Sphere colourings and Bell inequalities

Adrian Kent\textsuperscript{1,2} and Damián Pitalúa-García\textsuperscript{1}

\textsuperscript{1}Centre for Quantum Information and Foundations, DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

\textsuperscript{2}Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada.

(Dated: July 2013 (revised August 2013))

We consider the quantum and local hidden variables (LHV) correlations obtained by measuring a pair of qubits by projections defined by randomly chosen axes separated by an angle \( \theta \). LHV’s predict binary colourings of the Bloch sphere with antipodal points oppositely coloured. We prove Bell inequalities separating the LHV predictions from the singlet quantum correlations for \( \theta \in (0, \frac{\pi}{2}) \).

We raise and explore the hypothesis that, for a continuous range of \( \theta > 0 \), the maximum LHV anticorrelation is obtained by assigning to each qubit a colouring with one hemisphere black and the other white.

According to quantum theory, space-like separated experiments performed on entangled particles can produce outcomes whose correlations violate Bell inequalities\textsuperscript{1} that would be satisfied if the experiments could be described by local hidden variable theories (LHVT). Many experiments have tested the quantum prediction of non-local causality (e.g.\textsuperscript{2–4}). The observed violations of Bell inequalities are consistent with quantum theory. They refute LHVT with overwhelmingly high degrees of confidence, modulo some known loopholes that arise from experimental conditions. We define the correlation \( C(A, B) = \int_A d\lambda \rho(\lambda) a(A, \lambda) b(B, \lambda) \). Such correlations satisfy the CHSH inequality\textsuperscript{11}: \( I_2 = |C(0, 0) + C(1, 1) + C(1, 0) - C(0, 1)| \leq 2 \).

Consider for definiteness the EPR-Bohm experiment performed on spin-\( \frac{1}{2} \) particles in the singlet state \( |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \). Alice and Bob measure their particle spin projection along the direction \( \vec{a}_A \) and \( \vec{b}_B \), respectively. As before, Alice and Bob choose a measurement from a set of two elements, that is, \( A, B \in \{0, 1\} \). In general, the vectors \( \vec{a}_A \) and \( \vec{b}_B \) can point along any direction in 3-dimensional Euclidean space, and the sets of their possible values define Bloch spheres \( S^2 \). The correlation predicted by quantum theory is \( Q(\theta) = -\cos \theta \), where \( \cos \theta = \vec{a}_A \cdot \vec{b}_B \). Sets of measurement axes can be found for which the quantum correlations violate the CHSH inequality, \( I_2^{QM} > 2 \), up to the Cirel’son\textsuperscript{14} bound \( I_2^{QM} \leq 2\sqrt{2} \).

When Alice’s and Bob’s measurement choices belong to a set of \( N \) possible elements, the correlations predicted by LHVT satisfy the Braunstein-Caves inequality\textsuperscript{15}:

\begin{equation}
I_N = \left| \sum_{k=0}^{N-1} C(k, k) + \sum_{k=0}^{N-2} C(k+1, k) - C(0, N-1) \right| \leq 2N-2.
\end{equation}

The CHSH inequality is a special case of the Braunstein-Caves inequality with \( N = 2 \). We are interested here in exploring Bell inequalities that generalize the CHSH and Braunstein-Caves inequalities, in the following sense. Instead of restricting Alice’s and Bob’s measurement choices to a finite set, we allow them to choose any spin measurement axes, \( \vec{a} \) and \( \vec{b} \). However, we constrain these axes to be separated by a fixed angle \( \theta \), so \( \cos \theta = \vec{a} \cdot \vec{b} \). The maximal violation of the Braunstein-Caves inequality by quantum correlations, given by \( I_N^{QM} = 2N\cos(\frac{\theta}{2}) \textsuperscript{10}) \), arises for fixed sets

\textsuperscript{1}A.P.A.Kent@damtp.cam.ac.uk

\textsuperscript{2}D.Pitalua-Garcia@damtp.cam.ac.uk
of pairs of axis choices that satisfy this constraint with \( \theta = \frac{\pi}{2} \). We consider experiments where pairs of axes separated by \( \theta \) are chosen randomly and where \( \theta \) is unrestricted. Aside from the theoretical interest, one practical motivation is to explore simple Bell tests that could allow quantum theory and LHVT to be distinguished more efficiently, particularly in the adversarial context of quantum cryptography. Here an eavesdropper may be trying to spoof quantum correlations, and one would like to ensure that such attacks can be detected efficiently by being able to choose (without committing in advance) from a large range of possible tests that involve no preferred axis choices.

We explore LHVT in which Alice’s and Bob’s spin measurement results are given by \( a(\bar{a}, \lambda) \) and \( b(\bar{b}, \lambda) \), respectively; where \( \lambda \) is a local hidden variable common to both particles. For fixed \( \lambda \), we can describe the functions \( a \) and \( b \) by two binary (black and white) colourings of spheres, associated to \( \lambda \) and \( b \), respectively, where black (white) represents the outcome ‘1’ (‘-1’). Different sphere colourings are associated with different values of \( \lambda \). To look at specific cases, we drop the \( \lambda \)-dependence and include a label \( x \) that indicates a particular pair of colouring functions \( a_x(\bar{a}) \) and \( b_x(\bar{b}) \).

Measuring spin along \( \vec{a} \) with outcome 1 (-1) is equivalent to measuring spin along \( -\vec{a} \) with outcome -1 (1), and so \( a \) and \( b \) are antipodal functions:

\[
a_x(\bar{a}) = -a_x(-\bar{a}), \quad b_x(\bar{b}) = -b_x(-\bar{b}),
\]

for all \( \bar{a}, \bar{b} \in \mathbb{S}^2 \). We define \( \mathcal{X} \) as the set of all colourings \( x \) satisfying the antipodal property, Eq. (2). For example, a simple colouring of the spheres satisfying the antipodal property is colouring 1, in which, for one sphere, one hemisphere is completely black and the other one is completely white, and the colouring is reversed for the other sphere (see Fig. 1).

The correlation for outcomes of measurements about randomly chosen axes separated by \( \theta \) for the pair of colouring functions labelled by \( x \) is

\[
C_x(\theta) = \frac{1}{8\pi^2} \int_{\mathbb{S}^2} dA a_x(\bar{a}) \int_0^{2\pi} d\omega b_x(\bar{b}),
\]

where \( dA \) is the area element of the sphere corresponding to Alice’s axis \( \vec{a} \) and \( \omega \) is an angle in the range \([0, 2\pi]\) along the circle described by Bob’s axis \( \vec{b} \) with an angle \( \theta \) respective to \( \vec{a} \). A general correlation is of the form

\[
C(\theta) = \int_{\mathcal{X}} dx \mu(x) C_x(\theta), \text{ where } \mu(x) \text{ is a probability distribution over } \mathcal{X}.
\]

If all colourings \( x \in \mathcal{X} \) satisfy \( Q_{\rho_L}(\theta) < C^L(\theta) \leq C_x(\theta) \) or \( C_x(\theta) \leq C^U(\theta) < Q_{\rho_U}(\theta) \) for quantum correlations \( Q_{\rho_L}(\theta) \) and \( Q_{\rho_U}(\theta) \) obtained with particular two qubit states \( \rho_L \) and \( \rho_U \), and some identifiable lower and upper bounds, \( C^L(\theta) \) and \( C^U(\theta) \), respectively, then a general correlation \( C(\theta) \) must satisfy the same inequalities. Our aim here is to explore this possibility via intuitive arguments and numerical and analytic results. We focus on the case \( \rho_L = |\Psi^-\rangle \langle \Psi^-| \), for which \( Q_{\rho_L}(\theta) \equiv Q(\theta) = -\cos \theta \), which is the maximum quantum anticorrelation for a given angle \( \theta \) (see Supplemental Material for details and related questions). We begin with some suggestive observations.

We consider colouring functions \( x \in \mathcal{X} \) for which the anticorrelation when Alice and Bob choose the same measurement, averaged uniformly over all axis choices is

\[
P(a_x = -b_x|\theta = 0) = 1 - \gamma.
\]

In general, \( 0 \leq \gamma \leq 1 \). We first consider small values of \( \gamma \) and seek Bell inequalities distinguishing quantum correlations for the singlet from classical correlations for which an anticorrelation is observed with probability \( 1-\gamma \) when the same measurement axis is chosen on both sides. Experimentally, we can verify quantum nonlocality using these results if we carry out nonlocality tests that include some frequency of anticorrelation tests about a randomly chosen axis (chosen independently for each test). The anticorrelation tests allow statistical bounds on \( \gamma \), which imply statistical tests of nonlocality via the \( \gamma \)-dependent Bell inequalities.

In the limiting case \( \gamma = 0 \), we have

\[
a_x (\bar{a}) = -b_x (\bar{a}),
\]

for all \( \bar{a} \in \mathbb{S}^2 \) and \( x \in \mathcal{X} \). This case is quite interesting theoretically, in that one might hope to prove stronger results assuming perfect anticorrelation. We describe some numerical explorations of this case below.

Second, for any pair of colourings \( x \in \mathcal{X} \) and \( \theta \in [0, \pi] \), we have \( C_x(\pi - \theta) = -C_x(\theta) \). This can be seen as follows. For a fixed \( \bar{a} \), the circle with angle \( \theta = \theta' \) around the axis \( \bar{a} \), defined by the angle \( \omega \) in Eq. (3) contains a point \( \bar{b} \) that is antipodal to a point on the circle with angle \( \theta = \pi - \theta' \) around \( \bar{a} \). Since the colouring is antipodal, we have that the value of the integral \( \int_0^{2\pi} d\omega b_x(\bar{b}) \) in Eq. (3) for \( \theta = \theta' \) is the negative of the corresponding integral for \( \theta = \pi - \theta' \). It follows that \( C_x(\pi - \theta') = -C_x(\theta') \). Therefore, in the rest of this paper, we restrict to consider correlations for the range \( \theta \in [0, \pi] \) unless otherwise stated. From the previous argument, we have \( C_x(\frac{\pi}{2}) = -C_x(\frac{\pi}{2}) \), which implies that \( C_x(\frac{\pi}{2}) = 0 \). We also have that \( C_x(0) = 1 - 2P(a_x = -b_x|\theta = 0) \), so that the LHVT we consider give \( C_x(0) = -1 + 2\gamma \). The LHV correlations given by Eqs. (3) and (4) in the case \( \gamma = 0 \) thus coincide with the quantum singlet state correlations for \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \), where \( Q(0) = C_x(0) = -1 \) and \( Q(\frac{\pi}{2}) = C_x(\frac{\pi}{2}) = 0 \).

Third, consider colouring 1, defined above. We have \( C_1(\theta) = -(1 - \frac{\pi}{2\theta}) \), for \( \theta \in [0, \frac{\pi}{2}] \). That is, \( C_1(\theta) \) linearly interpolates between the values at \( C_1(0) = -1 \), which is common to all colourings with \( \gamma = 0 \), and \( C_1(\frac{\pi}{2}) = 0 \), which is common to all colourings, and we have \( 0 > C_1(\theta) > Q(\theta) \) for \( \theta \in (0, \frac{\pi}{2}) \).

Then we have the following lemmas, whose proofs are given in the Supplemental Material.
Lemma 1. For any colouring \( x \in X \) satisfying Eq. (4) and any \( \theta \in (0, \frac{\pi}{2}) \), we have \(-1 + \frac{3}{2} \gamma \leq C_x(\theta) \leq \frac{1}{2} + \frac{\pi}{2} \gamma\).

Remark 1. Unsurprisingly, since small \( \gamma \) implies near-perfect anticorrelation at \( \theta = 0 \), we see that for \( \theta \in (0, \frac{\pi}{2}) \) and \( \gamma \) small there are no colourings with very strong correlations. However, strong anticorrelations are possible for small \( \theta \). We are interested in bounding these.

Lemma 2. For any colouring \( x \in X \) satisfying Eq. (4), any integer \( N > 2 \) and any \( \theta \in \left[ \frac{\pi}{N}, \frac{\pi}{N+1} \right) \), we have \( C_x(\theta) \geq C_1(\frac{\pi}{N}) - 2\gamma \).

Remark 2. In other words, for small \( \theta \), \( C_1(\theta) \) is very close to the maximal possible anticorrelation for LHVT when \( \gamma \ll \theta \).

Geometric intuitions also suggest bounds on \( C_x(\theta) \) that are maximised by colouring 1 for small \( \theta \); see Supplemental Material [1].

These various observations motivate us to explore what we call the Weak Hemispherical Colouring Maximality Hypothesis (WHCMH).

WHCMH. There exists an angle \( \theta_{w}^{\text{max}} \in (0, \frac{\pi}{2}) \) such that for every colouring \( x \in X \) with \( \gamma = 0 \) and every angle \( \theta \in [0, \theta_{w}^{\text{max}}] \), \( C_x(\theta) \geq C_1(\theta) \).

The WHCMH considers models with perfect anticorrelation for \( \theta = 0 \), because we are interested in distinguishing LHVT models from the quantum singlet state, which produces perfect anticorrelations for \( \theta = 0 \). Of course, there is a symmetry in the space of LHVT models given by exchanging the colours of one qubit’s sphere, which maps \( \gamma \rightarrow 1 - \gamma \) and \( C_x(\theta) \rightarrow -C_x(\theta) \). The WHCMH thus also implies that \( C_x(\theta) \leq -C_1(\theta) \) for all colourings \( x \in X \) with \( \gamma = 1 \).

It is also interesting to investigate stronger versions of the WHCMH and related questions. For instance, is it the case that for every angle \( \theta \in (\theta_{w}^{\text{max}}, \frac{\pi}{2}) \) there exists a colouring \( x' \in X \) with \( \gamma = 0 \) such that \( C_{x'}(\theta) < C_1(\theta) \)? And does this hypothesis still hold true (not necessarily for the same \( \theta_{w}^{\text{max}} \)) if we consider general local hidden variable models corresponding to independently chosen colourings for the two qubits, not constrained by any choice of the correlation parameter \( \gamma \)?

The following theorem and lemmas, whose proofs are presented in the Supplemental Material, give some relevant bounds.

Theorem. For any colouring \( x \in X \), any integer \( N \geq 2 \) and any \( \theta \in \left[ \frac{\pi}{N}, \frac{\pi}{N+1} \right) \), we have \( C_1(\frac{\pi}{N}) \leq C_x(\theta) \leq -C_1(\frac{\pi}{N}) \).

Remark 3. In particular, for small \( \theta \), \(-C_1(\theta)\) and \( C_1(\theta) \) are very close to the maximal possible correlation and anticorrelation for any LHVT, respectively.

Lemma 3. If any colouring \( x \in X \) obeys \( C_x(\theta) < C_1(\theta) \) \( \left( C_x(\theta) > -C_1(\theta) \right) \) for some \( \theta \in \left( \frac{\pi}{2M+1}, \frac{\pi}{M} \right] \) and an integer \( M \geq 2 \) then there are angles \( \theta_j = \frac{\pi}{M+1} - \theta \) with \( j = 1, 2, \ldots, M - 1 \), which satisfy \( 0 \leq \theta_j < \theta \) if \( j < \frac{M}{2} + 1 \), and \( \frac{\pi}{2} > \theta_j > \theta \) if \( j \geq \frac{M}{2} + 1 \), such that \( C_x(\theta_j) < C_1(\theta_j) \left( C_x(\theta_j) < -C_1(\theta_j) \right) \).

Remark 4. In this sense (at least), the anticorrelations defined by \( C_1 \) and the correlations defined by \(-C_1\) cannot be dominated by any other colourings.

Lemma 4. For any colouring \( x \in X \) and any \( \theta \in (0, \frac{\pi}{2}) \), we have \( Q(\theta) < C_x(\theta) < -Q(\theta) \).

Remark 5. This inequality separates all possible LHVT correlations \( C_x(\theta) \) from the singlet state quantum correlations \( Q(\theta) \) for all \( \theta \in (0, \frac{\pi}{2}) \).

The previous observations motivate the Strong Hemispherical Colouring Maximality Hypothesis (SHCMH).

SHCMH. There exists an angle \( \theta_{s}^{\text{max}} \in \left( 0, \frac{\pi}{2} \right) \) such that for every colouring \( x \in X \) and every angle \( \theta \in [0, \theta_{s}^{\text{max}}] \), \( C_1(\theta) \leq C_x(\theta) \leq -C_1(\theta) \).

Note that the SHCMH applies to all colourings, without any assumption of perfect anticorrelation for \( \theta = 0 \). If the SHCMH is true then so is the WHCMH. In this case, we have that \( \theta_{w}^{\text{max}} \leq \theta_{s}^{\text{max}} \). Thus, an upper bound on \( \theta_{w}^{\text{max}} \) implies an upper bound on \( \theta_{s}^{\text{max}} \).

We investigated the WHCMH numerically by computing the correlation \( C_x(\theta) \) for various colour functions that satisfy the antipodal property, Eq. (2), the condition [5], and that have azimuthal symmetry (see Fig. 1). Details of our numerical work is given in the Supplemental Material. Our numerical results are consistent with the WHCMH for \( \theta_{w}^{\text{max}} \leq 0.386\pi \), and with the SHCMH for \( \theta_{s}^{\text{max}} \leq 0.375\pi \), but do not give strong evidence for these values. Nor do the numerical results, per se, constitute compelling evidence for the WHCMH and SHCMH, although they confirm that the underlying intuitions hold for some simple colourings.

We have explored here what can be learned by carrying out local projective measurements about completely randomly chosen axes, separated by an angle \( \theta \), on a pair of qubits. This is not currently a standard way of testing for entanglement or nonlocality, but we have shown that it distinguishes quantum correlations from those predicted by local hidden variables for a wide range of \( \theta \). In particular, we find Bell inequalities for \( \theta \in (0, \frac{\pi}{2}) \), given by the theorem, which separate the singlet state quantum correlations from all LHVT correlations for \( \theta \in (0, \frac{\pi}{2}) \).
We have also explored hypotheses that would refine and unify these results further: the weak and strong hemispherical colouring maximality hypotheses. These state that the LHV defined by the simplest spherical colouring, with opposite hemispheres coloured oppositely, maximizes the LHV anticorrelations for a continuous range of $\theta \geq 0$, either among LHVs with perfect anticorrelation at $\theta = 0$ (the weak case) or without any restriction (the strong case).

We should note here that the intuition supporting the WHCMH relates specifically to colourings in two or more dimensions, where there seems no obvious way of constructing colourings that vary over small scales in a way that is regular enough to produce very strong (anti-) correlations for small $\theta$.

On the other hand, the one-dimensional analogue of the WHCMH – that the strongest anticorrelations for colourings on the circle arise from colouring opposite half-circles oppositely – is easily seen to be false. For $n$ odd, the colouring $a(\epsilon) = -b(\epsilon) = (-1)^{\lfloor \pi \epsilon \rfloor / \pi}$ with $\epsilon \in [0, 2\pi]$ is antipodal and is perfectly anticorrelated for $\theta = 2\pi / n$.

That said, this distinction between one and higher dimensions is consistent with what is known about other colouring problems in geometric combinatorics \cite{17,18}. The intuition that colouring 1 should be optimal because it solves the isoperimetric problem of finding the coloured region with half the area of the sphere that has shortest boundary remains suggestive. Verifying the WHCMH and the SHCMH look at first sight like simple classical problems in geometry and combinatorics that can be stated quite independently of quantum theory. They have many interesting generalisations \cite{19}. Nonetheless, as far as we are aware, these questions have not been seriously studied by pure mathematicians to date, although some intriguing relatively recent results \cite{17,18} on colourings in $\mathbb{R}^n$ encourage hope that proof methods could indeed be found. We thus simply state the WHCMH and the SHCMH as interesting and seemingly plausible hypotheses to be investigated further rather than offering them as conjectures, preferring to reserve the latter terms for propositions for which very compelling evidence has been amassed.

We would like to stress what we see as a key insight deserving further exploration, namely that stronger and more general Bell inequalities could in principle be proven by results about continuous colourings, rather than restricting to colourings of discrete sets. While we have focussed on the simplest case of projective measurements of pairs of qubits, this observation of course applies far more generally. We hope our work will stimulate further investigation of the WHCMH and the SHCMH and related colouring problems, which seem very interesting in their own right, and in developing further this intriguing link between pretty and natural questions in geometric combinatorics and measures of quantum nonlocality.

After completing this work, our attention was drawn to a related question considered in \cite{20}; see Supplemental Material \cite{12} for discussion.

I. SUPPLEMENTAL MATERIAL

A. Proofs of the theorem and lemmas

1. Proof of Lemma \[12\]

From the CHSH inequality,

$$| C(0,0) + C(1, 1) + C(1, 0) - C(0, 1) | \leq 2, \quad (6)$$

in the case in which the measurements $A = 0, A = 1$ and $B = 0$ correspond to projections on states with Bloch vectors separated from each other by the same angle $\theta \in \left(0, \frac{2\pi}{n}\right]$. Bob’s measurement $B = 1$ is the same as Alice’s measurement $A = 0$ and the outcomes are described by LHVT satisfying Eq. \[3\] of the main text, we obtain after averaging over random rotations of the Bloch sphere that $|3C_{x}(\theta) - C_{x}(0)| \leq 2$. Then, the result follows because, as shown in the main text, we have $C_{x}(0) = -1 + 2\gamma$. \[\Box\]

2. Proof of Lemma \[13\]

From the Braunstein-Caves inequality, Eq. \[1\] of the main text, we have that

$$I_{N} = \left| \sum_{k=0}^{N-1} C(k, k) + \sum_{k=0}^{N-1} C(k + 1, k) \right| \leq 2N - 2, \quad (7)$$

with the convention that measurement choice $N$ is measurement choice 0 with reversed outcomes. We consider the case in which Alice’s and Bob’s measurement $k$ are the same, for $k = 0, 1, \ldots, N - 1$ and $N \geq 2$, and their outcomes are described by LHVT satisfying Eqs. \[3\] and \[4\] of the main text, which then also satisfy $\gamma_{x}(0) = -1 + 2\gamma$. If we take measurement $k$ to be the projection onto the state $|\xi_{k}\rangle$ so that the states $\{|\xi_{k}\rangle\}_{k=0}^{N-1}$ are along a great circle on the Bloch sphere with a separation angle $\theta = \frac{\pi}{N}$ between $|\xi_{k}\rangle$ and $|\xi_{k+1}\rangle$ for $k = 0, 1, \ldots, N - 2$, for example $|\xi_{k}\rangle = \cos\left(\frac{\pi k}{N}\right)|0\rangle + \sin\left(\frac{\pi k}{N}\right)|1\rangle$, and average over random rotations of the Bloch sphere, this gives

$$|NC_{x}(0) + NC_{x}(\theta)| \leq 2N - 2. \quad (8)$$

ACKNOWLEDGMENTS

We thank Boris Bukh for very helpful discussions and for drawing our attention to Refs. \[12\] \[18\]. AK was partially supported by a grant from the John Templeton Foundation and by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. DPG thanks Tony Short and Berry Groisman for helpful discussions, and acknowledges financial support from CONACYT México and partial support from Gobierno de Veracruz.
Since $C_x(0) = -1 + 2\gamma$, it follows that $C_x(\frac{\pi}{N}) \geq -1 + \frac{2\gamma}{N} - 2\gamma = C_1(\frac{\pi}{N}) - 2\gamma$. Similarly, if we take the states $\{\ket{x_k}\}_{k=0}^{N-1}$ to be along a zigzag path crossing a great circle on the Bloch sphere with a separation angle $\theta > \frac{\pi}{N}$ between $\ket{x_k}$ and $\ket{x_{k+1}}$ for $k = 0, 1, \ldots, N - 2$, in such a way that the angle separation between $\ket{x_{N-1}}$ and the state with Bloch vector antiparallel to that one of $\ket{0}$ is also $\theta$ (see Fig. 2), we obtain after averaging over random rotations of the Bloch sphere that $C_x(\theta) \geq -1 + \frac{2\gamma}{N} - 2\gamma = C_1(\frac{\pi}{N}) - 2\gamma$.

4. Proof of Lemma 

Consider a colouring $x \in \mathcal{X}$ and an angle $\theta \in (\frac{\pi}{M+1}, \frac{\pi}{M})$ for an integer $M \geq 2$ such that $C_x(\theta) < C_1(\theta)$ or $C_x(\theta) > -C_1(\theta)$. From the theorem and the fact that $C_x(\frac{\pi}{2}) = C_1(\frac{\pi}{2}) = 0$, it must be that $\theta \neq \frac{\pi}{M}$ if $M$ is even. We define the angles $\theta_j \equiv \frac{\pi}{M+1-j} - \theta$ with $j = 1, 2, \ldots, M - 1$. Considering the cases $M$ even and $M$ odd, and using that $\theta \neq \frac{\pi}{M}$ if $M$ is even, it is straightforward to obtain that $0 \leq \theta_j < \theta$ if $j < \frac{M}{2} + 1$ and $\frac{\pi}{M} > \theta_j > \theta$ if $j \geq \frac{M}{2} + 1$. Now consider the Braunstein-Caves inequality, Eq. (1) of the main text, in the case in which Alice’s and Bob’s measurements’ outcomes are described by LHVT satisfying Eq. (3) of the main text. Let Alice’s and Bob’s measurements $k$ to correspond to the projections onto the states $\ket{x_k}$ and $\ket{\chi_k}$, respectively, for $k = 0, 1, \ldots, N - 1$ and $N \geq 2$. Let the angle along the great circle in the Bloch sphere passing through the states $\ket{x_k}$ and $\ket{\chi_k}$ be $\theta$, for $k = 0, 1, \ldots, N - 1$. Similarly, let the angle along the great circle passing through $\ket{\chi_k}$ and $\ket{\chi_{k+1}}$ be $\theta$ for $k = 0, 1, \ldots, N - 1$, with the convention that the state $\ket{\chi_N}$ has Bloch vector antiparallel to that one of $\ket{0}$. If $\theta = \frac{\pi}{2N}$, the states are on the same great circle beginning at $\ket{0}$ and ending at $\ket{\chi_N}$. If $\theta > \frac{\pi}{2N}$, the states can be accommodated on a zigzag path crossing the great circle that goes from $\ket{0}$ to $\ket{\chi_N}$ (see Fig. 3). Thus, from the Braunstein-Caves inequality, after averaging over random rotations of the Bloch sphere, we have $C_1(\frac{\pi}{2N}) = -1 + \frac{1}{N} \leq C_x(\theta) \leq -1 - \frac{1}{N} = -C_1(\frac{\pi}{2N})$, for $\theta \geq \frac{\pi}{2N}$.
0, 1, . . . , N − 1. From the Braunstein-Caves inequality, after averaging over random rotations of the Bloch sphere, we obtain \(-1 + \frac{1}{N} \leq \frac{1}{2} (C_1(\phi_2) + C_1(\phi_1)) \leq 1 - \frac{1}{N}\). Since the average angle \(\bar{\theta} \equiv \frac{1}{2}(\theta_1 + \theta_2)\) satisfies \(\bar{\theta} = \frac{2}{N(N+1)} = \frac{\pi}{2N}\) and \(C_1(\frac{\pi}{2N}) = -1 + \frac{1}{N}\), we have \(C_1(\bar{\theta}) \leq \frac{1}{2} (C_2(\phi_2) + C_2(\phi_1)) \leq -C_1(\bar{\theta})\). Since \(C_1(\phi)\) is a linear function of \(\phi\), it follows that \(C_2(\phi_2) > C_1(\phi_1)\) if \(C_2(\phi) < C_1(\phi)\). Similarly, \(C_2(\phi_2) < -C_1(\phi_1)\) if \(C_2(\phi) > -C_1(\phi)\).

5. Proof of Lemma 4

Let \(x \in \mathcal{X}\) be any colouring and \(\phi \in (0, \frac{\pi}{2})\). We first consider the case \(\phi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)\). From the theorem, we have \(C_1(\phi) \leq C_2(\phi) \leq C_1(\phi)\). The quantum correlation for the singlet state is \(Q(\phi) = -\cos \phi\). Since \(Q(\phi)\) is a strictly increasing function of \(\phi\), we have \(Q(\phi) < Q(\frac{\pi}{4}) = -\frac{1}{2} = \frac{C_1(\phi)}{C_1(\phi)}\) for \(\phi < \frac{\pi}{4}\). Therefore, \(Q(\phi) < C_2(\phi) < -Q(\phi)\) for \(\phi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)\). Similarly, it is easy to see that \(Q(\phi) < C_2(\phi) < -Q(\phi)\) for \(\phi \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right)\). Now we consider the case \(\phi \in (0, \frac{\pi}{4})\). We define \(\rho = \frac{1}{N} \frac{1}{N} \). It follows that \(\bar{x} = \frac{\pi}{3N-1}\) for any integer \(N \geq 4\). From the theorem, we have \(-1 + \frac{1}{N} = C_1(\frac{\pi}{3N-1}) \leq C_2(\phi) \leq -C_1(\frac{\pi}{3N-1}) = -1 + \frac{1}{N}\). From the Taylor series \(Q(\phi) = -1\frac{\phi^2}{3} - \frac{\phi^3}{4} + \frac{\phi^4}{5} \ldots\), it is easy to see that \(Q(\phi) > -1 + \frac{2}{2(N-1)}\) for \(0 < \phi < \sqrt{30}\). Thus, we have \(Q(\frac{\pi}{3N-1}) > -1 + \frac{2}{2(N-1)}\). Since \(N^2 > (\frac{\pi}{8} + 2)N - 1\), it follows that \((N - 1)^2 > \frac{\pi}{8} N\), which implies \(-1 + \frac{1}{N} > -1 + \frac{1}{2} (\frac{\pi}{3N-1})\). It follows that \(C_2(\phi) > Q(\frac{\pi}{3N-1})\). Since \(Q(\phi)\) is a strictly increasing function of \(\phi\) and \(\phi < \frac{\pi}{3(N-1)}\), we have \(Q(\frac{\pi}{3N-1}) > Q(\phi)\). Thus, we have \(C_2(\phi) > Q(\phi)\). Similarly, we have \(C_2(\phi) < -Q(\phi)\).

B. Geometric intuitions

Consider simple colourings, in which a set of (not necessarily connected) piecewise differentiable curves of finite total length separate black and white regions. (Points lying on these curves may have either colour.) Intuition suggests that, for small \(\gamma\) and simple colourings with \(\gamma = 0\), the quantity \(1 + C_2(\theta)\), which measures the deviation from pure anticorrelation, should be bounded by a quantity roughly proportional to the length of the boundary between the black and white areas of the sphere colouring \(x \in \mathcal{X}\). Since colouring 1 has the smallest such boundary (the equator), this might suggest that \(C_2(\theta) \geq C_1(\theta)\) for small \(\theta\) and for all simple colourings \(x \in \mathcal{X}\) with \(\gamma = 0\). Intuition also suggests that any non-simple colouring will produce less anticorrelation than the optimal simple colouring, because regions in which black and white colours alternate with arbitrarily small separation tend to wash out anticorrelation. These intuitive arguments are clearly not rigorous as currently formulated. For example, they ignore the possibility of sequences of colourings \(C_i(\theta)\) and angles \(\theta_i \to 0\) such that \(C_i(\theta_i) < C_1(\theta_i)\), while \(\lim_{\theta \to 0} (C_i(\theta) - C_1(\theta)) > 0\) for all \(i\). Still, they are suggestive, at least in generating hypotheses to be investigated.

C. Related questions for exploration

An interesting related question is, for an arbitrary two qubit state \(\rho\) and qubit projective measurements performed by Alice and Bob corresponding to random Bloch vectors separated by an angle \(\pi\), what are the maximum values of the quantum correlations and anticorrelations \(Q_{\rho}(\phi)\), and which states achieve them? We show that the maximum quantum anticorrelations and correlations are \(Q_{\rho}(\phi) = -\cos \phi\), achieved by the singlet state \(\rho = |\Psi^-\rangle\langle\Psi^-|\), and \(Q_{\rho}(\phi) = \frac{1}{2} \cos \phi\), achieved by the other Bell states, \(\rho = |\Psi^+\rangle\langle\Psi^+|\) and \(\rho = |\Psi^\mp\rangle\langle\Psi^\mp|\), respectively. This result follows because, as we show below, we have
\[-\cos \phi \leq Q_{\rho}(\phi) \leq \frac{1}{3} \cos \phi.
\]

Another related question that we do not explore further here is, for a fixed given angle \(\phi\) separating Alice’s and Bob’s measurement axes, what are the maximum correlations and anticorrelations, if in addition to the two qubit state \(\rho\), Alice and Bob have other resources? For example, Alice and Bob could have an arbitrary entangled state on which they perform arbitrary local quantum operations and measurements. In a different scenario, Alice and Bob could have some amount of classical or quantum communication. Another possibility is for Alice and Bob to share arbitrary no-signalling resources, not necessarily quantum, with no communication allowed. It is interesting to note that in this case, the no-signalling principle does not restrict the value of the correlations, because \(C = \pm 1\) are achieved for all \(\phi\) by a generalization of the PR-box [21], which is given by the following non-signalling outcome probabilities: \(P(\pm \bar{a}, \bar{b}) = P(\pm \bar{a}, \bar{b}) = \frac{1}{2}, P(\pm \bar{a}, \bar{b}) = P(\pm \bar{a}, \bar{b}) = 0\) for all \(\bar{a}, \bar{b} \in \mathbb{S}^2\). Different variations of the task described above with continuous parameters can be investigated.

As mentioned above, some interesting related questions involving non-local games with continuous inputs have been considered in [20]. In particular, in the third game considered in [20], Alice and Bob are given uniformly distributed Bloch sphere vectors, \(\vec{r}_A\) and \(\vec{r}_B\), and aim to maximise the probability of producing outputs that are anticorrelated if \(\vec{r}_A \cdot \vec{r}_B \geq 0\) or correlated if \(\vec{r}_A \cdot \vec{r}_B < 0\). Aharon et al. suggest that the LHV strategy defined by opposite hemispherical colourings is optimal, though they give no argument. They also suggest that the quantum strategy given by sharing a singlet and carrying out measurements corresponding to the input vectors is optimal, based on evidence from semi-definite
programming. Equation (9) shows that this is the case for all \(\theta\), and so in particular for the average advantage in the game considered, if Alice and Bob are restricted to outputs defined by projective measurements on a shared pair of qubits. Our earlier results also prove that there is a quantum advantage for all \(\theta\) in the range \(0 < \theta < \frac{\pi}{2}\), and hence for many versions of this game defined by a variety of probability distributions for the inputs.

We show Eq. (9) below. First, we compute the average outcome probabilities when Alice and Bob apply local projective measurements on a two qubit state \(\rho\), for measurement bases defined by Bloch vectors separated by an angle \(\theta\). The average is taken over random rotations of these vectors in the Bloch sphere, subject to the angle separation \(\theta\). Then, we compute the quantum correlations.

Consider a fixed pair of pure qubit states \(|0\rangle\) and \(|\chi\rangle = \cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle\) for Alice’s and Bob’s measurements corresponding to outcomes ‘+1’, respectively. A general state for Bob’s measurement separated by an angle \(\theta\) with respect to a fixed state \(|0\rangle\) for Alice’s measurement is obtained by applying the unitary \(R_x(\omega)\) that corresponds to a rotation of an angle \(\omega \in [0, 2\pi]\) around the z axis in the Bloch sphere, which only adds a phase to the state \(|0\rangle\). Then, after applying \(R_x(\omega)\), a general pure product state \(|\xi_\theta\rangle \otimes |\chi_\theta\rangle\) of two qubits with Bloch vectors separated by an angle \(\theta\) is obtained by applying the unitary \(R_x(\phi)R_y(\epsilon)\) that rotates the Bloch sphere around the y axis by an angle \(\epsilon \in [0, \pi]\) and then around the z axis by an angle \(\phi \in [0, 2\pi]\).

Thus, we have \(|\xi_\theta\rangle \otimes |\chi_\theta\rangle = U_{\phi,\epsilon,\omega}|0\rangle \otimes U_{\phi,\epsilon,\omega}|\chi\rangle\), with \(U_{\phi,\epsilon,\omega} = R_x(\phi)R_y(\epsilon)R_x(\omega)\). This is a general unitary acting on a qubit, up to a global phase. Therefore, we can parameterize this unitary by the Haar measure \(\mu\) on SU(2), hence, we have \(|\xi_\theta\rangle \otimes |\chi_\theta\rangle = U_{\mu}|0\rangle \otimes U_{\mu}|\chi\rangle\).

After taking the average, the probability that both Alice and Bob obtain the outcome ‘+1’ is

\[
P(+ | + \theta) = \int d\mu Tr\left( \rho \langle \xi_\theta | \xi_\theta \rangle \langle \chi_\theta | \chi_\theta \rangle \right)
\]

\[
= \int d\mu Tr\left( \rho (U_\mu \otimes U_\mu) \langle 0 | \langle 0 | \langle \chi | \chi \rangle \langle \mu | \mu \rangle \right)
\]

\[
= Tr\left( \int d\mu (U_\mu \otimes U_\mu) \rho (U_\mu \otimes U_\mu) \langle 0 | \langle 0 | \langle \chi | \chi \rangle \langle \mu | \mu \rangle \right)
\]

\[
= Tr\left( \tilde{\rho} \langle 0 | \langle 0 | \langle \chi | \chi \rangle \langle \mu | \mu \rangle \right),
\]

(10)

where in the third line we used the linearity and the ciclicity of the trace and in the fourth line we used the definition \(\tilde{\rho} \equiv \int d\mu (U_\mu \otimes U_\mu) \rho (U_\mu \otimes U_\mu)\). The state \(\tilde{\rho}\) is invariant under a unitary transformation \(U \otimes U\), for any \(U \in SU(2)\). The only states with this symmetry are the Werner states [22], which for the two qubit case have the general form

\[
\tilde{\rho} = r |\Psi^\perp\rangle \langle \Psi^\perp | + \frac{1 - r}{3} (|\Psi^+\rangle \langle \Psi^+ | + |\Phi^+\rangle \langle \Phi^+ | + |\Phi^-\rangle \langle \Phi^- |),
\]

(11)

with \(0 \leq r \leq 1\). Thus, from Eqs. (10) and (11), we obtain

\[
P(+ | + \theta) = \frac{1 - r}{3} + \frac{4r - 1}{6} \sin^2 \left( \frac{\theta}{2} \right).
\]

(12)

Since the projectors corresponding to Alice and Bob obtaining outcomes ‘-1’ are obtained by a unitary transformation of the form \(U \otimes U\) on \(|0\rangle \otimes |\chi\rangle\), with \(U \in SU(2)\), then from Eq. (10) we see that after integrating over the Haar measure on SU(2), we obtain \(P(- | - \theta) = P(+ | + \theta)\).

Thus, the average quantum correlation is \(Q_\rho(\theta) = 4P(+ | + \theta) - 1\), which from Eq. (12) gives

\[
Q_\rho(\theta) = - \left( \frac{4r - 1}{3} \right) \cos \theta.
\]

(13)

Then, Eq. (9) follows because \(0 \leq r \leq 1\).

D. Numerical results

We investigated the WHCMH numerically by computing the correlation \(C_x(\theta)\) for various colouring functions that satisfy the antipodal property (Eq. (2) of the main text) the condition [4] in the main text, and that have azimuthal symmetry. These colourings are illustrated in Fig. 1 of the main text and defined in section III D 1.

We define \((\epsilon, \phi)\) as the the spherical coordinates of \(\vec{a}\) and \((\alpha, \beta)\) as those of \(\vec{b}\), where \(\epsilon, \alpha \in [0, \pi]\) are angles from the north pole and \(\beta, \phi \in [0, 2\pi]\) are azimuthal angles. The vectors \(\vec{a}\) and \(\vec{b}\) are separated by a fixed angle \(\theta\). The set of possible values of \(\vec{b}\) around the fixed axis \(\vec{a}\) generate a circle parameterized by an angle \(\omega\) (see Fig. 4). The spherical coordinates \((\alpha, \beta)\) for a point \(\vec{b}\) with angular coordinate \(\omega\) on this circle are:

\[
\alpha = \arccos (\cos \theta \cos \epsilon - \sin \sin \epsilon \cos \omega),
\]

\[
\beta = \left[ \phi + k_\omega \arccos \left( \frac{\cos \epsilon \sin \phi \cos \omega + \sin \epsilon \cos \theta}{\sin \alpha} \right) \right] \text{mod} 2\pi,
\]

(14)

(15)

where \(k_\omega = 1\) if \(0 \leq \omega \leq \pi\) and \(k_\omega = -1\) if \(\pi < \omega \leq 2\pi\).

Notice that \(\beta\) is undefined for \(\alpha \in \{0, \pi\}\).

Equations (14) and (15) were used to compute the double integral in Eq. 3 of the main text. The integral with respect to the angle \(\omega\) was performed analytically. Thus, the correlations \(C_x(\theta)\) were reduced to a sum of terms that include single integrals with respect to the polar angles \(\epsilon\); the obtained expressions are given in section II D 2.

The single integrals with respect to \(\epsilon\) were computed numerically with a program using the software ‘Mathematica’, which we provide as Supplemental Material.

Our results are plotted in Fig. 5; they are consistent with the WHCMH. They also show that \(\theta_{\text{max}}^w < \frac{\pi}{2}\), because they show that there exists a colouring \(x'\) with \(C_x(\theta) < C_1(\theta)\) for some angles \(\theta \in (0, \frac{\pi}{2})\), namely colouring 3 for angles \(\theta \in \{0.405\pi, \frac{7\pi}{12}\}\).

Another interesting result is that there exist colourings that produce correlations \(C_x(\theta) < Q(\theta)\) for \(\theta\) close
in which case we have that \( \theta \in \{ \frac{\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2} \} \). It is interesting to find other colourings whose correlations satisfy \( C_x(\theta) < C_1(\theta) \) and \( C_x(\theta) < Q(\theta) \) for angles \( \theta \) closer to zero. For this purpose, we consider colouring \( 3_5 \), which is defined in section 1D1 and consists of a small variation of \( 3 \) in terms of the parameter \( \delta \). Colouring \( 3_5 \) reduces to colouring \( 3 \) if \( \delta = 0 \). For values of \( \delta \) in the range \( \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \), we obtained that the smallest angle \( \theta \) for which \( C_{3_5}(\theta) < C_1(\theta) \) is achieved for \( \delta = -0.038\pi \), in which case we have that \( C_{3_5(-0.038\pi)}(\theta) < C_1(\theta) \) for \( \theta \in \left[ 0.386\pi, \frac{7\pi}{2} \right) \). We also obtained that the smallest angle \( \theta \) for which \( C_{3_5}(\theta) < Q(\theta) \) is achieved for \( \delta = -0.046\pi \), in which case we have that \( C_{3_5(-0.046\pi)}(\theta) < Q(\theta) \) for \( \theta \in \left[ 0.431\pi, \frac{7\pi}{2} \right) \) (see Fig. 5).

Our numerical results imply the bound \( \theta_{\text{max}}^{\text{w}} \leq 0.386\pi \). They also imply that \( \theta_{\text{max}}^{\text{w}} \leq 0.375\pi \), because \( C_2(\theta) > C_1(\theta) \) for \( \theta \in \left( 0, 0.375\pi, \frac{7\pi}{2} \right) \), and \( C_1(\theta) < C_2(\theta) \leq -C_1(\theta) \) for \( \theta \in 2, 3, 4, 3_3 \) and \( \theta \in \left[ 0, 0.375\pi \right] \).

In order to confirm analytically the numerical observation that there exist colouring functions \( x \in X \) such that \( C_x(\theta) < Q(\theta) \) for \( \theta \) close to \( \frac{7\pi}{2} \), we computed \( C_3 \left( \frac{\pi}{2} - \tau \right) \) for \( 0 \leq \tau \ll 1 \) to order \( O(\tau^2) \). The computation is presented in section 1D3. We obtain

\[
C_3 \left( \frac{\pi}{2} - \tau \right) = -1.5\tau + \mathcal{O}(\tau^2). \tag{16}
\]

On the other hand, the quantum correlation gives \( Q \left( \frac{\pi}{2} - \tau \right) = -\cos \left( \frac{\pi}{2} - \tau \right) = -\tau + \mathcal{O}(\tau^3) \). Thus, we see that for \( \tau \) small enough, indeed \( C_3 \left( \frac{\pi}{2} - \tau \right) < Q \left( \frac{\pi}{2} - \tau \right) \).

Further numerical investigations of the WHCMH and SHCMH might well shed further light on the questions we explore here. For example, one could define an antipodal colouring function \( x \) as the sign of a sum of spherical harmonics, \( \text{sgn} \left( \sum_{m=-l}^{l} \sum_{\ell=0}^{L} a_{lm} Y_{lm}(\epsilon, \phi) \right) \), where the coefficients \( a_{lm} \) are variable parameters, and then search for...
the minimum value of $C_x(\theta)$, for any given $\theta$, among such functions by optimizing with respect to the $a_{lm}$. As an ansatz, one might assume that components corresponding to spherical harmonics that oscillate rapidly compared to $\theta$ are relatively negligible, given that the colourings defined by such functions contain black and white areas small compared to $\theta$ everywhere on the sphere, giving a contribution to the correlation very close to zero. This would allow searches over a finite set of parameters, for any given $\theta$, while the ansatz itself can be tested by finding how the maximum changes with increasing $L$.

1. Definitions of the colouring functions

In general, a colouring function $a_x$ with azimuthal symmetry can be defined in terms of the set $E_x$ in which it takes the value 1 as follows:

$$a_x(\epsilon) = \begin{cases} 1 & \text{if } \epsilon \in E_x, \\ -1 & \text{if } \epsilon \in [0, \pi]/E_x, \end{cases}$$

(Eq. (2) of the main text) and the constraint \([5]\) in the main text to reduce the correlation given by Eq. (3) of the main text to:

$$C_x(\theta) = -\frac{1}{\pi} \int_0^{\pi} \sin \epsilon a_x(\epsilon) \int_0^{\pi} d\omega a_x[\alpha(\theta, \epsilon, \omega)],$$

(19)

where $\alpha(\theta, \epsilon, \omega)$ is given by Eq. (14). We computed the integral with respect to $\omega$ in the previous expression. We define the function

$$\chi(\theta, a, b, \alpha) \equiv \frac{2}{\pi} \int_a^b d\epsilon \sin \epsilon \arccos \left( \frac{\cos \theta \cos \epsilon - \cos \alpha}{\sin \theta \sin \epsilon} \right),$$

(20)

where $a, b, \alpha \in [0, \pi]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. We obtained the following expressions for the correlations $C_x(\theta)$:

$$C_2(\theta) = \begin{cases} h_1^2(\theta) & \text{if } \theta \in [0, \pi/4], \\ h_2^2(\theta) & \text{if } \theta \in (\pi/4, 3\pi/8], \\ h_3^2(\theta) & \text{if } \theta \in (3\pi/8, \pi/2], \\ h_4^2(\theta) & \text{if } \theta \in [0, \pi/6], \\ h_5^2(\theta) & \text{if } \theta \in (\pi/6, \pi/4], \\ h_6^2(\theta) & \text{if } \theta \in (\pi/4, \pi/3], \\ h_7^2(\theta) & \text{if } \theta \in (\pi/3, 5\pi/12], \\ h_8^2(\theta) & \text{if } \theta \in (5\pi/12, \pi/2], \end{cases}$$

$$C_3(\theta) = \begin{cases} h_1^3(\theta) & \text{if } \theta \in [0, \pi/8], \\ h_2^3(\theta) & \text{if } \theta \in (\pi/8, \pi/4], \\ h_3^3(\theta) & \text{if } \theta \in (\pi/4, \pi/3], \\ h_4^3(\theta) & \text{if } \theta \in (\pi/3, 5\pi/16], \\ h_5^3(\theta) & \text{if } \theta \in (5\pi/16, 3\pi/8], \end{cases}$$

$$C_4(\theta) = \begin{cases} h_1^4(\theta) & \text{if } \theta \in [0, \pi/8], \\ h_2^4(\theta) & \text{if } \theta \in (\pi/8, \pi/4], \\ h_3^4(\theta) & \text{if } \theta \in (\pi/4, \pi/3], \\ h_4^4(\theta) & \text{if } \theta \in (\pi/3, 5\pi/16], \\ h_5^4(\theta) & \text{if } \theta \in (5\pi/16, 3\pi/8], \end{cases}$$

$$C_3(\theta) = \begin{cases} r_1^3(\theta) & \text{if } \delta \in \left[\frac{\pi}{3}, \frac{\pi}{8}\right] \text{ and } \theta \in \left[\frac{\pi}{3}, \frac{\pi}{8} - \delta\right], \\ r_2^3(\theta) & \text{if } \delta \in \left[\frac{\pi}{8}, \frac{\pi}{18}\right] \text{ and } \theta \in \left[\frac{\pi}{8}, \frac{\pi}{18} - \delta\right], \\ r_3^3(\theta) & \text{if } \delta \in \left[\frac{\pi}{18}, \frac{\pi}{27}\right] \text{ and } \theta \in \left[\frac{\pi}{18}, \frac{\pi}{27} - \delta\right], \\ r_4^3(\theta) & \text{if } \delta \in \left[\frac{\pi}{27}, \frac{\pi}{45}\right] \text{ and } \theta \in \left[\frac{\pi}{27}, \frac{\pi}{45} - \delta\right], \end{cases}$$

$$C_4(\theta) = \begin{cases} r_1^4(\theta) & \text{if } \delta \in \left[\frac{\pi}{3}, \frac{\pi}{8}\right] \text{ and } \theta \in \left[\frac{\pi}{3}, \frac{\pi}{8} - \delta\right], \\ r_2^4(\theta) & \text{if } \delta \in \left[\frac{\pi}{8}, \frac{\pi}{18}\right] \text{ and } \theta \in \left[\frac{\pi}{8}, \frac{\pi}{18} - \delta\right], \\ r_3^4(\theta) & \text{if } \delta \in \left[\frac{\pi}{18}, \frac{\pi}{27}\right] \text{ and } \theta \in \left[\frac{\pi}{18}, \frac{\pi}{27} - \delta\right], \\ r_4^4(\theta) & \text{if } \delta \in \left[\frac{\pi}{27}, \frac{\pi}{45}\right] \text{ and } \theta \in \left[\frac{\pi}{27}, \frac{\pi}{45} - \delta\right], \\ r_5^4(\theta) & \text{if } \delta \in \left[\frac{\pi}{45}, \frac{\pi}{90}\right] \text{ and } \theta \in \left[\frac{\pi}{45}, \frac{\pi}{90} - \delta\right], \\ r_6^4(\theta) & \text{if } \delta \in \left[\frac{\pi}{90}, \frac{\pi}{180}\right] \text{ and } \theta \in \left[\frac{\pi}{90}, \frac{\pi}{180} - \delta\right], \end{cases}$$

(21)

2. Expressions for the correlations

We use the azimuthal symmetry of the colourings $x = 2, 3, 4, 3_\delta$ defined in section $[13]$ the antipodal property where
\( h_2^1(\theta) \equiv -1 + 2 \left[ \cos \left( \frac{\pi}{4} \right) - \cos \left( \frac{\pi}{4} + \theta \right) \right] + \chi \left( \theta, \frac{\pi}{4} - \theta, \frac{\pi}{4} + \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} - \theta, \frac{\pi}{4} + \frac{\pi}{4} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} + \frac{\pi}{2} \right), \)

\( h_2^2(\theta) \equiv 1 + 2 \left[ \cos \left( \frac{\pi}{4} \right) - \cos \left( \theta - \frac{\pi}{4} \right) \right] + \chi \left( \theta, \theta - \frac{\pi}{4}, \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} + \frac{\pi}{2} \right) + \chi \left( \theta, \frac{\pi}{4} + \frac{\pi}{4} \right), \)

\( h_2^3(\theta) \equiv 1 + 2 \left[ \cos \left( \frac{\pi}{4} \right) - \cos \left( \theta - \frac{\pi}{4} \right) \right] - \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} + \frac{\pi}{2} \right) + \chi \left( \theta, \theta - \frac{\pi}{4}, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{\pi}{4} + \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} + \frac{\pi}{4} \right) - \chi \left( \theta, \frac{3\pi}{4} - \theta, \frac{3\pi}{4} + \frac{3\pi}{4} \right); \)

\( h^1(\theta) \equiv -1 + 2 \left[ \cos \left( \frac{\pi}{6} \right) - \cos \left( \frac{\pi}{6} + \theta \right) + \cos \left( \frac{\pi}{3} \right) - \cos \left( \frac{\pi}{3} + \theta \right) \right] + \chi \left( \theta, \frac{\pi}{6} - \theta, \frac{\pi}{6} + \frac{\pi}{6} \right) - \chi \left( \theta, \frac{\pi}{6} - \theta, \frac{\pi}{6} + \frac{\pi}{6} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right), \)

\( h^2(\theta) \equiv 1 + 2 \left[ \cos \left( \frac{\pi}{6} \right) - \cos \left( \theta - \frac{\pi}{6} \right) + \cos \left( \frac{\pi}{6} + \theta \right) - \cos \left( \frac{\pi}{3} \right) \right] + \chi \left( \theta, \theta - \frac{\pi}{6}, \frac{\pi}{6} + \frac{\pi}{6} \right) - \chi \left( \theta, \frac{\pi}{3} - \theta, \frac{\pi}{6} \right) + \chi \left( \theta, \theta - \frac{\pi}{6}, \frac{\pi}{6} + \frac{\pi}{6} \right) + \chi \left( \theta, \frac{\pi}{3} - \theta, \frac{\pi}{6} \right) - \chi \left( \theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} \right), \)

\( h^3(\theta) \equiv 1 + 2 \left[ \cos \left( \frac{\pi}{6} \right) - \cos \left( \theta - \frac{\pi}{6} \right) + \cos \left( \frac{\pi}{6} + \theta \right) - \cos \left( \frac{\pi}{3} \right) \right] - \chi \left( \theta, \frac{\pi}{3} - \theta, \frac{\pi}{6} + \frac{\pi}{6} \right) + \chi \left( \theta, \theta - \frac{\pi}{6}, \frac{\pi}{6} + \frac{\pi}{6} \right) + \chi \left( \theta, \frac{\pi}{3} - \theta, \frac{\pi}{6} \right) - \chi \left( \theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} \right) - \chi \left( \theta, \frac{\pi}{3} - \theta, \frac{\pi}{6} \right) - \chi \left( \theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} \right), \)

\( h_4^1(\theta) \equiv -1 + 2 \left[ \cos \left( \theta - \frac{\pi}{3} \right) - \cos \left( \frac{\pi}{3} \right) + \cos \left( \frac{\pi}{6} \right) + \cos \left( \theta - \frac{\pi}{6} \right) - \cos \left( \frac{\pi}{3} \right) \right] - \chi \left( \theta, \theta - \frac{\pi}{3}, \frac{\pi}{6} + \frac{\pi}{3} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} + \frac{\pi}{2} \right) + \chi \left( \theta, \frac{\pi}{6} + \frac{\pi}{3} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} + \frac{\pi}{2} \right), \)

\( h_4^2(\theta) \equiv 1 + 2 \left[ \cos \left( \theta - \frac{\pi}{3} \right) - \cos \left( \theta - \frac{\pi}{6} \right) + \cos \left( \frac{\pi}{6} \right) + \cos \left( \theta - \frac{\pi}{6} \right) - \cos \left( \frac{\pi}{3} \right) \right] - \chi \left( \theta, \theta - \frac{\pi}{3}, \frac{\pi}{6} + \frac{\pi}{3} \right) + \chi \left( \theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} \right) - \chi \left( \theta, \frac{\pi}{3} + \frac{\pi}{2} \right), \)

\( h_4^3(\theta) \equiv 1 + 2 \left[ \cos \left( \theta - \frac{\pi}{3} \right) - \cos \left( \theta - \frac{\pi}{6} \right) + \cos \left( \frac{\pi}{6} \right) + \cos \left( \theta - \frac{\pi}{6} \right) - \cos \left( \frac{\pi}{3} \right) \right] + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} + \frac{\pi}{2} \right) - \chi \left( \theta, \theta - \frac{\pi}{3}, \frac{\pi}{6} + \frac{\pi}{3} \right) + \chi \left( \theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} \right) - \chi \left( \theta, \frac{\pi}{3} + \frac{\pi}{2} \right) - \chi \left( \theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} \right), \)

\( h_4^4(\theta) \equiv -1 + 2 \left[ \cos \left( \theta - \frac{\pi}{3} \right) - \cos \left( \theta - \frac{\pi}{6} \right) + \cos \left( \frac{\pi}{6} \right) + \cos \left( \theta - \frac{\pi}{6} \right) - \cos \left( \frac{\pi}{3} \right) \right] + \chi \left( \theta, \frac{\pi}{6} + \frac{\pi}{3} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} + \frac{\pi}{2} \right) + \chi \left( \theta, \frac{\pi}{3} + \frac{\pi}{2} \right) + \chi \left( \theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} \right). \)
\[ h_1^1(\theta) = -1 + 2 \left[ \cos \left( \frac{\pi}{8} \right) - \cos \left( \frac{\pi}{8} + \theta \right) + \cos \left( \frac{\pi}{4} \right) - \cos \left( \frac{\pi}{4} + \theta \right) + \cos \left( \frac{3\pi}{8} \right) - \cos \left( \frac{3\pi}{8} + \theta \right) \right] + \chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8}, \frac{\pi}{8} \right) \]
\[ -\chi \left( \theta, \frac{\pi}{8} + \theta, \frac{\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} - \theta, \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{3\pi}{8} - \theta, \frac{3\pi}{8} \right) \]
\[ -\chi \left( \theta, \frac{3\pi}{8}, \frac{3\pi}{8} + \theta \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right), \]
\[ h_2^1(\theta) = 1 + 2 \left[ \cos \left( \frac{\pi}{8} \right) - \cos \left( \frac{\pi}{4} \right) + \cos \left( \frac{\pi}{4} + \theta \right) - \cos \left( \frac{\pi}{4} + \theta \right) - \cos \left( \frac{3\pi}{8} \right) + \cos \left( \frac{3\pi}{8} \right) \right] + \chi \left( \theta, \theta, \frac{\pi}{8}, \frac{\pi}{8}, \frac{\pi}{8} \right) \]
\[ -\chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} - \theta, \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{3\pi}{8} - \theta, \frac{3\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right) \]
\[ h_3^1(\theta) = 1 + 2 \left[ \cos \left( \frac{\pi}{8} \right) - \cos \left( \frac{\pi}{8} \right) + \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{3\pi}{8} \right) + \cos \left( \frac{3\pi}{8} \right) \right] + \chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8}, \frac{\pi}{8} \right) \]
\[ -\chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{3\pi}{8} - \theta, \frac{3\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right) \]
\[ -\chi \left( \theta, \frac{3\pi}{8}, \frac{3\pi}{8} + \theta \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right), \]
\[ h_1^2(\theta) = -1 + 2 \left[ \cos \left( \frac{\pi}{4} \right) - \cos \left( \frac{\pi}{8} + \theta \right) + \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{3\pi}{8} \right) + \cos \left( \frac{3\pi}{8} \right) \right] + \chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8}, \frac{3\pi}{8} \right) \]
\[ -\chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8} \right) - \chi \left( \theta, \frac{\pi}{8} + \theta, \frac{\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{3\pi}{8} - \theta, \frac{3\pi}{8} \right) - \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) \]
\[ +\chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right) \]
\[ h_2^2(\theta) = 1 + 2 \left[ \cos \left( \frac{\pi}{4} \right) - \cos \left( \frac{\pi}{8} + \theta \right) + \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{3\pi}{8} \right) + \cos \left( \frac{3\pi}{8} \right) \right] + \chi \left( \theta, \theta, \frac{\pi}{8}, \frac{\pi}{8}, \frac{\pi}{8} \right) \]
\[ -\chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{3\pi}{8} - \theta, \frac{3\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right) \]
\[ -\chi \left( \theta, \frac{3\pi}{8}, \frac{3\pi}{8} + \theta \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right), \]
\[ h_3^2(\theta) = 1 + 2 \left[ \cos \left( \frac{\pi}{8} \right) - \cos \left( \frac{\pi}{8} \right) + \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{\pi}{8} + \theta \right) - \cos \left( \frac{3\pi}{8} \right) + \cos \left( \frac{3\pi}{8} \right) \right] + \chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8}, \frac{3\pi}{8} \right) \]
\[ -\chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) - \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{3\pi}{8} - \theta, \frac{3\pi}{8} \right) + \chi \left( \theta, \frac{\pi}{4} + \theta, \frac{\pi}{4} \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right) \]
\[ -\chi \left( \theta, \frac{3\pi}{8}, \frac{3\pi}{8} + \theta \right) + \chi \left( \theta, \frac{\pi}{2} - \theta, \frac{\pi}{2} \right), \]
\[ \text{where } \chi \left( \theta, \theta, \frac{\pi}{8}, \frac{\pi}{8}, \frac{\pi}{8} \right), \chi \left( \theta, \frac{\pi}{8} - \theta, \frac{\pi}{8}, \frac{\pi}{8} \right) \text{ are functions defined in the text.} \]
\[ r_3^1(\theta) = -1 + 2 \left[ \cos\left(\theta - \frac{\pi}{3}\right) - \cos\left(\frac{\pi}{6} + \delta\right) + \cos\left(\theta - \frac{\pi}{6} - \delta\right) - \cos\left(\frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{6} + \delta\right) \right] \]
\[ -\chi(\theta, \theta - \frac{\pi}{3}, \frac{\pi}{6} + \delta, \frac{\pi}{3}) + \chi(\theta, \frac{\pi}{6} + \delta, \frac{\pi}{3}, \frac{\pi}{3}) - \chi(\theta, \frac{\pi}{2} - \theta, \frac{\pi}{3}, \frac{\pi}{2}) - \chi(\theta, \theta - \frac{\pi}{6} - \delta, \frac{\pi}{3}, \frac{\pi}{6} + \delta) \]
\[ + \chi\left(\theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3}, \frac{2\pi}{3}\right) - \chi\left(\theta, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) + \chi\left(\theta, \frac{\pi}{2} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) \]
\[ + \chi\left(\theta, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) - \chi\left(\theta, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) + \chi\left(\theta, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) \]
\[ + \chi\left(\theta, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) - \chi\left(\theta, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) + \chi\left(\theta, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right) \]

3. Proof of Equation (16)

We expand \( C_3\left(\frac{\pi}{2} - \tau\right) \) in its Taylor series to obtain

\[ C_3\left(\frac{\pi}{2} - \tau\right) = C_3\left(\frac{\pi}{2}\right) + \tau \left[ \frac{d}{d\tau} C_3\left(\frac{\pi}{2} - \tau\right) \right]_{\tau=0} + \mathcal{O}(\tau^2). \]

Let \( 0 \leq \tau \ll 1 \). We show Eq. (16):

\[ C_3\left(\frac{\pi}{2} - \tau\right) = -1.5\tau + \mathcal{O}(\tau^2). \]
As shown in the main text, the correlation satisfies $C_x(\frac{\tau}{2}) = 0$ for any pair of colourings labelled by $x$ that we consider. Thus, we have that $C_3(\frac{\tau}{2}) = 0$. From Eq. (21), we have that $C_3(\frac{\tau}{2} - \tau) = h_3(\frac{\tau}{2} - \tau)$ for $0 \leq \tau \ll 1$. Thus, we only need to show that we have that

$$\int_a^b d\xi(\theta, \epsilon, \alpha),$$

where

$$\xi(\theta, \epsilon, \alpha) \equiv \frac{2}{\pi} \sin \epsilon \arccos \left( \frac{\cos \theta \cos \epsilon - \cos \alpha}{\sin \theta \sin \epsilon} \right),$$

as defined by Eq. (20). Differentiating the function $\chi$, we obtain

$$\frac{d}{d\theta} h_3(\theta) \bigg|_{\theta = \pi/2} = 1.5.$$

The function $h_3(\theta)$ has terms of the form

$$\chi(\theta, a, b, \alpha) \equiv \int_a^b d\xi(\theta, \epsilon, \alpha),$$

as defined by Eq. (20). Differentiating the function $\chi$, we obtain

$$\frac{d}{d\theta} \chi(\theta, a, b, \alpha) = \xi(\theta, b, \alpha) \frac{db}{d\theta} - \xi(\theta, a, \alpha) \frac{da}{d\theta} + \int_a^b d\epsilon \frac{\partial}{\partial \theta} \xi(\theta, \epsilon, \alpha).$$

We have that

$$\frac{d}{d\theta} \chi(\theta, a, b, \alpha) = \xi(\theta, b, \alpha) \frac{db}{d\theta} - \xi(\theta, a, \alpha) \frac{da}{d\theta} + \int_a^b d\epsilon \frac{\partial}{\partial \theta} \xi(\theta, \epsilon, \alpha).$$

We obtain that

$$\frac{2}{\pi} \int_a^b \frac{d\epsilon}{\sqrt{1 - \left(\frac{\cos \alpha}{\sin \epsilon}\right)^2}} = \mu(a, b, \alpha),$$

where

$$\mu(a, b, \alpha) \equiv \frac{2}{\pi} \left( \sqrt{\sin^2 b - \cos^2 \alpha} - \sqrt{\sin^2 a - \cos^2 \alpha} \right),$$

for $\cos^2 \alpha \leq \sin^2 b$ and $\cos^2 \alpha \leq \sin^2 a$. We define

$$\nu(\epsilon, \alpha) \equiv \xi \left( \frac{\pi}{2}, \epsilon, \alpha \right).$$

From the definition of $h_3(\theta)$ given in section 1D2 and Eqs. (27) – (31), it is straightforward to obtain that

$$\left[ \frac{d}{d\theta} h_3(\theta) \right]_{\theta = \pi/2} = \frac{1}{\pi} \left[ 6 - 4 \left( \sqrt{3} - \sqrt{2} \right) \right] = 1.5,$$

as claimed.

[1] J. Bell, Physics 1, 195 (1964).
[2] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).
[3] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 81, 5039 (1998).
[4] W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, Phys. Rev. Lett. 81, 3563 (1998). N. Gisin and H. Zbinden, Phys. Lett. A 264, 103 (1999).
[5] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Nature (London) 409, 791 (2001).
[6] D. N. Matsukevich, P. Maunz, D. L. Moehring, S. Olmschenk, and C. Monroe, Phys. Rev. Lett. 100, 150404 (2008).
[7] D. Sålart, A. Baas, J. A. W. van Houwelingen, N. Gisin, and H. Zbinden, Phys. Rev. Lett. 100, 220404 (2008).
[8] M. Giustina, A. Mech, S. Ramelow, B. Wittmann, J. Kofler, J. Beyer, A. Lita, B. Calkins, T. Ger-
rits, S. W. Nam, R. Ursin, and A. Zeilinger, Nature (London) 497, 227 (2013)
[9] P. M. Pearle, Phys Rev. D. 2, 1418 (1970)
[10] A. Kent, Phys. Rev. A 72, 012107 (2005).
[11] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[12] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[13] D. Bohm, Quantum Theory (Prentice-Hall, 1951).
[14] B. S. Cirel’son, Lett. Math. Phys. 4, 93 (1980).
[15] S. L. Braunstein and C. M. Caves, Ann. Phys. 202, 22 (1990).
[16] S. Wehner, Phys. Rev. A 73, 022110 (2006).
[17] B. Bukh, Geometric and Functional Analysis 18, 668 (2008).
[18] F. M. De Oliveira Filho and F. Vallentin, J. Eur. Math. Soc. 12, 1417 (2010).
[19] For example, among non-antipodal bipartite colourings of the sphere in which the black region has area $A < 2\pi$, which colouring(s) produce maximal correlation? Or, consider a general region $R$ of volume $V$ in $\mathbb{R}^n$, and define $p_\epsilon(R)$ to be the probability that, given a randomly chosen point $x \in R$, and a randomly chosen point $y$ such that $d(x, y) = \epsilon$, we find that $y \in R$. Do the balls maximize this probability, for any given sufficiently small $\epsilon$?
[20] N. Aharon, S. Machnes, B. Reznik, J. Silman, and L. Vaidman, Nat. Comput. 12, 5 (2013).
[21] S. Popescu and D. Rohrlich, Found. Phys. 24, 379 (1994).
[22] R. F. Werner, Phys. Rev. A 40, 4277 (1989).