General relativistic dynamics applied to the rotation curves of galaxies

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Abstract

We extend our general relativistic analysis of galactic rotation curves with galaxies NGC 2841, NGC 2903 and NGC 5033. As before, we employ the solution of the Einstein field equations of general relativity with an expansion in Bessel functions. As in our earlier studies, the fits to the data are found to be very precise and the calculated baryonic masses are lower than those based upon Newtonian gravity. Also as in our previous studies, the galactic radii at which the optical luminosities terminate are seen to correlate with densities near $10^{-21.75}$ kg·m$^{-3}$. This concordance lends further support to the correctness of the procedure as well as providing a potentially valuable piece of information in the understanding of galactic evolution.

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1 Introduction

It is widely believed that Einstein’s theory of gravity is primarily a theory for very strong gravitational fields and that it provides only very small corrections to Newtonian predictions for weak fields. Indeed this is very often seen to be the case as in planetary motion studies. Since the gravitational fields within galaxies are generally weak and the velocities of the component stars are non-relativistic, general relativity was never brought into the analysis prior to our work in [1], [2] and [3]. What we had found was that
general relativity does indeed yield results that differ from those derived from Newtonian gravity for extended gravitationally bound systems, even for such weak-field systems. From the relatively flat galactic rotation curves that were first observed by V. Rubin, Newtonian gravity yielded only one possible conclusion: the galaxies must be encircled by vast spherical halos of “dark matter” with masses up to an order of magnitude larger than those indicated by their visible contents. Even earlier, F. Zwicky had proposed that vast quantities of dark matter were required to propel the galaxies in galactic clusters such as in the Coma Cluster. While there have been several ideas proposed regarding the physical constitution of this mysterious matter, no direct detection has ever been achieved. Moreover, while probably the majority of investigators are convinced of its reality, a variety of researchers have come to doubt its very existence. In recent years, the dark matter issue has evolved into one of the most important problems in theoretical physics, astrophysics and cosmology.

Our own attitude is guided by “Occam’s razor”. Unless such a new element (and a very bizarre one at that) is seen to be absolutely necessary, we prefer to work within the framework of its presumed absence. An essential point is this: the known data from galactic rotation curves can be accommodated with at most relatively little extra matter when the analysis is performed with Einstein’s as opposed to Newton’s gravity and it is almost universally believed that Einstein’s general relativity is our best theory of gravity. However, it must be stressed that Einstein’s theory is much richer than Newton’s theory. General relativity can also accommodate large halos of extra encircling matter and we have determined the criterion for deducing the extent, if any, of the presence of extra matter [4] [5]. It is determined by the degree of velocity dispersion for stars that lie above and below the galactic symmetry plane; the greater the degree of velocity dispersion that is indicated, the lesser the degree of indicated extra matter. Hopefully such dispersion data will eventually be available in sufficient quantity and quality to make such a determination possible.

In the meantime, we have continued our analysis of rotation curves, again fitting the known data with the appropriate solutions of the Einstein field equations. We see the pattern continuing of smaller indicated galactic masses than those calculated on the basis of Newtonian theory. As well, we see a continuation of the correlation of the onset of optical luminosities of the new galaxies studied at the same approximate value of average density, namely $10^{-21.75} \text{ kg m}^{-3}$. It is remarkable that Einstein’s theory has within it the
power to reveal such a trend solely on the basis of rotation curve data. It is significant that the optical thresholds occur at quite a variety of distances from the galactic axes of rotation so it is not as if we are dealing with a “cookie cutter” situation. This confluence of results fortifies the trust in the methodology.

While various researchers are now turning to gravitational lensing in the search for evidence for dark matter, probably the majority of researchers regard the flat galactic rotation curves as the key indicators. Some have indicated that it is evidence garnered from microwave early-universe observations that is most compelling but there are problems associated with the dark matter theory of galaxy formation. For example, this theory permits 10 to 100 times as many small galaxies as those which are actually observed.

Clearly the issue is of paramount importance given that dark matter is said to comprise the dominant constituent of an extended galactic mass by multiple factors. The dark matter enigma has influenced particle theorists to devise acceptable candidates for its constitution. At the present time, various researchers are hoping to discover dark matter particles in the LHC (Large Hadron Collider) experiments at CERN in Geneva. While physicists and astrophysicists have pondered the dark matter issue, other researchers have devised new theories of gravity to account for the observations (see for example [6, 7, 8, 9]). However the latter approaches, however imaginative, have met with understandable skepticism, as they have been devised solely for the purpose of the task at hand. General relativity was the existing preferred theory of gravity long before dark matter was even contemplated and it remains the preferred theory to this day. General relativity has been successful in every test that it has encountered, going beyond Newtonian theory where required. Therefore, should it actually transcend Newtonian gravity in resolving the galactic rotation curve issue, it would greatly alter the understanding of some basic aspects in physics and astrophysics.

In dismissing general relativity in favour of Newtonian gravitational theory for the study of galactic dynamics, insufficient attention has been paid to the fact that the stars that compose the galaxies are essentially in motion under gravity alone (“gravitationally bound”). It had been known for many years, in fact since the time of Eddington, that the gravitationally bound problem in general relativity is an intrinsically non-linear problem even when the conditions are such that the field is weak and the motions are non-relativistic, at least in the time-dependent case. Most significantly, we have found that under these conditions, the general relativistic analysis of
the problem is also non-linear for the stationary (non-time-dependent) case at hand. Thus the intrinsically linear Newtonian-based approach used prior to our work has been inadequate for the description of the galactic dynamics and Einstein’s general relativity must be brought into the analysis within the framework of established gravitational theory. This is an essential departure from conventional thinking on the subject and it leads to major consequences.

Since our initial posting [1], many colleagues have offered their comments and criticisms. To provide some perspective, we briefly address the most common areas of contention in Section 5. The totality of the issues known to us have been discussed in [2], [3], [4] and [5]. In those papers, we developed the theory in more detail and applied it to three galaxies and the Milky Way. We also developed a new observational discriminator for assessing the degree, if any, of external matter that may lie beyond the visible/HI regions. This is determined by examining the galactic rotation curves at different galactic latitudes, bringing into consideration the global dynamical structure of the galaxy. It is well to emphasize that the demand for global consistency applies not only to our own but also to all proposed models and theories.

2 Axially-Symmetric Model Galaxy

In the interests of self-containment, we set out the essentials of the general relativistic mathematical structure for the problem. When we consider the complexity of the detailed structure of a typical galaxy with its arms and irregular density variations, it becomes clear that the modeling within the context of the complicated theory of general relativity demands some simplifications, at least at this point in our facility in handling this rich and demanding theory. As long as the essence of the structure is captured, these simplifications are justified and we can derive valuable information. To capture the essence, we consider an idealized model that lends itself to detailed analysis, a time-independent uniformly rotating fluid without pressure and symmetric about its axis of rotation. In generality, the stationary axially symmetric metric in consort with the model can be described in the form

$$ds^2 = -e^{\nu-w}(udz^2 + dr^2) - r^2e^{-w}d\phi^2 + e^{w}(cdt - Nd\phi)^2$$  \hspace{1cm} (1)

where $u$, $\nu$, $w$ and $N$ are functions of cylindrical polar coordinates $r$, $z$. It is easy to show that to the order required, $u$ can be taken to be unity. It is
simplest to work in the frame that is comoving with the matter,

\[ U^i = \delta^i_0 \]  

(2)

where \( U^i \) is the four-velocity. This is reminiscent of the standard approach that is followed for FRW (Friedmann-Robertson-Walker) cosmologies. However, the FRW spacetimes are homogeneous and they are not stationary. The comoving approach was taken in the pioneering paper by van Stockum [11] who set \( w = 0 \) from the outset. Interestingly, the geodesic equations imply that \( w = constant \) (which can be taken to be zero as in [11]) even for the exact Einstein field equations as studied in [11]. In fact the requirement that \( w = 0 \) can be seen directly using [2] and the metric equation \( g_{ik}U^iU^k = 1 \) [2]. As we discussed in our previous work, we perform a purely local \((r, z)\) held fixed at each point when taking differentials) transformation. It is to be noted that this local transformation is used only to deduce the connection between \( N \) and \( \omega \) (and hence \( V \)). All subsequent work continues in the original unbarred comoving frame. The localized transformation is

\[ \tilde{\phi} = \phi + \omega(r, z) t \]  

(3)

which is adjusted to locally diagonalize the metric. In this way, we are able to determine the local angular velocity \( \omega \) and the tangential velocity \( V \) as

\[ \omega = \frac{Nce^w}{r\,e^{-w} - \frac{Ne^w}{r}} \approx \frac{Nc}{r} \]  

(4)

\[ V = \omega r \]  

(5)

with the approximate velocity value \( Nc/r \) applicable for the weak fields under consideration. The Einstein field equations to order \( G^1 \) with \( w \) kept for later comparison, are

\[ 2r\nu_r + N^2_r - N^2_z = 0, \]

\[ r\nu_z + N_rN_z = 0, \]

\[ N^2_r + N^2_z + 2r^2(\nu_{rr} + \nu_{zz}) = 0, \]

\[ N_{rr} + N_{zz} - \frac{N_r}{r} = 0, \]

\[ \left(w_{rr} + w_{zz} + \frac{w_r}{r}\right) + \frac{3}{4}r^{-2}(N^2_r + N^2_z) \]

\[ + \frac{N}{r^2}\left(N_{rr} + N_{zz} - \frac{N_r}{r}\right) - \frac{1}{2}(\nu_{rr} + \nu_{zz}) = 8\pi G\rho/c^2 \]  

(7)
where $G$ is the gravitational constant and $\rho$ is the mass density. Subscripts indicate partial differentiation with respect to the indicated variable. Fortunately, these rather complicated equations are easily combined to give

$$\nabla^2 w + \frac{N^2_r + N^2_z}{r^2} = \frac{8\pi G \rho}{c^2}$$

(8)

where the first term is the flat-space Laplacian in cylindrical polar coordinates

$$\nabla^2 w \equiv w_{rr} + w_{zz} + \frac{w_r}{r}$$

(9)

and $\nu$ is determined by simple integration.

From the application of the freely gravitating constraint (i.e. stress-free motion) and the requirement that $w = 0$ which arises from the choice of comoving coordinates, the field equations for $N$ and $\rho$ in this globally dust distribution are reduced to

$$N_{rr} + N_{zz} - \frac{N_r}{r} = 0$$

(10)

$$\frac{N^2_r + N^2_z}{r^2} = \frac{8\pi G \rho}{c^2}.$$  

(11)

Note that with the minus sign in (10), $N$ does not satisfy the Laplace equation. Note also that from both the field equation for $\rho$ and the expression for $\omega$ that $N$ is of order $G^{1/2}$. This is a point that has been misunderstood by some of our critics, leading them to erroneous conclusions. (We discuss this briefly later in the paper.)

The non-linearity of the galactic dynamical problem is evident through the non-linear relation between the functions $\rho$ and $N$. While we have eliminated $w$ either by using the geodesic equations to get (11) or by the metric equation and the choice of comoving coordinates, $N$ cannot be eliminated and hence non-linearity is intrinsic to the study of the galactic dynamics. Rotation under freely gravitating motion is the key factor at play in the present problem.

It is worthy of note that (10) can be expressed as

$$\nabla^2 \Phi = 0$$

(12)

where

$$\Phi \equiv \int \frac{N}{r} dr.$$  

(13)
Thus, flat-space harmonic functions \( \Phi \) are the generators of the axially symmetric stationary pressure-free weak fields that we seek. (In fact Winicour [16] has shown that all such sources, even when the fields are strong, are generated by such flat-space harmonic functions.) We refer to these as “generating potentials”. It is noteworthy that these generating potentials play a different role in general relativity than do the ordinary potentials of Newtonian gravitational theory even though both functions are harmonic. Using (5) and (13), we have the expression for the tangential velocity of the distribution,

\[
V = c \frac{N}{r} = c \frac{\partial \Phi}{\partial r}.
\]  

(14)

In Newtonian gravity, the potential gradients relate to acceleration whereas here, they relate to velocity. These potentials do not play the same role in the two theories of gravity.

We now have the necessary mathematical framework in place.

3 Connecting the Observed Galactic Rotation Curves to the Model

We first consider the ideal strategy for galactic modeling, given the nature of the equations at hand. Since the field equation for \( \rho \) is non-linear, the simpler way to proceed is to first find the required generating potential \( \Phi \) and from this, derive an appropriate function \( N \) for the galaxy that is being analyzed. With \( N \) found, (11) yields the density distribution. If this agrees with the observations, the efficacy of the approach is established. This is in the reverse order of the standard approach to solving gravitational problems but it is most efficient in this formalism because of the existence of one linear field equation.

For any given galaxy, a suitable set of composing functions for the series that satisfies the linear equation is required. Once found, this yields the generating potential. With cylindrical polar coordinates, it is simplest to use separation of variables leading to the following solution for \( \Phi \) in (12):

\[
\Phi = C e^{-k|z|} J_0(kr)
\]  

(15)

where \( J_0 \) is the Bessel function \( m = 0 \) of Bessel \( J_m(kr) \) and \( C \) is an arbitrary constant (see for example [18]). Using this form of solution, the absolute
value of \( z \) must be used to provide the proper reflection of the distribution for negative \( z \). While this produces a discontinuity in \( N_z \) at \( z = 0 \), it is important to note that in the problem at hand, this discontinuity is consistent with the general case of having a density \( z \)-gradient discontinuity at the plane of reflection symmetry. This point has been the subject of considerable attention in the literature. We will return to this issue later.

The beauty of a linear equation is that it allows for linear superposition of solutions. From (12) we express the general solution of this form as the linear superposition

\[
\Phi = \sum_n C_n e^{-k_n|z|} J_0(k_n r).
\]

We choose integer \( n \) as required to the level of accuracy that we wish to achieve. From (16) and (14), the tangential velocity is

\[
V = -c \sum_n k_n C_n e^{-k_n|z|} J_1(k_n r).
\]

For this, we have used the Bessel relation \( dJ_0(x)/dx = -J_1(x) \) (see, e.g. [19]). From (14), we see that if \( N \) should exceed \( r \), the velocity would exceed \( c \). This does not occur because with our choice of separable solutions, the velocity is given by (17). As \( r \) approaches 0, this function falls as \( r^1 \) (i.e. \( N \) approaches \( r = 0 \) as \( r^2 \)) and so \( V \) falls to 0 as we see as well in the plots of the rotation curves. Thus the potential problem referred to by Wiltshire [17] is not present in our case. As well, for large \( r \), the Bessel functions fall as \( 1/\sqrt{r} \) and hence the velocity goes to 0 for large \( r \). In general, this would still lead to a large amount of matter external to the galaxy. However, by our selecting the right multiplying coefficients in the Bessel solution sequence, we can achieve consistency with a very limited amount of external matter, far less than that which is indicated on the basis of Newtonian gravity. We have shown this by fitting fictitious faster-than-1/\( \sqrt{r} \) fall-off data for the extension of the observed points in the velocity profile (see [4]).

We choose the \( k_n \) so that the \( J_0(k_n r) \) terms are orthogonal to each other. The Bessel functions \( J_0(kr) \) satisfy the orthogonality relation:

\[
\int_0^1 J_0(k_n r)J_0(k_m r)rdr \propto \delta_{mn}\]

(18)

where \( k_n \) are the zeros of \( J_0 \) at the limits of integration. With only 10 functions with parameters \( C_n, n \in \{1 \ldots 10\} \), we have been able to achieve
an excellent fit to the velocity curve for the Milky Way and for NGC 3031, NGC 3198 and NGC 7331 (see [4]). In this paper, we advance beyond what we achieved in our earlier work.

It should be noted that the $J_1(x)$ Bessel functions are 0 at $x = 0$ and oscillate with decreasing amplitude, falling as $1/\sqrt{x}$ asymptotically [19]. However, this alone does not assure a realistic fall-off of matter. We have dealt at length with this issue in [4] (see also [5]). Also, our curves drop as $r$ approaches 0. This is in contrast to the Mestel [10] and MOND [6, 7, 8] curves that are flat everywhere.

¿From (14) and (17), the $N$ function is determined in detail and from (11), the density distribution follows. Most significantly, our correlation of the flat velocity curve is achieved with disk mass up to an order of magnitude smaller than the halo mass of dark matter proposed by earlier studies. (See e.g.[20, 21]).

It is to be emphasized that general relativity does not distinguish between the luminous and non-luminous contributions. The deduced $\rho$ density distribution is derived from the totality of the two. Any substantial amount of ordinary baryonic non-luminous matter would necessarily lie in the flattened region relatively close to $z = 0$ because this is the region of significant density $\rho$ and would be due to dead stars, planets, neutron stars and other normal non-luminous baryonic matter debris. Each term within the series has $z$-dependence of the form $e^{-k_n|z|}$ which causes the steep density fall-off profile.

This fortifies the picture of a standard galactic essentially flattened disk-like shape as opposed to a halo sphere. From the evidence provided thus far by rotation curves [1] [2] [3], there is no support for the widely accepted notion of the necessity for massive halos of dark matter surrounding visible galactic disks: conventional gravitational theory, namely general relativity, can account for the observed flat galactic rotation curves linked to essentially flattened disks with no evident need for dark matter, at least not with the velocity distribution data presently at our disposal.

We recall from our earlier work, general relativity in conjunction with the figures provided by Kent [12] for optical intensity curves and our log density profiles for three galaxies NGC 3031, NGC 3198 and NGC 7331 provide a common thread: all three galaxies indicate the threshold density for the onset of visible galactic light as we probed in the radial direction at approximately $10^{-21.75}$ kg·m$^{-3}$. This led us to hypothesize that this density is the universal optical luminosity threshold for galaxies as tracked in the radial direction. In
what follows, we will present further evidence in support of this hypothesis from the investigation of three additional galaxies. As well, the radius at which the optical luminosity fall-off occurs can be predicted for other sources using this particular density parameter. The predicted optical luminosity fall-off for the Milky Way is at a radius of 19-21 Kpc based upon the density threshold indicator that we have determined. It will be interesting to test this indicated range according to our hypothesis with more refined data that the astronomers will be able to accumulate in the future.

4 Analysis of Galaxies NGC 2841, NGC 2903 and NGC 5033

Velocity curve-fits and density profiles (see figures) for galaxies NGC 2841, NGC 2903 and NGC 5033 were obtained using the methods outlined in Section 3. As in [4], only ten parameters were used per data set, the values for which were obtained via least-squares minimization. (Data tables follow below.) The value for the radius at which the visible light is first observed was obtained from Kent [12] for galaxies NGC 2903 and NGC 5033, and was obtained from Macri et al [13] for NGC 2841.

| $-C_n \times k_n$ | $k_n$ (Kpc$^{-1}$) |
|------------------|------------------|
| 0.001263951207   | 0.06680070994    |
| 0.0003509253903  | 0.153355031      |
| 0.0002902041947  | 0.240381309      |
| 0.0001233680182  | 0.3275426233     |
| 0.0001966403398  | 0.4147477141     |
| 0.0001053122057  | 0.5019739991     |
| 0.00009540622257 | 0.589211286      |
| 0.0001110222640  | 0.6764575425     |
| 0.00006439084160 | 0.7637077537     |
| 0.00007895785801 | 0.8509612908     |

Table 1: Curve-fitted coefficients for NGC 2841
Figure 1: Velocity curve-fit and derived density for NGC 2841
Figure 2: Velocity curve-fit and derived density for NGC 2903
Figure 3: Velocity curve-fit and derived density for NGC 5033
Table 2: Curve-fitted coefficients for NGC 2903

| $-C_n \cdot k_n$ | $k_n$ (Kpc$^{-1}$) |
|------------------|--------------------|
| 0.0009976025857  | 0.09619302231      |
| 0.0002429098737  | 0.2208031244       |
| 0.0002257927027  | 0.3461491165       |
| 0.0001463861599  | 0.4716613776       |
| 0.0001087456698  | 0.5972367083       |
| 0.00007440663732 | 0.7228425587       |
| 0.00008677052850 | 0.848654652        |
| 0.00002425953621 | 0.9740988612       |
| 0.00006760519096 | 1.099739165        |
| 3.255331982*10^{-6} | 1.225384259       |

Table 3: Curve-fitted coefficients for NGC 5033

| $-C_n \cdot k_n$ | $k_n$ (Kpc$^{-1}$) |
|------------------|--------------------|
| 0.001263951207  | 0.06680070994      |
| 0.0003509253903 | 0.1533355031       |
| 0.0002902041947 | 0.2403813309       |
| 0.0001233680182 | 0.3275426233       |
| 0.0001966403398 | 0.4147477141       |
| 0.0001053122057 | 0.5019739991       |
| 0.00009540622257 | 0.5892121286     |
| 0.0001110222640 | 0.6764575425       |
| 0.00006439084160 | 0.7637077537     |
| 0.00007895785801 | 0.8509612908       |

While the visible light profile for NGC 2841 terminates at $r = 17.0$ Kpc, the HI profile extends to 40 Kpc. The density at $z = 0$, $r = 17.0$ Kpc, i.e. the critical density, was found to be $10^{-21.60} \text{ kg/m}^3$. Integrating through the HI outer region to $r = 40$ Kpc yields a total mass of $51.5 \times 10^{10} M_\odot$. This value is considerably less than that obtained by Kent [12] of $74.6 \times 10^{10} M_\odot$. For NGC 2903, we found the critical density to be $10^{-21.59} \text{ kg/m}^3$ and the mass to be $8.9 \times 10^{10} M_\odot$. Kent [12] found a value of $19.4 \times 10^{10} M_\odot$ for
this galaxy, more than double the model value. Finally, using rotation curve data from NGC 5033 the value for the critical density was determined to be $10^{-21.83} \text{kg/m}^3$ and the mass to be $22.2 \times 10^{10} M_\odot$, while Kent found the mass to be $37.1 \times 10^{10} M_\odot$.

Thus, as before, with nothing more than Einstein’s theory of relativity and the astronomical data of rotation curves, we have consistently found a value in the range of $10^{-21.6} \text{kg/m}^3$ to $10^{-21.8} \text{kg/m}^3$ for the threshold density of the onset of galactic optical luminosity. It is well to contemplate the sequence of analysis: Firstly, from Doppler shifts of spectral lines, we deduce the velocities of stars as a function of distance from the galactic axis of rotation. Secondly, the solution from Einstein’s theory of gravity uses this data to produce a map of galactic averaged matter density as a function of this distance. Thirdly, we note from astronomical observations the distance at which the optical luminosity cuts out and from the density plot, we note the density at which this occurs. This distance is different for each galaxy yet remarkably, the density in each of now six cases is approximately the same, namely in the $10^{-21.75} \text{kg/m}^3$ range. This remarkable concordance fortifies one’s faith in the correctness of the procedure.

## 5 Perspectives

It is useful to summarize some of the history relating to our work. We had challenged the traditional view that Newtonian gravity was the adequate theory to describe the motions of stars in the galaxies, arguing that general relativity was required within the context of accepted gravitational physics. We had shown that the known galactic rotation curves could fit into the framework of general relativity without the demands for vast halos of dark matter surrounding the galaxies. Other gravity theories had been specifically designed to account for the motions without invoking dark matter. Einstein’s
theory stands firmly on its own, predating all such efforts.

Understandably, our work was problematic for the many researchers who had relied upon traditional gravity theory for their research. They could ignore the alternative theories as being outside of their sphere of accepted science but general relativity was accepted by virtually all researchers as the very best theory of gravity. Thus our work was the subject of intense scrutiny and criticism. It is to be emphasized that we have responded to all the issues of criticism and to this point, as far as we are aware, there has not been any new rebuttal offered by any of these critics. As well, there has been some work by others lending support to our findings.

To round out a unified picture of our work, it is worth a brief recount of the key areas of criticism. The complete analysis is contained in our earlier papers and in the book [5]. A persistent issue concerns the nature of the matter distribution. Some critics have noted that given the existence of the discontinuity of $N_z$ that we had pointed to in [1], a significant surface tensor $S_i^k$ can be constructed whose integral represents an accumulation of mass above and beyond the volume integral of the regular continuous mass density.

To analyze this claim, we calculated the surface mass that was said to be present in the four galaxies that we had first studied in [1], [2], [3]. This was done by integrating this supposed surface density. For each galaxy, we calculated a numerical value slightly less than the mass that we had derived from the volume integral of our continuous mass density distribution using (17), (14) and (11). Upon reflection, we realized that the surface integral of this supposed singular layer is merely a mathematical construct that indirectly describes most of the continuously distributed mass by means of the Gauss divergence theorem. The reason that the surface contributions are slightly less in each case derives from the small contributions that are absent from the remaining surfaces to bound the entire volume. The totality of bounding surfaces are required for the application of the Gauss theorem.

The essential point is that singularities have to be interpreted properly. As a useful example, consider the simple case of a purely vacuum Schwarzschild solution with mass $m$ for $r > a$ and Minkowski space for $r < a$. Since it is a vacuum solution, the only place for the mass to reside is on the surface at $r = a$. The surface layer construct finds this mass at $r = a$. However in our present galactic modeling case, the supposed surface layer construct merely recovers the volume-distributed mass in a different manner, i.e. by the Gauss theorem. By contrast, in the Schwarzschild shell example,
there is no volume distribution of mass present. The surface layer calculation indicates the mass value of a continuous distribution of matter that could have been squeezed together to form the shell. Clearly, this example connects directly with our interpretation of the galactic model singularity. Actually, had we followed the standard prescription to calculate mass with the surface layers, we would have had to ascribe a negative value to it as others have claimed. However, as Bondi has discussed in his works, negative mass repels other masses, be they of positive or negative mass and our careful investigations have shown that test particles are invariably attracted rather than repelled at positions near the discontinuous surface. This negates the negative mass surface layer claim.

Our model consists of dust with reflection symmetry about the $z = 0$ plane. The density naturally increases symmetrically as this plane is approached from above and from below with the same absolute value of density gradient but of opposite sign from symmetry. In all generality, the density $z$-gradient will be different from zero as this plane is approached and because of reflection symmetry, this gradient will of necessity be discontinuous. This is the physical origin of the singularity. In our case, the singularity is benign, with a ready physical explanation.

Regarding another issue, some researchers misunderstood our work by not appreciating the effect of rotation on the perturbation sequence. Typically, these authors present the well-known expression of the field equations in the harmonic gauge in Cartesian coordinates. From the standard description of the post-Newtonian perturbation scheme, they conclude that the solution to the galactic problem must be the usual Newtonian one and that all corrections must be of higher order. Firstly, we did not use this scheme (as noted as well in [15]). We used cylindrical polar coordinates comoving with the matter. For the gravitationally bound system, the metric components are of different orders in $G$. This is a key point that was overlooked by some of our critics. The novel aspect for our problem is that the lowest order equation, of order $G^{1/2}$, has zero on the RHS and the second equation that would normally be the Newtonian Poisson equation, differs in that it has non-linear terms. Thus, the structure of our solution does not proceed as in the standard approach referred to by our critics. In the standard approach, the lowest order base solution is the Newtonian solution whereas in the galactic problem, the lowest order equation is the Laplace equation for which an order $G^{1/2}$ solution is necessary and the next order (order $G^1$) equation for
the density
\[
\frac{N_r^2 + N_z^2}{r^2} = \frac{8\pi G \rho}{c^2}
\]
has non-linear terms in the metric in the form of the squares of the derivatives of an order \(G^{1/2}\) metric tensor component \(N\). Thus, our situation is unlike standard iterative perturbation scheme applications.

A key point is that the equations have an inherent non-linearity as a result of the fact that the metric components are of different orders and the different orders are a necessary consequence of the problem being a gravitationally bound one with rotation.

The authors of [14] arrive at our equations (6), (7) (with \(w\) set to zero) apart from the exponential \(\nu\) factor which they later note can be taken to be a constant scaling factor and find the same order of magnitude reduction of galactic mass that we had found [1] starting from their exact solution class. This provides some vindication for our analysis.

While we have shown that the presently available data on galactic rotation curves can be explained without vast halos of dark matter surrounding the galaxies, more complete data above and below the (approximate) galactic symmetry plane could reveal the presence of more matter than is presently observed.

Clearly it is important to approach the dark matter issue in as many ways as possible. After all, from a purely formal point of view, general relativity should be able to model vastly extended distributions of pressure-free fluids in rotation. In this vein, we have constructed a test in principle that relies upon data in the \textit{visible/HI} regime thus making it particularly useful. When we examine the different possible fall-offs as in [3], we see that different profiles beyond the HI region imply different mass accumulations in those external regions. Carrying these back with continuity into the visible/HI region, we find that the extent of the velocity dispersion as we track curves at different non-zero \(z\) values depends on the assumed external velocity profile fall-off. (See, for example, the dispersion figure in [3].) With sufficient data, it should be possible, at least in principle, to provide limits on the extent of extra matter that might lie outside of the visible/HI region. To this point, we have only the data provided in [22] [23] [24] but far more data will be required to provide an adequate discriminating test.
6 Concluding Comments

Readers are often confused as to how our results could be so different from Newtonian predictions. They note that the dynamic solar system analysis proceeds very accurately on the basis that the planets move in almost the same manner using general relativity as that deduced by Newtonian dynamics and the observations of the planetary motions confirm this. However there is an essential difference between the solar system dynamics and the galactic dynamics. In the case of the solar system, the primary source of gravity is the sun and the planets are treated as test particles in this field apart from contributing minor perturbations when the slight changes are being sought. The planets respond to the field of the sun but their own gravitational contributions are not retained since they are so small. By contrast, in the galaxy problem, the source of the field is the combined rotating mass of all of the freely-gravitating elements themselves that compose the galaxy. There is no one single dominant contributor in the galactic problem. The observed elements in the galactic case produce the field; they do not merely respond to the field as they do in the planetary problem.

We have seen that the non-linearity for the computation of density distribution inherent in the Einstein field equations for a stationary axially-symmetric pressure-free mass distribution, even in the case of weak fields, leads to the incorporation of the known galactic velocity curves with little or no extra dark matter. This is as opposed to the curves that had been derived on the basis of Newtonian gravitational theory that demanded the presence of vast extra reservoirs of dark matter. It is worth repeating for emphasis: the results were consistent with the observations of velocity as a function of radius plotted as a rise followed by an essentially flat extended region and no halo of dark matter with multiples of the normally computed galactic mass was required to achieve them. The density distribution that is revealed thereby is one of an essentially flattened disk without an accompanying overwhelmingly massive vastly extended dark matter halo. With the “dark” matter being associated with the disk which is itself visible, it is natural to regard the non-luminous material as normal baryonic matter.

To some extent we have had to assume extensions of matter distribution with assumptions regarding ultimate velocity fall-offs beyond that which is actually observed in order to make comparisons [3]. In the course of these investigations, we have seen that these can readily yield a picture of galactic structure without huge extended very massive halos of dark matter. It would
be helpful if new data beyond those presently available would be produced. This would help tie down the complete physical picture.

Of particular interest is that we have within our grasp a criterion for determining the extent, if of any significance, of extra matter beyond the visible and HI regions of a galaxy. It is possible in principle to determine this with data solely within the visible/HI region by plotting the velocity dispersion of rotation curves for various $z$ values. This is an attractive area for future research. In particular, it expands the demands upon not only our galactic model but also upon any other proposed model by other researchers. It asks for consistency between observation and theoretical prediction for the overall averaged picture of stellar motions within the galaxy.

We have seen with now six galaxies that Einstein’s general relativity has within it the power to reveal what is looking more and more like a general truth, that there is a consistent average density for galaxies at the point where their visible light first appears. This is a power that is lacking in the much simpler theory of gravity of Newton.

It is to be noted that we have also analyzed an intrinsically dynamic free-fall model, that of an idealized Coma Cluster of galaxies [25]. This work has the following attractive features:

a) It makes use of an exact solution of the Einstein equations, hence removing any issues arising from the application of approximations.

b) Being free of any singularities of any kind during the weak-field period of analysis, it eliminates any possibility of the kind of objections that we have discussed in the present paper.

c) It demonstrates that for weak fields and bodies with $V << c$, Einstein’s theory leads to results that differ from those of Newton’s theory.

This work further reveals the greater richness of general relativity in dealing with issues in galactic dynamics.

We share the belief of many that the scientific method has been most successful when guided by Occam’s razor, that new elements should not be introduced into a theory unless absolutely necessary. If dark matter should turn out to be another case similar to the ether of the 19th Century, it is well for us to determine this sooner rather than later.

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