The sliding mode control approach design for nonholonomic mobile robots based on non-negative piecewise predefined-time control law

Lixiong Lin1, Peixin Wu1, Bingwei He1, Yanjie Chen1, Jiachun Zheng2, Xiafu Peng3

1 School of Mechanical Engineering and Automation, Fuzhou University, Fujian, P. R. China
2 School of Information Engineering, Jimei University, Fujian, P. R. China
3 Department of Automation, Xiamen University, Fujian, P. R. China

Correspondence
Xiafu Peng, Department of Automation, Xiamen University, Fujian 361005, P. R. China.
Email: xfpeng@xmu.edu.cn

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Abstract

In this paper, a novel trajectory tracking control method of nonholonomic mobile robots based on the non-negative piecewise predefined-time theorem is proposed. The idea of cascade control is used to divide the posture error system of the mobile robot into two subsystems. Firstly, for the first-order subsystem, an active predefined-time controller is designed to realize that the angle error system converges and stabilizes to zero within a given time, which is preset in advance. Secondly, a novel predefined-time sliding mode controller is designed for the second-order subsystem, which adds a constant to compensate for the influence of singularity. Moreover, compared with the existing fixed-time control algorithm, the control scheme proposed in this paper provides a more accurate upper bound of the settling time estimation. For convenience, the complex expression of the settling time estimation is transformed into an adjustable parameter. Furthermore, the stability of the two developed controllers is analyzed and some conditions for selecting parameters are given. Finally, the simulation results show the feasibility and correctness of the proposed control algorithm.

1 | INTRODUCTION

With the development of artificial intelligence, intelligent robots have a great contribution to humanity. Mobile robot has the advantages of simple structure, easy control, and flexible movement, so it has great significance in the intelligent society. There are many types of mobile robots widely applied in various fields, such as sorting robots, rescue robots, service robots and so on [1–3]. To realize the movement of mobile robots to follow the desired trajectory, it is necessary to design an appropriate controller to control the robots. Hence, the problem of robot control has always attracted people's attention.

Trajectory tracking is still a hot topic in robotics research [4–6]. Wheeled mobile robot trajectory tracking control means that the robot tracks a desired trajectory by the controller and moves stably along the desired trajectory. For this problem, researchers have proposed some control schemes to make the actual trajectory approximate to the desired trajectory. In [7], Takanori et al. used an adaptive backstepping approach to design an adaptive tracking controller. In order to achieve better tracking, Chang et al. designed an uncoupled linear Proportional-Integral-Derivative (PID) controller and gave Explicit conditions for selecting the control gains in [8]. Taking advantage of the anti-interference performance of the sliding mode control, Goswami developed a PID controller with an adjustable gain switching to reduce the influence of disturbance in [9]. To further investigate trajectory tracking, sliding mode control (SMC) [10–12] has attracted wide attention due to its fast response, simple design, and good robustness. Chwa proposed a new sliding-mode control method for mobile robots with kinematics in two-dimensional polar coordinates in [13]. Based on the cascade control idea [14], a smooth time-varying feedback control law has been introduced to guarantee the global exponential convergence in [15].
Note that the above control methods only consider the global asymptotic stability [16], which means that the convergence of error system of mobile robot is relatively slow and infinite. Therefore, it is necessary to investigate a controller based on finite-time stability. For uncertain chained form systems, Hong et al. [17] proposed a new switch control strategy with the help of the regularity properties of a homogeneous function [18]. In [19], a finite-time controller was presented which was based on the disturbance observer and was combined with the terminal sliding mode (TSM) control approach. It overcomes the limitation that the sliding mode control cannot drive the system state to converge in finite time. Compared with the traditional finite-time controller [20–22], Zhang et al. designed a global finite-time tracking controller combined with sliding mode control in [23], which added a linear term to increase the speed of convergence. Continuous distributed finite-time tracking controllers have been applied to multi-robot systems in [24]. As we all know, the limitation of the finite-time control law is that the settling time is dependent on the initial values of the error system. Hence, Polyakov first proposed the fixed-time stability theory [25]. Huang et al.[26] proposed a fixed-time trajectory tracking method, which used higher-degree terms to provide a faster convergence rate to define the upper bound of the settling time. For the leader-follower consensus problem, Defoort et al. [27] proposed a fixed-time stability strategy based on local information exchange and explicitly specified the convergence time. Singularity is a common problem in the development of fixed-time controllers for second-order systems. To overcome it, the researchers [28, 29] added a saturation function in the design process of the controller. Although the settling time of fixed-time stability is only related to the controller’s parameters, there is no simple way to set parameters for users to adjust it [30, 31]. In fact, the complex expression of the settling time estimation can be transformed into an adjustable parameter. Sanchez-Torres [32] proposed a new theorem named predefined-time stability to overcome these difficulties. However, the predefined time control of mobile robots has rarely been investigated [33–35].

Motivated by the above discussion, we propose a new global predefined-time trajectory tracking control law for mobile robots. Firstly, the idea of cascade control design is applied to divide the error system into a first-order subsystem and a second-order system. Besides, the piecewise Lyapunov function is used to accurately estimate the settling time. Secondly, one subsystem is quickly stabilized by the predefined-time control law. Another subsystem combines the sliding mode control method to make the system stable smoothly and avoid the singularity of the controller. Finally, it can be concluded that the expected settling time for the mobile robot is obtained by setting the tuning parameters in advance through our study.

The rest of the paper is arranged as follows. In Section 2, we introduce some basic preliminaries and problem descriptions. Firstly, the kinematic model of a two-wheeled nonholonomic mobile robot is defined. Subsequently, the description of the trajectory tracking control problem provides a mathematical model for the controller design. Finally, some lemmas and definitions used in this paper are given.

2 | PRELIMINARIES

In this section, we introduce some basic preliminaries and problem descriptions. Firstly, the kinematic model of a two-wheeled nonholonomic mobile robot is defined. Subsequently, the description of the trajectory tracking control problem provides a mathematical model for the controller design. Finally, some lemmas and definitions used in this paper are given.

2.1 | Kinematic model

In this paper, we consider a nonholonomic mobile robot which is made up of two driving wheels and two driven wheels as shown in Figure 1. The center of mass of the mobile robot is the geometric center $P$, represented as $(x_P, y_P)$. Define the deflection angle of the mobile robot between the forward direction and the $X$-axis of the global coordinate system as $\theta$. Assume that the mobile robot has no slipping influence, the velocity component parallel to the $Y_0$-axis is 0. Therefore, we can obtain the constraint equation as

$$\dot{x}\sin \theta - \dot{y}\cos \theta = 0. \quad (1)$$

The kinematic model is given by

$$\dot{\mathbf{p}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} \quad (2)$$

where the control input $\mathbf{u} = [v, w]^T$, $v$ and $w$ represent the forward velocity and angular velocity of the mobile robot, respectively.
2.2 The description of the trajectory tracking control problem

The purpose of the trajectory tracking control is to design a forward velocity controller and angular velocity controller to realize the actual posture \( p = (x, y, \theta) \) tracking the reference posture \( \dot{p}_d = (x_d, y_d, \theta_d)^T \).

Assume that the angle between the local coordinate system \( X_0 - Y_0 \) and the global coordinate system \( X - Y \) is \( \theta \). Then, according to the coordinate transformation formula, the posture error system of the mobile robot is described as [4]:

\[
p_e = \begin{pmatrix} x_e \\ y_e \\ \theta_e \\ \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \end{pmatrix},
\]

where the state vector \( p_e = (x_e, y_e, \theta_e)^T \) denotes the posture error.

Taking the derivative of \( p_e \), we can obtain the posture error dynamics as follows:

\[
\dot{p}_e = \begin{pmatrix} x_e \\ y_e \\ \theta_e \\ \end{pmatrix} = \begin{pmatrix} y_e \omega - v_e \cos \theta_e \\ -v_e \omega + y_e \sin \theta_e \\ v_e \end{pmatrix},
\]

where \( v_e \) and \( \omega \) denote the desired forward velocity and angular velocity, respectively.

2.3 Lemmas and definitions

For convenience, in this paper, we set

\[
sig(x)^\alpha = \text{sgn}(x) |x|^\alpha, \quad \alpha > 0,
\]

where \( x \in \mathbb{R} \), \( \text{sgn}(\cdot) \) denotes the sign function and \( |x| \) denotes the absolute of \( x \).

A nonlinear system is defined as

\[
x = g(t, x), \quad x(0) = x_0,
\]

where \( x \in \mathbb{R}^n \) denotes the state vector and \( g : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) represents a nonlinear function.

**Definition 1.** [25]. For nonlinear system (6), if it satisfies two conditions: (i) The system is stable in finite-time. (ii) The settling time \( T(x_0) \) is globally bounded and independent of the initial state of the system, that is, \( \exists T_{\text{max}} > 0 : T(x_0) \leq T_{\text{max}}, \forall x_0 \in \mathbb{R}^n \). It is said that the origin of the system (6) is fixed-time stable.

**Lemma 1.** [25]. Consider the following differential equation:

\[
y = -\alpha y^\alpha - \beta y^\beta,
\]

where \( \alpha \) and \( \beta \) are positive constants. \( p, q, m, n \) are odd integers and satisfy \( 0 < p < q, m > n > 0 \). The settling time \( T(x_0) \) is bounded by

\[
T(x_0) < T_{\text{max}} \leq \frac{1}{\alpha} \frac{q}{m - p} + \frac{1}{\beta} \frac{n}{m - n},
\]

**Definition 2.** [32]. The origin of the system (6) is defined to be a globally predefined-time stable equilibrium point if it can converge to zero in finite-time and the upper bound of settling time can be set by the user. The settling time is described as

\[
T(x_0) \leq T_\epsilon, \quad \forall x_0 \in \mathbb{R}^n,
\]

where \( T_\epsilon \) is a positive constant named tuning parameter.

**Lemma 2.** [34]. Suppose that the continuous radially unbounded function \( V : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\} \) satisfies

\[
(1) \ V(x) = 0 \text{ when } x = 0;
\]

\[
(2) \text{ For any } V(x) > 0, \text{ there exist } b_1, b_2, T_1, T_2 > 0, c_1, c_2 \geq 0,
\]

\[
p > 0 \text{ and } q > 1 \text{ such that:}
\]

\[
V \leq \begin{cases}
\frac{c_1}{T_1} (b_1 V^p + c_1), & \text{when } 0 < V \leq 1, \\
\frac{c_2}{T_2} (b_2 V^q + c_2), & \text{when } V > 1,
\end{cases}
\]

where \( T_1 \) and \( T_2 \) are user-defined parameters and

\[
C_\alpha = \begin{cases}
\frac{1}{b_1 (1 - p)} \cdot \left[ \left( \frac{b_1^p + c_1^p}{c_1^p} \right)^{1-p} - \frac{c_1^p}{c_1} \right], & \text{if } p < 1, \\
\frac{1}{b_1} \ln \frac{b_1 + c_1}{c_1}, & \text{if } p = 1, \\
\frac{2^{p-1} - \left( \frac{b_1^p + c_1^p}{c_1^p} \right)^{1-p} - \frac{c_1^p}{c_1}}{\left( \frac{b_1^p + c_1^p}{c_1^p} \right)^{p-1}}, & \text{if } p > 1
\end{cases}
\]

and

\[
C_\beta = \frac{2^{q-1} - \left( \frac{b_2^q + c_2^q}{c_2^q} \right)^{1-q} - \frac{c_2^q}{c_2}}{b_2 (q-1)},
\]

then the zero solution of nonlinear systems (6) can realize predefined-time stability, and the settling time \( T_\epsilon \) is:

\[
T_\epsilon = T_1 + T_2.
\]

**Remark 1.** Note that the settling time of fixed-time stability only depends on the system parameters, but it is difficult to find an explicit relation between the system parameters and the fixed time, which leads to being difficult to predict or adjust the
stable time of the system. For convenience, the predefined-time stability is introduced in this paper, which transforms the complex expression of the settling time estimation into an adjustable parameter. Then, the system will achieve uniform global stability within the predefined time $T_c$.

3 | CONTROLLER DESIGN AND ANALYSIS

In this section, the error system is divided into two subsystems by cascade control [14]. For the two subsystems, a predefined-time forward velocity controller and an angular velocity controller are designed, respectively. Finally, the stability analysis of the two subsystems are given.

3.1 Design of angular control law

Based on the idea of cascade system and the piecewise function, we consider the first-order subsystem as:

$$\dot{\theta}_c = w_d - w.$$  \hspace{1cm} (10)

**Theorem 1.** The angular velocity controller is designed as

$$w = \begin{cases} 
    x_d + \frac{C_{01}}{T_{01}} \left[ a_0 \sin(\theta) \, \frac{n_0}{m_0} + \text{sgn}(\theta) \cdot e_0 \right], & 0 \leq |\theta| < 1 \\
    x_d + \frac{C_{02}}{T_{02}} \left[ b_0 \sin(\theta) \, \frac{n_0}{m_0} + \text{sgn}(\theta) \cdot d_0 \right], & |\theta| \geq 1
\end{cases}$$  \hspace{1cm} (11)

where $a_0, b_0 > 0, e_0, d_0 \geq 0, p_0, q_0, m_0, n_0$ are positive odd integers satisfying $p_0 < q_0$ and $m_0 > n_0$. We set $\mu_0 = \frac{p_0}{q_0}, \nu_0 = \frac{m_0}{n_0}$ and

$$C_{01} = \frac{1}{a_0^2} \left( \frac{1}{\mu_0} + \frac{1}{\nu_0} \right)^{1-\mu_0} \left( 1 - \mu_0 \right),$$

$$C_{02} = \frac{2^{\nu_0-1}}{b_0^{\nu_0} (\nu_0 - 1)} \left( \frac{1}{\mu_0} + \frac{1}{\nu_0} \right)^{1-\nu_0}.$$  \hspace{1cm} (12)

$T_{01}$ and $T_{02}$ are the settling time expected by the user. The tracking error $\theta_c$ will converge to zero in predefined-time, and the settling time is satisfied

$$T_{01} < T_{\text{max}} := T_{01} + T_{02}. $$

**Proof.** Further, Equation (11) is transformed into

$$\dot{\theta}_c = \begin{cases} 
    - \frac{C_{01}}{T_{01}} \left[ a_0 \sin(\theta) \, \frac{n_0}{m_0} + \text{sgn}(\theta) \cdot e_0 \right], & 0 \leq |\theta| < 1 \\
    - \frac{C_{02}}{T_{02}} \left[ b_0 \sin(\theta) \, \frac{n_0}{m_0} + \text{sgn}(\theta) \cdot d_0 \right], & |\theta| \geq 1
\end{cases}.$$  \hspace{1cm} (13)

Lyapunov function is designed as follows

$$V(\theta) = |\theta|,$$  \hspace{1cm} (14)

(1) By Lemma 2, when $0 \leq |\theta| < 1$, and taking the derivative of (13), we have

$$V'(\theta) = \text{sgn}(\theta) \cdot \dot{\theta}_c,$$

$$= \frac{C_{01}}{T_{01}} \left[ a_0 \sin(\theta) \, \frac{n_0}{m_0} + \text{sgn}(\theta) \cdot e_0 \right],$$  \hspace{1cm} (15)

If $\theta = 0$, we have $V'(\theta) = 0$; if $\theta \neq 0$, it is obvious that $V'(\theta) = \frac{C_{01}}{T_{01}} (a_0 \sin(\theta) \, \frac{n_0}{m_0} + \text{sgn}(\theta) \cdot e_0)$. Based on Lemma 2, we conclude that the angular tracking errors $\theta_c$ will converge to zero in predefined-time $T_{\text{01}} = T_{01} + T_{02}$. Proof is completed.

**Remark 2.** For the first-order subsystem, the angular velocity controller is designed based on the predefined-time control theory to achieve the angular error system converge to zero. Compared with the traditional fixed-time theorem [25, 31], this method has three advantages: (i) The estimated settling time is closer to the actual settling time when the control law is divided into two parts; (ii) Add a constant to achieve faster convergence; (iii) The settling time of the angular velocity can be obtained in advance by setting tuning parameters.

3.2 Design of the forward velocity control law

Consider the second-order subsystem as follows:

$$\dot{x}_c = y_c \, w_d - v + v_d,$$  \hspace{1cm} (16)

$$\dot{y}_c = -x_c \, w_d.$$  \hspace{1cm} (17)
Remark 3. With the angular velocity controller (11), the angular error $\dot{\theta}$, will converge to zero in predefined-time, that is, $\theta = 0$. Based on the cascade control theory, the posture error system (4) is considered as a second-order subsystem and transformed into (16) and (17).

Theorem 2. Supposing there exist

$$V_c = \begin{cases} \frac{C_{c11}}{T_{c11} \cdot w_d} \left\{ a_1 \frac{b_1}{q_1} \cdot \text{sgn}(y) + \frac{m_1}{n_1} \cdot \frac{q_1}{n_1} \cdot e_1 \right\}, & 0 \leq |y_c| < 1 \\ \frac{C_{c12}}{T_{c12} \cdot w_d} \left\{ b_1 \frac{b_1}{q_1} \cdot \text{sgn}(y) + \frac{m_1}{n_1} \cdot d_1 \right\}, & |y_c| \geq 1 \end{cases},$$

such that the system (17) is global predefined-time stable, where $a_1, b_1, c_1, d_1 > 0, p_1, q_1, m_1, n_1$ are positive odd integers satisfying $p_1 = q_1, m_1 > n_1 > 0$. We set $\mu_1 = \frac{b_1}{q_1}, \nu_1 = \frac{m_1}{n_1}$ and

$$C_{c11} = \frac{1}{a_1} \ln \left( \frac{a_1 + \epsilon_1}{\epsilon_1} \right),$$

$$C_{c12} = \frac{2^{\nu_1 - 1}}{b_1^{\nu_1}} \left( \frac{1}{b_1^{\nu_1}} + \frac{1}{a_1^{\nu_1}} \right)^{1-\nu_1}.$$

Then, $T_{c11}$ and $T_{c12}$ indicate the settling time expected by user and

$$T_{c1} < T_{\text{mov1}} := T_{c11} + T_{c12}.$$

Proof. Substituting (18) into (17), we can obtain

$$\dot{y}_c = \begin{cases} -\frac{C_{c11}}{T_{c11}} \left\{ a_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y) + \frac{q_1}{n_1} \cdot e_1 \right\}, & 0 \leq |y_c| < 1 \\ -\frac{C_{c12}}{T_{c12}} \left\{ b_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y) + \frac{m_1}{n_1} \cdot d_1 \right\}, & |y_c| \geq 1 \end{cases}.$$

To analyze the stability of system (17), we design Lyapunov function as

$$V(y_c) = |y_c|.$$

(1) If $0 \leq |y_c| < 1$, the derivative of Lyapunov function $\dot{V}(y_c)$ is given by

$$\dot{V}(y_c) = \text{sgn}(y_c) \cdot y_c = \text{sgn}(y_c) \cdot \frac{C_{c11}}{T_{c11}} \left\{ a_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y_c) + \frac{q_1}{n_1} \cdot e_1 \right\}.$$

If $y_c = 0$, we have $\dot{V}(y_c) = 0$; if $y_c \neq 0$, it is obvious that $\dot{V}(y_c) = -\frac{C_{c11}}{T_{c11}} (a_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot \frac{q_1}{n_1} \cdot e_1)$.

(2) If $|y_c| \geq 1$, the derivative of Lyapunov function $\dot{V}(y_c)$ is given by

$$\dot{V}(y_c) = \text{sgn}(y_c) \cdot \frac{C_{c12}}{T_{c12}} \left\{ b_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y_c) + \frac{m_1}{n_1} \cdot d_1 \right\}.$$

Since $|y_c| > 1$, it is obvious that $\dot{V}(y_c) = -\frac{C_{c12}}{T_{c12}} (b_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot d_1)$. Based on Lemma 2, we conclude that the tracking error $y_c$ will converge to zero in predefined-time $T_{c1} = T_{c11} + T_{c12}$. Proof is completed.

The sliding manifold is defined by

$$s(t) = \begin{cases} \frac{C_{c11}}{T_{c11} \cdot w_d} \left\{ a_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y) + \frac{q_1}{n_1} \cdot e_1 \right\}, & 0 \leq |y_c| < 1 \\ \frac{C_{c12}}{T_{c12} \cdot w_d} \left\{ b_1 \frac{b_1}{q_1} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y) + \frac{m_1}{n_1} \cdot d_1 \right\}, & |y_c| \geq 1 \end{cases}.$$

The derivative of sliding manifold $s$ is designed as

$$\dot{s} = \begin{cases} \dot{x}_c + \frac{C_{c11}}{T_{c11}} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y_c) \cdot \frac{q_1}{n_1} \cdot e_1, & 0 \leq |y_c| < 1 \\ \dot{x}_c + \frac{C_{c12}}{T_{c12}} \cdot \frac{m_1}{n_1} \cdot \text{sgn}(y_c) \cdot \frac{m_1}{n_1} \cdot d_1, & |y_c| \geq 1 \end{cases}.$$

It is worth noting that the global stability of the mobile robot can be guaranteed when the velocity controllers realize the state $y_c$ and the terminal sliding manifold converge to zero at the same time in predefined-time. Next, we design an appropriate velocity controller based on predefined-time sliding mode control (FTSMC) scheme and prove the stability of the subsystem.

Remark 4. The idea of sliding mode control has great significance in the second-order system. In this paper, the velocity controller of the mobile robot is designed by the idea of FTSMC and the state feedback $\dot{x}_c$ synchronizes with the state $y_c$ in predefined-time. Through the analysis of Theorem 2, it can be obtained that $y_c$ converges to zero in predefined-time, therefore, $x_c$ also converges to zero in predefined-time.
Theorem 3. Consider the second-order subsystem (16) and (17), the velocity controllers of mobile robot are designed as

\[
v = \begin{cases} 
    w_d y_d + v_d + \frac{C_{c11}}{T_{c11}} a_1 x_1 + \frac{C_{c21}}{T_{c21}} \cdot a_2 \text{sgn}(s) \frac{w_d}{a_2}, & 0 \leq |s| < 1 \\
    w_d y_d + v_d + \frac{C_{c12}}{T_{c12}} \cdot b_2 \text{sgn}(s) \frac{w_d}{a_2}, & |s| \geq 1 
    \end{cases}
\]

such that the second-order subsystem (16) and (17) are global predefined-time stable, where \(a_2, b_2 > 0, p_2, q_2, m_2, n_2\) are positive odd integers satisfying \(p_2 < q_2, m_2 > n_2\). We set \(\mu_2 = \frac{p_2}{q_2}, \nu_2 = \frac{m_2}{a_2}\) and

\[
C_{c11} = \frac{1}{a_2(1 - \mu_2)}, \\
C_{c21} = C_{c22} = \frac{\text{frac}2q2 - 1}2. \\
\]

\(T_{c21}\) and \(T_{c22}\) indicate the settling time expected by user and

\[
T_{c2} = T_{\text{max}2} := T_{c21} + T_{c22}. 
\]

Proof. Lyapunov function is designed as follows

\[
V(s) = |s|, 
\]

(1) If \(0 \leq |s| < 1\), the derivative of Lyapunov function \(V(s)\) is given by

\[
\dot{V}(s) = \text{sgn}(s) \cdot \dot{s}, \\
= \text{sgn}(s)(\dot{x}_d - \frac{C_{c12}}{T_{c12}} b_1(y_d - x_1)), \\
= \text{sgn}(s)(y_d v_d + v_d - v + \frac{C_{c12}}{T_{c12}} b_1 m_{n1} x_1^{m_{n1} - 1} x_1). 
\]

Take velocity controllers (23) into

\[
\dot{V}(s) = \text{sgn}(s) \cdot \frac{C_{c22}}{T_{c22}} b_2 \text{sgn}(s) \frac{w_d}{a_2}, \\
= -\frac{C_{c22}}{T_{c22}} b_2 \text{sgn}(s) \cdot |s| \frac{w_d}{a_2}. 
\]

Since \(|s| > 1\), it is obvious that \(\dot{V}(s) = -\frac{C_{c22}}{T_{c22}} b_2 V\frac{w_d}{a_2}\). Based on Lemma 2, then the sliding manifold will converge to zero in predefined-time \(T_{c2} = T_{c21} + T_{c22}\). Proof is completed.

Remark 5. From Theorem 3, in order to guarantee that the sliding manifold \(s\) is stable within predefined-time, \(i\) is specified by

\[
i = \left\{ \begin{array}{ll}
-\frac{C_{c21}}{T_{c21}} \cdot a_2 \text{sgn}(s) \frac{w_d}{a_2}, & 0 \leq |s| < 1 \\
-\frac{C_{c22}}{T_{c22}} \cdot b_2 \text{sgn}(s) \frac{w_d}{a_2}, & |s| \geq 1 
\end{array} \right.
\]

It is well-known that the chattering problem is caused by the construction of the control law \(i\) by the sign function. In this paper, the continuous function is used instead of the sign function to eliminate the chattering phenomenon effectively.

Remark 6. It is worth noting that the second-order subsystem has two predefined-time parameters \(T_{c1}\) and \(T_{c2}\). When \(T_{c1} > T_{c2}\), the terminal sliding manifold \(s\) converges to zero at first, so the settling time will be reduced. In other words, the second-order subsystem can reach a steady state as long as \(x_1\) and \(y_1\) converge to zero.

Remark 7. Polyakov proposed the fixed-time control law in [25]. Unfortunately, this control law is not always well defined in the second-order subsystem since there exist singularity when \(y_d = 0\). Therefore, the controller designed by Polyakov’s control law cannot guarantee the input value is bounded. To solve the singularity at the equilibrium point, we set the exponent \(\frac{p_1}{q_1} = 1\) and add a constant \(\epsilon_1\).

Remark 8. In theory, the choice of controller parameters is arbitrary. It should consider the requirements of the actual system. However, in practice, the system parameters are limited by the maximum value, such as

\[
|w| \leq w_{\text{max}}, |i| \leq i_{\text{max}}. 
\]
where \( w_{\text{max}} \) and \( v_{\text{max}} \) represent the maximum angular velocity and maximum velocity of the mobile robot respectively. Therefore, the choice of predefined-time \( T_c \) should combine with the controller gains to achieve the predefined-time stability within the maximum value of the system parameters.

4 SIMULATION

In this section, a simulation example of a wheeled mobile robot is presented to verify the feasibility and stability of the proposed predefined-time sliding mode control design scheme.

The desired forward velocity and the angular velocity are given by \( v_d = 5.0 \) and \( \omega_d = 1.0 \), respectively. Then the mobile robot makes a uniform circular motion and the desired posture \( q_d = [x_d, y_d, \theta_d]^T \) is specified as

\[
\begin{align*}
x_d(t) &= r \cos \theta_d(t) \\
y_d(t) &= r \sin \theta_d(t) \\
\theta_d(t) &= \omega_d t
\end{align*}
\]

where \( r = v_d / \omega_d \) is the radius of the circular trajectory.

In this simulation, we define the initial posture errors as \([x_e(0), y_e(0), \theta_e(0)]^T = [2, 2, 0.5]^T\) and employ the control laws (11) and (23), where the controller parameters are chosen as \( a_0 = 1.3, b_0 = 1, c_0 = 0.1, d_0 = 0.1, \mu_0 = 5, q_0 = 7, m_0 = 11, n_0 = 7, a_1 = 2, b_1 = 2, c_1 = 0.01, d_1 = 0.01, p_1 = 1, q_1 = 1, n_1 = 9, n_2 = 7, a_2 = 1, b_2 = 1, p_2 = 3, q_2 = 5, m_2 = 3, n_2 = 3\). We can calculated that

\[
C_{01} = \frac{1}{a_0^{\mu_0} (1 - \mu_0)} \left( \left( \frac{1}{a_0^{\mu_0}} + \frac{1}{\mu_0} \right)^{1-\mu_0} - \frac{1}{\mu_0} \right) = 1.748,
\]

\[
C_{102} = \frac{2^{v_0-1}}{b_0^{v_0}} \left( \frac{1}{b_0^{\lambda_0} + a_0^{\mu_0}} \right) = 2.469,
\]

\[
C_{111} = \frac{1}{a_1} \ln \frac{a_1 + c_1}{c_1} = 2.652,
\]

\[
C_{112} = \frac{2^{v_1-1}}{b_1^{v_1}} \left( \frac{1}{b_1^{\lambda_1} + a_1^{\mu_1}} \right) = 2.469,
\]

\[
C_{21} = \frac{1}{a_2(1 - \mu_2)} = 2.5,
\]

\[
C_{22} = \frac{2^{v_2-1}}{b_2^{v_2}} = 2.38.
\]

To verify the settling time adjustability of the proposed controller, we perform simulations in two cases, and the simulation results are depicted in Figures 2–5.

Case (a). When \( T_{01} = T_{11} = T_{12} = 2, T_{21} = T_{22} = 4, \) then \( T_0 = 2, T_1 = 4, T_2 = 4 \), the simulation results are shown as Figures 2(b)–5(b). The actual trajectory tracking the reference trajectory is shown in Figure 2(a). As shown in Figure 3(a), it is appreciable that the angular error is enforced by the controller (11) to converge to zero in 0.777 < \( T_0 \). Similarly, it can be seen that the X-axis and Y-axis errors are enforced by the controller (23) to converge to zero in 2.331 < \( T_1 \) and 2.209 < \( T_2 \), respectively. Figure 4(a) illustrates the input curve of the angular velocity controller, and the control input \( \nu \) approximates to the reference angular velocity at 0.777. Figure 5(a) illustrates the input curve of the forward velocity controller, and the control input \( \nu \) approximates to the reference forward velocity at 2.331.

Case (b). When \( T_{01} = T_{02} = 2, T_{11} = T_{12} = 4, T_{21} = T_{22} = 4, \) then \( T_0 = 4, T_1 = 8, T_2 = 8 \), the simulation results are shown as Figures 2(b)–5(b). The actual trajectory tracking the reference trajectory is shown in Figure 2(b). As shown in
Figure 3(b), it is appreciable that the angular error is enforced by controller (11) to converge to zero in $1.553 < T_c$. Similarly, it can be seen that the $X$-axis and $Y$-axis errors are enforced by the controller (23) to converge to zero in $4.594 < T_c$ and $4.335 < T_c$, respectively. Figure 4(b) illustrates the input curve of the angular velocity controller, and the control input $\omega$ approximate to the reference angular velocity at 1.553. Figure 5(b) illustrates the input curve of the forward velocity controller, and the control input $v$ approximate to the reference forward velocity at 4.594. It concludes that the system can achieve stability in different tuning parameters.

As we all know, the design of the predefined-time controller depends on the actual system and the settling time of the system is independent of the initial conditions. In Figure 6, some different initial conditions $[x_e(0), y_e(0), \theta_e(0)]^T = [1, 1, \frac{\pi}{3}]^T, [-1, 2, \frac{\pi}{3}]^T, [-1, -2, \frac{2\pi}{3}]^T, [-2, 1.5, -\frac{\pi}{3}]^T$ are defined. It is worth noting that the proposed controllers (11) and (23) drive the posture error system (4) to converge to zero before the predefined-time $T_c$. In addition, the predefined-time $T_c$ can be selected by users during the design process of the controllers and the simulation results are shown in Figure 6(a) with $T_c = 2, T_c = 4, T_c = 4$ and Figure 6(b) with
Control input $v$ of mobile robot (a) when $T_c_0 = 2$, $T_c_1 = 4$, $T_c_2 = 4$. (b) when $T_c_0 = 4$, $T_c_1 = 8$, $T_c_2 = 8$.

Trajectory tracking errors for several different initial conditions (a) when $T_c_0 = 2$, $T_c_1 = 4$, $T_c_2 = 4$. (b) when $T_c_0 = 4$, $T_c_1 = 8$, $T_c_2 = 8$.

Clearly, the settling time of the system can be obtained in advance.

Furthermore, to verify that our control law has a faster convergence rate than the existing control law, the control law of angular velocity is designed as

$$w = w_d + a_0 \text{sig}(\theta) + b_0 \text{sig}(\dot{\theta})$$

which is proposed in [26, 30]. In this paper, we can remove the predefined tuning parameters of controllers (11) and (23) to obtain the fixed-time stability of mobile robots. Then we simulate the closed-loop system under the two different fixed-time angular velocity controllers, and the comparative simulation result is plotted in Figure 7. Meanwhile, the integral square error (ISE) values are calculated as shown in Figure 8. The ISE value of our method is smaller than the existing method. Therefore, our method has better performance indicators.

Remark 9. Through the above numerical simulation, it can be concluded that the error system of the mobile robot has a fast convergence rate, and the second-order sliding mode control has no singularity. By setting different tuning parameters, the mobile robot can reach a stable state within the desired time. The system is more flexible and has better robustness.
5 1 CONCLUSION

For the nonholonomic mobile robots, we have proposed a new trajectory tracking control method based on the non-negative piecewise predefined-time theorem. The posture error system was divided into two subsystems by using the idea of cascaded control. For the angular velocity subsystem, a controller has been designed to realize that the angle error system converges and stabilizes to zero in predefined-time. For the forward velocity error subsystem, a novel sliding mode control approach has been developed, which combines with the predefined-time theorem. Compared with the existing fixed-time trajectory tracking of mobile robots, our method has several contributions as follows:

1. In this paper, the control law adds a constant to increase the convergence rate and compensate for the influence of singularity of the second-order nonlinear system.
2. In the process of sliding mode control design, we use the continuous function instead of the piecewise function to eliminate the chattering phenomenon.
3. This method transforms the complex expression of the setting time estimation into an adjustable parameter, which is convenient for users to predict the stable time of the trajectory tracking of the mobile robot by selecting the tuning parameters.

The simulation results show the feasibility and correctness of the proposed control algorithm. In the future, we will consider the external unknown disturbances into the mobile robot systems and apply them in actual experiments.

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ORCID

Lixiong Lin https://orcid.org/0000-0002-9829-5358

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