Abstract. The Hermite functions are an orthonormal basis of the space of square integrable functions with favourable approximation properties. Allowing for a flexible localization in position and momentum, the Hagedorn wavepackets generalize the Hermite functions also to several dimensions. Using Hagedorn’s raising and lowering operators, we derive explicit formulas and recurrence relations for the Wigner and FBI transform of the wavepackets and show their relation to the Laguerre polynomials.

1. Introduction

The Hermite functions are an important member of the family of special functions. They form an orthonormal basis of the space of square integrable functions on the real line. The Hermite functions are eigenfunctions of the Fourier transform and of the harmonic oscillator. They can be generated either by raising and lowering operators or by a three-term recurrence relation. And one could name more of their distinguished properties.

The Hagedorn wavepackets [H81, H85, H98] generalize the Hermite functions to several space dimensions while adding more flexibility in terms of position and momentum localization. Moreover, they allow for a positive scale parameter \( \varepsilon > 0 \) setting the wavepackets’ width at the order of \( \sqrt{\varepsilon} \) and their wavelength at the order of \( \varepsilon \). In particular, they are constructed from the complex Gaussian wavepacket

\[
\phi_0^\varepsilon(x) = (\pi \varepsilon)^{-d/4} \det(Q)^{-1/2} \exp\left(\frac{1}{2\varepsilon}(x - q)^T P Q^{-1} (x - q) + \frac{i}{\varepsilon} p^T (x - q)\right)
\]

centered in position \( q \in \mathbb{R}^d \) and momentum \( p \in \mathbb{R}^d \), where the real and imaginary parts of the complex matrices \( Q, P \in \mathbb{C}^{d \times d} \) satisfy a symplecticity condition. In this way, the Gaussian has a position density \( |\phi_0^\varepsilon(x)|^2 \), which is proportional to the multivariate normal distribution with mean \( q \) and covariance matrix \( \frac{i}{\varepsilon} Q Q^* \). The corresponding momentum distribution \( |F_\varepsilon \phi^\varepsilon(\xi)|^2 \), defined via the \( \varepsilon \)-scaled Fourier transform

\[
F_\varepsilon \phi^\varepsilon(\xi) = (2\pi \varepsilon)^{-d/2} \int_{\mathbb{R}^d} \phi(x) e^{-ix^T \xi / \varepsilon} \, dx, \quad \xi \in \mathbb{R}^d,
\]
is proportional to the normal distribution with mean \( p \) and covariance \( \frac{i}{\varepsilon} P P^* \). The familiar raising operator of the Hermite functions is generalized to

\[
A^\dagger = \frac{1}{\sqrt{\varepsilon}} \left( P^* (x - q) - Q^* (-i \varepsilon \nabla_x - p) \right),
\]
and we obtain the \( k \)th Hagedorn wavepacket \( \phi_k^\varepsilon \) by the \( k \)-fold application of the raising operator to the initial Gaussian, that is,

\[
\phi_k^\varepsilon = \frac{1}{\sqrt{k!}} (A^\dagger)^k \phi_0^\varepsilon, \quad k \in \mathbb{N}^d.
\]
In view of a construction process, which balances the position and the momentum parameters, we ask for transforms of the Hagedorn wavepackets, which treat position and momentum or time and frequency variables simultaneously. That is, we aim at the Wigner transform and the Fourier-Bros-Iagolnitzer (FBI) transform of the Hagedorn wavepackets, paying due attention to the ladder operator approach.

The \( \varepsilon \)-scaled Wigner transform of two Schwartz functions \( \varphi, \psi : \mathbb{R}^d \to \mathbb{C} \) is the Fourier transform of their correlation function,

\[
W^\varepsilon(\varphi, \psi)(x, \xi) = (2\pi \varepsilon)^{-d} \int_{\mathbb{R}^d} \overline{\varphi(x + \frac{\xi}{2\varepsilon})} \psi(x - \frac{\xi}{2\varepsilon}) e^{iy\xi / \varepsilon} dy, \quad (x, \xi) \in \mathbb{R}^{2d},
\]

see for example [FG, Chapter 1.8] for a discussion of basic properties or [Br] for the interpretation as a musical score. The Wigner transform maps Schwartz functions to Schwartz functions on phase space, while respecting orthogonality in the sense that

\[
\langle W^\varepsilon(\varphi_1, \psi_1), W^\varepsilon(\varphi_2, \psi_2) \rangle_{L^2(\mathbb{R}^{2d})} = \langle \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}^d)} \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{R}^d)}
\]

for all Schwartz functions \( \varphi_1, \varphi_2, \psi_1, \psi_2 : \mathbb{R}^d \to \mathbb{C} \). On the diagonal, the Wigner function \( W^\varepsilon(\varphi) = W^\varepsilon(\varphi, \varphi) \) is real-valued with the position and momentum density as marginals,

\[
|\varphi(x)|^2 = \int_{\mathbb{R}^d} W^\varepsilon(\varphi)(x, \xi) d\xi, \quad |\mathcal{F}^\varepsilon \varphi(\xi)|^2 = \int_{\mathbb{R}^d} W^\varepsilon(\varphi)(x, \xi) dx.
\]

However, \( W^\varepsilon(\varphi) \) is not a probability density on phase space, since it might attain negative values. For example, odd functions \( \varphi \) satisfy \( W^\varepsilon(\varphi)(0, 0) = (-2\pi \varepsilon)^{-d} ||\varphi||^2 \), and also the Wigner transforms of the Hagedorn wavepackets, except for the initial Gaussian \( \varphi_0 \), attain negative values. This lack of positivity can be cured by the convolution with the properly \( \varepsilon \)-scaled Gaussian phase space function \( G^\varepsilon(z) = (\pi \varepsilon)^{-d} \exp\left(-\frac{1}{\varepsilon} |z|^2\right) \), \( z \in \mathbb{R}^{2d} \). The resulting positive transform

\[
\mathcal{H}^\varepsilon(\varphi) = G^\varepsilon \ast W^\varepsilon(\varphi)
\]

is the so-called Husimi transform \( \mathcal{H}^\varepsilon(\varphi) : \mathbb{R}^{2d} \to [0, \infty[ \). The Husimi transform can also be deduced from the Fourier-Bros-Iagolnitzer (FBI) transform, which is defined as the inner product with the Gaussian wavepacket

\[
g^\varepsilon_{\varphi}(y) = (\pi \varepsilon)^{-d/4} \exp\left(-\frac{1}{\varepsilon} |y|^2 + \frac{\xi^T}{\varepsilon}(y - x)\right), \quad y \in \mathbb{R}^d,
\]

centered in the phase space point \( (x, \xi) \in \mathbb{R}^{2d} \). That is,

\[
T^\varepsilon(\varphi)(x, \xi) = (2\pi \varepsilon)^{-d/2} \langle \varphi, g^\varepsilon_{\varphi} \rangle, \quad (x, \xi) \in \mathbb{R}^{2d},
\]

see for example [FG, Chapter 3.3]. Then, the Husimi transform appears as the modulus squared of the FBI transform, that is, \( \mathcal{H}^\varepsilon(\varphi)(x, \xi) = |T^\varepsilon(\varphi)(x, \xi)|^2 \) for all \( (x, \xi) \in \mathbb{R}^{2d} \), which immediately reveals positivity.

Our study of the Hagedorn wavepackets from the phase space point of view proceeds as follows. In Section \( \text{II} \) we give a brief account on Hermite functions and recall a polynomial sum rule for computing the Wigner and FBI transform of the Hermite functions. In Section \( \text{III} \) we review the construction process of the Hagedorn wavepackets, discuss their relation to the Hermite and Laguerre polynomials, and provide a Rodriguez type formula. Section \( \text{IV} \) then derives explicit formulas for the Wigner and FBI transform of the Hagedorn wavepackets and recasts the three-term recurrence relation of the Hagedorn wavepackets on the Wigner level. In the appendix, Section \( \text{A} \) presents another characterization of the Hagedorn wavepackets based on the polar decomposition of one of the width matrices, while \( \text{B} \) reformulates the ladder operators in Weyl quantization.
2. Hermite functions

Hermite functions are Hermite polynomials times a Gaussian. They can be generated from the Gaussian

\[ \varphi_0(x) = \pi^{-1/4} \exp(-\frac{1}{2}x^2), \quad x \in \mathbb{R}, \]

using the ladder operators \( A^\dagger = \frac{1}{\sqrt{2}}(x - \nabla_x) \),

\[ \varphi_{k+1} = \frac{1}{\sqrt{k+1}} A^\dagger \varphi_k, \quad k \in \mathbb{N}. \]

The formal adjoint \( A = \frac{1}{\sqrt{2}}(x + \nabla_x) \) of the ladder operator \( A^\dagger \) allows to descend within the Hermite functions,

\[ \varphi_k = \frac{1}{\sqrt{k+1}} A \varphi_{k+1}, \quad k \in \mathbb{N}. \]

**Remark 1.** The Hermite functions are eigenfunctions of the Fourier transform and the harmonic oscillator \( \frac{1}{2}(AA^\dagger + A^\dagger A) = \frac{1}{2}(-\Delta_x + x^2) \),

\[ \mathcal{F} \varphi_k = (-i)^k \varphi_k, \quad \frac{1}{2}(AA^\dagger + A^\dagger A) \varphi_k = (k + \frac{1}{2}) \varphi_k \]

for all \( k \in \mathbb{N} \). Moreover, the family \( \{ \varphi_k \mid k \in \mathbb{N} \} \) forms an orthonormal basis of the Hilbert space \( L^2(\mathbb{R}) \), see for example [Tu, §7.8]. They enjoy the following approximation property: Let \( K \in \mathbb{N} \), \( s \leq K \) and \( f : \mathbb{R}^d \to \mathbb{C} \) a Schwartz function. Then,

\[ \| f - \sum_{k<K} \langle \varphi_k, f \rangle \varphi_k \|_{L^2(\mathbb{R})} \leq (K(K-1) \cdots (K-s+1))^{-1/2} \| A^s f \|_{L^2(\mathbb{R})}, \]

see [Lu, Theorem 1.2].

2.1. Hermite polynomials. Alternatively, the Hermite functions can be written as

\[ \varphi_k(x) = \frac{1}{\sqrt{2^{k} k!}} h_k(x) \varphi_0(x), \quad x \in \mathbb{R}, \]

with

\[ h_k(x) = \exp(x^2) (-\frac{d}{dx})^k \exp(-x^2) \]

\[ = \sum_{j=0}^{[k/2]} \frac{k!}{j!(k-2j)!} (-1)^j (2x)^{k-2j} \]

the \( k \)th Hermite polynomial. Starting from \( h_0 = 1 \), the Hermite polynomials can be generated by repeated application of the ladder operator \( B^\dagger = 2x - \nabla_x \),

\[ h_{k+1} = B^\dagger h_k, \quad k \in \mathbb{N}, \]

or from the three-term recurrence relation

\[ h_{k+1}(x) = 2x h_k(x) - 2kh_{k-1}(x), \quad k \geq 1. \]

The orthonormality of the Hermite functions implies for the Hermite polynomials

\[ \int_{\mathbb{R}} h_k(x) h_l(x) e^{-x^2} dx = \sqrt{\pi} 2^k k! \delta_{k,l}, \quad k, l \in \mathbb{N}. \]
2.2. Integral formulas. The Hermite polynomials satisfy several beautiful integral formulas. Those, which we employ for the phase space transformation of the Hermite functions, can be deduced from the following sum rule

\[
\begin{equation}
    h_k(x + z) = \sum_{j=0}^{k} \binom{k}{j} (2z)^{k-j} h_j(x), \quad x, z \in \mathbb{C},
\end{equation}
\]

which is due to [Fo]. We refer to Proposition 3 later on for a proof.

**Proposition 1.** Let \( k \leq l \) and \( h_k \) and \( h_l \) be the \( k \)th and \( l \)th Hermite polynomial. Then, for \( z_1, z_2 \in \mathbb{C} \),

\[
\begin{equation}
    \int_{\mathbb{R}} h_k(x + z_1) h_l(x + z_2) e^{-x^2} dx = \sqrt{\pi} 2^l l! z_2^{l-k} L_k^{(l-k)}(-2z_1z_2),
\end{equation}
\]

where

\[
\begin{equation}
    L_k^{(\gamma)}(x) = \sum_{j=0}^{k} (-1)^j \binom{k + \gamma}{k - j} x^j / j!, \quad k \in \mathbb{N}
\end{equation}
\]

are the Laguerre polynomials associated with \( \gamma \in \mathbb{R} \). In particular,

\[
\begin{equation}
    \int_{\mathbb{R}} h_l(x + z_2) e^{-x^2} dx = \sqrt{\pi} 2^l l!
\end{equation}
\]

**Proof.** We write

\[
\begin{align*}
    \int_{\mathbb{R}} h_k(x + z_1) h_l(x + z_2) e^{-x^2} dx &= \sum_{j=0}^{k} \binom{k}{j} \binom{l}{j'} (2z_1)^{k-j} (2z_2)^{l-j'} \int_{\mathbb{R}} h_j(x) h_{j'}(x) e^{-x^2} dx.
\end{align*}
\]

From the orthogonality condition [5] we then deduce

\[
\begin{align*}
    \int_{\mathbb{R}} h_k(x + z_1) h_l(x + z_2) e^{-x^2} dx &= \sum_{j=0}^{k} \binom{k}{j} \binom{l}{j'} (2z_1)^{k-j} (2z_2)^{l-j} \sqrt{\pi} 2^l j! \sqrt{\pi} 2^l l!
\end{align*}
\]

\[
\begin{align*}
    &= \sqrt{\pi} 2^l l! z_2^{l-k} \sum_{j=0}^{k} \binom{k}{j} \binom{l}{j'} (2z_1 z_2)^{k-j} \frac{l!}{(l-j)!} = \sqrt{\pi} 2^l l! z_2^{l-k} L_k^{(l-k)}(-2z_1 z_2).
\end{align*}
\]

2.3. Phase space transforms. The Hermite-Laguerre connection of Proposition 1 translates to the Wigner transform of Hermite functions. For alternative proofs, see [Fo] Chapter 1.9 or [Tu] Chapter 1.3.

**Corollary 1.** If \( \varphi_k, \varphi_l \) are the \( k \)th and \( l \)th Hermite function, then the Wigner function is

\[
\begin{align*}
    \mathcal{W}^l(\varphi_k, \varphi_l)(x, \xi) &= \begin{cases} 
    \frac{(-1)^k}{\pi} \sqrt{2^{l-k}} \sqrt{\frac{\pi}{2}} e^{-|z|^2} L_k^{(l-k)}(2|z|^2), & k \leq l, \\
    \frac{(-1)^l}{\pi} \sqrt{2^{k-l}} \sqrt{\frac{\pi}{2}} e^{-|z|^2} L_l^{(k-l)}(2|z|^2), & l \leq k,
    \end{cases}
\end{align*}
\]

with \( z = x + i\xi \) for \( x, \xi \in \mathbb{R} \). In particular,

\[
\begin{align*}
    \mathcal{W}^l(\varphi_k)(x, \xi) &= \frac{(-1)^k}{\pi} e^{-|z|^2} L_k^{(0)}(2|z|^2).
\end{align*}
\]
Proof. We compute

\[
W^1(\varphi_k, \varphi_l)(x, \xi) = \frac{1}{2\pi^{3/2}} \frac{1}{\sqrt{2^{k+l}k!l!}} \int_{\mathbb{R}} h_k(x + \frac{y}{2})h_l(x - \frac{y}{2})e^{-(x^2 + (y/2)^2)}e^{iy\xi}dy
\]

where we have used \(h_t(-x) = (-1)^t h_t(x)\). Changing the variable as \(y/2 - ix = \eta\), we perform a contour integration in the complex plane. Analyticity and exponential decay of the integrand then provide

\[
W^1(\varphi_k, \varphi_l)(x, \xi) = \frac{(-1)^l}{\pi^{3/2}} \frac{e^{-|z|^2}}{\sqrt{2^{k+l}k!l!}} \int_{\mathbb{R}} h_k(\eta + z)h_l(\eta - \bar{z})e^{-\eta^2}d\eta.
\]

The integral formula (7) concludes the proof. \(\square\)

The integral formula (7) of Proposition 1 provides the FBI and the Husimi transform of Hermite functions, see [Fl, §2].

Corollary 2. Let \(\varphi_k\) be the \(k\)th Hermite function. Then, the FBI transform is

\[
\mathcal{T}^1(\varphi_k)(x, \xi) = \frac{e^{-\frac{1}{4}(x^2 + \xi^2)}}{\sqrt{2^{k+1}k!}} e^{-\frac{1}{2}|z|^2}, \quad x, \xi \in \mathbb{R},
\]

with \(z = x + i\xi\). Consequently, the Husimi transform is

\[
\mathcal{H}^1(\varphi_k)(x, \xi) = \frac{1}{\pi^{2k+1}2^{k+1}k!} e^{-\frac{1}{2}|z|^2}.
\]

Proof. We apply the integral formula (7) to obtain

\[
\mathcal{T}^1(\varphi_k)(x, \xi) = \frac{1}{\pi \sqrt{2^{k+1}k!}} \int_{\mathbb{R}} h_k(y) e^{-\frac{1}{2}y^2} e^{-\frac{1}{2}(y-x)^2} e^{iy\xi}e^{-y\eta} dy
\]

\[
= \frac{e^{-\frac{1}{2}x^2}e^{-\frac{1}{2}(x^2 + \xi^2)}}{\pi \sqrt{2^{k+1}k!}} \int_{\mathbb{R}} h_k(y) e^{-(y-\frac{1}{2}(x+i\xi))^2} dy
\]

\[
= \frac{e^{-\frac{1}{2}x^2}e^{-\frac{1}{2}(x^2 + \xi^2)}}{\sqrt{\pi 2^{k+1}k!}} (x + i\xi)^k.
\]

Remark 2. We note that both the Wigner transform \(W^1(\varphi_k)\) and the Husimi transform \(\mathcal{H}^1(\varphi_k)\) of the \(k\)th Hermite function \(\varphi_k\) are radially symmetric, that is, they are functions of the energy variable \(|z|^2 = x^2 + \xi^2\), see also [HC] §2.4.

3. HAGEDORN WAVEPACKETS

In [H98], George Hagedorn devised parametrized ladder operators which allow for a beautiful generalization of the Hermite functions also in several space dimensions: Let \(z > 0\) be a scale parameter, \(q, p \in \mathbb{R}^d\) and \(C = CT \in \mathbb{C}^{d \times d}\) a complex symmetric matrix with positive definite imaginary part \(\text{Im}(C) > 0\).

Remark 3. Any complex symmetric matrix \(C \in \mathbb{C}^{d \times d}\) with \(\text{Im}(C) > 0\) can be written as \(C = PQ^{-1}\), where \(P, Q \in \mathbb{C}^{d \times d}\) are invertible matrices satisfying

\[
Q^TP - P^TQ = 0, \quad Q^TP - P^TQ = 2\text{Id}.
\]

This condition is equivalent to

\[
Y = \begin{pmatrix} \text{Re}(Q) & \text{Im}(Q) \\ \text{Re}(P) & \text{Im}(P) \end{pmatrix} \text{ is symplectic:} \quad Y^TJY = J = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}.
\]
Conversely, any pair of matrices $P, Q \in \mathbb{C}^{d \times d}$ satisfying (10) is invertible and defines via $C = PQ^{-1}$ a complex symmetric matrix with
\[
\text{Im}(C) = (QQ^*)^{-1} > 0,
\]
see [Lu ] §5, Lemma 1.1]. The applications of the Hagedorn wavepackets especially to quantum dynamics emphasize the importance of allowing both matrices $Q$ and $P$ to be complex. We note, that Hagedorn uses $A$ and $iB$ for $Q$ and $P$ in his work, while we adopt the more recent notation of [FGL, Lu].

Let $Q, P \in \mathbb{C}^{d \times d}$ be matrices satisfying (10). Then, the Hermite ladder operators $A^\dagger = \frac{i}{\sqrt{2x}}(x - \nabla_x)$ and $A = \frac{i}{\sqrt{2x}}(x + \nabla_x)$ generalize as
\[
A^\dagger[q, p, Q, P] = \frac{i}{\sqrt{2x}}(P^*(x - q) - Q^*(-i\epsilon \nabla_x - p)),
\]
\[
A[q, p, Q, P] = -\frac{i}{\sqrt{2x}}(P^T(x - q) - Q^T(-i\epsilon \nabla_x - p)).
\]
Before employing this ladder, we summarize some useful linear algebra.

**Lemma 1.** Let $\epsilon > 0$, $q, p \in \mathbb{R}^d$, and $Q, P \in \mathbb{C}^{d \times d}$ be matrices satisfying (10). Then, $QQ^*$ and $PP^*$ are real symmetric matrices. Moreover, the components of $A^\dagger[q, p, Q, P] = (A^\dagger)^d_{j=1}$ commute, that is,
\[
[A^\dagger_j, A^\dagger_k] = A^\dagger_j A^\dagger_k - A^\dagger_k A^\dagger_j = 0
\]
for $j, k = 1, \ldots, d$.

**Proof.** Since $C = PQ^{-1}$ is complex symmetric with $\text{Im}(C) = (QQ^*)^{-1} > 0$, we have $QQ^* \in \mathbb{R}^{d \times d}$ and $QQ^* = (QQ^*)^T$. Moreover,
\[
PP^* = CQQ^*\overline{C} = \text{Re}(C)\text{Im}(C)^{-1}\text{Re}(C) + \text{Im}(C) \in \mathbb{R}^{d \times d}, \quad PP^* = (PP^*)^T.
\]
Finally,
\[
\left[A^\dagger_j, A^\dagger_k\right] = \frac{i}{2\epsilon} \left( \sum_{l=1}^d \overline{p}_{lj} x_l - \overline{q}_{lj}(-i\epsilon \partial_{x_l}), \sum_{m=1}^d \overline{q}_{mk} x_m - \overline{p}_{mk}(-i\epsilon \partial_{x_m}) \right)
\]
\[
= -\frac{i}{2\epsilon} \sum_{l,m=1}^d \left( \overline{p}_{lj} \overline{q}_{mk}[x_l, -i\epsilon \partial_{x_m}] + \overline{q}_{lj} \overline{p}_{mk}[-i\epsilon \partial_{x_l}, x_m] \right)
\]
\[
= \frac{i}{\epsilon} \left( -P^\dagger \overline{Q} + Q^\dagger \overline{P} \right)_{jk} = 0
\]
due to the canonical commutator relation $\frac{i}{\epsilon^2}[x_j, -i\epsilon \partial_{x_k}] = \delta_{jk}$ and the matrix condition (10). \qed

The next step, is to generalize the zeroth order Hermite function $\varphi_0$ as the complex Gaussian wavepacket
\[
\varphi_0^\epsilon(x) = \varphi_0[q, p, Q, P](x) = (\pi \epsilon)^{-d/4} \det(Q)^{-1/2} \exp(\frac{i}{2\epsilon^2}(x - q)^T PQ^{-1}(x - q) + \frac{i}{\epsilon}p^T(x - q))
\]
for $x \in \mathbb{R}^d$. We note, that the vectors $q, p \in \mathbb{R}^d$ and the matrices $QQ^*, PP^* \in \mathbb{R}^{d \times d}$ provide the centers and the width of the corresponding position and momentum densities,
\[
|\varphi_0^\epsilon(x)|^2 = (\pi \epsilon)^{-d/2} \det(Q)^{-1} \exp\left( -\frac{1}{2}(x - q)^T (QQ^*)^{-1}(x - q) \right),
\]
\[
|\mathcal{F}^\epsilon \varphi_0^\epsilon(\xi)|^2 = (\pi \epsilon)^{-d/2} \det(P)^{-1} \exp\left( -\frac{1}{2}(\xi - p)^T (PP^*)^{-1}(\xi - p) \right)
\]
for $x, \xi \in \mathbb{R}^d$. With the ladder operator $A^\dagger = A^\dagger[q, p, Q, P]$, the Hagedorn wavepackets $\varphi_k^\epsilon = \varphi_k[q, p, Q, P]$ are defined recursively via
\[
\varphi_{k+e_j}^\epsilon = \frac{1}{\sqrt{k+1}} A^\dagger_j \varphi_k^\epsilon, \quad k \in \mathbb{N}^d,
\]
Moreover, we check
\[ \rho_k = \frac{1}{\sqrt{2}} A_j \varphi_k. \]

The harmonic oscillator equation of the Hermite functions generalizes to
\[ \frac{1}{2} \sum_{j=1}^d (A_j A_j^* + A_j^* A_j) \varphi_k = (|k| + \frac{1}{2}) \varphi_k, \quad k \in \mathbb{N}^d. \]

That is, every Hagedorn wavepacket \( \varphi_k \), \( k \in \mathbb{N}^d \), is an eigenfunction of an harmonic oscillator for the eigenvalue \(|k| + \frac{1}{2}\), see [H98, Theorem 3.3].

**Remark 4.** The scaled Fourier transform of the \( k \)-th Hagedorn wavepacket satisfies
\[ \mathcal{F} \varphi_k[q, p, Q, P] = (\frac{-i}{\varepsilon})^{|k|} e^{-i p^T q / \varepsilon} \varphi_k[p - q, -p - q], \quad k \in \mathbb{N}^d, \]
see [H98, §2] for an effortless ladder proof.

The Hagedorn wave packets behave conveniently under modulations and translations, which will be useful for computing their Wigner and FBI transform.

**Lemma 2.** Let \( \varphi, \psi : \mathbb{R}^d \to \mathbb{C} \) be Schwartz functions and
\[ \varphi_{x, \xi} : \mathbb{R}^d \to \mathbb{C}, \quad \varphi_{x, \xi}(y) = e^{\frac{-i}{\varepsilon} T y} \varphi(y + x) \]
for \((x, \xi) \in \mathbb{R}^{2d}\). Then,
\[ \mathcal{W} \psi(x, \xi)(0, 0) = \mathcal{W} \psi(\varphi_{x, \xi}, \psi_{x, \xi})(0, 0), \quad \mathcal{T} \psi(x, \xi) = \mathcal{T} \psi(\varphi_{x, \xi})(0, 0). \]
Moreover, for \( k \in \mathbb{N}^d \), \( q, p \in \mathbb{R}^d \), and \( Q, P \in \mathbb{C}^d \times d \) satisfying \( T H \) we have
\[ (\varphi_k[q, p, Q, P])_{x, \xi} = e^{\frac{-i}{\varepsilon} T(q-x)} \varphi_k[q - x, p - \xi, Q, P]. \]

**Proof.** We check
\[ \mathcal{W} \psi(\varphi_{x, \xi}, \psi_{x, \xi})(0, 0) = \left(2\pi\varepsilon\right)^{-d} \int_{\mathbb{R}^d} \overline{\psi}_{x, \xi}(\frac{q}{\varepsilon}) \psi_{x, \xi}(\frac{q}{\varepsilon}) dy \]
\[ = \left(2\pi\varepsilon\right)^{-d} \int_{\mathbb{R}^d} \overline{\psi}(x + \frac{q}{\varepsilon}) \psi(x + \frac{q}{\varepsilon}) e^{i p^T \xi / \varepsilon} dy \]
\[ = \mathcal{W} \psi(\varphi, \psi)(x, \xi) \]
and
\[ \mathcal{T} \psi(x, \xi) = \left(2\pi\varepsilon\right)^{-d/2} \left(2\pi\varepsilon\right)^{-d/4} \int_{\mathbb{R}^d} \overline{\psi}(y) e^{\frac{-i y^T (x-y)}{\varepsilon}} dy \]
\[ = \left(2\pi\varepsilon\right)^{-d/2} \left(2\pi\varepsilon\right)^{-d/4} \int_{\mathbb{R}^d} \overline{\psi}(y + x) e^{\frac{-i y^T (x-y)}{\varepsilon}} dy \]
\[ = \mathcal{T} \psi(\varphi_{x, \xi})(0, 0). \]
Moreover, \( (\varphi_{0}[q, p, Q, P])_{x, \xi} = e^{-\frac{1}{\varepsilon} T(q-x)} \varphi_{0}[q - x, p - \xi, Q, P] \) and
\[ (A^\dagger[q, p, Q, P] \varphi)_{x, \xi} = A^\dagger[q - x, p - \xi, Q, P] \varphi_{x, \xi} \]
for all Schwartz functions \( \varphi \). Therefore, for all \( k \in \mathbb{N}^d \),
\[ (\varphi_k[q, p, Q, P])_{x, \xi} = \frac{1}{\sqrt{A^\dagger}} \left( (A^\dagger[q, p, Q, P]) \varphi_{0}[q, p, Q, P] \right)_{x, \xi} \]
\[ = e^{-\frac{i}{\varepsilon} T(q-x)} \varphi_k[q - x, p - \xi, Q, P]. \]
\[ \square \]
3.1. Hermite polynomials. The Hagedorn wavepackets are known to satisfy the three-term recurrence relation

\[
\left( \sqrt{k_j + 1} \varphi_{k+e_j}^k(x) \right)^d_{j=1} = \\
\sqrt{2} Q^{-1}(x - q) \varphi_k^k(x) - Q^{-1} Q \left( \sqrt{k_j} \varphi_{k-e_j}^k(x) \right)^d_{j=1},
\]

see [2, Chapter V.2]. Therefore, \( \varphi_k^k = \varphi_k^k[q, p, Q, P] \) is the product of a polynomial of degree \( |k| \) with the complex Gaussian \( \varphi_0^0 \). However, we can deduce some more information.

**Proposition 2.** Let \( \varepsilon > 0 \), \( q, p \in \mathbb{R}^d \), and \( Q, P \in \mathbb{C}^{d \times d} \) satisfy (14). The kth Hagedorn wavepacket \( \varphi_k^k = \varphi_k^k[q, p, Q, P] \), \( k \in \mathbb{N}^d \), can be written as

\[
\varphi_k^k(x) = \frac{1}{\sqrt{2^k k!}} p_k^k(x) \varphi_0^0(x), \quad x \in \mathbb{R}^d,
\]

where \( p_k^k \) is a multivariate polynomial of degree \( |k| \) generated by the recursion \( p_0^0 = 1 \), \( p_{k+e_j}^k = B_j^1 p_k^k \) for \( j = 1, \ldots, d \), with

\[
B_j^1 = \frac{2}{\sqrt{\varepsilon}} Q^{-1} \delta_{p} - \frac{1}{\sqrt{\varepsilon}} Q^*(\varepsilon \nabla_x).
\]

The components of \( B_j^1 = (B_j^1)' \) commute, that is, \( [B_j^1, B_j'^1] = 0 \) for \( j, j' = 1, \ldots, d \). Moreover,

\[
p_k^k(-x) = (-1)^{|k|} p_k^k(x + 2q), \quad x \in \mathbb{R}^d.
\]

If \( Q = Q^T \in \mathbb{R}^{d \times d} \), then

\[
p_k^k(x) = \prod_{j=1}^d h_{k_j}(\sqrt{\varepsilon} (Q^{-1}(x - q)))_j, \quad x \in \mathbb{R}^d.
\]

**Proof.** We denote \( A^1 = A_q^1[q, p, Q, P] \). We first consider the special case \( Q = Q^T \in \mathbb{R}^{d \times d} \) and compute

\[
Q^*(-i\varepsilon \nabla_x - p) \prod_{j=1}^d h_{k_j}(y_j) \varphi_0^0(x)
\]

\[
= -i \sqrt{\varepsilon} Q^* Q^{-T} \left( h_{k_1}(y_1) \prod_{j \neq 1} h_{k_j}(y_j) \right) \varphi_0^0(x)
\]

\[
+ Q^* C(x - q) \prod_{j=1}^d h_{k_j}(y_j) \varphi_0^0(x)
\]

with \( C = PQ^{-1} \). Since \( P^* Q - Q^* C Q = -2i \text{Id} \), we obtain

\[
A^1 \prod_{j=1}^d h_{k_j}(y_j) \varphi_0^0(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
(2y_1 h_{k_1}(y_1) - h_{k_1}'(y_1)) \prod_{j \neq 1} h_{k_j}(y_j) \\
(2y_d h_{k_d}(y_d) - h_{k_d}'(y_d)) \prod_{j \neq d} h_{k_j}(y_j)
\end{array} \right) \varphi_0^0(x)
\]

\[
= \frac{1}{\sqrt{2}} \left( \begin{array}{c}
(2y_1 h_{k_1}(y_1) - h_{k_1}'(y_1)) \prod_{j \neq 1} h_{k_j}(y_j) \\
(2y_d h_{k_d}(y_d) - h_{k_d}'(y_d)) \prod_{j \neq d} h_{k_j}(y_j)
\end{array} \right) \varphi_0^0(x).
\]

Assuming, that the claimed identity (15) holds for \( k \in \mathbb{N}^d \), we derive

\[
\varphi_{k+e_j}^k(x) = \frac{1}{\sqrt{k_j + 1}} A_j^1 \varphi_k^k(x) = \frac{1}{\sqrt{2^{|k|+1+k+e_j|}}} h_{k_j+1}(y_1) \prod_{j \neq j} h_{k_j}(y_j) \varphi_0^0(x).
\]
For general $Q \in \mathbb{C}^{d \times d}$, we observe that
\[ Q^*(−iε\nabla_x − p)p_k^e(x)\varphi_0^e(x) = (Q^*(−iε\nabla_x)p_k^e)(x)\varphi_0^e(x) + Q^*C(x − q)p_k^e(x)\varphi_0^e(x) \]
and
\[ A_j^1p_k^e(x)\varphi_0^e(x) = \frac{1}{\sqrt{x_{k+1}}}(P^* − Q^*C)(x − q)p_k^e(x) − (−iε\nabla_x)p_k^e(x) \]
\[ = \frac{1}{\sqrt{x_{k+1}}}(\frac{2}{\sqrt{x}}Q^{-1}(x − q)p_k^e(x) − \frac{1}{\sqrt{x}}Q^*(−iε\nabla_x)p_k^e(x))\varphi_0^e(x). \]
Assuming that equation (13) holds for $k$, we have
\[ \varphi_{k+e_j}^e = \frac{1}{\sqrt{x_{k+1}}}A_j^1\varphi_k^e = \frac{1}{\sqrt{2x_{k+1}(k+e_j)}}(B_j^1p_k^e)\varphi_0^e. \]
Moreover,
\[ [B_j^1, B_j^2] = -2i\sum_{i,m=1}^{d} (Q^{-1}Q_jy^m[x_i, −i\varepsilon\partial_{x_m}] − Q_j\tilde{Q}^{-1}[-i\varepsilon\partial_{x_i}, x_m]) = 2\varepsilon(Q^{-1}Q^*Q^{-T})_{jj'} = 0, \]
where we have used that $[x_i, −i\varepsilon\partial_{x_m}] = i\varepsilon\delta_{im}$ and $QQ^* = (QQ^*)^T = \overline{QQ}^T$. For proving the symmetry relation, we argue once more inductively. We have
\[ p_{k+e_j}^e(x) = \frac{1}{\sqrt{x}}(Q^{-1}(−x − q))_j p_k^e(x) - \varepsilon((Q^*\nabla_x)_j p_k^e)(−x) = \frac{1}{2}(Q^{-1}(−x − q))_j (−1)^{|k|}p_k^e(x + 2q) + \sqrt{\varepsilon}(Q^*\nabla_x)_j (−1)^{|k|}p_k^e(x + 2q) = (−1)^{|k|+1}p_{k+e_j}^e(x + 2q). \]

Remark 5. In the univariate case, Proposition 3 and 6 provide the original definition of $\varphi_k^e[p, q, Q, P]$ given in [HS1] §1,
\[ \varphi_k^e[p, q, Q, P](x) = \frac{(Q/Q)^k}{\sqrt{2\pi k!}}\int \varphi_0^e[p, q, Q, P](x). \]
For $Q = Q^T \in \mathbb{R}^{d \times d}$, the description of Hagedorn wavepackets as rotated, scaled versions of harmonic oscillator eigenfunctions is mentioned in [HS5] Remark 4 of §1. For general $Q \in \mathbb{C}^{d \times d}$, there is a more complicated relation, see Proposition 1.

3.2. Laguerre polynomials. The polynomial ladder of Proposition 3 allows to generalize the Hermite polynomial’s sum rule (3) and the integral connection to the Laguerre polynomials. A similar argumentation for generalized univariate Hermite polynomials can be found in [DM] Proposition 2.

Proposition 3. Let $x > 0$, $q, p \in \mathbb{R}^d$, and $Q, P \in \mathbb{C}^{d \times d}$ satisfy (10). Let $k, l \in \mathbb{N}^d$, $k \leq l$, and $p_k^e, p_l^e$ be the $k$th and $l$th polynomials defined in Proposition 3. Then, for $x, y, z \in \mathbb{C}^d$,
\[ p_k^e(x + z) = \sum_{\nu \leq k} \binom{k}{\nu} \left(\frac{2}{\sqrt{x}}Q^{-1}z\right)^{k-\nu}p_\nu^e(x) \]
and
\[ \int_{\mathbb{R}^d} p_k^e(x + y)p_l^e(x + z)|\varphi_0^e(x)|^2dx = \prod_{j=1}^{d} L_{k_j,l_j}\left(\frac{1}{\sqrt{x}}(Q^{-1}y)_j, \frac{1}{\sqrt{x}}(Q^{-1}z)_j\right) \]
with
\[
L_{m,n}(\eta, \zeta) = \begin{cases} 
2^n m! \zeta^{m-n} L_m^{(n-m)}(-2\eta \zeta), & m \leq n, \\
2^n n! \zeta^{m-n} L_n^{(m-n)}(-2\eta \zeta), & n \leq m.
\end{cases}
\]

Proof. For \( z \in \mathbb{C}^d \) we use the translation \((\tau_z f)(x) = f(x + z)\) and observe
\[
\tau_z \circ B^\dagger = (B^\dagger + \frac{x}{\sqrt{z}} Q^{-1} z) \circ \tau_z.
\]
Inductively, we obtain
\[
\tau_z \circ (B^\dagger)^k = (B^\dagger + \frac{x}{\sqrt{z}} Q^{-1} z)^k \circ \tau_z = \left( \sum_{\nu \leq k} \binom{k}{\nu} \left( \frac{2}{\sqrt{z}} Q^{-1} z \right)^{k-\nu} (B^\dagger)^\nu \right) \circ \tau_z
\]
for all \( k \in \mathbb{N}^d \), which applied to \( p_0^z \equiv 1 \) yields the claimed sum rule. Therefore,
\[
\int_{\mathbb{R}^d} p^z(x + y)p^z(x + z)|\varphi_0^z(x)|^2 \, dx = \sum_{\nu \leq k} \binom{k}{\nu} \frac{2^\nu}{\sqrt{z}} Q^{-1} z \nu \delta_{\nu, \nu'} \]
where we have used the orthogonality relation
\[
\int_{\mathbb{R}^d} p^z_\nu(x)p^z_{\nu'}(x)|\varphi_0^z(x)|^2 \, dx = 2^\nu \delta_{\nu, \nu'} \delta_{\nu, \nu'}.
\]
If \( k_j \leq l_j \), then
\[
\sum_{\nu_j = 0}^{k_j} \frac{k_j l_j l_j^!}{(k_j - \nu_j)!(l_j - \nu_j)!\nu_j^!} 2^{\nu_j} \left( \frac{2}{\sqrt{z}} Q^{-1} y \right)^{k_j - \nu_j} \left( \frac{2}{\sqrt{z}} Q^{-1} z \right)^{l_j - \nu_j}
\]
by the monomial representation of the Laguerre polynomials \([\text{8}]\). The analogous argument for \( k_j \geq l_j \) concludes the proof. \( \square \)

3.3. Rodriguez formula. The Hagedorn wavepackets’ three-term recurrence formula \([\text{12}]\) rewrites on the polynomial level as
\[
(\overset{d}{\bigotimes} p_{k_j + e_j}(x))_{j=1}^d = \frac{2}{\sqrt{z}} Q^{-1} (x - q)p_k^z(x) - 2Q^{-1} Q \left( k_j p_{k_j - e_j}(x) \right)_{j=1}^d.
\]
Together with the following Rodriguez-type formula we obtain another integration formula for the polynomials.

**Proposition 4.** Let \( \varepsilon > 0 \), \( q, p \in \mathbb{R}^d \), and \( Q, P \in \mathbb{C}^{d \times d} \) satisfy \([\text{10}]\). Let \( k \in \mathbb{N}^d \), and \( p_k^z \) be the \( k \)th polynomial defined in Proposition\([\text{3}]\). Then,
\[
p_k^z(x) = |\varphi_0^z(x)|^{-2}(-\sqrt{z} Q^* \nabla_x)^k|\varphi_0^z(x)|^2, \quad x \in \mathbb{R}^d.
\]
Let \( M = \frac{1}{2}(\mathbb{I} + \varepsilon \nabla_x) \). Then, \( \mathbb{I} + Q^T M Q \) is invertible, and we have for all \( z \in \mathbb{C}^d \)
\[
\int_{\mathbb{R}^d} p^z_k(x + z) e^{-\frac{1}{2}(x-q)^T(\text{Im}(M) + iM)(x-q)} \, dx = \sum_{\nu \leq k} \binom{k}{\nu} \left( \frac{2}{\sqrt{z}} Q^{-1} z \right)^{k-\nu} c_\nu
\]
with
\[
c_\nu = 0, \quad (c_{\nu + e_j})_{j=1}^d = -2(\mathbb{I} + Q^T M Q)^{-1} Q^T M Q(\nu_{\nu} c_{\nu - e_j})_{j=1}^d
\]
for $\nu \in \mathbb{N}^d$ with $|\nu|$ odd. Moreover, $M = 0$ is equivalent to $C = \text{Id}$. In this case,
\[
\int_{\mathbb{R}^d} p_k(x+z)e^{-\frac{1}{\varepsilon}(x-q)^T\text{Im}(C)(x-q)}dx = (\pi\varepsilon)^{d/2}\text{det}(Q) \left(\frac{2}{\sqrt{\varepsilon}}Q^{-1}z\right)^{k}.
\]

**Proof.** We set $q_k^\nu(x) = |\varphi_0^\nu(x)|^{-2}(-\sqrt{\varepsilon}Q^*\nabla_x)^k|\varphi_0^\nu(x)|^2$ and verify, that $q_k^\nu$ satisfies the three-term recurrence [6]. By the Leibniz rule,
\[
(q_{k+\epsilon_j}^\nu(x))_j^d = |\varphi_0^\nu(x)|^{-2}((-\sqrt{\varepsilon}Q^*\nabla_x)^{k+\epsilon_j})_j^d |\varphi_0^\nu(x)|^2
\]
\[
= |\varphi_0^\nu(x)|^{-2}((-\sqrt{\varepsilon}Q^*\nabla_x)^k \frac{2}{\sqrt{\varepsilon}}Q^{-1}(x-q))|\varphi_0^\nu(x)|^2
\]
\[
= |\varphi_0^\nu(x)|^{-2} \sum_{\nu \leq k} \binom{k}{\nu} (-\sqrt{\varepsilon}Q^*\nabla_x)^\nu \frac{2}{\sqrt{\varepsilon}}Q^{-1}(x-q)(-\sqrt{\varepsilon}Q^*\nabla_x)^{k-\nu}|\varphi_0^\nu(x)|^2
\]
\[
= 2Q^{-1}|\varphi_0^\nu(x)|^{-2} \sum_{j=1}^d k_j ((Q^*\nabla_x)x)(-\sqrt{\varepsilon}Q^*\nabla_x)^{k-\epsilon_j} |\varphi_0^\nu(x)|^2
\]
\[
= \frac{2}{\sqrt{\varepsilon}}Q^{-1}(x-q)q_k^{\nu}(x) - 2Q^{-1}Q\left(k_jq_{k-\epsilon_j}(x)\right)_j^d.
\]
Moreover,
\[
(\text{Id} + Q^TMQ)^T = \frac{2}{\sqrt{\varepsilon}}\text{Id} + \frac{2}{\sqrt{\varepsilon}}Q^*CQ = \frac{2}{\sqrt{\varepsilon}}\text{Id} + \frac{2}{\sqrt{\varepsilon}}Q^*P = \text{Id} + \frac{2}{\sqrt{\varepsilon}}\text{Re}(Q^*P),
\]
where we have used the matrix condition [10]. Hence, $\text{Id} + Q^TMQ$ is invertible. For proving the claimed integral formula, we use the sum rule (16) and obtain
\[
\int_{\mathbb{R}^d} p_k(x+z)e^{-\frac{1}{\varepsilon}(x-q)^T\text{Im}(C)+M}(x-q)dx = \sum_{\nu \leq k} \binom{k}{\nu} \left(\frac{2}{\sqrt{\varepsilon}}Q^{-1}z\right)^{k-\nu} c_{\nu}
\]
with
\[
c_{\nu} = \int_{\mathbb{R}^d} p_k(x)e^{-\frac{1}{\varepsilon}(x-q)^T\text{Im}(C)+M}(x-q)dx.
\]
By the symmetry relation [14], $c_{\nu} = 0$ for $|\nu|$ odd. Moreover, by the Rodriguez formula and the three-term recurrence,
\[
(c_{\nu+\epsilon_j})_j^d = \int_{\mathbb{R}^d} e^{-\frac{1}{\varepsilon}(x-q)^T\text{M}(x-q)} \left((-\sqrt{\varepsilon}Q^T\nabla_x)^{\nu+\epsilon_j}\right)_j^d |\varphi_0^\nu(x)|^2 dx
\]
\[
= - \int_{\mathbb{R}^d} \frac{2}{\sqrt{\varepsilon}}Q^TM(x-q)e^{-\frac{1}{\varepsilon}(x-q)^T\text{M}(x-q)}(-\sqrt{\varepsilon}Q^T\nabla_x)^{\nu}|\varphi_0^\nu(x)|^2 dx
\]
\[
= -Q^TMQ\int_{\mathbb{R}^d} \frac{2}{\sqrt{\varepsilon}}Q^{-1}(x-q)p_k(x)e^{-\frac{1}{\varepsilon}(x-q)^T\text{Im}(C)+M}(x-q)dx
\]
\[
= -Q^TMQ(c_{\nu+\epsilon_j})_j^d = -2Q^TMQ(c_{\nu+\epsilon_j})_j^d = -2Q^TMQ(c_{\nu+\epsilon_j})_j^d.
\]
\]

**Remark 6.** In the univariate case, the previous integral formula allows for simplifications. We set $\frac{1}{\alpha_1} = \text{Im}(C) + M$ and $\alpha_2 = Q^2M/(1 + |Q|^2M)$ to obtain
\[
\int_{\mathbb{R}} p_k(x+z)e^{-\frac{1}{\alpha_1}(x-q)^2}dx = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \left(\frac{2}{\sqrt{\varepsilon}}Q^{-1}z\right)^{k-2j} c_j
\]
with
\[
c_j = -2\alpha_2(2j-1)c_{j-1} = 2^{j}(-\alpha_2)^j(2j-1) \cdot (2j-3) \cdots 1 \cdot c_{0}
\]
\[
= \sqrt{\pi\alpha_1}(-\alpha_2)^j(2j)!/j.
\]
The monomial representation of the Hermite polynomials \([7]\) then implies
\[
\int_{\mathbb{R}} p_k(x + z)e^{-\frac{1}{2\varepsilon^2}(x-q)^2} dx = \sqrt{\frac{2\pi\varepsilon}{\alpha_1\alpha_2}} \frac{k!}{\alpha_1\alpha_2} h_k\left(\frac{1}{\sqrt{2\pi\varepsilon}} z\right).
\]
For the Hermite polynomials, this formula is due to [16].

4. Hagedorn wavepackets in phase space

Our studies so far have provided two integral formulas for the polynomial part of the Hagedorn wavepackets. Now we apply them for computing the Wigner and FBI transform.

4.1. Wigner transform. Proposition’s integral connection to the Laguerre polynomials allows us to write the Wigner function of the Hagedorn wavepackets in terms of Gaussians and Laguerre polynomials depending on the complex vector
\[
z(x, \xi) = -i \left( P^T (x - q) - Q^T (\xi - p) \right), \quad x, \xi \in \mathbb{R}^d.
\]

Theorem 1. Let \( \varepsilon > 0, q, p \in \mathbb{R}^d \) and \( Q, P \in \mathbb{C}^{d \times d} \) satisfy (17). Then the scaled Wigner function of the \( k \)th and the \( l \)th Hagedorn wavepacket \( \varphi_k^\varepsilon \) and \( \varphi_l^\varepsilon \) satisfies
\[
W^\varepsilon(\varphi_k^\varepsilon, \varphi_l^\varepsilon)(x, \xi) = (\pi \varepsilon)^{d-1} e^{-\frac{1}{2} |z|^2} \left( \frac{-1}{\sqrt{2|k|+|l|}} \right)^{\frac{d}{2}} \prod_{j=1}^{d} L_{k_j, l_j} \left( \frac{1}{\sqrt{\varepsilon z_j}} \right)
\]
with \( z = -i \left( P^T (x - q) - Q^T (\xi - p) \right) \) for \( (x, \xi) \in \mathbb{R}^{2d} \) and
\[
L_{m, n}(\zeta) \equiv \begin{cases} 2^n m! L_{m-n}^{(2|\zeta|^2)} (2|\zeta|^2), & m \leq n, \\ 2^m n! (\zeta)^{m-n} L_{n-m}^{(2|\zeta|^2)} (2|\zeta|^2), & n \leq m. \end{cases}
\]
In particular,
\[
W^\varepsilon(\varphi_k^\varepsilon, \varphi_l^\varepsilon)(0, 0) = (\pi \varepsilon)^d e^{-\frac{1}{2} |z|^2} \prod_{j=1}^{d} L_{k_j, l_j}^{(0)} \left( \frac{z_j}{\sqrt{\varepsilon}} \right)
\]

Proof. We first study the Wigner transform in the origin. By Proposition 8
\[
W^\varepsilon(\varphi_k^\varepsilon, \varphi_l^\varepsilon)(0, 0)
= (\pi \varepsilon)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_k(y) p_l^\varepsilon(-y) \varphi_k^\varepsilon(y) \varphi_l^\varepsilon(-y) dy
= (\pi \varepsilon)^{-d} \left( \frac{-1}{\sqrt{2|k|+|l|}} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_k(\eta - q) p_l^\varepsilon(\eta + q) \varphi_k^\varepsilon(\eta - q) \varphi_l^\varepsilon(\eta + q) d\eta.
\]
The idea is to simplify the integrand such that the Laguerre connection of Proposition 8 can be applied. We compute
\[
\varphi_l^\varepsilon(\eta - q) \varphi_l^\varepsilon(-\eta - q)
= (\pi \varepsilon)^{-d/2} \det(Q)^{-1} e^{-\frac{1}{2|q|}(\eta - 2q)^T C(\eta - 2q)} + \frac{1}{2|q|} \eta^T C^2 q - \frac{2}{|q|} p^T (\eta - q)

= (\pi \varepsilon)^{-d/2} \det(Q)^{-1} e^{-\frac{1}{2} \eta^T C^2 q + \frac{2}{2|q|} \eta^T C q - \frac{2}{2|q|} \eta^T C q + \frac{2}{|q|} \eta^T C q - \frac{2}{|q|} \eta^T C q - \frac{2}{|q|} \eta^T C q}

The complex vector
\[
z = -i \left( P^T (-q) - Q^T (-p) \right)
\]
satisfies \( z = i Q^T (Cq - p) \), since \( C \) is complex symmetric. Due to \( \text{Im}(C) = (QQ^*)^{-1} \) and \( QQ^* = (QQ^*)^T = Q C Q^T \), we have \( C z = -q + i|q|^2 (\text{Re}(C) q - p) \). Moreover,
\[
|z|^2 = \bar{z}^T z = q^T \text{Im}(C) q + (\text{Re}(C) q - p)^T |q|^2 (\text{Re}(C) q - p).
\]
We set \( w = -i |\mathcal{Q}z| = -i |Q^2 (\text{Re}(C)q - p) | \) and compute
\[
|\varphi_0^\epsilon(\eta + w)|^2 = (\pi \epsilon)^{-d/2} \det(\mathcal{Q})^{-1} e^{-\frac{1}{2}(\eta + w - q)^T \text{Im}(\mathcal{Q}) (\eta + w - q)}
\]
\[
= (\pi \epsilon)^{-d/2} \det(\mathcal{Q})^{-1} e^{-\frac{1}{2}q^T \text{Im}(\mathcal{Q}) q + \frac{1}{2} (\text{Re}(C)q - p)^T |Q|^2 (\text{Re}(C)q - p)}
\]
\[
e^{\frac{1}{\epsilon} \eta^T \text{Im}(\mathcal{Q}) \eta + \frac{2}{\epsilon} \eta^T \text{Im}(\mathcal{Q}) \eta + \frac{2}{\epsilon} (\eta - q)^T (\text{Re}(C)q - p)}.
\]
Therefore,
\[
\mathcal{V}_{\epsilon}(\eta + q) = e^{-\frac{1}{\epsilon} |z|^2} |\varphi_0^\epsilon(\eta + w)|^2
\]
and
\[
\mathcal{W}_{\epsilon}(\varphi_0^\epsilon, \varphi_0^\epsilon)(0, 0) = (\pi \epsilon)^{-d} \frac{(-1)|w - \frac{1}{\epsilon} |z|^2|^d}{\sqrt{2^{k+d} |z|^d}} \prod_{j=1}^d \mathcal{L}_{k_j, l_j} \left( -\frac{1}{\sqrt{\epsilon}} (\mathcal{Q}^{-1} q + w)_{lj}, \frac{1}{\sqrt{\epsilon}} (\mathcal{Q}^{-1} q - w)_{lj} \right)
\]
\[
= (\pi \epsilon)^{-d} \frac{(-1)|w - \frac{1}{\epsilon} |z|^2|^d}{\sqrt{2^{k+d} |z|^d}} \prod_{j=1}^d \mathcal{L}_{k_j, l_j} \left( \frac{1}{\sqrt{\epsilon}} z_j, -\frac{1}{\sqrt{\epsilon}} \right),
\]
due to analyticity and exponential decay of the integrand. By Proposition 3, we use Lemma 2 and obtain
\[
\mathcal{W}_{\epsilon}(\varphi_0^\epsilon, \varphi_0^\epsilon)(x, \xi) = \mathcal{W}_{\epsilon}(\varphi_0^\epsilon[q - x, p - \xi, Q, P], \varphi_0^\epsilon[q - x, p - \xi, Q, P])(0, 0).
\]
\[
\square
\]
To generalize the observation, that the Hermite function’s Wigner function only depends on the energy variable \(|z|^2\), see Remark 2, one has to combine the Hagedorn wavepackets for the \(|k|\)th eigenspace. We set \( \Phi_{k|}^\epsilon = \Phi_{k|}^\epsilon[q, p, Q, P] \),
\[
\Phi_{k|}^\epsilon(x) = (\varphi_{k_j}^\epsilon(x), \ldots, \varphi_{k_l}^\epsilon(x))_{j_1 + \cdots + l_\alpha = |k|} \in \mathbb{C}^N, \quad x \in \mathbb{R}^d,
\]
and move to the associated matrix-valued Wigner transform
\[
\mathcal{W}_{\epsilon}(\Phi_{k|}^\epsilon, \Phi_{k|}^\epsilon)(x, \xi) = (2\pi \epsilon)^{-d} \int_{\mathbb{R}^d} \mathcal{V}_{\epsilon}[k|x + \frac{\xi}{2}] \Phi_{k|}^\epsilon(x - \frac{\xi}{2})^T e^{iy^T \xi / \epsilon} dy \in \mathbb{C}^{N \times N}.
\]

**Corollary 3.** Let \( \epsilon > 0 \), \( q, p \in \mathbb{R}^d \) and \( Q, P \in \mathbb{C}^{d \times d} \) satisfy (17). Then, the Hagedorn wavepackets’ vector for the \(|k|\)th eigenspace, \( \Phi_{k|}^\epsilon = \Phi_{k|}^\epsilon[q, p, Q, P], \) \( k \in \mathbb{N}^d \), satisfies
\[
\text{tr} \left( \mathcal{W}_{\epsilon}(\Phi_{k|}^\epsilon, \Phi_{k|}^\epsilon)(x, \xi) \right) = (\frac{-1}{\pi \epsilon})^d e^{-\frac{1}{\epsilon} |z|^2} L^{(d-1)}(\frac{2}{\epsilon} |z|^2)
\]
with \( z = -i (PT(x - q) - QT(\xi - p)) \) for \( (x, \xi) \in \mathbb{R}^{2d} \).
4.2. Three-term recurrence relation. The Hermite functions’ ladder operators can be translated to the Wigner function level, see [T1, Theorem 1.3.3]. The same is possible for the Hagedorn wavepackets and provides a useful three-term recurrence in phase space.

**Theorem 2.** Let \( \varepsilon > 0 \), \( q, p \in \mathbb{R}^d \), and \( Q, P \in \mathbb{C}^{d \times d} \) satisfy (11). Then, the Wigner transform \( W_{kl}^\varepsilon = W^\varepsilon(\varphi_k^\varepsilon, \varphi_l^\varepsilon) \) of the \( k \)th and \( l \)th Hagedorn wavepackets \( \varphi_k^\varepsilon = \varphi_k^\varepsilon[q, Q, P] \) and \( \varphi_l^\varepsilon = \varphi_l^\varepsilon[q, p, Q] \), \( k, l \in \mathbb{N}^d \), satisfies

\[
K^j W_{kl}^\varepsilon = 2\sqrt{K_{j+1}} W_{k+l,j}^\varepsilon, \quad K_l W_{kl}^\varepsilon = 2\sqrt{K_j} W_{k-l,j}^\varepsilon,
\]

\[
L^j W_{kl}^\varepsilon = 2\sqrt{L_{j+1}} W_{k+l,j}^\varepsilon, \quad L_l W_{kl}^\varepsilon = 2\sqrt{L_j} W_{k-l,j}^\varepsilon,
\]

for \( j = 1, \ldots, d \) with

\[
K^\dagger = -\frac{i}{\sqrt{2\pi}} \left( P^T (2(x - q) + (-i\varepsilon \nabla_x)) - Q^T (2(\xi - p) + (-i\varepsilon \nabla_x)) \right),
\]

\[
K = -\frac{i}{\sqrt{2\pi}} \left( P^* (2(x - q) + (-i\varepsilon \nabla_x)) - Q^* (2(\xi - p) + (-i\varepsilon \nabla_x)) \right),
\]

and

\[
L^\dagger = -\frac{i}{\sqrt{2\pi}} \left( P^T (2(x - q) + (-i\varepsilon \nabla_x)) - Q^* (2(\xi - p) + (-i\varepsilon \nabla_x)) \right),
\]

\[
L = -\frac{i}{\sqrt{2\pi}} \left( P^* (2(x - q) + (-i\varepsilon \nabla_x)) - Q^T (2(\xi - p) + (-i\varepsilon \nabla_x)) \right).
\]

**Proof.** We compute

\[
(-i\varepsilon \nabla_x) W_{kl}^\varepsilon(x, \xi) = -(2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} (-i\varepsilon \nabla_x - p) \varphi_k^\varepsilon(x + \frac{\xi}{2}) \varphi_l^\varepsilon(x - \frac{\xi}{2}) e^{\frac{i}{\varepsilon} T y} dy
\]

\[
+ (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \varphi_k^\varepsilon(x + \frac{\xi}{2}) (-i\varepsilon \nabla_x - p) \varphi_l^\varepsilon(x - \frac{\xi}{2}) e^{\frac{i}{\varepsilon} T y} dy
\]

and

\[
(-i\varepsilon \nabla_\xi) W_{kl}^\varepsilon(x, \xi) = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \varphi_k^\varepsilon(x + \frac{\xi}{2} - q) \varphi_l^\varepsilon(x + \frac{\xi}{2}) e^{\frac{i}{\varepsilon} T y} dy
\]

\[
- (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \varphi_k^\varepsilon(x + \frac{\xi}{2}) (x - \frac{\xi}{2} - q) \varphi_l^\varepsilon(x - \frac{\xi}{2}) e^{\frac{i}{\varepsilon} T y} dy.
\]

Since

\[
A^\dagger = -\frac{i}{\sqrt{2\pi}} \left( P^* (x - q) - Q^* (-i\varepsilon \nabla_x - p) \right),
\]

\[
A = -\frac{i}{\sqrt{2\pi}} \left( P^T (x - q) - Q^T (-i\varepsilon \nabla_x - p) \right),
\]

we obtain

\[
-\frac{i}{\sqrt{2\pi}} \left( P^* (-i\varepsilon \nabla_x) + Q^* (-i\varepsilon \nabla_\xi) \right) W_{kl}^\varepsilon(x, \xi) = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \left( A \varphi_k^\varepsilon(x + \frac{\xi}{2}) \varphi_l^\varepsilon(x - \frac{\xi}{2}) + \varphi_k^\varepsilon(x + \frac{\xi}{2}) A^\dagger \varphi_l^\varepsilon(x - \frac{\xi}{2}) \right) e^{\frac{i}{\varepsilon} T y} dy
\]
and
\[ \frac{1}{\sqrt{2\pi}} (P^T(-i\xi \nabla_x) + Q^T(-i\xi \nabla_x)) W_k^\xi(x, \xi) = \] 
\[ (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \left( A_1 \varphi_k^\xi(x + \frac{\xi}{2}) \varphi_f^\xi(x - \frac{\xi}{2}) + \varphi_k^\xi(x + \frac{\xi}{2}) A \varphi_f^\xi(x - \frac{\xi}{2}) \right) e^{\frac{i}{\varepsilon} \xi^T y} dy. \]

Using \( A_1^\xi \varphi_k^\xi = \sqrt{k_j + 1} \varphi_{k+e_j}^\xi \) and \( A \varphi_k^\xi = \sqrt{k_j} \varphi_{k-e_j}^\xi \), we arrive at
\[ -\frac{1}{\sqrt{2\pi}} (P^T(-i\xi \nabla_x) + Q^T(-i\xi \nabla_x)) W_k^\xi = \sqrt{k_j} W_k^\xi, \xi - e_j + \sqrt{l_j} + 1 W_{k,l+e_j}^\xi. \]
\[ \frac{1}{\sqrt{2\pi}} (P^T(-i\xi \nabla_x) + Q^T(-i\xi \nabla_x)) W_k^\xi \] 
\[ = \sqrt{k_j + 1} W_{k+e_j,l}^\xi + \sqrt{l_j} W_{k,l-e_j}^\xi. \]

Moreover,
\[ 2(x - q) W_k^\xi(x, \xi) = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \left( (x + \frac{\xi}{2} - q) \varphi_k^\xi(x + \frac{\xi}{2}) \varphi_f^\xi(x - \frac{\xi}{2}) + \varphi_k^\xi(x + \frac{\xi}{2}) (x - \frac{\xi}{2} - q) \right) e^{\frac{i}{\varepsilon} \xi^T y} dy \]
\[ + (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \left( \varphi_k^\xi(x + \frac{\xi}{2}) (x - \frac{\xi}{2} - q) \varphi_f^\xi(x - \frac{\xi}{2}) + \varphi_k^\xi(x + \frac{\xi}{2}) e^{\frac{i}{\varepsilon} \xi^T y} \right) dy. \]

Since \( \xi e^{\frac{i}{\varepsilon} \xi^T y} = -i \xi \nabla_y e^{\frac{i}{\varepsilon} \xi^T y} \), an integration by parts yields
\[ 2(x - q) W_k^\xi(x, \xi) = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \left( -i \xi \nabla_x - p \right) \varphi_k^\xi(x + \frac{\xi}{2}) \varphi_f^\xi(x - \frac{\xi}{2}) e^{\frac{i}{\varepsilon} \xi^T y} dy \]
\[ + (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \varphi_k^\xi(x + \frac{\xi}{2}) (-i \xi \nabla_x - p) \varphi_f^\xi(x - \frac{\xi}{2}) e^{\frac{i}{\varepsilon} \xi^T y} dy, \]

Consequently,
\[ -\frac{1}{\sqrt{2\pi}} (2P^T(x - q) - 2Q^T(\xi - p)) W_k^\xi(x, \xi) = \] 
\[ (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \left( -A_1 \varphi_k^\xi(x + \frac{\xi}{2}) \varphi_f^\xi(x - \frac{\xi}{2}) + \varphi_k^\xi(x + \frac{\xi}{2}) A_1 \varphi_f^\xi(x - \frac{\xi}{2}) \right) e^{\frac{i}{\varepsilon} \xi^T y} dy \]
and
\[ -\frac{1}{\sqrt{2\pi}} (2P^T(x - q) - 2Q^T(\xi - p)) W_k^\xi(x, \xi) = \] 
\[ (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \left( -A \varphi_k^\xi(x + \frac{\xi}{2}) \varphi_f^\xi(x - \frac{\xi}{2}) + \varphi_k^\xi(x + \frac{\xi}{2}) A \varphi_f^\xi(x - \frac{\xi}{2}) \right) e^{\frac{i}{\varepsilon} \xi^T y} dy \]

This implies
\[ -\frac{1}{\sqrt{2\pi}} (2P^T(x - q) - 2Q^T(\xi - p)) W_k^\xi(x, \xi) \]
\[ = -\sqrt{k_j} W_{k-e_j,l}^\xi + \sqrt{l_j} + 1 W_{k+e_j,l}^\xi, \]
\[ -\frac{1}{\sqrt{2\pi}} (2P^T(x - q) - 2Q^T(\xi - p)) W_k^\xi(x, \xi) \]
\[ = -\sqrt{k_j} + 1 W_{k+e_j,l}^\xi + \sqrt{l_j} W_{k-l-e_j}^\xi. \]

Adding and subtracting the above raising and lowering identities for \( W_k^\xi \) gives the operators \( K^\xi, K^\xi, L^\xi, L^\xi \), and \( L^\xi \).

The phase space ladder allows to reformulate the Hagedorn wavepackets’ three-term recurrence relation (12) for the Wigner transform.

**Corollary 4.** Let \( \varepsilon > 0, q, p \in \mathbb{R}^d \), and \( Q, P \in \mathbb{C}^{d \times d} \) be matrices satisfying (10). Then, the Wigner transform \( W_k^\xi = \mathcal{W}^\xi(\varphi_k^\xi, \varphi_f^\xi) \) of the kth and the lth Hagedorn wavepackets \( \varphi_k^\xi = \varphi_k^\xi[q, p, Q, P] \) and \( \varphi_f^\xi = \varphi_f^\xi[q, p, Q, P] \), \( k, l \in \mathbb{N}^d \), satisfies
\[ \left( \sqrt{k_j + 1} W_{k+e_j,l}^\xi(x, \xi) \right)_{j=1}^d = -\sqrt{z} W_k^\xi(x, \xi) + \left( \sqrt{l_j} W_{k-l-e_j}^\xi(x, \xi) \right)_{j=1}^d \]
\[ \left( \sqrt{l_j} + 1 W_{k+e_j,l}^\xi(x, \xi) \right)_{j=1}^d = \sqrt{z} W_k^\xi(x, \xi) + \left( \sqrt{k_j} W_{k-l-e_j}^\xi(x, \xi) \right)_{j=1}^d \]
with \( z = -i (P^T(x - q) - Q^T(\xi - p)) \) for \( (x, \xi) \in \mathbb{R}^{2d} \).
Proof. We observe that

\[
\begin{align*}
\left( K_j - L_j \right) W_{kl}^c &= \frac{2}{\sqrt{2\pi}} \left( 2P^T(x - q) - 2Q^T(\xi - p) \right)_{j} W_{kl}^c \\
&= 2\sqrt{k_j} + 1W_{kl+1}^c - 2\sqrt{l_j}W_{kl-1}^c,
\end{align*}
\]

\[
\begin{align*}
\left( L_j - K_j \right) W_{kl}^c &= \frac{2}{\sqrt{2\pi}} \left( 2P^*(x - q) - 2Q^*(\xi - p) \right)_{j} W_{kl}^c \\
&= 2\sqrt{l_j} + 1W_{kl+1}^c - 2\sqrt{k_j}W_{kl-1}^c.
\end{align*}
\]

\[
\square
\]

The numerical computation of the Wigner transform of a Schwartz function \( \psi : \mathbb{R}^d \rightarrow \mathbb{C} \) for a set of phase space points \( \zeta_1, \ldots, \zeta_N \in \mathbb{R}^{2d} \) is notoriously difficult, since a direct approach poses the numerical quadrature of \( N \) Fourier integrals in possibly high dimensions,

\[
W^c(\psi, \psi)(x, \xi) = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \overline{\psi}(x + \frac{n}{2})\psi(x - \frac{n}{2})e^{i\varepsilon T\xi / \varepsilon} dy
\]

for \( (x, \xi) \in \{ \zeta_1, \ldots, \zeta_N \} \). The three-term recurrence of Corollary 4 might provide an efficient alternative, especially when a large number \( N \gg 1 \) of evaluations is required: Choosing suitable \( q, p \in \mathbb{R}^d, Q, P \in \mathbb{C}^{d \times d} \) satisfying \( \text{H1}, K \in \mathbb{N} \), and the hyperbolic multi-index set

\[
\mathcal{K} = \{ k \in \mathbb{N}^d : \prod_{j=1}^d (1 + k_j) \leq K \}.
\]

Then, one approximates

\[
\psi \approx \sum_{k \in \mathcal{K}} c_k \varphi^c_k, \quad c_k = \int_{\mathbb{R}^d} \overline{\varphi^c_k(y)}\psi(y)dy
\]

with \( \varphi^c_k = \varphi^c_k[q, p, Q, P] \) for \( k \in \mathcal{K} \). With literally the same proof as for the Hermite functions [L99, Theorem 1.5], the approximation error can be estimated as

\[
\|\psi - \sum_{k \in \mathcal{K}} c_k \varphi^c_k \|_{L^2(\mathbb{R}^d)} \leq C_{s,d}K^{-s/2} \max_{\|\sigma\|_{\infty} \leq s} \|A[q, p, Q, P]^s\psi\|_{L^2(\mathbb{R}^d)}
\]

for fixed \( s \in \mathbb{N} \) and every Schwartz function \( \psi : \mathbb{R}^d \rightarrow \mathbb{C} \). From the Wigner transform’s bilinearity and the orthogonality relation \( \text{H1} \) we deduce

\[
\begin{align*}
\|W^c(\varphi_1) - W^c(\varphi_2)\|_{L^2(\mathbb{R}^{2d})} &\leq \|W^c(\varphi_1 - \varphi_2)\|_{L^2(\mathbb{R}^{2d})} + \|W^c(\varphi_1 - \varphi_2)\|_{L^2(\mathbb{R}^{2d})} \\
&= \|\varphi_1 - \varphi_2\|_{L^2(\mathbb{R}^d)} (\|\varphi_1\|_{L^2(\mathbb{R}^d)} + \|\varphi_2\|_{L^2(\mathbb{R}^d)})
\end{align*}
\]

for all Schwartz functions \( \varphi_1, \varphi_2 : \mathbb{R}^d \rightarrow \mathbb{C} \). Therefore, evaluating the Wigner function of \( \psi \) via

\[
W^c(\psi, \psi) \approx \sum_{k, l \in \mathcal{K}} \overline{c_k c_l}W^c(\varphi^c_k, \varphi^c_l),
\]

together with the three-term recurrence relation of Corollary 4, we inherit the approximation accuracy of \( O(K^{-s/2}) \).

4.3. FBI transform. Our general findings for the FBI transform of the Hagedorn wavepackets are less beautiful than those for the Wigner function.
Proposition 5. Let \( \epsilon > 0, q, p \in \mathbb{R}^d \) and \( Q, P \in \mathbb{C}^{d \times d} \) satisfy (17). Let \( \varphi_k^z = \varphi_k^z[q, p, Q, P] \) be the \( k \)-th Hagedorn wavepacket. Then the scaled FBI transform is

\[
T^\varepsilon(\varphi_k^z)(x, \xi) = e^{\frac{i}{\varepsilon}(x, \xi) - \frac{1}{2\varepsilon}w^T(iC + Id)w} \sum_{\nu \leq k} \left( \frac{1}{\sqrt{\varepsilon}}Q^{-1}w \right)^k \epsilon^{-\nu} c_{\nu}
\]

with \( w = (iC + Id)^{-1}((x - q) + i(\xi - p)) \) for \( (x, \xi) \in \mathbb{R}^{2d} \). In particular, if \( C = \text{Id} \), then

\[
T^\varepsilon(\varphi_k^z)(x, \xi) = e^{\frac{i}{\varepsilon}(x, \xi) - \frac{1}{2\varepsilon}w^T(iC + Id)w} \sum_{\nu \leq k} \left( \frac{1}{\sqrt{\varepsilon}}Q^{-1}w \right)^k \epsilon^{-\nu} c_{\nu}
\]

and

\[
H^\varepsilon(\varphi_k^z, \varphi_k^z)(x, \xi) = \left( \frac{1}{\varepsilon} \right)^{d/2} e^{\frac{i}{\varepsilon}(x, \xi) - \frac{1}{2\varepsilon}w^T(iC + Id)w} e^{\frac{1}{\varepsilon}w^T(iC + Id)w} - \frac{1}{\varepsilon}|w|^2 dy.
\]

Proof. We start for \( (x, \xi) = (0, 0) \) and obtain

\[
T^\varepsilon(\varphi_k^z)(0, 0) = \frac{\det(Q)^{-1/2}}{\varepsilon^{d/2} (2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi_k(y) e^{-\frac{i}{\varepsilon}(y - q)^T C(y - q) - \frac{1}{2\varepsilon}|y|^2} dy.
\]

We compute

\[
-\frac{1}{\varepsilon}(y - q)^T C(y - q) - \frac{1}{\varepsilon}p^T(y - q) - \frac{1}{\varepsilon}|y|^2
\]

\[
= -\frac{1}{\varepsilon}(y - q)^T (iC + Id)(y - q) - \frac{1}{\varepsilon}(ip + q)^T(y - q) - \frac{1}{\varepsilon}|y|^2
\]

\[
= -\frac{1}{\varepsilon}(y - q - w)^T (iC + Id)(y - q - w) + \frac{1}{\varepsilon}w^T(iC + Id)w - \frac{1}{\varepsilon}|w|^2
\]

with \( w = -(iC + Id)^{-1}(ip + q) \). Therefore,

\[
T^\varepsilon(\varphi_k^z)(0, 0) = \frac{\det(Q)^{-1/2}}{\varepsilon^{d/2} (2\pi)^{d/2}} e^{-\frac{1}{\varepsilon}|w|^2} \int_{\mathbb{R}^d} \phi_k(y + w) e^{\frac{1}{\varepsilon}w^T(iC + Id)(y - q) dy}.
\]

Since \( \frac{1}{2}(iC + Id) = \text{Im}(C) + M \), we have by Proposition 4

\[
T^\varepsilon(\varphi_k^z)(0, 0) = \frac{\det(Q)^{-1/2}}{\varepsilon^{d/2} (2\pi)^{d/2}} e^{-\frac{1}{\varepsilon}|w|^2} \int_{\mathbb{R}^d} \phi_k(y + w) e^{\frac{1}{\varepsilon}w^T(iC + Id)(y - q) dy}.
\]

For arbitrary \( (x, \xi) \in \mathbb{R}^{2d} \), we use Lemma 2 and obtain

\[
T^\varepsilon(\varphi_k^z)(x, \xi) = e^{\frac{i}{\varepsilon}(x, \xi) - \frac{1}{2\varepsilon}w^T(iC + Id)w} \sum_{\nu \leq k} \left( \frac{1}{\sqrt{\varepsilon}}Q^{-1}w \right)^k \epsilon^{-\nu} c_{\nu}.
\]

In the special case \( C = \text{Id} \), we have \( QQ^* = \text{Id} \), \( P = 0 \), \( w = \frac{1}{2}((x - q) + i(\xi - p)) \) and \( c_{\nu} = 0 \) for \( \nu \neq 0 \). Consequently,

\[
T^\varepsilon(\varphi_k^z)(x, \xi) = \frac{e^{\frac{i}{\varepsilon}(x, \xi) - \frac{1}{2\varepsilon}w^T(iC + Id)w}}{\varepsilon^{d/2} (2\pi)^{d/2}} e^{-\frac{1}{\varepsilon}|w|^2} \sum_{\nu \leq k} \left( \frac{1}{\sqrt{\varepsilon}}Q^{-1}w \right)^k \epsilon^{-\nu} c_{\nu}.
\]

with \( z = (x - q) + i(\xi - p) \).
Appendix A. Polar decomposition

The following observations are not needed for computing the phase space transforms of the Hagedorn wavepackets. However, they shed further light on the relation to the Hermite polynomials in the general case of invertible $Q \in \mathbb{C}^{d \times d}$.

Lemma 3. Let $\varepsilon > 0$, $q, p \in \mathbb{R}^d$, and $Q, P \in \mathbb{C}^{d \times d}$ be matrices satisfying (10). If we decompose $Q = \vert Q \vert U^*$ with $\vert Q \vert = (Q^*Q)^{1/2}$ and $U \in \mathbb{C}^{d \times d}$ a unitary matrix, then $PQ^{-1} = (PU)\vert Q \vert^{-1}$, and the pair $\vert Q \vert$, $PU$ satisfies condition (10). Moreover, (19) $A^\dagger[q, p, Q, P] = UA^\dagger[q, p, \vert Q \vert, PU]$.

Proof. We observe

\[
0 = Q^TP - P^TQ = (\vert Q \vert U^*)^TP - P^T\vert Q \vert U^*
\]

\[
2i\text{Id} = Q^*P - P^*Q = (\vert Q \vert U^*)^*P - P^*\vert Q \vert U^*
\]

Therefore, $\vert Q \vert^T(PU) - (PU)^T\vert Q \vert = 0$ and $\vert Q \vert^*(PU) - (PU)^*\vert Q \vert = 2i\text{Id}$. For the ladder operators we compute

\[
A^\dagger[q, p, \vert Q \vert, PU] = \frac{1}{\sqrt{2\pi}}(\vert PU \vert^\dagger \text{op}_x(x - q) - \vert Q \vert^* \text{op}_x(\xi - p)) = \frac{1}{\sqrt{2\pi}}U^*(P^* \text{op}_x(x - q) - Q^* \text{op}_x(\xi - p)) = U^*A^\dagger[q, p, Q, P].
\]

The unitary relation (19) of the ladder operators provides information on the Hagedorn wavepackets associated with indizes of the same modulus: We first enumerate the multi-indizes $k \in \mathbb{N}^d$ of equal modulus redundantly by setting $\tilde{v}_0 = (0, \ldots, 0) \in \mathbb{N}^d$,

\[
\tilde{v}_{|k|+1} = \text{vec}
\begin{pmatrix}
\tilde{v}_{|k|,1} + e_1 & \cdots & \tilde{v}_{|k|,1} + e_d \\
\vdots & \ddots & \vdots \\
\tilde{v}_{|k|,d|k|} + e_1 & \cdots & \tilde{v}_{|k|,d|k|} + e_d
\end{pmatrix}, \quad \forall k \in \mathbb{N},
\]

such that $\tilde{v}_{|k|+1}$ is a vector of length $d^{|k|+1}$, whose entries are multi-indizes in $\mathbb{N}^d$. Then, we mark repeated occurrences of multi-indizes by setting

\[
\nu_{|k|,j} = \begin{cases}
\infty & \exists j' < j : \tilde{v}_{|k|,j'} = \tilde{v}_{|k|,j} \\
\tilde{v}_{|k|,j} & \text{otherwise},
\end{cases}
\]

for $j = 1, \ldots, d^{|k|}$. This redundant book-keeping allows to reformulate the creation process (11) on the level of the $|k|$th eigenspace as $\tilde{\varphi}_0 = \tilde{\varphi}_0^\xi$,

\[
\tilde{\varphi}_|k|+1 = \text{vec}
\begin{pmatrix}
\frac{1}{\sqrt{(|\nu_{|k|,1}|+1)}}A^\dagger_{|k|,1}\tilde{\varphi}_{|k|,1} & \cdots & \frac{1}{\sqrt{(|\nu_{|k|,1}|+1)}}A^\dagger_{|k|,1}\tilde{\varphi}_{|k|,1} \\
\vdots & \ddots & \vdots \\
\frac{1}{\sqrt{(|\nu_{|k|,d|k|}|+1)}}A^\dagger_{|k|,d|k|}\tilde{\varphi}_{|k|,d|k|} & \cdots & \frac{1}{\sqrt{(|\nu_{|k|,d|k|}|+1)}}A^\dagger_{|k|,d|k|}\tilde{\varphi}_{|k|,d|k|}
\end{pmatrix}.
\]

We note that the normalization with $1/\sqrt{(\nu_{|k|,j})+1}$ produces a zero whenever a multi-index is repeated in the enumeration.

Proposition 6. Let $\varepsilon > 0$, $q, p \in \mathbb{R}^d$, and $Q, P \in \mathbb{C}^{d \times d}$ satisfy (10). We decompose $Q = \vert Q \vert U^*$ with unitary $U \in \mathbb{C}^{d \times d}$. Then, $\vert Q \vert = \vert Q \vert^T \in \mathbb{R}^{d \times d}$, and (20) $\tilde{\varphi}_|k| \otimes \tilde{\varphi}_|k| \otimes [q, p, \vert Q \vert, PU]$, where $U \otimes \cdots \otimes U \in \mathbb{C}^{d^{|k|} \times d^{|k|}}$ denotes the $|k|$-fold Kronecker product of $U$ with itself.
Proof. We set
\[ A^\dagger = A^\dagger[q, p, Q, P], \quad \vec{\varphi}_{[k]} = \vec{\varphi}_{[k]}[q, p, Q, P], \]
\[ D^\dagger = A^\dagger[q, p, |Q|, PU], \quad \vec{\psi}_{[k]} = \vec{\varphi}_{[k]}[q, p, |Q|, PU]. \]
By the relation (19), \( A_j^\dagger = (UD^\dagger)_j = u_j^T D^\dagger \) with \( u_1, \ldots, u_d \in \mathbb{C}^d \) the row vectors of \( U \). Moreover, for arbitrary \( W \in \mathbb{C}^{m \times m}, w \in \mathbb{C}^m \), and \( a \in \mathbb{C}^d \)
\[ \begin{pmatrix} (u_1^T a)^w w \\ \vdots \\ (u_d^T a)^w w \end{pmatrix} = \begin{pmatrix} (u_1 a_1)^w w + \cdots + (u_d a_d)^w w \\ \vdots \\ (u_1 a_1)^w w + \cdots + (u_d a_d)^w w \end{pmatrix} = \begin{pmatrix} a_1 w \\ \vdots \\ a_d w \end{pmatrix} = U \otimes W \begin{pmatrix} a_1 w \\ \vdots \\ a_d w \end{pmatrix}. \]
Assuming that the claimed identity (20) holds for \( |k| \), we therefore obtain
\[ \vec{\varphi}_{[k]+1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} A_{\text{op}}^\dagger \vec{\varphi}_{|k|, 1} \\ \vdots \\ A_d^\dagger \vec{\varphi}_{|k|, d+1} \end{pmatrix} = \begin{pmatrix} (u_1^T D)^\dagger U^{|k|} \vec{\psi}_{[k]} \\ \vdots \\ (u_d^T D)^\dagger U^{|k|} \vec{\psi}_{[k]} \end{pmatrix} = \begin{pmatrix} u_1 \otimes U^{|k|} \vec{\psi}_{[k]+1} \\ \vdots \\ u_d \otimes U^{|k|} \vec{\psi}_{[k]+1} \end{pmatrix}. \]

\[ \Box \]

Appendix B. Weyl quantization

The raising and lowering operators \( A^\dagger \) and \( A \) as well as the generalized harmonic oscillator \( \frac{1}{2} (A^T A^\dagger + (A^\dagger A)^T A) \) of the Hagedorn wavepackets can be viewed as Weyl quantized operators obtained from smooth phase space functions of subquadratic growth \( a : \mathbb{R}^{2d} \rightarrow \mathbb{C}^d \),
\[ (\text{op}_\varepsilon(a) \varphi)(x) = (2 \pi \varepsilon)^{-d} \int_{\mathbb{R}^{2d}} a(x + y, \xi) e^{i T(x + y) \xi / \varepsilon} \varphi(y) dyd\xi \]
for Schwartz functions \( \varphi : \mathbb{R}^d \rightarrow \mathbb{C} \). The emerging phase space function
\[ z(x, \xi) = -i (P^T(x - q) - Q^T(\xi - p)) \]
generalizes the complex number \( z(x, \xi) = x + i \xi \), which characterizes the Wigner and FBI transform of the Hermite functions. It also appears in the Wigner function of the Hagedorn wavepackets.

Lemma 4. Let \( \varepsilon > 0, q, p \in \mathbb{R}^d, \) and \( Q, P \in \mathbb{C}^{d \times d} \) be matrices satisfying (17). Let \( A^\dagger = A^\dagger[q, p, Q, P] \) and \( A = A[q, p, Q, P] \). Then,
\[ A^\dagger = \frac{1}{\sqrt{2\varepsilon}} \text{op}_\varepsilon(z), \quad A = \frac{1}{\sqrt{2\varepsilon}} \text{op}_\varepsilon(z), \quad \frac{1}{2} \sum_{j=1}^d (A_j A_j^\dagger + A_j^\dagger A_j) = \frac{1}{2\varepsilon} \text{op}_\varepsilon(|z|^2) \]
with \( z(x, \xi) = -i (P^T(x - q) - Q^T(\xi - p)) \) for \( x, \xi \in \mathbb{R}^d \) and \(|z|^2 = z^T z\).
Proof. We observe

\[ |z|^2 = (x - q)^T P |x - q|^2 + (\xi - p)^T |\xi - p|^2 - (x - q)^T M (\xi - p) \]

with \( M = PQ^* + QP^T \in \mathbb{R}^{d \times d} \). Moreover,

\[
\frac{1}{2} (A^T A + (A^1)^T A) = \\
\frac{1}{2} \left( \text{op}_\varepsilon (x - q)^T |P|^2 \text{op}_\varepsilon (x - q) + \text{op}_\varepsilon (\xi - p)^T |Q|^2 \text{op}_\varepsilon (\xi - p) \right) \\
- \text{op}_\varepsilon (x - q)^T M \text{op}_\varepsilon (\xi - p) - (M \text{op}_\varepsilon (\xi - p))^T \text{op}_\varepsilon (x - q) \\
\]

and

\[
\text{op}_\varepsilon (x - q)^T M \text{op}_\varepsilon (\xi - p) = \text{op}_\varepsilon ((x - q)^T M (\xi - p)) - \varepsilon \frac{1}{2} \text{tr} M, \\
(M \text{op}_\varepsilon (\xi - p))^T \text{op}_\varepsilon (x - q) = \text{op}_\varepsilon ((x - q)^T M (\xi - p)) + \varepsilon \frac{1}{2} \text{tr} M. \\
\]

\[ \square \]

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