Modified dynamics of weakly coupled BEC’s Josephson Junction (BJJ)

Yu-ping Huang, Zhen-sheng Yuan, Lin-fan Zhu, Lin-jiao Luo, Xiao-jing Liu, Ke-zun Xu
Laboratory of Bond Selective Chemistry, Department of Modern Physics,
University of Science and Technology of China, Hefei, Anhui 230027, China
(Dated: October 25, 2018)

The tunnelling quantum dynamics of bimodal BJJ system is modified through introducing an equilibrium condition, which is based on the assumption that the BJJ is tend to keep on its ground state (with a lowest energy) during the oscillation. The tunnelling dynamics of BJJ with symmetric and asymmetric traps is discussed through numerically solving the modified equations. Stationary states are found to exist in the both BJJs. Compared to previous works, the macroscopic quantum self trapping (MQST) is auto-avoided. Meanwhile, it is revealed that the BJJ oscillates with its inherent frequency which is only related to the Josephson energy, which has been testified experimentally in other contexts.

PACS numbers: 03.75.Fi, 74.50.+r, 05.30.Jp, 32.80.Pj

I. INTRODUCTION

Since Bose-Einstein condensation was first detected in 1995 [1], there have been rapid and critical developments in experimental techniques [2, 3, 4, 5]. In 1997, interference fringes in two overlapping condensates were observed [2]. Consequently in 1998, the superposition of condensed atoms in different hyperfine levels was created [4, 5]. And the evolution of the relative phase of two coupled condensates was measured by interferometry techniques [3, 4]. In 2001, the direct observation of an oscillating atomic current in one-dimensional array Josephson junctions was realized [3]. With these precise manipulation of Bose-Einstein condensates (BEC’s), it has enhanced the possibility of tailoring the new quantum systems to a degree not possible with other quantum systems, like superfluid. And the studies of spatial coherence naturally raise the question of measurement and exploration of temporal phase coherence between two condensates.

Theoretically, with the approximation of non-interacting atoms and small-amplitude Josephson oscillations, some aspects of temporal phase coherence have been investigated in the context of BEC by means of Josephson junctions [3, 4]. Yet many important features of this subject remained to be explored. A non-excitation system of interacting bosons confined within an external potential can be described by a macroscopic wave function, which has the meaning of an order parameter and satisfies the nonlinear Gross-Pitaevskii (GP) equation [10]. The semiclassical tunnelling quantum dynamics using GP equation has been studied [11]. And many interesting phenomena such as macroscopic quantum self-trapping (MQST) [11, 12, 13] and π states [11, 12] were predicted in a weakly coupled double BEC. In those works, the discrete nonlinear Schrödinger equation (DNLSE) has served as a powerful tool for describing the boson Josephson junction (BJJ) [2, 3]. But so far, many investigations have cast doubts on the validity of the DNLSE for conserved quasiparticles interacting with a boson field [14]. The objections are not to the GP equation itself, but to the quasiparticle approximation instead. The earlier results at the condition of short time scales obtained through the DNLSE in the context of coupled quasiparticle-boson systems were verified by the fully quantum version of the BJJ tunnelling model [14, 15]. But on long time scales, the MQST in symmetric traps, which is predicted by DNLSE, was found to be destroyed in fully quantum dynamics [14].

As revealed in other context [14], the MQST (or z-symmetry breaking) in a translationally invariant Hamiltonian is an artifact of the factorization assumption inherent in the semiclassical dynamics. In the DNLSE procedure, atoms are assumed to be localized in either of the traps with discrete wavefunctions. Then, the quantum states for the total BJJ system is the superposition of those eigenstates in each trap individually. Furthermore, the population distribution is determined through the calculation of time dependent Schrödinger equation. Compared to the fully quantum dynamics, in the semiclassical description, the quantum thermodynamic equilibrium for BJJ system is overlooked. And a stationary BJJ is not restricted to its ground state with a lowest energy. A consequent result is the artifact of MQST, which would not break down in the semiclassical description even on the condition of long time scales.

Compared to previous works, the BJJ is assumed to occupy its ground state in present study. And a corresponding lowest energy condition was deduced, based on which the semiclassical dynamics was modified. The numerical calculation of the modified dynamics revealed that the MQST was auto-avoided. Also, it was found that the BJJ oscillated with an inherent frequency, which was only related to the Josephson energy.

In the following Sec. II, the equilibrium behavior of BJJ is discussed, and the lowest energy state is found under a condition of a fixed relationship between the population distribution and the quantum states of BEC’s. In Sec. III the semiclassical dynamics is modified by introducing the stationary condition into the DNLSE procedure. In Sec. IV the tunnelling dynamics for BJJ with symmetric and asymmetric traps is discussed and some
numercial results are given.

II. THE EQUILIBRIUM BEHAVIOR OF WEAKLY COUPLED BJJ

The effective many-body Hamiltonian describing atomic BEC in a double-well trapping potential \( V_{\text{trap}}(r) \) can be written in the second-quantization form as:

\[
\hat{H} = \int d^3r \hat{\Psi}^\dagger \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} \right] \hat{\Psi} + \frac{U_0}{2} \int d^3r \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \hat{\Psi}. \tag{1}
\]

Here \( m \) is the atomic mass and \( U_0 = 4\pi a^2 \hbar^2 / m \) (\( a \) denotes the s-wave scattering length, measuring the strength of the two-body atomic interaction). The Heisenberg atomic field operators \( \hat{\Psi} \) and \( \hat{\Psi} \) satisfy the standard bosonic commutation relation \( [\hat{\Psi}^\dagger(r,t), \hat{\Psi}(r',t)] = \delta(r - r') \). In the quasiparticle approximation, one can expand the field operators \( \hat{\Psi} \) in terms of two local modes

\[
\hat{\Psi}(r,t) \approx \sum_{i=1,2} \hat{a}_i(t) \psi_i(r). \tag{2}
\]

Here \( \{\hat{a}_i(t), \hat{a}_i^\dagger(t)\} = \delta_{ij} \), and \( \psi_i(r) \) stands for the local mode function of either well and satisfies

\[
\int d^3r \psi_i^*(r) \psi_i(r) \approx \delta_{ij}. \tag{3}
\]

Substituting Eq. (2) into Hamiltonian (1), the two-mode approximation of \( \hat{H} \) is yielded

\[
\hat{H} = \sum_{i=1,2} \left( E_i^0 \hat{a}_i^\dagger \hat{a}_i + \lambda_i \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \hat{a}_i \right) - (J \hat{a}_1^\dagger \hat{a}_2 + J^* \hat{a}_2 \hat{a}_1^\dagger), \tag{4}
\]

where the parameters are estimated by [12]

\[
E_i^0 = \int d^3r \psi_i^* \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} \right] \psi_i, \\
\lambda_i = \frac{U_0}{2} \int d^3r |\psi_i|^4, \tag{5}
\]

\[
J = -\int d^3r \left[ \frac{\hbar^2}{2m} \nabla \psi_1^\dagger \cdot \nabla \psi_2 + V_{\text{trap}} \psi_1^\dagger \psi_2^\dagger \right].
\]

Here the interactions between atoms in different wells are neglected as in the weakly coupled BEC’s. Defining two local number operators \( \hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \), it is easy to verify that the total number operator \( \hat{N} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 = N_T \) represents a conserved quantity. After neglecting the constant term, the two-mode Hamiltonian \( \hat{H} \) can be rewritten as [12]:

\[
\hat{H}' = E_c (\hat{n} - \hat{n}_g)^2 - (J \hat{a}_1^\dagger \hat{a}_2 + J^* \hat{a}_2 \hat{a}_1^\dagger), \\
\hat{n} = (\hat{n}_1 - \hat{n}_2)/2, \\
E_c = \lambda_1 + \lambda_2, \tag{6}
\]

\[
\hat{n}_g = \frac{1}{2E_c} [(E_2^0 - E_1^0) + (N_T - 1)(\lambda_1 - \lambda_2)].
\]

Here \( \hat{n} \) is the number difference operator, \( E_c \) is the “charging energy” and \( \hat{n}_g \) is known as the “gate charge” [10].

Generally, the BJJ system woks in Fock regime, which gives \( E_c \gg J \), and Hamiltonian (6) can be well approximated by

\[
\hat{H}' = E_c (\hat{n} - \hat{n}_g)^2. \tag{7}
\]

A BJJ system in ground state should yield a minimized energy (or say a maximized entropy) and the Hamiltonian (7) should be approximate zero, which gives

\[
\hat{n} = \hat{n}_g. \tag{8}
\]

Eq. (8) denotes the relationship between the population distribution and “gate charge” of an stationary BJJ system in its energy ground state. It is important to notice that the “gate charge”, which is related to the eigenstates of BEC in the either trap, is determined by two components, i.e. the boson-field interaction and self interaction, as shown in Eq. (9). In the present work, the number difference \( \hat{n} = \hat{n}_g \) is not necessarily an integer, and the artifact of the factorization of assumption inherent in the semiclassical dynamics is automatically avoided. In this work, the boson-field interaction energy is considered as independent of the number difference \( n \), while the self interaction energy is shifted by the atom numbers in either trap [10].

III. MODIFIED TUNNELLING DYNAMICS

In the DNLSE procedure, the wavefunction Eq. (2) is rewritten in the quasiparticle form, with

\[
\Psi(r,t) = \sum_{i=1,2} \phi_i(t) \psi_i(r), \\
\phi_{1,2}(t) = \sqrt{N_{1,2}(t)} e^{i \theta_{1,2}(t)}, \tag{9}
\]

where \( N \) and \( \theta \) denote the population and phase shift of BJJ, and \( \phi_i \) represents the local mode function \( \hat{a}_i(t) \). Then, the Hamiltonian (6) is reformulated as

\[
i\hbar \frac{\partial \phi_1}{\partial t} = (E_1^0 + \lambda_1 N_1) \phi_1 - J \phi_2, \\
i\hbar \frac{\partial \phi_2}{\partial t} = (E_2^0 + \lambda_2 N_2) \phi_2 - J \phi_1. \tag{10}
\]
Defining the fractional population imbalance and relative phase as

\[ z(t) = (N_1(t) - N_2(t))/N_T, \]
\[ \beta(t) = \theta_1(t) - \theta_2(t). \]

Eq. (10) becomes

\[ \dot{z}(t) = -\sqrt{1 - z(t)^2}\sin[\beta(t)], \]
\[ \dot{\beta}(t) = \Delta E/(2J) + \frac{z(t)}{\sqrt{1 - z(t)^2}}\cos[\beta(t)]. \]

And the total conserved energy is given by

\[ \Delta E = (E_1^0 - E_2^0)/2. \]

Here \( t \) has been rescaled to a dimensionless time \( t2J/h \). And \( \Delta E \) denotes the particle energy difference between atoms in either traps, which acts as an asymmetric parameter in the tunnelling quantum dynamic of BJJ. In present work, the BJJ system is assumed to stay at its ground state. So the stationary condition Eq. (8) should be met. By introducing Eq. (8), Eq. (12) should be modified. It was found that the dynamic equations keep its form as shown by Eq. (12), but with the energy difference modified to

\[ \Delta E = (E_1^0 - E_2^0)/2. \]

And the total conserved energy is given by

\[ H = -\frac{1}{2J} \Delta E \cdot z - \sqrt{1 - z^2}\cos \beta. \]

Eq. (12) can be rewritten in the Hamiltonian form

\[ \frac{\partial}{\partial t} z = -\frac{\partial H}{\partial \beta}, \]
\[ \frac{\partial}{\partial t} \beta = \frac{\partial H}{\partial z}. \]

In deducing Eq. (14), we have put forward the fact that at low oscillation frequency, the BJJ system would stay in its lowest energy state, say ground state, and the population distribution would be related to the “gate charge”. In the previous tunnelling quantum dynamic description of BJJ, the GP equation describing the mean-field dynamics of a BEC is reformed to the bimodal discrete nonlinear Schrödinger equation, and the calculation is carried out by solving self-consistent equation in each trap respectively. It would bring about artifact since the equilibrium behavior has been overlooked. And a consequent result of previous tunnelling dynamics is the MQST. Generally, MQST is based on the condition that the initial conserved energy

\[ H_0 = H[z(0), \beta(0)] > 1, \]

where \( z(0) \) and \( \beta(0) \) are the initial values of \( z \) and \( \beta \) respectively. In the earlier work, the energy gap \( \Delta E \) is the function of population imbalance \( z \), which would result in some certain initial value of \( z \) satisfying Eq. (10). Thus the BJJ system would maintain in MQST state even at the condition of long time scales. Compared to the previous work, a equilibrium condition Eq. (5) was introduced to modify DNLSE procedure in present work. It was found the artifact induced by DNLSE is auto-avoided because in the BJJ with symmetric traps the expression \( H_0 = -\sqrt{1 - z^2}\cos \beta \leq 1 \) is satisfied by any given \( z(0) \) and \( \beta(0) \). Naturally, because the tunnelling quantum dynamics behavior is restricted by equilibrium condition, it is a self-consist result of Eq. (8) and Eq. (12) that the oscillation frequency should be independent of initial condition \( z(0) \) and \( \beta(0) \).

IV. TUNNELING DYNAMICS OF BJJ WITH SYMMETRIC AND ASYMMETRIC TRAPS

A. The symmetric trap case

For a symmetric BEC Josephson Junction (BJJ), the motion equations (12) is given by

\[ \dot{z}(t) = -\sqrt{1 - z(t)^2}\sin[\beta(t)], \]
\[ \dot{\beta}(t) = \frac{z(t)}{\sqrt{1 - z(t)^2}}\cos[\beta(t)], \]

with a conserved energy \( H = -\sqrt{1 - z^2}\cos \beta \). The ground state is a symmetric stationary solution of Eq. (14), with a conserved energy and \( E_+ = 1 \) and

\[ \beta = 2n\pi, \]
\[ z = 0. \]

The next stationary state with higher energy \( E_+ = 1 \) is an antisymmetric solution with

\[ \beta = (2n + 1)\pi, \]
\[ z = 0. \]

The imbalanced initial population, i.e. \( z(0) \neq 0 \), would result in an oscillating solution of Eq. (17), as shown in Fig. 4. It is seen that for a given \( z(0) \), the population imbalance oscillates around it’s zero-value, and the \( z \)-symmetry of bi-mode BJJ is preserved. Meanwhile, the oscillation amplitude \( A \) is determined by the initial population imbalance \( z(0) \), i.e. \( A = |z(0)| \).

B. The asymmetric trap case

When the trap is asymmetric, the energy difference is not equal to zero, i.e. \( \Delta E \neq 0 \). The numerical solution of
Eq. (12) is given in Fig. 2 with the trap asymmetry number $\Delta E/2J = 1$ and $z(0) = 0, 0.4,$ and 0.7 respectively. It is shown that the population imbalance $z(t)$ oscillates around its mean value $\bar{z}$ with an amplitude $A = |z(0) - \bar{z}|$. Also the $z(t)$ plotted against relative phase $\beta(t)$ is given in Fig. 3. It is expected that when $z(0) = \bar{z} = 0.447$, oscillating amplitude would be zero, i.e. $A = 0$, and the BJJ system should arrive at a stationary state. The corresponding ground state is obtained with the symmetric phases,

$$\beta = 2n\pi,$$
$$z = \bar{z}. \quad (20)$$

The next stationary state with higher conserved energy is given by

$$\beta = (2n + 1)\pi, \quad z = \bar{z}, \quad (21)$$

with antisymmetric phases. Moreover, as shown in Eq. (12), the stationary state may be arrived only on the condition of $z(0) = \bar{z}$, and is independent of its initial relative phases.

It is important to notice that, both in Fig. 2 and Fig. 3 the oscillation frequency $f$ is independent of either the trap asymmetry number $\Delta E/2J$ or the initial value $z(0)$. It is the inherent character of BJJ, which is only related to the Josephson energy $J$, with

$$f \propto \frac{\hbar}{J} \quad (22)$$

Since Josephson energy is determined by the interwell potential height, it is experimentally feasible to testify the modified dynamics through Eq. (22). In fact, it has provided a testification of the present scheme.\[18\]

\section{Conclusions}

In conclusion, the tunnelling quantum dynamics of bi-modal BJJ system is modified under an assumption that the BEC's is tend to keep on the ground state with a lowest energy. And the lowest energy is determined by the population distribution. The dynamics of BJJ with symmetric and asymmetric traps is discussed with the modified equations. Compared with previous works, the MQST is auto-avoided in this work, as has been testified.
by previous experimental works. Also, it is revealed that the BJJ oscillates with its inherent frequency, which is only determined by the Josephson energy. This has been testified experimentally. To the end, it is necessary to point out that it remains to be seen how quasiparticle excitation and energy dissipation of BEC would affect the tunnelling dynamics.

Acknowledgments

The discussion with Prof. Zeng-bing Chen is very helpful to this work. And supports by National Nature Science Fund of China (10134010) and the Youth Foundation of the University of Science and Technology of China are gratefully acknowledged.

[1] M. H. Anderson, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198(1995); K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, ibid. bf 75, 1687(1995); E. Cornell, ibid. 81, 1539 (1998).
[2] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science 275, 637(1997).
[3] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle, Phys. Rev. Lett. 80, 2027(1998).
[4] D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, S. Inouye, J. Stenger, and W. Ketterle , e-print cond-mat/9805022
[5] D. S. Hall et al., Phys. Rev. Lett. 81, 1543(1998).
[6] F. S. Cataliotti, S. Burger, C. Fort et.al, Science 293, 843(2001).
[7] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 81, 1539(1998).
[8] J. Javanainen, Phys. Rev. Lett. 57, 3164(1986).
[9] F. Dalfovo, L. Pitaevskii, and S. Stringari, Phys. Rev. A 54, 4213(1996).
[10] F. Dalfovo, S. Giogini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463(1999).
[11] S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A 59, 620(1999).
[12] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A 55, 4318(1997).
[13] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. 79, 4950(1997).
[14] S. Raghavan, A. Smerzi, and V. M. Kenkre, Phys. Rev. A 60, R1787(1999).
[15] Zeng-Bing Chen and Yong-De Zhang, Phys. Rev. A 65, 022318(2002).
[16] Y. Makhlin, G. Schon, ans A. Shnirman, Reb. Mod. Phys. 73, 357(2001).
[17] J. Williams, R. Walser, J. Cooper, E. Cooper, and M. Holland, Phys. Rev. A 59, R31(1999).
[18] F. S. Cataliotti, S. Burger, C. Fort, P.Maddaloni, ct.al Science, 293, 843(2001).