Properties of the Spatial Sections of the Space-Time of a Rotating System.

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Abstract

We study the symmetry group of the geodesic equations of the spatial solutions of the space-time generated by a noninertial rotating system of reference. It is a seven dimensional Lie group, which is neither solvable nor nilpotent. The variational symmetries form a five dimensional solvable subgroup. Using the symplectic structure on the cotangent bundle we study the resulting Hamiltonian system, which is closely related to the geodesic flow on the spatial sections. We have also studied some intrinsic and extrinsic geometrical properties of the spatial sections.

Keywords

1 Introduction

In Classical Mechanics and prerelativistic science, space and time are separated and independent. The time is absolute but the space is not. In the theory of Relativity both space and time are relative, yet there is a combination of them which is absolute. In this sense this theory introduces the concept of space-time. This invariant combination can be expressed by the metric. In the Special Theory of Relativity the space-time manifold is flat and its metric with respect to an inertial system of reference takes the well known Lorentzian form

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]  

(1)

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In the General Theory of Relativity the presence of a gravitational field affects the geometry of space-time. Here the space-time manifold is curved. Actually the gravitational field is the curvature of the space-time and Einstein’s equations relate the distribution of mass and energy with the curvature of space-time.

The cornerstone of the general theory is the principle of equivalence, which states that every gravitational field is locally (but not globally) equivalent to a noninertial system of reference. If there is no gravitational field but we decide to use a noninertial system of reference the expression of the metric will not be the above Lorentzian metric of Eq. (1), but still all the components of the Riemann tensor will be zero, since there is a global transformation from the inertial system to the noninertial that we are using. So with respect to a noninertial system of coordinates the space-time is still flat, but on the other hand the space itself may be curved. What we will study here are some geometrical properties of the spatial part of the space-time with respect to a rotating coordinate system with constant angular velocity $\omega$ around the $z$-axis. This is one of the simplest possible cases. Using similar techniques we can study more complicate problems resulting either from a more complicate system of reference, or because of the presence of a gravitational field. These problems can help us to understand better the principle of equivalence as well as the relation between gravity and the noninertial systems. We can also study the possibility of changes in the topology of the spatial part of the space-time.

Substituting $x$ and $y$ by $x \cos \omega t - y \sin \omega t$ and $x \sin \omega t + y \cos \omega t$ in Eq. (1) we obtain:

$$d\mathbf{s}^2 = [\omega^2 (x^2 + y^2) - c^2] dt^2 + dx^2 + dy^2 + dz^2 - 2\omega y dx dt + 2\omega x dy dt$$  \hspace{1cm} (2)

The geometry of the space with respect to this rotating observer is not Euclidean since the relation between the circumference and the diameter of a circle is not any more $\pi$. Due to the Lorentz contraction along the circumference this is larger than $\pi$ by a factor of $(1 - \omega^2 r^2/c^2)^{1/2}$. In cylindrical coordinates the above metric becomes:

$$d\mathbf{s}^2 = (\omega^2 r^2 - c^2) dt^2 + 2\omega r^2 d\phi dt + dz^2 + r^2 d\phi^2 + dr^2$$ \hspace{1cm} (3)

In this paper we have found the intrinsic and extrinsic curvatures of the spatial sections of the above space-time. They are both negative and blow up where the coordinate system we are using ceases to have physical meaning. We have also found the symmetry group of the geodesic equations and its variational subgroup. The first one is neither solvable nor nilpotent. On the other hand the variational subgroup is solvable. Using the Hamilton-Jacobi theory for the Hamiltonian system which is induced on the cotangent bundle we have found the expression of the geodesic flow in terms of the affine time $t$.

Account of the theory of Relativity can be found for example in the books of Misner, Thorne and Wheeler [11], Weinberg [16], O’Neill [13], Hawking and Ellis [7] and Landau and Lifschitz [8].
The geometry of spatial sections.

Let the metric of a space-time be given by:

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \]  \hspace{1cm} (4)

The Greek indices take the values 0, 1, 2, 3 and the Latin indices the values 1, 2, 3. The index zero corresponds to the time of the event and \( x^1, x^2, x^3 \) determine its location in space. As a result we have:

\[ ds^2 = g_{00} dx^0 dx^0 + 2 g_{0i} dx^0 dx^i + g_{ij} dx^i dx^j \]  \hspace{1cm} (5)

to study the spatial sections of the space-time manifold we must consider the set of all events that are simultaneous to a given event. Using the null geodesics we can prove that two nearby events are simultaneous if they satisfy the condition \( (dx)_0 = 0 \). As a result the metric on the spatial sections in contravariant form is

\[ dl^2 = g^{ij} dx_i dx_j \]

and in covariant form it becomes

\[ dl^2 = \left( g^{-1}_{0i} \frac{g_{00}}{g_{00}} \right) dx^i dx^j \]  \hspace{1cm} (6)

Applying this to Eqs. (2) and (3) we get the metric on the spatial sections of a noninertial rotating system in cartesian coordinates in the form:

\[ dl^2 = (1 + Ay^2) dx^2 - 2Axy dy dx + (1 + Ax^2) dy^2 + dz^2 \] \hspace{1cm} (7)

In cylindrical coordinates we have

\[ dl^2 = dr^2 + \frac{r^2}{1 - \omega^2 r^2/c^2} d\phi^2 + dz^2 \] \hspace{1cm} (8)

In Eq. (7) the factor \( A \) is given by:

\[ A = \frac{\omega^2}{c^2 - \omega^2 (x^2 + y^2)} \]

The metric in Eq. (8) is static so it can be used to define and to calculate finite distances and the result will not depend on the cosmic curve connecting the two points. On the other hand we must bear in mind that the concept of simultaneous events which is the foundation of the positive definite spatial metric of Eq. (6) is only a local one and it may not be possible to define simultaneous events in a consistent way over the whole manifold since \( g_{0i} \neq 0 \)

These arguments indicate that the concept of a spatial slice is maybe more complicate than the concept of a hypersurface for example and here may appear some questions of topological character.

We turn now to some geometrical properties of the metric (8). The \( z \)-axis is a geodesic parametrized by arclength that is the same with the coordinate \( z \). This
is not true for the $x, y, r$ or $\phi$ coordinates. This is a space of negative curvature. The Riemann tensor has only 6 functionally independent components that are all zero except $R_{1212}$, which is given by:

$$
R_{1212} = -\frac{3c^4\omega^2r^2}{(c^2 - \omega^2r^2)^3}
$$

(9)

Here we use the notation $x^1 = r, x^2 = \phi$ and $x^3 = z$. We can easily calculate the scalar curvature and get the following result:

$$
R = g^{ki}g^{lj}R_{klij} = -\frac{6c^2\omega^2}{(c^2 - \omega^2r^2)^2}
$$

(10)

This is negative and blows up when $r = c/\omega$. This is exactly where the coordinates of the rotating system cease to have meaning since beyond that distance we have speeds higher than the speed of light in vacuum. Using the metric (8) we can get for the element of the arclength along the circle with both $z, r$ constant the expression:

$$
dl = \frac{cr}{\sqrt{c^2 - \omega^2r^2}}d\phi
$$

from where we get that the ratio of the circumference to the diameter is greater than $\pi$ by the Lorentz factor.

The intrinsic geometry does not give any information about how the spatial slices are impended in the four dimensional space-time. Geometrically the impending can be found by studying the unit normal vector on the spatial sections and is somehow represented by the extrinsic curvature, which is a symmetric tensor parallel to the spatial slices. The extrinsic curvature is closely related to the Weingarten map, which is the Jacobian of the Gauss map and can be calculated as the covariant derivative of the unit normal along the spatial slices. The metric (4) in the canonical 1 + 3 language takes the form:

$$
ds^2 = g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - N^2 dt^2 = (g_{ij}N^iN^j - N^2)dt^2
$$

$$
+ g_{ij}N^j dx^i dt + g_{ij}N^i dx^j dt + g_{ij}dx^i dx^j
$$

(11)

where $N^i$ are the shift functions and $N$ is the lapse function. Comparing this expression with Eq. (8) we get the shift and lapse functions of our space-time:

$$
N^r = N^z = 0
$$

(12)

$$
N^\phi = \omega
$$

(13)

$$
N = c
$$

(14)

So the contravariant normal vector connecting the spatial sections that correspond to time $t$ and $t + dt$ is

$$(dt, 0, -\omega dt, 0)$$
and its proper length is $c dt$ as we should expect. So the unit contravariant vector is
\[ \eta^\alpha = \left( \frac{1}{c}, 0, -\frac{\omega}{c}, 0 \right) \] (15)

In covariant form is
\[ \eta_\alpha = (c, 0, 0, 0) \] (16)

Since the metric $g_{ij}$ is static, the tensor of the extrinsic curvature can be calculated using the following relation:
\[ K_{ij} = \frac{1}{2N} \left( N_j{}^i + N_i{}^j \right) \] (17)

where the semicolon represents covariant differentiation with respect to the metric of the spatial sections. We can easily calculate the quantities $N_r, N_\phi$ and $N_z$ and we obtain:
\[ N_r = N_z = 0 \] (18)
\[ N_\phi = \frac{\omega c^2 r^2}{c^2 - \omega^2 r^2} \] (19)

Using these relations in (17) we can prove that all the components of the extrinsic curvature are zero except the component $K_{r\phi}$, which is given by:
\[ K_{r\phi} = \frac{-\omega c^2 r}{(c^2 - \omega^2 r^2)^{3/2}} \] (20)

and it has again a singularity at the positions $r = c/\omega$. The Riemannian tensor $R_{ijkl}$ and the above extrinsic curvature tensor are related by the Gauss-Codazzi equations.

3 The symmetry group of the geodesics.

Here we shall study the Lie point symmetries and the variational symmetries of the geodesic equations. A symmetry group of a system of differential equations is a group acting on the space of independent and dependent variables in such a way that solutions are mapped into other solutions. A local group of transformations is a variational symmetry if it necessarily leaves the variational integral invariant. Every variational group is a symmetry group of the corresponding Euler-Lagrange equations but the opposite is not true. The symmetry approach to differential equations can be found, for example, in the books of Olver [12], Bluman and Cole [3], Bluman and Kumei [4], Fushchich and Nikitin [6] and Obsiannikov [14]. In our problem the geodesics satisfy the following equations:
\[ \frac{d^2 x^j}{ds^2} + \Gamma^j_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \] (21)

where $\Gamma^j_{jk}$ are the Christoffel symbols and $s$ is the length of the geodesics, which we use to parametrize them. Here we do not have to worry about null geodesics.
since the metric (8) on the spatial sections is positive definite. Because of the form of this metric the Christoffel symbols are given by the following relations:

\[ \Gamma^j_{jk} = 0 \]  
\[ \Gamma^j_{ii} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^j} \]  
\[ \Gamma^i_{ij} = \frac{1}{2} \frac{\partial \ln g_{ii}}{\partial x^j} \]  
\[ \Gamma^i_{ii} = \frac{1}{2} \frac{\partial \ln g_{ii}}{\partial x^i} \]  

where the indices \( i, j, k \) are all different. Using these in the metric (8) we get the differential equations of the geodesics in the form:

\[ \frac{d^2 r}{ds^2} - \frac{c^2 r}{(c^2 - \omega^2 r^2)^2} \left( \frac{d \phi}{ds} \right)^2 = 0 \]  
\[ \frac{d^2 \phi}{ds^2} + \frac{2c^2}{r(c^2 - \omega^2 r^2)} \frac{d \phi}{ds} \frac{dr}{ds} = 0 \]  
\[ \frac{d^2 z}{dr^2} = 0 \]  

We can also find the geodesics by considering the partial differential equation

\[ g^{ij} \frac{\partial w}{\partial x^i} \frac{\partial w}{\partial x^j} = \left( \frac{\partial w}{\partial r} \right)^2 + \frac{c^2 - \omega^2 r^2}{c^2 r^2} \left( \frac{\partial w}{\partial \phi} \right)^2 + \left( \frac{\partial w}{\partial z} \right) = 1 \]  

A complete solution of this equation has the form

\[ w(r, \phi, z, a_1, a_2) = k \]

where \( k \) is a constant. The equations of the geodesics in parametric form take the form:

\[ \frac{\partial w}{\partial a_1} = b_1 \]
\[ \frac{\partial w}{\partial a_2} = b_2 \]

where \( b_1 \) and \( b_2 \) are arbitrary constants.

Using the method of Lie groups we shall study the symmetry group of the geodesics. A vector field \( G \) acting on the space of independent and dependent variables is a symmetry of the equations (26)-(28) if and only if it satisfies the relations

\[ \text{pr}^{(2)} G(Eq26) = \text{pr}^{(2)} G(Eq27) = \text{pr}^{(2)} G(Eq28) = 0 \]

where \( \text{pr}^{(2)} G \) is the second prolongation or extension of the vector field \( G \). If

\[ G = \sum \frac{\partial}{\partial s} + R \frac{\partial}{\partial r} + \Phi \frac{\partial}{\partial \phi} + Z \frac{\partial}{\partial z} \]
then

$$p r^{(2)} G = G + (\ddot{R} - \dddot{\Sigma}) \frac{\partial}{\partial r} + (\Phi - \dddot{\Sigma}) \frac{\partial}{\partial \phi} + (\dot{Z} - \dddot{\Sigma}) \frac{\partial}{\partial \bar{z}}$$

$$+ (\ddot{R} - \dddot{\Sigma} - 2\dddot{\Sigma}) \frac{\partial}{\partial r} + (\Phi - \dddot{\Sigma} - 2\dddot{\Sigma}) \frac{\partial}{\partial \phi}$$

$$+ (\dot{Z} - \dddot{\Sigma} - 2\dddot{\Sigma}) \frac{\partial}{\partial \bar{z}}$$

(31)

If we apply this to the geodesic equations (26)-(27) we get the following conditions:

$$\ddot{R} - \dddot{\Sigma} \dot{r} - 2\dddot{\Sigma} \dot{r} - \frac{2c^4 r}{(c^2 - \omega^4 r^2)^2} \dot{\phi} (\Phi - \dddot{\phi})$$

$$- \frac{c^4 (c^2 + 3\omega^2 r^2)}{(c^2 - \omega^4 r^2)^3} R \dddot{\phi}^2 = 0$$

(32)

$$\dddot{\Phi} - \dddot{\Sigma} - 2\dddot{\Sigma} \dot{\phi} + \frac{2c^2}{r(c^2 - \omega^4 r^2)} \dot{\phi} (\ddot{R} - \dddot{\Sigma} \dot{r})$$

$$+ \frac{2c^2}{r(c^2 - \omega^4 r^2)} (\ddot{\Phi} - \dddot{\Sigma} \dot{\phi}) - \frac{2c^2 (c^2 - 3\omega^2 r^2)}{r^2 (c^2 - \omega^4 r^2)^2} R \dddot{\phi} \dot{r} = 0$$

$$\dot{Z} - \dddot{\Sigma} \dot{z} - 2\dddot{\Sigma} \dot{z} = 0$$

(33)

Expanding Eq. (34) and using Eqs. (26)-(28) we obtain:

$$Z_{ss} + 2Z_{sr} \dot{r} + 2Z_{s\phi} \dot{\phi} + 2Z_{sz} \dot{z} + 2Z_{r\phi} \dot{\phi} + 2Z_{r\dot{z}} \dot{r} \dot{z} + 2Z_{\phi z} \dot{\phi} \dot{z}$$

$$+ Z_{r a(r) \dot{r}^2} - Z_{\phi b(r) \dot{\phi} \dot{r}} + Z_{rr} \dot{r}^2 + Z_{\phi \phi} \dot{\phi}^2 + Z_{zz} \dot{z}^2 - \Sigma_{ss} \dot{z}^2$$

$$- 2\Sigma_{s\phi} \dot{z} \dot{\phi} - 2\Sigma_{s\dot{z}} \dot{z} \dot{\phi} + 2\Sigma_{\phi z} \dot{\phi} \dot{z} - 2\Sigma_{\phi \phi} \dot{\phi} \dot{z}^2 - 2\Sigma_{\phi \dot{z}} \dot{\phi} \dot{z}^2$$

$$- \Sigma_{\phi \phi} \dot{\phi} \dot{z} - \Sigma_{\phi \phi} \dot{\phi}^2 - \Sigma_{s z} \dot{z}^2 = 0$$

(35)

The method we used is similar with the one in [9] and [15]. Condition (35) after a long calculation gives the following for $\Sigma$ and $Z$:

$$\Sigma = \frac{1}{2} c_1 s^2 + c_2 s + c_3 + z(c_5 s + c_6)$$

(36)

$$Z = c_5 z^2 + c_8 z + c_9 + s(\frac{1}{2} c_1 z + c_7)$$

(37)

where all the $c_i$ are arbitrary constants. Using the results above in Eqs. (32) and (33) after a long but straightforward calculation we obtain the following final result:

$$\Sigma = c_2 s + c_3 z + c_4$$

(38)

$$R = 0$$

(39)

$$\Phi = c_1$$

(40)
\[ Z = c_5 s + c_6 z + c_7 \]  

(41)

So the symmetries of the geodesic equations form a seven dimensional group with generators

\[ x_1' = \partial_\phi, x_2' = s \partial_s, x_3' = z \partial_s, x_4' = \partial_s, x_5' = s \partial_z, x_6' = z \partial_z, x_7' = \partial_z \]

The multiplication table for this Lie group is given by the relations:

\[ [\vec{x}_2', \vec{x}_3'] = -\vec{x}_3', [\vec{x}_2', \vec{x}_4'] = -\vec{x}_4', [\vec{x}_5', \vec{x}_2'] = -\vec{x}_5', [\vec{x}_3', \vec{x}_5'] = \vec{x}_6' - \vec{x}_2', [\vec{x}_3', \vec{x}_6'] = -\vec{x}_3', [\vec{x}_3', \vec{x}_7'] = -\vec{x}_4', [\vec{x}_4', \vec{x}_5'] = \vec{x}_7', [\vec{x}_5', \vec{x}_6'] = -\vec{x}_3', [\vec{x}_6', \vec{x}_7'] = -\vec{x}_7' \]

The subgroup \( g = [G, G] \) is five dimensional and is generated by the vectors \( \vec{x}_2' - \vec{x}_6', \vec{x}_3', \vec{x}_4', \vec{x}_5' \) and \( \vec{x}_7' \).

If we calculate the group \([g, g]\) we will see that this is the same as \( g \). So the derived series of ideals terminates. So the symmetry group of the geodesics is neither nilpotent, nor solvable, in contrary to systems like the Maxwell-Bloch [5] or the Hilbert-Einstein equations in Cosmology [15].

Using Noether’s theorems we can check which of these symmetries are variational and also we can find the corresponding conserved quantities. A vector field \( \vec{X} \) is a variational symmetry of a variational problem with Lagrangian \( L \) if and only if it satisfies the following condition:

\[ pr^{(1)} \vec{X} + L \vec{\nabla} \cdot \vec{\xi} = 0 \]  

(42)

where

\[ \vec{X} = \sum_{i=1}^{n} \lambda^i \frac{\partial}{\partial t^i} + \sum_{j=1}^{q} \psi^j \frac{\partial}{\partial u^j} \]  

(43)

and

\[ \vec{\xi} = \sum_{i=1}^{n} \lambda^i \frac{\partial}{\partial t^i} \]  

(44)

Every variational symmetry \( \vec{X} \) generates a conservation law, which can be written in the form:

\[ \vec{\nabla} \cdot \vec{P} = \sum_{j=1}^{q} Q_j \cdot E_j(L) \]  

(45)

where \( E_j(L) \) are the Euler-Lagrange operators acting on the Lagrangian and \( Q_j \) are the characteristics of \( \vec{X} \). In our case \( L \) is given by Eq. (8). Using the conditions above we can prove that only the infinitesimal generators \( \vec{X}_i \) with \( i = 1, 2, 3, 4 \) and 7 are variational symmetries. They form the Lie subgroup \( g_v \) of the full symmetry group. We can easily check that

\[ [g_v, g_v] = [g_v, [g_v, g_v]] = \{X_3', X_4'\} \]

and since \([X_3', X_4'] = 0\) we infer that the group \( g_v \) is solvable but not nilpotent. So we have the following theorem:
Theorem 1. The Group of symmetries of the geodesic equations of the spatial sections of the four dimensional space-time with respect to a uniformly rotating noninertial system of reference is a seven dimensional Lie group, which is neither nilpotent, nor solvable. on the other hand the rotational symmetries form a five dimensional subgroup, which is solvable but not nilpotent.

The equations of Killing, which can be written in the form

$$\zeta_{j;}^{,i} + \zeta_{i;}^{,j} = 0$$

(46)

determine the infinitesimal generators of the isometry group of a space. Here the semicolon represents covariant differentiation with respect to the metric(8).

If in Eqs. (38)-(41) we set $\Sigma = c_5 = 0$ we get the vector

$$c_1 \frac{\partial}{\partial \phi} + (c_6 z + c_7) \frac{\partial}{\partial z}$$

which acts only on the coordinates $r, \phi$ and $z$ of the spatial sections. This is of course a local symmetry of the geodesic equations so it preserves the geodesics. If we set $c_6 = c_7 = 0$ we get the vector

$$\frac{\partial}{\partial \phi}$$

which not only preserves the geodesics, but is also a Killing vector. A second Killing vector can be found if we set $c_6 = c_1 = 0$ and it is given by

$$\frac{\partial}{\partial z}$$

Finally for $c_1 = c_6 = 0$ we get the vector

$$z \frac{\partial}{\partial z}$$

which preserves the geodesics but it is not a Killing vector because it does not satisfy

$$\frac{\partial \xi_z}{\partial z} = 0$$

which is one of the Killing’s equations. Taking into account the fact that every Killing vector preserves the geodesics we infer that the above arguments indicate that the spatial sections have only two Killing vectors, which are the ones shown above.

4 The geodesics from the Hamiltonian point of view

Here we shall study the geodesics using the canonical symplectic structure on the cotangent bundle of the spatial sections [1], [2].
If $M$ is a Riemannian manifold with metric $g_{ij}$ we consider the Hamiltonian $H$ with local expression

$$H(x^i, p_i) = \frac{1}{2} g^{ij}(x^k)p_ip_j$$

where $x^i, p_i$ are canonical coordinates of the contangent bundle $T^*M$.

We can easily prove that the projections of the inertial curves of $H$ onto $M$ are the geodesics of $g_{ij}$ parameterized by their affine parameter $t$. Hamilton’s equations can be written in the form:

$$\dot{x}^i = g^{il}p_l$$  \hspace{1cm} (47)

$$\dot{p}_i = -\frac{1}{2} \frac{\partial g^{kl}}{\partial x^i} p_k p_l$$  \hspace{1cm} (48)

Inverting Eq. (47) we get

$$p_l = g_{il} \dot{x}^i$$  \hspace{1cm} (49)

Differentiating Eq. (47) and using Eqs. (47) and (49) we can find

$$\ddot{x}^i = -\frac{1}{2} g^{il} \frac{\partial g^{mn}}{\partial x^l} p_mp_n + \frac{\partial g^{il}}{\partial x^k} g^{km} p_mp_l$$  \hspace{1cm} (50)

Using Eq. (49) and after some algebra we end up with:

$$\ddot{x}^i = g^{il} \left[ \frac{\partial g_{jk}}{2 \partial x^l} \dot{x}^k \dot{x}^j - \frac{\partial g_{lj}}{\partial x^k} \dot{x}^l \dot{x}^j \right]$$  \hspace{1cm} (51)

which is actually

$$\ddot{x}^i = g^{il} \left[ \frac{\partial g_{jk}}{2 \partial x^l} \dot{x}^k \dot{x}^j - \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{lk}}{\partial x^j} \right] \dot{x}^k \dot{x}^j$$  \hspace{1cm} (52)

Taking into account the definition of Christoffel symbols it becomes clear that here we have the geodesic Eqs.(21).

Using the metric (8) we get the Hamiltonian in the form:

$$H(x^i, p_i) = \frac{1}{2} p_r^2 + \frac{1}{2} \frac{c^2 - \omega^2 r^2}{c^2 r^2} p_\phi^2 + \frac{1}{2} p_z^2$$  \hspace{1cm} (53)

We can easily prove that this Hamiltonian is separable since it satisfies the Levi-Civita conditions [10]. Thus we can solve this system using the Hamilton-Jacobi method. Hamilton’s characteristic function takes the form

$$W = W_1(r) + p_\phi \phi + p_z z$$  \hspace{1cm} (54)

and the Hamilton-Jacobi equation becomes

$$\frac{1}{2} \left( \frac{dW_1}{dr} \right)^2 + \frac{1}{2} \frac{c^2 - \omega^2 r^2}{c^2 r^2} p_\phi^2 + \frac{1}{2} p_z^2 = E$$  \hspace{1cm} (55)
where $p_\phi, p_z$ and $E$ are constants. Integrating we get $W$

$$W = \frac{1}{c} \int \frac{1}{r} \sqrt{(2Ec^2 - c^2 p_z^2 + \omega^2 p_\phi^2)r^2 - c^2 p_\phi^2} dr + p_\phi \phi + p_z z$$  (56)

At this stage we can write down the solution in the form

$$t + c_1 = \frac{\partial W}{\partial E}$$  (57)

$$c_2 = \frac{\partial W}{\partial p_\phi}$$  (58)

$$c_3 = \frac{\partial W}{\partial p_z}$$  (59)

where $t$ is the affine parameter of the geodesics and $c_1, c_2$ and $c_3$ are arbitrary constants. Using Eq. (55) and after some integrations we end up with the following results:

$$t + c_1 = \frac{c}{A} \sqrt{Ar^2 - B}$$  (60)

$$c_2 = \phi + \frac{\omega^2 p_\phi \sqrt{Ar^2 - B}}{c} + \tan^{-1} \left[ \frac{c p_\phi}{\sqrt{Ar^2 - B}} \right]$$  (61)

$$c_3 = z - \frac{c p_z}{A} \sqrt{Ar^2 - B}$$  (62)

where $A$ and $B$ are given by:

$$A = 2Ec^2 - c^2 p_z^2 + \omega^2 p_\phi^2$$  (63)

$$B = c^2 p_\phi^2$$  (64)

In the Maxwell-Bloch system we found a symplectic realization [5] and using it we have found the action angle variables and then we integrated the system using elliptic functions. In contrary as we have seen the above system can be integrated using elementary functions.

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