"Double swallow-tail" singularity and glass-glass transition in a quasibinary system.

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The system with the square shoulder (SS) potential is considered in the frame of Mode Coupling Theory (MCT) approach. An approximation for the structure factor is used that emphasizes the quasibinary character of the system. The qualitative phase diagram is constructed that includes continuous and discontinuous glass–glass transitions. The phase diagram is governed by two swallow tails connected with two $A_4$ singularities.

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A large number of papers dedicated to the liquid-glass transition have been published during last decades. They include results of real experiments, computer simulations and different theoretical approaches. Nowadays the most popular and the most cited of the various theories of glasses are based on mean–field replica approach [1] and the so called Random First Order Transition theory (RFOT) [2], both based on analogies with the well-developed equilibrium statistical mechanics of spin glasses. Numerous results were obtained in the framework of the so-called mode-coupling theory (MCT) (see, e.g., the pioneering work [3], the review [4], and also the detailed presentation in the recent monograph [5]). Although to-day the investigations of dynamical heterogeneities (see e.g. [8]) in glassy systems make favourable for a long time the only consistent theory describing details of the transitions in supercooled liquids. Despite suffering of a shift of the actual glass transition have been published during last decades.

In MCT, the system dynamics is described in terms of the Fourier transform of the system density. The autocorrelation function satisfies the equation

$$\frac{\partial^2 \Phi_q(t)}{\partial t^2} + \nu_q \frac{\partial \Phi_q(t)}{\partial t} + \Omega^2_q \Phi_q(t) + \Omega^2_q \int_0^t dt' m_q(t-t') \frac{\partial \Phi_q(t')}{\partial t'} = 0, \quad (1)$$

where $\nu_q$ is white noise and $\Omega_q$ the characteristic frequency. The memory function $m_q(t)$ has the form

$$m_q(t) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} V_{q,k} \Phi_{\bar{k}}(t) \Phi_{\bar{q}-\bar{k}}(t) \quad (2)$$

The interaction potential of the system particles is included in the vertex function

$$V_{q,k} = \rho S_q S_{\bar{k}} \bar{S}_{\bar{q}-\bar{k}} [\bar{q}^2 c_k + \bar{q}(\bar{q} - \bar{k}) c_{\bar{q}-\bar{k}}] / q^4 \quad (3)$$

through the static structure factor $S_q$ of liquid and the direct correlation function $c_q \bar{c}_{\bar{q}}$. These two quantities are related:

$$S_q = 1/(1 - \rho c_q). \quad (4)$$

The behavior of the solution of Eq. (1) at large times determines relaxation processes in the system [2]. As $t \to \infty$, the algebraic equation for the limit correlation function $f_q = \Phi_q(\infty)$:

$$\frac{f_q}{1 - f_q} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} V_{q,k} f_k f_{\bar{q}-\bar{k}} \quad (5)$$

can be obtained from Eq. (1). Eq. (5) always has the trivial solution $f_q = 0$ corresponding to the liquid (ergodic) phase. Eq. (5) can also have a nonzero solution $f_q > 0$ corresponding to a nonergodic glass phase. If $f_q = 0$ is a bifurcation point of Eq. (5) the liquid-glass transition is continuous, if no, then $f_q$ changes jumpwise from zero to a certain value $f_q^*$ at the transition point. Hence, the value $f_q^*$ can be regarded as an order parameter (or a nonergodicity parameter) for the liquid-glass transition.
The MCT was first used to describe the transition to the glass state in the system with the hard-core potential in \([1]\) and then in a large number of various systems. Its applicability was confirmed experimentally (see \([2, 3, 10]\)). At the same time, papers where the possibility of describing the glass-glass transition for certain potentials in the MCT framework have recently appeared: in this case, the glass characterized by the nonergodicity parameter \(f_q^{(1)}\) transforms jumpwise into another glass with the nonergodicity parameter \(f_q^{(2)}\). Such a transition was predicted for systems with a potential consisting of a solid core and a very narrow and deep attractive well in \([11, 14]\). In this case, the first glass state is determined by repulsion as for the system of hard spheres, while the second glass state is determined by attraction. It is the competition between these two states that determines the glass-glass transition. The glass-glass transition line continues the liquid-glass transition line smoothly to the glass region and ends at a third-order bifurcation point.

Now we consider a system in which the attracting well is changed for a repulsive step - SS system. This system and its smoothed versions were widely studied \([6, 7, 15–27]\). It was shown that a series of unusual phenomena are observed in the system, including maxima on the melting curve and structural transitions in the solid phase. In addition, thermodynamic anomalies like ones observed in water were discovered in the system with a smoothed version of the potential.

The potential of the system under consideration has the form

\[
\Phi(r) = \begin{cases} 
\infty, & r \leq d \\
\varepsilon, & d < r \leq \sigma \\
0, & r > \sigma 
\end{cases}
\]  

where \(d\) is the diameter of the hard core and \(\sigma\) and \(\varepsilon\) are the diameter and height of the repulsive step. In the papers \([25, 27]\) the MCT equations obtained for this system using different approximations for the static structure factor were solved numerically and slightly different results were obtained, drastically depending on the approximation.

It can be seen that all the variants of the structure factor used in the mentioned papers have an obvious common feature: the first peak, corresponding to the diameter of the repulsive step, decreases with the increase of the density, and the second peak, determined by the hard core, increases. In this paper, we try to demonstrate the main features of this quasibinary system in the frame of a very simple approximation for the structure factor \(S(q)\) \([6]\):

\[
S(q) \approx 1 + A\delta(q - k_1) + B\delta(q - k_2),
\]  

where \(k_1\) and \(k_2\) are the positions of the first and second maxima of the structure factor. The parameters \(A\) and \(B\) are determined by the area under the corresponding maxima \([6]\). As it was shown in Ref. \([6]\), this approximation works well for low densities and gives qualitatively correct result for the high density region. Our aim is to obtain a schematic phase diagram in terms of three control parameters

\[
x = \frac{S(k_2)k_2^2}{S(k_1)k_1^2},
\]

\[
a = \frac{S(k_1)k_1}{8\pi^2\rho},
\]

\[
b = \frac{S(k_2)k_2}{8\pi^2\rho}.
\]  

Substituting approximation \((7)\) in Eq. \((5)\), we obtain an approximate system of equations for the nonergodicity parameters \(f(k_1)\) and \(f(k_2)\):

\[
\frac{f(k_1)}{1-f(k_1)} = \frac{S(k_1)k_1}{8\pi^2\rho} \left( Af(k_1) + \frac{k_2}{k_1} B f(k_2) \right)^2
\]

\[
\frac{f(k_2)}{1-f(k_2)} = \frac{S(k_2)k_2^2}{8\pi^2k_1^2\rho} \left( Af(k_2) + \frac{k_2}{k_1} B f(k_2) \right)^2
\]  

The system can be considered as a mixture of hard spheres with diameters \(d\) and \(\sigma\) \([6]\). It can be seen \([6]\), that the glass consisting of particles with the larger diameter \(\sigma\) exists for small densities, while the glass consisting of hard spheres with the diameter \(d\) prevails for high densities. If these two types of local glass structure are taken into account, then the question arises, whether the glass-glass transition between these states exists. Indeed, an analysis of Eqs. \((11)\) shows that there is a fourth-order singularity point \(A_4\) for this system. This singularity corresponds to the division of the space of three control parameters into three parts corresponding to the absence of nontrivial solution of the equations, to the part with two solutions, and the remaining part with four solutions, by the surface of the figure called "swallow tail" (see \([28, 30]\)).

Our approximation emphasizes the quasibinary nature of the system and enables one to obtain a qualitative phase diagram of the system. The approximation leads to the new kind of the symmetry of the equations and to a degeneracy, both having a physical ground. The duplication of the singularities demonstrates one of the consequences of the degeneracy - we obtain "double swallow tail".

Despite a lot of works (see, for example, \([5, 25, 31, 32]\)) where the higher order singularities in the MCT were discussed, the degenerate cases and the multiplication of singularities were not considered in the literature.

Let us note that although our results are of qualitative character, the obtained picture can be considered in as a
scenario for “the ideal glass-glass transition” in quasibinary systems.

We now pass directly to the problem of finding the bifurcation points of Eqs. (11). As is well known (see, e.g. [33]) the uniqueness of solution of two functional equations

\[ A \] fails at the points where the determinant of derivarives || det \( \frac{\partial}{\partial z^k} \)| is zero.

Let us use the standard MCT notations (see [5]) and rewrite the system (11) in the following form:

\[
\begin{align*}
\frac{f_1}{1-f_1} &= (af_1 + bf_2)^2 = [F_1], \\
\frac{f_2}{1-f_2} &= x(af_1 + bf_2)^2 = [F_2],
\end{align*}
\]

where it is easy to see that these equations are invariant under the transformation

\[ f_1 \to f'_2; f_2 \to f'_1, \]

\[ a \to \sqrt{x}b'; b \to \sqrt{x}a'; x \to 1/x'. \]

This means that for every solution of (12) there will be another one and the same can be said about every point \( A_k \).

At the bifurcation points we have || det\( \delta_{qk} - A_{qk} \)|| = 0, where

\[
\begin{align*}
A_{11} &= (1-f_1)^2F_{1,1}', \\
A_{12} &= (1-f_1)(1-f_2)F_{1,2}', \\
A_{21} &= (1-f_2)^2(1-f_2)F_{2,1}', \\
A_{22} &= (1-f_2)^2F_{2,2}'.
\end{align*}
\]

Here \( F_{1,1}' = 2ay; F_{1,2}' = 2by; F_{2,1}' = 2axy; F_{2,2}' = 2bxy. \)

The determinant || det \( A_{qk} \)|| = 0, and this fact is the consequence of the mentioned degeneracy. In fact, the matrix \( A_{qk} \) eigenvalues in our case are:

\[ \lambda_1 = 0, \lambda_2 = A_{11} + A_{22} \]

and the bifurcation condition can be written in the form

\[ (1-f_1)^22ay + (1-f_2)^22bxy = 1 \]

(14)

To find the points of bifurcation of the system it is technically convenient to use the new variable defined as follows:

\[ y = af_1 + bf_2. \]

The equations (12) can be rewritten in terms of \( y \). Now

\[ f_1 = \frac{y^2}{1+y^2}; f_2 = \frac{xy^2}{1+xy^2} \]

so that

\[ y = \frac{ay^2}{1+y^2} + \frac{bxy^2}{1+xy^2} \]

(15)

The derivative of this equation relative to \( y \) gives

\[ 1 = \frac{2ay}{(1+y^2)^2} + \frac{2bxy}{(1+xy^2)^2} \]

(16)

This equation coincides with the bifurcation condition (13), if one takes into account that

\[
\begin{align*}
1 - f_1 &= \frac{1}{1+y^2}, \\
1 - f_2 &= \frac{1}{1+xy^2}.
\end{align*}
\]

Now the problem of obtaining the bifurcation points of initial equations reduces to the system of equations (15)-(16), that is the equations

\[ \tilde{\Psi}(y) = 0; \quad \tilde{\Psi}'_y = 0 \]

with

\[ \tilde{\Psi}(y) = y - \frac{ay^2}{1+y^2} + \frac{bxy^2}{1+xy^2}. \]

Let us make two remarks. First, it is easy to see that the point \( y = 0 \) is not a bifurcation point of our equations, so that one can use \( \tilde{\Psi}(y) \) instead of \( \tilde{\Psi}'(y) \) for our purpose. Second, as far as the bifurcation properties are concerned it is possible to consider only the numerator as the main functional. All the equations for singularities (including higher orders) for these functionals will differ only by additive terms proportional to the lower order derivatives, which are zero. So, we can use the following functional:

\[ \Psi(y, a, b, x) = ay(1+xy^2) + bxy(1+y^2) - (1+y^2)(1+xy^2). \]

(17)

We have for the \( A_2 \) singularities:

\[
\begin{align*}
\Psi(y, a, b, x) &= 0; \\
\Psi'_y(y, a, b, x) &= 0;
\end{align*}
\]

(18)

for the \( A_3 \) singularities:

\[
\begin{align*}
\Psi(y, a, b, x) &= 0; \\
\Psi'_y(y, a, b, x) &= 0; \\
\Psi''_y(y, a, b, x)(y) &= 0;
\end{align*}
\]

(19)
and for the $A_4$ singularity:

$$\Psi(y, a, b, x) = 0;$$

$$\Psi'(y, a, b, x) = 0;$$

$$\Psi''(y, a, b, x)(y) = 0;$$

$$\Psi'''(y, a, b, x)(y) = 0;$$  \hspace{1cm} (20)

while $\Psi^{IV}(y, a, b, x) \neq 0$.

The explicit forms for these derivatives are:

$$\Psi'(y, a, b, x)(y) = a(1 + 3xy^2) + bx(1 + 3y^2) - 2y(1 + x + 2xy^2),$$  \hspace{1cm} (21)

$$\Psi''(y, a, b, x)(y) = 6xy(a + b) - 2 - 2x - 12xy^2,$$  \hspace{1cm} (22)

$$\Psi'''(y, a, b, x)(y) = 6x(a + b) - 24xy,$$  \hspace{1cm} (23)

and $\Psi^{IV}(y, a, b, x) = -24x$.

Technically it is convenient to proceed in the following way. The equations (18) with (17) and (21) are two linear equations for the variables $a$ and $b$ which can be easily solved:

$$a = \frac{(1 - xy^2)(1 + y^2)^2}{2y^3(1 - x)};$$

$$b = \frac{-(1 - y^2)(1 + xy^2)^2}{2xy^3(1 - x)}.\hspace{1cm} (24)$$

Substituting of these $a$ and $b$ in (22) and (23) gives:

$$\Psi'' \sim \mu_2 = -6 + 2y^2(1 + x) - 6xy^4 \hspace{1cm} (25)$$

and

$$\Psi''' \sim \mu_3 = 1 + (1 + x)y^2 - 5xy^4 \hspace{1cm} (26)$$

The points $\{a, b\}$ (at fixed values of $x$) for which there are some real solutions $y > 0$ satisfying both of the equations (21) make the lines of points $A_2$. In fact, at the fixed value of $x$ and the set of $a$ the first of the equations (21) was solved. The obtained value of $y$ at the fixed $x$ then give value of $b$ through the second equation (24). Those points $(a, b, x)$ for which in addition to the existence of $y$, the values $x$ and $y$ are related through (20), present the line of $A_3$ singularity.

It is easy to solve the system of equations $\mu_2 = 0$ and $\mu_3 = 0$ and to obtain two solutions for the $A_4$ singularity:

$x_{01} = 0.0294373, y_{01} = 2.41421$ and $x_{02} = 33.9706, y_{02} = 0.4142$. The corresponding values of $a$ and $b$ are determined by (23): $a_{01} = 1.41421; b_{01} = 8.24264$ and $a_{02} = 1.41421, b_{02} = 0.24264$ with $b_{01} \sqrt{x_{01}} = b_{02} \sqrt{x_{02}}$.

In Fig. 1 the lines of $A_2$ points at $x = 0.002$ are presented, as well as two $A_3$ points and the crossing point $B$. The similar pictures can be obtained for any value of $x$ up to $x_0$ corresponding to the $A_4$ singularity. The section at $x_{01}$ is shown in Fig. 2. The sections for $x$ between $x_{01}$ and $x_{02}$ present smooth curves. At $x = x_{02}$ the figure is similar to Fig. 1 and for $x > x_{02}$ - similar to Fig. 1. For given values of $x$ and $y$ it is easy to obtain $f_1$ and $f_2$.

Taking into account the maximum principle (see [5]) for the functions $f_1$ and $f_2$, one can show that the transition lines are $BA_1^1$ for $x < x_{01}$ and $BA_2^2$ for $x > x_{02}$ (see Fig. 1). These lines are related to jumpwise glass-glass transitions. So, due to symmetry property we obtain two swallow tails for the surfaces of $A_2$ singularities and two $A_4$ singularities. Each swallow tail separates one from another three parts of the 3D space of the variables $x, a, b$.

In one (curved pyramid) there are 4 solutions of the initial equation $\Psi(y, a, b, x) = 0$; in the neighboring part there are 2 solutions and in the remainder - no solution. The number of positive solutions, we are interested in, may be different from the mentioned. This remaining part (for example, for $x = 0.002$ it corresponds approxi-
have \( f_1 > f_2 \) and \( f_2 > f_1 \) for \( x > 1 \). On the surfaces shown on the figure by red color (two curved triangles) there is a jump of functions \( f_i \) and we have the discontinuous glass-glass transitions. In Fig. 4 we present \( f_1 \) and \( f_2 \) as functions of \( b \) for \( x = 0.002 \) and \( a = 1.9 \), which are discontinuous when passing through the red curved triangle. If the value of \( x \) is greater than \( x_{02} \) we have the analogous behavior with exchanged roles of \( f_1 \) and \( f_2 \). The overall picture is symmetric in the variables \( a \) and \( b\sqrt{x} \). The plane of variables \( a, b \) for \( x = 1 \) (\( x = \frac{S(k_2)}{S(k_1)} \frac{k_f^2}{k_s^2} \)) is the boundary between two glass phases: with \( f_1 > f_2 \) and with \( f_2 > f_1 \). This transition is continuous. The behavior of the functions \( f_i \) in this case is shown in Fig. 5.

The surfaces of the discontinuous glass-glass transitions cross the surface of liquid-glass transition and end in the point of the \( A_4 \) singularities. This crossing line is one of the part of the curved triangles, the other being the line of \( A_3 \) singularities.

The obtained qualitative picture is in good agreement with the results of the MCT approach with Roger-Young approximation for the direct correlation function \( C(r) \) or with numerically obtained \( C(r) \) for binary system \[27\]. It is also in good agreement with MD approach to the system \[27\].

To obtain quantitative phase diagram, one has to model the dependence of \( a, b, x \) on the physical variables and add the ”tail” to the structure factor. The second part of such a program can be done in the following way. Let us introduce in Eqs. (12) a small third term of the same structure. We have now instead of (12) the following system:

\[
\begin{align*}
\frac{f_1}{1 - f_1} &= (af_1 + bf_2 + \alpha f_3)^2; \\
\frac{f_2}{1 - f_2} &= x(af_1 + bf_2 + \alpha f_3)^2; \\
\frac{f_3}{1 - f_3} &= z(af_1 + bf_2 + \alpha f_3)^2. 
\end{align*}
\tag{27}
\]
It is very important to note, that in this case the overall topology of the bifurcation surfaces is changed entirely, we will obtain the singularity of the higher order $A_6$, and the problem will be complicated to great extend (see [34]). Again we have the multiplication of singularities. Now there will be six points $A_6$ due to the invariance of the Eq. (27) connected with the following transformations:

$$f_1 \rightarrow f'_1, f_2 \rightarrow f'_2, f_3 \rightarrow f'_3, f_4 \rightarrow f'_4$$

$$a \rightarrow \sqrt{a'b'}, b \rightarrow \sqrt{a'a'}, c \rightarrow \sqrt{a'c'}, x \rightarrow 1/x', z \rightarrow z'/x'$$

$$f_1 \rightarrow f'_1, f_2 \rightarrow f'_2, f_3 \rightarrow f'_3, f_4 \rightarrow f'_4$$

$$a \rightarrow \sqrt{a'a'}, b \rightarrow \sqrt{b'b'}, c \rightarrow \sqrt{a'c'}, x \rightarrow x'/z', z \rightarrow 1/z'$$

$$f_2 \rightarrow f'_2, f_3 \rightarrow f'_3, f_4 \rightarrow f'_4$$

$$a \rightarrow a', b \rightarrow b', c \rightarrow c', x \rightarrow x', z \rightarrow z'$$

$$f_1 \rightarrow f'_1, f_2 \rightarrow f'_2, f_3 \rightarrow f'_3, f_4 \rightarrow f'_4$$

$$a \sqrt{a} \rightarrow b', b \sqrt{a} \rightarrow a', c \sqrt{a} \rightarrow c', x \rightarrow z'/x', 1/x'$$

The coefficients $f_3$ enters the equations with the multiplier $\alpha$. This means that we can define $f_3$ in terms of unperturbed values and use the third of the Eqs. (27) as a definition of $f_3$. In fact, as it can be seen from the consideration below, the actual form of the perturbation term is not important. For example, we can add the function $f_4$ with the small coefficient $\beta$. Now we can use our previous result as the unperturbed one in order to write the new functional $\hat{\Psi}(w, a, b, x)$ depending on the parameters $\alpha f_3$ and $\beta f_4$, where $\alpha, \beta << 1$ and

$$f_3 = \frac{z_{60}^2}{1 + z_{60}^2}, f_4 = \frac{t y_{60}^2}{1 + t y_{60}^2}$$

In this case, for the $A_4$ singularity we have the conditions: $\hat{\Psi}_w(w_0, \ldots) = 0, \hat{\Psi}_w'(w_0, \ldots) = 0, \hat{\Psi}_w''(w_0, \ldots) = 0, \hat{\Psi}_w'''(w_0, \ldots) = 0$. Let us expand the functional $\hat{\Psi}$ around the solution for $A_4$ singularity of $\Psi$, that is around $\Psi(y_0, a_0, b_0, x_0)$, up to the first order in $\alpha, \beta$. Now instead of the relation (29) we have:

$$w = \frac{a w^2}{1 + w^2} + \frac{b x w^2}{1 + x w^2} + \alpha f_3 + \beta f_4; \quad (28)$$

Within the proposed accuracy, it is irrelevant whether $f_3$ and $f_4$ are written in terms of $w$ or $y_0$. So, we consider the functional

$$\hat{\Psi}(y, a, b, x) = \gamma + \Psi(y, a, b, x) \quad (29)$$

with $\Psi(y, a, b, x)$ given by (17) and $\gamma = (\alpha f_3 + \beta f_4)/y_0$.

Let us note, that for $\gamma$ we can use functions taken at the point $A_4$. In this approximate formulation, the topology of the problem is not changed: we only have the shift of the singularity point. The zero solution for $w$ also exists. Now let us put $y = y_0 + \gamma y_1, x = x_0 + \gamma x_1, a = a_0 + \gamma a_1, b = b_0 + \gamma b_1$, expand the functional up to first order in $\gamma$ and find the shifted point of singularity $A_4$. The equations of zeroth order are fulfilled, and we obtain the system of 4 linear equations for the variables $x_1, a_1, b_1$ and $y_1$:

$$\hat{\Psi}(y) = \Psi(0) + \gamma y_1 \Psi'_y(0) + \gamma y'_1(0)x_1 + \gamma a_1 \Psi'_a(0) + \gamma b_1 \Psi'_b(0) + \gamma = 0;$$

$$\hat{\Psi}'(y) = \Psi'_y(0) + \gamma y_1 \Psi''_{yy}(0) + \gamma y''_1(0)x_1 + \gamma a_1 \Psi''_{aa}(0) + \gamma b_1 \Psi''_{bb}(0) + \gamma = 0;$$

$$\hat{\Psi}''(y) = \Psi''_{yy}(0) + \gamma y_1 \Psi''_{y.yy}(0) + \gamma y''_1(0)x_1 + \gamma a_1 \Psi''_{aa}(0) + \gamma b_1 \Psi''_{bb}(0) + \gamma = 0;$$

$$\hat{\Psi}'''(y) = \Psi'''_{yy}(0) + \gamma y_1 \Psi'''_{y.yyy}(0) + \gamma y''_1(0)x_1 + \gamma a_1 \Psi'''_{aa}(0) + \gamma b_1 \Psi'''_{bb}(0) + \gamma = 0. \quad (30)$$

This system has the form

$$A_{11} x_1 + A_{12} a_1 + A_{13} b_1 + A_{14} y_1 = -1;$$

$$A_{21} x_1 + A_{22} a_1 + A_{23} b_1 + A_{24} y_1 = 0;$$

$$A_{31} x_1 + A_{32} a_1 + A_{33} b_1 + A_{34} y_1 = 0;$$

$$A_{41} x_1 + A_{42} a_1 + A_{43} b_1 + A_{44} y_1 = 0;$$

The coefficients $A_{ij}$ can be easily obtained by taking derivatives of the equations (17) and (21)-(24) at the points $A_4$. We have:

$$A_{11} = by + 3y^4 - y^2; A_{12} = y + xy^3;$$

$$A_{13} = xy + xy^3; A_{14} = 0;$$

$$A_{21} = b + 8y^3 - 2y; A_{22} = 1 + 3xy^2;$$

$$A_{23} = x + 3xy^2; A_{24} = 0;$$

$$A_{31} = 12y^2 - 2; A_{32} = 6xy;$$

$$A_{33} = 6xy; A_{34} = 0;$$

$$A_{41} = 0; A_{42} = 6x;$$

$$A_{43} = 6x; A_{44} = -24x.$$
Now we can solve the system and obtain the values $x_1$, $a_1$, $b_1$, and $y_1$. For example,

$$x_1 = -\frac{3}{y_0(1 - 3y_0)}.$$  \hfill (31)

This value is positive for the point $x_0 = x_{01}, y_0 = y_{01}$ and is negative for $x_0 = x_{02}, y_0 = y_{02}$. So, the points $A_4$ move one towards another, and it is this result that we were going to show: this fact makes our qualitative results closer to numerical observations.

Let us recall, that in fact the topology of the bifurcation surfaces has to be changed essentially if the higher degrees of $f_i$ are taken into account.

The equations defining four variables $x_0, a_0, b_0, y_0$ at the point $A_4$ for $\gamma = 0$ are formally the same as the equations for $A_4$ in the first order in $\gamma$. This means that we can continue our iteration process up to $\gamma^2$. Here $\gamma^2$ denote the terms of the second order in $\alpha, \beta, \gamma$, which has the form

$$\left(\alpha f_3 + \beta f_4\right)^2,$$

where $f_3$ and $f_4$ are written up to first order in $\gamma$. This new shift is also of the form (31) but with $y_0$ changed for $y_0 + \gamma y_1$. This means that for small $\gamma$ the points $A_4$ continue to come nearer to one another.

It should be noted, that the obtained results do not contradict to Ref. [25], and one can show just in the same way as it is done in [31], that in our model with $A_4$-singularities the relaxation is logarithmic. The influence of multiplication of singularities on the relaxation characteristics will be considered in a separate paper.

To conclude, the SS system is considered in the frame of Mode Coupling Theory approach. An approximation for the structure factor is used that emphasizes the quasibinary character of the system. The approximation enables to obtain in an analytic way the solution of the MCT equations for the infinite time limit and focus on the symmetry of the problem. The bifurcation singularity is $A_4$, and, taking into account the symmetry, gives the “double swallow tail”. The qualitative phase diagram is constructed in the variables

$$x = \frac{S(k_2)k_2^2}{S(k_1)k_1^2}, a = \frac{S(k_1)k_1}{8\pi^2 \rho}, b = \frac{S(k_2)k_2}{8\pi^2 \rho}$$

where $k_1$ and $k_2$ are the locations of two maxima of the structure factor. Besides the liquid-glass transition, the phase diagram includes a plane $x = 1$ of the continuous glass-glass transitions, and the discontinuous glass-glass transitions occur for $x < x_{01}$ and $x > x_{02}$.

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