SELBERG ZETA-FUNCTION ASSOCIATED TO COMPACT RIEHMANN SURFACE IS PRIME

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Abstract. Let $Z(s)$ be the Selberg zeta-function associated to a compact Riemann surface. We consider decompositions $Z(s) = f(h(s))$, where $f$ and $h$ are meromorphic functions, and show that such decompositions can only be trivial.

1. Introduction

We continue the investigation of decompositions of the Selberg zeta-function which was started in Garunkštis and Steuding [6]. First we reproduce required definitions. Let $s = \sigma + it$ be a complex variable and $X$ a compact Riemann surface of genus $g \geq 2$ with constant negative curvature $-1$. The surface $X$ can be written as a quotient $\Gamma \backslash H$, where $\Gamma \subset \text{PSL}(2, \mathbb{R})$ is a strictly hyperbolic Fuchsian group and $H$ is the upper half-plane of $\mathbb{C}$. Then the Selberg zeta-function associated with $X = \Gamma \backslash H$ is defined by (see Hejhal [8, §2.4, Definition 4.1])

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} \left(1 - N(P_0)^{-s-k}\right). \quad (1.1)$$

Here $\{P_0\}$ is the conjugacy class of a primitive hyperbolic element $P_0$ of $\Gamma$ and $N(P_0) = \alpha^2$ if the eigenvalues of $P_0$ are $\alpha$ and $\alpha^{-1}$ with $|\alpha| > 1$. Equation (1.1) defines the Selberg zeta-function in the half-plane $\sigma > 1$. The function $Z(s)$ can be extended to an entire function (see [8] §2.4, Theorem 4.25).

Definition 1.1 (Gross [7], Chuang and Yang [1 Section 3.2], [6]). Let $F$ be a meromorphic function. Then an expression

$$F(z) = f(h(z)), \quad (1.2)$$

where $f$ is meromorphic and $h$ is entire ($h$ may be meromorphic when $f$ is a rational function), is called a decomposition of $F$, with $f$ and $h$ as its left and right components, respectively. $F$ is said to be prime in the sense of a decomposition

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if for every representation of $F$ of the form (1.2) we have that either $f$ or $h$ is linear. If every representation of $F$ of the form (1.2) implies that $f$ is rational or $h$ is a polynomial, we say that $F$ is pseudo-prime in the sense of a decomposition. Furthermore, $F$ is said to be left-prime (right-prime) if every factorization (1.2) implies that $f$ is linear whenever $h$ is transcendental ($h$ is linear whenever $f$ is transcendental).

Liao and Yang [10] showed that the Riemann zeta-function is prime. In [6] the following theorem is proved.

Theorem A. The Selberg zeta-function $Z$ associated with a compact Riemann surface of genus $g$ is pseudo-prime and right-prime. Moreover, if $Z(s) = f(h(s))$, where $f$ is rational and $h$ is meromorphic, then $f$ is a polynomial of degree $k$, where $k$ divides $2g - 2$, and $h$ is an entire function.

Here we complete Theorem A.

Theorem 1.2. The Selberg zeta-function $Z$ associated with a compact Riemann surface of genus $g \geq 2$ is prime.

Theorem 1.2 follows from Theorem A, the property that $Z(s)$ has a simple zero at $s = 1$ ([8 § 2.4, Theorem 4.11]), and the following lemma.

Lemma 1.3. If there exist a polynomial $P$ and an entire function $h$ such that $Z(s) = P(h(s))$ then the polynomial $P$ has only one root in the complex plane (counting without multiplicities).

The proof of Lemma 1.3 is based on the distribution of zeros of $Z(s) - a$, $a \in \mathbb{C}$, (such zeros are called $a$-points of $Z(s)$) and of zeros of $Z'(s)$ in the left half-plane of $\mathbb{C}$. These zeros are described below.

The Selberg zeta-function $Z(s)$ has trivial zeros at integers $s = -n$, $n \geq 1$, of multiplicity $(2g - 2)(2n + 1)$; at $s = 0$ with multiplicity $2g - 1$; and an already mentioned zero at $s = 1$ with multiplicity 1 (see [8 § 2.4, Theorem 4.11], also for nontrivial zeros).

For the trivial zeros of $Z'(s)$, Theorem 1 from [5] together with the equality $Z'(s) = Z(\overline{s})$ give the following proposition.

Proposition 1.4.

(i) There is $\sigma_0 \geq 1$ such that $Z'(s) \neq 0$ in $\sigma \geq \sigma_0$;
(ii) the function $Z'(s)$ has zeros at $s = n$ of multiplicity $(2g - 2)(1 - 2n) - 1$ for any $n \leq -1$, and at $s = 0$ of multiplicity $2g - 2$.

Moreover, for any $0 < \varepsilon < 1/2$, there is a constant $n_0 = n_0(\varepsilon) \leq 1$ such that
(iii) $Z'(s)$ has a simple real zero in the disc $|s + 1/2 - n| \leq \varepsilon$ for any $n \leq n_0$;
(iv) $Z'(s)$ has no other zeros in $\sigma \leq n_0$ except those mentioned in (ii) and (iii).

For more about the zeros of the derivative of the Selberg zeta-function see [4][11][12].

For the $a$-points of $Z(s)$ we will prove the following two statements.
Proposition 1.5. Let \( b > 0 \) and \( 1/6 < r < 1/2 \). Then there exists a negative number \( N = N(Z,b,r) \) such that, for \( a \in \mathbb{C}, 0 < |a| \leq b \), the function \( Z(s) - a \) has \((2g-2)(1-2n)\) simple zeros in \(|s-n| < r\), where \( n < N \) are integers. Furthermore, \( Z(s) - a \) has no other zeros in \( \sigma < N \).

On the other hand, Proposition 1.4 implies that, for sufficiently large negative \( n \), a neighborhood of \( n + 1/2 \) contains a double zero of \( Z(s) - Z(n + 1/2) \).

Using Proposition 1.5 and the particular kind of polynomials \( P(z) = z^k + C \) we can easily demonstrate the main idea of the proof of Lemma 1.3. Indeed, let

\[
Z(s) = h(s)^k + C,
\]

where \( C \neq 0 \) and \( h(s) \) is an entire function. Then all zeros of \( Z(s) - C \) are at least of order \( k \). By Proposition 1.5 we see that \( k = 1 \) and Lemma 1.3 is true for this particular kind of polynomials. To consider the general case we will need the following consequence of Proposition 1.5.

Corollary 1.6. Let \( a : [0,1] \rightarrow \mathbb{C} \setminus 0 \) be a continuous function. Then for any sufficiently large negative \( n \) there are \((2g-2)(2n+1)\) continuous functions \( s_j : [0,1] \rightarrow \mathbb{C} \) such that, for each \( j \), we have \( Z(s_j(x)) = a(x) \), \(|s_j(x) - n| < 1/3\), and \( s_j(x) \neq s_m(x) \) if \( j \neq m \) and \( x \in [0,1] \).

In the last corollary, \( 1/3 \) can be replaced by any number \( r \), \( 1/6 < r < 1/2 \).

Various properties of \( a \)-points of Selberg zeta-functions were considered in [2, 3]. The next section contains the proofs of Proposition 1.5, Corollary 1.6, and Lemma 1.3.

2. Proofs

Proof of Proposition 1.5. We have (see [8, § 2.4, Theorem 4.12])

\[
Z(s) = f(s)Z(1 - s),
\]

where

\[
f(s) = \exp \left( \text{area}(X) \int_0^{s-1/2} v \tan(\pi v) \, dv \right).
\]

It is known ([8, Lemma 6]) that, for \( t \geq 0 \) and \( s \) not an integer,

\[
\int_0^{s-1/2} v \tan(\pi v) \, dv = \frac{i(s - 1/2)^2}{2} - \frac{s - 1/2}{\pi} \log(1 + e^{2\pi i(s-1/2)}) + \frac{i}{2\pi^2} \text{Li}_2(-e^{2\pi i(s-1/2)}) + \frac{i}{24},
\]

where the integration is along the straight line segment joining the origin to \( s-1/2 \) if \( s \) is not on the real line; if \( s \) is on the real line, and not an integer, we define the integral by the requirement of continuity as \( s \) is approached from the upper half-plane; furthermore, the branch of the logarithm is chosen such that

\[-\pi/2 \leq \Im \log(1 + e^{2\pi i(s-1/2)}) \leq \pi/2.\]
Then, for \( \sigma \to -\infty \),
\[
|f(s)| = \exp \left( \text{area}(X) \left( -(\sigma - 1/2)t - \frac{\sigma - 1/2}{\pi} \log |1 + e^{2i\pi(s-1/2)}| + O(|t| + 1) \right) \right)
\] (2.2)
uniformly in \( t \geq 0 \). Let
\[
g(\sigma, t) = t + \frac{1}{\pi} \log |1 + e^{2i\pi(\sigma-1/2+it)}|.
\]
We will observe that there is \( \delta_r > 0 \) such that
\[
g(\sigma, t) > \delta_r,
\] (2.3)
where \( s = \sigma + it \) lies on the semicircle \( |s-n| = r, t \geq 0, n \in \mathbb{Z}, \) and \( 1/6 < r < 1/2 \).
Note that \( g(x+n,t) = g(-x+n,t), x \in \mathbb{R} \). Thus it is enough to prove (2.3) for the following quarter of the circle: \( |s-n| = r, t \geq 0, 0 \leq \sigma - n \leq r \), which we parametrize by \( t = x, \sigma = \sqrt{r^2 - x^2} + n, x \in [0,r] \). Consequently we consider the function
\[
q(x) = g(\sqrt{r^2 - x^2} + n, x).
\]
Straightforward calculations show that \( q(0) > 0 \) and \( q'(x) > 0 \) for \( 0 \leq x \leq r, 1/6 < r < 1/2 \). This establishes the inequality (2.3).

Hence, for any given real positive number \( Y \) and \( 1/6 < r < 1/2 \), there is a negative number \( M = M(Y, r) \) such that
\[
|f(s)| = \exp \left( \text{area}(X) \left( -(\sigma - 1/2)g(\sigma, t) + O(|t| + 1) \right) \right) > Y,
\]
if \( |s-n| = r, t \geq 0, \) and \( n < M \). The Dirichlet series expansion of \( Z(s) \) yields
\[
|Z(s)| > 1/2
\] (2.4)
if \( \sigma \) is sufficiently large. Note that
\[
\overline{Z(s)} = Z(\overline{s}).
\] (2.5)

Then Rouché’s theorem gives that for sufficiently large negative \( n \) the functions \( Z(s) \) and \( Z(s) - a \) have the same number of zeros in the disc \( |s-n| \leq r \). In this disc \( Z(s) \) has only one distinct zero at \( s = n \) and clearly \( Z(n) \neq a \). This, \( 1/6 < r < 1/2 \), and Proposition 1.4 give that \( Z(s) - a \) and \( (Z(s) - a)' = Z'(s) \) have no common zeros in \( \sigma < N \). Accordingly, all zeros of \( Z(s) - a \) located in \( |s-n| \leq r \) are simple.

It remains to show that for any sufficiently large negative \( n \) the area \( \{ s : |s-n| > r, n-1/2 \leq \sigma \leq n + 1/2 \} \) is free from zeros of \( Z(s) - a \). This follows by the inequalities \( \partial g(\sigma, t)/\partial t > 0 \) if \( t > 0, \sigma \in \mathbb{R} \) and \( g(\sigma, 0) > 0 \) if \( |\sigma - 1/2 - n| < 1/3 \), together with formulas (2.1)–(2.5). Proposition 1.5 is proved. \( \Box \)
Lemma 2.1. If the polynomial $P(z)$ has at least two different roots, then there is a nonzero constant $c$ such that $P(z) - c$ has a multiple root.

Proof. Let $\deg P = k \geq 2$. Conversely to the statement of the lemma, suppose that the roots of $P(z) - c$ are simple for all $c \neq 0$. Then $(P(z) - c)' = P'(z)$ has no common roots with $P(z) - c$ for any $c \neq 0$. Therefore, for any root $z'_j$, $j \in \{1, \ldots, k-1\}$, of $P'(z)$, we have $P'(z'_j) = 0$. This is possible only if $P(z) = a(z-z'_1)^k$ and $z'_j = z'_1$, for all $j \in \{2, \ldots, k-1\}$. The contradiction obtained proves the lemma. □

Proof of Corollary 1.6. By Proposition 1.5 for any large negative $n$ and fixed $x \in [0,1]$, there are exactly $(2g-2)(n+1)$ simple zeros $s_j(x)$ of $Z(s) - a(x)$ in the disc $|s-n| < 1/3$. Then the corollary follows by the implicit function theorem ([9, Theorem 2.4.1]) from which we see that $Z(s)$ is a one-to-one function in some neighborhood of each $s_j(x)$, $j = 1, \ldots, (2g-2)(n+1)$, $x \in [0,1]$. □

Proof of Lemma 1.3. Note that $P$ cannot be a constant polynomial. To obtain a contradiction, assume that $Z(s) = P(h(s))$ and the polynomial $P$, $\deg P = k$, has at least two different roots. Then Lemma 2.1 implies the existence of $a_1$ such that $P'(a_1) = 0$ and $P(a_1) \neq 0$. Therefore we can write

$$P(z) - P(a_1) = d(z-a_1)^{k_1} \cdots (z-a_m)^{k_m}, \quad (2.6)$$

where $k_1 \geq 2$ and $k_1 + \cdots + k_m = k$. In view of Proposition 1.5 there are infinitely many zeros of $Z(s) - P(a_1)$ each of which lies at a distance smaller than $1/3$ from some negative integer. Thus there are an infinite subset $S$ of these zeros and $a_j$ defined by (2.6) such that $h(\rho) - a_j = 0$ for $\rho \in S$. If $k_j \geq 2$ then the zeros $\rho$ are multiple zeros of $Z(s) - P(a_1)$ and this contradicts Proposition 1.5. Hence $k_j = 1$, $P'(a_j) \neq 0$, and by (2.6) we see that $j \geq 2$. Therefore there is a continuous function $a : [0,1] \rightarrow \mathbb{C}$, such that $a(0) = a_j$, $a(1) = a_1$, and

$$P'(a(x)) \neq 0 \quad \text{for} \quad x \in [0,1). \quad (2.7)$$

By Corollary 1.6 there is a continuous function $\psi : [0,1] \rightarrow \mathbb{C}$ such that $\psi(0) \in S$,

$$Z(\psi(x)) = P(a(x)), \quad (2.8)$$

and, for $x \in [0,1]$ and some large negative integer $n$,

$$|\psi(x) - n| < 1/3.$$ 

Note that $Z(\psi(x)) = P(h(\psi(x)))$. To get the contradiction we will show that $h(\psi(1)) = a_1$. By (2.8)

$$P(h(\psi(x))) = P(a(x)).$$

In view of (2.7) we have that $P(z)$ is a one-to-one function in a sufficiently small neighborhood of any $a(x), x \in [0,1)$. Then $h(\psi(0)) = a(0)$ leads to the equality $h(\psi(x)) = a(x)$ for $x \in [0,1)$. Continuity gives $h(\psi(1)) = a(1) = a_1$ and thus $z = \psi(1)$ is a multiple zero of $Z(z) - Z(\psi(1))$. This contradicts Proposition 1.5 and proves Lemma 1.3. □
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