CONVEXITY AND SINGULARITIES OF CURVATURE EQUATIONS IN CONFORMAL GEOMETRY

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Abstract. We define a generalization of convex functions, which we call $\delta$-convex functions, and show they must satisfy interior Hölder and $W^{1,p}$ estimates. As an application, we consider solutions of a certain class of fully nonlinear equations in conformal geometry with isolated singularities, in the case of non-negative Ricci curvature. We prove that such solutions either extend to a Hölder continuous function across the singularity, or else have the same singular behavior as the fundamental solution of the conformal Laplacian. We also obtain various removable singularity theorems for these equations.

1. Introduction

Our goal in this paper is to understand the behavior of solutions to certain geometric PDEs with isolated point singularities. Since the relevant equations are fully nonlinear we impose an additional condition on our solutions, known as admissibility, which guarantees that the resulting equations are elliptic. Although the solutions are defined on deleted neighborhoods of Riemannian manifolds, in the course of our analysis we are naturally lead to the study of functions locally defined on Euclidean space which satisfy a certain convexity condition. This notion of convexity is weaker than the more familiar one of $k$-convexity, and has the additional advantage of being a linear condition.

Let us begin by recalling some basic definitions. For $1 \leq k \leq n$, denote by $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ the $k$-th elementary symmetric polynomial, and $\Gamma_{\sigma_k} \subset \mathbb{R}^n$ the component of $\{x \in \mathbb{R}^n | \sigma_k(x) > 0\}$ containing the positive cone $\{x \in \mathbb{R}^n | x_1 > 0, ..., x_n > 0\}$. A function $u \in C^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open, is $k$-convex in $\Omega$ if

$$ D^2 u \in \Gamma_{\sigma_k} $$

at each point in $\Omega$. This condition naturally arises in the study of the Hessian equations

$$ \sigma_k(D^2 u) = f(x). $$

In particular, (1.2) is elliptic provided $u$ is $k$-convex; see [Gar59], [CNS85], and [TW99], for example.

To introduce our new notion of convexity we need to define a family of nested cones $\Gamma_\delta \subset \mathbb{R}^n$:

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**Definition 1.1.** Let $\delta \in \mathbb{R}$. The $n$-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_\delta$ if and only if

\begin{equation}
\lambda_i > -\delta \sum_{j=1}^{n} \lambda_j \quad \forall \ 1 \leq i \leq n,
\end{equation}

and we define $\overline{\Gamma_\delta}$ to be the closure of $\Gamma_\delta$.

Note that $\Gamma_{\delta_1} \subset \Gamma_{\delta_2}$ whenever $\delta_1 < \delta_2$, and also that if $\lambda \in \Gamma_\delta$ for $\delta > -1/n$, then

\begin{equation}
\sum_{i=1}^{n} \lambda_i > 0.
\end{equation}

**Definition 1.2.** Let $W \in C^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open. We say that $W$ is strictly $\delta$-convex in $\Omega$ if

\begin{equation}
D^2W \in \Gamma_\delta
\end{equation}

at each point in $\Omega$, and $W$ is $\delta$-convex in $\Omega$ if

\begin{equation}
D^2W \in \overline{\Gamma_\delta}
\end{equation}

at each point in $\Omega$.

For $k > n/2$, a $k$-convex function is $\delta$-convex for $\delta = \frac{n-k}{n(k-1)}$. This fact is implicit in the work of Trudinger-Wang [TW99], where they proved a priori estimates for $k$-convex functions. However, they did not express it in these terms. Our main results about $\delta$-convex functions are corresponding Hölder and $W^{1,p}$ estimates:

**Theorem 1.3.** Let $0 \leq \delta < \frac{1}{n-2}$, $\Omega \subset \mathbb{R}^n$ be open, and assume $W \in C^2(\Omega)$ is $\delta$-convex. Then on any domain $\Omega' \subset \subset \Omega$, $W$ satisfies

\begin{equation}
\|W\|_{C^\gamma(\Omega')} \leq C_1 \int_{\Omega} |W|,
\end{equation}

where

\begin{equation}
\gamma = \frac{1 + (2 - n)\delta}{1 + \delta},
\end{equation}

and $C_1$ depends only upon $\Omega'$, $\Omega$, and $\delta$. Furthermore, for $p < p_\delta = \frac{n(1+\delta)}{(n-1)\delta}$,

\begin{equation}
\left\{ \int_{\Omega'} |\nabla W|^p \right\}^{1/p} \leq C_2 \int_{\Omega} |W|,
\end{equation}

and $C_2$ depends only upon $\Omega'$, $\Omega$, and $p$.

**Remark 1.4.** Theorem 1.3 is a generalization of the classical theorem stating that convex functions are Lipschitz [DG92, Chapter 6]. For $k$-convex functions, the constants simplify to $\gamma = 2 - n/k$ and $p_\delta = \frac{nk}{n-k}$, and our estimates agree with the estimates of Trudinger and Wang [TW99]. While the proof of Theorem 1.3 is based on the work of Trudinger-Wang, we emphasize that only the condition of $\delta$-convexity is used, which is weaker than $k$-convexity.
1.1. Conformal Geometry. We now turn to the fully nonlinear equations in conformal geometry which motivated this work. To begin, let \((M^n, g)\) be a smooth, closed Riemannian manifold of dimension \(n\). We denote the Ricci tensor of \(g\) by \(\text{Ric}\) and the scalar curvature by \(R\). In addition, the Weyl-Schouten tensor is defined by

\[
A = \frac{1}{(n-2)} \left( \text{Ric} - \frac{1}{2(n-1)} Rg \right).
\]

\(A\) is a symmetric \((0, 2)\)-tensor; using the Riemannian metric we can associate the dual tensor of type \((1, 1)\) denoted \(g^{-1} A\). In classical language, \(g^{-1} A\) is obtained from \(A\) by “raising an index.” The tensor \(g^{-1} A\) can also be viewed as a symmetric linear transformation of the tangent space at each point; thus it has \(n\) real eigenvalues.

Let \(g_u = e^{-2u} g\) be a conformal metric and let \(A_u\) denote the Weyl-Schouten tensor of \(g_u\). In this paper we will consider singular solutions of equations of the form

\[
F(g_u^{-1} A_u) = f(x),
\]

where \(F\) is a real-valued function of \(n\)-variables, \(F(g_u^{-1} A_u)\) means \(F\) applied to the eigenvalues of \(g_u^{-1} A_u\), and \(f(x)\) is a given function. Since \(A_u\) is related to \(A\) by the formula

\[
A_u = A + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g
\]

(see [Via02]), in general (1.11) will be a fully nonlinear equation of second order.

An example of particular importance is when \(F = \sigma^{1/k}_k\):

\[
\sigma^{1/k}_k(g_u^{-1} A_u) = f(x).
\]

(1.13)

Following the conventions of our previous papers [GV04a], [GV04b] we use \(g\) (not \(g_u\)) to raise the index in \(A_u\). That is, we interpret \(A_u\) as a bilinear form on the tangent space with inner product \(g\) (instead of \(g_u\)), and understand \(\sigma_k(\cdot)\) to mean \(\sigma_k\) applied to the eigenvalues of \(g^{-1} A_u\). Using this convention, equation (1.13) becomes

\[
\sigma^{1/k}_k(A_u) = f(x)e^{-2u},
\]

or, by (1.12),

\[
\sigma^{1/k}_k(A + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g) = f(x)e^{-2u}.
\]

(1.15)

Note that when \(k = 1\), then \(\sigma_1(g^{-1} A) = \text{trace}(A) = \frac{1}{2(n-1)} R\). Therefore, (1.15) is the scalar curvature equation.

As in the example of the Hessian equations (1.2), given an open set \(\Omega \subset M^n\) and a solution \(u \in C^2(\Omega)\) of (1.15), \(u\) is an elliptic solution if the eigenvalues of \(A_u\) are in \(\Gamma_{\sigma_k}\) at each point of \(\Omega\).

As we observed above, if \(A_u \in \Gamma_{\sigma_k}\) then \(A_u \in \Gamma_{\delta}\) for some \(\delta = \delta(k, n) > 0\). Given an open set \(\Omega \subset M^n\) and \(u \in C^2(\Omega)\), if the eigenvalues of \(A_u\) are in \(\Gamma_{\delta}\) at each point of \(\Omega\), we then say that \(u\) is strictly \(\delta\)-admissible in \(\Omega\); if the eigenvalues are in the closure \(\overline{\Gamma_{\delta}}\), then we say that \(u\) is \(\delta\)-admissible in \(\Omega\).

It turns out that \(\delta\)-admissibility has an important geometric consequence: If the eigenvalues of the Schouten tensor \(A_g\) are in \(\Gamma_{\delta}\) at each point of \(M^n\), then (1.4) for
\( \delta > -1/n \) implies the scalar curvature of \((M^n, g)\) is positive, while \([1.13]\) for \( \delta < \frac{1}{n-2} \) implies the Ricci curvature is positive. In fact,

\[
(1.16) \quad \text{Ric}_g - [1 + (2 - n)\delta]\sigma_1(A_g)g \geq 0.
\]

This fact is crucial in our analysis.

When considering the more general equation \([1.11]\) we need to impose various structural conditions on the function \( F \) and its domain. Suppose

\[
F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}
\]

with \( F \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \), where \( \Gamma \subset \mathbb{R}^n \) is an open, symmetric, convex cone. In addition, we assume

(i) \( F \) is symmetric, concave, and homogeneous of degree one.

(ii) \( F > 0 \) in \( \Gamma \), and \( F = 0 \) on \( \partial \Gamma \).

(iii) \( F \) is elliptic: \( F_{\lambda_i}(\lambda) > 0 \) for each \( 1 \leq i \leq n \), \( \lambda \in \Gamma \).

(iv) \( \Gamma \supset \Gamma_\sigma_n \), and there exists a constant \( 0 \leq \delta < \frac{1}{n-2} \) such that \( \Gamma \subset \Gamma_\delta \).

For \( F \) satisfying (i) – (iv), consider the equation

\[
(1.18) \quad F(A_u) = f(x)e^{-2u}.
\]

Given an open set \( \Omega \subset M^n \) and a solution \( u \in C^2(\Omega) \) of \([1.18]\), \( u \) is an elliptic solution if the eigenvalues of \( A_u \) are in \( \Gamma \) at each point of \( \Omega \). We then say that \( u \) is strictly \( \Gamma \)-admissible (or just strictly admissible). By (iv), any strictly \( \Gamma \)-admissible solution is strictly \( \delta \)-admissible. We will also be interested in solutions of \([1.18]\) with \( f(x) \geq 0 \) and \( A_u \in \overline{\Gamma} \). In this case equation \([1.18]\) may be degenerate elliptic; correspondingly we say such solutions are \( \Gamma \)-admissible (or just admissible), and therefore \( \delta \)-admissible.

Some examples of interest are

**Example 1.** Let

\[
(1.19) \quad F(A_u) = \sigma^{1/k}(A_u) = f(x)e^{-2u}
\]

with \( \Gamma = \Gamma_{\sigma_k}, k > n/2 \). Since \( k > n/2 \), by \([GVW03]\) we find that the eigenvalues of \( A_u \) satisfy inequality \([1.13]\) with

\[
0 \leq \delta = \frac{n - k}{n(k - 1)} < \frac{1}{n - 2}.
\]

**Example 2.** Let \( 1 \leq l < k \) and \( k > n/2 \), and consider

\[
(1.20) \quad F(A_u) = \left( \sigma_k(A_u) \right)^{1/k} \sigma_l(A_u)^{1/l} = f(x)e^{-2u}.
\]

In this case we also take \( \Gamma = \Gamma_{\sigma_k} \).
Example 3. For $\tau \leq 1$ let
\[ A^\tau = \frac{1}{(n-2)} \left( Ric - \frac{\tau}{2(n-1)} Rg \right), \]
and consider the equation
\[ F(A_u) = \sigma_1^{1/k} (A_u^{\tau}) = f(x)e^{-2u}. \]
By (1.12), this is equivalent to the fully nonlinear equation
\[ \sigma_k^{1/k} \left( A^\tau + \nabla^2 u + \frac{1-\tau}{n-2} (\Delta u) g + du \otimes du - \frac{2-\tau}{2} |\nabla u|^2 g \right) = f(x)e^{-2u}. \]

In the Appendix of [GV04b] we showed that the results of [GVW03] imply the existence of $\tau_0 = \tau_0(n,k) > 0$ and $\delta_0 = \delta(k,n) < \frac{1}{n-2}$ so that if $1 \geq \tau > \tau_0(n,k)$ and $A_g^\tau \in \Gamma_{\sigma_k}$ with $k > n/2$, then $A_g \in \Gamma_\delta$ with $\delta = \delta_0$. When we refer to this example, we will tacitly make the assumption that $k > n/2$ and $1 \geq \tau > \tau_0(n,k)$.

In this paper we study solutions of (1.18) with isolated point singularities. Thus, we assume $u$ is a solution of (1.18) in $\Omega = B(O, r_0) \setminus \{O\}$, and attempt to understand the behavior of $u(x)$ as $x \to O$.

**Theorem 1.5.** Let $u \in C^4(\Omega)$ be an admissible solution of (1.18) in $\Omega = B(O, r_0) \setminus \{O\}$, with $f(x) \equiv 0$ near $O$.

(i) If
\[ \liminf_{x \to O} u(x) > -\infty, \]
then $u$ can be extended to a Hölder continuous function $u^* \in C^\gamma(B(O, r_0))$, with
\[ \gamma = \frac{1 + (2-n)\delta}{1 + \delta}. \]

(ii) If
\[ \liminf_{x \to O} u(x) = -\infty, \]
then there is a constant $C > 0$ such that for all $x \in \Omega$,
\[ 2 \log d(x) - C \leq u(x) \leq 2 \log d(x) + C. \]

**Remark 1.6.** In view of Proposition 4.1 below (which only uses the sign of the scalar curvature), a singular solution is necessarily bounded from above.

**Remark 1.7.** In case $u$ satisfies (1.19) or (1.20) (in particular, when $\Gamma = \Gamma_{\sigma_k}$ with $k > n/2$), then
\[ \gamma = 2 - n/k > 0. \]
If $u$ satisfies (1.21) with $k > n/2$ and $\tau > \tau_0 = \frac{2(n-k)}{n}$, then
\[ \gamma = \frac{(n-2)(2k-2n+n\tau)}{n-2k+kn-n\tau} > 0. \]
Remark 1.8. We have a stronger statement of Theorem 1.5 in the locally conformally flat case; see Theorem 1.17 below.

Remark 1.9. It is instructive to compare Theorem 1.5 (i) with the recent classification of radial solutions of the $\sigma_k$-curvature equations carried out by Chang-Han-Yang [CHY05]. When $k > n/2$ they show the existence of a solution to (1.19) in $C^{2-n/k}(S^n)$, but whose second derivative blows up at an isolated point. Therefore, Theorem 1.5 (i) is optimal.

Remark 1.10. To provide some context for the conclusions of part (ii) in Theorem 1.5, consider the case of the sphere with the round metric $(S^n, g_0)$. Fix a point $O \in S^n$, and let $G$ denote the Green’s function for the conformal Laplacian $L$ with pole at $O$. $G$ satisfies

$$LG = 0 \quad \text{on} \quad S^n \setminus \{O\},$$
$$G(x) \sim \text{dist}(x, O)^{2-n}.$$

The the conformal manifold $(S^n \setminus \{O\}, G^{(n-2)/(n+2)} g_0)$ is actually isometric to Euclidean space; consequently the function $u = -\frac{2}{(n-2)} \log G$ satisfies

$$F(A_0 + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g_0) = 0$$

on $S^n \setminus \{O\}$. Thus, (1.25) says, in essence, that a solution of (1.18) with isolated singularity must blow up at the same rate as the fundamental solution of the conformal Laplacian. In fact, using the results from Section 7 of [GV04b], it is possible to show that (1.25) implies furthermore that the background metric is the Euclidean metric, and $u(x) \equiv 2\log r + C$. We omit the details, but the main idea is to look at $(M^n \setminus \{O\}, g_u)$ as a complete manifold with non-negative Ricci curvature outside of a compact set. An adaptation of the arguments in Section 7 of [GV04b] imply that the end is asymptotically flat, and applying a version of Bishop’s volume comparison theorem, we find that $(M^n \setminus \{O\}, g_u)$ is isometric to Euclidean space outside of a compact set.

For the lower bound in (1.29) above, we can weaken the regularity assumption to $u \in C^3$, but we must specialize to the $F$ given in the above examples:

**Theorem 1.11.** Let $u \in C^3(\Omega)$ be an admissible solution of either (1.19), (1.20), or (1.21) in $\Omega = B(O, r_0) \setminus \{O\}$, with $f(x) \equiv 0$ near $O$.

If

$$\liminf_{x \to O} u(x) = -\infty,$$

then there is a constant $C > 0$ such that for all $x \in \Omega$,

$$2\log d(x) - C \leq u(x).$$

**Remark 1.12.** The proof of the estimate in (1.29) requires a local gradient estimate for $C^3$ solutions, which is currently only known for the special cases in Examples 1–3.
We conjecture it is true for general \( F \). If \( u \in C^4 \), then a local \( C^2 \)-estimate for general \( F \) follows from Sophie Chen’s work [Che05], which was used in Theorem 1.5. This is discussed in detail in Section 6.

1.2. Relation to the Existence Theory. Theorem 1.5 is in fact a corollary of a much more general result about the local behavior of admissible functions. This result depends upon an explicit but rather subtle relationship between admissibility and \( \delta \)-convexity; see Section 3.1. Moreover, this generalization of Theorem 1.5 holds under weaker regularity assumptions—an important consideration for certain applications, for reasons we now explain.

Aside from its intrinsic interest, the study of solutions with isolated singularities is central to the study of a priori estimates for solutions of (1.18), and the related problem of analyzing the blow-up of sequences of solutions. Both of these topics were treated in our previous paper [GV04b], where we proved a general existence result for solutions of (1.18) assuming properties (i) − (iv) above and certain a priori estimates are satisfied.

Precisely because singular solutions often appear as limits of smooth ones, there is an additional technical difficulty that often arises. Namely, the limit may only be in \( C^{1,1}_{\text{loc}} \) and satisfy (1.18) almost everywhere. For example, in [GV04b], a divergent sequence of solutions \( \{u_i\} \) to (1.18) is rescaled by defining \( v_i = u_i + \tau_i \), where \( \{\tau_i\} \) is a sequence of numbers with \( \tau_i \to +\infty \) as \( i \to \infty \). Each \( v_i \) is also a solution of (1.18), but with \( f_i(x) = e^{-2\tau_i} f(x) \). Now, the sequence \( \{v_i\} \) converges (away from a finite point set \( \Sigma \)), but the limit \( v \in C^{1,1}_{\text{loc}}(M^n \setminus \Sigma) \) is a (degenerate) admissible solution of (1.18) with \( f(x) \equiv 0 \). Thus, when studying singular solutions of (1.18) it is natural to impose the weakest possible regularity.

A similar construction, by the way, was carried out in Schoen’s work on the Yamabe problem [Sch89]. In this case, a divergent minimizing sequence for the Yamabe functional is rescaled, and a subsequence converges (away from a finite point set \( \Sigma \)) to a solution of

\[
L h = 0 \quad \text{on} \quad M^n \setminus \Sigma,
\]

where \( L = \Delta - \frac{(n-2)}{4(n-1)} R \). The important difference here is that while \( h \) is a singular solution of (1.30), it is smooth away from the singular points. This allows one to apply the results of Serrin [Ser56] and Gilbarg-Serrin [GS56], who classified \( C^2 \)-solutions of (1.30) with isolated singularities: in fact, \( h \) must be a linear combination of fundamental solutions of the conformal laplacian.

In general, to understand the behavior of solutions near isolated singularities some form of the Harnack inequality seems essential, as it was in the work of Gilbarg-Serrin for the semilinear case. While Harnack inequalities have been established for solutions of (1.18) (see [Via02], [LL02], [GW03b], [Che05]) they all assume at the very least \( u \in C^3 \) and \( f \in C^1 \), for the simple reason that the proofs rely on differentiating the equation. In our existence work [GV04b] described above, we were able to show that the singular solution \( v \) satisfied a Harnack inequality by using the fact it was the limit of smooth solutions, each of which satisfied the local gradient bounds proved
by Guan and Wang [GW03]. But for an arbitrary solution of (1.18) in, say, $C^{1,1}_{\text{loc}}$, it remains an open question whether one can obtain such an estimate.

1.3. Scale-Invariant Estimates. To clarify precisely what is lacking for weak solutions of (1.18), we introduce the following terminology:

**Definition 1.13.** Let $B = B(O, r_0) \subset M^n$ and $\Omega = B \setminus \{O\}$. For $x \in \Omega$, let $d(x)$ denote the distance to $O$. We say that $u \in C^{1,1}_{\text{loc}}(\Omega)$ satisfies a scale-invariant $C^1$-estimate if there is a constant $C$ such that

$$|\nabla u(x)| \leq \frac{C}{d(x)},$$

(1.31)

for every $x \in \Omega$. We say that $u \in C^{1,1}_{\text{loc}}(\Omega)$ satisfies a scale-invariant $C^2$-estimate if there is a constant $C$ such that

$$|\nabla u(x)|^2 + |\nabla^2 u(x)| \leq \frac{C}{d(x)^2}$$

(1.32)

for almost every $x \in \Omega$.

**Remark 1.14.** Suppose $u$ is a solution of (1.18) on $\mathbb{R}^n \setminus \{0\}$ with $f(x) \equiv 0$. If $u$ satisfies (1.32), then so does $u_\lambda(x) = u(\lambda x)$ for any $\lambda > 0$ (with the same constant $C$). This is why (1.32) is called a “scale-invariant” estimate. For example, consider $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ given by

$$u(x) = \log |x|^2.$$  

Then $u$ is a solution of (1.18) with $f(x) \equiv 0$. Moreover, $|\nabla u_\lambda(x)| = |\nabla u(\lambda x)|$.

We will postpone for now the question of when a scale-invariant estimate can be verified. Instead, we will first restate our main result for $C^{1,1}_{\text{loc}}$-solutions of (1.18), with (1.31) and (1.32) as additional assumptions:

**Theorem 1.15.** Suppose $u \in C^{1,1}_{\text{loc}}$ satisfies a scale-invariant $C^2$-estimate, and $A_u \in \Gamma_\delta$ almost everywhere in $\Omega = B(O, r_0) \setminus \{O\}$, where $0 \leq \delta < \frac{1}{n-2}$. Then the conclusions of Theorem 1.5 hold.

When the background metric is locally conformally flat, we only need a scale-invariant $C^1$-estimate:

**Theorem 1.16.** Assume $(M, g)$ is locally conformally flat. Suppose $u \in C^{1,1}_{\text{loc}}$ satisfies a scale-invariant $C^1$-estimate, and $A_u \in \Gamma_\delta$ almost everywhere in $\Omega = B(O, r_0) \setminus \{O\}$, where $0 \leq \delta < \frac{1}{n-2}$. Then the conclusions of Theorem 1.5 hold.

In fact, part (i) of Theorem 1.5 holds without assuming a scale-invariant estimate; for part (ii), we can verify a one-sided bound:

**Theorem 1.17.** Assume $(M, g)$ is locally conformally flat. Suppose $u \in C^{1,1}_{\text{loc}}$ with $A_u \in \Gamma_\delta$ almost everywhere in $\Omega = B(O, r_0) \setminus \{O\}$, where $0 \leq \delta < \frac{1}{n-2}$.
(i) If
\[
\liminf_{x \to O} u > -\infty,
\]
then \( u \) can be extended to a Hölder continuous function \( u^* \in C^{1,1}_{\text{loc}}(\Omega) \cap C^\gamma(B(O, r_0)) \), with \( \gamma \) given by (1.23).

(ii) If
\[
\liminf_{x \to O} u(x) = -\infty,
\]
then there is a constant \( C > 0 \) such that for all \( x \in \Omega \),
\[
u(x) \leq 2 \log d(x) + C.
(1.35)
\]

Remark 1.18. The main reason we have a stronger statement in the locally conformally flat case is roughly that, in normal coordinates, a general Riemannian metric will be close to Euclidean only to second order, while in the locally conformally flat case, we can find a conformal metric which is exactly Euclidean in a neighborhood of a point. More precisely, if we let \( \{x^i\} \) denote normal coordinates (with respect to the background metric \( g \)) centered at \( O \), the cone condition \( A_u \in \Gamma_\delta \) is equivalent to
\[
g^{ij} \left( \partial_i \partial_j u - \Gamma^k_{ij} \partial_k u + u_i u_j - \frac{1}{2} g^{pq} \partial_p u \partial_q u g_{ij} + A_{ij} \right) \in \Gamma_\delta
\]
a.e. in \( \Omega \). In normal coordinates, \( g^{ij} = \delta^{ij} + O(|x|^2) \), and \( |\Gamma^k_{ij}| = O(|x|) \) as \( |x| \to 0 \).
In particular, we find error terms of the form \( O(|x|^2) (\partial_i \partial_j u) \) as \( |x| \to 0 \), which could be unbounded without a scale-invariant \( C^2 \)-estimate on \( u \).

We now turn to the question: when does a solution of (1.18) satisfy a scale-invariant \( C^2 \)-estimate? If \( u \in C^4(\Omega) \) is a solution of (1.18), then the local estimates proved by Sophie Chen [Che05] can be used to verify a scale-invariant \( C^2 \)-estimate. If \( u \in C^3(\Omega) \) and is a solution of either (1.19), (1.20), or (1.21), then the local estimates of solutions established in [GW03b], [GW03a], [LL03], and [GV03] can be used to to verify the scale-invariant \( C^1 \)-estimate:

**Proposition 1.19.** (i) Let \( u \in C^4(\Omega) \) be an admissible solution of (1.18) in \( \Omega = B(O, r_0) \setminus \{O\} \). If \( f \equiv 0 \) in a neighborhood of \( O \), then \( u \) satisfies the scale-invariant \( C^2 \)-estimate (1.32).

(ii) Let \( u \in C^3(\Omega) \) be an admissible solution of either (1.19), (1.20), or (1.21) in \( \Omega = B(O, r_0) \setminus \{O\} \). If \( f \equiv 0 \) in a neighborhood of \( O \), then \( u \) satisfies the scale-invariant \( C^1 \)-estimate (1.31).

**Remark 1.20.** In particular, Theorem 1.15 follows from Proposition 1.19 and Theorem 1.16.
1.4. Hölder extension. A geometrically natural condition to consider is that of finite volume. For example, suppose \( u \in C^{1,1}_{\text{loc}} \) satisfies the hypotheses of Theorem 1.15 and the volume of \( g_u = e^{-2u}g \) is finite:

\[
\text{Vol}_g(\Omega) = \int_{\Omega} e^{-nu} d\text{vol}_g < \infty.
\]

Then by examining the integrand in (1.37), it is clear that \( u \) cannot satisfy (1.25). Consequently, we have

**Corollary 1.21.** Let \( 0 \leq \delta < \frac{1}{n-2} \), \( u \in C^{1,1}_{\text{loc}} \) satisfy \( A_u \in \overline{\Gamma}_\delta \) almost everywhere in \( \Omega = B(O,r_0) \setminus \{0\} \). Assume that \( g \) is locally conformally flat and \( u \) satisfies a scale-invariant \( C^{1,1} \)-estimate, or that \( u \) satisfies a scale-invariant \( C^2 \)-estimate in the general case. If the volume of the conformal metric \( g_u = e^{-2u}g \) is finite, then \( u \) can be extended to a Hölder continuous function \( u^* \in C^{1,1}_{\text{loc}}(\Omega) \cap C^\gamma(B(O,r_0)) \), where \( \gamma \) is given in (1.23).

In addition, finite volume actually implies a scale-invariant estimate:

**Theorem 1.22.** Let \( u \in C^4 \). Assume that \( u \) is an admissible solution of (1.18) in \( \Omega = B(O,r_0) \setminus \{O\} \). If the volume of the conformal metric \( g_u = e^{-2u}g \) is finite; i.e., if (1.37) is satisfied, then \( u \) satisfies a scale-invariant \( C^2 \)-estimate. Consequently, by Corollary 1.21, \( u \) can be extended to a Hölder continuous function \( u^* \in C^\gamma(B(O,r_0)) \).

In the locally conformally flat case, when \( u \) is a solution of one of the special examples (1.19)–(1.21), then we can slightly weaken the regularity assumption:

**Theorem 1.23.** Let \( (M^n, g) \) be locally conformally flat and \( u \in C^3 \). Assume that \( u \) is an admissible solution of either (1.19), (1.20), or (1.21) in \( \Omega = B(O,r_0) \setminus \{O\} \). If the volume of the conformal metric \( g_u = e^{-2u}g \) is finite then \( u \) satisfies a scale-invariant \( C^{1,1} \)-estimate. Consequently, by Corollary 1.21, \( u \) can be extended to a Hölder continuous function \( u^* \in C^\gamma(B(O,r_0)) \).

In [Gon04b] Gonzalez studied the behavior of solutions to (1.15), \( k < n/2 \), with isolated singularities. She proved that \( C^3 \)-solutions with finite volume are bounded across the singularity. In related work, Han [Han04] proved local \( L^\infty \)-estimates for \( W^{2,2} \)-solutions of (1.18) when \( k = 2 \) and \( n = 4 \), assuming a smallness condition on the volume.

In a subsequent paper [Gon04a] Gonzalez considered a subcritical version of (1.15):

\[
\sigma_k^{1/k}(A_u) = f(x)e^{-2\beta u},
\]

where \( \beta < 1 \). Solutions of (1.38) with isolated singularities are either bounded or satisfy a (sharp) growth condition near the singularity analogous to (1.25).

In the conformally flat case, when \( f(x) \geq c_0 > 0 \) near the singularity \( O \) we can also rule out blow-up:

**Theorem 1.24.** Let \( (M^n, g) \) be locally conformally flat and \( u \in C^3 \) a strictly admissible solution of either (1.19), (1.20), or (1.21) in \( \Omega = B(O,r_0) \setminus \{O\} \), with \( f(x) \geq c_0 > 0 \) near \( O \). Then \( u \) can be extended to a Hölder continuous function \( u^* \in C^3(\Omega) \cap C^\gamma(B(O,r_0)) \).
Remark 1.25. This is proved in Section 7.2. We conjecture that Theorem 1.24 is true without the local conformal flatness assumption.

1.5. Hölder and $L^p$-estimates. Our final results are some $W^{1,p}$- and Hölder-estimates for $\delta$-admissible functions:

**Theorem 1.26.** Let $0 \leq \delta < \frac{1}{n-2}$, $u \in C^{1,1}_{loc}$ satisfy $A_u \in \Gamma_\delta$ almost everywhere in $B(O, r_0)$. Let $v = e^{\beta u}$, where

$$\beta = \frac{1 + (2 - n)\delta}{2(1 + \delta)}.$$  

Then for $p < p_\delta = \frac{n(1+\delta)}{(n-1)\delta}$,

$$\left\{ \int_{B(x_0, r/2)} |\nabla v|^p \right\}^{1/p} \leq C \int_{B(x_0, r)} |v|,$$

where $C = C(r, p, \|g\|_{C^2})$. Consequently, if $\gamma = \frac{1 + (2 - n)\delta}{1 + \delta}$, then by the Sobolev embedding theorem

$$\|v\|_{C^\alpha(B(x_0, r/2))} \leq C \int_{B(x_0, r)} |v|,$$

for any $\alpha < \gamma$, where $C = C(r, \alpha, n, \|g\|_{C^2})$. If $g$ is locally conformally flat, then we may take $\alpha = \gamma$ in (1.41).

We mention a related body of work which considers solutions of (1.15) defined on subdomains in the sphere. By the work of Schoen and Yau [SY88], such solutions arise when considering complete, conformally flat (admissible) metrics. The goal is to study the singular set and derive estimates for the Hausdorff dimension (see [CHY04], [Gon05], [GLW05]).

In closing, we would also like to mention the preprints of Yanyan Li [Li05a, Li05b], and Trudinger-Wang [TW05] which contain closely related results regarding the above Hölder estimates, and asymptotic behaviour of solutions at singularities.

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2. Hölder estimates for $\delta$-convex functions

We begin by giving the proof of the Hölder estimate (1.7) in Theorem 1.3. The $W^{1,p}$ estimate (1.9) will be proved later in Section 8. First, notice that the function

$$G(x) = C|x - y|^{\gamma}$$

is a solution of

$$F_\delta(D^2G) \equiv \det^{1/n} \left( D^2G + \delta(\Delta G) I \right) = 0$$
on \( R^n - \{y\} \), with \( D^2G \in \Gamma_{\sigma_n} \), for
\[
(2.3) \quad \gamma = \frac{1 + (2 - n)\delta}{1 + \delta}.
\]
To see this, by scaling and translation, assume that \( C = 1 \), and \( y = 0 \), then
\[
(2.4) \quad \partial_i \partial_j G = \gamma (\gamma - 2)|x|^{\gamma - 4}x_i x_j + \gamma |x|^{\gamma - 2} \delta_{ij},
\]
\[
(2.5) \quad \Delta G = (\gamma (\gamma - 2) + n\gamma)|x|^{\gamma - 2}.
\]
A computation shows that
\[
\partial_i \partial_j G + \delta(\Delta G) \delta_{ij} = \gamma (\gamma - 2)|x|^{\gamma - 4}x_i x_j
\]
\[
+ \left( \gamma + \delta(\gamma (\gamma - 2) + n\gamma) \right)|x|^{\gamma - 2} \delta_{ij},
\]
\[
= \gamma (\gamma - 2)|x|^{\gamma - 4}x_i x_j - \gamma (\gamma - 2)|x|^{\gamma - 2} \delta_{ij}.
\]
This has one zero eigenvalue, and the other eigenvalues are all equal to \( \gamma (2 - \gamma) > 0 \), so \( (2.2) \) follows, with \( D^2G \in \Gamma_{\sigma_n} \).

Equation \( (2.2) \) is a fully nonlinear (degenerate) elliptic equation, with concave \( F_\delta \) and ellipticity cone \( \Gamma_{\sigma_n} \). Let \( y \in B, r > 0 \), and define on \( B(y, R) \setminus \{y\} \),
\[
(2.6) \quad \tilde{W}(x) = \frac{W(x) - W(y)}{\text{osc}_{B_R(y)} W},
\]
\[
(2.7) \quad G(x) = \left( \frac{|x - y|}{R} \right)^\gamma.
\]
Choose \( \epsilon > 0 \); we have
\[
(2.8) \quad F_\delta(D^2(\tilde{W}(x) - \epsilon)) > 0 \text{ on } B(y, R) \setminus B(y, r),
\]
\[
(2.9) \quad F_\delta(D^2G) = 0 \text{ on } B(y, R) \setminus B(y, r),
\]
\[
(2.10) \quad \tilde{W} - \epsilon \leq G \text{ on } \partial(B(y, R) \setminus B(y, r)),
\]
for \( r \) sufficiently small. Note that \( (2.8) \) is strictly elliptic, but \( (2.9) \) is degenerate elliptic. By Theorem \( \text{[GT83 \ 17.1]} \), the difference \( \tilde{W}(x) - \epsilon - G \) satisfies a linear strictly elliptic equation. From the maximum principle,
\[
(2.11) \quad \tilde{W} - \epsilon \leq G \text{ on } B(y, R) \setminus B(y, r).
\]
Since \( \epsilon > 0 \) is arbitrary, for \( x \in B_R(y) \setminus \{y\} \) we obtain
\[
(2.12) \quad W(x) - W(y) \leq (\text{osc}_{B_R(y)} W) \left( \frac{|x - y|}{R} \right)^\gamma.
\]
The estimate \( (1.7) \) then follows by a standard interpolation argument; see \( \text{[TW99]} \).
3. Admissibility and $\delta$-Convexity

The next result explains the relationship between admissibility and $\delta$-convexity:

**Theorem 3.1.** Suppose $A_u \in \Gamma_{\delta}$, where $A_u$ is given by (1.12). Let $v = e^{\beta u}$, where

\[ \beta = \frac{1 + (2 - n)\delta}{2(1 + \delta)}. \]

Then

\[ \nabla^2 v + \beta v A_g \in \Gamma_{\delta}. \]

In particular, if $g_u = e^{-2u} ds^2$, where $ds^2$ is the flat metric on $\mathbb{R}^n$, then $v$ is $\delta$-convex.

**Proof.** Since $\log v = \beta u$, we have

\[ \beta \nabla u = \frac{\nabla v}{v}, \]

\[ \beta \nabla^2 u = \frac{\nabla^2 v}{v} - \frac{dv \otimes dv}{v^2}. \]

Letting $\alpha = \beta^{-1}$ and using (1.12), we obtain

\[ A_v = A + \alpha \frac{\nabla^2 v}{v} + (\alpha^2 - \alpha) \frac{1}{v^2} dv \otimes dv - \frac{1}{2} \frac{\alpha^2}{v^2} |\nabla v|^2 g. \]

In terms of $v$, the admissibility condition $A_v \in \Gamma_{\delta}$ implies

\[ vA_g + \alpha v \nabla^2 v + (\alpha^2 - \alpha) dv \otimes dv - \frac{\alpha^2}{2} |\nabla v|^2 g \in \Gamma_{\delta}. \]

Now examine the gradient terms, which are proportional to:

\[ (\alpha - 1) dv \otimes dv - \frac{\alpha}{2} |\nabla v|^2 g. \]

In terms of $\delta$, this is

\[ \frac{1 + n\delta}{1 + (2 - n)\delta} dv \otimes dv - \frac{1 + \delta}{1 + (2 - n)\delta} |\nabla v|^2 g, \]

which is proportional to

\[ \frac{1 + n\delta}{1 + \delta} dv \otimes dv - |\nabla v|^2 g. \]

The eigenvalues of this tensor are

\[ \left( \frac{(n - 1)\delta}{1 + \delta}, -1, \ldots, -1 \right) |\nabla v|^2, \]

and the trace is

\[ -\frac{n - 1}{1 + \delta} |\nabla v|^2. \]
Clearly, this implies that (3.7) belongs to $-\overline{\Gamma}_\delta$. Since $\overline{\Gamma}_\delta$ is a convex cone, it follows that

$$\nabla^2 v + \beta v A_g \in \overline{\Gamma}_\delta.$$ (3.9)

\[\square\]

4. Preliminary estimates for singular solutions

In this section we prove some technical results which will be used in the proofs of the main theorems.

4.1. Pointwise estimates. The following Proposition gives an upper bound for solutions in $B(O, r_0) \setminus \{O\}$, assuming only that the scalar curvature is non-negative.

**Proposition 4.1.** Let $u \in C^{1,1}_\text{loc}(\Omega)$ satisfy

$$\sigma_1(A + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g) \geq 0$$ (4.1)

almost everywhere in $\Omega = B(O, r_0) \setminus \{O\}$. Then

$$\sup_{\Omega} u < +\infty.$$ (4.2)

**Proof.** The inequality (4.1) is simply

$$\Delta u - \frac{(n-2)}{2} |\nabla u|^2 + \sigma_1(A) \geq 0.$$ a.e. in $\Omega$, or

$$\Delta u \geq \frac{(n-2)}{2} |\nabla u|^2 - \sigma_1(A)$$ a.e. in $\Omega$.

Our first observation is

**Lemma 4.2.** Let $B = B(O, r_0)$. Then $u \in W^{1,2}(B)$.

**Proof.** For $\epsilon > 0$ small, let $\eta_\epsilon$ denote a cut-off function supported in $\Omega$ satisfying

$$\eta_\epsilon(x) = \begin{cases} 0 & x \in B(O, \epsilon), \\ 1 & x \in B(O, r_0/4) \setminus B(O, 2\epsilon), \\ 0 & x \in B \setminus B(O, r_0/2), \end{cases}$$ (4.4)

and $|\nabla \eta_\epsilon| \leq C/\epsilon$. Since (4.3) holds a.e. on $\Omega$ and $\eta_\epsilon^2$ is supported in $\Omega$ we have

$$\int \eta_\epsilon^2 \Delta u \geq \int \frac{(n-2)}{2} |\nabla u|^2 \eta_\epsilon^2 - \int \eta_\epsilon^2 \sigma_1(A).$$ (4.5)

Integrating by parts,

$$-\int 2(\nabla \eta_\epsilon, \nabla u) \eta_\epsilon \geq \int \frac{(n-2)}{2} |\nabla u|^2 \eta_\epsilon^2 - \int \eta_\epsilon^2 \sigma_1(A).$$
Using the inequality
\[-\int 2 \langle \nabla \eta, \nabla u \rangle \eta \leq \int \frac{(n-2)}{4} |\nabla u|^2 \eta^2 + \int \frac{4}{(n-2)} |\nabla \eta|^2,
\]
we conclude
\[\int \frac{(n-2)}{2} |\nabla u|^2 \eta^2 - \int \eta^2 \sigma_1(A) \leq \int \frac{(n-2)}{4} |\nabla u|^2 \eta^2 + \int \frac{4}{(n-2)} |\nabla \eta|^2,
\]
which implies
\[\int |\nabla u|^2 \eta^2 \leq C \int \left[ |\nabla \eta|^2 + \eta^2 \right].\]
Note that
\[\int |\nabla \eta|^2 \leq C \epsilon^2 \int_{B(O,2\epsilon) \setminus B(O,\epsilon)} \leq C \epsilon^{(n-2)} \to 0\]
as $\epsilon \to 0$. Therefore, letting $\epsilon \to 0$ in (4.6), we get
\[\int_B |\nabla u|^2 \leq C.\]

(4.7)

To prove that $u \in L^2(B)$ we apply the Poincare inequality, which states
\[\int_B \varphi^2 \leq \frac{1}{\lambda_1} \int_B |\nabla \varphi|^2\]
for all $\varphi \in W^{1,2}_0(B)$, where $\lambda_1$ is the first (Dirichlet) eigenvalue of $-\Delta$ on $B$. For $k \geq 1$, define
\[u_k(x) = \begin{cases} k & \text{if } u(x) \geq k, \\ u(x) & \text{if } k \geq u(x) \geq -k, \\ -k & \text{if } u(x) \leq -k. \end{cases}\]

(4.9)

Let $\zeta$ be another cut-off function supported in $B$, this time satisfying
\[\zeta(x) \equiv 1 \quad \forall x \in B(O,r_0/4).\]

(4.10)

For each $k \geq 1$, (4.7) implies $|\nabla u_k| \in L^2(B)$, and since $u_k$ is bounded, it follows that $u_k \in W^{1,2}(B)$. Therefore, by (4.8),
\[\int_B \zeta^2 u_k^2 \leq \frac{1}{\lambda_1} \int_B |\nabla (\zeta u_k)|^2 \]
\[\leq 2 \left[ \int_B \zeta^2 |\nabla u_k|^2 + \int_B u_k^2 |\nabla \zeta|^2 \right].\]

(4.11)

By (4.10), $|\nabla \zeta| \equiv 0$ on $B(O,r_0/4)$, so
\[\int_B u_k^2 |\nabla \zeta|^2 = \int_{B \setminus B(O,r_0/4)} u_k^2 |\nabla \zeta|^2 \leq C,
\]
because \( u \in C^{1,1}_{\text{loc}}(\Omega) \) and is therefore locally bounded. Also, by (4.11),
\[
\int_B \zeta^2 |\nabla u_k|^2 \leq C
\]
independent of \( k \). From (4.11) we conclude
\[
\int_{B(0,r_0/4)} u_k^2 \leq C,
\]
and from the Monotone Convergence Theorem it follows
\[
\int_B u_k^2 \leq C.
\]

**Lemma 4.3.** \( u \) satisfies
\[
\Delta u \geq -\sigma_1(A)
\]
in the \( W^{1,2} \)-sense on \( B = B(O, r_0) \); i.e., for each non-negative \( \varphi \in C^1_0(B) \),
\[
\int -\langle \nabla u, \nabla \varphi \rangle \geq \int -\sigma_1(A) \varphi.
\]

**Proof.** For \( \epsilon > 0 \) small, let \( \eta_\epsilon \) denote the cut-off function defined in (4.4), and let \( \varphi \in C^1_0(B) \) be non-negative. Since (4.3) holds on \( \Omega \) and \( \eta_\epsilon^2 \varphi \) is supported in \( \Omega \) we have
\[
\int \eta_\epsilon^2 \varphi \Delta u \geq \int \frac{(n-2)}{2} |\nabla u|^2 \eta_\epsilon^2 \varphi - \int \eta_\epsilon^2 \varphi \sigma_1(A).
\]
Integrating by parts,
\[
\int -\langle \nabla \varphi, \nabla u \rangle \eta_\epsilon^2 \geq \int 2 (\nabla \eta_\epsilon, \nabla u) \eta_\epsilon \varphi \geq \int \frac{(n-2)}{2} |\nabla u|^2 \eta_\epsilon^2 \varphi - \int \eta_\epsilon^2 \varphi \sigma_1(A),
\]
which we rewrite as
\[
\int -\langle \nabla \varphi, \nabla u \rangle \eta_\epsilon^2 \geq \int \left[ 2 (\nabla \eta_\epsilon, \nabla u) \eta_\epsilon \varphi + \frac{(n-2)}{2} |\nabla u|^2 \eta_\epsilon^2 \varphi \right] - \int \eta_\epsilon^2 \varphi \sigma_1(A).
\]
By the Lebesgue Dominated convergence theorem, for the first and last integrals in (4.15) we have
\[
\int -\langle \nabla \varphi, \nabla u \rangle \eta_\epsilon^2 \rightarrow \int -\langle \nabla \varphi, \nabla u \rangle,
\]
and
\[
\int \eta_\epsilon^2 \varphi \sigma_1(A) \rightarrow \int \varphi \sigma_1(A)
\]
as \( \epsilon \rightarrow 0 \). We estimate the middle integral in the following way: First,
\[
\int 2 (\nabla \eta_\epsilon, \nabla u) \eta_\epsilon \varphi \geq \int -\frac{(n-2)}{2} |\nabla u|^2 \eta_\epsilon^2 \varphi - \int \frac{2}{(n-2)} \varphi |\nabla \eta_\epsilon|^2.
\]
Therefore,
\[
\int \left[ 2(\nabla \eta, \nabla u) \eta \varphi + \frac{(n-2)}{2} |\nabla u|^2 \eta^2 \varphi \right] \geq \int -C_n \varphi |\nabla \eta|^2.
\]
Note that
\[
\int \varphi |\nabla \eta|^2 \leq C \epsilon^{-2} \int_{B(O,2 \epsilon) \setminus B(O, \epsilon)} \varphi \leq C \epsilon^{(n-2)} \to 0
\]
as \( \epsilon \to 0 \). Substituting (4.16), (4.17), and (4.18) into (4.15), we get (4.13). \( \square \)

To complete the proof of the Proposition, we refer to Theorem 8.17 of Gilbarg-Trudinger [GT83], which implies that any \( W^{1,2} \)-solution of (4.12) satisfies
\[
\sup_{B(O, r_0/2)} u \leq C(r_0, g) \left( \|u\|_{L^2(B(O, r_0))} + C(g) \right).
\]
Thus, the desired bound follows from Lemma 4.2. \( \square \)

While Proposition 4.1 gives an upper bound on solutions, there are examples of singular solutions for which a lower bound fails to hold. The next result controls the rate at which \( u(x) \) can go to \( -\infty \), provided \( u \) satisfies (1.31).

**Proposition 4.4.** Let \( u \in C^{1,1}_{loc}(\Omega) \) satisfy
\[
\sigma_1(A + \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g) \geq 0
\]
a.e. in \( \Omega = B(O, r_0) \setminus \{O\} \). Assume \( u \) satisfies the scale-invariant gradient estimate (1.31). Then there is a constant \( C > 0 \) such that
\[
u(x) \geq 2 \log d(x) - C.
\]

The Proof of Proposition 4.4 is essentially contained in Proposition 6.1 of [GV04b]. The only difference is the regularity assumed: we need to show the same argument applies to \( C^{1,1} \)-solutions.

**Proof.** As we observed above, \( u \in C^{1,1}_{loc}(\Omega) \) satisfies the inequality
\[
\Delta u \geq \frac{(n-2)}{2} |\nabla u|^2 - \sigma_1(A)
\]
a.e. in \( \Omega \). Let
\[
w = e^{-\frac{(n-2)}{2} u}.
\]
Then \( w \in C^{1,1}_{loc}(\Omega) \), and a simple calculation shows that \( w \) satisfies
\[
Lw = \Delta w - \frac{(n-2)}{4(n-1)} Rw \leq 0
\]
a.e. in $\Omega$. Let $\Gamma$ denote the Green’s function for $L$ with pole at $O$. Since $\Gamma(x) \sim d(x)^{2-n}$, it suffices to prove
\begin{equation}
(4.23) \quad w(x) \leq C \Gamma(x), \quad x \in \Omega
\end{equation}
for some constant $C$. To this end, consider the function
\begin{equation}
(4.24) \quad G(x) = \frac{w(x)}{\Gamma(x)}.
\end{equation}
It follows from (4.22) and the definition of $\Gamma$ that
\begin{equation}
(4.25) \quad \Delta G \leq -2\langle \nabla G, \frac{\nabla \Gamma}{\Gamma} \rangle
\end{equation}
a.e. in $\Omega$. Since clearly $G \in C^{1,1}_{\text{loc}}(\Omega)$, it follows that (4.26) holds in a $W^{1,2}$-sense in $\Omega$.

Now, fix $r > 0$ small, and let $\Omega_r = B \setminus B(O, r)$. By the strong maximum principle [GT83, Theorem 8.19], $G$ cannot attain an interior minimum in $\Omega_r$ unless it is constant; of course, if $G$ were constant then (4.23) would follow immediately. Therefore, assume $G$ attains its minimum on $\partial \Omega_r = \partial B(O, r_0) \cup \partial B(O, r)$; in fact, assume it is attained on $\partial B(O, r)$.

Since we are assuming $u$ satisfies a scale-invariant gradient estimate, given any two points $x, y \in \partial B(O, r)$ we have
\begin{equation}
|u(x) - u(y)| \leq Cr \max_{\partial B(O, r)} |\nabla u| \leq C.
\end{equation}
From this inequality it follows that $w$—and hence $G$—satisfies a Harnack inequality:
\begin{equation}
(4.26) \quad \max_{\partial B(O, r)} G \leq C \min_{\partial B(O, r)} G.
\end{equation}
Therefore,
\begin{equation}
\min_{\Omega_r} G = \min_{\partial \Omega_r} G \geq C^{-1} \max_{\partial B(O, r)} G.
\end{equation}
In case the minimum of $G$ is attained on $\partial B(O, r_0)$, we can apply the same argument. In either case, we conclude
\begin{equation}
(4.27) \quad \max_{\partial \Omega_r} G \leq C \min_{\Omega_r} G.
\end{equation}
If we choose a point $x_0 \in \Omega$, then (4.27) implies
\begin{equation}
(4.28) \quad \max_{\partial \Omega_r} G \leq C \min_{\Omega_r} G \leq CG(x_0) = C',
\end{equation}
Since (4.28) holds for all $r > 0$ small, it follows that $G$ is uniformly bounded in $\Omega$. This completes the proof.
4.2. **An integral estimate.** The final result of this section is an integral estimate which will be used in the proof of Theorem 1.17.

**Proposition 4.5.** Let $u \in C^{1,1}_{\text{loc}}$ satisfy $A_u \in \Gamma_\delta$ almost everywhere in $\Omega = B(O, r_0) \setminus \{O\}$. Let

$$v = e^{\beta u},$$

where $\beta$ is defined in (1.39). Then $v$ satisfies

$$\int_\Omega |\nabla v|^n \leq C(\delta, n, g).$$

**Proof.** This estimate is actually a corollary of Theorem 3.7 of [GV04b]:

**Theorem 4.6.** ([GV04b, Theorem 3.7]) Let $u \in C^{1,1}_{\text{loc}}(A(\frac{1}{2}r_1, 2r_2))$, where $O \in M^n$ and $A(\frac{1}{2}r_1, 2r_2)$ denotes the annulus $A(\frac{1}{2}r_1, 2r_2) \equiv B(O, 2r_2) \setminus B(O, \frac{1}{2}r_1)$, with $0 < r_1 < r_2$. Assume $g_u = e^{-2u}g$ satisfies

$$Ric(g_u) - 2\delta_0\sigma_1(A_u)g \geq 0$$

almost everywhere in $A(\frac{1}{2}r_1, 2r_2)$ for some $0 \leq \delta_0 < \frac{1}{2}$. Define

$$\alpha_0 = \frac{(n - 2)}{(1 - 2\delta)} \delta_0 \geq 0.$$ 

Then given any $\alpha > \alpha_0$, there are constants $p \geq n$ and $C = C((\alpha - \alpha_0)^{-1}, n) > 0$ such that

$$\int_{A(r_1,r_2)} |\nabla u|^p e^{\alpha u} dv_g \leq C \left( \int_{A(\frac{1}{2}r_1, 2r_2)} |Ric_g|^{p/2} e^{\alpha u} dv_g \right. $$

$$+ r_1^{-p} \int_{A(\frac{1}{2}r_1, r_1)} e^{\alpha u} dv_g + r_2^{-p} \int_{A(r_2, 2r_2)} e^{\alpha u} dv_g \).$$

In fact, we can take

$$p = n + 2\alpha_0 \geq n.$$ 

**Remark 4.7.** Keeping with the conventions of this paper we will omit the volume form $dv_g$.

Let $\delta_0$ and $\delta$ be related by the formula

$$\delta = \frac{1 - 2\delta_0}{n - 2},$$

or equivalently,

$$2\delta_0 = 1 + (2 - n)\delta.$$ 

Inequality (4.30) for $\delta_0$ is equivalent to saying that $A_u \in \Gamma_\delta$. Since $A_u \in \Gamma_\delta$ almost everywhere in $\Omega = B(O, r_0) \setminus \{O\}$, inequality (4.30) holds for $\delta_0$ as defined in (4.35).
Clearly, the inequality (4.30) then also holds for $\delta_0 = 0$, and hence $\alpha_0 = 0$ and $p = n$. Letting $r_2 = \frac{1}{2} r_0$ and $r_1 = r < \frac{1}{2} r_0$, from (4.32) we have
\[
\int_{A(r,r_0/2)} |\nabla u|^n e^{\alpha u} \leq C(\alpha^{-1}, r_0, g) \left( \int_{A(r/2,r_0)} e^{\alpha u} + r^{-n} \int_{A(r/2,r)} e^{\alpha u} + \int_{A(r_0/2,r_0)} e^{\alpha u} \right),
\]
for any $\alpha > 0$. By Proposition 4.1, $u$ is bounded above on $\Omega$, and therefore $e^{\alpha u} \leq C$ on $\Omega$. Also, notice the middle integral on the right-hand side of (4.36) is uniformly bounded:
\[
r^{-n} \int_{A(r/2,r)} e^{\alpha u} \leq C r^{-n} \int_{A(r/2,r)} \leq C r^{-n} (cr^n) \leq C.
\]
Consequently, for all $\alpha > 0$ we have
\[
\int_{A(r,r_0/2)} |\nabla u|^n e^{\alpha u} \leq C(\alpha^{-1}, r_0, g)
\]
(4.37) independent of $r$. Letting $r \to 0$ we obtain
\[
\int_\Omega |\nabla u|^n e^{\alpha u} \leq C(\alpha^{-1}, r_0, g).
\]
(4.38)
If $v = e^{\beta u}$, then
\[
|\nabla v|^n = \beta^n |\nabla u|^n e^{\beta nu}.
\]
Therefore, taking $\alpha = \beta n$ in (4.38) we get (4.29).

5. The Proof of Theorems 1.15 and 1.17

To prove Theorems 1.15 and 1.17 we assume $u \in C^{1,1}_{loc}(\Omega)$ with $A_u \in \Gamma_\delta$ almost everywhere. For part (i), we further assume
\[
\liminf_{x \to O} u > -\infty,
\]
and that either $g$ is LCF, or that $u$ satisfies a scale-invariant $C^2$-estimate. In each case we wish to show that $u$ can be extended to a Hölder continuous function $u^* \in C^{1,1}_{loc}(\Omega) \cap C^\gamma(B(O, r_0))$, where $\gamma$ is defined in (1.23). Note that (5.1) along with (1.2) imply
\[
\sup_\Omega |u| < \infty.
\]
As in the proof of Theorem 3.1 let \( v = e^{\beta u} \), where \( \beta \) is defined in (1.39). By (5.2), \( v \in C^{1,1}_{loc}(\Omega) \) satisfies
\[
0 < c_0 \leq v(x) \leq c_0^{-1}. \tag{5.3}
\]
First, consider the locally conformally flat case. Let \( \{x^i\} \) denote conformally flat coordinates centered at \( O \); in this coordinate system (3.2) is equivalent to
\[
(\partial_i \partial_j v + \beta v A_{ij}) \in \Gamma_\delta \tag{5.4}
\]
a.e. in \( \Omega \). Define
\[
W(x) = v(x) + \Lambda |x|^2, \tag{5.5}
\]
where \( |x|^2 = \sum_i (x^i)^2 \) and \( \Lambda >> 0 \) is a large constant. Then
\[
\partial_i \partial_j W = \partial_i \partial_j v + 2\Lambda \delta_{ij} = \partial_i \partial_j v + \beta v A_{ij} + 2\Lambda \delta_{ij} - \beta v A_{ij}.
\]
Therefore, by (5.3) and (5.4), we can choose \( \Lambda >> 0 \) large enough so that
\[
\partial_i \partial_j W \in \Gamma_\delta \tag{5.6}
\]
a.e., in a deleted neighborhood of \( O \).

Using the coordinates \( \{x^i\} \) we can identify a neighborhood of \( O \in M^n \) with a neighborhood \( U \) of the origin \( 0 \in \mathbb{R}^n \). Following [TW99], let \( \rho \in C^\infty_0 \) be a spherically symmetric mollifier satisfying \( \rho(x) \geq 0, \rho(x) = 1 \) for \( |x| < 1, \rho(x) = 0 \) for \( |x| > 2 \), and \( \int \rho = 1 \). Define the mollification of \( W \) by
\[
W_h(x) = h^{-n} \int \rho\left(\frac{x-y}{h}\right)W(y)dy. \tag{5.7}
\]
Let \( U' \) be a subset of \( U \), such that the \( h \)-neighborhood of \( U' \) is also contained in \( U \).

**Proposition 5.1.** \( W_h : U' \to \mathbb{R} \) is a smooth, bounded, strictly \( \delta \)-convex function.

**Proof.** The smoothness of \( W_h \) follows from elementary properties of convolutions. We let \( x \in U \) with \( d(x, \partial U) > h \) and \( r > 0 \) a small number. By the divergence theorem,
\[
D_{ij}W_h(x) = \int_U D_{ij}\rho_h(x-y)W(y)dy
\]
\[
= \int_{B(0, r)} D_{ij}\rho_h(x-y)W(y)dy + \int_{U \setminus B(0, r)} D_{ij}\rho_h(x-y)W(y)dy
\]
\[
= \int_{B(0, r)} D_{ij}\rho_h(x-y)W(y)dy - \int_{U \setminus B(0, r)} D_i\rho_h(x-y)D_j W(y)dy
\]
\[
+ \oint_{\partial B(0, r)} \nu_j D_i\rho_h(x-y)W(y)dS(y),
\]
where \( \{\nu_j\} \) are the components of the outward unit normal to \( \partial B(0, r) \). Since \( W \) is bounded, as \( r \to 0 \) we obtain
\[
D_{ij}W_h(x) = - \int_U D_i\rho_h(x-y)D_j W(y)dy. \tag{5.8}
\]
Applying the divergence theorem again, (5.9)
\[ D_{ij}W_h(x) = -\int_{B(0,r)} D_i \rho_h(x - y) D_j W(y) dy - \int_{U \setminus B(0,r)} D_i \rho_h(x - y) D_j W(y) dy \]
\[ = -\int_{B(0,r)} D_i \rho_h(x - y) D_j W(y) dy + \int_{U \setminus B(0,r)} \rho_h(x - y) D_{ij} W(y) dy \]
\[ - \oint_{\partial B(0,r)} \rho_h(x - y) D_j W(y) dS(y). \]

**Lemma 5.2.** There is a sequence \( r_i \to 0 \) such that (5.10)
\[ \left| \int_{\partial B(0,r_i)} \rho_h(x - y) \nu_i D_j W(y) dS(y) \right| \to 0, \]
and (5.11)
\[ \left| \int_{B(0,r_i)} D_i \rho_h(x - y) D_j W(y) dy \right| \to 0, \]
as \( i \to \infty \).

**Proof.** By Proposition 4.5, \( v \) satisfies (5.12)
\[ \int_{\Omega} |\nabla v|^n \leq C(\delta, n, g). \]
This implies (5.13)
\[ \int_{U} |D W(y)|^n dy \leq C. \]
It follows from the co-area formula that there is a sequence of radii \( r_i \to 0 \) such that (5.14)
\[ \int_{\partial B(0,r_i)} |D W(y)|^n dS(y) \leq C/r_i. \]
Therefore, by Hölder’s inequality,
\[ \left| \int_{\partial B(0,r_i)} \rho_h(x - y) \nu_i D_j W(y) dS(y) \right| \leq C \int_{\partial B(0,r_i)} |D W(y)| dS(y) \]
\[ \leq C r_i^{(n-1)/n} \left( \int_{\partial B(0,r_i)} |D W(y)|^n dS(y) \right)^{1/n} \]
\[ \leq C r_i^{n-2}, \]
and (5.10) follows.

For the same sequence of radii, using (5.13) and Hölder’s inequality we have
\[ \left| \int_{B(0,r_i)} D_i \rho_h(x - y) D_j W(y) dy \right| \leq C \int_{B(0,r_i)} |D W(y)| dy \]
\[ \leq C r_i^{n-1} \left( \int_{B(0,r_i)} |D W(y)|^n dy \right)^{1/n} \]
\[ \leq C r_i^{n-1}. \]
Taking $r = r_i$ in (5.9) and letting $i \to \infty$ we obtain
\begin{equation}
D_{ij} W_h(x) = \int_U \rho_h(x-y) D_{ij} W(y) dy.
\end{equation}
Since $D^2 W \in \Gamma_\delta$ almost everywhere in $U$, it follows that $D_{ij} W_h(x) \in \Gamma_\delta$. Consequently, $W_h$ is a smooth, strictly $\delta$-convex function on $U'$.

\textbf{Proposition 5.3.} $W$ has a $C^\gamma$-Hölder continuous extension across the origin.

\textbf{Proof.} From Theorem 1.3 and the Arzela-Ascoli Theorem, $W_{h_i} \to \overline{W}$ uniformly in the $C^{\gamma'}$-norm, $\gamma' < \gamma$, for some sequence $h_i \to 0$, and $\overline{W} \in C^{\gamma}(B)$. Define $W(0) = \overline{W}(0)$. From general properties of mollification, $W_{h_i} \to W$ on $B \setminus \{0\}$. Therefore $W = \overline{W}$ in $B$.

Since $v = W - \Lambda |x|^2$, the same holds for $v$ and (by the definition of $v$) for $u$ as well.

Turning to the non-$LCF$ case, we now assume $u$ satisfies a scale-invariant $C^2$-estimate. In view of (5.2) and inequality (1.32), $v$ satisfies
\begin{equation}
|\nabla^2 v(x)| + |\nabla v(x)|^2 \leq c_2 d(x)^{-2}
\end{equation}
for almost every $x \in \Omega$.

Let $\{x^i\}$ denote normal coordinates (with respect to the background metric $g$) centered at $O$. In this coordinate system (3.2) is equivalent to
\begin{equation}
g^{il} \left( \partial_i \partial_j v - \Gamma^k_{ij} \partial_k v + \beta v A_{ij} \right) \in \Gamma_\delta
\end{equation}
a.e. in $\Omega$. In normal coordinates, $g^{il} = \delta^{il} + O(|x|^2)$, and $|\Gamma^k_{ij}| = O(|x|)$ as $|x| \to 0$, so using (5.3) and (5.16) we conclude
\begin{equation}
\partial_i \partial_j v + B_{ij} \in \Gamma_\delta,
\end{equation}
where $B_{ij}$ is uniformly bounded:
\begin{equation}
\|B_{ij}\| < C_1.
\end{equation}
As before, we let
\begin{equation}
W(x) = v(x) + \Lambda |x|^2.
\end{equation}
Then
\begin{align*}
\partial_i \partial_j W &= \partial_i \partial_j v + 2\Lambda \delta_{ij} \\
&= \partial_i \partial_j v + B_{ij} + 2\Lambda \delta_{ij} - B_{ij}.
\end{align*}
Therefore, by (5.18) and (5.19) we can choose $\Lambda >> 0$ large enough so that
\begin{equation}
\partial_i \partial_j W \in \Gamma_\delta
\end{equation}
a.e., in a deleted neighborhood of $O$. The rest of the proof proceeds exactly as in the $LCF$-case (note however that Lemma 5.2 is much easier to prove under the assumption (1.31)). This completes the proof of part (i) of Theorem 1.15 and Theorem 1.17.
To prove part (ii) of Theorems 1.15 and 1.17, we assume
\[ \liminf_{x \to O} u = -\infty, \] (5.21)
and first assume that either \( g \) is \( LCF \); or that \( u \) satisfies (1.32). In each case we wish to show that \( u \) obeys the growth estimate
\[ u(x) \leq 2 \log d(x) + C. \] (5.22)
To this end, we appeal to part (i), in which we showed that the function \( v = e^{\beta u} \) can be extended to a Hölder continuous function \( v^* \in C^{1,1}_{loc}(\Omega) \cap C^\gamma(B(O, r_0)) \), where \( \gamma \) is given by (1.23). Therefore,
\[ |v^*(x) - v^*(y)| \leq Cd(x, y)\gamma \]
for all \( x, y \) near \( O \). Rewriting this in terms of \( u \), we have
\[ |e^{\beta u(x)} - e^{\beta u(y)}| \leq Cd(x, y)\gamma \]
for all \( x, y \in \Omega \). Since \( \beta = \gamma/2 \), this implies
\[ |e^{u(x)} - e^{u(y)}| \leq Cd(x, y)^2. \] (5.23)
From assumption (5.21), there exists a sequence of points \( y_i \in \Omega \) with \( y_i \to O \) and \( u(y_i) \to -\infty \) as \( i \to \infty \). Taking \( y = y_i \) in (5.23) and letting \( i \to \infty \) we obtain
\[ e^{u(x)} \leq Cd(x)^2, \]
which implies
\[ u(x) \leq 2 \log d(x) + C. \] (5.24)
To complete the proof of Theorems 1.15 and 1.17 we turn our attention to the lower inequality (1.25). Since in both cases we are assuming \( u \) satisfies scale-invariant \( C^1 \)-estimate it follows from Proposition 4.4 that
\[ u(x) \geq 2 \log d(x) - C. \] (5.25)

Remark 5.4. The method in this section simplifies somewhat our previous proof of the above estimate (5.24) which was given in Section 6 of [GV04b]. However, the methods are in essence the same in that they are both based on some version of the maximum principle.

6. THE SCALE-ININVARIANT ESTIMATES

In this section we verify the scale-invariant estimates (1.31) for \( C^3 \)-solutions and (1.32) for \( C^4 \)-solutions subject to various assumptions. These results follow from various local estimates for solutions, along with a scaling argument.

The local gradient and \( C^2 \)-estimates for (1.19) and (1.20) were first proved in [GW03b] and [GLW04]. This work was extended to local gradient and \( C^2 \)-estimates for (1.21) in [LL03]. For the case of general symmetric functions \( F \) satisfying assumptions (i) – (iv) in the introduction, local \( C^2 \)-estimates for \( C^4 \) solutions were proved recently in [Che05]. We note that the local gradient estimates for \( C^3 \) solutions are currently not known for general \( F \), but we conjecture these to be true.
We say that \( u \in C^2 \) is \( k \)-admissible if \( A_u \in \Gamma_{\sigma_k} \) (this is not to be confused with the notion of \( \delta \)-admissibility defined above). For equations \( \text{(1.19)} \) and \( \text{(1.20)} \), the scale-invariant estimates are based on the following results of Guan-Wang and Guan-Lin-Wang:

**Theorem 6.1.** (Theorem 1.1 of \([GW03b]\), Theorem 1 of \([GLW04]\)) Let \( u \in C^3(M^n) \) be a \( k \)-admissible solution of \( \text{(1.19)} \) or \( \text{(1.20)} \) in \( B(x_0, \rho) \), where \( x_0 \in M^n \) and \( \rho > 0 \). Then there is a constant

\[
C_0 = C_0(k, n, \rho, \|g\|_{C^2(B(x_0, \rho))}, \|f\|_{C^1(B(x_0, \rho))}),
\]

such that

\[
|\nabla u|^2(x) \leq C_0 \left( 1 + e^{-2\inf_{B(x_0, \rho)} u} \right)
\]

for all \( x \in B(x_0, \rho/2) \).

Let \( u \in C^4(M^n) \) be a \( k \)-admissible solution of \( \text{(1.19)} \) in \( B(x_0, \rho) \), where \( x_0 \in M^n \) and \( \rho > 0 \). Then there is a constant

\[
C_0 = C_0(k, n, \rho, \|g\|_{C^3(B(x_0, \rho))}, \|f\|_{C^2(B(x_0, \rho))}),
\]

such that

\[
|\nabla^2 u|(x) + |\nabla u|^2(x) \leq C_0 \left( 1 + e^{-2\inf_{B(x_0, \rho)} u} \right)
\]

for all \( x \in B(x_0, \rho/2) \).

We first observe that when \( f(x) \equiv 0 \) in \( \text{(1.19)} \) or \( \text{(1.20)} \), then there is no exponential term in the estimate \( \text{(6.1)} \). We will only verify this explicitly for solutions of \( \text{(1.19)} \), but the argument for solutions of \( \text{(1.20)} \) is essentially identical.

**Corollary 6.2.** Let \( u \in C^3(M^n) \) be a \( k \)-admissible solution of \( \text{(1.19)} \) in \( B(x_0, \rho) \) with \( f(x) \equiv 0 \). Then there is a constant

\[
C_0 = C_0(k, n, \rho, \|g\|_{C^2(B(x_0, \rho))})
\]

such that

\[
|\nabla u|^2(x) \leq C_0
\]

for all \( x \in B(x_0, \rho/2) \). In fact,

\[
C_0 = C_1 \rho^{-2},
\]

where \( C_1 = C_1(k, n, \|g\|_{C^2(B(x_0, \rho))}) \).

Let \( u \in C^4(M^n) \) be a \( k \)-admissible solution of \( \text{(1.19)} \) in \( B(x_0, \rho) \) with \( f(x) \equiv 0 \). Then there is a constant

\[
C_0 = C_0(k, n, \rho, \|g\|_{C^3(B(x_0, \rho))})
\]

such that

\[
|\nabla^2 u|(x) + |\nabla u|^2(x) \leq C_0
\]

for all \( x \in B(x_0, \rho/2) \). In fact,

\[
C_0 = C_1 \rho^{-2},
\]
where \( C_1 = C_1(k, n, \| g \|_{C^3(B(x_0, \rho))}) \).

**Proof.** If we imitate the proof of Guan and Wang, one can trace the origin of the exponential term in (6.1) to two places: inequalities (2.20) and (3.10) in \([GW03b]\). These inequalities appear when estimating the gradient and Hessian terms respectively.

For the gradient term, Guan and Wang estimate

\[
T_1 = \sum_l F_l u_l = \sum_l (fe^{-2u})_l u_l = \sum_l e^{-2u} (f_l u_l - 2 f u_l^2),
\]

where the subscript \( l \) denotes \( \frac{\partial}{\partial x_l} \). They do so in the following way:

\[
T_1 = \sum_l F_l u_l = \sum_l e^{-2u} (f_l u_l - 2 f u_l^2) \geq -C(1 + e^{-2u})|\nabla u|^2,
\]

thus the appearance of the exponential term in (6.1). Of course, if \( f \equiv 0 \) then \( T_1 \equiv 0 \).

While estimating the Hessian a similar term appears in (3.10) of \([GW03b]\): \( T_2 = \sum \rho F_{ll} \), where \( \rho \) is a cut-off function supported in \( B(0, r_0) \). Unfortunately, a typographical error appears when estimating \( T_2 \); the first line of (3.10) should read

\[
T_2 = \sum_l \rho F_{ll} = \sum_l \rho (f e^{-2u})_l = \sum_l \rho (f_l - 4 f_l u_l + 4 f u_l^2 - 2 f u_l) e^{-2u}.
\]

In any case, once again if \( f \equiv 0 \) then \( T_2 \equiv 0 \), and the Hessian estimate no longer contributes an exponential term. Consequently, inequality (6.5) holds.

To prove (6.6) we need to specify the dependence of \( C_0 \) on the radius of the ball \( \rho \); that is, we need to show

\[
|\nabla^2 u(x)| + |\nabla u|^2(x) \leq C_1(g)/\rho^2.
\]

However, such an estimate follows from an elementary scaling argument; see Lemma 3.3 of \([GV04b]\) for a detailed explanation.

From Corollary 6.2 it follows that the scale-invariant estimate (1.32) holds for solutions of (1.19) when \( f \equiv 0 \):

**Corollary 6.3.** Let \( u \in C^3(\Omega) \) be a (degenerate) admissible solution of (1.19) in \( \Omega = B(O, r_0) \setminus \{O\} \) with \( f \equiv 0 \) in a neighborhood of \( O \). Then \( u \) satisfies (1.31).

Let \( u \in C^4(\Omega) \) be a (degenerate) admissible solution of (1.19) in \( \Omega = B(O, r_0) \setminus \{O\} \) with \( f \equiv 0 \) in a neighborhood of \( O \). Then \( u \) satisfies (1.32).
Proof. For $x \in \Omega$ close enough to $O$, $u$ is a solution of (1.19) in the ball $B = B(x, d(x)/2)$, where $d(x) = d(O, x)$. By Corollary 6.2, if $u$ is $C^3$, then $u$ satisfies
\[ |\nabla u|^2(x) \leq Cd(x)^{-2}, \]
and if $u$ is $C^4$, then $u$ satisfies
\[ |\nabla^2 u|(x) + |\nabla u|^2(x) \leq Cd(x)^{-2}. \]

6.1. The Proof of Proposition 1.19. If one traces through the local estimates of Guan-Lin-Wang for the quotient equations (1.20) ([GLW04]), and in [GW04], or the estimates for solutions of (1.21) in [LL03], or the estimates for general $F$ in [Che05], then in all cases the presence of the exponential term comes from the right-hand side of the equation. Therefore, when $f \equiv 0$, the same argument presented above leads to the scale-invariant estimates for solutions of these equations. \qed

7. H"{o}lder Extension

Next we consider the case of finite volume metrics. As a preliminary observation, we note that a corollary of the local estimates is an $\epsilon$-regularity result:

Proposition 7.1. (See Proposition 3.6 of [GW03b] and Proposition 3.4 of [GV04b])

(i) Let $u \in C^3(B(x_0, \rho))$ be an admissible solution of (1.19), (1.20), or (1.21). Then there exist constants $\epsilon_0 > 0$ and $C = C(\epsilon_0)$ such that if
\[ \int_{B(x_0, \rho)} e^{-nu} d\text{vol}_g \leq \epsilon_0, \]
then
\[ \inf_{B(x_0, \rho/2)} u \geq -C_2 + \log \rho. \]
Consequently, there is a constant
\[ C_3 = C_3(k, n, \epsilon_0, \|g\|_{C^3(B(x_0, \rho))}), \]
such that
\[ |\nabla u|^2(x) \leq C_3\rho^{-2} \]
for all $x \in B(x_0, \rho/4)$.

(ii) Let $u \in C^4(B(x_0, \rho))$ be an admissible solution of (1.17). Then there exist constants $\epsilon_0 > 0$ and $C = C(\epsilon_0)$ such that if
\[ \int_{B(x_0, \rho)} e^{-nu} d\text{vol}_g \leq \epsilon_0, \]
then
\[ \inf_{B(x_0, \rho/2)} u \geq -C_2 + \log \rho. \]
Consequently, there is a constant
\[ C_4 = C_4(k, n, \epsilon_0, \|g\|_{C^4(B(x_0, \rho))}), \]
such that
\[ |\nabla^2 u(x)| + |\nabla u(x)|^2 \leq C_4 \rho^{-2} \]
for all \( x \in B(x_0, \rho/4) \).

Proof. For solutions of (1.19), (1.20), and (1.21), this was proved in Proposition 3.6 of [GW03b] and Proposition 3.4 of [GV04b]. For \( C^4 \) solutions of (1.18), this follows from the same method of proof, using the local \( C^2 \)-estimates from [Che05]. \( \square \)

7.1. The Proof of Theorems 1.22 and 1.23 Suppose \( u \in C^4 \) satisfies the hypotheses of Theorem 1.22. Since the volume of \( g_u = e^{-2u}g \) is finite,
\[ \text{Vol}(g_u) = \int_{\Omega} e^{-nu} < \infty. \]

Therefore, there is a radius \( r_1 = r_1(\epsilon_0) \) such that for all \( x \in \Omega \) with \( d(x) < r_1 \),
\[ \int_{B(x,d(x)/2)} e^{-nu} < \epsilon_0. \]

From (7.5) and (7.6) it follows that
\[ u(x) \geq -C + \log d(x), \quad |\nabla^2 u(x)| + |\nabla u(x)|^2 \leq Cd(x)^{-2}. \]

This proves Theorem 1.22. In the case that \( u \in C^3 \) and \( F \) is as in cases (1.19), (1.20), or (1.21), the above argument yields the scale invariant gradient estimate (1.31), which proves Theorem 1.23. \( \square \)

7.2. The Proof of Theorem 1.24 In Theorem 1.24 we assume that \( g \) is LCF and \( u \in C^3 \) is a (strictly) admissible solution of either (1.19), (1.20), or (1.21) in \( \Omega = B(O, r_0) \setminus \{O\} \), with \( f(x) \geq c_0 > 0 \) near \( O \). The goal is to show that \( u \) can be extended to a Hölder continuous function \( u^* \in C^3(\Omega) \cap C^\gamma(B(O, r_0)) \).

We first observe that the Ricci curvature of \( g_u = e^{-2u}g \) is strictly positive:
\[ \text{Ric}(g_u) \geq c_1 g_u, \]
where \( c_1 = c_1(\min f) > 0 \). This follows from [GVW03]; see Lemma 4.1 and the remark thereafter of [GV04b] for a proof.

Now, in view of Theorem 1.17 (i), if
\[ \liminf_{x \to 0} u(x) > -\infty \]
then we are done. Therefore, we must have
\[ \liminf_{x \to 0} u(x) = -\infty, \]
and from part (ii) of Theorem 1.17 it follows that
\[ u(x) \leq 2 \log d(x) + C. \]
Note that since \( f \) is not identically zero, the scale invariant gradient estimate is not necessarily satisfied, so we only get the upper inequality. To summarize the idea of the proof, we first show that (7.10) implies \( g_u \) has geodesics of arbitrary length. However, since \( g_u \) has strictly positive Ricci curvature, this will yield a contradiction, and it will follow that (7.9) must hold.

To analyze the behavior of \( g_u \) near the singularity let \( \{x^i\} \) denote conformally flat coordinates centered at \( O \), so that \( g_{ij} = \delta_{ij} \) in \( \Omega = B(O, r_0) \setminus \{O\} \). Let us write the metric as follows:

\[
g_u = e^{-2u} g = e^{-2\Psi} |x|^{-4} g = e^{-2\Psi} g_\ast \tag{7.11}
\]

where

\[
\Psi = u - 2 \log |x|, \tag{7.12}
\]

\[
g_\ast = |x|^{-4} g. \tag{7.13}
\]

Changing coordinates to \( z^j = |x|^{-2} x^j \), which are defined on \( \mathbb{R}^n \setminus B(0, r_0) \), it is easy to see that

\[
(g_\ast)_{ij}(z) = \delta_{ij}.
\]

Therefore,

\[
g_u = e^{-2\Psi} \delta_{ij} \geq e^{-2C} \delta_{ij}. \tag{7.14}
\]

The inverted \( z \)-coordinates are only defined on the complement of a ball, so let us extend \( \Psi \) arbitrarily to a function defined on all of \( \mathbb{R}^n \), and consider the metric \( \tilde{g} = e^{-2\Psi} \delta_{ij} \).

**Lemma 7.2.** The metric \((\mathbb{R}^n, \tilde{g})\) is geodesically complete.

**Proof.** Let \( x_0 \in \mathbb{R}^n \), and let \( \zeta(t) \) be a unit-speed geodesic with \( \zeta(0) = x_0 \). Assume the maximal domain of definition of \( \zeta \) is \( [0, T) \); we want to show that \( T = \infty \). We will make use of the following property of maximal geodesics: \( \zeta : [0, T) \rightarrow \mathbb{R}^n \) must leave every compact subset of \( \mathbb{R}^n \) as \( t \rightarrow T \). That is, given any compact subset \( K \subset \mathbb{R}^n \), there exists a \( t_K \) such that \( \zeta(t) \in \mathbb{R}^n \setminus K \) for all \( t > t_K \). For a simple proof of this fact, see [Pet98, page 109].

Therefore, without loss of generality, we may assume there is a time \( 0 < a < T \) such that \( \zeta(t) \in \mathbb{R}^n \setminus B(0, R) \) for \( t \geq a \). For \( t \geq a \), by (7.14)

\[
|\dot{\zeta}(t)|^2_{g_u} = (g_u)_{ij}(\zeta(t))\dot{\zeta}(t)^i \dot{\zeta}(t)^j \\
\geq (e^{-2C_1} \delta_{ij}) \dot{\zeta}(t)^i \dot{\zeta}(t)^j \\
\geq e^{-2C_1} |\dot{\zeta}(t)|^2_0. \tag{7.15}
\]
Now let \( b \in (a, T) \), so \( \zeta(b) \in \mathbb{R}^n \setminus B(0, R) \). Since \( \zeta \) has unit speed, the length of \( \zeta([0, b]) \) is given by

\[
b = L(\zeta([0, b])) = \int_0^b |\dot{\zeta}(t)|_{g_w} dt = \int_0^a |\dot{\zeta}(t)|_{g_w} dt + \int_a^b |\dot{\zeta}(t)|_{g_w} dt \\
\geq a + e^{-C_2} \int_a^b |\dot{\zeta}(t)|_0 dt.
\]

(7.16)

Since segments minimize distance in the Euclidean metric, we have

\[
\int_a^b |\dot{\zeta}(t)|_0 dt \geq |\zeta(b) - \zeta(a)|_0.
\]

Therefore,

\[
b - a \geq e^{-C_1} |\zeta(b) - \zeta(a)|_0.
\]

(7.17)

Now, recall that given any compact set \( K \subset \mathbb{R}^n \), there must be a time \( t_K \) with \( \zeta(t) \in \mathbb{R}^n \setminus K \) for \( t > t_K \). Therefore, by choosing a large enough compact set we can arrange so that \( \zeta(b) \in \mathbb{R}^n \setminus K \) and \( |\zeta(b) - \zeta(a)|_0 \) is as large as we like. By (7.17), this means we can choose \( b \) as large as we like, i.e., \( T = \infty \). It follows that \((\mathbb{R}^n, \tilde{g})\) is geodesically complete.

To finish the proof, \( \tilde{g} \) is a complete \( C^3 \) metric on \( \mathbb{R}^n \) which has strictly positive Ricci curvature outside of a compact set. By Myers’ Theorem (see [Pet98]), this is impossible. More precisely, take any constant \( N > 0 \), and choose a point \( y \) in \( \mathbb{R}^n \) with \( |y|_0 > N + R \). By the Hopf Rinow Theorem (which is valid for \( C^3 \) metrics), there exists a unit speed minimizing geodesic \( \zeta(t) : [0, d(x_0, y)] \rightarrow \mathbb{R}^n \) with respect to the metric \( \tilde{g} \) with \( \zeta(0) = 0 \). Choose the smallest time \( a \) so that \( \zeta(t) \in \mathbb{R}^n \setminus B(0, R) \) for \( t \geq a \). From (7.17), we have

\[
d(x_0, y) - a \geq e^{-C_1} |y - R|_0 \geq e^{-C_1} N.
\]

(7.18)

We can therefore find a minimizing geodesic in \( \mathbb{R}^n \setminus B(0, R) \) with arbitrarily long length. But from our assumption, together with Newton’s inequality, and the estimate (iv), \( \tilde{g} \) has strictly positive Ricci curvature \( \text{Ric} > cg > 0 \), on \( \mathbb{R}^n \setminus B(0, R) \), so Myer’s Theorem gives a upper bound on the length (depending only upon \( c \)), which is a contradiction.

**Remark 7.3.** We remark why this proof does not work in the general (non-locally conformally flat) case, even if we assume \( u \in C^4 \). The main point is that since \( f \) is not identically zero, a scale-invariant \( C^2 \)-estimate does not necessarily hold (see Section 6 below). In the course of the proof we used Theorem 1.17 which requires local conformal flatness, while Theorem 1.15 in the general case requires a scale-invariant \( C^2 \)-estimate to hold at the singularity. Also, Theorem 1.17 only requires the solution to be \( C^{1,1}_{\text{loc}} \), but we must assume \( C^3 \) to be able to apply the tools from Riemannian geometry that were used above.
8. Integral and Hölder estimates for admissible functions

In this section we prove Theorem 1.26. Let \( u \in C^{1,1}_{\text{loc}} \) satisfy \( A_u \in \Gamma_\delta \) almost everywhere in \( B(O, r_0) \), and once again denote
\[
v = e^{\beta u},
\]
where \( \beta \) is defined in (1.39). We first prove (1.40), from which the Hölder estimate (1.41) follows. Afterwards we will consider the conformally flat case.

**Proposition 8.1.** Let
\[
p_0 = 2 + \frac{1}{\delta} > n.
\]
Then given any \( q < p_0 - 1 \), there is a constant \( C = C(\delta, q, n, \mu, g) \) such that
\[
\| \nabla v \|_{L^q(B(x_0, r/2))} \leq C (1 + r^{-1}) \| \nabla v \|_{L^q(B(x_0, 3r/4))} + C \| v \|_{L^q(B(x_0, r))}.
\]

**Proof.** We first show that (3.2) implies that \( v \) satisfies a certain inequality with respect to the \( p \)-laplacian:

**Lemma 8.2.** Let \( p_0 \) be given by (8.1). Then
\[
\nabla_i (|\nabla v|^{p_0 - 2} \nabla_i v) \geq -C |\nabla v|^{p_0 - 2} \quad \text{a.e., where } C = C(\delta, n, g).
\]

**Proof.** First, note that (3.2) implies
\[
\nabla^2 v + \delta \Delta v g + \mu vg \geq 0 \quad \text{a.e.}
\]
for some constant \( \mu = \mu(\delta, n, g) > 0 \). Now, a simple calculation gives
\[
\nabla_i (|\nabla v|^{p_0 - 2} \nabla_i v) = (p_0 - 2) \nabla^2 v (\nabla v, \nabla v) |\nabla v|^{p_0 - 4} + \Delta v |\nabla v|^{p_0 - 2}
\]
\[
= (p_0 - 2) \left[ \nabla^2 v + \frac{1}{(p_0 - 2)} \Delta v g + \mu vg \right] (\nabla v, \nabla v) |\nabla v|^{p_0 - 4}
\]
\[
- \mu(p_0 - 2) |\nabla v|^{p_0 - 2} v.
\]
Since \( p_0 \) satisfies
\[
\frac{1}{(p_0 - 2)} = \delta,
\]
from (8.4) we conclude
\[
\nabla_i (|\nabla v|^{p_0 - 2} \nabla_i v) \geq -C |\nabla v|^{p_0 - 2} v.
\]

Now, suppose \( 2 < p < p_0 \). Then
\[
|\nabla v|^{2-p} \nabla_i (|\nabla v|^{p_0 - 2} \nabla v_i) = \frac{p - 2}{p_0 - 2} |\nabla v|^{2-p} \nabla_i (|\nabla v|^{p_0 - 2} \nabla v_i) + \frac{p_0 - p}{p_0 - 2} \Delta v.
\]
By Lemma 8.2 this implies
\[
|\nabla v|^{2-p} \nabla_i (|\nabla v|^{p-2} \nabla v_i) \geq \frac{p_0 - p}{p_0 - 2} \Delta v - C v,
\]
which we rewrite as
\[
\nabla_i (|\nabla v|^{p-2} \nabla v_i) \geq \frac{p_0 - p}{p_0 - 2} |\nabla v|^{2-p} \Delta v - C |\nabla v|^{2-p} v.
\]

**Lemma 8.3.** Suppose \( v \in C^{1,1}_{\text{loc}}(B(x_0, r)) \) satisfies (8.4). Then
\[
\Delta v \geq c_0 |\nabla^2 v| - C_1 v
\]
for positive constants \( c_0, C_1 \) depending on \( \delta, n, \) and \( g. \)

**Proof.** Choose any point \( P \in B(x_0, r) \) at which (8.4) holds. Let \( \{\nu_1, \nu_2, \ldots, \nu_n\} \) denote the eigenvalues of \( \nabla^2 v(P) \). Then \( \Delta v(P) = \nu_1 + \cdots + \nu_n \), and (8.4) implies
\[
\nu_i + \delta \Delta v(P) + \mu v(P) \geq 0
\]
for each \( 1 \leq i \leq n \). Summing this inequality for \( i \neq j \)
\[
\sum_{i \neq j} \nu_i + (n - 1)\delta \Delta v(P) + \mu(n - 1)v(P) \geq 0,
\]
which gives
\[
-\nu_j + [1 + (n - 1)\delta] \Delta v(P) + \mu(n - 1)v(P) \geq 0.
\]
From (8.9) and (8.11) we conclude that each eigenvalue satisfies
\[
[1 + (n - 1)\delta] \Delta v(P) + Cv(P) \geq \nu_j \geq -\delta \Delta v(P) - Cv(P).
\]
Inequality (8.8) follows immediately. \( \square \)

By (8.7) and (8.8),
\[
\nabla_i (|\nabla v|^{p-2} \nabla v_i) \geq c_0' |\nabla v|^{2-p} |\nabla^2 v| - C |\nabla v|^{2-p} v
\]
for positive constants \( c_0', C \). Let \( \eta \in C^\infty_0 \) be a smooth, non-negative cut-off function supported in \( B = B(x_0, r) \), with \( \eta(x) = 1 \) in \( B(x_0, r/2), \eta(x) = 0 \) on \( B \setminus B(x_0, \frac{3}{4}r) \), and \( |\nabla \eta| \leq Cr^{-1} \). Multiplying both sides of (8.13) by \( \eta \), integrating and applying the divergence theorem gives
\[
\int \eta |\nabla^2 v| |\nabla v|^{p-2} \leq C \int |\nabla \eta| |\nabla v|^{p-1} + C \int \eta |\nabla v|^{p-2} v.
\]
By Hölder’s inequality,
\[
\int \eta |\nabla v|^{p-2} v \leq \left( \int \eta |\nabla v|^{p-1} \right)^{(p-2)/(p-1)} \left( \int \eta v^{p-1} \right)^{1/(p-1)}.
\]
Therefore, by the properties of \( \eta \),
\[
\int_{B(x_0, r/2)} |\nabla^2 v| |\nabla v|^{p-2} \leq C (1 + r^{-1}) \int_{B(x_0, 3r/4)} |\nabla v|^{p-1} + C \int_{B(x_0, r)} v^{p-1}.
\]
By the Sobolev imbedding theorem,
\begin{equation}
\left( \int_{B(x_0, r/2)} |\nabla v|^{\frac{n}{n-r}(p-1)} \right)^{(n-1)/n} \leq C \int_{B(x_0, r/2)} |\nabla|\nabla v|^{p-1} + C r^{-1} \int_{B(x_0, r/2)} |\nabla v|^{p-1}
\end{equation}
\begin{equation}
\leq C \int_{B(x_0, r/2)} |\nabla^2 v| |\nabla v|^{p-2} + C r^{-1} \int_{B(x_0, r/2)} |\nabla v|^{p-1}
\end{equation}
\begin{equation}
\leq C (1 + r^{-1}) \int_{B(x_0, 3r/4)} |\nabla v|^{p-1} + C \int_{B(x_0, r)} v^{p-1}.
\end{equation}

Taking \( q = p - 1 \), this completes the proof of Proposition 8.1. \( \square \)

**Corollary 8.4.** We have
\begin{equation}
\int_{B(x_0, r/2)} |\nabla v|^p \leq C \left( \int_{B(x_0, r)} v^2 \right)^{p/2}
\end{equation}
for any \( p < p_\delta = \frac{n}{n-1} (p_0 - 1) \).

**Proof.** By tracing (8.4) we see that \( v \) satisfies the linear elliptic inequality
\( \Delta v \geq -C v \)
in \( B(x_0, r) \). Therefore,
\begin{equation}
\int_{B(x_0, 3r/4)} |\nabla v|^2 \leq C \int_{B(x_0, r)} v^2.
\end{equation}

Taking \( p = 3 < p_0 \) in (8.15) and using (8.17) gives
\begin{equation}
\left( \int_{B(x_0, r/2)} |\nabla v|^{(n-1)/n} \right) \leq C \int_{B(x_0, r)} v^2.
\end{equation}

If we now take \( p = 1 + 2n/(n-1) \) in (8.15) and continue iterating, we obtain the bound
\begin{equation}
\int_{B(x_0, r/2)} |\nabla v|^p \leq C \left( \int_{B(x_0, r)} v^2 \right)^{p/2}
\end{equation}
for any \( p < \frac{n}{n-1} (p_0 - 1) \). One can easily check that
\[ p_\delta = \frac{n}{n-1} (p_0 - 1) = \frac{n(1 + \delta)}{(n-1) \delta} > n. \]

The estimate (1.40) follows from (8.18) by interpolation. This gives the Hölder estimate (1.41):
\[ \|v\|_{C^\alpha(B(x_0, r/2))} \leq C \int_{B(x_0, r)} |v|; \]
for any \( \alpha < \gamma = \frac{1+(2-n)\delta}{1+\delta} \).
To complete the proof of Theorem 1.26 it only remains to show why in the \( LCF \) case we can take \( \alpha = \gamma \) in (1.41). However, this follows from the proof of Theorem 1.15: recall that in conformally flat coordinates \( \{ x^i \} \) inequality 3.2 is equivalent to
\[
(\partial_i \partial_j v + \beta v A_{ij}) \in \Gamma_{\delta}
\]
a.e. Defining \( W(x) = v(x) + \Lambda |x|^2 \) as before, with \( \Lambda >> 0 \) sufficiently large, we can verify that \( W \) is a \( \delta \)-convex. After convolving, we obtain a smooth \( \delta \)-convex mollification \( W_h \), and by Theorem 1.3 \( W_h \) satisfies the estimate (1.41) with \( \alpha = \gamma \). Letting \( h \to 0 \), we obtain the same estimate for \( W \) and \( v \).

Remark 8.5. In the locally conformally flat case, we note the proof of (1.40) only requires the \( \delta \)-convexity of \( v \), so the estimate (1.9) follows.

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