Wavelet Modeling of Control Stochastic Systems at Complex Shock Disturbances

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Abstract: This article is devoted to the development of methodological supports and experimental software tools for accuracy analysis and information processing in control stochastic systems (CStS) with complex shock disturbances (ShD) by means of wavelet Haar–Galerkin technologies. Basic new results include methods and algorithms of stochastic covariance analysis and modeling on the basis of the Galerkin method and wavelet expansion for linear, linear with parametric noises, and quasilinear CStS with ShD. Results are illustrated by an information-control system at ShD. New stochastic effects accumulation for systematic and random errors are detected and investigated.

Keywords: complex shock disturbances; covariance modeling; Haar wavelet; Haar–Galerkin method; multi-echelon shock; control stochastic systems

1. Introduction

Control stochastic systems (CStS) described by nonstationary linear stochastic differential equations (SDE) in the Ito sense with parametric noises are the most adequate models at single shock disturbance (ShD) and Gaussian and non-Gaussian multi-ShD. The exact accuracy theory of stochastic processes for such SDE is developed in [1,2]. For SIS with impulse, ShD asymptotic theory is given in [3,4]. Nowadays wavelet methods are intensively applied to the problem of deterministic and stochastic numerical analysis and modeling. In the past few decades, a broad range of numerical methods based on Haar wavelet methods achieved great success [5]. These methods are simple in sense versatility and flexibility and possess less computational cost for accuracy problems. A wavelet is a numerical mathematical function used to divide a given function into different scale components. The usage and study of wavelets has attained its modern growth due to mathematical analysis of the wavelet in [6–8]. The concept of multiresolution analysis (MRA) was developed in [9]. In [10], z method to construct wavelets with compact support and scale function was presented. A review of basic properties of wavelets and MRA is given in [11]. Among the wavelet families, which are described by an analytical expression, special attention deserves the Haar wavelets. Haar wavelets are very effective and popular for solving ordinary differential equations [11–15]. The wavelet solution of integral and evolutionary equation is discussed in [16–21]. The application of a wavelet for canonical expansions of random functions and SDE has been suggested in [22] and developed [23,24]. In [25,26], a combination of Haar wavelets and the Galerkin method [22] numerical solution of linear equations were presented. Development of the Haar–Galerkin method and experimental software tools for accuracy analysis of CStS at single ShD is given in [19,20].

Let us consider modeling the wavelet Haar–Galerkin methodology and experimental software tools for covariance accuracy analysis of CStS at complex multi-ShD.
2. Stochastic Systems at Complex Shock Disturbances

Let us consider the differential stochastic system described by the following Ito vector equation:

\[ \dot{X} = a^h(X, U, t) + b^h(X, U, t)V^h, X(t_0) = X_0. \]  

(1)

Here, \( X \) (\( \dim X = p \)) is the Markovian state vector, \( V^h \) is the shock white noise (non-Gaussian in the general case), and \( a^h = a^h(X, U, t) \) and \( b^h = b^h(X, U, t) \) are known functions defined by ShD.

For single ShD in cases where Equation (1) satisfies the “filter hypotheses”, nonlinear effects are not able and it is enough to use linear or linear with parametric noises models:

\[ \dot{X} = a^hX + a^h_0 + b^hV^h, X(t_0) = X_0. \]  

(2)

\[ \dot{X} = a^hX + a^h_0 + \left( b^h + \sum_{n=1}^{p} b_n X_n \right)V^h, X(t_0) = X_0. \]  

(3)

In case of independent multi-echelon ShD at time moments, it is necessary to input additional items:

\[ a^h(X, U, t) = a^h_0(t, U) + a^h_1(t, U)X + \mu \bar{F}_{0t}, \]  

(4)

\[ b^h(X, U, t) = b^h_0(t, U) + \sum_{n=1}^{p} b^h_n(t, U)X_n + \sqrt{\mu \bar{F}_{1t}}. \]  

(5)

Here, \( \mu \bar{F}_{0t} \) and \( \sqrt{\mu \bar{F}_{1t}} \) are small functions depending on \( t \) and \( U \). At practice functions \( a^h_0, a^h_1, b^h_0, b^h_1 \) are approximated by formulae:

\[ a^h_0(t, U) = \sum_{k} A^0_k \psi_k^h (t, U, T_k), \]  

(6)

\[ a^h_1(t, U) = \sum_{k} A^1_k \psi_k^h (t, U, T_k). \]  

(7)

\[ b^h_0(t, U) = \sum_{k} B^0_k \psi_k^h (t, U, T_k), \]  

(8)

\[ b^h_1(t, U) = \sum_{k} B^1_k \psi_k^h (t, U, T_k). \]  

(9)

The shock white noise \( V^h \) in Equation (2) may be non-Gaussian in general cases, whereas \( V^h \) in Equation (3) is strictly Gaussian.

The analytical off-line modeling Equations (2)–(5) at given ShD and system parameters \( U \) are based on known methods [1,2].

3. Wavelet Covariance Modeling at Additive Complex ShD

3.1. Let Us Consider Linear Nonstationary Differential StS

\[ \dot{Y}_1 = a^1Y_1 + a^1_0 + b^1X^h, Y_1(t_0) = Y_{10}, \]  

(10)

\[ X^h = \sum_{k} [A_k \varphi_k(t, T_k) + B_k Y_2]. \]  

(11)

\[ \dot{Y}_2 = a^2Y_2 + a^2_0 + b^2V, Y_2(t_0) = Y_{20}. \]  

(12)

Here, \( Y_1 \) is the state vector; \( \dim Y_1 = n_1 \), \( Y_2 \) is the vector of the forming filter \( \dim Y_2 = n_2 \). \( X^h \) is the additive complex ShD; \( a^1, a^2, a^1_0, a^2_0 \) are coefficients of corresponding dimensions; \( A_k \) and \( B_k, T_k \) are random variables; \( \varphi_k(t, T_k) \) is the impulse forming functions; \( V \) is non-Gaussian in the general-case white noise, \( \dim V = n_0 \), with the matrix intensity \( v \). Using:
\[ Y = [Y_1^T \quad Y_2^T]^T, a_0 = \begin{bmatrix} a_0^1 + \sum_k A_k \varphi_k(t, T_k) \end{bmatrix}, a = \begin{bmatrix} a^1 \sum_k B_k \end{bmatrix}, b = \begin{bmatrix} 0 \ b^2 \end{bmatrix} \]

we come to the following linear vector stochastic equation:

\[
\dot{Y} = a^{sh}Y + a_0^{sh} + b^{sh}V, Y(t_0) = Y_0
\]  (9)

Using Equation (9), \(a = a^{sh}, a_0 = a_0^{sh}, b = b^{sh}\), we have the following set of equations for the mathematical expectation \(m_y\), covariance matrix \(K_y\), and matrix of covariance functions \(K_y(t_1, t_2)\) [1,2]:

\[
m_y = am_y + a_0, m_y(t_0) = m_0;
\]  (10)

\[
K_y = aK_y + K_ya^T + bb^T, K(t_0) = K_{y0};
\]  (11)

\[
\frac{\partial K_y(t_1, t_2)}{\partial t_2} = K_y(t_1, t_2)a(t_2)^T, K_y(t_1, t_1) = K_y(t_1), t_1 < t_2;
\]  (12)

\[
K_y(t_1, t_2) = K_y(t_2, t_1)^T, t_1 > t_2;
\]  (13)

Let us consider Equation (9) on the time interval \([t_0, T]\). After a change of variables at

\[
\bar{t} = \frac{(t-t_0)}{(T-t_0)}
\]  (14)

we transform Equation (14) to the following:

\[
\bar{Y} = \bar{a}(\bar{t})\bar{Y} + \bar{a}_0(\bar{t}) + \bar{b}(\bar{t})\bar{V}(\bar{t})
\]  (15)

at \(\bar{t} \in [0,1], \bar{V}(0) = Y_0\) and white noise \(\bar{V}(\bar{t})\) with intensity \(\bar{v}(\bar{t}) = v((T-t_0)\bar{t} + t_0)\),

where:

\[
\bar{Y}(\bar{t}) = Y((T-t_0)\bar{t} + t_0), \bar{V}(\bar{t}) = V((T-t_0)\bar{t} + t_0).
\]

\[
\bar{a}(\bar{t}) = (T-t_0)a((T-t_0)\bar{t} + t_0), \bar{a}_0(\bar{t}) = (T-t_0)a_0((T-t_0)\bar{t} + t_0).
\]

\[
\bar{b}(\bar{t}) = (T-t_0)b((T-t_0)\bar{t} + t_0).
\]

Here, prime is differential by \(\bar{t}\). Furthermore, briefly, we use \(t = \bar{t}\).

3.2. For Solving Equation

\[
\bar{m}' = \bar{a}\bar{m} + \bar{a}_0, \bar{m}(0) = m_0
\]  (16)

defining \(\bar{m} = E\bar{Y}\), we apply [15,16] at condition for functions \(\bar{m}_h, \bar{m}_h, \bar{a}_{hk}, \bar{a}_{hh} (h, k=1,2,\ldots,p)\) belonging to \(L^2[0,1]\); as shown in [13], these functions can be expanded into convergent wavelet series. Following [15], we define the orthonormal basis of the Haar wavelet:

\[
w_1(t) = \varphi(t) = \varphi_{00}(t) = \begin{cases} 1 & \text{at } t \in [0,1), \\ 0 & \text{at } t \notin [0,1). \end{cases}
\]  (17)
\[ w_2(t) = \varphi(t) = \psi_{00}(t) = \begin{cases} 1 & \text{at } t \in \left[0, \frac{1}{2}\right), \\ -1 & \text{at } t \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{at } t \not\in [0,1), \end{cases} \] (18)

\[ w_i(t) = \varphi_{jk}(t) = \begin{cases} \sqrt{2^j} & \text{at } t \in \left[\frac{k}{l}, \frac{k+0.5}{l}\right), \\ -\sqrt{2^j} & \text{at } t \in \left[\frac{k+0.5}{l}, \frac{k+1}{l}\right), \\ 0 & \text{at } t \not\in \left[\frac{k}{l}, \frac{k+1}{l}\right). \end{cases} \] (19)

Here, \( \varphi = \varphi(t) \) is the scaling function, \( \psi = \psi(t) \) is the maternal wavelet
\[
\psi_{jk} = \sqrt{2^j} \psi(2^j t - k),
\]
(20)

\( K = 0, 1, ..., l - 1; l = 2^j; j = 1, 2, ..., J; L = 2 \cdot 2^j; i = l + k + 1; l = 3, 4, ..., L; j \) is the maximal resolution level.

Then, we introduce integrals:
\[ p_i(t) = \int_0^t w_i(t) dt \quad (i = 1, 2, ..., L), \] (21)

where:
\[ p_1(t) = \begin{cases} t & \text{at } t \in [0,1), \\ 0 & \text{at } t \not\in [0,1), \end{cases} \]
\[ p_i(t) = \begin{cases} \sqrt{2^j} \left(t - \frac{k}{l}\right) & \text{at } t \in \left[\frac{k}{l}, \frac{k+0.5}{l}\right), \\ -\sqrt{2^j} \left(k + 1 - t\right) & \text{at } t \in \left[\frac{k+0.5}{l}, \frac{k+1}{l}\right), \\ 0 & \text{at } t \not\in \left[\frac{k}{l}, \frac{k+1}{l}\right), \end{cases} \] (i = 2, 3, ..., L).

For every component, \( m_h = m_h(t), h = 1, ..., p; \) Equation (16) gives expressions:
\[ m_h = \sum_{l=1}^L c_h(t) m_l(t) + \bar{a}_{oh}(t), \quad m_h(0) = m_{oh}. \] (22)

Expanding \( \bar{m}_h \) in the form of Haar wavelet expansion (WLE),
\[ \bar{m}_h = \sum_{l=1}^L c_h(t) w_l(t), \] (23)

we obtain from (22):
\[ m_h(t) = \int_0^t \sum_{l=1}^L c_h(t) w_l(t) \, d\tau + m_{oh} = \sum_{l=1}^L c_h(t) p_l(t) + m_{oh} \quad (h = 1, ..., p). \] (24)

After substituting Equations (23), (24) into (22), we have expression:
\[ \sum_{l=1}^L c_h(t) w_l(t) = \sum_{k=1}^p \sum_{l=1}^L c_{hl}(\bar{a}_{hk}(t) p_l(t), w_s(t)) + \sum_{k=1}^p m_{oh}(\bar{a}_{hk}(t), w_s(t)) + (\bar{a}_{oh}(t), w_s(t)) \] (25)

Projecting (25) on \( w_s(t) \), we receive \((L \times p)\) linear algebraic Equations
\[
\sum_{i=1}^L c_{hs}(w_i(t), w_s(t)) = \sum_{k=1}^p \sum_{l=1}^L c_{hl} \left(\bar{a}_{hk}(t) p_l(t), w_s(t)\right) + \sum_{k=1}^p m_{oh} \left(\bar{a}_{hk}(t), w_s(t)\right) + \left(\bar{a}_{oh}(t), w_s(t)\right) \] (26)

\( (s = 1, 2, ..., L; h = 1, 2, ..., p). \)

Due to orthonormality \( w_s(t) \), we obtain from (26) the following Equations:
\[ c_{hs} = \sum_{k=1}^p \sum_{l=1}^L c_{hl} \left(\bar{a}_{hk}(t) p_l(t), w_s(t)\right) + \sum_{k=1}^p m_{oh} \left(\bar{a}_{hk}(t), w_s(t)\right) + \left(\bar{a}_{oh}(t), w_s(t)\right) \] (27)
Then, we expand functions $\tilde{a}_{hk}(t)p_i(t)$, $\tilde{a}_{oh}(t)$ in the form of WLE:

$$\tilde{a}_{hk}(t)p_i(t) = \sum_{j=1}^{l_i} g_{j}^{hk}w_j(t) \quad (l = 1, 2, \ldots, l; h, k = 1, 2, \ldots, p),$$

$$\tilde{a}_{hk}(t) = \sum_{j=1}^{l_i} q_{j}^{hk}w_j(t) \quad (h, k = 1, 2, \ldots, p),$$

$$\tilde{a}_{oh}(t) = \sum_{j=1}^{l_i} \rho_{j}^{h}w_j(t) \quad (h = 1, 2, \ldots, p),$$

where:

$$g_{j}^{hk} = \langle \tilde{a}_{hk}(t)p_i(t), w_j(t) \rangle = \int_{0}^{1} \tilde{a}_{hk}(\tau)p_i(\tau)w_j(\tau)d\tau,$$

$$q_{j}^{hk} = \langle \tilde{a}_{hk}(t), w_j(t) \rangle = \int_{0}^{1} \tilde{a}_{hk}(\tau)w_j(\tau)d\tau,$$

$$\rho_{j}^{h} = \langle \tilde{a}_{oh}(t), w_j(t) \rangle = \int_{0}^{1} \tilde{a}_{oh}(\tau)w_j(\tau)d\tau.$$

Equation (27) may be presented in resultant form:

$$c_{hs} = \sum_{l=1}^{p} \sum_{l_i=1}^{l_i} c_{h,i}g_{h,i}^{l} + \sum_{k=1}^{p} m_{oh}q_{k}^{h} + \rho_{j}^{h} (s = 1, 2, \ldots, L; h = 1, 2, \ldots, p).$$

So we obtain the following algorithm.

**Theorem 1.** Let the following conditions be satisfied:

- Equations (6)–(8) are reducible to Equation (9) and Equation (15) for composed vector $Y$;
- Scalar functions $\tilde{m}_{h}$, $\tilde{m}_{h}$, $\tilde{a}_{hk}$, $\tilde{a}_{oh}$ (h, k=1,2,...,p) belong to space $L^2[0,1]$;
- In space $L^2[0,1]$ the wavelet basic Haar is defined by formulae (17)–(20).

Then, the solution of Equation (22) for component $\tilde{m}_{h}$ of composed vector $\tilde{Y}(t)$ is as follows:

$$\tilde{m}_{h}(t) = \sum_{l=1}^{p} \sum_{l_i=1}^{l_i} c_{h,i}p_i(t) + m_{oh} (h = 1, \ldots, p),$$

where functions $p_i(t)$ are defined by (21), coefficients $c_{h,i}$ are the solutions of linear algebraic Equation (34).

3.3. From Equation (11), for Every Element $K_{rk} = K_{rk}(t)$ Covariance Matrix $\tilde{K} = E[(\tilde{Y} - \tilde{m})^T(\tilde{Y} - \tilde{m})]$, We Have the Following Ordinary Differential Equation

$$\tilde{K}_{rk}(t) = \sum_{h=1}^{p} \tilde{a}_{rh}(h)\tilde{K}_{hr}(t) + \sum_{h=1}^{p} K_{rh}(t)\tilde{a}_{rh}(t) + \sum_{h=1}^{p} \sum_{s=1}^{n} \tilde{b}_{rh}(s)\tilde{v}_{hs}(t)\tilde{b}_{rh}(t),$$

Due to the $\tilde{K}$ symmetry, it is sufficient to compose Equations for $r_1 = 1, p$, $r_2 = \tilde{r}_2$. In this case, elements $\tilde{K}_{hr}$ at $h > r_2$ are replaced by $\tilde{K}_{hr}$ and $\tilde{K}_{rh}$ at $r_1 > h$ by $\tilde{K}_{hr}$. As a result, we have only $(p + 1)/2$ equations. Let us introduce notation:

$$B_{rk}(t) = \sum_{h=1}^{p} \sum_{s=1}^{n} \tilde{b}_{rh}(s)\tilde{v}_{hs}(t)\tilde{b}_{rh}(t)$$

Then, analogously, from Section 3.2, we obtain the following formulae:

$$\tilde{K}_{rk}(t) = \sum_{l=1}^{l_i} c_{l}^{r_{1}r_{2}}w_{l}(t),$$

$$c_{l}^{r_{1}r_{2}} = \int_{0}^{1} \tilde{K}_{rk}(\tau)w_{l}(\tau)d\tau,$$

$$\tilde{K}_{rk} = \sum_{l=1}^{l_i} c_{l}^{r_{1}r_{2}}p_{l}(t) + K_{0r}.$$
\[ \sum_{i=1}^{L} c_i^{r_1r_2} w_i(t) = \sum_{h=1}^{L} \sum_{i=1}^{L} c_i^{hr_2} \tilde{a}_{r_1h}(t) p_i(t) + \sum_{h=1}^{L} K_{0r_1h} \tilde{a}_{r_1h}(t) + \\
\quad + \sum_{i=1}^{L} \sum_{h=1}^{L} c_i^{r_2h} \tilde{r}_{r_2h}(t) p_i(t) + \sum_{h=1}^{L} K_{0r_2h} \tilde{r}_{r_2h}(t) + B_{r_1r_2}(t). \]  
(41)

Protecting (41) on basis \( w_s(t) \), we receive \( p(p + 1)L/2 \) equations for \( c_i^{r_1r_2} \):

\[ c_s^{r_1r_2} = \sum_{i=1}^{L} \sum_{h=1}^{L} c_i^{hr_2} g_i^{r_1r_2h} + \sum_{h=1}^{L} K_{0hr_2} g_i^{r_1r_2h} + \sum_{h=1}^{L} K_{0r_2h} q_s^{r_2h} + p_s^{r_1r_2}, \]  
(42)

where:

\[ \rho_s^{r_1r_2} = \int_0^1 B_{r_1r_2}(t) w_s(t) \, dt. \]

So, we come to the following algorithm:

**Theorem 2.** Let the conditions of Theorem 1 and additional conditions be considered:

- Equations for elements of covariance matrix Equation (11) are reduced to Equation (36);
- The scalar functions \( \bar{K}_{r_1r_2}(t), B_{r_1r_2}(t) \) \((r_1, r_2 = 1, p)\) belonging to space \( L^2[0,1] \) are fulfilled.

Then, the solution of Equation (36) for elements \( \bar{K}_{r_1r_2} \) of the covariance matrix of the composed vector \( \bar{Y} \) is expressed by Equation (40), where \( p_i(t) \) is defined by (21), coefficients \( c_i^{r_1r_2} \) being the solution of linear algebraic Equation (42).

3.4. From Equation (12) FOR Every Element \( \bar{K}_{r_1r_2}(t_1, t_2) \) of Matrix \( \bar{K}(t_1, t_2) = E[(\bar{Y}(t_1) - \bar{m}(t_1))(\bar{Y}(t_2) - \bar{m}(t_2))]^T \) at \( t = t_2 \) We Have the Following Ordinary Equations with Corresponding Initial Condition:

\[ \frac{\partial \bar{K}_{r_1r_2}(t_1, t_2)}{\partial t_2} = \sum_{h=1}^{L} \bar{K}_{r_1h}(t_1, t_2) \tilde{a}_{r_2h}(t_2), \]  
(43)

\[ \bar{K}_{r_1r_2}(t_1, t_2) = \bar{K}_{r_1r_2}(t_1) \quad (r_1, r_2 = 1, p; t_1, t_2 \in [0,1]; t_1 < t_2). \]  
(44)

Analogously, from Sections 3.2 and 3.3, we obtain the following expressions:

\[ \frac{\partial \bar{K}_{r_1r_2}(t_1, t_2)}{\partial t_2} = \sum_{i=1}^{L} d_i^{r_1r_2} w_i(t_2), \]  
(45)

\[ d_i^{r_1r_2} = \int_0^1 \frac{\partial \bar{K}_{r_1r_2}(t_1, t_2)}{\partial t_2} w_i(t_2) \, dt_2, \]  
(46)

\[ \bar{K}_{r_1r_2}(t_1, t_2) = \sum_{i=1}^{L} d_i^{r_1r_2} p_i(t_2) + \bar{K}_{r_1r_2}(t_1), \]  
(47)

\[ \sum_{i=1}^{L} d_i^{r_1r_2} w_i(t_2) = \sum_{h=1}^{L} \tilde{a}_{r_2h}(t_2) \left[ \sum_{i=1}^{L} d_i^{r_1h} p_i(t_2) + \bar{K}_{r_1h}(t_2) \right] \]  
(48)

After projecting (47) on basis \( w_s(t_2) \) and tauing into consideration on Expressions (31), (32), we have a system of \( p^2 \times L \) linear equations for coefficients \( d_i^{r_1r_2} \):

\[ d_s^{r_1r_2} = \sum_{i=1}^{L} \sum_{h=1}^{L} d_i^{r_1h} g_s^{r_2hi} + \sum_{h=1}^{L} \bar{K}_{r_1h}(t_2) q_s^{r_2h} \quad (r_1, r_2 = 1, p; s = 1, L). \]  
(49)

**Theorem 3.** Let the conditions of Theorems 1, 2, and additional conditions be considered:

- Equations (12) may be reduced to Equations (43) and (44);
- functions \( \frac{\partial \bar{K}_{r_1r_2}(t_1, t_2)}{\partial t_2}, \bar{K}_{r_1r_2}(t_1, t_2) \) \((r_1, r_2 = 1, p)\) at \( t = t_2 \) belong to space \( L^2[0,1] \) at fixed \( t_1 \in [0,1] \) all valid.
Then, the solution of Equations (43), (44) for elements $\overline{R}_{r_1r_2}(t_1, t_2)$ ($r_1, r_2 = \overline{1, p}$) of the covariance matrix of the composed vector $\overline{Y}$ for $t_1 < t_2$ is as follows: (47). Here, $p_j(t)$ is defined by (21), coefficients $d_{i}^{r_1r_2}$ are defined by (49), and the values of $\overline{R}_{r_1r_2}(t_1)$ are taken from Theorem 2.

4. Wavelet Covariance Modeling at Parametric Complex Shd

Let us consider the Ito linear with parametric noises nonstationary differential StS:
\begin{align*}
\overline{Y} &= a Y + a_{0} + X^{zh}, \\
X^{zh} &= X_1^{zh} + X_2^{zh}, X_1^{zh} = \sum_{k} A_{k} \psi_{k} + \psi_{k} V^{zh}, \\
X_2^{zh} &= \sum_{h=1}^{p} \sum_{k} \alpha_{k} Y_{h} (A_{k} \psi_{k} + \psi_{k} V^{zh}).
\end{align*}

Here, $V^{zh}$ is Gaussian while noise with intensity matrix $\sigma^{zh} = \sigma(t)$. Equations (50) and (51) may be transformed into the following form:
\begin{align*}
\overline{Y} &= a^{zh} Y + a_{0}^{zh} + \left( b_{0}^{zh} + \sum_{h=1}^{p} b_{h}^{zh} Y_{h} \right) V^{zh}, Y(t_0) = Y_0.
\end{align*}

For Equations (52) and (53), the following covariance equations are valid [1,2]:
\begin{align*}
m &= am + a_{0}, m(t_0) = m_0; \\
\dot{K} &= aK + Ka^{\tau} + b_{0} v_{0}^{\tau} + \sum_{h=1}^{p} \left( b_{h} v_{h}^{\tau} + b_{0} v_{h}^{\tau} m_{h} + \sum_{h=1}^{p} b_{h} v_{h}^{\tau} (m_{h} m_{h} + K_{h}) \right), K(t_0) = K_{0};
\end{align*}

where:
\begin{align*}
a_{0}^{zh} &= a_{0} + \sum_{k} A_{k} \psi_{k}, a^{zh} Y = a Y + \sum_{h=1}^{p} \sum_{k} A_{k} \psi_{k} Y_{h}, \\
b_{0}^{zh} &= \sum_{k} \psi_{k}, \sum_{h=1}^{p} b_{h}^{zh} Y^{zh} = \sum_{h=1}^{p} \sum_{k} \psi_{k} \psi_{h} Y_{h} V^{zh}.
\end{align*}

The equation for mathematical expectation $m = E \overline{Y}$ is separated from equations for covariance characteristics and is defined by Equation (16) (see Theorem 1).

The equations for elements $\overline{K}_{r_1r_2} = \overline{K}_{r_1r_2}(t), t \in [0,1]$, due to their symmetry ($r_1 = 1, p; r_2 = r_{p-n}$), have the form:
\begin{align*}
\overline{K}_{r_1r_2}(t) &= \sum_{i=1}^{p} \overline{a}_{r_1i}(t) \overline{K}_{r_2i}(t) + \sum_{i=1}^{p} \overline{a}_{r_2i}(t) \overline{K}_{r_1i}(t) + \sum_{i=1}^{n} \sum_{j=1}^{p} \overline{b}_{r_1s_1j}(t) \overline{Y}_{s_2j}(t) \overline{b}_{r_2s_1j}(t) + \sum_{k=1}^{p} \overline{m}_{k}(t) \left[ \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \overline{b}_{r_1s_1j}(t) \overline{Y}_{s_2j}(t) \overline{b}_{r_2s_2j}(t) + \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \overline{b}_{r_1s_1j}(t) \overline{Y}_{s_2j}(t) \overline{b}_{r_2s_2j}(t) \right] + \ldots
\end{align*}
\[ + \sum_{k=1}^{p} \sum_{l=1}^{p} \left( (\bar{m}_k(t)\bar{m}_l(t) + \bar{K}_{kl}(t)) \right) \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \bar{b}_r^{0s_1}(t) \bar{b}_r^{0s_2}(t), \bar{K}_{r_1r_2}(t_0) = K_{r_1r_2}^0. \] 

Using notations:

\[ B_{r_1r_2}^{00}(t) = \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \bar{b}_r^{0s_1}(t) \bar{b}_r^{0s_2}(t) \]

\[ B_{r_1r_2}^{0k}(t) = \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \bar{b}_r^{ks_1}(t) \bar{b}_r^{ks_2}(t) \]

\[ B_{r_1r_2}^{k0}(t) = \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \bar{b}_r^{0s_1}(t) \bar{b}_r^{ks_2}(t) \]

\[ B_{r_1r_2}^{kl}(t) = \sum_{s_1=1}^{n} \sum_{s_2=1}^{n} \bar{b}_r^{ks_1}(t) \bar{b}_r^{ks_2}(t) \]

we transform Equation (58) into:

\[ K_{r_1r_2}^{0}(t) = \sum_{l=1}^{L} c_{r_1r_2l}(t) K_{l0}(t) + \sum_{l=1}^{L} c_{r_1r_2l}(t) B_{r_1r_2l}(t) \]

After, the Haar expanding left hand of Equation (58) is equal to:

\[ \bar{K}_{r_1r_2}(t) = \int_0^1 \bar{K}_{r_1r_2}(t) \, dt \]

we obtain the solution (59):

\[ \bar{K}_{r_1r_2}(t) = \sum_{l=1}^{L} c_{r_1r_2l}(t) K_{l0}(t) + K_{r_1r_2}^0. \]

Substituting Equations (60) and (61) into (58), we have the following results:

\[ \sum_{i=1}^{L} c_{r_1r_2i}(t) p_i(t) = \sum_{i=1}^{L} c_{r_1r_2i}(t) p_i(t) + \sum_{i=1}^{L} \bar{a}_{r_1i}(t) K_{i0}(t) + \]

\[ + \sum_{i=1}^{L} \sum_{k=1}^{p} c_{k1i} B_{r_1r_2k}(t) p_{i}(t) + \sum_{k=1}^{p} m_{k}^0 \sum_{l=1}^{L} c_{kl} B_{r_1r_2l}(t) p_{i}(t) + \]

At last taking into account the notations:

\[ \bar{a}_{ik}(t)p_{i}(t) = \sum_{j=1}^{L} g_{j}^{ki} w_{j}(t), \]

\[ \bar{a}_{ik}(t) = \sum_{j=1}^{L} q_{j}^{ik} w_{j}(t), \]

\[ B_{r_1r_2}^{00}(t) = \sum_{j=1}^{L} h_{r_1r_2j} w_{j}(t), \]

\[ B_{r_1r_2}^{0k}(t) + B_{r_1r_2}^{k0}(t) = \sum_{j=1}^{L} h_{r_1r_2j} w_{j}(t), \]
\[ B^{ki}_{r_1r_2}(t) = \sum_{j=1}^{L} h^{k}_{r_1r_2j} w_j(t), \]
\[ [B^{k0}_{r_1r_2}(t) + B^{0k}_{r_1r_2}] p_l(t) = \sum_{j=1}^{L} u^{0k}_{r_1r_2j} w_j(t), \]
\[ B^{ki}_{r_1r_2}(t) p_l(t) = \sum_{j=1}^{L} u^{kli}_{r_1r_2j} w_j(t), \]
\[ B^{ki}_{r_1r_2}(t) p_i(t) = \sum_{j=1}^{L} u^{kli}_{r_1r_2j} w_j(t) \]

And projecting (62) on wavelet basis \( w_0(t) \), we obtain the \( p(p+1)\overline{L}/2 \) linear algebraic Equation relatively on coefficients \( c^{r_1r_2}_{s} \) (\( r_1 = \frac{1}{p}; \ r_2 = \frac{1}{p}; \ s = \frac{1}{L} \)):

\[ c^{r_1r_2}_{s} = \sum_{l=1}^{L} \sum_{l=1}^{L} c^{lr_1}_{i} g^{r_2}_{s} + \sum_{l=1}^{L} q^{r_1}_{i} K^{r_2}_{s} + \sum_{l=1}^{L} \sum_{l=1}^{L} c^{r_1}_{i} g^{r_2}_{s} + \sum_{l=1}^{L} q^{r_1}_{i} K^{r_2}_{s} + h^{00}_{s} + \sum_{k=1}^{p} \sum_{l=1}^{L} c^{k}_{l} D^{r_1r_2}_{s} \]
\[ + \sum_{k=1}^{p} \sum_{l=1}^{L} m^{0}_{k} \sum_{l=1}^{L} c^{k}_{l} D^{r_1r_2}_{s} + \sum_{k=1}^{p} \sum_{l=1}^{L} m^{0}_{k} \sum_{l=1}^{L} c^{k}_{l} D^{r_1r_2}_{s} \]
\[ + \sum_{k=1}^{p} \sum_{l=1}^{L} m^{0}_{k} \sum_{l=1}^{L} c^{k}_{l} D^{r_1r_2}_{s} \]
\[ (63) \]

**Theorem 4.** Let the conditions of Theorem 1 and:

- Equations (55) are reduced to Equations (58);
- scalar functions \( a_{ik}(t), b^{0k}_{r_1r_2}(t), B^{k0}_{r_1r_2}(t), B^{0k}_{r_1r_2}(t), [B^{k0}_{r_1r_2}(t) + B^{0k}_{r_1r_2}], B^{ki}_{r_1r_2}(t), [B^{k0}_{r_1r_2}(t) + B^{0k}_{r_1r_2}] p_l(t), B^{ki}_{r_1r_2}(t) p_i(t), B^{ki}_{r_1r_2}(t) p_i(t) p_i(t) \) \( (l, k, r_1, r_2 = \frac{1}{p}; \ i, i_1, i_2 = \frac{1}{L}, \ L) \) belong to space \( L^2[0,1] \).

Then, the solutions of Equation (58) is expressed by Equations (61) and (63).

For solution of Equation (56), we use Theorem 3.

**5. Examples**

5.1. Let Consider the Information-Control System (ICS) of the Third Order at ShD Described by the Following Equations:

\[ \dot{Y}_1 = Y_2, \]
\[ \dot{Y}_2 = -\omega_2^2 Y_1 - 2\varepsilon\omega_c Y_2 + (S + n_{t^{sh}} + V_{t^{sh}})(\cos\alpha + Y_1\sin\alpha), \]
\[ \dot{Y}_3 = Y_1, \]
\[ Y_i(t_0) = Y_i(0) (i = 1, 2, 3). \]

Here, \( Y_i \) represents the components of the state vector \( Y = [Y_1 \ Y_2 \ Y_3]^T \), \( \alpha, \varepsilon, \omega_c, S \) are the system parameters; \( n_{t^{sh}} \) is the rectangular impulse function at time moment \( t^{sh} \), \( \alpha \) is the angle between acceleration \((S + n_{t^{sh}} + V_{t^{sh}})\) and longitudinal ICS axis; \( V_{t^{sh}} \) is the white noise with intensity \( \nu_t^{sh} \). The equation may be presented in the vector form:

\[ \dot{Y} = a_0^{sh} + a^{sh} Y + (b_0^{sh} + b_1^{sh} Y) V_t^{sh}, \]

where:

\[ a_0^{sh} = \begin{bmatrix} 0 \\ (S + n_{t^{sh}})\cos\alpha \end{bmatrix}, \quad a^{sh} = \begin{bmatrix} 0 \\ -\nu_t^{sh} \\ -2\varepsilon\omega_c \end{bmatrix}, \]
\[ b_0^{sh} = \begin{bmatrix} \nu_t^{sh} \\ \cos\alpha \end{bmatrix}, \ b_1^{sh} = \begin{bmatrix} 0 \\ \sin\alpha \end{bmatrix}, \quad \nu_t^{sh} = \omega_c^2 - (S + n_{t^{sh}})\sin\alpha. \]
Equations for mathematical expectations $m_i = EY_i$ and elements of covariance matrix $k_{ij} = E[(Y_i - m_i)(Y_j - m_j)]$ are as follows:

\[
\begin{align*}
m_1 &= m_2, \\
m_2 &= a_{21}^h m_1 + a_{22}^h m_2 + a_{02}^h, \\
m_3 &= m_1 \\
m_i(t_0) &= m_{i0}(i = 1,2,3); \\
k_{11} &= 2k_{12}, \\
k_{12} &= a_{21}^h k_{11} + a_{22}^h k_{12} + k_{22}, \\
k_{13} &= k_{11} + k_{23}, \\
k_{22} &= 2a_{21}^h k_{12} + 2a_{22}^h k_{22} + \nu_t^h \cos^2 \alpha + \nu_t^h m_1 \sin 2 \alpha + \nu_t^h (m_1^2 + k_{11}) \sin^2 \alpha, \\
k_{23} &= k_{12} + a_{22}^h k_{13} + a_{23}^h k_{23}, \\
k_{33} &= 2k_{13}, \\
k_{ij}(t_0) &= k_{ij0}(i,j = 1,2,3).
\end{align*}
\]

Typical Figures 1 and 2 for ICS with parameters $J = 5; L = 64; \varepsilon = 0.7; \omega_c = 1; S = 1; t^{sh} = 0.135$;

\[
n_{t}^{sh} = \begin{cases} 
10, & t \in [t_{sh}; t_{sh} + 1], \\
0, & t \not\in [t_{sh}; t_{sh} + 1].
\end{cases}
\]

\[
\nu_t^{sh} = \begin{cases} 
6, & t \in [t_{sh}; t_{sh} + 1], \\
0, & t \not\in [t_{sh}; t_{sh} + 1].
\end{cases}
\]

show the accumulation effect of systematic and random errors for output variable $Y_1$.

Typical Figure 1a,b for ICS with parameter $\alpha = 45^\circ$. Typical Figure 2a,b—with parameter $\alpha = 225^\circ$.

So, for single ShD and $0^\circ \leq \alpha \leq 180^\circ$, the following conclusions are valid:

- Systematic restricted error takes place for variables $m_2, m_3$, whereas for variable $m_1$, we have an accumulation growing effect (linear systematic drift);
- A random restricted effect takes place for variables $k_{ij} (i,j = 2,3)$, whereas for $k_{i1}, k_{12}, k_{13}$, we have only the accumulation effect.

Figure 1. (a) Graphs $m_1, m_2, m_3$ at $\alpha = 45^\circ$; (b) Graphs $k_{11}, k_{12}, k_{22}$ at $\alpha = 45^\circ$.
5.2. Let Us Consider the Information-Control System (ICS) of the Third Order at ShD Described by the Following Equations:

\[ Y'_1 = Y_2, \]
\[ Y'_2 = -\omega \varphi_\alpha Y_1 + \left( S + \sum_{k=1}^{N} \varphi_k(t_k)(n_i^{sh} + V_i^{sh}) \right)(\cos \alpha + Y_1 \sin \alpha), \]
\[ Y'_3 = Y_1, \]
\[ Y_i(t_0) = Y_{i0} (i = 1, 2, 3). \]

Here, \( Y_i \) represents the components of the state vector \( Y = [Y_1, Y_2, Y_3]^T \), \( \alpha, \varepsilon, \omega, S, \varphi_k(t_k), N \) is the system parameters; \( n_i^{sh} \) is rectangular impulse function at time moment \( t_k \); \( \alpha \) is the angle between acceleration \( S + \sum_{k=1}^{N} \varphi_k(t_k)(n_i^{sh} + V_i^{sh}) \) and longitudinal ICS axis; \( V_i^{sh} \) is the white noise with intensity \( v_i^{sh} \). The equation may be presented in the vector form:

\[ \dot{Y} = a_0^{sh} + a^{sh}Y + (b_0^{sh} + b_1^{sh}Y)V_i^{sh}, \]

where:

\[ a_0^{sh} = \begin{bmatrix} 0 \\ S + \sum_{k=1}^{N} \varphi_k(t_k)n_i^{sh} \end{bmatrix} \cos \alpha, \quad a^{sh} = \begin{bmatrix} 0 \\ -\Omega^{sh} \\ -2\varepsilon \omega \end{bmatrix}, \]
\[ b_0^{sh} = \begin{bmatrix} \cos \alpha \sum_{k=1}^{N} \varphi_k(t_k) \\ 0 \end{bmatrix}, \quad b_1^{sh} = \begin{bmatrix} \sin \alpha \sum_{k=1}^{N} \varphi_k(t_k) \\ 0 \end{bmatrix}, \]
\[ \Omega^{sh} = \omega^2 - (S + \sum_{k=1}^{N} \varphi_k(t_k)n_i^{sh}) \sin \alpha. \]

Equations for mathematical expectations \( m_i = EY_i \) and elements of covariance matrix \( k_{ij} = E[(Y_i - m_i)(Y_j - m_j)] \) are as follows:

\[ \dot{m}_1 = m_2, \]
\[ \dot{m}_2 = a_2^{sh}m_1 + a_3^{sh}m_2 + a_4^{sh}, \]
\[ m_3 = m_1, \]
\[ m_i(t_0) = m_{i0} (i = 1, 2, 3); \]
\[ \dot{k}_{11} = 2k_{12}, \]
\[ \dot{k}_{12} = a^{sh}_{21}k_{11} + a^{sh}_{22}k_{12} + k_{22}, \]
\[ \dot{k}_{13} = k_{11} + k_{23}, \]
\[ \dot{k}_{22} = 2a^{sh}_{21}k_{12} + 2a^{sh}_{22}k_{22} + v^{sh}_t [(b^{sh}_{02})^2 + 2b^{sh}_{02}b^{sh}_{12}m_1 + (b^{sh}_{12})^2(m_1^2 + k_{11})], \]
\[ \dot{k}_{23} = k_{12} + a^{sh}_{21}k_{13} + a^{sh}_{22}k_{23}, \]
\[ \dot{k}_{33} = 2k_{13}, \]
\[ k_{ij}(t_0) = \delta_{ij} (i,j = 1,2,3). \]

Typical Figures 3–10 for ICS with parameters \( J = 5; L = 64; \varepsilon = 0.7; \omega_c = 1; S = 1; \)
\( t_k = 0.135; \theta_k = t_k + 2(k - 1); n^{sh}_t = 10; v_t^{sh} = 6; k = 1/N. \)
\[ \varphi_k(t_k) = \begin{cases} 1, & t \in [t_k; t_{k+1}] \\ 0, & t \not\in [t_k; t_k + 1] \end{cases}. \]

show the accumulation effect of systematic and random errors for output variable \( Y_1. \)

Typical Figure 3a,b for ICS with parameters \( a = 0°, N = 1. \) Typical Figure 4a,b— with parameters \( a = 180°, N = 1. \)

Typical Figure 5a,b for ICS with parameters \( a = 0°, N = 2. \) Typical Figure 6a,b— with parameters \( a = 180°, N = 2. \)

Typical Figure 7a,b for ICS with parameters \( a = 0°, N = 4. \) Typical Figure 8a,b— with parameters \( a = 180°, N = 4. \)

Typical Figure 9a,b for ICS with parameter \( a = 0°, N = 10. \) Typical Figure 10a,b— with parameters \( a = 180°, N = 10. \)

So, for multi-echelon ShD and \( 0° \leq a \leq 180°, \) we have the nonlinear systematic and random accumulation and drift effects for the corresponding variables.

At fixed \( N \) and various \( a, \) we have different quality graphs. So, at \( N \to \infty \) mathematical expectations, \( m_1, m_2 \) are restricted, whereas \( m_3 \) is not restricted. Variances \( k_{11}, k_{22} \) are restricted and \( k_{33} \) is not restricted.

![Figure 3. (a) Graphs \( m_1, m_2, m_3 \) at \( a = 0°, N = 1; \) (b) Graphs \( k_{11}, k_{12}, k_{22} \) at \( a = 0°, N = 1. \)
Figure 4. (a) Graphs $m_1, m_2, m_3$ at $\alpha = 180^\circ, N = 1$; (b) Graphs $k_{11}, k_{12}, k_{22}$ at $\alpha = 180^\circ, N = 1$.

Figure 5. (a) Graphs $m_1, m_2, m_3$ at $\alpha = 0^\circ, N = 2$; (b) Graphs $k_{11}, k_{12}, k_{22}$ at $\alpha = 0^\circ, N = 2$.

Figure 6. (a) Graphs $m_1, m_2, m_3$ at $\alpha = 180^\circ, N = 2$; (b) Graphs $k_{11}, k_{12}, k_{22}$ at $\alpha = 180^\circ, N = 2$. 
Figure 7. (a) Graphs $m_1$, $m_2$, $m_3$ at $\alpha = 0^\circ, N = 4$; (b) Graphs $k_{11}$, $k_{12}$, $k_{22}$ at $\alpha = 0^\circ, N = 4$.

Figure 8. (a) Graphs $m_1$, $m_2$, $m_3$ at $\alpha = 180^\circ, N = 4$; (b) Graphs $k_{11}$, $k_{12}$, $k_{22}$ at $\alpha = 180^\circ, N = 4$.

Figure 9. (a) Graphs $m_1$, $m_2$, $m_3$ at $\alpha = 0^\circ, N = 10$; (b) Graphs $k_{11}$, $k_{12}$, $k_{22}$ at $\alpha = 0^\circ, N = 10$. 
Figure 10. (a) Graphs $m_1$, $m_2$, $m_3$ at $\alpha = 180^\circ, N = 10$; (b) Graphs $k_{11}$, $k_{12}$, $k_{22}$ at $\alpha = 180^\circ, N = 10$.

6. Conclusions

For control stochastic systems at nonstationary shock disturbances described by linear stochastic differential equations with stochastic parametric noises, corresponding modeling methodological support and experimental software tools are developed. The methodology is based on the deterministic Haar–Galerkin algorithms for the solution of equations for the mathematical expectation, covariance matrix, and matrix of covariance functions.

Original new results include methods and algorithms of stochastic covariance analysis and modeling on the basis of the Galerkin method and wavelet expansion for linear, linear with parametric noises, and quasilinear control stochastic systems with complex shock disturbances. A new methodology may be called quick “analytical numerical modeling”. This methodology does not use Monte Carlo methods.

For the accuracy confirmation wavelet methodology, two special examples in the form of an information-control system at single and multi-echelon shock disturbances was presented. New nondeterministic error accumulations and drift effects are detected. Future works include nonlinear covariance analysis and probabilistic distributions problems in the field of nonlinear stochastic systems and dependent multi-shock disturbances.

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Abbreviations

CStS Control Stochastic Systems
ICS Information–Control System
MRA MultiResolution Analysis
SDE Stochastic Differential Equations
ShD Shock Disturbances
WLE Wavelet Expansion

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