An extension of Fourier analysis for the $n$-torus in the magnetic field and its application to spectral analysis of the magnetic Laplacian

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Abstract

We solved the Schrödinger equation for a particle in a uniform magnetic field in the $n$-dimensional torus. We obtained a complete set of solutions for a broad class of problems; the torus $T^n = \mathbb{R}^n/\Lambda$ is defined as a quotient of the Euclidean space $\mathbb{R}^n$ by an arbitrary $n$-dimensional lattice $\Lambda$. The lattice is not necessary either cubic or rectangular. The magnetic field is also arbitrary. However, we restrict ourselves within potential-free problems; the Schrödinger operator is assumed to be the Laplace operator defined with the covariant derivative. We defined an algebra that characterizes the symmetry of the Laplacian and named it the magnetic algebra. We proved that the space of functions on which the Laplacian acts is an irreducible representation space of the magnetic algebra. In this sense the magnetic algebra completely characterizes the quantum mechanics in the magnetic torus. We developed a new method for Fourier analysis for the magnetic torus and used it to solve the eigenvalue problem of the Laplacian. All the eigenfunctions are given in explicit forms.

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1 Introduction

In this paper we solve the Schrödinger equation for a particle in a uniform magnetic field in an $n$-dimensional torus. The problem looks plain at first sight but actually it turns out to be a hard problem. Hence we begin this paper by a quick explanation of the problem. After that we will describe our strategy to solve it. Subsequently we will briefly mention studies by other people and describe our motivation of this study. At the end of Introduction we will give guides for quick access to main results of this paper.

An $n$-dimensional torus, or $n$-torus, is defined as $T^n = \mathbb{R}^n / \mathbb{Z}^n$. In the coordinate a point $(t^1, \ldots, t^j + 1, \ldots, t^n)$ is identified with $(t^1, \ldots, t^j, \ldots, t^n)$ for each $j = 1, \ldots, n$. The eigenvalue problem of the ordinary Laplacian in the torus is the equation

$$- \Delta f = - \sum_{j=1}^{n} \left( \frac{\partial}{\partial t^j} \right)^2 f = \varepsilon f \quad (1.1)$$

with the periodic boundary condition

$$f(t^1, \ldots, t^j + 1, \ldots, t^n) = f(t^1, \ldots, t^j, \ldots, t^n). \quad (1.2)$$

The eigenvalue problem can be immediately solved by Fourier expansion. A plane-wave function

$$\chi_k(t^1, \ldots, t^n) = e^{2\pi i \sum_{j=1}^{n} k_j t^j} \quad (1.3)$$

with quantized momenta $k_j \in \mathbb{Z}$ is a solution. The whole set of eigenfunctions $\{ \chi_k \mid (k_1, \ldots, k_n) \in \mathbb{Z}^n \}$ constitutes a complete orthonormal set of the space of periodic functions over the torus. This is a well-known result.

In this paper we would like to solve an eigenvalue problem of the magnetic Laplacian; the magnetic Laplacian is defined by replacing the partial derivative in the ordinary Laplacian by a covariant derivative as

$$\Delta f = \sum_{j,l=1}^{n} g^{jl} \left( \frac{\partial}{\partial t^j} - 2\pi i A_j \right) \left( \frac{\partial}{\partial t^l} - 2\pi i A_l \right) f. \quad (1.4)$$

Here $A_l$ is a component of the $U(1)$ gauge field

$$A_l = \frac{1}{2} \sum_{j=1}^{n} \phi_{jl} t^j + \alpha_l \quad (1.5)$$

with integers $\{ \phi_{jl} = -\phi_{lj} \}$ and real numbers $\{ \alpha_j \}$. The gauge field $A = \sum_{l=1}^{n} A_l \, dt^l$ generates a uniform magnetic field $B = dA = (1/2) \sum_{j,l=1}^{n} \phi_{jl} \, dt^j \wedge dt^l$. Moreover, we would like to consider a general oblique torus $T^n = \mathbb{R}^n / \Lambda$; $\Lambda$ is an $n$-dimensional lattice. Edges of the unit cell of the lattice do not necessarily cross at a right angle and they do not necessarily have a same length. Hence we introduce a metric $g^{jl}$ in the definition of the magnetic Laplacian.
to take inclined and stretched or shortened unit cells into account. The eigenvalue problem of (1.4) is accompanied by the condition
\[ f(t^1, \ldots, t^{j+1}, \ldots, t^n) = e^{\pi i \sum k=1^n \phi_{jk} t^k} f(t^1, \ldots, t^j, \ldots, t^n), \quad (1.6) \]
which we call a twisted periodic condition. Thus the plain problem (1.1) with (1.2) is generalized to the ‘magnetic’ problem (1.4) with (1.6). At first glance it looks rather straightforward to generalize the problem in this way but it is actually highly nontrivial and difficult to generalize the solution.

Let us see where the difficulty lies. In the case of the ordinary Laplacian, the plane-wave solution (1.3) is a simultaneous eigenfunction of the momentum operators
\[ p_j = -i \frac{\partial}{\partial t^j} \quad (j = 1, \ldots, n) \quad (1.7) \]
as \( p_j \chi_k = 2\pi k_j \chi_k \). The Laplacian can be expressed in terms of the momentum operators as
\[ -\Delta = \sum_{j=1}^n (p_j)^2 \]
and, of course, it commutes with the momentum operators. Thus integers \((k_1, \ldots, k_n)\) are ‘good’ quantum numbers. Then the whole set of simultaneous eigenfunctions \(\{\chi_k\}\) forms the complete solutions of the Laplacian problem. This is the way how Fourier analysis works. However, when we turn to the magnetic Laplacian, we may seek for a simultaneous eigenfunction of ‘magnetic’ momentum operators
\[ p_j = -i \left( \frac{\partial}{\partial t^j} - 2\pi i A_j \right) \quad (j = 1, \ldots, n). \quad (1.8) \]
But such a simultaneous eigenfunction does not exist because magnetic momenta do not commute with each other and instead exhibit commutators
\[ [p_j, p_l] = 2\pi i \phi_{jl}. \quad (1.9) \]
The magnetic Laplacian can be still expressed in terms of the magnetic momentum operators as \(-\Delta = \sum_{j=1}^n (p_j)^2\) but it does not commute with \(p_j\). Hence, the strategy of ordinary Fourier analysis does not work well for the magnetic Laplacian.

To solve the problem we developed a new method, which we call Fourier analysis for the magnetic torus. This is a main subject of this paper. Let us describe our strategy: First, we will find a group of operators that commute with magnetic momentum operators. We call the group a magnetic translation group. Second, we enlarge a family of operators to define an algebra, which includes magnetic momenta and magnetic translations as its elements. We call the algebra a magnetic algebra and construct its representations. Third, we show that the space of twisted periodic functions over the torus is actually an irreducible representation space of the magnetic algebra. By diagonalizing a maximal commutative subalgebra of the magnetic algebra we obtain a complete orthonormal set of twisted periodic functions. This set of orthogonal functions provides a kind of unitary transformation as the
set of plane-wave functions provides the Fourier transformation which bridges between the momentum space and the real space. We note that it is easy to diagonalize the Laplacian in the momentum space. Finally, we get a whole set of eigenfunctions in the real space by applying the unitary transformation. In this procedure the third step is the hardest part and is actually accomplished by lengthy cumbersome calculations. However, the strategy is clear.

We would like to briefly review studies by other people on spectral analysis in magnetic field. Brown [1] first examined the symmetry structure of the Schrödinger equation for an electron in a lattice in a uniform magnetic field and found that the symmetry is described by a noncommutative discrete translation group. At the almost same time Zak [2] also found the same symmetry structure and named the group a magnetic translation group (MTG). Zak [3] immediately built a representation theory of the MTG in the three-dimensional lattice. From the viewpoint of functional analysis, Avron, Herbst, and Simon have been studying spectral problems of the Schrödinger operators in a magnetic field in a series of papers [4, 5, 6]. Dubrovin and Novikov [7, 8] studied the spectrum of the Pauli operator in a two-dimensional lattice with a periodic magnetic field and intensively analyzed the gap structure above the ground state. Florek [9, 10] constructed tensor product representations of the MTG to analyze a three-particle system in a lattice in a magnetic field. Kuwabara [11, 12] has been studying quantum-classical correspondence from the viewpoint of spectral geometry. For example, he [11] proved that if the whole set of level spacings of the quantum spectrum is not dense in $\mathbb{R}$, every trajectory of the corresponding classical particle is a closed orbit. Arai [13] found a quantum plane and quantum group structure in the quantum system in a singular magnetic field. Thus we can see that quantum mechanics in a magnetic field has been an active research area.

Our study on quantum mechanics in magnetic fields originates from studies of extra-dimension models of the space-time. In extra-dimension models the space-time is assumed to be a base space of a fiber bundle with a compact fiber or a noncompact fiber. The history of extra-dimension models is rather old, but an interest in these models is recently renewed as Arkani-Hamed, Dimopoulos, and Dvali [14] pointed out that the extra-dimension model may solve the hierarchy problem of high energy physics. Inspired with extra-dimension models we [15] built a model which has a circle $S^1$ as a fiber over an any-dimensional space-times $\mathbb{R}^{D-1}$. Then we found that a twisted boundary condition in the $S^1$-direction causes spontaneous breaking of the translational symmetry. Based on this observation, we [16] proposed a new mechanism of supersymmetry breaking. Next we [17, 18] built a model which has a two-dimensional sphere $S^2$ as a fiber over the four-dimensional space-times $\mathbb{R}^4$. We solved dynamics in the sphere in a magnetic monopole background and then found that the monopole induces spontaneous breaking of the rotational symmetry and the $CP$ symmetry. We also built a model which has an $n$-dimensional torus as a fiber and tried to
analyze dynamics in the torus in a background magnetic field. However, its analysis was not a straightforward task. Then we studied the symmetry structure of quantum mechanics in the torus in the magnetic field. We \[19\] constructed the MTG in the \(n\)-torus and classified irreducible representations of the MTG.

Armed with these tools we are now ready to solve the spectral problem in the \(n\)-torus \(T^n = \mathbb{R}^n/\Lambda\). We decide to solve the problem exhaustively; in our treatment the dimensions of the torus is taken to be arbitrary, lengths and angles of edges of the unit cell of \(\Lambda\) are arbitrary, and an arbitrary constant magnetic field is applied to the torus. Thus we aim to solve the widest class of quantum mechanics in the \(n\)-torus in uniform magnetic fields.

For busy readers here we give guides for quick access to main results. In Sec. 2 we provide a geometric setting to define the problem. The problem to be solved is the eigenvalue problem of the magnetic Laplacian \( (2.13) \) with the twist condition \((2.5)\). In Sec. 3 we find a family of operators that commute with the covariant derivative. Actually they are composition of ordinary displacements and gauge transformations as shown at \((3.2)\). These displacement vectors form a restricted family of vectors as shown at \((3.8)\). These displacement operators generate the magnetic translation group (MTG), which is noncommutative as shown at \((3.12)\). Along \((3.16)-(3.28)\) we construct irreducible representations of the MTG. In Sec. 4 we introduce a coordinate system, which will be revealed to be useful later. In Sec. 5 we define the magnetic algebra by adding differential operators \((5.2)-(5.4)\) and multiplicative operators \((5.9)\) to the MTG. Then we construct and classify irreducible representations of the magnetic algebra. Sec. 6 is devoted to calculation of simultaneous eigenfunctions \((6.8)\) of a maximal commutative subalgebra of the magnetic algebra. Then we obtain a complete orthonormal set of functions over the magnetic torus, which provide an extension of Fourier analysis for the magnetic torus. This is one of main products of this paper. In Sec. 7 by applying this method we solve the original problem, the eigenvalue problem of the magnetic Laplacian. There we obtain a whole set of eigenfunctions \((7.11)\) and eigenvalues \((7.12)\). These are main results of this paper.

## 2 Gauge field in the torus

Let \(t = (t^1, \ldots, t^n)\) denote a coordinate of an \(n\)-dimensional torus \(T^n = \mathbb{R}^n/\mathbb{Z}^n\). Namely, a point \((t^1, \ldots, \overline{t^j+1}, \ldots, t^n)\) is identified with \((t^1, \ldots, \overline{t^j}, \ldots, t^n)\) in \(T^n\). A uniform magnetic field is generated by the gauge field

\[
A = \sum_{k=1}^{n} A_k \, dt^k = \frac{1}{2} \sum_{j,k=1}^{n} \phi_{jk} \, t^j \, dt^k + \sum_{k=1}^{n} \alpha_k \, dt^k.
\]  

(2.1)

Here \(\{\phi_{jk} = -\phi_{kj}\}\) and \(\{\alpha_j\}\) are real constants. Then the magnetic field is

\[
B = dA = \frac{1}{2} \sum_{j,k=1}^{n} \phi_{jk} \, dt^j \wedge dt^k.
\]  

(2.2)
Therefore, the number $\phi_{jk}$ represents magnetic flux which penetrates the $(t^j, t^k)$-face of the torus. We call the array of numbers $(\phi_{jk})$ a magnetic flux matrix.

Let us introduce a complex scalar field $f$ in the torus. The scalar field couples to the gauge field via the covariant derivative

$$Df = df - 2\pi i Af.$$  (2.3)

We put the coefficient $2\pi i$ in front of $A$ for later convenience. Topology of the torus imposes a boundary condition on the scalar field. The gauge field itself is not a periodic function on $\mathbb{R}^n$ but it changes its form as

$$A(t^1, \ldots, t^j + 1, \ldots, t^n) = A(t^1, \ldots, t^j, \ldots, t^n) + \frac{1}{2} \sum_{k=1}^n \phi_{jk} dt^k.$$  (2.4)

Therefore, if we make the gauge transformation

$$f(t^1, \ldots, t^j + 1, \ldots, t^n) = e^{\pi i \sum_{k=1}^n \phi_{jk} t^k} f(t^1, \ldots, t^j, \ldots, t^n),$$  (2.5)

the covariant derivative (2.3) remains covariant as

$$Df(t^1, \ldots, t^j + 1, \ldots, t^n) = e^{\pi i \sum_{k=1}^n \phi_{jk} t^k} Df(t^1, \ldots, t^j, \ldots, t^n).$$  (2.6)

We call the condition (2.5) a twisted periodic condition. There are two ways to bring a point $(t^1, \ldots, t^j + 1, \ldots, t^k + 1, \ldots, t^n)$ to $(t^1, \ldots, t^j, \ldots, t^k, \ldots, t^n)$. The first way is

$$f(t^1, \ldots, t^j + 1, \ldots, t^k + 1, \ldots, t^n) = e^{\pi i \sum_{l=1}^n \phi_{jl} t^l} \sum_{l=1}^n \phi_{kl} t^l f(t^1, \ldots, t^j, \ldots, t^n).$$  (2.7)

The other way is

$$f(t^1, \ldots, t^j + 1, \ldots, t^k + 1, \ldots, t^n) = \sum_{l=1}^n \phi_{kl} t^l f(t^1, \ldots, t^j + 1, \ldots, t^k, \ldots, t^n).$$  (2.8)

To make these two expressions coincide we need to have

$$e^{\pi i \phi_{jk}} = e^{\pi i \phi_{kj}},$$

namely

$$e^{\pi i (\phi_{jk} - \phi_{kj})} = e^{2\pi i \phi_{jk}} = 1.$$  (2.9)

Therefore, compatibility of the periodic conditions (2.7) and (2.8) demands that $\phi_{jk}$ is an integer. Hence, the magnetic flux through each face of the torus is quantized. We call the torus where the magnetic field has been introduced a magnetic torus.
Since two displacements \( t^j \mapsto t^j + 1 \) and \( t^k \mapsto t^k + 1 \) are commutative, we can write the twisted periodic condition (2.5) in a more general form
\[
f(t + m) = e^{\pi i \sum_{j,k=1}^{n} \phi_{jk} m^j t^k} f(t)
\] (2.10)
with an arbitrary \( m = (m^1, \ldots, m^n) \in \mathbb{Z}^n \). An inner product of two twisted periodic functions \( f(t) \) and \( g(t) \) is defined by
\[
\langle f | g \rangle = \int_0^1 dt^1 \cdots \int_0^1 dt^n f^* (t) g(t).
\] (2.11)
Equipped with this inner product the space of twisted periodic functions becomes a Hilbert space.

To define the Laplacian we need to introduce a metric into the torus. Let \( A \) be an \( n \)-dimensional lattice in the Euclidean space \( \mathbb{R}^n \). We equip the torus \( T^n \) with a Riemannian structure by identifying \( T^n \) with the quotient space \( \mathbb{R}^n / A \). Let \( \{ u_1, \ldots, u_n \} \) be a set of vectors that generates the lattice \( A \). Their inner products is denoted by
\[
g_{jk} = \langle u_j, u_k \rangle
\] (2.12)
and its inverse is denoted by \( g^{jk} \). Then the magnetic Laplacian is defined as
\[
\Delta f = \sum_{j,k=1}^{n} g^{jk} \left( \frac{\partial}{\partial t^j} - 2\pi i A_j \right) \left( \frac{\partial}{\partial t^k} - 2\pi i A_k \right) f.
\] (2.13)
It is also referred as the Bochner Laplacian in literature. The purpose of this paper is to solve the eigenvalue problem of the magnetic Laplacian accompanied by the twisted periodic condition (2.5).

### 3 Magnetic translation group

Our goal is to find a complete set of eigenvalues and eigenfunctions of the magnetic Laplacian (2.13) as announced above. A royal road to solving an eigenvalue problem is to detect symmetry. In this section we determine a group of operators that commute with the Laplacian and construct irreducible representations of the group.

The vector space \( \mathbb{R}^n \) acts on the torus as isometries. However, the gauge field restricts the admissible class of vectors as seen below. An arbitrary vector \( v \in \mathbb{R}^n \) displaces the gauge field (2.1) as
\[
A(t) \mapsto A(t - v) = A(t) - d \left( \frac{1}{2} \sum_{j,k=1}^{n} \phi_{jk} v^j t^k \right).
\] (3.1)
If we perform a gauge transformation of the scalar field simultaneously with the displacement
\[
f(t) \mapsto f'(t) = (U(v) f)(t) = e^{\pi i \sum_{j,k=1}^{n} \phi_{jk} v^j t^k} f(t - v),
\] (3.2)
then the covariant derivative is changed covariantly
\begin{equation}
Df(t) \mapsto Df'(t) = e^{\pi i \sum_{j,k=1}^{n} \phi_{jk} v^j t^k} (Df)(t - v).
\end{equation}
In other words, the transformation $U(v)$ commutes with the covariant derivative as
\begin{equation}
(DU(v)f)(t) = (U(v)Df)(t).
\end{equation}
Hence it commutes with the magnetic Laplacian, which is defined in terms of the covariant derivative. The operator $U(v)$ is unitary with respect to the inner product (2.11).

The displaced function (3.2) also must satisfy the twisted periodic condition. If the original function $f$ satisfies the condition (2.10), the displaced function changes its form as
\begin{equation}
f'(t + m) = e^{2\pi i \sum_{j,k=1}^{n} \phi_{jk} v^j m^k} e^{\pi i \sum_{j,k=1}^{n} \phi_{jk} m^j t^k} f'(t)
\end{equation}
for $m \in \mathbb{Z}^n$. Thus the displaced function satisfies the condition (2.10) if and only if
\begin{equation}
\sum_{j,k=1}^{n} \phi_{jk} v^j m^k
\end{equation}
is an integer for an arbitrary $m \in \mathbb{Z}^n$. In other words,
\begin{equation}
\sum_{j=1}^{n} \phi_{jk} v^j (k = 1, \ldots, n)
\end{equation}
must be an integer. We call such a restricted vector $v$ a magnetic shift. The set of magnetic shifts forms an Abelian group
\begin{equation}
\Omega^n = \{v \in \mathbb{R}^n \mid \phi v \in \mathbb{Z}^n\}.
\end{equation}
There is a sequence of Abelian subgroups $\mathbb{Z}^n \subset \Omega^n \subset \mathbb{R}^n$. In particular, an integer vector $m \in \mathbb{Z}^n$ induces a displacement
\begin{equation}
(U(m)f)(t) = e^{\pi i \sum_{j,k=1}^{n} \phi_{jk} m^j t^k} f(t - m).
\end{equation}
However, owing to the twisted periodic condition (2.10), this is reduced to the identity transformation
\begin{equation}
(U(m)f)(t) = f(t).
\end{equation}
Thus we conclude that the group $G$ of effective transformations is generated by
\begin{equation}
\{U(v) \mid v \in \Omega^n / \mathbb{Z}^n\}.
\end{equation}
We call the group $G$ the magnetic translation group (MTG). A product of transformations is
\begin{equation}
U(v)U(w) = e^{\pi i \sum_{j,k=1}^{n} \phi_{jk} v^j w^k} U(v + w).
\end{equation}
Their commutator is
\[ U(v)U(w)U(-v)U(-w) = e^{2\pi i \sum_{j,k=1}^{n} \phi_{jk} v^j w^k}. \]  
(3.12)

We can say that the MTG is a central extension of the Abelian group \( \Omega^n / \mathbb{Z}^n \) by \( U(1) \).

We can express the MTG in a standard form. Let \( L(n, \mathbb{Z}) \) denote a group of the \( n \)-dimensional matrices \( \{ S \} \) of integers such that \( \det S = \pm 1 \). The matrix \( S \in L(n, \mathbb{Z}) \) acts on \( t \in \mathbb{R}^n \) by \( t \mapsto St \) and this action induces an automorphism of the torus \( T^n = \mathbb{R}^n / \mathbb{Z}^n \).

It also induces a transformation of the magnetic flux matrix as
\[ \phi_{jk} \mapsto \phi'_{jk} = \sum_{l,p=1}^{n} \phi_{lp} S^l_j S^p_k. \]  
(3.13)

The Frobenius lemma [20] tells that for any integral antisymmetric matrix there exists a transformation to bring it into a standard form
\[
\begin{pmatrix}
0 & q_1 & 0 & \cdots & 0 \\
-q_1 & 0 & q_2 & \cdots & 0 \\
0 & -q_2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -q_m & 0 \\
-q_m & 0 & \cdots & 0 & 0 \\
\end{pmatrix},
\]  
(3.14)

where \( \{ q_i \} \) are positive integers and they constitutes a sequence \( q_1 | q_2 | \cdots | q_m \), which implies \( q_i \) divides \( q_{i+1} \). For example, we may have a sequence \( 3 | 6 | 12 | 48 \). Of course, \( 2m \leq n \). The vector subspace of the zero eigenvalue of the matrix \( \phi \) has dimensions \( n - 2m \) and it is called null directions. In the following we suppose that the flux matrix is in the standard form \( (3.14) \).

Now we can write the magnetic shifts (3.8) in a more explicit form. Let \( \{ e_1, \ldots, e_n \} \) be the standard basis of \( \mathbb{R}^n \) in the \((t^1, \ldots, t^n)\)-coordinate. Then any magnetic shift is uniquely expressed as
\[ v = \sum_{j=1}^{m} \left( \frac{s_{2j-1}}{q_j} e_{2j-1} + \frac{s_{2j}}{q_j} e_{2j} \right) + \sum_{k=1}^{n-2m} \theta_{2m+k} e_{2m+k} \]  
(3.15)

with integers \( \{ s_1, \ldots, s_{2m} \} \) and real numbers \( \{ \theta_{2m+1}, \ldots, \theta_n \} \). Namely, the magnetic shifts are generated by \( \{(1/q_j)e_{2j-1}, (1/q_j)e_{2j} \mid j = 1, \ldots, m \} \) with integral coefficients and \( \{ e_{2m+k} \mid k = 1, \ldots, n - 2m \} \) with real coefficients. Hence, if the flux matrix \( \phi \) has null directions \( n - 2m > 0 \), the MTG has a continuous component. Otherwise, the MTG is a completely discrete group.
Here we summarize our discussion; the MTG is generated by the unitary operators

$$U_j = U \left( \frac{1}{q_j} e_{2j-1} \right), \quad V_j = U \left( \frac{1}{q_j} e_{2j} \right) \quad (j = 1, \ldots, m)$$

(3.16)

and

$$W_k(\theta) = U (\theta e_{2m+k}) \quad (k = 1, \ldots, n - 2m).$$

(3.17)

According to (3.9), (3.11), (3.12) and (3.14), these generators satisfy the following relations

$$(U_j)^{q_j} = (V_j)^{q_j} = 1,$$  

(3.18)

$$U_j V_j U_j^{-1} V_j^{-1} = e^{2\pi i/q_j},$$

(3.19)

$$W_k(1) = 1,$$  

(3.20)

$$W_k(\theta) W_k(\theta') = W_k(\theta + \theta')$$

(3.21)

and other trivial commutators.

To solve the eigenvalue problem of the magnetic Laplacian we need to prepare the whole set of irreducible representations of the MTG. Let \{\{r_1, \ldots, r_m, d_1, \ldots, d_{n-2m}\}\} be elements of a representation space that are labeled by

$$r_j \in \mathbb{Z}/\mathbb{Z}_{q_j} \quad (j = 1, \ldots, m),$$

(3.22)

$$d_k \in \mathbb{Z} \quad (k = 1, \ldots, n - 2m).$$

(3.23)

Then the generators (3.16) and (3.17) are represented by

$$U_j |r_1, \ldots, r_j, \ldots, r_m; d_1, \ldots, d_{n-2m}\rangle = |r_1, \ldots, r_j + 1, \ldots, r_m; d_1, \ldots, d_{n-2m}\rangle,$$

(3.24)

$$V_j |r_1, \ldots, r_j, \ldots, r_m; d_1, \ldots, d_{n-2m}\rangle = e^{-2\pi i r_j/q_j} |r_1, \ldots, r_j, \ldots, r_m; d_1, \ldots, d_{n-2m}\rangle,$$

(3.25)

$$W_k(\theta) |r_1, \ldots, r_j, \ldots, r_m; d_1, \ldots, d_{n-2m}\rangle = e^{-2\pi i d_k \theta} |r_1, \ldots, r_j, \ldots, r_m; d_1, \ldots, d_{n-2m}\rangle.$$  

(3.26)

Thus

$$\mathcal{H}_d = \bigoplus_{r_1 = 0}^{q_1} \cdots \bigoplus_{r_m = 0}^{q_m} C |r_1, \ldots, r_m; d_1, \ldots, d_{n-2m}\rangle$$

(3.27)

provides an irreducible representation space of the MTG. Its dimension is

$$\dim \mathcal{H}_d = q_1 \times q_2 \times \cdots \times q_m.$$  

(3.28)

The labels \(r_j\) and \(d_k\) in (3.22) and (3.23) will become good quantum numbers for the Laplacian (2.13). The dimension (3.28) will give the degree of degeneracy of each eigenvalue.

### 4 Diagonalization of the magnetic field strength

To solve the eigenvalue problem of the magnetic Laplacian we need to take the metric (2.12) into account. Remember that the basis \{\(u_1, \ldots, u_n\}\} generates the lattice \(A\) in the Euclidean...
space $R^n$ and that the torus is isometric to $R^n/\Lambda$. Let $x = (x^1, \ldots, x^n)$ be an orthonormal coordinate of $R^n$. It is related to the normalized coordinate $t = (t^1, \ldots, t^n)$ via

$$x = t^1 u_1 + \cdots + t^n u_n = U t. \quad (4.1)$$

In these coordinates the magnetic field (2.2) is expressed as

$$B = \frac{1}{2} \sum_{j,k=1}^n B_{jk} \, dx^j \wedge dx^k = \frac{1}{2} \sum_{j,k,l,p=1}^n B_{jk} \, u^j_i u^k_p \, dt^l \wedge dt^p = \frac{1}{2} \sum_{l,p=1}^n \phi_{lp} \, dt^l \wedge dt^p. \quad (4.2)$$

The number $B_{jk}$ is areal density of magnetic flux which penetrates the $(x^j, x^k)$-plane. If we perform a coordinate transformation

$$x = Ry \quad (4.3)$$

by an orthogonal transformation $R \in O(n, R)$, the components of the field strength is transformed as

$$B = \frac{1}{2} \sum_{j,k=1}^n B_{jk} \, dx^j \wedge dx^k = \frac{1}{2} \sum_{j,k,l,p=1}^n B_{jk} \, R^j_R^k \, dy^l \wedge dy^p. \quad (4.4)$$

By a suitable orthogonal transformation the field strength matrix $(B_{jk})$ can be brought into a standard form

$$\nu = t^R B R = \begin{pmatrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \nu_2 \\ -\nu_2 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & \nu_m \\ -\nu_m & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{pmatrix} \begin{pmatrix} \nu_2 & \cdots & \nu_m \\ -\nu_m & \cdots & 0 \end{pmatrix}, \quad (4.5)$$

where $\{\nu_j\}$ are positive real numbers. The $(t^1, \ldots, t^n)$-coordinate system block-diagonalizes the magnetic flux in the form of (3.14) while $(y^1, \ldots, y^n)$ block-diagonalizes the magnetic field strength in the form of (4.5). The transformations (4.1) and (4.3) are combined into

$$y = t^R x = t^R U t = Lt. \quad (4.6)$$

Then we obtain relations among the matrices

$$\phi = t^R B U = t^R R \nu^R U = t^L \nu L. \quad (4.7)$$

The phase factor in (3.2) is rewritten as

$$\sum_{j,k=1}^n v^j \phi_{jk} t^k = \sum_{j,k,l,p=1}^n v^j L^j_j \nu_{lp} L^p_k t^k = \sum_{j,l,p=1}^n v^j L^j_j \nu_{lp} y^p = \sum_{j=1}^n \sum_{l=1}^m v^j (L^{2j-1}_j \nu_l y^{2j-1} - L^{2j}_j \nu_l y^{2j-1}). \quad (4.8)$$
The gauge field (2.1) is expressed in the y-coordinate as

\[ A = \frac{1}{2} \sum_{j=1}^{m} \nu_j (y^{2j-1} dy^{2j} - y^{2j} dy^{2j-1}) + \sum_{j,k=1}^{n} \alpha_j (L^{-1})^j_k dy^k. \] (4.9)

We put

\[ \beta_k = \sum_{j=1}^{n} \alpha_j (L^{-1})^j_k \] (4.10)

for later use.

Actually, we can choose a transformation matrix \( L \) that has zeros in this pattern

\[ L = \begin{pmatrix} L_{2p-1}^{2i-1} & L_{2q}^{2i-1} & L_{2m+r}^{2i-1} \\ L_{2p-1}^{2i} & L_{2q}^{2i} & L_{2m+r}^{2i} \\ L_{2p-1}^{2i+k} & L_{2q}^{2i+k} & L_{2m+r}^{2i+k} \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \] (4.11)

with \( i, j, p, q = 1, \ldots, m \) and \( k, r = 1, \ldots, n - 2m \). The inverse matrix \( L^{-1} \) also has the same pattern of zeros. This distribution of zeros is proved in the appendix A. Then we can rewrite the magnetic shift operators (3.16) and (3.17) with a help of (4.8) in the y-coordinate as

\[ U_j f(y^i) = e^{\pi i} \sum_{l=1}^{m} \nu_l (L_{2j-1}^{2i-1} - L_{2j-1}^{2i} \nu_l y^{2l-1} - L_{2j-1}^{2i} \nu_l y^{2l-1}) f(y^i - L_{2j-1}^i (1/q_j)), \] (4.12)

\[ V_j f(y^i) = e^{-\pi i} \sum_{l=1}^{m} \nu_l (L_{2j}^{2i} y^{2l-1} - L_{2j}^{2i} y^{2l-1}) f(y^i - L_{2j}^i (1/q_j)), \] (4.13)

\[ W_k(\theta) f(y^i) = f(y^i - L_{2m+k}^i \theta) \] (4.14)

for \( j = 1, \ldots, m \) and \( k = 1, \ldots, n - 2m \). From (4.11) and (4.7) we get a formula

\[ \nu_j L_{2j-1}^{2i-1} = \sum_{l,p=1}^{m} \nu_l L_{2j-1}^{2i-1} L_{2p}^{2i} (L^{-1})_{2i}^{2p} \]
\[ = \sum_{l,p=1}^{m} \left( L_{2j-1}^{2i-1} \nu_l L_{2p}^{2i} - L_{2j-1}^{2i} \nu_l L_{2p}^{2i-1} \right) (L^{-1})_{2i}^{2p} \]
\[ = \sum_{p=1}^{m} q_j \delta_{jp} (L^{-1})_{2i}^{2p} \]
\[ = q_j (L^{-1})_{2i}^{2j}, \] (4.15)

which will be repeatedly used later.

5 Magnetic algebra

In this section we introduce new operators which act on twisted periodic functions. These new operators and the operators in the MTG generate an algebra, which we call a magnetic algebra. We construct and classify its irreducible representations. In the next section we will prove that the space of twisted periodic functions is actually an irreducible representation of the magnetic algebra. In this sense, the magnetic algebra completely characterizes the quantum mechanics in the magnetic torus.
Now we introduce a family of Hermite operators. Expanding the covariant derivative \((2.3)\) in terms of the \(y\)-coordinate

\[
Df = i \sum_{l=1}^{n} P_{l} f \, dy^{l},
\]

we define differential operators

\[
P_{2j-1} = -i \left( \frac{\partial}{\partial y^{2j-1}} + \pi i \nu_{j} y^{2j} - 2\pi i \beta_{2j-1} \right)
\]

\[
P_{2j} = -i \left( \frac{\partial}{\partial y^{2j}} - \pi i \nu_{j} y^{2j-1} - 2\pi i \beta_{2j} \right) \quad (j = 1, \ldots, m)
\]

\[
P_{2m+k} = -i \left( \frac{\partial}{\partial y^{2m+k}} - 2\pi i \beta_{2m+k} \right) \quad (k = 1, \ldots, n - 2m).
\]

These are Hermitian with respect to the inner product \((2.11)\). Since \((y^{1}, \ldots, y^{n})\) is an orthonormal coordinate, the Laplacian \((2.13)\) becomes

\[
-\Delta f = \sum_{i=1}^{n} (P_{i})^{2} f.
\]

Nontrivial commutators among \(P\)’s are

\[
[P_{2j-1}, P_{2j}] = 2\pi i \nu_{j} \quad (j = 1, \ldots, m).
\]

The other commutators vanish. We call the operators \(\{P_{2j-1}, P_{2j}\}\) transverse momenta while we call the operators \(\{P_{2m+k}\}\) longitudinal momenta. Since the covariant derivative commutes with the magnetic shifts as seen at \((3.4)\), the momentum operators \(\{P_{i}\}\) commute with the shift operators \(\{U_{j}, V_{j}, W_{k}\}\), which are defined at \((3.16)\) and \((3.17)\).

Next we introduce another family of unitary operators. For this purpose we need an observation; in the \(t\)-coordinate that expresses the magnetic flux matrix in the standard form \((3.14)\), \(t^{2m+k} \quad (k = 1, \ldots, n - 2m)\) are genuine cyclic coordinates. That is to say, the twisted periodic function \((2.5)\) is periodic with respect to these coordinates as

\[
f(t^{1}, \ldots, t^{2m+k} + 1, \ldots, t^{n}) = f(t^{1}, \ldots, t^{2m+k}, \ldots, t^{n}) \quad (k = 1, \ldots, n - 2m)
\]

and the magnetic shift \((3.2)\) is reduced to an ordinary continuous shift

\[
(W_{k}(\theta)f)(t^{1}, \ldots, t^{2m+k}, \ldots, t^{n}) = f(t^{1}, \ldots, t^{2m+k} - \theta, \ldots, t^{n}).
\]

Then we define an operator \(T_{k}^{i}\) for each \(k = 1, \ldots, n - 2m\) which acts on \(f\) by multiplication

\[
(T_{k}^{i} f)(t) = e^{2\pi i t^{2m+k}} f(t) = e^{2\pi i \sum_{i=1}^{n} (L_{i}^{-1})^{2m+k}_{i} y^{i}} f(t).
\]
Here we used the inverse $L^{-1}$ of the coordinate transformation (4.10). The operators $\{T^k\}$ are unitary operators with respect to the inner product (2.11). They satisfy

$$W_k(\theta)T^k = e^{-2\pi i \theta} T^k W_k(\theta), \quad (5.10)$$

$$[P_i, T^k] = 2\pi (L^{-1})^i_{2m+k} T^k \quad (i = 1, \ldots, n; \ k = 1, \ldots, n-2m) \quad (5.11)$$

and commute with the other generators of the MTG.

Combining all the operators introduced above we define an algebra $\mathcal{A}$ with the generators $\{P_i, U_j, V_j, W_k(\theta), T^k \mid i = 1, \ldots, n; \ j = 1, \ldots, m; \ k = 1, \ldots, n-2m; \ \theta \in \mathbb{R}\}$ and with the relations (3.18)-(3.21), (5.6), (5.10), (5.11) and other trivial commutators. We call the algebra $\mathcal{A}$ a magnetic algebra. In the following we will construct all the irreducible representations of the algebra $\mathcal{A}$ and classify their unitary equivalence classes.

A subset of generators $\{P_{2j-1}, P_{2m+l}, V_j, W_i(\theta) \mid j = 1, \ldots, m; \ l = 1, \ldots, n-2m; \ \theta \in \mathbb{R}\}$ generates a maximal Abelian subalgebra of $\mathcal{A}$. Hence these generators are simultaneously diagonalizable. Their simultaneous eigenstate $|k, r, d\rangle$ is labeled by $r_j \in \mathbb{Z}/4\mathbb{Z}$ of (3.22) and $d_l \in \mathbb{Z}$ of (3.23) with new labels

$$k_{2j-1}, k_{2m+l} \in \mathbb{R}. \quad (5.12)$$

The generators $\{P_{2j-1}, P_{2m+l}\}$ act on these states as

$$P_{2j-1}|k, r, d\rangle = 2\pi k_{2j-1}|k, r, d\rangle, \quad (5.13)$$

$$P_{2m+l}|k, r, d\rangle = 2\pi k_{2m+l}|k, r, d\rangle, \quad (5.14)$$

and $\{V_j, W_l(\theta)\}$ act as (3.25), (3.26), respectively. The coefficient $2\pi$ was put for later convenience. Other generators $\{P_{2j}, T^l \mid j = 1, \ldots, m; \ l = 1, \ldots, n-2m\}$ act on the states as

$$\langle \psi | P_{2j} | k, r, d\rangle = i\nu_j \frac{\partial}{\partial k_{2j-1}} \langle \psi | k, r, d\rangle, \quad (5.15)$$

$$T^l | k_i, r, d_k \rangle = e^{2\pi i \sum_{j=1}^{m} (k_{2j-1} + \Delta^l k_{2j-1}) \nu_j + \sum_{p=1}^{n-2m} Z^p d_p} | k_i + \Delta^l k_i, r, d_k + \delta^l_k \rangle, \quad (5.16)$$

with $\Delta^l k_i = (L^{-1})^i_{2m+l}$. We will determine the $(n-2m) \times (n-2m)$ matrix $Z^p$ soon later. Here $|\psi\rangle$ represents an arbitrary state. The rests $\{U_j \mid j = 1, \ldots, m\}$ act as (3.24). Therefore, in an irreducible representation space the eigenvalues $k_{2m+i}$ are linked to $d_l$ via

$$k_{2m+i} = \sum_{l=1}^{n-2m} d_l(L^{-1})^i_{2m+i} - \beta_{2m+i}$$

$$= \sum_{l=1}^{n-2m} (d_l - \alpha_{2m+l})(L^{-1})^i_{2m+i} \quad (i = 1, \ldots, n-2m). \quad (5.17)$$

Here the real number $\beta_{2m+i}$ coincides with the one that appeared in (3.4). We also used the relations (4.10) and (4.11).
The matrix $Z^ll$ is determined by the condition $T^llT^l = T^lT^ll$. The action of $T^llT^l$ gives

$$
T^llT^l|k_i, r, d_k\rangle = e^{2\pi i (\sum_{j=1}^{m}(k_{2j-1} + \Delta^l k_{2j-1}) \Delta^l k_{2j}/\nu_j + \sum_{p=1}^{n-2m} Z^llp d_p)}
$$

while the action of $T^lT^ll$ gives

$$
T^lT^ll|k_i, r, d_k\rangle = e^{2\pi i (\sum_{j=1}^{m}(k_{2j-1} + \Delta^l k_{2j-1}) \Delta^l k_{2j}/\nu_j + \sum_{p=1}^{n-2m} Z^l'p d_p)}
$$

A general solution of the above equation is

$$
|k_i + \Delta^l k_i + \Delta^l' k_i, r, d_k + \delta_k^l + \delta_k'^l\rangle
$$

To give $T^llT^l = T^lT^ll$ the matrix $Z^ll'$ must satisfy

$$
\sum_{j=1}^{m}(\Delta^l k_{2j-1} \Delta^l' k_{2j}/\nu_j) + Z^ll = \sum_{j=1}^{m}(\Delta^l k_{2j-1} \Delta^l k_{2j}/\nu_j) + Z^ll'.
$$

A general solution of the above equation is

$$
Z^ll' = \sum_{j=1}^{m}(\Delta^l k_{2j-1} \Delta^l' k_{2j}/\nu_j) + S^ll'.
$$

Here we leave an arbitrary symmetric matrix $S^ll' = S'^ll$ yet undetermined. Actually any choice of $S^ll'$ results in an equivalent representation, and therefore we take $S^ll' = 0$.

A restricted set of vectors $\{|k, r, d\rangle\}$ that are labeled by the mutually independent parameters

$$
k_{2j-1} \in \mathbb{R}, \ r_j \in \mathbb{Z}/\mathbb{Z}_{q_j}, \ d_l \in \mathbb{Z} \quad (j = 1, \ldots, m; \ l = 1, \ldots, n - 2m)
$$

spans a Hilbert space $\mathcal{H}_\alpha$ for each fixed value of $(\alpha_{2m+1}, \alpha_{2m+2}, \ldots, \alpha_n)$. Thus we conclude that a unitary equivalence class of irreducible representations of the algebra $\mathcal{A}$ has one-to-one correspondence with the parameter $(\alpha_{2m+1}, \alpha_{2m+2}, \ldots, \alpha_n) \in \mathbb{R}^{n-2m}/\mathbb{Z}^{n-2m}$.

### 6 Fourier analysis for the magnetic torus

Now let us turn to the space of twisted periodic functions. It is a representation space of the magnetic algebra. We will calculate the whole family of eigenfunctions of the maximal Abelian subalgebra of the magnetic algebra. These eigenfunctions $\chi_{k,r,d}(t^1, \ldots, t^n) = \langle t|k, r, d\rangle$ satisfy

$$
P_{2j-1} \chi_{k,r,d} = -i \left( \frac{\partial}{\partial y_{2j-1}} + \pi i \nu_j y_{2j} - 2\pi i \beta_{2j-1} \right) \chi_{k,r,d} = 2\pi k_{2j-1} \chi_{k,r,d}
$$
where $c$ is a common normalization constant. The coefficients $\gamma_j$ are given later at (6.13). The fact that the eigenfunctions are uniquely determined up to the common coefficient $c$ implies that the space of twisted periodic functions is an irreducible representation space of the magnetic algebra. Consequently, the eigenfunctions (6.8) constitute a complete orthonormal set of the space of twisted periodic functions over the torus. This is one of main results of this paper. Hence an arbitrary twisted periodic function over the torus can be expanded as

$$f(y^1, \ldots, y^n) = \sum_{k,r,d} \lambda_{k,r,d} \chi_{k,r,d}(y^1, \ldots, y^n)$$

with unique coefficients $\lambda_{k,r,d}$. Therefore, the complete set $\{\chi_{k,r,d}\}$ provides a new basis for Fourier analysis in the magnetic torus.
In the rest of this section we give detailed lengthy calculations to prove the above statements. The reader may skip them to the next section, where we calculate solutions of the eigenvalue problem of the magnetic Laplacian using the main result (6.8). First, a simultaneous solution of (6.1) and (6.3) is the eigenvalue problem of the magnetic Laplacian using the main result (6.8). First, a simultaneous solution of (6.1) and (6.3) is

\[ \chi_{k,r,d}(y^1, y^2, \ldots, y^n) = e^{2\pi i \sum_{j=1}^{m} \{(k_{2j-1} + \beta_{2j-1})y^{2j-1} - (1/2)\nu_j y^{2j-1} y^{2j}\}} \\
+ e^{2\pi i \sum_{j=1}^{m-2} d_j (L^{-1})^{2m+j}_{2m+l} y^{2m+l}} \phi_{k,r,d}(y^2, y^4, \ldots, y^{2m}), \quad (6.10) \]

where \( \phi_{k,r,d}(y^2, y^4, \ldots, y^{2m}) \) is an arbitrary function to be specified later.

Next let us turn to the other equation (6.5). Using (4.13) we can rewrite (6.5) as

\[ e^{-\pi i \sum_{l=1}^{m} (1/q_j) L_{2j}^{2l} \nu_l y^{2l-1}} \chi_{k,r,d}(y^i - L_{2j}^{2l}/q_j) = e^{-2\pi i \gamma_j/q_j} \chi_{k,r,d}(y^i). \quad (6.11) \]

As discussed at (4.11) we have taken the matrix \( L \) such that \( L_{2j}^{2l-1} = 0 \). Therefore, when (6.10) is substituted, the LHS of (6.11) becomes

\[ e^{-\pi i \sum_{l=1}^{m} (1/q_j) L_{2j}^{2l} \nu_l y^{2l-1}} \chi_{k,r,d}(y^i - L_{2j}^{2l}/q_j) = e^{2\pi i \sum_{l=1}^{m-2} d_p (L^{-1})^{2l}_{2m+l} y^{2m+l}} e^{-2\pi i \sum_{p,l=1}^{m-2} d_p (L^{-1})^{2m+p}_{2m+l} L_{2j}^{2l}/q_j} \phi_{k,r,d}(y^i - L_{2j}^{2l}/q_j). \quad (6.12) \]

Hence, if we put

\[ \gamma_j = \sum_{p,l=1}^{m-2} d_p (L^{-1})^{2l}_{2m+l} L_{2j}^{2l}, \quad (6.13) \]

(6.11) implies that

\[ e^{-2\pi i \gamma_j/q_j} \phi_{k,r,d}(y^i - L_{2j}^{2l}/q_j) = e^{-2\pi i \gamma_j/q_j} \phi_{k,r,d}(y^i). \quad (6.14) \]

If we introduce another coordinate system \((z^1, z^2, \ldots, z^m)\) which is related to \((y^2, y^4, \ldots, y^{2m})\) via

\[ y^i = \sum_{j=1}^{m} L_{2j}^{2l} (z^j/q_j), \quad (6.15) \]

then (6.14) is rewritten as

\[ \phi_{k,r,d}(z^1, \ldots, z^j - 1, \ldots, z^m) = e^{2\pi i (\gamma_j - r_j)/q_j} \phi_{k,r,d}(z^1, \ldots, z^j, \ldots, z^m). \quad (6.16) \]

Moreover, if we put

\[ \psi_{k,r,d}(z^1, \ldots, z^m) = e^{2\pi i \sum_{j=1}^{m-1} (\gamma_j - r_j) z^j/q_j} \phi_{k,r,d}(z^1, \ldots, z^m), \quad (6.17) \]

then (6.16) implies that

\[ \psi_{k,r,d}(z^1, \ldots, z^j - 1, \ldots, z^m) = \psi_{k,r,d}(z^1, \ldots, z^j, \ldots, z^m). \quad (6.18) \]
Hence $\psi_{k,r,d}$ is a periodic function with the period 1 and can be expanded in a Fourier series

$$\psi_{k,r,d}(z^1, \ldots, z^j, \ldots, z^m) = \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} c_{k,r,d,\sigma} e^{2\pi i \sum_{j=1}^{m} \sigma_j z^j}. \quad (6.19)$$

Note that the inverse transformation of (6.15) is given by

$$(z^j / q_j) = \sum_{i=1}^{m} (L^{-1})_{2i}^j y_{2i}. \quad (6.20)$$

Combining the above equations we can write down the eigenfunction (6.10) in a more specific form as

$$\chi_{k,r,d}(y^1, y^2, \ldots, y^n) = e^{2\pi i \sum_{j=1}^{m} \{(k_{2j-1} + \beta_{2j-1})y^{2j-1} - (1/2)\nu_j y^{2j-1} y^{2j}\}} e^{2\pi i \sum_{j,l=1}^{m} \{d_j (L^{-1})_{2m+1}^j y^{2m+1} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} c_{k,r,d,\sigma} e^{2\pi i \sum_{j=1}^{m} (q_j \sigma_j + r_j - \gamma_j) (L^{-1})_{2i}^j y_{2i}. \quad (6.21)$$

Moreover, referring to (4.11) and (4.14), we can see that (6.21) satisfies (6.6). Thus we have seen that $\chi_{k,r,d}$ is a simultaneous eigenfunction of $\{P_{2j-1}, P_{2m+1}, V_j, W_l(\theta)\}$ as announced above.

The remaining task is to solve (6.2), (3.4), and (6.7). Let us begin with (6.2). The LHS of (6.2) is

$$-i \left( \frac{\partial}{\partial y^{2j}} - \pi i \nu_j y^{2j-1} - 2\pi i \beta_{2j} \right) \chi_{k,r,d}$$

$$= \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} 2\pi \left( -\frac{1}{2} \nu_j y^{2j-1} + \sum_{i=1}^{m} (q_i \sigma_i + r_i - \gamma_i) (L^{-1})_{2i}^j y^{2j-1} - \frac{1}{2} \nu_j y^{2j-1} - \beta_{2j} \right) e^{2\pi i \sum_{j,l=1}^{m} \{d_j (L^{-1})_{2m+1}^j y^{2m+1} \} c_{k,r,d,\sigma} e^{2\pi i \sum_{i=1}^{m} (q_i \sigma_i + r_i - \gamma_i) (L^{-1})_{2i}^j y^{2j}}.$$

On the other hand, the RHS of (6.2) is

$$i \nu_j \frac{\partial}{\partial k_{2j-1}} \chi_{k,r,d} = i \nu_j e^{2\pi i \sum_{i=1}^{m} \{(k_{2i-1} + \beta_{2i-1})y^{2i-1} - (1/2)\nu_i y^{2i-1} y^{2i}\}} e^{2\pi i \sum_{i=1}^{m} \{d_i (L^{-1})_{2m+1}^i y^{2m+1} \} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} \left( \frac{\partial c_{k,r,d,\sigma}}{\partial k_{2j-1}} + 2\pi i y^{2j-1} c_{k,r,d,\sigma} \right) e^{2\pi i \sum_{i=1}^{m} (q_i \sigma_i + r_i - \gamma_i) (L^{-1})_{2i}^j y^{2j}}.$$

Therefore, we have an equation

$$i \nu_j \frac{\partial c_{k,r,d,\sigma}}{\partial k_{2j-1}} = 2\pi \left( \sum_{i=1}^{m} (q_i \sigma_i + r_i - \gamma_i) (L^{-1})_{2i}^j y^{2j} - \beta_{2j} \right) c_{k,r,d,\sigma} \quad (6.22)$$

and get its solution

$$c_{k,r,d,\sigma} = c_{0,r,d,\sigma} e^{-2\pi i \sum_{i=1}^{m} (q_i \sigma_i + r_i - \gamma_i) (L^{-1})_{2i}^j k_{2j-1}/\nu_j - \sum_{j=1}^{m} \beta_{2j} k_{2j-1}/\nu_j}, \quad (6.23)$$
Thus \((6.21)\) becomes
\[
\chi_{k,r,d}(y^1, y^2, \ldots, y^n) = e^{2\pi i \sum_{j=1}^m \{ (k_{j-1} + \beta_{j-1})y^{2j-1} + k_{j-1}\beta_{j-1}/\nu_j - (1/2)\nu_j y^{2j-1}y^{2j} \}}
\]
\[
e^{2\pi i \sum_{j,l=1}^{n-2m} d_{j,l}(L^{-1})_2^{2m+i} y^{2m+i}}
\sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^\infty c_{0,r,d,\sigma} e^{2\pi i \sum_{j,l=1}^m (q_j \sigma_j + r_j - \gamma_j)(L^{-1})_2^{2l}(y^{2l-1}k_{2l-1}/\nu_j)}.
\] (6.24)

Next we turn to \((6.4)\). With the aid of \((4.12)\) and \((4.15)\) the LHS of \((6.4)\) becomes
\[
e^{2\pi i \sum_{j,l=1}^m (1/q_j)(L^{-1})_2^{2l-1}y^{2l-1} - (1/2)\nu_j y^{2l-1}} \chi_{k,r,d}(y^j - L^{2j-1}/q_j)
= e^{2\pi i \sum_{j,l=1}^m \{ -\beta_{2j-1}L^{2j-1}/q_j \} - (1/2)(L^{-1})_2^{2l-1}L^{2j-1}/q_j \} e^{-2\pi i \sum_{j,l=1}^m (L^{-1})_2^{2m+i}L^{2m+i}/q_j}
\]
\[
e^{2\pi i \sum_{j,l=1}^m d_j(L^{-1})_2^{2m+i} y^{2m+i}}
\sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^\infty c_{0,r,d,\sigma} e^{2\pi i \sum_{j,l=1}^m (q_j \sigma_j + r_j - \gamma_j)(L^{-1})_2^{2l}(y^{2l-1}k_{2l-1}/\nu_j)}
\] (6.25)

To make this coincide with the RHS of \((6.4)\) we have a recursive equation
\[
e^{2\pi i \sum_{j,l=1}^m \{ -\beta_{2j-1}L^{2j-1}/q_j \} - (1/2)(L^{-1})_2^{2l-1}L^{2j-1}/q_j \} e^{-2\pi i \sum_{j,l=1}^m d_j(L^{-1})_2^{2m+i}L^{2m+i}/q_j}
\]
\[
e^{-2\pi i \sum_{j,l=1}^m c_{0,r,d,\sigma} (q_j \sigma_j + r_j - \gamma_j)(L^{-1})_2^{2l}(y^{2l-1}k_{2l-1}/\nu_j)} = c_{0,r,d,\sigma} = c_{0,(r+1),d,\sigma}.
\] (6.26)

Here \((r+1)\) means \((r_i + \delta_{ij})\). If we define an \(m \times m\) matrix
\[
Y_{ij} = \sum_{l=1}^m (L^{2l-1})_2^{2l}L^{2j-1}/q_j = \sum_{l=1}^m \nu_l L^{2l-1}_2 L^{2j-1}/(q_i q_j),
\] (6.27)

it is symmetric as
\[
Y_{ij} - Y_{ji} = \sum_{l=1}^m (L^{2l-1}_2 \nu_l L^{2j-1}_2 - L^{2l-1}_2 \nu_l L^{2j-1}_2)/(q_i q_j) = \phi_{2i-1,2j-1}/(q_i q_j) = 0
\] (6.28)

by virtue of \((4.7)\). Then the solution of \((6.26)\) is
\[
c_{0,r,d,\sigma} = c_{0,0,d,\sigma} e^{-2\pi i \sum_{j,l=1}^m \beta_{2j-1}L^{2j-1}r_j/q_j}
\]
\[
e^{-2\pi i \sum_{j,l=1}^m d_l(L^{-1})_2^{2m+i}L^{2m+i}r_j/q_j}
\]
\[
e^{-2\pi i \sum_{i,j,l=1}^m (q_i \sigma_i + (1/2)r_i - \gamma_i)(L^{-1})_2^{2l}(y^{2l-1}k_{2l-1}/q_j)}.
\] (6.29)

Therefore, \((6.24)\) becomes
\[
\chi_{k,r,d}(y^1, y^2, \ldots, y^n) = e^{2\pi i \sum_{j,l=1}^m \{ (k_{j-1} + \beta_{j-1})y^{2j-1} + k_{j-1}\beta_{j-1}/\nu_j - (1/2)\nu_j y^{2j-1}y^{2j} \}}
\]
\[
e^{2\pi i \sum_{i,j,l=1}^{n-2m} d_{i,j,l}(L^{-1})_2^{2m+i} y^{2m+i}}
\sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^\infty c_{0,0,d,\sigma}
\]
\[
e^{2\pi i \sum_{j,l=1}^m (q_j \sigma_j + r_j - \gamma_j)(L^{-1})_2^{2l}(y^{2l-1}k_{2l-1}/q_j)} - \sum_{i,j,l=1}^m \nu_i L^{2l-1}_2 L^{2j-1}(r_i/q_i) r_j/q_j)
\] (6.30)
Since \((U_j)^{q_j} = 1\) by (3.18), the substitution \(r_j \mapsto r_j + q_j\) must leave \(\chi_{k,r,d}\) invariant. This substitution gives

\[
\chi_{k,(r+q),d}(y) = e^{2\pi i \sum_{j=1}^{m} \{ (k_{2j-1} + \beta_{2j-1}) y^{2j-1} + k_{2j+1} \beta_{2j+1} / y^{2j+1} - (1/2) y_j y^{2j+1} y^{2j} \}} \nonumber
\]

\[
e^{2\pi i \sum_{i=1}^{n} \{ d_i (L_i - 1)^{2m+i} y^{2m+i} - \sum_{p=1}^{m} L_{2p} - 1 \} \rho_p / q_p \}} \nonumber
\]

\[
e^{-2\pi i \sum_{i=1}^{m} \beta_{2i-1} L_{2i-1} / y_j} e^{2\pi i \sum_{i=1}^{m} \nu_i L_{2i-1} / y_j} \nonumber
\]

\[
e^{-2\pi i \sum_{i=1}^{m} \beta_{2i-1} L_{2i-1} / y_j} \nonumber
\]

\[
e^{-2\pi i \sum_{i=1}^{m} \sigma_i \sigma_{i+1} \ldots \sigma_m} \nonumber
\]

\[
e^{2\pi i \sum_{i=1}^{m} \nu_i L_{2i-1} / y_j} \nonumber
\]

\[
e^{2\pi i \sum_{i=1}^{m} \nu_i L_{2i-1} / y_j} \nonumber
\]

\[
\tag{6.31}
\]

Here \((\sigma - 1)\) is an abbreviation of \((\sigma_i - \delta_{ij})\). We used (4.15). Then we have another recursive equation

\[
c_{0,0,d,\sigma} = c_{0,0,d,\sigma(1)} e^{-2\pi i \sum_{i=1}^{n-2m} d_i (L_i - 1)^{2m+i} L_{2i-1} / y_j} e^{-2\pi i \sum_{i=1}^{m} \beta_{2i-1} L_{2i-1} / y_j} \nonumber
\]

\[
e^{2\pi i \{ \sum_{i=1}^{m} \nu_i L_{2i-1} / y_j(r_i / q_i) + \sum_{p=1}^{m} L_{2p} - 1 \} \rho_p / q_p \}} \nonumber
\]

\[
e^{-2\pi i \sum_{i=1}^{m} \sigma_i \sigma_{i+1} \ldots \sigma_m} \nonumber
\]

\[
e^{2\pi i \sum_{i=1}^{m} \nu_i L_{2i-1} / y_j} \nonumber
\]

\[
\tag{6.32}
\]

Remember that \(q_i q_j Y_{ij} = \sum_{j=1}^{m} \nu_i L_{2i-1} L_{2j-1} = q_j q_i Y_{ji}\) is symmetric. The solution of (6.32) is

\[
c_{0,0,d,\sigma} = c_{0,0,d,0} e^{-2\pi i \sum_{i=1}^{n-2m} \sum_{j=1}^{m} d_i (L_i - 1)^{2m+i} L_{2j-1} / y_j \sigma_j} e^{-2\pi i \sum_{i=1}^{m} \beta_{2i-1} L_{2i-1} \sigma_j} \nonumber
\]

\[
e^{2\pi i \{ \sum_{i=1}^{m} \nu_i L_{2i-1} / y_j(r_i / q_i) \sigma_j + \sum_{p=1}^{m} L_{2p} - 1 \} \rho_p / q_p \}} \nonumber
\]

\[
e^{-2\pi i \sum_{i=1}^{m} \sigma_i \sigma_{i+1} \ldots \sigma_m} \nonumber
\]

\[
\tag{6.33}
\]

Substituting it into (6.30) and using (4.15), we get

\[
\chi_{k,r,d}(y) = e^{-\pi i \sum_{j=1}^{m} \nu_j y^{2j-1} y^{2j}} \nonumber
\]

\[
\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m = -\infty}^{\infty} c_{0,0,d,0} \nonumber
\]

\[
e^{2\pi i \sum_{i,j=1}^{m} \nu_i L_{2i-1} / y_j(r_i / q_i)} \nonumber
\]

\[
e^{2\pi i \sum_{j=1}^{m} \beta_{2j-1} y_{2j-1} - \sum_{i=1}^{m} L_{2i-1} / y_j(q_i \sigma_i + r_i) / q_i} \nonumber
\]

\[
e^{2\pi i \sum_{i,j=1}^{m} \nu_i L_{2i-1} / y_j(r_i / q_i)} \nonumber
\]

\[
e^{2\pi i \sum_{i=1}^{m} \beta_{2i-1} L_{2i-1} / y_j} \nonumber
\]

\[
e^{2\pi i \sum_{i=1}^{m} \nu_i L_{2i-1} / y_j} \nonumber
\]

\[
\tag{6.34}
\]

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Finally, we are going to solve (6.7). Its LHS becomes
\[ e^{2\pi i t^{2m+1}} = e^{-\pi i \sum_{j=1}^{m} \nu_j y^{2j-1} y^{2j}} \]
\[ \sum_{\sigma_1,\sigma_2,\ldots,\sigma_m = -\infty}^{\infty} C_{0,0,d,0} \]
\[ e^{2\pi i \sum_{i,j,p=1}^{m} (q_i \sigma_i + r_i)(q_j \sigma_j + r_j)/(q_i q_j)} \]
\[ e^{2\pi i \sum_{j=1}^{m} \beta_{2j-1} \{ y^{2j-1} - \sum_{i=1}^{m} L_{2i-1}^{2j-1} (q_i \sigma_i + r_i)/q_i \}} \]
\[ e^{2\pi i \sum_{j=1}^{m} (q_j \sigma_j + r_j - \gamma_j + \sum_{i=1}^{m} (L^{-1})_{2i}^{2m+1} L_{2j}^{2j} (y^{2j-1} - \sum_{i=1}^{m} L_{2i-1}^{2j-1} (q_i \sigma_i + r_i)/q_i)} \]
\[ e^{2\pi i \sum_{p=1}^{m} (L^{-1})_{2p}^{2m+1} L_{2p-1}^{2j-1} (q_i \sigma_i + r_i)/q_i} \]
\[ e^{2\pi i \sum_{i=1}^{m} (d_i + \delta_{p}^{i}) (y^{2m+1} - \sum_{j=1}^{m} L_{2j-1}^{2m+1} (q_j \sigma_j + r_j)/q_j) \]
\[ e^{2\pi i \sum_{p=1}^{m} \sum_{j=1}^{m} (L^{-1})_{2i}^{2m+1} L_{2j-1}^{2j} (q_j \sigma_j + r_j)/q_j} \]
\[ e^{2\pi i \sum_{j=1}^{m} (k_{2j-1} + (L^{-1})_{2j}^{2m+1} (y^{2j-1} + \beta_{2j} + \gamma_j - \sum_{p=1}^{m} L_{2p-1}^{2j-1} (q_p \sigma_p + r_p - \gamma_p)/q_p) \]
\[ e^{-2\pi i \sum_{j=1}^{m} (L^{-1})_{2j}^{2m+1} (\beta_{2j}/\nu_j - \sum_{p=1}^{m} L_{2p-1}^{2j-1} (q_p \sigma_p + r_p - \gamma_p)/q_p)} \].
(6.35)

We put \( \Delta^l k_i = (L^{-1})_{i}^{2m+1} \) as before. The change \( d_p \mapsto d_p + \delta^l_p \) causes a change of \( \gamma_j \) as
\[ \Delta^l \gamma_j = \sum_{i=1}^{n-2m} (L^{-1})_{i}^{2m+l} L_{2j}^{2j} \]
\[ = \sum_{i=1}^{n} (L^{-1})_{i}^{2m+l} L_{2j}^{2j} - \sum_{i=1}^{m} (L^{-1})_{2i-1}^{2m+l} L_{2j}^{2j} - \sum_{i=1}^{m} (L^{-1})_{2i}^{2m+l} L_{2j}^{2j} \]
\[ = -\sum_{p,i=1}^{m} L_{2p-1}^{2j-1} \Delta^l \gamma_p/q_p \]
\[ = -\sum_{p,i=1}^{m} (L^{-1})_{2p}^{2j} L_{2i}^{2j} \]
\[ = -\sum_{i=1}^{m} (L^{-1})_{2i}^{2m+l} L_{2j}^{2j} \]
\[ = -(L^{-1})_{2j}^{2m+l} / \nu_j \]
\[ = -\Delta^l k_{2j}/\nu_j \).
(6.36)

via (6.13) with (4.11). Moreover, we can see that
\[ \sum_{p=1}^{m} L_{2p-1}^{2j-1} \Delta^l \gamma_p/q_p = -\sum_{p,i=1}^{m} L_{2p-1}^{2j-1} (L^{-1})_{2i}^{2m+l} L_{2i}^{2j} / q_p \]
\[ = -\sum_{p,i=1}^{m} (L^{-1})_{2p}^{2j} L_{2i}^{2j} / \nu_j \]
\[ = -\sum_{i=1}^{m} (L^{-1})_{2i}^{2m+l} L_{2j}^{2j} / \nu_j \]
\[ = -\Delta^l k_{2j}/\nu_j \).
(6.37)

Therefore (6.35) becomes
\[ e^{2\pi i t^{2m+1}} \chi_{k,r,d} = e^{2\pi i \sum_{j=1}^{m} (k_{2j-1} + \Delta k_{2j-1}) \Delta k_{2j}/\nu_j} \]
\[ e^{-2\pi i \sum_{j=1}^{m} (L^{-1})_{2j}^{2m+l} \beta_{2j}/\nu_j + \sum_{p=1}^{m} L_{2p-1}^{2j-1} \gamma_p/q_p} \]
\[ e^{-\pi i \sum_{j=1}^{m} \nu_j y^{2j-1} y^{2j}} \]
Finally, we have proved that the set of functions
functions over the torus is an irreducible representation space of the magnetic algebra
Thus we reach
\[ \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} c_{0,0,d,0} \]
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)

In the course of calculation we used the fact \( \sum_{p=1}^n (L^{-1})^{2m+l} L^p_{2j-1} = 0 \). With the aid of (4.15), (6.36) and the definition (5.21) we can deduce that
\[ \sum_{j,p=1}^m (L^{-1})^{2m+l} L^p_{2j-1} \gamma_p / q_p = - \sum_{i=1}^{n-2m} Z^{\nu} d_i. \]
(6.39)
Thus we reach
\[ e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j) \]
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)

To satisfy (6.7) we meet another recursive equation
\[ c_{0,0,(d+1),0} = e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j) c_{0,0,d,0} \]
and we get the solution
\[ c_{0,0,d,0} = e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j) c_{0,0,0,0}. \]
(6.42)
Substituting it into (6.34) we reach the final result
\[ \chi_{k,r,d}(y) = \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} c_{0,0,0,0} \]
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)
e^{2\pi i} \sum_{j,k,d} \nu_p L^{2p+1} L_{2j-1} (q, q_j, r_j) / (q, q_j)

This is the result (6.8) announced previously. The eigenfunction \( \chi_{k,r,d} \) is determined up to a unique normalization constant \( c_{0,0,0,0} \). Thus we conclude that the space of twisted periodic functions over the torus is an irreducible representation space of the magnetic algebra \( \mathcal{A} \). Finally, we have proved that the set of functions \( \{ \chi_{k,r,d} \} \) is a complete orthonormal set in the space of twisted periodic functions over the torus as announced at (6.9).
7 Eigenfunctions of the magnetic Laplacian

In this section we will write down explicitly solutions of the eigenvalue problem of the magnetic Laplacian (2.13) or (3.5), which is expressed in the \( y \)-coordinate as

\[
\Delta f = \sum_{j=1}^{m} \left( \frac{\partial}{\partial y^{2j-1}} + \pi \nu_j y^{2j} - 2\pi i \beta_{2j-1} \right)^2 f + \left( \frac{\partial}{\partial y^{2j}} - \pi \nu_j y^{2j-1} - 2\pi i \beta_{2j} \right)^2 f \]  
+ \sum_{k=1}^{n-2m} \left( \frac{\partial}{\partial y^{2m+k}} - 2\pi i \beta_{2m+k} \right)^2 f.  
\tag{7.1}
\]

The function \( f \) must satisfy the twisted periodic condition (2.5). We will obtain solutions using the Fourier analysis that is developed in the previous section.

Here we would like to describe the outline of our method. As mentioned in the previous section, the Laplacian commutes with the magnetic shift operators, \( \{ U_j, V_j, W_k \} \). Hence the labels \( (r,d) = (r_1, r_2, \ldots, r_m, d_1, d_2, \ldots, d_{n-2m}) \), which are defined in (3.22)-(3.26), are ‘good’ quantum numbers. Moreover, the Laplacian commutes with the longitudinal momentum operators \( \{ P_{2m+k} \} \). Hence the corresponding momentum eigenvalues \( \{ k_{2m+k} \} \) are also good quantum numbers and are related to the labels \( d_l \) via (5.17). On the other hand, the Laplacian does not commute with the transverse momentum operators \( \{ P_{2j-1}, P_{2j} \} \). Hence the transverse momentum eigenvalues \( \{ k_{2j-1} \} \) do not remain good quantum numbers. The Laplacian admits a new set of good quantum numbers \( (n_1, n_2, \ldots, n_m) \), which will be introduced soon later. It will be revealed that eigenfunctions of the Laplacian are actually matrix elements of a unitary transformation,

\[
\psi_{n,r,d}(k_1, k_3, \ldots, k_{2m-1}) = \langle k, r, d | n, r, d \rangle,  
\tag{7.2}
\]

which relates the quantum numbers \( n \)’s to \( k \)’s. In the \( k \)-space it is rather easy to get eigenfunctions by the standard method of a harmonic oscillator. On the other hand, the set of eigenfunctions of the momenta and magnetic shifts,

\[
\chi_{k,r,d}(y^1, y^2, \ldots, y^n) = \langle y | k, r, d \rangle,  
\tag{7.3}
\]

plays a role a unitary transformation which bridges between the momentum space and the real space like the usual Fourier transformation. Hence the Laplacian eigenfunctions are transformed into the \( y \)-coordinate representations by

\[
\psi_{n,r,d}(y^1, y^2, \ldots, y^n) = \langle y | n, r, d \rangle = \int_{-\infty}^{\infty} dk_1 dk_3 \cdots dk_{2m-1} \langle y | k, r, d \rangle \langle k, r, d | n, r, d \rangle.  
\tag{7.4}
\]

This will give the desired result.

Now let us carry out the program outlined above. We define creation and annihilation operators associated with the transverse momenta as

\[
a_j^\dagger = \frac{1}{\sqrt{4\pi \nu_j}} (P_{2j-1} - iP_{2j}), \quad a_j = \frac{1}{\sqrt{4\pi \nu_j}} (P_{2j-1} + iP_{2j}) \quad (j = 1, \ldots, m).  
\tag{7.5}
\]
It is easily verified that \([a_j, a_k^\dagger] = \delta_{jk}\). Then the Laplacian \((5.5)\) becomes

\[
-\Delta = \sum_{j=1}^{m} 4\pi \nu_j \left( a_j^\dagger a_j + \frac{1}{2} \right) + \sum_{k=1}^{n-2m} (P_{2m+k})^2. \tag{7.6}
\]

The eigenstate \(|\Omega\rangle\) for the lowest eigenvalue satisfies

\[
0 = \langle k | a_j | \Omega \rangle = \frac{1}{\sqrt{4\pi \nu_j}} \langle k | (P_{2j-1} + i P_{2j}) | \Omega \rangle = \frac{1}{\sqrt{4\pi \nu_j}} \left( 2\pi k_{2j-1} + \nu_j \frac{\partial}{\partial k_{2j-1}} \right) \langle k | \Omega \rangle. \tag{7.7}
\]

Here we used \((5.13)\) and \((5.13)\). The solution is

\[
\langle k | \Omega \rangle = e^{-\pi \sum_{j=1}^{m} (k_{2j-1})^2 / \nu_j}. \tag{7.8}
\]

States for higher eigenvalues are generated by creation operators as

\[
\langle k | n \rangle = \frac{1}{\sqrt{n_1! \cdots n_m!}} \langle k | (a_1^\dagger)^{n_1} \cdots (a_m^\dagger)^{n_m} | \Omega \rangle = \frac{1}{\sqrt{n_1! \cdots n_m!}} \prod_{j=1}^{m} \left[ \frac{1}{\sqrt{4\pi \nu_j}} (2\pi k_{2j-1} - \nu_j \frac{\partial}{\partial k_{2j-1}}) \right]^{n_j} \langle k | \Omega \rangle = \frac{1}{\sqrt{n_1! \cdots n_m!}} e^{\pi \sum_{j=1}^{m} (k_{2j-1})^2 / \nu_j} \prod_{j=1}^{m} \left[ \frac{1}{\sqrt{4\pi \nu_j}} (-\nu_j \frac{\partial}{\partial k_{2j-1}}) \right]^{n_j} e^{-2\pi \sum_{j=1}^{m} (k_{2j-1})^2 / \nu_j} \tag{7.9}
\]

for \(n_1, n_2, \ldots, n_m = 0, 1, 2, \ldots\). We are suppressing other labels \((r, d)\). In the appendix B we prove that

\[
\int_{-\infty}^{\infty} dk \ e^{2\pi ikz} \cdot e^{\pi k^2 / \nu} \left(-\nu \frac{\partial}{\partial k}\right)^n e^{-2\pi k^2 / \nu} = \sqrt{\nu} e^{\pi \nu z^2} \left(-i \frac{\partial}{\partial z}\right)^n e^{-2\pi \nu z^2} = \sqrt{\nu} \left(i \sqrt{2\pi \nu}\right)^n e^{-\pi \nu z^2} H_n(z \sqrt{2\pi \nu}). \tag{7.10}
\]

In the second line \(H_n(\xi)\) is the \(n\)-th Hermite polynomial. Substituting \((6.8)\) and \((7.9)\) into \((7.4)\) and applying \((7.10)\) we obtain

\[
\langle y | n, r, d \rangle = \int_{-\infty}^{\infty} dk_1 dk_3 \cdots d_{2m-1} \langle y | k, r, d \rangle \langle k, r, d | n, r, d \rangle = c e^{-\pi i \sum_{j=1}^{m} n_j (L_{2j-1}^2)^\nu_j} e^{-\pi i \sum_{j=1}^{m} \nu_j y^{2j-1} y^{2j}} \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} e^{\pi i \sum_{j=1}^{m} \nu_j L_{2j-1}^{2j-1} (q_j \sigma_j + r_j)(q_j \sigma_j + r_j)/q_j} e^{2\pi i \sum_{j=1}^{m} \beta_{2j-1} (y^{2j-1} - \sum_{l=1}^{m} L_{2j-1}^{2j-1} (q_j \sigma_j + r_j)/q_j)}.
\]

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This is the main result of this paper. Using (5.17), we can calculate eigenvalues of the Laplacian (7.6) as

\[ e^{2\pi i \sum_{j,l=1}^{m} (q_j \sigma_j + r_j - \gamma_j)(L^{-1})^{2j}/2!} \{ y^{2l} \sum_{i=1}^{m} L_{2i-1}^2(q_i \sigma_i + r_i)/q_i \} \]

\[ e^{2\pi i \sum_{j,l=1}^{n-2m} d_k (L^{-1})^{2m+k} (y^{2m+l} - \sum_{i=1}^{m} L_{2i-1}^2 q_i \sigma_i + r_i - \gamma_i)/q_k \} \]

\[ \prod_{j=1}^{m} \left[ \sqrt{\frac{n_j}{n_j!}} \left( \frac{i}{\sqrt{2}} \right)^{n_j} e^{-\pi n_j (y^{2j-1} + \beta_{2j}/\nu_j - \sum_{i=1}^{m} L_{2i-1}^2 (q_i \sigma_i + r_i - \gamma_i)/q_i)^2} \right] \]

\[ H_{n_j} \left( \sqrt{2\pi} \nu_j \left( y^{2j-1} + \beta_{2j}/\nu_j - \sum_{i=1}^{m} L_{2i-1}^2 (q_i \sigma_i + r_i - \gamma_i)/q_i \right) \right) \]

(7.11)

This is the main result of this paper. Using (5.17), we can calculate eigenvalues of the Laplacian (7.6) as

\[-\frac{1}{2} \Delta \psi_{n,r,d} = \left[ \sum_{j=1}^{m} 2\pi \nu_j \left( n_j + \frac{1}{2} \right) + \frac{1}{2} \sum_{k=1}^{n-2m} \left\{ \sum_{l=1}^{2m} 2\pi (d_l - \alpha_{2m+l}) (L^{-1})^{2m+l} \right\}^2 \right] \psi_{n,r,d} \]

(7.12)

Eigenvalues depend on quantum numbers \( n_1, n_2, \ldots, n_m = 0, 1, 2, \ldots \) and \( d_1, d_2, \ldots, d_{n-2m} = 0, \pm 1, \pm 2, \ldots \) but not on \( r = (r_1, r_2, \ldots, r_m) \in \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_m} \). Thus each eigenvalue is degenerated by \( q_1 q_2 \cdots q_m \) folds as predicted in (3.28). If a ratio \( \nu_i/\nu_j (i \neq j) \) is rational, degeneracy happens more. On the other hand, if \( \alpha_{m+l} = 0 \), the eigenvalue for \(-d_l \) coincides with the one for \( d_l \). If \( \alpha_{2m+l} = 1/2 \), the eigenvalue for \((-d_l + 1) \) coincides with the one for \( d_l \). Moreover, for specific values of \( (L^{-1})^{2m+l} \) we may meet more multi-fold degeneracy.

Let us discuss physical meanings of the eigenvalue (7.12). It is energy of an electrically-charged particle moving in the magnetic field in the torus. In (7.12) we put the coefficient \(-1/2 \) in front of \( \Delta \) to adjust the equation to the conventional Schrödinger equation.

In the context of classical mechanics, the particle exhibits a cyclic motion with the frequency \( \nu_j \) in the \((y^{2j-1}, y^{2j})\)-plane for each \( j = 1, 2, \ldots, m \). And it exhibits a uniform straight motion along the \( y^{2m+k} \)-axis. The whole motion is a superposition of those cyclic and straight motions. When we turn to quantum mechanics, energy of the cyclic motion is quantized and results in the so-called Landau level \( 2\pi \nu_j (n_j + 1/2) \). On the other hand, the longitudinal momentum \( P_{2m+k} \) associated with the straight motion is quantized to be \( 2\pi k_{2m+k} = 2\pi \sum_{l=1}^{n-2m} (d_l - \alpha_{2m+l}) (L^{-1})^{2m+l} \) with integers \((d_1, d_2, \ldots, d_{n-2m})\) as explained in (5.17). Along the course of the straight motion the particle flies around the torus and picks up the so-called Aharonov-Bohm effect. Then the momentum is shifted by the Aharonov-Bohm parameters \((\beta_{2m+1}, \ldots, \beta_n)\). Accordingly, the kinetic energy of the straight motion is also quantized. The total energy is then given as (7.12).

Moreover, let us examine meanings of other Aharonov-Bohm parameters \((\beta_1, \ldots, \beta_{2m})\). These do not affect the energy (7.12) and hence they have a geometric significance rather than a physical significance. To understand their meaning we rewrite the eigenfunction (6.43) as

\[ \psi_{k,r,d}(y) = c e^{-2\pi i \sum_{j=1}^{m} \beta_{2j-1} L_{2j-1}^{-1} \gamma_{2j}/q_i} e^{-2\pi i \sum_{j=1}^{m} \sum_{l=1}^{n-2m} d_l (L^{-1})^{2m+l} \beta_{2j}/\nu_j} \]

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\[ e^{-\pi i \sum_{j=1}^{m} \nu_j y^{2j-1} y^j} \]
\[ e^{2\pi i \sum_{j=1}^{m} \beta_{2j-1} y^{2j-1}} \]
\[ \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty} \]
\[ e^{\pi i \sum_{j,i=1}^{m} \nu_j L^{2i-1}_{2j-1} L^{2j-1}_{2i-1} (q_i \sigma_i + r_i) (q_j \sigma_j + r_j) / (q_i q_j)} \]
\[ e^{2\pi i \sum_{j,i=1}^{m} (q_i \sigma_j + r_j - \gamma_j) (L^{-1})^{2j-1}_{2i-1} y^{2i-1} / \nu_j - \sum_{i=1}^{m} L^{2i-1}_{2j-1} (q_i \sigma_i + r_i) / q_j} \]
\[ e^{2\pi i \sum_{k,l=1}^{n-2m} d_k (L^{-1})^{2m+k}_{2m+l} \{ y^{2m+i} - \sum_{i=1}^{m} L^{2m+i}_{2i-1} (q_i \sigma_i + r_i) / q_i \}} \]
\[ e^{2\pi i \sum_{j=1}^{m} k_{2j-1} (y^{2j-1} + \beta_{2j}) / \nu_j - \sum_{i=1}^{m} L^{2i-1}_{2j-1} (q_i \sigma_i + r_i - \gamma_i) / q_i} \].

From (7.10) we have
\[ \sum_{j,i=1}^{m} \beta_{2j-1} L^{2i-1}_{2j-1} \gamma_i / q_i = - \sum_{j=1}^{m} \sum_{i=1}^{m} d_i (L^{-1})^{2m+i}_{2j-1} \beta_{2j-1} \nu_j. \]

Then (7.13) is rewritten as
\[ \chi_{k,r,d}(y) = e^{2\pi i \sum_{j=1}^{m} d_j (L^{-1})^{2m+i}_{2j-1} \beta_{2j-1}} / \nu_j \]
\[ e^{-\pi i \sum_{j=1}^{m} \nu_j y^{2j-1} y^j} \]
\[ e^{2\pi i \sum_{j=1}^{m} \beta_{2j-1} y^{2j-1}} \]
\[ \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty} \]
\[ e^{\pi i \sum_{j,i=1}^{m} \nu_j L^{2i-1}_{2j-1} L^{2j-1}_{2i-1} (q_i \sigma_i + r_i) (q_j \sigma_j + r_j) / (q_i q_j)} \]
\[ e^{2\pi i \sum_{j,i=1}^{m} (q_i \sigma_j + r_j - \gamma_j) (L^{-1})^{2j-1}_{2i-1} y^{2i-1} / \nu_j - \sum_{i=1}^{m} L^{2i-1}_{2j-1} (q_i \sigma_i + r_i) / q_j} \]
\[ e^{2\pi i \sum_{k,l=1}^{n-2m} d_k (L^{-1})^{2m+k}_{2m+l} \{ y^{2m+i} - \sum_{i=1}^{m} L^{2m+i}_{2i-1} (q_i \sigma_i + r_i) / q_i \}} \]
\[ e^{2\pi i \sum_{j=1}^{m} k_{2j-1} (y^{2j-1} + \beta_{2j}) / \nu_j - \sum_{i=1}^{m} L^{2i-1}_{2j-1} (q_i \sigma_i + r_i - \gamma_i) / q_i} \].

Thus we can see that the parameters \((\beta_1, \ldots, \beta_{2m})\) induce a displacement
\[ (y^{2j-1}, y^j) \rightarrow (y^{2j-1} + \beta_{2j} / \nu_j, y^{2j} - \beta_{2j-1} / \nu_j) \]
of the profile \(|\chi_{k,r,d}(y)|\). This is the geometric significance of the transverse Aharonov-Bohm parameters.

\section{Conclusion}

Here we summarize our discussions. As well-known, a \(U(1)\) gauge field replaces the partial derivative by the covariant derivative and generates a magnetic field. As a natural extension of the eigenvalue problem of the ordinary Laplacian in the \(n\)-torus, we formulated the eigenvalue problem of the magnetic Laplacian. The ordinary Laplacian admits continuous Abelian symmetry and therefore the usual Fourier analysis is applicable. However, the
magnetic Laplacian does not admit continuous Abelian symmetry and therefore the usual Fourier analysis is not applicable to it. Hence, we developed an alternative method, which became an extension of the usual Fourier analysis. We identified symmetry structure of the magnetic Laplacian and defined the magnetic translation group (MTG), which is discrete and non-Abelian in general. Moreover, we defined the magnetic algebra by extending the MTG. We proved that the space of functions on which the magnetic Laplacian acts is an irreducible representation space of the magnetic algebra. By diagonalizing the maximal Abelian subalgebra of the magnetic algebra we obtained a complete orthonormal set of functions \( \{ \chi_{k,r,d}(y) \} \) over the magnetic torus; those functions are labeled by a set of good quantum numbers \((k, r, d)\). It was rather easy to diagonalize the magnetic Laplacian in the \( k \)-space representation. Applying a unitary transformation by \( \{ \chi_{k,r,d}(y) \} \) to the eigenstate of the magnetic Laplacian, we finally obtained the eigenfunction in the \( y \)-space representation. The eigenvalues of the magnetic Laplacian were naturally interpreted as sums of energies of cyclic motions in the transverse directions to the magnetic field and energies of linear motions in the longitudinal direction to it.

New results of this paper are the definition and representations of the magnetic algebra, the proof of irreducibility of the space of twisted periodic functions as a representation space of the magnetic algebra, the complete orthogonal set of functions (6.8) which provides a basis of the extended Fourier analysis, the eigenfunctions (7.11) of the magnetic Laplacian in explicit forms, and the eigenvalues (7.12).

Before closing this paper we would like to discuss briefly possible directions for further development. We treated only the Laplace operator in this paper but for application to physics it is more desirable to treat the Schrödinger operator

\[
H = -\frac{1}{2} \Delta + V ,
\]

which has a potential energy term \( V \). The potential \( V \) is a periodic function; in the \( t \)-coordinate it satisfies \( V(t^1, \ldots, t^i+1, \ldots, t^n) = V(t^1, \ldots, t^i, \ldots, t^n) \) for each \( i \). It acts on the twisted periodic function \( f(t) \) by multiplication. To take the potential term into account we may introduce new operators \( X^k \) by

\[
(X^j f)(t) = e^{2\pi i t^j} f(t) \quad (j = 1, 2, \ldots, 2m),
\]

which belong to the same family of operators \( T^k \) of (5.9). Then any periodic potential operator can be expanded as

\[
V(t^1, \ldots, t^n) = \sum_{\sigma_1, \sigma_2, \ldots, \sigma_n = -\infty}^{\infty} c_{\sigma} e^{2\pi i (\sigma_1 t^1 + \cdots + \sigma_n t^n)}
= \sum_{\sigma_1, \sigma_2, \ldots, \sigma_n = -\infty}^{\infty} c_{\sigma} (X^1)^{\sigma_1} \cdots (X^{2m})^{\sigma_2 m} (T^1)^{\sigma_2 m + 1} \cdots (T^{n-2m})^{\sigma_n}.
\]
We can easily calculate commutators of $X$’s with other operators to get an algebra which is an extension of the magnetic algebra. The resulted algebra is isomorphic to the so-called noncommutative torus [21] although we do not yet examine these relation thoroughly.

Another direction for future development is to solve an eigenvalue problem of the Dirac operator in the $n$-torus in the background magnetic field. We also construct supersymmetric field theory, which have both scalar and spinor fields as its constituents to pursue a new mechanism of supersymmetry breaking.

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**Appendix A. Distribution of zeros**

As announced at (4.11) we prove existence of a transformation matrix $L$ that has zeros in the pattern

$$L = \begin{pmatrix}
L_{2i-1}^{2p-1} & L_{2i-1}^{2q} & L_{2i-1}^{2m+r} \\
L_{2p-1}^{2j} & L_{2q}^{2j} & L_{2m+r}^{2j} \\
L_{2m+k}^{2p-1} & L_{2q}^{2m+k} & L_{2m+r}^{2m+k}
\end{pmatrix} = \begin{pmatrix}
* & 0 & 0 \\
* & * & 0 \\
* & * & *
\end{pmatrix}$$

(A.1)

with $i, j, p, q = 1, \ldots, m$ and $k, r = 1, \ldots, n - 2m$.

In $\mathbb{R}^n$ we have an anti-symmetric bilinear form $B$. We say that a vector $u$ is longitudinal with respect to $B$ if it satisfies

$$B(u, v) = 0$$

(A.2)

for arbitrary $v \in \mathbb{R}^n$. We call

$$M^0 = \{ u \in \mathbb{R}^n | \forall v \in \mathbb{R}^n, B(u, v) = 0 \}$$

(A.3)

a longitudinal vector subspace. Let $(t^1, t^2, \ldots, t^n)$ be the coordinate system that expresses $B$ in the standard form

$$B = \frac{1}{2} \sum_{i,j=1}^{n} \phi_{ij} dt^i \wedge dt^j = \sum_{j=1}^{m} q_j dt^{2j-1} \wedge dt^{2j}.$$  

(A.4)

as in (3.14). Let $(y^1, y^2, \ldots, y^n)$ be another coordinate system that is related to $(t^1, t^2, \ldots, t^n)$ by a linear transformation $y^i = \sum_{j=1}^{n} L_{ij} t^j$. The matrix $L$ is not yet specified. Basis vectors
generated by these coordinates are related as
\[ \frac{\partial}{\partial t^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial t^j} \quad \frac{\partial}{\partial y^i} = \sum_{i=1}^{n} L^i_j \frac{\partial}{\partial t^i} \]
(A.5)
\[ \frac{\partial}{\partial t^j} = \sum_{i=1}^{n} \frac{\partial t^i}{\partial y^j} \quad \frac{\partial}{\partial y^i} = \sum_{i=1}^{n} (L^{-1})^i_j \frac{\partial}{\partial t^i} \]
(A.6)

Define vector subspaces \( M^- \) and \( M^+ \) of \( \mathbb{R}^n \) as
\[ M^- = \mathbb{R} \frac{\partial}{\partial t^1} \oplus \mathbb{R} \frac{\partial}{\partial t^3} \oplus \cdots \oplus \mathbb{R} \frac{\partial}{\partial t^{2m-1}}, \]
(A.7)
\[ M^+ = \mathbb{R} \frac{\partial}{\partial t^2} \oplus \mathbb{R} \frac{\partial}{\partial t^4} \oplus \cdots \oplus \mathbb{R} \frac{\partial}{\partial t^{2m}}. \]
(A.8)

Then \( M^- \oplus M^+ \oplus M^0 = \mathbb{R}^n \). Now let us remember that \( \mathbb{R}^n \) is equipped with inner product structure. Then the two-form \( B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) can be regarded as an antisymmetric operator \( \hat{B} : \mathbb{R}^n \to \mathbb{R}^n \). As a square of a linear operator \( \hat{B}^2 \) is well-defined and becomes a symmetric operator and therefore is diagonalizable by an orthogonal transformation. \( \hat{B}^2 \) has non-positive eigenvalues. The eigenspace \( W^0 \) associated with the zero eigenvalue of \( \hat{B}^2 \) coincides with \( M^0 \). Of course,
\[ \left\{ \frac{\partial}{\partial t^{2m+1}}, \frac{\partial}{\partial t^{2m+2}}, \cdots, \frac{\partial}{\partial t^n} \right\} \]
(A.9)
is a basis of \( M^0 = W^0 \). We take an orthonormal basis
\[ \left\{ \frac{\partial}{\partial y^{2m+1}}, \frac{\partial}{\partial y^{2m+2}}, \cdots, \frac{\partial}{\partial y^n} \right\} \]
(A.10)
of \( W^0 \). This implies that
\[ L^{2i-1}_{2m+r} = L^{2j}_{2m+r} = (L^{-1})^{2i-1}_{2m+r} = (L^{-1})^{2j}_{2m+r} = 0 \]
(A.11)
in (A.5) and (A.6).

Let us step into a difficult part of the proof. Each eigenspace associated with each nonzero eigenvalue of \( \hat{B}^2 \) can be decomposed into two-dimensional subspaces such that each two-dimensional subspace \( W_j \) is irreducible with respect to the action of \( \hat{B} \). Thus we get an orthogonal decomposition
\[ \mathbb{R}^n = W_1 \perp W_2 \perp \cdots \perp W_m \perp W^0. \]
(A.12)

Next we define vector subspaces
\[ W^+_j = W_j \cap (M^+ \oplus M^0), \]
(A.13)
\[ W^-_j = W_j \cap (M^+ \oplus M^0)^\perp, \]
(A.14)
then we can show that both $W_j^+$ and $W_j^-$ have one dimension. First, note that $\dim W_j^+ + \dim W_j^- = \dim W_j = 2$. If $\dim W_j^+ = 2$, $W_j^+$ coincides with $W_j$ itself. Since the two-form $B$ is degenerated on $(M^+ \oplus M^0)$, it must be degenerated also on $W_j = W_j^+ \subset (M^+ \oplus M^0)$. This contradicts the fact that $W_j$ is irreducible with respect to $\hat{B}$. On the other hand, if $\dim W_j^- = 2$, $W_j^-$ coincides with $W_j$ itself. Then we can take an arbitrary one-dimensional subspace $M_{-1} \subset W_j = W_j^- \subset (M^+ \oplus M^0)$. This contradicts the fact that $W_j$ is irreducible with respect to $\hat{B}$. Hence we conclude that $\dim W_j^+ = \dim W_j^- = 1$.

We take a normalized vector $\partial/\partial y^{2j}$ of $W_j^+$. And we take another normalized vector $\partial/\partial y^{2j-1}$ of $W_j^-$ such that $\nu_j = B(\partial/\partial y^{2j-1}, \partial/\partial y^{2j}) > 0$ for each $j = 1, \ldots, m$. Then we obtain a complete orthonormal basis \{\partial/\partial y^i | i = 1, 2, \ldots, n\} that expresses $B$ in the standard form $B = \sum_{j=1}^{m} \nu_j dy^{2j-1} \wedge dy^{2j}$.

Since $\partial/\partial y^{2q} \in W_q^+ \subset M^+ \oplus M^0$, we can say that

$$ (L^{-1})_{2q}^{2q-1} = 0 $$(A.15)

in (A.6). By an elementary argument of linear algebra we can say that

$$ L_{2q}^{2q-1} = 0. $$ (A.16)

The proof is over.

**Appendix B. Fourier transformation of the Hermite polynomials**

In our convention the Hermite polynomial is defined as

$$ H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}. $$ (B.1)

The formula (L.10) can be deduced by a partial integration and a change of variables as

$$ \int_{-\infty}^{\infty} dk \, e^{2\pi ik z} \cdot e^{\pi k^2 / \nu} \left( -\frac{\partial}{\partial k} \right)^n e^{-2\pi k^2 / \nu} $$

$$ = \int_{-\infty}^{\infty} dk \, e^{-2\pi k^2 / \nu} \left( \frac{\partial}{\partial k} \right)^n e^{2\pi ik z} \cdot e^{\pi k^2 / \nu} $$

$$ = e^{\pi \nu z^2} \int_{-\infty}^{\infty} dk \, e^{-2\pi k^2 / \nu} \left( \frac{\partial}{\partial k} \right)^n e^{\pi (k+i\nu z)^2 / \nu} $$

$$ = e^{\pi \nu z^2} \int_{-\infty}^{\infty} dk \, e^{-2\pi k^2 / \nu} \left( -\frac{i}{\nu} \frac{\partial}{\partial z} \right)^n e^{\pi (k+i\nu z)^2 / \nu} $$

$$ = e^{\pi \nu z^2} \left( -\frac{i}{\nu} \frac{\partial}{\partial z} \right)^n \int_{-\infty}^{\infty} dk \, e^{-\pi (k-i\nu z)^2 / \nu - 2\pi \nu z^2} $$

$$ = e^{\pi \nu z^2} \left( -\frac{i}{\nu} \frac{\partial}{\partial z} \right)^n \sqrt{\nu} e^{-2\pi \nu z^2}. $$ (B.2)
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