Sparse reconstruction in spin systems I: iid spins

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Abstract

For a sequence of Boolean functions $f_n : \{-1,1\}^{V_n} \rightarrow \{-1,1\}$, defined on increasing configuration spaces of random inputs, we say that there is sparse reconstruction if there is a sequence of subsets $U_n \subseteq V_n$ of the coordinates satisfying $|U_n| = o(|V_n|)$ such that knowing the coordinates in $U_n$ gives us a non-vanishing amount of information about the value of $f_n$.

We first show that, if the underlying measure is a product measure, then no sparse reconstruction is possible for any sequence of transitive functions. We discuss the question in different frameworks, measuring information content in $L^2$ and with entropy. We also highlight some interesting connections with cooperative game theory. Beyond transitive functions, we show that the left-right crossing event for critical planar percolation on the square lattice does not admit sparse reconstruction either. Some of these results answer questions posed by Itai Benjamini.

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1 Introduction and main results

Consider some random variables $X_V := \{X_v : v \in V\}$ on some probability space $(\Omega^V, \mathbb{P})$, where $\mathbb{P}$ is not necessarily a product measure, and a function $f : \Omega^V \to \mathbb{R}$, often the indicator function of an event. How much information does a subset $X_U$ of the input variables has about the output $f(X_V)$? There are several possible approaches to formulate this question precisely; here we are focusing on the following one (and the connections to some others will be reviewed in Subsection 3.1). Depending on the measure $\mathbb{P}$ and the function $f$, when is it possible that knowing $X_U$ for a small but carefully chosen subset $U$ (specified in advance, independently of the values of the variables) will give enough information to estimate $f(X_V)$?

First of all, we need to measure the amount of information we gain about $f(X_V)$ by learning a subset of the coordinate values of a configuration. For a subset $U \subseteq V$, let $\mathcal{F}_U$ denote the $\sigma$-algebra generated by $X_U$.

**Definition 1.1 (L²-clue).** Let $f : (\Omega^V, \mathbb{P}) \to \mathbb{R}$ and $U \subseteq V$. Then,

$$\text{clue}(f \mid U) := \frac{\text{Var}(E[f \mid \mathcal{F}_U])}{\text{Var}(f)}.$$  \hfill (1.1)

This notion of $\text{clue}_f(U)$ quantifies the proportion of the total variance of $f$ attributed to the variance of the function projected onto $\mathcal{F}_U$. The clue is always a number between 0 and 1, as an orthogonal projection can only decrease the variance. It is also worth noting that

$$\text{clue}(f \mid U) = \frac{\text{Cov}^2(f, E[f \mid \mathcal{F}_U])}{\text{Var}(f) \text{Var}(E[f \mid \mathcal{F}_U])} = \text{Corr}^2(f, E[f \mid \mathcal{F}_U]),$$  \hfill (1.2)

using that $\text{Cov}(f, E[f \mid \mathcal{F}_U]) = \text{Var}(E[f \mid \mathcal{F}_U])$.

This concept first appeared under this name in [GPS10], where $f$ was always the indicator function of some crossing event in critical planar percolation, and, among many other results, the following [BKS99, Conjecture 5.1] was proved. Let $\mathbb{P}$ be the product measure in which every edge in the box $[n]^2$ of the square lattice $\mathbb{Z}^2$ is deleted with probability $1/2$ independently; let $f_n$ be the indicator of having a left-to-right crossing in $[n]^2$, and let $U_n$ be the set of vertical edges in $[n]^2$. Then, $\text{clue}(f_n \mid U_n) \to 0$, as $n \to \infty$. The present project was started by Itai Benjamini asking the question: in critical planar percolation, does $\text{clue}(f_n \mid U_n) \to 0$ hold for every sequence of subsets with $|U_n| = o(n^2)$? The results of [GPS10] give an affirmative answer for many such sequences $\{U_n\}$, but not for all. And how about other natural Boolean functions in place of percolation events, still with iid measures $\mathbb{P}_n$?

**Definition 1.2 (Sparse reconstruction).** Consider a sequence $f_n : (\Omega^V, \mathbb{P}_n) \to \mathbb{R}$. We say that there is sparse reconstruction for $f_n$ w.r.t. $\mathbb{P}_n$ if there is a sequence of subsets $U_n \subseteq V_n$ with $\lim_n |U_n|/|V_n| = 0$ such that $\liminf_n \text{clue}(f_n \mid U_n) > 0$.

In this paper, our main focus will be on product measures $\mathbb{P}_n$. If $f_n$ depends only on a small proportion of the variables (e.g., dictators and juntas), then sparse reconstruction is obviously possible. But what happens if all the variables play an equal role, i.e., if there is some transitive group action $\Gamma_n \curvearrowright V_n$ for every $n$ under which both the measure $\mathbb{P}_n$ and the function $f_n$ are invariant? Here is our answer for iid sequences in a probability space $(\Omega, \pi)$.

**Theorem 1.1 (Clue of transitive functions).** Let $f \in L^2(\Omega^V, \pi^{\otimes V})$, and suppose that there is a subgroup $\Gamma \leq \text{Sym}_V$ acting on $V$ transitive such that $f$ is invariant under its action. If $U \subseteq V$, then

$$\text{clue}(f \mid U) \leq \frac{|U|}{|V|}.$$  

In particular, sparse reconstruction for transitive functions of iid variables is not possible.
We will first give a proof for the case when \( \pi \) is the uniform measure on \( \{-1, 1\} \), using the Fourier spectral sample, introduced in [GPS10]. We will then generalize this proof to general product measures using the Efron-Stein decomposition from [ES81] (see also [OD14, Section 8.3]). Given these notions of a spectral sample, the proof turns out to be surprisingly simple (see Section 2). However, it is quite rigid, using transitivity in an essential way. For instance, one cannot replace transitivity by the condition that each variable has the same small influence, as shown by the example of Remark 2.3 below. That example is even quasi-transitive (with two orbits, one being much larger than the other). We can fully avoid transitivity only if \( U \) is not a carefully chosen deterministic set, but is random:

**Theorem 1.2** (No reconstruction from sparse random sets). Let \( f \in L^2(\Omega^V, \pi^{\otimes V}) \) be any function. Let \( \mathcal{U} \) be a random subset of \( V \), independent of \( \pi^{\otimes V} \). Then

\[
\mathbb{E}[\text{clue}(f \mid \mathcal{U})] \leq \delta(\mathcal{U}),
\]

where \( \delta(\mathcal{U}) := \max_{j \in V} \mathbb{P}[j \in \mathcal{U}] \) is called the revealment of \( \mathcal{U} \).

This notion of revealment was introduced in [SS10] for randomized algorithms computing Boolean functions by asking bits one-by-one, allowed to use the information already obtained in choosing which bit to ask next, along with extra randomness. Many interesting functions are known to have small revealment algorithms computing their values, and the key discovery of Schramm and Steif in [SS10] was that such functions are necessarily noise sensitive. Although this is not the usual definition, noise sensitivity can be defined in terms of clue, and then the result of [SS10] can be stated as follows: if \( f_n \) can be computed with a randomized algorithm with revealment \( \delta_n \), and \( B^{1-\eta_n} \) is an iid Bernoulli(1 - \( \eta_n \)) subset of \( V_n \), with \( \eta_n/\sqrt{\delta_n} \to \infty \), then

\[
\mathbb{E}[\text{clue}(f_n \mid B^{1-\eta_n})] \to 0.
\]

If \( \mathcal{U} \) was a small revealment subset, independent of \( \pi^{\otimes V} \), which had a clue close to 1 about \( f \), then asking the bits in \( \mathcal{U} \) would be a randomized algorithm that approximately computes \( f \), hence the above theorem from [SS10] would say that \( f \) is noise sensitive. But then, even the high density random set \( B^{1-\eta} \) would have a small clue, so do we not get a contradiction to \( \mathcal{U} \) having large clue, obtaining a proof of Theorem 1.2 immediately from [SS10]? The answer is “no”, for two reasons. One, getting from non-vanishing clue to a clue close to 1 does not seem to be an obvious matter (see Question 7.3 at the end of the paper). Two, a small revealment random set \( \mathcal{U} \) might be quite different from an iid Bernoulli random set. Consider, e.g., one of the standard noise sensitive Boolean functions \( \text{Tribes}_k \), where \( k^2 \) fair bits form \( 2^k \) tribes of size \( k \) each, and the output is 1 iff there is at least one tribe where all bits are 1. Here, asking all the bits in a fixed small proportion of the tribes gives a small but non-vanishing clue as \( k \to \infty \), in contrast with asking an iid Bernoulli subset of the bits, with any density that is bounded away from 1. (See also Question 7.5.)

The small revealment theorem of [SS10] nevertheless suggests that if we want interesting non-transitive functions for which sparse reconstruction is possible, then we should probably look for noise-sensitive examples. A central example in the theory (see [GS15]) is left-to-right crossing in critical planar percolation. Here a lot is known about the spectral sample [GPS10], but still, the proof of Theorem 1.1 does not generalize in a straightforward way. We will nevertheless answer Benjamini’s question by proving in Section 6 that there is no sparse reconstruction, with an argument going in the opposite direction: we will produce a translation-invariant function from left-to-right crossings, and will show that if crossings had sparse reconstruction, then this transitive function would also have, contradicting Theorem 1.1.

**Theorem 1.3** (No sparse reconstruction in percolation). There is no sparse reconstruction for \( f_n \), being the indicator of left-to-right crossing in the box \([n]^2\) in critical bond percolation on \( \mathbb{Z}^2 \).
The same argument works for left-to-right crossing in the $n \times n$ rhombus $R_{n Renderer error: Unknown option '72x126'.
The proof of Theorem 1.5 is combinatorial, by induction, quite similar to the proof of Shearer’s inequality. See Section 5.

Our motivation for defining different notions of clue and proving the corresponding small clue theorems was not just abstract curiosity. In forthcoming work, we will study sparse reconstruction for non-iid measures \( P \) — primarily the Ising model and factor of iid spin systems. As we have explained, a small clue theorem may be considered as a baby noise sensitivity result. However, discrete Fourier analysis breaks down for non-iid measures, hence it is highly desirable to come up with possible replacements. For instance, we will prove a small clue theorem for any high temperature Curie-Weiss model (the Ising model on the complete graph) using the I-clue and a version of Theorem 1.4.

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2. \( L^2 \)-clue and sparse reconstruction for transitive functions

2.1 The Fourier-Walsh expansion and the Spectral Sample

We introduce a function transform on the hypercube which turns out to be an essential tool in the analysis of Boolean functions. We still consider the uniform measure \( P_{1/2} := (\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{1}) \otimes V_n \). We can introduce the natural inner product \((f,g) = E[f g]\) on the space of real functions on the hypercube.

**Definition 2.1** (Fourier-Walsh expansion). For any \( f \in L^2(\{-1,1\}^V, P_{1/2}) \) and \( \omega \in \{-1,1\}^V \)

\[
f(\omega) = \sum_{S \subseteq V} \hat{f}(S) \chi_S(\omega), \quad \chi_S(\omega) := \prod_{i \in S} \omega_i \quad \text{(and } \chi_S(\emptyset) := 1). \tag{2.1}\]

This is in fact the Fourier transform, the event space naturally identified with the group \( \mathbb{Z}_2^V \) by assigning a generator \( g_x \) to every \( x \in V \). The functions \( \chi_S \) are in fact the characters of \( \mathbb{Z}_2^V \).

It is straightforward to check that the functions \( \chi_S \) form an orthonormal basis with respect to the inner product, so Parseval’s formula applies and therefore

\[
\sum_{S \subseteq V} \hat{f}(S)^2 = \|f\|^2.
\]

Noting that \( \hat{f}(\emptyset) = E[f] \), we also have

\[
\text{Var}(f) = \sum_{\emptyset \neq S \subseteq V} \hat{f}(S)^2. \tag{2.2}
\]

For a subset \( T \subseteq V \) let us denote by \( \mathcal{F}_T \) the \( \sigma \)-algebra generated by the bits in \( T \). So \( \mathcal{F}_T \) expresses knowing the coordinates in \( T \). It turns out that the conditional expectation of any function \( f : \{1,1\}^n \to \mathbb{R} \) with respect to \( \mathcal{F}_T \) can be expressed in terms of the squared Fourier-Walsh expansion coefficients; see [GS15]:

\[
E[f | \mathcal{F}_T] = \sum_{S \subseteq T} \hat{f}(S) \chi_S. \tag{2.3}
\]

The proof is fairly simple: we only need to observe that, if \( S \subseteq T \), then \( E[\chi_S | \mathcal{F}_T] = \chi_S \), and in any other case \( E[\chi_S | \mathcal{F}_T] = 0 \).
Using (2.2) we get a concise spectral expression for the variance of the conditional expectation:

$$\text{Var}(\mathbb{E}[f \mid \mathcal{F}_T]) = \sum_{\emptyset \neq S \subseteq T} \hat{f}(S)^2.$$  \hfill (2.4)

It turns out to be useful to think about the squared Fourier coefficients $\hat{f}(S)^2$ as the measure on all the subsets of the spins. We usually normalize this measure to get a probability measure. The random subset $\mathcal{S}_f$ distributed accordingly is called the spectral sample.

**Definition 2.2 (Spectral sample).** Let $f \in L^2(\{-1, 1\}^V, \mathbb{P}_{1/2})$. The spectral sample $\mathcal{S}_f$ of $f$ is a random subset of $V$ chosen according to the distribution

$$\mathbb{P}[\mathcal{S}_f = S] = \frac{\hat{f}(S)^2}{\|f\|^2}, \text{ for any } S \subseteq V.$$  \hfill (2.5)

The advantage of this concept is that it introduces a rather compact language, where some important concepts admit straightforward translations. The notion of clue, in particular, translates well to the Spectral Sample language. Using (2.2) and (2.4) we get that

$$\text{clue}(f \mid U) = \mathbb{P}[\mathcal{S}_f \subseteq U \mid \mathcal{S}_f \neq \emptyset].$$

This equation, as we shall see, is one of the key observations in the proof of Theorem 2.1.

### 2.2 No sparse reconstruction for transitive functions of fair coins

The following theorem provides a sharp upper bound on the clue of not only Boolean, but general real-valued transitive functions. The proof is surprisingly short and it demonstrates the power of the notion of spectral sample in an impressive way.

**Theorem 2.1 (Clue of transitive functions).** If $f : \{-1, 1\}^V \rightarrow \mathbb{R}$ transitive, $U \subseteq V$, then

$$\text{clue}(f \mid U) \leq \frac{|U|}{|V|}.$$  \hfill (2.6)

**Proof.** Let $X$ be a uniformly random element from the spectral sample $\mathcal{S}_f$ of $f$ conditioned on being non-empty. Because $f$ is transitive, $X$ is uniform on $V$. Using (2.5) we get the following:

$$\text{clue}(f \mid U) = \mathbb{P}[\mathcal{S} \subseteq U \mid \mathcal{S} \neq \emptyset] \leq \tilde{\mathbb{P}}[X \in U] = \sum_{u \in U} \tilde{\mathbb{P}}[X = u] = \frac{|U|}{|V|},$$

where $\tilde{\mathbb{P}}$ denotes the probability measure conditioned on $\{\mathcal{S} \neq \emptyset\}$. \hfill $\square$

**Remark 2.2.** The bound in Theorem 2.1 is sharp, as it is testified by the function $\sum_{v \in V} \omega_v$.

It is worth to emphasize that the result does not only apply for sequences of Boolean functions, but also for any sequences of real-valued functions, no matter bounded or not.

**Remark 2.3.** There is no obvious way to relax the condition of transitivity. Let $f : \{-1, 1\}^V \rightarrow \{-1, 1\}$ and $j \in V$. The influence of the coordinate $j$ is $I_j(f) := \mathbb{P}[f(\omega) \neq f(\omega^j)]$, where $\omega^j$ is the same as $\omega$ except its $j$th coordinate is flipped.

We now sketch an example of a sequence of Boolean functions where the influences $I_j(f_n)$ are (almost) equal for every $n$, however there is a sparse subset of coordinates $U_n$ (i.e., $\lim_n \frac{|U_n|}{|V_n|} = 0$) such that $\lim_n \text{clue}(f_n \mid U_n) = 1$. 

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6
Let $a_n$ be a sequence of integers such that $a_n \rightarrow \infty$. Let us define the asymmetric majority functions

$$\text{Maj}_n^{a_n}(\omega) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \omega(i) > a_n \sqrt{n} \\ -1 & \text{if } \sum_{i=1}^{n} \omega(i) < a_n \sqrt{n}. \end{cases}$$

We can choose $a_n$ in such a way that for some small $\epsilon > 0$

$$I_i(\text{Maj}_n^{a_n}) = \frac{\binom{n}{n/2+2a_n\sqrt{n}}}{2^n} \sim \frac{1}{n^{2/3}}$$

holds.

Furthermore, define the Boolean function $\text{Tribes}_n: \{-1, 1\}^n \rightarrow \{0, 1\}$ as follows: we group the bits in $k$ $l$-element subsets, these are the so-called tribes. The function takes on 1 if there is a tribe $T$ such that for every $i \in T: \omega(i) = 1$, and 0 otherwise.

$\text{Tribes}_{n,k}$ is known to be balanced if $l_n = \log n - \log \log n$ and $k_n = n/l_n$. Let us denote this balanced version of the tribes on $n$ bits by $\text{Tribes}_n$. An easy calculation shows that $I_i(\text{Tribes}_n) \sim \frac{\log n}{n}$.

Take a disjoint union $V_n = M_n \cup T_n$, with $|M_n| = m_n$ and $|T_n| = t_n$. Now we define our function as follows:

$$f_n := \begin{cases} \text{Maj}_n^{a_n}(\omega_{M_n}) & \text{if } \text{Tribes}(\omega_{T_n}) = 1 \\ \text{Maj}_{-a_n}(\omega_{M_n}) & \text{if } \text{Tribes}(\omega_{T_n}) = -1. \end{cases}$$

We adjust the size of $M_n$ and $T_n$ in such a way that the influence of each coordinate is the same. So we have the equation $\frac{\log t_n}{m_n} = \frac{1}{m_n}$, or equivalently

$$m_n = \left( \frac{t_n}{\log t_n} \right)^{3/2}.$$

So the density of $T_n$ goes to 0 compared to $|V_n| = t_n + m_n$. At the same time, from the Central Limit Theorem it is clear that $\lim_n \mathbb{P}[\text{Maj}_{m_n}^{a_n} = 1] = 0$ and $\lim_n \mathbb{P}[\text{Maj}_{-m_n}^{-a_n} = 1] = 1$. Consequently, $\lim_n \text{clue}(f_n | T_n) = 1$.

Remark 2.4. We point out an interpretation of the random element $X$ of the spectral sample appearing in the proof of Theorem 2.1. This setup also has some interesting connections with one of the key lemmas in Chatterjee’s book on superconcentration and chaos [Cha14].

For a function $f: \{-1, 1\}^V \rightarrow \mathbb{R}$ we define the stability of $f$ at level $p$ as

$$\text{Stab}_f(p) := \sum_{\emptyset \neq S \subseteq V} \hat{f}(S)^2 p^{|S|}.$$

(This is a small modification of the definition in [OD14].) Let us denote by $\omega^{1-p}$ the random vector which we obtain from $\omega$ by resampling each of its bits independently with probability $1 - p$. With this notation, we clearly have $\text{Stab}_f(p) = \text{Cov}(f(\omega), f(\omega^{1-p}))$.

At the same time, it is also the expected clue of a Bernoulli random set of coordinates $B^p$ of density $p$: $\frac{\text{Stab}_f(p)}{\text{Var}(f)} = \mathbb{E}[\text{clue}(f | B^p)]$.

Stability can be generalized as a polynomial of $|V|$ variables. Then the quantity

$$\frac{\text{Stab}_f(x)}{\text{Var}(f)} = \frac{1}{\text{Var}(f)} \sum_{\emptyset \neq S \subseteq V} \hat{f}(S)^2 \prod_{i \in S} x_i$$

can be interpreted as the expected clue of a random subset where the bit $i$ is selected with probability $x_i$, independently from other bits.
Denote by $\mathbf{p}$ the vector with all of its coordinates is equal to $p$ and for a $j \in V$ take the partial derivative of $\text{Stab}_f(\mathbf{p})$ with respect to the $j$th coordinate. We obtain that

$$\frac{\partial \text{Stab}_f(\mathbf{p})}{\partial p_j} = \sum_{S \ni j} \hat{f}(S)^2 p^{|S| - 1}.$$ 

Now here is the relationship with $X$, the uniformly random element of the spectral sample:

$$\int_0^1 \frac{\partial \text{Stab}_f(\mathbf{p})}{\partial p_j} dp = \sum_{S \ni j} \hat{f}(S)^2 \frac{1}{|S|} = \text{Var}(f) \mathbb{P}[X = j].$$

The above quantity can be understood as the average increase in clue over all $p$ values, induced by a small increase in the probability of selecting $j$ into the random set. This interpretation becomes even more explicit in the cooperative game theory framework (see Proposition 5.1 below).

Now we get to the connection with Chatterjee’s work. Let $f, g : \{-1, 1\}^V \rightarrow \{-1, 1\}$ be monotone Boolean functions. Let us denote by $\omega^{1-p}$ is the We start by expressing $\mathbb{P}[j \in \mathcal{P}_f(\omega) \cap \mathcal{P}_g(\omega^{1-p})]$ in terms of the Fourier-Walsh transform.

Observe that for any monotone $f : \{-1, 1\}^V \rightarrow \{-1, 1\}$, we have

$$\nabla_j f(\omega) = f(\omega|\omega_j = 1) - f(\omega|\omega_j = -1) = \sum_{S \ni j} \hat{f}(S) \chi_{S \cap j}(\omega).$$

As $j$ is in $\mathcal{P}_f(\omega)$ if and only if $\nabla_j f(\omega) = 2$ and otherwise $\nabla_j f(\omega) = 0$, we get that

$$\mathbb{1}_{i \in \mathcal{P}_f(\omega)} = \frac{1}{2} \sum_{S \ni j} \hat{f}(S) \chi_{S \cap j}(\omega).$$

Now recall that

$$\mathbb{E}[\chi_T(\omega) \chi_S(\omega^{1-p})] = \begin{cases} 0 & \text{if } T \neq S, \\ p^{|S|} & \text{if } T = S, \end{cases}$$

and thus, whenever $f$ and $g$ are monotone, we have

$$\mathbb{P}[j \in \mathcal{P}_f(\omega) \cap \mathcal{P}_g(\omega^{1-p})] = \mathbb{E}[\mathbb{1}_{j \in \mathcal{P}_f(\omega)} \mathbb{1}_{j \in \mathcal{P}_g(\omega^{1-p})}] = \frac{1}{4} \sum_{S \ni j} \hat{f}(S) \hat{g}(S) p^{|S| - 1}. \quad (2.8)$$

(We note that this formula is almost a generalization of Lemma 2.7 in [RS18].) Using that $\sum_{j \in V} \mathbb{P}[j \in \mathcal{P}_f(\omega) \cap \mathcal{P}_g(\omega^{1-p})] = \mathbb{E}[|\mathcal{P}_f(\omega) \cap \mathcal{P}_g(\omega^{1-p})|]$, we get from (2.8) that

$$\int_0^1 \mathbb{E}[|\mathcal{P}_f(\omega) \cap \mathcal{P}_g(\omega^{1-p})|] dp = \frac{1}{4} \sum_{j \in V} \left( \sum_{S \ni j} \hat{f}(S) \hat{g}(S) \frac{1}{|S|} \right) = \frac{1}{4} \text{Cov}(f, g). \quad (2.9)$$

This is essentially a special case of Lemma 2.1 from [Cha14] (referred to as “covariance lemma”), where the Markov process is the random walk on the hypercube. At the same time, setting $g = f$, by (2.7) we have

$$\mathbb{P}[X = j] = \frac{1}{\text{Var}(f)} \int_0^1 \frac{\partial \text{Stab}_f(\mathbf{p})}{\partial p_j} dp = \frac{4}{\text{Var}(f)} \int_0^1 \mathbb{P}[j \in \mathcal{P}_f(\omega) \cap \mathcal{P}_f(\omega^{1-p})] dp,$$

which is a coordinate-wise localized version of the covariance lemma.

We also note that the threshold saddle vertex introduced in [Riv19+] for level set percolation is a random pivotal vertex, so it might be considered as a real (non-Fourier) space analogue of our $X$, and it is also closely related to Chatterjee’s covariance lemma.
2.3 No sparse reconstruction in general product measures

One may ask whether a result similar to Theorem 1.1 can be derived in case we replace the \([-1,1]\) space in the domain with something more complicated or if we replace the product measure with some other measure. A natural idea in this direction is to try to generalize the concept of spectral sample. We might take again Equation (2.5) as a starting point.

Observe that the quantity \(\text{clue}(f|U)\) is well defined for any \(U \subseteq V\) on any product space \(X^V\), no matter what the probability measure is. So one could try to use equation (2.5) as the definition for a generalised spectral sample. As the probabilities \(P[\mathcal{S} \subseteq U]\) are known for all \(U\), one can also calculate the probabilities \(P[\mathcal{S} = T]\) for all \(T\). Once we have this generalised spectral sample in hand (depending on the function, the space and the underlying measure) we might be able to repeat the argument in the proof of Theorem 2.1.

Unfortunately this strategy fails in general. The problem is that nothing guarantees that the quantities \(P[\mathcal{S} = T]\) that we get from the Möbius inversion are non-negative. Nevertheless, in case the underlying measure is a product measure, the above strategy works as the quantities \(P[\mathcal{S} = T]\) turn out to be non-negative. As we will show, this follows directly from the so-called Efron-Stein decomposition ([OD14], Section 8.2), a generalization of the Fourier-Walsh transform for product measures.

We will need the following simple observation, which turns out to be crucial. In fact, as we shall see Efron-Stein decomposition as well as the possibility of a spectral sample, ultimately depends on Fubini’s Theorem.

**Lemma 2.5.** Let \(f \in L^2(\Omega^n,\pi^\otimes n)\) and let \(K, L \subseteq [n] \). Then
\[
E[E[f | \mathcal{F}_L] | \mathcal{F}_K] = E[f | \mathcal{F}_{L \cap K}].
\]

**Proof.** Rewriting the conditional expectations as integral, and using Fubini’s theorem,
\[
\int_{X^K} \left( \int_{X^L} f(X_L, x_{L^c} \, dx_{L^c}) \right) \, dx_{K^c} = \int_{X^{K \cup L^c}} f(X_{L \cap K}, x_{K^c \cup L^c}) \, dx_{K^c \cup L^c}.
\]

**Theorem 2.6** (Efron-Stein decomposition [ES81]). For any \(f \in L^2(\Omega^n,\pi^\otimes n)\), there is a unique decomposition
\[
f = \sum_{S \subseteq [n]} f^S,
\]
where \(f^S\) is a function that depends only on the coordinates in \(S\), and \((f^S, f^T) = 0\) whenever \(S \neq T\).

For completeness, we include a proof, following the ideas from [OD14], but with our notation.

**Proof.** Observe that in case we assume that the sought decomposition exists, we have (like in the case of the hypercube, see (2.3)),
\[
E[f | \mathcal{F}_T] = \sum_{S \subseteq T} f^S.
\]
Indeed this is clear from the fact that \(E[f^S | \mathcal{F}_T]\) is \(f^S\) in case \(S \subseteq T\) (as in this case \(f^S\) is \(\mathcal{F}_T\)-measurable) and 0 otherwise (because of the orthogonality of the decomposition).

We can use this to find the functions \(f^S\) via a Möbius inversion (in this case, an exclusion-inclusion principle) from the conditional expectations. So define
\[
f^S := \sum_{L \subseteq S} (-1)^{S-L} E[f | \mathcal{F}_L].
\]
It is obvious from the construction that \( f = T \) only depends on coordinates in \( T \). So what is left to show is that \( f = T \) and \( f = S \) are if they are orthogonal, whenever \( T \neq S \). First we show that if \( g \) is \( \mathcal{F}_U \)-measurable and \( S \setminus T \neq \emptyset \) then \( f = T \) and \( g \) are orthogonal. We can pick an \( i \in S \setminus T \) and write the above inner product as

\[
\mathbb{E}[g f = S] = \sum_{L \subseteq S \setminus \{i\}} (-1)^{S - L} \mathbb{E}[g \mathbb{E}[f | \mathcal{F}_L]] - \mathbb{E}[g \mathbb{E}[f | \mathcal{F}_{L \cup \{i\}}]],
\]

using that \((-1)^{S - L} \) and \((-1)^{S - L \cup \{i\}} \) has opposite signs. Conditioning on \( T \) and after on \( L \) before taking the expectation and applying Lemma 2.5 twice gives that

\[
\mathbb{E}[g \mathbb{E}[f | \mathcal{F}_L]] = \mathbb{E}\left[ \mathbb{E}[g | \mathcal{F}_{T \cap L}] \mathbb{E}[f | \mathcal{F}_{T \cap L}] \right] = \\
\mathbb{E}\left[ \mathbb{E}[g | \mathcal{F}_{T \cap (L \cup \{i\})}] \mathbb{E}[f | \mathcal{F}_{T \cap (L \cup \{i\})}] \right] = \mathbb{E}[g \mathbb{E}[f | \mathcal{F}_{L \cup \{i\}}]].
\]

We used that \( T \cap (L \cup \{i\}) = T \cap L \), since \( i \notin L \) and \( i \notin T \). This shows that \( \mathbb{E}[g f = S] = 0 \). From this to \( \mathbb{E}[f = T f = S] \) and switching the roles, it follows \( \mathbb{E}[f = T f = S] = 0 \) if either \( S \setminus T \neq \emptyset \) or \( T \setminus S \neq \emptyset \) which is equivalent to \( T \neq S \).

Observe that this is indeed a generalization of the Fourier-Walsh transform, with \( f = S = \hat{f}(S) \chi_S \). What is important for our purpose is that we can again define a Spectral Sample

\[ \mathbb{P}[\mathcal{S} = S] := \frac{\|f = S\|^2}{n} \]

for every square-integrable function, as in the case of the hypercube and thus Theorem 2.1 generalizes for product measures.

**Theorem 2.7** (Small clue theorem for random sets). Let \( f \in L^2(\Omega^\gamma, \mu^\gamma) \) and suppose that there is a \( G \leq S_n \) acting on the \( n \) copies of \( \Omega \) transitively such that \( f \) is invariant under its action. If \( U \subseteq [n] \), then

\[ \text{clue}(f | U) \leq \frac{|U|}{n}. \]

The proof is exactly the same as for Theorem 2.1, the only difference being that we need to use the Efron-Stein decomposition instead of the Fourier-Walsh transform to build the spectral sample.

We close this section by giving a variant of Theorem 2.7 which avoids the notion of transitivity altogether.

**Proposition 2.8** (No reconstruction from sparse random set). Let \( f \in L^2(\Omega^V, \pi^V) \) be any function. Let \( \mathcal{U} \) be a random subset of \( V \), independent of \( \pi^V \). Then

\[ \mathbb{E}[\text{clue}(f | \mathcal{U})] \leq \delta(\mathcal{U}), \]

where \( \delta(\mathcal{U}) := \max_{j \in V} \mathbb{P}[j \in \mathcal{U}] \) is called the revelation of \( \mathcal{U} \).

**Proof.** The proof basically repeats the proof of Theorem 2.1. Generate the spectral sample \( \mathcal{S}_f \) independently of \( \mathcal{U} \), and let \( X \) be a uniformly random element from \( \mathcal{S}_f \) conditioned on being non-empty; \( \mathbb{P} \) denotes the respective conditional probability measure. We use again (2.5), and it is easy to verify that \( \mathbb{E}[\text{clue}(f | \mathcal{U})] = \mathbb{P}[\mathcal{S} \subseteq \mathcal{U} | \mathcal{S} \neq \emptyset] \). Therefore, using that \( \mathcal{U} \) is independent from \( \pi^V \), we get

\[ \mathbb{E}[\text{clue}(f | \mathcal{U})] \leq \mathbb{P}[X \in \mathcal{U}] = \sum_{j \in [n]} \mathbb{P}[X = j, j \in \mathcal{U}] = \\
\sum_{j \in [n]} \mathbb{P}[X = j] \mathbb{P}[j \in \mathcal{U}] \leq \delta(\mathcal{U}) \sum_{j \in [n]} \mathbb{P}[X = j] = \delta(\mathcal{U}). \]

\[ \square \]
This statement can be read in such a way that in product measure there is no reconstruction for any function from a sparse random subset \( U \) (that is, for which \( \delta(U) \to 0 \)).

Note that this proposition implies Theorem 2.7. For a transitive function \( f \), if there was sparse reconstruction from a small subset \( U \), then any translate of \( U \) would work equally well. So, a uniform random translate of \( U \) would be a small revealment subset that had large clue about \( f \), contradicting Proposition 2.8.

3 Other approaches to measuring “clue”

3.1 Significance and influence of subsets

We would like to make a small detour to discuss some possible alternatives to “clue” as defined in Definition 1.1. Given a Boolean function \( f : \{-1, 1\}^V \to \{-1, 1\} \) and an underlying probability measure \( \mathbb{P} \), we want to quantify the amount of information a subset of the coordinates gives us about the function \( f \). We will denote the size of the coordinate set \( V \) by \( n \).

We start with a sort of dual to clue.

Definition 3.1. The significance of a subset \( U \subseteq V \) is

\[
\text{sig}(f | U) = \frac{\mathbb{E}[\text{Var}(f | \mathcal{F}_{U^c})]}{\text{Var}(f)}
\]

We call it a dual because we have \( \text{sig}(f | U) = 1 - \text{clue}(f | U^c) \). It expresses how much information we are still missing on average if we know the values of the bits outside of \( U \). We have the following description of \( \text{sig}(f | U) \) in terms of the spectral sample:

\[
\text{sig}(f | U) = \mathbb{P}[\mathcal{S}_f \cap U \neq \emptyset].
\]

This shows that for product measures \( \text{sig}(f | U) \geq \text{clue}(f | U) \). In general, this inequality does not hold: \( \text{sig}(f | U) < \text{clue}(f | U) \) whenever \( \text{clue}(f | U) + \text{clue}(f | U^c) > 1 \), which can easily happen if the underlying measure has lots of dependencies. Also, Theorem 2.1 is not true if we replace \( \text{clue} \) by \( \text{sig} \). For example, any subset \( U \subseteq V \) has significance 1 with respect to the parity function \( \chi_V \), which is obviously transitive. The famous “It Ain’t Over Till It’s Over” Theorem proved in [MOO05] says that, for sequences of functions with low maximal influence, for arbitrary small (but fixed) \( \epsilon \) the average significance of a Bernoulli random subset of level \( 1 - \epsilon \) is not vanishing.

We mention a similar concept introduced in [BL89]. For a subset \( U \subseteq V \) the influence of \( U \) is defined as follows:

\[
I(f | U) = \mathbb{P}[f \text{ is not determined by the bits on } U^c]
\]

Influence is much weaker than \( \text{sig} \), in the sense that it is easier to have high influence than to have high significance: it is clear from the definition that, for any underlying measure, \( I(f | U) \geq \text{sig}(f | U) \). Like in social choice theory, one may think about coordinates as individual agents trying to influence the value (outcome) of \( f \) by the values of the respective bits. In this framework the influence of a subset quantifies the probability that the set of agents in \( U \) can change the value of \( f \) by coordinating their values. While in this setting coordinates are allowed to cooperate, the significance rather quantifies the average gain of information (measured in variance) for a uniformly random configuration of \( U \).

We can again take the dual concept of influence, the combinatorial equivalent of clue, which is the probability that the subset \( U \) is a witness. For a Boolean function \( f : \{-1, 1\}^V \to \{-1, 1\} \)
and a configuration \( \omega \in \{-1, 1\}^V \) a subset \( W \subseteq V \) is a witness for \( f \) if \( \omega_W \) already decides the value of \( f \).

\[
W(f \mid U) = \mathbb{P}[f \text{ is determined by the bits on } U]
\]

Obviously, \( I(f \mid U) \geq W(f \mid U) \) holds irrespective of what the measure is, and also \( I(f \mid U) \geq \text{sig}(f \mid U) \), which entails after taking the dual in both sides, that \( \text{clue}(f \mid U) \geq W(f \mid U) \).

There are still many questions to be investigated. For the left-right crossing event \( \text{LR}_n \) for critical planar percolation, when \( U_n \) is a sub-square, it is proved in [GPS10] that \( I(\text{LR}_n \mid U_n) \asymp \text{sig}(\text{LR}_n \mid U_n) \). For \( \text{Maj}_n \) on the other hand, this is not the case. As it is easy to check, \( I(\text{Maj}_n \mid U) \gg \text{sig}(\text{Maj}_n \mid U) \) for any sequence of subsets with constant density.

**Question 3.1.** Characterise sequences of Boolean functions such that for any sequence of subsets \( U_n \) with constant density \( I(f_n \mid U_n) \gg \text{sig}(f_n \mid U_n) \) holds, or where \( I(f_n \mid U_n) \asymp \text{sig}(f_n \mid U_n) \), respectively.

### 3.2 Clue via distances between probability measures

In this section we introduce a somewhat different approach to measure the amount of information of a subset of coordinates about a Boolean function.

Let us consider the usual configuration space \( \{-1, 1\}^V \) endowed with a probability measure \( \mu \). Clearly, any Boolean function \( f : \{-1, 1\}^V \rightarrow \{0, 1\} \) can be interpreted as the density function of the measure \( \mu \) conditioned on the set of configurations \( \{\omega \in \{-1, 1\}^V : f(\omega) = 1\} \) (after normalizing with \( \mathbb{E}[f] \)).

More generally, every \( f : \{-1, 1\}^V \rightarrow \mathbb{R}_{\geq 0} \) with \( \mathbb{E}[f] > 0 \) can be interpreted as a density, and can be used to define another probability measure on the same space by

\[
\nu[\omega] := \frac{1}{\mathbb{E}[f]} f(\omega) \mu[\omega], \quad \omega \in \{-1, 1\}^n.
\]

(3.1)

Now that we identified our function with a probability measure, we can express clue in terms of distances of probability measures. Here we will consider three possible metrics: the total variation distance, the \( L^2 \) distance and, in Section 3.3, an information theoretic distance.

We fix a (non-trivial) Boolean function \( f \) and let us introduce the notation \( \mu^1 \) and \( \mu^0 \) for the measures \( \mu \) conditioned on the set \( \{f(\omega) = 1\} \) and \( \{f(\omega) = 0\} \), respectively. Furthermore, let \( \mu[f(\omega) = 1] = p \). So we have \( \mu = (1-p)\mu^0 + p\mu^1 \) and as a consequence,

\[
\frac{d\mu^1}{d\mu} = \frac{1}{p} f \quad \text{and} \quad \frac{d\mu^0}{d\mu} = \frac{1}{1-p} (1-f).
\]

We will need the measures \( \mu[U], \mu^1[U] \) and \( \mu^0[U] \), which are the marginals (projections) of the respective measures on the subset of coordinates \( U \subseteq V \). It is straightforward to check that

\[
\frac{d\mu^1_U}{d\mu_U} = \frac{1}{p} \mathbb{E}[f|\mathcal{F}_U] \quad \text{and} \quad \frac{d\mu^0_U}{d\mu_U} = \frac{1}{1-p} (1 - \mathbb{E}[f|\mathcal{F}_U]).
\]

In addition, we still have \( \mu_U = (1-p)\mu^0_U + p\mu^1_U \).

We introduce two meaningful ways of measuring the clue, irrespective from the particular notion of distance \( D(\cdot, \cdot) \). In the first version, the total information content of the function is measured through its distance from the original measure \( \mu \):

\[
\text{clue}_{\text{asymp}}(f \mid U) = \frac{D(\mu[U], \mu^1[U])}{D(\mu, \mu^1)}.
\]

(3.2)
In the alternative, symmetric version, which is the setup used in [Per99] Chapter 16, we express the information content with the distance between the measures $\mu^1$ and $\mu^0$:

$$\text{clue}_{\text{sym}}(f \mid U) = \frac{D(\mu^0 \mid_U, \mu^1 \mid_U)}{D(\mu^0, \mu^1)}.$$ 

This version has the (desirable) general property that $\text{clue}_{\text{sym}}(1_A \mid U) = \text{clue}_{\text{sym}}(1_{A^c} \mid U)$ for any $U \subseteq V$. The asymmetric version, however, has the advantage that — in contrast with the symmetric one — it trivially extends to all functions $f : \{-1, 1\}^n \rightarrow \mathbb{R}_{\geq 0}$.

Now we shall discuss two notions of distances between probability measures. The squared $L^2$ distance between two measures $\nu, \theta \ll \mu$ is given by

$$D^2_{\text{sym}}(\nu, \theta) := \frac{1}{4} \int_{\Omega} (f - g)^2 d\mu.$$

where $f = d\nu/d\mu$ and $g = d\theta/d\mu$. We shall denote the corresponding clues by $\text{clue}^{L^2}_{\text{sym}}$ and $\text{clue}^{L^2}_{\text{sym}}$, respectively. A simple calculation shows that

$$D^2_{\text{sym}}(\mu, \nu) = \frac{1}{4} \int_{\Omega} \left(1 - \frac{f}{p}\right)^2 d\mu = \frac{\text{Var}(f)}{4p^2},$$

where $p = \mathbb{E}[f]$. and, whenever $\mu = (1-p)\mu^0 + p\mu^1$ we have

$$D^2_{\text{sym}}(\mu^0, \mu^1) = \frac{1}{4} \int_{\Omega} \left(\frac{f}{p} - 1 - f - p\right)^2 d\mu = \frac{\text{Var}(f)}{4p^2(1-p)^2}.$$\hfill (3.4)

Therefore, using (3.3) and (3.4) for both numerator and denominator we get that $\text{clue}^{L^2}_{\text{sym}}(f \mid U) = \text{clue}^{L^2}_{\text{sym}}(f \mid U) = \text{clue}(f \mid U)$ thus it eventually does not matter whether we choose the symmetric or the asymmetric definition of $\text{clue}^{L^2}$.

The total variation distance (or $L^1$ distance) is defined as

$$D_{TV}(\nu, \theta) := \frac{1}{2} \sum_{\omega \in \Omega} |\nu[\omega] - \theta[\omega]| = \frac{1}{2} \int_{\Omega} |f - g| d\mu.$$

Calculations similar to the $L^2$ distance case give that

$$D_{TV}(\mu, \theta) = \frac{\mathbb{E}[|f - \mathbb{E}[f]|]}{2\mathbb{E}[f]},$$

and

$$D_{TV}(\mu^0, \mu^1) = \frac{\mathbb{E}[|f - \mathbb{E}[f]|]}{2\mathbb{E}[f](\mathbb{E}[f] - 1)}.$$

So

$$\text{clue}^{TV}(f \mid U) = \frac{\mathbb{E}[|\mathbb{E}[f]F_U| - \mathbb{E}[f]|]}{\mathbb{E}[|f - \mathbb{E}[f]|]},$$\hfill (3.6)

again irrespective from which of the two variants we use.

Without going into details we mention that again one can define a dual notion by $\text{sig}^{TV}(f \mid U) = 1 - \text{clue}^{TV}(f \mid U^c)$.

Let us compare $\text{clue}$ and $\text{clue}^{TV}$ for Boolean functions.

**Proposition 3.2.** Let $f : \{-1, 1\}^V \rightarrow \{0, 1\}$ and $U \subseteq V$ then

$$\frac{p_{\min}}{2} \text{clue}(f \mid U) \leq \text{clue}^{TV}(f \mid U) \leq \frac{2}{p_{\min}} \text{clue}(f \mid U),$$

where $p_{\min} := \min(\mu[f = 1], \mu[f = 0])$ where $p = \mathbb{E}[f]$. 

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Proof. First note that $\mu[f = 1]f + \mu[f = 0]g = 1$ and thus $|f - g| \leq \max(1/p, 1/(1-p)) = 1/p_{\min}$ so

$$D_2^2(\mu^0, \mu^1) = \sum_\omega \mu[\omega](f(\omega) - g(\omega))^2 \leq \frac{1}{p_{\min}} \sum_\omega \mu[\omega]|f(\omega) - g(\omega)| = \frac{1}{p_{\min}} D_{TV}(\mu^0, \mu^1).$$

For the second inequality we use the Cauchy-Schwartz inequality to get

$$D_{TV}(\mu^0, \mu^1) = \frac{1}{2} \sum_\omega \sqrt{\mu[\omega]} \sqrt{\mu[\omega]} |f(\omega) - g(\omega)| \leq \frac{1}{2} \sum_\omega \mu[\omega] (f(\omega) - g(\omega))^2 = 2D_2^2(\mu^0, \mu^1).$$

We emphasize that the above result is true irrespective of the underlying measure. In particular, Theorem 2.1 implies that, for non-degenerate Boolean functions there is still no sparse reconstruction on product measures if we replace clue by clue$_{TV}$.

### 3.3 Clue via entropy

Our setup remains the same, but we formulate it in a somewhat different way. Let $\{X_v : v \in V\}$ be a set of real-valued discrete random variables defined on a common probability space. As in the Introduction, for $S \subseteq V$ we let $X_S := \{X_j : j \in S\}$. The variables $\{X_v : v \in V\}$ obviously play the role of the coordinates of Section 2. Let $f : \mathbb{R}^V \to \mathbb{R}$ and let $Z = f(X_V)$. In this section we are going to discuss an alternative way of measuring the amount of information a subset $S \subseteq V$ of coordinates contains about the function $f$. In the sequel we use concepts from information theory and define an information-theoretic clue accordingly.

Our main interest is still the special case where the variables $X_v$ and $Z$ are binary valued variables (spins) (the case $f : \{-1, 1\}^V \to \{0, 1\}$), but the arguments we present here work in this slightly more general framework.

For sake of completeness we start with some classical definitions.

For a (possibly vector valued) random variable (or a probability distribution) entropy measures the amount of randomness in the distribution or information in a sample.

**Definition 3.2** (Entropy). Let $X$ be a discrete random variable. Then the entropy of $X$ is

$$H(X) = -\sum_{x \in \text{Ran}(X)} \mu[X = x] \log \mu[X = x].$$

We will also need the concept of conditional entropy. The entropy of $X$ conditioned on the random variable $Y$ expresses how much randomness remains in $X$ on average if we learn the value of $Y$.

**Definition 3.3** (Conditional entropy). Let $X$ and $Y$ be two discrete random variables defined on the same probability space. The conditional entropy of $X$ given $Y$, denoted by $H(X \mid Y)$, is the expected entropy of the conditional distribution of $X$ given $Y$.

The mutual information quantifies the common information present in two variables. In a way it measures how far the joint distribution of the two variables is from being independent.
Definition 3.4 (Mutual Information). Let $X$ and $Y$ be two discrete random variables defined on the same probability space. Suppose that $H(X)$ and $H(Y)$ are both finite. Then the mutual information between $X$ and $Y$ is:

$$I(X : Y) = H(X) + H(Y) - H(X, Y) = H(X) - H(X|Y). \tag{3.7}$$

Now comes the definition of clue in this framework.

Definition 3.5 (I-clue). Let $\{X_v : v \in V\}$ be a finite family of discrete real valued random variables defined on the same probability space, and for some $f : R^V \rightarrow R$ let us consider the random variable $Z = f(X_V)$. The information theoretic clue (I-clue) of $f$ with respect to $U \subseteq V$ is

$$\text{clue}^I(f \mid U) = \frac{I(Z : X_U)}{I(Z : X_V)} = \frac{I(Z : X_U)}{H(Z)}.$$

Note that if $Z$ is $X_U$-measurable then $H(Z|X_U) = 0$, and therefore $I(Z : X_U) = H(Z)$, while if $Z$ is independent from $X_U$ then $I(Z : X) = 0$, in accordance with what we expect from a clue-type notion.

As for the cases discussed before, here too we can introduce the dual (which expresses again how much information we are missing if we do not know the coordinates in $U$):

$$\text{sig}^I(f \mid U) = 1 - \frac{I(Z : X_U)}{H(Z)} = \frac{H(Z \mid X_U)}{H(Z)}.$$

In the sequel we show that $\text{clue}^I$ can also be interpreted via distances of probability measures. For this, we will need the following definition.

Definition 3.6 (Kullback-Leibler divergence). Let $\mu$ and $\nu$ measures on the same discrete probability space $\Omega$, where $\nu \ll \mu$. The relative entropy between $\nu$ and $\mu$ is

$$D_{KL}(\nu||\mu) = -\sum_{x \in \Omega} \nu(x) \log \frac{\nu(x)}{\mu(x)}.$$

The KL divergence, although it also means to express a concept of distance between two distributions, is not a metric. In particular, it is neither symmetric, nor does it satisfy the triangle inequality.

One can easily check that

$$D_{KL}(\mu||\nu) = \text{Ent}(f),$$

where $f = d\nu/d\mu$ as before, and $\text{Ent}(f) := \mathbb{E}[f \log f] - \mathbb{E}[f] \log \mathbb{E}[f]$ (the expectation is taken with respect to $\mu$ and, in case $f(\omega) = 0$, we have, by continuity, $f(\omega) \log f(\omega) = 0$).

The mutual information can be expressed through Kullback-Leibler (KL) divergence. As in Section 3.2, we have $f = d\mu^1/d\mu$ and $g = d\mu^0/d\mu$, $Z = f(\omega)$. Then

$$I(Z : X_{[\omega]}) = \mathbb{E}[D_{KL}(\mu^Z \mid \mu)] = \mu\text{Ent}(f) + (1 - \mu)\text{Ent}(g).$$

Here $\mu$ denotes the basic measure on the configuration space $\{-1, 1\}^V$ and $\mu^Z$ is the measure which is $\mu^0$ when $Z = 0$ and $\mu^1$, when $Z = 1$.

We introduce another information theoretic distance (defined in Chapter 16 of [Per99] for $p = 1/2$) as follows: $D^p_I(\mu^1, \mu^0) := \mathbb{E}[D_{KL}(\mu^Z \mid \mu)]$. Since $I(Z : X_U) = \mathbb{E}[D_{KL}(\mu^Z \mid u \mid \mu)]$ we can write

$$\text{clue}^I(f \mid U) = \frac{\mathbb{E}[D_{KL}(\mu^Z \mid u \mid \mu|U)]}{\mathbb{E}[D_{KL}(\mu^Z \mid \mu|U)]} = \frac{D^0_I(\mu^0|U, \mu^1|U)}{D^1_I(\mu^0, \mu^1)}.$$
In Section 3.2 we introduced symmetric and asymmetric versions of clue. It is clear that clue is a symmetric one (as it is symmetric under swapping \(\mu_0\) with \(\mu_1\)). Although we cannot directly use (3.2) here, we can take the \(D_{KL}(\mu^1||\mu)\) instead of taking the expected \(D_{KL}\) of \(\mu^Z\) from the reference measure. Hence we get:

\[
\text{clue}^{KL}(f \mid U) := \frac{D_{KL}(\mu^1||\mu)}{D_{KL}(\mu^1||\mu)} = \frac{\text{Ent}(\mathbb{E}[Z|U])}{\text{Ent}(Z)}
\]

Note that the second ratio can be used for any \(f \geq 0\), in which case in the first ratio we have, instead of \(\mu^1\), the \(f\)-biased measure \(\nu\) from (3.1). One may wonder why we do not use \(D_{KL}\) as the defining measure of distance between probability measures. Indeed, it works for the asymmetric version, but the symmetric version fails as \(D_{KL}(\mu^0, \mu^1) = \infty\) since \(\mu^0\) and \(\mu^1\) are singular.

We mention that \(\text{Ent}(Z)\) and \(\text{Var}(Z)\) together with the respective concepts of clue can be examined in the general framework of \(\Phi\)-entropies (see for example [BLM13], Chapter 14 and 15). The main idea is that for any convex function \(\Phi\), many important properties of \(\Phi\)-entropy for every integrable random variable \(X\), by

\[
\mathcal{H}_\Phi(X) = \mathbb{E}[\Phi(X)] - \Phi(\mathbb{E}[X]).
\]

It turns out that under some general analytic conditions on \(\Phi\) many important properties we require from an information measure remains valid for \(\mathcal{H}_\Phi(X)\) (for example, it is always non-negative because of Jensen’s inequality). In particular, we get \(\mathcal{H}_\Phi(X) = \text{Var}(X)\) when \(\Phi(x) = x^2\) and \(\mathcal{H}_\Phi(X) = \text{Ent}(X)\) with \(\Phi(x) = x \log x\).

The following proposition shows that for non-degenerate sequences of Boolean functions, sparse reconstruction with respect to clue and clue\(^f\) are equivalent.

**Proposition 3.3.** Let \(\mu\) be a measure on \([-1, 1]^V\) and \(\{\sigma_v : v \in V\}\) a spin system distributed according to \(\mu\). Let \(f : \{-1, 1\}^V \rightarrow \{0, 1\}\) and \(U \subseteq V\). Then,

\[
\mathbb{E}[f]^2(1 - \mathbb{E}[f])^2 \text{clue}^f(f \mid U) \leq \text{clue}(f \mid U) \leq \frac{1}{p_{\min}} \text{clue}^f(f \mid U),
\]

where \(p_{\min} := \min(\mu[f = 1], \mu[f = 0])\).

**Proof.** For the first inequality we use the setup of Section 3.2 and follow the idea sketched in the proof of Lemma 16.5 (ii) in [Per99]. So let \(\mu^1\) and \(\mu^0\) measures on \(\Omega\) such that \(\mu = p \mu^1 + (1-p)\mu^0\) and let \(f := d\mu^1/d\mu\) and \(g := d\mu^0/d\mu\). We are going to show that

\[
D_I(\mu^0, \mu^1) \leq 4D_2^f(\mu^0, \mu^1).
\]

We can rewrite the left hand side as

\[
p\text{Ent}(f) + (1 - p)\text{Ent}(g) = \int_\Omega \frac{1 + \psi}{2} \log \frac{1 + \psi}{2p} + \frac{1 - \psi}{2} \log \frac{1 - \psi}{2(1-p)} d\mu,
\]

where \(\psi = pf - (1-p)g\). Separate the integral and using that \(\frac{1+\psi}{2} = pf\) and \(\frac{1-\psi}{2} = (1-p)g\) we get

\[
D_I(\mu^0, \mu^1) = \int_\Omega \frac{1 + \psi}{2} \log (1 + \psi) + \frac{1 - \psi}{2} \log (1 - \psi) d\mu
\]

\[
- \int_\Omega pf \log (2p) + (1 - p)g \log (2(1-p)) d\mu.
\]
Using that the entropy of a binary random variable can be at most $\log 2$. Finally, since $\log(1 + x) \leq x$ for all $x \in \mathbb{R}$ (here log is the natural logarithm) we obtain:

$$D_I(\mu^0, \mu^1) \leq \int_{\Omega} \frac{1 + X}{2} \log (1 + X) + \frac{1 - X}{2} \log (1 - X) d\mu \leq \int_{\Omega} X^2 d\mu \leq \max(p, 1-p)^2 \int_{\Omega} (f - g)^2 d\mu \leq 4D^2(x^0, \mu^1).$$

Applying (3.4) to $E[Z | F_U]$ this gives that

$$I(Z, \sigma_U) \leq \frac{\text{Var}(E[Z | F_U])}{E[f^2(1 - E[f])]^2}. \quad (3.9)$$

Now we turn to the second inequality. We show that more generally, whenever $f : \{-1,1\}^n \rightarrow [-1,1]$ we have

$$\text{Var}(E[Z | F_U]) \leq 2I(Z, \sigma_U). \quad (3.10)$$

Our argument follows Lemma 4.4 in [Tao05]. First we fix some notations. Let $z$ be in the range of $f$ and $u \in \{-1,1\}^U$. Then

$$p_z := \mu[Z = z], \quad p_u := \mu[\sigma_U = u], \quad p_{z|u} := \mu[Z = z | \sigma_U = u].$$

Now, with this notation we have

$$\text{Var}(E[Z | F_U]) = \sum_{u \in \{-1,1\}^U} p_u (E[Z] - E[Z | \sigma_U = u])^2$$

and for a fixed $u \in \{-1,1\}^U$

$$\left( E[Z] - E[Z | \sigma_U = u] \right)^2 = \sum_z (p_z - p_{z|u})^2 \leq \sum_z (p_z - p_{z|u})^2.$$ 

So we get that

$$\text{Var}(E[Z | F_U]) \leq \sum_{u \in \{-1,1\}^U} p_u \sum_z (p_z - p_{z|u})^2. \quad (3.11)$$

With the notation $h(x) := -x \log x$ for $x \in [0,1]$ (where $h(0) := 0$) we can write the mutual information as

$$I(Z, \sigma_U) = H(Z) - H(Z | \sigma_U) = \sum_z \left( h(p_z) - \sum_{u \in \{-1,1\}^U} p_u h(p_{z|u}) \right). \quad (3.12)$$

Using linear Taylor expansion with error term around $p_z$ for $h(p_{z|u})$, we get the following estimate:

$$h(p_{z|u}) = h(p_z) + h'(p_z)(p_{z|u} - p_z) - \frac{1}{2p_z^*}(p_{z|u} - p_z)^2,$$

with some $p_z^*$ between $p_{z|u}$ and $p_z$, using for the error term that $h''(x) = -\frac{1}{x^2}$. Substituting this estimate into (3.12), we observe that the terms with $h'(p_z)$ cancel, since for any $z \in \{0,1\}$ we have $\sum_{u \in \{-1,1\}^U} p_u (p_{z|u} - p_z) = p_z - p_{z|u} = 0$. Therefore we obtain that

$$\sum_{u \in \{-1,1\}^U} p_u \sum_z \frac{(p_z - p_{z|u})^2}{p_z^*} = 2I(Z, \sigma_U).$$
As $0 < p_{z|u}^* < 1$ we can conclude, using (3.11) that

$$\text{Var}(\mathbb{E}[Z | \mathcal{F}_U]) \leq \sum_{u \in \{-1,1\}^U} p_u \sum_z \frac{(P_z - p_{z|u})^2}{p_{z|u}^*} \leq 2I(Z, \sigma_U).$$

Finally, in order to get a stronger bound we calculate a sharper inequality between the entropy and the variance in the denominators. Here it is more convenient if our random variable takes on $\{-1,1\}$ so we consider $Z' = 2f(\sigma) - 1$ The entropy of $Z'$ can be expressed as a function of $x = \mathbb{E}[Z']$. A quadratic Taylor expansion around 0 gives the following asymptotics:

$$H(Z') = -\left(\frac{1-x}{2} \log \frac{1-x}{2} + \frac{1+x}{2} \log \frac{1+x}{2}\right) = 1 - \frac{1}{\ln 4} x^2 - O(x^4),$$

where $\log$ denotes base 2 logarithm.

Then $|\mathbb{E}[Z']| = 1 - 2p_{\min}$. At the same time, a simple calculation shows that if $|\mathbb{E}[Z']| \leq 1 - c$, or equivalently, if $c \leq 2p_{\min}$ we have

$$1 - \frac{1}{\ln 4} \mathbb{E}[Z']^2 \leq \frac{1}{c}(1 - \mathbb{E}[Z']^2).$$

Therefore, noting that $1 - \mathbb{E}[Z']^2 = \text{Var}(Z') = 4\text{Var}(Z)$, we have

$$H(Z) \leq 1 - \frac{1}{\ln 4} \mathbb{E}[Z']^2 \leq \frac{2}{p_{\min}} \text{Var}(Z). \quad (3.13)$$

Now putting together (3.10), (3.9) and (3.13) we get the statement. \qed

4 Sparse reconstruction with respect to I-clue and KL-clue

In this section we show some analogues of Theorem 2.7 for the I-clue and KL-clue. We note that the following theorem, as well as the definition of I-clue only works well in the discrete case, as the continuous counterpart of entropy, differential entropy has some drawbacks (for example, it can be negative). The notation and setup follows Subsection 3.3.

**Theorem 4.1.** Let $\{X_v : v \in V\}$ be discrete valued i.i.d. random variables with finite entropy. Let $f : \Omega^V \rightarrow \mathbb{R}$ be a transitive function and $Z = f(\{X_v : v \in V\})$. Then

$$\text{clue}^I(f | U) \leq \frac{|U|}{|V|}. \quad (4.1)$$

For the proof we will use the following well-known inequality which finds numerous applications in combinatorics. For a proof see, for example, [LP16, Theorem 6.28].

**Theorem 4.2** (Shearer’s inequality [CGFS86]). Let $X_1, X_2, \ldots, X_n$ random variables defined on the same probability space. Let $S_1, S_2, \ldots, S_L$ be subsets of $[n]$ such that for every $i \in [n]$ there are at least $k$ among $S_1, S_2, \ldots, S_L$ containing $i$. Then

$$kH(X_{[n]}) \leq \sum_{j=1}^L H(X_{S_j}).$$

First we need the following consequence of Shearer’s inequality.

**Lemma 4.3.** Suppose $X_1, X_2, \ldots, X_n$ are independent, and $Z = f(X_1, \ldots, X_n)$. Let $S_1, \ldots, S_L$ be a system of subsets of $[n]$ such that each $i \in [n]$ appears in at most $k$ sets. Then

$$\sum_{j=1}^L I(Z : X_{S_j}) \leq kI(Z : X_{[n]}). \quad (4.2)$$
Proof. Without loss of generality we can assume that each \(i\) appears in exactly \(k\) sets. Indeed, if this is not the case, we can always add some additional subset so that this condition is satisfied. While adding new sets the right hand side of the inequality does not change and the left hand side can only increase.

Since the variables \(X_i\) are independent:

\[
\sum_{j=1}^{L} H(X_{S_j}) = \sum_{j} \sum_{i \in S_j} H(X_i) = k \sum_{i \in [n]} H(X_i) = kH(X_{[n]}).
\] (4.3)

On the other hand, using Shearer’s inequality,

\[
-\sum_{j=1}^{L} H(X_{S_j} | Z) \leq -kH(X_{[n]} | Z).
\] (4.4)

Using that \(I(Z : X_{S_j}) = H(X_{S_j}) - H(X_{S_j} | Z)\), adding up (4.3) and (4.4) completes the proof. \(\square\)

Now the proof of the clue-theorem:

**Proof.** Recall that we have the iid measure \(P\), and the function \(f\) is invariant under some transitive action of a group \(G\). Let \(U \subseteq V\) be arbitrary. Then, for each \(g \in G\),

\[
I(Z : X_U) = I(Z : X_{U^g}),
\]

where \(U^g = \{ug : g \in G\}\).

Observe that \(v \in U^g \iff \exists u \in U\) with \(vg^{-1} = u\). For each pair of \(v \in V\) and \(u \in U\) there are \(|G_v|\) such \(g\), where \(G_v\) is the stabilizer subgroup of \(G\) at \(v\). (Since the action is transitive such a \(g\) exists, moreover the cardinality of the stabilizer subgroup \(G_v\) is the same for every \(v \in V\).) The conclusion is that each \(v \in V\) appears in exactly \(|U||G_v|\) translated version of \(U\). Applying Lemma 4.3 gives

\[
|G| I(Z : X_U) = \sum_{g \in G} I(Z : X_{U^g}) \leq |U||G_v| I(Z : X_V) = |U||G_v| H(Z),
\]

which is what we wanted since \(|G| = n|G_v|\) by the orbit-stabilizer theorem. \(\square\)

Observe that for any non-degenerate sequence of Boolean functions, sparse reconstruction with respect to I-clue is equivalent to sparse reconstruction with respect to the original, \(L^2\) version (irrespective of the underlying measure). This follows from Proposition 3.3. Nevertheless, in case \(\{f_n\}\) is degenerate Boolean, or non-Boolean, Proposition 3.3 does not help us compare the sequences of clues and I-clues. This raises the following question:

**Question 4.4.** Is there a sequence of functions \(f_n \in L^2([-1,1]^V_n, \pi_n \otimes V_n)\) and a corresponding sequence of subsets \(U_n \subseteq V_n\) such that

1. \(\text{clue}^I(f_n | U_n) \prec \text{clue}(f_n | U_n)\)
2. \(\text{clue}^I(f_n | U_n) \succ \text{clue}(f_n | U_n)\) ?

What is the answer if we ask \(f_n\) to be Boolean for all \(n \in \mathbb{N}\)? What is the answer if we allow for non-product measures on \([-1,1]^V_n\)?
At the same time, it is remarkable that for product measures we have the exact same general for the clue and I-clue of general (possibly degenerate or \( \mathbb{R} \)-valued) transitive functions. In particular, we emphasize that Theorem 2.7 and Theorem 4.1 are not implying one another.

Interestingly enough, along the same logic one can prove the respective version of Theorem 2.7 and Theorem 4.1 for clue \( KL \). We should emphasize that, in contrast with mutual information, relative entropy is a concept that remains meaningful for continuous random variables as well. So Theorem 4.6 holds for all product measures, just like Theorem 2.7. The proof relies on the following Shearer-type inequality:

**Lemma 4.5.** Let \( \mathbb{P} \) be a product measure, and \( \mu \) another probability measure on the same space satisfying \( \mu \ll \mathbb{P} \). Let \( S_1, \ldots, S_L \) be a system of subsets of \( V \) such that each \( i \in V \) appears in at most \( k \) sets. Then

\[
\sum_{j=1}^{L} D(\mu_{S_j} || \mathbb{P}_{S_j}) \leq kD(\mu || \mathbb{P}).
\]

In our application, of course \( \mu \) is the measure with density \( f \). It is easy to recognise that Lemma 4.5 is a close relative of Lemma 4.3. The proof of this Lemma is also a straightforward consequence of Shearer’s inequality (Theorem 4.2); for a proof see [GLSS12]. The corresponding clue theorem follows in the same way as Lemma 4.3 implies Theorem 4.1.

**Theorem 4.6.** Let \( \{X_v : v \in V\} \) be \( \Omega \)-valued iid random variables. Let \( f : \Omega^n \to \mathbb{R} \) be a transitive function and \( Z = f(\{X_v : v \in V\}) \). Then

\[
\text{clue}^{KL}(f \mid U) \leq \frac{|U|}{n}.
\]

It is worth noting that for sequences of transitive Boolean functions on the hypercube there is no sparse reconstruction no matter which version of clue we wish to choose. Indeed, clue\( TV \) and \( W \) (witness) are dominated by clue so Theorem 2.1 applies, while for clue\( I \) and clue\( KL \) it has been shown in the present section (Theorem 4.1 and Theorem 4.6).

5 Sparse reconstruction and cooperative game theory

The field of cooperative game theory (for an introduction see, for example, [BDT08] or [PS07]) starts with the following setup: there is a set of players which we denote by \( V \) here (to be consistent with our previous notation), and the game is defined by assigning a positive real number \( v(S) \) to every subset \( S \) of the players. Usually it is assumed that \( v(\emptyset) = 0 \). The function \( v : 2^V \to \mathbb{R} \) is referred to as the characteristic function. This aims to model a situation where individuals can gain profit, but the profit may change (typically increases) in case certain individuals cooperate and form a coalition. Thus \( v(S) \) is the joint payoff of the individuals in \( S \) provided that they cooperate.

Cooperative game theory is mostly concerned with finding some sort of fair distribution of the payoff given the characteristic function \( v \). One of these concepts is the Shapley value, introduced in [Sha53], which distributes the payoff based on the average marginal contribution of the individuals.

**Definition 5.1** (Shapley value).

\[
\phi_i(v) = \frac{1}{|V|} \sum_{S \subseteq V \setminus \{i\}} \frac{v(S \cup \{i\}) - v(S)}{|S|}.
\]

(5.1)
A straightforward calculation shows that \( \sum_{i \in V} \phi_i(v) = v(V) \). So the Shapley-value is indeed the distribution of the payoff of the grand coalition. In general, this is not true for smaller coalitions \( S \subset V \).

Observe that for a given \( f : \{-1,1\}^V \rightarrow \{-1,1\} \) we can define a cooperative game via \( v_f(U) := \text{Var}[E[f \mid \mathcal{F}_U]] \) for any \( U \subseteq V \). Besides fitting the mathematical definition, it also fits into the interpretation of the theory. It is a sort of information game, where the payoff depends on how accurately we know a piece of information (represented by the value of the function). Each individual possesses one piece of information (the value of the corresponding coordinate) but only together they determine the valuable piece of information.

In the proof of Theorem 2.1 we introduced the random element \( X \) of the index set, which is a uniformly random element of the Spectral Sample. In fact, \( X \) is distributed according to the Shapley value.

**Proposition 5.1.** Let \( f : \{-1,1\}^V \rightarrow \mathbb{R} \). Then

\[
\frac{\phi_i(v_f)}{v_f(V)} = \mathbb{P}[X = i].
\]

**Proof.** Without loss of generality we may assume that \( \text{Var}(f) = 1 \). Let \( n = |V| \). First, observe that

\[
\mathbb{P}[X = u] = \sum_{S \supseteq u} \hat{f}(S)^2 \frac{1}{|S|}.
\]

Now we calculate \( \phi_i(v_f) \) via Fourier-Walsh expansion and show that it equals to \( \mathbb{P}[X = u] \).

Using that \( v_f(S) = \sum_{T \subseteq S} \hat{f}(T)^2 \) we get that

\[
\phi_i(v) = \frac{1}{n} \sum_{S \subseteq V \setminus \{i\}} \sum_{T \subseteq S} \hat{f}(T \cup \{i\})^2 = \frac{1}{n} \sum_{T \subseteq V \setminus \{i\}} \hat{f}(T \cup \{i\})^2 \sum_{S \subseteq |n| \setminus \{i\} : T \subseteq S} \frac{1}{|S|}.
\]

For a fixed \( T \) there are \( \binom{n-1-|T|}{k-|T|} \) \( k \)-element subsets \( S \) which contain \( T \). Therefore we have

\[
\phi_i(v) = \frac{1}{n} \sum_{T \subseteq V \setminus \{i\}} \hat{f}(T \cup \{i\})^2 \sum_{k=|T|}^{n-1} \binom{n-1-|T|}{k-|T|} \binom{k}{|T|} \binom{n-1}{k}.
\]

With some elementary manipulation of the binomial coefficients we get that

\[
\binom{n-1-|T|}{k-|T|} = \binom{k}{|T|} \binom{n}{|T|+1}.
\]

Now we apply the so called hockey-stick identity — \( \sum_{k=|T|}^{n-1} \binom{k}{|T|} = \binom{n}{|T|+1} \) — and we get the desired formula. \( \square \)

Given how naturally the Shapley value arises in the proof of Theorem 2.1, it is perhaps not surprising that there is a proof that does not use Fourier-Walsh transform, only simple concepts from cooperative game theory and combinatorics. The advantage of this approach is that it makes it more clear the conditions under which a clue theorem like Theorem 2.7 or Theorem 4.1 can be true. It should also be noted that this approach entails both the \( L^2 \) and the entropy version of the theorem.

We need to introduce another concept of fair distribution which is related to our topic. The core, introduced in [Gil59], defines those distributions of the profit in which every coalition of players gets in total at least as much as they deserve (according to the characteristic function).
Definition 5.2 (Core). The core of a cooperative game $v$ with set of players $V$ is the set $C(v) \subseteq \mathbb{R}^{|V|}$ such that $x \in C(v)$ if and only if

$$\sum_{i \in V} x_i = v(V),$$

and for every $S \subseteq V$

$$\sum_{i \in S} x_i \geq v(S).$$

We have the following simple observation.

Lemma 5.2. Let $v$ be a transitive game. If the Shapley value vector $\phi(v)$ is in the core $C(v)$ then for every $S \subseteq V$

$$v(S) \leq \frac{|S|}{|V|} v(V).$$

Proof. For transitive games, obviously $\phi_i(v) = \frac{v(V)}{|V|}$. Using that $\phi(v) \in C(v)$, we get that

$$v(S) \leq \sum_{i \in S} \phi_i(v) = \frac{|S|}{|V|} v(V).$$

We are going to show that a class of cooperative games, the so-called convex games, satisfy the conditions of Lemma 5.2. This concept was also first studied by Shapley (see [Sha71]).

Definition 5.3 (Convex games). A cooperative game $v$ is convex if the characteristic function is supermodular. That is, for every subset of players $S, T \subseteq [n]$

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{5.2}$$

Recall that with any function $f$ on a product space we can associate a game $v_f$ by $v_f(U) := \text{Var}[E[f \mid F_U]]$. We have another game if we define the information we gain about $Z = f(X_V)$ via information theoretic concepts (see Definition 3.5):

$$v'_f(S) = I(Z : X_S).$$

It is not difficult to see that for product measures, both $v_f$ and $v'_f$ are convex games. The entropy version is immediate from the submodularity of entropy, which can be written as:

$$-H(X_S|Z) - H(X_T|Z) \leq -H(X_{S\cap T}|Z) - H(X_{S\cup T}|Z).$$

Using that for independent variables the submodularity inequality is sharp we get

$$H(X_S) - H(X_S|Z) + H(X_T) - H(X_T|Z) \leq H(X_{S\cup T}) - H(X_{S\cap T}|Z) + H(X_{S\cap T}) - H(X_{S\cup T}|Z).$$

For the $L^2$ version, the supermodularity of $\text{Var}[E[f \mid F_U]]$ follows easily from the spectral description. Here we present an argument that does not require Fourier-Walsh expansion or Efron-Stein decomposition.

Proposition 5.3. Let $f : X^V \rightarrow \mathbb{R}$, where $X^V$ is endowed with a product measure. The set function (cooperative game) $v_f(S) = \text{Var}[E[f \mid F_S]]$ for $(S \subseteq V)$ is supermodular (convex).
Proof. First observe that whenever \( S \subseteq T \) then \( \mathbb{E}[\mathbb{E}[f | \mathcal{F}_T] | \mathcal{F}_S] = \mathbb{E}[f | \mathcal{F}_S] \) by the towering property, and, using that conditional expectation is an orthogonal projection, we get that

\[
\text{Var}(\mathbb{E}[f | \mathcal{F}_T]) - \text{Var}(\mathbb{E}[f | \mathcal{F}_S]) = \text{Var}(\mathbb{E}[f | \mathcal{F}_T] - \mathbb{E}[f | \mathcal{F}_S]),
\]

and therefore (5.2) can be rewritten as

\[
\text{Var}(\mathbb{E}[f | \mathcal{F}_T] - \mathbb{E}[f | \mathcal{F}_{S \cap T}]) \leq \text{Var}(\mathbb{E}[f | \mathcal{F}_{S \cup T}]) - \mathbb{E}[f | \mathcal{F}_S]).
\] (5.3)

Fix \( S, T \subseteq V \) such that \( S \subseteq T \). Using Lemma 2.5 for \((T \setminus S)^c\) and \( T \), we get

\[
\mathbb{E}[f | \mathcal{F}_S] = \mathbb{E}[\mathbb{E}[f | \mathcal{F}_{(T \setminus S)^c}] | \mathcal{F}_T].
\]

Note that this is the only place in the argument where the fact that the underlying measure is a product measure is exploited.

This identity allows us to write \( \mathbb{E}[f | \mathcal{F}_{S \cap T}] = \mathbb{E}[\mathbb{E}[f | \mathcal{F}_{(T \setminus S)^c}] | \mathcal{F}_T] \) and \( \mathbb{E}[f | \mathcal{F}_S] = \mathbb{E}[\mathbb{E}[f | \mathcal{F}_{(S \cup T)^c}] | \mathcal{F}_{S \cup T}] \). Since \( T \setminus (S \cap T) = (S \cup T) \setminus S = T \setminus S \), (5.3) becomes

\[
\text{Var}(\mathbb{E}[f - \mathbb{E}[f | \mathcal{F}_{(T \setminus S)^c}] | \mathcal{F}_T]) \leq \text{Var}(\mathbb{E}[f - \mathbb{E}[f | \mathcal{F}_{(T \setminus S)^c}] | \mathcal{F}_{S \cup T}]),
\]

which always holds, because orthogonal projection cannot increase the variance (\( L^2 \)-norm).

The subgame \( v_U \) denotes the game \( v \) with its domain restricted to the subset \( U \subseteq [n] \).

**Lemma 5.4.** If \( v \) is a convex and transitive game, then \( S \subseteq T \) implies

\[
\phi_i(v_S) \leq \phi_i(v_T).
\]

**Proof.** We are going to show this when \( T = S \cup \{j\} \). Let \( |S| = k \). We have

\[
\phi_i(v_S) = \frac{1}{k} \sum_{L \subseteq S \setminus \{i\}} v(L \cup \{i\}) - v(L) \leq \frac{1}{k+1} \sum_{L \subseteq T \setminus \{i\}} v(L \cup \{i\}) - v(L)
\]

and

\[
\phi_i(v_T) = \frac{1}{k+1} \left( \sum_{L \subseteq S \setminus \{i\}} \frac{v(L \cup \{i\}) - v(L)}{\binom{k}{|L|}} + \sum_{L \subseteq S \setminus \{i\}} \frac{v(L \cup \{i,j\}) - v(L \cup \{j\})}{\binom{k+1}{|L|+1}} \right).
\]

It is a straightforward calculation to verify that for any \( l \leq k \)

\[
\frac{1}{k} \frac{1}{l} = \frac{1}{k+1} \left( \frac{1}{l} + \frac{1}{l+1} \right)
\]

and therefore, using that by supermodularity, \( v(L \cup \{i\}) - v(L) \leq v(L \cup \{i,j\}) - v(L \cup \{j\}) \), we get

\[
\phi_i(v_S) = \frac{1}{k+1} \left( \sum_{L \subseteq S \setminus \{i\}} \frac{v(L \cup \{i\}) - v(L)}{\binom{k}{|L|}} + \sum_{L \subseteq S \setminus \{i\}} \frac{v(L \cup \{i,j\}) - v(L)}{\binom{k+1}{|L|+1}} \right) \leq \phi_i(v_{S \cup \{j\}}).
\]

**Corollary 5.5.** Let \( \mu \) be a measure on \( \Omega^V \) with \( |V| = n \) and let \( f \in L^2(\Omega^V, \mu) \) transitive. If the set function \( v_f(S) = \text{clue}(f | S) \) is supermodular (convex) (here clue may stand for any of the previously defined clue notions) we have

\[
\text{clue}(f | U) \leq \frac{|U|}{n}.
\]
Proof. Lemma 5.4 implies that, if \( v \) is a convex game, then for any \( S \subset V \)

\[
v(S) = \sum_{i \in S} \phi_i(v_S) \leq \sum_{i \in U} \phi_i(v), \tag{5.4}
\]

and therefore \( \phi(v) \) is indeed in the core. So Lemma 5.2 applies and we get the statement. \( \Box \)

By Proposition 5.3 for any real function \( f \) the game \( v_f \) and \( v_f^I \) are convex whenever the underlying measure is a product measure, so Corollary 5.5 can be seen as a common proof of Theorem 2.7 and Theorem 4.1. Apart from product measures, however, we do not know any other example where the condition of supermodularity of \( v_f \) is satisfied. We note that in order to show that there is no sparse reconstruction for transitive functions under some sequence of measures it would suffice to show the weaker condition that for some fixed \( k \in \mathbb{N} \) the set function \( v_f^k(S) := (\text{clue}(f | S))^k \) is supermodular. Indeed, in that case by Corollary 5.5 we would get that \( \text{clue}(f | S) \leq (|U|/n)^{1/k} \).

Observe that for a transitive game with a non-empty core the Shapley value, i.e., the uniform vector, will always be in the core. It is because the core is convex and itself is invariant under the group action. Therefore, one could weaken the condition of Proposition 5.2 by only requiring the non-emptiness of the core. A classical result in Cooperative Game Theory (see for example [BDT08] Theorem 2.4) gives necessary and sufficient conditions for this. It has to be said, however, that on a practical level, the conditions of this theorem are not very easy to verify.

**Theorem 5.6** (Bondareva-Shapley). The core of the game \( v \) is non-empty if and only if for every \( \alpha : 2^V \setminus \emptyset \to [0, 1] \) such that for every \( i \in V \)

\[
\sum_{S \subseteq V : i \in S} \alpha(S) = 1
\]

it holds that

\[
\sum_{S \subseteq V} \alpha(S)v(S) \leq v(V).
\]

### 6 Sparse reconstruction for planar percolation

#### 6.1 Basics and the result

Bernoulli bond (or site) percolation at level \( p \) on a graph \( G \) means the random graph obtained by deleting every edge (or vertex) of a graph with probability \( 1 - p \). Here we only mention some basic concepts and results. For a background on percolation theory, criticality and other concepts, we advise the reader to consult [Gr99] in general, [Wer09] in two dimensions, and [GS15] with a focus on noise sensitivity questions.

In case \( G \) is infinite, we are interested whether for a particular value of \( p \) the arising random graph contains an infinite connected component. A simple coupling argument shows that this event is monotone increasing in \( p \) and thus we an introduce the critical value \( p_c = \inf \{ p : \mathbb{P}_p(\exists \infty \text{ cluster}) = 1 \} \).

Throughout this section we consider critical Bernoulli edge percolation on the square lattice (so every edge is open with probability \( p = 1/2 \), independently). Our main focus will be the left-to-right crossing \( \text{LR}_n \) on the \((n-1) \times n \) rectangle. This is the event that there exists a path consisting of open edges between the leftmost and the rightmost vertices of the rectangle.

It is known that when \( p = 1/2 \) then \( \mathbb{P}[\text{LR}_n] \) tends to \( 1/2 \) as \( n \to \infty \). It is also standard that every percolation configuration on a square (or, in fact, on any planar) lattice induces a percolation configuration on the dual lattice. On the dual lattice the sites are the faces of the lattice and
two faces are connected in configuration if the two faces are bordered with an edge which is closed in the original percolation and the the \((n-1) \times n\) rectangle has the important property that it is isomorphic to its dual graph.

It turns out that the critical model in many graphs displays interesting, fractal-like features. There is a universality principle coming from statistical physics which connects the behaviour of various models around their phase transition. For example, physicists believe that percolation on any “nice” planar lattice \(G\), at the critical point \(p_c(G)\), describes the same “ideal” percolation, only in possibly different frames.

This principle suggests the existence of so-called critical exponents, which describe the probability of important observables of the percolation at the phase transition (i.e., at \(p = p_c\)) via universal power laws. Physicists can calculate the value of these exponents and believe that these values are universal in the sense above. Nevertheless, from the point of view of the mathematician, little is actually known.

Nevertheless, there have been some important developments in the last few decades. The main breakthrough was by Smirnov [Smi01], who showed that in the case of the triangular lattice the universality conjecture of the physicists holds, in particular, the value of the critical exponents is as predicted. For the square lattice, however (and for any planar lattice) no similar result has been proved.

We now introduce a family of critical exponents that play an important role in the sequel. The 1-arm event on \(\mathbb{Z}^2\) — we only consider this lattice, but the arm events can be defined for any planar lattice — \(A_1(R)\) is the event that there is a path of open edges from 0 to a site (vertex) which is at graph distance \(R\) away from the origin. The event \(A_1(r, R)\) is the event that there is path of open edges starting somewhere in distance at most \(r\) from 0 and ending at a site which is at distance \(R\) from the origin. It is conjectured based on the above universality principle that on any reasonable lattice (on \(\mathbb{Z}^2\), in particular) \(\alpha_1(r, R) := \mathbb{P}[A_1(R)] \approx \left( \frac{r}{R} \right)^{5/48 + o(1)}\). This is only known for site percolation on the triangular lattice (See [Smi01]). Up until today, this is the only lattice where the value of this (and many other) exponents are verified.

In our proof we are going to use some other events, the 2-arm and 3-arm events in a half plane, which we denote by \(A_2^+(r, R)\) and \(A_3^+(r, R)\). The 2-arm event says that there is a path of open edges and a dual path of closed edges in the positive half plane \(\mathbb{Z} \times \mathbb{N}\) starting at distance \(r\) from the origin and reaching until distance \(R\). The 3-arm event is that there are two paths of open edges in the positive half plane starting at distance \(r\) from the origin and reaching until distance \(R\), and the two open arms are separated with a similar dual arm consisting of closed edges. The exponents for \(A_2^+(r, R)\) and \(A_3^+(r, R)\) are known for \(\mathbb{Z}^2\). There is a combinatorial argument that does not rely on the universality conjecture. Experience shows that in general fractional arm exponents are hard, while integer arm exponents are approachable.

**Proposition 6.1 ([LSW02]).** For the \(\mathbb{Z}^2\) lattice,

\[
\alpha_2^+(r, R) := \mathbb{P}[A_2^+(r, R)] \asymp \left( \frac{r}{R} \right),
\]

and

\[
\alpha_3^+(r, R) := \mathbb{P}[A_3^+(r, R)] \asymp \left( \frac{r}{R} \right)^2.
\]

In this section we are going to show that the left-right crossing event in critical planar percolation cannot be reconstructed from a sparse subset of spins, by this answering a question posed by Itai Benjamini.

We shall use the framework of Boolean functions, since an edge percolation configuration can be naturally identified with an \(\omega \in \{-1, 1\}^E\), where \(E\) is the edge set of the graph on which we percolate. In our case, \(LR_n : \{-1, 1\}^{E(R_n)} \rightarrow \{-1, 1\}\) will denote the indicator function of
the left-to-right crossing event, where $R_n$ is the $(n-1) \times n$ rectangle. We consider the critical probability $p = 1/2$, thus we have the uniform measure on $\{-1, 1\}^{E(R_n)}$ which we shall denote by $\mathbb{P}_n$. Our result is the following:

**Theorem 6.2.** Let $LR_n : \{\{-1, 1\}^{E(R_n)}, \mathbb{P}_n\} \rightarrow \{-1, 1\}$ be the left-right crossing event as above. Then there is no sparse reconstruction for $LR_n$, that is for any sequence $U_n \subset E(R_n)$ with $\lim_{n \rightarrow \infty} |U_n|/|E(R_n)| = 0$, we have

$$\lim_{n \rightarrow \infty} \text{cure}(LR_n | U_n) = 0.$$

In the sequel we will use a slight modification of this setup. We are going to embed the $(n-1) \times n$ square into the torus $\mathbb{Z}_n^2$ and think about the crossing event as a Boolean function $LR_n : \{\{-1, 1\}^{E(\mathbb{Z}_n^2)}, \mathbb{P}_n\} \rightarrow \{-1, 1\}$. For $LR_n$ we simply ignore the extra edges of the torus, that is if $e \notin E(R)$ then the value of $\omega_e$ does not influence $LR_n$. The reason for this embedding is that we shall use the symmetries of the torus, in order to make us of the results of the previous section. In particular, we can translate $LR_n$ with elements $\mathbb{Z}_n^2$ and still get a function defined on $\{\{-1, 1\}^{E(\mathbb{Z}_n^2)}$. We are going to argue that the left-right crossing event is in some sense not far from being transitive and Lemma 6.6, a variant of Theorem 2.1 can be applied.

Here is a brief summary of what we are going to do: let us denote by $LR_n$ the characteristic function of the left-right crossing event. We will show that for every $\epsilon$ there is a corresponding sublattice $H \subseteq \mathbb{Z}_n^2$ the size of which only depends on $\epsilon$ with the following property: $M^H[LR_n]$, the average of the $LR_n$ translates on the $H$ lattice, is close to a $\mathbb{Z}_n^2$-invariant function $M[LR_n]$ in the sense that $\text{Corr}(M^H[LR_n], M[LR_n]) \geq 1 - O(\sqrt{\epsilon})$. A straightforward variant of Theorem 2.1 implies that, $M[LR_n]$ though not transitive, admits almost the same bound as transitive functions. Moreover, Lemma 6.4 tells us that in case two functions are highly correlated their clues with respect to any particular subset is also close.

Now if the crossing event $LR_n$ had uniformly positive clue with respect to some sequence of subsets $U_n$, the function $M^H[LR_n]$ would also have high clue with respect to the union of the original subset $U_n$ and its $H$-translates, which is still small since the size of $H$ does not grow with $n$. But this is impossible because then in turn $M[LR_n]$, being highly correlated with $M^H[LR_n]$, would also have had uniformly positive clue with respect to a sparse sequence of subsets, which is in contradiction with Lemma 6.6.

### 6.2 Projections and clue

In this section we present some general results about projections that are necessary for the proof of Theorem 6.2. The upcoming two simple lemmas estimate how much a projection can distort correlations. The geometric intuition is that in case the correlation of two functions is high and the projection is not too 'radical' (meaning here that it does not decrease the norm drastically), then the projection will roughly preserve the correlation. Note that these results are completely general, i.e., we do not make use of the fact that the underlying measure is a product measure.

The space of functions over a given configuration space $\{-1, 1\}^V$ and a corresponding probability measure (the uniform measure in this case) can be endowed with a Hilbert space structure via the inner product $\langle f, g \rangle := \mathbb{E}[fg]$. In order to state the following Lemmas in full generality, we introduce a generalization of clue for closed linear subspaces, making use of the Hilbert space structure. It is the logical extension of clue as it was defined previously for $\sigma$-algebras (See Definition 1.1).

Let $\mu$ be a probability measure and let $\mathcal{H}$ be a closed subspace of $L^2(S, \mu)$. (In our applications we always have $S = \{-1, 1\}^V$ for some finite set $V$.) Denote by $P_{\mathcal{H}}$ the orthogonal
projection onto this subspace. For any \( f \in L^2(S, \mu) \) we define the clue of \( f \) with respect to the subspace \( \mathcal{H} \) as
\[
\text{clue}(f \mid \mathcal{H}) = \frac{\text{Var}(P_{\mathcal{H}}[f])}{\text{Var}(f)}.
\]

**Lemma 6.3.** Let \( f, g \in L^2(S, \mu) \) satisfying
\[
\text{Corr}(f, g) \geq 1 - \epsilon.
\]
Let \( \mathcal{H} \) be a subspace of \( L^2(S, \mu) \) and let us denote by \( P \) the orthogonal projection onto this subspace. Assume that
\[
\text{clue}(f \mid \mathcal{H}) \geq c, \text{ and } \text{clue}(g \mid \mathcal{H}) \geq c.
\]
Then
\[
\text{Corr}(P[f], P[g]) \geq 1 - \frac{\epsilon}{c}.
\]

**Proof.** Without loss of generality we may assume that \( E[f] = E[g] = 0 \) and \( \text{Var}(f) = \text{Var}(g) = 1 \), since both clue and correlation are invariant under affine transformations. As in this case they are equivalent, we may use \( \| \cdot \|^2 \) instead of the variance, depending on the context.

Using that \( \text{Var}(f) = \text{Var}(g) = 1 \), we get
\[
\| f - g \|^2 = \text{Var}(f - g) = \text{Var}(f) + \text{Var}(g) - 2\sqrt{\text{Var}(f)\text{Var}(g)}\text{Corr}(f, g) = 2(1 - \text{Corr}(f, g)) \leq 2\epsilon.
\]

In a similar fashion, we get for the respective projections that
\[
\| P[f] - P[g] \|^2 = \text{Var}(P[f] - P[g])
= \text{Var}(P[f]) + \text{Var}(P[g]) - 2\text{Cov}(P[f], P[g])
= \sqrt{\text{Var}(P[f])\text{Var}(P[g])} \left( \sqrt{\frac{\text{Var}(P[f])}{\text{Var}(P[g])}} + \sqrt{\frac{\text{Var}(P[g])}{\text{Var}(P[f])}} - 2\text{Corr}(P[f], P[g]) \right) .
\]

Using first that \( \text{Var}(f) = \text{Var}(g) = 1 \) and after our assumption, we get
\[
\sqrt{\text{Var}(P[f])\text{Var}(P[g])} = \sqrt{\frac{\text{Var}(P[f])}{\text{Var}(f)} \text{Var}(P[g])} \geq c .
\]

On the other hand
\[
\sqrt{\frac{\text{Var}(P[f])}{\text{Var}(P[g])}} + \sqrt{\frac{\text{Var}(P[g])}{\text{Var}(P[f])}} \geq 2 ,
\]
so we conclude that
\[
\| P[f] - P[g] \|^2 \geq 2c(1 - \text{Corr}(P[f], P[g])).
\]

Finally, putting together estimates for \( \| f - g \|^2 \) and \( \| P[f] - P[g] \|^2 \) and using that \( P \), being a projection, cannot increase the \( L^2 \) norm, we conclude that
\[
2\epsilon \geq \| f - g \|^2 \geq \| P[f] - P[g] \|^2 \geq 2c(1 - \text{Corr}(P[f], P[g])).
\]

After reordering this inequality the statement follows.

**Lemma 6.4.** Let \( f, g \in L^2(S, \mu) \) with
\[
\text{Corr}(f, g) \geq 1 - \epsilon.
\]
Let $P$ denote the orthogonal projection onto the subspace $\mathcal{H}$ of $L^2(S,\mu)$ and suppose that
\[
\text{clue}(f \mid \mathcal{H}) \geq c.
\]
Under these conditions,
\[
\text{clue}(g \mid \mathcal{H}) \geq c - 2\epsilon
\]
and
\[
\text{Corr}(P[f],P[g]) \geq 1 - \frac{\epsilon}{c - 2\epsilon}.
\]

Proof. Again, without loss of generality we may assume that $E[f] = E[g] = 0$ and $\text{Var}(f) = \text{Var}(g) = 1$ and therefore we may use $\| \cdot \|^2$ instead of variance, like previously.

Using that $P[f]$ is the closest point to $f$ in $\mathcal{H}$ for every $h \in \mathcal{H}$
\[
\| f - P[f] \|^2 \leq \| f - h \|^2.
\]
Therefore, with the triangle inequality we get
\[
\| g - P[g] \|^2 \leq \| g - P[f] \|^2 \leq \| g - f \|^2 + \| f - P[f] \|^2.
\]
(6.1)

Recall that, since $P$ is an orthogonal projection, for every $f \in L^2(S,\mu)$ we have
\[
\| P[f] \|^2 + \| f - P[f] \|^2 = \| f \|^2
\]
(6.2)
As in Lemma 6.3, $\text{Corr}(f,g) \geq 1 - \epsilon$ implies $\| g - f \|^2 \leq 2\epsilon$.

On the other hand, by our assumptions,
\[
\text{clue}(f \mid \mathcal{H}) = \frac{\| P[f] \|^2}{\| f \|^2} = \| P[f] \|^2 \geq c.
\]
Thus (6.2) shows that $\| f - P[f] \|^2 \leq 1 - c$. Plugging the estimates into (6.1) we can write (using that dividing by $\| g \|^2 = 1$ does not change the equation)
\[
\frac{\| g - P[g] \|^2}{\| g \|^2} \leq 2\epsilon + (1 - c).
\]
Using (6.2) again, we get
\[
1 - \text{clue}(g \mid U) \leq 2\epsilon + 1 - c,
\]
from which $\text{clue}(g \mid \mathcal{H}) \geq (c - 2\epsilon)$ is immediate.

We can apply Lemma 6.3 to get that $\text{Corr}(P[f],P[g]) \geq 1 - \frac{\epsilon}{c - 2\epsilon}$. \hfill \square

6.3 No sparse reconstruction for critical planar percolation

Let $0 < \delta < 1$. For a $t \in \mathbb{Z}^2_n$ we will denote the rectangle $t + [-\lfloor \delta n \rfloor, \lfloor \delta n \rfloor]^2 \subset \mathbb{Z}^2_n$ by $R_\delta(t)$. It is straightforward to see that $4(\delta n - 1)^2 \leq |R_\delta(t)| \leq 4(\delta n)^2$.

Lemma 6.5. Let $R_\delta := R_\delta(0) = [-\lfloor \delta n \rfloor, \lfloor \delta n \rfloor]^2$ as above. Then there is a $K > 0$ such that for every $d_1,d_2 \in R_\delta$
\[
\text{Corr}(LR_n^{d_1}, LR_n^{d_2}) \geq 1 - K\delta
\]

Proof. Let $d \in R_\delta$. We are going to show that
\[
\mathbb{P}[LR_n \neq LR_n^{d_1}] \leq O(\delta).
\]
Figure 6.1: If $LR \neq LR^d_n$, then we have a 3-arm event in a half plane.

From this the statement of the lemma follows. Indeed, for any $d_1, d_2 \in R_\delta$

$$\text{Corr}(LR^d_1, LR^d_2) = 1 - 2P[LR^d_1 \neq LR^d_2] = 1 - 2P[LR_n \neq LR^d_{n-d}] \geq 1 - O(\delta).$$

Let us assume that $d = (0,t)$. Observe (see Figure 6.1) that the event \{LR_n \neq LR^d_n\} basically entails a 3-arm event in a half plane from radius $O(\delta n)$ to a distance of order $n$.

More precisely, cut the torus into a square so that the distance of the resulting left and right boundaries from the disagreement band is at least $n/3$, and consider an exploration interface between open and closed edges from the corner of this square until reaching the disagreement band, at a $\delta n$-box $B$. If the distance of $B$ from the boundary of the square is $r$, then we have a half-plane 3-arm event from distance $n\delta$ to $r$, and a quarter-plane 2-arm event from a distance of order $r$ to $n$. For the quarter-plane probabilities we can use the obvious bound

$$\alpha_2^{++}(r, R) \leq \alpha_2^+(r, R).$$

Using a dyadic division $r \in [n\delta 2^j, n\delta 2^{j+1})$ for the possible values of $r$, and using that at each scale there are order $2^j$ possible locations for $B$, we have, by the arm probability bounds in Proposition 6.1:

$$P[LR_n \neq LR^d_n] \leq O(1) \sum_{j=1}^{\log(1/\delta)} \alpha_3^+(n\delta, n\delta 2^j) \alpha_2^{++}(n\delta 2^j, n) 2^j \leq O(1) \sum_{j=1}^{\log(1/\delta)} (2^j)^{-2} (\delta 2^j)^2 2^j \approx \delta^2 \sum_{j=1}^{\log(1/\delta)} 2^j \approx \delta,$$

as we claimed.

In case $d = (t, 0)$ we have exactly the same argument exploiting the $\pi/2$ rotational symmetry of the model (switching to the dual lattice and using that $LR_n$ does not happen if and only if there is a dual up-down crossing).

The case of a general $d \in R_\delta$ now easily follows. If \{LR_n \neq LR^d_n\}, then either \{LR_n \neq LR^d_{dx}\} or \{LR_n \neq LR^d_{dy}\}, where $d_x$ and $d_y$ are the projections of $d$ onto the first and the second coordinates, respectively.

As a consequence, $P[LR_n \neq LR^d_n] \leq P[LR_n \neq LR^d_{dx}] + P[LR_n \neq LR^d_{dy}] \leq O(\delta).$ \hfill $\square$

Our proof uses the idea that the percolation crossing event is almost invariant under a quasi-transitive group action, so we define a linear operator that maps any function to an invariant
we define the sublattice $H$ that $L \Gamma = \mathbb{Z}^2$. Importantly, $\Gamma$ is the orthogonal projection onto the space of $\mathbb{Z}^2$-invariant functions. In our case we have $\Gamma = \mathbb{Z}^2_n$ and we will write $M[LR]$ for $M^{\mathbb{Z}^2_n}[LR]$, the average of all translates of the left-right crossing event. Importantly, $M[LR]$ is $\mathbb{Z}^2_n$-invariant. It is not quite transitive, since $\mathbb{Z}^2_n$ does not act transitively on the edge set of the torus, but Lemma 6.6 shows that for our purpose it is close enough to being transitive. In case the averaging is done according to some subset $H \subset \mathbb{Z}^2_n$ we obviously indicate it in the superscript.

For a number $\delta > 0$ we shall also consider a coarser lattice of mesh size $\delta$. More precisely, we define the sublattice $H_\delta := \{[n\delta], 2[n\delta], \ldots, L[n\delta]\}^2$, where $L$ is the largest integer such that $L[n\delta] < n$ (that is, $L = \lfloor \frac{n}{n\delta} \rfloor$). Obviously, $\frac{n}{n\delta} - 1 \leq \frac{n}{n\delta} - 1 \leq L \leq \frac{n}{n\delta} \leq \frac{n}{n\delta - 1}$, so $1/\delta - 1 \leq L \leq 1/\delta + O(1/n)$ and therefore

$$\left(1 - \frac{1}{\delta}\right)^2 \leq |H_\delta| \leq \frac{1}{\delta^2} + O \left(\frac{1}{\delta n}\right).$$

Now we show that $M[LR]$, although not transitive, satisfies a clue bound similar to the one in Theorem 2.1.

**Lemma 6.6.** For any $U \subseteq E(\mathbb{Z}^2_n)$ we have

$$\text{clue}(M[LR] | U) \leq 2 \frac{|U|}{n^2}.$$

**Proof.** The action of $\mathbb{Z}^2_n$ on $E(\mathbb{Z}^2_n)$ has two orbits: the set of vertical edges $E_v$ and the set of horizontal edges $E_h$. Let $X$ be a uniformly random element from the spectrum of $M[LR]$. Since $M[LR]$ is invariant under this action for any $e_1, e_2 \in E_v$ we have $\mathbb{P}[X = e_1] = \mathbb{P}[X = e_2]$ and similarly for $e_1, e_2 \in E_h$ it holds that $\mathbb{P}[X = e_1] = \mathbb{P}[X = e_2]$.

So $\sum_{e \in E_v} \mathbb{P}[X = e] = n^2/2 \mathbb{P}[X = e] \leq 1$, and the same goes for horizontal edges This implies that for any $e \in E(\mathbb{Z}^2_n)$ $\mathbb{P}[X = e] \leq 2/n^2$. Therefore reproducing the argument in the proof Theorem 2.1 we get

$$\text{clue}(M[LR] | U) = \sum_{e \in U} \mathbb{P}[X = e] = 2 \frac{|U|}{n^2}.$$

\[\square\]

In the following lemma we show that $M[LR]$ and $M^{H_\delta}[LR]$, the average of the translates of the crossing event over the $\delta n$-lattice, are highly correlated.

**Lemma 6.7.** Let $\delta > 0$. Then

$$\text{Corr}(M[LR], M^{H_\delta}[LR]) \geq 1 - O(\sqrt{\delta}).$$
Proof. We consider a new spin system $\sigma$ on the $\mathbb{Z}_n^2$ torus which is a factor of the uniform Bernoulli percolation on the edges. Namely, at every vertex $v \in \mathbb{Z}_n^2$, we set $\sigma_v = L_{\mathbb{R}^2}^v$.

The outline of the proof is as follows: First we observe that for any $\delta n \times \delta n$ square on the $\delta$-lattice the value of $\sigma$ is the same on the four vertices of the square, with probability $1 - O(\delta)$. For a fixed configuration we call a square on the $\delta n$-lattice good if this is the case, bad otherwise.

The second step is to show that the event that there exists a point $t$ inside a good square such that $\sigma_t$ differs from the value of $\sigma$ on the vertices of the square also happens with probability at most $1 - O(\delta)$. These two claims together suffice to show that the average on the $\delta n$-lattice already gives a good approximation about the average on the entire torus $\mathbb{Z}_n^2$.

Define the event $A := \{ \sigma_0 = \sigma_{(0,n\delta)} = \sigma_{(n\delta,0)} = \sigma_{(n\delta,n\delta)} \}$. By Lemma 6.5 and the union bound,

$$\mathbb{P}[A^c] \leq 2(\mathbb{P}[\sigma_0 \neq \sigma_{(0,n\delta)}] + \mathbb{P}[\sigma_0 \neq \sigma_{(n\delta,0)}]) \leq O(\delta).$$

Because of the translation invariance of the measure, this means that we have:

$$\mathbb{E}[|\text{bad } \delta n\text{-squares}|] \leq O(\delta) |H_\delta| \leq O(\delta) O\left(\frac{1}{\delta^2}\right) = O\left(\frac{1}{\delta}\right). \quad (6.4)$$

Let $B$ denote the event that for every $t \in \mathbb{Z}_n^2 \cap [0, \delta n]^2$ the values $\sigma_t$ are the same. We are going to show that $\mathbb{P}[B^c \cap \{ [0, \delta n]^2 \text{ is a good square} \}] \leq O(\delta)$. In other words, if a square on the $\delta n$ lattice has the same value on all of the four vertices of the square then with high probability this is the value everywhere inside the square.

First, observe that the event $B^c \cap \{ [0, \delta n]^2 \text{ is a good square} \}$ implies the existence of an alternating triple $t_1, t_2, t_3$ on a vertical or horizontal line segment of length at most $n\delta$ on the torus such that $\sigma_{t_1} = \sigma_{t_3}$ but $\sigma_{t_1} \neq \sigma_{t_2}$. Indeed, if there is a $t$ on the boundary of the square such that $\sigma_t \neq \sigma_{v_t}$, the statement is true. If this is not the case, there is a vertex $t$ inside the square $\sigma_t \neq \sigma_{v_t}$, but for any vertex $b$ on the boundary of the square $\sigma_t \neq \sigma_b$ so again the statement holds.

This configuration, in a similar way to Lemma 6.5, implies the existence of two 3-arm events in two disjoint half planes both from distance $\delta n$ to order $n$ (see Figure 6.2), and this enables us to give an upper bound on the probability that the box $[0, \delta n]^2$ is good, but there is some vertex with a different value of the crossing event inside the box.

![Figure 6.2: $\sigma_{t_1} \neq \sigma_{t_2}$ and $\sigma_{t_2} \neq \sigma_{t_3}$ results in two 3-arm events in two disjoint half planes](image)

As in the proof of Lemma 6.5, both of the 3-arm events starts in of $1/\delta$ different $\delta n \times \delta n$-boxes (as one of the coordinate of the starting box is fixed). Thus, ignoring the boundary effects, and using the union bound and Proposition 6.1:

$$\begin{align*}
\mathbb{P}[B^c \cap [0, \delta n]^2 \text{ is a good square}] &\leq O(1/\delta^2) (\alpha_3^+(\delta, O(1)))^2 \\
&= O(1/\delta^2) O(\delta^2) = O(\delta^2).
\end{align*} \quad (6.5)$$

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Let us call a $\delta n \times \delta n$ square perfect, if it is good and for any $t$ in the box the $\sigma_t$ values are the same as those in the vertices of the square. A square is imperfect, if it is not perfect. Therefore, using (6.4) and (6.5) we get that

$$E[|\text{imperfect squares}|] = E[|\text{bad squares}|] + E[|\text{good, but not perfect squares}|] \leq O(1/\delta) + O(1) = O(1/\delta)$$

We are now ready estimate the correlation. We first use Markov’s inequality to bound the probability that at least $\sqrt{\delta}$ ratio of all squares are imperfect:

$$P[|\text{imperfect squares}| > \sqrt{\delta}(1/\delta)^2] = P[|\text{imperfect squares}| > \delta^{-3/2}] \leq \frac{O(1/\delta)}{\delta^{-3/2}} = O(\sqrt{\delta}).$$

Note that the respective averages restricted to a perfect square are equal, so the difference between $M[LR]$ and $M^{H_\delta}[LR]$ comes only from imperfect squares. Therefore, on the event $\{ |\text{imperfect squares}| \leq \sqrt{\delta}(1/\delta)^2 \}$ we have

$$|M[LR] - M^{H_\delta}[LR]| \leq O(\sqrt{\delta}),$$

using that the averages $M[LR]$ and $M^{H_\delta}[LR]$ on any set are between $-1$ and $1$, and thus the respective differences are at most $2$ on imperfect squares. So

$$P[|M[LR] - M^{H_\delta}[LR]| \leq \sqrt{\delta}] \geq 1 - O(\sqrt{\delta}),$$

which implies

$$\text{Corr}(M[LR], M^{H_\delta}[LR]) \geq 1 - O(\sqrt{\delta}),$$

again because $|M[LR] - M^{H_\delta}[LR]| \leq 2$.

Now we are ready to prove the main result of this section.

**Proof of Theorem 6.2.** Let $U_n \subseteq \mathbb{Z}_n^2$ be a sparse sequence of subsets, i.e., $\lim_{n \to \infty} \frac{|U_n|}{n^2} = 0$. Indirectly, we assume that there is a $c > 0$ such that clue$(LR_n \mid U_n) > c$ for every large $n$.

We start by giving an outline of the proof. Fix an arbitrary small $\delta > 0$. Using the indirect assumption that there is a sparse sequence of subsets with clue greater than $c > 0$, we are going to show that the average of the translated crossing events on the $\delta$-lattice $M^{H_\delta}[LR_n]$ also has clue greater than $c' > 0$ for a larger, but still sparse sequence of subsets $U^\delta_n$ (where $c'$ depends on $\delta$, but not on $n$).

At the same time Lemma 6.7 shows that the average of the translates on the $\delta$-lattice and the average of all translates $M[LR_n]$ are highly correlated. Therefore, the same sequence of sparse subsets also gives us positive amount of clue about $M[LR_n]$. Nevertheless, this is in contradiction with Lemma 6.6, which claims that a sequence of sparse subsets cannot give asymptotically positive clue about $M[LR_n]$.

For a given $\delta$, we define the set $U^\delta_n = \bigcup_{t \in H_\delta} U^t$, where $U^t = \{ u + t : u \in U \}$. So $U^\delta_n$ is the union of all $H_\delta$-translates of $U$. Clearly, clue$(LR_n \mid U^\delta_n) \geq c$. We shall choose the appropriate value of $\delta$ at the end of the proof.
As the Bernoulli measure is $\mathbb{Z}_n^2$-invariant, we clearly have $\text{Var}(L_n) = \text{Var}(L_n^\theta)$ for every $t \in \mathbb{Z}_n^2$ and therefore

$$\text{Var}(M^{H_\delta}[L_n]) = \frac{1}{|H_\delta|^2} \sum_{h,g \in H_\delta} \text{Cov}(L_n^h, L_n^g) = \text{Var}(L_n) \frac{1}{|H_\delta|^2} \sum_{h,g \in H_\delta} \text{Corr}(L_n^h, L_n^g) \leq \text{Var}(L_n).$$

We are now ready to bound $\text{clue}(M^{H_\delta}[L_n])$ from below. We will denote by $P$ the projection (conditional expectation, from the probabilistic point of view) onto $\mathcal{F}_{U_n^\delta}$. Let $h_1$ and $h_2 \in \mathbb{Z}_n^2$.

Then

$$\text{Cov}(P[L_n^{h_1}], P[L_n^{h_2}]) \leq \sqrt{\text{Var}(P[L_n^{h_1}])} \sqrt{\text{Var}(P[L_n^{h_2}])} \text{Corr}(P[L_n^{h_1}], P[L_n^{h_2}]) \overset{(6.6)}{=} \sqrt{\text{clue}(L_n^{h_1}) \text{clue}(L_n^{h_2})} \text{Corr}(P[L_n^{h_1}], P[L_n^{h_2}]) \geq c \text{Corr}(P[L_n^{h_1}], P[L_n^{h_2}]),$$

where used that $\text{clue}(L_n^{h_1}) > c$ for any $h \in H_\delta$.

We fix another grid size $\theta$, which is coarser than $\delta$, so $0 < \delta < \theta$. Now we have

$$\text{clue}(M^{H_\delta}[L_n]) \leq \frac{\text{Var}(P[M^{H_\delta}[L_n]])}{\text{Var}(M^{H_\delta}[L_n])} \overset{(6.7)}{=} \frac{1}{|H_\delta|^2} \sum_{h_1, h_2 \in H_\delta} \text{Cov}(P[L_n^{h_1}], P[L_n^{h_2}]) \text{Var}(L_n) \geq \frac{c}{|H_\delta|^2} \sum_{h_1, h_2 \in H_\delta} \text{Corr}(P[L_n^{h_1}], P[L_n^{h_2}]) \geq \frac{c}{|H_\delta|^2} \sum_{h \in H_\delta} \sum_{d \in R_\delta(h) \cap H_\delta} \text{Corr}(P[L_n^{h}], P[L_n^{d}]).$$

We remind the reader that $R_\theta(h)$ is the square with side length $2\theta n$ around $h$. In the estimation above we first used the upper bound for $\text{Var}(M^{H_\delta}[L_n])$ after (6.6) and finally that $L_n$ is monotone, and therefore, by the FKG-inequality $\text{Cov}(P[L_n^{h_1}], P[L_n^{h_2}]) \geq 0$.

By Lemma 6.5 there exists a $K > 0$ such that

$$\text{Corr}(L_n^{h}, L_n^{d}) \geq 1 - K \theta$$

for every $h \in \mathbb{Z}_n^2$ and $d \in R_\theta(h)$. Applying Lemma 6.3 for $L_n^{h}$, $L_n^{d}$ and $P$, and choosing $\theta$ small enough so that $2K \theta < c/2$, we get that

$$\text{Corr}(P[L_n^{h}], P[L_n^{d}]) \geq 1 - \frac{K \theta}{c - 2K \theta} \geq 1 - \frac{2 \theta \theta}{c},$$

Plugging this back into (6.7), and using that $|H_\delta| = \frac{1}{\delta^2}$ and $|R_\theta(h) \cap H_\delta| = |R_\theta \cap H_\delta| = 4\theta^2/\delta^2$, 33
and thus \(|R_\theta(h) \cap H_\delta|/|H_\delta| = \theta^2\) for any \(h\), we obtain the following bound:

\[
\text{clue}(M^{H_\delta}[LR_n] \mid U^\delta_n) \geq \frac{c}{|H_\delta|^2} |R_\theta H_\delta| \left(1 - \frac{2K \theta}{c}\right)
\]

\[
= \frac{|R_\theta H_\delta|}{|H_\delta|^2} \left(1 - \frac{2K \theta}{c}\right)
\]

\[
= \theta^2(c - 2K \theta)
\]

\[
\geq \theta^2 \frac{c}{2}.
\]

At the same time, by Lemma 6.7, there is some \(L > 0\) such that

\[
\text{Corr}(M^{H_\delta}[LR_n], M[LR_n]) \geq 1 - L \sqrt{\delta}.
\]

Now choose \(\delta \leq \theta^4 \frac{c^2}{16L}\) so that \(L \sqrt{\delta} \leq \theta^2 \frac{c}{4}\). Applying Lemma 6.4 again with \(M^{H_\delta}[LR_n]\) and \(M[LR_n]\) we get from (6.8) that for all \(n \in \mathbb{N}\)

\[
\text{clue}(M[LR_n] \mid U^\delta_n) \geq \theta^2 \frac{c}{2} - L \sqrt{\delta} \geq \theta^2 \frac{c}{4}.
\]

But \(|U^\delta_n| = \frac{1}{2^2}|U_n| = o(n^2)\) and by Lemma 6.6 \(\text{clue}(M[LR_n] \mid U^\delta_n) \to 0\), which is in contradiction with (6.9).

\[
\square
\]

7 Some open problems

Here we collect the open problems raised somewhere in the paper and some further ones. We have not thought thoroughly about these questions, but we would definitely be interested in the answers.

From Subsection 3.1, we have the following question relating significance and influence; see the discussion there.

**Question 7.1.** Characterise sequences of Boolean functions such that for any sequence of subsets with constant density \(I(f_n \mid U_n) \gg \text{sig}(f_n \mid U_n)\) holds, or where \(I(f_n \mid U_n) \asymp \text{sig}(f_n \mid U_n)\), respectively.

From Section 4, comparing \(L^2\)-clue and I-clue:

**Question 7.2.** Is there a sequence of functions \(f_n \in L^2(\{-1, 1\}^V_n, \pi_n \circ \pi_n)\) and a corresponding sequence of subsets \(U_n \subseteq V_n\) such that

1. \(\text{clue}^f(f_n \mid U_n) \ll \text{clue}(f_n \mid U_n)\)
2. \(\text{clue}^f(f_n \mid U_n) \gg \text{clue}(f_n \mid U_n)\)?

What is the answer if we ask \(f_n\) to be Boolean for all \(n \in \mathbb{N}\)? What is the answer if we allow for non-product measures on \(\{-1, 1\}^V_n\)?

Staying in the setup of iid measures, let us define the clue-profile:

\[
\text{cp}_f(s) := \inf \left\{u : \exists U \subseteq V \text{ with } |U|/|V| \leq u \text{ and } \text{clue}(f \mid U) \geq s\right\}.
\]

**Question 7.3.** Can we say anything general about the shape of this monotone increasing function \(\text{cp}_f : [0, 1] \to [0, 1]\), beyond \(\text{cp}_f(s) \leq s\)? Almost anything is possible, as in [ASP17]? When do we have a sharp threshold, as for monotone graph properties [Fri05] or in the cutoff phenomenon for random walk mixing times [AD86, BHP17]?
A specific question is the following:

**Question 7.4.** Does every monotone Boolean function has a constant $c_f > 0$ such that

$$cp_f(s) > cf_s, \quad \text{for all } s \in [0,1]?$$

*Might there be even a universal constant $0 < c < c_f$?*

The Schramm-Steif small revealment noise sensitivity theorem [SS10] does not have a converse: there are monotone noise-sensitive functions without a small revealment algorithm [GS15, Section 8.6]. The following question would aim at a similar converse in terms of clue:

**Question 7.5.** Is it true for every sequence of monotone noise-sensitive functions $f_n$ that, for any $\delta > 0$ fixed,

$$cp_{f_n}(\delta) \gg E[\text{clue}(f_n | B^\delta)]?$$

Another specific question in the flavour of Question 7.3 is the following. We feel it should be easy, but we have not found the answer yet:

**Question 7.6.** The O’Donnell-Servedio bound [ODS04] says that if an adaptive randomized algorithm computes a Boolean function $f$, then the expected number of bits examined is at least

$$(\sum_i I_i(f))^2 = (E|\mathcal{F}f|)^2,$$

where $I_i(f)$ is the influence of the $i^{th}$ bit, defined in Remark 2.3. Does a lower bound of the same order of magnitude hold also if we just want an answer that has a uniformly positive correlation with $f$?

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