Fermions with a bounded and discrete mass spectrum

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Abstract

A mechanism for determining fermion masses in four spacetime dimensions is presented, which uses a scalar-field domain wall extending in a fifth spacelike dimension and a special choice of Yukawa coupling constants. A bounded and discrete fermion mass spectrum is obtained analytically for spinors localized in the fifth dimension. These particular mass values depend on a combination of the absolute value of the Yukawa coupling constant and the parameters of the scalar potential. A similar mechanism for a finite mass spectrum may apply to $(1+1)$-dimensional fermions relevant to condensed matter physics.

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I. INTRODUCTION

There are at least two ways to determine fermion masses explicitly: Dirac’s quantization of the spinor mass over a four-dimensional space embedded in a five-dimensional projective space [1] and the Kaluza–Klein model with a fifth periodic compact dimension (see, e.g., Ref. [2] and references therein). Both of these fermion mass spectra are discrete but unbounded ($|m_n| \to \infty$ for $n \to \infty$).

In this article, we consider another possibility by adding one extra spacelike open dimension to the standard Minkowski spacetime and introducing Yukawa couplings between the scalar and the Dirac fermions. Our suggestion relies on having a scalar-field domain-wall background in the five-dimensional spacetime and making a special choice for the values of the two Yukawa coupling constants (they must be the opposite of each other). A similar five-dimensional setup has, of course, been used in previous studies [2, 3]. Restricting to localized fermion solutions in the background of the scalar-field domain wall, we find a discrete and bounded fermion mass spectrum.

These three ways of calculating fermion masses have one thing in common: the dimension of spacetime is increased from four to five or more. Note that this fifth extra dimension may be essentially different from the four dimensions of standard Minkowski spacetime. In more general terms, an infinite number of degrees of freedom has been added to the four-dimensional theory, even though the theory considered in the end applies again to four spacetime dimensions.

The outline of our short paper is as follows. In Sec. II we establish our notation. In Sec. III we define the theory and look for nonsingular localized fermion solutions in the background of a scalar-field domain wall. The main results of this section are the bounded fermion mass spectra (3.22) and (3.24), together with the corresponding wave functions (3.23) and (3.25). A further result of this section is the vanishing backreaction (3.28) of the fermions on the scalar-field domain wall, at least as far as the classical dynamical equations are concerned. In Sec. IV we present some further remarks and comment on the possible relevance to condensed matter physics. Appendix A contains the details of the domain-wall fermion solutions for a generic ratio of the Yukawa and Higgs coupling constants. Appendix B recalls, for completeness, the massless chiral spinor solution [4].

Before we start, it may be worthwhile to emphasize that our main interest is in finding a mechanism responsible for a bounded and discrete fermion mass spectrum (for earlier results in the brane-world context, see, e.g., Ref. [5]). Even though our elementary discussion is far removed from reality (fermions of a chiral gauge theory are not considered) and heuristic to a certain extent (leaving open the proper role and interpretation of the additional degrees of freedom, naively those from an extra dimension), we believe that there is some interest in finding quite generally that a bounded and discrete fermion mass spectrum may result from a simple normalization condition.
II. NOTATION

We take \( w \) to denote the extra spacelike coordinate and keep \( x^\mu \) for the coordinates of the usual four-dimensional Minkowski spacetime. Latin indices \( a, b, \ldots \) refer to all five spacetime coordinates, while Greek indices \( \mu, \nu, \ldots \) leave out the fifth coordinate \( w \). With the spacetime coordinates

\[
(x^a) = (x^0, x^1, x^2, x^3, w),
\]

the metric of the flat five-dimensional spacetime \( \mathcal{M}_5 \) is given by

\[
(g_{ab}) = \text{diag}(1, -1, -1, -1, -1).
\]

Next, we specify the \( 2 \times 2 \) Pauli and the \( 4 \times 4 \) Dirac matrices to be used:

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix},
\]

where \( \mathbb{I} \) is the rank-two identity matrix. The five-dimensional gamma matrices corresponding to the metric (2.1b) are then

\[
\Gamma^\mu = \gamma^\mu, \quad \Gamma^5 = -i\gamma^5.
\]

The factor \( i \) in \( \Gamma^5 \) is to keep up with our choice of metric and the minus sign has been chosen for the sake of convenience.

Throughout, natural units are used with \( \hbar = c = 1 \).

III. MECHANISM

A. Theory

Our starting point is the following classical Lagrange density over the five-dimensional spacetime \( \mathcal{M}_5 \):

\[
\mathcal{L}_5 = \bar{\Psi} i \bar{\gamma} \gamma \Psi + \bar{\Omega} i \bar{\gamma} \gamma \Omega - f \bar{\Psi} \gamma \gamma \phi + f \bar{\Omega} \gamma \gamma \phi + \frac{1}{2} \partial_a \phi \partial^a \phi - \frac{\lambda^2}{2} \left( \phi^2 - M^2 \right)^2.
\]

This Lagrange density describes two spinor fields \( \Psi \) and \( \Omega \) coupled to a real Higgs-like scalar field \( \phi \) (the Yukawa coupling constant \( f \) of the \( \Omega \) scalar is taken positive, \( f > 0 \)). All fields are defined over a five-dimensional flat spacetime. The slashed differential operator in (3.1) is explicitly \( \bar{\gamma} \equiv \Gamma^a \partial_a \), with the Dirac matrices \( \Gamma^a \) as given by (2.3).
As is clear from the preceding paragraph, we consider the fields $\phi$, $\Psi$, and $\Omega$ to be c-number functions. The relevance of our results to quantum field theory will be briefly discussed in Sec. III F.

It is also possible to add gauge fields or even a dynamical metric field, also considered to be c-number functions. The simplest possibility is to add a $U(1)$ gauge field and to have opposite electric charges for the two spinors $\Psi$ and $\Omega$ and zero electric charge for the scalar $\phi$. The present paper, however, keeps the types of fields to a minimum: scalar and spinor.

The two spinor fields $\Psi$ and $\Omega$ are introduced with strictly opposite Yukawa coupling constants in (3.1), for the sake of obtaining exact classical solutions later on (specifically, the two spinor-field source terms in the scalar field equation will cancel). The positive parameters $M$, $f$, and $\lambda$ in (3.1) are the vacuum expectation value of the scalar field and the coupling constants in the usual four-dimensional sense. In the five-dimensional theory, these parameters acquire different mass dimensions compared to their counterparts in the four-dimensional theory. For instance, $\lambda$ then has mass dimension $-1/2$ and $M$ has mass dimension $3/2$. Thus, $\lambda M$ may serve as a mass parameter.

The quantum theory corresponding to (3.1) is, most likely, nonrenormalizable, given the presence of negative-mass-dimension couplings such as $\lambda^2$ for the quartic scalar interaction term. This is perhaps one more reason to consider the fifth dimension to be essentially different than the four of standard Minkowski spacetime, but this suggestion will not be pursued further in this paper.

**B. Scalar-field domain-wall solution**

From the Lagrange density (3.1), the classical scalar field equation is

$$-\partial_a \partial^a \phi - 2 \lambda^2 \left( \phi^2 - M^2 \right) \phi = f \left( \bar{\Psi} \Psi - \bar{\Omega} \Omega \right).$$

As will be demonstrated in Sec. III F, the contributions of the c-number fields $\Psi$ and $\Omega$ on the right-hand side of the scalar field equation (3.2) can be made to cancel each other, so we drop them for the moment. Assuming $\phi$ to depend only on the fifth coordinate $w$, Eq. (3.2) then becomes

$$\partial_w^2 \phi(w) = 2 \lambda^2 \left[ \phi(w)^2 - M^2 \right] \phi(w),$$

with a domain-wall solution

$$\phi(w) = M \tanh(c w),$$

$$c \equiv \lambda M,$$

where the constant $c$ has mass dimension 1. In order to have this domain-wall solution, it is necessary to choose the fifth dimension to be spacelike (see, e.g., Ref. [2] for further discussion).
In the following subsections, we focus on localized normalizable spinor solutions in the background of a scalar-field domain wall. There are, of course, also scattering-type solutions \[2\], but they have higher energies and reach out to \(|w| = \infty\).

### C. Spinor Ansätze

In the background of the scalar domain-wall solution \((3.4)\), we have for the spinors
\[
(i\gamma^\mu \partial_\mu + i\Gamma^5 \partial_w) \Psi = +f M \tanh(c w) \Psi ,
\]
\[(i\gamma^\mu \partial_\mu + i\Gamma^5 \partial_w) \Omega = -f M \tanh(c w) \Omega .
\]

From these last two equations, it is clear that \(f M\) has mass dimension 1.

If we now write \(\Psi\) and \(\Omega\) in terms of two-component spinors,
\[
\Psi = \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega_l \\ \omega_r \end{pmatrix},
\]
(3.6)

Eq. (3.5a) for the \(\Psi\) field becomes
\[
 i\sigma^\mu \partial_\mu \psi_r - \partial_w \psi_l = f M \tanh(c w) \psi_l ,
\]
\[
 i\overline{\sigma}^\mu \partial_\mu \psi_l + \partial_w \psi_r = f M \tanh(c w) \psi_r ,
\]
(3.7a)

where the index \(\mu\) runs over 0,1,2,3 (corresponding to the four coordinates of the usual Minkowski spacetime) and
\[
\sigma^\mu \equiv (I, \sigma^1, \sigma^2, \sigma^3) ,
\]
(3.8a)
\[
\overline{\sigma}^\mu \equiv (I, -\sigma^1, -\sigma^2, -\sigma^3) .
\]
(3.8b)

Equation (3.5b) for the \(\Omega\) field acquires a similar form as (3.7), with \((\psi_l, \psi_r)\) replaced by \((\omega_l, \omega_r)\) and \(f\) by \(-f\).

The next step is to make the following Ansatz (separation of variables):
\[
\psi_l(x^0, x^1, x^2, x^3, w) = v_l(w) \chi(x^0, x^1, x^2, x^3) ,
\]
(3.9a)
\[
\psi_r(x^0, x^1, x^2, x^3, w) = v_r(w) \xi(x^0, x^1, x^2, x^3) .
\]
(3.9b)

For later use, we already give the corresponding Ansatz for the two-spinors of the \(\Omega\) field:
\[
\omega_l(x^0, x^1, x^2, x^3, w) = v'_l(w) \chi'(x^0, x^1, x^2, x^3) ,
\]
(3.10a)
\[
\omega_r(x^0, x^1, x^2, x^3, w) = v'_r(w) \xi'(x^0, x^1, x^2, x^3) .
\]
(3.10b)

Continuing with the discussion of the \(\Psi\) field and substituting the Ansatz \((3.9)\) into Eq. \((3.7)\), we obtain:
\[
i\sigma^\mu \partial_\mu \chi(x) v_l(w) - \partial_w v_l(w) \chi(x) = f M \tanh(c w) v_l(w) \chi(x) ,
\]
(3.11a)
\[
i\overline{\sigma}^\mu \partial_\mu \chi(x) v_r(w) + \partial_w v_r(w) \xi(x) = f M \tanh(c w) v_r(w) \xi(x) ,
\]
(3.11b)
with function argument \((x)\) standing for \((x^0, x^1, x^2, x^3)\).

We make one more \textit{Ansatz} at this point:

\[
\begin{align*}
  i\sigma^\mu \partial_\mu \xi(x) &= m_4 \chi(x), \\
  i\sigma^\mu \partial_\mu \chi(x) &= m_4 \xi(x),
\end{align*}
\]  

(3.12a)

(3.12b)

together with the standard normalization condition on the two-spinors

\[
  \chi^\dagger(x) \xi(x) + \xi^\dagger(x) \chi(x) = 2 |m_4|. 
\]  

(3.12c)

We could have assigned two different parameters at the places of \(m_4\) in Eqs. \((3.12a)\) and \((3.12b)\). However, if we wish to interpret \(\chi\) and \(\xi\) as the left-handed and right-handed components of the same massive Dirac spinor in four-dimensional Minkowski spacetime, it is necessary to have a unique mass value \(m_4 \neq 0\) in Eqs. \((3.12a)\) and \((3.12b)\). Equation \((3.11)\) is now reduced to the following set of coupled equations:

\[
\begin{align*}
  -\partial_w v_l(w) + m_4 v_r(w) &= f M \tanh(c w) v_l(w), \\
  +\partial_w v_r(w) + m_4 v_l(w) &= f M \tanh(c w) v_r(w).
\end{align*}
\]  

(3.13a)

(3.13b)

In order to solve this last set of equations, we make a change of variable

\[
s \equiv \tanh(c w),
\]  

(3.14)

so that \(\partial_w = c (1 - s^2) \partial_s\). The equations then take a more recognizable form:

\[
\begin{align*}
  - (1 - s^2) \partial_s v_l(s) + m_f v_r(s) &= F s v_l(s), \\
  + (1 - s^2) \partial_s v_r(s) + m_f v_l(s) &= F s v_r(s),
\end{align*}
\]  

(3.15a)

(3.15b)

where the new parameters are

\[
m_f \equiv m_4 / c \equiv m_4 / (\lambda M),
\]  

(3.16a)

\[
F \equiv f / \lambda.
\]  

(3.16b)

Note that both \(m_f\) and \(F\) are dimensionless.

From the first equation \((3.15a)\) we can express \(v_r\) in terms of \(v_l\) and \(\partial_s v_l\). Then, we substitute this expression for \(v_r\) into the second equation \((3.15b)\) and obtain

\[
\begin{align*}
  (1 - s^2) \frac{d^2 v_l(s)}{ds^2} - 2s \frac{dv_l(s)}{ds} + \left[ F(F+1) - \frac{F^2 - m_f^2}{1 - s^2} \right] v_l(s) &= 0. \\
\end{align*}
\]  

(3.17)

A similar equation for \(v_r\) is obtained as:

\[
\begin{align*}
  (1 - s^2) \frac{d^2 v_r(s)}{ds^2} - 2s \frac{dv_r(s)}{ds} + \left[ F(F-1) - \frac{F^2 - m_f^2}{1 - s^2} \right] v_r(s) &= 0.
\end{align*}
\]  

(3.18)

The above two uncoupled equations will be solved in the next subsection and Appendix A.
D. Spinor solutions and fermion mass spectrum

The two equations (3.17) and (3.18) belong to the class of Legendre equations [6–8], with degree $\nu$ and order $\mu$ taking the values

$$\nu \in \{F, F-1\},$$

$$\mu = \pm \sqrt{F^2 - m_f^2}.$$  \hspace{1cm} (3.19a)

$$\mu = \pm \sqrt{F^2 - m_f^2}.$$  \hspace{1cm} (3.19b)

These equations admit square-integrable solutions on the open interval $(-1, 1)$ with the measure induced by the reparametrization (3.14) if and only if the absolute values of the degree $\nu$ and order $\mu$ differ by integers and the degree is bounded from below, $\nu > 1/2$. For simplicity, we focus on the case of integer degree in this subsection. The case of noninteger degree is discussed in Appendix A, which also gives the standard form (A1) of the Legendre equation and the corresponding definitions of the degree $\nu$ and the order $\mu$.

Assuming the degree to be integer and restricting to square-integrable solutions, the possible absolute values for the degree and order are given by (cf. Table 4.8.2 in Ref. [8])

$$F = l \in \mathbb{N}_0 \setminus \{0, 1\},$$

$$\sqrt{F^2 - m_f^2} = m,$$

$$m \in \{1, 2, \ldots, l-2, l-1\},$$  \hspace{1cm} (3.20a)

$$\sqrt{F^2 - m_f^2} = m,$$  \hspace{1cm} (3.20b)

$$m \in \{1, 2, \ldots, l-2, l-1\},$$  \hspace{1cm} (3.20c)

where some of the usual integers have been omitted (specifically, $l = 0, 1$ and $m = 0, l$).

The reason for omitting certain integers in (3.20) is twofold: first, there are two Legendre equations to consider simultaneously [namely Eqs. (3.17) and (3.18)] and, second, the relevant solutions [the associated Legendre functions of the first kind, $P^m_l(s)$] are required to have a finite norm for an $s$–measure equal to $dw/ds = 1/(1 - s^2)$. Specifically, we have the following normalization of the relevant solutions:

$$\int_{-1}^{1} \frac{ds}{(1 - s^2)} \left( P^m_l(s) \right)^2 = \frac{(l + m)!}{m \ (l - m)!},$$

for positive integers $l$ and $m$. In particular, the degree $m$ of the solution $P^m_l(s)$ cannot be 0; otherwise, the norm of the corresponding Legendre function would be infinite for the $s$–measure inherited from the five-dimensional spacetime.

According to Eqs. (3.12a) and (3.12b), we interpret $m_f^4$ entering the $m_f$ definition (3.16a) as the inertial mass of a fundamental fermion propagating in the usual four-dimensional Minkowski spacetime. The above conditions (3.20) then imply that, since $F \equiv f/\lambda$ is a fixed integer $N$, there is only a finite number of fundamental fermions. The masses $m_4$ of
these fundamental fermions take the following values:

\[ m_4/(\lambda M) \in \left\{ \pm \sqrt{N^2-1^2}, \pm \sqrt{N^2-2^2}, \pm \sqrt{N^2-3^2}, \ldots, \pm \sqrt{N^2-(N-1)^2} \right\}, \quad (3.22a) \]

\[ f/\lambda = N, \quad (3.22b) \]

\[ N \in \{2, 3, 4, \ldots\}. \quad (3.22c) \]

The corresponding solutions of Eqs. (3.17) and (3.18) are:

\[ v_l(s) \propto P_F^\pm \sqrt{F^2-m_f^2}(s), \quad (3.23a) \]

\[ v_r(s) \propto P_{F-1}^\pm \sqrt{F^2-m_f^2}(s), \quad (3.23b) \]

with \( s = s(w) \) as defined by Eq. (3.14) and \( F \equiv f/\lambda \) and \( m_f \equiv m_4/(\lambda M) \) taking values according to (3.22).

It is interesting to note that the values 0 and \( fM \) for \( m_4 \) do not appear in the fermion mass spectrum (3.22a) due to the normalizability condition on \( v_r(w) \) and \( v_l(w) \). Since \( v_r(s) \) is given by (3.23b), \( m_f \) cannot be 0 [the function \( P_F^m \) would have \( |m| > f \)]. The same expression for \( v_r(s) \) also tells us that \( m_f \) cannot be equal to \( F \) [\( P_{F-1}^0 \) is not normalizable for the relevant measure, according to Eq. (3.21)]. Hence, both the minimal mass value \( (m_4 = 0) \) and the maximal mass value \( (m_4 = fM = F\lambda M) \) are not present in the mass spectrum of the localized Dirac spinors (there does exist a massless spinor solution, but it is chiral; see Appendix [B]). As to the range of \( |m_4|/(\lambda M) \), the highest value is \( N\sqrt{1-1/N^2} \) and the lowest value is \( \sqrt{2N-1} \), so that the mass gap increases as the ratio of coupling constants \( f/\lambda = N \) grows.

With the wave functions \( v_l(w) \) and \( v_r(w) \) entering the chiral fields (3.9a) and (3.9b), it is also possible to distinguish left-handed and right-handed fermions, because the corresponding solutions (3.23) are different. This means that the left-handed and right-handed fermions are localized differently in the fifth dimension. This is not altogether surprising since the different chiralities trace back to the fact that left-handed and right-handed fermions correspond to different eigenvalues of the \( \Gamma^5 \) matrix, namely \( -1 \) and \( +1 \). But \( \Gamma^5 \) is also the Dirac gamma matrix of the fifth dimension. So, the double role of \( \Gamma^5 \) brings together chirality and the fifth dimension (see also Sec. [IV] for further comments). In the discussion up until now, we have used one specific domain-wall solution (3.4) with a particular direction and this direction treats the left and right chiralities differently in a particular way. Using the other domain-wall solution obtained by \( \phi \rightarrow -\phi \), would switch the roles of left and right chirality.

The square-integrable solutions for the case of noninteger degree \( \nu \) are detailed in Appendix [A]. Applying these solutions to the two particular Legendre equations for \( v_r(w) \) and \( v_l(w) \) as given by Eqs. (3.17) and (3.18), the following fermion mass spectrum is obtained:

\[ m_4/(\lambda M) \in \left\{ \pm \sqrt{2F \times 1-1^2}, \pm \sqrt{2F \times 2-2^2}, \ldots, \pm \sqrt{2F \times [F] - ([F])^2} \right\}, \quad (3.24) \]
where $F \equiv f/\lambda > 3/2$ is a positive noninteger number and $\lfloor F \rfloor$ denotes the largest integer that is not greater than $F$. The corresponding wave functions are

$$v_l(s) \propto P_F^{-\sqrt{F^2-m_f^2}}(s), \quad (3.25a)$$

$$v_r(s) \propto P_{F-1}^{-\sqrt{F^2-m_f^2}}(s), \quad (3.25b)$$

again for noninteger $F > 3/2$ and with $m_f \equiv m_4/(\lambda M)$ from Eq. (3.24).

Three remarks are in order. First, the mass value 0 again does not appear in the spectrum (3.24). The reason is that, for both spinor equations to have square-integrable solutions, it is necessary that $\sqrt{F^2-(m_4/\lambda M)^2} \leq F - 1$. Second, the mass value 0 does appear for a chiral solution [4], whose basic structure is recalled in Appendix B. Third, taking the spectrum from Eq. (3.24) as it stands and letting $F$ approach an integer $N \geq 2$ from below reproduces precisely the masses from Eq. (3.22a).

E. Consistency of scalar and spinor solutions

We now complete the discussion on the exactness and consistency of the classical solutions. Let us perform the same calculation for the $\Omega$ field as for the $\Psi$ field. The $\Omega$ versions of Eqs. (3.7) and (3.11) are obtained by replacing $f$ by $-f$. Equally, replacing $F$ in Eqs. (3.18) and (3.17) by $-F$ gives the equations for $\Omega$. For a given $\Psi$ solution with functions $v_l(w)$, $v_r(w)$, $\xi(x)$, and $\chi(x)$, the corresponding $\Omega$ solution has the following primed wave functions:

$$v'_l(w) = v_r(w), \quad (3.26a)$$

$$v'_r(w) = v_l(w), \quad (3.26b)$$

and similar relations for the primed two-spinor fields as for the unprimed two-spinor fields:

$$i\sigma^\mu \partial_\mu \xi'(x) = m_4 \chi'(x), \quad (3.27a)$$

$$i\sigma^\mu \partial_\mu \chi'(x) = m_4 \xi'(x), \quad (3.27b)$$

$$\chi'^\dagger(x) \xi'(x) + \xi'^\dagger(x) \chi'(x) = 2|m_4|, \quad (3.27c)$$

where the same mass value $m_4$ has been taken as for the unprimed two-spinor fields in (3.12).

Next, we expand the spinor-field source term of the classical scalar field equation (3.2) and see that the two contributions cancel as follows:

$$f \left( \bar{\Psi}\Psi - \bar{\Omega}\Omega \right) = f \left[ \bar{\psi}_l^\dagger \psi_r + \bar{\psi}_r^\dagger \psi_l - \bar{\omega}_l^\dagger \omega_r - \bar{\omega}_r^\dagger \omega_l \right]$$

$$= f \left[ v_l v_r (\chi^\dagger \xi + \xi^\dagger \chi) - v'_l v'_r (\chi'^\dagger \xi' + \xi'^\dagger \chi') \right]$$

$$= 0, \quad (3.28)$$

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where the second equality uses the fact that the wave functions $v_{l,r}$ and $v'_{l,r}$ are real and the third equality follows from (3.26), (3.12c), and (3.27c). Actually, even if we had flipped the sign of $m_4$ in Eqs. (3.27a) and (3.27b), the third equality in (3.28) would still hold true.

In short, the scalar domain wall (3.4) and the obtained spinor fields (3.6) are exact solutions of the combined classical field equations (3.2) and (3.5). These spinor fields have chiral components (3.9) and (3.10) with wave functions given by associated Legendre functions (3.23), (3.25), and (3.26).

F. Remarks on the quantum theory

The discussion of Sec. III up to and including Sec. III E has been for $c$-number fields $\phi$, $\Psi$, and $\Omega$. For the quantum theory, these fields become operators and the discussion can follow that of the Jackiw–Rebbi paper [4]. We now use the following states:

$$|P, \pm, \pm; p, m_4; k, \tilde{m}_4\rangle,$$

(3.29)
corresponding to a single soliton (domain wall) in five-dimensional Minkowski spacetime with total four-momentum $P_\mu$ and fourfold degeneracy [4] due to the $\Psi$ and $\Omega$ zero modes, to which are bound a single $\Psi$ fermion with asymptotic four-momentum $p_\mu$ and mass $m_4$ and a single $\Omega$ fermion with asymptotic four-momentum $k_\mu$ and mass $\tilde{m}_4$.

We next consider the operator equation (3.2), leaving aside issues of operator ordering, and take the diagonal matrix element between the $\cdots$ soliton with $\Psi$ and $\Omega$ bound fermions of equal mass, $\tilde{m}_4 = m_4$. The result is a partial differential equation for the scalar form factor, with a source term on the right-hand side obtained from the following matrix element:

$$\langle P', -, -; p, m_4; k, m_4| f (\bar{\Psi}\Psi - \bar{\Omega}\Omega) | P, -, -; p, m_4; k, m_4\rangle,$$

(3.30)
where $\Psi$ and $\Omega$ are quantum fields (denoted by bold symbols). For sufficiently low soliton and fermion momenta, the leading contribution to the matrix element (3.30) is given by a Fourier integral of the right-hand side of (3.28) in terms of spinor form factors $\psi_{l,r}$ and $\omega_{l,r}$, which then cancel as shown by the further steps in (3.28).

Much more can and must be said on the quantum theory, but that lies outside the scope of the present article, which deals solely with solutions of the classical fields equations.

IV. DISCUSSION

The key input of the mechanism presented in Sec. III is the requirement that the spinor wave function be nonsingular and localized in the fifth dimension (nonlocalized scattering-type solutions exist but are not considered here). More precisely, the wave functions of the left-handed and right-handed spinors have extra normalizable factors which are nonsingular.
functions of the fifth coordinate \( w \). This normalization condition together with the existence of the fifth dimension bring about two important consequences: first, the bounded and discrete mass spectrum of the fermions in the four-dimensional Minkowski spacetime [specifically, the mass spectrum is given by Eq. (3.22a) for an integer ratio (3.22b) of coupling constants or by Eq. (3.24) for a noninteger coupling-constant ratio larger than \( 3/2 \) and, second, a hard-wired difference of left-handed and right-handed fermions [the difference being due to wave functions with different associated Legendre functions (3.23) and (3.25)]. Note that having the chirality of four-dimensional fermions distinguished by their position in an extradimensional direction is precisely what has been used to construct models of chiral lattice fermions [3].

It is also clear that the tangent hyperbolic function from the domain wall plays a special role in the discussion of Sec. III. Recall that Pöschl and Teller [9] have already studied a large class of sinus and hyperbolic-sinus potentials, whose corresponding Schrödinger equations exhibit similar spectra. We may now ask the following question: is the tangent hyperbolic function absolutely necessary for obtaining a bounded and discrete spectrum of fermion masses? We conjecture that the answer is negative, based on following argument.

From the construction in Sec. III, it is readily seen that, as long as we take the Ansätze (3.9) and (3.12), and then require \( v_l(w) \) and \( v_r(w) \) to be localized functions in the fifth dimension, the spectrum would be necessarily discrete. Some additional computations are needed to demonstrate that the mass spectrum is bounded for localized spinor solutions. Replace \( M \tanh(cw) \) in Eq. (3.13) by a general function \( \Phi(w) \). We then arrive at the following coupled equations:

\[
\begin{align*}
- \partial_w v_l + m_4 v_r &= f \Phi v_l, \\
+ \partial_w v_r + m_4 v_l &= f \Phi v_r,
\end{align*}
\]

(4.1a)  (4.1b)

which give the following uncoupled equations:

\[
\begin{align*}
- \partial_w^2 v_l + (f^2 \Phi^2 - f \partial_w \Phi) v_l &= m_4^2 v_l, \\
- \partial_w^2 v_r + (f^2 \Phi^2 + f \partial_w \Phi) v_r &= m_4^2 v_r.
\end{align*}
\]

(4.2a)  (4.2b)

These last two equations give the eigenfunctions of two Schrödinger-type equations with potentials proportional to \( f^2 \Phi^2 \mp f \partial_w \Phi \) and energy eigenvalue proportional to \( m_4^2 \).

In order to have localized wave functions from the Schrödinger-type equations (4.2), there must be a deep enough potential-energy well. If the scalar background field \( \Phi(w) \) approaches different finite values at \( \pm \infty \), the derivatives vanish asymptotically. Then \( \Phi^2 \) may provide finite-height edges of a potential-energy well. In order to obtain localized states, the energy of the “particle” (essentially \( m_4^2 \)) cannot exceed the minimum height of the edges of the potential-energy well. Hence, the mass spectrum is bounded. However, due to the different signs in front of the terms \( \partial_w \Phi \) in the Schrödinger potentials of (4.2), the mass spectra for left-handed and right-handed spinors will, in general, not overlap. Without overlap, there
would be no combined solutions to Eqs. (4.1) and (4.2). It is indeed a pleasant surprise that the domain-wall solution of the scalar field equation can produce two largely overlapping spectra. There are, in principle, other potentials $\Phi(w)$ which can produce overlapping spectra and they can be expected to give different numerical predictions for the fermion masses. Further investigation of the pair of Schrödinger-type equations (4.2) is needed (see also the discussion in Ref. [5]).

As a final remark, we note that the same type of analysis applies to (1+1)-dimensional fermions moving along a domain wall in 2+1 spacetime dimensions. [Specifically, taking the three-dimensional gamma matrices as $\Gamma^0 = \sigma^1$, $\Gamma^1 = i\sigma^2$, and $\Gamma^2 = i\sigma^3$, the calculation in 1+2 dimensions directly parallels the one in 1+4 dimensions.] The challenge for condensed matter physics is to provide a suitable domain wall (or an equivalent trapping mechanism) and to tune the two Yukawa coupling constants to appropriate values.

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Appendix A: Integrable solutions with noninteger Legendre degree

It is clear from Ref. [8] that $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are two linearly independent solutions of the Legendre equation:

$$\left(1 - x^2\right) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - x^2}\right] y = 0,$$

where the degree $\nu$ and the order $\mu$ are assumed to be real noninteger numbers. In particular, $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are real solutions on the real interval $(-1, 1)$. By this restriction of the domain (“on the cut”), the solutions are singularity free. The goal here is to show that there exist square-integrable solutions $y(x)$ to (A1) with respect to the following measure and domain:

$$\int_{-1}^{1} \frac{dx}{1 - x^2} |y|^2.$$  

(A2)

Throughout this Appendix, square-integrability is always for this measure and domain, and for the noninteger parameters $\nu$ and $\mu$. Recall from Sec. III that the degree is given by $\nu = F$ for the case of left-handed fermions and by $\nu = F - 1$ for the case of right-handed fermions, while both cases correspond to the same order $\mu = \pm (F^2 - m_f^2)^{1/2}$.

Since the solutions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are singularity free on the real interval $(-1, 1)$, their square-integrability depends solely on the behavior at the two ends of the interval [namely,
−1 and 1], because the measure diverges at these two ends. Without loss of generality, ν is assumed to be a positive noninteger. We start with the solution $P^\mu_\nu(x)$ and discuss three results.

The first result is that μ must be negative for $P^\mu_\nu(x)$ to be square-integrable. This is because the behavior of $P^\mu_\nu(x)$ near $x = -1$ is problematic for $\mu > 0$. Namely, the first equation on p. 197 in Ref. [8] says that, for $\mu > 0$ and $x \sim -1^+$,

$$P^\mu_\nu(x) \sim -2^{\mu/2} \sin(\pi \nu) \pi^{-1} \Gamma(\mu) (1 + x)^{-\mu/2},$$  \hspace{1cm} (A3)

where $\Gamma(\mu)$ is the Euler Gamma function for argument $\mu$. Since $\mu$ is positive, $\Gamma(\mu)$ is finite. Hence, the numerical factor in front of $(1 + x)^{-\mu/2}$ is finite. Since the exponent of the $(1 + x)$ term is negative for $\mu > 0$, $P^\mu_\nu(x)$ diverges at $x \sim -1^+$ (the superscript ‘+’ here means “just above” −1) and is not square-integrable on the chosen domain. Thus, the only possibility is $\mu < 0$ (recall that $\mu$ has been assumed to be noninteger in this Appendix).

The second result is that $\nu + \mu$ must be a non-negative integer for $P^\mu_\nu(x)$ to be square-integrable. Again, it is the behavior of $P^\mu_\nu(x)$ near $x = -1$ that causes problems for the case $\nu + \mu < 0$. The second equation on p. 197 in Ref. [8] says that, for $\mu < 0$ and $x \sim -1^+$,

$$P^\mu_\nu(x) \sim 2^{-\mu/2} \frac{\Gamma(-\mu)}{\Gamma(1 + \nu - \mu) \Gamma(-\nu - \mu)} (1 + x)^{\mu/2}.$$  \hspace{1cm} (A4)

Now $\Gamma(-\nu - \mu)$ is finite for $\nu + \mu < 0$. Since $\mu$ is negative, $\Gamma(-\mu)$ is also finite. Hence, all numerical factors in front of $(1 + x)^{\mu/2}$ are finite. For $\mu < 0$, $(1 + x)^{\mu/2}$ goes to infinity as $x$ approaches $-1$. Then $P^\mu_\nu(x)$ cannot be square-integrable on the domain for the case considered. The only possibility left is for the other case, with $\mu < 0$ and $\nu + \mu \geq 0$.

If $\nu + \mu$ is greater than or equal to 0 but not an integer, $\Gamma(-\nu - \mu)$ is still finite and the other numerical factors in Eq. (A4), too. Thus $P^\mu_\nu(x)$ cannot be square-integrable on the domain. However, if $\nu + \mu$ is a non-negative integer, $\Gamma(-\nu - \mu)$ becomes infinite and independent from $x$. Since $\Gamma(-\nu - \mu)$ appears in the denominator in the above equation, it may cancel the divergence from $(1 + x)^{\mu/2}$. Thus, it is possible that $P^\mu_\nu(x)$ becomes square-integrable on the domain, which is, in fact, to be discussed as the next result.

The third result is that $P^\mu_\nu(x)$ is square-integrable if $\mu$ is less than 0 and $\nu + \mu$ is a non-negative integer. If $\nu + \mu$ is an integer, $P^\mu_\nu(x)$ is either odd or even on the domain ($-1, 1$), as implied by the 7th equation on p. 170 of Ref. [8]:

$$P^\mu_\nu(-\hat{x}) = P^\mu_\nu(\hat{x}) \cos[\pi(\nu + \mu)] - 2\pi^{-1} Q^\mu_\nu(\hat{x}) \sin[\pi(\nu + \mu)]$$

$$= \pm P^\mu_\nu(\hat{x}),$$  \hspace{1cm} (A5)

for $\hat{x} \in (0, 1)$. In addition, the second equation on p. 192 in Ref. [8] reads, for the case $\mu < 0$ and $\nu + \mu \in \{1, 2, 3, \ldots\}$,

$$\int_0^1 (1 - x^2)^{-1} [P^\mu_\nu(x)]^2 dx = -\frac{1}{2} \mu^{-1} \frac{\Gamma(1 + \nu + \mu)}{\Gamma(1 + \nu - \mu)}.$$  \hspace{1cm} (A6)
According to Sec. 3.12 of Ref. [6], the above integral is originally due to Barnes. Combining the last two equations, one obtains

\[
\int_{-1}^{1} (1 - x^2)^{-1} \left[ P_{\nu}(x) \right]^2 dx = -\mu^{-1} \frac{\Gamma(1 + \nu + \mu)}{\Gamma(1 + \nu - \mu)} .
\]  

(A7)

Note that the right-hand side of the last equation is finite, making \( P_{\nu}(x) \) square-integrable for \( \mu < 0 \) and \( \nu + \mu \in \{1, 2, 3, \ldots\} \). Remark also that the structure of the right-hand side of Eq. (A7) corresponds to that of Eq. (3.21).

If \( \nu + \mu = 0 \), we can use Eq. (8.6.17) from Ref. [7], which reads

\[
P_{-\nu}(\cos \theta) = \frac{(\sin \theta)^\nu}{2^\nu \Gamma(\nu + 1)} ,
\]  

(A8)

where \( \cos \theta \) is a parametrization of \( x \), with \( \theta \in (0, \pi) \). It is now clear that \( P_{-\nu}(x) \) is square-integrable if and only if \( \nu > 1/2 \). The condition \( \nu > 1/2 \) translates into \( F > 1/2 \) for left-handed fermions and into \( F > 3/2 \) for right-handed fermions.

To sum up, for the case that both \( \mu \) and \( \nu \) are noninteger, the necessary and sufficient condition for \( P_{\nu}(x) \) to be square-integrable, with respect to the chosen measure and domain, is that \( \mu \) is negative, \( \nu \) is larger than \( 1/2 \), and \( \mu + \nu \) is a non-negative integer. In short, \( P_{\nu}(x) \) is square-integrable for a given degree \( \nu \) with

\[
\nu \in \mathbb{R}\setminus\mathbb{Z} ,
\]  

(A9a)

\[
\nu > 1/2 ,
\]  

(A9b)

only if the order \( \mu \) is given by

\[
\mu \in \{-\nu, -\nu + 1, -\nu + 2, \ldots, -\nu + [\nu]\} ,
\]  

(A9c)

where \( [\nu] \) denotes the largest integer that is not greater than \( \nu \) (in the mathematics literature, \([x]\) is called the floor or \textit{Entier} function of the real number \( x \)).

It is also necessary to check if \( Q_{\nu}(x) \) in certain cases is square-integrable with respect to the chosen measure. The check is done in a similar way as for \( P_{\nu}(x) \). The answer is affirmative, only if both \( \nu \) and \( \mu \) are positive half odd integers and \( \mu < \nu \). For these cases, the fourth equation on p. 170 in Ref. [8] says

\[
Q_{-\nu}^{-\mu}(x) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[ Q_{\nu}^\mu(x) \cos(\pi \mu) + \frac{\pi}{2} P_{\nu}^\mu(x) \sin(\pi \mu) \right] \\
= \pm \frac{\pi}{2} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} P_{\nu}^\mu(x) ,
\]  

(A10)

and the conclusion is that \( P_{-\nu}^{-\mu}(x) \) and \( Q_{\nu}^\mu(x) \) give essentially the same solution of (A1). Thus, these cases are already included in the previous discussion, up to a change of sign for \( \mu \).
It should be pointed out that linear combinations of $P^\mu_\nu(x)$ and $Q^\mu_\nu(x)$ have not been checked. We assume that this gives no additional interesting cases.

Considering the two general Legendre equations for $v_r(w)$ and $v_l(w)$, we obtain the fermion mass spectrum as given by Eq. (3.24) in the main text.

**Appendix B: Massless spinor solution**

Another type of spinor solution holds for the massless case, $m_f = 0$ in Eqs. (3.15). Taking a positive coupling-constant ratio, $F \equiv f/\lambda > 0$, it is readily verified that a solution is given by

\begin{align*}
v_l(s) &\propto (1 - s^2)^{F/2}, \\
v_r(s) &= 0,
\end{align*}

(B1a)

(B1b)

where $v_l(s)$ is square-integrable. A nonzero solution for $v_r(s)$ does exist, proportional to $(1 - s^2)^{-F/2}$, but is not square-integrable. The solution (B1) corresponds to a chiral spinor and is well known in the literature (cf. Refs. [2–4]).

Strictly speaking, the solution (B1b) is not localized on the domain wall at $s = 0$. For this reason, we have not included the mass value 0 in the discussion of Sec. III D, but massless chiral spinors are definitely part of the low-energy physics.

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