Free Vibration analysis of bimodular composite material laminated curved beam

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Abstract. In this paper, the free vibration analysis of simply supported bimodular composite material laminated curved beam has been carried out using equivalent stiffness method. The Non-dimensional free vibration frequencies for positive half of vibration cycle and for negative half of vibration cycle have been presented for different ratio and laminated scheme. The analysis is based on classical beam theory. It is observed that the percentage difference of free vibration frequencies obtained from different equivalent stiffness method is more for angle ply laminated beam and for cross ply laminated beam the percentage difference is very less.

1. Introduction

There are some composite materials which exhibit different behavior in tension and compression as shown in figure 1 [1]. These materials are called bimodular composite material. A few examples of such materials are aramid rubber, polyester rubber, carbon-carbon composite, bone and soft tissues etc. The analysis of bimodular material laminated structure is little bit difficult as compare to unimodular composite material laminated structure. Timoshenko [2] was the first person to discuss about the bimodular materials. The major issue of bimodular material is the assignment of material properties which makes the analysis difficult. Jones [1], Bert [3], Papazoglou [4] and Khan et al. [5] have suggested different material models to assign proper material properties for the analysis of bimodular material laminated structural element. It has been noticed from the literature available[6-8] in this field that the Bert’s material model is used mostly and almost all the analysis were confined to static, stability and free vibration analysis of plates. The free and forced vibration analysis of plates, has been carried out by Patel et al. [9]. They have used their own model and have presented a comparative study of Bert’s model and their own model. The research paper on bimodular composite material laminated beam is meager. In this paper the free vibration analysis of bimodular composite material laminated curved beam has been carried out using classical beam theory and the stiffness parameters [A, B, D] have been calculated using different equivalent stiffness method.

Figure 1. Stress-strain curve of bimodular material
2. Methodology
A thin composite simply supported curved beam having radius of curvature R width b and thickness h as shown in figure 2. The assumed displacement field is:

\[ u(x, y, z, t) = u_0(x, t) - z \frac{\partial w(x, t)}{\partial x} \]

\[ v(x, y, z, t) = 0 \]

\[ w(x, y, z, t) = w(x, t) \]

The strain displacement relation is assumed as:

\[ \varepsilon = (\varepsilon_0 + z\kappa) \]

where,

\[ \varepsilon_0 = \frac{\partial u_0}{\partial x} + \frac{w_0}{R} \] and \[ \kappa = -\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{R} \frac{\partial u_0}{\partial x} \]

![Figure 2. Geometry and coordinate system of simply supported laminated curved beam](image)

Using Hamilton’s principles the equation of motions of curved beam can be written as [10]:

\[ \frac{\partial N}{\partial x} \frac{Q}{R} = I_1 \frac{\partial^2 u_0}{\partial t^2} - p_x \]

\[ -\frac{N}{R} \frac{\partial Q}{\partial x} = I_1 \frac{\partial^2 w_0}{\partial t^2} - p_z \]

where,

\[ Q = \frac{\partial M}{\partial x} \]

Where \( p_x, p_z \) are external axial and normal forces respectively and for \( n \) numbers of lamina and

\[ I_1 = \sum_{k=1}^{N} b \rho^k (h_k - h_{k-1}) \]
The resultant force and moment are calculated as:

$$[N, M] = b \int_{-h/2}^{h/2} [1, z] \alpha dz$$

The resultant force and moment can be rewritten in terms of various stiffness parameter ($A_{11}$: Extensional stiffness, $B_{11}$: Bending stretching coupling stiffness and $D_{11}$: Bending stiffness) and primary variables as:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{bmatrix} \epsilon_0 \\ \kappa \end{bmatrix}$$

(6)

2.1. Solution methodology [10]

Considering simply supported boundary condition the general solutions is assumed as:

$$[u_0, w_0] = \sum_{m=1}^{M} [A_m \sin(\alpha_m x), C_m \cos(\alpha_m x)] \sin \omega t$$

(7)

where, $\alpha_m = \frac{m \pi}{a}$, $A_m$, $C_m$ are constant, $m$ is integer and $a$ is length of the beam.

The external transverse force($p_x$) is expanded in Fourier series and expressed as:

$$p_x = \sum_{m=1}^{M} p_{cm} \cos(\alpha_m x) \sin \omega t, \quad \text{where, } p_{cm} = \frac{2}{a} \int p_x \cos(\alpha_m x) dx$$

(8)

Using equation (6) to (9), equation (3) and (4) can be rewritten as:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} + \omega^2 \begin{bmatrix} I_1 \\ 0 \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} + \begin{bmatrix} p_{cm} \\ -p_{cm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(9)

where,

$$C_{11} = -(A_{11} + 2B_{11}/R + D_{11}/R^2) \alpha_m^2, \quad C_{22} = D_{11} \alpha_m^4 + 2(B_{11}/R) \alpha_m^2 + (A_{11}/R^2)$$

$$C_{21} = -C_{12} = B_{11} \alpha_m^3 + \alpha_m [(A_{11}/R) + (B_{11}/R^2)] + (D_{11}/R) \alpha_m^3$$

2.2. The equivalent stiffness method

The stiffness parameters ($A_{11}$, $B_{11}$, $D_{11}$) are calculated using different equivalent stiffness method

2.2.1. Vinson-Sierakowski equivalent stiffness method (VS) [11]

In this method the equivalent stiffness parameters are calculated as:

$$[A_{11}, B_{11}, D_{11}] = \sum_{k=1}^{N} b E_x \left[ (h_k - h_{k-1}), \frac{1}{2} (h_k^3 - h_{k-1}^3), \frac{1}{3} (h_k^4 - h_{k-1}^4) \right]$$

(10)

where,

$$\frac{1}{E_x} = \cos^4 \theta^k + \left[ \frac{1}{G_{12n}} - \frac{2 \nu_{12n}}{E_x} \right] \cos^4 \theta^k \sin^2 \theta^k + \frac{\sin^4 \theta^k}{E_{22n}}$$
Here, $E_{11}^k$, $E_{22}^k$, $G_{12}^k$, $\nu_{12}^k$ and $\theta^k$ are young’s modulus in fiber direction, young’s modulus transverse to fiber direction, in-plane shear deformation, in-plane poison ratio and ply-angle for $k$th layer respectively.

### 2.2.2. Rios- Chan equivalent stiffness method (RC) [12]

In this method the entire [ABD] matrix, relates the in-plane force and moment resultants to the strain, is calculated. The relationship among in-plane force and moment resultants and strains can be written as:

$$
\begin{bmatrix}
N_x \
N_y \
N_{xy} \
M_x \
M_y \
M_{xy}
\end{bmatrix} = 
\begin{bmatrix}
A \
B \
D
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \
\varepsilon_y \
\gamma_{xy} \
\kappa_x \
\kappa_y \
2\kappa_{xy}
\end{bmatrix}
$$

(11)

All components of the [ABD] can be calculated for unimodular material as given [14] and for bimodular materials the required modification is to be done. The equivalent stiffness parameters are calculated as:

$$
A_{11} = \frac{1}{a_{11} - \frac{b_{11}^2}{d_{11}}}, \quad B_{11} = \frac{1}{b_{11} - \frac{a_{11}d_{11}}{b_{11}}}, \quad D_{11} = \frac{1}{d_{11} - \frac{b_{11}^2}{a_{11}}}
$$

(12)

Where, $a_{11}=J_{11}$, $b_{11}=J_{14}$, $d_{11}=J_{44}$ and $[J]=[ABD]^{-1}$

For free vibration the value of axial and normal force are zero and then from above equation we obtained natural frequency. For bimodular material, there are two natural frequency, positive half cycle natural frequency and negative half cycle natural frequency. After obtained the value of natural frequency non-dimensional frequency is obtained by:

$$
[\Omega_1, \Omega_2] = [\omega_1, \omega_2] a^2 \sqrt{\frac{12\rho}{E_i h^2}}
$$

(13)

$E_{ic}$ is fibre direction modulus of elasticity in compression.

## 3. Results and discussion

For the present analysis we have considered aramid rubber and polyester rubber composite materials. The material properties of aramid rubber and polyester rubber are given in table 1. In this analysis a rectangular cross section simply supported curved beam having 1m length ($a$), 0.025m width ($b$) and 0.01m height ($h$) is considered. The convergence study of non-dimensional positive half cycle frequency for [0]$^4$ laminated bimodular composite has been presented in table 2 for $a/R =1$. It is observed that the converged value of frequency has been obtained in forth iteration. In table 3 the non-dimensional positive and negative half cycle free vibration frequencies for fundamental mode of vibration have been presented for cross-ply laminated beam for different $a/R$ ratio. It is observed that the percentage difference of positive and negative half cycle frequencies is very less and also there is no difference between the results of VS and RC methods. The non-dimensional frequency decreases as $a/R$ increases for almost all cases for both the materials.

### Table 1. Material properties of aramid rubber and polyester rubber [7]

| Materials properties | Properties of aramid rubber | Properties of polyester rubber |
|----------------------|-----------------------------|-------------------------------|
|                      | Tension(GPa) | Compression(GPa) | Tension(GPa) | Compression(GPa) |
| $E_1$                | 3.58        | 0.012            | 0.617        | 0.0369            |
| $E_2$                | 0.00909     | 0.012            | 0.008        | 0.0106            |
The non-dimensional frequencies decreases as a/R increases. The difference between the results obtained from VS and RC methods are not same. The difference between positive and negative half cycle frequencies decreases as a/R increases. The difference between the results obtained from VS and RC method decreases as a/R increases.

4. Conclusion
The following conclusion are drawn

Table 2. Convergence study of frequency for [0]₄ laminated bimodular composite

| Iteration No | 1 | 2 | 3 | 4 | 5 |
|--------------|---|---|---|---|---|
| Non-dimensional Frequency | 54.056 | 31.027 | 22.481 | 20.794 | 20.794 |

Table 3. Non-dimensional frequencies for [0/90], laminated curved beam for different a/R ratio.

| a/R | Ω₁ (VS) | Ω₂ (RC) | Ω₁ (VS) | Ω₂ (RC) | Ω₁ (VS) | Ω₂ (RC) |
|-----|---------|---------|---------|---------|---------|---------|
| 0.0 | 18.6571 | 18.6572 | 18.6572 | 14.1653 | 14.1789 | 14.1653 |
| 0.1 | 18.6123 | 18.6124 | 18.6456 | 14.1344 | 14.1480 | 14.1533 |
| 0.2 | 18.5136 | 18.5136 | 18.5799 | 14.0610 | 14.0746 | 14.0985 |
| 0.3 | 18.3697 | 18.3697 | 18.4686 | 13.9468 | 13.9602 | 14.0023 |
| 0.5 | 18.0554 | 18.0555 | 18.2135 | 13.6070 | 13.6201 | 13.6955 |
| 0.8 | 18.5320 | 18.5323 | 18.6960 | 12.8986 | 12.9110 | 13.0242 |
| 1.0 | 20.4796 | 20.4802 | 20.5780 | 12.3929 | 12.4047 | 12.5329 |

Table 4. Non-dimensional frequencies for [30/60] laminate curved beam for different a/R ratio.

| a/R | Ω₁ (VS) | Ω₂ (RC) | Ω₁ (VS) | Ω₂ (RC) | Ω₁ (VS) | Ω₂ (RC) |
|-----|---------|---------|---------|---------|---------|---------|
| 0.0 | 10.3502 | 12.5788 | 8.8227 | 8.8344 | 5.1282 | 6.6606 |
| 0.1 | 10.3331 | 12.5601 | 8.8090 | 8.8208 | 5.1199 | 6.6505 |
| 0.2 | 10.2849 | 12.5035 | 8.7686 | 8.7804 | 5.0961 | 6.6201 |
| 0.3 | 10.2059 | 12.4094 | 8.7021 | 8.7139 | 5.0571 | 6.5697 |
| 0.5 | 9.9599 | 12.1139 | 8.4936 | 8.5054 | 4.9354 | 6.4116 |
| 0.8 | 9.3942 | 11.4297 | 8.0124 | 8.0238 | 4.6552 | 6.0459 |
| 1.0 | 8.9118 | 10.8437 | 7.6006 | 7.6116 | 4.4160 | 5.7329 |

In Table 4 the non-dimensional frequencies are presented for angle ply laminated beam. The positive and negative half cycle frequencies are not same as cross ply laminated beam. The results obtained from VS and RC methods are not same. The difference between positive and negative half cycle frequencies decreases as a/R increases. The difference between the results obtained from VS and RC method decreases as a/R increases.
1. For cross-ply laminated beam the positive and negative half cycle frequencies are almost same. The VS and RC method give the same results.

2. For angle-ply laminated beam the positive and negative half cycle frequencies are not same. The VS and RC method give the different results.

3. The difference between positive and negative half cycle frequencies decreases as a/R increases for angle-ply laminated beam.

4. The difference between the results obtained from VS and RC method decreases as a/R increases.

5. **Symbols and notation**

| Symbol | Definition |
|--------|------------|
| $a$, $b$, $h$, $R$ | Dimensions of curved beam |
| $N$, $M$, $Q$ | Normal force, bending moment and shear force respectively |
| $u$, $u_0$ | Axial displacement at arbitrary point and mid-surface respectively |
| $v$ | Displacement in Y-direction |
| $w$, $w_0$ | Transverse displacement at arbitrary point and mid-surface respectively |
| $\varepsilon$, $\varepsilon_0$ | Strain at arbitrary point and mid-surface respectively |
| $\kappa$ | Curvature change |
| $\omega_1$, $\omega_2$ | Positive and negative frequency respectively |
| $\Omega_1$, $\Omega_2$ | Positive and negative non-dimensional frequency respectively |

6. **References**

[1] Jones R M and Morgan H S “Bending and extension of cross-ply laminates with different moduli in tension and compression”, *Computers Structures* 11, pp 181-190, 1980

[2] Timoshenko S *Strength of materials, part II, Advanced Theory and Problems*, 2nd edition, van Nostrand, *Princeton New Jersey* pp 362-369, 1941

[3] Bert C W “Models for fibrous composites with different properties in tension and Compression”, *Journal of Engineering Materials and Technology, Transactions of the ASME* 99H pp 344-349, 1977

[4] Papazoglou V J and Tsouvalis N G “Mechanical behaviour of bimodular laminated plates”, *composite structure*, 17 pp 1-22, 1991

[5] Khan K, Patel B P and Nath Y “Dynamic characteristics of bimodular laminated panel using an efficient layerwise theory”, *Composite structure* 132 pp 759-771, 2015

[6] Tran A D and Bert CW “Bending of thick beams of bimodular materials”, *composite structure* 15 pp 627-642, 1982

[7] Kang K, Bert C W “Vibration analysis of shear deformable circular arches by the differential quadrature method” *Journal of sound and vibration*, 181 pp 353-360, 1995

[8] Zeng H, Bert C W “Vibration analysis of tapered bar using differential transformation” *Journal of sound and vibration*, 242 pp 737-739, 2001

[9] Patel B P, Gupta S S and Sarda R “Free flexural vibration behaviour of bimodular material angle-ply laminated composite plates”, *Journal of sound and vibration* 286 pp 167-186, 2005

[10] Qatu M S “theories and analysis of thin and moderately thick laminated composite curved beam”, *Int. J. solid structure*, 30 pp 2743-2756, 1993

[11] Vinson J R and Sierakowski R L,”The behavior of structures composed of composite material “ *Kluwer Academic Publication. ISBN*, 2009

[12] Rios G and Chan W SA,”unified analysis of stiffener reinforced composite beams with arbitrary cross-section”, *Ph.D Dissertation (University of Texas at Arlington)*, 2009