WASSERSTEIN STABILITY FOR PERSISTENCE DIAGRAMS

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Abstract. The stability of persistence diagrams is among the most important results in applied and computational topology. Most results in the literature phrase stability in terms of the bottleneck distance between diagrams and the $\infty$-norm of perturbations. This has two main implications: it makes the space of persistence diagrams rather pathological and it often provides very pessimistic bounds with respect to outliers. In this paper, we provide new stability results with respect to the $p$-Wasserstein distance between persistence diagrams. This includes an elementary proof for the setting of functions on sufficiently finite spaces in terms of the $p$-norm of the perturbations, along with an algebraic framework for $p$-Wasserstein distance which extends the results to wider class of modules. We also apply the results to a wide range of applications in topological data analysis (TDA) including topological summaries, persistence transforms and the special but important case of Vietoris-Rips complexes.

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1. Introduction

Persistent (co)homology has been the subject of extensive study in applied topology. Roughly speaking it is a homology theory for filtrations or filtered spaces. A landmark result in applied topology is that persistent (co)homology and more importantly persistence diagram are stable with respect to perturbations of the input filtration. The classical result states:

Theorem 1.1 ([17]). Let $X$ be a triangulable space with continuous tame functions $f, g : X \to \mathbb{R}$. Then the persistence diagrams $\text{Dgm}(f)$ and $\text{Dgm}(g)$ for their sublevel set filtrations satisfy

$$d_B(\text{Dgm}(f), \text{Dgm}(g)) \leq ||f - g||_\infty$$

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where \(d_B(\cdot)\) represents the bottleneck distance. 

This result has been generalized to algebraic \(^2\) and categorical settings \(^7\), with recent work strongly aimed at multiparameter and more general settings, particularly where classical notions of persistence diagrams do not exist. Here we study the \(p\)-Wasserstein stability of persistence diagrams for \(1 \leq p \leq \infty\). This has been far less studied, with existing results almost exclusively in terms of the classical results relating \textit{interleaving distances} between filtrations and the \(\infty\)-Wasserstein distance, i.e. bottleneck distance. Upper bounds on the \(p\)-Wasserstein distances are less common and often rely on bottleneck stability resulting in pessimistic bounds. Furthermore, there is often a requirement for \(p\) to be sufficiently large. However \(p\)-Wasserstein distances for small values of \(p\) (i.e. \(p = 1, 2\) rather than \(p = \infty\)) are important for a number of reasons. From the applications side, the \(2\)-Wasserstein distance on persistence diagrams has been found to be much more effective than bottleneck distance.

One of the main difficulties in establishing a \(p\)-Wasserstein bound is that we cannot use \textit{interleavings} between persistent modules, which are a key tool in persistence stability results and allow us to relate the bottleneck distance between persistence diagrams to the interleaving distance between persistence modules. Indeed, at first there does not seem to be a natural algebraic description of Wasserstein distance in the same way as there is for bottleneck distance. Here we take a fundamentally different approach to proving \(p\)-Wasserstein stability, which, at its core, focuses on a cellular \(p\)-Wasserstein stability theorem. The proof exploits the local correspondences between coordinates of the points in the persistence diagram with critical cells in a filtration over a cellular complex. This cellular \(p\)-Wasserstein stability theorem then can be modified and applied to a variety of settings to prove a range of stability theorems. The stability of linear representations of persistent homology are usually stated as upper bounds in terms of the \(1\)-Wasserstein distance, for which the pre-existing stability results for bottleneck and \(p\)-Wasserstein distances cannot be applied. An important corollary of this paper is that we also get stability results for these linear representations.

Using the cellular stability theorem as inspiration, we also develop the algebraic theory to deal with the \(p\)-Wasserstein distance for persistence modules. The description of the distance is different to the notion of an interleaving which is closer to the single morphism characterization from \(^2\). Our generalized description recovers this result in the special case of bottleneck distance which is the same as the \(\infty\)-Wasserstein distance.

Using this theory, we prove the stability results in a more general setting, albeit with more technically involved proofs compared to the cellular stability proof. We also show a Minkowski-type of result connecting what we call the norm of a persistence modules, also called total persistence, in a short exact sequence\(^4\). The upper bound follows naturally from our study of the Wasserstein distance, but surprisingly, there is a lower bound as well, giving a geometric result for extensions of persistence modules for all \(p \geq 1\).

In summary, the main contributions of this paper are:

1. A cellular \(p\)-Wasserstein stability theorem, which also provides a simplified proof of bottleneck stability of diagrams for finite complexes.
2. The application the above theorem to produce stability theorems for a number of applications, including grey-scale images, persistent homology transforms, and Vietoris-Rips filtrations. As the \(2\)-Wasserstein distance is widely used in applications, this addresses a significant gap in the applied topology literature.
3. An algebraic formulation of the \(p\)-Wasserstein distance and prove stability which applies to pointwise finite dimensional modules with an additional technical condition.
4. Upper and lower bounds relating the \(p\)-Wasserstein distance of diagrams of modules in a short exact sequence.
5. Sufficient algebraic conditions which ensure that our algebraic definition yields a pseudo-distance without depending on the structure of persistence modules.

The paper can be thought of as split into two main parts. After introducing the relevant preliminaries, the first part of the paper is focused on results which are immediately relevant for applications in topological data analysis. We present a number of examples which illustrate the inherent instabilities in persistence diagrams, followed by general cellular stability theorem. We then discuss applications to the important cases of image analysis, persistent homology transforms, Vietoris-Rips filtrations, and other topological summaries. The results are focused on explicit bounds for finite cases, which are the main interest in applications.

\(^1\)We note that this is not a norm in the strict sense as persistence modules do not have a vector space structure.
The second part of the paper provides an algebraic perspective on Wasserstein stability. This includes a formulation which we show is equivalent to the Wasserstein distance for persistence diagrams but which applies to a wider class of modules. The main idea can be thought of the algebraic generalization of the cellular stability proof. This section is not required for the applications and so can be skipped by readers more interested in the practical implications of the results. Likewise, the algebraically minded reader need not go through the applications of the cellular stability theorem.

2. Preliminaries

As described above, we distinguish between settings where the underlying objects are finite and more general settings where the objects may be infinite but still sufficiently nice. The finite setting provides a clear illustration of the ideas and are often sufficient for many applications. In finite settings, we restrict ourselves to a restricted class of functions over a finite CW-complex denoted by $K$. We do not require any additional structure (e.g. simplicial, cubical, etc.). Exceptions are for the analysis of particular applications, e.g. cubical complexes for images (Section 5.1) and the simplicial structure of Vietoris-Rips complexes (Section 5.3). In the algebraic section we consider persistence modules more generally.

Definition 2.1. A persistence module $F$ is a collection of vector spaces $\{F_\alpha\}_{\alpha \in \mathbb{R}}$ along with transition maps $\psi^{\beta}_\alpha : F_\alpha \rightarrow F_\beta$ for all $\alpha \leq \beta$ such that $\psi^{\beta}_\alpha$ is the identity for all $\alpha$ and $\psi^{\beta}_\alpha \circ \psi^{\gamma}_\beta = \psi^{\gamma}_\alpha$ whenever $\alpha < \beta < \gamma$. If $F_\alpha$ is finite dimensional for all $\alpha$, then we say $F$ is pointwise finite dimensional (or p.f.d.).

One of the most common ways persistence modules arise is via filtrations of finite CW-complexes, denoted by $K$, especially those associated to functions. Without loss of generality, when considering functions we restrict ourselves to sublevel sets: for $f : K \rightarrow \mathbb{R}$, the corresponding sublevel set filtration $\{K_\alpha\}_{\alpha \in \mathbb{R}}$ with

$$K_\alpha = \{ \sigma | f(\sigma) \leq \alpha \}.$$ 

From the definition, it is clear we only consider functions which are piecewise constant on the interior of cells, that is

$$f(\sigma) = \sup_{x \in \sigma} f(x).$$

We further will require that our functions are monotone which means that if $\tau$ is a face of $\sigma$ then $f(\tau) \leq f(\sigma)$. This monotone assumption greatly simplifies the exposition as this ensures that all sublevel sets are (closed) CW-complexes. This is the most common setting in applications of persistence. This is more restrictive than the definition in [23], which only includes the condition that the space monotonically non-decreasing and importantly excludes piecewise linear (PL) functions. However, in most cases one can find a piecewise constant function which results in an isomorphic persistence module to the module arising from the PL function.

These conditions are sufficient to allow us to construct a filtered cellular chain complex. Applying the homology functor over a field to the filtered chain complex, we obtain the corresponding persistence module, denoted $\{H_k(K_\alpha)\}_{\alpha \in \mathbb{R}}$ with the maps $\psi^{\beta}_\alpha : H_k(K_\alpha) \rightarrow H_k(K_\beta)$, induced by the inclusions $K_\alpha \hookrightarrow K_\beta$ for all $\alpha \leq \beta$.

One could generalize the results in Section 2 to a more general setting, such as constructible functions. However the increase in technical complications yields relatively little gain, in light of the algebraic framework in Section 2 where we work directly with persistence modules. We restrict ourselves to the class of p.f.d. modules due to the result below. We note that given a piecewise constant filtration over a finite CW-complex, the resulting persistence module is pointwise finite dimensional.

Theorem 2.2 ([20] Theorem 1.1). A p.f.d. persistence module admits an interval decomposition. That is, the module can be decomposed into rank one summands:

$$\bigoplus_x I(b(x), d(x))$$

which are unique up to isomorphism. This is referred to as a persistence barcode and we refer to the elements as bars or intervals.

We can associate to each interval $I(b(x), d(x))$ a point in $\mathbb{R}^2$ with the first coordinate $b(x)$ and second coordinate $d(x)$. We refer to $b(x)$ as the birth time and $d(x)$ as the death time. By taking the union over all the intervals in an interval decomposition of a persistence module we obtain a multiset of points in $\mathbb{R}^2_+ = \{(a, b) | a \leq b\}$. We refer to this multiset of points as the persistence diagram.
In classical sublevelset persistent homology, all intervals are of form closed-open \([b, d]\). However, there are four types of intervals which may appear in the decomposition of persistence modules \([13]\): open-open – \((b, d)\), open-closed – \((b, d]\), closed-open – \([b, d)\), and closed-closed – \([b, d]\). All the results in this paper apply to the four types of intervals hence our choice of notation. This follows from the observation that different types of intervals are indistinguishable with respect to Wasserstein distance, i.e. the Wasserstein distance between intervals of different types is 0. We note however that the results do not apply as-is for zig-zag modules as this would already require modifying the definition of persistence module (Definition 2.1).

**Remark 2.3.** For p.f.d. modules, the decomposition may contain an uncountable number of intervals. However, there can only be countably many intervals where \(b(x) < d(x)\). As in the case of different interval types, see \([13]\), removing intervals where \(b(x) = d(x)\) (which form an ephemeral submodule \([12]\)) induces a Wasserstein distance of 0. Hence, we may assume the decomposition has countably many intervals.

Throughout the paper, we will often be interested in persistence modules arising from filtrations of chain complexes. In this case, the persistence module is naturally equipped with a grading in dimension, hence we introduce the following notation.

**Definition 2.4.** Given a graded persistence module \(F = \bigoplus_{k \in \mathbb{R}} F_k\), let \(\text{Dgm}_k(F)\) denote the \(k\)-th persistence diagram of \(F_k\). Taking the grading over dimension, we denote

\[
\text{Dgm}(F) = \bigoplus_k \text{Dgm}_k(F).
\]

**Remark 2.5.** If we are considering an abstract persistence module, e.g. not arising from chain complexes, or the grading is not used, we omit it.

Our main focus is the Wasserstein distance between diagrams. The Wasserstein distance is a form of optimal transport metric. We can consider all possible transportation plans for moving the points within one persistence diagram to a different one. The transportation plans between persistence diagrams are called matchings. Each of these transportation plans has a cost and the distance becomes the infimum of these costs. To define our potential transportation plans we need to introduce an abstract element which we call the diagonal and denote by \(\Delta\). This abstract diagonal element can be philosophically thought of as an empty interval.

**Definition 2.6.** Set \(\mathbb{R}\) to be the extension of the real line to include \(\infty\) and set \(\mathbb{R}^2 \times \{b, d\} = \{(b, d) \in \mathbb{R}^2 \mid b \leq d\}\). Let \(X\) and \(Y\) be persistence diagrams represented by countable multisets in \(\mathbb{R}^2\). A matching \(M\) between \(X\) and \(Y\) is a subset of \(\{X \cup \Delta\} \times \{Y \cup \Delta\}\) such that every element in \(X\) and \(Y\) appears in exactly one pair. The abstract diagonal element \(\Delta\) can appear in many pairs.

As the points in \(X\) and \(Y\) lie in \(\mathbb{R}^2\) we can use the \(l_p\) distance in the plane. We can define an \(l_p\) distance between a point in \(\mathbb{R}^2\) to \(\Delta\) by taking the perpendicular distance, that is

\[
\|(a, b) - \Delta\|_p = \inf_{t \in \mathbb{R}} \|(a, b) - (t, t)\|_p = \|\Delta\|_p = \left(\left(\frac{a + b}{2}\right) - \left(\frac{a + b}{2}\right)\right)\|_p = \frac{2}{p}\|b - a\|
\]

and \(\|(a, b) - \Delta\|_\infty = \frac{a + b}{2}\). Furthermore we say that \(\|\Delta - \Delta\|_p = 0\) for all \(p\) and \(\|(a, \infty) - (b, \infty)\|_p = |a - b|\).

**Definition 2.7.** Given two diagrams, \(\text{Dgm}_k(F)\) and \(\text{Dgm}_k(G)\), the \((p, q)\)-Wasserstein distance is

\[
W_{p,q}(\text{Dgm}_k(F), \text{Dgm}_k(G)) = \inf_{M} \left(\frac{\sum_{(x,y) \in M} \|x - y\|_q^p}{\sum_{x \in X} \|x - y\|_q^p}\right)^{\frac{1}{p}}
\]

where \(M \subset \{(\text{Dgm}_k(F) \cup \Delta) \times \{\text{Dgm}_k(G) \cup \Delta\}\}\) is a matching. The total \((p, q)\)-Wasserstein distance is defined as

\[
W_{p,q}(\text{Dgm}(F), \text{Dgm}(G)) = \left(\sum_k \left(W_{p,q}(\text{Dgm}_k(F), \text{Dgm}_k(G))^p\right)^{\frac{1}{p}}\right)^{\frac{1}{p}}
\]

This is a standard approach to defining the distances and for more details, we refer the reader to \([32, 40, 39]\). 

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For all fixed \( p \), all \( W_{p,q} \) are bi-Lipschitz equivalent. Hence, we focus on the case \( p = q \), which we denote by \( W_p \). Taking the limit \( p \to \infty \) recovers the bottleneck distance
\[
W_\infty(Dgm(\mathcal{F}), Dgm(\mathcal{G})) = \inf \sup \sup_{\mathcal{M}} ||x - M(x)||_\infty
\]
It is worth commenting on the relative strength of stability results for different \( p \). The following statements illustrate that bottleneck stability is the weakest form of stability. We first note the following lemma\(^2\), whose proof can be found in Appendix A.

**Lemma 2.8.** For any \( p' \leq p \), given persistence diagrams \( Dgm(\mathcal{F}) \) and \( Dgm(\mathcal{G}) \),
\[
W_p(Dgm(\mathcal{F}), Dgm(\mathcal{G})) \leq W_{p'}(Dgm(\mathcal{F}), Dgm(\mathcal{G})).
\]

This lemma implies that when bounding the \( p \)-Wasserstein distance from above, the smaller \( p \) is, the stronger a stability result. An important quantity for understanding the Wasserstein distance more abstractly is the norm of a persistence module:

**Definition 2.9.** Let the \( p \)-norm of a p.f.d. module \( \mathcal{F} \) be the \( p \)-th root of the sum of the \( p \)-th power of the lengths of the bars, i.e.
\[
||\mathcal{F}||_p = \left( \sum_{x \in Dgm(\mathcal{F})} \ell(x)^p \right)^{\frac{1}{p}}
\]
where \( \ell(x) = d(x) - b(x) \), i.e. the length of a bar.

This quantity is also called the total persistence and is a natural quantity going back to [24], where it was observed that the squared total persistence or as we refer to it, the squared 2-norm is precisely the running time for the incremental algorithm for computing a persistence diagram. By analogy, we refer to this as the norm of a persistence module/diagram, although we do not assume the properties of a norm.

The remainder of this section consists of algebraic preliminaries for the category of persistence modules. The reader who is either familiar with this material or is interested primarily in the applications of Wasserstein stability, may skip the remainder of this section as these concepts are used in Section 7.

When studying the stability of persistence modules it is critical to understand the morphisms between persistence modules. The following facts are standard and are included for completeness.

**Scholium 2.10.** Given a morphism between persistence modules, its kernel, image, and cokernel are well-defined and are persistence modules themselves. This was studied extensively in [2], where the following facts were proven.

- A monic morphism between persistence modules, denoted by \( \mathcal{A} \xrightarrow{f} \mathcal{B} \), induces an injective multiset map between the death times of \( \mathcal{A} \) and the death times of \( \mathcal{B} \). This induces an injective set map denoted \( f_\star \) from the indecomposables of \( \mathcal{A} \) to the indecomposables of \( \mathcal{B} \).
- An epic morphism between persistence modules, denoted by \( \mathcal{A} \xleftarrow{g} \mathcal{B} \), induces an injective multiset map between the birth times of \( \mathcal{B} \) and the birth times of \( \mathcal{A} \). This induces an injective set map denoted \( g^\star \) from the indecomposables of \( \mathcal{B} \) to the indecomposables of \( \mathcal{A} \).

We make extensive use of these induced matchings. Finally, we note that standard constructions such as the pullback and pushout are well-defined. Furthermore, a prior application of these properties including a discussion of short exact sequences of persistence modules, can be found in [27].

An important assumption is that we assume that all modules have a common parameterization – in many cases, e.g. Vietoris-Rips filtrations, this is a natural assumption and it avoids problems which arising in comparing persistence modules and diagrams in more general settings. An important point in this paper is that all the morphisms we consider, unless specifically stated, are ungraded with respect to this parameterization. Consider a morphism between persistence modules, \( f : \mathcal{A} \to \mathcal{B} \) and let \( \mathcal{A}_\alpha \) denote the vector space at \( \alpha \). For any \( [x] \in \mathcal{A}_\alpha \),
\[
[x] \mapsto f([x]) \in \mathcal{B}_\alpha.
\]
Note that interleaving maps do not satisfy this condition. For further discussion of this notion in the context of bottleneck distance, see [27].

\(^2\)This is a standard result but the proof in the appendix is included for completeness for the reader.
While interval decompositions exist for all p.f.d. persistence modules, these may not be sufficiently well-behaved for our purposes. As an intermediate step in our proofs, we will often consider finitely generated modules. Hence to obtain the general result, we give a construction for approximating p.f.d. modules by finitely generated modules subject to the following technical condition of bounded $p$-norm.

**Definition 2.11.** If $\mathcal{F}$ is p.f.d. persistence module, we say $\mathcal{F}$ has bounded $p$-norm if

$$\sum_{x \in \text{Dgm}(\mathcal{F})} \ell(x)^p < \infty$$

Observe that the p.f.d. condition guarantees that there are a countable number of intervals. We focus most of our discussion on finite intervals. Infinite intervals are technically simpler as they correspond to freely generated summands and so the condition that there only a countable number is sufficient.

Our goal is to approximate the submodule consisting of finite intervals with a finitely generated module. We will make use of the following construction in Section 7.

**Lemma 2.12.** Let $\mathcal{F}$ be a p.f.d. module with bounded $p$-norm. For any $\varepsilon > 0$, there exists a finitely generated module $\mathcal{F}'$ with a monic morphism $i_\mathcal{F} : \mathcal{F}' \hookrightarrow \mathcal{F}$ and epic morphism $q_\mathcal{F} : \mathcal{F} \twoheadrightarrow \mathcal{F}'$ such that

$$W_p(\text{Dgm}(\mathcal{F}), \text{Dgm}(\mathcal{F}')) \leq \varepsilon, \quad \|\ker i_\mathcal{F}\|_p \leq \varepsilon, \quad \|\text{coker} q_\mathcal{F}\|_p \leq \varepsilon.$$ 

**Proof.** The proof is constructive – without loss of generality, assume that there are a countable number of intervals. Sort the intervals in the decomposition of $\mathcal{F}$ by decreasing length, i.e. $i \leq j$ implies $(d_i - b_i) \geq (d_j - b_j)$, breaking ties arbitrarily. Since $\mathcal{F}$ has bounded $p$-norm, the sum of the $p$-lengths is finite and there exists a constant $K$, such that for any $\varepsilon > 0$,

$$\sum_{i=K}^{\infty} (d_i - b_i)^p \leq \varepsilon^p$$

Define

$$\mathcal{F}' = \bigoplus_{i=1}^{K-1} \mathbb{I}\{b_i, d_i\}.$$

where the interval type is chosen to be the same as the corresponding interval in $\mathcal{F}$. To bound the Wasserstein distance, consider the matching which sends all intervals with index $i > K$ to the diagonal. The cost for matching each interval $\{b_i, d_i\}$ is $(d_j - b_j)^p$. By assumption, the sum over all the intervals matched to the diagonal has a cost less that $\varepsilon^p$. The morphisms $i_\mathcal{F}$ and $q_\mathcal{F}$ are the obvious morphisms $i_\mathcal{F}$ mapping all intervals after index $K$ to 0. The bound on norms then again follows by assumption. $\square$

We conclude this section with the following remark.

**Remark 2.13.** The central notion of the paper, the Wasserstein distance is defined in several different settings, so we often omit the term Wasserstein. For example, we refer to the above simply as the diagram distance (Definition 2.7), the distance between points embedded in $\mathbb{R}^d$ as the point set distance (Definition 5.6), and the algebraic notion as the module distance (Definition 7.7).

Furthermore, there are several different notions of points we consider: points in the persistence diagrams, elements of a point set in $\mathbb{R}^d$, and it will be useful for exposition to consider the vertices of a Vietoris-Rips complex as points. To minimize confusion, we restrict the term points to refer to persistence diagrams, preferring the term vertices for the more geometric notion. For a complete list of notation, see Appendix C.

3. Existing stability results and their limitations

As already mentioned, almost all stability results involve the bottleneck distance between persistence diagrams. A complete overview of these results is beyond the scope of this paper and we direct the reader to [14] for the stability of geometric constructions and [8] for the categorical foundations of $\infty$-Wasserstein stability. This should not be considered as a complete list as there is a large body of work on stability which we do not address review here.

3While this has the drawback of resulting in references to vertices in a point set, we feel this is a good compromise
3.1. Lipschitz functions on compact manifolds. The most relevant related work to the results presented in this paper can be found in [18]. To the best of our knowledge, this paper contains the main existing stability result for bounding the \((p \neq \infty)\)-Wasserstein distance between two persistence diagrams. It is for the setting of sub-level set filtrations of Lipschitz functions.

**Theorem 3.1** (Wasserstein Stability Theorem [18]). Let \(X\) be a triangulable, compact metric space that implies bounded degree-\(k\) total persistence, for \(k \geq 1\), and let \(f, g : X \rightarrow \mathbb{R}\) be two tame Lipschitz functions. Then

\[
W_p(f, g) \leq C^{1/p} \|f - g\|_\infty^{1 - \frac{1}{p}}
\]

for all \(p \geq k\), where \(C = C_X \max\{\Lip(f)^k, \Lip(g)^k\}\) and \(C_X\) is a constant dependent on \(X\).

To put our results into context, it is worthwhile understanding the limitations of this theorem. We will find lower bounds on \(C_X\) and \(k\), restricting ourselves to the case where \(X\) is a compact \(d\)-dimensional manifold. An important aspect is the bounded degree-\(k\) total persistence which will force this stability result to only hold for sufficiently large \(p\).

**Definition 3.2** ([18]). A metric space \(X\) has bounded degree-\(k\)-total persistence if there exists a constant \(C_X\) that depends only on \(X\) such that

\[
\|\Dgm(X(f))\|_k^k < C_X
\]

for every tame function \(f\) with Lipschitz constant \(\Lip(f) \leq 1\).

To construct a counterexample for functions over manifolds we will use a function which is the sum of functions with supports over disjoint balls of small radius. It would easy to adapt this to construct counterexamples for stratified spaces.

**Lemma 3.3.** Given an \(d\)-dimensional compact Riemannian manifold \(X\) and \(r > 0\) small enough, there exists a packing of \(\left\lfloor \frac{\text{vol}(X)}{\kappa_0 \omega_d 2^{d+1}} \right\rfloor\) disjoint balls of radius \(r\) in \(X\), where \(\omega_d\) is the volume of the \(d\)-dimensional Euclidean ball and \(\kappa_0\) is a constant which depends on the infimum of the scalar curvature of \(X\).

**Proof.** Since we have restricted ourselves to compact manifolds, there is a finite lower bound on the curvature of \(X\). Lemma 3.3 in [15] establishes a lower bound on the volume of a ball of radius \(r\). The result then follows from standard arguments involving packing and covering numbers. \(\square\)

**Lemma 3.4.** Let \(X\) be an \(d\)-dimensional compact manifold. If \(X\) has bounded degree-\(k\)-total persistence then \(k \geq d\).

**Proof.** We will prove this via a counterexample when \(k < d\). Let \(P = \{p_1, \ldots, p_N\}\) be the centers of a packing of \(N = \left\lfloor \frac{\text{vol}(X)}{\kappa_0 \omega_d 2^{d+1}} \right\rfloor\) disjoint balls of radius \(r\) in \(X\) (such a packing exists by Lemma 3.3). Set \(T_{r,p}\) to be a teepee shaped function about \(p\) with height \(r\), with \(T_{r,p}(x) = \max\{r - d(x, p), 0\}\). Then we consider functions \(f_r = \sum_{i=1}^N T_{r,p_i}\)(see Figure 1). Observe that \(f\) is \(1\)-Lipschitz. Then,

\[
\|\Dgm(X(f))\|_k^k = \sum_{i=1}^N r^k = \left\lfloor \frac{\text{vol}(X)}{\kappa_0 \omega_d 2^{d+1}} \right\rfloor r^k = O(r^{k-d})
\]

When \(k < d\) then this cannot be uniformly bounded from above for all small enough \(r > 0\). \(\square\)

![Figure 1](image-url)

**Figure 1.** (Left) The teepee function and (right) sum of teepee functions from Lemma 3.4.

The same example used in the above lemma, coupled with \(g\) the zero function provides a lower bound on \(C_X\). We see that \(C_X\) grows linearly as a function of the volume of \(X\).

Other papers that prove Wasserstein stability results are few and far between. In [15], Chen and Edelsbrunner consider non-Lipschitz functions on non-compact spaces, using scale-space diffusion. They focus on convergence properties as opposed to stability but also attain some Wasserstein stability results. Crucially, just as in the Lipschitz case, this \(p\)-Wasserstein stability only holds for \(p > d\) where \(d\) is the dimension of the domain. The condition \(p > d\) also appears in stability results for Čech filtrations, or equivalently distance filtrations, for point clouds in \(\mathbb{R}^d\).
3.2. **Erroneous appeals to previous p-Wasserstein stability results.** Unfortunately, the Lipschitz Wasserstein stability theorem in [18] appears to be one of the most misunderstood and miscited results within the field of topological data analysis. Common errors include using a small \( p \) (often 1 or 2) for high dimensional data, assertions that the persistence diagrams depend Lipschitz-continuously on data and applying the theorem to Vietoris-Rips filtrations. Luckily, many of the erroneous applications can now be covered by the stability results in this paper. Rather than discuss individual examples, in Section 6 we examine the consequences for topological summaries, which are increasingly the most common way to apply persistence diagrams to data.

4. **Cellular Wasserstein Stability**

We begin with a result mirroring the classical stability theorem by bounding the differences at the chain level, which will induce an upper bound on the \( p \)-Wasserstein distance between the corresponding diagrams. As stated in Section 2 we remind the reader that \( K \) is a finite CW-complex and \( f : K \to \mathbb{R} \) is a monotone function on the complex so all sublevel sets are subcomplexes. The persistence module associated to the sublevel set filtration of such a function, denoted \( \text{Dgm}(K(f)) \), is automatically pointwise finite dimensional. Since we shall be working with a fixed complex \( K \), we shorten our notation to \( \text{Dgm}(f) \).

**Definition 4.1.** The \( L^p \) norm of a function \( f : K \to \mathbb{R} \) is given by
\[
\|f\|_p^p = \sum_{\sigma \in K} |f(\sigma)|^p.
\]

This induces a distance between functions.

**Definition 4.2.** The \( L^p \) distance between two monotone functions \( f, g : K \to \mathbb{R} \) is given by
\[
\|f - g\|_p^p = \sum_{\sigma \in K} |f(\sigma) - g(\sigma)|^p.
\]

Note that this notion of the \( L^p \) distance between functions is analogous to the \( L^p \) distance for functions over discrete sets where the sum here is over the discrete set of cells. It remains an open problem to determine the relationship between the simplicial norm and the more classical functional \( p \)-norm.

**Open Problem 4.3.** Do there exist reasonable conditions on the underlying space and the function to relate the cellular \( p \)-norm to the more common functional \( p \)-norm, e.g. the integral of \( f^p \) over the space?

**Remark 4.4.** Given that we restrict ourselves to piecewise constant monotone functions, the only additional condition we require is that the underlying complex is finite. We have chosen to present the results in this way, so that it is clear that it applies to common settings including simplicial and cubical complexes.

The main idea in the proof of cellular Wasserstein stability is to bound the Wasserstein distance by considering a straightline homotopy between \( f \) and \( g \). We split the straightline homotopy into finitely many sub-intervals where a local result will hold, and then collect together the summands for the final desired inequality. By focusing on small enough sub-intervals we can exploit a consistent correspondence between the coordinates of the points in the persistence diagram with critical cells in the filtration. Though we phrase the proof in a different way, this idea first appeared in [19] to show a bottleneck stability result for vineyards. Indeed the idea of tracking of points in the persistence diagram is fundamental to the definition of vineyards. We refer the reader to [19] for more details and an algorithmic perspective on tracking critical simplices.

**Definition 4.5.** \( f : K \to \mathbb{R} \) be a monotone function. We define the partial order on \( K \) induced by \( f \) by \( \sigma \preceq \tau \) whenever \( f(\sigma) \leq f(\tau) \), or \( f(\sigma) = f(\tau) \) and \( \sigma \subseteq \tau \).

**Definition 4.6.** A partial order \( P \) embeds into partial order \( Q \) if there exists an injective order preserving map \( P \to Q \).

We first consider the easy case by bounding the distance between functions where the ordering of cells does not change.

**Lemma 4.7.** Let \( f, g : K \to \mathbb{R} \) be monotone functions over a CW complex \( K \) such that the partial orders induced by \( f \) and \( g \) embed into a common total order, then
\[
W_p(\text{Dgm}(f_\sigma), \text{Dgm}(g_\sigma)) \leq \|f_\sigma - g_\sigma\|_p.
\]
If we fix the homological dimension $k$ then

$$W_p(\text{Dgm}_k(f), \text{Dgm}_k(g))^p \leq \sum_{\dim(\sigma) \in \{k, k+1\}} |f(\sigma) - g(\sigma)|^p.$$  

Proof. By assumption, we may fix one total order $\leq$ for $f$ and $g$. The algorithm for computing persistent homology for $f$ and $g$ partitions the cells in $K$ into pairs $(\sigma_i, \tau_i)$ and singletons $(\omega_j, \emptyset)$ such that each cell appears exactly once. The partition of the cells is dependent only on the total order which is common for both $f$ and $g$. The intervals of the persistence modules for $f$ and $g$ are then given by $\{(f(\sigma_i), f(\tau_i)) \in B \}$ for $i \in A$ and $\{(f(\sigma_i), f(\tau_i)) \in \Delta \}$ for $i \in B$. Let $A = \{i : f(\tau_i) > f(\sigma_i) \text{ and } g(\tau_i) > g(\sigma_i)\}$, $B = \{i : f(\tau_i) > f(\sigma_i) \text{ and } g(\tau_i) = g(\sigma_i)\}$ and $C = \{i : f(\tau_i) = f(\sigma_i) \text{ and } g(\tau_i) > g(\sigma_i)\}$. Construct a matching $\mathbf{M} \subset \text{Dgm}(f) \cup \Delta \times (\text{Dgm}(g) \cup \Delta)$ given by the union of the sets

$$\{(f(\sigma_i), f(\tau_i)), (g(\sigma_i), g(\tau_i)) : i \in A\}$$

$$\{(f(\sigma_i), f(\tau_i)), (\Delta) : i \in B\}$$

$$\{(\Delta), (g(\sigma_i), g(\tau_i)) : i \in C\}$$

$$\{(f(\omega_j), (g(\omega_j), \emptyset))\}.$$

Note that for $i \in B$ we have $\|(f(\sigma_i), f(\tau_i)) - (\Delta)\|^p \leq |f(\sigma_i) - g(\sigma_i)|^p + |f(\tau_i) - g(\tau_i)|^p$ as $(g(\sigma_i), g(\tau_i))$ is on the diagonal, and for $i \in C$ we have $\|\Delta - (g(\sigma_i), g(\tau_i))\|^p \leq |f(\sigma_i) - g(\sigma_i)|^p + |f(\tau_i) - g(\tau_i)|^p$ as $(f(\sigma_i), f(\tau_i))$ is on the diagonal. The $p$-th power of the cost of this matching $\mathbf{M}$ is thus bounded by

$$\sum_{\sigma} |f(\sigma) - g(\sigma)|^p$$

as every cell appears at most once. Since the $p$-Wasserstein distance is the smallest possible matching cost, we conclude that

$$W_p(\text{Dgm}(f), \text{Dgm}(g))^p \leq \|f - g\|^p.$$

The proof for when we restrict to homology dimension $k$ follows from the observation that only addition of the $k$-cells and the $k+1$-cells can affect homology in dimension $k$. $\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{A linear interpolation between functions $f_a$ and $f_c$ can be subdivided into intervals where the ordering does not change in the interior of the interval. If the underlying space is a finite CW complex, the number of such intervals is finite.}
\end{figure}

To complete the proof we observe that any straightline homotopy may be divided into intervals where the ordering does not change, see Figure 2. Since our underlying space is a finite CW complex, the number of such intervals must also be finite. This implies one of our main theorems:

**Theorem 4.8 (Cellular Wasserstein Stability Theorem).** Let $f, g : K \to \mathbb{R}$ be monotone functions. Then

$$W_p(\text{Dgm}(f), \text{Dgm}(g)) \leq \|f - g\|_p.$$  

If we fix a homology dimension $k$ then

$$W_p(\text{Dgm}_k(f), \text{Dgm}_k(g))^p \leq \sum_{\dim(\sigma) \in \{k, k+1\}} |f(\sigma) - g(\sigma)|^p.$$
Proof. Let \( f_t : K \rightarrow \mathbb{R} \) be the linear interpolation between \( f \) and \( g \) as \( t \) varies. That is, for \( t \in [0, 1] \) and \( \sigma \in K \), let \( f_t(\sigma) = (1-t)f(\sigma) + tg(\sigma) \). Observe that \( f_t \) is monotone for all \( t \) and that for \( 0 \leq a \leq a' \leq 1 \) we have \( \|f_{a'} - f_a\|_p = |a' - a|\|f - g\|_p \).

Each of the functions \( t \mapsto f_t(\sigma) \) is linear which implies that \( f_t(\sigma) = f_t(\hat{\sigma}) \) for two or more values of \( t \) if and only if \( f(\sigma) = f(\hat{\sigma}) \) and \( g(\sigma) = g(\hat{\sigma}) \), in which case \( f_t(\sigma) = f_t(\hat{\sigma}) \) for all \( t \in [0, 1] \).

There are only finitely many values \( t = a_1, a_2, \ldots, a_n \in (0, 1) \), sorted in increasing value, where there exists \( \sigma, \hat{\sigma} \) with \( f_t(\sigma) = f_t(\hat{\sigma}) \) but \( f(\sigma) \neq f(\hat{\sigma}) \). Set \( a_0 = 0, a_{n+1} = 1 \). In each of the intervals \((a_i, a_{i+1})\), the ordering of the function values is consistent, with possible equalities. Hence, there exists a consistent total ordering which is compatible with all the induced partial orders on \( K \) for any choice in \((a_i, a_{i+1})\). This choice of total ordering will also be compatible with the partial orders induced at \( a_i \) and \( a_{i+1} \). As noted above, if two simplices have equal function values, at any point in the open interval, they have the equal function values over the entire interval. We can therefore apply Lemma \[4.7\] for with \( f = f_{a_i} \) and \( g = f_{a_{i+1}} \).

\[
W_p(Dgm(f), Dgm(g)) \leq \sum_{i=0}^{n} W_p(Dgm(f_{a_i}), Dgm(f_{a_{i+1}}))
\]

\[
\leq \sum_{i=0}^{n} \|f_{a_i} - f_{a_{i+1}}\|_p = \sum_{i=0}^{n} (a_{i+1} - a_i)\|f - g\|_p = \|f - g\|_p
\]

The proof for when we restrict to homology dimension \( k \) is highly analogous, using the corresponding bound in Lemma \[4.7\] restricting to homology dimension \( k \). \( \square \)

5. Applications

We present some applications of the results of the cellular Wasserstein stability theorem. Sublevel set filtrations of grayscale images and persistent homology transforms of different geometric embeddings of the same simplicial complex are both cases which involve height functions determined by vertex values. We will prove Lipschitz stability in terms of the \( l_p \) norms over the set of vertices, where the Lipschitz constants are bounded by the number of cells in the links of each vertex. We also will prove some immediate corollaries for stability of Rips filtrations.

5.1. Stability of the sublevel set filtrations of grayscale images. Our first application is for the stability of grayscale images. The natural application is to two dimensional images, however we will state our results for more general \( d \)-dimensional images. An image is a real-valued piecewise constant function where each pixel/voxel is assigned a value. There are two main methods in the literature for creating a filtration of cubical complexes from a grayscale image.

Method 1. We can create a cubical complexes from a 2D image where each pixel corresponds to a 2-dimensional cubical cell. The edges correspond to sides of the pixels, and vertices to the corners. This construction naturally extends to higher dimensional images. There is a natural sublevel set filtration induced on the complex: the image defines values for the maximal dimensional cells (i.e. pixels/voxels) and the function values for lower dimensional cells are given as the minimum value over all cofaces.

Method 2. We can also consider the dual of the cubical complex in Method 1, which is again a cubical complex. In a 2D image we have a vertex for each pixel and an edge for each pair of neighbouring pixels (not including diagonals), and 2-cells where four pixels intersect. This construction naturally extends to higher dimensional images. We can build a filtration on this cubical complex by setting the values on the vertices as those of the pixel/voxel values provided, and setting the function values for higher dimensional cells as the maximum value over all faces.

It is worth noting that the sublevel set filtrations for these two methods can result in substantially different persistent homology. This difference stems from whether diagonally neighbouring pixels are considered connected. However, applying Theorem \[3\] separately to both methods obtains stability for both methods individually.

Theorem 5.1. Let \( f \) and \( g \) be the grayscale functions of two images of the same dimensions over the same grid of pixels. Let \( \hat{f} \) and \( \hat{g} \) be the corresponding monotone functions on the underlying cubical
complex generated by either Method 1 or 2 (both \( \hat{f} \) and \( \hat{g} \) using the same method). Then

\[
W_p(Dgm(\hat{f}), Dgm(\hat{g})) \leq \left( \sum_{i=0}^{d} 2^{d-i} \binom{d}{i} \right) ||f - g||_p
\]

Proof. Let us suppose we are using Method 1 for constructing our persistence diagrams. As the underlying space is a cubical complex, changing the function value of a maximal cell can affect all of the lower dimensional cells it contains. Each \( d \)-dimensional hypercube contains \( 2^{d-k} \binom{d}{k} \) \( k \)-dimensional hypercubes on its boundary. Summing up over all dimensions yields a bound on how many cell-values change when we change the value of a pixel. Applying Theorem \( 4.8 \) yields the result.

The proof for Method 2 is similar. Changing the function value of a vertex can affect all of the higher dimensional cofaces. There are at most \( 2^k \binom{d}{k} \) possibly affected \( k \)-dimensional cells and applying Theorem \( 4.8 \) completes the proof. \( \square \)

5.2. Stability of persistent homology transforms. The study of persistent homology transforms is a relatively recent development in the persistent homology literature \cite{11, 25, 21} with applications to statistical shape analysis. Given an embedded shape \( M \subset \mathbb{R}^n \), every unit vector \( v \) corresponds to a height function in direction \( v \),

\[
h_v : M \rightarrow \mathbb{R}, \quad h_v : x \mapsto \langle x, v \rangle.
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product. The resulting \( k \)-dimensional persistence diagram computed by filtering \( M \) by the sub-level sets of \( h_v \), is denoted \( \text{Dgm}_k(h_v^M) \). This diagram records geometric information from the perspective of direction \( v \). As \( v \) changes, the persistent homology classes track geometric features in \( M \). The key insight behind the persistent homology transform (PHT) is we do not need to choose a specific direction; by considering the persistent homology from every direction, we obtain a surprising stability of persistent homology transforms.

Definition 5.4. The Persistent Homology Transform PHT of a constructible set \( M \in CS(\mathbb{R}^d) \) is the map \( \text{PHT}(M) : S^{d-1} \rightarrow \text{Dgm}^d \) that sends a direction to the set of persistent diagrams gotten by filtering \( M \) in the direction of \( v \):

\[
\text{PHT}(M) : v \mapsto (\text{Dgm}_0(h_v^M), \text{Dgm}_1(h_v^M), \ldots, \text{Dgm}_{d-1}(h_v^M))
\]

where \( h_v^M : M \rightarrow \mathbb{R}, \ h_v^M(x) = \langle x, v \rangle \) is the height function on \( M \) in direction \( v \). Letting the set \( M \) vary gives us the map

\[
\text{PHT} : CS(\mathbb{R}^d) \rightarrow C^0(S^{d-1}, \text{Dgm}^d),
\]

where \( C^0(S^{d-1}, \text{Dgm}^d) \) is the set of continuous functions from \( S^{d-1} \) to \( \text{Dgm}^d \), the latter being equipped with some Wasserstein \( p \)-distance.

The persistent homology transform is a complete descriptor of constructible sets; for \( M_1, M_2 \subset \mathbb{R}^d \), \( \text{PHT}(M_1) = \text{PHT}(M_2) \) implies \( M_1 = M_2 \) as subsets of \( \mathbb{R}^d \). This was originally proved in \cite{11} for piecewise linear compact subsets in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), and then the more general proof was given in \cite{21} and independently in \cite{28}. Here we restrict ourselves to different embeddings of the same simplicial complex where the embeddings are determined linearly by the placement of the vertices. We call such shapes the geometric vertex embedding of a finite simplicial complex and notably they are always a constructible set.

Definition 5.3. Let \( K \) be a finite simplicial complex with vertex set \( V \). For \( f : V \rightarrow \mathbb{R}^d \) we can define a piece-wise linear extension of \( f : K \rightarrow \mathbb{R}^d \) by setting \( f(\sum a_i v_i) = \sum a_i f(v_i) \). We call \( f : K \rightarrow \mathbb{R}^d \) a geometric vertex embedding of \( K \) if \( f(K) \) is a geometric realisation of \( K \) (i.e. no self-intersections).

We can define a metric on the space of persistent homology transforms by considering the appropriate integrals of Wasserstein distances in each direction. We obtain a different distance for each \( p \in [1, \infty] \)

Definition 5.4. For \( p \in [1, \infty) \), and constructible sets \( M_1, M_2 \subset \mathbb{R}^d \) we can define a \( p \)-PHT distance between \( M_1, M_2 \) by

\[
d_p^{\text{PHT}}(M_1, M_2) = \left( \int_{S^{d-1}} W_p(Dgm(h_v^{M_1}), Dgm(h_v^{M_2}))^p \, dv \right)^{1/p}.
\]
We can use the cellular Wasserstein stability result to prove a stability theorem for the persistent homology transforms of different vertex embeddings of the same simplicial complex.

**Theorem 5.5.** Fix a simplicial complex $K$ with vertex set $V$. Let $C_{p,d} = 2\omega_{d-2} \int_0^{2\pi} \cos^p(\theta) \sin^{d-2}(\theta) d\theta$ where $\omega_{d-2}$ is the area of the unit sphere $S^{d-2}$ and

$$C_K = \max_{v \in V} |\{\sigma \in K | v \in \sigma\}|.$$

Let $f, g : K \to \mathbb{R}^d$ be different geometric vertex embeddings of $K$. Then

$$d_p^{\text{PHT}}(f(K), g(K)) \leq \left( C_K C_{p,d} \sum_{v \in V} \|f(v) - g(v)\|_p^2 \right)^{1/p}.$$

**Proof.** Define functions $k_w^f : K \to \mathbb{R}$ by setting

$$k_w^f(v_0, \ldots, v_n) = \max\{h_w(f(v_0), h_w(f(v_1)), \ldots, h_w(f(v_n))\},$$

and $k_w^g : K \to \mathbb{R}$ similarly. As discussed in [21], the sublevel set filtrations of $k_w^f$ and $h_w^{f(K)}$ have the same persistent homology. Similarly, $k_w^g$ and $h_w^{g(K)}$ give the same sub-level set persistent homology. By Theorem 4.8 we know that

$$W_p(Dgm(k_w^f), Dgm(k_w^g))^p \leq \sum_{\Delta \in K} |k_w^f(\Delta) - k_w^g(\Delta)|^p.$$

For any finite set $X$,

$$\max_{x \in X} f(x) - \max_{y \in X} g(y) \leq \max_{x \in X} |f(x) - g(x)|$$

which implies

$$\sum_{\sigma \in K} |k_w^f(\sigma) - k_w^g(\sigma)|^p \leq \sum_{\sigma \in K} \max_{v \in \sigma} \{|k_w^f(v) - k_w^g(v)|^p\} \leq C_K \sum_{v \in V} |k_w^f(v) - k_w^g(v)|^p.$$

But $k_w^f(v) = \langle w, f(v) \rangle$ and $k_w^g(v) = \langle w, g(v) \rangle$ which implies

$$\sum_{\sigma \in K} |k_w^f(\sigma) - k_w^g(\sigma)|^p \leq C_K \sum_{v \in V} |\langle w, f(v) - g(v) \rangle|^p.$$

Hence

$$d_p^{\text{PHT}}(f(K), g(K))^p = \int_{S^{d-1}} W_p(Dgm(h_w^f), Dgm(h_w^g))^p dw \leq C_K \sum_{v \in V} |\langle w, f(v) - g(v) \rangle|^p dw \leq C_K \sum_{v \in V} \int_{S^{d-1}} |\langle w, f(v) - g(v) \rangle|^p dw.$$

Let $e_1$ denote the vector with 1 in the first coordinate and 0 in every other coordinate. For each $u \in \mathbb{R}^d$ we have $\int_{S^{d-1}} |\langle w, u \rangle|^p dw = \|u\|_2^p \int_{S^{d-1}} |\langle w, e_1 \rangle|^p dw$. This implies

$$d_p^{\text{PHT}}(f(K), g(K))^p \leq C_K \sum_{v \in V} \|f(v) - g(v)\|_2^p \int_{S^{d-1}} |\langle w, e_1 \rangle|^p dw = C_K \sum_{v \in V} \|f(v) - g(v)\|_2^p 2 \int_0^{2\pi} \cos^p(\theta) \omega_{d-2} \sin^{d-2}(\theta) d\theta = C_K C_{p,d} \sum_{v \in V} \|f(v) - g(v)\|_2^p.$$

In particular $C_{p,3} = \frac{4\pi}{p+1}$, $C_{p,2} \leq 2$ for all $p$, and $C_{1,d} = \frac{2\omega_{d-2}}{d-1}$. \qed
5.3. Stability results for Rips complexes. The goal of this section is to bound the change in Dgm(\(\mathcal{R}(P)\)) as the underlying point set \(P\) changes, so we first find the appropriate distance between point sets. We will first state the definition of the Wasserstein distances between measures. This views each point cloud as a sum of point masses. In order for this distance to be defined we require that the point sets have same cardinality. This because if there are different numbers of points then the total masses of the measures are different and no transport plan can be formed between the two measures.

**Definition 5.6.** Let \(P_0\) and \(P_1\) be two finite point sets in \(\mathbb{R}^d\) and assume \(|P_0| = |P_1|\). Define the point set Wasserstein distance between them as

\[
W^\text{point cloud}_p(P_0, P_1) = \inf_{\phi} \left( \sum_{v \in P_0} ||v - \phi(v)||^p \right)^{1/p}.
\]

where \(\phi\) is a bijection.

Since we are dealing with finite sets this definition is equivalent to the classical Wasserstein distance between the measures \(\mu_0\) and \(\mu_1\) where \(\mu_i = \sum_{x \in P_i} \delta_x\).

Before stating our stability results let us first recall some basic definitions.

**Definition 5.7.** Given a point cloud \(P \subset \mathbb{R}^d\), the Vietoris-Rips complex is the simplicial complex \(\mathcal{R}_k(P)\) where a \(k\)-simplex is a subsets of \(k + 1\) points \(\{v_1, \ldots, v_{k+1}\}\) such that \(||v_i - v_j|| \leq \delta\) for all \(i, j = 1, \ldots, k + 1\).

We implicitly use the identification of the vertices of \(\mathcal{R}(P)\) and the points of \(P\). By varying \(\delta\), we obtain a filtration.

**Definition 5.8.** The Vietoris-Rips filtration (or simply Rips filtration) of a point set \(P\) is the filtration \(\{\mathcal{R}_k(P)\}\) induced by ranging \(\delta\) from 0 to \(\infty\). The corresponding persistence diagram is denoted Dgm(\(\mathcal{R}(P)\)).

**Theorem 5.9.** Fix \(M > 0\). For all \(p \geq 1\), for all \(k\), and all point clouds \(P_0, P_1\) with \(|P_0|, |P_1| = M\) we have

\[
W_p(\text{Dgm}_k(\mathcal{R}(P_0)), \text{Dgm}_k(\mathcal{R}(P_1))) \leq 2 \left( \frac{M - 1}{k} \right)^{1/p} W^\text{point cloud}_p(P_0, P_1)
\]

where \(\text{Dgm}_k(\mathcal{R}(P_0))\) and \(\text{Dgm}_k(\mathcal{R}(P_1))\) are the \(k\)-dimensional persistence diagrams for the Vietoris-Rips filtration on the point sets \(P_0\) and \(P_1\) respectively. Furthermore,

\[
W_p(\text{Dgm}(\mathcal{R}(P_0)), \text{Dgm}(\mathcal{R}(P_1))) \leq 2^{M/p+1} W^\text{point cloud}_p(P_0, P_1).
\]

**Proof.** Let \(\phi: P_0 \to P_1\) be a bijection which achieves the minimum of

\[
W^\text{point cloud}_p(P_0, P_1) = \inf_{\phi} \left( \sum_{v \in P_0} ||v - \phi(v)||^p \right)^{1/p}.
\]

Relabel the points in \(P_0 = \{x_1, \ldots, x_M\}\) and \(P_1 = \{y_1, \ldots, y_M\}\) so that \(\phi(x_i) = y_i\). Let \(K\) be the complete simplicial complex on \(M\) vertices \(\{v_1, \ldots, v_M\}\). Define a functions \(f, g: K \to \mathbb{R}\) by \(f([v_{i_0}, v_{i_1}, \ldots, v_{i_k}])\) the time when \([x_{i_0}, x_{i_1}, \ldots, x_{i_k}]\) is included in \(\mathcal{R}(P_0)\) and \(g([v_{i_0}, v_{i_1}, \ldots, v_{i_k}])\) the time when \([y_{i_0}, y_{i_1}, \ldots, y_{i_k}]\) is included in \(\mathcal{R}(P_1)\).

Suppose for now that \(k \geq 1\). Then

\[
|f([v_{i_0}, v_{i_1}, \ldots, v_{i_k}]) - g([v_{i_0}, v_{i_1}, \ldots, v_{i_k}])| = \max_{j,l} \{||x_j - x_{i_l}|| - ||y_j - y_{i_l}||\} \leq \max_{j,l} ||x_j - x_{i_l}|| - ||y_j - y_{i_l}||.
\]

By the triangle inequality \(||x_j - x_{i_l}|| - ||y_j - y_{i_l}|| < ||x_j - y_j|| + ||x_{i_l} - y_{i_l}||\). This implies

\[
|f([v_{i_0}, v_{i_1}, \ldots, v_{i_k}]) - g([v_{i_0}, v_{i_1}, \ldots, v_{i_k}])| \leq \max_{j,l} \{||x_j - y_j|| + ||x_{i_l} - y_{i_l}|| \leq 2 \max_{j,l} \{||x_j - y_j||\}
\]

Since \(K\) is the complete simplicial complex over \(M\) vertices, each edge \([v_i, v_j]\) appears in \(\binom{M-2}{k-1}\) \(k\)-simplices (we only need to decide which extra \(k - 1\) vertices to include).
Using the cellular stability theorem,
\[ W_p(Dgm_k(R(P_0)), Dgm_k(R(P_1))) \leq \sum_{i} \left( \frac{M - 2}{k - 1} \right)^2 \| x_i - y_i \|^p + \sum_{i} \left( \frac{M - 2}{k} \right)^2 \| x_i - y_i \|^p \]
\[ \leq 2^p \left( \frac{M - 1}{k} \right)^2 \text{Wasserstein}_{\text{point cloud}}(P_0, P_1)^p \]

For \( k = 0 \) the calculations are even easier as the vertex values are all 0.
\[ W_p(Dgm_0(R(P_0)), Dgm_0(R(P_1))) \leq \sum_{i<j} |f([v_i, v_j]) - g([v_i, v_j])|^p \]
\[ = \sum_{i<j} \| x_i - x_j \| - \| y_i - y_j \| |x_i - y_i|^p \]
\[ \leq \sum_{i<j} (2 \| x_i - y_i \|^p) \]
\[ = 2^p \text{Wasserstein}_{\text{point cloud}}(P_0, P_1)^p \]

To prove the second part, we again use the cellular stability theorem to compute
\[ W_p(Dgm_k(R(P_0)), Dgm_k(R(P_1))) \leq \sum_{k=0}^{M} \sum_{i} \left( \frac{M}{k} \right)^2 \| x_i - y_i \|^p \]
\[ \leq 2^p 2^M \text{Wasserstein}_{\text{point cloud}}(P_0, P_1)^p. \]

\[ \square \]

6. Consequences for topological summaries

A common theme in Topological Data Analysis is to use topological invariants to process and summarise input data. We will call the features we compute via topology as topological summaries. Many topological summaries have associated metrics that allow us to form metric spaces (or sometimes pseudo-metric spaces). Most of this paper concerns the topological summary of the persistence diagram, and topological summaries have associated metrics that allow us to form metric spaces (or sometimes pseudo-metric spaces). Most of this paper concerns the topological summary of the persistence diagram, and the space of persistence diagrams \( \mathcal{D} \) can be equipped with any of the \( p \)-Wasserstein metrics. Most examples of topological summaries contain the same information as persistence diagrams (such as persistence images) or strictly less (such as Betti curves) and as such we can form a functions from the space of persistence diagrams to this corresponding space of this topological summary. For example, we can compute the persistent image from a persistence diagram by placing a Gaussian kernel on each point in each persistence diagram.

Stability results for topological summary statistics computed from persistent homology bound the distance between the summaries from above in terms of the \( p \)-Wasserstein distance of the corresponding persistence diagrams, most often using 1-Wasserstein distance between the input persistence diagrams. As \( W_1(X, Y) \geq W_p(X, Y) \) for all \( p \geq 1 \), the 1-Wasserstein distance provides the largest upper bound on the distance between topological summaries amongst the Wasserstein metrics. However, most bounds on the distance between persistence diagrams in terms of geometric measures of the difference, e.g. Hausdorff distance, are with respect to the bottleneck distance, i.e. the \( \infty \)-Wasserstein distance. This provides the smallest lower bound with respect to the distortion measure of the input. Therefore, these results cannot be combined to provide a stability result. However, as we provide upper bounds on the 1-Wasserstein distance between the diagrams, we obtain immediate corollaries for stability results of these topological summary statistics in terms of the input data. In this section, we present some of the positive results that follow.
Corollary 6.1. Let $(X,d)$ be a metric space of topological summaries which are computable from persistence diagrams and $T : D \to X$ the function that computes this topological summary from a persistence diagram. Suppose that there exists a $C_T > 0$ such that
\[
d(T(X), T(Y)) \leq C_T W_1(X, Y)
\]
for all persistence diagrams $X, Y$. If $f, g$ are monotone functions over cellular complex $K$, with $T(Dgm(f))$ and $T(Dgm(g))$ the corresponding topological summaries for the sub-level filtrations of $f$ and $g$ respectively then
\[
d(T(Dgm(f)), T(Dgm(g))) \leq C_T \|f - g\|_{K,1}.
\]

The proof follows directly from the earlier stability results in this paper.

It can be directly applied to a number of topological summaries already in the literature where the condition of $d(T(X), T(Y)) \leq C_T W_1(X, Y)$ for all persistence diagrams $X, Y$ has already been established. This includes

- (1) sliced Wasserstein kernel, $C_T = 1$, see [11]
- (2) persistent images, $C_T = 1$, see [1]
- (3) persistent scale space, $C_T = 1$, see [35, 31]
- (4) weighted Betti curves see [30, 42, 16]
- (5) learned/optimized representations [28, 29],
- (6) persistent homology rank function [36], $C_T = 1$ (see Corollary 6.5).

Related results for topological summaries constructed via grey scale images, the persistent homology transform, and Rips complexes follow from the theorems in Sections 5.

In the rest of this section, we examine in more detail Lipschitz stability as relates to linear representations of persistence diagrams, providing necessary conditions. We also consider persistence landscapes which are one of the most common forms of non-linear representations. We prove negative Lipschitz stability results for all $L^p$ function norms of persistence landscapes where $p < \infty$.

6.1. Linear representations of persistence diagrams. A growingly common form of topological statistic are linear representations of persistence diagrams. Examples of linear representations include persistence images, persistent rank functions and weighted Betti curves. We view persistence diagrams as measures over the plane (and call these persistence measures) and then have a function from the plane to some Banach space. The resulting linear representation is the integral of these functions over the persistence measure.

Definition 6.2. Let $B$ be a Banach space. A linear representation is a function $\Phi : D \to B$ such that $\Phi(\mu) = \int_{\mathbb{R}^2} f(x)d\mu(x)$ for some $f : \mathbb{R}^2+ \to B$. Here we view persistence diagram $X$ as a measure $\mu_X = \sum_{x \in X} \delta_x$.

As these topological summaries lie in Banach spaces, often even Hilbert spaces, the number of statistical methods available for analysis increases. Often these constructions of linear representations are justified as maintaining relevant persistence homology information because of stability with respect to 1-Wasserstein distances of the original persistence diagrams.

Lipschitz stability with respect to 1-Wasserstein distance of diagrams has been shown for a number of linear representations, see for example persistence scale space kernel [35] and persistence images [10]. Related theoretic bounds for distances between general linear representations are in [22], where they prove conditions when linear representations are continuous with respect to Wasserstein distances and provide a Lipschitz bound for the supremum norm between linear representations in terms of the 1-Wasserstein distance. hence, in this section, we focus on Lipschitz stability for linear representations into general Banach spaces.

For completeness we recall the necessary and sufficient conditions for Lipschitz stability. Note that all the $L_q$ metrics over $\mathbb{R}^2+ \cup \Delta$ are bi-Lipschitz equivalent up to a slight change in constant. For the sake of clarity we will restrict the case of the $L_1$ metric on $\mathbb{R}^2+ \cup \Delta$.

Theorem 6.3. Let $\Phi : D \to B$ be a non-trivial linear representation constructed via $f : \mathbb{R}^2+ \to B$. Then $\Phi$ is Lipschitz continuous with respect to $W_p$ with constant $C$ if and only if $p = 1$ and $f$ is Lipschitz continuous with constant $C$ and limiting to 0 on the diagonal.

Proof. Let us first assume that $\Phi : D \to B$ is Lipschitz continuous with respect to $W_p$ with constant $C$. We will show that $p = 1$ by way of contradiction. Let $x \in \mathbb{R}^2+ \text{ with } f(x) \neq 0$. Set $X$ to be the
persistence diagram consisting of \( k \) copies of \( x \), and \( Y \) the persistence diagram containing no off-diagonal points. Now
\[
W_p(X,Y) = (k\|x - \Delta\|_p^p)^{1/p} = k^{1/p}\|x - \Delta\|_p.
\]
In contrast \( \|\Phi(X) - \Phi(Y)\|_B = \|\Phi(X)\|_B = \|k \cdot f(x)\|_B = k\|f(x)\|_B. \)

By assumption we have
\[
k\|f(x)\|_B \leq Ck^{1/p}\|x - \Delta\|_p
\]
for all \( k \) which clearly creates a contradiction if \( p > 1 \).

From now on we set \( p = 1 \). To show \( f \) is Lipschitz, let \( x, y \in \mathbb{R}^{2+} \cup \Delta \) and set \( X \) and \( Y \) to be the persistence diagrams containing only the diagonal alongside \( x \) and \( y \) respectively. We have
\[
\|f(x) - f(y)\|_B = \|\Phi(x) - \Phi(y)\|_B \leq \text{CW}_1(X, Y) \leq C\|x - y\|_1
\]
where the first inequality follows by assumption and the second because \( \phi(x) = y \) determines a matching (which may not necessarily be optimal).

To prove the other direction, suppose \( \|f(x) - f(y)\|_B \leq C\|x - y\|_1 \) for all \( x, y \in \mathbb{R}^{2+} \cup \Delta \) and \( f(\Delta) = 0 \). Let \( X, Y \) be persistence diagrams and let \( M \) be a matching between them.
\[
\|\Phi(X) - \Phi(Y)\|_B = \left\| \sum_{x \in X} f(x) - \sum_{y \in Y} f(y) \right\|_B \leq \sum_{(x,y) \in M} \|f(x) - f(y)\|_B \\
\leq \sum_{(x,y) \in M} \|f(x) - f(y)\|_B \leq C \\|x - y\|_1
\]
This holds for all matchings \( M \) and hence \( \|\Phi(X) - \Phi(Y)\|_B \leq \text{CW}_1(X, Y) \).

**Definition 6.4.** We define the \( k \)-th dimensional persistent homology rank function corresponding to the filtration \( K \) to be:
\[
\beta_k(K) : \mathbb{R}^{2+} \to \mathbb{Z} \\
(a, b) \mapsto \text{rk} \text{Im}(H_k(K_a) \to H_k(K_b))
\]
where \( K_a \) is the filtration at \( a \). Given a weighting function \( \phi \) over \( \mathbb{R}^{2+} \) we can define an \( L^q \) function distance function by
\[
d_q(f, h) = \left( \int_{x<y} |f - h|^q \phi(y - x) \, dx \, dy \right)^{1/q}
\]
Following \( [35] \) we use \( \phi(t) = e^{-t} \).

Persistent homology rank functions lie in the space of Banach space real valued functions over \( \mathbb{R}^{2+} \) viewed as a measure space with density function \( \phi(y - x) \). The \( q \)-norm for this Banach space is \( \|g\|_q^q = \int_{x<y} |g|^q \phi(y - x) \, dx \, dy \).

Although each persistent homology rank function can only have integer values at each point in \( \mathbb{R}^{2+} \) we wish to consider the larger space of real valued functions in order to be able to perform statistical methods, such as computing means, or allowing rescaling for normalisation.

**Corollary 6.5.** Persistent homology rank functions with \( L^q \) weighted metric are Lipschitz continuous with respect to the \( p \)-Wasserstein distances between diagrams if and only if \( q = p = 1 \). In this case, the Lipschitz constant is 1.

**Proof.** We can see that the persistent homology rank function is a linear representation of persistence diagrams. Define \( f : \mathbb{R}^{2+} \to B \) by \( f(a, b) = 1_{\{(x,y) : a \leq x \leq y \leq b\}}. \) Then we can observe that for any diagram \( X \) we have \( \beta(X) = \sum_{x \in X} f(x). \)

Since persistent homology rank functions are linear representations we can apply Theorem 6.3. We automatically get the requirement that \( p = 1 \). We will next show that we will also need \( q = 1 \) through a counterexample.

Let \( x_1 \leq x_2 \leq y \) and consider the persistent homology rank functions constructed from the persistence diagrams containing a single off-diagonal point \((x_1, y)\) and \((x_2, y)\) respectively. Note that the 1-Wasserstein distance between these persistence diagrams is \( x_2 - x_1 \). The \( q \)-distance between these persistent homology rank functions is
\[
\|f(x_1, y) - f(x_2, y)\|_q = \left( \int_{x_1}^{x_2} \int_t^y e^{t-s}ds \right)^{1/q} \, dt = ((x_2 - x_1) - e^{x_2-y} + e^{x_1-y})^{1/q}.
\]
For each fixed $x_1, x_2$ we can consider the limit as $y$ goes to infinity. This limit is
$$\lim_{y \to \infty} \|f(x_1, y) - f(x_2, y)\|_q = (x_2 - x_1)^{1/q}$$

If the persistent homology rank functions with $L^q$ weighted metric are Lipschitz continuous then there is some $C$ such that for all $x_1 \leq x_2 < y$ we have $\|f(x_1, y) - f(x_2, y)\|_q \leq C(x_2 - x_1)$. Combining with the limit above we have $(x_2 - x_1)^{1/q} \leq C(x_2 - x_1)$ for all $x_1 < x_2$. If $q > 1$ there is no constant $C$ that will satisfy this and thus by contradiction we see that is the persistent homology rank functions with $L^q$ weighted metric are Lipschitz continuous with respect to the $p$-Wasserstein distances between diagrams then $q = 1$.

All that remains to be shown is that for $f : \mathbb{R}^2+ \to B$ by $f(a, b) = 1_{(x, y) : a \leq x \leq y \leq b}$ we have
$$\|f(x_1, y_1) - f(x_2, y_2)\|_1 \leq |x_1 - x_2| + |y_1 - y_2|.$$  
Without loss of generality assume $x_1 \leq x_2$. If $x_2 \leq y_1$ then
$$\|f(x_1, y_1) - f(x_2, y_2)\|_1 \leq \|f(x_1, y_1) - f(x_2, y_1)\|_1 + \|f(x_2, y_1) - f(x_2, y_2)\|_1.$$  
Using the integrals above we see that $\|f(x_1, y_1) - f(x_2, y_1)\|_1 \leq |x_2 - x_1|$ and analogously that $\|f(x_2, y_1) - f(x_2, y_2)\|_1 \leq |y_2 - y_1|$. Together they imply that $\|f(x_1, y_1) - f(x_2, y_2)\|_1 \leq (x_1 - y_1) - (x_2 - y_2)$.

If $x_2 > y_1$ then the supports of $f(x_1, y_1)$ and $f(x_2, y_2)$ are disjoint. Routine calculations show that $\|f(x_1, y_1) - f(x_2, y_2)\|_1 \leq |y - x|$. In this scenario, $(x_1, y_1) - (x_2, y_2) \geq |y_1 - x_1| - |y_2 - x_2|$ and hence $\|f(x_1, y_1) - f(x_2, y_2)\|_1 \leq \|f(x_1, y_1) - (x_2, y_2)\|_1.$

\section{6.2. Persistence Landscapes are Not Lipschitz Stable.}

Landscapes \cite{landscapes2017} were among the first functional proposals for persistence diagrams and remain among the most popular in practice.

\begin{definition}
The persistence landscape of persistence module $M$ is the function $\lambda : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ defined by $\lambda(k, t)(M) = \max\{h \geq 0 \mid \text{rk}(M(t - h \leq t + h)) \geq k\}$.

We call $\lambda(k, \cdot)(M)$ the $k$-th persistence landscape.
\end{definition}

The $L^q$ distance between persistence landscapes is defined as the sum over $k$ of the $L^q$ distances between the $k$-th persistence landscape. Let $p_k(f)$ denote the $k$-th persistence landscape for sublevel persistence diagram for $f$.

Unlike the linear functionals in the previous subsection, there is no Lipschitz nor even Hölder stability with respect to the $p$-Wasserstein distances of their corresponding persistence diagrams.

\begin{theorem}
Let $(\mathcal{D}, W_p)$ denote the space of persistence diagrams with the $W_p$ metric and let $(PL_L^q)$ denote the space of persistence landscapes with the $L^q$ metric. For all $q \in [1, \infty)$, the function $p : (\mathcal{D}, W_p) \to (PL, L^q)$ which sends each persistence diagram to its corresponding persistence landscape is not Hölder continuous.
\end{theorem}

\begin{proof}
Let $X$ and $Y$ be the persistence diagrams with one off-diagonal point at $(0, a)$ and $(0, a - r)$ respectively, where $r \ll a$. The first persistence landscapes for $p(X)$ and $p(Y)$ are both a triangle function. These are centred at $a/2$ and $(a - r)/2$ respectively. We can compute that $p(X) - p(Y)$ is a trapezium shape:

\begin{equation*}
(p(X) - p(Y))(t) = \begin{cases} 
  t & \text{for } t \in [(a - r)/2, a/2] \\
  r & \text{for } t \in [a/2, a - r] \\
  a - t & \text{for } t \in [a - r, a] \\
  0 & \text{otherwise}
\end{cases}
\end{equation*}

When $a \gg r$, the contribution of the integral over $[a/2, a - r]$ will dominate the $L^q$ distance between $p(X)$ and $p(Y)$. The function distance is bounded below by

$$\|p(X) - p(Y)\|_q > \left( \int_{a/2, a - r} r^q \, dt \right)^{1/q} = (a/2 - r)^{1/q}.$$ 

We also know that for $r \ll a$ the optimal matching between $X$ and $Y$ sends the point at $(0, a)$ to $(0, a - r)$ and hence $W_p(X, Y) = r$ for all $p \in [1, \infty]$. For a Hölder stability result to hold we would need there to be $a, C > 0$ such that $\|p(X) - p(Y)\|_q \leq CW_p(X, Y)^a$ for all $X, Y \in \mathcal{D}$.

This would imply

$$r(a/2 - r)^{1/q} \leq Cr^a$$

for all $a \gg r$. 

By setting \( r \) small and \( a \) large we can make the left hand side arbitrarily large and the right hand side arbitrarily small which provides a contradiction regardless of the choice of \( q, C \) and \( \alpha \). This means there cannot be any Hölder continuity when \( q \neq \infty \). \( \square \)

**Porism 6.8.** Let \( M \) be a simplicial complex containing at least one edge. Let \((X, L^p)\) denote the space of monotone functions over \( M \) with the \( L^p \) metric. For all \( p, q \in [1, \infty] \), the function \( PL : (X, d_{L^p}) \to (pl, L^q) \) which sends each function to the persistence landscape of its sublevel set filtration is not Hölder continuous.

**Proof.** We prove by creating an example of a pair of function that produce the persistence diagrams in Theorem 6.7. Fix an edge \([x_1, x_2]\) in \( M \). Set \( f([x_1]) = 0, f([x_2]) = 0, g([x_1, x_2]) = a - r \) and \( g(\tau) = a \) for all other cells \( \tau \in M \). Note that \( ||f - g||_p = r \) for all \( r \in [0, a] \). The persistence diagram of the sublevel set filtrations of \( f \) and \( g \) are the same inequalities as before. \( \square \)

**Remark 6.9.** It is worth noting that [6] does have a limited version of Wasserstein stability using [18]. This corollary states that for \( X \) a triangulable, compact metric space that implies bounded degree-k total persistence for some real number \( k \geq 1 \), and \( f, g \) two tame Lipschitz functions we have

\[
||PL(f) - PL(g)||_p \leq C ||f - g||_{\infty}^\frac{k}{p}
\]

for all \( p \geq k \), where

\[
C = C_{X,k} ||f||_\infty (\text{Lip}(f))^k + \text{Lip}(g)^k + C_{X,k+1} \frac{1}{p+1} (\text{Lip}(f)^{k+1} + \text{Lip}(g)^{k+1})
\]

See Section 3 for some limitations in terms of \( k \) and \( C_{X,k} \).

7. **Algebraic Wasserstein Stability**

In this section we give an approach to Wasserstein stability at the algebraic level. For bottleneck distance this has yielded important insights, and this new approach opens the possibility to studying Wasserstein stability without an underlying chain complex as well as relaxing the finiteness conditions. There are two significant obstructions. A key tool in understanding bottleneck stability is the concept of interleaving between persistence modules. The corresponding interleaving distance naturally corresponds to a sup-norm as morphisms must be defined over the whole space, and also the representation of an interleaving by a single morphism \([2]\) cannot be used since its effect on Wasserstein distance cannot be controlled. Instead, we construct an alternative intermediate object which interpolates between the two persistence modules without reference to the underlying chain complex and filtration. The section is organized into four parts: in the first part, we give our definition of algebraic distance and show that it is equivalent to the diagram distance. We then prove an additional bound on extensions of persistence modules which may be of independent interest, we rederive the cellular stability result in the algebraic setting, i.e. two functions over one chain complex (not necessarily finite). Finally, we discuss how the algebraic distance could be defined in other contexts. For convenience we recall the following set up from Section 2:

- We are concerned with the category of p.f.d. persistence modules with a common parameterization.
- We only consider ungraded morphisms between persistence modules, i.e. given a morphism \( f : \mathcal{A} \to \mathcal{B} \), for any \( x \in \mathcal{A}_\alpha \), \( x \mapsto f(x) \in \mathcal{B}_\alpha \), where \( x \) denotes a basis element.
- For readability, we mention freely generated submodules, i.e. infinite intervals, only where necessary to avoid confusion. As p.f.d. modules have decompositions, the modules may be split into summands consisting of either only finite or only infinite intervals, these two cases can be dealt with separately, or the resulting distance is infinite and the desired results hold trivially.

Throughout this section we make extensive use of the algebraic properties of persistence modules as well as standard constructions such as short exact sequences. For a more in-depth description of the algebraic structure of persistence modules see [2] and for a discussion of short exact sequences of persistence modules, see [27]. Turning to the Wasserstein distance, recall that the definition for persistence diagrams is defined as

\[
W_p(Dgm(A), Dgm(B)) = \inf_{M} \left( \sum_{x \in Dgm(A)} |b(x) - b(M(x))|^p + |d(x) - d(M(x))|^p \right)^{\frac{1}{p}}
\]
Each matching gives a transport map, moving points from \( \text{Dgm}(A) \) to \( \text{Dgm}(B) \) or to the diagonal (or from the diagonal to \( \text{Dgm}(B) \)). To avoid confusion, we will refer to the above as the diagram or matching distance, in contrast to the algebraic distance. A matching that achieves the infimum is called an optimal transport map.

**Remark 7.1.** Note that a distance of zero does not imply isomorphism of modules, as points on the diagonal do not contribute to the norm as they are zero length. Furthermore, an infinite distance is also possible. Hence, the \( p \)-Wasserstein distance is an extended pseudo-distance on the set of persistence modules. It can be made into an extended distance by considering the observable category of persistence modules \([12]\). We refer the reader to \([9]\) for a more complete account of different variations of persistence modules. For readability, we refer to \( W_p \) simply as a distance.

Our goal is to define an algebraic distance which yields the same distance in the case of interval decompositions. Our approach is similar in spirit to the single morphism characterization of interleaving from \([2]\). Unfortunately, a matching is generally not realizable as an ungraded morphism. This is overcome in \([2]\) by employing a shift operator which ensures that a morphism exists. This approach cannot be applied in our setting as the shift operator will incur a transport cost which is proportional to the number of points in the diagram. So rather than look for a single morphism, we look for an intermediate object with morphisms to the modules we are comparing.

**Definition 7.2.** A span of two persistence modules \( A \) and \( B \) is a triple \((C, \varphi, \psi)\) in a diagram

\[
\begin{array}{c}
C \\
\varphi \downarrow \\
A \\
\psi \\
B
\end{array}
\]

This idea has appeared in the persistence literature several times, e.g. \([13]\). There exists a dual notion with morphisms from \( A \) and \( B \) to a third module (cospan).

**Definition 7.3.** We define the class of zero modules, denoted \( 0 \), as any module which has no intervals of positive length in its decomposition.

**Remark 7.4.** All modules in \( 0 \) modules are at a Wasserstein distance of 0. As an abuse of notation, we will use the notation \( 0 \) when referring to a module in this class.

We recall the definition of the \( p \)-norm of a persistence module (Definition \([2,9]\)).

\[
\|A\|_p = \left( \sum_{x \in \text{Dgm}(A)} \ell(x)^p \right)^{\frac{1}{p}}.
\]

This is directly related to the \( p \)-Wasserstein diagram distance to the trivial or \( 0 \) persistence module. The transport distance to the diagonal matches to the midpoint of each bar for all \( p \geq 1 \). This introduces a multiplicative factor when relating the two.

\[
W_p(\text{Dgm}(A), 0) = 2^{-\frac{1}{2p}} \|A\|_p.
\]

We will phrase the results in terms of \( \|A\|_p \) to avoid extra constants in the expressions.

**Remark 7.5.** In the above definition, if a module has any infinite intervals then they have an infinite norm. Our statements still hold in these cases, as it may happen modules with infinite norms have a finite distance.

We now define the main quantity which we study in section.

**Definition 7.6.** Let \((C, \varphi, \psi)\) be a span between two persistence modules \( A \) and \( B \). Given a \( p \)-norm on modules, we define the transport cost of the span to be

\[
ce(\varphi, \psi) = \| \ker \varphi \oplus \text{coker} \varphi \oplus \ker \psi \oplus \text{coker} \psi \|_p.
\]

We may now define the algebraic Wasserstein distance:

**Definition 7.7.** The algebraic \( p \)-Wasserstein distance between persistence modules is

\[
W^\text{alg}_p(A, B) = \inf_{(C, \varphi, \psi)} e(\varphi, \psi).
\]
Our main result here will be to show that in the case of p.f.d. modules, the algebraic and diagram Wasserstein distances agree. It is non-trivial that the above actually defines a distance. In the case of p.f.d. modules, the equivalence we prove implies this result. However, in Section 7.3 we provide sufficient algebraic conditions for $W^\text{alg}_p(\mathcal{A}, \mathcal{B})$ to define a distance.

We begin with the following qualitative result – note that this holds for arbitrary p.f.d. modules with finite p-norm.

**Lemma 7.8.** If $\mathcal{A} \rightarrow \mathcal{B}$ or dually $\mathcal{B} \rightarrow \mathcal{A}$ implies $||\mathcal{A}||_p \leq ||\mathcal{B}||_p$.

**Proof.** Recall that $\mathcal{A} \rightarrow \mathcal{B}$ implies that there is an injective set map $f_*$ from the intervals of $\mathcal{A}$ to the intervals of $\mathcal{B}$. This is equivalent to a set map $\text{Dgm}(\mathcal{A}) \rightarrow \text{Dgm}(\mathcal{B})$.

$$||\mathcal{A}||_p^p = \sum_{x \in \text{Dgm}(\mathcal{A})} (d(x) - b(x))^p \leq \sum_{x \in \text{Dgm}(\mathcal{A})} (d(f_*(x)) - b(f_*(x)))^p \leq \sum_{x \in \text{im} f_*} (d(f_*(x)) - b(f_*(x)))^p + \sum_{x' \in \text{B} - \text{im} f_*} (d(x') - b(x'))^p \leq ||\mathcal{B}||_p^p.$$

Note that since $f_*$ is a set map, $\text{im} f_* \subseteq \text{Dgm}(\mathcal{B})$.

Every interval in $\mathcal{A}$ maps to an interval which is has the same death time but must have an equal or earlier birth time (the first inequality). This follows from the fact that $f$ is a homomorphism of persistence modules. Hence, each term in the summation for $\mathcal{A}$ is dominated by the corresponding term in $\mathcal{B}$, implying the inequality. The proof for $\mathcal{B} \rightarrow \mathcal{A}$ is similar as there is an injective set map from the intervals of $\mathcal{A}$ to the intervals of $\mathcal{B}$.

To obtain quantitative bounds, we use the fact that we can consider short exact sequences as transport maps. We begin by defining two interpolations from a persistence module $\mathcal{A}$ to $\mathcal{0}$: one which sends the births to the deaths and the other sends deaths to births. Let $\mathcal{A}$ be p.f.d. persistence module with the decomposition $\bigoplus_{x} \{b(x), d(x)\}$. We omit reference to the interval types as valid choices can easily be determined.

**Definition 7.9.** For $t \in [0, 1]$, define

$$\mathcal{A}_t = \bigoplus_{x \in \text{Dgm}(\mathcal{A})} \mathbb{I}\{b(x), tb(x) + (1-t)d(x)\}.$$  

We refer this to as the death-birth interpolation.

**Definition 7.10.** For $t \in [0, 1]$, define

$$\mathcal{A}_s = \bigoplus_{x \in \text{Dgm}(\mathcal{A})} \mathbb{I}\{(1-t)b(x) + td(x), d(x)\}.$$  

We refer this to as the birth-death interpolation.

For intuition, in Figure 4 we show two types of interpolations on a point in a persistence diagram. In both cases, $\mathcal{A}_0 = \mathcal{A}$ by construction and $\mathcal{A}_1$ is in the class $\mathcal{0}$ since $\mathcal{A}_1$ only has points on the diagonal (ephemeral classes). We abuse notation and write $\mathcal{A}_t \simeq \mathcal{0}$. In the death-birth interpolation, there is a map $\mathcal{A}_s \rightarrow \mathcal{A}_t$ for $s < t$ and in the birth-death interpolation, there is a map $\mathcal{A}_t \rightarrow \mathcal{A}_s$ for $s < t$.

**Observation 7.11.** Constructing a death-birth or birth-death interpolation requires that the module has finite norm. Hence, we cannot interpolate an infinite interval to $\mathcal{0}$. While somewhat counter-intuitive, a simple calculation shows that the distance between two freely generated modules, i.e. with infinite interval summands, may be finite. However, in this case, the module which we interpolate to $\mathcal{0}$ must consist of only finite intervals.

We first show that the above interpolation can be pushed forward or pulled back via monomorphisms and epimorphisms respectively. These are based on standard constructions [13] and their properties. Some of the statements will be obvious to experts, but we include for completeness.
Lemma 7.12. Given a monomorphism \( \varphi : A \to B \) and the death-birth interpolation on \( A \), for any \( t \in [0, 1] \), there exists \( B_t \) such that the following commutative diagram exists:

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A_t & \xrightarrow{i} & B_t \\
\downarrow & & \downarrow \\
0 & \xrightarrow{j} & \coker \varphi
\end{array}
\]

Furthermore, if there exists a short exact sequence

\[
0 \to A \xhookrightarrow{i} B \xrightarrow{\varphi} C \to 0,
\]

there exists a short exact sequence

\[
0 \to A_t \xhookrightarrow{i} B_t \xrightarrow{\varphi} C \to 0.
\]

Proof. First, we define \( B_t \) via the pushout

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A_t & \xrightarrow{i} & B_t \\
\downarrow & & \downarrow \\
0 & \xrightarrow{j} & \coker \varphi
\end{array}
\]

We observe the outer rectangle commutes since \( \varphi(A) \to \coker \varphi \) is trivial by definition. The upper square commutes by the construction of the pushout and the lower square commutes by the universality of the colimit. Since pushouts preserve epimorphisms, \( j \) is epic. To show that \( i \) is injective, and hence, monic, consider any non-trivial element \( \alpha \in A_t \). There exists a \( \gamma \in A \) such that \( \psi(\gamma) = \alpha \) and since \( \varphi \) is monic, \( \varphi(\gamma) \neq 0 \). Hence for each \( \alpha \), the image of \( i \) is \( (\alpha \oplus \varphi(\gamma))/(\alpha \sim \varphi(\gamma)) \neq 0 \). Through a diagram chase, it can directly be verified that for any two non-trivial elements in \( A_t \), \( \alpha \neq \alpha' \), they do not map to the same equivalence class, showing that \( i \) is injective. To prove the second part of the lemma, we extend the pushout diagram to \( C \).

Since \( \zeta \circ \varphi = 0 \) by exactness, the outer diagram commutes and by the universality of push outs \( h \) exists and is unique. The fact that \( h \) is epic follows from \( \zeta = h \circ j \), and \( \zeta \) is epic by assumption. The inclusion \( \text{im } i \subseteq \ker h \) follows by commutativity \( (h \circ i = 0) \). To show \( \ker h \subseteq \text{im } i \), choose a lift \([x]\) of an element in \( \ker h \) through \( j \). By commutativity, \( \zeta([x]) = 0 \), hence by exactness \([x] \in \text{im } \varphi \). By commutativity, \( j([x]) \) must be contained in \( \text{im } i \), completing the proof. \( \square \)

Lemma 7.13. Given an epimorphism \( \varphi : A \twoheadrightarrow B \) and the birth-death interpolation on \( B \), for any \( t \in [0, 1] \), there exists \( A_t \) such that the following commutative diagram exists:

\[
\begin{array}{ccc}
\ker \varphi & \to & 0 \\
\downarrow & & \downarrow \\
A_t & \xrightarrow{i} & B_t \\
\downarrow & & \downarrow \\
A & \xrightarrow{\varphi} & B
\end{array}
\]

Furthermore, if there exists a short exact sequence

\[
0 \to C \xhookrightarrow{i} A \xrightarrow{\varphi} B \to 0,
\]

there exists a short exact sequence

\[
0 \to C \xhookrightarrow{i} A_t \xrightarrow{\varphi} B_t \to 0.
\]

Proof. First, we define \( A_t \) via the pullback

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A_t & \xrightarrow{i} & B_t \\
\downarrow & & \downarrow \\
C & \xrightarrow{\zeta} & B
\end{array}
\]
We observe the outer rectangle commutes since ker $\varphi \to A$ is trivial by definition. The lower square commutes by the construction of the pullback, and the upper square commutes by the universality of the pullback. Since pullbacks preserve monomorphisms, $j$ is a monomorphism. To show that $i$ is epic, we show that it is a surjection. For any non-trivial element $\alpha \in B_t$, $\psi(\alpha)$ is non-trivial. Since $\varphi$ is epic, there exists $\gamma \in A$ such that $\varphi(\gamma) = \psi(\alpha)$. Hence, there exists a non-trivial element $(\gamma, \alpha) \in A_t$, hence the map is surjective. The proof of the second part of the lemma proceeds as similarly to the proof of Lemma 7.12. Consider the diagram

\[
\begin{array}{ccc}
A \ar[r]^-{\varphi} & B \\
\downarrow^{\psi} & \downarrow^{\psi} \\
A_t \ar[u]_{\iota} & B_t \ar[l]_{j}
\end{array}
\]

By exactness $\varphi \circ \zeta = 0$, hence the outer diagram commutes and by the universality of pullbacks, $h$ exists and is unique. Furthermore, by commutativity, $\zeta = j \circ h$, so $h$ is monic. im $h \subseteq \ker i$ follows from the fact that commutativity implies that $i \circ h = 0$. To show ker $i \subseteq \text{im } h$, consider a non-trivial $x \in \text{ker } i$. Since $j$ is monic, $j(x) \neq 0$, and $\varphi \circ j(x) = 0$ by commutativity. Hence $j(x) \in \text{im } \zeta$ and so we conclude $x \in \text{im } h$. \hfill $\square$

Based on these interpolations, we mirror the proofs in Section 4 to relate the algebraic and matching distances. We use the fact that the norm of the module is equal to the corresponding distance induced by interpolating the module to the 0 module. In the proofs, we make extensive use of the fact that monomorphisms map death times to death times and epimorphisms map birth times to birth times \cite{2, 0, 22}. Hence, there always exists a matching between indecomposables and this matching is unique if the death times (respectively birth times) are unique.

**Lemma 7.14.** Let $A$ be a module consisting of a single non-zero summand $[b, d]$. Given a monomorphism $A \xrightarrow{\varphi} B$ and a death-birth interpolation or dually $B \xrightarrow{\psi} A$ and a birth-death interpolation. There exists a decomposition of $[0, 1]$ into a finite number of disjoint open intervals where

1. the closure of the intervals cover $[0, 1]$,
2. For any $t \in [0, 1]$, $\varphi_{t}$ (or $\psi_{t}$) within each interval the induced matchings ($\varphi_{t}$) or ($\psi_{t}$) are constant.

**Proof.** We first prove the case of a monomorphism – choosing the interval as a basis for $A$, we apply $\varphi$ and obtain the image which is again an interval $[b, d]$. This is a linear combination of generators in $B$ (generators in the interval basis of $B$). The main claim is that this linear combination of generators is finite. This follows from the p.f.d. assumption. For any $s \in (b, d)$, there corresponding vector space is finite dimensional. This follows since all birth times must occur at or before $b$ by the properties of persistence module homomorphisms and all the death times at or before $d$ but after $b$. If this is not the case the morphism does not commute with the internal morphisms of the module. Since all births must come before the first death, it follows that if linear combination were infinite there would exist an $s \in (b, d)$ where $B_{s}$ would be infinite dimensional contradicting the p.f.d. assumption. To complete the proof, we observe that as we change the death time, only this linear combination can be affected (one can directly verify this in the construction), which implies that $\varphi_{t}$ is only non-constant when $d$ has equality with a finite number of the death times.

The proof for the epimorphism is similar. We claim that there is a finite number of bars which map to $a \in A$. Formally say that the space generated by the set $\{g_{i}\}$ as such that $\psi(g_{i}) = a$ is finite dimensional. By the same argument as above we can deduce that the deaths must occur at or after $d$ and all the births at or after $b$, implying that in the corresponding bars the births must come before the first death (which is $d$) since $\psi$ is a persistence homomorphism. \hfill $\square$
Lemma 7.15. Given an epimorphism between p.f.d. persistence modules \( f : A \to B \) such that \( \ker f \) is finitely generated,

\[
W_p(Dgm(A), Dgm(B)) \leq ||\ker f||_p.
\]

Proof. Consider the following short exact sequence

\[ 0 \to \ker f \xrightarrow{\phi} A \xrightarrow{f} B \to 0 \]

It simplifies the argument to add an arbitrarily small positive perturbation to the death times in \( A \) that appear in \( \ker f \), so we can assume that the death times in \( \ker f \) are unique. At each point, we only consider classes in \( \ker f \) which are not on the diagonal. For brevity, we also refer to the indecomposables as bars (rather than intervals or indecomposables).

We consider a death-birth interpolation on \( \ker f \). Let \( f_t : A_t \to B \) as in Lemma 7.12 for \( t = [0, 1] \). By [2] Theorem 4.2, there exists an injective set map between the summands of \( \ker f \). Denote this map \( \phi_t \). The proof mirrors the proof of the cellular stability theorem. Consider an interval \( t \in (a_i, a_{i+1}) \), such that the ordering of the death times in \( A_t \) does not change between \( a_i \) and \( a_{i+1} \). Therefore, \( \phi_t \) is a fixed set map for all \( t \in (a_i, a_{i+1}) \).

As only the death times in the image of \( \phi \) change from \( A_{a_i} \) to \( A_{a_{i+1}} \),

\[
W_p(Dgm(A_{a_i}), Dgm(A_{a_{i+1}})) \leq \sum_{x \in Dgm(\ker f)} (d(\phi_t(x)^{a_i}) - d(\phi_t(x)^{a_{i+1}}))^p
\]

The death time for each bar is linear as a function of \( t \). The proof mirrors the proof of the cellular case. Now we note that the interpolation on \( \ker f \) is linear so we can consider intervals where the ordering is consistent.

Assuming there are finitely many such intervals which cover \([0, 1]\) except at a finite number of times, where we have equality of death times. These can be covered taking one-sided limits. Hence summing up over the intervals, just as in Theorem 4.8, we obtain an upper bound of

\[
W_p(Dgm(\ker f_0), Dgm(\ker f_1)) = ||\ker f||_p.
\]

Since \( W_p(Dgm(A_1), Dgm(B)) = 0 \), it follows that Wasserstein distance can only be smaller, yielding the result.

If \( A \) is finitely generated, the set of death times is finite and so the existence of the decomposition of \([0, 1]\) into intervals where the matching is consistent is immediate, just as in Theorem 4.8. However, if \( A \) is p.f.d., we are only guaranteed to have a countable number of death times, however we will show it is sufficient for \( \ker f \) to be finitely generated. This follows from Lemma 7.14. Since each bar may only introduce a finite number of non-constant points and there are a finite number of bars in \( \ker f \) (by the assumption it is finitely generated), the result follows.

There are two subtleties which we point out in the proof above:

1. If \( A \) is freely generated, the epimorphism is either an isomorphism or the distance is infinite. From the perspective of persistence diagram this is correct, as the epimorphism indicates that the birth times are the same, but the fact that the morphism is not an isomorphism implies that the essential classes are not the same and so an infinite distance is appropriate.

2. It is insufficient in the above proof for \( \ker f \) to be p.f.d. If \( \ker f \) contains a countable rather than finite number of bars (finitely generated), it is straightforward to construct examples where open intervals in \([0, 1]\) where \( \phi_t \) is constant do not exist, even though each bar only has a finite number of “pairing switches,” the proof relies on doing the interpolation entire module at once – hence there may still be an infinite number of “pairing switches.” We remove the requirement of \( \ker f \) to be finitely generated in Lemma 7.21.

We also have the dual statement.

Lemma 7.16. Given a monomorphism between p.f.d. persistence modules \( f : A \hookrightarrow B \), where \( \text{coker } f \) is finitely generated,

\[
W_p(Dgm(A), Dgm(B)) \leq ||\text{coker } f||_p.
\]

\(^4\)We avoid using the term intervals here, to avoid confusion with the interval decomposition of the modules.
Proof. The proof is dual to Lemma 7.15 using a birth-death interpolation on coker $f$. The proof of finiteness is also dual, with only a finite number of bars in $B$ mapping to a single bar in coker $f$, by the p.f.d. assumption. The remainder of the proof is the same with the only difference on equality of birth rather than death times.

Notice that if $A$ is free, the monomorphism implies that $B$ must also be free. Furthermore, the cokernel precisely captures the difference in birth times.

Porism 7.17. Given a monomorphism between freely generated p.f.d. persistence modules $f : A \hookrightarrow B$,

$$W^p_p(Dgm(A), Dgm(B)) \leq ||\text{coker } f||_p.$$  

Proof. If the cokernel contains an infinite interval, then the norm is infinite and the statement holds trivially. If $||\text{coker } f||_p$ is finite, then the linear interpolation in the proof of Lemma 7.16 may be applied even in the case of countably many intervals. This is because as $B$ is freely generated, the induced matching remains consistent throughout the interpolation and so there need only be a countable, rather than finite number of intervals.

Remark 7.18. We do not explicitly deal with a module which contain both finite and infinite intervals. However, since we restrict to p.f.d. modules and a decomposition exists, we can study the matchings between only finite intervals or only infinite intervals. This is because while a morphism can induce a matching between an infinite interval and finite interval, the corresponding distance in this case is infinite.

Remark 7.19. Lemmas 7.15 and 7.16 depend critically on the induced matchings from epic and monic morphisms respectively. While we have chosen to be as explicit as possible in the proof, we note that the results on matchings follow from [2, Theorem 5.7], which shows that while the matchings are functorial in the subcategories where morphisms are either all monomorphisms or all epimorphisms. This enables us to relate the distance of the interpolation to the distance it induces.

We now extend Lemmas 7.16 and 7.15 to remove the assumption of finitely generated kernel and cokernel respectively. We will show that any bounded $p$-norm p.f.d. module can be approximated by a finitely generated module. First, we define the matching norm induced by a monomorphism $f : A \hookrightarrow B$

$$W^f_p(Dgm(A), Dgm(B)) = \left( \sum_{x \in Dgm(A)} (b(x) - b(f_*(x)))^p + \sum_{x' \in Dgm(B)\setminus\text{im } f_*} d(x')^p + b(x')^p \right)^{1/p},$$

where $f_*$ is the injective set map between intervals induced by $f$. Note that as $f$ is a monomorphism the matched death times coincide. Dually given an epimorphism $g : B \twoheadrightarrow A$, the induced matching $p$-cost is

$$W^g_p(Dgm(A), Dgm(B)) = \left( \sum_{x \in Dgm(A)} (d(g^*(x)) - d(x))^p + \sum_{x' \in Dgm(B)\setminus\text{im } g^*} d(x')^p + b(x')^p \right)^{1/p},$$
where \( g^* \) is the injective set map induced by \( g \) (which we remind the reader goes in opposite direction to \( g \)). We note that this intermediate construction is only used in the proof of the lemma below.

**Lemma 7.20.** Consider three p.f.d. persistence modules such that \( A \xrightarrow{f} B \xrightarrow{g} C \) or dually \( C \xrightarrow{g} B \xrightarrow{f} A \).

\[
W_p(Dgm(B), Dgm(C)) \leq W_p^h(Dgm(A), Dgm(C)),
\]

where \( h = g_* \circ f_* \) if \( f \) and \( g \) are epic, and \( h = g^* \circ f^* \) if \( f \) and \( g \) are monic.

**Proof.** The proof is similar to Lemma 7.8. We prove the case of monic \( f \) and \( g \) as the epic case is identical. As the Wasserstein distance is the infimum over all matchings, the matching cost is an upper bound, i.e.

\[
W_p(Dgm(B), Dgm(C)) \leq W_p^g(Dgm(B), Dgm(C)).
\]

To complete the proof we show \( W_p^g(Dgm(B), Dgm(C)) \leq W_p^g f(Dgm(A), Dgm(C)) \). This requires two inequalities. We first show a lower bound for \( W_p^g f(Dgm(A), Dgm(C)) \).

\[
\sum_{x \in Dgm(A)} (|b(x) - b(h(x))|^p) \geq \sum_{x \in Dgm(A)} (|b(f_*(x)) - b(g_* f_*(x))|^p)
\]

(5)

The first inequality follows from the fact that \( b(x) \geq b(f_*(x)) \geq b(h(x)) \) since the set maps are induced by persistence module homomorphisms. The equality is implied by the injectivity of \( f_* \).

Our second inequality states that a matching reduces the overall cost.

\[
\sum_{x \in Dgm(C) \setminus \text{im } h} d(x)^p + b(x)^p = \sum_{x \in \text{im } g_* \setminus \text{im } h} d(x)^p + b(x)^p + \sum_{x \in Dgm(C) \setminus \text{im } g_*} d(x')^p + b(x')^p
\]

(6)

The first equality simply divides the sum into the parts matched by \( g_* \) and those which are unmatched. The next inequality follows for any \( a, b \geq 0 \) and \( p \geq 1 \), \((a - b)^p \leq a^p + b^p \). The last inequality follows since for any \( x' \in Dgm(B) \), \( d(g_*(x')) = d(x') \geq b(x') \geq b_*(x') \).

Together these imply the required inequality.

\[
W_p^g f(Dgm(A), Dgm(C)) = \sum_{x \in Dgm(A)} (|b(x) - b(h(x))|^p) + \sum_{x \in Dgm(C) \setminus \text{im } h} d(x)^p + b(x)^p
\]

by Eq. (6)

\[
\geq \sum_{\hat{x} \in \text{im } f_*} (|b(\hat{x}) - b(g_*(\hat{x}))|^p) + \sum_{x \in Dgm(C) \setminus \text{im } h} d(x')^p + b(x')^p
\]

by Eq. (6)

\[
\geq \sum_{\hat{x} \in \text{im } f_*} (|b(\hat{x}) - b(g_*(\hat{x}))|^p) + \sum_{x \in Dgm(B) \setminus \text{im } f_*} (|b(x') - b(g_*(x'))|^p) + \sum_{x \in Dgm(C) \setminus \text{im } g_*} d(x''|^p) + b(x''^p)
\]

\[
= W_p^g(Dgm(B), Dgm(C)).
\]

where the last equality follows from the observation that the three sums partition \( Dgm(C) \).

**Lemma 7.21.** Given an epimorphism between p.f.d. persistence modules \( f : A \rightarrow B \),

\[
W_p(Dgm(A), Dgm(B)) \leq || \ker f ||_p
\]

**Proof.** If \( \ker f \) does not have bounded \( p \)-norm, the statement holds trivially as the upper bound is infinite. Hence we can assume \( || \ker f ||_p < \infty \). By Lemma 2.12 there exists a finitely generated approximation
to \( \ker f \) which we denote \( \overline{\ker f} \) such that \( W_p(\ker f, \ker f) < \varepsilon \). There exists a commutative diagram

\[
\begin{array}{ccc}
\ker f & \xrightarrow{\psi} & A \\
\downarrow & & \downarrow \\
\ker f & \xrightarrow{\phi} & A'
\end{array}
\]

where \( A' = A/\ker(\phi \circ \psi) \). This module can be described explicitly,

\[
A' = \bigoplus_{x \in \text{Dgm}(\ker f) - \text{im} \phi^*} \{ b(i_\star(x), b(x)) \} \oplus \bigoplus_{x' \in \text{Dgm}(\ker f)} \{ b(i_\star \circ \phi^*(x')), d(i_\star \circ \phi^*(x')) \}.
\]

The intervals which are quotiented out in obtaining the finitely generated approximation are shortened.

The diagram commutes by universality. We note \( j \) is epic since pushouts preserve epimorphisms and \( h \) must be epic since \( h \circ j = \psi \). By Lemma 7.20, \( A \twoheadrightarrow A'' \twoheadrightarrow A' \) implies that

\[
W_p(\text{Dgm}(A), \text{Dgm}(A'')) \leq W_p(\text{Dgm}(A), \text{Dgm}(A')) \leq \varepsilon.
\]

The second inequality follows from Lemma 7.15 and the last inequality is by construction. Taking the limit of \( \varepsilon \) to 0 yields the result.

The dual statement follows similarly.

**Lemma 7.22.** Given an monomorphism between p.f.d. persistence modules \( f : A \hookrightarrow B \),

\[
W_p(\text{Dgm}(A), \text{Dgm}(B)) \leq ||\text{coker } f||_p.
\]

**Proof.** If \( \text{coker } f \) does not have bounded \( p \)-norm, the statement holds trivially as the upper bound is infinite. Hence we can assume \( ||\text{coker } f||_p < \infty \). By Lemma 2.12, there exists a finitely generated approximation to \( \text{coker } f \) which we denote \( \text{coker } f' \) such that \( W_p(\text{coker } f', \text{coker } f) < \varepsilon \). There exists a commutative diagram

\[
\begin{array}{ccc}
B' & \xrightarrow{j'} & \text{coker } f \\
\downarrow & & \downarrow \\
B & \xrightarrow{j} & \text{coker } f
\end{array}
\]

We define \( B' \) explicitly:

\[
B' = \bigoplus_{x \in \text{Dgm}(\text{coker } f) - \text{im } \psi} \{ d(x), d(j^\star(x)) \} \oplus \bigoplus_{x' \in \text{Dgm}(\text{coker } f)} \{ b(j^\star \circ \psi_*(x')), d(j^\star \circ \psi_*(x')) \}.
\]

We can directly check that \( B' \) is a submodule of \( B \) since we shorten intervals by increasing the birth time and that the matching \( p \)-cost \( W_p(\text{Dgm}(B, B')) \leq \varepsilon \). Commutativity can also be directly verified.
As before, to apply interpolation with respect to \( f \), we must construct the pullback \( B'' \).

![Diagram](image)

(10)

Commutativity follows from universality, and \( i \circ h = \varphi \) implies that \( h \) is monic. Since \( B' \hookrightarrow B'' \hookrightarrow B \), by Lemma 7.20 it follows that \( W_p(Dgm(B), Dgm(B'')) \leq \varepsilon \). Since \( \ker f \) is finitely generated we may apply Lemma 7.16 to

\[
0 \to A \hookrightarrow B'' \to \ker f \to 0
\]

\[
W_p(Dgm(A), Dgm(B)) \leq W_p(Dgm(A), Dgm(B'')) + \varepsilon 
\]

\[
\leq ||\ker f||_p + \varepsilon 
\]

\[
\leq ||\ker f||_p + 2\varepsilon.
\]

□

We combine the above results to obtain statements about short exact sequences.

**Theorem 7.23.** Given a short exact sequence of p.f.d. persistence modules

\[
0 \to A \to B \to C \to 0
\]

then

(i) \( W_p(Dgm(A), Dgm(B)) \leq ||C||_p \),

(ii) \( W_p(Dgm(B), Dgm(C)) \leq ||A||_p \).

**Proof.** Given a monic map \( A \to B \), there is a short exact sequence

\[
0 \to A \to B \to \ker f \to 0,
\]

applying Lemma 7.22 we obtain (i). Given an epic map \( B \to C \), there is a short exact sequence

\[
0 \to \ker f \to B \to C \to 0,
\]

applying Lemma 7.21 we obtain (ii).

□

As an immediate corollary,

**Corollary 7.24.** Given a morphism between persistence modules \( f : A \to B \),

\[
W_p(Dgm(A), Dgm(B)) \leq ||\ker f \oplus \ker f||_p.
\]

**Proof.** First note that for \( p < \infty \),

\[
||\ker f \oplus \ker f||_p = ||\ker f||_p^p + ||\ker f||_p^p,
\]

and for \( p = \infty \)

\[
||\ker f \oplus \ker f||_\infty = \max\{||\ker f||_\infty, ||\ker f||_\infty\}.
\]

We construct the following two short exact sequences

\[
0 \to \ker f \to A \xrightarrow{f} \im f \to 0,
\]

\[
0 \to \im f \to B \to \ker f \to 0.
\]

Theorem 7.23 implies

\[
W_p(Dgm(A), Dgm(\im f))^p \leq ||\ker f||_p^p,
\]

\[
W_p(Dgm(B), Dgm(\im f))^p \leq ||\ker f||_p^p.
\]

We note that \( ||\ker f||_p^p \) bounds the change in birth times, i.e. the death times are unchanged and \( ||\ker f||_p^p \) bounds the change in death times, i.e. the birth times are unchanged. Comparing with Equation 4 we deduce the result.

□
By Corollary 7.24, Proof.

Given modules \( \mathcal{A}, \mathcal{B} \) and a span \((\mathcal{C}, \varphi, \psi)\), for \( p < \infty \)

\[
W_p(\text{Dgm}(\mathcal{A}), \text{Dgm}(\mathcal{B}))^p \leq ||\ker \varphi||_p^p + ||\ker \psi||_p^p + ||\text{coker} \varphi||_p^p + ||\text{coker} \psi||_p^p.
\]

Remark 7.25. Note that the weaker result

\[
W_p(\text{Dgm}(\mathcal{A}), \text{Dgm}(\mathcal{B})) \leq ||\ker f||_p + ||\text{coker} f||_p,
\]

follows directly from the triangle inequality for the Wasserstein distance.

Example 7.26. One is tempted to use the realization of a morphism between persistence modules as a matching between indecomposables in order realize the transport map. Unfortunately, this is not the case. Consider

\[
\mathcal{A} \cong [x_2, x_4], \quad \mathcal{B} \cong [x_1, x_4] \oplus [x_2, x_3], \quad \mathcal{C} \cong [x_1, x_3],
\]

with \( x_1 < x_2 < x_3 < x_4 \). Furthermore, let \([a]\) denote the lone summand in \( \mathcal{A} \), \([c]\) the lone summand in \( \mathcal{C} \), and \([b_1] = [x_1, x_4] \) and \([b_2] = [x_2, x_3] \). These three modules can be put into a short exact sequence

\[
0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0
\]

with the maps given explicitly by

\[
f([a]) \mapsto [b_1] + [b_2], \quad g([b_1]) \mapsto [c], \quad g([b_2]) \mapsto -[c].
\]

The maps and the example described below are shown in Figure 4. Note that the image of the first morphism is \([x_2, x_4]\) and the image of the second is \([x_1, x_3]\) as required by injectivity and surjectivity of the short exact sequence. This is an example of a non-trivial extension, i.e. \( \mathcal{B} \) is not isomorphic to \( \mathcal{A} \oplus \mathcal{C} \).

In realizing the transport map from \( \mathcal{B} \) to \( \mathcal{C} \), when interpolating from \( x_4 \) to \( x_3 \), the longer bar, i.e. \([x_1, x_4]\) becomes shorter. That is, for any \( t \in [x_3, x_4] \), the longer bar \( ([b_1]_t) \) in \( \mathcal{B} \) is given by \([x_1, t]\). When \( t = x_3 \), there is a pairing switch and upon further interpolating \( \mathcal{A} \) to \( \mathcal{B} \), the shorter bar \( ([x_2, x_3]_t) \) becomes shorter. That is, for \( t \in [x_2, x_3] \), \([b_2]_t \) is given by \([x_2, t]\) with \([b_1]_t \) unchanged.

One can view this as either a pairing switch in between death times, or that the pairing in \( \mathcal{B} \) changes – by the elder rule the relation maps to the youngest generator and so maps to the summand born at \( x_2 \) rather than the one born at \( x_1 \). Note that this precisely mirrors the tracking which occurs in the cellular proof of stability.

We are now ready to prove our main result. Note that the case of \( p = \infty \) is the stability result from \[2\]. We remark that the proof below can be modified to cover the case \( p = \infty \) as well.

Theorem 7.27. Given modules \( \mathcal{A}, \mathcal{B} \) and a span \((\mathcal{C}, \varphi, \psi)\), for \( p < \infty \)

\[
W_p(\text{Dgm}(\mathcal{A}), \text{Dgm}(\mathcal{B}))^p \leq ||\ker \varphi||_p^p + ||\ker \psi||_p^p + ||\text{coker} \varphi||_p^p + ||\text{coker} \psi||_p^p.
\]

Proof. By Corollary 7.24

\[
W_p(\text{Dgm}(\mathcal{C}), \text{Dgm}(\mathcal{A}))^p \leq ||\ker \varphi||_p^p + ||\text{coker} \varphi||_p^p,
\]
\[ W_p(\text{Dgm}(\mathcal{C}), \text{Dgm}(\mathcal{B}))^p \leq ||\ker \psi||_p^p + ||\coker \psi||_p^p. \]

Since we have the \( p \)-th power, we cannot use the triangle inequality and so it remains to prove that we can add these expressions to obtain the result. We observe that \( \ker \varphi \) as a transport map increases birth times from \( \mathcal{A} \), while \( \coker \psi \) decreases birth times of \( \mathcal{C} \). By considering the short exact sequences with \( \text{im} \varphi \), we can infer a matching of birth and death times in \( \mathcal{A} \), \( \mathcal{B} \) and \( \mathcal{C} \). That is, every birth time in \( \mathcal{A} \) matches a birth time in \( \mathcal{B} \) matches a birth time in \( \mathcal{C} \) in \( \mathcal{A} \).

\[ \phi \text{ can add these expressions to obtain the result. We observe that coker} \ \phi \ \text{decreases birth times of} \ \mathcal{C} \text{ which corresponds to an empty interval. We construct} \ \mathcal{A} \text{ with im} \ \phi \ \text{birth times from} \ \mathcal{A} \text{.} \]

Since we have the \( p < \infty \) consider \( \mathcal{A} \text{ of the matching} \mathcal{C} \text{ which corresponds to an empty interval. We construct} \ \mathcal{A} \text{ with im} \ \phi \text{birth times from} \ \mathcal{A} \text{.} \]

The argument is similar for death times.

We now show that any matching produces a span with a matching algebraic distance. Again we consider \( p < \infty \) to simplify the statement of the Theorem, but remark that the proof holds for this case as well.

**Theorem 7.28.** Let \( \mathcal{M} \) denote a matching between two diagrams corresponding to modules \( \mathcal{A} \) and \( \mathcal{B} \). For \( p < \infty \), every matching \( \mathcal{M} \) induces a span \( (\mathcal{C}, \varphi, \psi) \) such that the \( p \)-Wasserstein transportation cost of the matching \( \mathcal{M} \) is

\[ \sum_{(x,y) \in \mathcal{M}} |b(x) - b(y)|^p + |d(x) - d(y)|^p = ||\ker \varphi||_p^p + ||\coker \varphi||_p^p + ||\ker \psi||_p^p + ||\coker \psi||_p^p. \]

**Proof.** We show that for two persistence module \( \mathcal{A} \) and \( \mathcal{B} \), any matching between the corresponding persistence diagrams induces a span which induces the same distance. We do this by explicitly constructing \( \mathcal{C} \).

We can assume that

\[ \mathcal{A} = \bigoplus_{(b_i, d_i) \in \text{Dgm}(\mathcal{A})} [b_i, d_i], \]

with half-open intervals. If not, the structure theorem of p.f.d. persistence modules and tells us that there is a morphism \( \alpha : \bigoplus_{(b_i, d_i) \in \text{Dgm}(\mathcal{A})} \mathcal{I}(b_i, d_i) \rightarrow \mathcal{A} \) such that \( \ker \alpha \) and \( \text{coker} \alpha \) contains only intervals over single points which occur when the endpoints of the intervals in the decomposition differ. We can then cover these other interval shapes by composing \( \phi \) with \( \alpha \). Similarly, we can assume \( \mathcal{B} = \bigoplus_{(b_i, d_i) \in \text{Dgm}(\mathcal{A})} \mathcal{I}(b_i, d_i) \).

There are six cases which cover all possible types of pairs within a matching as shown in the following persistence diagram. We observe that a morphism exists between points if the target point is below and to the left of the source point.

![Persistence Diagram](image)

When we match a point \( (b, d) \) to the diagonal we are effectively matching it to the point \( \left( \frac{b+d}{2}, \frac{b+d}{2} \right) \), which corresponds to an empty interval. We construct \( \mathcal{C} \) by taking a direct sum over the pairs in the matching \( \mathcal{M} \);

\[ \mathcal{C} = \bigoplus_{((b_A, d_A), (b_B, d_B)) \in \mathcal{M}} \mathcal{I}([\max\{b_A, b_B\}, \max\{d_A, d_B\}]). \]
For example, we contribute $I(b_A, d_A)$ in case (i) and contribute $I(\frac{d_A + b_A}{2}, d_A)$ in case (v). We consider the obvious morphisms $\phi: \mathcal{C} \to \mathcal{A}$ and $\psi: \mathcal{C} \to \mathcal{B}$ sending the generator of $I(\max\{b_A, b_B\}, \max\{d_A, d_B\})$ to that of $I(b_A, d_A)$ and $I(b_B, d_B)$ respectively. The cokernels and kernels of $\phi$ and $\psi$ are generated by the shifts in birth and deaths times respectively with

$$\ker \phi = \bigoplus I(d_A, d_B)_{(b_A, d_A), (b_B, d_B)} \in \mathcal{M} \quad d_A < d_B$$

$$\ker \psi = \bigoplus I(d_B, d_A)_{(b_A, d_A), (b_B, d_B)} \in \mathcal{M} \quad d_A > d_B$$

$$\text{coker} \phi = \bigoplus I(b_A, b_B)_{(b_A, d_A), (b_B, d_B)} \in \mathcal{M} \quad b_A < b_B$$

$$\text{coker} \psi = \bigoplus I(b_B, b_A)_{(b_A, d_A), (b_B, d_B)} \in \mathcal{M} \quad b_B < b_A$$

When we compute the sum of the $p$-th powers of the $p$-norms we see that

$$\|\ker \phi\|_p^p + \|\ker \psi\|_p^p + \|\text{coker} \phi\|_p^p + \|\text{coker} \psi\|_p^p = \sum_{(b_A, d_A), (b_B, d_B)} |d_A - d_B|^p + |b_A - b_B|^p,$$

which is precisely the cost for the transportation map $\mathcal{M}$. \hfill \Box

Together Theorems 7.27 and 7.28 imply that the diagram distance is equivalent to the algebraic distance.

7.1. A Lower Bound for Short Exact Sequences. In the proof of equivalence between algebraic and diagram distance, the triangle inequality between terms in a short exact sequence follow naturally. In this section, we prove a more surprising result. We prove a lower bound for the norm of the middle term of a short exact sequence in terms of the other two terms. This is connected to the space of extensions of persistence modules. One consequence of our result is that the trivial extension has the smallest possible $p$-norm.

**Lemma 7.29.** Given a short exact sequence of persistence modules

$$0 \to \mathcal{A} \overset{\phi}{\rightarrow} \mathcal{B} \overset{\psi}{\rightarrow} \mathcal{C} \to 0,$$

then

$$\|\mathcal{A} \oplus \mathcal{C}\|_p \leq \|\mathcal{B}\|_p.$$

Before proving the case of general $p$, we prove two special cases $p = 1$ and $p = \infty$ whose proof is straightforward.

**Lemma 7.30.** Given a short exact sequence of persistence modules

$$0 \to \mathcal{A} \overset{\phi}{\rightarrow} \mathcal{B} \overset{\psi}{\rightarrow} \mathcal{C} \to 0$$

then

$$\|\mathcal{A} \oplus \mathcal{C}\|_1 = \|\mathcal{B}\|_1$$

**Proof.** Observe that for any module $\mathcal{F}$, $\|\mathcal{F}\|_1 = \int_\mathbb{R} \text{rk}(\mathcal{F}_t)dt$. We then have

$$\|\mathcal{A} \oplus \mathcal{C}\|_1 = \int_\mathbb{R} \text{rk}(\mathcal{A} \oplus \mathcal{C}_t)dt = \int_\mathbb{R} \text{rk}(\mathcal{A}_t) + \text{rk}(\mathcal{C}_t)dt = \int_\mathbb{R} \text{rk}(\mathcal{B}_t)dt = \|\mathcal{B}\|_1,$$

where the third equality holds by exactness restricted to each $t \in \mathbb{R}$. \hfill \Box

**Lemma 7.31.** Given a short exact sequence of persistence modules

$$0 \to \mathcal{A} \overset{\phi}{\rightarrow} \mathcal{B} \overset{\psi}{\rightarrow} \mathcal{C} \to 0$$

then

$$\|\mathcal{A} \oplus \mathcal{C}\|_\infty \leq \|\mathcal{B}\|_\infty.$$
Proof. Assume that
\[ ||A \oplus C||_\ell^\infty > ||B||_\ell^\infty. \]
There must exist a summand in either \( A \) or \( C \) which is more persistent than any summand in \( B \). Consider the case where a summand in \( A \) achieves the norm, i.e. is the most persistent bar. By the injectivity of \( \varphi \), the image must be at least as persistent contradicting the assumption. Alternatively, if the norm is achieved in \( C \), the surjectivity of the \( \psi \) again contradicts the assumption since there must exist a summand which is at least as persistent as any summand in \( C \).

\[ \square \]

Figure 4 illustrates this contradiction. We now prove the general result.

Proof of Lemma 7.29. We first assume that \( A \) and \( C \) are finitely generated. Recall that in homological algebra the short exact sequence
\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]
is known as an extension of \( C \) by \( A \). To prove the result, we show that all extensions have a larger norm than the trivial extension, i.e. \( A \oplus C \).

We do this iteratively, beginning with the trivial extension and construct a sequence of extensions, each increasing the norm. First, we recount some basic facts from homological algebra. The equivalence classes of extensions are in 1-1 correspondence with \( \text{Ext}^1(C, A) \), where the 0 element corresponds to the trivial extension, see [43](Theorem 3.4.3). The \( \text{Ext} \)-functor is a standard construction in homological algebra and we refer readers to [43, Definition 2.52]. Taking a projective resolution of \( C \),
\[ 0 \rightarrow R_C \rightarrow G_C \rightarrow C \rightarrow 0, \]
where \( R_C \) and \( G_C \) are freely generated and for persistence modules correspond to deaths and births respectively. That is \( R_C \) is the space of boundaries and \( G_C \) is the space of cycles. Applying \( \text{Hom}(-, A) \), we obtain the following exact sequence
\[ \text{Hom}(G_C, A) \rightarrow \text{Hom}(R_C, A) \rightarrow \text{Ext}^1(C, A) \rightarrow 0. \]
Thus for any \( B \), there exists a homomorphism \( \gamma : R_C \rightarrow A \), such that
\[ \begin{array}{ccc}
0 & \rightarrow & R_C \\
\downarrow \gamma & & \downarrow j \\
0 & \rightarrow & A \\
\end{array} \]
where \( B \) is the pushout
\[ R_C \xrightarrow{(\gamma, j)} A \oplus G_C, \]
or equivalently
\[ B = \text{coker}(\gamma, j). \]
As \( \gamma = 0 \) corresponds to \( A \oplus C \). We denote
\[ \gamma_0 = 0, \quad B_0 = A \oplus C. \]
Let \( \gamma_n \) denote the homomorphism which corresponds to \( B \), where \( n \) is the rank of \( R_C \). Order the generators in order of increasing birth time. Define \( \gamma_i \) as the restriction of \( \gamma_n \) to the first \( i \) generators, sending the remaining ones to 0. By construction each corresponding \( B_i \) is an extension.

We now prove that each step increases the norm. Recall, we have
\[ R_C \xrightarrow{(\gamma, j)} A \oplus G_C \rightarrow B_i \rightarrow 0, \]
where \( G_C \) is free and \( A \) is not free but is finitely generated. Going from \( \gamma_{i-1} \) to \( \gamma_i \) potentially changes one of the relations, resulting in a different pairing. This is called a cascade [19]. Let \( r(i) \) denote the added relation. If this does not change the pairing we continue to \( i + 1 \). If it does, let \( (g(i), r(i)) \) denote the resulting indecomposable. Observe that all other affected generators satisfy
\[ g(i) > g_1(i) > g_2(i) > \ldots > g_k(i), \tag{11} \]
and the affected relations satisfy
\[ r(i) < r_1(i) < g_2(i) < \ldots < r_k(i). \tag{12} \]
Note that these may come from \( R_C \), \( A \), or both.

Here we assume a familiarity with the standard persistence algorithm based on ordered Gaussian elimination [44]. Simulating the persistence algorithm between generators and relations – the map
between relations and generators as a matrix with the rows indexed by generators sorted in filtration order top-down and the columns indexed by relations sorted in filtration order left to right – a summand is a pivot of relations and generators such that the pivot is the lowest non-zero entry in the corresponding column. See Figure 5 for an illustration. As the algorithm proceeds left to right, selecting pivots by the lowest non-zero entry, a change in a column can only affect columns which are to the right of it and generators which are above the row of the pivot.

The resulting indecomposables are \((q_i, r_i)\). This can be shown inductively. Consider the \(k \times k\) matrix of the affected generators and relations. Given the first pivot, we can zero out the column above the pivot using row operations and zero out the row using column operations. Hence, we are left with a \((k - 1) \times (k - 1)\) matrix where the pivot must again be in the lower left hand corner.

We note that this structure can also be shown using Nested-Disjoint Lemma [33, p.30], which describes the possible pairing changes which can occur when the relative orders of relations and generators are changed.

This pairing maximizes the norm over all possible pairings. This follows from the Rearrangement inequality [38, p.78] for \(p = 2\). We prove the case for general \(p\) in Corollary [32, see Appendix B]. This implies that
\[
||B_i||_p \leq ||B_{i+1}||_p,
\]
completing the proof for finitely generated modules.

To conclude the proof for the p.f.d. case, we provide an approximation argument similar to Lemmas [7.21] and [7.22]. If \(B\) does not have bounded \(p\)-norm, the result holds trivially. If \(B\) has bounded \(p\)-norm we can consider a finitely generated \(\varepsilon\)-approximation of \(B\) denoted by \(B'\). We can construct the following commutative diagram
\[
t_0 \rightarrow \mathcal{A} \overset{i}{\longrightarrow} \mathcal{B} \overset{j}{\longrightarrow} \mathcal{C} \overset{\psi}{\longrightarrow} 0
\]
\[
0 \rightarrow \text{im } q_B \circ i \overset{q_B}{\longrightarrow} \mathcal{B}' \overset{\text{coker}(j|_{\mathcal{B}'})}{\longrightarrow} 0
\]
where \text{coker}(j|_{\mathcal{B}'}) is cokernel of the restriction of \(j\) to \(\mathcal{B}'\). This is well defined since by Lemma 2.12, there is also a monomorphism \(\mathcal{B}' \hookrightarrow \mathcal{B}\). We observe that by Lemma 7.6
\[
||\ker \phi||_p \leq ||\ker q_B||_p \leq \varepsilon,
\]
\[
||\ker \psi||_p \leq ||\ker q_B||_p \leq \varepsilon.
\]
We can use this to bound the \(p\)-norm of the direct sum. For \(\varepsilon\) sufficiently small, there exists a constant \(K\) independent of \(\varepsilon\) such that
\[
||\mathcal{A} \oplus \mathcal{C}||_p^p = ||\mathcal{A}||_p^p + ||\mathcal{C}||_p^p \leq (||\text{im } q_B||_p + \varepsilon)^p + (||\text{coker}(j|_{\mathcal{B}'})||_p + \varepsilon)^p
\]
\[
\leq ||\text{im } q_B \oplus \text{coker}(j|_{\mathcal{B}'})||_p^p + K\varepsilon.
\]
Taking the \(p\)-th root of the above expression yields
\[
(||\text{im } q_B \oplus \text{coker}(j|_{\mathcal{B}'})||_p^p + K\varepsilon)^{1/p} \leq ||\text{im } q_B \oplus \text{coker}(j|_{\mathcal{B}'})||_p + K'\varepsilon.
\]
Since \(p > 1\) the above is a concave function, we can upper bound the expression by a linear function where the slope is some constant \(K'\). Applying the result for finitely generated modules from above
\[
||\mathcal{A} \oplus \mathcal{C}||_p \leq ||\text{im } q_B \oplus \text{coker}(j|_{\mathcal{B}'})||_p + K'\varepsilon \leq ||\mathcal{B}'||_p + K'\varepsilon \leq ||\mathcal{B}||_p + K'\varepsilon.
\]
Taking the limit \(\varepsilon \to 0\) completes the proof. \(\square\)

**Remark 7.32.** This together with the triangle inequality (from the equivalence with the Wasserstein distance on diagrams) gives a Minkowski-type bound related to short exact sequences of persistence modules,
\[
||\mathcal{A} \oplus \mathcal{C}||_p \leq ||\mathcal{B}||_p \leq ||\mathcal{A}||_p + ||\mathcal{C}||_p,
\]
where the second inequality follows from the triangle inequality. We note that if \(p < \infty\), then
\[
||\mathcal{A} \oplus \mathcal{C}||_p^p = ||\mathcal{A}||_p^p + ||\mathcal{C}||_p^p,
\]
which gives
\[
(||\mathcal{A}||_p^p + ||\mathcal{C}||_p^p)^{1/p} \leq ||\mathcal{B}||_p \leq ||\mathcal{A}||_p + ||\mathcal{C}||_p.
\]

\(^5^\)We remark that the case of infinite bars is not interesting in this case as it can be shown that they must split trivially.
Figure 5. The generators are sorted in increasing filtration order top to bottom and
the relations are sorted in filtration order left to right as a matrix. (Left) The added
relation to \( \gamma_i \) can create a new pivot (shown by the entry 1) in \( B \). The shaded region
shows the relations and generators which can be affected. Generators after \( g \) since the
pivot is defined as the lowest non-zero entry. The relations before \( r \) cannot be affected
as those columns remain reduces. (Right) Since we only show the affected relations and
generators, the matrix after reduction must have the following shape, as each new pivot
can only affect the upper right-hand submatrix.

or equivalently
\[
||A||_p + ||C||_p \leq ||B||_p \leq (||A||_p + ||C||_p)^p.
\]

7.2. Application of Algebraic Stability. Here we show how the algebraic framework can be used to
obtain the results in Section 4 directly. Here we reprove Theorem 4.8 directly on the chain complexes.
There are not necessarily finite, but the induced persistence modules must be p.f.d.

**Theorem 7.33.** Given \( f, g : K \to \mathbb{R} \), let \( F \) and \( G \) denote the persistence modules corresponding to the
respective sub-level set filtrations, then if \( F \) and \( G \) are p.f.d.
\[
W_p(Dgm(F), Dgm(G)) \leq ||f - g||_p
\]

**Proof.** Consider the resulting filtered chain complexes \( C_k(f) \) and \( C_k(g) \), where the filtrations are induced
by the sub-level sets of the functions \( f \) and \( g \). We observe that the filtered chain complexes can be
considered as persistence modules, where each simplex generates a bar. We then directly construct
the span at the chain level. For the span, we consider the sub-level set filtration induced by the function
max(\( f, g \)). We obtain the following diagram graded by dimension,

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C_k(\max(f, g)) & \xrightarrow{i_k} & C_k(f) & \xrightarrow{j_k} & C_k(g) & \rightarrow & \text{coker } i_k & \rightarrow & 0 \\
\downarrow & & & & & \downarrow & & \downarrow & \\
0 & \rightarrow & \text{coker } j_k & \rightarrow & 0
\end{array}
\]

where the maps \( i_k \) and \( j_k \) map a chain to itself. It can be directly verified that these chain maps respect
the filtration and that
\[
||\text{coker } i_k||_p^p = \sum_{\sigma \in K, \dim(\sigma) = k} (\max(f, g)(\sigma) - f(\sigma))^p,
\]
\[
||\text{coker } j_k||_p^p = \sum_{\sigma \in K, \dim(\sigma) = k} (\max(f, g)(\sigma) - g(\sigma))^p.
\]
Applying the homology functors to the two short exact sequences for all $k$, we obtain two long exact sequences.

\[
\begin{align*}
\cdots & \xrightarrow{\delta_{k+1}} H_k(\max(f, g)) \xrightarrow{j^*_k} \mathcal{G} \xrightarrow{\delta_k} H_k(\coker j_k) \xrightarrow{j^*_{k-1}} \cdots, \\
\cdots & \xrightarrow{\delta'_{k+1}} H_k(\max(f, g)) \xrightarrow{i^*_k} \mathcal{F} \xrightarrow{\delta'_k} H_k(\coker i_k) \xrightarrow{i^*_{k-1}} \cdots.
\end{align*}
\]

Recall that $\coker i_k$ and $\coker j_k$ are chain complexes and since homology is a subquotient, we may define surjective maps

\[
\coker i_k \twoheadrightarrow H_k(\coker i_k) \quad \text{and} \quad \coker j_k \twoheadrightarrow H_k(\coker j_k).
\]

By Lemma 7.8 it follows that

\[
||H_k(\coker i_k)||_p \leq ||\coker i_k||_p \quad \text{and} \quad ||H_k(\coker j_k)||_p \leq ||\coker j_k||_p.
\]

Fixing the homological dimension $k$, we can extract the relevant terms from the long exact sequences into the following diagram

\[
\begin{array}{ccc}
0 & \downarrow & 0 \\
\ker j^*_k & \downarrow & \ker i^*_k \\
0 \xrightarrow{\ker j^*_k} \ker i^*_k \xrightarrow{i^*_k} H_k(\max(f, g)) \xrightarrow{i^*_k} \mathcal{F} \xrightarrow{\delta'_k} \coker j^*_k \xrightarrow{\delta'_k} 0 \\
\downarrow & \downarrow & \downarrow \\
\mathcal{G} & \xrightarrow{i^*_k} & \coker j^*_k \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

By exactness of the sequences in Equation 14

\[
\coker \delta_{k+1} \cong \ker i^*_k \quad \text{im} \delta_k \cong \ker i^*_k 
\coker \delta_{k+1} \cong \ker i^*_k \quad \text{im} \delta_k \cong \ker i^*_k.
\]

Inserting these isomorphisms into Theorem 7.27

\[
W_p(\mathcal{F}, \mathcal{G})^p \leq \left\| \bigoplus_k \ker i^*_k \oplus \coker i^*_k \oplus \ker j^*_k \oplus \coker j^*_k \right\|^p_p
\]

\[
= \left\| \bigoplus_k \coker \delta_k \oplus \text{im} \delta_k \oplus \coker \delta'_k \oplus \text{im} \delta'_k \right\|^p_p.
\]

Using the short exact sequences,

\[
0 \rightarrow \ker \delta_k \rightarrow H_k(\coker i_k) \rightarrow \text{im} \delta_k \rightarrow 0,
\]

\[
0 \rightarrow \ker \delta'_k \rightarrow H_k(\coker j_k) \rightarrow \text{im} \delta'_k \rightarrow 0.
\]

and Lemma 7.29 yields

\[
|| \ker \delta_{k-1} \oplus \text{im} \delta_k ||_p^p \leq ||H_k(\coker i_k)||_p^p,
\]

\[
|| \ker \delta'_{k-1} \oplus \text{im} \delta'_k ||_p^p \leq ||H_k(\coker j_k)||_p^p.
\]
Rearranging the terms in Equation 16 and using Equation 17,
\[
W_p(\mathcal{F}, \mathcal{G})^p \leq \left| \bigoplus_k \text{coker } \delta_k \oplus \text{im } \delta_k \oplus \text{coker } \delta'_k \oplus \text{im } \delta'_k \right|^p
\]
\[
\leq \left| \bigoplus_k H_k(\text{coker } i_k) \oplus H_k(\text{coker } j_k) \right|^p
\]
\[
\leq \left| \bigoplus_k \text{coker } i_k \oplus \text{coker } j_k \right|^p
\]
\[
= \sum_{\sigma \in K} (\max(f, g)(\sigma) - f(\sigma))^p + (\max(f, g)(\sigma) - g(\sigma))^p
\]
\[
= \sum_{\sigma \in K} (f(\sigma) - g(\sigma))^p = \|f - g\|_p^p.
\]
The third inequality follows from Equation 15 and the final equality follows from Equation 13 and observing that for any simplex, at least one of the two terms
\[
\max(f, g)(\sigma) - f(\sigma) \quad \text{and} \quad \max(f, g)(\sigma) - g(\sigma),
\]
must be zero.

The case of \( p = \infty \) follows similarly by considering the maximum rather than the sum.

\[\square\]

Remark 7.34. The bound is equivalent to Theorem 7.3. To achieve this, we required Lemma 7.2. We remark that without this result a slightly weaker bound can be achieved using the following inequalities
\[
\|\ker i_k\|_p \leq \|H_{k+1}(\text{coker } i_{k+1})\|_p \quad \text{and} \quad \|\text{coker } i_k\|_p \leq \|H_k(\text{coker } i_k)\|_p
\]
\[
\|\ker j_k\|_p \leq \|H_{k+1}(\text{coker } j_{k+1})\|_p \quad \text{and} \quad \|\text{coker } j_k\|_p \leq \|H_k(\text{coker } j_k)\|_p,
\]
which yields
\[
W_p(\text{Dgm}(f), \text{Dgm}(g))^p \leq \sum_k (2\|\ker i_k\|_p^p + 2\|\ker j_k\|_p^p) = 2\|f - g\|_p^p.
\]

7.3. Properties of Algebraic Distance. Finally we explore the requirements for the algebraic distance to define an (extended) pseudometric in more general settings. While this follows for p.f.d. one-parameter persistence modules by the Equivalence Theorem 7.28. This has been extensively explored in the context of noise systems [37] and amplitudes [26]. We refer the reader to these papers for a more complete discussion in a variety of general settings. Rather, in this section we consider the special case of the algebraic distance in terms of the norm of a module. Formally this can be seen as a function such that
\[
\|\langle A \rangle : A \mapsto \mathbb{R}^\geq 0 \cup \{\infty\}
\]
The requirements for non-negativity is obvious and even in simple cases the need to allow for the possibility of an infinite distance often arises. We also require the norm to have the following properties:

1. **Zero Module**: The zero module 0 has norm 0, i.e., \( \langle 0 \rangle = 0 \).
2. **Monotonicity**: Given \( \mathcal{A} \hookrightarrow \mathcal{B} \) or \( \mathcal{B} \rightarrow \mathcal{A} \), implies
   \[
   \langle \mathcal{A} \rangle \leq \langle \mathcal{B} \rangle
   \]
3. **Direct Sums**: The norm must be well-behaved under a direct sum: if \( \langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle \), then for any \( \mathcal{C} \),
   \[
   \langle \mathcal{A} \oplus \mathcal{C} \rangle = \langle \mathcal{B} \oplus \mathcal{C} \rangle
   \]
4. **Subadditivity**: Given a short exact sequence,
   \[
   0 \to \mathcal{A} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{C} \to 0,
   \]
   implies
   \[
   \langle \mathcal{B} \rangle \leq \langle \mathcal{A} \rangle + \langle \mathcal{C} \rangle.
   \]
We remark on the relation of the above conditions and amplitudes and noise systems. We observe that the above three conditions are more restrictive than amplitudes – which only require monotonicity and subadditivity [26, Definition 3.1]. We remark however that noise systems may be taken to closed under direct sums, which is sufficient to deduce the above property (see [37, Proposition 7.1]).

Recall that a pseudometric \( d(\cdot, \cdot) \) must satisfy:
For the algebraic distance, (i) holds from our assumption on $p$-norms. Likewise, for (ii), for a module $A$, we can construct a span $(A, \text{id}, \text{id})$. As the kernels and cokernels are trivial, the result holds by the assumption the norm of the zero module and the behaviour of the norms under direct sums. Likewise, the definition of a span is symmetric, which implies (iii).

All that remains is to verify the triangle inequality. To prove this we will require one additional property of $p$-norms.

(5) **Duality**: For every short exact sequence
\[ 0 \to A \to B \to C \to 0, \]
we can find a short exact sequence
\[ 0 \to D \to E \to F \to 0, \]
such that
\[ \langle A \rangle = \langle F \rangle, \quad \langle B \rangle = \langle E \rangle, \quad \langle C \rangle = \langle D \rangle. \]
This condition seems extraneous but will often hold in most cases we consider. Perhaps the most obvious example includes taking duality between homology and cohomology. In the case of vector spaces, one can use the non-canonical isomorphism between a vector space and its dual. We leave the categorical foundations of the algebraic distance for future work (as well as the question of whether this is needed).

We conclude this section by showing that if the $p$-norm satisfies the above conditions, then the algebraic distance satisfies the triangle inequality.

**Lemma 7.35.** Given a span $(\mathcal{E}, i, j)$ between $A$ and $B$ and a span $(\mathcal{F}, r, s)$ between $B$ and $C$. There exists a span $(\mathcal{G}, \varphi, \psi)$ such that
\[ c(\varphi, \psi) \leq c(i, j) + c(r, s), \]
where
\[ c(i, j) = \langle \ker i \oplus \ker j \oplus \coker i \oplus \coker j \rangle, \]
\[ c(r, s) = \langle \ker r \oplus \ker s \oplus \coker r \oplus \coker s \rangle. \]
This implies that the algebraic Wasserstein distance induced by $\langle \cdot \rangle$ satisfies the triangle inequality.

**Proof.** We do this constructively. By assumption, we construct $(\mathcal{G}, \varphi, \psi)$ such that
\[ c(\varphi, \psi) \leq c(i, j) + c(r, s). \]
Define $\mathcal{G} = \ker(j + r) \subseteq \mathcal{E} \oplus \mathcal{F}$, which yields the following diagram

\[ \begin{array}{ccc}
\mathcal{G} & \xrightarrow{\beta} & \mathcal{F} \\
\alpha \downarrow & & \downarrow r \\
\mathcal{E} & \xrightarrow{j} & \mathcal{B} \\
\varphi \downarrow & & \downarrow s \\
A & \xrightarrow{i} & \mathcal{B}
\end{array} \]

By definition $\varphi = i \circ \alpha$ and $\psi = s \circ \beta$, so there short exact sequences
\[ 0 \to \ker \alpha \leftrightarrow \ker \varphi \leftrightarrow \alpha(\ker \varphi) \to 0, \]
\[ 0 \to \ker \beta \leftrightarrow \ker \psi \leftrightarrow \beta(\ker \psi) \to 0. \]
Furthermore since $\text{im} \varphi \subseteq \text{im} i$ and $\text{im} \psi \subseteq \text{im} s$, there are short exact sequences
\[ 0 \to \text{im} i / \text{im} \varphi \to \text{coker} \varphi \to \text{coker} i \to 0, \]
\[ 0 \to \text{im} s / \text{im} \psi \to \text{coker} \psi \to \text{coker} \ell \to 0. \]
By the Duality Property (5), for the short exact sequence
\[ 0 \to \ker \beta \leftrightarrow \ker \psi \leftrightarrow \beta(\ker \psi) \to 0, \]
there exists a short exact sequence such that
\[ 0 \to X \to Y \to Z \to 0, \]
such that
\[ \langle X \rangle = \langle \beta(\ker \psi) \rangle, \quad \langle Y \rangle = \langle \ker \psi \rangle, \quad \langle Z \rangle = \langle \ker \beta \rangle. \]
Applying the same to
\[ 0 \to \text{im } s/\text{im } \psi \to \text{coker } \psi \to \text{coker } s \to 0, \]
and taking the direct sum over all four sequences and using Direct Sum Property (3)
\[ \langle \ker \varphi \oplus \ker \psi \oplus \text{coker } \varphi \oplus \text{coker } \psi \rangle \leq \langle \ker \alpha \oplus \beta(\ker \psi) \oplus \text{im } i/\text{im } \varphi \oplus \text{coker } s \rangle + \langle \alpha(\ker \varphi) \oplus \ker \beta \oplus \text{coker } i \oplus \text{im } s/\text{im } \psi \rangle. \]
We observe that there are injective maps \( \alpha(\ker \varphi) \to \ker i \) and \( \beta(\ker \psi) \to \ker s \), so by the Monotonicity Property (2),
\[ \langle \alpha(\ker \varphi) \rangle \leq \langle \ker i \rangle \quad \quad \langle \beta(\ker \psi) \rangle \leq \langle \ker s \rangle. \]
By construction, \( \beta \) restricted to \( \ker \alpha \) is injective. Commutativity implies that \( \beta(\ker \alpha) \subseteq \ker r \). Hence,
\[ \langle \ker \alpha \rangle \leq \langle \ker r \rangle. \]
Likewise,
\[ \langle \ker \beta \rangle \leq \langle \ker j \rangle. \]
We now claim that \( \langle \text{im } i/\text{im } \varphi \rangle \leq \langle \text{coker } r \rangle \). Pick a non-trivial element \( x \in \text{im } i/\text{im } \varphi \). There exists a lift of \( x \) to \( E \) denoted \( y \). By construction, \( \ker j \subseteq \text{im } \alpha \), so any lift has the property that \( j(y) \) is non-trivial. Finally, if \( j(y) \in \text{im } r \) it would again imply that \( y \in \text{im } \alpha \). We conclude that \( j(y) \in \text{coker } r \). Hence
\[ \langle \text{im } i/\text{im } \varphi \rangle \leq \langle \text{coker } r \rangle. \]
Likewise,
\[ \langle \text{im } s/\text{im } \psi \rangle \leq \langle \text{coker } j \rangle. \]
Substituting the inequalities and using the Subadditive Property (4), we conclude
\[ \langle \ker \varphi \oplus \ker \psi \oplus \text{coker } \varphi \oplus \text{coker } \psi \rangle \leq \langle \ker i \oplus \ker j \oplus \text{coker } i \oplus \text{coker } j \rangle + \langle \ker r \oplus \ker s \oplus \text{coker } r \oplus \text{coker } s \rangle. \]
which proves the result. \( \square \)

8. Discussion

We have investigated Wasserstein stability for persistence diagrams which has a scarcity of results despite becoming increasingly important for applications. While we have presented numerous results (far more than we originally intended), we believe this is a starting point for further investigation. Below we outline possible further questions and directions.

- **Cellular Stability Theorem:** This surprisingly straightforward proof can be extended to other settings of interval decomposable modules, e.g. zig-zag persistence [10], exact and weakly-exact multiparameter modules [5]. Another interesting direction is to consider implication for the study of random functions, e.g. discrete Gaussian random fields.

- **Algebraic Wasserstein Stability:** The algebraic formulation is a clear step toward understanding Wasserstein stability in more general settings where interval decompositions do not exist, including but not limited to multiparameter persistence, sheaf-based persistence, or even more general categorical or algebraic settings. However, there are multiple obstacles before the techniques developed here can be applied, including what is a suitable definition of norm. There have been some suggested following a previous version of this paper [3] [20].

  The notion of an interpolating object is reminiscent of erosion distance [34], it would be interesting to understand the precise relationship between the two notions. Furthermore, while the interpolating object is convenient for stating the results, most approximation results for persistence diagrams use interleaving. In the future work we will describe a notion of interleaving for Wasserstein distance based on the theory of partial maps.
• **Applications:** In addition to providing stability bounds for a number of topological summaries, can the results be used for better understanding the behaviour under a small number of outliers. There are also important questions on the stability of distance based filtrations. While explicit bounds independent of the point set size may not exist, the configurations are specific and do not occur if there sufficient randomness, just as on a Poisson point process the expected size of the Delaunay complex is linear rather $n^{\frac{d}{d+2}}$. How one could quantify this to obtain better bounds is an interesting and important question.

The Wasserstein bounds are potentially useful for investigation of stochastic topology where the 2-Wasserstein distance is most commonly used. While the bottleneck distance is far too coarse, we believe the combination of lower and upper bounds on the persistence norm can produce new local-to-global techniques for understanding random phenomena.

Finally, this opens up an avenue for bridging persistence and spectral analysis. With $p = 2$, the Cellular Stability Theorem implies a stability in terms of eigenfunction expansions of functions. The obvious result relating squared differences of coefficients with distances between diagrams is limited to fixed, sufficiently nice triangulations. However, we intend to investigate this in future work.

### 9. Funding

KT is the recipient of an Australian Research Council Discovery Early Career Award (project number DE200100056) funded by the Australian Government.

### 10. Declarations

10.1. **Conflicts of Interest.** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Lemma A.1 (Lemma 2.8). For any \( p' \leq p \), \( W_p(X, Y) \leq W_{p'}(X, Y) \).

Proof. For any matching between diagrams, \( M : X \to Y \), we have

\[
\left( \sum_{x \in X} (d(x) - d(M(x)))^p + (b(x) - b(M(x)))^p \right)^{1/p} \\
= \left( \sum_{x \in X} (d(x) - d(M(x)))^{pp'/p} + (b(x) - b(M(x)))^{pp'/p} \right)^{1/p'} \\
\leq \left( \sum_{x \in X} (d(x) - d(M(x)))^{pp'/p} + (b(x) - b(M(x)))^{pp'/p} \right)^{1/p'} \\
= \left( \sum_{x \in X} (d(x) - d(M(x)))^{pp'/p} + (b(x) - b(M(x)))^{pp'} \right)^{1/p'}
\]

To prove the inequality, we show that for \( 0 < a \leq 1 \)

\[(x + y)^a \leq x^a + y^a\]

This follows from the fact that the above inequality holds if and only if, \((1 + t)^a \leq 1 + t^a\) with \( 0 \leq t \leq 1 \). The result follows since the function \((1 + t)^a - (1 + t)^a\) is positive. Taking the derivative of this function, we obtain \( (t^{a-1} - (1 + t)^{a-1}) \). Since \( 0 < a < 1 \), the derivative is positive and at \( t = 0 \), the function is \( 0 \), hence the function is positive. Since the inequality is true for any matching begin with the matching for \( p' \), the resulting matching induces a smaller norm for \( p \), and since the distance is infimum over all matchings the Wasserstein distance is smaller. \( \square \)

Appendix B. Proof of the Rearrangement Inequality for \( p \geq 1 \)

Lemma B.1. Given a sequence

\[ a_n \leq a_{n-1} \leq \ldots \leq a_1 \leq b_1 \leq b_2 \leq \ldots \leq b_n \]

and an increasing convex function \( f \), the cost of a bijection \( T : \{b_i\} \mapsto \{a_j\} \), is

\[ f(T) = \sum_{i=1}^{n} f(b_i - T(b_i)). \]

The matching \( b_i \mapsto a_i \) minimizes this sum.

Proof. This is similar to the proof of the rearrangement inequality in [38 (p. 78)]. Begin with any bijection \( T = T_0 \). Define \( T_i \), in terms of \( T_{i-1} \) by performing an inversion:

\[
T_i(b_j) = \begin{cases} 
  a_j & j = i \\
  T_{i-1}(b_j) & j = T^{-1}_{i-1}(a_i) \\
  T_{i-1}(b_j) & \text{else}
\end{cases}
\]

This pairs \( b_i \) to \( a_i \), and pairs \( T_{i-1}(b_i) \) with the \( b_j \) which was previously paired with \( a_i \), i.e. \( T_{i-1}^{-1}(a_i) \). The transpositions pair \( b_i \) to \( a_i \) from the middle of the sequence outward. Since all pairs with index less than \( i \) align, i.e. \( T_i(b_j) = a_j \) for all \( j \leq i \), it follows that

\[ T(b_i) \leq a_i \leq b_i \leq T^{-1}(a_i). \]

Now we show that \( f(T_i) \leq f(T_{i+1}) \). If the bijection does not change from \( i \) to \( i + 1 \), i.e. \( b_i = T_{i-1}^{-1}(a_i) \), then the statement is trivial. Let \( j \) denote the index of \( T(b_i) \) and \( j' \) the index of \( T^{-1}(a_i) \). We must show

\[ f(b_j - a_i) + f(b_i - a_j) \leq f(b_{j'} - a_j) + f(b_i - a_i). \]

As we are only concerned with differences, without loss of generality we can set \( a_j = 0 \) and rearranging terms gives

\[ f(b_i) - f(b_i - a_i) \leq f(b_{j'}) - f(b_{j'} - f(a_i)). \]
Using the substitutions
\[ x_1 = b_i - a_i, \quad x_2 = b_j - a_i \quad d = a_i \]
Therefore, we must prove that
\[ f(x_1 + d) - f(x_1) \leq f(x_2 + d) - f(x_2). \]
A classical fact about convex functions is that the quantity
\[ \frac{f(y) - f(z)}{y - z} \]
is monotonically non-decreasing in both \( y \) and \( z \). Since the function is increasing, the quantity is always positive for \( y \leq z \). This implies
\[ \frac{f(x_1 + d) - f(x_1)}{d} \leq \frac{f(x_2 + d) - f(x_1)}{d + x_2 - x_1} \leq \frac{f(x_2 + d) - f(x_2)}{d}. \]
As \( d > 0 \), this implies Eq. 18. \( \square \)

**Corollary B.2.** For any sequence as above and \( p \geq 1 \), the identity matching minimizes
\[ \sum_{i=1}^{n} (b_i - T(b_i))^p \]

**Proof.** As the exponentiation function \( x^p \) is an increasing convex function for non-negative \( x \), Lemma B.1 implies the result. \( \square \)

### Appendix C. Notation

- \( K \) – a finite CW complex
- \( \text{Dgm}_k(K,f) \) or \( \text{Dgm}_k(f) \) – the \( k \)-th dimensional persistence diagram for the filtration induced on \( K \) by a monotone function \( f \).
- \( \text{Dgm}(K, f) \) or \( \text{Dgm}(f) \) – \( \bigoplus_k \text{Dgm}_k(K,f) \) if coming from a chain complex
- \( \mathcal{F} \) – a persistence module
- \( \text{Dgm}(\mathcal{F}) \) – the persistence diagram of a persistence module
- \( \check{C}(\mathcal{P}) \) – the Čech complex
- \( \mathcal{R}(\mathcal{P}) \) – the Vietoris-Rips complex, where \( \mathcal{R}_\delta \) refers to the complex at parameter \( \delta \)
- \( W_p \) – the \( p \)-th Wasserstein distance
- \( W_{p, \text{alg}} \) – the \( p \)-th algebraic Wasserstein distance
- \( C_k(\cdot) \) – \( k \)-dimensional chain complex
- \( H_k(\cdot) \) – \( k \)-dimensional homology
- \( T \) – a bijective matching between sets
- \( C \) – a correspondence
- \( \hookrightarrow \) – monomorphism
- \( \twoheadrightarrow \) – epimorphism
- \( \text{rk} \) – the rank
- \( v, w \) – geometric points
- \( x \) – points in persistence diagrams
- \( b(x) \) – birth time of point \( x \) in a persistence diagram
- \( d(x) \) – death time of point \( x \) in a persistence diagram
- \( \ell \) – lifetime of point \( x \), i.e. \( d(x) - b(x) \)

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