Unified Analysis for Variational Time Discretizations of Higher Order and Higher Regularity Applied to Non-stiff ODEs

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Abstract

We present a unified analysis for a family of variational time discretization methods, including discontinuous Galerkin methods and continuous Galerkin–Petrov methods, applied to non-stiff initial value problems. Besides the well-definedness of the methods, the global error and superconvergence properties are analyzed under rather weak abstract assumptions which also allow considerations of a wide variety of quadrature formulas. Numerical experiments illustrate and support the theoretical results.

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1 Introduction

We study variational time discretization methods applied to non-stiff initial value problems of the form

\[ u'(t) = f(t, u(t)), \quad u(t_0) = u_0 \in \mathbb{R}^d, \quad (1.1) \]

where \( f \) is supposed to be sufficiently smooth and to satisfy a Lipschitz condition on the second variable with constant \( L > 0 \). Here, an initial value \( u_0 \) is given at \( t = t_0 \). The system of ordinary differential equations (ODEs) is considered on a time interval \( I = (t_0, t_0 + T] \) with arbitrary but fixed length \( T > 0 \).

Probably the most popular variational discretization schemes for solving (1.1) numerically are discontinuous Galerkin (dG) and continuous Galerkin–Petrov (cGP) methods. Both have been studied in the literature for many decades. However, an important part for the design of a then fully computable discretization, namely the use of quadrature methods for approximate integration, is often only marginally considered or the assumptions on the quadrature formulas are quite restrictive and, thus, allow a small variability of quadrature formulas only. So, for example, the number of quadrature points is supposed to equal the dimension of the local ansatz space of the discrete solution, cf. [6, 5], or the number of independent variational conditions, cf. [13, 12]. Often also the investigations are restricted to special Gauss, Gauss–Radau, or Gauss–Lobatto formulas. A quite general setting is considered in [8] where so-called internal and external quadrature formulas are studied. External quadrature means that integrals over products of \( f \) with test functions are approximated by quadrature rules. If \( f \) is replaced by a polynomial approximation before (numerical) integration then internal quadrature formulas appear. Since in [8] the dynamical behavior of the schemes is analyzed, it is assumed that at least for Dahlquist’s test problem \( u' = \lambda u \) all integrals are integrated exactly which again is quite restrictive.

In this paper we try to keep the requirements on the approximation operators involved in the definition of the discrete method as low as possible. Our scope is to figure out how the

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approximation properties of these (external and internal) operators affect the error behavior of the numerical solution. Thereby our studies are not restricted to the dG or cGP method but cover the whole family of variational discretization methods recently introduced in \[3, 2\]. The used assumptions are quite general and abstract in order enable the study of a wide variety of methods. Especially, we allow that the right-hand side is approximated by a composition of various interpolation operators (interpolation cascade) or that the quadrature formulas also use derivative values. Therefore all variational methods considered in \[2\] and their convergence results can be verified by a rigorous and unified error analysis in the non-stiff case now.

The paper is structured as follows. In Section 2 the variational time discretization methods are formulated. The well-definedness of the method formulation is studied in Section 3 where the existence of unique discrete solutions is proven under appropriate conditions. Section 4 is devoted to the error analysis. Here, at first, the global error is bounded by a sum of rather basic quantities describing the approximation properties of the involved operators. Thereafter, superconvergence results in the time mesh nodes are derived in Section 5. Finally, in Section 6 we use numerical experiments to illustrate the convergence behavior of the variational time discretization methods and show that the proven estimates are sharp.

Notation: In this paper \(C\) denotes a generic constant independent of the time mesh interval length. To describe the vector-valued case \((d > 1)\) in an easy way, let \((\cdot, \cdot)\) be the standard inner product and \(\| \cdot \|\) the Euclidean norm on \(\mathbb{R}^d\). Besides let \(c_j\), \(1 \leq j \leq d\), be the \(j\)th standard unit vector in \(\mathbb{R}^d\).

For an arbitrary interval \(J\) and positive integers \(q\), the spaces of continuous and \(k\) times continuously differentiable \(\mathbb{R}^q\)-valued functions on \(J\) are written as \(C(J, \mathbb{R}^q)\) and \(C^k(J, \mathbb{R}^q)\), respectively. Furthermore, the space of square-integrable \(\mathbb{R}^q\)-valued functions shall be denoted by \(L^2(J, \mathbb{R}^q)\) or, for convenience, sometimes also by \(C^{-1}(J, \mathbb{R}^q)\). For non-negative integers \(s\), we write \(P_s(J, \mathbb{R}^q)\) for the space of \(\mathbb{R}^q\)-valued polynomials of degree less than or equal to \(s\). For \(q = 1\), we sometimes omit \(\mathbb{R}^q\). Further notation will be introduced later at the beginning of the sections where it is needed.

## 2 Formulation of the methods

The interval \(I\) is decomposed by

\[
t_0 < t_1 < \cdots < t_{N-1} < t_N = t_0 + T
\]

into \(N\) disjoint subintervals \(I_n := (t_{n-1}, t_n]\), where \(n = 1, \ldots, N\). Furthermore, we set

\[
\tau_n := t_n - t_{n-1}, \quad \tau := \max_{1 \leq n \leq N} \tau_n.
\]

For convenience and to simplify the notation in some proofs, we assume \(\tau_n \leq 1\) which is not really a restriction since we are anyway interested in the asymptotic error behavior for \(\tau \to 0\). For any piecewise continuous functions \(v\) we define by

\[
v(t_{n}^+) := \lim_{t \to t_{n}^+} v(t), \quad v(t_{n}^-) := \lim_{t \to t_{n}^-} v(t), \quad [v]_n := v(t_{n}^+) - v(t_{n}^-)
\]

the one-sided limits and the jump of \(v\) at \(t_n\).

We now present a slight generalization of the variational time discretization methods \(VTD_k^r\) investigated in \[2\]. Let \(r, k \in \mathbb{Z}, 0 \leq k \leq r\). Then the local version \((I_n,\text{-problem})\) of the numerical method reads as follows:

Given \(U(t_{n-1}) \in \mathbb{R}^d\), find \(U \in P_r(I_n, \mathbb{R}^d)\) such that

\[
\begin{align*}
U(t_{n-1}^+) &= U(t_{n-1}^-), & \text{if } k \geq 1, \\
U^{(i+1)}(t_{n}^-) &= \left. \frac{d^i}{dt^i} (f(t, U(t))) \right|_{t=t_n}, & \text{if } k \geq 2, i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1, \\
U^{(i+1)}(t_{n-1}^+) &= \left. \frac{d^i}{dt^i} (f(t, U(t))) \right|_{t=t_{n-1}^-}, & \text{if } k \geq 3, i = 0, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor - 1,
\end{align*}
\]
and

\[
\mathcal{J}_n \left( \left( U', \varphi \right) \right) + \delta_{0,k} \left( \left[ U \right]_{n-1} \varphi(t_{n-1}') \right) = \mathcal{J}_n \left( \left( \mathcal{I}_n f \left( \cdot, U \left( \cdot \right) \right), \varphi \right) \right) \quad \forall \varphi \in P_{r-k}(I_n, \mathbb{R}^d) \tag{2.1d}
\]

where \( U(t_0') = u_0 \) and \( \delta_{i,j} \) is the Kronecker symbol.

Here, \( \mathcal{J}_n \) denotes an integrator that typically represents either the integral over \( I_n \) or the application of a quadrature formula for approximate integration. Details will be described later on. Moreover, \( \mathcal{I}_n \) could be used to model some projection/interpolation of \( f \) or the usage of some special (internal) quadrature rules even if \( \mathcal{J}_n \) is just the integral.

### 3 Existence and Uniqueness

First of all, we agree that both \( \mathcal{J}_n \) and \( \mathcal{I}_n \) are local versions (obtained by transformation) of appropriate linear operators \( \hat{\mathcal{J}} \) and \( \hat{\mathcal{I}} \) given on the closure of the reference interval \((-1, 1]\). However, \( \mathcal{J}_n \) is an approximation of the integral operator while \( \mathcal{I}_n \) approximates the identity operator. Thus, the transformations scale quite differently. More precisely, let

\[
T_n : (-1, 1] \rightarrow I_n, \quad \hat{t} \mapsto \frac{t_n + t_{n-1}}{2} + \frac{\tau_n}{2} \hat{t},
\]

(3.1)

denote the affine transformation that maps the reference interval \((-1, 1]\) on an arbitrary mesh interval \( I_n = [t_{n-1}, t_n] \). Furthermore, let \( k_{\mathcal{J}} \) and \( k_{\mathcal{I}} \) be the smallest non-negative integers such that \( \hat{\mathcal{J}} \) and \( \hat{\mathcal{I}} \) are well-defined for functions in \( C^{k_{\mathcal{J}}}([-1, 1]) \) and \( C^{k_{\mathcal{I}}}([-1, 1]) \), respectively. Then we have for all \( \varphi \in C^{k_{\mathcal{J}}}([T_n, \mathbb{R}^d]) \) and for all \( \psi \in C^{k_{\mathcal{I}}}([\hat{T}_n, \mathbb{R}^d]) \) that

\[
\mathcal{J}_n[\varphi] = \hat{\mathcal{J}}[\varphi \circ T_n](T_n)' = \frac{\tau_n}{2} \hat{\mathcal{J}}[\varphi \circ T_n] \quad \text{and} \quad \mathcal{I}_n \psi = (\hat{\mathcal{I}}(\psi \circ T_n)) \circ T_n^{-1}.
\]

Of course, these operators act component-wise when applied to vector-valued functions. Moreover, we suppose for all non-negative integers \( l \) that \( \hat{\mathcal{I}} \hat{\psi} \in C^l([-1, 1]) \) holds for all \( \hat{\psi} \in C^{\max(k_{\mathcal{J}} - 1)}([-1, 1]), \) i.e., \( \hat{\mathcal{I}} \hat{\psi} \) is at least as smooth as \( \hat{\psi} \).

The study of existence and uniqueness of solutions to (2.1) as well as the later error analysis is strongly connected with the following operator.

Let \( \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} : C^{k_{\mathcal{J}} + 1}([-1, 1]) \rightarrow P_1([-1, 1]) \) where \( k_{\mathcal{J}} := \max \left\{ \left\lfloor \frac{k}{2} \right\rfloor - 1, k_{\mathcal{J}}, k_{\mathcal{I}} \right\} \) be defined by

\[
\left( \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} \hat{\psi} \right)(-1^+) = \hat{\psi}(-1^+), \quad \text{if } k \geq 1, i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \tag{3.2a}
\]

\[
\left( \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} \hat{\psi} \right)(+1^-) = \hat{\psi}(+1^-), \quad \text{if } k \geq 1, i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \tag{3.2b}
\]

\[
\hat{\mathcal{J}} \left[ \left( \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} \hat{\psi} \right)(\hat{t}) \right] + \delta_{0,k} \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} \hat{\psi}(-1^+) \hat{\psi}(-1^+) = \hat{\mathcal{J}} \left( \hat{\mathcal{I}}(\hat{\psi}) \hat{\psi} \right) + \delta_{0,k} \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} \hat{\psi}(-1^+) \hat{\psi}(-1^+) \quad \forall \hat{\psi} \in P_{r-k}([-1, 1]). \tag{3.2c}
\]

**Assumption 1**

As before let \( r, k \in \mathbb{Z}, 0 \leq k \leq r \), be the parameters of the method. The integrator \( \hat{\mathcal{J}} \) is such that \( \hat{\psi} \in P_{r-\max(1, k)}([-1, 1]) \) and

\[
\hat{\mathcal{J}} \left[ (1 - \hat{t}) \left( 1 + \hat{t} \right) \left\lfloor \frac{\hat{t}}{1 + \hat{t}} \right\rfloor \hat{\psi} \right] = 0 \quad \forall \hat{\psi} \in P_{r-\max(1, k)}([-1, 1])
\]

imply \( \hat{\psi} \equiv 0 \). Note that the absolute value in the exponent is needed only for \( k = 0 \).

**Lemma 3.1**

Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r \). Suppose that Assumption 1 holds. Then \( \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} \) given by (3.2) is well-defined. If \( \hat{\mathcal{I}} \) preserves polynomials up to degree \( \bar{r} \) then \( \hat{\mathcal{J}}^{\mathcal{J}, \mathcal{I}} \) preserves polynomials up to degree \( \min\{\bar{r} + 1, r\} \).
Proof: We have to study existence and uniqueness of the approximation. We need to determine $r + 1$ coefficients of a polynomial in $P_r([-1, 1])$. To this we have
\[(\lceil \frac{k-1}{2} \rceil + 1) + \lfloor \frac{k}{2} \rfloor + (r - k + 1) = \frac{k}{2} + \frac{k-1}{2} - \frac{1}{2} + 2 + r - k = r + 1\]
linear conditions. So, it remains to prove that for $\hat{v} \equiv 0$ we obtain $\hat{\mathcal{J}}_{I'}\hat{v} \equiv 0$ as uniquely defined approximation. The conditions with $\hat{v} \equiv 0$ read as follows
\[
\begin{align*}
(\hat{\mathcal{J}}_{I'}\hat{v})^{(i)}(-1^+) &= 0, & \text{if } k \geq 1, i = 0, \ldots, \lfloor \frac{k-1}{2} \rfloor, & \text{(3.3a)} \\
(\hat{\mathcal{J}}_{I'}\hat{v})^{(i)}(+1^-) &= 0, & \text{if } k \geq 2, i = 1, \ldots, \lfloor \frac{k}{2} \rfloor. & \text{(3.3b)} \\
\hat{\mathcal{J}} \left[ (\hat{\mathcal{J}}_{I'}\hat{v})' \right] + \delta_{0,k}\hat{\mathcal{J}}_{I'}\hat{v}(-1^+) &= 0 & \forall \hat{v} \in P_{r-k}([-1, 1]). & \text{(3.3c)}
\end{align*}
\]
Now, there are two cases. For $k = 0$ testing in (3.3c) with $\hat{v} = (1 + i)\hat{\psi}$, $\hat{\psi} \in P_{r-1}([-1, 1])$, we obtain $\hat{\mathcal{J}} \left[ (\hat{\mathcal{J}}_{I'}\hat{v})' \right] = 0$. So, by Assumption 1 we get $(\hat{\mathcal{J}}_{I'}\hat{v})' \equiv 0$. Using this and choosing in (3.3c) the test function $\hat{\psi} \equiv 1$, it furthermore follows $\hat{\mathcal{J}}_{I'}\hat{v}(-1^+) = 0$ which then yields $\hat{\mathcal{J}}_{I'}\hat{v} \equiv 0$.

Otherwise, for $k \geq 1$ we see from (3.3a), (3.3b) that $(\hat{\mathcal{J}}_{I'}\hat{v})'(i) = (1 - i)\lfloor \frac{\hat{v}}{2} \rfloor (1 + i)\lfloor \hat{v} \rfloor \hat{\psi}(i)$ with a polynomial $\hat{\psi} \in P_{r-k}([-1, 1])$. Because of Assumption 1 the variational condition (3.3c) implies that $\hat{\psi} \equiv 0$. Thus, $(\hat{\mathcal{J}}_{I'}\hat{v})' \equiv 0$. Additionally using (3.3a) for $i = 0$, it again follows that $\hat{\mathcal{J}}_{I'}\hat{v} \equiv 0$.

Hence, either way $\hat{\mathcal{J}}_{I'}\hat{v} \equiv 0$ is the unique approximation of $\hat{v} \equiv 0$ and so in general the approximation is always uniquely determined.

Using the uniqueness of the approximation, the second statement can be easily verified. \qed

Lemma 3.2
Let $r \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $0 \leq k \leq r$. Suppose that Assumption 1 holds. Then the mapping
\[
\hat{\psi} \mapsto \left( \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \left\| \hat{\psi}^{(i)}(-1^+) \right\| + \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \left\| \hat{\psi}^{(i)}(+1^-) \right\| + \sum_{i=0}^{r-k} \left\| \hat{\mathcal{J}} \left[ \hat{\psi}(i) (1 + i)^\gamma \right] \right\| \right)
\]
defines a norm on $P_{r-k}([-1, 1], \mathbb{R}^d)$.

Proof: The absolute homogeneity and the triangle inequality follow from the linearity of the integrator and of the derivatives along with the respective properties of the norm $\left\| \cdot \right\|$. Finally, to show the positive definiteness, we can adapt the arguments that are used in the proof of Lemma 3.1 to show $(\hat{\mathcal{J}}_{I'}\hat{v})' \equiv 0$ if $\hat{v} \equiv 0$. Note that precisely those $r$ conditions of (3.2) that uniquely define $(\hat{\mathcal{J}}_{I'}\hat{v})'$ also appear on the right-hand side of the mapping (3.4). \qed

Lemma 3.3
Let $r \in \mathbb{N}_0$. Suppose that Assumption 1 holds. Then the mapping
\[
\hat{\psi} \mapsto \sum_{i=0}^{r} \left\| \hat{\mathcal{J}} \left[ \hat{\psi}'(i) (1 + i)^\gamma \right] + \delta_{0,i}\hat{\psi}(-1^+) \right\|
\]
defines a norm on $P_r([-1, 1], \mathbb{R}^d)$.

Proof: As in the proof of Lemma 3.2, the absolute homogeneity and the triangle inequality are clear. Moreover, by an adaption of the $k = 0$ case of the proof of Lemma 3.1 we get the positive definiteness. \qed

Recall from (3.1) the affine transformation $T_n$ mapping the reference interval $(-1, 1)$ on an arbitrary mesh interval $I_n = (t_{n-1}, t_n)$ Then, of course, any $\varphi \in P_r(I_n, \mathbb{R}^d)$ is associated with a polynomial $\hat{\varphi} \in P_r((-1, 1], \mathbb{R}^d)$ by
\[
\hat{\varphi}(i) := (\varphi \circ T_n)(i) = \varphi(T_n(i)) \quad \forall i \in (-1, 1).
\]
The chain rule yields
\[
\hat{\varphi}'(\hat{t}) = \varphi'(t)T_n'(\hat{t}) = \frac{\tau_n}{2} \varphi'(t) \quad \text{with} \quad t = T_n(\hat{t}) = \frac{t_n + t_{n-1}}{2} + \frac{\tau_n}{2} \hat{t},
\]
where \(\hat{\varphi}\) and \(T_n\) are differentiated with respect to \(\hat{t}\) and \(\varphi\) is differentiated with respect to \(t\). The inverse of \(T_n\) and its derivative are given by
\[
T_n^{-1}(t) = \frac{2t - (t_n + t_{n-1})}{\tau_n}, \quad (T_n^{-1})'(t) = \frac{2}{\tau_n}
\]
for all \(t \in I_n\).

**Lemma 3.4**

Let \(r, k \in \mathbb{Z}, 0 \leq k \leq r\). Suppose that Assumption 1 holds. Then there is a positive constant \(\kappa_{r,k}\), independent of \(\tau_n\), such that
\[
\int_{I_n} \|\varphi'(t)\| \, dt \leq \tau_n \sup_{t \in I_n} \|\varphi'(t)\|
\]
\[
\leq \kappa_{r,k} \left( \sum_{i=1}^{\lfloor \frac{r-k}{2} \rfloor} (\frac{\tau_n}{2})^i \|\varphi^{(i)}(t_{n-1}^+)\| + \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} (\frac{\tau_n}{2})^i \|\varphi^{(i)}(t_n^-)\| + \sum_{i=\delta_{0,k}}^{r-k} \|\mathcal{F}_n[\varphi'(t) (1 + T_n^{-1}(t))^i]\| \right)
\]
and
\[
\sup_{t \in I_n} \|\varphi(t)\| \leq \min \{ \|\varphi(t_{n-1}^+)\|, \|\varphi(t_n^-)\| \} + \int_{I_n} \|\varphi'(t)\| \, dt
\]
for all \(\varphi \in P_r(I_n, \mathbb{R}^d)\).

**Proof:** Since the statement is obviously true for \(r = 0\), we can assume \(r \geq 1\) in the following. By the fundamental theorem of calculus we have for \(t \in I_n\) that
\[
\varphi(t) = \varphi(t_{n-1}^+) + \int_{t_{n-1}}^t \varphi'(s) \, ds = \varphi(t_n^-) - \int_t^{t_n} \varphi'(s) \, ds.
\]
Therefore, it follows that
\[
\sup_{t \in I_n} \|\varphi(t)\| \leq \|\varphi(t_{n-1}^+)\| + \sup_{t \in I_n} \left\| \int_{t_{n-1}}^t \varphi'(s) \, ds \right\| \leq \|\varphi(t_{n-1}^+)\| + \int_{I_n} \|\varphi'(s)\| \, ds
\]
and analogously
\[
\sup_{t \in I_n} \|\varphi(t)\| \leq \|\varphi(t_n^-)\| + \int_{I_n} \|\varphi'(s)\| \, ds
\]
which already gives the second statement.

On the finite dimensional function space \(P_{r-1}((-1, 1], \mathbb{R}^d)\), both
\[
\hat{\psi} \mapsto \sup_{\hat{t} \in (-1, 1]} \|\hat{\psi}(\hat{t})\| \quad \text{and} \quad (3.4)
\]
define equivalent norms. Hence, for all \(\hat{\psi} \in P_{r-1}((-1, 1], \mathbb{R}^d)\) we have
\[
\sup_{\hat{t} \in (-1, 1]} \|\hat{\psi}(\hat{t})\| \leq \frac{\kappa_{r,k}}{2} \left( \sum_{i=0}^{\lfloor \frac{r-k-1}{2} \rfloor} \|\hat{\psi}^{(i)}(1)^{+}\| + \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \|\hat{\psi}^{(i)}(-1)^-\| + \sum_{i=\delta_{0,k}}^{r-k} \|\hat{F}[\hat{\psi}(\hat{t}) (1 + \hat{t})^i]\| \right)
\]
where \(\kappa_{r,k}\) is a positive constant that is independent of \(\tau_n\).
So, using the transformation \( T_n \) given in (3.1), the last estimate together with integral transformations (which analogously also hold for the integrator) and the chain rule imply that

\[
\int_{I_n} \| \varphi'(t) \| \, dt \leq \tau_n \sup_{t \in I_n} \| \varphi'(t) \| = \tau_n \sup_{t \in (-1,1)} \| \varphi(\dot{i}) \| = 2 \left( \sup_{t \in (-1,1)} \| \varphi'(\dot{i}) \| \right)
\]

\[
\leq \kappa_{r,k} \left( \sum_{i=1}^{r} \| \varphi^{(i)}(-1^+) \| + \sum_{i=1}^{r} \| \varphi^{(i)}(+1^-) \| + \sum_{i=d_{0,k}}^{r} \| \widehat{\varphi}'(\dot{i}) (1 + \dot{i})^1 \| \right)
\]

\[
= \kappa_{r,k} \left( \sum_{i=1}^{r} (\frac{\tau_n}{\tau})^{\dot{i}} \| \varphi^{(i)}(t_{n-1}^+) \| + \sum_{i=1}^{r} (\frac{\tau_n}{\tau})^{\dot{i}} \| \varphi^{(i)}(t_n^-) \| + \sum_{i=d_{0,k}}^{r} \| \widehat{\varphi}'(t_n^-) \| \right)
\]

for all \( \varphi \in P_r(I_n, \mathbb{R}^d) \). This completes the proof. \( \square \)

**Lemma 3.5**

*Let \( r \in \mathbb{N}_0 \). Suppose that Assumption 1 holds. Then there is a positive constant \( \kappa_r \), independent of \( \tau_n \), such that*

\[
\sup_{t \in I_n} \| \varphi(t) \| \leq \kappa_r \sup_{i \in I_n} \| \widehat{\varphi}'(t_n^-) (1 + T_n^{-1}(t_n^+)) \| + \delta_{0,i} \varphi(t_{n-1}^-)
\]

*for all \( \varphi \in P_r(I_n, \mathbb{R}^d) \).*

**Proof:** Similar to the proof of Lemma 3.4 the statement follows from Lemma 3.3 using the norm equivalence on the finite dimensional space \( P_r((-1,1], \mathbb{R}^d) \) along with transformations via \( T_n \) given in (3.1) between the reference interval \((-1,1]\) and the actual interval \( I_n \). \( \square \)

In the following, we assume that the inverse inequalities

\[
\sup_{t \in I_n} \| V^{(i)}(t) \| \leq C_{\text{inv}} \left( \frac{\tau_n}{\tau} \right)^{i} \sup_{t \in I_n} \| V(t) \| \quad \forall V \in P_r(I_n, \mathbb{R}^d), i \in \mathbb{N}, \quad (3.7)
\]

hold true where \( C_{\text{inv}} > 0 \) is independent of \( \tau_n \) but may depend on \( l \) and \( r \). The proof is standard and uses transformations to together with norm equivalences on finite dimensional spaces on the reference interval.

For the proof of the existence and uniqueness of solutions of (2.1) we need some more assumptions.

**Assumption 2**

The reference integrator \( \widehat{\mathcal{F}} \) satisfies

\[
\left\| \widehat{\mathcal{F}}(\varphi) \right\| \leq C_0 \sup_{i \in [-1,1]} \| \varphi^{(j)}(i) \| \quad \forall \varphi \in C^{k_s}([-1,1], \mathbb{R}^d)
\]

where as before \( k_s \geq 0 \) is the smallest integer such that \( \widehat{\mathcal{F}} \) is well-defined on \( C^{k_s}([-1,1]) \). This means by transformation that

\[
\left\| \mathcal{F}_n(\varphi) \right\| \leq C_0 \left( \frac{\tau_n}{\tau} \right)^{k_s} \sup_{i \in I_n} \| \varphi^{(j)}(i) \| \quad \forall \varphi \in C^{k_s}(T_n, \mathbb{R}^d)
\]

holds for the local integrators \( \mathcal{F}_n, 1 \leq n \leq N \).

**Assumption 3**

Let \( 0 \leq l \leq k_s \). The reference approximation operator \( \widehat{\mathcal{T}} \) satisfies

\[
\sup_{i \in [-1,1]} \left\| \left( \widehat{\mathcal{T}} \varphi \right)^{(l)}(i) \right\| \leq C_1 \sup_{j=0}^{\max\{k_s, l\}} \| \varphi^{(j)}(i) \| \quad \forall \varphi \in C^{\max\{k_s, l\}}([-1,1], \mathbb{R}^d)
\]
where as before $k_T \geq 0$ is the smallest integer such that $\mathcal{I}$ is well-defined on $C^{k_T}([-1,1])$. This means by transformation that

$$\sup_{t \in I_n} \| (\mathcal{I}_n \varphi)^{(j)}(t) \| \leq \mathcal{C}_1 \sum_{j=0}^{\max{k_T,l}} (\mathcal{Z}_n)^{j-l} \sup_{t \in I_n} \| \varphi^{(j)}(t) \| \quad \forall \varphi \in C^{\max{k_T,l}}(\mathcal{T}_n, \mathbb{R}^d)$$

holds for the local approximation operators $\mathcal{I}_n$, $1 \leq n \leq N$.

**Assumption 4**

For $0 \leq i \leq k_J = \max \{ \lfloor \frac{k}{2} \rfloor - 1, k_R, k_T \}$, it holds for sufficiently smooth functions $v, w$ that

$$\left\| \frac{d^i}{dt^i} \left( f(t,v(t)) - f(t,w(t)) \right) \right\|_{L^\infty} \leq \mathcal{C}_2 \sum_{l=0}^{i} \| (v-w)^{(l)}(s) \| \quad \text{for a.e. } s \in \mathcal{T} = [t_0, t_0 + T].$$

Here $\mathcal{C}_2$ depends on $k_J$ and $f$.

**Remark 3.6**

Sufficient conditions for Assumption 4 would be, e.g.,

(i) for $k_J = 0$: $f$ satisfies a Lipschitz condition on the second variable with constant $L > 0$,

(ii) for $k_J \geq 1$: $f$ is affine linear in $u$, i.e., $f(t,u(t)) = A(t)u(t) + b(t)$, and $\| A(\cdot) \|_{C^{k_J}} < \infty$. Then the inequality follows from Leibniz’ rule for the $i$th derivative.

(iii) In the literature, see e.g. [11, p. 74], there also appear conditions of the form

$$\sup_{t \in I, y \in \mathbb{R}^d} \left\| \frac{\partial}{\partial y} f^{(i)}(t, y) \right\| < \infty, \quad 0 \leq i \leq k_J,$$

where $f^{(i)}$ denotes the $i$th total derivative of $f$ with respect to $t$ in the sense of [11, p. 65]. These conditions may be weaker in some cases.

Since in general the constant $\mathcal{C}_2$ is somewhat connected to the Lipschitz constant and, thus, to the stiffness of the ode system, the dependence on this constant shall be studied very thoroughly in the analysis.

Now, we shall investigate the solvability of the local problem (2.1).

**Theorem 3.7 (Existence and uniqueness)**

Let $r, k \in \mathbb{Z}$, $0 \leq k \leq r$. We suppose that Assumptions 1, 2, 3, and 4 hold. Then there is a constant $\gamma_{r,k} > 0$ multiplicatively depending on $\mathcal{C}_2^{-1}$ but independent of $n$ such that problem (2.1) has a unique solution for all $1 \leq n \leq N$ when $\tau_n \leq \gamma_{r,k}$.

**Proof:** Since the single local problems can be solved one after another, it suffices to prove that (2.1) determines a unique solution for a given initial value $U_{n-1}^-$. In order to show this, we use the auxiliary mapping $g : P_r(I_n, \mathbb{R}^d) \to P_r(I_n, \mathbb{R}^d)$ given by

$$g(U)(t_{n-1}^+) = U_{n-1}^-, \quad \text{if } k \geq 1,$$

$$g(U)^{(i+1)}(t_n^-) = \frac{d^i}{dt^i} \left( f(t,U(t)) \right) \bigg|_{t=t_n^-}, \quad \text{if } k \geq 2, \text{ } i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1,$$

$$g(U)^{(i+1)}(t_{n-1}^-) = \frac{d^i}{dt^i} \left( f(t,U(t)) \right) \bigg|_{t=t_{n-1}^-}, \quad \text{if } k \geq 3, \text{ } i = 0, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor - 1,$$

and

$$\mathcal{J}_n \left[ \left( g(U)'(t), \varphi(t) \right) \right] + \delta_{0,k} \left( g(U)(t_{n-1}^+), \varphi(t_{n-1}^+) \right)
\mathcal{J}_n \left[ \left( \mathcal{I}_n f(t,U(t)), \varphi(t) \right) \right] + \delta_{0,k} \left( U_{n-1}^-, \varphi(t_{n-1}^-) \right) \quad \forall \varphi \in P_{r-k}(I_n, \mathbb{R}^d). \quad (3.8d)$$


Since all these conditions are linear, \( g(U) \) is uniquely determined by (3.8) if and only if we have the unique solvability of the corresponding homogeneous problem, i.e., find \( V \in P_r(I_n, \mathbb{R}^d) \) such that

\[
\begin{align*}
V(t_{n-1}^+) &= 0, & \text{if } k \geq 1, \\
V^{(i+1)}(t_{n-1}^-) &= 0, & \text{if } k \geq 2, \ i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1, \\
V^{(i+1)}(t_{n-1}^+) &= 0, & \text{if } k \geq 3, \ i = 0, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor - 1,
\end{align*}
\]

(3.9a)

and

\[
\mathcal{J}_n \left[ \left( V(t), \varphi(t) \right) \right] + \delta_{0,k} \left( V(t_{n-1}^-), \varphi(t_{n-1}^-) \right) = 0 \quad \forall \varphi \in P_{r-k}(I_n, \mathbb{R}^d).
\]

(3.9d)

Analogously to the proof of Lemma 3.1, we obtain that the conditions of (3.9) allow the trivial solution \( V \equiv 0 \) only. So the homogeneous problem is uniquely solvable and, thus, the mapping \( g \) is well-defined by (3.8).

It can be easily seen that for given \( U_{n-1}^- \) every fixed point of \( g \) is a solution of the local problem (2.1) and vice versa. To get the existence of a unique fixed point, Banach’s fixed point theorem shall be applied. Therefore, we will prove for \( \tau_n \leq \gamma_{r,k} \) with \( \gamma_{r,k} > 0 \) to be defined that \( g \) is on \( \{ U \in P_r(I_n, \mathbb{R}^d) : U(t_{n-1}^+) = U_{n-1}^- \text{ if } k \geq 1 \} \) a contraction with respect to the supremum norm.

So, let \( V, W \in P_r(I_n, \mathbb{R}^d) \) and additionally assume \( V(t_{n-1}^+) = W(t_{n-1}^+) = U_{n-1}^- \) if \( k \geq 1 \). Since \( g(V) - g(W) \in P_r(I_n, \mathbb{R}^d) \), Lemma 3.4 (if \( k \geq 1 \)) or Lemma 3.5 (if \( k = 0 \)), respectively, yields

\[
\begin{align*}
\sup_{t \in I_n} \| g(V) - g(W) \| & \leq \kappa_{r,k} \left( \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left( \frac{\tau_n}{2} \right)^i \left\| (g(V) - g(W))^{(i)}(t_{n-1}^-) \right\| + \sum_{i=0}^{\frac{k}{2}} \left( \frac{\tau_n}{2} \right)^i \left\| (g(V) - g(W))^{(i)}(t_n^-) \right\| \\
+ \left( \sum_{i=0}^{r-k} \mathcal{J}_n \left[ (g(V) - g(W))^i(t) (1 + T_n^{-1}(t))^i \right] + \delta_{0,k} \delta_{0,k}(g(V) - g(W))(t_{n-1}^-) \right) \right)
\end{align*}
\]

(3.10)

where

\[
\kappa_{r,k} := \begin{cases} \kappa_r, & k = 0, \\ \kappa_{r,k}, & k > 0. \end{cases}
\]

Here, for \( k \geq 1 \) we also used that \( (g(V) - g(W))(t_{n-1}^-) = (V - W)(t_{n-1}^-)(t_{n-1}^-) = 0 \) by (3.8a). Involving (3.8b) and (3.8c) the sums (I) and (II) can be rewritten as

\[
\begin{align*}
(\text{I}) + (\text{II}) &= \frac{\tau_n}{2} \left( \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left( \frac{\tau_n}{2} \right)^i \left\| \frac{d^i}{dt^i} \left( f(t, V(t)) \right) \right|_{t=t_{n-1}^-} \right) \\
&\quad + \left( \sum_{i=0}^{\frac{k}{2}} \left( \frac{\tau_n}{2} \right)^i \left\| \frac{d^i}{dt^i} \left( f(t, V(t)) \right) \right|_{t=t_{n}^-} \right).
\end{align*}
\]

Thus by Assumption 4 we conclude

\[
(\text{I}) + (\text{II}) \leq \mathcal{C}_2 \frac{\tau_n}{2} \left( \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left( \frac{\tau_n}{2} \right)^i \left\| (V - W)^{(i)}(t_{n-1}^-) \right\| + \sum_{i=0}^{\frac{k}{2}} \left( \frac{\tau_n}{2} \right)^i \left\| (V - W)^{(i)}(t_{n-1}^-) \right\| \right)
\]

\[
\leq \mathcal{C}_2 \frac{k}{2} \frac{\tau_n}{2} \left( \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left( \frac{\tau_n}{2} \right)^i \left\| (V - W)^{(i)}(t_{n-1}^-) \right\| + \sum_{i=0}^{\frac{k}{2}} \left( \frac{\tau_n}{2} \right)^i \left\| (V - W)^{(i)}(t_{n-1}^-) \right\| \right).
\]
The appearing sums can be bounded by the inverse inequalities \((3.7)\) in the following way
\[
\sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor - 1} \left( \frac{T}{2} \right)^l \| (V - W)^{(l)}(t_n) \| \leq \sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor - 1} \left( \frac{T}{2} \right)^l \sup_{t \in I_n} \| (V - W)^{(l)}(t) \|
\]
\[
\leq \left\lfloor \frac{k}{2} \right\rfloor - 1 \sum_{l=0}^{\left\lfloor \frac{k}{2} \right\rfloor - 1} \left( \frac{T}{2} \right)^l C_{\text{inv}} \left( \frac{T}{2} \right)^{-l} \sup_{t \in I_n} \| (V - W)(t) \| \leq C_{\text{inv}} \left\lfloor \frac{k}{2} \right\rfloor \sup_{t \in I_n} \| (V - W)(t) \| .
\]

The argumentation for the second sum is analogue. Altogether, we obtain
\[
(I) + (II) \leq C_2 C_{\text{inv}} \left\lfloor \frac{k}{2} \right\rfloor (k - 1) \frac{T_n}{2} \sup_{t \in I_n} \| (V - W)(t) \|. \tag{3.11}
\]

We now consider the third sum of \((3.10)\). Exploiting \((3.8d)\) we gain that
\[
(III) = \sum_{i=0}^{r-k} \| \mathcal{J}_n \left[ (I_n f(t, V(t)) - I_n f(t, W(t))) (1 + T_n^{-1}(t))^j \right] \|
\]
where we used test functions of the form \(\varphi(t) = (1 + T_n^{-1}(t))^j \varepsilon_j, \ j = 1, \ldots, d\), in order to derive the component-wise identity needed here. Applying the estimate of Assumption 2 it follows
\[
\| \mathcal{J}_n \left[ (I_n f(t, V(t)) - I_n f(t, W(t))) (1 + T_n^{-1}(t))^j \right] \|
\]
\[
\leq C_0 \frac{T_n}{2} \sum_{j=0}^{k_g} \left( \frac{T}{2} \right)^j \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} \left( I_n f(t, V(t)) - I_n f(t, W(t)) \right) (1 + T_n^{-1}(t))^j \right\|
\]
\[
\leq C_0 \frac{T_n}{2} \sum_{j=0}^{k_g} \sum_{l=0}^{j} \left( \frac{T}{2} \right)^l \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} \left( I_n f(t, V(t)) - I_n f(t, W(t)) \right) \sup_{t \in I_n} \left\| (1 + T_n^{-1}(t))^j \right\| \right\|
\]
where Leibniz’ rule for jth derivative was exploited for the last inequality. Obviously, \((1 + T_n^{-1}(t))^j\) is a polynomial of degree \(i\) in \(t\) and one easily shows
\[
((1 + T_n^{-1}(t))^j)^{(j-l)} = \begin{cases} \left( \frac{2}{T_n} \right)^{j-l} (1 + T_n^{-1}(t))^{(j-l)}, & 0 \leq j - l \leq i, \\ 0, & i < j - l. \end{cases}
\]

Moreover, by construction \(1 + T_n^{-1}(t) \in (0, 2)\) for all \(t \in I_n\). Hence, we get
\[
\| \mathcal{J}_n \left[ (I_n f(t, V(t)) - I_n f(t, W(t))) (1 + T_n^{-1}(t))^j \right] \|
\]
\[
\leq C_0 \frac{T_n}{2} \sum_{j=0}^{k_g} \sum_{l=0}^{\min\{i + 1, k_g\}} \left( \frac{T}{2} \right)^l \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} \left( I_n f(t, V(t)) - I_n f(t, W(t)) \right) \right\|
\]
\[
= C_0 \frac{T_n}{2} \sum_{j=0}^{k_g} \sum_{l=0}^{\min\{i + 1, k_g\}} \left( \frac{T}{2} \right)^l \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} \left( I_n f(t, V(t)) - I_n f(t, W(t)) \right) \right\|
\]
\[
\leq C_0 \frac{T_n}{2} \sum_{j=0}^{k_g} \sum_{l=0}^{\min\{i + 1, k_g\}} \left( \frac{T}{2} \right)^l \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} \left( f(t, V(t)) - f(t, W(t)) \right) \right\|
\]
\[
\leq C_0 \frac{T_n}{2} \sum_{j=0}^{k_g} \sum_{l=0}^{\min\{i + 1, k_g\}} \left( \frac{T}{2} \right)^l \sum_{j=0}^{\min\{i + 1, k_g\}} \left( \frac{T}{2} \right)^j \frac{d^j}{dt^j} \left( f(t, V(t)) - f(t, W(t)) \right) \right\|
\]
\[
= C_0 \frac{T_n}{2} \sum_{j=0}^{k_g} \sum_{l=0}^{\min\{i + 1, k_g\}} \left( \frac{T}{2} \right)^l \sum_{j=0}^{\min\{i + 1, k_g\}} \left( \frac{T}{2} \right)^j \frac{d^j}{dt^j} \left( f(t, V(t)) - f(t, W(t)) \right) \right\|
\]
The backmost double sum $C_{Σ,i}$ can be simplified and estimated for example as follows

$$C_{Σ,i} = \sum_{l=0}^{k_g} \min\{k_g, k_g - 1\} \sum_{j=0}^{\min\{k_g, k_g - 1\}} \binom{j}{l} \frac{(\frac{r}{n})^l}{(l)!} 2^{-j} \leq \frac{k_g!}{(l)!} \sum_{j=0}^{\min\{k_g, k_g - 1\}} \binom{j}{l} 2^{-j} = \frac{k_g!}{(l)!} \sum_{j=0}^{l} 2^{-j} = (k_g + 1)! 3^l. \quad (3.12)$$

By Assumption 4 we then obtain that

$$( III ) \leq c_0 c_1 c_2 \frac{\tau_n}{2} \sum_{l=0}^{\max\{k_g, k_g\}} \binom{l}{j} \sum_{t \in I_u} \sup_{t} \| (V - W)(t) \| \sum_{i=0}^{r-k} \max\{k_g, k_g\} \binom{l}{j} \sup_{t \in I_u} \| (V - W)(t) \| \sum_{j=0}^{\max\{k_g, k_g\}} \binom{j}{l} 2^{j-l} C_{Σ,i}$$

$$\leq c_0 c_1 c_2 (\max\{k_g, k_g\} + 1)(k_g + 1)! \frac{(3r-k+1 - 1) \tau_n}{2} \sum_{l=0}^{\max\{k_g, k_g\}} \binom{l}{j} \sup_{t \in I_u} \| (V - W)(t) \|. \quad (3.13)$$

So, applying the inverse inequalities (3.7) we conclude similar to the estimation of (I) and (II) that

$$( III ) \leq c_0 c_1 c_2 c_{inv} (\max\{k_g, k_g\} + 1)^2 (k_g + 1)! \frac{(3r-k+1 - 1) \tau_n}{2} \sup_{t \in I_u} \| (V - W)(t) \|. \quad (3.13)$$

Now, combining (3.10), (3.11), and (3.13), we get

$$\sup_{t \in I_u} \| g(V) - g(W) \| \leq c_{r,k} c_2 (C(C_{inv}, k) + C(c_0, c_1, c_{inv}, k_g, k_g, r, k)) \frac{\tau_n}{2} \sup_{t \in I_u} \| (V - W)(t) \|.$$

Hence, for sufficiently small $\tau_n \leq c_{r,k}$ where $c_{r,k}$ multiplicatively depends on $c_2^{-1}$ but is independent of $n$, the mapping $g$ is on $\{ U \in P_r(I_n, \mathbb{R}^d) : U(t_{n-1}^+) = U(t_{n-1}^-) \}$ a contraction with respect to the supremum norm. By Banach’s fixed point theorem we have the existence of a unique fixed point which is just the unique solution of the local problem (2.1).}

### 4 Error analysis

In order to prove an error estimate we shall reuse the operator $\tilde{\mathcal{J}}^n$ introduced in the previous section. It is used to define a local approximation on the interval $I_n$.

**Lemma 4.1 (Approximation operator)**

Let $r, k \in \mathbb{Z}$ with $0 \leq k \leq r$. Moreover, suppose that Assumption 1 holds. Then, the operator $\mathcal{J}_n^{\tilde{\mathcal{J}}} : C^{k+1}(\mathcal{T}_n, \mathbb{R}^d) \rightarrow P_r(I_n, \mathbb{R}^d)$ given by

$$\left( \mathcal{J}_n^{\tilde{\mathcal{J}}} v \right)(t_{n-1}^+) = v^i(t_{n-1}^+), \quad \text{if } k \geq 1, i = 0, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor,$$

$$\left( \mathcal{J}_n^{\tilde{\mathcal{J}}} v \right)(t_{n}^-) = \bar{v}^i(t_{n}^-), \quad \text{if } k \geq 2, i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor,$$

$$\mathcal{J}_n \left[ \left( \mathcal{J}_n^{\tilde{\mathcal{J}}} v \right)'(t), \varphi(t) \right] = \delta_{0,k} \mathcal{J}_n^{\tilde{\mathcal{J}}} v'(t_{n-1}^-) \varphi(t_{n-1}^-)$$

$$= \mathcal{J}_n \left[ (I_n)(t_{n}'), \varphi(t) \right] + \delta_{0,k} v(t_{n}^+) \varphi(t_{n}^+), \quad \forall \varphi \in P_{r-k}(I_n, \mathbb{R}^d) \quad (4.1c)$$

is well-defined.

**Proof:** The existence and uniqueness of the approximation is a direct consequence of Lemma 3.1.\[\square\]

For convenience, we define for $v \in C^{k+1}(\mathcal{T}, \mathbb{R}^d)$ a global approximation operator $\mathcal{J}^n^{\tilde{\mathcal{J}}} v$ by combining the local approximations on the mesh intervals, i.e., $\left( \mathcal{J}^n^{\tilde{\mathcal{J}}} v \right)|_{I_n} = \mathcal{J}_n^{\tilde{\mathcal{J}}} (v|_{I_n}) \in P_r(I_n, \mathbb{R}^d)$ for $n = 1, \ldots, N$ and setting $\mathcal{J}^n^{\tilde{\mathcal{J}}} v(t_0^-) = v(t_0^+) = v(t_0)$. Note that even for $k \geq 1$ in general $\mathcal{J}^n^{\tilde{\mathcal{J}}} v$ is not globally continuous.
Lemma 4.2
Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r, \) and suppose that Assumptions 1, 2, and 3 hold. Then for \( 1 \leq n \leq N \) and \( 0 \leq l \leq r \) we have for all \( v \in C^{k+l+1}(\mathcal{T}_n, \mathbb{R}^d) \)

\[
\sup_{t \in I_n} \| (\mathcal{J}_n^\delta I v)^{(l)}(t) \| \leq C \sum_{j=0}^{k+l} \left( \frac{\tau^n}{\mathcal{K}^n} \right)^{j-l} \sup_{t \in I_n} \| \varphi^{(j)}(t) \|
\]

Proof: To derive the estimate, an inverse inequality, which yields the factor \( \left( \frac{\tau^n}{\mathcal{K}^n} \right)^{-l} \), is applied first. The remaining term, \( \sup_{t \in I_n} \| \mathcal{J}_n^\delta I v(t) \| \), can be bounded on the reference interval independent of \( \tau_n \). The factors \( \left( \frac{\tau^n}{\mathcal{K}^n} \right)^j \) then result from transformation.

For the argument used to prove the error estimate it we need additional assumptions. In detail, compared to Theorem 3.7 we replace Assumption 3 by Assumption 5a or 5b since derivatives can be handled given in certain points, but not their supremum. Furthermore, we exploit in the error analysis an auxiliary interpolation operator \( \mathcal{I}_{\text{app}} \) defined below, see Definition 4.4, which amongst others is based on these assumptions.

For brevity the Assumptions 5a and 5b below shall be stated directly for the local operators \( \mathcal{J}_n \) and \( \mathcal{I}_n \). However, appropriate properties of \( \mathcal{J} \) and \( \mathcal{I} \) guarantee these assumptions by transformation, cf. Assumptions 2 and 3.

Assumption 5a
For \( 1 \leq n \leq N \) and \( 0 \leq l \leq k_\delta \) it holds \( \mathcal{I}_n \varphi \in C^l(\mathcal{T}_n, \mathbb{R}^d) \) and there are disjoint points \( \tilde{t}_m^\delta, m = 0, \ldots, K_\delta \), in the reference interval \([-1, 1]\) such that

\[
\left( \frac{\tau^n}{\mathcal{K}^n} \right)^l \sup_{t \in I_n} \| (\mathcal{I}_n \varphi)^{(l)}(t) \| \leq C_1 \left[ \frac{\tau^n}{2} \sum_{m=0}^{K_\delta} \sum_{j=0}^{\mathcal{K}^n_m} \left( \frac{\tau^n}{\mathcal{K}^n_m} \right)^j \| \varphi^{(j)}(t_{n,m}) \| \right] + C_2 \sup_{t \in I_n} \| \varphi(t) \| \quad \forall \varphi \in C^{k_\delta}(\mathcal{T}_n, \mathbb{R}^d)
\]

where \( t_{n,m}^\delta := \frac{\lambda_n + \lambda_{n+1}}{2} + \frac{\tau^n}{\mathcal{K}^n_m} \tilde{t}_m^\delta \). Note that then typically \( k_\delta = \max\{ \tilde{K}^\delta_m : m = 0, \ldots, K_\delta \} \).

Assumption 5b
For \( 1 \leq n \leq N \), there are disjoint points \( \tilde{t}_m^\delta, m = 0, \ldots, K_\delta \), in the reference interval \([-1, 1]\) such that

\[
\| \mathcal{J}_n \varphi \| \leq \tilde{C}_1 \left[ \frac{\tau^n}{2} \sum_{m=0}^{K_\delta} \sum_{j=0}^{\mathcal{K}^n_m} \left( \frac{\tau^n}{\mathcal{K}^n_m} \right)^j \| \varphi^{(j)}(t_{n,m}) \| \right] + \tilde{C}_2 \sup_{t \in I_n} \| \varphi(t) \| \quad \forall \varphi \in C^{k_\delta}(\mathcal{T}_n, \mathbb{R}^d)
\]

where \( t_{n,m}^\delta := \frac{\lambda_n + \lambda_{n+1}}{2} + \frac{\tau^n}{\mathcal{K}^n_m} \tilde{t}_m^\delta \). Note that then typically \( k_\delta = \max\{ \tilde{K}^\delta_m : m = 0, \ldots, K_\delta \} \).

Moreover, for \( 1 \leq n \leq N \) assume that there are disjoint points \( \tilde{t}_m^\delta, m = 0, \ldots, K_\delta \), in the reference interval \([-1, 1]\) such that

\[
\sum_{m=0}^{K_\delta} \sum_{j=0}^{\mathcal{K}^n_m} \left( \frac{\tau^n}{\mathcal{K}^n_m} \right)^j \| (\mathcal{I}_n \varphi)^{(j)}(t_{n,m}) \| + \sup_{t \in I_n} \| \mathcal{I}_n \varphi(t) \|
\]

\[
\leq \tilde{C}_1 \left[ \frac{\tau^n}{2} \sum_{m=0}^{K_\delta} \sum_{j=0}^{\mathcal{K}^n_m} \left( \frac{\tau^n}{\mathcal{K}^n_m} \right)^j \| \varphi^{(j)}(t_{n,m}) \| \right] + \tilde{C}_2 \sup_{t \in I_n} \| \varphi(t) \| \quad \forall \varphi \in C^{\max\{k_\delta, k_\gamma\}}(\mathcal{T}_n, \mathbb{R}^d)
\]

where \( t_{n,m}^\gamma := \frac{\lambda_n + \lambda_{n+1}}{2} + \frac{\tau^n}{\mathcal{K}^n_m} \tilde{t}_m^\gamma \).

Remark 4.3
Assumption 5a is satisfied if \( \mathcal{I}_n \) is a polynomial approximation operator whose defining degrees of freedom only use derivatives in certain points, as for example Hermite interpolation operators. Together with Assumption 2 the term \( \| \mathcal{J}_n \mathcal{I}_n \varphi \| \) can be estimated by the supremum of \( \| \varphi \| \) and certain point values of derivatives of \( \varphi \).

However, Assumption 5a is not satisfied if \( \mathcal{I}_n = \text{Id} \) and \( k_\gamma > 0 \). In order to enable a similar estimate for \( \| \mathcal{J}_n \mathcal{I}_n \varphi \| \) also in this case, Assumption 5b is formulated. Here the requirements on
the operator should satisfy the following conditions: Let \( I_n \) and \( J_n \) be satisfied. Denote by 

\[
\begin{align*}
(T_n \varphi)^{(l)}(t_n^{-}) &= \varphi^{(l)}(t_n^{-}) \quad \text{for } 0 \leq l \leq \left\lfloor \frac{k}{l} \right\rfloor - 1, \\
(T_n \varphi)^{(l)}(t_{n-1}^{+}) &= \varphi^{(l)}(t_{n-1}^{+}) \quad \text{for } 0 \leq l \leq \left\lfloor \frac{k}{l} \right\rfloor - 1.
\end{align*}
\]  

(4.2)

Moreover, we demand that 

\[
(T_n \varphi)^{(l)}(t_{n,m}) = \varphi^{(l)}(t_{n,m}) \quad \text{for } 0 \leq m \leq K, 0 \leq l \leq \tilde{K}.
\]  

(4.3a)

with \( t_{n,m} := \frac{tn}{m} + \frac{tn}{m} \) where the points \( t_{n,m} \) are those of Assumption 5a or 5b, respectively. 

If (4.2) and (4.3a) provide independent interpolation conditions and \( r^{\text{app}} < r + 1 \), then we choose \( r + 1 - r^{\text{app}} \) further points \( t_{n,m} \in (-1,1) \setminus \{t_{n,j} : j = 0, \ldots, K\} \), \( m = K + 1, \ldots, K + r + 1 - r^{\text{app}} \), and demand 

\[
(T_n \varphi)^{(l)}(t_{n,m}) = \varphi^{(l)}(t_{n,m}) \quad \text{for } K + 1 \leq m \leq K + r + 1 - r^{\text{app}}
\]  

(4.3b)

where again \( t_{n,m} := \frac{tn}{m} + \frac{tn}{m} \). We agree that \( T_n \varphi \) is applied component-wise to vector-valued functions. So, overall the conditions (4.2) and (4.3) uniquely define an Hermite-type interpolation operator of ansatz order \( \max \{r^{\text{app}} - 1, r\} \).

\[\]

Now, we are able to prove an abstract error estimate.

**Theorem 4.5**

Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r \). We suppose that Assumptions 1, 2, and 4 hold. Moreover, let Assumption 5a or 5b be satisfied. Denote by \( u \) and \( U \) the solutions of (1.1) and (2.1), respectively. Then we have for \( 1 \leq n \leq N \), sufficiently small \( \tau \), and \( l = 0, 1 \) that

\[
\sup_{t \in I_n} \| (u - U)^{(l)}(t) \| \leq C \max_{1 \leq \nu \leq n} \left( \sup_{t \in I_{n,\nu}} \| (\text{Id} - T_n^\nu) u(t) \| + \sum_{j=0}^{l} \sup_{t \in I_{n,\nu}} \| (u - J_{\nu}^\nu U)^{(j)}(t) \| \right)
\]

\[
+ C \max_{1 \leq \nu \leq n} \tau_{\nu}^{-1} \| (u - J_{\nu}^\nu U)(t_{n,\nu}) \|, \tag{4.4}
\]

where the constants \( C \) in general exponentially depend on the product of \( T \) and \( \mathcal{E}_2 \).

**Proof:** In order to shorten the notation of the proof, we set 

\[
E_n := \sup_{t \in I_n} \| u(t) - U(t) \|. 
\]

Using the approximation \( J_{\nu}^\nu u \), which has been introduced directly below Lemma 4.1, we split the error in two parts

\[
\eta(t) := u(t) - J_{\nu}^\nu u(t), \quad \zeta(t) := J_{\nu}^\nu u(t) - U(t). 
\]

Then the triangle inequality yields

\[
E_n = \sup_{t \in I_n} \| u(t) - U(t) \| \leq \sup_{t \in I_n} \| \eta(t) \| + \sup_{t \in I_n} \| \zeta(t) \|. \tag{4.4}
\]

The first term on the right-hand side is the approximation error of the operator \( J_{\nu}^\nu \) and shall be analyzed separately later on. It remains to study the second term.

Since \( \zeta \) is continuously differentiable on each time interval \( I_{\nu, \nu} = 1, \ldots, N \), we have for \( t \in I_{\nu, \nu} \)

\[
\zeta(t) = \zeta(t_{\nu, \nu}) + \sum_{\nu=1}^{n-1} \left( \int_{I_{\nu, \nu}} \zeta'(s) \, ds + \left[ \zeta \right]_{n-1} \right) + \left( \int_{I_{n-1}}^{t} \zeta'(s) \, ds + \left[ \zeta \right]_{n-1} \right)
\]

\[
\leq \sum_{\nu=1}^{n} \left( \int_{I_{\nu, \nu}} \left| \zeta'(s) \right| \, ds + \left| \zeta \right|_{n-1} \right). 
\]

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Here, the integrals and the modulus have to be applied component-wise. Furthermore, \( \zeta(t_0^-) = 0 \) due to \( U(t_0^+) = u_0 = u(t_0^-) \) and \( J^{\mathcal{F}, \mathcal{I}} u(t_0^-) = u(t_0^-) \), see (2.1) and directly below Lemma 4.1. Hence, by the triangle inequality it follows

\[
\sup_{t \in I_n} \|\zeta(t)\| \leq \left\| \sum_{\nu = 1}^{n} \left( \int_{I_{\nu}} |\zeta'(s)| \, ds + [\zeta]_{\nu-1} \right) \right\| \leq \sum_{\nu = 1}^{n} \left( \| \int_{I_{\nu}} |\zeta'(s)| \, ds \| + \| [\zeta]_{\nu-1} \| \right) \\
\leq \sum_{\nu = 1}^{n} \left( \int_{I_{\nu}} \|\zeta'(s)\| \, ds + \| [\zeta]_{\nu-1} \| \right). \quad (4.5)
\]

We start analyzing the integral term of (4.5). For \( 1 \leq n \leq N \) and because of \( \zeta |_{I_n} \in \mathcal{P}_r(I_n, \mathbb{R}^d) \) we can apply Lemma 3.4 to get

\[
\int_{I_n} \|\zeta'(t)\| \, dt \leq \tau_n \sup_{t \in I_n} \|\zeta'(t)\| \quad (4.6)
\]  

\[
\leq \kappa_{r,k} \left( \sum_{i = 1}^{\frac{|k| + 1}{2}} (\frac{\tau_n}{2})^i \left\| \zeta^{(i)}(t_{n-1}^-) \right\| + \sum_{i = 1}^{\frac{|k|}{2}} (\frac{\tau_n}{2})^i \left\| \zeta^{(i)}(t_n^-) \right\| + \sum_{i = \delta_{0,k}}^{r-k} \left\| \mathcal{F}_n [\zeta'(t) (1 + T_n^{-1}(t))^i] \right\| \right). \quad (\text{I})
\]

The right-hand side terms are now studied separately.

The summands of (I) are basically of the form \( (\frac{\tau_n}{2})^{i+1} \left\| \zeta^{(i+1)}(\tilde{t}) \right\| \) where \( \tilde{t} \in \{ t_{n-1}^-, t_n^+ \} \), \( i \geq 0 \).

The definitions of \( J^{\mathcal{F}, \mathcal{I}} u \) and \( U \), especially (4.1a), (4.1b) and (2.1b), (2.1c), together with the \( i \)th derivative of the ode system (1.1) yield for the appearing summands

\[
\zeta^{(i+1)}(\tilde{t}) = (J^{\mathcal{F}, \mathcal{I}} u - U)^{(i+1)}(\tilde{t}) = (u - U)^{(i+1)}(\tilde{t}) = \frac{d^i}{dt^i} \left( f(t,u(t)) - f(t,U(t)) \right) \bigg|_{t = \tilde{t}}
\]

for \( 0 \leq i \leq \bar{k} \) where

\[
\bar{k} = \bar{k}(\tilde{t}) = \begin{cases} \frac{|k|}{2} - 1, & \text{if } \tilde{t} = t_n^+, \\ \frac{|k|}{2} - 1, & \text{if } \tilde{t} = t_{n-1}^-. \end{cases}
\]

denotes the decremented upper limits of the sums in (I) depending on \( \tilde{t} \). By Assumption 4.1 we obtain

\[
\sum_{i = 0}^{\bar{k}} (\frac{\tau_n}{2})^{i+1} \left\| \zeta^{(i+1)}(\tilde{t}) \right\| \leq \frac{\tau_n}{2} \sum_{i = 0}^{\bar{k}} (\frac{\tau_n}{2})^i \left\| \frac{d^i}{dt^i} \left( f(t,u(t)) - f(t,U(t)) \right) \bigg|_{t = \tilde{t}} \right\| \\
\leq \mathcal{C}_2 \frac{\tau_n}{2} \sum_{i = 0}^{\bar{k}} (\frac{\tau_n}{2})^i \sum_{l = 0}^{i} \left\| u - U \right\|^{(i)}(\tilde{t}) \\
\leq \mathcal{C}_2 \max\{0, (\bar{k} + 1)\} \frac{\tau_n}{2} \sum_{i = 0}^{\bar{k}} (\frac{\tau_n}{2})^i \left\| (u - U)^{(i)}(\tilde{t}) \right\|.
\]

Exploiting the Hermite interpolation operator \( \mathcal{I}_{n}^{\text{app}} \) that especially preserves derivatives up to order \( \frac{|k|}{2} - 1 \) in \( t_n^+ \) and up to order \( \frac{|k|}{2} - 1 \) in \( t_{n-1}^- \), see (4.2), as well as invoking the inverse inequality (3.7), we find for \( 0 \leq l \leq \bar{k} \):

\[
\| (u - U)^{(l)}(\tilde{t}) \| = \| (\mathcal{I}_{n}^{\text{app}} u - U)^{(l)}(\tilde{t}) \| \leq \sup_{t \in I_n} \| (\mathcal{I}_{n}^{\text{app}} u - U)^{(l)}(t) \| \\
\leq C_{\text{inv}} \left( \frac{\tau_n}{2} \right)^{-l} \sup_{t \in I_n} \| (\mathcal{I}_{n}^{\text{app}} u - U)(t) \| \\
\leq C_{\text{inv}} \left( \frac{\tau_n}{2} \right)^{-l} \left( \sup_{t \in I_n} \| (\mathcal{I}_{n}^{\text{app}} u - U)(t) \| + \sup_{t \in I_n} \| (u - U)(t) \| \right). \quad (4.7)
\]
Overall this implies
\[(I) \leq C_2C_{\text{inv}} \left( \max\{0, \left[ \frac{k}{2} \right]\}^2 + \frac{k+1}{2} \right) \frac{\tau_n}{2} \left( \sup_{t \in I_n} \| (u - T_{\text{app}} u) (t) \| + E_n \right). \tag{4.8} \]

In order to bound (II) in (4.6) we first of all note that by definition of $\mathcal{J}_n^{\mathcal{A}} u$, especially (4.1c), we gain for all $\varphi \in P_{r-\gamma}(I_n, \mathbb{R}^d)$ that
\[
\mathcal{J}_n \left[ \left( (\mathcal{J}_n^{\mathcal{A}} u)'(t), \varphi(t) \right) \right] + \delta_{0,k} \left( (\mathcal{J}_n^{\mathcal{A}} u)(t_{n-1}^+, \varphi(t_{n-1}^+)) \right) \\
= \mathcal{J}_n \left[ \left( I_n u'(t), \varphi(t) \right) \right] + \delta_{0,k} \left( u(t_{n-1}^+), \varphi(t_{n-1}^+) \right) \\
= \mathcal{J}_n \left[ \left( I_n f(t, u(t)), \varphi(t) \right) \right] + \delta_{0,k} \left( u(t_{n-1}^+), \varphi(t_{n-1}^+) \right),
\tag{4.9}
\]
where the initial value problem (1.1) and the continuity of $u$ are used in the last identity. So, subtracting the variational equation of the local problem (2.1d) we obtain for all $\varphi \in P_{r-\gamma}(I_n, \mathbb{R}^d)$
\[
\mathcal{J}_n \left[ \left( \zeta'(t), \varphi(t) \right) \right] + \delta_{0,k} \left( \zeta(t_{n-1}^+), \varphi(t_{n-1}^+) \right) \\
= \mathcal{J}_n \left[ \left( (\mathcal{J}_n^{\mathcal{A}} u)'(t) - U'(t), \varphi(t) \right) \right] + \delta_{0,k} \left( (\mathcal{J}_n^{\mathcal{A}} u - U)(t_{n-1}^+), \varphi(t_{n-1}^+) \right) \\
= \mathcal{J}_n \left[ \left( I_n f(t, u(t)) - I_n f(t, U(t)), \varphi(t) \right) \right] + \delta_{0,k} \left( (u - U)(t_{n-1}^+), \varphi(t_{n-1}^+) \right).
\]
Using test functions of the form $\varphi(t) = (1 + T_{n-1}^{-1}(t))^j e_j, j = 1, \ldots, d$, we get by a component-wise derivation that
\[(II) = \sum_{i=\delta_{0,k}}^{r-\gamma} \| \mathcal{J}_n \left[ \zeta'(t) (1 + T_{n-1}^{-1}(t))^j \right] \| = \sum_{i=\delta_{0,k}}^{r-\gamma} \| \mathcal{J}_n \left[ \left( I_n f(t, u(t)) - I_n f(t, U(t)) \right) (1 + T_{n-1}^{-1}(t))^j \right] \|. \tag{4.10}\]

For the summands on the right-hand side, we consider two different cases. If Assumption 5a holds, we first conclude by Assumption 2 and Leibniz’ rule for the $j$th derivative that
\[
\| \mathcal{J}_n \left[ \left( I_n f(t, u(t)) - I_n f(t, U(t)) \right) (1 + T_{n-1}^{-1}(t))^j \right] \|
\leq \mathcal{C}_0 \frac{\tau_n}{2} \sum_{i=0}^{k_j} \left( \frac{r}{2} \right)^j \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} \left( I_n f(t, u(t)) - I_n f(t, U(t)) \right) (1 + T_{n-1}^{-1}(t))^j \right\|
\leq \mathcal{C}_0 \frac{\tau_n}{2} \sum_{i=0}^{k_j} \sum_{j=0}^{k_j} \left( \frac{r}{2} \right)^j \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} \left( I_n f(t, u(t)) - I_n f(t, U(t)) \right) \right\| \sup_{t \in I_n} \left\| (1 + T_{n-1}^{-1}(t))^j \right\| \sum_{j=0}^{\min\{i, k_j+1\}} \left( \frac{r}{2} \right)^j \left( \frac{r}{2} \right)^j \|
\leq \mathcal{C}_0 \frac{\tau_n}{2} \left\| \left( I_n f(t, u(t)) - I_n f(t, U(t)) \right) \right\| \sup_{t \in I_n} \left\| (1 + T_{n-1}^{-1}(t))^j \right\| \sum_{j=0}^{\min\{i, k_j+1\}} \left( \frac{r}{2} \right)^j \left( \frac{r}{2} \right)^j \|
\]
where we refer for details regarding the derivation of the last identity to the proof of Theorem 3.7. Then involving the estimate of Assumption 5a, we obtain that
\[
\| \mathcal{J}_n \left[ \left( I_n f(t, u(t)) - I_n f(t, U(t)) \right) \right] \|
\leq \mathcal{C}_0 C_{\Sigma} \frac{\tau_n}{2} \left( \mathcal{C}_{1,1} \sum_{m=0}^{k_j} \sum_{l=0}^{k_j} \left( \frac{r}{2} \right)^j \left\| \frac{d^j}{dt^j} \left( f(t, u(t)) - f(t, U(t)) \right) \right\| \right) \sup_{t \in I_n} \left\| f(t, u(t)) - f(t, U(t)) \right\|
\]
with $C_{\Sigma,i}$ from (3.12). A quite similar argumentation also works when only Assumption 5b holds. In this case we gain

$$\left\| \mathcal{J}_n \left[ (I_n f(t,u(t)) - I_n f(t, U(t))) (1 + T_n^{-1}(t))^i \right] \right\|$$

$$\leq \bar{c}_0 \frac{\tau_n c}{2} \sum_{m=0}^{K^x} \sum_{j=0}^{\tilde{K}_m^x} \left( \frac{\tau_n}{i} \right)^j \left\| \frac{d^j}{dt^j} \left( (I_n f(t,u(t)) - I_n f(t, U(t))) (1 + T_n^{-1}(t))^i \right) \right\|_{t=t^x_{n,m}}$$

$$+ \bar{c}_0 \frac{\tau_n c}{2} \sup_{t \in I_n} \left\| (I_n f(t,u(t)) - I_n f(t, U(t))) (1 + T_n^{-1}(t))^i \right\|$$

$$\leq \bar{c}_0 \frac{\tau_n c}{2} \sum_{m=0}^{K^x} \sum_{j=0}^{\tilde{K}_m^x} \left( \frac{\tau_n}{i} \right)^j \left\| \frac{d^j}{dt^j} \left( I_n f(t,u(t)) - I_n f(t, U(t)) \right) \right\|_{t=t^x_{n,m}} \sum_{n=0}^{(i+1)^2} (1 + \frac{\tau_n c}{2})^{2i-j}$$

$$+ \bar{c}_0 \frac{\tau_n c}{2} \sup_{t \in I_n} \left\| (I_n f(t,u(t)) - I_n f(t, U(t))) \right\| 2^i$$

$$\leq \max \{ \bar{c}_0, k, 3^i, \bar{c}_0, 2^{2i} \} \frac{\tau_n c}{2} \sum_{m=0}^{K^x} \sum_{j=0}^{\tilde{K}_m^x} \left( \frac{\tau_n}{i} \right)^j \left\| \frac{d^j}{dt^j} \left( f(t,u(t)) - f(t, U(t)) \right) \right\|_{t=t^x_{n,m}}$$

$$+ \bar{c}_1 \sup_{t \in I_n} \left\| (f(t,u(t)) - f(t, U(t))) \right\| .$$

So, either way applying Assumption 4 gives

$$\left\| \mathcal{J}_n \left[ (I_n f(t,u(t)) - I_n f(t, U(t))) (1 + T_n^{-1}(t))^i \right] \right\|$$

$$\leq C \bar{c}_2 \frac{\tau_n c}{2} \left( \sum_{m=0}^{K^x} \sum_{j=0}^{\tilde{K}_m^x} \left( \frac{\tau_n}{i} \right)^j \left\| (u-U)^{(\tilde{r})}(t^x_{n,m}) \right\| + \sup_{t \in I_n} \left\| (u-U)(t) \right\| \right)$$

$$\leq C \bar{c}_2 \frac{\tau_n c}{2} \left( \max_{0 \leq m \leq K^x} \left\{ \tilde{K}_m^x + 1 \right\} \sum_{m=0}^{K^x} \sum_{j=0}^{\tilde{K}_m^x} \left( \frac{\tau_n}{i} \right)^j \left\| (u-U)^{(\tilde{r})}(t^x_{n,m}) \right\| + \sup_{t \in I_n} \left\| (u-U)(t) \right\| \right).$$

The terms that include derivatives can be estimated similar to (4.7) since by definition the interpolation operator $T_n^{app}$ also preserves the particular derivatives at the particular points $t^x_{n,m}$, see (3.4a). Therefore, altogether this results in

$$\left\| \mathcal{J}_n \left[ (I_n f(t,u(t)) - I_n f(t, U(t))) (1 + T_n^{-1}(t))^i \right] \right\| \leq C \bar{c}_2 \frac{\tau_n c}{2} \left( \sup_{t \in I_n} \left\| (Id - T_n^{app})u(t) \right\| + E_n \right)$$

(4.11)

and, hence, together with (4.10) we obtain

$$\left\| \mathcal{J}_n \left[ (I_n f(t,u(t)) - I_n f(t, U(t))) (1 + T_n^{-1}(t))^i \right] \right\| \leq \bar{c}_3 \frac{\tau_n c}{2} \left( \sup_{t \in I_n} \left\| (Id - T_n^{app})u(t) \right\| + E_n \right).$$

(4.12)

So, combining (4.6) with (4.8) and (4.12), we have already shown that

$$\int_{I_n} \left\| \zeta'(t) \right\| dt \leq \tau_n \sup_{t \in I_n} \left\| \zeta'(t) \right\| \leq C \bar{c}_2 \frac{\tau_n c}{2} \left( \sup_{t \in I_n} \left\| (Id - T_n^{app})u(t) \right\| + E_n \right)$$

(4.13)

for all $1 \leq n \leq N$.

Next, we analyze the jump term of (4.5). First, we have a closer look at $[\zeta]_{n-1}$ for $1 \leq n \leq N$. There are two cases. If $k \geq 1$, the discrete solution $U$ is globally continuous due to (2.1a). So,

$$[\zeta]_{n-1} = [\mathcal{J}^x u]_{n-1} - [U]_{n-1} = \mathcal{J}^x u(t_{n-1}^+) - \mathcal{J}^x u(t_{n-1}^-) = u(t_{n-1}^+) - u(t_{n-1}^-) \mathcal{J}^x u(t_{n-1}^-)$$

$$= u(t_{n-1}^-) - \mathcal{J}^x u(t_{n-1}^-)$$
where also (4.1a) and the continuity of $u$ was used. If $k = 0$ we rewrite the jump term as follows
\[
\begin{align*}
[\zeta]_{n-1} &= \mathcal{J}_n[\zeta'(t)] + \zeta(t^+_{n-1}) - \zeta(t^-_{n-1}) - \mathcal{J}_n[\zeta'(t)] \\
&= \mathcal{J}_n[\mathcal{I}_n f(t, u(t)) - \mathcal{I}_n f(t, U(t))] + (u - U)(t^-_{n-1}) - (\mathcal{J}^\mathcal{I} u - U)(t^-_{n-1}) - \mathcal{J}_n[\zeta'(t)] \\
&= u(t^-_{n-1}) - \mathcal{J}^\mathcal{I} u(t^-_{n-1}) + \mathcal{J}_n[\mathcal{I}_n f(t, u(t)) - \mathcal{I}_n f(t, U(t))] - \mathcal{J}_n[\zeta'(t)]
\end{align*}
\]
where we exploited (4.9). The integrator terms on the right-hand side can be bounded using already known estimates. Indeed, (4.11) with $i = 0$ yields
\[
\|\mathcal{J}_n[(\mathcal{I}_n f(t, u(t)) - \mathcal{I}_n f(t, U(t)))]\| \leq C\varepsilon_2 \frac{\tau_n}{2} \left( \sup_{t \in I_n} \|\text{Id} - \mathcal{I}^{\text{app}}_n u(t)\| + E_n \right).
\]
Furthermore, combining Assumption 2, the inverse inequality (3.7), and (4.13) we gain
\[
\|\mathcal{J}_n[\zeta'(t)]\| \leq \varepsilon_0 \frac{\tau_n}{2} \sum_{j=0}^{k_0} \left( \varepsilon_j \right)^j \sup_{t \in I_n} \|\zeta^{(j+1)}(t)\| \leq \varepsilon_0 C_{\text{inv}} (k_0 + 1) \frac{\tau_n}{2} \sup_{t \in I_n} \|\zeta'(t)\| \leq C\varepsilon_2 \frac{\tau_n}{2} \left( \sup_{t \in I_n} \|\text{Id} - \mathcal{I}^{\text{app}}_n u(t)\| + E_n \right).
\]
Hence, in both cases we have
\[
\begin{align*}
\|\zeta_{n-1}\| &\leq \|u(t^-_{n-1}) - \mathcal{J}^\mathcal{I} u(t^-_{n-1})\| + \delta_0 k \left( \|\mathcal{J}_n[\mathcal{I}_n f(t, u(t)) - \mathcal{I}_n f(t, U(t))]\| + \|\mathcal{J}_n[\zeta'(t)]\| \right) \\
&\leq \|u(t^-_{n-1}) - \mathcal{J}^\mathcal{I} u(t^-_{n-1})\| + \delta_0 k \varepsilon_2 \frac{\tau_n}{2} \left( \sup_{t \in I_n} \|\text{Id} - \mathcal{I}^{\text{app}}_n u(t)\| + E_n \right),
\end{align*}
\]
(4.14)
Summarizing, we get from (4.4), (4.5), (4.13), and (4.14) for $1 \leq n \leq N$
\[
E_n \leq \sup_{t \in I_n} \|\eta(t)\| + \sup_{t \in I_n} \|\zeta(t)\| \leq \sup_{t \in I_n} \|\eta(t)\| + \sum_{\nu=1}^{n} \left( \left( \int_{I_{\nu}} \|\zeta'(s)\| \, ds + \|\zeta_{\nu-1}\| \right) \sup_{t \in I_{\nu}} \|\text{Id} - \mathcal{I}^{\text{app}}_{\nu} u(t)\| + E_{\nu} \right) + \sum_{\nu=1}^{n} \|u(t^-_{\nu-1}) - \mathcal{J}^\mathcal{I} u(t^-_{\nu-1})\|.
\]
Note that $C$ is independent of $T$ but especially depends multiplicatively on the Lipschitz constant of $f$ (hidden in $C_2$). For $\tau_n$ sufficiently small ($C\tau_n/2 < 1$), the $E_n$ term of the right-hand side can be absorbed on the left. After that the application of a variant of the discrete Gronwall lemma, see [15, Lemma 1.4.2, p. 14], applied with
\[
k_s = \frac{\tilde{C}\tau_1}{1 - \tilde{C}\tau_1} (1 - \delta_{0,s}), \quad \phi_s = E_s, \quad g_0 = 0,
\]
\[
p_s = \frac{1}{1 - \tilde{C}\tau_1} \left( \sup_{t \in (t_0, t_{s+1})} \|\eta(t)\| - \sup_{t \in (t_0, t_{s+1})} \|\eta(t)\| \right) + \tilde{C} \tau_{s+1} \sup_{t \in I_{s+1}} \|\text{Id} - \mathcal{I}^{\text{app}}_{s+1} u(t)\| + \|u(t^-_{s}) - \mathcal{J}^\mathcal{I} u(t^-_{s})\|),
\]
yields
\[
E_n \leq \exp \left( \sum_{s=1}^{n-1} \frac{\tilde{C}\tau_s}{1 - \tilde{C}\tau_s} \right) \left( \frac{1}{1 - \tilde{C}\tau_1} \left( \max_{1 \leq \nu \leq n} \left( \sup_{t \in I_{\nu}} \|\eta(t)\| \right) + \sum_{\nu=1}^{n} \left( \tilde{C}\tau_{\nu} \sup_{t \in I_{\nu}} \|\text{Id} - \mathcal{I}^{\text{app}}_{\nu} u(t)\| + \|u(t^-_{\nu-1}) - \mathcal{J}^\mathcal{I} u(t^-_{\nu-1})\| \right) \right) \right)
\leq \exp \left( \tilde{C}(t_n - t_0)/2 \right) \left( \frac{1}{1 - \tilde{C}\tau_1} \left( \max_{1 \leq \nu \leq n} \left( \sup_{t \in I_{\nu}} \|\text{Id} - \mathcal{I}^{\text{app}}_{\nu} u(t)\| \right) \right) \right)
\leq \max_{1 \leq \nu \leq n} \left( \sup_{t \in I_{\nu}} \|\text{Id} - \mathcal{J}^\mathcal{I} u(t)\| \right) + (t_n - t_0) \max_{1 \leq s \leq n-1} \left( \tau_{s-1} \|u(t^-_{s}) - \mathcal{J}^\mathcal{I} u(t^-_{s})\| \right) \right)
where we also used that \( u(t_0^-) = u_0 = \mathcal{J}^{\mathcal{T}} u(t_0^-) \). This already is the wanted statement for \( l = 0 \).

Especially note the exponential dependency on \( T \) and the Lipschitz constant of \( f \) (hidden in \( C, \mathcal{C}_2 \)).

Finally, we shall derive a bound for the first derivative of the error. Recalling the estimate for \( \zeta' \) in (4.13) it follows

\[
\sup_{t \in I_n} \|(u - U)'(t)\| \leq \sup_{t \in I_n} \|(u - \mathcal{J}^{\mathcal{T}} u)'(t)\| + \sup_{t \in I_n} \|(\mathcal{J}^{\mathcal{T}} u - U)'(t)\|
\]

\[
\leq \sup_{t \in I_n} \|(u - \mathcal{J}^{\mathcal{T}} u)'(t)\| + C \mathcal{C}_2 \left( \sup_{t \in I_n} \|\Id - \mathcal{T}^{\text{app}}\| u(t) \| + E_n \right).
\]

Using the already known estimate for \( E_n \), the statement for \( l = 1 \) follows. \( \square \)

**Remark 4.6**

Based on Theorem 4.5 we can also prove abstract estimates for higher order derivatives of the error between the solution \( u \) of (1.1) and the discrete solution \( U \) of (2.1). Of course, we gain that

\[
\sup_{t \in I_n} \|(u - U)^{(l)}(t)\| \leq \sup_{t \in I_n} \|(u - \mathcal{J}^{\mathcal{T}} u)^{(l)}(t)\| + \sup_{t \in I_n} \|(\mathcal{J}^{\mathcal{T}} u - U)^{(l)}(t)\|
\]

\[
\leq \sup_{t \in I_n} \|(u - \mathcal{J}^{\mathcal{T}} u)^{(l)}(t)\| + C_{\text{inv}} \left( \tau^{-l} \sup_{t \in I_n} \|(\mathcal{J}^{\mathcal{T}} u - U)(t)\| \right)
\]

\[
\leq \sup_{t \in I_n} \|(u - \mathcal{J}^{\mathcal{T}} u)^{(l)}(t)\| + C_{\text{inv}} \left( \tau^{-l} \sup_{t \in I_n} \|(\mathcal{J}^{\mathcal{T}} u - U)(t)\| + \sup_{t \in I_n} \|(u - U)(t)\| \right)
\]

where an inverse inequality was used. However, since we only have a non-local error estimate for \( \sup_{t \in I_n} \|(u - U)(t)\| \) we cannot expect that the inverse of the local time step length can be compensated in general. So, usually we additionally need to assume that \( \tau \leq \tau_{n+1} \) for all \( n \) or alternatively that the mesh is quasi uniform (\( \tau/n \leq C \) for all \( n \)) to obtain a proper estimate. \( \clubsuit \)

**Remark 4.7**

In the proof of Theorem 4.5 stiffness of the problem would be critical at several points.

Indeed, for large Lipschitz constants (hidden in \( \mathcal{C}_2 \) and so in \( \hat{C} \)) the needed inequality \( \hat{C} \tau/n < 1 \) would force very small time step lengths. For semidiscretizations in space of time-space problems, where the Lipschitz constant is typically proportional to \( h^{-2} \) with \( h \) denoting the spatial mesh parameter, this would cause upper bounds on the time step length with respect to \( h \) similar to CFL conditions.

Moreover, since the error constant \( C \) exponentially depends on \( \mathcal{C}_2 \), this constant would be excessively large for stiff problems such that the error estimate would be useless in this case. \( \clubsuit \)

Of course, Theorem 4.5 provides an abstract bound for the error of the variational time discretization method. However, the order of convergence still is not clear. Since \( \mathcal{T}^{\text{app}} \) is an Hermite-type interpolator of polynomial ansatz order larger than or equal to \( r \) its approximation order is at least \( r + 1 \). It remains to prove suitable bounds on the error of the approximation operator \( \mathcal{J}^{\mathcal{T}} \).

**Definition 4.8 (Approximation orders of \( \mathcal{G}_n \) and \( \mathcal{I}_n \))**

Let \( r_{\mathcal{G}} \), \( r_{\mathcal{G},r} \), \( r_{\mathcal{I}} \), and \( r_{\mathcal{I},r,i} \in \mathbb{N}_0 \cup \{-1, \infty\} \) denote the largest numbers such that

\[
\int_{I_n} \varphi(t) \, dt = \mathcal{G}_n[\varphi] \quad \forall \varphi \in P_{r_{\mathcal{G},r}}(I_n),
\]

\[
\int_{I_n} \varphi(t) \, dt = \int_{I_n} \mathcal{I}_n \varphi(t) \, dt \quad \forall \varphi \in P_{r_{\mathcal{G}}}(I_n),
\]

\[
\varphi = \mathcal{I}_n \varphi \quad \forall \varphi \in P_{r_{\mathcal{I}}}(I_n),
\]

\[
\mathcal{G}_n[\varphi \psi_i] = \mathcal{G}_n[\mathcal{I}_n \varphi] \psi_i \quad \forall \varphi \in P_{r_{\mathcal{G},r,i}}(I_n), \psi_i \in P_{r_{\mathcal{I}}}(I_n).
\]

Here, \( P_{-1}(I_n) \) is interpreted as \( \{0\} \), in which case the respective operator does not provide the corresponding approximation property. For convenience, set \( r_{\mathcal{I}} : = r_{\mathcal{I},r,k} \). Note that \( r_{\mathcal{G}} \geq r_{\mathcal{G},r} \geq r_{\mathcal{I}} \) and \( r_{\mathcal{I},r,i} \geq r_{\mathcal{I}} \) hold by definition.
Using standard techniques the above quantities can be connected with certain approximation estimates. For example, let \( r \in \mathbb{N}_0 \) then together with Assumption 2 we find that for arbitrary \( \varphi \in C^{\max \{k_J, \min \{r, r_J + 1\}\}}(I_n, \mathbb{R}^d) \)

\[
\left\| \int_{I_n} \varphi(t) \, dt - J_n[\varphi] \right\| \leq C \frac{\tau_n^{\alpha \{k_J, \min \{r, r_J + 1\}\}}}{2} \sum_{j=\min \{r, r_J + 1\}}^{\max \{k_J, \min \{r, r_J + 1\}\}} (\frac{\tau_n}{2})^j \sup_{t \in I_n} \|\varphi^{(j)}(t)\|. \tag{4.15}
\]

Furthermore, together with Assumption 3 it follows for arbitrary \( \varphi \in C^{\max \{k_J, \min \{r, r_J + 1\}\}}(T_n, \mathbb{R}^d) \)

\[
\left\| \int_{I_n} \varphi(t) - \mathcal{I}_n \varphi(t) \, dt \right\| \leq C \frac{\tau_n^{\alpha \{k_J, \min \{r, r_J + 1\}\}}}{2} \sum_{j=\min \{r, r_J + 1\}}^{\max \{k_J, \min \{r, r_J + 1\}\}} (\frac{\tau_n}{2})^j \sup_{t \in I_n} \|\varphi^{(j)}(t)\|. \tag{4.16}
\]

**Lemma 4.9**

Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r \), and suppose that Assumptions 1, 2, and 3 hold. Furthermore, let \( l, \tilde{r} \in \mathbb{N}_0 \) and define

\[ j_{\min, r} := \min \{\tilde{r}, r + 1, r_J + 2\}, \quad j_{\max, r} := \max \{k_J + 1, l, j_{\min, r}\}. \]

If \( v \in C^{j_{\max, r}}(I_n, \mathbb{R}^d) \) then the error estimate

\[
\sup_{t \in I_n} \left\| (v - J_n^{r, J} v)^{(l)}(t) \right\| \leq C \sum_{j=j_{\min, r}}^{j_{\max, r}} \left( \frac{\tau_n}{2} \right)^{j-l} \sup_{t \in I_n} \|v^{(j)}(t)\| =: T_n^{r, J, l}(v) \tag{4.17}
\]

holds with a constant \( C \) independent of \( \tau_n \).

**Proof:** The error estimate follows from standard approximation theory since \( J_n^{r, J} \) preserves under the given assumptions polynomials up to degree \( \min \{r, r_J + 1\} \). Here also the stability estimate for \( J_n^{r, J} \), cf. Lemma 4.2, which motivates the upper summation bound \( j_{\max, r} \). \( \square \)

As we shall see below, compared to the pointwise estimate of Lemma 4.9, the estimate for the approximation error of \( J_n^{r, J} \) in the mesh points \( t_n \) can be even improved in some cases. However, to this end we need some further knowledge on the approximation property connected with the quantity \( r_J^{J, J} \). Besides, the respective result presented in the following lemma will be also used later in the superconvergence analysis.

**Lemma 4.10**

Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r \). Suppose that the Assumptions 2 and 3 hold. Moreover, assume that \( \psi_i \in P_i(I_n, \mathbb{R}), i \in \mathbb{N}_0 \), satisfies \( \sup_{t \in I_n} \|\psi_i^{(j)}(t)\| \leq C \tau_n^{\alpha - j} \) for all \( j \in \mathbb{N}_0 \). Let \( \tilde{r} \in \mathbb{N}_0 \) and define

\[ j_{\min, i, r}^{*, J} := \min \{\tilde{r}, r_J + 1\}, \quad j_{\max, i, r}^{*, J} := \max \{k_J, k, j_{\min, i, r}^{*, J}\}. \]

Then, we have for \( \varphi \in C^{j_{\max, i, r}^{*, J}}(I_n, \mathbb{R}^d) \) the bound

\[
\| J_n[\varphi - \mathcal{I}_n \varphi(t) \psi_i(t)] \| \leq C \tau_n^{j_{\max, i, r}^{*, J}} \sum_{j=j_{\min, i, r}^{*, J}}^{j_{\max, i, r}^{*, J}} \left( \frac{\tau_n}{2} \right)^{j} \sup_{t \in I_n} \|\varphi^{(j)}(t)\| \tag{4.18}
\]

with a constant \( C \) independent of \( \tau_n \).

**Proof:** Let \( \tilde{\varphi} \in P_{\min \{r - 1, r_J\}}(I_n, \mathbb{R}^d) \) be arbitrarily chosen. We start rewriting the left-hand side of the wanted inequality as follows

\[
J_n[\varphi - \mathcal{I}_n \varphi(t) \psi_i(t)] = J_n[\varphi - \tilde{\varphi}(t) \psi_i(t)] + J_n[\tilde{\varphi} - \mathcal{I}_n \tilde{\varphi}(t) \psi_i(t)] + J_n[\mathcal{I}_n (\tilde{\varphi} - \varphi)(t) \psi_i(t)].
\]

Since the second summand on the right-hand side vanishes by definition of \( r_J^{J, J} \), we find

\[
\| J_n[\varphi - \mathcal{I}_n \varphi(t) \psi_i(t)] \| \leq \| J_n[\varphi - \tilde{\varphi}(t) \psi_i(t)] \| + \| J_n[\mathcal{I}_n (\tilde{\varphi} - \varphi)(t) \psi_i(t)] \|.
\]

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Exploiting Assumption 2, the Leibniz rule for the jth derivative, and the given bound for \( \psi_i^{(j)} \), we gain
\[
\| J_n [ (\varphi - I_n \varphi)(t) \psi_i (t) ] \|
\leq C \frac{\tau_n^2}{2} \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{j} \left( \sup_{t \in I_n} \| (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| + \sup_{t \in I_n} \| (I_n (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| \right)
\]
\[
\leq C \frac{\tau_n^2}{2} \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{j} \left( \sup_{t \in I_n} \| (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| + \sup_{t \in I_n} \| (I_n (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| \right)
\]
\[
\leq C \frac{\tau_n^2}{2} \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{j} \left( \sup_{t \in I_n} \| (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| + \sup_{t \in I_n} \| (I_n (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| \right)
\]

Further, using Assumption 3 which yields
\[
\sup_{t \in I_n} \| (I_n (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| \leq C \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{j} \sup_{t \in I_n} \| (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \|,
\]
we conclude that
\[
\| J_n [ (\varphi - I_n \varphi)(t) \psi_i (t) ] \|
\leq C \frac{\tau_n^2}{2} \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{j} \sup_{t \in I_n} \| (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \|
\]
\[
\leq C \frac{\tau_n^2}{2} \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{j} \sup_{t \in I_n} \| (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \|
\]

for all \( \tilde{\varphi} \in P_{\max\{r - 1, r_{2,i}^\phi\}}(I_n, \mathbb{R}^d) \).

Choosing \( \tilde{\varphi} \) as the Taylor polynomial of \( \varphi \) at \( (t_{n-1} + t_n)/2 \) of degree \( \min\{r - 1, r_{2,i}^\phi\} \), we have that
\[
\sup_{t \in I_n} \| (\varphi - \tilde{\varphi}) \psi_i^{(j)} (t) \| \leq C \left( \frac{\tau_n}{2} \right)^{\max\{j, \min\{r, r_{2,i}^\phi, +1\}\}} \sup_{t \in I_n} \| \psi_i^{(\max\{j, \min\{r, r_{2,i}^\phi, +1\}\})} (t) \|.
\]

So, overall it follows
\[
\| J_n [ (\varphi - I_n \varphi)(t) \psi_i (t) ] \|
\leq C \frac{\tau_n^2}{2} \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{\max\{j, \min\{r, r_{2,i}^\phi, +1\}\}} \sup_{t \in I_n} \| \psi_i^{(\max\{j, \min\{r, r_{2,i}^\phi, +1\}\})} (t) \|
\]
\[
\leq C \frac{\tau_n^2}{2} \sum_{j=0}^{k_g} \left( \frac{\tau_n}{2} \right)^{\max\{j, \min\{r, r_{2,i}^\phi, +1\}\}} \sup_{t \in I_n} \| \psi_i^{(j)} (t) \|
\]

which completes the proof. \( \square \)

Now, we can state and prove the improved estimate for the approximation error of \( J_{n,i}^\phi \) in the mesh points \( t_{n,i}^\phi \).
Lemma 4.11
Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r \). Suppose that the Assumptions 1, 2, and 3 hold. Moreover, assume that \( \max\{r_{ex}, r_{I} + 1\} \geq r - 1 \). Let \( \tilde{r} \in \mathbb{N}_0 \) and define
\[
J_{\min, \tilde{r}} := \min\{\tilde{r}, \max\{r_{ex} + 1, \min\{r, r_{I} + 1\} + 1, r_{I,0} + 2\}\}, \quad J_{\max, \tilde{r}} := \max\{k, r, J_{\min, \tilde{r}}\}.
\]
Then, provided \( v \in C^{\max, \tilde{r}}(I_n, \mathbb{R}^d) \), the estimate
\[
\left\| (v - J_{\hat{I}}v)(t_n) \right\| \leq C \sum_{j=J_{\min, \tilde{r}}}^{J_{\max, \tilde{r}}} \left( \frac{\tau_n}{2} \right)^j \sup_{t \in I_n} \|v^{(j)}(t)\| \leq \tau_{(4.18)}[v]
\]
holds for \( 1 \leq n \leq N \) where the constant \( C \) is independent of \( \tau_n \).

Proof: We start rewriting the term to be estimated. From the fundamental theorem of calculus and exploiting the definition (4.1) of \( J_{\hat{I}}^r \) we gain
\[
(v - J_{\hat{I}}^r v)(t_n) = (v - J_{\hat{I}}^r v)(t_{n-1}) + \int_{I_n} (v - J_{\hat{I}}^r v)'(t) dt = \delta_{n,k}(v - J_{\hat{I}}^r v)(t_{n-1}) + J_{\hat{I}}[v' - (J_{\hat{I}}^r v)'] - J_{\hat{I}}[v' - (J_{\hat{I}}^r v)'] + \int_{I_n} (v - J_{\hat{I}}^r v)'(t) dt.
\]
The first difference on the right-hand side can be estimated by (4.15). We obtain
\[
\left\| \int_{I_n} v'(t) dt - J_{\hat{I}}[v'] \right\| \leq C\frac{\tau_n}{2} \sum_{j=0}^{\max\{k, \min\{r-1, r_{ex} + 1\}\}} \left( \frac{\tau_n}{2} \right)^j \sup_{t \in I_n} \|v^{(j+1)}(t)\|.
\]
In order to bound the second difference, we apply Lemma 4.10 with \( i = 0, \psi_0 \equiv 1 \). It follows
\[
\left\| J_{\hat{I}}[v' - I_n v'] \right\| \leq C\frac{\tau_n}{2} \sum_{j=0}^{\max\{k, k_2, \min\{r-1, r_{I,0} + 1\}\}} \left( \frac{\tau_n}{2} \right)^j \sup_{t \in I_n} \|v^{(j+1)}(t)\|.
\]
Finally, if \( r_{ex} \geq r - 1 \) the third difference vanishes since then \( (J_{\hat{I}}^r v)' \in P_{r-1}(I_n, \mathbb{R}^d) \) is integrated exactly. If \( r_{I} + 1 \geq r - 1 \), i.e., \( J_{\hat{I}}^r \) preserves polynomials up to degree \( r - 1 \), we obtain from combining (4.15) and (4.17) that
\[
\left\| \int_{I_n} (J_{\hat{I}}^r v)'(t) dt - J_{\hat{I}}[(J_{\hat{I}}^r v)'] \right\| \leq C\frac{\tau_n}{2} \sum_{j=\min\{r-1, r_{ex} + 1\}}^{\min\{r-1, r_{I} + 1, r_{ex} + 1\}} \left( \frac{\tau_n}{2} \right)^j \sup_{t \in I_n} \|v^{(j)}(t)\| \sup_{t \in I_n} \|v^{(j+1)}(t)\| + \int_{I_n} (v - J_{\hat{I}}^r v)'(t) dt \leq \frac{\tau_{(4.17)}[v]}{2}.
\]
Let Assumption 5a or 5b be satisfied. Denoting by \( r, r_{\alpha}, r_{\alpha,0} \) respectively, we have for \( r \)
\[
\| (v - J_n^{r} v)(t_n) \| \leq C \sup_{t \in I_n} \|(u - U)(t)\| \leq C \max\{k, r, r_{\alpha} + 2, r_{\alpha,0} + 2\} \sum_{j = \min\{r, r_{\alpha} + 2, r_{\alpha,0} + 2\}}^{\max\{k, r, r_{\alpha} + 2, r_{\alpha,0} + 2\}} (\frac{\tau}{2})^j \sup_{t \in I_n} \|(u^{(j)}(t)\|.
\]
Therefore, recalling (4.17) which in some cases may provide a better estimate, we conclude the wanted statement. Here also note that always \( r_{I,r-k}^{\frac{r_{\alpha}}{r}} = r_{I,r-k}^{\frac{r_{\alpha}}{r}} \geq \min\{r, r_{I}^{\frac{r_{\alpha}}{r}}\} \).

Finally, summarizing the above results, we now want to list the proven convergence orders.

**Corollary 4.12**

Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r, \) and \( l \in \{0, 1\} \). Suppose that Assumptions 1, 2, 3, and 4 hold. Moreover, let Assumption 5a or 5b be satisfied. Denoting by \( u \) and \( U \) the solutions of (1.1) and (2.1), respectively, we have for \( 1 \leq n \leq N \)
\[
\sup_{t \in I_n} \|(u - U)(t)\| \leq C \max\{r, r_{\alpha}^{\frac{r_{\alpha}}{r}} + 1\},
\]
with \( r_{\alpha}^{\frac{r_{\alpha}}{r}} \) as defined in Definition 4.8. If we furthermore suppose that \( \max\{r_{ex}, r_{I}^{\frac{r_{\alpha}}{r}} + 1\} \geq r - 1, \) then we even have
\[
\sup_{t \in I_n} \|(u - U)(t)\| \leq C \max\{r, r_{\alpha}^{\frac{r_{\alpha}}{r}} + 1, r_{\alpha,0}^{\frac{r_{\alpha}}{r}} + 1, \max\{r_{ex}, r_{I}^{\frac{r_{\alpha}}{r}} + 1, \max\{r, r_{\alpha}^{\frac{r_{\alpha}}{r}} + 1\}\} \}
\]
as improved \( L^{\infty} \) estimate.

If \( \max\{r_{ex}, r_{I}^{\frac{r_{\alpha}}{r}} + 1\} \geq r - 1 \) is satisfied, we obtain formally
\[
\sup_{t \in I_n} \|(u - U)(t)\| \leq C \max\{r, r_{\alpha}^{\frac{r_{\alpha}}{r}} + 1, r_{\alpha,0}^{\frac{r_{\alpha}}{r}} + 1, \max\{r_{ex}, r_{I}^{\frac{r_{\alpha}}{r}} + 1, \max\{r, r_{\alpha}^{\frac{r_{\alpha}}{r}} + 1\}\} \}
\]
for the \( W^{1,\infty} \) seminorm. However, this gives the same convergence order as (4.19).

Since the quantity \( r_{\alpha}^{\frac{r_{\alpha}}{r}} = r_{I}^{\frac{r_{\alpha}}{r}} \) used in the lemmas and the corollary above is quite abstract, we want to provide some lower bound for \( r_{I,i}^{\frac{r_{\alpha}}{r}} \) based on the more familiar quantities \( r_{I}, r_{\alpha}^{\frac{r_{\alpha}}{r}} \), and \( r_{\alpha}^{\frac{r_{\alpha}}{r}} \). However, for this result, we need to formulate some further assumptions.

**Assumption 6**

The operator \( I_n \) is a projection onto the space of polynomials of maximal degree \( r_{I} < \infty \), i.e.,
\[
I_n : C^{kz}(\overline{T}_n) \rightarrow P_{r_{I}}(\overline{T}_n) \text{ and } I_n \varphi = \varphi \text{ for all } \varphi \in P_{r_{I}}(\overline{T}_n), \text{ or } I_n \text{ is the identity characterized by setting } r_{I} = \infty.
\]

**Assumption 7**

Given \( \ell \in \mathbb{N}_0 \), the identity
\[
\int_{I_n} I_n((\varphi - I_n \varphi)\psi)(t) \ dt = 0 \quad \forall \psi \in P_{\ell}(I_n)
\]
holds for all \( \varphi \in C^{kz}(\overline{T}_n) \). Note that Assumption 6 implies (4.21) for \( \ell = 0 \). If \( I_n \) is either the identity or an Hermite-type interpolation operator then condition (4.21) holds for any \( \ell \in \mathbb{N}_0 \).

We now give the lower bounds for \( r_{I,i}^{\frac{r_{\alpha}}{r}} \).

**Lemma 4.13**

Let \( r, k \in \mathbb{Z}, 0 \leq k \leq r, \) and \( i \in \mathbb{N}_0 \). Then, it always holds \( r_{I,i}^{\frac{r_{\alpha}}{r}} \geq r_{I} \). Supposing that Assumption 6 is fulfilled, we even get
\[
r_{I,i}^{\frac{r_{\alpha}}{r}} \geq \max\{r_{I}, \min\{r_{ex} - i, r_{I,i}^{\frac{r_{\alpha}}{r}}\}\}
\]
If \( I_n \) additionally satisfies Assumption 7 (with parameter \( \ell = i \)), we simply have
\[
r_{I,i}^{\frac{r_{\alpha}}{r}} \geq \max\{r_{I}, \min\{r_{ex}, r_{ex}^{\frac{r_{\alpha}}{r}}\} - i\}
\]
since then $r_{I_j,t}^f \geq \max\{r_{I_j}, r_{ex}^f - i\}$.

Furthermore, under the weaker assumption that $I = I^1 \circ \ldots \circ I^l$ is a composition of several operators $I^j$, $1 \leq j \leq l$, that all by themselves satisfy the Assumptions 6 and 7 (with parameter $\ell = i$), we still find

$$r_{I_j,t}^f \geq \min_{j \in M, \cup\{l\}} \{ \max\{r_{I_j}, r_{ex}^f - i\} \}$$

where $M_i := \{ j \in N : 1 \leq j \leq l - 1, \max\{r_{I_j}, r_{ex}^f - i\} < \min_{j+1 \leq m \leq \{r_{I_m}\}} \{r_{I_m}\} \}$.

**Proof:** Since by definition $\varphi = I_n[\varphi \psi]$ for $\varphi \in P_{I_2}(I_n)$, it also holds $J_n[\varphi \psi] = J_n[(I_n[\varphi \psi])]$ for $\varphi \in P_{I_2}(I_n)$ and sufficiently smooth $\psi$. This always implies $r_{I_j,t}^f \geq r_{I_2}$.

Now, assume that $\min\{r_{ex}^f - i, r_{I_j,t}^f\} > r_{I_2}$ and let $\varphi \in P_{\min\{r_{ex}^f - i, r_{I_j,t}^f\}}(I_n)$. Then Assumption 6 yields that $I_n[\varphi \psi] \in P_{\min\{r_{ex}^f - i, r_{I_j,t}^f\}}(I_n)$. Therefore, since $J_n[\varphi \psi]$ is exact for polynomials up to degree $r_{ex}^f$ and recalling the definition of $r_{I_j,t}^f$, it follows

$$J_n[(\varphi - I_n[\varphi \psi])\psi_1] = \int_{I_n} (\varphi - I_n[\varphi \psi])\psi_1 dt = 0 \quad \text{for all} \; \psi_1 \in P_I(I_n).$$

Hence, $r_{I_j,t}^f \geq \min\{r_{ex}^f - i, r_{I_j,t}^f\}$, if $\min\{r_{ex}^f - i, r_{I_j,t}^f\} > r_{I_2}$. So, altogether under Assumption 6 we have shown that $r_{I_j,t}^f \geq \max\{r_{I_j}, \min\{r_{ex}^f - i, r_{I_j,t}^f\}\}$.

Finally, we assume that $I = I^1 \circ \ldots \circ I^l$ where each $I^j$ satisfies the Assumptions 6 and 7. Let $\varphi \in P_{I_2}(I_n)$ with $\ell \geq 0$ to be specified later and $\psi_1 \in P_I(I_n)$, then

$$\int_{I_n} (\varphi - I_n[\varphi \psi])\psi_1 dt$$

$$= \int_{I_n} (I_n - I_n[\varphi])\psi_1 dt + \sum_{j=1}^{l-1} \int_{I_n} ((I_n - I_n)[I_n[\varphi]])\psi_1 dt$$

$$= \int_{I_n} (I_n - I_n[\varphi])\psi_1 dt + \sum_{j \in M, \cup\{l\}} \int_{I_n} (I_n - I_n)[I_n[\varphi]]\psi_1 dt$$

$$= \int_{I_n} (I_n - I_n[\varphi])((I_n - I_n)[I_n[\varphi]])\psi_1 dt + \sum_{j \in M, \cup\{l\}} \int_{I_n} (I_n - I_n)((I_n - I_n)[I_n[\varphi]])\psi_1 dt$$

$$+ \int_{I_n} (I_n - I_n[\varphi])((I_n - I_n)[I_n[\varphi]])\psi_1 dt + \sum_{j \in M, \cup\{l\}} \int_{I_n} (I_n - I_n)((I_n - I_n)[I_n[\varphi]])\psi_1 dt.$$

Because of (4.21), both terms in the last line vanish. Here note that actually (4.21) is only needed for $I_n$ with $j \in M, \cup\{l\}$ and $\varphi \in P_{\min\{r_{I_2} \cup\{l\}}}) \{I_n\}$. Moreover, since

$$((I_n - I_n)[I_n[\varphi]])\psi_1 \in P_{\min\{r_{I_2} \cup\{l\}}}) \{I_n\}$$

also those summands of the penultimate line with $\max\{r_{I_j}, r_{ex}^f - i\} \geq \min_{j+1 \leq m \leq l} \{r_{I_m}\}$ vanish. Hence, we obtain

$$\int_{I_n} (\varphi - I_n[\varphi \psi])\psi_1 dt$$

$$= \int_{I_n} (I_n - I_n[\varphi])(\varphi\psi_1) dt + \sum_{j \in M, \cup\{l\}} \int_{I_n} (I_n - I_n)[I_n[\varphi]]\psi_1 dt.$$

From this identity, we easily find that $r_{I_j,t}^f \geq \min_{j \in M, \cup\{l\}} \{ \max\{r_{I_j}, r_{ex}^f - i\} \}$. Note that here Assumption 6 is exploited to guarantee that for a polynomial $\varphi$ the degree of $I_n[\varphi]$ is never greater than that of $\varphi$. 

\[\square\]
5 Superconvergence analysis

In order to prove superconvergence in time mesh points, we shall exploit a special representation of the discrete problem (2.1). However, to this end we will define the approximation operator \(\tilde{P} : C^0([-1, 1]) \to P_{r-1}([-1, 1])\) by

\[
\begin{align*}
(\tilde{P}^\mathcal{I}_E v)^{(i)}(t^{n-1}_i) &= \tilde{v}^{(i)}(t^{n-1}_i), & \text{if } k \geq 3, i = 0, \ldots, \left\lceil \frac{k-1}{2} \right\rceil - 1, \quad (5.1a) \\
(\tilde{P}^\mathcal{I}_E v)^{(i)}(t^n_i) &= \tilde{v}^{(i)}(t^n_i), & \text{if } k \geq 2, i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1, \quad (5.1b) \\
\tilde{G}^E (\tilde{P}^\mathcal{I}_E \tilde{v}) &= \tilde{G}^E (\tilde{I} \tilde{v}), & \forall \tilde{v} \in P_{r-k}([-1, 1]) \text{ with } \delta_{0, k} \tilde{v}(+1) = 0. \quad (5.1c)
\end{align*}
\]

Note that \(\tilde{P}^\mathcal{I}_E\) is connected to \(\tilde{G}^E\) in such a way that \(\tilde{P}^\mathcal{I}_E (\tilde{v}^\prime) = (\tilde{G}^E \tilde{v})^\prime\) holds true. Analogously to the respective lemmas for \(\tilde{G}^E\) (Lemmas 3.1, 4.1, and 4.9), we also get the following properties and estimates for the operator \(\tilde{P}^\mathcal{I}_E\).

**Lemma 5.1**

Let \(r \in \mathbb{N} \) and \(k \in \mathbb{Z}\) such that \(0 \leq k \leq r\). Suppose that Assumption 1 holds. Then \(\tilde{P}^\mathcal{I}_E\) given by (5.1) is well-defined. If \(\tilde{I}\) preserves polynomials up to degree \(\tilde{r}\) then \(\tilde{P}^\mathcal{I}_E\) preserves polynomials up to degree \(\min \{\tilde{r}, r - 1\}\).

**Lemma 5.2 (Approximation operator)**

Let \(r \in \mathbb{N}\) and \(k \in \mathbb{Z}\) such that \(0 \leq k \leq r\). Moreover, suppose that Assumption 1 holds. Then the operator \(P^\mathcal{I}_E : C^k([-1, 1]) \to P_{r-1}([-1, 1])\) given by

\[
\begin{align*}
(P^\mathcal{I}_E v)^{(i)}(t^{n-1}_i) &= v^{(i)}(t^{n-1}_i), & \text{if } k \geq 3, i = 0, \ldots, \left\lceil \frac{k-1}{2} \right\rceil - 1, \quad (5.2a) \\
(P^\mathcal{I}_E v)^{(i)}(t^n_i) &= v^{(i)}(t^n_i), & \text{if } k \geq 2, i = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor - 1, \quad (5.2b) \\
\tilde{G}_n^E ([P^\mathcal{I}_E v(t), \varphi(t)]) &= \tilde{G}_n^E ([I_n v(t), \varphi(t)]) & \forall \varphi \in P_{r-k}([-1, 1]) \text{ with } \delta_{0, k} \varphi(t^n_i) = 0, \quad (5.2c)
\end{align*}
\]

is well-defined.

Furthermore, let Assumptions 2 and 3 hold. For \(l, \tilde{r} \in \mathbb{N}_0\), we define

\[j_{\min, \tilde{r}} := \min \{\tilde{r}, r, r^2 + 1\}, \quad j_{\max, l, \tilde{r}} := \max \{k, j, j_{\min, \tilde{r}}\}.\]

Then, we have for \(v \in C^{j_{\max, l, \tilde{r}}}([-1, 1])\) the error estimates

\[
\sup_{t \in I_n} \| (v - P^\mathcal{I}_E v)^{(i)}(t) \| \leq C \sum_{j=j_{\min, \tilde{r}}}^{j_{\max, l, \tilde{r}}} (\frac{r}{2})^{j-i} \sup_{t \in I_n} \| v^{(j)}(t) \| =: T_{(5.3)}^E [v]
\]

with a constant \(C\) independent of \(\tau_n\).

In order to cover also the setting \(r = k\), we agree that in this case the operators \(\tilde{P}^\mathcal{I}_E\) and \(P^\mathcal{I}_E\) always return the zero function.

Inspecting the definition of the discrete solution \(U\) of problem (2.1) and (5.2), we find that

\[U'|_{I_n} = P^\mathcal{I}_E f(\cdot, U(\cdot)).\]

Therefore, since for \(k \geq 1\) furthermore \(U(t^n_i) = U^n_{i-1}\) with \(U^n_{i-1}\) given, we also see that the discretization method fits into the unified framework of [1] for \(k \geq 1\).

**Theorem 5.3 (Superconvergence estimate)**

Let \(r, k \in \mathbb{Z}\), \(0 \leq k \leq r\). Suppose that the Assumptions 1, 2, and 3 hold. Moreover, denote by \(u\) and \(U\) the solutions of (1.1) and (2.1), respectively. Suppose that \(\tau\) (for \(\tau\) sufficiently small) the global error \(\sup_{t \in I} \| (u - U)(t) \|\), as well as \(U\) and all of its derivatives, can be bounded independent of the mesh parameter. Then we have

\[
\| (u - U)(t^n_i) \| \leq C(f, u) \left( \sup_{t \in I} \| (u - U)(t) \|^2 + r^{\min(2r-k+1, r_{\max, l, \tilde{r}} + 1)} \right) \quad (5.5)
\]

where \(r_{\max, l, \tilde{r}} := \min \{0, \tilde{r} - k \} \{r_{\max, l, \tilde{r}} + 1\} \).
Proof: In order to prove the wanted statement, we firstly derive an estimate for the error \( e(t) = (u - U)(t) \) at \( t_n^- \) provided that we already have a suitable bound at \( t_n^- \). To this end, we adapt some basic ideas known from the superconvergence proof for collocation methods, see e.g. [9, Theorem II.1.5, p. 28]. So, we consider the local discrete solution \( U|_{I_n} \) as solution \( \tilde{U} \) of the perturbed initial value problem

\[
\tilde{U}'(t) = f(t, \tilde{U}(t)) + \text{def}(t) \quad \text{in} \quad I_n, \quad \tilde{U}(t_n^-) = U(t_n^- + 1),
\]

where \( \text{def}(t) := U'(t) - f(t, U(t)) \) denotes the defect. Since \( u \) solves (1.1), we find for all \( t \in I_n \)

\[
U'(t) - u'(t) = f(t, U(t)) - f(t, u(t)) + \text{def}(t)
\]

\[
= \int_0^1 \frac{\partial}{\partial v} f(t, u(t) + s(U(t) - u(t))) \left( U(t) - u(t) \right) \, ds + \text{def}(t)
\]

\[
= \frac{\partial}{\partial v} f(t, u(t))(U(t) - u(t)) + \text{def}(t)
\]

\[
+ \int_0^1 \int_0^s \left( U(t) - u(t) \right) T \frac{\partial^2}{\partial v^2} f(t, u(t) + \tilde{s}(U(t) - u(t))) \left( U(t) - u(t) \right) \, d\tilde{s} \, ds.
\]

Here \( f \) is interpreted as function of \((t, u)\) such that \( \frac{\partial}{\partial v} f \) for example is the derivative of \( f \) with respect to the second (function) argument. Also note that we used for the last identity that

\[
\frac{\partial}{\partial v} f(t, u(t) + s(U(t) - u(t))) = \frac{\partial}{\partial v} f(t, u(t)) + \int_0^s \left( U(t) - u(t) \right) T \frac{\partial^2}{\partial v^2} f(t, u(t) + \tilde{s}(U(t) - u(t))) \, d\tilde{s}.
\]

Shortly, we have

\[
e'(t) = \frac{\partial}{\partial v} f(t, u(t)) e(t) - \text{def}(t) - \text{rem}(t) \quad \forall t \in I_n.
\]

Therefore, the variation of constants formula, cf. e.g. [10, Theorem I.112, p. 66], yields

\[
e(t) = R(t, t_n^+)(e(t_n^-)) - \int_{t_n^-}^t R(t, s)(\text{def}(s) + \text{rem}(s)) \, ds \quad \forall t \in I_n
\]

where \( R(t, s) \) is the resolvent of the homogeneous differential equation \( y'(t) = \frac{\partial}{\partial v} f(t, u(t)) y(t) \) for initial values given at \( s \). Using that \( u \) is continuous as well as \( U \) for \( k \geq 1 \) we conclude

\[
e(t_n^-) = R(t_n^-, t_n^+)(e(t_n^-)) - \delta_{0,k} R(t_n^-, t_n^-)(U|_{I_n}) - \int_{t_n^-}^t R(t_n^-, s)(\text{def}(s) + \text{rem}(s)) \, ds. \tag{5.6}
\]

After splitting the right-hand side in an appropriate way, we shall study the single terms separately.

First, for the term including \( e(t_n^-) \) we gain

\[
R(t_n^-, t_n^+)(e(t_n^-)) = \left( \frac{\partial}{\partial t} R(s, t_n^-)(\tilde{s}) \right) \left. e(t_n^-) \right|_{s=0}
\]

which implies

\[
\|R(t_n^-, t_n^+)(e(t_n^-))\| \leq \left( 1 + \tau_n \sup_{s \in I_n} \left\| \frac{\partial}{\partial t} R(s, t_n^-)(\tilde{s}) \right\| \right) \|e(t_n^-)\| \leq (1 + C\tau_n) \|e(t_n^-)\|. \tag{5.7}
\]

Furthermore, the term including the remainder term \( \text{rem}(\cdot) \) can be bounded as follows

\[
\left\| \int_{I_n} R(t_n^-, s) \, \text{rem}(s) \, ds \right\| \leq \tau_n \sup_{s \in I_n} \|R(t_n^-, s)\| \sup_{s \in I_n} \|\text{rem}(s)\|,
\]

\[
\leq \tau_n \sup_{s \in I_n} \|R(t_n^-, s)\| \left( \frac{1}{2} \sup_{s \in I_n} \sup_{\tilde{s} \in [0,1]} \left\| \frac{\partial}{\partial v} f(s, u(s) - \tilde{\epsilon}e(s)) \right\| \right) \sup_{s \in I_n} \|e(s)\|^2
\]

\[
\leq C\tau_n \sup_{s \in I_n} \|e(s)\|^2. \tag{5.8}
\]
Here, for the last step we also exploited that $u(s) - \tilde{s}e(s)$ is in a bounded neighborhood of $u(s)$ for all $s \in I_n$, $\tilde{s} \in [0,1]$, since by assumption $\|e(s)\| \leq C$.

Finally, the remaining terms are considered. A Taylor series expansion of $R(t_n^-, s)$ with respect to $s$ in $t_n^{*-\frac{1}{2}}$ gives

$$R(t_n^-, s) = \sum_{i=0}^{r-k} \frac{(s - t_n^{-\frac{1}{2}})^i}{i!} \frac{\partial^i}{\partial s^i} R(t_n^-, t_n^{*-\frac{1}{2}}) + \int_{t_n^{*-\frac{1}{2}}}^s (s - t_n^{-\frac{1}{2}})^{r-k} \frac{\partial^{r-k+1}}{\partial s^{r-k+1}} R(t_n^-, \tilde{s}) \, d\tilde{s}.$$  

This formula motivates the following decomposition

$$\delta_{0,k} R(t_n^-, t_n^{*-\frac{1}{2}}) [U]_{n-1} + \int_{I_n} R(t_n^-, s) \, ds (5.9)$$

$$= \sum_{i=0}^{r-k} \frac{1}{i!} \frac{\partial^i}{\partial s^i} R(t_n^-, t_n^{*-\frac{1}{2}}) \left( \int_{I_n} (s - t_n^{-\frac{1}{2}})^{i} (U'(s) - f(s, U(s))) \, ds + \delta_{0,k} \delta_{0,1} [U]_{n-1} \right)$$

$$+ \int_{I_n} \int_{t_n^{*-\frac{1}{2}}}^s (s - t_n^{-\frac{1}{2}})^{r-k} \frac{\partial^{r-k+1}}{\partial s^{r-k+1}} R(t_n^-, \tilde{s}) \, d\tilde{s} (U'(s) - f(s, U(s))) \, ds .$$

We start with the second term and bound (II) as follows

$$\|\text{(II)}\| \leq \int_{I_n} \int_{I_n} \frac{(s - t_n^{-\frac{1}{2}})^{r-k}}{(r-k)!} \frac{\partial^{r-k+1}}{\partial s^{r-k+1}} R(t_n^-, \tilde{s}) \, d\tilde{s} \|U'(s) - f(s, U(s))\| \, ds$$

$$\leq \tau_n \frac{\tau_n^{r-k+1}}{(r-k)!} \sup_{\tilde{s} \in I_n} \left\| \frac{\partial^{r-k+1}}{\partial s^{r-k+1}} R(t_n^-, \tilde{s}) \right\| \sup_{s \in I_n} \|U'(s) - f(s, U(s))\| .$$

Because of (5.4) the last term on the right-hand side can be rewritten as an approximation error. Indeed, we have

$$\sup_{s \in I_n} \|U'(s) - f(s, U(s))\| = \sup_{s \in I_n} \|(1 - \mathcal{P}_n) f(s, U(s))\| \leq \tau_{(5.3)}^{\min(r, r^2, r^2+1)} \|f(\cdot, U(\cdot))\|$$

where we used the estimate of (5.3) with $v(\cdot) = f(\cdot, U(\cdot))$. This results in

$$\|\text{(II)}\| \leq C \tau_n^{\frac{\min(r, r^2, r^2+1)}{2} + r-k+1} \max_{j \geq 0} \left\| \frac{\partial^{j}}{\partial s^{j}} f(s, U(s)) \right\| .$$

In order to estimate (I) for $0 \leq i \leq r-k$, we split the term as

$$(I) = \int_{I_n} (s - t_n^{-\frac{1}{2}})^{i} (U'(s) - f(s, U(s))) \, ds + \delta_{0,k} \delta_{0,1} [U]_{n-1}$$

$$= \left( \int_{I_n} (s - t_n^{-\frac{1}{2}})^{i} (U'(s) - f(s, U(s))) \, ds - \mathcal{J}_n [(s - t_n^{-\frac{1}{2}})^{i} (U'(s) - f(s, U(s)))] \right)$$

$$+ \mathcal{J}_n [(s - t_n^{-\frac{1}{2}})^{i} (U'(s) - f(s, U(s)))] + \delta_{0,k} \delta_{0,1} [U]_{n-1}$$

where (2.1d) was used for the last identity. The first given difference on the right-hand side can
be bounded by (4.15). We gain, additionally applying Leibniz’ rule for the $j$th derivative,

$$\left\| \int_{I_n} (s - t_{n-1})^j(U'(s) - f(s, U(s))) \, ds - J_n[(s - t_{n-1})^j(U'(s) - f(s, U(s)))] \right\|$$

$$\leq C \frac{T_n}{2} \max \{ k, \min \{ r, \frac{r}{r_{\infty}^2} + 1 \} \} \sum_{j=0}^{\max \{ k, \min \{ r, \frac{r}{r_{\infty}^2} + 1 \} \} + 1} \left( \frac{\tau_n}{T_n} \right)^j \sup_{s \in I_n} \left\| \frac{d^j}{ds^j} (s - t_{n-1})^j(U'(s) - f(s, U(s))) \right\|$$

$$\leq C \frac{T_n}{2} \sum_{j=0}^{\max \{ k, \min \{ r, \frac{r}{r_{\infty}^2} + 1 \} \} + 1} \left( \frac{\tau_n}{T_n} \right)^j \sum_{m=\max \{ 0, j-i \}}^{j} \sup_{s \in I_n} \left\| \frac{d^m}{ds^m} (U'(s) - f(s, U(s))) \right\|$$

Recalling (5.4), the last supremum could be further estimated by (5.3) with $v(\cdot) = f(\cdot, U(\cdot))$. More detailed, it holds

$$\sup_{s \in I_n} \left\| \frac{d^m}{ds^m} (U'(s) - f(s, U(s))) \right\| \leq \left\| \sup_{s \in I_n} \left\| \frac{d^m}{ds^m} (P_{\infty} T - \text{Id}) f(s, U(s)) \right\| \leq T_r^m \left\| f(\cdot, U(\cdot)) \right\|, \text{ if } m \leq \bar{r} = \min \{ \bar{r}, r, \frac{r}{r_{\infty}^2} + 1 \}, \right.$$  

otherwise. Note that disregarding regularity aspects the choice $\bar{r} = \min \{ \bar{r}, r, \frac{r}{r_{\infty}^2} + 1 \}$ in Lemma 5.2 is allowed. Therefore, we have

$$\left\| \int_{I_n} (s - t_{n-1})^j(U'(s) - f(s, U(s))) \, ds - J_n[(s - t_{n-1})^j(U'(s) - f(s, U(s)))] \right\|$$

$$\leq C \frac{T_n}{2} \max \{ \min \{ r, \frac{r}{r_{\infty}^2} + 1 \}, \min \{ r, \frac{r}{r_{\infty}^2} + 1 \} + 1 \} \sum_{j=0}^{\max \{ k, \min \{ r, \frac{r}{r_{\infty}^2} + 1 \} \} + 1} \sup_{s \in I_n} \left\| U^{(j+1)}(s) \right\| + \left\| \frac{d^j}{ds^j} f(s, U(s)) \right\|.$$  

Factoring out $\tau_n^i$, the second term on the right-hand side of (5.10) can be analyzed by Lemma 4.10 with $\varphi(t) = f(t, U(t))$ and $\psi(t) = \left( \frac{T_{n-1}}{\tau_n^i} \right)^i$. Then we obtain

$$\left\| J_n[(s - t_{n-1})^j(\text{Id} - I_n) f(s, U(s))] \right\| \leq C \frac{T_n}{2} \sum_{j=0}^{\max \{ r, \frac{r}{r_{\infty}^2} + 1 \} + 1} \sup_{s \in I_n} \left\| \frac{d^j}{ds^j} f(s, U(s)) \right\|$$

with a constant $C$ independent of $\tau_n$ where

$$j_{\min, i, r} = \min \{ r, \frac{r}{r_{\infty}^2} + 1 \}, \quad j_{\max, i, r} = \max \{ k, r, j^*_{\min, i, r} \}.$$  

So, altogether we have

$$\left\| (I) \right\| \leq C \frac{T_n}{2} \sum_{j=0}^{\max \{ r, \frac{r}{r_{\infty}^2} + 1 \} + 1} \left( \sup_{s \in I_n} \left\| U^{(j+1)}(s) \right\| + \sup_{s \in I_n} \left\| \frac{d^j}{ds^j} f(s, U(s)) \right\| \right)$$
where for abbreviation we set

\[ r_{t,s} := \min \{ \bar{r} + i, r_{I,s} + i + 1, \max \{ \min \{ \bar{r}, r_{ex} + 1 \}, \min \{ r, r_{I} + 1 \} + i \} \}, \]

\[ j_{\text{max},i,r} := \max \{ k, \min \{ \bar{r}, r_{I,s} + 1 \}, \min \{ \bar{r}, r_{I} + 1 \}, \min \{ \bar{r}, r_{ex} + 1 \} \}. \]

Combining (5.9) with the above estimates for (I) and (II), we gain for \( \bar{r} \) sufficiently large

\[
\left\| \delta_{0,h} R(t_n, t_{n-1}) [U]_{n-1} + \int_{t_{n-1}}^{t_n} R(t,s) \text{det}(s) \, ds \right\| \leq C \frac{\tau_n}{2} \min \{ 2r - k + 1, r_{I} + 1 \} \right\}
\]

with \( r_{\text{var}} := \min_{0 \leq k \leq \bar{r}} \{ r_{I,s} + i \} \) and

\[ G_n := \sum_{i=0}^{r - k + 1} \sup_{\bar{s} \in I_n} \left\| \frac{\partial}{\partial t} \left( \delta_{0,h} R(t_n, \bar{s}) \right) \right\| \left( \sup_{s \in I_n} \left\| U^{(j+1)}(s) \right\| + \sup_{s \in I_n} \left\| \frac{d^j}{ds^j} f(s, U(s)) \right\| \right). \]

Here note that because of \( r_{\text{var}} \leq r_{I,s} \leq r - k = r_{I} + r - k \) the second term in the minimum on the right-hand side of (5.11) could be dropped. Therefore, incorporating (5.7), (5.8), and (5.11) in (5.6) gives

\[
\left\| e(t_n) \right\| \leq \left( 1 + \tau_n \sup_{\bar{s} \in I_n} \left\| \frac{\partial}{\partial t} \left( \delta_{0,h} R(\bar{s}, t_{n-1}) \right) \right\| \right) \left\| e(t_{n-1}) \right\| + C \tau_n \sup_{s \in I_n} \left\| e(s) \right\|^2
\]

\[ + C \frac{\tau_n}{2} \min \{ 2r - k + 1, \min \{ r_{ex} + 1, \min \{ r, r_{I} + 1 \} \} \} G_n. \]

A variant of the discrete Gronwall lemma, see [7, Proposition 3.3], applied with

\[ a_n = \left\| e(t_n) \right\|, \quad 0 = \theta, \quad \lambda_n = \sup_{\bar{s} \in I_n} \left\| \frac{\partial}{\partial t} \left( \delta_{0,h} R(\bar{s}, t_{n-1}) \right) \right\|, \]

\[ g_n = C \sup_{s \in I_n} \left\| e(s) \right\|^2 + C \frac{\tau_n}{2} \min \{ 2r - k + 1, \min \{ r_{ex} + 1, \min \{ r, r_{I} + 1 \} \} \} G_n, \]

then yields

\[
\left\| e(t_n) \right\| \leq \left( 1 + \lambda_n \tau_j \right) \left( \prod_{j=1}^{\nu} (1 + \lambda_j \tau_j) \right)
\]

\[ + C \sum_{\nu=1}^{\nu} \sup_{s \in I_\nu} \left\| e(s) \right\|^2 + \frac{1}{2} \min \{ 2r - k + 1, \min \{ r_{ex} + 1, \min \{ r, r_{I} + 1 \} \} \} G_\nu \left( \prod_{j=\nu}^{\nu} (1 + \lambda_j \tau_j) \right). \]

From the well known inequality \( (1 + x) \leq e^x \), it follows

\[
\prod_{j=1}^{\nu} (1 + \lambda_j \tau_j) \leq \prod_{j=1}^{\nu} (1 + \lambda_j \tau_j) \leq \exp \left( \sum_{\nu=1}^{\nu} \lambda_\nu \tau_\nu \right) \leq \exp \left( (t_n - t_0) \max_{\nu=1,\ldots,n} \lambda_\nu \right) \cdot \]

Hence, we conclude that

\[
\left\| e(t_n) \right\| \leq C \left( t_n - t_0 \right) \exp \left( (t_n - t_0) \max_{\nu=1,\ldots,n} \lambda_\nu \right)
\]

\[ \max_{\nu=1,\ldots,n} \sup_{s \in I_\nu} \left\| e(s) \right\|^2 + \tau_\nu \min \{ 2r - k + 1, \min \{ r_{ex} + 1, \min \{ r, r_{I} + 1 \} \} \} G_\nu \]

where we also used \( e(t_0) = 0 \).

It remains a small technical detail to verify that \( G_\nu \) can be uniformly bounded independent of \( \tau_\nu \). The term depends on partial derivatives of \( f \), derivatives of \( R \), and on the derivatives of the discrete solution \( U \), thus, potentially also on the mesh parameter. However, \( U \) can be uniformly bounded by assumption. So, we are done.
**Remark 5.4**

Using an alternative argument (inspired by the proof of [10, Theorem II.7.9, pp. 212/213]) that is based on the application of the nonlinear variation-of-constants formula [10, Corollary I.14.6, p. 97], it can be shown that for $1 \leq k \leq r$ the term $\sup_{t \in I_n} \| (u - U)(t) \|^2$ in (5.5) is not necessary and can be dropped.

However, for $k = 0$ the alternative proof is much more complicated and in general only guarantees a worse superconvergence estimate than Theorem 5.3. Moreover, for all $k$ the notation gets more involved.

**Lemma 5.5**

Suppose that Assumption 1 holds along with an estimate similar to (5.3) that at least guarantees approximation order $r - 1$ for $P_n^{j,x}$ (e.g. if $r_j^m \geq r - 2$). In addition, let the solutions $u$ of (1.1) and $U$ of (2.1) satisfy $\sup_{t \in I_n} \| (u - U)(t) \| \leq C$ for some constant $C$ independent of the mesh parameter. Then we have

$$\sup_{t \in I_n} \| U^{(l)}(t) \| \leq C(f, u)$$

for all $0 \leq l \leq r$.

**Proof:** We first of all note that the assumed estimate for the local error implies

$$\sup_{t \in I_n} \| U(t) \| \leq \sup_{t \in I_n} \| u(t) \| + \sup_{t \in I_n} \| u(t) - U(t) \| \leq C(f, u).$$

Using this as basis we can prove the wanted estimates by induction, exploiting the identity (5.4). For $0 \leq l \leq r - 1$ we proceed as follows

$$\sup_{t \in I_n} \| U^{(l+1)}(t) \| = \sup_{t \in I_n} \left\| \frac{d^l}{dt^l} P_n^{j,x} f(t, U(t)) \right\|$$

$$\leq \sup_{t \in I_n} \left\| \frac{d^l}{dt^l} f(t, U(t)) \right\| + \sup_{t \in I_n} \left\| \frac{d^l}{dt^l} (\text{Id} - P_n^{j,x}) f(t, U(t)) \right\|.$$  

The first term on the right-hand side contains derivatives of $U$ up to maximal order $l$, so can be bounded by $C(f, u)$ due to the induction hypothesis. It remains to study the second term.

Using (5.3) we obtain

$$\sup_{t \in I_n} \left\| \frac{d^l}{dt^l} (\text{Id} - P_n^{j,x}) f(t, U(t)) \right\| \leq C \sum_{j = j_{\min}}^{j_{\max}} \tau_{j-n}^{j-l} \sup_{t \in I_n} \left\| \frac{d^j}{dt^j} f(t, U(t)) \right\|$$

(5.12)

with some non-negative integer constants $r - 1 \leq j_{\min} \leq j_{\max}$. Now, according to a generalization of Faà di Bruno’s formula, see [14, Theorem 2.1] for details, we know that

$$\frac{d^l}{dt^l} f(t, U(t)) = \sum_{Q} C_Q \prod_{m = 0}^{p_m} \frac{\partial^{\sum_{i = 0}^{d} p_m} f}{\partial u_1^{p_1} \ldots \partial u_d^{p_d}}(t, U(t)) \prod_{i = 1}^{j} \left( \frac{d^i}{dt^i} \right)^{q_m} \prod_{m = 1}^{d} (U^{(i)}(t))^{q_m}$$

where the sum is over the non-negative integer solutions $Q = (q_{im})_{1 \leq i \leq j, 0 \leq m \leq d}$ of

$$\sum_{i = 1}^{j} i \left( \sum_{m = 0}^{d} q_{im} \right) = j$$

(5.13)

and

$$C_Q := \frac{j!}{\prod_{m = 0}^{d} (m!)^{\sum_{i = 0}^{d} q_{im}}} \prod_{m = 1}^{d} q_{im}!, \quad p_m := \sum_{i = 1}^{j} q_{im}.$$ 

Note that this formula especially implies that derivatives of $U$ appear up to maximal order $j$.
Now, taking into account that \( \frac{dq}{dt} = \delta_{1,i}, \ i \geq 1 \), the above summands can be bounded by

\[
C \left\| \frac{\partial^{d} \sum_{m=0}^{p} f^{m}}{\partial u_{1}^{m} \cdots \partial u_{d}^{m}} (t, U(t)) \right\| \prod_{i=1}^{j} \prod_{m=1}^{d} |U_{m}^{(i)}(t)|^{q_{im}}
\]

where the double product can be further estimated as

\[
\prod_{i=1}^{j} \prod_{m=1}^{d} |U_{m}^{(i)}(t)|^{q_{im}} \leq \prod_{i=1}^{j} \prod_{m=1}^{d} \|U^{(i)}(t)\|^{q_{im}} = \prod_{i=1}^{j} \|U^{(i)}(t)\|^{\sum_{m=1}^{d} q_{im}}.
\]

So, applying an inverse inequality multiple times, more precisely at most \( j \) times because of (5.13), we obtain

\[
\prod_{i=1}^{j} \|U^{(i)}(t)\|^{\sum_{m=1}^{d} q_{im}} \leq \prod_{i=1}^{j} \left( C_{\text{inv}} \left( \frac{\tilde{T}}{\tau_{n}} \right)^{-i} \sup_{t \in I_{n}} \|U(t)\| \right)^{\sum_{m=1}^{d} q_{im}} = C \left( \frac{\tilde{T}}{\tau_{n}} \right)^{-\sum_{i=1}^{j} (\sum_{m=1}^{d} q_{im})} \sup_{t \in I_{n}} \|U(t)\|.
\]

Note that this also implies that after applying an inverse inequality at most \( j - l \) times, the right-hand side only depends on derivatives of \( U \) of order less than or equal to \( l \). Therefore the summands on the right-hand side of (5.12) can be estimated (independent of \( \tau_{n} \)) by terms that contain derivatives of \( U \) up to maximal order \( l \). Hence, again by the induction hypotheses we gain the upper bound \( C(f, u) \).

\[ \square \]

Summarizing the above observations, our analysis guarantees the following estimates in the time mesh points.

**Corollary 5.6**

Let \( r, k \in \mathbb{Z}, \ 0 \leq k \leq r \). Suppose that Assumptions 1, 2, 3, and 4 hold. Moreover, let Assumption 5a or 5b be satisfied. Let \( u \) and \( U \) denote the solutions of (1.1) and (2.1), respectively. Then, if \( r_{f}^{2} \geq r - 2 \), we have for \( 1 \leq n \leq N \)

\[
\|(u - U)(t_{n}^{-})\| \leq C \left( \max_{k} \{ 2r - k + 1, r_{f}^{2} + 1, \max\{ r_{\varphi}^{2} + 1, \min\{ r_{\varphi}^{2} + 1 \} \} \} + \delta_{0,k}2r_{f}^{2} + 1 \right),
\]

(5.14)

with \( r_{f}^{2} := \min_{0 \leq i \leq r} \{ r_{f, i}^{2} + i \}, \ r_{f}^{2} = r_{f, r-k}^{2}, \) and \( r_{f, i}^{2} \) as defined in Definition 4.8.

If \( r_{f}^{2} < r - 2 \), we in general cannot ensure the uniform boundedness of \( U \) and its derivatives since Lemma 5.5 does not hold. Then we only have

\[
\|(u - U)(t_{n}^{-})\| \leq \sup_{t \in I_{n}} \|(u - U)(t)\|
\]

where we refer to Corollary 4.12 for bounds on the right-hand side term.

### 6 Numerical experiments

We consider the initial value problem

\[
\begin{pmatrix}
u_{1}(t) \\
u_{2}(t)
\end{pmatrix} = \begin{pmatrix}-u_{2}(t) - u_{2}(t) & u_{1}(t) - u_{1}(t)u_{2}(t)\end{pmatrix}, \quad t \in (0, 32), \quad u(0) = \begin{pmatrix}1/2 \\
0\end{pmatrix}.
\]

of a system of nonlinear ordinary differential equations which has

\[
u_{1}(t) = \frac{\cos t}{2 + \sin t}, \quad u_{2}(t) = \frac{\sin t}{2 + \sin t}
\]

as solution.
The appearing nonlinear systems within each time step were solved by Newton’s method where we applied a Taylor expansion of the inherited data from the previous time interval to calculate an initial guess for all unknowns on the current interval. If higher order derivatives were needed at initial time \( t_0 = 0 \), we apply

\[
\begin{align*}
  u^{(0)}(t_0) &:= u_0, \\
  u^{(1)}(t_0) &:= f(t_0, u(t_0)), \\
  u^{(j)}(t_0) &:= \frac{d^{j-1}}{dt^{j-1}} f(t, u(t)) \bigg|_{t=t_0}, \quad j \geq 3,
\end{align*}
\]

based on the ode system (1.1) and its derivatives.

By considering different choices for \( \mathcal{J}_n \) and \( \mathcal{I}_n \), we will show that our theory provides sharp bounds on the convergence order. Since \( \mathcal{J}_n \) and \( \mathcal{I}_n \) are obtained from \( \hat{\mathcal{J}} \) and \( \hat{\mathcal{I}} \) via transformation, only the reference operators will be specified.

Each integrator \( \hat{\mathcal{J}} \) that has been used in our calculations is based on Lagrangian interpolation with respect to a specific node set \( P_{\hat{\mathcal{J}}} \). Hence, we have \( k_{\hat{\mathcal{J}}} = 0 \). The interpolation operator \( \hat{\mathcal{I}} \) is of Lagrange-type and uses the node set \( P_{\hat{\mathcal{I}}} \). This means that \( k_{\hat{\mathcal{I}}} = 0 \). Both node sets will be given for each of our test cases. Since often nodes of quadrature formulas are used, we will write for instance “left Gauss–Radau(\( k \))” to indicate that the nodes of the left-sided Gauss–Radau formula with \( k \) points have been used. All upcoming settings fulfill Assumption 1.

For all test cases, the method \( \text{VTD}_n^k \), which is \( \text{cGP-C}^1(6) \), was applied as discretization. All calculations were carried out with the software Julia [4] using the floating point data type \text{BigFloat} with 512 bits. Errors will be measured in the norms

\[
\| \varphi \|_{L^\infty} := \sup_{t \in I} \| \varphi(t) \|, \quad \| \varphi \|_{\ell^\infty} := \max_{1 \leq n \leq N} \| \varphi(t_n^0) \|
\]

where \( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^d \).

### 6.1 Case group 1

**Case**

Choosing integration and interpolation according to

\[
P_{\hat{\mathcal{J}}} = \left\{ -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right\}, \quad P_{\hat{\mathcal{I}}} = \text{left Gauss–Radau}(3)
\]

leads to \( r_{x_{\hat{\mathcal{J}}}} = 3 \) and \( r_{x_{\hat{\mathcal{I}}}} = 2 \). Hence, the condition \( \max \{ r_{x_{\hat{\mathcal{J}}}}, r_{x_{\hat{\mathcal{I}}}^+} + 1 \} \geq r - 1 \) in Corollary 4.12 is violated and the expected convergence orders for both the \( L^\infty \) norm and the \( W^{1,\infty} \) seminorm are given by \( \min \{ r, r_{x_{\hat{\mathcal{I}}}^+} + 1 \} = 3 \), see (4.19). It can be seen from Table 1 that the theoretical predictions are met by the numerical experiments. Moreover, in accordance with Corollary 5.6 the \( \ell^\infty \) convergence order is also just 3. This means that the uniform boundedness of \( \sup_{t \in I} \| U^{(j)}(t) \| \) required by Theorem 5.3, which cannot be guaranteed because of \( r_{x_{\hat{\mathcal{I}}}^+} + 1 < r - 1 \), is violated since otherwise (5.5) would yield order 4.

The condition \( \max \{ r_{x_{\hat{\mathcal{J}}}}, r_{x_{\hat{\mathcal{I}}}^+} + 1 \} \geq r - 1 \) of Corollary 4.12 will be fulfilled for all coming cases. Hence, the computations should show the convergence order given by (4.20) for the \( L^\infty \) norm.

| \( N \) | \( ||e||_{L^\infty} \) ord | \( ||e||_{L^\infty} \) ord | \( ||e||_{\ell^\infty} \) ord |
|-----|------|------|------|
| 32  | 1.648-03 | 1.126-02 | 9.439-04 |
| 64  | 1.413-04 | 1.423-03 | 2.98  | 1.046-04 | 3.17 |
| 128 | 1.408-05 | 3.33  | 1.876-04 | 2.92  | 1.264-05 | 3.05 |
| 256 | 1.615-06 | 3.12  | 2.369-05 | 2.99  | 1.559-06 | 3.02 |
| 512 | 1.961-07 | 3.04  | 2.965-06 | 3.00  | 1.937-07 | 3.01 |
| 1024| 2.426-08 | 3.01  | 3.711-07 | 3.00  | 2.415-08 | 3.00 |

Table 1: Errors and convergence orders in different norms for case 1.
6.2 Case group 2

This group of cases provides choices for $P_{\hat{g}}$ and $P_2$ such that the $L_\infty$ convergence order is limited by the maximum expression inside the outer minimum in (4.20). In addition, the presented cases will show that each of the three terms occurring in the maximum term can limit the convergence order. We will indicate in the following the limiting term in boldface.

Case 2a

The choices

$$P_{\hat{g}} = \{-\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, \frac{2}{3}\}, \quad P_2 = \text{Gauss}(5)$$

provide the $L_\infty$ convergence order $\min\{7, 6, 6, \max\{4, \min\{6, 5\}\}\} = 5$ where the convergence order is limited by the second term inside the inner minimum. We see from Table 2 that the experimental order of convergence is 6, i.e., one order higher than expected. This behavior can be explained by a closer look to Lemma 4.11. The proof there guarantees in this case that $\|v - J_n^{\delta, I}v\|(t^-) = 0$ for all $v \in P_6(\mathcal{L}_n)$. However, due to symmetry reasons, it holds $\int_{t^-}^{t^+} (v - J_n^{\delta, I}v)(t) dt = 0$ for all $v \in P_6(\mathcal{L}_n)$ which implies that $\|v - J_n^{\delta, I}v\|(t^-) = 0$ even for $v \in P_6(\mathcal{L}_n)$. Thus, the convergence order of the limiting term is actually better than predicted.

Taking the same setting for $P_{\hat{g}}$ but using

$$P_2 = \{-\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, \frac{2}{3}\},$$

the convergence order predicted by (4.20) is again $\min\{7, 6, 6, \max\{4, \min\{6, 5\}\}\} = 5$. The limitation is here also caused by the second argument of the inner minimum. Table 2 shows under case 2a* that this convergence order is obtained in the numerical experiments. Note that here the interpolation points just were chosen such that still $r^2_{\hat{g}, 0} = 5 > 4 = r^2_{\hat{g}}$, i.e., especially $\hat{G}[(\hat{v} - \hat{J}^{\delta, I}\hat{v})] = \hat{G}[\hat{v}' - \hat{I}(\hat{v}')] = 0$ for $\hat{v}(\hat{t}) = \hat{t}^6$, but $\int_{\hat{t}^-}^{\hat{t}^+} (\hat{v} - \hat{J}^{\delta, I}\hat{v})(\hat{t}) d\hat{t} \neq 0$ for $\hat{v}(\hat{t}) = \hat{t}^6$ which ensures that the estimate of Lemma 4.11 is sharp.

Case 2b

If we set

$$P_{\hat{g}} = \{-\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, \frac{2}{3}\}, \quad P_2 = \{-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\},$$

the expected convergence order is $\min\{7, \infty, \infty, \max\{4, \min\{6, \infty\}\}\} = 6$ where the limitation comes from the first argument of the inner minimum. The numerical results given in Table 2 clearly show this convergence order.

Case 2c

Choosing

$$P_{\hat{g}} = \{-1, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\}, \quad P_2 = \{-1, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\},$$

results in the convergence order $\min\{7, \infty, \infty, \max\{6, \min\{6, \infty\}\}\} = 6$. The first argument of the maximum acts as limitation. The numerical results in Table 2 show that the expected convergence order is obtained. Note that it is not possible that $r_{\text{ex}}^g + 1$ is the only limiting term since the structure of (4.20) implies that $\min\{r + 1, r_{\text{ex}}^g + 2\} \geq r_{\text{ex}} + 1 \geq \min\{r, r_{\text{ex}}^g + 1\}$ if $r_{\text{ex}} + 1$ is limiting. Hence, the integer $r_{\text{ex}} + 1$ coincides either with $\min\{r + 1, r_{\text{ex}}^g + 2\}$ or $\min\{r, r_{\text{ex}}^g + 1\}$.

Table 2: Errors and convergence orders in the $L_\infty$ norm for the cases of group 2.

| $N$ | case 2a   | case 2a*  | case 2b   | case 2c   |
|-----|-----------|-----------|-----------|-----------|
| 32  | 8.482-06  | 7.745e-06 | 8.910-07  | 1.996-06  |
| 64  | 1.477-07  | 5.834-07  | 9.394-09  | 2.792-08  |
| 128 | 2.398-09  | 5.955-09  | 1.081-10  | 4.251-10  |
| 256 | 3.759-11  | 5.222-11  | 1.465-12  | 6.604-12  |
| 512 | 5.862-13  | 6.000-13  | 2.175-14  | 1.034-13  |
| 1024| 9.160-15 | 6.000-15  | 3.344-16  | 1.617-15  |

| theor | 5   | 5   | 6   | 6   |

Order of $v$.
6.3 Case group 3

This group of cases studies the convergence orders in the $L^\infty$ norm and the $W^{1,\infty}$ seminorm. The presented choices will show that each of the first three expressions in the outer minimum in (4.20) can bound the $L^\infty$ convergence order. Moreover, the cases will demonstrate that the convergence order in the $W^{1,\infty}$ seminorm can be limited by both occurring terms in (4.19). Again, the limiting numbers will be given in boldface.

Case 3a

The choice

$$P_{\mathcal{I}} = \text{Gauss}(6), \quad P_{\mathcal{I}} = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$$

results in

$$L^\infty \text{ order } = \min\{7, 6, 5, \max\{12, \min\{6, 5\}\}\} = 5, \quad W^{1,\infty} \text{ order } = \min\{6, 5\} = 5.$$

Hence, the third argument in the outer minimum determines the convergence order for the $L^\infty$ norm while the second argument of the minimum limits the convergence order of the $W^{1,\infty}$ seminorm.

The numerical results in Table 3 provide the predicted convergence orders.

| N    | $\|e\|_{L^\infty}$ | ord | $\|e'\|_{L^\infty}$ | ord | $\|e\|_{W^{1,\infty}}$ | ord | $\|e'\|_{L^\infty}$ | ord | $\|e'\|_{W^{1,\infty}}$ | ord |
|------|------------------|-----|------------------|-----|------------------|-----|------------------|-----|------------------|-----|
| 32   | 2.30605          | 3.19304 | 1.24203          | 4.94 | 1.71702          |      | 5.60407          | 3.00 |                  |      |
| 64   | 3.78607          | 5.93  | 1.04405          | 4.94 | 2.16903          | 2.98 |                  |      |                  |      |
| 128  | 7.26609          | 5.70  | 3.41107          | 4.94 | 2.84204          | 2.93 |                  |      |                  |      |
| 256  | 1.71610          | 5.40  | 1.07208          | 4.99 | 3.58105          | 2.99 |                  |      |                  |      |
| 512  | 4.68812          | 5.19  | 3.35310          | 5.00 | 4.47906          | 3.00 |                  |      |                  |      |
| 1024 | 1.40013          | 5.07  | 1.04811          | 5.00 | 5.60407          | 3.00 |                  |      |                  |      |
| Theo |                  |      |                  |      | 5               |      |                  |      |                  |      |

Table 3: Errors and convergence orders in different norms for the cases 3a and 3b.

Case 3b

Setting

$$P_{\mathcal{I}} = \text{Gauss}(6), \quad P_{\mathcal{I}} = \text{left Gauss–Radau}(3)$$

gives

$$L^\infty \text{ order } = \min\{7, 4, 5, \max\{12, \min\{6, 3\}\}\} = 4, \quad W^{1,\infty} \text{ order } = \min\{6, 3\} = 3.$$

The convergence order in the $W^{1,\infty}$ seminorm is again determined by the second argument of the corresponding minimum. The limitation of the convergence order of the $L^\infty$ norm is caused by the second term. We clearly see from Table 3 that the expected convergence orders are obtained by the numerical simulations.

Case 3c

If we take

$$P_{\mathcal{I}} = \text{Gauss}(6), \quad P_{\mathcal{I}} = \text{Gauss}(5),$$

we get

$$L^\infty \text{ order } = \min\{7, 8, 10, \max\{12, \min\{6, 7\}\}\} = 7, \quad W^{1,\infty} \text{ order } = \min\{6, 7\} = 6,$$

$$\ell^\infty \text{ order } = \min\{10, 10, \max\{12, \min\{6, 7\}\}\} = 10.$$

The convergence orders of the $L^\infty$ norm and the $W^{1,\infty}$ seminorm are limited by the first argument in the corresponding minimum expressions. The numerical results in Table 4 indicate that the predicted orders are achieved. The additionally presented results in the $\ell^\infty$ show also the predicted behavior. Note that all three error expressions show the optimal convergence orders which are also obtained if exact integration is used and $\mathcal{I}$ is the identity operator.
Table 4: Errors and convergence orders in different norms for case 3c.

| N  | $\|e\|_{L^{\infty}}$ | ord  | $\|e\|_{L^{\infty}}$ | ord  | $\|e\|_{\ell^{\infty}}$ | ord  |
|----|----------------------|------|----------------------|------|-------------------------|------|
| 32 | 7.603-07             | 1.669-05 | 7.587-10             |      |
| 64 | 6.524-09             | 6.86  | 2.870-07             | 5.86 | 7.087-13                 | 10.06 |
| 128| 5.181-11             | 6.98  | 4.556-09             | 5.98 | 6.862-16                 | 10.01 |
| 256| 4.068-13             | 6.99  | 7.155-11             | 5.99 | 6.869-19                 | 10.00 |
| 512| 3.181-15             | 7.00  | 1.119-12             | 6.00 | 6.529-22                 | 10.00 |
| 1024|2.486-17              | 7.00  | 1.749-14             | 6.00 | 6.375-25                 | 10.00 |
| theo|7  | 6  | 10  |

6.4 Case group 4

This group of cases studies the superconvergence. Hence, we will restrict ourselves to cases where the convergence order in the $L^{\infty}$ norm suggested by (5.14) is strictly greater than the convergence order in the $L^{\infty}$ norm given by (4.20). We will show for this situation that the first two arguments in the minimum in (5.14) and the first argument inside the maximum there can limit the $\ell^{\infty}$ convergence order. We remind that the limiting term will be written in boldface.

Case 4a

The choice

$$P_{\hat{j}} = \text{Gauss}(6), \quad P_{\hat{T}} = \text{Gauss-Lobatto}(5)$$

leads to

$$L^{\infty} \text{ order } = \min\{7, 6, 8, \max\{12, \min\{6, 5\}\}\} = 6, \quad \ell^{\infty} \text{ order } = \min\{10, 8, \max\{12, \min\{6, 5\}\}\} = 8,$$

see (4.20) and (5.14). The convergence order in $\ell^{\infty}$ is bounded by the second argument of the minimum expression. As viewable in Table 5, the expected convergence orders are obtained.

Case 4b

Setting

$$P_{\hat{j}} = \text{Gauss}(6), \quad P_{\hat{T}} = \text{Gauss}(6),$$

the convergence orders

$$L^{\infty} \text{ order } = \min\{7, \infty, \infty, \max\{12, \min\{6, \infty\}\}\} = 7, \quad \ell^{\infty} \text{ order } = \min\{10, \infty, \max\{12, \min\{6, \infty\}\}\} = 10$$

are expected by our theory. Hence, the convergence order in the $\ell^{\infty}$ norm is limited by the first term inside the minimum in (5.14). The numerical results coincide with our predictions.

Case 4c

Taking

$$P_{\hat{j}} = \text{Gauss}(4), \quad P_{\hat{T}} = \text{Gauss}(4)$$

provides

$$L^{\infty} \text{ order } = \min\{7, \infty, \infty, \max\{8, \min\{6, \infty\}\}\} = 7, \quad \ell^{\infty} \text{ order } = \min\{10, \infty, \max\{8, \min\{6, \infty\}\}\} = 8,$$

compare (4.20) and (5.14). The limitation of the $\ell^{\infty}$ convergence is caused by the first argument of the maximum in (5.14). The results in Table 5 show clearly the superconvergence since the convergence order in the $\ell^{\infty}$ norm is one higher than the convergence order in the $L^{\infty}$ norm.
Table 5: Errors and convergence orders in different norms for the cases of group 4.

|        | case 4a |        | case 4b |        | case 4c |        | case 4d |
|--------|---------|--------|---------|--------|---------|--------|---------|
| N      | $\|e\|_{L^\infty}$ | ord | $\|e\|_{\ell^\infty}$ | ord | $\|e\|_{L^\infty}$ | ord | $\|e\|_{\ell^\infty}$ | ord |
| 32     | 1.058-05 | 8.552-08 | 3.348-10 | 8.00  | 7.560-07 | 5.899-10 | 3.348-10 | 8.00  |
| 64     | 1.717-07 | 5.94   | 1.310-12 | 8.00  | 6.529-09 | 6.86   | 6.512-10 | 10.04 |
| 128    | 2.835-09 | 5.92   | 2.005-17 | 8.00  | 5.183-11 | 6.98   | 5.192-10 | 10.01 |
| 256    | 4.464-11 | 5.92   | 1.310-12 | 8.00  | 4.068-13 | 6.99   | 5.318-10 | 10.00 |
| 512    | 6.981-13 | 6.00   | 2.005-17 | 8.00  | 3.181-15 | 7.00   | 5.192-10 | 10.00 |
| 1024   | 1.092-14 | 6.00   | 7.833-20 | 8.00  | 2.486-17 | 7.00   | 5.070-25 | 10.00 |
|        | theo    | 6      | 8       |        | theo    | 7      | 10      |
| N      | $\|e\|_{L^\infty}$ | ord | $\|e\|_{\ell^\infty}$ | ord | $\|e\|_{L^\infty}$ | ord | $\|e\|_{\ell^\infty}$ | ord |
| 32     | 1.617-06 | 4.037-08 | 7.603-04 | 10.00 |
| 64     | 1.387-08 | 6.86   | 1.654-10 | 7.93  |
| 128    | 1.101-10 | 6.98   | 6.522-13 | 7.99  |
| 256    | 8.654-13 | 6.99   | 2.565-15 | 7.99  |
| 512    | 6.759-15 | 7.00   | 1.002-17 | 8.00  |
| 1024   | 5.283-17 | 7.00   | 3.916-20 | 8.00  |
|        | theo    | 7      | 8       |        | theo    | 4      | 4       |

**Case 4d**

Choosing $P_I = \text{Gauss}(6)$, $P_I = \text{Gauss}(3)$,

the estimates (4.20) and (5.14) suggest

- $L^\infty$ order = \min \{7, 4, 6, \max \{12, \min \{6, 3\}\}\} = 4,
- $\ell^\infty$ order = \min \{10, 6, \max \{12, \min \{6, 3\}\}\} = 6.

However, since the uniform boundedness of $\sup_{t \in I} \|U(t)\|$ assumed in Theorem 5.3 cannot be ensured by Lemma 5.5 due to $r = 3 < 5 = r - 1$, we actually do not expect any superconvergence. These expectations are confirmed by the numerical results given in Table 5. They show that for both the $L^\infty$ and the $\ell^\infty$ norm convergence order 4 is obtained.

### 6.5 Summary

The experimentally obtained and theoretically predicted convergence orders for all cases and all considered norms are collected in Table 6. The experimental orders of convergence were calculated using the results obtained for 256 and 512 time steps.

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Table 6: Errors and convergence orders in different norms for all cases.

| case | $\|e\|_{L^\infty}$ | $\|e'\|_{L^\infty}$ | $\|e\|_{\ell^\infty}$ |
|------|-------------------|-------------------|-------------------|
| 1    | 3.04              | 3.00              | 3.01              |
| 2a   | 6.00              | 5.00              | 6.00              |
| 2a*  | 5.08              | 5.00              | 5.00              |
| 2b   | 6.07              | 6.00              | 6.00              |
| 2c   | 6.00              | 5.99              | 6.00              |
| 3a   | 5.19              | 5.00              | 5.02              |
| 3b   | 4.02              | 3.00              | 4.12              |
| 3c   | 7.00              | 6.00              | 10.00             |
| 4a   | 6.00              | 5.00              | 8.00              |
| 4b   | 7.00              | 6.00              | 10.00             |
| 4c   | 7.00              | 6.00              | 8.00              |
| 4d   | 4.00              | 3.00              | 4.00              |

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