AN ERDŐS–RÉVÉSZ TYPE LAW OF THE ITERATED LOGARITHM FOR REFLECTED FRACTIONAL BROWNIAN MOTION

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Abstract. Let $B_H = \{B_H(t) : t \in \mathbb{R}\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. For the stationary storage process $Q_{B_H}(t) = \sup_{-\infty < s \leq t} (B_H(t) - B_H(s) - (t - s))$, $t \geq 0$, we provide a tractable criterion for assessing whether, for any positive, non-decreasing function $f$, $\mathbb{P}(Q_{B_H}(t) > f(t) \ i.o.)$ equals 0 or 1. Using this criterion we find that, for a family of functions $f_p(t)$, such that $\xi_p(t) = \mathbb{P}(\sup_{s \in [0,f_p(t)]}Q_{B_H}(s) > f_p(t))/f_p(t) = \mathcal{C}(t \log^{-1} p)^{-1}$, for some $\mathcal{C} > 0$, $\mathbb{P}(Q_{B_H}(t) > f_p(t) \ i.o.) = 1_{\{p \geq 2\}}$. Consequently, with $Q_{B_H}(t) = \sup\{s : 0 \leq s \leq t, Q_{B_H}(s) \geq f_p(s)\}$, for $p \geq 0$, $\lim_{t \to \infty} \xi_p(t) = \infty$ and $\limsup_{t \to \infty}(\xi_p(t) - t) = 0$ a.s. Complementary, we prove an Erdős–Rényi type law of the iterated logarithm lower bound on $\xi_p(t)$, i.e., $\liminf_{t \to \infty}(\xi_p(t) - 0)/h_p(t) = -1$ a.s., $p > 1$; $\liminf_{t \to \infty} \log(\xi_p(t)/t)/(h_p(t)/t) = -1$ a.s., $p \in (0,1]$, where $h_p(t) = (1/z_p(t)) p \log \log t$.

1. Introduction and Main Results

The analysis of properties of reflected stochastic processes, being developed in the context of classical Skorohod problems and their applications to queueing theory, risk theory and financial mathematics, is an actively investigated field of applied probability. In this paper we analyze 0-1 properties of a class of such processes, that due to its importance in queueing theory (and dual risk theory) gained substantial interest; see, e.g., [1, 2, 13, 14] or novel works on $\gamma$-reflected Gaussian processes [7, 12].

Consider a reflected (at 0) fractional Brownian motion with drift $Q_{B_H} = \{Q_{B_H}(t) : t \geq 0\}$, given by the following formula

$$Q_{B_H}(t) = B_H(t) - ct + \max\left(0, \inf_{s \in [0,t]} (B_H(s) - cs)\right),$$

where $c > 0$ and $B_H = \{B_H(t) : t \in \mathbb{R}\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, i.e., a centered Gaussian process with covariance function $\mathbb{Cov}(B_H(t), B_H(s)) = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$. We focus on the investigation of the long-time behavior of the unique stationary solution of (1), which has the following representation

$$Q_{B_H}(t) = \sup_{-\infty < s \leq t} (B_H(t) - B_H(s) - c(t-s)).$$

With no loss of generality in the reminder of this paper we assume that the drift parameter $c \equiv 1$. An important stimulus to analyze the distributional properties of $Q_{B_H}$ and its functionals stems from the Gaussian fluid queueing theory, where the stationary buffer content process in a queue which is fed by $B_H$ and emptied with constant rate $c = 1$ is described by (2); see e.g. [13]. In particular, in the seminal paper by Hüsler and Piterbarg [8] the exact asymptotics of one dimensional marginal distributions of $Q_{B_H}$ was derived; see also [3, 4, 6] for results on more general Gaussian input processes.

The purpose of this paper is to investigate the asymptotic 0-1 behavior of the processes $Q_{B_H}$. Our first contribution is an analog of the classical finding of Watanabe [18], where an asymptotic 0-1 type of behavior for centered stationary Gaussian processes was analyzed.

Theorem 1. For all functions $f(t)$ that are positive and nondecreasing on some interval $[T, \infty)$, it follows that

$$\mathbb{P}(Q_{B_H}(t) > f(t) \ i.o.) = 0 \ or \ 1,$$

according as the integral

$$I_f := \int_T^\infty \frac{1}{f(u)} \mathbb{P}\left(\sup_{t \in [0,f(u)]} Q_{B_H}(t) > f(u)\right) \, du$$

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is finite or infinite.

The exact asymptotics, as \( u \) grows large, of the probability in \( \mathcal{J}_f \) was found by Piterbarg [14, Theorem 7]. Namely, for any \( T > 0 \),

\[
(3) \quad \mathbb{P} \left( \sup_{t \in [0,T]} Q_{B_H}(t) > f(u) \right) = \sqrt{\pi a^2 b^{-\frac{1}{2}} H_{B_H}^2 T(v_f(u)) \Phi^{-1}(v_f(u))(1 + o(1))}, \quad u \to \infty,
\]

where \( v_f(u) = Af^{1-H}(u), \Phi(u) = 1 - \Phi(u) \), \( \Phi \) is the distribution function of the unit normal law and the constants \( a, b, A, \mathcal{H}_{B_H} \) are given explicitly in Section 2. Since relation (3) also holds when \( T = T(u) \to 0 \), provided that \( T(u)/(f(u))^{(1-H)/H} \to \infty \), we have that for \( H \in (0, \frac{1}{2}) \), as \( u \to \infty \),

\[
\frac{1}{f(u)} \mathbb{P} \left( \sup_{t \in [0,f(u)]} Q_{B_H}(t) > f(u) \right) \sim \mathbb{P} \left( \sup_{t \in [0,1]} Q_{B_H}(t) > f(u) \right).
\]

Theorem 1 provides a tractable criterion for settling the dichotomy of \( \mathbb{P}(Q_{B_H}(t) > f(t)) \text{ i.o.} \). For instance, let \( C_H = (2(1-H)^2 - H)/(2H(1-H)) \) and

\[
(4) \quad f_p(s) = \left( \frac{2}{A^2} (\log s + (1 + C_H - p) \log_2 s) \right)^{\frac{1}{1-p}} \quad p \in \mathbb{R}, H \in (0, 1).
\]

One can check that, as \( u \to \infty \),

\[
(5) \quad \frac{1}{f_p(u)} \mathbb{P} \left( \sup_{t \in [0,f_p(u)]} Q_{B_H}(t) > f_p(u) \right) = \frac{a^2 b^{-\frac{1}{2}} H_{B_H}^2 A^{1-p} 2C_H (u \log^{1-p} u)^{-1}(1 + o(1))}{\sqrt{2}}
\]

Hence, for any \( p \in \mathbb{R} \),

\[
\mathbb{P}(Q_{B_H}(t) > f_p(t) \text{ i.o.}) = \begin{cases} 1 & \text{if } p \geq 0, \\ 0 & \text{if } p < 0. \end{cases}
\]

Corollary 1. For any \( H \in (0, 1) \),

\[
\lim_{t \to \infty} \sup_{t \to \infty} \frac{Q_{B_H}(t)}{(\log t)^{1-p}} = \left( \frac{2}{A^2} \right)^{\frac{1}{1-p}} \text{ a.s.}
\]

This result extends findings of Zeevi and Glynn [19, Theorem 1], where it was proven that the above convergence holds weakly as well as in \( L_p \) for all \( p \in [1, \infty) \).

Now consider the process \( \xi_p = \{\xi_p(t) : t \geq 0\} \) defined as

\[
\xi_p(t) = \sup \{s : 0 \leq s \leq t, Q_{B_H}(s) \geq f_p(s)\}.
\]

Since \( \mathcal{J}_{f_p} = \infty \) for \( p \geq 0 \), from Theorem 1 it follows that

\[
\lim_{t \to \infty} \xi_p(t) = \infty \quad \text{a.s.} \quad \text{and} \quad \lim_{t \to \infty} (\xi_p(t) - t) = 0 \quad \text{a.s.}
\]

Let, cf. (5),

\[
h_p(t) = p \left( \frac{1}{f_p(t)} \mathbb{P} \left( \sup_{s \in [0,f_p(t)]} Q_{B_H}(s) > f_p(t) \right) \right)^{-1} \log_2 t.
\]

The second contribution of this paper is an Erdős–Révész type of law of the iterated logarithm for the process \( \xi_p \). We refer to Shao [16] for more background and references on Erdős–Révész type law of the iterated logarithm and a related result for centered stationary Gaussian processes; see also Dębicki and Kosiński [5] for extensions to order statistics.

Theorem 2. If \( p > 1 \), then

\[
\lim_{t \to \infty} \frac{\xi_p(t) - t}{h_p(t)} = -1 \quad \text{a.s.}
\]

If \( p \in (0, 1] \), then

\[
\lim_{t \to \infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t^2} = -1 \quad \text{a.s.}
\]
Now, let us complementary put \( \eta_p = \{ \eta_p(t) : t \geq 0 \} \), where
\[
\eta_p(t) = \inf \{ s \geq t : Q_{B_H}(s) \geq f_p(s) \}.
\]
Since
\[
\mathbb{P} (\xi_p(t) - t \leq -x) = \mathbb{P} \left( \sup_{s \in [t-x,t]} \frac{Q_{B_H}(s)}{f_p(s)} < 1 \right)
\]
and
\[
\mathbb{P} (z - \eta_p(z) \leq -x) = \mathbb{P} \left( \sup_{s \in [z,x+z]} \frac{Q_{B_H}(s)}{f_p(s)} < 1 \right),
\]
then it follows that
\[
\liminf_{t \to \infty} \frac{\xi_p(t) - t}{h_p(t)} = \liminf_{z \to \infty} \frac{z - \eta_p(z)}{h_p(z)}.
\]

Theorem 2 shows that for \( t \) big enough, there exists an \( s \) in \([t - h_p(t), t]\) (as well as in \([t, t + h_p(t)]\) by (6)) such that \( Q_{B_H}(s) \geq f_p(s) \) and that the length \( h_p(t) \) of the interval is the smallest possible. This shines new light on results, which are intrinsically connected with Gumbel limit theorems; see, e.g., [11], where the function \( h_p(t) \) plays crucial role. We shall pursue this elsewhere.

The paper is organized as follows. In Section 2 we introduce some useful properties of storage processes fed by fractional Brownian motion. In Section 3 we provide a collection of basic results on how to interpret extremes of the storage process \( Q_{B_H} \) as extremes of a Gaussian field related to the fractional Brownian motion \( B_H \). Furthermore, in Section 4 we prove lemmas, which constitute building blocks of the proofs of the main results.

2. Properties of the storage process

In this section we introduce some notation and state some properties of the supremum of the process \( Q_{B_H} \) as derived in [10, 14]. We begin with the relation

\[
\mathbb{P} \left( \sup_{t \in [0,T]} Q_{B_H}(t) > u \right) = \mathbb{P} \left( \sup_{s \in [0,T]/\nu} Z_u(s, \tau) > u^{1-H} \right), \quad \text{for any} \quad T > 0,
\]
where, with \( \nu(\tau) = \tau^{-H} + \tau^{1-H} \),
\[
Z_u(s, \tau) := B_H(u(\tau + s)) - B_H(su) \frac{\tau^{H} u^{H} \nu(\tau)}{\tau^{H} u^{H} \nu(\tau)}
\]
is a Gaussian field. Note that the self-similarity property of \( B_H \) implies that the field \( Z_u \) has the same distribution for any \( u \). Thus, we do not use \( u \) as an additional parameter in the following notation whenever it is not needed; let \( Z(s, \tau) := Z_1(s, \tau) \). Furthermore, the field \( Z(s, \tau) \) is stationary in \( s \), but not in \( \tau \). The variance \( \sigma_Z^2(\tau) \) of the field \( Z(s, \tau) \) equals \( \nu^{-2}(\tau) \) and \( \sigma_Z(\tau) \) has a single maximum point at
\[
\tau_0 = \frac{H}{1 - H}.
\]

Taylor expansion leads to
\[
\sigma_Z(\tau) = \frac{1}{A} - \frac{B}{2A^2}(\tau - \tau_0)^2 + O((\tau - \tau_0)^3),
\]
as \( \tau \to \tau_0 \), where
\[
A = \frac{1}{1 - H} \left( \frac{H}{1 - H} \right)^{-H} = \nu(\tau_0),
\]
\[
B = H \left( \frac{H}{1 - H} \right)^{-H-2} = \nu''(\tau_0).
\]

Let us define the correlation function of the process \( Z_u \) as follows
\[
r_{u,w}(s, \tau, s', \tau') := \mathbb{E} Z_u(s, \tau) Z_u(s', \tau') \nu(\tau) \nu(\tau')
\]
\[
= \frac{|us - u's'|^{2H}}{2|wu'w'\tau'|^H} \left( \left| 1 + \frac{u\tau}{(us - u's')} \right|^{2H} - 1 + \frac{(w\tau - uu'\tau')}{(us - u's')} \right)^{2H} + \left( 1 - \frac{u\tau'}{(us - u's')} \right)^{2H} - 1).
\]
By series expansion we find for any fixed $\tau_1 < \tau_0 < \tau_2$ and $\tau, \tau'$ with $0 < \tau_1 < \tau, \tau' < \tau_2 < \infty$,  
\[ |r_{u,v}(s, \tau, s', \tau')| \leq \frac{|us - us'|^{2H}}{(u\tau u'\tau')^H} 2H |2H - 1||us - u's'|^{-2} (u\tau u'\tau'), \quad 2H \neq 1, \]
provided that $|\frac{u}{us - u's'}|$ and $|\frac{u'}{us - u's'}|$ are sufficiently small. For $2H = 1$, we have $r_{u,v}(s, \tau, s', \tau') = 0$ since the increments of Brownian motion on disjoint intervals are independent. Therefore, there exists a positive constant $C$ depending only on $H, \tau_1$ and $\tau_2$. Similarly, from (8) it follows that for any fixed $M$ there exists $\delta \in (0, 1)$ such that  
\[ 0 < \delta \leq \inf_{|\tau - \tau'| \leq m} \frac{r_{u,v}(s, \tau, s', \tau') \leq \sup_{|\tau - \tau'| \leq m} r_{u,v}(s, \tau, s', \tau') \leq 1 - \delta < 1, \]
for sufficiently small $m$.

2.1. Asymptotics. Due to the following lemma, while analyzing tail asymptotics of the supremum of $Z$, we can restrict the considered domain of $(s, \tau)$ to a strip with $|\tau - \tau_0| \leq \log v/v$.

**Lemma 1** (Piterbarg [14], Lemma 2 and 4). There exists a positive constant $C$ such that for any $v, T > 0$,  
\[ \mathbb{P} \left( \sup_{|\tau - \tau_0| \geq \log v/v} AZ(s, \tau) > v \right) \leq CT^{2H} \exp \left( -\frac{1}{2} v^2 - b \log^2 v \right), \]
where $b = B/(2A)$. Furthermore, for any $T > 0$, with $a = 1/(2\tau_0^2 H)$, as $v \to \infty$,  
\[ \mathbb{P} \left( \sup_{|\tau - \tau_0| \leq \log v/v} AZ(s, \tau) > v \right) = \sqrt{\pi a} b^{-\frac{1}{2}} H_{B_H}^2 T v^{-\frac{1}{H}} - 2^{-2H} (v(1 + o(1)), \]
where  
\[ H_{B_H} = \lim_{T \to \infty} T^{-1} \mathbb{E} \left( \sup_{t \in [0, T]} \left( \sqrt{B_H(t) - t^{2H}} \right) \right) \in (0, \infty), \]
is the so-called Pickands' constant. This holds also for $T = v^{-1/H'}$, with $1 > H' > H$.

Hüsler and Piterbarg [9, Corollary 2] showed that the above actually holds true for $T$ depending on $v$ such that $v^{-1/H'} < T < \exp(cv^2)$, for any $H' \in (H, 1)$ and $c \in (0, \frac{1}{2})$.

2.2. Discretization. Let $\tau^*(v) = \log v/v$ and $J(v) = \{ \tau : |\tau - \tau_0| \leq \tau^*(v) \}$. For a fixed $T, \theta > 0$ and some $v > 0$, let us define a discretization of the set $[0, T] \times J(v)$ as follows  
\[ s_{l} = lq(v), \quad 0 \leq l \leq L, \quad L = \lfloor T/q(v) \rfloor, \quad q(v) = \theta v^{-\frac{1}{H}}, \]
\[ \tau_n = \tau_0 + nq(v), \quad 0 \leq n \leq N, \quad N = \lfloor \tau^*(v)/q(v) \rfloor. \]
Along the same lines as in [10, Lemma 6] we get the following lemma.

**Lemma 2**. There exist positive constants $K_1, K_2, v_0 > 0$, such that, for any $\theta > 0$ and $v \geq v_0$,  
\[ \mathbb{P} \left( \max_{0 \leq l \leq L} AZ(s_{l}, \tau_n) \leq v - \theta v^{-\frac{1}{H}}, \sup_{s \in [0, T]} AZ(s, \tau) > v \right) \leq K_1 v^{-\frac{1}{H}} - 2^{-2H} (v(1 + o(1)), \]
Finally, it is possible to approximate tail asymptotics of the supremum of $Z$ on the strip $[0, T] \times J(v)$ by maximum taken over discrete time points. The proof of the following lemma follows line-by-line the same as the proof of [14, Lemma 4] and thus we omit it. Similar result can be found in, e.g., [10, Lemma 7].
Lemma 3. For any $T, \theta > 0$, as $v \to \infty$,
\[
\mathbb{P} \left( \max_{0 \leq n \leq \lceil TV \rceil} AZ(s_i, \tau_n) > v \right) = \sqrt{\pi} a b^{-\frac{1}{2}} \left( \mathcal{H}_{B_\theta}^0 \right)^2 T v^{-1} \Psi(v) (1 + o(1)),
\]
where $\mathcal{H}_{B_\theta}^0 = \lim_{S \to \infty} S^{-1} \mathbb{E} \exp \left( \sup_{t \in [0, S]} \left( \sqrt{2} B(t) - t^2 H \right) \right)$.

It follows easily that $\mathcal{H}_{B_\theta}^0 \to H_{B_\theta}$ as $\theta \to 0$, so that the above asymptotics is the same as in Lemma 1 when the discretization parameter $\theta$ decreases to zero so that the number of discretization points grows to infinity.

3. Auxiliary Lemmas

We begin with some auxiliary lemmas that are later needed in the proofs. The first lemma is a slightly modified version of [11, Theorem 4.2.1].

Lemma 4 (Berman’s inequality). Suppose that $\xi_1, \ldots, \xi_n$ are normal random variables with correlation matrix $\Lambda^1 = (\Lambda_{ij}^1)$ and $\eta_1, \ldots, \eta_n$ similarly with correlation matrix $\Lambda^0 = (\Lambda_{ij}^0)$. Let $\sigma(\xi) = \sigma(\eta) \in (0,1]$, $\rho_{i,j} = \max(|\Lambda_{ij}^1|, |\Lambda_{ij}^0|)$ and $u_i$ be real numbers, $i = 1, \ldots, n$. Then,
\[
\mathbb{P} \left( \bigcap_{j=1}^n \{ \xi_j \leq u_j \} \right) = \mathbb{P} \left( \bigcap_{j=1}^n \{ \eta_j \leq u_j \} \right) \leq \frac{1}{2\pi} \sum_{1 \leq i < j \leq n} \left( (\Lambda_{ij}^1 - \Lambda_{ij}^0)^+ (1 - \rho_{i,j}^2) \right)^{\frac{1}{2}} \exp \left( - \frac{u_i^2 + u_j^2}{2(1 + \rho_{i,j})} \right).
\]

The following lemma is a general form of the Borel-Cantelli lemma; cf. [17].

Lemma 5 (Borel-Cantelli lemma). Consider a sequence of events $\{E_k\}_{k=0}^\infty$. If
\[
\sum_{k=0}^\infty \mathbb{P} (E_k) < \infty,
\]
then $\mathbb{P} (E_n \ i.o.) = 0$. Whereas, if
\[
\sum_{k=0}^\infty \mathbb{P} (E_k) = \infty \quad \text{and} \quad \liminf_{n \to \infty} \frac{n-1}{\sum_{k=1}^n \mathbb{P} (E_k)} \leq 1,
\]
then $\mathbb{P} (E_n \ i.o.) = 1$.

Lemma 6. For any $\varepsilon \in (0,1)$, there exist positive constants $K$ and $\rho$ depending only on $\varepsilon, H, p$ and $\lambda$ such that
\[
\mathbb{P} \left( \sup_{S < t \leq T} \frac{Q_{B_\theta}(t)}{f_p(t)} \leq 1 \right) \leq \exp \left( - \frac{(1 - \varepsilon)}{(1 + \varepsilon)} \int_{S + f_p(S)}^T \frac{1}{f_p(u)} \mathbb{P} \left( \sup_{t \in [0, f_p(u)]} Q_{B_\theta}(t) > f_p(u) \right) du \right) + KS^{-\rho},
\]
for any $T - f_p(S) \geq S \geq K$, with $f_p(T)/f_p(S) \leq C$ and $C$ being some universal positive constant.

Proof. Let $\varepsilon \in (0,1)$ be some positive constant. For the reminder of the proof let $K$ and $\rho$ be two positive constants depending only on $\varepsilon, H, p$ and $\lambda$ that may differ from line to line. For any $k \geq 0$ put $s_0 = S$, $y_0 = f_p(s_0)$, $t_0 = s_0 + y_0$, $x_0 = f_p(t_0)$ and
\[
s_k = t_{k-1} + \varepsilon x_{k-1}, \quad y_k = f_p(s_k), \quad t_k = s_k + y_k, \quad x_k = f_p(t_k), \quad i_k = \frac{I_k}{x_k}, \quad \tilde{i}_k = \frac{\tilde{I}_k}{x_k}, \quad |I_k| = |\tilde{I}_k| = \frac{y_k}{x_k}.
\]

From this construction, it is easy to see that the intervals $I_k$ are disjoint. Furthermore, $\delta(I_k, I_{k+1}) = \varepsilon x_k$, and $1 - \varepsilon \leq y_k/x_k \leq 1$, for any $k \geq 0$ and sufficiently large $S$. Note that, for any $k \geq 0$, $|I_k| \sim f_p(S)$ as $S$ grows large, therefore if $T(S, \varepsilon)$ is the smallest number of intervals $\{I_k\}$ needed to cover $[S, T]$, then $T(S, \varepsilon) \leq [(T - S)/(f_p(S)(1 + \varepsilon))]$. Moreover, since $f_p(T)/f_p(S)$ is bounded by the constant $C > 0$ not depending on $S$ and $\varepsilon$, it follows that, $x_k/x_i \leq C$ for any $0 \leq i < k \leq T(S, \varepsilon)$.
Now let us introduce a discretization of the set $\hat{I}_k \times J(v_k)$ as in Section 2.2. That is, for some $\theta > 0$, define grid points

$$s_{k,l} = \hat{s}_k + lq_k, \quad 0 \leq l \leq L_k, \quad L_k = [(1 - \varepsilon)/q_k], \quad q_k = \theta v_k^2, \quad \tau_{k,n} = \tau_0 + nq_k, \quad 0 \leq |n| \leq N_k, \quad N_k = [\tau^*(v_k)/q_k].$$

Since $f_p$ is an increasing function, it easily follows that,

$$P \left( \sup_{S \leq t \leq T} \frac{Q_B(t)}{f_p(t)} \leq 1 \right) \leq P \left( \bigcap_{k=0}^{T(S,c)} \left\{ \sup_{t \in I_k} Q_B(t) \leq x_k \right\} \right) \leq P \left( \bigcap_{k=0}^{T(S,c)} \left\{ \sup_{s \in I_k / x_k, \tau \in J(v_k)} AZ_{x_k}(s, \tau) \leq v_k \right\} \right) \leq \prod_{k=0}^{T(S,c)} P \left( \max_{0 \leq |n| \leq N_k} AZ_{x_k}(s_{k,l}, \tau_{k,n}) \leq v_k \right) \leq \sum_{0 \leq k < T(S,c)} C_{k,t} =: P_1 + P_2,$

where the last inequality follows from Berman’s inequality with

$$C_{k,t} = \sum_{0 \leq k < T(S,c)} \sum_{0 \leq |n| \leq N_k} \left| r_{x_k,x_l}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m}) \right| \exp \left( \frac{1}{1 + \left| r_{x_k,x_l}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m}) \right|} \right).$$

**Estimate of $P_1$.**

Note that we can use the fact that $Z_{x_k}$ has the same distribution as $Z_1 \equiv Z$ for any $x_k$. Since the process $Z$ is stationary with respect to the first variable, from Lemma 3, for any $\varepsilon \in (0, 1)$, sufficiently large $S$ and small $\theta$,

$$P_1 \leq \exp \left( -\sum_{k=0}^{T(S,c)} \max_{0 \leq |n| \leq N_k} AZ_{x_k}(s_{k,l}, \tau_{k,n}) > v_k \right) \leq \exp \left( -(1 - \varepsilon) \sum_{k=0}^{T(S,c)} P \left( \sup_{(s, \tau) \in I_k \times J(v_k)} AZ(s, \tau) > v_k \right) \right).$$

Then, by (7) combined with (3),

$$P_1 \leq \exp \left( -\left(1 - \frac{\varepsilon}{2}\right) \sum_{k=0}^{T(S,c)} \max_{0 \leq |n| \leq N_k} P \left( \sup_{(s, \tau) \in I_k \times J(v_k)} AZ(s, \tau) > v_k \right) \right) = \exp \left( -\left(1 - \frac{\varepsilon}{2}\right) \sum_{k=0}^{T(S,c)} \max_{t \in [0, f_p(t_k)]} P \left( \sup_{(s, \tau) \in I_k \times J(v_k)} Q_B(t) > f_p(t_k) \right) \right) \leq \exp \left( -\left(1 - \varepsilon\right) \sum_{k=0}^{T(S,c)} \max_{t \in [0, f_p(t_k)]} P \left( \sup_{(s, \tau) \in I_k \times J(v_k)} Q_B(t) > f_p(t_k) \right) \right) \leq \exp \left( \frac{1 - \varepsilon}{1 + \varepsilon} \int_{S + f_p(S)}^{T} \frac{1}{f_p(u)} \max_{t \in [0, f_p(u)]} P \left( \sup_{(s, \tau) \in I_k \times J(v_k)} Q_B(t) > f_p(u) \right) \, du \right).$$

**Estimate of $P_2$.**

For any $0 \leq t < k \leq T(S,c), 0 \leq l \leq L_k, 0 \leq p \leq L_t$, we have

$$x_k s_{k,l} - x_t s_{t,p} = (s_k + x_k l q_k) - (s_t + x_t q p) \geq \sum_{i=t}^{k-1} (y_i + \varepsilon x_i) + x_k l q_k - x_t q p \geq \sum_{i=t}^{k-1} (y_i + \varepsilon x_i) - x_t (1 - \varepsilon) \geq (y_t + \varepsilon x_t)(k - t) - x_t (1 - \varepsilon) \geq x_t (k - t) \varepsilon,$$
where the last inequality holds provided that $k - t \geq s_0$ with $s_0$ sufficiently large. Therefore, c.f. (9),

$$ r_{k,t}^* := \sup_{0 \leq t \leq L_i, 0 \leq p \leq L_i, |m| \leq N_i} |r_{x_k,x_t}(s_k,l, \tau_{k,n}, s_t,p, \tau_{t,m})| \leq r^*(k-t) \varepsilon \leq K(k-t)^{-\lambda} \leq \min(1, \lambda)/4. $$

Moreover, from (10) it follows that, for any $0 \leq k - t \leq s_0$, there exists a constant $\zeta \in (0, 1)$ depending only on $\varepsilon$ such that for sufficiently large $S$,

$$ \sup_{0 \leq t \leq L_i, 0 \leq p \leq L_i, |m| \leq N_i} |r_{x_k,x_t}(s_k,l, \tau_{k,n}, s_t,p, \tau_{t,m})| \leq \zeta < 1. $$

Finally, recall that $N_k \leq L_k \leq \theta^{-1}v_k^{\frac{1}{2}}(1)$ and $\exp(-v_k^2/2) = (t_k \log(1+G_{i,n}-p))^{-1}$, c.f. (4),(12), so that

$$ P_2 \leq \frac{4}{\sqrt{1-\zeta}^2} \sum_{0 \leq t \leq k \leq T(S,e)} L_k L_t N_k N_t r_{k,t}^* \exp \left( -\frac{v_k^2 + v_t^2}{2(1 + r_{k,t}^*)} \right) $$

$$ \leq K \left( \sum_{0 \leq k - t \leq s_0} + \sum_{k - t > s_0} \right) \exp \left( -\frac{v_k^2 + v_t^2}{2(1 + \lambda)} \right) $$

$$ \leq K \sum_{k=0}^{\infty} v_k^{\frac{1}{2}} \exp \left( -\frac{v_k^2}{1 + \lambda} \right) + \sum_{k - t > s_0} v_k^{\frac{1}{2}} v_t^{\frac{1}{2}}(k-t)^{-\lambda} \exp \left( -\frac{v_k^2 + v_t^2}{2(1 + \lambda)} \right) $$

$$ \leq K \left( \sum_{k=0}^{\infty} k^{-\frac{1}{2}} t_k^{-\frac{1}{2}} + \sum_{k - t > s_0} k^{-\frac{1}{2}} t_k^{-\frac{1}{2}} (k-t)^{-\lambda} \right) $$

$$ \leq KS^{-p}, $$

where the last inequality follows from basic algebra.

Let $S > 0$ be any fixed number, $a_0 = S, y_0 = f_p(a_0)$ and $b_0 = a_0 + y_0$. For $i > 0$, define

$$ a_i = b_{i-1}, \quad y_i = f_p(a_i), \quad b_i = a_i + y_i, \quad M_i = (a_i, b_i), \quad v_i = Ay_i^{-H}, $$

$$ t_i = M_i/y_i = (\tilde{a}_i, \tilde{b}_i). $$

From this construction it is easy to see that the intervals $M_i$ are disjoint, $\bigcup_{i=0}^T M_i = (S, b_i)$ and $|\tilde{M}_i| = 1$. Now let us introduce a discretization of the set $\tilde{M}_i \times J(v_i)$ as in Section 2.2. That is, for some $\theta > 0$, define grid points

$$ s_{i,l} = \tilde{a}_i + lq_i, \quad 0 \leq l \leq L_i, \quad L_i = [1/q_i], \quad q_i = \theta v_i^{\frac{1}{2}}, $$

$$ \tau_{i,n} = \tau_0 + nq_i, \quad 0 \leq |n| \leq N_i, \quad N_i = [\tau^*(v_i)/q_i]. $$

With the above notation, we have the following lemma.

**Lemma 7.** For any $\varepsilon \in (0, 1)$ there exist positive constants $K$ and $\rho$ depending only on $\varepsilon, H, p$ and $\lambda$ such that, with $\theta_i = v_i^{-4/H}$,

$$ \mathbb{P} \left( \left| T - S - f_p(S) \right| \max_{i=0}^T A\mathbb{E}_{s_{i,l}, \tau_{i,n}}(s_{i,l}, \tau_{i,n}) \leq v_i - \frac{\theta_i}{v_i} \right) $$

$$ \geq \frac{1}{4} \exp \left( -1 + \varepsilon \right) \int_{S} T \mathbb{P} \left( \sup_{t \in [0, f_p(u)]} Q_{B,H}(t) > f_p(u) \right) du - KS^{-\rho}, $$

for any $T - f_p(S) \geq S \geq K$, with $f_p(T)/f_p(S) \leq C$ and $C$ being some universal positive constant.
Proof. Put \( \hat{\epsilon}_i = v_i - \frac{\theta_i^2}{v_i} \) and \( I = [(T - S)/f_p(S)] \). Similarly as in the proof of Lemma 6 we find that Berman’s inequality implies

\[
P \left( \bigcup_{i=0}^{I} \left\{ \max_{0 \leq l \leq L_i} \max_{0 \leq n \leq N_i} AZ_{y_i}(s_{i,l}, r_{i,n}) \leq v_i - \frac{\theta_i^2}{v_i} \right\} \right) \geq \frac{1}{2} \exp \left( -\sum_{i=0}^{I} \mathbb{P} \left( \sup_{s \in M_i, r \in J(v_i)} AZ(s, r) > v_i - \frac{\theta_i^2}{v_i} \right) \right)
\]

where

\[
D_{i,j} = \frac{1}{2 \pi} \sum_{0 \leq l \leq L_i} \sum_{0 \leq p \leq L_j} \sum_{|m| \leq N_j} \exp \left( -\frac{1}{2} (\hat{u}_i^2 + \hat{u}_j^2) \right)
\]

with

\[
\tilde{r}_{y_i,v_i}(s_{i,l}, v_i, s_{j,p}, s_{j,p}, r_{j,m}) = -r_{y_i,v_i}(s_{i,l}, v_i, s_{j,p}, s_{j,p}, r_{j,m})
\]

Estimate of \( P'_1 \).

By Lemma 1 the correction term \( \theta_i^2/v_i \) does not change the order of the asymptotics of the tail of \( Z \). Furthermore, the tail asymptotics of the supremum on the strip \((s, r) \in M_i \times J(v_i)\) are of the same order if \( \tau \geq 0 \). Hence, for every \( \epsilon > 0 \), following the same lines of reasoning as in the estimation of \( P_1 \) in Lemma 6,

\[
P'_1 \geq \frac{1}{2} \exp \left( -\sum_{i=0}^{I} \mathbb{P} \left( \sup_{s \in M_i, r \in J(v_i)} AZ(s, r) > v_i - \frac{\theta_i^2}{v_i} \right) \right)
\]

Estimate of \( P'_2 \).

Clearly, for \( j \geq i + 2 \) and any \( 0 \leq l \leq L_i, 0 \leq p \leq L_j \); c.f. (13),

\[
y_j s_j - y_i s_i = a_j + y_j p_{q_j} - (a_i + y_i q_i) \geq (j - i - 1)y_i,
\]

so that by (9), for any \( 0 \leq i < j \leq I \),

\[
r_{i,j}^* := \sup_{0 \leq l \leq L_i, 0 \leq p \leq L_j, 0 \leq n \leq N_i, 0 \leq m \leq N_j} |\tilde{r}_{y_i,v_i}(s_{i,l}, v_i, s_{j,p}, s_{j,p}, r_{j,m})| \leq r^*(j - i - 1) \leq r^*(1) < 1.
\]

On the other hand, by (10), there exist positive constants \( s_0 \), such that for sufficiently large \( S \),

\[
|\tilde{r}_{y_i,v_i}(s_{i,l}, v_i, s_{j,p}, s_{j,p}, r_{j,m})| = 0, \quad \text{if} \quad j = i + 1, \quad |y_j s_j - y_i s_i|/y_i \leq s_0,
\]

\[
|\tilde{r}_{y_i,v_i}(s_{i,l}, v_i, s_{j,p}, s_{j,p}, r_{j,m})| \leq r^*(s_0) < 1, \quad \text{if} \quad j = i + 1, \quad |y_j s_j - y_i s_i|/y_j > s_0.
\]
Therefore, by (15)–(17) we obtain
\[ P_2' \leq \sum_{0 \leq i \leq t-1} \sum_{0 \leq l \leq L_i} \sum_{|n| \leq N_i} \frac{1}{\sqrt{1-r^*(s_0)}} \exp \left( -\frac{\frac{1}{2}(v_i^2 + v_j^2)}{1 + r^*(s_0)} \right) + \sum_{0 \leq i \leq t-2} \sum_{0 \leq l \leq L_i} \sum_{|n| \leq N_i} \frac{r^*(j-i-1)}{\sqrt{1-r^*(s_0)}} \exp \left( -\frac{\frac{1}{2}(v_i^2 + v_j^2)}{1 + r^*(j-i-1)} \right). \]

Completely similarly to the estimation of \( P_2 \) in the proof of Lemma 6, we can get that there exist positive constants \( K \) and \( \rho \) such that, for sufficiently large \( S \),
\[ P_2' \leq KS^{-\rho}. \]

The next lemma is a straightforward modification of [18, Lemma 3.1 and Lemma 4.1], see also [15, Lemma 1.4].

Lemma 8. If Theorem 1 is true under the additional condition, that for large \( t \),
\[ \left( \frac{2}{A^2} \log t \right)^{1/2(1-H)} \leq f(t) \leq \left( \frac{3}{A^2} \log t \right)^{1/2(1-H)}, \]

it is true without the additional condition.

4. Proof of the main results

Proof of Theorem 1. Note that the case \( \mathcal{F}_f < \infty \) is straightforward and does not need any additional knowledge on the process \( Q_{BH} \) apart from the stationarity property. Indeed, consider the sequence of intervals \( M_i \) as in Lemma 7. Then, for any \( \varepsilon > 0 \) and sufficiently large \( T \),
\[ \sum_{k=[T]+1}^{\infty} \mathbb{P} \left( \sup_{t \in M_k} Q_{BH}(t) > f(a_k) \right) = \sum_{k=[T]}^{\infty} \mathbb{P} \left( \sup_{t \in [0,f(b_k)]} Q_{BH}(t) > f(b_k) \right) \leq \mathcal{F}_f < \infty, \]
and the Borel-Cantelli lemma completes this part of the proof since \( f \) is an increasing function.

Now let \( f \) be an increasing function such that \( \mathcal{F}_f = \infty \). Using the same notation as in Lemma 6 with \( f \) instead of \( f_p \), we find that, for any \( S, \varepsilon, \theta > 0 \),
\[ \mathbb{P} \left( Q_{BH}(s) > f(s) \text{ i.o.} \right) \geq \mathbb{P} \left\{ \left( \sup_{t \in K_k} Q_{BH}(t) > f(t_k) \right) \text{ i.o.} \right\} \geq \mathbb{P} \left\{ \left\{ \max_{0 \leq l \leq L_k} A_{Z_{x_k}}(s_{k,l}, \tau_{k,n}) > v_k \right\} \text{ i.o.} \right\}. \]

Let
\[ E_k = \left\{ \max_{0 \leq l \leq L_k} A_{Z_{x_k}}(s_{k,l}, \tau_{k,n}) \leq v_k \right\}. \]

For sufficiently large \( S \) and \( \theta \); c.f. estimation of \( P_1 \), we get
\[ \sum_{k=0}^{\infty} \mathbb{P} \left( E_k^c \right) \geq \frac{(1-\varepsilon)}{(1+\varepsilon)} \int_{S+f(s)}^{\infty} \frac{1}{f(u)} \mathbb{P} \left( \sup_{t \in [0,f(u)]} Q_{BH}(t) > f(u) \right) \, du = \infty. \]

Note that
\[ 1 - \mathbb{P} \left( E_i^c \text{ i.o.} \right) = \lim_{m \to \infty} \prod_{k=m}^{\infty} \mathbb{P} \left( E_k \right) + \lim_{m \to \infty} \left( \mathbb{P} \left( \bigcap_{k=m}^{\infty} E_k \right) - \prod_{k=m}^{\infty} \mathbb{P} \left( E_k \right) \right). \]

The first limit equals to zero as a consequence of (19). The second limit equals to zero because of the asymptotic independence of the events \( E_k \). Indeed, there exist positive constants \( K \) and \( \rho \), depending only on \( H, \varepsilon, \lambda \), such that for any \( n > m \),
\[ A_{m,n} = \left| \mathbb{P} \left( \bigcap_{k=m}^{n} E_k \right) - \prod_{k=m}^{n} \mathbb{P} \left( E_k \right) \right| \leq K(S+m)^{-\rho}, \]
by the same calculations as in the estimate of $P_2$ in Lemma 6 after realizing that, by Lemma 8, we might restrict ourselves to the case when (18) holds. Therefore $\mathbb{P}(E^c_i \text{ i.o.}) = 1$, which completes the proof. □

**Proof of Theorem 2** In order to make the proof more transparent we divide it on several steps.

**Step 1.** Let $p > 1$. Then, for every $\varepsilon \in (0, \frac{1}{2})$,

$$
\liminf_{t \to \infty} \frac{\xi_p(t) - t}{h_p(t)} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.}
$$

**Proof.** Let $\{T_k : k \geq 1\}$ be a sequence such that $T_k \to \infty$, as $k \to \infty$. Put $S_k = T_k - (1 + 2\varepsilon)^2 h_p(T_k)$.

Since $h_p(t) = O(t \log^{1-p} t \log_2 t)$, then, for $p > 1$, $S_k \sim T_k$, as $k \to \infty$, and from Lemma 6 it follows that

$$
\mathbb{P}\left(\frac{\xi_p(T_k) - T_k}{h_p(T_k)} \leq -(1 + 2\varepsilon)^2\right) = \mathbb{P}(\xi_p(T_k) \leq S_k) = \mathbb{P}\left(\sup_{s_k < t \leq T_k} \frac{Q_{B_p}(t)}{f_p(t)} < 1\right)
$$

\[ \leq \exp\left(-\frac{(1 - \varepsilon)}{1 + \varepsilon} \int_{S_k + f_p(S_k)}^{T_k} \frac{1}{f_p(u)} \mathbb{P}\left(\sup_{t \in [0, f_p(u)]} Q_{B_p}(t) > f_p(u)\right) du\right) + 2K T_k^{-\rho}.
\]

Moreover, as $k \to \infty$,

$$
\int_{S_k + f_p(S_k)}^{T_k} \frac{1}{f_p(u)} \mathbb{P}\left(\sup_{t \in [0, f_p(u)]} Q_{B_p}(t) > f_p(u)\right) du \sim (1 + 2\varepsilon)^2 h_p(T_k) \frac{1}{f_p(T_k)} \mathbb{P}\left(\sup_{t \in [0, f_p(T_k)]} Q_{B_p}(t) > f_p(T_k)\right) = (1 + 2\varepsilon)^2 p \log_2 T_k.
$$

Now take $T_k = \exp(k^{1/p})$. Then,

$$
\sum_{k=0}^{\infty} \mathbb{P}(\xi_p(T_k) \leq S_k) \leq 2K \sum_{k=0}^{\infty} k^{-(1+\varepsilon/2)} < \infty.
$$

Hence, by the Borel-Cantelli lemma, we have

$$
\liminf_{k \to \infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \geq -(1 + 2\varepsilon)^2 \quad \text{a.s.}
$$

Since $\xi_p(t)$ is a non-decreasing random function of $t$, for every $T_k \leq t \leq T_{k+1}$, we have

$$
\frac{\xi_p(t) - t}{h_p(t)} \geq \frac{\xi_p(T_k) - T_k}{h_p(T_k)} - \frac{T_{k+1} - T_k}{h_p(T_k)}.
$$

For $p > 1$ elementary calculus implies

$$
\lim_{k \to \infty} \frac{T_{k+1} - T_k}{h_p(T_k)} = 0,
$$

so that

$$
\liminf_{k \to \infty} \frac{\xi_p(t) - t}{h_p(t)} \geq \liminf_{k \to \infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \quad \text{a.s.,}
$$

which completes the proof of this step. □

**Step 2.** Let $p > 1$. Then, for every $\varepsilon \in (0, 1)$,

$$
\liminf_{t \to \infty} \frac{\xi_p(t) - t}{h_p(t)} \leq -(1 - \varepsilon) \quad \text{a.s.}
$$

**Proof.** As in the proof of the lower bound (Step 1), we put

$$
T_k = \exp(k^{1+\varepsilon/2}/p), \quad S_k = T_k - (1 - \varepsilon) h_p(T_k), \quad k \geq 1.
$$

Let

$$
B_k = \{\xi_p(T_k) \leq S_k\} = \left\{\sup_{s_k < t \leq T_k} \frac{Q_{B_p}(t)}{f_p(t)} < 1\right\}.
$$

It suffices to show $\mathbb{P}(B_n \text{ i.o.}) = 1$, that is

$$
\lim_{m \to \infty} \mathbb{P}\left(\bigcup_{k=m}^{\infty} B_k\right) = 1.
$$
Let 
\[ a^k_i = S_k, \quad y^0 = f_p(a^0), \quad b^k_i = a^k_i + y^k, \]
\[ a^k_i = b_{i-1}^k, \quad y^k_i = f_p(a^k_i), \quad b^k_i = a^k_i + y^k_i, \quad M^k_i = (a^k_i, b^k_i), \quad v^k_i = A(y^k_i)^{1-H}, \quad \bar{M}^k_i = \bar{a}^k_i, \bar{b}^k_i. \]

Define \( J_k \) to be the biggest number such that \( b^k_{J_k} \leq T_k \) and \( b^k_{J_k} > T_k \). In what follows let \( b^k_{J_k} \) be redefined to \( T_k \). Note that \( J_k \leq \lceil (T_k - S_k) / f_p(S_k) \rceil \).

Since \( f_p \) is an increasing function,
\[ B_k = \bigcap_{i=0}^{J_k} \left\{ \sup_{t \in M^k_i} Q_{B^k_i}(t) < 1 \right\} \supset \bigcap_{i=0}^{J_k} \left\{ \sup_{t \in M^k_i} Q_{B^k_i}(t) < y^k_i \right\} = \bigcap_{i=0}^{J_k} \left\{ \sup_{s, \tau \geq 0} AZ_{y^k_i}(s, \tau) < v^k_i \right\}. \]

Analogously to (14), define a discretization of the set \( \bar{M}^k_i \times J(v^k_i) \) as follows
\[ s^k_{i,l} = \bar{a}^k_i + l\theta_i^k, \quad 0 \leq l \leq L^k_i, \quad L^k_i = \lceil 1 / q_i^k \rceil, \quad \alpha^k_i = \theta_i^k(v^k_i)^{1-H}, \quad \theta_i^k = (v^k_i)^{-\frac{1}{H}}, \quad \tau^k_{i,m} = \tau_0 + n\theta_i^k, \quad 0 \leq |n| \leq N^k_i, \quad N^k_i = \lceil \tau / (v^k_i) / q_i^k \rceil. \]

Finally, let
\[ A_k = \bigcup_{i=0}^{J_k} \left\{ \max_{0 \leq l \leq L^k_i} AZ_{y^k_i}(s^k_{i,l}, \tau^k_{i,m}) \leq v^k_i - \frac{(\theta_i^k)^{\frac{1}{H}}}{v^k_i} \right\}. \]

Observe that
\[ P \left( \bigcup_{k=m}^{\infty} A_k \right) \leq P \left( \bigcup_{k=m}^{\infty} B_k \right) + \sum_{k=m}^{\infty} P(A_k \cap B^c_k). \]

Furthermore,
\[ \sum_{k=m}^{\infty} P(A_k \cap B^c_k) \leq \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} P \left( \max_{0 \leq l \leq L^k_i} AZ_{y^k_i}(s^k_{i,l}, \tau^k_{i,m}) \leq v^k_i - \frac{(\theta_i^k)^{\frac{1}{H}}}{v^k_i}, \sup_{s \in M^k_i} AZ_{y^k_i}(s, \tau) \geq v^k_i \right) \]
\[ \leq \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} P \left( \max_{0 \leq l \leq L^k_i} AZ_{y^k_i}(s^k_{i,l}, \tau^k_{i,m}) \leq v^k_i - \frac{(\theta_i^k)^{\frac{1}{H}}}{v^k_i}, \sup_{s \in M^k_i} AZ_{y^k_i}(s, \tau) \geq v^k_i \right) \]
\[ + \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \sup_{s \in M^k_i} AZ_{y^k_i}(s, \tau) \geq v^k_i \}

(23)

By Lemma 2, for sufficiently large \( m \) and some \( K_1, K_2 > 0 \), the first sum is bounded from above by
\[ K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} (v^k_i)^{2M^{-1}M} \exp \left( -\frac{(v^k_i)^2}{2} - \frac{(v^k_i)^4}{2K_1} \right) \leq K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \frac{(\log a^k_i)^{1-H}}{a^k_i^{1+H}} \exp \left( -\frac{(v^2_i)^2}{2K_2} \right) \]
\[ \leq K \sum_{k=m}^{\infty} (S_k + f_p(S_k))^{-2} \leq K \sum_{k=m}^{\infty} (S_k)^{-1} \leq K m^{-4}. \]

Note that by (11), for sufficiently large \( m \), the term in (23) is bounded from above by
\[ K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} (v^k_i)^{\frac{1}{H}} \exp \left( -\frac{1}{2}(v^k_i)^2 - b \log^2 v^k_i \right) \leq K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \frac{1}{a^k_i^{1+H} a^k_i} \leq K \sum_{k=m}^{\infty} \sum_{i=0}^{J_k} \frac{1}{i \log^{1+H} i} \]
\[ \leq K \sum_{k=m}^{\infty} (\log(S_k))^{-p} \leq K \sum_{k=m}^{\infty} k^{-(1+p)} \leq K m^{-p}. \]

Therefore
\[ \lim_{m \to \infty} \sum_{k=m}^{\infty} P(A_k \cap B^c_k) = 0 \]
and
\[
\lim_{m \to \infty} \mathbb{P}\left( \bigcup_{k=m}^{\infty} B_k \right) \geq \lim_{m \to \infty} \mathbb{P}\left( \bigcup_{k=m}^{\infty} A_k \right).
\]

In order to complete the proof of (22) we only need to show that
\[
P(A_n \text{ i.o.}) = 1.
\]

Similarly to (20), we have
\[
\int_{S_k} \frac{1}{f_p(u)} \mathbb{P}\left( \sup_{t \in [0, f_p(u)]} Q_{B_n}(t) > f_p(u) \right) \, du \sim (1 - \varepsilon)p \log_2 T_k.
\]

Now from Lemma 7 it follows that
\[
P(A_k) \geq \frac{1}{4} \exp\left( - (1 - \varepsilon^2)p \log_2 T_k \right) - KS_k^{-p} \geq \frac{1}{8} k^{-1-\varepsilon^t},
\]
for every \( k \) sufficiently large. Hence,
\[
\sum_{k=1}^{\infty} P(A_k) = \infty.
\]

Applying Berman’s inequality, we get for \( t < k \)
\[
P(A_k, A_t) \leq P(A_k) P(A_t) + Q_{k,t},
\]
where
\[
Q_{k,t} = \sum_{0 \leq i \leq j, 0 \leq \ell \leq L^k} \sum_{0 \leq l \leq J, 0 \leq p \leq L^l} \sum_{|n| \leq N^k, |m| \leq N^l} |r_{y_i^k, y_j^k}(s_i^k, \tau_n^k, s_p^l, \tau_m^l)| \exp\left( - \frac{(v_i^k - (v_j^k)^{-3})^2 + (v_j^l - (v_i^l)^{-3})^2}{2(1 + T_{k} - T_{k-1})} \right).
\]

For any \( 0 \leq i \leq J_k, 0 \leq j \leq J_t, 0 \leq l \leq L^k, 0 \leq p \leq L^l \), and \( t < k \),
\[
y_i^k s_i^k - y_j^l s_p^l = a_i^k + y_i^k q_i^k - \left( a_j^l + y_j^l p_j^l \right) \geq S_k - T_1 \geq S_k - T_{k-1} \geq \frac{1}{2} (T_k - T_{k-1}),
\]
where the last inequality holds for \( k \) large enough, since
\[
\frac{S_{k+1} - T_k}{T_k - T_{k-1}} \sim 1, \quad \text{as } k \to \infty.
\]

Thus, for sufficiently large \( k \) and every \( 0 \leq t < k \), c.f. (9),
\[
\sup_{0 \leq i \leq j, 0 \leq \ell \leq L^k, 0 \leq l \leq L^l, 0 \leq p \leq L^l} \left| r_{y_i^k, y_j^k}(s_i^k, \tau_n^k, s_p^l, \tau_m^l) \right| \leq \sup_{0 \leq i \leq j} \sup_{0 \leq \ell \leq L^k} \left( T_k - T_{k-1} \right)^{-\lambda} \frac{1}{2} \log(a^k_i) \leq K \left( T_k - T_{k-1} \right)^{-\lambda/2} \leq \frac{\min(1, \lambda)}{32}.
\]

Therefore, for some generic constant \( K \) not depending on \( k \) and \( t \) which may vary between lines, for every \( t < k \) sufficiently large,
\[
Q_{k,t} \leq K \sum_{0 \leq i \leq j, 0 \leq \ell \leq L^k} L_i^k L_{\ell}^k N_i^k N_{\ell}^k (T_k - T_{k-1})^{-\lambda/2} \exp\left( - \frac{(v_i^k)^2 + (v_j^l)^2}{2(1 + \frac{1}{8})} \right)
\]
\[
\leq K (T_k - T_{k-1})^{-\lambda/2} \left( L_i^k L_{\ell}^l \sum_{0 \leq i \leq j, 0 \leq \ell \leq L^k} \left( a_i^k \log(1 + C_n - p a_i^k) \right)^{-\frac{1}{1 + \frac{1}{16}}} \left( a_j^l \log(1 + C_n - p a_j^l) \right)^{-\frac{1}{1 + \frac{1}{16}}} \right)
\]
\[
\leq K(T_k - T_{k-1})^{-\lambda/2} \log^{10} T_k (T_k)^{1 + \frac{3}{16}} (T_1)^{1 + \frac{3}{16}}
\]
\[
\leq KT_k^{-\lambda/8} \leq K \exp(-\lambda k^{(1+\varepsilon^t)/p} / 8).
\]
Hence, we have
\[ (27) \sum_{0 \leq t < k < \infty} Q_{k,t} < \infty. \]

Now (24) follows from (25)-(27) and the general form of the Borel-Cantelli lemma. \( \square \)

**Step 3.** If \( p = 1 \), then for every \( \epsilon \in (0, \frac{1}{4}) \)
\[ (28) \liminf_{t \to \infty} \frac{\log (\xi_p(t)/t)}{h_p(t)/t} \geq -(1 + 2\epsilon)^2 \text{ a.s.} \]

and
\[ (29) \liminf_{t \to \infty} \frac{\log (\xi_p(t)/t)}{h_p(t)/t} \leq -(1 - \epsilon) \text{ a.s.} \]

**Proof.** Put \( T_k = \exp(k), \ S_k = T_k \exp\left(-(1 + 2\epsilon)^2 h_p(T_k)\right). \)

Proceeding the same as in the proof of (21), one can obtain that
\[ \liminf_{k \to \infty} \frac{\log (\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \geq -(1 + 2\epsilon)^2 \text{ a.s.} \]

On the other hand it is clear that
\[ \liminf_{t \to \infty} \frac{\log (\xi_p(t)/t)}{h_p(t)/t} \leq \liminf_{k \to \infty} \frac{\log (\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \text{ a.s.,} \]

since
\[ \liminf_{k \to \infty} \frac{\log (T_k/T_{k+1})}{h_p(T_k)/T_k} = 0. \]

This proves (28).

Let \( T_k = \exp \left(k^{1+\epsilon^2}\right), \ S_k = T_k \exp\left(-(1 - \epsilon)h_p(T_k)\right). \)

Noting that
\[ \frac{S_{k+1} - T_k}{S_{k+1}} \sim 1, \quad \text{as } k \to \infty, \]

following along the same lines as in the proof of (22), we also have
\[ \liminf_{k \to \infty} \frac{\log (\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \leq -(1 - \epsilon) \text{ a.s.,} \]

which proves (29). \( \square \)

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