INVARIANTS OF PARTITIONS AND REPRESENTATIVE ELEMENTS

BAO SHOU† AND QIAO WU∗

Abstract. The symbol invariant is used to describe the Springer correspondence for the classical groups by Lusztig. And the fingerprint invariant can be used to describe the Kazhdan-Lusztig map. They are invariants of rigid semisimple operators described by pairs of partitions (λ′, λ′′). We construct a nice representative element of the rigid semisimple operators with the same symbol invariant. The fingerprint of the representative element can be obtained immediately. We also discuss the representative element of rigid semisimple operator with the same fingerprint invariant. Our construction can be regarded as the maps between these two invariants.

Contents

1. Introduction 1
2. Rigid Partitions in the $B_\alpha$, $C_\alpha$, and $D_\alpha$ theories 2
3. Symbol invariant of partitions
   3.1. Symbol invariant and the construction 3
   3.2. Representative element of rigid semisimple operators with the same symbol 4
4. Fingerprint invariant of partitions
   4.1. Fingerprint invariant 7
   4.2. Construction of the fingerprint invariant 8
   4.3. Representative element of rigid semisimple operators with the same fingerprint 10
   4.4. Fingerprint of the representative element $(\lambda', \lambda'')_R$ 11
5. $C_\alpha$ and $D_\alpha$ theories 11

References 11

1. Introduction

A partition $\lambda$ of the positive integer $n$ is a decomposition $\sum_{i=1}^{l} \lambda_i = n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$), with the length $l$. There is an one to one correspondence between the partitions and Young tableaux. Young tableaux occur in a number of branches of mathematics and physics. They are also tools for the construction of the eigenstates of Hamiltonian System [28] [29] [30] and label surface operators [3].

Surface operators are two-dimensional defects, which are a natural generalisations of the 't Hooft operators. In [2], Gukov and Witten initiated a study of surface operators in $\mathcal{N} = 4$ super Yang-Mills theories. The $S$-duality [1] assert that

$$S : (G, \tau) \to (G^L, -1/n_\mathfrak{g} \tau),$$

where $n_\mathfrak{g}$ is 2 for $F_4$, 3 for $G_2$, and 1 for other semisimple classical groups [2]; $\tau$ is usual gauge coupling constant and $G^L$ is Langlands dual group of $G$. In [3], Gukov and Witten extended their earlier analysis and identified a subclass of surface operators called 'rigid' surface operators expected to be closed under $S$-duality. There are two types rigid surface operators: unipotent and semisimple. The rigid semisimple surface operators are labelled by pairs of partitions $(\lambda', \lambda'')$. Unipotent rigid surface operators arise when one of the partitions is empty.

2010 Mathematics Subject Classification. 05E10.

Key words and phrases. Partition, fingerprint, Kazhdan-Lusztig map, construction, rigid semisimple operator.
In [4], Wyllard made some explicit proposals for how the S-duality maps should act on rigid surface operators based on the invariants of partitions. The invariant fingerprint based on Kazhdan-Lusztig map for the classical groups [20, 27]. The Kazhdan-Lusztig map is a map from the unipotent conjugacy classes to the set of conjugacy classes of the Weyl group. It can be extended to the case of rigid semisimple conjugacy classes [22]. The rigid semisimple conjugacy classes and the conjugacy classes of the Weyl group are described by pairs of partitions $\lambda, \lambda''$ and pairs of partitions $[\alpha; \beta]$, respectively. The fingerprint invariant is a map between these two classes of objects. There is another invariant symbol related to the the Springer correspondence [1].

Compared to the fingerprint, the symbol invariant is much easier to be calculated and more convenient to find the S-duality maps of surface operators. In [22], we find a new subclass of rigid surface operators related by S-duality. In [10], we found a construction of symbol for the rigid partitions in the $B_n$, $C_n$, and $D_n$ theories. Another construction of symbol is given in [13]. We discuss the basic properties of fingerprint and its constructions in [13].

A problematic mismatch in the total number of rigid surface operators between the $B_n$ and the $C_n$ theories was pointed out in [3]. In [22], we clear up cause of this discrepancy and construct all the $B_n/C_n$ rigid surface operators which can not have a dual. The fingerprint invariant is assumed to be equivalent to the symbol invariant [4]. In [14], we prove the symbol invariant of partitions implies the fingerprint invariant of partitions. The constructions of the symbol invariant and the fingerprint invariant in previous works play a significant role in the proof. This problem suggest us to find map between two invariants.

The following is an outline of this article. In Section 2, we summary some basic results related to the rigid partition in [4]. In Section 3, we introduce the definition and the construction of symbol in [10]. Then we present the construction of the representative element $(\lambda', \lambda'')_R$ of rigid surface operators with the given symbol. In Section 4, we give the constructions of the fingerprint invariant in [13]. We calculate the fingerprint of the rigid surface operators with the fingerprint given. our construction can be regarded as the maps between the fingerprint invariant and symbol invariant. In Section 5, we discuss the realization of the above results for the $C_n$ and $D_n$ theories.

We will focus on theories with gauge groups $SO(2n)$ which are Langlands self-duality and the gauge groups $Sp(2n)$ whose Langlands dual group is $SO(2n + 1)$.

2. Rigid Partitions in the $B_n$, $C_n$, and $D_n$ theories

In this section, we introduce the rigid partitions in the $B_n$, $C_n$, and $D_n$ theories. For the $B_n$($D_n$) theories, unipotent conjugacy classes are in one-to-one correspondence with partitions of $2n+1$(2n) where all even integers appear an even number of times. For the $C_n$ theories, unipotent conjugacy classes are in one-to-one correspondence with partitions $2n$ for which all odd integers appear an even number of times. If it has no gaps (i.e. $\lambda_i - \lambda_{i+1} \leq 1$ for all $i$) and no odd (even) integer appears exactly twice, a partition in the $B_n$ or $D_n$($C_n$) theories is called rigid. We will focus on rigid partition in this paper, which correspond to rigid operators. The operators will refer to rigid operators in the rest of paper. The addition rule of two partitions $\lambda$ and $\mu$ is defined by the additions of each part $\lambda_i + \mu_i$.

The following facts play important roles in this study.

Proposition 2.1. The longest row in a rigid $B_n$ partition always contains an odd number of boxes. The following two rows of the first row are either both of odd length or both of even length. This pairwise pattern then continues. If the Young tableau has an even number of rows the row of shortest length has to be even.

Remark: If the last row of the partition is odd, the number of rows is odd.

Proposition 2.2. For a rigid $D_n$ partition, the longest row always contains an even number of boxes. And the following two rows are either both of even length or both of odd length. This pairwise pattern then continue. If the Young tableau has an even number of rows the row of the shortest length has to be even.

Proposition 2.3. For a rigid $C_n$ partition, the longest two rows both contain either an even or an odd number number of boxes. This pairwise pattern then continues. If the Young tableau has an odd number of rows the row of shortest length has contain an even number of boxes.

Examples of partitions in the $B_n$, $D_n$, and $C_n$ theories:
3. Symbol invariant of partitions

In this section, we introduce the definition of symbol and its construction [16].

3.1. Symbol invariant and the construction

In [16], we proposed equivalent definitions of symbol for the C_n and D_n theories which are consistent with that for the B_n theory as much as possible.

Definition 1. The symbol of a partition in the B_n, C_n, and D_n theories.

- For the B_n theory: first we add l – k to the kth part of the partition. Next we arrange the odd parts of the sequence l – k + λ_k and the even parts in an increasing sequence 2f_i + 1 and in an increasing sequence 2g_i, respectively. Then we calculate the terms

\[ \alpha_i = f_i - i + 1 \quad \beta_i = g_i - i + 1. \]

Finally we write the symbol as

\[ \left( \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \cdots \end{array} \right). \]

- For the C_n theory:
  1: If the length of partition is even, compute the symbol as in the B_n case, and then append an extra 0 on the left of the top row of the symbol.
  2: If the length of the partition is odd, first append an extra 0 as the last part of the partition. Then compute the symbol as in the B_n case. Finally, we delete a 0 in the first entry of the bottom row of the symbol.

- For the D_n theory: first append an extra 0 as the last part of the partition, and then compute the symbol as in the B_n case. We delete two 0’s in the first two entries of the bottom row of the symbol.

| Parity of row | Parity of i + t + 1 | Contribution to symbol | L |
|---------------|---------------------|------------------------|---|
| odd          | even               | \( \begin{pmatrix} 0 & 0 & \cdots & 1 & 1 \cdots 1 \\ \cdots & 0 & \cdots & 0 & \cdots 0 \end{pmatrix} \) | \( \frac{1}{2} (\sum_{k=1}^{m} n_k + 1) \) |
| even         | odd                | \( \begin{pmatrix} 0 & 0 & \cdots & 1 & 1 \cdots 1 \\ \cdots & 0 & \cdots & 0 & \cdots 0 \end{pmatrix} \) | \( \frac{1}{2} (\sum_{k=1}^{m} n_k) \) |
| even         | even               | \( \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \cdots & 0 & \cdots & 0 & 1 & 1 \cdots 1 \end{pmatrix} \) | \( \frac{1}{2} (\sum_{k=1}^{m} n_k) \) |
| odd          | odd                | \( \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \cdots & 0 & \cdots & 0 & 1 & 1 \cdots 1 \end{pmatrix} \) | \( \frac{1}{2} (\sum_{k=1}^{m} n_k - 1) \) |

Table 1. Contribution to symbol of the \( i \)th row (\( B_n(t = -1), C_n(t = 0), \) and \( D_n(t = 1) \)).
In [16], we give the construction of symbol by Table 1. We determine the contribution to symbol for each row of a partition in the different theories uniformly. For rigid semisimple operators $(\lambda', \lambda'')$, one can construct the symbol of them by calculating the symbols for both $\lambda'$ and $\lambda''$, then add the entries that are 'in the same place' of these two results. An example illustrates the addition rule:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 2
\end{pmatrix} +
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 2 \\
1 & 2 & 2 & 2 & 2 & 3
\end{pmatrix}.
\]

### 3.2. Representative element of rigid semisimple operators with the same symbol

In this section, we propose an algorithm to construct a representative element of the rigid semisimple operators $(\lambda', \lambda'')$ with the same symbol invariant. This representative element have a nice structure and is unique. Especially, its fingerprint can be obtained directly in Section 4.4.

According to Table 1, the contribution to symbol of each row of partition have the following form

(3.1)\[
\begin{pmatrix}
0 & 0 & \cdots & 1 & 1 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots
\end{pmatrix}
\]

or

(3.2)\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 1 & 1 & \cdots
\end{pmatrix}
\]

The contribution (3.1) and contribution (3.2) can be represented by a line above a dashed line and a line under it, respectively. The length of the line corresponds to the number of '1' in the formulas (3.1) and (3.2). By using this method, the symbol invariant of a partition can be visualized as shown in Fig.(2). The contributions of rows of different partitions are represented by different colour. The red lines and the black lines represent the contributions of the rows of the partition $\lambda'$ and $\lambda''$, respectively.

![Figure 2. Visualization of symbol of the rigid semisimple operator $(\lambda', \lambda'')$.](image)

The representative element is constructed as follows. First, the longest line of the symbol must contributed by the first row of one of the two factors of the rigid semisimple operators. Without loss of generality, we assume it is contributed by the first row of $\lambda''$. Since the first row of $\lambda''$ is even, its contributions to symbol is on the top row of symbol according to Table 1. So the longest line a is above the dotted line as shown in Fig.(2). From the remaining lines, we chose the longest lines above and under the dotted line to combine the first pairwise rows of $\lambda''$. Without loss of generality, we assume the former one is longer than the latter. So these two lines b, c of symbol are contributed by odd pairwise rows of $\lambda''$ as shown in Fig.(3).

There may be a line d longer than b and shorter than c as shown in Fig.4. The line d must be contributed by a row of the $B_{n}$ partition $\lambda'$. Since the first row of $\lambda'$ is odd, its contribution to symbol is on the bottom row of symbol, a contradiction. So the contributions of other rows of $\lambda''$ are shorter than these three lines a, b, and c. Repeating the above procedure, we determine another two lines d and e of symbol contributed by pairwise row of $\lambda''$ as shown in Fig.(5).
Figure 3. The visualization of the first three rows of $\lambda''$.

Figure 4. The line $d$ would not be existent since the first line contributed by the first row of $\lambda'$ is under the dashed line.

Figure 5. The visualization of the first five rows of $\lambda''$.

For the next pair lines of symbol, the line $f$ is longer than $h$, which means they are contributed by two even rows of $\lambda''$. If a line $g$ satisfy $h < g < f$, it must be contributed by the first row of the partition $\lambda'$ as shown in Fig. 4.

These procedure continue until we determine rows of $\lambda'$ or $\lambda''$ for all lines of the symbol. The following lemma make sure we would reach a consistent result.

**Lemma 3.1.** There would not be two successive lines in the same part of the symbol, excepting the first two rows of a partition.
Figure 6. Visualization of contributions of symbol.

Proof. If there are two successive lines $b$ and $d$ in the same row of the symbol, then there must be another two lines $c$ and $e$ in the other part as shown in Fig. 6(a). The lines $b$ and $c$ are pairwise rows. And the lines $d$ and $e$ are pairwise rows. The partition corresponding to Fig. 6(a) are shown in Fig. 6(b). The length of the row $d$ is longer than the length of the row $c$, which is a contradiction. □

Figure 7. (a) The lines $b$ and $d$ are two successive lines in the same row of symbol. (b), The partition correspond to the symbol.

Figure 8. The representative element $(\lambda', \lambda'')_R$ and $\lambda$. The gray rows are rows of the partition $\lambda'$.

According to the above algorithm, we get a representative element $(\lambda', \lambda'')_R$ and $\lambda = \lambda' + \lambda''$ as shown in Fig. 8. For the block III, both the rows $A$ and $B$ are two rows of pairwise rows of the partition $\lambda'$. Note that the row $A$ and row $B$ are between the two rows of pairwise rows of $\lambda''$, which is a common feature of the rows of the partitions $\lambda'$ in $\lambda$. 
According to Table 1, the two lines \( b \) and \( c \) in Fig. (a) are contributed by even pairwise rows of \( \lambda'' \). And the two lines \( d \) and \( e \) are contributed by odd pairwise rows of \( \lambda'' \). According to the algorithm to construct the representative element, the rows \( A \) and \( B \) are two rows of pairwise rows of the partition \( \lambda' \) and are restricted by the rows \( b, c \) and the rows \( d, e \), respectively. The length of the row \( A \) is shorter than the length of the row \( b \) and longer than that of the row \( c \). And the length of the row \( B \) is shorter than the length of the row \( d \) and longer than that of the row \( e \). The line \( A \) is contributed by the first row of even pairwise rows of \( \lambda' \), and the line \( B \) is contributed by the second row of the even pairwise rows. Another example is shown in Fig. (b). The row \( A \) is contributed by the first row of odd pairwise rows, which have the same parity with the rows \( b, c \). The row \( B \) is contributed by the second row of the odd pairwise rows, which have the opposite parity with the rows \( d, e \).

![Figure 9](image-url)

**Figure 9.** The rows \( A \) and \( B \) are pairwise rows of \( \lambda' \).

Summary, for pairwise rows of the partition \( \lambda' \) in \( \lambda \), the parity of the first row is the same with the pairwise rows of \( \lambda'' \) in the block \( A \), while the parity of the second row would be opposite with the pairwise rows of \( \lambda'' \) in the block \( B \) as shown in Fig. 9, which corresponds to the block III in Fig. (8).

4. Fingerprint invariant of partitions

In the first two subsections, we would like to introduce the construction of the fingerprint in [18]. Then we try to find a representative element of the rigid semisimple operators with the same fingerprint and calculate symbol of it. Finally, we give the fingerprint of the representative element \( (\lambda', \lambda'')_R \) in previous section. In this way, we find a map between the symbol invariant and the fingerprint invariant.

4.1. Fingerprint invariant

For rigid semisimple surface operator \( (\lambda', \lambda'') \), we introduce the definition of fingerprint as follows. First, add the two partitions \( \lambda = \lambda' + \lambda'' \), and then calculate the partition \( \mu = Sp(\lambda) \)

\[
\mu_i = Sp(\lambda)_{ii} = \begin{cases} 
\lambda_i + p_{\lambda}(i) & \text{if } \lambda_i \text{ is odd and } \lambda_i \neq \lambda_i - p_{\lambda}(i), \\
\lambda_i & \text{otherwise}
\end{cases}
\]  

Next define the function \( \tau \) from an even positive integer \( m \) to \( \pm 1 \) as follows. For a partition in the \( B_n \) and \( D_n \) theories, \( \tau(m) \) is \(-1\) if at least one \( \mu_i \) such that \( \mu_i = m \) and either of the following three conditions is satisfied.

\[
\begin{align*}
(i) & \quad \mu_i \neq \lambda_i \\
(ii) & \quad i \sum_{k=1}^{i} \mu_k \neq i \sum_{k=1}^{i} \lambda_k \\
(iii)_{SO} & \quad \lambda'_i \text{ is odd.}
\end{align*}
\]  

Otherwise \( \tau \) is \( 1 \). For the partitions in the \( C_n \) theory, the definition is exactly the same except the condition \( (iii)_{SO} \) is replaced by

\[
(iii)_{Sp} \quad \lambda'_i \text{ is even.}
\]
Finally construct a pair of partitions \( [\alpha; \beta] \). For each pair of parts of \( \mu \) both equal to \( a \), satisfying \( \tau(a) = 1 \), retain one part \( a \) as a part of the partition \( \alpha \). For each part of \( \mu \) of size \( 2b \), satisfying \( \tau(2b) = -1 \), retain \( b \) as a part of the partition \( \beta \).

Note that \( \mu_i \neq \lambda_i \) only happen at the end of a row. According to the above definition of fingerprint, we have the following important lemma.

**Lemma 4.1.** Under the map \( \mu \), the change of rows depend on the parity of the row and the sign of \( p_\lambda(i) \).

| Parity of row | Sign | Change |
|---------------|------|--------|
| odd           |      | \( \mu_i = \lambda_i - 1 \) |
| even          |      | \( \mu_i = \lambda_i + 1 \) |
| even          | +    | \( \mu_i = \lambda_i \) |
| odd           | +    | \( \mu_i = \lambda_i \) |

![Figure 10](image.png)

**Figure 10.** The difference of the height of the \((j-1)\)th row and the \((j-3)\)th row is two, violating the rigid condition \( \lambda_i - \lambda_{i+1} \leq 1 \).

It is easy to prove the following fact.

**Proposition 4.1.** \( \text{[14]} \) For the partition \( \lambda \) with \( \lambda_i - \lambda_{i+1} \leq 1 \), the condition \( \text{(ii)} \) imply \( \text{(i)} \).

We would give an example where the condition \( \text{(ii)} \) works. As shown in Fig. (10), the heights of the \((j-1)\)th row, the \((j-3)\)th row, and the \((j-5)\)th are even. And the difference of the height of the \((j-1)\)th row and the \((j-3)\)th row violate the rigid condition \( \lambda_i - \lambda_{i+1} \leq 1 \) as well as that of the height of the \((j-3)\)th row and the \((j-5)\)th row. The part \((j-3)\) satisfy the condition \( \text{[14]}(\text{ii}) \) but do not satisfy the condition \( \text{(i)} \).

In this study, the neglect of the condition \( \text{(ii)} \) in the definition of fingerprint would not change the conclusions in this paper. In the rest of the paper, we would focus on the partition \( \lambda = \lambda' + \lambda'' \) with \( \lambda_i - \lambda_{i+1} \leq 1 \).

### 4.2. Construction of the fingerprint invariant

In this subsection, we introduce the construction of fingerprint of rigid \((\lambda', \lambda'')\) in the \( B_n \) theory. We use a simplified model which is enough for the construction of the fingerprint of the representative element \((\lambda', \lambda'')_\text{c1}\) in next subsection. For the \( C_n \) and \( D_n \) theories, we can get the same results with minor modifications.

**Operators** \( \mu_{c11}, \mu_{c12}, \mu_{c21}, \text{ and } \mu_{c22} \)

The partition \( \lambda = \lambda' + \lambda'' \) is constructed by inserting rows of \( \lambda' \) into \( \lambda'' \) one by one. Without confusion, the image of the map \( \mu \) is also denoted as \( \mu \). The partition \( \lambda = \lambda' + \lambda'' \) is decomposed into several blocks such as blocks \( I, II, III \) as shown in Fig. (11).

The \( III \) type block consist of pairwise rows of \( \lambda' \) and \( \lambda'' \) as shown in Fig. (11). The upper boundary and the lower boundary of the block consist of a row of \( \lambda' \) and one pairwise rows of \( \lambda'' \).

When the pairwise rows \( b, c \) of \( \lambda' \) are even, we define four operators \( \mu_{c11}, \mu_{c12}, \mu_{c21}, \text{ and } \mu_{c22} \) according to the positions of the rows of \( \lambda' \) inserted into \( \lambda'' \), as shown in Fig. (12). The first and last row of the block do not change under the map \( \mu \) as well as the even pairwise rows of \( \lambda'' \). We append a box to the first row of odd pairwise rows of \( \lambda'' \) and delete a box at the end of the second row. If \( b \) is not the first row of the block, it will behave as the first row of pairwise rows of \( \lambda'' \) as shown in Fig. (11). If \( c \) is not the last row of the block, it will behave as the second row of pairwise rows of \( \lambda'' \). When the pairwise rows of \( \lambda' \) inserted into \( \lambda'' \) are odd, we define four operators \( \mu_{o11}, \mu_{o12}, \mu_{o21}, \text{ and } \mu_{o22} \) similarly. However the odd rows and the even rows interchange roles in this
We can also consider III type block consisting of one pairwise rows $\lambda''$ and pairwise rows of $\lambda'$. The II type block only consist of pairwise rows of $\lambda'$ or $\lambda''$ as shown in Fig. [13] which do not change under the map $\mu$. The I type block consist of pairwise rows of $\lambda''$, besides of the first row $a$ of $\lambda'$ and the first row of $\lambda''$. We define two operators $\mu_{e_1}$ and $\mu_{e_2}$ correspond to row $a$ above the pairwise rows in
Fig. (14)(a) and between the pairwise rows in Fig. (14)(b), respectively. The last row of the block and the even pairwise rows of $\lambda''$ do not change under the map $\mu$. We append a box to the first row of odd pairwise rows of $\lambda''$ and delete a box at the end of the second row of odd pairwise rows. If $a$ is not the last row of the block, it will behave as the second row of pairwise rows of $\lambda''$ as shown in Fig. (14)(b). We can also consider $I$ type block consist of pairwise rows of $\lambda$, besides of the first rows of $\lambda'$ and $\lambda''$. Similarly, we define two operators $\mu_1$ and $\mu_2$. However the odd rows and the even rows interchange roles in this case. There is no $I$ type block of the $\lambda$ in the $C_n$ theory. So the fingerprint of the $C_n$ rigid semisimple operators is a simplified version of that in the $B_n$ and $D_n$ theories.

4.3. Representative element of rigid semisimple operators with the same fingerprint

With the fingerprint given, it is hope to find a natural representative element $(\lambda', \lambda'')_R$ which similar to the representative element $(\lambda', \lambda'')_R$. Since pair of fingerprint and $\lambda$ would determine the rigid semisimple operator $(\lambda', \lambda'')_R$, we would find a representative element $\lambda_r$ of $\lambda$.

An element $\lambda_r$ for $\lambda' + \lambda''$ can be constructed from the fingerprint invariant naturally, with the same fingerprint invariant $[\alpha, \beta]$. For each part $a$ of the partition $\alpha$ retain $a^2$. From the integers so obtained form the partition $\lambda'$. For each part $b$ of the partition $\beta$ retain $2b$. From the integers so obtained form the partition $\lambda''$. Then $\mu_r = \lambda_r + \lambda''$ is the image of $\lambda$ under the map $\mu$. All the partition $\lambda$ for rigid semisimple operator $(\lambda', \lambda'')$ with the same fingerprint leads to it by the map $\mu$.

A naive guess is that there always exist a operator $(\lambda', \lambda'')_R$ and $\mu_r = \lambda + \lambda''$. Unfortunately, it is wrong. We would give a counterexample as follows. The $B_6$ operator $(2^21^9, \emptyset)$ is the only operator with the dimension of 20. Its image under the map $\mu$ is

$$2^21^9 \rightarrow 2^21^8$$

where $\lambda \neq \mu$, a contradiction. However, for the rigid semisimple operator with $\lambda = \mu$, their symbol have nice properties.

Symbol of $\mu_r$:

The operator $\mu_r$ in Fig. (13) suggest that the possible structure of the partition $\mu_r$. It consist of pairwise rows and like a partition in the $C_n$ theory. Since the rigid semisimple operator can be obtain by pair of $[\alpha, \beta]$ and $\mu_r$, of which the symbol can be obtained directly according to Table 1.

1This is another invariant of rigid semisimple operators which contain less information than symbol invariant and fingerprint invariant.
4.4. Fingerprint of the representative element \((\lambda', \lambda'')_R\)

The representative semisimple operator \((\lambda', \lambda'')_R\) and the corresponding partition \(\lambda = \lambda' + \lambda''\) are shown in Fig. (8). There are three type of blocks in the representative element \(\lambda_R\) which are the I, II, and III type blocks.

- For the I type block as shown in Fig. (15), the \(\mu\) partition is given by the operator \(\mu_1\) in Fig. (12) since \(l_3\) is odd. The height of the last row of the block is even which satisfy the condition (iii) but do not satisfy the condition (i). The other rows of the block do not satisfy the condition (iii) but the even rows do change under the map \(\mu_1\) thus satisfying the condition (i). And the odd rows do change under the map \(\mu_{e1}\) thus satisfying the condition (i).

- For the II type block as shown in Fig. (8), the \(\mu\) partition is given by the operator \(\mu_e\) in Fig. (12). The rows of the block do not change under this operator, thus not satisfying the condition (i). However the block satisfy the condition (iii).

- For the III type block as shown in Fig. (8), the \(\mu\) partition is given by the operator \(\mu_{e21}\) or \(\mu_{o21}\) in Fig. (12), according to the parities of the pairwise rows \(A, B\). The height of the last row of the block is even. It satisfy the condition (iii) but do not satisfy the condition (i). The other rows of the block do not satisfy the condition (iii) but the even rows do change under the map \(\mu_{o21}\), thus satisfying the condition (i). And the odd rows do change under the map \(\mu_{e21}\), thus satisfying the condition (i).

![Figure 15. Block I of the representative element.](image)

5. \(C_n\) and \(D_n\) theories

For the rigid semisimple operators in the \(C_n\) and \(D_n\) theories, we can get the similar results by the same strategy in previous sections with minor modifications.

**Fingerprint of \(\lambda_R\):** Now we discuss the fingerprint invariant of the representative element \((\lambda', \lambda'')_R\) for the rigid semisimple operators in the \(C_n\) and \(D_n\) theories.

The first two rows of \(C_n\) partitions are pairwise rows according to Proposition 2.3. Thus there is no I type block of the \(\lambda\) in the \(C_n\) theory. So the fingerprint of the \(C_n\) rigid semisimple operators is a simplified version of that in the \(B_n\) and \(D_n\) theories. The other processes to calculate the fingerprint of \(\lambda_R\) is exact the same with that of the \(B_n\) rigid semisimple operator.

The first row of \(D_n\) partitions is even and not of pairwise rows according to Proposition 2.2. For the I type block of the \(\lambda\) in the \(D_n\) theory, there are only \(\mu_{e1}\) and \(\mu_{e2}\) operators. The other processes to calculate the fingerprint of \(\lambda_R\) is exact the same with that of the \(B_n\) rigid semisimple operator.

**Symbol of \(\mu_r\):** we will discuss the symbol invariant of the element \(\mu_r\).

The operator \(\mu_r\) in Fig. (13) suggest that the possible structure of the partition \(\mu_r\). It consist of pairwise rows and like a partition in the \(C_n\) theory. The rigid semisimple operator can be obtain by pair of \([\alpha, \beta]\) and \(\mu_r\), of which the symbol can be obtained directly according to Table 1.

References

[1] D. H. Collingwood and W. M. McGovern, Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold, 1993.

[2] S. Gukov and E. Witten, Gauge theory, ramification, and the geometric Langlands program, arXiv:hep-th/0612073

[3] S. Gukov and E. Witten, Rigid surface operators, arXiv:0804.1501
[4] N. Wyllard, Rigid surface operators and S-duality: some proposals, arXiv: 0901.1833
[5] B. Shou, Symbol, Rigid surface operators and S-duality, preprint, 26pp, arXiv: 1708.07388
[6] G. Lusztig, A class of irreducible representations of a Weyl group, Indag.Math, 41(1979), 323-335.
[7] G. Lusztig, Characters of reductive groups over a finite field, Princeton, 1984.
[8] N. Spaltenstein, Order relations on conjugacy classes and the Kazhdan-Lusztig map, Math. Ann., 292 (1992) 281.
[9] C. Montonen and D. I. Olive, “Magnetic monopoles as gauge particles?,” Phys. Lett. 177 (1977) 117.
[10] P. Goddard, J. Nuyts, and D. I. Olive, Gauge theories and magnetic charge, Nucl. Phys., B 125 (1977) 1.
[11] P. C. Argyres, A. Kapustin, and N. Seiberg, On S-duality for non-simply-laced gauge groups, JHEP, 06 (2006) 043, arXiv:hep-th/0603048.
[12] J. Gomis and S. Matsuura, Bubbling surface operators and S-duality, JHEP, 06 (2007) 025, arXiv:0704.1657.
[13] N. Drukker, J. Gomis, and S. Matsuura, Probing $\mathcal{N} = 4$ SYM with surface operators, JHEP, 10 (2008) 048, arXiv:0805.4199.
[14] S. Gukov, Surfaces Operators, arXiv:1412.7145.
[15] B. Shou, Solutions of Kapustin-Witten equations for ADE-type groups, preprint, 27pp, arXiv:1708.07388.
[16] B. Shou, Symbol Invariant of Partition and Construction, preprint, 31pp, arXiv:1708.07084.
[17] B. Shou, and Q. Wu, Construction of the Symbol Invariant of Partition, preprint, 31pp, arXiv:1708.07090.
[18] B. Shou, and Q. Wu, Fingerprint Invariant of Partitions and Construction, preprint, 23pp, arXiv:1711.03443.
[19] B. Shou, Invariants of Partitions, in preparation.
[20] M. Henningson and N. Wyllard, Low-energy spectrum of $\mathcal{N} = 4$ super-Yang-Mills on $T^3$: flat connections, bound states at threshold, and S-duality, JHEP, 06 (2007), arXiv:hep-th/0703172.
[21] M. Henningson and N. Wyllard, Bound states in $\mathcal{N} = 4$ SYM on $T^3$: Spin(2n) and the exceptional groups, JHEP, 07 (2007) 084, arXiv:0706.2803.
[22] M. Henningson and N. Wyllard, Zero-energy states of $\mathcal{N} = 4$ SYM on $T^3$: S-duality and the matching class group, JHEP, 04 (2008) 066, arXiv:0802.0660.
[23] B. Shou, Symbol, Surface operators and S-duality, preprint 27pp, arXiv: 1708.07388.
[24] B. Shou, Rigid Surface operators and Symbol Invariant of Partitions, preprint 23pp, arXiv: 1708.07388.
[25] N. Spaltenstein, Order relations on conjugacy classes and the Kazhdan-Lusztig map, Math. Ann., 292 (1992) 281.
[26] G. Lusztig, Characters of reductive groups over a finite field, Princeton, 1984.
[27] G. Lusztig, A class of irreducible representations of a Weyl group, Indag.Math, 41(1979), 323-335.
[28] B. Shou, J.F. Wu and M. Yu, AGT conjecture and AFLT states: a complete construction, preprint, 28 pp., arXiv:1107.4784.
[29] B. Shou, J.F. Wu and M. Yu, Construction of AFLT States by Reflection Method and Recursion Formula, Communications in Theoretical Physics, 61 (2014) 56-68.
[30] Z.S. Liu, B. Shou, J.F. Wu, Y.Y. Xu and M. Yu, Construction of AFLT States for $W_n \otimes H$, Symmetry, Analytic Continuation and Integrability on AGT Relation, Communications in Theoretical Physics, 63 (2015) 487-498.
[31] N. Nekrasov, A. Okounkov, Seiberg-Witten theory and random partitions, The Unity of Mathematics, 525-596, arXiv:hep-th/0306238.

* College of Logistics and E-commerce, Zhejiang Wanli University, No.8 South Qianhu Road, Ningbo 315100, P.R.China
† Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China
E-mail address: *10920005@zju.edu.cn, † bsoul@zju.edu.cn