On the characters of pro-$p$ groups of finite rank

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1 Introduction

We will say that a pro-$p$ group $G$ is **perfect** if all derived subgroups $G^{(k)}$ are open. Note that a pro-$p$ group is perfect if and only if

$$\lambda_i(G) = |\{\lambda \in \text{Irr}(G)| \lambda(1) = p^i\}|$$

is finite for any $i \geq 0$. Here $\text{Irr}(G)$ is the set of the characters of the irreducible smooth complex representations of $G$. When $G$ is uniform (a torsion-free powerful pro-$p$ group), $G$ is perfect if and only if $[G,G]$ is open in $G$.

In this note we shall investigate the function

$$\zeta_{ch}^G(s) = \sum_{i=0}^{\infty} \lambda_i(G)p^{-is} = \sum_{\lambda \in \text{Irr}(G)} |\lambda(1)|^{-s},$$

when $G$ is perfect $p$-adic analytic pro-$p$ group. The main result is the following theorem.

**Theorem 1.1.** Let $G$ be a perfect $p$-adic analytic pro-$p$ group. Then $\zeta_{ch}^G(s)$ is a rational function in $p^s$ if one of the following conditions holds:

1. $G$ is a uniform pro-$p$ group or
2. $p > 2$.

The proof of this theorem is based on the correspondence between the characters of a uniform pro-$p$ group and the orbits of the action of the group on the dual of its Lie algebra. This result is an analogue of the Kirillov theory, introduced first in the context of nilpotent Lie groups and then used in many other situations (see [7]). The correspondence is quite explicit and it also gives the exact formula for characters in some cases. It permits us to obtain a stronger version of Theorem 1.1 in these cases.

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We introduce a generalization of the notion of a powerful pro \( p \)-group: we say that a pro \( p \)-group \( G \) is \( k \)-\textbf{powerful} if \([G,G] \leq G^p^k\). We say that a pro-\( p \) group is \( k \)-\textbf{uniform} if \( G \) is \( k \)-powerful without torsion. We have the following generalization of Theorem 1.1.

\textbf{Theorem 1.2.} Let \( N \) be a \( 1 \)-uniform (uniform) pro-\( p \) group if \( p \geq 5 \) and \( 2 \)-uniform pro-\( p \) group if \( p = 3 \). Then for any \( g \in N \),

\[ \sum_{\lambda \in \text{Irr}(G)} \lambda(g)|\lambda(1)|^{-s} \]

is a rational function in \( p^s \).

In Section 2 we describe Howe’s version of Kirillov’s correspondence. The correspondence permits us “linearize” the problem and apply the potent tools of \( p \)-adic integration which we introduce in Section 3. In Section 4 we prove Theorem 1.1(1) and Theorem 1.2. Section 5 is dedicated to some results from the character theory of finite groups. We apply these results in Section 6, where we prove Theorem 1.1(2).

The notation is standard. If \( G \) is a profinite group, then \( \text{Irr}(G) \) denotes the set of the (complex) irreducible smooth (not only linear) characters of \( G \). If \( L \) is a \( \mathbb{Z}_p \)-Lie algebra of finite rank, then \( \text{Irr}(L) \) is the set of the irreducible characters of the additive group of \( L \) and \( L^* = \text{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p) \) is the dual of \( L \). The set \( \text{Ch}(G) \) is the set of the admissible smooth characters of \( G \). So any element of \( \text{Ch}(G) \) is a finite sum of elements from \( \text{Irr}(G) \). The scalar product of character we denote by \( < , > \). If \( N \) is a normal group of finite index and \( \chi \in \text{Irr}(N) \), then

\[ \text{Irr}(G|\chi) = \{ \lambda \in \text{Irr}(G) | < \lambda_N , \chi > \neq 0 \} \]

and \( \text{Ch}(G|\chi) \) is the set of the finite sums of elements from \( \text{Irr}(G|\chi) \). We will say that characters from \( \text{Ch}(G|\chi) \) lie over \( \chi \). We will use \( [ , ]_l \) to denote the Lie brackets and \( [ , ]_G \) for the group commutator. If \( g \) is an element of a group, \( o(g) \) will mean the order of element \( g \). The order of a non-zero complex number is its order in \( \mathbb{C}^* \).

\section{Correspondence between characters and coadjoint orbits}

In this section we describe the correspondence between characters of a uniform pro-\( p \) group \( N \) and coadjoint orbits of action of \( N \) on the dual of the Lie algebra associated with \( N \). This section is based on the paper \[5\], where almost all proofs of the results of this section can be found (at least in the case \( p > 2 \)).

Remind that if \( N \) is a uniform pro-\( p \) group, then from Lazard (see, for example, \[4\]) we know that we can associate a \( \mathbb{Z}_p \)-Lie algebra \( L \) with \( N \). This Lie algebra can be identified with \( N \) as a set and the Lie operations are defined by

\[ g + h = \lim_{n \to \infty} (g^p^n h^p^n)^{p^{-n}}, \quad [g,h]_L = \lim_{n \to \infty} [g^p^n, h^p^n]_G^{p^{-n}}. \]
Since \((L, +)\) is a pro-
\(p\) group, we can consider the set \(\text{Irr}(L)\). Note that \(N\) acts on \(L\) by conjugation, whence \(N\) also acts on \(\text{Irr}(L)\). It is easy to see that all \(N\)-orbits of the last action are finite.

**Theorem 2.1.** Let \(N\) be a 1-uniform (uniform) pro-
\(p\) group if \(p \geq 5\) and 2-uniform pro-
\(p\) group if \(p = 3\) and let \(\Omega\) be a \(N\)-orbit in \(\text{Irr}(L)\). Define \(\Phi_{\Omega} : N \to \mathbb{C}\) by means of
\[
\Phi_{\Omega}(u) = |\Omega|^{-\frac{1}{2}} \sum_{\omega \in \Omega} \omega(u).
\]
Then \(\Phi_{\Omega} \in \text{Irr}(N)\) and all characters of \(G\) have this form. Moreover, two different orbits \(\Omega_1\) and \(\Omega_2\) give two different \(\Phi_{\Omega_1}\) and \(\Phi_{\Omega_2}\).

If \(N\) satisfies the hypothesis of the previous theorem and it is also perfect, then we obtain that \(\lambda_i(N)\) is equal to the number of \(N\)-orbits in \(\text{Irr}(L)\) of size \(p^{2i}\). Next theorem shows that this is true for all uniform pro-
\(p\) groups.

**Theorem 2.2.** Let \(N\) be a uniform pro-
\(p\) group, \(\alpha \in \text{Irr}(N)\) and \(\Omega\) a \(N\)-orbit in \(\text{Irr}(L)\). If \(\langle \alpha, \Phi_{\Omega} \rangle \neq 0\), then \(\alpha(1) = \Phi_{\Omega}(1)\). In particular, if \(N\) is perfect, then \(\lambda_i(G)\) is equal to the number of \(N\)-orbits in \(\text{Irr}(L)\) of size \(p^{2i}\).

If \(w \in \text{Irr}(L, +)\), we define
\[
B_w(l, k) = w([l, k]_\mathbb{L}), \ l, k \in L.
\]
Then \(B_w\) is a bilinear form on \(L\). Put \(\text{Rad}(w) = \{l \in L | B_w(l, L) = 1\}\). An important point is that \(\text{Rad}(w)\) is equal to \(\text{St}_{\mathcal{N}}(w) = \{g \in N | w^g = w\}\). Hence by the previous theorem, we have that if \(N\) is a uniform pro-
\(p\) group then
\[
\zeta^b_{\mathcal{N}}(s) = \sum_{w \in \text{Irr}(L)} |L : \text{Rad}(w)|^{-\frac{1}{2}(s-2)}.
\]
Therefore we are interested in the function \(\zeta(s) = \sum_{w \in \text{Irr}(L)} |L : \text{Rad}(w)|^{-s}\).

### 3 P-adic integration

In this section we give an explanation of the notion of \(p\)-adic integral and we introduce the facts that we will use later. More detailed discussion of this subject can be found in [1, 2, 4].

We use the standard notation for \(p\)-adic sets. \(|\cdot|\) is the standard \(p\)-adic valuation on \(\mathbb{Q}_p\): if \(a \in p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p\), entonces \(|a| = p^{-k}\). \(\mu\) will be the Haar measure on \(\mathbb{Q}_p^\times\). We always suppose that \(\mu(\mathbb{Z}_p^\times) = 1\).

Let \(X = (X_1, \ldots, X_M)\) be \(M\) commuting indeterminates and let \(\mathbb{Q}_p[[X]]\) denote the set of formal power series over \(\mathbb{Q}_p\). We define the following subsets of \(\mathbb{Q}_p[[X]]\):

1. \(\mathbb{Z}_p[[X]]\) denotes the set of power series over \(\mathbb{Z}_p\);
2. \(\mathbb{Q}_p(X)\) consists of all formal power series \(\sum_i a_i X^i\) such that \(|a_i| \to 0\) as \(|i| \to \infty\).
3. \( Z_p \{ X \} = Z_p[[X]] \cap \mathbb{Q}_p \{ X \} \).
We define the function \( D : \mathbb{Z}_p^2 \to \mathbb{Z}_p \) by
\[
D(x, y) = \begin{cases} 
\frac{x}{y} & \text{if } |x| \leq |y| \text{ and } y \neq 0, \\
0 & \text{otherwise.} 
\end{cases}
\]

For \( n > 0 \) we define \( P_n \) to be the set of nonzero \( n \)th powers in \( \mathbb{Z}_p \).

We define the following language considered in [2]:

Let \( L_{an}^D \) be the language with logical symbol \( \forall, \lor, \land, = \), a contable number of variables \( X_i \) and

1. an \( m \)-place operation symbol \( F \) for each \( F(X) \in Z_p \{ X \} \), \( m \geq 0 \);
2. a binary operation symbol \( D \);
3. a unitary relation symbol \( P_n \) for each \( n \geq 2 \).

Each formula \( \phi(x_1, \cdots, x_M) \) in the language \( L_{an}^D \) defines a subset
\[
M_\phi = \{ x \in Z_p^M | \phi(x) \text{ is true in } \mathbb{Z}_p \},
\]
where we interpret

1. each \( F \in Z_p \{ X \} \), as a function \( f: \mathbb{Z}_p^m \to \mathbb{Z}_p \) defined by \( f(x) = F(x) \);
2. the binary operation symbol \( D \) as the function \( D \);
3. \( P_n(x) \) to be true if \( x \in P_n \).

We call such subset \( M_\phi \) definable. A function \( f: V \to \mathbb{Z}_p \) is called definable if its graph is definable subset. Note that, in particular, a definable function is bounded.

With each pair of definable functions \( f_1: \mathbb{Z}_p^M \to \mathbb{Z}_p, f_2: \mathbb{Z}_p^M \to \mathbb{Z}_p \) and the definable subset \( U \) of \( \mathbb{Z}_p^M \) we associate the following function:
\[
I(f_1, f_2, U, s) = \int_U |f_1(x)|^s |f_2(x)| d\mu.
\]
We shall call this function definable integral.

**Theorem 3.1.** Suppose that \( I(f_1, f_2, U, s) \) is a definable integral. Then

1. \( U \) is measurable, and
2. \( I(f_1, f_2, U, s) \) is a rational function in \( p^{-s} \).
4 The uniform case

We begin this section with some known facts about endomorphisms of $\mathbb{Z}_p^n$. Let $A, B \in M_n(\mathbb{Z}_p)$ be two matrices over $\mathbb{Z}_p$. We write $A \sim B$ if there are two invertible matrices $C_1$ and $C_2$ over $\mathbb{Z}_p$ such that $C_1AC_2 = B$.

If $A = (a_{ij})_{1 \leq i,j \leq n}$ and $U \subseteq \{1, \cdots, n\}$ we put $g_U(A) = \det[(a_{ij})_{i,j \in U}]$.

For any $1 \leq i \leq n$ we fix an order on subsets of $\{1, \cdots, n\}$ with $i$ elements. We put $h_0(A) = 1$ and let for $1 \leq i \leq n$ $h_i(A)$ be $g_U(A)$ such that $|U| = i$ and for any $U' \subseteq \{1, \cdots, n\}$ with $|U'| = i$ we have $|g_{U'}(A)| < |g_U(A)|$ or $|g_{U'}(A)| = |g_U(A)|$ and $U \leq U'$.

**Lemma 4.1.** Let $A \in M_n(\mathbb{Z}_p)$ be a matrix over $\mathbb{Z}_p$. Then

1. There are $s_1, \cdots, s_n \in \mathbb{Z}_p$, satisfying $|s_i| \geq |s_{i+1}|$ for any $1 \leq i \leq n$, such that $A \sim \text{diag}(s_1, \cdots, s_n)$. Moreover, if $A \sim \text{diag}(t_1, \cdots, t_n)$ with $|t_i| \geq |t_{i+1}|$ for any $1 \leq i \leq n$, then $|t_i| = |s_i|$.

2. For any $0 \leq i \leq n-1$, $|D(h_i(A), h_{i-1}(A))| \geq |D(h_{i+1}(A), h_i(A))|$ and

$$A \sim \text{diag}(h_1(A), D(h_2(A), h_1(A)), \cdots, D(h_n(A), h_{n-1}(A))).$$

**Proof.** It is enough to observe that $A \sim \text{diag}(s_1, \cdots, s_n)$ if and only if $\mathbb{Z}_p^n/A(\mathbb{Z}_p^n) \cong \bigoplus_{i=1}^n \mathbb{Z}_p/s_i\mathbb{Z}_p$. □

It is not difficult to see that if $A \sim B$ then $|h_i(A)| = |h_i(B)|$. Therefore, sometimes we will speak about $|h_i(\phi)|$, where $\phi \in \text{Hom}_{\mathbb{Z}_p}(M_1, M_2)$ and $M_1 \cong M_2 \cong \mathbb{Z}_p^n$. Hence the previous lemma implies the following result

**Lemma 4.2.** Let $M_1 \cong M_2 \cong \mathbb{Z}_p^n$ be two $\mathbb{Z}_p$-modules, $0 \neq z \in \mathbb{Z}_p$ and $\phi \in \text{Hom}_{\mathbb{Z}_p}(M_1, M_2)$. Then $|M_1/\phi^{-1}(zM_2)| =

\begin{align*}
\begin{cases}
1 & \text{if } |z| > |h_1(\phi)|, \\
|z|^{-k}|h_k(\phi)| & \text{if } |D(h_k(\phi), h_{k-1}(\phi))| \geq |z| > |D(h_{k+1}(\phi), h_k(\phi))|, 1 \leq k \leq n-1 \\
|z|^{-n}|h_n(\phi)| & \text{if } |z| \leq |D(h_n(\phi), h_{n-1}(\phi))|.
\end{cases}
\end{align*}

**Proof.** By the previous lemma, we can choose bases of $M_1$ and $M_2$ such that the matrix $A$ associated with $\phi$ in these bases is diagonal and equal to

$$\text{diag}(h_1(A), D(h_2(A), h_1(A)), \cdots, D(h_n(A), h_{n-1}(A))).$$

Since $|h_i(A)| = |h_i(\phi)|$ for all $i$, we obtain the lemma. □

Now, let $N$ be a uniform pro-$p$ group and $L$ the Lie algebra associated with $N$. Put $L^* = \text{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$. Since $N$ acts on $L$, $L^*$ has a structure of $N$-module.

Note that $\text{Irr}(L)$ is the direct limit of $\text{Irr}(L/p^iL)$. For each $i \geq 0$ we fix $\theta_i$ a $p^i$th primitive root of 1 in $\mathbb{C}$ and we construct a $N$-homomorphism $\Phi_i : L^* \rightarrow \text{Irr}(L/p^iL)$ in the following way:

$$\Phi_i(m)(l + p^iL) = \theta_i^{m(l)}.$$

The kernel of $\Phi_i$ is equal to $p^iL^*$ and so $\Phi_i$ is surjective.
Proof. Let \( t \) be a \( L \)-module.

Lemma 4.4. Let \( \Phi_t \) be the map defined in the following way: if \( l, k \) are non-negative integers and \( z \) is a \( \mathbb{Z}_p \)-module, then
\[
\Phi_t(z)(l, k) = (m([k, l])).
\]

Therefore, \( \{ \Phi_t \} \) can be also constructed in a canonical way.

Remark 4.3. Note also that \( L^* \) is \( N \)-isomorphic to the inverse limit of \( (\text{Irr}(L/p^kL), \phi_{k,j}) \) where the homomorphism \( \phi_{k,j} : \text{Irr}(L/p^kL) \to \text{Irr}(L/p^jL) \) \((k > j)_i\) is defined by means of
\[
\phi_{k,j}(m)(l + p^iL) = m(p^{j-i}l + p^kL).
\]

Therefore, \( \{ \Phi_t \} \) can be also constructed in a canonical way.

For any \( 0 \neq z \in \mathbb{Z}_p \) such that \( |z| = p^{-i} \) we define \( \Phi_z = \Phi_t \).

Let \( \Psi : L^* \to \text{Hom}_{\mathbb{Z}_p}(L, L^*) \) be the map defined in the following way: if \( l, k \) are non-negative integers and \( m \) is a \( \mathbb{Z}_p \)-module, then
\[
\Psi(m)(l)(k) = (m([k, l])).
\]

Lemma 4.4. Let \( m \in L^* \) and \( 0 \neq z \in \mathbb{Z}_p \). Then
\[
\text{Rad}(\Phi_z(m)) = (\Psi(m))^{-1}(zL^*).
\]

Proof. Let \( |z| = p^{-i} \). Then we have the following series of equivalent propositions:
\[
x \in \text{Rad}(\Phi_z(m)) \iff \Phi_z(m)([y, x]) = 1 \forall y \in L \iff \\
\theta^m_1(y, x) = 1 \forall y \in L \iff m([x, y]) \in z\mathbb{Z}_p \forall y \in L \iff \\
\Psi(m)(x)(y) \in z\mathbb{Z}_p \forall y \in L \iff \\
\text{Rad}(\Phi_z(m)) \subseteq (\Psi(m))^{-1}(zL^*).
\]

\(\square\)

Let \( \{e_1, \cdots, e_n\} \) be a basis of \( L \) and \( \{f_1, \cdots, f_n\} \) a basis of \( L^* \). So, any element \( a \) from \( L \) or \( L^* \) is identified with a vector \((a_1, \cdots, a_n)\) and we can view \( \text{Hom}_{\mathbb{Z}_p}(L, L^*) \) as \( \mathbb{M}_n(\mathbb{Z}_p) \). Then the entries of \( \Psi(a), a \in L^* \) are linear functions on \( \{a_i\} \) and \( \text{g}_v(\Psi(a)) \) are polynomials on \( \{a_i\} \) for every \( U \subseteq \{1, \cdots, n\} \). This implies that \( h_i(\Psi(a)) \) are definable functions in \( \mathbb{L}_n^{an} \).

Define the sets \( W = (L^* \setminus pL^*) \times (p\mathbb{Z}_p \setminus \{0\}) \),
\[
W_0 = \{(a, z) \in W \mid |z| > |h_1(\Psi(a))|\},
\]
for any \( 1 \leq k \leq n - 1 \),
\[
W_k = \{(a, z) \in W \mid |D(h_k(\Psi(a)), h_{k-1}(\Psi(a)))| \geq |z| > |D(h_{k+1}(\Psi(a)), h_k(\Psi(a)))|\}
\]
and
\[
W_n = \{(a, z) \in W \mid |z| \leq |D(h_n(\Psi(a)), h_{n-1}(\Psi(a)))|\}.
\]
If \( (a, z) \in W_k \) define \( \alpha(a, z) = D(z^k, h_k(\Psi(a))) \). Then \( \alpha \) is a definable function on \( W \). Using two previous lemmas, we obtain that

Corollary 4.5. Let \( (a, z) \in W \). Then \( |L : \text{Rad}(\Phi_z(a))| = |\alpha(a, z)|^{-1} \).

Now, suppose that \( N \) is perfect. In this case we have the next important property:
Lemma 4.6. Let $N$ be a perfect uniform pro-$p$ group and $L$ the Lie algebra associated with $N$. Then the number

$$m(L) = \min \{ k | p^k L \subseteq [L, L] \}$$

is finite and for any $w \in \text{Irr}(L)$ we have $o(w) \leq |L : \text{Rad}(w)|p^{m(L)}$.

Proof. The finiteness of $m(L)$ follows from the perfectness of $L$.

Since $p^{m(L)} L \leq [L, L]_L$, $|L : \text{Rad}(w)|p^{m(L)} L \leq [L, \text{Rad}(w)]_L$. Hence

$$w([L : \text{Rad}(w)]p^{m(L)} L) = 1,$$

and so $|L : \text{Rad}(w)|p^{m(L)} \geq o(w)$. 

Corollary 4.7. Let $(a, z) \in W$. Then

$$|p^{m(L)} \alpha(a, z)| \leq |z|.$$ 

Proof. Note that if $w = \Phi_z(a)$, then $|z| = o(w)^{-1}$. 

Theorem 4.8. Let $N$ be a perfect uniform pro-$p$ group. Then $\zeta_N^p(s)$ is a rational function in $p^s$.

Proof. Let $L$ be the Lie algebra associated with $N$ and $n$ the $\mathbb{Z}_p$-rank of $L$. First note that if $(a, z) \in W$ and $w = \Phi_z(a)$, then $|z| = o(w)^{-1}$. Hence

$$\mu(\{(a, z) \in W | \Phi_z(a) = w\}) = (p-1)p^{-1}o(w)^{-(n+1)}.$$ 

Therefore

$$\zeta(s) - 1 = \sum_{1_L \neq w \in \text{Irr}(L)} |L : \text{Rad}(w)|^{-s} =$$

$$\sum_{1_L \neq w \in \text{Irr}(L)} \int_{(a,t) \in W, \Phi_z(a) = w} \frac{p(p-1)^{-1}}{|z|^{(n+1)}} |L : \text{Rad}(\Phi_z(a))|^{-s} dadz =$$

$$p(p-1)^{-1} \int_W \frac{1}{|z|^{(n+1)}} |L : \text{Rad}(\Phi_z(a))|^{-s} dadz =$$

$$p(p-1)^{-1} \int_W |z|^{-(n+1)} |\alpha(a, z)|^s dadz.$$ 

By Corollary 4.7, we have that if $(a, z) \in W$, then

$$|z|^{-1} = |z|^{-1} |p^{m(L)} \alpha(a, z)||p^{m(L)} \alpha(a, z)|^{-1} =$$

$$|D(p^{m(L)} \alpha(a, z), z)||p^{m(L)} \alpha(a, z)|^{-1}.$$ 

Therefore we have:

$$\int_W |z|^{-(n+1)} |\alpha(a, z)|^s dadz =$$

$$p^{m(L)(n+1)} \left( \int_W |D(p^{m(L)} \alpha(a, z), z)|^{n+1} |\alpha(a, z)|^{s-n-1} dadz \right)$$
Now from Theorem 3.1 we obtain that the last integral is a rational function in \( p^s \), whence \( \zeta(s) \) is a rational function in \( p^s \). Note that since \( \zeta(s) = \sum a_i p^{-2i s} \), we have that \( \zeta(s) \) is a rational function in \( p^{2s} \). We conclude from the last paragraph of Section 2 that \( \zeta_N^h(s) \) is a rational function in \( p^s \).

Let \( N \) be a perfect uniform pro-\( p \) group, \( g \in N \) and let \( \mu_i \) be the sum of all irreducible characters of \( N \) of degree \( p^i \). Then

\[
\sum_{\lambda \in \text{Irr}(G)} \lambda(g) |\lambda(1)|^{-s} = \sum_{i=0}^{\infty} \mu_i(g) p^{-si}.
\]

Therefore the following theorem implies Theorem 1.2.

**Theorem 4.9.** Let \( N \) be a 1-uniform (uniform) pro-\( p \) group if \( p \geq 5 \) and 2-uniform pro-\( p \) group if \( p = 3 \). Then for any \( g \in N \),

\[
\sum_{i=0}^{\infty} \mu_i(g) p^{-si}
\]

is a rational function in \( p^s \).

**Proof.** By Theorem 2.1,

\[
\mu_i = \sum_{|\Omega| = p^{2i}} \Phi_\Omega = p^{-i} \sum_{w \in \text{Irr}(L), |L: \text{Rad}(w)| = p^{2i}} w.
\]

Hence

\[
\sum_{i=0}^{\infty} \mu_i(g) p^{-si} = \sum_{w \in \text{Irr}(L)} w(g) |L: \text{Rad}(w)|^{-(s+2)/2}.
\]

Let \( \theta_m \) be a \( p^m \)-th primitive root of 1. If the order of \( w \) is equal to \( p^m \geq p \), then

\[
\sum_{\sigma \in \text{Gal}(\mathbb{Q}(\theta_m)/\mathbb{Q})} w^\sigma(g) = \begin{cases} (p - 1)p^{m-1} & \text{if } g \in \text{Ker } w \\ -p^{m-1} & \text{if } o(w(g)) = p \\ 0 & \text{if } o(w(g)) > p. \end{cases}
\]

Therefore we have

\[
\sum_{i=0}^{\infty} \mu_i(g) p^{-si} = \sum_{w \in \text{Irr}(L), o(w(g)) = 1} |L: \text{Rad}(w)|^{-(s+2)/2} - \sum_{p-1}^{p-1} \sum_{w \in \text{Irr}(L), o(w(g)) = p} |L: \text{Rad}(w)|^{-(s+2)/2}.
\]

Now, note that the sets \( W_1 = \{(a, z) \in W | a(g) \equiv 0 \pmod{z} \} \) and \( W_2 = \{(a, z) \in W | pa(g) \equiv 0 \pmod{z} \} \) are definable. Following the proof of the previous theorem, we obtain that

\[
\sum_{w \in \text{Irr}(L), o(w(g)) = 1} |L: \text{Rad}(w)|^{-s} - 1 = p(p - 1)^{-1} \int_{W_1} |z|^{-(n+1)} |\alpha(a, z)|^s da dz,
\]
and
\[ \sum_{w \in \text{Irr}(L), \ o(w(g))=p} |L : \text{Rad}(w)|^{-s} = p(p-1)^{-1} \int_{W_2 \setminus W_1} |z|^{-(n+1)}|\alpha(a, z)|^s \text{d}adz. \]

Hence, as in the previous theorem, we conclude that \[ \sum_{i=0}^{\infty} \mu_i(g)p^{-si} \] is a rational function in \( p^s \).

5 Character triples

Let \( G \) be a group, \( N \) a normal subgroup of \( G \) of finite index and \( \theta \in \text{Irr}(N) \) \( G \)-invariant irreducible character of \( N \). Under these hypotheses we say that \( (G, N, \theta) \) is a character triple. We refer the reader to \([6, \text{Section 11}]\) for general discussion on character triples. The main idea consist in replacing \( (G, N, \theta) \) by another character triple \( (\Gamma, A, \lambda) \) in which \( \Gamma/A \cong G/N \) and \( \lambda \) is linear. Moreover, the character theory of extensions of \( \lambda \) on \( \Gamma \) and \( \theta \) on \( G \) is the same. (The next definition explains what this means exactly.)

Definition 5.1. Let \( (G, N, \theta) \) and \( (H, M, \phi) \) be character triples and \( \tau: H/M \to G/N \) be an isomorphism. For every such \( T \), suppose there exists a map \( \delta_T: \text{Ch}(T|\theta) \to \text{Ch}(T|\phi) \) such that the following conditions hold for \( T, K \) with \( M \leq K \leq T \leq H \) and \( \chi, \psi \in \text{Ch}(T|\theta) \).

1. \( \delta_T(\chi + \psi) = \delta_T(\chi) + \delta_T(\psi) \);
2. \( \langle \chi, \psi \rangle = \langle \delta_T(\chi), \delta_T(\psi) \rangle \);
3. \( \delta_K(\chi_K) = (\delta_T(\chi)K^\tau) \);
4. \( \delta_T(\chi\beta) = \delta_T(\chi)\beta^\tau \) for \( \beta \in \text{Irr}(T/M) \).

Let \( \delta \) denote the union of the maps \( \delta_T \). Then \( (\tau, \delta) \) is an isomorphism from \( (G, N, \theta) \) to \( (H, M, \phi) \).

Now, let \( p > 2 \) and \( G \) be a \( p \)-adic analytic pro-\( p \) group. We know that \( G \) has a 2-uniform subgroup \( N \) of finite index. Let \( L \) be the Lie algebra associated with \( N \). Let \( \chi \) be an irreducible character of \( N \). Then, from Theorem 2.4 it follows that there exists a \( G \)-orbit \( \Omega \) in \( \text{Irr}(L) \), such that \( \chi = \Phi_\Omega \). Let \( w \in \Omega \). By [2, Theorem 6.11], in order to understand the character theory of extensions of \( \chi \) on \( G \) we should investigate the character theory of extensions of \( \chi \) on \( I = I_G(\chi) = \{ g \in G | g^\varrho = \chi \} \). This problem is solved in the next theorem.

Theorem 5.2. Let \( G, N, I, L, \chi, \Omega \) and \( w \) as before. Then \( w_{\text{St}_N(w)} \in \text{Irr}(\text{St}_N(w)) \) and \( (I, N, \chi) \) is isomorphic to \( (\text{St}_G(w), \text{St}_N(w), w_{\text{St}_N(w)}) \).
This result is practically Proposition 1.2 from [5]. But, since we use a different notation, we present a sketch of the proof.

**Proof.** First, by [5, Lemma 1.4], there exists a subgroup $A$ of $N$ such that

1. $A$ is a uniform pro-$p$ group and so it is also a Lie subalgebra of $L$;
2. $B_w(A, A) = 1$ and $A$ is maximal with this property;
3. $w_A$ is also a groups character of $A$ respect multiplication and $\chi = (w_A)^N$;
4. $A$ is $\text{St}_G(w)$-invariant.

Put $S = \text{St}_G(w)A$. Then we have that $I = SN$ and $A = S \cap N$. Note that $w_A$ is $S$-invariant. First, we will see that $(I, N, \chi)$ and $(S, A, w_A)$ are isomorphic.

Let $\tau$ be the natural isomorphism between $S/A = SN/N$ and $I/N = SN/N$. $T$ a subgroup of $S$, which contains $A$, and $T^\tau$ the inverse image in $I$ of $\tau(T/A)$. Hence $T^\tau = TN$. Define $\delta_T$: $\text{Ch}(T|w_A) \to \text{Ch}(T^\tau|\chi)$ by means of

$$\delta_T(\alpha) = \alpha_{T^\tau}, \alpha \in \text{Ch}(T|w_A).$$

Since $w_N^N = \chi$ and $N$ fixes $\chi$, $\delta_T$ sends $\text{Ch}(T|w_A)$ to $\text{Ch}(T^\tau|\chi)$.

We have to check four properties of Definition 5.1. The fist one is clear. In order to prove the second one, observe that

$$(w_A)^{T^\tau} = ((w_A)^N)^{T^\tau} = \chi^{T^\tau}.$$ 

In particular,

$$< (w_A)^{T^\tau}, (w_A)^{T^\tau} >= < \chi^{T^\tau}, \chi^{T^\tau} >= < \chi, (\chi^{T^\tau})^N >= |T^\tau/N|.$$ 

On the other hand,

$$(w_A)^{T^\tau} = ((w_A)^T)^{T^\tau} = (\sum_{\lambda \in \text{Irr}(T|w_A)} \lambda(1)\lambda)^{T^\tau} = \sum_{\lambda \in \text{Irr}(T|w_A)} \lambda(1)^{T^\tau}.$$ 

Hence we have

$$|T^\tau/N| = < (w_A)^{T^\tau}, (w_A)^{T^\tau} >= \sum_{\lambda, \mu \in \text{Irr}(T|w_A)} \lambda(1)\mu(1) < \chi^{T^\tau}, \mu^{T^\tau} >= \sum_{\lambda \in \text{Irr}(T|w_A)} \lambda(1)^2 = |T/A| = |T^\tau/N|.$$ 

This implies that

$$< \lambda^{T^\tau}, \mu^{T^\tau} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}.$$ 

But this is exactly the second condition.

The third property is a direct consequence of [6] Problem 5.2 and the forth one can be obtained from [6, Problem 5.3], bearing in mind that $\beta^\tau = \beta$.

Now we will see that $(S, A, w_A)$ and $(\text{St}_G(w), \text{St}_N(w), w_{\text{St}_N(w)})$ are isomorphic. Since $\text{St}_N(w) = \text{St}_G(w) \cap A$ we have that $S/A$ and $\text{St}_G(w)/\text{St}_N(w)$ are isomorphic. Let $\tau$ be the natural isomorphism between $S/A$ and $\text{St}_G(w)/\text{St}_N(w)$,
Let $Q$ be a finite group and $F$ be a free group on $|Q|$ variables (so $F$ is generated by $x_q$, $q \in Q$). Define an homomorphism $\phi: F \to Q$ by means of $\phi(x_q) = q$. Let $H$ be the kernel of this homomorphism. Put $\tilde{F} = F/[H,F]$ and $H = H/[H,F]$. Then $H$ is an abelian group of finite rank. Hence we can write $H = A \oplus B$ where $A$ is torsion-free and $B$ is a torsion group. Let $\beta$ and $\alpha$ be $F$-invariant linear characters of $H$. Since they are $F$-invariant, we can see $\alpha$ and $\beta$ as characters of $\tilde{H}$. We will need the following criterion.

**Lemma 5.3.** With the previous notation suppose that for every $b \in B \cap [\tilde{F}, \tilde{F}]$ the orders of $\alpha(b)$ and $\beta(b)$ are same. Then the triples $(F,H,\alpha)$ and $(F,H,\beta)$ are isomorphic.

**Proof.** We split the proof in a number of steps.

**Step 1.** Let $\gamma$ be a linear character of $\tilde{H}$ which is trivial on $B \cap [\tilde{F}, \tilde{F}]$. Then there a character $\tau$ of $F$ such that $\tau_H = \gamma$.

Note that the commutator of $\tilde{F}$ is finite, so $[\tilde{F}, \tilde{F}] \cap \tilde{H} = B \cap [\tilde{F}, \tilde{F}]$. Hence $\gamma$ is also trivial on $[\tilde{F}, \tilde{F}] \cap \tilde{H}$. Therefore we can see $\gamma$ as a character of $C = H/([\tilde{F}, \tilde{F}] \cap \tilde{H})$. But $C$ can be seen as a subgroup of $\tilde{F}/[\tilde{F}, \tilde{F}]$ and so we can extend $\gamma$ on $\tilde{F}/[\tilde{F}, \tilde{F}]$.

**Step 2.** Let $\gamma$ be a linear character of $\tilde{H}$ which is trivial on $B \cap [\tilde{F}, \tilde{F}]$. Then $(F,H,\alpha)$ and $(F,H,\alpha \gamma)$ are isomorphic.

We will indicate the isomorphism. By the previous step there exists $\tau \in \text{Irr}(F)$ which extends $\gamma$. Let $T$ be a subgroup of $F$ which contains $H$ and $\chi$ a character of $T$ lying over $\alpha$. We put $\delta_T(\chi) = \chi \tau_T$. It is not difficult to see that it is really isomorphism of triples.

**Step 3.** Final step.

The conditions of lemma imply that $\alpha_{B \cap [\tilde{F}, \tilde{F}]}$ and $\beta_{B \cap [\tilde{F}, \tilde{F}]}$ are conjugated by an element $\sigma$ of the Galois group of $Q$. Hence $\alpha = \beta^\sigma \gamma$, where $\gamma$ is a linear character of $\tilde{H}$ which is trivial on $B \cap [\tilde{F}, \tilde{F}]$.

By the previous step, $(F,H,\alpha)$ and $(F,H,\beta^\sigma)$ are isomorphic. On the other hand, $(F,H,\beta)$ and $(F,H,\beta^\sigma)$ are also isomorphic.

Now let $\{s_i = s_i(x_q | q \in Q)\} \subset [F,F]$, be a finite set of elements of $F$ such that $B \cap [\tilde{F}, \tilde{F}] = \{s_i[H,F]\}$. We denote by $S_Q$ the set $\{s_i\}$.

Now, let $N \to G \xrightarrow{\phi} Q$ be an extension of $Q$, i.e. $\phi$ is a surjective homomorphism with kernel $N$. Let $\{t_q | q \in Q\}$ be a transversal for $N$ in $G$, such that $\phi(t_q) = q$. Let $\alpha$ be a $G$-invariant linear character of $N$. Define a vector

$$S_Q(G,N,\phi,\alpha) = (o(\alpha(s_1(t_q | q \in Q)))), \ldots, o(\alpha(s_k(t_q | q \in Q)))),$$

where $k = |S_Q|$ and $o(r)$ means the order of $r$. 

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Corollary 5.4. 1. The vector $S_Q(G, N, \phi, \alpha)$ does not depend on the choice of transversal for $N$ in $G$.

2. There exists only finite number of possibilities for $S_Q(G, N, \phi, \alpha)$.

3. If $N_1 \to G_1 \overset{\phi_1}{\to} Q_1$ and $N_2 \to G_2 \overset{\phi_2}{\to} Q_2$ are two extensions of $Q$, $\alpha_i$ is a $G_i$-invariant linear character of $N_i$ for $i = 1, 2$ and $S_Q(G_1, N_1, \phi_1, \alpha_1) = S_Q(G_2, N_2, \phi_2, \alpha_2)$, then $(G_1, N_1, \alpha_1)$ and $(G_2, N_2, \alpha_2)$ are isomorphic.

6 The general case

Throughout of this section we suppose that $p > 2$ and $G$ be a $p$-adic analytic pro-$p$. We can find an open normal 2-uniform subgroup $N$ of $G$. Let $L$ be the Lie algebra associated with $N$. Since $G$ acts on $L$, $G$ also acts on $L^*$. We will use the notation of Section 4. Let $\{e_1, \cdots, e_n\}$ be a basis of $L$ and $\{f_1, \cdots, f_n\}$ a basis of $L^*$. So, any element $a$ from $L$ or $L^*$ is identified with a vector $(a_1, \cdots, a_n)$ from $\mathbb{Z}_p^n$.

Fix a right transversal $(t_i | i = 1, \ldots, m)$ for $N$ in $G$ and suppose that $K = \bigcup_{i=1}^m t_i N$.

Note that $g_i(a) = a^{t_i}$, $a \in L^*$ is a linear function of $a$ and $f(a, g) = a^g$, $a \in L^*, g \in L$ is an analytic function of $(a, g)$. Define a formula $F_i$ in the language $L_{\alpha}^{\text{an}}$ in the following way.

$$F_i := \exists g ~ g_i(a) \equiv f(a, g)( \text{ mod } z).$$

Hence $W_K$ is equal to the following definable set

$$\{ (a, z) \in W | F_1(a, z) \& \cdots \& F_n(a, z) \& \neg F_{1+n}(a, z) \& \cdots \& \neg F_m(z, a) \text{ is true in } \mathbb{Z}_p \}.$$
Lemma 6.2. The set $W_{K,v} = \{(a, z) \in W_K | \Phi_2(a) \in \text{Irr}(L)_{K,v}\}$ is definable.

Proof. Note that $a(s_i(y_qa_q)|q \in Q))$ is an analytic map from $L^* \times L^Q$ to $\mathbb{Z}_p$ and for any $q \in Q$, the function $g_q(a) = a^{q^i}$, $a \in L^*$ is a linear function of $a$. Also the function $f(a, g) = a^q$, $a \in L^*$, $g \in L$ is an analytic function of $(a, g)$. Define the following formula $G_{K,v}$ in $L^{an}_G$:

\[
G_{K,v} : = \exists (n_q | q \in Q) \forall q \in Q \ g_q(a) \equiv f(a, -n_q)( \mod z) \& \n (\forall j \ (a(s_j(y_qa_q))) \equiv 0 ( \mod z) \& v = 1) \n (a(s_j(y_qa_q))v \equiv 0 ( \mod pz)).
\]

In the first row of the formula we find a transversal $(y_qa_q|q \in Q)$ for $\text{St}_N(\Phi_2(a))$ in $\text{St}_G(\Phi_2(a))$ and in the second and third we check the condition $V(a, z) = v$. Hence we have $W_{K,v} = \{(a, z) \in W_K | G_{K,v}(a, z) \text{ is true in } \mathbb{Z}_p\}$. Hence $W_{K,v}$ is definable.

Now, we are ready to prove the following theorem.

Theorem 6.3. Let $p > 2$ and $G$ be a $p$-adic analytic perfect pro-$p$ group. Then $\zeta^{ch}_{G}(s)$ is a rational function in $p^s$.

Proof. We conserve the previous notation. For any $N \leq K \leq G$ we define

\[
\zeta^{ch}_{G,K}(s) = \sum_{\lambda \in \text{Irr}(G|\chi), \chi \in \text{Irr}(N), I_G(\chi) = K} |\lambda(1)|^{-s}.
\]

Note that if $\lambda \in \text{Irr}(G|\chi), \chi \in \text{Irr}(N)$ and $I_G(\chi) = K$ then $\lambda$ lies over $|G/K|$ irreducible characters of $N$. Hence we have

\[
\zeta^{ch}_{G}(s) = \sum_{N \leq K \leq G} \frac{1}{|G : K|} \zeta^{ch}_{G,K}(s).
\]

So, we should prove the rationality of $\zeta^{ch}_{G,K}(s)$ for each $K$. Note that

\[
\zeta^{ch}_{G,K}(s) = |G/K|^{-s} \sum_{\chi \in \text{Irr}(N), I_G(\chi) = K} f_{\chi} |\chi(1)|^{-s}, \quad (6.1)
\]

where $f_{\chi} = \sum_{\lambda \in \text{Irr}(K|\chi)} |\lambda(1)|^{-s}$. Consider an arbitrary character $\chi \in \text{Irr}(N)$ such that $I_G(\chi) = K$. Then, from Theorem 5.3 it follows that there exists a $G$-orbit $\Omega$ in $\text{Irr}(L)$, such that $\chi = \Phi_2$. Let $w \in \Omega$. Since $I_G(\chi) = K$, we have $w \in \text{Irr}(L)_{K}$. By Theorem 6.2 $(K, N, \chi)$ is isomorphic to $(\text{St}_G(w), \text{St}_N(w), w_{\text{St}_N(w)})$. Hence

\[
f_{\chi} = \sum_{\lambda \in \text{Irr}(\text{St}_G(w)|w_{\text{St}_N(w)})} |\lambda(1)|^{-s}.
\]

For $w \in \text{Irr}(L)$ define $f_w = \sum_{\lambda \in \text{Irr}(\text{St}_G(w)|w_{\text{St}_N(w)})} |\lambda(1)|^{-s}$. Then the equality (6.1) can be rewritten as

\[
\zeta^{ch}_{G,K}(s) = |G/K|^{-s} \sum_{w \in \text{Irr}(L)_{K}} f_w |L : \text{Rad}(w)|^{-(s-2)/2}.
\]
Now, note that if $w_1, w_2 \in \text{Irr}(L)_{K,v}$ then, by Corollary 5.4, $(\text{St}_G(w_1), \text{St}_N(w_1), (w_1)_{\text{St}_N(w_1)})$ and $(\text{St}_G(w_2), \text{St}_N(w_2), (w_2)_{\text{St}_N(w_2)})$ are isomorphic and, in particular, $f_{w_1} = f_{w_2}$. Hence in order to prove the rationality of $\zeta_{\text{ch}}^{\text{ch}}(s)$ it is enough to prove that $\sum_{w \in \text{Irr}(L)_{K,v}} |L : \text{Rad}(w)|^{-s}$ is a rational function in $p^{-2s}$.

We do it in the same way as we proved Theorem 4.8 because we have that

$$
\sum_{w \in \text{Irr}(L)_{K,v}} |L : \text{Rad}(w)|^{-s} = p(p-1)^{-1} \int_{W_{K,v}} |z|^{-(n+1)} |\alpha(a,z)|^s \, da \, dz.
$$

The last integral we can transform to a definable integral using the argument of the proof of Theorem 4.8 because $W_{K,v}$ is a definable set by Lemma 6.2.

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