Almost optimal searching of maximal subrepetitions in a word

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Abstract

For $0 < \delta < 1$ a $\delta$-subrepetition in a word is a factor which exponent is less than 2 but is not less than $1 + \delta$ (the exponent of the factor is the ratio of the factor length to its minimal period). The $\delta$-subrepetition is maximal if it cannot be extended to the left or to the right by at least one letter with preserving its minimal period. In the paper we propose an algorithm for searching all maximal $\delta$-subrepetitions in a word of length $n$ in $O(\frac{n}{\delta} \log \frac{1}{\delta})$ time (the lower bound for this time is $\Omega(\frac{n}{\delta})$).

Let $w = w[1]w[2] \ldots w[n]$ be an arbitrary word of length $|w| = n$. A fragment $w[i] \ldots w[j]$ of $w$, where $1 \leq i \leq j \leq n$, is called a factor of $w$ and is denoted by $w[i..j]$. Note that this factor can be considered either as a word itself or as the fragment $w[i] \ldots w[j]$ of $w$. So for factors we have two different notions of equality: factors can be equal as the same fragment of the word $w$ or as the same word. To avoid this ambiguity, we use two different notations: if two factors $u$ and $v$ of $w$ are the same word (the same fragment of $w$) we will write $u = v$ ($u \equiv v$). For any $i = 1, \ldots, n$ the factor $w[1..i]$ ($w[i..n]$) is called a prefix (a suffix) of $w$. By positions in $w$ we mean the order numbers $1, 2, \ldots, n$ of letters of the word $w$. For any factor $v \equiv w[i..j]$ of $w$ the positions $i$ and $j$ are called start position of $v$ and endind position of $v$ and denoted by $\text{beg}(v)$ and $\text{end}(v)$ respectively. For any two factors $u$, $v$ of $w$ the factor $u$ is contained in $v$ if $\text{beg}(v) \leq \text{beg}(u)$ and $\text{end}(u) \leq \text{end}(v)$. Two factors $u$ and $v$ of $w$ such that $\text{beg}(u) \leq \text{beg}(v)$ are called overlapped if $\text{beg}(v) \leq \text{end}(u) + 1$. The intersection of the overlapped factors $u$ and $v$ is the factor $w[\text{beg}(v) .. \text{end}(u)]$ (if $\text{beg}(v) = \text{end}(u) + 1$, the intersection of $u$ and $v$ is assumed to be an empty word). The length of the intersection of the overlapped factors $u$ and $v$ is called the overlap of $u$ and $v$. The union of the overlapped factors $u$ and $v$ is the factor $w[i..j]$ where $i = \min(\text{beg}(u), \text{beg}(v))$, $j = \max(\text{end}(u), \text{end}(v))$. If some word $u$ is equal to a factor $v$ of $w$ then $v$ is called an occurrence of $u$ in $w$.

A positive integer $p$ is called a period of $w$ if $w[i] = w[i + p]$ for each $i = 1, \ldots, n - p$. We denote by $p(w)$ the minimal period of $w$ and by $e(w)$ the
ratio $|w|/p(w)$ which is called the exponent of $w$. Farther we use the following well-known fact which is usually called the periodicity lemma.

**Lemma 1.** If a word $w$ has two periods $p, q$, and $|w| \geq p + q$, then $\gcd(p, q)$ is also a period of $w$.

The periodicity lemma is actually a weaker version of the Fine and Wilf theorem (see [12]). Using the periodicity lemma, it is easy to obtain

**Proposition 1.** Let $q$ be a period of a word $w$ such that $|w| \geq 2q$. Then $q$ is divisible by $p(w)$.

We will also use the following evident fact.

**Proposition 2.** If two overlapped factors of a word have the same period $p$ and the overlap of these factors is not less than $p$ then $p$ is a period of the union of these factors.

A word is called primitive if its exponent is not an integer greater than 1. For primitive words the following well-known fact takes place (see, e.g., [3]).

**Lemma 2** (primitivity lemma). If $u$ is a primitive word, then the word $uu$ has no occurrences of $u$ which are neither prefix nor suffix of $uu$.

Words $r$ such that $e(r) \geq 2$ are called repetitions. A repetition in a word is called maximal if this repetition cannot be extended to the left nor to the right in the word by at least one letter while preserving its minimal period. More precisely, a repetition $r$ in a word $w$ is called maximal if it satisfies the following maximality conditions:

1. if $\text{beg}(r) > 1$, then $w[\text{beg}(r) - 1] \neq w[\text{beg}(r) + p(r) - 1]$,
2. if $\text{end}(r) < n$, then $w[\text{end}(r) - p(r) + 1] \neq w[\text{end}(r) + 1]$.

For example, word $ababaabaaababab$ has 6 maximal repetitions: $ababa$, $abaabaa$, $aabaaaba$, $aa$, $aaa$ and $ababab$. Maximal repetitions are usually called runs in the literature. By $\mathcal{R}(w)$ we denote the set of all maximal repetitions in the word $w$. For any natural $n$ we define also $R(n) = \max_{|w|=n} |\mathcal{R}(w)|$ and $E(n) = \max_{|w|=n} \sum_{r \in \mathcal{R}(w)} e(r)$.

The possible number of maximal repetitions was actively investigated in the literature. In [4] the linear upper bound $E(n) = O(n)$ is proved which implies obviously that $R(n) = O(n)$. Due to a series of papers (see, e.g. [5]) more precise upper bounds on $E(n)$ and $R(n)$ have been obtained. A breakthrough in this direction was made in [6] where the bounds $R(n) < n$, $E(n) < 3n$ are proved. To our knowledge, the best upper bound $|\mathcal{R}(w)| \leq \frac{163}{164}n$ for binary words $w$ of length $n$ is shown in [7] and the best lower bounds $2.035n$ on $E(n)$ and $0.9445757n$ on $R(n)$ are obtained respectively in [8] and [9]. Some results on the average number of runs in arbitrary words are obtained in [10,11].

In the paper we consider words over (polynomially bounded) integer alphabet, i.e. words over alphabet which consists of nonnegative integers bounded
by some polynomial of the length of words. So farther by \( w \) we will mean an arbitrary word of length \( n \) over integer alphabet.

The problem of finding of all maximal repetitions in words was also actively investigated in the literature. An \( O(n) \)-time algorithm for finding all runs in a word of length \( n \) was proposed in [3] for the case of constant-size alphabet. This result was generalized to the case of words over integer alphabet in [12]. Another \( O(n) \)-time algorithm for the case of integer alphabet, based on a different approach, has been proposed in [19]. Algorithm for solving the problem in the case of an unbounded linearly-ordered alphabet was proposed in [13]. This algorithm was improved in [14, 15]. Finally, a linear time algorithm for this case was proposed in [10]. The obtained results can be summarized in the following two theorems.

**Theorem 1.** The number of maximal repetitions in \( w \) is \( O(n) \), and all these repetitions with their minimal periods can be computed in \( O(n) \) time.

**Theorem 2.** The sum \( \sum_{r \in R(w)} e(r) \) of exponents of all maximal repetitions in \( w \) is \( O(n) \).

Let \( r \) be a repetition in \( w \). We call any factor of \( w \) which has the length \( p(r) \) and is contained in \( r \) a cyclic root of \( r \). The cyclic root which is the prefix of \( r \) is called prefix cyclic root of \( r \). It follows from the minimality of the period \( p(r) \) that any cyclic root of \( r \) is a primitive word. So the following proposition can be easily obtained from Lemma 2.

**Proposition 3.** Two cyclic roots \( x', x'' \) of a repetition \( r \) are equal if and only if \( \text{beg}(x') \equiv \text{beg}(x'') \pmod{p(r)} \) (\( \text{end}(x') \equiv \text{end}(x'') \pmod{p(r)} \)).

Since all roots of any repetition are primitive, any repetition \( r \) has \( p(r) \) different cyclic roots which are cyclic rotations of each other. The lexicographically minimal root among these cyclic roots is called Lyndon root of the repetition \( r \). Let \( x \) be the leftmost occurrence of the Lyndon root in the repetition \( r \). Then the difference \( \text{beg}(x) - \text{beg}(r) \) is denoted by \( a(r) \). Two repetitions with the same minimal period are called repetitions with the same cyclic roots if they have the same set of distinct cyclic roots. Note that repetitions with the same cyclic roots has the same Lyndon root.

Let \( r', r'' \) be maximal repetitions with the same cyclic roots, \( p \) be the minimal period of \( r' \) and \( r'' \), and \( x', x'' \) be cyclic roots of repetitions \( r', r'' \) respectively. Note that \( \text{beg}(r') + a(r') \) and \( \text{beg}(r'') + a(r'') \) are the starting positions of the leftmost Lyndon roots of repetitions \( r', r'' \) respectively. Denote by \( \delta' \) (\( \delta'' \)) the residue of \( \text{beg}(x') - (\text{beg}(r') + a(r')) \) (\( \text{beg}(x'') - (\text{beg}(r'') + a(r'')) \)) in modulo \( p \). Using Proposition 3 it is easy to see that \( x' = x'' \) if and only if \( \delta' = \delta'' \). Thus, we obtain the following fact.

**Proposition 4.** Let \( r', r'' \) be maximal repetitions with the same cyclic roots, \( p \) be the minimal period of \( r' \) and \( r'' \), and \( x', x'' \) be cyclic roots of repetitions \( r', r'' \) respectively. Then \( x' = x'' \) if and only if

\[
\text{beg}(x') - (\text{beg}(r') + a(r')) \equiv \text{beg}(x'') - (\text{beg}(r'') + a(r'')) \pmod{p}.
\]
Farther we will use double-linked lists of all maximal repetitions with the same cyclic roots in the order of increasing of their starting positions. According to [17], these lists can be computed for the word $w$ in $O(n)$ time. It is also shown in [17] that for all maximal repetitions $r$ in $w$ the values $a(r)$ can be computed in $O(n)$ total time.

We will also use the following facts for overlaps of maximal repetitions (see, e.g. [18]).

**Proposition 5.** The overlap of any two different maximal repetitions with the same minimal period $p$ is smaller than $p$.

**Proposition 6.** The overlap of any two different maximal repetitions $r'$ and $r''$ is smaller than $p(r') + p(r'')$.

A natural generalization of repetitions are factors with exponents strictly less than 2. We will call such factors subrepetitions. More precisely, a factor $r$ is called a subrepetition if $1 < e(r) < 2$. Analogously to repetitions, a subrepetition $r$ in $w$ is called maximal if $r$ satisfies the maximality conditions, i.e. if $r$ cannot be extended to the left nor to the right in $w$ by at least one letter while preserving its minimal period. For any $\delta$ such that such that $0 < \delta < 1$ a subrepetition $r$ is called $(\delta\text{-subrepetition})$ if $e(r) \geq 1 + \delta$. It is shown below that the number of maximal $(\delta\text{-subrepetitions})$ in a word of length $n$ is $O(n/\delta)$.

In this paper the following problem is investigated.

**Problem 1.** For a given value $\delta$ find in a given word $w$ of length $n$ all maximal $\delta$-subrepetitions.

Before in [19] two algorithms for solving of Problem 1 was proposed: the first algorithm has $O(\frac{n \log\log n}{\delta^2})$ time complexity and the second algorithm has $O(n \log n + \frac{n}{\delta^2} \log \frac{1}{\delta})$ expected time complexity. In [20] the expected time of the second algorithm was improved to the linear bound $O(\frac{n}{\delta^2} \log \frac{1}{\delta})$. Using the results of [21, 22], this time can be farther improved to $O(\frac{n}{\delta^2} \log \frac{1}{\delta})$. In [23], it is shown that all subrepetitions with the largest exponent (over all subrepetitions) in an overlap-free string can be found in $O(n)$ time for a constant-size alphabet. In this paper we propose an alternative deterministic algorithm for solving of Problem 1 in $O(\frac{n}{\delta} \log \frac{1}{\delta})$ time.

1 Repeats

Another regular structures in a word which are closely related to repetitions and subrepetitions are repeats. In general case, a repeat $\sigma$ in the word $w$ is a pair $u', u''$ of equal factors of the word $w$ where $\text{beg}(u') < \text{beg}(u'')$. The factors $u', u''$ are called copies of the repeat $\sigma$, in particular, $u'$ is called the left copy of $\sigma$ and $u'$ is called the right copy of $\sigma$. The length of copies of $\sigma$ is denoted by

\[1\text{In the paper for convenience we assume actually that }\delta < 1 - \varepsilon \text{ for some fixed } \varepsilon.\]
The difference $\text{beg}(u') - \text{beg}(u'')$ is called the period of the repeat $\sigma$ and is denoted by $p(\sigma)$. The minimal factor $w[\text{beg}(u')..\text{end}(u'')]$ containing the both copies $u', u''$ of $\sigma$ will be denoted by fact($\sigma$). Note that for different repeats $\sigma'$ and $\sigma''$ we have actually fact($\sigma'$) = fact($\sigma''$). Note also that $p(\sigma)$ is a period of fact($\sigma$), but the minimal period of fact($\sigma$) can be less than $p(\sigma)$. By the starting position (the ending position) of gapped $\sigma$ we will mean the starting position (the ending position) of fact($\sigma$). We will say that a maximal repeat $\sigma$ is contained in some factor $w'$ of the word if the factor fact($\sigma$) is contained in $w'$. A repeat is called overlapped if its copies are overlapped factors, otherwise the repeat is called gapped. In other words, the repeat $\sigma$ is gapped if fact($\sigma$) can be represented in the form $u'vu''$ where $v$ is a nonempty word called the gap of the repeat $\sigma$. For any $\alpha > 1$ a gapped repeat $\sigma$ is called $\alpha$-gapped if $p(\sigma) \leq \alpha c(\sigma)$.

Analogously to repetitions and subrepetitions, we can also introduce a notion of maximal repeats. A repeat $\sigma$ with left and right copies $u', u''$ in $w$ is called maximal if it satisfies the following conditions:

1. if $\text{beg}(u') > 1$, then $w[\text{beg}(u') - 1] \neq w[\text{beg}(u'') - 1]$,
2. if $\text{end}(u'') < n$, then $w[\text{end}(u') + 1] \neq w[\text{end}(u'') + 1]$.

In other words, a repeat in a word is called maximal if its copies cannot be extended to the left nor to the right in the word by at least one letter while preserving its period. Note that any repeat can be extended to uniquely defined maximal repeat with the same period. In particular, any $\alpha$-gapped repeat can be extended to uniquely defined maximal $\alpha$-gapped or overlapped repeat. In [21, 22] the following fact on maximal $\alpha$-gapped repeats was obtained.

**Theorem 3.** The number of maximal $\alpha$-gapped repeats in $w$ is $O(\alpha n)$, and all these repeats can be computed in $O(\alpha n)$ time.

In [22] the more precise upper bound $18\alpha n$ on the number of maximal $\alpha$-gapped repeats in $w$ was actually obtained. A tighter bound on this number was obtained later in [24]. An algorithm which finds in each position of word longest gapped repeats satisfying additional restrictions is proposed in [25].

Let $\sigma$ be an overlapped repeat with with left and right copies $u', u''$ in $w$. Note that in this case the period $p(\sigma)$ of fact($\sigma$) is not greater than the half of $|\text{fact}(\sigma)|$, so fact($\sigma$) is a repetition. Let $p$ be the minimal period of fact($\sigma$). By Proposition 1 we have that $p$ is a divisor of $p(\sigma)$. Let $\text{beg}(\text{fact}(\sigma)) = \text{beg}(u') > 1$. Since $\sigma$ is a maximal repeat, we have that

$$w[\text{beg}(\text{fact}(\sigma)) - 1] = w[\text{beg}(u') - 1] \neq w[\text{beg}(u'') - 1] = w[\text{beg}(\text{fact}(\sigma)) + p(\sigma) - 1].$$

On the other hand, since the period $p$ of fact($\sigma$) is a divisor of $p(\sigma)$, we have that $w[\text{beg}(\text{fact}(\sigma)) + p - 1] = w[\text{beg}(\text{fact}(\sigma)) + p(\sigma) - 1]$. Thus, we obtain that $w[\text{beg}(\text{fact}(\sigma)) - 1] \neq w[\text{beg}(\text{fact}(\sigma)) + p(\sigma) - 1].$ In analogous way we can obtain that if $\text{end}(\text{fact}(\sigma)) < n$ then $w[\text{beg}(\text{fact}(\sigma)) + 1] \neq w[\text{beg}(\text{fact}(\sigma)) - p + 1].$ Thus, we conclude that fact($\sigma$) is a maximal repetition whose minimal period is a divisor of $p(\sigma)$. We will denote this repetition by rep($\sigma$). Now let $r$ be
a maximal repetition in \( w \). Then we can consider the repeat \( \sigma \) with left and right copies \( w[\text{beg}(r)..\text{end}(r) - p(r)] \) and \( w[\text{beg}(r) + p(r)..\text{end}(r)] \). It is easy to note that \( \sigma \) is a maximal overlapped repeat such that \( p(\sigma) = p(r) \). We will call the repeat \( \sigma \) the principal repeat of the repetition \( r \). Principal repeats of maximal repetitions will be also called reprincipal repeats. Note that any reprincipal repeat \( \sigma \) is the principal repeats of the repetition \( \text{rep}(\sigma) \), so for any maximal repetition we have that this repetition and the principal repeat of this repetition are uniquely defined by each other. Therefore, we have one-to-one correspondence between all maximal repetitions and all reprincipal repeats in a word. Thus, in any word the number of reprincipal repeats is equal to the number of maximal repetitions. We have the following fact for reprincipal repeats.

**Proposition 7.** The number of reprincipal repeats in \( w \) is \( O(n) \), and all these repeats can be computed in \( O(n) \) time.

**Proof.** Recall that the number of reprincipal repeats in \( w \) is equal to the number of maximal repetitions in \( w \), so, by Theorem 1 this number is \( O(n) \). To compute all reprincipal repeats in \( w \), first we find all maximal repetitions in \( w \). By Theorem 1 it can be done in time \( O(n) \). Then for each found maximal repetition \( r \) we compute the principal repeat of \( r \) as repeat \( (u', v') \) where \( \text{beg}(u') = \text{beg}(r) \), \( \text{end}(u') = \text{end}(r) - p(r) \), \( \text{beg}(v') = \text{beg}(r) + p(r) \), \( \text{end}(v') = \text{end}(r) \). It can be computed in constant time. Therefore, the total time of computing of all reprincipal repeats from found maximal repetitions is proportional to the number of these repetitions, so this time is \( O(n) \). Thus, the total time of computing of all reprincipal repeats in \( w \) is \( O(n) \).

Now let \( r \) be a maximal \( \delta \)-subrepetition in \( w \). Then we can consider a gapped repeat \( \sigma \) with the left copy \( w[\text{beg}(r)..\text{end}(r) - p(r)] \) and the right copy \( w[\text{beg}(r) + p(r)..\text{end}(r)] \). It is easy to see that \( \sigma \) is a maximal gapped repeat with the period \( p(\sigma) = p(r) \). Moreover, we have

\[
c(\sigma) = \vert\sigma\vert - p(r) \geq (1 + \delta)p(r) - p(r) = \delta p(r) = \delta p(\sigma),
\]

so \( p(\sigma) \leq c(\sigma)/\delta \). Thus, \( \sigma \) is actually a maximal \( \frac{1}{\delta} \)-gapped repeat. We call the repeat \( \sigma \) the principal repeat of the subrepetition \( r \). Thus, for any maximal \( \delta \)-subrepetition there exists the principal repeat of this subrepetition. On the other hand, a maximal gapped repeat may not be the principal repeat of any maximal subrepetition. For example, in the word shown in Fig. 1 we can consider the maximal subrepetition \( r \) with the minimal period 7. Note that the gapped repeat \( u'vv' \) is the principal repeat of \( r \) while the gapped repeat \( \overline{\tau \tau \tau} \) is not the principal repeat of \( r \), so the repeat \( \overline{\tau \tau \tau} \) is not the principal repeat of any maximal subrepetition. Thus, the repeat \( u'vv' \) is a principal repeat and the repeat \( \overline{u'v} \) is not a principal repeat. Note that for any maximal subrepetition we have that this subrepetition and the principal repeat of this subrepetition are uniquely defined by each other, and we have one-to-one correspondence between all maximal \( \delta \)-subrepetitions and all principal maximal \( \frac{1}{\delta} \)-gapped repeats in a word. Thus, Problem 1 can be reformulated in the following way.
**Problem 2.** For a given value $\delta$ find in a given word $w$ of length $n$ all principal maximal $1/\delta$-gapped repeats.

We obtain also that in any word the number of maximal $\delta$-subrepetitions is not greater than the number of maximal $1/\delta$-gapped repeats. Thus, Theorem 3 implies the following upper bound on the number of maximal $\delta$-subrepetitions in a word.

**Proposition 8.** The number of maximal $\delta$-subrepetitions in $w$ is $O(n/\delta)$.

![Figure 1: Principal and non principal repeats.](image)

Not also that for any principal overlapped or gapped repeats we have the following obvious fact.

**Proposition 9.** A maximal repeat $\sigma$ is principal if and only if the minimal period of $\text{fact}(\sigma)$ is equal to $p(\sigma)$.

Let $M$ be a set of maximal repeats in the word $w$. Note that a maximal repeat is uniquely defined by its starting position together with its period. So we can represent the set $M$ by lists $ML_t$ for $t = 1, 2, \ldots, n$ where $ML_t$ is the list containing all these repeats with the starting position $t$ in the order of increasing of their periods. Using the bucket sorting, all the lists $ML_t$ can be computed from the set $M$ in total $O(n + |M|)$ time. Moreover, all the lists $ML_t$ can be traversed in total $O(n + |M|)$ time. We will call the lists $ML_t$ start position lists for the set $M$.

### 2 Covering of repeats

We say that a maximal repeat $\sigma$ is covered by another maximal repeat $\sigma'$ if $\text{fact}(\sigma)$ is contained in $\text{fact}(\sigma')$ and $p(\sigma') < p(\sigma)$. Note that the introduced notion of covering of repeats satisfies the transitivity property: if some maximal repeat $\sigma'$ is covered by some maximal repeat $\sigma''$ and the maximal repeat $\sigma''$ is covered by some maximal repeat $\sigma'''$ then $\sigma'$ is covered by $\sigma'''$. The following auxiliary facts can be easily checked.

**Proposition 10.** A maximal $\alpha$-gapped repeat can be covered only by $\alpha$-gapped or overlapped repeats.
Proposition 11. If a maximal repeat $\sigma$ is covered by a maximal repeat $\sigma'$ then the left copy of $\sigma$ is contained in the left copy of $\sigma'$ and the right copy of $\sigma$ is contained in the right copy of $\sigma'$.

In analogous way, a maximal repeat $\sigma$ is covered by a maximal repetition $r$ if $\text{fact}(\sigma)$ is contained in $r$ and $p(r) < p(\sigma)$. Note that the repeat $\sigma$ is covered by $r$ if and only if $\sigma$ is covered by the principal repeat of $r$. Note also that any maximal overlapped repeat coincides as factor with some maximal repetition whose period is a divisor of the period of this repeat. So any maximal repeat covered by a maximal overlapped repeat is covered also by the maximal repetition coinciding with this overlapped repeat. Thus we have the following fact.

Proposition 12. Any maximal repeat covered by a maximal overlapped repeat is covered also by the principal repeat of some maximal repetition.

Let a maximal repeat $\sigma$ be covered by a maximal repeat $\sigma'$. Then the factor $\text{fact}(\sigma)$ has the period $p(\sigma') < p(\sigma)$, so, by Proposition 9, the repeat $\sigma$ is not principal. Hence principal repeats are not covered by other maximal repeats. Now let a maximal repeat $\sigma$ be not principal, i.e. $\sigma$ is contained in some repetition or subrepetition $r$ such that $p(r) < p(\sigma)$. In this case $\sigma$ is covered by the principal repeat of $r$. Therefore, if $\sigma$ is not covered by any other maximal repeat then $\sigma$ is principal. Thus, we obtain the following fact.

Proposition 13. A maximal repeat is principal if and only if it is not covered by any other maximal repeat.

Using Propositions 13 and 10, Problem 2 can be reformulated in the following way.

Problem 3. For a given value $\delta$ find in a given word $w$ of length $n$ all maximal $1/\delta$-gapped repeats which are not covered by other maximal repeats.

3 Periodic and generated repeats

Repeat $\sigma$ is called periodic if the copies of $\sigma$ are repetitions with minimal period not greater than $c(\sigma)/3$, otherwise $\sigma$ is called nonperiodic. Let $\sigma$ be a maximal periodic repeat with a left copy $u'$ and a right copy $u''$. Since $u'$ and $u''$ are repetitions, these repetitions can be extended to some maximal repetitions $r'$ and $r''$ with the same cyclic roots. Let $r'$ and $r''$ be different maximal repetitions. Then we say that $\sigma$ is represented by the pair of maximal repetitions $r', r''$ or, more briefly, $\sigma$ is birepresented. The pair of maximal repetitions $r', r''$ will be called left if $|r'| \leq |r''|$, otherwise it is called right. We will also call the repeat $\sigma$ left (right) birepresented if $\sigma$ is represented by a left (right) pair of maximal repetitions. If the maximal repetitions $r'$ and $r''$ are the same repetition $r$, we say that $\sigma$ is represented by the maximal repetition $r$.

A maximal repeat $\sigma$ is called to be generated by a maximal repetition $r$ if $\text{beg}(\sigma) = \text{beg}(r)$, $\text{end}(\sigma) = \text{end}(r)$, and $p(\sigma)$ is divisible by $p(r)$. For maximal periodic repeats represented by maximal repetitions we have the following fact.
Proposition 14. Any maximal periodic repeat represented by a maximal repetition is generated by this maximal repetition.

Proof. Let $\sigma$ be a maximal periodic repeat represented by a maximal repetition $r$, and $u'$, $u''$ be respectively the left and right copies of $\sigma$. Note that, since $\sigma$ is periodic, the length of $u'$ and $u''$ is greater than $p(r)$. Denote by $x'$, $x''$ the cyclic roots of $r$ which are prefixes of $u'$ and $u''$ respectively. Since $x' = x''$, by Proposition 3 we have $\text{beg}(x') \equiv \text{beg}(x'') \pmod{p(r)}$, so $\text{beg}(u') \equiv \text{beg}(u'') \pmod{p(r)}$. Thus, $p(\sigma) = \text{beg}(u'') - \text{beg}(u')$ is divisible by $p(r)$. Let $\text{beg}(\sigma) = \text{beg}(u') > \text{beg}(r)$. Then the both symbols $w[\text{beg}(u') - 1]$ and $w[\text{beg}(u'') - 1]$ are contained in $r$ and the difference between the positions of these symbols is divisible by $p(r)$. So $w[\text{beg}(u') - 1] = w[\text{beg}(u'') - 1]$ which contradicts that $\sigma$ is a maximal repeat. Thus, $\text{beg}(\sigma) = \text{beg}(r)$. In an analogous way, we have that $\text{end}(\sigma) = \text{end}(r)$. Thus, $\sigma$ is generated by $r$.

Note that any maximal repetition $r$ generates no more than $e(r)/2$ gapped repeats and, knowing values $\text{beg}(r)$, $\text{end}(r)$ and $p(r)$, we can compute all these repeats in $O(e(r))$ time. We can check each of these repeats in constant time if this repeat is $\alpha$-gapped. Thus, we can compute all maximal $\alpha$-gapped repeats generated by $r$ in $O(e(r))$ time. So we have the following simple procedure of computing of all maximal $\alpha$-gapped repeats generated by maximal repetitions in $w$. First we find all maximal repetitions in $w$. According to Theorem 1 it can be done in $O(n)$ time, and the total number of these repetitions is $O(n)$. Then for each found repetition $r$ we compute all maximal $\alpha$-gapped repeats generated by $r$ in $O(e(r))$ time. The total time of these procedure is $O(\sum_{r \in R(w)} e(r))$, so, by Theorem 2 this time is $O(n)$. Note also that, by Theorem 2 the number of computed repeats is $O(n)$. Thus, we obtain the following fact.

Proposition 15. The number of maximal $\alpha$-gapped repeats generated by maximal repetitions in $w$ is $O(n)$, and all these repeats can be computed in $O(n)$ time.

4 Birepresented gapped repeats

Let the maximal repeat $\sigma$ with left and right copies $u'$ and $u''$ be represented by a left pair of maximal repetitions $r', r''$ with the same cyclic roots, and $p$ be the minimal period of $r'$ and $r''$. Assume that $\text{beg}(u') > \text{beg}(r')$ and $\text{beg}(u'') > \text{beg}(r'')$. Then $w[\text{beg}(u') - 1]$ and $w[\text{beg}(u'') - 1]$ are contained in $r'$ and $r''$ respectively, so

$$u'[p] = w[\text{beg}(u') - 1 + p] = w[\text{beg}(u') - 1],$$
$$u''[p] = w[\text{beg}(u'') - 1 + p] = w[\text{beg}(u'') - 1].$$

Thus, from $u'[p] = u''[p]$ we have $w[\text{beg}(u') - 1] = w[\text{beg}(u'') - 1]$ which contradicts that $\sigma$ is maximal. Therefore, we have $\text{beg}(u') = \text{beg}(r')$ or $\text{beg}(u'') = \text{beg}(r'')$. Analogously, it can be shown that $\text{end}(u') = \text{end}(r')$ or $\text{end}(u'') = \text{end}(r'')$. Note that, since $|r'| \leq |r''|$, the case of $\text{beg}(u'') = \text{beg}(r'')$
and \( \text{end}(u'') = \text{end}(r'') \) implies that \( \text{beg}(u') = \text{beg}(r') \) and \( \text{end}(u') = \text{end}(r') \). Taking it into account, generally, we can consider separately the three following cases:

1. \( \text{beg}(u') > \text{beg}(r') \), \( \text{end}(u') = \text{end}(r') \) and \( \text{beg}(u'') = \text{beg}(r'') \);
2. \( \text{beg}(u') = \text{beg}(r') \), \( \text{end}(u') = \text{end}(r') \) and \( \text{end}(u'') < \text{end}(r'') \);
3. \( \text{beg}(u') = \text{beg}(r') \) and \( \text{end}(u'') = \text{end}(r'') \).

We will call the repeat \( \sigma \) repeat of first type in the case 1, repeat of second type in the case 2, and repeat of third type in the case 3.

First consider the case 1. Let \( \sigma' \), \( \sigma'' \) be two different maximal repeats of first type represented by the left pair of maximal repetitions \( r', r'' \). If \( \text{beg}(\sigma') = \text{beg}(\sigma'') \) then it is easy to note that \( \sigma' \) and \( \sigma'' \) are the same repeat, so \( \text{beg}(\sigma') \neq \text{beg}(\sigma'') \). Let \( x', x'' \) be the prefixes of length \( p \) in the left copies of repeats \( \sigma' \) and \( \sigma'' \) respectively. Note that \( x', x'' \) are cyclic roots of \( r' \) which are equal to the prefix cyclic root of \( r'' \), so \( x' = x'' \). Thus, by Proposition 3 we have that the difference \( \text{beg}(x') - \text{beg}(x'') = \text{beg}(\sigma') - \text{beg}(\sigma'') \) is divisible by \( p \). Moreover, by definition of repeats of first type, we have that both values \( \text{beg}(\sigma'), \text{beg}(\sigma'') \) are in the segment

\[
\text{beg}(r') < \text{beg}(\sigma'), \text{beg}(\sigma'') \leq \text{end}(r') - 3p + 1.
\]

Thus, the starting positions of all maximal repeats of first type represented by the pair \( r', r'' \) form in the segment \( [\text{beg}(r') + 1; \text{end}(r') - 3p + 1] \) an arithmetic progression of numbers with the step \( p \). Let \( t' \) be the maximal number in this progression and \( k' \) be the number of all maximal repeats of first type represented by the pair \( r', r'' \). Then we can consider the numbers \( t', t' - p, t' - 2p, \ldots, t' - (k' - 1)p \) of this progression in descending order as the starting positions of the corresponding repeats. In this way, we consider the set of all maximal repeats of first type represented by the pair \( r', r'' \) as \( \{\sigma'_1, \sigma'_2, \ldots, \sigma'_{k'}\} \) where \( \text{beg}(\sigma'_j) = t' - (j - 1)p \) for \( j = 1, 2, \ldots, k' \). Note that \( \text{beg}(\sigma'_j) = \text{beg}(\sigma'_1) - (j - 1)p \),\( p(\sigma'_j) = p(\sigma'_1) + (j - 1)p \) and \( \text{end}(\sigma'_j) = \text{end}(\sigma'_1) + (j - 1)p \) for \( j = 1, 2, \ldots, k' \).

Now consider the case 2. Let \( \sigma', \sigma'' \) be two different maximal repeats of second type represented by the left pair of maximal repetitions \( r', r'' \), and \( \hat{u}', \hat{u}'' \) be the right copies of repeats \( \sigma' \), \( \sigma'' \) respectively. If \( \text{beg}(\hat{u}') = \text{beg}(\hat{u}'') \) then \( \sigma' \) and \( \sigma'' \) are the same repeat, so \( \text{beg}(\hat{u}') \neq \text{beg}(\hat{u}'') \). Let \( x', x'' \) be the prefixes of length \( p \) in \( \sigma' \) and \( \sigma'' \) respectively. Note that \( x', x'' \) are cyclic roots of \( r' \) which are equal to the prefix cyclic root of \( r'' \), so \( x' = x'' \). Thus, by Proposition 3 we have that the difference \( \text{beg}(\hat{u}') - \text{beg}(\hat{u}'') \) is divisible by \( p \). Moreover, by definition of repeats of second type, we have that both values \( \text{beg}(\hat{u}'), \text{beg}(\hat{u}'') \) are in the segment

\[
\text{beg}(r'') \leq \text{beg}(\hat{u}'), \text{beg}(\hat{u}'') \leq \text{end}(r'') - |r'|.
\]

Thus, the starting positions of right copies of all maximal repeats of second type represented by the pair \( r', r'' \) form in the segment \( [\text{beg}(r''); \text{end}(r'') - |r'|] \) an arithmetic progression of numbers with the step \( p \). Let \( t'' \) be the minimal number in this progression and \( k'' \) be the number of all maximal repeats of second type represented by the pair \( r', r'' \). Then we can consider the numbers

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Proposition 3. Let \( x \) be the root of \( \sigma \) of \( \sigma \) copy of \( \sigma \) such that \( \sigma \), \( \sigma \) be the leftmost cyclic root to the right of \( x \). Consider \( \sigma', \sigma'' \) be two different maximal repeats of third type represented by the left pair of maximal repetitions \( r', r'' \), and \( \hat{u}', \hat{u}'' \) be the right copies of repeats \( \sigma', \sigma'' \) respectively. Analogously to case 2, it can be shown that \( \text{beg}(\hat{u}') \neq \text{beg}(\hat{u}'') \) and the difference \( \text{beg}(\hat{u}') - \text{beg}(\hat{u}'') \) is divisible by \( p \). Thus, the starting positions of right copies of all maximal repeats of third type represented by the pair \( r', r'' \) form an arithmetic progression of numbers with the step \( p \). Let \( t\) be the minimal number in this progression and \( k\) be the number of all maximal repeats of third type represented by the pair \( r', r'' \). Then we can consider the numbers \( t + p, t + 2p, \ldots, t + (k - 1)p \) of this progression in ascending order as the starting positions of the right copies of the corresponding repeats.

In this way, we consider the set of all maximal repeats of second type represented by the pair \( r', r'' \) as \( \{\sigma'_1, \sigma'_2, \ldots, \sigma'_{k'}\} \) where the starting position of the right copy of \( \sigma_j'' \) is \( t'' + (j - 1)p \) for \( j = 1, 2, \ldots, k'' \). Note that \( \text{beg}(\sigma_j'') \) = \( \text{beg}(r') \), \( p(\sigma_j'') = p(\sigma_j') + (j - 1)p \) and \( \text{end}(\sigma_j'') = \text{end}(\sigma_j') + (j - 1)p \) for \( j = 1, 2, \ldots, k'' \).

Finally, consider the case 3. Let \( \sigma', \sigma'' \) be two different maximal repeats of third type represented by the left pair of maximal repetitions \( r', r'' \), and \( \hat{u}', \hat{u}'' \) be the right copies of repeats \( \sigma', \sigma'' \) respectively. Analogously to case 2, it can be shown that \( \text{beg}(\hat{u}') \neq \text{beg}(\hat{u}'') \) and the difference \( \text{beg}(\hat{u}') - \text{beg}(\hat{u}'') \) is divisible by \( p \). Thus, the starting positions of right copies of all maximal repeats of third type represented by the pair \( r', r'' \) form in the segment \( [\text{end}(r'') - |r'| + 1, \text{end}(r'') - 3p + 1] \) an arithmetic progression of numbers with the step \( p \).

Let \( t'' \) be the minimal number in this progression and \( k'' \) be the number of all maximal repeats of third type represented by the pair \( r', r'' \). Then we can consider the numbers \( t'' + p, t'' + 2p, \ldots, t'' + (k'' - 1)p \) of this progression in ascending order as the starting positions of the right copies of the corresponding repeats.

In this way, we consider the set of all maximal repeats of third type represented by the pair \( r', r'' \) as \( \{\sigma''_1, \sigma''_2, \ldots, \sigma''_{k''}\} \) where the starting position of the right copy of \( \sigma''_j \) is \( t'' + (j - 1)p \) for \( j = 1, 2, \ldots, k'' \). Note that \( \text{beg}(\sigma''_j) = \text{beg}(r') \), \( p(\sigma''_j) = p(\sigma''_{j'}) + (j - 1)p \) and \( \text{end}(\sigma''_j) = \text{end}(r'') \) for \( j = 1, 2, \ldots, k'' \). The repeat \( \sigma''_j \) will be called dominating, and other repeats \( \{\sigma''_2, \ldots, \sigma''_{k''}\} \) will be called nondominating.

Consider additionally the repeats \( \sigma''_{j'}, \sigma''_{j''} \). Let \( u'_0 \) and \( u''_0 \) be left and right copies of \( \sigma'_{j'}, \sigma''_{j''} \) and \( u'_1 \) and \( u''_1 \) be left and right copies of \( \sigma''_{j''} \). Consider in \( r' \) the prefix \( x' \) of length \( p \) which is a cyclic root of \( r' \). Since \( x' \) is a prefix of \( u'_1 \), we have also in \( u''_1 \) the prefix \( x'' \) of length \( p \) which is a cyclic root of \( r'' \) and is equal to \( x' \). Note that \( x'' \) is a factor of \( u''_0 \), so we can also consider in \( u'_0 \) the factor \( x''' \) corresponding to \( x'' \). Note also that \( x''' \) is a cyclic root of \( r' \) such that \( x''' = x' \). Moreover, \( x''' \) has to be in \( r' \) the leftmost cyclic root to the right of \( x' \), so, by Proposition 3, \( \text{beg}(x''') = \text{beg}(x') + p \). We have also that \( \text{beg}(x''') = \text{beg}(x') + p(\sigma''_1) \) and \( \text{beg}(x''') = \text{beg}(x') - p(\sigma'_{j'}') \), so \( \text{beg}(x''') = \text{beg}(x') + p(\sigma''_1) - p(\sigma'_{j'}') \). Thus, \( p(\sigma''_1) - p(\sigma'_{j'}') = p \), and

\[
\text{end}(\sigma''_1) = \text{end}(r') + p(\sigma''_1) = \text{end}(r') + p(\sigma''_1) + p = \text{end}(\sigma'_{j'}) + p.
\]

Consider also the repeats \( \sigma'_{j''}, \sigma''_{j''} \). Let \( u'_{0} \) and \( u''_{0} \) be left and right copies of \( \sigma'_{j''}, \sigma''_{j''} \) and \( u'_1 \) and \( u''_1 \) be left and right copies of \( \sigma''_{j''} \). Consider in \( r' \) the prefix \( x' \) of length \( p \) which is a cyclic root of \( r' \). Since \( x' \) is a prefix of \( u'_{0} \), we have in \( u''_1 \) the prefix \( x'' \) of length \( p \) which is a cyclic root of \( r'' \) and is equal to \( x' \). Since \( x' \) is a prefix of \( u'_{1} \), we have also in \( u''_1 \) the prefix \( x''' \) of length \( p \) which is a cyclic root of \( r'' \) and is equal to \( x' \). Thus, \( x'' \) and \( x''' \) are two equal cyclic roots of \( r'' \). Note that \( x''' \) has to be in \( r' \) the leftmost cyclic root to the right of \( x' \), so, by Proposition 3, \( \text{beg}(x'''') = \text{beg}(x'') + p \). Therefore,

\[
\begin{align*}
p(\sigma''_{j''}) &= \text{beg}(x''') - \text{beg}(u'_1) = \text{beg}(x''') - \text{beg}(u'_{0}) = \text{beg}(x''') + p - \text{beg}(u'_{0}) \\
&= p(\sigma''_{j''}) + p.
\end{align*}
\]
Now we can join all the repeats represented by the pair of repetitions \( r', r'' \) into the sequence of repeats \( \Psi = \sigma_1, \sigma_2, \ldots, \sigma_{k' + k'' + k'''} \) where the repeats of first type, the repeats of second type and the repeats of third type are inserted consecutively, i.e. \( \sigma_j = \sigma'_j \) for \( j = 1, 2, \ldots, k' \), \( \sigma_{k' + j} = \sigma''_j \) for \( j = 1, 2, \ldots, k'' \), and \( \sigma_{k' + k'' + j} = \sigma'''_j \) for \( j = 1, 2, \ldots, k''' \). From the above observations we have that \( p(\hat{\sigma}_{j+1}) = p(\hat{\sigma}_j) + p \) for \( j = 1, 2, \ldots, k' + k'' + k''' - 1 \), \( \text{beg}(\hat{\sigma}_{j+1}) < \text{beg}(\hat{\sigma}_j) \) for \( j = 1, 2, \ldots, k' \), \( \text{beg}(\hat{\sigma}_j) = \text{beg}(r') \) for \( j = k' + 1, k' + 2, \ldots, k' + k'' + k''' \), \( \text{end}(\hat{\sigma}_{j+1}) > \text{end}(\hat{\sigma}_j) \) for \( j = 1, 2, \ldots, k' + k'' \), and \( \text{end}(\hat{\sigma}_j) = \text{end}(r'') \) for \( j = k' + k'' + 1, k' + k'' + 2, \ldots, k' + k'' + k''' \). Note also that \( \text{end}(r''') = \text{end}(\hat{\sigma}_{k' + k'' + 1}) \leq \text{end}(\hat{\sigma}_{k' + k''}) + p \). Since \( p(\hat{\sigma}_j') < p(\hat{\sigma}_j'') \) for \( j' < j'' \), any repeat \( \hat{\sigma}_j \) can not be covered by repeats from \( \Psi \) with greater indexes. Moreover, since \( |\hat{\sigma}_j'\sigma_j''| < |\hat{\sigma}_j''| \) for \( j' < j'' < k' + k'' + 1 \), any repeat \( \hat{\sigma}_j \) for \( j < k' + k'' + 1 \) can not be covered by repeats from \( \Psi \) with smaller indexes. Thus, any repeat \( \hat{\sigma}_j \) for \( j < k' + k'' + 1 \) can not be covered by other repeats from \( \Psi \). On the other hand, all repeats \( \hat{\sigma}_j \) for \( j > k' + k'' + 1 \) which are actually nondominating repeats of third type are covered by the repeat \( \hat{\sigma}_{k' + k'' + 1} \) which is actually the dominating repeat of third type. Thus we have the following fact.

**Proposition 16.** A maximal repeat \( \sigma \) represented by a left pair of maximal repetitions is covered by another repeat represented by the same pair of repetitions if and only if \( \sigma \) is a nondominating repeat of third type.

From the left pair of maximal repetitions \( r', r'' \) one can compute effectively all maximal \( \alpha \)-gapped periodic repeats represented by the pair \( r', r'' \).

**Lemma 3.** Let \( r', r'' \) be a left pair of maximal repetitions with the same cyclic roots such that \( p(r') = p(r''), \text{beg}(r'), \text{beg}(r''), \text{end}(r'), \text{end}(r''), a(r'), a(r'') \) be known. Then all maximal \( \alpha \)-gapped periodic repeats represented by the pair \( r', r'' \) can be computed in time \( O(1 + s) \) where \( s \) is the number of the maximal \( \alpha \)-gapped periodic repeats represented by the pair \( r', r'' \).

**Proof.** First we compute in constant time sequence \( \Psi \) of repeats where by computing of \( \Psi \) we mean computing of formulas by which any repeat \( \hat{\sigma}_j \) from \( \Psi \) can be computed in constant time. Since any maximal repeat \( \sigma \) is defined uniquely by the values \( p(\sigma), \text{beg}(\sigma) \) and \( \text{end}(\sigma) \), we will compute actually for any repeat \( \hat{\sigma}_j \) from \( \Psi \) the values \( p(\hat{\sigma}_j), \text{beg}(\hat{\sigma}_j) \) and \( \text{end}(\hat{\sigma}_j) \).

Let \( p \) be the minimal period of \( r' \) and \( r'' \), and \( x'' \) be the prefix cyclic root of \( r'' \). Denote by \( f' \) the starting position of the leftmost cyclic root of \( r' \) which is equal to \( x'' \) and is not a prefix of \( r' \). Taking into account Proposition 3, it can be checked that

\[
f' = \begin{cases} 
\text{beg}(r') + a(r') - a(r'') & \text{if } a(r') > a(r''); \\
\text{beg}(r') + a(r') - a(r'') + p & \text{if } a(r') \leq a(r''). 
\end{cases}
\]

Note that \( f' \) has to be actually the starting position of the repeat \( \hat{\sigma}_{k'} \) from \( \Psi \), and in this case \( c(\hat{\sigma}_{k'}) = \text{end}(r') - f' + 1 \). Thus, if \( \text{end}(r') - f' + 1 < 3p \), we can conclude that \( k' = 0 \), otherwise \( k' > 0 \).
Let \( \text{end}(r') - f' + 1 \geq 3p \), i.e. \( k' > 0 \). Denote by \( f_1 \) the starting position of \( \sigma_1 \). Note that \( f_1 \) is the starting position of the rightmost cyclic root of \( r' \) which is equal to \( x'' \) and such that \( c(\sigma_1) = \text{end}(r') - f_1 + 1 \geq 3p \). Thus, taking into account Proposition \( \text{K} \) we have that \( f_1 \) is the greatest number such that \( f_1 - f' \) is divisible by \( p \) and \( f_1 \leq \text{end}(r') - 3p + 1 \). Note that \( f_1 \) can be computed in constant time. Then \( \text{beg}(\sigma_1) = f_1, \ p(\sigma_1) = \text{beg}(r'') - f_1 \) and \( \text{end}(\sigma_1) = \text{end}(r') + p(\sigma_1) \). Note also that

\[
f' = \text{beg}(\hat{\sigma}_k') = \text{beg}(\hat{\sigma}_1) - (k' - 1)p = f_1 - (k' - 1)p,
\]

so

\[
k' = \frac{f_1 - f'}{p} + 1, \tag{1}
\]

and for any \( j = 1, 2, \ldots, k' \) we have \( \text{beg}(\hat{\sigma}_j) = \text{beg}(\hat{\sigma}_1) - (j - 1)p, \ \text{end}(\hat{\sigma}_j) = \text{end}(\hat{\sigma}_1) + (j - 1)p, \ p(\hat{\sigma}_j) = p(\hat{\sigma}_1) + (j - 1)p \) and \( c(\hat{\sigma}_j) = c(\hat{\sigma}_1) + (j - 1)p \). Denote \( \hat{k} = k' + k'' \). As shown above, \( \hat{k} \) is the greatest number such that \( \text{end}(\hat{\sigma}_1) + p(\hat{k} - 1) < \text{end}(r'') \), so

\[
\hat{k} = \left\lfloor \frac{1}{p}(\text{end}(r'') - \text{end}(\hat{\sigma}_1) - 1) \right\rfloor + 1,
\]

and \( k'' = \hat{k} - k' \). If \( k'' > 0 \), from above observations for \( j = 1, 2, \ldots, k'' \) we have \( \text{beg}(\hat{\sigma}_{k' + j}) = \text{beg}(r'), \ \text{end}(\hat{\sigma}_{k' + j}) = \text{end}(\hat{\sigma}_{k'}) + jp, \ p(\hat{\sigma}_{k' + j}) = p(\hat{\sigma}_{k'}) + jp \) and \( c(\hat{\sigma}_{k' + j}) = |r'|. \) Moreover, the repeat \( \hat{\sigma}_{k + 1} \) from \( \Psi \) has to satisfy the conditions \( \text{beg}(\hat{\sigma}_{k + 1}) = \text{beg}(r'), \ \text{end}(\hat{\sigma}_{k + 1}) = \text{end}(r''), \ p(\hat{\sigma}_{k + 1}) = p(\hat{\sigma}_k) + p. \) Note that in this case

\[
c(\hat{\sigma}_{k' + 1}) = |\hat{\sigma}_{k' + 1}| - p(\hat{\sigma}_{k' + 1}) = \text{end}(r'') - \text{beg}(r') + 1 - p(\hat{\sigma}_k) - p.
\]

Thus, if

\[
\text{end}(r'') - \text{beg}(r') + 1 - p(\hat{\sigma}_k) - p < 3p,
\]

we conclude that \( k''' = 0 \), otherwise \( k''' > 0 \). Let \( k''' > 0 \). Note that \( k''' \) is the greatest number such that

\[
c(\hat{\sigma}_{k + k''}) = c(\hat{\sigma}_{k' + 1}) - p(k''' - 1) \geq 3p,
\]

i.e.

\[
\text{end}(r'') - \text{beg}(r') + 1 - p(\hat{\sigma}_k) - p k''' \geq 3p.
\]

Thus, \( k''' = \left\lfloor \frac{1}{p}(\text{end}(r'') - \text{beg}(r') + 1 - p(\hat{\sigma}_k)) \right\rfloor - 3 \), and for any \( j = 1, 2, \ldots, k''' \) we have \( \text{beg}(\hat{\sigma}_{k + j}) = \text{beg}(r'), \ \text{end}(\hat{\sigma}_{k + j}) = \text{end}(r''), \ p(\hat{\sigma}_{k + j}) = p(\hat{\sigma}_{k + 1}) + (j - 1)p \) and \( c(\hat{\sigma}_{k + j}) = c(\hat{\sigma}_{k' + 1}) - (j - 1)p. \)

Now consider the case \( k' = 0 \), i.e. \( \text{end}(r') - f' + 1 < 3p \) and the pair \( r', r'' \) represents no repeats of first type. Since all repeats of second or third type represented by the pair \( r', r'' \) must have starting position \( \text{beg}(r') \), the equality \( \text{beg}(\sigma_1) = \text{beg}(r') \) must be hold for \( \hat{\sigma}_1 \). Denote by \( \hat{\sigma}_0 \) the repeat with the starting position \( f' \) and the period \( \text{beg}(r'') - f' \). We proved above that
Therefore, in this case all repeats from $\Psi$ are gapped repeats. Further we note that $c(\sigma_1) = \min(|r'|, |r'| + f' - \beta(r') - p)$. Thus, we have $\beta(\sigma_j) = \beta(\sigma_j + j - 1)p$, $p(\sigma_j + j - 1)p < 3p$, $f' \leq \beta(r') + p$, and $|r'| - p < 3p$. Note also that $c(\sigma_{k+2})$ has to be equal to $c(\sigma_{k+1}) - p \leq |r'| - p$, so $c(\sigma_{k+2})$ has to be less than $3p$. Thus, we have $k'' \leq 1$. First consider the case $|r'| + f' - \beta(r') - p$, i.e. $\sigma_1$ is a repeat of second type, so $end(\sigma_1) = end(r') + p(\sigma_1)$. Then $k''$ can be computed in constant time as the greatest number such that $end(\sigma_1) + (k'' - 1)p < end(r')$, and for any $j = 1, 2, \ldots, k''$ we have $beg(\sigma_j) = beg(r')$, $end(\sigma_j) = end(\sigma_1) + (j - 1)p$, $p(\sigma_j) = p(\sigma_1) + (j - 1)p$ and $c(\sigma_j) = |r'|$. Moreover, we obtain that $\sigma_{k''+1}$ must satisfy the conditions $beg(\sigma_{k''+1}) = beg(r')$, $end(\sigma_{k''+1}) = end(r')$ and $p(\sigma_{k''+1}) = c(\sigma_{k''+1}) = end(r') - beg(r') + 1 - p(\sigma_{k''+1}) - p$.

Thus, if $end(r') - beg(r') + 1 - p(\sigma_{k''+1}) - p < 3p$, we conclude that $k'' = 0$. Otherwise, taking into account $k'' \leq 1$, we obtain that $k'' = 1$ and $beg(\sigma_{k''+1}) = beg(r')$, $end(\sigma_{k''+1}) = end(r')$ and $p(\sigma_{k''+1}) = p(\sigma_{k''}) + p$. Finally consider the case $|r'| > |r'| + f' - \beta(r') - p$, i.e. $k'' = 0$ and $\sigma_1$ is a repeat of third type, so $end(\sigma_1) = end(r')$. Taking into account $k'' \leq 1$, we obtain in this case that $\sigma_1$ is a unique repeat in $\Psi$. Thus, in any case $\Psi$ can be computed in constant time.

Now we need to select from $\Psi$ all $\alpha$-gapped repeats, i.e. all gapped repeats $\hat{\sigma}$ such that $\frac{\|\hat{\sigma}\|}{\|\sigma\|} \leq \alpha$. If $\hat{k} = 0$, i.e. $k' = k'' = 0$, as shown above, we have a trivial case when $\Psi$ contains no more than one repeat. So farther we assume that $\hat{k} > 0$. First we consider the case when $\hat{\sigma}_1$ is a gapped repeat. Note that for any repeats $\hat{\sigma}_j$ and $\hat{\sigma}_{j+1}$ from $\Psi$ the ending position of the left copy of $\hat{\sigma}_{j+1}$ is not greater than the ending position of the left copy of $\hat{\sigma}_j$ and the starting position of the right copy of $\hat{\sigma}_{j+1}$ is not less than the right copy of $\hat{\sigma}_j$. Thus, if $\hat{\sigma}_j$ is a gapped repeat then all repeats $\hat{\sigma}_l$ for $l > j$ are also gapped repeats. Therefore, in this case all repeats from $\Psi$ are gapped repeats. Further we note that for any $j$ such that $k' < j < k' + k'' + k'''$ we have $c(\hat{\sigma}_j) \geq c(\hat{\sigma}_{j+1})$ and $p(\hat{\sigma}_j) < p(\hat{\sigma}_{j+1})$, so

$$
p(\hat{\sigma}_{j+1}) = p(\hat{\sigma}_j) + p = \frac{p(\hat{\sigma}_j) + p}{c(\hat{\sigma}_j)} < \frac{p(\hat{\sigma}_{j+1})}{c(\hat{\sigma}_{j+1})} = \frac{\beta(\sigma_{k''+1})}{c(\sigma_{k''+1})} < \frac{\beta(\sigma_{k''+1} + k'' + k''')}{c(\sigma_{k''+1} + k'' + k''')}. \tag{2}
$$

Now consider a repeat $\hat{\sigma}_j$ from $\Psi$ such that $1 \leq j < k'$. Since $\hat{\sigma}_j$ is a gapped repeat, we have $c(\sigma_j) < c(\sigma_j)$ and, as shown above, $c(\sigma_{j+1}) = c(\sigma_j) + p$, $p(\sigma_{j+1}) = p(\sigma_j) + p$. Hence

$$
p(\sigma_{j+1}) = p(\sigma_j) + p = \frac{p(\sigma_j) + p}{c(\sigma_j)} < \frac{p(\sigma_{j+1})}{c(\sigma_{j+1})} = \frac{\beta(\sigma_{k''+1})}{c(\sigma_{k''+1})} < \frac{\beta(\sigma_{k''+1} + k'' + k''')}{c(\sigma_{k''+1} + k'' + k''')}. \tag{3}
$$

Thus, we have $\frac{p(\sigma_1)}{c(\sigma_1)} > \frac{p(\sigma_2)}{c(\sigma_2)} > \cdots > \frac{p(\sigma_{k'})}{c(\sigma_{k'})}$. 

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From inequalities (2) and (3) we conclude that all the α-gapped repeats from Ψ have to form a continuous segment $\hat{\sigma}_1, \hat{\sigma}_{l+1}, \ldots, \hat{\sigma}_m$ in Ψ. Thus, for efficient finding of all these repeats we need to compute the indexes $l$ and $m$.

First we compute $l$. Let $k' > 0$ and $\frac{p(\hat{\sigma}_1)}{\alpha c(\hat{\sigma}_1)} \leq \alpha$. Then we obviously obtain $l = 1$.

Now let $k' > 0$ and $\frac{p(\hat{\sigma}_1)}{\alpha c(\hat{\sigma}_1)} > \alpha \geq \frac{p(\hat{\sigma}_k)}{\alpha c(\hat{\sigma}_k)}$. Then, taking into account (3), we have that $l$ is smallest number such that $\frac{p(\hat{\sigma}_1)}{\alpha c(\hat{\sigma}_1)} \leq \alpha$. Since this number can not be greater than $k'$, as shown above, we have also $p(\hat{\sigma}_1) = p(\hat{\sigma}_1) + p(l - 1)$ and $c(\hat{\sigma}_1) = c(\hat{\sigma}_1) + p(l - 1)$. Thus, $l$ is smallest number such that $\frac{p(\hat{\sigma}_1) + p(l - 1)}{\alpha c(\hat{\sigma}_1) + p(l - 1)} \leq \alpha$.

It can be easily computed that

$$l = 1 + \left\lfloor \frac{p(\hat{\sigma}_1) - \alpha c(\hat{\sigma}_1)}{p(\alpha - 1)} \right\rfloor.$$ 

Now let $k' = 0$ or $\alpha < \frac{p(\hat{\sigma}_k)}{\alpha c(\hat{\sigma}_k)}$. In this case, if there exists the repeat $\hat{\sigma}_{k'+1}$ in Ψ and $\frac{p(\hat{\sigma}_{k'+1})}{c(\hat{\sigma}_{k'+1})} \leq \alpha$ then we have $l = k' + 1$, otherwise from (2) we obtain that there are no α-gapped repeats in Ψ.

Now we compute $m$. If $\frac{p(\hat{\sigma}_{k+m'})}{c(\hat{\sigma}_{k+m'})} \leq \alpha$ then we obviously have $m = \hat{k} + k''$. So farther we assume that $\frac{p(\hat{\sigma}_{k+m'})}{c(\hat{\sigma}_{k+m'})} > \alpha$. Let $k'' > 0$ and $\frac{p(\hat{\sigma}_{k+m'})}{c(\hat{\sigma}_{k+m'})} \leq \alpha < \frac{p(\hat{\sigma}_{k+m'})}{c(\hat{\sigma}_{k+m'})}$. Then, taking into account (2), we have that $m$ is equal to the greatest number $\hat{k} + m'$ for $k'' > m' \geq 1$ such that $\frac{p(\hat{\sigma}_{k+m'})}{c(\hat{\sigma}_{k+m'})} \leq \alpha$. As shown above, we have also $p(\hat{\sigma}_{k+m'}) = p(\hat{\sigma}_{k+1}) + (m' - 1)p$ and $c(\hat{\sigma}_{k+m'}) = c(\hat{\sigma}_{k+1}) - (m' - 1)p$. Thus, $m$ is equal to the greatest number $\hat{k} + m'$ for $k'' > m' \geq 1$ such that $\frac{p(\hat{\sigma}_{k+1}) + (m' - 1)p}{\alpha c(\hat{\sigma}_{k+1}) - (m' - 1)p} \leq \alpha$. It can be easily computed that

$$m = \hat{k} + 1 + \left\lfloor \frac{\alpha c(\hat{\sigma}_{k+1}) - p(\hat{\sigma}_{k+1})}{p(\alpha - 1)} \right\rfloor.$$ 

Now let $k'' > 0$ and $\frac{p(\hat{\sigma}_k)}{c(\hat{\sigma}_k)} \leq \alpha < \frac{p(\hat{\sigma}_{k+1})}{c(\hat{\sigma}_{k+1})}$. Then we obviously have that $m = \hat{k}$.

Now we can assume that $\alpha < \frac{p(\hat{\sigma}_k)}{c(\hat{\sigma}_k)}$ and, if $k'' > 0$, $\alpha < \frac{p(\hat{\sigma}_{k+1})}{c(\hat{\sigma}_{k+1})}$. Let $k'' > 0$ and $\frac{p(\hat{\sigma}_{k+m'})}{c(\hat{\sigma}_{k+m'})} \leq \alpha < \frac{p(\hat{\sigma}_k)}{c(\hat{\sigma}_k)}$. Then, taking into account (2), we have that $m$ is equal to the greatest number $k' + m''$ for $k'' > m'' \geq 1$ such that $\frac{p(\hat{\sigma}_{k'+m''})}{c(\hat{\sigma}_{k'+m''})} \leq \alpha$. From above observations we obtain that $p(\hat{\sigma}_{k'+m''}) = p(\hat{\sigma}_{k+1}) + (m'' - 1)p$ and $c(\hat{\sigma}_{k'+m''}) = \lfloor r' \rfloor$. Thus, $m$ is equal to the greatest number $k' + m''$ for $k'' > m'' \geq 1$ such that $\frac{p(\hat{\sigma}_{k+1}) + (m'' - 1)p}{\lfloor r' \rfloor} \leq \alpha$. It can be easily computed that

$$m = k' + 1 + \left\lfloor \frac{\alpha \lfloor r' \rfloor - p(\hat{\sigma}_{k+1})}{p} \right\rfloor.$$
Now we assume additionally that, if \( k'' > 0, \alpha < \frac{p(\hat{\sigma}_{k''+1})}{c(\hat{\sigma}_{k''+1})} \) In this case, if \( k' > 0 \) and \( \alpha \geq \frac{p(\hat{\sigma}_j)}{c(\hat{\sigma}_j)} \) then \( m = k' \), otherwise we can conclude that \( \Psi \) has no \( \alpha \)-gapped repeats. Thus, \( l \) and \( m \) can be computed in constant time, so all \( \alpha \)-gapped repeats in \( \Psi \) can be computed in the required time.

Finally, we consider the case when \( \hat{\sigma}_1 \) is an overlapped repeat. Denote for any \( j = 1, 2, \ldots, k' + k \) by \( \hat{u}_j' \) and \( \hat{u}_j'' \) the left and right copies of the repeat \( \hat{\sigma}_j \).

Let \( k' > 0 \), i.e. \( \hat{\sigma}_1 \) is a repeat of first type. Note that for any repeat \( \hat{\sigma}_j \) of first type we have \( \text{end}(\hat{u}_j') = \text{end}(\hat{u}_1') \) and \( \text{beg}(\hat{u}_j'') = \text{beg}(\hat{u}_1'') \). Therefore, if \( \hat{\sigma}_1 \) is an overlapped repeat, then any repeat \( \hat{\sigma}_j \) of first type is also an overlapped repeat.

Thus, in the considered case we obtain that all repeats of first type from \( \Psi \) are overlapped repeats, i.e. only repeats of second or third types from \( \Psi \) can be gapped repeats. Now consider in \( \Psi \) a repeat \( \hat{\sigma}_j \) such that \( j \geq k' + 2 \). As shown above, we have

\[
\text{beg}(\hat{u}_j') \geq \text{beg}(\hat{u}_{k'+2}'') = \text{beg}(\hat{u}_{k'+1}'') + p \geq \text{beg}(r''') + p.
\]

On the other hand, we have \( \text{end}(\hat{u}_j') \leq \text{end}(r') \). By Proposition 5 the overlap of repetitions \( r' \) and \( r''' \) is less than \( p \), so \( \text{end}(r') < \text{beg}(r''') + (p - 1) \). Thus, we obtain that \( \text{beg}(\hat{u}_j'') \geq \text{beg}(r''') + p \) and \( \text{end}(\hat{u}_j'') < \text{beg}(r''') + (p - 1) \), so \( \hat{\sigma}_j \) is a gapped repeats. Hence all repeats \( \hat{\sigma}_j \) from \( \Psi \) such that \( j \geq k' + 2 \) are gapped repeats. Thus, all gapped repeats from \( \Psi \) are repeats \( \hat{\sigma}_1, \hat{\sigma}_{k' + l}, \ldots, \hat{\sigma}_{k' + k} \) where \( l = k' + 1 \) if \( \hat{\sigma}_{k' + 1} \) is a gapped repeat and \( l = k' + 2 \) otherwise. Since one can check in constant time if the repeat \( \hat{\sigma}_{k' + 1} \) is gapped, the value \( l \) can be computed in constant time. Taking into account inequalities (2), we conclude that

\[
\frac{p(\hat{\sigma}_1)}{c(\hat{\sigma}_1)} < \frac{p(\hat{\sigma}_{k'+1})}{c(\hat{\sigma}_{k'+1})} < \ldots < \frac{p(\hat{\sigma}_{k'+k})}{c(\hat{\sigma}_{k'+k})}.
\]

Thus, if \( \frac{p(\hat{\sigma}_j)}{c(\hat{\sigma}_j)} > \alpha \) then there are no \( \alpha \)-gapped repeats in \( \Psi \); otherwise all \( \alpha \)-gapped repeats in \( \Psi \) form a continuous segment \( \hat{\sigma}_1, \hat{\sigma}_{k'+1}, \ldots, \hat{\sigma}_{m} \) where the index \( m \) can be computed in constant time analogously the considered above case when \( \hat{\sigma}_1 \) is a gapped repeat. Therefore, all \( \alpha \)-gapped repeats in \( \Psi \) can be computed in the required time in any case.

**Lemma 4.** Let \( r', r'' \) be a left pair of maximal repetitions with the same cyclic roots such that \( p(r') = p(r''), \text{beg}(r'), \text{beg}(r''), \text{end}(r'), \text{end}(r''), a(r'), a(r'') \) be known. Then all maximal \( \alpha \)-gapped nondominating repeats of third type represented by the pair \( r', r'' \) can be computed in time \( O(1 + s) \) where \( s \) is the number of the maximal \( \alpha \)-gapped nondominating repeats of third type represented by the pair \( r', r'' \).

**Proof.** According to the proof of the previous lemma, all maximal \( \alpha \)-gapped periodic repeats represented by the pair \( r', r'' \) form a continuous segment \( \hat{\sigma}_l, \hat{\sigma}_{l+1}, \ldots, \hat{\sigma}_m \) in \( \Psi \) where the indexes \( l, m \) can be computed in constant time. Note that a repeat \( \hat{\sigma}_j \) from \( \Psi \) is a nondominating repeat of third type if and only if \( j > k' + k'' + 1 \). Thus, in order to find all maximal \( \alpha \)-gapped nondominating repeats of third type represented by the pair \( r', r'' \), one needs to select from
For each maximal repetition \( r \), obviously in time \( O(1 + s) \) where \( s \) is the number of the selected repeats. □

Let \( r', r'' \) be a left pair of maximal repetitions representing some \( \alpha \)-gapped repeat \( \sigma \) with left and right copies \( u' \) and \( u'' \) and gap \( v \). Note that the distance \( \text{beg}(r'') - (\text{end}(r') + 1) \) between the repetitions \( r' \) and \( r'' \) can not be greater than the gap length \(|v|\) which is not greater that \((\alpha - 1)|u'| \leq (\alpha - 1)|r'| \). Thus, 
\[
\text{beg}(r'') - (\text{end}(r') + 1) \leq (\alpha - 1)|r'|, \\
\text{beg}(r'') - \text{beg}(r') = (\text{beg}(r'') - (\text{end}(r') + 1)) + |r'| \leq (\alpha - 1)|r'| + |r'| = \alpha|r'|. 
\]

If \( \text{beg}(r'') - \text{beg}(r') \leq \alpha|r'| \), we will say that the repetition \( r'' \) is \( \alpha \)-close to the repetition \( r' \) from the right. Thus, we obtain the following fact.

**Proposition 17.** If a left pair \( r', r'' \) of maximal repetitions represents at least one maximal \( \alpha \)-gapped repeat then \( r'' \) is \( \alpha \)-close to \( r' \) from the right.

We use also the following proposition.

**Proposition 18.** Any maximal repetition \( r' \) has no more than \( 2\alpha \) maximal repetitions which are \( \alpha \)-close to \( r' \) from the right.

**Proof.** Let \( r''_1, r''_2, \ldots, r''_s \) be all maximal repetitions which are \( \alpha \)-close to \( r' \) from the right in the increasing order of their starting positions:
\[
\text{beg}(r') < \text{beg}(r''_1) < \text{beg}(r''_2) < \ldots < \text{beg}(r''_s).
\]
For convenience denote \( r' \) by \( r'_0 \) and consider two consecutive repeats \( r''_j \) and \( r''_{j+1} \) for \( j = 0, 1, \ldots, s - 1 \). Since \( p(r''_j) = p(r''_{j+1}) = p(r') \), by Proposition 5, the overlap of \( r''_j \) and \( r''_{j+1} \) is less than \( p(r') \), so
\[
\text{beg}(r''_{j+1}) > \text{beg}(r''_j) + (|r''_j| - p(r')) \geq \text{beg}(r''_j) + (|r'| - p(r')) \geq \text{beg}(r''_j) + |r'|/2.
\]
Thus, we obtain that
\[
\text{beg}(r''_0) + s|r'|/2 < \text{beg}(r''_s) \leq \text{beg}(r') + \alpha|r'| = \text{beg}(r''_0) + \alpha|r'|,
\]
so \( s|r'|/2 < \alpha|r'| \), i.e. \( s < 2\alpha \).

Farther by computing of a periodic repeat we will mean that the minimal period of copies of this repeat is additionally computed.

**Lemma 5.** All left birepresented maximal \( \alpha \)-gapped periodic repeats in \( w \) can be computed in time \( O(n\alpha) \).

**Proof.** First we find all maximal repetitions in \( w \). According to Theorem 11 it can be done in \( O(n) \) time, and the total number of these repetitions is \( O(n) \). For each maximal repetition \( r \) in \( w \) we compute the values \( p(r) \), \( \text{beg}(r) \), \( \text{end}(r) \), \( a(r) \). Moreover, we divide all maximal repetitions in \( w \) into the subsets of all repetitions with the same cyclic roots, and represent each of these subsets as a double-linked list \( PRS R_j \) of the subset repetitions in the order of increasing of their starting positions. According to [17], it can be done in \( O(n) \) time.
We also rearrange repetitions in each list $PRS R_j$ in the order of nondecreasing of their lengths (the repetitions of the same length are arranged in the order of increasing of their starting positions). The rearranged list $PRS R_j$ will be denoted by $LRS R_j$. Using the bucket sorting, all the lists $LRS R_j$ can be computed from the lists $PRS R_j$ in total $O(n)$ time.

For each $j$ we compute separately all maximal $\alpha$-gapped periodic repeats represented by left pairs of repetitions from $PRS R_j$. The computation is performed as follows. We consider consecutively all repetitions from $LRS R_j$. For each repetition $r'$ from $LRS R_j$ we compute all maximal $\alpha$-gapped periodic repeats represented by left pairs $r', r''$ of repetitions where $r''$ is a repetition from $PRS R_j$. Before the computation we assume that all repetitions which precede the repetition $r'$ in the list $LRS R_j$ are already removed from the current list $PRS R_j$. Note that, in order to compute the required repeats for the repetition $r'$, we need to consider all the repetitions $r''$ from $PRS R_j$ such that the pair of maximal repetitions $r', r''$ is left and represents at least one maximal $\alpha$-gapped periodic repeat. According to Proposition 17 such repetitions $r''$ have to be $\alpha$-close to $r'$ from the right. Thus, we actually need to consider only repetitions $r''$ from $PRS R_j$ which are $\alpha$-close to $r'$ from the right. Note that, since the lengths of all these repetitions are not less than the length of $r'$ and the starting positions of all these repetitions are greater than the starting position of $r'$, all these repetitions follows to $r'$ in the list $LRS R_j$, so they are presented in the current list $PRS R_j$. Moreover, they follow to $r'$ in $PRS R_j$.

Recall that the current list $PRS R_j$ contains no repetitions of length less than the length of $r'$. So in the current list $PRS R_j$ between the repetition $r'$ and the repetitions which are $\alpha$-close to $r'$ from the right there are no any other repetitions. Thus, all the repetitions which are $\alpha$-close to $r'$ from the right form in the list $PRS R_j = (r'_1, r'_2, \ldots, r'_{\lambda})$ some continuous segment $(r''_1, r''_2, \ldots, r''_{\mu})$ which follows immediately to $r'$, i.e. $r' = r_{\lambda-1}$. Therefore, proceeding from the repetition $r'$, each of the repetitions which are $\alpha$-close to $r'$ from the right can be found in the current list $PRS R_j$ in constant time. After finding of each repetition $r''$ from these repetitions we compute all maximal $\alpha$-gapped periodic repeats represented by the left pair $r', r''$. According to Lemma 3, it can be done in time $O(1 + s)$ where $s$ is the number of computed repeats. The minimal periods of copies of all these computed repeats are defined as the minimal period of $r'$. Thus, the treating of each of the repetitions which are $\alpha$-close to $r'$ from the right can be done in time $O(1 + s)$. Since, according to Lemma 3 the number of maximal repetitions which are $\alpha$-close to $r'$ from the right is not greater than $2\alpha$, the total treating of all these repetitions can be done in time $O(\alpha + s')$ where $s'$ is the total number of computed repeats for all these repetitions. Then we remove the considered repetition $r'$ from the double-linked list $PRS R_j$. It can be done in constant time. Thus, the considering of the repetition $r'$ can be performed in time $O(\alpha + s')$. Therefore, since the number of all maximal repetitions is $O(n)$, the total time of considering of all maximal repetitions from all lists $LRS R_j$ is $O(n\alpha + s'')$ where $s''$ is the total number of computed repeats. Since each computed repeat is a maximal $\alpha$-gapped repeat, the number $s''$ is not greater than the total number of maximal $\alpha$-gapped repeats in $w$ which
is $O(n\alpha)$ by Theorem 3 so $s'' = O(n\alpha)$. Thus, the total time of considering of all maximal repetitions from all lists $LRSR_j$ (which is actually the time of computing all left birepresented maximal $\alpha$-gapped periodic repeats in $u$) is $s'' = O(n\alpha)$.

Analogously to proving of Lemma 6 from Lemma 4 we can also prove the following lemma.

**Lemma 6.** All left birepresented maximal $\alpha$-gapped nondominating repeats of third type in $w$ can be computed in time $O(n\alpha)$.

Now consider the case of periodic repeats represented by right pairs of maximal repetitions. Let the maximal repeat $\sigma$ be represented by the right pair of maximal repetitions $r', r''$ with the same cyclic roots and the minimal period $p$.

Analogously to the case of repeats represented by left pairs of repetitions, we can show that $\sigma$ can satisfy one of the three following cases:
1. $\text{end}(u') = \text{end}(r'), \text{beg}(u'') = \text{beg}(r'')$ and $\text{end}(u'') < \text{end}(r'')$;
2. $\text{beg}(u') > \text{beg}(r')$, $\text{beg}(u'') = \text{beg}(r'')$ and $\text{end}(u'') = \text{end}(r'')$;
3. $\text{beg}(u') = \text{beg}(r')$ and $\text{end}(u'') = \text{end}(r'')$.

We will call the repeat $\sigma$ repeat of first type in the case 1, repeat of second type in the case 2, and repeat of third type in the case 3.

Analogously to the case of repeats represented by left pairs of repetitions, we can also show that starting positions of the right copies of all maximal repeats of third type represented by the pair $r', r''$ form an arithmetic progression $t''', t'''+p, t'''+2p, \ldots, t'''+(k'''-1)p$ where $k'''$ is the number of all maximal repeats of third type represented by the pair $r', r''$. We will call the repeat of third type with the starting position $t'''$ of its right copy dominating repeat, all the other repeats of third type will be called nondominating. Analogously to Proposition 16 we can prove that a maximal repeat $\sigma$ represented by a right pair of maximal repetitions is covered by another repeat represented by the same pair of repetitions if and only if $\sigma$ is a nondominating repeat of third type. Taking it into account together with Proposition 16 we obtain the following fact.

**Corollary 1.** A maximal periodic repeat $\sigma$ represented by a pair of maximal repetitions is covered by another periodic repeat represented by the same pair of repetitions if and only if $\sigma$ is a nondominating repeat of third type.

Analogously to Lemmas 5 and 6 we can also prove the similar facts for repeats represented by right pairs of maximal repetitions.

**Lemma 7.** All right birepresented maximal $\alpha$-gapped periodic repeats in $w$ can be computed in time $O(n\alpha)$.

**Lemma 8.** All right birepresented maximal $\alpha$-gapped nondominating repeats of third type in $w$ can be computed in time $O(n\alpha)$.

From Lemmas 6 and 8 we obtain the following corollary.

**Corollary 2.** All birepresented maximal $\alpha$-gapped nondominating repeats of third type in $w$ can be computed in time $O(n\alpha)$. 
Lemma 9. All maximal $\alpha$-gapped periodic repeats in $w$ can be computed in time $O(n\alpha)$.

Proof. By Lemmas 5 and 7 we can find in time $O(n\alpha)$ all birepresented maximal $\alpha$-gapped periodic repeats in $w$, so for finding of all maximal $\alpha$-gapped periodic repeats in $w$ we need to compute additionally in $w$ all maximal $\alpha$-gapped periodic repeats represented by maximal repetitions. By Proposition 14 maximal $\alpha$-gapped periodic repeats represented by maximal repetitions are generated by these maximal repetitions. So for finding of all these repeats we can compute initially all maximal $\alpha$-gapped repeats generated by maximal repetitions in $w$. By Proposition 15 the number of such repeats is $O(n)$, and all these repeats can be computed in $O(n)$ time. Note that a maximal repeat $\sigma$ generated by a maximal repetition $r$ is a periodic repeat represented by $r$ if and only if $p(r) \leq c(\sigma)/3$. Moreover, in this case $p(r)$ is the minimal period of copies of $\sigma$. So for each of the computed repeats we can check in constant time if this repeat is a periodic repeat represented by a maximal repetition and can define in this case the minimal period of copies of this repeat. Thus, all maximal $\alpha$-gapped repeats generated by maximal repetitions in $w$ can be computed in $O(n)$ time, so the total time of computing of all maximal $\alpha$-gapped repeats generated by maximal repetitions in $w$ is $O(n\alpha)$. \qed

5 $\alpha$-periodic repeats

Repeat $\sigma$ is called $\alpha$-periodic if the minimal period of copies of $\sigma$ is not greater than $\frac{2c(\sigma)}{3\alpha}$, otherwise $\sigma$ is called $\alpha$-nonperiodic. Note that for any $\alpha$-periodic maximal repeat $\sigma$ which is $\alpha$-gapped or overlapped we have $c(\sigma) \geq p(\sigma)/\alpha \geq 3p$ where $p$ is the minimal period of copies of $\sigma$. So any $\alpha$-periodic maximal $\alpha$-gapped or overlapped repeat is a periodic repeat. Thus, all $\alpha$-periodic maximal $\alpha$-gapped or overlapped repeats can be classified as repeats represented by single maximal repetitions or repeats represented by pairs of maximal repetitions.

Proposition 19. Let $r$ be a maximal repetition such that $e(r) \geq 3$. Then the principal repeat of $r$ is $\alpha$-nonperiodic.

Proof. Let $\sigma$ be the principal repeat of $r$. Assume that $\sigma$ is $\alpha$-periodic, i.e. the copies of $\sigma$ are repetitions with minimal period $p'$ not greater than $\frac{2c(\sigma)}{3\alpha} = \frac{p(\sigma)}{3\alpha}$, so $p' < p(r)/3 < p(r)$. Since $e(r) \geq 3$, i.e. $|r| \geq 3p(r)$, we have that the length $c(\sigma)$ of copies of $\sigma$ is not less than $2p(r) > p(r) + p'$. Therefore, since the both $p(r)$ and $p'$ are periods of copies of $\sigma$, by periodicity lemma we obtain that $\gcd(p(r), p')$ is also a period of copies of $\sigma$. Since $p'$ is the minimal period of copies of $\sigma$, we conclude that $p' = \gcd(p(r), p')$, i.e. $p'$ is a divisor of $p(r)$. In this case we obtain that $p'$ is also a period of $r$ which contradicts that $p(r)$ is the minimal period of $r$. \qed

Let $\sigma$ be a $\alpha$-periodic maximal repeat represented by some maximal repetition $r$. Note that in this case $\sigma$ is covered by $r$, i.e. is covered by the principal repeat of $r$. Since the repeat $\sigma$ is a periodic repeat with the minimal period $p(r)$
of its copies, i.e. the length of its copies is not less than $3p(r)$, and $r$ contains copies of $\sigma$, we have that $e(r) \geq 3$. Thus, we have the following corollary from Proposition 19.

**Corollary 3.** Any $\alpha$-periodic maximal repeat represented by some maximal repetition is covered by the $\alpha$-nonperiodic principal repeat of this repetition.

Our algorithm is based on the following lemma.

**Lemma 10.** Let a maximal $\alpha$-gapped repeat $\sigma$ is covered by some $\alpha$-periodic maximal repeat and is not covered by any $\alpha$-nonperiodic principal repeat of maximal repetition. Then $\sigma$ is a periodic birepresented nondominating repeat of third type.

**Proof.** Let $\sigma$ be covered by an $\alpha$-periodic maximal repeat $\sigma'$. By Proposition 10 $\sigma'$ is an $\alpha$-gapped or overlapped repeat, so $\sigma'$ is a periodic repeat. If $\sigma'$ is represented by a maximal repetition, then, by Corollary 3 $\sigma'$ is covered by the $\alpha$-nonperiodic principal repeat of this repetition, so $\sigma$ is also covered by the $\alpha$-nonperiodic principal repeat of this repetition which contradicts conditions of the lemma. Thus, $\sigma'$ is represented by a pair of maximal repetitions $r', r''$. Let $p'$ the minimal period of these repetitions. Note that, by Proposition 11 the left copy of $\sigma$ is contained in the left copy of $\sigma'$ and the right copy of $\sigma$ is contained in the right copy of $\sigma'$. Hence the left copy of $\sigma$ is contained in $r'$ and the right copy of $\sigma$ is contained in $r''$, so $p'$ is a period of copies of $\sigma$. Moreover, we have that

$$c(\sigma) \geq \frac{p(\sigma)}{\alpha} > \frac{p(\sigma')}{\alpha} \geq 3p'. $$

Thus, $\sigma$ is a periodic repeat. Denote by $p$ the minimal period of copies of $\sigma$. Let $p < p'$. Since $c(\sigma) > 3p' > p' + p$, by periodicity lemma we have that $\gcd(p', p)$ is also a period of copies of $\sigma$ which has to be equal to $p$. Thus, $p$ is a divisor of $p'$, so in the case $p < p'$ we obtain that the cyclic roots of repetitions $r', r''$ are not primitive. Therefore, $p' = p$, i.e. $p'$ is the minimal period of copies of $\sigma$, so $\sigma$ is represented by the pair of repetitions $r', r''$. Thus, we have that $\sigma$ is a periodic repeat represented by the pair $r', r''$ and is covered by the periodic repeat $\sigma'$ represented by the same pair $r', r''$. So, by Corollary 3 $\sigma$ is a nondominating repeat of third type.

**Lemma 11.** All maximal $\alpha$-gapped $\alpha$-periodic repeats in $w$ can be computed in time $O(n\alpha)$.

**Proof.** Recall that any maximal $\alpha$-gapped $\alpha$-periodic repeat is a periodic repeat, so for finding of all maximal $\alpha$-gapped $\alpha$-periodic repeats we can compute initially all maximal $\alpha$-gapped periodic repeats in $w$. By Lemma 9 it can be done in time $O(n\alpha)$. Moreover, by Theorem 9 the number of computed repeats is $O(n\alpha)$. Recall that for each computed repeat we compute additionally the minimal period of copies of this repeat, so we can check in constant time if this repeat is $\alpha$-periodic. Thus, in $O(n\alpha)$ time we can select from the computed repeats all maximal $\alpha$-gapped $\alpha$-periodic repeats in $w$. The total time of the proposed procedure for computing of the required repeats is $O(n\alpha)$.
Now we consider maximal overlapped periodic repeats represented by pairs of maximal repetitions. Note that such repeats can be represented only by pairs of overlapped maximal repetitions.

**Lemma 12.** Let \( r', r'' \) be a left pair of maximal overlapped repetitions with the same cyclic roots such that \( p(r') = p(r'') \), \( \text{beg}(r') \), \( \text{beg}(r'') \), \( \text{end}(r') \), \( \text{end}(r'') \), \( a(r') \), \( a(r'') \) be known. Then the number of maximal overlapped periodic repeats represented by the pair \( r', r'' \) is less than \( e(r') \), and all these repeats can be computed in time \( O(e(r')) \).

**Proof.** It is shown in the proof of Lemma 5 that in the set \( \Psi \) all repeats \( \sigma_j \) such that \( j \geq k' + 2 \) are gapped repeats, so only repeats \( \sigma_1, \sigma_2, \ldots, \sigma_{k'+1} \) can be overlapped repeats, and, moreover, all these \( k' + 1 \) repeats can be computed in \( O(1 + k') \) time. It is also shown that repeats \( \sigma_1, \sigma_2, \ldots, \sigma_{k'} \) are overlapped if and only if repeat \( \sigma_1 \) is overlapped. Thus, in constant time we can select from repeats \( \sigma_1, \sigma_2, \ldots, \sigma_{k'+1} \) all overlapped repeats. It follows from equation (1) that \( k' \leq \frac{r_i^j}{p(r_j)} - 2 = e(r') - 2 \). Thus, the number the selected repeats is not greater than \( 1 + k' < e(r') \), and the total time of computing of these repeats is \( O(1 + k') = O(e(r')) \).  

**Lemma 13.** The number of maximal overlapped periodic repeats represented by left pairs of maximal repetitions is \( O(n) \), and all these repeats can be computed in \( O(n) \) time.

**Proof.** Analogously to the proof of Lemma 5 first we find all maximal repetitions in \( w \), for each maximal repetition \( r \) in \( w \) we compute the values \( p(r) \), \( \text{beg}(r) \), \( \text{end}(r) \), \( a(r) \), and, moreover, we divide all maximal repetitions in \( w \) into the subsets of all repetitions with the same cyclic roots and represent each of these subsets as a list \( PRSR_j \) of the subset repetitions in the order of increasing of their starting positions. As shown in the proof of Lemma 5 it can be done in \( O(n) \) time. Then we consider each list \( PRSR_j \). Let \( PRSR_j \) consists of consecutive repetitions \( (r_1^{(j)}), r_2^{(j)}, \ldots, r_q^{(j)} \). According to Proposition 5 for each repetition \( r_i^{(j)} \) the overlaps of \( r_i^{(j)} \) with repetitions \( r_{i-1}^{(j)} \) and \( r_{i+1}^{(j)} \) are less than \( p(r_i^{(j)}) \), so repetitions \( r_{i-1}^{(j)} \) and \( r_{i+1}^{(j)} \) can not be overlapped. Thus, only pairs \( r_i^{(j)}, r_{i+1}^{(j)} \) can be overlapped pairs of repetitions representing maximal overlapped repeats. Therefore, we traverse the list \( PRSR_j \) for finding all overlapped left pairs \( r_i^{(j)}, r_{i+1}^{(j)} \) of repetitions. Note that the total number of maximal repetitions in \( w \) is \( O(n) \), so the total time of traversing of all lists \( PRSR_j \) is \( O(n) \). For each found pair \( r_i^{(j)}, r_{i+1}^{(j)} \) we compute all maximal overlapped periodic repeats represented by this pair. By Lemma 12 the number of these repeats is less than \( e(r_i^{(j)}) \), and all these repeats can be computed in time \( O(e(r_i^{(j)})) \). For the computed repeats the minimal periods of copies of these repeats are defined as the minimal period of \( r_i^{(j)} \). Note that for each maximal repetition \( r \) in \( w \) there can be only one left pair \( r_i^{(j)}, r_{i+1}^{(j)} \) of repetitions such that \( r \equiv r_i^{(j)} \), so the total number of maximal overlapped periodic repeats represented by all left
pairs \( r^{(j)}_i, r^{(j)}_{i+1} \) of repetitions in all lists \( PRSR_j \) is less than \( \sum_{r \in R(w)} e(r) \), so, by Theorem 2 this number is \( O(n) \). By the same reason, the total time of computing of all maximal overlapped periodic repeats represented by all left pairs \( r^{(j)}_i, r^{(j)}_{i+1} \) of repetitions in all lists \( PRSR_j \) is \( O(\sum_{r \in R(w)} e(r)) \), so, by Theorem 2 this time is \( O(n) \). Thus, the total time of the considered procedure for computing of the required repeats is \( O(n) \).

We can also prove the analogous lemma for right pairs of maximal repetitions.

**Lemma 14.** The number of maximal overlapped periodic repeats represented by right pairs of maximal repetitions is \( O(n) \), and all these repeats can be computed in \( O(n) \) time.

From Lemmas 13 and 14 we directly obtain the corollary.

**Corollary 4.** The number of maximal overlapped periodic birepresented repeats is \( O(n) \), and all these repeats can be computed in \( O(n) \) time.

**Lemma 15.** The number of maximal overlapped \( \alpha \)-periodic birepresented repeats is \( O(n) \), and all these repeats can be computed in \( O(n) \) time.

**Proof.** Recall that any maximal overlapped \( \alpha \)-periodic repeat is a periodic repeat, so for finding of all maximal overlapped \( \alpha \)-periodic birepresented repeats we can compute initially all maximal overlapped periodic birepresented repeats in \( w \). By Corollary 4, the number of such repeats is \( O(n) \), and all these repeats can be computed in \( O(n) \) time. Since for each computed repeat we compute additionally the minimal period of copies of this repeat, we can check in constant time if this repeat is \( \alpha \)-periodic. Thus, in \( O(n) \) time we can select from the computed repeats all maximal overlapped \( \alpha \)-periodic birepresented repeats in \( w \). The total time of the proposed procedure for computing of the required repeats is \( O(n) \).

**Lemma 16.** The number of reprincipal \( \alpha \)-periodic birepresented repeats is \( O(n) \), and all these repeats can be computed in \( O(n) \) time.

**Proof.** Recall that all reprincipal repeats are maximal overlapped repeats, so for computing of all reprincipal \( \alpha \)-periodic birepresented repeats in \( w \) we can select them from all maximal overlapped \( \alpha \)-periodic birepresented repeats in \( w \). By Lemma 14 the number of all maximal overlapped \( \alpha \)-periodic birepresented repeats is \( O(n) \), and these repeats can be computed in \( O(n) \) time. By Proposition 4 the number of reprincipal repeats in \( w \) is also \( O(n) \), and all these repeats can be computed in \( O(n) \) time. In order to select all reprincipal repeats from maximal overlapped \( \alpha \)-periodic birepresented repeats, we represent the set all the computed maximal overlapped \( \alpha \)-periodic birepresented repeats by start position lists \( MOBRL_i \). By the same way, we represent the set of all the computed reprincipal repeats by start position lists \( PRL_i \). All the lists \( MOBRL_i \) and \( PRL_i \) can be computed in time \( O(n + S) \) where \( S \) is the total size of all lists \( MOBRL_i \) and \( PRL_i \), so, since this total size is \( O(n) \), the time of
computing of all the lists $MOBRL_t$ and $PRL_t$ is $O(n)$. Then, in order to select the required reprincipal repeats, we traverse simultaneously lists $MOBRL_t$ and $PRL_t$ for $t = 1, 2, \ldots, n$. It can be also done in time $O(n + S) = O(n)$. Thus, the total time of the proposed procedure for computing of the required repeats is $O(n)$.

Now we consider reprincipal repeats represented by maximal repetitions.

**Proposition 20.** The principal repeat of a maximal repetition can not be represented by another maximal repetition.

**Proof.** Let $\sigma$ be the principal periodic repeat of a maximal repetition $r$. Assume that $\sigma$ is represented by some another maximal repetition $r'$. Note that in this case $p(r') \leq p(r)$ and $r$ is contained in $r'$, i.e. the length of the overlap of the repetitions $r$ and $r'$ is not less than $2p(r)$. It contradicts Proposition 6.

**Corollary 5.** Any reprincipal $\alpha$-periodic repeat is a birepresented repeat.

**Proof.** Let $\sigma$ be the principal $\alpha$-periodic repeat of some maximal repetition $r$. Note that, as shown above, $\sigma$ is a periodic repeat. Assume that $\sigma$ is represented by some maximal repetition. By Proposition 20, $\sigma$ can be represented only by repetition $r$. In this case we have that $p(\sigma) = p(r)$ and the minimal period of copies of $\sigma$ is $p(r)$, so the minimal period of copies of $\sigma$ is greater than $\frac{p(\sigma)}{\alpha}$ which contradicts that $\sigma$ is a $\alpha$-periodic repeat.

From Lemma 16 and Corollary 5 we obtain immediately the following fact.

**Corollary 6.** The number of reprincipal $\alpha$-periodic repeats is $O(n)$, and all these repeats can be computed in $O(n)$ time.

### 6 Algorithm for solving of Problem 3

Let $\alpha = 1/\delta$. We compute initially the set $GR$ of all maximal $\alpha$-gapped repeats in $w$ and the set $PR$ of all reprincipal repeats in $w$. By Theorem 3 we have that $|GR| = O(n\alpha)$ and $GR$ can be computed in $O(n\alpha)$ time. Moreover, by Proposition 4 we have that $|PR| = O(n)$ and $PR$ can be computed in $O(n)$ time. Recall that our goal is to exclude from $GR$ all repeats which are covered by other maximal repeats. Using Propositions 10 and 12 we conclude that we need actually to exclude from $GR$ all repeats which are covered by other maximal gapped repeats from $GR$ or reprincipal repeats. Denote by $GR^*$ the set of all repeats from $GR$ which are not covered by other repeats from $GR$ or reprincipal repeats. In these terms, our goal is to compute the set $GR^*$.

Note that all maximal repeats generated by a maximal repetition except the principal repeat of this repetition are covered by this repetition and so are covered by the principal repeat of this repetition. Recall also that principal repeats of maximal repetitions are overlapped repeats, so gapped repeats cannot be reprincipal repeats. Thus, maximal gapped repeats generated by a maximal repetition are covered by the principal repeat of this repetition and so have to be excluded from $GR$. At the first stage we exclude from $GR$ all maximal
\(\alpha\)-gapped repeats generated by maximal repetitions. By Proposition 15, the number of these repeats is \(O(n)\), and all these repeats can be computed in \(O(n)\) time. Denote the set of all the computed repeats by \(GR\). In order to exclude from \(GR\) the repeats of the set \(CR\), we represent \(GR\) by start position lists \(GRL_t\). These lists can be computed in time \(O(n + |GR|) = O(\alpha n)\). By the same way, we represent the set \(CR\) by start position lists \(CRL_t\). These lists can be computed in time \(O(n + |CR|) = O(n)\). Then all the computed repeats which is contained in \(GR\) can be excluded from \(GR\) by simultaneous traversing of lists \(GRL_t\) and \(CRL_t\) in time \(O(n + |GR| + |CR|) = O(\alpha n)\).

Denote by \(GR'\) the resulting set of all repeats from \(GR\) which remain after the first stage. Recall that all repeats from \(GR\) which are removed at the first stage are covered by reprincipal repeats. Therefore, any repeat from \(GR\) which is covered by a repeat \(\sigma\) removed at the first stage is covered also by some reprincipal repeat covering the repeat \(\sigma\). Thus, in order to compute the set \(GR'^*\), we can remove from \(GR'\) all repeats which are covered by other repeats from \(GR'\) or reprincipal repeats. Denote by \(GR\) the set of all repeats from \(GR'\) together with all reprincipal repeats in \(w\). In these terms, in order to compute the set \(GR'^*\), we remove from \(GR'\) all repeats which are covered by other repeats from \(GR\).

At the second stage we remove from \(GR'\) all repeats which are covered by \(\alpha\)-nonperiodic repeats from \(GR\). For this purpose, first we compute the set \(AGR\) of all \(\alpha\)-periodic repeats from \(GR\) and the set \(APR\) of all \(\alpha\)-periodic reprincipal repeats in \(w\). By Lemma 11 the set \(AGR\) can be computed in \(O(n\alpha)\) time, and, by Corollary 13 the set \(APR\) can be computed in \(O(n)\) time. Note that after performing of the first stage the set \(GR'\) is represented by its start position lists \(GRL'_t\). The set \(AGR\) can be also represented by start position lists \(AGRL_t\) which can be computed in \(O(n + |AGR|) = O(n\alpha)\) time. Then, using simultaneous traversing of lists \(GRL'_t\) and \(AGRL_t\), we mark all \(\alpha\)-nonperiodic repeats from \(GR'\) as \(\alpha\)-nonperiodic (each repeat from \(GRL'_t\) which is not contained in \(AGRL_t\) is marked as \(\alpha\)-nonperiodic). Moreover, we compute the set \(NPR\) of all \(\alpha\)-nonperiodic reprincipal repeats in \(w\) by removing from the set \(PR\) all repeats from \(APR\). To perform thus removing, we also represent the sets \(PR\) and \(APR\) by their start position lists \(PRL_t\) and \(APRL_t\). All these lists can be computed in time \(O(n + |PR| + |APR|) = O(n)\). Then we compute all repeats from \(NPR\) by simultaneous traversing of lists \(PRL_t\) and \(APRL_t\) in total time \(O(n + |PR| + |APR|) = O(n)\). Note that the computed set \(NPR\) is also represented by its start position lists \(NPRL_t\). Then we compute the set \(GR' = GR' \cup NPR\) by merging the lists \(GRL'_t\) and \(NPRL_t\) into the start position lists \(\overline{GRL}'_t\) for this set. It can be done by simultaneous traversing of lists \(GRL'_t\) and \(NPRL_t\) in time \(O(n + |GR'| + |NPRL|) = O(\alpha n)\). During the merging of repeats into the lists \(\overline{GRL}'_t\) we also mark these repeats as gapped or as reprincipal. Note also that \(|GR'| = |GR'| + |NPRL|) = O(\alpha n)\).

For \(i = 1, 2, \ldots, \lfloor \log_2(n - 1) \rfloor\) we denote by \(RQ_i\) the subset of all repeats \(\sigma\) from \(GR'\) such that \(2^i \leq p(\sigma) < 2^{i+1}\), i.e. \(\lfloor \log_2 p(\sigma) \rfloor = i\).

**Proposition 21.** Let a maximal gapped repeat \(\sigma\) be covered by a maximal gapped
repeat \( \sigma' \). Then \( p(\sigma) \geq p(\sigma') > p(\sigma)/2 \).

**Proof.** Note that \( p(\sigma) < |\sigma| \leq |\sigma'| < 2p(\sigma') \), so \( p(\sigma') > p(\sigma)/2 \). The inequality \( p(\sigma) \geq p(\sigma') \) is obvious. \( \square \)

**Corollary 7.** Let a gapped repeat \( \sigma \) from \( RQ_i \) be covered by a gapped repeat \( \sigma' \) from \( GR' \). Then \( \sigma' \) is contained in \( RQ_i \) or \( RQ_{i-1} \).

**Proposition 22.** Let a gapped repeat \( \sigma \) from \( RQ_i \) be covered by a reprincipal repeat \( \sigma' \) from \( RQ_v \). Then \( i \geq i' \geq i - \lfloor \log_2 \alpha \rfloor \).

**Proof.** It is obvious that \( i \geq i' \). Assume that \( i' < i - \lfloor \log_2 \alpha \rfloor \). Note that for any \( \sigma \) from \( RQ_i \) and any \( \sigma' \) from \( RQ_v \) we have \( p(\sigma) > 2^{s-i-1}p(\sigma') \), so in this case \( p(\sigma) > 2^{\lfloor \log_2 \alpha \rfloor}p(\sigma') \geq \alpha p(\sigma') \). Hence, \( c(\sigma) \geq p(\sigma)/\alpha > p(\sigma') \). Let \( u', u'' \) be left and right copies of \( \sigma \). Since \( \sigma \) is covered by \( \sigma' \), the repeat \( \sigma \) is contained in the repetition \( \text{rep}(\sigma') \) with the minimal period \( p(\sigma') \), so the both copies \( u', u'' \) are contained in \( \text{rep}(\sigma') \) and \( |u'| = |u''| > p(\sigma') \). Consider the prefixes of length \( p(\sigma') \) in \( u' \) and \( u'' \). Since these prefixes are equal cyclic roots of \( \text{rep}(\sigma') \), by Proposition 21 we have \( \text{beg}(u') \equiv \text{beg}(u'') \pmod{p(\sigma')} \), so \( p(\sigma) = \text{beg}(u'') - \text{beg}(u') \) is divisible by \( p(\sigma') \), i.e. \( p(\sigma) \) is divisible by the minimal period \( p(\sigma') \). Moreover, \( \text{beg}(u') > \text{beg}(\text{rep}(\sigma')) \) then both symbols \( w[\text{beg}(u') - 1] \) and \( w[\text{beg}(u'') - 1] \) are contained in \( \text{rep}(\sigma') \), so \( w[\text{beg}(u') - 1] = w[\text{beg}(u'') - 1] \), i.e. \( \sigma \) can be extended to the left which contradicts that \( \sigma \) is a maximal repeat. Thus, \( \text{beg}(u') = \text{beg}(\text{rep}(\sigma')) \). It can be analogously proved that \( \text{end}(u') = \text{end}(\text{rep}(\sigma')) \). Thus, the repeat \( \sigma \) is generated by the repetition \( \text{rep}(\sigma') \) which contradicts that \( \sigma \in GR' \). \( \square \)

Summing up Corollary 7 and Proposition 22 we obtain the following fact.

**Corollary 8.** Let a gapped repeat \( \sigma \) from \( RQ_i \) be covered by a repeat \( \sigma' \) from \( RQ_v \). Then \( i \geq i' \geq i - \lfloor \log_2 \alpha \rfloor \).

For finding all repeats which have to be removed at the second stage, for each starting position \( t = 1, 2, \ldots, n \) we compute consecutively such repeats starting at position \( t \). Note that such repeats starting at position \( t \) can be covered only by \( \alpha \)-nonperiodic repeats from \( GR' \) which are starting at position not greater than \( t \) and ending at position greater than \( t \). Denote the set of all these \( \alpha \)-nonperiodic repeats by \( SQ \) and put \( SQ_i = SQ \cap RQ_i \) for \( i = 1, 2, \ldots, \lfloor \log_2 (n - 1) \rfloor \). Note that if for some repeat \( \sigma \) from \( SQ_i \) there exists a repeat \( \sigma' \) in \( SQ \) such that \( \text{end}(\sigma) \leq \text{end}(\sigma') \) and \( p(\sigma') \leq p(\sigma) \) then \( \sigma \) can be excluded from consideration. The remaining repeats from \( SQ \) form a sequence \( LQ_1 = \sigma_1, \sigma_2, \ldots, \sigma_s \) such that \( \text{end}(\sigma_1) < \text{end}(\sigma_2) < \ldots < \text{end}(\sigma_s) \) and \( p(\sigma_1) < p(\sigma_2) < \ldots < p(\sigma_s) \). In order to perform effective search in this sequence, we present \( LQ_i \) as AVL-tree \( LQT_i \). For each \( i \) we compute also the value \( \text{lep}_i \) which is the maximum of ending positions of the last repeats in sequences \( LQ_{i-1}, LQ_{i-2}, \ldots, LQ_{i-\lfloor \log_2 \alpha \rfloor} \).

Let \( LQ_i = \sigma_1, \sigma_2, \ldots, \sigma_s \). For any repeat \( \sigma \) we define \( \text{pe}(\sigma) = p(\sigma) + \text{end}(\sigma) \). From \( \text{end}(\sigma_1) < \text{end}(\sigma_2) < \ldots < \text{end}(\sigma_s) \) and \( p(\sigma_1) < p(\sigma_2) < \ldots < p(\sigma_s) \) we have \( \text{pe}(\sigma_1) < \text{pe}(\sigma_2) < \ldots < \text{pe}(\sigma_s) \).
Lemma 17. For each \( j \leq s - 2 \) the inequality \( pe(\sigma_{j+2}) > pe(\sigma_j) + \frac{\Delta p_3}{3\alpha} \) is valid.

Proof. Denote for convenience by \( u_1', u_2', u_3' \) the left copies of \( \sigma_j, \sigma_{j+1}, \sigma_{j+2} \), and by \( u_1'', u_2'', u_3'' \) the right copies of \( \sigma_j, \sigma_{j+1}, \sigma_{j+2} \). For \( k = 1, 2, 3 \) denote \( p_k = p(\sigma_{j+k-1}), e_k = end(\sigma_{j+k-1}), pe_k = p_k + e_k \) and for \( k = 2, 3 \) denote also \( \Delta p_k = p_k - p_1, \Delta e_k = e_k - e_1, \Delta pe_k = pe_k - pe_1 \).

Assume that \( pe_3 \leq pe_1 + \frac{p_3}{3\alpha} \), i.e. \( \Delta pe_2 < \Delta pe_3 \leq \frac{p_1}{3\alpha} \). Consider separately the following three cases.

1. Let \( u_3'' \) is contained in \( u_2'' \). So \( u_2'' \) contains the factor \( \hat{u}_1' \) corresponding to the factor \( u_3'' \) in \( u_2'' \) such that \( end(u_1') - end(\hat{u}_1') = \Delta p_2 \). Thus, since \( u_1' = \hat{u}_1' \), we obtain that \( u_1' \) has the period \( \Delta p_2 < \Delta pe_2 < \frac{p_1}{3\alpha} \) which contradicts that \( \sigma_j \) is \( \alpha \)-nonperiodic.

2. Now let \( u_2'' \) is not contained in \( u_2'' \), i.e. \( \text{beg}(u_2'') < \text{beg}(u_2'') \), and \( \Delta e_2 \leq \Delta p_2 \) which implies \( \text{end}(u_2') \leq \text{end}(u_1') \). Let \( u'' \) be the intersection of \( u_1'' \) and \( u_2'' \). Since \( u'' \) is a factor of \( u_1'' \) and \( u_2'' \), \( u'' \) there are corresponding factors \( u' \) in \( u_1'' \) and \( \hat{u}' \) in \( u_2'' \). Note that \( \Delta pe_2 < \Delta pe_3 < \frac{p_1}{3\alpha} \) which contradicts that \( \sigma_{j+1} \) is \( \alpha \)-nonperiodic.

3. Finally, let \( u_1'' \) is not contained in \( u_2'' \) and \( \Delta e_2 > \Delta p_2 \) which implies \( \text{end}(u_2'') \geq \text{end}(u_1') \). As in case 2, we define the factors \( u'' \) and \( u' \), and show that both these factors have period \( \Delta p_2 \). Note that \( u'' \) is a prefix of \( u_1'' \), so \( u' \) is a prefix of \( u_1'' \). Thus, \( \text{end}(u'') = \text{end}(u_1') \) . Consider the symbol \( w[\text{end}(u_1') + 1] \). Since \( \sigma_1 \) is maximal, we have \( w[\text{end}(u_1') + 1] \neq w[\text{end}(u_1'') + 1] \). Moreover, since \( \text{end}(u_2'') > \text{end}(u_1') \), the symbol \( w[\text{end}(u_1') + 1] \) is contained in \( u_2'' \), so \( u_2'' \) contains the corresponding symbol \( w[\text{end}(u_1') + 1] = w[\text{end}(u_1'') + \Delta p_2 + 1] \) which is equal to \( w[\text{end}(u_1') + 1] \). Thus, we obtain that \( w[\text{end}(u_1'') + 1] \neq w[\text{end}(u_1') + \Delta p_2 + 1] \). Let \( w[\text{end}(u_1'') + 1] \) be not contained in \( u_2'' \), i.e. \( \text{beg}(u_2'') > \text{end}(u_1'') + 1 \). In this case we have that

\[
\Delta e_3 > |u_2''| - \frac{p_3}{3\alpha} \geq \frac{p_1}{3\alpha} > \frac{p_1}{3\alpha} > 2\Delta p_2.
\]

Therefore, the intersection of \( u' \) and \( \hat{u}' \) has length greater than \( \Delta p_2 \), so, by Proposition 2 the union \( \hat{u} \) of factors \( u' \) and \( \hat{u}' \) has also the period \( \Delta p_2 \). Note that \( u_2'' \) is contained in \( \hat{u} \), so \( u_2'' \) also has the period \( \Delta p_2 < \frac{p_1}{3\alpha} < \frac{p_1}{3\alpha} \) which contradicts that \( \sigma_{j+1} \) is \( \alpha \)-nonperiodic.
Note that $\Delta p_3 > \Delta p_2$, so $\text{end}(u'_1) - \Delta p_3 + \Delta p_2 + 1 \leq \text{end}(u'_1)$. Thus, in this case we have

$$\text{beg}(u') \leq \text{end}(u'_1) - \Delta p_3 + 1 < \text{end}(u'_1) - \Delta p_3 + \Delta p_2 + 1 \leq \text{end}(u'_1) = \text{end}(u'),$$

i.e. the both unequal symbols $w[\text{end}(u'_1) - \Delta p_3 + 1]$ and $w[\text{end}(u'_1) - \Delta p_3 + \Delta p_2 + 1]$ are contained in $u'$ which contradicts that $u'$ has period $\Delta p_2$. Thus,

$$\text{end}(u'_1) - \Delta p_3 + 1 < \text{beg}(u') = \text{end}(u') - |u'| + 1 = \text{end}(u'_1) - |u'| + 1,$$

so $\Delta p_3 > |u'| = |u''|$. By relation (4), we have $|u''| > \frac{2p_1}{3\alpha}$. Thus, $\Delta p_3 > \frac{2p_1}{3\alpha}$ which contradicts that $\Delta p_3 < \frac{2p_1}{3\alpha}$.

Since we obtained contradictions in all considered cases, the lemma is proved.  

\begin{lemma}
$\text{end}(\sigma_{s-1}) < t + 2^{i+2}$.
\end{lemma}

\begin{proof}
Assume by contradiction that $\text{end}(\sigma_{s-1}) \geq t + 2^{i+2}$. Note that in this case $|\sigma_{s-1}| > 2^{i+2} = 2p(\sigma_{s-1})$, so $\sigma_{s-1}$ is a overlapped repeat. Since $\text{end}(\sigma_s) > \text{end}(\sigma_{s-1})$, by the same reason, $\sigma_s$ is also a overlapped repeat. Note that the intersection of repetitions $\text{rep}(\sigma_s)$ and $\text{rep}(\sigma_{s-1})$ contains the factor $w[t..t+2^{i+2}]$, so the overlap of these repetitions is greater than $2^{i+2} = p(\sigma_{s-1}) + p(\sigma_s)$ which contradicts Proposition 17 i.e. the lemma is proved.

Using Lemma 18, we have that

$$t + 2^i < pe(\sigma_1) < pe(\sigma_2) < \ldots < pe(\sigma_{s-1}) < t + 3 \cdot 2^{i+1},$$

and, by Lemma 17 for each $j < s - 2$ we have

$$pe(\sigma_{j+2}) - pe(\sigma_j) > \frac{p(\sigma_j)}{3\alpha} \geq \frac{2^i}{3\alpha}.$$

Thus, we obtain that $s - 1 = O(\alpha)$, so $s = O(\alpha)$. Here we state this fact.

\begin{corollary}
$|LQT_i| = O(\alpha)$ for any $i$.
\end{corollary}

The step of the algorithm for a starting position $t$ is as follows. First we remove from trees $LQT_i$ all repeats ending at position $t$ if such repeats exist. In order to perform effectively this removing, for each starting position $t$ we can maintain a double linked list containing all ending at position $t$ repeats from trees $LQT_i$. In this case all repeats which have to be removed from trees $LQT_i$ can be found in time proportional to the number of these repeats. Then we consider consecutively all repeats in list $GRL'_t$. Let $\sigma$ be a current considered gapped repeat from $GRL'_t$, and $|\log_2 p(\sigma)| = i$. By Corollary 17 $\sigma$ can be covered only by repeats from $RQ_i$ where $i \geq i' \geq i - |\log_2 \alpha|$. Note that $\sigma$ is covered by a repeat from $LQT_{i'}$ such that $i - 1 \geq i' \geq i - |\log_2 \alpha|$ if and only if $\text{end}(\sigma) \leq \text{lep}_i$. Thus, in this case we remove $\sigma$ from $GRL'_t$. Otherwise, we check if $\sigma$ is covered by a repeat from $LQT_i$. For this purpose we search in $LQT_i$ the maximal $k$ such that $p(\sigma_k) < p(\sigma)$. If such $k$ exist and $\text{end}(\sigma) \leq \text{end}(\sigma_k)$ then
perform at most one search in some tree $LQT_i$. Otherwise, if $\sigma$ is a $\alpha$-nonperiodic repeat, we insert $\sigma$ in $LQT_i$ between $\sigma_k$ and $\sigma_{k+1}$. After that we remove from $LQT_i$ all $\sigma_j$ such that $j > k$ and $\text{end}(\sigma_j) \leq \text{end}(\sigma)$. If it is necessary, we update the values $\text{lep}_{i+1}$, $\text{lep}_{i+2}$, ..., $\text{lep}_{i+\lceil \log_2 \alpha \rceil}$ which can depend on the value $\text{end}(\sigma)$. Now let $\sigma$ be a current considered reprincipal repeat from $\widehat{GRL}'_i$. In this case we remove $\sigma$ from $\widehat{GRL}'_i$ and insert $\sigma$ in $LQT_i$ where $i = \lfloor \log_2 p(\sigma) \rfloor$ by the same way as we insert above a $\alpha$-nonperiodic gapped repeat which is checked to be not covered. Then we proceed to the next repeat in $GRL'_i$.

Note that during the described procedure for each repeat $\sigma$ from $\widehat{GR}'$ we perform at most one search in some tree $LQT_i$, at most one insertion of $\sigma$ in $LQT_i$ and at most one deletion of $\sigma$ from $LQT_i$. All these operations can be performed in $O(\log |LQ|)$ time. Since, by Corollary 9, $|LQ_i| = O(\alpha)$, we obtain that all these operations can be performed in $O(\log \alpha)$ time. Moreover, after insertion of $\sigma$ no more than $\lceil \log_2 \alpha \rceil$ values $\text{lep}_i$ can be updated. It can be also performed in $O(\log \alpha)$ time. All the other operations over $\sigma$ required for the described procedure can be performed in constant time. Thus, each repeat from $\widehat{GR}'$ can be treated in $O(\log \alpha)$ time, so the total time of the described procedure for all starting positions $t$ is $O(n + |\widehat{GR}'| \log \alpha)$. Recall that $|\widehat{GR}'| = O(n \alpha)$, so the total time of the described procedure is $O(n \alpha \log \alpha)$.

Note that after the second stage we removed from $\widehat{GR}'$ all reprincipal repeats and all repeats from $GR'$ which are covered by $\alpha$-nonperiodic repeats from $\widehat{GR}$, so after the second stage the set $GR'$ consists of all repeats from $\widehat{GR}'$ which are not covered by $\alpha$-nonperiodic repeats from $\widehat{GR}$. Note also that $|\widehat{GR}'| = O(n \alpha)$.

At the third stage we compute the set $GR^*$ by removing from the set $\widehat{GR}'$ all repeats which are not contained in $GR^*$. Let $\sigma$ be a repeat from $\widehat{GR}'$ which is not contained in $GR^*$, i.e. $\sigma$ be covered some other repeat from $GR$. Since $\widehat{GR}'$ consists of repeats which are not covered by $\alpha$-nonperiodic repeats from $\widehat{GR}$, the repeat $\sigma$ can be covered only by an $\alpha$-periodic repeat from $\widehat{GR}$. Moreover, $\sigma$ can not be covered by any $\alpha$-nonperiodic reprincipal repeat, since, otherwise, $\sigma$ has to be removed from $\widehat{GR}'$ at the second stage. Thus, $\sigma$ is covered by some $\alpha$-periodic repeat from $\widehat{GR}$ and is not covered by any $\alpha$-nonperiodic reprincipal repeat. Therefore, by Lemma 11, $\sigma$ is a periodic birepresented nondominating repeat of third type. On the other hand, if $\sigma$ is a periodic birepresented nondominating repeat of third type from $\widehat{GR}'$, then, by Corollary 5, $\sigma$ is covered by another periodic repeat represented by the same pair of repetitions, so $\sigma$ is not contained in $GR^*$. Thus, a repeat from $\widehat{GR}'$ is not contained in $GR^*$ if and only if this repeat is a periodic birepresented nondominating repeat of third type. Hence for computing the set $GR^*$ we remove from $\widehat{GR}'$ all periodic birepresented nondominating repeats of third type. Recall that $\widehat{GR}'$ consists of maximal $\alpha$-gapped repeats, so we need actually to remove from $\widehat{GR}'$ all periodic birepresented maximal $\alpha$-gapped nondominating repeat of third type. For this purpose, we compute the set $BANR$ of all periodic birepresented maximal $\alpha$-gapped nondominating repeats of third type in $w$. By Corollary 5, this set can be computed in time $O(n \alpha)$. Moreover, since the set $BANR$ is a subset of
the set $GR$, we have $|BANR| = O(n\alpha)$. Then we represent the set $BANR$ by its start position lists $BANRL_t$ in time $O(n + |BANR|) = O(n\alpha)$. Finally, we remove from $GR'$ all repeats from $BANR$ by simultaneous traversing of lists $\tilde{GR}L_t$ and $BANRL_t$ in time $O(n + |\tilde{GR}'| + |BANR|) = O(n\alpha)$. As a result, we obtain the required set $GR^*$. The total time of the third stage procedure is $O(n\alpha)$.

Summing up the times of all the procedures of the proposed algorithm, we obtain that the total time of the algorithm is $O(n\alpha \log \alpha) = O(\frac{n}{\delta} \log \frac{1}{\delta})$. Thus, Problem 3 can be resolved in $O(\frac{n}{\delta} \log \frac{1}{\delta})$ time. Since, as shown above, Problem 3 is equivalent to Problem 1 we conclude the following main result of our paper.

**Theorem 4.** All maximal $\delta$-subrepetitions in a given word of length $n$ over integer alphabet can be found in time $O(\frac{n}{\delta} \log \frac{1}{\delta})$.

7 Conclusion

In the paper we proposed an algorithm for finding of all maximal $\delta$-subrepetitions in a given word of length $n$ in time $O(\frac{n}{\delta} \log \frac{1}{\delta})$. By Proposition 8, the number of all maximal $\delta$-subrepetitions in a word of length $n$ is $O(\frac{n}{\delta})$, so the considered problem could be presumably resolved in $O(\frac{n}{\delta})$ time. Thus, finding of all maximal $\delta$-subrepetitions in a given word of length $n$ in time $O(\frac{n}{\delta})$ is still an open problem.

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