LONG TIME STABILITY OF KAM TORI FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this paper, we prove the long time stability of KAM tori for the nonlinear Schrödinger equation on the torus with arbitrary dimensions.

1. Introduction

In this paper, we consider the long time stability of the invariant tori for the $d$-dimensional nonlinear Schrödinger (NLS) equation

$$-i\dot{u} = -\Delta u + V(x) \ast u + \varepsilon \frac{\partial F}{\partial \bar{u}}(|u|^2), \quad u = u(t, x)$$

under the periodic boundary condition $x \in \mathbb{T}^d, d \geq 1$. The convolution function $V : \mathbb{T}^d \to \mathbb{C}$ is analytic and the Fourier coefficient $\hat{V}(a)$ takes real value, when expanding $V$ into Fourier series $V(x) = \sum_{a \in \mathbb{Z}^d} \hat{V}(a)e^{i\langle a, x \rangle}$. The nonlinearity $F$ is real analytic.

The NLS equation (1.1) is a Hamiltonian PDE. The KAM theory is a well-known approach to establish the existence of the invariant tori for Hamiltonian PDEs. The invariant tori so constructed are often referred to as the KAM tori. The “KAM for PDE” theory started in late 1980’s and originally applied to the one spatial dimensional PDEs, which is now well understood. See for example [Kuk93, Kuk00, KP96, Pos96a, Pos96b, Way90, LY11, BBP13, BBHM18] and the references therein.

However, the KAM theory for space-multidimensional Hamiltonian PDEs is at its early stage. The first breakthrough was made by Bourgain [Bou98] on the two dimensional NLS equation, in which he developed Craig and Wayne’s scheme on periodic problems. Using new techniques of Green’s function estimates in the spectral theory, Bourgain proved the persistence of invariant tori for space-multidimensional NLS and nonlinear wave (NLW) equations [Bou95]. The above mentioned method is now known as the Craig-Wayne-Bourgain (CWB) method. See also [BB13, BB20, Wan16, Wan19, Wan20] and the references therein. The classical KAM approach for space-multidimensional Hamiltonian PDEs was developed by Eliasson and Kuksin [EK10] on NLS equation. They take a sequence of symplectic transformations such that the transformed Hamiltonian guarantees the existence of the invariant tori. Moreover, the KAM approach in [EK10] also provides the reducibility and linear stability of the obtained invariant tori. See also [EGK16, GY06, GXY11, PP13, PP12, Yua21] for the KAM approach on the space-multidimensional PDEs.

To ensure that the obtained KAM tori can be observed in physics and other real applications, one has to prove that those KAM tori are stable in some sense, among which the simplest one is the linear stability. Let us recall the definition of the linear stability of the invariant tori. Consider a nonlinear differential equation

$$\dot{x} = X(x),$$

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which has an invariant torus $\mathcal{T}$ carrying the quasi-periodic flow $x(t) = x_0(t)$. We say that the invariant torus $\mathcal{T}$ is linearly stable if the equilibrium of the linearized equation

$$\dot{y} = DX(x_0(t))y$$

of $\text{(1.2)}$ along $\mathcal{T}$ is Lyapunov stable. A more general definition is that $\mathcal{T}$ is linearly stable if all the Lyapunov exponents of $\mathcal{T}$ equal to zero.

The classical KAM tori are linearly stable as a matter of the KAM approach. To see this, we consider a Hamiltonian perturbation

$$H = N + P = N + P^{\text{low}} + P^{\text{high}}$$

of the integrable part

$$N = \langle \omega, y \rangle + \sum_{j \in \mathbb{Z}^d} \Omega_j z_j \bar{z}_j,$$

where

$$P^{\text{low}} = R^y + \langle R^y, y \rangle + \langle R^z, z \rangle + \langle R^{\bar{z} \bar{z}}, \bar{z} \rangle + \langle R^{\bar{z} z}, z \rangle + \langle R^{\bar{z} \bar{z} z}, \bar{z} \rangle + \langle R^{\bar{z} z \bar{z}}, \bar{z} \rangle$$

and

$$P^{\text{high}} = O(||y||^2 + ||z||: ||y|| + ||z||^3).$$

The classical KAM approach (for $d = 1$) aims at taking a sequence of symplectic transformations to eliminate all terms in $P^{\text{low}}$, except for the averages $\langle \hat{R}v(0), y \rangle$ and $\sum_{i=\bar{j}} R^{\bar{z} \bar{z}}(0)z_i \bar{z}_j$. In particular, the quadratic terms in $P^{\text{low}}$ are reduced to $\sum_{i=\bar{j}} \hat{R}^{\bar{z} \bar{z}}(0)z_i \bar{z}_j$ of constant coefficients, which can be put into the integrable part $N$ for the next iteration. In this way, the linearized equation of the obtained KAM tori can be reduced to

$$i\dot{z} = (\Omega + \varepsilon[R])z,$$

where $\Omega = \text{diag}(\Omega_j : j \in \mathbb{Z})$ and $[R] = \text{diag}((R^{\bar{z} \bar{z}})^\wedge(0) : i = j \in \mathbb{Z})$ are diagonal and constant. Obviously, the equilibrium $z = 0$ of $\text{(1.3)}$ is Lyapunov stable, and thus the KAM tori are linearly stable.

Unfortunately, there is a difficulty in extending the classical KAM approach for $d = 1$ to the case of $d > 1$. Taking NLS equation for example, the normal frequency satisfies $\Omega_j \sim |j|^d, j \in \mathbb{Z}^d$ after writing NLS equation into an infinitely dimensional Hamiltonian system as above. It follows that the normal frequencies may have unbounded multiplicities since $\# \{j' \in \mathbb{Z}^d : |j'| = |j| \} \sim |j|^{d-1} \to \infty$ as $|j| \to \infty$. This feature leads to serious resonances in solving the homological equations, which might impede the convergence of the symplectic transformations. Eliasson-Kuksin [EK10] analyzed carefully the separation property of the normal frequencies and provided insight on the Töplitz-Lipschitz property of the Hamiltonian.

Using the “super Newton iteration” (rather than the usual Newton iteration in KAM theory for $d = 1$), they succeeded in eliminating $P^{\text{low}}$, but leaving an infinitely dimensional block-diagonal and constant matrix in the quadratic term of $z, \bar{z}$ behind. As a result, they proved that there are plenty of KAM tori for NLS equation with $d > 1$, whose all Lyapunov exponents equal to zero and hence are linearly stable. As for NLW equation, the normal frequency $\Omega_j = |j| = \sqrt{j_1^2 + \cdots + j_d^2}$ does not have a good separation property like NLS equation.

Although Bourgain [Bou05] had applied the CWB method to prove the existence of KAM tori for NLW equation with $d > 1$, the linear stability of those KAM tori remains open. Recently, by modifying the CWB method, the authors [HSSY20] obtained not only the existence but also the linear stability of the KAM tori for the Hamiltonian system with finite degrees of freedom.
In many cases, we cannot determine the stability of the nonlinear system from its linearized equation directly. Typically, in linear plane dynamical systems, a center equilibrium can become a focus after certain perturbation. There are also examples that a linearly stable model can be triggered by an initial perturbation to exhibit chaotic dynamics [GG94]. This prompts us to study the nonlinear stability, among which the long time stability is of particular interest in PDEs. In finitely dimensional Hamiltonian system, the best result concerning the long time stability is the Nekhoroshev estimate [Neh77]. Consider a $n$ degree of freedom Hamiltonian $H = N(y) + \varepsilon R(y, x)$, where $(y, x) \in \mathbb{R}^n \times \mathbb{T}^n$ is the action-angle variable. Assume the functions $N$ and $R$ are analytic in $(y, x)$ in some open domain. The Nekhoroshev estimate tells us that the variation of the actions of all orbits remains small over a finite, but exponentially long time interval. More precisely, for sufficiently small $\varepsilon$, one has

$$|y(t) - y(0)| \lesssim \varepsilon^n \quad \text{for } |t| \lesssim \exp(\varepsilon^{-b}),$$

where the constants $a, b$ depend on the degree of the freedom. In particular, if $N$ is convex, one can get $a = b = \frac{1}{4}$. See Pöschel [Pos93]. Noticing also that the instabilities such as Arnold diffusion [Arn64] may occur with the degree of freedom $n \geq 3$ and transfer of energy may appear in NLS equation [CKS91], one should not expect some orbits are stable forever. Consequently, it is reasonable to apply the Nekhoroshev estimate on the long time stability of orbits to describe the stability of the Hamiltonian system.

For NLS equation (or generally the Hamiltonian PDEs), the degree of freedom of the Hamiltonian is infinite. One immediately gets that $a = b = 0$ and the Nekhoroshev estimate in (1.4) no longer works. Instead, Bourgain [Bou96] suggested investigating the long time behavior of orbits in the neighborhood of the equilibrium and relaxed the stable time interval from $|t| < \exp(-\varepsilon^d)$ to $|t| < \varepsilon^{-M}$ for large $M$. From then on, there are lots of literature devoted to the long time stability of the equilibrium for Hamiltonian PDEs. See [Bam03, DS04, BG06, FG13, YZ14, BMP20, CLW20, CMW20, BG21]. We emphasize that in [BG06] Bambusi and Grébert introduced the tame property of the vector field, which simplifies the proof considerably.

In contrast to the equilibrium, the KAM tori are much more complicated solutions of NLS equation. It is known that the KAM tori are sup-exponentially long time stable for the finitely dimensional Hamiltonian system [BFG88]. For Hamiltonian PDEs, the study of the long time stability of KAM tori is limited to the case of $d = 1$. For instance, [CLY16] and [CGL13] studied the long time stability of the KAM tori for NLS and NLW equations, respectively. For cubic defocusing NLS equation on $\mathbb{T}^2$, Maspero-Procesi [MP18] studied the large-time stability (with the stable time interval $|t| \leq \delta^{-2}$) of small finite gap solutions, which depend only on one spatial variable. For $d > 2$, as far as we know, there seems no results in this respect. The main result of the present paper is that the majority of the KAM tori obtained by Eliasson-Kuksin [EK10] are stable in long time $|t| < \delta^{-M}$ for large $M$. More precisely, we have the following theorem.

**Theorem 1.1.** Under the assumptions for equation (1.1), if $\varepsilon > 0$ is sufficiently small, then for typical $V$ (in the sense of measure), the nonlinear Schrödinger equation (1.1) possesses a linearly stable KAM torus $\mathcal{T} = \mathcal{T}_V$ in the Sobolev space $H^p(\mathbb{T}^d)$. Moreover, letting $M \approx \varepsilon^{-\frac{d}{4}}$ and $p \geq 80(4d)^d(M + 7)^d + 1$, there exists a small $\delta_0$ depending on $p, M$ and $\dim \mathcal{T}$ such that for any $0 < \delta < \delta_0$ and any solution $u(t, x)$ of (1.1) with the initial datum $u(0, \cdot)$ satisfying

$$d_{H^p(\mathbb{T}^d)}(u(0, \cdot), \mathcal{T}) := \inf_{w \in \mathcal{T}} \|u(0, \cdot) - w\|_{H^p(\mathbb{T}^d)} < \delta,$$

we have

$$d_{H^p(\mathbb{T}^d)}(u(t, \cdot), \mathcal{T}) < 2\delta, \quad \forall |t| < \delta^{-M}.$$
In other words, the KAM tori for the nonlinear Schrödinger equation \((1.1)\) are stable in long time.

Theorem 1.1 consists of two results (Theorem 2.12 and Theorem 2.13) after writing NLS equation as the infinitely dimensional Hamiltonian system. By excluding some parameters, we establish the KAM theorem (see Theorem 2.12) to guarantee the existence and linear stability of the KAM tori, which have already been obtained in \([\text{EK}10]\). However, to study the long time stability of the KAM tori, we modify the proof in \([\text{EK}10]\) by taking Kolomogorov’s iterative scheme such that the transformed Hamiltonian after the KAM iteration is still defined on an open domain. By the further parameter exclusion, we show the long time stability of the majority of the obtained KAM tori (see Theorem 2.13), by establishing the partial normal form of the transformed Hamiltonian.

The proof of Theorem 1.1 is given at the end of section 2. We clarify the main ideas in the proof of the theorem.

i) Write \((1.1)\) as an infinitely dimensional Hamiltonian system

\[
H = \sum_{a \in \mathcal{A}} \omega_a r_a + \frac{1}{2} \sum_{a \in \mathcal{L}} \Omega_a (\xi_a^2 + \eta_a^2) + f
\]

and split

\[
f = f^{\text{low}} + f^{\text{high}}.
\]

The KAM approach developed in \([\text{EK}10]\) aims at eliminating \(f^{\text{low}}\), but leaving some resonant terms. In this way, the transformed Hamiltonian \(H_\infty = H \circ \psi_\infty\) takes the form of

\[
H_\infty = \langle \omega', r \rangle + \frac{1}{2} \langle \zeta, (\Omega + Q)\zeta \rangle + f^{\text{high}}_\infty.
\]

One sees that \(r = 0, \zeta = 0\) is the KAM torus for \(H_\infty\).

Recall that the domain \(D(\mu_j, \sigma_j)\) of the symplectic transformation \(\psi_j\) in \([\text{EK}10]\) degenerates into a singleton since \(\mu_j \to 0, \sigma_j \to 0\) as \(j \to \infty\). Since \(\psi_j\) is quadratic in \(r\) and \(\zeta\), one surely can extend the domain of \(\psi_j\) to the initial domain \(D(\mu_0, \sigma_0)\). In addition, one can take \(D(\mu_0, \sigma_0)\) as the domain of the vector field for the linearized equation of the Hamiltonian \(H_\infty\), based on which one is able to study the Lyapunov stability of the equilibrium on \(D(\mu_0, \sigma_0)\). Along this line, Eliasson-Kuksin \([\text{EK}10]\) developed a powerful KAM approach for NLS equation with \(d > 1\) to show not only the existence of the KAM tori, but also their linear stability.

However, when studying the long time stability of those KAM tori, we have to take the domain of the high order term \(f^{\text{high}}_\infty\) into consideration. Since the domain \(\cap_{j=1}^{\infty} D(\mu_j, \sigma_j)\) of \(\psi_\infty\) is a singleton, we can only define \(f^{\text{high}}_\infty\) on \(D(0, 0)\), on which the KAM tori are indeed constructed for the original NLS equation. We emphasize that the domain of \(f^{\text{high}}_\infty\) usually cannot be extended to \(D(\mu_0, \sigma_0)\) since \(f^{\text{high}}_\infty\) is not a polynomial function. For that reason, we introduce Kolmogorov’s iterative scheme in the framework of \([\text{EK}10]\) by modifying the homological equations such that \(D(\mu_j, \sigma_j) \supset D(\frac{\mu_j}{2^j}, \frac{\sigma_j}{2^j})\) for all \(j\). Then we can define \(f^{\text{high}}_\infty\) on an open set \(D(\frac{\mu_0}{2^j}, \frac{\sigma_0}{2^j})\) to take normal form computations.

ii) To establish the long time stability of the KAM tori so constructed, we shall take symplectic transformation of \(f^{\text{high}}_\infty\) to obtain a suitable Birkhoff normal form. We will not put the frequency shift produced in the symplectic transformation into the homological equations, and we will finally get

\[
f^{\text{high}}_\infty; \text{new} = O(\|z\|^{M+1}).
\]
It then follows that the KAM torus \((r = 0, z = 0)\) is stable in a long time interval of length \(\delta^{-M}\).

In this process, the tame property for space-multidimensional NLS equation can be preserved during the KAM iteration. Moreover, we will take the advantage of the momentum conservation. The corresponding Hamiltonian \(f\) consists of monomials

\[
e^{i(k \cdot r)} \prod_{a \in A} r_a^n \prod_{a \in \mathcal{L}} u_a^l v_a^m
\]
satisfying

\[
- \sum_{a \in A} k_a a + \sum_{a \in \mathcal{L}} (l_a - m_a) a = 0.
\]

We need to verify momentum conservation in the KAM iteration. The persistence under Poisson bracket can be checked directly. Since the homological equations are of constant coefficients, the persistence under solving homological equations can also be directly checked. By the momentum conservation, we can deal with the frequency shift to establish the long time stability.

We end up this section with several remarks.

**Remark 1.2.** In this paper, we benefit a lot from the momentum conservation, which comes from the \(x\)-independent nonlinearity \(F\). See also [GY06, PP15, PP12]. For the general case, there are extra difficulties in dealing with a block-diagonal shift of frequency.

**Remark 1.3.** As mentioned before, the existence of KAM tori (quasi-periodic solution) for space-multidimensional NLW equation can be obtained by the CWB method [Bou05]. However, on the one hand, a counterpart of KAM approach for NLW equation like Eliasson-Kuksin [EK10] (on NLS equation) is still not available. See [EGK16]. On the other hand, the CWB method does not provide a normal form of the Hamiltonian in the neighborhood of the KAM torus. As a result, the linear stability of KAM tori (quasiperiodic solutions) for NLW equation with \(d > 1\) is not clear, let alone the long time stability.

The paper is organized as follows. In section 2 we introduce some notations as the preliminary and present our main results. In section 3 we formulate and solve the homological equation. In section 4 we prove the KAM theorem to show the existence of the KAM tori for NLS equation. In section 5 we construct a partial normal form to show the long time stability of the obtained KAM tori.

## 2. Main results

In this section, we present the main results of the infinitely dimensional Hamiltonian system. To begin with, we introduce some notations as the preliminary.

### 2.1. Preliminary

In this part, we collect some notations and definitions, which are frequently used throughout the paper. In subsection 2.1.1 we write NLS equation as an infinitely dimensional Hamiltonian system. In subsection 2.1.2 we introduce the tame property of the Hamiltonian vector field. In subsection 2.1.3 we introduce the Töplitz-Lipschitz property. Finally, in subsection 2.1.4 we introduce the normal form matrix.

#### 2.1.1. Hamiltonian formulation of NLS equation

In order to prove Theorem 1.1 we write the nonlinear Schrödinger equation (1.1) as an infinitely dimensional Hamiltonian system. We keep the notations consistent with those in [EK10].
Write
\[ u(x) = \sum_{a \in \mathbb{Z}^d} u_a e^{i(a \cdot x)} , \quad \overline{u(x)} = \sum_{a \in \mathbb{Z}^d} v_a e^{i(-a \cdot x)} , \]
and let
\[ \zeta_a = \left( \begin{array}{c} \xi_a \\ \eta_a \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} u_a + v_a \\ -i(u_a - v_a) \end{array} \right) . \]
Then the nonlinear Schrödinger equation (1.1) becomes a real Hamiltonian system with the symplectic structure \( \partial \xi \wedge \partial \eta \) and the Hamiltonian
\[ \frac{1}{2} \sum_{a \in \mathbb{Z}^d} (|a|^2 + \tilde{V}(a))(\xi_a^2 + \eta_a^2) + \varepsilon \int_{\mathbb{T}^d} F(|u(x)|^2) dx . \]
Let \( \mathcal{A} \) be a finite subset of \( \mathbb{Z}^d \) and \( \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A} \). Introduce action-angle variables \( (\varphi_a, r_a), a \in \mathcal{A}, \)
\[ \xi_a = \sqrt{2(r_a + q_a)} \cos \varphi_a, \quad \eta_a = \sqrt{2(r_a + q_a)} \sin \varphi_a, \quad q_a > 0 . \]
Let
\[ \omega_a = |a|^2 + \tilde{V}(a), a \in \mathcal{A} \quad \text{and} \quad \Omega_a = |a|^2 + \tilde{V}(a), a \in \mathcal{L} . \]
We have the Hamiltonian
\[ h + f = \sum_{a \in \mathcal{A}} \omega_a r_a + \frac{1}{2} \sum_{a \in \mathcal{L}} \Omega_a (\xi_a^2 + \eta_a^2) + \varepsilon \int_{\mathbb{T}^d} F(|u(x)|^2) dx . \]
Assume \( f \) is real analytic on
\[ D(\rho, \mu, \sigma) = \{ (\varphi, r, \xi, \eta) \in (\mathbb{C}/2\pi\mathbb{Z})^4 \times \mathbb{C}^4 \times l_2^2 : |3\varphi| \leq \rho, |r| \leq \mu, \| \xi \|_p \leq \sigma \} , \]
where
\[ \| \zeta \|_p^2 = \sum_{a \in \mathcal{L}} (|\xi_a|^2 + |\eta_a|^2) a |a \|^{2p}, \quad (a) = \max(|a|, 1) . \]

2.1.2. The \( p \)-tame norm of the Hamiltonian vector field. In this paper, \( \| \cdot \| \) is an operator norm or \( l^2 \) norm. \( | \cdot | \) will in general denote a sup norm. For \( a \in \mathbb{Z}^d \), we use \( |a| \) for the \( l^2 \) norm. Let \( \mathcal{A} \) be a finite subset of \( \mathbb{Z}^d \) and \( \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A} \). Denote \( \langle \zeta, \zeta' \rangle = \sum (\xi_a \xi'_a + \eta_a \eta'_a) \) and \( J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) . \)

For \( \gamma \geq 0 \), we denote
\[ l_{p, \gamma}^2 = \{ \zeta = (\xi, \eta) \in \mathbb{C}^{2d} : \| \zeta \|_{p, \gamma} < \infty \} , \]
where
\[ \| \zeta \|_{p, \gamma}^2 = \sum_{a \in \mathcal{L}} (|\xi_a|^2 + |\eta_a|^2) e^{2\gamma |a|} a |a \|^{2p}, \quad (a) = \max(|a|, 1) . \]
When \( \gamma = 0 \), we simply write \( l_{p}^2 \) and \( \| \zeta \|_p \). The phase space of the Hamiltonian dynamical system is defined by
\[ \mathcal{P}^p = (\mathbb{C}/2\pi\mathbb{Z})^4 \times \mathbb{C}^4 \times l_{p}^2 . \]

Let \( U \subset \mathbb{R}^{2d} \) be a parameter set with positive measure (in the sense of Gauss or Kolmogorov). We define \( p \)-tame norm as in [CLY16].

**Definition 2.1.** Let
\[ D(\rho) = \{ \varphi \in (\mathbb{C}/2\pi\mathbb{Z})^4 : |3\varphi| \leq \rho \} , \]
and \( f : D(\rho) \times U \to \mathbb{C} \) be analytic in \( \varphi \in D(\rho) \) and \( C^1 \) (in the sense of Whitney) in \( w \in U \) with
\[ f(\varphi; w) = \sum_{k \in \mathbb{Z}^4} \hat{f}(k; w) e^{i(k \cdot \varphi)} . \]
Define the norm
\[
\|f\|_{D(p) \times U} = \sup_{w \in U, a \in \mathbb{Z}_d} \sum_{k \in \mathbb{Z}^A} \left( |f(k; w)| + |\partial_{w,a} f(k; w)| \right) e^{|k|p}.
\]

**Definition 2.2.** Let
\[
D(\rho, \mu) = \{(\varphi, r, \zeta) \in (\mathbb{C}/2\pi\mathbb{Z})^A \times \mathbb{C}^A : |3\varphi|, |r| \leq \rho, |\zeta| \leq \mu \},
\]
and \( f : D(\rho, \mu) \times U \to \mathbb{C} \) be analytic in \( (\varphi, r) \in D(\rho, \mu) \) and \( C^1 \) in \( w \in U \) with
\[
f(\varphi, r; w) = \sum_{\alpha \in \mathbb{N}^A} f^\alpha(\varphi; w) r^\alpha.
\]
Define the norm
\[
\|f\|_{D(\rho, \mu) \times U} = \sum_{\alpha \in \mathbb{N}^A} \|f^\alpha(\varphi; w)\|_{D(\rho, \mu) \times U} |\alpha|.
\]

**Definition 2.3.** Let
\[
D(\rho, \mu, \sigma) = \{(\varphi, r, \zeta) \in (\mathbb{C}/2\pi\mathbb{Z})^A \times \mathbb{C}^A \times \mathbb{C}^\beta : |3\varphi|, |r| \leq \rho, |\zeta|, |\zeta| \leq \mu, |\sigma| \leq \sigma \},
\]
and \( f : D(\rho, \mu, \sigma) \times U \to \mathbb{C} \) be analytic in \( (\varphi, r, \zeta) \in D(\rho, \mu, \sigma) \) and \( C^1 \) in \( w \in U \) with
\[
f(\varphi, r, \zeta; w) = \sum_{\alpha \in \mathbb{N}^A, \beta \in \mathbb{N}^2} f^{\alpha \beta}(\varphi; w) r^\alpha \zeta^\beta,
\]
where \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}, \mathcal{L}_{-1} = \mathcal{L} \). For \( a \in \mathcal{L}_1, \zeta_a = \xi_a \), and for \( a \in \mathcal{L}, \zeta_a = \eta_a \). Define the modulus
\[
[f]_{D(\rho, \mu) \times U}(\zeta) = \sum_{\beta \in \mathbb{N}^2} \|f^\beta(\varphi, r; w)\|_{D(\rho, \mu) \times U} \zeta^\beta,
\]
where
\[
f^\beta(\varphi, r; w) = \sum_{\alpha \in \mathbb{N}^A} f^{\alpha \beta}(\varphi; w) r^\alpha.
\]

For a homogeneous polynomial \( f(\zeta) \) of degree \( h > 0 \), it is associated with a symmetric \( h \)-linear form \( \hat{f}(\zeta^{(1)}, \ldots, \zeta^{(h)}) \) such that \( \hat{f}(\zeta, \ldots, \zeta) = f(\zeta) \). For a monomial
\[
f(\zeta) = f^\beta \zeta^\beta = f^\beta \zeta_{j_1} \cdots \zeta_{j_h},
\]
define
\[
\hat{f}(\zeta^{(1)}, \ldots, \zeta^{(h)}) = \hat{f}^\beta \zeta^\beta = \frac{1}{h!} \sum_{\tau_h} f^\beta(\tau_h^{(1)}) \cdots \zeta_{j_h}^{(h)}
\]
where \( \tau_h \) is an \( h \)-permutation. For a homogeneous polynomial
\[
f(\zeta) = \sum_{|\beta| = h} f^\beta \zeta^\beta,
\]
define
\[
\hat{f}(\zeta^{(1)}, \ldots, \zeta^{(h)}) = \sum_{|\beta| = h} \hat{f}^\beta \zeta^\beta.
\]
Now we can define \( p \)-tame norm of a Hamiltonian vector field. We first consider a Hamiltonian
\[
f(\varphi, r, \zeta; w) = f_h = \sum_{\alpha \in \mathbb{N}^A, \beta \in \mathbb{N}^2, |\beta| = h} f_h^{\alpha \beta}(\varphi; w) r^\alpha \zeta^\beta.
\]
Let $f_\zeta = (f_\eta, -f_\zeta)$, and the Hamiltonian vector field $X_f$ is $(f_r, -f_\phi, f_\zeta)$. For $h \geq 1$, denote

$$\| (\zeta^h) \|_{p,1} = \frac{1}{h} \sum_{j=1}^h \| \zeta(1) \|_1 \cdots \| \zeta^{(j-1)} \|_1 \| \zeta^{(j)} \|_p \| \zeta^{(j+1)} \|_1 \cdots \| \zeta^{(h)} \|_1.$$  

**Definition 2.4.** Let

$$||| f_\zeta \|||_{p,D(p,\mu) \times U}^T = \left\{ \begin{array}{ll} \sup_{0 \neq \zeta^{(i)} \in U, 1 \leq j \leq h-1} \frac{\| h_{\zeta}^{(j)} \|_{D(p,\mu) \times U} \| (\zeta^{(i)} \cdots \zeta^{(h-1)}) \|_p}{\| (\zeta^{(i)} \cdots \zeta^{(h-1)}) \|_p}, & h \geq 2 \\ \| f_\zeta \|_{D(p,\mu) \times U}, & h = 0, 1. \end{array} \right.$$  

Define the $p$-tame norm of $f_\zeta$ by

$$||| f_\zeta \|||_{p,D(p,\mu,\sigma) \times U} = \max(||| f_\zeta \|||_{p,D(p,\mu) \times U}, ||| f_\zeta \|||_{p,D(p,\mu) \times U}) \sigma^{-h-1}.$$  

**Definition 2.5.** Let

$$||| f_r \|||_{D(p,\mu) \times U} = \left\{ \begin{array}{ll} \sup_{0 \neq \zeta^{(i)} \in U, 1 \leq j \leq h-1} \frac{||| f_r \|||_{D(p,\mu) \times U}}{||| (\zeta^{(i)} \cdots \zeta^{(h-1)}) \|||_1}, & h \geq 1 \\ \| f_r \|_{D(p,\mu) \times U}, & h = 0. \end{array} \right.$$  

Define the norm of $f_r$ by

$$||| f_r \|||_{D(p,\mu,\sigma) \times U} = ||| f_r \|||_{D(p,\mu) \times U} \sigma^{-h}.$$  

The norm of $f_\phi$ is defined as that of $f_r$.

**Definition 2.6.** Define the $p$-tame norm of the Hamiltonian vector field $X_f$ by

$$||| X_f \|||_{p,D(p,\mu,\sigma) \times U} = ||| f_r \|||_{D(p,\mu,\sigma) \times U} + \frac{1}{\mu} ||| f_\phi \|||_{D(p,\mu,\sigma) \times U} + \frac{1}{\sigma} ||| f_\zeta \|||_{p,D(p,\mu,\sigma) \times U}.$$  

**Definition 2.7.** For a Hamiltonian

$$f(\varphi, r, \zeta; w) = \sum_{h \geq 0} f_h, \quad f_h = \sum_{\alpha \in \mathbb{N}^A, \beta \in \mathbb{N}^C, |\beta| = h} f_h^{\alpha \beta}(\varphi; w) r^\alpha \zeta^\beta,$$

define the $p$-tame norm of the Hamiltonian vector field $X_f$ by

$$||| X_f \|||_{p,D(p,\mu,\sigma) \times U} = \sum_{h \geq 0} ||| X_f \|||_{p,D(p,\mu,\sigma) \times U}.$$  

**Remark 2.8.** The $p$-tame norm can also be defined in complex coordinates

$$z = \left( \begin{array}{c} u \\ v \end{array} \right) = C^{-1} \left( \begin{array}{c} \xi \\ \eta \end{array} \right), \quad C = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -i & i \end{array} \right).$$

Following the proof of Theorem 3.1 in [CLY16], we have the following proposition.

**Proposition 2.9.** If $0 < \tau < \rho, 0 < \tau' < \frac{\rho}{2}$, then

$$||| X_{(f,g)} \|||_{p,D(\rho-\tau,(\sigma-\tau)^2,\sigma-\tau') \times U} \leq C \max \left( \frac{1}{\tau}, \frac{\sigma}{\tau'} \right) \||| X_f \|||_{p,D(\rho,\sigma^2,\sigma) \times U} \||| X_g \|||_{p,D(\rho,\sigma^2,\sigma) \times U},$$

where $C > 0$ is a constant depending on $\# A$.

We define the weighted norm of the Hamiltonian vector field $X_f$ by

$$||| X_f \|||_{p,\rho, D(p,\mu,\sigma) \times U} = \sup_{(\varphi, r, \zeta; w) \in D(p,\mu,\sigma) \times U} ||| X_f \|||_{p,\rho, D(p,\mu,\sigma) \times U},$$

where

$$||| X_f \|||_{p,\rho, D(p,\mu,\sigma) \times U} = ||| f_r \|||_{p,\rho} + \frac{1}{\mu} ||| f_\phi \|||_{p,\rho} + \frac{1}{\sigma} ||| f_\zeta \|||_{p,\rho}.$$
Following the proof of Theorem 3.5 in [CLY16], we have

\[ |||X_f|||_{\mathcal{P}_D(\rho,\mu,\sigma)\times U} \leq |||X_f|||_{\mathcal{P}_D(\rho,\mu,\sigma)\times U}.\]

2.1.3. Töplitz-Lipschitz property. Recall the definition of Töplitz-Lipschitz matrices in [EK10]. Let \( gl(2, \mathbb{C}) \) be the space of all complex \( 2 \times 2 \)-matrices. For \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in gl(2, \mathbb{C}) \), denote \( \pi A = \frac{1}{2} \begin{pmatrix} a + d & b - c \\ c - b & a + d \end{pmatrix} \) and \( [A] = \begin{pmatrix} |a| & |b| \\ |c| & |d| \end{pmatrix} \). Now consider an infinite-dimensional \( gl(2, \mathbb{C}) \)-valued matrix

\[ A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C}), (a, b) \mapsto A^b_a.\]

For any \( \mathcal{D} \subset \mathcal{L} \times \mathcal{L} \), define

\[ |A|_\mathcal{D} = \sup_{(a,b) \in \mathcal{D}} ||A^b_a||,\]

where \( || \cdot || \) is the operator norm. Define \( (\pi A)^b_a = \pi A^b_a \) and \( (\mathcal{E}^\pm A)^b_a = [A^b_a]e^{\gamma(a \mp b)}. \) Define the norm

\[ |A|_\gamma = \max(|\mathcal{E}^+ \pi A|_{\mathcal{L} \times \mathcal{L}}, |\mathcal{E}^- (1 - \pi) A|_{\mathcal{L} \times \mathcal{L}}).\]

Define

\[ T^+_\Delta A = A \big|_{\{(a,b) \in \mathcal{L} \times \mathcal{L} : |a-b| \leq \Delta\}}, T^-_\Delta A = T^+_\Delta \pi A + T^-_\Delta (1 - \pi) A.\]

A matrix \( A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C}) \) is Töplitz at \( \infty \), if for all \( a, b, c \), the two limits

\[ \lim_{t \to +\infty} A^a_b c = A^c_a(\pm, c).\]

For \( c \neq 0 \), define \( (\mathcal{M}_c A)^b_a = \left(\max\left(\frac{|a|}{|c|}, \frac{|b|}{|c|}\right) + 1\right) [A^b_a]. \) For \( \Lambda \geq 0 \), define the Lipschitz domain \( D^+_{\Lambda}(c) \subset \mathcal{L} \times \mathcal{L} \) be the set of all \( (a, b) \) such that there exist \( a’, b’ \in \mathbb{Z}^d, t \geq 0 \) such that

\[ |a = a’ + tc| \geq \Lambda(|a’| + |c|)|c|, \quad |b = b’ + tc| \geq \Lambda(|b’| + |c|)|c|, \quad \frac{|a|}{|c|}, \frac{|b|}{|c|} \geq 2\Lambda^2.\]

Define \( (a, b) \in D^+_{\Lambda}(c) \) if and only if \( (a, -b) \in D^-_{\Lambda}(c). \) Define the Lipschitz constants

\[ \text{Lip}_{\Lambda, \gamma}^+, A = \sup_c |\mathcal{E}^+ \mathcal{M}_c (A - A(\pm, c))|_{D^+_{\Lambda}(c)},\]

and the Lipschitz norm

\[ \langle A \rangle_{\Lambda, \gamma} = \max(\text{Lip}_{\Lambda, \gamma}^+ \pi A, \text{Lip}_{\Lambda, \gamma}^- (1 - \pi) A) + |A|_\gamma.\]

For \( d = 2 \), the matrix \( A \) is Töplitz-Lipschitz if it is Töplitz at \( \infty \) and \( \langle A \rangle_{\Lambda, \gamma} < \infty \) for some \( \Lambda, \gamma \).

For \( d > 2 \), we can define Töplitz-Lipschitz matrices inductively (see Section 2.4 in [EK10]).

**Definition 2.10.** Let

\[ D^\gamma(\sigma) = \{ \zeta \in l^2_{p, \gamma} : ||\zeta||_{p, \gamma} \leq \sigma \},\]

and \( f : D^0(\sigma) \to \mathbb{C} \) be real analytic. We say that \( f \) is Töplitz at \( \infty \) if \( \partial_\zeta^2 f(\zeta) \) is Töplitz at \( \infty \) for all \( \zeta \in D^0(\sigma) \). Define the norm \( [f]_{\Lambda, \gamma, \sigma} \) to be the smallest \( C \) such that

\[ |f(\zeta)| \leq C, \quad \forall \zeta \in D^0(\sigma),\]

\[ ||\partial_\zeta f(\zeta)||_{p, \gamma} \leq \frac{C}{\sigma}, \quad \langle \partial_\zeta^2 f(\zeta) \rangle_{\Lambda, \gamma} \leq \frac{C}{\sigma^2}, \quad \forall \zeta \in D^\gamma(\sigma), \forall \gamma' \leq \gamma.\]
Definition 2.11. Let $A(w) : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$ be $C^1$ in $w \in U$. Define

$$|A|_{\gamma;U} = \sup_{w \in U}(|A(w)|_{\gamma}, |\partial_w A(w)|_{\gamma}).$$

If $A(w), \partial_w A(w)$ are Töplitz at $\infty$ for all $w \in U$, define

$$\langle A \rangle_{\Delta,\gamma;U} = \sup_{w \in U}(|A(w)|_{\Delta,\gamma}, (\partial_w A(w))_{\Delta,\gamma}).$$

When $\gamma = 0$, we simply write $|A|_U, \langle A \rangle_{\Delta,U}.$

2.1.4. The Normal form matrix. For $\Delta \geq 0$, define an equivalence relation on $\mathcal{L}$ generated by the pre-equivalence relation

$$a \sim b \iff |a| = |b|, |a - b| \leq \Delta.$$ 

Let $[a]_{\Delta}$ be the equivalence class (block) of $a$ and $\mathcal{E}_{\Delta}$ be the set of equivalence classes. Let $d_{\Delta}$ be the supremum of all block diameters, then by Proposition 4.1 in [EK10], $d_{\Delta} \leq \Delta^{\frac{d+1}{2}}$.

A matrix $A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$ is on normal form, denoted $\mathcal{N}F_{\Delta}$, if $A$ is real valued, symmetric, $\pi A = A$ and block-diagonal over $\mathcal{E}_{\Delta}$, i.e., $A_a^{b} = 0, \forall [a]_{\Delta} \not= [b]_{\Delta}$. A matrix $Q : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is on normal form, denoted $\mathcal{N}F_{\Delta}$, if $Q$ is Hermitian and block-diagonal over $\mathcal{E}_{\Delta}$.

For a normal form matrix $A$,

$$\frac{1}{2} \langle \zeta, A \zeta \rangle = \frac{1}{2} \langle \xi, A_1 \xi \rangle + \langle \xi, A_2 \eta \rangle + \frac{1}{2} \langle \eta, A_1 \eta \rangle,$$

where $A_1 + iA_2$ is a Hermitian matrix. Let

$$z = \left( \begin{array}{c} u \\ v \end{array} \right) = C^{-1} \left( \begin{array}{c} \xi \\ \eta \end{array} \right), C = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -i & i \end{array} \right),$$

and define $C^T AC : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$ by $(C^T AC)^b_a = C^T A_b^a C$. If $A$ is on normal form, then

$$\frac{1}{2} \langle z, C^T ACz \rangle = \langle u, Qv \rangle,$$

where $Q$ is the normal form matrix associated to $A$.

2.2. Main results. Let

$$h(r, \zeta; w) = \langle \omega(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H(w)) \zeta \rangle,$$

where $\Omega(w)$ is a real diagonal matrix with diagonal elements $\Omega_a(w)I$, $H(w), \partial_w H(w)$ are Töplitz at $\infty$ and $\mathcal{N}F_{\Delta}$ for all $w \in U$.

Assume

(2.1) $\partial_w \omega_b(w) = \delta_{ab}, a \in \mathbb{Z}^d, b \in \mathcal{A}, w \in U,$

(2.2) $\partial_w \Omega_b(w) = \delta_{ab}, a \in \mathbb{Z}^d, b \in \mathcal{L}, w \in U.$

Assume there exist constants $c_1, c_2, c_3, c_4, c_5 > 0$ such that

(2.3) $|\Omega_a(w) - |a|^2| \leq c_1 e^{-c_2 |a|}, a \in \mathcal{L}, w \in U,$

(2.4) $|\Omega_a(w)| \geq c_3, a \in \mathcal{L}, w \in U,$

(2.5) $|\Omega_a(w) + \Omega_b(w)| \geq c_3, a, b \in \mathcal{L}, w \in U,$

(2.6) $|\Omega_a(w) - \Omega_b(w)| \geq c_3, |a| \not= |b|, a, b \in \mathcal{L}, w \in U,$
Theorem 2.12. Consider the Hamiltonian \( h + f \), where

\[
h(r, \zeta; w) = \langle \omega(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H(w))\zeta \rangle
\]

satisfy (2.14)-(2.19), \( H(w), \partial_w H(w) \) are Töplitz at \( \infty \) and \( \mathcal{NF}_\Delta \) for all \( w \in U \).

(2.10) \[ \| X_f \|_{P, D(\rho, \mu, \sigma) \times U} \leq \varepsilon, \]

(2.11) \[ \| f \|_{\Delta, \gamma, D(\rho, \mu, \sigma)} \leq \varepsilon. \]

Assume \( \gamma, \sigma, \rho, \mu < 1, \Lambda, \Delta \geq 3, \rho = \sigma, \mu = \sigma^2, d\Delta \gamma \leq 1 \). Then there is a subset \( U_\infty \subset U \) such that if

\[
\varepsilon \leq \min \left(\gamma, \rho, \frac{1}{\Delta \cdot \frac{1}{\Lambda}}\right)^{\exp},
\]

then for all \( w \in U_\infty \), there is a real analytic symplectic map

\[
\Phi : D(\frac{\rho}{2}, \frac{\mu}{2}) \rightarrow D(\rho, \mu, \sigma)
\]

such that

\[
(h + f) \circ \Phi = h_\infty + f_\infty,
\]

where

\[
h_\infty = \langle \omega_\infty(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H_\infty(w))\zeta \rangle,
\]

\[
f_\infty = O(|r|^2 + |r||\zeta| + ||\zeta||_p^3)
\]

with the estimates

(2.12) \[ \| X_{f_\infty} \|_{P, D(\frac{\rho}{2}, \frac{\mu}{2}) \times U_\infty} \leq c \varepsilon^{\frac{3}{p}}, \]

(2.13) \[ |\omega_\infty(w) - \omega(w)| + |\partial_w (\omega_\infty(w) - \omega(w))| \leq c \varepsilon^{\frac{3}{p}}, \]

(2.14) \[ \| H_\infty(w) - H(w) \| + \| \partial_w (H_\infty(w) - H(w)) \| \leq c \varepsilon^{\frac{3}{p}}, \]

(2.15) \[ \text{meas}(U \setminus U_\infty) \leq \varepsilon^{\exp'}. \]

The exponents \( \exp, \exp' \) depend on \( d, \#A, p \), and the constant \( c \) depends on \( d, \#A, p, c_1, \ldots, c_5 \).

The proof of Theorem 2.12 is delayed to section 4.

Theorem 2.13. Given any \( 1 \leq M \leq (4c \varepsilon^{\frac{3}{p}})^{-1} \) and \( p \geq 80(4d)^{2d}(M + 7)^4 + 1 \), there exists a set \( \tilde{U} \subset U_\infty \) such that for any \( \delta > 0 \) and \( w \in \tilde{U} \), the KAM tori obtained in Theorem 2.12 is stable in long time \( \delta^{-M} \). Moreover, \( \text{meas}(U_\infty \setminus \tilde{U}) \leq \delta^{\exp} \), where the positive exponent \( \exp \) depends on \( d, \#A, p, M \).
The proof of Theorem 2.13 is given at the end of section 5. Based on Theorem 2.12 and Theorem 2.13, we are able to prove Theorem 1.1 on the long time stability of the KAM tori for NLS equation.

**Proof of Theorem 1.1** Recall the Hamiltonian formulation of NLS equation (1.1) in subsection 2.1.1. Let \( \omega = |a|^2 + \hat{V}(a), a \in \mathcal{A}, \Omega = |a|^2 + \hat{V}(a), a \in \mathcal{L} \), and take \( w_0 = \hat{V}(a) \). Then we have

\[
h = \sum_{a \in \mathcal{A}} \omega_a r_a + \frac{1}{2} \sum_{a \in \mathcal{L}} \Omega_a (\xi_a^2 + \eta_a^2), \quad f = \varepsilon \int_{\mathbb{T}^d} F(|u(x)|^2) dx.
\]

The Töplitz-Lipschitz property of \( f \) follows from Theorem 7.2 in \( \text{EK10} \) and the tame property follows from Section 3.5 in \( \text{BG06} \). By Theorem 2.12, if \( \varepsilon > 0 \) is sufficiently small, then for typical \( V \) (in the sense of measure), the \( d \)-dimensional nonlinear Schrödinger equation (1.1) has a quasi-periodic solution. By Theorem 2.13 assume \( u_0(t, x) \) with initial value \( u_0(0, x) \) is a quasi-periodic solution for the equation (1.1), then for any solution \( u(t, x) \) with initial value \( u(0, x) \) satisfying

\[
\|u(0, \cdot) - u_0(0, \cdot)\|_{H^p(\mathbb{T}^d)} < \delta, \quad \forall \ 0 < \delta \ll 1,
\]

we have

\[
\|u(t, \cdot) - u_0(t, \cdot)\|_{H^p(\mathbb{T}^d)} < C\delta, \quad \forall \ 0 < |t| < \delta^{-M}.
\]

In other words, the obtained KAM tori for the nonlinear Schrödinger equation (1.1) are of long time stability. \( \square \)

3. The homological equations

In this section, we formulate and solve the homological equation in the KAM iteration. To obtain an open and uniform domain for the transformed Hamiltonian, we apply Kolmogorov’s iterative scheme. As a result, the homological equation is complicated than that in \( \text{EK10} \), but it can be solved by the method developed in \( \text{EK10} \).

Write

\[
f(\varphi, r, \zeta; w) = f^{\text{low}} + f^{\text{high}},
\]

where

\[
f^{\text{low}} = f^\varphi + f^0 + f^1 + f^2 = \hat{F}^\varphi(\varphi; w) + \langle F_0(\varphi; w), r \rangle + \langle F_1(\varphi; w), \zeta \rangle + \frac{1}{2} \langle F_2(\varphi; w)\zeta, \zeta \rangle.
\]

Define

\[
\mathcal{T}_\Delta f^{\text{low}} = \sum_{|k| \leq \Delta} \left( \hat{F}^\varphi(k; w) + \langle \hat{F}_0(k; w), r \rangle + \langle \hat{F}_1(k; w), \zeta \rangle + \frac{1}{2} \langle \mathcal{T}_\Delta \hat{F}_2(k; w)\zeta, \zeta \rangle \right) e^{i(k, \varphi)}.
\]

Let \( \Delta' > 1 \) and \( 0 < \kappa < 1 \). Assume there exists \( \mathcal{U}' \subset \mathcal{U} \) such that for all \( w \in \mathcal{U}' \), \( 0 < |k| \leq \Delta' \), the following properties hold:

- Diophantine condition:
  \[
  |\langle k, \omega(w) \rangle| \geq \kappa;
  \]

- The first Melnikov condition:
  \[
  |\langle k, \omega(w) \rangle + \alpha(w)\rangle | \geq \kappa, \quad \forall \ \alpha(w) \in \text{spec}((\Omega + H)(w))[a]_{\Delta}, \quad \forall \ [a]_{\Delta};
  \]

- The second Melnikov condition with the same sign:
  \[
  |\langle k, \omega(w) \rangle + \alpha(w) + \beta(w)\rangle | \geq \kappa, \quad \forall \ \left\{ \begin{array}{l}
  \alpha(w) \in \text{spec}((\Omega + H)(w))[a]_{\Delta}, \\
  \beta(w) \in \text{spec}((\Omega + H)(w))[b]_{\Delta},
  \end{array} \right\}, \quad \forall \ [a]_{\Delta}, [b]_{\Delta};
  \]
The second Melnikov condition with opposite signs:

\begin{equation}
|\langle k, \omega(w) \rangle + \alpha(w) - \beta(w) | \geq \kappa
\end{equation}

\[\forall \left\{ \begin{array}{l} \alpha(w) \in \text{spec}((\Omega + H)(w))[a, \Delta], \\
\beta(w) \in \text{spec}((\Omega + H)(w))[b, \Delta], \end{array} \right. \]

and for any \text{dist}([a, \Delta], [b, \Delta]) \leq \Delta' + 2d\Delta.

We have the following result on the solution of the homological equation.

**Proposition 3.1.** Consider the Hamiltonian \( h + f \), where

\[ h(r, \zeta; w) = \langle \omega(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H(w))\zeta \rangle \]

satisfy (2.1) - (2.3). \( H(w), \partial_w H(w) \) are Töplitz at \( \infty \) and \( \mathcal{N} \mathcal{F}_\Delta \) for all \( w \in U \),

\[ f(\varphi, r, \zeta; w) = f^{\text{low}} + f^{\text{high}} \]

satisfy

\begin{equation}
|||X^{\text{low}}|||_{p, D(\rho, \mu, \sigma) \times U} \leq \epsilon, \quad |||X^{\text{high}}|||_{p, D(\rho, \mu, \sigma) \times U} \leq 1,
\end{equation}

\begin{equation}
[f^{\text{low}}]_{(\Lambda, \gamma, \sigma; U, \rho, \mu} \leq \epsilon, \quad [f^{\text{high}}]_{(\Lambda, \gamma, \sigma; U, \rho, \mu} \leq 1.
\end{equation}

Assume \( \gamma, \sigma, \mu < 1 \), \( \Lambda, \Delta \geq 3 \), \( \rho = \sigma, \mu = \sigma^2 \), \( d\Delta \gamma \leq 1 \). Let \( U' \subset U \) satisfy (3.1) - (3.4).

Then for all \( w \in U' \), the homological equation

\[ \{ h(s), \zeta \} = -\mathcal{T}_{\Delta} f^{\text{low}} - \mathcal{T}_{\Delta} \{ f^{\text{high}}, s \}^{\text{low}} + h_1 \]

has solutions

\begin{equation}
s(\varphi, r, \zeta; w) = s^{\text{low}} = s^0 + s^1 + s^2,
\end{equation}

\begin{equation}
h_1(r, \zeta; w) = a_1(w) + \langle \chi_1(w), r \rangle + \frac{1}{2} \langle \zeta, H_1(w)\zeta \rangle
\end{equation}

with the estimates

\begin{equation}
|||X^{\varphi}|||_{p, D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{\epsilon}{\tau \kappa^2},
\end{equation}

\begin{equation}
|||X^{s}|||_{p, D(\rho - 3\tau, (\sigma - 3\tau)^2, (\sigma - 3\tau)) \times U'} \leq \frac{d\Delta^2 \epsilon}{\rho \kappa^4},
\end{equation}

\begin{equation}
|||X^{\sigma}|||_{p, D(\rho - 5\tau, (\sigma - 5\tau)^2, (\sigma - 5\tau)) \times U'} \leq \frac{d\Delta^2 \epsilon}{\rho \kappa^6},
\end{equation}

\begin{equation}
|||X^{\zeta}|||_{p, D(\rho - 5\tau, (\sigma - 5\tau)^2, (\sigma - 5\tau)) \times U'} \leq \frac{d\Delta^2 \epsilon}{\rho \kappa^8},
\end{equation}

\begin{equation}
|||X^{h_1}|||_{p, D(\rho - 5\tau, (\sigma - 5\tau)^2, (\sigma - 5\tau)) \times U'} \leq \frac{d\Delta^2 \epsilon}{\rho \kappa^6},
\end{equation}

where \( 0 < \tau < \frac{\rho}{100}, a \leq b \) means there exists a constant \( c > 0 \) depending on \( d, \# A, p, c_1, \cdots, c_5 \) such that \( a \leq cb \).

The new Hamiltonian

\begin{equation}
(h + f) \circ X^t_t |_{t=1} = h + h_1 + f_1
\end{equation}
with
\[
f_1 = (1 - T_{\Delta'}) f^{\text{low}} + f^{\text{high}} + (1 - T_{\Delta'}) \{f^{\text{high}}, s\}^{\text{low}} + \{f^{\text{high}}, s\}^{\text{high}} \\
+ \int_0^1 (1 - t) \{\{h, s\}, s\} \circ X^t_s dt + \int_0^1 \{f^{\text{low}}, s\} \circ X^t_s dt \\
+ \int_0^1 (1 - t) \{f^{\text{high}}, s\}, s\} \circ X^t_s dt
\]
(3.17)
satisfies

\[
|||X^t_{f^{\text{low}}}||_{T, p, D(\rho - \sigma r, (\sigma - \sigma r)^2, \sigma - \sigma r) \times U'} \leq \frac{d_A^0}{k^4 r^3 A + 1} e^{-\frac{\gamma}{\Delta t}} + \frac{(\Delta t')^{\exp} e^{-\frac{\gamma}{\Delta t}}}{\sigma d_A p + 1} + \frac{d_A^0 d_A^2}{\tau^3 r^2 k^1},
\]
(3.18)
and

\[
|||X^t_{f^{\text{high}}}||_{T, p, D(\rho - \sigma r, (\sigma - \sigma r)^2, \sigma - \sigma r) \times U'} \leq 1 + \frac{d_A^0}{\tau^3 r^2 k^1} + \frac{d_A^0 d_A^2}{\tau^3 r^2 k^1},
\]
(3.19)

where the exponent \(\exp\) depends on \(d, \#A, p\).

Moreover, the following estimates hold.

i) The solution \(s\) and the remainder \(h\) satisfy

\[
[s]_{t, \Delta + 2, \gamma, \sigma, \rho, \rho', \mu, \mu'} \leq \frac{1}{k^2} (\Delta t')^{\exp} \frac{1}{\rho - \rho'} \left( \frac{1}{\sigma - \sigma'} + \frac{1}{\mu - \mu'} \right) \frac{1}{\mu},
\]
(3.20)
\[
[h]_{t, \Delta + 2, \gamma, \sigma, \rho, \rho', \mu, \mu'} \leq \frac{1}{k^2} (\Delta t')^{\exp} \frac{1}{\rho - \rho'} \left( \frac{1}{\sigma - \sigma'} + \frac{1}{\mu - \mu'} \right) \frac{1}{\mu},
\]
(3.21)

where \(\rho' < \rho, \mu' < \mu, \sigma' < \sigma, \Delta' \geq \text{cte. max}(\Lambda, d_A^2, d_A^{\Delta})\), and the constant \(\text{cte. is the one in [EK10, Proposition 6.7].}\)

ii) There is the measure estimate

\[
\text{meas}(U \setminus U') \leq \min(\Lambda, \Delta, \Delta')^{\exp} (\frac{1}{k^2})^\mu.
\]
(3.22)
The measure estimate \(3.22\) follows directly from \([EK10\text{, Proposition 6.6 and 6.7]}\), which is not repeated here. The rest of this section is devoted to the proof of Proposition \(5.1\). For the sake of notations, we shall not indicate the dependence on the parameter \(w\) of functions when it is known from the text.

3.1. Formulation of the homological equations. Write

\[
f^{\text{low}} = f^0 + f^1 + f^2 = F(\phi; w) + \langle F_0(\phi; w), r \rangle + \langle F_1(\phi; w), \zeta \rangle + \frac{1}{2} \langle F_2(\phi; w) \zeta, \zeta \rangle,
\]
\[
s^{\text{low}} = s^0 + s^1 + s^2 = S(\phi; w) + \langle S_0(\phi; w), r \rangle + \langle S_1(\phi; w), \zeta \rangle + \frac{1}{2} \langle S_2(\phi; w) \zeta, \zeta \rangle.
\]

By the calculations in \([CLY16\text{, Section 4.1.2]}\), we obtain

\[
\{f^{\text{high}}, s\}^{\text{low}} = \{f^{\text{high}}, s\}^0 + \{f^{\text{high}}, s\}^1 + \{f^{\text{high}}, s\}^2,
\]
(3.23)
\[
\{f^{\text{high}}, s\}^0 = \{f^{\text{high}}, s^0 + s^1\}^0, \quad \{f^{\text{high}}, s\}^1 = \{f^{\text{high}}, s^2\}^1,
\]
(3.24)
\[
\{f^{\text{high}}, s\}^0 = \{f^{\text{high}}, s^0 + s^1\}^0, \quad \{f^{\text{high}}, s\}^1 = \{f^{\text{high}}, s^2\}^1.
\]
(3.25)

Let \(g = \{f^{\text{high}}, s\}\). Write

\[
g^{\text{low}} = g^0 + g^1 + g^2 = \langle G_0(\phi; w), r \rangle + \langle G_1(\phi; w), \zeta \rangle + \frac{1}{2} \langle G_2(\phi; w) \zeta, \zeta \rangle.
\]
In Fourier modes, the homological equation (3.7) decomposes into

\[(3.26)\]  
\[-i \langle k, \omega(w) \rangle \hat{S}^\varphi(k; w) = -\hat{F}^\varphi(k; w) + \delta_0^k a_1(w),\]

\[(3.27)\]  
\[-i \langle k, \omega(w) \rangle \hat{S}_1(k; w) + J(\Omega(w) + H(w)) \hat{S}_1(k; w) = -\hat{F}_1(k; w) - \hat{G}_1(k; w),\]

\[(3.28)\]  
\[-i \langle k, \omega(w) \rangle \hat{S}_0(k; w) = -\hat{F}_0(k; w) - \hat{G}_0(k; w) + \delta_0^k \chi_1(w),\]

\[(3.29)\]  
\[-i \langle k, \omega(w) \rangle \hat{S}_2(k; w) + (\Omega(w) + H(w)) J \hat{S}_2(k; w) - \hat{S}_1(k; w) J(\Omega(w) + H(w)) = -\hat{F}_2(k; w) - \hat{G}_2(k; w) + \delta_0^k H_1(w).\]

We solve the equations (3.26)-(3.29) in the order (3.26) \(\rightarrow\) (3.27) \(\rightarrow\) (3.28) \(\rightarrow\) (3.29).

### 3.2. Solution of the homological equation (3.26)

The homological equation (3.26) is very standard in the KAM theory. From (3.26), we obtain

\[a_1(w) = \hat{F}^\varphi(0; w) \quad \text{and} \quad \hat{S}^\varphi(k; w) = \frac{\hat{F}^\varphi(k; w)}{i \langle k, \omega(w) \rangle}, \quad k \neq 0.\]

By the Diophantine condition (3.1), we have

\[(3.30)\]  
\[|\hat{S}^\varphi(k; w)| \leq \frac{1}{\kappa} |\hat{F}^\varphi(k; w)|.\]

Differentiating (3.26), we obtain a similar homological equation

\[(3.31)\]  
\[-i \partial_w \langle k, \omega(w) \rangle \hat{S}^\varphi(k; w) - i \langle k, \omega(w) \rangle \partial_w \hat{S}^\varphi(k; w) = -\partial_w \hat{F}^\varphi(k; w)\]

for \(\partial_w \hat{S}^\varphi(k; w)\) and there is also

\[|\partial_w \hat{S}^\varphi(k; w)| \leq \frac{1}{\kappa} (|k| : |\hat{S}^\varphi(k; w)| + |\partial_w \hat{F}^\varphi(k; w)|),\]

which together with (3.30) implies

\[|\hat{S}^\varphi(k; w)| + |\partial_w \hat{S}^\varphi(k; w)| \leq \frac{|k| + 1}{\kappa^2} (|\hat{F}^\varphi(k; w)| + |\partial_w \hat{F}^\varphi(k; w)|).\]

It follows that

\[\|\hat{S}^\varphi\|_{D(\rho-\tau) \times U'} = \sum_{k \in \mathbb{Z}^A} (|\hat{S}^\varphi(k; w)| + |\partial_w \hat{S}^\varphi(k; w)|) e^{\langle k \rangle (\rho-\tau)}\]

\[\leq \sum_{k \in \mathbb{Z}^A} \frac{|k| + 1}{\kappa^2} (|\hat{F}^\varphi(k; w)| + |\partial_w \hat{F}^\varphi(k; w)|) e^{\langle k \rangle (\rho-\tau)}\]

\[\leq \frac{1}{\tau \kappa^2} \|\hat{F}^\varphi\|_{D(\rho) \times U'}.\]

As a result, we have

\[(3.32)\]  
\[\|\|X^\varphi\|_{p, D(\rho-\tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\tau \kappa^2} \|\|X^\varphi\|_{p, D(\rho, \sigma^2, \sigma) \times U'}.\]
3.3. Solution of the homological equation (3.27). For simplicity, we write (3.27) as
\[ i(k, \omega(w)) S + J(\Omega + H) S = F + G. \]

Transforming into complex coordinates
\[ z = \begin{pmatrix} u \\ v \end{pmatrix} = C^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \]
and letting \( S' = C^{-1} S = \begin{pmatrix} S'_1 \\ S'_2 \end{pmatrix} \), \( F' = C^{-1} F = \begin{pmatrix} F'_1 \\ F'_2 \end{pmatrix} \), \( G' = C^{-1} G = \begin{pmatrix} G'_1 \\ G'_2 \end{pmatrix} \), we obtain the equivalent equations
\[
\begin{aligned}
&i(k, \omega(w)) S'_1 - i(\Omega + H^T) S'_1 = F'_1 + G'_1, \\
&i(k, \omega(w)) S'_2 + i(\Omega + H) S'_2 = F'_2 + G'_2.
\end{aligned}
\]

Solution of (3.33). We only solve \( S'_1 \) in (3.33) since \( S'_2 \) can be solved accordingly. By the first Melnikov condition (3.2), we have
\[ \| S'_1[a] \| \leq \frac{1}{\kappa} \| F'_1[a] + G'_1[a] \|. \]

Using similar arguments to that of \( S' \), we get
\[ \| S'_1[a] \| + \| \partial_w S'_1[a] \| \leq \frac{|k| + 1}{\kappa^2} \left( \| F'_1[a] + G'_1[a] \| + \| \partial_w F'_1[a] + \partial_w G'_1[a] \| \right) \]

Estimate of \( s_1^1 \). Recall
\[ s^1 = \sum_{a \in \mathcal{L}} (S'_{1a}(\varphi; w) u_a + S'_{2a}(\varphi; w) v_a), \]
and consider
\[ |s_1^1|_{D(\rho-3\tau, \sigma^2) \times U'}(z) = \sum_{a \in \mathcal{L}} \left( \| S'_{1a}(\varphi; w) \|_{D(\rho-3\tau) \times U'} u_a + \| S'_{2a}(\varphi; w) \|_{D(\rho-3\tau) \times U'} v_a \right). \]

For \( z = (u, v) \), define \( \tilde{z} = (\tilde{u}, \tilde{v}) \) such that for all \( a \in [a]_\Delta \), \( \tilde{u}_a = \| u[a] \|, \tilde{v}_a = \| v[a] \|. \) By (3.34), we see the first sum in \( s_1^1 \) satisfies
\[ \begin{aligned}
&| \sum_{a \in \mathcal{L}} \| S'_{1a}(\varphi) \|_{D(\rho-3\tau) \times U'} u_a | \\
\leq & \sum_{a \in \mathcal{L}} \sum_{k \in \mathbb{Z}^2} \sum_{[a]_\Delta \in \mathcal{E}_{\Delta}} \sum_{a \in [a]_\Delta} \left( |S'_{1a}(\varphi; k)| + \| \partial_w S'_{1a}(\varphi; k) \| \cdot e^{(\rho-3\tau)|k|} \cdot |u_a| \right) \\
\leq & \sum_{a \in \mathcal{L}} \sum_{k \in \mathbb{Z}^2} \frac{|k| + 1}{\kappa^2} \left( |F'_{1a}(\varphi; k) + G'_{1a}(\varphi; k)| + \| \partial_w F'_{1a}(\varphi; k) + \partial_w G'_{1a}(\varphi; k) \| \right) \tilde{u}_a e^{(\rho-3\tau)|k|} \\
\leq & \frac{1}{\tau \kappa^2} \sum_{a \in \mathcal{L}} \| F'_{1a}(\varphi) + G'_{1a}(\varphi) \|_{D(\rho-2\tau) \times U'} \tilde{u}_a.
\end{aligned} \]

Similarly, we have
\[ \begin{aligned}
&| \sum_{a \in \mathcal{L}} \| S'_{2a}(\varphi; w) \|_{D(\rho-3\tau) \times U'} v_a | \leq \frac{1}{\tau \kappa^2} \sum_{a \in \mathcal{L}} \| F'_{2a}(\varphi; w) + G'_{2a}(\varphi; w) \|_{D(\rho-2\tau) \times U'} \tilde{v}_a.
\end{aligned} \]
Moreover, we see from
\[ (3.36) \]
then we immediately get
\[ (3.35) \]
and \[ \parallel \tilde{z} \parallel_1 \leq d^4 \parallel z \parallel_1 \] that
\[ (3.35) \]
Estimate of \( s_1 \). Consider
\[ \parallel s_1 \parallel^T_{p, D(p-3\tau, \sigma^2)} \leq \parallel s_1 \parallel^T_{D(p-3\tau, \sigma^2)} \parallel p, \]
\[ \parallel s_1 \parallel^T_{D(p-3\tau, \sigma^2)} \parallel p \]
\[ = \sum_{a \in L}(\parallel s_1^T_{a} (\varphi) \parallel^2_{D(p-3\tau)} + \parallel s_2^T_{a} (\varphi) \parallel^2_{D(p-3\tau)} + \parallel s_3^T_{a} (\varphi) \parallel^2_{D(p-3\tau)})(\varphi)^2. \]
By \[ (3.35) \], we have
\[ \left( \sum_{a \in [a]\Delta} \parallel S_1^T_{a} (\varphi) \parallel^2_{D(p-3\tau)} \right)^{\frac{1}{2}} \leq \left| \sum_{a \in [a]\Delta} \left( \sum_{k \in Z^A} \left( \parallel \dot{S}_{1a}^T_{1a} (k) \parallel^2_{D(p-3\tau)} + \parallel \partial_w \dot{S}_{1a}^T_{1a} (k) \parallel^2 \right) e^{(p-3\tau)|k|} \right) \right| \]
\[ \leq \sum_{k \in Z^A} \sum_{a \in [a]\Delta} \parallel \dot{S}_{1a}^T_{1a} (k) \parallel^2_{D(p-3\tau)} + \parallel \partial_w \dot{S}_{1a}^T_{1a} (k) \parallel^2 \] \[ \leq \sum_{k \in Z^A} \sum_{a \in [a]\Delta} \parallel \dot{F}_{1a}^T (\varphi) + G_{1a}^T (\varphi) \parallel^2_{D(p-2\tau)} \]
which implies the first sum in \[ s_1 \] satisfies
\[ \sum_{a \in [a]\Delta} \parallel S_1^T_{1a} (\varphi; w) \parallel^2_{D(p-3\tau) \times U', \langle a \rangle} 2 \parallel p \leq \left( \frac{d^4}{\tau K} \right)^2 \sum_{a \in [a]\Delta} \parallel F_1^T (\varphi; w) + G_{1a}^T (\varphi; w) \parallel^2_{D(p-2\tau) \times U', \langle a \rangle} 2 \parallel p. \]
Similarly, the other sum in \[ s_1 \] satisfies
\[ \sum_{a \in [a]\Delta} \parallel S_2^T_{1a} (\varphi; w) \parallel^2_{D(p-3\tau) \times U', \langle a \rangle} 2 \parallel p \leq \left( \frac{d^4}{\tau K} \right)^2 \sum_{a \in [a]\Delta} \parallel F_2^T (\varphi; w) + G_{2a}^T (\varphi; w) \parallel^2_{D(p-2\tau) \times U', \langle a \rangle} 2 \parallel p. \]
Then we immediately get
\[ (3.36) \]
\[ |||s_1^T|||_{p, D(p-3\tau, \sigma^2) \times U'} \leq \left( \frac{d^4}{\tau K} \right)^2 ||| f_1^T + g_1^T |||_{p, D(p-2\tau, \sigma^2) \times U'}. \]
Combining \[ (3.35) \] and \[ (3.36) \], we have
\[ (3.37) \]
3.4. Solution of the homological equations \(3.28\)–\(3.29\). Solving equation \(3.28\) as equation \(3.26\), we obtain

\[
\|\|X_{\varphi}\|\|_{p,D(\rho-5\tau,\sigma^2,\sigma)^*U^*}^T \leq \frac{1}{\tau K^2} \|\|X_{f^0 + \varphi^0}\|\|_{p,D(\rho-4\tau,\sigma^2,\sigma)^*U^*}^T.
\]

Now we consider the equation \(3.29\). For simplicity, we write \(3.29\) as

\[
i(\varphi, \omega(w))S + (\Omega + H)JS = S(\Omega + H) = F + G - H_1.
\]

Changing into complex coordinates \(z = C^{-1}\) and letting \(S' = C^T S C = \left(\begin{array}{ccc} S_1' & S_2' & S_3' \end{array}\right)\), \(F' = C^T F C, G' = C^T G C, H'_1 = C^T H_1 C\), we obtain the equivalent equations as follows

\[
i(\varphi, \omega(w))S_1' + (\Omega + H)S_1' + iS_1' (\Omega + H^T) = F_1' + G_1',
\]

\[
i(\varphi, \omega(w))S_2' + (\Omega + H)S_2' - iS_2' (\Omega + H) = F_2' + G_2' - H_2',
\]

\[
i(\varphi, \omega(w))S_3' - (\Omega + H)S_3' - iS_3' (\Omega + H) = F_3' + G_3'.
\]

Solutions of \(3.39\)–\(3.42\). Consider first \(S_2'\) in \(3.40\)–\(3.41\). When \(k \neq 0\), we have \(H_{12}' = 0\). In a similar way as before, we obtain from the second Melnikov condition \(3.3\) that

\[
\|S_2'^{[b]a}\|_\Delta + \|\partial_\varphi S_2'^{[b]a}\|_\Delta \leq \begin{cases} \frac{|k|}{\kappa^2} (\|F_2^{[b]a}\|_\Delta + G_2^{[b]a}\|_\Delta) + \|\partial_\varphi F_2^{[b]a}\|_\Delta + \|\partial_\varphi G_2^{[b]a}\|_\Delta) \end{cases}.
\]

When \(k = 0\), the above estimate also holds since \(H_{12}' = (F_2' + G_2')\). Using the second Melnikov condition \(3.3\) with the same sign, we can also solve \(S_1'\) and \(S_3'\). In conclusion, we have

\[
\|S_2'^{[b]a}\|_\Delta + \|\partial_\varphi S_2'^{[b]a}\|_\Delta \leq \begin{cases} \frac{|k|}{\kappa^2} (\|F_2^{[b]a}\|_\Delta + G_2^{[b]a}\|_\Delta) + \|\partial_\varphi F_2^{[b]a}\|_\Delta + \|\partial_\varphi G_2^{[b]a}\|_\Delta) \end{cases},
\]

with \(\nu \in \{1, 2, 3\}\).

Estimate of \(s_\varphi^2\). Recalling

\[
s_\varphi^2 = \frac{1}{2} \sum_{a, b \in \mathcal{L}} (S_{1a}^{[b]} (\varphi; w) u_a u_b + 2S_{2a}^{[b]} (\varphi; w) u_a v_b + S_{3a}^{[b]} (\varphi; w) v_a v_b),
\]

we consider

\[
\|s_\varphi^2\|_{D(\rho-5\tau,\sigma^2) \times U'} (z) = \frac{1}{2} \sum_{a, b \in \mathcal{L}} ((S_{1a}^{[b]} (\varphi; w))_{D(\rho-5\tau) \times U'} u_a u_b + 2\|S_{2a}^{[b]} (\varphi; w)\|_{D(\rho-5\tau) \times U'} u_a v_b + \|S_{3a}^{[b]} (\varphi; w)\|_{D(\rho-5\tau) \times U'} v_a v_b),
\]

and the associated multilinear form \(s_\varphi^2\)

\[
(s_\varphi^2)_{D(\rho-5\tau,\sigma^2) \times U'} (z^{(1)}, z^{(2)}) = \frac{1}{2} \sum_{a, b \in \mathcal{L}} ((S_{1a}^{[b]} (\varphi; w))_{D(\rho-5\tau) \times U'} u_a^{(1)} u_b^{(2)} + 2\|S_{2a}^{[b]} (\varphi; w)\|_{D(\rho-5\tau) \times U'} u_a^{(1)} v_b^{(2)} + \|S_{3a}^{[b]} (\varphi; w)\|_{D(\rho-5\tau) \times U'} v_a^{(1)} v_b^{(2)}).
\]
By (3.43), we know that $|\sum_{a \in [a], b \in [b]} S^{fib}_{1a2}(\varphi; u) \|_{D(\rho-5\tau) \times U'} \sum_{a \in [a], b \in [b]} u_a^{(1)} u_b^{(2)}| \leq \frac{d^d}{\tau k^2} \sum_{a \in [a], b \in [b]} \| F^{fib}_{1a2}(\varphi) + G^{fib}_{1a2}(\varphi) \|_{D(\rho-4\tau) \times U'} \bar{u}_a^{(1)} \bar{u}_b^{(2)},$

which implies

$$\left| \sum_{a, b \in L} \| S^{fib}_{1a2}(\varphi) \|_{D(\rho-5\tau) \times U'} u_a^{(1)} u_b^{(2)} \right| \leq \frac{1}{\tau k^2} \sum_{a, b \in L} \| F^{fib}_{1a2}(\varphi) + G^{fib}_{1a2}(\varphi) \|_{D(\rho-4\tau) \times U'} \bar{u}_a^{(1)} \bar{u}_b^{(2)}.$$

There are similar estimates for the other three summations in the R.H.S. of (3.44).

Then we have

$$\| s^2_p \|_{D(\rho-5\tau, \sigma^2) \times U'} (z^{(1)}, z^{(2)}) \| \leq \frac{1}{\tau k^2} \left[ \check{f}_p^2 + g_p^2 \right]_{D(\rho-4\tau, \sigma^2) \times U'} (\check{z}^{(1)}, \check{z}^{(2)}).$$

Since

$$\| s^2_p \|_{D(\rho-5\tau, \sigma^2) \times U'} = \sup_{0 \neq z \in \mathbb{F}_p^2} \frac{\| s^2_p \|_{D(\rho-5\tau, \sigma^2) \times U'} (z^{(1)}, z^{(2)})}{\| z^{(1)} \|_1 \| z^{(2)} \|_1},$$

we see from $\| \check{z} \|_1 \leq d^d \| z \|_1$ that

$$\| s^2_p \|_{D(\rho-5\tau, \sigma^2) \times U'} \leq \frac{d^d}{\tau k^2} \| F^2_p + g^2_p \|_{D(\rho-4\tau, \sigma^2) \times U'}.$$

**Estimate of $s^2_p$.** Now we estimate

$$\| s^2_p \|_{p, D(\rho-5\tau, \sigma^2) \times U'} \leq \frac{d^{2d}}{\tau k^2} \| F^2_p + g^2_p \|_{D(\rho-4\tau, \sigma^2) \times U'},$$

in which $\| s^2_p \|_{D(\rho-5\tau, \sigma^2) \times U'}$ equals to the following sum

$$\sum_{a \in L} \left( \sum_{b \in L} \left( \| S_{1a2}^{fib}(\varphi; u) \|_{D(\rho-5\tau) \times U'} u_b + \| S_{1b2}^{fib}(\varphi; u) \|_{D(\rho-5\tau) \times U'} v_b \right)^2 \right)^{\frac{1}{2}} \langle u \rangle^{2p} \label{3.46}$$

$$+ \sum_{a \in L} \left( \sum_{b \in L} \left( \| S_{2a2}^{fib}(\varphi; u) \|_{D(\rho-5\tau) \times U'} u_b + \| S_{2b2}^{fib}(\varphi; u) \|_{D(\rho-5\tau) \times U'} v_b \right)^2 \right)^{\frac{1}{2}} \langle u \rangle^{2p}.$$
\begin{align*}
= & \left[ \sum_{a \in [a]} \left( \sum_{k \in \mathcal{A}} \sum_{b \in \mathcal{L}} \left( \| \hat{S}^{th}_{2a}(k) + |\partial_w \hat{S}^{th}_{2a}(k)| \right) e^{(\rho-5\tau)|k|} |u_b| \right)^2 \right]^{\frac{1}{2}} \\
\leq & \sum_{k \in \mathcal{A}} \left[ \sum_{a \in [a]} \left( \sum_{b \in \mathcal{L}} \left( \| \hat{S}^{th}_{2a}(k) + |\partial_w \hat{S}^{th}_{2a}(k)| \right)^2 \right) |u_b| \right] e^{(\rho-5\tau)|k|} \\
\leq & \sum_{k \in \mathcal{A}} \sum_{a \in [a]} \sum_{b \in \mathcal{L}} \left( \| \hat{S}^{th}_{2a}(k) + |\partial_w \hat{S}^{th}_{2a}(k)| \right) e^{(\rho-5\tau)|k|} \\
\leq & \sum_{k \in \mathcal{A}} \frac{1}{\kappa^2} \sum_{a \in [a]} \sum_{b \in \mathcal{L}} \| F^{th}_{1a}(\varphi) + G^{th}_{1a}(\varphi) \|_{D(\rho-4\tau) \times \Gamma} \tilde{u}_b.
\end{align*}

As a result, we have

\begin{align*}
\sum_{a \in [a]_\Delta} \left( \sum_{b \in \mathcal{L}} \| \hat{S}^{th}_{2a}(\varphi) \|_{D(\rho-5\tau) \times \Gamma} |u_b| \right)^2 \leq \left( \frac{\delta_d^4}{\kappa^2} \right)^2 \sum_{a \in [a]_\Delta} \left( \sum_{b \in \mathcal{L}} \| F^{th}_{1a}(\varphi) + G^{th}_{1a}(\varphi) \|_{D(\rho-4\tau) \times \Gamma} \tilde{u}_b \right)^2.
\end{align*}

Similar estimates hold for the other three summations in (3.46). Then we have

\begin{equation}
\| s^2 \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \| \varphi \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \| \leq \frac{d^2}{\kappa^2} \| f^2 + g^2 \|_{D(\rho-4\tau,\sigma^2) \times \Gamma} \| \varphi \|_{D(\rho-4\tau,\sigma^2) \times \Gamma},
\end{equation}

which together with \( \| \tilde{z} \|_p \leq d^2 \| \tilde{z} \|_p \) implies

\begin{equation}
\| s^2 \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \| \varphi \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \leq \frac{d^2}{\kappa^2} \| f^2 + g^2 \|_{D(\rho-4\tau,\sigma^2) \times \Gamma} \| \varphi \|_{D(\rho-4\tau,\sigma^2) \times \Gamma}.
\end{equation}

Finally, combining (3.45) and (3.47), we have

\begin{equation}
\| X_{s^2} \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \leq \frac{d^2}{\kappa^2} \| X_{f^2 + g^2} \|_{D(\rho-4\tau,\sigma^2) \times \Gamma}.
\end{equation}

3.5. Verification of the estimates (3.10) - (3.15). In this part, we shall verify the estimates of the vector fields \( X_s \) and \( X_{h_1} \).

Recall that \( s = s^0 + s^1 + s^2 = s^0 + s^1 + s^2 \). By (3.3) and (3.32), we have

\begin{equation}
\| X_{s^0} \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \leq \frac{1}{\kappa^2} \| X_{s^0} \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \leq \frac{\epsilon}{\kappa^2},
\end{equation}

which together with Proposition 2.9 and (3.5) implies

\begin{equation}
\| X_{s^2} \|_{D(\rho-5\tau,\sigma^2) \times \Gamma} \leq \frac{\epsilon}{\kappa^2}.
\end{equation}
Moreover, the vector field \( X \) (3.19). Using Taylor’s formula, we obtain from the homological equation (3.7) that

\[
\| X \|_{p,D(\rho-3\tau,|\sigma-3\tau|^2,|\sigma-3\tau|) \times U'} \leq \frac{d^3}{\tau \kappa^2} d|X_f|_{p,D(\rho-3\tau,|\sigma-3\tau|^2,|\sigma-3\tau|) \times U'} + \| X_{g'} \|_{p,D(\rho-3\tau,|\sigma-3\tau|^2,|\sigma-3\tau|) \times U'} \leq \frac{d^3}{\tau \kappa^2} \varepsilon.
\]

Similar to \( X_{s^1} \), we can estimate \( X_{s^0} \) and \( X_{s^2} \) in sequence and finally get

\[
\| X_s \|_{p,D(\rho-5\tau,|\sigma-5\tau|^2,|\sigma-5\tau|) \times U'} \leq \frac{d^3}{\tau \kappa^2} \varepsilon.
\]

Moreover, the vector field \( X_{h_1} \) satisfies

\[
\| X_{h_1} \|_{p,D(\rho-5\tau,|\sigma-5\tau|^2,|\sigma-5\tau|) \times U'} \leq \| X_{f^0+g^0} \|_{p,D(\rho-5\tau,|\sigma-5\tau|^2,|\sigma-5\tau|) \times U'} + \| X_{f^2+g^2} \|_{p,D(\rho-5\tau,|\sigma-5\tau|^2,|\sigma-5\tau|) \times U'} \leq \frac{d^3}{\tau \kappa^2} \varepsilon.
\]

3.6. Estimate of the new Hamiltonian. In this part, we shall verify the properties (3.16)-(3.19). Using Taylor’s formula, we obtain from the homological equation (3.7) that

\[
(h + f) \circ X^t_{s} |_{t=1} = (h + f^{low} + f^{high}) \circ X^t_{s} |_{t=1}
\]

\[
= h + \{h, s\} + \int_0^1 (1-t) \{ \{h, s\}, s\} \circ X_s^t dt + f^{low} + \int_0^1 \{ f^{low}, s\} \circ X_s^t dt
\]

\[
+f^{high} + \{ f^{high}, s\} + \int_0^1 (1-t) \{ \{f^{high}, s\}, s\} \circ X_s^t dt
\]

\[
= h + h_1 + (1 - T_{\Delta'}) f^{low} + f^{high} + (1 - T_{\Delta'}) \{ f^{high}, s\}^{low} + \{ f^{high}, s\}^{high}
\]

\[
+ \int_0^1 (1-t) \{ \{h, s\}, s\} \circ X_s^t dt + \int_0^1 \{ f^{low}, s\} \circ X_s^t dt + \int_0^1 (1-t) \{ \{f^{high}, s\}, s\} \circ X_s^t dt.
\]

Then we have \( f_1 = f_1^{low} + f_1^{high} \), where

\[
f_1^{low} = (1 - T_{\Delta'}) f^{low} + (1 - T_{\Delta'}) \{ f^{high}, s\}^{low} + \left( \int_0^1 (1-t) \{ \{h, s\}, s\} \circ X_s^t dt \right)^{low}
\]

\[
\quad + \left( \int_0^1 \{ f^{low}, s\} \circ X_s^t dt \right)^{low} + \left( \int_0^1 (1-t) \{ \{f^{high}, s\}, s\} \circ X_s^t dt \right)^{low},
\]

\[
f_1^{high} = f^{high} + \{ f^{high}, s\}^{high} + \left( \int_0^1 (1-t) \{ \{h, s\}, s\} \circ X_s^t dt \right)^{high}
\]

\[
\quad + \left( \int_0^1 \{ f^{low}, s\} \circ X_s^t dt \right)^{high} + \left( \int_0^1 (1-t) \{ \{f^{high}, s\}, s\} \circ X_s^t dt \right)^{high}.
\]

Estimate of the vector field generated by \( (1 - T_{\Delta'}) f^{low} \). Observing that

\[
\| (1 - T_{\Delta'}) F_{\varphi} \|_{D(\rho-\tau) \times U'} \leq \sum_{|k| > \Delta'} \| (\dot{F}_{\varphi}(k; w) + |\partial_w F_{\varphi}(k; w)|) e^{|k| (\rho-\tau)} \|_{D(\rho) \times U'},
\]

\[
\leq \sum_{|k| > \Delta'} e^{-\tau|k|} \| F_{\varphi} \|_{D(\rho) \times U'} \leq \frac{1}{\tau \# A} e^{-\frac{1}{2} \tau \Delta'} \| F_{\varphi} \|_{D(\rho) \times U'},
\]
we obtain
\[
\left\| \left( 1 - \mathcal{T}_\Delta \right) f \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'},
\]

The same estimates hold for \( \left\| \left( 1 - \mathcal{T}_\Delta \right) f \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \) and \( \left\| \left( 1 - \mathcal{T}_\Delta \right) f \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \).

Then we turn to
\[
\left( 1 - \mathcal{T}_\Delta \right) f^2 = (1 - \mathcal{T}^1_\Delta) f^2 + (1 - \mathcal{T}^2_\Delta) f^2,
\]

where
\[
(1 - \mathcal{T}^1_\Delta) f^2 = \frac{1}{2} \langle \zeta, (1 - \mathcal{T}_\Delta) F_2(\varphi; w) \zeta \rangle, \quad (1 - \mathcal{T}^2_\Delta) f^2 = \frac{1}{2} \sum |k| > \Delta' \langle \zeta, \mathcal{T}_\Delta \hat{F}_2(k; w) \zeta \rangle e^{i(k, \varphi)}.
\]

It is easy to see
\[
(3.52) \quad \left\| \left( 1 - \mathcal{T}^2_\Delta \right) f^2 \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'} \varepsilon.
\]

By (3.52), we have
\[
\sup_{(\varphi, w) \in D(\rho) \times U} \langle [F_2(\varphi; w)]_\gamma, |\partial_u F_2(\varphi; w)|_\gamma \rangle \leq \frac{\varepsilon}{\sigma^2}.
\]

Hence
\[
\left| \hat{F}_{2\alpha}(k; w) \right| + |\partial_w \hat{F}^{\alpha}_{2\alpha}(k; w)| \leq \frac{\varepsilon}{\sigma^2} e^{-\gamma |a-b|-\rho |k|},
\]

which implies
\[
\| F^{\alpha}_{2\alpha}(\varphi; w) \|_{D(\rho - \tau) \times U} = \sum_{k \in \mathcal{A}} \left( |\hat{F}^{\alpha}_{2\alpha}(k; w)| + |\partial_w \hat{F}^{\alpha}_{2\alpha}(k; w)| \right) e^{(\rho - \tau) |k|}
\]
\[
\leq \sum_{k \in \mathcal{A}} |k| e^{-\tau |k|} \frac{\varepsilon}{\sigma^2} e^{-\gamma |a-b|} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'}.
\]

Using Young’s inequality (2) in (3.52), we obtain
\[
\left\| \left( 1 - \mathcal{T}_\Delta \right) f \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'},
\]

which together with (3.52) leads to
\[
\left\| \left( 1 - \mathcal{T}_\Delta \right) f \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'}.
\]

In conclusion, we have
\[
\left\langle \left( 1 - \mathcal{T}_\Delta \right) f \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'}.
\]

**Estimate of the vector field generated by** \((1 - \mathcal{T}_\Delta) g_{low}^\text{low} = (1 - \mathcal{T}_\Delta) \{ f_{high}^\text{high}, s \}_{low}^\text{low} \). From equation (3.26), we obtain
\[
(3.54) \quad \left\langle \left( 1 - \mathcal{T}_\Delta \right) f_{\alpha} \right\|_{p,D(\rho - \tau, \sigma^2, \sigma) \times U'} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'}.
\]

Applying [EK10, Proposition 6.6] to equation (3.27), we obtain
\[
\| \mathcal{S}_1(k; \cdot) \|_{p; \gamma; U} \leq \frac{1}{\gamma^{d+p}} \frac{1}{\# \mathcal{A} + 1} \frac{\varepsilon}{\sigma^2} e^{-\frac{1}{2} \gamma \Delta'}.
\]

Noticing that
\[
\{ f_{high}^\text{high}, s^\alpha \} = -\langle \partial_s f_{high}^\text{high}, \partial_s s^\alpha \rangle,
\]
it follows from (3.6), (3.54) and [EK10, Equation (42)] that

\[
(3.55) \quad [\{ f_{\text{high}}, s^1 \}_\Lambda, \gamma, \sigma; U', \rho(1), \mu(1)] \leq \frac{1}{\rho - \rho(1)} \frac{1}{\mu - \mu(1)} \frac{1}{\kappa^2} (\Delta')^{\exp \varepsilon},
\]

which together with (3.6) and (3.23) implies

\[
[s^1]_{\Lambda, \gamma, \sigma; U', \rho(1), \mu(1)} \leq \frac{1}{\kappa^4} (\Delta')^{\exp \varepsilon} \frac{1}{\rho - \rho(1)} \frac{1}{\mu - \mu(1)} \varepsilon.
\]

Since \( s^1 \) is independent of \( r \), there is

\[
(3.56) \quad [s^1]_{\Lambda, \gamma, \sigma; U', \rho(1), \mu} \leq \frac{1}{\kappa^4} (\Delta')^{\exp \varepsilon} \frac{1}{\rho - \rho(1)} \frac{1}{\mu} \varepsilon.
\]

Next we estimate

\[
\{ f_{\text{high}}, s^1 \} = -\langle \partial_r f_{\text{high}}, \partial \phi s^1 \rangle + \langle \partial \zeta f_{\text{high}}, J \partial \zeta s^1 \rangle.
\]

By (3.6), (3.56) and the Cauchy estimate (42) in [EK10], we have

\[
[(\partial_r f_{\text{high}}, \partial \phi s^1)]_{\Lambda, \gamma, \sigma; U', \rho(2), \mu(1)} \leq \frac{1}{\kappa^4} (\Delta')^{\exp \varepsilon} \frac{1}{\rho - \rho(1)} \frac{1}{\rho(1)} \frac{1}{\mu - \mu(1)} \frac{1}{\mu} \varepsilon.
\]

Applying further Proposition 3.1 (ii) in [EK10], we have

\[
[(\partial \zeta f_{\text{high}}, J \partial \zeta s^1)]_{\Lambda, \gamma, \sigma; U', \rho(1), \mu} \leq \frac{1}{\kappa^4} (\Delta')^{\exp \varepsilon} \frac{1}{\rho - \rho(1)} \frac{1}{\sigma - \sigma(1)} \frac{1}{\mu} \varepsilon,
\]

which implies

\[
[\{ f_{\text{high}}, s^1 \}]_{\Lambda, \gamma, \sigma; U', \rho(1), \mu} \leq \frac{1}{\kappa^4} (\Delta')^{\exp \varepsilon} \delta, \quad \text{where}
\]

\[
\delta = \frac{1}{\rho - \rho(1)} \left( \frac{1}{\sigma - \sigma(1)} \frac{1}{\mu} + \frac{1}{\rho(1)} \frac{1}{\mu - \mu(1)} \right) \frac{1}{\mu}.
\]

By (3.23) and (3.55), we have

\[
(3.58) \quad [g_{\text{low}}]_{\Lambda, \gamma, \sigma; U', \rho(1), \mu} \leq \frac{1}{\kappa^4} (\Delta')^{\exp \varepsilon} \delta, \quad \text{which implies}
\]

\[
\sup_{(\varphi, w) \in D(\rho(2))^2 \times U'} \left\{ |G_2(\varphi; w)|_{\gamma}, \left| \partial w G_2(\varphi; w)_{\gamma} \right| \right\} \leq \frac{(\Delta')^{\exp \varepsilon}}{(\sigma(1))^2} \delta',
\]

where

\[
\delta' = \frac{1}{\rho - \rho(1)} \left( \frac{1}{\sigma - \sigma(1)} \frac{1}{\mu} + \frac{1}{\rho(1)} \frac{1}{\mu - \mu(1)} \right) \frac{1}{\mu}.
\]

By (3.23) and (3.48), we have

\[
(3.60) \quad \| X g_{\text{low}} \|_{p, D(\rho - 4 \tau, \rho - 4 \tau)^2 \times U'} \leq \frac{\delta'}{\tau^2} \varepsilon.
\]
Following the proof of (3.53), we get from (3.59) and (3.60) that
\[
\rho (3.62)
\]
By Proposition 2.9, (3.5), (3.49), we have
\[
\text{which implies}
\]
From (3.28), (3.6) and (3.58), we obtain
\[
\text{Estimate of the vector field } X_{f^{	ext{low},w}}. \text{ Using (3.50), (3.7), (3.60) and (3.60), we have}
\]
\[
\text{which together with Proposition 2.9 and (3.49) implies}
\]
\[
\text{By Proposition 2.9, (3.5), (3.49), we have}
\]
\[
\text{which implies}
\]
\[
\text{Then applying Theorem 3.3 in [CLY16], we have}
\]
\[
\text{By (3.53), (3.61), (3.63) and (3.51), we have}
\]
\[
\text{Estimate of the vector field } X_{f^{	ext{high},w}}. \text{ By (3.5), (3.6), (3.63) and (3.51), we have}
\]
\[
\text{3.7. Verification of (3.20)-(3.21). In this part, we shall verify the estimates (3.20)-(3.21). From (3.22), (3.6) and (3.58), we obtain}
\]
\[
\text{Using (3.5), (3.62), (3.63) and (3.51), we have}
\]
\[
\text{which implies}
\]
\[
\text{Estimate of the vector field } X_{f^{	ext{high},w}}. \text{ By (3.5), (3.6), (3.63) and (3.51), we have}
\]
\[
\text{3.7. Verification of (3.20)-(3.21). In this part, we shall verify the estimates (3.20)-(3.21). From (3.22), (3.6) and (3.58), we obtain}
\]
\[
\text{Using (3.5), (3.62), (3.63) and (3.51), we have}
\]
\[
\text{which implies}
\]
\[
\text{Estimate of the vector field } X_{f^{	ext{high},w}}. \text{ By (3.5), (3.6), (3.63) and (3.51), we have}
\]
\[
\text{3.7. Verification of (3.20)-(3.21). In this part, we shall verify the estimates (3.20)-(3.21). From (3.22), (3.6) and (3.58), we obtain}
\]
\[
\text{Using (3.5), (3.62), (3.63) and (3.51), we have}
\]
\[
\text{which implies}
\]
\[
\text{Estimate of the vector field } X_{f^{	ext{high},w}}. \text{ By (3.5), (3.6), (3.63) and (3.51), we have}
\]
where \( \delta \) is given by (3.57). Since \( s^0 \) is independent of \( \zeta \), we obtain

\[
(3.64) \quad [s^0]_{\Lambda, \gamma, \sigma, U', \rho^2, \mu^1} \lesssim \frac{1}{K^5} (\Delta \Delta' \exp \frac{1}{\rho - \rho^1}) \left( \frac{1}{\sigma^2} + \frac{1}{\rho^1} - \frac{1}{\rho^2} \right) \frac{\lambda}{\mu}. 
\]

Applying Proposition 6.7 in [EK10] to equation (3.29), it follows from (3.64) and (3.58) that

\[
(3.65) \quad [s]_{\Lambda', \gamma, \sigma, U', \rho^2, \mu^1} \lesssim \frac{1}{K^4} (\Delta \Delta' \exp \delta \varepsilon, 
\]

Using (3.54), (3.56), (3.64) and (3.65), we obtain

\[
(s)_{\Lambda', \gamma, \sigma, U', \rho^2, \mu^1} \lesssim \frac{1}{K^4} (\Delta \Delta' \exp \delta \varepsilon, 
\]

This completes the proof of Proposition 3.1.

4. PROOF OF THE KAM THEOREM 2.12

This section is devoted to the proof of Theorem 2.12. In subsection 4.1, we take the normal form computations. In subsection 4.2, we establish and prove the KAM iterative lemma, based on which Theorem 2.12 is an immediate result.

4.1. The normal form computation. For \( \rho_+ < \rho, \gamma_+ < \gamma \), let

\[
\Delta' = 80 (\log \frac{1}{\varepsilon})^2 \min(\gamma - \gamma_+, \rho - \rho_+), 
\]

and \( n = \lfloor \log \frac{1}{\varepsilon} \rfloor \). Assume \( \rho = \sigma, \mu = \sigma^2, d \Delta \gamma \leq 1 \). For \( 1 \leq j \leq n \), let

\[
\varepsilon_j = \frac{\varepsilon}{K 2^n}, \quad \varepsilon_0 = \varepsilon, 
\]

\[
\gamma_j = \gamma - j \frac{\gamma - \gamma_+}{n}, \quad \gamma_0 = \gamma, 
\]

\[
\rho_j = \rho - j \frac{\rho - \rho_+}{n}, \quad \rho_0 = \rho, 
\]

\[
\sigma_j = \sigma - j \frac{\sigma - \sigma_+}{n}, \quad \sigma_0 = \sigma, 
\]

\[
\mu_j = \sigma_j^2, \quad \mu_0 = \mu, 
\]

\[
A_j = A_{j-1} + d_\Delta + 30, \quad A_0 = \text{cte. max}(\Lambda, d_\Delta^2, d_\Delta'), 
\]

where the constant cte. is the one in Proposition 6.7 in [EK10].

We have the following lemma.

**Lemma 4.1.** For \( 0 \leq j < n \), consider the Hamiltonian \( h + h_1 + \cdots + h_j + f_j \), where

\[
h(r, \zeta; w) = \langle \omega(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H(w)) \zeta \rangle 
\]

satisfy (2.11), (2.19). \( H(w), \partial_w H(w) \) are Töplitz at \( \infty \) and \( N F_{\Delta} \) for all \( w \in U \). Let \( U' \subset U \) satisfy (3.1)-(3.4). For all \( w \in U' \),

\[
h_j = a_j(w) + \langle \chi_j(w), r \rangle + \frac{1}{2} \langle \zeta, H_j(w) \zeta \rangle, 
\]

\[
f_j = f_j^{\text{low}} + f_j^{\text{high}}
\]
satisfy

\[ \| X_{f_{j+1}} \|_{p, D(\rho_{j+1}, \mu_{j+1}, \sigma_{j+1}) \times U'} \leq \beta^j \varepsilon_j, \quad \| X_{f_{j+1}^{\text{high}}} \|_{p, D(\rho_{j+1}, \mu_{j+1}, \sigma_{j+1}) \times U'} \leq 1, \]

\[ [f_{j+1}^{\text{low}}]_{A_{j+1}, \gamma_{j+1}, \sigma_{j+1}, U', \rho_{j+1}, \mu_{j+1}} \leq \beta^{j+1} \varepsilon_{j+1}, \quad [f_{j+1}^{\text{high}}]_{A_{j+1}, \gamma_{j+1}, \sigma_{j+1}, U', \rho_{j+1}, \mu_{j+1}} \leq \beta^{j+1} \varepsilon_{j+1} \]

for some

\[ \beta \leq \max \left( \frac{1}{\gamma - \gamma_+}, \frac{1}{\rho - \rho_+}, \frac{1}{\Delta, \Lambda, \log \frac{1}{\varepsilon}} \right) \exp_1. \]

Then there exists an exponent \( \exp_2 \) such that

\[ \varepsilon \leq \kappa_{20} \min \left( \frac{1}{\gamma - \gamma_+}, \frac{1}{\rho - \rho_+}, \frac{1}{\Delta}, \frac{1}{\Lambda}, \frac{1}{\log \frac{1}{\varepsilon}} \right) \exp_2, \]

then for all \( w \in U' \), there is a real analytic symplectic map \( \Phi_j \) such that

\[ (h + h_1 + \cdots + h_j + f_j) \circ \Phi_j = h + h_1 + \cdots + h_{j+1} + f_{j+1}, \]

with the estimates

\[ \| X_{f_{j+1}} \|_{p, D(\rho_{j+1}, \mu_{j+1}, \sigma_{j+1}) \times U'} \leq \beta^{j+1} \varepsilon_{j+1}, \]

\[ \| X_{f_{j+1}^{\text{high}}} \|_{p, D(\rho_{j+1}, \mu_{j+1}, \sigma_{j+1}) \times U'} \leq 1 + \frac{1}{\kappa_6} \beta^{j+1} \varepsilon_j + \beta^{j+1} \varepsilon_{j+1}, \]

\[ [f_{j+1}^{\text{low}}]_{A_{j+1}, \gamma_{j+1}, \sigma_{j+1}, U', \rho_{j+1}, \mu_{j+1}} \leq \beta^{j+1} \varepsilon_{j+1}, \]

\[ [f_{j+1}^{\text{high}}]_{A_{j+1}, \gamma_{j+1}, \sigma_{j+1}, U', \rho_{j+1}, \mu_{j+1}} \leq 1 + \frac{1}{\kappa_7} \beta^{j+1} \varepsilon_j + \beta^{j+1} \varepsilon_{j+1}, \]

where the exponents \( \exp_1, \exp_2 \) depend on \( d, \#A, p \).

**Proof.** By Proposition 3.1, we can solve the homological equation

\[ (h, s_j) = -\mathcal{T}_{\Delta} f_{j+1} + \mathcal{T}_{\Delta} (f_{j}^{\text{high}}, s_j)^{\text{low}} + h_{j+1} \]

with the estimates

\[ [s_j] \left\{ A_{j+d} + 2, \gamma_j, \sigma_j^{(1)}; U', \rho_j^{(1)}, \mu_j^{(1)} \right\} \leq \frac{1}{\kappa_9} (\Delta \Lambda')^{\exp_1} \delta_1 \varepsilon_j, \]

\[ [h_{j+1}] \left\{ A_{j+d} + 2, \gamma_j, \sigma_j^{(1)}; U', \rho_j^{(1)}, \mu_j^{(1)} \right\} \leq \frac{1}{\kappa_9} (\Delta \Lambda')^{\exp_1} \delta_1 \varepsilon_j, \]

where

\[ \delta_1 = \frac{1}{\rho_j - \rho_j^{(1)}} \left( \frac{1}{\sigma_j - \sigma_j^{(1)}} + \frac{1}{\mu_j - \mu_j^{(1)}} \right) \frac{1}{\mu_j} \beta^j. \]
Step 1: computation of \( f_{j+1} \). In this step, we compute the new perturbation \( f_{j+1} = f_{j+1}^{low} + f_{j+1}^{high} \). Using Taylor’s formula, by the homological equation (4.7), we obtain

\[
(h + h_1 + \cdots + h_j + f_j) \circ X_{s_j}^t \big|_{t=1} = h + h_1 + \cdots + h_{j+1} + \frac{1}{2} \int_0^1 \{ (h, s_j), s_j \} \circ X_{s_j}^t \, dt + \frac{1}{2} \int_0^1 \{ h_1 + \cdots + h_j, s_j \} \circ X_{s_j}^t \, dt + \frac{1}{2} \int_0^1 \{ f_j^{low}, s_j \} \circ X_{s_j}^t \, dt.
\]

Let \( \Phi_j = X_{s_j}^t \big|_{t=1} \) and

\[
(h + h_1 + \cdots + h_j + f_j) \circ X_{s_j}^t \big|_{t=1} = h + h_1 + \cdots + h_{j+1} + f_{j+1}.
\]

Then we get from (4.7) and (4.9) that

\[
f_{j+1} = (1 - T_{A'} f_j^{low} + f_j^{high} + (1 - T_{A'}) \{ f_j^{high}, s_j \}^{low} + \{ f_j^{high}, s_j \}^{high}
\]

\[
+ \int_0^1 (1-t) \{ -T_{A'} f_j^{low}, s_j \} \circ X_{s_j}^t \, dt + \int_0^1 (1-t) \{ -T_{A'} \{ f_j^{high}, s_j \}^{low}, s_j \} \circ X_{s_j}^t \, dt
\]

\[
+ \int_0^1 \{ h_1 + \cdots + h_j + (1-t)h_{j+1}, s_j \} \circ X_{s_j}^t \, dt + \int_0^1 \{ f_j^{low}, s_j \} \circ X_{s_j}^t \, dt
\]

As a result, there is

\[
f_{j+1}^{low} = (1 - T_{A'} f_j^{low} + (1 - T_{A'}) \{ f_j^{high}, s_j \}^{low} + \left( \int_0^1 \{ f_j^{low}, s_j \} \circ X_{s_j}^t \right)^{low}
\]

\[
+ \left( \int_0^1 (1-t) \{ -T_{A'} f_j^{low}, s_j \} \circ X_{s_j}^t \right)^{low}
\]

\[
+ \left( \int_0^1 (1-t) \{ -T_{A'} \{ f_j^{high}, s_j \}^{low}, s_j \} \circ X_{s_j}^t \right)^{low}
\]

\[
+ \left( \int_0^1 (1-t) \{ f_j^{high}, s_j \} \circ X_{s_j}^t \right)^{low}
\]

\[
+ \left( \int_0^1 \{ h_1 + \cdots + h_j + (1-t)h_{j+1}, s_j \} \circ X_{s_j}^t \right)^{low},
\]
and
\[
\begin{align*}
    f_{j+1}^{\text{high}} &= f_j^{\text{high}} + \{f_j^{\text{high}}, s_j\}^{\text{high}} + \left(\int_0^1 (1-t) \{\mathcal{T} f_j^{\text{low}}, s_j\} \circ X_t^{\text{high}} dt\right) \\
    &+ \left(\int_0^1 (1-t) \{\mathcal{T} f_j^{\text{high}}, s_j\}^{\text{low}} \circ X_t^{\text{low}} dt\right) \\
    &+ \left(\int_0^1 (1-t) \{f_j^{\text{low}}, s_j\} \circ X_t^{\text{high}} dt\right) \\
    &+ \left(\int_0^1 \{f_j^{\text{low}}, s_j\} \circ X_t^{\text{low}} dt\right) \\
    &+ \left(\int_0^1 \{h_1 + \cdots + h_j + (1-t)h_{j+1}, s_j\} \circ X_t^{\text{high}} dt\right).
\end{align*}
\]

**Step 2: estimates of** $f_{j+1}^{\text{low}}$ **and** $f_{j+1}^{\text{high}}$. In this part, we verify the estimates \[4.5\] and \[4.6\]. The various estimates of the poisson brackets are based on [8] and [10], Proposition 3.3.

To begin with, we see from \[4.2\] that
\[
(4.10) \quad \left[(1 - \mathcal{T} \Delta) f_j^{\text{low}}\right] \left\{\Lambda_j, \gamma_j^{(1)}, \mu_j \right\} \lesssim \left[\left(\frac{1}{\rho_j - \rho_j^{(1)}}\right)^{\#A} e^{-\frac{1}{2}(\rho_j - \rho_j^{(1)})\Delta'} + e^{-(\gamma_j - \gamma_j^{(1)})\Delta'}\right] \beta^j \varepsilon_j.
\]

By \[4.5\], there is
\[
(4.11) \quad \left[\{f_j^{\text{high}}, s_j\}^{\text{low}}\right] \left\{\Lambda_j, \gamma_j^{(1)}, \mu_j \right\} \lesssim \frac{1}{\kappa^4} (\Delta \Delta') \exp \delta_1 \varepsilon_j,
\]

and thus
\[
(4.12) \quad \left[(1 - \mathcal{T} \Delta) \{f_j^{\text{high}}, s_j\}^{\text{low}}\right] \left\{\Lambda_j, \gamma_j^{(1)}, \mu_j \right\} \lesssim \left[\left(\frac{1}{\rho_j - \rho_j^{(1)}}\right)^{\#A} e^{-\frac{1}{2}(\rho_j - \rho_j^{(2)})\Delta'} + e^{-(\gamma_j - \gamma_j^{(1)})\Delta'}\right] \frac{1}{\kappa^4} (\Delta \Delta') \exp \delta_1 \varepsilon_j.
\]

Let
\[
\delta_2 = (\Lambda_j + d_\Delta + 2)^2 \left(\frac{1}{\gamma_j - \gamma_j^{(1)}}\right)^{d+1} \frac{1}{\sigma_j^{(1)} - \sigma_j^{(2)}} \frac{1}{\sigma_j^{(1)}} + \frac{1}{\rho_j - \rho_j^{(2)}} \frac{1}{\mu_j^{(1)} - \mu_j^{(2)}} \frac{1}{\mu_j^{(1)}}.
\]

By \[4.2\], \[4.5\] and [10], Proposition 3.3, we have
\[
(4.13) \quad \left[\{f_j^{\text{high}}, s_j\}^{\text{low}}\right] \left\{\Lambda_j + d_\Delta + 5, \gamma_j^{(1)}(\sigma_j^{(2)})\right\} \lesssim \delta_2 \frac{1}{\kappa^4} (\Delta \Delta') \exp \delta_1 \varepsilon_j,
\]

and
\[
(4.14) \quad \left[\{f_j^{\text{low}}, s_j\}\right] \left\{\Lambda_j + d_\Delta + 5, \gamma_j^{(1)}(\sigma_j^{(2)})\right\} \lesssim \delta_2 \frac{1}{\kappa^4} (\Delta \Delta') \exp \delta_1 \beta^j \varepsilon_j.
\]
Moreover, we obtain from (4.8), (4.11) and [EK10, Proposition 3.3] that

\[(4.15) \quad \{ \{ f_j^{\text{high}}, s_j \} \}_{\text{low}, s_j}^{\text{low}, s_j} \{ \Lambda_j + d_{\Delta} + 5 \gamma_j^{(2)} \sigma_j^{(3)} \}_{U', \rho_j^{(2)}}, \mu_j^{(2)} \} \leq \delta_2 \frac{1}{\kappa_1} \exp \delta_1 \varepsilon_j^2.
\]

Let

\[
\delta_3 = (\Lambda_j + d_{\Delta} + 5)^2 \left( \frac{1}{\gamma_j^{(1)} - \gamma_j^{(2)}} \right)^{d+1} \frac{1}{\sigma_j^{(2)} - \sigma_j^{(3)}} \frac{1}{\sigma_j^{(2)} + 1} + \frac{1}{\rho_j^{(2)} - \rho_j^{(3)}} \frac{1}{\mu_j^{(2)} - \mu_j^{(3)}}.
\]

By (4.12), we have

\[
\frac{\{ h_{l+1}, s_j \}}{\{ f_j^{\text{high}}, s_j \}_{U', \rho_j^{(2)}}, \mu_j^{(2)}} \leq \delta_3 \frac{1}{\kappa_1} \exp \delta_1 \varepsilon_j^2,
\]

and

\[
\{ f_j^{\text{low}}, s_j \}_{\text{low}, s_j}^{\text{low}, s_j} \{ \Lambda_j + d_{\Delta} + 5 \gamma_j^{(2)} \sigma_j^{(3)} \}_{U', \rho_j^{(2)}}, \mu_j^{(2)} \} \leq \frac{1}{\kappa_1} \exp \delta_1 \varepsilon_j^2.
\]

Take

\[
\rho_j^{(l)} = \rho_j - \frac{l}{4} (\rho_j - \rho_{j+1}), \quad \gamma_j^{(l)} = \gamma_j - \frac{l}{4} (\gamma_j - \gamma_{j+1}),
\]

\[
\sigma_j^{(l)} = \sigma_j - \frac{l}{4} (\sigma_j - \sigma_{j+1}), \quad \mu_j^{(l)} = (\sigma_j^{(l)})^2,
\]

where \( l = 1, 2, 3, 4 \). By (4.10), we have

\[(4.15) \quad \{ 1 - T_{\Delta} \} f_j^{\text{low}} \Lambda_j \gamma_j^{(1)} \sigma_j U', \rho_j^{(1)}, \mu_j \leq \beta \varepsilon \beta^j \varepsilon_j \leq \beta^{j+1} \varepsilon_j + 1.
\]

By (4.12), we have

\[
\{ 1 - T_{\Delta} \} f_j^{\text{high}, s_j} \Lambda_j \gamma_j^{(1)} \sigma_j U', \rho_j^{(1)}, \mu_j \leq \frac{1}{\kappa_1} \beta \varepsilon \beta^j \varepsilon_j \leq \beta^{j+1} \varepsilon_j + 1,
\]

\[
\{ f_j^{\text{low}, s_j} \}_{\text{low}, s_j}^{\text{low}, s_j} \{ \Lambda_j + d_{\Delta} + 5 \gamma_j^{(2)} \}, \sigma_j^{(2)} U', \rho_j^{(2)}}, \mu_j^{(2)} \} \leq \frac{1}{\kappa_1} \beta \varepsilon \beta^j \varepsilon_j \leq \beta^{j+1} \varepsilon_j + 1,
\]

\[
\{ f_j^{\text{low}, s_j} \}_{\text{low}, s_j}^{\text{low}, s_j} \{ \Lambda_j + d_{\Delta} + 5 \gamma_j^{(2)} \sigma_j^{(3)} U', \rho_j^{(3)}}, \mu_j^{(3)} \} \leq \frac{1}{\kappa_1} \beta \varepsilon \beta^j \varepsilon_j \leq \beta^{j+1} \varepsilon_j + 1,
\]

\[
\{ h_1 + \cdots + h_j + (1 - t)h_{j+1}, s_j \}_{\text{low}, s_j}^{\text{low}, s_j} \{ \Lambda_j + d_{\Delta} + 5 \gamma_j^{(2)} \sigma_j^{(3)} U', \rho_j^{(3)}}, \mu_j^{(3)} \} \leq \frac{1}{\kappa_1} \beta \varepsilon \beta^j \varepsilon_j \leq \beta^{j+1} \varepsilon_j + 1,
\]

where \( \rho_j, \sigma_j, \mu_j \) are the projections of \( \theta_j, \sigma_j, \mu_j \) onto the low modes.
which imply the following properties

\[
\begin{align*}
\int_0^1 (1-t)\{-T\mathcal{A}_{j+1}\mathcal{E}_{j}^{\text{low}}, s_j\} \circ X_s^t \, dt &\leq \beta^{j+1}_1 \varepsilon_{j+1}, \\
\int_0^1 \{f_j^{\text{low}}, s_j\} \circ X_s^t \, dt &\leq \beta^{j+1}_1 \varepsilon_{j+1}, \\
\int_0^1 (1-t)\{-T\mathcal{A}_{j+1}\mathcal{E}_{j}^{\text{high}}, s_j\} \circ X_s^t \, dt &\leq \beta^{j+1}_1 \varepsilon_{j+1}, \\
\int_0^1 \{h_1 + \cdots + h_j + (1-t)h_{j+1}, s_j\} \circ X_s^t \, dt &\leq \beta^{j+1}_1 \varepsilon_{j+1}, \\
\int_0^1 (1-t)\{f_j^{\text{high}}, s_j\} \circ X_s^t \, dt &\leq \beta^{j+1}_1 \varepsilon_{j+1}.
\end{align*}
\]

Then from the definition of \( f^{\text{low}}_{j+1} \) and \( f^{\text{high}}_{j+1} \), we get the desired estimates

\[
\begin{align*}
[f^{\text{low}}_{j+1}]_{\mathcal{A}_{j+1}, \mathcal{E}_{j+1}: U', \rho_{j+1}, \mu_{j+1}} &\leq \beta^{j+1}_1 \varepsilon_{j+1}, \\
[f^{\text{high}}_{j+1}]_{\mathcal{A}_{j+1}, \mathcal{E}_{j+1}: U', \rho_{j+1}, \mu_{j+1}} &\leq 1 + \frac{1}{\kappa^2} \beta^{j+1}_1 \varepsilon_{j+1} + \beta^{j+1}_1 \varepsilon_{j+1}.
\end{align*}
\]

**Step 3: estimates of the vector fields** \( X_{f^{\text{low}}_{j+1}} \) and \( X_{f^{\text{high}}_{j+1}} \). By (3.49) and (3.50), we have

\[
|||X_{s_j}|||_{P, D(\rho_j^{(1)}, \mu_j^{(1)}, \sigma_j^{(1)}) \times U'} \leq \frac{1}{\kappa^3} \Delta^{\exp} \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^5 \beta^j \varepsilon_j,
\]

(4.18)

\[
|||X_{h_{j+1}}|||_{P, D(\rho_j^{(1)}, \rho_j^{(1)}, \sigma_j^{(1)}) \times U'} \leq \frac{1}{\kappa^3} \Delta^{\exp} \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^4 \beta^j \varepsilon_j.
\]

Using (3.53), we have

\[
|||X_{(1-\mathcal{A}_{j})f^{\text{low}}_{j+1}}|||_{P, D(\rho_j^{(1)}, \mu_j^{(1)}, \sigma_j^{(1)}) \times U'} \leq \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^{#A} e^{-\frac{1}{2}(\rho_j - \rho_j^{(1)})\Delta'} \beta^j \varepsilon_j
\]

(4.19)

\[
+ \frac{1}{\gamma_{d+p}} \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^{#A+1} \frac{1}{\sigma_j} e^{-\frac{1}{2}\gamma_j \Delta'} \beta^j \varepsilon_j \leq \beta^{j+1}_1 \varepsilon_{j+1}.
\]

By (3.60) and (3.61), we have

\[
|||X_{f^{\text{high}}_{j+1}, s_j}|||_{P, D(\rho_j^{(1)}, \mu_j^{(1)}, \sigma_j^{(1)}) \times U'} \leq \frac{1}{\kappa^4} \Delta^{\exp} \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^4 \beta^j \varepsilon_j,
\]

(4.20)

and

\[
|||X_{(1-\mathcal{A}_{j})f^{\text{high}}_{j+1}, s_j}|||_{P, D(\rho_j^{(1)}, \mu_j^{(1)}, \sigma_j^{(1)}) \times U'} \leq \frac{1}{\kappa^4} \Delta^{\exp} \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^4 \beta^j \varepsilon_j
\]

(4.21)

\[
\leq \frac{1}{\kappa^4} (\Delta')^{\exp} \left( \frac{1}{\rho_j - \rho_j^{(1)}} \right)^{#A+4} \left[ e^{-\frac{1}{2}(\rho_j - \rho_j^{(1)})\Delta'} \frac{1}{\gamma_{d+p}} \frac{1}{\sigma_j} e^{-\frac{1}{2}\gamma_j \Delta'} \right] \beta^j \varepsilon_j \leq \beta^{j+1}_1 \varepsilon_{j+1}.
\]
By (4.1), (4.18) and Proposition 2.9, we have

\[ \| X_{f_j}^{[\text{low}, s_j]} \|_{\rho_j, \mu_j, \sigma_j} \leq \frac{1}{\rho_j - \rho_j^{(2)}} \Delta^{(1)} \beta_j e_j, \]

\[ \| X_{f_j}^{[\text{high}, s_j]} \|_{\rho_j, \mu_j, \sigma_j} \leq \frac{1}{\rho_j - \rho_j^{(2)}} \Delta^{(1)} \beta_j e_j. \]

Using (3.6), we have

\[ \| X_{\{h, s_j\}} \|_{\rho_j, \mu_j, \sigma_j} \leq \frac{1}{\rho_j - \rho_j^{(1)}} \Delta^{(1)} \beta_j e_j. \]

By (4.18), (4.23), (4.24), using Proposition 2.9 we have

\[ \| X_{\{f_j^{[\text{low}}, s_j\}} \|_{\rho_j, \mu_j, \sigma_j} \leq \frac{1}{\rho_j - \rho_j^{(2)}} \Delta^{(1)} \beta_j e_j. \]

By (4.22) and (4.25), we see from [CLY16, Theorem 3.3] that

\[ \| X_{f^{[\text{low}}, j, s_j]} \|_{\rho_j, \mu_j, \sigma_j} \leq \beta_j e_j. \]

\[ \| X_{f^{[\text{high}}, j, s_j]} \|_{\rho_j, \mu_j, \sigma_j} \leq \beta_j e_j. \]

Finally, we obtain from the definitions of \( f^{[\text{low}}, j+1 \) and \( f^{[\text{high}}, j+1 \) that

\[ \| X_{f^{[\text{low}}, j+1]} \|_{\rho_j, \mu_j, \sigma_j} \leq \beta_j e_j. \]

This completes the proof of Lemma 4.1. \( \square \)

4.2. The KAM iterative lemma. Assume \( \rho = \sigma, \mu = \sigma^2, d\Delta \gamma \leq 1 \). For \( m \geq 0 \), let

\[ \varepsilon_m = e^{-d\Delta_m}, \varepsilon_0 = \varepsilon, \]

\[ \vartheta_m = \frac{\sum_{j=1}^m \vartheta_j}{\sum_{j=1}^m 1}, \vartheta_0 = 0, \]

\[ \rho_m = (1 - \vartheta_m) \rho, \rho_0 = \rho, \]

\[ \sigma_m = (1 - \vartheta_m) \sigma, \sigma_0 = \sigma, \]

\[ \mu_m = \sigma_m^2, \mu_0 = \mu, \]

\[ \gamma_m = d^{-1}_m, \gamma_0 = \min(\gamma, d^{-1}_m), \]

\[ \Delta_m = \Delta_m - \Delta_m \gamma - \Delta_m \gamma, \]

\[ \Delta_m = \Delta_m - \Delta_m \gamma - \Delta_m \gamma, \]

\[ \Delta_m = \Delta_m - \Delta_m \gamma - \Delta_m \gamma, \]

\[ \Delta_m = \Delta_m - \Delta_m \gamma - \Delta_m \gamma, \]

\[ \Delta_m = \Delta_m - \Delta_m \gamma - \Delta_m \gamma, \]
\[ \Delta_m = 80(\log \frac{1}{\varepsilon^{-1}})^2 \frac{1}{\min(\gamma_{m-1}, \rho_{m-1} - \rho_m)}, \quad \Delta_0 = \Delta, \]

where the constant \( cte \) is the one in Proposition 6.7 in \[EK10\].

We have the following KAM iterative lemma.

**Lemma 4.2.** For \( m \geq 0 \), consider the Hamiltonian \( h_m + f_m \), where

\[ h_m = \langle \omega_m(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H_m(w)) \zeta \rangle, \]

\( H_m(w) \), \( \partial_w H_m(w) \) are Töplitz at \( \infty \) and \( \mathcal{N}_\mathcal{F}_m \) for all \( w \in U_m \),

\[ f_m = f_m^{low} + f_m^{high} \]

satisfy

\[ |||X_{f_m^{low}}|||_{p,D(\rho_m, \mu_m, \sigma_m) \times U_m} \leq \varepsilon_m, \quad |||X_{f_m^{high}}|||_{p,D(\rho_m, \mu_m, \sigma_m) \times U_m} \leq \varepsilon + \sum_{j=1}^{m} \varepsilon_j^{\frac{3}{2}}, \]

\[ \Delta_m, \gamma_m, \sigma_m, U_m, \rho_m, \mu_m \leq \varepsilon_m, \quad \Delta_{m+1}, \gamma_{m+1}, \sigma_{m+1}, U_{m+1}, \rho_{m+1}, \mu_{m+1} \leq \varepsilon + \sum_{j=1}^{m} \varepsilon_j^{\frac{3}{2}}. \]

Assume for all \( w \in U_m \),

\[ |\omega_m(w) - \omega(w)| + |\partial_w(\omega_m(w) - \omega(w))| \leq \sum_{j=1}^{m} \varepsilon_j^{\frac{3}{2}}, \]

\[ \|H_m - H\|_{U_m} + \langle H_m - H \rangle_{\Lambda_m, U_m} \leq \sum_{j=1}^{m} \varepsilon_j^{\frac{3}{2}}. \]

Then there is a subset \( U_{m+1} \subset U_m \) such that if

\[ \varepsilon \leq \min \left( \gamma, \rho, \frac{1}{\Delta}, \frac{1}{\Lambda} \right)^{exp}, \]

then for all \( w \in U_{m+1} \), there is a real analytic symplectic map \( \Phi_m \) such that

\( (h_m + f_m) \circ \Phi_m = h_{m+1} + f_{m+1} \)

with the estimates

\[ |||X_{f_{m+1}^{low}}|||_{p,D(\rho_{m+1}, \mu_{m+1}, \sigma_{m+1}) \times U_{m+1}} \leq \varepsilon_{m+1}, \]

\[ |||X_{f_{m+1}^{high}}|||_{p,D(\rho_{m+1}, \mu_{m+1}, \sigma_{m+1}) \times U_{m+1}} \leq \varepsilon + \sum_{j=1}^{m+1} \varepsilon_j^{\frac{3}{2}}, \]

\[ \Delta_{m+1}, \gamma_{m+1}, \sigma_{m+1}, U_{m+1}, \rho_{m+1}, \mu_{m+1} \leq \varepsilon_{m+1}; \]

\[ |\omega_{m+1}(w) - \omega(w)| + |\partial_w(\omega_{m+1}(w) - \omega(w))| \leq \sum_{j=1}^{m+1} \varepsilon_j^{\frac{3}{2}}, \]

\[ \|H_{m+1} - H\|_{U_{m+1}} + \langle H_{m+1} - H \rangle_{\Lambda_{m+1}, U_{m+1}} \leq \sum_{j=1}^{m+1} \varepsilon_j^{\frac{3}{2}}, \]

\[ \text{meas}(U_m \setminus U_{m+1}) \leq \varepsilon_{m+1}^{exp'}. \]
where the exponents $\exp, \exp'$ depend on $d, \#A, p$.

**Proof.** Take $\kappa^{20} = \varepsilon \frac{1}{16}$ in Lemma 4.1, there is a real analytic symplectic map $\Phi$ such that

\[(h + f) \circ \Phi = h + h_1 + \cdots + h_n + f_n.\]

Let $h_+ = h + h_1 + \cdots + h_n$, $f_+ = f_n$. The KAM iterative lemma follows immediately from Lemma 4.1. \qed

5. **LONG TIME STABILITY OF THE KAM TORI**

In this section, we prove Theorem 2.13 on the long time stability of the KAM tori. By momentum conservation, the frequency shift is diagonal. We will construct a partial normal form of order $M + 2$ based on $h_{\tilde{m}} + f_{\tilde{m}}^{\text{high}}$, where

\[h_{\tilde{m}} = \langle \omega_{\tilde{m}}(w), r \rangle + \frac{1}{2} \langle \zeta, (\Omega(w) + H_{\tilde{m}}(w)) \zeta \rangle,\]

$H_{\tilde{m}}(w), \partial_u H_{\tilde{m}}(w)$ are Töplitz at $\infty$ and $\mathcal{N}^\Delta_{\tilde{m}}$ for all $w \in U_{\infty}$.

We change to complex coordinates

\[z = \begin{pmatrix} u \\ v \end{pmatrix} = C^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix},\]

then

\[h_{\tilde{m}} = \langle \omega_{\tilde{m}}(w), r \rangle + \langle u, (\Omega(w) + H_{\tilde{m}}(w)) v \rangle.\]

Let $\mathcal{B} = \{ a \in \mathcal{L} : \|a\| \leq N \}$, $\tilde{u} = (u_a)_{a \in \mathcal{B}}, \tilde{v} = (v_a)_{a \in \mathcal{B}}, \check{u} = (u_a)_{a \in \mathcal{L} \setminus \mathcal{B}}, \check{v} = (v_a)_{a \in \mathcal{L} \setminus \mathcal{B}}$. Write

\[h_{\tilde{m}} = \langle \omega_{\tilde{m}}(w), r \rangle + \sum_{a \in \mathcal{B}} \tilde{\lambda}_a(w) u_a v_a + \langle \check{u}, (\Omega(w) + H_{\tilde{m}}(w)) \check{v} \rangle,\]

where $\tilde{\lambda}_a(w) \in \text{spec}(\Omega(w) + H_{\tilde{m}}(w))_{\mathcal{B}}$.

Let $\tilde{\Delta} > 1$ and $0 < \tilde{k} < 1$. Assume there exists $\tilde{U} \subset U_{\infty}$ such that for all $w \in \tilde{U}$, $|k| \leq \tilde{\Delta}$, $|\check{l}| \leq M + 2$, $|k| + |\check{l}| \neq 0$, the following conditions hold:

- **Diophantine condition:**

\[(5.1) \quad |\langle k, \omega_{\tilde{m}}(w) \rangle + \langle \check{l}, \tilde{\lambda}(w) \rangle| \geq \frac{\tilde{k}}{4M N^{(4d)^{\text{th}(\tilde{l}+4)^2}}} ;\]

- **The first Melnikov condition:**

\[(5.2) \quad |\langle k, \omega_{\tilde{m}}(w) \rangle + \langle \check{l}, \tilde{\lambda}(w) \rangle + \alpha(w)| \geq \frac{\tilde{k}}{4M N^{(4d)^{\text{th}(\tilde{l}+4)^2}}} ;\]

for any $\alpha(w) \in \text{spec}(((\Omega + H_{\tilde{m}}(w))_{[a]_{\Delta_{\tilde{m}}}})$ and any $[a]_{\Delta_{\tilde{m}}}$;

- **The second Melnikov condition with the same sign:**

\[(5.3) \quad |\langle k, \omega_{\tilde{m}}(w) \rangle + \langle \check{l}, \tilde{\lambda}(w) \rangle + \alpha(w) + \beta(w)| \geq \frac{\tilde{k}}{4M N^{(4d)^{\text{th}(\tilde{l}+4)^2}}} ;\]

for any $\alpha(w) \in \text{spec}(((\Omega + H_{\tilde{m}}(w))_{[a]_{\Delta_{\tilde{m}}}})$, $\beta(w) \in \text{spec}(((\Omega + H_{\tilde{m}}(w))_{[b]_{\Delta_{\tilde{m}}}})$ and any $[a]_{\Delta_{\tilde{m}}}, [b]_{\Delta_{\tilde{m}}}$;

- **The second Melnikov condition with the opposite signs:**

\[(5.4) \quad |\langle k, \omega_{\tilde{m}}(w) \rangle + \langle \check{l}, \tilde{\lambda}(w) \rangle + \alpha(w) - \beta(w)| \geq \frac{\tilde{k}}{4M N^{(4d)^{\text{th}(\tilde{l}+4)^2}}} ;\]

for any $\alpha(w) \in \text{spec}(((\Omega + H_{\tilde{m}}(w))_{[a]_{\Delta_{\tilde{m}}}})$, $\beta(w) \in \text{spec}(((\Omega + H_{\tilde{m}}(w))_{[b]_{\Delta_{\tilde{m}}}})$, and any $\text{dist}([a]_{\Delta_{\tilde{m}}}, [b]_{\Delta_{\tilde{m}}}) \leq \tilde{\Delta} + 2d_{\Delta_{\tilde{m}}}$.
We have the following lemma. For the sake of notations, we denote

\[ \sum_{(1)} = \sum_{2|\alpha|+|\beta|+|\nu|=j} \quad \sum_{(2)} = \sum_{2|\alpha|+|\beta|+|\nu|=j-1} \quad \sum_{(3)} = \sum_{2|\alpha|+|\beta|+|\nu|=j-2}. \]

Lemma 5.1. For \(2 \leq j_0 \leq M+1\), consider the partial normal form of order \(j_0\)

\[ T_{j_0} = h_m + Z_{j_0} + P_{j_0} + R_{j_0} + Q_{j_0}, \]

with

\[ Z_{j_0} = \sum_{3 \leq j \leq j_0} Z_{j_0}, \quad P_{j_0} = \sum_{j \geq j_0 + 1} P_{j_0}, \quad R_{j_0} = \sum_{3 \leq j \leq j_0} R_{j_0}, \quad Q_{j_0} = \sum_{j \geq 3} Q_{j_0}, \]

where

- \(Z_{j_0}\) equals to
  \[ \sum_{2|\alpha|+|\beta|+|\nu|=j} Z_{j_0}^{\alpha \beta \nu}(w) r^\alpha \bar{u}^\beta \bar{v}^\nu, \]

- \(P_{j_0}\) equals to
  \[ \sum_{2|\alpha|+|\beta|+|\nu|=j} P_{j_0}^{\alpha \beta \nu}(w) r^\alpha \bar{u}^\beta \bar{v}^\nu \]
  \[ + \sum_{2|\alpha|+|\beta|+|\nu|=j} \langle \bar{u}, P_{j_0}^{\alpha \beta \nu \delta}(w) \rangle r^\alpha \bar{u}^\beta \bar{v}^\nu \]
  \[ + \sum_{2|\alpha|+|\beta|+|\nu|=j} \langle \bar{u}, P_{j_0}^{\alpha \beta \nu \delta}(w) \rangle r^\alpha \bar{u}^\beta \bar{v}^\nu \]

- \(R_{j_0}\) equals to
  \[ \sum_{2|\alpha|+|\beta|+|\nu|=j} R_{j_0}^{\alpha \beta \nu \delta}(w) r^\alpha \bar{u}^\beta \bar{v}^\nu \]
  \[ + \sum_{2|\alpha|+|\beta|+|\nu|=j} \langle \bar{u}, R_{j_0}^{\alpha \beta \nu \delta}(w) \rangle r^\alpha \bar{u}^\beta \bar{v}^\nu \]
  \[ + \sum_{2|\alpha|+|\beta|+|\nu|=j} \langle \bar{u}, R_{j_0}^{\alpha \beta \nu \delta}(w) \rangle r^\alpha \bar{u}^\beta \bar{v}^\nu \]

- \(Q_{j_0}\) equals to
  \[ \sum_{2|\alpha|+|\beta|+|\nu|=j} Q_{j_0}^{\alpha \beta \nu \delta}(w) r^\alpha \bar{u}^\beta \bar{v}^\nu \bar{w}^\nu \]

Let \(\rho' = \frac{p}{12M}\), \(\delta' = \frac{\delta}{12}\) and

\[ D_{j_0} = D \left( \frac{p}{2} - 3(j_0 - 2)\rho', (5\delta - 2(j_0 - 2)\delta')^2, 5\delta - 2(j_0 - 2)\delta' \right), \]

\[ D_{j_0}' = D \left( \frac{p}{2} - 3(j_0 - 2) + 1\rho', (5\delta - 2(j_0 - 2)\delta')^2, 5\delta - 2(j_0 - 2)\delta' \right). \]

Assume

\[ |||X_{j_0}|||^T_{p, D_{j_0} \times \bar{U}} \leq \delta \left( \frac{1}{\kappa^2} \Delta_m^\exp N^{(4d)(j_0+4)^2} \delta \right)^{\frac{j_0 - 3}{3}}, \]

(5.5)

\[ |||X_{j_0}|||^T_{p, D_{j_0} \times \bar{U}} \leq \delta \left( \frac{1}{\kappa^2} \Delta_m^\exp N^{(4d)(j_0+4)^2} \delta \right)^{\frac{j_0 - 3}{3}}, \]

(5.6)
\[
\begin{align*}
(5.7) \quad \|X_{P_{j_0}}\|_{p,D_{j_0} \times \hat{U}} & \leq \delta \left( \frac{1}{k^2} \Delta^\exp \, N^{(4d)^{4d}(j_0+5)^2} \delta \right)^{j-3}, \\
(5.8) \quad \|X_{Q_{j_0}}\|_{p,D_{j_0} \times \hat{U}} & \leq \delta \left( \frac{1}{k^2} \Delta^\exp \, N^{(4d)^{4d}(j_0+5)^2} \delta \right)^{j-3},
\end{align*}
\]

where \( a \leq b \) means there is a constant \( c > 0 \) depending on \( \rho, M, d, \# \mathcal{A}, p, c_1, c_2, c_3, c_4, c_5 \) such that \( a \leq cb \), the exponent \( \exp \) depends on \( d, \# \mathcal{A}, p \).

Then there is a symplectic map \( \Psi_{j_0} \) such that
\[
T_{j_0+1} = T_{j_0} \circ \Psi_{j_0} = h_{\hat{m}} + Z_{j_0+1} + P_{j_0+1} + R_{j_0+1} + Q_{j_0+1},
\]
which is given exactly by the formula of \( T_{j_0} \) but with \( j_0 + 1 \) in place of \( j_0 \). Moreover, the estimates (5.5) - (5.8) also hold with \( j_0 + 1 \) in place of \( j_0 \).

**Proof.** Consider the homological equation
\[
(5.9) \quad \{h_{\hat{m}}, F_{j_0} \} = -T_{\Delta} P_{j_0(j_0+1)} + \hat{Z}_{j_0},
\]
where
\[
T_{\Delta} P_{j_0(j_0+1)} = \sum_{|k| \leq \Delta} \left[ \sum_{2|\alpha|+|\beta|+|\nu|=j_0+1} \hat{P}_{j_0}^{\alpha \beta \nu \bar{\omega}}(k; w) r_\alpha \bar{u}^\beta \bar{v}^\nu \\
+ \sum_{2|\alpha|+|\beta|+|\nu|=j_0+1} \left( \langle \hat{u}, \hat{P}_{j_0}^{\alpha \beta \nu \bar{u}}(k; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu + \langle \hat{v}, \hat{P}_{j_0}^{\alpha \beta \nu \bar{u}}(k; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu \right) \\
+ \sum_{2|\alpha|+|\beta|+|\nu|=j_0-1} \left( \frac{1}{2} \langle \hat{u}, \hat{P}_{j_0}^{\alpha \beta \nu \bar{u}}(k; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu + \frac{1}{2} \langle \hat{v}, \hat{P}_{j_0}^{\alpha \beta \nu \bar{v}}(k; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu \right) \right].
\]

Let
\[
F_{j_0}(\varphi, r, u, v; w) = \sum_{2|\alpha|+|\beta|+|\nu|=j_0+1} F_{j_0}^{\alpha \beta \nu \bar{\omega}}(\varphi; w) r_\alpha \bar{u}^\beta \bar{v}^\nu \\
+ \sum_{2|\alpha|+|\beta|+|\nu|=j_0+1} \langle \hat{u}, F_{j_0}^{\alpha \beta \nu \bar{u}}(\varphi; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu + \langle \hat{v}, F_{j_0}^{\alpha \beta \nu \bar{u}}(\varphi; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu \\
+ \sum_{2|\alpha|+|\beta|+|\nu|=j_0-1} \frac{1}{2} \langle \hat{u}, F_{j_0}^{\alpha \beta \nu \bar{u}}(\varphi; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu + \frac{1}{2} \langle \hat{v}, F_{j_0}^{\alpha \beta \nu \bar{v}}(\varphi; w) \rangle r_\alpha \bar{u}^\beta \bar{v}^\nu.
\]

In Fourier modes, we have
\[
\begin{align*}
(5.10) \quad -i(\langle k, \omega_{\hat{m}} \rangle + (\beta - v, \lambda)) \hat{F}_{j_0}^{\alpha \beta \nu \bar{\omega}}(k) = -\hat{P}_{j_0}^{\alpha \beta \nu \bar{\omega}}(k) + \delta_0^k \delta_\nu \delta_\beta \hat{Z}_{j_0}, \\
(5.11) \quad -i(\langle k, \omega_{\hat{m}} \rangle + (\beta - v, \lambda)) \hat{F}_{j_0}^{\alpha \beta \nu \bar{u}}(k) - i(\Omega + H_{\hat{m}}) \hat{F}_{j_0}^{\alpha \beta \nu \bar{u}}(k) = -\hat{P}_{j_0}^{\alpha \beta \nu \bar{u}}(k), \\
(5.12) \quad -i(\langle k, \omega_{\hat{m}} \rangle + (\beta - v, \lambda)) \hat{F}_{j_0}^{\alpha \beta \nu \bar{v}}(k) + i(\Omega + H_{\hat{m}}^T) \hat{F}_{j_0}^{\alpha \beta \nu \bar{v}}(k) = -\hat{P}_{j_0}^{\alpha \beta \nu \bar{v}}(k), \\
(5.13) \quad -i(\langle k, \omega_{\hat{m}} \rangle + (\beta - v, \lambda)) \hat{F}_{j_0}^{\alpha \beta \nu \bar{u} \bar{u}}(k) - i(\Omega + H_{\hat{m}}) \hat{F}_{j_0}^{\alpha \beta \nu \bar{u} \bar{u}}(k) \\
- i(\hat{F}_{j_0}^{\alpha \beta \nu \bar{u}}(k)(\Omega + H_{\hat{m}})) = -\hat{P}_{j_0}^{\alpha \beta \nu \bar{u} \bar{u}}(k),
\end{align*}
\]
\(\begin{align*}
&\quad -i\langle k, \omega_\tilde{m} \rangle + (\beta - \nu, \tilde{x}) \hat{E}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k}) + i(\Omega + H^T_{j_0})\hat{E}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k}) \\
&\quad + i\hat{E}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k})(\Omega + H_{j_0}) = -\hat{P}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k}),
\end{align*}\)

(5.14)

\(\begin{align*}
&\quad -i\langle k, \omega_\tilde{m} \rangle + (\beta - \nu, \tilde{x}) \hat{E}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k}) + i\hat{E}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k})(\Omega + H_{j_0}) \\
&\quad - i(\Omega + H_{j_0})\hat{E}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k}) = -\hat{P}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}(\tilde{k}) + \delta_0^{\beta} \tilde{Z}_{j_0}^{\alpha \beta \mu \nu \tilde{v} \tilde{b}}.
\end{align*}\)

(5.15)

We solve equations (5.10)-(5.15) as in Proposition 3.1. We have

\(\hat{Z}_{j_0}(t, u; v; w) = \sum_{2|\alpha + 2|\beta = j_0 + 1} \hat{P}_{j_0}^{\alpha \beta \gamma \delta}(0; w)\alpha \tilde{u}^\beta \tilde{v}^\gamma \tilde{w}^\delta \)

(5.16)

Let \(\hat{Z}_{j_0} = X_{j_0}^1|_{t=1}\). Using Taylor’s formula, there is

\(\hat{T}_{j_0+1} = \hat{T}_{j_0} \circ \hat{Z}_{j_0} = (h_{\tilde{m}} + Z_{j_0} + P_{j_0} + R_{j_0} + Q_{j_0}) \circ X_{j_0}^1|_{t=1}

\)

(5.17)

\(\begin{align*}
&\quad = h_{\tilde{m}} + \{h_{\tilde{m}}, F_{j_0}\} + \int_0^1 (1 - t)\{\{h_{\tilde{m}}, F_{j_0}\}, F_{j_0}\} \circ X_{j_0}^1 dt \\
&\quad + Z_{j_0} + \int_0^1 \{Z_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt + P_{j_0} \circ X_{j_0}^1 dt + \int_0^1 \{P_{j_0}, (j_0+1), F_{j_0}\} \circ X_{j_0}^1 dt \\
&\quad + P_{j_0} - P_{j_0}(j_0+1) + \int_0^1 \{P_{j_0} - P_{j_0}(j_0+1), F_{j_0}\} \circ X_{j_0}^1 dt \\
&\quad + R_{j_0} + \int_0^1 \{R_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt + Q_{j_0} + \int_0^1 \{Q_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt.
\end{align*}\)

(5.18)

By (5.14) and (5.15), we have

\(\hat{T}_{j_0+1} = h_{\tilde{m}} + Z_{j_0} + \hat{Z}_{j_0} + \int_0^1 (1 - t)\{\{h_{\tilde{m}}, F_{j_0}\}, F_{j_0}\} \circ X_{j_0}^1 dt \\
\quad + \int_0^1 \{Z_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt + \int_0^1 \{P_{j_0}(j_0+1), F_{j_0}\} \circ X_{j_0}^1 dt \\
\quad + P_{j_0} - P_{j_0}(j_0+1) + \int_0^1 \{P_{j_0} - P_{j_0}(j_0+1), F_{j_0}\} \circ X_{j_0}^1 dt \\
\quad + R_{j_0} + (1 - T_\Delta)P_{j_0}(j_0+1) + \int_0^1 \{R_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt + Q_{j_0} \\
\quad + \int_0^1 \{Q_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt.
\)

Hence

\(Z_{j_0+1} = Z_{j_0} + \hat{Z}_{j_0},\)

(5.19)

\(R_{j_0+1} = R_{j_0} + (1 - T_\Delta)P_{j_0}(j_0+1),\)

(5.20)

and

\(P_{j_0+1} + Q_{j_0+1} = \int_0^1 (1 - t)\{\{h_{\tilde{m}}, F_{j_0}\}, F_{j_0}\} \circ X_{j_0}^1 dt + \int_0^1 \{Z_{j_0} + P_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt \\
\quad + \int_0^1 \{R_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt + P_{j_0} - P_{j_0}(j_0+1) + Q_{j_0} + \int_0^1 \{Q_{j_0}, F_{j_0}\} \circ X_{j_0}^1 dt.
\)

(5.21)
By (5.1)-(5.4) and (5.7), following the proof of Proposition 3.1, we have

\[
\| X_{F_{j_0}} \|_T^{(p,D_{j_0})} \lesssim \frac{1}{\kappa^2} \Delta_{t_0}^{\exp} N^{(4d)^2} (j_0+5)^3 \| X_{F_{j_0}} \|_T^{(p,D_{j_0})} \lesssim \left( \frac{1}{\kappa^2} \Delta_{t_0}^{\exp} N^{(4d)^2} (j_0+5)^2 \right)^{j_0-1} .
\]

(5.22)

Write

\[
Z_{j_0+1} = \sum_{3 \leq j \leq j_0+1} Z_{(j_0+1),j},
\]

where

\[
Z_{(j_0+1),j} = Z_{j_0,j}, \quad 3 \leq j \leq j_0,
\]

\[
Z_{(j_0+1),j_0+1} = Z_{j_0}.
\]

For \(3 \leq j \leq j_0\), using (5.5), we have

\[
\| X_{Z_{(j_0+1),j}} \|_T^{(p,D_{j_0+1})} \lesssim \delta \left( \frac{1}{\kappa^2} \Delta_{t_0}^{\exp} N^{(4d)^2} (j_0+5)^2 \right)^{j_0-3} .
\]

(5.23)

By (5.7) and (5.10), we have

\[
\| X_{Z_{j_0}} \|_T^{(p,D_{j_0+1})} \lesssim \delta \left( \frac{1}{\kappa^2} \Delta_{t_0}^{\exp} N^{(4d)^2} (j_0+5)^2 \right)^{j_0-2} .
\]

(5.24)

Then the estimate of (5.19) follows from (5.23) and (5.24).

Write

\[
R_{j_0+1} = \sum_{3 \leq j \leq j_0+1} R_{(j_0+1),j},
\]

where

\[
R_{(j_0+1),j} = R_{j_0,j}, \quad 3 \leq j \leq j_0,
\]

\[
R_{(j_0+1),j_0+1} = (1 - T_{\Delta}) P_{j_0}(j_0+1).
\]

For \(3 \leq j \leq j_0\), using (5.6), we have

\[
\| X_{R_{(j_0+1),j}} \|_T^{(p,D_{j_0+1})} \lesssim \delta \left( \frac{1}{\kappa^2} \Delta_{t_0}^{\exp} N^{(4d)^2} (j_0+5)^2 \right)^{j_0-3} .
\]

(5.25)

By (5.7), we have

\[
\| X_{R_{(j_0+1),j_0+1}} \|_T^{(p,D_{j_0+1})} \lesssim \delta \left( \frac{1}{\kappa^2} \Delta_{t_0}^{\exp} N^{(4d)^2} (j_0+5)^2 \right)^{j_0-2} .
\]

(5.26)

By (5.24) and (5.26), we obtain the estimate of (5.20).

Let

\[
h_{m}^{(j)}(\varphi, r, u, v; w) = \sum_{2|\alpha|+|\beta|+|\nu|+|\mu|+|\nu| = j_0-1+2} h_{m}^{(j)}(\varphi, r, u, v; w) \alpha \beta \gamma \delta \epsilon \zeta \eta \zeta \eta ,
\]

\[
\text{and}
\]

\[
\int_{0}^{1} (1-t) \{ h_{m}, F_{j_0} \circ X_{F_{j_0}} \} dt = \sum_{j \geq 2} \frac{1}{j} h_{m}^{(j)} .
\]

(5.27)
By (5.7), (5.9) and (5.10), we have
\[(5.28)\] \[\|X_{h_m^{(j)}}\|_{p,D,j_0} \leq \delta \left( \frac{1}{\kappa^2} \Delta_m^{\exp N(4d)^4d(j_0+5)^2} \delta \right)^{j_0-2}.\]

For \( j \geq 2 \), let \( \rho_j = \frac{\theta}{\delta_j M} \), \( \delta_j = \frac{\delta}{2\delta_j M} \). By (5.22), (5.28) and Proposition 2.9, we have
\[(5.29)\]
\[\frac{1}{j!} \|X_{h_m^{(j)}}\|_{p,D,j_0+1} \leq \delta \left( C \max \left( \frac{1}{\rho_j}, \frac{\delta}{\delta_j} \right) \right)^j \left( \|X_{F_{j_0}}\|_{p,D,j_0} \right)^{j-1} \|X_{h_m^{(j)}}\|_{p,D,j_0} \]
\[\leq \frac{1}{j!} \delta \left( \frac{1}{\kappa^2} \Delta_m \exp N(4d)^4d(j_0+5)^2 \delta \right)^{j_0-2} \left( C \frac{1}{\kappa^2} \Delta_m \exp N(4d)^4d(j_0+5)^2 \delta \right)^{j_0-1} \]
\[\leq \delta \left( \frac{1}{\kappa^2} \Delta_m \exp N(4d)^4d(j_0+6)^2 \delta \right)^{j(j_0-1)-1},\]

since \( j, j_0 \geq 2, j(j_0-1) + 2 \geq 2j_0 \geq j_0 + 2 \). By (5.29), we obtain the estimate of (5.27).

Let \( W_i = Z_{j_{i+}}^{0}, 3 \leq i \leq j_0, W_i = P_{j_{i+}}^{0}, i \geq j_0 + 1 \).

We have
\[Z_{j_0} + P_{j_0} = \sum_{i \geq 3} W_i.\]

Let
\[W_i^{(0)} = W_i, W_i^{(j)} = \{W_i^{(j-1)}, F_{j_0}\}, j \geq 1.\]

We have
\[W_i^{(j)}(\varphi, r, u, v; w) = \sum_{2|\alpha|+|\beta|+|\nu|+|\mu|+|\nu| = j(j_0-1)+1} W_i^{(j)}(\varphi; u) r^\alpha u^\beta v^\gamma w^\mu \nu,\]

and
\[(5.30)\]
\[\int_{0}^{1} \{Z_{j_0} + P_{j_0}, F_{j_0} \} \circ X_{F_{j_0}}^{t} dt = \sum_{i \geq 3} \sum_{j \geq 1} \frac{1}{j!} W_i^{(j)}.\]

Following the proof of (5.29), we have
\[(5.31)\]
\[\frac{1}{j!} \|X_{W_i^{(j)}}\|_{p,D,j_0+1} \leq \delta \left( \frac{1}{\kappa^2} \Delta_m \exp N(4d)^4d(j_0+6)^2 \delta \right)^{j(j_0-1)+i-3},\]

since \( j \geq 1, j_0 \geq 2, i \geq 3, j(j_0-1) + i \geq j_0 + 2 \). By (5.31), we obtain the estimate of (5.30).

Using the same method, we can estimate
\[\int_{0}^{1} \{R_{j_0}, F_{j_0} \} \circ X_{F_{j_0}}^{t} dt + \int_{0}^{1} \{Q_{j_0}, F_{j_0} \} \circ X_{F_{j_0}}^{t} dt.\]

Hence, we obtain the estimate of (5.21). 

Now we prove Theorem 2.13 concerning the long time stability of the KAM tori of the infinitely dimensional Hamiltonian system.

**Proof of Theorem 2.13** Since the difference between \( h_\infty + f_\infty \) and \( h_m + f_{m}^{high} \) is \( \varepsilon^\delta \), if we choose \( m \) such that \( \varepsilon^\delta \sim \delta^{M+1} \), then we can construct a partial normal form of order \( M + 2 \) based on \( h_m + f_{m}^{high} \). 

□
By Lemma 5.1, there is a symplectic map $\Psi$ such that

$$(h_{\tilde{m}} + f^{\text{high}}_{\tilde{m}}) \circ \Psi = h_{\tilde{m}} + Z_{M+2} + P_{M+2} + R_{M+2} + Q_{M+2}.$$ 

Taking $N = \delta^{-\frac{d-1}{4}}$, $\tilde{\kappa} = \delta^{-\frac{1}{4}}$, we have

$$|||X_{P_{M+2}}|||^T_{p,D_{(\frac{x}{\delta},(\delta^4)^2,4\delta^4)} \times \tilde{U}} \leq \delta \left( \frac{1}{\kappa^2} \Delta_m \exp \left( \frac{4d^3(M+7)^2}{\delta} \right) \right) \leq \delta^{M+\frac{1}{4}}.$$

Taking $\tilde{\Delta} = 800(M(\log \frac{1}{\delta})^2 \frac{1}{\min(\gamma_{\tilde{m}},\rho)})$, we have

$$|||X_{R_{M+2}}|||^T_{p,D_{(\frac{x}{\delta},(\delta^4)^2,4\delta^4)} \times \tilde{U}} \leq \delta^{M+\frac{1}{4}}.$$

Since

$$\|\tilde{\varepsilon}_1\| = \left( \sum_{|a| > N} |\tilde{\varepsilon}_a|^2 |a|^2 \right)^{\frac{1}{2}} = \left( \sum_{|a| > N} |\tilde{\varepsilon}_a|^2 \frac{|a|^{2p}}{|a|^{2p-2}} \right)^{\frac{1}{2}} \leq \|\tilde{\varepsilon}_1\|_p \leq \delta^{M+1} \|\tilde{\varepsilon}_1\|_p,$$

we have

$$|||X_{Q_{M+2}}|||^T_{p,D_{(\frac{x}{\delta},(\delta^4)^2,4\delta^4)} \times \tilde{U}} \leq \delta^{M+\frac{1}{4}}.$$

As done in Section 5.3 of [CLY16], we can prove the long time stability for the KAM tori obtained in Theorem 2.12. The measure estimate can be done as in Theorem 2.12. \qed

References

[Arn64] V. I. Arnold. Instability of dynamical systems with many degrees of freedom. Dokl. Akad. Nauk SSSR, 156:9–12, 1964.

[Bam03] D. Bambusi. Birkhoff normal form for some nonlinear PDEs. Comm. Math. Phys., 234(2):253–285, 2003.

[BB13] M. Berti and P. Bolle. Quasi-periodic solutions with Sobolev regularity of NLS on $\mathbb{T}^d$ with a multiplicative potential. J. Eur. Math. Soc. (JEMS), 15(1):229–286, 2013.

[BB20] M. Berti and P. Bolle. Quasi-periodic solutions of nonlinear wave equations on the $d$-dimensional torus. EMS Monographs in Mathematics. EMS Publishing House, Berlin, [2020] ©2020.

[BBHM18] P. Baldi, M. Berti, E. Haus, and R. Montalto. Time quasi-periodic gravity water waves in finite depth. Invent. Math., 214(2):739–911, 2018.

[BBP13] M. Berti, L. Biasco, and M. Procesi. KAM theory for the Hamiltonian derivative wave equation. Ann. Sci. Éc. Norm. Supér. (4), 46(2):301–373 (2013), 2013.

[BFG88] G. Benettin, J. Fröhlich, and A. Giorgilli. A Nekhoroshev-type theorem for Hamiltonian systems with infinitely many degrees of freedom. Comm. Math. Phys., 119(1):95–108, 1988.

[BG06] D. Bambusi and B. Grébert. Birkhoff normal form for partial differential equations with tame modulus. Duke Math. J., 135(3):507–567, 2006.

[BG21] J. Bernard and B. Grébert. Long time dynamics for generalized Korteweg-de Vries and Benjamin-Ono equations. Arch. Ration. Mech. Anal., 241(3):1139–1241, 2021.

[BMP20] L. Biasco, J. E. Massetti, and M. Procesi. An abstract Birkhoff normal form theorem and exponential type stability of the 1d NLS. Comm. Math. Phys., 375(3):2089–2153, 2020.

[Bou96] J. Bourgain. Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations. Geom. Funct. Anal., 6(2):201–230, 1996.
[Bou98] J. Bourgain. Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. *Ann. of Math. (2)*, 148(2):363–439, 1998.

[Bou05] J. Bourgain. *Green’s function estimates for lattice Schrödinger operators and applications*, volume 158 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2005.

[CGL15] H. Cong, M. Gao, and J. Liu. Long time stability of KAM tori for nonlinear wave equation. *J. Differential Equations*, 258(8):2823–2846, 2015.

[CKS+10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.*, 181(1):39–113, 2010.

[CLW20] H. Cong, C. Liu, and P. Wang. A Nekhoroshev type theorem for the nonlinear wave equation. *J. Differential Equations*, 269(4):3855–3889, 2020.

[CLY16] H. Cong, J. Liu, and X. Yuan. Stability of KAM tori for nonlinear Schrödinger equation. *Mem. Amer. Math. Soc.*, 239(1134):vii+85, 2016.

[CMW20] H. Cong, L. Mi, and P. Wang. A Nekhoroshev type theorem for the derivative nonlinear Schrödinger equation. *J. Differential Equations*, 268(9):5207–5256, 2020.

[DS04] J.-M. Delort and J. Szeftel. Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres. *Int. Math. Res. Not.*, (37):1897–1966, 2004.

[EGK16] L. H. Eliasson, B. Grébert, and S. B. Kuksin. KAM for the nonlinear beam equation. *Geom. Funct. Anal.*, 26(6):1588–1715, 2016.

[EK10] L. H. Eliasson and S. B. Kuksin. KAM for the nonlinear Schrödinger equation. *Ann. of Math. (2)*, 172(1):371–435, 2010.

[FG13] E. Faou and B. Grébert. A Nekhoroshev-type theorem for the nonlinear Schrödinger equation on the torus. *Anal. PDE*, 6(6):1243–1262, 2013.

[GG94] T. Gebhardt and S. Grossmann. Chaos transition despite linear stability. *Phys. Rev. E*, 50:3705–3711, 1994.

[GXY11] J. Geng, X. Xu, and J. You. An infinite dimensional KAM theorem and its application to the two dimensional cubic Schrödinger equation. *Adv. Math.*, 226(6):5361–5402, 2011.

[GY06] J. Geng and J. You. A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces. *Comm. Math. Phys.*, 262(2):343–372, 2006.

[HSSY20] X. He, J. Shi, Y. Shi, and X. Yuan. On linear stability of KAM tori via the Craig-Wayne-Bourgain method. 2020. arXiv 2003.01487 [math.DS].

[KP96] S. B. Kuksin and J. Pöschel. Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. *Ann. of Math. (2)*, 143(1):149–179, 1996.

[Kuk93] S. B. Kuksin. *Nearly integrable infinite-dimensional Hamiltonian systems*, volume 1556 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1993.

[Kuk00] S. B. Kuksin. *Analysis of Hamiltonian PDEs*, volume 19 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000.

[LY11] J. Liu and X. Yuan. A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations. *Comm. Math. Phys.*, 307(3):629–673, 2011.

[MP18] A. Maspero and M. Procesi. Long time stability of small finite gap solutions of the cubic nonlinear Schrödinger equation on $T^2$. *J. Differential Equations*, 265(7):3212–3309, 2018.

[Neh77] N. N. Nehorošev. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. *Uspehi Mat. Nauk*, 32(6(198)):5–66, 287, 1977.
LONG TIME STABILITY

[Pos93] J. Pöschel. Nekhoroshev estimates for quasi-convex Hamiltonian systems. *Math. Z.*, 213(2):187–216, 1993.

[Pos96a] J. Pöschel. A KAM-theorem for some nonlinear partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(1):119–148, 1996.

[Pos96b] J. Pöschel. Quasi-periodic solutions for a nonlinear wave equation. *Comment. Math. Helv.*, 71(2):269–296, 1996.

[PP12] M. Procesi and C. Procesi. A normal form for the Schrödinger equation with analytic non-linearities. *Comm. Math. Phys.*, 312(2):501–557, 2012.

[PP15] C. Procesi and M. Procesi. A KAM algorithm for the resonant non-linear Schrödinger equation. *Adv. Math.*, 272:399–470, 2015.

[Wan16] W.-M. Wang. Energy supercritical nonlinear Schrödinger equations: quasiperiodic solutions. *Duke Math. J.*, 165(6):1129–1192, 2016.

[Wan19] W.-M. Wang. Quasi-periodic solutions to nonlinear PDEs. In *Harmonic analysis and wave equations*, volume 23 of *Ser. Contemp. Appl. Math. CAM*, pages 127–175. Higher Ed. Press, Beijing, 2019.

[Wan20] W.-M. Wang. Space quasi-periodic standing waves for nonlinear Schrödinger equations. *Comm. Math. Phys.*, 378(2):783–806, 2020.

[Way90] C. E. Wayne. Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. *Comm. Math. Phys.*, 127(3):479–528, 1990.

[Yua21] X. Yuan. KAM theorem with normal frequencies of finite limit points for some shallow water equations. *Comm. Pure Appl. Math.*, 74(6):1193–1281, 2021.

[YZ14] X. Yuan and J. Zhang. Long time stability of Hamiltonian partial differential equations. *SIAM J. Math. Anal.*, 46(5):3176–3222, 2014.

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