On the minimum distance between masses of relative equilibria of the $n$-body problem

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January 15, 2014

Abstract

We prove that if for relative equilibrium solutions of a generalisation of the $n$-body problem of celestial mechanics the masses and rotation are given, then the minimum distance between the point masses of such a relative equilibrium has a universal lower bound that is not equal to zero. We furthermore prove that the set of such relative equilibria is compact.

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1 Introduction

By the $n$-body problem we mean the problem of deducing the dynamics of $n$ point masses with time dependent coordinates $q_1, \ldots, q_n \in \mathbb{R}^k$, $k \geq 2$ and respective masses $m_1, \ldots, m_n$ as described by the system of differential equations

$$\ddot{q}_i = \sum_{j=1, j \neq i}^n m_j (q_j - q_i) \|q_j - q_i\|^{2a}, \quad n \geq 3, \quad a < -\frac{1}{2}. \quad (1.1)$$

If $a = -\frac{3}{2}$ and $k = 3$, then we speak of the classical $n$-body problem. We call any solution to such a problem where the $q_1, \ldots, q_n$ describe a rotating configuration of points a relative equilibrium and the set of all such configurations that are equivalent under rotation and scalar multiplication a class of relative equilibria.

Steve Smale conjectured on his famous list (see [12]) after Wintner (see [15]) that for the classical case, if the equilibria are induced by a plane rotation, the number of classes of relative equilibria is finite, if the masses $m_1, \ldots, m_n$ are given. This problem is still open for $n > 5$ and was solved for $n = 3$ by A. Wintner (see [15]), $n = 4$ by M. Hampton and R. Moeckel (see [5]) and for $n = 5$ by A. Albouy and V. Kaloshin (see [3]). Results on the finiteness of subclasses of relative equilibria can be found in [6], [7], [8] and [9]. G. Roberts showed in [10] that for the classical five-body problem, if one of the masses is negative, a continuum of relative equilibria exists. As a potential step towards a proof of Smale’s problem, M. Shub showed in [11] that the set of all classes of relative equilibria, provided they have the same set of masses, is compact. Moreover, Shub proved, again in [11], that if the rotation inducing the equilibria is given as well, then there exists a universal nonzero, minimal distance that the point masses lie apart from each other. For further background information and a more detailed overview regarding Smale’s problem, see [1], [5], [14] and [10] and the references therein.

In this paper, as a logical next step after Shub’s work in [11], we prove Shub’s results when using (1.1) instead of the classical $n$-body problem. Specifically, we prove that

**Theorem 1.1.** Consider the set $R_{A,m_1,\ldots,m_n}$ of all relative equilibria with rotation matrix $T_k(\vec{A}t)$ and masses $m_1, \ldots, m_n$ (see Definition [2.1]). Then there
exists a constant $c \in \mathbb{R}^0$ such that for all relative equilibria $\{T_k(\overrightarrow{At})Q_i\}_{i=1}^n$ in the set $R_{A,m_1,...,m_n}$, we have that $\|Q_i - Q_j\| > c$ for all $i, j \in \{1,...,n\}, i \neq j$.

and consequently that

**Corollary 1.2.** Consider the set $R_{A,m_1,...,m_n}$ of all relative equilibria with rotation matrix $T_k(\overrightarrow{At})$ and masses $m_1,..., m_n$ (see Definition 2.1). Then there exists a $C \in \mathbb{R}^0$ such that for all relative equilibria $\{T_k(\overrightarrow{At})Q_i\}_{i=1}^n$ in the set $R_{A,m_1,...,m_n}$, we have that $\|Q_i\| < C$ for all $i \in \{1,...,n\}$.

In order to prove Theorem 1.1 and Corollary 1.2 we will first formulate needed definitions, a criterion for relative equilibria and a lemma related to relative equilibria, which will be done in section 2. Then we will prove Theorem 1.1 in section 3 and Corollary 1.2 in section 4.

## 2 Background Theory

Before getting to the lemma that will form the backbone of our theorems, we will have to formulate a criterion, for which we will have to adopt the following definition:

Let

$$T(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and define for any $p$-dimensional vector-valued function

$$\overrightarrow{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_p \end{pmatrix}$$
a $k \times k$ diagonal block matrix $T_k(\vec{\theta})$ as

$$T_k(\vec{\theta}) = \begin{cases} 
\begin{pmatrix}
T(\theta_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & T(\theta_p)
\end{pmatrix} & \text{if } k = 2p, \\
\begin{pmatrix}
T(\theta_1) & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & T(\theta_p) & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix} & \text{if } k = 2p + 1
\end{cases}. $$

Then

**Definition 2.1.** Let $n \in \mathbb{N}$, let $q_i, i \in \{1, \ldots, n\}$ solve (1.1), let

$$\vec{A} = \begin{pmatrix}
A_1 \\
\vdots \\
A_p
\end{pmatrix} \in \mathbb{R}_{>0}^p,$$

$Q_1, \ldots, Q_n \in \mathbb{R}^k$ and let $q_i(t) = T_k(\vec{A}t)Q_i, i = 1, \ldots, n$. Then we say that $q_1, \ldots, q_n$ form a *relative equilibrium* with rotation matrix $T_k(\vec{A}t)$.

Inserting the expressions for $q_1, \ldots, q_n$ as described in Definition 2.1 into (1.1) and using that for any $x \in \mathbb{R}^k, \|T_k(\vec{A}t)x\| = \|x\|$ and that $(T_k(\vec{A}t))'' = -A^2T_k(\vec{A}t)$ where $A$ is the diagonal matrix

$$A = \begin{cases} 
\begin{pmatrix}
A_1 & 0 & \cdots & 0 & 0 \\
0 & A_1 & \cdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A_p & 0 \\
0 & \cdots & 0 & 0 & A_p
\end{pmatrix} & \text{if } k = 2p, \\
\begin{pmatrix}
A_1 & 0 & \cdots & 0 & 0 \\
0 & A_1 & \cdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A_p & 0 \\
0 & \cdots & 0 & 0 & A_p
\end{pmatrix} & \text{if } k = 2p + 1
\end{cases}. $$
we get

**Criterion 1.** For any relative equilibrium solution \(\{T_k(\vec{A}t)Q_i\}_{i=1}^n\) of (1.1) as described in Definition 2.1, we have that

\[
A^2 Q_i = \sum_{j=1, j\neq i}^n m_j (Q_i - Q_j) \|Q_i - Q_j\|^{2a}, \quad i \in \{1, ..., n\}.
\]

Before being able to prove Theorem 1.1 and Corollary 1.2, we will need the following lemma:

**Lemma 2.2.** Let \(q_1 = T_k(\vec{A}t)Q_1, ..., q_n = T_k(\vec{A}t)Q_n\) be a relative equilibrium according to Definition 2.1. If for \(i, l \in \{1, ..., n\}\) we write

\[
R_{il} = \sum_{j=2}^l m_j (Q_i - Q_j) \|Q_i - Q_j\|^{2a},
\]

then

\[
A^2 \sum_{i=2}^l m_i (Q_1 - Q_i) = \left( \sum_{i=1}^l m_i \right) \sum_{j=2}^l m_j (Q_1 - Q_j) \|Q_1 - Q_j\|^{2a} + \sum_{i=2}^l m_i (R_{1l} - R_{il}),
\]

(2.1)

**Proof.** For \(i \in \{2, ..., l\}\), using Criterion 1 we get

\[
A^2 (Q_1 - Q_i) = \sum_{j=2}^l m_j (Q_1 - Q_j) \|Q_1 - Q_j\|^{2a} - \sum_{j=1, j\neq i}^l m_j (Q_i - Q_j) \|Q_i - Q_i\|^{2a} + R_{1l} - R_{il}
\]

\[
= \sum_{j=2}^l m_j (Q_1 - Q_j) \|Q_1 - Q_j\|^{2a} - \sum_{j=2, j\neq i}^l m_j (Q_i - Q_j) \|Q_i - Q_j\|^{2a}
\]

\[
+ m_1 (Q_1 - Q_i) \|Q_1 - Q_i\|^{2a} + R_{1l} - R_{il}.
\]

(2.2)

Note that

\[
\sum_{i=2}^l \sum_{j=2, j\neq i}^l m_i m_j (Q_i - Q_j) \|Q_i - Q_j\|^{2a} = 0,
\]
so multiplying both sides of (2.2) with \( m_i \) and then summing both sides over \( i \) from 2 to \( l \) gives

\[
A^2 \sum_{i=2}^{l} m_i (Q_1 - Q_i) = \left( \sum_{i=2}^{l} m_i \right) \sum_{j=2}^{l} m_j (Q_1 - Q_j) \|Q_1 - Q_j\|^{2a} - 0
\]

(2.3)

\[
+ \sum_{i=2}^{l} m_1 m_i (Q_1 - Q_i) \|Q_1 - Q_i\|^{2a} + \sum_{i=2}^{l} m_i (R_{1i} - R_{il})
\]

\[
= \left( \sum_{i=1}^{l} m_i \right) \sum_{j=2}^{l} m_j (Q_1 - Q_j) \|Q_1 - Q_j\|^{2a} + \sum_{i=2}^{l} m_i (R_{1i} - R_{il}).
\]

(2.4)

We now have all that is needed to prove our main theorem.

3 Proof of Theorem 1.1

Proof. Assume that the contrary is true. Then there exist sequences \( \{Q_{ir}\}_{r=1}^{\infty} \) and relative equilibria \( q_{ir}(t) = T_k(\vec{A}t)Q_{ir}, i \in \{1,...,n\} \) for which we may assume, if we renumber the \( Q_{ir} \) in terms of \( i \) and take subsequences if necessary, the following:

1. There exist sequences \( \{Q_{1r}\}_{r=1}^{\infty}, \{Q_{lr}\}_{r=1}^{\infty}, l \leq n \) such that \( \|Q_{ir} - Q_{jr}\| \) goes to zero for \( r \) going to infinity if \( i, j \in \{1,\ldots, l\} \).

2. \( \|Q_{ir} - Q_{jr}\| \) does not go to zero for \( r \) going to infinity if \( i \in \{1,\ldots, l\} \) and \( j \in \{l+1,\ldots, n\} \).

3. \( Q_{1r},...,Q_{lr} \) do not go to zero, as any solution of (2.1) is determined up to rotation and translation, so by translating \( Q_{1r},...,Q_{lr} \) if necessary, we may assume that \( Q_{1r},...,Q_{lr} \) do not go to zero.

4. \( \|Q_{1r} - Q_{lr}\| \geq \|Q_{ir} - Q_{jr}\| \) for all \( i, j \in \{1,\ldots, l\} \), for all \( r \in \mathbb{N} \).

Note that for any \( i \in \{1,\ldots, l\} \) the vectors \( Q_{1r} - Q_{lr}, Q_{1r} - Q_{ir} \) and \( Q_{ir} - Q_{lr} \) either form a triangle with \( \|Q_{1r} - Q_{lr}\| \) the length of its longest side, or the three
of them align, meaning the angles between them are zero. Consequently, the angle between $Q_{1r} - Q_{tr}$ and $Q_{1r} - Q_{ir}$ is smaller than $\frac{\pi}{2}$. Let $\beta_{1lr}$ be the angle between $Q_{1r} - Q_{lr}$ and $Q_{1r} - Q_{ir}$. If there are $i$ such that $\lim_{r \to \infty} \beta_{1lr} < \frac{1}{2}\pi$, then taking inner products on both sides of (2.3) with $\frac{Q_{1r} - Q_{lr}}{\|Q_{1r} - Q_{lr}\|}$ and then letting $r$ go to infinity gives a contradiction. As $\lim_{r \to \infty} \beta_{1lr} = 0$, there is at least one such an $i$. This completes the proof.

4 Proof of Corollary 1.2

Proof. Assume the contrary to be true. Then there exist sequences $\{Q_{ir}\}_{r=1}^{\infty}$, $i \in \{1, \ldots, n\}$ for which $q_{ir}(t) = T_{k}(\bar{A}t)Q_{ir}$ define relative equilibrium solutions of (1.1) and for which there has to be at least one sequence $\{Q_{ir}\}_{r=1}^{\infty}$ that is unbounded. Taking subsequences and renumbering the $Q_{ir}$ in terms of $i$ if necessary, we may assume that $\{Q_{1r}\}_{r=1}^{\infty}$ is unbounded. By Criterion 1,

$$A^2Q_{1r} = \sum_{j=2}^{n} m_j(Q_{1r} - Q_{jr})\|Q_{1r} - Q_{jr}\|^{2a}. \quad (4.1)$$

As the left-hand side of (4.1) is unbounded, the right-hand side must be unbounded as well, which means that there must be $j \in \{2, \ldots, n\}$ for which

$$m_j(Q_{1r} - Q_{jr})\|Q_{1r} - Q_{jr}\|^{2a}$$

is unbounded if we let $r$ go to infinity. But as

$$\|m_j(Q_{1r} - Q_{jr})\|Q_{1r} - Q_{jr}\|^{2a}\| = m_j\|Q_{1r} - Q_{jr}\|^{2a+1},$$

that means that $\|Q_{1r} - Q_{jr}\|$ goes to zero for $r$ going to infinity, which is impossible by Theorem 1.1. This completes the proof.

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