NONLINEAR SCHRÖDINGER EQUATION WITH COULOMB POTENTIAL

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ABSTRACT. In this paper, we study the Cauchy problem for the nonlinear Schrödinger equations with Coulomb potential
\[ i\partial_t u + \Delta u + \frac{K}{|x|} u = \lambda |u|^{p-1} u \]
with \(1 < p \leq 5\) on \(\mathbb{R}^3\). We mainly consider the influence of the long range potential \(K|x|^{-1}\) on the existence theory and scattering theory for nonlinear Schrödinger equation. In particular, we prove the global existence when the Coulomb potential is attractive, i.e. \(K > 0\) and scattering theory when the Coulomb potential is repulsive i.e. \(K \leq 0\). The argument is based on the interaction Morawetz-type inequalities and the equivalence of Sobolev norms.

Key Words: Nonlinear Schrödinger equation; global well-posedness; blow-up; scattering.

AMS Classification: 35P25, 35Q55, 47J35.

1. Introduction

We study the initial-value problem for the nonlinear Schrödinger equations with Coulomb potential
\[
\begin{aligned}
(i\partial_t - L_K)u &= \lambda f(|u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
u(0, x) &= u_0(x) \in H^1(\mathbb{R}^3), \quad x \in \mathbb{R}^3,
\end{aligned}
\]
(1.1)
where \(u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}\), \(L_K = -\Delta - \frac{K}{|x|}\) with \(K \in \mathbb{R}\), \(f(|u|^2) = |u|^{p-1}\), and \(\lambda \in \{\pm 1\}\) with \(\lambda = 1\) known as the defocusing case and \(\lambda = -1\) as the focusing case.

The study of the operator \(L_K = -\Delta - K|x|^{-1}\) with the Coulomb potential originates from both the physical and mathematical interests. In particular, \(K\) is positive, this operator provides a quantum mechanical description of the Coulomb force between two charged particles and corresponds to having an external attractive long-range potential due to the presence of a positively charged atomic nucleus. We refer the reader to [34, 40] for work on these more models of the hydrogen atom in quantum physics fields.

The mathematical interest in these equations however comes from the operator theory with a long range decay potential and the dispersive behavior of the solution. Note that \(|x|^{-1} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)\), we know from [35] Theorem X.15 that \(L_K\) is essentially self-adjoint on \(C_0^\infty(\mathbb{R}^3)\) and self-adjoint on \(D(-\Delta)\). We refer the reader to [38, 44] for more theory of this operator. The nonlinear equation (1.1) and many variations aspects have been studied extensively in the literature. In particular, the existence of a unique strong global-in-time solution to (1.1) with Hartree nonlinearity \(f(|u|^2) = |x|^{-1} * |u|^2\) goes back to [6]. When \(K \leq 0\), the solution \(u(t)\) to (1.1) with the Hartree nonlinearity is studied in [11, 19] in which they proved the global existence and a decay rate for the solution; however, they
need the initial data in a weighted-\(L^2\) space. When \(K > 0\), Lenzmann and Lewin [31] proved a time average estimate holds for every \(R > 0\) such that

\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \int_{|x| \leq R} |u(t, x)|^2 \, dx \, dt \leq 4K
\]  

(1.2)

and

\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \int_{|x| \leq R} |\nabla u(t, x)|^2 \, dx \, dt \leq K^3
\]  

(1.3)

which is related to the RAGE theorem (see Reed-Simon [38]).

In this paper, we will study the Cauchy problem for the nonlinear Schrödinger equation (1.1) with initial data in energy space \(H^1(\mathbb{R}^3)\). The Cauchy problem, including the global existence and scattering theory, for the nonlinear Schrödinger equation without potential, i.e. \(K = 0\), has been intensively studied in [5, 17]. Due to the perturbation of the long range potential, many basic tools which were used to study the nonlinear Schrödinger equation are different even fails. We only have a local-in-time Strichartz estimate and global-in-time Strichartz estimate fails when \(K > 0\). We therefore show the solution of (1.1) is global existence but does not scatter. Fortunately, in the case \(K < 0\), Mizutani [35] recently obtained the global-in-time Strichartz estimate by employing several techniques from scattering theory such as the long time parametrix construction of Isozaki-Kitada type [23], propagation estimates and local decay estimates. In this repulsive case, we will establish an interaction Morawetz estimate for the defocusing case, which provides us a decay of the solution \(u\) to (1.1). Combining this with the global-in-time Strichartz estimate [35], we therefore obtain the scattering theory in the repulsive and defocusing cases.

It is worth mentioning that in the proof of scattering theory, we also need a chain rule which is established by proving the equivalence of the Sobolev norm from the heat kernel estimate, as we did in [26, 47]. Even though we obtain some results for this Cauchy problem, the whole picture of the nonlinear Schrödinger equation with the Coulomb potential is far to be completed, for example, the scattering theory in the energy-critical cases.

Equation (1.1) admits a number of symmetries in \(H^1(\mathbb{R}^3)\), explicitly:

* **Phase invariance**: if \(u(t, x)\) solves (1.1), then so does \(e^{i\gamma}u(t, x)\), \(\gamma \in \mathbb{R}\);

* **Time translation invariance**: if \(u(t, x)\) solves (1.1), then so does \(u(t + t_0, x + x_0)\), \((t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3\).

From the Ehrenfest law or direct computation, these symmetries induce invariances in the energy space, namely: mass

\[
M(u) = \int_{\mathbb{R}^3} |u(t, x)|^2 \, dx = M(u_0)
\]  

(1.4)

and energy

\[
E(u) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u|^2 - \frac{K}{2} |u|^2 + \frac{\lambda}{p + 1} |u|^{p+1} \right) \, dx.
\]  

(1.5)

Comparing with the classical Schrödinger equation (i.e. (1.1) with \(K = 0\)), equation (1.1) is not space translation invariance, which induces that the momentum

\[
P(u) := \text{Im} \int_{\mathbb{R}^3} \bar{u} \nabla u \, dx
\]
Moreover, one can also check that the only homogeneous $L^2$ that is left invariant under (1.7) is $\dot{u}$; we refer the reader to [4, 5, 8, 18, 39, 45] for defocusing case in the energy-subcritical sign between the kinetic energy and potential energy.

It is known that the defocusing case is different from the focusing one due to the opposite sign between the kinetic energy and potential energy. The problem is called energy-subcritical problem. The problem is known as energy-critical when $s \leq c$; to [12, 13, 14, 15, 20, 25, 28] for the focusing case. It is known that the defocusing case is different from the focusing one due to the opposite sign between the kinetic energy and potential energy.

In this paper, we mainly consider the influence of the long range potential $K|x|^{-1}$ on the existence theory and scattering theory for nonlinear Schrödinger equation. We will find some influences, e.g. global existence, are same as the result of (1.6); but, in particular $K > 0$, some results are quite different. For example, the solution is global existence no matter what sign of $K$, but it scatters when $K < 0$ but does not scatter when $K > 0$ even in the defocusing case.

As mentioned above the focusing case is different from the defocusing case. In the focusing case ($\lambda = -1$), we will also use the energy without potential

$$
E_0(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t,x)|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u(t,x)|^{p+1} \, dx,
$$

to give the threshold for global/blowup dichotomy. As the same argument as in [27, 33] considering NLS with an inverse square potential, in the case $K < 0$, we will consider the initial data below the threshold of the ground state $Q$ to the classical elliptic equation

$$
-\Delta Q + Q = Q^p, \quad 1 < p < 5
$$

due to the sharp constant in the Gagliardo-Nirenberg inequality

$$
\| f \|_{L^{p+1}} \leq C_K \| f \|_{L^2}^{\frac{p}{p-1}} \left( \sqrt{\nabla_x f} \right)_{L^2}^{\frac{3(p-1)}{4}} = C_K \| f \|_{L^2}^{\frac{p}{p-1}} \left( \| f \|_{H^1}^2 - K \int |f|^2 \, dx \right)^{\frac{3(p-1)}{4}}.
$$

Let $C_0$ be the sharp constant of the classical Gagliardo-Nirenberg inequality

$$
\| f \|_{L^{p+1}} \leq C_0 \| f \|_{L^2}^{\frac{p}{p-1}} \| f \|_{H^1}^{\frac{3(p-1)}{4}}.
$$

Then, we claim that $C_K = C_0$, it is well-known that equality in (1.10) with $K = 0$ is attained by $Q$, but we will see that equality in (1.10) with $K < 0$ is never attained. Indeed, by the sharp Gagliardo-Nirenberg inequality for (1.10), we find

$$
\lim_{n \to \infty} \frac{\| Q \|_{L^{p+1}}}{\| Q \|_{L^2}^{\frac{5}{2}}} \left( \| Q \|_{H^1}^2 - K \int |Q|^2 \, dx \right)^{\frac{3(p-1)}{4}} = \frac{\| Q \|_{L^{p+1}}}{\| Q \|_{L^2}^{\frac{5}{2}}} \| Q \|_{H^1}^{\frac{3(p-1)}{4}} = C_0.
$$
Thus, $C_0 \leq C_K$. However, for any $f \in H^1 \setminus \{0\}$ and $K < 0$, the standard Gagliardo-Nirenberg inequality implies

$$\|f\|_{L^{p+1}}^{p+1} \leq C_0 \|f\|_{L^2}^{\frac{2p}{p-1}} \|f\|_{L^2}^{\frac{2(p-1)}{p-1}} < C_0 \|f\|_{L^2}^{\frac{2}{p-1}} \|\sqrt{\Delta} f\|_{L^{p-1}}^{\frac{2}{p-1}}.$$ 

Thus $C_K = C_0$, and the last estimate also shows that equality is never attained.

In the energy-critical case ($s_c = 1$), we consider the ground state $W$ to be the elliptic equation

$$-\Delta W = W^5$$

due to the sharp constant in Sobolev embedding. We refer to [1] [16] [30] about the existence and uniqueness of the ground state.

Now, we state our main results. First, we consider the global well-posedness for the problem (1.1) under some restrictions. In the energy-subcritical case (i.e. $p - 1 < 4$), the global well-posedness will follow from local well-posedness theory and uniform kinetic energy control (1.11) gives immediately the global well-posedness. We refer to Theorem 1.1, we consider the ground state $W$ to be the elliptic equation

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**Theorem 1.1** (Global well-posedness). Let $K \in \mathbb{R}$ and $u_0 \in H^1(\mathbb{R}^3)$. Suppose that $0 < p - 1 \leq 4$ in the defocusing case $\lambda = 1$. While for the focusing case $\lambda = -1$, we assume that $0 < p - 1 < \frac{4}{3}$ (mass-subcritical) or

- If $p - 1 = \frac{4}{3}$ (mass-critical), assume $M(u_0) < M(Q)$.
- If $\frac{4}{3} < p - 1 < 4$ and $K < 0$, assume $M(u_0)^{1-s_c} E(u_0)^{s_c} < M(Q)^{1-s_c} E_0(Q)^{s_c}, \|u_0\|_{L^2}^{1-s_c} \|u_0\|_{H^1}^{s_c} < \|Q\|_{L^2}^{1-s_c} \|Q\|_{H^1}^{s_c}.$

**(1.12)**

- If $p - 1 = 4$ (energy-critical) and $K < 0$, assume that $u_0$ is radial and

$$E(u_0) < E_0(W), \|u_0\|_{H^1} < \|W\|_{H^1}.$$ **(1.13)**

Then, there exists a unique global solution $u(t, x)$ to (1.1) such that

$$\|u\|_{L^4(I, H^{1-s_c})} \leq C(\|u_0\|_{H^1}, |I|),$$ **(1.14)**

for any $I \subset \mathbb{R}$ compact and $(q, r) \in A_0$ admissible defined below.

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1. For $K < 0$ and $\lambda = -1$, we remark that under the assumption $M(u_0)^{1-s_c} E(u_0)^{s_c} < M(Q)^{1-s_c} E_0(Q)^{s_c}$, the condition $\|u_0\|_{L^2}^{1-s_c} \|u_0\|_{H^1}^{s_c} < \|Q\|_{L^2}^{1-s_c} \|Q\|_{H^1}^{s_c}$ is equivalent to $\|u_0\|_{L^2}^{1-s_c} \|u_0\|_{H^1}^{s_c} - K \|x\|_{L^2}^{-\frac{s_c}{2}} \|u_0\|_{L^2}^{2} < \|Q\|_{L^2}^{1-s_c} \|Q\|_{H^1}^{s_c}$, see Remark 1.2.

2. Here the restriction $K < 0$ induces us to utilize the result of Kenig-Merle [25] in which one needs a radial initial data.
Remark 1.2. The global existence is almost completed in the defocusing case regardless of whether in the repulsive or attractive case. The focusing case is more complicated and the following blow up result below is a supplement of this global existence.

Next, for the global solution $u$ to equation (1.1), we want to study the long-time behavior of the solution, such as scattering theory. We say that a global solution $u$ to (1.1) scatters, if there exist $u_{\pm} \in H^{1}_{x}(\mathbb{R}^{3})$ such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{-i t \Delta} u_{\pm}\|_{H^{1}_{x}(\mathbb{R}^{3})} = 0.$$ 

From the argument as in the proof of well-posedness theory, we know that one can regard the long-range potential term $K |x| u$ as the nonlinear perturbation term(it looks like the cubic nonlinear term $|u|^{2} u$ from scaling analysis). However, by Reed-Simon [38], we know that the limits

$$s - \lim_{t \to \pm \infty} e^{i t \Delta} e^{i t \Delta} \in L^{2}(\mathbb{R}^{3})$$

do not exist. Therefore, we can not regard the potential term $K |x| u$ as the nonlinear perturbation in the scattering theory. We refer the reader to several different constructions of wave operators in the long-range case, such as momentum approach [22], Isozaki-Kitada method [23] and position approach [10, 46].

On the other hand, the standard arguments show that the scattering is equivalent to the global Strichartz-norm boundedness ($\|u(t)\|_{L^{q}_{t}(\mathbb{R}; L^{r}_{x}(\mathbb{R}^{3}))} < +\infty$) provided that we have the global-in time Strichartz estimate. However, in the attractive case, i.e. $K > 0$, the global-in-time Strichartz estimate does not hold, see Subsection 2.2 below. Thus, we don’t know whether the solution $u$ to (1.1) with $K > 0$ scatters or not even for the small initial data. While for the repulsive case, i.e $K < 0$, the global-in-time Strichartz estimates were recently established by Mizutani [35]. Then, combining with Sobolev norm equivalence (1.17) below, one can easily obtain the scattering result for the small initial data. For the general initial data, we will get the scattering result in the defocusing energy-subcritical case ($\lambda = 1, p < 5$) by establishing the interaction Morawetz estimate, which gives a global Strichartz-norm boundedness.

In the case $K > 0$, we know from [3, Lemma 6] that there is a positive solution $f(x) \in H^{2}$ of the elliptic equation

$$- \Delta f - \frac{K}{|x|} f + f + f^{p} = 0. \quad (1.15)$$

This implies that there is a soliton $u(t, x) := e^{i t \Delta} f(x)$ solves (1.1) with $\lambda = 1$. We remark that such soliton is global but not scatters. Equation (1.15) arises in the Thomas-Fermi-von Weizsacker (TFW) theory of atoms and molecules [2, 32] without electronic repulsion. There, $K |x|^{-1}$ is the electric potential due to a fixed nucleus of atomic number $K$ located at the origin, $f(x)^{2}$ stands for the electronic density and $\int f(x)^{2} dx$ is the total number of electrons.

While for the case $K \leq 0$, we will derive the quadratic Morawetz identity for (1.1) and then establish the following interaction Morawetz estimate for $\lambda = 1$

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{3}} |u(t, x)|^{4} dx dt \leq C M(u_{0}) \sup_{t \in \mathbb{R}} \|u(t, \cdot)\|_{H^{1/2}_{x}}^{2}, \quad (1.16)$$
which provides us a decay of the solution $u$ to (1.1). Combining this with Strichartz estimate and Leibniz rule obtained by the following Sobolev norm equivalence
\[
\|\sqrt{1 + \mathcal{L}_K f}\|_{L^p(\mathbb{R}^3)} \simeq \|\sqrt{1 - \Delta f}\|_{L^p(\mathbb{R}^3)}, \quad 1 < p < 3, \tag{1.17}
\]
we establish the scattering theory as follows.

**Theorem 1.3** (Scattering theory). Let $K \leq 0$, $\frac{4}{3} < p - 1 < 4$, $\lambda = 1$ and $u_0 \in H^1(\mathbb{R}^3)$. Then, there exists a global solution $u$ to (1.1), and the solution $u$ scatters in the sense that there exists $u_{\pm} \in H^1(\mathbb{R}^3)$ such that
\[
\lim_{t \to \pm \infty} \|u(t, \cdot) - e^{-it\mathcal{L}_K} u_{\pm}\|_{H^1(\mathbb{R}^3)} = 0. \tag{1.18}
\]

In the focusing case, i.e $\lambda = -1$, by the classical Virial argument, one can obtain the blow-up result for the negative energy.

**Theorem 1.4** (Blow-up result). Let $K \in \mathbb{R}$, $\frac{4}{3} < p - 1 \leq 4$, $\lambda = -1$.

(i) Let $u_0 \in \Sigma := \{u_0 \in H^1, x u_0 \in L^2\}$. Then, the solution $u$ to (1.1) blows up in both time direction, in one of the three cases:

1. $C(E(u_0), M(u_0)) < 0$;
2. $C(E(u_0), M(u_0)) = 0$, $y'(0) < 0$;
3. $C(E(u_0), M(u_0)) > 0$, $y'(0)^2 \geq 24(p - 1)C(E(u_0), M(u_0))\|x\|_{L^2(\mathbb{R}^3)}^2$;

where
\[
y'(0) = 4\text{Im} \int_{\mathbb{R}^3} x \cdot \nabla u_0 \bar{u}_0 \, dx,
\]
and
\[
C(E(u_0), M(u_0)) := \begin{cases} E(u_0) & \text{if } K \leq 0 \smallskip \\ E(u_0) + \frac{3K^2}{2(3p - 7(p - 1))} M(u_0) & \text{if } K > 0 \end{cases}. \tag{1.19}
\]

(ii) Let $u_0 \in H^1(\mathbb{R}^3)$ be radial, and assume that $C(E(u_0), M(u_0)) < 0$. Then, the solution $u$ to (1.1) blows up in both time direction.

The paper is organized as follows. In Section 2, as a preliminaries, we give some notation, recall the Strichartz estimate and prove the Sobolev space equivalence. Section 3 is devoted to proving global well-posedness, i.e Theorem 1.1. We show the interaction Morawetz-type estimates in Section 4, and we utilize such Morawetz-type estimates and the equivalence of Sobolev norm to prove Theorem 1.3. Finally, we use the Virial argument to obtain the blow-up result (Theorem 1.4) in Section 5.

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2. Preliminaries

In this section, we first introduce some notation, and then recall the Strichartz estimates. We conclude this section by showing the Sobolev space equivalence between the operator $\mathcal{L}_K$ and Laplacian operator $-\Delta$. 

2.1. Notations. First, we give some notations which will be used throughout this paper. To simplify the expression of our inequalities, we introduce some symbols \( \preceq, \simeq, \ll \). If \( X, Y \) are nonnegative quantities, we use \( X \preceq Y \) or \( X = O(Y) \) to denote the estimate \( X \leq CY \) for some \( C \), and \( X \sim Y \) to denote the estimate \( X \preceq Y \preceq X \). We denote \( a \pm \epsilon \) to be any quantity of the form \( a \pm \epsilon \) for any \( \epsilon > 0 \).

For a spacetime slab \( I \times \mathbb{R}^3 \), we write \( L^q_1 L^r_x(I \times \mathbb{R}^3) \) for the Banach space of functions \( u : I \times \mathbb{R}^3 \to \mathbb{C} \) equipped with the norm

\[
\|u\|_{L^q_1 L^r_x(I \times \mathbb{R}^3)} := \left( \int_I \|u(t, \cdot)\|_{L^r_x(\mathbb{R}^3)}^{1/q} \right),
\]

with the usual adjustments when \( q \) or \( r \) is infinity. When \( q = r \), we abbreviate \( L^q_1 L^q_x = L^q_{1,x} \). We will also often abbreviate \( \|f\|_{L^q_x} \) to \( \|f\|_{L^q_x} \). For \( 1 \leq r \leq \infty \), we use \( r' \) to denote the dual exponent to \( r \), i.e. the solution to \( \frac{1}{r} + \frac{1}{r'} = 1 \).

The Fourier transform on \( \mathbb{R}^3 \) is defined by

\[
\hat{f}(\xi) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx,
\]
giving rise to the fractional differentiation operators \( |\nabla|^s \) and \( \langle \nabla \rangle^s \), defined by

\[
|\nabla|^s f(\xi) := |\xi|^s f(\xi), \quad \langle \nabla \rangle^s f(\xi) := \langle \xi \rangle^s f(\xi),
\]

where \( \langle \xi \rangle := 1 + |\xi| \). This helps us to define the homogeneous and inhomogeneous Sobolev norms

\[
\|u\|_{\tilde{W}^{s,p}_x(\mathbb{R}^3)} = \||\nabla|^s u\|_{L^p_x}, \quad \|u\|_{W^{s,p}_x(\mathbb{R}^3)} = \|\langle \nabla \rangle^s u\|_{L^p_x}.
\]

Especially, for \( p = 2 \), we denote \( \tilde{W}^{s,p}_x(\mathbb{R}^3) = \tilde{H}^s(\mathbb{R}^3) \) and \( W^{s,p}_x(\mathbb{R}^3) = H^s(\mathbb{R}^3) \).

Next, we recall the well-known Lorentz space and some properties of this space for our purpose. Given a measurable function \( f : \mathbb{R}^3 \to \mathbb{C} \), define the distribution function of \( f \) as

\[
f_*(t) = \mu(\{ x \in \mathbb{R}^3 : |f(x)| > t \}), \quad t > 0
\]

and its rearrangement function as

\[
f^*(s) = \inf\{ t : f_*(t) \leq s \}.
\]

For \( 1 \leq p < \infty \) and \( 1 \leq r \leq \infty \), define the Lorentz quasi-norm

\[
\|f\|_{L^{p,r}(\mathbb{R}^3)} = \begin{cases} 
\left( \int_0^\infty (s^{\frac{1}{r}} f^*(s))^r \frac{ds}{s} \right)^{1/r}, & 1 \leq r < \infty; \\
\sup_{s>0} s^{\frac{1}{r}} f^*(s), & r = \infty.
\end{cases}
\]

The Lorentz space \( L^{p,r}(\mathbb{R}^3) \) denotes the space of complex-valued measurable functions \( f \) on \( \mathbb{R}^3 \) such that its quasi-norm \( \|f\|_{L^{p,r}(\mathbb{R}^3)} \) is finite. From this characterization, \( L^{p,\infty}(\mathbb{R}^3) \) is the usual weak \( L^p \) space, \( L^{p,p}(\mathbb{R}^3) = L^p(\mathbb{R}^3) \) and \( L^{p,r}(\mathbb{R}^3) \subset L^{p,\tilde{r}}(\mathbb{R}^3) \) with \( r < \tilde{r} \).

We refer to O’Neil [36] for the following Hölder inequality in Lorentz space.

**Proposition 2.1** (Hölder’s inequality in Lorentz space). Let \( 1 \leq p, p_0, p_1 < \infty \) and \( 1 \leq r, r_0, r_1 \leq \infty \), then

\[
\|fg\|_{L^{p,r}} \leq C\|f\|_{L^{p_0,r_0}}\|g\|_{L^{p_1,r_1}}, \quad \frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}, \quad \frac{1}{r} = \frac{1}{r_0} + \frac{1}{r_1}.
\]
2. Strichartz estimate. It is well known that the Strichartz estimate is very useful in the study of the nonlinear dispersive equations. To state the result, we define
\[ \Lambda_0 = \{(q, r) : \frac{2}{q} = 3\left(\frac{1}{r} - \frac{1}{2}\right), q, r \geq 2\}. \quad (2.2) \]

**Theorem 2.2** (Local-in-time Strichartz estimate). Let \( K \in \mathbb{R} \) and \( \mathcal{L}_K \) be as above. For \((q, r) \in \Lambda_0\), there holds
\[ \|e^{it\mathcal{L}_K} f\|_{L_t^q(I, L_x^r)} \leq C(|I|)\|f\|_{L_x^2}. \quad (2.3) \]

**Proof.** The proof is based on a perturbation argument. Let \( u(t, x) = e^{it\mathcal{L}_K} f \), then \( u \) satisfies that
\[ i\partial_t u + \Delta u = -\frac{K}{|x|} u, \quad u(0, x) = f(x) \]
We regard the Coulomb potential as an inhomogeneous term, hence we have by Duhamel’s formula
\[ e^{it\mathcal{L}_K} f = u(t) = e^{it\Delta} f + iK \int_0^t e^{i(t-s)\Delta} \frac{u}{|x|} \, dx. \]
For our purpose, we recall the inhomogeneous Strichartz estimate without potential on Lorentz space.

**Lemma 2.3** (Strichartz estimate for \( e^{it\Delta} \), [21, 37]). For \((q, r), (q_1, r_1) \in \Lambda_0\), we have
\[ \|e^{it\Delta} f\|_{L_t^q(I, L_x^{r_1})} \leq C\|f\|_{L_x^{r_2}}; \]
\[ \left\| \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L_t^q(I, L_x^{r_1})} \leq C\|F(t, x)\|_{L_t^{q_1}(I, L_x^{r_1})}, \quad (2.4) \]
where \( \frac{1}{q} + \frac{1}{r} = 1 \).

Using the above lemma, we show that
\[ \|e^{it\mathcal{L}_K} f\|_{L_t^q(I, L_x^r)} \leq C\|f\|_{L_x^2} + |K| \left\| \int_0^t e^{i(t-s)\Delta} \frac{u}{|x|} \, dx \right\|_{L_t^q(I, L_x^r)}. \]
We use the above inhomogeneous Strichartz estimate to obtain
\[ \left\| \int_0^t e^{i(t-s)\Delta} \frac{u}{|x|} \, dx \right\|_{L_t^q(I, L_x^r)} \leq \|K\left\| \int_0^t e^{i(t-s)\Delta} \frac{u}{|x|} \, dx \right\|_{L_t^{q_1}(I, L_x^{r_1})} \]
\[ \leq C\left\| \frac{u}{|x|} \right\|_{L_t^{q_1}(I, L_x^{r_1})} \leq C|I|^{\frac{1}{q}} \|u\|_{L_t^{q_1}} \|u\|_{L_x^{r_1}} \]
\[ \leq C(|I|)\|f\|_{L_x^2} \]
where we use the mass conservation in the last inequality. Therefore we prove (2.3). \qed

It is nature to ask whether the global-in-time Strichartz estimate holds or not. The answer is that the global-in-time Strichartz estimate does not hold in the attractive case \( K > 0 \) but holds in the repulsive case \( K \leq 0 \).

To see the attractive case, a simple computation shows
\[ \Delta(e^{-c|x|}) = c^2 e^{-c|x|} - \frac{2}{|x|} c e^{-c|x|}. \]
Let $c_K = K/2$, this implies
\[ \mathcal{L}_K(e^{-c_K|x|}) = \left( -\Delta - \frac{K}{|x|} \right) (e^{-c_K|x|}) = -\left( \frac{K}{2} \right)^2 e^{-c_K|x|}. \]

Then, the function $u(t, x) = e^{it \frac{K}{2}} (e^{-c_K|x|})$ with $c_K = K/2$ solves the linear equation $i\partial_t u - \mathcal{L}_K u = 0$ and $u_0(x) = e^{-c_K|x|} \in L^2(\mathbb{R}^3)$ when $K > 0$. However,
\[ \|u(t, x)\|_{L_t^r(\mathbb{R}, L_x^s(\mathbb{R}^3))} = +\infty. \] (2.5)

In the repulsive Coulomb potential case, Mizutani [35] recently proved the global-in-time Strichartz estimate, where the proof employs several techniques from linear scattering theory such as the long time parametrix construction of Isozaki-Kitada type [23], propagation estimates and local decay estimates.

**Theorem 2.4 (Global-in-time Strichartz estimate, [35]).** For $(q, r), (q_1, r_1) \in \Lambda_0$ and $K < 0$, there holds
\[ \|e^{it \mathcal{L}_K} f\|_{L_t^q(\mathbb{R}, L_x^r(\mathbb{R}^3))} \leq C \|f\|_{L_t^1(\mathbb{R}, L_x^1(\mathbb{R}^3))}, \] (2.6)
and
\[ \left\| \int_0^t e^{i(t-s) \mathcal{L}_K} F(s) \, ds \right\|_{L_t^q(\mathbb{R}, L_x^r(\mathbb{R}^3))} \leq C \|F\|_{L_t^{q_1}(\mathbb{R}, L_x^{r_1}(\mathbb{R}^3))}. \] (2.7)

### 2.3. Fractional product rule.

As mentioned in the introduction, we need the following fractional chain rule in the proof of scattering theory when $K < 0$. The $L^p$-product rule for fractional derivatives in Euclidean spaces
\[ \|(-\Delta)^{\frac{s}{2}}(fg)\|_{L^p(\mathbb{R}^3)} \lesssim \|(-\Delta)^{\frac{s}{2}} f\|_{L^{p_1}(\mathbb{R}^3)} \|g\|_{L^{p_2}(\mathbb{R}^3)} \]
\[ + \|f\|_{L^{q_1}(\mathbb{R}^3)} \|(-\Delta)^{\frac{s}{2}} g\|_{L^{q_2}(\mathbb{R}^3)}, \]
was first proved by Christ and Weinstein [9]. Here $1 < p, p_1, p_2, q_1, q_2 < \infty$, $s \geq 0$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$. Similarly, we have the following for the operator $\mathcal{L}_K$ with $K < 0$.

**Lemma 2.5 (Fractional product rule).** Fix $K < 0$ and let $\mathcal{L}_K$ be as above. Then for all $f, g \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ we have
\[ \|\sqrt{1 + \mathcal{L}_K}(fg)\|_{L^p(\mathbb{R}^3)} \lesssim \|\sqrt{1 + \mathcal{L}_K} f\|_{L^{p_1}(\mathbb{R}^3)} \|g\|_{L^{p_2}(\mathbb{R}^3)} + \|f\|_{L^{q_1}(\mathbb{R}^3)} \|\sqrt{1 + \mathcal{L}_K} g\|_{L^{q_2}(\mathbb{R}^3)}, \]
for any exponents satisfying $1 < p, p_1, q_2 < 3$, $1 < p_2, q_1 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

This is a consequence of the equivalence of Sobolev norm
\[ \|f\|_{L^p(\mathbb{R}^3)} + \|\nabla f\|_{L^p(\mathbb{R}^3)} \sim \|\sqrt{1 + \mathcal{L}_K} f\|_{L^p(\mathbb{R}^3)}, \quad 1 < p < 3, \]
which will be proved in the next subsection.

### 2.4. Sobolev space equivalence.

In this subsection, we study the relationship between Sobolev space adapted with Laplacian operator perturbed by Coulomb potential and classical Laplacian operator, that is, for suitable $s$ and $p$ such that
\[ \|\mathcal{L}_K f\|_{L^p(\mathbb{R}^3)} \sim \|\nabla f\|_{L^p(\mathbb{R}^3)} \]
(2.8)
where $\langle a \rangle = (1 + |a|^2)^{1/2}$. To this end, we recall the heat kernel estimate
Lemma 2.6 (Heat kernel). Let $K < 0$ and let $\mathcal{L}_K$ be as above. Then there exist constants $C, c > 0$ such that

$$0 \leq e^{-t\mathcal{L}_K(x, y)} \leq Ct^{-3/2}e^{-\frac{|x-y|^2}{4t}}. \quad (2.9)$$

**Proof.** Since $K < 0$, then $\mathcal{L}_K = -\Delta + V(x)$ with a positive positive $V = -K|x|^{-1}$. It is easy to verify that $V \in L^2_{\text{loc}}(\mathbb{R}^3)$. It is well known that [29], e.g. see [29]. Indeed, one can use the estimate of the fundamental solution of the elliptic operator $\mathcal{L}_K + \lambda$ with non-negative parameter $\lambda$ in Shen [42] to obtain the heat kernel estimate.

Lemma 2.7 (Sobolev norm equivalence). Let $K < 0$, $1 < p < 3$ and $0 < s \leq 2$. There holds

$$\left\|(1 + \mathcal{L}_K)^{\frac{s}{p}} f \right\|_{L^p(\mathbb{R}^3)} \sim \left\|(1 - \Delta)^{\frac{s}{p}} f \right\|_{L^p(\mathbb{R}^3)}. \quad (2.10)$$

**Proof.** The proof is classical and follows from heat kernel estimate and Stein complex interpolation. We refer to Y. Hong[21] or the authors [47], but we give a complete proof for convenience.

First, we consider $s = 2$. Using the Hardy inequality [47] Lemma 2.6 with $p < 3$, we obtain

$$\left\|\frac{f}{|x|} \right\|_{L^p} \lesssim \|f \mathcal{L}_K f\|_{L^p}.$$ 

Hence,

$$\left\|\frac{f}{|x|} \right\|_{L^p} \lesssim \|f \mathcal{L}_K f\|_{L^p}.$$ 

This implies (2.10) with $s = 2$.

Next, since the heat kernel operator $e^{-t(1+\mathcal{L}_K)}$ obeys the Gaussian heat kernel estimate, we have by Sikora-Wright [41]

$$\left\| (1 - \Delta)^{ib} f \right\|_{L^p} + \left\| (1 + \mathcal{L}_K)^{ib} f \right\|_{L^p} \lesssim (b)^{\frac{3}{2}}, \quad \forall \ b \in \mathbb{R}, \ \forall \ 1 < p < +\infty.$$ 

Let $z = a + ib$, define

$$T_z = (1 + \mathcal{L}_K)^{z} (1 - \Delta)^{-z}, \quad G_z = (1 - \Delta)^{z} (1 + \mathcal{L}_K)^{-z}.$$ 

Then we have that for $1 < p < 3$

$$\|T_{1+ib}\|_{L^p \to L^p} \leq (b)^{3} \|\mathcal{L}_K (1 - \Delta)^{-1}\|_{L^p \to L^p} \leq C(b)^3.$$ 

This shows that

$$\left\| (1 - \Delta)^{\frac{z}{2}} f \right\|_{L^p} \lesssim (\text{Im} z)^{\frac{3}{2}} \left\| (1 + \mathcal{L}_K)^{\frac{z}{2}} f \right\|_{L^p}$$

$$\left\| (1 + \mathcal{L}_K)^{\frac{z}{2}} f \right\|_{L^p} \lesssim (\text{Im} z)^{\frac{3}{2}} \left\| (1 - \Delta)^{\frac{z}{2}} f \right\|_{L^p}.$$
holds for $1 < p < +\infty$ when $\Re z = 0$ and for $1 < p < 3$ when $\Re z = 1$. Therefore, (2.10) follows by the Stein complex interpolation.

□

3. Global well-posedness

In this section, we prove the well-posedness for equation (1.1) including local and global well-posedness. In this part, we only use the classical Strichartz estimate for the Schrödinger equation without potential $i\partial_t u - \Delta u = 0$ on Lorentz space.

In the energy-subcritical case (i.e. $p - 1 < 4$), the global well-posedness will follow from local well-posedness theory and uniform kinetic energy control

3.1. Local well-posedness for energy-subcritical: $s_c < 1$.

**Theorem 3.1** (Local well-posedness, energy-subcritical). Let $K \in \mathbb{R}$, $0 < p - 1 < 4$ and $u_0 \in H^1(\mathbb{R}^3)$. Then there exists $T = T(\|u_0\|_{H^1}) > 0$ such that the equation (1.1) with initial data $u_0$ has a unique solution $u$ with

$$u \in C(I; H^1(\mathbb{R}^3)) \cap L^q_t(I, W^{1,r_0}(\mathbb{R}^3)), \quad I = [0, T],$$

where $(q_0, r_0) = \left(\frac{4(p+1)}{p(p-1)}, p + 1\right) \in \Lambda_0$.

**Proof.** Define the map

$$\Phi(u(t)) := e^{it\Delta}u_0 + i \int_0^t e^{i(t-s)\Delta} \left(\frac{K}{|x|}u - \lambda |u|^{p-1}u\right)(s) \, ds,$$

with $I = [0, T]$

$$B(I) = \{ u \in Y(I) = C(I, H^1(\mathbb{R}^3)) \cap L^q_t(I, W^{1,r_0}), \|u\|_{Y(I)} \leq 2C\|u_0\|_{H^1}\},$$

and the metric $d(u, v) = \|u - v\|_{L^q_t(I, L^r(I, L^s(\mathbb{R}^3)))}$.

For $u \in B(I)$, we have by Strichartz estimate [2,4]

$$\|\Phi(u)\|_{Y(I)} \leq C\|u_0\|_{H^1} + C\|\langle \nabla \rangle \left(\frac{K}{|x|}u\right)\|_{L^q_t L^{\frac{q}{2}}(\mathbb{R}^3)} + C\|\langle \nabla \rangle (|u|^{p-1}u)\|_{L^q_t L^{\frac{q}{p-1}}(\mathbb{R}^3)}$$

$$\leq C\|u_0\|_{H^1} + C_1T^{\frac{1}{2}}\|u\|_{L^r_t H^1} + C_1T^{1-\frac{2}{p}}\|u\|_{L^r_t (I, L^{r_0})}^{p-1}\|u\|_{L^r_t (I, L^{r_0})}$$

$$\leq C\|u_0\|_{H^1} + C_1T^{\frac{1}{2}}\|u\|_{L^r_t H^1} + C_1T^{1-\frac{2}{p}}\|u\|_{Y(I)}^{r_0}$$

$$\leq C\|u_0\|_{H^1} + 2CC_1T^{\frac{1}{2}}\|u_0\|_{H^1} + 2CC_1T^{\frac{5-s}{2(r-1)}}\|u_0\|_{H^1} (2C\|u_0\|_{H^1})^{p-1}$$

$$\leq 2C\|u_0\|_{H^1}$$

by taking $T$ small such that

$$2C_1T^{\frac{1}{2}} + 2C_1T^{\frac{5-s}{2(r-1)}} (2C\|u_0\|_{H^1})^{p-1} \leq 1.$$
On the other hand, for \( u, v \in B(I) \), we get by Strichartz estimate
\[
d(\Phi(u), \Phi(v)) = \left\| \int_0^t e^{i(t-s)\Delta} \frac{K}{|x|} (u-v) - (|u|^{p-1}u - |v|^{p-1}v) \right\|_{L^2_x(I, L^p_y)} \leq C \left\| \frac{u-v}{|x|} \right\|_{L^2_x L^{-\frac{1}{2}}_y} + C \left\| |u|^{p-1}u - |v|^{p-1}v \right\|_{L^2_x(I, L^p_y)} \leq C T^{\frac{1}{2}} \left\| u - v \right\|_{L^\infty_t L^2_x} + C T^{\frac{5-p}{2(p-1)}} \left\| u - v \right\|_{L^p_x(I, L^\infty_y)} \left\| (u, v) \right\|_{H^1_x(I, L^p_y)} \leq \frac{1}{2} d(u, v)
\]
by taking \( T \) small such that
\[
C T^{\frac{1}{2}} + 4 C T^{\frac{5-p}{2(p-1)}} (2C \left\| u_0 \right\|_{H^1})^{p-1} \leq \frac{1}{2}.
\]

A standard fixed point argument gives a unique local solution \( u : [0, T] \times \mathbb{R}^3 \to \mathbb{C} \) to \( \text{(1.1)} \).

\[\square\]

3.2. Global well-posedness for energy-subcritical: \( s_c < 1 \). By the local well-posedness theory and mass conservation, the global well-posedness will follow from the uniform kinetic energy control
\[
\sup_{t \in I} \left\| u(t) \right\|_{H^1_x} \leq C(E(u_0), M(u_0)). \tag{3.4}
\]

We argue the following several cases.

**Case 1: the defocusing case, i.e. \( \lambda = 1 \).** In the defocusing case, we have the uniform bound
\[
\left\| u(t, \cdot) \right\|_{H^1_x} \leq C(M(u_0), E(u_0)). \tag{3.5}
\]
In fact, we have by Hardy’s inequality and Young’s inequality
\[
\int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \, dx \leq C \left\| u \right\|_{H^{\frac{3}{2}}}^2 \leq C \left\| u \right\|_{L^2_x} \left\| u \right\|_{H^1_x} \leq \frac{1}{2|K|} \left\| u \right\|_{H^1_x}^2 + 2 C^2 |K| \left\| u \right\|_{L^2_x}^2, \tag{3.6}
\]
which implies
\[
E(u_0) = E(u) \geq \frac{1}{4} \int_{\mathbb{R}^3} |
abla u(t)|^2 \, dx - C^2 |K| M(u_0)
\]
and hence
\[
\left\| u(t) \right\|_{H^1_x}^2 \leq C_1 M(u_0) + 4 E(u_0).
\]
Therefore we can extend the local existence to be a global one.

**Case 2: \( \lambda = -1, 0 < p - 1 < \frac{4}{3} \).** In this case, we have by Gagliardo-Nirenberg inequality and Young’s inequality
\[
\left\| u \right\|_{L^{p+1}_x} \leq C \left\| u \right\|_{L^2_x}^{\frac{5-p}{2}} \left\| u \right\|_{H^{\frac{5}{2}}_x}^{\frac{3(p-1)}{2}} \leq C_1 M(u_0)^{\frac{5-p}{2}} + \frac{p+1}{8} \left\| u \right\|_{H^1_x}^2.
\]
This together with (3.6) implies
\[
E(u_0) = E(u) \geq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla u(t)|^2 \, dx - C^2 |K| M(u_0) - \frac{C_1}{p+1} M(u_0)^{\frac{5-p}{2}}
\]
and so
\[
\left\| u(t) \right\|_{H^1_x}^2 \leq C_1 M(u_0) + 8 E(u_0).
\]
Thus we can obtain the global existence by extending the local solution.
Case 3: \( \lambda = -1, \ p = \frac{7}{3}, \ M(u_0) < M(Q) \). For the mass-critical equation:
\[
\dot{i} \partial_t u + \Delta u + \frac{K}{|x|} u + |u|^4 u = 0.
\]
From (3.6), we obtain
\[
\frac{3}{10} \int \frac{|u|^4}{|x|} \ dx \leq \frac{1}{2} \left| \frac{\|u\|^2_{L^2} + C^2|K|}{\varepsilon} \right| M(u_0).
\]
One the other hand, we have by the sharp Gagliardo-Nirenberg inequality
\[
\frac{3}{10} \int \frac{|u|^4}{|x|} \ dx \leq \frac{1}{2} \left( \frac{\|u\|^4_{L^2}}{\|Q\|_{L^2}^2} \right)^\frac{1}{4} \|\nabla u\|^2_{L^2}.
\]
Hence,
\[
E(u_0) \geq \frac{1}{2} \|\nabla u(t, \cdot)\|^2_{L^2} \left( 1 - \varepsilon - \left( \frac{\|u\|^2_{L^2}}{\|Q\|_{L^2}^2} \right)^\frac{1}{4} \right) \frac{C^2|K|}{\varepsilon} M(u_0)
\]
\[
\geq \frac{1}{4} \|\nabla u(t, \cdot)\|^2_{L^2} \left( 1 - \left( \frac{\|u\|^2_{L^2}}{\|Q\|_{L^2}^2} \right)^\frac{1}{4} \right) \frac{C^2|K|}{\varepsilon} M(u_0).
\]
This shows
\[
\|u(t, \cdot)\|_{L^\infty H^s_x} \leq C(M(u_0), E(u_0)).
\]

Case 4: \( \lambda = -1, \ K < 0, \ \frac{7}{3} < p - 1 < 4 \). In this case, we assume that
\[
M(u_0)^{1-s_c} E(u_0)^{s_c} < M(Q)^{1-s_c} E_0(Q)^{s_c}, \ \|u_0\|^{1-s_c}_{L^2} \|u_0\|^{s_c}_{H^1} < \|Q\|^{1-s_c}_{L^2} \|Q\|^{s_c}_{H^1}.
\]
Then, there exists \( \delta > 0 \) such that
\[
M(u_0)^{1-s_c} E(u_0)^{s_c} \leq (1 - \delta) M(Q)^{1-s_c} E_0(Q)^{s_c}.
\]
By the sharp Gagliardo-Nirenberg inequality, we have
\[
\|f\|_{L^{p+1}_{x,t}} \leq C_0 \|f\|_{L^2_x}^\frac{5-p}{2} \|f\|_{H^s_t}^{\frac{3(p-1)}{2}}, \quad (3.7)
\]
with the sharp constant
\[
C_0 \|Q\|_{L^2_x}^{(1-s_c)(p-1)} \|Q\|_{H^s_t}^{(p-1)} = \frac{2(p + 1)}{3(p - 1)}, \quad (3.8)
\]
This shows for \( K < 0 \)
\[
(1 - \delta) M(Q)^{1-s_c} E_0(Q)^{s_c} \geq M(u)^{1-s_c} E(u)^{s_c}
\]
\[
\geq \|u(t)\|^{(1-s_c)}_{L^2_x} \left( \frac{1}{2} \|u(t)\|_{H^1_t}^2 - \frac{C_0}{p + 1} \|u(t)\|_{L^2_x}^{\frac{5-p}{2}} \|u(t)\|_{H^s_t}^{\frac{3(p-1)}{2}} \right)^{s_c}
\]
for any \( t \in I \). This together with
\[
E_0(Q) = \frac{3p - 7}{6(p-1)} \|Q\|_{H^s}^2 = \frac{3p - 7}{4(p+1)} \|Q\|_{L^{p+1}_{x,t}}^{p+1}
\]
implies that
\[
(1 - \delta)^\frac{1}{s_c} \geq \frac{3(p-1)}{3p-7} \left( \frac{\|u(t)\|^{1-s_c}_{L^2_x} \|u(t)\|^{s_c}_{H^1_t}}{\|Q\|^{1-s_c}_{L^2_x} \|Q\|^{s_c}_{H^1_t}} \right)^\frac{1}{s_c} - \frac{2}{3p-7} \left( \frac{\|u(t)\|^{1-s_c}_{L^2_x} \|u(t)\|^{s_c}_{H^1_t}}{\|Q\|^{1-s_c}_{L^2_x} \|Q\|^{s_c}_{H^1_t}} \right)^\frac{1}{s_c} \quad (3.9)
\]
Using a continuity argument, together with the observation that
\[
(1 - \delta)^\frac{1}{s_c} \geq \frac{3(p-1)}{3p-7} y^{\frac{1}{s_c}} - \frac{2}{3p-7} y^{\frac{1}{s_c}}(p-1) \implies |y| \geq \delta' \quad \text{for some} \quad \delta' = \delta'(\delta) > 0,
\]
we obtain
\[
\|u(t)\|_{L^2}^{1-s_c} \|u(t)\|_{H^1}^{s_c} < \|Q\|_{L^2}^{1-s_c} \|Q\|_{H^1}^{s_c}, \quad \forall t \in I. \tag{3.10}
\]
In sum, we obtain the uniform kinetic energy control in the maximal life-span. Therefore, we conclude the proof of Theorem 3.1.

**Remark 3.2.** (i) For $K < 0$ and $\lambda = -1$, we remark that under the assumption $M(u_0)^{1-s_c} E(u_0)^{s_c} \leq (1 - \delta) M(Q)^{1-s_c} E_0(Q)^{s_c}$ for some $\delta > 0$, the condition
\[
\|u_0\|_{L^2}^{1-s_c} \|u_0\|_{H^1}^{s_c} < \|Q\|_{L^2}^{1-s_c} \|Q\|_{H^1}^{s_c} \tag{3.11}
\]
is equivalent to
\[
\|u_0\|_{L^2}^{1-s_c} \left(\|u_0\|_{H^1}^2 - K \|x|^{-\frac{s}{2}} u_0\|_{L^2}^2\right)^{\frac{s_c}{2}} < \|Q\|_{L^2}^{1-s_c} \|Q\|_{H^1}^{s_c} \tag{3.12}
\]
We take $s_c = \frac{1}{2}$ for example. In this case, we have $p = 3$, and the ground state $Q$ solves
\[-\Delta Q + Q = Q^3.\]
A simple computation shows that
\[
E_0(Q_0) = \frac{1}{6} \|Q_0\|_{H^1}^2 = \frac{1}{8} \|Q_0\|_{L^4}^4 = \frac{1}{2} \|Q_0\|_{L^2}^2 \tag{3.13}
\]
and
\[
C_0 := \frac{\|Q\|_{H^1}^4}{\|Q\|_{L^2}^4 \|Q\|_{H^1}^3} = \frac{4}{3} \frac{1}{\|Q\|_{L^2}^2 \|Q\|_{H^1}}. \tag{3.14}
\]
Since $K < 0$, it is easy to get (3.11) from (3.12). Now, we assume (3.11). By the sharp Gagliardo-Nirenberg’s inequality
\[
\|u\|_{L^4}^4 \leq C_0 \|u\|_{L^2} \|u\|_{H^1}^3
\]
and using (3.14), we obtain
\[
M(u_0) E(u_0) \geq \frac{1}{2} \|u_0\|_{L^2}^2 \left(\|u_0\|_{H^1}^2 - K \|x|^{-\frac{s}{2}} u_0\|_{L^2}^2\right) \geq \frac{1}{4} \|u_0\|_{L^2}^2 \|u_0\|_{H^1}^4 \tag{3.15}
\]
This together with the assumption $M(u_0) E(u_0) \leq (1 - \delta) M(Q) E_0(Q)$ and (3.13) yields that
\[
\frac{1}{2} \|u_0\|_{L^2}^2 \left(\|u_0\|_{H^1}^2 - K \|x|^{-\frac{s}{2}} u_0\|_{L^2}^2\right) \leq M(u_0) E(u_0) + \frac{1}{3} \|Q\|_{L^2}^2 \|Q\|_{H^1}^3 \leq (1 - \delta) M(Q) E_0(Q) + \frac{1}{3} \|Q\|_{L^2}^2 \|Q\|_{H^1}^3 \leq \frac{3 - \delta}{6} \|Q\|_{L^2}^2 \|Q\|_{H^1}^3.
\]
And so
\[
\|u_0\|_{L^2}^2 \left(\|u_0\|_{H^1}^2 - K \|x|^{-\frac{s}{2}} u_0\|_{L^2}^2\right) < \|Q\|_{L^2}^2 \|Q\|_{H^1}^3.
\]
(iii) By the same argument as in (i), for $K < 0$, $\lambda = -1$ and $p = 5$, under the assumption $E(u_0) < E_0(W)$, the condition

$$
\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}
$$

is equivalent to

$$
\|u_0\|_{\dot{H}^1}^2 - K\|x|^{-\frac{4}{p}}u_0\|_{L^2}^2 < \|W\|_{\dot{H}^1}^2.
$$

3.3. Global well-posedness for energy-critical: $s_c = 1$ and $K < 0$. We will show the global well-posedness by controlling global kinetic energy and proving “good local well-posedness” as in Zhang [48]. More precisely, we will show that there exists a small constant $T = T(\|u_0\|_{\dot{H}^1})$ such that (1.1) is well-posed on $[0, T]$, which is so-called “good local well-posed”. On the other hand, since the equation in (1.1) is time translation invariant, this “good local well-posed” combining with the global kinetic energy control gives immediately the global well-posedness.

**Step 1. global kinetic energy.** For the defocusing case ($\lambda = 1$), it follows from Case 1 in Subsection 3.2 that

$$
\sup_{t \in I} \|u(t, \cdot)\|_{\dot{H}^1}^2 \leq C_1 M(u_0) + 4E(u_0).
$$

While for the focusing case ($\lambda = -1$) and $K < 0$, under the restriction

$$
E(u_0) < E_0(W), \quad \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1},
$$

we easily obtain

$$
E_0(u_0) < E_0(W), \quad \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}.
$$

Hence, we have by coercivity as in [25]

$$
\sup_{t \in I} \|u(t)\|_{\dot{H}^1} < cE_0(u) < cE(u_0) < cE_0(W).
$$

And so we derive the global kinetic energy.

**Step 2: good local well-posedness.** To obtain it, we first introduce several spaces and give estimates of the nonlinearities in terms of these spaces. For a time slab $I \subset \mathbb{R}$, we define

$$
\dot{X}^0_1 := L_{t,x}^{\frac{10}{3}} \cap L_t^{10} L_x^{\frac{30}{7}}(I \times \mathbb{R}^3), \quad \dot{X}^1_1 := \{ f: \nabla f \in \dot{X}^0_1 \}, \quad X^1_1 = \dot{X}^0_1 \cap \dot{X}^1_1.
$$

Then, we have by Hölder’s inequality and Sobolev embedding

$$
\|\nabla_i (u^k u^{4-k})\|_{L_{t,x}^4(I \times \mathbb{R}^3)} \lesssim \|u\|_{X^1_1}^{p-1}\|u\|_{X^1_1},
$$

for $i = 0, 1$, and

$$
\|\langle\nabla\rangle (\frac{|v|}{|x|})\|_{L_{t,x}^{\frac{2}{3}}(I \times \mathbb{R}^3)} \lesssim C|I|^\frac{2}{3}\|u\|_{L_{t,x}^\infty(I \times \mathbb{R}^3)}.
$$

Now, it follows from [3] for the defocusing case ($\lambda = 1$) and [25] for the focusing case ($\lambda = -1$) under the assumption (3.18) and $u_0$ radial that the Cauchy problem

$$
\begin{cases}
  i\partial_tv + \Delta v = |v|^4v, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
  v(0) = u_0,
\end{cases}
$$

is globally well-posed and the global solution $v$ satisfies the estimate

$$
\|v\|_{L^\infty(\mathbb{R}; L^1_x)} \leq C(\|u_0\|_{\dot{H}^1}), \quad \|v\|_{L^\infty(\mathbb{R}; L^r_x)} \leq C(\|u_0\|_{\dot{H}^1})\|u_0\|_{L^2}
$$

(3.23)
for all \((g, r) \in \Lambda_0\). So to recover \(u\) on the time interval \([0, T]\), where \(T\) is a small constant to be specified later, it’s sufficient to solve the difference equation of \(\omega\) with \(0\)-data initial on the time interval \([0, T]\),

\[
\begin{cases}
\dot{\omega} + \Delta \omega = -\frac{K}{|x|}(v + \omega) - \lambda|v + \omega|^q(v + \omega) + \lambda|v|^4 v \\
\omega(0) = 0.
\end{cases}
\] (3.24)

In order to solve (3.24), we subdivide \([0, T]\) into finite subintervals such that on each subinterval, the influence of \(v\) to the problem (3.24) is very small.

Let \(\epsilon\) be a small constant, from (3.23), it allows us to divide \(\mathbb{R}\) into subintervals \(I_0, \ldots, I_{J-1}\) such that on each \(I_j\),

\[
\|v\|_{X^1(I_j)} \sim \epsilon, \quad 0 \leq j \leq J - 1 \quad \text{with} \quad J \leq C(\|u_0\|_{H^1}, \epsilon).
\] (3.25)

So without loss of generality and renaming the intervals if necessary, we can write

\[
[0, T] = \bigcup_{j=0}^{J'} I_j, \quad I_j = [t_j, t_{j+1}]
\]

with \(J' \leq J\) and on each \(I_j\)

\[
\|v\|_{X^1(I_j)} \lesssim \epsilon.
\] (3.25)

Now we begin to solve the difference equation (3.24) on each \(I_j\) by inductive arguments. More precisely, we show that for each \(0 \leq j \leq J' - 1\), there exists a unique solution \(\omega\) to (3.24) on \(I_j\) such that

\[
\|\omega\|_{X^1(I_j)} + \|\omega\|_{L^\infty(I_j; H^1)} \leq (2C)^J T^{\frac{q}{2}}.
\] (3.26)

We mainly utilize the induction argument. Assume (3.24) has been solved on \(I_{j-1}\) and the solution \(\omega\) satisfies the bound (3.26) until to \(j - 1\), it is enough to derive the bound of the \(\omega\) on \(I_j\).

Define the solution map

\[
\Phi(\omega(t)) = e^{i(t-t_j)\Delta w(t_j)} + \int_{t_j}^t e^{i(t-s)\Delta} \left( \frac{K}{|x|}(v + \omega) + \lambda|v + \omega|^q(v + \omega) - \lambda|v|^4 v \right)(s) ds
\]

and a set

\[
B = \{ \omega : \|\omega\|_{L^\infty(I_j; H^1)} + \|\omega\|_{X^1(I_j)} \leq (2C)^J T^{\frac{q}{2}} \}
\]

and the norm \(\| : \|_B\) is taken as the same as the one in the capital bracket. Then it suffices to show that \(B\) is stable and the solution map \(\Phi\) is contractive under the weak topology \(X^0(I_j) \cap L^\infty(I_j, L^2_x)\). Actually, it follows from the Strichartz estimate on Lorentz space and (3.20), (3.21) that

\[
\|\Phi(\omega)\|_B \lesssim \|\omega(t_j)\|_{H^1} + \|v + w\|^q(v + w) - |v|^4 v\|_{L^\infty(I_j; W^{1, q}_x)} + \|\langle \nabla \rangle (\frac{K}{|x|}(v + \omega))\|_{L^2(I_j; L^{\frac{q}{2}})}
\]

\[
\lesssim \|\omega(t_j)\|_{H^1} + \sum_{i=0}^4 \|v\|_{X^1(I_j)}^{5-i} \|\omega\|_{X^1(I_j)}^{i} + T^{\frac{q}{2}} \|v + \omega\|_{L^\infty(I_j; H^1)}
\]

Thus, (3.23) and (3.25) gives

\[
\|\Phi(\omega)\|_B \leq C \left( \|\omega(t_j)\|_{H^1} + \sum_{i=0}^4 \epsilon^i \|\omega\|_{X^1(I_j)}^{5-i} + CT^{\frac{q}{2}} + T^{\frac{q}{2}} \|\omega\|_{L^\infty(I_j; H^1)} \right).
\]
Plugging the inductive assumption $\|\omega(t_j)\|_{H^1} \leq (2C)^{j-1}T^\frac{3}{4}$, we see that for $\omega \in B$,

$$
\|\Phi(\omega)\|_B \leq C[(2C)^{j-1} + C\epsilon^4T^\frac{3}{4} + (2C)^jT^\frac{3}{2}]T^\frac{3}{4} \quad (3.27)
$$

$$
+ C\sum_{i=0}^3((2C)^jT^\frac{3}{4})^{5-i}\epsilon^i \quad (3.28)
$$

Thus we can choose $\epsilon$ and $T$ small depending only on the Strichartz constant such that

$$3.27 \leq \frac{3}{4}(2C)^jT^\frac{3}{4}. \quad (3.27)$$

Fix this $\epsilon$, $3.28$ is a higher order term with respect to the quantity $T^\frac{3}{4}$, we have

$$3.28 \leq \frac{1}{4}(2C)^jT^\frac{3}{4}, \quad \text{(3.28)}$$

which is available by choosing $T$ small enough. Of course $T$ will depend on $j$, however, since $j \leq J' - 1 \leq C(\|u_0\|_{H^1})$, we can choose $T$ to be a small constant depending only on $\|u_0\|_{H^1}$ and $\epsilon$, therefore is uniform in the process of induction. Hence

$$\|\Phi(\omega)\|_B \leq (2C)^jT^\frac{3}{4}. \quad \text{(3.27)}$$

On the other hand, by a similarly argument as before, we have, for $\omega_1, \omega_2 \in B$

$$
\|\Phi(\omega_1) - \Phi(\omega_2)\|_{\dot{X}_0^1(I_j) \cap L^\infty_t(I_j, L_x^\infty)} \leq C\|\frac{\omega_1 - \omega_2}{|x|}\|_{L^2_t(I_j; L_x^2)} + C\|v + \omega_1|^4(v + \omega_1) - |v + \omega_2|^4(v + \omega_2)\|_{L^2_{-1}(I_j \times \mathbb{R}^3)}
$$

$$
\leq CT^\frac{3}{4}\|\omega_1 - \omega_2\|_{L^\infty_t(I_j, L_x^\infty)} + C\|\omega_1 - \omega_2\|_{\dot{X}_0^1(I_j)} (\|v\|_{\dot{X}_0^1(I_j)} + \|\omega_1\|_{\dot{X}_0^1(I_j)} + \|\omega_2\|_{\dot{X}_0^1(I_j)})
$$

$$
\leq \|\omega_1 - \omega_2\|_{\dot{X}_0^1(I_j) \cap L^\infty_t(I_j, L_x^\infty)} (CT^\frac{3}{4} + \epsilon^4 + 2(2C)^jT^\frac{3}{4})
$$

which allows us to derive

$$
\|\Phi(\omega_1) - \Phi(\omega_2)\|_{\dot{X}_0^1(I_j) \cap L^\infty_t(I_j, L_x^\infty)} \leq \frac{1}{2}\|\omega_1 - \omega_2\|_{\dot{X}_0^1(I_j) \cap L^\infty_t(I_j, L_x^\infty)}
$$

by taking $\epsilon, T$ small such that

$$CT^\frac{3}{4} + \epsilon^4 + 2(2C)^jT^\frac{3}{4} \leq \frac{1}{4}. \quad \text{(3.27)}$$

A standard fixed point argument gives a unique solution $\omega$ of $3.24$ on $I_j$ which satisfies the bound $3.26$. Finally, we get a unique solution of $3.24$ on $[0, T]$ such that

$$
\|\omega\|_{\dot{X}_0^1([0, T])} \leq \sum_{j=0}^{J'-1} \|\omega\|_{\dot{X}_0^1(I_j)} \leq \sum_{j=0}^{J'-1} (2C)^jT^\frac{3}{4} \leq C(2C)^jT^\frac{3}{4} \leq C.
$$

Since on $[0, T]$, $u = v + \omega$, we obtain a unique solution to $1.1$ on $[0, T]$ such that

$$
\|u\|_{\dot{X}_0^1([0, T])} \leq \|\omega\|_{\dot{X}_0^1([0, T])} + \|v\|_{\dot{X}_0^1([0, T])} \leq C(\|u_0\|_{H^1}).
$$

As we mentioned before, this “good local well-posedness” combining with the “global kinetic energy control” as in Step 1 gives finally the global well-posedness. However, since the solution is connected one interval by another, it does not have global space-time bound. In the following, we will discuss the defocussing case, in which the global solution have the enough decay to imply scattering.
4. Morawetz estimate and scattering theory

In this section, we establish an interaction Morawetz estimate and the scattering theory in Theorem 1.3. In the whole of the section, we are in the defocusing case with repulsive potential, that is, \( K < 0 \) and \( \lambda = 1 \).

4.1. Morawetz estimate. In this subsection, we establish the interaction Morawetz estimate for (1.1) with \( K < 0 \) and \( \lambda = 1 \).

Lemma 4.1. Let \( u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} \) solve \( i\partial_t u + \Delta u + V(x)u = N, \) and \( N\bar{u} \in \mathbb{R} \). Given a smooth weight \( w : \mathbb{R}^3 \to \mathbb{R} \) and a (sufficiently smooth and decaying) solution \( u \) to (1.1), we define

\[
I(t, w) = \int_{\mathbb{R}^3} w(x)|u(t, x)|^2 \, dx.
\]

Then, we have

\[
\partial_t I(t, w) = 2 \text{Im} \int_{\mathbb{R}^3} \bar{u} \nabla u \cdot \nabla w \, dx,
\]

(4.1)

\[
\partial_u I(t, w) = - \int_{\mathbb{R}^3} |u|^2 \Delta^2 w \, dx + 4 \text{Re} \int \partial_j u \partial_k \bar{u} \partial_j \partial_k w
\]

\[
+ \int_{\mathbb{R}^3} |u|^2 \nabla V \cdot \nabla w \, dx + 2 \text{Re} \int (N \nabla \bar{u} - \bar{u} \nabla N) \cdot \nabla w \, dx.
\]

(4.2)

Proof. First, note that

\[
\partial_t u = i\Delta u + iV(x)u - iN,
\]

(4.3)

we get

\[
\partial_t I(t, w) = 2 \text{Re} \int_{\mathbb{R}^3} w(x) \partial_t u \bar{u} \, dx
\]

\[
= 2 \text{Re} \int_{\mathbb{R}^3} w(x)(i\Delta u + iV(x)u - iN) \bar{u} \, dx
\]

\[
= - 2 \text{Im} \int_{\mathbb{R}^3} w(x) \Delta u \bar{u} \, dx
\]

\[
= 2 \text{Im} \int_{\mathbb{R}^3} \bar{u} \nabla u \cdot \nabla w \, dx.
\]
Furthermore,
\[
\partial_t I(t, w) = 2 \text{Im} \int_{\mathbb{R}^3} \bar{u}_t \nabla u \cdot \nabla w \, dx + 2 \text{Im} \int_{\mathbb{R}^3} \bar{u} \nabla u_t \cdot \nabla w \, dx
\]
\[
= 2 \text{Im} \int_{\mathbb{R}^3} (-i \Delta \bar{u} - i V(x) \bar{u} + i \hat{N}) \nabla u \cdot \nabla w \, dx
\]
\[
+ 2 \text{Im} \int_{\mathbb{R}^3} \bar{u} \nabla (i \Delta u + i V(x) u - i \hat{N}) \cdot \nabla w \, dx
\]
\[
= 2 \text{Re} \int_{\mathbb{R}^3} (-\Delta \bar{u} \nabla u + \bar{u} \nabla \Delta u) \cdot \nabla w \, dx
\]
\[
+ 2 \text{Re} \int_{\mathbb{R}^3} (\bar{u} \nabla (V u) - V \bar{u} \nabla u) \cdot \nabla w \, dx
\]
\[
+ 2 \text{Re} \int_{\mathbb{R}^3} (N \nabla \bar{u} - \bar{u} \nabla N) \cdot \nabla w \, dx.
\]

\[\square\]

**Remark 4.2.** (i) For \( N = \lambda |u|^{p-1} u, \) so \( \hat{N} \bar{u} = \lambda |u|^{p+1} \in \mathbb{R}, \) then one has
\[
2 \text{Re} \int_{\mathbb{R}^3} (N \nabla \bar{u} - \bar{u} \nabla N) \cdot \nabla w \, dx = \lambda \frac{2(p-1)}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \Delta w \, dx.
\]

(ii) For \( N = |u|^{p-1} u, \ V(x) = \frac{K}{|x|}, \) and \( w \) being radial, we have
\[
\partial_t I(t, w) = -\int_{\mathbb{R}^3} |u|^2 \Delta^2 w \, dx + 4 \text{Re} \int \partial_j u \partial_k \bar{u} \partial_j \partial_k w
\]
\[
- K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} \partial_r w \, dx + \frac{2(p-1)}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \Delta w \, dx.
\]

As a consequence, we obtain the following classical Morawetz estimate by taking \( w(x) = |x|. \)

**Lemma 4.3** (Classical Morawetz estimate). Let \( u : I \times \mathbb{R}^3 \to \mathbb{C} \) solve (1.1) with \( \lambda = 1. \) Then,
\[
\frac{d}{dt} \text{Im} \int_{\mathbb{R}^3} \frac{x}{|x|} \cdot \nabla u \, dx = c|u(t, 0)|^2 + 2 \int_{\mathbb{R}^3} \frac{|
abla u|^2}{|x|} \, dx
\]
\[
- \frac{K}{2} \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} \partial_r w \, dx + \int_{\mathbb{R}^3} \frac{|u|^4}{|x|} \, dx.
\]

Moreover, we have for \( K < 0 \)
\[
\int_{I} \int_{\mathbb{R}^3} \left( \frac{|u|^2}{|x|^2} + \frac{|u|^4}{|x|^4} \right) \, dx \, dt \leq C \sup_{t \in I} \| u(t, \cdot) \|_{H^{\frac{1}{2}}}^2.
\]

Next, we establish the interaction Morawetz estimate for (1.1) with \( K < 0 \) and \( \lambda = 1 \) as the case that \( K = 0 \) in [7].
Theorem 4.4 (Interaction Morawetz estimate). Let $u: I \times \mathbb{R}^3 \to \mathbb{C}$ solve $i\partial_t u + \Delta u + \frac{K}{|x|} u = |u|^{p-1} u$. Then, for $K < 0$, we have

$$
\left\| u \right\|^2_{L^2_t(L^1(I; L^4(\mathbb{R}^3)))} \leq C \left\| u(t_0) \right\|_{L^2} \sup_{t \in I} \|u(t)\|_{H^1}.
$$

(4.6)

Proof. We consider the NLS equation in the form of

$$
i\partial_t u + \Delta u = gu
$$

(4.7)

where $g = g(\rho, |x|)$ is a real function of $\rho = |u|^2 = 2T_{00}$ and $|x|$. We first recall the conservation laws for free Schrödinger in Tao [43]

$$
\partial_t T_{00} + \partial_j T_{0j} = 0,
$$

$$
\partial_t T_{0j} + \partial_k T_{jk} = 0,
$$

where the mass density quantity $T_{00}$ is defined by $T_{00} = \frac{1}{2}|u|^2$, the mass current and the momentum density quantity $T_{0j} = T_{j0}$ is given by $T_{0j} = T_{j0} = \text{Im}(\bar{u}\partial_j u)$, and the quantity $T_{jk}$ is

$$
T_{jk} = 2\text{Re} (\partial_j u \partial_k \bar{u}) - \frac{1}{2}\delta_{jk} \Delta (|u|^2),
$$

(4.8)

for all $j, k = 1, \ldots, n$, and $\delta_{jk}$ is the Kroncker delta. Note that the kinetic terms are unchanged, we see that for (4.7)

$$
\partial_t T_{00} + \partial_j T_{0j} = 0,
$$

$$
\partial_t T_{0j} + \partial_k T_{jk} = -\rho \partial_j g.
$$

(4.9)

By the density argument, we may assume sufficient smoothness and decay at infinity of the solutions to the calculation and in particular to the integrations by parts. Let $h$ be a sufficiently regular real even function defined in $\mathbb{R}^3$, e.g. $h = |x|$. The starting point is the auxiliary quantity

$$
J = \frac{1}{2}\langle |u|^2, h * |u|^2 \rangle = 2\langle T_{00}, h * T_{00} \rangle.
$$

Define the quadratic Morawetz quantity $M = \frac{1}{2}\partial_t J$. Hence we can precisely rewrite

$$
M = -\frac{1}{2}\langle \partial_j T_{0j}, h * T_{00} \rangle - \frac{1}{2}\langle T_{00}, h * \partial_j T_{0j} \rangle = -\langle T_{00}, \partial_j h * T_{0j} \rangle.
$$

(4.10)

By (4.10) and integration by parts, we have

$$
\partial_t M = \langle \partial_k T_{0k}, \partial_j h * T_{0j} \rangle - \langle T_{00}, \partial_j h * \partial_k T_{0j} \rangle
$$

$$
= -\sum_{j,k=1}^n \langle T_{0j}, \partial_j h * T_{0j} \rangle + \langle T_{00}, \partial_j h * T_{jk} \rangle + \langle \rho, \partial_j h * (\rho \partial_k g) \rangle.
$$

For our purpose, we note that

$$
\sum_{j,k=1}^n \langle T_{0k}, \partial_j h * T_{0j} \rangle = \langle \text{Im}(\bar{u}\nabla u), \nabla^2 h * \text{Im}(\bar{u}\nabla u) \rangle
$$

$$
= \langle \bar{u}\nabla u, \nabla^2 h * \bar{u}\nabla u \rangle - \langle \text{Re}(\bar{u}\nabla u), \nabla^2 h * \text{Re}(\bar{u}\nabla u) \rangle.
$$

(4.11)

Therefore it yields that

$$
\partial_t M = \langle \text{Re}(\bar{u}\nabla u), \nabla^2 h * \text{Re}(\bar{u}\nabla u) \rangle - \langle \bar{u}\nabla u, \nabla^2 h * \bar{u}\nabla u \rangle
$$

$$
+ \langle \bar{u}\nabla u, \partial_j h * \text{Re}(\partial_j u \partial_k \bar{u}) - \frac{1}{2}\delta_{jk} \Delta (|u|^2) \rangle + \langle \rho, \partial_j h * (\rho \partial_k g) \rangle.
$$

From the observation

$$
-\langle \bar{u}\nabla u, \partial_j h * \delta_{jk} \Delta (|u|^2) \rangle = \langle \nabla (|u|^2), \Delta h * \nabla (|u|^2) \rangle,
$$

the desired result is achieved.
we write
\[ \partial_t M = \frac{1}{2} \langle \nabla \rho, \Delta h \ast \nabla \rho \rangle + R + \langle \rho, \partial_j h \ast (\rho \partial_j g) \rangle, \]
where \( R \) is given by
\[ R = \langle \bar{u} u, \nabla^2 h \ast (\nabla \bar{u} \nabla u) \rangle - \langle \bar{u} \nabla u, \nabla^2 h \ast \bar{u} \nabla u \rangle \]
\[ = \frac{1}{2} \int \left( \bar{u}(x) \nabla \bar{u}(y) - \bar{u}(y) \nabla \bar{u}(x) \right) \nabla^2 h(x - y) \left( u(x) \nabla u(y) - u(y) \nabla u(x) \right) dxdy. \]

Since the Hessian of \( h \) is positive definite, we have \( R \geq 0 \). Integrating over time in an interval \([t_1, t_2] \subset I \) yields
\[ \int_{t_1}^{t_2} \left( \frac{1}{2} \langle \nabla \rho, \Delta h \ast \nabla \rho \rangle + \langle \rho, \partial_j h \ast (\rho \partial_j g) \rangle + R \right) dt = -\langle T_{00}, \partial_j h \ast T_{0j} \rangle \bigg|_{t=t_1}^{t=t_2}. \]

From now on, we choose \( h(x) = |x| \). One can follow the arguments in [7] to bound the right hand by the quantity
\[ \left| \text{Im} \int \int_{R^3 \times R^3} |u(x)|^2 \frac{x - y}{|x - y|} \bar{u}(y) \nabla u(y) dx dy \right| \leq C \sup_{t \in I} \|u(t)\|_{L^2}^2 \|u(t)\|_{H^{1/2}}^2. \]

Therefore we conclude
\[ \int_{t_1}^{t_2} \langle \rho, \partial_j h \ast (\rho \partial_j g) \rangle dt + \|u\|_{L^4(I; L^4(R^3))}^2 \leq C \sup_{t \in I} \|u(t)\|_{L^2} \|u(t)\|_{H^{1/2}}. \]

Now we consider the term
\[ P := \langle \rho, \nabla h \ast (\rho \nabla g) \rangle. \]
Consider \( g(\rho, |x|) = \rho^{(p-1)/2} + V(x) \), then we can write \( P = P_1 + P_2 \) where
\[ P_1 = \langle \rho, \nabla h \ast (\rho \nabla (\rho^{(p-1)/2})) \rangle = \frac{p-1}{p+1} \langle \rho, \Delta h \ast \rho^{(p+1)/2} \rangle \geq 0 \]
and
\[ P_2 = \int \int \rho(x) \nabla h(x - y) \rho(y) \nabla (V(y)) dx dy \]
\[ = \int \int |u(x)|^2 \frac{(x - y) \cdot y}{|x - y|} |y|^3 |u(y)|^2 \, dx \, dy. \]

By using the Morawetz estimate (4.3)
\[ \int \int_{R^3} \frac{|u|^2}{|x|^2} \, dx \, dt \leq C \sup_{t \in I} \|u\|_{H^{1/2}}^2, \]
one has
\[ |P_2| \leq \|u_0\|_{L^2}^2 \sup_{t \in I} \|u\|_{H^{1/2}}^2. \]
And so, we conclude the proof of Theorem 4.4

**Remark 4.5.** By the same argument as above, one can extend the Coulomb potential \( V(x) = \frac{K}{|x|} \) to \( V(x) \) satisfies the following argument: first, we have by Morawetz estimate
\[ \int \int_{R^3} |u|^2 \frac{x}{|x|} \cdot \nabla V \, dx \, dt \leq C \sup_{t \in I} \|u\|_{H^{1/2}}^2. \]
As in (4.15), we are reduced to estimate the term
\[ \int_I \int_{\mathbb{R}^3} |u|^2 \frac{1}{2} \nabla V \, dx \, dt. \]
Therefore, we can extend \( V(x) \) satisfying
\[ \frac{x}{|x|} \cdot \nabla V \geq c|\nabla V|, \]
with some positive constant \( c \).

4.2. Scattering theory. Now we use the global-in-time interaction Morawetz estimate (1.6)
\[ \|u\|^2_{L^1_t(\mathbb{R};L^1_x(\mathbb{R}^3))} \leq C \|u_0\|_{L^1} \sup_{t \in \mathbb{R}} \|u(t)\|_{H^\frac{1}{2}}, \quad (4.16) \]
to prove the scattering theory part of Theorem 1.3. Since the construction of the wave operator is standard, we only show the asymptotic completeness.

Let \( u \) be a global solution to (1.1). Let \( \eta > 0 \) be a small constant to be chosen later and split \( \mathbb{R} \) into \( L = L(\|u_0\|_{H^1}) \) finite subintervals \( I_j = [t_j, t_{j+1}] \) such that
\[ \|u\|_{L^1_t(I_j \times \mathbb{R}^3)} \leq \eta. \quad (4.17) \]
Define
\[ \|\langle \nabla \rangle u\|_{S^0(I_j)} := \sup_{(q,r) \in A_0} \|\langle \nabla \rangle u\|_{L^2_q L^r_t(I_j \times \mathbb{R}^3)}. \]
Using the Strichartz estimate and Sobolev norm equivalence (2.10), we obtain
\[ \|\langle \nabla \rangle u\|_{S^0(I_j)} \lesssim \|u(t_j)\|_{H^1} + \|\langle \nabla \rangle(|u|^{p-1} u)\|_{L^2_q L^r_t(I_j \times \mathbb{R}^3)}. \quad (4.18) \]
Let \( \epsilon > 0 \) be determined later, and \( r_\epsilon = \frac{6}{4-\epsilon} \). On the other hand, we use the Leibniz rule and Hölder’s inequality to obtain
\[ \|\langle \nabla \rangle(|u|^{p-1} u)\|_{L^2_q L^{\frac{2}{p-1}}(I_j;L^\infty_x)} \lesssim \|\langle \nabla \rangle u\|_{L^2_q L^{\frac{2}{p-1}}(I_j;L^\infty_x)} \|u\|_{L^\infty_q L^{\frac{2(p-1)+2}{p-1}}(I_j;L^\infty_x)} \|u\|_{L^\infty_q L^{\frac{2(p-1)+2}{p-1}}(I_j;L^\infty_x)}. \]
Taking \( \epsilon = 2_+ \), and so \( r_\epsilon = 3_- \). If \( p \in \left( \frac{5}{4}, 4 \right] \), then \( 2(p-1)(2+\epsilon)/\epsilon > 4 \) and \( 2 \leq \frac{3(p-1)(2+\epsilon)}{4+\epsilon} \leq 6 \). Therefore we interchange \( \alpha \) to obtain
\[ \|u\|_{L^2_q L^{\frac{2(p-1)(2+\epsilon)}{4+\epsilon}}(I_j;L^\infty_x)} \lesssim \|u\|^\alpha_{L^1_t L^2_x(I_j \times \mathbb{R}^3)} \|u\|^\beta_{L^\infty_q L^2_x(I_j \times \mathbb{R}^3)} \|u\|^\gamma_{L^2_q L^2_x(I_j \times \mathbb{R}^3)}; \]
where \( \alpha > 0, \beta, \gamma \geq 0 \) satisfy \( \alpha + \beta + \gamma = 1 \)
\[ \left( \frac{2(p-1)(2+\epsilon)}{4+\epsilon} \right) \alpha + \beta + \gamma = \left( \frac{2(p-1)(2+\epsilon)}{4+\epsilon} \right) \frac{2}{4+\epsilon} + \frac{3}{4+\epsilon} = \frac{4}{4+\epsilon} + \frac{3}{4+\epsilon} = \frac{4}{4+\epsilon} + \frac{3}{4+\epsilon}. \]
Hence
\[ \|\langle \nabla \rangle(|u|^{p-1} u)\|_{L^2_q L^{\frac{2p}{p-1}}(I_j;L^\infty_x)} \lesssim \|\langle \nabla \rangle u\|_{L^2_q L^{\frac{2p}{p-1}}(I_j;L^\infty_x)} \|u\|^\alpha_{L^1_t L^2_x(I_j \times \mathbb{R}^3)} \|u\|^{(\beta+\gamma)(p-1)}_{L^1_t H^\frac{1}{2}_x(I_j \times \mathbb{R}^3)} \leq C\eta^{p-1}\|\langle \nabla \rangle u\|_{S^0(I_j)}. \]
Plugging this into (4.18) and noting that \( \alpha(p-1) > 0 \), we can choose \( \eta \) to be small enough such that
\[ \|\langle \nabla \rangle u\|_{S^0(I_j)} \leq C(E,M,\eta). \]
Hence we have by the finiteness of $L$
\[
\|⟨∇⟩u\|_{S^0(\mathbb{R})} \leq C(E, M, \eta, L). \tag{4.19}
\]

If $p \in (4, 5)$, we use interpolation to show that
\[
\|u\|_{L^p_t L^p_x(I \times \mathbb{R}^3)} \leq C\|u\|_{L^p_t H^p_x(I \times \mathbb{R}^3)}^{\beta} \|u\|_{L^p_t L^p_x(I \times \mathbb{R}^3)}^{\gamma},
\]
where $\alpha > 0, \beta, \gamma \geq 0$ satisfy $\alpha + \beta + \gamma = 1$ and
\[
\begin{align*}
\frac{2(\beta+1)}{\beta+1} & = \frac{\alpha}{4} + \frac{\beta}{\infty} + \frac{\gamma}{2}, \\
\frac{3(\beta+1)}{\beta+1} & = \frac{\alpha}{4} + \frac{\beta}{6} + \frac{\gamma}{18}.
\end{align*}
\]

It is easy to solve these equations for $p \in (4, 5)$. Since $r_\epsilon \in [2, 3_\pm]$ for $\epsilon = 2_\pm$, we have
\[
\|⟨∇⟩(\|u|^{p-1}u)\|_{L^2_t L^2_x} \lesssim \|⟨∇⟩u\|_{L^{2+\epsilon}_t L^{2+\epsilon}_x} \|u\|_{L^p_t H^p_x(I \times \mathbb{R}^3)}^{\alpha(p-1)} \|u\|_{L^p_t L^p_x(I \times \mathbb{R}^3)}^{\beta(p-1)} \|⟨∇⟩u\|_{S^0(I \times \mathbb{R}^3)}^{\gamma(p-1)}
\]
\[
\leq C\theta^{\alpha(p-1)}\|⟨∇⟩u\|_{S^0(I \times \mathbb{R}^3)}^{1+\gamma(p-1)}.
\]

Hence arguing as above we obtain (4.19).

Finally, we utilize (4.19) to show asymptotic completeness. We need to prove that there exist unique $u_+\pm$ such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\mathcal{L}_K}u_\pm\|_{H^1} = 0.
\]

By time reversal symmetry, it suffices to prove this for positive times. For $t > 0$, we will show that $v(t) := e^{-it\mathcal{L}_K}u(t)$ converges in $H^1_+$ as $t \to +\infty$, and denote $u_+$ to be the limit. In fact, we obtain by Duhamel’s formula
\[
v(t) = u_0 - i \int_0^t e^{-i\tau \mathcal{L}_K} (|u|^{p-1}u)(\tau) d\tau. \tag{4.20}
\]

Hence, for $0 < t_1 < t_2$, we have
\[
v(t_2) - v(t_1) = -i \int_{t_1}^{t_2} e^{-i\tau \mathcal{L}_K} (|u|^{p-1}u)(\tau) d\tau.
\]

Arguing as before, we deduce that for some $\alpha > 0, \beta \geq 1$
\[
\|v(t_2) - v(t_1)\|_{H^1(\mathbb{R}^3)} = \left\| \int_{t_1}^{t_2} e^{-i\tau \mathcal{L}_K} (|u|^{p-1}u)(\tau) d\tau \right\|_{H^1(\mathbb{R}^3)}
\]
\[
\lesssim \|⟨∇⟩(|u|^{p-1}u)\|_{L^2_t L^2_x([t_1, t_2] \times \mathbb{R}^3)}
\]
\[
\lesssim \|u\|_{L^{p}_t L^{p}_x([t_1, t_2] \times \mathbb{R}^3)} \|⟨∇⟩u\|_{S^0([t_1, t_2])}^{\beta}
\]
\[
\to 0 \quad \text{as} \quad t_1, t_2 \to +\infty.
\]

As $t$ tends to $+\infty$, the limitation of (4.20) is well defined. In particular, we find the asymptotic state
\[
u_+ = u_0 - i \int_0^\infty e^{-i\tau \mathcal{L}_K} (|u|^{p-1}u)(\tau) d\tau.
\]

Therefore, we conclude the proof of Theorem 1.3.
5. Blow up

In this section, we study the blow up behavior of the solution in the focusing case, i.e. \( \lambda = -1 \). In the case that \( K > 0 \), we will use the sharp Hardy’s inequality and Young’s inequality to obtain

\[
\int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \, dx \leq \left( \int |u|^2 \, dx \right)^{\frac{1}{2}} \left( \int \frac{|u|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \\
\leq 2\|u\|_{L^2} \|u\|_{\dot{H}^1} \\
\leq \frac{1}{C_{p,K}} \|u\|_{L^2}^2 + C_{p,K} \|u\|_{\dot{H}^1}^2,
\]

(5.1)

for any \( C_{p,K} > 0 \).

From Remark 4.2, it follows that for \( w \) radial function, we have

\[
\partial_t \int_{\mathbb{R}^3} w(x)|u|^2 \, dx = - \int_{\mathbb{R}^3} |u|^2 \Delta^2 w \, dx + 4\text{Re} \int_{\mathbb{R}^3} \partial_j u \partial_{j,k} \overline{u} \partial_{j,k} w \\
- K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \partial_j w \, dx - \frac{2(p-1)}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \Delta w \, dx.
\]

(5.2)

**Case 1:** \( u_0 \in \Sigma \). By taking \( w(x) = |x|^2 \), we obtain

**Corollary 5.1.** Let \( u \) solve (1.1), then we have

\[
\frac{d^2}{dt^2} \int_{\mathbb{R}^3} |x|^2 |u(t,x)|^2 \, dx = 8 \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - 2K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \, dx - \frac{12(p-1)}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx \\
= 12(p-1)E(u) - 2(3p-7) \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + 6K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \, dx.
\]

Let \( I = [0, T] \) be the maximal interval of existence. Let

\[
y(t) := \int |x|^2 |u(t,x)|^2 \, dx,
\]

then for \( t \in I \)

\[
y'(t) = 4\text{Im} \int_{\mathbb{R}^3} x \cdot \nabla u \bar{u} \, dx.
\]

By Corollary 5.1 and (5.1) with \( C_{p,K} = \frac{3p-7}{3K} \) when \( K > 0 \), we get

\[
y''(t) \leq 12(p-1)C(E,M) := \begin{cases} 
12(p-1)E(u_0) & \text{if } K \leq 0 \\
12(p-1)E(u_0) + \frac{18K^2}{3p-7}M(u_0) & \text{if } K > 0.
\end{cases}
\]

(5.3)

Hence

\[
y(t) \leq 6(p-1)C(E,M)t^2 + y'(0)t + y(0).
\]

which implies \( I \) is finite provided that

(i) \( C(E,M) < 0 \); (ii) \( C(E,M) = 0, y'(0) < 0 \); (iii) \( C(E,M) > 0, y'(0)^2 \geq 24(p-1)C(E,M)y(0) \).

In fact, in the above conditions, we have \( T < +\infty \) and

\[
\lim_{t \to T} y(t) = 0
\]

this together with

\[
\|u(0)\|_2^2 = \|u(t)\|_2^2 \leq \|x|u(t)|\|_{L^2} \|u(t)|\|_{\dot{H}^1}.
\]
implies
\[
\lim_{t \to T} \|u(t)\|_{\dot{H}^1} = +\infty. \tag{5.4}
\]

**Case 2:** \(u_0 \in H^1_{rad}(\mathbb{R}^3)\). Let \(\phi\) be a smooth, radial function satisfying \(|\partial_r^2 \phi(r)| \leq 2, \phi(r) = r^2\) for \(r \leq 1\), and \(\phi(r) = 0\) for \(r \geq 3\). For \(R \geq 1\), we define \(\phi_R(x) = R^2 \phi(\frac{r}{R})\) and \(V_R(x) = \int_{\mathbb{R}^3} \phi_R(\mathbf{r})|u(t, \mathbf{r})|^2 d\mathbf{r}\).

Let \(u(t, x)\) be a radial solution to (1.1), then by a direct computation, we have by (5.2)
\[
\partial_t V_R(x) = 2\text{Im} \int_{\mathbb{R}^3} [\overline{\mathbf{r}} \partial_j u(t, \mathbf{r})] \partial_j \phi_R(\mathbf{r}) d\mathbf{r}, \tag{5.5}
\]
and
\[
\partial^2_t V_R(x) = 4\text{Re} \int_{\mathbb{R}^3} \partial_j u \partial_k \bar{u} \partial_j \partial_k \phi_R - \int_{\mathbb{R}^3} |u|^2 \Delta^2 \phi_R \, dx
\]
\[
- K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} \partial_R \phi_R \, dx - \frac{2(p - 1)}{p + 1} \int_{\mathbb{R}^3} |u|^{p+1} \Delta \phi_R \, dx
\]
\[
= 4 \int_{\mathbb{R}^3} \phi_R' \Delta u^2 \, dx - K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} \phi_R' \, dx - \frac{12(p - 1)}{p + 1} \int_{\mathbb{R}^3} |u|^{p+1} \Delta \phi_R \, dx
\]
\[
- 4 \int_{\mathbb{R}^3} |\nabla u|^2 (2 - \phi_R') \, dx + K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} (2 - \phi_R') \, dx + \frac{2(p - 1)}{p + 1} \int_{\mathbb{R}^3} |u|^{p+1} (6 - \Delta \phi_R) \, dx
\]
\[
\leq 12(p - 1)E(u) - 2(3p - 7)\|\nabla u\|_{L^2}^2 + 6K \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \, dx
\]
\[
- 4 \int_{\mathbb{R}^3} |\nabla u|^2 (2 - \phi_R') \, dx + C \int_{|x| \geq R} \left( \frac{|u|^2}{R} + |u|^{p+1} \right) \, dx.
\]

By the radial Sobolev inequality, we have
\[
\|f\|_{L^\infty(|x| \geq R)} \leq \frac{C}{R} \|f\|^\frac{1}{2}_{L^2(|x| \geq R)} \|\nabla f\|^\frac{1}{2}_{L^2(|x| \geq R)}.
\]

Therefore, by mass conservation and Young's inequality, we know that for any \(\epsilon > 0\) there exist sufficiently large \(R\) such that for \(K \leq 0\)
\[
\partial^2_t V(t) \leq 12(p - 1)E(u) - 2(3p - 7 - \epsilon)\|u\|^2_{\dot{H}^1} + \epsilon^2
\]
\[
\leq 12(p - 1)E(u_0) + \epsilon^2,
\]
and for \(K > 0\) by using (5.1) with \(C_{p,K} = \frac{2p - 7 - \delta}{3p - 7}\) and \(0 < \delta \ll 1\)
\[
\partial^2_t V(t) \leq 12(p - 1)E(u) - (\delta - \epsilon)\|u\|^2_{\dot{H}^1} + \frac{18K^2}{3p - 7 - \delta} M(u) + \epsilon^2
\]
\[
\leq 12(p - 1)E(u_0) + \frac{18K^2}{3p - 7 - \delta} M(u_0) + \epsilon^2,
\]
for any \(3p - 7 > \delta > \epsilon > 0\).

Finally, if we choose \(\epsilon\) sufficient small, we can obtain
\[
\partial^2_t V_R(t) \leq \begin{cases} 
6(p - 1)E(u_0), & \text{if } K \leq 0 \\
6(p - 1)E(u_0) + \frac{9K^2}{3p - 7 - \delta} M(u_0), & \text{if } K > 0,
\end{cases} \tag{5.6}
\]
which implies that $u$ blows up in finite time by the same argument as Case 1, since for the case $K > 0$, the assumption

$$E(u_0) + \frac{3K^2}{2(3p-7)(p-1)} M(u_0) < 0,$$

shows that there exists $0 < \delta \ll 1$ such that

$$6(p-1)E(u_0) + \frac{9K^2}{3p-7-\delta} M(u_0) < 0.$$ 

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