Conservation laws and scattering for de Sitter classical particles

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Abstract
Starting from an intrinsic geometric characterization of de Sitter timelike and lightlike geodesics we give a new description of the conserved quantities associated with classical free particles on the de Sitter manifold. These quantities allow for a natural discussion of classical pointlike scattering and decay processes. We also provide an intrinsic definition of the energy of a classical de Sitter particle and discuss its different expressions in various local coordinate systems and their relations with earlier definitions found in the literature.

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1. Introduction
Since the first pioneering observations of the luminosity-red shift relation of distant type Ia supernovae [1–4] it has been accepted as an established fact that the expansion of the universe is accelerating. This circumstance could be interpreted by saying that there exists some kind of agent, dubbed dark energy, which exerts an overall repulsive effect on ordinary matter (both visible and dark). This repulsion has long since overcome the mutual attraction of the various parts of the latter, thereby being responsible for the present accelerated expansion. The nature of dark energy is to date entirely mysterious. The only facts we know with reasonable certainty are that dark energy contributes today an amount of about 73% (the exact figure depending on the cosmological model adopted) to the total energy content of the universe, and that its spatial distribution is compatible with perfect uniformity.
The simplest possible explanation for dark energy which can be put forward is to assume that it is just a universal constant, the so-called cosmological constant, denoted \( \Lambda \). If we espouse this point of view, this would mean that the background arena for all natural phenomena, once all physical matter–energy has been ideally removed, is not the familiar flat Minkowski spacetime \( M^{(1,3)} \). Instead, it consists of the maximally symmetric de Sitter spacetime \( dS_4 \) whose radius \( R \) is related to \( \Lambda \) by the equation \( R = \sqrt{3/\Lambda} \). The actual value of \( \Lambda \) is extremely small in astrophysical and also in galactic terms (\( \Lambda \simeq 10^{-56} \text{ cm}^{-2} \)), so that cosmic expansion has no significant effect, say, on the structure of a typical galaxy, such a structure being essentially controlled by the material (in all its forms) composing the galaxy itself, by the mutual gravitational attraction of the galaxy’s parts and by the galaxy’s angular momentum. On the other hand, \( \Lambda \) has an essential effect on the distribution of matter on large cosmic scales, such as on the structure of the cobweb pattern of filaments and voids characterizing the arrangements of galaxies and galaxy clusters in the universe.

It is not our purpose here to deal with the now long-standing problem of the nature of dark energy and why the dark energy content of the universe is, at the present epoch, comparable with the universe’s ordinary matter content (see, e.g., the reviews [5–7]). Instead, we adhere to the simple working hypothesis that the cosmological constant is a true universal constant, just like the speed of light and Planck constant are, say.

In the approximation in which the effects of gravitation on the geometry of spacetime can, at least locally, be neglected, the presence of a cosmological constant would naturally lead to the problem of the formulation of the theory of special relativity in the presence of a universal residual constant background curvature, namely of a de Sitter relativity in place of the customary flat Minkowski one. Then, the symmetry group of the theory would be the de Sitter group \( SO(1, 4) \) (the Lorentz group in five dimensions) instead of the Poincaré group, which is the contraction [8] of the latter arising in the limit \( \Lambda \to 0 \). A considerable amount of work has already been performed in this direction, both in the classical [9–16] as well as in the quantum domain [17–22]. However, to our knowledge, at an elementary level a systematic treatment of particle kinematics and dynamics in de Sitter spacetime is still lacking.

In this paper, we contribute to filling this gap by providing a description of the free motion of classical particles and particle collisions in terms of an intrinsic characterization of the associated conservation laws. We adhere, of course, to the geodesic hypothesis [23]: the worldline of a free particle is a geodesic in spacetime. In the de Sitter universe, timelike and lightlike geodesics can be fully and economically characterized by using the closest analogue to Minkowski momentum space that is available: this is the lightcone of the five-dimensional Minkowski space \( M^{(1,4)} \) in which the de Sitter universe can be represented as an embedded four-dimensional one-sheeted hyperboloid. The relevant conserved quantities associated with free motion can themselves be expressed in terms of the same lightlike five-vectors, as we do here. Then, it turns out that, in a given particle collision, the conservation of energy and momentum of ingoing and outgoing particles at the collision point can be expressed in terms of the corresponding one particle conserved quantities before and after the collision.

The structure of the paper is as follows. In section 2, we first recall the expression of the generators of the de Sitter symmetry group in terms of the flat coordinates of the five-dimensional ambient Minkowski space. Then, by using the Noether theorem applied to the invariant action of a free massive particle, we derive the set of the associated conserved quantities \( K \). Of such conserved quantities we give two different intrinsic characterizations. One in terms of the two lightlike vectors \( \xi \) and \( \eta \) of \( M^{(1,4)} \) that uniquely identify the given timelike geodesic. The other one in terms of either one of such vectors and of a given point of the geodesic. These characterizations are independent of the choice of any particular coordinate patch on the de Sitter manifold \( dS_4 \). We also find the corresponding formulae for
lightlike geodesics. We add a few remarks on how this scheme could turn out to be useful for quantization and briefly discuss a de Sitter classical analogue of the Hawking–Unruh effect.

In section 3, we describe particle collisions and decays in terms of the conserved quantities introduced earlier. Precisely, we re-express the conservation of the total energy–momentum at the point of a collision as a conservation law for the total invariants $K$. In particular, the conservation equations can be given a perspicuous expression which involves explicitly the collision point. The conservation of the invariants $K$ allows us to relate the values of the energy and momentum at the point of collision to their values at any observation point. We do this by providing an explicit formula, valid both for massive and massless particles, which indeed relates the energy–momentum vector at two arbitrary points on the geodesic. In particular, this formula applied to photons yields the well-known frequency redshift relation.

Section 4 is devoted to the definition of the energy of a free particle, both massive and massless, by comparison of the corresponding geodesic to the reference geodesic associated with a localized observer. This definition is itself intrinsic and does not make reference to any particular coordinate patch. However, we also give the explicit expression of the energy in terms of some specific coordinate choices on the de Sitter manifold: flat, spherical and static coordinates, the first two having cosmological significance, the third chosen for its relevance in black hole physics. To our knowledge, this definition of the energy of a de Sitter particle first appeared in [16]; it was introduced there in yet another set of coordinates, the stereographic ones. The value and the novelty of our approach resides in the fact that we have given a coordinate independent meaning to the definition of energy.

Section 5 is devoted to a possible definition of particle momentum. While the definition of energy is intrinsic, being purely related to a reference geodesic, any possible definition of momentum is unavoidably linked to a choice of a coordinate system. Nevertheless, we examine reasonable expressions of momenta corresponding to different choices of coordinates. The consistency of these definitions is set in evidence by the fact that in the flat Minkowski limit all these choices converge to the correct flat momenta expression.

We end with several concluding remarks.

2. Conservation laws for de Sitter motion

In what follows we will present our results by making reference to the (physical) four-dimensional de Sitter spacetime. However, as will be evident from the discussion, our formulae are completely general and valid in any dimension just by replacing 4 by $d$ (and 5 by $d + 1$).

The four-dimensional de Sitter spacetime $dS_4$ can be realized as the one-sheeted hyperboloid with equation

$$dS_4 = \{ X \in M^{(1,4)}, X^2 = X \cdot X = \eta_{AB} X^A X^B = -R^2 \} \quad (1)$$

embedded in the five-dimensional Minkowski spacetime $M^{(1,4)}$ where a Lorentzian coordinate system has been chosen: $X = X^A \epsilon_A$ and whose metric is given by $\eta_{AB} = \text{diag}\{1, -1, -1, -1\}$ in any Lorentzian frame. The geometry of the de Sitter spacetime is induced by restriction of the metric of the ambient spacetime to the manifold,

$$ds^2 = (\eta_{AB} dX^A dX^B)_{dS_4} \quad (2)$$

This is the maximally symmetric solution of the cosmological Einstein equations in vacuo provided that $R = \sqrt{3/\Lambda}$, with $\Lambda > 0$. The corresponding isometry group (the relativity
group of $dS_4$ is $SO(1,4)$, i.e. the Lorentz group of the ambient spacetime $M^{(1,4)}$ which is generated by the following ten Killing vector fields:\footnote{The restriction of these operators to the de Sitter manifold is well defined. This can be shown by introducing the projection operator $h$ and the tangential derivative $D$ as follows:}

$$L_{AB} = \left( X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} \right)_{dS_4}. \tag{3}$$

Since the group acts transitively on the manifold $dS_4$ it is useful to select a reference point (the origin) in $dS_4$ as follows:

$$X_0 = (0, 0, 0, 0, R). \tag{4}$$

Now consider a classical massive particle on the de Sitter universe. The usual action for geodesical (free) motion can be written by using the coordinates of the ambient five-dimensional spacetime as follows:

$$S = -mc \int \left[ (V^2)^\frac{1}{2} + a(X^2 + R^2) \right] d\lambda; \tag{5}$$

here $\lambda \rightarrow X(\lambda)$ is a parametrized timelike curve subject to the constraint $X^2(\lambda) = -R^2$ as enforced by the Lagrange multiplier $a$; $V^A(\lambda) = dX^A/d\lambda$ is the corresponding velocity. $V^A(\lambda)$ is tangent to the curve $X(\lambda)$ and therefore orthogonal to the vector $X(\lambda)$ (in the ambient space sense). The condition of tangentiality $X \cdot V = 0$ has to be imposed also on the initial conditions when solving the equations of motion. Consider now the generic infinitesimal isometry of $dS_4$

$$X_A \mapsto X_A + \omega_{AB} X^B, \tag{6}$$

where $\omega_{AB}$ are antisymmetric infinitesimal parameters. The action is invariant under (6) and using the Noether theorem we find ten quantities that are conserved along the timelike geodesics,

$$K_{AB} = \frac{m(X_A V_B - X_B V_A)}{R \sqrt{V^2}} = \frac{m}{R} (X_A W_B - X_B W_A) \tag{7}$$

$$= \frac{1}{R} (X_A \Pi_B - X_B \Pi_A);$$

$W^A = dX^A/d\tau$ is the Minkowskian five-velocity relative to the proper time $d\tau = ds/c$ and $\Pi^A = m dX^A/d\tau$ the corresponding Minkowskian five-momentum. Of these ten quantities only six are independent. Indeed, in order to specify a geodesic completely one must for example assign the proper initial conditions, namely the initial point on $dS_4$ and the initial velocity (at $\tau = 0$, say). We note that the quantities (7) are of course defined also along particle trajectories which are not geodesics. However, in this case not all of them (if any) will be constants of the motion.

We now derive two alternative intrinsic characterizations of the conserved quantities $K_{AB}$ which are independent on the choice of any particular coordinate patch on $dS_4$. We do this by exploiting an elementary way to describe the de Sitter timelike geodesics: in complete analogy with the great circles of a sphere that are constructed by intersecting the sphere with planes $\ldots$

\footnote{The restriction of these operators to the de Sitter manifold is well defined. This can be shown by introducing the projection operator $h$ and the tangential derivative $D$ as follows:}

$$h^{AB} = \eta^{AB} + \frac{X^A X^B}{R^2}, \quad D^A = h^{AB} \partial_B = \partial^A + \frac{X^A}{R^2} X \cdot \partial.$$
containing its center, the de Sitter timelike geodesics can be obtained as intersections between dS4 and two planes containing the origin of \( M^{(1,4)} \) and having three independent spacelike normals. Each such two plane also intersects the forward lightcone in \( M^{(1,4)} \) (the asymptotic cone),

\[
C^+ = \{ X \in M^{(1,4)}, X^2 = 0, X^0 > 0 \},
\]

along two of its generatrices. Any two future directed null vectors \( \xi \) and \( \eta \) lying on such generatrices (see figure 1) can be used to parametrize the corresponding geodesic in terms of the proper time as follows \[22\]:

\[
X(\tau) = R \frac{\xi e^{-\tau} - \eta e^{\tau}}{\sqrt{2\xi \cdot \eta}}.
\]

Then, by inserting (9) into (7) we find that the conserved quantities have a very simple expression, homogeneous of degree zero, in the components of the vectors \( \xi \) and \( \eta \),

\[
K_{AB} = mc \frac{\xi A \eta B - \eta A \xi B}{\xi \cdot \eta}.
\]

These numbers also coincide with the components of the 2-form

\[
K = K_{(\xi,\eta)} = mc \frac{\xi \wedge \eta}{\xi \cdot \eta},
\]

in the frame \( \{ \epsilon_A \} \) that has been chosen in the ambient space. We normalize the dimensionless vectors \( \xi \) and \( \eta \) according with

\[
\xi \cdot \eta = \frac{2m^2}{k^2},
\]

\[6 \xi \] and \( \eta \) denote here the covariant 1-forms associated with the null vectors; we use the same symbol for a vector and its dual.
where $k$ is a constant with the dimensions of a mass whose value can be fixed according to the specific convenience. With this normalization, formulae (9)–(11) write, respectively,

$$X(\tau) = \frac{kr}{2m}(\xi e^{-\frac{c\tau}{R}} - \eta e^{-\frac{c\tau}{R}}),$$  \hspace{1cm} (13)$$

$$K_{AB} = \frac{k^2 c}{2m}(\xi_A \eta_B - \xi_B \eta_A), \hspace{1cm} K_{(\xi,\eta)} = \frac{k^2 c}{2m}(\xi \wedge \eta).$$  \hspace{1cm} (14)$$

The replacements $\xi \rightarrow \mu \xi, \eta \rightarrow \frac{1}{\mu} \eta (\mu > 0)$, do not alter (12). As (13) shows, they do however shift the origin of the $\tau$ variable. Therefore, the normalizations of $\xi$ and $\eta$ are fixed separately by equation (12) (in which a given choice has been made for the positive constant $k$) and by selecting the point on the geodesic corresponding to zero proper time. With these qualifications it turns out that the pair $(\xi, \eta)$ depends on six independent parameters. Then (14) shows once more that only six of the ten constants of motion $K_{AB}$ are independent (in the appendix we illustrate this fact with an explicit example).

Formula (11) (or equivalently formula (14)) provides our first intrinsic characterizations of the constants $K_{AB}$. The second characterization that we display brings about an arbitrary fixed point $X(\tau)$ on the geodesics. Indeed, from (13) one has the relation

$$\eta = \xi - \frac{2m}{kr} \bar{X},$$  \hspace{1cm} (15)$$

where $\bar{X} = X(0)$, which allows us to rewrite the geodesic (13) in the alternative form

$$X(\tau) = \bar{X} e^{-\frac{c\tau}{R}} + \frac{kr}{m} \sinh \frac{c\tau}{R}.$$  \hspace{1cm} (16)$$

Inserting (15) into (14) and using (16) gives

$$K_{AB} = \frac{kc}{R} (X_A(0)\xi_B - X_B(0)\xi_A) = \frac{kc}{R} e^{-\frac{c\tau}{R}} (X_A(\tau)\xi_B - X_B(\tau)\xi_A).$$  \hspace{1cm} (17)$$

As before, we can introduce the tensor

$$K = K_{\xi,\bar{X}} = \frac{kc}{R} e^{-\frac{c\tau}{R}} (X(\tau) \wedge \xi).$$  \hspace{1cm} (18)$$

The normalization (12) and equation (15) imply

$$\xi \cdot \bar{X} = -\frac{Rm}{k}.$$  \hspace{1cm} (19)$$

The tensor $K$ in its two alternative expressions (11) and (18) will play an important role in the following.

To perform the massless limit we set

$$m = k\epsilon, \hspace{1cm} \tau = \frac{\sigma}{c} \epsilon$$  \hspace{1cm} (20)$$

and let $\epsilon \rightarrow 0$ in equations (16) and (19), thus obtaining the parametrization of a lightlike geodesic,

$$X(\sigma) = \bar{X} + \xi \sigma \hspace{1cm} \text{with} \hspace{1cm} \xi \cdot \bar{X} = 0,$$  \hspace{1cm} (21)$$

where $\sigma$ is an affine parameter. Therefore, a lightlike geodesic is characterized by one lightlike vector which is parallel to the geodesic and by the choice of an initial event that uniquely selects the particular geodesic among the infinitely many pointing in that direction. The conserved quantities are still given by formula (17),

$$K_{AB} = \frac{kc}{R} (X_A(0)\xi_B - X_B(0)\xi_A) = \frac{1}{R} (X_A \Pi_B - X_B \Pi_A).$$  \hspace{1cm} (22)$$
where $\Pi^A = k e \frac{dX^A}{dr}$ is the Minkowskian five-momentum of the zero mass particle. There is of course no analogue of formulae (11) and (14) because $\xi$ and $\eta$ coincide in the massless limit.

An alternative standard way to arrive at formulae (22) starts from rewriting the action for a massive particle in the first-order formalism,

$$S[e, \gamma] = \frac{k}{2} \int_{\gamma} \left[ \frac{1}{e} V^2 + a(X^2 + R^2) \right] \, d\lambda + \frac{1}{2k} m^2 c^2 \int_{\gamma} e \, d\lambda,$$

(23)

where $e$ is a function of $\lambda$ and $k$ is once more a constant with the dimensions of a mass. The equations of motion are obtained by varying the action with respect to $e$ and to the curve $\gamma$. The action for massless particles is obtained by setting $m = 0$ in equation (23) and the corresponding equations of motion are

$$V^2 = 0, \quad \frac{d}{d\lambda} \left( \frac{1}{e} V^4 \right) = 0.$$

(24)

Introducing an affine parameter $\sigma$ such that $d\sigma = c e(\lambda) \, d\lambda$, the general solution is (21).

The conserved quantities can be determined as before by means of the Noether theorem, giving (22).

2.1. Remarks on quantization

The setup that we have just described can also be employed to provide a fresh look at de Sitter quantum mechanics and field theory. Indeed, the variables on the cone in $\mathbb{M}^{1,4}$ that we have been using to describe the geodesics can be employed to parametrize the phase-space pertinent to elementary systems. Then one can invoke his/her favorite method, like geometric quantization [24, 25] or the method of coadjoint orbits [26] to obtain a quantum description of such elementary systems. In doing this, a substantial difference will arise when quantization deals with massless particles and the method will fail to provide a de Sitter covariant theory. This problem has been known for a long time and has fairly profound implications for cosmological structure formation (since it gives rise to the scale invariant spectrum of inflation [27–29]) and for the dynamical restoration of spontaneously broken continuous symmetries [30]. These effects are genuine quantum phenomena and have no classical counterpart. We shall not investigate geometric quantization further here and leave it for future work.

However, we can at least provide here an heuristic example of a connection between the classical and quantum counterparts of a typical physical effect arising in the de Sitter manifold. For simplicity, we discuss this in two-dimensional de Sitter spacetime. Specifically, consider a ‘rigid’ (here one-dimensional) box of length $2L$ containing initially a uniform distribution of a large number of identical point particles of mass $m$ which are all at rest. ‘Rigid’ means here that we assume the internal forces holding the box together to prevent it from participating unhindered to the de Sitter expansion, so that a local geodesic observer $O$ comoving with the box sees that the spatial extension of the latter does not change in time. In other words, the walls of the box are not receding away during the cosmic expansion. In particular, if we assume the observer $O$ to sit at the midpoint of the box, the worldlines of the endpoints of the box will not be geodesics and the box itself will shrink compared to the comoving spatial coordinates. As to ‘initially at rest’ it means that, at a given initial time, all the particles inside the box are assumed to move with zero velocity in the comoving frame. Then, because of the expansion of the universe, the observer $O$ will see the particles move away from each other and eventually start hitting the walls of the box, bounce back and collide with each other. By virialization, the result is that (after a long time and in a nonrelativistic framework) they will reach a thermodynamic equilibrium at a temperature $T_c = H^2 L^2 m/2.26 k$, where $H$ is the Hubble constant and $k$ is the Boltzmann constant (see appendix B).
Now consider a quantum scalar field in $dS_2$, which we assume to be in its ground state (the de Sitter vacuum [17, 31]). Due to the interaction with the spacetime curvature the vacuum fluctuations generate real particles with a thermal spectrum at temperature $T_q = \hbar c/(2\pi R k)$ [19, 32]. It seems reasonable to assume that the lightest allowed mass (the $dS$ mass) for such a field is the one corresponding to the Compton wavelength of the order of the de Sitter radius $R$, giving $m = m_{dS} = \hbar/c R$ which would correspond to a quantum temperature $T_q = m_{dS}^2/4\pi^2 k \simeq m_{dS} R^2 H^2/k$. This is of the same order of magnitude of the classical temperature $T_c = H^2 R^2 m_{dS}/2.26k$ calculated before, for classical particles of mass $m_{dS}$ in a box extending to the cosmological horizon. This allows us to interpret the classical temperature $T_c = H^2 R^2 m_{dS}/2.26k$ as a classical analogue, in de Sitter spacetime, of the Hawking–Unruh effect.

We have derived the above analogy for a two-dimensional de Sitter spacetime. However, the above considerations can be easily extended to $dS_4$ provided the classical particles are taken to be rigid spheres of some small but nonzero radius.

In addition, the expression for $T_c$ has been derived under the assumption that the classical particles are nonrelativistic. Strictly speaking, this approximation is not justified. Indeed, since $R \approx 10^{28}$ cm, the quantum de Sitter temperature $T_q$ is of the order of $10^{-29}$ K, whereas the de Sitter mass $m_{dS}$ is of the order $10^{-65}$ g. Therefore, the average velocity of our de Sitter particles at the de Sitter temperature is comparable with the speed of light, so that they are highly relativistic. This fact can be readily understood by noting that, as the walls of the box approach the cosmological horizon, their speed relative to the comoving coordinates approaches $c$. Therefore, when the first of the de Sitter particle hits the wall it bounces back with a highly relativistic speed, which is then transmitted to the other de Sitter particles through particle collisions and further collisions with the walls themselves. Nonetheless, since we are only concerned with orders of magnitude, we still claim that our crude estimates relating classical and quantum de Sitter temperatures are justified.

Finally note that our effective vacuum particles can in some sense be viewed upon as the lightest detectable particles in a de Sitter background. Indeed, in such a background the largest uncertainty in position is $\Delta x \simeq R$, whereas $\Delta p \simeq mc$, so that the Heisenberg principle gives $m \gtrsim \hbar/c R$. It is not clear what this exactly means, though one may boldly suggest that quantum effects in de Sitter forbid the existence of lighter particles. More presumably, it may mean that the semiclassical description of lightest particles is too naive and that strong quantum effects do come into play.

### 3. Collisions and decays

We consider the collision of two ingoing particles which gives rise to the production of a certain number of outgoing particles,

$$b_1 + b_2 \rightarrow c_1 + c_2 + \cdots + c_N. \quad (25)$$

The particles $b_i$, with masses $m_i$, are described by geodesic curves ending at the collision point $\bar{X}$, which is also the starting point of the $N$ geodesics describing the outgoing particles $c_f$ with masses $m_f$. We assume the collision point to be the common zero of the proper time of all particles involved in the process, namely $\bar{X} = X_i(0) = X_f(0)$. Denoting by $(\chi_i, \zeta_i)$ and $(\xi_f, \eta_f)$ the pairs of normalized null vectors parametrizing the ingoing and outgoing particles we have

$$\zeta_i = \chi_i - \frac{2m_i}{k_i R} \bar{X}, \quad i = 1, 2; \quad \eta_f = \xi_f - \frac{2m_f}{k_f R} \bar{X}, \quad f = 1, 2, \ldots, N. \quad (26)$$
and the quantities which are conserved along each geodesic are
\[ K_i = \frac{k_i c}{R} \hat{X} \wedge \chi_i, \quad i = 1, 2; \quad K_f = \frac{k_f c}{R} \hat{X} \wedge \xi_f, \quad f = 1, 2, \ldots, N. \]  
(27)

Solving the collision problem amounts to finding the outgoing vectors \( \xi_f \) given the ingoing ones \( \chi_i \). At the collision point the total covariant energy–momentum four-vector must be conserved,
\[ \pi^{\mu}_1 + \pi^{\mu}_2 = \sum_{j=1}^{M} \pi^{\mu}_j. \]  
(28)

Here we have introduced a local coordinate system \( x^\mu, \mu = 0, 1, 2, 3 \), so that \( \pi^{\mu}_1 = mx^\mu \) (respectively, \( \pi^{\mu}_2 = kc x^\mu \)) for any given massive particle of timelike (respectively, massless particle of lightlike) worldline \( x^\mu(\tau) \) (respectively \( x^\mu(\sigma) \)) on \( dS_4 \). In terms of the embedding in \( M(1,4) \), at the collision point \( \hat{X} \) we have, for a given particle,
\[ K_{AB}|_{\hat{X}=\hat{X}} = \frac{1}{R} \left( X_A \frac{\partial X_B}{\partial x^\mu} - X_B \frac{\partial X_A}{\partial x^\mu} \right) \bigg|_{x=\bar{x}} \pi^{\mu}. \]  
(29)

where \( X_A = X_A(x^\mu(\tau)) \) or \( X_A = X_A(x^\mu(\sigma)) \) depending on whether the particle is massive or massless. By summing over all ingoing and outgoing particles and using (28) we find the simple relation
\[ K_1 + K_2 = \sum_{f=1}^{N} K_f. \]  
(30)

Similarly, for the decay \( b \rightarrow c_1 + c_2 + \cdots + c_N \) of a single particle
\[ K = \sum_{f=1}^{N} K_f. \]  
(31)

Note that \( K_{AB}K^{AB} = -2m^2c^2 \). This relation replaces in \( dS_4 \) the Minkowskian one \( \pi_{\mu}\pi^{\mu} = m^2c^2 \). Then, choosing the normalization constants \( k_i \) and \( k_f \) equal for all particles, equation (27) allows us to write the conservation equations (30) and (31) respectively as
\[ \left( \chi_1 + \chi_2 - \sum_{f=1}^{N} \xi_f \right) \wedge \hat{X} = 0, \]  
(32)
\[ \left( \chi - \sum_{f=1}^{N} \xi_f \right) \wedge \hat{X} = 0. \]  
(33)

Though equations (30) and (32) are equivalent to equation (28) they have the advantage of being expressed in an intrinsic form. To further clarify their meaning it is interesting to find the explicit expressions of the null vectors \( \chi_i \) and \( \xi_f \) corresponding to a particular choice of the collision event \( \hat{X} \). For example, choosing \( \hat{X} = X_0 \) equation (32) becomes equivalent to
\[ \chi_1^\mu + \chi_2^\mu = \sum_{f=1}^{N} \xi_f^\mu, \quad \mu = 0, 1, 2, 3. \]  
(34)

From equation (26)
\[ \xi^\mu = \chi^\mu, \quad \mu = 0, 1, 2, 3, \quad \text{and} \quad \xi^4 = \chi^4 = \frac{2m}{k}. \]  
(35)
(we have omitted the index \(i = 1, 2\) for notational simplicity). Since \(\chi\) and \(\zeta\) are null vectors, if \(m \neq 0\) this relation implies \(\chi^4 = -\zeta^4 = \frac{m^4}{k^2}\). Therefore, we have

\[
\chi = \left(\chi^0, \bar{\chi}, \frac{m}{k}\right), \quad \zeta = \left(\chi^0, \bar{\chi}, -\frac{m}{k}\right),
\]

with

\[
\left(\chi^0\right)^2 - \left(\bar{\chi}\right)^2 = \frac{m^2}{k^2}.
\]

By using the parametrization (13) it follows that

\[
m \frac{dX^\mu}{d\tau}\bigg|_{\tau=0} = kc \chi^\mu = q^\mu, \quad m \frac{dX^4}{d\tau}\bigg|_{\tau=0} = 0,
\]

with

\[
q^2 = (q^0)^2 - \left(\bar{q}\right)^2 = m^2 c^2.
\]

In a small neighborhood of \(X_0\) in \(dS_4\) we choose local coordinates \(x^\mu\) defined by \(x^\mu = X^\mu, \mu = 0, 1, 2, 3\). Since the plane \(X^4 = R\) is tangent to \(dS_4\) at \(X_0\) we have \(\partial X_0/\partial x^\mu|_{X_0} = 0\) so that, at \(X_0\), the metric of \(dS_4\), expressed in terms of the coordinates \(x^\mu\), is given by \(ds^2|_{X_0} = (\eta_{AB} dX^A dX^B)|_{dS_4, X_0} = \eta_{\mu\nu} dx^\mu dx^\nu\). Then \(x^\mu\) are locally Lorentzian at \(X_0\) and \(dX^\mu/d\tau|_{\tau=0} = dx^\mu/d\tau|_{\tau=0}\) where \(x^\mu(\tau)\) is the parametrization of the geodesic at \(X_0\). Hence, equation (38) tells us that \(q^\mu\) can be interpreted as the components (in the chosen frame) of the Lorentzian four-momentum at \(X_0\) of the particle moving along the geodesic \(x^\mu(\tau)\). This interpretation applies to zero mass particles as well.

Then, denoting by \(q_i^\mu\) and \(\tilde{q}_f^\mu\), respectively, the four-momenta at the collision point \(\bar{X} = X_0\) of the incoming and outgoing particles relative to the coordinates \(x^\mu\) we have

\[
\chi_i = \frac{1}{kc} \left(q_i^0, \bar{q}_i, m_i c\right), \quad i = 1, 2,
\]

and

\[
\xi_f = \frac{1}{kc} \left(\tilde{q}_f^0, \bar{\tilde{q}}_f, \tilde{m}_f c\right), \quad f = 1, 2, \ldots, N
\]

and the conservation equation (34) becomes

\[
q_1^\mu + q_2^\mu = \sum_{f=1}^{N} \tilde{q}_f^\mu
\]

expressing once more the equivalence of (32)–(28). Similar considerations apply to the decay (31).

The expressions of the incoming and outgoing null vectors \(\chi_i\) and \(\xi_f\) in the general case, when the collision point \(\bar{X}\) is arbitrary, can be obtained by applying to (40) and (41) an arbitrary five-dimensional Lorentz transformation.

In conclusion, it is worth noting that since any Lorentzian manifold is locally inertial, at the classical level the conservation laws in de Sitter point particle collisions express nothing more than the usual total energy–momentum conservation in the process, so that \(\Lambda\) plays no role here. The situation is drastically different in the quantum case due essentially to the spread of wave packets. For example, in de Sitter particle decay the decay amplitude depends on \(\Lambda\) and the presence of curvature allows in some cases for a nonzero probability for an unstable particle of mass \(m\) to decay into particles whose total mass is larger than \(m\), a process which is strictly forbidden in Minkowski spacetime due to energy–momentum conservation [20].
Finally, as regards the geodesic motion of a single particle, it is important to remark that the explicit expressions
\[ \xi = \frac{1}{k_c}(q^0, \vec{q}, mc), \quad \eta = \frac{1}{k_c}(q^0, -\vec{q}, -mc), \] (43)
of the components of the pair of normalized null vectors \( \xi \) and \( \eta \) characterizing the particle geodesic \( X(\tau) \) when \( \bar{X} \) is chosen at the origin (4) as well as their corresponding expressions for arbitrary \( \bar{X} \), which are obtained by applying to (43) a suitable five-dimensional Lorentz transformation, depend solely on the choice of \( \bar{X} \) and do not make reference to any particular local coordinate system on \( dS_4 \). Instead the introduction of one such suitable system about \( \bar{X} \) is made necessary for the correct physical interpretation of the components of \( \xi \) and \( \eta \).

3.1. Detection

In a collision process any outgoing particle is not detected, and its properties measured, at a collision point \( \bar{X} \). Instead, the detection takes place at some other event far away from \( \bar{X} \). In particular, if we measure the energy and the momentum of the particle, we need a formula which relates these quantities at the point of measurement to the same quantities at the production point. To avoid being monotonous we illustrate the procedure with a lively example. Consider the \( pp \) scattering
\[ p + p \rightarrow p + p + a + b + c, \]
and suppose that we are searching for an intermediate process
\[ p + p \rightarrow p + p + Z \rightarrow p + p + a + b + c, \] (44)
where \( Z \) is a massive particle decaying into the triple \( a, b, c \) with a very short lifetime, so that it cannot be directly detected. Then, by (31)
\[ K_Z = K_a + K_b + K_c, \]
so that
\[ 2K_Z^2 := (K_{aAB} + K_{bAB} + K_{cAB})(K_a^{AB} + K_b^{AB} + K_c^{AB}) = -2m_Z^2c^2 \]
must hold. Assume we look at a large number of such processes and that we are able to measure experimentally \( K_{aAB}, K_{bAB} \) and \( K_{cAB} \) in each individual process. Then, plotting the number density of processes \( \frac{dn}{dK} \) as a function of the invariant mass \( Q_Z := \sqrt{-K_Z^2} \), we should find a resonance at \( Q_Z = m_Zc \). As a well-known example of a reaction of the type (44) we may mention the process
\[ p + p \rightarrow p + p + Z \rightarrow p + p + \pi^+ + \pi^- + \pi^0, \]
where \( Z \) can either be one of the mesons \( \eta(547) \) or \( \omega(782) \) or some broader resonance (see, e.g., [33]).

The experimental problem of measuring the quantities \( K_a, K_b, K_c \) could be tackled as follows. To fix ideas, consider just one particle, which we suppose to detect at an event whose local coordinates are \( x^\mu_1 \). Barring intrinsic indeterminacies the detection measures the position \( x^\mu_1 \) and the momentum \( \pi^\mu(\tau_1) \). Then \( K_{aAB} \) is determined by (29) as
\[ K_{aAB} = K_{aAB}|_{x=x_1} = \frac{1}{R} \left( X_A \frac{\partial X_B}{\partial x^\mu} - X_B \frac{\partial X_A}{\partial x^\mu} \right)|_{x=x_1} \pi^\mu(\tau_1). \] (45)
This formula can be used to relate the covariant momentum at the point of measurement to the one at the collision point. Indeed, if \( x^\mu_0 \) are the local coordinates of the collision event and \( \pi^\mu(0) \) the covariant momentum of the particle at the same point, then
\[ K_{aAB}|_{x=x_1} \left. = \frac{1}{R} \left( X_A \frac{\partial X_B}{\partial x^\mu} - X_B \frac{\partial X_A}{\partial x^\mu} \right) \right|_{x=x_0} \pi^\mu(0). \] (46)
By multiplying both sides of this equation by 
\[ \frac{1}{R} \left( X^A \frac{\partial X^B}{\partial x^\nu} - X^B \frac{\partial X^A}{\partial x^\nu} \right) \bigg|_{x=x_0}, \]
and summing over A and B we find
\[ \pi^\mu(0) = G^\mu_\nu(x_0, x_1) \pi^\nu(t_1). \] (47)

Here
\[ G^\mu_\nu(x_0, x_1) = -\frac{1}{2R^2} g^{\rho\sigma}(x_0) \left( X^A \frac{\partial X^B}{\partial x^\rho} - X^B \frac{\partial X^A}{\partial x^\rho} \right) \bigg|_{x=x_0} \left( X^_A \frac{\partial X^_B}{\partial x^\sigma} - X^_B \frac{\partial X^_A}{\partial x^\sigma} \right) \bigg|_{x=x_1}, \]
where
\[ g_{\mu\nu}(x) = \eta_{AB} \frac{\partial X^A}{\partial x^\mu} \frac{\partial X^B}{\partial x^\nu} \]
is the metric on $dS_4$ in the given coordinates. Formulae (29)–(31) and (47) hold for lightlike particles as well.

As an example, choose the local coordinates $x^\mu = \{c t, x^i\}$ to be the flat ones,
\[ X(t, x^i) = \begin{cases} X^0 = R \sinh \frac{ct}{R} + \frac{x^i}{2R} e^\frac{i}{R}, \\ X^i = e^\frac{i}{R} x^i, \\ X^4 = R \cosh \frac{ct}{R} - \frac{x^2}{2R} e^\frac{i}{R}. \end{cases} \] (48)

If for simplicity we restrict ourselves to the two-dimensional case in flat coordinates and choose $x_0 = (0, 0)$ and $x_1 = (ct, x)$ we find
\[ G^\mu_\nu(x_0, x_1) = \begin{pmatrix} \frac{1}{R} & \frac{e^\frac{i}{R} x^i}{R} \\ \frac{e^\frac{i}{R}}{R} & \cosh \frac{ct}{R} + \frac{x^2}{2R} e^\frac{i}{R} \end{pmatrix}. \] (49)

In particular, consider the case of a photon transmitted from $x_0$ to $x_1$. To express the momenta in terms of inertial frames at rest in each point with respect to the given local coordinates, we introduce the zweibein $e^0 = c dt$, $e^i = e^\frac{i}{R} dx$. The inertial energy–momentum $\hat{\pi}^\mu$ has components $\hat{\pi}^0 = \pi^0$ and $\hat{\pi}^1 = e^\frac{i}{R} \pi^1$. In particular, in $x_0$, $\hat{\pi}^\mu(0) = \pi^\mu(0)$ and
\[ \hat{\pi}^0(0) = \hat{\pi}^1(0) = \frac{\hbar \nu_0}{c}, \]
\[ \hat{\pi}^0(0) = \hat{\pi}^1(0) = \frac{\hbar \nu}{c}. \] (50)

(51)

Obviously $x_1$ cannot be any point, but must lie on a lightlike geodesic starting from $x_0$. It can be easily found putting
\[ \hat{\pi}^0(0) = \hat{\pi}^1(0) = \frac{\hbar \nu_0}{c}, \]
in (50) and (51) and solving for $x = x(t)$. This gives
\[ x(t) = R(1 - e^{-\frac{ct}{R}}). \]

Using this in (50) we finally obtain
\[ \nu = e^{-\frac{ct}{R}} \nu_0. \] (52)

This is the redshift measured by the observer at $x_1$: the photon emitted with frequency $\nu_0$ at $x_0$ is perceived as a photon of frequency $\nu$ by the observer at $x_1$. 

4. Energy

In Einstein’s special relativity the energy of a particle is defined (and measured) relative to an arbitrary given Lorentz frame, it being the zero component of a four-vector. In physical terms, a Lorentz frame can be seen as an ideal global network of (free) particles relatively at rest and carrying clocks that stay forever synchronized. This picture does not extend to the de Sitter case where frames are defined only locally.

However, the maximal symmetry of the de Sitter universe allows for the energy of a pointlike particle to be defined relative to just one reference massive free particle understood conventionally to be at rest (the sharply localized observer). Below we will compare this definition with the ones obtained in various coordinate patches by a more standard Lagrangian approach.

The procedure amounts to fixing arbitrarily a timelike reference geodesic (the geodesic of the particle ‘at rest’). Let us denote by \( u \) and \( v \) the future oriented null vectors which identify such a geodesic; the energy of the free particle \( (9) \) with respect to the reference geodesic is defined as follows:

\[
E = E(\xi, \eta) (u, v) = -\frac{cK(\xi, \eta) (u, v)}{u \cdot v}.
\]

We have that

\[
E(u,v)(\xi, \eta) = E(\xi,\eta) (u, v)
\]

which can be interpreted as the symmetry between the active and passive point of view. In particular, the proper energy is

\[
E(\xi,\eta) (\xi, \eta) = mc^2,
\]

as it should be. To further elaborate this definition let us choose an origin \( \bar{Y} = Y(0) \) on the reference geodesic and denote by \( \lambda \) the scalar such that

\[
u \cdot \bar{Y} = 2\lambda^2, \quad v = u - \frac{2\lambda \bar{Y}}{R}.
\]

As before, fixing \( \lambda \) removes the scale arbitrariness in the choice of \( u \) and \( v \) and it follows that

\[
Y(\tau) = R \frac{u e^\frac{\bar{Y}}{\lambda} - v e^{-\frac{\bar{Y}}{\lambda}}}{2\lambda} = \bar{Y} e^{-\frac{\bar{Y}}{\lambda}} + \frac{Ru}{\lambda} \sinh \frac{c\tau}{R}.
\]

Proper times in (16) and (55) are of course not to be confused. Taking into account equations (15) and (54) it follows that

\[
E = -mc^2 (\xi \wedge \eta)(u, v) \frac{(\xi \cdot \eta)(u \cdot v)}{(\xi \cdot \bar{Y})(u \cdot \bar{Y})} = -\frac{ke^2}{\lambda R^2} (\xi \wedge \bar{X})(u, \bar{Y}).
\]

Finally, by inserting into this expression equation (19) and the analogous relation

\[
\lambda = -\frac{1}{R} (u \cdot \bar{Y})
\]

we get the expression

\[
E = mc^2 \frac{u \cdot \bar{X})(\xi \cdot \bar{Y}) - (\bar{X} \cdot \bar{Y})(\xi \cdot u)}{(\xi \cdot \bar{X})(u \cdot \bar{Y})}.
\]

which is an alternative form of (53).

We now wish to refer the energy \( E \) of the particle to a given coordinate patch. This can be done as follows. Suppose a local frame \((t, x')\) has been selected so that the embedding of \( dS_4 \) in \( M^{(1,4)} \) is given by \( X^A(P) = X^A(t, \bar{x}) \); to fix ideas let us perform this choice so that the event \( t = 0, \bar{x} = 0 \) is the "origin" \( X_0 \) of the de Sitter manifold.

Then, we define the energy \( E \) of a particle relative to the given frame as the energy of the particle w.r.t. the particle (observer) at rest at the origin, i.e. w.r.t. the reference geodesic passing through the origin with zero velocity \((\bar{x}(0) = 0, \frac{d\bar{x}}{dt}(0) = 0)\). We work out a few explicit examples.
4.1. Flat coordinates

The flat coordinate system \( \{t, x^i\} \) is defined by (48). In these coordinates the de Sitter geometry is that of a flat exponentially expanding Friedmann universe,

\[
ds^2 = c^2 dt^2 - e^{2ct/R} \delta_{ij} dx^i dx^j = c^2 dt^2 - a^2(t) \delta_{ij} dx^i dx^j.
\]  (59)

The reference geodesic (55) through the origin \( Y(t = 0, x^i = 0) = (0, 0, 0, 0, R) \) with zero velocity is uniquely associated with the choices \( \bar{Y} = (0, 0, 0, 0, R) \) and \( u = \lambda(1, 0, 0, 0, 1) \).

With such a choice for \( \bar{Y} \) and \( u \), equation (58) is explicitly written as follows:

\[
E = \frac{k c^2 R}{\xi} (\xi^0 \dot{X}^4 - \dot{\xi}^4 \dot{X}^0).
\]  (60)

Noting that \( \xi = \frac{m}{k R} \left( \dot{X} + \frac{R}{c} \frac{dX(0)}{d\tau} \right) \) and using equation (48) we readily find

\[
E = mc^2 \frac{dt}{d\tau} - \frac{c}{R} p_i = \frac{mc^2}{\sqrt{1 - a^2(t) v^2 c^2}} - \frac{c}{R} x^i p_i,
\]  (62)

where we have set

\[
v^i = \frac{dx^i}{dt}, \quad p_i = -mc^{2ct/R} \frac{dx^i}{d\tau} = -\frac{ma^2(t) v^i}{\sqrt{1 - a^2(t) v^2 c^2}}.
\]  (63)

In section 5 we will show that

\[
p^i = -\frac{1}{a^2(t)} p_i = \frac{mv^i}{\sqrt{1 - a^2(t) v^2 c^2}}
\]  (64)

can be interpreted as the de Sitter version of the linear momentum (in flat coordinates). In the limit \( R \to \infty \), (62) and (64) go over into the usual Minkowskian expressions of the energy and momentum.

That (62) can be interpreted as the correct de Sitter energy of the particle is confirmed by noting that it is the conserved quantity associated with the invariance of the particle action (5) under time translation. Indeed, since in flat coordinates the spatial distances dilate in the course of time by the exponential factor \( e^{ct/R} \), the expression of an infinitesimal symmetry under time evolution is

\[
t \to t + \epsilon, \quad x^i \to x^i - \frac{c}{R} x^i \epsilon.
\]  (65)

The action

\[
S = -mc \int \sqrt{1 - e^{ct/R} \frac{v^iv^j}{c^2} \delta_{ij}} dt
\]  (66)

is invariant under (65) and, by Noether’s theorem, the corresponding constant quantity is precisely (62).

For a massless particle, using equation (20) we find

\[
E = k e^3 \frac{d\tau}{d\sigma} - \frac{c}{R} v^i p_i,
\]  (67)

where we have set

\[
p_i = -k e^2 \frac{v^i}{(d\chi/d\sigma)}.
\]  (68)
In particular, note that \( \frac{d\vec{x}}{dt} \cdot d\vec{x}/dt = c^2 e^{-2\xi} \). To find the relation between \( t \) and \( \sigma \), we take the derivative with respect to \( \sigma \) of the relation defining the cosmic time \( X^0 + X^4 = Re^{\frac{\xi}{R}} \), and use (21) to obtain

\[
\frac{dt}{d\sigma} = \frac{1}{c} e^{-\frac{\xi}{R}} (\xi^0 + \xi^4) .
\]  

(69)

Inserting this into the expression of the energy we find

\[
E = kc^2(\xi^0 + \xi^4) e^{-\frac{\xi}{R}} - \frac{c}{R} \frac{\xi}{R} p_i , \quad p_i = -k(\xi^0 + \xi^4) e^{\frac{\xi}{R}} \frac{dx_i}{dt} .
\]  

(70)

In the flat limit \( R \to \infty \) we have \( E \to kc^2(\xi^0 + \xi^4) \) so that, if we associate a frequency to the de Sitter massless particle, we have

\[
h \nu = kc^2(\xi^0 + \xi^4)
\]

and finally

\[
E = h \nu , \quad \vec{p} = \frac{h \nu}{c} \vec{n} .
\]  

(72)

4.2. Spherical coordinates

Let \( \{ t, \omega^\alpha \} , \alpha = 1, \ldots, 4 \), be such that

\[
\begin{align*}
X^0 &= R \sinh \frac{ct}{R} , \\
X^\alpha &= R \cosh \frac{ct}{R} \omega^\alpha , 
\end{align*}
\]  

(73)

where \( \omega^\alpha \delta_{\alpha\beta} = 1 \), that is the \( \omega^\alpha \) is a vector on the sphere \( S^3 \) of unit radius. Concretely

\[
\begin{align*}
\omega^1 &= \sin \chi^1 \sin \chi^2 \cos \chi^3 , \\
\omega^2 &= \sin \chi^1 \sin \chi^2 \sin \chi^3 , \\
\omega^3 &= \sin \chi^1 \cos \chi^2 , \\
\omega^4 &= \cos \chi^1 .
\end{align*}
\]  

(74)

This coordinate system covers the whole de Sitter manifold. The geometry is that of a closed Friedmann universe undergoing an epoch of exponential contraction which is followed by an epoch of exponential expansion:

\[
ds^2 = c^2 dt^2 - R^2 \cosh^2 \frac{ct}{R} \left[ (d\chi^1)^2 + \sin^2 \chi^1 (d\chi^2)^2 + \sin^2 \chi^2 (d\chi^3)^2 \right] .
\]  

(75)

The initial point and the lightlike vector identifying the reference geodesic are once more \( \vec{Y} = X_0 \) and \( u = \lambda(1, 0, 0, 0, 1) \), and therefore the energy is again given by equation (60).

Expressing \( \xi \) in terms of \( X \) and \( dX/dt \) it follows that

\[
E = mc^2 \frac{\omega^\alpha - \frac{R}{c^2} v^\alpha \sinh \frac{2ct}{R}}{\sqrt{1 - \frac{R^2}{c^2} \cosh^2 \frac{ct}{R} v^\alpha v^\beta \delta_{\alpha\beta}}} ,
\]  

(76)

where \( \alpha, \beta = 1, 2, 3, 4 \) and \( v^\alpha = d\omega^\alpha /dt \).

Again, expression (76) can be recovered as the conserved quantity associated with a time translation plus a rescaling of the \( \omega \)’s that together leave invariant the action

\[
S = -mc^2 \int \sqrt{1 - \frac{R^2}{c^2} \cosh^2 \frac{ct}{R} \Omega_{ij} w^i w^j } \, dt ,
\]  

(77)

where \( d\Omega^2 = \Omega_{ij} \, d\chi^i \, d\chi^j \) and \( w^i = \frac{dx^i}{dt} \).
4.3. Static (black hole) coordinates

This is the coordinate system originally introduced by de Sitter in his 1917 paper [34]. It describes a portion of the de Sitter manifold as follows:

\[
\begin{align*}
X^0 &= R \sqrt{1 - \frac{r^2}{R^2}} \sinh \frac{\alpha t}{R}, \\
X^i &= r^i, \quad i = 1, 2, 3, \\
X^4 &= R \sqrt{1 - \frac{r^2}{R^2}} \cosh \frac{\alpha t}{R},
\end{align*}
\]

(78)

where \( r^2 = \sum_{i=1}^{3} r^i r^i \). With these coordinates the metric exhibits a bifurcate Killing horizon at \( r = R \),

\[
ds^2 = \left(1 - \frac{r^2}{R^2}\right)c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2).
\]

(79)

We choose the same origin and reference geodesic as before and find

\[
E = mc^2 \left(1 - \frac{r^2}{R^2}\right) \frac{1}{\sqrt{1 - \frac{r^2}{R^2} - \frac{(\mathbf{r} \cdot \dot{\mathbf{r}})^2}{(R^2 - r^2)c^2} - \frac{\dot{r}^2}{c^2}}}
\]

(80)

In these coordinates, the action for a massive free particle is

\[
S = -mc^2 \int \sqrt{1 - \frac{r^2}{R^2} - \frac{(\mathbf{r} \cdot \dot{\mathbf{r}})^2}{(R^2 - r^2)c^2} - \frac{\dot{r}^2}{c^2}} \, dt.
\]

(81)

Here the dot means derivation with respect to \( t \). This action is invariant under time translations and the associated conserved energy coincides with (80).

We leave it as an exercise to find the analogue of expressions (76) and (80) for massless particles. As mentioned in the introduction, a fourth example can be found in [16] where the expression of the energy is given in terms of stereographic coordinates.

5. A possible definition of momentum

Whereas, as shown in section 4, the energy of a particle can be defined relative to an arbitrary fixed reference geodesic, and therefore in a frame-independent manner, no similar characterization can be given for the linear momentum. Instead, a definition of momentum for a de Sitter particle necessarily requires the selection of some coordinate system. In Minkowskian relativity, energy and momentum are defined as the conserved quantities associated with the invariance of the action respectively under infinitesimal inertial time and space translations. In addition, in the de Sitter case, for a class of reference frames which in the limit \( R \to \infty \) become inertial, we have seen that the particle energy arises again as the conserved quantity associated with time translations (depending on the choice of the particular coordinate system, the latter may or may not act on the space coordinates as well). Therefore, it is natural to attempt to define the de Sitter momentum in any such frame as the conserved vector quantity which is associated with infinitesimal space translations. This requires fixing the origin of the coordinate system, since, due to the presence of curvature, changing the origin affects the definition of space translations. Of course, in order for such a definition of momentum to be consistent, one must be sure to recover the usual Minkowskian momentum in the limit \( R \to \infty \). Then we search for Lorentz transformations in the embedding spacetime \( M(1, 4) \) which generate spatial translations of the origin, once the latter has been identified. Specifically, let \((t, x')\) be local coordinates, \(X^A(t, x')\) the embedding functions and \(O \equiv X^A(0)\) the origin. We consider the
submanifold defined by \( t = 0 \). It defines a hypersurface of \( dS_4 \) which identifies an osculating hyperplane in \( O \). Any infinitesimal Lorentz transformation which leaves the osculating plane invariant defines an infinitesimal translation of \( O \) which is transverse to the reference geodesic defining the energy. As stated above we use such transformations to define the momentum. We illustrate again the construction for the coordinate systems of section 4.

5.1. Flat coordinates

In this case, the \( t = 0 \) surface defines the osculating plane \( X^0 + X^4 = R \). The Lorentz transformations leaving this hyperplane invariant are generated by the infinitesimal rotations of the form \( v \wedge v_i \), where \( v = (1, 0, 0, 0, -1) \) and \( v_i = (0, \vec{e}_i, 0) \), where \( \vec{e}_i \) is the standard basis of \( \mathbb{R}^3 \). Then the momentum is

\[
p_i := K(v, v_i),
\]

which expressed in coordinates gives

\[
p_i = -mc^2e^\frac{x^i}{R} \frac{dx^i}{d\tau}.
\]

This expression coincides with (63) and corresponds to the conserved quantities associated with the invariance of the action under spatial translations \( x^i \mapsto x^i + a^i \). For a massless particle we should use \( \tilde{K} \) in place of \( K \), finding the second of formulae (71) as conserved momentum. Note that we have a scale ambiguity in choosing the vectors \( v_i, v \). However such ambiguity can be fixed by requiring that we obtain the usual Minkowskian expression in the \( R \to \infty \) limit.

5.2. Spherical coordinates

In spherical coordinates the \( t = 0 \) slice corresponds to \( X^0 = 0 \). We have \( v = (0, 0, 0, 0, 1) \) and we can choose \( v_i \) as before. Then

\[
K(v, v_i) = mc^2 \frac{e_v v_i - e_i e_v}{\xi \cdot \eta},
\]

and, in term of the coordinates,

\[
p_i = mR \cosh^2\frac{ct}{R} \left( \omega_4 \frac{d\omega_4}{d\tau} - \omega_i \frac{d\omega_i}{d\tau} \right).
\]

The flat limit leaving the origin invariant can be easily performed and it gives the correct momentum for Minkowski spacetime.

5.3. Black hole coordinates

Again the \( t = 0 \) slice defines the osculating hyperplane \( X^0 = 0 \) and \( K(v, v_i) \) is once more given by (84). Then

\[
p_i = m \left( \sqrt{1 - \frac{r_i^2}{R^2}} \cosh\frac{ct}{R} - \sqrt{1 - \frac{r_i^2}{R^2}} \sinh\frac{ct}{R} - \frac{1}{\sqrt{1 - \frac{r_i^2}{R^2}}} \sum_{k=1}^{3} r_k \frac{dr_k}{d\tau} \cosh\frac{ct}{R} \right).
\]
6. Conclusions

In this paper, we have taken the stance that in the absence of gravitation the spacetime arena in which all physical phenomena take place is the (maximally symmetric) four-dimensional de Sitter manifold $dS_4$. Then, barring discrete spacetime operations such as space reflection and time reversal, which are not exact symmetries of nature, the corresponding relativity group is the connected component of $SO(1, 4)$, the (ten-dimensional) de Sitter group. This attitude springs naturally from very basic properties of all natural phenomena, which so far have never been experimentally contradicted. To wit, experiments and observations in the whole realms of physics, astrophysics and cosmology point to the fact that local laws governing natural phenomena do not depend on time and on the location in space (spacetime homogeneity) and that no direction in space is privileged compared to any other (isotropy of space). In addition, all experiments also point to the local validity of the principle of inertia, which states that no operational meaning whatsoever can be given to a notion of absolute rest (invariance under boosts). Therefore, the laws of physics should possess a ten parameter invariance group acting on the four-dimensional spacetime continuum, which embodies all above symmetries.

Working at the Lie algebra level it has been shown by Bacry and Lévy-Leblond in a remarkable paper published almost 40 years ago [35] that such a group is uniquely determined up to two undetermined parameters, a velocity scale (the invariant speed $c$) and a length scale (the cosmological constant $\Lambda$). The actual values of these parameters in nature are of course not fixed by the basic symmetries and must be found experimentally. And, indeed, though limiting values of $c$ and $\Lambda$ (such as $c = 0$, $\infty$ and/or $\Lambda = 0$, $\infty$) cannot be excluded a priori, it comes as no surprise that the values of $c$ and $\Lambda$ determined by the observations are well defined and finite. It would be surprising if it were otherwise! It is a different (and to some extent metaphysical) question why $c$ and $\Lambda$ have the values they have and not others. However we are not concerned with this problem here.

It is then clear that, if Minkowski space should be replaced by de Sitter space and, correspondingly, the Poincaré group by the de Sitter group, one is naturally led to a reformulation of the theory of special relativity on these grounds [10–14]. However, compared to the Minkowski case, this task presents certain complications which are essentially connected with the fact that in the de Sitter case there exists no class of privileged reference frames as are the Minkowskian inertial ones. Indeed, the associated coordinate systems of any such hypothetical class of equivalent frames should respect the basic symmetries of the spacetime manifold. In particular, the homogeneity requirement would imply the coordinate transformation between any two such equivalent frames to be affine [36], and this is impossible if the underlying manifold is curved. Therefore, the absence of a privileged class of equivalent frames suggests that, in de Sitter relativity, it would be desirable, whenever possible, to characterize significant physical quantities in an intrinsic way, namely in a manner independent of the choice of any particular coordinate patch. In this paper, we have accomplished this for any set of independent conserved quantities along the geodesic motion of a free de Sitter particle and for the overall conservation of the total constants of the motion in any particle collision. In particular, we have also been able to give an intrinsic definition of the energy of a de Sitter particle, as the energy of any such particle relative to an arbitrary selected reference particle chosen conventionally to be at rest. In this respect, it is important to stress that in the same way as there is a unique Lorentzian (i.e. relativistic) generalization of the kinetic energy of a Galilean (i.e. nonrelativistic) particle, the de Sitter energy is the unique de Sitter generalization of the Lorentzian energy, which arises from the appearance of an intrinsic residual curvature of the spacetime manifold.
We remark that, due to the smallness of $\Lambda$, the actual corrections to Einstein’s special relativity which are brought about by the presence of the cosmological constant, such as for instance those embodied in formula (47), are utterly tiny at scales of laboratory experiments performed on Earth or in space. Specifically, formulae (50) and (51) tell us that to first order in $1/R$ the four-momentum of a particle traveling a distance $x$ is altered by a relative amount of the order $x/R$. In particular, for example, since $R \simeq 10^{28}$ cm, the order of magnitude of the fractional frequency shift of a photon traveling a distance $x$ in the de Sitter universe is

$$\frac{\Delta \nu}{\nu} \simeq 10^{-25} \quad \text{for} \quad x \simeq 10 \text{ m}$$

(87)

and

$$\frac{\Delta \nu}{\nu} \simeq 10^{-16} \quad \text{for} \quad x \simeq 10 \text{ million km.}$$

(88)

The figure $\Delta \pi/\pi \simeq 10^{-25}$ would be relevant also for particles produced in a collision in a particle accelerator such as the Tevatron or LHC since the distance between the collision point and the detector is typically of the order of a several meters. By comparison, we recall that in the classical experiment by Pound and Rebka [37–39] devised to measure the gravitational shift of a photon falling in the Earth’s gravitational field we have

$$\frac{\Delta \nu}{\nu} \simeq 10^{-15},$$

(89)

whereas in experiments with atomic clocks which monitor the variation of the gravitational potential of the Sun at the location of the Earth between perihelion and aphelion [40] we have

$$\frac{\Delta \nu}{\nu} \simeq 10^{-10}.$$

(90)

The precision with which the value of $\Delta \nu/\nu$ has been measured in the Pound and Rebka experiment is of the order of 1%, whereas values of $\Delta \nu/\nu$ in the range of the figure of formula (90) in experiments with atomic clocks as are mentioned above are now tested with a precision of the order of almost one part per million. This shows that, whereas there is no chance to realistically test formula (47) for $\Lambda$ in the gravitational field of the Earth, even for falls of thousands of kilometers, comparison of the ticks of atomic clocks set in suitable eccentric orbits around the Sun may in principle be able to reveal an effect due to $\Lambda$ in a not too unforeseeable future. Indeed, in such a hypothetical case, due to the known periodicities it should be possible to filter out the effects from all other contributions, gravitational and not.

Finally, we remark that while at the present epoch the effects due to $\Lambda$ are tangible only at cosmological scales, they might have been essential, even at microscopic distances, during the period of inflationary expansion in the very early universe, when the effective de Sitter radius was extremely small ($\simeq 10^{-27} - 10^{-28}$ cm), at which time, however, quantum effects are expected to have been dominant [41].

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We are indebted to F Hehl who stimulated our interest in the subject.

Appendix A

We show in this appendix, by constructing an explicit example, that the components of the future oriented lightlike five-vectors $\xi$ and $\eta$ which identify a given timelike geodesic are fixed by the assignment of the values of six independent parameters once $\xi$ and $\eta$ have been
separately normalized according to equation (12) and the selection of the point of the geodesic corresponding to zero proper time. Precisely, let $x^\mu = \{ct, r^i\}$ be a set of local coordinates on $dS_4$ and consider a timelike geodesic passing through $\vec{r}_0$ at time $t = 0$ with velocity $\vec{v}_0 = d\vec{r}/dt|_{t=0}$. Parametrizing the geodesic as in (9) and choosing a suitable normalization for $\xi$ and $\eta$, we can express these vectors in terms of the initial conditions $\vec{r}_0$ and $\vec{v}_0$. Indeed

$$X(x^\mu(t)) = R \frac{\xi e^{\tau(t)} - \eta e^{-\tau(t)}}{\sqrt{2\xi \cdot \eta}}. \quad (A.1)$$

Then, setting $t = 0$ and assuming $\tau(0) = 0$, we find

$$X(x_0) = \frac{R}{\sqrt{2\xi \cdot \eta}} (\xi - \eta) \quad (A.2)$$

and, taking the derivatives with respect to $t$ at $t = 0$ we get

$$\left.\frac{dX}{dt}\right|_{t=0} = \partial_\mu X(x_0) \frac{v^\mu}{c t_0} = \frac{c}{\sqrt{2\xi \cdot \eta}} (\xi + \eta) t_0. \quad (A.3)$$

Choosing $k = m$ in (12), and solving with respect to $\xi$ and $\eta$ we find

$$\xi = \frac{\partial_\mu X(x_0) v^\mu_0}{c t_0} + \frac{X(x_0)}{R}, \quad \eta = \frac{-\partial_\mu X(x_0) v^\mu_0}{c t_0} - \frac{X(x_0)}{R}, \quad (A.4)$$

where $v^0 = c$, $v^i = \dot{x}^i$ and

$$c t_0 = \sqrt{\left.\frac{ds^2}{dt}\right|_{t=0}}. \quad (A.5)$$

The condition $\xi \cdot \eta = 2$ and the choice of the initial point $X(x_0)$ on the geodesic fixes the separate normalizations of $\xi$ and $\eta$ and, as a consequence, formulae (A.4) express $\xi$ and $\eta$ as functions of $\vec{r}_0$ and $\vec{v}_0$.

For example, if we choose static coordinates we have explicitly

$$\xi^i = \frac{1}{c t_0} v^i_0 + \frac{r^i_0}{R}, \quad (A.6)$$

$$\eta^i = \frac{1}{c t_0} v^i_0 - \frac{r^i_0}{R}, \quad i = 1, 2, 3 \quad (A.7)$$

and similar formulae for $\xi^0, \eta^0, \xi^4$ and $\eta^4$, where

$$t_0 = \left.\frac{dr}{dt}\right|_{t=0} = \sqrt{1 - \frac{r^2_0}{R^2} - \frac{v^2_0}{c^2} - \frac{(\vec{r}_0 \cdot \vec{v}_0)^2}{c^2(R^2 - r^2_0)}} = \frac{m c^2}{E} \left(1 - \frac{r^2_0}{R^2}\right). \quad (A.8)$$

Note in particular that $ds^2$ is positive along timelike curves so that setting

$$\bar{\rho} = \frac{1}{R} \vec{r}, \quad \bar{\beta} = \frac{1}{c} \vec{v}, \quad (A.9)$$

we have

$$1 - \rho^2 - \beta^2 - \rho^2 \beta^2 \cos^2 \theta > 0, \quad (A.10)$$

where $\theta$ is the angle between $\vec{r}$ and $\vec{v}$ (and $\bar{\rho}, \bar{\beta}$ and $\theta$ are functions of $t$). Then

$$\beta < \frac{1 - \rho^2}{\sqrt{1 - \rho^2 \sin^2 \theta}}. \quad (A.11)$$
In particular for lightlike geodesics we find
\[ \beta = \frac{1 - \rho^2}{\sqrt{1 - \rho^2 \sin^2 \theta}}, \tag{A.12} \]
so that massless particles have velocities
\[ 1 - \rho^2 \leq \beta \leq \sqrt{1 - \rho^2}, \tag{A.13} \]
depending on the angle between the velocity and the position vector.

Appendix B

In one dimension the de Sitter line element in flat coordinates is given by
\[ ds^2 = c^2 dt^2 - e^{2\tau} dx^2 = c^2 \, dr^2 - a^2(t) \, dx^2. \tag{B.1} \]
Then, denoting by \( y \) the spatial coordinate with respect to which the rigid box has fixed length \( 2L \), we have
\[ y = x \, e^{\tau} \tag{B.2} \]
so that the \( j \)th particle of our classical comoving gas drifts toward one of the walls of the box according to the equation
\[ y_j(t) = y_{0j} \, e^{\tau}, \quad y_{0j} = L \frac{j}{N}, \quad j = 0, \pm 1, \pm 2, \ldots, \pm (N - 1). \tag{B.3} \]
Hence the \( \pm (N - 1) \) th particle, which is the one initially closest to the right (left) wall, reaches the latter (and bounces back elastically) at the time
\[ t_{N-1} = \frac{R}{c} \log \frac{N}{N-1} \tag{B.4} \]
thereby colliding after a while with the \( \pm (N - 2) \) th particle, thus starting the thermalization process through further collisions with the other particles. Now, the equation of motion of particle \( j \) is
\[ \ddot{y}_j = \frac{c^2}{R^2} y_j = 0 \tag{B.5} \]
corresponding to a potential
\[ V(y) = -\frac{1}{2} \frac{c^2}{R^2} y^2. \tag{B.6} \]
Therefore, after some transient time the gas virializes itself eventually reaching a thermal state at a temperature \( T \) given by
\[ \frac{1}{2} kT = \langle K \rangle = m |\langle V \rangle| = m \frac{1}{\int_{-L}^{L} \rho(y) \, dy} \int_{-L}^{L} |V(y)| \rho(y) \, dy, \tag{B.7} \]
where \( K \) is the kinetic energy and \( \rho(y) \) is the mass density of the gas at equilibrium. Since our gas is an ideal one its equation of state is
\[ m p(y) = \rho(y) kT. \tag{B.8} \]
Then, eliminating the pressure from (B.8) and the Euler equation
\[ \frac{dV(y)}{dy} = \frac{1}{\rho(y)} \frac{dp(y)}{dy}, \tag{B.9} \]
we get
\[- \frac{dV(y)}{dy} = \frac{kT}{m} \frac{d}{dy} \log \rho(y)\] (B.10)
from which
\[\frac{kT}{m} \log \frac{\rho(y)}{\rho(0)} = |V(y)| = \frac{c^2}{2R^2} y^2.\] (B.11)
Hence
\[\rho(y) = \rho(0) e^{\frac{mc^2}{2kT} y^2}\] (B.12)
so that
\[kT = \frac{mc^2}{R^2} \int_{-L}^L dy \frac{\rho(y) e^{\frac{mc^2}{2kT} y^2}}{\rho(y)}\] (B.13)
which can be written as
\[\int_0^{\sqrt{\frac{mR}{2kT}}} dw w^2 e^{w^2} = \frac{1}{2} \int_0^{\sqrt{\frac{mR}{2kT}}} dw e^{w^2}.\] (B.14)
The solution of this equation is
\[\frac{L}{R} \sqrt{\frac{mc^2}{2kT}} = 1.063,\] (B.15)
which can be written as
\[T \simeq \frac{mc^2 L^2}{2R^2 k (1.063)^2} \simeq \frac{mH^2 L^2}{2.26 K}.\] (B.16)

References

[1] Perlmutter S et al 1999 Measurements of omega and lambda from 42 high-redshift supernovae Astrophys. J. 517 565–86
[2] Riess A G et al 1998 Observational evidence from supernovae for an accelerating universe and a cosmological constant Astron. J. 116 1009–38
[3] Riess A G et al 2001 The farthest known supernova: support for an accelerating universe and a glimpse of the epoch of deceleration Astrophys. J. 560 49–71
[4] Spergel D N et al 2003 First year Wilkinson microwave anisotropy probe observations: determination of cosmological parameters Astrophys. J. Suppl. 148 175
[5] Sahni V and Starobinsky A A 2000 The case for a positive cosmological lambda-term Int. J. Mod. Phys. D 9 373–444
[6] Padmanabhan T 2003 Cosmological constant: the weight of the vacuum Phys. Rep. 380 235–320
[7] Peebles P J E and Ratra B 2003 The cosmological constant and dark energy Rev. Mod. Phys. 75 559–606
[8] Inoue E and Wigner E P 1953 On the contraction of groups and their represenations Proc. Natl Acad. Sci. 39 510–24
[9] Abbott L F and Deser S 1982 Stability of gravity with a cosmological constant Nucl. Phys. B 195 76–96
[10] Aldrovandi R, Beltran Almeida J P and Pereira J G 2004 Cosmological term and fundamental physics Int. J. Mod. Phys. D 13 2241–8
[11] Aldrovandi R, Beltran Almeida J P and Pereira J G 2007 de Sitter special relativity Class. Quantum Grav. 24 1353–8
[12] Guo H-Y, Huang C-G, Xu Z and Zhou B 2004 On Beltrami model of de Sitter spacetime Mod. Phys. Lett. A 19 1701–10
[13] Guo H-Y, Huang C-G, Xu Z and Zhou B 2004 On special relativity with cosmological constant Phys. Lett. A 331 1–7

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[14] Guo H-Y, Zhou B, Tian Y and Xu Z 2007 The triality of conformal extensions of three kinds of special relativity Phys. Rev. D 75 026006
[15] Kowalski-Glikman J and Nowak S 2003 Doubly special relativity and de Sitter space Class. Quantum Grav. 20 4799–816
[16] Gursey F G 1965 Introduction to the de Sitter group Group Theoretical Concepts and Methods in Elementary Particle Physics ed F G Gursey (New York: Gordon and Breach)
[17] Birrell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge: Cambridge University Press)
[18] Bros J and Moschella U 1996 Two-point functions and quantum fields in de Sitter universe Rev. Math. Phys. 8 327–92
[19] Bros J, Epstein H and Moschella U 1998 Analyticity properties and thermal effects for general quantum field theory on the de Sitter space-time Commun. Math. Phys. 196 535–70
[20] Bros J, Epstein H and Moschella U 2008 The lifetime of a massive particle in a de Sitter universe J. Cosmol. Astropart. Phys. JCAP02(2008)003
[21] Moschella U 2006 The de Sitter and anti-de Sitter sightseeing tour Einstein, 1905–2005 (Progress in Mathematical Physics vol 47) ed T Damour, O Darrigol, B Duplantier and V Rivesseau (Basel: Birkhauser)
[22] Moschella U 2007 Particles and fields on the de Sitter universe AIP Conf. Proc. 910 396–411
[23] Synge J L 1960 Relativity: The General Theory (Amsterdam: North-Holland)
[24] Souriau J M 1970 Structure des Systèmes Dynamiques (Paris: Dunod)
[25] Konstant B 1970 Quantization and Unitary Representations. Lecture Notes in Mathematics (Berlin: Springer)
[26] Kirillov A A 1970 Elements of the Theory of Representations (Berlin: Springer)
[27] Harrison E R 1970 Fluctuations at the threshold of classical cosmology Phys. Rev. D 1 2726–30
[28] Zeldovich Y B 1972 A hypothesis, unifying the structure and the entropy of the universe Mon. Not. R. Astron. Soc. 160 1–3
[29] Mukhanov V F, Feldman H A and Brandenberger R H 1992 Theory of cosmological perturbations: I. Classical perturbations: II. Quantum theory of perturbations. III. Extensions Phys. Rep. 215 203–333
[30] Ratra B 1985 Restoration of spontaneously broken continuous symmetries in de Sitter space-time Phys. Rev. D 31 1931–55
[31] Bunch T S and Davies P C W 1978 Quantum field theory in de Sitter space: renormalization by point splitting Proc. R. Soc. Lond. A 360 117–34
[32] Gibbons G W and Hawking S W 1977 Cosmological event horizons, thermodynamics, and particle creation Phys. Rev. D 15 2738–51
[33] Barberis D et al 1998 A study of the centrally produced $\pi^+\pi^-\pi^0$ channel in $pp$ interactions at 450-gevc Phys. Lett. B 422 399
[34] de Sitter W 1917 On the curvature of space Proc. K. Ned. Akad. Wet. 20 229
[35] Bary H and Lévy-Leblond J M 1968 Possible kinematics J. Math. Phys. 9 1605–14
[36] Lévy-Leblond J M 1976 One more derivation of the Lorentz transformation Am. J. Phys. 44 271
[37] Pound R V and Rebka G A J 1960 Apparent weight of photons Phys. Rev. Lett. 4 337–41
[38] Pound R V and Snider J L 1964 Effect of gravity on nuclear resonance Phys. Rev. Lett. 13 539–40
[39] Pound R V and Snider J L 1965 Effect of gravity on gamma radiation Phys. Rev. B 140 788–804
[40] Ashby N et al 2007 Testing local position invariance with four cesium-fountain primary frequency standards and four nist hydrogen masers Phys. Rev. Lett. 98 070802
[41] Linde A D 2005 Particle physics and inflationary cosmology Preprint hep-th/0503203