Cohomology groups for projection point patterns

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Abstract. Aperiodic point sets (or tilings) which can be obtained by the method of cut and projection from higher dimensional periodic sets play an important role for the description of quasicrystals. Their topological invariants can be computed using the higher dimensional periodic structure. We report on the results obtained for the cohomology groups of projection point patterns supplemented by explicit calculations made by F. Gähler for many well-known icosahedral tilings.

1. Introduction

Among the basic input data for the description of an aperiodic solid is the point set $T$ of its equilibrium atomic positions. It is a discrete subset of $\mathbb{R}^d$ which enjoys a number of properties depending on the particular nature of the solid \cite{BZ00}. For systems which show diffractive behaviour, like quasicrystals, one expects $T$ to have certain repetivity properties: e.g. $r$-patches, which are intersections of $r$-balls with $T$, repeat in a relatively dense way. $T$ is then a more or less regular pattern. For ideal quasicrystals there is a preferred construction for such patterns: by projection out of a higher dimensional periodic structure. These are the projection point patterns we are interested in here.

We consider topological invariants for point patterns of $\mathbb{R}^d$. These invariants are defined as the cohomology groups of the pattern groupoid. For projection point patterns matters simplify enormously because the pattern groupoid is equivalent to a groupoid defined by a Cantor dynamical system $(X,\mathbb{Z}^d)$ which can be explicitly described. As a consequence, we only have to deal with the cohomology of the group $\mathbb{Z}^d$ with coefficients in the integer valued continuous functions $C(X,\mathbb{Z})$. Furthermore, these cohomology groups are isomorphic to the (unordered) $K$-groups of the $C^*$-algebra associated with the pattern groupoid.

The invariants yield an important step towards the classification of point patterns. Moreover, the invariants play a role for the labelling of the gaps in the spectrum of a Hamiltonian describing the particle motion in the solid. The $C^*$-algebra of the pattern groupoid is the algebra of observables in the tight binding
representation. The tight binding Hamiltonian belongs to it and so do its spectral projections associated with gaps, i.e. projections on (generalized) eigenstates of all energies up to the gap. The $K_0$-class of such a projection furnishes a label for the gap which is stable under perturbations. An additional input from physics is a trace on the $C^*$-algebra coinciding in the physically relevant representation with the trace per unit volume. It induces a homomorphism from the $K_0$-group of the algebra to $\mathbb{R}$ and the values of the integrated density of states on gaps belong to its image. This theory is due to Bellissard \cite{Bel92}. The present results do not yet complete the gap-labelling because they do not contain information about the order and the image of tracial states. For recent results in the latter direction see \cite{BKL00}.

In the coming section we briefly describe the hull-construction of a point set. After that we link this construction with the non-commutative approach using $C^*$-algebras and $K$-theory. In Section 4 we introduce the point sets on which we focus our attention here, presenting the general results in Section 5 and the examples in Section 6. The results of Section 5 (together with the first example of Table 1) are taken from \cite{FHK00} to which we refer for all proofs and missing explanations. The remaining examples have been computed by Franz Gähler.

2. Hull of a point set and its dynamical system

Point sets in $\mathbb{R}^d$ give rise to topologically highly non-trivial spaces and dynamical systems by means of the so-called hull construction: Let $B_r$ be the ball of radius $r$ around the origin $0 \in \mathbb{R}^d$, $\partial B_r$ its boundary, and $B_r(T):=(B_r \cap T) \cup \partial B_r$ called the $r$-patch of the set $T \subset \mathbb{R}^d$. Then

$$D(T, T') = \inf \{ \frac{1}{r+1} | d_H(B_r(T), B_r(T')) < \frac{1}{r} \},$$

where $d_H$ is the Hausdorff metric, is a metric on the set of all closed subsets of $\mathbb{R}^d$. Thus two sets are close if they coincide on a large ball around the origin up to a small discrepancy. We are interested in the topology defined by that metric. The group $\mathbb{R}^d$, acting on a subset of $\mathbb{R}^d$ by translation, acts continuously. The (continuous) hull of $T$ is the $D$-completion $MT$ of the orbit of $T$ under the action of $\mathbb{R}^d$. This action extends to the closure and so we define the dynamical system $(MT, \mathbb{R}^d)$ associated to $T$. This dynamical system is the starting point for many investigations, see e.g. \cite{Rud89, RW92, AP98}.

The space $MT$ is compact under very general conditions, in particular if $T$ is a Delone set. But it may be quite complicated. It consists of orbits of point sets each one being homeomorphic to a copy of $\mathbb{R}^d$. The closed subset $\Omega T = \{ T \in MT \mid 0 \in T \}$ intersects each orbit and this intersection is transversal in the sense that very small translations move any point of $\Omega T$ outside it. If the dimension $d$ is 1 we can reduce by Poincaré’s construction the continuous dynamical system $(MT, \mathbb{R})$ to a discrete one $(\Omega T, \mathbb{Z})$ without loss of topological information (the first return map yielding the action of $\mathbb{Z}$). If $d$ is larger we cannot in general find an action of $\mathbb{Z}^d$ on $\Omega T$ which generalizes Poincaré’s construction but have to work with an $r$-discrete groupoid (the pattern groupoid) instead. Projection point patterns, however, fall into the category of point sets for which exist another transversal $X$ such that $(MT, \mathbb{R}^d)$ can be reduced to a discrete dynamical system $(X, \mathbb{Z}^d)$ without loss of

\footnote{A projection method pattern as in \cite{FHK00} is a little bit more general than a projection point pattern in that it allows for additional decorations.}
topological information. Another way of putting this is that $MT$ is the mapping torus of $(X, \mathbb{Z}^d)$, i.e. $MT = X \times \mathbb{R}^d/\sim$ where $(x, y + a) \sim (\alpha_a(x), y)$ for all $a \in \mathbb{Z}^d$ and $\alpha$ denotes the $\mathbb{Z}^d$ action.

3. $C^*$-algebras, $K$-groups and cohomology

Topological dynamical systems give rise to $C^*$-algebras by the crossed product construction. In our context we have two such algebras, $C(MT) \rtimes \mathbb{R}^d$ and $C(X) \rtimes \mathbb{Z}^d$. They are strongly Morita equivalent and therefore they have isomorphic $K$-theory (up to scale). As an aside we mention that there is yet another construction of a $C^*$-algebra from a point set $T$, the groupoid $C^*$-algebra of the pattern groupoid. This $C^*$-algebra, which exists for rather general point sets, is strongly Morita equivalent to $C(MT) \rtimes \mathbb{R}^d$ as well. It has the interpretation as algebra of observables for the tight binding approximation, for a review see [KP00].

The mere existence of the transversal $X$ allows one to connect the $K$-theory with the cohomology of the group $\mathbb{Z}^d$. By Connes’ Thom isomorphism [Bla86], the $K$-groups of the crossed product $C(X) \rtimes \mathbb{Z}^d$ are isomorphic to those of the continuous hull,

\begin{equation}
K_n(C(X) \rtimes \mathbb{Z}^d) \cong K_{n-d}(C(MT)).
\end{equation}

A cofiltration of $MT$ gives rise to a spectral sequence whose $E_2$-term is isomorphic to $H^*(\mathbb{Z}^d, K_*(C(X)))$, the cohomology of $\mathbb{Z}^d$ with coefficients in $K_*(C(X))$. Forrest and Hunton [FH99] have established that, if $X$ is homeomorphic to the Cantor set, the spectral sequence collapses at the $E_2$-term and

\begin{equation}
K_j(C(MT)) \cong \bigoplus_j H^{2j+1}(\mathbb{Z}^d, C(X, \mathbb{Z})).
\end{equation}

For projection point patterns, $X$ is indeed homeomorphic to the Cantor set and the machinery of spectral sequences proves also to be useful in explicitly computing the cohomology-groups. Below we present the results of a refined analysis for canonical projection method patterns or tilings [FHK00]. We note, however, that this approach fails to give information about the order on $K_0$ and the ranges of tracial states.

4. Projection point patterns

Projection point patterns are obtained by cut and projection from higher-dimensional periodic structures, see e.g. [KD86]. Let $\Lambda$ be a rank $N$ lattice spanning $\mathbb{R}^N$ and $E$ be a linear subspace intersecting $\Lambda$ only trivially. Let $E^\perp$ be a complimentary subspace and denote by $\pi : \mathbb{R}^N \to E$ and $\pi^\perp : \mathbb{R}^N \to E^\perp$ the projections with kernel $E^\perp$ and $E$, respectively. Finally let $K \subset E^\perp$ be a compact subset which is the closure of its interior, called the acceptance domain.

$P(K) := \{\pi(x) | x \in \Lambda, \pi^\perp(x) \in K\}$

is the projection point pattern defined by the data $(\Lambda, E, K)$. There is a canonical choice for $K$, namely the projection of the unit cell for $\Lambda$ under $\pi^\perp$, but also fractal $K$ is of interest (although we will have nothing to say about this case). The euclidian closure of $\pi^\perp(\Lambda)$ can be written as $V + \Delta$, where $V$ is a linear subspace of $E^\perp$ and $\Delta$ a discrete subgroup spanning a transversal to $V$ in $E^\perp$.

Often one finds it convenient to consider tilings instead of patterns. The vertices of the canonical projection method tiling form the projection pattern with canonical
A clearer picture of the construction of canonical projection method tilings arises from a formulation based on dualization [KS89]. Here one starts with an \( N \)-dimensional periodic polyhedral complex and the tiles of the tiling consist of the projection of those \( d \)-faces of the complex which satisfy an acceptance condition. To formulate this condition one needs a dual complex. In particular, there is a duality involved here, and if one interchanges the complex with its dual one obtains the dual tiling.

For projection patterns we have a very explicit description of the hull and of the dynamical system \((X, \mathbb{Z}^d)\) which we now describe. Let \( \Gamma = \pi^+(A) \cap V \). It acts on \( V \) by translation. Further let \( X = (V, \mathbb{Z}^d) \). We formulate this condition one needs a dual complex. In particular, there is a duality projection of those \( N \)-spaces and their translates under the action by translation \( \Gamma \). The rather simple dynamical system \((V, \Gamma)\) extends to a dynamical system \((\overline{V}, \Gamma)\) which coincides with the old one on the dense \( G_\delta \)-set \( V \setminus S \). \( \overline{V} \) is locally a Cantor set and obtained from \( V \) upon disconnecting it along the points of \( S \). In the most interesting cases, the set \( S \) can be described as follows: there is a finite set \( W \) of (affine) hyperplanes of \( V \) such that \( S \) is the union of their translates under the natural action of \( \Gamma \). We call these planes and their translates singular planes. In the canonical case, \( W \) consists simply of the spaces spanned by the boundary faces of the acceptance domain \( K \), i.e. \( S = \bigcup_{w \in W, x \in \Gamma} (W + x) \). In general, \( S \) may not be such a union but we have to restrict our attention to that case. (In the formulation based on dualization one can also identify acceptance domains, these are the projections of the duals of the \( d \)-faces onto \( E^\perp \).) Finally we obtain \((X, \mathbb{Z}^d)\) from \((\overline{V}, \Gamma)\) upon splitting \( \Gamma = \mathbb{Z}^{\dim V} \oplus \mathbb{Z}^d \) in such a way that \( \mathbb{Z}^{\dim V} \) spans \( V \) and setting \( X = \overline{V}/\mathbb{Z}^{\dim V} \). Then \( \Gamma \) induces an action of \( \Gamma/\mathbb{Z}^{\dim V} \simeq \mathbb{Z}^d \) on \( X \). We summarize this in the following set up.

**Set up 4.1.** We consider data \((V, \Gamma, W)\), a dense lattice \( \Gamma \) of finite rank in an euclidian space \( V \) with a finite family \( W = \{W_i\}_i \) of affine hyperplanes whose normals span \( V \). We make the additional assumption that these normals form an indecomposable set in the sense that \( W \) cannot be written as a union \( W = W_1 \cup W_2 \) such that the normals of \( W_1 \) span complimentary spaces \( V_i \). We then define the dynamical system \((X, \mathbb{Z}^d)\) from \((\overline{V}, \Gamma)\) upon splitting \( \Gamma = \mathbb{Z}^{\dim V} \oplus \mathbb{Z}^d \) as above.

### 5. Cohomology groups for projection point patterns

The cohomology groups \( H^*(\mathbb{Z}^d, C(X, \mathbb{Z})) \) depend on the geometry and combinatorics of the intersections of the singular planes, i.e. of (affine) subspaces of the form

\[
\bigcap_{(W, x) \in A} (W + x)
\]

where \( A \) is some finite subset of \( W \times \Gamma \). We call such a space a singular \( l \)-space if its dimension is \( l \). Let \( P_l \) be the set of singular \( l \)-spaces and denote the orbit space under the action by translation \( I_l := P_l/\Gamma \). The stabilizer \( \{ x \in \Gamma \mid \hat{\Theta} + x = \hat{\Theta} \} \) of a singular \( l \)-space \( \hat{\Theta} \) depends only on the orbit class \( \Theta \in I_l \) of \( \hat{\Theta} \) and we denote it \( \Gamma^\Theta \). Fix \( \hat{\Theta} \in P_k \), \( l < k < \dim V \) and let \( P_l^{\hat{\Theta}} := \{ \Psi \in P_l \mid \Psi \subset \hat{\Theta} \} \). Then \( \Gamma^\Theta \) (the orbit class of \( \hat{\Theta} \)) acts on \( P_l^{\hat{\Theta}} \) and we let \( I_l^{\hat{\Theta}} := P_l^{\hat{\Theta}} / \Gamma^\Theta \). We can naturally identify \( I_l^{\hat{\Theta}} \) with \( I_l^{\hat{\Theta}'} \) if \( \hat{\Theta} \) and \( \hat{\Theta}' \) belong to the same \( \Gamma \)-orbit and so we define \( I_l^{\Theta} \), for the class \( \Theta \in I_k \). \( I_l^{\Theta} \subset I_l \) consists of those orbits of singular \( l \)-spaces which have a representative that lies in a singular space of class \( \Theta \). Finally we use the notation

\[
L_l = |I_l|, \quad L_l^{\Theta} = |I_l^{\Theta}|.
\]
Theorem 5.1. Given data \((V, \Gamma, W)\) as in (4.1). If \(L_0\) is finite \(H^p(\mathbb{Z}^d, C(X, \mathbb{Z}))\) is a finitely generated free abelian group, i.e. \(H^{p+q}(\mathbb{Z}^d, C(X, \mathbb{Z})) \cong \mathbb{Z}^{D_p}\) for finite \(D_p\). If \(L_0\) is infinite then \(H^d(\mathbb{Z}^d, C(X, \mathbb{Z}))\) is infinitely generated.

For better comparison with [FHK00] we wrote \(D_p = \text{rank} H^{d-p}(\mathbb{Z}^d, C(X, \mathbb{Z}))\). It is easily seen that \(D_p = 0\) for \(p < 0\) or \(p > d\).

Theorem 5.2. Given data \((V, \Gamma, W)\) as in (4.1). If \(L_0\) is finite the rank of the stabilizer \(\Gamma_{\Theta}\) depends only on the dimension \(\dim \Theta\) of the plane it stabilizes, i.e.
\[
\text{rank} \Gamma_{\Theta} = \nu \dim \Theta
\]
where \(\nu = \frac{\dim V}{\dim \Theta}\). In particular, \(\nu\) is a natural number.

The Euler characteristic is defined as
\[
e = \sum_{p} (-1)^p D_p.
\]
We can determine it for arbitrary codimension. Define a singular sequence to be a (finite) sequence \(c = \Theta_1, \Theta_2, ..., \Theta_k\) of \(\Gamma\)-orbits of singular spaces strictly ascending in the sense that \(\Theta_j \in I_{\Theta_{j+1}}\), \(\dim \Theta_j < \dim \Theta_{j+1}\), and \(\dim \Theta_1 = 0\). The length of the chain \(c\) is \(k\), written \(|c| = k\).

Theorem 5.3. Given data \((V, \Gamma, W)\) as in (4.1) with \(L_0\) finite. Then the Euler characteristic is
\[
e = \sum (-1)^{|c| + \dim V}
\]
where the sum is over all singular chains \(c\).

Finally we present explicit formulae for the ranks \(D_p\) in case the codimension is smaller or equal to 3. If \(\{M_i : i \in I\}\) is a family of submodules of some bigger module we denote by \(\langle M_i : i \in I \rangle\) their span. For a finitely generated lattice \(G\) we let \(\Lambda G\) be the exterior ring it generates.

Theorem 5.4. Given data \((V, \Gamma, W)\) as in (4.1) with \(L_0\) finite. We have
\[
dim V = 1
\]
\[
D_p = \binom{\nu}{p+1}, \quad p > 0
\]
\[
D_0 = (\nu - 1) + e,
\]
\[
dim V = 2
\]
\[
D_p = \binom{2\nu}{p+2} + L_1 \binom{\nu}{p+1} - r_{p+1} - r_p, \quad p > 0,
\]
\[
D_0 = \left(\frac{2\nu}{2}\right) - 2\nu + 1 + L_1 (\nu - 1) + e - r_1,
\]
where \(r_p = \text{rank}(\Lambda_{p+1} \Gamma^\alpha : \alpha \in I_1)\),
\[
dim V = 3
\]
\[
D_p = \binom{3\nu}{p+3} + L_2 \binom{2\nu}{p+2} + \tilde{L}_1 \binom{\nu}{p+1} - R_p - R_{p+1}, \quad p > 0
\]
\[
D_0 = \sum_{j=0}^{3} (-1)^j \binom{3\nu}{3-j} + L_2 \sum_{j=0}^{2} (-1)^j \binom{2\nu}{2-j} + \tilde{L}_1 \sum_{j=0}^{1} (-1)^j \binom{\nu}{1-j} + e - R_1
\]
where \( \tilde{L}_1 = -L_1 + \sum_{\alpha \in I_2} L^{\alpha}_1 \) and

\[
R_p = \text{rank}\langle \Lambda_{p+2}^{\alpha} : \alpha \in I_2 \rangle - \text{rank}\langle \Lambda_{p+1}^{\Theta} : \Theta \in I_1 \rangle + \sum_{\alpha \in I_2} \text{rank}\langle \Lambda_{p+1}^{\Theta} : \Theta \in I_1^{\alpha} \rangle.
\]

6. Examples

The above formulae for the ranks of the cohomology groups can be evaluated with a computer using a derivative of a program to compute Wyckoff positions of crystallographic space groups [EGN97]. It was already used to calculate the cohomology groups of codimension 2 tilings [GK99]. Franz Gähler used it lately to calculate these groups for codimension 3 tilings, cf. Table 1. Apart from the first example, the Ammann-Kramer tiling, these results have not been published and the authors thank Franz Gähler for his permission to present them here.

The tilings we look at here belong to a collection of icosahedral tilings which is described in [KP95] and we refer the reader for details and notation to that article. The first three tilings of Table 1 are obtained by the variant of the cut and projection method which is based on dualization. The fourth tiling, the Danzer tiling, has originally been defined by a substitution [Dan89]. It is equivalent to a tiling of Socolar and Steinhardt [SS86]. What is important here is that the hull of Danzer’s tiling can also be described by data \((\Gamma, V, W)\). The lattice \( \Gamma \) is the icosahedral projection of \( \mathbb{Z}^6 \) in the first case and of the root lattice \( \mathbb{D}_6 \) in the three others. In particular, \( V \) and the tiling are in all cases both 3-dimensional.

The Ammann-Kramer tiling, \( T^{(P)} \), is the canonical projection method tiling obtained from the integer lattice \( \mathbb{Z}^6 \). Its acceptance domain (the icosahedral projection of the unit cube) is the triacontrahedron. We refer to [PHK00] for a description of the singular spaces. This tiling is sometimes referred to as 3-dimensional Penrose tiling. Since the Voronoi complex of \( \mathbb{Z}^6 \) and its dual are identical (up to a shift) there is only one tiling of \( P \)-type.

The second tiling of Table 1 and its dual (the third) are derived from \( \mathbb{D}_6 \). For the canonical \( \mathbb{D}_6 \)-tiling, \( T^{(2F)} \), the singular planes are the lattice planes which are perpendicular to the 3-fold and 5-fold axes of the lattice whereas the singular planes for the dual canonical \( \mathbb{D}_6 \)-tiling, \( T^*^{(2F)} \), are those perpendicular to the 2-fold axes.

The Danzer tiling can be locally derived from the canonical \( \mathbb{D}_6 \)-tiling. Its relevant data differ from the latter tiling only in the set of singular planes. These consist for the Danzer tiling only of the lattice planes which are perpendicular to the 5-fold axes. Note that this tiling has surprisingly small cohomology groups.

Table 1. Cohomology groups of various icosahedral tilings. Also the quantities which enter into Theorem 5.4 are given.

| Tiling        | \( H^0 \) | \( H^1 \) | \( H^2 \) | \( H^3 \) | \( L_0 \) | \( e \) | \( L_1 \) | \( \tilde{L}_1 \) | \( L_2 \) | \( R_1 \) | \( R_2 \) |
|--------------|----------|----------|----------|----------|---------|------|---------|---------|---------|-------|-------|
| Ammann-Kramer | \( \mathbb{Z}^{12} \) | \( \mathbb{Z}^{71} \) | \( \mathbb{Z}^{180} \) | 32       | 120     | 46   | 74      | 15      | 69      | 9     |
| canonical \( \mathbb{D}_6 \) | \( \mathbb{Z}^{13} \) | \( \mathbb{Z}^{72} \) | \( \mathbb{Z}^{205} \) | 56       | 145     | 45   | 75      | 16      | 73      | 9     |
| dual canon. \( \mathbb{D}_6 \) | \( \mathbb{Z}^{12} \) | \( \mathbb{Z}^{101} \) | \( \mathbb{Z}^{330} \) | 64       | 240     | 76   | 104     | 15      | 69      | 9     |
| Danzer       | \( \mathbb{Z}^{7} \) | \( \mathbb{Z}^{16} \) | \( \mathbb{Z}^{20} \) | 1        | 10      | 15   | 6       | 33      | 5      |       |
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