Solutions of Inhomogeneous Perturbed Generalized Moisil–Teodorescu System and Maxwell’s Equations in Euclidean Space

Juan Bory-Reyes and Marco Antonio Pérez-de la Rosa

Abstract. In this paper, based on a proposed notion of generalized conjugate harmonic pairs in the framework of complex Clifford analysis, necessary and sufficient conditions for the solvability of inhomogeneous perturbed generalized Moisil–Teodorescu systems in higher dimensional Euclidean spaces are proved. As an application, we derive corresponding solvability conditions for the inhomogeneous Maxwell’s equations.

Mathematics Subject Classification. 30G35, 47F05, 47G10, 35Q60.

Keywords. Clifford analysis, Moisil–Teodorescu system, Maxwell’s equations, Conjugate harmonic pairs.

1. Introduction

Clifford analysis is the study of properties of solutions of the first-order, vector-valued Dirac operator $\partial_x$ acting on functions defined on Euclidean spaces $\mathbb{R}^{m+1}$ ($m \geq 2$) with values in the corresponding real or complex Clifford algebra, that will be denoted below by $\mathbb{R}_{0,m+1}$ and $\mathbb{C}_{0,m+1}$ respectively. Thereby, this function theory may be considered as an elegant way of extending the theory of holomorphic functions in the complex plane to higher dimension and it provides at the same time a refinement of the theory of harmonic functions.

Clifford analysis is centered around the notion of monogenic function, i.e. a null solution of $\partial_x$. It is, however, often important (and interesting) to consider special types of solutions obtained by considering functions taking values in suitable subspaces of the real or complex Clifford algebras (so be the case).
As is established in [10], there exists an isomorphism between the Cartan algebra of differential forms and the algebra of multivector functions in Clifford analysis. In particular, the action of the operator $d - d^*$, where $d$ and $d^*$ are the differential and codifferential operators (the standard de Rham differential and its adjoint) respectively, on the space of smooth $k$-forms is identified with the action (on the right) of the Dirac operator, which plays the role of the Cauchy–Riemann operator on the space of smooth $k$-vector fields. Meanwhile the action of the operator $d + d^*$ is identified with the action (on the left) of the Dirac operator. A smooth differential form belonging to the kernel of $d + d^*$ was called in [12,31] self-conjugate differential form.

The Moisil–Teodorescu elliptic system of equations of first order in $\mathbb{R}^3$ is a vector valued analogue of Cauchy–Riemann system [37].

In the context of real Clifford analysis a generalized Moisil–Teodorescu systems of type $(r, p, q)$ was introduced in [1], where some general properties of solutions to this system have been investigated. Afterwards, there was a growing interest in the study and better understanding of properties of solutions of generalized Moisil–Teodorescu systems, see for instance [5,7,14–17,30,43–45]. Some special cases of these systems are well known and well understood.

Following the identification mentioned before, a subsystem of generalized Moisil–Teodorescu systems leads to a subsystem of self-conjugate differential forms and vice versa.

To deal with the inhomogeneous generalized Moisil–Teodorescu systems in [6] the authors embedded the systems in an appropriate real Clifford analysis setting. Necessary and sufficient conditions for the solvability of inhomogeneous systems are provided and its general solution described and consequently some results in the literature are re-obtained, such as those given in [2–4,39,40].

The present paper is devoted to give explicit general solution of the inhomogeneous generalized perturbed Moisil–Teodorescu system in the framework of complex Clifford analysis upon the usage of a notion of generalized harmonic conjugates pairs.

A wealth of information about the subject of the nontrivial connection between Maxwell’s electrodynamics, Clifford algebras and, in fact, expressing the monogenicity of a certain Clifford algebra-valued function can be found in the literature, see for instance [11,21,26–29,31–36,41,46]. The common feature of the method is to represent the Maxwell’s equations in a Dirac like form.

To illustrate the application of the main result, we establish solvability conditions for the inhomogeneous Maxwell’s equations. Our study is based on the complex Clifford algebra-based form of the Dirac equation and the Maxwell’s equations proposed in [32].
2. Rudiments of Clifford Analysis

The section provides a brief exposition of the basic notions and terminology of Clifford analysis aimed at readers who are unfamiliar with this function theory. Standard references are the monographs [8,18,23–25].

Let $\mathbb{R}^{0,m+1}$ be the real vector space $\mathbb{R}^{m+1}$ equipped with a quadratic form of signature $(0,m+1)$ and let $e_0, e_1, e_2, \ldots, e_m$ be an orthogonal basis of $\mathbb{R}^{0,m+1}$.

The real Clifford algebra $\mathbb{R}_{0,m+1}$ with generators $e_0, e_1, e_2, \ldots, e_m$, subject to the basic multiplication rules

\[ e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad i, j = 0, 1, \ldots, m, \quad i < j, \]

is a real linear associative but non-commutative algebra with identity 1, having dimension $2^{m+1}$ and containing $\mathbb{R}$ and $\mathbb{R}^{m+1}$ as subspaces.

The complex Clifford algebra $\mathbb{C}_{0,m+1}$ constructed over $\mathbb{R}^{0,m+1}$, is a linear associative algebra over $\mathbb{C}$, has dimension $2^{m+1}$ meaning that one takes the same standard basis as for $\mathbb{R}^{0,m+1}$, with the same multiplication rules, however allowing for complex constants. Indeed, an element of $\mathbb{C}_{0,m+1}$ may be written as $a = \sum A a_A e_A$, where $a_A$ are complex constants and $A$ runs over all the possible ordered sets

\[ A = \{i_1, \ldots, i_s\}, \quad 0 \leq i_1 < i_2 < \cdots < i_s \leq m, \quad \text{or} \ A = \emptyset, \]

and

\[ (e_A : |A| = s, \ s = 0, 1, \ldots, m + 1), \ e_A = e_{i_1} e_{i_2} \cdots e_{i_s}, e_{\emptyset} = 1. \]

Also, a conjugation is defined as the unique linear morphism of $\mathbb{C}_{0,m+2}^{(0)}$ with $\bar{e}_j = -e_j, \ j = 0, 1, \ldots, m + 1$. Notice that for any basic element $e_A$ with $|A| = s, \bar{e}_A = (-1)^{s(s+1)/2} e_A$.

One of the basic properties relied upon in building up the $\mathbb{C}_{0,m+1}$-valued continuously differentiable function theory in domains of $\mathbb{R}^{m+1}$ is the fact that the Dirac operator $\partial_x$ in $\mathbb{R}^{m+1}$ factorizes the Laplacian $\Delta_x$ through the relation $\partial^2_x = -\Delta_x$, where $\partial_x = \sum_{i=0}^m e_i \partial x_i, \ x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$.

For technical reasons to become clear below (see [32]), we embed everything into a larger Clifford algebra, say $\mathbb{R}_{0,m+1} \subseteq \mathbb{C}_{0,m+1} \subseteq \mathbb{C}_{0,m+2}$. Fix $\alpha \in \mathbb{C}$ and set

\[ \partial_{x,\alpha} = \partial_x + \alpha e_{m+1}, \]

then $-\partial^2_{x,\alpha} = \Delta_x + \alpha^2$, the Helmholtz operator.

If $F$ is a $\mathbb{C}_{0,m+2}$-valued function defined in an open subset $\Omega \subset \mathbb{R}^{m+1}$, set

\[ \partial_{x,\alpha} F := \sum_{i=0}^m e_i \frac{\partial F}{\partial x_i} + \alpha e_{m+1} F, \]

and

\[ F \partial_{x,\alpha} := \sum_{i=0}^m \frac{\partial F}{\partial x_i} e_i + \alpha F e_{m+1}. \]
Definition 2.1. Let $S$ be a subspace of $\mathbb{C}_{0,m+2}$ and $F : \Omega \to S$, whose components are of class $C^1$ in $\Omega$. Then $F$ is called left $\alpha$-monogenic (or $\alpha$-monogenic for short) in $\Omega$ if $\partial_{x,\alpha} F = 0$ in $\Omega$ and is called right $\alpha$-monogenic in $\Omega$ if $F \partial_{x,\alpha} = 0$ in $\Omega$. The corresponding classes will be denoted by $\mathcal{M}_\alpha(\Omega, S)$ and $\mathcal{M}_{\alpha,r}(\Omega, S)$, respectively.

A function $F : \Omega \to S$, whose components are of class $C^2$ in $\Omega$ is called $\alpha^2$-metaharmonic if $(\Delta_x + \alpha^2) F = 0$. The corresponding class will be denoted by $\mathcal{H}_{\alpha^2}(\Omega, S)$.

Remark 2.2. Observe that, by the factorization $-\partial^2_{x,\alpha} = \Delta_x + \alpha^2$, each component of a $\alpha$-monogenic function is annihilated by the Helmholtz operator and $\mathcal{M}_\alpha(\Omega, S) \subset \mathcal{H}_{\alpha^2}(\Omega, S)$ for any subspace $S$ of $\mathbb{C}_{0,m+2}$.

Following [32,33], for $x \neq 0$ and $\alpha \in \mathbb{C}$ with $\text{Im}(\alpha) > 0$ the expression

$$\Lambda_\alpha(x) := -\frac{1}{(4\pi)^{m+1}} \int_0^{+\infty} \exp\left(\frac{\alpha^2 t - |x|^2}{4t}\right) \frac{dt}{t^{m+1}},$$

is the fundamental solution of the Helmholtz operator $\Delta_x + \alpha^2$ in $\mathbb{R}^{m+1}$.

A direct consequence of the factorization of the Helmholtz operator is that

$$E_\alpha(x) := -\partial_{x,\alpha} \Lambda_\alpha(x) = -\sum_{i=0}^m e_i \frac{\partial \Lambda_\alpha}{\partial x_i} - \alpha e_{m+1} \Lambda_\alpha,$$

is the fundamental solution of the perturbed Dirac operator.

For example, if $m = 2$ then $\Lambda_\alpha(x) = -\exp(i\alpha |x|)/4\pi |x|$ so that for $x \in \mathbb{R}^3$

$$E_\alpha(x) = -\frac{1}{4\pi} \frac{\exp(i\alpha |x|)}{|x|^3} \left[ \bar{x} (i\alpha |x| - 1) - \alpha |x|^2 e_4 \right].$$

By Proposition 3.1 and Lemma 3.2 from [33] we have that

$$E_\alpha(x) = \begin{cases} \frac{1}{\sigma_{m+1} |x|^{m+1}} - \alpha e_{m+1} \Lambda_\alpha(x) + O \left(|x|^{-m+2}\right) & \text{as } |x| \to 0, \\ O \left(\exp\{-\text{Im}\alpha |x|\}\right) & \text{as } |x| \to +\infty, \end{cases}$$

where $\sigma_{m+1}$ is the area of the sphere in $\mathbb{R}^{m+1}$. For explicit formulae in a different setting we refer the reader to [48].

On the other side, recall that the space $\mathbb{C}^{(s)}_{0,m+2}$ of s-vectors in $\mathbb{C}_{0,m+2}$ ($0 \leq s \leq m + 2$) is defined by

$$\mathbb{C}^{(s)}_{0,m+2} = \text{span}_\mathbb{C}(e_A : |A| = s). \quad (2.1)$$

Notice, in particular, that for $s = 0$, $\mathbb{C}^{(0)}_{0,m+2} \cong \mathbb{C}$.

For $0 \leq s \leq m + 2$ fixed, the space $\mathbb{C}^{(s)}_{0,m+2}$ of s-vectors lead to the decomposition

$$\mathbb{C}_{0,m+2} = \bigoplus_{s=0}^{m+2} \mathbb{C}^{(s)}_{0,m+2}, \quad (2.2)$$

and the associated projection operators $[\ ]_s : \mathbb{C}_{0,m+2} \mapsto \mathbb{C}^{(s)}_{0,m+2}$. 
Notice that by writing \( E_\alpha(x) = E_1(x) + e_{m+1}E_2(x) \) with \( E_1(x) := \frac{1}{\sigma_{m+1}|x|^{m+1}} \) and \( E_2(x) := -\alpha\Lambda_\alpha(x) \) by letting \(|x| \to 0\), we have that \( E_1 \) is \( \mathcal{C}_{0,m+2}^{(1)} \)-valued while \( E_2 \) is \( \mathcal{C}_{0,m+2}^{(0)} \)-valued.

An element \( x = (x_0, x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+2} \) is usually identified with \( x = \sum_{i=0}^{m+1} e_i x_i \in \mathbb{R}^{0,m+2} \).

For \( x, y \in \mathcal{C}_{0,m+2}^{(1)} \), the product \( xy \) splits in two parts, namely
\[
xy = x \bullet y + x \wedge y, \tag{2.3}
\]
where \( x \bullet y = [xy]_0 \) is the scalar part of \( xy \) and \( x \wedge y = [xy]_2 \) is the 2-vector or bivector part of \( xy \) which are given by
\[
x \bullet y = -\sum_{i=0}^{m+1} x_i y_i,
\]
and
\[
x \wedge y = \sum_{i<j} e_i e_j (x_i y_j - x_j y_i).
\]

More generally, for \( x \in \mathcal{C}_{0,m+2}^{(1)} \) and \( v \in \mathcal{C}_{0,m+2}^{(s)} \), \((0 < s < m + 2)\), we have that the product \( xv \) decomposes into
\[
xv = x \bullet v + x \wedge v,
\]
where
\[
x \bullet v = [xv]_{s-1} = \frac{1}{2} (xv - (-1)^s vx), \tag{2.4}
\]
and
\[
x \wedge v = [xv]_{s+1} = \frac{1}{2} (xv + (-1)^s vx). \tag{2.5}
\]

3. Generalized Perturbed Moisil–Teodorescu Systems

Let \( r, p, q \in \mathbb{N} \) with \( 0 \leq r \leq m + 2, 0 \leq p \leq q \) and \( r + 2q \leq m + 2 \). Exploring further the multivector structure of \( \mathcal{C}_{0,m+2}^{(r,p,q)} \) one may also write
\[
\mathcal{C}_{0,m+2}^{(r,p,q)} = \bigoplus_{j=p}^{q} \mathcal{C}_{0,m+2}^{(r+2j)}.
\]

If a \( \mathcal{C}_{0,m+2}^{(r+1,p,q)} \bigoplus e_{m+1} \mathcal{C}_{0,m+2}^{(r,p,q)} \)-valued smooth function \( F \) defined in an open subset \( \Omega \subset \mathbb{R}^{m+1} \) is decomposed following
\[
F = \sum_{j=p}^{q} \bigoplus F^{(r+2j+1)} + e_{m+1} \sum_{j=p}^{q} \bigoplus F^{(r+2j)}
= \sum_{j=p}^{q} \bigoplus F^{(r+2j+1)} + \sum_{j=p}^{q} \bigoplus (-1)^{r+2j} F^{(r+2j)} e_{m+1},
\]
then in $\Omega$:

$$\partial_{x,\alpha} F = 0 \quad \text{if and only if,}$$

$$\begin{cases}
\partial_x^{-} F^{r+2p+1} - \alpha F^{r+2p} = 0, \\
\partial_x^{+} F^{r+2j+1} + \partial_x^{-} F^{r+2(j+1)+1} - \alpha F^{r+2(j+1)} = 0, & j = p, \ldots, q - 1; \\
\partial_x^{+} F^{r+2q+1} = 0, \\
\partial_x^{-} (-1)^{r+2p} F^{r+2p} = 0, \\
\partial_x^{+} (-1)^{r+2j} F^{r+2j} + \partial_x^{-} (-1)^{r+2(j+1)} F^{r+2(j+1)} + \alpha (-1)^{r+2j+1} F^{r+2j+1} = 0, & j = p, \ldots, q - 1; \\
\partial_x^{+} (-1)^{2+2q} F^{r+2q} + \alpha (-1)^{r+2q+1} F^{r+2q+1} = 0,
\end{cases}$$

(3.1)

where the differential operators $\partial_x^{+}$ and $\partial_x^{-}$ act on smooth $\mathbb{C}^{(s)}_{0,m+2}$-valued functions $F^s$ in $\Omega$ as

$$\partial_x^{+} F^s = \frac{1}{2} (\partial_x F^s - (-1)^s F^s \partial_x),$$

(3.2)

and

$$\partial_x^{-} F^s = \frac{1}{2} (\partial_x F^s + (-1)^s F^s \partial_x).$$

(3.3)

It is perhaps worth remarking that $\partial_x^{+} F^s$ is $\mathbb{C}^{(s+1)}_{0,m+2}$-valued while $\partial_x^{-} F^s$ is $\mathbb{C}^{(s-1)}_{0,m+2}$-valued.

The system (3.1) generalizes that of [6] and is called generalized perturbed Moisil–Teodorescu system of type $(r,p,q)$.

In [6] is pointed out that for $\alpha = 0$ the system (3.1) includes some basic systems of first order linear partial differential equations as particular cases. For example, if $p = 0$, $q = 1$ and $0 \leq r \leq m+1$ fixed, the system (3.1) reduces to the Moisil–Teodorescu system in $\mathbb{R}^{m+1}$ introduced in [5]. If $p = q = 0$ and $0 < r < m+1$ fixed, the system (3.1) reduces to the generalized Riesz system $\partial_x F^r = 0$; its solutions are called harmonic multi-vector fields. If $r = 0$, $p = 0$ and $m+1 = 3$ then $q = 1$, the original Moisil–Teodorescu system introduced in [42] is re-obtained. If $r = 0$, $p = 0$ and $m+1 = 4$ then $q = 2$ and one obtains the Fueter system in $\mathbb{R}^4$ for so-called left regular functions of quaternion variable; it lies at the basis of quaternionic analysis (see [22,47]).

4. The Inhomogeneous Dirac Equation

From now on we assume $\Omega$ to be a Lipschitz domain in $\mathbb{R}^{m+1}$, i.e., a domain whose boundary $\Gamma$ is given locally by the graph of a real valued Lipschitz function, after an appropriate rotation of coordinates.

The fundamental tool for solving the inhomogeneous perturbed Dirac equation (commonly called the $\overline{\partial}$-problem)

$$\partial_{x,\alpha} F = G,$$

(4.1)

is the Borel–Pompeiu integral formula (see below) which is named after the French and Romanian mathematicians Émile Borel (1871–1956) and Dimitrie Pompeiu (1873–1954), respectively.
For bounded \( F \in C^0(\Omega; \mathbb{C}_{0,m+2}) \), we consider the Teodorescu and the Cauchy-type operators associated to the Cauchy kernel \( E_\alpha \), i.e.,
\[
T_\Omega[F](x) := \int_\Omega E_\alpha(x - y)F(y)dy, \quad x \in \mathbb{R}^{m+2},
\]
and by
\[
C_\Gamma[F](x) := -\int_\Gamma E_\alpha(x - y)n(y)F(y)d\Gamma_y, \quad x \notin \Gamma,
\]
where \( n(y) = \sum_{i=0}^{m+1} e_in_i(y) \) is the outward pointing unit normal to \( \Gamma \) at \( y \in \Gamma \).

**Lemma 4.1.** Let \( F \in C^1(\Omega; \mathbb{C}_{0,m+2}) \cap C^0(\Omega \cup \Gamma; \mathbb{C}_{0,m+2}) \) and \( \alpha \in \mathbb{C} \) with \( \text{Im}(\alpha) > 0 \). Then we have
\[
(i) \quad C_\Gamma[F](x) + T_\Omega[\partial_{x,\alpha}F](x) = \begin{cases} F(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^{m+1} \setminus (\Omega \cup \Gamma). \end{cases} \tag{4.2}
\]
\[
(ii) \quad \partial_{x,\alpha}T_\Omega[F](x) = \begin{cases} F(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^{m+1} \setminus (\Omega \cup \Gamma). \end{cases} \tag{4.3}
\]
\[
(iii) \quad \partial_{x,\alpha}C_\Gamma[F](x) = 0, \quad x \in \Omega \cup (\mathbb{R}^{m+1} \setminus (\Omega \cup \Gamma)). \tag{4.4}
\]

For a proof of the Borel–Pompeiu formula (4.2) see, e.g. [48].

**Remark 4.2.** The Borel–Pompeiu formula (4.2) solves the inhomogeneous perturbed Dirac equation (4.1) in the standard way and the general solution is given by
\[
F = T_\Omega[G] + H, \tag{4.5}
\]
where \( H \in M_\alpha(\Omega; \mathbb{C}_{0,m+2}) \).

5. Generalized Conjugate Metaharmonic Pairs

The notion of conjugate harmonic functions in the complex plane is well-known. This concept has been generalized to higher dimensional setting in the framework of Clifford analysis, see for instance [6,9,23,38,42].

In this section, we introduce a new generalization of notion of conjugate harmonic functions in a Clifford setting, based on a certain splitting of the Clifford algebra.

**Definition 5.1.** Let
\[
F_1 = \sum_{j=p}^{q} F^{r+2j+1} + e_{m+1} \sum_{j=p}^{q} F^{r+2j},
\]
in $H_{\alpha^2}\left(\Omega; \mathbb{C}_{0,m+2}^{(r+1,p,g)} \bigoplus e_{m+1}\mathbb{C}_{0,m+2}^{(r,p,g)}\right)$. An

$$F_2 = \left(F^{r+2p-1} + F^{r+2q+3}\right) + e_{m+1} \left(F^{r+2p-2} + F^{r+2q+2}\right),$$

in $H_{\alpha^2}\left(\Omega; \left(\mathbb{C}_{0,m+2}^{(r+2p-1)} \bigoplus \mathbb{C}_{0,m+2}^{(r+2q+3)}\right) \bigoplus e_{m+1} \left(\mathbb{C}_{0,m+2}^{(r+2p-2)} \bigoplus \mathbb{C}_{0,m+2}^{(r+2q+2)}\right)\right)$ is called hyper-conjugate metaharmonic to $F_1$ if

$$F_1 + F_2 \in \mathcal{M}_\alpha(\Omega; \mathbb{C}_{0,m+2}).$$

The pair $(F_1, F_2)$ is then called a pair of hyper-conjugate metaharmonic functions.

6. Main Result

We are in condition to state and prove our main result.

**Theorem 6.1.** Let $G \in C\left(\Omega; \sum_{j=p}^{q+1} \mathbb{C}_{0,m+2}^{(r+2j)} + \sum_{j=p}^{q+1} \mathbb{C}_{0,m+2}^{(r+2j-1)}e_{m+1}\right)$ and $\alpha \in \mathbb{C}$ with $\text{Im}(\alpha) > 0$. The inhomogeneous generalized perturbed Moisil–Teodorescu system

$$\begin{align*}
\partial_x^{-} F^{r+2p+1} - \alpha F^{r+2p} &= G^{r+2p}, \\
\partial_x^+ F^{r+2j+1} + \partial_x^- F^{r+2(j+1)+1} - \alpha F^{r+2(j+1)} &= G^{r+2j+2}, \\
\partial_x^+ F^{r+2q+1} &= G^{r+2q+2}, \\
\partial_x^{-} (-1)^{r+2p} F^{r+2p} &= G^{r+2p-1}, \\
\partial_x^+ (-1)^{r+2j} F^{r+2j} + \partial_x^- (-1)^{r+2(j+1)} F^{r+2(j+1)} + \alpha(-1)^{r+2j+1} F^{r+2j+1} &= G^{r+2j+1}, \\
\partial_x^+ (-1)^{r+2q} F^{r+2q} + \alpha(-1)^{r+2q+1} F^{r+2q+1} &= G^{r+2q+1},
\end{align*}$$

(6.1)

where $G^s \in C^1\left(\Omega; \mathbb{C}_{0,m+2}^{(s)} \cap \mathbb{C}^0\left(\Omega \cup \Gamma; \mathbb{C}_{0,m+2}^{(s)}\right)\right)$, has a solution if and only if for the $\left(\mathbb{C}_{0,m+2}^{(r+2p-1)} \bigoplus \mathbb{C}_{0,m+2}^{(r+2q+3)}\right) \bigoplus e_{m+1} \left(\mathbb{C}_{0,m+2}^{(r+2p-2)} \bigoplus \mathbb{C}_{0,m+2}^{(r+2q+2)}\right)$-valued function

$$P := \left(\int E_1 \bullet G^{r+2p} dy + (-1)^{r+2p} \int E_2 G^{r+2p-1} dy + \int E_1 \wedge G^{r+2q+2} dy\right)$$

$$+ e_{m+1} \left((-1)^{r+2p} \int E_1 \bullet G^{r+2p-1} dy + \int E_1 \wedge G^{r+2q+1} dy + \int E_2 G^{r+2q+2} dy\right),$$

(A) either is identically zero;
(B) or has a hyper-conjugate metaharmonic function.

If it is true, then the general solution of (6.1) is given by:
(A*) either
\[
F = \sum_{j=p}^{q} \left( \left( \int E_1 \land G^{r+2j} \, dy + \int E_1 \bullet G^{r+2j+2} \, dy + (-1)^{r+2j+2} \int E_2 G^{r+2j+1} \, dy \right)
\right.
\]
\[
+ \varepsilon_{m+1} \left( \int E_1 \land G^{r+2j-1} \, dy + (-1)^{r+2j+2} \int E_1 \bullet G^{r+2j+1} \, dy
\right.
\]
\[
+ \left. \int E_2 G^{r+2j} \, dy \right) \right] + \hat{H}_1,
\]
\[
(6.2)
\]
(B*) or
\[
F = \sum_{j=p}^{q} \left( \left( \int E_1 \land G^{r+2j} \, dy + \int E_1 \bullet G^{r+2j+2} \, dy
\right.
\right.
\]
\[
+ (-1)^{r+2j+2} \int E_2 G^{r+2j+1} \, dy
\]
\[
\left. + \varepsilon_{m+1} \left( \int E_1 \land G^{r+2j-1} \, dy + (-1)^{r+2j+2} \int E_1 \bullet G^{r+2j+1} \, dy
\right.
\right.
\]
\[
+ \left. \int E_2 G^{r+2j} \, dy \right) \right] + \hat{H}_1 + \hat{H}_1,
\]
\[
(6.3)
\]
where \( \hat{H}_1 \) is a metaharmonic hyper-conjugate of \(-P\), and \( \hat{H}_1 \) is an arbitrary \( \alpha \)-monogenic function.

Proof. First, notice that system (6.1) is a restriction of (4.1), for
\[
F = \sum_{j=p}^{q} F^{r+2j+1} + \varepsilon_{m+1} \sum_{j=p}^{q} F^{r+2j},
\]
and with
\[
G = \sum_{j=p}^{q+1} G^{r+2j} + \sum_{j=p}^{q+1} G^{r+2j-1} \varepsilon_{m+1}.
\]
Let \( F \) be such a solution. Looking at the components of (4.5) one has:
\[
\begin{align*}
0 &= \int E_1 \bullet G^{r+2p} \, dy + (-1)^{r+2p} \int E_2 G^{r+2p-1} \, dy + H^{r+2p-1}, \\
F^{r+2j+1} &= \int E_1 \land G^{r+2j} \, dy + \int E_1 \bullet G^{r+2j+2} \, dy \\
& \quad + (-1)^{r+2j+2} \int E_2 G^{r+2j+1} \, dy + H^{r+2j+1}, \quad j = p, \ldots, q, \\
0 &= \int E_1 \land G^{r+2q+2} \, dy + H^{r+2q+3}, \\
F^{r+2j} &= \int E_1 \land G^{r+2j-1} \, dy + (-1)^{r+2j+2} \int E_1 \bullet G^{r+2j+1} \, dy \\
& \quad + \int E_2 G^{r+2j} \, dy + H^{r+2j}, \quad j = p, \ldots, q, \\
0 &= \int E_1 \land G^{r+2q+1} \, dy + \int E_2 G^{r+2q+2} \, dy + H^{r+2q+2},
\end{align*}
\]
where \((H_1, H_2)\) is a hyper-conjugate metaharmonic pair with
\[
H_1 := \sum_{j=p}^{q} H^{r+2j+1} + \varepsilon_{m+1} \sum_{j=p}^{q} H^{r+2j},
\]
and

\[ H_2 := (H^{r+2p-1} + H^{r+2q+3}) + e_{m+1} (H^{r+2p-2} + H^{r+2q+2}). \]

For the case of \( P \) being the zero function, one obtains that \( \hat{H}_1 \) becomes also the zero function, and both the necessity of (A) and the formula (6.2) is proved.

For the case of \( P \) not being identically zero one has

\[
H_2 := - \left( \int E_1 \cdot G^{r+2p} d\gamma + (-1)^{r+2p} \int E_2 G^{r+2p-1} d\gamma + \int E_1 \wedge G^{r+2q+2} d\gamma \right) - e_{m+1} \left( (-1)^{r+2p} \int E_1 \cdot G^{r+2p-1} d\gamma + \int E_1 \wedge G^{r+2q+1} d\gamma + \int E_2 G^{r+2q+2} d\gamma \right),
\]

meaning the existence of the hyper-conjugate metaharmonic function to

\[
- \left( \int E_1 \cdot G^{r+2p} d\gamma + (-1)^{r+2p} \int E_2 G^{r+2p-1} d\gamma + \int E_1 \wedge G^{r+2q+2} d\gamma \right) - e_{m+1} \left( (-1)^{r+2p} \int E_1 \cdot G^{r+2p-1} d\gamma + \int E_1 \wedge G^{r+2q+1} d\gamma + \int E_2 G^{r+2q+2} d\gamma \right),
\]

which (whenever it exists) will be denote by \( \tilde{H}_1 \). Thus, the necessity of (B) is also proved together with formula (6.3).

Finally, for the “sufficiency part” one has to reverse the reasoning. If (A) is true the one can take \( H_2 \) as the zero function implying \( H_1 \) to be an arbitrary \( \alpha \)-monogenic function; thus we arrive at (6.2). If (B) is true then one can take in (6.4) the function \( H_2 \) to be \( H_2 = -P \); hence, one can write \( H_1 \) as \( H_1 = \hat{H}_1 + \tilde{H}_1 \) for any arbitrary \( \alpha \)-monogenic \( \hat{H}_1 \); thus, one obtains (6.3), and the theorem is proved.

\[ \square \]

Remark 6.2. Our approach provides a rather new generalization and certain consolidation of the hyper-complex method (by means of techniques based on harmonic conjugates) described in [2–4,6,13,19,20,39,40] concerning the solvability of the \( \overline{\partial} \)-problem in spaces of functions satisfying different systems in the framework of (complex) Clifford and Quaternionic analysis.

7. The Inhomogenous Maxwell’s Equations

To keep the exposition self-contained let us repeat key observations from [32, pag. 1613]. Let \( E \) be a \( r \)-vector and \( H \) a \((r+1)\)-vector with \( 0 \leq r \leq m + 1 \),
defined in some open subset of $\mathbb{R}^{m+1}$. To these, we associate an $C_{0,m+2}$-valued function $M$ by setting

$$M := H - ie_{m+1}E = H + i(-1)^{r+1}Ee_{m+1}. \quad (7.1)$$

**Remark 7.1.** Observe that

$$\partial_{x,\alpha}M = i(-1)^{r+1} \left( \partial^+_xE + \partial^-xE - i\alpha H \right) e_{m+1} + \left( \partial^+_xH + \partial^-xH + i\alpha E \right).$$

**Proposition 7.2.** The function $M$ defined in (7.1) is (left or right) $\alpha$-monogenic if and only if $E$ and $H$ satisfy the Maxwell’s equations

$$\begin{cases} 
\partial^+_xE - i\alpha H = 0, \\
\partial^-xE = 0, \\
\partial^-xH + i\alpha E = 0, \\
\partial^+_xH = 0.
\end{cases} \quad (7.2)$$

Moreover, for $x \in \mathbb{R}^{m+1} \setminus \{0\}$ the function $M$ satisfies the radiation condition

$$\left(1 - ie_{m+1} \frac{x}{|x|}\right) M(x) = o \left(|x|^{-m/2}\right) \quad \text{as} \quad |x| \to \infty,$$

if and only if $E$ and $H$ satisfy the Silver-Müller-type radiation conditions

$$\begin{aligned}
\frac{x}{|x|} \wedge E - H &= o \left(|x|^{-m/2}\right) \quad \text{as} \quad |x| \to \infty, \\
\frac{x}{|x|} \cdot H - E &= o \left(|x|^{-m/2}\right) \quad \text{as} \quad |x| \to \infty.
\end{aligned} \quad (7.3)$$

**Remark 7.3.** If $\alpha$ is non-zero then, because $(\partial^+_x)^2 = (\partial^-_x)^2 = 0$, the equations $\partial^-_xE = 0$ and $\partial^+_xH = 0$ become superfluous. Nonetheless, as for $\alpha = 0$ Maxwell’s equations decouple (i.e. $E$ and $H$ become unrelated), it is precisely this case for which these two equations are relevant. Also, when $m + 1 = 3$, $r = 1$ (and $\alpha \neq 0$), the formulae (7.2) reduce to the more familiar system of equations

$$\begin{cases} 
\nabla \times E - i\alpha H = 0, \\
\nabla \times H + i\alpha E = 0.
\end{cases} \quad (7.5)$$

**Theorem 7.4.** Let $G \in C \left(\Omega; \sum_{j=0}^1 \bigoplus \mathbb{C}^{(r+2j)}_{0,m+2} + \sum_{j=0}^1 \bigoplus \mathbb{C}^{(r+2j-1)}_{0,m+2}e_{m+1}\right)$ and $\alpha \in \mathbb{C}$ with $\text{Im}(\alpha) > 0$. The inhomogeneous Maxwell’s equations

$$\begin{cases} 
\partial^+_xE - i\alpha H = G^{r+1}, \\
\partial^-_xE = G^{r-1}, \\
\partial^-xH + i\alpha E = G^r, \\
\partial^+_xH = G^{r+2},
\end{cases} \quad (7.6)$$
where \( G^* \in C^1(\Omega; C_{0,m+2}^{(s)}(\Omega \cup \Gamma; C_{0,m+2}^{(s)}) \) has a solution if and only if for the \( C_0^{(s)}(\Omega; C^{(s)}_{0,m+2} + C_0^{(s)}(\Omega \cup \Gamma; C^{(s)}_{0,m+2}) \)-valued function

\[
P := \left( \int E_1 \cdot G^r dy + (-1)^r \int E_2 G^{r-1} dy + \int E_1 \wedge G^{r+2} dy \right) + \epsilon_{m+1} \left( (-1)^r \int E_1 \cdot G^{r-1} dy + \int E_1 \wedge G^{r+1} dy + \int E_2 G^{r+2} dy \right),
\]

(A) either is identically zero;
(B) or has a hyper-conjugate metaharmonic function.

If it is true, then the general solution of (7.6) is given by:

(A*) either

\[
M = H - i\epsilon_{m+1} E = H + i(-1)^{r+1} E \epsilon_{m+1}
\]

\[
= \left( \int E_1 \wedge G^r dy + \int E_1 \cdot G^{r+2} dy + (-1)^{r+2} \int E_2 G^{r+1} dy \right)
\]

\[
+ \epsilon_{m+1} \left( \int E_1 \wedge G^{r-1} dy + (-1)^{r+2} \int E_1 \cdot G^{r+1} dy + \int E_2 G^r dy \right)
\]

\[
+ \hat{H}_1,
\]

(B*) or

\[
M = H - i\epsilon_{m+1} E = H + i(-1)^{r+1} E \epsilon_{m+1}
\]

\[
= \left( \int E_1 \wedge G^r dy + \int E_1 \cdot G^{r+2} dy + (-1)^{r+2} \int E_2 G^{r+1} dy \right)
\]

\[
+ \epsilon_{m+1} \left( \int E_1 \wedge G^{r-1} dy + (-1)^{r+2} \int E_1 \cdot G^{r+1} dy + \int E_2 G^r dy \right)
\]

\[
+ \hat{H}_1 + \tilde{H}_1,
\]

where \( \tilde{H}_1 \) is a metaharmonic hyper-conjugate of \(-P\), and \( \hat{H}_1 \) is an arbitrary \( \alpha \)-monogenic function.

**Proof.** It is a direct consequence of the Theorem 6.1 by choosing \( p = q = 0 \) and \( F = M \).

**Remark 7.5.** Observe that, under the assumptions of the previous Theorem, it is possible to obtain both the electric field and the magnetic field separately; say if \( P \) is identically zero, then

\[
H = \int E_1 \wedge G^r dy + \int E_1 \cdot G^{r+2} dy + (-1)^{r+2} \int E_2 G^{r+1} dy + \left[ \hat{H}_1 \right]_{r+1},
\]

\[
E = i \left( \int E_1 \wedge G^{r-1} dy + (-1)^{r+2} \int E_1 \cdot G^{r+1} dy + \int E_2 G^r dy + \left[ \hat{H}_1 \right]_r \right),
\]
while if $P$ has a hyper-conjugate metaharmonic function, then

$$H = \int E_1 \wedge G^r dy + \int E_1 \bullet G^{r+2} dy + (-1)^{r+2} \int E_2 G^{r+1} dy$$

$$+ \left[ \hat{H}_1 \right]_{r+1} + \left[ \tilde{H}_1 \right]_{r+1},$$

$$E = i \left( \int E_1 \wedge G^{r-1} dy + (-1)^r \int E_1 \bullet G^{r+1} dy + \int E_2 G^r dy +

+ \left[ \hat{H}_1 \right]_r + \left[ \tilde{H}_1 \right]_r \right).$$

Acknowledgements

The authors wish to thank Instituto Politécnico Nacional and Fundación Universidad de las Américas Puebla, for partial financial support. We also thank the Reviewers for their comments and careful review, which helped improve the manuscript.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Abreu Blaya, R., Bory Reyes, J., Delanghe, R., Sommen, F.: Generalized Moisil–Théodoresco systems and Cauchy integral decompositions. Int. J. Math. Math. Sci. 2008, Article ID746946 (2008)

[2] Abreu Blaya, R., Bory Reyes, J., Luna-Elizarrarás, M.E., Shapiro, M.: $\bar{\partial}$-problem in domains of $\mathbb{C}^2$ in terms of hyper-conjugate harmonic functions. Complex Var. Ellipt. Equ. 57(7–8), 743–749 (2012)

[3] Abreu Blaya, R., Bory Reyes, J.: $\bar{\partial}$-problem for an overdetermined system con two higher dimensional variables. Arch. Math. (Basel) 97(6), 579–586 (2011)

[4] Bory Reyes, J., Abreu Blaya, R., Pérez-de la Rosa, M.A., Schneider, B.: A quaternionic treatment of inhomogeneous Cauchy–Riemann type systems in some traditional theories. Complex Anal. Oper. Theory 11(5), 1017–1034 (2017)

[5] Bory Reyes, J., Delanghe, R.: On the structure of solutions of the Moisil–Théodoresco system in Euclidean space. Adv. Appl. Clifford Algebras 19(1), 15–28 (2009)

[6] Bory-Reyes, J., Pérez-de la Rosa, M.A.: Solutions of inhomogeneous generalized Moisil–Teodorescu systems in Euclidean space. Adv. Appl. Clifford Algebras 29(2), Paper No. 27 (2019)

[7] Brackx, F., Delanghe, R., De Schepper, H.: Hardy spaces of solutions of generalized Riesz and Moisil–Teodorescu systems. Complex Var. Ellipt. Equ. 57(7–8), 771–785 (2012)

[8] Brackx, F., Delanghe, R., Sommen, F.: Clifford Analysis. Pitman Publishers, Boston (1982)

[9] Brackx, F., Delanghe, R., Sommen, F.: On conjugate harmonic functions in Euclidean space. Math. Methods Appl. Sci. 25, 1553–1562 (2002)
[10] Brackx, F., Delanghe, R., Sommen, F.: Differential forms and/or multi-vector functions. Cubo 7(2), 139–169 (2005)
[11] Chisholm, M.: Such Silver Currents: The Story of William and Lucy Clifford 1845–1929, p. 28. Lutterworth Press, Cambridge (2002)
[12] Cialdea, A.: On the theory of self-conjugate differential forms. Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996). Atti Sem. Mat. Fis. Univ. Modena 46(suppl.), 595–620 (1998)
[13] Colombo, F., Luna-Elizarrarás, M.E., Sabadini, I., Shapiro, M., Struppa, D.C.: A quaternionic treatment of the inhomogeneous div-rot system. Mosc. Math. J. 12(1), 37–48 (2012)
[14] Delanghe, R.: On homogeneous polynomial solutions of the Riesz system and their harmonic potentials. Complex Var. Ellipt. Equ. 52(10–11), 1047–1061 (2007)
[15] Delanghe, R.: On Moisil–Théodoresco systems in Euclidean space. AIP Conf. Proc. 1048(1), 17–20 (2008)
[16] Delanghe, R.: On homogeneous polynomial solutions of generalized Moisil–Théodoresco systems in Euclidean space. Cubo 12(2), 145–167 (2010)
[17] Delanghe, R., Lávicka, R., Souček, V.: On polynomial solutions of generalized Moisil–Théodoresco systems and Hodge–de Rham systems. Adv. Appl. Clifford Algebras 21(3), 521–530 (2011)
[18] Delanghe, R., Sommen, F., Souček, V.: Clifford Algebra and Spinor-Valued Functions—A Function Theory for the Dirac Operator. Kluwer Academic Publishers, Dordrecht (1992)
[19] Delgado, B.B., Porter, M.R.: General solution of the inhomogeneous div-curl system and consequences. Adv. Appl. Clifford Algebras 27(4), 3015–3037 (2017)
[20] Delgado, B.B., Kravchenko, V.V.: A right inverse operator for curl + λ curl + λ and applications. Adv. Appl. Clifford Algebras 29(3), 29:40 (2019)
[21] Franssens, G.R.: Introducing Clifford Analysis as the Natural Tool for Electromagnetic Research. PIERS Proceedings, Moscow, Russia, August 19–23, pp. 112–116 (2012)
[22] Fueter, R.: Die Funktionentheorie der Differentialgleichungen Δu = 0 und ΔΔu = 0 mit vier reellen Variablen. Comment. Math. Helv. 7(1), 307–330 (1934)
[23] Gilbert, J., Murray, M.: Clifford Algebras and Dirac Operators in Harmonic Analysis. Cambrigde University Press, Cambridge (1991)
[24] Gürlebeck, K., Sprössig, W.: Quaternionic and Clifford Calculus for Physicists and Engineers. Mathematical Methods in Practice, Wiley (1997)
[25] Gürlebeck, K., Habetha, K., Sprössig, W.: Holomorphic Functions in the Plane and n-Dimensional Space. Birkhäuser, Basel (2008)
[26] Hernandez-Herrera, A.: Higher dimensional transmission problems for Dirac operators on Lipschitz domains. J. Math. Anal. Appl. 478(2), 499–525 (2019)
[27] Imaeda, K.: A new formulation of classical electrodynamics. Nuovo Cimento B (11) 32(1), 138–162 (1976)
[28] Jancewicz, B.: Multivectors and Clifford Algebra in Electrodynamics. World Scientific, Teaneck (1988)
[29] Kravchenko, V.V.: Applied quaternionic analysis. Maxwell’s system and Dirac’s equation. In: Tutshkie, W. (ed.) Functional-Analytic and Complex Methods, Their Interactions, and Applications to Partial Differential Equations. World Scientific, pp. 143–160 (2001)
[30] Lavicka, R.: Orthogonal Appell bases for Hodge–de Rham systems in Euclidean spaces. Adv. Appl. Clifford Algebras 23(1), 113–124 (2013)
[31] Malaspaia, A.: The Rudin–Carleson theorem for non-homogeneous differential forms. Int. J. Pure Appl. Math. 1(2), 203–215 (2002)
[32] McIntosh, A., Mitrea, M.: Clifford algebras and Maxwell’s equations in Lipschitz domains. Math. Methods Appl. Sci. 22(18), 1599–1620 (1999)
[33] Mitrea, M.: Boundary value problems and Hardy spaces associated to the Helmholtz equation in Lipschitz domains. J. Math. Anal. Appl. 202(3), 819–842 (1996)
[34] Mitrea, M.: Boundary value problems for Dirac operators and Maxwell’s equations in nonsmooth domains. Math. Methods Appl. Sci. 25(16–18), 1355–1369 (2002)
[35] Mitrea, M.: Generalized Dirac operators on non-smooth manifolds and Maxwell’s equations. J. Fourier Anal. Appl. 7(3), 207–256 (2001)
[36] Mohazzabi, P., Wielenberg, N.J., Alexander, G.C.: A new formulation of Maxwell’s equations in Clifford algebra. J. Appl. Math. Phys. 5, 1575–1588 (2017)
[37] Moisil, G.C., Theodorescu, N.: Fonction holomophic dans l’espace. Bul. Soc. St. Cluj 6, 177–194 (1931)
[38] Nolder, C.A.: Conjugate harmonic functions and Clifford algebras. J. Math. Anal. Appl. 302(1), 137–142 (2005)
[39] Porter, M.R., Shapiro, M., Vasilevski, N.L.: Quaternionic differential and integral operators and the ∂-problem. J. Nat. Geom. 6(2), 101–124 (1994)
[40] Porter, M.R., Shapiro, M., Vasilevski, N.L.: On the analogue of the ∂-problem in quaternionic analysis. In: Clifford Algebras and Their Applications in Mathematical Physics (Deinze, 1993), Fundamental Theories of Physics, vol. 55, pp. 167–173. Kluwer Academic Publishers Group, Dordrecht (1993)
[41] Seagar, A.: Application of Geometric Algebra to Electromagnetic Scattering. The Clifford–Cauchy–Dirac Technique. Springer, Singapore (2016)
[42] Shapiro, M.: On the conjugate harmonic functions of M. Riesz-E. Stein-G. Weiss. Topics in complex analysis, differential geometry and mathematical physics (St. Konstantin, 1996), 8–32. World Sci. Publ, River Edge (1997)
[43] Sirkka-Liisa, E., Heikki, O.: On Hodge–de Rham systems in hyperbolic Clifford analysis. AIP Conf. Proc. 1558, 492–495 (2013)
[44] Soucek, V.: On massless Field equation in higher dimensions. In: Gürlebeck, K., Körke, C. (eds.)18th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, Weimar, 07–09 July (2009)
[45] Soucek, V.: Representation theory in Clifford analysis. In: Alpay, D. (ed.) Operator Theory. Springer, Basel, pp. 1509–1547 (2015)
[46] Spörgg, W.: Maxwell’s equations in Clifford calculus framework—an overview on the development. Finite or infinite dimensional complex analysis and applications, pp. 85–100. Adv. Complex Anal. Appl., vol. 2. Kluwer Acad. Publ., Dordrecht (2004)
[47] Sudbery, A.: Quaternionic analysis. Math. Proc. Camb. Philos. Soc. 85(2), 199–224 (1979)

[48] Zhenyuan, X.A.: Function theory for the operator $D - \lambda$. Complex Var. Ellipt. Equ. 16(1), 27–42 (1991)

Juan Bory-Reyes
ESIME-Zacatenco
Instituto Politécnico Nacional
07738 Mexico City
Mexico
e-mail: juanboryreyes@yahoo.com

Marco Antonio Pérez-de la Rosa
Department of Actuarial Sciences, Physics and Mathematics
Universidad de las Américas Puebla
San Andrés Cholula
72810 Puebla
Mexico
e-mail: marco.perez@udlap.mx

Received: October 15, 2020.
Accepted: March 18, 2021.