Some basic aspects of quantum phase transitions

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Abstract

Several basic problems of the theory of quantum phase transitions are reviewed. The effect of the quantum correlations on the phase transition properties is considered with the help of basic models of statistical physics. The effect of quenched disorder on the quantum phase transitions is also discussed. The review is performed within the framework of the thermodynamic scaling theory and by the most general methods of statistical physics for the treatment of phase transitions: general length-scale arguments, exact solutions, mean field approximation, Hubbard-Stratonovich transformation, Feynman path integral approach, and renormalization group in the field theoretical variant. Some new ideas and results are presented. Outstanding theoretical problems are mentioned.

1. Introduction

1.1. Historical notes

The experimental and theoretical research of quantum phase transitions and quantum critical phenomena is a branch of the statistical physics of a rapidly growing importance for the explanation of essential features of low dimensional fermion and spin systems, dilute Bose fluids, superconductors, quantum Hall systems, ferroelectrics [1 2 3 4 5 6 7 8].

The quantum statistics has a substantial influence on the critical behaviour near various (multi)critical points of low temperature and zero temperature continuous phase transitions as well as on thermodynamic and correlation properties near equilibrium points of low-temperature first order phase transitions [9 10 11]. This involves the quantum statistical physics in the field of a particular type of phase transitions - the quantum
phase transitions. The thermodynamics and the correlation phenomena near such phase transitions are essentially affected by quantum mechanical effects.

It seems convenient to begin this review with brief historical notes. Although the interest in quantum effects on the critical behaviour in many-body systems dated from the dawn of the quantum statistical physics, the theory of quantum phase transitions and, in particular, the theory of continuous quantum phase transitions (alias, quantum critical phenomena) received a substantial development after the late sixties of the previous century owing to the application of fruitful ideas of universality and scaling by renormalization group (RG) methods; see, e.g., Refs. [1, 12, 13, 14, 15, 16]. The contemporary theory of these phenomena has two distinct periods of development.

The first, classical period, began in the seventies of the previous century with the pioneering papers by Pfeuty and Elliott [17], Rechester [18], Young [19], and the particularly important work of Hertz [20]. This period of a relatively quiet research based on quantum field and statistical methods, including RG, continued till 1986-1987 when the number of papers on quantum critical phenomena has abruptly increased due to the great interest in heavy-fermion [8] and high-temperature superconductors [21], low-dimensional magnetic systems [6], quantum Hall effect [8], metal-insulator transition problems [22], and dilute Bose gases and Bose fluids [1, 15, 23, 24].

The new period of research [2, 3, 4, 5, 6, 8] after 1986-1987 seems to be much more intensive but one should not forget that this renewed research in an extended area of problems and their applications relies very much on the classical results obtained during the first period. We shall partly consider some classical results together with related developments and applications accomplished up to now.  

1.2. Scope and aims of the review

Numerous real systems, for example, magnets, ferroelectrics, quantum Hall systems, dilute Bose gases, low-dimensional Josephson-junction arrays, heavy-fermion and high-temperature superconductors exhibit phase transition lines which extend to low and up to zero-temperature phase transition points, namely, to temperatures of a strong quantum statistical degeneration [1, 2, 3, 4, 5, 6, 7, 8, 21, 22, 23, 24]. In this temperature region the quantum statistical correlations can, in certain cases and under certain experimental conditions, produce observable effects on the low- and zero-temperature phase transition. The task of the theory is to predict and describe the quantum effect on the thermodynamic and correlation properties near such phase transitions. This difficult task remains among the outstanding problems of the quantum statistical physics, despite the remarkable success achieved in this field of research during the last 30-35 years.

We shall not expand our discussion over the great variety of low-temperature phase transitions. Rather we shall focus our attention on several aspects of continuous and, mainly, second order quantum phase transitions, i.e. quantum critical phenomena. For our aims
we shall use general thermodynamic arguments and results for basic and relatively simple statistical models.

Despite of the fact that in real systems at low temperatures the first order phase transitions occur more frequently, quantum phase transitions of first order are less investigated and few results are available for them. Perhaps, this is due to the fact that the ordering phenomena and thermal fluctuations at equilibrium points of discontinuous (first-order) phase transitions have finite length scales and the scaling methods cannot be applied. That is why the quantum fluctuations at first order phase transitions should have a stronger effect on the phase transition properties than at points of continuous phase transitions. This interesting topic is also beyond the scope of the paper.

Our attention will be focussed on the quantum behaviour of Bose gases which reveals the main features of the quantum phase transitions in various systems. Furthermore, our consideration is closely related to the problem of Bose-Einstein condensation (BEC) of noninteracting bosons (ideal Bose gas, shortly, IBG) and the superfluidity of interacting bosons (nonideal Bose gas; NBG) \textsuperscript{9, 10, 11}. We shall be particularly interested in the properties of usual and disordered Bose fluids in a close vicinity of the $\lambda$-point. General problems of quantum phase transitions are also reviewed. Along with this review we shall present several new aspects of the quantum phase transitions. The new ideas and results will be summarized in our concluding remarks.

In Sec. 2 we perform a phenomenological investigation of several topics concerning the change (crossover) of the usual classical (“high-temperature”) critical behaviour when the critical temperature is lowered to the range of the quantum degeneration up to zero. The lowering of the critical temperature is referred to as “low-temperature” and “zero-temperature” limiting cases of the critical behaviour. Numerous experimental and theoretical studies indicate that a rather nontrivial and general “high-low” temperature crossover (HLTC) exists, namely, that the high-temperature and low-temperature critical phenomena are quite different from each other. The first fundamental question is whether this HLTC is a result of low-temperature and zero-temperature limits themselves or is produced by quantum effects.

The problem can be solved provided both the universal and nonuniversal properties of the low-temperature critical behaviour are thoroughly investigated. Note, that some particular properties of the system or special experimental conditions can quell the quantum correlations and then the low-temperature critical behaviour will remain totally or partially classical up to the zero temperature. Therefore, the quantum critical behaviour is a particular case of low-temperature and zero-temperature critical behaviour. The classical-to-quantum dimensional crossover (CQC) \textsuperscript{17, 19, 20} is also discussed as a particular case of HLTC. For our aims in Sec. 2 we apply a new scaling scheme for quantum systems \textsuperscript{5} which is an alternative of the standard quantum scaling, developed for the first time in Ref. \textsuperscript{23} (see, also, Refs. \textsuperscript{6, 18, 21, 24}).
In Sec. 3 IBG at constant density and constant pressure is considered. The general methods for the investigation of systems of interacting bosons are reviewed in Sec. 4. In Sec. 5 the main RG results for NBG are discussed. In Sec. 6 the transverse Ising model (TIM) is considered with the help of the mean-field (MF) approximation, the consideration of the Ginzburg critical region, and RG. The results in Secs. 3, 5 and 6 confirm the scaling description outlined in Sec. 2. Disorder effects are discussed in Sec. 7. In Sec. 8, the notion about the break down of universality for quantum critical phenomena and a classification of these phenomena based on this notion are established. The concluding remarks are presented in Sec. 9. We presume that the reader is well acquainted with the basis of the scaling theory of phase transitions and RG.

There is an overwhelming amount of papers published in this field and we cannot enumerate and discuss all of them within a manageable length of this review. We have tried to present the most important original papers, comprehensive review articles and books.

2. Phenomenology and quantum phase transitions

2.1. General criterion for quantum effects on phase transition

The quantum effects (alias, quantum correlations, often called also quantum fluctuations and even quantum pseudo-interactions) due to the overlap of particles wave functions exert an influence on the thermodynamic and correlation properties near the phase transition points in many body systems provided the de Broglie thermal wavelength

$$\lambda(T) = \left( \frac{2\pi \hbar^2}{mk_B T} \right)^{\theta},$$

is greater than the correlation length

$$\xi(T) = \xi_0(T_c)|t(T)|^{-\nu},$$

of the thermal fluctuations:

$$\rho = \frac{\lambda}{\xi} > 1.$$  \(1\)

In Eqs. (1) - (2),

$$0 \leq |t(T)| = \frac{|T - T_c|}{T_c} \ll 1,$$

defines a broad vicinity of the (multi)critical point \(T_c\), in which the phase transition phenomena occur, \(\xi_0 \equiv \xi(T_c)\) is the so-called “zero temperature correlation length,” the critical exponent \(\nu > 0\) describes the behaviour of the correlation length \(\xi(T)\) in the phase transition region \(4\), \(\theta > 0\) is the exponent for the thermal wavelength. We have to note that usually, \(\theta = 1/2\) \(10\) but in order to comprise all possible quantum statistical models we consider \(\theta > 0\); see, e.g., \(11\). In Eq. (1), the parameter \(m\) denotes either the real particles mass or an effective mass \(m_{\text{eff}} \sim \hbar^2/c(g)\) of composite bosons (boson excitations).
which represents the effect of some interaction constant $g$; below the suffix “eff” will be omitted (see also Secs. 3.2 and 4.1).

We must keep in mind that the correlation length $\xi(T)$ describes only classical phenomena. The length $\xi_0$ is a nonuniversal quantity and can be specified only for concrete systems (models) which means that it cannot be presented in a concrete mathematical form before the choice of the concrete model (Hamiltonian) is done and, in particular cases, as for example IBG, after the thermodynamic analysis of the system is fulfilled. That is why, at this general stage of consideration we do not give a mathematical formula for $\xi_0$. But we must emphasize that $\xi_0$ depends on intrinsic parameters of the system and, especially, on the critical temperature $T_c$. The term “zero-temperature correlation length,” that is sometimes used for the scaling amplitude $\xi_0$ of $\xi(T)$ is an indication of the simple fact that $t(0) = 1$ and, hence, $\xi(t = 1) = \xi_0$.

It will be shown in Sec. 4.3 that $\theta = 1/z$, where $z$ is the so-called dynamical critical exponent which describes the intrinsic (quantum) dynamics of the system [20]. It can be stated as a theorem that the relation $\theta = 1/z$ is valid always when the dynamics of the quantum system [20] is not influenced by other time-dependent phenomena [12, 25].

It has been already mentioned that $T_c$ is a (multi)critical temperature, i.e., it denotes the phase transition point of quantum or classical phase transitions of second and higher order, i.e., continuous phase transitions. For the first order phase transitions, however, Eqs. (2) and (4) can be also used. In this case $T_c$ is a characteristic temperature of the system near the equilibrium phase transition temperature $T_{eq} \neq T_c$: $(|T - T_{eq}|/T_{eq}) = t_{eq}(T) \sim |t| \ll 1$. The phase transition phenomena, including the metastable states are located in the temperature domain $|t_{eq}(T)| \sim t_{eq}(T_c) |1|$. The infinitesimally small vicinity $|t(T) \to 0|$ of critical points $T_c$ of continuous phase transitions is often referred to as an “asymptotic critical region” and the behaviour in this small domain is usually called an “asymptotic critical behaviour.”

Phase transitions which satisfy the criterion (3) in the vicinity of their equilibrium phase transition points $T_c$ are called quantum phase transitions. This criterion has been introduced for the first time by M. Suzuki [26] for quantum critical phenomena (phenomena at continuous quantum phase transitions) and here we find reasonable to extend it to all quantum phase transitions. The condition (3) is very similar to the respective one for quantum fluids, for them there is a requirement for the thermal length $\lambda$ to exceed the mean interparticle distance (the lattice constant in crystal bodies): $a = \rho^{1/d} = (N/V)^{1/d}$ [9, 10, 27]; $N$ is the particle number, $V = (L_1...L_d)$ is the volume of the system, and $d \geq 0$ is the spatial dimensionality.

The criterion (3) is a direct result of the fundamental notion that the phase transition is a (quasi)macroscopic phenomenon and, hence, phenomena at length scales shorter than the length scale $\xi$ of the classical fluctuations are irrelevant [26]. It is confirmed by the available studies of quantum statistical models. The criterion (3) yields both the necessary
and the sufficient conditions for the appearance of a quantum phase transition but the circumstances, under which this can actually happen remain hidden in its generality.

It seems at first glance that the condition (3) can be more easily fulfilled for first order transitions, where \( \xi(T_{eq}) < \infty \). However, the answer of the question whether the inequality (3) can be satisfied even in this case depends on the values of the parameters \( \xi_0 \) and \( \lambda_0 = (2\pi \hbar^2/mk_B)^\theta = \lambda T^\theta \) for the concrete substance of interest. The same problem exists for the continuous phase transitions. Hereafter we shall discuss continuous phase transitions, where \( \xi(T) \) is always infinite.

2.2. Quantum critical region

The vicinity (4) of the critical point \( T_c \), where the phase transition phenomena occur and the Landau expansion of Gibbs free energy in powers of the fluctuation order parameter \( \phi(\vec{r}) \) is valid, includes both the MF domain of description and the Ginzburg critical region of strong fluctuations very close to \( T_c \)[1, 11]. This vicinity of \( T_c \), \( \xi(T) \gg \xi_0(T_c) \), where the classical (thermal) fluctuations and ordering phenomena occur, hereafter will be called “classical transition region” or, shortly, “transition region”. For some types of phase transitions the transition region may be defined by \( |t(T)| < 1 \) rather than by the strong inequality (4).

While the correlation length \( \xi(T) \) does not describe the MF domain of ordering and, moreover, this quantity cannot be defined within the standard MF approximation, the inequality \( \xi(T) \gg \xi_0(T) \), equivalent to \( t(T) \ll 1 \), does include the temperature domain of the phase transition phenomena and can be used in our discussion. We shall distinguish between the Ginzburg critical region \[\boxplus\boxtimes\] and the much larger classical transition region defined above. The subdomain of the transition region, where the quantum fluctuations have an essential influence on the phase transition properties will be called a “quantum transition (sub)domain.” At a next stage of the present consideration, this picture will be generalised to include additional thermodynamic parameters.

From Eq. (1), the condition (3) and \( \xi_0(T_c) \geq a_0 \) – a condition which is always satisfied in real systems, we see that a quantum critical phenomenon may occur only in the regime of quantum degeneration of the system \( (\lambda > a) \), i.e., near low- and zero-temperature critical points, where conditions (3) and (4) are simultaneously satisfied. Here we accept as a definition that we shall use the term “quantum critical phenomenon” to indicate the fact that the quantum effects do penetrate in the transition region \( |t| \ll 1 \).

Quantum critical phenomena, that is, quantum effects in the “transition region”, may occur in condensed matter systems and gases, where the critical line \( T_c(X) \) extends over the low-temperature region to the zero-temperature critical point \( T_c(X_0) = 0 \) by variations of an additional thermodynamic parameter \( X \). Depending on the particular system, the parameter \( X \) may represent some auxiliary intrinsic interactions which suppress the main interaction responsible for the phase transition, or other physical quantities (density,
Figure 1: (a) Low temperature part of a critical line with zero temperature critical point $T_c(X_0) = 0$. The shaded part $(a - X_0 - b)$ of the transition region, marked by the lines 1 and 2, corresponds to a low temperature classical behaviour. (b) Low temperature critical line with $X_0 = 0$; domains $(a - X_0 - b)$ and $1 - 0 - 2$ coincide.

pressure, concentration of impurities, etc.) having the same effect; see Fig. 1, where such critical lines are depicted for $X_0 > 0$ and $X_0 = 0$. The slope of the curve $T_c(X)$ and the fact that it is tilted to the left are irrelevant to the present discussion. In certain systems, the critical line $T_c(X)$ is tilted to the right and then the maximal value of $T_c$ will be at some $X > X_0$ (see Sec. 6.2).

We should keep in mind that Eqs. (2) and (4) describe only the temperature dependence of the characteristic length $\xi$. In general, the correlation length $\xi$ depends on the parameter $X$ as well: $\xi = \xi(T, X)$. The dependence of thermodynamic and correlation quantities on $X$ must be considered on the same footing as the dependence on $T$. The reason is that, as seen from Fig. 1, the phase transitions at $T_c(X)$ or, equivalently, at $X_c(T)$, can occur both by variations of $T$ at fixed $X$ ($T$–driven transitions) and by variations of $X$ at fixed $T$ ($X$–driven transition). Phase transitions produced by both $T$– and $X$–variations are also possible. We suppose that their main properties can be clarified by the investigation of $T$– and $X$–transitions.

These notes are important for the evaluation of that part of the transition region where the quantum effects penetrate, i.e., in the quantum transition region. The inequality (4) defines the transition region of the critical phenomena produced by $T$–transitions and, hence, the temperature transition width $[T_{>}(X) - T_{<}(X)]$ along the $T$–axis satisfies the condition $[T_{>}(X) - T_{<}(X)] \ll T_c(X)$ for each $X = X_c$; here, $T_{>}(X) > T_{<}(X)$. As $T_c(X) \to 0$ for $X \to X_0$, the temperature transition width vanishes at the point $(T, X) = (0, X_0)$. The shape of this temperature transition region is similar to the form of the shaded domains $a - X_0 - b$ and $1-0-2$ in Figs. 1a,b.
However, the transition region along the \( X \)-axis is produced by \( X \)-transitions and this effect should be considered as well. That is why the notion for the transition region should be generalized to include the \( X \)-driven phenomena, too. At any given \( T = T_c \), \( \xi(T_c, X) \to \infty \) for \( X \to X_c \) because of the general criticality assumption: \( \xi(T_c, X_c) = \infty \). Therefore, one can represent \( \xi(T_c, X) \) in a general scaling form, namely, \( \xi(T_c, X) = \tilde{\xi}_0 \tilde{t}^{-\nu} \), where \( \tilde{t}(X) = |X - X_c|/X_c \) is the \( X \)-distance from \( X_c \), \( \nu \geq 0 \) is the respective critical exponent, and \( \tilde{\xi}_0 \equiv \tilde{\xi}_0(X_c) = [\xi(T_c, \tilde{t}(0))] > 0 \) is the scaling amplitude. Now one can define the \( X \)-transition interval \( [X_>(T) - X_<\leq(T)] \ll X_c(T) \) for each \( T = T_c \); here \( X_>(T) > X_<\leq(T) \). This interval will vanish, only if \( X_c(T) = 0 \) at some critical temperature \( (T_c, 0) \). The latter case is illustrated in Fig. 1a by the high-temperature critical point \( (T_c, 0) \), and in Fig. 1b, where \( X_c(0) = 0 \).

The total transition region \( (T, X)_t \) in the vicinity of the critical line \( T_c(X) \) includes both \( T \)-transition and \( X \)-transition widths and can be represented as a two-dimensional domain of shape similar to the form of the domain confined between the lines 1 and 2 in Figs. 1a,b. The picture of the transition region \( (T, X)_t \) outlined above will be valid provided the scaling amplitude \( \xi_0(T_c) \) is finite for all \( T_c \geq 0 \). Remember that \( (T, X)_t \) will be a classical transition region till the quantum effects are ignored in our consideration.

The next question is whether the quantum correlations, if properly taken into account, will affect the thermodynamic properties in the transition region \( (T, X)_t \). When the transition region \( (T, X)_t \) vanishes, as this is the case for some zero temperature critical points, the quantum fluctuations will fill up the whole nearest vicinity of the critical point and then we have an entirely quantum phase transition. The problem is nontrivial when the classical region \( (T, X)_t \) exists.

There are three scenarios:

(i) The quantum effects are relevant in the whole region \( (T, X)_t \). Then the latter is a quantum transition region, where quantum phase transition phenomena occur.

(ii) The quantum effects are relevant in a part (quantum subdomain) of the total transition region. This will be enough to include the phase transition in the quantum ones.

(iii) The quantum effects are irrelevant anywhere in \( (T, X)_t \) and, hence, the phase transition is completely classical.

The variants (i)-(iii) are shown in Fig. 1a,b. The shaded domain \( a - X_0 - b \) in Fig. 1a represents the classical subdomain of the total transition region \( (T, X)_t \), where the classical (thermal) fluctuations dominate and the phase transition is completely classical. The unshaded domain between the lines 1 and 2 is the quantum transition subdomain. Here one can find both cases (i)-(ii) depending on whether \( T > 0 \) or \( T = 0 \) and, whether one is interested in \( T \)- or \( X \)-driven phase transitions. In Fig. 1b the classical domain \( a - X_0 - b \) coincides with the total transition domain 1-0-2 which is an illustration of the classical variant (i).
In Secs. 2.3-2.4 we shall investigate the variants (i)-(iii) with the help of general scaling arguments. For a methodical convenience, we shall discuss the $T$–transition region along the $T$–axis and the $X$–transition region along the $X$-axis separately, although they are a result of one and the same reason – thermal and quantum fluctuation and ordering phenomena.

2.3. $T$–driven transitions

According to Eqs. (1) and (2), $\xi \to \infty$ for $T \to T_c > 0$, but $\lambda$ remains finite. Therefore, the close vicinity of finite temperature critical points $T_c > 0$ will always exhibit a classical behaviour. In particular, this is true for the so-called asymptotic critical behaviour corresponding to the infinitesimally narrow distance ($|t| \to 0$) from $T_c$.

Using the criterion (3) as well as Eqs. (1) and (2), it is easy to show that quantum critical phenomena will occur, i.e., the quantum effects will penetrate in the (temperature) critical region $|t(T)| \ll 1$, if

$$T_c(\frac{\xi_0}{\lambda})^{1/\nu} < |(T - T_c)| \ll T_c.$$  \hspace{1cm} (5)

The conditions (5) are well defined for $T_c > 0$, including the zero-temperature limit $T_c \to 0$, provided $T \to 0$ too, so that $|t| < 1$ is satisfied. The size of the quantum subdomain ($\Delta T)_Q = |T^* - T_c|$ above and below $T_c$ is given by $(\Delta T)_Q = T_c^{(1+\theta/\nu)}(\frac{\xi_0}{\lambda_0})^{1/\nu}$, where $\lambda_0 = \lambda(T)T^\theta$. The size of this temperature interval depends on the behaviour of the function $\xi_0(T_c)$.

Obviously, the quantum portion (5) of the transition region will exist, if

$$\xi_0(T_c) < \lambda(T),$$  \hspace{1cm} (6)

which corresponds to a moderate ($\lambda > a$) or strong ($\lambda \gg a$) quantum degeneration for ($\xi_0 > a$) and ($\xi_0 \gg a$), respectively. The condition (6) can be written in the form $T^\theta < |\lambda_0/\xi_0(T_c)|$.

The inequality (6) may bring more information about the quantum criticality corresponding to $T$–driven transitions, if we remember that within the present description ($|t| \ll 1$ or $|t| < 1$), we can freely substitute $T$ with $T_c$ in all formulae except that for $t(T)$. Conversely, the substitution of $T_c$ with $T$ will be also allowed in the whole transition domain (4), if the nominator of $t(T)$ is not affected. Besides, the latter variant of the theory seems to be more closely connected with the original form of the free energy [11], which is derived by standard statistical methods from the microscopic Hamiltonian of the system (see also Sec. 4).

The further consideration depends on the way, in which we shall treat the nonuniversal length $\xi_0(T_c) > a$. When $T_c$ is decreased to some extent, depending on the specific properties of a given system, $\xi_0(T_c)$ grows and this is described by the relation

$$\xi_0(T_c) = \xi_{00}T_c^{-\nu_0},$$  \hspace{1cm} (7)
where $a < \xi_{00} < \infty$ is a new (low-temperature) scaling amplitude and $0 \leq \nu_0 \leq \infty$ is a new (low-temperature) exponent; $\nu_0$ must be nonnegative number because of the criticality at low $T_c$.

The value of exponent $\nu_0$ and the range of temperatures, where the low-temperature dependence (7) will be valid, can be evaluated from the properties of system. The quantities $\xi_{00} > a$ and $\nu_0 > 0$ appear as a result of a gradual low-temperature crossover in the critical behaviour owing to $T_c$ lowering.

The temperature, $T$, is not sufficient for the description of the low-temperature crossover and the entire investigation can be performed in terms of both relevant thermodynamic parameters $T$ and $X$. The low temperature critical behaviour limit should be given by the ratio $[T_c(X)/T_c(0)] \ll 1$ or, equivalently, by the ratio $[g(X)/g_0] \ll 1$, where $g_0 = g(0)$ is the interaction responsible for the $T$-driven transition and $g(X)$ describes the effective decrease of this interaction owing to the auxiliary parameter $X$; in certain cases, $g(X) \sim T_c(X)$. These remarks, involving the term “interaction” in our phenomenological analysis, are not restricted to systems of interacting particles only; see Sec. 3, where the phase transition is produced rather by the Bose statistics and global thermodynamic constraints than by direct interparticle interactions.

Having in mind these notes we continue our analysis of the condition (6) by substituting $\lambda(T)$ with $\lambda(T_c)$ and $\xi_0(T_c)$ with the scaling form (7):

$$\frac{\xi_{00}}{\lambda_0} < T_c^{\nu_0 - \theta}.$$  \hspace{1cm} (8)

When the critical temperature is lowered enough, the condition (8) will be broken unless

$$\theta > \nu_0.$$  \hspace{1cm} (9)

If the inequality (9) is satisfied, the quantum critical phenomena will occur at low-temperatures in certain part (5) of the temperature interval (4) and, moreover, for $T_c \to 0$, they will prevail in the asymptotic critical behaviour $[t(T) \to 0]$, too.

If the criterion (9) is fulfilled, the classical fluctuations will be completely irrelevant to the critical behaviour at $T_c = 0$. Their effect at $T_c \sim 0$ (extremely low-temperature critical points) will be restricted in a negligibly narrow vicinity of $T_c$, which is practically inaccessible to experiments. In both cases, $T_c \sim 0$ and $T_c = 0$, the experiment should observe quantum critical phenomena only. For $\theta = \nu_0$, the quantum correlations will have an effect on the pre-asymptotic low-temperature critical behaviour outside the small distance $T_c(\xi_{00}/\lambda_0)^{1/\nu}$ from $T_c$, provided $(\xi_{00}/\lambda_0) < 1$, i.e., in case of convenient nonuniversal properties of the particular system. Quantum critical phenomena will not exist at all, if $\theta < \nu_0$. In this case the critical behaviour remains classical.

The same results can be obtained by an alternative consideration which is consistent with our discussion about the possible change of factors $T$ with $T_c$, and vice versa. The scaling
law (2) has the following alternative definition:

\[ \xi(T) = \tilde{\xi}_0(T)|\tilde{t}(T)|^{-\nu}, \tag{10} \]

where \( \tilde{t}(T) = (T - T_c)/T \). The quantity \( \tilde{\xi}_0(T) \) defined by Eq. (10) can be called “zero–\( T_c \) correlation length”; \( \tilde{t}(T_c = 0) = 1 \). In the high-temperature and, to some extent, in the low temperature range of critical temperatures \( T_c \), Eq. (10) will give the same leading scaling dependence as Eq. (2) because the corrections to the scaling are of the order \( O(|t|^{1-\nu}) \).

In the limit \( T \to 0 \) however the quantity \( \tilde{t}(T) \) is not singular, whereas the correlation length \( \xi(T) \) should tend to infinity at \( T_c = 0 \). This means that the decreasing of the temperature will unavoidably yield a HLTC. The latter consists in a gradual temperature-driven transformation of the \( \tilde{t}(T) \)–singularity in Eq. (10) to a new singularity – the divergence of the amplitude \( \tilde{\xi}_0(T) \) at \( T_c = 0 \).

Let us suppose that \( \tilde{\xi}_0(T) \) obeys a scaling law with respect to \( T \) having the form (7). Then one can immediately rederive a condition for temperature \( T \) which will be identical to the condition (8) for \( T_c \) and, hence, the criterion (9) will be straightforwardly confirmed. Note, that the new (low temperature) singularity may arise with respect to the variable \( \tilde{t}(X) \) rather than towards \( T \) (see an example in Sec. 6.2).

The singularity (7) is a mere indication for an existence of a scaling law for \( \xi(T) \) of the type \( \xi(T) \sim \tilde{\xi}_0(T) = \tilde{\xi}_00/T^{\nu_0} \) corresponding to zero critical temperatures \( (T_c = 0) \). In the low-temperature limiting case, the critical behaviour singularities are developed as singularities of the scaling amplitude while at high temperatures this scaling amplitude is a slow varying function of temperature. It is worth noting that the present phenomenological consideration, including the ansatz (7) for \( \xi_0(T_c) \) and the related one for \( \xi_0(T) \), is supported by results from RG calculations for particular systems \[28\] and experiments on ferroelectric phase transitions \[29, 30\].

2.4. \( X \)–driven transitions

In general, the properties of \( X \)–driven transitions are quite different from those of \( T \)–driven transitions. In particular, essential differences may be expected in the high-temperature \( (X \sim 0) \) and low-temperature \( (X \sim X_0) \) ranges of temperatures. By substituting \( T \) with \( X \) and, of course, \( T_c \) with \( X_c \) in (2) and (4), one can perform the phenomenological analysis (Sec. 2.3) in terms of the variable \( X \). This analysis directly yields a form of scaling law for the correlation length \( \xi(T, X) \) which is identical to that related with \( T \)–transitions in the high-temperature region where \( X \sim 0 \). The only difference may come from the values of critical exponents which describe the scaling laws with respect to \( t(T) \) and \( \tilde{t}(X) \). There is no reason to suppose that the exponents towards \( t(T) \) and \( \tilde{t}(X) \) should be equal.

In fact, the critical behaviour in the low-temperature limit described in Sec. 2.3 has a similarity with high-temperature \( X \)–transitions because \( X_c \sim 0 \) at high temperatures. A conformity of type \( T \leftrightarrow X \) concerning the form of scaling laws and the way, in which
the low-temperature limit (for $T$–transitions) and the high-temperature limit (for $X$–transitions) are developed from the corresponding scaling amplitudes certainly exists. However, except for particular systems, a total correspondence including the values of respective critical exponents and scaling amplitudes cannot be expected; note, that the length $\lambda(T)$ has no analog in terms of $X$. In contrast, there is no such similarity between the low-temperature $T$– and $X$–transitions.

For $X$–transitions, the criterion (3) gives

$$|\tilde{t}(X)| > \left[\frac{\xi_0(X_c)}{\lambda(T)}\right]^{1/\nu},$$

where all quantities are defined by the change of $T$ with $X$ in (2) and (4). Having in mind Eq. (1) and the fact that the critical region (lines 1 and 2 in Fig. 1) along the $X$–axis is defined by $|\tilde{t}(X)| \ll 1$, it becomes evident, that in the low-temperature range ($X \sim X_0$), where $[\xi_0(X_c)/\lambda(T) < 1]$, the quantum subdomain gradually enlarges when the temperature is decreased or, equivalently, when $X_c(T)$ approaches $X_0$. In the zero-temperature limit ($T \to 0$) or, equivalently, for $X_c(T) \to X_0$, this subdomain fills up totally the transition region.

In the range of temperatures, where $[\xi_0(X_c)/\lambda(T)] > 1$, the critical behaviour is totally classical. We should have in mind that the latter condition for a total classical criticality can be easily satisfied in the temperature range of quantum degeneration in real systems with a large zero-temperature correlation length ($\xi_0 \gg a$).

At high temperatures, where $X_c \sim 0$, the scaling of correlation length $\xi$ can be conveniently investigated by a scaling law with respect to $X$ or $(X - X_c)$, instead of $\tilde{t}(X)$. The problems in the treatment of $X$–transitions in the high-temperature limit are analogous to those for the low-temperature $T$–transitions and the analysis can be carried out by following the ideas presented in Sec. 2.3.

A quite special situation should exist when $X_0 = 0$; see Fig. 1b. In this case both $T$ and $X$ are equal to zero at the zero-temperature critical point, where the correlation length $\xi(T, X)$ may exhibit scaling dependence on both $t(T)$ and $\tilde{t}(X)$. Obviously, such zero-temperature critical points will offer less opportunity for the quantum critical phenomena observation or, in some cases, this opportunity may not exist at all. To emphasize that the lack of any quantum critical phenomenon may be expected at such zero-temperature critical points, we have depicted in Fig. 1b a classical transition region which completely fills up the total transition region $(T, X)_t$, confined between the lines 1 and 2.

However, there exist systems, where the critical temperature $T_c$ depends on $X$ and tends to zero for $X \to X_0 = 0$ but the correlation length $\xi$ does not show any divergence with respect to the parameter $X$. In this case the function $\xi(T, X)$, for all possible $T$ and $X$, has the general form $\xi[t(T), T_c(X))]$, which describes only the scaling law with respect to
$t(T)$ as given by Eq. (2) or Eq. (10). The transition region width along the $T$–axis tends to zero for $T_c \to 0$ and a transition region with respect to the parameter $X$ does not exist at all. Thus the lines 1 and 2 in Fig. 1b will definitely terminate at $T = X = 0$; see the dashed line 0–2. Certainly, the $X$–transitions at low temperatures offer much more favourable conditions for quantum critical phenomena, although the practical observation of such transitions in certain systems, where the parameter $X$ cannot be gradually varied, is almost impossible.

We end the discussion of the total and quantum transition regions by two remarks. In some systems at extremely low and zero temperature, the quantum transition region might be larger than the total classical region. Then quantum critical phenomena may happen outside the classical transition region. In this case, quantum critical phenomena may occur even if the size of the classical transition region tends to zero (the case depicted in Fig.1b). The second remark is about the possibility for a further generalization of this consideration by assuming that the symbol $X$ represents more than one thermodynamic parameters: $X = (X_1, ...)$. This generalization is straightforward.

2.5. Crossovers

We have shown that the low-temperature critical behaviour can be either quantum or classical depending on the specific (nonuniversal) properties of a particular system. Besides, there are no general arguments indicating that high-temperature and low temperature critical properties should be equivalent. Thus we may suppose that, in general, classical high-temperature classical low-temperature and quantum critical properties present three different types of critical behaviour. These three types of critical phenomena can be included in the framework of a quite general notion for a “high-low temperature crossover” (HLTC) of the critical behaviour which corresponds to the change of the critical properties (or of some of them) when the thermal length $\lambda$ varies from $\lambda < a$ at high temperatures to $\lambda > a$ (up to $\lambda \gg a$) at sufficiently low temperatures, and vice versa. A special case of HLTC is the classical ($\rho < 1$) to quantum ($\rho > 1$) crossover (CQC) which describes the difference between classical high-temperature phenomena and the quantum critical phenomena at $T \to 0$ [17, 19, 20]. This CQC is treated by statistical methods (Secs. 4-7).

The experiment has a finite accuracy and omits very narrow temperature intervals such as the asymptotic classical regions in the close vicinity of “almost–zero-temperature” critical points ($T_c \sim 0$). This should be taken into account in interpretations of experiments. Because of the zero temperature unattainability, quantum critical phenomena produced by zero-temperature $X$–transitions cannot be observed in a real experiment but certain low-temperature $X$–transitions might be experimentally investigated. On account of limitations in the accuracy of the equilibrium temperature measurement, the very narrow classical region will remain unattainable for experimental studies and, hence, the experiment will reproduce results which, to some extent, will give an information about
asymptotic quantum critical phenomena at zero temperature.

3. Ideal Bose gas

3.1. Preliminary notes

The model of IBG describes the phenomenon of Bose–Einstein condensation (BEC) \([1, 11, 10, 11]\). BEC of IBG is important for the understanding of quantum phenomena in many areas of physics, in particular, in superfluid helium liquids \([9]\), excitonic phases in semiconductors \([31]\), the recently discovered BEC of dilute Bose gases in magneto-optical traps \([32, 33, 34, 35, 36]\) although in these and other real systems the interparticle interactions do affect the thermodynamic and correlation properties.

We find reasonable to emphasize that the IBG does not contain interparticle interactions but it possesses two other properties which may cause a phase transition. Firstly, the quantum-statistical correlations act as attractive “pseudo-interactions.” This can be seen from the negative sign of the first quantum correction to the equation of state of the ideal classical (Boltzman) gas coming from the Bose statistics \([10, 11]\). Secondly, the constraints imposed on the system usually have an effect similar to that of some interaction. Let us remember that the thermodynamics of IBG is ruled by constraints (a constant density, or, the less common constraint of a constant pressure and, why not, a constraint of constant temperature, which has been never investigated). These constraints are a simple experimental requirement and are used to define the chemical potential as a function of the temperature within the framework of the grand canonical ensemble \([10, 11]\).

Besides, we have to emphasize that the ground states of IBG and NBG are quite different in their physical properties. While IBG produces the originally predicted by A. Einstein BEC, a very small interaction is needed to induce a superfluid ground state in NBG as demonstrated theoretically in a rigorous way by H. H. Bogoliubov \([37]\) (see, also, Ref. \([9]\)).

Another problem is the description of the phase transitions in Bose gases: from one side, the phase transition to BEC in IBG at finite and zero temperatures and, from the other side, the so-called \(\lambda\)-transition, i.e., the phase transition to a superfluid state in NBG. These phase transitions exhibit quite different properties. Here we shall review the results from Refs. \([39, 10, 11, 12]\), where BEC in IBG has been investigated within the framework of the scaling theory of phase transitions \([11, 12, 13]\).

3.2. Thermodynamic equations

The thermodynamic properties of \(d\)-dimensional IBG, including BEC, are described by equations for the grand canonical potential \(\Omega\) and the number density \(\rho = (N/V)\) of bosons \([9, 10]\):

\[
\Omega = k_B T \sum_k \ln \left[ 1 - e^{-\beta \varepsilon(k)} \right],
\]

(12)
and
\[\rho = \frac{1}{V} \sum_{\vec{k}} \langle a^+_k a_{-\vec{k}} \rangle,\]
respectively. Here the brackets \(\langle \rangle\) denote a statistical averaging, \(\beta = 1/k_B T\), \(a^+_k\) and \(a_{-\vec{k}}\) are the creation and annihilation Bose operators for a plane-wave state of wave vector \(\vec{k} = (k_i; \ i = 1, \ldots, d)\), \(V = (L_1 \ldots L_d)\) is the volume of the gas. The self-energy \(\varepsilon(k) = \varepsilon_0(k) + r\) is represented by the energy spectrum \(\varepsilon_0(k) = \hbar^2 k^2 / 2m\) of free (noninteracting) real particles or excitations, and the chemical potential \(\mu = -r \leq 0; k = |\vec{k}|\).

Throughout the paper we shall use periodic boundary conditions. The wave vector components \(k_i = 2\pi l_i / L_i; (l_i = 0, \pm 1, \ldots)\), are given by the spatial dimensions \(L_i\). They are supposed to be much larger than any characteristic length of the system, for example, \(L_i \gg (\xi, \lambda)\). In this case it is usually said that the dimensions \(L_i\) are “infinite”. The stated condition allows to pass to the continuum limit, i.e., from a summation in Eqs. (12) and (13) to the corresponding integration. The quite uncommon case of composite Bose excitations due to long-range interactions in electron and magnetic systems can be included in the consideration by the generalization \(\varepsilon_0(k) = \hbar^2 k^\sigma / 2m, (0 < \sigma < 2)\) of energy spectrum \(\varepsilon_0(k)\); see, e.g., Refs. [1, 44, 45]. A simple dimensional analysis of the exponent \(\beta \varepsilon(k)\) in Eq. (12) shows that \(\xi = (\hbar^2 / 2mr)^{1/\sigma}\).

With the help of the Bose distribution
\[n(k) = \langle a^+_k a_{-\vec{k}} \rangle = \frac{1}{e^{\beta \varepsilon(k)} - 1},\] Eq. (13) will take the form
\[\rho = \frac{1}{V} \sum_{\vec{k}} \frac{1}{e^{\beta \varepsilon(k)} - 1}.\]

The correct investigation of IBG thermodynamics in the continuum limit implies to take the density \(\rho_0 = (N_0/V) = n(0)\) of bosons with zero wave numbers out the sum (15). This is important for obtaining a correct description of the properties below finite temperature critical points \(T_c > 0\). For all other cases this separation is redundant. As we shall be mainly interested in the latter case, we shall avoid the mentioned separation; see Ref. [42].

Substituting the \(\vec{k}\)–summation in Eqs. (12) and (13) by a \(d\)–dimensional integration,
\[\frac{1}{V} \sum_{\vec{k}} \rightarrow \int \frac{d^d k}{(2\pi)^d} \equiv \int_0^\infty dk \ k^{d-1},\]
\([K_d = 2^{1-d} / \pi^{d/2} \Gamma(d/2)]\), we obtain
\[P = k_B T A \lambda^{-d} g_{(d+1)} \left( \frac{r}{k_B T} \right)\]
\[ \rho = A \lambda^{-d} g_{(d/\sigma)} \left( \frac{r}{k_B T} \right). \]

Here \( P = -\left( \Omega / V \right) \) is the pressure, \( \lambda = (2 \pi \hbar^2 / mk_B T)^{1/\sigma} \) is the thermal wavelength with an exponent \( \theta = 1/\sigma \), c.f. Eq. (1). The function \( g_{\nu}(\kappa) \) and the parameter \( A \) which enter in above equations are given by the expressions,

\[ g_{\nu}(\kappa) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} x^{\nu-1} e^{x+\kappa - 1} dx, \]

and

\[ A = \frac{2^{1-d+2d/\sigma} \Gamma(d/\sigma)}{\sigma \pi^{d(1/2-1/\sigma)} \Gamma(d/2)}, \]

(for \( \sigma = 2, A = 1 \)).

The BE condensate is a coherent state of a macroscopic number \( N_0 \sim N \) of bosons with a momentum \( \hbar k = 0 \). The critical temperature \( T_c \) of the transition to BEC in the momentum (\( \vec{k} \)) space is defined by the equation \( r(T_c)/T_c = 0 \). The BEC order parameter \( \phi_0 = \langle a_0^+ \rangle / \sqrt{V} \) is related to the square root of the number density \( \rho_0 = (N_0/V) \) of the condensate bosons: \( |\phi_0| = \sqrt{\rho_0} \). In the present problem, the ratio (3) takes the form

\[ \varrho = \left( \frac{4\pi r}{k_B T} \right)^{1/\sigma}. \]

The IBG critical properties can be investigated with the help of Eqs. (17) and (18). They depend on the thermodynamic conditions imposed on IBG, i.e., on the way, in which the chemical potential \( \mu = -r \) is determined.

We shall briefly consider three cases: a constant pressure \( P \) [11], a constant density \( \rho \) and spatial dimensions \( d > \sigma \) [39-40], and a constant density at dimensions \( d < \sigma \) [42]. In all cases the critical regime is defined by the condition \( r \leq k_B T \) and all critical phenomena are almost \( r \sim k_B T \) or completely \( r \ll k_B T \) classical; c.f. Eq. (21).

3.3. Constant pressure

The parameter \( r \) is calculated from Eq. (17) as a function of \( T \) and \( P \). The result \( \mu(T, P) \) is substituted in Eq. (18) and, hence, one obtains the equation of state \( f(T, \rho, P) = 0 \). The solution \( T_c(P) \) of Eq. (17) with \( r = 0 \) yields the critical temperature

\[ T_c(P) = \left[ \frac{\lambda_0^d P}{\zeta(d/\sigma + 1) A k_B} \right]^{\sigma/(d+\sigma)}. \]

A characteristic feature of the phase transition at constant \( P \) is that the critical temperature is finite \( (T_c > 0) \) for all dimensions \( d > 0 \) and \( P > 0 \). The zero temperature critical point exists only in the limit \( P \to 0 \). This phase transition does not exhibit a crossover
to a low-temperature behaviour when the pressure $P$ tends to zero because the critical exponents remain unchanged. There is a crossover, however, from Gaussian exponents at dimensions $d > \sigma$ to low-dimensional ($d < \sigma$) Gaussian exponents. This crossover cannot be considered as HLTC or CQC.

3.4. Constant density

At a constant density $\rho = (N/V)$, the equilibrium chemical potential $\mu(T, \rho)$ is obtained as a solution of Eq. (18). The equation of state is given by Eq. (17) after the substitution of solution $r(T, \rho)$. Eq. (18) with $r = 0$ yields

$$T_c(\rho) = \left[ \frac{\lambda^d \rho}{\zeta(d/\sigma)A} \right]^{\sigma/d}$$

and, therefore, $T_c(\rho) > 0$ for $d > \sigma$, provided $\rho > 0$, whereas $T_c = 0$ for $d \leq \sigma$; note, that the zeta function $\zeta(d/\sigma)$ tends to infinity when the ratio $d/\sigma$ decreases to unity.

The finite temperature critical behaviour ($T_c > 0$) is identical to that of the classical Berlin-Kac [43] spherical model [39, 40] (for the spherical model see, e.g., Ref. [44], and the brief note in Sec. 4.1). The reason for this behaviour is in the constant density $\rho$ condition which, as is seen from Eq. (13), is equivalent to the mean spherical constraint on the variations of mean densities $\langle a_k^+ a_k \rangle / V$. The critical exponents are shown in Table 1, including the case $d > 2\sigma$ when they take Gaussian values [11, 12]. All critical exponents are given by their usual notations and definitions; see, e.g., Refs. [12, 15].

| $T_c$   | $d/\sigma$ | $\eta$ | $\alpha$ | $\alpha_s$ | $\gamma$ | $\tilde{\gamma}$ | $\nu$ |
|--------|------------|--------|----------|------------|----------|-------------------|-------|
| $T_c > 0$ | $d \geq 2\sigma$ | 2 $-$ $\sigma$ | 0 | $\alpha_s$ | $\gamma$ | $\tilde{\gamma}$ | $\nu$ |
| $T_c > 0$ | $\sigma < d < 2\sigma$ | 2 $-$ $\sigma$ | 0 | $(d - 2\sigma)/(d - \sigma)$ | $\sigma/(d - \sigma)$ | $1/(d - \sigma)$ |
| $T_c = 0$ | 1 | 2 $-$ $\sigma$ | -1 | - | $\infty$ | $\infty$ | $\infty$ |
| $T_c = 0$ | 0 $<$ $d < \sigma$ | 2 $-$ $\sigma$ | $-d/\sigma$ | - | $\sigma/(\sigma - d)$ | $d/(\sigma - d)$ | $1/(\sigma - d)$ |

The critical exponent $\gamma$ describes the off-diagonal susceptibility

$$\chi = -\left( \frac{\partial \Omega}{\partial h \partial h^*} \right)_{h=0} \sim (1/r)$$

where $h$ is a complex number -- a fictitious external field conjugate to the Bose operator $(\sum_k a_k^+)$. For $T_c = 0$, a new critical exponent $\tilde{\gamma}$ can be introduced to describe the density $n(0) \approx k_B T_c \chi$. For $d < \sigma$, the values of $\gamma$ from Table 1 and the relation $\tilde{\gamma} = (\gamma - 1)$ yield: $\tilde{\gamma} = d(\sigma - d)$ for $d < \sigma$, and $\tilde{\gamma} = \infty$ for $d = \sigma$. The exponent $\tilde{\gamma}$ does not exist for $T_c > 0$, where the density $n(0) \sim \chi$ is described by the usual exponent $\gamma$.

In Table 1, the exponent $\alpha_s$ is an auxiliary exponent intended to give a more detailed description of IBG specific heat for $d > \sigma$. For $d > \sigma$, the specific heat $C(T)$ of IBG and
spherical model can be represented in the form
\[ C(T) = C_0 + C'(T - T_c)^{-\alpha_s}, \] (25)
where \( C_0 \) is a regular constant. This behaviour corresponds to the zero value of the usual specific heat exponent \( \alpha \), defined by the scaling law \( C(T) = C^{(0)}|T - T_c|^\alpha \), where \( C^{(0)} \) is the scaling amplitude. Usually several shapes of \( C(T) \) are usually ascribed to the value \( \alpha = 0 \), namely, a logarithmic divergence \( \ln|T_c - T|/T_c \), a finite jump as is in the MF theory, and cusps of different forms. In order to avoid this arbitrariness, Fisher [15] introduced the second term in Eq. (25). The values of \( \alpha_s \) in the Table 1 are negative and this next-to-leading term in \( C(T) \) is always small but the derivative \( dC(T)/dT \) is divergent at \( T_c \) and this is an information about the cusp shape at \( T_c \).

The essential difference between the properties of the phase transition to BEC in IBG and the usual second order phase transitions can be easily seen by the comparison of the Landau free energy \[ F(\varphi) = V f(\varphi) \] of a standard second order phase transition, \( f = (r_0\varphi^2 + \varphi^4) \) with the free energy density \( f(\varphi) \) of the Bose gas at constant density \( \rho \), given by \( f = (r_0 + \varphi^2)^3 \), where \( r_0 \sim t(T) \). The simple form of these free energies corresponds to the suitable choice of units for \( f \) and the order parameter \( \varphi \). The functions \( f(\varphi) \) are shown in Figs. 2a and 2b for several values of the parameter \( r_0 \). While the well defined minima of \( f \) in Fig. 2a describe the ordered phase of usual second order phase transitions BEC is described by the inflection points \( \partial f/\partial \varphi = \partial^2 f/\partial \varphi^2 = 0, \partial^3 f/\partial \varphi^3 > 0 \) of the \( f \)-curves shown in Fig. (2b). At these inflection points the \( f \)-curves intersect the \( \varphi \)-axis; the regions where \( f < 0 \), see Fig. (2b), correspond to a positive value of the chemical potential and, hence, are unphysical [40].

Certainly, the properties of IBG below the condensation point \( r_0 < 0 \) also correspond to a continuous phase transition which is quite different from the standard second order transition. Note, that the coefficient of the \( \varphi^4 \)-term in the free energy density of the Bose gas, \( f = (r_0 + \varphi^2)^3 \), is proportional to \( t \) and tends to zero when \( t \to 0 \). This behaviour of the fourth-order term is quite similar to that of the Landau expansion for tricritical phenomena [1]. But the behaviour of IBG differs from that at standard tricritical points [1], where the temperature dependence of the Landau coefficients is different.

3.5. Zero temperature condensation

The zero-temperature \( (T_c = 0) \) BEC at a constant density was investigated in Ref. [12]. This condensation is possible at a finite constant density \( (\rho > 0) \) and a low spatial dimensionality \( d \leq \sigma \). The results for the critical exponents at \( d = \sigma \) and \( 0 < d < \sigma \) are shown in Table 1. These critical exponents describe the scaling laws with respect to the variations of temperature, for example, \( C(T) = C_0/T^\alpha \).

It can be seen from Table 1 that the zero-temperature condensation at a constant density exhibits quite unusual critical properties. The critical exponents \( \gamma \) and \( \nu \) for \( d < \sigma \) can be obtained from the familiar values at \( \sigma < d < 2\sigma \) by the change of sign of \( (d - \sigma) \). The
Figure 2: (a) The function $f(\varphi)$ for a standard second order phase transition for: $r_0 = 1.0$ (curve 1), $r_0 = 0.5$ (2), $r_0 = -0.5$ (3), $r_0 = -1.0$ (4). (b) The function $f(\varphi)$ for the IBG at constant density (the curves 1-4 correspond to the same values of $r_0$ as given for Fig.2a).

Fisher exponent $\eta$ has its usual value but the exponent $\alpha$ is negative rather than zero as is for $d > \sigma$. For zero-temperature $T$–driven transitions (Sec. 2.3), as is in our case of $d \leq \sigma$ spatial dimensions, the negative values of the critical exponent $\alpha$ are consistent with the Nernst theorem. When $d > \sigma$, $\alpha = 0$ and this theorem is satisfied in the limit $T_c(\rho) \to 0$ of an extreme dilution ($\rho \sim 0$) because the scaling amplitude $C_0$ tends to zero.

The infinite values of the exponents $\nu$ and $\gamma$ at $d = \sigma$ indicate the exponential divergence of the correlation length $\xi = (\hbar^2/2mr)^{1/\sigma}$ and the susceptibility $\chi = (1/r)$ at $T_c = 0$. These exponential divergences are known from the mechanism of critical fluctuations in classical systems at their lower borderline (critical) dimensionality $d_L$. Here the exponential divergence of $\xi$ and $\chi$ comes from the same low dimensional effect. The exponents given in the last two lines of Table 1 can be referred to as low dimensional spherical exponents.

The next question is why this zero-temperature critical behaviour exists. It is intuitively clear that the low dimensional ($d \leq \sigma$) IBG should have a ground state of a total $\vec{k}$–space condensation ($N_0 = N$) at $T = 0$ which means that the solution $T_c = 0$ of the equation $[r(T_c)/T_c] = 0$ should exist. For $T \sim 0$, $(r/k_BT) \ll 1$ and, hence, the contributions to the sum (15) are given by terms with low energies $\epsilon(k) \sim r$, i.e., with small wave numbers $k$.

There is an onset of BEC at $T \sim 0$, where the bigger part of bosons are at states $\epsilon(k) \sim 0$ but the macroscopic condensation occurs only at $T = 0$ \[12\]. At $T = 0$ the order parameter $\phi_0$ jumps from zero to $|\phi_0| = \rho^{1/2}$ and all $N$ bosons enter in the ground state ($k = 0$). This peculiar critical behaviour results from the constant density $\rho$ condition.
(the spherical constraint).

The same behaviour can be obtained from the classical spherical model. Although the $\vec{k}$–space condensation is a pure quantum effect due to the Bose statistics, the critical behaviour around critical points ($T_c \geq 0$) is totally ruled by classical fluctuation effects described by the classical spherical model.

The IBG critical behaviour can be exactly derived from thermodynamic Eqs. (12) and (15) because this model is exactly solvable. Sometimes the RG analysis of such exactly solvable models is also useful, for example, in studies of the crossover of the critical behaviour of IBG to that of NBG [23].

Note that a zero-temperature BEC ($T_c = 0$) is possible also in the limit $\rho \to 0$ (extreme dilution) for the case of constant density, and in the limit $P \to 0$ for the case of constant pressure for each spatial dimensionality $D \geq 0$. As mentioned in Sec. 3.3 at these onset transitions there is no HLTC or CQC.

3.6. Classical and quantum critical regimes

The critical behaviour of IBG in a close vicinity ($r \ll k_B T$) of phase transition points ($T_c \geq 0$) to BEC of IBG is definitely classical. All our calculations have been performed by keeping the leading power in $(r/k_B T)$ in the corresponding series for Bose integrals (19) in powers of $(r/k_B T) < 1$. This approximation has been used to determine the leading scaling behaviour, namely, the scaling laws. This essentially classical expansion gives the critical regime ($\mu \sim 0$).

The transition region (Sec. 2.2) where the phase transition phenomena occur, can be defined for a broad interval of temperatures around $T_c \geq 0$, i.e., by $r < k_B T$. In this region but not very near ($r \ll k_B T$) to $T_c$ we must take into account some next-to-leading terms in the same series expansion of the Bose integral (19) in powers of $(r/k_B T) < 1$. These secondary terms have a quantum origin and, therefore, taking several of them as corrections to the leading scaling powers, we shall obtain the quantum corrections to the main scaling behaviour.

Thus the classical region $(a–0–b)$ discussed in Sec. 2 can be defined by the inequality $r \ll k_B T$. For both $T_c(P)$ and $T_c(\rho)$ this region will look like the shaded wing $1–0–2$ in Fig. 1b; see the dashed line $0–2$. The quantum region will be outside this vicinity of critical line, in the rest part of the critical domain ($r < k_B T$) discussed in Sec. 2.

The parameters $P$ and $\rho$ rule the critical temperature but they do not participate in the scaling laws. Thus they have not a direct effect on the critical region size. This is a particular IBG property. The size of the classical critical region is reduced with the decrease of $P$ or $\rho$, i.e., of $T_c$. This is valid for both widths of classical ($r \ll k_B T$) and total ($r < k_B T$) critical regions along the $X$–axis; $X = (P, \rho)$. In fact, at fixed $X$ we should consider the inequalities $r < k_B T_c(X)$ and $r \ll k_B T_c(X)$. When $T_c(X)$ is lowered
the critical regions width along the $X$–axis will decrease to zero for $X \to 0$.

The variations of the overall density $\rho$ in the ground state $[T = T_c(\rho) = 0]$ of the low-dimensional ($d \leq \sigma$) IBG does not produce critical effects, because the condensate density is equal to $\rho$ (all particles in IBG are in the condensate). If the zero-temperature critical point, containing the complete BE condensate of IBG is approached by $T$–variations in the disordered phase, the observed critical phenomena will be mainly classical on account of the real phase transition to BEC at $T_c = 0$.

The phase diagram of the zero-temperature BEC in low dimensional systems at a constant density (Sec. 3.4) is very simple because the critical line coincides with the $\rho$–axis. The critical line $[T_c(\rho) = 0, 0 < \rho < \infty]$ can be approached only by $T$–variations and the critical phenomena are purely classical.

Finally we shall note that the criterion (9) is not satisfied for the above mentioned transitions. There is no real CQC in IBG because of the lack of fluctuation interactions but we can consider the crossover from the high-temperature to the zero-temperature at a constant density $\rho > 0$ caused by the change of the dimensionality from $d > \sigma$ to $d < \sigma$. This formal dependence describes the difference between the high–dimensional ($d > \sigma$) critical phenomena at finite critical temperatures and those of zero-temperature BEC at a constant density $\rho > 0$ and low dimensionality ($d < \sigma$).

4. Interacting bosons

4.1. General methods

The interacting bosons can be investigated by propagator expansions within the Green functions method [9, 16, 47, 48]. The infinite perturbation series are truncated on the basis of several approximations. The perturbative approach breaks down in a close (asymptotic) vicinity of the critical point because of the strong fluctuation interactions. In this situation, after accomplishing the MF analysis, the strong fluctuation interactions in the Ginzburg fluctuation region [11] around the critical point and their effects on the asymptotic critical behaviour are investigated by RG [12, 13].

The RG method consists of a suitable choice of a length-scale transformation combined with the so-called loop expansion [11, 13] (see also Sec. 4.2). In the framework of the general RG approach the MF approximation is equivalent to the so-called “tree approximation” - the lowest order theory within the loop expansion. Certain aims as, for example, the evaluation of the size of the fluctuation effects by a calculation of the Ginzburg critical region size or investigations of qualitative features of fluctuation effects in the pre-asymptotic transition region, can be achieved by suitable perturbation calculations without RG (see, e.g., Ref. [17, 18], and Sec. 6.3). This outline of general investigation methods of critical phenomena is common for both classical and quantum statistical models.

For the interacting bosons we must use the Hamiltonian $H[\hat{\psi}(\vec{r})]$ in terms of second–
quantized Bose field operators $\hat{\psi}(\vec{r})$ which for practical calculations are expanded in terms of annihilation and creation operators mentioned in Sec. 3.1. Within this formulation which gives the opportunity to use the powerful Green functions method the quantum effects are “hidden” in the commutation relations for the field operators $\hat{\psi}(\vec{r})$ and, hence, in the time ordering procedure under the trace for partition function and statistical averages. For finite temperatures the “time” ordering procedure, i.e., the account of quantum effects, is performed with the help of an auxiliary “time” $\tau$, reciprocal to the temperature: $(0 \leq \tau \leq \hbar \beta)$. This quantity is called an “imaginary time” because of the relation $\tau = it$ with the real time $t$.

Alternatively, the theory can be formulated by the Feynman path integrals [49, 50, 51, 52]. With the help of the coherent state representation the grand canonical partition function can be written as a functional integral over a $(c-$number) field $\phi(x)$ depending on a $(d + 1)-$dimensional vector $x = (\tau, \vec{r})$ in restricted “time”–space: $[(0 \leq \tau \leq \hbar \beta), \vec{r} \in V]$. So, we may introduce a $(d + 1)-$dimensional “volume” $V_{d+1} = (\beta L_1...L_d)$, where $\beta(= L_0)$ is the finite size (“thickness”) of the time–space “hyperslab”. Therefore, the quantum effects bring an extra dimension $\hbar \beta$ along the new $(\tau-)$ axis. At sufficiently high temperature (classical limit), the $\tau-$dimension collapses to zero and the behaviour of the system is classical. Of course, the same classical limit can be achieved by setting $\hbar = 0$. At sufficiently low temperatures the $\tau-$dimension is large and becomes infinite at the absolute zero ($T = 0$). At such temperatures the classical limit ($\hbar \to 0$) does not produce reliable results.

The field $\phi(x)$ represents the order parameter, which describes the ordering below the phase transition temperature. The equilibrium ordering is given by the statistical average $<\phi(x)>$. The fluctuation phenomena are represented by the fluctuation part $\delta\phi(x) = [\phi(x) - <\phi(x)>]$ of the (nonequilibrium) field $\phi(x)$. In usual cases, for example, when the phase transition is a result of a spontaneous breaking of the global discrete or continuous symmetry, the average $<\phi(x)>$ does not depend on $x$, i.e. we have a uniform equilibrium order parameter. For the sake of simplicity we shall often use this example.

The complex Bose field $\phi(x)$ is sometimes referred to as a “classical field”. This term points to the difference from the field operators which obey commutation rules. In fact, the classical fluctuation field is $\phi(0, \vec{r}) \equiv \phi(\vec{r})$ and corresponds to the classical limit ($\hbar = 0$). The $\tau-$variations of the field $\phi(x)$ are created by quantum fluctuations. The way, in which the quantum fluctuations take part in this picture is very similar to that, in which the classical fluctuations appear as spatial variations of the field $\phi(x)$.

Furthermore, in order to describe a large variety of systems one may consider $\phi(x)$ as a $(n/2)-$component complex vector field: $\phi(x) = \{\phi_\alpha(x), \alpha = 1,...,n/2\}$, where the components $\phi_\alpha$ are complex functions of $x$ and $n$ is an even positive number depending on the symmetry of the system, or, more precisely, on the symmetry of the possible ground states (orderings). The number $n$ is often called a “symmetry index” or a “number of
order parameter components.”

For certain problems [12, 13] the field theories of n-component real fields are equivalent to the theories of (n/2)-component complex fields. The classical systems are usually described by real scalar or vector fields whereas the quantum systems are represented either by real field components or by complex field components of twice less number (the choice depends on the aims of the particular investigation). Thus one may compare the results for quantum systems with a symmetry index of ordering n/2 to the results for classical systems with a symmetry index n. Moreover, this correspondence can be extended to all integer numbers n > 0. Then a complex field with n’ = n/2 components will correspond to a classical Ising system for n = 1, i.e., n’ = 1/2. A Heisenberg ferromagnet (n = 3) will be described by a complex vector field of n’ = 3/2 components.

On account of the analytical relation τ = it the purely quantum fluctuations reveal the intrinsic quantum dynamics of the system. The dependence of physical quantities on real time t is obtained by an analytical continuation (τ → it) of the results from calculations to the real time axis t. While in classical systems the static and dynamic phase transition properties are considered separately, in the framework of the quantum statistical mechanics these phenomena have a unified treatment. The intrinsic quantum dynamics contained in the quantum statistical correlations is important for the dynamical critical properties of Bose systems; see, e.g., Refs. [53, 54, 55, 56]. Note, that this intrinsic quantum dynamics can be substantially changed under the action of external time-dependent potential(s) [12, 25].

Adopting periodic boundary conditions along the τ–axis, φ(τ, ⃗r) = φ(τ + ℏβ, ⃗r), and having in mind the periodic boundary conditions along the spatial axes (Sec. 3.2) we can expand the field components φα(x) in Fourier series

\[ \phi_\alpha(x) = \frac{1}{\sqrt{\beta V}} \sum_q e^{i q x} \phi_\alpha(q), \]  

where the (d + 1)–dimensional vector q = (ωl, ⃗k) is given by the Matsubara frequencies

\[ \omega_l = 2\pi l k_B T / \hbar, \]  

where l = (0, ±1, ...); qx = (ωlτ + ⃗k ⋅ ⃗r). Setting ωl = 0 in Eq. (26) we neglect the quantum fluctuations and the amplitudes [φ(0, ⃗k)/√β → φ(⃗k)] will describe classical fluctuations only. In the x–representation this corresponds to the limit β → 0, in which the system size ("thickness") along the τ–direction tends to zero.

The functional formulation of Bose systems in terms of (c–number) complex functions φ(x) is performed directly for the original microscopic second-quantized Hamiltonian because the commutation relations of the Bose operators ψ(⃗r) have a direct classical limit [9, 49, 50]. The microscopic Hamiltonians of fermionic and spin systems are transformed to effective Bose Hamiltonians with the help of Hubbard-Stratonovich transformations and Feynman path integration [2, 20, 50, 51, 52]. The effective field Bose Hamiltonians have the same meaning and role in the theory of quantum phase transitions as the effective
field Hamiltonians for classical systems [11,13,45]. In both cases of classical and quantum systems the effective field Hamiltonians is not exact counterpart of the respective original Hamiltonians but they provide a correct description of the (quasi)macroscopic phenomena which is enough for a reliable treatment of phase transition problems. The lack of an entire correspondence with the microscopic model in nontrivial cases of interacting particles is a result of certain forms of coarse graining which are unavoidable product of the known theoretical techniques of derivation of effective field models.

As our interest is focussed on standard second order phase transitions we shall consider \( \phi^4 \)–effective Bose “Hamiltonians” \( H \) of fermionic and spin systems, as well as, the same type microscopic Hamiltonians of genuine Bose systems. In fact, the “Hamiltonian” \( H \) that we shall investigate is related to the action \( S \) of the system by \( H = -S \); here we follow a conventional terminology used in a number of papers; see, e.g., Ref. [20].

A quite general quantum effective Hamiltonian \( H = \beta H \) comprising a number of systems can be written in the form

\[
H[\phi] = \sum_{\alpha,q} G_0^{-1}(q)|\phi_\alpha(q)|^2 + \frac{v}{2\beta V} \sum_{\alpha,\beta, q_1,q_2,q_3} \phi_\alpha^*(q_1)\phi_\beta^*(q_2)\phi_\alpha(q_3)\phi_\beta(q_1 + q_2 - q_3),
\]

(27)

where \( v > 0 \) is the interaction constant. The (bare) correlation (Green) function \( G_0(q) = \langle |\phi_\alpha(q)|^2 \rangle_0 \) is given by

\[
G_0^{-1}(q) = i\omega_l + \varepsilon(k)
\]

(28)

for interacting real bosons and transverse XY model [71, 72]. For other effective Bose models of quantum systems the bare correlation function \( G_0(q) \) can be written in the form [20]

\[
G_0^{-1}(q) = \frac{|\omega_l|^m}{k^m} + ck^\sigma + r
\]

(29)

with positive exponents \( m, m' \), \( 0 < \sigma \leq 2 \), and Landau parameters \( c > 0 \) and \( r \sim t(T) \) or \( r \sim t(X) \). The parameter \( c \) can be always presented as \( c = \hbar^2/2m(g) \) where the effective mass \( m(g) \) of the composite bosons (Bose excitations) depends on some interaction constant \( g \).

The microscopically formulated Bose Hamiltonians contain an upper cutoff \( \Lambda = (\pi/a) \) corresponding to the Brillouin zone ends \( -\pi/a < k_i < \pi/a \). An example of such Hamiltonians is considered in Sec. 5. The effective quasimacroscopic Hamiltonians derived from microscopic Hamiltonians contain only large–scale spatial fluctuations of the field \( \phi(q) \) and, hence, the corresponding cutoff \( \Lambda \) is relatively small: \( 0 < \Lambda \ll (\pi/a) \) [45].

An example of a quasimacroscopic (effective field) Hamiltonian, derived by a Hubbard-Stratonivich transformation from the microscopic TIM Hamiltonian, is discussed in Sec. 6. As far as we are discussing critical phenomena \( (\xi \gg a, \lambda \gg a) \) the precise value of the cutoff \( \Lambda \) is not important. It is, however, important to have the conditions \( \xi > (1/\Lambda) \) and \( \lambda > (1/\Lambda) \) satisfied and then, the relevant phenomena included into consideration.
The statistical treatment implies the calculation of the generalized grand canonical partition function
\[ Z(T, \tilde{a}) = \int \mathcal{D}\phi e^{-\mathcal{H}[\phi(x)]}, \] (30)
which gives the Gibbs free energy
\[ \Omega = -\beta \ln Z(T, \tilde{a}). \] (31)

In Eq. (30), \( \tilde{a} \) denotes the Hamiltonian parameters \((c, r, v)\), \( \int \mathcal{D}\phi \) denotes the functional integration over all allowed field \( \phi(x) \) configurations
\[ \int \prod_{\alpha=1}^{n/2} \prod_{x \in V_{(d+1)}} d\phi^*_\alpha(x) d\phi_\alpha(x), \] (32)
or, in the \( \vec{k} \)-space,
\[ \int \prod_{\alpha, q} d\phi^*_\alpha(q) d\phi_\alpha(q) \] (33)
over all allowed Fourier amplitudes \( \phi_\alpha(q) \). Constraints on the field configurations lead to a critical behaviour change. Within the present functional formulation the constraint of a constant density (13) is given by the mean spherical condition
\[ \frac{nN}{2} = \sum_{\alpha, q} \langle |\phi_\alpha(q)|^2 \rangle. \] (34)

Therefore, the mean square of the \( Nn/2 \) dimensional complex vector \( \vec{\phi} = [\phi_\alpha(q)] \) (with components given by all possible \( \alpha \) and \( q \)) describes a sphere of radius \( Nn/2 \). In the \( \phi^4 \)-Hamiltonian of the nonideal Bose gas (NBG) given by Eqs. (27) and (28) with \( r = -\mu \), this constraint can be taken into account in the so-called large-\( n \) limit \((n \to \infty)\) \[1, 12\].

Let us note, that for certain problems \[57\] the effect of constraints like (34) can be taken into account in Eq. (27) by adding an auxiliary \( \phi^4 \)-interaction term: \( \sim \tilde{u}\phi^4 \). In contrast to the \( u \)-interaction terms \( \phi^*_\alpha(q_1)\phi^*_\beta(q_2)\phi_\alpha(q_3)\phi_\beta(q_1 + q_2 - q_3) \) in Eq. (34), the auxiliary interaction is given by two summation \( q \)-vectors: \( \phi^*_\alpha(q_1)\phi^*_\beta(q_2)\phi_\alpha(q_1)\phi_\beta(q_2) \) \[57\].

The practical calculations are carried out by a substitution of the summation over the wave vector components \( k_i \) with an integration as shown by Eq. (16), provided the corresponding dimensions \( L_i \) satisfy the criterion \( L_i \gg \xi \) for a quasi-infiniteness. The summation over the Matsubara frequencies \( \omega_l \) can be substituted with the integration
\[ \frac{1}{\beta} \sum_{\omega_l} \rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)}, \] (35)
provided the temperature \( T \) is low enough or, in an exact mathematical sense, if \( T \to 0 \).
In order to determine the condition, under which the integration (35) can be used without substantial errors in the final results, let us consider the model (27) corresponding to the correlation function $G_0(q)$ given by Eq. (29) with $m' = 0$ and $m = 1$. Then the $\omega_l$-dependent modes $\phi(\omega_l \neq 0, \vec{k})$ in Eq. (27) will yield relevant contributions to the partition function, if they have a relatively high statistical weight. This may happen provided $\omega_1 = 2\pi k_B T < r$ near the critical point $r \sim 0$. In terms of characteristic lengths $\lambda = (4\pi c/k_B T)^{1/\sigma}$, and $\xi = (c/r)^{1/\sigma}$ we have $\lambda > (8\pi^2)^{1/\sigma} \xi$ which is consistent with the criterion (3); here we have used Eq. (2) and $c = \hbar^2/2m$.

If the criterion (3) is satisfied, we can substitute the summation over $\omega_l$ with an integration as shown by the rule (35) without introducing a substantial error in the calculations. If the criterion (3) is not satisfied, the frequencies $\omega_l$ can be neglected and the quantum fluctuations ignored.

There exists a well developed theory [50, 58, 59, 60, 61, 62] of the $\lambda$-transition in $^4$He which can be applied to other Bose fluids with critical temperatures $T_c > 0$. We shall not dwell on this important topic. Our attention will be concentrated on the case $T_c \to 0$ when the asymptotic scaling is influenced by quantum effects.

4.2. Notes about renormalization group

A number of problems in the scope of this article are usually solved by RG in the $q$-space, i.e., by field theoretical variants of RG [1, 12, 13, 16]. However, because of the simultaneous presence of more than one effect (quantum and thermal fluctuations, disorder, anisotropy or gauge-field effects) a comprehensive RG investigation of quantum systems can be performed by widely applicable and not extremely sophisticated variants of RG such as, for example, the $\epsilon$- or the $(1/n)$-expansion [1, 12, 13]. The latter expansion is convenient for the calculation of critical exponents but is quite hard as a method of investigation of the RG transformation [12].

In the $\epsilon = (d_U - d)$-expansion the difference $(d_U - d)$ is used as a small parameter; $d_U$ is the so-called upper critical (borderline) dimensionality, above which the fluctuations are irrelevant and the system is described by the MF approximation. For the classical variant ($\omega_l = 0$) of the Hamiltonian (27) we have $d_U = 2\sigma$ but for other Hamiltonians $d_U$ may have another value [1]. Within the $\epsilon$-expansion the physical quantities can be calculated as series in powers of $\epsilon$ and the order of accuracy of the calculation (first, second, etc. order in $\epsilon$) coincides with the order of the loop expansion which has been taken into account [1, 13], that is, the calculated order in $\epsilon$ exactly corresponds to the respective order in the loop expansion. The method works for dimensionalities $d > d_L$, where $d_L$ is the so-called lower critical (borderline) dimensionality, below which the ordering is destroyed by the fluctuation effects; for the Hamiltonian (27) with ($\omega_l = 0$) we have $d_L = \sigma$ [1].

The $\epsilon$- expansion has been widely used in investigations of critical phenomena and in the
remainder of this paper we shall discuss results obtained by this method. Note, that the $\epsilon-$expansion is asymptotic and all reliable predictions about the possible types of fixed points (FPs) of the RG differential equations $[1,13]$, or, equivalently, of the Wilson-Fisher recursion relations $[1,12,63,64,65,66]$ are obtained within the one-loop approximation (equivalent to the first order in the $\epsilon-$expansion).

The next order of the theory, i.e., the two-loop order, which is equivalent to the second order in the $\epsilon-$expansion, provides better quantitative results for the critical exponents, corresponding to the FPs, but this and even the higher orders in the loop expansion do not essentially contribute to the understanding of the main qualitative features of the system, such as the possible types of critical behaviour and their stability properties. The same is valid for the precise determination of the domains of attraction of simultaneously existing stable FPs of complex Hamiltonians describing more than one interaction effect; this point has been discussed in Ref. $[67]$ on the basis of a quite complex model.

The above notes justify our approach to the classical and quantum critical phenomena in complex systems with competing effects, for which we shall often use results in first order of $\epsilon$. We would like to emphasize an important but not widely accepted feature of the modern RG methods, namely, that the lowest (one-loop) results reveal the entire picture of the possible critical phenomena whereas the higher-order considerations in the loop expansion are useful for other aims: the improvement of numerical results for the critical exponents and also the investigation of the asymptotic nature of the RG perturbation-like method(s).

Bearing in mind that the most often used expansions in the theoretical physics are asymptotic we should not take the asymptotic nature of the $\epsilon-$expansion as a great disadvantage, or, as a reason for an unreliability of the $\epsilon-$results. The experience accumulated by the research done during the last 20 years firmly indicates that the $\epsilon-$expansion results are meaningful and quite useful, provided they are consistently analyzed and interpreted $[67]$.

The main types of critical behaviour which can be described by a concrete model (Hamiltonian) are given by the FPs of the respective RG equations. Different FPs describe different types of critical behaviour and, hence, different critical exponents. These exponents are obtained from the relevant and irrelevant stability exponents of the FPs. If a FP appears of order $O(\epsilon^2)$ but does not exist within the one-loop (first order in $\epsilon$) RG equations, it cannot be accepted as a new reliable object of investigation and the reason lies in the asymptotic nature of the $\epsilon-$expansion.

At relatively high spatial dimensionalities $d > d_U$ which usually do not correspond to real systems the RG equations have one stable FP, the so-called Gaussian FP (GFP). This FP describes the MF behaviour for $d > d_U$. The same GFP is unstable for $d < d_U$ and the stable critical behaviour in this domain of spatial dimensionalities, which includes the real system dimensionalities ($d = 1, 2, 3$), is described by another stable FP. This is usually a nontrivial FP which gives $\epsilon-$corrections to the MF values of the critical exponents, i.e.,
this FP yields a nontrivial (non-MF) critical behaviour. Sometimes such FP is called the Wilson FP \cite{63, 64, 65} but the most often used terms are either Heisenberg FP (for systems with a continuous symmetry, $n > 1$) or Ising FP (for systems with a discrete symmetry, $n = 1$). When the system has a XY symmetry ($n = 2$) the nontrivial FP is called XY FP. GFP has zero coordinates ($r_G = v_G = 0$) in the $(r, v)$ parameter space of the Hamiltonian but the nontrivial FPs have nonzero coordinates which are given to first or higher order in $\epsilon$, depending on the order of the loop approximation used in the particular investigation.

The possible types of critical behaviour predicted by RG are classified in universality classes. For a given structure of the Hamiltonian, one can define the universality classes by the couple $(d, n)$, that is, the critical behaviour depends on the dimensionality $d < d_U$ and the symmetry index (number of components of the order parameter field $\phi$) $n$. But one must keep in mind that the change of the mathematical structure of the Hamiltonian usually leads to another double series $(d, n)$ of universality classes. The equivalence of the critical behaviour of many substances, described by the same Hamiltonian structure and the couple $(d, n)$ is recognized as a property of the universality of critical phenomena. The MF behaviour at $d > d_U$ is superuniversal because it does not depend on $d$ and $n$.

Below the upper critical dimensionality $d_U$ more than one FPs may exist and be stable in different domains (domains of attraction) of the Hamiltonian parameter space. Usually this happens for complex systems with competing effects. The Hamiltonians of such systems include more than one interaction term. The simultaneous stability of more than one FP is a manifestation of the simultaneous action of more than one fluctuation interaction or the presence of other effects which interfere with the main fluctuation interaction; the latter is usually represented by a $\phi^4$-term in the Hamiltonian as shown in Eq. (27). This situation is quite common in the RG analysis of Hamiltonians with competing effects. Now one is faced with the hard task to determine precisely the FPs domains of attraction which is often necessary for the correct prediction of the critical behaviour. We have already mentioned that for quite important cases this cannot be reliably done even in orders higher than the first order in $\epsilon$ \cite{67}. In other cases, the effects competing the main fluctuation interaction ($\phi^4$) are hidden in additional modes like the modes $\phi(\omega_l, \vec{k})$ with $\omega_l \neq 0$ of the quantum fluctuations, and under certain circumstances, these modes may have an essential effect on the critical behaviour.

The RG predictions are uncertain when there are no stable FPs. This may happen for a definite domain of the Hamiltonian parameter space or, in extreme cases (see Sec. 7), for the whole parameter space. The lack of stable FP is usually interpreted as a lack of a stable (multi)critical behaviour of usual type. Sometimes, the absence of stable FP can be considered either as a signal for a first order phase transition or as a more particular type of a continuous phase transition. The reliable prediction of the phase transition type in this situation requires additional heuristic arguments and investigations outside the
4.3. Classical-to-quantum dimensional crossover

We have already mentioned that the $d$-dimensional quantum systems resemble $D = (d + 1)$-dimensional classical ones. This is valid for the model (29) with $m' = 0$ and $m = \sigma$. By the formal substitution $k_B T = c^{1/\sigma} / L_0$ in $\omega_i$, where $L_0 \equiv \lambda = (c^{1/\sigma} / k_B T)$ is the thermal wavelength corresponding to the model (29), we obtain a $D = (d + 1)$-dimensional momentum $q = [q_i; i = 0, 1, \ldots, d]$ with components $q_i = (2\pi l_i / L_i)$. The representation gives a total formal correspondence between the present time–space geometry and a hyperslab with $(d + 1)$ spatial dimensions. This formal analogy was revealed for the first time in Refs. [68, 69] and further explored in Ref. [70]. Using the formal analogy we have shown that for this effective model the notation (1) yields the exponent $\theta = 1$.

The temperature dependence $\lambda = \lambda_0 / T^{1/\sigma}$, as in IBG, is valid for other effective models, for example, for the model (28) and that given by Eq. (29) with $m' = 0$ and $m = 1$. Therefore, the form of the scaling law $\lambda = \lambda_0 / T^\theta$ depends on the values of exponents $m$, $m'$ and $\sigma$: $\theta = \theta(m, m', \sigma)$. A simple dimensional analysis of Eqs. (28) and (29) shows that $\theta = 1/z$. The same result ($\theta = 1/z$) can be obtained by the RG rescaling transformation [1, 12, 13]. Then the criterion (9) becomes $z\nu_0 < 1$. Usually the (bare) values of exponents $\theta$ and $\nu = 1/\sigma$ of $\lambda$ and $\xi$, respectively, receive perturbation corrections which are calculated with the help of RG.

It has been shown by Hertz [20] with the help of the above mentioned method as well by a RG analysis that

$$z = \frac{\sigma + m'}{m}$$

and, hence, that the real CQC is given by

$$D = d + z$$

CQC is always associated with the dimensional crossover (37) which was revealed before Hertz by Pfeuty and Elliott [17] and by Young [19] on the basis of TIM (for the latter, $m' = 0$, $\sigma = 2$, $m = 2$, and $z = 1$). Note, that the upper borderline dimensionality changes from its classical values $d_U$ to the “quantum” value $d_U^0 = (d_U - z)$ and the lower borderline dimensionality becomes $d_L^0 = \max[0, d_L - z]$.

Thus we should expect that the quantum critical phenomena in $d$-dimensional (quantum) systems are described by the universality class $(D, n)$ known for $D = (d + z)$-dimensional classical systems. If the shift $d \rightarrow D$ of spatial dimensionality $d$ is the only result of quantum effects, we can say that the quantum critical phenomena in the particular $d$-dimensional quantum system (or a class of systems) obey a form of universality, namely, that they are identical to the critical phenomena in the corresponding $D$-dimensional classical system (or class of systems). Under the term “corresponding classical system”
of the quantum model (27) we understand the system described by the classical variant ($\omega_l = 0$) of Eq. (27).

This definition of the universality of quantum critical phenomena allows a comparison between the critical properties of classical and quantum systems. There are other points of view on the definition of the universality for quantum critical phenomena. For example, sometimes it is said that the quantum critical phenomena in a system are universal, if a classical critical behaviour can be found at the same or another spatial dimensionality which describes the same critical phenomena [74, 75, 76]. This point is important, for example, in the discussion of the extraordinary zero-temperature critical behaviour of NBG (Sec. 5).

5. Nonideal Bose gas

5.1. Preliminary notes

The first studies [74, 75, 76, 77] of NBG by RG were performed in the framework of the operator formalism [9, 10, 46, 47]. The direct RG applications to the second-quantized NBG Hamiltonian lead to a scaling law for the commutation relations of the field operators $\hat{\psi}(\vec{r})$. In Refs. [74, 75, 76, 77] this was interpreted as a rescaling of the boson mass $m$ which grows with the successive RG transformations. The fact that the quantity, to which such a rescaling should be ascribed is the thermal length $\lambda$ or, equivalently, the factor $(1/mT)$ associated with it, has become clear later. However, this circumstance was undoubtedly irrelevant to the main conclusion of these works, namely, that the finite temperature ($T_c > 0$) BEC should exhibit a universal critical behaviour corresponding to the universality class ($d, 2$) of classical XY model. The latter result has been confirmed and extensively discussed in other studies; see e.g., [78, 79, 80, 81, 82, 83, 84, 85]. An error in the calculations [74, 75] which is irrelevant for finite critical points ($T_c > 0$) but very important for the zero-temperature critical behaviour at $T_c = 0$) has been pointed out in Ref. [81] and noticed also in Ref. [23]. Other relatively early papers on critical properties of real bosons systems are mentioned in Refs. [1, 5, 23].

The zero-temperature critical behaviour of NBG was treated for the first time in Ref. [86] in the one-loop approximation and despite of the correct result for the critical exponents there are errors in the derivation of the RG equations and their FPs. In Ref. [87] the correct calculation of the self-energy function, the Fisher exponent $\eta$ and the dynamical critical exponent $z$ was made in the two-loop approximation.

The Wilson–Fisher recursion relations [1, 12] appropriate for a correct and thorough treatment of the critical properties of NBG at low and zero temperature critical points have been derived in Ref. [81] and in Secs. 5.2–5.6 we shall follow this approach. The perturbation series and the diagrammatic representation of perturbation terms are standard and we shall not dwell on technical details; for a more instructive explanation of the corresponding calculations, see Refs. [1, 81, 88].
Because of the model particular properties the low-temperature limit gives an extraordinary opportunity for an exact summation in all orders of the loop expansion and, therefore, for obtaining of the low-temperature critical behaviour without approximations. This was demonstrated for the first time in Ref. [81]. However, the exact result [81] has been overlooked in subsequent papers [23, 89, 90, 91] where the same low-temperature critical behaviour was rediscovered in the one–loop approximation and applied to dilute Bose systems. In Ref. [89], the Gaussian-like critical behaviour [81] to one-loop order (see also Sec. 5.3) was very conveniently interpreted as a form of “quasi-universality.” In Ref. [23, 91] the RG recursion relations [81] were re-established in the form of one-loop RG differential equations and solved by the Rudnick–Nelson [92] method of integration.

More recently, the simple structure of the perturbation series discussed below, was found also in the nonrelativistic scalar field theory and appropriately called the “scaling anomaly” [93]. Sachdev and co-workers [94] introduced another appropriate term – “zero scale–factor universality” – for the same exact solution [81]. These authors overlooked the opportunity to take advantage of perturbation series exact summability (see below) but recently the exact result has been rediscovered and lengthy explained by one of them without any reference to the original source (see Ref. [6]). The same disadvantage characterizes the results of other papers [23, 92] because the exact solution [81] is not familiar to the authors [95, 96].

Here we shall use the functional formulation of NBG (see, e.g., Ref. [50]) which seems to be more convenient for RG studies [79, 81]. We shall be mainly interested in the zero-temperature critical behaviour of NBG [81]. The relation to other model systems with similar properties and recent research will be also mentioned (Sec. 5.7). The functional NBG formulation is given by Eqs. (27) and (28) where the parameter $r = -\mu$ is related to the (bare) unrenormalized chemical potential $\mu \leq 0$. In order to avoid lengthy mathematical formulae we shall set $\sigma = 2$. The generalization of results to $0 < \sigma \leq 2$ is not difficult. The critical behaviour is investigated with respect to variations of the chemical potential $\mu$ from the value $\mu_c = \mu(T_c)$, corresponding to the critical point (for IBG, $\mu_c = 0$). A treatment of the phase transition in NBG at a constant density with the help of variations of the quantity $t \sim (T - T_c)$ has been performed in Ref. [85].

5.2. Renormalization group equations

The RG recursion relations within the one–loop approximation (to first order in $\epsilon = d_U - d$) are [81]:

$$k'_i = bk_i, \quad \omega'_i = b^{2-\eta}\omega_i, \quad m' = b^\eta m,$$

(38)

$$r' = b^{2-\eta}\left[r + \frac{1}{2}(n+2)(v/\beta)I_1(r)\right],$$

(39)

and

$$(v/\beta)' = b^{4-d-2\eta}\left\{(v/\beta) - \frac{1}{2}(v/\beta)^2\left[(n+6)I_2(r) + 2\tilde{I}_2(r)\right]\right\}. $$

(40)
Here we shall write in an explicit form the quantities

\[ I_1(r) = \beta K_d \int_{\Lambda/b}^{\Lambda} dk \, k^{d-1} n(k) , \]  

(41)

\[ I_2(r) = \left( \frac{\beta}{2} \right)^2 K_d \int_{\Lambda/b}^{\Lambda} dk \frac{k^{d-1}}{\sinh^2[\beta \varepsilon(k)/2]} , \]  

(42)

and

\[ \tilde{I}_2(r) = \frac{\beta}{2} K_d \int_{\Lambda/b}^{\Lambda} dk \, k^{d-1} \frac{\cosh[\beta \varepsilon(k)/2]}{\varepsilon(k)} . \]  

(43)

In the above expressions the integration is performed up to an upper cutoff \( \Lambda \ll (\pi/a) \); \( b > 1 \) is the RG rescaling factor \[1, 12\].

In contrast to other RG studies here the problem for the cutoff \( \Lambda \) is not trivial \[23, 81\]. The problem arises in the treatment of the dilute \((d > 2)\)-dimensional Bose gas in the low temperature limit \((T_c \to 0)\) which corresponds to an extreme dilution \( \rho \to 0 \); see Eq. (23) which indicates that \( \Lambda \sim (\pi/a) \to 0 \) for \( \rho = a^{-d} \to 0 \). It seems at first sight that the interparticle distance \( a \) cannot be used in the definition of the finite cutoff \( \Lambda \) or as a length unit of dimensionless characteristic lengths \( \xi/a \) and \( \lambda/a \). For this reason, the fourth (and last) characteristic length in the problem – the finite scattering length \( v = (\hbar^2/m) a_{sc} \) \[9, 10\] was discussed as a possible inverse cutoff, \((1/\Lambda) \sim a_{sc}\), for the first time in Ref. \[81\] and later used also in Ref. \[23\]; for \( d \neq 3 \), \( v \sim (\hbar^2/m) a_{sc}^{(d-2)} \).

Let us note, however, that the RG investigation can be reliably performed by the standard cutoff \( \Lambda = \pi/a \), although \( a \) tends to infinity for the zero temperature critical behaviour. In fact, according to Eq. (23), \( \lambda > a \) for all temperatures in the interval \( 0 \leq T < A_0(d/2)T_c(\rho) \) including the close vicinity of \( T_c \) where \( \xi > a \), too; note that \( A_0(d/2) > 1 \). In the temperature range of interest the characteristic lengths \( \xi \) and \( \lambda \) are always greater than \( a \) and, therefore, there is no danger of shortcomings in the description; see also the brief discussion after Eq. (50) in Sec. 5.3.

The summation over the frequencies \( \omega_l \) of the internal lines of perturbation diagrams formed by the legs of one and the same Hamiltonian is performed with the help of the rule

\[ \sum_{\omega_l} e^{(+0)i\omega_l} G_0(q) , \]  

(44)

where \((+0)\) denotes a positive number which is set equal to zero after the calculation. The correlation function \( G_0(q) \) is given by Eq. (28); c.f. Refs. \[11, 12\], where the Green function \( G_0(q) \) has an opposite sign.
The summation (44) leads to a difference between the integrand $n(k)$ in $I_1(r)$ and the corresponding integrand in the RG relations presented in Refs. [86] but coincides with the corresponding result in Refs. [74, 75, 78]. Moreover, the integrand in $\tilde{I}_2(r)$ differs from the corresponding integrand in the RG equations in Ref. [74, 75] but coincides with that in Ref. [86] as mentioned for the first time in Ref. [81]. The mentioned errors in papers before 1981 do not have an effect on the results for the critical behaviour in the classical limit ($\rho < 1, T_c \neq 0$). However, in the low-temperature limiting case ($T_c \to 0$), these errors lead to wrong results for the FP coordinates and the rational extension of the loop expansion in higher than first order.

Owing to this formulation of the RG treatment the relation (38) for $\omega_l$ should be referred to the temperature:

$$T' = b^{2-\eta}T.$$  (45)

For the same reason the mass $m$ should be kept invariant. We shall proceed with this choice up to the end of this subsection, where we shall propose a more elegant scheme of scaling. The mass $(m-)$ invariance implies $\eta = 0$. This is the usual result within the one–loop approximation. Apart from quite special cases the exponent value $\eta = 0$ receives $\epsilon$–corrections of order $O(\epsilon^2)$, i. e., in the two– and higher–loop approximations.

The task is to solve RG Eqs. (38)–(40). This means to obtain FPs and investigate their stability. Eq. (45) shows two temperature FP values: $T_C^* = \infty$ (classical) and $T_Q^* = 0$ (quantum). The infinite FP value $T_C^*$ of the temperature has nothing in common with infinite temperatures. Rather it describes the classical critical behaviour near finite temperature critical points $T_c > 0$.

Remember that the classical limit of quantum Hamiltonians (27) is taken, in a strict mathematical sense, by decreasing the interval $[0, \beta]$ of variations of Matsubara time $\tau$ to zero which corresponds to the limit $T \to \infty$. Obviously, the FP coordinate $T_Q^* = 0$ should correspond to the low-temperature regime. As FPs lie on the critical surface $[\xi(T_c) = \infty]$ in the Hamiltonian parameter space it is clear that the zero FP value corresponds to a zero-temperature critical point ($T_c = 0$). However, it can be shown that for “high-temperature” critical points the parameter $T$ is absolutely redundant together with the relation (45) and the corresponding infinite FP value of temperature (see Sec. 5.5). The infinite FP value $T_C^* = \infty$ merely indicates that the zero- temperature critical behaviour is unstable towards temperature fluctuations.

Eq. (38)–(40) can be solved analytically in two limiting cases:

(i) $\beta \varepsilon(k) \ll 1$ (high-temperature behaviour),

(ii) $\beta \varepsilon(k) \gg 1$ (low-temperature behaviour).

These conditions will be discussed in Sec. 5.4. Here we shall take the leading terms of the integrals (41)–(43) for each of these limiting cases and perform the formal analysis of RG equations.
The limit (i) yields the classical universality as shown in Refs. [74, 75]. In this case ($\epsilon = 4 - d$) the FP value of the temperature is $T^*_C = \infty$. This allows small values of the factor $\beta$ in the condition (i). The RG relations have two stable FPs. For dimensionalities $d > 4$, the Gaussian FP with $(r, u)$–coordinates $r_G = u_G = 0$ is stable and describes the usual MF behaviour; $u = v/\beta$. For dimensionalities $2 < d < 4$, the Heisenberg FP is stable. The coordinates of HFP are given by

$$r_H = - \frac{(n + 2)}{2(n + 8)} \frac{h^2 \Lambda^2}{2m} \epsilon, \quad u_H = \frac{16\pi^2}{(n + 8)} \left( \frac{h^2}{2m} \right)^2 \epsilon. \quad (46)$$

As the temperature FP coordinate is infinity the FP value $v_H$ of the parameter $v = \beta u$ is equal to zero because the respective FP value $u_H$ of $u$ is finite, as given by Eq. (46). This behaviour of the interaction parameter ($v$ or $u$) is not strange because it precisely describes the behaviour of the system at high temperatures where the modes $\phi(q)$ with $\omega_l \neq 0$ can be neglected. Ignoring the quantum fluctuations we obtain a classical Hamiltonian with an interaction constant of the form $u = v/\beta$ which plays the role of an interaction constant in the usual classical Hamiltonians ($H = \beta H$).

However, this is not the final answer. As shown in Sec. 4, we must substitute the modes $\phi(0, \vec{k})$ with the true classical modes $\sqrt{\beta} \phi(\vec{k})$ and this yields a factor $\beta \approx (1/k_B T_c)$ in front of parameters in the $\phi^2$–part as well as the square of the same factor in front of the interaction constant. So, in the classical variant of the theory the actual interaction constant is $\beta v$. Therefore, the parameter $u = (v/\beta)$ is the interaction in the classical Hamiltonian in terms of the field $\phi(0, \vec{k})$ and this parameter transforms to $u = \beta v$ when the classical Hamiltonian is written by the field $\phi(\vec{k}) = \phi(0, \vec{k})/\sqrt{\beta}$.

This discussion can be used to explain the infinite value of FP coordinates of temperature: $T^*_G = T^*_H \equiv T^*_C = \infty$. The field $\phi(0, \vec{k})$ has a rescaling factor $b$ (for $\eta = 0$). The field crossover [$\phi(0, \vec{k}) = \sqrt{\beta} \phi(\vec{k})$] implies that this rescaling factor should be associated with the proper classical field $\phi(\vec{k})$ rather than with the temperature. This is just what happens in the classical limit (i) where the relation (45) does not exist at all. Therefore, in the limiting case (i) the temperature is a redundant parameter and the relation (45) can be suspended from the high-temperature analysis although it remains in the general RG scheme. Up to the stage, for which the investigation is performed in terms of the field $\phi(0, \vec{k})$, i.e., in the low-temperature (classical or quantum) regime we must consider the relation (45) and the FP temperature value $T^*_Q = 0$.

5.3. Low temperature behaviour

In the low-temperature limiting case (ii), Eqs. (38)–(40) are solved with the help of $\epsilon = (2 - d)$–expansion. This expansion reflects the dimensional CQC which is given by $D = (d + 2)$. The upper critical dimensionality $d_U$ is changed from $d_U = 4$ for the classical case (i) to $d_U = 2$ for the quantum case (ii) [20]. Up to now there is no evidence that the case (ii) describes quantum critical phenomena but it is clear that the low-temperature limit
can be taken within the RG scheme and that it will bring to a new critical behaviour. Within RG this limit always exists because of the lower cutoff \((\Lambda/b) > 0\) which permits the condition (ii) irrespective of the value of the ratio \(r/k_B T\); see also Eq. (21). For the case (ii) the integral \(I_2(r)\) tends exponentially to zero but the integral \(\tilde{I}_2(r)\) is finite and exhibits a power–law infrared divergence for \(d < 2\). This yields the borderline value \(d_U = 2\). Performing standard calculations we obtain Eqs. (39) and (40) in a simple form

\[
r' = b^2 r, \quad v' = b^2 - d v(1 - a_0 v),
\]

where

\[
a_0 \equiv \left[ \tilde{I}_2(r) \right]_{\beta \varepsilon(k) \to \infty} = \frac{1}{4\pi} \int_{\Lambda/b}^{\infty} \frac{k \, dk}{\varepsilon(k)}.
\]

The straightforward calculation gives

\[
a_0 = \frac{m}{2\pi^2 \hbar^2} \ln b.
\]

The relevant parameters are \(T\) and \(r\). The variations of these parameters near the FP values \(T^* = r^* = 0\) drive the system away from the zero-temperature critical state. The inclusion of the temperature \(T\) as a second relevant parameter corresponds to a real physical situation, namely, that the zero-temperature critical state is approached when both \(T\) and \(r\) tend to zero. The zero-temperature critical behaviour is described by two stable FPs: the Gaussian FP \((T_G = r_G = v_G = 0)\), which is stable for \(d > 2\), and the Gaussian–like FP \([81]\),

\[
T_{Gl} = r_{Gl} = 0, \quad v_{Gl} = \frac{2\pi^2 \hbar^2}{m} \varepsilon.
\]

The equation for \(v_{Gl}\) shows that the renormalized value of s–wave scattering length is \(a_{sc} \sim (1/\varepsilon)^{1/\epsilon}\), \(\epsilon = (2 - d)\). The scattering length \(a_{sc} \to \infty\) for the important case \(\epsilon \to 0\) of 2\(d\) Bose fluids and, therefore, the cutoff \(\Lambda \sim (1/a_{sc})\) applied in Ref. [23] seems to be inconvenient.

In the remainder of this Section we shall discuss the nontrivial Gaussian–like FP which will be called GIFP. GIFP has Gaussian values for the main critical exponents: \(\eta = 0\), \(\nu = 1/2\), \(z = 2\). But the interaction parameter gives a correction-to-scaling exponent and this circumstance does not allow to put this critical behaviour in the Gaussian universality class (such an incorrect conclusion has been made in Ref. [87]). Note, that the Gaussian universality class is sometimes denoted by \(n = -2\) (see, e.g., Ref. [72]).

In the present case the \(\epsilon\)–analysis can be extended to any order in \(\epsilon\) [81] and this exceptional case will be discussed below. The problem has an exact solution [81] because of the great simplification of the perturbation series in the limit (ii). In this limit, all self–energy perturbation contributions (see Fig. 3b) exponentially tend to zero. In each diagrammatic
Figure 3: (a) A tree diagram denoting the interaction part of the Bose Hamiltonian. (b) The compact self-energy diagram which is equal to zero in the limit $T \to 0$ [the thick loop denotes the full (renormalized) Green function $G(q)$]. (c) An example of a diagram from the perturbation series for the interaction vertex which gives a zero contribution in the zero temperature limit. (d) The infinite ladder series of diagrams which yields the geometric progression (51).

(perturbation) term in the self-energy function [1, 2, 50] all poles in the frequency ($\omega$) lie in the complex upper-half plane and, hence, the contour of the respective frequency integral can be deformed into the lower-half plane to give zero. Therefore, in this limiting case, no corrections to the MF values of the critical exponents describing the main scaling laws will appear to any order of the loop expansion.

Furthermore, in the same zero-temperature limit (ii) the bigger part of perturbation contributions to the interaction vertex $v'$ tend to zero, too. For example, the diagrams of type shown in Fig. 3c do not give contribution. The only exception is the ladder series shown in Fig. 3d. This is the so-called superconductivity channel for the interaction vertex where the Green function lines are oriented in one and the same direction; for more details, see Refs. [1, 88]. The final result of the summation of the ladder in Fig. 3d is an infinite geometric progression [81]

$$v' = b^{2-d} \frac{v}{1 + a_0 v}. \quad (51)$$

The above consideration proves that within the grand canonical formalism the dynamical and static critical exponents of main scaling behaviour described by G1FP have Gaussian values. This is true for the scaling laws written in terms of the chemical potential $|\mu| = r$. As the parameter $r$ does not receive a renormalization it remains equal to that of IBG. The zero temperature critical point is defined by $[r(T_c)/T_c] = 0$ as is for IBG (Sec. 3).

The FP value $v_{Gl}$ can be obtained from Eq. (51) to any order in $\epsilon = (2-d)$. To do this one should expand the integral (48) and Eq. (51) to the corresponding order in $\epsilon$ by having in mind that the FP value $v_{Gl}(\epsilon)$ have to be also expanded. For the sake of convenience
in calculations to second order in \( \epsilon \), the FP value \( v_{\text{Gl}}(\epsilon) \) is usually written in the form

\[
v_{\text{Gl}}(\epsilon) = \frac{\hbar^2 \epsilon}{mK_d} [1 + \epsilon \ln \Lambda]
\] (52)

Expanding \( K_d \) in \( \epsilon \) we have

\[
v_{\text{Gl}}(\epsilon) = 2\pi^2 \frac{m}{\epsilon} \left\{ 1 + \epsilon \left[ \frac{c_E}{2} + \ln \left( \frac{\Lambda}{\sqrt{4\pi}} \right) \right] \right\},
\] (53)

(\( c_E \) is the Euler constant). The FP coordinate \( v_{\text{Gl}} \) receives corrections in all orders of the \( \epsilon \) expansion. Within the one-loop approximation, the stability exponent \( y_v \) associated with the interaction parameter \( v \) has a negative value \( y_v = -\epsilon = (d - 2) \) for \( 0 < d < 2 \). This exponent does not receive higher order \( \epsilon \) corrections and, therefore, the value \( (d - 2) \) is exact.

This completes the proof \[81\]. The above results have been confirmed by Monte Carlo calculations for one-dimensional interacting bosons \[97\].

We should emphasize that GlFP is non-Gaussian FP. It is a product of the thermal and quantum fluctuation interactions represented by the interaction parameter \( v \). In fact, GlFP is conjugate to actual GFP (\( v_G = 0 \)) within the same RG analysis. Moreover, the total critical behaviour includes corrections to the main scaling laws which are described by GlFP. The interpretation \[87\] of results within the classical universality class \((d + 2, n = -2)\) has no heuristic significance. As we have already mentioned this interpretation is incorrect because of the correction-to-scaling exponent coming from the interaction. Therefore, GlFP constitutes a new universality class of zero-temperature QCP which has numerous applications, in particular, to fluids of real bosonic atoms and XY magnets (Sec. 5.7).

5.4. Breakdown of quantum universality

GlFP and the critical behaviour described by it are quite unusual. Let us mention that here the term “usual zero-temperature FP” means a zero-temperature FP which can be obtained from the respective stable finite temperature FP of the same system after substituting \( \epsilon = (d_U - d) \) with \( \epsilon = (d_U - d - z_0) \) in the formulae for the coordinates and the attached critical exponents of the latter. Therefore, there is a clear form of conformity between the finite temperature FP and the respective zero-temperature FP which we temporarily call “usual” zero-temperature FP. This point of view is consistent with the definition of universality of quantum critical phenomena given in Sec. 4.3.

If all zero-temperature FP are usual in the above mentioned sense then one does not need to define new universality classes for quantum critical phenomena. The universality classes determined by the known finite-temperature FPs can be used, after the account of the dimensional shift due to the CQC, to describe the possible types of quantum critical phenomena, too. If this scheme works, we can introduce the term “universality”
of quantum critical phenomena by understanding this as a quantum universality which coincides with the classical universality at a shifted effective dimensionality of the space (see also Sec. 4.3). Then a zero-temperature FP obeying the mentioned correspondence with the respective finite-temperature FP can be called “FP describing quantum critical universality” or, shortly, “quantum universality FP.”

Certainly, the GlFP is not such usual or quantum universality FP. It does not obey the present definition of quantum universality although it comes as a result of CQC in NBG. According to Fisher and Hohenberg [89] the GlFP describes a “quasiuniversality” associated with the existence of an almost marginal line in the Hamiltonian parameter space \((T, \mu, v)\) defined by \(T = \mu = 0\) and a nonzero interaction parameter \(v\) which can take positive values.

This is in contrast with other zero-temperature FP revealed by J. Hertz [20] which are usual in the above mentioned aspect and obey the quantum universality. Therefore, there is a number of real systems which do not satisfy the widely accepted notion that the main effect of the quantum fluctuations on the zero-temperature critical behaviour is CQC. Such systems are the Bose fluids and the XY magnets (Sec. 7) where the quantum universality concept is not valid.

5.5. Discussion of limiting cases

The limiting cases (i) and (ii) are analyzed under the general condition \((\xi/\Lambda) \gg 1\). For a microscopic model, \(\Lambda \sim (\pi/a)\) and, hence, we have \((\xi/a) \gg 1\). Then the inequality (i) yields \(\lambda \ll a\), i.e., the high-temperature condition. The values \(\lambda \ll a\) have a statistical meaning in the continuum limit \((V/a^d) \to \infty\).

The high-temperature condition is certainly satisfied near the infinite FP value \(T^*_C = \infty\) and well below it. The condition (ii) is valid either in the classical low-temperature critical region

\[ a \ll \lambda < \xi \, ; \quad (54) \]

or in quantum one, given by

\[ a \ll \xi < \lambda \, ; \quad (55) \]

c.f., the general criterion (3). The inequalities (54) and (55) can be written in terms of parameters \(T\) and \(r\) which enter in the RG relations. The condition (54) will be valid near and at low-temperature FPs given by the zero values of \(T\) and \(r\) provided in the zero-temperature limit \(T \to 0\) we have \((r/k_BT) \to 0\). In the same limit, the quantum condition (55) will be valid, if \((r/k_BT) \to \infty\). In particular, to answer the question which of these two types of the low-temperature critical behaviour is described by GlFP (50), we must know the value of the ratio \((r_{Gl}/T_{Gl})\). If it is greater than unity, GlFP will describe quantum critical phenomena but if it is less than unity, the low dimensional \((d \leq 2)\) low-temperature critical behaviour will be classical.

A standard example is given by the system of interacting bosons at a constant density (the
case of noninteracting particles has been discussed in Sec. 3.4). As the perturbation series does not yield self–energy contributions at all the parameter $r$ for ideal and interacting Bose systems is the same. The value of the correlation length critical exponent $\nu$, cited in Table 1, is greater than the exponent $1/\sigma$ of thermal length $\lambda$. So in the zero-temperature limit the ratio $(r/k_BT) \sim (\lambda/\xi)^2$ will tend to zero; c.f. Eq. (21). Therefore, the low-temperature critical behaviour described by GIP will be classical which is true for $T$–driven transitions (Sec. 2.3). The $\rho$–driven transitions will exhibit quantum critical effects. These results are consistent with the general criterion (9).

In the same way one can show that the critical effects near the phase transition points of interacting bosons under the condition of a very low constant pressure will be influenced by quantum effects (see also Sec.3.3). The application of the results from Sec. 5.3 to dilute Bose systems in the low density limit $\rho \to 0$ will be briefly discussed in Sec. 5.7.

The NBG properties at finite low temperatures are very similar to those of IBG at constant density (Sec. 3.4). It is then convenient to investigate the corrections to the (spherical) Hartree limit by the standard $1/n$–expansion \[12\]. The alternative approach \[57\] by a reduced Landau-Ginzburg free energy function has been mentioned in Sec. 4.1.

5.6. Formulation by thermal wavelength

An alternative RG treatment can be performed by the transformation

$$\sqrt{\hbar^2/2m\phi(q)} \to \phi(q)$$

in the Bose Hamiltonian (27)–(28). In the new notations for the theory parameters the mass $m$ is absent and the Matsubara frequency $\omega_l$ is substituted by $8\pi^2l/\lambda^2$, i.e., by $(1/\lambda^2)$. The parameters $r$ and $v$ are multiplied by factors $(2m/\hbar^2)$ and $(2m/\hbar^2)^2$, respectively. The important point is that the recursion relation (38) for the mass $m$ and the recursion relation (45) for the temperature are now substituted by

$$1 = b^n,$$

and

$$\lambda' = b^{-1}\lambda,$$

respectively. Eq. (58) can be written in the form

$$m'T' = b^2mT.$$
different from the suggested in Ref. 74 75. From Eq. (59) one cannot conclude anything about the individual behaviour of $T$ and $m$ towards the RG transformation. Rather the correct conclusion is that the thermal wavelength $\lambda$ is renormalized, as given by Eq. (58), and that RG transformation will drive $\lambda$ to zero, unless it is at the high-temperature FP value $\lambda_C^* = 0$; another FP value of $\lambda$ is $\lambda_Q^* = \infty$.

Note that the renormalized thermal wavelength $\lambda = \lambda(b)$ does not belong to the parametric space $(r, v)$ of the Hamiltonian. Hence, the temperature $T$ is also excluded from this space. It is convenient to consider these parameters as describing HLTC. The renormalized wavelength $\lambda(b)$ drives the system from the high temperatures ($\lambda \ll a$) to the zero temperature ($\lambda \gg a$) Gaussian–like critical behaviour; see Eq. (58). The rescaling factor cannot be taken as a crossover parameter because of the RG restriction $\epsilon \ln b < 1$. So the RG flows from the usual Heisenberg FP where $\lambda_C^* = 0$ to GIPF where $\lambda_Q^* = \infty$ are produced by successive RG transformations. If the temperature is equal to zero, the system will be exactly at GIPF and the FP Hamiltonian is invariant towards RG. The system remains in the ground state of total BEC ($\rho_0 = \rho$). For any $T > 0$, the RG flow will drive the system to the usual classical behaviour. Now it is not difficult to rederive all RG results in the present formulation by avoiding the unnecessary mass–invariance condition and the unnatural HT FP value $T_C^* = \infty$.

5.7. Related topics and applications

As the systems undergoing zero-temperature phase transitions are numerous here we shall mainly discuss the applicability of the results from Sec. 5.3 to real systems and topics having some relationship with these results.

5.7.1. Magnetic and ferroelectric systems

Experiments 98 99 100 101 on quasi-one-dimensional antiferromagnets with integer spin in an external magnetic field $\vec{H}$, mainly intended to a search of Haldane gap 102, reveal a continuous phase transition to a nonzero ground state magnetization which occurs at the absolute zero ($T_c = 0, H_c > 0$). Theoretical investigations 103 104 105 demonstrate that this phenomenon is a type of BEC of magnons with spin $S = 1$. The critical properties in the vicinity of the transition point $(0, H_c)$ are described by GIPF (Sec. 5.3), as shown in Ref. 94, where low-temperature ($T \sim 0$) features and the phase diagram of these spin chain antiferromagnets have been investigated.

The results for interacting real bosons are straightforwardly extended to systems described by the quantum XY model in a transverse field and, in particular, to XY magnets without a time reversal invariance. RG calculations to the one–loop order in Refs. 71 72 have drawn the attention to the possibility for a Gaussian zero-temperature critical behaviour of this model. In contrast to systems with different symmetry, for example, uniaxial ferromagnets and ferroelectrics described the TIM (Sec. 6), in XY systems above the critical transverse field the spins at $T = 0$ have maximal projections on the field axis and
there are no quantum spin fluctuations at all. Below the critical field, in the ferromagnetic phase, such quantum fluctuations exist; for the ground state properties of spin systems see Ref. [107].

In Ref. [71] the real symmetry of XY systems has been neglected and the obtained RG equations depart from the correct Eqs. (38) - (43) just as the RG equations in Ref. [56] (see Sec. 5.2). The ignored symmetry properties are related to the particular form of the bare correlation function $G_0(q)$ which has been obtained in Refs. [72, 107, 108] exactly of the form (28). The main interaction term in the effective Hamiltonian of the XY model is of the form given by Eq. (27), but there are also additional interaction terms of type $|\phi|^{2m}$; $m > 2$. It has been shown in Ref. [109], that the additional interaction terms do not change the critical behaviour predicted with the help of the $\phi^4$–Hamiltonian (27)–(28) and, therefore, the RG results from Secs. 5.2–5.6, can be straightforwardly applied to XY systems; see, e.g., Ref. [73].

We should emphasize that the transverse field in the XY model plays the role of the auxiliary parameter $X$, introduced in Sec. 2. The $X$–driven transitions considered in Sec. 2.4 are easily performed in XY systems by variations of the transverse field around its critical value at a fixed low temperature. Similar field variations in uniaxial ferroelectrics [18, 110] and ferromagnets [11, 12] described by TIM are discussed in Sec. 6. The measurements in real substances should indicate the Gaussian–like critical behaviour in XY systems and the standard zero-temperature critical behaviour described by the universality Ising quantum FP of TIM [20]. The results for the quantum critical behaviour described by GLFP have been applied [24] also to low-dimensional quantum antiferromagnets.

Therefore, according to our considerations in Sec. 2 as well as the concrete studies of nonuniversal characteristics of XY systems [106, 107, 111], quantum critical phenomena should be observed in low-temperature critical experiments on low dimensional XY ferromagnets. The arguments presented so far demonstrate that the low-temperature phase transitions in XY systems are a promising area of experimental studies of the Gaussian–like critical behaviour. Besides, respective experiments on uniaxial ferromagnets and ferroelectrics described by TIM are expected to confirm the universal quantum critical behaviour.

5.7.2. Quantum Hall liquid and superconductivity

The results from Sec. 5.3 can be applied to the (fractional) quantum Hall liquids (see. e.g., [52, 113]. This problem has been investigated by Schakel [114, 115]; see, also Ref. [4]. The quantum Hall liquid was considered [114, 115] in the framework of the effective (quasimacroscopic) Chern-Simons-Ginzburg-Landau (CSGL) theory which describes the fractional quantum Hall effect (FQHE) [52, 113]. This effective field theory is based on the $\phi^4$–theory given by Eqs. (27) and (28) but the difference is that the Bose field $\phi(x)$ interacts with the sum of the vector potential $\vec{A}(\vec{r})$ of the external magnetic field $\vec{H}$ and the Chern-Simons gauge field which is related to internal degrees of freedom (composite
particles featuring the FQHE). It has been shown that the basic features of the quantum phase transition in this system are described with the help of the exact solution presented in Sec. 5.3 (see also Sec. 7.6.2 for a discussion of disorder effects).

The relationship between the superconductivity and the quantum criticality seems to be a new important issue in strongly correlated materials, including heavy-fermion and high-$T_c$ cuprate superconductors, in particular, in cases of unconventional $d-$ and $p-$wave Cooper pairs (see, e.g., Refs. [116, 117]). The problem has been mentioned for the first time within the framework of the resonating-valence-bond theory of high-temperature superconductors [118, 119]; see also Ref. [21] for a discussion of quantum critical points in superconductors. Here we shall draw the attention to several new results.

Barzyukin and Gor’kov [120] have recently shown that in two-dimensional superconductors, the phase transition line to the Larkin-Ovchinnikov-Fulde-Ferrell state may extend up to $T_c = 0$ at a certain value of the external magnetic field. If the theory is reliable, this phase transition line may give a new type of quantum (multi)critical point, or, a zero-temperature first-order transition.

Other magnetic field induced quantum phase transitions were established experimentally in heavy-fermion superconductors such as Ce-based (CeCu$_{6-x}$Ag$_x$ [121, 122] and CeCoIn$_5$ [123]) and YbRh$_2$Si$_2$ [124] compounds (see also Ref. [125]). The bilayer ruthenate Sr$_3$Ru$_2$O$_7$ exhibits a low temperature metamagnetism [126]. Assuming that for a certain value of the external magnetic field the phase transition line may reach the absolute zero, theoretical arguments [127] have been presented in favour of a weakly first order quantum phase transition in Sr$_3$Ru$_2$O$_7$.

A phenomenological model [128], that proposes a coexistence of the ferromagnetism and the unconventional superconducting state of spin-triplet Cooper pairs in heavy-fermion compounds, yields, in accord with experiments [129, 130, 131], a two pressure-driven quantum phase transitions at the same zero-temperature multicritical point: a transition from ferromagnetic to paramagnetic phase and a transition from superconducting to normal metal.

The theory [132] and the experiment [133] on ultrathin superconducting wires demonstrate the crossover from thermally activated phase slips (a product of thermal fluctuations) and quantum phase slips, activated by the quantum fluctuations. This crossover very much resembles the general scaling picture of effective HLTC and, in particular, CQC at nonzero temperature, outside an unattainable by experiment, small asymptotic vicinity of the critical point (see the discussion in Sec. 2). Dissipation-driven quantum phase transitions in (quasi-)one-dimensional arrays of Josephson junctions [134] are a subject of intensive experimental [135] and theoretical [136, 137] research.

A Monte Carlo calculation [138] based on a two-dimensional boson Hubbard model, equivalent to an anisotropic spin-1/2 XXZ model with definite parameters, reveals a phase dia-
gram with a quantum \((T = 0)\) superfluid-solid phase transition. As mentioned in Ref. [138] there is a striking qualitative similarity of this phase diagram to those of fermionic \(^3\)He and bosonic \(^4\)He two-dimensional systems.

5.7.3. **Dilute Bose fluids**

At the end of this discussion let us come back to the systems of real boson particles. A potential application of the nonuniversal critical behaviour described in Sec. 5.3 is to thin films of \(^4\)He [139, 140]. An interesting opportunity exists for an experimental investigation of the crossover of the critical behaviour of IBG to the critical behaviour of NBG by a careful controlled overall density \(\rho\) of bosons up to the dilute limit \((\rho \ll 1)\) in \(^4\)He [141, 142, 143, 144, 145] and spin-polarized hydrogen [146]. The critical fluctuations of the superfluid order parameter \(< \phi(x) >\) can be accounted by the experimental data for the behaviour of the superfluid density \(\rho_s \sim < |\phi|^2 >\).

Experiments following this idea have been performed [141, 142] for \(^4\)He condensed in Vicor which is a highly connected semiregular sponge-like glass. In other experiments [143, 144, 145] another porous medium (Silica Gel) has been used. This is again a medium which consists of randomly distributed three-dimensional \((d = 3)\) networks of interconnected pores and channels of diameter approximately 0.005 \(\mu\)m (the pore size can be varied from sample to sample). In these experiments the critical temperature \(T_c\) of the superfluid phase transition is observed to decrease with the decrease of the density \(\rho\) of \(^4\)He. The respective curves can be reliably extrapolated to zero \(T_c(\rho = \rho_c)\) for a nonzero critical density \((\rho_c > 0)\).

The experiments [141, 142, 143, 144, 145] have been extensively analyzed in theoretical works [23, 24, 90, 148, 149, 150, 91] based on scaling and RG methods. All mentioned porous media are a source of disorder, so that certain properties of the Bose fluid in such media should be described with the methods of the theory of disordered systems (Sec. 7). However, some main features can be discussed by the means of pure systems where no disorder effects are present.

As pointed out in Ref. [24] such systems undergo a superfluid onset transition as the density \(\rho\) increases at \(T = 0\) and, moreover, this phase transition may be related with the theory presented in the preceding Secs. 5.2-5.6. A similar conclusion has been made in Refs. [148, 91] where the empirical data is interpreted with the help of the crossover from the critical behaviour of NBG to that of IBG. These notes demonstrate that the experiments are close to a confirmation of the Gaussian-like critical behaviour established in Sec. 5.3, provided the respective porous medium can be considered as a factor leading to a decrease of the (effective) spatial dimensionality of the Bose fluid contained in it up to \(d \leq 2\). Similar experiments can be made on thin superfluid films.

Following Ref. [24] we shall briefly mention some properties of this experimentally observed phase transition in \(^4\)He placed in Vycor. While the experimental feasibility of
the probing low-temperature behaviour stops short of $T = 0$, the superfluid transition at sufficiently low temperatures will be dominated by GLFP until the very close vicinity of the critical density $\rho_c$ is approached. The crossover from the zero-temperature critical behaviour described by GLFP and the finite-temperature critical behaviour corresponding to $T_c > 0$, described by the standard XY FP($n = 2$) has been treated in Ref. [24] as a standard multicritical-to-critical crossover (within this interpretation the zero-temperature critical point is considered as a multicritical point).

We should draw the attention to a quantum ($T = 0$) phase transition [151] to superfluid state in $^3$He in aerogel at a nonzero pressure and a critical density of $^3$He $\rho_c > 0$. Unlike bulk $^3$He which is superfluid at all pressures (densities) between zero and the melting pressure, $^3$He in aerogel will be not superfluid, unless the density of $^3$He exceeds a critical value $\rho_c > 0$. Some of the problems of the quantum phase transition revealed in Ref. [151] are similar to that for the respective phase transition in $^4$He, discussed in a few lines above (for details, see Ref. [151]).

6. Transverse Ising model

6.1. Microscopic and field models

Here we shall consider the quantum transverse Ising model with the aim to reveal the properties of HLTC. We shall use MF, the lowest-order perturbation theory and Ginzburg criterion considerations and RG.

TIM describes essential features of phase transitions in ferromagnets with a strong uniaxial anisotropy [152] and displacive phase transitions in certain types of quantum ferroelectrics [18, 153, 154, 155, 110]; for experiments, see, e.g., Refs. [29, 156] (for other applications of TIM, see Refs. [157, 158]). Here we shall mention the MF results [152], the cumulant expansion [157], the RG investigations [19, 20, 154, 155], the description of CQC in Hartree limit [70, 159] and the formal interrelationship between the finite–size crossover in systems of slab geometry and CQC in TIM [70].

TIM is given by the Hamiltonian [152]

$$H = -\frac{1}{2} \sum_{ij} J_{ij} S_x^i S_x^j - \Gamma \sum_i S_x^i ,$$

(60)

where $S_{\gamma}, \gamma = (x,y,z)$, are the components of spins with a magnitude $S = 1$, $J_{ij}$ is the exchange interaction and $\Gamma$ is the transverse field magnitude. We shall assume that TIM is defined on a $d$–dimensional regular lattice with a lattice spacing $a = 1$. The assumption for nearest–neighbour ($nn$) interactions which we shall use to define the parameters of the effective Hamiltonian, does not restrict the generality of results in this Sec.

The fluctuation field Hamiltonian of TIM can be derived from the microscopic spin model (60) by the Hubbard-Stratonovich transformation [19, 70, 159]. This Hamiltonian
(\mathcal{H} = H/T; k_B = 1) is given in the form

\[ \mathcal{H}[\phi] = \frac{1}{2} \sum_{\alpha,q} G_0^{-1}(q) |\phi_\alpha(q)|^2 + \frac{u_0}{V} \sum_{\alpha,\beta q_1 q_2 q_3} \phi^*_\alpha(q_1) \phi^*_\beta(q_2) \phi_\alpha(q_3) \phi_\beta(q_1 + q_2 - q_3), \]

(61)

where \( \phi(q) \) is a real scalar field with a (bare) correlation function in the form

\[ G_0^{-1}(q) = |\omega_l|^2 + k^2 + t_0. \]

(62)

The parameters \( t_0 \) and \( u_0 \) are given by the expressions,

\[ t_0 = \left[ 1 - J \th \left( \frac{\Gamma}{T} \right) \right], \]

(63)

and

\[ u_0 = \frac{J^2 T}{8 \Gamma^3} \left[ \th \left( \frac{\Gamma}{T} \right) - \frac{\Gamma}{T} + \frac{\Gamma}{T} \th \left( \frac{\Gamma}{T} \right) \right]. \]

(64)

In Eq. (64), \( J \equiv J(0) = 2dJ_0 \) is a product of the number \( z = 2d \) of nearest neighbour \( nn \) spins and the constant \( J_0 \) of the single \( (i - j) \) \( nn \) interaction (\( J_0 = J_{ij} \) for \( nn \) sites \( i \) and \( j \)).

It is convenient to use units, in which all quantities in the effective Hamiltonian \( \mathcal{H} \) are dimensionless. The dimensionless wave components \( k_i \) are given by \( k_i = 2\pi \kappa l_i / L_i \), where

\[ \kappa = \sqrt{\frac{J \th (\Gamma/T)}{2d\Gamma}}. \]

(65)

In the critical region \( t_0 < 1, \kappa \sim 1 \). The upper cutoff for the wave numbers \( k \) is \( \Lambda = \gamma \pi \kappa \), where \( \gamma \) is a small number (\( \gamma \ll 1 \)) [45].

The characteristic lengths \( \xi \) and \( \lambda \) are given by \( \xi = 1/\sqrt{|t_0|} \) and \( \lambda = (\Gamma/T) \). We shall consider only the low temperature domain defined by \( \lambda \gg a \). In this case, \( u_0 = (J^2 T/8 \Gamma^3) \).

6.2. Mean field approximation

Within the standard MF approximation we shall consider only the uniform mode \( \phi(0) \) of the field \( \phi \), that is the \( q \)-dependent (classical and quantum) fluctuations will be ignored. It will be more convenient to use the MF free energy \( \Omega = T \mathcal{H}[\phi(0)] \) instead of the dimensionless MF free energy \( \mathcal{H}[\phi(0)] \). This choice, after a change of the nonequilibrium order parameter from \( \phi(0) \) to \( \phi_0 = (T/V)^{1/2} \phi(0) \), will induce an extra factor \( 1/T \) in front of \( \phi_0^4 \)-term of free energy \( \Omega \) which makes possible to avoid difficulties in our further analysis connected with the definition of the order parameter \( \phi_0 \) at \( T = 0 \). The problem is that the parameter \( u_0 \) is proportional to \( T \), see Eq. (64), and the investigation with the original parameter \( \phi(0) \) will give for the equilibrium order parameter \( \phi(0) \sim (1/T)^{1/2} \) which is divergent for \( T \to 0 \). This is an example of order parameter HLTC.

The Gibbs thermodynamic potential in the form

\[ \Omega = V \left[ \frac{t_0}{2} \phi_0^2 + \frac{u_0}{T} \phi_0^4 \right] \]

(66)
allows a correct MF analysis at low temperatures. The analysis of the free energy (66) can be done straightforwardly and we shall not enter in details. The critical line (Fig. 4a) is defined by $t_0(T_c, \Gamma) = 0$, i.e.,

$$T_c(\Gamma) = \frac{\Gamma}{\text{Arth}(\Gamma/J)}$$  \hspace{1cm} (67)

or, equivalently, by $t_0(T, \Gamma_c) = 0$, which yields

$$\frac{\Gamma_c}{J} = \text{th} \left( \frac{\Gamma_c}{T} \right).$$  \hspace{1cm} (68)

Despite of the simple form of Eq. (66) the MF critical properties of TIM cannot be investigated analytically for the whole curve $T_c(\Gamma)$ because of the quite complex dependence of parameters $t_0$ and $u_0$ on $T$ and $\Gamma$.

We shall consider the low temperature critical behaviour in the vicinity ($|\Gamma - J| < J$) of zero temperature critical point $[T_c(J) = 0, \Gamma_c(0) = J]$ and the neighbour critical points with coordinates $T_c \ll J$ and $\Gamma_c \sim J$. In this low temperature region we can distinguish four types of phase transitions along the lines: $0J, AJ, aa$, and $bb$; see Fig. 4b. We shall suppose that the couples of parallel lines are very near to each other. Following the terms introduced in Sec. 2, the transitions along the lines $AJ$ and $aa$ are $T$–transitions whereas the transitions along the lines $0J$ and $bb$ can be thought of as $\Gamma$–transitions. The results can be compared with those for the high temperature $T$– and $X$–transitions in the vicinity of the critical point $T_c(0)$; see the respective lines $0J$, $bb$, $AJ$, and $aa$ in Fig. 4b.

In MF we shall consider the critical exponents $\beta$, $\gamma$, and $\nu$, of the order parameter $|\phi_0| = (T|t_0|/4u_0)^{1/2}$, the susceptibility $\chi = 1/|\vec{t}_0|$, and the correlation length $\xi = 1/|\vec{t}_0|^{1/2}$.
respectively, where \( \bar{t}_0 = t_0 \) for \( t_0 > 0 \), and \( \bar{t}_0 = 2t_0 \) for \( t_0 < 0 \). Obviously, the behaviour of these quantities with respect to variations of \( T \) and \( \Gamma \) depends on the behaviour of \( t_0 \). Note, that parameter \( t_0 \) as given by Eq. (63) does not change in the Gaussian approximation for the quantum and classical fluctuations. This makes possible to extend the consideration of the parameter \( t_0 \) in the paraphase (\( \phi_0 = 0 \)), where the results should be interpreted as pure fluctuation effects.

The summary of results reads:

**Line 0J:** We obtain \( t_0 \approx (\Gamma - J)/\Gamma, \phi_0 \sim |t_0|^{1/2}, \chi \sim \Gamma/|\Gamma - J| \) and, therefore, \( \gamma = 1 \) and \( \beta = 1/2 \). There exists a full coincidence between the standard MF behaviour with respect to variations of \( T \) around \( J \) at \( \Gamma = 0 \) and the present critical behaviour due to variations of \( \Gamma \) around \( J \) at \( T = 0 \). This correspondence can be written as \( T \leftrightarrow \Gamma \).

**Line bb:** The parameter \( t_0 \) is

\[
t_0 \approx \frac{J(\Gamma - \Gamma_c)}{\Gamma_c^2} (1 - \frac{4\Gamma_c}{T} e^{-2\Gamma_c/T} + ...) .
\] (69)

Here \( t_0 \) tends to the value \( (\Gamma - J)/\Gamma \) when \( \Gamma_c(T) \to J \) for \( T \to 0 \). The critical exponents are the same as those along the line 0J. The exponential correction in Eq. (69) can be neglected.

**Line AJ:** There is no equilibrium ordering along this line but there is a criticality in the paraphase. The susceptibility above the zero-temperature critical (\( \Gamma_c = J \)) is

\[
t_0 = 2e^{-2J/T} .
\] (70)

This exponential behaviour corresponds to the critical exponents \( \gamma = \nu = \infty \).

**Line aa:** Along this line

\[
t_0 \approx \frac{4J|T - T_c|}{T_c^2} e^{-2\Gamma/T_c}
\] (71)

and, therefore, for both \( T < T_c \) and \( T > T_c \), the susceptibility \( \chi \sim 1/|t_0| \) obeys the scaling law \( \chi = \chi_0/|T - T_c| \) with an exponentially increasing scaling amplitude \( \chi_0 \). There is a smooth crossover between the pure exponential behaviour (70) and the power law (71). As \( (u_0/T) \approx J^2/8\Gamma^3 \) for \( \Gamma \sim J \gg T \), we have \( |\phi_0| \sim |t_0|^{1/2} \), where \( t_0 \) is given by Eq. (71). The order parameter exponent \( \beta \) has the classical value \( \beta = 1/2 \) but the scaling amplitude \( \sim T_c^{-1}\exp(-\Gamma/T_c) \) of the order parameter \( \phi_0 \) exponentially decreases for \( T_c \to 0 \); for a comparison of these low-temperature MF properties with high-temperature ones; see Ref. [160].

The Eqs. (7), (70) and (71) yield a crossover exponent \( \nu_0 = \infty \). Perhaps, this value of \( \nu_0 \) will be different provided the fluctuation phenomena are taken into account.

6.3. Lowest order perturbation theory
The \((T, \Gamma)\) domains of validity of the MF results in Sec. 6.2 can be investigated by the Ginzburg criterion; see, e.g., Ref. [1]. Here we shall use the standard derivation of this criterion from the first–order perturbation contribution to the “self–energy” \(t_0\):

\[ \hat{t}_0 = t_0 + 12u_0A_1(t_0) \]  

(72)

with

\[ A_1(t_0) = \int \frac{d^d k}{(2\pi)^d} S_1(t_0, k) , \]

(73)

where

\[ S_1(t_0, k) = \frac{\Gamma \text{cth}[(\Gamma / T)\sqrt{k^2 + t_0}]}{T \sqrt{k^2 + t_0}} . \]

(74)

Neglecting a term of order \(u_0^2\) we can substitute \(t_0\) in the intergal \(A_1(t_0)\) by \(\hat{t}_0\). The Ginzburg criterion for the validity of the MF results will be given in the general form

\[ |\hat{t}_0| > 12u_0[A_1(0) - A_1(\hat{t}_0)] . \]

(75)

The same criterion defines the validity of the Gaussian approximation of noninteracting fluctuations above and below the critical point. The general criterion (75) is valid for both the paramagnetic \((\hat{t}_0 > 0)\) and the ferromagnetic \((\hat{t}_0 < 0)\) phases; in the ferromagnetic phase a factor \(1/2\) should be added to the r.h.s. of (75).

Obviously the results from this first–order perturbation approximation depend on the properties of the integral (73). We shall write this integral in the form

\[ A_1(\lambda, \xi) = K_d\lambda^{2-d} \int_0^{\Lambda} dy y^{d-1} \frac{\text{cth}\sqrt{y^2 + \xi^2}}{\sqrt{y^2 + \xi^2}} , \]

(76)

where \(y = \lambda k\), and \(\xi\) is given by Eq. (3). For high temperatures \((\lambda < 1, \xi \ll 1)\) the main contributions in the integral are from the small wave numbers \(y = \lambda k \ll 1\), and the approximation \(\text{cthy} \sim 1/y\) yields the standard perturbation integral \(A_1\) known from the classical theory (the Curie–Weiss limit). One can rederive the classical theory \((\omega_l = 0)\) by using the high-temperature value of the interaction parameter: \(u_0 = (J^2/12T^2)\); see Eq. (64) for \(\Gamma \ll T\).

In the low-temperature range of temperatures \((\lambda \gg 1, \lambda \Lambda \sim 1)\) we can distinguish between the quantum limit \(\xi \gg 1\) and the classical limit \(\xi \ll 1\). In the quantum limit \((\xi \gg 1)\) the \(\text{coth}\) can be approximated with unity by neglecting exponentially small correction terms. They are very similar to those given by Eq. (69)–(71). These corrections enter in the renormalized parameter \(\hat{t}_0\) and give small \((\sim u_0)\) corrections to the coefficients of the corresponding exponential terms in \(t_0\); see Eqs. (70) and (71).

Let us denote the integral which is obtained from \(A_1\) for \(\text{cthy} \sim 1\) by \(A_{01}\). The difference \(u_0[A_1 - A_{01}]\) has been estimated [18] to be of order \(\lambda^{-2} \sim T^2\) for 3d–systems; note that
in the low-temperature limit, \( u_0 \sim T \). This type of temperature corrections to the pure quantum limit has been widely used in interpretations of experimental results for quantum ferroelectrics \[29, 156\]. The same corrections were also derived in Refs. \[154, 155\]; for a calculation in the Hartree limit, see Ref. \[161\].

At extremely low but finite temperature \( (T \sim 0) \) one may try to take into account the temperature corrections to the critical behaviour at the absolute zero \( (T = 0) \) as an alternative to the present consideration, where we take the low temperature limit in the general self-energy integral \( (76) \). Taking into account the temperature corrections to the zero-temperature limiting case is equivalent to an estimation of the difference arising from the substitution of summation over the Matsubara frequencies with an integration according to the rule \( (35) \). It is clear that such corrections will come from the counterterm series in powers of \( \rho^{-2} \) in the Euler–Maclaurin summation formula. Therefore, these corrections could not be given by \( d \)-dependent powers in \( T \) as is in the Hartree limit considered in Ref. \[161\].

The temperature corrections are important for the phase transition properties along lines like \( AJ \) and \( aa \), where \( \Gamma \sim J \). Within the present lowest order perturbation theory the corrections in powers of \( T \) should be considered small. However, it has been proven by the “parquet” summation \[18\] that they are quite big and essentially influence the critical behaviour in the classical low-temperature region \( (\rho < 1, \lambda \gg 1) \).

Now we shall consider the Ginzburg criterion \( (75) \). In the quantum limit we must substitute the integral \( A_1 \) with \( A_{01} \). As a result of CQC, the integral \( A_{01} \) yields the upper and lower borderline dimensionalities: \( d_U = 3 \) and \( d_L = 1 \). In order to simplify the calculations we shall consider the case \( d = 2 \) which shows the main features of the quantum critical behaviour for all dimensionalities \( 1 < d < 3 \).

The straightforward calculation gives the criterion \( (75) \) in a simple form:

\[
|t_0| > \left( \frac{3}{4\pi} \right)^2 \left( \frac{J}{\Gamma} \right)^4 .
\]

This criterion cannot be applied to the \( T \)-transitions along the lines \( AJ \) and \( aa \) where the behaviour is classical \( (\lambda < \xi) \). The reason is that the strong quantum condition \( (\rho \gg 1) \) has been used in the derivation of inequality \( (77) \).

Along the line \( bb \) we obtain the criterion

\[
|\Gamma - \Gamma_c| > 0.06\Gamma_c \left( \frac{J}{\Gamma_c} \right)^3 \sim 10^{-2}\Gamma_c .
\]

If we set in \( (78) \) \( \Gamma_c = J \), we shall find a criterion along the line \( 0J \). Because of the very sharp slope of the transition curve \( T_c(\Gamma) \) near \( \Gamma = J \), the quantum criterions along the lines \( bb \) and \( 0J \) are practically the same. The Ginzburg critical region \( \sim 10^{-2}\Gamma_c \) given by \( (78) \) is well established. It enlarges at \( T > 0 \) by \( T \)-correction terms.
The quantum condition (3) along the line $bb$ can be written in the simple form $J(|\Gamma - \Gamma_c|) \gg T^2$ provided $(\Gamma/\Gamma_c)^2 \approx 1$. As $J \gg T$, this condition becomes

$$|\Gamma - \Gamma_c| > T,$$

which is obviously consistent with (78). The condition (79) shows that the quantum region approaches the critical point $\Gamma_c$ when $T$ decreases to zero. The whole surrounding of the zero-temperature critical point $(\Gamma_c = J)$ is influenced by quantum effects which produce the quantum fluctuation region.

A more detailed analysis including the onset of thermal fluctuations can be obtained by the exact calculation of the integral $A_1$ at $d = 2$:

$$A_1 = \frac{1}{2\pi} \ln \frac{\sh \sqrt{\lambda \Lambda + \varrho^2}}{\sh \varrho}.$$

The approximation $\sh y \sim \exp(y)/2$ corresponds to $A_1 \approx A_{01}$. The thermal corrections are obtained in powers of $(\varrho^2/\lambda \Lambda)$ which, together with the factor $u_0 \sim T$, yields the lowest order correction term to $u_0 A_{01}$ of the type $T^2 |t_0|$.

The asymptotic critical behaviour corresponding to the $T$–transitions will not exhibit quantum critical phenomena but rather a low-temperature classical behaviour which is different from the high-temperature one. The only quantum effect is related with CQC at zero-temperature $\Gamma$–transitions (Sec. 6.4). A study of the Ginzburg critical region for zero-temperature structural phase transitions has been presented in Ref. [162]. The results are consistent with the present analysis.

6.4. Renormalization group arguments

The RG recursion relation for the interaction parameter $u_0$ in the one–loop approximation will be

$$u'_0 = b^{4-d} u_0 [1 - 36 u_0 A_2(0, b)],$$

where

$$A_2(t_0, b) = \frac{\partial A_1(t_0, b)}{\partial t_0}.$$  

In Eq. (82) $A_1(t_0, b)$ is the integral (73) with lower $(0 < b^{-1} < 1)$ and upper $(\Lambda = 1)$ cutoffs of the wave number $k$. The other recursion relations are

$$t'_0 = b^{4-d} [t_0 + 12 u_0 A_1(t_0, b)],$$

and

$$\lambda' = b^{-z} \lambda,$$

where the dynamical critical exponent $z$ is equal to unity in this order of the theory.

In the high-temperature region ($\lambda < a$) the integrals $A_1$ and $A_2$ can be substituted with the classical integrals by setting $\csc h(y) \sim 1/y$. In this case the integral $A_2(0, b)$ has a
logarithmic infrared divergence at the upper critical dimensionality \(d_U = 4\). For \(d = 4\), \(A_2(0, b) = K_d \ln b\).

Further, using the standard RG analysis \([20]\) one reveals the usual universality class of the critical behaviour of the classical Ising model (\(\Gamma = 0\)). Besides, there is a possibility to perform a calculation of dynamical critical exponent \(z\) and, hence, to reveal the quantum dynamics of TIM \([163]\). The \(\epsilon\) corrections to \(z\) are calculated from the \(q\)-dependent self-energy diagrams in two- and higher-loop approximations. These corrections are small compared with the \(\epsilon\)-corrections to the static exponents, for example, \(\nu\) and \(\gamma\).

The low temperature limit is treated as shown in Sec. 6.2. In this case we should make the approximation \(c_{thy} \sim 1\) in Eqs. (81)–(83). Thus we recover the dimensional CQC: \(d \to (d + 1) = D\). The analysis in \(\epsilon = (3 - d)\) yields results for the quantum critical behaviour corresponding to the nontrivial Ising universality class \((D, 1)\) \([20]\). For all dimensionalities \(d \geq 3\) the QC behaviour will be described by the classical MF universality class, while for dimensionalities \(1 < d < 3\) the QC behaviour will be nontrivial. This result is straightforwardly generalized for a \(n\)-component real field with a bare correlation function of the form (62).

Thus we obtain that in TIM CQC satisfies the universality property discussed in Sec. 4.2. An important point in the mechanism of this crossover is that the interaction parameter \(u_0\) in the recursion relation (81) is changed to the parameter \(v_0 = (u_0/\lambda)\):

\[
v_0' = b^{3-d}v_0[1 - 36v_0K_3\ln b].
\] (85)

The quantum effects play the crucial role for CQC in TIM. If the RG equations are treated in the classical scheme, as shown in Refs. \([153, 155]\), the transformation of Eq. (81) for \(u_0\) to that for \(v_0\) cannot be performed in the way shown by Eq. (85). An extra–factor \(\lambda^{-1}\) that remains in the second term of Eq. (85) leads to the prediction of a Gaussian critical behaviour at low temperatures for dimensionalities \(d > 2\).

The RG investigation \([154, 155]\) of the critical behaviour along the onset \(T\)-transition at \(T_c = 0\) yields critical exponents \(\gamma = 2\nu = 2\) instead of classical exponents \(\gamma = 2\nu = 1\) (here we ignore \(\epsilon\)-corrections). These exponents do not depend on the dimensionality \(d\) as is in the Hartree limit \([161]\). For \(t_0 = 0\), i.e., on the line \(AJ\), the integral \(A_1\) has a logarithmic divergence which is an evidence of strong thermal fluctuations. Perhaps this nonuniversal critical exponents of the zero temperature \(T\)-transition are the outcome of the simultaneous effect of classical and quantum fluctuations.

The HLTC of the \(T\)-transitions from classical critical exponents at high temperatures to low-temperature critical exponents at the zero temperature phase transition \((T_c = 0)\), \(\gamma = 2\nu = 2\), shows that \(z\nu_0 = 1\). Therefore, within this accuracy of the theory one cannot conclude whether the criterion (9) is satisfied or not. These bare values of \(z\) and \(\nu_0\) have \(\epsilon\) corrections from RG and we should have in mind that the correction to \(z\) is of order \(O(\epsilon^2)\).
whereas that to $\nu$ is of first order in $\epsilon$ with a positive sign. Therefore the criterion (9) is not satisfied and, hence, the asymptotic critical behaviour along the onset $T-$transition to the absolute zero ($T_c = 0$) is classical.

Another important feature of the quantum critical behaviour is that it is unstable with respect to any perturbation of the temperature from zero. This feature is common to the quantum critical phenomena in all systems. Of course, the quantum critical phenomena will belong to the nontrivial ($d-$dependent) class of critical behaviour in the quantum critical region shown by Eq. (78). The zero-temperature transitions outside this narrow region will exhibit the usual MF behaviour. It becomes clear from this picture that like in the XY model discussed in Sec. 5.6, the uniaxial ferroelectrics and ferromagnets described by TIM, are convenient for the experimental observation of quantum critical phenomena near $\Gamma-$transitions at low and extremely low temperatures.

7. Disorder effects

Here we shall review investigations on the effects of quenched disorder of type “random critical temperature” which is caused by randomly distributed (quenched) impurities and inhomogeneities. The main results for the quantum critical behaviour in systems with disorder of type “random field” will also be mentioned (for the random field disorder see, e.g., Refs. [1, 164, 165]).

The critical behaviour in disordered systems obeys the Harris criterion derived by rather general arguments. This criterion states that the disorder will be irrelevant to the (classical) criticality, if the following inequality between the specific heat exponent $\alpha$ and the correlation length exponent $\nu$ takes place: $(\alpha \nu - 2) > 0$. One can check that this criterion is satisfied in all examples considered below.

The first RG investigation of the quantum critical behaviour in disordered quantum systems reveals an instability of the usual quantum critical phenomena at $T = 0$ towards the disorder of type random impurities. The instability has been proven for a quite general quantum field model which describes the quantum phase transitions in almost all known quantum systems. This problem is closely related with the description of the localization in real superfluids. Here we shall consider the instability phenomenon, RG methods of stabilization of the quantum criticality and the related problems of localization and superfluid-insulator transition as well as the formation of “glass-like” phases in disordered Bose fluids.

7.1. Random impurities

The disorder of type random impurities changes the local interaction responsible for the phase transition and, hence, the critical temperature which depends on the sites of the crystal lattice. In the continuum limit the local (nonequilibrium) critical temperature depends on the spatial vector $\vec{r}$. This fact is taken into account in the effective Hamiltonian by an additional $\phi^2-$term containing a random function $\varphi(\vec{r})$ that obeys the Gaussian
distribution
\[ g(\vec{r}, \vec{r}') \equiv [\varphi_\alpha(\vec{r}) \varphi_{\alpha'}(\vec{r}')]_R = \bar{\Delta} \delta_{\alpha\alpha'} \delta(\vec{r} - \vec{r}') , \] (86)

or, in the \( \vec{k} \)-space,
\[ g(\vec{k}, \vec{k}') \equiv [\varphi_\alpha(\vec{k}) \varphi_{\alpha'}(\vec{k}')]_R = \Delta \delta_{\alpha\alpha'} \delta(\vec{k} + \vec{k}', 0) . \] (87)

Here \([\ ]_R\) denotes the operation of averaging (hereafter, the suffix “R” will stand for quantities related to disordered systems). We shall often use the notations \( g_{\alpha\alpha'} = \delta_{\alpha\alpha'} g \) and \( \tilde{g}_{\alpha\alpha'} = \delta_{\alpha\alpha'} \tilde{g} \).

The distribution function (86) or respectively, \( \tilde{g}(\vec{r}, \vec{r}') \), which is often called “random correlation function,” describes quenched impurities with the so-called short-range random correlations. Sometimes, this type of random impurities is referred to as “\( \delta \)–correlations” or, even, “\( \delta \)–impurities” because of their mathematical expression by the \( \delta \)–function valid in the continuum limit.

For the long-range random correlations, the \( \delta \)–function in Eq. (86) should be substituted by a function of type \( f(R) = 1/R^{\Theta} \), where \( R = |\vec{R}| = |\vec{r} - \vec{r}'| \), \( 0 < \Theta < d \). In this case, the random distribution (correlation) function is given by \( \tilde{g}(\vec{r}, \vec{r}') \equiv g(R) \) and
\[ g(R) = \frac{\bar{\Delta}}{R^{\Theta}} . \] (88)

A justification of this definition of long-range correlations is presented in an investigation [168] of the random-field problem [1, 164, 165]. The Fourier transformations of various shapes of random correlation functions \( g(R) \) with powerwise or more complex dependence on the distance \( R \) are considered in details, and the Fourier transforms \( g(k, \vec{k}') = \delta(\vec{k} + \vec{k}')g(k) \) are given for various shapes of \( g(R) \) [168, 169]. Note, that in the general case of random correlations, different from the \( \delta \)–correlations given by Eq. (86), the parameters \( \bar{\Delta} \) and \( \Delta \) in Eqs. (86) and (87) can be different from each other but both of them should be nonnegative on account of the requirement for the stability of the Gaussian distribution.

An alternative way of representation [168, 169] of the correlation function \( g(R) \) from Eq. (88) seems to be more convenient for several considerations (see Sec. 7.4.1). This representation is given by the exponent \( \Theta' = (d - \Theta) \), \( d \geq \Theta' > 0 \) and is suitable for the scaling and RG analysis in the \( \vec{k} \)-space.

In order to describe the disorder of type random impurities a new term should be added to the quantum Hamiltonian (27) in the form
\[ \mathcal{H}_R[\phi] = \frac{1}{\sqrt{V}} \sum_{\alpha, \omega; \vec{k}_1, \vec{k}_2} \varphi(\vec{k}_1 - \vec{k}_2) \phi^*_{\alpha}(\omega, \vec{k}_1) \phi_{\alpha}(\omega, \vec{k}_2) . \] (89)

This term can be derived by the Hubbard–Stratonovich transformation from the microscopic model of the concrete system or, alternatively, postulated and used in further investigations of disorder effects.
The sum of the Hamiltonian parts (27) and (89), i.e., the total Hamiltonian

\[ \mathcal{H}_{\text{tot}}(\phi) = \mathcal{H}(\phi) + \mathcal{H}_R(\phi), \] (90)

describes a great amount of quantum systems with quenched (randomly distributed) impurities, often called “impure systems.” As usual, we shall discuss also the critical behaviour of the corresponding classical systems \((\omega_l \equiv 0)\). The respective pure systems are those, for which the Hamiltonian part \(\mathcal{H}_R(\phi)\) is equal to zero \([\varphi(\mathbf{K}) \equiv 0, \text{or}, g(R) = 0, \text{which gives the same results for the thermodynamic and correlation properties}]\).

The thermodynamic behaviour strongly depends on the properties of the random potential represented by the random function \(\varphi(\mathbf{r})\). The random potential \(\varphi(\mathbf{r})\) is not a thermodynamic variable, that is why, the theoretical treatment will include two steps. At first one should try to calculate the thermal averages as functionals of \(\varphi(\mathbf{r})\). As a second step the averaging over the random function is done with the help of Eqs. (86)–(88). As a result one obtains the thermodynamic quantities as functions of the disorder parameter \(\Delta\); see, e.g., [1, 12, 164, 165].

Alternatively, an ansatz called the “replica-trick” can be used (see, e.g., [164, 165, 166]). In perturbative and perturbative-like investigations as, for example RG, this trick should always give the same results as the method of direct perturbation treatment of disorder. This fact has a simple explanation. The replica index \((m)\) appears in the perturbation terms through polynomials and, hence, there is no problem in the analytical continuation of the results in the limit \(m \to 0\). The limit must be taken at the end of the replica-trick calculation in order to retrieve the original disordered system [164, 165, 166]. In some non-perturbative calculations, however, the same analytical continuation and the justification of the limit \(m \to 0\) are not very easy; see, e.g., Ref. [164, 165, 166]. A recent theory of the thermodynamic behaviour of disordered superfluids below the finite-temperature critical point \((T_c > 0)\) has been developed by Lopatin and Vinokur [172] on the basis of the Beliaev theory [173, 174] and the standard replica trick treatment of \(\delta-\)correlated random impurities.

The results reviewed in the remainder of this paper are obtained by RG and provide an information about the asymptotic critical behaviour, namely, the critical behaviour in an infinitesimally close vicinity of the critical point. Note, that less information is available about the MF behaviour of systems with quenched disorder and the pre-asymptotic critical behaviour.

7.2. Results for classical and quantum models at finite temperatures

The RG investigation of quantum effective Hamiltonians (27)–(29) with the additional term (88) for all possible values of \(m, m’\) and \(\sigma\) was published for the first time in Refs. [170, 171]. Here we shall review the results from Refs. [170, 171] together with the necessary information from other papers; see, e.g., Refs. [12, 164, 165, 166].

7.2.1. Some results for classical systems
In systems with symmetry index $n > 4$ the disorder effects are irrelevant to the critical behaviour at finite temperature critical points ($T_c > 0$). The finite temperature critical behaviour of systems with symmetry indices $n > 4$ is described by the usual Heisenberg FP of the corresponding pure systems ($\Delta = 0$) (for $n = 1$ this FP is usually called Ising FP) \cite{1,12,13}. Here we shall often refer to this FP as “pure” FP or, shortly, PFP and even P. Note, that PFP exists also for some values $n < 4$ but for such systems this FP is unstable towards disorder ($\Delta > 0$).

For systems with $n < 4$ the disorder effects essentially influence the critical behaviour at finite critical points ($T_c > 0$); see, e.g., Ref. \cite{12}. In this case the critical behaviour is described by a “random” FP established for the first time by Lubensky \cite{175}; hereafter referred to as RFP, or Lubensky RFP, or shortly denoted by “R”.

The effect of the random impurities on the critical behaviour in Ising-like systems ($n = 1$) is quite peculiar (see, e.g., Refs. \cite{12,165}). The RG equations exhibit a special form of degeneration and, as a consequence, the usual perturbation expansion breaks down and should be conveniently modified. For example, the usual $\epsilon$–expansion should be substituted with an expansion in non-integer powers ($\sim \epsilon^{p/2}$, $p = 1, 2, ...$) of $\epsilon = (d_U - d)$.

The modified $\epsilon$–expansion yields special FP which attracts the RG flows through spiral paths. Such FPs are often called “focuses” or focal FPs. These FPs give complex values for the correction-to-scaling exponents \cite{12} which result in an oscillatory behaviour of the corrections to the scaling laws. Perhaps, this extraordinary aspect of the impure critical behaviour is a feature of systems where an anisotropic ordering appears below the critical temperature or other anisotropy effects are present, as shown in Ref. \cite{67} for a wide class of anisotropic impure systems. We shall not dwell on the case $n = 1$ (for this case, see, e.g., Refs. \cite{1,12}) rather we shall discuss systems with continuous symmetry ($n > 1$). Let us, however, mention that several FPs of focal type are discussed in Secs. 7.4 and 7.5 for other reasons which are associated with the effective increase of the upper critical dimensionality $d_U^R$ of the disordered system ($d_U^R > d_U$).

### 7.2.2. Quantum systems at finite temperature

The picture outlined in Sec. 7.2.1 for the asymptotic critical behaviour of classical systems does not change for quantum systems at finite critical points ($T_c > 0$). The additional information which can be obtained for the finite temperature critical behaviour of disordered quantum systems is about the quantum critical dynamics, in particular, about the value of dynamical critical exponent $z$. The results \cite{170,171} for the dynamical critical exponent $z$ of finite temperature critical points can be written in the form:

$$z = \sigma + \frac{(4 - n)}{8(n - 1)}\epsilon, \quad 1 < n < 4,$$

(91)

where $\epsilon = (4 - d)$. Eq. (91) has been obtained for the $XY$ model and Bose systems which are described by the bare correlation function (28). In the case of models that have
Hamiltonians (27) with the correlation function (29), the dynamical exponent has been obtained \[170, 171\] in the form

\[z = \frac{\sigma}{m} + \frac{(4 - n)}{8(n - 1)} \epsilon, \quad m' = 0, \quad 1 < n < 4.\]  

(92)

For \(m' > 0\) the exponent \(z\) is given by Eq. (36). In the latter case the critical exponent \(z\) has no correction to the first order in \(\epsilon\,[171]\). These results correspond to RFP, namely, when the disorder is relevant. The same results \[171\] have been confirmed for disordered itinerant antiferromagnets \[176, 177\].

Eq. (91) coincides with that obtained in Ref. \[178\] for disordered classical systems within an approach based on a time dependent Landau–Ginzburg equation \[12, 25\]. The results (91) and (92) demonstrate that the dynamical critical exponent \(z\) in systems with random impurities has an \(\epsilon\)–correction in the one–loop (first order in \(\epsilon = 4 - d\)) approximation whereas the dynamical exponent of the corresponding pure system (\(\Delta = 0\)) has an \(\epsilon\)–correction of order \(O(\epsilon^2)\). This is a direct consequence of the fact that the random function \(\varphi(\vec{k})\) does not depend on the Matsubara frequency \(\omega_l\).

7.3. Instability of the quantum critical behaviour in disordered systems

The zero-temperature critical behaviour exhibits an instability with respect to the quenched disorder \[170, 171\]. The mechanism of the instability is related to CQC. Let us consider the RG equations corresponding to this problem \[171\]. In the limit \(T_c \to 0\) the parameters of the Hamiltonian obey the lowest order RG transformations of the type \[170, 171\]:

\[v' = b^{2\sigma-d-z_0}v,\]  

(93)

and

\[\Delta' = b^{2\sigma-d}\Delta,\]  

(94)

where \(z_0\) denotes the bare value \(z(0)\) of the dynamical exponent \(z(\epsilon)\). The ”tree” approximation used in Eqs. (93) and (94) is sufficient for our present discussion (the one-loop RG equations have been derived in Ref. \[171\]).

Because of CQC the interaction parameter \(v\) is relevant to the zero-temperature critical behaviour in spatial dimensionalities \(d < (2\sigma - z_0)\), whereas the disorder parameter \(\Delta\) is relevant to \(d < 2\sigma\). This is readily seen from Eqs. (93) and (94). The difference in the upper critical dimensionalities of the parameters \(v\) and \(\Delta\) is produced by CQC and the lack of \(\omega_l\)–dependence of the random function \(\varphi(\vec{k})\). On one side, at the classical borderline dimensionality \(d_U = 2\sigma\), the parameter \(v\) is irrelevant and, hence, the RG equations in \(d = (4 - \epsilon)\) dimensions describe a simple Gaussian instability towards the disorder parameter \(\Delta\). On the other side, the zero-temperature \(\epsilon\)–expansion in terms of \(\epsilon_0 = (2\sigma - d - z_0)\) yields pure FPs of the type \(\Delta^* = 0\) that are unstable with respect to the disorder parameter \(\Delta\) \[171\]. In these two variants of the theory FPs of corresponding
Figure 5: A scheme representing four relevant FPs (P, R, Gl, Un) in disordered systems.

pure system are unstable with respect to disorder effects for dimensionalities less than the upper borderline dimensionality $d_U$ [171].

Moreover, the disorder itself does not generate new stable FPs in the quantum limit ($T_c \rightarrow 0$). This is a clear indication for the lack of a standard (pure or random) zero-temperature critical behaviour and, therefore, one should expect either some unconventional (multi)critical behaviour in these systems at zero temperature or a first-order phase transition.

The zero-temperature and, perhaps, low-temperature instability of the standard critical behaviour in pure systems against disorder effects of type quenched impurities is probably related with the instability of the usual ground state of the respective system and, hence, with localization effects in real disordered superfluids (see Sec. 7.7.3). Therefore, it seems important to have a good notion about the relevant FPs which govern the impure quantum systems. Using the results discussed up to now we can distinguish between four types of critical behaviour of pure and disordered Bose fluids and XY systems described by the Eqs. (27), (28), and (90). They are given by four FPs: (i) Pure FP (P) corresponding to $T_c > 0$ and $\Delta = 0$ (lack of disorder), (ii) Gaussian-like FP (Gl), corresponding to $T_c = 0$ and $\Delta = 0$, Random FP (R) - for $T_c > 0$ and $\Delta > 0$, and Unstable FP (Un) corresponding to the case of zero-temperature critical point ($T_c = 0$) of disordered system $\Delta > 0$. These main four FPs are shown in Fig. 5 at their places in the parameter space ($\Delta, T$).

For other quantum systems, for which Eq. (29) should be used instead of Eq. (28), the diagram in Fig. 5 remains valid, provided the Gl FP is substituted with the usual zero-temperature FP corresponding to CQC in pure systems [20]. Thus we have two types of instability of the quantum critical behaviour towards random impurities: (a) instability of the quantum critical non-universal behaviour represented by G1FP, and (b) instability of the quantum critical universal behaviour described by the universality PFP (see Sec. 5.4).

The effect of quenched (random) impurities on the quantum tricritical behaviour has been also investigated [179]. The RG results demonstrate an instability of the pure quantum
critical behaviour towards the disorder and lack of any new stable FP of the RG equations [179]. Therefore, the usual critical and tricritical points exhibit an instability due to CQC which takes place in the pure subsystem (the subsystem described by $\Delta = 0$) but does not affect the “disorder subsystem,” described by the Hamiltonian part (90).

The random field disorder is itself a source of a dimensional crossover [1, 12, 164, 165]. This is another type of quenched disorder that may occur in real experiments. It has been shown [168, 169] that the outcome of the simultaneous action of quantum and random field effects at zero temperature is again an instability of the quantum critical behaviour below the upper borderline dimensionality (the latter dimensionality is higher than $2\sigma$ for the presence of short- or long-range correlations of the random fields [168, 169]).

7.4. Long-range random correlations and extended impurities

7.4.1. Long-range random correlations.

The quantum critical behaviour is unstable also towards disorder effects produced by random impurities with long-range random correlations considered for the first time by Weinrib and Halperin [180] (for the calculation of the critical exponent $\eta$ in this case, see Ref. [181]). The quenched disorder is described again by the Hamiltonian part (90) but the correlation function $g(R)$ is given by Eq. (88).

The Fourier transform of the random correlation function (88), to the leading order in the wave vector $\mathbf{k}$, has the form $g(k) = \Delta_1 > 0$ for $\Theta \geq d$, and

$$g(k) = \Delta_2 k^{\Theta-d}$$

for $\Theta < d$. These two cases have been combined in Ref. [180], where the RG analysis is carried out by the correlation function

$$g(k) = \Delta_1 + \Delta_2 k^{\Theta-d},$$

which contains both short-range ($\Delta_2 = 0$) and long-range ($\Delta_1 = 0, \Theta < d$) random correlations between the quenched impurities.

For classical models ($\omega_l = 0$) [180] the RG analysis gives new stable FP of a focal type with complex eigenvalues, leading to oscillating corrections to scaling laws and consistent with the Harris criterion [167]. Obviously, the same features will remain valid for quantum systems at finite critical temperatures $T_c > 0$. As there is no spatial anisotropy in this model and, moreover, no anisotropy is present at all in the $n-$ dimensional space of the order parameter vector, we are faced with an example when focal FP, describing oscillating corrections to the scaling laws can appear also in cases of a total lack of anisotropy. The only reason, for which such a focal FP appears in this class of systems is the disorder-driven dimensional crossover associated with the upper borderline dimensionality ($d_{cU}^R > d_{cU}$) as explained in more details at a next stage of our consideration.
Let us consider why the classical impure behaviour is proven to be stable by RG investigations, at least for a rather large class of systems \[180\]. This is a result of an ansatz, by which a double $\epsilon-$expansion is introduced in the RG studies \[180\]. However, without this ansatz, namely, by following the standard RG scheme, one should obtain an instability of the critical behaviour of the impure system as a result of a lack of any stable FPs of the RG equations in the nontrivial dimensionality range $d^R_L < d < d^R_U$.

The reason for the instability is in the dominant role of the disorder parameter $\Delta_2$ associated with the long-range random correlations. The presence of long-range random correlations simply quells the the fluctuation interaction effects represented by the parameter $v$ as well as the short-range random correlations, represented by the parameter $\Delta_1$. The behaviour of the system very much resembles the behaviour of noninteracting fluctuations (free field; $v = 0$) in a random potential.

In order to clarify this problem one may perform a simple dimensional analysis \[1, 13\] of the Hamiltonian, or, equivalently, to perform the RG rescaling within the simple “tree” approximation. The result of the RG rescaling procedure is

$$v' = b^{2\sigma - d} v, \quad \Delta_1 = b^{2\sigma - d} \Delta_1,$$

and

$$\Delta_2 = b^{2\sigma - \Theta} \Delta_2.$$  \hspace{1cm} (97)

If the difference $(d - \Theta) > 0$ is of order unity, as is for real systems, an instability of the classical critical behaviour corresponding to the pure system ($\Delta_1 = \Delta_2 = 0$) will occur and, moreover, a new stable critical behaviour will not appear at all. The reason is that the parameter $\Delta_2$ will always be relevant and the perturbation contributions which are supposed to be of order $\epsilon$ cannot compensate the growth of this parameter to infinity.

The ansatz introduced for the first time in Ref. \[180\] for this type of systems consists in the assumption that, at least during the formal derivation and analysis of the RG equations, $(2\sigma - \Theta) = \delta$ will be a small quantity of order $\epsilon = (2\sigma - d)$; remember, that the upper critical dimensionality of the pure system is $d_U = 2\sigma$. Then one performs a double $(\epsilon, \delta)$ which yields a new stable FP of focal type \[180\] and complex exponents for the correction-to-scaling laws which describe the critical behaviour of systems with long-range random correlations.

The formal assumption that $\delta \equiv (2\sigma - \Theta) \ll 1$ makes possible to develop a double $\epsilon-$expansion but the form of Eq. (98) does not allow an easy way to obtain the upper borderline dimensionality $d^R_U$ of the disordered system, which seems to be higher than that ($d_U = 2\sigma$) of the pure system on account of the particular effect of the parameter $\Delta_2$. This is a peculiar feature of the above described approach because usually the tree approximation immediately yields the upper critical dimensionality. The reason is in the lack of a $d$-dependence in the scaling transformation (98). Thus one is left with the opportunity to determine the dimensionality $d^R_U$ from the singularities of the one-loop
integrals rather than from the lowest order tree approximation. Therefore, one begins with the derivation of the RG equations for a general dimensionality $d$ and only after that one has to determine the critical dimensionality $d_{RU}^R$, at which RG equations are to be expanded.

The formal comparison of the scaling $(b–)$ factors in Eqs. (97) and (98) shows that $(2\sigma - \Theta) = [\epsilon - (d - \Theta)]$ and, hence, one may speculate that the upper dimensionality $d_{RU}^R$ of the disordered system is raised with respect to $d_U$ by $(d - \Theta)$ but this argument, although correct (see below), cannot be used to determine $d_{RU}^R$ as a number independent of $d$. The reason for this difficulty is in the choice of the exponent $\Theta$ that describes the $R$–dependence in Eq. (88). If we use the exponent $\Theta' = (d - \Theta)$, which has been already introduced in our analysis, the Eqs. (88), (95), (96), and (98) will change in an obvious way. For example, the $k$–dependence in the second term in the r.h.s. of Eq. (96) will be written as $(1/k^{\Theta'})$, and the $b –$ factor in Eq. (98) will take the form $b^{(2\sigma + \Theta' - d)}$. Now we can be certain that $d_{RU}^R$ is exactly equal to $(2\sigma + \Theta')$. The difference $(d_{RU}^R - d_U)$ is equal to $\Theta' = (d - \Theta) = (\delta - \epsilon) > 0$. Note, that the focal FP revealed in Ref. [180] has a singularity just at $\delta = \epsilon$.

The main result of the RG analysis within the double $\epsilon$–expansion – the stabilization of the quantum critical behaviour in disordered systems of special type, has been achieved by an ansatz which does not affect the $(1/T)$–axis of the total $(\tau, \vec{x})$ space of the critical events. It is easy to see that CQC in the limit $(T \rightarrow 0)$ will cause an instability of all available finite temperature FPs in these systems.

### 7.4.2. Extended impurities

Up to now we have discussed random “point” impurities, i.e. impurities localized at single points $\vec{r}$. The disorder of type extended (non-point) random impurities is described by infinitely-ranged random correlations along one or more spatial dimensions, say, $\tilde{d} < d$, and short–range random correlations along the other $(d - \tilde{d})$ dimensions [183, 184, 185, 186].

For example, in systems with a slab geometry the appropriate choice of extended impurities is the one–dimensional (linear) impurities oriented along the direction of the small size $L_0$ (the slab thickness) and randomly distributed along the other spatial directions [182].

The extended quenched impurities are described by a modification of the model for short- and long-range correlated point impurities, in which these finite-range correlations along the “small size” $L_0$, for instance, along the $z$–axis, are substituted with infinite–range correlations. The random correlations along the long dimensions of the slab, i.e., along the $x$– and $y$–axes, are kept of the short-range type. The length scale of “the infinite-range” correlations is much larger than the correlation length $\xi$ and the thickness $L_0$ of the slab. So, the strongly correlated along the small size point impurities behave like continuous uniform strings. In regard to the critical behaviour this disorder acts like point impurities with a short–range random distribution along the large (infinite) dimensions $L_i$ and an
uniform distribution along the small size $L_0$ [182].

In infinite systems one may consider: (i) straight lines of impurities or straight dislocation lines of random orientation, (ii) two-dimensional (plane, $d = 2$) defects, and finally, (iii) randomly distributed impurities or defects of dimensionality $\bar{d} < d$.

Now the important question is about the way of description of infinite-range correlations. This is simple because one takes the exponent $\Theta$ of the correlation function (88) to tend to zero ($\Theta \to 0$) along the directions in the $\bar{d}$–dimensional subspace of extended impurities. Then the correlations along this directions will be of infinite range but uniform. The body will be homogeneous (aligned) along this directions and the disorder will be present only along the rest part of spatial directions $(d - \bar{d})$. Now we can suppose either short or long range random correlations along these directions and let us choose, as in many papers, the former opportunity. Then we shall have no defects (ideal order) along chosen axes of number $\bar{d}$ and a short-range disorder of type random impurities along the rest $(d - \bar{d})$ axes of the $d$–dimensional coordinate space. The respective correlation function of type (86) is given in the form [183, 184, 185, 186]

$$g(\vec{r}, \vec{r}') = \bar{\Delta} \delta^{d-\bar{d}}(\vec{r} - \vec{r}') . \tag{99}$$

The case when $d = 2$ and $\bar{d} = 1$ corresponds to the exactly solvable McCoy-Wu lattice model [187, 188, 189] of disorder along one of the spatial directions for the two-dimensional Ising model. We must emphasize that the mentioned correspondence is only in some general features.

In all cases the extended impurities will generate a critical behaviour instability for dimensionalities less than $\bar{d} < d_{R^U} = (2 \sigma + \bar{d})$. The mechanism of this instability lies in the appearance of the finite shift $\bar{d} = (d_{R^U} - d_{U})$ of the upper borderline dimensionality, i.e., the reason is similar to that producing instability by long-range random correlations (Sec. 7.4.2). In fact, the raised dimensionality $d_{R^U}$ corresponds to the disorder parameter $\bar{\Delta}$ and the critical dimensionality associated with the fluctuation interaction $v$ remains the same as in the pure system $(d_{U} = 2 \sigma)$. This means that the disorder effect represented by the parameter $\bar{\Delta}$ suppresses the fluctuation interaction $\sim v \phi^4$ in Eq. (27) just like in the case of long-range random correlations described in Sec. 7.4.2 and, as a result, stable FP is lacking and the critical behaviour is unstable.

The instability is overcomed again by an ansatz of a suitably chosen double $\epsilon$–expansion [183, 184, 185, 186], namely under the supposition that $\bar{d}$ is a small quantity $\bar{d} \sim \epsilon$ and can be treated on the same footing as the expansion parameter $\epsilon = (2 \sigma - d)$. Thus using $\bar{d} = \epsilon_d$ as a second small parameter in the double $(\epsilon, \epsilon_d)$–expansion, the RG theory [183, 184, 185] gives a stable finite temperature critical behaviour by stable FP of a focal type. This (Dorogovtsev) FP is different from focal FP discussed in Sec. 7.4.1 and describes the critical behaviour with complex corrections corresponding to the extended impurities with a random distribution (99). Despite of the number of similarities between the critical
phenomena, from one hand, in systems with point impurities with long-range random correlations and, from the other hand, in systems with extended impurities with short-range random correlations there are also essential differences. These differences are mainly in theoretical aspects, namely, in particular features which create methodical problems. From an experimental point of view, however, the effects of these types of disorder on the critical behaviour can hardly be distinguished.

The stability of the critical behaviour in disordered systems with extended impurities vanishes when the quantum limit \((T \rightarrow 0)\) is taken and CQC occurs. The reason is again in the appearance of CQC which produces an effective suppression of the fluctuation interaction represented by the parameter \(v\).

7.4.3. Summary

We can conclude that the quantum critical behaviour is unstable against all relevant forms of disorder used in the theoretical physics for the description of real systems. The instability of the quantum critical behaviour in pure systems with respect to disorder effects of several types considered in Secs. 7.3 and 7.4 is a result of dimensional CQC that suppresses the thermal fluctuation interactions but does not affect the disorder. The reason is simply in the fact that the quenched disorder is considered as time \((\tau-)\) independent.

The similarity between dimensional CQC and the dimensional shift \((d^R_U - d_U)\) of the upper borderline dimensionality established in certain types of disordered classical systems can be used in the development of a unified approach to the description of dimensional crossover phenomena in quantum systems. It has been pointed out [185] that the related with the time \((\tau-)\) extra dimensionality of the quantum Ising model can be supposed throughout a double \((\epsilon, \epsilon_d)\)–expansion equal to \(\epsilon_d\) in a complete analogy with the case of linear random impurities in classical models.

7.5. The stability ansatz about the quantum criticality in impure systems

The idea of Boyanovsky and Cardy [185] mentioned at the end of Sec. 7.5, has been applied by Weichman and Kim [190] for solving the problem for the instability of the critical behaviour at \(T = 0\). In Ref. [190] disordered superfluids with random impurities have been investigated. These systems are described by the sum the Hamiltonian parts (27) and (88), and the bare Green function, given by Eq. (28). But the results in Ref. [190] can be applied beyond the model of disordered Bose fluids and include other quantum systems. Note, that an artificial upper cutoff \(\Lambda_\omega\) for the Matsubata frequencies \(\omega_l\) has been used in this investigation [190] but this cutoff is avoided in a later paper [115].

Weichman and Kim [190] considered linear impurities along the time \((\tau-)\) axis instead of the spatial ones. Thus the impure strings along the time direction are of size approximately given by \(\beta = 1/T\). For \(\lambda \sim 1/\sqrt{T} \rightarrow 0\) (for simplicity, here we set \(\theta = 1/2\)), when the temperature time \((\tau-)\) dimension reduces to zero and one can take the contin-
uum limit along the “β-axis”, these lines shrink to randomly distributed points in the 
$d$–dimensional volume of the system. For $T \to 0$, however, the “length” $β$ of these lines 
tends to infinity. Note, that the consideration based on the exponent $θ = 1/2$ corresponds 
to a dynamical exponent $z = 2$ and, hence, to all systems described by the Hamiltonian 
(27) with exponents $σ$, $m'$, and $m$, that satisfy Eq. (36) for $z = 2$. But the general 
features of these considerations are valid for all systems described by Eqs. (27), (28), and (29) 
and the respective disordered systems given by the Hamiltonian (90). 

In fact, within the framework of the problem, considered in Ref. [190] and later in Ref. [115], 
the dynamical critical exponent $z$ is equal to two and, hence, the term “lines” 
should be used, if we consider a $(d + 1)$–dimensional space–time($τ$). However, if we 
consider this space–time representation equivalent to $(d + 2)$ dimensional space, as is in CQC 
dimensional crossover for Bose fluids, we should stretch the “lines” (or strings) of “length” 
$β \sim λ^2$ in planes of area $λ × λ$. In general, we shall have hyperplanes of dimensionality 
z and “area” $∼ λ^z$. For $λ \to 0$ the randomly distributed objects will shrink to randomly 
distributed point impurities. These are two different geometrical representations of the 
same phenomenon. The formal analysis can be performed with the additional small parameter $ε'_d$ 
which runs from zero to unity and thus describes the classical ($ε'_d = 0$) and 
the quantum ($ε'_d = z$) limits of Bose system with point impurities. Alternatively one may 
consider the small parameter $ε_d = ε'_d/z$ [190] [115]. The double $ε$–expansion of the RG 
equations again yields stable focal FP. 

In result, the following interesting picture of the Bose systems critical behaviour is ob-
tained [190]. In the high-temperature range, where $λ < a$, the critical behaviour is 
described by the high-temperature RFP [175] corresponding to short–range (point–like) 
impurities, i.e, Lubensky RFP (see also our notations defined in Sec. 7.2.1). In the zero-
temperature limit ($T \to 0$), the impure critical behaviour described by Lubensky RFP under-
goes a crossover to a critical behaviour governed by extended (plane) impurities. This 
zero-temperature critical behaviour is described by the corresponding zero-temperature 
variant of Dorogovtsev RFP with complex stability exponents [183] [184]. In the Hamilto-
nian parameter space ($β$, $ν$, $Δ$), Lubensky RFP is conjugate to Heisenberg PFP ($Δ^* = 0$) 
and both of them are far from the zero-temperature plane $β = ∞$. Dorogovtsev RFP is 
conjugate to GIJP ($Δ_{GI} = T_{GI} = 0$) considered in details in Section 5.3. Like Heisenberg 
PFP, GIJP is stable only for pure Bose fluids (for real bosons, $n = 2$). For any $Δ > 0$ 
and $T > 0$, the RG flows tend to the finite-temperature Lubensky RFP, whereas the 
zero-temperature Dorogovtsev RFP will be attainable, if only the system is at $T = 0$. 

These four conjugate FPs describe the critical phenomena in disordered superfluids and 
all disordered quantum systems presented by the Hamiltonian given by Eqs. (27), (28), 
and (90). One may visualize this scheme by a picture similar to that given in Fig. 5. 

For the rest part of disordered quantum systems, namely, those described by the bare 
Green function from Eq. (29), the same scheme works again provided one substitutes
the nonuniversality GiFP with the respective universality PFP (see also Sec. 5.4). Then the zero-temperature pure critical behaviour obeys the exact universality rule (Sec. 4), that is, the \( d \)-dimensional zero-temperature critical behaviour is described exactly by the finite-temperature universality class \((d + z, n)\), to which the system belongs (see also Sec. 5.4). The same is valid for the respective disordered systems. In this case, the disordered quantum system enters in the high-temperature universality class \((d + z, n)\) of the respective classical system \((\omega_l = 0)\).

7.6. Related problems

7.6.1. Disordered ferro- and antiferromagnets

An approach identical to the Weichman-Kim approach [190] (Sec. 7.5) has been used in an investigation of the quantum phase transition in disordered itinerant quantum antiferromagnets [176, 177]. In the notations given by Eqs. (27)–(28) these systems correspond to Eq. (29) with \( m' = 0, m = 1 \). Note, that in Ref. [176] the result (92) for the dynamical critical exponent \( z \) has been generalized with the help of \((\epsilon, \epsilon_d)\)-expansion.

The picture of the quantum phase transition properties of pure and disordered itinerant ferromagnets given in the theoretical treatment [191, 192, 193] is worth mentioning. The energy spectrum \( \varepsilon(k) \) of the fluctuation mode (the magnetization) in the effective field Hamiltonian depends on the spatial dimensionality \( d \): \( \varepsilon(k) \sim k^{(d-2)} \). The quenched disorder of type random impurities produces a diffusive electron dynamics which induces an effective long-range electron spin interaction of the form \( 1/R^{2(1-d)} \) and this gives a \( d \)-dependent spectrum of the type \( 1/k^{(2-d)} \). The bare Green function \( G(q) \) of the same magnetization fluctuating modes is described with the help of the terms given by Eq. (29) for \( m' = 2, m = 1 \), and \( \sigma = \min(2, d-2) \) and an additional term of type \( k^2 \) which is often redundant. The \( d \)-dependence of the energy spectrum of the fluctuating order parameter leads to a completely new, and in some aspect, quite non-universal picture of the critical behaviour in the most interesting domain of spatial dimensionalities \((d \leq 4)\).

In order to evaluate this result one should be acquainted with the behaviour of the respective pure system, where the disorder is not present. By the same method of treatment - a derivation of an effective field Hamiltonian from the microscopic electronic Hamiltonian with the help of the Hubbard-Stratonovich transformation and by a generalization of the Hertz [20] approach the respective energy spectrum of the critical magnetic modes in pure itinerant ferromagnets was found again to be \( d \)-dependent [193]. Now the bare Green function \( G_0(q) \) will be given by Eq. (29), provided the exponents have the values \( m' = 1, m = 1, \sigma = 2 \) and, moreover, an additional term of \( c'k^{(d-1)} \) with \( c' > 0 \) is added to the r.h.s. of the same expression for \( G_0(q) \). As a result, a new universality class of critical behaviour arises at dimensionalities \( d \leq 3 \). This problem has a further development in Ref. [194] with the conclusion for a fluctuation-induced first order phase transition at dimensionalities \( d \leq 3 \).
For details about the motivation and the results of this challenging direction of research, see the original papers [191, 192, 193]. One should have in mind that the Hubbard-Stratonovich transformations as a general method of establishing a correspondence between a microscopically formulated Hamiltonian and its effective field (quasi-macroscopic) counterpart have shortcomings unless one is interested only on the asymptotical long-wavelength limit (see, e.g., Refs. [45, 195]). On the other hand let us mention that the integration out of critical or auxiliary-to-critical modes in complex models almost always generates $d-$dependent terms in the spectrum of the principal fluctuating mode describing the phase transition of interest to the particular study. In usual cases of effective field Hamiltonians with more than one fluctuating fields this procedure of a further reduction of the description is rarely correct.

A density matrix RG theory [196] was applied to the one-dimensional (chain) random-exchange spin-1/2 XXZ model [197]. The interplay of quantum fluctuations and disorder results in a disorder-induced quantum phase transition that exhibits a nonuniversal behaviour of the spin correlations.

### 7.6.2. Disorder in quantum Hall liquids

A RG investigation of the effect of point random impurities with short-range correlations of FQHE systems was done by Schakel [115] (see, also, Ref. [4]). The effective field CSGL theory, discussed in Sec. 5.7.2, has been considered in Ref. [115] in a reduced variant, in which the gauge fields are omitted and the Bose field $\phi(q)$, which interacts only with the random mode $\varphi(\vec{k})$ subjected to $\delta-$correlations is described by Eqs. (27), (28), (86), (89) and (90). In this case one retrieves the instability [171] (Sec. 7.3) and in order to ensure a delocalization transition, one is forced to look for a stabilization of the ground state. Note, that the gauge fields (the Chern-Simons field and the vector potential of the magnetic field) cause other fluctuation effects (see, e.g., Ref. [1, 202, 203]) that additionally lead to an instability of RG FPs. Thus one cannot easily distinguish between the effects producing the instability of the quantum critical behaviour within the framework of the complete CSGL theory. Note, that the reduction to a $\phi^4$ theory makes possible the investigation of the net effect of the disorder on FQHE.

In order to restore the stability of the quantum critical behaviour in this disordered system, Schakel [115] has proposed a method which departs from the Weichman-Kim approach [190] (see also Sec. 7.5). The main disadvantage of the Weichman-Kim treatment, as mentioned in Ref. [115], is the introduced by these authors an upper cutoff for the Matsubara frequencies. Note, that such a cutoff, after the paper of Hertz [20], is used by a number of authors. On the one hand, this cutoff is considered as irrelevant for the asymptotic critical behaviour which is governed by low-frequency fluctuation modes, but on the other hand, as mentioned in Ref. [115], the same cutoff is difficult to justify as it would imply a notion for a “discrete” rather than continuous imaginary time ($\tau -$) variable.
In the Schakel approach the frequency cutoff is avoided and the convergence of the frequency sums (35) in the perturbation terms is saved by introducing a noninteger imaginary time \((\tau - \epsilon_d)\) dimensionality \(\epsilon_d\): \(0 \leq \epsilon_d \leq 1\). The value \(\epsilon_d = 1\) describes the quantum limit \((T \to 0)\), when the imaginary time \(\tau \in [0, 1/T] \) runs over the values in the whole interval \([0, \infty]\). In this limit one can apply the standard rule (35). When \(\epsilon_d = 0\) we deal with the respective classical system where the temperature is sufficiently high and the \(\tau\)-dependence of the fluctuation field \(\phi(x)\) can be safely ignored. The intermediate values \(0 < \epsilon_d < 1\) simulate a form of an approximate interpolation between the two limiting cases.

The respective double \((\epsilon, \epsilon_d)\)-expansion reveals stable FP corresponding to the quantum regime \((T = 0)\) of the disordered system (for details, see Ref. [115]). In some aspects this FP is similar to that found by Weichman and Kim [190].

The Schakel approach works also for pure systems. In this case, the variations of the dimensionality \(\epsilon_d\) can be used to describe the classical-to-quantum crossover of the critical behaviour. Note, that this interpolation procedure is rather approximate, and the description is exact only for the asymptotic cases \((\epsilon_d = 0, 1)\). The approximation is a result of the substitution of the frequency sums in the perturbative terms by \(\epsilon_d\)-dimensional frequency integrations. This problem is important for all studies based on double \(\epsilon\)-expansions and has been investigated in details in Ref. [182] (see also Sec. 7.6.3). Perhaps, this approximation is one of the reasons for the singularity separating the quantum and classical regimes within the Schakel approach [115].

**7.6.3. Thin films with quenched impurities**

The formal conformity between CQC in quantum systems and the finite size crossover in systems of slab geometry [198, 199, 200, 201] mentioned in Sec. 4.3 as well as ideas discussed in Secs. 7.4-7.5 have been used in Ref. [182] in the study of disordered thin films with quenched point and extended (line) impurities. The method of work is similar to the Weichman-Kim [190] and Schakel [115] methods. The investigation has been carried out by a double \(\epsilon\)-expansion and reveals stable FPs which describe the critical behaviour of the disordered films.

The second small parameter in the double \(\epsilon\)-expansion is the quantity \(\delta \in [0, 1]\) which comes from an ansatz, discussed also in Sec. 7.6.2, and is used to define a substitution of lattice sums over the wave vector components with a \(\delta\)-fold integrals. The ansatz is not new and defines a \(\delta\)-integration that has two slightly different variants as shown in Ref. [182].

The method of integration over wave vectors with noninteger dimensionality lies in the basis of all double \(\epsilon\)-expansions discussed in this review. In Ref. [182] the two small parameters \(\tilde{\epsilon}\) and \(\delta\) have been chosen in the form: \(\tilde{\epsilon} = (4 - D_{\text{eff}})\) and \(\delta = (D_{\text{eff}} - d)\), where \(d_U = 4, D_{\text{eff}} = (d + \delta)\) is the effective (not the physical) spatial dimensionality of the slab.
(film). Obviously, the quantity $D_{\text{eff}}$ should depend on the ratio $y = (L_0/\xi)$, where $L_0$ is the thickness of the film. This dependence can be ascribed to $\delta: \delta(y)$. Thus the RG treatment in Ref. [182] has been performed by a double ($\epsilon, \delta$)–expansion, where the small parameters are defined as deviations from the effective dimensionality $D_{\text{eff}}$ of the system.

As shown in Ref. [182] the $\delta$–integration as a method of calculation of perturbation terms introduces a systematic error of a finite magnitude, except for the limiting cases $\delta = 0, 1$. While the RG studies are related with infrared divergences this finite error produces only incorrect values of the FP coordinates but does not affect the physically measurable quantities as the critical exponents. Since the precise location of the FP coordinates is important for specific intermediate RG calculations but is not important for the physical predictions, the RG results obtained so far by double $\epsilon$–expansions are reliable.

Furthermore, one may try to find a functional relation between the noninteger dimensionality $\delta$– and the ratio $y = (L_0/\xi)$. Obviously, the dependence $\delta(y)$ should have the properties: $\delta(y) \to 0$ for $y \to 0$, and $\delta(y) \to 1$ for $y \to \infty$. The method used in Ref. [182] confirms this property but the attempt to obtain an analytical dependence $\delta(y)$ have not been successful. Moreover, the method does not provide a sufficient accuracy of the results for the intermediate values $y \sim 1$ corresponding to $\delta \sim 0.5$.

The comparison of the RG results for disordered thin films [182] with the respective results for models of disordered quantum systems shows that there should be a formal equivalence between the universality features of the critical behaviour in thin films and that in quantum systems such as ferroelectrics and magnets described by TIM; see Eq. (29) for $\sigma = 2$, $m = 2$, and $m' = 0$. This correspondence is valid for both pure [70] and impure films [182], and is given by the formal replacement of the film thickness $L_0$ with $1/T$.

7.6.4. Superfluid-insulator and metal-insulator phase transitions

The instability of the quantum critical behaviour with respect to the disorder can be interpreted as a signal for phenomena of localization [204, 205, 206]. Huang and Meng [204] have shown by the methods of the pseudopotential and the Bogoliubov transformation that at zero temperature the random impurities can deplete the Bose condensate, though not completely. On the other hand, the random impurities generate an amount of normal fluid equal to a part of the condensate depletion.

A paper by Giamarchi and Schulz [205] also provides reliable theoretical predictions about localization-delocalization phase transitions in disordered superfluids. This paper is mainly devoted to the localization problem in one-dimensional electronic systems (metal-insulator transition [2, 207]) but the analogy with the localization-delocalization problem (superfluid-insulator transition [24, 148]) in superfluids is also considered. A variant of RG is developed [205] that is convenient for the treatment of the simultaneous effect of interelectron interactions and disorder. The phase diagram of the one-dimensional metal is deduced from the stability properties of the RG equations.
Remember that the interelectron interaction alone can produce a localization of electrons and this is the Mott localization. On the other hand, the disorder alone can induce the Anderson localization of the eigenstates. The interplay between these two localization mechanisms is a matter of an intensive research. The analogy between the superconducting-insulator transition in metals and the superfluid-insulator transition in Bose superfluids was discussed in Refs. 24, 148, 205. The superconducting-insulator phase transition in disordered thin films and wires was investigated in Refs. 208, 209. As noted in Ref. 24, the outcome of the competition between the interparticle interactions and the disorder may result, under certain circumstances, in a gapless insulating “Bose-glass” phase. It is supposed that this may happen when the interparticle interaction is sufficiently strong 24.

A generalization of the Halperin-Lubensky-Ma paper 202 for magnetic fluctuation effects on the order of the phase transition in pure superconductors was made for the zero-temperature superconductor-insulator phase transition in one- and two-dimensional pure 210 and disordered 211 superconductors (a δ−correlated disorder of type random impurities). It has been found that the zero-temperature phase transition is continuous and the system behaves like a normal metal up to the phase transition point (the resistance has a finite nonzero value at T = 0). A recent theoretical study 212 of the superconductor-insulator transition in a two-dimensional array of Josephson junctions with random couplings demonstrates that the previously predicted Bose-glass phase 24, 213 is a metal state that has a well defined zero-temperature limit for the conductivity.

In Ref. 206 one-dimensional Bose system described by a Hamiltonian of type (27) has been studied by Monte Carlo techniques with the aim to clarify the competition between the strong interaction and the random impurities. The model considered in Ref. 206 is a lattice version of the field theory given by Eqs. (27), (28), (89), and (90), only the random distribution function is somewhat different. The main result from this Monte Carlo investigation is the numerical justification of the Bose-glass phase proposed in Ref. 24 and the prediction of a new Anderson-type insulating phase (“Anderson glass”) in the weak coupling regime. The existence of two insulating phases of glassy type was proposed for the first time in Ref. 205.

8. Classification of quantum phase transitions

Here we shall enumerate the main types of quantum phase transitions. The general classification in two types: first order (discontinuous) and continuous phase transitions known from the classical theory remains valid also for the zero temperature phase transitions. Both basic types of phase transitions are possible at zero temperature in accord with the Nernst theorem.

We should have in mind that the quantum effects affect more essentially the close vicinity of low- and zero-temperature equilibrium points of first order phase transitions rather than low- and zero-temperature critical points of continuous phase transitions. While the
correlation length at critical points is infinite and the classical fluctuations dominate at any finite \( T_c > 0 \) critical point, at the equilibrium points of the first order transitions this length is finite and the quantum effects may compete the thermal fluctuations up to an infinitesimal vicinity of the transition point. The outcome of this competition between the classical and quantum fluctuations depends on nonuniversal properties of the system and should be checked for each particular investigation.

The low- and zero-temperature phase transitions may be either quantum or classical ones, depending on whether the quantum correlations have an essential effect on the phase transition properties or the classical effects prevail. General thermodynamic and scaling arguments (Sec. 2) as well as statistical considerations (Secs. 3 and 6) show that the zero temperature limit itself does not guarantee a quantum effect on the zero-temperature phase transition. This is valid both for discontinuous and continuous phase transitions.

Moreover, the zero temperature phase transitions can be classified as \( T \)-driven and \( X \)-driven transitions and the properties of these two types of transitions are quite different. Note, that there is a wide class of systems as, for example, Bose fluids, which do not exhibit \( X \)-driven transitions (Secs. 3 and 5).

The continuous quantum phase transitions, i.e. the quantum critical phenomena can be classified in two large groups:

1. universal, and
2. nonuniversal.

If all universal characteristic features (critical exponents, including the correction to scaling exponents) of a quantum system (model) at spatial dimensionality \( d \) are identical to the respective quantities characterizing the correspondent classical system (model) at spatial dimensionalities \( d + z \), where \( z \) is the dynamical critical exponent, then the quantum critical phenomena in this quantum system are from the group (1). The “correspondent classical model” of a quantum field model of the type (27) is that given by setting all Matsubara frequencies in Eq. (27) equal to zero. Therefore, the correspondent model is the same (the usual \( \phi^4 \)-theory) for the whole set of quantum models defined by Eqs. (27), (28), (29), and the possible values of the exponents \( m \), and \( m' \). For quantum lattice models, one should take the classical limit in the proper way in order to obtain the correspondent classical model. Within RG, the phenomena of the type (1) are described by zero-temperature FPs which are identical in properties to some respective (conjugate) finite temperature FP and CQC is the only quantum effect on the zero-temperature critical behaviour.

If the quantum critical phenomena are not in group (1) they belong to the group (2). The examples discussed in details in this review are the quantum critical phenomena in Bose fluids and XY magnets, described by the Gaussian-like FP (Secs. 5.3-5.4).
To the best of our knowledge this classification scheme is published for the first time in this review. The nonuniversality of quantum critical phenomena in Bose fluids and XY systems has been pointed out for the first time in Ref. [88] and then applied in Ref. [163] for the interpretation of results for a particular problem of structural phase transitions. With the help of various convenient terms of the authors choice the phenomenon of nonuniversality is confirmed by researchers in this field as already mentioned in Sec. 5.1.

9. Concluding remarks

The following results and notes can be enumerated at the end of our review:

1. The complete investigation of quantum phase transitions naturally includes the phase transitions at low- and extremely low temperatures and cannot be restricted to zero-temperature phase transitions only. The quantum phase transitions are a selected part of the low-temperature (including zero-temperature) phase transitions.

2. The properties of $T-$driven and $X-$driven phase transitions at low and zero temperature differ substantially from each other. The quantum effects on the phase transition properties are mainly exhibited at $X-$driven phase transitions at extremely low or zero temperature. There is, however, a conformity between the low-temperature $T-$driven continuous phase transitions and the high-temperature $X-$driven continuous phase transitions, as noticed in Ref. [5] and Sec. 2.4.

3. The first-order quantum phase transitions with sufficiently low equilibrium transition temperatures $T_{eq}$ have not been studied enough although they occur more frequently in crystal bodies and should be drastically affected by the quantum fluctuations.

4. The phenomenological scaling theory of continuous quantum phase transitions enters in the class of theories of dimensional crossover and takes advantage from similar theories as, for example, the scaling theory of finite size systems. Note, that the crossover exponent $\nu_0$, introduced in Sec. 2.3 can be compared with similar exponents in alternative approaches to finite-size scaling and quantum criticality.

5. Two types of crossover phenomena can be defined in the low-temperature limit: HLTC and CQC. The former does not necessarily include quantum effects on the critical behaviour. CQC is the product of quantum effects.

6. The performance of zero-temperature limit in the scaling equations does not itself guarantee quantum effects on the phase transition properties. Rather a phase transition can be considered quantum provided the general criterion (3), or, equivalently, the criterion (9) is satisfied in the transition region. Asymptotic quantum critical behaviour occurs unless the respective criterion, (3) or (9), is obeyed in the asymptotic vicinity of the critical point. This is quite strong requirement and may be fulfilled in certain systems with suitable nonuniversal parameters only at their zero temperature (multi)critical points ($T_c = 0$). However, for the finite experimental accuracy, apparent quantum effects
may be observed also at sufficiently low-temperature but finite transition points ($T_c > 0$).

7. The scaling behaviour of basic models of statistical physics does not exhibit quantum effects in the zero temperature limit but rather follows the criteria for classical critical behaviour. This is valid for certain regimes of performance of the phase transition (Secs. 3 and 6).

8. The interacting Bose fluids and XY magnets exhibit a nonuniversal quantum critical behaviour. The RG investigations indicate a break down of a well definite form of universality in these systems, i.e., the phenomenon of quantum universality breaking takes place.

9. A quantum non-universality occurs also in disordered quantum systems, where it appears as an instability of the zero-temperature critical behaviour towards all known mechanisms of disorder. All RG predictions for the critical properties of disordered systems at finite temperatures break down at zero temperature where the dimensional crossover lowers the critical dimensionality of the fluctuation interactions.

10. There is a formal analogy between three physically different types of crossover phenomena: CQC, the (finite-size) crossover from two-to-three dimensional systems, and the crossover from point to extended random impurities in disordered systems.

11. The available results about quantum phase transitions are not enough to build up a satisfactory picture of these intriguing phenomena.

12. In this review we have tried to show some of the outstanding problems and present a unified and consistent framework of the available results. It has been shown that the understanding of the quantum phase transitions as “zero-temperature phase transitions” is a restricted and incomplete point of view. Besides, the concept of CQC as a main result of the quantum effects on the critical behaviour is also unsatisfactory and should be complemented by the concept of the quantum universality breaking.

13. A substantial interest from both theoretical and practical points of view are the so-called $T-$driven phase transitions in the ultra-low temperature scale ($T \sim 0$). Their description is an outstanding problem. Among the unresolved problems are the equation of state for quantum phase transitions, in particular, in Bose fluids, where one should represent the behaviour of the system in terms of the density of particles and the temperature in the quantum limit ($T \to 0$).

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Figure captures:

Fig. 1. (a) Low temperature part of a critical line with zero temperature critical point \( T_c(X_0) = 0 \). The shaded part \((a - X_0 - b)\) of the transition region, marked by the lines 1 and 2, corresponds to a low temperature classical behaviour. (b) Low temperature critical line with \( X_0 = 0 \); domains \((a - X_0 - b)\) and \(1 - 0 - 2\) coincide.

Fig. 2. (a) The function \( f(\varphi) \) for a standard second order phase transition for: \( r_0 = 1.0 \) (curve 1), \( r_0 = 0.5 \) (2), \( r_0 = -0.5 \) (3), \( r_0 = -1.0 \) (4). (b) The function \( f(\varphi) \) for the IBG at constant density (the curves 1-4 correspond to the same values of \( r_0 \) as given for Fig.2a).

Fig. 3. (a) A tree diagram denoting the interaction part of the Bose Hamiltonian. (b) The compact self-energy diagram which is equal to zero in the limit \( T \to 0 \) [the thick loop denotes the full (renormalized) Green function \( G(q) \)]. (c) An example of a diagram from the perturbation series for the interaction vertex which gives a zero contribution in the zero temperature limit. (d) The infinite ladder series of diagrams which yields the geometric progression (51).

Fig. 4. (a) The graphical representation of Eq. (67) for \( J = 1 \). (b) The high- and low-temperature parts of the curve in Fig.4a with an indication of \( T^- \) and \( \Gamma^- \) transitions (see the text).

Fig. 5. A scheme representing four relevant FPs (P, R, Gl, Un) in disordered systems.