A SURVEY ON THE BICANONICAL MAP OF SURFACES WITH $p_g = 0$ AND $K^2 \geq 2$

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Abstract. We give an up-to-date overview of the known results on the bicanonical map of surfaces of general type with $p_g = 0$ and $K^2 \geq 2$.

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1. Introduction

Many examples of complex surfaces of general type with $p_g = q = 0$ are known, but a detailed classification is still lacking, despite much progress in the theory of algebraic surfaces. Surfaces of general type are often studied using properties of their canonical curves. If a surface has $p_g = 0$, then there are of course no such curves, and it is natural to look instead at the bicanonical system, which is not empty.

In this survey we describe the present (December 2001) state of knowledge about the bicanonical map for minimal surfaces of general type with $p_g = 0$ and $K^2 \geq 2$.

We do not consider the case $K_S^2 = 1$ (the so called numerical Godeaux surfaces) because, in what concerns the bicanonical map, this case is special (see [2]). We just remark that the numerical Godeaux surfaces are somewhat better understood than the other surfaces of general type with $p_g = 0$ and we refer to the paper [CT] and its bibliography.

This survey is organized as follows: in section 2 we discuss the dimension of the bicanonical image and in section 3 the base points of $|2K_S|$. In section 4 we present the bounds on the degree of the bicanonical map for $K_S^2 \geq 2$. In section 5 we discuss the surfaces that occur as bicanonical images, whilst in section 6 we describe a few relevant examples. Finally in section 7 we present some classification results and in section 8 we present a list of open problems.

For each of the results presented we only give a very rough sketch of the proof, referring to the relevant papers for the missing details.
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Notation and conventions. We work over the complex numbers; all varieties are assumed to be compact and algebraic. We do not distinguish between line bundles and divisors on a smooth variety, using the additive and the multiplicative notation interchangeably. Linear equivalence is denoted by $\equiv$. The rest of the notation is standard in algebraic geometry.

2. The dimension of the bicanonical image

Let $S$ be a minimal complex surface of general type with $p_g(S) = 0$. It is well known that:

- $q(S) = 0$,
- $1 \leq K_S^2 \leq 9$
- $P_2(S) := h^0(S, 2K_S) = 1 + K_S^2$.

We denote by $\varphi : S \to \mathbb{P}^{K_S^2}$ the bicanonical map of $S$ and by $\Sigma$ the image of $\varphi$. The first question one asks about the bicanonical map is what is the dimension of $\Sigma$. For $K_S^2 = 1$, one has $P_2(S) = 2$ and so $\Sigma$ is a curve. For $K_S^2 \geq 2$, the answer was given by Xiao Gang in the mid-eighties:

**Theorem 2.1.** (Xiao Gang, [X1]) Let $S$ be a minimal complex surface of general type with $p_g(S) = 0$. If $K_S^2 \geq 2$ then the image of the bicanonical map of $S$ is a surface.

**Sketch of proof.** We just explain the main ideas and refer the reader to [X1] for details.

By contradiction, suppose that $|2K_S|$ is composed with a pencil. Then one can show that necessarily $2K_S \equiv aF + Z$, where $|F|$ is a base point free genus 2 pencil and $a = K_S^2$. Using the very precise description of Horikawa ([Ho]) for the reducible fibres of a genus 2 fibration in terms of $K_S^2$ and $\chi(O_S)$, one shows that for $K_S^2 \geq 3$ the surface $S$ does not contain a genus 2 fibration, since otherwise the components of the reducible fibres would give $b_2(S)$ or more independent classes in $H^2(S, \mathbb{Z})$. For $K_S^2 = 2$, $S$ can have a genus 2 fibration with general fibre $F$, but one can use the same type of argument to show that it is impossible to decompose $2K_S$ as $2K_S \equiv 2F + Z$, i.e. that $|2K_S|$ is not composed with $|F|$.

\[\square\]
So the surfaces with $K^2_S = 1$ and $p_g = 0$, the numerical Godeaux, are in a class of their own. We just mention here that there is intensive work in progress on this subject by F. Catanese and R. Pignatelli and by Y. Lee, using in particular the bicanonical fibration. As already mentioned in the introduction, for more facts on numerical Godeaux one can see the paper [CP], which has also a very complete list of references.

3. The base points of the bicanonical system

Recall that, while for $p_g(S) > 0$ the bicanonical map is defined at every point of $S$ ([Bq], [Re], [F], CC2, cf. [Ci]), for $p_g(S) = 0$ it is still unknown whether $\varphi$ is always a morphism. For $K^2_S \geq 5$ we have:

**Theorem 3.1.** (Reider, [Re]) Let $S$ be a minimal surface of general type with $p_g = 0$ and let $\varphi: S \to \mathbb{P}^{K^2_S}$ be the bicanonical map.

If $K^2_S \geq 5$, then $\varphi$ is a morphism.

*Remark 1* This is a particular case of Reider’s theorem ([Re]) about adjoint systems, which only applies if $K^2_S \geq 5$.

*Remark 2* As far as we know, for all the known examples of surfaces of general type with $2 \leq K^2_S \leq 4$ and $p_g = 0$ the bicanonical map is a morphism.

For $K^2_S = 4$, Lin Weng ([W]) has proven that the base locus of the bicanonical system contains no $−2$–curve. This result has later been improved by Langer:

**Theorem 3.2.** (Langer, [La]) Let $S$ be a minimal surface of general type with $K^2_S = 4$ and $p_g(S) = 0$. Then the system $|2K_S|$ has no fixed component.

Still in the case $K^2_S = 4$, F. Catanese and F. Tovena ([CT]) and D. Kotschick ([Kd]) have related the existence of base points of the bicanonical system to properties of the fundamental group of the surface. Since the statements are quite technical, we just quote here the following consequence of their results:

**Theorem 3.3.** (Catanese-Tovena, [CT], Kotschick, [Kd]) Let $S$ be a minimal surface of general type with $K^2_S = 4$ and $p_g(S) = 0$. If $H^2(\pi_1(S), \mathbb{Z}_2) = 0$, then the bicanonical system $|2K_S|$ is base point free.
4. The degree

Once one knows that for $K^2 \geq 2$ the bicanonical image of a surface $S$ of general type with $p_g = 0$ is a surface, it is natural to look for bounds on the degree $d$ of the bicanonical map $\varphi$.

If $K^2_S = 2$, the bicanonical image is $\mathbb{P}^2$ and therefore $d \geq 2$. On the other hand, $(2K_S)^2 = 8$ implies $\deg \varphi \leq 8$, with equality holding if and only if $\varphi$ is a morphism. All the known examples with $K^2_S = 2$ have $\deg \varphi = 8$.

For higher values of $K^2_S$ we have:

**Theorem 4.1.** ([M]) Let $S$ be a minimal complex surface of general type such that $p_g(S) = 0$, $K^2_S \geq 3$ and let $\varphi: S \to \mathbb{P}^{K^2_S}$ be the bicanonical map of $S$. Then the degree of $\varphi$ is at most 5.

If $\varphi$ is a morphism (in particular if $K^2_S \geq 5$) then the degree of $\varphi$ is at most 4.

**Sketch of proof.** (See [M] for the complete proof). Let $d$ be the degree of $\varphi$ and let $m$ be the degree of the bicanonical image $\Sigma \subset \mathbb{P}^{K^2_S}$. Since $\Sigma$ is a non-degenerate surface in $\mathbb{P}^{K^2_S}$, one has $m \geq K^2_S - 1$. Write $|2K_S| = |M| + F$, where $M$ and $F$ are the moving part and the fixed part of the system, respectively. Notice that, if $F \neq 0$, then $M^2 < (2K_S)^2$, by the 2-connectedness of the bicanonical divisors. So we have $4K^2_S \geq md$ and equality holds if and only if $\varphi$ is a morphism. By an easy calculation we see that to prove the theorem it is enough to exclude the possibilities $K^2_S = 5$, $d = 5$, and $K^2_S = 3$, $d = 6$.

This is done by using the classification of surfaces of degree $n - 1$ in $\mathbb{P}^n$ (see [Nag]) to find the possibilities for $\Sigma$. Then, using the geometry of $\Sigma$, one is able to build irregular double covers of $S$, which in turn, by a theorem of De Franchis ([DF], see also [CC1]), yield special fibrations on $S$. Finally, with different “twists” for each case, the existence of such a fibration leads to a contradiction.

**Remark** As mentioned above, for all known examples of surfaces with $K^2_S > 1$ the bicanonical map is a morphism, and so the bound 5 of the theorem above may not be effective for $K^2_S = 3, 4$.

On the other hand, the bound 4, if $\varphi$ is a morphism, is effective, as shown by the Burniat surfaces with $K^2_S = 3, \ldots, 6$ ([Bu], [Pe], see also Example 3 of §3).

For high values of $K^2_S$ these bounds can be improved.

**Theorem 4.2.** ([MP]) Let $S$ be a minimal complex surface of general type such that $p_g(S) = 0$ and let $\varphi: S \to \mathbb{P}^{K^2_S}$ be the bicanonical map of $S$. Then one has the following bounds on $d := \deg \varphi$: [Further details can be added here based on the actual content of the document.]
i) if \( K^2_S = 9 \), then \( d = 1 \);
ii) if \( K^2_S = 7, 8 \), then \( d \leq 2 \);

**Sketch of proof.** (See [MP1] for the proof). For \( K^2_S = 9 \), one has \( c_2(S) = 3 \) and so \( b_2(S) = 1 \). The assertion is proven by combining Reider’s theorem ([Re]) and the fact that \( H^2(S, \mathbb{Q}) \) is generated by the class of an ample divisor \( D \) with \( D^2 = 1 \).

For \( K^2_S = 7 \), \( \varphi \) is a morphism, and therefore the degree \( d \) of \( \varphi \) is either 1, 2, or 4. If \( d = 4 \), then the bicanonical image \( \Sigma \) is a linearly normal surface of degree 7 in \( \mathbb{P}^7 \) with \( p_g = q = 0 \), and so it is the anti-canonical image of \( \mathbb{P}^2 \) blown-up at two points \( P, Q \). Combining the information obtained from the geometry of \( \Sigma \) and the fact that the second Betti number of \( S \) is small (\( b_2(S) = 3 \)), it is possible to find a contradiction, which shows that \( d = 4 \) does not occur.

For \( K^2_S = 8 \), the technique of proof is analogous. \( \square \)

**Remark** The bounds in this theorem are effective, since there are examples of minimal surfaces with \( p_g = 0 \) and \( K^2_S = 7, 8 \) for which the bicanonical map has degree 2 (see Examples 1 and 2 in Section 3).

## 5. The image

Another natural question that arises is finding the possibilities for the image of the bicanonical map \( \varphi \), if \( \varphi \) is not birational. A priori, one only knows that the image of \( \varphi \) is a surface with \( p_g = q = 0 \). It turns out that it is possible to be more precise, as we will see in the next theorem.

**Theorem 5.1.** (Xiao Gang, [X2], [MP3]) Let \( S \) be a minimal complex surface of general type such that \( p_g(S) = 0 \) and \( K^2_S \geq 2 \) and let \( \varphi : S \to \Sigma \subset \mathbb{P}^{K^2_S} \) be the bicanonical map of \( S \). If \( \varphi \) is not birational, then either

i) \( \Sigma \) is a rational surface,

or

ii) \( K^2_S = 3 \), \( \varphi \) is a morphism of degree 2 and \( \Sigma \subset \mathbb{P}^3 \) is an Enriques sextic.

**Sketch of proof.** (See [X2] and [MP3] for the proof). If the degree \( d \) of \( \varphi \) is bigger than 2, then \( \Sigma \) is a linearly normal surface with \( p_g = q = 0 \) and degree lesser than or equal to \( 2n - 2 \) in \( \mathbb{P}^n \), and so a rational surface. In [X2], Xiao Gang, using double covers techniques, showed that if \( d = 2 \) and \( \Sigma \) is not rational, then \( K^2_S \leq 4 \) and \( \Sigma \) is birationally equivalent to an Enriques surface with \( K^2_S + 4 \) nodes. The Enriques surfaces with 8 nodes are classified in [MP3]. Using the knowledge of the linear systems on these surfaces, one is able to determine precisely the surfaces of general type with \( p_g = 0 \) and \( K^2 = 4 \) whose bicanonical map factors
through a degree 2 map onto an Enriques surface. Such surfaces had
been previously constructed by D. Naie ([Nai]) and their bicanonical
map is of degree 4 onto a rational surface. Hence the possibility $K_S^2 = 4$
is excluded.

6. Some examples

In this section we give a quick description of some of the known
examples of surfaces with $p_g = 0$.

Example 1: Surfaces with $K_S^2 = 8$. All the examples known to us
of surfaces $S$ with $p_g = 0$ and $K_S^2 = 8$ are obtained by the following
construction, first suggested by Beauville (cf. [Be], [Do]). One takes
curves $C_1, C_2$ of genera $g_1$ and $g_2$, respectively, such that there exists a
group $G$ of order $(g_1 - 1)(g_2 - 1)$ that acts faithfully on both $C_1$ and $C_2$.
If the quotient curves $C_1/G$ and $C_2/G$ are rational and the diagonal
action of $G$ on $C_1 \times C_2$ is free, then $S := (C_1 \times C_2)/G$ is a minimal
surface of general type with $p_g = 0$ and $K_S^2 = 8$. By [Pa], the surfaces
with these invariants and bicanonical map of degree 2 are obtained by
this construction taking $C_1$ hyperelliptic of genus 3, and they belong
to four different families. Examples with birational bicanonical map
do exist. For instance, if one takes $C_1 = C_2$ to be the Fermat quintic,
then it is possible to let $G = \mathbb{Z}_5^2$ act on the two copies of the curve in
such a way that the diagonal action is free. The resulting surface has
birational bicanonical map.

Example 2: A surface with $K_S^2 = 7$. This example is due to Inoue
([In]), who constructs it by taking the quotient of a complete intersection
inside the product of four elliptic curves by a group isomorphic to $\mathbb{Z}_5^2$ acting freely. Alternatively, this surface can be constructed as
a $\mathbb{Z}_2^2$–cover of a rational surface with 4 nodes (see [MP1]). The bi-
canonical map has degree 2 and the bicanonical involution belongs to
the Galois group of the cover. The quotient of this surface by the
bicanonical involution is a rational surface with 11 nodes.

Example 3: Burniat surfaces. These surfaces were discovered by
Burniat ([Bu]) and studied later by Peters ([Pa]). They are obtained
by taking $\mathbb{Z}_2^2$–covers of the plane branched on a configuration of lines
as shown in Figure 1 in such a way that the images of the divisorial
components of the fixed loci of the three nonzero elements of the Galois
group of the cover are $l_{12} + m_1^1 + m_1^4$, $l_{23} + m_2^1 + m_2^4$ and $l_{13} + m_3^1 + m_3^4$. For a
general choice of the lines $m_j^i$, the resulting surface $Y$ is singular above
the points $P_1, P_2, P_3$ and the minimal resolution of $Y$ is obtained by
taking base change with the blow–up $\mathbb{P} \to \mathbb{P}^2$ of the plane at $P_1, P_2, P_3$
and then normalizing. In this way one obtains a minimal surface $S$ with $p_g = 0$ and $K^2_S = 6$. The bicanonical map is the composition of the induced $\mathbb{Z}_2^2$–cover $S \to \hat{\mathbb{P}}$ with the embedding of $\hat{\mathbb{P}}$ as a Del Pezzo sextic in $\mathbb{P}^6$.

Examples with the same properties and with $2 \leq K^2_S \leq 5$ can be obtained by letting one or more subsets of 3 lines $m_i^j$ go through the same point.
7. The limit cases

Surprisingly, the limit cases of Theorem 4.1 and Theorem 4.2, namely $K_S^2 = 7, 8$, $\deg \varphi = 2$, and $K_S^2 = 6$, $\deg \varphi = 4$, can be described precisely, as we will see in the next two theorems.

**Theorem 7.1.** (MP4) Let $S$ be a minimal complex surface of general type such that $p_g(S) = 0$ and $K_S^2 = 7, 8$. Let $\varphi: S \to \mathbb{P}^{K_S^2}$ the bicanonical map of $S$ and assume that $\deg \varphi = 2$. Then $K_S$ is ample and:

i) there exists a fibration $f: S \to \mathbb{P}^1$ such that the general fibre $F$ of $f$ is a smooth hyperelliptic curve of genus 3 and the bicanonical involution induces the hyperelliptic involution on $F$;

ii) if $K_S^2 = 8$ then $f$ is isotrivial and the singular fibres of $f$ are 6 double fibres with smooth support, while if $K_S^2 = 7$ then $S$ has 5 double fibres and exactly one fibre with reducible support.

**Remark 1** For $K_S^2 = 8$, the fact that $f$ is an isotrivial fibration whose only singular fibres are double fibres with smooth support implies that $S$ is one of the Beauville surfaces (see §6, Example 1). Using this fact it is possible to give a complete classification of these surfaces (see [Pa]). They belong to 4 different types, and the surfaces of each type form an irreducible connected component of the moduli space of surfaces of general type. An interesting feature of these surfaces is that they are smooth minimal models of double covers of the plane branched on a curve with certain singularities, a construction that had been suggested by Du Val ([DV], see also [Ci]). The expected number of parameters of the branch curve of this construction is negative, hence it seems very difficult prove directly its existence, that instead follows “a posteriori” from the classification of [Pa].

**Remark 2** For $K_S^2 = 7$ the hyperelliptic fibration is not isotrivial and a complete classification seems out of reach. In the Inoue’s surface (see §6), which is the only known example with $K_S^2 = 7$ and $\deg \varphi = 2$, the unique fibre with reducible support of the fibration $f$ is one of the double fibres. In principle one would expect this to be a special situation, hence it would be interesting to find examples where the reducible fibre is not a double fibre.

**Sketch of proof.** (See [MP4] for the proof). Consider the quotient $Y$ of $S$ by the bicanonical involution $\sigma$. $Y$ is a rational surface whose only singularities are $\nu = K_S^2 + 4$ nodes, which correspond to the isolated fixed points of $\sigma$. The minimal resolution of $Y$ is a rational surface
X having $\nu \geq b_2(X) - 3$ disjoint $-2$-curves. Such surfaces are characterized in [DMH], where it is shown in particular that there exists a fibration $g: Y \to \Sigma$ with rational fibres and with $[\frac{\nu}{2}]$ double fibres. Now, using some geometrical reasoning, one shows that $g$ pulls back to a fibration $f: S \to \mathbb{P}^1$ such that the general fibre of $f$ is hyperelliptic of genus 3.

The fact that $K_S$ is ample follows trivially for $K^2_S = 8$ from Miyaoka’s results ([M]) on the existence of rational curves on surfaces. For $K^2_S = 7$, the non-existence of $-2$-curves on $S$ is obtained by analyzing the structure of the unique reducible fibre of $f$ and using the equality $b_2(S) = 3$.

**Theorem 7.2.** ([MP2]) Let $S$ be a minimal complex surface of general type such that $p_g(S) = 0$ and $K^2_S = 6$ and let $\varphi: S \to \mathbb{P}^{K^2_S}$ the bicanonical map of $S$. Then:

$$\deg \varphi = 4 \text{ if and only if } S \text{ is a Burniat surface.}$$

In particular, $K_S$ is ample.

**Sketch of proof.** (See [MP2] for the proof). The first step in the proof of this theorem consists in showing that the bicanonical image $\Sigma$ of $S$ is the non-singular Del Pezzo surface of degree 6 in $\mathbb{P}^6$. This is shown by a case by case exclusion of all the possible singular such $\Sigma$.

The second step consists in showing that the sides of the “hexagon” of $-1$-curves of $\Sigma$ (cf. Fig. 2) are in the branch locus of $\varphi$. Using the curves in $S$ which correspond to these sides, one produces a subgroup $H$ of $\text{Pic}(S)$ such that $H \cong \mathbb{Z}_3^2$.

By studying the étale covers of $S$ given by the nonzero elements of $H$, it is possible to show that the three pencils of $\Sigma$ corresponding to the lines through $P_1, P_2$ and $P_3$ pull-back in $S$ to three genus 3 hyperelliptic pencils, each having two irreducible double fibres beside the two reducible ones given by pairs of sides of the hexagon. The images of these irreducible fibres are the remaining components of the branch locus of $\varphi$ (see Fig. 2).

Finally, one verifies that the bicanonical map is composed with the three involutions of $S$ induced by the hyperelliptic pencils. It follows that the bicanonical map is a $\mathbb{Z}_2 \times \mathbb{Z}_2$–cover and $S$ is a Burniat surface.

Using Theorem 7.2, one can obtain very precise information on the geometry of the moduli space of surfaces with $p_g = 0$, $K^2_S = 6$ and bicanonical map of degree 4.
Theorem 7.3. (MP2) Smooth minimal surfaces of general type $S$ with $K_S^2 = 6$, $p_g(S) = 0$ and bicanonical map of degree 4 form an unirational 4-dimensional irreducible connected component of the moduli space of surfaces of general type.

Sketch of proof. (See [MP2] for the proof). By the semicontinuity of $\deg \varphi$ and by Theorem 4.1, the surfaces with $\deg \varphi = 4$ are a closed subset of the moduli space.

Using the theory of natural deformations of abelian covers, one constructs explicitly a smooth family $\mathcal{X} \to B$ of $\mathbb{Z}_2^2$-covers of the plane blown up at three non collinear points with the following properties:

i) $B$ is smooth and irreducible;

ii) for every $b \in B$ the fibre $X_b$ is a Burniat surface and every Burniat surface occurs as a fibre for some $b \in B$;

iii) the family $\mathcal{X}$ is complete at every point $b$ of $B$.

This shows that the Burniat surfaces are an irreducible open subset of the moduli space. Hence, in view of Theorem 7.2, the surfaces with $\deg \varphi = 4$ are an irreducible open and closed subset of the moduli space.

Remark 3 The limit cases for the degree of the bicanonical map have some common properties. First of all, by Theorem 7.1 and Theorem 7.2, they all have ample canonical class. In addition, all surfaces with $K_S^2 = 6$ and $\deg \varphi = 4$, all surfaces with $K_S^2 = 8$ and $\deg \varphi = 2$ (see [Pa]) and the Inoue surfaces with $K_S^2 = 7$ move in positive dimensional families, while the expected dimension of the moduli at the corresponding points is zero.

8. SOME QUESTIONS

Here we point out some questions that arise naturally from the results outlined in the previous sections.

Question 1 (cf. 3) Is the bicanonical map $\varphi$ of surfaces with $p_g = 0$ a morphism also for $2 \leq K_S^2 \leq 4$?

Question 2 (cf. Theorem 4.1) Is there a surface with $p_g = 0$, $K_S^2 = 3$ or $K_S^2 = 4$ and $\deg \varphi = 5$? Notice that for such a surface $\varphi$ cannot be a morphism.

Question 3 (cf. Theorem 4.1 and Theorem 7.2) Is there is a surface with $p_g = 0$, $K_S^2 = 6$ and $\deg \varphi = 3$?

Question 4 (cf. Theorem 7.2) Is it possible to characterize surfaces with $K_S^2 = 5$, $p_g = 0$ and $\deg \varphi = 4$?
Question 5 (cf. §7, Remark 2) Are the Inoue surfaces the only surfaces with \( p_g = 0, \, K^2 = 7 \) and non birational bicanonical map?

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