Staffans–Weiss perturbations for maximal $L^p$-regularity in Banach spaces

Ahmed Amansag, Hamid Bounit, Abderrahim Driouich and Said Hadd

Abstract. In this paper, we show that the concept of maximal $L^p$-regularity is stable under a large class of unbounded perturbations, namely Staffans–Weiss perturbations. To that purpose, we first prove that the analyticity of semigroups is preserved under this class of perturbations, which is a necessary condition for the maximal regularity. In UMD spaces, $R$-boundedness is exploited to give conditions guaranteeing the maximal regularity. For Banach spaces, a condition is imposed to prove maximal regularity. Moreover, we apply the obtained results to perturbed boundary value problems.

1. Introduction

In this paper, we are interested in the concept of maximal $L^p$-regularity for evolution equations. It helps to use the fixed point technique for the solution of quasi-linear problems. It also makes it possible to prove the well-posedness of non-autonomous equations. The importance of the subject has prompted many researchers to contribute to its development, see, e.g., [3–5,7], and the monograph [6]. We recall that if a linear operator $A : D(A) \subset X \to X$ generates a bounded analytic semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$, $0 < T < \infty$ and $p \in [1, +\infty]$, then we say that $A$ has maximal $L^p$-regularity (we write $A \in MR^p(0, T; X)$) if for any $f \in L^p([0, T], X)$, $u = T \ast f$, $Au$ and $\dot{u}$ are functions in $L^p([0, T], X)$.

Several approaches are available to prove the maximal $L^p$-regularity. In fact, in the Hilbert setting, as shown in [5], if $1 < p < \infty$ then $A$ has maximal regularity if and only if $A$ generates an analytic semigroup on $X$. However, it is observed that the analyticity is not sufficient in general Banach spaces. A characterization of the maximal $L^p$-regularity, $1 < p < \infty$, in UMD spaces in terms of $R$-boundedness of the set $\{sR(is, A) : s \neq 0\}$ is obtained in [21]. Another approach to dealing with the above problem is perturbation theory. One of the first works in this direction is reference [14]. Indeed, the authors considered perturbations $P : X_{\alpha} \to X_{\alpha-1}$ of $A$ that have small norm, where for $\beta \in [0, 1]$ we denote $X_{\beta} = D((-A)^{\beta})$. They proved that the maximal $L^p$-regularity is preserved under this class of perturbations in Banach spaces. The technique used is essentially based on estimating the $R$-bound of the resolvent of the part of $A_{\alpha-1} + P$ in $X$, where $A_{\alpha-1} : X_{\alpha} \to X_{\alpha-1}$ is an extension of $A$. Another important contribution to the perturbation approach for the...
maximal $L^p$-regularity is the work [12], which sheds more light on the results of [14]. They considered perturbations $P$ which are not necessarily defined on fractional domain spaces, which allows flexibility in applications. Such perturbations appear in the feedback theory of infinite-dimensional linear systems.

In the present work, we follow the same spirit as in [2] and [14] and propose perturbation results on maximal $L^p$-regularity. To introduce our results, consider two Banach spaces $X, U$ and three operators $A : X \supset D(A) \to X$, $B : U \to X_{-1}$ and $C : D(A) \to U$. We assume that $A$ is the generator of a $C_0$-semigroup $\mathbb{T} := (\mathbb{T}(t))_{t \geq 0}$ on $X$ such that $A \in \mathcal{M} \mathcal{R}_p(0, T; X)$ for a fixed $p \in [1, \infty)$. Using the feedback theory of regular linear systems in Salamon–Weiss sense, it is shown in [24] that the following operator

$$A^{cl} := A_{-1} + B \tilde{C} \quad \text{with} \quad D(A^{cl}) := \{ x \in D(\tilde{C}) : (A_{-1} + B \tilde{C}) x \in X \} ,$$

(1)
is a generator of a $C_0$-semigroup $\mathbb{T}^{cl}$ on $X$ (here $\tilde{C}$ is a suitable extension of $C$). In Theorem 2, we show that the semigroup $\mathbb{T}^{cl}$, generated by $A^{cl}$, is analytic, which is a necessary condition for the maximal $L^p$-regularity in general Banach spaces $X$. In Theorem 4, by assuming that $X$ and $U$ are UMD spaces and the families $\{s^{\alpha} R(\omega + is, A_{-1})B : s \neq 0 \}$ and $\{s^{1-\alpha} R(\omega + is, A) : s \neq 0 \}$ are $R$-bounded for some $\omega > \omega_0(A)$ and some $\alpha \in (0, 1)$, we show that $A^{cl}$ has maximal $L^p$-regularity for $p \in (1, \infty)$. The proof is based on the formula of the resolvent of the operator $A^{cl}$ and the fact that transfer functions associated with infinite-dimensional systems are $R$-bounded when indexed on the imaginary axis as proved in [12, p. 513]. In Theorem 5, under the assumptions that $X$ is a general Banach space and the application $(\lambda_0, \infty) \mapsto \lambda R(\lambda, A_{-1})B \in \mathcal{L}(U, X)$ is bounded for some $\lambda_0 > \omega_0(A)$, we prove that the operator $A^{cl}$ has maximal $L^p$-regularity for a fixed $p \in [1, \infty)$. The proof shows that if the following operator

$$(\mathcal{R} f)(t) := A \int_0^t \mathbb{T}(t - s) f(s)ds, \quad f \in C([0, T], D(A))$$

(2)

has a bounded extension to $L^p([0, T], X)$, then the operator

$$(\mathcal{R}^{cl} f)(t) := A^{cl} \int_0^t \mathbb{T}^{cl}(t - s) f(s)ds, \quad f \in C([0, T], D(A^{cl}))$$

(3)

also has a bounded extension to $L^p([0, T], X)$, so that $A^{cl}$ has maximal $L^p$-regularity. We mention that we have proved in [2] that in the case of bounded operators $C \in \mathcal{L}(X, U)$, the operator $A^{cl}$ inherits the maximal $L^p$-regularity from $A$ (without assuming any other condition).

As application of the above results, we consider closed operators that have unbounded perturbations in their domains (induced by perturbation of the boundary conditions). More precisely, we consider three Banach spaces $X, Z, U$ such that $Z \subset X$.
(dense and continuous embedding), and three operators $A_m : Z \to X$ and $G, \mathcal{E} : Z \to U$ such that $A_m$ is closed, $G$ is bounded and surjective and $\mathcal{E}$ is bounded. We assume that $A := A_m|_{D(A)}$ with domain $D(A) := \ker(G)$ is a generator of a $C_0$-semigroup $T := (T(t))_{t \geq 0}$ on $X$ such that $A \in MR_p(0, T; X)$ for a fixed $p \in [1, \infty)$, and look for conditions for which the following operator
\[
A := A_m|_{D(A)} \quad \text{with} \quad D(A) := \{ x \in Z : Gx = \mathcal{E}x \}.
\] (4)
is a generator on $X$ and satisfies $A \in MR_p(0, T; X)$. We recall that the generation property of $A$ is obtained in [11, Theorem 4.1]. Under suitable conditions, it is shown that there exist bounded operators $B : U \to X_{-1}$ and $\mathcal{E'} : D(\mathcal{E'}) \subset U$ such that $Z \subset D(\mathcal{E'})$ and $\mathcal{E'}|_Z = \mathcal{E}$, for which $A$ coincides with the operator $A^{cl}$ defined above, see Theorem 6. Therefore, we obtain in Corollaries 1 and 2 maximal $L^p$-regularity of the operator $A$.

**Notation.** Let $0 < T < \infty$, $p \in [1, \infty)$ and $q \in (1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $X$ is a Banach space, we denote by $L^p([0, T]; X)$ the space of all $X$-valued Bochner $L^p$-integrable functions. For any $\theta \in (0, \pi)$, $\Sigma_\theta$ is the following sector:
\[
\Sigma_\theta := \{ z \in \mathbb{C}^*; |\text{arg} z| < \theta \}.
\]
For any $\alpha \in \mathbb{R}$, the right half-plane is defined by
\[
\mathbb{C}_\alpha := \{ z \in \mathbb{C}; \text{Re} z > \alpha \}.
\]
Given a semigroup $T := (T(t))_{t \geq 0}$ generated by an operator $A : D(A) \subset X \to X$, we will always denote by $\omega_0(T)$ (or $\omega_0(A)$) the growth bound of this semigroup, that is
\[
\omega_0(A) := \inf \left\{ \omega \in \mathbb{R} : \text{there exists } M_\omega \geq 1 \text{ such that } \|T(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0 \right\}.
\]
The resolvent set of $A$ is denoted by $\rho(A)$. Preferably, we denote the resolvent operator of $A$ by $R(\lambda, A) := (\lambda - A)^{-1}$ for any $\lambda \in \rho(A)$, where the notation $\lambda - A$ means $\lambda I - A$.

### 2. Preliminaries

The purpose of this section is to recall from [24] the generation property of the operator $A^{cl}$ defined in (1). On a Banach spaces $X$ and $U$, we consider the operators $A : X \supset D(A) \to X$, $B : U \to X_{-1}$ and $C : D(A) \to U$ such that $A$ generates a $C_0$-semigroup $T$ on $X$, $X_{-1}$ is the completion of $X$ with respect to the norm $\| \cdot \|_{-1} := \| R(\mu, A) \cdot \| \text{ and } B$ and $C$ are bounded linear operators. The Yosida extension of $C$ for $A$ is the following operator
\[ D(C_A) := \left\{ x \in X : \lim_{\mu \to +\infty} C\mu R(\mu, A)x \text{ exists in } U \right\} \]
\[ C_A x := \lim_{\mu \to +\infty} C\mu R(\mu, A)x. \]

Observe that \( D(A) \subset D(C_A) \) and \( (C_A)|_{D(A)} = C \).

We consider the operators
\[ \Phi_t u := \int_0^t \mathbb{T}_-(t - s)Bu(s)ds \]
for all \( t \geq 0 \) and \( u \in L^p([0, +\infty), U) \). Clearly, \( \Phi_t : L^p([0, +\infty), U) \to X_- \) are linear bounded. We say that \( B \) is a \( p \)-admissible control operator for \( A \) if there exists \( \tau > 0 \) such that \( \Phi_t u \in X \) for any \( u \in L^p([0, +\infty), U) \). This implies that \( \Phi_t \in \mathcal{L}(L^p([0, +\infty), U), X) \) for any \( t \geq 0 \), see [23]. On the other hand, we say that \( C \) is a \( p \)-admissible operator for \( A \) if for some \( \alpha > 0 \) (hence all \( \alpha > 0 \)), there is a constant \( \gamma := \gamma(\alpha) > 0 \) such that for any \( x \in D(A) \),
\[ \int_0^\alpha \| C\mathbb{T}_t(x)\|_p dt \leq \gamma^p \| x \|_p. \]  
(5)

In this case, \( \mathbb{T}(t)x \in D(C_A) \) for a.e. \( t > 0 \) and all \( x \in X \), and the application \( \Psi x := C_A\mathbb{T}(\cdot)x \) defines a linear bounded application \( X \to L^p([0, \alpha], U) \) for any \( \alpha > 0 \), see [22]. For \( \alpha > 0 \), we define the space
\[ W^{2,p}_{0,\alpha}(U) := \left\{ u \in W^{2,p}([0, \alpha], U) : u(0) = 0 \right\}, \]
which is a dense space in \( L^p([0, \alpha], U) \). Without loss of generality we assume that \( 0 \in \rho(A) \). Now an integration by parts shows that for any \( t \in (0, \alpha) \) and \( u \in W^{2,p}_{0,\alpha}(U) \),
\[ \Phi_t u = R(0, A_\alpha)Bu(t) - R(0, A)\Phi_t \dot{u}, \]

Clearly, \(- R(0, A)\Phi_t \dot{u} \in D(A)\). If \( R(0, A_\alpha)Bu(t) \in D(C_A) \), then \( \Phi_t u \in D(C_A) \). This allows us to define the following operator
\[ (\mathbb{F}_\alpha u)(t) = C_A \Phi_t u, \quad u \in W^{2,p}_{0,\alpha}(U), \quad t \in [0, \alpha]. \]  
(6)

**Definition 1.** Let \( B \) and \( C \) be \( p \)-admissible control and observation operators for \( A \), respectively. We say that the triple \( (A, B, C) \) is regular if there exists \( \lambda \in \rho(A) \) such that \( R(\lambda, A_-)BU \subset D(C_A) \) and for any \( \alpha > 0 \) there exists a constant \( \vartheta_\alpha > 0 \) such that for all \( u \in W^{2,p}_{0,\alpha}(U) \),
\[ \| \mathbb{F}_\alpha u \|_{L^p([0,\alpha], U)} \leq \vartheta_\alpha \| u \|_{L^p([0,\alpha], U)}. \]  
(7)

If the triple \( (A, B, C) \) is regular, then by (7) and a density argument, the operator \( \mathbb{F}_\alpha \) has an extension \( \mathbb{F}_\alpha \in \mathcal{L}(L^p([0, \alpha], U)) \) for any \( \alpha > 0 \). As a consequence, there
is a unique operator $\mathbb{F} : L^p_{\text{loc}}([0, \infty), U) \rightarrow L^p_{\text{loc}}([0, \infty), U)$, called the extended input–output map (see for instance [25]), such that
\[ \mathbb{F}_\alpha u = \mathbb{F}u, \quad \text{for all } u \in L^p([0, \alpha], U). \]

The transfer function of the triple $(A, B, C)$ is given by
\[ H(\lambda) = C_A R(\lambda, A_{-1}) B, \quad (8) \]
for any $\lambda \in \mathbb{C}$ with $\text{Re}\, \lambda > \alpha > \omega_0(\mathbb{T})$, see [25]. In addition, we have
\[ \lim_{s \to +\infty} H(s)v = 0, \quad v \in U \quad (9) \]
The following result is proved in [19,24].

**Theorem 1.** Let $(A, B, C)$ be a regular triple with $1 \in \rho(\mathbb{F})$. Then, the operator defined by
\[ A^{cl} := A_{-1} + BC_A, \quad D(A^{cl}) = \{x \in D(C_A) : (A_{-1} + BC_A) x \in X\}, \]
generates a $C_0$-semigroup $(T^{cl}(t))_{t \geq 0}$ on $X$ satisfying for any $x \in X$, $T^{cl}(t)x \in D(C_A)$ for a.e. $t > 0,$
\[ \|C_A T^{cl}(\cdot)x\|_{L^p([0, \alpha], U)} \leq c_\alpha \|x\|, \]
\[ T^{cl}(t)x = T(t)x + \int_0^t T_{-1}(t-s) BC_A T^{cl}(s)x ds, \quad t \geq 0. \quad (10) \]
for any $t \geq 0$ and constants $\alpha > 0$ and $c_\alpha > 0$. Moreover, for any $\lambda \in \rho(A)$,
\[ \lambda \in \rho(A^{cl}) \Leftrightarrow 1 \in \rho(C_A R(\lambda, A_{-1}) B). \]
In this case,
\[ R(\lambda, A^{cl}) = R(\lambda, A) + R(\lambda, A_{-1}) B(I_U - C_A R(\lambda, A_{-1}) B)^{-1} C R(\lambda, A). \]

**Remark 1.** Let the assumptions of Theorem 1 be satisfied and let $f \in L^p_{\text{loc}}([0, +\infty), X)$. We set
\[ z(t) = \int_0^t T^{cl}(t-s)f(s)ds, \quad t \geq 0. \quad (11) \]
This function is the mild solution of the inhomogeneous Cauchy problem $\dot{z}(t) = A^{cl}z(t) + f(t), \ t > 0$ and $z(0) = 0$. From (10) and [10, Prop.3.3 and its proof], we have $z(t) \in D(C_A)$ for a.e. $t \geq 0$, and for any $\alpha > 0$,
\[ \|C_A z(\cdot)\|_{L^p([0, \alpha], U)} \leq c_\alpha \|f\|_{L^p([0, \alpha], X)}, \]
where $c_\alpha > 0$ is a constant independent of $f$. By using an approximation argument, one can see that
\[ z(t) = \int_0^t T(t-s)f(s)ds + \int_0^t T_{-1}(t-s) BC_A z(s)ds, \quad t \geq 0. \]

The following result gives the analyticity of the semigroup generated by the operator $A^{cl}$. 

Theorem 2. Assume that \((A, B, C)\) is a regular triple, \(1 \in \rho(\mathbb{R})\) and the semigroup generated by \(A\) is analytic. Then, \(A^{cl}\) defined by (1) is the generator of an analytic semigroup on \(X\).

Proof. According to Theorem 1, \(A^{cl}\) is a generator of a strongly continuous semigroup on \(X\). Now assume that \(A\) generates an analytic semigroup \(T\) on \(X\). Then, by [17, Theorem 12.31], there exist constants \(\beta \in \mathbb{R}\) and \(M_1 > 0\) such that \(C_{\beta} \subset \rho(A)\) and 

\[
\| R(\lambda, A) \| \leq \frac{M_1}{|\lambda - \beta|}, \quad \lambda \in \mathbb{C}_{\beta}.
\]  

(12)

In [15, Lemma 2.3], it is proved that the admissibility of \(C\) (and \(B\) by the same method) for \(A\), imply that there exists \(\gamma \in \pi_{\frac{\pi}{2}}\pi\) such that

\[
M_2 := \sup_{z \in \Sigma_{\gamma}} |z|^\frac{1}{q} \| CR(z, A) \| < +\infty,
\]

\[
M_3 := \sup_{z \in \Sigma_{\gamma}} |z|^\frac{1}{p} \| R(z, A^{-1}) B \| < +\infty,
\]

(13)

where \(\frac{1}{p} + \frac{1}{q} = 1\). Let \(\omega_1 := \max\{\omega_0(A), \omega_0(A^{cl})\}\). Due to Theorem 1, for any \(\lambda \in \mathbb{C}_{\omega_1}\), we have

\[
R(\lambda, A^{cl}) = R(\lambda, A) + R(\lambda, A^{-1}) B (I_U - C_{\Lambda} R(\lambda, A^{-1}) B)^{-1} C_{\Lambda} R(\lambda, A).
\]

(14)

On the other hand, \((I_U - C_{\Lambda} R(\lambda, A^{-1}) B)^{-1} = I + H^{cl}(\lambda),\) where \(H^{cl}\) is the transfer function of the (closed-loop) regular linear system generated by \((A^{cl}, B, C_{\Lambda})\). Hence, there exists \(\alpha > \omega_0(A^{cl})\) such that

\[
\nu := \sup_{\mathbb{C}_{\alpha}} \left\| (I_U - C_{\Lambda} R(\lambda, A^{-1}) B)^{-1} \right\| < \infty.
\]

(15)

Now let \(\omega_2 := \max\{0, \alpha, \beta, \omega_1\}\). Combining (12), (13), (14) and (15), we obtain

\[
\| R(\lambda, A^{cl}) \| \leq \frac{\tilde{M}}{|\lambda - \omega_2|},
\]

for all \(\lambda \in \mathbb{C}_{\omega_2}\), where \(\tilde{M} = M_1 + \nu M_2 M_3\). Finally, by virtue of [17, Theorem 12.31] the semigroup generated by \(A^{cl}\) is analytic. \(\square\)

3. Perturbation Theorems for maximal regularity

The aim of this section is to study the maximal \(L^p\)-regularity for the operator \(A^{cl}\) defined by (1). We will use the notation and the hypothesis from the previous section. In particular those assumptions for which \(A^{cl}\) is a generator (see Theorem 1).

The problem to be solved in this section is the following: We assume that \(A\) has maximal \(L^p\)-regularity and look for condition for which \(A^{cl}\) has the same property.
By the maximal $L^p$-regularity we mean for $T > 0$, $p \in [1, \infty)$ and $f \in L^p([0, T])$, the problem $\hat{z} = Az + f$, $z(0) = 0$ has a unique solution $z \in W^{1,p}([0, T], X) \cap L^p([0, T], D(A))$. In this case, we write $A \in MR_p(0, T; X)$. Due to the closed graph theorem, if $A \in MR_p(0, T; X)$ then

$$
\|\hat{z}\|_{L^p([0,T],X)} + \|z\|_{L^p([0,T],X)} + \|Az\|_{L^p([0,T],X)} \leq C\|f\|_{L^p([0,T],X)},
$$

(16)

for a constant $C > 0$ independent of $f$. On the other hand, $A \in MR_p(0, T; X)$ if and only if the operator $B$ defined in (2) has a bounded extension to $L^p([0, T], X)$ (denoted by the same symbol), see [20]. We also mention that if $X$ is a Hilbert space and $1 < p < \infty$, $A \in MR_p(0, T; X)$ is equivalent to $A$ generates an analytic semigroup on $X$, see [5]. However, in a general Banach space the analyticity of the corresponding semigroup is only a necessary condition. Furthermore if for some $p \in [1, \infty]$, $A \in MR_p(0, T; X)$ then for all $r \in (1, \infty)$, $T' > 0$, and $\lambda \in \mathbb{C}$, we have $A \in MR_r(0, T'; X)$ and $A + \lambda I_X \in MR_p(0, T; X)$, see [7]. This allows us to assume from now that $\omega_0(A) < 0$ (that is the semigroup generated by $A$ is uniformly exponentially stable).

In the first main result of this section, we will work on UMD spaces. We recall the definition of these spaces.

**Definition 2.** We say that a Banach space $X$ is a UMD space if for some (hence all) $p \in (1, \infty)$, $\mathcal{H} \in \mathcal{L}(L^p(\mathbb{R}, X))$ where

$$(\mathcal{H} f)(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|s| > \epsilon} \frac{f(t - s)}{s} \, ds, \quad t \in \mathbb{R}, \quad f \in \mathcal{S}(\mathbb{R}, X),$$

is the Hilbert transform and $\mathcal{S}(\mathbb{R}, X)$ is the $X$-valued Schwartz space.

Examples of classical UMD spaces are Hilbert spaces and $L^p$-spaces, where $p \in (1, \infty)$. It is to be noted that every UMD space is a reflexive space (see [1]).

**Definition 3.** A set $\tau \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \tau$, $x_1, \ldots, x_n \in X$,

$$
\int_0^1 \| \sum_{j=1}^n r_j(s)T_jx_j \|_Y ds \leq C \int_0^1 \| \sum_{j=1}^n r_j(s)x_j \|_X ds,
$$

where $(r_j)_{j \geq 1}$ is a sequence of independent $\{-1; 1\}$-valued random variables on $[0, 1]$ (e.g., Rademacher variables).

**Remark 2.** Let $(A, B, C)$ be a regular triple with transfer function $H(\cdot)$, see Sect. 2. Then, $\{H(is) : s \neq 0\}$ is $\mathcal{R}$-bounded, see [13, p.513].

The following remarkable result is due to Weis [21].

**Theorem 3.** Let $\mathbb{G}$ be the generator of a bounded analytic semigroup in a UMD space $X$. Then, $\mathbb{G}$ has maximal $L^p$-regularity for some (hence all) $p \in (1, \infty)$ if and only if the set $\{sR(is, \mathbb{G}) : s \neq 0\}$ is $\mathcal{R}$-bounded.
We are still working under the setting of Sect. 2 and we will study the maximal regularity of the operator $A^{cl}$ in the case of UMD spaces.

**Theorem 4.** Assume that $X$, $U$ are UMD spaces, $p \in (1, \infty)$, $(A, B, C)$ is a regular triple with $1 \in \rho(\mathbb{F})$ and $A$ generates a bounded analytic semigroup. Furthermore, assume that there exists $\omega > \omega_1$ such that the sets
\[\{s^{1\over p} R(\omega+i s, A_{-1}) B; s \neq 0 \}\] (17)
and
\[\{s^{1\over q} C R(\omega+i s, A); s \neq 0 \}\] (18)
are $\mathcal{R}$-bounded. If $A \in MR_p(0, T; X)$ then $A^{cl} \in MR_p(0, T; X)$.

**Proof.** Assume that $A \in MR_p(0, T; X)$ and let $\omega > \omega_1$ such that the sets $\{s^{1\over p} R(\omega + i s, A_{-1}) B; s \neq 0 \}$ and $\{s^{1\over q} C R(\omega + i s, A); s \neq 0 \}$ are $\mathcal{R}$-bounded. Denote $A^{cl,\omega} := -\omega + A^{cl}$ and $A^\omega := -\omega + A$ with domains $D(A^{cl,\omega}) = D(A^{cl})$ and $D(A^\omega) = D(A)$, respectively. These operators are generators of analytic semigroups on $X$. We first observe that $A^\omega \in MR_p(0, T; X)$. To prove our theorem it suffices to show that $A^{cl,\omega} \in MR_p(0, T; X)$. Clearly, $\omega_0(A^\omega) = \omega_0(A) - \omega < 0$ and $\omega_0(A^{cl,\omega}) = \omega_0(A^{cl}) - \omega < 0$, so that $i \mathbb{R} \setminus \{0\} \subset \rho(A^\omega) \cap \rho(A^{cl,\omega})$. It is not difficult to prove that $(A^\omega, B, C)$ is a regular linear system on $X, U, U$ and $1 \in \rho(\mathbb{F}^\omega)$. According to Theorem 1, the operator $A^{cl,\omega}$ is the generator of a $C_0$-semigroup. As in (14) we have
\[
s R(i s, A^{cl,\omega}) = s R(i s, A^\omega) + s^{1\over p} R(i s, A_{-1}^\omega) B(I - H^\omega(i s))^{-1} s^{1\over q} C R(i s, A^\omega) = s R(i s, A^\omega) + s^{1\over p} R(\omega + i s, A_{-1}) B(I - H^\omega(i s))^{-1} s^{1\over q} C R(\omega + i s, A),
\] (19)
where $H^\omega(\lambda) = C_A R(\lambda, A_{-1}^\omega) B$, $\lambda \in \rho(A)$, is the transfer function of the regular linear system generated by $(A^\omega, B, C)$. Using the assumptions, equation (19) and Theorem 3, it suffice to show that the set $\{(I - H^\omega(i s))^{-1} : s \neq 0 \}$ is $\mathcal{R}$-bounded. In fact, by [24], the triple $(A^{cl,\omega}, B, C_A)$ is regular linear system with transfer function
\[H^{cl,\omega}(i s) = (I - H^\omega(i s))^{-1} H^\omega(i s), \quad s \neq 0,
\]
which implies that
\[(I - H^\omega(i s))^{-1} = I_U + H^{cl,\omega}(i s), \quad s \neq 0.
\]
According to Remark 2, the set $\{H^{cl,\omega}(i s) : s \neq 0 \}$ is $\mathcal{R}$-bounded. Hence, $\{(I - H^\omega(i s))^{-1} : s \neq 0 \}$ is $\mathcal{R}$-bounded. This ends the proof. \hfill $\Box$

In the rest of this section, we assume that $X$ is a general Banach space. The following is the second main result of this section.
**Theorem 5.** Assume that \((A, B, C)\) is a regular triple, \(p \in [1, \infty), T > 0\) and \(A \in MR_p(0, T; X)\). Assume additionally that there exists \(\lambda_0 \in \mathbb{R}\) such that
\[
\kappa_0 := \sup_{\lambda > \lambda_0} \|\lambda R(\lambda, A_{-1}) B\| < +\infty. \tag{20}
\]

Then, \(A^{cl} \in MR_p(0, T; X)\).

**Proof.** As \(A \in MR_p(0, T; X)\), then the operator \(\mathcal{R}\) defined by (2) has a bounded extension \(\mathcal{R} \in \mathcal{L}(L^p([0, T], X))\). On the other hand, according to Theorem 2, \(A^{cl}\) generates an analytic semigroup \(T^{cl}\) on \(X\). Now let \(\mathcal{R}^{cl}\) be the operator defined by (3). Next we will prove that \(\mathcal{R}^{cl}\) admits a bounded extension to \(L^p([0, T]; X)\). To this end, consider the Yosida approximation operators of \(A^{cl}\),
\[
A_{n}^{cl} := nA^{cl} R(n, A^{cl}) = n^2 R(n, A^{cl}) - nI, \quad n \in \mathbb{N} \cap (\omega_0(A^{cl}), \infty).
\]

Using (14), we have, for a sufficiently large integer \(n\),
\[
A_{n}^{cl} = nA R(n, A) + n^2 R(n, A_{-1}) B(I - C_{A} R(n, A_{-1}) B)^{-1} C R(n, A). \tag{21}
\]

For each \(n > \omega_0(A^{cl})\), we define
\[
(\mathcal{R}_{n}^{cl} f)(t) := A_{n}^{cl} \int_{0}^{t} T^{cl}(t - s) f(s)ds, \quad f \in C([0, T], D(A^{cl})).
\]

From [8, Chap. III]), we have
\[
(\mathcal{R}^{cl} f)(t) = \lim_{n \to \infty} (\mathcal{R}_{n}^{cl} f)(t), \quad t \in [0, T].
\]

Let \(t \mapsto z(t)\) be the function defined by (11). By Remark 1,
\[
(\mathcal{R}_{n}^{cl} f)(t) = A_{n}^{cl} \left( \int_{0}^{t} T(t - s) f(s)ds + \int_{0}^{t} T_{-1}(t - s) C_{A} z(s)ds \right).
\]

Using (21), we have for large \(n\)
\[
(\mathcal{R}_{n}^{cl} f)(t) = nA R(n, A) \int_{0}^{t} T(t - s) f(s)ds
+ n^2 R(n, A_{-1}) B(I - C_{A} R(n, A_{-1}) B)^{-1} C R(n, A) \times \int_{0}^{t} T(t - s) f(s)ds
+ nA R(n, A) \int_{0}^{t} T_{-1}(t - s) B C_{A} z(s)ds
+ n^2 R(n, A_{-1}) B(I - C_{A} R(n, A_{-1}) B)^{-1} C R(n, A) \times \int_{0}^{t} T_{-1}(t - s) B C_{A} z(s)ds
:= I_{n}^{1}(t) + I_{n}^{2}(t) + I_{n}^{3}(t) + I_{n}^{4}(t).
\]
Let us estimate separately the four terms on the right of the above equality. We know that there exists $M > 0$ such that $\|nR(n, A)\| \leq M$. Maximal regularity of $A$ yields
\[
\int_0^T \|I_n^1(t)\|^p dt = \|\mathcal{R}(nR(n, A)f)\|_{L^p([0,T], X)}^p \leq c_T p M^p \|f\|_{L^p([0,T], X)}^p,
\]
for a constant $c_T > 0$ independent of $f$. On the other hand, combining (15), (20) and [10, Prop.3.3], we obtain
\[
\int_0^T \|I_n^2(t)\|^p dt = \int_0^T \left\| nR(n, A-1) B(I - C_T R(n, A-1) B)^{-1} C nR(n, A) \right\|^p dt \\
\times \left( \int_0^T \|T(t-s)f(s)ds\|^p dt \right) \\
\leq (\kappa_0 \nu)^p \int_0^T C \int_0^t \|T(t-s)nR(n, A)f(s)ds\|^p dt \\
\leq (\kappa_0 \nu \gamma_T)^p \|f\|_L^p, = c_1(T, p) \|f\|_L^p,
\]
where $c_1(T, p)$ is a constant depending on $T$ and $p$. Similarly,
\[
\int_0^T \|I_n^3(t)\|^p dt = \int_0^T \left\| A \int_0^t \|T(t-s)nR(n, A-1)BC_A z(s)ds\|^p dt \right\|^p dt \\
\leq (\nu \kappa_0)^p \|C_A z(\cdot)\|_{L^p([0,T], U)}^p \\
\leq c_2(T, p) \|f\|_L^p,
\]
due to (20) and Remark 1, where $c_2(T, p) > 0$ is a constant depending only on $T$ and $p$. Next, we estimate
\[
\int_0^T \|I_n^4(t)\|^p dt \\
= \int_0^T \left\| nR(n, A-1) B(I - C_A R(n, A-1) B)^{-1} C_A \right\|^p dt \\
\times \left( \int_0^t \|T(t-s)nR(n, A-1)BC_A z(s)ds\|^p dt \right) \\
\leq (\nu \kappa_0)^p \|C_A z(\cdot)\|_{L^p([0,T], U)}^p \\
\leq c_3(T, p) \|f\|_L^p,
\]
by (15), (20), [10, Prop.3.3] and Remark 1, where $c_3(T, p) > 0$ is a constant depending only on $T$ and $p$. Finally one can conclude that
\[
\int_0^T \|\mathcal{R}(n f)(t)\|^p dt \leq C(T, p) \|f\|_L^p,
\]
for some constant $C(T, p) > 0$ depending only on $T$ and $p$. 

We know that \( \| (\mathcal{R}^\text{cl}_n f)(t) \|_p \to \| (\mathcal{R}^\text{cl} f)(t) \|_p \) for all \( t \in [0, T] \), then by Fatou’s lemma we obtain
\[
\int_0^T \| (\mathcal{R}^\text{cl} f)(t) \|_p \, dt \leq \liminf_{n \to \infty} \int_0^T \| (\mathcal{R}^\text{cl}_n f)(t) \|_p \, dt \leq C(T, p) \| f \|_{L^p}.
\]
Thus, \( \mathcal{R}^\text{cl} \) can be extended to a bounded operator on \( L^p([0, T]; X) \).

\[ \square \]

4. Application to boundary perturbations

Consider on a Banach spaces \( X \) and \( U \) the following boundary value problem

\[
\begin{aligned}
\dot{z}(t) &= A_m z(t) + f(t), \quad t \in [0, T] \\
Gz(t) &= \mathcal{C}z(t), \quad t \in [0, T]
\end{aligned}
\]

where \( A_m : Z \subset X \to X \) is a closed linear operator with \( Z \) is densely and continuously embedded in \( X \) and \( G, \mathcal{C} : Z \to U \) are bounded linear operators such that \( G \) is surjective. Closedness of \( A_m \) implies (by the closed graph theorem) that the graph norm of \( A_m \) and the norm of \( Z \) are equivalent and that \( A_m \in \mathcal{L}(Z, X) \). We assume that the operator \( A := A_m|_{D(A)} \) with domain \( D(A) := \ker(G) \) is a generator of a \( C_0 \)-semigroup \( \mathbb{T} := (\mathbb{T}(t))_{t \geq 0} \) on \( X \).

In [9], it is proved that under the above setting, \( Z \) can be written as a direct sum of \( D(A) \) and \( \ker(\lambda - A_m) \), with \( \lambda \in \rho(A) \), and the operator \( (G|_{\ker(\lambda - A_m)})^{-1} \) exists, it is noted \( \mathbb{D}_\lambda \) and satisfies \( \mathbb{D}_\lambda \in \mathcal{L}(U, X) \). Now we set

\[
B := (\lambda - A_{-1})\mathbb{D}_\lambda \quad \text{and} \quad C := \mathcal{C}|D(A)
\]

for \( \lambda \in \rho(A) \). We mention that \( B \in \mathcal{L}(U, X_{-1}) \) and \( C \in \mathcal{L}(D(A), X) \) and \( B \) is independent of \( \lambda \), due to the resolvent equation (use [9, Lemma 1.3 (1.16)]).

The following result, proved in [11, Theorem 4.1], guarantees the generation property of the operator defined by (4).

**Theorem 6.** Let \( (A, B, C) \) be a regular triple with \( 1 \in \rho(F) \). Then, the operator \( (A, D(A)) \) defined by (4) coincides with the following operator

\[
A^\text{cl} := A_{-1} + BC_A, \quad D(A^\text{cl}) = \{ x \in D(C_A) : (A_{-1} + BC_A)x \in X \},
\]

which generates a \( C_0 \)-semigroup \( (\mathbb{T}^\text{cl}(t))_{t \geq 0} \) on \( X \).

Now, one can easily see that the following corollaries follow from Theorems 4 and 5.
Corollary 1. Assume that $X, U$ are UMD spaces, $p \in (1, \infty)$, $(A, B, C)$ is a regular triple with $1 \in \rho(F)$ and $A$ generates a bounded analytic semigroup. Furthermore, assume that there exists $\omega > \omega_1$ such that the sets
\[
\{ s^{\frac{1}{p}} R(\omega + is, A_{-1}) B ; s \neq 0 \}
\] (23)
and
\[
\{ s^{\frac{1}{q}} C R(\omega + is, A) ; s \neq 0 \}
\] (24)
are $\mathcal{R}$-bounded. If $A \in MR_p(0, T; X)$ then $A \in MR_p(0, T; X)$.

Corollary 2. Assume that $(A, B, C)$ is a regular triple, $p \in [1, \infty)$, $T > 0$ and $A \in MR_p(0, T; X)$. Assume additionally that there exists $\lambda_0 \in \mathbb{R}$ such that
\[
\kappa_0 := \sup_{\lambda > \lambda_0} \| \lambda D \lambda \| < +\infty.
\] (25)
Then, $A \in MR_p(0, T; X)$.

Remark 3. (i) Here, we compare Corollary 1 with the corresponding one in [12, Prop 6.5 and Coro 6.6]. It is to be noted that the comparison can only occur in the case of boundary perturbation and this is due to the fact that the operators $B$ in Corollary 1 and [12, Prop 6.5 and Coro 6.6] are injective and strictly unbounded. In [12], the authors assumed that the $R$-bound of the transfer function $H(\cdot) = C A R(\cdot, A_{-1}) B$ is less than 1 to obtain invertibility of $(I - H(\cdot))$ and $R$-boundedness of its inverse. In our result, we do not need this restrictive condition since $R$-boundedness of $(I - H(\cdot))^{-1}$ is guaranteed by the one of the transfer function of the closed-loop system and the invertibility is guaranteed by the condition $1 \in \rho(F)$. It is true that the regularity of $(A, B, C)$ can be seen as an extra condition when comparing these results, but when the “Weiss-Conjecture” is satisfied, one can prove that conditions of [12, Prop 6.5 and Coro 6.6] imply the regularity of $(A, B, C)$. In [15], Le Merdy showed that if we have some “square function estimates” for $A$ then admissibility is guaranteed by the condition of the “Weiss-Conjecture,” and, to the best of our knowledge, there is no PDE-example of a generator of an analytic semigroup that would not admit these square function estimates.

(ii) Due to Weis’ characterization in Theorem 3, the exponent $p$ in Theorem 4 and Corollary 1 is fixed in $(1, \infty)$, while in Theorem 5 and Corollary 2, $p$ is fixed in $[1, \infty)$.

(iii) It is to be noted that the condition (25) implies that the space $X$ is non-reflexive. In fact, this estimate is equivalent to $Range(B) \subset F_{1}^{A_{-1}}$ (see [16, Remark 10]) where $F_{1}^{A_{-1}}$ stands for the Favard space associated with $A_{-1}$. We recall that the control operator $B$ is strictly unbounded, i.e., $Range(B) \cap X = \{ 0 \}$, since it comes from the boundary (see [18]). So, if $X$ is reflexive then $F_{1}^{A_{-1}} = X$ and consequently $Range(B) \subset X$ which is contradictory since $B$ is not identically zero.
Acknowledgements

We would like to thank the editors and the referee whose detailed comments helped us to considerably improve the organization and the content of the paper.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

REFERENCES

[1] H. Amann. Linear and Quasilinear Parabolic Problems. Vol. I. Birkhäuser, (1995).
[2] A. Amansag, H. Bounit, A. Driouch and S. Hadd. On the maximal regularity for perturbed autonomous and non-autonomous evolution equations. J. Evol. Equ., 20 (2020), 165–190.
[3] P. Cannarsa and V. Vespri. On maximal $L^p$ regularity for the abstract Cauchy problem. Boll. Un. Mat. Ital., B5 (1986), 165–175.
[4] T. Coulhon and D. Lamberton. Régularité $L^p$ pour les équations d’évolution. Séminaire d’Analyse Fonctionnelle 1984/1985, Publ. Math., Univ. Paris VII 26 (1986), 155–165.
[5] L. De Simon, Un applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineare astratte del primo ordine. Rend. Sem. Mat., Univ. Padova, 99 (1964), 205–223.
[6] R. Denk, M. Hieber and J. Pruss, R-Boundedness. Fourier Multipliers and Problems of Elliptic and Parabolic Type. Memoirs Amer. Math. Soc., vol. 166, Amer. Math. Soc., Providence, R.I., (2003).
[7] G. Dore. Maximal regularity in $L^p$ spaces for an abstract Cauchy problem. Adv. Differ. Equat., 5 (2020), 293–322.
[8] K.-J. Engel and R. Nagel. One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, Berlin, Heidelberg, 2000.
[9] G. Greiner. Perturbing the boundary conditions of a generator. Houston J. Math., 13 (1987), 213–229.
[10] S. Hadd. Unbounded perturbations of $C_0$-semigroups on Banach spaces and applications. Semigroup Forum, 70 (2005), 451–465.
[11] S. Hadd, R. Manzo and A. Rhandi. Unbounded perturbations of the generator domain. Discrete & Continuous Dynamical Systems A, 35 (2015), 703–723.
[12] B.H. Haak and M. Haase and P.C. Kunstmann. Perturbation, interpolation and maximal regularity. Adv. Differential Equations, 11 (2006), 201–240.
[13] B.H. Haak and P.C. Kunstmann. Admissibility of Unbounded Operators and Wellposedness of Linear Systems in Banach Spaces. Integral Equations Operator Theory, 55 (2006), 497–533.
[14] P.C. Kunstmann and L. Weis. Perturbation theorems for maximal $L^p$-regularity, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 30 (2001), 415–435.
[15] C. Le Merdy. The Weiss conjecture for bounded analytic semigroups. J. Lond. Math. Soc., 67 (2003), 715–738.
[16] F. Maragh, H. Bounit, A. Fadili and H. Hammouri. On the admissible control operators for linear and bilinear systems and the Favard spaces. Bull. Belg. Math. Soc., Simon Stevin 21 (2014), 711–732.
[17] M. Renardy and R. C. Rogers. An introduction to partial differential equations, 2nd ed., Texts in Applied Mathematics, vol. 13, Springer-Verlag, New York, (2004).
[18] D. Salamon. Infinite-dimensional linear system with unbounded control and observation: a functional analytic approach. Trans. Amer. Math. Soc., 300 (1987), 383–431.
[19] O.J. Staffans. Well-Posed Linear Systems. Cambridge Univ. Press, Cambridge, 2005.
[20] L. Weis. A new approach to maximal $L^p$-regularity, Proc. 6th International Conference on Evolution Equations, G. Lumer and L. Weis, eds, Dekker, New York (2000), 195–214.
[21] L. Weis. Operator-valued Fourier multiplier theorems and maximal $L^p$-regularity, Math. Ann., 319 (2001), 735–758.
[22] G. Weiss. Admissible observation operators for linear semigroups. Israel J. Math., 65 (1989), 17–43.
[23] G. Weiss, Admissibility of unbounded control operators. SIAM J. Control Optim., 27 (1989), 527–545.
[24] G. Weiss. *Regular linear systems with feedback*. Math. Control Signals Sys., 7 (1994), 23–57.
[25] G. Weiss. *Transfer functions of regular linear systems. Part I: Characterization of regularity*. Trans. Amer. Math. Soc., 342 (1994), 827-854.

Ahmed Amansag, Hamid Bounit, Abderrahim Driouich and Said Hadd  
Department of Mathematics, Faculty of Sciences  
Ibn Zohr University  
Hay Dakhla  
BP 8106, 80000 Agadir  
Morocco  
E-mail: ahmed.amansag@uiz.ac.ma

Hamid Bounit  
E-mail: h.bounit@uiz.ac.ma

Abderrahim Driouich  
E-mail: a.driouich@uiz.ac.ma

Said Hadd  
E-mail: s.hadd@uiz.ac.ma

Accepted: 13 October 2021