Abstract. In this paper we study linear parabolic equations on a finite oriented starshaped network; the equations are coupled by transmission conditions set at the inner node, which do not impose continuity on the unknown. We consider this problem as a parabolic approximation of a set of first order linear transport equations on the network and we prove that, when the diffusion coefficient vanishes, the family of solutions converges to the unique solution to the first order equations and satisfies suitable transmission conditions at the inner node, which are determined by the parameters appearing in the parabolic transmission conditions.

1. Introduction

In this paper we study a vanishing viscosity approximation for linear first order transport equations on a finite starshaped network composed by $m$ arcs $I_i$, $i \in \mathcal{M}$, an inner node $N$ and $m$ outer nodes $e_i$, $i \in \mathcal{M}$; each arc is a bounded interval $I_i = (0, L_i)$. We set

$$\mathcal{I} = \{ i \in \mathcal{M} : I_i \text{ is an incoming arc in the node } N \}$$

$$\mathcal{O} = \{ i \in \mathcal{M} : I_i \text{ is an outgoing arc from the node } N \}$$

$\mathcal{I}$ and $\mathcal{O}$ are non empty sets.

On each arc $I_i$, $i \in \mathcal{M}$, we consider a linear first order transport equation

$$u_{it} = -\lambda_i u_{ix} \quad x \in I_i, \quad t > 0,$$

where $\lambda_i > 0$; each equation is complemented by the initial condition

$$u_i(x, 0) = u_{0i}(x) \in BV(I_i),$$

and, when $i \in \mathcal{I}$, by the boundary condition at the outer node $e_i$,

$$u_i(0, t) = B_i, \quad t > 0, \quad i \in \mathcal{I};$$

moreover the equations for $i \in \mathcal{M}$, are coupled by the following condition at the inner node $N$,

$$\sum_{i \in \mathcal{I}} \lambda_i u_i(L_i, t) = \sum_{i \in \mathcal{O}} \lambda_i u_i(0, t),$$

which establishes the conservation of the flux.
First order nonlinear conservation laws on networks with transmission conditions preserving the flux at the inner nodes have been largely studied by several authors, see for instance [13] and references therein for traffic models.

Here we focus on cases when equality (1.4) is achieved imposing at the node this kind of conditions
\begin{equation}
\lambda_i u_i(0, t) = \sum_{j \in \mathcal{I}} \gamma_{ij} \lambda_j u_j(L_j, t), \quad \forall i \in O,
\end{equation}
where
\begin{equation}
\gamma_{ij} \geq 0 \forall i \in O \quad \text{and} \quad \sum_{i \in O} \gamma_{ij} = 1 \forall j \in \mathcal{I}.
\end{equation}

We notice that conditions (1.5) form actually a set of transmission conditions if the parameters \( \gamma_{ij} \) satisfy
\begin{equation}
\text{for all } i \in O \text{ there exists at least one } j \in \mathcal{I} \text{ such that } \gamma_{ij} > 0.
\end{equation}

Set
\begin{align*}
L^p(\mathcal{A}) &= \{ f = (f_1, f_2, \ldots, f_m) : f_i \in L^p(I_i) \}, \\
W^{m,p}(\mathcal{A}) &= \{ f = (f_1, f_2, \ldots, f_m) : f_i \in W^{m,p}(I_i) \}, \\
BV(\mathcal{A}) &= \{ f = (f_1, f_2, \ldots, f_m) : f_i \in BV(I_i) \}, \\
BV(\mathcal{A} \times (0, T)) &= \{ f = (f_1, f_2, \ldots, f_m) : f_i \in BV(I_i \times (0, T)) \},
\end{align*}
and \( \| f \| = \sum_{i \in \mathcal{M}} \| f_i \| \).

**Definition 1.1.** For every \( u_0 \in BV(\mathcal{A}) \), a function \( u \in BV(\mathcal{A} \times (0, T)) \) is a solution of (1.1)-(1.3), (1.5) if, for all \( \phi_i \in C^0_0((0, L_i) \times [0, T]) \),
\begin{align*}
\int_0^T \int_{I_i} [u_i(\phi_{i,t} + \lambda_i \phi_{i,x})](x, t)dxdt &+ \int_{I_i} u_{i0}(x)\phi_i(x, 0)dx = -\lambda_i B_i \int_0^T \phi_i(0, t)dt, \quad i \in \mathcal{I}, \\
\int_0^T \int_{I_i} [u_i(\phi_{i,t} + \lambda_i \phi_{i,x})](x, t)dxdt &+ \int_{I_i} u_{i0}(x)\phi_i(x, 0)dx = \beta_i \left( -\lambda_i u_i^0(N, t) + \epsilon u_i^0_x(N, t) \right) = \sum_{j \in \mathcal{I}} K_{ij}(u_j^0(N, t) - u_i^0(N, t)), \quad t > 0, \quad i \in \mathcal{M}, \\
u_i^0(e_i, t) &= u_{i0}(e_i), \quad i \in \mathcal{M}, \quad t > 0,
\end{align*}
for every \( u_0 \in BV(\mathcal{A}) \), problem (1.1)-(1.3), (1.5) has a unique solution in the sense of the above definition; this fact is easily proved by taking into accout that each component \( u_i \) has null weak derivative in the direction \( z = (s, \lambda_i s) \) and then using the Stampacchia’s Lemma [24] and the Green’s formula for BV functions [26] (see also [20], Remark 3 and Theorem 3).

In this paper we deal with a vanishing viscosity approximation for (1.1)-(1.3), (1.5), by considering the following parabolic problem, where \( \epsilon \) is a variable parameter to be sent to zero,
\begin{align}
\left\{
\begin{array}{ll}
u^t_i(x, t) &= -\lambda_i u^t_i + \epsilon u^t_{i,x} \quad x \in I_i, \quad t > 0, \quad i \in \mathcal{M}, \\
u^t_i(x, 0) &= u^t_{i0}(x) \in W^{2,1}(I_i), \quad x \in I_i, \quad i \in \mathcal{M}, \\
\beta_i \left( -\lambda_i u^t_i(N, t) + \epsilon u^t_{i,x}(N, t) \right) &= \sum_{j \in \mathcal{I}} K_{ij}(u^t_j(N, t) - u^t_i(N, t)), \quad t > 0, \quad i \in \mathcal{M}, \\
u^t_i(e_i, t) &= u^t_{i0}(e_i), \quad i \in \mathcal{M}, \quad t > 0,
\end{array}\right.
\end{align}
where \( u'(N,t) \) denotes \( u'(0,t) \) if \( i \in \mathcal{O} \) and denotes \( u'(L_i,t) \) if \( i \in \mathcal{I} \), and \( u'(e_i,t) \), \( u'(0,t) \) denote \( u'(0,t) \), \( u'(0) \) if \( i \in \mathcal{I} \), and denote \( u'(L_i,t) \), \( u'(0) \) if \( i \in \mathcal{O} \). Besides, we assume

\[
(1.9) \quad K_{ij} \geq 0, \quad K_{ij} = K_{ji} \quad \forall i, j \in \mathcal{M}, \quad \beta_i = \begin{cases} 1 & i \in \mathcal{I} \\ -1 & i \in \mathcal{O} \end{cases}.
\]

Finally, the values \( u_0' \) satisfy the transmission conditions in (1.8) for all \( i \in \mathcal{M} \),

\[
(1.10) \quad \beta_i (-\lambda_i u_{0i}(N) + \epsilon u_{0e}(N)) = \sum_{j \in \mathcal{M}} K_{ij} (u_{0j}(N) - u_{0e}(N))
\]

and \( u_0' \) approximates \( u_0 \) for \( \epsilon \to 0 \) in a suitable way (see Section 3).

Transmission conditions in (1.8) imply that the sum of the fluxes incoming in the inner node \( N \) is equal to the one of the outgoing fluxes, thanks to the assumptions on the coefficients \( K_{ij} \); however these conditions impose no continuity condition on the solution at node, as assumed for instance in [13, 6, 0, 25], so that they seem to be more appropriate when dealing with mathematical models for movements of individuals, as in traffic flows or in biological phenomena involving bacteria or other microorganisms movements, where discontinuities at the nodes for the density functions are expected. This kind of conditions were introduced by Kedem-Katchalsky in [18] as permeability conditions in the description of passive transport through a biological membrane, see also [22, 4, 5] for various mathematical and related formulations; more recently they were proposed as transmission conditions for hyperbolic-parabolic and hyperbolic-elliptic systems describing movements of microorganism on networks, driven by chemotaxis [12, 13, 14, 15].

In the next section we are going to prove that problem (1.8)-(1.10) has a unique smooth solution \( u' \) in the following sense.

**Definition 1.2.** For every \( u''_0 \in W^{2,1}(\mathcal{A}) \) satisfying (1.17), a solution of (1.8) is a function \( u' \in C([0, +\infty); W^{2,1}(\mathcal{A})) \cap C^1([0, +\infty); L^1(\mathcal{A})) \) which solves the equation in (1.8) in strong sense and verifies the initial, boundary and transmission conditions in (1.8).

Moreover we shall prove some estimates uniform in \( \epsilon \) for \( u' \), assuming the following condition on the coefficients \( K_{ij} \) in (1.8),

\[
(1.11) \quad \text{for all } i \in \mathcal{I}, K_{ij} > 0 \text{ for some } j \in \mathcal{O} \text{ (at least one } j)\).
\]

The uniform estimates allow to use compacteness techniques and prove the convergence of each sequence \( \{u'_n\}_{n \in \mathbb{N}} \) if \( \epsilon_n \to 0 \), to the unique solution to problem (1.11) which satisfies (1.12), (1.3) and (1.5) with coefficients \( \gamma_{ij} \) determined by \( K_{ij} \) and \( \lambda_i \) (if \( j \in \mathcal{M} \)). The result is achieved provided that \( \{u'_n\}_{n \in \mathbb{N}} \) is a sequence approximating in suitable way the initial data \( u_0 \) and assuming the boundary data \( B_i \) on the incoming arcs; we show the actual possibility to obtain such approximation.

The vanishing viscosity for conservation laws on networks has been approached in several papers. In particular we refer to [6], where the authors approximate a conservation law on a starshaped network with unlimited arcs by means of a parabolic problem, enforcing a continuity condition at the inner node beside to conservation of the parabolic flux; also, we refer to [7] where a more general set of transmission conditions on the parabolic fluxes is considered, inspired by [12], which preserve the parabolic flux but allows discontinuities for the unknown; both the papers consider a nonlinear conservation law and the results concern the convergence of a subsequence of solutions of the parabolic problems to a solution of the conservation law satisfying the conservation of the flux at the inner node; the flux function \( f \) in the conservation law is such that \( f(1) = f(0) = 0 \) and...
initial data verify $0 \leq u_{0i} \leq 1$, so that a uniform $L^\infty$ bound for solutions to the parabolic problems holds thanks to comparison results, and compensated compactness arguments are used to prove strong convergence.

Here, thanks to the linearity of the problems, not only we can study in details the solvability of the regularized problem, but also we are able to give a more precise result of convergence: once the velocities $\lambda_i$ and the coefficients $K_{ij}$ in the conditions at the node $N$ in (1.8) have been set, we identify univocally the limit function as the solution to (1.1)-(1.3) satisfying (1.5) with specific $\gamma_{ij}$, so that the convergence result holds for all the sequence $u^n$, $\epsilon_n \to 0$. The structure of the transmission conditions in (1.8) is crucial in proving this result.

We remark that we could not obtain a uniform $L^\infty$ bound for the functions $\{u^n\}$ by means of comparison techiques since, in general, it is not easy to find supersolutions and subsolutions to the problem (1.8); as a matter of fact, the existence of solutions in the form $U_i(x,t) = C_i$ satisfying transmission conditions (1.8) is connected to the features of the coefficients matrix of the linear system

$$\sum_{j \in M} K_{ij}(C_j - C_i) - \beta_i \lambda_i C_i = 0, \quad i \in M;$$

moreover, although the equations in (1.8) are linear, if $u$ is a solution, the function $u + C$ no longer satisfies the transmission conditions, so that we are able to compare solutions only with the null one, i.e. the unique solution which is constant on the whole network and satisfies the transmission conditions in (1.8).

One may wonder if, every given set of parameters $\{\gamma_{ij}\}$ satisfying (1.6) and (1.7) admits at least a corresponding set $\{K_{ij}\}$ verifying (1.9), (1.11) so that problems (1.8) approximates the problem (1.1)-(1.5). In this direction, first we prove that under the condition

(1.12) for each $i \in \mathcal{O}$ there exists at least one $j \in \mathcal{I}$ such that $K_{ij} > 0$, problem (1.1)-(1.5), whose solution is the limit function of the sequences $u^n$ ($\epsilon_n \to 0$), actually verifies condition (1.7), i.e. (1.6) are actually transmission conditions. However, in the general case, we do not have the proof that to each given family of parameters $\gamma_{ij}$ satisfying (1.6), (1.7) corresponds at least one family of coefficients $K_{ij}$ in (1.8), satisfying (1.9), (1.11), which allow to achieve the approximation result, since this involves too heavy computations. In this paper, we are able to show how choosing the coefficients $K_{ij}$ in correspondence with the parameters $\gamma_{ij}$ only for some classes of networks and some classes of conditions (1.5).

The paper is organized as follows. In Section 2 we study the solvability of problem (1.8)-(1.10) by using the theory of semigroups. In Section 3 we derive a priori estimates for the solutions to (1.8)-(1.10), (1.11), which will turn out to be uniform in $\epsilon$ providing the family of initial data $u^n_0$ approximate $u_0$ in a suitable way. This approximation is described in Section 4, where, finally, we prove the convergence result for all the sequences of solutions of the parabolic problem to a determined solution of (1.1)-(1.4). In the last section we propose, for some classes of networks and some classes of transmission conditions (1.5), sets of parameters $K_{ij}$ which give rise to limit function satisfying (1.5) with predetermined coefficients $\gamma_{ij}$.

2. Existence of solutions for the parabolic problem

This section is devoted to the prove of existence and uniqueness of solutions to problem (1.8)-(1.10). It is convenient to rewrite the transmission conditions in (1.8)
in a more compact form, as follows
\begin{equation}
\beta_i (\lambda_i u_i^* (N, t) - \varepsilon u_i^* (N, t)) = \sum_{j \in \mathcal{M}} \alpha_{ij} u_j^* (N, t), \quad t > 0, \ i \in \mathcal{M}, \tag{2.1}
\end{equation}
where
\begin{equation}
\alpha_{ij} = -K_{ij} \leq 0 \quad \text{for } i \neq j, \quad \alpha_{ii} = \sum_{j \in \mathcal{M}, j \neq i} K_{ij} \geq 0; \tag{2.2}
\end{equation}
notice that \(\alpha_{ij} = \alpha_{ji}\), and \(\sum_{j \in \mathcal{M}} \alpha_{ij} = \sum_{i \in \mathcal{M}} \alpha_{ij} = 0\); moreover, if (2.11) holds, then \(\alpha_{ii} > 0\) for \(i \in \mathcal{I}\).

Let us consider \(\delta > v\) there exists a solution \(\theta > v\) for all \(A\) in a Banach space

Finally, we set
\begin{equation}
\beta_i (\lambda_i u_i^* (N, t) - \varepsilon u_i^* (N, t)) = \sum_{j \in \mathcal{M}} \alpha_{ij} u_j^* (N, t), \quad t > 0, \ i \in \mathcal{M}, \tag{2.7}
\end{equation}

and the operator \(A_0\) defined as follows
\begin{equation}
D(A_0) = \left\{ w \in W^{2,1} (A) : w(e_i) = 0 \ (i \in \mathcal{M}), \right. \left. \beta_i (\lambda_i w_i (N) - \varepsilon w_{ix} (N)) = \sum_{j \in \mathcal{M}} \alpha_{ij} w_j (N) \ (i \in \mathcal{M}) \right\}, \tag{2.3}
\end{equation}
\begin{equation}
A_0 w = \{ \varepsilon w_{ixx} - \lambda_i w_{ix} \}_{i \in \mathcal{M}}. \tag{2.4}
\end{equation}

If \(w^*\) is a solution to problem (1.8)-(1.10) then \(w^* (t) = u^* (t) - q\) satisfies the problem
\begin{equation}
\begin{cases}
\varepsilon^* (t) \in C(0, +\infty); D(A_0^*) \cap C^1 ([0, +\infty); L^1 (A)), \\
\varepsilon^* = A_0^* \varepsilon^* + F, \\
\varepsilon^* (x, 0) = u_0^* (x) - q (x) \in D(A_0^*),
\end{cases} \tag{2.5}
\end{equation}
where \(F (x) = \{ F_i (x) = \varepsilon q_i^* (x) - \lambda_i q_i (x) \}_{i \in \mathcal{M}}\).

We are going to prove the existence and uniqueness of solutions to problem (2.5) in order to prove the same results for problem (1.8)-(1.10).

The first step in the proof is showing that, thanks to conditions (2.2) on \(\alpha_{ij}\), the linear unbounded operator \(A_0^*\) is m-dissipative in \(L^1 (A)\), so that the existence of a unique solution \(w^* \in C([0, +\infty); D(A_0^*)) \cap C^1 ([0, +\infty); L^1 (A))\) for the homogeneous problem associated to (2.5) will result from the theory of linear semigroups (see [3]). We recall that, as defined in the same reference, a linear unbounded operator \(A\) in a Banach space \(X\) is dissipative if \(\| v - \theta Av \|_X \geq \| v \|_X\) for all \(v \in D(A)\) and for all \(\theta > 0\); moreover a linear unbounded operator \(A\) in a Banach space \(X\) is m-dissipative if it is dissipative and there exists \(\theta_0 > 0\), such that for all \(f \in X\), there exists a solution \(u\) of \(v - \theta_0 Av = f\).

In some of the proofs of the paper we are going to use the following functions. Let \(\delta > 0\); we set
\begin{equation}
g_\delta (w) = \begin{cases}
0 & \text{if } w \in (-\infty, 0], \\
\varepsilon (4\delta)^{-1} & \text{if } w \in (0, 2\delta], \\
w - \delta & \text{if } w \in (2\delta, +\infty). 
\end{cases} \tag{2.6}
\end{equation}

Notice that
\begin{equation}
\begin{aligned}
g_\delta \in C^1 (\mathbb{R}) & , \ g_\delta (w) \rightarrow g_{\delta \rightarrow 0} \ [w]^+ \ \text{pointwise} , \ |g_\delta (w)| \leq |w| , \\
g_\delta^1 (w) & \rightarrow g_{\delta \rightarrow 0} \chi_{[0, +\infty)} \text{pointwise} , \ |g_\delta^1 (w)| \leq 1 , \\
g_\delta^0 (w) & \rightarrow g_{\delta \rightarrow 0} \ 0 \ \text{pointwise} , \ g^0 (w)w \ \text{is bounded} .
\end{aligned} \tag{2.7}
\end{equation}

Finally, we set
\begin{equation}
\overline{g}_\delta (w) := g_\delta (w) + g_\delta (-w) \in C^1 (\mathbb{R}). \tag{2.8}
\end{equation}

**Proposition 2.1.** The operator \(A_0^*\) defined in (2.8) is dissipative in \(L^1 (A)\).
Proof. We have to prove that
\[ \sum_{i \in M} \| v_i - \theta (\epsilon v_{i,x} - \lambda_i v_i) \|_1 \geq \sum_{i \in M} \| v_i \|_1 \quad \forall \epsilon \in D(A^*_0), \quad \forall \theta > 0 . \]

Setting \( f_i := v_i - \theta (\epsilon v_{i,x} - \lambda_i v_i) \), then
\[
\sum_{i \in I} \| f_i \|_1 \geq \sum_{i \in I} \int_{I_i} f_i \overline{\gamma}_\delta(v_i) dx = \sum_{i \in I} \int_{I_i} (v_i \overline{\gamma}_\delta(v_i) + \theta (\lambda_i v_i - \epsilon v_{i,x}) \overline{\gamma}_\delta(v_i)) dx
\]
\[
= \sum_{i \in I} \int_{I_i} v_i \overline{\gamma}_\delta(v_i) dx + \theta \sum_{i \in I} \int_{I_i} (\epsilon v_{i,x} - \lambda_i v_i) \overline{\gamma}_\delta(v_i)) dx
\]
\[
+ \theta \sum_{i \in I} \left( \sum_{j \in M} \alpha_{ij} v_j(N) \right) \overline{\gamma}_\delta(v_i(N)) + \theta \sum_{i \in O} \left( \sum_{j \in M} \alpha_{ij} v_j(N) \right) \overline{\gamma}_\delta(v_i(N)) .
\]
Since \( v_i, v_{i,x} \in L^1(I_i) \), thanks to (2.10), when \( \delta \) goes to zero, using the dominated convergence theorem, we obtain
\[
\sum_{i \in M} \| f_i \|_1 \geq \sum_{i \in M} \| v_i \|_1 + \theta \sum_{i \in I} \sum_{j \in M} \alpha_{ij} |v_j(N)| \text{sgn}(v_j(N)) \text{sgn}(v_i(N))
\]
\[
= \sum_{i \in M} \| v_i \|_1 + \theta \sum_{i \in I} \sum_{j \in M} |v_j(N)| \sum_{i \in M} \alpha_{ij} \text{sgn}(v_j(N)) \text{sgn}(v_i(N)) .
\]
The conditions (2.2), which are valid thanks to the assumptions on the coefficients \( K_{ij} \), ensure that there holds
\[
\sum_{i \in M} \alpha_{ij} \text{sgn}(v_j(N)) \text{sgn}(v_i(N)) \geq 0 ,
\]
so the dissipativity of \( A^*_0 \) follows. \( \square \)

Proposition 2.2. The operator \( A^*_0 \) defined in (2.3) is m-dissipative in \( L^1(A) \).

Proof. Since the operator \( A^*_0 \) is dissipative, we have only to prove that for all \( f \in L^1(A) \) there exists a solution \( v \in D(A^*_0) \) of the equation \( v - A^*_0v = f \). Let \( f = \{ f_i \}_{i \in M} \); if \( f_i \in C^\infty_0(I_i) \) for all \( i \in M \), the solutions of the equation
\[
v''_i - \frac{\lambda_i}{\epsilon} v'_i - \frac{1}{\epsilon} v_i = \frac{f_i}{\epsilon} \quad \forall \epsilon \in I_i
\]
can be written
\[
v_i(x) = a_{i1} e^{a_{i1} x} + a_{i2} e^{a_{i2} x} + p_i(x)
\]
where
\[
a_{i1} = \frac{\lambda_i}{2} - \frac{1}{2} \sqrt{\lambda_i^2 + 4 \epsilon} < 0 , \quad a_{i2} = \frac{\lambda_i}{2} + \frac{1}{2} \sqrt{\lambda_i^2 + 4 \epsilon} > \lambda_i ,
\]
and \( p_i(x) \) is the particular solution to (2.8) which satisfies the conditions
\[
p_i(L) = p''_i(L) = 0 \quad \text{if} \quad i \in I , \quad p_i(0) = p'_i(0) = 0 \quad \text{if} \quad i \in O .
\]
We are going to prove that there exist $c_{i1}, c_{i2} \in \mathbb{R}$ ($i \in \mathcal{M}$) such that $v = \{v_i\}_{i \in \mathcal{M}}$ given by (2.13) satisfies the boundary and the transmission conditions in (2.3). The boundary conditions read

$$c_{i1} + c_{i2} + p_i(0) = 0 \quad i \in \mathcal{I},$$

(2.12)

$$c_{i1}e^{a_{i1}L_i} + c_{i2}e^{a_{i2}L_i} + p_i(L_i) = 0 \quad i \in \mathcal{O},$$

while the conditions at the inner node read

$$
\begin{aligned}
&\left\{ \begin{array}{l}
\lambda_i \left( c_{i1}e^{a_{i1}L_i} + c_{i2}e^{a_{i2}L_i} \right) - \epsilon \left( c_{i1}a_{i1}e^{a_{i1}L_i} + c_{i2}a_{i2}e^{a_{i2}L_i} \right) \\
= \sum_{j \in \mathcal{I}} \alpha_{ij} \left( e^{a_{j1}L_j} + j_{j2}e^{a_{j2}L_j} \right) + \sum_{j \in \mathcal{O}} \alpha_{ij} \left( c_{j1} + j_{j2} \right), \quad i \in \mathcal{I},
\end{array} \right.
\end{aligned}
$$

(2.13)

$$
\begin{aligned}
&\left\{ \begin{array}{l}
-\lambda_i \left( c_{i1} + c_{i2} \right) + \epsilon \left( c_{i1}a_{i1} + c_{i2}a_{i2} \right) = \sum_{j \in \mathcal{I}} \alpha_{ij} \left( c_{j1}e^{a_{j1}L_j} + j_{j2}e^{a_{j2}L_j} \right) \\
+ \sum_{j \in \mathcal{O}} \alpha_{ij} \left( c_{j1} + j_{j2} \right), \quad i \in \mathcal{O}.
\end{array} \right.
\end{aligned}
$$

Using the conditions (2.12) in (2.13) and setting

$$\mathcal{P}_i := e^{a_{i1}L_i}(-\lambda_i + c_{i1})p_i(L_i) - \sum_{j \in \mathcal{I}} \alpha_{ij}e^{a_{j1}L_j}p_j(0) - \sum_{j \in \mathcal{O}} \alpha_{ij}e^{-a_{j1}L_j}p_j(L_j), \quad i \in \mathcal{I},$$

$$\mathcal{P}_i := e^{-a_{i1}L_i}(\lambda_i - c_{i1})p_i(L_i) + \sum_{j \in \mathcal{I}} \alpha_{ij}e^{a_{j1}L_j}p_j(0) - \sum_{j \in \mathcal{O}} \alpha_{ij}e^{-a_{j1}L_j}p_j(L_j), \quad i \in \mathcal{O},$$

\{c_{i2}\}_{i \in \mathcal{M}} will result as solution of the linear systems of $m$ equations:

$$
\begin{aligned}
&\left\{ \begin{array}{l}
\left( e^{a_{i2}L_i} - \lambda_i \right)e^{a_{i2}L_i} + \epsilon \left( e^{a_{i1}L_i} - e^{a_{i2}L_i} \right) c_{i2} \\
+ \sum_{j \in \mathcal{I}, j \neq i} \alpha_{ij} \left( e^{a_{j2}L_j} - e^{a_{j1}L_j} \right) c_{j2} + \sum_{j \in \mathcal{O}} \alpha_{ij} \left( 1 - e^{a_{j2}L_j} \right) c_{j2} = -\mathcal{P}_i, \quad i \in \mathcal{I},
\end{array} \right.
\end{aligned}
$$

and the assumptions on $\alpha_{ij}$ imply that

$$
\begin{aligned}
&\sum_{j \in \mathcal{M}, j \neq i} |h_{ij}| = (e^{a_{i2}L_i} - e^{a_{i1}L_i}) \sum_{j \in \mathcal{M}, j \neq i} |\alpha_{ij}| \leq \alpha_{ii}(e^{a_{i2}L_i} - e^{a_{i1}L_i}) \\
&i \in \mathcal{I},
\end{aligned}
$$

$$
\begin{aligned}
&< (e^{a_{i2}L_i} - \lambda_i)e^{a_{i2}L_i} + \epsilon \left( e^{a_{i1}L_i} - e^{a_{i2}L_i} \right) = h_{ii}, \\
&\sum_{j \in \mathcal{M}, j \neq i} |h_{ij}| = (e^{a_{i2}L_i} - 1) \sum_{j \in \mathcal{M}, j \neq i} |\alpha_{ij}| \leq \alpha_{ii}(e^{a_{i2}L_i} - 1) \\
&i \in \mathcal{O},
\end{aligned}
$$

so that $H$ has strictly dominant diagonal. It follows that there exists a unique solution \{c_{i2}\}_{i \in \mathcal{M}} of the linear system, and \{c_{i1}\}_{i \in \mathcal{M}} can be determined using (2.12).
If \( f \in L^1(A) \) then we consider a sequence \( \{f_n\}_{n \in \mathbb{N}}, f_n \in C_0^\infty(I_i) \) (\( i \in \mathcal{M} \)), such that \( f_n \xrightarrow{n \to +\infty} f \) in \( L^1(A) \); moreover we consider the corresponding sequences \( \{p_n\}_{n \in \mathbb{N}} \), satisfying (2.11), and the functions

\[
 v_{n_i}(x) = c_{i1}^n e^{a_{i1}x} + c_{i2}^n e^{a_{i2}x} + p_{n_i}(x), \quad i \in \mathcal{M},
\]

solutions to (2.8), where \( f \) is replaced by \( f_n \).

\[
 v''_{n_i} - \frac{\lambda_i}{\epsilon} v'_{n_i} - \frac{1}{\epsilon} v_{n_i} = \frac{f_{n_i}}{\epsilon}, \quad x \in I_i, \quad i \in \mathcal{M}.
\]

The sequence \( \{v_n\}_{n \in \mathbb{N}} \) converges to some \( \overline{v} \) in \( L^1(A) \), since the operator \( A_0' \) is dissipative in \( L^1(A) \) so that \( \|v_n - v_{n_i}\|_1 \leq \|f_n - f\|_1 \), for \( n, n_i \in \mathbb{N} \).

We can express each \( p_{n_i} \), for \( i \in \mathcal{I} \), as follows

\[
p_{n_i}(x) = \frac{1}{a_{i1} - a_{i1}} \int_{L_i} f_{n_i}(y) \left( -e^{a_{i1}(x-y)} + e^{a_{i2}(x-y)} \right) dy,
\]

and a similar expression holds for \( i \in \mathcal{O} \); if we define, for \( i \in \mathcal{I} \)

\[
 \overline{p}_i(x) = \frac{1}{a_{i1} - a_{i1}} \int_{L_i} f_i(y) \left( -e^{a_{i1}(x-y)} + e^{a_{i2}(x-y)} \right) dy,
\]

we obtain

\[
 |p_{n_i}(x) - \overline{p}_i(x)| \leq \frac{1}{a_{i1} - a_{i1}} \int_{L_i} |f_{n_i}(y) - f_i(y)| \left( e^{a_{i2}(x-y)} - e^{a_{i1}(x-y)} \right) dy,
\]

so that \( \{p_{n_i}\} \) converges in \( C(I_i) \) to \( \overline{p}_i \) for each \( i \in \mathcal{I} \), and similarly, for each \( i \in \mathcal{O} \). This fact implies the convergence of the sequences \( \{c_i^n\}_{n \in \mathbb{N}} \) and \( \{c_i^1\}_{n \in \mathbb{N}} \), for all \( i \in \mathcal{M} \). Then

\[
 \overline{v}_i(x) = \overline{v}_1 e^{a_{i1}x} + \overline{v}_2 e^{a_{i2}x} + \overline{p}_i(x)
\]

and the claim is proved if we show that \( \overline{v} \in D(A_0') \) and it satisfies (2.8).

Since, for all \( i \in \mathcal{M} \),

\[
p'_{n_i}(x) = \frac{1}{a_{i1} - a_{i1}} \int_{L_i} f_{n_i}(y) \left( -a_{i1} e^{a_{i1}(x-y)} + a_{i2} e^{a_{i2}(x-y)} \right) dy,
\]

the functions \( \overline{v}_i \) belong to \( W^{1,1}(A) \) and

\[
 \overline{v}'_i(x) = \frac{1}{a_{i1} - a_{i1}} \int_{L_i} f_i(y) \left( -a_{i1} e^{a_{i1}(x-y)} + a_{i2} e^{a_{i2}(x-y)} \right) dy;
\]

moreover

\[
 v'_{n_i}(x) = a_{i1} c_{i1}^n e^{a_{i1}x} + a_{i2} c_{i2}^n e^{a_{i2}x} + p'_{n_i}(x),
\]

\[
 \overline{v}'_i(x) = a_{i1} \overline{c}_{i1} e^{a_{i1}x} + a_{i2} \overline{c}_{i2} e^{a_{i2}x} + \overline{p}'_i(x),
\]

so we infer that

\[
 v_{n_i} \xrightarrow{n \to +\infty} \overline{v}_i \quad \text{in} \quad W^{1,1}(I_i).
\]

From the equation (2.13) we obtain

\[
 - \int_0^{L_i} \overline{v}_i \phi' = \int_0^{L_i} \phi \left( \frac{\lambda_i}{\epsilon} \overline{v}_i \phi + \frac{1}{\epsilon} \overline{v}_i \right), \quad \phi \in C_0^1(I_i),
\]

so \( \overline{v}_i \in W^{2,1}(I_i) \), \( |v_{n_i} - \overline{v}_i| \to 0 \) in \( W^{2,1}(I_i) \) and \( \overline{v}_i \) satisfies (2.8) for all \( i \in \mathcal{M} \).

Finally, the convergence in \( W^{2,1}(I_i) \) implies that \( \overline{v} \) satisfies the boundary and transmission conditions.

The proof of existence and uniqueness of solutions to problem (2.8), where homogeneous boundary conditions are considered, is a direct consequence of the previous propositions.

**Proposition 2.3.** For every \( u_0' \in D(A_0') \), there exists a unique solution to problem (2.4).
Proof. For all $\epsilon > 0$ we easily see that $D(A_0^\epsilon)$ is dense in $L^1(A)$, in fact
\[
\{ f = (f_1, f_2, \ldots, f_m) : f_i \in C_c^\infty (I_i) \} \subset D(A_0^\epsilon) .
\]
Since $A^\epsilon$ is an m-dissipative operator in $L^1(A)$ and $D(A_0^\epsilon)$ is dense in $L^1(A)$, the
Hille-Yosida theorem implies that $A_0^\epsilon$ generates a linear semigroup $T_\epsilon$. Moreover,
$F_1 \in C_c^\infty (I_i)$, so the linear contraction semigroups theory applies to problem
\[
(2.3), \ (2.4) \quad \text{(see \cite{3}), providing the unique solution $v^\epsilon \in C([0, +\infty); D(A_0^\epsilon)) \cap C^1([0, +\infty); L^1(A)) ,}
\]
\[
v^\epsilon (t) = \tau_\epsilon (t) (u_0^\epsilon - q) + \int_0^t \tau_\epsilon (t-s) F ds . \]

The existence and uniqueness of solutions to problem \((1.8)-(1.10)\) immediately
follows from the above proposition, thanks to the position $v^\epsilon = v^\epsilon - q$ set at the
beginning of this section.

Theorem 2.1. For every $u_0^\epsilon \in W^{2, 1}(A)$ satisfying \((1.10)\) there exists a unique
solution $u^\epsilon$ to problem \((1.8)\) with
\[
u^\epsilon \in C([0, +\infty); W^{2, 1}(A)) \cap C^1([0, +\infty); L^1(A)) ,
\]
and this solution is given by

\[
u^\epsilon (t) = \tau_\epsilon (t) (u_0^\epsilon - q) + \int_0^t \tau_\epsilon (t-s) F ds + q .
\]

3. Uniform estimates for the solutions of the parabolic problem

In this section we obtain some estimates for the solutions to problems \((1.8)-(1.10)\), which will be uniform in $\epsilon$ provided that $u_0^\epsilon_i$ approximates $u_0$ in suitable
way when $\epsilon \to 0^+$ (see next Section).

Let $0 < \epsilon \leq 1$ and let $q$ be the function defined at the beginning of Section 2.

Proposition 3.1. Let $u^\epsilon$ be the solution to problem \((1.8)-(1.10)\); then for all $t > 0,$
\[
\| u^\epsilon (t) \|_{L^1(A)} \leq \| u_0^\epsilon \|_{L^1(A)} + \eta_1 t + \eta_2 ,
\]
where $\eta_1, \eta_2 \geq 0.$

Proof. From Theorem 2.1 we know that
\[
u^\epsilon (t) = \tau_\epsilon (t) (u_0^\epsilon - q) + \int_0^t \tau_\epsilon (t-s) F ds + q
\]
so, thanks to the properties of contraction semigroups,
\[
\| u^\epsilon \|_{L^1(A)} \leq \| u_0^\epsilon \|_{L^1(A)} + 2\| q \|_{L^1(A)} + t \sum_{i \in \mathcal{M}} \| \epsilon q_i^\epsilon (x) - \lambda_i q_i (x) \|_{L^1(A)} . \]

\[
\]

Using the properties of the semigroup $\tau_\epsilon$ we can also prove the following estimate
for $u_\epsilon^\epsilon.$

Proposition 3.2. Let $u^\epsilon$ be the solution to problem \((1.8)-(1.10)\), then for all $t > 0,$
\[
\| u_i^\epsilon (t) \|_{L^1(A)} \leq \sum_{i \in \mathcal{M}} \| - \lambda_i u_i^\epsilon \|_{L^1(A)} + \epsilon \| u_0^\epsilon \|_{L^1(A)} .
\]
Proof. Since \((u_0^e - q) \in D(A_0^e)\) the following equality holds

\[ A_0^e \tau_e(u_0^e - q) = \tau_e A_0^e(u_0^e - q), \]

hence, using (2.15) (see [3]), we can write

\[
\|u_1^e(t)\|_{L^1(A)} = \|\tau_e(t)(u_0^e - q)\|_{L^1(A)} + \|A_0^e \tau_e(t)(u_0^e - q) + \tau_e(t)F\|_{L^1(A)} \\
\leq \|A_0^e(u_0^e - q) + F\|_{L^1(A)} = \|\epsilon u_{0xx} - \lambda_i u_{0xx}\|_{L^1(A)}.
\]

Now we can prove the following estimate for \(u_1^e\).

**Proposition 3.3.** Let \((L_{ij})\) hold and let \(u^e\) be the solution to problem (1.8)-(1.10); then, for all \(T > 0\) and all \(i \in \mathcal{M}\), for small \(\epsilon\),

\[
\sup_{t \in [0,T]} \int_{I_i} |u_{1,xx}^e(x,t)| dx \leq C \sum_{j \in \mathcal{M}} \|u_{0j}^e\|_{W^{1,1}(I_j)} + C \sup_{t \in [0,T]} \sum_{j \in \mathcal{M}} \left( \|u_{1,t}^e(t)\|_{L^1(I_j)} + \|u_{2,j}^e(t)\|_{L^1(I_j)} \right),
\]

where \(C\) depends on \(\lambda_i, L_i, K_{ij}\) \((i, j \in \mathcal{M})\) and Sobolev constants.

**Proof.** Using the function (2.4) we can write, for each \(i \in \mathcal{M}\) and \(t < 0\),

\[
\int_{I_i} [u_{1,t}^e \tilde{g}_i(u_{1,xx}^e)](x,t) dx = \int_{I_i} [(cu_{1,xx} - \lambda_i u_{1,xx}) \tilde{g}_i(u_{1,xx})](x,t) dx
\]

whence

\[
\int_{I_i} \lambda_i |u_{1,xx}^e\tilde{g}_i(u_{1,xx})|(x,t) dx = \int_{I_i} \epsilon |u_{1,xx} \tilde{g}_i(u_{1,xx})|(L_i, t) - \int_{I_i} \epsilon |u_{1,xx} \tilde{g}_i(u_{1,xx})|(0, t)
\]

\[
- \int_{I_i} \epsilon |u_{1,xx} \tilde{g}_i(u_{1,xx})u_{1,xx}^e|(x,t) dx - \int_{I_i} |u_{1,xx} \tilde{g}_i(u_{1,xx})|(x,t) dx.
\]

Now, let \(\delta\) go to zero and we use the dominated convergence theorem, taking into account that \(u_{1,xx}^e, u_{1,xxx}^e, u_{1,tt}^e \in L^1(I_i)\) and the properties of \(\tilde{g}_i\) (see (2.4)). So we obtain

\[
\int_{I_i} \lambda_i |u_{1,xx}^e(t)| dx \leq \|u_{1,t}^e(t)\|_{L^1(I_i)} + \|u_{1,xx}^e(L_i, t)\|_{L^1(I_i)} - \|\epsilon u_{1,xx}^e(0, t)\|_{L^1(I_i)}
\]

\[
\leq \|u_{1,t}^e(t)\|_{L^1(I_i)} + |cu_{1,xx}(L_i, t) - \lambda_i u_{1,xx}(L_i, t)| + |\lambda_i u_{1,xx}^e(L_i, t)|
\]

\[
\leq \|u_{1,t}^e(t)\|_{L^1(I_i)} + C_0^0 |(cu_{1,xx} - \lambda_i u_{1,xx})(t)|_{W^{1,1}(I_i)} + |\lambda_i u_{1,xx}^e(L_i, t)|
\]

\[
\leq C_0^0 (\|u_{1,xx}(t)\|_{L^1(I_i)} + \|u_{1,tt}(t)\|_{L^1(I_i)}) + |\lambda_i u_{1,xx}^e(L_i, t)|,
\]

where we used the equation satisfied by \(u_{1,t}^e\); then, for small \(\epsilon\),

\[
\int_{I_i} |u_{1,xx}^e(t)| dx \leq C_0' (\|u_{1,xx}(t)\|_{L^1(I_i)} + \|u_{1,tt}(t)\|_{L^1(I_i)}) + |\lambda_i u_{1,xx}^e(L_i, t)|,
\]

where \(C_0', C_1, C_2^0\) depend on Sobolev constants and on \(\lambda_i\).

For \(i \in \mathcal{O}\), the claim easily follows from (3.2), since \(u_{1,xx}^e(L_i, t) = u_{0xx}^e(L_i, t)\).

As a consequence we obtain the \(L^\infty\)-bound for \(u_{1,xx}^e, i \in \mathcal{O}\),

\[
\|u_{1,t}^e(t)\|_{L^\infty(I_i)} \leq C_3^0 (\|u_{1,xx}^e(t)\|_{L^1(I_i)} + \|u_{1,tt}(t)\|_{L^1(I_i)} + \|u_{1,xx}^0\|_{W^{1,1}(I_i)}), \quad t > 0,
\]

where \(C_3^0\) depends on \(\lambda_i\) and Sobolev constants.
In order to obtain the estimate in the claim for the incoming arcs, we fix $\epsilon$ and
\[
\overline{U}_\epsilon := \max_{j \in J} \max_{t \in [0,T]} [u_j^+ (L_j, t)]^+ , \quad \underline{U}_\epsilon := - \max_{j \in J} \max_{t \in [0,T]} [u_j^- (L_j, t)]^- .
\]

If $\overline{U}_\epsilon > 0$, let $k \in I$ and $t \in [0,T]$ be such that $u_k^+ (L_k, t) = \overline{U}_\epsilon$; then, arguing as in (3.1), we have
\[
\left| \sum_{j \in \mathcal{M}} \alpha_{kj} u_j^+ (N, t) \right| = |[\epsilon u_k^+ - \lambda_k u_k^+] (L_k, t)| 
\leq C_0^k (\|u_k^+ (t)\|_{L^2 \{I_k\}} + \epsilon \|u_k^+ (t)\|_{L^1 \{I_k\}} + \lambda_k \|u_k^+ (t)\|_{L^1 \{I_k\}})
\]
whence
\[
(3.5)
\]
\[
\overline{U}_\epsilon \left( \alpha_{kk} + \sum_{j \in I, j \neq k} \alpha_{kj} \right) \leq \alpha_{kk} \overline{U}_\epsilon + \sum_{j \in I, j \neq k} \alpha_{kj} u_j^+ (L_j, t) = \sum_{j \in I} \alpha_{kj} u_j^+ (L_j, t) 
\leq \sum_{j \in \mathcal{O}} \alpha_{kj} u_j^+ (0, t) + C_0^k (\|u_k^+ (t)\|_{L^1 \{I_k\}} + \epsilon \|u_k^+ (t)\|_{L^1 \{I_k\}} + \lambda_k \|u_k^+ (t)\|_{L^1 \{I_k\}}) ;
\]

now we use (1.11) to say that $\left( \alpha_{kk} + \sum_{j \in I, j \neq k} \alpha_{kj} \right) > 0$, and (3.5) to obtain
\[
\overline{U}_\epsilon \leq C_4 \sum_{j \in \mathcal{O} \cup \{k\}} \left( \|u_j^+ (t)\|_{L^1 \{I_j\}} + \|u_j^+ (t)\|_{L^1 \{I_j\}} + \|u_0^+\|_{W^{1,1} \{I_j\}} \right) + C_0^k \|u_k^+ (t)\|_{L^1 \{I_k\}},
\]
where $C_4$ is a positive constant depending on Sobolev constants, on $\lambda_i$ and on the coefficients $\alpha_{ij} (i, j \in \mathcal{M})$.

As for the lower bound, if $\underline{U}_\epsilon < 0$ let $h \in I$ and $t \in [0,T]$ be such that $u_h^+ (L_h, t) = \underline{U}_\epsilon$; arguing as above we obtain
\[
\underline{U}_\epsilon \geq - C_5 \sum_{j \in \mathcal{O} \cup \{h\}} \left( \|u_j^+ (t)\|_{L^2 \{I_j\}} + \|u_j^+ (t)\|_{L^1 \{I_j\}} + \|u_0^+\|_{W^{1,1} \{I_j\}} \right) - C_0^h \|u_h^+ (t)\|_{L^1 \{I_h\}},
\]
where $C_5$ is a positive constant depending on Sobolev constants, on $\lambda_i$ and on the coefficients $\alpha_{ij} (i, j \in \mathcal{M})$.

So, we achieved the $L^\infty$-bound for $u_i^+ (L_i, t), i \in I$,
\[
(3.6)
\]
\[
\|u_i^+ (L_i, \cdot)\|_{L^\infty (0,T)} \leq C_0 \left( \epsilon \sum_{j \in J} \sup_{s \in [0,T]} \|u_j^+ (s)\|_{L^1 \{I_j\}} + \sum_{j \in \mathcal{M}} \|u_0^+\|_{W^{1,1} \{I_j\}} \right) + C_0 \sup_{s \in [0,T]} \sum_{j \in \mathcal{M}} \left( \|u_j^+ (s)\|_{L^1 \{I_j\}} + \|u_j^+ (s)\|_{L^1 \{I_j\}} \right) ,
\]
where $C_0$ depends on Sobolev constants, on $\lambda_i$ and on the coefficients $\alpha_{ij} (i, j \in \mathcal{M})$, which allows us to control the quantity $|\lambda_i u_i^+ (L_i, t)|$ in (3.2) also when $i \in I$, \right)
obtaining, for \( t \in [0, T] \),
\[
\int_{I_i} |u^\varepsilon_{tx}(x,t)| \, dx \leq C_2^2 \left( \|u_i(t)\|_{L^1(I_i)} + \|u_{it}(t)\|_{L^1(I_i)} \right)
+ \lambda_C \sup_{\varepsilon \in [0,T]} \sum_{j \in M} \|u^\varepsilon_j(s)\|_{L^1(I_j)} + \sum_{j \in M} \|u^\varepsilon_j(s)\|_{W^{1,1}(I_j)}
+ \lambda_C \sup_{\varepsilon \in [0,T]} \sum_{j \in M} \left( \|u^\varepsilon_j(s)\|_{L^1(I_j)} + \|u^\varepsilon_j(s)\|_{L^1(I_j)} \right) ;
\]
finally
\[
\sum_{i \in I} \sup_{t \in [0,T]} \int_{I_i} |u^\varepsilon_{tx}(x,t)| \, dx \leq C_7 \sum_{j \in M} \|u^\varepsilon_j(s)\|_{W^{1,1}(I_j)}
+ C_7 \sup_{t \in [0,T]} \sum_{j \in M} \left( \|u^\varepsilon_j(t)\|_{L^1(I_j)} + \|u^\varepsilon_j(t)\|_{L^1(I_j)} \right) + C_7 \sup_{t \in [0,T]} \|u^\varepsilon_j(t)\|_{L^1(I_j)} ;
\]
where \( C_7 \) depends on Sobolev constants, \( \lambda_C \), \( \alpha_{ij} \) (\( i, j \in M \)). For small \( \varepsilon \) we have the claim.

Finally we prove the following comparison result, relying on the fact that the functions \( U_i(x) = 0 \), for all \( x \in I_i \) and all \( i \in M \), satisfies the equations and the transmission conditions in (1.8).

**Proposition 3.4.** Let \( u^\ast \) be the solution to problem (1.8)-(1.10); if \( u^\varepsilon_0(i,.) \geq 0 \) for all \( i \in M \), then, for all \( t > 0 \),
\[
\sum_{i \in M} \int_{I_i} |u^\ast_i(x,t)| \, dx \leq \sum_{i \in M} \int_{I_i} |u^\varepsilon_0(x)| \, dx ;
\]
in particular, if \( u^\varepsilon_0(x) \geq 0 \) then \( u^\ast_i(x, t) \geq 0 \).

**Proof.** Following standard methods (for example see [8]), using the function (2.5) we can write
\[
\sum_{i \in M} \left( \int_{I_i} g_0(-u^\ast(x,t)) \, dx - \int_{I_i} g_0(-u^\ast(x,0)) \, dx \right) = \sum_{i \in M} \int_0^t \int_{I_i} \left( g_0(-u^\ast(x,s)) + \alpha_{ij} u^\ast_j(N,s) g_0(u^\ast_i(x,s)) \right) \, dx \, ds
+ \sum_{i \in M} \int_0^t \int_{I_i} \lambda_C g_0^\ast(-u^\ast(x,s))(u^\ast_i(x,s)) \, dx \, ds .
\]
Since \( u^\ast_i(t) \), \( u^\ast_j(t) \in L^1(I_i) \) and \( u^\ast_i(N) \in L^1(0,t) \) for all \( i \in M \), thanks to (2.6) we can let \( \delta \) go to zero using the dominated convergence theorem, to obtain
\[
\sum_{i \in M} \int_{I_i} |u^\ast_i(x,t)| \, dx = \sum_{i \in M} \int_{I_i} |u^\varepsilon_0(x)| \, dx
\leq \int_0^t \sum_{i \in M} \sum_{j \in M} \alpha_{ij} |u^\ast_j(N,s)| |u^\ast_j(N,s)| |u^\ast_i(N,s)| ds \leq 0
\]
since the assumptions on the coefficients ensure that
\[
\sum_{i \in M} \sum_{j \in M} \alpha_{ij} |u^\ast_j(N,s)| |u^\ast_i(N,s)| ds \leq 0 , \text{ for all } j \in M.
\]
\[ \square \]
4. Convergence

In this section we prove the vanishing viscosity approximation result. The first subsection is devoted to the proof that, when \( \epsilon_n \to 0 \), each sequence \( \{u^n\} \) of solutions to parabolic problems \( \text{(1.8)-(1.10)} \) has a subsequences converging to a solution \( u \) of \( \text{(1.8)-(1.10)} \) satisfying \( \text{(1.3)} \); this convergence result holds provided \( \{u^n_0\} \) approximates \( u_0 \) in the sense of the Lemma \( \text{(13)} \) below. In Subsection 4.2 we prove that such limit function \( u \) satisfies conditions \( \text{(1.5)} \), where \( \gamma_{ij} \) are uniquely determined by \( \lambda_i \) and \( K_{ij} \), so that, thanks to uniqueness of solution to \( \text{(1.11)-(1.13)} \), each sequence \( \{u^n\} \) converges to \( u \). In particular, if the family \( \{K_{ij}\} \) verifies \( \text{(1.20)} \), then \( \text{(1.7)} \) holds for the parameters \( \gamma_{ij} \), i.e. the conditions satisfied by the limit function \( u \) at the inner node \( N \) are actually transmission conditions.

4.1. Vanishing viscosity limit. Let us set

\[
D_\epsilon := \left\{ w \in W^{2,1}(A) : \ w_i(\epsilon_i) = B_i \text{ if } i \in I, \right. \\
\left. \beta_i (\lambda_i w_i(N) - \epsilon w'_i(N)) = \sum_{j \in M} \alpha_{ij} w_j(N), \ i \in M \right\},
\]

where \( \beta_i \) is defined in \( \text{(1.3)} \).

We will use the following lemma whose quite technical proof is included in the Appendix.

**Lemma 4.1.** Let \( \{\epsilon_n\}_{n \in \mathbb{N}} \) be any decreasing sequence such that \( \epsilon_n \to 0 \). For every \( v \in BV(A) \) there exists a sequence \( \{v_n\}_{n \in \mathbb{N}} \) converging to \( v \) in \( L^1(A) \) such that \( v_n \in D_{\epsilon_n} \) for each \( n \in \mathbb{N} \), \( \|v'_n\|_{L^1(A)} \) are bounded by a quantity dependent on \( \|v\|_{BV(A)} \) and independent of \( n \), and \( \epsilon_n \|v_n\|_{W^{2,1}(A)} \) are bounded independently of \( n \).

Let \( u_0 \in BV(A) \); in this section, for each decreasing sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) such that \( \epsilon_n \to 0 \), \( \{u^n_0\}_{n \in \mathbb{N}} \) denotes a sequence approximating \( u_0 \) as in Lemma \( \text{(1.11)} \).

Let \( \{u^n\}_{n \in \mathbb{N}} \) be the sequence of solutions to the following problems:

\[
\begin{cases}
  u^n_{1+} = -\lambda_i u^n_{xx} + \epsilon_n u^n_{1xx}, & x \in I_i, \ t \in [0,T], \ i \in M, \\
  u^n_1(x,0) = u^n_{01}(x) \in W^{2,1}(A), & x \in I_i, \ i \in M, \\
  \beta_i (\lambda_i u^n_i(N) + \epsilon_n u^n_{1x}(N,t)) = \sum_{j \in M} K_{ij} (u^n_j(N,t) - u^n_i(N,t)), & t \in [0,T], \\
  u^n_i(0,t) = B_i, & i \in I, \ u^n_i(L_i,t) = u^n_{0i}(L_i), & i \in O, \ t \in [0,T],
\end{cases}
\]

where \( K_{ij} \) satisfy \( \text{(1.9)} \) and \( u^n_{0i} \in D_{\epsilon_n} \).

The results in the previous section imply the following proposition.

**Proposition 4.1.** Let \( \text{(1.17)} \) hold, let \( u_0 \in BV(A) \) and \( B_i \in \mathbb{R} \) for \( i \in I; \) for all \( T > 0 \)

\[
\sup \{ \|u^n\|_{BV(A \times (0,T))} : n \in \mathbb{N} \} < +\infty.
\]

**Proof.** Let \( C_1 \geq \epsilon_n \|u^n_0\|_{W^{2,1}(A)} \) for all \( \epsilon_n \). Lemma \( \text{(4.1)} \) and Proposition \( \text{(5.1)} \) imply

\[
\|u^n(t)\|_{L^1(A)} \leq C_2
\]

for \( t \in [0,T] \), where \( C_2 \) depends on \( T \) and on \( \|u_0\|_{L^1(A)} \); Lemma \( \text{(4.1)} \) and Proposition \( \text{(5.2)} \) imply

\[
\|u^n(t)\|_{L^1(A)} \leq C_3
\]
for $t \in [0, T]$, where $C_3$ depends on $\|u_0\|_{BV(A)}$ and on the quantity $C_1$; finally, Propositions 3.3, 3.1 and 3.2 imply

$$\|u^{n_k}_t(t)\|_{L^1(A)} \leq C_4$$

for $t \in [0, T]$, where $C_4$ depends on $T$, $\|u_0\|_{BV(A)}$, the quantity $C_1, \lambda_i, K_{ik}$ $(i, k \in \mathcal{M})$ and Sobolev constants. The constants $C_i, i = 2, 3, 4$ are independent from $\epsilon_n$, so the claim is proved.

The first step in proving the convergence result is the argument of the next proposition.

**Proposition 4.2.** Let (1.11) hold, let $u_0 \in BV(A)$, $B_i \in \mathbb{R}$ for $i \in \mathcal{I}$. For all $T > 0$ and for each decreasing sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\epsilon_n \to 0$, the sequence $\{u^{n_k}\}_{n \in \mathbb{N}}$ of solutions to (4.2) admits a subsequence converging in $L^1((A \times (0, T))$ to a solution $u \in BV(A \times (0, T))$, for $n \to +\infty$. It is possible to consider its traces $u_i^{n_k}(N, t)$ converges weakly$^*$ in $L^\infty(0, T)$ to some functions $W_i^{N_k}(t)$; besides, for all $i \in \mathcal{O}$, the limit function $u$ satisfies

$$u_i(0, t) = W_i^{N_k}(t), \ t > 0.$$  

**Proof.** Thanks to the previous proposition, fixed $T > 0$, for every sequence $\{u^{n_k}\}$ $(\epsilon_n \to 0)$ there exist a subsequence $\{u^{n_k}\}$ converging in $L^1(A \times (0, T))$ to a function $u \in BV(A \times (0, T))$, for $n \to +\infty$. It is possible to consider its traces $u(x, 0) \in L^\infty(A)$ and $u_i(0, t), u_i(N_i, t) \in L^\infty(0, T)$ $(i \in \mathcal{M})$, and we know that

$$\lim_{t \to 0} \|u(\cdot, t) - u(\cdot, 0)\|_{L^1(A)} = 0,$$

$$\lim_{x \to 0} \|u_i(x, \cdot) - u_i(0, \cdot)\|_{L^1(0, T)}, \lim_{x \to L_i} \|u_i(x, \cdot) - u_i(L_i, \cdot)\|_{L^1(0, T)} = 0, \ i \in \mathcal{M},$$

see [1].

We are going to identify the function $u$. First we have, for $i \in \mathcal{I}$,

$$\int_0^T \int_{I_i} u_i^{n_k} \phi_{i2} + \lambda_i \iota \iota_{i} \iota_{i} \phi_{i2} - \epsilon_n \iota \iota_{i} \iota_{i} \phi_{i2} \ dx \ dt + \int_{I_i} u_i(0) \phi_i(x, 0) \ dx$$

$$= \int_0^T \phi_i(L_i, t) \sum_{j \in \mathcal{M}} \alpha_{ij} u_j^{n_k}(N, t) \ dt, \quad \forall \phi_i \in C_0^\infty([0, L_i] \times [0, T]),$$

and in similar way, for $i \in \mathcal{O}$,

$$\int_0^T \int_{I_i} u_i^{n_k} \phi_{i2} + \lambda_i \iota \iota_{i} \iota_{i} \phi_{i2} - \epsilon_n \iota \iota_{i} \iota_{i} \phi_{i2} \ dx \ dt + \int_{I_i} u_i(0) \phi_i(x, 0) \ dx$$

$$= \int_0^T \phi_i(0, t) \sum_{j \in \mathcal{M}} \alpha_{ij} u_j^{n_k}(N, t) \ dt, \quad \forall \phi_i \in C_0^\infty([0, L_i] \times [0, T]),$$

here we have used the expression (2.4) for the transmission conditions.

First, we recall that, thanks to Propositions 3.1, 3.2 and 3.3 for $i \in \mathcal{M}$,

$$\int_0^T \int_{I_i} \epsilon_n \iota \iota_{i} \iota_{i} \phi_{i2} \ dx \ dt \to k_{-+\infty} 0.$$ 

Then we notice that, since $u^{n_k} \in C([0, +\infty); W^{2,1}(A))$, thanks to Propositions 3.1, 3.2 the sequence of the traces $u_i^{n_k}(N, t)$ is uniformly bounded in $L^\infty(0, T)$, so that we can consider a subsequence converging weak$^*$ in $L^\infty(0, T)$ to a certain function $W_i^{N_k}(t)$, for each $i \in \mathcal{M}$. 

So, we can consider a subsequence of \( \{ \epsilon_{n_k} \} \), still denoted by \( \{ \epsilon_{n_k} \} \), and letting \( \epsilon_{n_k} \) go to zero in the above equalities, using the dominated convergence theorem and Lemma 4.1, we obtain

\[
\int_0^T \int_{I_i} [u_i \partial_t + \lambda_i u_i \partial_{x_i}] (x,t) \, dx \, dt + \int_{I_i} u_{i0}(x) \phi_i(x,0) \, dx \\
= \int_0^T \phi_i(L_i, t) \sum_{j \in M} \alpha_{ij} W^N_j(t) \, dt , \quad \forall \phi_i \in C^\infty_0((0,L_i) \times [0,T)) \quad i \in I,
\]

Moreover, let \( \zeta \in C^\infty_0(\mathbb{R}) \) be a suitable function such that \( \zeta(x) = 1 \) in a neighborhood of \( x = 0 \); we set \( \zeta^a(x) := \zeta \left( \frac{x}{a} \right) \); if the parameter \( a \) is positive and small, \( (1 - \zeta^a) \phi_i \in C^\infty_0((I_i \times [0,T)) \), then

\[
\mathcal{I}[\phi_i] := \int_0^T [u_i \partial_t + \lambda_i u_i \partial_{x_i}] (x,t) \, dx + \int_{I_i} u_{i0}(x) \phi_i(x,0) \, dx.
\]

Using test functions vanishing in \( \{ N \} \times [0,T) \), it immediately follows that \( u_i \) is a weak solution to the conservation law (1.1) and satisfies in weak sense the initial condition.

In order to prove that \( u \) satisfies the boundary conditions (4.3) and (4.4), we are going to follow the [1, 23], where, however, only boundary conditions were considered. Here we adapt their arguments to transmission conditions.

Let \( \phi_i \in C^\infty_0(I_i \times [0,T)) \); we set

\[
\mathcal{I}[\phi_i] := \int_0^T [u_i \partial_t + \lambda_i u_i \partial_{x_i}] (x,t) \, dx + \int_{I_i} u_{i0}(x) \phi_i(x,0) \, dx.
\]

Moreover, let \( \zeta \in C^\infty_0(\mathbb{R}) \) be a suitable function such that \( \zeta(x) = 1 \) in a neighborhood of \( x = 0 \); we set \( \zeta^a(x) := \zeta \left( \frac{x}{a} \right) \); if the parameter \( a \) is positive and small, \( (1 - \zeta^a) \phi_i \in C^\infty_0(I_i \times [0,T)) \), then

\[
\mathcal{I}[\phi_i] = \mathcal{I}[(1 - \zeta^a) \phi_i] + \mathcal{I}[\zeta^a \phi_i] = \mathcal{I}[\zeta^a \phi_i] = \lambda_i \int_0^T (u_i \phi_i(L_i, t) - u_i \phi_i(0, t)) \, dt ,
\]

where the last equality follows letting \( a \) go to zero and using the properties of the traces (23).

As a consequence of the above computations, by density argument, for all test functions \( \phi_i \in W^{1,1}_0(\mathbb{R} \times (-\infty,T)) \) the following formula holds, for \( i \in M \),

\[
\int_0^T \int_{I_i} [u_i \partial_t + \lambda_i u_i \partial_{x_i}] (x,t) \, dx \, dt + \int_{I_i} u_{i0}(x) \phi_i(x,0) \, dx \\
= \lambda_i \int_0^T (u_i \phi_i(L_i, t) - u_i \phi_i(0, t)) \, dt .
\]

First, using (4.8) with test functions \( \phi_i \) having null trace in \( \{ e_i \} \times (0,T) \) (we recall that \( e_i \) is the the boundary node of the arc \( I_i \), and (4.3), (4.4), we obtain

\[
\lambda_i u_i(L_i, t) = \sum_{j \in M} \alpha_{ij} W^N_j(t) \quad \text{a.e. in } (0,T) , \quad i \in I
\]

the previous equalities prove that \( u \) satisfies the conservation of the fluxes (1.4) at the inner node \( N \), since \( \sum_{i \in M} \alpha_{ij} = 0 \).
Then we use the technique in [23], considering at the inner node the functions $\mathcal{W}_i^N(t), i \in \mathcal{O}$, in place of prescribed boundary functions. We consider functions $w_i \in W^{1,1}(I_i \times [0, T]), i \in \mathcal{I}$, whose traces in $\{0\} \times (0, T)$ are $B_i$ and functions $w_i \in W^{1,1}(I_i \times [0, T]), i \in \mathcal{O}$, whose traces in $\{0\} \times (0, T)$ are $\mathcal{W}_i^N(t)$.

Let $E(\xi) = \xi^2$, then the following inequality holds

$$E(u_{ik}^{\epsilon_n}) \leq \epsilon_n (E(u_{ik}^{\epsilon_n}))_{xx} - \lambda_i (E(u_{ik}^{\epsilon_n}))_x, \quad i \in \mathcal{M},$$

and considering positive test functions $\phi_i$ vanishing in $\{L_i\} \times (0, T)$, we obtain

$$0 \leq \int_0^T \int_{I_i} \left[ E(u_{ik}^{\epsilon_n}) (\phi_{it} + \lambda_i \phi_{ix}) - \epsilon_n (E(u_{ik}^{\epsilon_n}))_x \phi_{ix} \right] (x, t) dx dt$$

$$+ \int_{I_i} E(u_0^{\epsilon_n}(x))\phi_i(x, 0) dx + \int_0^T \left[ \lambda_i E(u_{ik}^{\epsilon_n}) - \epsilon_n (E(u_{ik}^{\epsilon_n}))_x \right] \phi_i (0, t) dt,$$

for all $i \in \mathcal{M}$; moreover, using the equation satisfied by $u_{ik}^{\epsilon_n}$ and test functions $\psi_i = \phi_i E'(w_i)$, we have

$$0 = \int_0^T \int_{I_i} \left[ u_{ik}^{\epsilon_n} (\psi_{ix} + \lambda_i \phi_{ix}) - \epsilon_n u_{ik}^{\epsilon_n} \psi_{ix} \right] (x, t) dx dt$$

$$+ \int_{I_i} u_0^{\epsilon_n}(x)\psi_i(x, 0) dx + \int_0^T \left[ \lambda_i u_{ik}^{\epsilon_n} - \epsilon_n u_{ik}^{\epsilon_n} \psi_{ix} \right] \phi_i (0, t) dt.$$

Subtracting the last relation from the previous one we obtain

$$0 \leq \int_0^T \int_{I_i} \left[ E(u_{ik}^{\epsilon_n}) \phi_{it} - u_{ik}^{\epsilon_n} \psi_{ix} + \lambda_i \left( E(u_{ik}^{\epsilon_n}) \phi_{ix} - u_{ik}^{\epsilon_n} \psi_{ix} \right) \right] (x, t) dx dt$$

$$- \int_0^T \int_{I_i} \epsilon_n \left[ (E(u_{ik}^{\epsilon_n}))_x \phi_{ix} - u_{ik}^{\epsilon_n} \psi_{ix} \right] (x, t) dx dt$$

$$+ \int_{I_i} E(u_0^{\epsilon_n}(x)) - E'(w_i(x, 0)) u_0^{\epsilon_n}(x) \phi_i(x, 0) dx$$

$$+ \int_0^T \left[ \lambda_i \left( E(u_{ik}^{\epsilon_n}) - u_{ik}^{\epsilon_n} E'(w_i) \right) - \epsilon_n \left( E'(u_{ik}^{\epsilon_n}) - E'(w_i) \right) u_{ik}^{\epsilon_n} \right] \phi_i (0, t) dt.$$  

Now we let $k \to +\infty$ and using the dominated convergence theorem and taking into account that $\|u_{ik}^{\epsilon_n}(t)\|_{W^{1,1}(\mathcal{A})}$ and $\epsilon_n \|u_{ik}^{\epsilon_n}(t)\|_{W^{2,1}(\mathcal{A})}$ are bounded in $[0, T]$, uniformly in $n$, we obtain

$$0 \leq \int_0^T \int_{I_i} \left[ E(u_i) \phi_{it} - u_i \psi_{ix} + \lambda_i \left( E(u_i) \phi_{ix} - u_i \psi_{ix} \right) \right] (x, t) dx dt$$

$$+ \int_{I_i} (E(u_0(x)) - E'(w_i(x, 0)) u_0(x)) \phi_i(x, 0) dx$$

$$+ \lambda_i \int_0^T \left[ (E(w_i) - w_i E'(w_i)) \phi_i \right] (0, t) dt,$$

then [38] implies

$$0 \leq \int_0^T \int_{I_i} \left[ E(u_i) \phi_{it} + \lambda_i E(u_i) \phi_{ix} \right] (x, t) dx dt + \int_{I_i} E(u_0(x)) \phi_i(x, 0) dx$$

$$+ \lambda_i \int_0^T \left[ (E(w_i) + E'(w_i)(u_i - w_i)) \phi_i \right] (0, t) dt,$$

for all $i \in \mathcal{M}$.

Now, we consider positive functions $\eta_i \in C^0_0((\infty, L_i) \times [0, T])$ and the functions $\zeta_i$ previously introduced in this proof, with $0 \leq \zeta \leq 1$; in the above inequality we choose $\phi_i = \phi_i^\zeta := \eta_i \zeta_i$.
As \( \alpha \) goes to zero, as in \([23]\), we obtain
\[
0 \leq \lambda_i \int_0^T \left[ E(w_i) - E(u_i) + E'(w_i)(u_i - w_i) \right] \eta_i (0, t) dt, \quad \text{for all } i \in \mathcal{M};
\]
since the above inequality holds for all positive \( \eta_i \in C^1_0((-\infty, L_i) \times [0, T]) \), it follows that, for all \( i \in \mathcal{M} \),
\[
u_i^2(0, t) - w_i^2(0, t) \leq 2w_i(0, t)(u_i(0, t) - w_i(0, t)), \quad \text{a.e. } t \in (0, T).
\]
It is readily seen that the above relation must be an equality and gives \( u_i(0, t) = w_i(0, t) \) for a.e. \( t \), i.e.
\[
\tag{4.10}
0 \leq \lambda_i \int_0^T \left[ E(w_i) - E(u_i) + E'(w_i)(u_i - w_i) \right] \eta_i (0, t) dt, \quad \text{for all } i \in \mathcal{M}.
\]

We remark that the conditions \(4.10\) identify each limit function \( u_i \) as the unique solution to equation \(1.1\), with boundary conditions in \( x = 0 \) given by \(4.10\) (see Definition \(11\) and \(4.8\) with \( \phi_i(L_i, t) = 0 \)).

4.2. **Identification of the limit.** Now we are going to prove that all the sequences \( \{u^n\} \) converge to the same limit function, showing that the limit function \( u \) in the Proposition \(12\) does not depend on the particular subsequence.  

First we notice that the limit function \( u \) is univoquely determined on the incoming arcs \( I_i, i \in \mathcal{I} \), by the boundary and initial conditions for these arcs; moreover, we recall that the limit function \( u \) satisfies the equalities \(1.1\), which, taking into account the second equalities in \(1.10\), can be written in the following way
\[
\sum_{j \in \mathcal{I}} \alpha_{ij} W_j^N(t) + \sum_{j \in \mathcal{O}} \alpha_{ij} u_j(0, t) = \lambda_i u_i(L_i, t), \quad i \in \mathcal{I},
\]
\[
\sum_{j \in \mathcal{I}} \alpha_{ij} W_j^N(t) + (\alpha_{ii} + \lambda_i) u_i(0, t) + \sum_{j \in \mathcal{O}, j \neq i} \alpha_{ij} u_j(0, t) = 0, \quad i \in \mathcal{O},
\]
for a.e. \( t \in (0, T) \); using these equalities we are going to prove that the values \( u_j(0, t), j \in \mathcal{O} \), are determined by the values \( u_j(L_j, t), j \in \mathcal{I} \), by \(1.5\), where the parameters \( \gamma_i \), \( \alpha_{ij} \) depend only on \( \alpha_{ij} \) (i.e. \( K_{ij} \)) and \( \lambda_i \).

Let \( Q \) be the \( m \times m \) coefficients matrix of the linear system \(1.11\) for the unknowns \( W_i^N(t), u_i(0, t), i \in \mathcal{I}, l \in \mathcal{O} \); thanks to \(1.3\) and \(2.2\) this matrix has some useful properties we are going to prove.

We assume that \( \mathcal{I} = \{1, 2, ... m_I\} \) and \( \mathcal{O} = \{m_I + 1, m_I + 2, ..., m\} \) and we set \( m_{\mathcal{O}} = m - m_{\mathcal{I}} \). The matrix \( Q \) has the form
\[
Q = 
\begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1m_I} & \alpha_{1m_I+1} & \cdots & \alpha_{1m} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{m_I1} & \cdots & \alpha_{m_I m_I} & \alpha_{m_I m_I + 1} & \cdots & \alpha_{m_I m} \\
\alpha_{m_I 1} & \cdots & \alpha_{m_I m_I} & \alpha_{m_I m_I + 1} & \cdots & \alpha_{m_I m} \\
\alpha_{m + 1} & \cdots & \alpha_{m m_I} & \alpha_{m m_I + 1} & \cdots & \alpha_{m m} \\
\alpha_{m 1} & \cdots & \alpha_{m m_I} & \alpha_{m m_I + 1} & \cdots & \alpha_{m m} \\
\end{pmatrix}
\]

First we prove that the assumptions on \( K_{ij}, i, j \in \mathcal{M} \) (i.e. on \( \alpha_{ij} \), see \(2.2\)), imply that the matrix \( Q \) is nonsingular. We need some preliminary definitions and theorems, for whom we refer to \([17, 10]\).
Definition 4.1. Given a matrix \( P \in \mathbb{C}^{n \times n} \), \( P = \{p_{ij}\} \), and \( n \) points \( Y_i \) in the plane, the oriented graph associated to \( P \) is the graph obtained joining the points \( Y_i \) and \( Y_j \) with an oriented arc from \( Y_i \) to \( Y_j \), for all \( i, j \) such that \( p_{ij} \neq 0 \).

Definition 4.2. An oriented graph is strongly connected if any two nodes are connected by an oriented walk (i.e., a sequence of oriented arcs \( V_i \) and points \( Y_i \), such that \( V_i = (Y_i, Y_{i+1}) \)).

We are going to deal with the class of irreducible matrices. The definition of irreducible matrix can be found, for example, in [17, 16]; here, in order to apply Theorem 4.1 below, we are going to use the following characterization [17, 16].

Proposition 4.3. \( P \in \mathbb{C}^{n \times n} \) is an irreducible matrix if and only if its associated oriented graph is strongly connected.

Theorem 4.1. (Gershgorin theorems) Let \( P \in \mathbb{C}^{n \times n} \), \( P = \{p_{ij}\} \), and let

\[
J_P^i = \{z \in \mathbb{C} : |z - p_{ii}| \leq \sum_{j=1,j \neq i}^{n} |p_{ij}| \};
\]

then all the eigenvalues of \( P \) belongs to the set \( \bigcup_{i=1}^{n} J_P^i \). Moreover, if \( P \) is an irreducible matrix and \( \mu \) is an eigenvalue lying on the boundary of each disk \( J_P^i \) which contains it, then it lies on the boundaries of all disks \( J_P^i \), \( i = 1, ..., n \).

Assume condition (1.11). We notice that we can consider the case when the parameters \( K_{ij} \) (and, consequently, \( \alpha_{ij} \)) are such that the matrix \( Q \) is irreducible. If not, problem (1.11) can be splitted in two or more independent transmission problems, and for each one of them the corresponding coefficients matrix in system (1.11) is an irreducible matrix; so, Theorem 4.2 below can be proven separately for each independent problem.

Lemma 4.2. Let (1.11) hold. The matrix \( Q \) is nonsingular and \( \det Q > 0 \).

Proof. The matrix \( Q \) is symmetric and

\[
0 < \alpha_{ii} = \sum_{j \in M, j \neq i} |\alpha_{ij}| \quad \text{if} \quad i \in I, \quad \alpha_{ii} + \lambda_i > \sum_{j \in M, j \neq i} |\alpha_{ij}| \quad \text{if} \quad i \in O,
\]

thanks to (2.2), since \( \lambda_i > 0 \) for \( i \in M \); these facts and Theorem 4.1 imply that \( Q \) has real positive eigenvalues since \( J_i^Q \subseteq \{\mathbb{R}z \geq 0\} \) for \( i \in M \), and none eigenvalue can be zero since the origin does not belong to the disks \( J_i^Q \) for \( i \in O \).

The matrix \( Q \) is a \( M \)-matrix, according with the following definition [21].

Definition 4.3. A matrix \( P \in \mathbb{R}^{n \times n} \) which can be expressed in the form \( P = \sigma I - \overline{P} \), where \( \overline{P} = \{p_{ij}\} \) with \( p_{ij} \geq 0 \), \( 1 \leq i, j \leq n \), and \( \sigma \geq \rho(\overline{P}) \), the maximum of the moduli of the eigenvalues of \( \overline{P} \), is called an \( M \)-matrix.

It is easy to check that the matrix \( Q \) verifies the above definition with the position

\[
\sigma = \max\{\alpha_{ii}, \alpha_{jj} + \lambda_j : i \in I, j \in O\}.
\]

Non singular \( M \)-matrices have several properties; in particular all their principal minors are positive and their inverse matrices have non negative elements [21].

In the following lemma we prove further properties for the elements of the matrix \( Q^{-1} \).

Lemma 4.3. Let (1.11) hold and let \( Z = Q^{-1} = \{z_{ij}\}_{i,j \in \mathbb{N}} \). For all \( i \in M \), \( z_{ii} > 0 \); for \( i \neq j \), if \( \alpha_{ij} < 0 \) then \( z_{ij} > 0 \) (\( i, j \in M \)).
\textbf{Proof.} Let consider a submatrix of $Q$ obtained by deleting a set of corresponding rows and columns
\[
\begin{pmatrix}
\alpha_{k,k_1} & \ldots & \alpha_{k,k_n} & \alpha_{k,h_1} & \ldots & \alpha_{k,h_l} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{k_n,k_1} & \ldots & \alpha_{k_n,k_n} & \alpha_{k_n,h_1} & \ldots & \alpha_{k_n,h_l} \\
\alpha_{h,k_1} & \ldots & \alpha_{h,k_n} & \alpha_{h_1,h_1} + \lambda_{h_1} & \ldots & \alpha_{h_1,h_l} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{h_l,k_1} & \ldots & \alpha_{h_l,k_n} & \alpha_{h_l,h_1} & \ldots & \alpha_{h_l,h_l}
\end{pmatrix},
\]
where
\[
(4.12)\quad 0 \leq n \leq m^\mathcal{I}, \quad 0 \leq l \leq m^\mathcal{O},
\]
\[
k_i \in \mathcal{I} \text{ for } i = 1, 2, \ldots, n, \quad h_i \in \mathcal{O} \text{ for } i = 1, 2, \ldots, l,
\]
\[
k_i \neq k_j \text{ for } i, j = 1, 2, \ldots, n, \quad h_i \neq h_j \text{ for } i, j = 1, 2, \ldots, l;
\]
then we consider the submatrix obtained edging the above one with the $r$-th row and the $c$-th column of $Q$,
\[
H_{nl}^{rc} = \begin{pmatrix}
\alpha_{rc} & \alpha_{rk_1} & \ldots & \alpha_{rk_n} & \alpha_{rh_1} & \ldots & \alpha_{rh_l} \\
\alpha_{k_1,c} & \alpha_{k_1,k_1} & \ldots & \alpha_{k_1,k_n} & \alpha_{k_1,h_1} & \ldots & \alpha_{k_1,h_l} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{k_n,c} & \alpha_{k_n,k_1} & \ldots & \alpha_{k_n,k_n} & \alpha_{k_n,h_1} & \ldots & \alpha_{k_n,h_l} \\
\alpha_{h_1,c} & \alpha_{h_1,k_1} & \ldots & \alpha_{h_1,k_n} & \alpha_{h_1,h_1} + \lambda_{h_1} & \ldots & \alpha_{h_1,h_l} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{h_l,c} & \ldots & \alpha_{h_l,k_n} & \alpha_{h_l,h_1} & \ldots & \alpha_{h_l,h_l} + \lambda_{h_l}
\end{pmatrix},
\]
where
\[
(4.13)\quad r, c \in \mathcal{M}, \quad r \neq c, \quad k_i, h_j \neq r, c \text{ for } i = 1, 2, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots, l.
\]
First we are going to show that all the $H_{nl}^{rc}$-type matrices have non positive determinant, and that, if $\alpha_{rc} < 0$, then $\det H_{nl}^{rc} < 0$.

This fact is readily seen for all $n, l$ such that $n + l = 0, 1, 2$ (for any $r, c$ as above), using (2.2) and the positivity of the principal minors of $Q$.

In order to use the principle of induction to prove that $\det H_{nl}^{rc} \leq 0$ for all $l + n < m^\mathcal{I} + m^\mathcal{O}$ and $r, c, k_i, h_j$ satisfying (4.12), (4.13), we assume that $\det H_{nl}^{rc} \leq 0$ for all $n, l$ such that $n + l = \nu - 1 < m^\mathcal{I} + m^\mathcal{O} - 2$ (and any $k_i, h_j, r, c$ as in (4.12), (4.13)), and we compute $\det H_{nl}^{rc}$ when $n + l = \nu$ (and any $k_i, h_j, r, c$ as in (4.12), (4.13)):
\[
\det H_{nl}^{rc} = \alpha_{rc} \det M_{rc} + \sum_{j=1}^{n} \alpha_{rk_j} (-1)^j (\det M_{rk_j}) + \sum_{j=1}^{l} \alpha_{rh_j} (-1)^{n+j} (\det M_{rh_j}),
\]
where $M_{rc}$ is the matrix obtained removing the first line and the first column in $H_{nl}^{rc}$; $M_{rk_j}$ is the one obtained removing the first line and the $j + 1$-th column and $M_{rh_j}$ is the one obtained removing the first line and the $n + j + 1$-th column.

$M_{rk_j}$ is a $H_{n-1,l}^{k_{j,c}}$ matrix, while, for all $j = 2, \ldots, n$, $M_{rk_j}$ becomes a $H_{n-1,l}^{k_{j,c}}$ matrix provided $j - 1$ exchanges of rows, and $M_{rh_j}$ becomes a $H_{nl-1}^{rc}$ matrix provided $n + j - 1$ exchanges of rows.
Using the inductive assumption we have
\[ \det H_{nl} = \alpha_{rc} \det M_{rc} \]

\[ - \sum_{j=1}^{n} \alpha_{rkc} (-1)^j (-1)^{j-1} | \det M_{rkj} | - \sum_{j=1}^{l} \alpha_{rcj} (-1)^{n+j} (-1)^{n+j-1} | \det M_{rhcj} | \]

\[ = \alpha_{rc} \det M_{rc} + \sum_{j=1}^{n} \alpha_{rkc} | \det M_{rkj} | + \sum_{j=1}^{l} \alpha_{rcj} | \det M_{rhcj} | ; \]

\[ \det M_{rc} > 0, \text{ since it is a principal minor of } Q, \text{ so, thanks to (2.2), the principle of induction proves that } \det H_{nl} \leq 0 \text{ for all } r, c, n, l \text{ satisfying (4.12), (4.13). } \]

Finally, it is readily seen that \( \alpha_{rc} \) is univokely determined by the initial data \( i, \) \( j, \) \( i < j \), \( i, j \in \mathbb{N} \), or \( i, j \in \mathbb{Z} \), which implies \( z_{ij} > 0 \) if \( \alpha_{ij} < 0 \).

\[ z_{ij} = (\det Q)^{-1} (-1)^{i+j} \det Q_{ij} \]

where:
- \( Q_{12} \) reveals to be a \( H^{21}_{m_2-1} \) matrix if \( m_2 \geq 2 \) and a \( H^{21}_{0(m_2-1)} \) matrix if \( m_2 = 1 \); \( \det Q_{12} < 0 \) if \( \alpha_{12} < 0 \);
- provided \( j - 2 \) exchanges of rows, \( Q_{ij} \) becomes a \( H^{11}_{n_j} \) matrix, where \( n_j = n_1 + n_2 \); \( \det Q_{ij} = 0 \) if \( \alpha_{ij} < 0 \);
- in general, provided \( j - 2 \) exchanges of rows and \( l - 1 \) exchanges of columns, \( Q_{ij} \) becomes a \( H^{11}_{n_j} \) matrix, where \( n_j = \sum_{i=1}^{l} \alpha_{ij} < 0 \)

so

\[ z_{ij} = (\det Q)^{-1} (-1)^{i+j} \det Q_{ij} \]

\[ = (\det Q)^{-1} (-1)^{i+j} (-1)^{i+j-1} (\det Q_{ij}) = (\det Q)^{-1} | \det Q_{ij} | \]

which implies \( z_{ij} > 0 \) if \( \alpha_{ij} < 0 \).

\[ z_{ij}, i > j: \]

the result follows by the simmetry of \( Z \).

\[ \square \]

In the following theorem we prove our main convergence result.

**Theorem 4.2.** Let (1.7) hold, let \( u_0 \in BV(\mathcal{A}) \) and \( B_i \in \mathbb{R} \) for \( i \in \mathcal{I} \). There exist parameters \( \gamma_{ij} \), satisfying (1.6), univokely determined by \( \lambda_i \) and \( K_{ij} \), such that all the sequences \( \{u_n\}_{n \in \mathbb{N}} \) of solutions to problems (1.2) (\( \epsilon_n \to 0 \)) converge in \( L^1((\mathcal{A} \times (0,T)) \) to the solution of (1.7) - (1.9), for all \( T > 0 \). If (1.10) holds, then the parameters \( \gamma_{ij} \) satisfy (1.7).

**Proof.** Let \( \epsilon_n \to 0 \) and let \( u \) be the limit function in Proposition 4.2 on each arc \( I_i \) incoming in the node the function \( u_i \) is univokely determined by the initial data \( u_{ij} \) and the boundary ones \( B_i \), thanks to the first equalities (3.10) (see Section 1).

The matrix \( Q \) of the system \( 4.11 \) is nonsingular, then \( W_i^{\mathcal{O}}(t) \) and \( u_j(0,t), (i \in \mathcal{I}, j \in \mathcal{O}) \) are univokely determined by \( \lambda_i u_i(L_i,t), i \in \mathcal{I} \); in particular,

\[ \lambda_j u_j(0,t) = \sum_{i \in \mathcal{I}} \lambda_j \gamma_{ij} \lambda_i u_i(L_i,t), \quad j \in \mathcal{O}, \quad (4.14) \]
where $Z = Q^{-1} = \{z_{ij}\}_{i,j \in \mathbb{N}}$; so, on each outgoing arc $I_i$ the limit function $u_j$ is univokely determined by the initial datum $u_{0,j}$, by the initial data $u_0$ and the boundary ones $B_i$, for $i \in \mathcal{I}$, (see Section 1).

This fact prove that the set of possible limit functions for subsequences \( \{u^{e_n}\} \) contains only the solution to problem (1.1)-(1.3), satisfying (1.5) with $\gamma_{ij} = \lambda_i z_{ij}$ $(i \in \mathcal{O}, j \in \mathcal{I})$, in the sense of Definition 1.1.

The elements of $Z$ are non negative, since it is the inverse of a $M-$ matrix, so $\gamma_{ij} \geq 0$; moreover, equalities (1.9) and (1.11) easily imply that $\sum_{i \in \mathcal{O}} \gamma_{ij} = 1$. Finally, thanks to the previous lemma, if $\alpha_{ij} < 0$ $(i \in \mathcal{O}, j \in \mathcal{I})$, then $z_{ji} > 0$, so that condition (1.12) implies condition (1.7).

5. Approximation examples

In the previous section we proved that, if the family of parameters $K_{ij}$ appearing in the transmission conditions in (1.2) satisfies the constrains (1.9), (1.11) and (1.12), then, when $\epsilon_n \to 0$, problems (1.2) approximate the first order transport problem (1.1)-(1.5) with appropriate transmission coefficients $\gamma_{ij}$ satisfying (1.6), (1.7) (in the sense of Theorem 1.2). On the other hand, in the general case, we are unable to prove that, given some parameters $\gamma_{ij}$ satisfying (1.6), (1.7), it is possible to pick out some corresponding coefficients $K_{ij}$ satisfying (1.9), (1.11) (or, equivalently, $\alpha_{ij}$ satisfying (2.2)) and to build a sequence of linear parabolic problems as (1.2) approximating the problem (1.1)-(1.5). The main difficulty is in inverting some complicate matrices and this involves heavy computations. However, in this section we prove such result for some particular and quite general instances.

5.1. First we show that, when the transmission conditions (1.5) have the particular form
\begin{equation}
\lambda_i u_i(0,t) = \gamma_i \sum_{j \in \mathcal{I}} \lambda_j u_j(L_j, t) \quad \forall i \in \mathcal{O}, \quad \gamma_i > 0, \quad \sum_{i \in \mathcal{O}} \gamma_i = 1,
\end{equation}
it is possible to find families $\{K_{ij}\}$ satisfying (1.9), (1.11) in such a way the limit $u$ of the sequence of solutions of problems (1.2) satisfies conditions (5.1).

Let $\mathcal{I} = \{1, 2, ..., m_2\}$, $\mathcal{O} = \{m_2 + 1, m_2 + 2, ..., m\}$; we consider the following parabolic transmission conditions, which are particular cases of the ones in (1.8),
\begin{align*}
-\lambda_i u'_i(N, t) + \epsilon u''_{ix}(N, t) &= \sum_{j \in \mathcal{O}} k_j (u'_j(N, t) - u'_i(N, t)) \quad i \in \mathcal{I}, \\
\lambda_i u'_i(N, t) - \epsilon u''_{ix}(N, t) &= k_i \sum_{j \in \mathcal{I}} (u'_j(N, t) - u'_i(N, t)) \quad i \in \mathcal{O},
\end{align*}
where $k_j > 0$ $(j \in \mathcal{O})$; this kind of transmission conditions involves only the jumps between the solutions on each outgoing arc and the solutions on each incoming one.

The corresponding coefficient matrix $Q$ of the linear system (4.11) is
\[
Q = \begin{pmatrix}
Q_{11} & -k_{m_2+1} & -k_{m_2+2} & \cdots & -k_m \\
-k_{m_2+1} & \ddots & \ddots & \ddots & \ddots \\
-k_{m_2+2} & \ddots & \ddots & \ddots & \ddots \\
\cdots & \ddots & \ddots & \ddots & \ddots \\
-k_m & \cdots & \cdots & \cdots & Q_{22}
\end{pmatrix},
\]
where $Q_{11}, Q_{22}$ are diagonal matrices,

\[
Q_{11} = \left( \sum_{j \in O} k_j \right) I, \quad Q_{22} = \text{diag}\{m_I k_i + \lambda_i, \ i \in O\}.
\]

We consider the second line in (4.11); in this case it gives

\[
(5.2) \quad (m_I k_i + \lambda_i) u_i(0, t) = k_i \sum_{j \in I} W^N_j (t), \quad i \in O;
\]

then, summing the first $m_I$ equations of system (4.11) we obtain

\[
(5.3) \quad \sum_{j \in O} k_j \sum_{i \in I} W^N_i (t) + m_I \sum_{i \in O} -k_i u_i(0, t) = \sum_{i \in I} \lambda_i u_i(L_i, t);
\]

using (5.2) in (5.3) gives

\[
\lambda_i u_i(0, t) = \lambda_i k_i m_I k_i + \lambda_i \sum_{j \in O} \lambda_j \gamma_{ij} \sum_{i \in I} W^N_i (t) = \lambda_i u_i(L_i, t), \quad i \in O.
\]

It is easy to show that, for every set $\{\gamma_i : i \in O, \ 0 < \gamma_i < 1, \sum_{i \in O} \gamma_i = 1\}$, there exist sets $\{k_i : i \in O, \ k_i > 0\}$ satisfying

\[
(5.4) \quad \gamma_i = \frac{\lambda_i k_i}{\lambda_i + m_I k_i} \left( \sum_{j \in O} \frac{\lambda_j k_j}{m_I k_j + \lambda_j} \right)^{-1} \sum_{i \in I} \lambda_i u_i(L_i, t), \quad i \in O.
\]

For this, it is sufficient to fix $\theta > 0$ such that $m_I \theta \gamma_i < \lambda_i$ for all $i \in O$, and choose each $k_i$ in such a way

\[
\frac{\lambda_i k_i}{\lambda_i + m_I k_i} = \theta \gamma_i.
\]

Notice that, when $\mathcal{I} = \{1, \ldots, m - 1\}, \ O = \{m\}$, there is only the following way to set transmission conditions conserving the flux at the node,

\[
(5.5) \quad \lambda_m u_m(0, t) = \sum_{i \in I} \lambda_i u_i(L_i, t);
\]

for this reason, Theorem 4.2 ensures that, for this kind of networks, for all the families $\{K_{ij}\}$ satisfying (4.10), (4.11) the sequences of solutions to problems (4.2) converge the solution to problem (1.1)-(1.3), (5.5).

5.2. Another interesting case we can deal with, it is for networks with only two outgoing arcs, i.e.:

\[
\mathcal{I} = \{1, 2, \ldots, m\}, \ O = \{h_1, h_2\},
\]

with transmission conditions

\[
(5.6) \quad \gamma_{ij} \lambda_i u_i(0, t) = \sum_{j \in \mathcal{I}} \gamma_{ij} \lambda_j u_j(L_j, t) \quad \text{for} \quad i = h_1, h_2,
\]

where $\gamma_{h_1j} \in (0, 1)$ and, obviously, $\gamma_{h_2j} = 1 - \gamma_{h_1j}$, for $j \in \mathcal{I}$. Fixed a pair of families $\{\gamma_{h_1j}\}_{j \in \mathcal{I}}$ and $\{\gamma_{h_2j}\}_{j \in \mathcal{I}}$ satisfying these conditions, we are going to find
families \( \{ K_{ij} \} \) achieving our purpose. We impose the following conditions on some of coefficients \( K_{ij} \) involved in the transmission conditions in \([5.2]\),

\[
K_{ij} = 0 \quad \text{if} \quad i \neq j, \quad i, j \in \mathcal{I} \quad \text{or} \quad i, j \in \mathcal{O}, \quad K_{ih_2} = k > 0 \quad \forall \ i \in \mathcal{I};
\]

the corresponding coefficient matrix of the linear system \([4.11]\) is

\[
Q = \begin{pmatrix}
K_{1h_1} + k & 0 & \cdots & 0 & -K_{1h_1} & -k \\
0 & K_{2h_1} + k & \cdots & 0 & -K_{2h_1} & -k \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & K_{m_2h_1} + k & -K_{m_2h_1} & -k \\
-K_{1h_1} & -K_{2h_1} & \cdots & -K_{m_2h_1} & \sum_{i \in \mathcal{I}} K_{ih_1} + \lambda_{h_1} & 0 \\
-k & -k & \cdots & -k & 0 & m_2k + \lambda_{h_2}
\end{pmatrix}.
\]

In the following we use the notations

\[
u_i := u_i(L_i, t), \quad W_i^N := W_i^N(t) \quad \text{for} \ i \in \mathcal{I}, \quad u_{h_1} := u_{h_1}(0, t), u_{h_2} := u_{h_2}(0, t).
\]

The first \( m_2 \) equations of system \([4.11]\) in this case give

\[
(K_{ih_1} + k)W_i^N = K_{ih_1}u_{h_1} + ku_{h_2} + \lambda_{h_1}u_i \quad i \in \mathcal{I};
\]

using the above relations in the penultimate equation in system \([4.11]\) we obtain

\[
\left( \sum_{i \in \mathcal{I}} \left( \frac{-K_{ih_1}^2}{k + K_{ih_1}} + K_{ih_1} \right) \frac{1}{\lambda_{h_1}} + 1 \right) \lambda_{h_1} u_{h_1}
\]

\[
+ \sum_{i \in \mathcal{I}} \left( \frac{-K_{ih_1}}{k + K_{ih_1}} \right) \frac{k}{\lambda_{h_2}} u_{h_2} = \sum_{i \in \mathcal{I}} \frac{K_{ih_1}}{k + K_{ih_1}} \lambda_{h_1} u_i,
\]

then, using the conservation of the flux \([1.4]\), we have

\[
\left( \frac{k}{\lambda_{h_1}} + \frac{k}{\lambda_{h_2}} \right) \sum_{i \in \mathcal{I}} \frac{K_{ih_1}}{k + K_{ih_1}} + 1 \right) \lambda_{h_1} u_{h_1}
\]

\[
= \sum_{i \in \mathcal{I}} \left( \frac{k}{\lambda_{h_2}} \sum_{j \in \mathcal{I}} \left( \frac{K_{jih_1}}{k + K_{jih_1}} + \frac{K_{ih_1}}{k + K_{ih_1}} \right) \lambda_{h_1} u_i
\]

and the transmission conditions \([5.6]\) are verified if, for \( i \in \mathcal{I},
\]

\[
\gamma_{h_1,i} \left( \left( \frac{k}{\lambda_{h_1}} + \frac{k}{\lambda_{h_2}} \right) \sum_{i \in \mathcal{I}} \frac{K_{ih_1}}{k + K_{ih_1}} + 1 \right) - \left( \frac{k}{\lambda_{h_2}} \sum_{j \in \mathcal{I}} \left( \frac{K_{jih_1}}{k + K_{jih_1}} + \frac{K_{ih_1}}{k + K_{ih_1}} \right) \lambda_{h_1} u_i \right) = 0.
\]

We set

\[
\theta(y) = \frac{y}{k + y}, \quad \theta_i = \theta(K_{ih_1}), \quad i \in \mathcal{I};
\]

so, for any set \( \{ \gamma_{h_1,i} \}_{i \in \mathcal{I}} \), we are looking for coefficients \( K_{ih_1} > 0 \) and \( k > 0 \) verifying

\[
k \left( -\gamma_{h_1,i}(\lambda_{h_1} + \lambda_{h_2}) + \lambda_{h_1} \right) \sum_{j \in \mathcal{I}} \theta_j + \theta_i \lambda_{h_1} \lambda_{h_2} = \gamma_{h_1,i}^{-1} \lambda_{h_1} \lambda_{h_2}, \quad i \in \mathcal{I}.
\]

We consider the linear system for the unknowns \( X_i \)

\[
k \left( -\gamma_{h_1,i}(\lambda_{h_1} + \lambda_{h_2}) + \lambda_{h_1} \right) \sum_{j \in \mathcal{I}} X_j + X_i \lambda_{h_1} \lambda_{h_2} = \gamma_{h_1,i} \lambda_{h_1} \lambda_{h_2}, \quad i \in \mathcal{I};
\]

for small \( k \) the system has dominant diagonal, so it has a unique solution \( \{ X_i \}_{i \in \mathcal{I}} \).

Notice that for \( y > 0 \) the function \( \theta \) increases and \( \theta(y) \in (0, 1) \), then we are going to show that \( 0 < X_i < 1 \) for all \( i \in \mathcal{I} \).
We sum the equations in (5.9)
\[ \sum_{j \in I} X_j \left( k \sum_{i \in I} (-\gamma_{i1}(\lambda_{1i} + \lambda_{2i}) + \lambda_{1i} \lambda_{2i}) \right) = \lambda_{1i} \lambda_{2i} \sum_{i \in I} \gamma_{i1}i \]
and we use the above equality in (5.9)
\[ X_i = \gamma_{i1}i - k \left( \lambda_{1i} \lambda_{2i} + k \sum_{j \in I} (-\gamma_{j1}(\lambda_{1j} + \lambda_{4j}) + \lambda_{4j}) \right), \quad i \in I; \]
now, since each 0 < γ_{i1} < 1, it is possible to choose k so small to have 0 < X_i < 1 for all i ∈ I.

It follows that, to each set \( \{ \gamma_{i1} \}_{i \in I}, \gamma_{i1} \in (0, 1) \), corresponds a small value \( k_0 \), such that, for each 0 < k < k_0 there exist \( K_{i1}, i \in I \), verifying (5.8).

6. Appendix

Proof of Lemma 4.1. Let v ∈ BV(\( A \)). We consider a sequence \( \{w_n\}_{n \in \mathbb{N}} \) such that \( w_n \in C^1(I_i) \cap W^{2,1}(I_i) \) and
\[
\begin{align*}
||w_n - v||_{L^1(A)} & \to n \to \infty 0, \\
||w_n'||_{L^1(I_i)} & \leq TV^0_{\gamma} (v_i) \forall i \in \mathcal{M}, \\
\epsilon_n \parallel w_n \parallel_{W^{2,1}(A)} & \leq C_1,
\end{align*}
\]
where \( C_1 \) is a quantity independent from \( n \) and, for \( f \in BV(A) \),
\[
TV^0_{\gamma} (f_i) = \sup \left\{ \int_{I_i} f_i \phi' dx : \phi \in C^0_{\gamma}(I_i), |\phi| \leq 1 \right\}
\]
Then we introduce the polynomials \( p_n \) defined on the network,
\[
p_n(x) = a_n x^3 + b_n x^2 + c_n x + d_n, \quad x \in I_i,
\]
whose coefficients have to be determined. When \( i \in \mathcal{O} \), we impose the conditions
\[
\begin{align*}
p_n(0) &= w_n(0), \\
\epsilon_n p'_n(0) &= \sum_{j \in \mathcal{M}} \alpha_{ij} w_n(N) + \lambda_i w_n(0),
\end{align*}
\]
where
\[
\delta_n = \epsilon_n^\theta, \quad \theta > 1,
\]
and we define the sequence \( \{v_n\}_{n \in \mathbb{N}} \) on the outgoing arcs
\[
v_{n1}(x) = \begin{cases} 
w_n(x) & x \in [\delta_n, L_i] \\
p_n(x) & x \in [0, \delta_n]
\end{cases}, \quad i \in \mathcal{O}.
\]
When \( i \in I \) we define
\[
v_{n1}(x) = \begin{cases} 
w_n(x) & x \in [\epsilon_n, L_i - \delta_n] \\
p_n(x) & x \in [L_i - \delta_n, L_i] \\
r_n(x) & x \in [0, \epsilon_n]
\end{cases}, \quad i \in I,
\]
where \( p_n(x) \) are the polynomials in (6.12) whose coefficients are determined by
\[
\begin{align*}
p_n(L_i) &= w_n(L_i), \\
\epsilon_n p'_n(L_i) &= \sum_{j \in \mathcal{M}} \alpha_{ij} w_n(N) + \lambda_i w_n(L_i),
\end{align*}
\]

\[
\begin{align*}
p_n(L_i - \delta_n) &= w_n(L_i - \delta_n), \\
p'_n(L_i - \delta_n) &= w'_n(L_i - \delta_n),
\end{align*}
\]
and \( r_{ni}(x) = \mu_n^i x^2 + \nu_n^i x + \rho_n^i \), where the coefficients are determined by the following conditions

\[
\begin{align*}
(6.6) & \ \\
& \begin{cases}
  r_{ni}(0) = B_i , \\
  r_{ni}(\epsilon_n) = w_{ni}(\epsilon_n) , \\
  r'_{ni}(\epsilon_n) = w'_{ni}(\epsilon_n) .
\end{cases}
\end{align*}
\]

For all \( n \in \mathbb{N} \), \( v_n \in D_{\epsilon_n} \), since \( v_{ni} \in C^1(I_n) \) for all \( i \in I \), \( \nu_n^i \in L^1(A) \), the transmission conditions at the internal node are verified, thanks to the first two equalities in (6.3) and (6.5), and also the boundary ones, thanks to the first condition in (6.6).

If \( i \in O \), the conditions in (6.3) imply

\[
(6.7) \quad d_n^i = w_{ni}(0) , \quad \epsilon_n c_n^i = \sum_{j \in I} \alpha_{ij} w_{nj}(N) + \lambda_{i} w_{ni}(0)
\]

so that

\[
(6.8) \quad |d_n^i| , \epsilon_n |c_n^i| \leq C_0 , \quad n \in \mathbb{N} ,
\]

where \( C_0 \) is a quantity independent from \( n \), thanks to (6.1). The conditions in (6.3) also imply

\[
(6.9) \quad |b_n^i \delta_n^i| = | - 2 \epsilon_n \delta_n + 3(w_{ni}(\delta_n) - w_{ni}(0)) - \delta_n w'_{ni}(\delta_n)|
\]

\[
(6.10) \quad |a_n^i \delta_n^2| = | \epsilon_n \delta_n - 2(w_{ni}(\delta_n) - w_{ni}(0)) + \delta_n w'_{ni}(\delta_n)| ;
\]

notice that the quantities in (6.9) and (6.10) go to zero when \( n \) goes to infinity, since \( w_{ni} \) is continuous and \( \delta_n w'_{ni}(\delta_n) \) is infinitesimal thanks to (6.3) and (6.1).

Now we have

\[
\|p_n\|_{L^1(0,\delta_n)} \leq \left| a_n^i \frac{\delta_n^2}{3} \right| + \left| b_n^i \frac{\delta_n^3}{2} \right| + |d_n^i| \delta_n
\]

\[
\|p_n'\|_{L^1(0,\delta_n)} \leq |a_n^i \delta_n^3| + |b_n^i \delta_n^2| + |c_n^i| \delta_n .
\]

Similar computations can be made when \( i \in I \), so that, thanks to (6.8), (6.9), (6.10),

\[
(6.11) \quad \sum_{i \in O} \|p_n\|_{W^{1,1}(0,\delta_n)} + \sum_{i \in I} \|p_n\|_{W^{1,1}(I_n)} \to n \to +\infty 0 .
\]

As regard to \( r_n \), using conditions in (6.6) we see that

\[
(6.12) \quad \rho_n^i = B_i , \quad \mu_n^i \epsilon_n^2 + \nu_n^i \epsilon_n + B_i = w_{ni}(\epsilon_n) , \quad 2\mu_n^i \epsilon_n + \nu_n^i = w'_{ni}(\epsilon_n) ,
\]

so we have, thanks to (6.1),

\[
|\nu_n \epsilon_n| = | - 2B_i + 2w_{ni}(\epsilon_n) - w'_{ni}(\epsilon_n)| \leq C_\nu , \quad C_\nu \text{ independent from } n ,
\]

\[
|\mu_n^{i} \epsilon_n^2| = |B_i + w'_{ni}(\epsilon_n) - w_{ni}(\epsilon_n)| \leq C_\mu , \quad C_\mu \text{ independent from } n ,
\]

which imply

\[
(6.12) \quad \sum_{i \in I} \|r_n\|_{L^1(0,\epsilon_n)} \leq |\mu_n^i \frac{\epsilon_n^2}{3}| + |\nu_n^i \frac{\epsilon_n}{2}| + |\rho_n^i| \epsilon_n \to n \to +\infty 0 ,
\]

and

\[
(6.13) \quad \sum_{i \in I} \|r_n'\|_{L^1(0,\epsilon_n)} \leq |\mu_n^{i} \epsilon_n^2| + |\nu_n^{i} \epsilon_n| \leq C_r , \quad C_r \text{ independent from } n .
\]

Thanks to (6.1), (6.11), (6.12) we obtain

\[
(6.14) \quad \|v_n - v\|_{L^1(A)} \to n \to +\infty 0 .
\]
Now we are going to prove that, for \( i \in \mathcal{M} \),

\[
\|v_n\|_{L^1(I_i)} \leq TV_0^{L^1}(v_i) + C_{BV}, \quad C_{BV} \text{ independent from } n
\]

for \( i \in \mathcal{I} \)

\[
\int_{I_i} |v_{n,i}'| dx \leq \|p_{n,i}'\|_{L^1(I_i - \delta_n, L_i)} + \|r_{n,i}'\|_{L^1(0,\varepsilon_n)} + \|w_{n,i}'\|_{L^1(I_i)} ,
\]

and similar computations can be made for the outgoing arcs, so that, using (6.1) and (6.11)-(6.13) we obtain (6.15), which implies that

\[
\|v_n\|_{W^{1,1}(A)} \leq C_2 , \quad \text{for all } n \in \mathbb{N} ,
\]

where \( C_2 \) depends on \( \|v\|_{L^1(\mathcal{A})} \) and \( TV_0^{L^1}(v_i), i \in \mathcal{M} \).

We have only to prove that \( \varepsilon_n\|v_n\|_{W^{2,1}(\mathcal{A})} \) is bounded independently of \( n \). For \( i \in \mathcal{O} \),

\[
\varepsilon_n\|p_{n,i}'\|_{L^1(0,\delta_n)} \leq \varepsilon_n(3|a_n^i|\delta_n^2 + 2|b_n^i|\delta_n);
\]

from (6.9), (6.10), (6.8) and (6.1) we know that

\[
\varepsilon_n|b_n^i|\delta_n = \varepsilon_n - 2\delta_n + \frac{3}{\delta_n}(w_{n,i}(\delta_n) - w_{n,i}(0)) - w_{n,i}'(\delta_n))| \leq 2C_0 + \varepsilon_n 4C_3\|v_{n,i}\|_{W^{2,1}(I_i)} \leq 2C_0 + 4C_3C_1
\]

and

\[
\varepsilon_n|a_n^i|\delta_n^2 = \varepsilon_n \left| r_n^i - 2\frac{w_{n,i}(\delta_n) - w_{n,i}(0)}{\delta_n} + w_{n,i}'(\delta_n) \right| \leq C_0 + 3\varepsilon_nC_3\|v_{n,i}\|_{W^{2,1}(I_i)} \leq C_0 + 3C_3C_1 ,
\]

where \( C_3 \) depends on Sobolev constants. Similar estimates can be obtained for \( i \in \mathcal{I} \), to conclude that there exists \( C_3 > 0 \) such that

\[
\varepsilon_n\|p_{n,i}'\|_{W^{2,1}(0,\delta_n)}, \varepsilon_n\|p_{n,j}'\|_{W^{2,1}(L_i, \varepsilon_n, L_j)} \leq C_3 , \quad \forall n \in \mathbb{N} , i \in \mathcal{O} , j \in \mathcal{I} ,
\]

(see also (6.1)) moreover, for \( i \in \mathcal{I} \), we know that

\[
\varepsilon_n\|v_{n,i}''\|_{L^1(0,\varepsilon_n)} \leq 2\varepsilon_n^2|\mu_n| \leq 2C_\mu .
\]

Since \( \varepsilon_n\|w_{n,i}\|_{W^{2,1}(A)} \leq C_1, \) for all \( n \in \mathbb{N} \), we conclude that \( \varepsilon_n\|v_n\|_{W^{2,1}(\mathcal{A})} \leq C , \) for all \( n \in \mathbb{N} \), where \( C \) depends on \( C_0, C_1, C_2, C_3, C_\mu \).

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