Generalized Donaldson–Thomas theory over fields $\mathbb{K} \neq \mathbb{C}$

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Abstract

Generalized Donaldson–Thomas invariants $\overline{DT}^\alpha(\tau)$ defined by Joyce and Song [64] are rational numbers which ‘count’ both $\tau$-stable and $\tau$-semistable coherent sheaves with Chern character $\alpha$ on a Calabi–Yau 3-fold $X$, where $\tau$ denotes Gieseker stability for some ample line bundle on $X$. The $\overline{DT}^\alpha(\tau)$ are defined for all classes $\alpha$, and are equal to the classical $DT^\alpha(\tau)$ defined by Thomas [121] when it is defined. They are unchanged under deformations of $X$, and transform by a wall-crossing formula under change of stability condition $\tau$. Joyce and Song use gauge theory and transcendental complex analytic methods, so that their theory of generalized Donaldson–Thomas invariants is valid only in the complex case. This also forces them to put constraints on the Calabi–Yau 3-fold they can define generalized Donaldson–Thomas invariants for.

This paper will propose a new algebraic method extending the theory to algebraically closed fields $\mathbb{K}$ of characteristic zero, and partly to triangulated categories and for non necessarily compact Calabi–Yau 3-folds under some hypothesis.

It will describe the local structure of the moduli stack $\mathcal{M}$ of (complexes of) coherent sheaves on $X$, showing that an atlas for $\mathcal{M}$ carries the structure of a $GL(n, \mathbb{K})$-invariant d-critical locus in the sense of [63] and thus it may be written locally as the zero locus of a regular function defined on an étale neighborhood in the tangent space of $\mathcal{M}$ and use this to deduce identities on the Behrend function $\nu_{\mathcal{M}}$.

Moreover, when $\mathbb{K} = \mathbb{C}$, [64, Thm. 4.9] uses the integral Hodge conjecture result by Voisin for Calabi–Yau 3-folds over $\mathbb{C}$ to show that the numerical Grothendieck group $K_{num}(\text{coh}(X))$ is unchanged under deformations of $X$. This is important for the results that $\overline{DT}^\alpha(\tau)$ for $\alpha \in K_{num}(\text{coh}(X))$ are invariant under deformations of $X$, even to make sense. We will provide an algebraic proof of that result, characterizing the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a globally constant lattice described using the Picard scheme.

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Introduction

In the following we will summarize some background material on Donaldson–Thomas theory which permits to allocate our problem and state the main result. After that, we outline the contents of the sections. Expert readers can skip the first introductory part.

Notations and conventions

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A Calabi–Yau 3-fold is a smooth projective 3-fold $X$ over $\mathbb{C}$ or $\mathbb{K}$, with trivial canonical bundle $K_X$ and $H^1(C_X) = 0$. Fix a very ample line bundle $C_X(1)$ on $X$, and let $\tau$ be Gieseker stability on the abelian category of coherent sheaves coh$(X)$ on $X$ with respect to $O_X(1)$. If $E$ is a coherent sheaf on $X$ then the class $[E] \in K_{num}(\text{coh}(X))$ is in effect the Chern character $\text{ch}(E)$ of $E$ in the Chow ring $A^*(X)_Q$ as in [30]. For a class $\alpha$ in the numerical Grothendieck group $K_{num}(\text{coh}(X))$, write $\mathcal{M}_{st}^{\alpha}(\tau), \mathcal{M}^{\alpha}_{st}(\tau)$ for the coarse moduli schemes of $\tau$-(semi)stable sheaves $E$ with class $[E] = \alpha$. Then $\mathcal{M}_{st}^{\alpha}(\tau)$ is a projective $\mathbb{C}$ or $\mathbb{K}$-scheme whose points correspond to S-equivalence classes of $\tau$-semistable sheaves, and $\mathcal{M}^{\alpha}_{st}(\tau)$ is an open subscheme of $\mathcal{M}_{st}^{\alpha}(\tau)$ whose points correspond to isomorphism classes of $\tau$-stable sheaves. Write $\mathfrak{M}$ for the moduli stack of coherent sheaves $E$ on $X$. It is an Artin $\mathbb{C}$ or $\mathbb{K}$-stack, locally of finite type and has affine geometric stabilizers. For $\alpha \in K_{num}(\text{coh}(X))$, write $\mathfrak{M}^{\alpha}$ for the open and closed substack of $\mathfrak{M}$ with $[E] = \alpha$ in $K_{num}(\text{coh}(X))$. Write $\mathfrak{M}^{\alpha}_{st}(\tau), \mathfrak{M}^{\alpha}_{st}(\tau)$ for the substacks of $\tau$-(semi)stable sheaves $E$ in class $[E] = \alpha$, which are finite type open substacks of $\mathfrak{M}^{\alpha}$.

Historical overview

In 1998, Thomas [121], following his proposal with Donaldson [23], motivates a holomorphic Casson invariant and defines the Donaldson–Thomas invariants $DT^{\alpha}(\tau)$ which are integers ‘counting’ $\tau$-stable coherent sheaves with Chern character $\alpha$ on a Calabi–Yau 3-fold $X$ over $\mathbb{K}$, where $\tau$ denotes Gieseker stability for some ample line bundle on $X$. Mathematically, and in ‘modern’ terms, he found that $\mathcal{M}_{st}^{\alpha}(\tau)$ is endowed with a symmetric obstruction theory and defined

$$DT^{\alpha}(\tau) = \int_{[\mathcal{M}_{st}^{\alpha}(\tau)]_{\vir}} 1$$

which is mathematical reflection of the heuristic that views $\mathcal{M}_{st}^{\alpha}(\tau)$ as the critical locus of the holomorphic Chern-Simons functional and the shadow of a more deeper ‘derived’ geometry. A crucial result is that the invariants are unchanged under deformations of the underlying geometry of $X$. Finally we remark that the conventional definition of Thomas [121] works only for classes $\alpha$ containing no strictly $\tau$-semistable sheaves and this permits to work just with schemes rather than stacks as the stable moduli scheme itself already encodes all the information about the Ext groups, and thus about the tangent-obstruction complex of the moduli functor.
In 2005, Behrend [4] proved a virtual Gauss–Bonnet theorem which in particular yields that Donaldson–Thomas type invariants can be written as a weighted Euler characterstic

\[ DT^\alpha(\tau) = \chi(M^{\text{st}}_{\alpha st}(\tau), \nu_{M^{\text{st}}_{\alpha st}(\tau)}) \]

of the stable moduli scheme \( M^{\text{st}}_{\alpha st}(\tau) \) by a constructible function \( \nu_{M^{\text{st}}_{\alpha st}(\tau)} \), as a consequence known in literature as the Behrend function. It depends only on the scheme structure of \( M^{\text{st}}_{\alpha st}(\tau) \), and it is convenient to think about it as a multiplicity function. An important moral is that it is better to ‘count’ points in a moduli scheme by the weighted Euler characteristic rather than the unweighted one as it often gives answers unchanged under deformations of the underlying geometry. It is worth to point out that this equation is local, and ‘motivic’, and makes sense even for non-proper finite type \( K \)-schemes. Anyway, using this formula to generalize the classical picture by defining the Donaldson–Thomas invariants as \( \chi(M^{\text{st}}_{\alpha st}(\tau), \nu_{M^{\text{st}}_{\alpha st}(\tau)}) \) when \( M^{\text{st}}_{\alpha st}(\tau) \neq M^{\text{st}}_{\alpha st}(\tau) \) is not a good idea as in the case there are strictly \( \tau \)-semistable sheaves, the moduli scheme \( M^{\text{st}}_{\alpha st}(\tau) \) is no more a good model and suggest that schemes are no more ‘enough’ to extend the theory.

The crucial work by Behrend [4] suggests that Donaldson–Thomas invariants can be written as motivic invariants, like those studied by Joyce in \([54, 59]\), and so it raises the possibility that one can extend the results of \([54, 59]\) to Donaldson–Thomas invariants by including Behrend functions as weights.

Thus, in 2005, Joyce and Song \([64]\) proposed a theory of generalized Donaldson–Thomas invariants \( DT^\alpha(\tau) \). They are rational numbers which ‘count’ both \( \tau \)-stable and \( \tau \)-semistable coherent sheaves with Chern character \( \alpha \) on a compact Calabi–Yau 3-fold \( X \) over \( \mathbb{C} \); strictly \( \tau \)-semistable sheaves must be counted with complicated rational weights. The \( DT^\alpha(\tau) \) are defined for all classes \( \alpha \), and are equal to \( DT^\alpha(\tau) \) when it is defined. They are unchanged under deformations of \( X \), and transform by a wall-crossing formula under change of stability condition \( \tau \). The theory is valid also for compactly supported coherent sheaves on \emph{compactly embeddable} noncompact Calabi–Yau 3-folds in the complex analytic topology.

To prove all this they study the local structure of the moduli stack \( \mathcal{M} \) of coherent sheaves on \( X \). They first show that \( \mathcal{M} \) is Zariski locally isomorphic to the moduli stack \( \mathcal{M}_{\text{et}} \) of algebraic vector bundles on \( X \). Then they use gauge theory on complex vector bundles and transcendental complex analytic methods to show that an atlas for \( \mathcal{M} \) may be written locally in the complex analytic topology as \( \text{Crit}(f) \) for \( f: U \to \mathbb{C} \) a holomorphic function on a complex manifold \( U \). They use this to deduce identities on the Behrend function \( \nu_{\mathcal{M}} \) through the Milnor fibre description of Behrend functions. These identities

\[
\nu_{\mathcal{M}}(E_1 \oplus E_2) = (-1)^{\chi([E_1],[E_2])} \nu_{\mathcal{M}}(E_1) \nu_{\mathcal{M}}(E_2),
\]

\[
\int_{|\lambda| \in \mathfrak{p}(\text{Ext}^1(E_1, E_1))} \nu_{\mathcal{M}}(F) d\chi \quad - \quad \int_{|\mu| \in \mathfrak{p}(\text{Ext}^1(E_1, E_2))} \nu_{\mathcal{M}}(D) d\chi = (e_{21} - e_{12}) \nu_{\mathcal{M}}(E_1 \oplus E_2),
\]

where \( e_{21} = \dim \text{Ext}^1(E_2, E_1) \) and \( e_{12} = \dim \text{Ext}^1(E_1, E_2) \) for \( E_1, E_2 \in \text{coh}(X) \), are crucial for the whole program of Joyce and Song, which is based on the idea that Behrend’s approach should be integrated with Joyce’s theory \([54, 59]\). As the proof uses gauge theory and transcendental methods, it works only over \( \mathbb{C} \) and forces them to put constraints on the Calabi–Yau 3-fold they can define generalized Donaldson–Thomas invariants for. Finally, in \([64] \S 4.5\), when \( K = \mathbb{C} \), the Chern character embeds \( K^{\text{num}}(\text{coh}(X)) \) in \( H^{\text{even}}(X; \mathbb{Q}) \), and the Voisin Hodge conjecture result \([134]\) for Calabi–Yau over \( \mathbb{C} \) completely characterize its image. They use this to show \( K^{\text{num}}(\text{coh}(X)) \) is unchanged under deformations of \( X \). This is important for the \( DT^\alpha(\tau) \) with \( \alpha \in K^{\text{num}}(\text{coh}(X)) \) to be invariant under deformations of \( X \) even to make sense.

In 2008 and 2010, with two subsequent papers \([74, 75]\), Kontsevich and Soibelman also studied generalizations of Donaldson–Thomas invariants, both in the direction of motivic and categorified Donaldson–Thomas invariants.

In \([74]\), they proposed a very general version of the theory, which, very roughly speaking, can be outlined saying that, supposing for the sake of simplicity that \( M^{\text{st}}_{\alpha st}(\tau) = M^{\text{st}}_{\alpha st}(\tau) \), their oversimplified idea is to define \emph{motivic Donaldson–Thomas invariants}

\[
DT^\alpha_{\text{mot}} = \Upsilon(M^{\text{st}}_{\alpha st}(\tau), \nu_{\text{mot}}),
\]

3
would be a natural cohomology group of $M$ the hypercohomology functional. Following this philosophy in which perverse sheaves are categorification of constructible functions, as the critical locus of a regular function. Moreover, using the notion of $Yau$ $3$-fold carries the structure of an algebraic d-critical stack and it is given locally in the Zariski topology objects by Joyce, and uses this theory to apply powerful results of derived algebraic geometry as in [103, 125–130] for the moduli stack $M$.

Thus, the very basic idea in Kontsevich and Soibelman’s paper is to define some kind of ‘generalized cohomology’ $\nu$ almost closed 1-form in the sense of [4], which turned out later to be a wrong direction to follow.

In [74], Kontsevich and Soibelman exposed the categorified version of Donaldson–Thomas theory. To fix ideas, suppose again $M_{\text{st}}(\tau) = M_{\text{sc}}(\tau)$. Following Thomas’ argument [121], one can, heuristically, think of $\nu_{M_{\text{st}}(\tau)}$ as the Euler characteristic of the perverse sheaf of vanishing cycles $\mathcal{P}$ of the holomorphic Chern-Simons functional. Following this philosophy in which perverse sheaves are categorification of constructible functions, the hypercohomology

$$H^*(M_{\text{st}}(\tau); \mathcal{P}|_{M_{\text{st}}(\tau)})$$

would be a natural cohomology group of $M_{\text{st}}(\tau)$ whose Euler characteristic is the Donaldson–Thomas invariant. Thus, the very basic idea in Kontsevich and Soibelman’s paper is to define some kind of ‘generalized cohomology’ for the moduli stack $M$ as a kind of Ringel–Hall algebra.

In 2013, a sequence of five papers [11–13, 17, 63] developed a theory of d-critical loci, a new class of geometric objects by Joyce, and uses this theory to apply powerful results of derived algebraic geometry as in [103, 125, 130] to Donaldson–Thomas theory. It is shown that the moduli stack of (complexes of) coherent sheaves on a Calabi–Yau 3-fold carries the structure of an algebraic d-critical stack and it is given locally in the Zariski topology as the critical locus of a regular function. Moreover, using the notion of orientation data, they construct a natural perverse sheaf and a natural motive on the moduli stack $M$, thus answering a long-standing question in the problem of categorification. See [3] for a detailed discussion.

The main result and its implications

Following Joyce and Song’s proposal, the aim of this paper is to provide an extension of the theory of generalized Donaldson–Thomas invariants in [64] to algebraically closed fields $K$ of characteristic zero. Our argument provides the algebraic analogue of [63] Thm 5.5, [64] Thm 5.11 and [64] Cor. 5.28 which are enough to extend [64] at least for compact Calabi–Yau 3-folds. Unfortunately, to extend the whole project to complexes of sheaves and to compactly supported sheaves on a noncompact quasi-projective Calabi–Yau 3-fold, we would need other results also from derived algebraic geometry which we do not have at the present. We hope to come back on this point in a future work.

We will show that an atlas for $M$ near $[E] \in M(K)$ may be written locally in the étale topology as the zero locus $f^{-1}(0)$ for a $G$-invariant regular function $f$ defined on an étale neighborhood of $0 \in \mathfrak{u}(K)$ in the affine $K$-space $\text{Ext}^1(E, E)$, where $G$ is a maximal torus of $\text{Aut}(E)$.

Based on this picture, we give an algebraic proof of the Behrend function identities. We point out that our approach is actually valid much more generally for any stack which is locally a global quotient, and we actually do not use any particular properties of coherent sheaves on Calabi–Yau 3-folds. In the past, the author tried a picture in which the moduli stack of coherent sheaves was locally described as a zero locus of an algebraic almost closed 1-form in the sense of [4], which turned out later to be a wrong direction to follow.
Finally, we will study the deformation invariance properties of $\bar{DT}^\alpha(\tau)$ under changes of the underlying geometry of $X$, characterizing a globally constant lattice containing the image through the Chern character of $K_{\text{num}}(\text{coh}(X))$ and in which classes $\alpha$ vary.

The implications are quite exciting and far-reaching. Our algebraic method could lead to the extension of generalized Donaldson–Thomas theory to the derived categorical context. The plan to extend from abelian to derived categories the theory of Joyce and Song [64] starts by reinterpreting the series of papers by Joyce [54–61] in this new general setup. In particular:

(a) Defining configurations in triangulated categories $\mathcal{T}$ requires to replace the exact sequences by distinguished triangles.

(b) Constructing moduli stacks of objects and configurations in $\mathcal{T}$. Again, the theory of derived algebraic geometry [103, 125, 130] can give us a satisfactory answer.

(c) Defining stability conditions on triangulated categories can be approached using Bridgeland’s results, and its extension by Gorodentscev et al., which combines Bridgeland’s idea with Rudakov’s definition for abelian categories [107]. Since Joyce’s stability conditions [56] are based on Rudakov, the modifications should be straightforward.

(d) The ‘nonfunctoriality of the cone’ in triangulated categories causes that the triangulated category versions of some operations on configurations are defined up to isomorphism, but not canonically, which yields that corresponding diagrams may be commutative, but not Cartesian as in the abelian case. In particular, one loses the associativity of the Ringel-Hall algebra of stack functions, which is a crucial object in Joyce and Song framework. We expect that derived Hall algebra approach of Toën [126] resolve this issue. See also [86].

We expect that a well-behaved theory of invariants counting $\tau$-semistable objects in triangulated categories in the style of Joyce’s theory exists, and we hope to come back on it in a future work.

**Outstanding problems and recent research**

Donaldson–Thomas theory depicted in this picture is promising and the literature based on the sketched milestones [4, 64, 74, 75, 121] is vast. Although several interesting developments have been achieved, there are many outstanding problems and a whole final picture overcoming these problems and related conjectures is far reaching.

In 2003, Maulik, Nekrasov, Okounkov and Pandharipande [91, 95] stated the celebrated MNOP conjecture in which Donaldson–Thomas invariants for sheaves of rank one have been conjectured to have deep connections with Gromov–Witten theory of Calabi–Yau 3-folds, but also with Gopakumar–Vafa invariants and Pandharipande–Thomas invariants [102]. Even if some results on this conjectural equivalence of theories of curve counting invariants (Bridgeland [15, 16], Stoppa and Thomas [119], Toda [123]), the MNOP conjecture is still unproved. Moreover, very little is known about the ‘meaning’ of higher rank Donaldson–Thomas invariants. In the same work, [94, Conj.1], they formulated a conjecture on values of the virtual count of $\text{Hilb}^dX$ (Donaldson–Thomas counting of dimension zero sheaves), that has now been established and different proofs are given by Behrend and Fantechi [6], Levine and Pandharipande [80] and Li [81].

In [64, Questions 4.18, 5.7, 5.10, 5.12, 6.29] Joyce and Song pointed out some outstanding problems of their theory and suggest new methods to deal with them. Some of those questions have been answered with new methods as in [11, 13, 17, 63]. However, the main limitation of Joyce and Song’s approach is due to the fact that they work using gauge theory and transcendental complex analytic methods, which causes the theory is valid only over the complex numbers and puts restrictions on the Calabi–Yau which they can define the theory for, and they deal with abelian rather than triangulated categories. This limits the usefulness of their theory as, for many applications, especially to physics, one needs triangulated categories. Moreover, in [64, §6], Joyce and Song, following Kontsevich and Soibelman [74, §2.5 & §7.1], and from ideas similar to Aspinwall–Morrison computation for a Calabi–Yau 3-fold, defined the $BPS$ invariants $\hat{DT}^\alpha(\tau)$, also generalizations of Donaldson–Thomas invariants, and conjectured to be integers for certain $\tau$. There are some evidences on this fact [64, §6],
but the problem is still open. Finally, in [64] §7, they extended their generalized Donaldson–Thomas theory to abelian categories of representations of a quiver $Q$ with relations coming from a superpotential on $Q$, and connected their ideas with the already existing literature on noncommutative Donaldson–Thomas invariants and on invariants counting quiver representations (just to cite some names: Bryan, Ginzburg, Hanany, Nagao, Nakajima, Reineke, Szendrői, and Young). This is an active area of research in representation theory.

There is a seething big area of research which aims to extend Donaldson–Thomas theory in the derived categorical framework. For a long time there was the problem to prove that the moduli space of complexes of sheaves can be given as a critical locus, similarly to the moduli space of sheaves. In 2006, Behrend and Getzler [9] announced a development in this direction, which various papers in literature refers to (e.g. Toda [123,124]), but the paper has not yet been published. It says that the formal potential function $f$ for the cyclic dg Lie algebra $L$ coming from the Schur objects in the derived category of coherent sheaves on Calabi–Yau 3-folds can be made to be convergent over a local neighborhood of the origin. In [124 Conj. 1.2], Toda formulates the derived categorical analog of [64 Thm. 5.5] and then Hua announced in [48] a joint work with Behrend [10] about the construction of the derived moduli space of complexes of coherent sheaves. In [48], Hua gives a construction of the global Chern-Simons functions for toric Calabi–Yau stacks of dimension three using strong exceptional collections. The moduli spaces of sheaves on such stacks can be identified with critical loci of these functions. Still in the direction of derived categorical context, Chang and Li [19] defined recently a semi–perfect obstruction theory and used it to construct virtual cycles of moduli of derived objects on Calabi–Yau 3-folds. In an other paper with Kiem [70], Li studied stable objects in derived category using a ‘$C^*$-intrinsic blowup’ strategy. Finally, in 2013, the author et al. in [11] completely answered the issue of presenting the moduli stack as a critical locus, and this opens now the question about possibilities to extend the whole project in [64] to triangulated categories, the main difficulty of which would be to provide a generalization of wall-crossing formulæ from abelian to triangulated categories, in the style of Joyce [54–61].

This discussion enlightens the fact that beyond this theory there is some deeper ‘derived’ geometry: as the deformation theory of coherent sheaves concerns the Ext groups, one way to talk about different geometric structures on moduli spaces is to ask what information they store about the Ext groups. For instance, in Koutsevich and Soibelman’s context, an interesting problem, among others, is finding what kind of geometric structure on moduli spaces of coherent sheaves on a Calabi–Yau 3-fold $X$ would be the most appropriate for doing motivic and categorified Donaldson–Thomas theory. As a consequence, a natural question would be to ask if derived algebraic geometry has something again to say about a theory of Donaldson–Thomas invariants for Calabi–Yau $m$-folds for $m > 3$, and what is the most suitable geometric structure to develop the theory, see Corollary ??.

Finally, due to both many unproved conjectures and exciting results, Kontsevich and Soibelman’s motivic and categorified theory brings to life a fervid area of research (just to cite some investigators: Behrend, Bryan, Davison, Dimca, Mozgovoy, Nagao and Szendrői). In the present work, we will not discuss much more this area, but we will come back to Kontsevich and Soibelman’s theory later.

Outline of the paper

The paper begins with a section of background material on obstruction theories and conventional definition of Donaldson–Thomas theory, Behrend functions and Behrend’s approach to Donaldson–Thomas theory and, finally, Joyce and Song’s and Kontsevich and Soibelman’s generalization of Donaldson–Thomas theory. This mainly aims to provide a soft introduction to Donaldson–Thomas theory and more specifically to Joyce’s theory and the scenery in which the following sections take place.

Subsection [1.1] will briefly recall material from [5], [82] and then [121]. This should provide a general picture about obstruction theories and the classical Donaldson–Thomas invariants. To say that a scheme $X$ has an obstruction theory means, very roughly speaking, that one is given locally on $X$ an equivalence class of morphisms of vector bundles such that at each point the kernel of the induced linear map of vector spaces is the tangent space to $X$, and the cokernel is a space of obstructions. Following Donaldson and Thomas [23, §3], Thomas [121] motivates a holomorphic Casson invariant counting bundles on a Calabi–Yau 3-fold. He develops the deformation theory necessary to obtain the virtual moduli cycles in moduli spaces of stable sheaves whose higher obstruction groups vanish, which allows him to define the holomorphic Casson invariant of a Calabi–Yau 3-fold $X$ and prove it is deformation invariant. Heuristically, the Donaldson–Thomas moduli space is the
critical set of the holomorphic Chern–Simons functional and the Donaldson–Thomas invariant is a holomorphic analogue of the Casson invariant.

Subsection 1.2 provides a more eclectic presentation of the Behrend function. The first part will review the microlocal approach to defining it, with a discussion on the attempt to categorify Donaldson–Thomas theory. In particular the section describes the bridge between perverse sheaves and vanishing cycles on one hand, and Miilnor fibres and Behrend functions on the other. Thus, if \( \mathfrak{M} \) is the Donaldson–Thomas moduli space of stable sheaves, one can, heuristically, think of \( \nu_{\mathfrak{M}} \) as the Euler characteristic of the perverse sheaf of vanishing cycles of the holomorphic Chern-Simons functional. Following this philosophy in which perverse sheaves are categorification of constructible functions, the section outline the categorification program for Donaldson–Thomas theory. Then, in the second part, the Euler characteristic weighted by the Behrend function is compared to the unweighted Euler characteristic, motivating the necessity to introduce the Behrend function as a multiplicity function. Finally, some properties are listed, in particular the Behrend approach to the Donaldson–Thomas invariants as weighted Euler characteristics and the formula in the complex setting of the Behrend function through linking numbers, which guarantee a more useful expression also in the case it is not known if the scheme admitting a symmetric obstruction theory can locally be written as the critical locus of a regular function on a smooth scheme. This is done introducing the definition of almost closed 1-forms. We point out that Pandharipande and Thomas [102] give examples which are zeroes of almost closed 1-forms, but are not locally critical loci, and this is the main indication that almost closed 1-forms are not ‘enough’ to develop our whole program.

Subsection 1.3 combines some results of Joyce’s series of papers [54–59] with Behrend’s approach to Donaldson–Thomas theory and describes how Joyce and Song developed the theory of generalized Donaldson–Thomas invariants in [61]. The idea behind the entire project is that one should insert the Behrend function \( \nu_{\mathfrak{M}} \) of the moduli stack \( \mathfrak{M} \) of coherent sheaves as a weight in the Joyce’s program. A good introduction to the book is provided by Joyce in [61]. Then, a concluding remark presents a sketch on Kontsevich and Soibelman’s generalization of Donaldson–Thomas theory. As the present paper is mainly concentrated on Joyce and Song’s approach, the remark focuses on analogies and differences between the two theories rather than going into a detailed explanation of Kontsevich and Soibelman’s program, both because it is beyond the author’s competence and it is not directly involved in the results presented here.

Sections 2–3 present briefly the main application in Donaldson–Thomas theory coming from the vast project described below in which the numerical Grothendieck group injects through the Chern character and Thomas [102] give examples which are zeroes of almost closed 1-forms, but are not locally critical loci, and this is the main indication that almost closed 1-forms are not ‘enough’ to develop our whole program.

Section 4 states our main results, including the description of the local structure of the moduli stack of coherent sheaves on a Calabi–Yau 3-fold, the Behrend function identities and the deformation invariance of the theory. The section explains why and where Joyce and Song use the restriction \( K = \mathbb{C} \) in [61] and how our results overcome this restriction: [1.2] provides algebraic analogues of [61] Thm. 5.5 [61] Thm. 5.11. Finally [1.3] provides the analogue of [61] Cor. 5.28 which yields the deformation invariance over \( K \) of the generalized Donaldson–Thomas invariants \( DT^\alpha(\tau) \) defined for classes \( \alpha \) varying in a deformation invariant lattice \( A_X \) described below in which the numerical Grothendieck group injects through the Chern character map. The section culminates in Theorem 1.4 which summarizes all these ideas.

Subsection 1.4 proves the Behrend function identities above using the existence of a \( T \)-equivariant \( d \)-critical chart in the sense of [63] for each given point \( E \) of \( \mathfrak{M} \), where \( T \subset G \) is a maximal torus in \( G \), a maximal torus of \( \text{Aut}(E) \). This gives us the local description of the stack as a critical locus for a \( T \)-invariant regular function \( f \).
defined on a smooth scheme $U \subset \text{Ext}^1(E, E)$. This method is valid for every locally global quotient stack, and in particular it provides the required local description of the moduli stack (Theorem 4.11). Note that we actually would not need the assumption of the local quotient structure if we wanted to restrict just to sheaves on Calabi–Yau 3-folds, as this would follow from the standard method for constructing coarse moduli schemes of semistable coherent sheaves as in Huybrechts and Lehn [50]. More precisely, one can find a ‘good’ local atlas for $\mathcal{M}$ which is a $\mathbb{G}$-invariant, locally closed $\mathbb{K}$-subscheme in the Grothendieck’s Quot Scheme $\text{Quot}_X(\mathbb{K}^P(n) \otimes \mathcal{O}_X(-n), P)$, explained in [50] §2.2, which parametrizes quotients $\mathbb{K}^P(n) \otimes \mathcal{O}_X(-n) \to E'$, where $E'$ has Hilbert polynomial $P$, and which is acted on by the $\mathbb{K}$-group $\text{GL}(P(n), \mathbb{K})$. From [64] it turns out that the proof of the first Behrend identity is reduced to an identity between the Behrend function of the zero locus of $df$, which is a $\mathbb{K}^*$-scheme, and the Behrend function of the fixed part of this zero locus, that is

$$\nu_{df^{-1}(0)}(p) = (-1)^{\dim(T_p(df^{-1}(0)) - \dim(T_p(df^{-1}(0))^T)} \nu_{df^{-1}(0)^T}(p),$$

where $p$ is a point in the $\mathbb{K}^*$-fixed point locus $(df^{-1}(0))^T$. This relation is a generalization of the result in [6] to the case that $p$ is not necessarily an isolated fixed point of the action and $\mathbb{K}$ is a general algebraically closed field of characteristic zero. This argument is a different approach from the one suggested in a work by Li–Qin [83], where there they use some properties of the Thom classes of vector bundles. The first Behrend function identity over algebraically closed fields $\mathbb{K}$ of characteristic zero follows from a trick in the argument of the second Behrend function identity proof, which is directly proved over $\mathbb{K}$, and is based on Theorem 1.8 which is the algebraic version of [64] Thm 4.11].

Subsection 4.3 yields that it is possible to extend [64] Cor. 5.28 on the deformation invariance of the generalized Donaldson–Thomas invariants in the compact case to algebraically closed fields $\mathbb{K}$ of characteristic zero. First of all, using existence results by Grothendieck and Artin, and smoothness and properness properties of the relative Picard scheme in a family of Calabi–Yau 3-folds, one proves that the Picard groups form a local system. Moreover, it is a local system with finite monodromy, so it can be made trivial after passing to a finite étale cover of the base scheme, as formulated in the Theorem which is the algebraic generalization of [64] Thm. 4.21], and which studies the monodromy of the Picard scheme in a family instead of the numerical Grothendieck group. Finally, Theorem 4.4 proves that $\Lambda_X$ is deformation invariant and the Chern character gives an injective morphism $\mathbb{K}^\text{num}(\text{coh}(X)) \to \Lambda_X$. Our $\text{DT}^\alpha(r)$ will be defined for classes $\alpha \in \Lambda_X$.

Section 5 sketches some implications of the theory and proposes new ideas for further research, in particular in the direction of derived categorical framework trying to establish a theory of generalized Donaldson–Thomas invariants for objects in the derived category of coherent sheaves, and for non necessarily compact Calabi–Yau 3-folds.

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1 Donaldson–Thomas theory: background material

This section should be conceived as background picture in which next new sections should be allocated. The competent reader can skip directly to [4].
1.1 Obstruction theories and Donaldson–Thomas type invariants

This section will briefly recall material from [5], [82] and then [121] which provide both important notions used in the sequel and a hopefully interesting picture of Donaldson–Thomas theory.

1.1.1 Obstruction theories

Suppose that $X$ is a subscheme of a smooth scheme $M$, cut out by a section $s$ of a rank $r$ vector bundle $E \to M$. Then the expected dimension, or virtual dimension, of $X$ is $n - r$, the dimension it would have if the section $s$ was transverse. If it is not transverse, one wants to take a correct $(n - r)$-cycle on $X$. As the section $s$ induces a cone in $E|_X$, one may then intersect this cone with the zero section of $X$ inside $E$ to get a cycle of expected dimension on $X$. The key observation is that one works entirely on $X$ and not in the ambient scheme $M$. The deformation theory of the moduli problem is often endowed with the infinitesimal version of $s : M \to E$ on $X$, namely the linearization of $s$, yielding the following exact sequence:

$$0 \longrightarrow TX \longrightarrow TM|_X \stackrel{ds}{\longrightarrow} E|_X \longrightarrow Ob \longrightarrow 0,$$

for some cokernel $Ob$, which in the moduli problem becomes the obstruction sheaf.

Moduli spaces in algebraic geometry often have an expected dimension at each point, which is a lower bound for the dimension at that point. Sometimes it may not coincide with the actual dimension of the moduli space and sometimes it is not possible to get a space of the expected dimension. When one has a moduli space $X$ one obtains numerical invariants by integrating certain cohomology classes over the virtual moduli cycle, a class of the expected dimension in its Chow ring.

One example is the moduli space of torsion-free, semi-stable vector bundles on a surface which yields the Donaldson theory and which provides a set of differential invariants of 4-manifolds. Another one is the moduli space of stable maps from curves of genus $g$ to a fixed projective variety which yields the Gromov–Witten invariants, a kind of generalization of the classical enumerative invariant which counts the number of algebraic curves with appropriate constraints in a variety. In both cases, these invariants are intersection theories on the moduli spaces, respectively, of vector bundles over the surfaces, and of stable maps from curves to a variety. The fundamental class is the core of an intersection theory. However, for Gromov–Witten invariants for example, one cannot take the fundamental class of the whole moduli space directly. The virtual moduli cycle, roughly speaking, plays the role of the fundamental class in an appropriate “good” intersection theory.

A nice picture to start with is the following situation: when the expected dimension does not coincide with the actual dimension of the moduli space, one may view this as if the moduli space is a subspace of an ‘ambient’ space cut out by a set of ‘equations’ whose vanishing loci do not meet transversely. Such a situation is well understood in the following setting described in the Introduction of [82]: let $X$, $Y$ and $W$ be smooth varieties, $X,Y \to W$ and let $Z = X \times_W Y$. Then $[X] \cdot [Y]$, the intersection of the cycle $[X]$ and $[Y]$, is a cycle in $A_*W$ of dimension $\dim X + \dim Y - \dim W$. When $\dim Z = \dim X + \dim Y - \dim W$, then $[Z] = [X] \cdot [Y]$. Otherwise, $[Z]$ may not be $[X] \cdot [Y]$. The excess intersection theory gives that one can find a cycle in $A_*Z$ so that it is $[X] \cdot [Y]$. One may view this cycle as the virtual cycle of $Z$ representing $[X] \cdot [Y]$. Following Fulton–MacPherson’s normal cone construction (in [30],[82]), this cycle is the image of the cycle of the normal cone to $Z$ in $X$, denoted by $C_{Z,X}$, under the Gysin homomorphism $s^* : A_*(C_{Y/Z} \times_Y Z) \to A_*Z$, where $s : Z \to C_{Y/Z}$ is the zero section. This theory does not apply directly to moduli schemes, since, except for some isolated cases, it is impossible to find pairs $X \to W$ and $Y \to W$ for smooth $X$, $Y$ and $W$ so that $X \times_W Y$ is the moduli space and $[X] \cdot [Y]$ so defined is the virtual moduli cycle one needs.

Behrend and Fantechi [5] and Li and Tian [82] give two different approaches to deal with this. Very briefly, the strategy to Li and Tian’s approach in [82] is that rather than trying to find an embedding of the moduli space into some ambient space, they will construct a cone in a vector bundle directly, say $C \subset V$, over the moduli space and then define the virtual moduli cycle to be $s^*|C$, where $s$ is the zero section of $V$. The pair $C \subset V$ will be constructed based on a choice of the tangent-obstruction complex of the moduli functor. The construction commutes with Gysin maps and carries a good invariance property.

In [5] Behrend and Fantechi introduce the notion of cone stacks over a scheme $X$ (or more generally for Deligne–Mumford stacks). These are Artin stacks which are locally the quotient of a cone by a vector bundle acting on it. They call a cone abelian if it is defined as $\text{Spec} \text{Sym } \mathcal{F}$, where $\mathcal{F}$ is a coherent sheaf on $X$. Every
cone is contained as a closed subcone in a minimal abelian one, which is called its \textit{abelian hull}. The notions of being abelian and of abelian hull generalize immediately to cone stacks. Then, for a complex $E^\bullet$ in the derived category $D(X)$ of quasi-coherent sheaves on $X$ which satisfies some suitable assumptions (denoted by $(\ast)$, see Definition 1.1), there is an associated abelian cone stack $h^1/h^0((E^\bullet)^\vee)$. In particular the cotangent complex $L^\bullet_X$ of $X$ constructed by Illusie [52] (a helpful review is given in Illusie [53] §1) satisfies condition $(\ast)$, so one can define the abelian cone stack $\mathfrak{N}_X := h^1/h^0((L^\bullet_X)^\vee)$, the intrinsic normal sheaf. More directly, $\mathfrak{N}_X$ is constructed as follows: étale locally on $X$, embed an open set $U$ of $X$ in a smooth scheme $W$, and take the stack quotient of the normal sheaf (viewed as abelian cone) $N_{U/W}$ by the natural action of $TW_{|U}$. One can glue these abelian cone stacks together to get $\mathfrak{N}_X$. The intrinsic normal cone $\mathfrak{E}_X$ is the closed subcone stack of $\mathfrak{N}_X$ defined by replacing $N_{U/W}$ by the normal cone $C_{U/W}$ in the previous construction. In particular, the intrinsic normal sheaf $\mathfrak{N}_X$ of $X$ carries the obstructions for deformations of affine $X$-schemes. With this motivation, they introduce the notion of \textit{obstruction theory} for $X$. To say that $X$ has an obstruction theory means, very roughly speaking, that one is given locally on $X$ an equivalence class of morphisms of vector bundles such that at each point the kernel of the induced linear map of vector spaces is the tangent space to $X$, and the cokernel is a space of obstructions. That is, this is an object $E^\bullet$ in the derived category together with a morphism $E^\bullet \to L^\bullet_X$, satisfying Condition $(\ast)$ and such that the induced map $\mathfrak{N}_X \to h^1/h^0((E^\bullet)^\vee)$ is a closed immersion. One denotes the sheaf $h^1(E^\vee)$ by $\text{Ob}$, the obstruction sheaf of the obstruction theory. It contains the obstructions to the smoothness of $X$. When an obstruction theory $E^\bullet$ is \textit{perfect}, $\mathfrak{E} = h^1/h^0((E^\bullet)^\vee)$ is a vector bundle stack. Once an obstruction theory is given, with the additional technical assumption that it admits a global resolution, one can define a universal fundamental class of the expected dimension: one has a vector bundle stack $\mathfrak{E}$ with a closed subcone stack $\mathfrak{E}_X$, and to define the universal fundamental class of $X$ with respect to $E^\bullet$ one simply intersects $\mathfrak{E}_X$ with the zero section of $\mathfrak{E}$. To get round of the problem of dealing with Chow groups for Artin stacks, Behrend and Fantechi choose to assume that $E^\bullet$ is globally given by a homomorphism of vector bundles $F^{-1} \to F^0$. Then $\mathfrak{E}_X$ gives rise to a cone $C$ in $F_1 = F^{-1} \vee 0$ and one intersects $C$ with the zero section of $F_1$ (see [77] for a statement without this assumption).

So, recall the following definitions from Behrend and Fantechi [4–6]:

\textbf{Definition 1.1.} Let $Y$ be a $\mathbb{K}$-scheme, and $D(Y)$ the derived category of quasi-coherent sheaves on $Y$.

(a) A complex $E^\bullet \in D(Y)$ is \textit{perfect of perfect amplitude contained in} $[a, b]$, if étale locally on $Y$, $E^\bullet$ is quasi-isomorphic to a complex of locally free sheaves of finite rank in degrees $a, a+1, \ldots, b$.

(b) A complex $E^\bullet \in D(Y)$ \textit{satisfies condition $(\ast)$} if

(i) $h^i(E^\bullet) = 0$ for all $i > 0$,

(ii) $h^0(E^\bullet)$ is coherent for $i = 0, -1$.

(c) An \textit{obstruction theory} for $Y$ is a morphism $\varphi : E^\bullet \to L_Y$ in $D(Y)$, where $L_Y = L_Y/\text{Spec} \mathbb{K}$ is the cotangent complex of $Y$, and $E$ satisfies condition $(\ast)$, and $h^0(\varphi)$ is an isomorphism, and $h^{-1}(\varphi)$ is an epimorphism.

(d) An obstruction theory $\varphi : E^\bullet \to L_Y$ is called \textit{perfect} if $E^\bullet$ is perfect of perfect amplitude contained in $[-1, 0]$.

(e) A perfect obstruction theory $\varphi : E^\bullet \to L_Y$ on $Y$ is called \textit{symmetric} if there exists an isomorphism $\vartheta : E^\bullet \to E^\bullet[1]$, such that $\vartheta^\vee[1] = \vartheta$. Here $E^\bullet[1] = R\text{Hom}(E^\bullet, \mathcal{O}_Y)$ is the dual of $E^\bullet$, and $\vartheta^\vee$ the dual morphism of $\vartheta$.

(f) If moreover $Y$ is a scheme with a $G$-action, where $G$ is an algebraic group, an \textit{equivariant} perfect obstruction theory is a morphism $E^\bullet \to L_Y$ in the category $D(Y)^G$, which is a perfect obstruction theory as a morphism in $D(Y)$ (this definition is originally due to Graber–Pandharipande [39]). Here $D(Y)^G$ denotes the derived category of the abelian category of $G$-equivariant quasi-coherent $\mathcal{O}_Y$-modules.

(g) A \textit{symmetric equivariant} obstruction theory (or an \textit{equivariant symmetric} obstruction theory) is a pair $(E^\bullet \to L_Y, E^\bullet \to E^\bullet[1])$ of morphisms in the category $D(Y)^G$, such that $E^\bullet \to L_Y$ is an equivariant perfect obstruction theory and $\vartheta : E^\bullet \to E^\bullet[1]$ is an isomorphism satisfying $\vartheta^\vee[1] = \vartheta$ in $D(Y)^G$. Note that this is more than requiring that the obstruction theory be equivariant and symmetric, separately, as said in [6].

If instead $Y \xrightarrow{\psi} U$ is a morphism of $\mathbb{K}$-schemes, so $Y$ is a $U$-scheme, we define \textit{relative} perfect obstruction theories $\phi : E^\bullet \to L_{Y/U}$ in the obvious way.
Behrend and Fantechi [5, Th. 4.5] prove the following theorem, which both explains the term obstruction theory and provides a criterion for verification in practice:

**Theorem 1.2.** The following two conditions are equivalent for $E^* \in D(Y)$ satisfying condition (§).

(a) The morphism $\phi : E^* \to L_Y$ is an obstruction theory.

(b) Suppose that we are given a square-zero extension $\mathcal{T}$ of $T$ with ideal sheaf $J$, with $T, \mathcal{T}$ affine, and a morphism $g : T \to Y$. The morphism $\phi$ induces an element $\phi^*(\omega(g)) \in \text{Ext}^1(g^*E^*, J)$ from $\omega(g) \in \text{Ext}^1(g^*L_Y, J)$ by composition. Then $\phi^*(\omega(g))$ vanishes if and only if there exists an extension $\mathcal{F}$ of $g$. If it vanishes, then the set of extensions form a torsor under $\text{Hom}(g^*E^*, J)$.

Some examples can be found in [6]: Lagrangian intersections, sheaves on Calabi–Yau 3-folds, stable maps to Calabi–Yau 3-folds. Next section will concentrate on Donaldson–Thomas obstruction theory as in [121].

### 1.1.2 Donaldson–Thomas invariants of Calabi–Yau 3-folds

Donaldson–Thomas invariants $DT^\alpha(\tau)$ were defined by Richard Thomas [121], following a proposal of Donaldson and Thomas [23, §3]. They are the virtual counts of stable sheaves on Calabi–Yau 3-folds $X$. Starting from the formal picture in which a Calabi–Yau $n$-fold is the complex analogue of an oriented real $n$-manifold, and a Fano with a fixed smooth anticanonical divisor is the analogue of a manifold with boundary, Thomas motivates a holomorphic Casson invariant counting bundles on a Calabi–Yau 3-fold. He develops the deformation theory necessary to obtain the virtual moduli cycles in moduli spaces of stable sheaves whose higher obstruction groups vanish which allows to define the holomorphic Casson invariant of a Calabi–Yau 3-fold $X$, prove it is deformation invariant, and compute it explicitly in some examples. Thus, heuristically, the Donaldson–Thomas moduli space is the critical set of the holomorphic Chern-Simons functional and the Donaldson–Thomas invariant is a holomorphic analogue of the Casson invariant.

Mathematically, Donaldson–Thomas invariants are constructed as follows. Deformation theory gives rise to a perfect obstruction theory [5] (or a tangent-obstruction complex in the language of [82]) on the moduli space of stable sheaves $\mathcal{M}_{st}^\alpha(\tau)$. Recall that Thomas supposes $\mathcal{M}_{st}^\alpha(\tau) = \mathcal{M}_{ss}^\alpha(\tau)$, that is, there are no strictly semistable sheaves $E$ in class $\alpha$, which implies the properness of $\mathcal{M}_{st}^\alpha(\tau)$. As Thomas points out in [121], the obstruction sheaf is equal to $\Omega_{\mathcal{M}_{st}^\alpha(\tau)}$, the sheaf of Kähler differentials, and hence the tangents $T_{\mathcal{M}_{st}^\alpha(\tau)}$ are dual to the obstructions. This expresses a certain symmetry of the obstruction theory on $\mathcal{M}_{st}^\alpha(\tau)$ and is a mathematical reflection of the heuristic that views $\mathcal{M}_{st}^\alpha(\tau)$ as the critical locus of a holomorphic functional. Associated to the perfect obstruction theory is the virtual fundamental class, an element of the Chow group $\mathcal{M}_{st}^\alpha(\tau)$ of algebraic cycles modulo rational equivalence on $\mathcal{M}_{st}^\alpha(\tau)$. One implication of the symmetry of the obstruction theory is the fact that the virtual fundamental class $[\mathcal{M}_{st}^\alpha(\tau)]^\text{vir}$ is of degree zero. It can hence be integrated over the proper space of stable sheaves to an integer, the Donaldson–Thomas invariant or ‘virtual count’ of $\mathcal{M}_{st}^\alpha(\tau)$

$$DT^\alpha(\tau) = \int_{[\mathcal{M}_{st}^\alpha(\tau)]^\text{vir}} 1.$$  \tag{1.1}$$

In fact Thomas did not define invariants $DT^\alpha(\tau)$ counting sheaves with fixed class $\alpha \in K^{\text{num}}(\text{coh}(X))$, but coarser invariants $DT^P(\tau)$ counting sheaves with fixed Hilbert polynomial $P(t) \in \mathbb{Q}[t]$. Thus

$$\mathcal{M}_{ss}^\alpha(\tau) = \prod_{\alpha : P_\alpha = P} \mathcal{M}_{ss}^\alpha(\tau) \leadsto DT^P(\tau) = \sum_{\alpha \in K^{\text{num}}(\text{coh}(X)) : P_\alpha = P} DT^\alpha(\tau),$$

is the relationship with Joyce and Song’s version $DT^\alpha(\tau)$ reviewed in [1.3] where the r.h.s. has only finitely many nonzero terms in the sum. Here, Thomas’ main result [121, §3, 3]:

**Theorem 1.3.** For each Hilbert polynomial $P(t)$, the invariant $DT^P(\tau)$ is unchanged by continuous deformations of the underlying Calabi–Yau 3-fold $X$ over $\mathbb{K}$.

The same proof shows that $DT^\alpha(\tau)$ for $\alpha \in K^{\text{num}}(\text{coh}(X))$ is deformation-invariant, provided it is known that the group $K^{\text{num}}(\text{coh}(X))$ is deformation-invariant, so that this statement makes sense. This issue is discussed in [64, §4.5]. There, it is shown that when $\mathbb{K} = \mathbb{C}$ one can describe $K^{\text{num}}(\text{coh}(X))$ in terms of cohomology
groups $H^*(X; \mathbb{Z}), H^*(X; \mathbb{Q})$, so that $K_{num}(\text{coh}(X))$ is manifestly deformation-invariant, and therefore $DT^\alpha(\tau)$ is also deformation-invariant. Theorem 4.19 crucially uses the integral Hodge conjecture result by [134] for Calabi–Yau 3-folds over $\mathbb{C}$. In [64] Rmk 4.20(c)], Joyce and Song propose to extend that description over an algebraically closed base field $K$ of characteristic zero by replacing $H^*(X; \mathbb{Q})$ by the algebraic de Rham cohomology $H^\text{dR}_c(X)$ of Hartshorne [46]. For $X$ a smooth projective $K$-scheme, $H^\text{dR}_c(X)$ is a finite-dimensional vector space over $K$. There is a Chern character map $\text{ch}: K_{num}(\text{coh}(X)) \hookrightarrow H^\text{num}_{dR}(X)$. In [46] §4, Hartshorne considers how $H^\text{dR}_c(X_t)$ varies in families $X_t : t \in T$, and defines a Gauss–Manin connection, which makes sense of $H^\text{dR}_c(X_t)$ being locally constant in $t$. In [4, §3] we will use another idea to characterize the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a globally constant lattice described using the Picard scheme.

Next section will introduce the Behrend function and the work done by Behrend in [4], which has been crucial for the development of Donaldson–Thomas theory.

### 1.2 Microlocal geometry and the Behrend function

This section briefly explains Behrend’s approach [4] to Donaldson–Thomas invariants as Euler characteristics of moduli schemes weighted by the Behrend function. It was introduced by Behrend [4] for finite type $\mathbb{C}$-schemes $X$; in [64] §4.1 it has been generalized to Artin $\mathbb{K}$-stacks. Behrend functions are also defined for complex analytic spaces $X_{an}$, and the Behrend function of a $\mathbb{C}$-scheme $X$ coincides with that of the underlying complex analytic space $X_{an}$. The theory is also valid for $\mathbb{K}$-schemes acted on by a reductive linear algebraic group. A good reference for this section, other than the original paper by Behrend [4], are [64] §4 and [100] for the equivariant version.

#### 1.2.1 Microlocal approach to the Behrend function

In [4], Behrend suggests a microlocal approach to the problem. The first part of the discussion describes how the Behrend function is defined while the second part, although not detailed and not directly involved in the rest of the paper, aim to give a more complete picture.

**The definition of the Behrend function.** Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and $X$ a finite type $\mathbb{K}$-scheme. Suppose $X \hookrightarrow M$ is an embedding of $X$ as a closed subscheme of a smooth $\mathbb{K}$-scheme $M$. Then one has a commutative diagram

$$
\begin{array}{ccc}
Z_*(X) & \xrightarrow{\text{Eu}} & \text{CF}_{\mathbb{Z}}(X) \\
eq & & \downarrow s_{A_0}^M \sim \\
A_0(X) & \xrightarrow{\text{Ch}} & L_X(M)
\end{array}
$$

where the two horizontal arrows are isomorphisms. Here $Z_*(X)$ denotes the group of algebraic cycles on $X$, as in Fulton [30], and $\text{CF}_{\mathbb{Z}}(X)$ the group of $\mathbb{Z}$-valued constructible functions on $X$ in the sense of [53]. The local Euler obstruction is a group isomorphism $\text{Eu}: Z_*(X) \rightarrow \text{CF}_{\mathbb{Z}}(X)$. The local Euler obstruction was first defined by MacPherson [90] to solve the problem of existence of covariantly functorial Chern classes, answering thus a Deligne–Grothendieck conjecture when $\mathbb{K} = \mathbb{C}$, using complex analysis, but Gonzalez–Sprinberg [38] provides an alternative algebraic definition which works over any algebraically closed field $\mathbb{K}$ of characteristic zero. It is the obstruction to extending a certain section of the tautological bundle on the Nash blowup. More precisely, if $V$ is a prime cycle on $X$, the constructible function $\text{Eu}(V)$ is given by

$$
\text{Eu}(V) : x \mapsto \int_{\mu^{-1}(x)} c(\hat{T}) \cap s(\mu^{-1}(x), \hat{V}),
$$

where $\mu : \hat{V} \rightarrow V$ is the Nash blowup of $V$, $\hat{T}$ the dual of the universal quotient bundle, $c$ the total Chern class and $s$ the Segre class of the normal cone to a closed immersion. Kennedy [68] Lem. 4] proves that $\text{Eu}(V)$ is constructible.

As pointed out in the next section, it is worth observing that independently, at about the same time, Kashiwara proved an index theorem over $\mathbb{C}$ for a holonomic $\mathcal{D}$-module relating its local Euler characteristic and
the local Euler obstruction with respect to an appropriate stratification (see \cite{36} for details). It coincides with the one defined above and this is equivalent to saying that the diagram (1.4) below commutes.

Observe that this part of the diagram exists without the embedding into $M$ and is sufficient to give the definition of the Behrend function as follow. Let $C_{X/M}$ be the normal cone of $X$ in $M$, as in \cite{23} p.73, and $\pi : C_{X/M} \to X$ the projection. As in \cite{4} §1.1, define a cycle $\mathcal{C}_{X/M} \in Z_*(X)$ by

$$\mathcal{C}_{X/M} = \sum_{C'} (-1)^{\dim \pi(C')} \text{mult}(C') \pi(C'),$$

where the sum is over all irreducible components $C'$ of $C_{X/M}$. It turns out that $\mathcal{C}_{X/M}$ depends only on $X$, and not on the embedding $X \hookrightarrow M$. Behrend \cite[Prop. 1.1]{3} proves that given a finite type $\mathbb{K}$-scheme $X$, there exists a unique cycle $\mathcal{C}_X \in Z_*(X)$, such that for any étale map $\varphi : U \to X$ for a $\mathbb{K}$-scheme $U$ and any closed embedding $U \hookrightarrow M$ into a smooth $\mathbb{K}$-scheme $M$, one has $\varphi^*(\mathcal{C}_X) = \mathcal{C}_{U/M}$ in $Z_*(U)$. If $X$ is a subscheme of a smooth $M$ one takes $U = X$ and get $\mathcal{C}_X = \mathcal{C}_{X/M}$. Behrend calls $\mathcal{C}_X$ the signed support of the intrinsic normal cone, or the distinguished cycle of $X$. For each finite type $\mathbb{K}$-scheme $X$, define the Behrend function $\nu_X$ in $\text{CF}_Z(X)$ by $\nu_X = \text{Eu}(\mathcal{C}_X)$, as in Behrend \cite[§1.2]{4}.

For completeness, the section now describes the other side of the diagram (1.2), which yields another possible way to define the Behrend function. Write $\mathcal{L}_X(M)$ for the free abelian group generated by closed, irreducible, reduced, conical Lagrangian, $\mathbb{K}$-subvariety of $\Omega_M$ lying over cycles contained in $X$. The isomorphism $\text{Ch} : \text{CF}_Z(X) \to \mathcal{L}_X(M)$ maps a constructible function to its characteristic cycle, which is a conic Lagrangian cycle on $\Omega_M$ supported inside $X$ defined in the following way. Consider the commutative diagram of group isomorphisms that fits in the diagram (1.2):

$$Z_*(M) \xrightarrow{\text{Eu}} \text{CF}_Z(M) \xrightarrow{\text{Ch}} \mathcal{L}(M).$$

(1.3)

Here $L : Z_*(M) \to \mathcal{L}(M)$ is defined on any prime cycle $V$ by $L : V \to (-1)^{\dim(V)} \ell(V)$, where $\ell(V)$ is the closure of the conormal bundle of any nonsingular dense open subset of $V$. Then $\text{Eu}$, $L$ are isomorphisms, and the characteristic cycle map $\text{Ch} : \text{CF}_Z(M) \to \mathcal{L}(M) \subset Z_{\dim M}(\Omega_M)$ is defined to be the unique isomorphism making (1.3) commute. In the complex case Ginsburg \cite{39} describes the inverse of this map as intersection multiplicity between two conical Lagrangian cycles. This formula is crucial in \cite[§4.3]{4}, where Behrend gives an expression for the Behrend function in terms of linking numbers, which has a validity also in the case it is not known if a scheme admitting a symmetric obstruction theory can locally be written as the critical locus of a regular function on a smooth scheme (Theorem 1.12). See also \cite[Ex. 19.2.4]{30}.

The maps to $A_0(X)$ are the degree zero Chern-Mather class, the degree zero Schwartz-MacPherson Chern class, and the intersection with the zero section, respectively. The Mather class is a homomorphism $c^M : Z_*(X) \to A_0(X)$, whose definition is a globalization of the construction of the local Euler obstruction. One has $c^M(V) = \mu_s(c(T) \cap [V])$, for a prime cycle $V$ of degree $p$ on $X$ with the same notation as above. For the expression in terms of normal cones, see for example \cite[§1]{108}. Applying $c^M$ to the cycle $\mathcal{C}_X$, one obtains the Aluffi class $\alpha_X = c^M(\mathcal{C}_X) \in A_0(X)$ defined in \cite{4}. If $X$ is smooth, its Aluffi class equals $\alpha_X = \epsilon(\mathcal{O}_X) \cap [X]$.

Now given a symmetric obstruction theory on $X$, the cone of curvilinear obstructions $cv \hookrightarrow ob = \mathcal{O}_X$, pulls back to a cone in $\Omega_{M|X}$ via the epimorphism $\Omega_{M|X} \twoheadrightarrow \mathcal{O}_X$. Via the embedding $\Omega_{M|X} \hookrightarrow \Omega_M$ one obtains a conic subscheme $C \hookrightarrow \Omega_M$, the obstruction cone for the embedding $X \hookrightarrow M$. Behrend proves that the virtual fundamental class is $[X]^{vir} = 0^f[C]$. The key fact is that $C$ is Lagrangian. Because of this, there exists a unique constructible function $\nu_X$ on $X$ such that $\text{Ch}(\nu_X) = [C]$ and $c_0^M(\nu_X) = [X]^{vir}$. Then Theorem 1.9 below follows as an application of MacPherson’s theorem \cite{99} (or equivalently from the microlocal index theorem of Kashiwara \cite{65}), which one can think of as a kind of generalization of the Gauss–Bonnet theorem to singular schemes. See Theorem 1.9 below for its validity over $\mathbb{K}$. The cycle $\mathcal{C}_X$ such that $\text{Eu}(\mathcal{C}_X) = \nu_X$ is as defined above, the (signed) support of the intrinsic normal cone of $X$. The Aluffi class $\alpha_X = c^M(\mathcal{C}_X) = c^M(\nu_X)$ has thus the property that its degree zero component is the virtual fundamental class of any symmetric obstruction theory on $X$.

In the case $\mathbb{K} = \mathbb{C}$, using MacPherson’s complex analytic definition of the local Euler obstruction \cite{99}, the definition of $\nu_X$ makes sense in the framework of complex analytic geometry, and so Behrend functions can
be defined for complex analytic spaces \(X_{\text{an}}\). Thus, as in [64] Prop. 4.2 one has that if \(X\) is a finite type \(\mathbb{K}\)-scheme, then the Behrend function \(\nu_X\) is a well-defined \(\mathbb{Z}\)-valued constructible function on \(X\), in the Zariski topology. If \(Y\) is a complex analytic space then the Behrend function \(\nu_Y\) is a well-defined \(\mathbb{Z}\)-valued locally constructible function on \(Y\), in the analytic topology. Finally, if \(X\) is a finite type \(\mathbb{C}\)-scheme, with underlying complex analytic space \(X_{\text{an}}\), then the algebraic Behrend function \(\nu_X\) and the analytic Behrend function \(\nu_{X_{\text{an}}}\) coincide. In particular, \(\nu_X\) depends only on the complex analytic space \(X_{\text{an}}\) underlying \(X\), locally in the analytic topology. Finally, the definition of Behrend functions is valid over \(\mathbb{K}\)-schemes, algebraic \(\mathbb{K}\)-spaces and Artin \(\mathbb{K}\)-stacks, locally of finite type (see [64] Prop. 4.4).

**Categorifying the theory.** What follows will not be needed to understand the rest of the paper. We include this material both for completeness, as it underlies the theory of Behrend functions, and also because it is one of the main application of the whole program [11][13][17][63] explained in [33].

For this paragraph, restrict to \(\mathbb{K} = \mathbb{C}\) for simplicity. There exists a sophisticated modern theory of linear partial differential equations on a smooth complex algebraic variety \(X\), sometimes called microlocal analysis, because it involves analysis on the cotangent bundle \(T^*X\); this yields a theory which is invariant with respect to the action of the whole group of canonical transformation of \(T^*X\) while the usual theory is only invariant under the subgroup induced by diffeomorphism of \(X\). It is sometimes called \(\mathcal{D}\)-module theory, because it involves sheaves of modules \(\mathcal{M}\) over the sheaf of rings of holomorphic linear partial differential operators of finite order \(\mathcal{D} = \mathcal{D}_X\); these rings are noncommutative, left and right Noetherian, and have finite global homological dimension. It is also sometimes called algebraic analysis because it involves such algebraic constructions as \(\text{Ext}^i_{\mathcal{D}}(\mathcal{M},\mathcal{N})\). The theory as it is known today grew out of the work done in the 1960s by the school of Mikio Sato in Japan. During the 1970’s, one of the central themes in \(\mathcal{D}\)-module theory was David Hilbert’s twenty-first problem, now called the *Riemann-Hilbert problem*. A generalization of it may be stated as the problem to solve the Riemann-Hilbert correspondence, which, roughly speaking, describes the nature of the correspondence between a system of differential equations and its solutions. A comprehensive reference is the book of Kashiwara and Shapira [65], while an interesting eclectic vision on the subject is provided by Ginsburg [36]. One has the following commutative diagram:

\[
\begin{array}{ccc}
\text{(perverse) constructible sheaves} & \overset{\sim}{\longrightarrow} & \text{(regular) holonomic modules} \\
\chi & \longleftarrow & \chi \\
\text{constructible functions} & \overset{\sim}{\longrightarrow} & \text{Lagrangian cycles in } T^*X.
\end{array}
\]

Recall that here \(SS\) denotes the characteristic cycle map which to a \(\mathcal{D}\)-module \(\mathcal{M}\) associates its characteristic cycle. It is the formal linear combination of irreducible components of the characteristic variety (the support of the graded sheaf \(\text{gr}\mathcal{M}\) associated to \(\mathcal{M}\)) counted with their multiplicities. It looks like

\[
SS(\mathcal{M}) = \sum m_\alpha(\mathcal{M}) \cdot T^*_{X_\alpha}X
\]

for a stratification \(\{X_\alpha\}\) of \(X\), where \(m_\alpha(\mathcal{M})\) are positive integers and \(T^*_{X_\alpha}X\) is the closure of the conormal bundle \(T^*_{X_\alpha}X\). Each component of the characteristic variety has dimension at least \(\dim(X)\). A \(\mathcal{D}\)-module \(\mathcal{M}\) is called holonomic if its characteristic variety is pure of dimension \(\dim(X)\). To have also regular singularities means, very roughly speaking, that the system is determined by its principal symbol.

So, to a holonomic system it has been associated an object of microlocal nature, the characteristic cycle. On the other side, the Riemann-Hilbert correspondence associates to a holonomic system \(\mathcal{M}\) its *De Rham complex*,

\[
\text{DR}(\mathcal{M}) : 0 \rightarrow \Omega^0(\mathcal{M}) \overset{d}{\rightarrow} \Omega^1(\mathcal{M}) \overset{d}{\rightarrow} \ldots \overset{d}{\rightarrow} \Omega^{\dim(X)}(\mathcal{M}) \overset{d}{\rightarrow} 0,
\]

where \(\Omega^p(\mathcal{M})\) is the sheaf of \(\mathcal{M}\)-valued \(p\)-forms on \(X\) and \(d\) is the differential defined by Cartan formula. As an object in the derived category it can be expressed as \(\text{DR}(\mathcal{M}) = R \text{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[\dim(X)]\). If \(\mathcal{M}\) is holonomic, \(\text{DR}(\mathcal{M})\) is constructible and determines \(\mathcal{M}\) provided that the latter has regular singularities. Recall the following definition (see also [64] §4):
**Definition 1.4.** Let $X$ be a complex analytic space. Consider sheaves of $\mathbb{Q}$-modules $\mathcal{C}$ on $X$. Note that these are not coherent sheaves, which are sheaves of $\mathcal{O}_X$-modules. A sheaf $\mathcal{C}$ is called constructible if there is a locally finite stratification $X = \bigcup_{j \in J} X_j$ of $X$ in the complex analytic topology, such that $\mathcal{C}|_{X_j}$ is a $\mathbb{Q}$-local system for all $j \in J$, and all the stalks $\mathcal{C}_x$ for $x \in X$ are finite-dimensional $\mathbb{Q}$-vector spaces. A complex $\mathcal{C}^*$ of sheaves of $\mathbb{Q}$-modules on $X$ is called constructible if all its cohomology sheaves $H^i(\mathcal{C}^*)$ for $i \in \mathbb{Z}$ are constructible. Write $D^b_{\text{con}}(X)$ for the bounded derived category of constructible complexes on $X$. It is a triangulated category. By [21 Thm. 4.1.5], $D^b_{\text{con}}(X)$ is closed under Grothendieck’s “six operations on sheaves” $R\varphi_* , R\varphi^! , \varphi^! , \varphi^* , R\mathcal{H}om , \otimes$. The perverse sheaves on $X$ are a particular abelian subcategory $\text{Per}(X)$ in $D^b_{\text{con}}(X)$, which is the heart of a $t$-structure on $D^b_{\text{con}}(X)$. So perverse sheaves are actually complexes of sheaves, not sheaves, on $X$. The category $\text{Per}(X)$ is noetherian and locally artinian, and is artinian if $X$ is of finite type, so every perverse sheaf has (locally) a unique filtration whose quotients are simple perverse sheaves; and the simple perverse sheaves can be described completely in terms of irreducible local systems on irreducible subvarieties in $X$.

Now, given a constructible sheaf $\mathcal{C}^*$ there is associated a constructible function on $X$: define a map $\chi_X : \text{Obj}(D^b_{\text{con}}(X)) \to \text{CF}^\mathbb{Z}(X)$ by taking Euler characteristics of the cohomology of stalks of complexes, given by

$$\chi_X(\mathcal{C}^* ) : x \mapsto \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathcal{H}^k(\mathcal{C}^*)_x .$$

Since distinguished triangles in $D^b_{\text{con}}(X)$ give long exact sequences on cohomology of stalks $\mathcal{H}^k(-)_x$, this $\chi_X$ is additive over distinguished triangles, and so descends to a group morphism $\chi_X : K_0(D^b_{\text{con}}(X)) \to \text{CF}^\mathbb{Z}(X)$. These maps $\chi_X : \text{Obj}(D^b_{\text{con}}(X)) \to \text{CF}^\mathbb{Z}(X)$ and $\chi_X : K_0(D^b_{\text{con}}(X)) \to \text{CF}^\mathbb{Z}(X)$ are surjective, since $\text{CF}^\mathbb{Z}(X)$ is spanned by the characteristic functions of closed analytic cycles $Y$ in $X$, and each such $Y$ lifts to a perverse sheaf in $D^b_{\text{con}}(X)$. In category-theoretic terms, $X \to D^b_{\text{con}}(X)$ is a functor $D^b_{\text{con}}$ from complex analytic spaces to triangulated categories, and $X \to \text{CF}^\mathbb{Z}(X)$ is a functor $\text{CF}^\mathbb{Z}$ from complex analytic spaces to abelian groups, and $X \to \chi_X$ is a natural transformation $\chi$ from $D^b_{\text{con}}$ to $\text{CF}^\mathbb{Z}$.

For a holonomic $\mathcal{D}$-module $\mathcal{M}$ one sets $\chi(x, \mathcal{M}) = \chi(x, DR(\mathcal{M}))$. Thus, if $\mathcal{M}$ is a regular holonomic $\mathcal{D}$-module on $X \subset \mathcal{M}$, with $\mathcal{M}$ smooth, whose characteristic cycle is $[\mathcal{C}_{X/M}]$, then

$$\nu_X(P) = \sum_i (-1)^i \dim \mathcal{H}^i_{\{P\}}(X, \mathcal{M}_{DR}) ,$$

for any point $P \in M$. Here $\mathcal{H}^i_{\{P\}}$ denotes cohomology with supports in the subscheme $\{P\} \to M$ and $\mathcal{M}_{DR}$ denotes the perverse sheaf associated to $\mathcal{M}$ via the Riemann-Hilbert correspondence, as incarnated, for example, by the De Rham complex $DR(\mathcal{M})$. At the moment, Kai Behrend is attempting to give explicit constructions in some cases (see [8]).

In the case $X$ is the critical scheme of a regular function $f$ on a smooth scheme $M$, Behrend [4] gives the following expression for the Behrend function due to Parusinski and Pragacz [104]. This formula has been crucial in [64]. For the definition of the Milnor fibres for holomorphic functions on complex analytic spaces and the a review on vanishing cycles a survey paper on the subject is Massey [91], and three books are Kashiwara and Schapira [65], Dimca [21], and Schürmann [112]. Over the field $\mathbb{C}$, Saito’s theory of mixed Hodge modules [110] provides a generalization of the theory of perverse sheaves with more structure, which may also be a context in which to generalize Donaldson–Thomas theory.

**Theorem 1.5.** Let $U$ be a complex manifold of dimension $n$, and $f : U \to \mathbb{C}$ a holomorphic function, and define $X$ to be the complex analytic space $\text{Crit}(f)$ contained in $U_0 = f^{-1}(\{0\})$. Then the Behrend function $\nu_X$ of $X$ is given by

$$\nu_X(x) = (-1)^{\dim U} (1 - \chi(\mathcal{M}F_f(x))) \quad \text{for } x \in X .$$

Moreover, the perverse sheaf of vanishing cycles $\phi_f(\mathbb{Q}[n-1])$ on $U_0$ is supported on $X$, and

$$\chi_U(\phi_f(\mathbb{Q}[n-1]))(x) = \begin{cases} \nu_X(x), & x \in X, \\ 0, & x \in U_0 \setminus X, \end{cases}$$

where $\nu_X$ is the Behrend function of the complex analytic space $X$. 

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Thus, if \( X \) is the Donaldson–Thomas moduli space of stable sheaves, one can, heuristically, think of \( \nu_X \) as the Euler characteristic of the perverse sheaf of vanishing cycles of the holomorphic Chern-Simons functional.

In [64] Question 4.18, 5.7, Joyce and Song ask the following question.

**Question 1.6.** (a) Let \( X \) be a Calabi–Yau 3-fold over \( \mathbb{C} \), and write \( \mathcal{M}_{si} \) for the coarse moduli space of simple coherent sheaves on \( X \). Does there exist a natural perverse sheaf \( \mathcal{P} \) on \( \mathcal{M}_{si} \), with \( \chi_{\mathcal{M}_{si}}(\mathcal{P}) = \nu_{\mathcal{M}_{si}} \), which is locally isomorphic to \( \phi_f(Q[\dim U - 1]) \) for \( f \) as in [64] Thm. 5.4)?

(b) Is there also some Artin stack stack of \( \mathcal{P} \) in (a) for the moduli stack \( \mathcal{M} \), locally isomorphic to \( \phi_f(Q[\dim U - 1]) \) for \( f \) as in Theorem 1.13 below?

(c) Let \( M \) be a complex manifold, \( \omega \) an almost closed holomorphic \((1,0)\)-form on \( M \) as defined below, and \( X = \omega^{-1}(0) \) as a complex analytic subspace of \( M \). Can one define a natural perverse sheaf \( \mathcal{P} \) supported on \( X \), with \( \chi_X(\mathcal{P}) = \nu_X \), such that \( \mathcal{P} \cong \phi_f(Q[\dim U - 1]) \) when \( \omega = df \) for \( f : M \rightarrow \mathbb{C} \) holomorphic? Are there generalizations to the algebraic setting?

One can also ask Question 1.6 for Saito’s mixed Hodge modules [110]. If the answer to Question 1.6(a) is yes, it would provide a way of categorifying (conventional) Donaldson–Thomas invariants \( DT^a(\tau) \). That is \( H^*(\mathcal{M}_{si}^\tau; \mathcal{P}|_{\mathcal{M}_{si}^\tau}) \) would be a natural cohomology group of the stable moduli scheme \( \mathcal{M}_{si}^\tau \) whose Euler characteristic is the Donaldson–Thomas invariant. This question is also crucial for the programme of Kontsevich–Soibelman [74] to extend Donaldson–Thomas invariants of Calabi–Yau 3-folds to other motivic invariants. as discussed in [64] Remark 5.8. We will explain in [4] how this question has been resolved.

### 1.2.2 The Behrend function and its characterization

Here we will point out some important results and properties of the Behrend function.

**Behrend function as a multiplicity function in the weighted Euler characteristic.** It is worth to report here [64] §1.2 which provides a good way to think of Behrend functions as multiplicity functions. If \( X \) is a finite type \( \mathbb{C} \)-scheme then the Euler characteristic \( \chi(X) \) ‘counts’ points without multiplicity, so that each point of \( X(\mathbb{C}) \) contributes 1 to \( \chi(X) \). If \( X_{\text{red}} \) is the underlying reduced \( \mathbb{C} \)-scheme then \( X_{\text{red}}(\mathbb{C}) = X(\mathbb{C}) \), so \( \chi(X_{\text{red}}) = \chi(X) \), and \( \chi(X) \) does not see non-reduced behaviour in \( X \). However, the weighted Euler characteristic \( \chi(X, \nu_X) \) ‘counts’ each \( x \in X(\mathbb{C}) \) weighted by its multiplicity \( \nu_X(x) \). The Behrend function \( \nu_X \) detects non-reduced behaviour, so in general \( \chi(X, \nu_X) \neq \chi(X_{\text{red}}, \nu_{X_{\text{red}}}) \). For example, let \( X \) be the \( k \)-fold point \( \text{Spec}(\mathbb{C}[z]/(z^k)) \) for \( k \geq 1 \). Then \( X(\mathbb{C}) \) is a single point \( x \) with \( \nu_X(x) = k \), so \( \chi(X) = 1 = \chi(X_{\text{red}}, \nu_{X_{\text{red}}}) \), but \( \chi(X, \nu_X) = k \).

An important moral of [4] is that (at least in moduli problems with symmetric obstruction theories, such as Donaldson–Thomas theory) it is better to ‘count’ points in a moduli scheme \( \mathcal{M} \) by the weighted Euler characteristic \( \chi(\mathcal{M}, \nu_X) \) than by the unweighted Euler characteristic \( \chi(\mathcal{M}) \). One reason is that \( \chi(\mathcal{M}, \nu_X) \) often gives answers unchanged under deformations of the underlying geometry, but \( \chi(\mathcal{M}) \) does not. For example, consider the family of \( \mathbb{C} \)-schemes \( X_t = \text{Spec}(\mathbb{C}[z]/(z^2-t^2)) \) for \( t \in \mathbb{C} \). Then \( X_t \) is two reduced points \( \pm t \) for \( t \neq 0 \), and a double point when \( t = 0 \). So as above we find that \( \chi(X_t, \nu_{X_t}) = 2 \) for all \( t \), which is deformation-invariant, but \( \chi(X_t) = 2 \) for \( t \neq 0 \) and 1 for \( t = 0 \), which is not deformation-invariant.

**Properties of the Behrend function.** Here are some important properties of Behrend functions. They are proved by Behrend [4] §1.2 & Prop. 1.5 when \( \mathbb{K} = \mathbb{C} \), but his proof is valid for general \( \mathbb{K} \).

**Theorem 1.7.** Let \( X, Y \) be Artin \( \mathbb{K} \)-stacks locally of finite type. Then:

(i) If \( X \) is smooth of dimension \( n \) then \( \nu_X \equiv (-1)^n \).

(ii) If \( \varphi : X \rightarrow Y \) is smooth with relative dimension \( n \) then \( \nu_X \equiv (-1)^n \varphi^*(\nu_Y) \).

(iii) \( \nu_{X \times Y} \equiv \nu_X \square \nu_Y \), where \( (\nu_X \square \nu_Y)(x, y) = \nu_X(x)\nu_Y(y) \).

Let us recall [64] Thm 4.11. It is stated using the Milnor fibre, but its proof works algebraically over \( \mathbb{K} \).
Theorem 1.8. Let $U$ be a smooth $\mathbb{K}$-variety, $f : U \to \mathbb{A}^1_\mathbb{K}$ a regular function over $U$, and $V$ a smooth $\mathbb{K}$-subvariety of $U$, and $v \in V \cap \text{Crit}(f)$. Define $\tilde{U}$ to be the blowup of $U$ along $V$, with blowup map $\pi : \tilde{U} \to U$, and set $\tilde{f} = f \circ \pi : \tilde{U} \to \mathbb{A}^1_\mathbb{K}$. Then $\pi^{-1}(v) = \mathbb{P}(T_v U/T_v V)$ is contained in $\text{Crit}(\tilde{f})$, and

$$\nu_{\text{Crit}(f)}(v) = \int_{w \in \mathbb{P}(T_v U/T_v V)} \nu_{\text{Crit}(\tilde{f})}(w) \, d\chi + (-1)^{\dim U - \dim V} (1 - \dim U + \dim V) \nu_{\text{Crit}(\tilde{f}|_V)}(v),$$

where $w \mapsto \nu_{\text{Crit}(\tilde{f})}(w)$ is a constructible function on $\mathbb{P}(T_v U/T_v V)$, and the integral is the Euler characteristic of $\mathbb{P}(T_v U/T_v V)$ weighted by this.

One can see the next result as a kind of virtual Gauss–Bonnet formula. It is crucial for Donaldson–Thomas theory. It is proved by Behrend [1], Th. 4.18 when $\mathbb{K} = \mathbb{C}$, but his proof is valid for general $\mathbb{K}$. It depends crucially on [4] Prop. 1.12 which again depend on an application of MacPherson’s theorem [90] over $\mathbb{C}$ but valid over general $\mathbb{K}$ thanks to Kennedy [68] and the definition of the Euler characteristic over algebraically closed field $\mathbb{K}$ of characteristic zero given by Joyce [54]. See also an independent construction of the Schwartz–MacPherson Chern class given by Aluffi [2].

Theorem 1.9. Let $X$ a proper $\mathbb{K}$-scheme with a symmetric obstruction theory, and $[X]^\text{vir} \in A_0(X)$ the corresponding virtual class. Then

$$\int_{[X]^\text{vir}} 1 = \chi(X, \nu_X) \in \mathbb{Z},$$

where $\chi(X, \nu_X) = \int_{X(\mathbb{K})} \nu_X \, d\chi$ is the Euler characteristic of $X$ weighted by the Behrend function $\nu_X$ of $X$. In particular, $\int_{[X]^\text{vir}} 1$ depends only on the $\mathbb{K}$-scheme structure of $X$, not on the choice of symmetric obstruction theory.

Theorem 1.9 implies that $DT^\alpha(\tau)$ in (1.13) is given by

$$DT^\alpha(\tau) = \chi(M^\alpha_{\text{st}}(\tau), \nu_{\mathcal{M}^\alpha_{\text{st}}(\tau)}).$$

(1.7)

There is a big difference between the two equations (1.1) and (1.7) defining Donaldson–Thomas invariants. Equation (1.1) is non-local, and non-motivic, and makes sense only if $M^\alpha_{\text{st}}(\tau)$ is a proper $\mathbb{K}$-scheme. But (1.7) is local, and (in a sense) motivic, and makes sense for arbitrary finite type $\mathbb{K}$-schemes $M^\alpha_{\text{st}}(\tau)$. In fact, one could take (1.7) to be the definition of Donaldson–Thomas invariants even when $M^\alpha_{\text{st}}(\tau) \neq \mathcal{M}^\alpha_{\text{st}}(\tau)$, but in [64] §6.5 Joyce and Song argued that this is not a good idea, as then $DT^\alpha(\tau)$ would not be unchanged under deformations of $X$. In [54] §6.5 Joyce and Song say:

‘Equation (1.7) was the inspiration for this book. It shows that Donaldson–Thomas invariants $DT^\alpha(\tau)$ can be written as motivic invariants, like those studied in [56]–[60], and so it raises the possibility that we can extend the results of [56]–[60] to Donaldson–Thomas invariants by including Behrend functions as weights.’

Almost closed 1-forms. In [102] Pandharipande and Thomas give a counterexample to the idea that every scheme admitting a symmetric obstruction theory can locally be written as the critical locus of a regular function on a smooth scheme. This limits the usefulness of the above formula for $\nu_X(x)$ in terms of the Milnor fibre. Here is the more general approach due to Behrend [4], which the author tried to use to give a strictly algebraic proof on the Behrend function identities, but later this proof turned out to be not completely correct.

Definition 1.10. Let $\mathbb{K}$ be an algebraically closed field, and $M$ a smooth $\mathbb{K}$-scheme. Let $\omega$ be an algebraic 1-form on $M$, that is, $\omega \in H^0(T^* M)$. Call $\omega$ almost closed if $d\omega$ is a section of $I_\omega \cdot \Lambda^2 T^* M$, where $I_\omega$ is the ideal sheaf of the zero locus $\omega^{-1}(0)$ of $\omega$. Equivalently, $d\omega|_{\omega^{-1}(0)}$ is zero as a section of $\Lambda^2 T^* M|_{\omega^{-1}(0)}$. In (étale) local coordinates $(z_1, \ldots, z_n)$ on $M$, if

$$\omega = f_1 dz_1 + \cdots + f_n dz_n,$$

then $\omega$ is almost closed provided

$$\frac{\partial f_k}{\partial z_j} \equiv \frac{\partial f_k}{\partial z_j} \mod (f_1, \ldots, f_n).$$
Let $M$ be a smooth Deligne–Mumford stack and $\omega$ an almost closed 1-form on $M$ with zero locus $X = Z(\omega)$. It is a general principle, that a section of a vector bundle defines a perfect obstruction theory for the zero locus of the section. This obstruction theory is given by

\[
\begin{array}{c}
\begin{pmatrix}
T_{M|_X} & d\omega^\vee \\
\omega^\vee & \Omega_{M|_X}
\end{pmatrix}
\end{array}
\xrightarrow{\begin{array}{l}
d \\
1
\end{array}}
\begin{array}{c}
\begin{pmatrix}
I/I^2 \\
\Omega_{M|_X}
\end{pmatrix}
\end{array}
\quad (1.8)
\]

This obstruction theory is symmetric, in a canonical way, because under the assumption that $\omega$ is almost closed one has that $d \circ \omega^\vee$ is self-dual, as a homomorphism of vector bundles over $X$.

Behrend [4, Prop. 3.14] proves a kind of converse of that, by a proof valid for general $K$, which says that, at least locally, every symmetric obstruction theory is given in this way by an almost closed 1-form.

**Proposition 1.11.** Let $K$ be an algebraically closed field, and $X$ a $K$-scheme with a symmetric obstruction theory. Then $X$ may be covered by Zariski open sets $Y \subseteq X$ such that there exists a smooth $K$-scheme $M$, an almost closed 1-form $\omega$ on $M$, and an isomorphism of $K$-schemes $Y \cong \omega^{-1}(0)$.

Restricting to $K = C$, Behrend [4, Prop. 4.22] gives an expression for the Behrend function of the zero locus of an almost closed 1-form as a *linking number*. It is possible to use it to give an algebraic proof of the first Behrend identity over $C$.

**Proposition 1.12.** Let $M$ be a smooth scheme and $\omega$ an almost closed 1-form on $M$, and let $Y = \omega^{-1}(0)$ be the scheme-theoretic zero locus of $\omega$. Fix $p$ a closed point in $Y$, choose étale coordinates $(x_1, \ldots, x_n)$ on $M$ around $p$ with $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ the associated canonical coordinates for $T^*M$. Write $\omega = \sum_{i=1}^n f_i dx_i$ in these coordinates. One can identify $T^*M$ near $p$ with $\mathbb{C}^{2n}$. Then for all $\eta \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$ with $0 < |\eta| < \epsilon < 1$ one has

\[
\nu_Y(p) = L_{S_\epsilon}(\Gamma_{\eta^{-1}\omega} \cap S_\epsilon \cap \Delta) \quad (1.9)
\]

where

- $S_\epsilon = \{(x_1, \ldots, x_n) \in \mathbb{C}^{2n} : |x_1|^2 + \cdots + |x_n|^2 = \epsilon^2\}$ is the sphere of radius $\epsilon$ in $\mathbb{C}^{2n}$,
- $\Gamma_{\eta^{-1}\omega}$ is the graph of $\eta^{-1}\omega$ regarded locally as a complex submanifold of $\mathbb{C}^{2n}$ of real dimension $2n$ oriented so that $M \rightarrow \Omega_M$ is orientation preserving and defined by the equations $\{\eta \eta_i = f_i(x)\}$,
- $\Delta = \{(x_1, \ldots, x_n) \in \mathbb{C}^{2n} : p_j = \bar{x}_j, j = 1, \ldots, n\}$, i.e. the image of the smooth map $M \rightarrow \Omega_M$ given by the section $\partial \eta$ of $\Omega_M$, with

\[
\varrho = \sum_{i} x_i \bar{x}_i + \sum_{i} p_i \bar{p}_i
\]

the square of the distance function defined on $\Omega_M$ by the choice of coordinates of real dimension $2n$,
- $L_{S_\epsilon}(\ ,\ )$ is the linking number of two disjoint, closed, oriented $(n-1)$-submanifolds in $S_\epsilon$.

We remark here that $\Delta$ is not a complex submanifold, but only a real submanifold. Thus, there are no good generalizations of $\Delta$ to other fields $K$.

### 1.3 Generalizations of Donaldson–Thomas theory

Next it will be briefly reviewed how the theory of generalized Donaldson–Thomas invariants has been developed, starting from the series of papers [54–60] about constructible functions, stack functions, Ringel–Hall algebras, counting invariants for Calabi–Yau 3-folds, and wall-crossing and then summarizing the main results in [64] including the definition of generalized Donaldson–Thomas invariants $DT^n(\tau) \in \mathbb{Q}$, their deformation-invariance, and wall-crossing formulae under change of stability condition $\tau$. In the sequel, there are two paragraphs on
statements and a sketch of proofs of the theorems [64, Thm 5.5] and [64, Thm 5.11] on which this paper is concentrated. We conclude with a brief and rough remark on Kontsevich and Soibelman’s parallel approach to Donaldson–Thomas theory [74], focusing more on analogies and differences with Joyce and Song’s construction [64] rather than going into a detailed exposition. This choice is due to the fact that for the present paper we do not need it.

### 1.3.1 Brief sketch of background from [54–60]

Here it will be recalled a few important ideas from [54–60]. They deal with Artin stacks rather than coarse moduli schemes, as in [121]. Let \( X \) be a Calabi–Yau 3-fold over \( \mathbb{C} \), and write \( \mathcal{M} \) for the moduli stack of all coherent sheaves \( E \) on \( X \). It is an Artin \( \mathbb{C} \)-stack.

The ring of stack functions \( \text{SF}(\mathcal{M}) \) in [55] is basically the Grothendieck group \( K_0(\text{St}_{\mathbb{A}}) \) of the 2-category \( \text{St}_{\mathbb{A}} \) of stacks over \( \mathcal{M} \). That is, \( \text{SF}(\mathcal{M}) \) is generated by isomorphism classes \( \left[ (\mathcal{M}, \rho) \right] \) of representable 1-morphisms \( \rho : \mathcal{R} \to \mathcal{M} \) for \( \mathcal{R} \) a finite type Artin \( \mathbb{C} \)-stack, with the relation

\[
\left[ (\mathcal{R}, \rho) \right] = \left[ (\mathcal{S}, \rho|_{\mathcal{S}}) \right] + \left[ (\mathcal{R} \setminus \mathcal{S}, \rho|_{\mathcal{R} \setminus \mathcal{S}}) \right]
\]

when \( \mathcal{S} \) is a closed \( \mathbb{C} \)-substack of \( \mathcal{R} \). In [55] Joyce studies different kinds of stack function spaces with other choices of generators and relations, and operations on these spaces. These include projections \( \Pi^i_n : \text{SF}(\mathcal{M}) \to \text{SF}(\mathcal{M}) \) to stack functions of virtual rank \( n \), which act on \( \left[ (\mathcal{M}, \rho) \right] \) by modifying \( \mathcal{R} \) depending on its stabilizer groups.

In [57, §5.2] he defines a Ringel–Hall type algebra \( \text{SF}_{\text{al}}(\mathcal{M}) \) of stack functions with algebra stabilizers on \( \mathcal{M} \), with an associative, non-commutative multiplication \( \ast \) and in [57, §5.2] he defines a Lie subalgebra \( \text{SF}_{\text{al}}^{\text{ind}}(\mathcal{M}) \) of stack functions supported on virtual indecomposables. In [57, §6.5] he defines an explicit Lie algebra \( L(X) \) to be the \( \mathbb{Q} \)-vector space with basis of symbols \( \lambda^\alpha \) for \( \alpha \in K^{\text{num}}(\text{coh}(X)) \), with Lie bracket

\[
[\lambda^\alpha, \lambda^\beta] = \chi(\alpha, \beta)\lambda^{\alpha+\beta},
\]

for \( \alpha, \beta \in K^{\text{num}}(\text{coh}(X)) \), where \( \chi(\cdot, \cdot) \) is the Euler form on \( K^{\text{num}}(\text{coh}(X)) \) defined as follows:

\[
\chi([E], [F]) = \sum_{i \geq 0} (-1)^i \dim \text{Ext}^i(E, F)
\]

for all \( E, F \in \text{coh}(X) \). As \( X \) is a Calabi–Yau 3-fold, \( \chi \) is antisymmetric, so (1.10) satisfies the Jacobi identity and makes \( L(X) \) into an infinite-dimensional Lie algebra over \( \mathbb{Q} \).

Then in [57, §6.6] Joyce defines a Lie algebra morphism \( \Psi : \text{SF}_{\text{al}}^{\text{ind}}(\mathcal{M}) \to L(X) \), which, roughly speaking, is of the form

\[
\Psi(f) = \sum_{\alpha \in K^{\text{num}}(\text{coh}(X))} \chi^{\text{st}}(f|\alpha^\alpha),
\]

where \( f = \sum_{i=1}^m c_i[(\mathcal{R}_i, \rho_i)] \) is a stack function on \( M \), and \( \mathcal{M}^\alpha \) is the substack in \( \mathcal{M} \) of sheaves \( E \) with class \( \alpha \), and \( \chi^{\text{st}} \) is a kind of stack-theoretic Euler characteristic. But in fact the definition of \( \Psi \), and the proof that \( \Psi \) is a Lie algebra morphism, are highly nontrivial, and use many ideas from [54, 55, 57], including those of ‘virtual rank’ and ‘virtual indecomposable’. The problem is that the obvious definition of \( \chi^{\text{st}} \) usually involves dividing by zero, so defining (1.12) in a way that makes sense is quite subtle. The proof that \( \Psi \) is a Lie algebra morphism uses Serre duality and the assumption that \( X \) is a Calabi–Yau 3-fold.

Now let \( \tau \) be a stability condition on \( \text{coh}(X) \), such as Gieseker stability. Then one has open, finite type substacks \( \mathcal{M}_n^\alpha(\tau), \mathcal{M}_n^{\alpha_1, \ldots, \alpha_n}(\tau) \) in \( \mathcal{M} \) of \( \tau \)-(semi)stable sheaves \( E \) in class \( \alpha \), for all \( \alpha \in K^{\text{num}}(\text{coh}(X)) \). Write \( \delta_n^\alpha(\tau) \) for the characteristic function of \( \mathcal{M}_n^\alpha(\tau) \), in the sense of stack functions [55]. Then \( \delta_n^\alpha(\tau) \in \text{SF}_{\text{al}}(\mathcal{M}) \). In [58, §8], Joyce defines elements \( e^\alpha(\tau) \) in \( \text{SF}_{\text{al}}(\mathcal{M}) \) by

\[
e^\alpha(\tau) = \sum_{n \geq 1, \alpha_1, \ldots, \alpha_n \in K^{\text{num}}(\text{coh}(X)) : \alpha_1 + \cdots + \alpha_n = \alpha, \tau(\alpha_i) = \tau(\alpha)} \frac{(-1)^{n-1}}{n} \delta_n^{\alpha_1}(\tau) \ast \delta_n^{\alpha_2}(\tau) \ast \cdots \ast \delta_n^{\alpha_n}(\tau),
\]
where \(*\) is the Ringel–Hall multiplication in \(\text{SF}_{\text{al}}(\mathcal{M})\). Then [58, Thm. 8.7] shows that \(\tilde{e}^\alpha(\tau)\) lies in the Lie subalgebra \(\text{SF}^{\text{ind}}_{\text{al}}(\mathcal{M})\), a nontrivial result. Thus one can apply the Lie algebra morphism \(\Psi\) to \(\tilde{e}^\alpha(\tau)\). In [59, §6.6] he defines invariants \(J^\alpha(\tau) \in \mathbb{Q}\) for all \(\alpha \in K^{\text{num}}(\text{coh}(X))\) by
\[
\Psi(\tilde{e}^\alpha(\tau)) = J^\alpha(\tau) \lambda^\alpha.
\]

These \(J^\alpha(\tau)\) are rational numbers ‘counting’ \(\tau\)-semistable sheaves \(E\) in class \(\alpha\). When \(\mathcal{M}^\alpha_{\text{ss}}(\tau) = \mathcal{M}^\alpha_{\text{ss}}(\tau)\) then \(J^\alpha(\tau) = \chi(\mathcal{M}^\alpha_{\text{ss}}(\tau))\), that is, \(J^\alpha(\tau)\) is the naïve Euler characteristic of the moduli space \(\mathcal{M}^\alpha_{\text{ss}}(\tau)\). This is \textit{not} weighted by the Behrend function \(\nu_{\text{M}^\alpha_{\text{ss}}(\tau)}\), and so in general does not coincide with the Donaldson–Thomas invariant \(DT^\alpha(\tau)\) in (1.10). As the \(J^\alpha(\tau)\) do not include Behrend functions, they do not count semistable sheaves with multiplicity, and so they will not in general be unchanged under deformations of the underlying Calabi–Yau 3-fold, as Donaldson– Thomas invariants are. However, the \(J^\alpha(\tau)\) do have very good properties under change of stability condition. In [59] Joyce shows that if \(\tau, \tilde{\tau}\) are two stability conditions on \(\text{coh}(X)\), then it is possible to write \(\tilde{e}^\alpha(\tilde{\tau})\) in terms of a (complicated) explicit formula involving the \(\tilde{e}^\beta(\tau)\) for \(\beta \in K^{\text{num}}(\text{coh}(X))\) and the Lie bracket in \(\text{SF}^{\text{ind}}_{\text{al}}(\mathcal{M})\). Applying the Lie algebra morphism \(\Psi\) shows that \(J^\alpha(\tilde{\tau})\lambda^\alpha\) may be written in terms of the \(J^\beta(\tau)\lambda^\beta\) and the Lie bracket in \(L(X)\), and hence [59, Thm. 6.28] yields an explicit transformation law for the \(J^\alpha(\tau)\) under change of stability condition. In [60] he shows how to encode invariants \(J^\alpha(\tau)\) satisfying a transformation law in generating functions on a complex manifold of stability conditions, which are both holomorphic and continuous, despite the discontinuous wall-crossing behaviour of the \(J^\alpha(\tau)\).

1.3.2 Summary of the main results from [64]

The basic idea behind the project developed in [64] is that the Behrend function \(\nu_{\mathcal{M}}\) of the moduli stack \(\mathcal{M}\) of coherent sheaves in \(X\) should be inserted as a weight in the programme of [54–60] summarized in [1.3.1]. Thus one will obtain weighted versions \(\tilde{\Psi}\) of the Lie algebra morphism \(\Psi\) of (1.12), and \(DT^\alpha(\tau)\) of the counting invariant \(J^\alpha(\tau) \in \mathbb{Q}\) in (1.14). Here is how this is worked out in [64].

Joyce and Song define a modification \(\tilde{L}(X)\) of the Lie algebra \(L(X)\) above, the \(\mathbb{Q}\)-vector space with basis of symbols \(\tilde{\lambda}^\alpha\) for \(\alpha \in K^{\text{num}}(\text{coh}(X))\), with Lie bracket
\[
[\tilde{\lambda}^\alpha, \tilde{\lambda}^\beta] = (-1)^{\langle \alpha, \beta \rangle} \tilde{\chi}(\alpha, \beta) \tilde{\lambda}^{\alpha+\beta},
\]
which is (1.12) with a sign change. Then they define a \textit{Lie algebra morphism} \(\tilde{\Psi} : \text{SF}^{\text{ind}}_{\text{al}}(\mathcal{M}) \to \tilde{L}(X)\). Roughly speaking this is of the form
\[
\tilde{\Psi}(f) = \sum_{\alpha \in K^{\text{num}}(\text{coh}(X))} \lambda^{\text{stk}}(f_{|\mathcal{M}^\alpha_{\text{al}}}, \nu_{\mathcal{M}^\alpha_{\text{al}}}) \tilde{\lambda}^\alpha,
\]
that is, in (1.12) we replace the stack-theoretic Euler characteristic \(\chi^{\text{stk}}\) with a stack-theoretic Euler characteristic weighted by the Behrend function \(\nu_{\mathcal{M}}\). The proof that \(\tilde{\Psi}\) is a Lie algebra morphism combines the proof in [57] that \(\Psi\) is a Lie algebra morphism with the two \textit{Behrend function identities} (1.16)–(1.17) proved in [64, thm. 5.11] and reported below. Proving (1.16)–(1.17) requires a deep understanding of the local structure of the moduli stack \(\mathcal{M}\), which is of interest in itself. First they show using a composition of \textit{Seidel–Thomas twists} by \(O_X(-n)\) for \(n \gg 0\) that \(\mathcal{M}\) is locally 1-isomorphic to the moduli stack \(\text{Vec}_{\mathbb{C}}\) of vector bundles on \(X\). Then they prove that near \([E] \in \text{Vec}_{\mathbb{C}}(\mathbb{C})\), an atlas for \(\text{Vec}_{\mathbb{C}}\) can be written locally in the complex analytic topology in the form \(\text{Crit} (f)\) for \(f : U \to \mathbb{C}\) a holomorphic function on an open set \(U\) in \(\text{Ext}^1(E, E)\). These \(U, f\) are \textit{not algebraic}, they are constructed using gauge theory on the complex vector bundle \(E\) over \(X\) and transcendental methods. Finally, they deduce (1.16)–(1.17) using the Milnor fibre expression (1.5) for Behrend functions applied to these \(U, f\).

Before going on with the review of Joyce and Song’s program, it is worth to stop for a while on some details about [64, Thm. 5.5] and [64, Thm. 5.11], the statements of the theorems and how they prove it. This will be useful later on in §?.

\textbf{Gauge theory and transcendental complex analytic geometry from [64].} In [64, Thm. 5.5] Joyce and Song give a local characterization of an atlas for the moduli stack \(\mathcal{M}\) as the critical points of a holomorphic function on a complex manifold. The statement and a sketch of its proof are reported below. Some background references are Kobayashi [73, §VII.3], Lübke and Teleman [87, §4.1 & §4.3], Friedman and Morgan [29, §4.1–§4.2] and Miyajima [93].
Theorem 1.13. Let $X$ be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on $X$. Suppose $E$ is a coherent sheaf on $X$, so that $[E] \in \mathfrak{M}(\mathbb{C})$. Let $G$ be a maximal reductive subgroup in $\text{Aut}(E)$, and $G^c$ its complexification. Then $G^c$ is an algebraic $\mathbb{C}$-subgroup of $\text{Aut}(E)$, a maximal reductive subgroup, and $G^c = \text{Aut}(E)$ if and only if $\text{Aut}(E)$ is reductive. There exists a quasiprojective $\mathbb{C}$-scheme $S$, an action of $G^c$ on $S$, a point $s \in S(\mathbb{C})$ fixed by $G^c$, and a $1$-morphism of Artin $\mathbb{C}$-stacks $\Phi : [S/G^c] \to \mathfrak{M}$, which is smooth of relative dimension $\dim \text{Aut}(E) - \dim G^c$, where $[S/G^c]$ is the quotient stack, such that $\Phi(sG^c) = [E]$, the induced morphism on stabilizer groups $\Phi_* : \text{Iso}_{[S/G^c]}(sG^c) \to \text{Iso}_{\mathfrak{M}}([E])$ is the natural morphism $G^c \hookrightarrow \text{Aut}(E) \cong \text{Iso}_{\mathfrak{M}}([E])$, and $\Phi|_{sG^c} : T_s S \cong T_s G^c[S/G^c] \to T_{[E]\mathfrak{M}} \cong \text{Ext}^1(E,E)$ is an isomorphism. Furthermore, $S$ parametrizes a formally versal family $(S,D)$ of coherent sheaves on $X$, equivariant under the action of $G^c$ on $S$, with fibre $D_s \cong E$ at $s$. If $\text{Aut}(E)$ is reductive then $\Phi$ is étale.

Write $S_{\text{an}}$ for the complex analytic space underlying the $\mathbb{C}$-scheme $S$. Then there exists an open neighbourhood $U$ of $0$ in $\text{Ext}^1(E,E)$ in the analytic topology, a holomorphic function $f : U \to \mathbb{C}$ with $f(0) = \partial f|_0 = 0$, an open neighbourhood $V$ of $s$ in $S_{\text{an}}$, and an isomorphism of complex analytic spaces $\Xi : \text{Crit}(f) \to V$, such that $\Xi(0) = s$ and $d\Xi|_0 : T_0 \text{Crit}(f) \to T_s V$ is the inverse of $d\Phi|_{sG^c} : T_s S \to \text{Ext}^1(E,E)$. Moreover we can choose $U, f, V$ to be $G^c$-invariant, and $\Xi$ to be $G^c$-equivariant.

In [64, Thm. 1.13] gives Joyce and Song the possibility to use the Milnor fibre formula (1.5) for the Behrend function of $\text{Crit}(f)$ to study the Behrend function $\nu_{\mathfrak{M}}$, crucially used in proving Behrend identities. The proof of Theorem 1.13 comes in two parts. First it is shown in [63, §8] that $\mathfrak{M}$ near $[E]$ is locally isomorphic, as an Artin $\mathbb{C}$-stack, to the moduli stack $\mathfrak{ Vect}$ of algebraic vector bundles on $X$ near $[E']$ for some vector bundle $E' \to X$. The proof uses algebraic geometry, and is valid for $X$ a Calabi–Yau $m$-fold for any $m > 0$ over any algebraically closed field $\mathbb{K}$. The local morphism $\mathfrak{M} \to \mathfrak{ Vect}$ is the composition of shifts and $m$ Seidel–Thomas twists by $\mathcal{O}_X(-n)$ for $n > 0$. Thus it is enough to prove Theorem 1.14 with $\mathfrak{ Vect}$ in place of $\mathfrak{M}$. This is done in [64, §9] using gauge theory on vector bundles over $X$. An interesting motivation for this approach could be found in [23, §3] and [121, §2]. Let $E \to X$ be a fixed complex (not holomorphic) vector bundle over $X$. Write $\mathfrak{A}$ for the finite-dimensional affine space of smooth semi-invariants ($\partial$-operators) on $E$, and $\mathfrak{G}$ for the finite-dimensional Lie group of \textit{smooth} gauge transformations of $E$. Then $\mathfrak{G}$ acts on $\mathfrak{A}$, and $\mathfrak{B} = \mathfrak{A} / \mathfrak{G}$ is the space of gauge-equivalence classes of semi-invariants on $E$. Fix $\partial_E$ in $\mathfrak{A}$ coming from a holomorphic vector bundle structure on $E$. Then points in $\mathfrak{A}$ are of the form $\partial_E + A$ for $A \in C^\infty(\mathcal{E}nd(E) \otimes \mathcal{A}^0, T^*X)$, and $\partial_E + A$ makes $E$ into a holomorphic vector bundle if $F^{0,2}_A = \partial_E A + A \wedge A$ is zero in $C^\infty(\mathcal{E}nd(E) \otimes \mathcal{A}^0, T^*X)$. Thus, the moduli space (stack) of holomorphic vector bundle structures on $E$ is isomorphic to $\{ \partial_E + A \in \mathfrak{A} : F^{0,2}_A = 0 \} / \mathfrak{G}$. In [121], it is observed that when $X$ is a Calabi–Yau 3-fold, there is a natural holomorphic function $CS : \mathfrak{A} \to \mathbb{C}$ called the \textit{holomorphic Chern–Simons functional}, invariant under $\mathfrak{G}$ up to addition of constants, such that $\{ \partial_E + A \in \mathfrak{A} : F^{0,2}_A = 0 \}$ is the critical locus of $CS$. Thus, $\mathfrak{Vect}$ is (informally) locally the critical points of a holomorphic function $CS$ on an infinite-dimensional complex stack $\mathfrak{B} = \mathfrak{A} / \mathfrak{G}$. To prove Theorem 1.13 Joyce and Song show that one can find a finite-dimensional complex submanifold $U$ in $\mathfrak{A}$ and a finite-dimensional complex Lie subgroup $G^c$ in $\mathfrak{G}$ preserving $U$ such that the theorem holds with $f = CS|_U$. These $U, f$ are not algebraic, they are constructed using gauge theory on the complex vector bundle $E$ over $X$ and transcendent methods.

The Behrend function identities from [64]. In [64, Thm. 5.11] Behrend function identities are proven: they are the crucial step to define the Lie algebra morphism $\hat{\Phi}$ below and then the generalized Donaldson–Thomas invariants:

Theorem 1.14. Let $X$ be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on $X$. The Behrend function $\nu_{\mathfrak{M}} : \mathfrak{M}(\mathbb{C}) \to \mathbb{Z}$ is a natural locally constructible function on $\mathfrak{M}$. For all $E_1, E_2 \in \text{coh}(X)$, it satisfies:

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{(n_1)(n_2)} \nu_{\mathfrak{M}}(E_1) \nu_{\mathfrak{M}}(E_2),$$

$$\int_{\{\lambda \in \mathfrak{P}(\text{Ext}^1(E_1, E_2)) : \lambda \leftrightarrow 0 \to E_1 \to F \to E_2 \to 0\}} \nu_{\mathfrak{M}}(F) \, d\chi - \int_{\{\mu \in \mathfrak{P}(\text{Ext}^1(E_1, E_2)) : \mu \leftrightarrow 0 \to E_2 \to D \to E_1 \to 0\}} \nu_{\mathfrak{M}}(D) \, d\chi = (e_{12} - e_{11}) \nu_{\mathfrak{M}}(E_1 \oplus E_2),$$
For $\alpha$ similar to Pandharipande–Thomas invariants \cite{102}. Let $\epsilon$ is the Euler form as in \[1.11\] the correspondence between $|\lambda| \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$ and $F \in \text{coh}(X)$ is that $|\lambda| \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$ lifts to some $0 \neq \lambda \in \text{Ext}^1(E_2, E_1)$, which corresponds to a short exact sequence $0 \to E_1 \to F \to E_2 \to 0$ in $\text{coh}(X)$ in the usual way. The function $|\lambda| \mapsto \nu_{\mathfrak{M}}(F)$ is a constructible function $\mathbb{P}(\text{Ext}^1(E_2, E_1)) \to \mathbb{Z}$, and the integrals in \[1.17\] are integrals of constructible functions using the Euler characteristic as measure.

Joyce and Song prove Theorem \[1.14\] using Theorem \[1.13\] and the Milnor fibre description of Behrend functions from \[1.3\]. They apply Theorem \[1.13\] to $E = E_1 \oplus E_2$, and take the maximal reductive subgroup $G$ of $\text{Aut}(E)$ to contain the subgroup $\{\text{id}_{E_1} + \lambda \text{id}_{E_2} : \lambda \in U(1)\}$, so that $G^c$ contains $\{\text{id}_{E_1} + \lambda \text{id}_{E_2} : \lambda \in \mathbb{G}_m\}$. Equations \[1.16\] and \[1.17\] are proved by a kind of localization using this $\mathbb{G}_m$-action on $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$. More precisely, Theorem \[1.13\] gives an atlas for $\mathfrak{M}$ near $E$ as Crit($f$) near 0, where $f$ is a holomorphic function defined near 0 on $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ and $f$ is invariant under the action of $T = \{\text{id}_{E_1} + \lambda \text{id}_{E_2} : \lambda \in U(1)\}$ on $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ by conjugation. The fixed points of $T$ on $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ are $\text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2)$ and heuristically one can says that the restriction of $f$ to these fixed points is $f_1 + f_2$, where $f_j$ is defined near 0 in $\text{Ext}^1(E_j, E_j)$ and Crit($f_j$) is an atlas for $\mathfrak{M}$ near $E_j$. The Milnor fibre $MF_f(0)$ is invariant under $T$, so by localization one has

$$\chi(MF_f(0)) = \chi(MF_f(0)^T) = \chi(MF_{f_1 + f_2}(0)).$$

A product property of Behrend functions, which may be seen as a kind of Thom-Sebastiani theorem, gives

$$1 - \chi(MF_{f_1 + f_2}(0)) = (1 - \chi(MF_{f_1}(0)))(1 - \chi(MF_{f_2}(0))).$$

Then the identity \[1.10\] follows from Theorem \[1.3\]

$$\nu_{\mathfrak{M}}(E) = (-1)^{\dim \text{Ext}^1(E, E) - \dim \text{Hom}(E, E)}(1 - \chi(MF_f(0))),$$

and the analogues for $E_1$ and $E_2$. Equation \[1.17\] uses a more involved argument to do with the Milnor fibres of $f$ at non-fixed points of the $U(1)$-action. The proof of Theorem \[1.14\] uses gauge theory, and transcendental complex analytic geometry methods, and is valid only over $\mathbb{K} = \mathbb{C}$. However, as pointed out in \[64\] Question 5.12, Theorem \[1.14\] makes sense as a statement in algebraic geometry, for Calabi–Yau 3-folds over $\mathbb{K}$.

In \[64\] §5, Joyce and Song then define generalized Donaldson–Thomas invariants $DT^\alpha(\tau) \in \mathbb{Q}$ by

$$\hat{\Psi}(e^\alpha(\tau)) = -DT^\alpha(\tau)\lambda^\alpha,$$

as in \[1.14\]. When $\mathcal{M}_\text{st}^\alpha(\tau) = \mathcal{M}_\text{st}^\alpha(\tau)$ then $e^\alpha(\tau) = \delta_\text{st}^\alpha(\tau)$, and \[1.15\] gives

$$\hat{\Psi}(e^\alpha(\tau)) = \chi_{\text{st}}(\mathcal{M}_\text{st}^\alpha(\tau), \nu_{\mathfrak{M}_\text{st}^\alpha(\tau)})\lambda^\alpha.$$ \hfill (1.19)

The projection $\pi : \mathfrak{M}_\text{st}^\alpha(\tau) \to \mathcal{M}_\text{st}^\alpha(\tau)$ from the moduli stack to the coarse moduli scheme is smooth of dimension $-1$, so $\nu_{\mathfrak{M}_\text{st}^\alpha(\tau)} = -\pi^*(\nu_{\mathcal{M}_\text{st}^\alpha(\tau)})$ by (ii) in \[1.2.2\] and comparing \[1.7\], \[1.18\]. \[1.19\] shows that $DT^\alpha(\tau) = DT^\alpha(\tau)$. But the new invariants $DT^\alpha(\tau)$ are also defined for $\alpha$ with $\mathcal{M}_\text{st}^\alpha(\tau) \neq \mathcal{M}_\text{st}^\alpha(\tau)$, when conventional Donaldson–Thomas invariants $DT^\alpha(\tau)$ are not defined.

Thanks to Theorem \[1.13\] and Theorem \[1.14\] $\hat{\Psi}$ is a Lie algebra morphism \[64\] §5.3, thus the change of stability condition formula for the $e^\alpha(\tau)$ in \[39\] implies a formula for the elements $-DT^\alpha(\tau)\lambda^\alpha$ in $L(X)$, and thus a transformation law for the invariants $DT^\alpha(\tau)$, using combinatorial coefficients.

To study the new invariants $DT^\alpha(\tau)$, it is helpful to introduce another family of invariants $P^\alpha,n(\tau')$, similar to Pandharipande–Thomas invariants \[102\]. Let $n \gg 0$ be fixed. A stable pair is a nonzero morphism $s : \mathcal{O}_X(-n) \to E$ in $\text{coh}(X)$ such that $E$ is $s$-semistable, and if Im $s \subset E' \subset E$ with $E' \neq E$ then $\tau([E']) < \tau([E])$. For $\alpha \in K_{\text{num}}(\text{coh}(X))$ and $n \gg 0$, the moduli space $\mathcal{M}_\text{st,n}^\alpha(\tau')$ of stable pairs $s : \mathcal{O}_X(-n) \to X$ with $|E| = \alpha$ is a fine moduli scheme, which is proper and has a symmetric obstruction theory. Joyce and Song define

$$P^\alpha,n(\tau') = \int_{|\mathcal{M}_\text{st,n}^\alpha(\tau')|_{\text{vir}}} 1 = \chi(\mathcal{M}_\text{st,n}^\alpha(\tau'), \nu_{\mathcal{M}_\text{st,n}^\alpha(\tau')}) \in \mathbb{Z},$$ \hfill (1.20)
where the second equality follows from Theorem 1.9. By a similar proof to that for Donaldson–Thomas invariants in [57], Joyce and Song find that $P_I^{\alpha,n}(\tau')$ is unchanged under deformations of the underlying Calabi–Yau 3-fold $X$. By a wall-crossing proof similar to that for $DT^\alpha(\tau)$, they show that $P_I^{\alpha,n}(\tau')$ can be written in terms of the $DT^\beta(\tau)$. As $P_I^{\alpha,n}(\tau')$ is deformation-invariant, one deduces from this relation by induction on rank $\alpha$ with $\dim \alpha$ fixed that $DT^\alpha(\tau)$ is also deformation-invariant.

The pair invariants $P_I^{\alpha,n}(\tau')$ are a useful tool for computing the $DT^\alpha(\tau)$ in examples in [64] §6. The method is to describe the moduli spaces $M^{\alpha,n}_{\text{stp}}(\tau')$ explicitly, and then use (1.20) to compute $P_I^{\alpha,n}(\tau')$, and their relation with $DT^\alpha(\tau)$ to deduce the values of $DT^\alpha(\tau)$. Their point of view is that the $DT^\alpha(\tau)$ are of primary interest, and the $P_I^{\alpha,n}(\tau')$ are secondary invariants, of less interest in themselves.

Motivic Donaldson–Thomas invariants: Kontsevich and Soibelman’s approach from [74]. Kontsevich and Soibelman in [74] also studied generalizations of Donaldson–Thomas invariants. They work in a more general context but their results are in great part based on conjectures. They consider derived categories of coherent sheaves, Bridgeland stability conditions [14], and general motivic invariants, whereas Joyce and Song work with abelian categories of coherent sheaves, Gieseker stability, and the Euler characteristic. Kontsevich and Soibelman’s motivic functions in the equivariant setting [74, §4.2], motivic Hall algebra [74, §6.1], motivic quantum torus [74] §6.2 and their algebra morphism to define Donaldson–Thomas invariants [74 Thm. 8] all have an analogue in Joyce and Song’s program.

It is worth to note here some points (see [64] §1.6 for the entire discussion).

(a) Joyce was probably the first to approach Donaldson–Thomas type invariants in an abstract categorical setting. He developed the technique of motivic stack functions and understood the relevance of motives to the counting problem [54–59]. The main limitation of his approach was due to the fact that he worked with abelian rather than triangulated categories. For many applications, especially to physics, one needs triangulated categories. The more recent theory of Joyce and Song [64] fixes some of these gaps and fits well with the general philosophy of [74] (and actually Joyce and Song use some ideas from Kontsevich and Soibelman). They deal with concrete examples of categories (e.g. the category of coherent sheaves) and construct numerical invariants via Behrend approach. It is difficult to prove that they are in fact invariants of triangulated categories which is manifest in [74].

(b) Kontsevich and Soibelman write their wall-crossing formulae in terms of products in a pro-nilpotent Lie group while Joyce and Song’s formulae are written in terms of combinatorial coefficients.

(c) Equations (1.16)–(1.17) are related to a conjecture of Kontsevich and Soibelman [74, Conj. 4] and its application in [74] §6.3, and could probably be deduced from it. Joyce and Song got the idea of proving (1.16)–(1.17) by localization using the $\mathbb{G}_m$-action on $\text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ from [74]. However, Kontsevich and Soibelman approach [74, Conj. 4] via formal power series and non-Archimedean geometry. Their analogue concerns the ‘motivic Milnor fibre’ of the formal power series $f$. Instead, in Theorem 1.13 Joyce and Song in effect first prove that they can choose the formal power series to be convergent, and then use ordinary differential geometry and Milnor fibres.

(d) While Joyce’s series of papers [54–59] develops the difficult idea of ‘virtual rank’ and ‘virtual indecomposables’, Kontsevich and Soibelman have no analogue of these. They come up against the problem (specialization from virtual Poincaré polynomial to Euler characteristic) this technology was designed to solve in the ‘absence of poles conjecture’ [74] §7.

Section 5 proposes new ideas for further research also in the direction of Kontsevich and Soibelman’s paper [74].

2 D-critical loci

We summarizes the theory of d-critical schemes and stacks introduced by Joyce [55]. There are two versions of the theory, complex analytic and algebraic d-critical loci, sometimes we give results for both the versions simultaneously, otherwise just briefly indicate the differences between the two, referring to [55] for details.
2.1 D-critical schemes

Let $X$ be a complex analytic space or a $K$-scheme. Then \[55\] Th. 2.1 & Prop. 2.3 associates a natural sheaf $S_X$ to $X$, such that, very briefly, sections of $S_X$ parametrize different ways of writing $X$ as $\text{Crit}(f)$ for $U$ a complex manifold or smooth $K$-scheme and $f : U \to \mathbb{C}$ holomorphic or $f : U \to \mathbb{A}^1$ regular. We refer to \[55\] Th. 2.1 & Prop. 2.3 for details. Just to give a bit more clear idea, we point out the following:

Remark 2.1. Suppose we have $U$ a complex manifold, $f : U \to \mathbb{C}$ an holomorphic, and $X = \text{Crit}(f)$, as a closed complex analytic subspace of $U$. Write $i : X \to U$ for the inclusion, and $I_{X,U} \subseteq i^{-1}(O_U)$ for the sheaf of ideals vanishing on $X \subseteq U$. Then we obtain a natural section $s \in H^0(S_X)$. Essentially $s = f + I_{X,U}$, where $I_{X,U} \subseteq O_U$ is the ideal generated by $df$. Note that $f|_X = f + I_{X,U}$, so $s$ determines $f|_X$. Basically, $s$ remembers all of the information about $f$ which makes sense intrinsically on $X$, rather than on the ambient space $U$.

Following \[55\] Def. 2.5 we define algebraic d-critical loci:

Definition 2.2. An (algebraic) d-critical locus over a field $K$ is a pair $(X,s)$, where $X$ is a $K$-scheme and $s \in H^0(S^0_X)$, such that for each $x \in X$, there exists a Zariski open neighbourhood $R$ of $x$, a smooth $K$-scheme $U$, a regular function $f : U \to \mathbb{A}^1 = K$, and a closed embedding $i : R \to U$, such that $i(R) = \text{Crit}(f)$ as $K$-subschemas of $U$, and $i|_{R,U}(s|_R) = i^{-1}(f) + i_{R,U}^*$. We call the quadruple $(R, U, f, i)$ a critical chart on $(X,s)$. If $U' \subseteq U$ is a Zariski open, and $R' = i^{-1}(U') \subseteq U$, $i' = i|_{R'} : R' \to U'$, and $f' = f|_{U'}$, then $(R', U', f', i')$ is a critical chart on $(X,s)$, and we call it a subchart of $(R, U, f, i)$, and we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let $(R, U, f, i), (S, V, g, j)$ be critical charts on $(X,s)$, with $R \subseteq S \subseteq X$. An embedding of $(R, U, f, i)$ in $(S, V, g, j)$ is a locally closed embedding $\Phi : U \to V$ such that $\Phi \circ i = j|_R$ and $f = g \circ \Phi$. As a shorthand we write $\Phi : (R, U, f, i) \to (S, V, g, j)$. If $\Phi : (R, U, f, i) \to (S, V, g, j)$ and $\Psi : (S, V, g, j) \to (T, W, h, k)$ are embeddings, then $\Psi \circ \Phi : (R, U, i, e) \to (T, W, h, k)$ is also an embedding.

A morphism $\phi : (X,s) \to (Y,t)$ of d-critical loci $(X,s),(Y,t)$ is a $K$-scheme morphism $\phi : X \to Y$ with $\phi^*(t) = s$. This makes d-critical loci into a category.

Remark 2.3. (a) For $(X,s)$ to be a (complex analytic or algebraic) d-critical locus places strong local restrictions on the singularities of $X$. For example, Behrend \[4\] notes that if $X$ has reduced local complete intersection singularities then locally it cannot be the zeroes of an almost closed 1-form on a smooth space, and hence not locally a critical locus, and Pandharipande and Thomas \[102\] give examples which are zeroes of almost closed 1-forms, but are not locally critical loci.

(b) If $X = \text{Crit}(f)$ for holomorphic $f : U \to \mathbb{C}$, then $f|_{X,\red}$ is locally constant, and we can write $f = f^0 + c$ uniquely near $X$ in $U$ for $f^0 : U \to \mathbb{C}$ holomorphic with $\text{Crit}(f^0) = X = \text{Crit}(f)$, $f^0|_{X,\red} = 0$, and $c : U \to \mathbb{C}$ locally constant with $c|_{X,\red} = f|_{X,\red}$. Defining d-critical loci using $s \in H^0(S^0_X)$ corresponds to remembering only the function $f^0$ near $X$ in $U$, and forgetting the locally constant function $f|_{X,\red} : X,\red \to \mathbb{C}$.

(c) In \[55\] ex. 2.16, Joyce shows a case in which the algebraic d-critical locus remembers more information, locally, than the symmetric obstruction theory. In \[55\] ex. 2.17, Joyce shows that the (symmetric) obstruction theory remembers global, non-local information which is forgotten by the algebraic d-critical locus.

(e) One could think about critical charts as Kuranishi neighbourhoods on a topological space, and embeddings as analogous to coordinate changes between Kuranishi neighbourhoods.

Here are \[55\] Prop.s 2.8, 2.30, Th.s 2.20, 2.28, Def. 2.31, Rem 2.32 & Cor. 2.33]:

Proposition 2.4. Let $\phi : X \to Y$ be a smooth morphism of $K$-schemes. Suppose $t \in H^0(S^0_Y)$, and set $s := \phi^*(t) \in H^0(S^0_X)$. If $(Y,t)$ is a d-critical locus, then $(X,s)$ is a d-critical locus, and $\phi : (X,s) \to (Y,t)$ is a morphism of d-critical loci. Conversely, if also $\phi : X \to Y$ is surjective, then $(X,s)$ a d-critical locus implies $(Y,t)$ is a d-critical locus.

Theorem 2.5. Suppose $(X,s)$ is an algebraic d-critical locus, and $(R, U, f, i), (S, V, g, j)$ are critical charts on $(X,s)$. Then for each $x \in R \cap S \subseteq X$ there exist subcharts $(R', U', f', i') \subseteq (R, U, f, i),(S', V', g', j') \subseteq (S, V, g, j)$ with $x \in R' \cap S' \subseteq X$, a critical chart $(T, W, h, k)$ on $(S', V', g', j')$, and embeddings $\Phi : (R', U', f', i') \to (T, W, h, k)$, $\Psi : (S', V', g', j') \to (T, W, h, k)$.
Theorem 2.6. Let \((X, s)\) be an algebraic d-critical locus, and \(X^\text{red} \subseteq X\) the associated reduced \(K\)-subscheme. Then there exists a line bundle \(K_{X, s}\) on \(X^\text{red}\) which we call the \textit{canonical bundle} of \((X, s)\), which is natural up to canonical isomorphism, and is characterized by the following properties:

(a) For each \(x \in X^\text{red}\), there is a canonical isomorphism
\[
\kappa_x : K_{X, s}|_x \xrightarrow{\cong} (\Lambda^{\text{top}}T^*_x X)^\otimes 2,
\] (2.1)
where \(T_x X\) is the Zariski tangent space of \(X\) at \(x\).

(b) If \((R, U, f, i)\) is a critical chart on \((X, s)\), there is a natural isomorphism
\[
\iota_{R, U, f, i} : K_{X, s}|_{R^\text{red}} \xrightarrow{\cong} i^*(K_U^\otimes 2)|_{R^\text{red}},
\] (2.2)
where \(K_U = \Lambda^{\dim U} T^* U\) is the canonical bundle of \(U\) in the usual sense.

(c) In the situation of (b), let \(x \in R\). Then we have an exact sequence
\[
0 \longrightarrow T_x X \xrightarrow{\text{dil}_x} T_{i(x)} U \xrightarrow{\text{Hess}_{i(x)} f} T^*_x U \xrightarrow{\text{dil}^*_x} T^*_x X \longrightarrow 0, \tag{2.3}
\]
and the following diagram commutes:
\[
\begin{array}{ccc}
K_{X, s}|_x & \xrightarrow{\kappa_x} & (\Lambda^{\text{top}}T^*_x X)^\otimes 2 \\
\downarrow{\iota_{R, U, f, i}|_x} & & \downarrow{\alpha_{x, R, U, f, i}} \\
K_U|_{i(x)}^\otimes 2,
\end{array}
\]
where \(\alpha_{x, R, U, f, i}\) is induced by taking top exterior powers in (2.3).

Proposition 2.7. Suppose \(\phi : (X, s) \to (Y, t)\) is a morphism of d-critical loci with \(\phi : X \to Y\) smooth, as in Proposition 2.4. The \textit{relative cotangent bundle} \(T^*_X/Y\) is a vector bundle of mixed rank on \(X\) in the exact sequence of coherent sheaves on \(X\):
\[
0 \longrightarrow \phi^*(T^* Y) \xrightarrow{\text{d}\phi^*} T^* X \xrightarrow{\text{dil}^*} T^*_X/Y \longrightarrow 0. \tag{2.4}
\]

There is a natural isomorphism of line bundles on \(X^\text{red}\):
\[
T_{\phi} : \phi^*|_{X^\text{red}}(K_{Y, t}) \otimes (\Lambda^{\text{top}} T^*_X/Y)|_{X^\text{red}} \xrightarrow{\cong} K_{X, s}, \tag{2.5}
\]
such that for each \(x \in X^\text{red}\) the following diagram of isomorphisms commutes:
\[
\begin{array}{ccc}
K_{Y, t}|_{\phi(x)} \otimes (\Lambda^{\text{top}} T^*_X/Y|_x)^\otimes 2 & \xrightarrow{\gamma_{\phi(x)}|_x} & K_{X, s}|_x \\
\downarrow{\kappa_{\phi(x)}\otimes \text{id}} & & \downarrow{\kappa_x} \\
(\Lambda^{\text{top}} T^*_{\phi(x)} Y)^\otimes 2 \otimes (\Lambda^{\text{top}} T^*_X/Y|_x)^\otimes 2 & \xrightarrow{\upsilon_{\phi(x)}^\otimes} & (\Lambda^{\text{top}} T^*_x X)^\otimes 2,
\end{array}
\] (2.6)

where \(\kappa_{\phi, x}\) and \(\upsilon_x\) are as in (2.1), and \(\upsilon_x : \Lambda^{\text{top}} T^*_{\phi(x)} Y \otimes \Lambda^{\text{top}} T^*_X/Y|_x \to \Lambda^{\text{top}} T^*_x X\) is obtained by restricting (2.4) to \(x\) and taking top exterior powers.

Definition 2.8. Let \((X, s)\) be an algebraic d-critical locus, and \(K_{X, s}\) its canonical bundle from Theorem 2.6.

An \textit{orientation} on \((X, s)\) is a choice of square root line bundle \(K_{X, s}^{1/2}\) for \(K_{X, s}\) on \(X^\text{red}\). That is, an orientation is a line bundle \(L\) on \(X^\text{red}\) together with an isomorphism \(L^\otimes 2 = L \otimes L \cong K_{X, s}\). A d-critical locus with an orientation will be called an \textit{oriented d-critical locus}.
**Remark 2.9.** In view of equation (2.1), one might hope to define a canonical orientation $K^{1/2}_{X,s}$ for a d-critical locus $(X,s)$ by $K^{1/2}_{X,s} = \Lambda^\text{top} T^*_x X$ for $x \in X^{\text{red}}$. However, this does not work, as the spaces $\Lambda^\text{top} T^*_x X$ do not vary continuously with $x \in X^{\text{red}}$ if $X$ is not smooth. An example in [55, Ex. 2.39] shows that d-critical loci need not admit orientations.

In the situation of Proposition 2.7, the factor $(\Lambda^\text{top} T^*_{X/Y})|_{X^{\text{red}}}$ has a natural square root $(\Lambda^\text{top} T^*_{X/Y})|_{X^{\text{red}}}$. Thus we deduce:

**Corollary 2.10.** Let $\phi : (X, s) \to (Y, t)$ be a morphism of d-critical loci with $\phi : X \to Y$ smooth. Then each orientation $K^{1/2}_{Y,t}$ for $(Y, t)$ lifts to a natural orientation $K^{1/2}_{X,s} = \phi|_{X^{\text{red}}}(K^{1/2}_{Y,t}) \otimes (\Lambda^\text{top} T^*_{X/Y})|_{X^{\text{red}}}$ for $(X, s)$.

### 2.2 D-critical stacks

In [55, §2.7–§2.8] Joyce extends the material of [2.4] from $k$-schemes to Artin $k$-stacks. We work in the context of the theory of sheaves on Artin stacks by Laumon and Moret-Bailly [78].

**Proposition 2.11** (Laumon and Moret-Bailly [78]). Let $X$ be an Artin $k$-stack. The category of sheaves of sets on $X$ in the lisse-étale topology is equivalent to the category $\text{Sh}(X)$ defined as follows:

**A** Objects $A$ of $\text{Sh}(X)$ comprise the following data:

(a) For each $k$-scheme $T$ and smooth 1-morphism $t : T \to X$ in $\text{Art}_k$, we are given a sheaf of sets $A(T, t)$ on $T$, in the étale topology.

(b) For each 2-commutative diagram in $\text{Art}_k$:

$$
\begin{array}{ccc}
\phi & \downarrow & u \\
T & \phi & \downarrow & u \\
& X,
\end{array}
$$

where $T, U$ are schemes and $t : T \to X$, $u : U \to X$ are smooth 1-morphisms in $\text{Art}_k$, we are given a morphism $A(\phi, \eta) : \phi^{-1}(A(U, u)) \to A(T, t)$ of étale sheaves of sets on $T$.

This data must satisfy the following conditions:

(i) If $\phi : T \to U$ in (b) is étale, then $A(\phi, \eta)$ is an isomorphism.

(ii) For each 2-commutative diagram in $\text{Art}_k$:

$$
\begin{array}{ccc}
\psi & \downarrow & v \\
U & \phi & \downarrow & u \\
& X,
\end{array}
$$

with $T, U, V$ schemes and $t, u, v$ smooth, we must have

$$
A(\psi \circ \phi, (\zeta \circ \eta) \circ \phi) = A(\phi, \eta) \circ \phi^{-1}(A(U, u)) \quad \text{as morphisms}
$$

$$(\psi \circ \phi)^{-1}(A(V, v)) = \phi^{-1} \circ \psi^{-1}(A(V, v)) \to A(T, t).
$$

**B** Morphisms $\alpha : A \to B$ of $\text{Sh}(X)$ comprise a morphism $\alpha(T, t) : A(T, t) \to B(T, t)$ of étale sheaves of sets on a scheme $T$ for all smooth 1-morphisms $t : T \to X$, such that for each diagram (2.7) in (b) the following commutes:

$$
\begin{array}{ccc}
\phi^{-1}(A(U, u)) & \xrightarrow{\alpha(T, t)} & A(T, t) \\
\phi^{-1}(\alpha(U, u)) & \otimes & \alpha(T, t) \\
\phi^{-1}(B(U, u)) & \xrightarrow{B(\phi, \eta)} & B(T, t).
\end{array}
$$
(C) Composition of morphisms \( A \to B \to C \) in \( \mathbf{Sh}(X) \) is \((\beta \circ \alpha)(T,t) = \beta(T,t) \circ \alpha(T,t)\). Identity morphisms \( \text{id}_A : A \to A \) are \( \text{id}_{\mathcal{A}(T,t)} \).

The analogue of all the above also holds for (étale) sheaves of \( \mathbb{K} \)-vector spaces, sheaves of \( \mathbb{K} \)-algebras, and so on, in place of (étale) sheaves of sets. Furthermore, the analogue of all the above holds for quasi-coherent sheaves, (or coherent sheaves, or vector bundles, or line bundles) on \( X \), where in (a) \( \mathcal{A}(T,t) \) becomes a quasi-coherent sheaf (or coherent sheaf, or vector bundle, or line bundle) on \( T \), in (b) we replace \( \phi^{-1}(\mathcal{A}(U,u)) \) by the pullback \( \phi^*(\mathcal{A}(U,u)) \) of quasi-coherent sheaves (etc.), and \( \mathcal{A}(\phi, \eta), \alpha(T,t) \) become morphisms of quasi-coherent sheaves (etc.) on \( T \).

We can also describe global sections of sheaves on Artin \( \mathbb{K} \)-stacks in the above framework: a global section \( s \in H^0(\mathcal{A}) \) of \( \mathcal{A} \) in part (A) assigns a global section \( s(T,t) \in H^0(\mathcal{A}(T,t)) \) of \( \mathcal{A}(T,t) \) on \( T \) for all smooth \( t : T \to X \) from a scheme \( T \), such that \( \mathcal{A}(\phi, \eta)^*(s(U,u)) = s(T,t) \) in \( H^0(\mathcal{A}(T,t)) \) for all 2-commutative diagrams \( (2.7) \) with \( t, u \) smooth.

In [55, Cor. 2.52] Joyce generalizes the sheaves \( S_X, S^0_X \) in \([2.1]\) to Artin \( \mathbb{K} \)-stacks:

**Proposition 2.12.** Let \( X \) be an Artin \( \mathbb{K} \)-stack, and write \( \mathbf{Sh}(X)_{\mathbb{K}\text{-alg}} \) and \( \mathbf{Sh}(X)_{\mathbb{K}\text{-vect}} \) for the categories of sheaves of \( \mathbb{K} \)-algebras and \( \mathbb{K} \)-vector spaces on \( X \) defined in Proposition \([2.11]\). Then:

(a) We may define canonical objects \( S_X \) in both \( \mathbf{Sh}(X)_{\mathbb{K}\text{-alg}} \) and \( \mathbf{Sh}(X)_{\mathbb{K}\text{-vect}} \) by \( S_X(T,t) := S_T \) for all smooth morphisms \( t : T \to X \) for \( T \in \mathbf{Sch}_\mathbb{K} \), for \( S_T \) as in \([2.1]\) taken to be a sheaf of \( \mathbb{K} \)-algebras (or \( \mathbb{K} \)-vector spaces) on \( T \) in the étale topology, and \( S_X(\phi, \eta) := \phi^* : \phi^{-1}(S_X(U,u)) = \phi^{-1}(S_U) \to S_T = S_X(T,t) \) for all 2-commutative diagrams \( (2.7) \) in \( \mathbf{Art}_\mathbb{K} \) with \( t, u \) smooth, where \( \phi^* \) is as in \([2.1]\).

(b) There is a natural decomposition \( S_X = \mathbb{K}_X \oplus S^0_X \) in \( \mathbf{Sh}(X)_{\mathbb{K}\text{-vect}} \) induced by the splitting \( S_X(T,t) = S_T = \mathbb{K}_T \oplus S^0_T \) in \([2.1]\) where \( \mathbb{K}_X \) is a sheaf of \( \mathbb{K} \)-subalgebras of \( S_X \) in \( \mathbf{Sh}(X)_{\mathbb{K}\text{-alg}} \), and \( S^0_X \) a sheaf of ideals in \( S_X \).

Here [55, Def. 2.53] is the generalization of Definition \([2.2]\) to Artin stacks.

**Definition 2.13.** A d-critical stack \( (X,s) \) is an Artin \( \mathbb{K} \)-stack \( X \) and a global section \( s \in H^0(S^0_X) \), where \( S^0_X \) is as in Proposition \([2.12]\) such that \( (T,s(T,t)) \) is an algebraic d-critical locus in the sense of Definition \([2.2]\) for all smooth morphisms \( t : T \to X \) with \( T \in \mathbf{Sch}_\mathbb{K} \).

Here is a convenient way to understand d-critical stacks \( (X,s) \) in terms of d-critical structures on an atlas \( t : T \to X \) for \( X \) from [55, Prop. 2.54].

**Proposition 2.14.** Let \( X \) be an Artin \( \mathbb{K} \)-stack, and \( t : T \to X \) a smooth atlas for \( X \). Then \( T \times_{t,X,t} T \) is equivalent to a \( \mathbb{K} \)-scheme \( U \) as \( t \) is representable and \( T \) a scheme, so we have a 2-Cartesian diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\pi_2} & T \\
\downarrow{\pi_1} & & \downarrow{\eta} \\
T & \xrightarrow{t} & X
\end{array}
\]

in \( \mathbf{Art}_\mathbb{K} \), with \( \pi_1, \pi_2 : U \to T \) smooth morphisms in \( \mathbf{Sch}_\mathbb{K} \). Also \( T, U, \pi_1, \pi_2 \) can be naturally completed to a smooth groupoid in \( \mathbf{Sch}_\mathbb{K} \), and \( X \) is equivalent in \( \mathbf{Art}_\mathbb{K} \) to the associated groupoid stack \([U \rightrightarrows T]\).

(i) Let \( S_X \) be as in Proposition \([2.12]\) and \( S_T, S_U \) be as in \([2.1]\) regarded as sheaves on \( T,U \) in the étale topology, and define \( \pi_i^* : \pi_i^{-1}(S_T) \to S_U \) as in \([2.1]\) for \( i = 1,2 \). Consider the map \( t^* : H^0(S_X) \to H^0(S_T) \) mapping \( t^* : s \mapsto s(T,t) \). This is injective, and induces a bijection

\[
t^* : H^0(S_X) \xrightarrow{\cong} \{ s' \in H^0(S_T) : \pi_1^*(s') = \pi_2^*(s') \text{ in } H^0(S_U) \}.
\]

The analogue holds for \( S^0_X, S^0_T, S^0_U \).

(ii) Suppose \( s \in H^0(S^0_X) \), so that \( t^*(s) \in H^0(S^0_T) \) with \( \pi_1^* \circ t^*(s) = \pi_2^* \circ t^*(s) \). Then \( (X,s) \) is a d-critical stack if and only if \((T,t^*(s)) \) is an algebraic d-critical locus, and then \((U, \pi_1^* \circ t^*(s)) \) is also an algebraic d-critical locus.
In [55 Ex. 2.55] we consider quotient stacks $X = [T/G]$.

**Example 2.15.** Suppose an algebraic $\mathbb{K}$-group $G$ acts on a $\mathbb{K}$-scheme $T$ with action $\mu : G \times T \rightarrow T$, and write $X$ for the quotient Artin $\mathbb{K}$-stack $[T/G]$. Then as in (2.8) there is a natural 2-Cartesian diagram

$$
\begin{array}{ccc}
G \times T & \xrightarrow{\mu} & T \\
\text{π}_T & \downarrow & \downarrow \text{π}_T' \\
T & \xrightarrow{t} & X = [T/G],
\end{array}
$$

where $t : T \rightarrow X$ is a smooth atlas for $X$. If $s' \in H^0(S^2_V)$ then $\pi_T^*(s') = \pi_T^*(s')$ in (2.9) becomes $\pi_T^*(s') = \mu^*(s')$ on $G \times T$, that is, $s'$ is $G$-invariant. Hence, Proposition 2.14 shows that $d$-critical structures $s$ on $X = [T/G]$ are in 1-1 correspondence with $G$-invariant $d$-critical structures $s'$ on $T$.

Here [55 Th. 2.56] is an analogue of Theorem 2.6.

**Theorem 2.16.** Let $(X, s)$ be a $d$-critical stack. Using the description of quasi-coherent sheaves on $X_{\text{red}}$ in Proposition 2.11 there is a line bundle $K_{X, s}$ on the reduced $\mathbb{K}$-substack $X_{\text{red}}$ of $X$ called the canonical bundle of $(X, s)$, unique up to canonical isomorphism, such that:

(a) For each point $x \in X_{\text{red}} \subseteq X$ we have a canonical isomorphism

$$
\kappa_x : K_{X, s}|_x \xrightarrow{\cong} (\Lambda_{\text{top}}^* T_{x}^*) \otimes (\Lambda_{\text{top}}^* \mathfrak{g}_x(X)) \otimes^\mathbb{L}^2,
$$

where $T_{x}^* X$ is the Zariski cotangent space of $X$ at $x$, and $\mathfrak{g}_x(X)$ the Lie algebra of the isotropy group (stabilizer group) $\text{Iso}_x(X)$ of $X$ at $x$.

(b) If $T$ is a $\mathbb{K}$-scheme and $t : T \rightarrow X$ a smooth 1-morphism, so that $t_{\text{red}} : T_{\text{red}} \rightarrow X_{\text{red}}$ is also smooth, then there is a natural isomorphism of line bundles on $T_{\text{red}}$:

$$
\Gamma_{T, t} : K_{X, s}(T_{\text{red}}, t_{\text{red}}) \xrightarrow{\cong} K_{T,s(T,t)} \otimes (\Lambda_{\text{top}}^* T_{T/X}^*) \otimes^\mathbb{L}^2.
$$

Here $(T, s(T,t))$ is an algebraic $d$-critical locus by Definition 2.13 and $K_{T,s(T,t)} \rightarrow T_{\text{red}}$ is its canonical bundle from Theorem 2.6.

(c) If $t : T \rightarrow X$ is a smooth 1-morphism, we have a distinguished triangle in $D_{\text{qcoh}}(T)$:

$$
t^*(\mathbb{L}_X) \xrightarrow{L_t} \mathbb{L}_T \xrightarrow{} T_{T/X}^* \xrightarrow{} t^*(\mathbb{L}_X)[1],
$$

where $\mathbb{L}_T, \mathbb{L}_X$ are the cotangent complexes of $T, X$, and $T_{T/X}^*$ the relative cotangent bundle of $t : T \rightarrow X$, a vector bundle of mixed rank on $T$. Let $p \in T_{\text{red}} \subseteq T$, so that $t(p) := t \circ p \in X$. Taking the long exact cohomology sequence of (2.12) and restricting to $p \in T$ gives an exact sequence

$$
0 \rightarrow T_{t(p)}^* X \rightarrow T_p^* T \rightarrow T_{T/X}^*|_p \rightarrow \mathfrak{g}_{t(p)}(X)^* \rightarrow 0.
$$

Then the following diagram commutes:

$$
\begin{array}{ccc}
K_{X, s}|_{t(p)} & \xrightarrow{\kappa_{t(p)}} & K_{X, s}(T_{\text{red}}, t_{\text{red}})|_p \\
\kappa_{t(p)} & \downarrow & \downarrow \kappa_{t(p)} \\
(\Lambda_{\text{top}}^* T_{t(p)}^*) \otimes (\Lambda_{\text{top}}^* \mathfrak{g}_{t(p)}(X)) \otimes^\mathbb{L}^2 & \xrightarrow{\alpha_p^2} & (\Lambda_{\text{top}}^* T_p^*) \otimes (\Lambda_{\text{top}}^* T_{T/X}^*) \otimes^\mathbb{L}^2,
\end{array}
$$

where $\kappa_p, \kappa_{t(p)}, \Gamma_{T, t}$ are as in (2.11), (2.10) and (2.11), respectively, and

$$
\alpha_p : \Lambda_{\text{top}}^* T_{t(p)}^* X \otimes \Lambda_{\text{top}}^* \mathfrak{g}_{t(p)}(X) \xrightarrow{\cong} \Lambda_{\text{top}}^* T_p^* T \otimes \Lambda_{\text{top}}^* T_{T/X}^*|_p^{-1}
$$

is induced by taking top exterior powers in (2.13).
Here \cite[Def. 2.57]{55} is the analogue of Definition 2.3.

**Definition 2.17.** Let \((X, s)\) be a d-critical stack, and \(K_{X, s}\) its canonical bundle from Theorem 2.16. An orientation on \((X, s)\) is a choice of square root line bundle \(K_{X, s}^{1/2}\) for \(K_{X, s}\) on \(X^{\text{red}}\). That is, an orientation is a line bundle \(L\) on \(X^{\text{red}}\), together with an isomorphism \(L^{\otimes 2} = L \otimes L \cong K_{X, s}\). A d-critical stack with an orientation will be called an oriented d-critical stack.

Let \((X, s)\) be an oriented d-critical stack. Then for each smooth \(t : T \to X\) we have a square root \(K_{X, s}^{1/2}(T^{\text{red}}, t^{\text{red}})\). Thus by \cite[(2.11)]{11}, \(K_{X, s}^{1/2}(T^{\text{red}}, t^{\text{red}}) \otimes (\Lambda^{\text{top}} L_{T/X})|_{T^{\text{red}}}^{\text{red}}\) is a square root for \(K_{T, s(T, t)}\). This proves \cite[Lem. 2.58]{55}:

**Lemma 2.18.** Let \((X, s)\) be a d-critical stack. Then an orientation \(K_{X, s}^{1/2}\) for \((X, s)\) determines a canonical orientation \(K_{T, s(T, t)}^{1/2}\) for the algebraic d-critical locus \((T, s(T, t))\), for all smooth \(t : T \to X\) with \(T\) a \(\mathbb{K}\)-scheme.

### 2.3 Equivariant d-critical loci

Here we summarize some results about group actions on algebraic d-critical loci from \cite{63}.

**Definition 2.19.** Let \((X, s)\) be an algebraic d-critical locus over \(\mathbb{K}\), and \(\mu : G \times X \to X\) an action of an algebraic \(\mathbb{K}\)-group \(G\) on the \(\mathbb{K}\)-scheme \(X\). We also write the action as \(\mu(\gamma) : X \to X\) for \(\gamma \in G\). We say that \((X, s)\) is \(G\)-invariant if \(\mu(\gamma)^*(s) = s\) for all \(\gamma \in G\), or equivalently, if \(\mu^*(s) = \pi_X^*(s)\) in \(H^0(G \times X, \mathcal{O}_G)\), where \(\pi_X : G \times X \to X\) is the projection.

Let \(\chi : G \to \mathbb{G}_m\) be a morphism of algebraic \(\mathbb{K}\)-groups, that is, a character of \(G\), where \(\mathbb{G}_m = \mathbb{K} \setminus \{0\}\) is the multiplicative group. We say that \((X, s)\) is \(G\)-equivariant, with character \(\chi\), if \(\mu(\gamma)^*(s) = \chi(\gamma) \cdot s\) for all \(\gamma \in G\), or equivalently, if \(\mu^*(s) = (\chi \circ \pi_G) \cdot (\pi_X^*(s))\) in \(H^0(G \times X, \mathcal{O}_G)\), where \(H^0(G \times X, \mathcal{O}_G) \ni \chi\) acts on \(H^0(G \times X, \mathcal{O}_G)\) by multiplication, as \(G\) is a smooth \(\mathbb{K}\)-scheme.

Suppose \((X, s)\) is \(G\)-invariant or \(G\)-equivariant, with \(\chi = 1\) in the \(G\)-invariant case. We call a critical chart \((R, U, f, i)\) on \((X, s)\) with a \(G\)-action \(\rho : G \times U \to U\) a \(G\)-equivariant critical chart if \(R \subseteq X\) is a \(G\)-invariant open subscheme, and \(i : R \to U\), \(f : U \to \mathbb{A}^1\) are equivariant with respect to the actions \(\mu|_{G \times R}, \rho, \chi\) of \(G\) on \(R, U, \mathbb{A}^1\), respectively.

We call a subchart \((R', U', f', i')\) \((R, U, f, i)\) a \(G\)-equivariant subchart if \(R' \subseteq R\) and \(U' \subseteq U\) are \(G\)-invariant open subschemes. Then \((R', U', f', i'), \rho'\) is a \(G\)-equivariant critical chart, where \(\rho' = \rho|_{G \times U'}\).

Note that \(X\) may not be covered by \(G\)-equivariant critical charts without extra assumptions on \(X, G\). We will restrict to the case when \(G\) is a torus, with a ‘good’ action on \(X\):

**Definition 2.20.** Let \(X\) be a \(\mathbb{K}\)-scheme, \(G\) an algebraic \(\mathbb{K}\)-torus, and \(\mu : G \times X \to X\) an action of \(G\) on \(X\). We call \(\mu\) a good action if \(X\) admits a Zariski open cover by \(G\)-invariant affine open \(\mathbb{K}\)-subschemes \(U \subseteq X\).

A torus-equivariant d-critical locus \((X, s)\) admits an open cover by equivariant critical charts if and only if the torus action is good:

**Proposition 2.21.** Let \((X, s)\) be an algebraic d-critical locus which is invariant or equivariant under the action \(\mu : G \times X \to X\) of an algebraic torus \(G\).

(a) If \(\mu\) is good then for all \(x \in X\) there exists a \(G\)-equivariant critical chart \((R, U, f, i), \rho\) on \((X, s)\) with \(x \in R\), and we may take \(\dim U = \dim T_x X\).

(b) Conversely, if for all \(x \in X\) there exists a \(G\)-equivariant critical chart \((R, U, f, i), \rho\) on \((X, s)\) with \(x \in R\), then \(\mu\) is good.

### 3 Derived symplectic structures in Donaldson–Thomas theory

We are now going to use derived algebraic geometry from \cite{103} and summarize the main results from the sequel \cite{11, 13, 17} and their consequences in Donaldson–Thomas theory. Some of them will not be used to prove our main results stated in \cite{41} but we will expose them as they contribute to a whole picture of the theory.
3.1 Symplectic derived schemes and critical loci

Here we summarizes the main results from [13]. The following is [13, Thm. 5.18].

**Theorem 3.1.** Let $X$ be a derived $\mathbb{K}$-scheme with $k$-shifted symplectic form $\omega$ for $k < 0$, and $x \in X$. Then there exists a standard form cdga $A$ over $\mathbb{K}$ which is minimal at $p \in \text{Spec} H^0(A)$ in the sense of [13, §4], a $k$-shifted symplectic form $\omega$ on $\text{Spec} A$, and a morphism $f : U = \text{Spec} A \to X$ with $f(p) = x$ and $f^*(\omega) \sim \omega$, such that if $k$ is odd or divisible by 4, then $f$ is Zariski open inclusion, and $A, \omega$ are in Darboux form, and if $k \equiv 2$ mod 4, then $f$ is étale, and $A, \omega$ are in strong Darboux form, as in [13, §5].

Let $Y$ be a Calabi–Yau $m$-fold over $\mathbb{K}$, that is, a smooth projective $\mathbb{K}$-scheme with $H^i(O_Y) = \mathbb{K}$ for $i = 0, m$ and $H^i(O_Y) = 0$ for $0 < i < m$. Suppose $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\text{coh}(Y)$, where we call $F \in \text{coh}(Y)$ simple if $\text{Hom}(F, F) = \mathbb{K}$. More generally, suppose $\mathcal{M}$ is a moduli $\mathbb{K}$-scheme of simple complexes of coherent sheaves in $D^b \text{coh}(Y)$, where we call $F^* \in D^b \text{coh}(Y)$ simple if $\text{Hom}(F^*, F^*) = \mathbb{K}$ and $\text{Ext}^c(F^*, F^*) = 0$. Such moduli spaces $\mathcal{M}$ are only known to be algebraic $\mathbb{K}$-spaces in general, but we assume $\mathcal{M}$ is a $\mathbb{K}$-scheme. Then $\mathcal{M} = t_0(\mathcal{M})$, for $\mathcal{M}$ the corresponding derived moduli $\mathbb{K}$-scheme. To make $\mathcal{M}, \mathcal{M}$ into schemes rather than stacks, we consider moduli of sheaves or complexes with fixed determinant. Then Pantev et al. [103, §2.1] prove $\mathcal{M}$ has a $(2 - m)$-shifted symplectic structure $\omega$, so Theorem 3.1 shows that $(\mathcal{M}, \omega)$ is Zariski locally modelled on $(\text{Spec} A, \omega)$, and $\mathcal{M}$ is Zariski locally modelled on $\text{Spec} H^0(A)$. In the case $m = 3$, so that $k = -1$, we get [13, Cor. 5.19]:

**Corollary 3.2.** Suppose $Y$ is a Calabi–Yau 3-fold over a field $\mathbb{K}$, and $\mathcal{M}$ is a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves, or simple complexes of coherent sheaves, on $Y$. Then for each $[F] \in \mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme $U$ with $\text{dim} U = \text{dim Ext}^4(F, F)$, a regular function $f : U \to \mathbb{A}^1$, and an isomorphism from $\text{Crit}(f)$ to $U$ to a Zariski open neighbourhood of $[F]$ in $\mathcal{M}$.

Here $\text{dim} U = \text{dim Ext}^4(F, F)$ comes from $A$ minimal at $p$ and $f(p) = [F]$ in Theorem 3.1. This is a new result in Donaldson–Thomas theory. We already explained that when $\mathbb{K} = \mathbb{C}$ and $\mathcal{M}$ is a moduli space of simple coherent sheaves on $Y$, using gauge theory and transcendental complex methods, Joyce and Song [64, Th. 5.4] prove that the underlying complex analytic space $\mathcal{M}^{\text{an}}$ of $\mathcal{M}$ is locally of the form $\text{Crit}(f)$ for $U$ a complex manifold and $f : U \to \mathbb{C}$ a holomorphic function. Behrend and Getzler announced the analogue of [64, Th. 5.4] for moduli of complexes in $D^b \text{coh}(Y)$, but the proof has not yet appeared. Over general $\mathbb{K}$, as in Kontsevich and Soibelman [73, §3.3] the formal neighbourhood $\mathcal{M}([F])$ of $\mathcal{M}$ at any $[F] \in \mathcal{M}$ is isomorphic to the critical locus $\text{Crit}(f)$ of a formal power series $f$ on $\text{Ext}^4(F, F)$ with only cubic and higher terms.

Here are [13, Thm. 6.6 & Cor. 6.7]:

**Theorem 3.3.** Suppose $(X, \omega)$ is a $-1$-shifted symplectic derived $\mathbb{K}$-scheme, and let $X = t_0(X)$ be the associated classical $\mathbb{K}$-scheme of $X$. Then $X$ extends uniquely to an algebraic $d$-critical locus $(X, s)$, with the property that whenever $(\text{Spec} A, \omega)$ is a $-1$-shifted symplectic derived $\mathbb{K}$-scheme in Darboux form with Hamiltonian $H \in A(0)$, as in [13, Exs. 5.8 & 5.15], and $f : \text{Spec} A \to X$ is an equivalence in $\text{dSch}_{\mathbb{K}}$ with a Zariski open derived $\mathbb{K}$-subscheme $R \subseteq X$ with $f^*(\omega) \sim \omega$, writing $U = \text{Spec} A(0), R = t_0(R), f = t_0(f)$ so that $H : U \to \mathbb{A}^1$ is regular and $f : \text{Crit}(H) \to \mathbb{A}^1$ is an isomorphism, for $\text{Crit}(H) \subseteq U$ the classical critical locus of $H$, then $(R, U, H, f^{-1})$ is a critical chart on $(X, s)$. The canonical bundle $K_{X, s}$ from Theorem 2.10 is naturally isomorphic to the determinant line bundle $\text{det}(\mathbb{L}_X|_{X^{\text{crit}}})$ of the cotangent complex $\mathbb{L}_X$ of $X$.

We can think of Theorem 3.3 as defining a truncation functor

$$F : \{\text{category of } -1\text{-shifted symplectic derived } \mathbb{K}\text{-schemes }(X, \omega)\} \to \{\text{category of algebraic } d\text{-critical loci } (X, s) \text{ over } \mathbb{K}\},$$

where the morphisms $f : (X, \omega) \to (Y, \omega')$ in the first line are (homotopy classes of) étale maps $f : X \to Y$ with $f^*(\omega') \sim \omega$, and the morphisms $f : (X, s) \to (Y, t)$ in the second line are étale maps $f : X \to Y$ with $f^*(t) = s$. In [13, Ex. 2.17] Joyce gives an example of $-1$-shifted symplectic derived schemes $(X, \omega), (Y, \omega')$, both global critical loci, such that $X, Y$ are not equivalent as derived $\mathbb{K}$-schemes, but their truncations $F(X, \omega), F(Y, \omega')$ are isomorphic as algebraic $d$-critical loci. Thus, the functor $F$ in (3.1) is not full.

Suppose again $Y$ is a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\text{coh}(Y)$. Then Thomas [121] defined a natural perfect obstruction theory $\phi : \mathcal{E}^* \to \mathbb{L}_\mathcal{M}$ on $\mathcal{M}$ in the

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sense of Behrend and Fantechi [5], and Behrend [4] showed that \( \phi : E^* \to L_M \) can be made into a symmetric obstruction theory. More generally, if \( M \) is a moduli \( \mathbb{K} \)-scheme of simple complexes of coherent sheaves in \( D^b\text{coh}(Y) \), then Huybrechts and Thomas [51] defined a natural symmetric obstruction theory on \( M \). Now in derived algebraic geometry \( M = t_0(M) \) for \( M \) the corresponding derived moduli \( \mathbb{K} \)-scheme, and the obstruction theory \( \phi : E^* \to L_M \) from [51][121] is \( L_{t_0} : L_M|_M \to L_M \). Panetev et al. [103] §2.1 prove \( M \) has a \(-1\)-shifted symplectic structure \( \omega \), and the symmetric structure on \( \phi : E^* \to L_M \) from [4] is \( \omega^0|_M \). So as for Corollary 3.2 Theorem 3.3 implies:

**Corollary 3.4.** Suppose \( Y \) is a Calabi–Yau 3-fold over \( \mathbb{K} \), and \( M \) is a classical moduli \( \mathbb{K} \)-scheme of simple coherent sheaves in \( \text{coh}(Y) \), or simple complexes of coherent sheaves in \( D^b\text{coh}(Y) \), with perfect obstruction theory \( \phi : E^* \to L_M \) as in Thomas [121] or Huybrechts and Thomas [51]. Then \( M \) extends naturally to an algebraic d-critical locus \((M,s)\). The canonical bundle \( K_{M,s} \) from Theorem 2.6 is naturally isomorphic to \( \det(E^*)|_{M_{\text{red}}} \).

### 3.2 Categorification using perverse sheaves and motives

Here we summarizes the main results from [12]. This particular section is not really used in the sequel, but it completes the discussion started in [12.2.1]. The following theorems are [12 Cor. 6.10 & Cor. 6.11]:

**Theorem 3.5.** Let \((X,\omega)\) be a \(-1\)-shifted symplectic derived scheme over \( \mathbb{C} \) in the sense of Panetev et al. [103] and \( X = t_0(X) \) the associated classical \( \mathbb{C} \)-scheme. Suppose we are given a square root \( \det(L_X)^{1/2} \) for \( \det(L_X)|_X \). Then we may define \( P_{X,\omega}^* \in \text{Perv}(X) \), uniquely up to canonical isomorphism, and isomorphisms \( \Sigma_X : P_{X,\omega}^* \to \mathbb{D}_{X}(P_{X,\omega}^*) \), \( T_{X,\omega} : P_{X,\omega}^* \to P_{X,\omega} \). The same applies for \( \mathcal{D} \)-modules and mixed Hodge modules on \( X \), and for l-adic perverse sheaves and \( \mathcal{D} \)-modules on \( X \) if \( X \) is over \( \mathbb{K} \) with \( \text{char}\mathbb{K} = 0 \).

**Theorem 3.6.** Let \( Y \) be a Calabi–Yau 3-fold over \( \mathbb{C} \), and \( M \) a classical moduli \( \mathbb{K} \)-scheme of simple coherent sheaves in \( \text{coh}(Y) \), or simple complexes of coherent sheaves in \( D^b\text{coh}(Y) \), with natural (symmetric) obstruction theory \( \phi : E^* \to L_M \) as in Behrend [4], Thomas [121], or Huybrechts and Thomas [51]. Suppose we are given a square root \( \det(E^*)^{1/2} \) for \( E^* \). Then we may define \( P_M^* \in \text{Perv}(M) \), uniquely up to canonical isomorphism, and isomorphisms \( \Sigma_M : P_M^* \to \mathbb{D}_{M}(P_M^*) \), \( T_M : P_M^* \to P_M^* \). The same applies for \( \mathcal{D} \)-modules and mixed Hodge modules on \( M \), and for l-adic perverse sheaves and \( \mathcal{D} \)-modules on \( M \) if \( Y, M \) are over \( \mathbb{K} \) with \( \text{char}\mathbb{K} = 0 \).

Theorem 3.6 is relevant to the categorification of Donaldson–Thomas theory as discussed in §1.2.1. As in [4 §1.2], the perverse sheaf \( P_{\mathcal{M}_0^a(\tau)}^* \) has pointwise Euler characteristic \( \chi(P_{\mathcal{M}_0^a(\tau)}) = \nu \). This implies that when \( A \) is a field, say \( A = \mathbb{Q} \), the (compactly-supported) hypercohomologies \( H^* (P_{\mathcal{M}_0^a(\tau)}^*) \), \( H^* \text{cs}(P_{\mathcal{M}_0^a(\tau)}^*) \) satisfy

\[
\sum_{k \in \mathbb{Z}} (-1)^k \dim H^k (P_{\mathcal{M}_0^a(\tau)}^*) = \sum_{k \in \mathbb{Z}} (-1)^k \dim H^k \text{cs}(P_{\mathcal{M}_0^a(\tau)}^*) = \chi(M_0^a(\tau), \nu) = DT^a(\tau),
\]

where \( H^k (P_{\mathcal{M}_0^a(\tau)}^*) \cong H^{-k} \text{cs}(P_{\mathcal{M}_0^a(\tau)}^*) \) by Verdier duality. That is, we have produced a natural graded \( \mathbb{Q} \)-vector space \( H^* (P_{\mathcal{M}_0^a(\tau)}^*) \), thought of as some kind of generalized cohomology of \( M_0^a(\tau) \), whose graded dimension is \( DT^a(\tau) \). This gives a new interpretation of the Donaldson–Thomas invariant \( DT^a(\tau) \).

In fact, as discussed at length in [120 §3], the first natural “refinement” or “quantization” direction of a Donaldson–Thomas invariant \( DT^a(\tau) \in \mathbb{Z} \) is not the Poincaré polynomial of this cohomology, but its weight polynomial

\[
w(\mathbb{H}^* (P_{\mathcal{M}_0^a(\tau)}^*), t) \in \mathbb{Z}[t^{\pm \frac{1}{2}}],
\]

defined using the mixed Hodge structure on the cohomology of the mixed Hodge module version of \( P_{\mathcal{M}_0^a(\tau)}^* \), which exists assuming that \( M_0^a(\tau) \) is projective.

The material above is related to work by other authors. The idea of categorifying Donaldson–Thomas invariants using perverse sheaves or \( \mathcal{D} \)-modules is probably first due to Behrend [4], and for Hilbert schemes \( \text{Hilb}^b(Y) \) of a Calabi–Yau 3-fold \( Y \) is discussed by Dinuca and Szendrői [22] and Behrend, Bryan and Szendrői [7] §3.4, using mixed Hodge modules. Corollary 3.6 answers a question of Joyce and Song [63 Question 5.7(a)].

As in [64][73] representations of quivers with superpotentials \((Q,W)\) give 3-Calabi–Yau triangulated categories, and one can define Donaldson–Thomas type invariants \( DT^a_{Q,W}(\tau) \) ‘counting’ such representations, which
are simple algebraic ‘toy models’ for Donaldson–Thomas invariants of Calabi–Yau 3-folds. Kontsevich and Soibelman [76] explain how to categorify these quiver invariants $DT_{Q,W}^\alpha(\tau)$, and define an associative multiplication on the categorification to make a Cohomological Hall Algebra. This paper was motivated by the aim of extending [76] to define Cohomological Hall Algebras for Calabi–Yau 3-folds.

The square root $\det(f^*)^{1/2}$ required in Corollary 3.10 corresponds roughly to orientation data in the work of Kontsevich and Soibelman [74, §5], [76].

Finally, we point out that Kiem and Li [71] have recently proved an analogue of Corollary 3.10 by complex analytic methods, beginning from Joyce and Song’s result [64, Th. 5.4], proved using gauge theory, that $\mathcal{M}_M^\alpha(\tau)$ is locally isomorphic to $\text{Crit}(f)$ as a complex analytic space, for $V$ a complex manifold and $f : V \to \mathbb{C}$ holomorphic.

Now, we summarizes the main results from [17]. The following theorems are [17, Cor. 5.12 & Cor. 5.13]:

**Theorem 3.7.** Let $(X, \omega)$ be a $-1$-shifted symplectic derived scheme over $\mathbb{K}$ in the sense of Panetev et al. [103], and $X = t_0(X)$ the associated classical $\mathbb{K}$-scheme, assumed of finite type. Suppose we are given a square root $\det(L_X)^{1/2}$ for $\det(L_X)_{|X}$. Then we may define a natural motive $MF_{X,\omega} \in \tilde{\mathcal{M}}$.

**Theorem 3.8.** Suppose $Y$ is a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ is a finite type moduli $\mathbb{K}$-scheme of simple coherent sheaves in $\text{coh}(Y)$, or simple complexes of coherent sheaves in $D^b\text{coh}(Y)$, with obstruction theory $\phi : E^* \to L_M$ as in Thomas [121] or Huybrechts and Thomas [51]. Suppose we are given a square root $\det(E^*)^{1/2}$ for $\det(E^*)$. Then we may define a natural motive $MF_{M} \in \tilde{\mathcal{M}}$.

Kontsevich and Soibelman define a motive over $\mathcal{M}_M^\alpha(\tau)$, by associating a formal power series to each (not necessarily closed) point, and taking its motivic Milnor fibre. The question of how these formal power series and motivic Milnor fibres vary in families over the base $\mathcal{M}_M^\alpha(\tau)$ is not really addressed in [74]. Corollary 3.8 answers this question, showing that Zariski locally in $\mathcal{M}_M^\alpha(\tau)$ we can take the formal power series and motivic Milnor fibres to all come from a regular function $f : U \to \mathbb{A}^1$ on a smooth $\mathbb{K}$-scheme $U$. As before, the square root $\det(E^*)^{1/2}$ required in Corollary 3.8 corresponds roughly to orientation data in Kontsevich and Soibelman [74, §5], [76].

### 3.3 Generalization to symplectic derived stacks

Here we summarizes the main results from [11]. The following theorems are [11, Cor. 2.11 & Cor. 2.12]:

**Theorem 3.9.** Let $(X, \omega_X)$ be a $-1$-shifted symplectic derived $\mathbb{K}$-stack, and $X = t_0(X)$ the corresponding classical $\mathbb{K}$-stack. Then for each $p \in X$ there exist a smooth $\mathbb{K}$-scheme $U$ with dimension $\dim H^0(L_X|_p)$, a point $t \in U$, a regular function $f : U \to \mathbb{A}^1$ with $d_{\text{dR}} f|_t = 0$, so that $T := \text{Crit}(f) \subseteq U$ is a closed $\mathbb{K}$-subscheme with $t \in T$, and a morphism $\phi : T \to X$ which is smooth of relative dimension $\dim H^1(L_X|_p)$, with $\phi(t) = p$. We may take $f_{|T_{\text{red}}} = 0$.

Thus, the underlying classical stack $X$ of a $-1$-shifted symplectic derived stack $(X, \omega_X)$ admits an atlas consisting of critical loci of regular functions on smooth schemes.

Now let $Y$ be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli stack of coherent sheaves $F$ on $Y$, or complexes $F^*$ in $D^b\text{coh}(Y)$ with $\text{Ext}^0(F^*,F^*) = 0$. Then $\mathcal{M} = t_0(\mathcal{M})$, for $\mathcal{M}$ the corresponding derived moduli stack. The (open) condition $\text{Ext}^0(F^*,F^*) = 0$ is needed to make $\mathcal{M}$ 1-geometric and 1-truncated (that is, a derived Artin stack, in our terminology); without it, $\mathcal{M}, \mathcal{M}$ would be a higher derived stack. Panetev et al. [103, §2.1] prove $\mathcal{M}$ has a $-1$-shifted symplectic structure $\omega_{\mathcal{M}}$. Applying Theorem 3.9 and using $H^1(L\mathcal{M}|_F) \cong \text{Ext}^{1-i}(F,F)^*$ yields a new result on classical 3-Calabi–Yau moduli stacks, the statement of which involves no derived geometry:

**Corollary 3.10.** Suppose $Y$ is a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves $F$, or more generally of complexes $F^*$ in $D^b\text{coh}(Y)$ with $\text{Ext}^0(F^*,F^*) = 0$. Then for each $[F] \in \mathcal{M}$, there exist a smooth $\mathbb{K}$-scheme $U$ with $\dim U = \dim \text{Ext}^1(F,F)$, a point $u \in U$, a regular function $f : U \to \mathbb{A}^1$ with $d_{\text{dR}} f|_u = 0$, and a morphism $\phi : \text{Crit}(f) \to \mathcal{M}$ which is smooth of relative dimension $\dim \text{Hom}(F,F)$, with $\phi(u) = [F]$.
This is an analogue of [13 Cor. 5.19]. When $\mathbb{K} = \mathbb{C}$, a related result for coherent sheaves only, with $U$ a complex manifold and $f$ a holomorphic function, was proved by Joyce and Song [64 Th. 5.5] using gauge theory and transcendental complex methods.

Here is [11 Thm. 3.18], a stack version of Theorem 3.3.

**Theorem 3.11.** Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $(X, \omega_X)$ a $-1$-shifted symplectic derived Artin $\mathbb{K}$-stack, and $X = t_0(X)$ the corresponding classical Artin $\mathbb{K}$-stack. Then there exists a unique $d$-critical structure $s \in H^0(S^0_U)$ on $X$, making $(X, s)$ into a $d$-critical stack, with the following properties:

(a) Let $f : U \to \mathbb{A}^1$, $T = \text{Crit}(f)$ and $\varphi : T \to X$ be as in Corollary 3.11 with $f|_{T_{\text{red}}} = 0$. There is a unique $s_T \in H^0(S^0_T)$ on $T$ with $\iota_T(s_T) = \varphi^{-1}(f) + I^2_{T, U}$, and $(T, s_T)$ is an algebraic $d$-critical locus. Then $s(T, \varphi) = s_T$ in $H^0(S^0_T)$.

(b) The canonical bundle $K_{X,s}$ of $(X, s)$ from Theorem 2.16 is naturally isomorphic to the restriction $\det(L_X)|_{X_{\text{red}}}$ to $X_{\text{red}} \subseteq X \subseteq X$ of the determinant line bundle $\det(L_X)$ of the cotangent complex $L_X$ of $X$.

We can think of Theorem 3.11 as defining a truncation functor

$$F : \left\{ \infty\text{-category of } -1\text{-shifted symplectic derived Artin } \mathbb{K}\text{-stacks } (X, \omega_X) \right\}$$

$$\to \left\{2\text{-category of } d\text{-critical stacks } (X, s) \text{ over } \mathbb{K} \right\}.$$

Let $Y$ be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves in $\text{coh}(Y)$, or complexes of coherent sheaves in $D^b\text{coh}(Y)$. There is a natural obstruction theory $\phi : \mathcal{E}^\bullet \to \mathcal{L}_\mathcal{M}$ on $\mathcal{M}$, where $\mathcal{E}^\bullet \in D_{\text{coh}}(\mathcal{M})$ is perfect in the interval $[-2, 1]$, and $h^i(\mathcal{E}^\bullet)_f \cong \text{Ext}^{1-i}(F, F)^*$ for each $\mathbb{K}$-point $F \in \mathcal{M}$, regarding $F$ as an object in $\text{coh}(Y)$ or $D^b\text{coh}(Y)$. Now in derived algebraic geometry $\mathcal{M} = t_0(\mathcal{M})$ for $\mathcal{M}$ the corresponding derived moduli $\mathbb{K}$-stack, and $\phi : \mathcal{E}^\bullet \to \mathcal{L}_\mathcal{M}$ is $\mathbb{L}_\mathcal{M} : L_{\mathcal{M}|\mathcal{M}} \to L_{\mathcal{M}}$. Panetz et al. [103 §2.1] prove $\mathcal{M}$ has a $-1$-shifted symplectic structure $\omega$. Thus Theorem 3.11 implies [11 Cor. 3.19]:

**Corollary 3.12.** Suppose $Y$ is a Calabi–Yau 3-fold over $\mathbb{K}$ of characteristic zero, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves $F$ in $\text{coh}(Y)$, or complexes of coherent sheaves $F^\bullet$ in $D^b\text{coh}(Y)$ with $\text{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathcal{L}_\mathcal{M}$. Then $\mathcal{M}$ extends naturally to an algebraic $d$-critical locus $(\mathcal{M}, s)$. The canonical bundle $K_{\mathcal{M},s}$ from Theorem 2.10 is naturally isomorphic to $\det(\mathcal{E}^\bullet)|_{\mathcal{M}_{\text{red}}}$.

Here is [11 Cor. 4.13], the stack version of Theorem 3.5.

**Theorem 3.13.** Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $(X, \omega)$ a $-1$-shifted symplectic derived Artin $\mathbb{K}$-stack, and $X = t_0(X)$ the associated classical Artin $\mathbb{K}$-stack. Suppose we are given a square root $\det(L_X)|_{\mathbb{X}}^{1/2}$. Then working in 1-adic perverse sheaves on stacks [14 §4] we may define a perverse sheaf $\tilde{P}_{X,\omega}^\bullet$ on $X$ uniquely up to canonical isomorphism, and Verdier duality and monodromy isomorphisms $\tilde{\Sigma}_{X,\omega} : \tilde{P}_{X,\omega}^\bullet \to D_X(\tilde{P}_{X,\omega}^\bullet)$ and $\tilde{T}_{X,\omega} : \tilde{P}_{X,\omega}^\bullet \to \tilde{P}_{X,\omega}^\bullet$. These are characterized by the fact that given a diagram

$$U = \text{Crit}(f : U \to \mathbb{A}^1) \leftarrow^i V \varphi \to X$$

such that $U$ is a smooth $\mathbb{K}$-scheme, $\varphi$ smooth of dimension $n$, $\mathbb{L}_{V/U} \cong \mathbb{T}_{V/X}[2]$, $\varphi^*(\omega_X) \sim \iota^*(\omega_U)$ for $\omega_U$ the natural $-1$-shifted symplectic structure on $U = \text{Crit}(f : U \to \mathbb{A}^1)$, and $\varphi^*(\det(L_X)|_{\mathbb{X}}^{1/2}) \cong \iota^*(K_U) \otimes \Lambda^n\mathbb{T}_{V/X}$, then $\varphi^*(\tilde{P}_{X,\omega})[n], \varphi^*(\tilde{\Sigma}_{X,\omega})[n], \varphi^*(\tilde{T}_{X,\omega})[n]$ are canonically isomorphic to $\iota^*(\mathcal{P}_{V,U,f}), \iota^*(\sigma_{U,f}), \iota^*(\tau_{U,f})$, for $\mathcal{P}_{V,U,f,\sigma_{U,f},\tau_{U,f}}$ as in [10]. The same applies in the other theories of perverse sheaves and $\mathcal{D}$-modules on stacks.

Here is [11 Cor. 4.14], the stack version of Theorem 3.6.

**Theorem 3.14.** Let $Y$ be a Calabi–Yau 3-fold over an algebraically closed field $\mathbb{K}$ of characteristic zero, and $\mathcal{M}$ a classical moduli $\mathbb{K}$-stack of coherent sheaves $F$ in $\text{coh}(Y)$, or complexes $F^\bullet$ in $D^b\text{coh}(Y)$ with $\text{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with obstruction theory $\phi : \mathcal{E}^\bullet \to \mathcal{L}_\mathcal{M}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)|^{1/2}$. Then working in $1$-adic perverse sheaves on stacks [14 §4], we may define a natural perverse sheaf $\tilde{P}_{\mathcal{M}}^\bullet \in \text{Perv}(\mathcal{M})$, and Verdier duality and monodromy isomorphisms $\tilde{\Sigma}_\mathcal{M} : \tilde{P}_{\mathcal{M}}^\bullet \to D_{\mathcal{M}}(\tilde{P}_{\mathcal{M}})$ and $\tilde{T}_\mathcal{M} : \tilde{P}_{\mathcal{M}}^\bullet \to \tilde{P}_{\mathcal{M}}^\bullet$. The pointwise Euler characteristic of $\tilde{P}_{\mathcal{M}}^\bullet$ is the Behrend function $\nu_{\mathcal{M}}$ of $\mathcal{M}$ from Joyce and Song [64 §4], so that $\tilde{P}_{\mathcal{M}}^\bullet$ is in effect a categorification of the Donaldson–Thomas theory of $\mathcal{M}$. The same applies in the other theories of perverse sheaves and $\mathcal{D}$-modules on stacks.
Here is [11, Cor. 5.16], the stack version of Theorem 3.15.

**Theorem 3.15.** Let $(X, \omega)$ be a $-1$-shifted symplectic derived Artin $\mathbb{K}$-stack in the sense of Panetev et al. [103], and $X = t_0(X)$ the associated classical Artin $\mathbb{K}$-stack, assumed of finite type and locally a global quotient. Suppose we are given a square root $\text{det}(L_X)^{1/2}$ for $\text{det}(L_X)\mid_X$. Then we may define a natural motive $MF_{X, \omega} \in \mathcal{M}_X^{\text{st}, \mu}$, which is characterized by the fact that given a diagram

$$
\begin{array}{ccc}
U & \xrightarrow{i} & V \\
\downarrow & & \downarrow \varphi \\
X & \xrightarrow{\gamma} & X 
\end{array}
$$

such that $U$ is a smooth $\mathbb{K}$-scheme, $\varphi$ is smooth of dimension $n$, $\text{det}(\mathcal{O}_U) \cong \mathcal{O}_U[2]$, $\varphi^*(\omega_X) \sim \nu(\omega_U)$ for $\omega_U$ the natural $-1$-shifted symplectic structure on $U = \text{Crit}(f : U \to \mathbb{A}^1)$, and $\varphi^*(\text{det}(L_X)^{1/2}) \cong i^*(K_U) \otimes \Lambda^n \mathcal{T}_V/V \cdot \mathcal{X}$, then $\varphi^*(MF_{X, \omega}) = \mathcal{L}^{n/2} \otimes i^*(MF_{U, f, \varphi})$ in $\mathcal{M}_V^{\text{st}, \mu}$.

Here is [11, Cor. 5.17], the stack version of Theorem 3.16.

**Theorem 3.16.** Let $Y$ be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathcal{M}$ a finite type classical moduli $\mathbb{K}$-stack of coherent sheaves in $\text{coh}(Y)$, with natural obstruction theory $\phi : \mathcal{E}^* \to \mathcal{M}_1$. Suppose we are given a square root $\text{det}(\mathcal{E}^*)^{1/2}$ for $\text{det}(\mathcal{E}^*)$. Then we may define a natural motive $MF_{\mathcal{M}} \in \mathcal{M}_{\mathcal{M}}^{\text{st}, \mu}$.

Theorem 3.16 is relevant to Kontsevich and Soibelman’s theory of motivic Donaldson–Thomas invariants [74]. Again, our square root $\text{det}(\mathcal{E}^*)^{1/2}$ roughly coincides with their orientation data [74, §5]. In [74, §6.2], given a finite type moduli stack $\mathcal{M}$ of coherent sheaves on a Calabi–Yau 3-fold $Y$ with orientation data, they define a motive $\int_{\mathcal{M}}^{\mathcal{M}}$ in a ring $D^b$ isomorphic to our $\mathcal{M}_{\mathcal{M}}^{\text{st}, \mu}$. We expect this should agree with $\pi_*(MF_{\mathcal{M}})$ in our notation, with $\pi : \mathcal{M} \to \text{Spec} \mathbb{K}$ the projection. This $\int_{\mathcal{M}}^{\mathcal{M}}$ is roughly the motivic Donaldson–Thomas invariant of $\mathcal{M}$. Their construction involves expressing $\mathcal{M}$ near each point in terms of the critical locus of a formal power series. Kontsevich and Soibelman’s constructions were partly conjectural, and our results may fill some gaps in their theory.

### 4 The main results

We will prove and use the algebraic analogue of Theorem 3.13 which we can state as follows:

**Theorem 4.1.** Let $X$ be a Calabi–Yau 3-fold over $\mathbb{K}$, and write $\mathfrak{M}$ for the moduli stack of coherent sheaves on $X$. Then for each $[E] \in \mathfrak{M}(\mathbb{K})$, there exists a smooth affine $\mathbb{K}$-scheme $U$, a point $p \in U(\mathbb{K})$, an étale morphism $u : U \to \text{Ext}^1(E, E)$ with $u(p) = 0$, a regular function $f : U \to \mathbb{A}^1$ with $f\mid_p = 0$ and a 1-morphism $\xi : \text{Crit}(f) \to \mathfrak{M}$ smooth of relative dimension dimension $\text{dim Aut}(E)$, with $\xi(p) = [E] \in \mathfrak{M}(\mathbb{K})$, such that if $\iota : \text{Ext}^1(E, E) \to T_{[E]} \mathfrak{M}$ is the natural isomorphism, then $d\xi\mid_p = \iota \circ du\mid_p : T_p U \to T_{[E]} \mathfrak{M}$. Moreover, let $G$ be a maximal algebraic torus in $\text{Aut}(E)$, acting on $\text{Ext}^1(E, E)$ by $\gamma : \epsilon \mapsto \gamma \circ \epsilon \circ \gamma^{-1}$. Then we can choose $u, p, u, f, \xi$ and $G$-action on $U$ such that $u$ is $G$-equivariant and $p, f$ are $G$-invariant, so that $\text{Crit}(f)$ is $G$-invariant, and $\xi : \text{Crit}(f) \to \mathfrak{M}$ factors through the projection $\text{Crit}(f) \to [\text{Crit}(f)/G]$.

Note that you can regard $u : U \to \text{Ext}^1(E, E)$ as an étale open neighbourhood of 0 in $\text{Ext}^1(E, E)$. Theorem 4.1 will be proved in [1.1.4] using [2]. Next, we will use this to prove the algebraic analogue of Theorem 1.13.

**Theorem 4.2.** Let $X$ be a Calabi–Yau 3-fold over $\mathbb{K}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on $X$. The Behrend function $\nu_{\mathfrak{M}} : \mathfrak{M}(\mathbb{K}) \to \mathbb{Z}$ is a naturally locally constructible function on $\mathfrak{M}$. For all $E_1, E_2 \in \text{coh}(X)$, it satisfies:

$$
\nu_{\mathfrak{M}}(E_1 + E_2) = (-1)^{\chi([E_1], [E_2])}\nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2),
$$

(4.1)

$$
\int_{[\lambda] \in \mathcal{P}(\text{Ext}^1(E_2, E_1))} \nu_{\mathfrak{M}}(E_1, E_2) = (e_{12} - e_{11}) \, \nu_{\mathfrak{M}}(E_1 + E_2),
$$

(4.2)
where \( e_{21} = \dim \text{Ext}^1(E_2, E_1) \) and \( e_{12} = \dim \text{Ext}^1(E_1, E_2) \) for \( E_1, E_2 \in \text{coh}(X) \). Here \( \chi([E_1], [E_2]) \) in (4.1) is the Euler form as in (1.11), and in (4.2) the correspondence between \( \lambda \in \mathbb{P}(\text{Ext}^1(E_2, E_1)) \) and \( F \in \text{coh}(X) \) is such that \( \lambda \in \mathbb{P}(\text{Ext}^1(E_2, E_1)) \) lifts to some \( 0 \neq \lambda \in \text{Ext}^1(E_2, E_1) \), which corresponds to a short exact sequence \( 0 \to E_1 \to F \to E_2 \to 0 \) in \( \text{coh}(X) \) in the usual way. The function \( \lambda \mapsto \nu_{\lambda}(F) \) is a constructible function \( \mathbb{P}(\text{Ext}^1(E_2, E_1)) \to \mathbb{Z} \), and the integrals in (4.2) are integrals of constructible functions using the Euler characteristic as measure.

As in §1.3, the identities (4.1)–(4.2) are crucial for the whole program in [64], and will be proved in §4.2.

In the next theorem, the condition that \( \text{Ext}^{<0}(E^*, E^*) = 0 \) is necessary for \( \mathfrak{M} \) to be an Artin stack, rather than a higher stack. Note that this condition is automatically satisfied by complexes \( E^* \) which are semistable in any stability condition, for example Bridgeland stability conditions [14]. Therefore to prove wall-crossing formulae for Donaldson-Thomas invariants in the derived category \( D^b \text{coh}(X) \) under change of stability condition by the “dominant stability condition” method of [57–60, 65], it is enough to know the Behrend function identities (4.1)–(4.2) for complexes \( E^* \) with \( \text{Ext}^{<0}(E^*, E^*) = 0 \), and we do not need to deal with complexes \( E^* \) with \( \text{Ext}^{<0}(E^*, E^*) \neq 0 \), or with higher stacks.

**Theorem 4.3.** Let \( X \) be a Calabi-Yau 3-fold over \( \mathbb{K} \), and write \( \mathfrak{M} \) for the moduli stack of complexes \( E^* \) in \( D^b \text{coh}(X) \) with \( \text{Ext}^{<0}(E^*, E^*) = 0 \). This is an Artin stack by [57]. Let \( [E^*] \in \mathfrak{M}(\mathbb{K}) \), and suppose that a Zariski open neighbourhood of \( [E^*] \) in \( \mathfrak{M}(\mathbb{K}) \) is equivalent to a global quotient \( [S/\text{GL}(n, \mathbb{K})] \) for \( S \) a \( K \)-scheme with a \( \text{GL}(n, \mathbb{K}) \)-action. Then the analogues of Theorems 4.1 and 4.2 hold with \( \mathfrak{M}, E^* \) in place of \( E, \mathfrak{M} \).

The condition on \( \mathfrak{M} \) that it should be *locally a global quotient*, is known for the moduli stack of coherent sheaves \( \mathfrak{M} \) using Quot schemes. A proof of that can be found in [64] §9.3, where Joyce and Song uses the standard method for constructing coarse moduli schemes of semistable coherent sheaves in Huybrechts and Lehn [60], adapting it for Artin stacks, and an argument similar to parts of that of Luna’s Étale Slice Theorem [88, §III]. However, this is not known for the moduli stack of complexes. The author expects Theorem 4.3 to hold without this technical assumption, but currently can’t prove it.

The proof of Theorem 4.3 is the same as the proof of Theorem 4.2 substituting sheaves with complexes of sheaves, and accordingly making the obvious modifications.

Finally, in §4.3 we will characterize the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a deformation invariant lattice described using the Picard group. First of all, using existence results, and smoothness and properness properties of the relative Picard scheme in a family of Calabi–Yau 3-folds, one proves that the Picard groups form a local system. Actually, it is a local system with finite monodromy, so it can be made trivial after passing to a finite étale cover of the base scheme, as formulated in the analogue of [64] Thm. 4.21], which studies the monodromy of the Picard scheme instead of the numerical Grothendieck group in a family. Then, Theorem 4.4, a substitute for [64] Thm. 4.19], which does not need the integral Hodge conjecture result by Voisin [134] for Calabi–Yau 3-folds over \( \mathbb{C} \) and which is valid over \( \mathbb{K} \), characterizes the numerical Grothendieck group of a Calabi–Yau 3-fold in terms of a globally constant lattice described using the Picard scheme:

**Theorem 4.4.** Let \( X \) be a Calabi–Yau 3-fold over \( \mathbb{K} \) with \( H^1(O_X) = 0 \). Define

\[
\Lambda_X = \{ (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \mid \lambda_0, \lambda_3 \in \mathbb{Q}, \lambda_1 \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \lambda_2 \in \text{Hom}(\text{Pic}(X), \mathbb{Q}) \text{ such that } \\
\lambda_0 \in \mathbb{Z}, \lambda_1 \in \text{Pic}(X)/\text{torsion}, \lambda_2 - \frac{1}{2} \lambda_1^2 \in \text{Hom}(\text{Pic}(X), \mathbb{Z}), \lambda_3 + \frac{1}{12} \lambda_1 \lambda_2 (TX) \in \mathbb{Z} \},
\]

where \( \lambda_1^2 \) is defined as the map \( \alpha \in \text{Pic}(X) \to \frac{1}{12} c_1(\lambda_1) \cdot c_1(\lambda_1) \cdot c_2(\alpha) \in A^3(X) \otimes_{\mathbb{Q}} \mathbb{Q} \), and \( \frac{1}{12} \lambda_1 c_2(TX) \) is defined as \( \frac{1}{12} c_1(\lambda_1) \cdot c_2(TX) \in A^3(X) \otimes_{\mathbb{Q}} \mathbb{Q} \). Then for any family of Calabi-Yau 3-folds \( \pi : X \to S \) over a connected base \( S \) with \( X = \pi^{-1}(s_0) \), the lattices \( \Lambda_X \), form a local system of abelian groups over \( S \) with fibre \( \Lambda_X \). Furthermore, the monodromy of this system lies in a finite subgroup of \( \text{Aut}(\Lambda_X) \), so after passing to an étale cover \( \tilde{S} \to S \) of \( S \), we can take the local system to be trivial, and coherently identify \( \Lambda_X \equiv \Lambda_X \) for all \( \tilde{s} \in \tilde{S} \). Finally, the Chern character gives an injective morphism \( \text{ch} : K^{\text{num}}(\text{coh}(X)) \to \Lambda_X \).

Following [64], this yields
Theorem 4.5. The generalized Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau)$ over $\mathbb{K}$ for $\alpha \in \Lambda_X$ are unchanged under deformations of the underlying Calabi–Yau 3-fold $X$, by which we mean the following: let $\mathcal{X} \to T$ a smooth projective morphism of algebraic $\mathbb{K}$-varieties $X, T$, with $T$ connected. Let $O_X(1)$ be a relative very ample line bundle for $X \to T$. For each $t \in T(\mathbb{K})$, write $X_t$ for the fibre $X \times_{\tau, T, t} \text{Spec} \mathbb{K}$ of $\tau$ over $t$, and $O_{X_t}(1)$ for $O_X(1)|_{X_t}$. Suppose that $X_t$ is a smooth Calabi–Yau 3-fold over $\mathbb{K}$ for all $t \in T(\mathbb{K})$, with $H^1(O_{X_t}) = 0$. Then the generalized Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau)_t$ are independent of $t \in T(\mathbb{K})$

More precisely, the isomorphism $\Lambda_{X_t} = \Lambda_X$ is canonical up to action of a finite group $\Gamma$, the monodromy on $T$, and $\bar{DT}^\alpha(\tau)_t$ are independent of the action of $\Gamma$ on $\alpha$, so whichever identification $\Lambda_{X_t} = \Lambda_X$ is chosen, it is still true $\bar{DT}^\alpha(\tau)_t$ independent of $t$.

Now, recall that in [64] Joyce and Song used the assumption that the base field is the field of complex numbers $\mathbb{K} = \mathbb{C}$ for the Calabi–Yau 3-fold $X$ in three main ways:

(a) Theorem 1.13 in [64.3] is proved using gauge theory and transcendental complex analytic methods, and work only over $\mathbb{K} = \mathbb{C}$. It is used to prove the Behrend function identities (1.16)–(1.17), which are vital for much of their results, including the wall crossing formula for the $\bar{DT}^\alpha(\tau)$, and the relation between $\bar{PT}^{\alpha,n}(\tau')$, $\bar{DT}^\alpha(\tau)$.

(b) In [64 §4.5], when $\mathbb{K} = \mathbb{C}$ the Chern character embeds $K^{num}(\text{coh}(X))$ in $H^\text{even}(X; \mathbb{Q})$, and they use this to show $K^{num}(\text{coh}(X))$ is unchanged under deformations of $X$. This is important for the results that $\bar{DT}^\alpha(\tau)$ and $\bar{PT}^{\alpha,n}(\tau')$ for $\alpha \in K^{num}(\text{coh}(X))$ are invariant under deformations of $X$ even to make sense.

(c) Their notion of ‘compactly embeddable’ noncompact Calabi–Yau 3-folds in [64 §6.7] is complex analytic and does not make sense for general $\mathbb{K}$. This constrains the noncompact Calabi–Yau 3-folds they can define generalized Donaldson–Thomas invariants for.

Now Theorem 1.1 and Theorem 4.2 extend the results in (a) over algebraically closed field $\mathbb{K}$ of characteristic zero. As noted in [54], constructible functions methods fail for $\mathbb{K}$ of positive characteristic. Because of this, the alternative descriptions (1.7) and (1.20), for $\bar{DT}^\alpha(\tau)$ and $\bar{PT}^{\alpha,n}(\tau')$ as weighted Euler characteristics, and the definition of $\bar{DT}^\alpha(\tau)$ in (1.3) cannot work in positive characteristic, so working over an algebraically closed field of characteristic zero is about as general as is reasonable.

The point (a) above has consequences also on (c), because Joyce and Song only need the notion of ‘compactly embeddable’ as their complex analytic proof of (1.16)–(1.17) requires $X$ compact. Unfortunately the given algebraic version of (1.16)–(1.17) in Theorem 4.2 uses results from derived algebraic geometry, and the author does not know if they apply also for compactly supported sheaves on a noncompact $X$. We can prove a version of that under some technical assumptions, as stated in §5. Observe, also, that in the noncompact case you cannot expect to have the deformation invariance property unless in some particular cases in which the moduli space is proper. The extension of (b) to $\mathbb{K}$ is given in Section 4.3 which yields Theorem 1.5 thanks to which it is possible to extend [64 Cor. 5.28] about the deformation invariance of the generalized Donaldson–Thomas invariants in the compact case to algebraically closed fields $\mathbb{K}$ of characteristic zero. Thus, this proves our main theorem:

Theorem 4.6. The theory of generalized Donaldson–Thomas invariants defined in [64] is valid over algebraically closed fields of characteristic zero.

Next, we will respectively prove Theorems 4.1, 4.2 and 4.4 in §4.1 §4.2 and §4.3.

4.1 Local description of the Donaldson–Thomas moduli space

Let us fix a moduli stack $\mathcal{M}$ which is locally a global quotient. In particular, $\mathcal{M}$ can be the moduli stack of coherent sheaves over a Calabi-Yau 3-fold $X$, so that the theory exposed in §4.1 and §4.2 applies.

The first step in order to proving Theorem 4.1 is to show the existence of a quasiprojective $\mathbb{K}$-scheme $S$, an action of $G$ on $S$, a point $s \in S(\mathbb{K})$ fixed by $G$, and a 1-morphism of Artin $\mathbb{K}$-stacks $\xi : [S/G] \to \mathcal{M}$, which is smooth of relative dimension $\dim \text{Aut}(E) - \dim G$, where $[S/G]$ is the quotient stack, such that $\xi(sG) = [E]$, the induced morphism on stabilizer groups $\xi_! : \text{Iso}_0([S/G])(sG) \to \text{Iso}_0([E])$ is the natural morphism $G \to \text{Aut}(E) \cong \text{Iso}_\mathcal{M}([E])$, and $d\xi|_{sG} : T_sS \cong T_{sG}[S/G] \to T_{[E]}\mathcal{M} \cong \text{Ext}^1(E, E)$ is an isomorphism.
As \( \mathfrak{M} \) is locally a global quotient, let’s say \( \mathfrak{M} \) is locally \([Q/H]\) with \( H = \text{GL}(n, \mathbb{K}) \), and a \( \mathbb{K} \)-scheme \( Q \) which is \( H \)-invariant, so that the projection \([Q/H] \to \mathfrak{M}\) is a 1-isomorphism with an open \( \mathbb{K} \)-substack \( \Sigma \) of \( \mathfrak{M} \). This 1-isomorphism identifies the stabilizer groups \( \text{Iso}_{\mathfrak{M}}([E]) = \text{Aut}(E) \) and \( \text{Iso}_Q([Q/H]) = \text{Stab}_H(s) \), and the Zariski tangent spaces \( T_{[E]}\mathfrak{M} \cong \text{Ext}^1(E, E) \) and \( T_{[Q/H]}\mathfrak{M} \cong T_sQ/T_s(h) \), so one has natural isomorphisms \( \text{Aut}(E) \cong \text{Stab}_H(s) \) and \( \text{Ext}^1(E, E) \cong T_sQ/T_s(h) \), and \( G \) is identified as a subgroup of \( H \).

To obtain the 1-morphism with the required properties, following [64] and Luna’s Étale Slice Theorem [SS] III, we obtain an atlas \( S \) as a \( G \)-invariant, locally closed \( \mathbb{K} \)-scheme in \( Q \) with \( s \in S(\mathbb{K}) \) such that \( T_sQ = T_sS \oplus T_s(h) \), and the morphism \( \mu : S \times H \to Q \) induced by the inclusion \( S \to Q \) and the \( H \)-action on \( Q \) is smooth of relative dimension \( \dim \text{Aut}(E) \). Here \( s \in Q(\mathbb{K}) \) project to the point \( sH \) in \( \Sigma(\mathbb{K}) \) identified with \( [E] \in \mathfrak{M}(\mathbb{K}) \) under the 1-isomorphism \( \Sigma \cong [Q/H] \) and \( G \), a \( \mathbb{K} \)-subgroup of \( \mathbb{K} \)-group \( H \), is as in the statement of Theorem 4.1, that is, a maximal torus in \( \text{Aut}(E) \). Since \( S \) is invariant under the \( \mathbb{K} \)-subgroup \( G \) of \( \mathbb{K} \)-group \( H \) acting on \( Q \), the inclusion \( i : S \to Q \) induces a representable 1-morphism of quotient stacks \( i_* : [S/G] \to [Q/H] \). In [64], Joyce and Song found that \( i_* \) is smooth of relative dimension \( \dim \text{Aut}(E) - \dim G \). Combining the 1-morphism \( i_* : [S/G] \to [Q/H] \), the 1-isomorphism \( \Sigma \cong [Q/H] \), and the open inclusion \( \Sigma \to \mathfrak{M} \), yields a 1-morphism \( \xi : [S/G] \to \mathfrak{M} \), as required for Theorem 4.1. This \( \xi \) is smooth of relative dimension \( \dim \text{Aut}(E) \), that is, \( \xi \) is smooth of relative dimension \( 0 \), that is, \( \xi \) is étale. The conditions that \( \xi(sG) = [E] \) and that \( \xi_* : \text{Iso}_{[S/G]}(sG) \to \text{Iso}_{\mathfrak{M}}([E]) \) is the natural \( G \to \text{Aut}(E) \cong \text{Iso}_{\mathfrak{M}}([E]) \) in Theorem 1.13 are immediate from the construction. Then \( \text{Ext}^1(E, E) \) is an isomorphism follows from \( T_{i([S/G])}[Q/H] \cong T_sQ/T_s(h) \).

In conclusion, we can summarize as follows: given a point \( [E] \in \mathfrak{M}(\mathbb{K}) \), that is an equivalence class of (a complex of) coherent sheaves, we will denote by \( G \) a maximal torus in \( \text{Aut}(E) \). As \( \mathfrak{M} \) is locally a global quotient, there exists an atlas \( S \), which is a scheme over \( \mathbb{K} \), and a smooth morphism \( \pi : S \to \mathfrak{M} \), with \( \pi \) smooth of relative dimension \( \dim G \). If \( x \in S \) is the point corresponding to \( [E \in \mathfrak{M}(\mathbb{K}) \), then \( \pi \) smooth of \( \dim G \) means that \( \pi \) has minimal dimension near \( x \), that is, \( \pi_*S = \text{Ext}^1(E, E) \). Moreover, the atlas \( S \) is endowed with a \( G \)-action, so that \( \pi \) descends to a morphism \([S/G] \to \mathfrak{M} \).

Note next that the maximal torus \( G \) acts on \( S \) preserving \( s \) and fixing \( x \). By replacing \( S \) by a \( G \)-equivariant étale open neighbourhood \( S' \) of \( s \), we can suppose \( S \) is affine. Then, from material in [2] and [3] we deduce that the atlas \( S' \) in the sense of Theorems 3.3 and 3.11 for the moduli stack \( \mathfrak{M} \) carries a d-critical locus structure \((S', s_{S'})\) which is \( \text{GL}(n, \mathbb{K}) \)-equivariant in the sense of [2.3].

Using Proposition 2.2.1 there exists a \( G \)-invariant critical chart \((R, U, f, i)\) in the sense of [2] for \((S, s)\) with \( x \in R \), and \( \dim U \) to be minimal so that \( T_{i(x)}U = T_xR = \text{Ext}^1(E, E) \).

Making \( U \) smaller if necessary, we can choose \( G \)-equivariant étale coordinates \( U \to \mathbb{A}^n = \text{Ext}^1(E, E) \) near \( i(x) \), sending \( i(x) \) to \( 0 \), and with \( T_{i(x)}U = \text{Ext}^1(E, E) \) the given identification. Then we can regard \( U \to \text{Ext}^1(E, E) \) as a \( G \)-equivariant étale open neighbourhood of \( 0 \) in \( \text{Ext}^1(E, E) \), which concludes the proof of Theorem 4.1.

### 4.2 Behrend function identities

Now we are ready to prove Theorem 4.2. Let \( X \) be a Calabi–Yau 3-fold over an algebraically closed field \( \mathbb{K} \) of characteristic zero, \( \mathfrak{M} \) the moduli stack of coherent sheaves on \( X \), and \( E_1, E_2 \) be coherent sheaves on \( X \). Set \( E = E_1 \oplus E_2 \). Using the splitting

\[
\text{Ext}^1(E, E) = \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2) \oplus \text{Ext}^1(E_1, E_2) \oplus \text{Ext}^1(E_2, E_1),
\]

write elements of \( \text{Ext}^1(E, E) \) as \((\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21})\) with \( \epsilon_{ij} \in \text{Ext}^1(E_i, E_j) \). For simplicity, we will write \( e_{ij} = \dim \text{Ext}^1(E_i, E_j) \). Choose a maximal torus \( G \) of \( \text{Aut}(E) \) which contains the subgroup \( T = \{\text{id}_{E_1} + \lambda \text{id}_{E_2} : \lambda \in \mathbb{G}_m\} \), which acts on \( \text{Ext}^1(E, E) \) by

\[
\lambda : (\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}) \mapsto (\epsilon_{11}, \epsilon_{22}, \lambda^{-1} \epsilon_{12}, \lambda \epsilon_{21}).
\]

Apply Theorem 4.1 with these \( E \) and \( G \). This gives an étale morphism \( u : U \to \text{Ext}^1(E, E) \) with \( U \) a smooth affine \( \mathbb{K} \)-scheme, and \( u(p) = 0 \), for \( p \in U(\mathbb{K}) \), a \( G \)-invariant regular function \( f : U \to \mathbb{A}^1_\mathbb{K} \) on \( U \) with
where one uses that \( \xi \) is smooth of relative dimension \( \dim \text{Aut}(E) \), and Theorem 1.7 to say that

\[
\nu_{\text{Crit}(f)} = (-1)^{\dim(\text{Aut}(E))} \xi^* \nu_M.
\]

On the other hand, the last part of the proof of (4.1) in [64, Section 10.1] uses algebraic methods and gives

\[
\nu_M(E_1 \oplus E_2) = (-1)^{\dim(\text{Aut}(E_1)) + \dim(\text{Aut}(E_2))} \nu_{\text{Crit}(f \circ g)}(0),
\]

where \( \nu_{\text{Crit}(f \circ g)}(0) = \nu_{\text{Crit}(f)}(0) = \nu_{\text{Crit}(f)}(0) \) and \( U \) is as in Theorem 4.1 and \( \text{Ext}^1(E, E)^G \) denotes the fixed point locus of \( \text{Ext}^1(E, E) \) for the \( G \)-action. Thus what actually remains to prove in order to establish identity (4.1) is

\[
\nu_{\text{Crit}(f)}(0) = (-1)^{\dim \text{Ext}^1(E_1, E_2) + \dim \text{Ext}^1(E_2, E_1)} \nu_{\text{Crit}(f \circ g)}(0).
\]

This is a generalization of a result in [6] over \( \mathbb{C} \) in the case of an isolated \( \mathbb{C} \)-fixed point. Combining equations (4.2), (4.6) and (4.7) and sorting out the signs as in [64, Section 10.1] proves equation (4.4). Equation (4.7) will be crucial also for the proof of the second Behrend identity (4.2).

Let us start by recalling an easy result similar to [64, Prop. 10.1], but now in the \( \text{étale} \) topology. Let \( u : U \to \text{Ext}^1(E, E) \) be the \( \text{étale} \) map as in (4.1) and \( p \in U \) such that \( u(p) = 0 \). We will consider points \( (0, 0, \epsilon, 0), (0, 0, 0, \epsilon_2) \in \text{Ext}^1(E, E) \) like basically points in \( U \). This is because we consider a unique lift \( \alpha(\epsilon_2) \) of \( (0, 0, \epsilon_2, 0) \in \text{Ext}^1(E, E) \) to \( U \), such that \( u(\alpha(\epsilon_2)) = (0, 0, \epsilon_2, 0) \) and \( \lim_{\lambda \to 0} \lambda \alpha(\epsilon_2) = p \), using that \( \lim_{\lambda \to 0} (0, 0, \lambda^{-1} \epsilon_2, 0) = (0, 0, 0, 0) \). So we can state the following result, for the proof of which we cite [64, Prop.10.1], with appropriate obvious modifications, working in the \( \text{étale} \) topology.

**Proposition 4.7.** Let \( \epsilon_1 \in \text{Ext}^1(E_1, E_2) \) and \( \epsilon_2 \in \text{Ext}^1(E_2, E_1) \). Then

(i) \( (0, 0, \epsilon_2), (0, 0, 0, \epsilon_2) \in \text{Crit}(f) \subseteq U \subseteq \text{Ext}^1(E, E) \), and \( (0, 0, \epsilon_2), (0, 0, 0, \epsilon_2) \in V \subseteq S(\mathbb{K}) \subseteq \text{Ext}^1(E, E) \);

(ii) \( \xi \) maps \( (0, 0, \epsilon_2, 0) \to (0, 0, \epsilon_2, 0) \) and \( (0, 0, 0, \epsilon_2) \to (0, 0, 0, \epsilon_2) \); and

(iii) the induced morphism on closed points \( S(\text{Aut}(E)) \to \mathbb{M}(\mathbb{K}) \) maps \( (0, 0, 0, \epsilon_2) \to [F] \) and \( [(0, 0, 0, \epsilon_2)] \to [F'] \), where the exact sequences \( 0 \to E_1 \to F \to E_2 \to 0 \) and \( 0 \to E_2 \to F' \to E_1 \to 0 \) in \( \text{coh}(X) \) correspond to \( \epsilon_2 \in \text{Ext}^1(E_1, E_2) \) and \( \epsilon_2 \in \text{Ext}^1(E_2, E_1) \), respectively.

Now use the idea in [64, §10.2]. Set \( U' = \{(\epsilon_{11}, \epsilon_{12}, \epsilon_{12}, \epsilon_{21}) \in U : \epsilon_{21} \neq 0 \} \), an open set in \( U \), and write \( V' \) for the submanifold of \( \{(\epsilon_{11}, \epsilon_{12}, \epsilon_{12}, \epsilon_{21}) \in U' \text{ with } \epsilon_{12} = 0 \} \). Let \( U' \) be the blowup of \( U' \) along \( V' \), with projection \( \pi' : U' \to U' \). Points of \( U' \) may be written \( (\epsilon_{11}, \epsilon_{12}, \lambda, \epsilon_{21}) \in \mathbb{P}(\text{Ext}^1(E_1, E_2)) \), and \( \lambda \in \mathbb{K} \), and \( \epsilon_{21} \neq 0 \). Write \( f' = f|_{U'} \) and \( f'' = f' \circ \pi' \). Then applying Theorem 1.8 to \( U', V', f', U', \pi', f' \) at the point \( (0, 0, 0, \epsilon_{21}) \in U' \) gives

\[
\nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21}) = \int_{[\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} \nu_{\text{Crit}(f_{|V'})}(0, 0, 0, \epsilon_{21}) \chi + (-1)^{\epsilon_{21}} (1 - \epsilon_{21}) \nu_{\text{Crit}(f_{|V'})}(0, 0, 0, \epsilon_{21}).
\]

(4.8)

Here \( \nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21}) \) is independent of the choice of \( \epsilon_{21} \) representing the point \( [\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_2, E_1)) \), and is a constructible function of \( [\epsilon_{21}] \), so the integrals in (4.8) are well-defined. Note that \( \nu_{\text{Crit}(f)} \) and the other Behrend functions in the sequel are nonzero just on the zero loci of the corresponding functions, so here and in the sequel the integrals over the whole \( \mathbb{P}(\text{Ext}^1(\ldots)) \) actually are just over the points that lie in these zero loci. Adopt this convention for the whole section.
Similarly consider the analogous situation exchanging the role of $\epsilon_{12}$ and $\epsilon_{21}$. Set $U'' = \{ (\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}) \in U : \epsilon_{12} \neq 0 \}$, an open set in $U$, and write $V'' = \{ (\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21}) \in U'' : \epsilon_{21} = 0 \}$. Let $\tilde{U}''$ be the blowup of $U''$ along $V''$, with projection $\pi'' : \tilde{U}'' \to U''$. Points of $\tilde{U}''$ may be written $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \epsilon_{21})$, where $[\epsilon_{21}] \in \mathbb{P}^\mathbb{P}(\text{Ext}^1(E_2, E_1))$, and $\lambda \in \mathbb{K}$, and $\epsilon_{12} \neq 0$. Write $f'' = f|_{U''}$ and $\tilde{f}'' = f'' \circ \pi''$. Similarly to the previous situation, we can apply Theorem 1.8 to $U'', \tilde{U}''$, $f'', \tilde{f}''$ at the point $0, 0, \epsilon_{12}, 0 \in U''$ which gives

$$\nu_{\text{Crit}(f)}(0, 0, \epsilon_{12}, 0) = \int_{[\epsilon_{21}] \in \mathbb{P}^\mathbb{P}(\text{Ext}^1(E_2, E_1))} \nu_{\text{Crit}(\tilde{f}'')(0, 0, \epsilon_{12}, 0, [\epsilon_{21}])} d\chi + (-1)^{\epsilon_{21}}(1 - \epsilon_{21})\nu_{\text{Crit}(f|_{\tilde{U}'})}(0, 0, \epsilon_{12}, 0).$$

(4.9)

Let $L_{12} \to \mathbb{P}(\text{Ext}^1(E_1, E_2))$ and $L_{21} \to \mathbb{P}(\text{Ext}^1(E_2, E_1))$ be the tautological line bundles, so that the fibre of $L_{12}$ over a point $[\epsilon_{12}]$ in $\mathbb{P}(\text{Ext}^1(E_1, E_2))$ is the 1-dimensional subspace $\{ \lambda \epsilon_{12} : \lambda \in \mathbb{K} \}$ in $\text{Ext}^1(E_1, E_2)$. Consider the fibre product

$$\begin{array}{ccc}
Z & \xrightarrow{\text{étale}} & \text{Ext}^1(E_1, E_1) \times \text{Ext}^1(E_2, E_2) \times (L_{12} \oplus L_{21}) \\
\downarrow & \nearrow & \downarrow \\
U & & \text{Ext}^1(E, E)
\end{array}$$

where the horizontal maps are étale morphisms. Informally, this defines $Z \subseteq \text{Ext}^1(E_1, E_1) \times \text{Ext}^1(E_2, E_2) \times (L_{12} \oplus L_{21})$ to be the étale open subset of points $(\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \lambda_1 \epsilon_{12}, \lambda_2 \epsilon_{21})$ for $\lambda_1 \in \mathbb{K}$, for which $(\epsilon_{21}, \epsilon_{22}, \lambda_1 \epsilon_{12}, \lambda_2 \epsilon_{21})$ lies in $U$. Observe that $Z$ contains both $\tilde{U}'$ and $\tilde{U}''$, which respectively have subspaces $\text{Crit}(\tilde{f}')$ and $\text{Crit}(\tilde{f}'')$.

Define also an étale open set of points $W \subseteq \text{Ext}^1(E_1, E_1) \times \text{Ext}^1(E_2, E_2) \times (L_{12} \oplus L_{21})$ fitting into the following cartesian square:

$$\begin{array}{ccc}
Z & \xrightarrow{\text{étale}} & \text{Ext}^1(E_1, E_1) \times \text{Ext}^1(E_2, E_2) \times (L_{12} \oplus L_{21}) \\
\downarrow & & \downarrow \\
W & \xrightarrow{\text{étale}} & \text{Ext}^1(E_1, E_1) \times \text{Ext}^1(E_2, E_2) \times (L_{12} \oplus L_{21})
\end{array}$$

where the line bundle $L_{12} \otimes L_{21} \to \mathbb{P}(\text{Ext}^1(E_1, E_2)) \times \mathbb{P}(\text{Ext}^1(E_2, E_1))$ has fibre over $([\epsilon_{12}], [\epsilon_{21}])$ which is $\{ \lambda \epsilon_{12} \otimes \epsilon_{21} : \lambda \in \mathbb{K} \}$. Write points of the total space of $L_{12} \otimes L_{21}$ as $([\epsilon_{12}], [\epsilon_{21}], \lambda \epsilon_{12} \otimes \epsilon_{21})$. Informally, $W$ is defined as the open subset of points $(\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], [\epsilon_{21}], \lambda \epsilon_{12} \otimes \epsilon_{21})$ for which $(\epsilon_{21}, \epsilon_{22}, \lambda \epsilon_{12} \epsilon_{21})$ lies in $U$. Since $U$ is $G$-invariant, this definition is independent of the choice of representatives $\epsilon_{12}, \epsilon_{21}$ for $[\epsilon_{12}], [\epsilon_{21}]$, since any other choice would replace $(\epsilon_{11}, \epsilon_{22}, \lambda \epsilon_{12}, \epsilon_{21})$ by $(\epsilon_{11}, \epsilon_{22}, \lambda \mu \epsilon_{12}, \mu^{-1} \epsilon_{21})$ for some $\mu \in \mathbb{G}_m$. The map $\Pi : Z \to W$ is étale equivalent to

$$\Pi' : (\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], \lambda_1 \epsilon_{12}, [\epsilon_{21}], \lambda_2 \epsilon_{21}) \mapsto (\epsilon_{11}, \epsilon_{22}, [\epsilon_{12}], [\epsilon_{21}], \lambda_1 \epsilon_{12} \otimes \epsilon_{21})$$

which is a smooth projection of relative dimension 1 except at the points such that $\lambda_1 = \lambda_2 = 0$. However it is smooth at $(0, \lambda_2)$ with $\lambda_2 \neq 0$ and similarly at $(\lambda_1, 0)$ with $\lambda_1 \neq 0$, that is, the two restrictions of $\Pi$ to $\tilde{U}'$ and $\tilde{U}''$ are both smooth of relative dimension 1.

Here is the crucial point: $\text{Crit}(\tilde{f}') \subseteq \tilde{U}'$ and $\text{Crit}(\tilde{f}'') \subseteq \tilde{U}''$ are $\mathbb{G}_m$-invariant subschemes, so there exists a subscheme $Q$ of $W$ such that $\text{Crit}(\tilde{f}') = \Pi^{-1}(Q) \cap \tilde{U}'$ and $\text{Crit}(\tilde{f}'') = \Pi^{-1}(Q) \cap \tilde{U}''$ and both $\Pi : \text{Crit}(\tilde{f}') \to Q$ and $\Pi : \tilde{W}'' \to Q$ are smooth of relative dimension 1. Thus Theorem 1.7 yields that $\nu_{\text{Crit}(\tilde{f}')} = -\Pi^*(\nu_Q)$ and $\nu_{\text{Crit}(\tilde{f}'')} = -\Pi^*(\nu_Q)$ and then

$$\nu_{\text{Crit}(\tilde{f}')}(0, 0, [\epsilon_{12}], 0, [\epsilon_{21}]) = -\nu_Q(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0) = \nu_{\text{Crit}(\tilde{f}'')}(0, 0, \epsilon_{12}, 0, [\epsilon_{21}]),$$

(4.10)

where the sign comes from the fact that the map $\Pi$ is smooth of relative dimension 1. Moreover observe that

$$\nu_{\text{Crit}(\tilde{f}'')} (0, 0, \epsilon_{12}, 0) = (-1)^{\epsilon_{21}} \nu_{\text{Crit}(\tilde{f}')} (0, 0, 0, 0),$$

(4.11)
This is because the $T$-invariance of $f$ imply that its values on $(\epsilon_{11}, \epsilon_{22}, 0, \epsilon_{21})$ and $(\epsilon_{11}, \epsilon_{22}, 0, 0)$ are the same and the projection $\text{Crit}(f|_{V'}) \to \text{Crit}(f|_{\mathcal{T}})$ is smooth of relative dimension $\epsilon_{21}$. For the same reason, one has

$$\nu_{\text{Crit}(f|_{V'})}(0, 0, \epsilon_{12}, 0) = (-1)^{\epsilon_{12}} \nu_{\text{Crit}(f|_{\mathcal{T}})}(0, 0, 0, 0).$$  

(4.12)

Now, substitute equations (4.10), (4.11) and (4.12) into (4.8) and (4.9). One gets

$$\nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21}) = -\int_{[\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E, E_1))} \nu_Q(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0) \, d\chi + (-1)^{\epsilon_{12}+\epsilon_{21}} (1 - \epsilon_{12}) \nu_{\text{Crit}(f|_{\mathcal{T}})}(0, 0, 0, 0),$$

(4.13)

$$\nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21}) = -\int_{[\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E, E_1))} \nu_Q(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0) \, d\chi + (-1)^{\epsilon_{12}+\epsilon_{21}} (1 - \epsilon_{21}) \nu_{\text{Crit}(f|_{\mathcal{T}})}(0, 0, 0, 0).$$

(4.14)

Finally integrating (4.13) over $[\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$ and (4.14) over $[\epsilon_{12}] \in \mathbb{P}(\text{Ext}^1(E, E_2))$, yields

$$\int_{[\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E_2, E_1))} \nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21}) \, d\chi = -\int_{([\epsilon_{12}], [\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E, E_2))) \times \mathbb{P}(\text{Ext}^1(E, E_2))} \nu_Q(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0) \, d\chi$$

\begin{equation}
+ (-1)^{\epsilon_{12}+\epsilon_{21}} (1 - \epsilon_{12}) \epsilon_{21} \nu_{\text{Crit}(f|_{\mathcal{T}})}(0),
\end{equation}

(4.15)

$$\int_{[\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E, E_2))} \nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{12}) \, d\chi = -\int_{([\epsilon_{12}], [\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E, E_2))) \times \mathbb{P}(\text{Ext}^1(E, E_2))} \nu_Q(0, 0, [\epsilon_{12}], [\epsilon_{21}], 0) \, d\chi$$

\begin{equation}
+ (-1)^{\epsilon_{12}+\epsilon_{21}} (1 - \epsilon_{21}) \epsilon_{12} \nu_{\text{Crit}(f|_{\mathcal{T}})}(0),
\end{equation}

(4.16)

since $\chi(\mathbb{P}(\text{Ext}^1(E_2, E_1))) = \epsilon_{21}$ and $\chi(\mathbb{P}(\text{Ext}^1(E_1, E_2))) = \epsilon_{12}$. Subtracting (4.15) from (4.16), gives

$$\int_{[\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E_2, E_1))} \nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21}) \, d\chi - \int_{[\epsilon_{12}]\in \mathbb{P}(\text{Ext}^1(E, E_2))} \nu_{\text{Crit}(f)}(0, 0, \epsilon_{12}, 0) \, d\chi =$$

\begin{equation}
(1 - 1)^{\epsilon_{12}+\epsilon_{21}} (1 - \epsilon_{21}) \epsilon_{12} \nu_{\text{Crit}(f|_{\mathcal{T}})}(0).
\end{equation}

(4.17)

Consider equation (4.17) applied substituting $\mathbb{P}(\text{Ext}^1(E_2, E_1) \oplus \mathbb{K})$ to $\mathbb{P}(\text{Ext}^1(E_2, E_1))$. This adds one dimension to $\text{Ext}^1(E, E)$. Denote $\tilde{f}$ the lift of $f$ to $\text{Ext}^1(E, E) \oplus \mathbb{K}$. In this case equation (4.17) becomes

$$\int_{[\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E_2, E_1) \oplus \mathbb{K})} \nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21} \oplus \lambda) \, d\chi - \int_{[\epsilon_{12}]\in \mathbb{P}(\text{Ext}^1(E, E_2))} \nu_{\text{Crit}(f)}(0, 0, \epsilon_{12}, 0) \, d\chi =$$

\begin{equation}
(1 - 1)^{1+\epsilon_{12}+\epsilon_{21}} (1 + \epsilon_{21} - \epsilon_{12}) \nu_{\text{Crit}(f|_{\mathcal{T}})}(0),
\end{equation}

(4.18)

Now, observe that $\nu_{\text{Crit}(f)} = -\nu_{\text{Crit}(\tilde{f})}$ from Theorem L.7 and $\nu_{\text{Crit}(\tilde{f})}(0) = \nu_{\text{Crit}(f|_{\mathcal{T}})}(0) = 0$ as $(\text{Ext}^1(E, E) \oplus \mathbb{K})^G = \text{Ext}^1(E, E)^G \oplus 0$ and the map $\text{Crit}(\tilde{f})^G \to \text{Crit}(f|_{\mathcal{T}})^G$ is étale. Thus

$$\int_{[\epsilon_{21}]\in \mathbb{P}(\text{Ext}^1(E_2, E_1))} \nu_{\text{Crit}(f)}(0, 0, 0, \epsilon_{21}) \, d\chi - \nu_{\text{Crit}(f)}(0, 0, 0, 0) + \int_{[\epsilon_{12}]\in \mathbb{P}(\text{Ext}^1(E, E_2))} \nu_{\text{Crit}(f)}(0, 0, \epsilon_{12}, 0) \, d\chi =$$

\begin{equation}
(1 - 1)^{1+\epsilon_{12}+\epsilon_{21}} (1 + \epsilon_{21} - \epsilon_{12}) \nu_{\text{Crit}(f|_{\mathcal{T}})}(0).
\end{equation}

(4.19)

Here, $\nu_{\text{Crit}(f)}(0)$ on the i.h.s. comes from the fact that the $\mathbb{G}_m$-action over $\mathbb{P}(\text{Ext}^1(E_2, E_1) \oplus \mathbb{K})$ fixes $\mathbb{P}(\text{Ext}^1(E_2, E_1))$ and $[0, 1]$; the free orbits of the $\mathbb{G}_m$-action contribute zero to the weighted Euler characteristic.
Then one uses that $\nu_{\text{Crit}(f)}$ valued over $[0,1]$ is equal to $-\nu_{\text{Crit}(f)}(0)$. Adding (4.17) and (4.19) yields (4.14), which concludes the proof of identity (4.1).

The conclusion of the proof of identity (4.2) is now easy. Let $0 \neq \epsilon_{21} \in \text{Ext}^1(E_2,E_1)$ correspond to the short exact sequence $0 \to E_1 \to F \to E_2 \to 0$ in $\text{coh}(X)$. Then

$$\nu_{\text{Crit}}(F) = (-1)^{\dim \text{Aut}(E)} \nu_{\text{Crit}(f)}(0,0,0,\epsilon_{21})$$

(4.20)

using $\xi_s : ((0,0,0,\epsilon_{21})) \to [F]$ from Proposition (4.7) and $\xi$ smooth of relative dimension $\dim(\text{Aut}(E))$. Substituting (4.20) and its analogue for $D$ in the place of $F$ into (4.2), using equation (4.10) and identity (4.7) to substitute for $\nu_{\text{Crit}}(E_1 \oplus E_2)$, and cancelling factors of $(-1)^{\dim \text{Aut}(E)}$, one gets that (4.2) is equivalent to (4.14), which concludes the proof.

### 4.3 Deformation invariance issue

Thomas’ original definition (1.1) of $DT^\alpha(\tau)$, and Joyce and Song’s definition (1.20) of the pair invariants $P_1^\alpha,\gamma(\tau')$, are both valid over $\mathbb{K}$. Joyce and Song suggest to solve problem (b) in [44] to work in [44] Rank 4.20 (e), replacing $H^*(X;\mathbb{Q})$ by the algebraic de Rham cohomology $H^*_{\text{dR}}(X)$ of Hartshorne [16]. Here we suggest another argument which is based on the theory of Picard schemes by Grothendieck [44, 45]. Other references are [37, 72]. Even if our argument will not prove that the numerical Grothendieck groups are deformation invariant, as this last fact depend deeply on the integral Hodge conjecture type result [134] which we are not able to prove in this more general context, we will however find a deformation invariant lattice $\Lambda_X$ containing its image through the Chern character map and define $DS^\alpha(\tau)$, for $\alpha \in \Lambda_X$, which will be deformation invariant.

To prove deformation-invariance we need to work not with a single Calabi–Yau 3-fold $X$ over $\mathbb{K}$, but with a family of Calabi–Yau 3-folds $X \to T$ over a base $\mathbb{K}$-scheme $T$. Taking $T = \text{Spec} \mathbb{K}$ recovers the case of one Calabi–Yau 3-fold. Here we are our assumptions and notation for such families. Let $X \to T$ be a smooth projective morphism of algebraic $\mathbb{K}$-varieties $X, T$, with $T$ connected. Let $O_X(1)$ be a relative very ample line bundle for $X \to T$. For each $t \in T(\mathbb{K})$, write $X_t$ for the fibre $X \times_{\tau_t} T \to \text{Spec} \mathbb{K}$ of $\tau$ over $t$, and $O_{X_t}(1)$ for $O_X(1)|_{X_t}$. Suppose that $X_t$ is a smooth Calabi–Yau 3-fold over $\mathbb{K}$ for all $t \in T(\mathbb{K})$, with $H^1(O_{X_t}) = 0$.

There are some important existence theorems which refine the original Grothendieck’s theorem [134 Thm. 3.1]. In [3 Thm. 7.3], Artin proves that given $f : X \to S$ a flat, proper, and finitely presented map of algebraic spaces cohomologically flat in dimension zero, then the relative Picard scheme $\text{Pic}_{X/S}$ exists as an algebraic space which is locally of finite presentation over $S$. Its fibres are the Picard schemes $\text{Pic}(X_t)$ of the fibres. They form a family whose total space is $\text{Pic}_{X/S}$. In [44 Prop. 2.10] Grothendieck shows that if $H^1(X_t, O_{X_t}) = 0$ for some $s \in S$, there exists a neighborhood $U$ of $s$ such that the scheme $\text{Pic}_{X/S}$ is smooth, and in this case $\dim(\text{Pic}(X_s)) = \dim(H^1(X_s, O_{X_s}))$.

In our case, $\text{Pic}_{X/T}$ exists and is smooth with 0-dimensional fibres which are the Picard schemes $\text{Pic}(X_t)$. Moreover the morphism $(\tau, P) : \text{Pic}_{X/T} \to T \times \mathbb{Q}[s]$, where $\tau$ is the projection to the base scheme and $P$ assigns to an isomorphism class of a line bundle $[L]$ its Hilbert polynomial $P_L(s)$ with respect to $O_X(1)$, is proper. This implies an upper semicontinuity result for $t \mapsto \dim(\text{Pic}(X_t))$ [45 Cor. 2.7]. These results yield that the Picard schemes $\text{Pic}(X_t)$ for $t \in T(\mathbb{K})$ are canonically isomorphic locally in $T(\mathbb{K})$. Observe that at the moment we don’t have canonical isomorphisms $\text{Pic}(X_t) \cong \text{Pic}(X)$ for all $t \in T(\mathbb{K})$ (this would be canonically isomorphic globally in $T(\mathbb{K})$). Instead, we mean that the groups $\text{Pic}(X_t)$ for $t \in T(\mathbb{K})$ form a local system of abelian groups over $T(\mathbb{K})$, with fibre $\text{Pic}(X)$.

When $\mathbb{K} = \mathbb{C}$, Joyce and Song proved [64 §4] that $K^{\text{num}}(\text{coh}(X_t))$ form a local system of abelian groups over $T(\mathbb{K})$, with fibre $K^{\text{num}}(\text{coh}(X))$. This means that in simply-connected regions of $T(\mathbb{C})$ in the complex analytic topology the $K^{\text{num}}(\text{coh}(X_t))$ are all canonically isomorphic, and isomorphic to $K(\text{coh}(X))$. But around loops in $T(\mathbb{C})$, this isomorphism with $K(\text{coh}(X))$ can change by monodromy, by an automorphism $\mu : K(\text{coh}(X)) \to K(\text{coh}(X))$ of $K(\text{coh}(X))$. In [64 Thm 4.21] they showed that the group of such monodromies $\mu$ is finite, and so it is possible to make it trivial by passing to a finite cover $\tilde{T}$ of $T$. If they worked instead with invariants $PI^\alpha,\gamma(\tau')$ counting pairs $s : O_X(-\nu) \to E$ in which $E$ has fixed Hilbert polynomial $P$, rather than fixed class $\alpha \in K^{\text{num}}(\text{coh}(X))$, as in Thomas’ original definition of Donaldson–Thomas invariants [121], then they could drop the assumption on $K^{\text{num}}(\text{coh}(X_t))$ in Theorem [64 Thm. 5.25].
Similarly, we now study monodromy phenomena for Pic(X₁) in families of smooth \( \mathbb{K} \)-schemes \( X \to T \) following the idea of [64, Thm. 4.21]. We find that we can always eliminate such monodromy by passing to a finite cover \( \tilde{T} \) of \( T \). This is crucial to prove deformation-invariance of the \( \Delta T^\alpha(\tau), P I^{\alpha,n}(\tau') \) in [61, §12].

**Theorem 4.8.** Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero, \( \varphi : X \to T \) a smooth projective morphism of \( \mathbb{K} \)-schemes with \( T \) connected, and \( O_X(1) \) a relative very ample line bundle on \( X \), so that for each \( t \in T(\mathbb{K}) \), the fibre \( X_t \) of \( \varphi \) is a smooth projective \( \mathbb{K} \)-scheme with very ample line bundle \( O_{X_t}(1) \). Suppose the Picard schemes Pic(X₁) are locally constant in \( T(\mathbb{K}) \), so that \( t \mapsto \text{Pic}(X_t) \) is a local system of abelian groups on \( T \). Fix a base point \( s \in T(\mathbb{K}) \), and let \( \Gamma \) be the monodromy group of Pic(Xₜ). Then \( \Gamma \) is a finite group. There exists a finite étale cover \( \pi : \tilde{T} \to T \) of degree \( |\Gamma| \), with \( \tilde{T} \) a connected \( \mathbb{K} \)-scheme, such that writing \( \tilde{X} = X \times_T \tilde{T} \) and \( \tilde{\varphi} : \tilde{X} \to \tilde{T} \) for the natural projection, with fibre \( \tilde{X}_t \) at \( \tilde{t} \in T(\mathbb{K}) \), then Pic(\( \tilde{X}_t \)) for all \( \tilde{t} \in T(\mathbb{K}) \) are all globally canonically isomorphic to Pic(Xₜ). That is, the local system \( \tilde{t} \mapsto \text{Pic}(\tilde{X}_t) \) on \( \tilde{T} \) is trivial.

**Proof.** As Pic(Xₜ) is finitely generated, one can choose classes \( [L_1], \ldots, [L_k] \in \text{Pic}(X_s) \) as generators. Let \( P_1, \ldots, P_k \) be the Hilbert polynomials respectively of \( [L_1], \ldots, [L_k] \) with respect to \( O_X(1) \). Let \( \gamma \in \Gamma \), and consider the images \( \gamma \cdot [L_i] \in \text{Pic}(X_s) \) for \( i = 1, \ldots, k \). As we assume \( O_X(1) \) is globally defined on \( T \), and does not change under monodromy, it follows that the Hilbert polynomials \( P_1, \ldots, P_k \) do not change under monodromy. Hence \( \gamma \cdot [L_i] \) has Hilbert polynomial \( P_i \). Again one uses properness to show that the set \( \text{Pic}^\gamma(X_s) \) composed by isomorphism classes of line bundles in Pic(Xₜ) with Hilbert polynomial \( P_i \) for some \( i = 1, \ldots, k \) is a finite set, that is, every \( P_i \) is the Hilbert polynomial of only finitely many classes \( [R_1], \ldots, [R_{n_i}] \) in Pic(Xₜ). It follows that for each \( \gamma \in \Gamma \) we have \( \gamma \cdot [L_i] \in \{ [R_1], \ldots, [R_{n_i}] \} \). So there are at most \( n_1 \cdots n_k \) possibilities for \( (\gamma \cdot [L_1], \ldots, \gamma \cdot [L_k]) \). But \( (\gamma \cdot [L_1], \ldots, \gamma \cdot [L_k]) \) determines \( \gamma \) as \( [L_1], \ldots, [L_k] \) generate Pic(Xₜ). Hence \( |\Gamma| \leq n_1 \cdots n_k \), and \( \Gamma \) is finite.

We can now construct an étale cover \( \pi : \tilde{T} \to T \) which is a principal \( \Gamma \)-bundle, and so has degree \( |\Gamma| \), such that the \( \mathbb{K} \)-points of \( \tilde{T} \) are pairs \( (\tilde{t}, \iota) \) where \( t \in T(\mathbb{K}) \) and \( \iota : \text{Pic}(X_t) \to \text{Pic}(X_{\tilde{t}}) \) is an isomorphism from the properness and smoothness argument above, and \( \Gamma \) acts freely on \( T(\mathbb{K}) \) by \( \gamma : (t, \iota) \mapsto (t, \gamma \circ \iota) \), so that the \( \Gamma \)-orbits correspond to points \( t \in T(\mathbb{K}) \). Then for \( \tilde{t} = (t, \iota) \) we have \( \tilde{X}_t = X_t \), with canonical isomorphism \( \iota : \text{Pic}(X_{\tilde{t}}) \to \text{Pic}(X_t) \).

So the conclusion is that from properness and smoothness argument, Pic(Xₜ) are canonically isomorphic locally in \( T(\mathbb{K}) \). But by Theorem 4.8 one can pass to a finite cover \( \tilde{T} \) of \( T \), so that the Pic(\( \tilde{X}_t \)) are canonically isomorphic globally in \( \tilde{T}(\mathbb{K}) \). So, replacing \( X, T \) by \( \tilde{X}, \tilde{T} \), we will assume from here that the Picard schemes Pic(Xₜ) for \( t \in T(\mathbb{K}) \) are all canonically isomorphic globally in \( T(\mathbb{K}) \), and we write Pic(X) for this group Pic(Xₜ) up to canonical isomorphism.

In Theorem [64, Thm. 4.19] Joyce and Song showed that when \( \mathbb{K} = \mathbb{C} \) and \( H^1(O_X) = 0 \) the numerical Grothendieck group \( K^{\text{num}}(\text{coh}(X)) \) is unchanged under small deformations of \( X \) up to canonical isomorphism. As we said, here we will not prove this result. So, the idea is to construct a globally constant lattice \( \Lambda_X \) using the global constancy of the Picard schemes such that there exist an inclusion \( K^{\text{num}}(\text{coh}(X)) \hookrightarrow \Lambda_X \). It could happen that the image of the numerical Grothendieck group varies with \( t \) as it has to do with the integral Hodge conjecture as in [64, Thm. 4.19], but this does not affect the deformation invariance of \( \Delta T^\alpha(\tau) \) as for them to be deformation invariant is enough to find a deformation invariant lattice in which the classes \( \alpha \) vary. Next, we describe such lattice \( \Lambda_X \) and explain how the numerical Grothendieck group \( K^{\text{num}}(\text{coh}(X)) \) is contained in it. Our idea follows [64, Thm. 4.19].

Let \( X \) be a Calabi–Yau 3-fold over \( \mathbb{K} \), with \( H^1(O_X) = 0 \) and consider the Chern character, as in Hartshorne [17]: for each \( E \in \text{coh}(X) \) we have the rank \( r(E) \in \mathbb{Z} \) and the Chern classes \( c_i(E) \in A^i(X) \) for \( i = 1, 2, 3 \). It is useful to organize these into the Chern character \( \text{ch}(E) \) in \( A^*(X)_Q \), where \( \text{ch}(E) = \text{ch}_0(E) + \text{ch}_1(E) + \text{ch}_2(E) + \text{ch}_3(E) \) with \( \text{ch}_0(E) = r(E), \quad \text{ch}_1(E) = c_1(E), \quad \text{ch}_2(E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E)), \quad \text{ch}_3(E) = \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) \).

By the Hirzebruch–Riemann–Roch Theorem [17, Th. A.4.1], the Euler form on coherent sheaves \( E, F \) is given in terms of their Chern characters by

\[
\chi([E], [F]) = \deg(\text{ch}(E)^\vee \cdot \text{ch}(F) \cdot \text{td}(TX)),
\]

(4.22)
where $(\cdot)_3$ denotes the component of degree 3 in $A^*(X)_Q$ and where $\text{td}(TX)$ is the Todd class of $TX$, which is $1 + \frac{1}{12}c_2(TX)$ as $X$ is a Calabi–Yau 3-fold, and $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)^\vee = (\lambda_0, -\lambda_1, \lambda_2, -\lambda_3)$, writing $(\lambda_0, \ldots, \lambda_3) \in A^*(X)$ with $\lambda_i \in A^i(X)$. Define:

$$\Lambda_X = \{ (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \mid \lambda_0, \lambda_3 \in \mathbb{Q}, \lambda_1 \in \text{Pic}(X) \otimes \mathbb{Q}, \lambda_2 \in \text{Hom} (\text{Pic}(X), \mathbb{Q}) \text{ such that }$$

$$\lambda_0 \in \mathbb{Z}, \lambda_1 \in \text{Pic}(X)/\text{torsion, } \lambda_2 - \frac{1}{2} \lambda_3^2 \in \text{Hom} (\text{Pic}(X), \mathbb{Z}), \lambda_3 + \frac{1}{12} \lambda_1 c_2(TX) \in \mathbb{Z} \},$$

where $\lambda_3^2$ is defined as the map $\alpha \in \text{Pic}(X) \to \frac{1}{12} c_1(\lambda_1) \cdot c_1(\lambda_1) \cdot c_1(\alpha) \in A^3(X)_Q \cong \mathbb{Q}$, and $\lambda_1 c_2(TX)$ is defined as $\frac{1}{12} c_1(\lambda_1) \cdot c_2(TX) \in A^3(X)_Q \cong \mathbb{Q}$. Theorem 4.4 states that $\Lambda_X$ is deformation invariant and the Chern character gives an injective morphism $\text{ch} : K^\text{num}(\text{coh}(X)) \to \Lambda_X$. The proof of Theorem 4.4 is straightforward:

**Proof.** The proof follows exactly as in [64] Thm. 4.19 and the fact the Picard scheme $\text{Pic}(X)$ is globally constant in families from the argument above yields that the lattice $\Lambda_X$ is deformation invariant. Moreover, the proof that $\text{ch}(K^\text{num}(\text{coh}(X))) \subseteq \Lambda_X$ is again as in [64] Thm. 4.19. Observe that we do not prove that $\text{ch}(K^\text{num}(\text{coh}(X))) = \Lambda_X$, fact which uses Voisin’s Hodge conjecture proof for Calabi–Yau 3-folds over $\mathbb{C}$ [134].

**Question 4.9.** Does Voisin’s result [134] work over $\mathbb{K}$ in terms of $\text{Hom}(\text{Pic}(X), \mathbb{Z})$?

This concludes the discussion of problem (b) in [64] and yields the deformation-invariance of $DT^\alpha(\tau), PI^{\alpha,n}(\tau')$ over $\mathbb{K}$.

## 5 Implications and conjectures

In this final section we sketch some exciting and far reaching implications of the theory and propose new ideas for further research. One proposal is in the direction of extending Donaldson–Thomas invariants to compactly supported coherent sheaves on noncompact quasi-projective Calabi–Yau 3-folds. A second idea is in the derived categorical framework trying to establish a theory of generalized Donaldson–Thomas invariants for objects in the derived category of coherent sheaves. Here we expose the problems and illustrate some possible approaches when known.

### 5.1 Noncompact Calabi–Yau 3-folds

We start by recalling the following definition from [64] Def. 6.27:

**Definition 5.1.** Let $X$ be a noncompact Calabi-Yau 3-fold over $\mathbb{C}$. We call $X$ compactly embeddable if whenever $K \subset X$ is a compact subset, in the analytic topology, there exists an open neighbourhood $U$ of $K$ in $X$ in the analytic topology, a compact Calabi-Yau 3-fold $Y$ over $\mathbb{C}$ with $H^1(\mathcal{O}_Y) = 0$, an open subset $V$ of $Y$ in the analytic topology, and an isomorphism of complex manifolds $\varphi : U \to V$.

Joyce and Song only need the notion of ‘compactly embeddable’ as their complex analytic proof of [1.10]–[1.17] recalled in [1.3.2] requires $X$ compact; but unfortunately the given algebraic version of [1.10]–[1.17] in Theorem 1.2 uses results from derived algebraic geometry [103, 125, 130], and the author does not know if they apply also for compactly supported sheaves on a noncompact $X$.

More precisely, in [103] it is shown that if $X$ is a projective Calabi-Yau $m$-fold then the derived moduli stack $\mathfrak{M}_{\text{Perf}(X)}$ of perfect complexes of coherent sheaves on $X$ is $(2 - m)$-shifted symplectic. It is not obvious that if $X$ is a quasi-projective Calabi-Yau $m$-fold, possibly noncompact, then the derived moduli stack $\mathfrak{M}_{\text{Perf}_c(X)}$ of perfect complexes on $X$ with compactly-supported cohomology is also $(2 - m)$-shifted symplectic.

At the present, we can state the following result. We thank Bertrand Toën for explaining this to us.

**Theorem 5.2.** Suppose $Z$ is smooth projective of dimension $m$, and $s \in H^0(K_Z^{-1})$, and $X \subset Z$ is Zariski open with $s$ nonvanishing on $X$, so that $X$ is a (generally non compact) quasi-projective Calabi-Yau $m$-fold. Then the derived moduli stack $\mathfrak{M}_{\text{Perf}_c(X)}$ of compactly-supported coherent sheaves on $X$, or of perfect complexes on $X$ with compactly-supported cohomology, is $(2 - m)$-shifted symplectic.
Proof. Let \( Z \) be smooth and projective of dimension \( m \), and \( s \) be any section of \( K_Z^{-1} \). Let \( Y \) be the derived scheme of zeros of \( s \) and \( X = Z \setminus Y \). Then, \( Y \) is equipped with a canonical \( O \)-orientation in the sense of \[103\] of dimension \( m - 1 \), so \( \mathcal{M}_{\text{Perf}}(Y) \) is \((2 - m - 1)\)-symplectic, even if \( Y \) is not smooth. The restriction map \( \mathcal{M}_{\text{Perf}}(Z) \to \mathcal{M}_{\text{Perf}}(Y) \) is moreover Lagrangian. The map \( * \to \mathcal{M}_{\text{Perf}}(Y) \), corresponding to the zero object is étale, and thus its pull-back provides a Lagrangian map \( \mathcal{M}_{\text{Perf}_{cs}}(X) \to * \), or, equivalently, a \((2 - m)\)-symplectic structure on \( \mathcal{M}_{\text{Perf}_{cs}}(X) \). Now if \( X' \) is open in \( X \), then \( \mathcal{M}_{\text{Perf}_{cs}}(X') \to \mathcal{M}_{\text{Perf}_{cs}}(X) \) is an open immersion, so \( \mathcal{M}_{\text{Perf}_{cs}}(X') \) is also \((2 - m)\)-symplectic.

We remark the following:

(a) We point out that the condition of Theorem 5.2 is similar to the compactly-embeddable condition in \[64\] Def. 6.27, but more general, as we do not require \( Z \) to be a Calabi-Yau.

(b) Observe that in the non-compact case we cannot expect to have the deformation invariance property unless in some particular cases in which the moduli space is proper.

(c) Note that we need the noncompact Calabi-Yau to be quasi-projective in order to have a quasi projective Quot scheme \[98\] Thm. 6.3.

We conclude the section with the following:

**Conjecture 5.3.** The theory of generalized Donaldson–Thomas invariants defined in \[64\] is valid over algebraically closed fields of characteristic zero for compactly supported coherent sheaves on noncompact quasi-projective Calabi–Yau 3-folds. In this last case, one can define \( \overline{DT}^{\alpha}(\tau) \) and prove the wall-crossing formulae and the relation with \( \Pi^{\alpha,n}(\tau') \) is still valid, while one loses the deformation invariance property and the properness of moduli spaces.

### 5.2 Derived categorical framework

Our algebraic method could lead to the extension of generalized Donaldson–Thomas theory to the derived categorical context. The plan to extend from abelian to derived categories the theory of Joyce and Song \[64\] starts by reinterpreting the series of papers by Joyce \[54–61\] in this new general setup. In particular:

(a) Defining configurations in triangulated categories \( \mathcal{T} \) requires to replace the exact sequences by distinguished triangles.

(b) Constructing moduli stacks of objects and configurations in \( \mathcal{T} \). Again, the theory of derived algebraic geometry \[103\][126][130] can give us a satisfactory answer.

(c) Defining stability conditions on triangulated categories can be approached using Bridgeland’s results, and its extension by Gorodentscev et al., which combines Bridgeland’s idea with Rudakov’s definition for abelian categories \[107\]. Since Joyce’s stability conditions \[50\] are based on Rudakov, the modifications should be straightforward.

(d) The ‘nonfunctoriality of the cone’ in triangulated categories causes that the triangulated category versions of some operations on configurations are defined up to isomorphism, but not canonically, which yields that corresponding diagrams may be commutative, but not Cartesian as in the abelian case. In particular, one loses the associativity of the Ringel-Hall algebra of stack functions, which is a crucial object in Joyce and Song framework. We expect that derived Hall algebra approach of Toën \[126\] resolve this issue. See also \[86\].

The list above does not represent a big difficulty. The main issues actually are: proving existence of Bridgeland stability conditions (or other type) on the derived category; proving that semistable moduli schemes and stacks are finite type (permissible), and proving that two stability conditions can be joined by a path of permissible stability conditions.

Theorem \[4.3\] is just one of the steps in developing this program. The author thus expects that a well-behaved theory of invariants counting \( \tau \)-semistable objects in triangulated categories in the style of Joyce’s theory exists, that is, Theorem \[4.0\] should be valid also in the derived categorical context.
Conjecture 5.4. The theory of generalized Donaldson–Thomas invariants defined in [64] is valid for complexes of coherent sheaves on Calabi-Yau 3-folds over algebraically closed fields of characteristic zero.

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