A general asymptotic decay lemma
for elliptic problems

Leon Simon∗

Introduction

Asymptotic decay and growth theorems are fundamental in the study of geometric variational problems. For example in the study of minimal surfaces the pioneering work of De Giorgi, Reifenberg, Almgren and Allard depended on proving appropriate asymptotic decay lemmas near “regular points.” In later work, asymptotic behavior near singularities has proved to be a key ingredient in attempts to understand the nature of the singular set.

While much has been achieved, nevertheless many basic questions concerning asymptotics near singularities remain open. For example, perhaps the most famous and basic of all open questions concerning asymptotics, there is the question of existence of a unique tangent cone for a minimal surface at each of its singular points—that is, the question of whether a singular minimal surface (or more generally a stationary integral varifold) is asymptotic to a cone on approach to each of its singular points.

We make no attempt here to give a systematic survey of the various works which address such questions, some references for which would include for example [Rei60], [DG61], [Alm66], [Alm00], [BDG69], [All72], [BG72], [All75], [SSY75], [Tay76], [Mir77], [HS79], [SS81], [Giu83], [Whi83], [Sim83], [Sim87], [AS88], [Cha88], [Sim92], [Sim93], [Sim95a], [Sim95b]. Rather here we will discuss one general, but technically reasonably straightforward, asymptotically decay lemma, in the hope that it will provide part (albeit a small part) of an effective introduction to the more technical works mentioned above.

The general asymptotic decay/growth theorem discussed here is applicable to various geometric variational problems, and gives general criteria for establishing growth and decay properties in the presence of singularities. The main results (Theorems 1,2 in §1) can be applied to positive supersolutions $u$ of equations of the form $\Delta_M u + b \cdot \nabla u + (q + a) u = 0$ with $q \geq 0$ and $a, b$ “small” perturbation terms, provided that the submanifold $M$ is part of a suitable “regular multiplicity 1 class” of submanifolds and, in the case of Theorem 2, provided that the pair $M, q$ is “asymptotically conic” in the appropriate sense. The terminology is made precise in §1 below.

One of the principal technical ingredients is the partial Harnack theory developed in §5, which is key to ensuring that “concentration of $L^p$-norm” does not occur. The main theorem (Theorem 1) is stated in §1 and proved in §6.

∗This paper is dedicated to S.-T. Yau on the occasion of his 60’th birthday, in recognition of his many contributions to the development of the field of geometric analysis, and in sincere appreciation of a friendship which has spanned several decades. The present work is a revision of an earlier (unpublished) preprint and has been supported by NSF grants DMS–0406209 & DMS–0104049 at Stanford University.
The applicability of the main theorem to interesting geometric problems is illustrated in \S 7, where we describe how the general theorem applies to give growth estimates for entire and exterior solutions of the minimal surface equation—i.e. for solutions of the minimal surface equation which are either $C^2$ on all of $\mathbb{R}^n$ or else $C^2$ on $\mathbb{R}^n \setminus \Omega$, where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, in case the gradient is unbounded. (If the gradient of an exterior solution is bounded then it has a limit at infinity by a result of Bers [Ber51] for $n = 2$ and by [Sim87] for $n \geq 3$; of course as pointed out in [BDG69], entire solutions of bounded gradient are actually linear by the $C^{1,\alpha}$ estimate for solutions of quasilinear elliptic equations ([GT83, Th. 13.1]), which gives $R^n |Du|_{\alpha, B_R} \leq C, C = C(n)$, whence by letting $R \to \infty$ we obtain $|Du|_{\alpha, \mathbb{R}^n} = 0$, i.e. $Du$ is constant on $\mathbb{R}^n$.) The result obtained in \S 7 is summarized in the following theorem:

**Theorem.** Suppose $\Omega$ is a bounded open subset of $\mathbb{R}^n$ and $u$ is a $C^2$ solution of the minimal surface equation on $\mathbb{R}^n \setminus \Omega$ such that $|Du|$ is not bounded. Then for each $\gamma < \gamma_0 \equiv \frac{n-3}{2} - \sqrt{(\frac{n-3}{2})^2 - (n-2)}$ there are constants $C, R_0 > 0$ (depending on $u$) such that

$$R^{-n} \int_{S_R} \nu_{n+1} d\mathcal{H}^n \leq CR^{-\gamma}, \quad \forall R \geq R_0$$

$$R^{-n} \int_{S_R} |Du| d\mathcal{H}^n \geq CR^{\gamma}, \quad \forall R \geq R_0.$$  

Here $S_R = \{(x, u(x)) \in (\mathbb{R}^n \setminus \Omega) \times \mathbb{R} : |x|^2 + u^2(x) < R^2\} \equiv G \cap \{(x, y) \in \mathbb{R}^{n+1} : |(x, y)| < R\}$, with $G = \{(x, u(x)) : x \in \mathbb{R}^n \setminus \Omega\}$ = graph $u$, $\nu_{n+1} = (1+|Du|^2)^{-1/2}(-Du, 1)$ is the $(n+1)$st component of the upward pointing unit normal $\nu = (1+|Du|^2)^{-1/2}(-Du, 1)$ of the graph $G$ (viewed as the restriction to $G$ of a function of $(x, y) \in \Omega \times \mathbb{R}$ which is independent of the variable $y$), and $\mathcal{H}^n$ is $n$-dimensional Hausdorff measure on $G$.

We prove the first inequality above in \S 7 as a consequence of the main decay estimate in Theorem 2 of \S 1 below. The second inequality is a consequence of the first by virtue of the Cauchy-Schwarz inequality and the fact that there is a fixed constant $C = C(n)$ such that $C^{-1} R^n \leq |S_R| \leq CR^n$ for all $R > 2 \text{diam } \Omega$.

The above theorem extends work of Caffarelli, Nirenberg, and Spruck [CNS90], Ecker & Huisken [EH90], and Nitsche [Nit89] with respect to the relevant growth exponent and also with respect to the information it gives with regard to the generality of the set of points $x$ where an inequality like $|Du(x)| \geq CR^\gamma$ must hold. In particular the exponent $\gamma_0$ in the above theorem is best possible in general because the original examples of non-linear entire solutions of the MSE constructed by [BDG69] (see also the discussion of [Sim89]) have exactly this growth. For further discussion we refer to [Sim08a].

Other applications of the main theorem here will be described elsewhere—see in particular [Sim08b].

1 Main Results

Let $\mathcal{P}$ be a collection of properly embedded $C^1$ submanifolds $P$ in $\mathbb{R}^N$ and corresponding to each $P \in \mathcal{P}$ we assume there is given an open subset $U_P$ of $\mathbb{R}^N$ which contains $P$. The collection $\mathcal{P}$ will be called a *regular multiplicity 1 class* if the following conditions are satisfied, in which we use the notation that $B_r(y)$ is the open ball in $\mathbb{R}^N$ (the
notation $B_{\rho}(y)$ being reserved for the closed ball):

1.1 (Reducibility of $\mathcal{P}$): $P \in \mathcal{P}$ and $\tilde{B}_{\rho}(y) \subset U_P$ with $P \cap \tilde{B}_{\rho}(y) \neq \emptyset \implies$ each connected component of $P \cap \tilde{B}_{\rho}(y)$ is also in $\mathcal{P}$ with $U_{P \cap \tilde{B}_{\rho}(y)} = \tilde{B}_{\rho}(y)$.

1.2 (Scale invariance of $\mathcal{P}$): $P \in \mathcal{P} \implies \eta_{\rho,\lambda}(P) \in \mathcal{P}$ for each $y \in \mathbb{R}^N$ and $\rho > 0$, and $U_{\eta_{\rho,\lambda}P} = \eta_{\rho,\lambda}(U_P)$; here $\eta_{\rho,\lambda} : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $\eta_{\rho,\lambda}(x) = \lambda^{-1}(x-y)$.

1.3 (Regularity property of $\mathcal{P}$): $\mathcal{H}^{n-2}(\{y \in P \cap B_{\theta}\}) < \infty \forall \theta \in (0,1)$, and for every sequence $\{P_k\} \subset \mathcal{P}$ with $U_{P_k} \supset \tilde{B}_1$ for each $k$, there is a subsequence of $\{P_k\}$ converging locally in $\tilde{B}_1$ in the Hausdorff distance sense to either the empty set or to some $P \in \mathcal{P}$ with $U_P \supset \tilde{B}_1$, and in the latter case we also require that locally, in a neighborhood of each compact subset $K \subset P \cap \tilde{B}_1$, the convergence holds in the $C^1$-sense that there is a fixed open set $U$ in $\mathbb{R}^N$ with $K \subset U$ and a sequence $\Psi_k$ of $C^1$ diffeomorphisms $U \to U$ with $\Psi_k$ converging to the identity map on $U$ in the $C^1$ norm and with $\Psi_k(P \cap U) = P_k \cap U$ for each sufficiently large $k$.

Remark: Notice that the above enables us to make good sense of statements like $f_k \to f$ locally in $L^p$ or locally in $C^0$ on $P$, even if the $f_k$ are actually defined on $P_k$ (with $P_k \to P$ as in 1.4) rather than on the fixed domain $P$. For example $f_k \to f$ locally in $C^0$ means that for each compact $K \subset P$ we have $f_k \circ \Psi_k|K$ converges uniformly to $f|K$, where $\Psi_k$ are as in 1.4.

$\mathcal{C}$ will denote the set of cones $\mathcal{C}$ in $\mathcal{P}$, so that $\mathcal{C} \subset \mathcal{C}$ means $U_{\mathcal{C}} = \mathbb{R}^N \setminus \{0\}$ and $\eta_{\rho,\lambda}\mathcal{C} = \mathcal{C} \forall \rho > 0$, where, here and subsequently (as in 1.2), $\eta_{\rho,\lambda}$ denotes the translation and scaling given by $\eta_{\rho,\lambda}(x) = \rho^{-1}(x-y)$.

We also let $\mathcal{E}$ be the corresponding class of $(n-1)$-dimensional submanifolds of $S^{N-1}$:

$$\mathcal{E} = \{\Sigma = \mathcal{C} \cap S^{N-1} : \mathcal{C} \subset \mathcal{C}\},$$

equipped with the Hausdorff distance metric $d$. Evidently, in view of 1.4, $\mathcal{E}$ is then a compact metric space. Subsequently we let

1.5 $\mathcal{C}_0$ and $\mathcal{E}_0$ denote compact subsets of $\mathcal{C}$, $\mathcal{E}$ respectively and correspondingly a collection

1.6 $\mathcal{Q}_0 = \{q_\Sigma\}_{\Sigma \in \mathcal{E}_0}$ with $q_\Sigma \geq 0$, $q_\Sigma$ locally bounded, measurable on $\Sigma$,

(i.e., the $q_\Sigma$ depend locally uniformly on $\Sigma$ with respect to the Hausdorff distance metric on $\mathcal{E}_0$). For $\Sigma \in \mathcal{E}_0$ and $q_\Sigma \in \mathcal{Q}_0$ as above, for each connected component $\Sigma_*$ of $\Sigma$ we let $\lambda_1(\Sigma_*)$ be the “minimum eigenvalue” of the operator $-(\Delta_\Sigma + q_\Sigma)$ on the component $\Sigma_*$. 

1.7 $\lambda_1(\Sigma_*) = \inf_{\zeta \in C_0^\infty(\Sigma_*), \|\zeta\|_{L^2(\Sigma_*)} = 1} \int_{\Sigma_*} \left(\|\nabla \zeta\|^2 - q_\Sigma \zeta^2\right).$
The reader should note that perhaps the word “eigenvalue” is misleading here since although the real number \( \lambda_1(\Sigma_*) \) exists, there may be no \( \varphi \in W^{1,2}(\Sigma_*) \) with \(- (\Delta_{\Sigma_*}, \varphi + \varphi_{\Sigma_*}) \varphi = \lambda_1(\Sigma_*) \varphi \) even weakly, on \( \Sigma_* \). Of course if \( \Sigma_* \) is a compact smooth manifold (i.e. sing \( \Sigma_* \) = \( \emptyset \)) then the usual Hilbert space applied in the space \( W^{1,2}(\Sigma_*) \) guarantees such a function \( \varphi \) does indeed exist, and in this case by elliptic regularity theory ([GT83, \S 8.8--8.10]) it will be continuous and everywhere non-zero on \( \Sigma_* \). In general, when sing \( \Sigma \neq \emptyset \), the De Giorgi Nash Moser theory does guarantee the existence of a positive \( \varphi \in W^{1,2}_{\text{loc}}(\Sigma_*) \cap L^p(\Sigma_* \setminus B^3/2) \) solution of the equation for \( p < \frac{6n}{n-2} \), as we discuss below.

With \( \lambda_1(\Sigma_*) \) as in \ref{1.7}, we define

\[ \lambda_1(\Sigma) = \max \{ \lambda_1(\Sigma_*): \Sigma_* \text{ is a connected component of } \Sigma \} \]

(notice that this makes sense, because, as we show in 2.4 of the next section, there are only finitely many connected components \( \Sigma_* \) of \( \Sigma \)), and we let

\[ \lambda_1(\mathcal{E}_0) = \sup_{\Sigma \in \mathcal{E}_0} \lambda_1(\Sigma). \]

The main theorems below relate to asymptotics for positive supersolutions \( u \) of suitable elliptic equations on various subdomains of \( M \in \mathcal{P} \). Specifically, we assume \( \tau \in (0, \frac{1}{4}] \) (to be specified in the main theorem) and \( U_M \supset B_{3/2} \setminus B_{\tau} \), and the main theorem (Theorem 1) below assumes \( u \) is given on \( M \cap B_{3/2} \setminus B_{\tau} \) with

\[ u \in W^{1,2}_{\text{loc}}(M \cap B_{3/2} \setminus B_{\tau}) \setminus \{0\}, \quad u \geq 0 \text{ a.e., } \Delta_M u + b \cdot \nabla_M u + (q + a) u \leq 0 \]

on \( M \cap B_{3/2} \setminus B_{\tau} \), where \( q \in L^{\infty}_{\text{loc}}(M \cap B_{3/2} \setminus B_{\tau}) \) with \( q \geq 0 \) and where \( a: M \cap B_{3/2} \setminus B_{\tau} \to \mathbb{R} \) and \( b: M \cap B_{3/2} \setminus B_{\tau} \to \mathbb{R}^n \) are given locally bounded measurable functions. Notice that since sing \( M = \overline{M} \setminus M \) is in general non-empty, the fact that \( q \in L^{\infty}_{\text{loc}} \) of course leaves open the possibility that \( q \) can be unbounded in the neighborhoods sing \( M \). Of course the inequality in \ref{1.9} is to be interpreted in the weak sense that

\[ \int_M \left( - \nabla_M u \cdot \nabla_M \zeta + b \cdot \nabla_M u \zeta + (q + a) \zeta u \right) \leq 0, \quad \zeta \in C^1_0(M \cap B_{3/2} \setminus B_{\tau}), \quad \zeta \geq 0. \]

Here \( \zeta \in C^1_0(M \cap B_{3/2} \setminus B_{\tau}) \) while sing \( M = \overline{M} \setminus M \), so support \( \zeta \cap \text{sing } M = \emptyset \), hence, in the above, and subsequently, there is no a-priori assumption on how \( u \) behaves on approach to sing \( M \).

The functions \( a, b \) should here be though of as “perturbation terms” and are included for reasons of generality. Such terms are not needed (and can be taken to be identically zero) in the application to solutions of the minimal surface equation discussed in §7. We shall in any case for the main theorems (Theorems 1--3) need to assume \( a, b \) small; we quantify this below.

The main growth theorem below considers the case when \( M \) is close to a cone \( C \in \mathcal{C}_0 \) in an annular region \( B_{3/2} \setminus B_{\tau} \) in the sense

\[ d(M \cap B_{3/2} \setminus B_{\tau}, C \cap B_{3/2} \setminus B_{\tau}) < \tau \]

where \( d \) is the Hausdorff distance metric for subsets of \( \mathbb{R}^N \). With \( \Sigma = C \cap S^{N-1} \) the corresponding submanifold in \( \mathcal{E}_0 \), we also need to assume that the perturbation terms
We are now ready to state the main growth theorem. In the statement,

\[
K = \{ x \in \mathbb{C} : \text{dist}(x, \text{sing } \mathbb{C}) \leq \gamma \} \cap B_{3/2} \setminus B_\tau \subset \mathbb{R}
\]

and if for every sequence \( \rho \) corresponding \( \gamma_1 \)

1.12 Definition:} For each \( (x, \Sigma) \in \mathcal{E} \), we shall give the proof of Theorem 1 in \( \mathcal{E} \). For the moment we establish a corollary of Theorem 1. This corollary applies to \((M, a, b, q)\) which are “asymptotically conic” either at 0 or \(\infty\) in the following sense:

\[
\gamma_0 = \left\{ \begin{array}{ll}
\frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\mathcal{E}_0)} & \text{if } \lambda_1(\mathcal{E}_0) \geq -\left(\frac{n-2}{2}\right)^2 \\
\lambda_1(\mathcal{E}_0) & \text{otherwise,}
\end{array} \right.
\]

where \(\lambda_1(\mathcal{E}_0)\) is as in 1.8.

**Theorem 1 (Main Growth Theorem).** For each \(\alpha, \beta > 0\), \(p \in [1, \frac{n+2}{2}]\), and \(\gamma < \gamma_0\) (\(\gamma_0\) as above), there is \(\tau = \tau(\gamma, p, C_0, \alpha, \beta) \in (\frac{\gamma}{\tau}, \frac{1}{\tau})\) and \(\rho = \rho(\gamma, p, C_0, \alpha, \beta) \in (2\tau, \frac{1}{\tau})\) such that if \(M \in \mathcal{P}, \mathbb{C} \subset C_0,\) and if 1.1–1.11 all hold, then

\[
\|u\|_{L^p(M \cap B_{3/2} \setminus B_\rho)} \geq \rho^{-\gamma} \|u\|_{L^p(M \cap B_1 \setminus B_{1/2})}.
\]

**Remarks:** (1) A key point here is that the constants \(\tau, \rho\) do not depend on the particular \(M, \mathbb{C}\) under consideration, so Theorem 1 can be applied uniformly across a large family of different \(M\) and \(\mathbb{C}\); this will be used in the proof of the corollaries below.

Of course the theorem still has content in the special case when \(\mathcal{E}_0\) consists of just one element \(\Sigma \in \mathcal{E}\), and in this case we have \(\lambda_1(\mathcal{E}_0) = \lambda_1(\Sigma)\).

(2) We shall show in 4.3 below that in fact under the hypotheses 1.9, 1.10, 1.11 we always automatically have a lower bound \(\lambda_1(\mathcal{E}_0) \geq -\left(\frac{n-2}{2}\right)^2 - \epsilon(\tau),\) with \(\epsilon(\tau) \downarrow 0\) as \(\tau \downarrow 0\), and of course trivially \(\lambda_1(\Sigma) \leq 0\) because \(q_{\Sigma} \geq 0\), so the constant \(\gamma_0\) in the above theorem is a well-defined real number in the interval \([0, \frac{n-2}{2}]\) and in fact for \(n \geq 3\) \(\gamma_0 > 0\) unless \(\lambda_1(\Sigma) = 0\), which evidently occurs only when \(q_{\Sigma} = 0\) a.e. on \(\Sigma\).

We shall give the proof of Theorem 1 in §6, after the necessary preliminaries are established. For the moment we establish a corollary of Theorem 1. This corollary applies to \((M, a, b, q)\) which are “asymptotically conic” either at 0 or \(\infty\) in the following sense:

\[
\sup_{\{x \in \mathbb{C}, \text{dist}(x, \text{sing } \mathbb{C}) \geq \gamma \} \cap B_{3/2} \setminus B_\tau} \left( \frac{r^2(|a|^{1/2} + |b|) \circ \Psi + |r^2 q \circ \Psi - q_{\Sigma}|}{\|a|^{1/2} + |b|\|_{L^{n+\alpha}(M \cap B_{3/2} \setminus B_\tau)}} \right) \leq \beta,
\]

where \(\Psi : U \to U\) is a \(C^1\) diffeomorphism of some open \(U\) containing the compact set

\[
\Psi(K) = \{ x \in M : \text{dist}(x, \text{sing } \mathbb{C}) \geq \tau \} \cap B_{3/2} \setminus B_\tau, \quad \text{and } \|\Psi - I\|_{C^1(U)} < \tau.
\]

In all that follows, \(L^p(\Omega)\) norms (with \(\Omega \subset M\)) always denote the normalized \(L^p\)-norm, with normalizing factor chosen so that the indicator function of \(\Omega\) has norm 1; thus

\[
\|f\|_{L^p(\Omega)} = \left( |\Omega|^{-1} \int_\Omega |f|^{p} \right)^{1/p}, \quad |\Omega| = H^n(\Omega).
\]

We are now ready to state the main growth theorem. In the statement,

\[
\gamma_0 = \left\{ \begin{array}{ll}
\frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\mathcal{E}_0)} & \text{if } \lambda_1(\mathcal{E}_0) \geq -\left(\frac{n-2}{2}\right)^2 \\
\lambda_1(\mathcal{E}_0) & \text{otherwise,}
\end{array} \right.
\]

where \(\lambda_1(\mathcal{E}_0)\) is as in 1.8.
set of cones in \( \mathcal{P} \) as discussed above), and also \( \rho_j^2 q(x_0 + \rho_j \cdot x) \to r^{-2} q_{\Sigma}(x) \) (\( \Sigma = C \cap S^{N-1} \)), uniformly on compact subsets of \( C \) (in the sense described in the remark following 1.4), where \( q_{\Sigma} \) is a non-negative locally bounded measurable function on \( \Sigma \).

\[
\lim_{j \to \infty} \rho_j^{-n/(n-\alpha)} \|a(x_0 + \rho_j \cdot x) + \rho_j \cdot b(x_0 + \rho_j \cdot x)\|_{L^{n-\alpha}(B_{\rho_j}(x_0) \setminus B_{\rho_j/2}(x_0))} < \infty \quad \text{and} \quad \rho_j^2 |a(x_0 + \rho_j \cdot x)| + |b(x_0 + \rho_j \cdot x)| \to 0 \text{ uniformly on compact subsets of } C.
\]

(b) Similarly \((M, q, a, b)\) is asymptotically conic at \( \infty \) if \( U_M \supset \mathbb{R}^N \setminus B_{R_0} \) for some \( R_0 > 0 \) and if for every sequence \( R_j \uparrow \infty \) there is a subsequence \( R_{j'} \), such that \( \eta_{R_{j'}} R_{j'} \to C \) in \( \mathbb{R}^N \setminus \{0\} \) (convergence in the sense of 1.4) for some cone \( C \in \mathcal{C} \), and also \( R_{j'}^2 q(R_{j'} \cdot x) \to r^{-2} q_{\Sigma}(x) \) (again \( \Sigma = C \cap S^{N-1} \)), uniformly on compact subsets of \( C \) (again in the sense described in the remark following 1.4), where \( q_{\Sigma} \) is a non-negative locally bounded measurable function on \( \Sigma \).

\[
\lim_{j \to \infty} \rho_{j'}^{-n/(n-\alpha)} \|a(R_{j'} \cdot x) + \rho_{j'} \cdot b(R_{j'} \cdot x)\|_{L^{n-\alpha}(M \cap B_{R_{j'}} \setminus B_{R_{j'}/2})} < \infty \quad \text{and} \quad R_{j'}^2 |a(R_{j'} \cdot x)| + |b(R_{j'} \cdot x)| \to 0 \text{ uniformly on compact subsets of } C.
\]

Notice that the definition here allows the possibility that the cone \( C \) may not be unique; that is, we may get different cones by taking different sequences \( \rho_j, \rho_{j'} \) in case (a) and different sequences \( R_j, R_{j'} \) in case (b). Any such cone \( C \) is called a tangent cone of \( M \) (“tangent cone at \( x_0 \)” in case (a) and “tangent cone at \( \infty \)” in case (b)).

We let \( \mathcal{C}(M, x_0) \) denote the (compact) set of all cones \( C \in \mathcal{C} \) which arise as in 1.12(a),(b) according as \( x_0 \in \text{sing } M \) or \( x_0 = \infty \) respectively, set

\[
\mathcal{C}_0 = \mathcal{C}(M, x_0), \quad \mathcal{E}_0 = \{ \Sigma = C \cap S^{N-1} : C \in \mathcal{C}_0 \},
\]

and

\[
\lambda_1(M, x_0) = \lambda_1(\mathcal{E}_0)
\]

with \( \lambda_1(\mathcal{E}_0) \) as in 1.8 with \( \mathcal{C}_0 = \mathcal{C}(M, x_0) \). Then we have a following corollary of Theorem 1 which, in view of the definition 1.12 of asymptotically conic, follows directly by applying Theorem 1 iteratively in the case when \( \mathcal{C}_0 = \mathcal{C}(M, x_0) \) and when the \( q_{\Sigma} \) corresponding to \( C \in \mathcal{C}_0 \) are the functions obtained as in 1.12.

\subsection*{Theorem 2}

Suppose 1.1–1.4 hold and \( p \in [1, \frac{n}{n-2}) \). If \( M \in \mathcal{P}, a, q : M \to \mathbb{R} \) with \( q \geq 0 \) a.e., \( b : M \to \mathbb{R}^N \), and either \( x_0 \in \text{sing } M \) or \( x_0 = \infty \), and \( (M, q, a, b) \) is asymptotically conic at \( x_0 \) in the sense of 1.12(a) in case \( x_0 \in \text{sing } M \) and in the sense of 1.12(b) in case \( x_0 = \infty \), if \( \gamma < \gamma_0 = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(M, x_0)} \), where \( \lambda_1(M, x_0) \) is as in 1.13, and if \( u \in W^{1,2}_{\text{loc}}(M) \setminus \{0\} \) is a non-negative supersolution of the equation \( \Delta_M u + b \cdot \nabla_M u + (q + a) u = 0 \) on \( M \), then there is \( r_0 = r_0(p, \gamma, M, q, a, b) > 0 \) such that

\[
\|u\|_{L^p(M \cap B_{r_0} \setminus B_{r_0/2})} \leq r^{-\gamma} \text{ for all } r \geq r_0 \text{ in case } x_0 = \infty
\]

\[
\|u\|_{L^p(M \cap B_{r_0}(x_0) \setminus B_{r_0/2}(x_0))} \geq r^{-\gamma} \text{ for all } r \leq r_0 \text{ in case } x_0 \in \text{sing } M;
\]

here we continue to use the notational convention that \( \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p} \), with \( |\Omega| = \mathcal{H}^n(\Omega) \).

\subsection*{Remark}

We emphasize again that there are no a-priori continuity or indeed integrability assumptions on \( u \); \( u \) is merely assumed to be non-negative a.e. and in \( W^{1,2}_{\text{loc}}(M) \setminus \{0\} \) on the open manifold \( M \) and to be a supersolution of the equation \( \Delta_M u + b \cdot \nabla_M u + (q + a) u = 0 \) locally weakly in \( M \). Of course, as the above discussion and the statement of the theorem already indicates, we can prove that \( u \) automatically
has integrability properties (for example part of the conclusion of the theorem is that $u$ is automatically in $L^p(M \cap B_r \setminus B_{r/2})$ if $p < \frac{n}{n-2}$).

**Proof of Theorem 2:** In case $x_0 \in \text{sing } M$, using the definition 1.12(a) we see that the hypotheses of Theorem 1 are satisfied, with $\eta_{x_0, p} M$ in place of $M$ and $u \circ \eta_{x_0, p}^{-1}$ in place of $u$, for all $\rho \leq r_0$, where $r_0 = r_0(p, \gamma, M, q, a, b) > 0$ and where $C_0 = C(M, x_0)$, the set of tangent cones of $M$ at $x_0$ as in the discussion preceding the statement of Theorem 2. Thus, by Theorem 1, $\|u\|_{L^p(M \cap B_{r_0}) \setminus B_{r_0/2}} \geq \tau^{-\gamma}\|u\|_{L^p(M \cap B_{r_0}) \setminus B_{r_0/2}}$, and, taking the choice $\rho = r_0\tau^j$ we obtain

$$\|u\|_{L^p(M \cap B_{r_j+1} \setminus B_{r_j+1/2})} \geq \tau^{-\gamma}\|u\|_{L^p(M \cap B_{r_j} \setminus B_{r_j/2})}, \quad j = 0, 1, 2, \ldots,$$

so by iteration we obtain the asymptotic stated in the theorem. The proof in case $x_0 = \infty$ is a similar iterative application of Theorem 1.

Notice in particular that the above theorem with $p = 1$ implies:

**Theorem 3.** Suppose $M \in \mathcal{P}$, $x_0 \in \text{sing } M$, $M, a, b, q$ is asymptotically conic at $x_0$ in the sense of 1.12(a) and suppose there exists $\rho > 0$ such that $u$ is a non-negative supersolution of the equation \( \Delta_M u + b \cdot \nabla u + (q + a)u = 0 \) in $M \cap B_\rho(x_0)$ with $\sup_{r \leq \rho} \tau^{-\gamma} \int_{M \setminus B_r(x_0)} (b \cdot \nabla u + (q + a)u) < 0$. Then $\liminf_{r \to 0} r^{-2-n} \int_{M_{r, \delta}} q = 0$ for each $\delta > 0$, where $M_{r, \delta} = \{x \in M \cap B_r(x_0) \setminus B_{r/2}(x_0) : \text{dist}(x, \text{sing } M) > \delta r\}$.

**Remark:** Thus in particular there cannot exist a bounded non-negative $W^{1,2}_{\text{loc}}(M \cap B_\rho(x_0))$ supersolution of the equation $\Delta_M u + b \cdot \nabla u + (q + a)u = 0$ if the function $q$ satisfies $\liminf_{r \to 0} r^{-2-n} \|q\|_{L^1(M_{r, \delta})} > 0$ for some $\delta > 0$.

## 2 Some preliminaries concerning the class $\mathcal{P}$

First we claim there are constants $\beta_1 = \beta_1(\mathcal{P})$, $\beta_2 = \beta_2(\mathcal{P}, \theta) > 0$ such that

$$\beta_1 \rho^n \leq \mathcal{H}^n(P \cap B_\rho(y)) \quad \text{and} \quad \mathcal{H}^n(P \cap B_\theta(y)) \leq \beta_2 \rho^n$$

for each $P \in \mathcal{P}$, $y \in \overline{P}$, $\theta \in (0, 1)$ and $\rho > 0$ with $B_\rho(y) \subset U_P$. The right inequality is in fact a direct consequence of the scale-invariance 1.2 and the first property in 1.4, and in view of 1.3 we then have that if $P_k$ is an arbitrary sequence in $\mathcal{P}$ with $U_{P_k} \supset B_1$ for each $k$ and if $P_k \to P \in \mathcal{P}$ in $\tilde{B}_1$ with $U_P \supset B_1$ then $\mathcal{H}^n(B_\theta \cap \{x \in P : \text{dist}(x, \text{sing } P) < \theta\}) \leq C\delta^2$ for each $\theta \in \{1/2, 1\}$. In view of the $C^1$ and Hausdorff distance sense convergence of 1.4, it evidently follows that $\mathcal{H}^n \sqcap P_k \to \mathcal{H}^n \sqcap P$ in $\tilde{B}_1$; that is $\int_{P_k} f \, d\mathcal{H}^n \to \int_P f \, d\mathcal{H}^n$ for each fixed continuous $f$ with compact support in $\tilde{B}_1$. Thus we have established

$$P, P_k \in \mathcal{P} \quad \text{with} \quad U_{P_k} \supset \tilde{B}_1 \quad \text{and} \quad P_k \to P \quad \text{in the sense of 1.4 in} \quad \tilde{B}_1 \quad \Rightarrow \quad \mathcal{H}^n \sqcap P_k \to \mathcal{H}^n \sqcap P \quad \text{in} \quad \tilde{B}_1.$$

To prove the left inequality in 2.1, suppose on the contrary that $\mathcal{H}^n(P_k \cap B_{\rho_k}(y_k)) < k^{-1}$, $k = 1, 2, \ldots$, with $y_k \in \overline{P_k}$ and $U_{P_k} \supset B_{\rho_k}(y_k)$. Then, with $\tilde{P}_k = \eta_{y_k, \rho_k} P_k$, we have by 1.2 that $\tilde{P}_k \in \mathcal{P}$ with $U_{\tilde{P}_k} \supset B_1$, $0 \in \text{the closure of } \tilde{P}_k$, and $\mathcal{H}^n(\tilde{P}_k \cap B_1) < k^{-1}$ for each $k$. Then by 1.4 there is $\tilde{P} \in \mathcal{P}$ and a subsequence $\tilde{P}_{k'} \to \tilde{P}$ in $\tilde{B}_1$ with $U_{\tilde{P}} \supset \tilde{B}_1$, with $\mathcal{H}^n(\tilde{P}) = 0$ (by 2.2) and with 0 in the closure of $\tilde{P}$ (by the Hausdorff distance
convergence), contradicting the assumption that all elements of $\mathcal{P}$ are $n$-dimensional submanifolds.

Notice that if $\theta \in \left[\frac{1}{2}, 1\right]$ is given, we can now bound the number of connected components $P_*$ of $P \cap \hat{B}_\theta(y)$ which intersect $B_{\delta\theta}(y)$ in case $U_P \supset B_{\theta}(y)$. Indeed, since for each such component $P_*$ we have $z \in P_* \cap B_{\theta\delta}(y)$, and hence $\hat{B}_{\frac{1}{2}(1-\theta)\delta}(z) \subset \hat{B}_{\delta\theta}(y) \subset \hat{B}_\theta(y) \subset U_{P_*}$, where $\hat{\theta} = \frac{1}{2\theta}$, the left inequality in 2.1 gives

$$\mathcal{H}^n(P_* \cap B_{\delta\theta}(y)) \geq \beta_1 \rho^n,$$

for suitable $\beta_1 = \beta_1(\theta, \mathcal{P})$ whereas the sum of $\mathcal{H}^n(P_* \cap B_{\theta\delta}(y))$ over all such components $P_*$ is $\leq \mathcal{H}^n(P \cap B_{\theta\delta}(y)) \leq \beta_2 \rho^n$ for some $\beta_2 = \beta_2(\theta, \mathcal{P})$ by the right inequality in 2.1, whence the number $Q$ of such components satisfies

$$Q \leq 1 + \beta_1^{-1} \beta_2.$$

Finally, we show that the conditions 1.1–1.4 are sufficient to give a "restricted Poincaré type" inequality on each $P \in \mathcal{P}$:

**2.5 Theorem.** Let $\mathcal{P}$ satisfy the conditions 1.1–1.4. Then for each $\theta_0 \in \left[\frac{1}{2}, 1\right]$ there are constants $C = C(\mathcal{P}, \theta_0) > 0, \delta = \delta(\mathcal{P}, \theta_0) \in (0, \frac{1}{2}]$ such that

$$\left(\int_{P \cap B_{\theta_0}} \varphi^\kappa\right)^{1/\kappa} \leq C \int_{P \cap B_1} |\nabla \varphi|,$$

whenever $P \in \mathcal{P}$ with $U_P \supset \hat{B}_1$ and $\varphi$ is a non-negative $C^1(P \cap \hat{B}_1)$ function satisfying the inequality $\mathcal{H}^n(\text{support } \varphi) < \delta$.

**2.6 Remarks:** (1) An examination of the proof will show that for this lemma it would suffice that $\mathcal{H}^{n-1}(\text{sing } P) = 0$ for each $P \in \mathcal{P}$ in place of the condition 1.3.

(2) By replacing $\varphi$ by $|\varphi|^{2(n-1)/(n-2)}$ for $n \geq 3$ and by $\varphi^{2q}$ for arbitrary $q > 1$ in case $n = 2$, and using the Hölder inequality on the right side, we see that the inequality of 2.5 admits the squared version

$$\left(\int_{P \cap B_{\theta_0}} \varphi^{2\kappa}\right)^{1/\kappa} \leq C \int_{P \cap B_1} |\nabla \varphi|^2,$$

with $\kappa = n/(n-2)$ and $C = C(\mathcal{P}, N)$ in case $n \geq 3$, and in case $n = 2$ the same with arbitrary $\kappa > 1$. (Of course we still require the restriction $\mathcal{H}^n(\text{support } \varphi) < \delta$ here.)

Before we begin the proof of 2.5 we observe that, using it in combination with a partition of unity for $B_1$ consisting of smooth functions, each of which has support in a set of diameter $\leq \delta$, we conclude the following.

**2.5′ Corollary.** If the hypotheses are as in 2.5, except that we drop the condition that $\mathcal{H}^n(\text{support } \varphi) < \delta$, then

$$\left(\int_{P \cap B_{\theta_0}} \varphi^{\kappa}\right)^{1/\kappa} \leq C \int_{P \cap B_1} (|\nabla \varphi| + |\varphi|).$$

**2.6′ Remark:** As in Remark 2.6(2), there is a squared version of the above inequality:

$$\left(\int_{P \cap B_{\theta_0}} \varphi^{2\kappa}\right)^{1/\kappa} \leq C \int_{P \cap B_1} (|\nabla \varphi|^2 + |\varphi|^2),$$
with $\kappa = n/(n-2)$ and $C = C(\mathcal{P}, N)$ in case $n \geq 3$, and in case $n = 2$ the same with arbitrary $\kappa > 1$.

**Proof of Theorem 2.5:** It is a well-known consequence of the coarea formula and the fact that $\mathcal{H}^{n-1}(\text{sing } P \cap \tilde{B}_1) = 0$ that such an inequality is equivalent to the fact that $\exists C = C(\mathcal{P}, \theta) > 0$ such that

$$\mathcal{H}^n(Q \cap B_{\theta_0})^{1/\kappa} \leq C \mathcal{H}^{n-1}(\partial Q \cap \tilde{B}_1)$$

whenever $Q$ is a relatively open subset of $P \cap \tilde{B}_1$ with $\mathcal{H}^n(Q) < \delta$, with boundary $\partial Q = \overline{Q} \setminus Q (\subset \mathcal{P})$ such that $\partial Q \cap P \cap \tilde{B}_1$ is locally $C^1$.

Take $\delta \in (0, \frac{1}{4}]$ such that $\delta^{1/2}$ is smaller than the volume $\omega_n$ of the unit ball in $\mathbb{R}^n$ and also smaller than the constant $\beta_1$ in 2.1, and assume (to get a contradiction) that $P_k$ is a sequence in $\mathcal{P}$ such that for each $k$ there is a relatively open subset $Q_k \subset P_k \cap B_1$ with $\mathcal{H}^n(Q_k) < \delta$ and with boundary $\partial Q_k \cap \tilde{B}_1$ such that $\mathcal{H}^{n-1}(\partial Q_k^\tau \cap \tilde{B}_1) < \frac{1}{k} (\mathcal{H}^n(Q_k \cap B_{\theta_0}))^{1/\kappa} \to 0$. Now for each point $y \in Q_k \cap B_{\theta_0}$, $\rho^{-n}\mathcal{H}^n(Q_k \cap B_p(y))$ has limiting value $\omega_n > \delta^{1/2}$ as $\rho \to 0$ and has value $\leq C(\theta_0)\delta < \delta^{1/2}$ when $\rho = \frac{1}{2}(1 - \theta_0)$, assuming $\delta$ small enough (depending on $\theta_0$), and $\lim_{\rho \to 0} \sigma^{-n}\mathcal{H}^n(Q_k \cap B_\sigma(y)) = \rho^{-n}\mathcal{H}^n(Q_k \cap B_p(y))$ for each $\rho \in (0, \frac{1}{4}(1 - \theta_0))$, so there is a smallest value $\rho(y, k)$ of $\rho \in (0, \frac{1}{2}(1 - \theta_0))$ such that $\mathcal{H}^n(Q_k \cap B_p(y)) \leq \delta^{1/2}\rho^2$. Thus

$$H^p(Q_k \cap B_{\rho(y, k)}(y)) = \delta^{1/2}\rho^n$$

and $H^p(Q_k \cap B_p(y)) \geq \delta^{1/2}\rho^n \forall \rho \in (0, \rho(y, k)]$

By the Besicovitch covering lemma there is a subcollection $\{B_{\rho(y, k)}(y_j, k)\}$ of such balls which covers $Q_k$ and which decomposes into a fixed number $J = J(N)$ of pairwise-disjoint subcollections. For each $k$ we must then have at least one of these balls, say $B_{\rho_k}(y_k)$, with

$$H^{n-1}(B_{\rho_k}(y_k) \cap \partial Q_k \cap \tilde{B}_1) \leq k^{-1/2}(H^n(Q_k \cap B_{\rho_k}(y_k)))^{1/\kappa},$$

because otherwise we would have

$$k^{-1/2}(H^n(Q_k \cap B_{\rho_k}(y_j, k)))^{1/\kappa} < H^{n-1}(B_{\rho_k}(y_j, k) \cap \partial Q_k \cap \tilde{B}_1)$$

for each $j$, and we could sum over $j$ to conclude that

$$H^n(Q_k \cap B_{\theta_0})^{1/\kappa} \leq \left( \sum_j H^n(Q_k \cap B_{\rho_k}(y_j, k)) \right)^{1/\kappa} \leq \left( \sum_j H^n(Q_k \cap B_{\rho_k}(y_j, k)) \right)^{1/\kappa} \leq k^{1/2}JH^{n-1}(\partial Q_k \cap \tilde{B}_1)$$

contrary to the original choice of $Q_k$ for sufficiently large $k$. (Notice that here we use the inequality $\left( \sum_j a_j \right)^{1/\kappa} \leq \sum_j a_j^{1/\kappa}$.)

Now with $y_k, \rho_k$ as in (2), let $Q_k' \equiv \eta_{y_k, \rho_k} Q_k$, and $P_k' \equiv \eta_{y_k, \rho_k} P_k$. By the compactness 1.4 we have a subsequence $P_k'' \to P$ in $\tilde{B}_1$, where $P \in \mathcal{P}$ with $U_P \subset \tilde{B}_1$. Let $\zeta = (\zeta^1, \ldots, \zeta^N)$ be a fixed $C^\infty_0(\tilde{B}_1; \mathbb{R}^N)$ function with support $\zeta[P]$ contained in a compact subset $K$ of $P \cap \tilde{B}_1$, and let $Q_k = \Psi_k(Q_k')$, as in the remark following 1.4, be $C^1$ on an open set $U$ containing $K$ and satisfy $\Psi_k(Q_k') \subset P$ and $\|\Psi_k-I\|_{C^1(U)} \to 0$ as $k \to \infty$. Then $H^{n-1}(\partial Q_k \cap \tilde{B}_1) \to 0$ and hence $\int_{Q_k'} \text{div}_P \zeta \to 0$, so that by the BV compactness theorem and the arbitrariness of $B_r(y)$ there is a measurable $Q \subset P$ and
a subsequence of $\tilde{Q}_k$ such that the indicator functions $\chi_{\tilde{Q}_k}$ converge strongly in $L^1$ on $P \cap \tilde{B}_1$ to $\chi_P$; then $\int_{\tilde{Q}} \div_P \zeta = \lim_{\tilde{Q} \to Q} \int_{\tilde{Q}} \div_P \zeta = 0$, which means that the indicator function of $Q$ is locally constant in $P \cap \tilde{B}_1$. Thus, up to a set of measure zero, $Q$ is a union of components of $P \cap \tilde{B}_1$. By virtue of 2.4 we have that at most finitely many components of $P \cap \tilde{B}_1$ can intersect the ball $B_{1/2}$, and hence at most finitely many of the components of $P \cap \tilde{B}_1$ which comprise $Q$ can intersect $B_{1/2}$. On the other hand by construction we arranged that $\tau^{-n}\mathcal{H}^m(Q \cap B_\tau) \in [\delta^{1/2}, \infty)$ for each $\tau < 1$, and hence there is at least one component $\tilde{P}$ of these finitely many components containing 0 in its closure. That is, there is a component $\tilde{P}$ of $P \cap \tilde{B}_1$ with $\tilde{P} \subset Q \cap \tilde{B}_1$ (up to a set of measure zero) and $0 \in$ the closure of $\tilde{P}$. But also by construction we have $\mathcal{H}^m(Q \cap \tilde{B}_1) \leq \delta^{1/2}$, which means that $\mathcal{H}^m(\tilde{P}) \leq \delta^{1/2}$. Since $\tilde{P} \in P$ with $U_{\tilde{P}} = \tilde{B}_1$ (by the reducibility hypothesis 1.1) with $0 \in$ closure $\tilde{P}$, this contradicts the bounds 2.1 since $\delta^{1/2} < \beta_1$, where $\beta_1$ is as in 2.1. Thus the proof is complete.

3 Stability Inequality

Here $M$ will denote a fixed element of $\mathcal{P}$ and $q$ will be a non-negative locally bounded measurable function on $M$. $u$ will denote a positive $W_{1,2}^1(M)$ supersolution of the equation $\Delta_M u + q u = 0$ with $q$ non-negative measurable and locally bounded on $M$. This means that

3.1 \[ \int_M (-\nabla u \cdot \nabla \zeta + q \zeta u) \leq 0, \quad \zeta \in C^1_c(M) \text{ with } \zeta \geq 0. \]

(Thus in the present section there are no perturbation terms $a, b$ as in 1.9.) We claim that then we have the “stability inequality”

3.2 \[ \int_M (|\nabla \zeta|^2 - q \zeta^2) \geq 0, \quad \zeta \in C^1_c(U_M), \]

and in particular that $\int_{M \cap K} q < \infty$ for each compact $K \subset U_M$. Notice that while the definition 3.1 requires $\zeta$ to vanish in a neighborhood of the singular set, the inequality 3.2 does not. To prove 3.2, first take any non-negative $\zeta \in C^1_c(U_M)$ and let $s_\delta : M \to [0, 1]$ be a smooth function with compact support in support $\zeta \cap M$ defined as follows: First use the definition of finite $\mathcal{H}^{n-2}$-measure and the compactness of support $\zeta \cap \text{sing } M$ there is a constant $\beta$ such that for each $\delta \in (0, 1)$ we can select a finite cover $\tilde{B}_{\rho_j/2}(y_j), j = 1, \ldots, Q$ of $\text{sing } M \cap \text{support } \zeta$ by balls with centers in $\text{sing } M$, $\sup_j \rho_j < \delta$ and $\sum_j \rho_j^{n-2} < \beta$. Next, for $j = 1, \ldots, Q$, let $\zeta_j$ be a smooth function on $M$ with $\zeta_j \equiv 0$ on $B_{\rho_j/2}(y_j), \zeta_j \equiv 1$ on $B_{\rho_j}(y_j), 0 \leq \zeta_j \leq 1$ everywhere, and $|\nabla \zeta_j| \leq 3\rho_j^{-1}$. Then we have $s_\delta = \min\{\zeta_1, \ldots, \zeta_Q\}$ we have $s_\delta(x) \equiv 1$ for $\text{dist}(x, \text{sing } M) > \delta$, $s_\delta \equiv 0$ in a neighborhood of $\text{sing } M \cap B_\rho(y)$, while $\int_{M \cap \text{support } \zeta} |\nabla s_\delta|^2 \leq C \sum_j \rho_j^{n-2} \leq C \beta$, where $C \leq \beta_2$ with $\beta_2$ as in 2.1. Now use 3.1 with $(\epsilon + u)^{-1}\zeta^2 s_\delta^2$ in place of $\zeta$. Since $\zeta s_\delta$ has compact support in $M$ this is a legitimate choice, and 3.1 gives

\[ \int_M \zeta^2 s_\delta^2 \left( \frac{w}{u + \epsilon} q + |\nabla w|^2 \right) \leq -2 \int_M (s_\delta \zeta |\nabla w| \cdot \nabla (s_\delta \zeta)) \]

with $w = \log(\epsilon + u)$. Using the Cauchy inequality $a \cdot b \leq |a|^2 + \frac{1}{4}|b|^2$ on the right side, we thus deduce that $\int_M \zeta^2 s_\delta^2 \frac{w}{u + \epsilon} q + |\nabla w|^2 \leq \int_M |\nabla (s_\delta \zeta)|^2 \leq C$ with constant $C$.
independent of $\delta$, so that by letting $\delta \downarrow 0$ we have $\int_M \zeta^2 |\nabla w|^2 < \infty$. On the other hand 3.1 implies
\[
\int_M \zeta^2 s_3^3 \left( \frac{u}{u + \epsilon} q + |\nabla w|^2 \right) \leq -2 \int_M (s_3^2 \zeta \nabla w \cdot \nabla \zeta) - 2 \int_M (s \zeta^2 \nabla w \cdot \nabla s_3).
\]
Letting $\delta \downarrow 0$ and using Cauchy-Schwarz to check that the last integral on the right tends to zero, and we then have
\[
\int_M \zeta^2 \left( \frac{u}{u + \epsilon} q + |\nabla w|^2 \right) \leq - \int_M ((2 \zeta \nabla w) \cdot \nabla \zeta)
\]
and so, letting $\epsilon \downarrow 0$ and using $ab \leq \frac{1}{4}a^2 + b^2$, we conclude 3.2 as claimed.

4 Compact classes of cones

The present section will be needed in the proof of Theorem 2 because we do not assume in the definition 1.12 of asymptotically conic that $M \in \mathcal{P}$ necessarily has a unique tangent cone at points $x_0 \in \text{sing} M$ or at $\infty$. To overcome this difficulty we shall use that fact if $M$ is as in Theorem 2 then the set of all possible tangent cones $C$ of $M$ arising as in 1.12 (at a point $x_0 \in \text{sing} M$ in case 1.12(a) or at $\infty$ in case 1.12(b)) is a compact subfamily of $\mathcal{P}$ with respect to the natural Hausdorff distance metric $d_1$ defined below in 4.1.

Let $C$ denote the set of all cones in $\mathcal{P}$ as in §1. As we mentioned in §1, the Hausdorff distance metric $d$ on $E = \{ \Sigma = C \cap S^{N-1} : C \in C \}$ makes $E$ into a compact metric space, and of course we can metrize $C$ by the metric $d_1$ given by $d_1(C_1, C_2) = d(\Sigma_1, \Sigma_2)$, where $\Sigma_j = C_j \cap S^{N-1}$, and then

4.1 $C$, equipped with the metric $d_1$, is a compact metric space.

Now as in §1 let $C_0$ be any fixed compact subset of $C$, let $E_0 = \{ C \cap S^{N-1} : C \in C_0 \}$, and for each $\Sigma \in E_0$ assume we have a non-negative $q_\Sigma$ such that the collection $Q_0$ of all such $q_\Sigma$ satisfies the compactness assumption of 1.6. In view of the compactness of $C_0$, we must then have for each $\tau > 0$

4.2 $q_\Sigma(\omega) \leq \Lambda_{C_0, \tau}, \quad \omega \in \Sigma_\tau,$

where $\Sigma_\tau = \{ x \in \Sigma : \text{dist}(x, \text{sing} \Sigma) > \tau \}$, and $\Lambda_{C_0, \tau}$ is a fixed constant depending only on $C_0$ and $\tau$, and not depending on the particular cone $C$.

For $\Sigma \in E_0$ we continue to define $\lambda_1(\Sigma)$ as in 1.7. We observe that for each $\tau \in (0, \frac{1}{2}]$ we have $\epsilon(\tau) = \epsilon(\tau, \Sigma, q_\Sigma) \downarrow 0$ as $\tau \downarrow 0$ such that

4.3 if $\Sigma \in E_0$ is such that $\exists$ a non-negative $u \in W^{1,2}_{\text{loc}}(C \cap \tilde{B}_1 \setminus B_\tau) \setminus \{ 0 \}$ with $\Delta_{C_\Sigma} u + r^{-2} q_\Sigma u \leq 0$ weakly on $C \cap \tilde{B}_1 \setminus B_\tau$, then $\lambda_1(\Sigma) \geq -\left( \frac{\alpha}{2} \right)^2 - \epsilon(\tau)$.

To prove this we use the stability inequality 3.2 with $\zeta(x) = \zeta_1(r) \zeta_2(\omega)$, where $r = |x|$, $\omega = |x|^{-1} x$, $\zeta_1 \in C^2_\text{loc}(\tau, 1)$, and $\zeta_2 \in C^1(\mathbb{R}^N \setminus \{ 0 \})$ homogeneous of degree zero with support $\zeta_2 \cap \text{sing} \Sigma = \emptyset$. Then 3.2 implies
\[
0 \leq \int_0^1 (\zeta_1'(r))^2 r^{n-1} dr \int_\Sigma \zeta_2^2(\omega) d\omega + \int_0^1 \zeta_2^2(r) r^{n-3} dr \int_\Sigma (|\nabla \zeta_2|^2 - q_\Sigma \zeta_2^2) d\omega.
\]
whence, taking inf over all $\zeta_1, \zeta_2$ with $L^2$ norms equal to 1, we conclude that

$$\lambda_1(\Sigma) \geq -\inf \int_0^1 \frac{(\zeta_1')^2 r^{n-1} dr}{\int_0^2 \zeta_1^2 r^{n-3} dr},$$

where the inf is over all $\zeta_1 \in C^1(\tau, 1)$ with compact non-empty support. It is a standard calculus fact that if $\tau = 0$ then the quantity on the right is exactly $-\left(\frac{\alpha}{2}\right)^2$ and hence in general it is $\geq -\left(\frac{\alpha}{2}\right)^2 - \epsilon(\tau)$ with $\epsilon(\tau) \downarrow 0$ as $\tau \downarrow 0$. This gives the required inequality $\lambda_1(\Sigma) \geq -\left(\frac{\alpha}{2}\right)^2 - \epsilon(\tau)$ as claimed.

The following lemma ensures we can always select a collection of eigenfunctions with good positivity properties on domains in $\Sigma \in \mathcal{E}_0$ and with eigenvalues not much bigger than the value $\lambda_1(\mathcal{E}_0) = \sup\{\lambda_1(\Sigma) : \Sigma \in \mathcal{E}_0\}$ of 1.8. In this lemma we use the notation that

$$\mathcal{E}_0(\Lambda) = \{\Sigma \in \mathcal{E}_0 : \lambda_1(\Sigma) \geq \Lambda\}$$

for $\Lambda \in \mathbb{R}$. Observe that $\mathcal{E}_0(\Lambda)$ is a closed (hence compact) subset of $\mathcal{E}_0$, which is easily checked by using the Rayleigh quotient definition 1.7 together with 1.6 and the local $C^1$ convergence described in 1.4.

**4.4 Lemma.** For each $\delta > 0$, $\Lambda \in \mathbb{R}$, $\exists \tau = \tau(\delta, \Lambda, \mathcal{E}_0, Q_0) > 0$ such that the following holds. For each $\Sigma \in \mathcal{E}_0(\Lambda)$ there are connected open $C^1$ domains $\Omega_1, \ldots, \Omega_Q$, $Q \leq Q_0 = Q_0(\mathcal{E}_0)$, with $\Omega_j \subset \Sigma$ and $\Omega_i \cap \Omega_j = \emptyset \forall i \neq j$, and corresponding non-negative $\varphi_{j,\delta} \in W_0^{1,2}(\Omega_j) \cap C^0(\Omega_j)$ with

$$\max \varphi_{j,\delta} = 1 \quad \text{and} \quad - (\Delta \Sigma \varphi_{j,\delta} + q_\Sigma \varphi_{j,\delta}) = \lambda_1(\Omega_j) \varphi_{j,\delta} \text{ weakly on } \Omega_j,$$

where $\lambda_1(\Omega_j) \leq \lambda_1(\mathcal{E}_0) + \delta$ for each $j = 1, \ldots, Q$

and

$$\mathcal{H}^n(\Sigma \setminus \bigcup_j \{x \in \Omega_j : \varphi_{j,\delta}(x) > \tau\}) < \delta.$$

**Remark:** An essential feature of the above lemma is that the constant $\tau$ does not depend on the particular $\Sigma \in \mathcal{E}_0(\Lambda)$ under consideration, so $\tau$ is chosen and then the lemma applies uniformly across the whole class $\mathcal{E}_0(\Lambda)$ and corresponding $Q_0$ (as in 1.6).

**Proof of Lemma 4.4:** If $\Sigma \in \mathcal{E}_0(\Lambda)$ with connected components $\Sigma_1, \ldots, \Sigma_Q$ (so that $Q \leq Q_0 = Q_0(\mathcal{E}_0)$ by 2.4), then for each sufficiently small $\tau > 0$ we can select connected open $C^1$ domains $\Omega_{j,\tau}$ with

$$\Sigma_{j,\tau} \subset \Omega_{j,\tau} \subset \overline{\Omega}_{j,\tau} \subset \Sigma_j,$$

where we use the notation $\Sigma_{j,\tau} = \{x \in \Sigma_j : \text{dist}(x, \text{sing } \Sigma_j) > \tau\}$. Then we can take $\varphi_{j,\tau} \in W_0^{1,2}(\Omega_{j,\tau}) \setminus \{0\}$ minimizing the Rayleigh quotient

$$\left(\int_{\Omega_{j,\tau}} \varphi^2 \right)^{-1} \int_{\Omega_{j,\tau}} (|\nabla \varphi|^2 - q_\Sigma \varphi^2)$$

over $\varphi \in W_0^{1,2}(\Omega_{j,\tau}) \setminus \{0\}$. Letting $\lambda_{j,\tau}$ denote the minimum value, we then have, for small enough $\tau = \tau(\Sigma, \delta)$, that $\varphi_{j,\tau}$ is non-negative a weak solution of the equation

$$-(\Delta_j \varphi_{j,\tau} + q_\Sigma \varphi_{j,\tau}) = \lambda_{j,\tau} \varphi_{j,\tau},$$
and, in the notation of 1.7, \( \lambda_{j,\tau} \to \lambda_1(\Sigma_j) \) as \( \tau \downarrow 0 \) for each \( j = 1, \ldots, Q \). Also by the De Giorgi Nash Moser theory ([GT83, §8.8–§8.10]) we know that \( \varphi_j^{(\tau)} \in W_0^{1,2}(\Omega_j, \tau) \) and that

\[
\sup_{\Omega_j, \tau} \varphi_j^{(\tau)} \leq C(\Sigma, \Lambda, \tau)\|\varphi_j^{(\tau)}\|_{L^2(\Omega_j, \tau)},
\]

\[
\inf_{\{x \in \Omega_j, \tau : \text{dist}(x, \partial \Omega_j) > \tau\}} \varphi_j^{(\tau)} \geq C(\Sigma, \tau)^{-1}\|\varphi_j^{(\tau)}\|_{L^2(\Omega_j, \tau)}.
\]

So we can normalize so that \( \max_{\Omega_j, \tau} \varphi_j^{(\tau)} = 1 \) and then \( \inf_{\{x \in \Omega_j, \tau : \text{dist}(x, \partial \Omega_j) > \tau\}} \varphi_j^{(\tau)} \geq \sigma \) with \( \sigma = \sigma(\Sigma, \tau) > 0 \), hence for sufficiently small \( \tau = \tau(\Sigma, q_{\Sigma}, \delta) \) and small enough \( \sigma = \sigma(\Sigma, \tau) > 0 \) we have

\[
\mathcal{H}^0(\Sigma \setminus (\bigcup_{j=1}^Q \{x \in \Omega_j, \tau : \varphi_j^{(\tau)} > \tau\})) < \delta.
\]

Thus with such a choice of \( \tau \) and \( \sigma \), we use the notation

\[
\Omega_j = \Omega_{j, \tau}, \quad \varphi_{j, \delta} = \varphi_j^{(\tau)},
\]

and then we have the required properties stated in the lemma except that the constant \( \tau \) depends on the particular \( \Sigma \in \mathcal{E}_0(\Lambda) \), i.e., \( \tau = \tau(\Sigma, q_{\Sigma}, \delta) \). But now observe that by 1.4 and 1.6 there is \( \epsilon = \epsilon(\Sigma, q_{\Sigma}) \) such that each \( \tilde{\Sigma} \) in the Hausdorff distance metric ball \( B_\epsilon(\Sigma) \subset \mathcal{E}_0(\Lambda) \) of radius \( \epsilon \) and center \( \Sigma \) is \( C^1 \) close to \( \Sigma \) in a neighborhood of the compact set \( \bigcup_{j=1}^Q \Gamma_j \subset \Sigma \), and correspondingly the function \( q_{\tilde{\Sigma}} \) is uniformly close to \( q_{\Sigma} \) in this neighborhood. Specifically, for any \( \eta > 0 \) and small enough \( \epsilon = \epsilon(\Sigma, q_{\Sigma}, \eta, \Omega_1, \ldots, \Omega_Q) > 0 \), there is a fixed \( U \), open in \( \mathbb{R}^N \), with \( \bigcup_{j=1}^Q \Gamma_j \subset U \) and for each \( \tilde{\Sigma} \in B_\epsilon(\Sigma) \) there is a \( C^1 \) map \( \tilde{\Psi} : U \to U \) with \( \|\tilde{\Psi} - I\|_{C^1} < \eta \) and such that

\[
\tilde{\Psi}(U \cap \Sigma) = U \cap \tilde{\Sigma}, \quad \max_{x \in U} \|q_{\Sigma} \circ \tilde{\Psi} - q_{\Sigma} - \lambda_{\Sigma, \Omega} \| < \eta \quad \text{and} \quad \tilde{\Omega}_j = \tilde{\Psi}(\Omega_j)
\]

have pairwise disjoint closures contained in \( \tilde{\Sigma} \) and a of distance \( < \eta \) from \( \Omega_j \) in the Hausdorff distance sense.

Thus, if we take \( \eta = \eta(\Sigma, q_{\Sigma}, \delta) \) is suitably small (technically depending also on the choice of \( \Omega_j \) for \( \Sigma \), but those domains are determined by \( \Sigma \) and \( \delta \) also) then we have, for all \( \tilde{\Sigma} \in B_\epsilon(\Sigma) \), open \( \Omega_j \subset \tilde{\Sigma} \) and functions \( \tilde{\varphi}_{j, \delta} \in W_0^{1,2}(\Omega_j) \cap C^0(\text{closure} \tilde{\Omega}_j) \) with

\[
\max \tilde{\varphi}_{j, \delta} = 1 \quad \text{and} \quad - (\Delta_{\tilde{\Sigma}} \tilde{\varphi}_{j, \delta} + q_{\Sigma} \tilde{\varphi}_{j, \delta}) = \lambda_1(\tilde{\Omega}_j) \tilde{\varphi}_{j, \delta} \text{ weakly on } \tilde{\Omega}_j,
\]

where

\[
\lambda_1(\tilde{\Omega}_j) \leq \lambda_1(\mathcal{E}_0) + 2\delta \text{ for each } j = 1, \ldots, Q
\]

and

\[
\mathcal{H}^0(\tilde{\Sigma} \setminus (\bigcup_j \{x \in \tilde{\Omega}_j : \tilde{\varphi}_{j, \delta}(x) > \tau\})) < 2\delta.
\]

with constant \( \tau = \tau(\Sigma, q_{\Sigma}, \delta) \). Since \( \mathcal{E}_0(\Lambda) \) is compact, we can select finitely many such balls \( B_\epsilon(\Sigma_k)(\Sigma_k), k = 1, \ldots, S, \Sigma = S(\Lambda, \mathcal{E}_0, \delta) \), such that \( \mathcal{E}_0(\Lambda) = \bigcup_{k=1}^S B_\epsilon(\Sigma_k)(\Sigma_k) \). Taking the minimum of the corresponding constants \( \tau(\Sigma_k, q_{\Sigma_k}, \delta)/2, k = 1, \ldots, S, \) we thus have the conclusion stated in the lemma.

**Remarks:** (1) Notice that in the above proof we first selected domains \( \Omega_j = \Omega_j(\Sigma, q_{\Sigma}) \) with each \( \tilde{\Omega}_j \) engulfing all but a thin boundary strip of one of the connected components of \( \Sigma \), but the reader should observe that the corresponding \( C^1 \) domains \( \tilde{\Omega}_j \subset \tilde{\Sigma} \) of the nearby \( \tilde{\Sigma} \in B_\epsilon(\Sigma_j)(\Sigma) \) may in fact be only a small fraction of one of the components of \( \tilde{\Sigma} \) (e.g. the union of two or more of the \( \tilde{\Omega}_j \) may be needed to encompass most of a
single component of $\tilde{\Sigma}$). This is because the nearby $\tilde{\Sigma}$ may have “necks” which shrink off on approach to $\Sigma$; so the union of several components of $\Sigma$ may be close to a single component of $\tilde{\Sigma}$.

(2) For $\Sigma \in \mathcal{E}_0(\Lambda)$ we let $\Omega_\Sigma = \bigcup_{j=1}^Q \Omega_j$ and we let $\varphi_\delta \in W^{1,2}_0(\Omega_\Sigma)$ be defined by

$$\varphi_\delta|_{\Omega_j} = \varphi_{j,\delta},$$

where $\Omega_j, \varphi_{j,\delta}, j = 1, \ldots, Q$ are as in Lemma 4.4. Then $\varphi_\delta$ satisfies

$$\int_{\Omega_\Sigma} \left( \nabla \cdot \nabla \varphi_\delta - (q_\Sigma + \lambda_{1,\delta}(\Sigma)) \varphi_\delta \right) \leq 0, \quad v \geq 0, \quad v \in W^{1,2}_0(\Omega_\Sigma),$$

(i.e. weakly satisfies $\Delta \varphi_\delta + (q_\Sigma + \lambda_{1,\delta}(\Sigma))\varphi_\delta \geq 0$ on $\Omega_\Sigma$), where

$$\lambda_{1,\delta}(\Sigma) = \max\{\lambda_1(\Omega_j) : j = 1, \ldots, Q\}$$

with $\lambda_1(\Omega_j)$ the first eigenvalue of $\Omega_j$ as in Lemma 4.4. We claim that in fact

$$\int_{\Omega_\Sigma} \left( \nabla \cdot \nabla \varphi_\delta - (q_\Sigma + \lambda_{1,\delta}(\Sigma)) \varphi_\delta \right) \leq 0$$

for any non-negative $C^1(\overline{\Omega_\Sigma})$ function $v$ (without the assumption that $v$ vanishes on $\partial \Omega_\Sigma$). We can check this by replacing $v$ in 4.5 by $v f_k \circ \varphi_\delta (\in W^{1,2}_0(\Omega_\Sigma))$, where $f_k$ is the piecewise linear function $f_k(t) = \max\{0, \min\{k(t - \frac{1}{2}), 1\}\}$. Since

$$\int_{\{x \in \Omega_\Sigma : \varphi_\delta(x) < \frac{2}{k}\}} |\nabla \varphi_\delta \cdot \nabla v| \leq C\|\varphi_\delta\|_{W^{1,2}(\Omega_\Sigma)}(\mathcal{H}^n(x \in \Omega_\Sigma : \varphi_\delta(x) < \frac{2}{k}))^{1/2} \to 0$$

and $\nabla \cdot \nabla (v f_k \circ \varphi_\delta) = f_k \circ \varphi_\delta \nabla \cdot \nabla v + v f_k'(\varphi_\delta)^2 \leq f_k \circ \varphi_\delta \nabla \cdot \nabla v$, we see that 4.5 gives 4.7 as claimed in the limit as $k \to \infty$.

Note also that in accordance with the conclusions of Lemma 4.4 for this $\delta > 0$ we have

$$\max_{\Omega_\Sigma} \varphi_\delta = 1 \quad \text{and} \quad - (\Delta \varphi_\delta + q_\Sigma \varphi_\delta) = \lambda_{1,\delta}(\Sigma) \varphi_\delta \quad \text{weakly on} \quad \Omega_\Sigma,$$

for suitable $\tau = \tau(\delta, \Lambda, \mathcal{E}_0, Q_0) > 0$ and all $\Sigma \in \mathcal{E}_0(\Lambda)$. There are also various circumstances which make it possible to prove upper bounds for $\lambda_1(\Sigma)$ to complement the lower bound 4.3. For example, if $\lambda_0 \in \mathbb{R}$, $C \in C_0$, $v \in W^{1,2}_0(\Sigma) \cap L^2(\Sigma) \setminus \{0\}$, $q_\Sigma v^2 \in L^1(\Sigma)$, and if $v$ is a subsolution of the equation $\Delta v + (q_\Sigma + \lambda_0) v = 0$ in the sense that $\int_{\Sigma} (\nabla v \cdot \nabla \zeta - (q_\Sigma + \lambda_0) v \zeta) \leq 0$, whenever $\zeta$ is a bounded non-negative $W^{1,2}$ function with compact support in $\Sigma$, and if $\mathcal{H}^{n-4}(\operatorname{sing} C) = 0$, then $\lambda_1(\Sigma) \leq \lambda_0$. To see this, we let $s_\delta$ be a function analogous to that used in the above discussion, except that we now choose the balls $B_{\rho_j/2}(y_j)$ to cover $\operatorname{sing} C$ and $\sum_j \rho_j^{n-4} < \delta$. Then using the above inequality with $v s_\delta^2$ in place of $\zeta$, we infer

$$\int_{\Sigma} s_\delta^2 (|\nabla v|^2 - (q_\Sigma + \lambda_0) v^2) \leq - \int_{\Sigma} (2v s_\delta \nabla v \cdot \nabla s_\delta)$$

and hence by the Cauchy-Schwarz inequality we infer first that

$$\int_{\Sigma} \left( \frac{1}{2} s_\delta^2 |\nabla v|^2 - (q_\Sigma + \lambda_0) v^2 \right) \leq C \int_{\Sigma} |\nabla s_\delta|^2 v^2 \leq C \left( \int_{\Sigma} |\nabla s_\delta|^4 \right)^{1/2} \left( \int_{\Sigma} v^4 \right)^{1/2}. $$
Since the right side $\to 0$ as $\delta \downarrow 0$ this then implies by Fatou’s lemma that $\int_{\Sigma} |\nabla v|^2 < \infty$, and then going back to the first inequality above we have

$$\int_{\Sigma} s_{\delta}^2 \left( (|\nabla v|^2 - (q_{\Sigma} + \lambda_0) v^2) \right) \leq \int_{\Sigma} \left( 2u s_{\delta} \nabla v \cdot \nabla s_{\delta} \right) - C \left( \int_{\Sigma} |\nabla v|^2 \right)^{1/2} \left( \int_{\Sigma} |\nabla s_{\delta}|^2 \right)^{1/2} \to 0 \text{ as } \delta \downarrow 0.$$ 

On the other hand $\int_{\Sigma} |\nabla (s_{\delta} v)|^2 = \int_{\Sigma} (s_{\delta}^2 |\nabla v|^2 + 2s_{\delta} v \nabla s_{\delta} \cdot \nabla v + v^2 |\nabla s_{\delta}|^2)$ and hence the above inequality implies

$$\limsup_{\delta \downarrow 0} \int_{\Sigma} \left( (\nabla (s_{\delta} v))^2 - (q_{\Sigma} + \lambda_0) (s_{\delta} v)^2 \right) \leq 0,$$

whence, since $\int_{\Sigma} (s_{\delta} v)^2 \to \int_{\Sigma} v^2 \in (0, \infty)$,

$$\limsup_{\delta \downarrow 0} \left( \int_{\Sigma} (s_{\delta} v)^2 \right)^{-1} \int_{\Sigma} \left( |\nabla (s_{\delta} v)|^2 - q_{\Sigma} (s_{\delta} v)^2 \right) \leq \lambda_0.$$ 

Thus we have proved that

$$\lambda_1(\Sigma) \leq \lambda_0$$

if $H^{n-1}(\text{sing } \mathbb{C}) = 0$ and if there is a non-negative $W^{1,2}_{\text{loc}}(\Sigma) \cap L^4(\Sigma) \setminus \{0\}$ subsolution of the equation $\Delta_{\Sigma} u + r^{-2} (q_{\Sigma} + \lambda_0) u = 0$ on $\mathbb{C}$.

## 5 A partial Harnack theory

Here $M$ will denote any fixed element of $\mathcal{P}$ and $a, b_1, \ldots, b_n$ will be measurable functions on $M$ with $|a|^{1/2} + |b|$ locally integrable on $M$, $b = (b_1, \ldots, b_n)$. We also need to assume $L^{n+\alpha}$ ($\alpha > 0$) bounds on the function $|a|^{1/2} + |b|$ on the various domains which arise here—the precise bounds needed will be stipulated in each case.

Recall that $u$ is a positive supersolution of the equation $\Delta_{\Sigma} u + b \cdot \nabla_{M} u + au = 0$ on $M$ means that $u > 0$ a.e. on $M$, $u \in W^{1,2}_{\text{loc}}(M)$ and

$$\int_{M} (au \zeta + b \cdot \nabla_{M} u \zeta - \nabla u \cdot \nabla \zeta) \leq 0$$

for each non-negative $C^1_c(M)$ function $\zeta$.

Of course by classical De Giorgi Nash theory (applied locally on the $C^1$ manifold $M$) there is no loss of generality in assuming that $u$ is positive on $M$ with local uniform positive lower bounds. Also if $p \in (0, 1)$ and if we use $\zeta u^{p-1}$ in place of $\zeta$ in this inequality, then, again letting $\epsilon \downarrow 0$, we get the inequality

$$\int_{M} \zeta \left( pa u^p + b \cdot \nabla_{M} u^p + \frac{4(1-p)}{p} |\nabla u^{p/2}|^2 \right) \leq \int_{M} \nabla u^p \cdot \nabla \zeta,$$

for each non-negative $\zeta \in C^1_c(M)$. Now $\nabla u^p = 2u^{p/2} \nabla u^{p/2}$, so if we replace $\zeta$ by $\zeta^2$ and use the Cauchy-Schwarz inequality then we obtain

$$\int_{M} \zeta^2 \left( \frac{1-p}{p} |\nabla u^{p/2}|^2 \right) \leq 8(1-p) \int_{M} u^p (|\nabla \zeta|^2 + (|a| + |b|^2) \zeta^2),$$

Now $u^p$ is a positive $C^1$ function on $M$. We also assume that $a, b_1, \ldots, b_n$ are locally Lipschitz on $M$ with $|a|^{1/2} + |b|$. We also assume that $\zeta$ is a non-negative $C^1$ function on $M$ such that $\zeta \equiv 1$ on $\{u > 0\}$.
for each $\zeta \in C^1_c(M)$. There is also a version of this for $p = 0$:

$$5.4 \quad \int_M \zeta^2(|\nabla w|^2) \leq 8 \int_M (|\nabla \zeta|^2 + (|a| + |b|)^2 \zeta^2),$$

where $w = \log u$, which is obtained by substituting $\zeta u^{-1}$ in place of $\zeta$ in 5.1.

Now we claim that 5.4 is valid for $\zeta \in C^1_c(U_M)$. That is, it is not necessary that $\zeta$
vanish in a neighborhood of sing $M$. Indeed to see that 5.4 is valid we simply replace
$\zeta$ by $\zeta s_\delta$ and let $\delta \downarrow 0$ as in our discussion of 3.2

We cannot do quite the same thing to justify the fact that 5.3 holds for any $\zeta \in C^1_c(U_M)$,
because $u^p$ is not necessarily bounded. However notice that if we take any $K > 0$ and
any $C^2$ concave increasing function $f_K$ with $f_K(t) = t$ for $0 \leq t \leq K$ and $f_K(t) \equiv K + \frac{1}{2}$
for $t > K + 1$, then by replacing $\zeta$ by $\zeta f_K(u)$ in 5.1 we obtain directly that $f_K(u)$ is
a supersolution of an equation of the same form. Thus we have the estimate 5.4 with
$u_K = f_K(u)$ in place of $u$ and with any $\zeta \in C^1_c(M)$:

$$\int_M \zeta^2(|\nabla u_K^{p/2}|^2) \leq C \int_M u_K^p (|\nabla \zeta|^2 + (|a| + |b|)^2 \zeta^2),$$

for any $\zeta \in C^1_c(M)$. Now with $s_\delta$ as in the discussion following 3.2, for any $z \in C^1_c(U_M)$
we can then substitute $\zeta s_\delta$ in place of $\zeta$ here. Since $u_K$ is bounded, we can then let
$\delta \downarrow 0$ (as we did to justify 3.2) to deduce that 5.3 holds for all $\zeta \in C^1_c(U_M)$, with $u_K$
in place of $u$. Then letting $K \uparrow \infty$, we deduce that 5.3 also holds for all $\zeta \in C^1_c(U_M)$
provided $u^p \in L^1(M \cap K)$ for each compact $K \subset U_M$. We show below in 5.7 that
indeed $u^p \in L^1(M \cap K)$ for each $p \in (0, \frac{n}{n-2})$.

5.5 Remark: Notice that in particular 5.2 says that $u^p$ is a supersolution of the
equation $\Delta_M u + b \cdot \nabla_M u + pau = 0$ locally in $M$ for each $p \in (0, 1)$.

Now assume that $y \in \overline{M} \cap U_M$, take any closed ball $B_p(y) \subset U_M$ and assume

$$5.6 \quad \rho^{-n/(n+\alpha)} |||a|||^{1/2} + |b| ||L^{n+\alpha}(M \cap B_p(y))|| \leq \beta$$

for some constants $\alpha, \beta > 0$. Recall that the weak Harnack theory for supersolutions
on domains in $\mathbb{R}^n$ says that, for any $\theta \in (0, 1)$, inf$B_p(y) \nu \geq C^{-1}(\rho^{-n} \int_{B_p(y)} u^p)^{1/p}$ for
positive supersolutions $\nu$ of the equation $\Delta u + b \cdot Du + au = 0$, with $C = C(n, \theta, p, \beta)$,
assuming $a, b$ satisfy an inequality like 5.6 in the Euclidean ball $B_p(y)$ (rather than in
$M \cap B_p(y)$). The proof of this requires not only a Sobolev inequality (analogous to
the inequality established for surfaces $P \in \mathcal{P}$ in 2.6) but also a Poincaré inequality,
and unfortunately in the present setting there is no such inequality (this would require
strong connectivity hypotheses on the submanifolds in the class $\mathcal{P}$), so we cannot
follow the $\mathbb{R}^n$ procedure to give a Harnack theory. Nevertheless, with only the Sobolev
inequality of Remark 2.6 and the modified Poincaré inequality of Remark 2.6 at our
disposal, we claim that it is still possible to prove the following:

5.7 Lemma. Suppose $\mathcal{P}$ is a regular multiplicity 1 class (so that 1.1–1.4 hold), $\theta \in
(0, 1)$, $p \in [1, \frac{n}{n-2})$, $M \in \mathcal{P}$, $y \in \overline{M}$ with $B_p(y) \subset U_M$, and $u$ is a positive supersolution
of the equation $\Delta_M u + b \cdot \nabla_M u + au = 0$ on $M \cap \bar{B}_p(y)$, where $a, b$ satisfy 5.6. Then
there is $\delta = \delta(\mathcal{P}, \alpha, \beta, \theta, p) \in (0, \frac{1}{2}]$ such that

$$\rho^{-n} \int_{M \cap B_p(y)} u^p \leq C \lambda, \quad C = C(\mathcal{P}, \alpha, \beta, \theta, p),$$
for any \( \lambda \) such that \( \mathcal{H}^n(\{x \in M \cap B_\rho(y) : u(x) > \lambda \}) \leq \delta \rho^n \).

**Proof:** We use some modifications of the relevant part of the De Giorgi Nash Moser theory ([GT83, §8.8–§8.10]). First, by rescaling, we can assume \( \rho = 1 \), so we aim to prove a bound for the \( L^p \) norm of \( u \) over the ball \( B_\theta(y) \), where \( \theta \in (0,1) \). For \( \lambda > 0 \) (fixed for the moment), let

\[
\tilde{w} = \min\{\max\{\log(u/\lambda), 0\}, K\},
\]

where \( K \geq 2 \) (we plan to let \( K \uparrow \infty \) eventually) and observe that for any \( q \geq 1 \), \( m, \gamma > 0 \) with \( 2m - \gamma > 2 \), and non-negative \( \zeta \in C^1(\tilde{B}_\theta(y)) \) satisfying

\[
(1) \quad \kappa \gamma = \gamma + 2, \quad \zeta \equiv 1 \text{ on } B_{\rho^2}, \quad 0 \leq \zeta \leq 1, \quad |\nabla \zeta| \leq C(\theta) \text{ on } B_1,
\]

where \( \kappa = n/(n-2) \) for \( n \geq 3 \) as in 2.5', 2.6' and \( \kappa > 2 \) is arbitrary in case \( n = 2 \). (So \( \gamma = n-2 \) if \( n \geq 3 \) and \( \gamma = 2/(\kappa - 1) \) if \( n = 2 \).) For the remainder of the proof we let \( C \) denote any constant

\[
C = C(\mathcal{P}, \alpha, \beta, m, N, \theta, p);
\]

it is important to keep track of the \( q \) dependence though, so that will be indicated explicitly at each stage of the proof. Then we can apply 2.6' with \( \varphi = \tilde{w}^{2q}\zeta^{2mq-\gamma} \) with \( q \geq 1 \), giving

\[
(2) \quad \left( \int (\tilde{w}\zeta^m)^{2q} \, d\mu \right)^{1/q} \leq C q^2 \int (|\nabla \tilde{w}|^2 \tilde{w}^{2q-2}\zeta^{2mq+2} + \tilde{w}^{2q}\zeta^{2qm}) \, d\mu,
\]

where \( \mu \) is the Borel measure defined by

\[
\mu = \zeta^{-\kappa \gamma} \mathcal{H}^n(M \cap B_1(y)) = \zeta^{-\gamma - 2} \mathcal{H}^n(M \cap B_1(y)).
\]

To handle the first term on the right of (2) we proceed slightly differently in the cases \( q \geq 2 \) and \( q \in [1,2) \). If \( q \geq 2 \), by replacing \( \zeta \) in 5.4 with \( \tilde{w}^{q-1}\zeta^{q_m-\gamma/2} \) then we get

\[
\int (\tilde{w}^{2q-2}\zeta^{2mq+2}|\nabla \tilde{w}|^2) \, d\mu \leq C q^2 \int (\tilde{w}^{2q-4}\zeta^{2mq+2}|\nabla \tilde{w}|^2 + A\tilde{w}^{2q-2}\zeta^{2mq}) \, d\mu,
\]

where

\[
A = 1 + (|a| + |b|^2)\zeta^2,
\]

and since \( \tilde{w}^{2q-2} \leq 1 + \tilde{w}^{2q} \) this gives

\[
(3) \quad \int (\tilde{w}^{2q-2}\zeta^{2mq+2}|\nabla \tilde{w}|^2) \, d\mu \leq C q^2 + C q^2 \int (\tilde{w}^{2q-4}\zeta^{2mq+2}|\nabla \tilde{w}|^2 + A\tilde{w}^{2q}\zeta^{2mq}) \, d\mu.
\]

For \( q \geq 2 \) Young’s inequality gives

\[
C q^2 \tilde{w}^{2q-4} \leq \frac{1}{2} \tilde{w}^{2q-2} + C q^{2q},
\]

and we thus get for any \( q \geq 2 \)

\[
(4) \quad \int (\tilde{w}^{2q-2}\zeta^{2mq+2}|\nabla \tilde{w}|^2) \, d\mu \leq C q^2 + C q^{2q} \int (\zeta^{2mq+2}|\nabla \tilde{w}|^2) \, d\mu + C q^2 \int (A\tilde{w}^{2q}\zeta^{2mq}) \, d\mu.
\]

On the other hand if \( q \in [1,2) \) then we can first use \( \tilde{w}^{2q-2} \leq 1 + \tilde{w}^2 \) and hence

\[
\int (\tilde{w}^{2q-2}\zeta^{2mq+2}|\nabla \tilde{w}|^2) \, d\mu \leq \int (\tilde{w}^2\zeta^{2mq+2}|\nabla \tilde{w}|^2) \, d\mu + \int (\zeta^{2mq+2}|\nabla \tilde{w}|^2) \, d\mu,
\]

and
and by replacing \( \zeta \) in 5.4 with \( \widetilde{w} \zeta^{m-\gamma/2} \) we obtain
\[
\int (\widetilde{w}^2 \zeta^{2mq+2} |\nabla \widetilde{w}|^2) \, d\mu \leq C \int (\zeta^{2mq+2} |\nabla \widetilde{w}|^2 + A \widetilde{w}^2 \zeta^{2mq}) \, d\mu,
\]
and hence since \( \widetilde{w}^2 \leq 1 + w^2q \) we again get (4), so in fact (4) is valid for any \( q \geq 1 \).

Another application of 5.4 (this time with \( \zeta^{mq-\gamma/2} \) in place of \( \zeta \)) gives
\[
\int (\zeta^{2mq+2} |\nabla \widetilde{w}|^2) \, d\mu \leq Cq^2 \int A \zeta^{2mq} \, d\mu \leq Cq^2,
\]
so (4) implies
\[
\int (\widetilde{w}^{2q-\gamma} \zeta^{2mq+2} |\nabla \widetilde{w}|^2) \, d\mu \leq C^q q^2 + Cq^2 \int A (\widetilde{w} \zeta^m)^{2q} \, d\mu
\]
for all \( q \geq 1 \).

Combining (5) with (2) we obtain
\[
\left( \int (\widetilde{w} \zeta^m)^{2q} \, d\mu \right)^{1/\kappa} \leq C^q q^2 + Cq^2 \int A (\widetilde{w} \zeta^m)^{2q} \, d\mu.
\]
Using the Hölder inequality and 5.6 again we then conclude that
\[
\left( \int (\widetilde{w} \zeta^m)^{2q} \, d\mu \right)^{1/\kappa} \leq C^q q^2 + Cq^2 \int (\widetilde{w} \zeta^m)^{2q} \, d\mu + Cq^2 \left( \int (\widetilde{w} \zeta^m)^{2\lambda} \, d\mu \right)^{1/\lambda}
\]
for some \( \lambda = \lambda(\alpha, n) \in (1, \kappa) \). Using the \( L^p \) interpolation inequality (with respect to the measure \( \mu \))
\[
\|f\|_{L^\lambda(\mu)} \leq \epsilon \|f\|_{L^\alpha(\mu)} + \epsilon^{-C(\kappa, \lambda)} \|f\|_{L^1(\mu)}
\]
with \( f = (\widetilde{w} \zeta^m)^{2q} \) then yields
\[
\left( \int (\widetilde{w} \zeta^m)^{2q} \, d\mu \right)^{1/\kappa} \leq C^q q^2 + Cq^2 \int (\widetilde{w} \zeta^m)^{2q} \, d\mu,
\]
and hence
\[
\Psi(q) \leq Cq + C^{1/q} q^{C/q} \Psi(q) \text{ with } \Psi(q) = \left( \int (\widetilde{w} \zeta^m)^{2q} \, d\mu \right)^{1/2q},
\]
for each \( q \geq 1 \). Replacing \( q \) by \( \nu \), \( \nu = 0, 1, 2, \ldots \), we obtain
\[
\Phi(\nu + 1) \leq C \kappa^\nu + C^{1/\kappa^\nu} \kappa^{\nu/\kappa^\nu} \Phi(\nu), \quad \nu = 0, 1, 2, \ldots, \text{ with } \Phi(\nu) = \Psi(\kappa^\nu).
\]
Iterating, and using the facts that \( \sum_{j=0}^\nu \kappa^j \leq C \kappa^\nu \) and \( \sum \kappa^{\nu - \nu} < \infty \), we get
\[
\Phi(\nu) \leq C \kappa^\nu + C \Phi(0).
\]
But
\[
\Phi(0) = \int (\widetilde{w} \zeta^m)^2 \, d\mu \leq \int_{M \cap B_6(y)} \widetilde{w}^2 \, d\mathcal{H}^n \leq C \int_{M \cap B_{1+\epsilon/2}(y)} |\nabla \widetilde{w}|^2 \, d\mathcal{H}^n \leq C
\]
by 2.6(2), 5.4, and 5.6, so in fact
\[
\Phi(\nu) \leq C \kappa^\nu, \quad \nu = 0, 1, \ldots,
\]
and then in view of (6) we conclude
\[ \int (\tilde{w}^{2q} \, \xi^m) \, d\mu \leq C^q q^{2q}, \quad q = 0, 1, \ldots, \]
and since \( \xi \equiv 1 \) on \( B_{\rho^2}(y) \), this gives
\[ \int_{M \cap B_{\rho^2}(y)} \tilde{w}^{2q} \, d\mathcal{H}^n \leq C^q q^{2q}, \quad q = 0, 1, \ldots. \]
Since \( q^{2q} \leq C^q (2q)! \) we can sum over \( q \) here to conclude that
\[ \int_{M \cap B_{\rho^2}(y)} e^{p_0 \tilde{w}} \, d\mathcal{H}^n \leq C \]
for some \( p_0 = p_0(\mathcal{P}, \alpha, \beta, \theta) \in (0, 1) \), and since \( \tilde{w} \uparrow \max\{\log(\frac{n}{\bar{a}}), 0\} \) as \( K \to \infty \), this implies
\[ (7) \quad \int_{M \cap B_{\rho^2}(y)} u^{p_0} \, d\mathcal{H}^n \leq C \lambda^{p_0}. \]

However 5.3 and the Sobolev inequality of Remark 2.6’ evidently imply that
\[ \left( \int_{M \cap B_{\rho^2}(y)} u^{\kappa p} \right)^{1/\kappa p} \leq C \left( \int_{M \cap B_{\rho}(y)} u^{p} \right)^{1/p}, \quad p \in (0, 1), \]
where \( C = C(\mathcal{P}, \alpha, \theta, \beta, p) \), and by finite iteration of this inequality we have
\[ (8) \quad \left( \int_{M \cap B_{\rho^2}(y)} u^{\kappa_1 \beta_1 p} \right)^{1/\kappa_1 \beta_1 p} \leq C \left( \int_{M \cap B_{\rho}(y)} u^{p} \right)^{1/p}, \quad \text{provided } \kappa_1 \beta_1^{-1} p \in (0, 1), \]
where \( C(\mathcal{P}, \alpha, j, p, \theta, \beta) \), which when used in combination with (7) evidently implies
\[ \int_{M \cap B_{\rho}(y)} u^{p} \, d\mathcal{H}^n \leq C \lambda^p \]
for each \( p \in (0, \kappa) \) and each \( \theta \in (0, 1) \), where \( C = C(\mathcal{P}, p, \alpha, \beta, \theta) \), thus completing the proof of 5.7.

We now want to show that 5.7 eliminates the possibility of concentration of \( L^p \) norm in regions of small measure. Specifically, we have the following corollary.

**5.8 Corollary.** *If the hypotheses are as in 5.7, then there are constants \( C = C(\theta, p, \mathcal{P}, \alpha, \beta) > 0 \) and \( \delta = \delta(\mathcal{P}, \theta, p, \alpha, \beta) \in (0, 1) \) such that*
\[ \|u\|_{L^p(M \cap B_{\rho}(y))} \leq C \|u\|_{L^1(M \cap B_{\rho}(y) \setminus \Omega_{\delta})} \]
*whenever \( \Omega_{\delta} \subset M \cap B_{\rho}(y) \) has \( \mathcal{H}^n \)-measure less than \( \delta \rho^n \).*

**Proof:** By change of variable \( x \mapsto \eta_{y, \rho}(x) \) we reduce to the case \( y = 0, \rho = 1 \), so \( \tilde{w} \equiv 1 \). If \( \Omega_{\delta} \) is any subset of \( M \cap B_1 \) of \( \mathcal{H}^n \)-measure less than \( \delta/2 \), where \( \delta \) is as in 2.5, and if for \( K > 1 \) we let \( A_K = \{ x \in M \cap B_1 \setminus \Omega_{\delta} : u(x) > K \|u\|_{L^1(M \cap B_1 \setminus \Omega_{\delta})} \} \), then
\[ \|u\|_{L^1(M \cap B_1 \setminus \Omega_{\delta})} \geq \|u\|_{L^1(A_K)} \geq K \mathcal{H}^n(A_K) \|u\|_{L^1(M \cap B_1 \setminus \Omega_{\delta})}, \]
so that $\mathcal{H}^n(A_K) \leq K^{-1}$ and hence with the choice $K = 2/\delta$ we get $\mathcal{H}^n(A_K) < \delta/2$. Thus with this $K$ we have

$$\mathcal{H}^n(\{x \in M \cap B_1 : u(x) > K\|u\|_{L^1(M \cap B_1 \setminus \Omega_\delta)}\})$$

$$\leq \mathcal{H}^n(\{x \in M \cap B_1 \setminus \Omega_\delta : u(x) > K\|u\|_{L^1(M \cap B_1 \setminus \Omega_\delta)}\}) + \mathcal{H}^n(\Omega_\delta)$$

$$= \mathcal{H}^n(A_K) + \mathcal{H}^n(\Omega_\delta) \leq \delta/2 + \delta/2 = \delta,$$

whence we can apply 5.7 with $\lambda = K\|u\|_{L^1(M \cap B_1 \setminus \Omega_\delta)}$, and this gives the required result.

6 Proof of Theorem 1

Suppose the theorem is false for some given $\alpha, \beta > 0$, $p \in \left[1, \frac{n}{n-2}\right)$, classes $\mathcal{P}, \mathcal{C}_0 \subset \mathcal{C}$ and $\gamma < \gamma_0$. Then for each choice of $\rho \in (0, \frac{1}{4})$ the theorem fails, so there is $\tau_k \downarrow 0$, $M_k \in \mathcal{P}$ with $U_{M_k} \supset B_{3/2} \setminus B_{\tau_k}$, $\mathcal{C}_k \subset \mathcal{C}_0$ such that 1.1–1.11 all hold with $M_k, \mathcal{C}_k, u_k, q_k, a_k, b_k, \tau_k$ in place of $M, \mathcal{C}, u, q, a, b, \tau$ respectively, yet such that

$$(1) \quad \|u_k\|_{L^p(M_k \cap B_\rho \setminus B_{\rho/2})} < \rho^{-7}\|u_k\|_{L^p(M_k \cap B_1 \setminus B_{1/2})}. $$

Thus

$$d(M_k \cap B_{3/2} \setminus B_{\tau_k}, \mathcal{C}_k \cap B_{3/2} \setminus B_{\tau_k}) \to 0$$

and by compactness of $\mathcal{E}_0$ we can pass to a subsequence and select $\mathcal{C} \in \mathcal{C}_0$ with

$$d(\mathcal{C}_k \cap B_{3/2}, \mathcal{C} \cap B_{3/2}) \to 0.$$

Let

$$\Sigma = \mathcal{C} \cap S^{N-1}.$$ 

Since $\Sigma$ has only finitely many connected components and the convergence of $M_k$ to $\mathcal{C}$ is in the $C^1$ sense of 1.4 near compact subsets of $\mathcal{C} \cap B_{3/2}$ we can use the classical Harnack theory of De Giorgi Nash Moser (i.e. [GT83, §8.8–§8.10]) applied locally to the solutions $u_k$ on $M_k$, and the local $W^{1,2}$ estimates of 5.3 for $u^p$ ($p < 1$), together with the Rellich compactness theorem, to assert that, for sufficiently small $\delta_0 = \delta_0(\Sigma) > 0$, a subsequence of the normalized sequence

$$\tilde{u}_k = \left\|u_k\right\|^{-1}_{L^1(\{x \in M_k \cap B_1 \setminus B_{1/2} : \text{dist}(x, \text{sing } \mathcal{C}) \geq \delta_0\})} u_k$$

converges (in the sense discussed in the remark following 1.4) strongly in $L^p$ for $p < \frac{n}{n-2}$, on compact subsets of $\mathcal{C} \cap B_{3/2}$ to a non-negative $u \in W^{1,2}_{\text{loc}}(\mathcal{C} \cap B_{3/2})$ which satisfies $u > 0$ on at least one connected component of $\mathcal{C}$, and

$$(2) \quad \Delta_{\mathcal{C}} u + r^{-2} q_{\mathcal{C}} u \leq 0$$

weakly on $\mathcal{C} \cap B_{3/2}$, where $r^{-2} q_{\mathcal{C}}$ is the uniform limit of $q_k$ on compact subsets of $\mathcal{C}$. Furthermore, by the inequality on the right of 2.1 and the fact that $\mathcal{H}^{n-2}(\text{sing } \mathcal{C} \cap B_{3/2}) < \infty$ there is $\delta(\tau) \downarrow 0$ as $\tau \downarrow 0$, with $\delta(\tau)$ not depending on $k$, such that

$$\mathcal{H}^n(M_k \cap \{x \in B_{3/2} \setminus B_{p/4} : \text{dist}(x, \text{sing } \mathcal{C}) \geq \tau\}) < \delta(\tau).$$
Then the partial Harnack theory (in particular Corollary 5.8) is applicable, ensuring that in fact we have the $L^1$ norm convergence

\[ \| \tilde{u}_k \|_{L^1(\mathbb{R}^n \setminus B_{2R}^\star)} \to \| u \|_{L^1(\mathbb{R}^n \setminus B_{2R}^\star)} \]

for each $R \in (\frac{3}{2}, 1)$ (and in particular this holds with $R = 1$ and $R = \rho$).

Observe that in view of (2) we can apply 4.3 with each $\tau > 0$ and so

\[ \lambda_1(\mathcal{E}_0) \geq \lambda_1(\mathbb{S}) \geq -\left( \frac{n-2}{2} \right)^2 \]

and hence $\gamma_0$ (in the statement of Theorem 1) is the smaller root of the quadratic equation $t^2 - (n-2)t - \lambda_1(\mathcal{E}_0)$. Thus if we take

\[ (4) \quad \mu = \frac{1}{2}(\gamma + \gamma_0), \]

then we have

\[ (5) \quad \gamma + C < \mu < \gamma_0 - C \quad \text{and} \quad \mu^2 - (n-2)\mu - \lambda_1(\mathcal{E}_0) \geq C, \]

with $C = C(n, \gamma, \mathcal{E}_0) > 0$.

Now let $\varphi_\delta, \lambda_{1,\delta}(\mathbb{S})$, and $\Omega_\Sigma \subset \mathbb{S}$ be as in 4.7. Notice that the weak form of (2) on $C \cap \tilde{B}_{3/2}$ is

\[ (6) \quad \int_0^{3/2} \int_\mathbb{S} \left( -u' \zeta' - r^{-2} \nabla u \cdot \nabla \zeta + r^{-2} q u \zeta \right) r^{n-1} \, dr \, d\omega \leq 0 \]

for any $\zeta \in C^\infty_c(\mathbb{S} \cap \tilde{B}_{3/2})$ with $\zeta \geq 0$, where $u'$ means $\frac{2u}{\partial r}$. Replacing $\zeta$ by $\zeta_1(\cdot)\varphi_\delta$ in (6) gives

\[ \int_0^{3/2} \int_\mathbb{S} \left( -u' \zeta_1' \varphi_\delta - r^{-2} \zeta_1 \nabla u \cdot \nabla \varphi_\delta + r^{-2} q u \zeta_1 \varphi_\delta \right) r^{n-1} \, dr \, d\omega \leq 0. \]

Using inequality 4.7 with $v(\omega) = u(\omega r)$ on the left, and writing $v_0(r) = \langle u(r), \varphi_\delta \rangle_{L^2(\Omega_\Sigma)}$, we then conclude

\[ - \int_0^{3/2} \zeta_1 v'_\delta r^{n-1} \, dr - \lambda_{1,\delta}(\Sigma) \int_0^{3/2} \zeta_1 v_\delta r^{n-3} \, dr \leq 0. \]

That is, weakly $v_\delta$ satisfies

\[ (7) \quad r^{1-n}(r^{n-1}v'_\delta)' - \lambda_{1,\delta}(\Sigma)r^{-2}v_\delta \leq 0, \quad r \in (0, 3/2) \]

Now with $\mu$ as in (5) let $w_\delta = r^\mu v_\delta$. Then $v_\delta = r^{-\mu}w_\delta$ and so

\[ v'_\delta = -\mu r^{-\mu-1}w_\delta + r^\mu w'_\delta, \quad v''_\delta = \mu(\mu + 1)r^{-\mu-2}w_\delta - 2\mu r^{-\mu-1}w'_\delta + r^{-\mu}w''_\delta. \]

Then substituting in the differential inequality (7) we get

\[ r^{-2}(\mu^2 - (n-2)\mu - \lambda_{1,\delta}(\Sigma))w_\delta + (n-1-2\mu)r^{-1}w'_\delta + w''_\delta \leq 0. \]

Since $\mu^2 - (n-2)\mu - \lambda_{1,\delta}(\Sigma) > 0$ (by (5)), we see that the previous implies

\[ r^{-(1+\beta)}(r^{1+\beta}w'_\delta)' \leq 0 \quad \text{(weakly for } r \in (0, 3/2)), \]

where $\beta = \frac{1}{2} - \frac{n}{2}$. This completes the proof of the asymptotic decay lemma.
where $\beta = n - 2 - 2\mu > 0$. This says that $w_\delta$ is a concave function of the new variable $s = r^{-\beta} \in ((3/2)^{-\beta}, \infty)$, and since $w_\delta$ is non-negative we see that then $w_\delta$ must be increasing with respect to the variable $s$; that is $w_\delta$ is a decreasing function of the variable $r$, so that

$$w_\delta(r_1) \geq w_\delta(r_2) \text{ for } 0 < r_1 < r_2 < 3/2,$$

which in terms of $u$ says

$$r_1^\mu \int_{\Sigma} u(r_1 \omega) \varphi_\delta(\omega) \geq r_2^\mu \int_{\Sigma} u(r_2 \omega) \varphi_\delta(\omega) \text{ for } 0 < r_1 < r_2 < 3/2.$$

Integrating over $(1/2, 1)$ with respect to the variable $r_1$ and over $(\rho/2, \rho)$ with respect to the variable $r_2$, we then conclude that

$$\rho^{-n} \int_{B_{\rho}\setminus B_{\rho/2}} u \varphi_\delta \geq 2^{-\mu-n} \int_{B_1\setminus B_{1/2}} u \varphi_\delta,$$

so that by 4.8 and 5.8 (applied with $M = C, p = 1$ and $a = b = 0$) we have

$$\rho^{-n} \int_{B_{\rho}\setminus B_{\rho/2}} u \geq C \rho^{-\mu} \int_{B_1\setminus B_{1/2}} u,$$

where $C = C(\gamma, \mathcal{P}, \mathcal{E}_0, \mathcal{Q}_0, \alpha, \beta)$ and provided $\delta = \delta(\gamma, \mathcal{P}, \mathcal{E}_0, \mathcal{Q}_0, \alpha, \beta) > 0$ is chosen suitably, and hence by the norm convergence (3) we have

$$\rho^{-n} \int_{B_{\rho}\setminus B_{\rho/2}} u_k \geq C \rho^{-\mu} \int_{B_1\setminus B_{1/2}} u_k$$

for all sufficiently large $k$, where $C = C(\gamma, \mathcal{P}, \mathcal{C}_0, \mathcal{Q}_0, \alpha, \beta)$, contradicting (1) in the case $p = 1$ for sufficiently small $\rho$ (depending only on $\gamma, \mathcal{P}, \mathcal{C}_0, \mathcal{Q}_0, \alpha, \beta$), by (5). This completes the proof in the case $p = 1$.

To handle the remaining $p \in (1, \frac{n}{n-2})$, observe that we could have integrated with respect to $r_2$ over $(1/4, 5/4)$ and also we can apply the Hölder inequality on the left side of (8), whence

$$\|u_k\|_{L^p(M_k \cap B_{\rho/2} \setminus B_{\rho/2})} \geq C \rho^{-\mu} \int_{M_k \cap B_{\rho/4} \setminus B_{1/4}} u_k$$

for any $p \in [1, \frac{n}{n-2})$. Then by applying inequality (8) in the proof of Lemma 5.7 (with a suitably scaled version of $M_k$ in place of $M$) we conclude

$$\|u_k\|_{L^p(M_k \cap B_{\rho/2} \setminus B_{\rho/2})} \geq C \rho^{-\mu} \|u_k\|_{L^p(M_k \cap B_{1/2} \setminus B_{1/2})}$$

for any $p \in [1, \frac{n}{n-2})$, with $C = C(p, \gamma, \mathcal{P}, \mathcal{C}_0, \mathcal{Q}_0, \alpha, \beta)$, which again contradicts (1) for sufficiently large $k$.

7 Application to growth estimates for exterior solutions

In this section we want to show how the main theorem applies to give lower growth estimates for entire and exterior solutions of the minimal surface equation. Thus we
assume that $u$ is $C^2$ and satisfies the minimal surface equation

\[ 7.1 \sum_{i=1}^{n} D_{i}\left(\frac{D_{i}u}{\sqrt{1+|Du|^2}}\right) = 0 \text{ in } \mathbb{R}^n \setminus B_1. \]

We need the non-trivial general facts given in the following two theorems:

**7.2 Theorem.** There is a regular multiplicity 1 class $\mathcal{P}$ (as in §1) with $N = n + 1$ such that $\mathcal{P}$ contains each minimal graph $G = \text{graph } u$, corresponding to a solution $u \in C^2(\Omega)$ of 7.1, where $\Omega$ is any open set in $\mathbb{R}^n$; in this case we always have that $U_G$ (the open set associated with $G \in \mathcal{P}$ as in §1) is just $\Omega \times \mathbb{R}$. Further the class $\mathcal{P}$ can be chosen so that the convergence $P_{\kappa_k} \to P$ of 1.4 is actually $C^\infty$ (i.e. $C^k$ for each $k$) on compact subsets of $P$ rather than merely $C^1$.

**Proof:** This follows from the De Giorgi theory of oriented boundaries of least area (also known as area minimizing hypersurfaces). For a clear exposition of this theory we refer to the book of Giusti [Giu83].

**7.3 Theorem.** If $G = \text{graph } u$, where $u \in C^2(\mathbb{R}^n \setminus B_1)$ satisfies the MSE on $\mathbb{R}^n \setminus B_1$, then $G$ is asymptotically conic in the sense of 1.12(b); that is for each sequence $\rho_k \to \infty$ there is a subsequence $\rho_{k_\ell}$ and a cone $C \in \mathcal{P}$ such that $\gamma_{0,\rho_{k_\ell}} G \to C$ in $\mathbb{R}^{n+1} \setminus \{0\}$ in the sense of 1.4. In fact in this case we have always that $C$ is cylindrical: $C = C_0 \times \mathbb{R}$ for some $(n-1)$-dimensional area minimizing cone $C_0 \subset \mathbb{R}^n$.

**Proof:** For the proof of this in the case when $u$ is an entire solution (i.e. when $u$ is defined over all of $\mathbb{R}^n$), see for example [Mir77] and [Giu83]; the proof for exterior solutions requires only a little more argument, and is described in [Sim87].

Next we recall that if $\nu = (\nu^1, \ldots, \nu^{n+1}) = \frac{(-Du_1, \ldots, -Du_n)}{\sqrt{1+|Du|^2}}$ is the upward pointing unit normal of $G$ (thought of as a function on $G$ rather a function in $\mathbb{R}^n \setminus B_1$), then $\nu^{n+1} \equiv e_{n+1} \cdot \nu$ satisfies the Jacobi-field equation

\[ 7.4 \Delta_G u + q_G u = 0, \]

where $q_G = |A_G|^2$ is the square length of the second fundamental form of $G$. Of course by the $C^\infty$ convergence of $G_{k_\ell}$ to $C = C_0 \times \mathbb{R}$, we trivially have that $\rho_{k_\ell}^2 q_{G_{k_\ell}} \to q_C$ uniformly on compact subsets of $C$, where $q_C = |A_C|^2 \equiv |A_{C_0}|^2$ is the square length of the second fundamental form of $C$, or, equivalently, $C_0$. Thus all the conditions for the application of the Theorem 2 of §1 do hold (with $a = 0, b = 0$ and $q = |A_G|^2$ in this case), and hence we conclude that

\[ R^{-n}||u||_{L^2(G \cap B_k \setminus B_{R/2})} \leq R^{-\gamma} \]

for all sufficiently large $R$ where $\gamma$ is any number less than $\gamma_0$, where $\gamma_0$ is inf $\gamma(\Sigma)$, where $\gamma(\Sigma) = \frac{2n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\Sigma)}$, and the inf is over all $\Sigma = C \cap S^n$ corresponding to all possible tangent cylinders $C = C_0 \times \mathbb{R}$ of $G$ at $\infty$; as in 1.8, $\lambda_1(\Sigma)$ is the first eigenvalue of the operator $-(\Delta_G + q_G)^\dagger$ in case $q_C$ is the square length of the second fundamental form of $\Sigma$.

\[ ^3 \text{Actually in 1.8 we defined } \lambda_1(\Sigma) \text{ to be maximum of } \lambda_1(\Sigma_*) \text{ over all the (finitely many) connected components } \Sigma_* \text{ of } \Sigma, \text{ but in this case by a result of Bombieri and Giusti } [BG72] \text{ we automatically have that } \Sigma \text{ is connected.} \]
We actually claim that the exponent \( \gamma_0 \) above can be computed in terms of the first eigenvalues of the cross-sectional cones as follows:

**Lemma.** \( \gamma(\Sigma) = \gamma(\Sigma_0) \), where \( \Sigma = (\mathbb{C} \times \mathbb{R}) \cap S^n \), \( \Sigma_0 = \mathbb{C}_0 \cap S^{n-1} \), \( \gamma(\Sigma) = \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 + \lambda_1(\Sigma)} \) and \( \gamma(\Sigma_0) = \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 + \lambda_1(\Sigma_0)} \).

**Proof:** For \( \delta > 0 \), select a \( C^\infty \) relatively open \( \Omega_\delta \subset \Sigma_0 \) with \( \{ \omega \in \Sigma_0 : \text{dist}(\omega, \text{sing} \Sigma_0) > \delta \} \subset \Omega_\delta \subset \Sigma_0 \subset \Sigma_\delta \) and let \( \varphi_\delta \) be the (smooth and positive) first eigenfunction for the operator \( -(\Delta_{\Sigma_0} + q_{\Sigma_0}) \) with zero Dirichlet data, i.e., \( \varphi_\delta = 0 \) on \( \partial \Omega_\delta \). Then \( \lambda_1(\Omega_\delta) \to \lambda_1(\Sigma_0) \) as \( \delta \to 0 \) and \( \lambda_1(\Omega_\delta) > \lambda_1(\Sigma_0) \) for every \( \delta > 0 \) and so for \( \delta \) sufficiently small we have \( -(\frac{n-3}{2})^2 < \lambda_\delta < 0 \), and in particular this means that \( \gamma^2 - (n-3)\gamma - \lambda_\delta \) has real roots, with smaller root \( \gamma_\delta \), \( \gamma_\delta = \frac{n-3}{2} - \sqrt{\left(\frac{n-3}{2}\right)^2 + \lambda_\delta} \), satisfying

\[
\gamma_\delta < \frac{n-3}{2} \quad \text{and} \quad \gamma_\delta^2 - (n-3)\gamma_\delta = \lambda_1(\Sigma_0).
\]

For \((x, y) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \), let \( r_0 = |x| \) and \( r = \sqrt{|x|^2 + y^2} \), and let

\[
\Lambda_\delta = (\mathbb{C}_0, \times \mathbb{R}) \cap S^n,
\]

where

\[
\mathbb{C}_0, \delta = \{ r\omega : r > 0, \omega \in \Omega_\delta \} \subset \mathbb{C}_0.
\]

Since \( \Delta_{\mathbb{C}_0} = r_0^{-2} \frac{\partial}{\partial r_0} (r_0^{-2} \frac{\partial}{\partial r_0}) + \frac{1}{r_0^2} \Delta_{\mathbb{C}_0} \), by direct computation we see that \( v_\delta \equiv r_0^{-\gamma_\delta} \varphi_\delta \) satisfies the equation

\[
\Delta_{\mathbb{C}_0} v_\delta + r_0^{-2} q_{\Sigma_0} v_\delta = 0
\]

and hence of course it also is a solution (independent of the \( y \)-variable) of the equation \( \Delta_{\mathbb{C}} v_\delta + r_0^{-2} q_{\Sigma_0} v_\delta = 0 \). Indeed since \( v_\delta \) can be written \( r^{-\gamma_\delta} (r/r_0)^{\gamma_\delta} \varphi_\delta \) and since the Laplacian on \( \mathbb{C} \) has the form \( r^{n-1} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \right) + \frac{1}{r} \Delta_{\mathbb{C}} \), we see that \( \Phi_\delta \equiv (r/r_0)^{\gamma_\delta} \varphi_\delta \) is a solution of the equation

\[
-(\Delta + q_{\Sigma}) \Phi_\delta = (\gamma_\delta^2 - (n-2)\gamma_\delta) \Phi_\delta \quad \text{on} \quad \Lambda_\delta,
\]

where \( q_{\Sigma} = r_0^{-2} q_{\Sigma_0} \). Also since \( \varphi_\delta \) is \( C^1 \) and \( \gamma_\delta < \frac{n-3}{2} \) it is readily checked that \( \Phi_\delta \in W^{1,2}_0(\Lambda_\delta) \). Then we can use a standard argument to assert that

\[
(\gamma_\delta^2 - (n-2)\gamma_\delta) = \lambda_1(\Lambda_\delta)
\]

where

\[
\lambda_1(\Lambda_\delta) = \inf_{v \in C^1_0(\Lambda_\delta) \setminus \{0\}} \frac{\|v\|^2_{L^2(\Lambda_\delta)}}{\int_{\Lambda_\delta} (|\nabla \Sigma v| - q_{\Sigma} v^2)},
\]

as follows. First note that because \( \Phi_\delta \in W^{1,2}_0(\Lambda_\delta) \) we can multiply by \( \Phi_\delta \) in (1) and integrate by parts, thus showing that

\[
\gamma_\delta^2 - (n-2)\gamma_\delta \geq \lambda_1(\Lambda_\delta).
\]

Take any smooth subdomain \( \Omega \subset \subset \Lambda_\delta \) and let \( \lambda_1(\Omega) \) be the minimum eigenvalue for \( -(\Delta + q_{\Sigma}) \) on \( \Omega \). We claim that then \( \lambda_1(\Omega) > \lambda = \gamma_\delta^2 - (n-2)\gamma_\delta \), because if \( \lambda_1(\Omega) \leq \lambda \) then, with \( \varphi \in C^1_0(\Omega) \) the positive smooth eigenfunction corresponding to the eigenvalue \( \lambda_1(\Omega) \), we would have

\[
\Delta_{\Sigma}(\Phi_\delta - \mu \varphi) + (q_{\Sigma} + \lambda)(\Phi_\delta - \mu \varphi) \leq 0 \quad \text{on} \quad \Omega
\]
for \( \mu > 0 \), and we could take \( \mu > 0 \) such that \( \Phi_{\delta} - \mu \varphi \) has a zero minimum in \( \Omega \), contradicting the maximum principle. Thus we must have \( \lambda_1(\Omega) > \lambda \) for all such \( \Omega \). Since \( \lambda_1(\Lambda_\delta) = \inf \lambda_1(\Omega) \) over all such \( \Omega \) we thus have \( \lambda_1(\Lambda_\delta) \geq \lambda \) and hence (by (4)) we must have (3) as claimed.

Then, by (1) and (3), \( \gamma_\delta \) is the (smaller) root of both the equation \( \gamma^2 - (n - 2)\gamma - \lambda_1(\Lambda_\delta) = 0 \) and also the equation \( \gamma^2 - (n - 3)\gamma - \lambda_1(\Omega_\delta) = 0 \), so that 7.5 follows by letting \( \delta \downarrow 0 \).

Finally we want to establish the bound \( \lambda_1(\Sigma_0) \leq -(n - 2) \) mentioned above:

7.6 Lemma. With \( \mathcal{C} = \mathbb{C}_0 \times \mathbb{R} \) any tangent cone at \( \infty \) for \( G \) and with \( \Sigma_0 = \mathbb{C}_0 \cap S^{n-1} \), we have \( \lambda_1(\Sigma_0) \leq -(n - 2) \).

Recall that by the identity of James Simons [Sim68] (see also [SSY75]) \( \Delta_{\Sigma_0}|A_{\Sigma_0}| + q_{\Sigma_0}|A_{\Sigma_0}| \geq (n - 2)|A_{\Sigma_0}| \), and this suggests that we should try to use 4.9. (Notice that the coefficient of the term on right side is indeed \( n - 2 \), because \( n - 2 \) is the dimension of \( \Sigma_0 \).) To make it possible to apply 4.9 we need also to recall that by [SSY75] we have the estimates

\[
\int_{\mathbb{C}_0} |A_{\mathbb{C}_0}|^p f^p \leq C \int_{\mathbb{C}_0} |\nabla f|^p
\]

for \( p \in [2, 4 + \sqrt{8/n}] \), provided \( f \in \mathcal{C}_C^\infty(\mathbb{C}_0) \). (Notice that the work of [SSY75] formally assumed sing \( \mathbb{C}_0 = \{0\} \), but the proof of course works without change in the case of an arbitrary singular set, so long as we assume, as we do here, that support \( f \) is a compact subset of the smooth manifold \( \mathbb{C}_0 \).) Now recall that in the present area minimizing case (see e.g. [Giu83], keeping in mind that the dimension of \( \mathbb{C}_0 \) is \( n - 1 \)) we have that dim \( \text{sing} \mathbb{C}_0 \leq n - 8 \), so that in particular

\[
\mathcal{H}^{n-5}(\text{sing} \mathbb{C}_0) = 0, \quad \text{hence} \quad \mathcal{H}^{n-6}(\text{sing} \Sigma_0) = 0.
\]

In view of (1) (with \( p = 4 \)) and (2) we can use precisely the same argument as in §4 (preceding 4.9) in order to deduce that (1) is also valid for any \( f \in \mathcal{C}_C^\infty(\overline{B_R \setminus B_\tau}) \) for any \( 0 < \tau < R < \infty \). Hence we conclude from (1) that \( A_{\mathbb{C}_0} \in L^4(\mathbb{C} \cap (B_2 \setminus B_{1/2})) \), and hence \( A_{\Sigma_0} \in L^4(\Sigma_0) \). Therefore we can apply 4.9 to conclude that

\[
\lambda_1(\Sigma_0) \leq -(n - 2)
\]

as claimed.

Thus with \( \mathcal{E}_0 = \{\mathbb{C}_0 \cap S^{n-1} : \mathbb{C}_0 \times \mathbb{R} \text{ is a tangent cylinder of } G \text{ at } \infty\} \), we have

\[
\lambda_1(\mathcal{E}_0) \leq -(n - 2)
\]

and hence by Lemma 7.5 we can apply the main decay estimate of Theorem 2 in §1 with \( \gamma_0 = \frac{n-3}{n-2} - \sqrt{\left(\frac{n-3}{n-2}\right)^2 - (n - 2)} \) to the solution \( u = \nu_{n+1} \) (as in 7.4), whence we conclude that for each \( \gamma < \gamma_0 \) there is a constant \( \rho_0 > 1 \) with

\[
\int_{S_R \setminus S_{R/2}} \nu_{n+1} \leq R^{-\gamma}, \quad R \geq \rho_0.
\]
On the other hand, by the Hölder inequality and the standard lower bound on the volume of the intersection of the graph $G$ with an $(n+1)$-dimensional ball we have

$$CR^n \leq \mathcal{H}^n(S_R \setminus S_{R/2})$$

$$\leq \int_{S_R \setminus S_{R/2}} \nu_{n+1} d\mathcal{H}^n \int_{S_R \setminus S_{R/2}} \sqrt{1 + |Du|^2} \ d\mathcal{H}^n$$

$$\leq CR^{-\gamma} \int_{S_R \setminus S_{R/2}} \sqrt{1 + |Du|^2} \ d\mathcal{H}^n,$$

whence

$$R^{-\gamma} \int_{S_R} |Du| \ d\mathcal{H}^n \geq CR^{-\gamma}, \ R \geq \rho_0,$$

as claimed in the introduction.

References

[All72] W. Allard, *On the first variation of a varifold*, Annals of Math. **95** (1972), 417–491.

[All75] ——, *On the first variation of a varifold—boundary behavior*, Annals of Math. **101** (1975), 418–446.

[Alm66] F. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein’s theorem*, Annals of Math. **84** (1966), 277–292.

[Alm00] ——, *Almgren’s big regularity paper. q-valued functions minimizing dirichlet’s integral and the regularity of area-minimizing rectifiable currents up to codimension 2*, World Scientific Monograph Series in Mathematics **1** (2000).

[AS88] D. Adams and L. Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana Univ. Math. J. **37** (1988), 225–254.

[BDG69] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.

[BDM69] E. Bombieri, E. De Giorgi, and M. Miranda, *Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche*, Arch. Rat. Mech. Anal. **32** (1969), 255–267.

[Ber51] L. Bers, *On isolated singularities of minimal surfaces*, Annals of Math. **53** (1951), 417–491.

[BG72] E. Bombieri and E. Giusti, *Harnack’s inequality for elliptic differential equations on minimal surfaces*, Invent. Math. **15** (1972), 24–46.

[Cha88] Sheldon X. Chang, *Two-dimensional area minimizing integral currents are classical minimal surfaces*, J. Amer. Math. Soc. **1** (1988), 699–778.

[CNS90] L. Caffarelli, L. Nirenberg, and J. Spruck, *On a form of Bernstein’s theorem*, Analyse mathématique et applications **55–56** (1990), .
A general asymptotic decay lemma

[DG61] E. De Giorgi, *Frontiere orientate di misura minima.*, Seminario Ann. Scuola Norm. Sup. Pisa (1961).

[EH90] K. Ecker and G. Huisken, *A Bernstein result for minimal graphs of controlled growth*, J. Differential Geom. 31 (1990), 397–400.

[Giu83] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston, 1983.

[GT83] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[HS79] R. Hardt and L. Simon, *Boundary regularity and embedded solutions of plateau’s problem*, Annals of Math. 110 (1979), 439–486.

[Mir77] M. Miranda, *Superficie minime illimitate*, Ann. Scuola Norm. Sup. Pisa, Ser. IV 4 (1977), 313–322.

[Nit89] J. Nitsche, *Lectures on minimal surfaces (vol 1)*, Cambridge University Press, 1989.

[Rei60] R.É. Reifenberg, *Solution of the plateau problem for m-dimensional surfaces of varying topological type*, Acta Math. 104 (1960), 1–92.

[Sim68] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. 88 (1968), 62–105.

[Sim83] L. Simon, *Asymptotics for a class of evolution equations, with applications to geometric problems*, Annals of Mathematics 118 (1983), 525–571.

[Sim87] , *Asymptotic behaviour of minimal graphs over exterior domains*, Ann. H. Poincaré 4 (1987), 231–242.

[Sim89] , *Entire solutions of the minimal surface equation*, J. Differential Geom. 30 (1989), 643–688.

[Sim93] , *Cylindrical tangent cones and the singular set of minimal submanifolds*, J. Differential Geom. 38 (1993), 585–652.

[Sim95a] , *Rectifiability of the singular set of energy minimizing maps*, Calculus of Variations and PDE 3 (1995), 1–66.

[Sim95b] , *Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy minimizing maps*, Surveys in differential geometry 2 (1995), 246–305.

[Sim08a] , *Lower growth estimates for solutions of the minimal surface equation*, In Preparation (2008).

[Sim08b] , *The symmetric minimal surface equation*, In Preparation (2008).

[SS81] R. Schoen and L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure and Appl. Math 34 (1981), 741–797.
| Reference | Author(s) | Title | Journal | Volume | Year | Pages |
|-----------|----------|-------|---------|--------|------|-------|
| [SSY75]   | R. Schoen, L. Simon, and S.-T. Yau | Curvature estimates for minimal hypersurfaces | Acta Math. | 134  | 1975 | 276–288 |
| [Tay76]   | Jean E. Taylor | The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces | Ann. of Math. (2) | 103  | 1976 | 489–539 |
| [Whi83]   | Brian White | Tangent cones to two-dimensional area-minimizing integral currents are unique | Duke Math. J. | 50  | 1983 | 143–160 |
| [Whi92]   | Brian White | Nonunique tangent maps at isolated singularities of harmonic maps | Bull. Amer. Math. Soc. (N.S.) | 26  | 1992 | 125–129 |