Erdős-Pósa property of minor-models with prescribed vertex sets

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Abstract

A minor-model of a graph $H$ in a graph $G$ is a subgraph of $G$ that can be contracted to $H$. We prove that for a positive integer $\ell$ and a non-empty planar graph $H$ with at least $\ell - 1$ connected components, there exists a function $f_{H,\ell}: \mathbb{N} \to \mathbb{R}$ satisfying the property that every graph $G$ with a family of vertex subsets $Z_1, \ldots, Z_m$ contains either $k$ pairwise vertex-disjoint minor-models of $H$ each intersecting at least $\ell$ sets among prescribed vertex sets, or a vertex subset of size at most $f_{H,\ell}(k)$ that meets all such minor-models of $H$. This function $f_{H,\ell}$ is independent with the number $m$ of given sets, and thus, our result generalizes Mader’s $\mathcal{S}$-path Theorem, by applying $\ell = 2$ and $H$ to be the one-vertex graph. We prove that such a function $f_{H,\ell}$ does not exist if $H$ consists of at most $\ell - 2$ connected components.

1 Introduction

A class $\mathcal{C}$ of graphs is said to have the Erdős-Pósa property if there exists a function $f$ satisfying the following property: for every graph $G$ and a positive integer $k$, either $G$ contains either $k$ pairwise vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{C}$ or a vertex set $T$ of size at most $f(k)$ such that $G - T$ has no subgraph isomorphic to a graph in $\mathcal{C}$. Erdős and Pósa [1] showed that the class of all cycles has the Erdős-Pósa property. Later, several variations of cycles having the Erdős-Pósa property have been investigated; for instance,
directed cycles [15], long cycles [5, 12, 16], cycles intersecting a prescribed vertex set [8, 13, 7], and holes [9]. We refer to a survey of the Erdős-Pósa property by Raymond and Thilikos [14] for more examples.

This property has been extended to a class of graphs that contains some fixed graph as a minor. A minor-model function of a graph $H$ in a graph $G$ is a function $\eta$ with the domain $V(H) \cup E(H)$, where

- for every $v \in V(H)$, $\eta(v)$ is a non-empty connected subgraph of $G$, all pairwise vertex-disjoint
- for every edge $e$ of $H$, $\eta(e)$ is an edge of $G$, all distinct
- for every edge $e = uv$ of $H$, if $u \neq v$ then $\eta(e)$ has one end in $V(\eta(u))$ and the other in $V(\eta(v))$; and if $u = v$, then $\eta(e)$ is an edge of $G$ with all ends in $V(\eta(v))$.

We call the image of such a function an $H$-minor-model in $G$, or shortly an $H$-model in $G$. We remark that an $H$-model is not necessarily a minimal subgraph that admits a minor-model function from $H$. For instance, when $H$ is the one-vertex graph, any connected subgraph is an $H$-model. As an application of Grid Minor Theorem, Robertson and Seymour [16] proved that the class of all $H$-models has the Erdős-Pósa property if and only if $H$ is planar.

Another remarkable result on packing and covering objects in a graph is Mader’s $S$-path Theorem [10]. Mader’s $S$-path Theorem states that for a family $S$ of vertex subsets of a graph $G$, $G$ contains either $k$ pairwise vertex-disjoint paths connecting two distinct sets in $S$, or a vertex subset of size at most $2k - 2$ that meets all such paths. An interesting point of this theorem is that the number of sets in $S$ does not affect on the bound $2k - 2$. A simplified proof was later given by Schrijver [19].

In this paper, we generalize Robertson and Seymour’s theorem on the Erdős-Pósa property of $H$-models and Mader’s $S$-path Theorem together. For a graph $G$ and a multiset $Z$ of vertex subsets of $G$, the pair $(G, Z)$ is called a rooted graph. For a positive integer $\ell$ and a family $Z$ of vertex subsets, an $H$-model $F$ is called an $(H, Z, \ell)$-model if there are at least $\ell$ distinct sets $Z$ of $Z$ that contains a vertex of $F$. A vertex set $S$ of $G$ is called an $(H, Z, \ell)$-deletion set if $G - S$ has no $(H, Z, \ell)$-models. For a graph $G$, we denote by $cc(G)$ the number of connected components of $G$.

We completely classify when the class of $(H, Z, \ell)$-models has the Erdős-Pósa property.

**Theorem 1.1.** For a positive integer $\ell$ and a non-empty planar graph $H$ with $cc(H) \geq \ell - 1$, there exists $f_{H,\ell} : \mathbb{N} \to \mathbb{R}$ satisfying the following property. Let $(G, Z)$ be a rooted graph and $k$ be a positive integer. Then $G$ contains either $k$ pairwise vertex-disjoint $(H, Z, \ell)$-models in $G$, or an $(H, Z, \ell)$-deletion set of size at most $f_{H,\ell}(k)$.

If $cc(H) \leq \ell - 2$, then such a function $f_{H,\ell}$ does not exist.

Together with the result of Robertson and Seymour [16] on $H$-models, we can reformulate as follows.
Theorem 1.2. The class of \((H, Z, \ell)\)-models has the Erdős-Pósa property if and only if \(H\) is planar and \(cc(H) \geq \ell - 1\).

By setting \(Z = \{V(G)\}\) and \(\ell = 1\), Theorem 1.1 contains the Erdős-Pósa property of \(H\)-models. We point out that the size of a deletion set in Theorem 1.1 does not depend on the number of given prescribed vertex sets in \(Z\). Thus it generalizes Mader’s \(S\)-path Theorem, when \(H\) is the one-vertex graph and \(\ell = 2\).

Here we give one example showing that the class of \((H, Z, \ell)\)-models does not satisfy the Erdős-Pósa property when \(H\) is planar, but consists of at most \(\ell - 2\) connected components. Let \(\ell = 3\) and \(H\) be a connected graph. Let \(G\) be an \((n \times n)\)-grid with sufficiently large \(n\), and let \(Z_1, Z_2, Z_3\) be the set of all vertices in the first column, the first row, and the last column, respectively, except corner vertices. See Figure 1. One can observe that there cannot exist two \(H\)-models in \(G\) meeting all of \(Z_1, Z_2, Z_3\) because \(H\) is connected. On the other hand, we may increase the minimum size of an \((H, \{Z_1, Z_2, Z_3\}, \ell)\)-deletion set as much as we can, by taking a sufficiently large grid. In Section 6 we will generalize this argument for all pairs \(H\) and \(\ell\) with \(cc(H) \leq \ell - 2\).

We remark that Bruhn, Joos, and Schaudt [2] considered labelled-minors and provided a characterization for 2-connected labelled graphs \(H\) where \(H\)-models intersecting a given set have the Erdős-Pósa property. However, their minor-models are minimal subgraphs containing a graph \(H\) as a minor. So the context is slightly different.

One of the main tools to prove Theorem 1.1 is the rooted variant of Grid Minor Theorem (Theorem 3.1). Note that just a large grid model may not contain many pairwise vertex-disjoint \((H, Z, \ell)\)-models. We investigate a proper notion, called a \((Z, k, \ell)\)-rooted grid model, which contains many disjoint \((H, Z, \ell)\)-models. Briefly, we show that every graph with a large grid model contains a \((Z, k, \ell)\)-rooted grid model, or a separation of small order separating most of sets of \(Z\) from the given grid model. Previously, Marx, Seymour, and Wollan [11] proved a similar result with one prescribed vertex set. The previous result was stated in terms of tangles [18], but we state in an elementary way without using tangles. The advantage of our formulation is that we do not need to define a relative tangle at each time, in the induction step.

In Section 4 we introduce a pure \((H, Z, \ell)\)-model that consists of a minimal set of connected components of an \((H, Z, \ell)\)-model intersecting \(\ell\) sets of \(Z\). A formal definition can be found in the beginning of Section 4. Note that in a pure \((H, Z, \ell)\)-model, each component has to intersect a set of \(Z\), while in an ordinary \((H, Z, \ell)\)-model, a component does not necessarily intersect a set of \(Z\). Because of this property, when we have a separation \((A, B)\) where \(B - V(A)\) contains few sets of \(Z\) but contains a large grid, we may find an irrelevant vertex to reduce the instance. Starting from this observation, we will obtain the Erdős-Pósa property for pure \((H, Z, \ell)\)-models (Theorem 4.1).

In Section 5 we obtain the Erdős-Pósa property for \((H, Z, \ell)\)-models, using the result for pure \((H, Z, \ell)\)-models. An observation is that every \((H, Z, \ell)\)-model contains a pure \((H, Z, \ell)\)-model, by taking components that essentially
Figure 1: There are no two pairwise vertex-disjoint $H$-models meeting all of $Z_1, Z_2, Z_3$.

hits $\ell$ sets of $Z$. So, any deletion set for pure $(H, Z, \ell)$-models hits all $(H, Z, \ell)$-models as well. When we have a situation that a given graph has large grid model (with a proper separation) and $k$ pairwise vertex-disjoint pure $(H, Z, \ell)$-models, we complete these models to $(H, Z, \ell)$-models, by taking rest components from the large grid model. This will complete the argument for Erdős-Pósa property of $(H, Z, \ell)$-models.

2 Preliminaries

All graphs in this paper are simple, finite and undirected. For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. For $S \subseteq V(G)$ and $v \in V(G)$, let $G - S$ be the graph obtained by removing all vertices in $S$, and let $G - v := G - \{v\}$. For two graphs $G$ and $H$, we define $G \cap H$ as the graph on the vertex set $V(G) \cap V(H)$ and the edge set $E(G) \cap E(H)$, and define $G \cup H$ analogously. A separation of order $k$ in a graph $G$ is a pair $(A, B)$ of subgraphs of $G$ such that $A \cup B = G$, $E(A) \cap E(B) = \emptyset$, and $|V(A \cap B)| = k$.

For two disjoint vertex subsets $A$ and $B$ of a graph $G$, we say that $A$ is complete to $B$ if for every vertex $v$ of $A$ and every vertex $w$ of $B$, $v$ is adjacent to $w$.

For a positive integer $n$, let $[n] := \{1, \ldots, n\}$. For two positive integers $m$ and $n$ with $m \leq n$, let $[m, n] := \{m, \ldots, n\}$. For a set $A$, we denote by $2^A$ the set of all subsets of $A$.

For positive integers $g$ and $h$, the $(g \times h)$-grid is the graph on the vertex set $\{v_{i,j} : i \in [g], j \in [h]\}$ where $v_{i,j}$ and $v_{i',j'}$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. For each $i \in [g]$, we call $\{v_{i,1}, \ldots, v_{i,h}\}$ the $i$-th row of $G$, and define its columns similarly. We denote by $G_g$ the $(g \times g)$-grid graph, and for a positive integer $\ell$, we denote by $\ell \cdot G_g$ the disjoint union of $\ell$ copies of $(g \times g)$-grid graphs.

A graph is planar if it can be embedded on the plane without crossing edges. A graph $H$ is a minor of $G$ if $H$ can be obtained from a subgraph of $G$.
by contracting edges. It is well known that $H$ is a minor of $G$ if and only if $G$ has an $H$-model.

Operations on multisets. A multiset is a set with allowing repetition of elements. In particular, for a rooted graph $(G, Z)$, we consider $Z$ as a multiset.

For a multiset $Z$, we denote by $|Z|$ the number of sets in $Z$, which counts elements with multiplicity, and does not count a possible empty set. For example, $||\{A, B, B, C, C, \emptyset\}|| = 5$. Note that for an ordinary set $A$, we use $|A|$ for the size of a set $A$ in a usual meaning.

Let $Z$ be a multiset of subsets of a set $A$. For a subset $B$ of $A$, we define $Z|_B := \{X \cap B : X \in Z\}$ and $Z\backslash B := \{X \setminus B : X \in Z\}$. For convenience, when $(G, Z)$ is a rooted graph and $H$ is a subgraph of $G$, we write $Z|_H := Z|_{V(H)}$ and $Z\backslash H := Z\backslash V(H)$.

Tree-width. A tree-decomposition of a graph $G$ is a pair $(T, \mathcal{B})$ of a tree $T$ and a family $\mathcal{B} = \{B_t\}_{t \in V(T)}$ of vertex sets $B_t \subseteq V(G)$, called bags, satisfying the following three conditions:

(T1) $V(G) = \bigcup_{t \in V(T)} B_t$.

(T2) For every edge $uv$ of $G$, there exists a vertex $t$ of $T$ such that $\{u, v\} \subseteq B_t$.

(T3) For $t_1, t_2$, and $t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever $t_2$ is on the path from $t_1$ to $t_3$.

The width of a tree-decomposition $(T, \mathcal{B})$ is $\max\{|B_t| - 1 : t \in V(T)\}$. The tree-width of $G$, denoted by $\text{tw}(G)$, is the minimum width over all tree-decompositions of $G$. Robertson and Seymour [16] showed that every graph of sufficiently large tree-width contains a big grid model.

Grid minor-models. In the course of Graph Minors Project, Robertson and Seymour [16] showed that every graph with sufficiently large tree-width contains a large grid as a minor. Our result is also based on this theorem.

Theorem 2.1 (Grid Minor Theorem [16]). For all $g \geq 1$, there exists $\kappa(g) \geq 1$ such that every graph of tree-width at least $\kappa(g)$ contains a minor isomorphic to $G_g$; in other words, it contains a $G_g$-model.

Suppose that a graph $G$ contains a $G_g$-model. When we remove $k$ vertices contained in the model, we can take a $G_{n-k}$-model in a special way. The following lemma describes how we can take a smaller grid model.
Lemma 2.2. Let $n > k \geq 0$. Let $G$ be a graph having a $G_n$-model $H$ with a model function $\eta$, and let $S \subseteq V(G)$ with $|S| = k$. Then $G - S$ contains a $G_{n-k}$-model contained in $H$ such that it contains all rows and columns of $H$ that does not contain a vertex of $S$.

Proof. Let $i_1 < i_2 < \cdots < i_a$ be the set of indices of all rows of $H$ that do not contain a vertex of $S$. Similarly, let $j_1 < j_2 < \cdots < j_b$ be the set of indices of all columns of $H$ that do not contain a vertex of $S$. Clearly, $a \geq n - k$ and $b \geq n - k$. Let $i_0 = j_0 = 0$.

We define that for $x \in [a]$ and $y \in [b]$,

$$\alpha(v_{x,y}) := \begin{cases} H[\bigcup_{i=1}^{a} \eta(v_{x',j_y})] & \text{if } x \in [a - 1] \text{ and } y \in [b - 1] \\ H[\bigcup_{i=1}^{a} \eta(v_{x',j_y})] & \text{if } x \in [a - 1] \text{ and } y = b \\ H[\bigcup_{i=1}^{a} \eta(v_{x',j_y})] & \text{if } x = a \text{ and } y \in [b - 1] \\ H[\bigcup_{i=1}^{a} \eta(v_{x',j_y})] & \text{if } x = a \text{ and } y = b. \end{cases}$$

These vertex-models with edges crossing between those models in $G_n$ induce a $G_{a,b}$-model such that it contains all rows and columns of $H$ that does not contain a vertex of $S$. By merging consecutive columns or rows if necessary, it also induces a $G_{n-k}$-model. \hfill \Box

3 Finding a rooted grid model

In this section, we introduce a grid model with some additional conditions, called a $(Z, k, \ell)$-rooted grid model. The advantage of this notion is that for a planar graph $H$ with at least $\ell - 1$ connected components, every $(Z, k, \ell)$-rooted grid model of sufficiently large order always contains many vertex-disjoint $(H, Z, \ell)$-models.

Let $(G, Z) = \{Z_i : i \in [m]\}$ be a rooted graph. For a positive integer $k$, a vertex set $\{w_i : i \in [n]\}$ in $G$ is said to admit a $(Z, k)$-partition if there exist a partition $L_1, \ldots, L_n$ of $\{w_i : i \in [n]\}$ and an injection $\gamma : [x] \to [m]$ such that for each $i \in [x]$, $|L_i| \leq k$ and $L_i \subseteq Z_{\gamma(i)}$. For positive integer $g, k, \ell$ with $g \geq k\ell$, we define a model function of a $(Z, k, \ell)$-rooted grid model of order $g$ as a model function $\eta$ of $G_g$ such that for each $i \in [k\ell]$, $V(\eta(v_{1,i}))$ contains a vertex $w_i$ and the root vertices of the model. The image of such a model $\eta$ is called a $(Z, k, \ell)$-rooted grid model of order $g$. A $(\emptyset, k, \ell)$-rooted grid model of order $g$ is just a model of $G_g$.

We prove the following.

Theorem 3.1. Let $g, k, \ell$, and $n$ be positive integers with $g \geq k\ell$ and $n \geq g(k^2\ell^2 + 1) + k\ell$. Every rooted graph $(G, Z)$ having a $G_n$-model contains either
(1) a separation $(A, B)$ of order less than $k(\ell - ||Z\setminus A||)$ where $||Z\setminus A|| \leq \ell - 1$ and $B - V(A)$ contains a $G_n-\{V(A,B)\}$-model, or

(2) a $(Z, k, \ell)$-rooted grid model of order $g$.

To prove Theorem 3.1, we prove a related variation of Menger’s theorem. For positive integers $k$ and $n$, a set of pairwise vertex-disjoint paths $P_1, \ldots, P_n$ from $\bigcup_{z \in Z} Z$ to $Y$ is called a $(Z, Y, k)$-linkage of order $n$ if the set of all end vertices of $P_1, \ldots, P_n$ in $\bigcup_{z \in Z} Z$ admits a $(Z, k)$-partition.

Proposition 3.2. Let $k$ and $\ell$ be positive integers. Every rooted graph $(G, Z)$ with $Y \subseteq V(G)$ contains either

(1) a separation $(A, B)$ of order less than $k(\ell - ||Z\setminus A||)$ such that $Y \subseteq V(B)$ and $||Z\setminus A|| \leq \ell - 1$, or

(2) a $(Z, Y, k)$-linkage of order $k\ell$.

Proof. Let $Z := \{Z_i : i \in [m]\}$. We obtain a graph $G'$ from $G$ as follows:

- for each $i \in [m]$, add a vertex set $W_i$ of size $k$ and make $W_i$ complete to $Z_i$.

It is not hard to observe that if there are $k\ell$ pairwise vertex-disjoint paths from $\bigcup_{i \in [m]} W_i$ to $Y$ in $G'$, then there is a $(Z, Y, k)$-linkage of order $k\ell$ in $G$. Thus, by Menger’s theorem, we may assume that there is a separation $(C, D)$ in $G'$ of order less than $k\ell$ with $\bigcup_{i \in [m]} W_i \subseteq V(C)$ and $Y \subseteq V(D)$. We claim that there is a separation $(A, B)$ described in (1).

If $Z_j \setminus V(C) \neq \emptyset$ for some $j \in [m]$, then one vertex of $Z_j$ is contained in $V(D) \setminus V(C)$, and thus, $W_j$ should be contained in $V(D)$ as $W_j$ is complete to $Z_j$. Since $(C, D)$ has order less than $k\ell$, we have

$$||Z\setminus C|| \leq \ell - 1.$$

Moreover, for every $j$ with $Z_j \setminus V(C) \neq \emptyset$, all vertices of $W_j$ are contained in $V(C \cap D)$, because $\bigcup_{i \in [m]} W_i \subseteq V(C)$. Thus, if we take a restriction $(C \cap G, D \cap G)$ of $(C, D)$ on $G$, then at least $k||Z\setminus C||$ vertices are removed from the separator $V(C \cap D)$. So, $(C \cap G, D \cap G)$ is a separation in $G$ of order less than $k(\ell - ||Z\setminus (C \cap G)||)$ such that $Y \subseteq V(D \cap G)$ and $||Z\setminus (C \cap G)|| = ||Z\setminus C|| \leq \ell - 1$.

Proof of Theorem 3.1. Let $Z := \{Z_i : i \in [m]\}$ and $H$ be the given $G_n$-model with a model function $\eta$. We say that the image $\bigcup_{v \in R} \eta(v)$ of a column $R$ of $G_n$ is a column of $H$. We will mark a set of columns of $H$ and use a constructive sequence of unmarked columns to construct the required grid model in (2).

Claim 1. For $t \leq \frac{n}{k\ell}$, $G$ contains a separation described in (1) or a sequence $(P_1, R_1), \ldots, (P_t, R_t)$ where

- $R_i$ is a set of $k\ell$ columns of $H$, and for $i \neq j$, $R_i$ and $R_j$ are disjoint,
Claim 2. There is no separation \( w \in G \) for some set \( T \) of \( k \ell \) vertices contained in pairwise distinct columns of \( R_1 \).

For every column \( R \notin \bigcup_{i \in [t]} R_i \), none of the paths in \( \bigcup_{i \in [t]} P_i \) meet \( R \).

Proof. We inductively find a sequence \((P_1, R_1), \ldots, (P_t, R_t)\) if a separation described in (1) does not exist. Suppose that there is such a sequence \((P_1, R_1), \ldots, (P_{t-1}, R_{t-1})\). Choose a set \( R \) of \( k \ell \) arbitrary columns not in \( \bigcup_{i \in [t-1]} R_i \). Such a set of columns exists as \( n \geq tk \ell \). We choose \( k \ell \) vertices from distinct columns of \( R \), and say \( T \). By Proposition 3.2, \( G \) contains a separation \((A, B)\) of order less than \( k(\ell - ||Z\setminus A||) \) where \( T \subseteq V(B) \) and \( ||Z\setminus A|| \leq \ell - 1 \), or a \((Z, T, k)\)-linkage of order \( k \ell \) in \( G \).

Suppose the former separation exists. By Lemma 2.2, \( G - V(A \cap B) \) contains a \( G_{n-V(A,B)} \)-model \( H' \), such that it contains all rows and columns of \( H \) that does not contain a vertex of \( A \cup B \). Note that \( |V(A \cap B)| < k \ell = |T| \).

Since we have chosen \( T \) from \( k \ell \) distinct columns of \( H \), \( H' \) contains a column that contains a vertex of \( T \) but does not contain a vertex of \( A \cap B \). Thus, this column is contained in \( H' \). It implies that \( H' \) is contained in \( B - V(A) \). As \( |V(A \cap B)| < k \ell \), it follows that \((A, B)\) is a separation described in (1), a contradiction.

Therefore, \( G \) contains a \((Z, T, k)\)-linkage \( P \) of order \( k \ell \). To guarantee the last condition of the claim, pick a path \( P \in P \), and if possible, shorten it such that its new second end vertex is in a column not in \( \bigcup_{i \in [t-1]} R_i \) and this column is different from where the second end vertices of the other paths in \( P \) are. We perform such shortenings as long as possible. If it is not possible to shorten anymore, then we let \( P_t = P \) and \( R_t \) to be the set of columns where the second end vertices of the paths in \( P \) are. Now, it is clear that the paths in \( P_t \) do not visit any column not in \( \bigcup_{i \in [t]} R_i \). \( \diamond \)

Let \((P_1, R_1), \ldots, (P_{k \ell}, R_{k \ell})\) be a sequence given by Claim 1. Since \( n \geq g(k^2 \ell^2 + 1) + k \ell \), there exists \( p \) with \( k \ell \leq p \leq n - g \) such that for every \( 1 \leq i \leq g \), the \((p + i)\)-th column is not in \( R \). Let

\[
X := \bigcup_{i \in [k \ell + 1, 2k \ell]} \eta(v_{i,p+1}), \quad D := \bigcup_{i \in [k \ell + 1, n]} \bigcup_{j \in [g]} \eta(v_{i,p+j}).
\]

Let \( G' \) be the graph obtained from \( G \) by deleting \( D \setminus X \) and contracting the set \( \eta(v_{i,p+1}) \) to a vertex \( w_{i,p+1} \) for each \( i \in [k \ell + 1, 2k \ell] \). Let \( W := \{ w_{i,p+1} : i \in [k \ell + 1, 2k \ell] \} \) and \( Z' := (Z \setminus V(G')) \setminus D \).

Claim 2. There is no separation \((A, B)\) in \( G' \) of order less than \( k(\ell - ||Z' \setminus A||) \) such that \( W \subseteq V(B) \) and \( ||Z' \setminus A|| \leq \ell - 1 \).

Proof. Assume that there is such a separation \((A, B)\) in \( G' \). Since the sets of columns \( R_1, \ldots, R_{k \ell} \) are disjoint and \( |V(A \cap B)| < k \ell \), there is an integer \( 1 \leq s \leq k \ell \) such that \( V(A \cap B) \) is disjoint from all of columns in
\(\mathcal{R}_s\). First suppose that all of columns in \(\mathcal{R}_s\) are contained in \(B - V(A)\). Since \(|V(A \cap B)| \leq k(\ell - |Z'\setminus A|) - 1\), among the paths in \(\mathcal{P}_s\), there are \(k|Z'\setminus A| + 1\) paths fully contained in \(B - V(A)\). On the other hand, by the definition of \((Z, T, k)\)-linkages, the set of all end vertices of paths in \(\mathcal{P}_s\) on the vertex set \(\bigcup_{Z \in Z} Z\) admits a \((Z, k)\)-partition, that is,

- there exist a partition \(L_1, \ldots, L_x\) of the end vertices in \(\bigcup_{i \in [m]} Z_i\) and an injection \(\gamma : [x] \to [m]\) where for each \(i \in [x]\), \(|L_i| \leq k\) and \(L_i\) is contained in \(Z_{\gamma(i)}\).

From the condition that each \(L_i\) has size at most \(k\), at least \(|Z'\setminus A|\) + 1 paths of \(\mathcal{P}_s\) are fully contained in \(B - V(A)\) and have end vertices in pairwise distinct sets of \(\{L_i : i \in [x]\}\). It means that \(|Z'\setminus A| \geq |Z'\setminus A| + 1\), which is a contradiction. We conclude that there is a column \(R\) in \(\mathcal{R}_s\) that is contained in \(A - V(B)\).

We observe that there are \(k\ell\) vertex-disjoint paths from \(R\) to \(W\) in \(G'\). If \(R\) is the \(i\)-th column of \(H\) for some \(i \leq p\), then we can simply use paths from \((k\ell + 1)\)-th, \(\ldots\), \((2k\ell)\)-th rows. Assume that \(R\) is the \(i\)-th column where \(i \geq p + 1\). Then for each \(t \in [k\ell]\), we construct an \(i\)-th path such that it starts in \(\eta(v_{ti,i})\), goes to \(\eta(v_{ti,p+t-k\ell})\) then goes to \(\eta(v_{t+1-P,1-t,p+t-k\ell})\), and then terminates in \(w_{2k\ell+1-t,p+1}\). This is possible because \(p \geq k\ell\). One of these \(k\ell\) paths is disjoint from \(V(A \cap B)\), hence there is a vertex of \(W\) in \(V(A)\setminus V(B)\), contradicting the assumption that \(W \subseteq V(B)\). \(\Box\)

Therefore, by Proposition 3.2, \(G'\) contains a \((Z', W, k)\)-linkage \(\mathcal{P}\) of order \(k\ell\). Let \(\eta'\) be a model of \(G_g\) where \(\eta'(v_{i,j}) = \eta(v_{k\ell+1,pi+j})\). This model can be extended by the paths in \(\mathcal{P}\), and it satisfies the conditions of the required model.

We show that every sufficiently large \((Z, k, \ell)\)-rooted grid model contains \(k\) pairwise vertex-disjoint \((H, Z, \ell)\)-models.

**Lemma 3.3.** Let \(k, \ell,\) and \(h\) be positive integers. Every rooted graph \((G, Z)\) having a \((Z, k, \ell)\)-rooted grid model of order \(k\ell(h + 2) + 1\) contains \(k\) pairwise vertex-disjoint \((\ell \cdot G_h, Z, \ell)\)-models. Moreover, if \(\ell \geq 2\), then \(G\) contains \(k\) pairwise vertex-disjoint \(((\ell - 1) \cdot G_h, Z, \ell)\)-models.

The usefulness of the \((Z, k)\)-partition is given in the next lemma.

**Lemma 3.4.** Let \((G, Z = \{Z_1, \ldots, Z_m\})\) be a rooted graph. Every \((Z, k)\)-partition of a vertex set \(\{w_1, \ldots, w_{k\ell}\}\) admits a partition \(I_1, \ldots, I_k\) of the index set \(\{1, \ldots, k\ell\}\) such that for each \(j \in [k]\),

- \(|I_j| = \ell,\)

- there is an injection \(\beta_j : I_j \to [m]\) where for each \(i \in I_j\), \(w_i\) is contained in \(Z_{\beta_j(i)}\), and
• there are two integers \(a_j, b_j \in I_j\) with \(a_j < b_j\) where there is no integer \(c\) in \(\bigcup_{i \in [j,k]} I_i\) with \(a_j < c < b_j\).

Proof. We inductively find such a partition \(I_1, \ldots, I_k\) of \(\{w_1, \ldots, w_{\ell k}\}\). If \(k = 1\), then by the definition of a \((Z, k)\)-partition, there is an injection \(\gamma : \{1, \ldots, \ell\} \rightarrow [m]\) where for each \(i \in [\ell]\), \(w_i \in Z_{\gamma(i)}\). Thus, \(I_1 = \{1, \ldots, \ell\}\) satisfies the property.

Let us assume that \(k \geq 2\), and let \(L_1, \ldots, L_x\) be a partition of \(\{w_1, \ldots, w_{\ell k}\}\) and \(\gamma : [x] \rightarrow [m]\) be an injection where for each \(i \in [x]\), \(|L_i| \leq k\) and \(L_i\) is contained in \(Z_{\gamma(i)}\).

Let us choose a vertex set \(S\) of size \(\ell\) from \(\bigcup_{j \in [x]} L_j\) so that

- for each \(j \in [x]\), \(|S \cap L_j| \leq 1\),
- if \(|L_i| = k\), then \(S \cap L_i \neq \emptyset\),
- there are two vertices \(w_p\) and \(w_{p+1}\) in \(S\) with consecutive indices.

If there is a set \(L_i\) with \(|L_i| = k\), then we first choose two consecutive vertices where one is contained in \(L_i\) and the other is not in \(L_i\), and then choose one vertex from each set \(L\) of \(\{L_1, \ldots, L_x\}\) with \(|L| = k\) that are not selected before.

If there is no set \(L_i\) with \(|L_i| = k\), then we choose any two consecutive vertices that are contained in two distinct sets of \(Z\). Note that the number of selected vertices cannot be larger than \(\ell\). If necessary, by putting some vertices from sets of \(\{L_1, \ldots, L_x\}\) which are not selected before, we can fill in \(S\) so that \(|S| = \ell\), there are two vertices with consecutive indices, and there is an injection \(\beta_1\) from the index set of \(S\) to \([m]\) where for each \(w_i \in S\), \(w_i\) is contained in \(Z_{\beta_1(i)}\).

By taking a restriction, we can obtain a partition \(I'_1, \ldots, I'_\ell\) of \(\{w_i : i \in [k\ell]\}\) \(\setminus S\) and an injection \(\gamma' : [x] \rightarrow [m]\) where \(|I'_i| \leq k - 1\) and \(I_i\) is contained in \(Z_{\gamma(i)}\) for each \(i \in [x]\). We give a new ordering \(v_1', \ldots, v'_{k(\ell - 1)}\) of the vertex set \(\{w_i : i \in [k\ell]\}\) \(\setminus S\) following the order in \(\{w_i : i \in [k\ell]\}\). Inductively, there exists a partition \(I'_2, \ldots, I'_k\) of \([(k - 1)\ell]\) such that for each \(j \in [2, k]\),

- \(|I'_j| = \ell - 1\) and there is an injection \(\beta_j : I'_j \rightarrow [m]\) where for each \(i \in I'_j\), \(w_i\) is contained in \(Z_{\beta_j(i)}\), and
- there are two integers \(a_j, b_j \in I'_j\) with \(a_j < b_j\) where there is no integer \(c\) in \(\bigcup_{i \leq j \leq k} I'_i\) with \(a_j < c < b_j\).

It gives a corresponding partition \(I_2, \ldots, I_k\) of the index set of \(\{w_i : i \in [k\ell]\}\) \(\setminus S\). Let \(I_1\) be the set of indices of vertices in \(S\). Then \(\{I_i : i \in [k]\}\) is a required partition.

Proof of Lemma 3.3. We first construct \(k\) pairwise vertex-disjoint \((\ell \cdot \mathcal{G}_h, Z, \ell)\)-imodels. Let \(\eta\) be the model function of the \((Z, k, \ell)\)-rooted grid model and let \(w_1, \ldots, w_{k\ell}\) be the root vertices of the model where \(w_i \in \eta(v_{i,1})\) for each \(i \in [k\ell]\). By Lemma 3.4, there exists a partition \(I_1, \ldots, I_k\) of \([k\ell]\) such that for each \(j \in [k]\).
Figure 2: This is a \((\{Z_1, Z_2, Z_3\}, 3, 2)\)-rooted grid model where \(Z_1 = \{v_1, v_3, v_4\}\), \(Z_2 = \{v_2, v_6\}\), \(Z_3 = \{v_5\}\). We can obtain a partition of \(\{1, \ldots, 6\}\) as \(I_1 = \{1, 2\}\), \(I_2 = \{4, 5\}\), \(I_3 = \{3, 6\}\). There are 3 vertex-disjoint \((2 \cdot G_i, Z, 2)\)-models \(G_1, G_2\), \(G_3\) such that each \(G_i\) consists of subgraphs containing vertices with indices in \(I_i\). Also, there are 3 vertex-disjoint \((G_i, Z, 2)\)-models using the three vertex-disjoint paths that connects two subgraphs in the same \(G_i\).

- \(|I_j| = \ell\) and there is an injection \(\beta_j : I_j \to [m]\) where for each \(i \in I_j\), \(w_i\) is contained in \(Z_{\beta_j(i)}\), and
- there are two integers \(a, b \in I_j\) with \(a < b\) where there is no integer \(c\) in \(\bigcup_{i \in [j, k]} I_i\) with \(a < c < b\).

For each \(j \in [k\ell]\), we define a subgraph \(Q_j\) of \(G_{k\ell(h+2)+1}\) as

\[
Q_j := (G_{k\ell(h+2)+1})[\{v_{a,j} : a \in [1, k\ell + 2 - j]\} \\
\cup \{v_{k\ell+2-j,b} : b \in [j, k\ell + 1 + h(j-1)]\} \\
\cup \{v_{a,k\ell+1+h(j-1)} : a \in [k\ell + 2 - j, k\ell + 2]\} \\
\cup \{v_{a,b} : a \in [k\ell + 2, k\ell + 1 + h], \\
\quad b \in [k\ell + 1 + h(j-1), k\ell + h(j)]\}].
\]

Note that each \(Q_j\) consists of a path from a vertex of \(\{w_1, \ldots, w_{k\ell}\}\) to a \(G_h\)-model. See Figure 2 for an illustration. For each \(j \in [k]\), let

\[G_j := \eta \left( \bigcup_{i \in I_j} V(Q_i) \right).\]

It is not hard to observe that each \(G_j\) is a \((\ell \cdot G_h, Z, \ell)\)-model. Thus, we obtain \(k\) pairwise vertex-disjoint \((\ell \cdot G_h, Z, \ell)\)-models.

Now, we prove the second statement. Let us assume that \(\ell \geq 2\). We first construct the subgraphs \(Q_j\) in the same way. Then, for each \(j \in [k]\), we add a path \(P_j\) as follows: \(P_j\) starts in \(v_{k\ell+1+h,k\ell+1+h(a_j-1)}\), goes to \(v_{k\ell+1+h+i,k\ell+1+h(a_j-1)}\)
then goes to $v_{k\ell+1+h+i,k\ell+1+h(b_j-1)}$, and then terminates in $v_{k\ell+1+h,k\ell+1+h(b_j-1)}$. Note that $P_1,\ldots,P_k$ are pairwise vertex-disjoint in $\mathcal{G}_{k\ell(h+2)+1}$ because of the second condition of the partition $I_1,\ldots,I_k$. So, for each $j \in [k]$, the union of $G_j \cup \eta(V(P_j))$ is a $((\ell-1) \cdot \mathcal{G}_h, Z, \ell)$-model, and thus, $(Z, k, \ell)$-rooted grid model of order $k\ell(h+2)+1$ contains $k$ pairwise vertex-disjoint $((\ell-1) \cdot \mathcal{G}_h, Z, \ell)$-models.

\section{Erdős-Pósa property of pure $(H, Z, \ell)$-models}

Let $(G, Z)$ be a rooted graph and $H$ be a graph. A subgraph $F$ of $G$ is called a pure $(H, Z, \ell)$-model if there exist a subset $\mathcal{H} = \{H_1,\ldots,H_t\}$ of the set of connected components of $H$ and a function $\alpha : \{1,\ldots,t\} \to 2^Z \setminus \{\emptyset\}$ such that

- $F$ is the image of a model $\eta$ of $H_1 \cup \cdots \cup H_t$ in $G$,
- for $Z \in \alpha(i)$, $\eta(V(H_i)) \cap Z \neq \emptyset$,
- $\alpha(1),\ldots,\alpha(t)$ are pairwise disjoint and $|\bigcup_{i \in [t]} \alpha(i)| = \ell$.

A pure $(H, Z, \ell)$-model can be seen as a minimal set of connected components of an $(H, Z, \ell)$-model for intersecting at least $\ell$ sets of $Z$. Because of the second and third conditions, every pure $(H, Z, \ell)$-model consists of the image of at most $\ell$ connected components of $H$. A pure $(H, Z, \ell)$-model may be an $(H, Z, \ell)$-model itself when $H$ consists of $\ell-1$ or $\ell$ connected components.

In this section, we establish the Erdős-Pósa property for pure $(H, Z, \ell)$-models. The reason for using the pure $(H, Z, \ell)$-models is that a procedure to find an irrelevant vertex in a graph of large tree-width works well for pure $(H, Z, \ell)$-models. Later, we will argue how the problem for $(H, Z, \ell)$-models can be reduced to the problem for pure $(H, Z, \ell)$-models. Generally, it is possible that $G$ has more than $k$ pairwise vertex-disjoint pure $(H, Z, \ell)$-models, even though it has no $k$ vertex-disjoint $(H, Z, \ell)$-models. So the relation is not very direct.

A vertex set $S$ of $G$ is called a pure $(H, Z, \ell)$-deletion set if $G-S$ has no pure $(H, Z, \ell)$-models.

**Theorem 4.1.** For a positive integer $\ell$ and a non-empty planar graph $H$ with $cc(H) \geq \ell-1$, there exists $f^H_{H,\ell} : \mathbb{N} \to \mathbb{R}$ satisfying the following property. Let $(G, Z)$ be a rooted graph and $k$ be a positive integer. Then $G$ contains either $k$ pairwise vertex-disjoint pure $(H, Z, \ell)$-models, or a pure $(H, Z, \ell)$-deletion set of size at most $f^H_{H,\ell}(k)$.

### 4.1 Erdős-Pósa property of $(H, Z, \ell)$-models in graphs of bounded tree-width

When the underlying graph has bounded tree-width, every class of graphs with at most $t$ connected components for some fixed $t$ has the Erdős-Pósa property. It follows from the Erdős-Pósa property of subgraphs in a tree consisting of at
most $d$ connected components for some fixed $d$, which was proved by Gyárfás and Lehel \cite{gyarfas1993}. Later, the bound on a hitting set was improved by Berger \cite{berger2000}.

For a tree $T$ and a positive integer $d$, a subgraph of $T$ is called a $d$-subtree if it consists of at most $d$ connected components.

**Theorem 4.2** (Berger \cite{berger2000}). Let $T$ be a tree and let $k$ and $d$ be positive integers. Let $F$ be a set of $d$-subtrees of $T$. Then $T$ contains either $k$ pairwise vertex-disjoint subgraphs in $F$, or a vertex set $S$ of size at most $(d^2 - d + 1)(k - 1)$ such that $T - S$ has no subgraphs in $F$.

**Proposition 4.3.** Let $k, h, \ell$, and $w$ be positive integers and let $H$ be a graph with $h$ vertices. Let $(G, Z)$ be a rooted graph with tree-width at most $w$. Then $G$ contains either $k$ pairwise vertex-disjoint $(H, Z, \ell)$-models, or an $(H, Z, \ell)$-deletion set of size at most $(w - 1)(h^2 - h + 1)(k - 1)$. Furthermore, the same statement holds for pure $(H, Z, \ell)$-models.

**Proof.** We first prove for usual $(H, Z, \ell)$-models. Let $\mathcal{H}$ be the class of all $(H, Z, \ell)$-models. Let $(T, \mathcal{B} := \{B_t\}_{t \in V(T)})$ be a tree-decomposition of $G$ of width at most $w$. For $v \in V(G)$, let $\mathcal{P}(v)$ denote the set of the vertices $t$ in $T$ such that $B_t$ contains $v$. From the definition of tree-decompositions, for each $v \in V(G)$, $T[\mathcal{P}(v)]$ is connected. If $xy$ is an edge in $G$, then there exists $B \in \mathcal{B}$ containing both $x$ and $y$. It implies that for a connected subgraph $F$ of $G$, $T[\bigcup_{x \in V(F)} \mathcal{P}(x)]$ is connected.

Let $\mathcal{F}$ be the family of sets $\bigcup_{x \in V(F)} \mathcal{P}(x)$ for all $F \in \mathcal{H}$. Observe that for $F_1, F_2 \in \mathcal{H}$ where $\bigcup_{x \in V(F_1)} \mathcal{P}(x)$ and $\bigcup_{x \in V(F_2)} \mathcal{P}(x)$ are disjoint, $F_1$ and $F_2$ are vertex-disjoint. For a set $S \in \mathcal{F}$, $T[S]$ consists of at most $h$ connected components, which means that it is a $h$-subtree. So, if $\mathcal{F}$ contains $k$ pairwise disjoint sets, then clearly, $\mathcal{H}$ contains $k$ pairwise vertex-disjoint subgraphs of $\mathcal{H}$ in $G$. Thus, we may assume that there are no $k$ pairwise disjoint subsets in $\mathcal{F}$.

Then by Theorem 4.2, $T$ has a vertex set $W$ of size at most $(h^2 - h + 1)(h - 1)$ such that $W$ meets all sets in $\mathcal{F}$. It implies that $\bigcup_{t \in W} B_t$ contains at most $(w - 1)(h^2 - h + 1)(k - 1)$ vertices and it meets all subgraphs in $\mathcal{H}$.

The same argument holds for pure $(H, Z, \ell)$-models. \hfill \Box

### 4.2 Reduction to graphs of bounded tree-width

Let $(G, Z)$ be a rooted graph and $H$ be a graph. Let $\tau^*_H(G, Z, \ell)$ be the minimum size of a pure $(H, Z, \ell)$-deletion set of $G$. A vertex $v$ of $G$ is called irrelevant for pure $(H, Z, \ell)$-models if $\tau^*_H(G, Z, \ell) = \tau^*_H(G - v, Z, \ell)$. If there is no confusion from the context, then we shortly say that $v$ is an irrelevant vertex.

In this subsection, we argue that for two simpler cases when $G$ has large tree-width, one can obtain an irrelevant vertex. This will help to show base cases of Theorem 4.1.

**Proposition 4.4.** Let $g, h$, and $\ell$ be positive integers and $x$ be a non-negative integer such that $g \geq (x^2 + 14hx + 2x + 1)(x^2 + 1)$. Let $(G, Z)$ be a rooted graph, and $H$ be a non-empty planar graph with $h$ vertices and $\text{cc}(H) \geq \ell - 1$,
and \((A, B)\) be a separation in \(G\) of order at most \(x\) such that \(||Z\setminus A|| = 0\) and \(B - V(A)\) contains a \(G'_\vartheta\)-model. Then there exists an irrelevant vertex for pure \((H, Z, \ell)\)-models.

**Proof.** We proceed it by induction on \(x\). If \(x = 0\), then a pure \((H, Z, \ell)\)-model cannot have a vertex of \(B - V(A)\) because every connected component of a pure \((H, Z, \ell)\)-model contains at least one vertex of \(\bigcup_{Z \in Z} Z\). Thus, every vertex in \(B - V(A)\) is irrelevant. We may assume that \(x \geq 1\).

Let \(W := V(A \cap B)\) and \(x' := x^2 + 14hx + 2x\). Note that \(g \geq x'(x^2 + 1) + x^2\).

By applying Theorem \(3.1\) to \(B\) with \((Z, g, k, \ell) \leftarrow ([W], x', |W|, 1)\), we deduce that \(B\) contains either

1. a separation \((A', B')\) of order less than \(|W|\) in \(B\) such that \(W \subseteq V(A')\) and \(B' - V(A')\) contains a \(G_{g-|V(A') \cap B'|}\)-model, or
2. a \(([W], |W|, 1)\)-rooted grid model of order \(x'\).

Suppose that there is a separation \((A', B')\) described in (1). Since \(W \subseteq V(A')\), \(((A - E(A \cap B)) \cup A', B')\) is a separation in \(G\) of order at most \(|W| - 1 \leq x - 1\) such that \(||Z\setminus (A \cup A')|| = 0\) and \(B' - V(A \cup A')\) contains a \(G_{g-|V(A \cup B')|}\)-model. Thus \(B' - V(A \cup A')\) contains a \(G_{g-(x-1)}\)-model. Note that

\[
g - (x - 1) \geq (x^2 + 14hx + 2x + 1)((x - 1)^2 + 1) \\
\geq ((x - 1)^2 + 14hx - 1 + 2(x - 1) + 1)((x - 1)^2 + 1).
\]

Thus, by induction hypothesis, \(G\) contains an irrelevant vertex.

So, we may assume that \(B\) contains a \(([W], |W|, 1)\)-rooted grid model of order \(x'\), say \(M\). This means that there is a model function \(\eta_{1} \in G_{M}^\vartheta\) for \(M\) such that for \(i \in [|W|]\), \(V(\eta_{1}(v_{1,i}))\) contains a vertex of \(W\). We choose a vertex \(w \in V(\eta_{1}(v_{1,x'}))\). We show that \(w\) is an irrelevant vertex. To show this, it is sufficient to check \(\tau^*_{H}(G, Z, \ell) \leq \tau^*_{H}(G - w, Z, \ell)\). Let \(T\) be a pure \((H, Z, \ell)\)-deletion set of \(G - w\). We claim that \(G\) contains a pure \((H, Z, \ell)\)-deletion set of size at most \(|T|\). If \(G - T\) has no pure \((H, Z, \ell)\)-models, then we are done. We may assume that \(G - T\) has a pure \((H, Z, \ell)\)-model. Let \(T_{A} := T \cap V(A)\) and \(T_{B} := T \cap V(B)\).

Let \(W'\) be a minimum size subset of \(W \setminus T\) such that \(G - (T_{A} \cup W')\) contains no pure \((H, Z, \ell)\)-models. Such a set exists, as \(G - (T_{A} \cup W)\) has no pure \((H, Z, \ell)\)-models. If \(|T' \setminus V(A)| \geq |W'|\), then \(T_{A} \cup W'\) is a pure \((H, Z, \ell)\)-deletion set of \(G\) of size at most \(|T'| \leq |T|\). So, we may assume that \(|T' \setminus V(A)| \leq |W'| - 1\).

We prove that \(G - (T \cup \{w\})\) has a pure \((H, Z, \ell)\)-model, which yields a contradiction.

**Claim 3.** \(G - (T \cup \{w\})\) has a pure \((H, Z, \ell)\)-model.

**Proof.** Note that all vertices of \(W\) are contained in the first row of \(M\) and \(w\) is contained in the last column of \(M\). See Figure \[8\]

Observe that for every \(i \in [|W| + 1, x']\), the \(i\)-th column of \(M\) does not contain a vertex of \(W\). Since \(w\) is in the last column of \(M\), there are at most
Figure 3: The image of $\eta_1$ in Proposition 4.4, which is a grey rectangle. The three paths from $W$ to the subgrid are $Q_1$, $Q_2$, and $Q_3$.

$x$ maximal sequence of columns of $M$ from the $(|W|+1)$-th column to the $x'$-th column, that have no vertices in $T \cup \{w\}$. Because

$$x' - |W| - |T \setminus (A)| - 1 \geq x(x + 14h + 2) - 2x = x(x + 14h),$$

there is a set of $x + 14h$ consecutive columns of $M$ that have no vertices in $T \cup W \cup \{w\}$.

A similar argument holds for rows as well. So, there is a set of $x + 14h$ consecutive rows of $M$ that have no vertices in $T \cup W \cup \{w\}$. It implies that there exist $1 \leq p \leq x' - 1 - (x + 14h)$ and $x \leq q \leq x' - 1 - (x + 14h)$ such that for each $1 \leq i, j \leq x + 14h$, the $(p + i)$-th row and the $(q + j)$-th column do not contain a vertex of $T \cup W \cup \{w\}$. We will use the grid model induced by $V(\eta_1(v_{p+i,q+j}))$'s for $1 \leq p \leq x' - (x + 14h)$ and $x \leq q \leq x' - 1 - (x + 14h)$. Let $G'$ be this subgraph.

Now, we observe that there are $|W|$ vertex-disjoint paths from $W$ to

$$\eta(v_{p+1,q+1}), \ldots, \eta(v_{p+|W|-q+1}).$$

We construct an $i$-th path such that it starts in $\eta_1(v_{1,i}) \cap W$, goes to $\eta_1(v_{p+1+i,|W|-i,q})$, and then terminates in $\eta_1(v_{p+1+|W|-i,q+1})$. Among those paths, there are $|W| - |T_B|$ paths $Q_1, \ldots, Q_{|W|-|T_B|}$ from $W \setminus T$ to $G'$ avoiding $T \cup \{w\}$. Let $C$ be the set of the end vertices of $Q_1, \ldots, Q_{|W|-|T_B|}$ in $W \setminus T$, and let $C' := (W \setminus T) \setminus C$.

Since

$$|C| = |W| - |T_B| = |W \setminus T| - |T \setminus (A)| \geq |W \setminus T| - (|W'|-1),$$

we have $|C'| \leq |W'|-1$. Therefore, by the choice of the set $W'$, $G - (T_A \cup C')$ contains a pure $(H, Z, \ell)$-model $F$ containing $w$. Note that each connected
Proposition 4.5. Let separation $p_H$ be irrelevant, and since $W$ is connected, then pure $G$-models are just $(H, 2, 2)$-models. We prove it by induction on $x$. If $x = 0$, then no pure $(H, Z, 2)$-models have a vertex of $B - V(A)$ as $H$ is connected. Thus, every vertex in $B - V(A)$ is irrelevant, and since $g > 1$, there is an irrelevant vertex in $B - V(A)$. Let us assume that $x > 1$. Also, if $||Z\setminus A|| = 0$, then by Proposition 3.4, $G$ contains an irrelevant vertex $w$ for pure $(H, Z, 2)$-models.

Proof. We prove it by induction on $x$. If $x = 0$, then no pure $(H, Z, 2)$-models have a vertex of $B - V(A)$ as $H$ is connected. Thus, every vertex in $B - V(A)$ is irrelevant, and since $g > 1$, there is an irrelevant vertex in $B - V(A)$. Let us assume that $x > 1$. Also, if $||Z\setminus A|| = 0$, then by Proposition 3.4, $G$ contains an irrelevant vertex $w$ for pure $(H, Z, 2)$-models.

Therefore, $\tau^*_H(G, Z, \ell) = \tau^*_H(G - w, Z, \ell)$ and we conclude that $w$ is an irrelevant vertex.

If $H$ is connected and $\ell = 2$, then we further analyze the case when $G$ has a separation $(A, B)$ with $||Z\setminus A|| = 1$. It will be needed for base cases. We remark that if $H$ is connected, then pure $(H, Z, 2)$-models are just $(H, 2, 2)$-models.

Proposition 4.5. Let $g$ and $h$ be positive integers and $x$ be a non-negative integer such that $g \geq 2(4x^2 + 14hx + 3x + 1)(4x^2 + 1)$. Let $(G, Z)$ be a rooted graph, $H$ be a non-empty connected planar graph with $h$ vertices, and $(A, B)$ be a separation in $G$ of order at most $x$ such that $||Z\setminus A|| \leq 1$ and $B - V(A)$ contains a $G_g$-model. Then $G$ contains an irrelevant vertex $w$ for pure $(H, Z, 2)$-models.

Proof. We prove it by induction on $x$. If $x = 0$, then no pure $(H, Z, 2)$-models have a vertex of $B - V(A)$ as $H$ is connected. Thus, every vertex in $B - V(A)$ is irrelevant, and since $g > 1$, there is an irrelevant vertex in $B - V(A)$. Let us assume that $x > 1$. Also, if $||Z\setminus A|| = 0$, then by Proposition 3.4, $G$ contains an irrelevant vertex $w$ for pure $(H, Z, 2)$-models.

Let $W := V(A \cap B)$ and $x' := 2(4x^2 + 14hx + 3x)$. Let us assume that $\{Z_j \in Z : Z_j \setminus S \neq \emptyset\} = \{Z_{a}\}$, and let $Y_a := Z_{a}\setminus V(A)$ and let $Z' := \{W, Y_a\}$. Note that $g \geq x'(4x^2 + 1) + 4x^2$. By applying Theorem 3.1 to $B$ (with $(Z, g, k, \ell) \rightarrow (Z', x', |W|, 2)$), $B$ contains either

1. a separation $(A', B')$ of order less than $|W|((2 - ||Z\setminus A'||) - 1)$ and $B' - V(A')$ contains $G_g - (V(A' \cup B')$-model, or
2. a $(Z', |W|, 2)$-rooted grid model of order $x'$. 

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Suppose there is a separation \((A', B')\) described in (1). Since \(|Z \setminus A'| \leq 1\), there are three possibilities; either \((Y_a \setminus V(A') \neq \emptyset \text{ and } W \subseteq V(A'))\), or \((Y_a \subseteq V(A') \text{ and } W \setminus V(A') \neq \emptyset)\), or \((Y_a \cup W \subseteq V(A'))\). In each case, we argue that there is an irrelevant vertex.

- **(Case 1. \(Y_a \setminus V(A') \neq \emptyset \text{ and } W \subseteq V(A').\))**
  Then \(((A - E(A \cap B)) \cup A', B')\) is a separation in \(G\) of order at most \(|W| - 1 \leq x - 1\), and \(|Z \setminus (A \cup A')| = 1\) and \(B' \setminus V(A \cup A')\) contains a \(G_{g-(x-1)}\)-model. Since
  \[
g - (x - 1) \geq 2(4x^2 + 14hx + 3x + 1)(4x^2 + 1) - (x - 1)
  \geq 2(4(x - 1)^2 + 14h(x - 1) + 3(x - 1) + 1)(4(x - 1)^2 + 1),
\]
  by induction hypothesis, \(G\) contains an irrelevant vertex.

- **(Case 2. \(Y_a \cup W \subseteq V(A').\))**
  Then \(((A - E(A \cap B)) \cup A', B')\) is a separation in \(G\) of order at most \(2|W| \leq 2x - 1\), and \(|Z \setminus (A \cup A')| = 0\) and \(B' \setminus V(A \cup A')\) contains a \(G_{g-(2x-1)}\)-model. Since
  \[
g - (2x - 1) \geq 2(4x^2 + 14hx + 3x + 1)(4x^2 + 1) - (2x - 1)
  \geq (8x^2 + 28hx + 6x + 2)(4x^2 + 1) - (2x - 1)
  \geq (4x^2 + 28hx)(4x^2 + 1)
  \geq (4x^2 + 28hx - 14h)(4x^2 - 4x + 2)
  = ((2x - 1)^2 + 14h(2x - 1) + 2(2x - 1) + 1)((2x - 1)^2 + 1),
\]
  by Proposition 14, \(G\) contains an irrelevant vertex.

- **(Case 3. \(Y_a \subseteq V(A') \text{ and } W \setminus V(A') \neq \emptyset.\))**
  In this case, \(((A - E(A \cap B)) \cup A', B')\) is a separation in \(G\) of order at most \(|W \cup V(A' \cap B')| \leq 2x - 1\) where \(|Z \setminus (A \cup A')| = 0\) and \(B' \setminus V(A \cup A')\) contains a \(G_{g-(2x-1)}\)-model. By the same reason in Case 2, \(G\) contains an irrelevant vertex.

Now, we assume that \(B\) contains a \((z', |W|, 2)\)-rooted grid model of order \(x'\), say \(M\). So, there is a model function \(\eta_1\) of \(G_{x'}\) for \(M\) such that for \(1 \leq i \leq 2|W|\), \(V(\eta_1(v_{1,i}))\) contains a vertex of \(W \cup Y_a\) and \(W \subseteq \{w_1, \ldots, w_{2|W|}\}\). We choose a vertex \(w\) in \(V(\eta_1(v_{x',x'}))\). We show that \(w\) is an irrelevant vertex. To show this, it is sufficient to check that \(\tau^{p}_{M}(G, Z, 2) \leq \tau^{p}_{N}(G - w, Z, 2)\). Let \(T\) be a pure \((H, Z, 2)\)-deletion set of \(G - w\). We claim that \(G\) contains a pure \((H, Z, f)\)-deletion set of size at most \(|T|\). If \(G - T\) has no pure \((H, Z, 2)\)-models, then we are done. We may assume that \(G - T\) has a pure \((H, Z, 2)\)-model. Let \(T_{A} := T \cap V(A)\) and \(T_{B} := T \cap V(B)\).

Let \(W'\) be a minimum size subset of \(W \setminus T\) such that \(G - (T_{A} \cup W')\) contains no pure \((H, Z, 2)\)-models. Such a set exists, because \(|Z \setminus A| = 1\) and thus
$G - (T_A \cup W)$ has no $(H, Z, 2)$-models. If $|T \setminus V(A)| \geq |W'|$, then $T_A \cup W'$ is a pure $(H, Z, 2)$-deletion set of size at most $|T|$. So, we may assume that $|T \setminus V(A)| \leq |W'| - 1$.

We claim that $G - (T \cup \{w\})$ has a pure $(H, Z, 2)$-model, which yields a contradiction.

**Claim 4.** $G - (T \cup \{w\})$ has a pure $(H, Z, 2)$-model.

**Proof.** Note that all vertices of $W$ are contained in the first row of $M$ and $w$ is contained in the last column of $M$.

Observe that for every $i \in [|W| + 1, x']$, the $i$-th column of $M$ does not contain a vertex of $W$. Since $w$ is in the last column of $M$, there are at most $x$ maximal sequence of columns of $M$ from the $(|W| + 1)$-th column to the $x'$-th column, that have no vertices in $T \cup \{w\}$. Because

$$x' - |W| - |T \setminus V(A)| - 1 \geq 2x(4x + 14h + 3) - 2x \geq x(14h + 3),$$

there is a set of $14h + 3$ consecutive columns of $M$ that have no vertices in $T \cup W \cup \{w\}$.

A similar argument holds for rows as well. So, there is a set of $14h + 3$ consecutive rows of $M$ that have no vertices in $T \cup W \cup \{w\}$. It implies that there exist $1 \leq p \leq x' - 1 - (14h + 3)$ and $1 \leq q \leq x' - 1 - (14h + 3)$ such that for each $1 \leq i, j \leq 14h + 3$, the $(p + i)$-th row and the $(q + j)$-th column do not contain a vertex of $T \cup W \cup \{w\}$. We will use the grid model induced by $V(\eta_1(v_{p+i,q+j})')$'s for $1 \leq p \leq x' - 1 - (14h + 3)$ and $1 \leq q \leq x' - 1 - (14h + 3)$. Let $G'$ be this subgraph.

Observe that there are $2|W|$ vertex-disjoint paths from $\{w_1, \ldots, w_{2|W|}\}$ to $\eta((v_{p+1,q+1}), \ldots, \eta((v_{p+2|W|,q+1}))$. Among them, there are at least $|W| - |T_B|$ paths $Q_1, \ldots, Q_{|W| - |T_B|}$ from $W \setminus T$ to $G'$ avoiding $T \cup \{w\}$. Similarly, among $|W|$ paths starting from $\{w_1, \ldots, w_{2|W|}\} \setminus W \subseteq Y$, there are at least $|W| - |T_B|$ paths from $\{w_1, \ldots, w_{2|W|}\} \setminus W$ to $G'$, avoiding $T \cup \{w\}$. Let $R$ be the one of the latter paths. Note that $R$ connects the set $Z_a$ and $G'$.

Let $C$ be the set of the end vertices of paths $Q_1, \ldots, Q_{|W| - |T_B|}$ contained in $W \setminus T$, and let $C' := (W \setminus T) \setminus C$. Since

$$|C| = |W| - |T_B| = |W \setminus T| - |T \setminus V(A)| \geq |W \setminus T| - (|W'| - 1),$$

we have $|C'| \leq |W'| - 1$. Therefore, by the choice of the set $W'$, there is an $(H, Z, 2)$-model in $G - (T_A \cup C')$. In particular, this model should intersect a set in $Z \setminus \{Z_a\}$. So there is a path $P$ from $\bigcup_{Z \in Z \setminus \{Z_a\}} Z$ to $G'$. Therefore, $G' \cup P \cup R$ contains an $(H, Z, 2)$-model, which is contained in $G - (T \cup \{w\})$. It contradicts our assumption that $G - (T \cup \{w\})$ contains no pure $(H, Z, 2)$-model. \hfill \qed

Therefore, $\tau^H_p(G, Z, 2) \leq \tau^H_p(G - w, Z, 2)$ and we conclude that $w$ is an irrelevant vertex. \hfill \qed
4.3 Separating a grid model from sets of \(Z\)

In the proof of Theorem 4.1 we will proceed by induction on \(\ell\). The following proposition shows that given a sufficiently large grid-model, we can find either many disjoint \((H, Z, \ell)\)-models, or a separation that separates a large grid-model from most of sets in \(Z\). The main difference from Theorem 3.1 is that here we also guarantee that when we have a latter separation, we keep having a rooted grid model with a smaller subset of \(Z\). This will help to reduce the instance into an instance with smaller \(\ell\) value so that one can apply the induction hypothesis.

**Proposition 4.6.** Let \(g, k, h, \) and \(\ell\) be positive integers with \(g \geq 2(k\ell(4h + 2) + 1)(k^2\ell^2 + 1)\) and let \(\ell^* := \ell - 1\) if \(\ell \geq 2\), and \(\ell^* := \ell\) otherwise. Every rooted graph \((G, Z)\) having a \(G_g\)-model contains either

1. a pairwise vertex-disjoint \((\ell^* \cdot G_{14h}, Z, \ell)\)-models, or
2. a separation \((A, B)\) of order less than \(k\ell^2\) such that \(|Z \setminus A| < \ell\) and \(B - V(A)\) contains a \(G_{g-k\ell^2}\)-model and also contains a \((Z \setminus A, k, |Z \setminus A|)\)-rooted grid model of order \(k\ell(4h + 2) + 1\).

**Proof.** We start with finding a separation \((A_0, B_0)\) in \(G\) of order less than \(k\ell\) where \(|Z \setminus A_0| \leq \ell - 1\) and \(B_0 - V(A_0)\) contains a \(G_{g-k\ell}\)-model. If \(|Z| \leq \ell - 1\), then the separation \((\emptyset, G)\) is such a separation. Suppose that \(|Z| \geq \ell\). Since \(g \geq (k\ell(4h + 2) + 1)(k^2\ell^2 + 1) + k\ell\), by Theorem 3.1 \(G\) contains either such a separation \((A_0, B_0)\), or a \((Z, k, \ell)\)-rooted grid model of order \(k\ell(4h + 2) + 1\). If it has a rooted grid model, then by Lemma 3.3 \(G\) contains \(k\) pairwise vertex-disjoint \((\ell^* \cdot G_{14h}, Z, \ell)\)-models. Therefore, we may assume that there is a separation \((A_0, B_0)\) in \(G\) of order less than \(k\ell\) where \(|Z \setminus A_0| \leq \ell - 1\) and \(B_0 - V(A_0)\) contains a \(G_{g-k\ell}\)-model.

We show the following.

**Claim 5.** \(G\) contains a separation \((A, B)\) with order less than \(k\ell^2\) such that \(|Z \setminus A| = \ell'\) for some \(0 \leq \ell' < \ell\), and \(B - V(A)\) contains a \((Z \setminus A, k, \ell')\)-rooted grid model of order \(k\ell(4h + 2) + 1\).

**Proof.** We recursively construct a sequence \((A_0, B_0), \ldots, (A_{\ell-1}, B_{\ell-1})\) such that for each \(0 \leq i \leq \ell - 1\),

- \((A_i, B_i)\) is a separation in \(G\) of order less than \(k \sum_{0 \leq j \leq i}(\ell - j)\),
- \(|Z \setminus A_i| \leq (\ell - 1) - i\), and
- \(B_i - V(A_i)\) contains a \(G_{g-|V(A_i \cup B_i)|}\)-model,

unless the separation described in the claim exists. If there is such a sequence, then \((A_{\ell-1}, B_{\ell-1})\) is a separation of order less than

\[
  k \sum_{0 \leq j \leq \ell - 1} (\ell - j) = k \sum_{1 \leq j \leq \ell} j = \frac{k\ell(\ell + 1)}{2} \leq k\ell^2
\]
such that \( ||Z \setminus A_{t-1}|| = 0 \) and \( B_{t-1} - V(A_{t-1}) \) contains a \( G_{g-k\ell^2} \)-model. 
Since a \( G_{g-k\ell^2} \)-model is a \((Z \setminus A_{t-1}, k, 0)\)-rooted grid model of order at least \( k\ell(14h+2)+1 \), \((A_{t-1}, B_{t-1})\) is a required separation.

Note that the sequence \((A_0, B_0)\) exists. Suppose that there is a such a sequence \((A_0, B_0), \ldots, (A_t, B_t)\) exists for some \( 0 \leq t < \ell - 1 \). If \( ||Z \setminus A_t|| < (\ell - 1) - t \), then by taking \((A_{t+1}, B_{t+1}) := (A_t, B_t)\), we have a required sequence \((A_0, B_0), \ldots, (A_{t+1}, B_{t+1})\). Thus, we may assume that \( ||Z \setminus A_t|| = (\ell - 1) - t \). Let \( Y := Z \setminus A_t \). Note that \( ||Y|| = (\ell - 1) - t \).

By Theorem 3.1, \( B_t \) contains either

1. a separation \((C, D)\) of order less than \( k(||Y|| - ||Y\setminus C||)\) where \( ||Y\setminus C|| \leq ||Y|| - 1 \) and \( D - V(C) \) contains a \( G_{g-|V(A_t \cap B_t)|-|V(C \cap D)|} \)-model, or
2. a \((Y, k, ||Y||)\)-rooted grid model of order \( k\ell(14h+2)+1 \).

If it has the latter rooted grid model, then \((A_t, B_t)\) is a required separation where \( t' = t \) and \( Z \setminus A_t = Y \).

Assume that we have a separation \((C, D)\) described in (1). Let \( A_{t+1} := A_t \cup C \) and \( B_{t+1} := D \). Observe that \((A_{t+1}, B_{t+1})\) is a separation in \( G \) such that

- \( ||Z \setminus A_{t+1}|| \leq ||Y|| - 1 = \ell - 1 - (t + 1) \), and
- \( B_{t+1} - V(A_{t+1}) \) contains a \( G_{g-|V(A_t \cap B_t)|-|V(C \cap D)|} \)-model.

Since \( |V(C \cap D)| < k||Y|| = k(\ell - 1 - t) \), \( |V(A_t \cap B_t)| + k(\ell - 1 - t) < k\sum_{0 \leq j \leq t+1} (\ell - j) \), as required. \( \Box \)

We conclude that \( G \) contains either \( k \) pairwise vertex-disjoint \((\ell^* \cdot G_{14h}, Z, \ell)\)-models, or a separation \((A, B)\) with order less than \( \frac{k\ell'(\ell+1)}{2} \) such that \( ||Z \setminus A|| = \ell' \) for some \( 0 \leq \ell' < \ell \) and \( B - V(A) \) contains a \((Z \setminus A, k, \ell')\)-rooted grid model of order \( k\ell(14h+2)+1 \).

\[ \text{Lemma 4.7.} \] Let \( \ell \) be a positive integer. Let \((G, Z)\) be a rooted graph such that there is a separation \((A, B)\) in \( G \) such that \( ||Z \setminus A|| = \ell' \) for some \( 1 \leq \ell' < \ell \). Let \( Z' \) be the multiset of all sets \( X \) in \( Z \) where \( X \subseteq V(A) \). If \( T \) is a pure \((H, Z', \ell - \ell')\)-deletion set of \( A - V(B) \), then \( T \cup V(A \cap B) \) is a pure \((H, Z, \ell)\)-deletion set of \( G \).

\[ \text{Proof.} \] Suppose that \( G - (T \cup V(A \cap B)) \) has a pure \((H, Z, \ell)\)-model \( F \). Since \( |Z \setminus A| = \ell', F \cap (A - V(B)) \) should meet at least \( \ell - \ell' \) sets of \( Z' \). It means that \( F \cap (A - V(B)) \) is a pure \((H, Z', \ell - \ell')\)-model. Since \( T \) meets all such models, it contradicts our assumption. We conclude that \( G - (T \cup V(A \cap B)) \) has no \((H, Z, \ell)\)-models. \( \Box \)

Now, we are ready to give the proof of Theorem 4.1.
Proof of Theorem 4.1. We recall that \( \kappa \) is the function in the Grid Minor Theorem. For a positive integer \( \ell \) and a non-empty planar graph \( H \) with \( cc(H) \geq \ell - 1 \) and \( h = V(H) \), we define that

\[
x = x_{H,\ell}(k) := k\ell^2
\]

\[
g = g_{H,\ell}(k) := 2(4x^2 + 14hx + 3x + 1)(4x^2 + 1) + x
\]

\[
\ell = \ell_{H,\ell}(k) := \begin{cases} \kappa(g_{H,\ell}(k))(h^2 - h + 1)(k - 1) & \text{if } \ell = 1 \\ \kappa(g_{H,\ell}(k))(h^2 - h + 1)(k - 1) + (\ell - 1)f_{H,\ell-1}(k) + k\ell^2 & \text{if } \ell \geq 2. \end{cases}
\]

Note that \( g_{H,\ell}(k) \geq 2(14hx + 2x + 1)(x^2 + 1) \), the function defined in Proposition 4.6. We will use the fact that \( G \) is a minimal counterexample where the theorem does not hold.

We prove the statement by induction on \( \ell + cc(H) \). We first show for two base cases.

- **(Case 1-1. \( \ell = 1 \))**
  
  From \( G \), we obtain a graph \( G' \) of tree-width at most \( \kappa(g) \) such that \( \tau_H(G', Z, \ell) = \tau_H(G', Z, \ell) \), unless \( G \) contains \( k \) pairwise vertex-disjoint pure \((H, Z, 1)\)-models. If \( G \) has tree-width at most \( \kappa(g) \), then we assign \( G' := G \). Suppose \( G \) has tree-width larger than \( \kappa(g) \). By Theorem 2.1 \( G \) contains a \( G_g \)-model.

  So, by Proposition 4.6 \( G \) contains either \( k \) pairwise vertex-disjoint pure \((H, Z, 1)\)-models, or a separation \((A, B)\) with order less than \( x \) such that \( |Z \setminus A| = 0 \) and \( B - V(A) \) contains a \( G_{x-x} \)-model. We may assume that we have the latter separation \((A, B)\). Then, since

  \[
g - x \geq (x^2 + 14hx + 2x + 1)(x^2 + 1),
\]

by recursively finding an irrelevant vertex until the resulting graph has no \( G_g \)-models, we can eventually output either \( k \) vertex-disjoint pure \((H, Z, 1)\)-models or a graph \( G' \) of tree-width at most \( \kappa(g) \) where \( \tau_H^+(G', Z, \ell) = \tau_H^+(G', Z, \ell) \). When we have the graph \( G' \), by Proposition 4.3 \( G' \) contains \( k \) pairwise vertex-disjoint \((H, Z, \ell)\)-models or a \((H, Z, \ell)\)-deletion set of size at most \((\kappa(g) + 1)(h^2 - h + 1)(k - 1) \). Thus, we conclude that \( \tau_H(G, Z, \ell) = \tau_H^+(G', Z, \ell) \leq (\kappa(g) + 1)(h^2 - h + 1)(k - 1) = f_{H,\ell}(k) \).

- **(Case 1-2. \( \ell = 2 \) and \( cc(H) = 1 \).)** This case is almost same as Case 1, but we use Proposition 4.5 instead of Proposition 4.4 to find an irrelevant vertex. This is possible because \( g - x \geq 2(4x^2 + 14hx + 3x + 1)(4x^2 + 1) \).

Now we assume that \( \ell \geq 2 \) and \( cc(H) \geq 2 \) if \( \ell = 2 \). We may assume that \( G \) has tree-width at least \( \kappa(g) \), otherwise we can apply Proposition 4.3. Now, we reduce the given instance in two ways; either reduce its tree-width or the parameter \( \ell \). By Theorem 2.1 \( G \) contains a \( G_g \)-model. Suppose that \( G \) is a minimal counterexample where the theorem does not hold.
By Proposition 4.6, \( G \) contains either \( k \) pairwise vertex-disjoint \((H, \mathcal{Z}, \ell)\)-models, or a separation \((A, B)\) with order less than \( x \) such that

- \(|\mathcal{Z}\setminus A| = \ell' < \ell\) and \( B - V(A) \) contains a \( \mathcal{G}_{\ell-x} \)-model and a \((\mathcal{Z}\setminus A, k, |\mathcal{Z}\setminus A|)\)-rooted grid model of order \( k\ell(14h + 2) + 1 \).

We may assume that we have the latter separation. Let \( \mathcal{Z}' \) be the multiset of all sets \( X \in \mathcal{Z} \) where \( X \subseteq V(A) \).

We observe the following.

- If \( \ell' = 0 \), then there is an irrelevant vertex \( v \) by Proposition 4.4.
- If \( \ell' = 1 \) and \( \ell = 2 \), then there is an irrelevant vertex \( v \) by Proposition 4.5.

Thus, in these cases, \( G - v \) satisfies the theorem because \( G \) is chosen as a minimal counterexample. But then \( G \) also satisfies the theorem as \( \tau^*_{H}(G, \mathcal{Z}, \ell) = \tau^*_{H}(G - v, \mathcal{Z}, \ell) \). Thus, we can assume that \( \ell' \geq 1 \) and \( \ell \geq 3 \) if \( \ell' = 1 \). We will argue that in the remaining part, one can reduce the instance into \( A - V(B) \) with the parameter \( \ell' - \ell \).

Recall that \( \text{cc}(H) \geq \ell - 1 \). We divide into two cases depending on \( \text{cc}(H) \geq \ell \) or not.

- (Case 2-1. \( \text{cc}(H) \geq \ell \)) Since \( B - V(A) \) contains a \((\mathcal{Z}\setminus A, k, \ell')\)-rooted grid model of order \( k\ell(14h + 2) + 1 \), by Lemma 3.3, \( B - V(A) \) contains \( k \) pairwise vertex-disjoint \((\ell' \cdot \mathcal{G}_{14h}, \mathcal{Z}\setminus A, \ell')\)-models. We will use this later.

Since \( \ell' > 0 \), we have \( \ell - \ell' < \ell \). So, we apply the induction hypothesis on \((A - V(B), \mathcal{Z}', \ell - \ell', k)\), and we have that \( A - V(B) \) contains either \( k \) vertex-disjoint pure \((H, \mathcal{Z}', \ell - \ell')\)-models, or a pure \((H, \mathcal{Z}', \ell - \ell')\)-deletion set \( T \) of size at most \( f^1_{H, \ell - \ell'}(k) \). If it outputs a deletion set \( T \), then by Lemma 4.7 \( G - (T \cup V(A \cap B)) \) has no pure \((H, \mathcal{Z}, \ell)\)-models. Since

\[
 f^1_{H, \ell - \ell'}(k) + k\ell^2 \leq f^1_{H, \ell - 1}(k) + k\ell^2 \leq f^1_{H, \ell}(k),
\]

\( T \cup V(A \cap B) \) is a required pure \((H, \mathcal{Z}, \ell)\)-deletion set in \( G \).

Suppose we have \( k \) pairwise vertex-disjoint pure \((H, \mathcal{Z}', \ell - \ell')\)-models in \( A - V(B) \). Since each model in \( A - V(B) \) consists of the image of at most \( \ell - \ell' \) connected components of \( H \) and \( H \) consists of at least \( \ell \) connected components, we can complete it into a pure \((H, \mathcal{Z}, \ell)\)-model by taking a relevant pure \((H, \mathcal{Z}\setminus A, \ell')\)-model from a \((\ell' \cdot \mathcal{G}_{14h}, \mathcal{Z}\setminus A, \ell')\)-model in \( B - V(A) \). For instance, if \( H_1, \ldots, H_\ell \) is a set of connected components of \( H \) and a pure \((H, \mathcal{Z}', \ell - \ell')\)-models in \( A - V(B) \) is the image of the union of \( H_1, \ldots, H_{\ell - \ell'} \), then we obtain images of \( H_{\ell - \ell' + 1}, \ldots, H_\ell \) from each connected component of the \( \ell' \cdot \mathcal{G}_{14h} \)-grid model. Therefore, \( G \) contains \( k \) pairwise vertex-disjoint pure \((H, \mathcal{Z}, \ell)\)-models.

- (Case 2-2. \( \text{cc}(H) = \ell - 1 \)) When \( \ell' \geq 2 \), we can prove as in Case 2-1. Note that every pure \((H, \mathcal{Z}', \ell-
\(\ell'\)-model is the image of at most \(\ell - \ell'\) connected components of \(H\), and there are \(k\) pairwise vertex-disjoint pure \((\ell' - 1) \cdot \mathcal{G}_{14h} \cdot \mathcal{Z} \cdot A, \ell'\)-models in \(B - V(A)\) by Lemma 3.3. Thus, we can return \(k\) pairwise vertex-disjoint pure \((H, \mathcal{Z}, \ell)\)-models, or a pure \((H, \mathcal{Z}, \ell)\)-deletion set of size at most \(f_{H,\ell}^1(k)\). But, we cannot do the same thing when \(\ell' = 1\), because we can only say that \(B - V(A)\) contains \(k\) pairwise vertex-disjoint pure \((\ell' - 1) \cdot \mathcal{G}_{14h} \cdot \mathcal{Z} \cdot A, \ell'\)-models, not pure \((\ell' - 1) \cdot \mathcal{G}_{14h} \cdot \mathcal{Z} \cdot A, \ell'\)-models. We may assume that \(\ell' = 1\). We may also assume that \(\ell \geq 3\) because \(H\) is connected when \(\ell = 2\) and this case was resolved in (Case 1-2).

Note that a pure \((H, \mathcal{Z}', \ell - 1)\)-model in \(A - V(B)\) may be the image of \(H\) itself. But, this cannot be a part of a pure \((H, \mathcal{Z}, \ell)\)-model of \(G\), and thus, we ignore it (the model already used all components to meet only \(\ell - 1\) sets of \(\mathcal{Z}\)). For this reason, we only consider pure \((H, \mathcal{Z}', \ell - 1)\)-models in \(A - V(B)\) that are not the images of \(H\). Since \(H\) consists of at least 2 connected components, we can observe that there are no \(k\) pairwise vertex-disjoint pure \((H, \mathcal{Z}', \ell - 1)\)-models in \(A - V(B)\) that are not the images of \(H\).

For each subgraph \(H'\) of \(H\) induced by its \(\ell - 2\) connected components, we apply induction hypothesis with \((A - V(B), \mathcal{Z}', \ell - 1, k)\). Since \(\text{cc}(H') < \text{cc}(H)\), we obtain that \(A - V(B)\) contains either \(k\) pairwise vertex-disjoint pure \((H', \mathcal{Z}', \ell - 1)\)-models or a pure \((H', \mathcal{Z}', \ell - 1)\)-deletion set \(T_{H'}\) of size at most \(f_{H',\ell-1}^1(k)\).

Suppose that for some subgraph \(H'\), we have \(k\) pairwise vertex-disjoint pure \((H', \mathcal{Z}', \ell - 1)\)-models in \(A - V(B)\). Then we can complete them into \(k\) pairwise vertex-disjoint pure \((H, \mathcal{Z}, \ell)\)-models in \(G\) using \(k\) pairwise vertex-disjoint \((\mathcal{G}_{14h}, \mathcal{Z}', 1)\)-models in \(B - V(A)\). Therefore, we may assume that for all possible subgraphs \(H'\) of \(H\) with \(\ell - 2\) connected components, we have a pure \((H', \mathcal{Z}', \ell - 1)\)-deletion set \(T_{H'}\) in \(A - V(B)\). Let \(T\) be the union of all such deletion sets \(T_{H'}\). Note that \(|T| \leq (\ell - 1)f_{H,\ell-1}^1(k)\).

We claim that \(G - (T \cup V(A \cap B))\) has no pure \((H, \mathcal{Z}, \ell)\)-models. Suppose \(G - (T \cup V(A \cap B))\) has a pure \((H, \mathcal{Z}, \ell)\)-model \(F\). First assume that \(F\) is fully contained in \(A - V(B)\). If \(F\) is the image of a subgraph \(H'\) of \(H\) consisting of its at most \(\ell - 2\) connected components, then there exists a subgraph \(H''\) of \(H\) induced by its exactly \(\ell - 2\) connected components where \(H''\) is a subgraph of \(H''\). Thus, by ignoring the set in \(\mathcal{Z}\) intersecting \(B - V(A)\), \(F\) contains a \((H'', \mathcal{Z}', \ell - 1)\)-model. But \(T_{H''}\) meets a vertex of \(F\), contradicting the assumption that \(V(F) \cap T = \emptyset\). Thus, we may assume that \(F\) is the image of \(H\). Since \(\ell \geq 3\) and \(H\) consists of \(\ell - 1\) connected components and \(F\) intersects \(\ell\) sets of \(\mathcal{Z}\), there should be a connected component of \(F\) intersecting exactly one set among those \(\ell\) sets. In other words, there is a subgraph \(H'\) of \(H\) consisting of its \(\ell - 2\) connected components where the subgraph of \(F\) induced by the image of \(H'\) intersects \(\ell - 2\) sets of \(\mathcal{Z}\). But this contradicts our assumption that \(V(F) \cap T_{H'} = \emptyset\).
So, we may assume that $F \setminus V(A) \neq \emptyset$. In this case, since $\ell' = 1$, only one connected component of $F$ can be contained in $B \setminus V(A)$. So, there are at most $\ell - 2$ connected components of $F \cap (A - V(B))$ whose union meets $\ell - 1$ sets of $Z'$. It means that $F \cap (A - V(B))$ is a pure $(H', Z', \ell - 1)$-model for some subgraph $H'$ of $H$ induced by its $\ell - 2$ connected components. Since $T$ meets all such models, we have a contradiction.

We conclude that $G - (T \cup (A \cap B))$ has no pure $(H, Z, \ell)$-models. Since $(\ell - 1)f_{H,\ell}^1(k) + k\ell^2 \leq f_{H,\ell}^1(k)$, we have a required pure $(H, Z, \ell)$-deletion set.

We conclude that $G$ contains either $k$ pairwise vertex-disjoint pure $(H, Z, \ell)$-models, or a pure $(H, Z, \ell)$-deletion set of size at most $f_{H,\ell}^1(k)$.

\[ \square \]

5 Packing and covering $(H, Z, \ell)$-models

Now we prove the main result of this paper.

**Theorem 1.1.** For a positive integer $\ell$ and a non-empty planar graph $H$ with $cc(H) \geq \ell - 1$, there exists $f_{H,\ell} : \mathbb{N} \to \mathbb{R}$ satisfying the following property. Let $(G, Z)$ be a rooted graph and $k$ be a positive integer. Then $G$ contains either $k$ pairwise vertex-disjoint $(H, Z, \ell)$-models in $G$, or an $(H, Z, \ell)$-deletion set of size at most $f_{H,\ell}^1(k)$.

**Proof.** We recall from the proof of Theorem 4.1 functions $x, g$, and $f^1$. For all positive integers $k, \ell$, and $h$,

\[ x = x_{H,\ell}(k) := k\ell^2 \]

\[ g = g_{H,\ell}(k) := 2(4x^2 + 14hx + 3x + 1)(4x^2 + 1) + x \]

\[ f_{H,\ell}^1(k) := \begin{cases} 
\kappa(g_{H,\ell}(k))(h^2 - h + 1)(k - 1) & \text{if } \ell = 1 \\
\kappa(g_{H,\ell}(k))(h^2 - h + 1)(k - 1) + (\ell - 1)f_{H,\ell}^1(k) + k\ell^2 & \text{if } \ell \geq 2.
\end{cases} \]

\[ f_{H,\ell}(k) := \ell f_{H,\ell}^1(k) + k\ell^2. \]

We observe that if $cc(H) = 1$, then pure $(H, Z, \ell)$-models are $(H, Z, \ell)$-models. Therefore, Theorem 4.1 implies the statement. We may assume that $cc(H) \geq 2$.

If $G$ has tree-width at most $\kappa(g)$, and by Proposition 4.3 $G$ contains either $k$ pairwise vertex-disjoint $H$-models or a $(H, Z, \ell)$-deletion set $T$ of size at most $(\kappa(g) + 1)(h^2 - h + 1)(k - 1) \leq f_{H,\ell}^1(k) \leq f_{H,\ell}(k)$. Therefore, we may assume that $G$ has tree-width larger than $\kappa(g)$. By Theorem 2.1 $G$ contains a $G_g$-model.

By Proposition 4.6 $G$ contains either $k$ pairwise vertex-disjoint $(\ell^* G_{14h}, Z, \ell)$-models, or a separation $(A, B)$ with order less than $x$ such that

- $||Z \setminus A|| = \ell'$ for some $0 \leq \ell' < \ell$ and $B - V(A)$ contains a $G_{g-\ell'}$-model, and contains a $(Z \setminus A, k, \ell')$-rooted grid model of order $k\ell(14h + 2) + 1$.
If $G$ has $k$ pairwise vertex-disjoint $(\ell^* \cdot G_{14h}, Z, \ell)$-models, then by Lemma 3.3 it has $k$ pairwise vertex-disjoint $(H, Z, \ell)$-models. Thus, we may assume that we have the latter separation.

If $\ell' \geq 1$, then we do a procedure similar to the proof in Theorem 4.1. But, when $\ell' = 0$ there was an irrelevant vertex argument for pure models. That argument cannot be extended to usual models. Instead, we can reduce the instance into an instance of packing and covering $k$ pairwise disjoint pure $(H, Z, \ell)$-models in $A - V(B)$.

- (Case 1. $\ell' = 0$.)
  We apply Theorem 4.1 to the instance $(A - V(B), Z, \ell, k)$ for pure $(H, Z, \ell)$-models. Then $A - V(B)$ contains either $k$ pairwise vertex-disjoint pure $(H, Z, \ell)$-models, or a pure $(H, Z, \ell)$-deletion set $T$ of size at most $f_{H, \ell}(k)$.
  Suppose that $A - V(B)$ has $k$ pairwise vertex-disjoint pure $(H, Z, \ell)$-models. Note that among those models, some of them is the image of $H$ itself, and some is not the image of $H$. Since $g - x \geq 14hk$, $B - V(A)$ contains $k$ vertex-disjoint $G_{14h}$-models. So, for those pure models in $A - V(B)$ that are not the images of $H$, we can complete them into $(H, Z, \ell)$-models, using $G_{14h}$-models in $B - V(A)$. Therefore, $G$ has $k$ pairwise vertex-disjoint $(H, Z, \ell)$-models.
  On the other hand, if we have a pure $(H, Z, \ell)$-deletion set $T$, then $G - (T \cup V(A \cap B))$ has no $(H, Z, \ell)$-models. This is because every $(H, Z, \ell)$-model in $G - V(A \cap B)$ contains a pure $(H, Z, \ell)$-model in $A - V(B)$ (by collecting only components hitting $\ell$ sets of $Z$). Since
  $$|T \cup V(A \cap B)| \leq f_{H, \ell}(k) + k\ell^2 \leq f_{H, \ell}(k),$$
  we have a required set.

Now, we assume that $\ell' \geq 1$. Let $Z'$ be the multiset of all sets $X$ in $Z$ where $X \subseteq V(A)$. We follow a similar procedure in Theorem 4.1. We divide cases depending on whether $cc(H) \geq \ell$ or not.

- (Case 2.1. $cc(H) \geq \ell$.)
  Since $B - V(A)$ contains a $(Z \setminus A, \ell', k)$-rooted grid model of order $kt(14h + 2) + 1$, by Lemma 3.3 $B - V(A)$ contains $k$ pairwise vertex-disjoint $(\ell', G_{14h}, Z \setminus A, \ell')$-models.
  We apply Theorem 4.1 to the instance $(A - V(B), Z', \ell - \ell', k)$. Then $A - V(B)$ contains either $k$ pairwise vertex-disjoint pure $(H, Z', \ell - \ell')$-models, or a pure $(H, Z', \ell - \ell')$-deletion set $T$ of size at most $f_{H, \ell - \ell'}(k)$.
  In the former case, we can obtain $k$ pairwise vertex-disjoint $(H, Z', \ell')$-models in $G$ using the $k$ pairwise vertex-disjoint $(\ell', G_{14h}, Z', \ell')$-models in $B - V(A)$. For the latter case, $G - (T \cup V(A \cap B))$ has no $(H, Z, \ell)$-models where $|T \cup V(A \cap B)| \leq f_{H, \ell - \ell'}(k) + k\ell^2 \leq f_{H, \ell}(k)$.

- (Case 2.2. $cc(H) \leq \ell - 1$.)
  If $\ell' \geq 2$, then we can obtain the same result as in Case 2.1, because
there are in fact \( k \) pairwise vertex-disjoint \( ((\ell' - 1) \cdot \mathcal{G}_{14h}, \mathcal{Z}', \ell') \)-models in \( B - V(A) \) by Lemma 3.3. We may assume that \( \ell' = 1 \). Since \( cc(H) \geq 2 \), we have \( \ell \gg 3 \).

Now, for each subgraph \( H' \) of \( H \) induced by its \( \ell - 2 \) connected components, we apply Theorem 4.1 to the instance \( (A - V(B), \mathcal{Z}', \ell - 1, k) \). Then we deduce that \( A - V(B) \) contains either \( k \) pairwise vertex-disjoint pure \( (H', \mathcal{Z}', \ell - 1) \)-models, or a pure \( (H', \mathcal{Z}', \ell - 1) \)-deletion set \( T_{H'} \) of size at most \( f_{H, \ell - 1}(k) \). If \( A - V(B) \) contains \( k \) pairwise vertex-disjoint pure \( (H', \mathcal{Z}', \ell - 1) \)-models for some subgraph \( H' \) of \( H \) induced by its \( \ell - 2 \) connected components, then we can complete them using \( (\mathcal{G}_{14h}, \mathcal{Z}', 1) \)-models in \( B - V(A) \). Therefore, we may assume that there is a pure \( (H', \mathcal{Z}', \ell - 1) \)-deletion set \( T_{H'} \) of size at most \( f_{H, \ell - 1}(k) \) in \( A - V(B) \) for all such subgraphs \( H' \). Let \( T \) be the union of all such deletion sets \( T_{H'} \). Note that \( |T| \leq (\ell - 1)f_{H, \ell - 1}(k) \).

We claim that \( G - (T \cup V(A \cap B)) \) has no \( (H, \mathcal{Z}, \ell) \)-models. Suppose that \( G - (T \cup V(A \cap B)) \) has an \( (H, \mathcal{Z}, \ell) \)-models \( F \). Since \( \ell' = 1 \), there are \( \ell - 2 \) connected components of \( F \cap (A - V(B)) \) that meet \( \ell - 1 \) sets of \( \mathcal{Z}' \), which contains a pure \( (H'', \mathcal{Z}', \ell - 1) \)-model for some subgraph \( H'' \) of \( H \) induced by \( \ell - 2 \) connected components of \( H \). Since \( T \) meets all such models, we conclude that \( G - (T \cup V(A \cap B)) \) has no \( (H, \mathcal{Z}, \ell) \)-models.

Since
\[
|T \cup V(A \cap B)| \leq (\ell - 1)f_{H, \ell - 1}(k) + k\ell^2 \leq f_{H, \ell}(k),
\]
we have a required \( (H, \mathcal{Z}, \ell) \)-deletion set.

We conclude that \( G \) contains either \( k \) pairwise vertex-disjoint \( (H, \mathcal{Z}, \ell) \)-models in \( G \), or an \( (H, \mathcal{Z}, \ell) \)-deletion set of size at most \( f(k, \ell, h) \).

\[\square\]

6 Examples when \( H \) has at most \( \ell - 2 \) connected components

In this section, we prove that if the number of connected components of \( H \) is at most \( \ell - 2 \), then the class of \( (H, \mathcal{Z}, \ell) \)-models does not have the Erdős-Pósa property.

**Proposition 6.1.** Let \( \ell \) be a positive integer and \( H \) be a non-empty planar graph with at most \( \ell - 2 \) connected components. Then the class of \( (H, \mathcal{Z}, \ell) \)-models does not have the Erdős-Pósa property.

**Proof.** We show that for every positive integer \( x \), there is a rooted graph \( (G, \mathcal{Z}) \) satisfying that
- \( G \) has one \( (H, \mathcal{Z}, \ell) \)-model, but no two vertex-disjoint \( (H, \mathcal{Z}, \ell) \)-models,
- for every vertex subset \( S \) of size at most \( x \), \( G - S \) has an \( (H, \mathcal{Z}, \ell) \)-model.

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For each \( \ell \) contain the image of one connected component of \( H \) in Figure 4. Especially, \( F \) is the image of one connected component of \( H \) which meets all sets of \( Z \) contained in \( G_1 \). However, since \( F_1 \cap G_1 \) and \( F_2 \cap G_1 \) are connected, they should intersect.

Now we claim that if \( n \geq 14h + x + 1 \) and \( S \) is a vertex set of size at most \( x \) in \( G \), then \( G - S \) contains an \((H, Z, \ell)\)-model. Suppose that \( n \geq 14h + x + 1 \) and \( S \) is a vertex set of size at most \( x \) in \( G \).

Let us fix \( 2 \leq j \leq t \). Then there exists \( 1 \leq p_j \leq n - (x + 1) \) and \( 0 \leq q_j \leq n - (x + 1) \) such that \( v_{p_j + a, q_j + b} \notin S \) for all \( 1 \leq a, b \leq 14h + x + 1 \). Let \( F_j \) be the subgraph of \( G_j \) induced by the vertex set \( \{v_{p_j + a, q_j + b} : 1 \leq a, b \leq x + 1 \} \). Clearly there are \( x + 1 \) pairwise vertex-disjoint paths from \( Z_{j+\ell-t} \) to \( \{v_{p_j + 1, q_j + b} : 1 \leq b \leq x + 1 \} \) and there is at least one path that does not meet \( S \). Since \( F_j \) contains an \( H_j \)-model, \( G_j - S \) contains an \( H_j \)-model having a vertex of \( Z_{j+\ell-t} \).

Figure 4: The construction for showing that \((H, Z, \ell)\)-models have no Erdős-Pósa property when the number of connected components of \( H \) is at most \( \ell - 2 \). This is an example when \( \ell = 8 \) and \( cc(H) = 5 \).
Similarly, in $G_1$, there exists $1 \leq p_1 \leq n - (x + 1)$ and $(\ell - t + 1)n \leq q_1 \leq (\ell - t + 1)n + n - (x + 1)$ such that $v^j_{p_1+a,q_1+b} \not\in S$ for all $1 \leq a, b \leq 14h + x + 1$. Let $F_1$ be the subgraph of $G_1$ induced by the vertex set $\{v^1_{p_1+a,q_1+b} : 1 \leq a, b \leq x + 1\}$. It is not hard to observe that there are $x + 1$ pairwise vertex-disjoint paths from $\{v^1_{p_1+a,q_1+1} : 1 \leq a \leq x + 1\}$ to each $Z_i$ for $1 \leq i \leq \ell - t + 1$. Therefore, $G_1 - S$ contains an $H_1$-model having a vertex of $Z_i$ for $1 \leq i \leq \ell - t + 1$. We conclude that $G - S$ has an $(H, Z, \ell)$-model, as required.

7 Conclusion

In this paper, we show that the class of $(H, Z, \ell)$-models has the Erdős-Pósa property if and only if $H$ is planar and $cc(H) \geq \ell - 1$. Among all the interesting results on the Erdős-Pósa property, some objects that intersect $\ell$ sets among given vertex sets have not much studied. For our result, we do not restrict an $H$-model to a minimal subgraph containing $H$-minor. What if we consider minimal $H$-models or $H$-subdivisions? Then it may be difficult to have such a nice characterization. As a first step to study these families, we pose one specific problem:

- Does the set of cycles intersecting at least two sets among given sets $Z_1, Z_2, \ldots, Z_m$ in a graph $G$ have the Erdős-Pósa property?

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