GLOBAL MARTINGALE SOLUTIONS FOR THE THREE-DIMENSIONAL
STOCHASTIC CHEMOTAXIS-NAVIER-STOKES SYSTEM

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Abstract. This paper studies the three-dimensional stochastic chemotaxis-Navier-Stokes (SCNS) system subjected to a Lévy-type random external force in bounded domain. Up to now, the existing results concerning global solvability of SCNS system mainly concentrated on the case of two spatial dimensions, little is known for the SCNS system in dimension three. We prove in present work that the three-dimensional SCNS system possesses at least one global martingale solution under proper assumptions, which is weak both in the analytical sense and in the stochastic sense. A new stochastic analogue of entropy-energy inequality and an uniform boundedness estimate are derived, which enable us to construct global-in-time approximate solutions from a properly regularized SCNS system via the Contraction Mapping Principle. The proof of the existence of martingale solution is based on the stochastic compactness method and an elaborate identification of the limits procedure, where the Jakubowski-Skorokhod Theorem is applied to deal with the phase spaces equipped with weak topology.

1. Introduction

1.1. Formulation of problem. The interaction of bacterial populations with a surrounding fluid in which the chemical substances is consumed has already been recognized by several authors [18,23,54]. This experiment intensively revealed complex facets of the spatio-temporal behavior in colonies of the aerobic species *Bacillus subtilis* when suspended in sessile water drops. On one hand, both the density of bacteria and the evolution of chemical substrates are changing over time corresponding to the flow of liquid environment; on the other hand, the motion of fluid is affected by certain external body force, which can be produced by different physical mechanism such as gravity, centrifugal, electric or magnetic forces as well as some uncertainties. Such a mutual chemotaxis & fluid interaction has been characterized accurately by virtue of the chemotaxis-Navier-Stokes (CNS) system introduced by Tuval et al. [54]. This paper is dedicated to the study of the following three-dimensional stochastic
chemotaxis-Navier-Stokes (SCNS) system affected by a Lévy-type random external force:

\[
\begin{align*}
\frac{dn}{dt} + u \cdot \nabla n &= D_n \Delta n - \text{div} (n \chi(c) \nabla c) dt, \quad \text{in } \mathbb{R}^+ \times D, \\
\frac{dc}{dt} + u \cdot \nabla c &= D_c \Delta c - n f(c) dt, \quad \text{in } \mathbb{R}^+ \times D, \\
\frac{du}{dt} + ((u \cdot \nabla) u + \nabla P) dt &= \delta \Delta u dt \\
+ n \nabla \Phi dt + h(t,u) dt + g(t,u) dW + \int_D L d\lambda, \quad \text{in } \mathbb{R}^+ \times D, \\
\text{div} u &= 0,
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0, \quad \text{in } \mathbb{R}^+ \times \partial D,
\end{align*}
\]

and the initial conditions

\[
\begin{align*}
n|_{t=0} = n_0, \quad c|_{t=0} = c_0, \quad u|_{t=0} = u_0, \quad \text{in } D.
\end{align*}
\]

Here, \( D \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary \( \partial D \), and \( \nu \) stands for the inward normal on the boundary. In (1.1), the unknowns are \( n = n(t,x) : \mathbb{R}^+ \times D \rightarrow \mathbb{R}^+ \), \( c = c(t,x) : \mathbb{R}^+ \times D \rightarrow \mathbb{R}^+ \), \( u(t,x) : \mathbb{R}^+ \times D \rightarrow \mathbb{R}^3 \) and \( P = P(t,x) : \mathbb{R}^+ \times D \rightarrow \mathbb{R} \), denoting the density of bacteria, concentration of substrate, velocity field of fluid and associated pressure, respectively. The positive constants \( D_n, D_c \) and \( \delta \) represents the diffusion coefficients for the bacteria, substrate and fluid, respectively. \( \chi(c) \) is the chemotactic sensitivity and \( f(c) \) is the consumption rate of the substrate by the bacteria. The term \( n \nabla \Phi \) denotes the external force exerted by the bacteria on the fluid through a given gravitational potential \( \Phi = \Phi(t,x) \).

Besides the gravity \( n \nabla \Phi \) caused by the mass of bacteria, the fluid dynamics behavior in (1.1) is also influenced by a Lévy-type random external force in the form of

\[
h(t,u) + g(t,u) \frac{dW}{dt} + \int_D L d\lambda.
\]

Specifically, \( h(t,u) \) represents the deterministic external force, \( W \) is a \((\mathcal{F}_t)\)-adapted \( d \)-dimensional continuous Wiener process on a given complete and filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\), and \( L d\lambda \) stands for the random external force exhibit jumps defined by

\[
L d\lambda \overset{\text{def}}{=} \begin{cases} 
K(u(x,t,\cdot),z) \tilde{\pi} dt dz, & \text{for } |z| < 1, \\
G(u(x,t,\cdot),z) \pi dt dz, & \text{for } |z| \geq 1.
\end{cases}
\]

Here, \( K \) and \( G \) are two functions satisfying the assumption \((A_4)\) below, \( \pi(dt,dz) \) denotes the time homogeneous Poisson random measure which is independent of \( W \), and \( \tilde{\pi}(dt,dz) = \pi(dt,dz) - \mu(dz) dt \) is the associated compensator with the intensity measure \( \mu(\cdot) = \mathbb{E}[\pi(1,\cdot)] \) satisfying

\[
\int_{\mathbb{R}^d \setminus \{0\}} (|z|^2 \wedge 1) \mu(dz) < \infty,
\]

where \( \mathbb{E}[X] = \int_{\Omega} X d\mathbb{P} \) defines the expectation of the random variable \( X \). More detailed descriptions can be found in the Subsection 1.3.
The main contribution of this paper is to present an affirmative answer to the question of global solvability for the three-dimensional SCNS system in a bounded domain. More precisely, under mild assumptions, we shall prove that the three-dimensional SCNS system (1.1) perturbed by both continuous and discontinuous random external forces possesses at least one global martingale solution. These solutions are weak in the analytical sense (derivatives exists only in the sense of distributions) and weak in the stochastic sense (the underlying probability space is not a priori given but part of the problem).

1.2. History. In this section, we overview the works related to the qualitative theory for the CNS system and SCNS system.

**CNS system in $\mathbb{R}^2$ or $\mathbb{R}^3$.** In recent years, the coupled CNS system (when $h, g$ and $L$ vanish in (1.1)) has received much attention. For $\mathbb{R}^3$, Duan, Lorz and Markowich [21] obtained the global existence of classical solutions near constant states, while for $\mathbb{R}^2$, they established the existence of global weak solutions provided that the external force is weak or the substrate concentration is small. Later in [53], Tan and Zhang obtained the optimal convergence rates of classical solutions for small initial perturbation around constant states, which partially improves the results in [21]. By virtue of a refined energy functional, Liu and Lorz [41] proved the global existence of weak solutions in two spatial dimensions with some conditions on the sensitivity and consumption rate. In [10], Chae, Kang and Lee established the local existence of regular solutions, and presented some blow-up criteria when the concentration equation is parabolic-type and hyperbolic-type, respectively. In the case that the initial mass is bounded and integrable, Kang and Kim [40] construct a solution to the CNS system such that the density of biological organism belongs to the absolutely continuous curves in the Wasserstein space. More recently, Jeong and Kang [37] investigated the local well-posedness and blow-up criteria in Sobolev spaces for both partially inviscid and fully inviscid cases by performing a weighted Gagliardo-Nirenberg-Sobolev inequality.

**CNS system in bounded domain.** The first work concerning the CNS system in bounded domain is due to Lorz [42], in which the existence of a local weak solution is proven by Schauder’s fixed point theory. If the domain is further assumed to be convex, by using some delicate entropy-energy estimates, Winkler [57] established the existence of a unique global classical solution with general initial data in two spatial dimensions, and he also provided an existence result in three dimensions without the effect of convection term. Note that the convex assumption on the region was then removed by Jiang, Wu and Zheng in [47]. Later in [58], Winkler studied the stability of solutions obtained in [57] in two-dimension case, and it is shown that the solution converges to a constant state as time goes to infinity. When the bounded domain $D \subseteq \mathbb{R}^3$ is convex and smooth, Winker [60] construct a global weak solution with suitable weak conditions on initial data by using an energy-type inequality. Moreover, it was proved recently in [61] that after some relaxation time, these weak solutions constructed in [60] enjoys further regularity properties and thereby complies with the concept of eventual energy solutions. The possibility for singularities to these weak energy solutions occur on small time-scales has also been shown to arise only on sets of
measure zero [62]. For the CNS system of production type, we refer to [4,30] and references therein for details.

**About SCNS system.** In order to account for the random environment surrounding the fluid at a macro- and microscopic level, the two-dimensional CNS system perturbed by random external force was considered by Zhai and Zhang [65]. Specifically, the authors investigated the CNS system with the fluid driven by a cylindrical Wiener process in a convex and bounded domain, and they affirmed the existence and uniqueness of both pathwise mild solutions and weak solutions with the help of an energy inequality. In comparison with its deterministic counterpart, the mathematical literature on the three-dimensional SCNS system is less developed, and as far as we know, [65] is the only result concerning the coupled SCNS system and in two spacial dimensions. For completeness, we would like to mention the works [28,34,44–46,51] in which the existence, uniqueness, and blow-up criteria for the decoupled stochastic Keller-Segel type systems have been discussed. Currently, the well-posed problem for the coupled three-dimensional SCNS system is still open, such as, the global existence in generalized solution frameworks. Motivated by the previous works [38, 60, 62, 65], we shall take a first attempt in this paper to investigate the global solvability of the three-dimensional SCNS system in a bounded domain.

1.3. **Main result.** To begin with, let us first give the rigorous definition of martingale weak solutions to SCNS system (1.1). Without loss of generality, we will take $D_n = D_c = \delta = 1$ throughout the paper.

**Definition 1.1.** A quantity

$$((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), W, n, c, u)$$

is called a **global martingale weak solution** to problem (1.1)-(1.3), provided:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
- $W$ is a $d$-dimensional $(\mathcal{F}_t)$-adapted Wiener process, and $\pi$ is a time homogeneous Poisson random measure over $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with intensity measure $\mu$;
- the triple $(n, c, u)$ is progressively measurable such that $\mathbb{P}$-a.s. $n \in L^2(\Omega; L^1_{loc}(0, \infty; L^1(D))), c \in L^2(\Omega; L^1_{loc}(0, \infty; W^{1,1}(D))), u \in L^2\left(\Omega; L^1_{loc}\left(0, \infty; \left(W^{1,1}_{0,\sigma}(D)\right)^3\right)\right)$,
  $$nf(c) \in L^1(0, T; L^1(D)), u \otimes u \in L^1(D \times [0, T]; \mathbb{R}^{3 \times 3}),$$
  $$n\chi(c) \nabla c, nu, cu \in L^1\left(0, T; (L^1(D))^3\right);$$
- the equation for bacterial density
  $$-\int_D n_0 \varphi|_{t=0} dx = \int_0^\infty \int_D (n\varphi_t + nu \cdot \nabla \varphi - \nabla n \cdot \nabla \varphi + n\chi(c)\nabla c \cdot \nabla \varphi) \, dx \, dt$$
  holds for all $\varphi \in C^\infty_0(D \times [0, \infty); \mathbb{R})$ $\mathbb{P}$-a.s.;
- the equation for substrate concentration
  $$-\int_D c_0 \varphi|_{t=0} dx = \int_0^\infty \int_D (c\varphi_t + cu \cdot \nabla \varphi - \nabla c \cdot \nabla \varphi - nf(c)\varphi) \, dx \, dt.$$
holds for all $\varphi \in C_0^\infty(D \times [0, \infty); \mathbb{R})$ $\mathbb{P}$-a.s.;

- the equation for velocity field

$$\begin{align*}
- \int_D u_0 \cdot \nabla \varphi|_{t=0} \, dx &= \int_0^\infty \int_D (u \cdot \nabla \varphi + u \otimes u : \nabla \varphi - \nabla u \cdot \nabla \varphi + n \nabla \Phi \cdot \varphi) \, dx \, dt \\
+ \int_0^\infty \int_D h(t, u) \varphi \, dx \, dt + \int_0^\infty \int_D g(t, u) \varphi \, dx \, dW \\
+ \int_0^\infty \int_{|z|<1} \int_D K(u(x, t-), z) \varphi \, dx \, \pi(dt, dz) + \int_0^\infty \int_{|z|\geq 1} \int_D G(u(x, t-), z) \varphi \, dx \, \pi(dt, dz)
\end{align*}$$

holds for all vector valued function $\varphi \in C_0^\infty(D \times [0, \infty); \mathbb{R}^3)$ with $\text{div} \varphi = 0$ $\mathbb{P}$-a.s.

Before giving the statement of the main result, let us introduce the following assumptions on the parameters and functions involved in the system (1.1).

(A_1) For the initial datum $n_0, c_0$ and $u_0$, we assume that

$$\begin{cases}
u_0 \in L_2^2(D), \ n_0 \in L \log L(D) \text{ is non-negative, } \ n_0 \neq 0, \\
c_0 \in L^\infty(D) \text{ is non-negative, } \sqrt{c_0} \in W^{1,2}(D),
\end{cases}$$

where $L \log L(D)$ denotes the standards Orlicz spaces associated with the Young function $\sigma \ln(1 + \sigma)$ on $(0, \infty)$, and $L_2^2(D) \overset{\text{def}}{=} \{ u \in L^2(D) ; \text{div} u = 0 \}$ stands for the solenoidal subspace of $L^2(D)$.

(A_2) For the signal consumption rate $f$, the chemotactic sensitivity $\chi$ and the potential $\Phi$ in SCNS (1.1), we assume that

$$\begin{cases}
\Phi : D \mapsto \mathbb{R} \text{ is Lipschitz continuous, i.e., } \Phi \in W^{1,\infty}(D; \mathbb{R}), \\
\chi : [0, \infty) \mapsto (0, \infty) \text{ is a } C^2 \text{ function,} \\
f : [0, \infty) \mapsto [0, \infty) \text{ is a } C^1 \text{ function with } f(0) = 0, \ f \geq 0 \text{ on } (0, \infty), \\
\left( \frac{f}{\chi} \right)' > 0, \ \left( \frac{f}{\chi} \right)'' \leq 0, \ \text{and } (f/\chi)' \geq 0 \text{ on } [0, \infty).
\end{cases}$$

(A_3) For any $\alpha \in \{0\} \cup [\frac{1}{2}, 1)$, there exists a positive constant $C$ such that

$$\begin{align*}
\|g(t, u)\|_{D(A^\alpha)} &\leq C \left( 1 + \|u\|_{D(A^\alpha)} \right), \\
\|h(t, u)\|_{D(A^\alpha)} &\leq C \left( 1 + \|u\|_{D(A^\alpha)} \right), \\
\|g(t, u_1) - g(t, u_2)\|_{D(A^\alpha)} &\leq C \|u_1 - u_2\|_{D(A^\alpha)}, \\
\|h(t, u_1) - h(t, u_2)\|_{D(A^\alpha)} &\leq C \|u_1 - u_2\|_{D(A^\alpha)}
\end{align*}$$

hold for all $u, u_1, u_2 \in D(A^\alpha)$, uniformly in time, where $A = -\mathcal{P}\Delta$ denotes the Stokes operator with domain $D(A) = W^{2,2}(D) \cap W^{1,2}_{0,\sigma}(D)$, and $\mathcal{P} : L^2(D) \mapsto L^2(D)$ is the Helmholtz projection.
(A₄) For any \( \alpha \in \{0\} \cup [\frac{1}{2}, 1) \), there exists a positive constant \( C \) such that
\[
\int_{|z|<1} \|K(u, z)\|_{D(A^\alpha)}^p \mu(dz) + \int_{|z|\geq 1} \|G(u, z)\|_{D(A^\alpha)}^p \mu(dz)
\leq C \left( 1 + \|u\|_{D(A^\alpha)}^p \right), \quad \text{for all } p \geq 2,
\]
\[
\int_{|z|<1} \|K(u_1, z) - K(u_2, z)\|_{D(A^\alpha)}^p \mu(dz) + \int_{|z|\geq 1} \|G(u_1, z) - G(u_2, z)\|_{D(A^\alpha)}^p \mu(dz)
\leq C \|u_1 - u_2\|_{D(A^\alpha)}^2
\]
hold uniformly in time for all \( u, u_1, u_2 \in D(A^\alpha) \).

The main result in this work can now be stated by the following theorem.

**Theorem 1.2.** Under the assumptions (A₁) – (A₄), the problem (1.1) – (1.3) possesses at least one global martingale weak solution \((\tilde{\Omega}, \tilde{F}, (\tilde{F}_t), \tilde{P}), \tilde{W}, \tilde{\pi}, \tilde{n}, \tilde{c}, \tilde{u})\) in the sense of Definition 1.1. Moreover, for any \( T > 0 \), the weak entropy-energy inequality
\begin{align}
- \int_0^T \phi'(t) \int_D \left( \tilde{n} \ln \tilde{n} + \frac{1}{2} |\nabla \tilde{u}|^2 + c^\dagger |\tilde{u}|^2 \right) dx dt & \\
+ \int_0^T \phi(t) \int_D \left( \frac{1}{2} \frac{|\nabla \tilde{n}|^2}{\tilde{n}} + d_1 \frac{|\nabla \tilde{c}|^4}{\tilde{c}} + d_2 \frac{|\Delta \tilde{c}|^2}{\tilde{c}} + \frac{c^\dagger}{2} |\nabla \tilde{u}|^2 \right) dx dt & \\
\leq \phi(0) \int_D \left( n_0 \ln n_0 + \frac{1}{2} |\nabla \Psi(c_0)|^2 + c^\dagger |u_0|^2 \right) dx & \\
+ C \int_0^T \phi(t) \left( \int_D |\tilde{u}(t)|^2 dx + 1 \right) dt + \int_0^T \phi(t) dM_E & \tag{1.6}
\end{align}
holds true \( \tilde{P}\)-almost surely for any deterministic test function \( \phi(t) \geq 0 \) with \( \phi(T) = 0 \), where
\[
\Psi(s) \overset{\text{def}}{=} \int_1^s \sqrt{\frac{\chi(f)}{f^2}} d\sigma, \quad \text{for } s > 0,
\]
\( M_E \) is a real-valued martingale with bounded \( p \)-th order momentum
\begin{equation}
\tilde{E} \sup_{t \in [0,T]} |M_E|^p \leq C \left[ \tilde{E} \left( \int_D \left( n_0 \ln n_0 + |\nabla \Psi(c_0)|^2 + |u_0|^2 \right) dx \right)^p + 1 \right], \tag{1.7}
\end{equation}
for any \( 1 \leq p < \infty \), and the positive constants \( d_1, d_2, c^\dagger, C \) depends only on \( p, c_0, f, \chi, \Phi \).

**Remark 1.3.** As far as we aware, Theorem 1.2 seems to be the first result to consider the global solvability for SCNS system in the three spatial dimensions, which extended the previous works in the deterministic case by Winkler [57, 60] and in the two-dimensional stochastic case by Zhai and Zhang [65]. It should also be mentioned that the convexity of the bounded domain \( D \) in Theorem 1.2 is unnecessary, which is technically assumed in [57, 60, 65]. Additionally, our result includes the effects of both continuous and discontinuous random external forces surrounding the fluid, which is more natural from a physical point of view, and we refer to [11, 19, 20, 29, 43, 48, 50, 61] for a few applications of Lévy processes in the stochastic hydrodynamics.
Remark 1.4. In dimension two, one can prove by classical Yamada-Watanabe Theorem that the SCNS system has a unique pathwise solution [65]. Whereas in dimension three, the uniqueness part can not be guaranteed. In fact, the uniqueness and smoothness of solutions to three-dimensional Navier-Stokes equations is still a millennium problem. In this aspect, we mention the recent breakthrough on the non-uniqueness of weak solutions to the stochastic Navier-Stokes equations based on the convex integration method [12, 31–33] which was introduced by De Lellis and Székelyhidi Jr. [16, 17] for Euler equations. This inspires us to pursue instead the martingale (weak in probability) weak solutions. In future, an interesting but challenging problem is to explore the regularity of solutions constructed in Theorem 1.2.

1.4. Outline of proof and main ideas. Theorem 1.2 will be proved through the following steps. First, we consider a regularized SCNS system (3.1) by applying Leray’s regularization and the weakly increasing approximation in [62, 63]. However, due to the irregular random external forces, the standard methodology used in [60] is failure for us to get the existence and uniqueness of approximate solutions. To overcome this, proper working spaces are selected such that the processes are bounded in space variable, and several cut-off operators, which depends on the size of $\|n_\epsilon\|_{\infty}$, $\|c_\epsilon\|_{W^{1,q}}$ and $\|A^\alpha u_\epsilon\|_{L^2}$, are used to deal with the nonlinear terms and stochastic integrals (cf. (3.2)). Based on a contraction mapping argument, one can prove the existence of a unique mild solution on any interval. Then we construct a maximal local mild solution $(n_\epsilon, c_\epsilon, u_\epsilon, \tilde{\tau}_\epsilon)$ to the regularized system (3.1) by removing the cut-off operators via a sequence of stopping times. Therein, the smoothing effect of heat semigroup and Stokes semigroup [25, 56] in a series of estimations plays an important role.

The second step is to show that the maximal local solution obtained in Lemma 3.3 is global in time, i.e., $\mathbb{P}[\tilde{\tau}_\epsilon = \infty] = 1$, where the key component is a newly derived stochastic analogue of entropy-energy type inequality and hence an uniform boundedness estimate for the approximate solutions $(n_\epsilon, c_\epsilon, u_\epsilon)$ within an unnecessarily convex region (see Lemma 4.5-4.6). Note that several stopping times should be introduced which assist us to establish some a priori estimates by a careful application of the BDG inequality. We remark that for the classical Keller-Segel system and the ones coupled to fluid, the entropy-energy dissipative structure constituted crucial ingredients in the development of global existence theories [21, 41, 55].

The third step is to take a limit as $\epsilon \to 0$ and prove the existence of global martingale solutions. At this stage, the main difficulty comes from the fact that the weak convergence inherited from the uniform bounds is too weak to proceeding a compactness argument as that in deterministic situation. We overcome this problem by adapting the original idea of Skorokhod [52] to investigating the tightness of approximate solutions, which can be carried out by applying the uniform energy estimate together with a generalized Aubin-Lions Lemma [48], the Flandoli’s Lemma [22] and a tightness criteria in $\mathcal{D}([0, T]; (W^{1,5}_{0, \sigma}(D))^d)$ due to Aldous [11,2]. Then with the Jakubowski-Skorokhod Theorem (which is valid for quasi-polish spaces such as the separable Banach spaces with weak topology) one can construct a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, on which a sequence of random
variables \((\hat{W}_{ej}, \hat{\pi}_{ej}, \hat{n}_{ej}, \hat{c}_{ej}, \hat{u}_{ej})_{j \geq 1}\) can be defined, which have the same distributions as the original ones and that in addition converge almost surely to an element \((\hat{W}, \hat{\pi}, \hat{n}, \hat{c}, \hat{u})\). Making use of the Bensoussan’s argument [5] but with a crucial generalization on dealing with the stochastic integrals, we show that \((\hat{\pi}_{ej}, \hat{c}_{ej}, \hat{u}_{ej})_{j \geq 1}\) satisfies the approximate system (3.1) under the new stochastic basis \(((\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}}), \hat{W}_{ej}, \hat{\pi}_{ej})\). With above results at hand, by carefully using the Uniform Integrability Theorem, the Vitali’s Convergence Theorem, the Dominated Convergence Theorem together with the almost surely convergence in Lemma 5.5, one can take the limit \(j \to \infty\) in (3.1) to verify that the quantity \((\hat{W}, \hat{\pi}, \hat{n}, \hat{c}, \hat{u})\) is indeed a global martingale weak solution to (1.1) in the sense of Definition 1.1.

1.5. Organization of the paper. In Section 2 we recall some basic concepts and useful lemmas from the stochastic analysis. Section 3 is devoted to the existence of maximal local mild solutions to the regularized SCNS system. In Section 4 we first verify that the solution constructed in Section 3 is indeed a variational solution associated to a Gelfand triple, then we derive a stochastic analogue of entropy-energy type inequality and also an uniform bounds for approximate solutions, based on which we show that the approximate solutions are global ones. Section 5 focuses on the identification of limits by using the stochastic compactness method and the Jakubowski-Skorokhod Theorem.

2. Preliminaries

Denote by \(W^{k,p}(D)\), \(k \in \mathbb{Z}, \ 0 < p \leq \infty\), the Sobolev spaces endowed with the norm \(\| \cdot \|_{W^{k,p}}\). For a Banach space \(X\), we use \(\| \cdot \|_{L^p(0,T;X)}\) as the norm of the space \(L^p(0,T;X)\).

For any Banach space \(V\) with its dual \(V^*\), we set \(\langle u, v \rangle \overset{\text{def}}{=} \langle u, v \rangle_{V,V^*}\), for any \(u \in V, v \in V^*\). The following Gagliardo-Nirenberg inequality will be frequently used.

**Lemma 2.1** ([9]). Let \(\mathcal{O} \subset \mathbb{R}^n\) be a measurable, bounded, open and connected domain satisfying the cone condition. Let \(1 \leq q \leq +\infty, \ 1 \leq r \leq +\infty, \ p \geq 1, \ j < m\) be nonnegative integers and \(\theta \in [0,1]\) such that the relation

\[
\frac{1}{p} = \frac{j}{n} + \theta \left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1 - \theta}{q}, \ \frac{j}{m} \leq \theta \leq 1
\]

holds. Assume that \(u \in L^q(\mathcal{O})\) such that \(D^m u \in L^r(\mathcal{O})\) and \(\sigma\) is arbitrary. Then,

\[
\| D^j u \|_{L^p(\mathcal{O})} \leq C \| D^m u \|_{L^r(\mathcal{O})}^{\theta} \| u \|_{L^q(\mathcal{O})}^{1 - \theta} + C \| u \|_{L^q(\mathcal{O})},
\]

where the constant \(C > 0\) is independent of \(u\).

In the following, for a fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we shall recall several basic concepts and useful analytical tools in stochastic analysis.

**Definition 2.2** ([3,6]). Let \(H\) be a Banach space. A stochastic process \(L = \{L(t), \ t \geq 0\}\) defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is called an \(H\)-valued Lévy process if the following conditions are satisfied:
(1) $L(0) = 0$ $\mathbb{P}$-a.s.;
(2) $L$ has independent increments, that is, if for each $n \in \mathbb{N}$ and each $0 \leq t_1 < t_2 \leq \cdots < t_{n+1} < \infty$ the random variables $(L(t_j) - L(t_{j-1}))$, $1 \leq j \leq n$ are independent;
(3) $L$ has stationary increments, that is, it has stationary increments if $L(t_{j+1}) - L(t_j) \overset{d}{=} L(t_{j+1} - t_j) - L(0);
(4) L$ is stochastically continuous, that is, $\lim_{t \to s} \mathbb{P}(|L(t) - L(s)|_H > \epsilon) = 0$, for all $\epsilon > 0$ and $s \geq 0$.

Let $\bar{\mathbb{N}} = \{0, 1, 2, \ldots, +\infty\}$, and $(E, \mathcal{E}) = (Z \times \mathbb{R}_+, Z \times \mathcal{B}(\mathbb{R}_+))$ be a measurable space. $\mathcal{P}_{\bar{\mathbb{N}}}(E)$ denotes the space of all $\bar{\mathbb{N}}$-valued measures on $(E, \mathcal{E})$, and $(\mathcal{P}_{\bar{\mathbb{N}}}(E), \mathcal{A})$ is the related measurable space with the $\sigma$-field $\mathcal{A}$ generated by the mappings $\mathcal{P}_{\bar{\mathbb{N}}} \ni \rho \mapsto \rho(\Gamma) \in \bar{\mathbb{N}}$, $\Gamma \in \mathcal{E}$.

**Definition 2.3** [3]. A Poisson random measure is a measurable map
$$\pi : (\Omega, \mathcal{F}) \mapsto (\mathcal{P}_{\bar{\mathbb{N}}}(E), \mathcal{A})$$
satisfying the following conditions:
(1) $\pi$ is independently scattered, that is, if the sets $B_j \in \mathcal{E}$, $j = 1, 2, \ldots, n$ are disjoint, then the random variables $\pi(B_j), j = 1, 2, \ldots, n$ are independent;
(2) for each $B \in \mathcal{E}$, $\pi(B)$ is a Poisson random variable with parameter $\mathbb{E}[\pi(B)];$
(3) for all $B \in \mathcal{Z}$ and $I \in \mathcal{B}(\mathbb{R}_+)$, $\mathbb{E}[\pi(B \times I)] = \mu(B)\text{Leb}(I)$, where $\text{Leb}(\cdot)$ is the standard Lebesgue measure, and $\mu$ is a non-negative measure on $\mathbb{Z}$;
(4) for each $U \in \mathcal{Z}$, the process $N(t, U) = \pi(U \times (0, t]), t \geq 0$ is $\mathcal{F}_t$-adapted, and if $t > s \geq 0$, then $N(t, U) - N(s, U) = \pi(U \times (s, t])$ is independent of $\mathcal{F}_s$.

Note that one can construct a corresponding Poisson random measure from a Lévy process, and vice versa (cf. [3]).

**Definition 2.4.** The family $\Lambda$ of probability measures on $(E, \mathcal{B}(E))$ is said to be tight if and only if for any $\epsilon > 0$, there exists a compact set $K_\epsilon \subset E$ such that $\mu(K_\epsilon) \geq 1 - \epsilon$, for all $\mu \in \Lambda$.

Let us denote by $\mathcal{D}([0, T]; E)$ the space of functions $u : [0, T] \to E$ that are right-continuous on $[0, T)$ and have left-limits at every point in $(0, T]$. The space $\mathcal{D}([0, T]; E)$ endowed with the Skorokhod topology is separable and metrizable (cf. Chapter 3 in [7]). The following lemma provides an effective condition to characterize the tightness of processes in $\mathcal{D}([0, T]; E)$.

**Lemma 2.5** ([1][2][48]. Let $E$ be a separable Banach space, and $(Y_k)_{k \geq 1}$ be a sequence of $E$-valued random variables. Assume that for every stopping time sequence $(\tau_k)_{k \geq 1}$ with $\tau_k \leq T$ and for every $\theta \geq 0$ the following condition holds
$$\sup_{k \geq 1} \mathbb{E} (\|Y_k(\tau_k + \theta) - Y_k(\tau_k)\|_E^\alpha) \leq C\theta^\epsilon,$$
for some $\alpha, \epsilon > 0$ and a constant $C > 0$. Then the distribution of $Y_k$ form a tight sequence on $\mathcal{D}([0, T]; E)$ endowed with the Skorokhod topology.

We now recall the following Burkholder-Davis-Gundy (BDG) inequality.
Lemma 2.6 (3). Let $T > 0$, for every fixed $p \geq 1$, there are two constants $0 < c_p \leq C_p$ depending only on $p$ such that for every real-valued square integrable càdlàg martingale $M_t$ with $M_0 = 0$, and for every $T \geq 0$,

$$c_p \mathbb{E}\left(\frac{\langle M \rangle_T}{T}\right) \leq \mathbb{E}\left(\max_{0 \leq t \leq T} |M_t|^p \right) \leq C_p \mathbb{E}\left(\frac{\langle M \rangle_T}{T}\right),$$

where $\langle M \rangle_t$ is the quadratic variation of $M_t$ on $[0, t]$.

The Vitali Convergence Theorem stated below is crucial for passing to the limit.

Lemma 2.7 (39). Let $(A_N)$ be a sequence of integrable functions defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $A_N \to A$ a.e. as $N \to \infty$ (or $A_N \to A$ in probability) for some integrable function $A$. Assume that there exist a $r > 1$ and a constant $C > 0$ such that $\mathbb{E} |A_N|^r \leq C$ for all $N \in \mathbb{N}$. Then $\mathbb{E} |A_N| \to \mathbb{E} |A|$ as $N \to \infty$.

Let us recall the Itô’s formula for general Lévy-type stochastic processes.

Lemma 2.8 (3). Let $Y = (X^i)$ be a $d$-dimensional semimartingale represented by

$$Y^i(t) = Y^i(0) + M^i(t) + A^i(t) + \int_0^t \int_{|s|<1} H^i(s,z) \pi(ds,dz) + \int_0^t \int_{|s|\geq1} K^i(s,z) \pi(ds,dz), \quad i = 1, \ldots, d,$$

where $M^i$ are continuous local martingales, $A^i$ are continuous adapted processes of finite variation, $H^i$ is predictable such that $\int_0^T \int_{0<|s|<1} |H^i(t,x)|^2 \mu(dx)dt < \infty \ \mathbb{P}\text{-a.s.}$, and $K^i$ are predictable processes. Then for any function $f \in C^2(\mathbb{R}^d)$, we have

$$f(Y(t)) = f(Y(0)) + \int_0^t \partial_s f(Y(s-)) dM^i(s) + \int_0^t \partial_i f(Y(s-)) dA^i(s) + \frac{1}{2} \int_0^t \partial_{ij} f(Y(s-)) d\langle M^i, M^j \rangle_s + \int_0^t \int_{|s|\geq1} (f(Y(s-)+K(s,z)) - f(Y(s-))) \pi(ds,dz) + \int_0^t \int_{|s|<1} (f(Y(s-)+H(s,z)) - f(Y(s-))) \pi(ds,dz) + \int_0^t \int_{|s|\geq1} (f(Y(s-)+H(s,z)) - f(Y(s-))+H^i(s,z)\partial_i f(Y(s-))) \mu(dz)ds,$$

where $\langle X, Y \rangle_t = \frac{1}{4} \{ \langle X + Y \rangle_t - \langle X - Y \rangle_t \}$ denotes the cross-variation process.

To cater to the Banach spaces with weak topology, the generalization of the classical Skorokhod Representation Theorem is required as shown in the following Lemma.

Lemma 2.9 (Jakubowski-Skorokhod, 36). Let $(\mu_n)_{n \in \mathbb{N}}$ be a family of probability laws on a quasi-Polish space $(\mathcal{Y}, \tau, (f_n)_{n \in \mathbb{N}})$ and let $\mathcal{F}$ be the $\sigma$-algebra generated by the maps $(f_n)_{n \in \mathbb{N}}$. 
Let \( (\mu_n)_{n \in \mathbb{N}} \) be a tight sequence of probability laws on \((\mathcal{Y}, \mathcal{S})\). Then there is a subsequence \( (\mu_{nk})_{k \in \mathbb{N}} \) such that the following holds. There is a probability space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\) and \(\mathcal{Y}\)-valued Borel measurable random variables \((\bar{X}_k)_{k \in \mathbb{N}}, \bar{X} : (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) \to (\mathcal{Y}, \mathcal{S})\) such that

1. The laws of \(\bar{X}_k\) under \(\bar{P}\) coincide with \(\mu_{nk}\), for any \(k \in \mathbb{N}\);
2. \(\bar{X}_k \to \bar{X}\) \(\bar{P}\)-a.s., as \(k \to \infty\);
3. The law of \(\bar{X}\) under \(\bar{P}\) is a Radon measure.

### 3. Local existence of approximate solutions

In this section, we first introduce a regularized SCNS system, and then prove the existence and uniqueness of maximal local mild solution in appropriate function spaces.

#### 3.1. Regularized SCNS system.

Suppose that \(\varrho \in C_0^\infty(\mathbb{R}^3)\) is a standard mollifier, for any \(\epsilon > 0\), consider the Leray’s regularisation defined by

\[
L_\epsilon u \overset{\text{def}}{=} \varrho \ast u = \frac{1}{\epsilon^3} \int_{\mathbb{R}^3} \varrho \left( \frac{x - y}{\epsilon} \right) u(y) dy,
\]

and utilize the weakly increasing approximation \(h_\epsilon\) given by \(h_\epsilon(s) \overset{\text{def}}{=} \frac{1}{\epsilon} \ln(1 + \epsilon s), s \geq 0\). Then the regularized SCNS system takes the form of

\[
\begin{aligned}
&dn_\epsilon + u_\epsilon \cdot \nabla n_\epsilon dt = \Delta n_\epsilon dt - \text{div} (n_\epsilon h'_\epsilon(n_\epsilon) = \nabla c_\epsilon dt, \\
dc_\epsilon + u_\epsilon \cdot \nabla c_\epsilon dt = \Delta c_\epsilon dt - h_\epsilon(n_\epsilon) f(c_\epsilon) dt, \\
du_\epsilon + \mathcal{P}(L_\epsilon u_\epsilon \cdot \nabla) u_\epsilon dt = -Au_\epsilon dt \\
\frac{\partial n_\epsilon}{\partial v} \bigg|_{\partial D} = \frac{\partial c_\epsilon}{\partial v} \bigg|_{\partial D} = 0, u_\epsilon \bigg|_{\partial D} = 0, \\
n_\epsilon|_{t=0} = n_{00}, \ c_\epsilon|_{t=0} = c_{00}, \ u_\epsilon|_{t=0} = u_{00}.
\end{aligned}
\]  

(3.1)

where the families of approximate initial datum \((n_{00}, c_{00}, u_{00})\) have the following properties:

\[
\begin{aligned}
0 \leq n_{00} &\in C_0^\infty(D), \ |n_{00}|_{L^1} = |n_0|_{L^1}, \ |n_{00} - n_0|_{L_{\log}(D)} \xrightarrow{\epsilon \to 0} 0; \\
0 \leq \sqrt{c_{00}} &\in C_0^\infty(D), \ |c_{00}|_{L^\infty} \leq |c_0|_{L^\infty}, \ \sqrt{c_{00}} - \sqrt{c_0} \xrightarrow{\epsilon \to 0} 0, \\
c_{00} &\xrightarrow{\epsilon \to 0} c_0 \text{ a.e. in } D; \\
u_{00} &\in C_{0,\sigma}^\infty(D), \ |u_{00}|_{L^2} \leq \|u_0\|_{L^2}, \ |u_{00} - u_0|_{1,2} \xrightarrow{\epsilon \to 0} 0, u_{00} \xrightarrow{\epsilon \to 0} u_0 \text{ a.e. in } D.
\end{aligned}
\]

Note that the above smooth functions can be found by standard mollification of \(n_0\) and \(c_0\) for the first two component (cf. [60]); the construction for the fluid component is based on the same idea (cf. Theorem III.4.1 in [24]).

Let us clarify the precise definition of a local mild solution to (3.1), which allows us to exploit the regularizing properties of the heat and Stokes semigroups.
Definition 3.1. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) be a given stochastic basis with complete right-continuous filtration, \(W\) be a \(d\)-dimensional \((\mathcal{F}_t)\)-Wiener process and \(\pi\) be a time homogeneous Poisson random measure with the intensity measure \(\mu(dz) \otimes dt\). A quadruplet \((n_\epsilon, c_\epsilon, u_\epsilon, \tau_\epsilon)\) is called a local mild solution to the system (3.1) provided:

- \(\tau_\epsilon\) is an a.s. strictly positive \((\mathcal{F}_t)\)-stopping time;
- the pair \((n_\epsilon, c_\epsilon)\) is a \(L^\infty(D) \times W^{1,q}(D)\)-valued \((\mathcal{F}_t)\)-progressively measurable process satisfying \(n_\epsilon(\cdot \wedge \tau_\epsilon) \geq 0, c_\epsilon(\cdot \wedge \tau_\epsilon) \geq 0\) \(\mathbb{P}\)-a.s., and
- \((n_\epsilon, c_\epsilon) \in L^2(\Omega; L^\infty(0,T;L^\infty(D))) \times L^2(\Omega; L^\infty(0,T;W^{1,q}(D))), \quad \text{for all } q > 3;
- the velocity \(u_\epsilon\) is a \(D(A^\alpha)\)-valued \(\mathcal{F}_t\)-progressively measurable process satisfying
  \[
  u_\epsilon \in L^2(\Omega; L^\infty(0,T;D(A^\alpha))), \quad \text{for all } \alpha \in (\frac{3}{4},1);
  \]
- there holds \(\mathbb{P}\)-a.s.
  \[
  n_\epsilon(t \wedge \tau_\epsilon) = e^{t-\Delta} n_{\epsilon_0} - \int_0^{t \wedge \tau_\epsilon} e^{(t-s)-\Delta} (u_\epsilon \cdot \nabla n_\epsilon + \text{div}(n_\epsilon h'(n_\epsilon)\chi(c_\epsilon)\nabla c_\epsilon)) ds,
  \]
  \[
  c_\epsilon(t \wedge \tau_\epsilon) = e^{t-\Delta} c_{\epsilon_0} - \int_0^{t \wedge \tau_\epsilon} e^{(t-s)-\Delta} (u_\epsilon \cdot \nabla c_\epsilon + h_n(n_\epsilon)f(c_\epsilon)) ds,
  \]
  \[
  u_\epsilon(t \wedge \tau_\epsilon) = e^{-t-\Delta} u_{\epsilon_0} - \int_0^{t \wedge \tau_\epsilon} e^{-(t-s)-\Delta} \mathcal{P}\left(\left[ L_{\epsilon} u_\epsilon \cdot \nabla \right] u_\epsilon - n_\epsilon \nabla \Phi - h(s, u_\epsilon) \right) ds
  + \int_0^{t \wedge \tau_\epsilon} e^{-(t-s)-\Delta} \mathcal{P} g(t, u_\epsilon) dW + \int_0^{t \wedge \tau_\epsilon} \int_Z e^{-(t-s)-\Delta} \mathcal{P} L_{\epsilon} d\lambda ds,
  \]
  for all \(t \geq 0\), where \(L_{\epsilon}\) is defined by replacing \(u\) by \(u_\epsilon\) in \(L\) (cf. (1.4)).

Here and in what follows, we denote by \((e^{t-\Delta})_{t \geq 0}\) the Neumann heat semigroup and by \((e^{-tA})_{t \geq 0}\) the Stokes semigroup with Dirichlet boundary data (cf. [25, 56]).

Definition 3.2. A quadruplet \((n_\epsilon, c_\epsilon, u_\epsilon, \tilde{\tau}_\epsilon)\) is called a maximal local mild solution to the system (3.1) provided:

- \(\tilde{\tau}_\epsilon\) is an \(\mathbb{P}\)-a.s. strictly positive \((\mathcal{F}_t)\)-stopping time;
- \((\tau_R)_{R \geq 1}\) is an increasing sequence of \((\mathcal{F}_t)\)-stopping times such that \(\tau_R < \tilde{\tau}_\epsilon\) on the set \([\tilde{\tau}_\epsilon < \infty]\), \(\lim_{R \to \infty} \tau_R = \tilde{\tau}_\epsilon\) \(\mathbb{P}\)-a.s., and
  \[
  \sup_{t \in [0,\tau_R]} \| (n_\epsilon(t), c_\epsilon(t), u_\epsilon(t)) \|_{L^\infty \times W^{1,q} \times D(A^\alpha)} \geq R \quad \text{on } [\tilde{\tau}_\epsilon < \infty];
  \]
- each quantity \((n_\epsilon, c_\epsilon, u_\epsilon, \tau_R)\) is a local mild solution in the sense of Definition 3.1.

Now we are ready to state the main result in this section.

Lemma 3.3. Suppose that the assumptions \((A_1) - (A_4)\) hold. Then the problem (3.1) possesses a unique maximal local mild solution \((n_\epsilon, c_\epsilon, u_\epsilon, \tilde{\tau}_\epsilon)\) in the sense of Definition 3.2.
3.2. Regularized system with truncation. Let $R > 0$, and $\varphi_R : [0, \infty) \to [0, 1]$ be a smooth cut-off function such that $\varphi_R(s) \equiv 1$ for $0 \leq s \leq R$ and $\varphi_R(s) \equiv 0$ for $s > 2R$. Then the regularized SCNS system with cut-off operators takes the following form:

$$
\begin{aligned}
\frac{dn_e + \varphi_R \left( \|u_e\|_{D(A^o)} \right) \varphi_R \left( \|n_e\|_{\infty} \right) u_e \cdot \nabla n_e dt}{dc_e + \varphi_R \left( \|u_e\|_{D(A^o)} \right) \varphi_R (\|c_e\|_{1,q}) u_e \cdot \nabla c_e dt} &= \Delta n_e dt - \varphi_R (\|n_e\|_{\infty}) \varphi_R (\|c_e\|_{1,q}) \text{div} \left( n_e h' (n_e) \chi(c_e) \nabla c_e \right) dt, \\
\frac{du_e + \varphi_R \left( \|u_e\|_{D(A^o)} \right) \mathcal{P} (L_e u_e \cdot \nabla) u_e dt}{df_e + \varphi_R \left( \|u_e\|_{D(A^o)} \right) \mathcal{P} (n_e \nabla \Phi) dt + \varphi_R (\|u_e\|_{D(A^o)}) \mathcal{P} h(t, u_e) dt} &= -A u_e dt + \varphi_R (\|n_e\|_{\infty}) \mathcal{P} (n_e \nabla \Phi) dt + \varphi_R (\|u_e\|_{D(A^o)}) \mathcal{P} h(t, u_e) dt \\
&+ \varphi_R (\|u_e\|_{D(A^o)}) \mathcal{P} g(t, u_e) dW + \int_{Z} \varphi_R (\|u_e\|_{D(A^o)}) \mathcal{P} L_e d\lambda,
\end{aligned}
$$

(3.2)

together with the initial-boundary conditions:

$$
\frac{\partial n_e}{\partial v} \bigg|_{\partial D} = \frac{\partial c_e}{\partial v} \bigg|_{\partial D} = u_e|_{\partial D} = 0, \quad n_e|_{t=0} = n_0, \quad c_e|_{t=0} = c_0, \quad u_e|_{t=0} = u_0.
$$

(3.3)

(3.4)

Here and in the proof of Lemma 3.4 below, we shall drop the subscript $R$ in $(n_{R,e}, c_{R,e}, u_{R,e})$ to avoid abundant notations. We remark that the concept of maximal local mild solutions to (3.2) can be defined as same as Definitions 3.1, 3.2 and we shall omit the details.

Lemma 3.4. Assume that the assumptions $(A_1) - (A_4)$ hold. Then for any $T > 0$, the problem (3.2) - (3.4) has a unique mild solution $(n_e, c_e, u_e)$ on $[0, T]$.

Proof. Step 0: The proof involves a reasoning based on the Banach fixed-point theorem. To this end, we introduce a Banach space $\mathcal{X}_T$ which consists of all $(\mathcal{F}_1)$-adapted processes with values in $L^\infty(D) \times W^{1,q}(D) \times D(A^o)$, namely,

$$
\mathcal{X}_T \overset{\text{def}}{=} L^2(\Omega; L^\infty(0, T; L^\infty(D))) \times L^2(\Omega; L^\infty(0, T; W^{1,q}(D))) \\
\times L^2(\Omega; L^\infty(0, T; D(A^o)) \), \quad \text{for any } q \in (3, \infty) \text{ and } \alpha \in \left( \frac{3}{4}, 1 \right),
$$

which is equipped with the norm $\|(n_e, c_e, u_e)\|^2_{\mathcal{X}_T} \overset{\text{def}}{=} \mathbb{E}(\|n_e\|^2_{L^\infty(0, T; L^\infty)}) + \mathbb{E}(\|c_e\|^2_{L^\infty(0, T; W^{1,q})}) + \mathbb{E}(\|u_e\|^2_{L^\infty(0, T; D(A^o))})$. With $r > 0$ to be specified below, we consider a closed ball in $\mathcal{X}_T$:

$\mathcal{B}_T(r) = \{ (n_e, c_e, u_e) \in \mathcal{X}_T; \|(n_e, c_e, u_e)\|^2_{\mathcal{X}_T} \leq r \}$. For any $(n_e, c_e, u_e) \in \mathcal{B}_T$, we explore a map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ given by

$$
\mathcal{F}_1(n_e, c_e, u_e)(t) \overset{\text{def}}{=} e^{t \Delta} n_{e0} - \int_{0}^{t} e^{(t-s) \Delta} \left( \varphi_R \left( \|u_e\|_{D(A^o)} \right) \varphi_R (\|n_e\|_{\infty}) u_e \cdot \nabla n_e \right) ds + \varphi_R (\|n_e\|_{\infty}) \varphi_R (\|c_e\|_{1,q}) \text{div} \left( n_e h' (n_e) \chi(c_e) \nabla c_e \right) ds,
$$

$$
\mathcal{F}_2(n_e, c_e, u_e)(t) \overset{\text{def}}{=} e^{t \Delta} c_{e0} - \int_{0}^{t} e^{(t-s) \Delta} \left( \varphi_R \left( \|u_e\|_{D(A^o)} \right) \varphi_R (\|c_e\|_{1,q}) u_e \cdot \nabla c_e \right) ds + \varphi_R (\|n_e\|_{\infty}) \varphi_R (\|c_e\|_{1,q}) h_e (n_e) f(c_e) ds,
$$

$$
\mathcal{F}_3(n_e, c_e, u_e)(t) \overset{\text{def}}{=} e^{t \Delta} u_{e0} - \int_{0}^{t} e^{(t-s) \Delta} \left( \varphi_R \left( \|u_e\|_{D(A^o)} \right) \varphi_R (\|n_e\|_{\infty}) u_e \cdot \nabla u_e \right) ds,
$$

(3.2)
and

\[
\mathcal{F}_3(n_\epsilon, c_\epsilon, u_\epsilon)(t) \stackrel{\text{def}}{=} e^{-tA} u_{00} - \int_0^t e^{-(t-s)A} \mathcal{P}(\varphi_R(||u_\epsilon||_{D(A^\alpha)}) (L_\epsilon u_\epsilon \cdot \nabla) u_\epsilon - \varphi_R(||n_\epsilon||_{\infty}) n_\epsilon \nabla \Phi - h(s, u_\epsilon)) ds
\]
\[
+ \int_0^t e^{-(t-s)A} \mathcal{P} g(s, u_\epsilon) dW + \int_0^t \int_Z e^{-(t-s)A} \mathcal{P} L_\epsilon d\lambda ds
\]
\[
\stackrel{\text{def}}{=} e^{-tA} u_{00} + \mathcal{F}_{31}(t) + \mathcal{F}_{32}(t) + \mathcal{F}_{33}(t).
\]

In next two steps, we need to show that the map \( \mathcal{F} \) satisfies the conditions for applying the contraction mapping principle.

**Step 1:** \( \mathcal{F} : \mathcal{X}_T \mapsto \mathcal{X}_T \) is well-defined. For \( \mathcal{F}_1(n_\epsilon, c_\epsilon, u_\epsilon) \), since \( q > 3 \), one can pick a \( \gamma \in (0, \frac{1}{2}) \) such that \( 2\gamma - \frac{3}{q} > 0 \) and \( D((1 - \Delta)^\gamma) \subset C(D) \) by the Sobolev embedding, where \((1 - \Delta)^\gamma \) denotes the Bessel potential. Applying the smooth effect of Neumann heat semigroup (cf. [59]) and the identity \( \div (u_\epsilon n_\epsilon) = u_\epsilon \cdot \nabla n_\epsilon \), we have for \( t \in [0, T] \)

\[
\|\mathcal{F}_1(n_\epsilon, c_\epsilon, u_\epsilon)(t)\|_{L^\infty}
\]
\[
\leq \|n_\epsilon\|_{L^\infty} + C \int_0^t (t-s)^{\frac{2\gamma+1}{2}} \varphi_R(||u_\epsilon||_{D(A^\alpha)}) \varphi_R(||n_\epsilon||_{\infty}) ||u_\epsilon n_\epsilon||_{L^q} ds
\]
\[
+ C \int_0^t (t-s)^{\frac{2\gamma+1}{2}} \varphi_R(||n_\epsilon||_{\infty}) \varphi_R(||c_\epsilon||_{1,q}) ||n_\epsilon h_\epsilon'(n_\epsilon) \chi(c_\epsilon) \nabla c_\epsilon||_{L^q} ds
\]
\[
\leq \|n_\epsilon\|_{L^\infty} + C R T^{\frac{1}{2} - \gamma} + C R T < \infty,
\]

where the second inequality used condition \((A_2)\) and the fact of \( 0 \leq h_\epsilon'(n_\epsilon) \leq 1 \).

For \( \mathcal{F}_2(n_\epsilon, c_\epsilon, u_\epsilon) \), by choosing a \( \delta \in (\frac{1}{2}, 1) \) such that \( 2\delta - \frac{3}{q} > 1 - \frac{3}{q} \), which makes sense of the estimate \( \|J\|_{W^{1,q}} \leq \|J\|((1 - \Delta)^\delta \cdot ||L^q||, we gain

\[
\|\mathcal{F}_2(n_\epsilon, c_\epsilon, u_\epsilon)(t)\|_{W^{1,q}}
\]
\[
\leq \|c_\epsilon\|_{W^{1,q}} + \int_0^t \left\| (1 - \Delta)^\delta e^{(t-s)\Delta} \left[ \varphi_R(||u_\epsilon||_{D(A^\alpha)}) \varphi_R(||c_\epsilon||_{1,q}) u_\epsilon \cdot \nabla c_\epsilon \\
+ \varphi_R(||n_\epsilon||_{\infty}) \varphi_R(||c_\epsilon||_{1,q}) h_\epsilon(n_\epsilon) f(c_\epsilon) \right]\right\|_{L^q} ds
\]
\[
\leq \|c_\epsilon\|_{W^{1,q}} + C T^{1 - \delta} < \infty, \quad \text{for all } t \in [0, T].
\]

Now let us estimate \( \mathcal{F}_3(n_\epsilon, c_\epsilon, u_\epsilon) \). For \( \mathcal{F}_{31}(t) \), it follows from the fact of \( \alpha \in (\frac{3}{4}, 1) \) that \( \|J\|_{W^{1,2}} \leq C \|J\|_{D(A^\alpha)}, \|J\|_{L^\infty} \leq C \|J\|_{D(A^\alpha)}. \) Since the operator \( L_\epsilon \) is bounded from \( L^2(D) \) into \( L^\infty(D) \), we deduce from the smooth effect of Stokes semigroup (cf. [25]) that

\[
\|A^\alpha \mathcal{F}_{31}(t)\|_{L^2} \leq \|A^\alpha u_0\|_{L^2} + \int_0^t (t-s)^{\alpha} \left[ \varphi_R(||u_\epsilon||_{D(A^\alpha)}) \|L_\epsilon u_\epsilon \cdot \nabla u_\epsilon\|_{L^2} \\
+ \varphi_R(||n_\epsilon||_{\infty}) ||n_\epsilon \nabla \Phi||_{L^2} + \varphi_R(||u_\epsilon||_{D(A^\alpha)}) (1 + ||u_\epsilon||_{L^2}) \right] ds
\]
\[
\leq ||u_0||_{D(A^\alpha)} + C T^{1 - \alpha} < \infty, \quad \text{for all } t \in [0, T].
\]
For $\mathcal{F}_{32}(t)$, it will be convenient to rewrite the integral into the evolution equation $A^\alpha \mathcal{F}_{32}(t) = \int_0^t A^{\alpha+1} \mathcal{F}_{32}(s)\,ds + \int_0^t \varphi_R \left( \|u_\epsilon\|_{D(A^\alpha)} \right) A^\alpha P g(t, u_\epsilon)\,dW$. By applying the BDG inequality, the boundedness of the projection $P$ as well as the assumption $(A_2)$ (note that $(\alpha, 1) \subset [\frac{3}{2}, 1]$), we obtain

\[
\begin{align*}
\mathbb{E} \sup_{t \in [0,T]} \|A^\alpha \mathcal{F}_{32}(t)\|_{L^2}^2 &\leq \mathbb{E} \int_0^T \varphi_R^2 \left( \|u_\epsilon\|_{D(A^\alpha)} \right) \|A^\alpha P g(t, u_\epsilon)\|_{L^2}^2 \,dt \\
+ 2\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \varphi_R \left( \|u_\epsilon\|_{D(A^\alpha)} \right) \langle A^\alpha \mathcal{F}_{32}, A^\alpha P g(t, u_\epsilon)\,dW \rangle \right| \\
&\leq CT + C\mathbb{E} \int_0^T \sup_{s \in [0,t]} \|A^\alpha \mathcal{F}_{32}(s)\|_{L^2}^2 \,dt,
\end{align*}
\]

which combined with the Gronwall inequality lead to

\[(3.8)\quad \mathbb{E} \sup_{t \in [0,T]} \|A^\alpha \mathcal{F}_{32}(t)\|_{L^2}^2 \leq CT.\]

For $\mathcal{F}_{33}(t)$, note that $A^\alpha \mathcal{F}_{33}$ satisfies the identity

\[
A^\alpha \mathcal{F}_{33}(t) = \int_0^t A^{\alpha+1} \mathcal{F}_{33}(s)\,ds + \int_0^t \int_{|z| < 1} \varphi_R \left( \|u_\epsilon\|_{D(A^\alpha)} \right) A^\alpha P K(u_\epsilon(x, s-), z) \tilde{\pi}(ds, dz) \\
+ \int_0^t \int_{|z| \geq 1} \varphi_R \left( \|u_\epsilon\|_{D(A^\alpha)} \right) A^\alpha P G(u_\epsilon(x, s-), z) \pi(ds, dz).
\]

By applying Itô’s formula (cf. Lemma 2.8) to the functional $\|A^\alpha \mathcal{F}_{33}(t)\|_{L^2}^2$, after integrating by parts and rearranging the integrals, we obtain

\[
\begin{align*}
&\|A^\alpha \mathcal{F}_{33}(t)\|_{L^2}^2 + 2 \int_0^t \|A^{\alpha+1} \mathcal{F}_{33}(s)\|_{L^2}^2 \,ds \\
&\leq 2 \left| \int_0^t \int_{|z| \geq 1} \varphi_R \left( \|u_\epsilon\|_{D(A^\alpha)} \right) \langle A^\alpha \mathcal{F}_{33}(s-), A^\alpha P G(u_\epsilon(s-), z) \rangle_{L^2} \mu(dz)\,ds \right| \\
&+ \int_0^t \int_{|z| < 1} \varphi_R^2 \left( \|u_\epsilon\|_{D(A^\alpha)} \right) \|A^\alpha P K(u_\epsilon(s-), z)\|_{L^2}^2 \mu(dz)\,ds \\
&+ \int_0^t \int_{|z| \geq 1} \varphi_R^2 \left( \|u_\epsilon\|_{D(A^\alpha)} \right) \|A^\alpha P G(u_\epsilon(x, s-), z)\|_{L^2}^2 \mu(dz)\,ds \\
&+ 2 \left| \int_0^t \int_{|z| < 1} \left( \varphi_R \left( \|u_\epsilon\|_{D(A^\alpha)} \right) \langle A^\alpha \mathcal{F}_{33}(s-), A^\alpha P K(u_\epsilon(s-), z) \rangle_{L^2} \right) \tilde{\pi}(ds, dz) \right| \\
&+ \varphi_R^2 \left( \|u_\epsilon\|_{D(A^\alpha)} \right) \|A^\alpha P G(u_\epsilon(x, s-), z)\|_{L^2}^2 \tilde{\pi}(ds, dz) \overset{\text{def}}{=} I_1(t) + \cdots + I_5(t).
\end{align*}
\]
Taking the supremum for $I_1(t)$ over interval $[0, T]$, and then taking the mathematical expectation, we get by assumption $(A_4)$ that

$$
\mathbb{E} \sup_{t \in [0, T]} I_1(t) \leq \mathbb{E} \int_0^T \| A^\alpha F_{33}(t) \|_{L^2}^2 ds + C_\mu \mathbb{E} \int_0^T \varphi_R^2 \left( \| u_\epsilon \|_{D(A^\alpha)} \right) \left( 1 + \| u_\epsilon \|_{D(A^\alpha)}^2 \right) ds \\
\leq \mathbb{E} \int_0^T \| A^\alpha F_{33}(s) \|_{L^2}^2 ds + CT.
$$

(3.10)

Here $C_\mu \overset{\text{def}}{=} \int_{|z| \geq 1} \mu(dz) \leq \int_{\mathbb{R} \setminus \{0\}} (|z|^2 \wedge 1) \mu(dz)$ is finite since $\mu$ is a Lévy measure (see (1.3)). For $I_2(t)$ and $I_3(t)$, we conclude that

$$
\mathbb{E} \sup_{t \in [0, T]} (I_2 + I_3)(t) \leq C \int_0^T \varphi_R^2 \left( \| u_\epsilon \|_{D(A^\alpha)} \right) \left( 1 + \| u_\epsilon \|_{D(A^\alpha)}^2 \right) ds \leq CT.
$$

(3.11)

For $I_4(t)$, on the one hand, we get by applying the Itô’s isometry that

$$
\mathbb{E} \sup_{t \in [0, T]} I_{41}(t) \leq C \mathbb{E} \left[ \sup_{t \in [0, T]} \| A^\alpha F_{33}(t) \|_{L^2} \left( \int_0^T \varphi_R^2 \left( \| u_\epsilon \|_{D(A^\alpha)} \right) \left( 1 + \| u_\epsilon \|_{D(A^\alpha)}^2 \right) ds \right)^{1/2} \right]
\leq \epsilon_1 \mathbb{E} \sup_{t \in [0, T]} \| A^\alpha F_{33}(t) \|_{L^2}^2 + C \int_0^T \varphi_R \left( \| u_\epsilon \|_{D(A^\alpha)} \right) \left( 1 + \| u_\epsilon \|_{D(A^\alpha)}^2 \right) ds
\leq \epsilon_1 \mathbb{E} \sup_{t \in [0, T]} \| A^\alpha F_{33}(t) \|_{L^2}^2 + CT, \quad \text{for any } \epsilon_1 > 0.
$$

On the other hand, it follows from the assumption on $K$ that

$$
\mathbb{E} \sup_{t \in [0, T]} I_{42}(t) \leq 2 \mathbb{E} \left( \int_0^T \int_{|z| < 1} \varphi_R^4 \left( \| u_\epsilon \|_{D(A^\alpha)} \right) \| A^\alpha K \|_{L^2}^2 \mu(dz) ds \right)^{1/2}
\leq C \mathbb{E} \left( \int_0^T \varphi_R^4 \left( \| u_\epsilon \|_{D(A^\alpha)} \right) \left( 1 + \| u_\epsilon \|_{D(A^\alpha)}^2 \right) ds \right)^{1/2} \leq CT^{1/2}.
$$

The last two inequalities imply that

$$
\mathbb{E} \sup_{t \in [0, T]} I_4(t) \leq \epsilon_1 \mathbb{E} \sup_{t \in [0, T]} \| A^\alpha F_{33}(t) \|_{L^2}^2 + CT + CT^{1/2}.
$$

(3.12)

Proceeding similarly, by using the Itô’s isometry and the assumption on $G$, one has

$$
\mathbb{E} \sup_{t \in [0, T]} I_5(t) \leq \epsilon_2 \mathbb{E} \sup_{t \in [0, T]} \| A^\alpha F_{33}(t) \|_{L^2}^2 + CT + CT^{1/2}, \quad \text{for any } \epsilon_2 > 0.
$$

(3.13)

Substituting the estimate (3.10)–(3.13) into (3.9), after choosing $\epsilon_1, \epsilon_2 > 0$ small enough such that $\epsilon_1 + \epsilon_2 \leq \frac{1}{2}$, we obtain

$$
\mathbb{E} \sup_{t \in [0, T]} \| A^\alpha F_{33}(t) \|_{L^2}^2 \leq \mathbb{E} \int_0^T \| A^\alpha F_{33}(s) \|_{L^2}^2 ds + C \left( T + T^{1/2} \right),
$$

which implies

$$
\mathbb{E} \sup_{t \in [0, T]} \| A^\alpha F_{33}(t) \|_{L^2}^2 \leq C \left( T + T^{1/2} \right).
$$

(3.14)
As a result, it follows from (3.7), (3.8) and (3.14) that
\[ \mathbb{E} \sup_{t \in [0,T]} \| A^\alpha \mathcal{F}(n_\epsilon, c_\epsilon, u_\epsilon)(t) \|_{L^2}^2 \leq 4 \| u_0 \|_{D(A^\alpha)}^2 + C \left( T^{1/2} + T + T^2 + T^{2-2\alpha} \right), \]
which together with (3.5) and (3.6) lead to
\[ \| \mathcal{F}(n_\epsilon, c_\epsilon, u_\epsilon) \|_{\mathcal{Y}_T}^2 \leq 3 \| n_0 \|_{L^\infty}^2 + 2 \| c_0 \|_{W^{1,\alpha}}^2 + 4 \| u_0 \|_{D(A^\alpha)}^2 \]
\[ + C \left( T + T^2 + T^{1-2\gamma} + T^{2-2\delta} + T^{2-2\alpha} \right). \]
Choosing $r = 6 \| n_0 \|_{L^\infty}^2 + 4 \| c_0 \|_{W^{1,\alpha}}^2 + 8 \| u_0 \|_{D(A^\alpha)}^2$ and then $T > 0$ small enough such that
\[ C(T + T^2 + T^{1-2\gamma} + T^{2-2\delta} + T^{2-2\alpha}) \leq \frac{r}{2}. \]
This shows that $\mathcal{F}$ is a map from the ball $B_T(r) \subseteq \mathcal{X}_T$ into itself.

**Step 2.** $\mathcal{F} : B_T(r) \mapsto B_T(r)$ is a contraction for $T > 0$ small enough. For any $(n_\epsilon, c_\epsilon, u_\epsilon)$, $(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon) \in B_T(r)$, and any $\gamma \in \left( \frac{2}{2\gamma}, \frac{1}{2} \right)$, we get for the $n_\epsilon$-equation that
\[ (3.15) \]
\[ \| \mathcal{F}_1(n_\epsilon, c_\epsilon, u_\epsilon)(t) - \mathcal{F}_1(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon)(t) \|_{L^\infty} \]
\[ \leq \int_0^t (t - s)^{-\gamma - \frac{1}{2}} \left\| \left[ \phi_R(\| u_\epsilon \|_{D(A^\alpha)}) \varphi_R(\| n_\epsilon \|_{L^\infty}) u_\epsilon n_\epsilon - \varphi_R(\| \bar{u}_\epsilon \|_{D(A^\alpha)}) \varphi_R(\| \bar{n}_\epsilon \|_{L^\infty}) \bar{u}_\epsilon \bar{n}_\epsilon \right] \right\|_{L^1} ds \]
\[ + \int_0^t (t - s)^{-\gamma - \frac{1}{2}} \left\| \left[ \varphi_R(\| n_\epsilon \|_{L^\infty}) \varphi_R(\| c_\epsilon \|_{L^1}) (\epsilon \nabla c_\epsilon) \right] \right\|_{L^1} ds \overset{\text{def}}{=} J_1(t) + J_2(t). \]
The term $J_1(t)$ can be handled in a similar manner as that in [63] and we have
\[ (3.16) \]
\[ J_1(t) \leq C \bar{R} T^{1-\gamma} \left( \| n_\epsilon - \bar{n}_\epsilon \|_{L^\infty(0,T,L^\infty)} + \| c_\epsilon - \bar{c}_\epsilon \|_{L^\infty(0,T,W^{1,\alpha})} \right). \]
Define
\[ G \overset{\text{def}}{=} \varphi_R(\| n_\epsilon \|_{L^\infty}) \varphi_R(\| c_\epsilon \|_{L^1}) \bar{n}_\epsilon \bar{c}_\epsilon - \varphi_R(\| \bar{n}_\epsilon \|_{L^\infty}) \varphi_R(\| \bar{c}_\epsilon \|_{L^1}) \bar{n}_\epsilon \bar{c}_\epsilon(\bar{c}_\epsilon) \nabla \bar{c}_\epsilon. \]
We will divide the estimate for $J_2$ into several cases.

- If $\max\{\| n_\epsilon \|_{L^\infty(0,T,L^\infty)}, \| c_\epsilon \|_{L^\infty(0,T,W^{1,\alpha})}\} > 2\bar{R}$ and $\max\{\| \bar{n}_\epsilon \|_{L^\infty(0,T,L^\infty)}, \| \bar{c}_\epsilon \|_{L^\infty(0,T,W^{1,\alpha})}\} > 2\bar{R}$, then $\| G \|_{L^1} = 0$.
- If $\max\{\| n_\epsilon \|_{L^\infty(0,T,L^\infty)}, \| c_\epsilon \|_{L^\infty(0,T,W^{1,\alpha})}\} > 2\bar{R}$ and $\| \bar{n}_\epsilon \|_{L^\infty(0,T,L^\infty)} \leq 2\bar{R}$, then
\[ \| G \|_{L^1} \leq \max_{s \in [0,C\bar{R}]} \left( \chi(s) \varphi_R(\| \bar{n}_\epsilon \|_{L^\infty}) \varphi_R(\| \bar{c}_\epsilon \|_{L^1}) \| \bar{n}_\epsilon \|_{L^2} \right)^{\frac{1}{2}} \| \nabla \bar{c}_\epsilon \|_{L^1} \]
\[ \leq 8\bar{R}^{3} \max_{s \in [0,C\bar{R}]} \chi(s) \left( \varphi_R(\| \bar{n}_\epsilon \|_{L^\infty}) - \varphi_R(\| n_\epsilon \|_{L^\infty}) \right) \leq C \| n_\epsilon - \bar{n}_\epsilon \|_{L^\infty(0,T,L^\infty)}, \]
where we used $\| \bar{c}_\epsilon \|_{L^\infty} \leq C \| \bar{c}_\epsilon \|_{W^{1,\alpha}} \leq C\bar{R}$.
- If $\max\{\| n_\epsilon \|_{L^\infty(0,T,L^\infty)}, \| c_\epsilon \|_{L^\infty(0,T,W^{1,\alpha})}\} > 2\bar{R}$ and $\| \bar{c}_\epsilon \|_{L^\infty(0,T,L^\infty)} \leq 2\bar{R}$, then
\[ \| G \|_{L^1} \leq \max_{s \in [0,C\bar{R}]} \left( \chi(s) \varphi_R(\| \bar{n}_\epsilon \|_{L^\infty}) \varphi_R(\| \bar{c}_\epsilon \|_{L^1}) \| \bar{n}_\epsilon \|_{L^2} \right)^{\frac{1}{2}} \| \nabla \bar{c}_\epsilon \|_{L^1} \leq C \| c_\epsilon - \bar{c}_\epsilon \|_{L^\infty(0,T,L^\infty)}. \]
- By symmetry, if $\max\{\| n_\epsilon \|_{L^\infty(0,T,L^\infty)}, \| \bar{c}_\epsilon \|_{L^\infty(0,T,W^{1,\alpha})}\} > 2\bar{R}$, $\| n_\epsilon \|_{L^\infty(0,T,L^\infty)} \leq 2\bar{R}$, then
\[ \| G \|_{L^1} \leq C \| n_\epsilon - \bar{n}_\epsilon \|_{L^\infty(0,T,L^\infty)}. \]
• By symmetry, if \( \max\{\|\tilde{n}_e\|_{L^\infty(0,T;L^\infty)}, \|\tilde{c}_e\|_{L^\infty((0,T)W^{1,q})}\} > 2R, \|c_e\|_{L^\infty(0,T;L^\infty)} \leq 2R \), then
  \[
  \|G\|_{L^q} \leq C\|c_e - \bar{c}_e\|_{L^\infty(0,T;L^\infty)}.
  \]

• If \( \max\{\|n_e\|_{L^\infty(0,T;L^\infty)}, \|c_e\|_{L^\infty((0,T)W^{1,q})}, \|\bar{n}_e\|_{L^\infty(0,T;L^\infty)}, \|\bar{c}_e\|_{L^\infty((0,T)W^{1,q})}\} \leq 2R \), then
  \[
  \|G\|_{L^q} \leq \varphi_R(\|n_e\|_\infty)\varphi_R(\|c_e\|_{L^\infty})|\bigl(\|n_e - \bar{n}_e\|_{L^\infty} + \|c_e - \bar{c}_e\|_{L^\infty}\bigr|
  + \|\nabla c_e - \nabla \bar{c}_e\|_{L^q} + \bigl|\varphi_R(\|n_e\|_\infty) - \varphi_R(\|\bar{n}_e\|_\infty)\bigr|igr)
  \]
  \[
  \leq C(\|n_e - \bar{n}_e\|_{L^\infty} + \|\nabla c_e - \nabla \bar{c}_e\|_{L^q}).
  \]

According to the definition of \( \varphi_R, h, \) and \( \chi, \) the Mean Value Theorem as well as the embedding \( W^{1,q}(D) \subset L^\infty(D) \) for \( q > 3 \), one can deduce that
  \[
  \|G\|_{L^q} \leq C_R,\chi \|n_e - \bar{n}_e\|_{L^\infty} + \|h'(n_e) - h'(\bar{n}_e)\|_{L^\infty} + \|\nabla c_e - \nabla \bar{c}_e\|_{L^q}.
  \]

We conclude from the discussion in \((I_{21}) - (I_{26})\) that
  \[
  J_2(t) \leq C \int_0^T (T - s)^{-\gamma - \frac{1}{2}} (\|n_e - \bar{n}_e\|_{L^\infty} + \|\nabla c_e - \nabla \bar{c}_e\|_{L^q}) ds
  \]
  \[
  \leq CT^{\frac{1}{2} - \gamma} \left( \|n_e - \bar{n}_e\|_{L^\infty(0,T;L^\infty)} + \|c_e - \bar{c}_e\|_{L^\infty(0,T;W^{1,q})} \right).
  \]

From estimates \((3.15) - (3.17)\), we obtain
  \[
  \mathbb{E} \sup_{t \in [0,T]} \left| \mathcal{F}_1(n_e, c_e, u_e) - \mathcal{F}_1(\bar{n}_e, \bar{c}_e, \bar{u}_e) \right|^2_{L^\infty} \leq CT^{1 - 2\gamma}
  \]
  \[
  \times \left( \mathbb{E}\|n_e - \bar{n}_e\|^2_{L^\infty(0,T;L^\infty)} + \mathbb{E}\|c_e - \bar{c}_e\|^2_{L^\infty(0,T;W^{1,q})} \right), \quad \text{for any } \gamma \in \left( \frac{3}{2q}, \frac{1}{2} \right).
  \]

Similarly for the \( n_e \)-equation, one can prove that
  \[
  \mathbb{E} \sup_{t \in [0,T]} \left| \mathcal{F}_2(n_e, c_e, u_e) - \mathcal{F}_2(\bar{n}_e, \bar{c}_e, \bar{u}_e) \right|_{W^{1,q}} \leq CT^{2 - 2\delta}
  \]
  \[
  \times \left( \mathbb{E}\|n_e - \bar{n}_e\|^2_{L^\infty(0,T;L^\infty)} + \mathbb{E}\|c_e - \bar{c}_e\|^2_{L^\infty(0,T;W^{1,q})} \right), \quad \text{for } \delta \in \left( \frac{1}{2}, 1 \right).
  \]
It remains to estimate \( \| \mathcal{F}_3(n_\epsilon, c_\epsilon, u_\epsilon)(t) - \mathcal{F}_3(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon)(t) \|_{D(A^{\alpha})} \) with respect to the \( u_\epsilon \)-equation. Thanks to the smooth effect of Stokes semigroup, we have
\[
\begin{align*}
\| A^\alpha (\mathcal{F}_3(n_\epsilon, c_\epsilon, u_\epsilon) - \mathcal{F}_3(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon))\|_{L^2} & \\
& \leq \int_0^t (t-s)^{-\alpha} \| \varphi_R (\| u_\epsilon \|_{D(A^{\alpha})} (L_\epsilon u_\epsilon \cdot \nabla) u_\epsilon - \varphi_R (\| \bar{u}_\epsilon \|_{D(A^{\alpha})} (L_\epsilon \bar{u}_\epsilon \cdot \nabla) \bar{u}_\epsilon) \|_{L^2} ds \\
& \quad + \int_0^t (t-s)^{-\alpha} \| \varphi_R (\| n_\epsilon \|_{L^\infty}) n_\epsilon \nabla \Phi - \varphi_R (\| \bar{n}_\epsilon \|_{L^\infty}) \bar{n}_\epsilon \nabla \Phi \|_{L^2} ds \\
& \quad + \int_0^t (t-s)^{-\alpha} \| \varphi_R (\| u_\epsilon \|_{L^\infty}) h(s, u_\epsilon) - \varphi_R (\| \bar{u}_\epsilon \|_{L^\infty}) h(s, \bar{u}_\epsilon) \|_{L^2} ds \\
& \quad + \left\| \int_0^t A^\alpha e^{-(t-s)A} P \left( \varphi_R (\| u_\epsilon \|_{D(A^{\alpha})}) g(t, u_\epsilon) - \varphi_R (\| \bar{u}_\epsilon \|_{D(A^{\alpha})}) g(t, \bar{u}_\epsilon) \right) dW \right\|_{L^2} \\
& \quad + \left\| \int_0^t \int_Z A^\alpha e^{-(t-s)A} P \left( \varphi_R (\| u_\epsilon \|_{D(A^{\alpha})}) L_\epsilon - \varphi_R (\| \bar{u}_\epsilon \|_{D(A^{\alpha})}) \bar{L}_\epsilon \right) d\lambda ds \right\|_{L^2} \\
& \quad \overset{\text{def}}{=} K_1(t) + \cdots + K_5(t) \|_{L^2}.
\end{align*}
\]

The estimate for \( K_1(t) \) will be considered in four cases.

- If \( \max \{ \| u_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))}, \| \bar{u}_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))} \} > 2R \), then \( K_1(t) \equiv 0 \).
- If \( \| u_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))} > 2R \) and \( \| \bar{u}_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))} \leq 2R \), then it follows from the embedding \( W^{2\alpha,2}(D) \subset W^{1,2}(D) \) with \( \alpha > \frac{3}{4} \) that
  \[
  K_1(t) \leq C \int_0^t (t-s)^{-\alpha} (\varphi_R (\| \bar{u}_\epsilon \|_{D(A^{\alpha})}) - \varphi_R (\| u_\epsilon \|_{D(A^{\alpha})})) \| \bar{u}_\epsilon \|_{L^2} A^\alpha \bar{u}_\epsilon \|_{L^2} ds \\
  \leq CT^{-\alpha} \| \bar{u}_\epsilon - u_\epsilon \|_{D(A^{\alpha})}.
  \]
- Similar to \( K_1(t) \), if \( \| u_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))} \leq 2R \) and \( \| \bar{u}_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))} > 2R \), then
  \[
  J_1(t) \leq CT^{-\alpha} \| \bar{u}_\epsilon - u_\epsilon \|_{D(A^{\alpha})}.
  \]
- If \( \max \{ \| u_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))}, \| \bar{u}_\epsilon \|_{L^\infty(0,T;D(A^{\alpha}))} \} \leq 2R \), then
  \[
  \begin{align*}
  & \| \varphi_R (\| u_\epsilon \|_{D(A^{\alpha})}) (L_\epsilon u_\epsilon \cdot \nabla) u_\epsilon - \varphi_R (\| \bar{u}_\epsilon \|_{D(A^{\alpha})}) (L_\epsilon \bar{u}_\epsilon \cdot \nabla) \bar{u}_\epsilon \|_{L^2} \\
  & \quad \leq C \| u_\epsilon - \bar{u}_\epsilon \|_{D(A^{\alpha})} (\| L_\epsilon u_\epsilon \cdot \nabla \|_{L^2} + \varphi_R (\| \bar{u}_\epsilon \|_{D(A^{\alpha})}) (\| L_\epsilon \bar{u}_\epsilon - L_\epsilon \bar{u}_\epsilon \|_{L^2} \| L_\epsilon \bar{u}_\epsilon \cdot \nabla (u_\epsilon - \bar{u}_\epsilon) \|_{L^2} \\
  & \quad \leq C \| u_\epsilon - \bar{u}_\epsilon \|_{D(A^{\alpha})},
  \end{align*}
  \]
  which implies that \( K_1(t) \leq CT^{-\alpha} \| \bar{u}_\epsilon - u_\epsilon \|_{D(A^{\alpha})} \).

In conclusion, we get
\[
\mathbb{E} \sup_{t \in [0,T]} |K_1(t)|^2 \leq CT^{2-2\alpha} \mathbb{E} \left( \| \bar{u}_\epsilon - u_\epsilon \|^2_{L^\infty(0,T;D(A^{\alpha}))} \right),
\]
for \( K_2(t) \), we prove by a similar method that
\[
\mathbb{E} \sup_{t \in [0,T]} |K_2(t)|^2 \leq CT^{2-2\alpha} \mathbb{E} \left( \| \bar{n}_\epsilon - n_\epsilon \|^2_{L^\infty(0,T;L^\infty)} \right).
\]
For $K_3(t)$, in virtue of the assumption on $h$, we have

$$|K_3(t)| \leq \int_0^t (t - s)^{-\alpha} |\varphi_R(\|u_\varepsilon\|_\infty) - \varphi_R(\|\bar{u}_\varepsilon\|_\infty)| (1 + \|u_\varepsilon\|_{L^2}^2) ds$$

$$+ \int_0^t (t - s)^{-\alpha} \varphi_R(\|\bar{u}_\varepsilon\|_\infty) \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^2} ds,$$

which implies that

$$\mathbb{E} \sup_{t \in [0,T]} |K_3(t)|^2 \leq C T^{2-2\alpha} \mathbb{E} \left( \|\bar{u}_\varepsilon - u_\varepsilon\|_{L^\infty(0,T; D(A^\alpha))}^2 \right).$$

(3.23)

For $K_4(t)$, we rewrite it as $dK_4(t) = AK_4(t) dt + A^\alpha h(t) dW$ with $K_4(0) = 0$, where $h(t) \equiv \mathcal{P} \left( \varphi_R \left( \|u_\varepsilon\|_{D(A^\alpha)} \right) g(t, u_\varepsilon) - \varphi_R \left( \|\bar{u}_\varepsilon\|_{D(A^\alpha)} \right) g(t, \bar{u}_\varepsilon) \right)$. By applying Itô’s formula and the BDG inequality, we get

$$\mathbb{E} \sup_{t \in [0,T]} \|K_4(t)\|_{L^2}^2 \leq \mathbb{E} \int_0^T \|A^\alpha h(t)\|_{L^2}^2 dt + 2 \mathbb{E} \left[ \sup_{t \in [0,T]} \|J_4(t)\|_{L^2} \left( \int_0^T \|A^\alpha h(t)\|_{L^2}^2 dt \right)^{1/2} \right]$$

$$\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0,T]} \|J_4(t)\|_{L^2}^2 + C \mathbb{E} \int_0^T \|A^\alpha h(t)\|_{L^2}^2 dt.$$

For the last term on the R.H.S., we have

$$\|A^\alpha h(t)\|_{L^2}^2 \leq 2 \left| \varphi_R \left( \|u_\varepsilon\|_{D(A^\alpha)} \right) - \varphi_R \left( \|\bar{u}_\varepsilon\|_{D(A^\alpha)} \right) \right|^2 \|A^\alpha g(t, u_\varepsilon)\|_{L^2}^2$$

$$+ 2 \varphi_R^2 \left( \|\bar{u}_\varepsilon\|_{D(A^\alpha)} \right) \|A^\alpha [g(t, u_\varepsilon) - g(t, \bar{u}_\varepsilon)]\|_{L^2}^2$$

$$\leq \frac{C}{R} \left( 1 + R^2 \right) \|u_\varepsilon - \bar{u}_\varepsilon\|_{D(A^\alpha)}^2,$$

It follows from the last estimate that

$$\mathbb{E} \sup_{t \in [0,T]} \|K_4(t)\|_{L^2}^2 \leq C T \mathbb{E} \left( \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(0,T; D(A^\alpha))}^2 \right).$$

(3.24)

To estimate $K_5(t)$, we first note that

$$K_5(t) = \int_0^t AK_5(s) ds + \int_0^t \int_{|z| < 1} A^\alpha \mathcal{P} V_1(s, z) \pi(ds, dz) + \int_0^t \int_{|z| \geq 1} A^\alpha \mathcal{P} V_2(s, z) \pi(ds, dz),$$

where

$$V_1(t, z) = \varphi_R \left( \|u_\varepsilon\|_{D(A^\alpha)} \right) K(u_\varepsilon(t-), z) - \varphi_R \left( \|\bar{u}_\varepsilon\|_{D(A^\alpha)} \right) K(\bar{u}_\varepsilon(t-), z),$$

$$V_2(t, z) = \varphi_R \left( \|u_\varepsilon\|_{D(A^\alpha)} \right) G(u_\varepsilon(t-), z) - \varphi_R \left( \|\bar{u}_\varepsilon\|_{D(A^\alpha)} \right) G(\bar{u}_\varepsilon(t-), z).$$

By using the assumptions on $K$ and $G$, one can verify that, for all $t \in [0, T]$

$$\int_{|z| < 1} \|V_1(t, z)\|^2_{D(A^\alpha)} \mu(dz) + \int_{|z| \geq 1} \|V_2(t, z)\|^2_{D(A^\alpha)} \mu(dz) \leq C \|u_\varepsilon - \bar{u}_\varepsilon\|^2_{D(A^\alpha)}.$$

(3.25)
By applying the Itô’s formula to $\|K_5(t)\|_{L^2}^2$, we get

$$
\|K_5(t)\|_{L^2}^2 \leq 2 \left| \int_0^t \int_{|z|>1} \langle J_5(s-), A^\alpha \mathcal{P} V_2(s, z) \rangle \mu(\mathrm{d}z) \mathrm{d}s \right|
+ \int_0^t \int_{|z|<1} \|A^\alpha \mathcal{P} V_1(s-, z)\|_{L^2}^2 \mu(\mathrm{d}z) \mathrm{d}s
+ \int_0^t \int_{|z|\geq 1} \|A^\alpha \mathcal{P} V_2(s-, z)\|_{L^2}^2 \mu(\mathrm{d}z) \mathrm{d}s
$$

(3.26)

+ $\left| \int_0^t \int_{|z|<1} (2\langle J_5(s-), A^\alpha \mathcal{P} V_1(s, z) \rangle + \|A^\alpha \mathcal{P} V_1(s-, z)\|_{L^2}^2) \tilde{\pi}(\mathrm{d}t, \mathrm{d}z) \right|
+ \left| \int_0^t \int_{|z|>1} (2\langle J_5(s-), A^\alpha \mathcal{P} V_2(s-, z) \rangle + \|A^\alpha \mathcal{P} V_2(s-, z)\|_{L^2}^2) \tilde{\pi}(\mathrm{d}t, \mathrm{d}z) \right|

$\overset{\text{def}}{=} L_1(t) + \cdots + L_5(t)$.

For $L_1(t)$, it follows from (3.25) and the Hölder inequality that

$$
L_1(t) \leq \int_0^t \|K_5(s)\|_{L^2}^2 \mathrm{d}s + \int_0^t \left| \int_{|z|\geq 1} \|A^\alpha \mathcal{P} V_2(s-, z)\|_{L^2} \mathrm{d}z \right|^2 \mathrm{d}s
\leq \int_0^t \|K_5(s)\|_{L^2}^2 \mathrm{d}s + C T \|\bar{u}_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(0,T;D(A^\alpha))}^2.
$$

By (3.25) again, we have

$$
L_2(t) + L_3(t) \leq C \int_0^T \|u_\varepsilon - \bar{u}_\varepsilon\|_{D(A^\alpha)}^2 \mathrm{d}s \leq C T \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(0,T;D(A^\alpha))}^2.
$$

For $L_4(t)$, by applying the BDG inequality, we get

$$
\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \int_{|z|<1} \langle K_5(s-), A^\alpha \mathcal{P} V_1(t, z) \rangle_{L^2} \tilde{\pi}(\mathrm{d}t, \mathrm{d}z) \right|
\leq \eta \mathbb{E} \sup_{t \in [0,T]} \|K_5(t)\|_{L^2} + C T \mathbb{E} \left( \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(0,T;D(A^\alpha))}^2 \right),
$$

and

$$
\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^T \int_{|z|<1} \|A^\alpha \mathcal{P} V_1(s-, z)\|_{L^2}^2 \tilde{\pi}(\mathrm{d}t, \mathrm{d}z) \right|
\leq \eta \mathbb{E} \sup_{t \in [0,T]} \|u_\varepsilon - \bar{u}_\varepsilon\|_{D(A^\alpha)}^2 + C T \mathbb{E} \left( \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(0,T;D(A^\alpha))}^2 \right),
$$

which implies that

$$
\mathbb{E} \sup_{t \in [0,T]} \|L_4(t)\|_{L^2} \leq 2\eta \mathbb{E} \sup_{t \in [0,T]} \|K_5(t)\|_{L^2} + C T \mathbb{E} \left( \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(0,T;D(A^\alpha))}^2 \right).
$$

In a similar manner, by assumptions on $G$ and (3.25), we also have

$$
\mathbb{E} \sup_{t \in [0,T]} \|L_5(t)\|_{L^2} \leq 2\eta \mathbb{E} \sup_{t \in [0,T]} \|K_5(t)\|_{L^2} + C T \mathbb{E} \|u_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(0,T;D(A^\alpha))}^2.
$$
Substituting the above estimates for $L_1(t) \sim L_5(t)$ into (3.26) and then choosing $\eta > 0$ small enough, we obtain

$$
E \sup_{t \in [0, T]} \|K_5(t)\|_{L^2}^2 \leq \int_0^T \|K_5(s)\|_{L^2}^2 ds + CTE\|u_\epsilon - \bar{u}_\epsilon\|_{L^\infty(0, T; D(A^\alpha))}^2.
$$

Applying the Gronwall inequality to the above inequality leads to

$$(3.27) \quad E \sup_{t \in [0, T]} \|K_5(t)\|_{L^2}^2 \leq CTE\|u_\epsilon - \bar{u}_\epsilon\|_{L^\infty(0, T; D(A^\alpha))}^2.$$ 

It follows from the estimates (3.21)-(3.24) and (3.27) that

$$
E \sup_{t \in [0, T]} \|A^\alpha (\mathcal{F}_3(n_\epsilon, c_\epsilon, u_\epsilon) - \mathcal{F}_3(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon))\|_{L^2}^2 \leq C(T + T^{2-2\alpha})
$$

$$
\times \left( E\|\bar{u}_\epsilon - u_\epsilon\|_{L^\infty(0, T; W^{1, q})} + E\|\bar{n}_\epsilon - n_\epsilon\|_{L^\infty(0, T; L^\infty)} + E\|u_\epsilon - \bar{u}_\epsilon\|_{L^\infty(0, T; D(A^\alpha))} \right),
$$

which together with (3.18) and (3.19) yields that

$$
\|\mathcal{F}(n_\epsilon, c_\epsilon, u_\epsilon) - \mathcal{F}(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon)\|_{X_T}^2 \leq C(T + T^{2-2\alpha} + T^{1-2\gamma} + T^{2-2\delta}) \|(n_\epsilon, c_\epsilon, u_\epsilon) - (\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon)\|_{X_T}^2.
$$

If we take $T_R > 0$ small enough such that $C(T + T^{2-2\alpha} + T^{1-2\gamma} + T^{2-2\delta}) \leq \frac{1}{2}$, then the map $\mathcal{F} : B_{T_R}(r) \mapsto B_{T_R}(r)$ formulates a contraction, and the Banach fixed point theorem implies that there must be a unique triple $(n_\epsilon, c_\epsilon, u_\epsilon) \in B_T(r)$ such that $(n_\epsilon, c_\epsilon, u_\epsilon)$ is a mild solution of (3.2) on the interval $[0, T_R]$. Moreover, observing that $T_R > 0$ is independent of the initial data $(n_0, c_0, u_0)$, one can repeat above arguments to (3.2) with the new initial data $(n_\epsilon(T_R), c_\epsilon(T_R), u_\epsilon(T_R))$. In this way, one can construct a unique mild solution on the intervals $[T_R, 2T_R], [2T_R, 3T_R], ..., $ which implies the existence and uniqueness of mild solution to (3.2) on $[0, T]$ for any $T > 0$. The proof of Lemma 3.4 is completed. \( \square \)

3.3. Removing the cut-off operator.

**Proof of Lemma 3.3** Assume that $(n_\epsilon, c_\epsilon, u_\epsilon, \tau^\epsilon)$ and $(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon, \bar{\tau}^\epsilon)$ are two local mild solutions to the system (3.1) with the same initial data. For $l > 0$, we set $\tau^{\epsilon}_l = \tau^\epsilon \wedge \bar{\tau}^\epsilon$, $\tau^{\epsilon}_{l, l} = \tau^{\epsilon}_l \wedge \bar{\tau}^\epsilon_l$, where

$$
\tau^\epsilon_l = \inf \{0 \leq t < \infty; \|(n_\epsilon, c_\epsilon, u_\epsilon)\|_{X_T} \geq l\} \wedge \tau^\epsilon,
$$

$$
\bar{\tau}^\epsilon_l = \inf \{0 \leq t < \infty; \|(\bar{n}_\epsilon, \bar{c}_\epsilon, \bar{u}_\epsilon)\|_{X_T} \geq l\} \wedge \bar{\tau}^\epsilon.
$$

Denote $n^*_\epsilon = n_\epsilon - \bar{n}_\epsilon$, $c^*_\epsilon = c_\epsilon - \bar{c}_\epsilon$, and $u^*_\epsilon = u_\epsilon - \bar{u}_\epsilon$. The proof of uniqueness result depends upon some energy estimates for $(n^*_\epsilon, c^*_\epsilon, u^*_\epsilon)$. For the difference $n^*_\epsilon$ of the random PDEs satisfied by $n_\epsilon$ and $\bar{n}_\epsilon$, we get by integrating by parts and the divergence-free condition
that, for any \( t \in [0, \tau^*_t \wedge T) \),

\[
\|n^*_t(t)\|_{L^2}^2 + \int_0^t \|\nabla n^*_t\|_{L^2}^2 ds \\
\leq \int_0^t \int_D |n^*_t u^*_t \cdot \nabla n_t| dx ds + \int_0^t \int_D |n^*_t h_t^\epsilon(n_t) \chi(c_t) \nabla n^*_t \cdot \nabla c_t| dx ds \\
+ \int_0^t \int_D |\nabla h_t^\epsilon(\pi)\chi(c_t) - \chi(\tau_0)| \nabla n^*_t \cdot \nabla c_t| dx ds \\
+ \int_0^t \int_D |\nabla h_t^\epsilon(\pi)\chi(\tau_0)\nabla n^*_t \cdot \nabla c_t| dx ds \overset{\text{def}}{=} N_1 + \cdots + N_5.
\]

(3.28)

For \( N_1 \), by using the embedding \( W^{1,q}(D) \subset L^\infty(D) \) for \( q > 3 \) and integrating by parts \( \int_D n^*_t u^*_t \cdot \nabla n_t dx = -\int_D n^*_t \nabla u^*_t \cdot \nabla n^*_t dx \) because \( \text{div} u^*_t = 0 \), we get for any \( \eta > 0 \),

\[
N_1 + N_2 \leq \eta \int_0^t \|\nabla n^*_t\|_{L^2}^2 ds + C \int_0^t (\|u^*_t\|_{L^2}^2 + \|n^*_t\|_{L^2}^2) ds, \quad t \in [0, \tau^*_t \wedge T).
\]

For \( N_3 \) and \( N_5 \), we get from the boundedness of \( h''(\cdot) \) that, for any \( \eta > 0 \),

\[
N_3 + N_5 \leq \eta \int_0^t \|\nabla n^*_t\|_{L^2}^2 ds + C \int_0^t (\|u^*_t\|_{L^2}^2 + \|n^*_t\|_{L^2}^2) ds, \quad t \in [0, \tau^*_t \wedge T).
\]

For \( N_4 \), recalling the following interpolation result: If \( B_0 \subset B \subset B_1 \) are three Banach spaces, the embedding \( B_0 \subset B \) being compact, \( B \subset B_1 \) being continuous, then for every \( \eta > 0 \) there is a constant \( C_\eta > 0 \) such that \( \|x\|_B \leq \eta \|x\|_{B_0} + C_\eta \|x\|_{B_1} \) for every \( x \in B_0 \). We apply this fact to \( W^{1,2}(D) \subset L^{\frac{2q}{q-2}}(D) \subset W^{-1,2}(D) \) with \( q > 3 \): For any \( \eta > 0 \), there exists a \( C_\eta > 0 \) such that

\[
\|f\|_{L^{\frac{2q}{q-2}}}^2 \leq \eta \|\nabla f\|_{L^2}^2 + C_\eta \|f\|_{L^2}^2.
\]

It then follows from the local Lipschitz continuity of \( \chi \) that

\[
N_4 \leq \eta \int_0^t \|\nabla n^*_t\|_{L^2}^2 ds + C \int_0^t \|\pi_t\|_{L^\infty}^2 \|c^*_t\|_{L^2}^{\frac{2q}{q-2}} \|\nabla c_t\|_{L^2}^2 ds \\
\leq \eta \int_0^t \|\nabla n^*_t\|_{L^2}^2 ds + C \int_0^t \|c^*_t\|_{L^{\frac{2q}{q-2}}}^2 ds \\
\leq \eta \int_0^t \|\nabla n^*_t\|_{L^2}^2 ds + \eta \int_0^t \|\nabla c^*_t\|_{L^2}^2 ds + C \int_0^t \|c^*_t\|_{L^2}^2 ds.
\]

Therefore plugging the estimates for \( N_1 - N_6 \) into (3.28) leads to

\[
\mathbb{E} \sup_{s \in [0, \tau^*_t \wedge T]} \|n^*_t(s)\|_{L^2}^2 + (1 - 3\eta) \int_0^{t \wedge \tau^*_t} \|\nabla n^*_t\|_{L^2}^2 ds \\
\leq \eta \int_0^{t \wedge \tau^*_t} \|\nabla c^*_t\|_{L^2}^2 ds + C \int_0^{t \wedge \tau^*_t} (\|u^*_t\|_{L^2}^2 + \|n^*_t\|_{L^2}^2 + \|c^*_t\|_{L^2}^2) ds.
\]
Proceeding similarly to the equations satisfied by $c_\epsilon$ and $\pi_\epsilon$, we obtain

\begin{equation}
\mathbb{E} \sup_{s \in [0, t \wedge \tau^*_{\epsilon,t}]} \| c_\epsilon^*(s) \|_{L^2}^2 + \int_0^{t \wedge \tau^*_{\epsilon,t}} \| \nabla c_\epsilon^*(s) \|_{L^2}^2 ds \leq C \int_0^{t \wedge \tau^*_{\epsilon,t}} (\| u_\epsilon^* \|_{L^2}^2 + \| n_\epsilon^* \|_{L^2}^2 + \| c_\epsilon^* \|_{L^2}^2) ds.
\end{equation}

To estimate $\| u_\epsilon^* \|_{L^2}$, we apply the Itô’s formula to infer that, for any $t \in [0, \tau^*_{\epsilon,t} \wedge T)$,

\begin{align*}
\| u_\epsilon^*(t) \|_{L^2}^2 + 2 \int_0^t \| A^1 u_\epsilon^*(s) \|_{L^2}^2 ds & \leq \int_0^t \| \mathcal{P}[g(s, u_\epsilon) - g(s, \pi_\epsilon)] \|_{L^2}^2 ds \\
& \quad + 2 \int_0^t \left\langle u_\epsilon^*(t), \mathcal{P}[(L_\epsilon u_\epsilon \cdot \nabla) u_\epsilon - L_\epsilon \pi_\epsilon \cdot \nabla \pi_\epsilon] - \mathcal{P}(n_\epsilon^* \nabla \Phi) \right\rangle_{L^2} ds \\
& \quad + \int_0^t \int_{|z| \leq 1} \| \mathcal{P}[K(u_\epsilon(x, s-), z) - K(\pi_\epsilon(x, s-), z)] \|_{L^2}^2 \pi(ds, dz) \\
& \quad + \int_0^t \int_{|z| > 1} \| \mathcal{P}[G(u_\epsilon(x, s-), z) - G(\pi_\epsilon(x, s-), z)] \|_{L^2}^2 \pi(ds, dz) \\
& \overset{\text{def}}{=} K_1(t) + \cdots + K_8(t).
\end{align*}

Now we shall estimate each integral terms on the R.H.S. of (3.31). For $K_1$, by Young’s inequality and the hypothesis on $g$ (taking $\alpha = 0$), we have

\begin{equation}
\mathbb{E} \sup_{s \in [0, t \wedge \tau^*_{\epsilon,t}]} K_1(s) \leq C \mathbb{E} \int_0^{t \wedge \tau^*_{\epsilon,t}} \| g(s, u_\epsilon) - g(s, \pi_\epsilon) \|_{L^2}^2 ds \leq C \mathbb{E} \int_0^{t \wedge \tau^*_{\epsilon,t}} \| u_\epsilon^* \|_{D(A^\alpha)}^2 ds.
\end{equation}

For $K_2$, by virtue of the identities $\langle u_\epsilon^*, (L_\epsilon \pi_\epsilon \cdot \nabla) u_\epsilon^* \rangle = 0$ and $\langle u_\epsilon^*, (L_\epsilon \pi_\epsilon \cdot \nabla) u_\epsilon^* \rangle = - (u_\epsilon^*, (L_\epsilon u_\epsilon \cdot \nabla) u_\epsilon^*)$ because $\text{div}(L_\epsilon \pi_\epsilon) = 0$, we get for any $\eta > 0$

\begin{align*}
\mathbb{E} \sup_{s \in [0, t \wedge \tau^*_{\epsilon,t}]} K_2(s) & \leq 2 \mathbb{E} \left( \sup_{t \in [0, t \wedge \tau^*_{\epsilon,t}]} \left| \int_0^{t \wedge \tau^*_{\epsilon,t}} \left\langle u_\epsilon^*, (L_\epsilon u_\epsilon^* \cdot \nabla) u_\epsilon + (L_\epsilon \pi_\epsilon \cdot \nabla) u_\epsilon^* \right\rangle_{L^2} ds \right| \right) \\
& \quad + 2 \mathbb{E} \int_0^{t \wedge \tau^*_{\epsilon,t}} \| u_\epsilon^* \|_{L^2} \| n_\epsilon^* \|_{L^2} \| \nabla \Phi \|_{L^\infty} ds \\
& \leq \eta \mathbb{E} \int_0^{t \wedge \tau^*_{\epsilon,t}} \| \nabla u_\epsilon^* \|_{L^2}^2 ds + C \mathbb{E} \int_0^{t \wedge \tau^*_{\epsilon,t}} \left( \| u_\epsilon^* \|_{L^2}^2 + \| n_\epsilon^* \|_{L^2}^2 \right) ds.
\end{align*}
For $K_3$, we have
\[
\mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} K_3(s) \leq C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \int_{|z| > 1} \left\| G(u_\varepsilon(x,s^-), z) - G(\overline{v}_\varepsilon(s^-), z) \right\|_{L^2}^2 ds dz
\]
\[
= C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \int_{|z| > 1} \left\| G(u_\varepsilon(x,s^-), z) - G(\overline{v}_\varepsilon(s^-), z) \right\|_{L^2}^2 \mu(dz) ds
\]
\[
\leq C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \|u^*_\varepsilon(s)\|_{L^2}^2 ds,
\]
and similarly
\[
\mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} (K_4(s) + K_5(s)) \leq C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \|u^*_\varepsilon(s)\|_{L^2}^2 ds.
\]
For $K_6$, the assumption on $g$ and BDG inequality imply that, for any $\eta > 0$,
\[
\mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} K_6(s) \leq C \mathbb{E} \left( \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} \|u^*_\varepsilon(s)\|_{L^2}^2 \int_0^{t \wedge \tau^*_\varepsilon} \|\mathcal{P}[g(s,u_\varepsilon) - g(s,\overline{v}_\varepsilon)]\|_{L^2}^2 ds \right)^{1/2}
\]
\[
\leq \eta \mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} \|u^*_\varepsilon(s)\|_{L^2}^2 + C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \|u^*_\varepsilon(s)\|_{L^2}^2 ds.
\]
For $K_7$, by using the assumption on $K$ and the BDG inequality, we obtain for any $\eta > 0$
\[
\mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} K_7(s)
\]
\[
\leq C \mathbb{E} \left[ \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} \|u^*_\varepsilon(s)\|_{L^2} \left( \int_0^{t \wedge \tau^*_\varepsilon} \int_{|z| \leq 1} \left\| K(u_\varepsilon(s^-), z) - K(\overline{v}_\varepsilon(s^-), z) \right\|_{L^2}^2 \mu(dz) ds \right)^{1/2} \right]
\]
\[
\leq \eta \mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} \|u^*_\varepsilon(s)\|_{L^2}^2 + C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \|u^*_\varepsilon(s)\|_{L^2}^2 ds.
\]
In a similar manner, the term $K_8$ can be estimated by
\[
\mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} K_8(s) \leq \eta \mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} \|u^*_\varepsilon(s)\|_{L^2}^2 + C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \|u^*_\varepsilon(s)\|_{L^2}^2 ds.
\]
Taking the supremum to (3.31) on time over $[0, t \wedge \tau^*_\varepsilon]$ and then taking expectation on both sides, after substituting the above estimates for $K_1 \sim K_8$ and choosing $\eta > 0$ small enough, we arrive at
\[
(3.32) \quad \mathbb{E} \sup_{s \in [0,t \wedge \tau^*_\varepsilon]} \|u^*_\varepsilon(s)\|_{L^2}^2 \leq C \mathbb{E} \int_0^{t \wedge \tau^*_\varepsilon} \left( \|u^*_\varepsilon(s)\|_{L^2}^2 + \|n^*_\varepsilon(s)\|_{L^2}^2 \right) ds.
\]
By virtue of the estimates (3.29), (3.30) and (3.32) and taking small $\eta > 26$, we obtain
\[
\mathbb{E} \sup_{s \in [0,t \land \tau_R]} \left( \|u^*_\epsilon(s)\|_{L^2}^2 + \|c^*_\epsilon(s)\|_{L^2}^2 + \|c^*_\epsilon(s)\|_{L^2}^2 \right)
\]
(3.33)
\[
\leq C \mathbb{E} \int_0^t \sup_{s \in [0,t \land \tau_R]} \left( \|u^*_\epsilon(s)\|_{L^2}^2 + \|n^*_\epsilon(s)\|_{L^2}^2 + \|c^*_\epsilon(s)\|_{L^2}^2 \right) \, ds.
\]
which combined with the Gronwall inequality imply that $\mathbb{E} \sup_{s \in [0,t \land \tau_R]} (\|u^*_\epsilon(s)\|_{L^2}^2 + \|c^*_\epsilon(s)\|_{L^2}^2 + \|c^*_\epsilon(s)\|_{L^2}^2) = 0$. By taking $t = \tau^*$ and letting $l \to \infty$, this proves the uniqueness part.

Now let us extend the solution $(n_\epsilon, c_\epsilon, u_\epsilon)$ to a maximal time of existence $\tau_\epsilon$. For any $R > 0$, we define
\[
\tau_R \equiv \begin{cases} 
\inf \left\{ 0 \leq t < \infty; \|c_\epsilon\|_{L^\infty([0,t];W^{1,q})} + \|n_\epsilon\|_{L^\infty([0,t];L^\infty)} + \|u_\epsilon\|_{L^\infty([0,t];D(A^\alpha))} \geq R \right\}, \\
+\infty, & \text{if the above set } \{ \cdots \} \text{ is empty.}
\end{cases}
\]
It is clear that $(\tau_R)_{R>0}$ is a sequence of stopping times, and since $\|c_\epsilon\|_{W^{1,q}} + \|n_\epsilon\|_{L^\infty} + \|u_\epsilon\|_{D(A^\alpha)}$ is bounded from above, $\tau_R > 0$ $\mathbb{P}$-a.s. for all $R$ large enough. According to the definition of cut-off operator, we see that the solution $(n_\epsilon, c_\epsilon, u_\epsilon)$ constructed in Lemma 3.4 restricted on $[0, \tau_R]$ is a local mild solution to the regularized system (3.1). Moreover, it follows from the uniqueness result that $\tau_R \leq \tau_{R+1}$ $\mathbb{P}$-a.s., and for $t \in [0, \tau_R]$, there holds $(n_{R+1,\epsilon}, c_{R+1,\epsilon}, u_{R+1,\epsilon}) = (n_{R,\epsilon}, c_{R,\epsilon}, u_{R,\epsilon})$.

Define a stopping time $\tilde{\tau}_\epsilon(\omega) = \lim_{R \to \infty} \tau_R(\omega)$, and a triplet $(n_\epsilon, c_\epsilon, u_\epsilon) = (n_{R,\epsilon}, c_{R,\epsilon}, u_{R,\epsilon})$ on $[0, \tau_R]$. One can conclude from the definition of $(\tau_R)_{R>0}$ that $(n_\epsilon, c_\epsilon, u_\epsilon, \tilde{\tau}_\epsilon)$ is actually a unique maximal local mild solution to (3.1). The proof of Lemma 3.3 is completed.

4. Global-in-time approximate solutions

In this section, we prove that the local solution constructed in Section 3 is actually a global one. The proof is based on a new entropy-energy estimate, which can be regarded as an extension of the deterministic counterpart but without convex condition on the domain. Let us begin with an observation on spatio-temporal regularity of solutions.

4.1. Variational formulation. For $q > 3$ and $\alpha \in (\frac{3}{4}, 1)$, we infer from Definition 3.1 that
\[
(c_\epsilon, u_\epsilon) \in L^2 \left( [0, \tilde{\tau}_\epsilon); W^{1,2}(D) \times (W^{1,2}_0(D))^3 \right), \quad \mathbb{P}-a.s.
\]
(4.1)
Since $n_\epsilon \in L^\infty([0, \tilde{\tau}_\epsilon); L^\infty(D))$, (4.1) implies that $u_\epsilon \cdot \nabla n_\epsilon = \text{div}(u_\epsilon n_\epsilon) \in L^2([0, \tilde{\tau}_\epsilon); L^2(D))$ and $\text{div}(n_\epsilon h_\epsilon(n_\epsilon)\chi(c_\epsilon) \nabla c_\epsilon) \in L^2([0, \tilde{\tau}_\epsilon); W^{-1, 2}(D))$. By making use of the $n_\epsilon$-equation in (3.1) and the maximal $L^p - L^q$ regularity for parabolic equations (cf. [26]), we obtain $n_\epsilon \in L^2([0, \tilde{\tau}_\epsilon); W^{1,2}(D))$, $\mathbb{P}$-a.s.

Setting
\[
V = W^{1,2}(D) \times W^{1,2}(D) \times (W^{1,2}_0(D))^3, \quad H = L^2(D) \times L^2(D) \times (L^2_0(D))^3.
\]
Then, $V \subset H \equiv H^* \subset V^*$ formulates a Gelfand inclusion, where $H$ is identified with its dual $H^*$ and $V^*$ is the dual space of $V$. As a consequence, the local mild solution $(n_\epsilon, c_\epsilon, u_\epsilon)$
constructed in Lemma 3.3 is actually a variational solution, and the variational formulation of (3.1) in \( \mathbf{V}^* \) is given by

\[(4.2a) \quad \langle n_\epsilon(t), \phi \rangle + \int_0^t \langle u_\epsilon \cdot \nabla n_\epsilon, \phi \rangle \, ds = \int_0^t \langle \Delta n_\epsilon, \phi \rangle \, ds - \int_0^t \langle \text{div} (n_\epsilon h_\epsilon(n_\epsilon)x(c_\epsilon)\nabla c_\epsilon), \phi \rangle \, ds,\]

\[(4.2b) \quad \langle c_\epsilon(t), \varphi \rangle + \int_0^t \langle u \cdot \nabla c_\epsilon, \varphi \rangle \, ds = \int_0^t \langle \Delta c_\epsilon, \varphi \rangle \, ds - \int_0^t \langle h_\epsilon(n_\epsilon)f(c_\epsilon), \varphi \rangle \, ds,\]

\[(4.2c) \quad \langle u_\epsilon(t), \psi \rangle + \int_0^t \langle P(L_\epsilon u_\epsilon \cdot \nabla) u_\epsilon, \psi \rangle \, ds = - \int_0^t \langle Au_\epsilon, \psi \rangle \, ds + \int_0^t \langle P(n_\epsilon \nabla \Phi), \psi \rangle \, ds + \int_0^t \langle Ph(s, u_\epsilon), \psi \rangle \, ds + \int_0^t \langle Pg(s, u_\epsilon) dW, \psi \rangle + \int_0^t \int_Z \langle PL_\epsilon, \psi \rangle \, d\lambda,\]

for any \((\phi, \varphi, \psi) \in \mathbf{V}\), and any \( t \in [0, T] \subset [0, \bar{\tau}_\epsilon), \mathbb{P}\)-a.s.

In view of the assumptions on \( g, K \) and \( G \) (see (A3)-(A4)), the solution \((n_\epsilon, c_\epsilon, u_\epsilon)\) can be regarded as a \( \mathbf{V}^* \)-valued càdlàg semimartingale with the stochastic integrals being local square integrable martingale. Thereby, a kind of Itô’s formula by Gyöngy and Krylov (cf. Theorem 1 in [27]; see also Theorem A.1 in [8] and Appendix C in [9]) in Banach spaces can be applied.

4.2. A new entropy-energy inequality. To begin with, let us recall an identity which was initially derived by Winkler [57] for the deterministic counterpart in the case of \( \mathbf{L}_\epsilon = \text{Id} \). Due to the divergence-free condition \( \text{div} u_\epsilon = 0 \) and the fact that the derivation of this identity mainly depends on the \( n_\epsilon \)-equation and \( c_\epsilon \)-equation, similar result also holds for the first two random PDEs in (3.1).

**Lemma 4.1.** For any given \( \epsilon \in (0, 1) \) and any \( T > 0 \), the solution

\[
\begin{align*}
\text{d} & \left( \int_D n_\epsilon \ln n_\epsilon \, dx + \frac{1}{2} \int_D |\nabla \Psi(c_\epsilon)|^2 \, dx \right) + \int_D \frac{|
abla n_\epsilon|^2}{n_\epsilon} \, dx \, dt + \int_D \theta(c_\epsilon)|D^2 \rho(c_\epsilon)|^2 \, dx \, dt \\
& = \int_D h_\epsilon(n_\epsilon) \left( \frac{f(c_\epsilon)\theta'(c_\epsilon)}{2\theta^2(c_\epsilon)} - \frac{f'(c_\epsilon)}{\theta(c_\epsilon)} \right) |\nabla c_\epsilon|^2 \, dx \, dt \\
& + \int_D \left( \frac{1}{\theta(c_\epsilon)} \Delta c_\epsilon - \frac{1}{2} \frac{\theta'(c_\epsilon)}{\theta^2(c_\epsilon)} |\nabla c_\epsilon|^2 \right) (u_\epsilon \cdot \nabla c_\epsilon) \, dx \, dt \\
& + \frac{1}{2} \int_D \frac{\theta''(c_\epsilon)}{\theta^2(c_\epsilon)} |\nabla c_\epsilon|^4 \, dx \, dt + \frac{1}{2} \int_{\partial D} \frac{1}{\theta(c_\epsilon)} \frac{\partial |\nabla c_\epsilon|^2}{\partial \nu} \, d\Sigma \, dt, \quad \mathbb{P}\text{-a.s.,}
\end{align*}
\]

for all \( t \in (0, T \wedge \bar{\tau}_\epsilon) \), where

\[
\theta(s) = \frac{f(s)}{\chi(s)}, \quad \Psi(s) = \int_1^s \frac{d\sigma}{\sqrt{\theta(\sigma)}} \quad \text{and} \quad \rho(s) = \int_1^s \frac{d\sigma}{\theta(\sigma)}.
\]

**Proof.** For almost all \( \omega \in \Omega \), thanks to the fact of \( \text{div} u_\epsilon = 0 \), one can apply the chain rule to \( \text{d}(n_\epsilon \ln n_\epsilon) \) and \( \text{d}|\nabla \Psi(c_\epsilon)|^2_{L^2} \) associated with the \( n_\epsilon \) and \( c_\epsilon \)-equation, respectively. Then the identity (4.3) can be obtained by performing a straightforward computation similar to [57]. We shall omit the details here. \( \Box \)
Remark 4.2. Employing the assumptions on $f, \chi$, and the fact that $t \mapsto \|c_e(\cdot,t)\|_{L^\infty}$ is nonincreasing, one can find two positive constants $C^- \leq C^+$ such that $C^- s \leq \theta(s) \leq C^+ s$ for all $s \in [0,\|c_0\|_{L^\infty}]$, and $\theta'(c_e) \geq \theta'(\|c_0\|_{L^\infty}) > 0$ for all $(x,t) \in D \times (0,\tilde{T}_e)$.

Lemma 4.3 (Mizoguchi-Souplet, [47]). Let $D \subset \mathbb{R}^d$, $d \geq 1$ be a bounded domain with a $C^2$ boundary. Let $w \in C^2(\overline{D})$ such that $\frac{\partial w}{\partial n} = 0$ on the boundary $\partial D$. Then
\begin{align*}
\frac{\partial |\nabla w|^2}{\partial n} \leq 2\kappa |\nabla w|^2 \quad \text{on } \partial D,
\end{align*}
for some constant $\kappa$ depending only on the domain $D$.

Lemma 4.4. Any solution $(n_e, c_e, u_e)$ to the problem (3.1) in the sense of Definition 3.1 satisfies that, for all $T > 0$
\begin{align}
(4.4) \quad n_e(x,t) &\geq 0, \quad c_e(x,t) \geq 0, \quad \forall t \in (0, T \wedge \tilde{T}_e), \quad x \in D, \ \mathbb{P}\text{-a.s.}, \\
(4.5) \quad \|n_e(\cdot,t)\|_{L^1} &\equiv \|n_0\|_{L^1}, \quad \|c_e(\cdot,t)\|_{L^\infty} \leq \|c_0\|_{L^\infty}, \quad \forall t \in (0, T \wedge \tilde{T}_e), \ \mathbb{P}\text{-a.s.},
\end{align}

Proof. Integrating the first equation in (3.1) and using the condition $\text{div} u_e = 0$, we obtain (4.4). Since $f > 0$ on $(0, \infty)$, $f(0) = 0$ by our assumption $(A_2)$ and $h_s(s) \geq 0$ for all $s > 0$, an application of the comparison principle to $c_e$-equation in (3.1) gives (4.5).

Lemma 4.5. Let $T > 0$ be arbitrary. Assume that conditions $(A_1)-(A_4)$ hold, then for any solution $(n_e, c_e, u_e)$ to (3.1) in the sense of Definition 3.1 satisfies
\begin{align}
(4.6) \quad \mathcal{E}[n_e, c_e, u_e](t) + \int_0^t \mathcal{D}[n_e, c_e, u_e](s) ds &\leq \mathcal{E}[n_{e_0}, c_{e_0}, u_{e_0}] + \mathcal{L}[n_e, c_e, u_e](t),
\end{align}
where
\begin{align*}
\mathcal{E}[n_e, c_e, u_e](t) &\overset{\text{def}}{=} \int_D \left( n_e \ln n_e + \frac{1}{2} |\nabla \Psi(c_e)|^2 + c^1 |u_e|^2 \right) dx, \\
\mathcal{D}[n_e, c_e, u_e](t) &\overset{\text{def}}{=} \int_D \left( \frac{1}{2} \frac{|\nabla n_e|^2}{n_e} + d_1 \frac{|\nabla c_e|^4}{c_e^3} + d_2 \frac{\Delta c_e^2}{c_e} + |\nabla u_e|^2 \right) dx,
\end{align*}
and
\begin{align*}
\mathcal{L}[n_e, c_e, u_e](t) &\overset{\text{def}}{=} C t + C \int_0^t |\Psi(c_e)|_{L^2}^2 ds + C \int_0^t |u_e(s)|_{L^2}^2 ds + 2 c^1 \int_0^t \langle u_e, \mathcal{P}g(s, u_e) dW \rangle \\
&\quad + c^1 \int_0^t \int_{|z| < 1} |K(u_e(s-), z)|_{L^2}^2 \pi(ds, dz) + c^1 \int_0^t \int_{|z| \geq 1} |G(u_e(s-), z)|_{L^2}^2 \pi(ds, dz) \\
&\quad + c^1 \int_0^t \int_{|z| < 1} 2(u_e, \mathcal{P}K(u_e(s-), z)) \tilde{\pi}(ds, dz) + c^1 \int_0^t \int_{|z| \geq 1} 2(u_e, \mathcal{P}G(u_e(s-), z)) \tilde{\pi}(ds, dz),
\end{align*}
for all $t \in (0, T \wedge \tilde{T}_e)$, with positive constants $d_1, d_2, c^1$ depending only on $\|c_0\|_{L^\infty}$ and the functions $f$ and $\chi$.

Proof. First, by virtue of the fact of $h_s(u_e) \geq 0$ and the assumptions on $f, \chi$, we infer that
\begin{align*}
\theta''(c_e) &\leq 0 \quad \text{and} \quad \frac{f'(c_e) \theta'(c_e)}{2 \theta^2(c_e)} - \frac{f'(c_e)}{\theta(c_e)} = -\frac{(\chi f)'(c_e)}{2 f(c_e)} \leq 0,
\end{align*}
which imply that the first and third terms on the R.H.S. of (4.3) can be neglected in estimating. In order to control the second term, we get by integrating by parts that

$$-\frac{1}{2} \int_D \frac{\theta'(c_\epsilon)}{\theta^2(c_\epsilon)} (u_\epsilon \cdot \nabla c_\epsilon) |\nabla c_\epsilon|^2 \, dx$$

(4.7)

$$= \frac{1}{2} \int_D \text{div} \left( \frac{u_\epsilon}{\theta(c_\epsilon)} \right) |\nabla c_\epsilon|^2 \, dx = - \sum_{i,j} \int_D \frac{1}{\theta(c_\epsilon)} u_\epsilon^i \partial_j c_\epsilon \partial_i \partial_j c_\epsilon \, dx,$$

and

$$\int_D \frac{1}{\theta(c_\epsilon)} \Delta c_\epsilon (u_\epsilon \cdot \nabla c_\epsilon) \, dx = \sum_{i,j} \int_D \frac{\theta'(c_\epsilon)}{\theta^2(c_\epsilon)} u_\epsilon^i \partial_j c_\epsilon |\partial_i c_\epsilon|^2 \, dx$$

(4.8)

$$- \sum_{i,j} \int_D \frac{1}{\theta(c_\epsilon)} \partial_i c_\epsilon \partial_i u_\epsilon^j \partial_j c_\epsilon \, dx - \sum_{i,j} \int_D \frac{1}{\theta(c_\epsilon)} \partial_i c_\epsilon u_\epsilon^j \partial_j c_\epsilon \, dx.$$

Moreover, let us recall useful functional inequality of Lemma 3.2 in [57] or Lemma 3.3 in [60], and then apply it to the function $\theta(c_\epsilon)$, we gain

$$\int_D \frac{\theta'(c_\epsilon)}{\theta^3(c_\epsilon)} |\nabla c_\epsilon|^4 \leq (2 + \sqrt{3})^2 \int_D \frac{\theta(c_\epsilon)}{\theta^2(c_\epsilon)} |D^2 \rho(c_\epsilon)|^2.$$

(4.9)

It then follows from identities (4.7)-(4.9) and the Young inequality that, for any $\delta > 0$,

$$\left| \int_D \left( \frac{1}{\theta(c_\epsilon)} \Delta c_\epsilon - \frac{1}{2} \frac{\theta'(c_\epsilon)}{\theta^2(c_\epsilon)} |\nabla c_\epsilon|^2 \right) (u_\epsilon \cdot \nabla c_\epsilon) \, dx \right| = \left| \sum_{i,j} \int_D \frac{1}{\theta(c_\epsilon)} \partial_i c_\epsilon \partial_i u_\epsilon \partial_j c_\epsilon \, dx \right|$$

(4.10)

$$\leq \delta \int_D \frac{\theta'(c_\epsilon)}{\theta^3(c_\epsilon)} |\nabla c_\epsilon|^4 \, dx + C_\delta \int_D \frac{\theta(c_\epsilon)}{\theta^2(c_\epsilon)} |\nabla u_\epsilon|^2 \, dx$$

$$\leq \delta (2 + \sqrt{3})^2 \int_D \frac{\theta(c_\epsilon)}{\theta^2(c_\epsilon)} |D^2 \rho(c_\epsilon)|^2 + C \int_D \frac{\theta(c_\epsilon)}{\theta^2(c_\epsilon)} |\nabla u_\epsilon|^2 \, dx.$$

For the boundary integral term on the R.H.S. of (4.3), we get from Lemma 4.3 and the Sobolev embedding theorem that, for any $r \in (0, \frac{1}{2})$,

$$\left| \frac{1}{2} \int_{\partial D} \frac{1}{\theta(c_\epsilon)} \frac{\partial |\nabla c_\epsilon|^2}{\partial \nu} \, d\Sigma \right|$$

(4.11)

$$\leq \kappa \int_{\partial D} \left| \frac{\nabla c_\epsilon}{\sqrt{\theta(c_\epsilon)}} \right|^2 \, d\Sigma \leq C \left\| \nabla \Psi(c_\epsilon) \right\|^2_{W^{r+\frac{1}{2}, 2}} = C \left\| \nabla \Psi(c_\epsilon) \right\|^2_{W^{r+\frac{1}{2}, 2}}.$$

By using an interpolation argument between $W^{r+\frac{1}{2}, 2}(D)$ and $L^2(D)$, and applying the estimate $\|w\|_{W^{2, 2}} \leq C(\|\Delta w\|_{L^2} + \|w\|_{L^2})$, we get for any $\eta > 0$

$$\left\| \nabla \Psi(c_\epsilon) \right\|^2_{W^{r+\frac{1}{2}, 2}} \leq C \left\| \Psi(c_\epsilon) \right\|^2_{W^{r+\frac{3}{2}, 2}} \left\| \Psi(c_\epsilon) \right\|^\frac{1}{2}_L \left\| \nabla \Psi(c_\epsilon) \right\|^\frac{1}{2}_L$$

(4.12)

$$\leq C \left( \|\Delta \Psi(c_\epsilon)\|^2_{L^2} \right)^{\frac{1}{2}+\frac{1}{2}} \left( \|\Psi(c_\epsilon)\|^2_{L^2} \right)^{\frac{1}{2}+\frac{1}{2}} + C \left\| \Psi(c_\epsilon) \right\|^2_{L^2}$$

$$\leq \eta \|\Delta \Psi(c_\epsilon)\|^2_{L^2} + C \|\Psi(c_\epsilon)\|^2_{L^2}.$$
A direct calculation for $\Delta \Psi(c_\epsilon)$ leads to
\[
\|\Delta \Psi(c_\epsilon)\|^2_{L^2} \leq 6 \int_D \theta(c_\epsilon)|D^2 \rho(c_\epsilon)|^2 dx + \frac{1}{2} \int_D \frac{\theta'(c_\epsilon)}{\theta'(c_\epsilon)} \|\nabla c_\epsilon\|^4 dx,
\]
where the first integral on the R.H.S. used the fact of $|\Delta f|^2 \leq 3|D^2 f|^2$. This estimate together with (4.11) and (4.12) lead to
\[
\frac{1}{2} \int_{\partial D} \frac{\partial |\nabla c_\epsilon|^2}{\partial \nu} d\Sigma \leq C\|\Psi(c_\epsilon)\|^2_{L^2} + \eta C \int_D \theta(c_\epsilon) |D^2 \rho(c_\epsilon)|^2 dx + \eta C \int_D \frac{\theta(c_\epsilon)}{\theta'(c_\epsilon)} |D^2 \rho(c_\epsilon)|^2 dx.
\]
Substituting the estimates (4.10) and (4.13) into (4.3), in view of Remark 4.3, we get
\[
d \int_D \left(n_\epsilon \ln n_\epsilon + \frac{1}{2} \|\nabla \Psi(c_\epsilon)\|^2\right) dx + \int_D \|\nabla u_\epsilon\|^2_{L^2} dx dt
\]
\[
+ \left(1 - \eta C - \eta C \theta'(|c_0|_{L^\infty}) - \delta(2 + \sqrt{3})^2 \theta'(|c_0|_{L^\infty})\right) \int_D \theta(c_\epsilon) |D^2 \rho(c_\epsilon)|^2 dx dt
\]
\[
\leq C \theta'(|c_0|_{L^\infty})|c_0|_{L^\infty} \int_D \|\nabla u_\epsilon\|^2_{L^2} dx + C \|\Psi(c_\epsilon(t))\|^2_{L^2} dt.
\]
Choosing $\delta, \eta > 0$ small enough such that $\frac{1}{2} \leq 1 - \eta C - \eta C \theta'(|c_0|_{L^\infty}) - \delta(2 + \sqrt{3})^2 \theta'(|c_0|_{L^\infty}) < 1$, and then integrating both sides of (4.14) over the interval $[0, t]$, we get
\[
\int_D \left(n_\epsilon \ln n_\epsilon + \frac{1}{2} \|\nabla \Psi(c_\epsilon)\|^2\right) dx + \int_0^t \int_D \|\nabla n_\epsilon\|^2_{L^2} dx ds + \frac{1}{2} \int_0^t \int_D \theta (c_\epsilon) |D^2 \rho(c_\epsilon)|^2 dx ds
\]
\[
\leq \int_D \left(n_0 \ln n_0 + \frac{1}{2} \|\nabla \Psi(c_\epsilon)\|^2\right) dx + c^\dagger \int_0^t \|\nabla u_\epsilon\|^2_{L^2} ds + C \int_0^t \|\Psi(c_\epsilon)\|^2_{L^2} ds,
\]
for any $t \in [0, T \wedge \tau_\epsilon]$, where $c^\dagger \equiv C \theta'(|c_0|_{L^\infty})|c_0|_{L^\infty}$. For the future argument, one can magnify $C > 0$ large enough such that $c^\dagger > 1$.

Now let us make use of the Itô’s formula to the functional $\|u_\epsilon(t)\|^2_{L^2}$, after integrating by parts, rearranging the terms and multiplying both sides by $c^\dagger$, we infer that
\[
c^\dagger \|u_\epsilon(t)\|^2_{L^2} + 2c^\dagger \int_0^t \|\nabla u_\epsilon(s)\|^2_{L^2} ds
\]
\[
= c^\dagger \|u_\epsilon(0)\|^2_{L^2} + 2c^\dagger \left(\int_0^t \langle u_\epsilon, \mathcal{P}(n_\epsilon, \nabla \Phi) \rangle ds + \int_0^t \langle u_\epsilon, \mathcal{P} h(s, u_\epsilon) \rangle ds + \int_0^t \|\mathcal{P} g(s, u_\epsilon)\|^2_{L^2} ds \right)
\]
\[
+ 2c^\dagger \left(\int_0^t \langle u_\epsilon, \mathcal{P} (s, u_\epsilon) dW \rangle + c^\dagger \int_0^t \|\mathcal{P} K(u_\epsilon(s), z)\|^2_{L^2} \pi(ds, dz) \right)
\]
\[
+ c^\dagger \int_0^t \|\mathcal{P} G(u_\epsilon(s), z)\|^2_{L^2} \pi(ds, dz) \right) + 2c^\dagger \int_0^t \langle u_\epsilon, \mathcal{P} K(u_\epsilon(s), z) \rangle \tilde{\pi}(ds, dz) + 2c^\dagger \int_0^t \langle u_\epsilon, \mathcal{P} G(u_\epsilon(s), z) \rangle \mu(dz) ds,
\]
for any $t \in [0, T \wedge \tau_e]$. Let us estimate the terms appearing on the R.H.S. of (4.16). By applying the assumption on $\Phi$, the Sobolev embedding $W^{1,2}(D) \subset L^6(D)$ as well as the Gagliardo-Nirenberg inequality

$$\|n_e\|_{L^6} \leq C \|\nabla \sqrt{n_e}\|_{L^2}^{\frac{1}{2}} \|\sqrt{n_e}\|_{L^2}^{\frac{1}{2}} + C \|\sqrt{n_e}\|_{L^2}^2 \leq C \left( \|\nabla \sqrt{n_e}\|_{L^2}^{\frac{1}{2}} + 1 \right),$$

which is sufficient to derive

$$2c^t \int_0^t \left\langle u_e, \mathcal{P}(n_e \nabla \Phi) \right\rangle_{L^2} ds \leq 2c^t \int_0^t \|u_e\|_{L^6} \|n_e\|_{L^6} \|\nabla \Phi\|_{L^\infty} ds$$

$$\leq C \int_0^t \|\nabla u_e\|_{L^2} \left( \|\nabla \sqrt{n_e}\|_{L^2}^{\frac{3}{2}} + 1 \right) ds$$

$$\leq \left( \frac{c^t}{2} - \frac{1}{2} \right) \int_0^t \|\nabla u_e\|_{L^2}^2 ds + 2 \int_0^t \|\nabla \sqrt{n_e}\|_{L^2}^2 ds + Ct,$$

for all $t \in [0, T \wedge \tau_e]$. By using the Sobolev embedding $W^{1,2}(D) \subset L^6(D)$, the assumption on $h$ and the Young inequality, we get

$$\left| 2c^t \int_0^t \left\langle u_e, \mathcal{P}h(s, u_e) \right\rangle ds \right| \leq 2c^t \int_0^t \|u_e\|_{L^6} \|\mathcal{P}h(s, u_e)\|_{L^6} ds$$

$$\leq \left( \frac{c^t}{2} - \frac{1}{2} \right) \int_0^t \|\nabla u_e\|_{L^2}^2 ds + C \int_0^t (\|u_e\|_{L^2}^2 + 1) ds.$$  

Applying the assumptions on $g$ and $G$, for all $t \in [0, T \wedge \tau_e]$, we get by Hölder inequality that

$$c^t \int_0^t \int_{|z| \geq 1} 2 \left\langle u_e, \mathcal{P}G(u_e(s), z) \right\rangle_{L^2} \mu(dz) ds + c^t \int_0^t \|\mathcal{P}g(s, u_e)\|_{L^2}^2 ds$$

$$\leq C \int_0^t \left( \|u_e(s)\|_{L^2} \int_{|z| \geq 1} \|\mathcal{P}G(u_e(s), z)\|_{L^2} \mu(dz) \right) ds + C \int_0^t (\|u_e\|_{L^2}^2 + 1) ds$$

$$\leq C \int_0^t \|u_e(s)\|_{L^2}^2 ds + Ct.$$

Putting (4.15)-(4.19) together, we obtain

$$\int_D \left( n_e \ln n_e + \frac{1}{2} |\nabla \Psi(c_e)|^2 + c^t |u_e|^2 \right) dx + \frac{1}{2} \int_0^t \int_D |\nabla n_e|^2 n_e dx ds$$

$$+ \frac{1}{2} \int_0^t \int_D \theta(c_e) |D^2 \rho(c_e)|^2 dx ds + \int_0^t \|\nabla u_e(s)\|_{L^2}^2 ds$$

$$\leq C t + \int_D \left( n_{e0} \ln n_{e0} + \frac{1}{2} |\nabla \Psi(c_{e0})|^2 + c^t |u_{e0}|^2 \right) dx + C \int_0^t \|\Psi(c_e)\|_{L^2}^2 ds$$

$$+ C \int_0^t \|u_e(s)\|_{L^2}^2 ds + 2c^t \int_0^t \left\langle u_e, \mathcal{P}g(s, u_e) dW \right\rangle + c^t \int_0^t \|\mathcal{P}K(u_e(s), z)\|_{L^2}^2 \pi(ds, dz)$$

$$+ c^t \int_0^t \int_{|z| \geq 1} \|\mathcal{P}G(u_e(s), z)\|_{L^2}^2 \pi(ds, dz) + 2c^t \int_0^t \int_{|z| < 1} \left\langle u_e, \mathcal{P}K(u_e(s), z) \right\rangle \tilde{\pi}(ds, dz)$$

$$+ 2c^t \int_0^t \int_{|z| \geq 1} \left\langle u_e, \mathcal{P}G(u_e(s), z) \right\rangle \tilde{\pi}(ds, dz).$$
To obtain the desired inequality, one need to estimate the integral \(\int_{D} \theta(c_\epsilon)|D^2\rho(c_\epsilon)|^2dx\) from below. Indeed, we get by integrating by parts and using the Hölder inequality that
\[
\int_{D} \theta(c_\epsilon)|D^2\rho(c_\epsilon)|^2dx \geq 1/2 \int_{D} \frac{\theta(c_\epsilon)}{c_\epsilon^2}|\partial_i\partial_j c_\epsilon|^2dx - \frac{1}{2} \int_{D} \frac{(\theta(c_\epsilon))^2}{\theta(c_\epsilon)}|\partial_i c_\epsilon \partial_j c_\epsilon|^2dx
\]
for some constants \(C', C'' > 0\). Using the estimate (4.9) and Remark 4.2, we gain
\[
\int_{D} \theta(c_\epsilon)|D^2\rho(c_\epsilon)|^2dx \geq C''' \int_{D} \frac{\nabla c_\epsilon}{c_\epsilon^2}dx,
\]
for some constant \(C''' > 0\). From the last two inequalities, we have
\[
\int_{D} \theta(c_\epsilon)|D^2\rho(c_\epsilon)|^2dx \geq C \left(\int_{D} \frac{\nabla c_\epsilon}{c_\epsilon^2}dx + \int_{D} \frac{\Delta c_\epsilon}{c_\epsilon}dx\right),
\]
which combined with (4.20) lead to the desired inequality. This completes the proof of Lemma 4.5.

\[\blacksquare\]

**Lemma 4.6.** Assume that the assumptions (A_1) – (A_4) hold. Then for any \(T > 0\), there exists a positive constant \(C > 0\) independent of \(\epsilon\) such that
\[
\mathbb{E} \left[\sup_{t \in [0, T \wedge \tau_\epsilon]} \int_{D} \left(n_\epsilon \ln n_\epsilon + \frac{1}{2} |\nabla \Psi(c_\epsilon)|^2 + c^\dagger |u_\epsilon|^2\right)dx\right]^p
\]
\[
+ \mathbb{E} \left[\frac{1}{2} \int_0^{T \wedge \tau_\epsilon} \int_{D} \frac{|\nabla n_\epsilon|^2}{n_\epsilon}dxdt + \int_0^{T \wedge \tau_\epsilon} \int_{D} \frac{\nabla c_\epsilon}{c_\epsilon^2}dxdt + \int_0^{T \wedge \tau_\epsilon} \int_{D} \frac{\Delta c_\epsilon}{c_\epsilon}dxdt
\]
\[
+ \frac{c^\dagger}{2} \int_0^{T \wedge \tau_\epsilon} \int_{D} |\nabla u_\epsilon|^2dxdt\right]^p
\]
\[
\leq C \left(\mathbb{E} \left[\int_{D} \left(n_\epsilon \ln n_\epsilon + \frac{1}{2} |\nabla \Psi(c_\epsilon)|^2 + c^\dagger |u_\epsilon|^2\right)dx\right]^p + 1\right),
\]
for all \(1 \leq p < \infty\), where the function \(\Psi\) is defined in Lemma 4.4 and \(C > 0\) depends only on \(p, c_0, f\) and \(\chi\).

**Proof.** The proof consists in explicitly estimating each integrals appearing on the R.H.S. of (4.6). First, by using the property for \(\theta(\cdot)\) and (4.5), we infer
\[
|\Psi(c_\epsilon)| = \left|\int_1^{c_\epsilon} \frac{d\sigma}{\sqrt{\theta(\sigma)}}\right| \leq 2/\sqrt{C} \left|\int_1^{c_\epsilon} \frac{d\sigma}{2\sqrt{\sigma}}\right| \leq \frac{2}{\sqrt{C}} \left(1 + \sqrt{\|c_0\|_{L^\infty}}\right),
\]
which combined with the Remark 4.2 imply that \(\mathbb{E} \sup_{t \in [0, T \wedge \tau_\epsilon]} C \int_0^t \|\Psi(c_\epsilon)\|^2_{L^2}ds \leq CT\). It then follows that
(4.22)
\[
\mathbb{E} \sup_{t \in [0, T \wedge \tau_\epsilon]} \left|Ct + C \int_0^t \|\Psi(c_\epsilon)\|^2_{L^2}ds + C \int_0^t \|u_\epsilon\|^2_{L^2}ds\right|^p \leq C \mathbb{E} \left(\int_0^T \|u_\epsilon\|^2_{L^2}ds\right)^p + CT^p.
\]
By applying the BDG inequality and the condition on $g$, we get for any $\eta > 0$

$$
\mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \left| 2c^\dagger \int_0^t \langle u_e, \mathcal{P}g(s, u_e) dW \rangle_{L^2} \right|^p 
$$

(4.23)

$$
\leq C \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \| u_e \|^2_{L^2} \left( \int_0^{T \wedge \tilde{\tau}_e} \| u_e \|^2_{L^2} dt + T \right) \right]^\frac{p}{2} 
$$

$$
\leq \eta \mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \| u_e \|^{2p}_{L^2} + C \mathbb{E} \left( \int_0^{T \wedge \tilde{\tau}_e} \| u_e \|^2_{L^2} dt \right)^p + CT^p. 
$$

Noting that the compensated Poisson random measure $\tilde{\pi}(ds, dz) = \pi(ds, dz) - \mu(dz)ds$, it follows from the BDG inequality and Young inequality that

$$
\mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \left| c^\dagger \int_0^t \int_{|z| < 1} \| K(u_e(s-), z) \|^2_{L^2} \pi(ds, dz) \right|^p 
$$

(4.24)

$$
\leq C \left[ \mathbb{E} \left( \int_0^{T \wedge \tilde{\tau}_e} \int_{|z| < 1} \| K(u_e(s-), z) \|^2_{L^2} \mu(dz)ds \right)^{\frac{p}{2}} + \mathbb{E} \left( \int_0^T (\| u_e \|^2_{L^2} + 1) ds \right)^p \right] 
$$

$$
\leq \eta \mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \left( \| u_e \|^{2p}_{L^2} + 1 \right)^p + C \mathbb{E} \left( \int_0^{T \wedge \tilde{\tau}_e} \| u_e \|^2_{L^2} dt \right)^p + CT^p. 
$$

Similarly, the integral including the functional $G$ can be estimated by

$$
\mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \left| c^\dagger \int_0^t \int_{|z| > 1} \| G(u_e(s-), z) \|^2_{L^2} \pi(ds, dz) \right|^p 
$$

(4.25)

$$
\leq \eta \mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \left( \| u_e \|^{2p}_{L^2} + 1 \right)^p + C \mathbb{E} \left( \int_0^{T \wedge \tilde{\tau}_e} \| u_e \|^2_{L^2} dt \right)^p + CT^p. 
$$

By virtue of the BDG inequality and the assumption on $K$, we deduce that

$$
\mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \left| 2c^\dagger \int_0^t \int_{|z| < 1} \langle u_e, \mathcal{P}K(u_e(s-), z) \rangle_{L^2} \pi(ds, dz) \right|^p 
$$

(4.26)

$$
\leq C \mathbb{E} \left( \int_0^{T \wedge \tilde{\tau}_e} \int_{|z| < 1} \| \langle u_e, \mathcal{P}K(u_e(s-), z) \rangle_{L^2} \|^2 \mu(dz)ds \right)^{\frac{p}{2}} 
$$

$$
\leq \eta \mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \| u_e \|^{2p}_{L^2} + C \mathbb{E} \left( \int_0^{T \wedge \tilde{\tau}_e} \left( \| u_e \|^2_{L^2} + 1 \right) ds \right)^p. 
$$

For the last term, we get by making use of the BDG inequality that

$$
\mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \left| c^\dagger \int_0^t \int_{|z| \geq 1} 2\langle u_e, \mathcal{P}G(u_e(s-), z) \rangle_{L^2} \pi(ds, dz) \right|^p 
$$

(4.27)

$$
\leq \eta \mathbb{E} \sup_{t \in [0, T \wedge \tilde{\tau}_e]} \| u_e \|^{2p}_{L^2} + C \mathbb{E} \left( \int_0^{T \wedge \tilde{\tau}_e} \| u_e \|^2_{L^2} dt \right)^p + CT^p. 
$$
Collecting the above estimates together, taking the supremum over the interval \([0, T]\) and then the mathematical expectation, we get
\[
\mathbb{E} \left( \sup_{t \in [0, T \wedge \bar{\tau}_e]} \mathcal{E}(n_e, c_e, u_e)(t) + \int_0^{T \wedge \bar{\tau}_e} \mathcal{D}(n_e, c_e, u_e)(s) \, ds \right)^p \leq 5\eta \mathbb{E} \sup_{t \in [0, T \wedge \bar{\tau}_e]} \|u_e(t)\|_{L^2}^{2p} + CE \left( \int_0^{T \wedge \bar{\tau}_e} \|u_e\|_{L^2}^2 \, dt \right)^p + C\mathbb{E} [\mathcal{E}(n_{o0}, c_{o0}, u_{o0})^p] + CT^p.
\] (4.28)

By choosing \(0 < \eta \leq \frac{c_1}{10}\), we deduce from (4.28), the basic inequality \(a^p + b^p \leq (a + b)^p \leq 2^{p-1}(a^p + b^p)\) as well as the Hölder inequality that
\[
\mathbb{E} \sup_{t \in [0, T \wedge \bar{\tau}_e]} \|u_e(t)\|_{L^2}^{2p} \leq C T^{p-1} \mathbb{E} \int_0^{T \wedge \bar{\tau}_e} \|u_e(s)\|_{L^2}^{2p} \, ds + C\mathbb{E} [\mathcal{E}(n_{o0}, c_{o0}, u_{o0})^p] + CT^p,
\]
which combined with the Gronwall inequality lead to
\[
\mathbb{E} \sup_{t \in [0, T \wedge \bar{\tau}_e]} \|u_e(t)\|_{L^2}^{2p} \leq e^{CT^{p-1}} C \left( \mathbb{E} [\mathcal{E}(n_{o0}, c_{o0}, u_{o0})^p] + T^p \right).
\]

Plugging the last estimate into (4.28) leads to
\[
\mathbb{E} \left( \sup_{t \in [0, T \wedge \bar{\tau}_e]} \mathcal{E}(n_e, c_e, u_e) + \int_0^{T \wedge \bar{\tau}_e} \mathcal{D}(n_e, c_e, u_e) \, ds \right)^p \leq C e^{CT^{p-1}} \left( \mathbb{E} [\mathcal{E}(n_{o0}, c_{o0}, u_{o0})^p] + T^p \right),
\]
which implies the desired energy estimate. The proof of Lemma 4.6 is completed. \(\square\)

4.3. Global existence of approximate solutions. Based on the uniform bounds in Lemma 4.6, it is sufficient to prove that the approximate solution is indeed global in time:

**Lemma 4.7.** For each \(\epsilon \in (0, 1)\), the approximate solution \((n_\epsilon, c_\epsilon, u_\epsilon, \bar{\tau}_\epsilon)\) to system (3.1) is global in time, that is, \(\mathbb{P}[\bar{\tau}_\epsilon = \infty] = 1\).

**Proof.** For any \(l > 0\), we define
\[
\tau_l \overset{\text{def}}{=} \inf \left\{ 0 \leq t < \infty; \max \left\{ \int_0^t \|\nabla c_\epsilon \|_{L^4}^4 \, ds, \|u_\epsilon(t)\|_{L^2}, \int_0^t \|\nabla u_\epsilon(s)\|_{L^2}^2 \, ds \right\} > l \right\},
\]
and \(+\infty\) if the above set is empty.

It is clear that \((\tau_l)_{l>0}\) is a sequence of nonnegative stopping times \(\mathbb{P}\text{-a.s.}\), and \(\tau_l \to \infty\) as \(l \to \infty\). Thanks to the uniform bound in Lemma 4.6, we deduce by (4.9) in Lemma 4.5 and the definition of \(\tau_l\) that
\[
\mathbb{E} \int_0^{T \wedge \tau_l} \|\nabla c_\epsilon(s)\|_{L^4}^4 \, ds \leq \|c_{0\infty}\|_{L^\infty}^4 \mathbb{E} \int_0^{T \wedge \tau_l} \left\| \nabla c_\epsilon(s) \right\|_{L^4}^4 \, ds \leq C.
\] (4.29)

Applying the chain rule to \(d\|n_\epsilon\|_{L^p}^p\) with \(p \in [2, 4]\), integrating by parts and using the divergence-free condition, we infer that for all \(t \in [0, T \wedge \tau_l]\)
\[
\|n_\epsilon(t)\|_{L^p}^p + p(p - 1) \int_0^t \int_D n_\epsilon^{p-2} |\nabla n_\epsilon|^2 \, dx \, ds
\]
\[
= \|n_{o0}\|_{L^p}^p + p(p - 1) \int_0^t \int_D n_\epsilon^{p-2} n_\epsilon h_\epsilon'(n_\epsilon) \chi(c_\epsilon) \nabla n_\epsilon \cdot \nabla c_\epsilon \, dx \, ds.
\] (4.30)
Noticing that $2(p - 2) \leq p$ for all $p \in [2, 4]$, $0 < n_c h'(n_c) \leq \frac{1}{\varepsilon}$ and $\|\chi(c_\varepsilon)\|_{L^\infty} \leq C$ by Remark [4.2], it follows that for $t \in [0, T \land \tau]$

$$E \left| p(p - 1) \int_D n_c^{p-2} n_c h'(n_c) \chi(c_\varepsilon) \nabla n_c \cdot \nabla c_\varepsilon \, dx \right|$$

$$\leq \frac{p(p - 1)}{2} E \int_D n_c^{p-2} |\nabla n_c|^2 \, dx + C \left( \int_D n_c^{2(p-2)} \, dx + E \int_D |\nabla c_\varepsilon|^4 \, dx \right)$$

$$\leq \frac{p(p - 1)}{2} E \int_D n_c^{p-2} |\nabla n_c|^2 \, dx + C E \int_D n_c \, dx + C,$$

which together with (4.29)-(4.30) and the Gronwall inequality yield that

$$E \sup_{t \in [0, T \land \tau]} \|n_c(t)\|_{L^p}^p + E \int_0^{T \land \tau} \int_D n_c^{p-2} |\nabla n_c|^2 \, dx \, ds \leq C (E \|n_0\|_{L^p}^p + 1).$$

By making use of the uniform bound (4.21), we now prove a boundedness for $u_\varepsilon$. First, basic properties for $L_\varepsilon$ imply that, for all $t \in [0, T \land \tau],

$$\|P(L_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon\|_{L^2}^2 \leq \|L_\varepsilon u_\varepsilon\|_{L^\infty}^2 \|\nabla u_\varepsilon\|_{L^2}^2 \leq C \|\nabla u_\varepsilon\|_{L^2}^2.$$ 

Applying the Itô's formula to $d\|A^{\frac{1}{2}} u_\varepsilon\|_{L^2}^2$, after integrating by parts and using (4.32) and the Young inequality, we infer that for all $t \in [0, T \land \tau]$

$$\left\|A^{\frac{1}{2}} u_\varepsilon(t)\right\|_{L^2}^2 + \int_0^t \|A u_\varepsilon\|_{L^2}^2 \, ds$$

$$\leq \left\|A^{\frac{1}{2}} u_\varepsilon\right\|_{L^2}^2 + C_l \int_0^t \left( \|A^{\frac{1}{2}} u_\varepsilon\|_{L^2}^2 + 1 \right) \, ds + 2 \left| \int_0^t \langle A^{\frac{1}{2}} u_\varepsilon, A^{\frac{1}{2}} g(s, u_\varepsilon) \, dW \rangle_{L^2} \right|$$

$$+ \left| \int_0^t \int_{|z| \leq 1} \left( \|\pi_1(u_\varepsilon(s-), z)\|_{L^2}^2 - 2 \langle A^{\frac{1}{2}} u_\varepsilon, A^{\frac{1}{2}} \pi_1(u_\varepsilon(s-), z) \rangle_{L^2} \right) \pi(ds, dz) \right|$$

$$+ \left| \int_0^t \int_{|z| > 1} \left( \|G(u_\varepsilon(s-), z)\|_{L^2}^2 - 2 \langle A^{\frac{1}{2}} u_\varepsilon, A^{\frac{1}{2}} G(u_\varepsilon(s-), z) \rangle_{L^2} \right) \pi(ds, dz) \right|,$$

where the estimate (4.21) with $p = 2$ is applied. By taking the supremum in time over $[0, T \land \tau]$ and then the mathematical expectation, we get by applying the BDG inequality and the assumptions on $g, K, G$ and $h$ that

$$\frac{1}{2} E \sup_{t \in [0, T \land \tau]} \left\|A^{\frac{1}{2}} u_\varepsilon(t)\right\|_{L^2}^2 + E \int_0^t \|A u_\varepsilon(s)\|_{L^2}^2 \, ds$$

$$\leq E \left\|A^{\frac{1}{2}} u_\varepsilon\right\|_{L^2}^2 + C_l E \int_0^{T \land \tau} \left( \|A^{\frac{1}{2}} u_\varepsilon(s)\|_{L^2}^2 + 1 \right) \, ds,$$

Thereby, by utilizing the equivalence between the norms $\|A^a u_\varepsilon\|_{L^2}$ and $\|(-\Delta)^a u_\varepsilon\|_{L^2}$ and then the Gronwall inequality, we gain

$$E \sup_{t \in [0, T \land \tau]} \|\nabla u_\varepsilon(t)\|_{L^2}^2 + E \int_0^{T \land \tau} \|\Delta u_\varepsilon(s)\|_{L^2}^2 \, ds \leq C.$$
For \( u \in \mathcal{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};H) \) and \( \zeta \in (0,T\wedge \tau) \), we have for any \( t \in [\zeta,T\wedge \tau] \)

\[
U_1(t) \leq C t^{-\alpha} \| u_0 \|_{\mathcal{L}^2} \leq C \| u_0 \|_{\mathcal{L}^2}, \quad \text{for all } t \in [\zeta,T\wedge \tau].
\]

For \( U_2(t) \), by applying the smoothing estimate of the Stokes semigroup (cf. [25]), we deduce from (4.34) that

\[
\mathbb{E} \sup_{t \in [0,T\wedge \tau]} U_2^2(t) \leq C \mathbb{E} \left( \sup_{t \in [0,T\wedge \tau]} \| \nabla u_\epsilon \|_{\mathcal{L}^2} \int_0^{T\wedge \tau} (T\wedge \tau - s)^{-\alpha} ds \right)^2 \leq C \frac{T^{1-\alpha}}{1-\alpha} \mathbb{E} \sup_{t \in [0,T\wedge \tau]} \| \nabla u_\epsilon \|_{\mathcal{L}^2}^2 \leq C.
\]

For \( U_3(t) \) and \( U_4(t) \), it follows from (4.21) with \( p = 2 \) and the condition on \( h \) that

\[
\mathbb{E} \sup_{t \in [0,T\wedge \tau]} (U_3 + U_4)^2(t) \leq C \mathbb{E} \left( \int_0^{T\wedge \tau} (T\wedge \tau - s)^{-\alpha} (\| u_\epsilon \|_{\mathcal{L}^2} + \| u_\epsilon \|_{\mathcal{L}^2} + 1) ds \right)^2 \leq C.
\]

The term \( U_5(t) \) can be treated similar to (3.8), and one can obtain for any \( t \in (0,T\wedge \tau] \)

\[
\mathbb{E} \sup_{t \in [0,t]} (U_5(t))^2 \leq C T + C \mathbb{E} \int_0^t \| A^\alpha u_\epsilon(s) \|_{\mathcal{L}^2}^2 ds.
\]

To estimate \( U_6(t) \), we observe that \( dU_6(t) = -AU_6(t)dt + \int_{|z| \leq 1} A^\alpha \mathcal{P}K(u_\epsilon(s-),z)\pi(ds,dz) \) with \( U_6(0) = 0 \). By making use of the Itô’s formula to \( \| U_6(t) \|_{\mathcal{L}^2}^2 \) and then integrating by parts leads to

\[
\| U_6(t) \|_{\mathcal{L}^2}^2 + 2 \int_0^t \| \nabla U_6 \|_{\mathcal{L}^2}^2 ds \leq 2 \left| \int_0^t \int_{|z| \leq 1} \langle U_6, A^\alpha \mathcal{P}K(u_\epsilon(s-),z) \rangle_{\mathcal{L}^2} \pi(ds,dz) \right| + \int_0^t \int_{|z| \leq 1} \| A^\alpha \mathcal{P}K(u_\epsilon(s-),z) \|_{\mathcal{L}^2}^2 \pi(ds,dz),
\]
for all \( t \in [0, T \land \tau_1] \). By taking the supremum in time over \( [0, T^*] \) and then mathematical expectation, it follows from the the assumption on \( K \) that, for any \( \eta > 0 \),

\[
\mathbb{E} \sup_{t \in [0, T \land \tau_1]} \|U_6(t)\|_{L^2}^2 + 2 \int_0^{T \land \tau_1} \|\nabla U_6(s)\|^2_{L^2} ds \\
\leq C \mathbb{E} \left[ \int_0^{T \land \tau_1} \int_{|z| \leq 1} \|A^\alpha \mathcal{P} K(u_\epsilon(s-), z)\|^2_{L^2} \mu(dz) ds \right]^\frac{1}{2} \\
+ \mathbb{E} \int_0^{T \land \tau_1} \int_{|z| \leq 1} \|A^\alpha \mathcal{P} K(u_\epsilon(s-), z)\|^2_{L^2} \mu(dz) ds \\
\leq \eta \mathbb{E} \sup_{t \in [0, T \land \tau_1]} \|U_6(t)\|_{L^2}^2 + C \mathbb{E} \int_0^{T \land \tau_1} (\|A^\alpha u_\epsilon(s)\|_{L^2}^2 + 1) ds.
\]

By choosing \( \eta > 0 \) small enough in the last estimate it holds

\[
\mathbb{E} \sup_{t \in [0, T \land \tau_1]} \|U_6(t)\|_{L^2}^2 \leq C \mathbb{E} \int_0^{T \land \tau_1} (\|A^\alpha u_\epsilon(s)\|_{L^2}^2 + 1) ds.
\]

Using the assumption on \( G \) and the BDG inequality, the stochastic integral \( U_7(t) \) can be estimated similar to \( U_6(t) \), and we infer that

\[
\mathbb{E} \sup_{t \in [0, T \land \tau_1]} \|U_7(t)\|_{L^2}^2 \leq C \mathbb{E} \int_0^{T \land \tau_1} (\|A^\alpha u_\epsilon(s)\|_{L^2}^2 + 1) ds.
\]

To obtain the desired estimate, we take the 2-th power on both sides of (4.35), the supremum in time over \( [0, T \land \tau_1] \), and apply expectations. Summarizing the previous estimates from \( U_1(t) \) to \( U_7(t) \), we obtain from the Gronwall inequality that

\[
(4.36) \quad \mathbb{E} \sup_{t \in [0, T \land \tau_1]} \|A^\alpha u_\epsilon(t)\|_{L^2}^2 \leq C.
\]

In view of the embedding \( D(A^\alpha) \subset L^\infty(D) \) with \( \frac{3}{4} < \alpha < 1 \), (4.36) implies that

\[
(4.37) \quad \|u_\epsilon(t)\|_{L^\infty} \leq C \|u_\epsilon(t)\|_{D(A^\alpha)} < \infty, \quad \mathbb{P}\text{-a.s.},
\]

for all \( t \in [\zeta, T \land \tau_1] \), for any \( \zeta \in (0, \tau_\epsilon) \).

Now we need to explore the evolution of the quantity \( \|\nabla c_\epsilon\|_{L^q} \) with \( q > 3 \). To this end, we employ \( \nabla \) to both sides of the variation-of-constants formula to obtain

\[
\nabla c_\epsilon(t) = \nabla e^{(t-\frac{s}{2})\Delta} c_\epsilon \left( \frac{\zeta}{2} \right) - \int_0^t \nabla e^{(t-s)\Delta} (u_\epsilon \cdot \nabla c_\epsilon + h_\epsilon(n_\epsilon) f(c_\epsilon)) ds, \quad t \in \left( \frac{\zeta}{2}, T \land \tau_1 \right].
\]
Taking advantage of the smooth effect of Neumann heat semigroup \([59]\), the bound \((4.33)\) and \((4.31)\), we get from the Duhamel’s formula of \(c_\epsilon\)-equation that
\[
\mathbb{E} \sup_{t \in [\zeta, T \wedge \tau]} \| \nabla c_\epsilon(t) \|_{L^4} \\
\leq C \zeta^{-\frac{\alpha}{2}} + C \mathbb{E} \left( \sup_{t \in [\frac{\zeta}{2}, T \wedge \tau]} \| n_\epsilon(t) \|_{L^4} \sup_{t \in [\zeta, T \wedge \tau]} \int_{\frac{\zeta}{2}}^{t} (t - s)^{-\frac{1}{2}} ds \right) \\
+ C \mathbb{E} \left( \sup_{t \in [\frac{\zeta}{2}, T \wedge \tau]} \| u_\epsilon \|_{L^\infty} \sup_{t \in [\zeta, T \wedge \tau]} \int_{\frac{\zeta}{2}}^{t} (t - s)^{-\frac{1}{2}} \| \nabla c_\epsilon(s) \|_{L^4} ds \right) \\
\leq C \sup_{t \in [\zeta, T]} \left( \int_{\frac{\zeta}{2}}^{t} (t - s)^{-\frac{3}{4}} ds \right)^{\frac{3}{2}} \mathbb{E} \int_{\frac{\zeta}{2}}^{t \wedge \tau} \| \nabla c_\epsilon(s) \|_{L^4}^4 ds + C \zeta^{-\frac{\alpha}{2}} + CT + C,
\]
which combined with \((4.29)\) imply that
\[
(4.38) \quad \mathbb{E} \sup_{t \in [\zeta, T \wedge \tau]} \| \nabla c_\epsilon(t) \|_{L^4} \leq C.
\]
Now we assert that it is possible to obtain an enhanced estimate than \((4.29)\). Setting the sectorial operator \(B = 1 - \Delta\) with homogeneous Neumann boundary condition, and rewriting the diffusion term \(\Delta c_\epsilon\) as \(c_\epsilon - Bc_\epsilon\). It then follows from the \(L^p\)-\(L^q\) estimates (cf. \([56]\) and the embedding \(W^{2,4}(D) \subset W^{1,\infty}(D)\)) with \(\beta \in (\frac{7}{8}, 1)\) that
\[
\| \nabla c_\epsilon(t) \|_{L^\infty} \leq \| B^\beta e^{-tB} c_0 \|_{L^4} + \int_{0}^{t} \| B^\beta e^{-(t-s)B} (u_\epsilon \cdot \nabla c_\epsilon - c_\epsilon + h_\epsilon(n_\epsilon)f(c_\epsilon)) \|_{L^4} ds \\
\leq Ct^{-\beta} \| c_0 \|_{L^4} + \int_{0}^{t} (t - s)^{-\beta} (\| \nabla c_\epsilon \|_{L^4} + \| c_\epsilon \|_{L^4} + \| n_\epsilon \|_{L^1}) ds,
\]
for all \(t \in [\zeta, T \wedge \tau]\), where we used the fact of \(h_\epsilon(n_\epsilon) \leq n_\epsilon\) and the uniform bound \((4.31)\). Then by taking the supremum and expectation, we deduce from \((4.31)\) and \((4.38)\) that
\[
(4.39) \quad \mathbb{E} \sup_{t \in [\zeta, T \wedge \tau]} \| \nabla c_\epsilon(t) \|_{L^\infty} \leq C.
\]
In view of the boundedness of \(h_\epsilon'(n_\epsilon)\) and \(\chi(c_\epsilon)\), and making use of the Duhamel’s formula for the \(n_\epsilon\)-equation, we deduce from \((4.31)\), \((4.36)\), \((4.39)\) that
\[
(4.40) \quad \mathbb{E} \sup_{t \in [\zeta, T \wedge \tau]} \| n_\epsilon(t) \|_{L^\infty} \leq \mathbb{E} \left( \sup_{t \in [\zeta, T \wedge \tau]} \left( t - \frac{\zeta}{2} \right)^{-\frac{2}{\alpha}} \| n_0 \|_{L^1} \right) \\
+ \mathbb{E} \sup_{t \in [\zeta, T \wedge \tau]} \int_{\frac{\zeta}{2}}^{t} \left( t - \frac{\zeta}{2} \right)^{-\frac{\alpha + 3}{2p}} \| u_\epsilon n_\epsilon + n_\epsilon h_\epsilon'(n_\epsilon) \chi(c_\epsilon) \nabla c_\epsilon \|_{L^p} ds \\
\leq C \| n_0 \|_{L^1} + C \mathbb{E} \int_{\frac{\zeta}{2}}^{T} \left( t - \frac{\zeta}{2} \right)^{-\frac{\alpha + 3}{2p}} (\| u_\epsilon \|_{L^\infty} + \| \nabla c_\epsilon \|_{L^\infty}) \| n_\epsilon \|_{L^p} ds \leq C.
\]
In view of \((4.36)\), \((4.39)\) and \((4.40)\), we have \(\sup_{t \in [\zeta, T \wedge \tau]} \| (n_\epsilon(t), c_\epsilon(t), u_\epsilon(t)) \|_{L^\infty \times W^{1,q} \times D(A^\alpha)} < \infty\ \mathbb{P}\text{-a.s.}\), which implies that the solution \((n_\epsilon, c_\epsilon, u_\epsilon)\) does not blow up at time \(t = T \wedge \tau\), for any \(T > 0\). Moreover, from the definition of maximal existence time \(\tau_\epsilon\), we have \(\tau_\epsilon \geq T \wedge \tau\),
for any \( T > 0 \) and \( l > 0 \). Therefore, by taking the limit \( l \to \infty \) shows that \( \mathbb{P}(\tilde{T}_\epsilon = \infty) = 1 \).

The proof of Lemma 5.7 is completed. \( \square \)

5. Identification of the limits

5.1. Further spatio-temporal regularity.

Lemma 5.1. Assume that \( T > 0 \) and \( p \geq 1 \). For any \( \epsilon \in (0, 1) \), let \((n_\epsilon, c_\epsilon, u_\epsilon)\) be solution to system \((3.1)\), then there exists a positive constant \( C \) independent of \( \epsilon \) such that

\[
\begin{align*}
(5.1a) & \quad \|\nabla \sqrt{n_\epsilon}\|_{L^p(\Omega; L^2(0,T;L^2))} \leq C, \\
(5.1b) & \quad \|n_\epsilon\|_{L^p(\Omega; L^2(0,T;W^{1,2}))} \leq C, \\
(5.1c) & \quad \|\nabla \sqrt{c_\epsilon}\|_{L^p(\Omega; L^4(0,T;L^4))} \leq C, \\
(5.1d) & \quad \|c_\epsilon\|_{L^p(\Omega; L^2(0,T;W^{1,2}))} \leq C, \\
(5.1e) & \quad \|u_\epsilon\|_{L^p(\Omega; L^\infty(0,T;L^2)} \cap L^p(\Omega; L^2(0,T;W^{1,2})) \leq C, \\
(5.1f) & \quad \|u_\epsilon\|_{L^p(\Omega; L^p(0,T;L^p))} \leq C.
\end{align*}
\]

Proof. The uniform bounds \((5.1a)\), \((5.1c)\), \((5.1d)\) and \((5.1e)\) are direct consequence of the uniform boundedness estimate in Lemma 4.6. The estimates \((5.1b)\) and \((5.1f)\) can be verified by utilizing the conservation of mass and properly making use of the Gagliardo-Nirenberg inequality, and we omit the details here. \( \square \)

Lemma 5.2. For any \( T > 0 \) and \( \epsilon \in (0, 1) \), there exists \( C > 0 \) independent of \( \epsilon \) such that

\[
\begin{align*}
(5.2) & \quad \mathbb{E} \left( \int_0^T \|\partial_t n_\epsilon\|_{(W^{2,q}(D))^*}^q \, dt \right)^{\frac{p}{q}} \leq C, \quad \text{for some } q > 3, \\
(5.3) & \quad \mathbb{E} \left( \int_0^T \|\partial_t \sqrt{c_\epsilon}\|_{(W^{1,\frac{3}{2}}(D))^*}^\frac{3}{2} \, dt \right)^{\frac{p}{\frac{3}{2}}} \leq C.
\end{align*}
\]

Proof. For any \( \varphi \in W^{2,q}(D) \), multiplying the \( n_\epsilon \)-equation by \( \varphi \) and then integrating with respect to \( x \) on \( D \), it follows from Young inequality that

\[
\left| \int_D \partial_t n_\epsilon(t) \varphi \, dx \right| \leq \int_D \left( u_\epsilon n_\epsilon - \nabla n_\epsilon + n_\epsilon h_\epsilon(n_\epsilon) \chi(c_\epsilon) \nabla c_\epsilon \right) \cdot \nabla \varphi \, dx
\]

\[
\leq C \left( \|u_\epsilon\|_{L^\infty}^{\frac{10}{3}} \|n_\epsilon\|^\frac{10}{3} \|\nabla n_\epsilon\|_{L^2}^2 + \|\nabla c_\epsilon\|_{L^2}^2 + \|n_\epsilon\|_{L^4}^4 + \|c_\epsilon\|_{L^4}^4 \right) \|\varphi\|_{W^{2,q}},
\]

where the last inequality used the Sobolev embedding \( W^{2,q}(D) \subset W^{1,\infty}(D) \) for \( q > 3 \). It then follows from the definition of the norm in \((W^{2,q}(D))^*\), the basic inequality \((a+b)^p \leq C(a^p+b^p)\) and Lemma 5.1 that

\[
\mathbb{E} \left( \int_0^T \|\partial_t n_\epsilon\|_{(W^{2,q})^*}^p \, ds \right)^{\frac{p}{q}} \leq C \mathbb{E} \left( \|u_\epsilon\|^p_{L^\frac{10}{3}(0,T;L^\frac{10}{3})} + \|n_\epsilon\|^p_{L^\frac{5}{3}(0,T;L^\frac{5}{3})} + \|\nabla n_\epsilon\|^p_{L^2(0,T;L^2)} + \|\nabla c_\epsilon\|^p_{L^4(0,T;L^4)} + 1 \right) \leq C,
\]

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which proves (5.2). To show (5.3), we first rewrite the $c_\epsilon$-equation in the following form
\[
d\sqrt{c_\epsilon} = -u_\epsilon \cdot \nabla \sqrt{c_\epsilon} dt + \frac{\Delta c_\epsilon}{2\sqrt{c_\epsilon}} dt - \frac{h_\epsilon(n_\epsilon)f(c_\epsilon)}{2\sqrt{c_\epsilon}} dt.
\]

Then multiplying both sides of above equation by any $\varphi \in W^{1,\frac{3}{2}}(D)$ and integrating by parts on $D$, we deduce that
\[
\left| \int_D \partial_t \sqrt{c_\epsilon} \varphi dx \right| \leq \left| \int_D \sqrt{c_\epsilon} u_\epsilon \cdot \nabla \varphi dx \right| + \left| \int_D \nabla \sqrt{c_\epsilon} \cdot \nabla \varphi dx \right| + \left| \int_D |\nabla \sqrt{c_\epsilon}|^2 \varphi dx \right|
\]
\[
+ \left| \int_D \frac{h_\epsilon(n_\epsilon)f(c_\epsilon)}{2\sqrt{c_\epsilon}} \varphi dx \right| \leq C \left( ||u_\epsilon||_{L^\infty}^{\frac{10}{3}} + ||\nabla \sqrt{c_\epsilon}||_{L^4}^4 + ||n_\epsilon||_{L^\infty}^{\frac{3}{2}} + 1 \right) ||\varphi||_{W^{1,\frac{3}{2}}},
\]
where we used the facts of
\[
||\nabla \sqrt{c_\epsilon}||_{L^4}^4 \leq C ||\nabla \sqrt{c_\epsilon}||_{L^4} ||\sqrt{c_\epsilon}||_{L^\infty}^4 \leq C ||\nabla \sqrt{c_\epsilon}||_{L^4},
\]
and by $f(0) = 0$,
\[
\left| \int_D \frac{h_\epsilon(n_\epsilon)f(c_\epsilon)}{2\sqrt{c_\epsilon}} \varphi dx \right| \leq C ||n_\epsilon||_{L^\infty} ||\beta c_\epsilon|| \sqrt{c_\epsilon} \leq C n_\epsilon, \quad \beta \in (0, 1).
\]

By (5.4) and Lemma 5.1 we get
\[
\mathbb{E} \left( \int_0^T ||\partial_t c_\epsilon||_{(W^{1,5})^*} ds \right)^p \leq \mathbb{E} \left[ \int_0^T \left( ||u_\epsilon||_{L^\infty}^{\frac{10}{3}} + ||\nabla \sqrt{c_\epsilon}||_{L^4}^4 + ||n_\epsilon||_{L^\infty}^{\frac{3}{2}} + 1 \right) ds \right]^p \leq C.
\]

The proof of Lemma 5.2 is finished.\qed

In order to obtain compactness result for the $u_\epsilon$-equation whose solution are not first-order differentiable, we use the fractional Sobolev spaces $W^{\alpha,p}(0, T; H)$: Let $p > 1$, $\alpha \in (0, 1)$ and $H$ be a separable Hilbert space, the space $W^{\alpha,p}(0, T; H)$ consists of all of the measurable functions $v \in L^p(0, T; H)$ endowed with the norm
\[
||v||_{W^{\alpha,p}(0, T; H)} \overset{\text{def}}{=} ||v||_{L^p(0, T; H)} + \int_0^T \int_0^T |v(t) - v(s)|^p |t - s|^{1+\alpha} dtds.
\]

**Lemma 5.3.** For any given $T > 0$ and $\alpha \in (0, \frac{1}{2})$, there exists a positive constant $C$ independent of $\epsilon$ such that
\[
\mathbb{E} \left( ||u_\epsilon||_{W^{\alpha,2}(0, T; D(A)^*)}^2 \right) \leq C.
\]

**Proof.** By making use of the Lemma 2.1 in [22] and (5.1e), one can deduce that for $p \geq 2$
\[
\mathbb{E} \left( \left| \int_0^T \mathcal{P}g(s, u_\epsilon(s))dW \right|_{W^{\alpha,p}(0, T; L^2)}^p \right) \leq C.
\]

Next we prove
\[
\mathbb{E} \left( \left| \int_0^T \int_{|z| < 1} \mathcal{P}K(u_\epsilon(x, s-), z)\tilde{\pi}(dz) ds \right|_{W^{\alpha,p}(0, T; L^2)}^p \right) \leq C.
\]
To this end, setting $\mathcal{J}(t) \overset{\text{def}}{=} \int_0^t \int_{|z| < 1} \mathcal{P}K(u_\epsilon(s-), z) \tilde{\pi}(ds, dz)$, it follows that

$$
\mathbb{E}\left(\|\mathcal{J}(t)\|_{W^{\alpha,p}(0,T; L_2^2)}^p\right) = \mathbb{E}\left(\|\mathcal{J}(t)\|_{L^p(0,T; L_2^2)}^p\right) + \mathbb{E} \int_0^T \int_0^T \frac{|\mathcal{J}(t) - \mathcal{J}(s)|^p}{|t - s|^{1+\alpha \rho}} \mathrm{d}s \mathrm{d}t \\
\overset{\text{def}}{=} I_1 + I_2.
$$

For $I_1$, by using the BDG inequality, the assumption on $K$ and the bound (5.1e), we have

$$
I_1 \leq C \mathbb{E} \left( \left( \int_0^T \int_{|z| < 1} \|\mathcal{P}K(u_\epsilon(s-), z)\|_{L_2^2}^2 \mu(dz)ds \right)^{\frac{p}{2}} \right) \leq C \mathbb{E} \sup_{t \in [0,T]} (\|u_\epsilon(s)\|_{L_2}^p + 1) \leq C.
$$

For $I_2$, by virtue of the BDG inequality, we deduce that

$$
I_2 = \mathbb{E} \int_0^T \int_0^T \frac{\int_{s \wedge t}^{s \vee t} \int_{|z| < 1} \mathcal{P}K(u_\epsilon(\sigma-), z) \tilde{\pi}(d\sigma, dz) |^p}{|t - s|^{1+\alpha \rho}} \mathrm{d}s \mathrm{d}t
$$

If $p = 2$, then it follows from the stochastic Fubini theorem that

$$
I_2 \leq 2 \mathbb{E} \int_0^T \int_0^T \mathbb{E} \left( \|u_\epsilon(\sigma)\|_{L_2}^2 + 1 \right) \mathrm{d}\sigma \mathrm{d}s \leq C \mathbb{E} \int_0^T (\|u_\epsilon(t)\|_{L_2}^2 + 1) \mathrm{d}t \leq C.
$$

If $p > 2$, by utilizing the embedding $W^{1,\frac{2}{p}}(0, T; \mathbb{R}) \subset W^{2\alpha,\frac{2}{p}}(0, T; \mathbb{R})$, we then get

$$
I_2 = \mathbb{E} \left( \left( \int_0^T \left( \|u_\epsilon\|_{L_2}^2 + 1 \right) ds \right)^{\frac{p}{2}} \right)_{W^{2\alpha,\frac{2}{p}}(0,T;\mathbb{R})}
$$

In either case, one obtain the uniform bound

$$(5.8)\quad \mathbb{E} \left( \|\mathcal{J}(t)\|_{W^{\alpha,p}(0,T; L_2^2)}^p\right) \leq C.
$$

Moreover, making use of the decomposition $\tilde{\pi}(ds, dz) = \pi(ds, dz) - \mu(dz)ds$ and assumptions on $G$, similar to estimates for $I(t)$, we get

$$(5.9)\quad \mathbb{E} \left[ \int_0^T \int_{|z| \geq 1} \mathcal{P}G(u_\epsilon(s-), z) \pi(ds, dz) \right]_{W^{\alpha,p}(0,T; L_2^2)}^p \leq C,
$$

To prove (5.5), in view of (5.6)-(5.9) and the $u_\epsilon$-equation, it is sufficient to estimate the term

$$
S_\epsilon(t) \overset{\text{def}}{=} u_0 - \int_0^t [\mathcal{P}(\mathbf{L}_\epsilon u_\epsilon \cdot \nabla) u_\epsilon - A u_\epsilon] ds + \int_0^t \mathcal{P}(n_\epsilon \nabla \Phi + h(s, u_\epsilon)) ds \quad \text{in } L^2(\Omega; W^{1,2}(0, T; D(A)^*)).
$$

Indeed, from the Sobolev embedding $L_2^2(D) \subset D(A)^*$ and $W^{1,2}(0, T) \subset W^{\alpha,2}(0, T)$, we have

$$
\mathbb{E} \left( \|S_\epsilon\|_{W^{1,2}(0,T; D(A)^*)}^2 \right) \leq C \left( \|u_0\| + \mathbb{E} \int_0^T \|\mathcal{P}(\mathbf{L}_\epsilon u_\epsilon \cdot \nabla) u_\epsilon\|_{D(A)}^2 dt + \mathbb{E} \int_0^T \|\mathcal{P}(n_\epsilon \nabla \Phi + h(t, u_\epsilon))\|_{L_2(D(A)^*)}^2 dt \right).
$$
For the second term on the R.H.S. of (5.10), we have
\[
\mathbb{E} \int_0^T \| u_e \|_{D(A)}^2 \, dt \leq C E \sup_{t \in [0,T]} \| u_e \|_{L^2}^2 + C \mathbb{E} \left[ \left( \int_0^T \| \nabla u_e \|_{L^2}^2 \, dt \right)^2 \right] \leq C.
\]
The third term on the R.H.S. of (5.10) can be estimated as
\[
\mathbb{E} \int_0^T \| A u_e \|_{D(A)}^2 \, dt \leq C \mathbb{E} \int_0^T \| \nabla u_e \|_{L^2}^2 \, dt \leq C.
\]
For the forth term, in view of the continuously embedding \( L^2(D) \subset L^1(D) \subset D(A)^* \), the conservation property (4.5) as well as the assumptions on \( h \) and \( \Phi \), we gain
\[
\mathbb{E} \int_0^T \| P(n_\epsilon \nabla \Phi + h(t, u_e)) \|_{D(A)}^2 \, dt \leq C \mathbb{E} \int_0^T \left( \| n_\epsilon \|_{L^1}^2 + \| u_e \|_{L^2}^2 + 1 \right) \, dt \leq C.
\]
Plugging the last three estimates into (5.10) leads to \( \mathbb{E} \left( \| S_\epsilon \|_{W^{1,2}(0,T;D(A)^*)}^2 \right) \leq C \), which implies the desired uniform bound (5.5).

5.2. Reconstruction of approximate solutions.

5.2.1. Tightness. Define the phase space
\[
\mathcal{X} \equiv \mathcal{X}_W \times \mathcal{X}_\pi \times \mathcal{X}_n \times \mathcal{X}_c \times \mathcal{X}_u,
\]
where
\[
\mathcal{X}_W \equiv C_{loc}([0,\infty);\mathbb{R}^d), \quad \mathcal{X}_\pi = \mathcal{M}_R(Z \times [0,T]),
\]
\[
\mathcal{X}_n \equiv \left( L^2_{loc}(0,\infty;W^{1,2}(D)), \text{weak} \right) \cap \left( L^2_{loc}(0,\infty;L^2(D)), \text{weak} \right),
\]
\[
\mathcal{X}_c \equiv L^2_{loc}(0,\infty;W^{1,2}(D)) \cap (L^\infty_{loc}(0,\infty;L^\infty(D)), \text{weak-star}),
\]
\[
\mathcal{X}_u \equiv D_{loc}([0,\infty);(W^{1,5}_{0,\sigma}(D))^*) \cap L^2_{loc}(0,\infty;L^2(D)) \cap (L^2_{loc}(0,\infty;W^{1,2}(D)), \text{weak}).
\]
Here, \((X, \text{weak})\) (\((X, \text{weak-star}), \text{resp.}\)) stands for the Banach space \( X \) equipped with the weak topology (weak-star topology, resp.). For any \( \epsilon \in (0,1) \), let \((n_\epsilon, c_\epsilon, u_\epsilon)\) be the global solution to the regularized system (3.1) guaranteed by Lemma 4.6.

Lemma 5.4. Denote by \( \Pi_\epsilon(\cdot) \) the joint distribution of \((W_\epsilon, \pi_\epsilon, n_\epsilon, c_\epsilon, u_\epsilon)\) on \( \mathcal{X} \) with respect to the measure \( \mathbb{P} \), that is,
\[
\Pi_\epsilon(B) \equiv \mathbb{P} \left[ \omega \in \Omega : (W_\epsilon(\omega,\cdot), \pi_\epsilon(\omega,\cdot), n_\epsilon(\omega,\cdot), c_\epsilon(\omega,\cdot), u_\epsilon(\omega,\cdot)) \in B \right], \quad \forall B \subseteq \mathcal{X}.
\]
Then, the sequence \((\Pi_\epsilon)_{\epsilon>0}\) is tight on \( \mathcal{X} \).

Proof. The proof is divided into five steps.

Step 1. It is an established fact that the space \( C([0,T];\mathbb{R}^d) \) is separable and metrizable by a complete metric, which implies that the sequence of probability measures, defined by \( \mu_{W_\epsilon}(\cdot) \equiv \mathbb{P}[W_\epsilon \in \cdot] \), is constantly equal to single element and is tight on \( C([0,T];\mathbb{R}^d) \). On the other hand, since the space \( \mathcal{P}_N(Z \times [0,T]) \) endowed with the Prohorov’s metric is also a
separable metric space (cf. Theorem 6.2 in [15]), it then holds that the distributions of the family $\mu_{\pi_c}(\cdot) \overset{\text{def}}{=} \mathbb{P}[\pi_c \in \cdot]$ are tight on $\mathcal{P}(\mathbb{Z} \times [0,T])$.

**Step 2.** We prove that the sequence $\mu_{n_c}(\cdot) \overset{\text{def}}{=} \mathbb{P}[n_c \in \cdot]$ is tight on $L^2(0,T;L^2(D))$. For this, we introduce the Banach space $Y_1$ with the norm

$$
\|y\|_{Y_1} \overset{\text{def}}{=} \|y\|_{L^q(0,T;W^{1,q}_0(D))} + \|\frac{dy}{dt}\|_{L^1(0,T;W^{2,q}(D))^*}, \quad q > 3.
$$

We choose $B_1(r)$ as a open ball with radius $r$ in $L^\frac{3}{2}(0,T;W^{1,\frac{3}{2}}(D)) \cap W^{1,1}(0,T;W^{2,q}(D))^*$, it follows from a generalized Aubin-Lions Lemma (cf. Lemma 4.3 in [48]) that the ball $B_1(r)$ is compact in $L^\frac{3}{2}(0,T;L^\frac{3}{2}(D))$. In view of the uniform bounds (5.1b) and (5.2), we infer that

$$
\mathbb{P}[n_c \notin B_1(r)] \leq \mathbb{P}\left[\|n_c\|_{L^\frac{3}{2}(0,T;W^{1,\frac{3}{2}}(D))} \geq \frac{r}{2}\right] + \mathbb{P}\left[\|n_c\|_{W^{1,1}(0,T;W^{2,q}(D))^*} \geq \frac{r}{2}\right]
\leq \frac{4}{r^2} \mathbb{E}\left(\|n_c\|_{L^\frac{3}{2}(0,T;W^{1,\frac{3}{2}}(D))}^2 + \|n_c\|_{W^{1,1}(0,T;W^{2,q}(D))^*}^2\right) \leq \frac{C}{r^2}.
$$

By choosing $\eta r^2 = C$ leads to $\inf_{\epsilon > 0} \mathbb{P}\{w; n_c(\omega, \cdot) \in B_1(r)\} > 1 - \eta$, which implies the tightness of $\mu_{n_c}$ on $L^\frac{3}{2}(0,T;L^\frac{3}{2}(D))$. The tightness of $\mu_{n_c}$ on $L^\frac{2}{3}(0,T;W^{1,3}(D))$, weak) is a direct consequence of the uniform bound (5.1b).

To show the tightness of $\mu_{n_c}$ on $L^\frac{3}{2}(0,T;L^\frac{3}{2}(D))$, weak), for any $r > 0$, we set

$$
\tilde{B}_1(r) = \{n_c \in L^\frac{3}{2}(0,T;L^\frac{3}{2}(D)); \|n_c\|_{L^\frac{3}{2}(0,T;L^\frac{3}{2}(D))} \leq r\}.
$$

By invoking the Gagliardo-Nirenberg inequality, we gain from (5.1a) that, for any $p \geq 1$,

$$
\mathbb{E}\left(\|n_c\|_{L^\frac{3}{2}(0,T;L^\frac{3}{2}(D))}^p\right) = \mathbb{E}\left(\int_0^T \|\nabla \sqrt{n_c}\|_{L^2(D)}^{\frac{9}{4}} \|n_c\|_{L^\frac{3}{2}(D)}^{\frac{9}{4}} dt\right)^{\frac{3p}{9}}
\leq C \mathbb{E}\left(\int_0^T \left(\|\nabla \sqrt{n_c}\|_{L^2(D)}^2 + \|n_c\|_{L^\frac{3}{2}(D)}^2\right) dt\right)^{\frac{3p}{9}}
\leq C \left(\int_0^T (\|\nabla \sqrt{n_c}\|_{L^2(D)}^2 + 1) dt\right)^p \leq C.
$$

Thereby, we get from (5.11) with $p = 2$ that

$$
\mathbb{P}\left[n_c \notin \tilde{B}_1(r)\right] = \mathbb{P}\left[\|n_c\|_{L^\frac{3}{2}(0,T;L^\frac{3}{2}(D))} > r\right] \leq \frac{\mathbb{E}\left(\|n_c\|_{L^\frac{3}{2}(0,T;L^\frac{3}{2}(D))}^2\right)}{r^2} \leq \frac{C}{r^2},
$$

which implies the desired result.

**Step 3.** To find a compact subset $\overline{B}_2(r) \in L^2(0,T;W^{1,2}(D))$ such that $\mu_{c_e}(B_2(r)) \overset{\text{def}}{=} \mathbb{P}[c_e \notin B_2(r)] < \eta$. Let us consider the Banach space $Y_2$ endowed with the norm

$$
\|y\|_{Y_2} \overset{\text{def}}{=} \|y\|_{L^2(0,T;W^{2,2}(D))} + \|\frac{dy}{dt}\|_{W^{1,2}(0,T;(W^{1,2}(D))^*)}.
$$
Choosing an open ball \( B_2(r) \) with radius \( r > 0 \) centered at 0 in \( L^2(0, T; W^{2,2}(D)) \cap L^\infty(0, T; (W^{1,\frac{5}{2}}(D))^*) \). The Aubin-Lions Lemma infer that the set \( B_2(r) \subset L^2(0, T; W^{1,2}(D)) \) is compact. Moreover, we get from (5.1d) and (5.3) and the Chebyshev inequality that

\[
\Pr[c_\varepsilon \notin B_2(r)] \leq \frac{4}{r^2} \mathbb{E} \left( \|c_\varepsilon\|^2 \|L^2(0, T; W^{2,2}(D)) \cap L^\infty(0, T; (W^{1,\frac{5}{2}}(D))^*)\| \right) \leq \frac{C}{r^2}.
\]

Choosing \( r > 0 \) large enough leads to \( \Pr[c_\varepsilon \in B_2(r)] \geq 1 - \varepsilon \), and hence \( (\mu_{c_\varepsilon})_{c > 0} \) is tight on \( L^2(0, T; W^{1,2}(D)) \). Moreover, by (4.6), any ball with finite radius in \( L^\infty(D \times [0, T]) \) is relatively compact in the weak-star topology. This proves the tightness of \( (\mu_{c_\varepsilon})_{c > 0} \) on \( \mathcal{X}_c \).

**Step 4.** To prove that the sequence \( \mu_{u_\varepsilon} \) is tight on \( \mathcal{D}([0, T]; (W^{1,\frac{5}{2}}(D))^*) \), we need to make use of Lemma 2.5. Indeed, it suffices to prove that, for any stopping times \( (\tau_k)_{k \geq 1} \) satisfying \( 0 \leq \tau_k \leq T \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that

\[
\mathbb{E} \left( \|u_\varepsilon(\tau_k + \theta) - u_\varepsilon(\tau_k)\|^2_{(W^{1,\frac{5}{2}}(D))^*} \right) \leq C\theta^{\frac{1}{2}}.
\]

Indeed, by integrating the \( u_\varepsilon \)-equation from 0 to \( t \) for any \( t \in (0, T] \), we obtain

\[
\begin{align*}
&u_\varepsilon(\tau_k + \theta) - u_\varepsilon(\tau_k) \\
&= -\int_{\tau_k}^{\tau_k + \theta} \mathcal{P}(L u_\varepsilon \cdot \nabla) u_\varepsilon dt - \int_{\tau_k}^{\tau_k + \theta} A u_\varepsilon dt + \int_{\tau_k}^{\tau_k + \theta} \mathcal{P}(n_\varepsilon \nabla \Phi + h(t, u_\varepsilon)) dt \\
&\quad + \int_{\tau_k}^{\tau_k + \theta} \mathcal{P}g(t, u_\varepsilon) dW + \int_{\tau_k}^{\tau_k + \theta} \int_{|z| < 1} \mathcal{P}K(u_\varepsilon(x, s-), z) \pi(ds, dz) \\
&\quad + \int_{\tau_k}^{\tau_k + \theta} \int_{|z| > 1} \mathcal{P}G(u_\varepsilon(x, s-), z) \pi(ds, dz) 
\end{align*}
\]

For \( D_1 \), by using the divergence-free condition and integrating by parts, we get form the Young inequality \( \|ab\|_2 \leq \|a\|_2 \|b\|_\infty \) that

\[
\mathbb{E} \left( \|D_1\|_{(W^{1,\frac{5}{2}}(D))^*} \right) = \mathbb{E} \sup_{\|\psi\|_{W^{1,\frac{5}{2}}(D)} = 1} \int_{D} \mathcal{P}(L u_\varepsilon \otimes u_\varepsilon) \cdot \nabla \psi(x) d\sigma \times d\sigma \leq C\theta^{\frac{1}{2}} \left( \mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{L^2}^2 + \mathbb{E} \int_0^T \|u_\varepsilon(t)\|_{L^2}^{\frac{10}{3}} dt \right) \leq C\theta^{\frac{1}{2}},
\]

where we used the uniform bounds (5.1d), (5.1e) and the fact of \( \|L u_\varepsilon\|_{L^2} \leq C\|u_\varepsilon\|_{L^2} \). Similarly, by using the embedding \( L^\infty(D) \subset L^2(D) \subset (W^{1,\frac{5}{2}}(D))^* \), we get

\[
\mathbb{E} \left( \|D_2\|_{(W^{1,\frac{5}{2}}(D))^*} \right) \leq C\theta^{\frac{1}{2}} \mathbb{E} \left( \int_0^T \|\nabla u_\varepsilon\|^2_{L^2} dt \right)^{\frac{1}{2}} \leq C\theta^{\frac{1}{2}},
\]

and

\[
\mathbb{E} \left( \|D_3\|_{(W^{1,\frac{5}{2}}(D))^*} \right) \leq C\theta \int_{\tau_k}^{\tau_k + \theta} (\|n_\varepsilon \nabla \Phi\|_{L^1} + \|u_\varepsilon\|_{L^2} + 1) dt \leq C\theta,
\]
For $D_4$, by using the BDG inequality, H"older inequality and the assumption on $g$, we obtain
\[
\mathbb{E}\left(\|D_4\|_{(W^{1,5}_{0,\sigma})^*}\right) \leq C\mathbb{E}\left[\int_{\tau_k}^{\tau_k+\theta} \left(\sup_{\|\psi\|_{W^{1,5}_{0,\sigma}}=1} \int_D \mathcal{P}g(s, u_\epsilon)\psi dx\right)^2 ds\right]^{\frac{1}{2}} \\
\leq C\mathbb{E}\left(\int_{\tau_k}^{\tau_k+\theta} \left(\|u_\epsilon(s)\|_{L^2}^2 + 1\right) ds\right)^{\frac{1}{2}} \leq C\theta^{\frac{3}{2}} \left(\mathbb{E} \sup_{t\in[0,T]} \|u_\epsilon\|_{L^2}^2 + 1\right) \leq C\theta^{\frac{3}{2}}.
\]

For $D_5$, in virtue of the BDG inequality, stochastic Fubini theorem and the assumption on $K$, we obtain
\[
\mathbb{E}\left(\|D_5\|_{(W^{1,5}_{0,\sigma})^*}\right) \leq C\mathbb{E}\left(\int_{\tau_k}^{\tau_k+\theta} \int_{|z|<1} \|\mathcal{P}K(u_\epsilon(s), z)\|_{L^2}^2 \mu(dz) ds\right)^{\frac{1}{2}} \\
\leq C\theta^{\frac{3}{2}} \mathbb{E}\left(\sup_{t\in[0,T]} \|u_\epsilon(s)\|_{L^2}^2 + 1\right) \leq C\theta^{\frac{3}{2}}.
\]

For $D_6$, recalling that $\pi(ds, dz) = \tilde{\pi}(ds, dz) - \mu(dz) ds$ is a martingale Poisson random measure, it follows that
\[
\mathbb{E}\left(\|D_6\|_{(W^{1,5}_{0,\sigma})^*}\right) \leq C\mathbb{E}\left[\int_{\tau_k}^{\tau_k+\theta} \int_{|z|>1} \sup_{\|\psi\|_{W^{1,5}_{0,\sigma}}=1} \int_D \mathcal{P}G(u_\epsilon(s), z)\psi dx \mu(dz) ds\right]^{\frac{1}{2}} \\
\leq C\mathbb{E}\left(\int_{\tau_k}^{\tau_k+\theta} \left(\|u_\epsilon(s)\|_{L^2}^2 + 1\right) ds\right) \leq C\theta.
\]

Substituting the estimates of $D_1 \sim D_6$ into (5.13) gives (5.12). Hence $(u_\epsilon)_{\epsilon>0}$ satisfy the Aldous condition [12] in $(W^{1,5}_{0,\sigma}(D))^*$, and Lemma 2.5 implies the tightness of $(u_\epsilon)_{\epsilon>0}$ on $\mathcal{D}([0,T]; (W^{1,5}_{0,\sigma}(D))^*)$.

**Step 5.** We show that the sequence $(\mu_{u_\epsilon})_{\epsilon>0}$ is also tight on $L^2([0,T]; L^2_\sigma(D))$. This can be done by introducing the Banach space $Y_3$ with the norm
\[
\|y\|_{Y_3} \overset{\text{def}}{=} \|y\|_{L^2(0,T; W^{1,2}_{0,\sigma}(D))} + \|y\|_{W^{\alpha,2}(0,T; D(A))^*},
\]
in terms of the Aubin-Lions Lemma (cf. [48]), we infer that any bounded closed ball of $L^2(0,T; W^{1,2}_{0,\sigma}(D)) \cap W^{\alpha,2}(0,T; D(A)^*)$ is compact in $L^2(0,T; L^2_\sigma(D))$. Therefore, by choosing an open ball $B_3(r)$ with radius $r > 0$ centered at 0 in $Y_3$, we find
\[
\mathbb{P}[u_\epsilon \notin B_3(r)] \leq \frac{4}{r^2} \mathbb{E}\left(\|u_\epsilon\|_{L^2(0,T; W^{1,2}_{0,\sigma}(D))}^2 \|u_\epsilon\|_{W^{\alpha,2}(0,T; D(A))^*}\right) \leq C\frac{1}{r^2}.
\]

Then one can finish the proof by choosing $r > 0$ large enough such that $\mathbb{P}[u_\epsilon \notin B_3(r)] < \eta$. The tightness of $(\mu_{u_\epsilon})_{\epsilon>0}$ on $(L^2_{loc}(0,\infty; W^{1,2}(D)), \text{weak})$ is obvious due to the bound (5.1e).

In conclusion, the joint distribution of the processes $(W_\epsilon, \pi_\epsilon, n_\epsilon, c_\epsilon, u_\epsilon)_{\epsilon>0}$ is tight on $\mathcal{X}$. The proof of Lemma 5.4 is completed. \qed
From Lemma 5.4, the Prohorov’s theorem (cf. Theorem 5.1 in [7]) tells us that \((\Pi_\epsilon)_{\epsilon > 0}\) is weakly compact in probability measure space, which implies that there exists a measure \(\Pi\) on \(\mathcal{X}\) such that, for a subsequence \((\epsilon_j)_{j \geq 1}\),

\[
\Pi_{\epsilon_j} \overset{\text{def}}{=} \mathbb{P} \circ (W_{\epsilon_j}, \pi_{\epsilon_j}, n_{\epsilon_j}, c_{\epsilon_j}, u_{\epsilon_j})^{-1} \to \Pi \text{ weakly as } j \to \infty.
\]

However, the weak convergence of \(u_\epsilon\) to \(u\) in distribution is too weak to construct solutions due to the missing of topology structure in random variable \(\omega\). Instead, we shall apply the Jakubowski-Skorokhod Theorem (cf. Lemma 2.9) to upgrade the weak convergence (5.14) to almost surely convergence in \(\mathcal{X}\) for newly-found random variables but defined on another probability space as a price.

**Lemma 5.5.** There exists a subsequence \((\epsilon_j)_{j \geq 1}\) whose limit is zero as \(j \to \infty\), a new probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), with the associated expectation denoted by \(\hat{\mathbb{E}}\), and \(\mathcal{X}\)-valued random variables \((\hat{W}_{\epsilon_j}, \hat{\pi}_{\epsilon_j}, \hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})_{j \geq 1}\)

\[
\begin{align*}
\text{a) } & \text{The laws of } (\hat{W}_{\epsilon_j}, \hat{\pi}_{\epsilon_j}, \hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}) \text{ and } (\hat{W}, \hat{\pi}, \hat{n}, \hat{c}, \hat{u}) \text{ are } \Pi_{\epsilon_j} \text{ and } \Pi, \text{ respectively;} \\
\text{b) } & (\hat{W}_{\epsilon_j}, \hat{\pi}_{\epsilon_j}) = (\hat{W}, \hat{\pi}) \text{ everywhere on } \hat{\Omega}; \\
\text{c) } & (\hat{W}_{\epsilon_j}, \hat{\pi}_{\epsilon_j}, \hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}) \text{ converges } \hat{\mathbb{P}}\text{-almost surely to } (\hat{W}, \hat{\pi}, \hat{n}, \hat{c}, \hat{u}) \text{ in the topology of } \mathcal{X}, \\
& \text{that is, } \hat{W}_{\epsilon_j} \to \hat{W}, \hat{\pi}_{\epsilon_j} \to \hat{\pi}, \text{ and} \\
& \hat{n}_{\epsilon_j} \to \hat{n} \text{ strongly in } \mathcal{X}_n, \hat{c}_{\epsilon_j} \to \hat{c} \text{ strongly in } \mathcal{X}_c, \hat{u}_{\epsilon_j} \to \hat{u} \text{ strongly in } \mathcal{X}_u, \hat{\mathbb{P}}\text{-a.s.;} \\
\text{d) } & \hat{W} \text{ is a Wiener process; } \hat{\pi} \text{ is a time homogeneous Poisson random measure on } \mathcal{B}(\mathcal{Z}) \times \mathcal{B}([0, T]) \text{ over } (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \text{ with intensity measure } d\Pi \otimes dt; \\
\text{e) } & \text{the quantity } (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}}), \hat{W}_{\epsilon_j}, \hat{\pi}_{\epsilon_j}, \hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}) \text{ is a martingale weak solution to the regularized system (3.1), which satisfies}
\end{align*}
\]

\[
\begin{align*}
(5.15a) & \quad \langle \hat{n}_{\epsilon_j}(t), \varphi \rangle = \langle \hat{n}_{\epsilon_j}(0), \varphi \rangle_{t=0} + \int_0^t \langle \hat{u}_{\epsilon_j}, \hat{n}_{\epsilon_j} - \nabla \hat{n}_{\epsilon_j} + \hat{n}_{\epsilon_j} \hat{h}_{\epsilon_j}(\hat{n}_{\epsilon_j}) \chi(\hat{c}_{\epsilon_j}) \nabla \hat{c}_{\epsilon_j}, \nabla \varphi \rangle ds, \\
(5.15b) & \quad \langle \hat{c}_{\epsilon_j}(t), \varphi \rangle = \langle \hat{c}_{\epsilon_j}(0), \varphi \rangle_{t=0} + \int_0^t \langle \hat{u}_{\epsilon_j}, \hat{c}_{\epsilon_j} - \nabla \hat{c}_{\epsilon_j}, \nabla \varphi \rangle ds - \int_0^t \langle \hat{h}_{\epsilon_j}(\hat{n}_{\epsilon_j}) f(\hat{c}_{\epsilon_j}), \varphi \rangle ds, \\
(5.15c) & \quad \langle \hat{u}_{\epsilon_j}(t), \psi \rangle = \langle \hat{u}_{\epsilon_j}(0), \psi \rangle_{t=0} + \int_0^t \langle \mathcal{P}(\hat{Y}_j, \hat{u}_{\epsilon_j} \otimes \hat{u}_{\epsilon_j}) - \nabla \hat{u}_{\epsilon_j}, \nabla \psi \rangle ds + \int_0^t \langle \mathcal{P}(\hat{n}_{\epsilon_j} \nabla \Phi), \psi \rangle ds \\
& \quad \quad + \int_0^t \langle \mathcal{P} h(t, \hat{u}_{\epsilon_j}), \psi \rangle ds + \int_0^t \langle \mathcal{P} g(t, \hat{u}_{\epsilon_j}) d\hat{W}_j, \psi \rangle + \int_Z \langle \mathcal{P} \hat{L}_{\epsilon_j}, \psi \rangle d\hat{\lambda}_{\epsilon_j},
\end{align*}
\]

for any \(\varphi \in C_0^\infty(D \times [0, \infty); \mathbb{R})\), and \(\psi \in C_0^\infty(D \times [0, \infty); \mathbb{R}^3)\) with \(\text{div} \psi = 0\).

**Remark 5.6.** By Lemma 5.5, since \((\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})\) in the new probability space has the same distributions as for \((n_{\epsilon_j}, c_{\epsilon_j}, u_{\epsilon_j})\), it is straightforward to show that \((\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})\) shares the same estimates in Lemma 5.1-5.3 under the expectation \(\hat{\mathbb{E}}\).
Proof of Lemma 5.5. The proof of \( a - d \) is standard (see e.g., [3][11][18]). For \( e \), the verification of the first two equations in (3.1) follows directly from Lemma 5.5, it remains to prove the \( u \)-equation. For all \( t \in [0, T] \) and \( \psi \in C^\infty_0(D \times [0, \infty); \mathbb{R}^3) \), we define

\[
\mathcal{L}_3(W, \pi, n, c, u)_t \overset{\text{def}}{=} \mathcal{L}_{31}(n, c, u)_t - \mathcal{L}_{32}(W, u)_t - \mathcal{L}_{33}(\pi, u)_t,
\]

where

\[
\mathcal{L}_{31}(n, c, u)_t \overset{\text{def}}{=} \langle u(t), \psi \rangle - \langle u(0), \psi \rangle + \int_0^t \langle \mathcal{P}(Y_{\epsilon_j} u \otimes u), \nabla \psi \rangle ds - \int_0^t \langle \nabla u, \nabla \psi \rangle ds
\]

\[
+ \int_0^t \langle \mathcal{P}(n \nabla \Phi), \psi \rangle ds - \int_0^t \langle \mathcal{P}h(t, u), \psi \rangle ds,
\]

\[
\mathcal{L}_{32}(W, u)_t \overset{\text{def}}{=} \int_0^t \langle \mathcal{P}g(t, u) dW, \psi \rangle,
\]

\[
\mathcal{L}_{33}(\pi, u)_t \overset{\text{def}}{=} \int_0^t \int_{|z| \leq 1} \langle \mathcal{P}K(u(s, z), \psi) \tilde{\pi}(ds, dz) + \int_0^t \int_{|z| \leq 1} \langle \mathcal{P}G(u(s, z), \psi) \pi(dz) ds.
\]

Since \( (W_{\epsilon_j}, \pi_{\epsilon_j}, n_{\epsilon_j}, c_{\epsilon_j}, u_{\epsilon_j}) \) satisfies (3.1), it is clear that \( \mathcal{L}_3(W_{\epsilon_j}, \pi_{\epsilon_j}, n_{\epsilon_j}, c_{\epsilon_j}, u_{\epsilon_j})_t = 0, \mathbb{P}\)-a.s. To complete the proof, it suffices to prove that

\[
\mathbb{E} \left[ \mathcal{L}_3(\tilde{W}_{\epsilon_j}, \tilde{\pi}_{\epsilon_j}, \tilde{n}_{\epsilon_j}, \tilde{c}_{\epsilon_j}, \tilde{u}_{\epsilon_j})_t \right] = \mathbb{E} \left[ \mathcal{L}_3(W_{\epsilon_j}, \pi_{\epsilon_j}, n_{\epsilon_j}, c_{\epsilon_j}, u_{\epsilon_j})_t \right].
\]

The argument will be divided into three parts.

(P1) The absolutely continuous part \( \mathcal{L}_{31} \) can be treated as before, and we obtain

\[
(5.16) \quad \mathbb{E} \left[ \mathcal{L}_{31}(\tilde{n}_{\epsilon_j}, \tilde{c}_{\epsilon_j}, \tilde{u}_{\epsilon_j})_t \right] = \mathbb{E} \left[ \mathcal{L}_{31}(n_{\epsilon_j}, c_{\epsilon_j}, u_{\epsilon_j})_t \right].
\]

(P2) Let \( \rho_k(t) \) be a standard mollifier, then we define

\[
[\mathcal{G}_k(u_{\epsilon_j})](s) \overset{\text{def}}{=} \int_0^t \rho_k(t - s) \mathcal{P}g(s, u_{\epsilon_j}(s)) ds,
\]

\( k \in \mathbb{N} \).

Apparently, \( \mathcal{G}_k(u) \) is a time-smooth function in \( L^2(\Omega; L^2(0, T; L_2(\mathbb{R}^d; L_2^2(D)))) \), which is of course a bounded variation function for \( t \in [0, T] \). Moreover, the properties of mollifiers imply that

\[
(5.17) \quad \| \mathcal{G}_k(u_{\epsilon_j}) \|_{L^2(\Omega; L^2(0, T; L_2(\mathbb{R}^d; L_2^2(D)))))} \leq C \| \mathcal{P}g(\cdot, u_{\epsilon_j}(\cdot)) \|_{L^2(\Omega; L^2(0, T; L_2(\mathbb{R}^d; L_2^2(D))))},
\]

and

\[
(5.18) \quad \mathcal{G}_k(u_{\epsilon_j}) \to \mathcal{P}g(s, u_{\epsilon_j}(s)) \text{ in } L^2(\Omega; L^2(0, T; L_2(\mathbb{R}^d; L_2^2(D)))) \text{, as } k \to \infty.
\]

Note that the Wiener process is continuous in time, by making use of the integrating by parts for Riemann-Stieltjes integral, we gain

\[
\mathcal{L}_{32}^k(W_{\epsilon_j}, u_{\epsilon_j})_t \overset{\text{def}}{=} \int_0^t \langle [\mathcal{G}_k(u_{\epsilon_j})](s) dW_{\epsilon_j}(s), \psi \rangle
\]

\[
= \langle [\mathcal{G}_k(u_{\epsilon_j})](t) W_{\epsilon_j}(t), \psi \rangle - \int_0^t \langle [\mathcal{G}_k(u)]'(s) W_{\epsilon_j}(s) ds, \psi \rangle,
\]

(5.19)
which can be viewed as a deterministic functional of $W_{\varepsilon_j}$. In virtue of the BDG inequality, (5.17)-(5.18) and the Dominated Convergence Theorem, we have
\[
E \left| \mathcal{L}_{32} \left( W_{\varepsilon_j}, \mu_{\varepsilon_j} \right)_t - \mathcal{L}_{32}^k \left( W_{\varepsilon_j}, \mu_{\varepsilon_j} \right)_t \right| \leq C \varepsilon \left( \int_0^t \| P \varepsilon \left( u_{\varepsilon_j}(s) \right) \|_{L_2(t; L^2)} \right)^{1/2} \rightarrow 0, \quad \text{as } k \rightarrow \infty.
\]

(P3) Now let us identify the stochastic integrals with jumps by borrowing some ideas from Cyr et al. [13, 14]. Note that by employing the well-known piecing out argument (or interlacing procedure) (cf. [3, 14, 35]), it suffices to consider the Lévy process possesses with jumps of small size, that is, $G \in 0 \in \mathcal{L}_{33} \left( \pi_{\varepsilon_j}, u_{\varepsilon_j} \right)_t$. To this end, we choose subsets $(Z_k)_{k \geq 1} \subseteq \{ z \in Z; |z| \leq 1 \}$ such that $Z_k \uparrow Z_0$ and $\mu (Z_k) < \infty$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, $\pi_k \equiv \pi |_{[0, \infty) \times Z_k}$ is a Poisson random measure on $Z$ corresponding to a $(\mathcal{F}_t)$-Poisson point process, and the intensity measure of $\pi_k$ is given by $\mu_k \equiv \mu |_{Z_k}$, which is a finite measure. Define
\[
\mathcal{L}_{33}^k \left( \pi_{\varepsilon_j}, u_{\varepsilon_j} \right)_t \overset{\text{def}}{=} \int_0^t \int_{Z_k} \langle PK (u_{\varepsilon_j}(s-), z), \psi \rangle 1_{Z_k}(z) \pi_{\varepsilon_j}(ds, dz)
\]
which can be viewed as a deterministic functional of $(\pi_{\varepsilon_j}, u_{\varepsilon_j})$. Moreover, since $Z_k \uparrow Z_0$, we deduce from the BDG inequality and the Dominated Convergence Theorem that
\[
E \left| \mathcal{L}_{33} \left( \pi_{\varepsilon_j}, u_{\varepsilon_j} \right)_t - \mathcal{L}_{33}^k \left( \pi_{\varepsilon_j}, u_{\varepsilon_j} \right)_t \right| = E \left| \int_0^t \int_{Z_0 \setminus Z_k} \langle PK (u_{\varepsilon_j}(s-), z), \psi \rangle 1_{Z_k}(z) \pi_{\varepsilon_j}(ds, dz) \right| \leq C \varepsilon \left( \int_0^t \int_{Z_0 \setminus Z_k} \| PK (u_{\varepsilon_j}(s-), z) \|_{L_2}^2 \mu (dz) ds \right)^{1/2} \rightarrow 0, \quad \text{as } k \rightarrow 0.
\]

From (P1)-(P3), for each $k \in \mathbb{N}$, one can introduce
\[
\mathcal{L}_{3}^k \left( W_{\varepsilon_j}, \pi_{\varepsilon_j}, n_{\varepsilon_j}, c_{\varepsilon_j}, u_{\varepsilon_j} \right)_t \overset{\text{def}}{=} \mathcal{L}_{31} \left( n_{\varepsilon_j}, c_{\varepsilon_j}, u_{\varepsilon_j} \right)_t - \mathcal{L}_{32}^k \left( W_{\varepsilon_j}, u_{\varepsilon_j} \right)_t - \mathcal{L}_{33}^k \left( \pi_{\varepsilon_j}, u_{\varepsilon_j} \right)_t
\]
which is actually a deterministic function on $\mathcal{X}$. Moreover, it follows from (5.16), (5.20) and (5.21) that
\[
E \left| \mathcal{L}_{3} \left( W_{\varepsilon_j}, \pi_{\varepsilon_j}, n_{\varepsilon_j}, c_{\varepsilon_j}, u_{\varepsilon_j} \right)_t - \mathcal{L}_{3}^k \left( W_{\varepsilon_j}, \pi_{\varepsilon_j}, n_{\varepsilon_j}, c_{\varepsilon_j}, u_{\varepsilon_j} \right)_t \right| \leq E \left| \mathcal{L}_{32} (W, u)_t - \mathcal{L}_{32}^k (W, u)_t \right| + E \left| \mathcal{L}_{33} (\pi, u)_t - \mathcal{L}_{33}^k (\pi, u)_t \right| \rightarrow 0, \quad \text{as } k \rightarrow 0.
\]
By the same reasoning, the random variable
\[ \mathcal{L}_3^k (\hat{W}_{\epsilon}, \hat{\pi}_{\epsilon}, \hat{n}_{\epsilon}, \hat{c}_{\epsilon}, \hat{u}_{\epsilon}) \]
where \( \mathcal{L}_3^k (\cdot) \) is provided in (5.22), is also a deterministic function such that
\[ (5.24) \quad \mathbb{E} \left[ \mathcal{L}_3^k (\hat{W}_{\epsilon}, \hat{\pi}_{\epsilon}, \hat{n}_{\epsilon}, \hat{c}_{\epsilon}, \hat{u}_{\epsilon}) \right] \rightarrow 0, \quad \text{as } k \rightarrow 0. \]

Since \( (\hat{W}_{\epsilon}, \hat{\pi}_{\epsilon}, \hat{n}_{\epsilon}, c_{\epsilon}, u_{\epsilon}) \) has the same distribution, it follows from the definition of \( \mathcal{L}_3^k (\cdot) \) that
\[ \mathbb{E} \left[ \mathcal{L}_3^k (\hat{W}_{\epsilon}, \hat{\pi}_{\epsilon}, \hat{n}_{\epsilon}, \hat{c}_{\epsilon}, \hat{u}_{\epsilon}) \right] = \mathbb{E} \left[ \mathcal{L}_3^k (W_{\epsilon}, \pi_{\epsilon}, n_{\epsilon}, c_{\epsilon}, u_{\epsilon}) \right], \]
which combined with (5.23) and (5.24) lead to the desired equality (5.152). Therefore, the quantity \( (\hat{W}_{\epsilon}, \hat{\pi}_{\epsilon}, \hat{n}_{\epsilon}, \hat{c}_{\epsilon}, \hat{u}_{\epsilon}) \) satisfies the \( u_{\epsilon} \)-equation. The proof of Lemma 5.5 is now completed. 

5.3. Recovering the original SPDEs. Now we have all in hand to complete the proof of Theorem 1.2. We begin by turning the probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \) into a stochastic basis \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\pi}, \hat{\mathbb{P}}, \hat{W}, \hat{\pi}) \) with natural filtration \( \hat{\mathcal{F}}_t \), that is, the smallest filtration with respect to which all the relevant processes are adapted,
\[ \hat{\mathcal{F}}_t = \sigma \left( \sigma (\hat{n}_{[0,t]}, \hat{\pi}_{[0,t]}, \hat{u}_{[0,t]}, \hat{W}_{[0,t]}, \hat{\pi}_{[0,t]}) \right) \cup \left\{ N \in \hat{\mathcal{F}}; \hat{\mathbb{P}}(N) = 0 \right\}, \quad t \in [0,T]. \]

Proof of Theorem 1.2 The proof is long and will be divided into several steps.

Step 1 (Identification of \( n \)-equation). In the first step, let us identify the limit for \( n \)-equation. To this end, we consider functionals
\[ m_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}) \]
and
\[ m(\hat{n}, \hat{c}, \hat{u}) \]
where \( (\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}) \) and \( (\hat{n}, \hat{c}, \hat{u}) \) are processes constructed in Lemma 5.5.

We Claim that, for all \( \psi \in C^\infty_0 (D \times [0, \infty); \mathbb{R}) \),
\[ (5.27a) \quad \lim_{j \to \infty} \hat{\mathbb{E}} \left[ \int_0^T \left| \langle \hat{n}_{\epsilon_j}(t) - \hat{n}(t), \psi \rangle \right| dt \right] = 0, \]
and
\[ (5.27b) \quad \lim_{j \to \infty} \hat{\mathbb{E}} \left[ \int_0^T \left| m_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}) - m(\hat{n}, \hat{c}, \hat{u}, \psi) \right| dt \right] = 0. \]
Moreover, since which implies by (5.11) that the sequence Hence by Vitali Convergence Theorem we infer that (5.27a) holds.

By utilizing (5.28) and the Young inequality, we gain

(5.28)

Proof of 5.27a. Observing that, by Lemma 5.5 \( \hat{n}_{\varepsilon_j} \to \hat{n} \) weakly in \( L^5_{loc}(0, \infty; L^\frac{5}{3}(D)) \) \( \hat{P} \)-a.s. For each \( T > 0 \), we have

\[
\int_0^T \int_D \hat{n}^\frac{5}{3}_{\varepsilon_j}(x,t) dx dt = \int_0^\infty \int_D 1_{D \times (0,T)} \hat{n}^\frac{5}{3}_{\varepsilon_j}(x,t) dx dt
\]

\[
\to \int_0^\infty \int_D 1_{D \times (0,T)} \hat{n}^\frac{5}{3}(x,t) dx dt = \int_0^T \int_D \hat{n}^\frac{5}{3}(x,t) dx dt \quad \text{as} \quad j \to \infty, \quad \hat{P} \text{-a.s.,}
\]

which implies that

(5.27a)

\( \hat{n}_{\varepsilon_j} \to \hat{n} \) strongly in \( L^5_{loc}(0, \infty; L^\frac{5}{3}(D)) \) as \( j \to \infty, \) \( \hat{P} \)-a.s.

By utilizing (5.28) and the Young inequality, we gain

\[
\int_0^T |\langle \hat{n}_{\varepsilon_j}(t) - \hat{n}(t), \psi \rangle|^{\frac{5}{3}} dt \leq \|\psi\|_{L^\infty(0,T;L^\frac{5}{3})}^5 \int_0^T \|\hat{n}_{\varepsilon_j}(t) - \hat{n}(t)\|_{L^\frac{5}{3}} dt \to 0 \quad \text{as} \quad j \to \infty, \quad \hat{P} \text{-a.s.}
\]

Moreover, since \( \hat{n}_{\varepsilon_j} \) satisfies the uniform bound (5.11), we have for any \( p \geq 1 \) that

\[
\hat{E} \left( \int_0^T |\langle \hat{n}_{\varepsilon_j}(t) - \hat{n}(t), \psi \rangle|^{\frac{5}{3}} dt \right)^p
\]

\[
\leq \|\psi\|_{L^\infty(0,T;L^\frac{5}{3})}^5 \left[ \hat{E} \left( \int_0^T \|\hat{n}_{\varepsilon_j}(t)\|_{L^\frac{5}{3}} dt \right)^p + \hat{E} \left( \int_0^T \|\hat{n}(t)\|_{L^\frac{5}{3}} dt \right)^p \right] \leq C,
\]

which implies by (5.11) that the sequence \( \int_0^T |\langle \hat{n}_{\varepsilon_j}(t) - \hat{n}(t), \psi \rangle|^{\frac{5}{3}} dt \) is uniformly integrable.

Hence by Vitali Convergence Theorem we infer that (5.27a) holds.

Proof of 5.27b. Note that, by Fubini’s theorem,

\[
\hat{E} \int_0^T \left| \mathbf{m}_{\varepsilon_j}(\hat{n}_{\varepsilon_j}, \hat{c}_{\varepsilon_j}, \hat{u}_{\varepsilon_j})_t - \mathbf{m}(\hat{n}, \hat{c}, \hat{u})_t, \psi \right| dt
\]

\[
= \int_0^T \hat{E} \left| \mathbf{m}_{\varepsilon_j}(\hat{n}_{\varepsilon_j}, \hat{c}_{\varepsilon_j}, \hat{u}_{\varepsilon_j})_t - \mathbf{m}(\hat{n}, \hat{c}, \hat{u})_t, \psi \right| dt.
\]

It suffices to prove that each term on the R.H.S. of (5.25) tends to the corresponding term on the R.H.S. of (5.26) in \( L^1(\Omega \times (0, T)) \). Similar to (5.27a), noting that \( \hat{n} \) is right-continuous at \( t = 0 \), we first have

(5.29)

\[
\hat{E} \left[ |\langle \hat{n}_{\varepsilon_j}(0) - \hat{n}(0), \psi|_{t=0} \rangle| \right] = 0.
\]

By c) of Lemma 5.5 we have \( \nabla \hat{n}_{\varepsilon_j} \to \nabla \hat{n} \) weakly in \( L^\frac{5}{3}_{loc}(0, \infty; L^\frac{5}{3}(D)) \), \( \hat{P} \)-a.s., which implies

(5.30)

\[
\int_0^T \langle \nabla \hat{n}_{\varepsilon_j}, \nabla \psi \rangle ds \to \int_0^T \langle \nabla \hat{n}, \nabla \psi \rangle ds \quad \text{as} \quad j \to \infty, \quad \hat{P} \text{-a.s.}
\]
By uniform bounds (5.1c), (5.11) and the Gagliardo-Nirenberg inequality, we have
\[
\hat{\mathbb{E}} \left( \int_0^T \| \hat{u}_{\epsilon_j}(t) \|_{L^{10}_{\xi}}^{10} \, dt \right)^p \leq C \hat{\mathbb{E}} \left[ \int_0^T \left( \| \nabla \hat{u}_{\epsilon_j}(t) \|_{L^2}^2 \| \hat{u}_{\epsilon_j}(t) \|_{L^2}^4 + \| \hat{u}_{\epsilon_j}(t) \|_{L^2}^{10} \right) \, dt \right]^p
\leq C \hat{\mathbb{E}} \left( \sup_{t \in [0,T]} \| \hat{u}_{\epsilon_j}(t) \|_{L^2}^{10} \right)^p + C \hat{\mathbb{E}} \left( \int_0^T \| \nabla \hat{u}_{\epsilon_j}(t) \|_{L^2}^2 \, dt \right)^p \leq C,
\]
which implies that \( \hat{u}_{\epsilon_j} \) is uniformly bounded in \( L^p \left( \Omega; L^{10}_{\xi}(0,T;L^{\frac{10}{3}}(D)) \right) \), and so there exists a subsequence (still denoted by itself) such that
\[
\hat{u}_{\epsilon_j} \rightarrow \hat{u} \quad \text{weakly in} \quad L^p \left( \Omega; L^{10}_{\xi}(0,T;L^{\frac{10}{3}}(D)) \right), \quad j \rightarrow \infty.
\]
In virtue of (5.31), (5.11) and (5.11), it follows from the Hölder inequality that
\[
\hat{\mathbb{E}} \left( \| \hat{u}_{\epsilon_j} \hat{n}_{\epsilon_j} \|_{L^1(0,T;L^1)}^p \right) \leq C \hat{\mathbb{E}} \left( \| \hat{u}_{\epsilon_j} \|_{L^{10}_{\xi}(0,T;L^{\frac{10}{3}})}^{2p} \right) + C \hat{\mathbb{E}} \left( \| \hat{n}_{\epsilon_j} \|_{L^2(0,T;L^2)}^{2p} \right) \leq C,
\]
which shows that \( \hat{u}_{\epsilon_j} \hat{n}_{\epsilon_j} \) is uniformly bounded in \( L^p \left( \Omega; L^1(0,T;L^1(D)) \right) \). There exists a subsequence (still denoted by itself) such that
\[
\hat{u}_{\epsilon_j} \hat{n}_{\epsilon_j} \rightarrow \hat{u} \hat{n} \quad \text{weakly in} \quad L^p \left( \Omega; L^1(0,T;L^1(D)) \right) \quad \text{as} \quad j \rightarrow \infty,
\]
which infers that \( \hat{u} \hat{n} \in L^1(0,T;L^1(D)) \) \( \hat{\mathbb{P}} \)-a.s., and \( \hat{\mathbb{E}} \left| \int_0^t \langle \hat{u}_{\epsilon_j} \hat{n}_{\epsilon_j} - \hat{u} \hat{n}, \nabla \psi \rangle \, ds \right| \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty. \)
By (5.32) and the Dominated Convergence Theorem, we gain
\[
\lim_{j \rightarrow \infty} \int_0^T \hat{\mathbb{E}} \left| \int_0^t \langle \hat{u}_{\epsilon_j} \hat{n}_{\epsilon_j} - \hat{u} \hat{n}, \nabla \psi \rangle \, ds \right| \, dt = 0.
\]
Now we show that
\[
\hat{\mathbb{E}} \left| \int_0^t \langle \hat{n}_{\epsilon_j} \hat{h}'(\hat{n}_{\epsilon_j}) \chi(\hat{c}_{\epsilon_j}) \nabla \hat{c}_{\epsilon_j} - \hat{n} \chi(\hat{c}) \nabla \hat{c}, \nabla \psi \rangle \, ds \right| \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.
\]
To do that, we make a decomposition
\[
\hat{h}'(\hat{n}_{\epsilon_j}) \chi(\hat{c}_{\epsilon_j}) \hat{c}_{\epsilon_j}^3 - \chi(\hat{c}) \hat{c}^3 = \left( \hat{h}'(\hat{n}_{\epsilon_j}) - 1 \right) \chi(\hat{c}_{\epsilon_j}) \hat{c}_{\epsilon_j}^3
+ (\hat{c}_{\epsilon_j} - \hat{c}) \chi' (\theta \hat{c}_{\epsilon_j} + (1 - \theta) \hat{c}) \hat{c}_{\epsilon_j}^3 + \left( \hat{c}_{\epsilon_j}^3 - \hat{c}^3 \right) \chi(\hat{c}),
\]
where the Mean Value Theorem is applied for some \( \theta \in (0,1) \). To verify (5.34), let us first show the convergence
\[
\hat{h}'(\hat{n}_{\epsilon_j}) \chi(\hat{c}_{\epsilon_j}) \hat{c}_{\epsilon_j}^3 \rightarrow \chi(\hat{c}) \hat{c}^3 \quad \text{strongly in} \quad L^{\frac{20}{3}} \left( 0,T;L^{\frac{20}{3}}(D) \right) \quad \text{as} \quad j \rightarrow \infty, \quad \hat{\mathbb{P}}\text{-a.s.}
\]
The proof is divided into three steps:
- From the definition of \( \hat{h}_{\epsilon_j} \) and \( \hat{n}_{\epsilon_j} \in L^\infty(0,T;L^\infty(D)) \) \( \hat{\mathbb{P}}\)-a.s., we infer that \( \hat{h}'(\hat{n}_{\epsilon_j}) - 1 = -\frac{\hat{c}_{\epsilon_j}}{1 + \epsilon_j \hat{n}_{\epsilon_j}} \rightarrow 0 \) as \( j \rightarrow \infty \) \( \hat{\mathbb{P}} \otimes dt \otimes dx \)-a.s., which together with \( \| \chi(\hat{c}_{\epsilon_j}) \hat{c}_{\epsilon_j}^3 \|_{L^\infty(0,T;L^\infty)} \leq C \) \( \hat{\mathbb{P}}\)-a.s. (see (4.3)) and the Dominated Convergence Theorem yield that
\[
\left\| \left( \hat{h}'(\hat{n}_{\epsilon_j}) - 1 \right) \chi(\hat{c}_{\epsilon_j}) \hat{c}_{\epsilon_j}^3 \right\|_{L^{\frac{20}{3}}(0,T;L^{\frac{20}{3}})} \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty, \quad \hat{\mathbb{P}}\text{-a.s.}
\]
Using the boundedness \( \|x'(\theta \hat{c}_{e_j} + (1-\theta)\hat{c})\hat{c}_{e_j}^{\frac{3}{4}} \|_{L^\infty([0,T];L^\infty)} \leq C \) \( \hat{P} \)-a.s. (see (4.5)), we have
\[
\left\| \left( \hat{c}_{e_j} - \hat{c} \right) x'(\theta \hat{c}_{e_j} + (1-\theta)\hat{c})\hat{c}_{e_j}^{\frac{3}{4}} \right\|_{L^\frac{20}{17}(0,T;L^\frac{20}{17})} \to 0 \quad \text{as} \quad j \to \infty, \quad \hat{P}-\text{a.s.}
\]

Since by Lemma 5.5 \( \hat{c}_{e_j} \to \hat{c} \) strongly in \( L^2_{\text{loc}}(0,\infty;L^2(D)) \) \( \hat{P} \)-a.s., and hence \( \hat{c}_{e_j} \to \hat{c} \) strongly in \( L^2(\Omega;L^2_{\text{loc}}(0,\infty;L^2(D))) \) by the uniform bound in Lemma 5.1 and the Vitali Convergence Theorem, there exists a subsequence denoted by itself such that
\[
\hat{c}_{e_j} \to \hat{c} \quad \hat{P} \otimes dt \otimes dx \text{-almost surely, as } j \to \infty.
\]
In terms of (5.36) and the continuity of \( \chi(\cdot) \), we have \( |\hat{c}_{e_j}^{\frac{3}{4}}(\chi(\hat{c}_{e_j}))|_{\frac{20}{3}} \to |\hat{c}^{\frac{3}{4}}\chi(\hat{c})|_{\frac{20}{3}} \) \( \hat{P} \otimes dt \otimes dx \)-almost surely as \( j \to \infty \). It follows from the boundedness of \( \hat{c} \) and \( \hat{c}_{e_j} \) as well as the Dominated Convergence Theorem that
\[
\int_0^T \int_D |\hat{c}_{e_j}^{\frac{3}{4}}(\chi(\hat{c}_{e_j}))|_{\frac{20}{3}}^2 dx \, dt \to \int_0^T \int_D |\hat{c}^{\frac{3}{4}}\chi(\hat{c})|_{\frac{20}{3}}^2 dx \, dt \quad \text{as } j \to \infty, \quad \hat{P}\text{-a.s.}
\]
Moreover, we conclude from c) in Lemma 5.5 that
\[
\hat{c}_{e_j}^{\frac{3}{4}}(\chi(\hat{c}_{e_j})) \to \hat{c}^{\frac{3}{4}}\chi(\hat{c}) \quad \text{weakly-star in } L^\infty_{\text{loc}}(0,\infty;L^\infty(D)) \quad \text{as } j \to \infty, \quad \hat{P}\text{-a.s.}
\]
and hence weakly in \( L^{\frac{20}{3}}(0,T;L^{\frac{20}{3}}(D)) \) \( \hat{P} \)-a.s., which together with (5.37) imply
\[
\left\| \left( \hat{c}_{e_j}^{\frac{3}{4}} - \hat{c}^{\frac{3}{4}} \right) \chi(\hat{c}) \right\|_{L^{\frac{20}{3}}(0,T;L^{\frac{20}{3}})} \to 0 \quad \text{as } j \to \infty, \quad \hat{P}-\text{a.s.}
\]
This proves (5.35).
Noting that
\[
\hat{n}_{e_j} h'_{e_j}(\hat{n}_{e_j}) \chi(\hat{c}_{e_j}) \nabla \hat{c}_{e_j} = 4\hat{n}_{e_j} \left( h'_{e_j}(\hat{n}_{e_j}) \chi(\hat{c}_{e_j}) \hat{c}_{e_j}^{\frac{3}{4}} \right) \nabla \hat{c}_{e_j}^{\frac{3}{4}},
\]
we get from (5.28), (5.35) and (5.16) that
\[
\hat{n}_{e_j} h'_{e_j}(\hat{n}_{e_j}) \chi(\hat{c}_{e_j}) \nabla \hat{c}_{e_j} \to \hat{n} \chi(\hat{c}) \nabla \hat{c} \quad \text{weakly in } L^1(0,T;L^1(D)) \quad \text{as } j \to \infty, \quad \hat{P}\text{-a.s.}
\]
Moreover, for any \( p \geq 1 \), Lemma 5.1 implies that
\[
\hat{E} \left[ \int_0^T \left( \int_0^T \langle \hat{n}_{e_j} h'_{e_j}(\hat{n}_{e_j}) \chi(\hat{c}_{e_j}) \nabla \hat{c}_{e_j}, \nabla \psi \rangle dt \right)^p \right] \leq C \hat{E} \left[ \int_0^T \|\hat{n}_{e_j}\|_{L^1} \, dt \right]^p \leq C \hat{E} \left[ \int_0^T \|\hat{n}_{e_j}\|_{L^\left(\frac{4}{3}\right)}^\frac{4}{3} \, dt \right]^\frac{3p}{4} \leq C,
\]
which together with (5.38) and the Vitali Convergence Theorem lead to (5.34). Hence we get
\[
\lim_{j \to \infty} \int_0^T \hat{E} \left[ \int_0^t \langle \hat{n}_{e_j} h'_{e_j}(\hat{n}_{e_j}) \chi(\hat{c}_{e_j}) \nabla \hat{c}_{e_j} - \hat{n} \chi(\hat{c}) \nabla \hat{c}, \nabla \psi \rangle \, ds \right] = 0.
\]
According to (5.29), (5.30), (5.33), (5.39), and noting that \((\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})\) is a martingale weak solution to (3.1), we gain
\[
\hat{E} \int_0^T |\langle \hat{n}(t), \psi \rangle - m(\hat{n}, \hat{c}, \hat{u})| \, dt = \lim_{j \to \infty} \hat{E} \int_0^T |\langle \hat{n}_{\epsilon_j}(t), \psi \rangle - m_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})| \, dt = 0,
\]
which implies that
\[
\langle \hat{n}(t), \psi \rangle = m(\hat{n}, \hat{c}, \hat{u})_t, \quad \text{for a.e. } t \in [0, T], \hat{P} \text{-a.s.}
\]
Since two càdlàg functions equal for a.e. \(t \in [0, T]\) must be equal for all \(t \in [0, T]\), the \(n\)-equation hold for all \(t \in [0, T]\), \(\hat{P}\)-a.s. The proof for \(n\)-equation is now completed.

**Step 2 (Identification of c-equation).** The identification of limit for the \(c\)-equation can be treated in a same manner as that for \(n\)-equation, where the main difficulty lies upon proving that \(\hat{c}_{\epsilon_j} \hat{n}_{\epsilon_j} \to \hat{c} \hat{n}\) strongly in \(L^1(0, T; L^1(D))\) and \(h_{\epsilon_j}(\hat{n}_{\epsilon_j}) f(\hat{c}_{\epsilon_j}) \to \hat{n} f(\hat{c})\) strongly in \(L^1(0, T; L^1(D))\) as \(j \to \infty\), \(\hat{P}\)-a.s. Here we only verify the later one for saving the space.

Indeed, we first get by continuity of \(f\) and Lemma 5.5 that
\[
(5.40) \quad f(\hat{c}_{\epsilon_j}) \to f(\hat{c}) \quad \text{weakly-star } L^\infty(0, T; L^\infty(D)) \text{ as } j \to \infty, \hat{P} \text{-a.s.,}
\]
It follows from the Lipschitz continuity property of \(h_{\epsilon_j}(\cdot)\) that
\[
\| h_{\epsilon_j}(\hat{n}_{\epsilon_j}) - \hat{n} \|_{L^\infty(0, T; L^\infty)} \leq \| h_{\epsilon_j}(\hat{n}_{\epsilon_j}) - h_{\epsilon_j}(\hat{n}) \|_{L^\infty(0, T; L^\infty)} + \| h_{\epsilon_j}(\hat{n}) - \hat{n} \|_{L^\infty(0, T; L^\infty)}
\]
\[
\leq \| \hat{n}_{\epsilon_j} - \hat{n} \|_{L^\infty(0, T; L^\infty)} + \| h_{\epsilon_j}(\hat{n}) - \hat{n} \|_{L^\infty(0, T; L^\infty)}.
\]
For the first term, we get by \(c\) of Lemma 5.5 that \(\| \hat{n}_{\epsilon_j} - \hat{n} \|_{L^\infty(0, T; L^\infty)} \to 0\) as \(j \to \infty, \hat{P}\)-a.s. For the second term, we use the property \(h_{\epsilon_j}(s) \to s\) as \(j \to \infty\) to get \(h_{\epsilon_j}(\hat{n}) \to \hat{n}\) \(\hat{P} \otimes dt \otimes dz\)-almost surely as \(j \to \infty\). Moreover, we get by the uniform bound (5.1c) that \(\| h_{\epsilon_j}(\hat{n}) \|_{L^\infty(0, T; L^\infty)} \leq \| \hat{n} \|_{L^\infty(0, T; L^\infty)} < \infty \hat{P}\)-a.s. Therefore, we get by Dominated Convergence Theorem that
\[
(5.41) \quad h_{\epsilon_j}(\hat{n}_{\epsilon_j}) \to \hat{n} \quad \text{strongly in } L^\infty(0, T; L^\infty) \text{ as } j \to \infty, \hat{P} \text{-a.s.}
\]
Note that (5.40) also infer that \(f(\hat{c}_{\epsilon_j}) \to f(\hat{c})\) weakly in \(L^5(0, T; L^5(D))\) as \(j \to \infty, \hat{P}\)-a.s., which combined with (5.41) leads to the desired result.

**Step 3 (Identification of u-equation).** For any \(\psi \in C_0^\infty(D \times [0, \infty); \mathbb{R}^3)\) with \(\text{div}\psi = 0\), we define
\[
\begin{align*}
k_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})_t \overset{\text{def}}{=} & \langle \hat{u}_{\epsilon_j}(0), \psi \rangle + \int_0^t \langle \mathcal{P}(L_{\epsilon_j} \hat{u}_{\epsilon_j} \otimes \hat{u}_{\epsilon_j}), \nabla \psi \rangle ds - \int_0^t \langle \nabla \hat{u}_{\epsilon_j}, \nabla \psi \rangle ds \\
+ & \int_0^t \langle \mathcal{P}(\hat{n}_{\epsilon_j} \nabla \Phi), \psi \rangle ds + \int_0^t \langle \mathcal{P} h(t, \hat{u}_{\epsilon_j}), \psi \rangle ds + \int_0^t \langle \mathcal{P} g(t, \hat{u}_{\epsilon_j}) d\hat{W}_{\epsilon_j}, \psi \rangle \\
+ & \int_0^t \int_{|z| < 1} \langle \mathcal{P} K(\hat{u}_{\epsilon_j}(s-), z), \psi \rangle \hat{n}_{\epsilon_j} (ds, dz) + \int_0^t \int_{|z| \geq 1} \langle \mathcal{P} G(\hat{u}_{\epsilon_j}(s-), z), \psi \rangle \hat{n}_{\epsilon_j} (dz) ds,
\end{align*}
\]
and
\[
\begin{aligned}
k(\hat{n}, \hat{c}, \hat{u})_t & \overset{\text{def}}{=} \langle \hat{u}(0), \psi \rangle + \int_0^t \langle \mathcal{P} \hat{u} \otimes \hat{u}, \nabla \psi \rangle ds - \int_0^t \langle \nabla \hat{u}, \nabla \psi \rangle ds + \int_0^t \langle \mathcal{P}(\hat{n} \nabla \Phi), \psi \rangle ds \\
& + \int_0^t \langle \mathcal{P}h(t, \hat{u}), \psi \rangle ds + \int_0^t \langle \mathcal{P}g(t, \hat{u})d\tilde{W}, \psi \rangle + \int_0^t \int_{|z| < 1} \langle \mathcal{P}K(\hat{u}(s-), z), \psi \rangle \tilde{\pi}(ds, dz) \\
& + \int_0^t \int_{|z| \geq 1} \langle \mathcal{P}G(\hat{u}(s-), z), \psi \rangle \tilde{\pi}(dz) ds.
\end{aligned}
\]

Apparently, since \((\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})\) is a martingale weak solution to the system \((3.1)\), there holds \(\langle \hat{u}_{\epsilon_j}(t), \psi \rangle = k_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})_t\) for all \(t \in [0, T]\) \(\hat{P}\)-a.s. In particular,
\begin{equation}
(5.42) \quad \hat{E} \int_0^T \| \langle \hat{u}_{\epsilon_j}(t), \psi \rangle - k_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})_t \| ^2 dt = 0.
\end{equation}

The Key points is to prove that, for any \(\psi \in C_0^\infty(D \times (0, T); \mathbb{R}^3)\) with \(\text{div} \psi = 0\),
\begin{align}
(5.43a) \quad & \lim_{j \to \infty} \hat{E} \int_0^T \| \langle \hat{u}_{\epsilon_j}(t) - \hat{u}(t), \psi \rangle \| ^2 dt = 0, \\
(5.43b) \quad & \lim_{j \to \infty} \hat{E} \int_0^T \| \langle k_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}) - k(\hat{n}, \hat{c}, \hat{u}) \rangle \| ^2 dt = 0.
\end{align}

Indeed, if \((5.43a) - (5.43b)\) hold, then we infer from \((5.42)\) that
\[
\hat{E} \int_0^T \| \langle \hat{u}(t), \psi \rangle - k(\hat{n}, \hat{c}, \hat{u})_t \| ^2 dt = \lim_{j \to \infty} \hat{E} \int_0^T \| \langle \hat{u}_{\epsilon_j}(t), \psi \rangle - k_{\epsilon_j}(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})_t \| ^2 dt = 0,
\]
which indicates that \(\langle \hat{u}(t), \psi \rangle = q(\hat{n}, \hat{c}, \hat{u})_t\) holds for a.e. \(t \in [0, T]\) \(\hat{P}\)-a.s. Since two càdlàg functions equal for a.e. \(t \in [0, T]\) must be equal for all \(t \in [0, T]\), we see that \(\hat{u}(\cdot)\) is a martingale solution for the \(u\)-equation for all \(t \in [0, T]\), \(\hat{P}\)-a.s. In the sequel, let us prove \((5.43a)\) and \((5.43b)\) respectively.

Proof of \((5.43a)\). By c) of Lemma 5.5, we have
\[
\int_0^T \| \langle \hat{u}_{\epsilon_j}(t) - \hat{u}(t), \psi \rangle \| ^2 dt \leq \| \psi \|^2_{L_\infty(0, T; L^2)} \int_0^T \| \hat{u}_{\epsilon_j}(t) - \hat{u}(t) \|^2_{L^2} dt \to 0 \quad \text{as} \ j \to \infty, \ \hat{P}\text{-a.s.}
\]
Moreover, it follows from \((5.1e)\) that
\[
\hat{E} \left( \int_0^T \| \langle \hat{u}_{\epsilon_j}(t) - \hat{u}(t), \psi \rangle \| ^2 dt \right)^p \leq C, \quad \text{for any} \ p \geq 1,
\]
which indicates the uniform integrability of \(\int_0^T \| \langle \hat{u}_{\epsilon_j}(t) - \hat{u}(t), \psi \rangle \| ^2 dt\), and hence by Vitali Convergence Theorem we get \((5.43a)\).
Proof of (5.43b). For the linear terms involved in $q(\hat{n}, \hat{c}, \hat{u})_t$, similar to Step 1, by using the Lemma 5.5 and Lemma 5.1, one can prove that
\begin{equation}
\lim_{j \to \infty} \mathbb{E} \left[ \left| (\hat{u}_{\epsilon_j}(0) - \hat{n}(0), \psi_{|t=0}) \right|^2 \right] = 0,
\end{equation}
(5.44)
\begin{equation}
\lim_{j \to \infty} \int_0^T \mathbb{E} \left| \int_0^t \langle \nabla \hat{u}_{\epsilon_j} - \nabla \hat{u}, \nabla \psi \rangle ds \right|^2 dt = 0,
\end{equation}
(5.45)
\begin{equation}
\lim_{j \to \infty} \int_0^T \mathbb{E} \left| \int_0^t \langle \mathcal{P}(\hat{n}_{\epsilon_j} \nabla \Phi) - \mathcal{P}(\hat{n} \nabla \Phi), \psi \rangle ds \right|^2 dt = 0.
\end{equation}
(5.46)

It remains to deal with the remaining nonlinear terms and the stochastic integrals involved in $q(\hat{n}, \hat{c}, \hat{u})_t$. The proof will be divided into two parts.

**Part 1:** We show the convergence of nonlinear terms, that is,
\begin{equation}
\lim_{j \to \infty} \int_0^T \mathbb{E} \left| \int_0^t \langle \mathbf{L}_{\epsilon_j} \hat{u}_{\epsilon_j} \otimes \hat{u}_{\epsilon_j} - \hat{u} \otimes \hat{u}, \nabla \psi \rangle ds \right|^2 dt = 0,
\end{equation}
(5.47)
\begin{equation}
\lim_{j \to \infty} \int_0^T \mathbb{E} \left| \int_0^t \langle h(t, \hat{u}_{\epsilon_j}) - h(t, \hat{u}), \nabla \psi \rangle ds \right|^2 dt = 0.
\end{equation}
(5.48)

The convergence of the convection term is based on the following fact:
\begin{equation}
\mathbf{L}_{\epsilon_j} \hat{u}_{\epsilon_j} \otimes \hat{u}_{\epsilon_j} \to \hat{u} \otimes \hat{u} \text{ strongly in } L^1(0, T; L^1(D)) \text{ as } j \to \infty, \hat{P}\text{-a.s.}
\end{equation}
(5.49)

Indeed, it suffices to prove that $\mathbf{L}_{\epsilon_j} \hat{u}_{\epsilon_j} \to \hat{u}$ strongly in $L^2(0, T; L^2(D))$ as $j \to \infty, \hat{P}\text{-a.s.}$, and $\hat{u}_{\epsilon_j} \to \hat{u}$ strongly in $L^2(0, T; L^2(D))$ as $j \to \infty, \hat{P}\text{-a.s.}$ The later conclusion is ensured by (c) of Lemma 5.5 and it remains to prove the former one.

Recalling the following properties:
\begin{equation}
\| \mathbf{L}_\epsilon f \|_{L^2} \leq \| f \|_{L^2}, \| \mathbf{L}_\epsilon f - f \|_{L^2} \to 0 \text{ as } \epsilon \to 0, \text{ for any } f \in L^2_\sigma(D).
\end{equation}
We have
\begin{equation}
\int_0^T \| \mathbf{L}_{\epsilon_j} \hat{u}_{\epsilon_j}(t) - \hat{u}(t) \|_{L^2}^2 dt \leq \int_0^T \| \hat{u}_{\epsilon_j}(t) - \hat{u}(t) \|_{L^2}^2 dt + \int_0^T \| \mathbf{L}_{\epsilon_j} \hat{u}(t) - \hat{u}(t) \|_{L^2}^2 dt.
\end{equation}
(5.50)

The first term on the R.H.S. of (5.50) converges to zero almost surely according to Lemma 5.5. For the second term, on the one hand we get by (5.50) that $\| \mathbf{L}_{\epsilon_j} \hat{u}(t) - \hat{u}(t) \|_{L^2}^2 \to 0$ for $\hat{P} \otimes dt$-almost all $(\omega, t) \in \hat{\Omega} \times [0, T]$; on the other hand, we get by (5.1.1)
\begin{equation}
\int_0^T \| \mathbf{L}_{\epsilon_j} \hat{u}(t) - \hat{u}(t) \|_{L^2}^2 dt \leq C \left( \| \mathbf{L}_{\epsilon_j} \hat{u} \|_{L^2(0, T; L^2)}^2 + \| \hat{u} \|_{L^2(0, T; L^2)}^2 \right)
\end{equation}
\begin{equation} \leq C \| \hat{u} \|_{L^2(0, T; L^2)}^2 < \infty, \hat{P}\text{-a.s.}
\end{equation}
An application of the Vitali Convergence Theorem leads to $\int_0^T \| \mathbf{L}_{\epsilon_j} \hat{u}(t) - \hat{u}(t) \|_{L^2}^2 dt \to 0$ $\hat{P}$-a.s. Therefore,
\begin{equation}
\int_0^T \| \mathbf{L}_{\epsilon_j} \hat{u}_{\epsilon_j}(t) - \hat{u}(t) \|_{L^2}^2 dt \to 0, \text{ as } j \to \infty, \hat{P}\text{-a.s.}
\end{equation}
This proves (5.49).
Moreover, by uniform bound (5.1e), we get for $p \geq 4$
$$
\mathbb{E} \left| \int_0^t \left< \mathbf{L}_{\varepsilon_j} \hat{u}_{\varepsilon_j} \otimes \hat{u}_{\varepsilon_j} - \hat{u} \otimes \hat{u}, \nabla \psi \right> \, ds \right|^p \leq C \mathbb{E} \left( \| \hat{u}_{\varepsilon_j} \|^p_{L^2(0,T;L^2)} + \| \hat{u} \|^p_{L^2(0,T;L^2)} \right) \leq C,
$$
which infers the uniform integrability of $| \int_0^t \left< \mathbf{L}_{\varepsilon_j} \hat{u}_{\varepsilon_j} \otimes \hat{u}_{\varepsilon_j} - \hat{u} \otimes \hat{u}, \nabla \psi \right> \, ds |$. In view of (5.49) and the Vitali Convergence Theorem, we get
$$
\mathbb{E} \left| \int_0^t \left< \mathbf{L}_{\varepsilon_j} \hat{u}_{\varepsilon_j} \otimes \hat{u}_{\varepsilon_j} - \hat{u} \otimes \hat{u}, \nabla \psi \right> \, ds \right|^2 \to 0 \quad \text{as} \quad j \to \infty.
$$
Making use of the Dominated Convergence Theorem, (5.51) implies (5.47).

Since $(\hat{u}_{\varepsilon_j})_{j \geq 1}$ is uniformly bounded in $L^2(\Omega; L^2_{\text{loc}}(0, \infty; L^2_\sigma(D)))$, the sequence $\left( \| \hat{u}_{\varepsilon_j} \|^2_{L^2(0,T;L^2)} \right)_{j \geq 1}$ is uniformly integrable, and the Vitali Convergence Theorem infers that
$$
\hat{u}_{\varepsilon_j} \to \hat{u} \quad \text{strongly in} \quad L^2(\Omega; L^2_{\text{loc}}(0, \infty; (L^2_\sigma(D))^3)) \quad \text{as} \quad j \to \infty.
$$
Based on (5.52), in view of the assumptions on $h$ and $g$, we get
$$
h(t, \hat{u}_{\varepsilon_j}) \to h(t, \hat{u}) \quad \text{strongly in} \quad L^2(\Omega; L^2_{\text{loc}}(0, \infty; L^2_\sigma(D))) \quad \text{as} \quad j \to \infty,
$$
which combined with the Dominated Convergence Theorem lead to (5.48).

**Part 2:** Let us deal with the convergence of stochastic integrals. From (5.52) and the assumptions on $g$, $K$ and $G$, there hold
$$
g(t, \hat{u}_{\varepsilon_j}) \to g(t, \hat{u}) \quad \text{strongly in} \quad L^2(\Omega; L^2_{\text{loc}}(0, \infty; L^2_\sigma(D))) \quad \text{as} \quad j \to \infty,
$$
and
$$
\lim_{j \to \infty} \mathbb{E} \int_0^T \int_{|z| < 1} \int_D |K(\hat{u}_{\varepsilon_j}, z) - K(\hat{u}, z)|^2 \, dx \, d\mu(z) \, dt = 0,
$$
$$
\lim_{j \to \infty} \mathbb{E} \int_0^T \int_{|z| > 1} \int_D |G(\hat{u}_{\varepsilon_j}, z) - G(\hat{u}, z)|^2 \, dx \, d\mu(z) \, dt = 0.
$$
By using the BDG inequality and the fact that $(\hat{W}_{\varepsilon_j}, \hat{\pi}_{\varepsilon_j}) = (\hat{W}, \hat{\pi})$, we get from (5.53) that
$$
\lim_{j \to \infty} \mathbb{E} \left| \int_0^t \left< \mathcal{P} g(t, \hat{u}_{\varepsilon_j}) - \mathcal{P} g(t, \hat{u}), \psi \right> \, d\hat{W} \right|^2 \leq C \lim_{j \to \infty} \mathbb{E} \int_0^T \| g(t, \hat{u}_{\varepsilon_j}) \to g(t, \hat{u}) \|^2_{L^2} \, dt = 0.
$$
On the other hand, we get by using (5.1e) and the BDG inequality that
$$
\int_0^T \left( \mathbb{E} \left| \int_0^t \left< \mathcal{P} g(s, \hat{u}_{\varepsilon_j}) - \mathcal{P} g(s, \hat{u}), \psi \right> \, d\hat{W} \right|^2 \right) \, dt
\leq C \int_0^T \left( \mathbb{E} \left| \int_0^t \left( |\langle \mathcal{P} g(s, \hat{u}_{\varepsilon_j}), \psi \rangle|^2 + |\langle \mathcal{P} g(s, \hat{u}), \psi \rangle|^2 \right) \, ds \right|^2 \right) \, dt
\leq C \mathbb{E} \sup_{t \in [0,T]} \left( \| \hat{u}_{\varepsilon_j}(s) \|^4 + \| \hat{u}(s) \|^4_{L^2} + 1 \right) \leq C,
$$
which implies the uniform integrability of $\left| \int_0^t \langle \mathcal{P}g(s, \hat{u}_{\epsilon_j}) - \mathcal{P}g(s, \hat{u}), \psi \rangle d\hat{W} \right|^2$ for $t \in [0, T]$, and it follows from Vitali Convergence Theorem that

$$\lim_{j \to \infty} \int_0^T \mathbb{E} \left[ \left| \int_0^t \langle \mathcal{P}g(t, \hat{u}_{\epsilon_j}) - \mathcal{P}g(t, \hat{u}), \psi \rangle d\hat{W} \right|^2 \right] dt = 0.$$  \hspace{1cm} (5.56)

For the stochastic integral with respect to $\tilde{\pi}$, similar to (5.56), one can derive that

$$\lim_{j \to \infty} \int_0^T \mathbb{E} \left[ \left| \int_0^t \langle \mathcal{P}K(\hat{u}_{\epsilon_j}(s-), z) - \mathcal{P}K(\hat{u}(s-), z), \psi \rangle \tilde{\pi}(ds, dz) \right|^2 \right] dt = 0.$$  \hspace{1cm} (5.57)

For the last integral with respect to $\tilde{\pi}$, by making use of the facts of $\tilde{\pi}(dt, dz) = \tilde{\pi}(dt, dz) - \mu(dz)dt$, the Itô’s isometry and the assumption on $G$, we have

\begin{align*}
\mathbb{E} \left[ \left| \int_0^t \int_{|z| < 1} \langle \mathcal{P}g(\hat{u}_{\epsilon_j}(s-), z) - \mathcal{P}g(\hat{u}(s-), z), \psi \rangle \tilde{\pi}(ds, dz) \right|^2 \right] & \\
& \leq \mathbb{E} \left[ \int_0^t \int_{|z| \geq 1} |\langle \mathcal{P}g(\hat{u}_{\epsilon_j}(s-), z) - \mathcal{P}g(\hat{u}(s-), z), \psi \rangle| \mu(dz) ds \right] \\
& \leq \mathbb{E} \int_0^t \|\hat{u}_{\epsilon_j} - \hat{u}\|^2_{L^2} ds \to 0 \text{ as } j \to \infty.
\end{align*}

By applying (5.1e) and assumption on $G$, we also have

\begin{align*}
\mathbb{E} \left[ \left| \int_0^t \int_{|z| \geq 1} \langle \mathcal{P}G(\hat{u}_{\epsilon_j}(s-), z) - \mathcal{P}G(\hat{u}(s-), z), \psi \rangle \tilde{\pi}(ds, dz) \right|^2 \right] & \\
& \leq C \mathbb{E} \int_0^t \left( \|G(\hat{u}_{\epsilon_j}(s-), z)\|^2_{L^2} + \|G(\hat{u}(s-), z)\|^2_{L^2} \right) \mu(dz) ds \leq C.
\end{align*}

Thereby it follows from the Dominated Convergence Theorem that

$$\lim_{j \to \infty} \int_0^T \mathbb{E} \left[ \left| \int_0^t \int_{|z| \geq 1} \langle \mathcal{P}G(\hat{u}_{\epsilon_j}(s-), z) - \mathcal{P}G(\hat{u}(s-), z), \psi \rangle \tilde{\pi}(ds, dz) \right|^2 \right] dt = 0.$$  \hspace{1cm} (5.58)

Putting (5.44), (5.46), (5.47), (5.48), (5.56) and (5.57) together leads to (5.43b), which implies that $(\hat{W}, \tilde{\pi}, \hat{n}, \hat{c}, \hat{u})$ satisfies the $u$-equation.

**Step 4 (Weak entropy-energy inequality).** Thanks to Lemma 5.5 and Lemma 4.5, the new random variables $(\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j})$ satisfy the entropy-energy inequality (4.6) on the new probability space (see Lemma 4.5 for details), which can also be rewritten in the differential form

$$d\mathcal{E}[\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}](t) + \mathcal{D}[\hat{n}_{\epsilon_j}, \hat{c}_{\epsilon_j}, \hat{u}_{\epsilon_j}](t)dt \leq C + \|\hat{u}_{\epsilon_j}(t)\|^2_{L^2} dt + d\mathcal{M}_E^\prime(t),$$  \hspace{1cm} (5.58)
where \( \mathcal{E} [\cdot] \) and \( \mathcal{D} [\cdot] \) defined as before, and \( \mathcal{M}^\varepsilon_j \) is a real-valued square integrable martingale (thanks to the uniform bounds in Lemma 5.5) given by
\[
\mathcal{M}^\varepsilon_j = 2c^\dagger \int_0^t \langle \hat{u}_{\varepsilon_j}, \mathcal{P} g(s, \hat{u}_{\varepsilon_j})d\hat{W}_{\varepsilon_j} \rangle
+ c^\dagger \int_0^t \int_{|z| < 1} \|K(\hat{u}_{\varepsilon_j}(s, -), z)\|_{L^2}^2 \tilde{\pi}_{\varepsilon_j}(ds, dz) + c^\dagger \int_0^t \int_{|z| < 1} 2(\hat{u}_{\varepsilon_j}, \mathcal{P} K(\hat{u}_{\varepsilon_j}(s, -), z)) \tilde{\pi}_{\varepsilon_j}(ds, dz)
+ c^\dagger \int_0^t \int_{|z| > 1} \|G(\hat{u}_{\varepsilon_j}(s, -), z)\|_{L^2}^2 \tilde{\pi}_{\varepsilon_j}(ds, dz) + c^\dagger \int_0^t \int_{|z| > 1} 2(\hat{u}_{\varepsilon_j}, \mathcal{P} G(\hat{u}_{\varepsilon_j}(s, -), z)) \tilde{\pi}_{\varepsilon_j}(ds, dz).
\]
We also denote by \( \mathcal{M}_E \) the associated limit process by removing all of the subscripts \( \varepsilon_j \) in the definition of \( \mathcal{M}^\varepsilon_j \). Note that, by Lemma 5.5, \( \mathcal{M}^\varepsilon_j \) is a square integrable martingale. Multiplying both sides of (5.58) by any deterministic smooth test function \( \phi(t) \geq 0 \) with \( \phi(T) = 0 \), and then integrating over \([0, T]\), we get
\[
-\int_0^T \phi'(t) \mathcal{E} \left[ \hat{n}_{\varepsilon_j}, \hat{c}_{\varepsilon_j}, \hat{u}_{\varepsilon_j} \right] (t)dt + \int_0^T \phi(t) \mathcal{D} \left[ \hat{n}_{\varepsilon_j}, \hat{c}_{\varepsilon_j}, \hat{u}_{\varepsilon_j} \right] (t)dt 
\leq \phi(0) \mathcal{E} \left[ \hat{n}_{\varepsilon_j}, \hat{c}_{\varepsilon_j}, \hat{u}_{\varepsilon_j} \right] (0) + \int_0^T \phi(t) \|\hat{u}_{\varepsilon_j}(t)\|_{L^2}^2 dt + C \int_0^T \phi(t) dt + \int_0^T \phi(t) d\mathcal{M}_E^\varepsilon_j.
\]
To take the limit in the first integral of (5.59), we first get by (5.28) and the inequality
\[-\frac{1}{e} \leq x \ln x \leq \frac{3}{2} x^{\frac{3}{2}} \text{ for all } x > 0\]
that
\[-\operatorname{Leb}(D) \frac{1}{e} \leq \int_D \hat{n}_{\varepsilon_j} \ln \hat{n}_{\varepsilon_j} dx \leq \frac{3}{2} \int_D \hat{n}_{\varepsilon_j}^\frac{3}{2} dx < \infty, \quad \hat{P}\text{-a.s.}\]
where \( \operatorname{Leb}(D) \) denotes the Lebesgue measure of \( D \). On the other hand, it follows from \( c \) of Lemma 5.5 that \( \|\hat{n}_{\varepsilon_j} - \hat{n}\|_{L^2} \to 0 \hat{P} \otimes dt\text{-almost surely. Thus by extracting a subsequence denoted by itself, we obtain}
\[
\int_D \hat{n}_{\varepsilon_j}(t) \ln \hat{n}_{\varepsilon_j}(t) dx \to \int_D \hat{n}(t) \ln \hat{n}(t) dx \quad \text{as } j \to \infty, \quad \hat{P} \otimes dt\text{-almost all } (\omega, t).
\]
Direct calculation shows that
\[
\Delta \Psi(c) = \frac{\Delta c}{\sqrt{\theta(c)}} + \frac{\theta'(c) - 2\theta(c)\nabla c}{2\theta^2(c)}|\nabla c|^2.
\]
Based on this identity, we deduce from (4.5), Remark 4.2 and Lemma 4.6 that \( \Psi(c) \in L^p(\Omega; L^2(0, T; W^{2,2}(D))) \), which together with the compact embedding \( L^2(0, T; W^{1,2}(D)) \subset L^2(0, T; L^2(D)) \) imply that \( \nabla \Psi(\hat{c}) \to \nabla \Psi(c) \) strongly in \( L^p(\Omega; L^2(0, T; L^2(D))) \), and so one can extract a subsequence denoted by itself such that
\[
\int_D \frac{1}{2} |\nabla \Psi(\hat{c}_j(t))|^2 dx \to \int_D \frac{1}{2} |\nabla \Psi(c(t))|^2 dx \quad \text{as } j \to \infty, \quad \hat{P} \otimes dt\text{-almost all } (\omega, t).
\]
Furthermore, by virtue of \( c \) of Lemma 5.5, we have
\[
\int_D |\hat{u}_{\varepsilon_j}(t)|^2 dx \to \int_D |\hat{u}(t)|^2 dx \quad \text{as } j \to \infty, \quad \hat{P} \otimes dt\text{-almost all } (\omega, t).
\]
From (5.60)-(5.62), we deduce that $\mathcal{E}[\tilde{n}_{\epsilon_j}, \tilde{c}_{\epsilon_j}, \tilde{u}_{\epsilon_j}] \to \mathcal{E}[\tilde{n}, \tilde{c}, \tilde{u}]$ as $j \to \infty$, $\hat{P} \otimes dt$-almost all $(\omega, t)$, which combined with the Dominated Convergence Theorem yield that
\[
- \int_0^T \phi'(t) \mathcal{E}[\tilde{n}_{\epsilon_j}, \tilde{c}_{\epsilon_j}, \tilde{u}_{\epsilon_j}] (t) dt \to - \int_0^T \phi'(t) \mathcal{E}[\tilde{n}, \tilde{c}, \tilde{u}] (t) dt \text{ as } j \to \infty, \quad \hat{P}\text{-a.s.}
\]
Similarly,
\[
\phi(0) \mathcal{E}[\tilde{n}_{\epsilon_j}, \tilde{c}_{\epsilon_j}, \tilde{u}_{\epsilon_j}] (0) \to \phi(0) \mathcal{E}[\tilde{n}, \tilde{c}, \tilde{u}] (0) \text{ as } j \to \infty, \quad \hat{P}\text{-a.s.}
\]
Taking the limit $j \to \infty$ in (5.59), and using the Fatou’s lemma, the lower-continuous of norms as well as Lemma 5.1, we get
\[
- \int_0^T \phi'(t) \mathcal{E}[\tilde{n}, \tilde{c}, \tilde{u}] (t) dt + \int_0^T \phi(t) \mathcal{D}[\tilde{n}, \tilde{c}, \tilde{u}] (t) dt \leq \phi(0) \mathcal{E}[\tilde{n}, \tilde{c}, \tilde{u}] (0)
\]
Moreover, by applying the BDG inequality, the uniform bounds in Lemma 5.1, the Lemma (5.63)\[\hat{E} \left[ \int_0^T \phi(t) d\mathcal{M}^{\phi}_{E} - \int_0^T \phi(t) d\mathcal{M}_{E} \right]^2 \to 0 \text{ as } j \to \infty, \quad \hat{P}\text{-a.s.}\]
which implies that, by extracting a subsequence (still denoted by itself), there holds
\[
\lim_{j \to \infty} \int_0^T \phi(t) d\mathcal{M}^{\phi}_{E} = \int_0^T \phi(t) d\mathcal{M}_{E}, \quad \hat{P}\text{-a.s.}
\]
The desired energy-type inequality is a consequence of (5.63) and (5.64). For the martingale $\mathcal{M}_{E}$, by applying the BDG inequality, the assumptions on $K$ and $G$ and the uniform bound in Lemma 5.1, we have
\[
\hat{E} \sup_{t \in [0,T]} |\mathcal{M}_{E}|^p \leq C \left[ \hat{E} \left( \int_0^T |\langle \tilde{u}, \mathcal{P}g(s, \tilde{u}) \rangle|^2 dt \right)^{\frac{p}{2}} + \hat{E} \left( \int_0^T (1 + \|\tilde{u}(s)\|_{L^2}^4) ds \right)^{\frac{p}{2}} \right.
\]
\[
+ \hat{E} \left( \int_0^T \int_{|z| > 1} |\langle \tilde{u}, \mathcal{P}G(\tilde{u}(s-), z) \rangle|^2 \mu(dz) ds \right)^{\frac{p}{2}}
\]
\[
+ \hat{E} \left( \int_0^T \int_{|z| < 1} |\langle \tilde{u}, \mathcal{P}K(\tilde{u}(s-), z) \rangle|^2 \mu(dz) ds \right)^{\frac{p}{2}} \right]
\]
\[
\leq C \hat{E} \left( \sup_{t \in [0,T]} \|\tilde{u}(t)\|_{L^2}^{2p} + 1 \right)
\]
\[
\leq C \left( \hat{E} \left[ \int_D \left( \tilde{n}(0) \ln \tilde{n}(0) + \frac{1}{2} |\nabla \Psi(\tilde{c}(0))|^2 + c^* |\tilde{u}(0)|^2 \right) dx \right]^p + 1 \right),
\]
which implies the inequality (1.7). The proof of Theorem 1.2 is now completed. \[\square\]
6. Conflict of Interest Statement

The authors declared that they have no conflicts of interest to this work.

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