An inverse problem for a heat equation with piecewise-constant thermal conductivity

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Abstract

The governing equation is \(u_t = \left(a(x)u_x\right)_x\), \(0 \leq x \leq 1\), \(t > 0\), \(u(x,0) = 0\), \(u(0,t) = 0\), \(a(1)u'(1,t) = f(t)\). The extra data are \(u(1,t) = g(t)\). It is assumed that \(a(x)\) is a piecewise-constant function, and \(f \neq 0\). It is proved that the function \(a(x)\) is uniquely defined by the above data. No restrictions on the number of discontinuity points of \(a(x)\) and on their locations are made. The number of discontinuity points is finite, but this number can be arbitrarily large.

If \(a(x) \in C^2[0,1]\), then a uniqueness theorem has been established earlier for multidimensional problem, \(x \in \mathbb{R}^n, n > 1\) (see MR1211417 (94e:35004)) for the stationary problem with infinitely many boundary data. The novel point in this work is the treatment of the discontinuous piecewise-constant function \(a(x)\) and the proof of Property C for a pair of the operators \(\{\ell_1, \ell_2\}\), where \(\ell_j := -\frac{d^2}{dx^2} + k^2 q_j^2(x)\), \(j = 1, 2\), and \(q_j^2(x) > 0\) are piecewise-constant functions, and for the pair \(\{L_1, L_2\}\), where \(L_j u := -[a_j(x)u'(x)]' + \lambda u\), \(j = 1, 2\), and \(a_j(x) > 0\) are piecewise-constant functions. Property C stands for completeness of the set of products of solutions of homogeneous differential equations (see MR1759536 (2001f:34048))

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1 Introduction

Let

\[
\begin{align*}
\dot{u} &= (a(x)u')', \quad 0 \leq x \leq 1, \quad t > 0, \quad u' := \frac{\partial u}{\partial x}, \quad \dot{u} := \frac{\partial u}{\partial t}, \\
0 &= u(x,0), \quad 0 = u(0,t), \quad a(1)u'(1,t) = f(t) \neq 0, \\
0 &= u(1,t) = g(t).
\end{align*}
\]

Problem (1)–(2) describes the heat transfer in a rod, \(a(x)\) is the heat conductivity, \(a(1)u'(1,t)\) is the heat flux, \(g(t)\) is the measurement, the extra data.

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The inverse problem (IP) is:

**IP:** Given \( f(t) \) and \( g(t) \) for all \( t > 0 \), find \( a(x) \).

**Assumption A:** \( a(x) \) is a piecewise-constant function, \( a(x) = a_j, \ x_j \leq x \leq x_{j+1}, \)

\( x_1 = 0, \ x_{n+1} = 1, \ 0 < c_0 \leq a_j \leq c_1, \ 1 \leq j \leq n. \)

This assumption holds throughout the paper and is not repeated. The set of piecewise-constant functions with finitely many discontinuity points is denoted by \( \Pi \).

If \( a(x) \in C^2 \), then the uniqueness of the solution to some multidimensional inverse problems has been proved in [3] (see also [2]). Problem (1)–(3) with \( a(x) \in C^2([0,1]) \) has been studied in [5], [6]. The treatment of discontinuous piecewise-constant \( a(x) \) is of interest in applications.

In [1] equation (1) with the conditions \( u(0,t) = u(1,t) = 0, \ u(x,0) = g(x), \) was studied, and the measured (extra) data were the values \( u(\xi_m,t), \forall t > 0, \ 1 \leq m \leq M, \ 0 \leq \xi_m \leq 1, \) where \( M = 3n, \) and \( n \) is the number of the discontinuity points of \( a(x). \) It was assumed in [1] that \( \min_j |x_j - x_{j+1}| \) is not too small. Under these assumptions the uniqueness theorem for the IP was proved in [1], and an algorithm for finding \( a(x) \) was proposed. The stability of this algorithm with respect to perturbations of the data was not studied in [1].

In our paper the extra data \( (3) \) consists of measurement, taken at one point, rather than at \( 3n \) points, and we impose no restrictions on \( \min_j |x_j - x_{j+1}|. \) Under these assumptions, which are much weaker than in [1], we prove the uniqueness of the solution to IP.

One of our main results is

**Theorem 1** The IP has at most one solution.

**Remark 1** The IP is ill-posed: small variations of the data \( \{f(t),g(t)\} \) in the \( C(0,\infty) \)–norm may lead to large variations of the coefficient \( a(x) \), or may lead to a problem which has no solutions. We assumed that the data are known for all \( t > 0 \). If one assumes that \( f(t) = 0 \) for \( t > T \), where \( T > 0 \) is an arbitrary fixed number, then the solution \( u(x,t) \) is an analytic function of \( t \) in the region \( t > T \). Therefore the data \( \{f(t),g(t)\}, \) known in the interval \( [0,T+\epsilon) \), where \( \epsilon > 0 \) is an arbitrary small fixed number, determine uniquely the data for all \( t > 0 \). Thus, if \( f(t) = 0 \) for \( t > T \), then the uniqueness theorem for the solution to IP remains valid if the data are known for \( t \in [0,T+\epsilon) \).

Let us formulate IP in an equivalent form.

Take the Laplace transform of the equation (1)–(3), denote

\[ v(x,\lambda) := Lu := \int_0^\infty e^{-\lambda t}u(x,t)dt, \]

and get:

\[
\begin{align*}
\lambda v - (a(x)v')' &= 0, \quad 0 \leq x \leq 1, \quad v(0,\lambda) = 0, \\
(4) \quad a(1)v'(1,\lambda) &= F(\lambda), \quad v(1,\lambda) = G(\lambda),
\end{align*}
\]

where \( F := Lf \) and \( G := Lg. \)

The IP can be reformulated as follows:
**IP:** Given $F(\lambda)$ and $G(\lambda)$ for all $\lambda > 0$, find $a(x)$.

Let us transform equation (4)-(5) to yet another equivalent form.

Let $a(x)v' := \psi$. Then (4)-(5) can be replaced by the following problem

\[-\psi'' + \lambda a^{-1}(x)\psi = 0, \quad \psi(1, \lambda) = F(\lambda), \quad \psi'(0, \lambda) = 0, \quad \psi'(1, \lambda) = \lambda G(\lambda).\]  \hfill (6)

The IP can be reformulated as follows:

**IP** Given $G(\lambda)$ and $F(\lambda)$, find $a^{-1}(x) := q^2(x)$.

Let

$$\ell \psi := -\psi'' + k^2 q^2(x)\psi = 0, \quad \lambda := k^2, \quad q^2(x) := a^{-1}(x), \quad c_1^{-1} \leq q^2(x) \leq c_0^{-1}. $$  \hfill (8)

Consider the following problems:

$$\ell_j \psi_j = 0, \quad \ell_j := -\frac{d^2}{dx^2} + k^2 q^2_j(x), \quad \psi'_j(0, k) = 0, \quad \psi_j(0, k) = 1, \quad j = 1, 2.$$  \hfill (9)

Our second main result is

**Theorem 2** The sets $\{\psi_1(x, k)\psi_2(x, k)\}_{k \geq 0}$ and $\{v_1'(x, \lambda)v_2'(x, \lambda)\}_{\lambda \geq 0}$, $k := \lambda^{1/2}$, are dense in the set $\Pi$ of piecewise-constant functions on $[0, 1]$.

**Remark 2** Theorem 2 says that if $h(x) \in \Pi$ and

$$\int_0^1 h(x)\psi_1(x, k)\psi_2(x, k)dx = 0, \quad \forall k \geq 0, $$  \hfill (10)

then $h = 0$. Similar conclusion holds if $\psi_j(x, k)$ is replaced by $v_j'(x, \lambda)$ in (11). Such a property of the pair of the operators $\{\ell_1, \ell_2\}$ is called Property C ([2], [4]).

Clearly if the set $\{\psi_1(x, k)\psi_2(x, k)\}_{k \geq 0}$ is dense in the set $\Pi$, then the set of products $\{v_1'(x, \lambda)v_2'(x, \lambda)\}_{\lambda \geq 0}$ is dense in the set $\Pi$.

In Section 2 proofs are given.

2 Proofs

2.1 Proof of Theorem 1

**Proof.** We prove this Theorem for the problem (4)-(5). Suppose there are $v_j$ and $a_j \in \Pi$, $j = 1, 2$, which solve problem (4)-(5), and let $w := v_1 - v_2$. Then

$$\lambda w - (a_1 w')' = (pv_2')', \quad p := a_1(x) - a_2(x),$$
$$w(0, \lambda) = 0, \quad w(1, \lambda) = 0, \quad a_1 v_1'(1, \lambda) = a_2 v_2'(1, \lambda).$$  \hfill (11)
Multiply (1) by $v_1$, a solution to equation (1) with $a = a_1$, and integrate over $[0, 1]$, and then by parts, to get

$$\int_0^1 p(x)v_2v'_1dx = pv_2v_1\bigg|_0^1 + a_1 wv_1\bigg|_0^1 - a_1 wv'_1\bigg|_0^1 = 0, \quad \forall \lambda > 0, \quad \lambda = k^2, \quad k > 0, \quad (13)$$

where we have used the conditions $w(0, \lambda) = w(1, \lambda) = 0$ and $a_1(1)v'_1(1, \lambda) = a_2(1)v'_2(1, \lambda)$.

Note that $v_2(x, \lambda)$ can be considered as an arbitrary solution to equation (4), up to a constant factor. The set $\{v'_1(x, \lambda)v'_2(x, \lambda)\}$ is dense in $\Pi$ by Theorem 2. Since $a_1(x) - a_2(x) := p(x) \in \Pi$, it follows from (13) that $p(x) = 0$. So $a_1 = a_2$. Theorem 1 is proved. □

### 2.2 Proof of Theorem 2

**Proof.** Let us prove completeness of the set of products $\{\psi_j(x, k)\}_{k \in \mathbb{Z}^+}$. Assume that $h \in \Pi$ and (10) holds. The function $\psi_j(x, k)$, $j = 1, 2$, are entire functions of $k$. This follows from the integral equation for $\psi_j$, which is an immediate consequence of equations (8)–(9):

$$\psi_j(x, k) = 1 + k^2 \int_0^x (x - s)q^2_j(s)\psi_j(s, k)ds, \quad x \geq 0, \quad j = 1, 2. \quad (14)$$

Equation (14) implies that for any fixed $k$ one has $\psi_j(x) := \psi_j(x, k) \geq 1, \forall x \in [0, 1]$, $j = 1, 2$, that $\psi'_j(x, k) \geq 0$, $\psi''_j(x, k) \geq 0$, and $\frac{d^m \psi_j(x, k)}{dk^m} \geq 0$ for all $m = 0, 1, 2, \ldots$.

Consequently, $\psi_j(x, k)$, $j = 1, 2$, are convex functions of $x$ on the semiaxis $x > 0$. Since $\psi_j(x, k)$, $j = 1, 2$, are positive, it follows from (14) that $\psi_j(x, k)$, $j = 1, 2$, are increasing functions with respect to both $x$ and $k$. So we have

$$\psi_j(x, k) > 0, \quad \psi'_j(x, k) > 0, \quad \psi''_j(x, k) > 0, \quad \forall k > 0, \quad j = 1, 2. \quad (15)$$

Assume $0 < x_{11} < x_{12} < \cdots < x_{1N_1} < 1$ and $0 < x_{21} < x_{22} < \cdots < x_{2N_2} < 1$ are discontinuity points of $a_1(x)$ and $a_2(x)$, respectively.

To derive from (10) that $h = 0$ it is sufficient to prove that $h(x) = 0, \forall x \in [x_0, 1]$, where $x_0 := \max(x_{1N_1}, x_{2N_2})$, because then one can prove similarly, in finitely many steps, that $h = 0$ on the whole interval $[0, 1]$ using the assumption $h \in \Pi$. We have

$$\psi''_j(x, k) = k^2 q^2_{jN_j}(x)\psi_j(x, k), \quad \forall k > 0, \quad \forall x \in [x_0, 1], \quad (16)$$

where $q_{jN_j}$ is the value of $q_j$ on the interval $[x_0, 1]$. From (16) one gets

$$\psi_j(x, k) = a_j(k)e^{kq_jN_j(x-x_0)} + b_j(k)e^{-kq_jN_j(x-x_0)}, \quad \forall k \geq 0, \quad j = 1, 2. \quad (17)$$

It follows from (15) and (17) that

$$\psi_j(x_0, k) = a_j(k) + b_j(k) > 0, \quad \psi'_j(x_0, k) = kq_{jN_j}[a_j(k) - b_j(k)] \geq 0, \quad (18)$$

4
\[
2a_j(k) = \psi_j(x_0, k) + \frac{\psi_j(x_0, k)}{kq_j N_j} > \psi_j(x_0, k).
\]

This implies
\[
a_j(k) \geq |b_j(k)| \geq 0, \quad \forall k > 0, \quad j = 1, 2.
\]

Since \( h \in \Pi \), one may assume without loss of generality that
\[
h(x) = C \geq 0, \quad \forall x \in [x_0, 1].
\]

It follows from (10) that
\[
- \int_0^{x_0} \psi_1(x, k)\psi_2(x, k)h(x)dx = \int_{x_0}^1 \psi_1(x, k)\psi_2(x, k)h(x)dx, \quad \forall k > 0.
\]

From (15), (17) and (20), one gets
\[
1 \leq \psi_j(x, k) \leq \psi_j(x_0, k) < 2a_j(k), \quad 0 \leq x \leq x_0, \quad \forall k > 0, \quad j = 1, 2.
\]

Therefore,
\[
\left| \int_0^{x_0} \psi_1(x, k)\psi_2(x, k)h(x)dx \right| \leq 4a_1(k)a_2(k) \int_0^{x_0} |h(x)|dx.
\]

From (20), (17) and (15) one obtains
\[
\psi_j(x, k) \geq a_j(k)[e^{kq_j N_j (x-x_0)} - e^{-kq_j N_j (x-x_0)}], \quad x \in [x_0, 1], \quad j = 1, 2.
\]

Take an arbitrary \( y \in (x_0, 1) \) and fix it. One has \( \psi_j(x, k) \geq \psi_j(y, k), \forall x \in [y, 1] \). Therefore,
\[
\int_{x_0}^1 \psi_1(x, k)\psi_2(x, k)h(x)dx \geq C(1-y)\psi_1(y, k)\psi_2(y, k), \quad \forall k > 0.
\]

This, (23), (22), and (24) imply the following inequalities:
\[
\infty > 4 \int_0^{x_0} |h(x)|dx \geq C(1-y)\frac{\psi_1(y, k)\psi_2(y, k)}{a_1(k)a_2(k)}, \quad \forall k > 0.
\]

It follows from (25) that
\[
\lim_{k \to \infty} \frac{\psi_j(y, k)}{a_j(k)} = \infty.
\]

Let \( k \to \infty \) in (27) and use (28) to conclude that \( C = 0 \) and, therefore, \( h(x) = 0 \) for \( x \in [x_0, 1] \). Similarly one proves that \( h(x) = 0 \) for all \( x \in [0, 1] \).

Theorem 2 is proved.
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