NUMERICAL TESTS OF THE KUGO-OJIMA
COLOR CONFINEMENT CRITERION

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The Kugo-Ojima color confinement criterion, which is based on the BRST symmetry of QCD is numerically tested by the lattice Landau gauge simulation. We first discuss the BRST symmetry on the lattice and the Gribov copy problem. The horizon function defined by Zwanziger based on the $U$-linear definition of the gauge field and that based on $\log U$ are compared. The parameter $u$ which is expected to be $-1/\delta_{ab}$ in the continuum theory was found to be $-0.75\delta_{ab}$ in the strong coupling region. There are about 20% difference between the two horizon functions, but in the weak coupling region, the expectation value of the horizon function is suggested to be zero or negative.

1 Introduction

There are essentially two aspects in the manifestation of color confinement in QCD. One aspect is the linear potential between quarks, which was conjectured by Gribov as a consequence of an enhancement of the singularity of the ghost propagator due to the restriction on the gauge field $A$.

Another aspect is the absence of free single colored particle state in the asymptotic Hilbert space, which culminates in the Kugo and Ojima color confinement criterion based on the BRST(Becchi-Rouet-Stora-Tyutin) symmetry: i.e. in the Landau gauge, a coefficient in the two-point function produced by the ghost, the antighost and the gauge field becomes $-\delta_{ab}$, where $a$ and $b$ specify the color in the adjoint representation. Analytical calculation of this value is extremely difficult and so far no verification was performed.

In 1994, Zwanziger’s developed a lattice QCD theory for Gribov ambiguity. He claimed that, if the restriction to the fundamental modular region is achieved, the gluon propagator at 0 momentum should vanish in the continuum limit. In the Lattice QCD test, we address the following problems: whether the gluon propagator is infrared finite, how singular the ghost propagator is in the infrared region, and whether the Kugo-Ojima color confinement criterion.
is satisfied.

2 The lattice simulation of the Landau gauge QCD and the Gribov problem

2.1 The path integral formulation of the gauge fixed theory in the presence of the Gribov copy (Fujikawa, Hirschfeld)

First we give a brief review of the path integral formulation of the gauge fixed theory by Fujikawa \(^4\) and Hirschfeld \(^5\). They discussed a possible situation which may give a way out of the gauge fixing degeneracy problem (the Gribov problem). Let us define a gauge unfixed partition function as,

\[ Z = \int dU e^{-\beta S(U)}. \]  

(1)

In derivation of the Faddeev-Popov (Faddeev-Popov) formula of the gauge fixed theory \( f(U) = 0 \), one considers the determinant function \( \Delta(U^g) \),

\[ \Delta(U^g) = \det \left( \frac{\partial f(U^g)}{\partial U^g} \right) = \det \left( \frac{\partial f(U^g)}{\partial A^g} \right) \]

where \( A_\mu = A_\mu(U) \), and one finds that \( \Delta(U^g) \) is indeed a function of \( U^g \), and looks at the integral on the gauge orbit, \( U^g \),

\[ N(U) \equiv \int dg \Delta(U^g) \delta(f(U^g)). \]

Obviously \( N(U) \) is an orbit function, that is, \( N(U^g) = N(U) \). At the intersection points of the gauge orbit \( U^g \) with the surface \( f(U) = 0 \), i.e., \( g = g_i(U) \), Gribov’s copies, the above delta function is transcribed to give

\[ N(U) = \int dg \sum_i \delta(g - g_i(U)) = \int dg \sum_i \text{sign}(\Delta(U^g)) \delta(g - g_i(U)). \]

These delta functions contribute 0 or \( \pm 1 \), and thus, if the orbit function \( N(U) \) is non vanishing over all orbits, \( N(U) \neq 0 \), then in use of the identity

\[ 1 = \frac{1}{N(U)} \int dg \Delta(U^g) \delta(f(U^g)), \]

the standard FP procedure applies, and one factors out the gauge volume and obtains the formula for expectation values of functions, \( F(U) \),

\[ \langle F(U) \rangle|_{\text{gauge}} = \frac{\int dU \Delta(U) \delta(f(U)) F(U) e^{-\beta S(U)}}{\int dU \Delta(U) e^{-\beta S(U)}} \frac{e^{-\beta S(U)}}{N(U)}. \]  

(2)
This formula allows the BRST formulation as in the following, and also derives a natural simulation algorithm of the gauge fixed theory.

2.2 The simulation algorithm for the gauge fixed theory (Mandula-Ogilvie) and the Gribov problem

Multiplying by the gauge volumes, the denominator and the numerator of the formula (2), respectively, one recovers the $Z$ in the denominator, and has in the numerator,

$$
\int dU \int dg \Delta(U) \delta(f(U))F(U)e^{-\beta S(U)}/N(U).
$$

In use of (2.1), one obtains the numerator as

$$
\int dU \sum_i \text{sign}(\Delta(U_i))F(U_i)e^{-\beta S(U)}/N(U)
$$

where $U_i$ is the $i$-th Gribov copy on the orbit $U$. This gives us the algorithm in the simulation

$$
\langle F(U) \rangle|_{f\text{ gauge}} = \frac{1}{Z}\int dU \sum_i \frac{\text{sign}(\Delta(U_i))F(U_i)}{N(U)}e^{-\beta S(U)} = \langle \bar{F}(U) \rangle
$$

where the last averaging $\langle \rangle$ is that of simulation, i.e., that with respect to the Boltzmann weight $e^{-\beta S}$, and $\bar{F}(U)$ is a sign weighted average of $F(U_i)$ on the gauge orbit $U$.

However, in case of gauge non invariant function $F(U)$, this formula would rather be interpreted as representing Gribov noise. Thus a modification of the gauge such that one chooses a unique copy among others on the orbit is favored, that is, a new gauge without Gribov copy, and in that case, the above formula is useful in practice.

2.3 The BRST formulation and the Gribov problem

The obvious standard FP formula allows BRST formulation,

$$
\langle F(U) \rangle|_{f\text{ gauge}} = \frac{1}{Z'}\int d\mu \exp(\delta \int \bar{c}f(U))F(U)e^{-\beta S(U)}/N(U),
$$

$$
Z' = \int d\mu \exp(\delta \int \bar{c}f(U))e^{-\beta S(U)}/N(U),
$$
where $\delta$ stands for BRST transformation, and the measure is defined as $d\mu = dU dB d\bar{c} c$, where $dB$ integration is performed on $R$ and $d\bar{c} c$ is suitably defined differentiations with respect to Grassmann numbers, $c$ and $\bar{c}$.

It was shown by Neuberger that if the gauge fixing function $f(U)$ is a smooth function of compact variables $U$, then the expectation value of gauge invariant function $F(U)$ becomes an indefinite form,

$$\langle F(U) \rangle |_{f \text{ gauge}} = \frac{0}{0},$$

which implies that all Gribov copies contribute to give total cancellation, in other words, the assumption that the above $N(U)$ is non-vanishing on all orbits, does not hold, and the formula (8) is totally meaningless.

An essential point of his argument is as follows. Let us consider a general expression

$$Z'(t) = \int d\mu \exp(t\delta \int \bar{c} f(U)) G(U)$$

with a gauge invariant function $G(U)$ as the Boltzmann weight. Then one finds that

$$\frac{dZ'}{dt}(t) = \int d\mu \delta \left( \int \bar{c} f(U) \right) \exp(t\delta \int \bar{c} f(U)) G(U)$$

can be written from nilpotency of $\delta$, i.e., $\delta^2 = 0$, and from $\delta G(U) = 0$ as,

$$\frac{dZ'}{dt}(t) = \int d\mu \delta \left( \int \bar{c} f(U) \right) \exp(t\delta \int \bar{c} f(U)) G(U) \bigg).$$

If $W(U, B, c, \bar{c})$ is an analytic function of the compact variables $U$, then one can show that

$$\int d\mu \delta W(U, B, c, \bar{c}) = 0.$$  \hspace{1cm} (12)

Thus it follows that $\frac{dZ'}{dt}(t) = 0$. Together with $Z'(0) = \int \cdots \cdots \bar{c} G(U) = 0$, one finds that $Z'(t) = 0$, so $Z'(1) = 0$. This concludes total cancellation of the Gribov copies of gauges given by analytic gauge functions of $U$. Thus one is forced to consider non analytic gauge functions of $U$ as desired gauge functions. As a one dimensional $U(1)$ toy example which avoids (12), one can consider $U = e^{iA}$, $A = \text{Im log } U$, where $A$ is not continuous at $U = -1$, and with definition $\delta A = c$, one finds that

$$\int dU dB d\bar{c} c \delta (e^{-B^2} \bar{c} A) = \int_{-\pi}^{\pi} dA \int_{-\infty}^{\infty} dB d\bar{c} c e^{-B^2} \bar{c} c \neq 0.$$  \hspace{1cm} (13)

Although this problem is still open in the lattice BRST formulation, use of non compact variables in gauge fixing functions may be helpful.
2.4 The Landau gauge and the Gribov problem

Now we focus on the Landau gauge in $SU(3)$ QCD, that is, $f(U) = \partial_\mu A_\mu(U)$, where there are some options of definition $A_\mu(U)$ as

1. $U$-linear one: $A_{x,\mu} = \frac{1}{2}(U_{x,\mu} - U_{x,\mu}^\dagger)$ traceless part.

2. use of exponential map: $U_{x,\mu} = \exp A_{x,\mu}$, $A_{x,\mu}^\dagger = -A_{x,\mu}$, where absolute values of all eigenvalues of $A_{x,\mu}$ do not exceed $4\pi/3$.

In the latter definition, $A_{x,\mu}(U)$ is not analytic with respect to compact variable $U_{x,\mu}$ contrary to the former one. In both cases the Landau gauge, $\partial A^g = 0$, can be characterized in use of optimizing functions $F_U(g)$ of $g$, such that $\delta F_U(g) = 0$ for any $\delta g$.

1. $U$-linear definition: $F_U(g) = \frac{2N}{2V(N^2 - 1)} \sum_{x,\mu} \left( 1 - \frac{1}{N} \text{Re tr} U_{x,\mu}^2 \right)$.

2. use of exponential map: $F_U(g) = ||A^g||^2 = \frac{1}{2V(N^2 - 1)} \sum_{x,\mu} \text{tr} \left( A^g_{x,\mu} A^g_{x,\mu}^\dagger \right)$, where $N$ is the number of colors and $d$ is the dimension. It is noteworthy that $||A^g||^2$ is a continuous function of compact variables $U^g$ in spite of non analytic property of $A_{x,\mu}(U^g)$.

Gribov copy is generic phenomena in both definitions, there exist a lot of local minima of $F_U(g)$ along the gauge orbit $U^g$. Thus the naive Landau gauge loses its rigid basis both in the theoretical and in the simulation view points for examination of gauge non invariant quantities such as gluon propagator, ghost propagator, etc.

The arguments in the preceding subsections formally applies for a corresponding gauge function containing non analytic Heaviside function. The gauge-fixing algorithm in the simulation is required to attain the absolute minimum of the $F_U(g)$ along the gauge orbit. But the global minimization is difficult in general and developing the efficient algorithm is still an open problem.

3 The Kugo-Ojima confinement criterion and the Gribov-Zwanziger’s theory

3.1 Kugo-Ojima’s theory

A sufficient condition of the color confinement given by Kugo and Ojima is that $u^{ab}$ defined by the two-point function of the FP ghost fields, $c(x), \bar{c}(y)$,
and $A_\nu(y)$,

$$\int e^{ip(x-y)}(0|TD_\mu c^a(x)g(A_\nu \times \bar{c})^b(y)|0)d^4x = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})u^{ab}(p^2) \quad (14)$$

satisfies $u^{ab}(0) = -\delta^{ab}$.

The corresponding Euclidean expression can be calculated by

$$\frac{1}{V} \sum_{x,y} e^{-ip(x-y)}\langle\text{tr} \left( \lambda^a \frac{1}{\partial D} [-A_\nu \lambda^b] \right) \rangle_{xy} = (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})u^{ab}(p^2), \quad (15)$$

where $\lambda^a$ is a normalized antihermmitian basis of Lie algebra, $V$ a lattice volume, and the ghost propagator is given by

$$\langle c^a x \bar{c}^b y \rangle = \langle \text{tr} \left( \lambda^a \frac{1}{\partial D} \lambda^b \right) \rangle_{xy}. \quad (16)$$

### 3.2 Zwanziger’s theory

The **fundamental modular region** $\Lambda$ is specified by the absolute minimum along the gauge orbits in the **Gribov region** $\Omega$.

$$\Lambda = \{ A ||A||^2 = \text{Min}_g ||A^g||^2 \}, \quad \Lambda \subset \Omega = \{ A | -\partial D \geq 0, \partial A = 0 \}. \quad (17)$$

Zwanziger relaxes the periodicity restriction on the gauge transformation $g$, and imposes larger periods than the original. Then some two points in the fundamental modular region $\Lambda$ may be bridged to be Gribov copies of each other, and one of them is not the absolute minimum of the minimizing function along the gauge orbit anymore. Surviving points as the absolute minimum consist of **core region** $\Xi (\Xi \subset \Lambda)$. In the so defined core region $\Xi$, a horizon function $H(U)$ given below is negative.

The Horizon function is defined as follows. Let two point tensor $G_{\mu\nu xy}^{ab}$ be

$$G_{\mu\nu xy}^{ab} = \text{tr} \left( \lambda^a \frac{1}{\partial D} [-D_\nu \lambda^b] \right)_{xy}. \quad (18)$$

Then $H(U)$ is given as

$$H(U) = \sum_{x,y,a} G_{\mu\nu xy}^{aa} - (N^2 - 1)E(U) \quad (19)$$

where $E(U)$ reads as follows:
1. in $U$-linear version, $E(U) = \sum \frac{1}{N} \text{Re tr} U_l$,

2. in $A = \log U$ version, $E(U) = 1 - \frac{1}{N^2 - 1} \sum_{l,a} \text{tr} \left( \chi^a \lambda^b S(A_l) \lambda^a \right)$,

where $A_l = \text{adj} A_l$, and $S(x) = \frac{x}{\text{th}(x/2)}$.

Let us define an average tensor $G_{\mu\nu xy}$ be

$$G_{\mu\nu xy} \delta_{ab} = \langle G_{ab \mu\nu xy} \rangle,$$

provided color symmetry is not broken. One sees that a Fourier transform of the average tensor, takes a form

$$G_{\mu\nu}(p) \delta_{ab} = \left( \frac{e}{d} \right) p_{\mu} p_{\nu} \delta_{ab} - \left( \delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) u_{ab}, \quad (20)$$

where $e = \langle E(U) \rangle / V$, and that it is related with the horizon function as

$$\frac{\langle H(U) \rangle}{V} = (N^2 - 1) \left[ \lim_{p \to 0} G_{\mu\mu}(p) - e \right]. \quad (21)$$

He defined the **augmented core region** $\Psi = \{ U : H(U) \leq 0 \} \cap \Omega$ ( $\Xi \subset \Psi \subset \Omega$ ). $\Psi$ and $\Lambda$ are qualitatively similar, and he defined the partition function $Z_\Psi$ in the path integral in use of the corresponding Landau gauge function $f_\Psi(U)$, and concluded in the infinite volume limit that

$$\lim_{V \to \infty} \frac{\langle H(U) \rangle}{V} = 0.$$

Putting Kugo-Ojima parameter as $u_{ab}(0) = -\delta_{ab} c$, one finds from (20), (21), that

$$(\frac{e}{d}) + (d - 1) c - e = (d - 1) \left( e - \frac{e}{d} \right) = 0,$$

which is called horizon condition. Since we can measure $c$ and $e$ by the lattice simulation, we can check to what extent Zwanziger’s horizon condition holds in our simulation. With respect to the value $e/d$, note that the classical vacuum is characterized by $e/d = 1$.

4 The numerical results

4.1 The formulation of lattice Landau gauge fixing

We define the gauge field on links as an element of $SU(3)$ Lie algebra as,

$$e^{A_{x,\mu}} = U_{x,\mu}, \quad \text{where } A_{x,\mu}^\dagger = -A_{x,\mu}. \quad (22)$$

We perform the gauge transformation as $e^{A_{x,\mu}} = g_x^1 e^{A_{x,\mu}} g_{x+\mu}$ and define $|\partial A| = \text{Max}_{x,\mu,a} |\partial A^a_x|$, $||\partial A||^2 = \frac{1}{V(N^2 - 1)} \sum \text{tr} \partial A_x^\dagger \partial A_x$. 

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The Landau gauge is realized by minimizing $\|A^\theta\|^2$ via a gauge transformation $g^\dagger U g$, where $g = e^{\epsilon}$. In order to obtain $\epsilon$, we switch the following two methods, depending on the current value of $|\partial A|$ in comparison to some critical parameter $|\partial A|_{cr}$.

1. When $|\partial A| > |\partial A|_{cr}$, $\epsilon_x = \frac{\eta'}{\|\partial A\|} \partial A_x$ with suitable parameter $1 < \eta' < 2.2$

2. When $|\partial A| < |\partial A|_{cr}$, $\epsilon = (-\partial_\mu D_\mu(A))^{-1} \eta \partial A$ where $1 < \eta < 2$ is a parameter.

The restriction to the fundamental modular region is not always achieved. But, we observed that the obtained norm $\|A\|$ is larger or smaller than that obtained after the smeared gauge fixing within 1% accuracy.

### 4.2 The Kugo-Ojima two-point function and the ghost propagator

The FP operator is

$$M[U] = - (\partial \cdot D(A)) = - (D(A) \cdot \partial),$$

where the definition of the gauge field is adopted as $U = e^A$, and the lattice covariant derivative $D_\mu(A) = \partial_\mu + Ad(A_\mu)$ is given in ref. 7.

The inverse, $M^{-1}[U] = (M_0 - M_1[U])^{-1}$, is calculated perturbatively by using the Green function of the Poisson equation $M_0^{-1} = (-\partial^2)^{-1}$ and $M_1 = \partial_\mu Ad(A_\mu(x))$.

The ghost propagator is infrared divergent and its singularity can be parameterized as $p^{-2(1+\alpha)}$, where $p^2 = \sum_{k_\mu} (4 \sin^2 \frac{\pi k_\mu}{L})$, $(-L/2 < k_\mu \leq L/2)$.

It depends on $\beta$ slightly, but its finite-size effect is small. These qualitative features are in agreement with the analysis of the Dyson-Schwinger equation 13.

We measured the lattice version of $|u^a_\alpha(0)|$ on $8^3 \times 16, 8^4 12^4$ and $16^4$ for $\beta = 5.5$ and 6. When $\beta = 6$ and the lattice size is small, the Polyakov loop distribution deviates from the uniform distribution. In this case, we perform the $Z_3$ rotation by multiplying the global phase $e^{\pm \frac{2\pi}{3} i}$ such that the distribution concentrate around one angle, before we measure the Kugo-Ojima two-point function.

We obtained that $u^a_\alpha(0)$ is consistent with $-c \delta^a_\alpha$, $c = 0.7$ in $SU(3)$ quenched simulation, $\beta = 5.5$, on $8^4, 12^4$ and $16^4$. 

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4.3 The gluon propagator

The gluon propagator is infrared finite. We parameterized the zero-temperature lattice data using the Stingl’s Factorised Denominator Rational Approximant (FDRA) method. The effective mass of the gluon in the analysis of $8^3 \times 16$ is found to be about 600 MeV.

The infrared finiteness is in accordance with the Kugo-Ojima color confinement mechanism. As stated in the their inverse Higgs mechanism theorem, if we have no mass less vector poles in all channels of the gauge field, $A^{a}_{\mu}$, and if the color symmetry is not broken at all, it follows that $1 + u = 0$.

5 Discussion and conclusions

We performed the first test of the Kugo-Ojima color confinement criterion in lattice Landau gauge. We observed that the $8^4, 12^4$ and $16^4$ lattice data. The data of $\beta = 5.5$ indicates that $u = -0.7$, and those of $\beta = 6.0$ are smaller by about 10%.

In the Zwanziger’s theory, the two-point function $G_{\mu\nu}(k^2)$ can be expressed in terms of the Kugo-Ojima two-point function as (20). Zwanziger’s horizon condition in the infinite volume limit reads as In terms of the Kugo-Ojima parameter $c$, the left hand side can be written as $(e/4) + 3c$, and the horizon condition is that $c = e/d$. In the table below $e_1$ and $e_2$ stand for $e$ in our $16^4$ lattice simulation of the first and the second option of the gauge fields, respectively. If the gauge fixing is performed so that it brings the configuration into the core region or the augmented core region and if the infinite volume
limit is considered somehow, then the legitimate check of the horizon condition can be done. The core gauge fixing is, however, difficult, and even impossible in general, which implies that the core gauge is literally not the gauge, and thus we give the direct results in the table.

| $\beta$ | c     | $G_{\mu\nu}(0)$ | $e_1/d$ | $e_2/d$ | $1/\tilde{Z}_3$ |
|---------|-------|-----------------|---------|---------|-----------------|
| 5.5     | 0.712(18) | 3.14(5)         | 0.783   | 0.657   | 0.78            |
| 6.0     | 0.628(70) | 2.88(17)        | 0.860(3)| 0.693(1)| 0.86            |

Although the renormalization factor of the ghost propagator $\tilde{Z}_3$ cannot be fitted precisely, its inverse is numerically close to $e_1/d$. Simulation data show in general that when $\beta$ becomes larger, $e$ becomes smaller, while $c$ has an opposite tendency. This fact itself does not necessarily disprove the horizon condition, but our preliminary data of $c$ which is calculated in the $A = \log U$ version already gives the zero-intersection of $c - e_2/d$ in the increase of $\beta$ from 5.5 to 6.

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