Summary of Delsarte’s “Nombre de Solutions des Équations Polynomiales sur un Corps Fini”

Wim van Dam*

February 5, 2020

Abstract
An English summary is given of Jean Delsarte’s article “Nombre de solutions des équations polynomiales sur un corps fini.”

Introduction
These notes grew out of my desire to check the details of the article:

• Jean Delsarte, “Nombre de solutions des équations polynomiales sur un corps fini”, Séminaire Bourbaki, Exposé 39:1–9, March 1951

Because I was only interested in the main result, I did not translate the Sections 2 and 4. The same notation and equation numbering is maintained and the original page numbering is included in the right margin of the text. Some typos are corrected and I added some potentially helpful comments in italic. An alternative proof of the result of Delsarte was published in [2]. This paper cites both Delsarte and the 1949 article [1] by Elza Furtado Gomida as sources for the original result. I make this translation public to increase the accessibility of Delsarte’s article. The current translation is by no means authoritative and does not contain any new results. Comments are welcome.

References
[1] Elza Furtado Gomida, “On the theorem of Artin-Weil”, Boletim da Sociedade de Matemática de São Paulo, Volume 4, pp. 1–18 (1949,1951)

[2] Neal Koblitz, “The number of points on certain families of hypersurfaces over finite fields”, Compositio Mathematica, Volume 48, No. 1, pp. 3–23 (1983)

*Massachusetts Institute of Technology, Center for Theoretical Physics, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA. email: vandam@mit.edu. This work is supported in part by funds provided by the U.S. Department of Energy and cooperative research agreement DF-FC02-94ER40818 and by a CMI postdoctoral fellowship.
Number of Solutions of Polynomial Equations over Finite Fields
by Jean Delsarte

1 Gauss Sums of Finite Fields
Let $K$ be a finite field $\mathbb{F}_q$ and $K^0$ the multiplicative group $\mathbb{F}_q^\times$. Also, let $\chi$ be a multiplicative character and $\psi$ a non-trivial additive character. The Gauss sum over $K$ is defined by
\[ g(\chi) := \sum_{x \in K} \chi(x)\psi(x) \]
where the $x$ could also run over $K^0$ as $\chi(0) = 0$. By changing $x$ into $tx$ with $t \in K^0$ we get
\[ g(\chi) = \chi(t) \sum_{x \in K} \chi(x)\psi(tx), \]
which shows how to convert to different additive characters $\psi$ by multiplying the Gauss sum by a know factor $\chi(t)$.

A classic results concerns the absolute value of the Gauss sum; we have
\[ g(\chi)\overline{g} = \sum_{x \in K} \sum_{y \in K^0} \chi(xy^{-1})\psi(x - y) \]
\[ = \sum_{x \in K^0} \chi(x) \sum_{y \in K^0} \psi((x - 1)y), \]
where the summation over $y \in K^0$ is $q - 1$ if $x = 1$, and $-1$ otherwise. Thus
\[ |g(\chi)| = \sqrt{q}. \]

2 Finite Extensions of Finite Fields: the Hasse-Davenport Theorem [...]

3 Some Enumerative Formulae
Let $E_s$ be the $s$-dimensional $K$ vector space $K^s$, view $E_s$ as a ring with pointwise addition and multiplication. Consider the variety defined by the equation
\[ F := \sum_{i=1}^r a_ix_1^{m_{i1}} \cdots x_s^{m_{is}} = 0 \]
with $a_i \in K^0$ for $i = 1, \ldots, r$. The size $q$ of $K$ is big, such that $q - 1$ does not divide any of the $m_{ij}$. Let $\psi$ be an additive character over $K$; we want to calculate the summation $S := \sum x \psi(\mathcal{F}(x))$, where the $x = (x_1, \ldots, x_s) \in E_s$. We start by calculating the sum $\bar{S}$ where we only sum over those $x$ with $x_j \in K^0$ (that is: $x \in E_s^0$). Let
\[
y_i = x_1^{m_{i1}} \cdots x_s^{m_{is}}, \tag{1}\]
for $i = 1, \ldots, r$, where $x = (x_1, \ldots, x_s)$ is an invertible element of the ring $E_s = K \times \cdots \times K$ (“Véronèse variety”). Equation 1 defines a group homomorphism from $E_s^0$ to $E_r^0$ (of the direct products of the multiplicative groups $K^0$). Let $d$ be the size of the kernel of this homomorphism and let $G$ be its image; then we have
\[
\bar{S} = \sum_{x \in E_r^0} \psi(\mathcal{F}(x)) = d \sum_{y \in G} \psi(ay) \tag{2}\]
with $a = (a_1, \ldots, a_r) \in E_r^0$ (the coefficients of $\mathcal{F}$). The product $ay$ is expressed in the ring $E_r$ and the additive character $\psi$ is extended to this ring (according to $\psi(ay) := \psi(a_1 y_1 + \cdots + a_r y_r) = \psi(a_1 y_1) \cdots \psi(a_r y_r)$).

Let $\chi$ be a multiplicative character of the group $E_r^0$; we have for $y = (y_1, \ldots, y_r) \in E_r^0$
\[
\chi(y) = \chi_1(y_1) \cdots \chi_r(y_r)
\]
where $(\chi_1, \ldots, \chi_r)$ is a system of $r$ multiplicative characters of $K$. Let us introduce the group $G$ (orthogonal to $\hat{G}$), which is the ensemble of characters on $E_r^0$ with $\chi(y) = 1$ for every $y \in G$. Such a character is constant on the cosets of $G$ in $E_r^0$. (The group $\hat{G}$ is the group of multiplicative characters on $E_r^0$; hence its size is $d(q - 1)^{r-s}$.) Now consider the sum
\[
T = \sum_{\chi \in G} \sum_{y \in E_r^0} \chi(y) \psi(ay). \tag{3}\]
For fixed $y$, the sum $\sum_{\chi} \chi$ is zero when $y$ is outside $G$, for $y$ in $G$ the sum is $|G| = d(q - 1)^{r-s}$. Therefore
\[
T = d(q - 1)^{r-s} \sum_{y \in G} \psi(ay),
\]
hence
\[
T = (q - 1)^{r-s} \bar{S},
\]
and finally
\[
\bar{S} = (q - 1)^{s-r} \sum_{y \in G} \sum_{y \in E_r^0} \chi(y) \psi(ay). \tag{4}\]
Consider again the Gauss sums, which we have defined for $K$ with an additive character $\psi$. For a multiplicative character $\chi = (\chi_1, \ldots, \chi_r)$ over $E^0$, define

$$G(\chi) = g(\chi_1) \cdots g(\chi_r).$$

Because $\psi(ay) = \psi(a_1 y_1) \cdots \psi(a_r y_r)$ we find (because of the earlier derived equality $\sum_{y_j} \chi_j(y_j) \psi(a_j y_j) = \bar{\chi}_j(a_j) g(\chi_j)$)

$$\sum_{y \in E^0} \chi(y) \psi(ay) = \bar{\chi}(a) G(\chi)$$

and hence

$$\bar{S} = (q-1)^s - \sum_{\chi \in \tilde{G}} \bar{\chi}(a) G(\chi).$$

**An application of the result.** Let us try to calculate the number of solutions of $F = 0$ in $E^0_s$, which gets denoted by $\bar{N}$. Let

$$\bar{S}(\psi) := \sum_{x \in E^0_s} \psi(F(x))$$

Now calculate the sum of values $\bar{S}(\psi)$ where $\psi$ ranges over all non-trivial additive characters over $K$:

$$\sum_{\psi} \bar{S}(\psi) = \sum_{\psi} \sum_{x \in E^0_s} \psi(F(x)).$$

For fixed $x$, the sum over the characters $\psi$ will be $-1$ if $F(x)$ is not 0, and $q-1$ if $F(x) = 0$, hence

$$\sum_{\psi} \bar{S}(\psi) = (q-1)\bar{N} - ((q-1)^s - \bar{N})$$

$$= q\bar{N} - (q-1)^s$$

because the number values $x \in E^0_s$ for which $F(x) \neq 0$ is $(q-1)^s - \bar{N}$. Now fix a nontrivial character $\psi_0$. For every other nontrivial character $\psi$ we have a $u \in K^0$ such that $\psi(x) = \psi_0(ux)$ for all $x \in K$ (thus establishing a bijection between the set of nontrivial characters and $K^0$). Moreover we have

$$g(\chi_1) = \sum_{x_1 \in K^0} \chi_1(x_1) \psi(x_1) = \sum_{x_1 \in K^0} \chi_1(x_1) \psi_0(ux_1),$$

or

$$g(\chi_1) = \bar{\chi}_1(u) g_0(\chi_1)$$

where $g_0$ is the Gauss sum where we used the additive character $\psi_0$. Similarly, one finds

$$G(\chi) = \lambda(u) G_0(\chi),$$
for the Gauss sums over $E_r$. (Here $\lambda$ is the multiplicative character over $K^0$ defined by the product $\lambda = \chi_1 \cdots \chi_r$.) Finally, one gets
\[ S(\psi) = (q-1)^{s-r} \sum_{\chi \in \tilde{G}} \lambda(u) \bar{\chi}(a) G_0(\chi). \]

If we want to sum this expression over all nontrivial $\psi$, it is sufficient to sum over all $u \in K^0$. If $\chi$ is a multiplicative character over $E_r^0$, then for the character $\lambda$ over $K^0$, we have that $\sum_{u \in K^0} \lambda(u)$ is 0 if $\lambda$ is nontrivial, and the sum is $q - 1$ if $\lambda$ is trivial. We thus have
\[ \sum_{\psi} \bar{S}(\psi) = (q-1)^{s-r+1} \sum_{\chi \in \tilde{G}^*} \chi(a) G_0(\chi) \quad (7) \]
where $\tilde{G}^*$ is the subgroup of $\tilde{G}$ under the restriction that the product $\chi_1 \cdots \chi_r$ is the trivial multiplicative character of $K^0$. “Without pain” we thus get the final result
\[ \bar{N} = \frac{1}{q} \left( (q-1)^{s} + (q-1)^{s-r+1} \sum_{\chi \in \tilde{G}^*} \bar{\chi}(a) G_0(\chi) \right). \quad (8) \]

4 The Artin-Weil Series [. . .]

References
[1] H. Davenport and H. Hasse, “Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen”, Journal für die Reine und Angewandte Mathematik, Volume 172, pp. 151–182 (1935)
[2] André Weil, “Numbers of Solutions of equations in finite fields”, Bulletin of the American Mathematical Society, Volume 55, pp. 497–508 (1949)
[3] André Weil, “Sur les courbes algébriques et les variétés qui s’en déduisent”, Actualités scientifiques et industrielles, no. 1041; Publications de l’Institut de mathématique de l’Université de Strasbourg, Volume 7 (1945), Hermann et Cie., Paris (1948)
[4] André Weil, “Variétés abéliennes et courbes algébriques”, Actualités scientifiques et industrielles, no. 1064; Publications de l’Institut de mathématique de l’Université de Strasbourg, Volume 8 (1946), Hermann et Cie. Paris (1948)