KOSZUL ALGEBRAS AND THE FROBENIUS ENDOMORPHISM

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Abstract. Let $R$ be a standard graded algebra over an $F$-finite field of characteristic $p > 0$. Let $\phi : R \to R$ be the Frobenius endomorphism. For each finitely generated graded $R$-module $M$, let $\phi M$ be the abelian group $M$ with an $R$-module structure induced by the Frobenius endomorphism. The $R$-module $\phi M$ has a natural grading given by $\deg x = j$ if $x \in M_{jp+i}$ for some $0 \leq i \leq p - 1$. In this paper, we prove that $R$ is Koszul if and only if there exists a non-zero finitely generated graded $R$-module $M$ such that $\text{reg}_R \phi M < \infty$. We derive this analog of Kunz’s regularity criterion in positive characteristic by developing a theory of Castelnuovo-Mumford regularity over certain homomorphisms between $N$-graded algebras.

1. Introduction

Let $k$ be an arbitrary field and $(R, m)$ be a standard graded commutative $k$-algebra, i.e., $R_0 = k$ and $R$ is finitely generated over $k$ by $R_1$ and $m = R_+$. We will also use $(R, m, k)$ to denote the algebra $R$. Let $M$ be a finitely generated graded $R$-module. The Castelnuovo-Mumford regularity of $M$ over $R$,

$$\text{reg}_R M = \sup \{ t : \text{Tor}_i^R(M, k)_{i+t} \neq 0 \text{ for some } i \},$$

is among the most important numerical invariants of $M$. When $R$ is a Koszul algebra, that is when $\text{reg}_R k = 0$, Avramov and Eisenbud in [3] showed that $\text{reg}_R M < \infty$ for all finitely generated $R$-modules $M$. Moreover, in [7], Avramov and Peeva showed that the finiteness of regularity of the residue field implies the Koszul property of $R$. It is a natural analog of the characterization of regular rings by Auslander, Buchsbaum and Serre. For a recent survey on Koszul algebras and related topics, we refer to [10].

Over a field of characteristic $p > 0$, Kunz in [17] showed that $\phi^e R$ can be taken to be the testing object for the regular property of the ring $R$. Recall that if $R$ is either a local ring or a standard graded $k$-algebra of characteristic $p > 0$, then the Frobenius morphism $\phi : R \to R$ is the ring homomorphism given by $\phi(a) = a^p$ for each $a \in R$. We say that $R$ is $F$-finite if $\phi$ is a finite morphism. Given $e \in \mathbb{Z}$, $e \geq 1$, denote $q = p^e$. For each $e \geq 1$, $\phi^e : R \to R$ mapping $a \in R$ to $a^q$ is also an endomorphism of $R$. For each $R$-module $M$, denote by $\phi^e M$ the abelian group $M$, considered as an $R$-module via the action of $\phi^e$. Concretely, this action is given by $a \cdot x = a^q x$.
for all $a \in R$, $x \in \phi^e M$. Recently, in [5], Avramov, Hochster, Iyengar and Yao gave a vast generalization of Kunz’s result. Namely, they showed in [5, Theorem 1.1] that $R$ is regular if there exists an $e > 0$ and a nonzero finitely generated (graded) $R$-module $M$ such that $\phi^e M$ has finite flat dimension; see also [19, Theorem 2]. For further information about the homological significance of Frobenius endomorphism, the reader may consult [13], [18], [20]. In this paper, we prove:

**Theorem 1.1.** Let $k$ be an $F$-finite field of characteristic $p > 0$ and $R$ be a standard graded $k$-algebra. Assume that there exists a non-zero finitely generated graded $R$-module $M$ and some $e > 0$ such that $\text{reg}_R \phi^e M < \infty$. Then $R$ is a Koszul algebra.

Note that regularity depends critically on the grading. So let us first introduce the natural grading on the module $\phi^e M$. Observe that $\phi^e$ is also a homogeneous ring homomorphism from $R$ to $R^{(q)}$, the $q$-th Veronese subring of $R$. For each $i = 0, 1, \ldots, q - 1$, define the module $V_i(q, M) = \bigoplus_{j \in \mathbb{Z}} M_{q+i}^j$ with the grading $\text{deg } x = j$ if $x \in M_{q+i}^j$. In other words, $V_i(q, M) = V_i(q, M)^0 \oplus V_i(q, M)^1 \oplus \cdots$. Similarly, $V_i(q, M)^{q+i}$ is isomorphic to $V_i(q, M)^0$ as an algebra. Hence, $\phi^e$ becomes a graded $R$-module (see Section 3.1), and so does $\phi^e M = \bigoplus_{i=0}^{q-1} V_i(q, M)$. We call this grading of $\phi^e M$ the *Veronese grading*. The regularity in Theorem 1.1 is computed with respect to the Veronese grading.

If $k$ is $F$-finite, it is easy to show that $\phi^e$ is a finite endomorphism for every $e \geq 1$; see 4.1. In this paper, we will improve Theorem 1.1 considerably by allowing $k$ to be of arbitrary characteristic and replacing the power $\phi^e$ of Frobenius by certain non-finite endomorphism of $R$. To afford such a statement, we introduce a broad generalization of the Castelnuovo-Mumford regularity, namely the regularity over a homomorphism. We will make use of the approach in [5] and [6] to construct the notion of regularity over a homomorphism. Let $\varphi : (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a homomorphism having an order $d \geq 1$ of graded rings, where each of $R$ and $S$ is generated in a single positive degree as an algebra. The condition $\varphi$ has order $d$ means that $\text{deg } \varphi (a) = d \text{deg } (a)$ for every homogeneous element $a \in R$. For each complex $M \in D^b (\text{Gr } S)$ (i.e., $H(M)$ is finitely generated and bounded below; see Section 2), we define in Section 4 the regularity of $M$ over $\varphi$. Denote this number by $\text{reg}_\varphi M$. The main technical result in this paper, which generalizes Theorem 1.1, is:

**Theorem 1.2.** Let $R$ be a standard graded $k$-algebra. Let $\psi : R \rightarrow R$ be an endomorphism of order at least 2. Assume that there exists a non-zero finitely generated graded $R$-module $M$ such that $\text{reg}_\psi M < \infty$. Then $R$ is a Koszul algebra.

See also Remark 7.1 for examples of endomorphisms of order at least 2 in characteristic 0.

Let us sketch the proof of Theorem 1.2, keeping attention to the special case of Theorem 1.1. Let $\text{char } k = p > 0$, $k$ be $F$-finite and $R$ a standard graded $k$-algebra with the Frobenius morphism $\phi$. In the situation of Theorem 1.1, one can think of the Veronese functor and the Veronese grading as tools to “homogenize” the Frobenius and the graded module $M$. Similarly, in the situation of Theorem 1.2, we can homogenize $M$ with respect to the action of $\psi$. Say, an endomorphism of order
$d$ gives us a homogeneous ring homomorphism from $R$ to $R^{(d)}$. By scalar restriction along $\psi$, $\psi M = \oplus_{i=0}^{d-1} V_i(d, M)$ equipped with the Veronese grading becomes a $R$-module; see Section 3. There is a kind of inverse process to “homogenize” the Frobenius endomorphism, achieved by what we call the fractional Veronese functor.

**Definition 1.3.** Let $R$ be an $\mathbb{N}$-graded algebra over $k$ which is generated in a single degree $g \geq 1$. Let $s \geq 1$ be a positive integer. Let $R^{(1/s)}$ be the ring $R$ with a new grading given by: $R^{(1/s)}_i = R_i$ for all $i \in \mathbb{Z}$ and $R^{(1/s)}_j = 0$ for all $j$ which is not divisible by $s$. Similarly, for each finitely generated graded $R$-module $M$, we define $M^{(1/s)}$. We call $M^{(1/s)}$ the $s$-th fractional Veronese of $M$.

If $\psi : R \rightarrow R$ is now an endomorphism of order $d \geq 2$, we have an induced homogeneous morphism $R^{(1/d)} \rightarrow R$. One of the main steps in the proof of Theorem 1.1 and the proof of Theorem 1.2 is to compare the regularity of $M$ over the homomorphism $\psi$, and the regularity of $\psi M$ over homomorphism $R \rightarrow R^{(d)}$. Theorem 5.2 carries out this comparison.

The study of endomorphisms of order $\geq 1$ and $R^{(1/s)}$ was first carried out in full generality by Koh and Lee in [16, p. 686]. Fractional Veronese keeps all the information about the ring $R$ and the module $M$, except for the grading. This is the difference between fractional Veronese and the Veronese functor.

In the next step in the proof of Theorem 1.2, we study the behavior of regularity along compositions of homomorphisms, following an idea in [5]. The result of that study (Theorem 5.4) promotes the following construction. For each complex of graded $R$-modules $M \in \mathcal{D}^f_+(\text{Gr } R)$, define inductively complexes $M^i \in \mathcal{D}^f_+(\text{Gr } R)$ as follows: $M^1 = M$, and for $i \geq 1$,

$$M^{i+1} = (M^i)^{(1/d)} \otimes_{R^{(1/d)}}^L M,$$

where $\otimes_{R}^L$ denote the left derived functor of the tensor product functor $\otimes_R$. Now application of Theorem 5.4 shows that if $M$ has finite regularity over $\psi$, then $\text{reg}_{\psi}^M M^i < \infty$ for each $i \geq 1$. By the comparison result mentioned above (Theorem 5.2), this also implies that $\text{reg}_{\widetilde{\psi}}^\psi \psi M^i < \infty$, where $\widetilde{\psi}$ is the homogeneous morphism $R \rightarrow R^{(d)}$ induced by $\psi$, and $\psi M^i \in \mathcal{D}^f_+(\text{Gr } R^{(d)})$ is equipped with the Veronese grading. Moreover, by a result of Backelin, $R^{(d)}$ is Koszul for large enough $i$. Together with our following characterization of Koszul algebras, we finish the proof of Theorem 1.2.

**Theorem 1.4.** Let $\varphi : (R, m, k) \rightarrow (S, n, l)$ be a homogeneous morphism of standard graded rings, where $S$ is a Koszul $l$-algebra. Assume that there exists a complex $M \in \mathcal{D}^f_+(\text{Gr } S)$ such that $M \not\simeq 0$ and $\text{reg}_\varphi M < \infty$. Then $R$ is a Koszul $k$-algebra.

This result, in itself a far-reaching generalization of Avramov-Peeva’s characterization of Koszul algebras among standard graded ones, is an analog of [1, Theorem 1].

The paper is organized as follow. We start with some background materials on graded homological algebra of complexes in Section 2. In Section 3, we investigate the two constructions which are important for later discussions, namely the Veronese gradings and the fractional Veronese functor. Theorem 3.3 points out a natural
functor from the graded category of the fractional Veronese to that of the original ring. This functor will be exploited in Section 5. In Section 4 we define the regularity over a non-homogeneous morphism. The main result in this section is Theorem 4.5 which implies (Remark 4.6) that for a homogeneous finite morphism, regularity over this morphism is the same as Castelnuovo-Mumford regularity computed over the source ring. In Section 5, we show that regularity over homomorphism behaves well with respect to certain factorizations of the given homomorphism. Then we prove the afore-mentioned Theorem 5.2. We prove in Theorem 5.4 a finiteness property of regularity with respect to compositions of homomorphisms. In Sections 6 and 7, we apply the study of regularity over homomorphism to the proofs of the main results. In the final section, we provide a partial analog of Theorem 1.1 for local rings and linearity defect and list some open questions relating to the Frobenius characterization of Koszul algebras.

2. Preliminaries

Let \((R, \mathfrak{m}, k)\) be an \(\mathbb{N}\)-graded \(k\)-algebra with graded maximal ideal \(\mathfrak{m}\), where \(k\) is a field. By convention, all the algebras in this paper are finitely generated. Modules are always finitely generated and identified with complexes concentrated in degree 0. The abelian category of chain complexes of graded modules and degree 0 homomorphisms of complexes over \(R\) is denoted by \(\text{Gr} R\). Let \(\text{Gr} R\) be its derived category. Let \(\text{Df} (\text{Gr} R)\) be the full subcategory of \(\text{D}(\text{Gr} R)\) consisting of complexes \(M\) such that \(H_i(M)\) is a finitely generated \(R\)-module for every \(i\). We say that \(M\) is homologically bounded below (or above) if \(H_i(M) = 0\) for \(i \ll 0\) (respectively, for \(i \gg 0\)). The full subcategory of \(\text{Df} (\text{Gr} R)\) consisting of homologically bounded below complexes of graded \(R\)-modules is denoted by \(\text{Df}^+ (\text{Gr} R)\). Similarly, the notation \(\text{Df}^- (\text{Gr} R)\) stands for the full subcategory of \(\text{Df} (\text{Gr} R)\) consisting of homologically bounded above complexes.

The notation \(\simeq\) signifies isomorphisms in \(\text{Df} (\text{Gr} R)\). Denote by \(\inf M\) the infimum of the set \(\{i : H_i(M) \neq 0\}\).

For a complex \(M \in \text{Df} (\text{Gr} R)\), let \(G\) be the minimal graded free resolution of \(M\) in \(\text{Df} (\text{Gr} R)\). Let \(G_i = \oplus_j R(-j)^{\beta_{i,j}(M)}\) for all \(i\). The numbers \(\beta_{i,j}(M)\) are called the graded Betti numbers of \(M\). The Poincaré series of a homologically bounded below complex is a good way to encode information about its Betti numbers. The Poincaré series of \(M\) is the formal power series \(P_M^R(t, y) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \beta_{i,j}(M) t^i y^j \in \mathbb{Z}[[t, y]]\). We refer to [9], [21] for more detailed treatments of homological algebra of complexes and to [14], [15] for the graded setting. See also [2] for in depth discussions of free resolutions.

Recall that if \(R\) is standard graded, the Castelnuovo-Mumford regularity over \(R\) of a complex \(M \in \text{Df} (\text{Gr} R)\) is defined by

\[
\text{reg}_R M = \sup\{j - i : \beta_{i,j}(M) \neq 0\}.
\]

Note that in the literature, sometimes \(\text{reg}_R M\) is referred to as the Ext-regularity and the Castelnuovo-Mumford regularity is defined as another invariant; see for example [14], [15]. However, we will not deal with that different notion in this paper.
The following simple lemma will be used several times in the sequel.

**Lemma 2.1.** Let $R$ be a standard graded $k$-algebra.

(i) If $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of graded $R$-modules, then there are inequalities

$$\text{reg}_R M \leq \max\{\text{reg}_R M', \text{reg}_R M''\},$$

$$\text{reg}_R M' \leq \max\{\text{reg}_R M, \text{reg}_R M'' + 1\},$$

$$\text{reg}_R M'' \leq \max\{\text{reg}_R M, \text{reg}_R M' - 1\}.$$ 

In particular, if $\text{reg}_R M' \geq \text{reg}_R M + 1$, then $\text{reg}_R M'' = \text{reg}_R M' - 1$.

(ii) (See [8, Lemma 2.2(b)].) Given an exact sequence of graded $R$-modules

$$\cdots \to M_i \to M_{i-1} \to \cdots \to M_0 \to N \to 0,$$

there is an inequality

$$\text{reg}_R N \leq \sup\{\text{reg}_R M_i - i : i \geq 0\}.$$ 

We record the following simple fact for later usage. Let $R \to S$ be a ring homomorphism. Every $S$-module is automatically an $R$-module by scalar restriction. Denote by $\mathcal{D}(R)$ the derived category of the category of $R$-complexes.

**Lemma 2.2.** The scalar restriction functor is an exact functor from the category $\mathcal{D}(S)$ to the category $\mathcal{D}(R)$.

### 3. Veronese gradings and fractional Veronese

When $\text{char } k = p > 0$, and $R$ is an $\mathbb{N}$-graded $k$-algebra, the Frobenius $\phi : R \to R$ given by $\phi(a) = a^p$ is a ring homomorphism. However, it is neither $k$-linear nor a homogeneous morphism of graded rings. There are two opposite ways to "homogenize" the Frobenius: considering Veronese subrings of $R$, and considering the fractional Veronese functor $(\cdot)^{(1/p)}$. To obtain more general results, we will work with a class of ring homomorphisms larger than the class of powers of the Frobenius, namely the class of homomorphisms with positive orders.

#### 3.1. Veronese gradings

Recall the following notion from the introduction.

**Definition 3.1.** Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a homomorphism of algebras such that $\deg \varphi(a) = d \deg(a)$ for every homogeneous element $x \in R$, for some constant $d \in \mathbb{Z}, d \geq 1$. Then we say that $d$ is the order of $\varphi$ and write order $\varphi = d$.

For example, the power $\phi^e$ of Frobenius endomorphism has order $q = p^e$ for all $e \geq 1$. Homogeneous morphisms are simply morphisms of order 1. Clearly if $\varphi$ is a homomorphism of order $d$, then we have an induced homogeneous morphism $R \to S(d)$.

Let $M$ be a graded $S$-module, define $^eM$ to be the following $R$-module $M$. Consider the $d$-th Veronese modules $V_i(d, M) = \bigoplus_{u \in \mathbb{Z}} M_{du+i}$ for $0 \leq i \leq d - 1$. These are graded modules over the Veronese ring $S(d)$ and are finitely generated if $M$ is finitely generated as $S$-module.
By scalar restriction along the homomorphism \( R \to S^{(d)}, \) each \( V_i(d, M) \) is a graded \( R \)-module. Therefore, \( \mathfrak{r}M = \bigoplus_{i=0}^{d-1} V_i(d, M) \) is also a graded \( R \)-module. We call this grading of \( \mathfrak{r}M \) the Veronese grading of \( \mathfrak{r}M. \) In the case \( \operatorname{char} k = p > 0, S = R \) and \( \varphi = \partial^\varphi, \) we obtain the Veronese grading for \( \varphi^\varphi M \) already mentioned in the introduction.

### 3.2. Fractional Veronese

Let \( R \) be an \( \mathbb{N} \)-graded algebra over \( k \) (\( \operatorname{char} k \) can be 0), such that \( R \) is generated in a single degree \( g \geq 1. \) Let \( M \) be a finitely generated graded \( R \)-module and \( s \geq 1 \) an integer. Recall from the introduction that \( R^{(1/s)} \) is the ring \( R \) with a new grading given by: \( R^{(1/s)}_{ni} = R_i \) for all \( i \in \mathbb{Z} \) and \( R^{(1/s)}_{j} = 0 \) for all \( j \) not divisible by \( s. \) Similar definition goes for \( M^{(1/s)} \), and this is a \( R^{(1/s)} \)-module. Furthermore, \( R^{(1/s)} \) is a \( k \)-algebra generated in degree \( sg, \) and the fractional Veronese functor \( (\cdot)^{(1/s)} \) from \( \mathcal{D}^f(\text{Gr} R) \) to \( \mathcal{D}^f(\text{Gr} R^{(1/s)}) \) is exact.

Note that if \( \varphi : R \to S \) has order \( d, \) then one gets an induced homogeneous ring homomorphism \( R^{(1/d)} \to S, \) denoted by \( \overline{\varphi}. \) Any \( S \)-module is naturally a module over \( R^{(1/d)} \). Given \( d \geq 1 \) and an endomorphism \( \psi : R \to R \) of order \( d, \) any graded \( R \)-module is naturally an \( R^{(1/d)} \)-module via \( \overline{\psi}. \) We will construct a functor relating graded \( R^{(1/d)} \)-modules and graded \( R \)-modules and regularity of complexes over those rings.

**Definition 3.2.** Let \( N \) be a graded \( R^{(1/d)} \)-module. The \( d \)-th Veronese modules of \( N \) are naturally \( R \)-graded via the induced action: if \( a \in R \) and \( x \in V_i(d, N) \) where \( 0 \leq i \leq d - 1, \) we define \( a \circ x = ax, \) where the multiplication with scalar on the right is that of \( R^{(1/d)} \). In fact, if \( a \in R_s \) and \( x \in N_{dt+i} \) (so \( \deg x = t \) in \( V_i(d, N) \)), then \( a \in R^{(1/d)}_{ds} \) and hence \( ax \in N_{d(s+t)+i}. \) Thus \( \deg(a \circ x) = s + t \) in \( V_i(d, N), \) as expected. We define \( \Phi(N) \) to be the graded \( R \)-module \( \bigoplus_{i=0}^{d-1} V_i(d, N) \) with the above \( \circ \) action. Then define the image via \( \Phi \) of a morphism between graded \( R^{(1/d)} \)-modules in a natural way.

For each complex \( N' \) of graded \( R^{(1/d)} \)-modules, define the complex of graded \( R \)-modules \( \Phi(N') \) in the natural way: the \( i \)-th module \( \Phi(N')_i \) is image via \( \Phi \) of the \( i \)-th module of \( N', \) and the new differentials are the images via \( \Phi \) of the old ones.

The important properties of \( \Phi \) are stated in the following:

**Theorem 3.3.** The functor \( \Phi \) is an exact functor from \( \mathcal{D}^f(\text{Gr} R^{(1/d)}) \) to \( \mathcal{D}^f(\text{Gr} R). \) It is a left inverse of fractional Veronese functor \( (\cdot)^{(1/d)} \). Moreover, \( \Phi(R^{(1/d)}) \) has a natural structure of a graded \( k \)-algebra and there is a homogeneous isomorphism \( \Phi(R^{(1/d)}) \cong R \) of graded \( k \)-algebras.

**Proof.** That \( \Phi \) is exact follows from Lemma 2.2 applying to the identity homomorphism \( R \to R^{(1/d)}. \) For any graded \( R \)-module \( M, \) it is easy to verify that \( \Phi(M^{(1/d)})_i = M_i \) for all \( i \in \mathbb{Z}. \) Hence \( \Phi \) is a left inverse of \( d \)-th fractional Veronese.

Given \( j \in \mathbb{Z}, \) there are equalities

\[
\Phi(R^{(1/d)})_j = \bigoplus_{i=0}^{d-1} (R^{(1/d)})_{jd+i} = (R^{(1/d)})_{jd} = R_j.
\]
The $k$-algebra structure of $R^{(1/d)}$ naturally induces a like structure for $\Phi(R^{(1/d)})$. Using the definition of the action $\sigma$, we conclude that $\Phi(R^{(1/d)}) \cong R$. Therefore the proof is completed. \hfill \Box

4. Regularity over homomorphisms

In this section, we begin studying the notion of regularity over a homomorphism. Since the main applications we have in mind are for standard graded algebras in positive characteristics, we restrict ourselves to working over $\mathbb{N}$-graded algebras which are generated in a single positive degree. Theoretically at least, our results are also applicable to characteristic zero; see Remark 7.1. We believe that it is possible to define regularity over homomorphisms for arbitrarily graded algebras, however the theory would be more elegant if we focus on algebras which are generated in a single degree. This claim will be backed up by most of the results in the current and the forthcoming sections. See also Question 8.5 for a related discussion.

There are two advantages of introducing regularity over a homomorphism. Firstly, it gives a close approximation of the regularity of the module $\psi M$, where $\psi : R \to R$ is an endomorphism of order $d \geq 1$. See Theorem 5.2(ii). Secondly, it behaves very well with respect to compositions of homomorphisms of positive orders; see Theorem 5.4. The second feature will be crucial to the proof of Theorem 1.2 in Section 7.

**Remark 4.1.** Assume that $\text{char } k = p > 0$ and $k$ is $F$-finite. For every $e > 0$, $\phi^e$ is a finite morphism. Namely, if $R = k[x_1, \ldots, x_n]/I$ is a presentation of $R$, where $I$ is a homogeneous ideal of the polynomial ring $k[x_1, \ldots, x_n]$, then $\phi^e R$ is generated by the classes of the generators of $k$ as $\phi(k)$-module and the classes of $x_1^{a_1} \cdots x_n^{a_n}$, where $0 \leq a_j \leq q - 1$ for all $j = 1, \ldots, n$.

In this section, we will work with a (not necessarily finite) homomorphism $\varphi : (R, m, k) \to (S, n, l)$ of $\mathbb{N}$-graded rings with order $d \geq 1$. Assume that $R$ is generated in degree $g$ and $S$ is generated in degree $h$, where $g, h \geq 1$. Let $M \in D^f_{\ell}(\text{Gr } S)$ be a complex of graded $S$-modules. We will define Betti numbers and Poincaré series of $M$ over the homomorphism $\varphi$, following the construction for local homomorphisms of local rings in [5, Section 2].

Recall that $\varphi$ induces a homogeneous homomorphism $\varphi : R^{(1/d)} \to S$ and $R^{(1/d)}$ is generated in degree $dg$. Since $S$ is generated in degree $h$, we infer that $dg = hc$ for some integer $c \geq 1$. By hypothesis, obviously $mS \subseteq n^c$, where $n^c \subseteq S$ denotes the $c$-th power of the ideal $n$. Let $x = x_1, \ldots, x_n$ be a minimal system of generators of the ideal $n^c$ modulo $mS$, such that $\text{deg } x_i = dg$ for all $i = 1, \ldots, n$. Note that $n^c$ is generated in degree $hc = dg$.

Let $M \in D^f_{\ell}(\text{Gr } S)$ be a complex. Denote by $K[x; M]$ the Koszul complex of $M$ with respect to the sequence $x$ in $S$ (see [6] for more information). Each graded $S$-module $H_i(k \otimes_{R^{(1/d)}} K[x; M]) \cong \text{Tor}^R_{i}((k, K[x; M])$ is finitely generated. Moreover, from [6, Lemma 1.5.6] it is also an $S/n^e$-module. Therefore we see that $\dim_{l} H_i(k \otimes_{R^{(1/d)}} K[x; M]) < \infty$ for all $i$. By definition, for all $i, j \in \mathbb{Z}$, the $(i, j)$ graded Betti number of $M$ over $\varphi$ is

$$\beta_{i,j}^{\varphi}(M) = \dim_{l} H_i(k \otimes_{R^{(1/d)}} K[x; M])_{j}.$$
Given $i$, for all but finitely many $j$, we have $\beta^\varphi_{i,j}(M) = 0$.

**Definition 4.2.** The *regularity of $M$ over the homomorphism* $\varphi$, denoted by $\text{reg}_\varphi M$, is defined as follow

$$\text{reg}_\varphi M = \sup \left\{ \frac{j - idg}{dg} : \beta^\varphi_{i,j}(M) \neq 0 \right\}.$$  

If $\varphi = \text{id}^R$ is the identity of $R$, we will denote $\text{reg}_{\text{id}^R} M$ simply by $\text{reg}_R M$. It is clear from the definition that for every $m \in \mathbb{Z}$,

$$\text{reg}_R M(-m) = \text{reg}_R M + \frac{m}{g}.$$  

If additionally $R$ is standard graded then $\beta^\varphi_{i,j}(M)$ is the usual $(i,j)$-Betti number of $M$ over $R$ and $\text{reg}_R M$ is the usual Castelnuovo-Mumford regularity of $M$.

The following proposition shows that graded Betti numbers and regularity of $M$ over $\varphi$ do not depend on the choice of minimal generators of $n^c/\text{m}S$. Moreover, we can compute the regularity of $M$ over $\varphi$ by choosing any (minimal or not) generating set of $n^c$ modulo $\text{m}S$ consisting of elements of degree $dg$.

**Proposition 4.3.** Let $z = z_1, \ldots, z_p$ be a sequence of elements having degree $dg$ which generates $n^c$ modulo $\text{m}S$. Denote by $\kappa$ the minimal number of homogeneous generators of $n^c/\text{m}S$. Denote $\beta^z_{i,j}(M) = \dim_k H_i(k \otimes_{R(1/dg)} K[z; M])_j$ for $i,j \in \mathbb{Z}$, and

$$\text{reg}_{\varphi,z} M = \sup \left\{ \frac{j - idg}{dg} : \beta^z_{i,j}(M) \neq 0 \right\},$$  

Then

$$P^\varphi_M(t,x)(1 + ty^{dg})^{p-\kappa} = \sum_i \sum_j \beta^z_{i,j}(M)t^iy^j,$$

and $\text{reg}_{\varphi,z} M = \text{reg}_\varphi M$.

**Proof.** The equality between formal series is a graded analog of [6, Proposition 4.3.1], with appropriate changes concerning the grading.

For the second part, we have that for all $i,j$,

$$\beta^z_{i,j}(M) = \sum_{u=0}^{p-\kappa} \binom{p-\kappa}{s} \beta^\varphi_{i-u,j-udg}.$$  

Therefore

$$\text{reg}_{\varphi,z} M = \sup \left\{ \frac{j - idg}{dg} : \beta^z_{i,j}(M) \neq 0 \right\} = \text{reg}_\varphi M,$$

as claimed.

The regularity over a homomorphism is unchanged by tensoring with appropriate Koszul complexes.

**Lemma 4.4.** For any finite sequence $\nu$ of elements of degree $dg$ in $S$, we have

$$\text{reg}_\varphi M = \text{reg}_\varphi K[\nu; M].$$
Proof. Let \( x \) be a minimal generating set of elements of degree \( dg \) for the ideal \( n^c/mS \). Then
\[
K[x; K[v; M]] \cong K[x; v; M],
\]
hence the conclusion follows by an application of Proposition 4.3. \( \square \)

The following theorem is the main result in this section. It is an analog of [6, Theorem 7.2.3]. The proof requires non-trivial modifications so we carry it out in detail.

**Theorem 4.5.** Let \( y \) be a finite sequence of elements of degree \( dg \) in \( S \). If the \( S \)-module
\[
\frac{H(M)}{yH(M) + mH(M)}
\]
has finite length, then there is an equality
\[
\text{reg}_\varphi M = \sup \left\{ \frac{j - idg}{dg} : H_i(k \otimes_{R^{(1/d)}} K[y; M])_j \neq 0 \right\}.
\]

**Remark 4.6.** If \( S/mS \) is an artinian ring, then we can choose \( y = \emptyset \) in Theorem 4.5. In particular,
\[
\text{reg}_\varphi M = \sup \left\{ \frac{j - idg}{dg} : H_i(k \otimes_{R^{(1/d)}} M)_j \neq 0 \right\}.
\]
If moreover, \( R \to S \) is a finite morphism then the last equality says that
\[
\text{reg}_\varphi M = \text{reg}_{R^{(1/d)}} M.
\]

We start by proving the following result; it was suggested by Lemma 1.2.3 in ibid.

**Lemma 4.7.** Let \( X(-s) \xrightarrow{\theta} X \to C \to 0 \) be a triangle in \( D^f(\text{Gr} S) \) where \( s \geq 1 \). Let \( i \) be an integer. Assume that the map \( H_i(\theta)^t : H_i(X)(-st) \to H_i(X) \) is zero for some \( t \geq 1 \). Assume further that \( H(X) \) has finite length. Then for each \( j \), there are inequalities
\[
\ell(H_i(C)_j) \leq \ell(H_i(X)_j) + \ell(H_{i-1}(X)_{j-s}),
\]
\[
\ell(H_i(X)_j) \leq \sum_{m=0}^{t-1} \ell(H_{i+1}(C)_{j+(m+1)s}),
\]
where for a finite length \( S \)-module \( N \), \( \ell(N) \) denotes \( \dim_k N \).

**Proof.** Denote \( \alpha_{i,j} = H_i(\theta)_j : H_i(X)_j \to H_i(X)_{j+s} \). For each \( j \), we have exact sequences
\[
0 \to \text{Coker} \alpha_{i,j-s} \to H_i(C)_j \to \text{Ker} \alpha_{i-1,j-s} \to 0,
\]
and
\[
0 \to \text{Ker} \alpha_{i,j-s} \to H_i(X)_{j-s} \xrightarrow{\alpha_{i,j-s}} H_i(X)_j \to \text{Coker} \alpha_{i,j-s} \to 0.
\]
So there are formulas
\[
\ell(H_i(C)_j) = \ell(\text{Coker} \alpha_{i,j-s}) + \ell(\text{Ker} \alpha_{i-1,j-s}),
\]
\[
\ell(\text{Coker} \alpha_{i,j-s}) \leq \ell(H_i(X)_j),
\]
\[
\ell(\text{Ker} \alpha_{i,j-s}) \leq \ell(H_i(X)_{j-s}).
\] (4.1)
From (4.1), it is clear that
\[ \ell(H_i(C)_j) \leq \ell(H_i(X)_j) + \ell(H_{i-1}(X)_{j-s}). \]
By hypothesis, \( \alpha_{i,j} + (t-1)s \circ \cdots \circ \alpha_{i,j} \circ \alpha_{i,j} = 0 \). Therefore examining the sequence
\[ H_i(X)_j \xrightarrow{\alpha_{i,j}} H_i(X)_{j+s} \rightarrow \cdots \rightarrow H_i(X)_{j+(t-1)s} \xrightarrow{\alpha_{i,j} + (t-1)s} H_i(X)_{j+ts}, \]
we obtain
\[ \ell(H_i(X)_j) \leq \sum_{m=0}^{t-1} \ell(\text{Ker} \alpha_{i,j+ms}). \]
Combining with (4.1), we finish the proof of the lemma. \( \Box \)

**Proof of Theorem 4.5.** Denote \( K = K[y; M] \). Using Lemma 4.4, we have \( \text{reg}_x M = \text{reg}_x K \). So it is enough to show
\[ \text{reg}_x K = \sup \left\{ \frac{j - \text{idg}}{dg} : H_i(k \otimes_{R(1/d)}^L K)_{j} \neq 0 \right\}. \]
(4.2)

Let \( x = x_1, \ldots, x_n \) be a minimal generating set of degree-\( dg \) elements of \( n^c \) modulo \( mS \). Denote \( K^{(\nu)} = K[x_1, \ldots, x_\nu; K] \) for all \( \nu = 0, 1, \ldots, n \). Arguments similar to the proof of [loc. cit., Theorem 7.2.3] show that:

(i) the length of the \( S \)-module \( H(K)/mH(K) \) is finite,
(ii) \( H_i(k \otimes_{R(1/d)}^L K^{(\nu)}) \) is a finite \( S \)-module for all \( i \in \mathbb{Z}, \nu = 0, 1, \ldots, n \),
(iii) for any fixed \( i \), there exists a number \( t_i \geq 1 \) such that \( H_i(k \otimes_{R(1/d)}^L K^{(\nu)}) \) is annihilated by \( n^t \) for all \( \nu = 0, 1, \ldots, n \).

For each \( \nu \), \( K^{(\nu+1)} \) is the mapping cone of the morphism \( \lambda^{(\nu)} : K^{(\nu)}(-dg) \rightarrow K^{(\nu)} \), where the latter is the multiplication by \( x_{j+1} \). Moreover, we have a triangle in \( D^f(\text{Gr} S) \)
\[ k \otimes_{R(1/d)}^L K^{(\nu)}(-dg) \xrightarrow{k \otimes_{R(1/d)}^L \lambda^{(\nu)}} k \otimes_{R(1/d)}^L K^{(\nu)} \rightarrow k \otimes_{R(1/d)}^L K^{(\nu+1)} \rightarrow \cdot \]
Denote \( \beta_{i,j}(K^{(\nu)}) = \dim_k H_i(k \otimes_{R(1/d)}^L K^{(\nu)})_j \) for each \( i, j \in \mathbb{Z} \), and
\[ r(\nu) = \sup \left\{ \frac{j - \text{idg}}{dg} : \beta_{i,j}(K^{(\nu)}) \neq 0 \right\}. \]
Applying Lemma 4.7, there are inequalities
\[ \beta_{i,j}(K^{(\nu+1)}) \leq \beta_{i,j}(K^{(\nu)}) + \beta_{i-1,j-dg}(K^{(\nu)}), \]
(4.3)
\[ \beta_{i,j}(K^{(\nu)}) \leq \sum_{m=0}^{t_i-1} \beta_{i+1,j+(m+1)dg}(K^{(\nu+1)}). \]
(4.4)

Now we will show that for all \( \nu \),
\[ r(\nu) = r(\nu + 1). \]
First, take any \( i, j \) such that \( (j - \text{idg})/dg > r(\nu) \). Then obviously
\[ \frac{(j - dg) - (i-1)dg}{dg} = \frac{j - \text{idg}}{dg} > r(\nu). \]
Therefore \( \beta_{i,j}(K(\nu)) = \beta_{i-1,j-dg}(K(\nu)) = 0 \). From (4.3), we also get \( \beta_{i,j}(K(\nu+1)) = 0 \).

Hence \( r(\nu + 1) \leq r(\nu) \).

Second, take any \( i, j \) such that \( (j - idg)/dg > r(\nu + 1) \). Then for all \( m \geq 0 \),

\[
\frac{j + (m + 1)dg - (i + 1)dg}{dg} = \frac{j - idg}{dg} + m \geq \frac{j - idg}{dg} > r(\nu + 1).
\]

So \( \beta_{i+1,j+(m+1)dg}(K(\nu+1)) = 0 \) for \( m = 0, \ldots, t_i - 1 \). Using (4.4), we infer that \( \beta_{i,j}(K(\nu)) = 0 \). Hence \( r(\nu) \leq r(\nu + 1) \). So \( r(\nu) = r(\nu + 1) \), as claimed.

All in all, we obtain \( r(0) = r(1) = \cdots = r(n) \). Finally, note that \( r(n) = \text{reg}_\varphi K \) and

\[
r(0) = \sup \left\{ \frac{j - idg}{dg} : H_i(k \otimes_{R(1/d)}^L K)_{\nu(j)} \neq 0 \right\},
\]

so (4.2) is proved. The proof of the theorem is now completed. \( \square \)

5. Factorizations and composition of homomorphisms

Let \( \varphi : (R, m, k) \to (S, n, l) \) be again a homomorphism of order \( d \), \( R \) is generated over \( k \) in degree \( g \) and \( S \) is generated over \( l \) in degree \( h \), where \( d, g, h \geq 1 \). In stead of working with \( \varphi \), we can work with a morphism from a certain polynomial extension of \( R \) to \( S \).

In detail, let \( c \in \mathbb{N} \) be such that \( dg = ch \). Let \( y_1, \ldots, y_m \) be a generating set of \( n \) modulo \( m \mathbb{S} \) for which \( \deg y_i = h \) for all \( i \). Let \( R[t_1, \ldots, t_m] \) be a polynomial extension of \( R \), \( t_i \) are variables of degree \( g \). Consider the morphism \( \varphi' : R[t_1, \ldots, t_m] \to S \) mapping \( t_i \) to \( y_i^c \) for all \( i \). We have a commutative diagram

\[
\begin{array}{ccc}
R[t_1, \ldots, t_m] & \xrightarrow{\varphi'} & S \\
\downarrow \varphi & & \\
R & \xrightarrow{\varphi} & S
\end{array}
\]

Clearly \( \varphi' \) has order \( d \) and \( S/(m \mathbb{S} + (t_1, \ldots, t_m)S) \) is artinian. We will call such a pair \((R[t_1, \ldots, t_m], \varphi')\) an artinian factorization of \( \varphi \). Regularity of an \( S \)-complex computed over \( \varphi \) is the same as that computed over a factorization of \( \varphi \) because of the following general statement.

**Theorem 5.1.** Let \( R[t_1, \ldots, t_m] \) be an arbitrary polynomial extension of \( R \) (where \( m \geq 1 \), the variables \( t_i \) have degree \( g \)). Let \( \varphi' : R[t_1, \ldots, t_m] \to S \) be a homomorphism of order \( d \) such that \( \varphi \) factors through \( \varphi' \). Then for any \( M \in \mathbb{D}_+(\text{Gr} \mathbb{S}) \), there is an equality

\[
\text{reg}_\varphi M = \text{reg}_{\varphi'} M.
\]

**Proof.** Let \( \tilde{\varphi} : R^{(1/d)} \to S \) be the induced homogeneous morphism. From the definition, clearly \( \text{reg}_\varphi M = \text{reg}_{\tilde{\varphi}} M \). Moreover, \( (R[t_1, \ldots, t_m])^{(1/d)} \cong R^{(1/d)}[t_1', \ldots, t_m'] \), a polynomial extension of \( R^{(1/d)} \), where the new variables \( t'_i \) have degree \( dg \). Therefore we can assume from the beginning that \( \varphi \) is a homogeneous morphism, namely \( d = 1 \). So \( g = hc \).
Let $F$ be the minimal graded free resolution of $k$ over $R$. Denote $Q = R[t_1, \ldots, t_m]$. Let $K[t; Q]$ be the Koszul complex of the sequence $t = t_1, \ldots, t_m$. We know that $K[t; Q]$ is the minimal graded free resolution of $R$ over $Q$.

Since $R \rightarrow Q$ is flat, we have the following isomorphisms in $\mathcal{D}^f(\text{Gr } Q)$

$$F \otimes^L_R K[t; Q] \simeq k \otimes^L_R K[t; Q] \simeq k \otimes^L_R R \simeq k.$$ 

Let $m$ be the graded maximal ideal of $Q$. Since $\varphi$ factors through $\varphi'$, we have $mS \subseteq mQ S$. Let $x$ be a finite sequence of elements of degree $g$ in $S$ which minimally generates $n^c$ modulo $mS$. Then we have isomorphisms in $\mathcal{D}^f(\text{Gr } Q)$:

$$k \otimes^L_Q K[x; M] \simeq (F \otimes^L_R K[t; Q]) \otimes^L_Q K[x; M] \simeq F \otimes^L_R (K[t; Q] \otimes^L_Q K[x; M]) \simeq F \otimes^L_R K[\varphi'(t), x; M] \simeq k \otimes^L_R K[x; K[\varphi'(t); M]].$$

Note that $x$ also generates $n^c$ modulo $mQ S$, therefore, Proposition 4.3 and Lemma 4.4 imply two equalities at two ends in the following strand

$$\text{reg}_{\varphi'} M = \text{reg}_{\varphi', x} M = \text{reg}_{\varphi} K[\varphi'(t); M] = \text{reg}_{\varphi} M.$$ 

The middle equation follows from the isomorphisms above. The proof of the theorem is now complete. $\square$

Now we will show the stability of regularity over a homomorphism with respect to taking fractional Veronese. The second part of the next result can be seen as a connection between working with the fractional Veronese, namely computing $\text{reg}_{\varphi} M$ and working with the Veronese gradings, namely computing $\text{reg}_{\hat{\varphi}}^g M$.

**Theorem 5.2.** The following statements hold true for any complex $M \in \mathcal{D}^f(\text{Gr } S)$.

(i) For any $t \geq 1$, denote by $\varphi^{(1/t)}$ the induced homomorphism $R^{(1/t)} \rightarrow S^{(1/t)}$. Then we have

$$\text{reg}_{\varphi} M = \text{reg}_{\varphi^{(1/t)}} M^{(1/t)}.$$ 

(ii) Denote by $\hat{\varphi}$ the induced homogeneous morphism $R \rightarrow S^{(d)}$. Then

$$\text{reg}_{\hat{\varphi}}^g M \leq \text{reg}_{\varphi} M \leq \text{reg}_{\hat{\varphi}}^g M + \frac{d - 1}{dg}.$$ 

**Proof.** For (i): let $x'$ be the sequence $x_1, \ldots, x_n$ where $x_1, \ldots, x_n$ are now elements of $S^{(1/t)}$. It is not hard to see that in $\mathcal{D}^f(\text{Gr } S^{(1/t)})$,

$$K[x'; M^{(1/t)}] = K[x; M]^{(1/t)}.$$ 

Denote $t^{\varphi}_i (M) = \sup \{ j : \beta^{\varphi}_i (M) \neq 0 \}$ for each $i \in \mathbb{Z}$. Using the exactness of the functor $(\cdot)^{(1/t)}$, we have

$$k \otimes^L_{R^{(1/d)}} K[x'; M^{(1/t)}] \simeq k^{(1/t)} \otimes^L_{R^{(1/d)}} K[x; M]^{(1/t)} \simeq (k \otimes^L_{R^{(1/d)}} K[x; M])^{(1/t)}.$$ 


Therefore $t^{(1/d)}_i(M^{(1/d)}) = t \cdot t^\varphi_i(M)$ for all $i$. The desired equality follows immediately.

For (ii): choose an artinian factorization $(R[t_1, \ldots, t_m], \varphi')$ of $\varphi$. By Theorem 5.1, we get $\text{reg}_\varphi M = \text{reg}_{\varphi'} M$ and $\text{reg}^\varphi M = \text{reg}^{\varphi'} M$. Therefore we can replace $R$ by $R[t_1, \ldots, t_m]$ and assume that $S/mS$ is artinian.

Applying Remark 4.6, we see that

$$
\text{reg}_\varphi M = \sup \left\{ \frac{j - idg}{dg} : H_i(k \otimes_{R^{(1/d)}} M)_j \neq 0 \right\},
$$

and

$$
\text{reg}^\varphi M = \sup \left\{ \frac{j - ig}{g} : H_i(k \otimes_R^L \varphi M)_j \neq 0 \right\}.
$$

Denote

$$
t_i = \sup \{ j : H_i(k \otimes_{R^{(1/d)}} M)_j \neq 0 \},
$$

$$
s_i = \sup \{ j : H_i(k \otimes_R^L \varphi M)_j \neq 0 \},
$$

for each $i$. Then $\text{reg}_\varphi M = \sup_{i \in \mathbb{Z}} \{(t_i - idg)/dg\}$. Let $G$ be the minimal graded free resolution of $k$ over $R^{(1/d)}$. Then by Theorem 3.3, $\Phi(G)$ is the minimal graded free resolution of $k$ over $R$. Hence

$$
\Phi(k \otimes_{R^{(1/d)}} M) = \Phi(G \otimes_{R^{(1/d)}} M) \simeq \Phi(G) \otimes_R \varphi M \simeq k \otimes_R^L \varphi M
$$

as $S^{(d)}$-complexes. Therefore $H_i(k \otimes_R^L \varphi M)_j \neq 0$ for some $j$ if and only if $H_i(k \otimes_{R^{(1/d)}} M)_{dj+r} \neq 0$ for some $0 \leq r \leq d-1$. In particular, we get

$$
s_i \leq \frac{t_i}{d} \leq s_i + \frac{d-1}{d},
$$

for each $i$. Combining with

$$
\text{reg}^\varphi M = \sup_{i \in \mathbb{Z}} \left\{ \frac{s_i - ig}{g} \right\},
$$

the conclusion follows.

Next, we consider the behavior of regularity over a homomorphism along polynomial extensions. Let $S[t]$ be a polynomial extension of $S$ where $\deg t = h$. Let $R[t']$ be a polynomial extension of $R$ where $\deg t' = g$. Let $\varphi[t]$ be the morphism $R[t'] \to S[t]$ which restricts to $\varphi$ on $R$ and $\varphi[t](t') = t^e$. Clearly $\varphi[t]$ also has order $d$. Denote $M[t] = M \otimes_S S[t]$ for each $M \in \mathcal{D}_+^l(\text{Gr} S)$. We have

**Proposition 5.3.** For any $M \in \mathcal{D}_+^l(\text{Gr} S)$, there is an equality

$$
\text{reg}_{\varphi[t]} M[t] = \text{reg}_{\varphi} M + \left(1 - \frac{h}{dg}\right).
$$
Proof. First, consider the case $S/mS$ is artinian. Then $S[t]/(m + (t'))S[t]$ is also artinian. We know from Remark 4.6 that

$$\text{reg}_\varphi M = \sup \left\{ \frac{j - idg}{dg} : H_1(k \otimes_{R^{(1/d)}}^L M)_j \neq 0 \right\},$$

where $R^{(1/d)}[z] \cong (R[t'])^{(1/d)}$ is a polynomial extension of $R^{(1/d)}$ with deg $z = dg$. Denote $A = R^{(1/d)}$.

Since $z$ maps to $t^c \in S[t]$, we have isomorphisms of graded free $S[z]$-modules $S[t] \cong \bigoplus_{j=0}^{c-1} S[z]t^j \cong \bigoplus_{j=0}^{c-1} S[z](-hj)$. Therefore $M[t] \cong \bigoplus_{j=0}^{c-1} (M[z])(-hj)$ as $S[z]$-complexes. We obtain that $k \otimes_{A[z]} M[t] \cong \bigoplus_{j=0}^{c-1} (k \otimes_{A[z]} M[z])(-hj)$. Let $F$ be the minimal graded $S$-free resolution of $k \otimes_A^{L} M$. Then because of the flatness of $S \to S[z]$, $F \otimes S S[z]$ is the minimal graded $S[z]$-free resolution of $k \otimes_{A[z]}^{L} M[z]$. In particular

$$\sup \{ j : H_i(k \otimes_{A}^{L} M)_j \neq 0 \} = \sup \{ j : H_i(k \otimes_{A[z]}^{L} M[z])_j \neq 0 \}.$$

Hence $\text{reg}_{\varphi[t]} M[t] = \text{reg}_\varphi M + (hc - 1)/dg$, as desired.

Now consider the general case. Let $(R[t'_1, \ldots, t'_m], \varphi')$ be an artinian factorization of $\varphi$. Then from Theorem 5.1, $\text{reg}_{\varphi[t]} M = \text{reg}_{\varphi'} M$. Denote by $(\varphi[t])'$ the induced homomorphism $R[t'_1, \ldots, t'_m, t'] \to S[t]$. The two morphisms $\varphi'[t]$ and $(\varphi[t])'$ are equal, so we have $\text{reg}_{\varphi'[t]} M[t] = \text{reg}_{(\varphi[t])'} M[t]$. Now applying the previous case for the morphism $\varphi'$, we obtain the desired conclusion. \hfill \Box

The next result is the key to construct certain complexes in the proof of Theorem 1.2.

**Theorem 5.4.** Let $(R', m', k') \xrightarrow{\pi'} (R, m, k) \xrightarrow{\pi} (S, n, l)$ be (non-homogeneous) homomorphisms of graded algebras such that order $\pi' = d'$ and order $\pi = d$. Assume that $R', R, S$ is generated as an algebra in the degree $g', g, h \geq 1$, respectively. Let $L \in D^b_{f}(\text{Gr} R)$ and $N \in D^b_{f}(\text{Gr} S)$ be such that $L, N \neq 0$, $\text{reg}_\pi L < \infty$ and $\text{reg}_\pi N < \infty$. Denote by $P$ the complex of graded $S$-modules $L^{(1/d)} \otimes_{R^{(1/d)}}^{L} N$. Then we also have

$$\text{reg}_{\pi \circ \pi'} P < \infty.$$

**Proof.** Replacing $\pi', \pi$ by suitable artinian factorizations and polynomial extensions, we can assume that $R/m'R$ and $S/mS$ are artinian rings.

Indeed, choose an artinian factorization $(R[t_1, \ldots, t_m], \tau)$ of $\pi$. Let $R'[z_1, \ldots, z_m]$ be a polynomial extension of $R'$ such that degree of each new variable is $g'$. Extend $\pi'$ to a homomorphism $\pi'' = \pi'[t_1, \ldots, t_m] : R'[z_1, \ldots, z_m] \to R[t_1, \ldots, t_m]$ mapping $z_i$ to $t'_i$, where $c$ is the unique integer such that $d'g' = gc$. Then $\pi''$ also has order $d'$. \hfill \Box
Choose an artinian factorization \((R'[y_1, \ldots, y_n, z_1, \ldots, z_m], \tau')\) of \(\pi''\). We have the following diagram.

\[
\begin{array}{ccc}
R'[y_1, \ldots, y_n, z_1, \ldots, z_m] & \xrightarrow{\tau'} & R'[z_1, \ldots, z_m] \\
& \xrightarrow{\pi''} & R[t_1, \ldots, t_m] \\
& & \xrightarrow{\pi} S
\end{array}
\]

Let \(L'' = L \otimes_R R[t_1, \ldots, t_m]\). We will show that the statement of our result is not affected if we replace \(R'\) by \(R'[y_1, \ldots, y_n, z_1, \ldots, z_m]\), \(R\) by \(R[t_1, \ldots, t_m]\) and \(L\) by \(L''\).

Firstly, from Theorem 5.1 and Proposition 5.3,

\[
\text{reg}_{\pi'} L'' = \text{reg}_{\pi''} L'' = \text{reg}_{\pi'} L + m(1 - g/d'g') < \infty
\]

and

\[
\text{reg}_{\pi} N = \text{reg}_{\tau'} N < \infty.
\]

We also have \(L''^{(1/d)} \cong L^{(1/d)} \otimes_{R^{(1/d)}} R^{(1/d)}[t'_1, \ldots, t'_m]\); the latter ring is a polynomial extension of \(R^{(1/d)}\) with variables of degree \(dq\). Denote by \(P'\) the \(S\)-complex \(L''^{(1/d)} \otimes_{R[1, \ldots, t_m]^{(1/d)}} N\), we have isomorphisms in \(D'((\mathcal{R} S))\):

\[
P' \cong (L^{(1/d)} \otimes_{R^{(1/d)}} R^{(1/d)}[t'_1, \ldots, t'_m]) \otimes_{R^{(1/d)}} N \cong L^{(1/d)} \otimes_{R^{(1/d)}} N \cong P.
\]

But \(\tau' \circ \tau\) is an artinian factorization of \(\pi' \circ \pi\), hence \(\text{reg}_{\pi' \circ \pi } P = \text{reg}_{\pi' \circ \pi } P'\). Therefore we can make the artinian assumptions from above.

Furthermore, replacing \(R\) by \(R^{(1/d')}\), \(R'\) by \(R^{(1/dd')}\), \(N\) by \(N^{(1/d')}\) and using Theorem 5.2, we can assume that \(d = d' = 1\). Now \(P = L \otimes_R N\) and we have to show that \(\text{reg}_{\pi' \circ \pi} P < \infty\).

From the associativity of derived tensor product

\[
(k' \otimes_{R'} L) \otimes_R N \cong k' \otimes_{R'} (L \otimes_R N),
\]

one obtains the standard spectral sequence

\[
\text{Tor}_p^R(\text{Tor}_q^{R'}(k', L), N) \Rightarrow \text{Tor}_p^{R'}(k', P).
\]

Since \(R/m'R, S/mS\) are artinian and \(\text{Tor}_p^{R'}(k', L)\) has finite length as an \(R\)-module, we have for all \(i, j \in \mathbb{Z}\):

\[
\dim \text{Tor}_i^{R'}(k', P)_j \leq \sum_{p+q=i} \dim \text{Tor}_p^R(\text{Tor}_q^{R'}(k', L), N)_j
\]

\[
\leq \sum_{p+q=i} \sum_{u} \dim_k \text{Tor}_q^R(k', L)_u \dim_l \text{Tor}_p^R(k(-u), N)_j
\]

\[
= \sum_{p+q=i} \sum_{u} \dim_k \text{Tor}_q^R(k', L)_u \dim_l \text{Tor}_p^R(k, N)_{j-u}.
\]
The second inequality follows by filtering $\text{Tor}_q^R(k', L)$ appropriately.

Define $t_i''(L) = \sup\{j : H_i(k \otimes L)_{j} \neq 0\}$ and similarly $t_i''(N), t_i''(P)$. From the last inequality, we get for all $i$,
\[
t_i''(P) \leq \sup_{p \geq \inf N} \{t_i''(L) + t_i''(N)\}
\leq \sup_{p \geq \inf N} \{g'(i - p) + g' \text{reg}_\pi L + gp + g \text{reg}_\pi N\}
\leq g'i + g' \text{reg}_\pi L + g \text{reg}_\pi N + (g - g') \inf N.
\]
The last inequality follows since $g' \geq g$, which in turn holds because $R' \to R$ is degree-preserving. Finally, from Remark 4.6 and the last string we conclude that
\[
\text{reg}_{\text{reg}} P \leq \text{reg}_{\text{reg}} L + \frac{g \text{reg}_\pi N + (g - g') \inf N}{g'} < \infty.
\]

\[\square\]

6. Proofs of Theorem 1.4

Recall that $R, S$ are standard graded algebras over $k, l$ respectively. The main work in the proof of Theorem 1.4 is done via the following statement, which was suggested to the authors by Srikanth Iyengar.

**Proposition 6.1.** For any complex $G \in \mathcal{D}^f_1(\text{Gr } R)$, we have
\[
\text{reg}_R G \leq \sup_{i \in \mathbb{Z}} \{\text{reg}_R H_i(G) - i\}.
\]

**Proof.** Since $H_i(G) = 0$ for $i < 0$, the minimal graded free resolution of $G$ can be chosen to be a bounded below complex $F$ with $F_i = 0$ for $i < \inf G$. Observe that for any $m \in \mathbb{Z}$, we have

(i) $F[m]$ is the minimal free resolution of $G[m]$, hence,
\[
\text{reg}_R G[m] = \sup_{i \in \mathbb{Z}} \{\text{reg}_R F[m]_i - i\} = \text{reg}_R G + m,
\]

(ii) $\text{reg}_R H_i(G[m]) - i = \text{reg}_R H_{i+m}(G) - (i + m) + m$, so
\[
\sup_{i \in \mathbb{Z}} \{\text{reg}_R H_i(G[m]) - i\} = \sup_{i \in \mathbb{Z}} \{\text{reg}_R H_i(G) - i\} + m.
\]

Therefore by replacing $G$ by $G[m]$, both sides of the inequality in question increase by $m$. Hence we can assume that $\inf G = 0$.

Denote $B_i = B_i(F) = \text{Im}(F_{i+1} \to F_i), Z_i = Z_i(F) = \text{Ker}(F_i \to F_{i-1})$ and $H_i = H_i(F) = H_i(G)$. Since $F$ is minimal, for each $i \geq 0$, $0 \to B_i \to F_i \to F_i/B_i \to 0$ is the beginning of the minimal graded free resolution of $F_i/B_i$. Therefore
\[
\text{reg}_R F_i \leq \text{reg}_R F_i/B_i,
\]
and
\[
\text{reg}_R B_i \leq \text{reg}_R F_i/B_i + 1.
\]
On the other hand, we have the following exact sequence
\[
0 \to H_i \to F_i/B_i \to B_{i-1} \to 0.
\]
Hence
\[ \text{reg}_R F_i/B_i \leq \max\{\text{reg}_R H_i, \text{reg}_R B_{i-1}\}. \]

Combining the last two inequalities, one has
\[ \text{reg}_R B_i \leq \max\{\text{reg}_R H_i + 1, \text{reg}_R B_{i-1} + 1\}. \]

By induction on \( i \geq 0 \), it is easy to see that
\[ \text{reg}_R B_i \leq \max\{\text{reg}_R H_i + 1, \text{reg}_R B_{i-1} + 1, \text{reg}_R H_{i-1} + 1, \ldots, \text{reg}_R H_0 + i + 1\}. \]

(6.3)

Now using (6.1), the sequence (6.2), and (6.3), we have for every \( i \geq 0 \),
\[ \text{reg}_R F_i \leq \text{reg}_R F_i/B_i \leq \max\{\text{reg}_R H_i, \text{reg}_R B_{i-1}\} \leq \max\{\text{reg}_R H_i, \text{reg}_R H_{i-1} + 1, \ldots, \text{reg}_R H_0 + i\}. \]

This implies that
\[ \text{reg}_R F_i - i \leq \max_{0 \leq j \leq i} \{\text{reg}_R H_j - j\}. \]

The proof of the result is now complete.

\[ \square \]

**Remark 6.2.** The strict inequality in Proposition 6.1 may happen. For example, take \( S = k[x]/(x^2) \). Consider the Koszul complex on \( x \) of \( S \):
\[ G : 0 \to S(-1) \xrightarrow{x} S \to 0. \]

\( G \) is the minimal resolution of \( G \) itself, so clearly \( \text{reg}_R G = 0 \). On the other hand, \( H_1(G) \cong (xS)(-1), H_0(G) = k \), hence \( \sup_{i \in \mathbb{Z}} \{\text{reg}_S H_i(G) - i\} = \text{reg}_S H_1(G) - 1 = 1 \).

At this point, we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** By abuse of notation, denote \( k = R/m \). Let \( y \) be minimal generators of \( n \) modulo \( mS \) and denote \( N = K[y; M] \). We have an equality between Poincaré series; see [4, Lemma 1.5.3]:
\[ P^S_{k \otimes_R N}(t, y) = P^R_{k}(t, y)P^S_{N}(t, y). \]

Denote \( t_i = \sup\{j : H_i(k \otimes_R N) j \neq 0\} \). From [6, Lemma 1.5.6], \( H_i(k \otimes_R N) \) is an \( S/n \)-module. But \( S \) is a Koszul algebra, hence \( \text{reg}_S H_i(k \otimes_R N) = t_i \). Together with Proposition 6.1, we have
\[ \text{reg}_S k \otimes_R N \leq \sup_{i \in \mathbb{Z}} \{\text{reg}_S H_i(k \otimes_R N) - i\} \]
\[ = \sup_{i \in \mathbb{Z}} \{t_i - i\} = \text{reg}_\varphi M. \]

The last equality follows from the definition of Castelnuovo-Mumford regularity. Combining with the equality of Poincaré series, the last inequality implies that \( \text{reg}_S N + \text{reg}_R k \leq \text{reg}_\varphi M \). Since \( N \in D^b_c(\text{Gr} S) \) and \( N \neq 0 \), we infer that \( \text{reg}_S N > -\infty \), and hence \( \text{reg}_R k < \infty \). By Avramov-Peeva’s theorem \( R \) is Koszul.

\[ \square \]

The proof of Theorem 1.4 also has the following corollary, which extends and gives a new proof to the main result of [3].
Corollary 6.3. Let $R \to S$ be a finite homomorphism of Koszul algebras over $k$. Let $M$ be an $S$-module. Then $\text{reg}_S M \leq \text{reg}_R M$.

Proof. Replacing $R$ by a suitable polynomial extension and using Theorem 5.1, we can assume that $R \to S$ is surjective. It is also harmless to assume that $\text{reg}_R M$ is finite (which is actually the case since $R$ is Koszul, but we will not recourse to this fact). Hence from the proof of Theorem 1.4 where $y$ is now $\emptyset$, we have

$$\text{reg}_S M = \text{reg}_S M + \text{reg}_R k \leq \text{reg}_R M,$$

as claimed. □

Denote by $\text{reg}_R$ the absolute Castelnuovo-Mumford regularity of $R$. We have:

Corollary 6.4. Let $R$ be standard graded over a field $k$, where $k$ is $F$-finite of characteristic $p > 0$. Let $e > 0$ be given. If for some $i$ with $q^i \geq (\text{reg}_R + 1)/2$, we have that $\text{reg}_R R^{(q^i)} < \infty$, then $R$ is a Koszul algebra.

Proof. For $q^i \geq (\text{reg}_R + 1)/2$, it is well-known that $R^{(q^i)}$ is a Koszul algebra; see [11, Theorem 2]. Now $\phi^e : R \to R^{(q^i)}$ is a finite homomorphism of standard graded rings. Since $k$ is $F$-finite and $\text{reg}_R R^{(q^i)} < \infty$, using Theorem 1.4 with $M = R^{(q^i)}$, we conclude that $R$ is also Koszul. □

Remark 6.5. There are several applications of Theorem 1.4. If $R \to S$ is a finite morphism of standard graded $k$-algebras such that $\text{reg}_R S < \infty$, and $S$ is a Koszul algebra then $R$ is also a Koszul algebra. A similar observation was made in [10, Theorem 3.2]. In particular, in the case $S = k$ and $R \to k$ is the canonical surjection, Theorem 1.4 gives the Avramov-Peeva’s characterization of Koszul algebras. (But one does not obtain a new proof to this last result.)

7. PROOFS OF THE MAIN RESULTS

Now we will prove Theorem 1.2. This theorem combines with Remark 4.6 and Theorem 5.2(ii) to give Theorem 1.1, by letting $\psi = \phi^e$. The construction via derived tensor in the following argument grew out of a beautiful idea in the proof of [5, Theorem 5.1].

Proof of Theorem 1.2. Let $d$ be the order of $\psi$. Define inductively a complex $M^i \in D^+_+(\text{Gr} R)$ as follow: $M^1 = M$, and for $i \geq 1$

$$M^{i+1} = (M^i)^{(1/d)} \otimes_{R^{(1/d)}} M.$$

For $i \geq 1$, the action of $R$ on $M^{i+1}$ is defined as follows. Consider the diagram $R \xrightarrow{\psi^i} R \xrightarrow{\psi} R$. In the derived tensor product, we view $M$ as a complex over the ring $R$ on the right and $M^i$ as a complex over the middle ring of the diagram. The action of $R$ on $M^{i+1} = (M^i)^{(1/d)} \otimes_{R^{(1/d)}} M$ is induced by the action of $R$ on the second variable.

It is easy to see that $M^i \neq 0$ for every $i \geq 1$. We will show by induction that for every $i \geq 1$, $\text{reg}_{\psi^i} M^i < \infty$. For $i = 1$, this is known from above. For $i \geq 2$,
it is enough to apply Theorem 5.4 for the homomorphisms $R \xrightarrow{\psi^{i-1}} R \xrightarrow{\psi} R$ and complexes $L = M^{i-1}$ and $N = M$.

Choose $d^i \geq (\text{reg } R + 1)/2$, then $R^{(d^i)}$ is Koszul; see [11, Theorem 2] and [10, Theorem 3.6 and Remark 3.7]. From Theorem 5.2(ii) we have

$$\text{reg}_{\psi^i} M^i \leq \text{reg}_{\psi^i} M < \infty.$$  

Applying Theorem 1.4 for the morphism $R \xrightarrow{\tilde{\psi}^i} R^{(d^i)}$ and the complex $\psi^i M^i$, we conclude that $R$ is Koszul.

\[ \square \]

**Remark 7.1.** The following example gives a class of graded algebras over a field $k$ of characteristic 0 which possess endomorphisms of order at least 2. Let $\Lambda$ be a positive affine monoid, namely, $\Lambda$ is a finitely generated submonoid of $\mathbb{N}^m$ for some $m \geq 1$. Let $R = k[\Lambda]$ be the affine monoid ring of $\Lambda$ with the inherited $\mathbb{N}^m$-grading. Let $a_1, \ldots, a_n$ be the minimal generating set of $\Lambda$, and denote by $t^{a_1}, \ldots, t^{a_n}$ the corresponding generators of $R$. For any $d \geq 2$, the endomorphism of $R$ mapping $k$ to itself and $t^{a_i}$ to $t^{da_i}$ for all $i = 1, \ldots, n$ is an endomorphism of order $d$.

As a consequence of Theorem 1.1, we have:

**Corollary 7.2.** Let $\text{char } k = p > 0$, $k$ be $F$-finite and $R$ be a standard graded $k$-algebra. If for some $e > 0$, $\text{reg}_R R^e R < \infty$, then $R$ is a Koszul algebra.

This result marked a starting point for the research in this paper. We came to this result via an entirely different approach; because of the simplicity of this approach, we record it below. The involving techniques were developed in [8, Section 5].

**Alternative proof of Corollary 7.2.** Denote $V_i = V_i(q, R)$ for $0 \leq i \leq q - 1$. By the hypothesis, $r = \text{reg}_R R^e R = \max\{\text{reg}_R V_i : i \in \{0, 1, \ldots, q - 1\}\} < \infty$. Let $M$ be a graded $R$-module. Firstly, we will show that

$$\text{reg}_R M^{(q)} \leq \left\lceil \frac{\text{reg}_R M}{q} \right\rceil + \ell.$$  

(7.1)

Of course, it suffices to consider the case $\text{reg}_R M < \infty$. Let $G_i$ be the minimal graded $R$-free resolution of $M$. Then $\text{reg} G_i \leq i + \ell$ for all $i \geq 0$, where $\ell = \text{reg}_R M$.

From the exactness of the complex $G^{(q)}_i \to M^{(q)} \to 0$ and Lemma 2.1(ii), we have

$$\text{reg}_R M^{(q)} \leq \sup\{\text{reg}_R G^{(q)}_i : i \geq 0\}.$$

(7.2)

Assume that $G_i = \bigoplus R(-j)^{\beta_{ij}}$. Observe that $R(-j)^{(q)} = V_{u_j}(-\lceil j/q \rceil)$ where $u_j = q\lceil j/q \rceil - j$. Thus we get

$$G^{(q)}_i = \bigoplus_j (V_{u_j}(-\lceil j/q \rceil))^{\beta_{kj}}.$$  

Hence it is clear that

$$\text{reg}_R G^{(q)}_i \leq \left\lceil \frac{\text{reg}_R G_i}{q} \right\rceil + \ell \leq \left\lceil \frac{i + \ell}{q} \right\rceil + \ell.$$  

(7.3)
Together with (7.2), this implies that
\[
\text{reg}_R M^{(q)} \leq \sup \left\{ \left\lceil \frac{i + \ell}{q} \right\rceil + r - i : i \geq 0 \right\}
\leq \left\lceil \frac{\ell}{q} \right\rceil + r = \left\lceil \frac{\text{reg}_R M}{q} \right\rceil + r,
\]
proving (7.1).

Note that \((R^{(q^i)})^{(q)} = R^{(q^{i+1})}\). Therefore using the inequality (7.1) and induction we have \(\text{reg}_R R^{(q^i)} < \infty\) for all \(i > 0\). Hence taking \(q^i \geq (\text{reg} R + 1)/2\), and applying Corollary 6.4, we conclude that \(R\) is also Koszul. \(\square\)

We close this section with some examples to illustrate Theorem 1.1.

**Example 7.3.** Let \(k = \mathbb{Z}/(2)\), \(R\) be defined by monomial relations. In the following, the notation \(V_i\) stands for \(V_i(2, R)\) for \(i = 0, 1\).

(i) If \(R = k[x, y]/(xy)\), then \(V_0 = R, V_1 = R/(y) \oplus R/(x)\) where the generators of the second module are \(\bar{x}, \bar{y}\), respectively. Therefore \(\text{reg}_R V_0 = \text{reg}_R V_1 = 0\). The ring \(R\) is Koszul.

(ii) More generally, consider \(R = Q/I_G = k[\Delta]\), where \(G\) is a finite simple graph on \(n\) vertices, \(Q = k[x_1, \ldots, x_n]\) and \(I_G = (x_ix_j : \{i, j\} \text{ is an edge of } G)\). Let \(\Delta\) be the associated simplicial complex, i.e. a subset \(C\) of \([n]\) belongs to \(\Delta\) if and only if \(\prod_{i \in C} x_i \notin I_G\). Then for \(i = 0, 1,\)
\[
V_i = \bigoplus_{0 \leq t \leq \frac{2n}{2r+1}} \bigoplus_{C \in \Delta \atop |C| = 2t+i} [R/(x_j : j \in N(C))](−t),
\]
where \(N(C)\) denotes the sets of vertices of \(G\) which are neighbors in \(G\) of some element in \(C\). By convention \(N(\emptyset) = \emptyset\). We prove the equality for \(i = 0\); similar arguments apply to the remaining case. Observe that \(V_0\) is the direct sum of the cyclic modules \(R \prod_{i \in C} x_i\), where \(C \in \Delta\) and \(|C|\) is even. Now for each such face \(C\), let \(t\) be the degree of \(\prod_{i \in C} x_i\) in \(V_0\), we have an isomorphism \(R \prod_{i \in C} x_i \cong [R/(x_j : j \in N(C))](−t)\). The formula for \(V_0\) is now proved.

It is well-known that \(R\) is Koszul and for every subset \(I\) of \([n]\), it holds that \(\text{reg}_R R/(x_i : i \in I) = 0\). Hence for \(i = 0, 1,\) we have
\[
\text{reg}_R V_i = \max \{t : \text{there exists } C \in \Delta \text{ such that } |C| = 2t + i\}.
\]

(iii) If \(R = k[x]/(x^3)\), then \(V_0 = k[x]/(x^2), V_1 = k\). The minimal graded free resolution of \(V_1 = k\) over \(R\) is
\[
T : \cdots \xrightarrow{x} R(−3) \xrightarrow{x^2} R(−1) \xrightarrow{x} R.
\]
Concretely \(T_{2i} = R(−3i), T_{2i+1} = R(−(3i+1))\) for all \(i \geq 0\). In particular, \(\text{reg}_R V_1 = \infty\) and \(R\) is not Koszul.
8. Remarks on the local case and open questions

We can try to extend our main result Theorem 1.1 to the local situation. First, let us recall some notations; the reader may consult [12] for more details. Let \((R, m, k)\) be a Noetherian local ring with maximal ideal \(m\) and residue field \(R/m \cong k\) or a standard graded \(k\)-algebra with graded maximal ideal \(m\). For each finitely generated \(R\)-module \(M\), let \(F\) be the minimal free resolution of \(M\). Then \(F\) has the filtration \(\{F^i F\}_{i \geq 0}\) where \(F^i F\) is the following complex

\[
\cdots \rightarrow F_{i+1} \rightarrow F_i \rightarrow m F_{i-1} \rightarrow \cdots \rightarrow m^i F_0 \rightarrow 0.
\]

The linear part of \(F\), denoted by \(\text{lin}^R F\) is the associated graded complex of the filtration \(\{F^i F\}_{i \geq 0}\). Concretely, \((\text{lin}^R F)_i = \text{gr}_m(F)(-i)\) for each \(i \geq 0\). The linearity defect \(\text{ld}_R M\) of \(M\) is defined by

\[
\text{ld}_R M = \sup \{ i : H_i(\text{lin}^R F) \neq 0 \}.
\]

The local ring (graded algebra) \(R\) is called Koszul if \(\text{ld}_R k = 0\). If \(R\) is standard graded, this is equivalent to the usual notion of a Koszul algebra. In contrast to the graded case, it is an open question whether for a local ring \((R, m, k)\), \(\text{ld}_R k < \infty\) implies that \(R\) is a Koszul ring; see [12, Question 1.14]. Our next result says that \(\text{ld}_R^{\phi} R < \infty\) for some \(e \geq 1\) implies \(\text{ld}_R k < \infty\).

First we introduce an invariant which is modelled after the number \(\nu(R)\) of Takahashi and Yoshino in [20, Section 3].

**Definition 8.1.** For each maximal sequence of \(R\)-regular elements \(y = y_1, \ldots, y_t \in m^2\), where \(t = \text{depth} R\), note that \(H^0_m(R/(y)) \cap m^s(R/(y)) = 0\) for \(s \gg 0\). Denote by \(\mathfrak{r}(R)\) the smallest number \(s\) such that \(H^0_m(R/(y)) \cap m^s(R/(y)) = 0\) for certain maximal sequence of \(R\)-regular elements \(y = y_1, \ldots, y_t \in m^2\).

Note that \(\nu(R) \leq \mathfrak{r}(R) < \infty\).

**Theorem 8.2.** Let \((R, m, k)\) be an \(F\)-finite local ring of characteristic \(p > 0\). If \(\text{ld}_R^{\phi} R < \infty\) for some \(e \geq \mathfrak{r}(R)\) then \(\text{ld}_R k < \infty\).

Note that one has the following useful lemma.

**Lemma 8.3.** Let \(M\) be a finitely generated \(R\)-module. Let \(x \in m^2\) be a regular element with respect to \(M\). Then

\[
\text{ld}_R M/xM = \text{ld}_R M + 1.
\]

**Proof.** Denote \(\text{ld}_R M = \ell\). Let \(G\) be the minimal free resolution of \(M\) over \(R\). The morphism \(\theta : G \xrightarrow{\phi} G\) lifts the morphism \(M \xrightarrow{x} M\). Using mapping cone on the exact sequence

\[
0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0,
\]

then \(W = G \oplus G[-1]\) is minimal free resolution of \(M/xM\) over \(R\).

Since \(x \in m^2\), one has \(\theta(G) \subseteq m^2 G\). Therefore \(\text{lin}^R(W) = \text{lin}^R G \oplus (\text{lin}^R G)[-1]\). Hence from \(\text{ld}_R M = \ell\), we get \(H_{i+1}(\text{lin}^R(W)) = H_i(\text{lin}^R G) \neq 0\) and \(H_i(\text{lin}^R(W)) = H_{i-1}(\text{lin}^R G) = 0\) for \(i \geq \ell + 2\). So \(\text{ld}_R M/xM = \ell + 1\), as desired. □
Proof of Theorem 8.2. We follow the scheme of proof of [20, Corollary 3.4]. Firstly, using arguments similar to Corollary 3.3 in loc. cit., one finds a maximal \( R \)-regular sequence \( \mathbf{y} \subseteq (\mathfrak{m}^2) \) such that \( k \) is isomorphic to a direct summand of the \( R \)-module \( \mathcal{O}R/(\mathbf{y}) \). Therefore \( \mathrm{ld}_R k \leq \mathrm{ld}_R \mathcal{O}R/(\mathbf{y}) \). Now from Lemma 8.3, we have

\[
\mathrm{ld}_R \mathcal{O}R/(\mathbf{y}) = \mathrm{ld}_R \mathcal{O}R + \text{depth } R.
\]

Therefore using the hypothesis, \( \mathrm{ld}_R k \leq \mathrm{ld}_R \mathcal{O}R + \text{depth } R < \infty \). The proof is now complete. \( \square \)

Finally, we introduce several open questions relating to the main results.

**Question 8.4.** Let \( R \) be a standard graded \( k \)-algebra where \( k \) is \( F \)-finite of positive characteristic \( p \). Is it true that if for some \( e > 0 \), \( \text{reg}_R R^{(e)} < \infty \), then \( R \) is Koszul?

Note that the analog of this question for regular rings is not true, that is, it can happen that \( \text{pd}_R R^{(e)} < \infty \) for every \( e > 0 \), but \( R \) is not regular. For example, let \( k = \mathbb{Z}/(p) \) and \( R = k[x, y]/(xy) \). It is not hard to check that \( \phi^e : R \rightarrow R^{(e)} \) is an isomorphism for every \( e > 0 \). This example also shows that the analog of Corollary 6.4 for regular rings and projective dimension does not hold. On the other hand, we do not know of any counterexample to Question 8.4.

The majority of results for Frobenius of local rings are also applied for contracting endomorphisms; see, e.g., [5]. An endomorphism \( \psi : (R, \mathfrak{m}) \rightarrow (R, \mathfrak{m}) \) is called contracting if for every (homogeneous) element \( a \in R \), the sequence \( \{\psi^i(a)\}_{i \geq 1} \) converges in the \( \mathfrak{m} \)-adic topology of \( R \). For example, Frobenius endomorphism and more generally, the endomorphisms of order at least 2 considered in this paper are contracting. The homomorphism \( \varphi : k[x, y] \rightarrow k[x, y] \) given by \( \varphi(x) = x^2, \varphi(y) = y^3 \), is a contracting endomorphism which has no order.

**Question 8.5.** Generalize Theorem 1.2 for contracting endomorphisms of a standard graded \( k \)-algebra \( R \).

For local rings, we wonder if the following improvement of Theorem 8.2 is true.

**Question 8.6.** Let \( (R, \mathfrak{m}, k) \) be an \( F \)-finite local ring of characteristic \( p \) > 0. Is it true that whenever \( \mathrm{ld}_R \mathcal{O}R < \infty \) for some \( e > 0 \), then \( \mathrm{ld}_R k < \infty \)?

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