Is the mean-field approximation so bad?
A simple generalization yielding realistic critical indices for 3D Ising-class systems

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Modification of the renormalization-group approach, invoking Stratonovich transformation at each step, is proposed to describe phase transitions in 3D Ising-class systems. The proposed method is closely related to the mean-field approximation. The low-order scheme works well for a wide thermal range, is consistent with a scaling hypothesis and predicts very reasonable values of critical indices.

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It is well understood since 1960’s that an essential increase of fluctuations accompanies a second-order phase transition. Exact results for 2D Ising model showed that Landau phenomenology was too crude to describe the vicinity of the transition point. Ginzburg demonstrated that it is correlations what breaks Landau approach down. He established the parameter which determines the thermal range for the so-called fluctuation region. According to the scaling hypothesis, the properties of a system in the fluctuation region are determined by the single quantity \(r_c\), being the correlation length for fluctuations. This parameter shows a power-law dependence on the external field \(h\) and on the thermal interval \(t\) from the transition: \(r_c \propto h^{-\mu}, r_c \propto t^{-\nu}\). Values of \(\mu\) and \(\nu\) are determined by the universal characteristics, like the dimensionality \(D\), the number of components of the order parameter, and the type of the interaction in the system (long-range or short-range). Other critical indices can be expressed via \(\mu\) and \(\nu\). For instance for the scalar order parameter \(\bar{x}\) one obtains \(\bar{x} \propto t^{-\beta}/s\), where \(\bar{x}\) is a cut-off and \(s > 1\) is a parameter. The main assumption of the method is that the potential energy of thus renormalized system can be truncated to certain simplified form, for example to the form of \(\phi^4\) model. The second step is a scale transformation. It is aimed to restore the cut-off for \(k\) back to \(k_0\) and preserve the original dispersion law for the long-wavelength modes.

The RG transformation should be applied recursively; its unstable fixed point corresponds to a phase transition. Lyapunov exponents for the vicinity of the fixed point determine the critical indices. The scaling hypothesis requires exactly two of the Lyapunov exponents to be positive, they correspond to the two independent critical indices.

With respect to the value of \(s\), two modifications of the RG scheme can be distinguished. Wilson’s \(\varepsilon\) expansion deals with \(s \gg 1\). The method is formally valid near the upper critical dimension \(D_{up}\) (for the Ising-class classical systems \(D_{up} = 4\)). The "most divergent" diagram series can be pointed out in this case. This gives the asymptotic expansion in terms of \(\varepsilon \equiv D_{up} - D\). In the continuous version of the scheme (so-called exact RG approach), \(s\) is infinitesimally close to unity. Certain decoupling (truncation) should be introduced in this case to solve the integro-differential equation for RG flow. An accuracy is then determined by a form of the decoupling used.

All the approaches mentioned above are very well-developed. They allow to calculate critical indices and universal combinations of amplitudes with a high accuracy. Progress is also achieved in the calculation of non-universal quantities; particularly a crossover between Gaussian (Landau-like) and critical behaviour is analyzed. However RG is applicable only for long-wavelength excitations since they can be described as a fluctuating field. Therefore, in a practical calculation, short-wavelength excitations should be first integrated out by means of some perturbation expansion, and then RG should be applied. The short- and the long-wavelength excitations should be separated ad hoc. Such a two-step procedure is not very elegant. At least from the methodological point of view it is desired to develop a scheme which gives all the information in a uniform manner.

It is interesting to note that calculations are much simpler outside the critical region. In fact, even the mean-field approximation (MFA) is usually sufficient for the qualitative analysis of this region. In MFA, the non-local interaction in the system is replaced by an effect of a self-consistent average field. The MFA predictions for the transition temperature and thermal behaviour of the order parameter in 3D case are quite reasonable. On the other hand MFA gives Landau set of indices and is not very accurate. Note that Landau approach itself has a mean-field nature.

In this paper, we present a simple generalization of MFA, performing well for the 3D Ising-class systems both nearby and far from the transition. Particularly the critical behaviour is reproduced correctly: the values of indices are predicted with a few-percent accuracy. Essentially, the present approach consists in the alternating application of the Stratonovich transformation and the most primitive version of the renormalization-group approach.

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transformation at a finite value of \( s \).

We consider a classical scalar field with an anharmonic local potential and a short-range harmonic interaction at temperature \( T \). The partition function of this system is

\[
Z = \int [Dx] e^{-V/T}; \quad (1)
\]

\[
V = \int U(x_r + \bar{x})dr + \sum_k \frac{\Omega_k |x_k|^2}{2}.
\]

Here \( U(x + \bar{x}) \) is an even on-site potential, \( x_k \) is a Fourier transform of the displacement \( x_r \) from the average position \( \bar{x}; \ \bar{x} \neq 0 \) below the transition temperature, and the average of \( x_r \) equals zero by definition. To make the definitions unambiguous, we put additional conditions \( U(\bar{x}) = 0; \Omega_{k=0} = 0 \). The wave vector \( k \) does not exceed the unitary cut-off: \( k^2 \leq 1 \). The case of a short-range interaction corresponds to the asymptotic behaviour

\[
\Omega_k = \omega k^2, k \to 0. \quad (2)
\]

The simplest way to integrate out the modes is to skip \( x_k \) with \( k > 1/s \). This corresponds to the zero-th approximation in the nonlinearity of \( U \). The subsequent scaling results in the RG transformation \( k \to sk, U \to s^3 U, x \to x/\sqrt{s} \). The defect of thus applied RG method comes from the increase of correlations (non-linear part of \( U \) grows). Therefore, after several iterative transformations, the RG procedure is not valid any more. Although the formal analysis of the fixed point \( U(x) = 0 \) can be done, it wrongly gives Landau-like critical indices.

To avoid the problem higher-order approximations and the truncation of the potential are used in known schemes described above (a very clear explanation is given in [10]). Here we use another procedure: the zero-th approximation is used, but the change of variables is made at each RG step. The change of variables reduces correlations and is aimed to compensate its increase at RG transformation.

At the first stage, we integrate out (omit) all the modes with \( \Omega_k > \bar{\Omega} \), where \( \bar{\Omega} = \frac{4}{\pi x} \int \Omega_k d^3k \) is an average of \( \Omega_k \).

For the rest of the modes, the Stratonovich identity is utilized:

\[
\exp \left( -\frac{(\Omega_k - \bar{\Omega}|x_k|^2)}{2T} \right) = \frac{\bar{\Omega} - \Omega_k}{\pi \Omega^2} \int \exp \left( -\frac{1}{T} \left( 2\bar{\Omega} \Re(x_k f_k^*) + \frac{\bar{\Omega} \Omega_k |f_k|^2}{\Omega - \Omega_k} \right) \right) df_k. \quad (3)
\]

Here the integration over the complex variable \( f_k = \Re(f_k) + i \Im(f_k) \) denotes the integration over the complex plane:

\[
\int df_k \equiv \int_{-\infty}^{\infty} d\Re(f_k) \int_{-\infty}^{\infty} d\Im(f_k).
\]

After (3) is substituted into (1), expressions \( \sum_k \Re(x_k f_k^*) = \int x_r f_r dr \) and \( \sum_k \bar{\Omega}|x_k|^2 = \int \bar{\Omega} x_r^2 \) can be collected in the exponent. Therefore

\[
Z \propto \int [Dx] \int [Df] \exp \left( -\frac{1}{T} \left( \int \left( U(x_r + \bar{x}) - \Omega_k f_r x_r + \frac{\bar{\Omega} x_r^2}{2} \right) dr + \sum_k \frac{\bar{\Omega} \Omega_k |f_k|^2}{2(\Omega - \Omega_k)} \right) \right), \quad (4)
\]

where \( f_r \) is the inverse Fourier transform of \( f_k \).

Now, we can integrate over \( x_r \). Let us introduce the function \( F(f) \) accordingly to the equation:

\[
\exp \left( -\frac{F(f + \bar{x}) - F_0}{T} \right) = \int \exp \left( -\frac{1}{T} \left( U(x + \bar{x}) + \frac{\bar{\Omega}(x - f)^2}{2} \right) \right) dx,
\]

where \( F_0 \) is a constant, defined from the condition \( F(\bar{x}) = 0 \).

With thus defined \( F \), the partition function takes the form

\[
Z \propto \int [Df] e^{-W/T}; \quad (6)
\]

\[
W = \int F(f_r + \bar{x})dr + \sum_k \frac{\bar{\Omega} \Omega_k |f_k|^2}{2(\Omega - \Omega_k)}. \]

Equation (6) is formally very similar to Eq.(4). Function \( F \) is even; the dispersion at \( k \to 0 \) coincides that of Eq.(4). One can also guess from Eq.(3) that as the average of \( x_r \) equals zero, the average of \( f_r \) also vanishes. Indeed, if this average would not be zero, \( f_{k=0} \) would fluctuate around a macroscopically large average value, and (4) would not be fulfilled.

We argue here that variables \( f_k \) are much less correlated, than \( x_k \). Let us consider the parabolic approximation for \( F \): \( F(\bar{x} + f) \approx \frac{\partial^2 F(\bar{x})}{\partial f^2} f + \frac{\partial F(\bar{x})}{\partial f} f^2 \). Variables \( f_k \) are uncorrelated in this approximation. Obviously \( < f > \)
vanishes only if \( \frac{\partial F(x)}{\partial f} = 0 \), that gives
\[
\frac{\bar{\Omega}}{T} = \int \frac{x \exp \left( -T^{-1} \left( U(x + \bar{x}) + \frac{1}{2} \bar{\Omega} x^2 \right) \right) dx}{\int \exp \left( -T^{-1} \left( U(x + \bar{x}) + \frac{1}{2} \bar{\Omega} x^2 \right) \right) dx}.
\]
(7)

This is exactly the mean-field equation of state for the system [3]. So, even the complete neglect of correlations in \( f_k \) still gives something reasonable. On the other hand, the parabolic approximation for \( U \) directly in Eq. (3) gives non-sense: since \( U(x) \) does not depend on temperature, the system does not show a transition at all in this approximation. Such a comparison convinces us that correlations in \( f_k \) are less essential, than in \( x_k \). As it was pointed above we expect that this can compensate an increase of correlations in RG transformation.

Now, we finish the RG transformation, operating with Eq. (3). All \( f_k \) with \( k > 1/s \) are to be integrated out (omitted in our approximation). After the scale transformation \( k \to sk \), \( f \to x/\sqrt{s} \) is done, the potential energy of the system takes the form
\[
\int U^{(1)}(x_r + \bar{x}^{(1)}) dx + \sum_k \frac{\Omega_k^{(1)} |x_k|^2}{2},
\]
where
\[
\Omega_k^{(1)} = \frac{s^2 \Omega_k}{\Omega - \Omega_k/s},
\]
(9)

\( U^{(1)}(x + \bar{x}^{(1)}) = s^3 F(x/\sqrt{s} + \bar{x}), \quad \bar{x}^{(1)} = \bar{x} \sqrt{s}. \)
(10)

Such a transformation should be applied recursively, like in the standard RG approach. The analysis can be easily performed numerically. The sequence \( \Omega_k^{(1)}, \Omega_k^{(2)}, \ldots \) is independent of \( U \) and \( T \). At every step a new \( \bar{\Omega}^{(n)} \) are to be calculated. After several transformations \( \bar{\Omega}^{(n)} \) and \( \bar{\Omega}^{(n)} \) converge to certain limit. Calculation from formula (9) gives
\[
\bar{\Omega}_k^{(\infty)} = \frac{3 \omega \bar{\Omega}_k^{(\infty)} (s^2 - 1) k^2}{s \bar{\Omega}_k^{(\infty)} (s^2 - 1) - \omega k^2},
\]
(11)

\( \bar{\Omega}_k^{(\infty)} \) is implicitly defined from the equation \( \bar{\Omega}_k^{(\infty)} = 0 \).

Now, let us consider the properties of the sequence \( U, U^{(1)}, U^{(2)}, \ldots \) at \( \bar{x} = 0 \). Transformation \( U^{(n)} \to U^{(n+1)} \) has a non-trivial unstable fixed point \( U_f(x) \). This function is shown in the upper panel of Fig.1. Consider small deviations from this fixed point: \( U^{(n)}(x) = U_f(x) + u_n(x) \). The linearization of the transformation \( U^{(n)} \) gives
\[
u = u^0 + s^3 \int u_n(x) A(x, y) dx;
\]
(12)
\[A = \exp \left( -U_f(x) - s^{-3} U_f(y) + F + \frac{\bar{\Omega}}{T} (x - y/\sqrt{s})^2 \right).\]

where \( u^0 \) stands to fulfill the condition \( u_{n+1}(0) = 0 \). The analysis of this operator shows that among its eigenvalues only two exceed unity. Denote them \( s^{\lambda_e} \) and \( s^{\lambda_o} \); they correspond to an even and odd eigenvector, respectively. Thus the theory is consistent with the scaling hypothesis, as it was mentioned in an introductory part.

Exponents \( \lambda_e \) and \( \lambda_o \) are the inverse of the critical indices \( \nu \) and \( \mu \), respectively [3]. In the ideal case, critical indices should be independent of the particular \( U(x) \) and \( \Omega_k \), as well as of the value of the factor \( s \). Indeed, after a proper re-scaling of the units, the values of \( T \) and \( \Omega \) drop out from Eq. (11). Therefore in our model \( \nu \) and \( \mu \) can depend only on \( s \). Further, it turns out that \( \mu = 2/5 \) at any \( s \), as it should be for the local-potential approximation [3]. What is about \( \nu \), it depends on \( s \) very weakly. The numerical result for \( \nu(s) \) is shown in the lower panel of Fig.1.

On the other hand it is important to point out that the crude universal properties of the model enter the transformation \( U^{(n)} \) essentially. The behaviour \( \Omega_k \to \infty k^2 \) corresponds to a short-range range interaction. Such behaviour and the dimensionality of the model determine the scale transformation, i.e. powers of \( s \) in Eq. (3). It is also directly reflected in the formula that the order parameter is a scalar.

For the more detailed analysis it is reasonable to consider an extreme value of \( \nu(s) \), because in this case small
cally, it is enough to calculate (compare with the derivation of formula (7)). Numeri-

cation both of the fluctuation region and of the system far

and a scalar order parameter. It gives a unified descrip-

tion correctly all the physics of the second-order phase

transitions in 3D systems with a short-range interaction

still performs good.

Note that Ising model is very non-linear, but the scheme

value of $\bar{\Omega}$ does not change the result obtained. To determine the

dependence of the order parameter $x$ on temperature for the 3D Ising model. The

model is defined as a discrete cubic lattice with $U(x) = \delta(x - 1) + \delta(x + 1)$ and the nearest-neighbors interaction

$\Omega_k = 3 - \cos \pi k_x - \cos \pi k_y - \cos \pi k_z$. The initial Brillouin

zone is a cube $-1 < k_{x,y,z} < 1$; at the first step the value

$\Omega = 3$ is used and all the modes with $|k| > 1/s$ are

integrated. The procedure for further steps is exactly as described above. To simplify the calculation, we put

$\Omega^{(n)} = \Omega^{(\infty)}$ at these steps; this simplification almost

does not change the result obtained. To determine the

value of $x$ we use the condition

$$\frac{\partial}{\partial x} U_n(x_n) = 0, n \to \infty$$

(compare with the derivation of formula (1)). Numeri-
cally, it is enough to calculate $U_n$ up to $n \approx 8$. The result for $x^2(T)$ is presented in Fig.2. The mean-field and experi-

mental (Monte-Carlo) data are sketched in the same figure for comparison. Far from the transition $U^{(1)}$ is almost parabolic, therefore the theory passes into MFA.

The critical behaviour occurs at the transition point. Note that Ising model is very non-linear, but the scheme still performs good.

We conclude that the presented low-order scheme de-

scribes correctly all the physics of the second-order phase transitions in 3D systems with a short-range interaction and a scalar order parameter. It gives a unified description both of the fluctuation region and of the system far

from the transition point. The method does not pretend to be very accurate; its advantage consists in simplicity. In principle, one can improve the accuracy by a higher-

order procedure for the integration in RG transformation. An applicability of the method for another universality

classes is the subject of further studies.

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**TABLE I:** Values of the critical indices obtained from the Lyapunov exponents for the transformation $[12]$ at $s = 2.5$. Results of the Landau theory and known values $[4]$ for 3D Ising model are given for the comparison.

| Index | Estimation from Eq.(12) | 3D Ising | Landau theory |
|-------|------------------------|----------|---------------|
| $\nu$ | 0.652                  | 0.63     | 1/2           |
| $\mu$ | 2/5                    | 0.403    | 1/3           |
| $\beta$ | 0.326               | 0.327    | 1/2           |

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**FIG. 2:** The dependence of the square of the order parameter on the temperature in the 3D Ising model. Solid curve: the present approach with $s = 2.5$; the dot curve: mean-field calculation; the dash curve: numerical results. Thin ticks show the tangents to $x^2(T)$ in the transition point.

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