^{-1}D_+ \hat{G}) (\hat{G}^{-1}D_- G) + (G^{-1}D_- \hat{G}) (\hat{G}^{-1}D_+ G) >$$

where $<,>$ is the non-degenerate scalar product on $\mathcal{A}$, and $D_{\pm}$ are the fermionic derivatives. $G(x,\theta)$ is a superfield that belongs to the supergroup $\mathcal{G}$ associated to $\mathcal{A}$: let me recall that when writing $G(x,\theta) = expV(x,\zeta)$, we will have $V(x,\theta) = V^a(x,\theta)t_a$ with $t_a$ a basis of $\mathcal{A}$ and $V^a(x,\theta)$ a bosonic (fermionic) superfield when $t_a$ is a commuting (anti-commuting) generator. I have also introduced the isomorphism $\hat{\cdot}$ of $\mathcal{A}$ that changes the sign in front of the fermionic superfields. The action of $\hat{\cdot}$ is also trivially extended to $\mathcal{G}$. I recall that I consider only the classical case. $S_0$ is invariant under the transformations

$$G(x_+,x_-,\theta_+\theta_-) \rightarrow L(x_-,\theta_-)G(x_+,x_-,\theta_+\theta_-)R(x_+,\theta_+) \quad \text{with} \quad L,R \in \mathcal{G}$$

The corresponding currents $J = \kappa D_- GG^{-1}$ and $\bar{J} = \kappa \hat{G}^{-1}D_+ G$ form two copies of the super Kac-Moody (KM) algebra based on $\mathcal{A}$. Note that $J$ and $\bar{J}$ are fermionic superfields. The equations of motion for $G$ are just the currents conservation laws:

$$D_- J = 0 \quad \text{and} \quad D_+ \bar{J} = 0$$

$S_0(G)$ is also invariant under superconformal transformations, generated by

$$T_0 = \frac{1}{3\kappa^2} Str(J\bar{J}J) + \frac{1}{2\kappa} Str(JD_- J) \quad \text{and} \quad \bar{T}_0 = \frac{1}{3\kappa^2} Str(\bar{J}D_+ \bar{J}) + \frac{1}{2\kappa} Str(\bar{J}D_+ \bar{J})$$

The superconformal properties of the currents are encoded in the superOPEs:

$$T(Z_-)J(X_-) = \frac{1}{2} \left( \frac{\eta_- - \theta_-}{Z_- - X_-} \right)^2 J(X_-) + \frac{1}{Z_- - X_-} D_- J(X_-) + \frac{\eta_- - \theta_-}{Z_- - X_-} D_-^2 J(X_-)$$

$$\bar{T}(Z_+)\bar{J}(X_+) = \frac{1}{2} \left( \frac{\eta_+ - \theta_+}{Z_+ - X_+} \right)^2 \bar{J}(X_+) + \frac{1}{Z_+ - X_+} D_+ \bar{J}(X_+) + \frac{\eta_+ - \theta_+}{Z_+ - X_+} D_+^2 \bar{J}(X_+)$$

$$T(Z_-)J(X_+) = 0$$

$$\bar{T}(Z_+)\bar{J}(X_-) = 0$$

with $Z_{\pm} = (z_{\pm},\eta_{\pm})$, $X_{\pm} = (x_{\pm},\theta_{\pm})$ and $Z - X = z - x - \eta\theta$. These OPEs show that $J$ (resp. $\bar{J}$) is a primary superfield of dimensions $\left(\frac{1}{2},0\right)$ (resp. $\left(0,\frac{1}{2}\right)$).

---

\[2\text{Strictly speaking, I should write Poisson Brackets instead of OPEs. It is nevertheless possible to use the OPEs formalism with the correspondence } \delta(x-x') \equiv \frac{1}{2\pi} \text{. Note however that these "classical" OPEs obey to the Leibniz rule, whereas the quantum ones do not.}\]
To define a constrained WZW models, one first introduces a grading operator $H$ of $\mathcal{A}$:

$$\mathcal{A} = \oplus_{j \in \frac{1}{2} \mathbb{Z}} \mathcal{A}_j, \quad \text{with} \quad [H, X_j] = jX_j \quad \forall X_j \in \mathcal{A}_j$$  

For the model to be well-defined \[2\], I will choose $H$ to be the Cartan operator of some $OSp(1|2)$ subsuperalgebra of $\mathcal{A}$. I will call

$$\mathcal{A}_+ = \oplus_{j > 0} \mathcal{A}_j \quad \text{and} \quad \mathcal{A}_- = \oplus_{j < 0} \mathcal{A}_j$$  

and $\mathcal{G}_{\pm,0}$ the subgroups associated to $\mathcal{A}_{\pm,0}$.

Then, one wants to gauge the left (right) action of $\mathcal{G}_+$ ($\mathcal{G}_-$). For that aim, one makes the transformation

$$G \rightarrow \alpha_+ G \alpha_- \quad \text{where} \quad \alpha_\pm \in \mathcal{G}_\pm$$

Using the Polyakov-Wiegmann relation

$$S_0(G_1 G_2) = S_0(G_1) + S_0(G_2) + \kappa \int d^2 x d^2 \theta < G_1^{-1} D_+ \hat{G}_1, (D_- G_2) G_2^{-1} >$$

we are led to

$$S_0(\alpha_+ G \alpha_-) = S_0(G) + \kappa \int d^2 x d^2 \theta \left\{ < A_+, (D_+ G) G^{-1} > + < G^{-1} D_+ \hat{G}, A_- > + < G^{-1} A_+ \hat{G}, A_- > \right\}$$

with the gauge superfields

$$A_+ = \alpha_+^{-1} D_+ \hat{\alpha}_+ \in \mathcal{A}_+ \quad \text{and} \quad A_- = (D_- \alpha_-) \alpha_-^{-1} \in \mathcal{A}_-$$

Finally, denoting by $F_\pm$ the fermionic roots of the $OSp(1|2)$ subsuperalgebra whose Cartan generator is $H$, the complete gauge-invariant action which leads to the non-Abelian superconformal Toda models reads:

$$S(G, A_+, A_-) = S_0(G) + \kappa \int d^2 x d^2 \theta \left\{ < A_+, (D_+ G) G^{-1} > - F_- > + < G^{-1} D_+ \hat{G} - F_+, A_- > + < G^{-1} A_+ \hat{G}, A_- > \right\}$$

This action is invariant under the superconformal transformations generated by

$$T_H(X) = T_0(X) + < H, D_-^2 J(X) > \quad \text{and} \quad \bar{T}_H(X) = \bar{T}_0(X) - < H, D_+^2 \bar{J}(X) >$$

Note that because of the new term in $T_H$, the different components of $J$ will have a superspin\[^3\] which depends on their grade under $H$:

$$J(X_-) = \sum_j J_j(X_-) t_j \quad \text{with} \quad \text{superspin}(J_j) = \left( \frac{1}{2} + j, 0 \right) \quad \text{if} \quad t_j \in \mathcal{A}_j$$

$$\bar{J}(X_+) = \sum_j \bar{J}_j(X_+) t_j \quad \text{with} \quad \text{superspin}(\bar{J}_j) = \left( 0, \frac{1}{2} - j \right) \quad \text{if} \quad t_j \in \mathcal{A}_j$$

\[^3\]I remind that a (chiral) superfield of superspin $s$ is constructed on one field of spin $s$ and one field of spin $s + \frac{1}{2}$. 

3
In particular, the $J$-components (resp. $\bar{J}$-components) of negative (resp. positive) grades will have a negative spin, so that they should be set to zero. This is exactly what is required by the equations of motion of $A_+$ and $A_-:$

$$J|_{A_-} = F_- \quad \text{and} \quad J|_{A_+} = F_+$$

Using a Gauss decomposition

$$G = G_+ G_0 G_- \quad \text{with} \quad G_+ \in \mathcal{G}_+, \ G_0 \in \mathcal{G}_0 \quad (2.13)$$

these constraints can be set in a nice way:

$$(D_- G_-)G_-^{-1} + \hat{G}_- A_- G_-^{-1} = \hat{G}_0^{-1} F_- G_0$$

$$G_+^{-1} D_+ \hat{G}_+ + G_+^{-1} A_+ \hat{G}_+ = G_0 F_+ \hat{G}_0^{-1}$$

The action is of course invariant under the gauge transformations:

$$\begin{align*}
G & \to L_+ G R_- & \text{with} & \quad L_+(X_+, X_-) \in \mathcal{G}_+, \ R_-(X_+, X_-) \in \mathcal{G}_- \\
A_+ & \to L_+ A_+ \hat{L}_+^{-1} - (D_+ \hat{L}_+) \hat{L}_+^{-1} \\
A_- & \to \hat{R}_-^{-1} A_- \hat{R}_- - \hat{R}_-^{-1} D_- R_-
\end{align*} \quad (2.14)$$

One can use this gauge invariance to set $A_+ = 0$ and $A_- = 0$. This partially fixes the gauge freedom. With the residual gauge transformations (associated to $L_+(X_+)$ and $R_-(X_-)$) two gauge fixing are in general used:

In the **diagonal gauge**, the currents read

$$\begin{align*}
J_{\text{diag}}(x_-, \theta_-) & = F_- + \sum_i \Phi^i(x_-, \theta_-) t_i & \text{with} \ t_i \ \text{basis of} \ \mathcal{B}_0 \\
\bar{J}_{\text{diag}}(x_+, \theta_+) & = F_+ + \sum_i \bar{\Phi}^i(x_+, \theta_+) t_i
\end{align*} \quad (2.15)$$

where $\mathcal{B}_0 = \mathcal{A}_0$ if $H$ is such that all the (fermionic) bosonic superfields have (half) integer superspin. If it is not the case, $\mathcal{B}_0 = \mathcal{A}_0 \oplus \mathcal{Q}_{\frac{1}{2}} \oplus \mathcal{Q}_{-\frac{1}{2}}$, with $\mathcal{Q}_{\pm \frac{1}{2}}$ subspaces of $\mathcal{A}_{\pm \frac{1}{2}}$ determined through a halving (see [4]).

The **highest weights**, or Drinfeld-Sokolov (DS) gauge [31]:

$$\begin{align*}
J_{\text{DS}}(x_-, \theta_-) & = F_- + \sum_a \mathcal{W}^a(x_-, \theta_-) e_a \quad \text{with} \ [F_+, e_a] = 0 \\
\bar{J}_{\text{DS}}(x_+, \theta_+) & = F_+ + \sum_a \bar{\mathcal{W}}^a(x_+, \theta_+) e'_a \quad \text{with} \ [F_-, e'_a] = 0
\end{align*} \quad (2.16, 2.17)$$

Let us look at the superconformal properties in each of these gauges. The $\Phi$’s and $\bar{\Phi}$’s are superfields of superspin $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively (because the $t_i$’s belong to $\mathcal{A}_0$). They are all primary, but the one associated to $H$ (because to the scalar product with $H$ in (2.12)). The $\mathcal{W}$’s (and $\bar{\mathcal{W}}$’s) are all primary superfields, the superspin of which being given by the $OSp(1|2)$ decomposition of the adjoint representation of $\mathcal{A}[\mathfrak{8}]$. One can show that the $\mathcal{W}$ superfields are gauge invariant polynomials [3], so that the Poisson bracket of two $\mathcal{W}$ generators closes (in the enveloping algebra of the $\mathcal{W}$’s): they form the superconformal $\mathcal{W}$ algebra. Going from one gauge to the other allows us to have a free super-fields realization of the super $\mathcal{W}$ generators (super-Miura transformation). All these points will be developed on an example in the section [4].
3 \textit{Sl}(2) reduction

Instead of using a superfield formalism, one can do the previous work with a field formalism, as for WZW models built on Lie algebras. Since we will deal with fields, the model will not be explicitly supersymmetric. To be precise, if I use "superconformal" to distinguish the "linearly supersymmetric” from the ”non-linearly supersymmetric”, the model will be conformally but not superconformally invariant. Thus, the super Virasoro algebra will not be a subalgebra of the \( W \) algebra under consideration. As the superconformal algebra is associated with \( OSp(1|2) \) algebra, whereas the conformal one relies on the \( Sl(2) \) algebra, it is easy to deduce that the reduction and the constraints will be associated with an \( Sl(2) \) embedding in \( A \). Of course, there will be strong connections between the two approaches: it has been already shown [10] that in the Abelian case the ”non superconformal” model can be obtained from the superconformal one by choosing a gauge for the supersymmetry. I will show that this property extends to any of the \( Sl(2) \) subalgebras that are the bosonic part of a \( OSp(1|2) \) superalgebra.

3.1 Generalised conformal Toda action

To construct a generalised Toda action with a \( Sl(2) \) reduction, we follow the same steps as in the previous section, replacing superfields by fields. I briefly recall the main steps.

We start with the WZW action built on the superalgebra \( A \):

\[
S_0(g) = \frac{\kappa}{2} \int d^2x \left< g^{-1} \partial_+ g, g^{-1} \partial_- g \right> + \int d^2x dt \left< g^{-1} \partial_t g, [g^{-1} \partial_+ g, g^{-1} \partial_- g] \right> \tag{3.18}
\]

\( S_0(g) \) is invariant under the KM transformations generated by \( j(x_-) = \kappa (\partial_- g) g^{-1} \) and \( \bar{j}(x_+) = \kappa g^{-1} \partial_+ g \). It is also conformally invariant with respect to \( t_0(x_-) = \frac{1}{2\kappa} tr(\bar{j} \partial_- j) \) and \( \bar{t}_0(x_+) = \frac{1}{2\kappa} tr(j \partial_+ \bar{j}) \).

Then, we choose a Cartan generator \( H \) of an \( Sl(2) \) subalgebra of \( A \), and gauge the (left) right action of \( (G_+) \). Then, denoting by \( E_\ell \) the roots of the \( Sl(2) \) subalgebra under consideration, the generalised conformal Toda action built on the superalgebra \( A \) reads:

\[
S(g, a_+, a_-) = S_0(g) - 2 \int d^2x \left< a_-, g^{-1} \partial_+ g + E_- > + (\partial_- g) g^{-1} - E_+, a_+ > + g^{-1} a_+ g, a_- \right> \tag{3.19}
\]

The conformal generators are

\[
t_H(x) = t_0(x) + < H, \partial_- j(x) > \quad \text{and} \quad \bar{t}_H(x) = \bar{t}_0(x)- < H, \partial_+ \bar{j} > \tag{3.20}
\]

Note that the spin of the components of \( j \) and \( \bar{j} \) are

\[
j(x_-) = \sum_\ell j_\ell(x_-) t_\ell \quad \text{with} \quad \text{spin}(j_\ell) = (1 + \ell, 0) \quad \text{if} \quad t_\ell \in A_\ell
\]

\[
\bar{j}(x_+) = \sum_\ell \bar{j}_\ell(x_+) t_\ell \quad \text{with} \quad \text{spin}(\bar{j}_\ell) = (0, 1 - \ell) \quad \text{if} \quad t_\ell \in A_\ell
\]

The \( W \) generators are defined in the DS gauge as:

\[
j(x_-) = E_- + \sum_a W^a(x_-) e_a \quad \text{with} \quad [E_+, e_a] = 0
\]

\[
\bar{j}(x_-) = E_+ + \sum_a \bar{W}^a(x_-) e'_a \quad \text{with} \quad [E_-, e'_a] = 0
\]
and the spin contents is given by the $Sl(2)$ decomposition of $\mathcal{A}$.

### 3.2 Wess-Zumino gauge for the supersymmetry

Now, I want to show how to relate the action (3.19) to the action (2.11). As the calculation is the same as in the Abelian case, I will just sketch the proof: for details, see [10]. I come back to the action (2.11). From the gauge transformation laws (2.14), it is easy to see that we can choose a gauge such that:

\begin{align*}
A_+|_{\theta=0} &= 0 \quad \text{and} \quad (D_+A_+)|_{\theta=0} = 0 \\
A_-|_{\theta=0} &= 0 \quad \text{and} \quad (D_-A_-)|_{\theta=0} = 0 \\
J|_{\theta=0} &\in (\mathcal{A}_- \oplus \mathcal{A}_0) \quad \text{and} \quad \bar{J}|_{\theta=0} \in (\mathcal{A}_+ \oplus \mathcal{A}_0)
\end{align*}

(3.21)

(3.22)

(3.23)

The fields $(D_+D_-A_+)|_{\theta=0}$ and $(D_+D_-A_-)|_{\theta=0}$ are auxiliary fields for the supersymmetry: their equation of motion together with (3.23) show that the restriction of $J|_{\theta=0}$ (resp. $\bar{J}|_{\theta=0}$) on $\mathcal{A}_+$ (resp. $\mathcal{A}_-$) is equal to $F_-$ (resp. $F_+$). Denoting $J|_{\theta=0} = F_- + \chi_-$ and $\bar{J}|_{\theta=0} = F_+ + \chi_+$ where $\chi_{\pm}$ are some fields of grade 0, one can write the component form of the action:

\begin{equation}
S = \frac{\kappa}{2} \int d^2x \left< g^{-1}\partial_+g, g^{-1}\partial_-g \right> + \int d^2x dt \left< g^{-1}\partial_+g, [g^{-1}\partial_+g, g^{-1}\partial_-g] \right> + \int d^2x \left\{ \left< a_-, g^{-1}\partial_+g + F_-^2 \right> + \left< (\partial_-g)g^{-1} - F_+^2, a_+ \right> + g^{-1}a_+g, a_- \right\} + \int d^2x \left\{ \left< \chi_-, \partial_+\chi_- \right> + \left< \chi_+, \partial_-\chi_+ \right> - 2 \left< a_+, \chi_+ \right>, F_+ \right> + 2 \left< \chi_+ \right>, F_- \right>, a_- \right>.
\end{equation}

(3.24)

where $g(x) = G(X)|_{\theta=0}$ is an element of the group built on $\mathcal{A}$, and $a_+ = (D_+\hat{A}_+)|_{\theta=0}$, $a_- = (D_-A_-)|_{\theta=0}$ are the physical gauge fields for the "non superconformal" action. Their equation of motions lead to constraints on the (non supersymmetric) currents $j(x) = (D_-J)|_{\theta=0} = (\partial_-g)g^{-1}$ and $\bar{j}(x) = (D_+\bar{J})|_{\theta=0} = g^{-1}\partial_+g$:

\begin{align*}
\left. j(x) \right|_{\mathcal{A}_-} &= \frac{1}{2} \left( F_- - [\chi_-, F_-] - ga_+g^{-1} \right) = E_- - [\chi_-, F_-] - ga_+g^{-1} \\
\left. \bar{j}(x) \right|_{\mathcal{A}_+} &= \frac{1}{2} \left( F_+ + [\chi_+, F_+] \right) - g^{-1}a_+g = E_+ + [\chi_+, F_+] - g^{-1}a_+g
\end{align*}

(3.25)

(3.26)

where $E_{\pm}$ are the (bosonic) roots of the $Sl(2)$ subalgebra whose Cartan generator is $H$. One recovers the usual constraints of a "non-superconformal" generalised Toda model\footnote{I denote by "non-superconformal" the actions that do not possess any linear supersymmetry invariance. The action (3.24) is (non linearly) supersymmetric, but the supersymmetry transformations have quadratic terms, as it was shown in [10].} implemented by the spin-$\frac{1}{2}$ fields that appear in the halving procedure [2].

Note that because $\chi_{\pm}$ belong to $\mathcal{A}_0$, they will have a spin $\frac{1}{2}$, but they may be bosons, depending on the characteristic (commuting or anti-commuting) of the $\mathcal{A}$-generators they are carried by. As far as spin $1/2$ fermions are concerned, it is known [12] that the corresponding
\( \mathcal{W} \) algebras can be factorized. However, for spin 1/2 bosons, although nothing has been proved until now, it seems that the factorization works as well.

Let us now fix the gauge \( a_+ = 0 = a_- \). Then,

\[
S = \frac{\kappa}{2} \int d^3x < g^{-1} \partial_+ g, g^{-1} \partial_- g > + \int d^2x dt < g^{-1} \partial g, [g^{-1} \partial_+ g, g^{-1} \partial_- g] > \\
+ \int d^2x \{ < \chi_- , \partial_\chi_- > + < \chi_+, \partial_\chi_+ > \}
\]

the component action is the sum of a usual ("non-superconformal") generalised Toda model built on \( A \) (in the gauge \( A = 0 \)) and of the action of free spin \( \frac{1}{2} \) fields. This means that the \( W \) algebra which is a symmetry of this generalised Toda model can be deduced from the superconformal \( \mathcal{W} \) algebra constructed from the action \((2.11)\) by factorizing out the spin \( \frac{1}{2} \) fields (bosons and fermions) \( \chi_\pm \). I will show below that the converse is true: when starting from a \( W \) algebra (obtained from the \( \mathcal{W} \) reduction of \( A \)), if one can make it superconformal by adding spin-\( \frac{1}{2} \) fields, then the \( \mathcal{W} \) subalgebra is the bosonic part of an \( OSp(1|2) \) sub-superalgebra of \( A \). In that case, the superconformal version of the \( W \) algebra is obtained by the \( OSp(1|2) \) reduction of \( A \).

3.3 \( \mathcal{W} \) decompositions of superalgebras and "non-superconformal" \( \mathcal{W} \) algebras

The way of decomposing a superalgebra \( A \) with respect to \( \mathcal{W} \) subalgebras is similar to the \( \mathcal{W} \) decomposition of fundamental and adjoint representations of Lie algebra (see [3] for details). One first decomposes (w.r.t \( \mathcal{W} \)) the fundamental representation of each simple Lie algebra entering in the bosonic part of \( A \). Then, the anticommuting generators being in their fundamental representations \((m, \bar{n}) + (\bar{m}, n)\) for \( \mathcal{W}(m|n) \), \( (m, n) \) for \( \mathcal{W}(m|n) \), and so on., one get their \( \mathcal{W} \) decompositions by a simple product of \( \mathcal{W} \) representations. Finally, the bosonic part decomposition is obtained as in the Lie algebras case. A good check of the results given in the previous section is to compare the \( \mathcal{W} \) and the \( OSp(1|2) \) decompositions when the \( \mathcal{W} \) is the bosonic part of \( OSp(1|2) \).

Let us study an example in the \( \mathcal{W}(m|n) \) case. I start with the principal embedding of \( \mathcal{W} \) in \( \mathcal{W}(p+1) \oplus \mathcal{W}(p) \), considered as a subalgebra of \( \mathcal{W}(m|n) \). We have the decompositions:

\[
(m, \bar{n}) + (\bar{m}, n) = 2 \left( D_{p/2} \oplus (m - p - 1)D_0 \right) \times \left( D_{(p-1)/2} \oplus (n - p)D_0 \right) \\
= 2 \oplus_{j=1/2}^{p-1/2} D_j \oplus 2(n - q - 1)D_{p/2} \oplus 2(m - p - 1)D_{(p-1)/2} \oplus 2(m - p - 1)(n - p - 2)D_0 \\
\mathcal{W}(m) \oplus \mathcal{W}(n) \oplus \mathcal{W}(1) = \left( D_{p/2} \oplus (m - p - 1)D_0 \right)^2 \oplus \left( D_{(p-1)/2} \oplus (n - p)D_0 \right)^2 \oplus D_0 \\
2 \oplus_{j=0}^{p-1} D_j \oplus D_p \oplus 2(m - p - 1)D_{p/2} \oplus 2(n - p - 2)D_{(p-1)/2} \oplus \\
\oplus \left[ (m - p - 1)^2 + (n - p - 2)^2 - 1 \right] D_0
\]

which can be transcribed into fields contents:

Bosonic fields : \( W_{p+1} \); \( 2W_j \ (j = 1, ..., p) \); \( 2(n - p)W_{(p+1)/2} \); \( 2(m - p - 1)W_{p/2+1} \)
\[ (m - p - 1)^2 + (n - p)^2 - 1 \] W_1 \quad (3.27)

Fermionic fields : 
\[ 2W_{j+1/2} \ (j = 1, \ldots, p) ; 2(m - p - 1)W_{(p+1)/2} ; 2(n - p)W_{p/2+1} \]
\[ 2(m - p - 1)(n - p)W_1 \quad (3.28) \]

Now, if one does the same work with an OSp(1/2) subalgebra which is the principal embedding of \( Sl(p + 1|p) \subset Sl(m|n) \), one gets:

\[ \text{Fund}^{Sl} = R_{p/2} \oplus (m - p - 1)R_0 \oplus (n - p)R_0^{\pi} \]
\[ \text{Adj} = \bigoplus_{j=0}^{p-1} (R_j^B \oplus R_{j+1/2}^F) \oplus R_p^B \oplus 2(m - p - 1)R_{p/2}^F \oplus 2(n - p)R_{p/2}^{B} \]
\[ \oplus [(m - p - 1)^2 + (n - p)^2 - 1] R_0^F \oplus 2(m - p - 1)(n - p)R_0^{B} \quad (3.29) \]

where \( R, R^\pi, R^F, R^B \) are OSp(1/2) representations, made of two \( Sl(2) \) representations: in 
the fundamental representation of \( Sl(m|n) \), \( R_j = (D_j, D_{j-1}) \) and \( R_0^\pi = (D_{j-1}, D_j) \) where 
the first term in the pair belongs to \( Sl(m) \) representation while the second is in \( Sl(n) \) one; in the 
adjoint representation of \( Sl(m|n) \), \( R_0^F = (D_q, D_{q-1}) \) and \( R_0^{B} = (D_q, D_{q-1}) \) with \( D_q \) built on 
anticommuting (resp. commuting) generators for \( R^F \) (resp. \( R^B \)). The adjoint decomposition gives rise to the following superspin contents for the super \( W \) algebra:

\[ \text{Bosonic superfields} : \ W_j \ (j = 1, \ldots, p) ; 2(n - p)W_{(p+1)/2} \]
\[ 2(m - p - 1)(n - p)W_{1/2} \quad (3.31) \]

\[ \text{Fermionic superfields} : \ W_{j+1/2} \ (j = 0, \ldots, p) ; 2(m - p - 1)W_{(p+1)/2} \]
\[ [(m - p - 1)^2 + (n - p)^2 - 1] W_{1/2} \quad (3.32) \]

In terms of fields, one writes the super \( W \) algebra as:

\[ \text{Bosonic fields} : \ W_{p+1} \ ; 2W_j \ (j = 1, \ldots, p) ; 2(n - p)W_{(p+1)/2} \ ; 2(m - p - 1)W_{p/2+1} \]
\[ 2(m - p - 1)(n - p)W_{1/2} ; [(m - p - 1)^2 + (n - p)^2 - 1] W_1 \quad (3.33) \]

\[ \text{Fermionic fields} : \ 2W_{j+1/2} \ (j = 1, \ldots, p) ; 2(m - p - 1)W_{(p+1)/2} \ ; 2(n - p)W_{p/2+1} \]
\[ [(m - p - 1)^2 + (n - p)^2] W_{1/2} \ ; 2(m - p - 1)(n - p)W_1 \quad (3.34) \]

As announced, the two approaches give the same fields contents, except for spin \( \frac{1}{2} \) fields that 
are absent from the \( Sl(2) \) reduction. This calculation can be done for all kind of OSp(1/2) 
embeddings in \( Sl(m|n) \), the result is the same. It extends for OSp(1|n) algebras by folding 
of \( Sl(m|n) \) ones\( ^5 \). Of course when the \( Sl(2) \) subalgebra is not the bosonic part of an 
OSp(1/2) superalgebra, the corresponding \( W \) will not be equivalent to one obtained from an 
OSp(1/2) algebra, but it will not possible to make it superconformal. In the table \( ^4 \) I give 
the basic \( Sl(2) \) embeddings that furnish a "virtually superconformal" \( W \) algebra, as well as 
the OSp(1/2) algebra which is in correspondence. Any "virtually superconformal" \( W \) algebra 
will be obtained from a (or a sum of) basic \( Sl(2) \).

\[ ^5 \text{One can also do the calculation by hand} \]
3.3.1 Quasi-Abelian $W$ algebras

To end this section, I give a class of $Sl(2)$ reductions that lead to "non-superconformal" $W$ algebras: the cases where the $Sl(2)$ is the principal embedding in the bosonic part of $\mathcal{A}$. These models are very close to the Abelian Toda models in the sense that $\mathcal{A}_0$ is generated by the Cartan subalgebra only. However, they are not Abelian, because one has to add some fermions of the $\mathcal{A}_\perp$ spaces to build the model (halving). Of course, for the $W$ algebra to be "non-superconformal", $\mathcal{A}$ must be a superalgebra with no principal $OSp(1|2)$ embedding. That is, one must choose $\mathcal{A}$ different from $Sl(m|n \pm 1)$, $OSp(2m \pm 1|2m)$, $OSp(2m|2m)$, $OSp(2m + 2|2m)$ and $D(2,1;\alpha)$.

Let us start with $Sl(m+1|n+1)$ ($m \neq n \pm 1$). The decomposition of the fundamental of an $Sl(p + 1)$ with respect to its principal embedding is $\varpi = D_{p/2}$, so that we have:

$$(m+1, n+1) + (m+1, n+1) = 2(D_{m/2} \times D_{n/2}) = 2 \oplus \bigoplus_{j=\lfloor m-n/2 \rfloor}^{(m+n)/2} D_j$$

$$Sl(m+1) \oplus Sl(n+1) \oplus U(1) = (D_{m/2} \times D_{m/2}) \oplus (D_{n/2} \times D_{n/2}) \oplus D_0$$

$$= \bigoplus_{j=1}^{m} D_j \oplus \bigoplus_{j=1}^{n} D_j \oplus D_0 \hspace{1cm} (3.35)$$

In the case of $OSp(2m+1|2n)$ algebras ($m \neq n, m-1$), we have

$$(2m+1, 2n) = D_m \times D_{n-1/2} = \bigoplus_{j=\lfloor m-n+1/2 \rfloor}^{m+n-1/2} D_j$$

$$SO(2m+1) \oplus Sp(2n) = (D_{m} \times D_{m})\big|_{\mathcal{A}_{\text{even}}} \oplus (D_{n-1/2} \times D_{n-1/2})\big|_{S}$$

$$= \bigoplus_{j=0}^{m-1} D_{2j+1} \oplus \bigoplus_{j=0}^{j=n-1} D_{2j+1} \hspace{1cm} (3.36)$$

Let us remark that the well-known $WB(0,n)$ is the subcase $m = 0$ in (3.36).

For $OSp(2m|2n)$ algebras, the decompositions read

$$(2m, 2n) = (D_{m-1} \oplus D_0) \times D_{n-1/2} = D_{n-1/2} \oplus \bigoplus_{j=\lfloor m-n+1/2 \rfloor}^{m+n-3/2} D_j$$

$$SO(2m) \oplus Sp(2n) = (D_{m-1} \oplus D_0) \times (D_{m-1} \oplus D_0)\big|_{\mathcal{A}_{\text{even}}} \oplus (D_{n-1/2} \times D_{n-1/2})\big|_{S}$$

$$= D_{m-1} \oplus \bigoplus_{j=0}^{m-2} D_{2j+1} \oplus \bigoplus_{j=0}^{j=n-1} D_{2j+1} \hspace{1cm} (3.37)$$

The calculation for $D(2,1;\alpha)$ is the same as for $OSp(4|2)$: although they are different, the $W$ algebras have the same superspin contents. For $G(3)$, the decomposition under the principal $Sl(2)$ of $G_2 \oplus Sl(2)$ leads to

$$(7, 2) = D_3 \times D_{1/2} = D_{7/2} \oplus D_{5/2}$$

$$G_2 \oplus Sl(2) = D_5 \oplus 2D_1$$

Finally, for $F(4)$, we get

$$(2, 8) = D_{1/2} \times (D_3 \oplus D_0) = D_{7/2} \oplus D_{5/2} \oplus D_{1/2}$$

$$Sl(2) \oplus SO(7) = D_5 \oplus D_3 \oplus 2D_1$$

The corresponding spin contents are gathered in the table [4]. Note that all these algebras have "good" statistic (bosons and fermions with usual spins).
4 Factorisation of fermions in $Sl(n+1|2)$ case

In this section I will show explicitly how the factorization of the spin $\frac{1}{2}$ fermions works on one basic example: the quadratic superconformal algebra built on $Sl(n+1|2)$ (i.e. the $\mathcal{W}[A(n, 1), A(1, 1)]$ algebra). This algebra has been already studied in [9] from the point of view of $Sl(2)$ reduction. I study the case of the $OSp(1|2)$ reduction. I will give a free superfields realization of the superconformal $\mathcal{W}$ algebra, and give its superOPEs. Then, I will factorize the spin 1/2 fermions and recover the free field realization given in [9] for $W$ algebra obtained by a reduction with respect to $Sl(2)$. To avoid boring repetitions, I will focus on the $J$ part, but the results are valid also for $\bar{J}$. I will not mention the subscript "−" anymore, and speak of the (super) spin $s$ instead of $(s, 0)$.

4.1 Constraints for the $Sl(n+1|2)$ quadratic superconformal algebra

The algebra corresponds to the second line of table 1 with $m = 1$ and $p = 0$. Let us first look at the superspin and spin contents. It is very similar to the example presented in section 3.3. The results is:

- Bosonic superfields : $2(n-1)\mathcal{W}_1$
- Fermionic superfields : $\mathcal{W}_{3/2}; (n-1)^2 \mathcal{W}_{1/2}$

(4.38)

We see that we will have a $Sl(n) \oplus U(1)$ super KM algebra. Looking at the results of [9], one may be surprised, since they get an $Sl(n+1) \oplus U(1)$ KM algebra. The difference comes from the superfields formalism: in this formalism, we have a super Virasoro subalgebra, but the price to pay is an (apparent) diminution of the KM part from $Sl(n+1)$ to (super) $Sl(n)$. The other spin 1 fields have been used to build supersymmetry partners of the spin 3/2 fields. Indeed, in terms of fields, one writes the super $\mathcal{W}$ algebra as:

- Bosonic fields : $W_2; (n+1)^2W_1$
- Fermionic fields : $2nW_{3/2}; n^2W_{1/2}$

(4.39) (4.40)

showing that we have the right number of fields to construct an $Sl(n+1) \oplus U(1)$ KM algebra.

Now, we start with an element $J$ in the super KM algebra of $Sl(n+1|2)$ and want to impose some constraints on it. The Cartan element for the $OSp(1|2)$ algebra under consideration is

$$H = \begin{pmatrix}
0 & 0 & 0 & 0 \\
: & : & : & : \\
0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & \frac{1}{2}
\end{pmatrix}$$

(4.41)
which leads to the grades\footnote{\( g \) is defined by \( g_{ij} = \text{grade}(E_{ij}) \) with \( E_{ij} \) the matrix basis \( (E_{ij})_{pq} = \delta_{ip}\delta_{jq} \).}

\[
g = \begin{pmatrix}
0 & \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\
\vdots & \vdots \\
-\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \\
\frac{1}{2} & \cdots & \frac{1}{2} & 0 & 1 \\
-\frac{1}{2} & \cdots & -\frac{1}{2} & -1 & 0
\end{pmatrix}
\] (4.42)

and the constrained current reads

\[
J_{\text{constr.}} = \begin{pmatrix}
* & 0 & * \\
\vdots & \vdots & \vdots \\
0 & * & 1 \\
* & \cdots & \cdots & * & * & 0 & * \\
0 & \cdots & 0 & -1 & 0 & *
\end{pmatrix}
\] (4.43)

The two gauges described in section\footnote{[1]} are:

\[
J_{\text{diag.}} = \begin{pmatrix}
\Phi_1 \Phi_2 1_{n+1} + M \\
\vdots \\
0 \\
\Phi_1 \\
0 \\
\Phi_2
\end{pmatrix}
\] with \( M \in Sl(n+1) \)

and

\[
J_{DS} = \begin{pmatrix}
0 & 0 & G_n \\
\vdots & \vdots & \vdots \\
0 & 0 & G_1 \\
G_n & \cdots & -G_0 \\
0 & \cdots & 0 & -1 & 0 & U & T
\end{pmatrix}
\] with \( N \in Sl(n) \)

where the highest weights gauge is defined with respect to the root

\[
F_+ = \begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & 1 \\
0 & \cdots & 0 & 1 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}
\] (4.44)
4.2 SuperOPEs of the \( \mathcal{W} \) algebra

First, I compute the OPEs of the \( \mathcal{W} \) algebra thanks to a supersymmetric version of the so-called soldering procedure \(^{[14]}\). Starting with an element \( \Lambda(X) \) in the super KM algebra \( Sl(n+1|2) \), we make an infinitesimal gauge transformation of \( J_{DS}(Z) \):

\[
\delta_{\Lambda} J_{DS}(Z) = [\Lambda, J_{DS}(Z)] + kD\Lambda
\] (4.45)

and ask this transformation to preserve the gauge: this imposes some relations between the matrix elements of \( \Lambda \) and allows us to determine some of the \( \Lambda_{jk} \)'s in terms of the \( \mathcal{W} \) generators and of the other \( \Lambda_{jk} \)'s. Then, replacing these parameters in the variation of \( J_{DS} \), we can compute its infinitesimal gauge transformation. Finally, from the relation

\[
\delta_{\Lambda} J_{DS}(Z) = \int dx d\theta \text{Str}(\dot{\Lambda} J_{DS}(X)) J_{DS}(Z)
\] (4.46)

we are able to deduce the super OPEs:

\[
T(Z)T(X) = k^2 \left\{ \frac{3}{2} \frac{\eta - \theta}{(Z - X)^2} T(X) + \frac{1}{2} \frac{\eta^2}{Z - X} DT(X) + \frac{\eta - \theta}{Z - X} D^2 T(X) \right\} + \frac{k^3}{(Z - X)^3}
\] (4.47)

\[
T(Z)G_0(X) = k^2 \left\{ \frac{\eta - \theta}{(Z - X)^2} G_0(X) + \frac{1}{2} \frac{\eta^2}{Z - X} DG_0(X) + \frac{\eta - \theta}{Z - X} D^2 G_0(X) \right\}
\] (4.48)

\[
T(Z)G_p(X) = k^2 \left\{ \frac{\eta - \theta}{(Z - X)^2} G_p(X) + \frac{1}{2} \frac{\eta^2}{Z - X} DG_p(X) + \frac{\eta - \theta}{Z - X} D^2 G_p(X) \right\} +
\]

\[
+ \frac{k/2}{Z - X} \left[ \frac{n - 1}{n} UG_p + (NG)_p \right] + \frac{\eta - \theta}{Z - X} \left[ \frac{k}{n} G_p DU + k[(DN)G]_p + (N^2G)_p \right]
\] (4.49)

\[
T(Z)\bar{G}_p(X) = k^2 \left\{ \frac{\eta - \theta}{(Z - X)^2} \bar{G}_p(X) + \frac{1}{2} \frac{\eta^2}{Z - X} D\bar{G}_p(X) + \frac{\eta - \theta}{Z - X} D^2 \bar{G}_p(X) \right\} +
\]

\[
- \frac{k/2}{Z - X} \left[ (GN)_p + \frac{n - 1}{n} U\bar{G}_p \right] + \frac{\eta - \theta}{Z - X} \left[ -k \frac{n - 1}{n} \bar{G}_p DU - k(DN)\bar{G}_p - (GN^2)_p \right]
\] (4.50)

\[
T(Z)U(X) = 0 \quad T(Z)N_{pq}(X) = 0
\] (4.51)

\[
G_0(Z)G_0(X) = \frac{k^2}{4} \left\{ 2 \frac{\eta - \theta}{Z - X} \frac{1}{k^2} T(X) + \frac{2k}{(Z - X)^2} \right\}
\] (4.52)

\[
G_0(Z)G_p(X) = \frac{k}{2} \left\{ \frac{2}{Z - X} G_p + \frac{\eta - \theta}{Z - X} DG_p \right\} + \frac{1}{2} \frac{\eta - \theta}{Z - X} (WG)_p
\] (4.53)

\[
G_0(Z)\bar{G}_p(X) = -\frac{k}{2} \left\{ \frac{2}{Z - X} \bar{G}_p + \frac{\eta - \theta}{Z - X} D\bar{G}_p \right\} + \frac{1}{2} \frac{\eta - \theta}{Z - X} (\bar{W}G)_p
\] (4.54)
\[ G_0(Z)U(X) = 0 \quad G_0(Z)N_{pq}(X) = 0 \quad (4.55) \]
\[ U(Z)G_p(X) = \frac{\eta - \theta}{Z - X} G_p(X) \quad U(Z)\bar{G}_p(X) = -\frac{\eta - \theta}{Z - X}\bar{G}_p(X) \quad (4.56) \]
\[ N_{pq}(Z)G_r(X) = \frac{\eta - \theta}{Z - X} \left( \delta_{qr} G_p(X) - \frac{1}{n} \delta_{pq} G_r(X) \right) \quad (4.57) \]
\[ N_{pq}(Z)\bar{G}_r(X) = -\frac{\eta - \theta}{Z - X} \left( \delta_{pr} \bar{G}_q(X) - \frac{1}{n} \delta_{pq} \bar{G}_r(X) \right) \quad (4.58) \]
\[ \frac{1}{k^2} G_p(Z)\bar{G}_q(X) = \frac{k \delta_{pq}}{(Z - X)^2} + \frac{\eta - \theta}{Z - X} \left\{ -\delta_{pq} T(X) + \frac{1}{k^2} (W^3)_{pq} - \frac{1}{k} (W D W)_{pq} \right\} + \right. \]
\[ -\frac{2 \eta - \theta}{k Z - X} W_{pq} G_0(X) - \frac{\delta_{pq}}{k} \left\{ \frac{2}{Z - X} G_0(X) + \frac{\eta - \theta}{Z - X} D G_0(X) \right\} + \right. \]
\[ -\frac{\eta - \theta}{(Z - X)^2} W_{pq}(X) + \frac{1}{Z - X} \left( \frac{1}{k} (W^2)_{pq}(X) - D W_{pq}(X) \right) + \right. \]
\[ -\frac{\eta - \theta}{Z - X} D^2 W_{pq}(X) \quad (4.59) \]
\[ U(Z)U(X) = -\frac{kn}{n - 1} \quad U(Z)N_{pq}(X) = 0 \quad (4.60) \]
\[ N_{pq}(Z)N_{rs}(X) = \frac{\eta - \theta}{Z - X} \left\{ \delta_{qr} N_{ps}(X) - \delta_{sp} N_{rq}(X) \right\} + \frac{k}{Z - X} \left( \delta_{ps} \delta_{qr} - \frac{1}{n} \delta_{rs} \delta_{pq} \right) \quad (4.61) \]

with the convention that the indices \( i, j, l, m \) run from 1 to \( n + 1 \), while the indices \( p, q, r, s \) run from 1 to \( n \) (and summation over repeated indices). I have also introduced:

\[ W_{pq}(X) = N_{pq}(X) - \frac{\eta - \theta}{n} U(X) \quad (4.62) \]

As announced in [8], we get an \( N = 2 \) (linear) superconformal algebra \( \{ T(Z), G_0(Z) \} \). This algebra commutes with the super KM algebra \( Sl(n) \oplus U(1) \) built on \( \{ U(Z), N_{pq}(Z) \} \). When adding the superstress energy tensor of this super KM algebra, all the fields become primary under the stress-energy tensor \( T_{tot} \):

\[ T_{tot}(Z) = \frac{1}{k^2} T(Z) + T_{Sl(n)}(Z) + T_{U(1)}(Z) \quad (4.63) \]
\[ T_{Sl(n)}(Z) = \frac{1}{3k^2} Str(N^3) + \frac{1}{2k} Str(NDN) \quad (4.64) \]
\[ T_{U(1)}(Z) = -\frac{1}{2k} n \quad (4.65) \]
\[ T_{tot}(Z)U(X) = \frac{1}{2} \eta - \theta \quad U(X) + \frac{1}{2} \frac{\eta - \theta}{Z - X} D U(X) + \frac{\eta - \theta}{Z - X} D^2 U(X) \quad (4.66) \]
\[ T_{tot}(Z)N_{pq}(X) = \frac{1}{2} \frac{\eta - \theta}{Z - X} N_{pq}(X) + \frac{1}{2} \frac{\eta - \theta}{Z - X} D N_{pq}(X) + \frac{\eta - \theta}{Z - X} D^2 N_{pq}(X) \quad (4.67) \]
\[ T_{tot}(Z)G_0(X) = \frac{\eta - \theta}{(Z - X)^2} G_0(X) + \frac{1}{2} \frac{\eta - \theta}{Z - X} D G_0(X) + \frac{\eta - \theta}{Z - X} D^2 G_0(X) \quad (4.68) \]
\[ T_{tot}(Z)G_p(X) = \frac{\eta - \theta}{(Z - X)^2} G_p(X) + \frac{1}{2} \frac{\eta - \theta}{Z - X} D G_p(X) + \frac{\eta - \theta}{Z - X} D^2 G_p(X) \quad (4.69) \]
\[ T_{\text{tot}}(Z) T_{\text{tot}}(X) = \frac{3}{2} \frac{\eta - \theta}{(Z - X)^2} T_{\text{tot}}(X) + \frac{1}{2} \frac{\eta - \theta}{Z - X} D T_{\text{tot}}(X) + \frac{\eta - \theta}{Z - X} D^2 T_{\text{tot}}(X) + \frac{k}{(Z - X)^3} \]

(4.70)

However, I failed in finding something to add to \( G_0(Z) \) for keeping the \( N = 2 \) linear superconformal algebra. It is somehow surprising, since the superfields \( G_p \) and \( \bar{G}_p \) have the right \( U(1) \) hypercharge to be \( N = 2 \) superfields\(^7\). With respect to \( T_{\text{tot}}(Z) \), the \( W \) algebra seems to be \( N = 1 \) but not \( N = 2 \) superconformal.

Now, from the above superOPEs (between superfields), one can deduce the OPEs (between fields), and, by comparison with the OPEs obtained in \( [9] \), one is able to give the combination to choose to get a factorized \( W \) algebra. Before doing it, I first compute the free superfield realization of the \( W \) algebra.

### 4.3 Free superfields realization

We start from \( J_{\text{diag}} \) and make a finite (residual) gauge transformation to get \( J_{\text{DS}} \):

\[ J_{\text{diag}} \rightarrow \hat{L} J_{\text{diag}} L^{-1} + k(DL)L^{-1} \equiv J_{\text{DS}} \quad \text{with} \quad L \in \mathcal{G}_+ \quad (4.71) \]

Demanding the transformed current to be in the DS-gauge fixes all the (super)parameters of \( L \) in terms of the free superfields of \( J_{\text{diag}} \). Putting these expressions into the transformed current gives the free superfields realization of the \( W \) algebra. I find:

\[
T(Z) = -(M^3)_{n+1,n+1} + (M^2)_{n+1,n+1} M_{n+1,n+1} + \frac{k}{2} \left( (DM) M + M DM \right)_{n+1,n+1} + \\
+ \left( -\sqrt{\frac{k}{2}} (M^2)_{n+1,n+1} + 2k \sqrt{\frac{n-1}{n+1} \Phi_+ M_{n+1,n+1}} \right) \Phi_- + \frac{k}{2} \frac{n-1}{n+1} \Phi_+ D \Phi_- + \\
+ k \sqrt{\frac{k}{2}} \frac{n-1}{n+1} \left( M_{n+1,n+1} D \Phi_+ + \Phi_+ D M_{n+1,n+1} \right) + k^2 \left( -\frac{1}{2} \Phi_- D \Phi_- - \sqrt{\frac{k}{2}} D^2 \Phi_- \right)
\]

\[
G_0(Z) = \frac{1}{2} (M^2)_{n+1,n+1} + \sqrt{\frac{k}{2}} \left( -M_{n+1,n+1} + \sqrt{\frac{k}{2}} \frac{n-1}{n+1} \Phi_+ \right) \Phi_- + \\
+ kD \left( -M_{n+1,n+1} + \sqrt{\frac{k}{2}} \frac{n-1}{n+1} \Phi_+ \right)
\]

\[
G_p(Z) = M_{p,q} M_{q,n+1} - \sqrt{\frac{k}{2}} M_{p,n+1} \left( \Phi_- - \sqrt{\frac{k}{2}} \frac{n-1}{n+1} \Phi_+ \right) - k D M_{p,n+1}
\]

\[
\bar{G}_p(Z) = -M_{n+1,q} M_{q,p} - \sqrt{\frac{k}{2}} \left( \Phi_- + \sqrt{\frac{k}{2}} \frac{n-1}{n+1} \Phi_+ \right) M_{n+1,p} + k D M_{n+1,p}
\]

\(^7\)I recall that a \( N = 1 \) superfield \( \Phi \) of superspin \( s \) can be seen as a \( N = 2 \) superfield of \( U(1) \) hypercharge \( \pm 2s \) if it transforms as \( G_0(Z) \Phi(X) = \pm \left\{ 2s \frac{\eta(X)}{X} + \frac{\eta - \theta}{X} D \Phi(X) \right\} \).
\[ N_{pq}(Z) = M_{pq} + \frac{1}{n} \delta_{pq} M_{n+1,n+1} \]

\[ U(Z) = -M_{n+1,n+1} + \sqrt{2k \frac{n^2}{n^2 - 1}} \Phi_+ \]

with \( \Phi_+ = \sqrt{\frac{1}{2k} \left( \Phi_1 + \Phi_2 \right)} \) and \( \Phi_- = \sqrt{\frac{1}{2k} \left( \Phi_1 - \Phi_2 \right)} \)

The total stress energy tensor is given by \( T_{tot}(Z) = \frac{1}{l^2} T(Z) + T_{Sl(n)}(Z) + T_{U(1)}(Z) \) with \( T_{Sl(n)}(Z) \) and \( T_{U(1)}(Z) \) given in (4.64-4.67). It writes

\[ T_{tot}(Z) = -\frac{1}{2} \Phi_+ D \Phi_+ - \frac{1}{2} \Phi_- D \Phi_- - \sqrt{\frac{k}{2}} D^2 \Phi_- + T_{Sl(n+1)}(Z) \]

\[ T_{Sl(n+1)}(Z) = \frac{1}{3k^2} Tr(M^3) + \frac{1}{2k} Tr(MDM) \]

From the form of \( T_{tot} \), one sees that \( \Phi_\pm \) are two free superfields of superspin \( \frac{1}{2} \), and that \( \Phi_+ \) is primary, whereas \( \Phi_- \) is not. This is in agreement with the last remark of section 2, since \( \Phi_- \) is carried by \( H \).

Using the superOPEs

\[ \Phi_\pm(Z)\Phi_\pm(X) = -\frac{1}{Z - X} \]

\[ \Phi_+(Z)\Phi_-(X) = 0 \]

\[ M_{ij}(Z)M_{lm}(X) = \frac{\eta - \theta}{Z - X} \{ \delta_{jl}M_{im}(X) - \delta_{im}M_{jl}(X) \} + \frac{k}{Z - X} \left( \delta_{jl}\delta_{im} - \frac{\delta_{ij}\delta_{lm}}{n + 1} \right) \]

and the above free superfields realization, one can compute the OPEs between the \( \mathcal{W} \) superfields, and check that we recover the superOPEs (4.47-4.61).

### 4.4 Factorisation of the fermions

To factorize out the fermions in the \( \mathcal{W} \) algebra, we have to use the fields expression (we want to fix the gauge for the supersymmetry). Let us first introduced some notation for the component fields of the \( \mathcal{W} \)-generators:

\[ T(Z) = \gamma(z) + \theta t(z) \]

\[ G_0(Z) = g_0(z) + \theta b_0(z) \]

\[ G_p(Z) = g_p(z) + \theta b_p(z) \]

\[ N_{pq}(Z) = \lambda_{pq}(z) + \theta \lambda_{pq}(z) \]

\[ U(Z) = c(z) + \theta u(z) \]

\[ \Phi_\pm(Z) = \psi_\pm(z) + \theta \varphi_\pm(z) \]

\[ M_{ij}(Z) = \mu_{ij}(z) + \theta \mu_{ij}(z) \]

The \( n^2 \) fermions that we want to factorize are the supersymmetric partners of the super KM \( Sl(n) \oplus U(1) \): \( \epsilon(z) \) and \( \lambda_{pq}(z) \) \( (p,q = 1,\ldots,n \text{ with } tr \lambda = 0) \). I will denote with a “\( o \)” the quantities built from the \( \mathcal{W} \) generators that (anti)commute with these fermions, and say that they are “free from fermions”. For instance, as \( N_{pq}(Z) \) and \( U(Z) \) anticommute and because \( U(Z) \) is a \( U(1) \) current, \( u(z) \) is free from fermions: \( \partial u(z) = u(z) \). On the contrary, it is (of course) impossible to construct with \( \epsilon(z) \) something free from fermions: \( \partial \epsilon = 0 \).
4.4.1 Computation of the factorized algebra

Let us start with the stress-energy tensor. \( t(z) \) is the stress energy tensor for the whole \( W \) algebra, and we want to keep from it the part that commutes with the fermions. For such a purpose, one just subtracts the stress energy tensor of the fermions:

\[
\overset{o}{t}(z) = t(z) + \frac{1}{2k} tr(\lambda(z) \partial \lambda(z)) - \frac{n-1}{2nk} \epsilon(z) \partial \epsilon(z) \quad (4.77)
\]

We can apply the same procedure to the \( Sl(n) \) KM algebra \( n_{pq}(z) \): we just have to subtract the fermionic realization of this algebra to get a KM algebra free from fermions:

\[
\overset{o}{n}_{pq}(z) = n_{pq}(z) - \frac{1}{k} \lambda_{pr}(z) \lambda_{rq}(z) \quad \text{and} \quad \overset{o}{u}(z) = u(z) \quad (4.78)
\]

Since we are dealing with classical fields, the level of \( \overset{o}{n} \) is the same as the one of \( n \): \( \overset{o}{k} = k \). The remaining spin 1 fields \( b_0(z) \), \( b_p(z) \) and \( \overset{o}{b}_p(z) \) do not required any work, since their OPEs show that they already commute with the fermions:

\[
\overset{o}{b}_0(z) = b_0(z) \quad ; \quad \overset{o}{b}_p(z) = b_p(z) \quad \text{and} \quad \overset{o}{b}_p(z) = \overset{o}{b}_p(z) \quad (4.79)
\]

Note that although \( \overset{o}{n}_{pq}(z) \), \( \overset{o}{b}_r(z) \), \( \overset{o}{b}_s(z) \), \( \overset{o}{b}_0(z) \) and \( \overset{o}{u}(z) \) form an \( Sl(n+1) \oplus U(1) \) KM algebra, they are not the usual basis for this algebra, since the \( Sl(n) \) part has been singled out. I give the connection with the usual basis in the next section.

Let us look at the spin 3/2 fields. \( g_0(z) \) is already free from fermions, as it can be seen on the superOPEs between \( G_0(Z) \) and \( N_{pq}(Z) \), \( U(Z) \). \( \gamma(z) \) being the generator of the linear supersymmetry, we deduce that the part we have to subtract is the just the supersymmetry generator of an \( Sl(n) \oplus U(1) \) superKM algebra.

\[
\overset{o}{\gamma}(z) = \gamma(z) + \frac{n-1}{nk} \epsilon(z) u(z) - \frac{2}{3k^2} tr(\lambda^3)(z) - \frac{1}{k} tr(\lambda(z)n(z)) \quad \text{and} \quad \overset{o}{g}_0(z) = g_0(z) \quad (4.80)
\]

The treatment of \( \gamma(z) \) differs from the one of \( t(z) \) because the former requires the whole \( Sl(n) \oplus U(1) \) algebra, while the latter involves only the fermions. This means that we cannot make the factorization at the superfield level, as it should be clear from the section 2. Finally, looking at the superOPEs of \( G_p(Z) \) and \( \overset{o}{G}_p(Z) \) with \( N_{pq}(Z) \) and \( U(Z) \), it is quite easy to deduce:

\[
\overset{o}{g}_p(z) = g_p(z) - \frac{1}{k} \lambda_{pq}(z) b_q(z) + \frac{n-1}{2nk} \epsilon(z) b_p(z) \quad (4.81)
\]
\[
\overset{o}{\overset{o}{g}}_p(z) = \overset{o}{g}_p(z) + \frac{1}{k} \overset{o}{b}_q(z) \lambda_{qp}(z) - \frac{n-1}{2nk} \overset{o}{b}_p(z) \epsilon(z) \quad (4.82)
\]

4.4.2 Comparison with the \( Sl(2) \) \( W \) algebra

Let us first compare our free fields with the ones of [9]. The spin contents in the diagonal gauge is the same for the two approaches. However, the fermions in [9] do not belongs to \( A_0 \).
(see section 3.2). Moreover, the fermions $\mu_{ij}(z)$ form a representation of the spin 1 bosons $m_{ij}(z)$, whereas in [9] the bosons commute with the fermions. Thus, even in the diagonal gauge, we have to make some combination to recover the fields of [9]. In this gauge, the task is easy. Using a superscript "JOK" (for Jens-Ole and Katsuhi) to denote the fields of [9], we get:

$$\chi^{JOK}_{n+1}(z) = \frac{1}{\sqrt{k}} \left( \mu_{n+1,n+1} + \sqrt{\frac{k}{2n+1}} \psi - \sqrt{\frac{k}{2n+1}} \psi_+ \right)$$

$$\tilde{\chi}^{JOK}_{n+1}(z) = \frac{1}{\sqrt{k}} \left( \mu_{n+1,n+1} - \sqrt{\frac{k}{2n+1}} \psi - \sqrt{\frac{k}{2n+1}} \psi_+ \right)$$

$$\tilde{j}^{JOK}_{ij}(z) = m_{ij}(z) - \frac{1}{k} \mu_{il}(z) \mu_{lj}(z)$$

For the $W$ generators, we first have to find the usual basis for the $Sl(n+1) \oplus U(1)$ KM algebra. The procedure is more or less the inverse of the one used to singled out the $Sl(n)$ subalgebra in $Sl(n+1)$. We use the $b$'s to complete the algebra, construct a $U(1)$ part that commutes with them, and relax the traceless condition for the $Sl(n)$ subalgebra. After some calculations, one finds:

$$J_{p,n+1}(z) = -\frac{1}{k} \tilde{b}_p(z)$$

$$J_{n+1,n+1}(z) = \frac{2n}{k(n+1)} \tilde{b}_0(z) - \frac{n-1}{n+1} \hat{u}(z)$$

$$v(z) = 2 \hat{u}(z) - \frac{2}{k} \tilde{b}_0(z)$$

Using the $\tilde{j}^{JOK}$-fields introduced in (4.83), one recover the expressions given in [9] with:

$$t^{JOK}(z) = \tilde{t}(z)$$

$$2G^{JOK}_{n+1}(z) = \sqrt{k} \tilde{\tau}(z) - \frac{2}{\sqrt{k}} \tilde{g}_0(z)$$

$$2\tilde{G}^{JOK}_{n+1}(z) = \sqrt{k} \tilde{\tau}(z) + \frac{2}{\sqrt{k}} \tilde{g}_0(z)$$

$$J^{JOK}_{ij}(z) = J_{ij}(z)$$

which end the proof of the equality between the factorized $\mathcal{W}$-algebra and the $W$-algebra obtained from $Sl(2)$ reduction.

One may wonder if it is possible to add more fermions to the $\tilde{j}^{JOK}$-algebra, so that the other non-linear supersymmetry generators $G_p$ and $\tilde{G}_p$ become linear. Let me first remark that the $W$ algebras we get have central extension terms, so that the linearization will be possible only up to $N = 4$ linear supersymmetries [13]. Moreover, the $OSp(1|2)$ reduction give rise to $N = 1, 2, 3, 4$ linear supersymmetry, depending on the choice of the $OSp(1|2)$ under consideration (see last section of [8] for more details). Thus, it seems clear that the factorization of spin 1/2 fields between $OSp(1|2)$ and $Sl(2)$ reduction will directly provide the maximal number of linear supersymmetries. However, the possibility for these linear supersymmetries to be associated to the total stress energy tensor is still an open question (see end of section 1.2).
5 Conclusion

In this paper, I have considered the Toda models constructed from the Hamiltonian reduction of a WZW theory built on a Lie superalgebra. I have shown that the action $S$ obtained by the reduction with respect to an $Sl(2)$ subalgebra can be set manifestly supersymmetric if and only if the $Sl(2)$ algebra is the bosonic part of an $OSp(1|2)$ superalgebra. The two actions $S$ and $S$ obtained from $Sl(2)$ and $OSp(1|2)$ reductions are related in the following way: starting from $S$, one first fixes the gauge for the supersymmetry and get an action $S_0$. Then, there is a choice of (KM-)gauge where $S_0$ and $S$ differs only by kinetic term of spin 1/2 fields (bosons and fermions). As a consequence, the $W$ algebra which is a symmetry of $S$ can be deduced from the super $W$ algebra, symmetry of $S$, by factorizing out the spin 1/2 fields. This relation has been explicitly carried out for the quadratic superconformal algebra built on $Sl(n + 1|2)$.

A lot of problems are still open on this subject. First, one can wonder whether this kind of relation is still valid at the quantum level. As the spin contents of the $W$ (and $\mathcal{W}$) algebras are not changed, one can already notice that these quantum algebras will differ only by spin 1/2 fields. However, a general treatment on the quantum actions is still to be done. The detailed study of ”quasi-Abelian” $W$ algebras has to be achieved both for classical and quantum case. One may also wonder if some $W$ algebras can be related to some others by a kind of factorization of higher spins: such studies are in progress [16]. Of course, the factorization of the bosonic spin 1/2 fields have to be considered in details. In particular, a proof of the factorization apart from the Toda action environment is needed. Note also that, even for spin 1/2 fermions, there exists no systematic way for doing the factorization: this problem will be solve in [16]. Finally, the total number of linear supersymmetries closing on the total stress energy tensor (or another weaker stress energy tensor) has to be studied.
Acknowledgements

I would like to thank K. Ito, J.O. Madsen and J.L. Petersen for a lot of fruitful discussions, and F. Delduc and P. Sorba for reading this manuscript and many advice.

References

[1] For a general review on $W$ algebras, see P. Bouwknegt and K. Schoutens, Phys. Reports 223 (1993) 183, and references therein.

[2] For a general review on non Abelian Toda theories, see L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Report 222 (1992) no 1, and references therein.

[3] A.N. Leznov and M.V. Saveliev, Comm. Math. Phys. 89 (1983) 59; D.A. Leites, M.V. Saveliev and V.V. Serganova, in Proceedings of the Third Yurmale Seminar (VUN Science, Utrecht, The netherlands, 1986).

[4] K. Hornfeck, Preprint KCL-TH-92-9/DFTT-70/92.

[5] H.G. Kausch, Ph.D thesis, Cambridge University (1991).

[6] W. Eholzer, A. Honecker and R. Hübel, preprint BONN-HE-93-08

[7] L. Fehér, L. O’Raifeartaigh, P. Ruelle and I. Tsutsui, Preprint BONN-HE-93-14/DIAS-STP-93-02.

[8] L. Frappat, E. Ragoucy and P. Sorba, preprint ENSLAPP-A-391/92, to be published in Comm. Math. Phys.

[9] K. Ito and J.O. Madsen, Phys. Lett. B283 (1992) 223.

[10] F. Delduc, P. Sorba and E. Ragoucy, Comm. Math. Phys. 146 (1992) 403.

[11] V.G. Drinfeld and V.V. Sokolov, Jour. Sov. Math. 30 (1985) 1975.

[12] P. Goddard and A. Schwimmer, Phys. Lett. B214 (1988) 209.

[13] K. Ito, Int. Journ. Mod. Phys. A7 (1992) 4885. G.M.T Watts, Nucl. Phys. B361 (1991) 311.

[14] A.M. Polyakov, Int. Journ. Mod. Phys. A5 (1990) 833.

[15] K. Schoutens, Nucl. Phys. B295 (1988) 634.

[16] F. Delduc, L. Frappat, E. Ragoucy, P. Sorba and F. Toppan, in preparation.
| Superalgebra | \(Sl(2)\) embedding | Corresponding \(OSp(1|2)\) |
|--------------|-------------------|-------------------------|
| \(Sl(n + 1|m + 1)\) | \(Sl(p + 1) \oplus Sl(p)\) \(p \leq (m, n - 1)\) & \(Sl(p + 1|p)\) |
| | \(Sl(p) \oplus Sl(p + 1)\) \(p \leq (m - 1, n)\) & \(Sl(p|p + 1)\) |
| \(OSp(m|2n)\) | \(SO(2p + 1) \oplus Sp(2p)\) \(p \leq ([m - 1]/2, n)\) & \(OSp(2p + 1|2p)\) |
| | \(SO(2p - 1) \oplus Sp(2p)\) \(p \leq ([m + 1]/2, n)\) & \(OSp(2p - 1|2p)\) |
| | \(SO(2p) \oplus Sp(2p)\) \(p \leq ([m]/2, n)\) & \(OSp(2p|2p)\) |
| | \(SO(2p + 2) \oplus Sp(2p)\) \(p \leq ([m - 2]/2, n)\) & \(OSp(2p + 2|2p)\) |
| | \(Sl(p + 1) \oplus Sl(p)\) \(p \leq ([m - 2]/2, 2n)\) & \(Sl(p + 1|p)\) |
| | \(Sl(p) \oplus Sl(p + 1)\) \(p \leq ([m]/2, 2n - 1)\) & \(Sl(p|p + 1)\) |
| \(G(3)\) | \(Sl(2)\) | \(Sl(2|1)\) |
| | \(Sl(2)'\) | \(Sl(2|1)\)' |
| | \(Sl(2)''\) | \(OSp(1|2)\) |
| | \(Sl(2) \oplus Sl(2)\) | \(OSp(3|2)\) |
| | \(Sl(2) \oplus Sl(2) \oplus Sl(2)\) | \(D(2, 1; 3)\) |
| \(F(4)\) | \(Sl(2)\) | \(Sl(2|1)\) |
| | \(Sl(2)'\) | \(Sl(1|2)\) |
| | \(Sl(2)''\) | \(OSp(2|2)\) |
| | \(Sl(2) \oplus Sl(2) \oplus Sl(2)\) | \(D(2, 1; 2)\) |
| \(D(2, 1; \alpha)\) | \(Sl(2)\) | \(Sl(2|1)\) |
| | \(Sl(2)'\) | \(OSp(2|2)\) |
| | \(Sl(2) \oplus Sl(2) \oplus Sl(2)\) | \(D(2, 1; \alpha)\) |

Table 1: Basic \(Sl(2)\) embeddings that lead to a "virtually superconformal" \(W\) algebra
| Superalgebra              | Spin contents (Bosons)                                      | Spin contents (Fermions)                                      |
|--------------------------|-------------------------------------------------------------|--------------------------------------------------------------|
| $Sl(m+1|n+1)$             | $W_j, \ (j = 2, \ldots, m+1)$                              | $2W_j \ (j = \frac{|m-n|}{2}, \ldots, \frac{m+n+2}{2})$     |
|                          | $W_1$                                                       |                                                              |
|                          | $W_j, \ (j = 2, \ldots, n+1)$                              |                                                              |
| $OSp(2m+1|2n)$            | $W_{2j}, \ (j = 1, \ldots, m)$                             | $W_j \ (j = \frac{2m-2n+1}{2} + 1, \ldots, \frac{2m+2n+1}{2})$ |
|                          | $W_{2j}, \ (j = 1, \ldots, n)$                             |                                                              |
| $OSp(2m|2n)$              | $W_{2j}, \ (j = 1, \ldots, m-1) \ ; \ W_m$                 | $W_j \ (j = \frac{2m-2n-1}{2W_{n+1/2}} + 1, \ldots, \frac{2m+2n-1}{2})$ |
|                          | $W_{2j}, \ (j = 1, \ldots, n)$                             |                                                              |
| $G(3)$                   | $W_6 \ ; \ 2W_2$                                           | $W_{9/2} \ ; \ W_{7/2}$                                     |
| $F(4)$                   | $W_6 \ ; \ W_4 \ ; \ 2W_2$                                 | $W_{9/2} \ ; \ W_{7/2} \ ; \ W_{3/2}$                      |

Table 2: Quasi-Abelian $W$ algebras