THE CATEGORY OF ORDERED BRATTELI DIAGRAMS

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Abstract. A category structure for ordered Bratteli diagrams is proposed in which isomorphism coincides with the notion of equivalence of Herman, Putnam, and Skau. It is shown that the natural one-to-one correspondence between the category of Cantor minimal systems and the category of simple properly ordered Bratteli diagrams is in fact an equivalence of categories. This gives a Bratteli–Vershik model for factor maps between Cantor minimal systems. We give a construction of factor maps between Cantor minimal systems in terms of suitable maps (called premorphisms) between the corresponding ordered Bratteli diagrams, and we show that every factor map between two Cantor minimal systems is obtained in this way. Moreover, solving a natural question, we are able to characterize Glasner and Weiss’s notion of weak orbit equivalence of Cantor minimal systems in terms of the corresponding C*-algebra crossed products.

1. Introduction

In 1972, Bratteli introduced what are now called Bratteli diagrams to study AF algebras [5]. He associated to each AF algebra an infinite directed graph (see Definition 2.1) and used these very effectively to study (and classify) AF algebras. Some attributes of an AF algebra (such as its ideal structure) can be read off directly from its Bratteli diagram.

The second author introduced the notion of dimension group and gave a classification of AF algebras using K-theory in 1976 [11], showing that the functor $\text{K}_0 : \text{AF} \to \text{DG}$, from the category of AF algebras with *-homomorphisms to the category of scaled dimension groups with order-preserving homomorphisms, is a strong classification functor (see also [12, Sections 5.1–5.3]).

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is called a classification functor ([12]) if $F(a) \cong F(b)$ implies $a \cong b$, for each $a, b \in \mathcal{C}$, and a strong classification functor if each isomorphism from $F(a)$ to $F(b)$ is the image of an isomorphism from $a$ to $b$.

In [1], the authors introduced the category $\text{BD}$ of Bratteli diagrams, isomorphisms of which coincide with the notion of equivalence of Bratteli diagrams introduced by Bratteli, to capture isomorphism of the corresponding AF algebras. We showed that the map $B : \text{AF} \to \text{BD}$, defined by Bratteli in [5] on objects, is in fact a functor. The fact that this is a strong classification functor [1, Theorem 3.11] is a functorial formulation of Bratteli’s classification of AF algebras in terms of diagrams, and completes his work from the classification functor point of view of [12].

Bratteli diagrams have been used to study certain dynamical systems. In 1981, Vershik used Bratteli diagrams to construct the so-called adic transformations [30, 29]. Based on his work (and the work of Power [24]), Herman, Putnam, and Skau introduced the notion of ordered Bratteli diagram, and associated a dynamical system to a properly ordered Bratteli diagram [20]. They showed that there is a one-to-one correspondence between properly ordered Bratteli diagrams and essentially minimal totally 2010 Mathematics Subject Classification. Primary: 37B05, 46M15; secondary: 37A20, 19K14.

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disconnected dynamical systems [20, Theorem 4.7]. In particular, each Cantor minimal system has a Bratteli–Vershik model. This correspondence was used effectively to study Cantor minimal systems and in particular to characterize what they called strong orbit equivalence in terms of isomorphism of dimension groups and the corresponding C*-algebra crossed products [20, 14, 17, 23]. (Simple orbit equivalence is also characterized in [21]. In the present work we do this for weak orbit equivalence.)

Most of the classification results concerning Cantor minimal systems and their associated ordered Bratteli diagrams and ordered K-groups, obtained up to now, only deal with isomorphism classes (see [14, 17, 20]). For instance, in [20] Herman, Putnam, and Skau, among other things, showed that two Cantor minimal systems $(X, \varphi)$ and $(Y, \psi)$ are conjugate if and only if their associated ordered Bratteli diagrams are equivalent. An obvious question is then whether one can realize factor maps (an important notion in dynamics) from $(X, \varphi)$ to $(Y, \psi)$ in terms of maps between the associated ordered Bratteli diagrams. In particular, one could ask if $(Y, \psi)$’s being a factor of $(X, \varphi)$ could be decided by looking at the corresponding ordered Bratteli diagrams. Sugisaki in [28] and Host and Glasner in [16] studied certain factor maps (for instance, almost one-to-one extensions) in terms of dimension groups (see also [13] and [9]). The functorial classification approach of [12] (finding classification functors—which are possibly full or faithful) is relevant to this question, as it takes general morphisms into account, and could lead, at least in certain cases, to a classification of morphisms. This is the main objective of the current paper. In particular, we obtain a functor $P$ from the category of Cantor minimal systems to the category of ordered Bratteli diagrams and its inverse functor $V$, leading to a model for factor maps between Cantor minimal systems. Having a model may have many applications. For instance, the classical result on the existence of the maximal rational equicontinuous factor for Cantor minimal systems (Theorem 4.12), and the uniqueness of a factor map onto an odometer (Proposition 4.14) follow easily from this model. (The characterization of almost one-to-one extensions and the study of finite-to-one factor maps will be given in the forthcoming paper [19].)

There is a close relation between Cantor minimal systems and certain C*-algebras. Indeed, to every Cantor minimal system $(X, \varphi)$ there is associated a C*-algebra crossed product $C(X) \rtimes_{\varphi} \mathbb{Z}$ with the same ordered $K_0$-group as that system [20]. One expects that every (equivalence) relation between two Cantor minimal systems has characterizations in terms of C*-algebra crossed products. This has been shown already for strong orbit equivalence by Giordano, Putnam, and Skau in [14], and for orbit equivalence by Lin in [21]. (See also [14] for characterizations of flip conjugacy and Kakutani (strong) orbit equivalence.) However, no characterization for the weak orbit equivalence of Glasner and Weiss in terms of C*-algebras was known. We use the notion of tracial equivalence in the sense of Lin [21] to achieve this goal (Theorem 5.1).

The structure of the paper is as follows. Following on the ideas of [1], we first propose a notion of morphism between ordered Bratteli diagrams and obtain the category $\text{OBD}$ of ordered Bratteli diagrams (Section 2). Isomorphism in this category coincides with equivalence in the sense of Herman, Putnam, and Skau. We show that the correspondence obtained by Herman, Putnam, and Skau in [20] is an equivalence of categories. More precisely, for the category $\text{SDS}$ of scaled essentially minimal totally disconnected dynamical systems (Definition 3.5), which contains the category of Cantor minimal systems, we construct a contravariant functor $P : \text{SDS} \to \text{OBD}$ (Section 3), leading to what might be viewed as a model for essentially minimal totally disconnected
dynamical systems and their morphisms. In particular, the functor \( P \) gives a Bratteli–Vershik model for factor maps between Cantor minimal systems, which we then use in the study of factors of such systems.

In Section 3, we show that the contravariant functor \( P : SDS \to OBD \) is full and faithful, and identify the (essential) range of this functor, as the class of properly ordered Bratteli diagrams \( OBD_{po} \). This gives an equivalence of categories \( P : SDS \to OBD_{po} \) (Theorem 3.10). We also construct an inverse to the functor \( P \), a contravariant functor \( V : OBD_{po} \to SDS \), which, naturally, is also an equivalence of categories. This latter functor gives us a handle on factor maps between Cantor minimal systems, by graphically constructing certain arrows (premorphisms) between the associated ordered Bratteli diagrams. This is in particular useful when one applies these functors to morphisms. In this way, one obtains a functorial formulation (including general morphisms) of the correspondence of [20] between properly ordered Bratteli diagrams and essentially minimal totally disconnected dynamical systems (Theorem 3.15).

In Section 4, we apply the results of Section 3 to certain subcategories of SDS. In particular, we show that the category of Cantor minimal systems is equivalent to the category of what Durand, Host, and Skau called properly ordered Bratteli diagrams; see [10] (Corollary 4.4 below). In Subsection 4.2, we focus on factors of Cantor minimal systems, to illustrate the use of our functorial machinery. We give concrete examples of the construction of factor maps using premorphisms. In particular, we reprove—by the technique of premorphisms—that every Cantor minimal system has a maximal odometer factor (Theorem 4.12). In [19] more applications of this technique are given. Indeed the notion of (ordered) premorphism enables us to construct desired factor maps by using an explicit graphical method.

In Section 5, we give an equivalent condition—in terms of the corresponding C*-algebra crossed products—for the weak orbit equivalence of Glasner and Weiss.

2. The Category of Ordered Bratteli Diagrams

In this section we propose a notion of morphism for the category \( OBD \) of ordered Bratteli diagrams. This construction is similar to the construction of the category of Bratteli diagrams, \( BD \), given in [1]. In particular, first we need a notion of (ordered) premorphism. We shall see that isomorphism in this category coincides with equivalence of ordered Bratteli diagrams, as defined by Herman, Putnam, and Skau in [20].

Let us first recall and fix some notation concerning Bratteli diagrams. See [3, 4, 8, 10, 20, 14, 1] for more information about (simple and non-simple) Bratteli diagrams.

Definition 2.1. A Bratteli diagram consists of a vertex set \( V \) and an edge set \( E \) satisfying the following conditions. We have a decomposition of \( V \) as a disjoint union \( V_0 \cup V_1 \cup \cdots \), where each \( V_n \) is finite and non-empty and \( V_0 \) has exactly one element, \( v_0 \). Similarly, \( E \) decomposes as a disjoint union \( E_1 \cup E_2 \cup \cdots \), where each \( E_n \) is finite and non-empty. Moreover, we have maps \( r, s : E \to V \) such that \( r(E_n) \subseteq V_n \) and \( s(E_n) \subseteq V_{n-1} \), \( n = 1, 2, 3, \ldots \) (\( r = \text{range}, s = \text{source} \)). We also assume that \( s^{-1}\{v\} \) is non-empty for all \( v \) in \( V \) and \( r^{-1}\{v\} \) is non-empty for all \( v \) in \( V \setminus V_0 \). Let us denote such a \( B \) by the diagram

\[
\begin{array}{ccc}
V_0 & E_1 & V_1 & E_2 & V_2 & E_3 & \cdots \\
0 & 1 & 2 & 3 & \cdots \\
\end{array}
\]

In the preceding definition, if we fix a total order on each \( V_n \), then to each edge set \( E_n \) a matrix \( M(E_n) \) is associated, called the multiplicity matrix of \( E_n \) (also called the “incidence matrix” [14]).
Let $k, l$ be integers with $0 \leq k < l$. Let $E_{k, l} = E_{k+1} \circ E_k \circ \cdots \circ E_l$ denote the set of all paths from $V_k$ to $V_l$; that is, the tuples $(e_{k+1}, \ldots, e_l)$ where $e_i \in E_i$, for $i = k+1, \ldots, l$, with $r(e_i) = s(e_{i+1})$, for $i = k+1, \ldots, l-1$. In particular, $E_{k, k} = \{(v, v) \mid v \in V_k\}$ is an edge set from $V_k$ to itself. We identify $E_{k, k}$ with its multiplicity matrix.

**Definition 2.2** ([10, 3, 4]). Let $B = (V, E)$ be a Bratteli diagram (as in Definition 2.1). $B$ is called simple if there exists a telescoping $(V', E')$ of $(V, E)$ such that the multiplicity matrices of $(V', E')$ have only non-zero elements at each level. In other words, $B$ is simple if for each $n \geq 0$ there is $m > n$ such that, for every $v \in V_n$ and every $w \in V_m$, there is a path in $E_{n,m}$ from $v$ to $w$.

**Definition 2.3.** An ordered Bratteli diagram is a Bratteli diagram $(V, E)$ as in Definition 2.1 together with an order relation $\geq$ on $E$ such that $e, e' \in E$ are comparable if, and only if, $r(e) = r(e')$. In other words, we have a linear order on each set $r^{-1}\{v\}$, for every $v \in V \setminus V_0$.

If $(V, E, \geq)$ is an ordered Bratteli diagram and $k, l$ are integers with $0 \leq k < l$, then the set $E_{k, l}$ may be given an induced (lexicographic) order [10, 20].

For an ordered Bratteli diagram $(V, E, \geq)$, denote by $E_{\text{max}}$ and $E_{\text{min}}$ the set of maximal and minimal edges of $E$, respectively.

**Definition 2.4.** Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. We say that $B$ is properly ordered if there are unique infinitely long paths in $E_{\text{max}}$ and $E_{\text{min}}$, that is, there is only one sequence $(e_1, e_2, \ldots)$ with each $e_i \in E_{\text{max}}$ and $s(e_{i+1}) = r(e_i)$, for all $i \geq 1$, and the same holds for $E_{\text{min}}$.

Note that, properly ordered Bratteli diagrams (in the sense of the preceding definition) are called essentially simple in [20, 10, 8]. We use the now standard term “properly ordered” (see, e.g., [3]).

Let us define the category of ordered Bratteli diagrams. We need a notion of (ordered) premorphism before considering the final notion of morphism. Denote by OBD the class of all ordered Bratteli diagrams.

**Definition 2.5.** Let $B = (V, E, \geq)$ and $C = (W, S, \geq)$ be ordered Bratteli diagrams. By an ordered premorphism (or just a premorphism if there is no confusion) $f : B \to C$ we mean a triple $(F, (f_n)_{n=0}^{\infty}, \geq)$ where $(f_n)_{n=0}^{\infty}$ is a cofinal (i.e., unbounded) sequence of positive integers with $f_0 = 0 \leq f_1 \leq f_2 \leq \cdots$, $F$ consists of a disjoint union $F_0 \cup F_1 \cup F_2 \cup \cdots$ together with a pair of range and source maps $r : F \to W$, $s : F \to V$, and $\geq$ is a partial order on $F$ such that:

1. each $F_0$ is a non-empty finite set, $s(F_n) \subseteq V_n$, $r(F_n) \subseteq W_{f_n}$, $F_0$ is a singleton, $s^{-1}\{v\}$ is non-empty for all $v \in V$, and $r^{-1}\{w\}$ is non-empty for all $w \in W$;
2. $e, e' \in F$ are comparable if and only if $r(e) = r(e')$, and $\geq$ is a linear order on $r^{-1}\{w\}$, for all $w \in W$;
3. the diagram of $f : B \to C$,

\[
\begin{array}{cccccc}
V_0 & \overset{E_1}{\longrightarrow} & V_1 & \overset{E_2}{\longrightarrow} & V_2 & \overset{E_3}{\longrightarrow} & \cdots \\
F_0 & \overset{F_1}{\longrightarrow} & F_1 & \overset{F_2}{\longrightarrow} & F_2 & \text{commutes. The (ordered) commutativity of the diagram of } f \text{ means that for each } n \geq 0, E_{n+1} \circ F_{n+1} \cong F_n \circ S_{f_n, f_{n+1}}, \text{ i.e., there is a (necessarily unique) bijective map }
\end{array}
\]


Figure 1. The difference between unordered commutativity (left diagram) and ordered commutativity (right diagram). In both diagrams the number of paths from $u$ to $w$ passing through $y$ and the number through $z$ are equal (which is one here), and the same for paths from $v$ to $w$. However, in the left diagram the source map is not preserved, since the first path ending in $w$ and passing through $y$ starts at $v$ while the first path ending in $w$ and passing through $z$ starts at $u$. In the right diagram the source map is preserved.

from $E_{n+1} \circ F_n$ to $F_n \circ S_{f_n,f_{n+1}}$ preserving the order and intertwining the respective source and range maps.

If $B$ and $C$ in the preceding definition are (unordered) Bratteli diagrams then a pair $f = (F,(f_n)_{n=0}^\infty)$ with the properties stated in the preceding definition (without Condition (2)) is called a premorphism from $B$ to $C$. Note that, in this case, we require only (unordered) commutativity of the diagram of $f$, that is, for each $n \geq 0$, each $v \in V_n$, and each $w \in W_{f_n+1}$, the number of paths from $v$ to $w$ passing through $W_{f_n}$ and the number through $V_{n+1}$ are equal. This is equivalent to saying that for any positive integer $n$, $M(F_{n+1})M(E_{n+1}) = M(S_{f_n,f_{n+1}})M(F_n)$.

We remark that the ordered commutativity required in Definition 2.5 is essential. In fact, if $f$ is a premorphism (i.e., only unordered commutativity holds), then one obtains a continuous map between the associated Bratteli compacta. However, if $f$ is an ordered premorphism (i.e., ordered commutativity holds), then one gets not only a continuous map but also a homomorphism between the associated dynamical systems (see Subsection 3.2). See Figure 1 for an illustrative example of ordered and unordered commutativity.

We give an illustrative example of an ordered premorphism in Figure 2. We use thick and curved arrows to depict the edges of premorphisms.

Example 2.6. In Figure 2, a premorphism $f : B \to C$ is depicted where $B$ is the odometer of type $(k_n)_{n=1}^\infty$ with $k_1 = 1$ and $k_n = 3$ for $n \geq 2$ (see Definition 4.8 below), and $C$ (with left-to-right order) is a Toeplitz system. The (ordered) commutativity needed in Definition 2.5 can be checked easily at each level. Note that since $B$ has only one vertex at each level, ordered commutativity (as in Definition 2.5) is the same as commutativity for $f$. As we will see in Subsection 3.2, applying the functor $\mathcal{V}$, we get a factor map $\mathcal{V}([f]) : \mathcal{V}(C) \to \mathcal{V}(B)$ as defined before Lemma 3.13. In fact, $\mathcal{V}(B)$ is the maximal rational equicontinuous factor of $\mathcal{V}(C)$ (see Theorem 4.12 below, and [15]).

In a way similar to [1], we define an isomorphism relation on the class of ordered premorphisms and we define the composition of two ordered premorphisms.

Definition 2.7. Let $B,C \in \text{OBD}$ and let $f,f' : B \to C$ be a pair of ordered premorphisms where $f = (F,(f_n)_{n=0}^\infty,\geq)$ and $f' = (F',(f'_n)_{n=0}^\infty,\geq')$. We shall say that $f$ is isomorphic to $f'$, and write $f \cong f'$, if $f_n = f'_n$, $n \geq 0$, and there is a bijective
map from $F$ to $F'$, preserving the order and the range and source maps. This is an equivalence relation on the class of ordered premorphisms from $B$ to $C$. We denote the equivalence class of $f$ by $[f]$. Let $B$, $C$, and $D$ be objects in $\text{OBD}$ and let $f : B \to C$ and $g : C \to D$ be ordered premorphisms; $f = (F, (f_n)_{n=0}^\infty, \geq)$, $g = (G, (g_n)_{n=0}^\infty, \geq)$, where $F = \bigcup_{n=0}^\infty F_n$ and $G = \bigcup_{n=0}^\infty G_n$ (disjoint unions). The composition of $f$ and $g$ is defined as $gf = (H, (h_n)_{n=0}^\infty, \geq)$, where $h_n = g_{f_n}$, $H = \bigcup_{n=0}^\infty H_n$, and $H_n = F_n \circ G_{f_n}$, $n \geq 0$ (i.e., the set of all paths from $s(F_n)$ to $r(G_{f_n})$. The partial order $\geq$ on $H$ is the induced lexicographic order. Also, set $gf = g\circ f$.

It is not hard to see that the class $\text{OBD}$, with ordered premorphisms modulo the relation of isomorphism (see above) is a category. We shall refer to this as the category of ordered Bratteli diagrams with ordered premorphisms. Two ordered Bratteli diagrams are isomorphic in the category $\text{OBD}$ with (ordered) premorphisms if, and only if, they are isomorphic in the sense of Herman, Putman, and Skau ([20]).

We define an equivalence relation on ordered premorphisms.

**Definition 2.8.** Let $B, C$ be ordered Bratteli diagrams and let $f, g : B \to C$ be ordered premorphisms with $B = (V, E, \geq)$, $C = (W, S, \geq)$, $f = (F, (f_n)_{n=0}^\infty, \geq)$, and $g = (G, (g_n)_{n=0}^\infty, \geq)$. We shall say that $f$ is *equivalent* to $g$, and write $f \sim g$, if there...
are sequences \((n_k)_{k=1}^\infty\) and \((m_k)_{k=1}^\infty\) of positive integers such that \(n_k < m_k < n_{k+1}\) for each \(k \geq 1\), and the diagram

\[
\begin{array}{cccc}
V_{n_1} & \rightarrow & V_{m_1} & \rightarrow & V_{n_2} & \rightarrow & V_{m_2} & \rightarrow & \cdots \\
\uparrow & & \downarrow & & \uparrow & & \downarrow & & \\
F_{n_1} & \rightarrow & G_{m_1} & \rightarrow & F_{n_2} & \rightarrow & G_{m_2} & \rightarrow & \cdots \\
W_{f_{n_1}} & \rightarrow & W_{g_{m_1}} & \rightarrow & W_{f_{n_2}} & \rightarrow & W_{g_{m_2}} & \rightarrow & \cdots
\end{array}
\]

is (ordered) commutative, i.e., each minimal square commutes: for each \(k \geq 1\),

\[
E_{n_k,m_k} \circ G_{m_k} \cong F_{n_k} \circ S_{f_{n_k} \circ g_{m_k}},
\]

\[
E_{m_k,n_{k+1}} \circ F_{n_{k+1}} \cong G_{m_k} \circ S_{g_{m_k} \circ f_{n_{k+1}}}
\]

It is easily checked that \(\sim\) is an equivalence relation on the class of ordered premorphisms from \(B\) to \(C\). Let us call the equivalence classes ordered morphisms, or if there is no confusion, just morphisms, in OBD. We shall denote the equivalence class of an ordered premorphism \(f : B \rightarrow C\) by \([f] : B \rightarrow C\), or, if there is no confusion, just by \(f\).

The composition of morphisms \([f] : B \rightarrow C\) and \([g] : C \rightarrow D\) is defined as \([gf] : B \rightarrow D\) where \(gf\) is the composition of ordered premorphisms (see Definition 2.7). This composition is well defined. The first statement of the next result is proved in a way similar to the proof of [1, Theorem 2.7]. The second statement is easy to prove.

**Proposition 2.9.** The class OBD, with (ordered) morphisms as defined above, is a category. Two ordered Bratteli diagrams are isomorphic in this category if and only if they are equivalent in the sense of Herman, Putnam, and Skau.

Let us refer to the category OBD with (ordered) morphisms as defined above as the category of ordered Bratteli diagrams.

We shall now give two alternative formulations of the definition of equivalence for premorphisms (Definition 2.8). The first one will be used in a number of places later.

**Definition 2.10.** Let \(f,g : B \rightarrow C\) be ordered premorphisms in OBD, with \(B = (V,E,\geq), \ C = (W,S,\geq), \ f = (F_n)_{n=0}^\infty, \ geq\), and \(g = (G_n)_{n=0}^\infty, \geq\). We shall say that \(f\) is equivalent to \(g\), in the second sense, if for each \(n \geq 0\) there is an \(m \geq f_n, g_n\) such that \(F_n \circ S_{f_{n,m}} \cong G_n \circ S_{g_{n,m}}\), and equivalent to \(g\), in the third sense, if for each \(n \geq 0\) and for each \(k \geq n\), there is an \(m \geq f_n, g_n\) such that \(F_n \circ S_{f_{n,m}} \cong E_{n,k} \circ G_k \circ S_{g_{k,m}}\).

Using an analogue of [1, Proposition 2.11], one can see that Definitions 2.8 and 2.10 are equivalent.

It might be noted that the category of Bratteli diagrams could also be described in terms of the general category construction of inductive limits starting from single-step Bratteli diagrams (see, e.g., [18]).

We close this section with another illustrative example of an ordered premorphism \(f\). We will construct the inverse of the morphism \([f]\) in Example 4.15. Thus, \([f]\) is an isomorphism in the category OBD.

**Example 2.11.** Consider the Chacon substitution system \((X, \varphi)\) described in [15], i.e., the substitution minimal system associated to the Chacon substitution \(0 \rightarrow 0010, \ 1 \rightarrow 1\). Let \(C = (W,S,\geq)\) be the Bratteli–Vershik model for \((X, \varphi)\) as explained in [15, Section 4.2]. The diagram \(C\) is drawn on the left in Figure 4, below. Let \(C' = (W',S',\geq')\) be the telescoping to the sequence \(0, 2, 3, 4, \ldots\) of \(C\). The diagram \(C'\) is drawn on the right in Figure 3. Let \(B = (V,E,\geq)\) be the properly ordered Bratteli diagram drawn on the left in Figure 3. One can check that \(f : B \rightarrow C\) in Figure 3 is an
ordered premorphism. In the notation of Definition 2.5, the multiplicity matrices are the following:
\[
M(E_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad M(E_n) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M(S'_1) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \quad M(S'_n) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \text{for } n \geq 2;
\]
\[
M(F_0) = (1), \quad M(F_n) = \begin{pmatrix} 7 & 4 \\ 7 & 7 \end{pmatrix}, \quad \text{for } n \geq 1.
\]
In fact, these two diagrams are equivalent (see [15, Section 4.2]). To show this, we will construct the inverse of the morphism \([f]\) in Example 4.15.
3. Functors between Categories of Dynamical Systems and Bratteli Diagrams

In this section, we shall construct two (contravariant) functors, \( P : SDS \to OBD_{po} \) and \( V : OBD_{po} \to SDS \), which are equivalences of categories and are inverse to each other. In particular, the functor \( P \) provides a model for factor maps between Cantor minimal systems and the functor \( V \) provides a method to construct factor maps between two Cantor minimal systems by drawing suitable arrows (i.e., ordered premorphisms) between their ordered Bratteli diagrams.

3.1. The functor \( P \) from SDS to OBD. In this subsection, we define the category of scaled essentially minimal totally disconnected dynamical systems SDS, and construct a functor \( P : SDS \to OBD_{po} \) which is an equivalence of categories (Theorem 3.10).

Here we are mainly interested in minimal (totally disconnected) systems, but almost all the results hold in a more general setting, namely, essentially minimal totally disconnected systems. The minimal case will be discussed more specifically in Section 4.

Definition 3.1 (cf. [20], Definition 1.2). Let \( X \) be a metrizable compact space, let \( \varphi \) be a homeomorphism of \( X \), and let \( x_0 \in X \). The triple \((X, \varphi, x_0)\) is called an essentially minimal dynamical system if the dynamical system \((X, \varphi)\) has a unique minimal (non-empty, closed, invariant) subset \( Y \) and \( x_0 \in Y \).

Recall that if moreover \( X \) is totally disconnected and has no isolated points, then \( X \) is homeomorphic to the Cantor set. There are of course essentially minimal totally disconnected dynamical systems which are not minimal. For example, the one-point compactification of a locally compact non-compact Cantor minimal system is essentially minimal but not minimal ([23]).

Definition 3.2. Let us define the category \( DS \) of essentially minimal totally disconnected dynamical systems as follows. The objects of this category are essentially minimal totally disconnected dynamical systems. Let \((X, \varphi, x_0)\) and \((Y, \psi, y_0)\) be in \( DS \). By a morphism \( \alpha : (X, \varphi, x_0) \to (Y, \psi, y_0) \) in \( DS \) we shall mean a homomorphism from the dynamical system \((X, \varphi)\) to \((Y, \psi)\) (i.e., a continuous map \( \alpha : X \to Y \) with \( \alpha \circ \varphi = \psi \circ \alpha \)) such that \( \alpha(x_0) = y_0 \).

Note that in the definition above, \( \alpha \) maps the unique minimal subset of \((X, \varphi)\) to that of \((Y, \psi)\). Also, isomorphism in the category \( DS \) coincides with pointed topological conjugacy introduced in [20]. We recall the notion of a Kakutani–Rokhlin partition [20].

Definition 3.3. Let \((X, \varphi, x_0)\) be an essentially minimal totally disconnected dynamical system. A Kakutani–Rokhlin partition for \((X, \varphi, x_0)\) is a partition \( P \) of \( X \) where

\[
P = \{Z(k, j) \mid k = 1, \ldots, K, \ j = 1, \ldots, J(k)\},
\]

in which \( K \) and \( J(1), \ldots, J(K) \) are non-zero positive integers and the \( Z(k, j) \) are non-empty clopen subsets of \( X \) with the following properties:

1. \( \varphi(Z(k, j)) = Z(k, j + 1) \) for all \( 1 \leq k \leq K \), and \( 1 \leq j < J(k) \);
2. setting \( Z = \bigcup Z(k, J(k)) \), one has \( x_0 \in Z \) and \( \varphi(Z) = \bigcup Z(k, 1) \).

For each \( 1 \leq k \leq K \), the set \( \{Z(k, j) \mid j = 1, \ldots, J(k)\} \) is called the \( k \)th tower of \( P \) with height \( J(k) \). The sets \( Z \) and \( \varphi(Z) \) are called the top and base of \( P \), respectively.

The following definition was used implicitly in [20].

Definition 3.4. Let \((X, \varphi, x_0)\) be an essentially minimal totally disconnected dynamical system. A system of Kakutani–Rokhlin partitions for \((X, \varphi, y)\) is a sequence \((P_n)_{n=0}^{\infty}\) of Kakutani–Rokhlin partitions for \( X \) such that \( P_0 = \{X\} \) and:
(1) if $Z_n$ denotes the top of $P_n$ for each $n \geq 1$, the sequence $(Z_n)_{n=1}^\infty$ is a decreasing sequence of clopen sets with intersection $\{x_0\}$;

(2) for all $n$, $P_n \leq P_{n+1}$, i.e., $P_{n+1}$ is a refinement of $P_n$;

(3) $\bigcup_{n=0}^\infty P_n$ is a basis for the topology of $X$.

**Definition 3.5.** By a scaled essentially minimal totally disconnected dynamical system we mean a quadruple $(X, \varphi, x_0, R)$ where $(X, \varphi, x_0)$ is an essentially minimal totally disconnected dynamical system and $R$ is a system of Kakutani–Rokhlin partitions for $(X, \varphi, x_0)$. The category of scaled essentially minimal totally disconnected dynamical systems $\text{SDS}$ is the category whose objects are the essentially minimal totally disconnected dynamical systems and whose morphisms are as follows. Let $(X, \varphi, x_0, R)$ and $(Y, \psi, y_0, S)$ be in $\text{SDS}$. By a morphism $\alpha : (X, \varphi, x_0, R) \to (Y, \psi, y_0, S)$ we mean a homomorphism between the dynamical systems $(X, \varphi)$ and $(Y, \psi)$ (i.e., a continuous map $\alpha : X \to Y$ with $\alpha \circ \varphi = \psi \circ \alpha$) such that $\alpha(x_0) = y_0$.

We shall need the following notation in a number of places.

**Notation.** Let $(X, \varphi, x_0, R)$ be an essentially minimal totally disconnected dynamical system and $P$ and $Q$ be a pair of Kakutani–Rokhlin partitions for it such that $P \leq Q$ (i.e., $Q$ is a refinement of $P$) and the top of $P$ contains the top of $Q$. Considering the towers of $P$ and $Q$ as vertices, we shall denote by $E(P, Q)$ the (ordered) edge set from $P$ to $Q$ defined as follows. We have an edge in $E(P, Q)$ each time a tower of $Q$ passes a tower of $P$; explicitly, $E(P, Q)$ contains all elements of the form $(S, T, k)$ where $S = \{Z_1, \ldots, Z_n\}$ and $T = \{Y_1, \ldots, Y_m\}$ are towers of $P$ and $Q$, respectively, and $k$ is a positive (i.e., non-negative) integer such that $Y_{k+j} \subseteq Z_j$ for all $1 \leq j \leq n$ (cf. [20, Section 4]). Note that $(S, T, k) \in E(P, Q)$ if and only if $Y_{k+j} \subseteq Z_j$ for some $1 \leq j \leq n$. Write $(S, T, k) \leq (S', T', k')$ if $T = T'$ and $k \leq k'$. This is an order relation on $E(P, Q)$, which is a total order on the subset of edges leading to a common vertex.

We shall need the following lemma. This is a topological version of [1, Lemma 3.4] (see Definition 2.5 for the notation $\Xi$). The proof is straightforward.

**Lemma 3.6.** Let $(X, \varphi, x_0)$ be an essentially minimal totally disconnected dynamical system and let $P_1$, $P_2$, and $P_3$ be Kakutani–Rokhlin partitions such that $P_1 \leq P_2 \leq P_3$ and the top of $P_i$ contains the top of $P_{i+1}$, for $i = 1, 2$. Then $E(P_1, P_3) \cong E(P_1, P_2) \circ E(P_2, P_3)$, i.e., the following diagram commutes, in the natural sense:

$$
\begin{array}{ccc}
P_1 & \xrightarrow{E(P_1, P_2)} & P_2 \\
\downarrow{E(P_1, P_3)} & & \downarrow{E(P_2, P_3)} \\
P_3
\end{array}
$$

Now we are ready to define the functor $\mathcal{P} : \text{SDS} \to \text{OBD}$. (The definition of this functor on objects was already given in [20, Section 4].)

Define the contravariant functor $\mathcal{P} : \text{SDS} \to \text{OBD}$ as follows. Let $(X, \varphi, x_0, R)$ be in $\text{SDS}$. Consider the ordered Bratteli diagram $\mathcal{P}(X, \varphi, x_0, R) = (V, E, \geq)$ constructed in [20, Section 4] for $(X, \varphi, x_0, R)$. Let $\mathcal{R}$ be as in Definition 3.4 and set $V_n = \{(n, T) \mid T$ is a tower of $P_n\}$, $n \geq 0$, and $V = \bigcup_{n=0}^\infty V_n$. Set $E_n = \{(n, S, T, k) \mid (S, T, k) \in E(P_{n-1}, P_n)\}$, $n \geq 1$, and $E = \bigcup_{n=1}^\infty E_n$. The order on $E$ is defined as the union of orderings on the $E_n$ as described just before Lemma 3.6.

Now let $(X, \varphi, x_0, R)$ and $(Y, \psi, y_0, S)$ be in $\text{SDS}$, where $\mathcal{R} = (P_n)_{n=0}^\infty$ and $\mathcal{S} = (Q_n)_{n=0}^\infty$ and let $\alpha : (X, \varphi, x_0, R) \to (Y, \psi, y_0, S)$ be a morphism in $\text{SDS}$, i.e., a continuous map $\alpha : X \to Y$ with $\alpha(x_0) = y_0$ and $\alpha \circ \varphi = \psi \circ \alpha$ (no relation to $\mathcal{R}$ and $\mathcal{S}$).
Define the (ordered) premorphism \( f = (F, (f_n)_{n=0}^{\infty}, \geq) \) from \( P(Y, \psi, y_0, S) = (W, S, \geq) \) to \( P(X, \varphi, x_0, R) = (V, E, \geq) \) as follows. Set \( f_0 = 0 \) and \( F_0 = \{0\} \), and suppose that we have chosen \( f_0, f_1, \ldots, f_n-1 \) and \( F_0, F_1, \ldots, F_{n-1} \). To define \( f_n \) and \( F_n \), observe that since \( Q_n \) is a Kakutani–Rokhlin partition for \((Y, \psi, y_0)\), the set \( \alpha^{-1}(Q_n) \) of inverse images of elements of \( Q_n \) is a Kakutani–Rokhlin partition for \((X, \varphi, x_0)\). By Properties (1) and (3) of Definition 3.4, there is an integer \( f_n \) with \( f_n \geq f_{n-1} \) such that \( \alpha^{-1}(Q_n) \leq P_{f_n} \), the top of \( \alpha^{-1}(Q_n) \) contains the top of \( P_{f_n} \), and the sequence \((f_n)_{n=0}^{\infty}\) is cofinal. Set

\[
F_n = \{(n, S, T, k) \mid (\alpha^{-1}(S), T, k) \in E(\alpha^{-1}(Q_n), P_{f_n})\}.
\]

There is a natural one-to-one correspondence between \( F_n \) and \( E(\alpha^{-1}(Q_n), P_{f_n}) \). Define the order on \( F_n \) to be the induced order from \( E(\alpha^{-1}(Q_n), P_{f_n}) \). This makes \( F_n \) an edge set from \( W_n \) to \( V_{f_n} \).

Continuing this procedure, we can obtain a cofinal sequence of integers \((f_n)_{n=0}^{\infty}\) with \( f_0 = 0 \leq f_1 \leq f_2 \leq \cdots \) and an edge set \( F = \bigcup_{n=0}^{\infty} F_n \) such that each \( F_n \) is an edge set from \( W_n \) to \( V_{f_n} \). The source and range maps are defined in the natural way, i.e., \( s(n, S, T, k) = (n, S) \) and \( r(n, S, T, k) = (f_n, T) \). The order \( \leq \) on \( F \) is the union of the orders on \( F_n \). Now set \( f = (F, (f_n)_{n=0}^{\infty}, \geq) \). Applying Lemma 3.6, we see that \( f : (W, S) \to (V, E) \) is an (ordered) premorphism. Set \( \mathcal{P}(\alpha) = [f] \), the equivalence class of \( f \). The following is immediate.

**Proposition 3.7.** The map \( \mathcal{P} : \text{SDS} \to \text{OBD} \) is a contravariant functor.

Next, we show that any premorphism between the Bratteli diagrams of two essentially minimal totally disconnected dynamical systems can be lifted to a homomorphism between them.

**Theorem 3.8.** The functor \( \mathcal{P} : \text{SDS} \to \text{OBD} \) is a full and faithful functor.

**Proof.** First let us show that \( \mathcal{P} \) is full. The idea is to reverse the procedure described above. Let \( X_1 = (X, \varphi, x_0, R) \) and \( X_2 = (Y, \psi, y_0, S) \) be in \( \text{SDS} \) and write \( \mathcal{P}(X_1) = (V, E) \) and \( \mathcal{P}(X_2) = (W, S) \). Let \( f : (W, S) \to (V, E) \) be an (ordered) premorphism. We must show that there is a morphism \( \alpha : X_1 \to X_2 \) with \( \mathcal{P}(\alpha) = [f] \).

Write \( f = (F, (f_n)_{n=0}^{\infty}, \geq), R = (R_n)_{n=0}^{\infty}, \) and \( S = (S_n)_{n=0}^{\infty} \). Let \( F = \bigcup_{n=0}^{\infty} F_n \) denote the decomposition of \( F \) according to Definition 2.5. For each \( n \geq 0 \), \( F_n \) fills the towers of \( P_{f_n} \), with the \( Q_n \)s, specifically, let \( T \) be a tower of \( P_{f_n} \). Let \( e_1, e_2, \ldots, e_k \) denote the edges in \( F_n \) with range \( (f_n, T) \) and \( e_1 < e_2 < \cdots < e_k \). Denote by \( S_n \), the tower of \( Q_n \) such that the source of \( e_i \) is \((n, S_n), 1 \leq i \leq k \). Then the height of \( T \) equals the sum of the heights of \( S_1, S_2, \ldots, S_k \) since \( f \) is a premorphism.

Choose \( x \) in \( X \). For each \( n \geq 0 \) there is an \( A_n \in P_{f_n} \) such that \( x \in A_n \). We have \( A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \). Since \( X \) is Hausdorff and \( \bigcup_{n=0}^{\infty} P_n \) is a basis for \( X \), we have \( \bigcap_{n=0}^{\infty} A_n = \{x\} \). Fix \( n \geq 0 \). For \( T_n \), the tower of \( P_{f_n} \) containing \( A_n \), by the preceding paragraph, there is a unique tower \( S_n \) in \( Q_n \) and a unique element \( B_n \) in \( S_n \) which corresponds to \( A_n \) when \( F_n \) fills \( T_n \) by the towers of \( Q_n \). We may construct \( \alpha \) in such a way that \( \alpha(A_n) \subseteq B_n \).

By Definition 2.5, for each \( n \geq 1 \), we have \( E_n \circ F_n \cong F_{n-1} \circ S_{f_{n-1}, f_n} \). Thus, \( B_n \subseteq B_{n-1} \). The set \( \bigcap_{n=0}^{\infty} B_n \) is a singleton, say with the element \( \alpha(x) \). This gives a map \( \alpha : X \to Y \).

Our construction yields \( \alpha(A_n) \subseteq B_n \), for \( n \geq 0 \). From this, it follows that \( \alpha \) is continuous. Let us show that \( \alpha(x_0) = y_0 \). Since \( y \) is in the top of each \( P_{f_n} \), \( \alpha(x_0) \) is in the top of each \( Q_n \). Now by Property (1) of Definition 3.4 we have \( \alpha(x_0) = y_0 \).

It is not hard to see that \( \alpha \circ \varphi = \psi \circ \alpha \). Let \( x \in X \setminus \{x_0\} \). Hence, \( \alpha : X_1 \to X_2 \) is a morphism in \( \text{SDS} \) and our construction shows that \( \alpha^{-1}(Q_n) \subseteq P_{f_n}, n \geq 0 \). Moreover,
the premorphism associated to \(\alpha\) for the sequence \((f_n)_{n=0}^{\infty}\) is obviously equivalent to \(f\). Hence, \(\mathcal{P}(\alpha) = [f]\). The proof of the faithfulness of \(\mathcal{P}\) is straightforward. \(\Box\)

Let us determine the essential range of the functor \(\mathcal{P} : \text{SDS} \to \text{OBD}\). Recall that the essential range of a functor is the subcategory of those objects in the codomain category which are isomorphic to objects in the range of the functor.

Let us denote by \(\text{OBD}_{po}\) the full subcategory of \(\text{OBD}\) consisting of all properly ordered Bratteli diagrams (see Definition 2.4).

**Lemma 3.9.** The essential range of \(\mathcal{P} : \text{SDS} \to \text{OBD}\) is \(\text{OBD}_{po}\).

**Proof.** Let \(X'\) be in \(\text{SDS}\). Then the ordered Bratteli diagram \(\mathcal{P}(X')\) is properly ordered [20, Section 4]. Now let \(B\) be an ordered Bratteli diagram which is isomorphic in \(\text{OBD}\) to \(\mathcal{P}(X')\), for some \(X' \in \text{SDS}\). By Proposition 2.9, \(B\) is equivalent to \(\mathcal{P}(X')\). By [20, Proposition 2.7], \(B\) is also properly ordered. Hence the essential range of \(\mathcal{P}\) is contained in \(\text{OBD}_{po}\). Now let \(B\) be a properly ordered Bratteli diagram. Denote by \((X, \varphi, x_0)\) the Vershik transformation associated to \(B\) [20, Section 3]. Fix a system of Kakutani–Rokhlin partitions \(\mathcal{R}\) for \((X, \varphi, x_0)\), which exists by [20, Theorem 4.2]. By [20, Theorem 4.6], \(B\) is equivalent to \(\mathcal{P}(X, \varphi, x_0, \mathcal{R})\). \(\Box\)

The following result follows from Theorem 3.8, Lemma 3.9, and [22, Theorem IV.4.1].

**Theorem 3.10.** The functor \(\mathcal{P} : \text{SDS} \to \text{OBD}_{po}\) is an equivalence of categories.

3.2. The functor \(\mathcal{V}\) from \(\text{OBD}_{po}\) to \(\text{SDS}\). In this subsection we shall construct the contravariant functor \(\mathcal{V} : \text{OBD}_{po} \to \text{SDS}\) which is the inverse of the functor \(\mathcal{P} : \text{SDS} \to \text{OBD}_{po}\).

The definition of \(\mathcal{V}\) on objects of \(\text{OBD}_{po}\) coincides with the construction in [20]. Our contribution is finding the way \(\mathcal{V}\) acts on morphisms. In particular, this gives a way to construct factor maps between two Cantor minimal systems by drawing suitable arrows between their ordered Bratteli diagrams.

Let \(B = (V, E, \geq)\) be a properly ordered Bratteli diagram. Denote by \(\mathcal{V}(B)\) the Bratteli–Vershik dynamical system associated to \(B\), as described in [20, Section 3] and [14, Section 3]. Recall that \(\mathcal{V}(B)\) is defined as follows. Let \(X_B\) denote the space of infinite paths, topologized by specifying a basis of open sets, namely the family of cylinder sets \(U(e_1, e_2, \ldots, e_k) = \{f_1, f_2, \ldots \mid f_i = e_i, 1 \leq i \leq k\}\). Denote by \(x_{\text{max}}\) and \(x_{\text{min}}\) the unique elements of \(E_{\text{max}}\) and \(E_{\text{min}}\), respectively. The homeomorphism \(\lambda_B : X_B \to X_B\), called the Vershik transformation, is defined in [20, Section 3]. Then \((X_B, \lambda_B, x_{\text{max}})\) is an essentially minimal totally disconnected dynamical system.

Let us recall from [20] the canonical system of Kakutani–Rokhlin partitions \(\mathcal{R}_B = (P_n)_{n=0}^{\infty}\) for \((X_B, \lambda_B, x_{\text{max}})\) such that \((X_B, \lambda_B, x_{\text{max}}, \mathcal{R}_B)\) is in \(\text{SDS}\). Set \(P_0 = \{X_B\}\). Fix \(n \geq 1\) and define \(P_n\) as follows. For each \(v \in V_n\) we have a tower \(T_v\) in \(P_n\). For each \((e_1, e_2, \ldots, e_n)\) in \(E_1 \circ E_2 \circ \cdots \circ E_n\) with \(r(e_n) = v\) we have an element \(U(e_1, e_2, \ldots, e_n)\), as defined above, in \(T_v\). Hence,

\[
P_n = \{U(e_1, e_2, \ldots, e_n) \mid (e_1, e_2, \ldots, e_n) \in E_1 \circ E_2 \circ \cdots \circ E_n\}.
\]

Note that each \(P_n\) is a Kakutani–Rokhlin partition and that \(\mathcal{R}_B = (P_n)_{n=0}^{\infty}\) satisfies the conditions of Definition 3.4, and hence is a system of Kakutani–Rokhlin partitions for \((X_B, \lambda_B, x_{\text{max}})\). Finally, set \(\mathcal{V}(B) = (X_B, \lambda_B, x_{\text{max}}, \mathcal{R}_B)\). To summarize:

**Proposition 3.11.** For each ordered Bratteli diagram \(B = (V, E, \geq)\), the system \(\mathcal{V}(B) = (X_B, \lambda_B, x_{\text{max}}, \mathcal{R}_B)\) is in \(\text{SDS}\).
Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. Define an ordered premorphism $f_B : B \to \mathcal{P}(V(B))$ as follows: $f_B = (f_B(n))_{n=0}^{\infty}$, where $F_B = \{(v, T_v) \mid v \in V\}$. The decomposition of $F_B$ is obtained by setting $F_{B,n} = \{(v, T_v) \mid v \in V_n\}$, $n \geq 0$. The source and range maps of $F_B$ are defined by $s(v, T_v) = v$ and $r(v, T_v) = T_v$. There is a unique way to define an order on $F_B$ as above, since $r^{-1}\{T_v\}$ is a singleton. It is not hard to see that $f_B : B \to \mathcal{P}(V(B))$ is an ordered premorphism, which turns to be an isomorphism in the category of ordered Bratteli diagrams with ordered premorphisms (see Definition 2.7). Denote by $\tau_B : B \to \mathcal{P}(V(B))$ the associated ordered morphism, i.e., $\tau_B = [f_B]$, which is an isomorphism in $\text{OBD}$.

Once one fixes the isomorphism $\tau_B = [f_B]$, for each $B$, there is a unique way to define $V : \text{OBD}_po \to \text{SDS}$ on morphisms to obtain the natural inverse of $\mathcal{P} : \text{SDS} \to \text{OBD}_po$ (see the proof of [22, Theorem IV.4.1] for details). In fact, let $h : B \to C$ be a morphism in $\text{OBD}_po$. Then $\tau_C h \tau_B^{-1} : \mathcal{P}(V(B)) \to \mathcal{P}(V(C))$ is a morphism in $\text{OBD}_po$. By Theorem 3.8, there is a unique morphism $\alpha : V(C) \to V(B)$ such that $\mathcal{P}(\alpha) = h$. Set $V(h) = \alpha$. We have almost finished showing the next result. All the required properties of the map $V$ follow from the equality $\mathcal{P}(V(f)) \tau_B = \tau_A h$ (cf. the proof of [22, Theorem IV.4.1]), which, in particular, gives $\tau : 1_{\text{OBD}_po} \cong \mathcal{P}$.

**Theorem 3.12.** The map $V : \text{OBD}_po \to \text{SDS}$ as defined above, is a contravariant functor which is an equivalence of categories and the unique (up to natural isomorphism) inverse of the functor $\mathcal{P} : \text{SDS} \to \text{OBD}_po$.

Let us examine how the functor $V$ acts on morphisms. Let $f : B \to C$ be an ordered premorphism as in Definition 2.5. Define a map $\alpha : X_C \to X_B$ as follows. Let $x = (s_1, s_2, \ldots)$ be in $X_C$, i.e., an infinite path in $S$. Define the path $\alpha(x) = (e_1, e_2, \ldots)$ in $X_B$ as follows. Fix $n \geq 1$. By Definition 2.5, the diagram

\[
\begin{array}{ccc}
V_0 & \xrightarrow{E_{0,n}} & V_n \\
V_0 & \downarrow{f_0} & \downarrow{f_n} \\
W_0 & \xrightarrow{S_{0,n}} & W_{f_n}
\end{array}
\]

commutes, that is, $F_0 \circ S_{0,f_n} \cong E_{0,n} \circ F_n$. Thus, there is a unique path $(e_1, e_2, \ldots, e_n, d_n)$ in $E_{0,n} \circ F_n$, corresponding to the path $(s_0, s_1, \ldots, s_{f_n})$ in $F_0 \circ S_{0,f_n}$, where $s_0$ is the unique element of $F_0$. We need to check that the first $n$ edges of the path associated to each $m > n$ coincide with the edges for $n$.

**Lemma 3.13.** With the above notation, let $m > n$ and consider the path $(e'_1, e'_2, \ldots, e'_n, d'_m)$ in $E_{0,m} \circ F_m$ associated to $m$ in the above construction. Then $e'_i = e_i$ for each $1 \leq i \leq n$.

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
V_0 & \xrightarrow{E_{0,n}} & V_n \\
V_0 & \downarrow{f_0} & \downarrow{f_n} \\
W_0 & \xrightarrow{S_{0,n}} & W_{f_n}
\end{array}
\]

Since $(e_1, e_2, \ldots, e_n, d_n)$ in $E_{0,n} \circ F_n$ is the unique path corresponding to $(s_0, s_1, \ldots, s_{f_n})$ in $F_0 \circ S_{0,f_n}$, we get $r(d_n) = r(s_{f_n})$. Thus the path

\[(3.1) \quad (e_1, e_2, \ldots, e_n, s_{f_n+1}, \ldots, s_{f_m})\]
in $E_{0,n} \circ F_n \circ S_{f_m,f_n}$ is the unique path corresponding to $(s_0, s_1, \ldots, s_{f_m})$ in $F_0 \circ S_{0,f_m}$ by the isomorphism $E_{0,n} \circ F_n \circ S_{f_m,f_n} \cong F_0 \circ S_{0,f_m}$.

Moreover, the path $(e'_1, e'_2, \ldots, e'_{m}, d'_{m})$ in $E_{0,m} \circ F_m$ is the unique path corresponding to the path $(s_0, s_1, \ldots, s_{f_m})$ by the isomorphism $E_{0,m} \circ F_m \cong F_0 \circ S_{0,f_m}$. Since the isomorphisms involved are unique (because of the order), by the isomorphism $E_{0,m} \circ F_m \cong E_{0,n} \circ F_n \circ S_{f_m,f_n}$, the path $(e'_1, e'_2, \ldots, e'_{m}, d'_{m})$ corresponds to the path in (3.1).

Let $(e''_{n+1}, e''_{n+2}, \ldots, e''_{m}, d''_{m})$ denote the unique path in $E_{n,m} \circ F_m$ corresponding to the path $(d_{n}, s_{f_n+1}, \ldots, s_{f_m})$ in $F_n \circ S_{f_n,f_m}$ by the isomorphism $E_{n,m} \circ F_m \cong F_n \circ S_{f_n,f_m}$. Then $r(e_n) = s(d_n) = s(e''_{n+1})$. Thus, the path

$$ (e_1, e_2, \ldots, e_n, e''_{n+1}, e''_{n+2}, \ldots, e''_{m}, d''_{m}) $$

corresponds to the path in (3.1) by the isomorphism $E_{0,m} \circ F_m \cong E_{0,n} \circ F_n \circ S_{f_m,f_n}$. Since the path $(e'_1, e'_2, \ldots, e'_{m}, d'_{m})$ also corresponds to the path (3.1) by the same isomorphism, we conclude that $(e'_1, e'_2, \ldots, e'_{m}, d'_{m}) = (e''_1, e''_2, \ldots, e''_m)$ is equal to the path in (3.2). Therefore, $e'_i = e''_i$ for $1 \leq i \leq n$, $e'_i = e''_i$ for $n + 1 \leq i \leq m$, and $d'_m = d''_m$. □

By the preceding lemma, one can define, without ambiguity, the path $\alpha(x) = (e_1, e_2, \ldots)$ in $X_B$ associated to the path $x = (s_1, s_2, \ldots)$ in $X_C$. (We will describe a second way to compute $\alpha(x)$ after Proposition 3.14.) We thus obtain a map $\alpha : X_C \to X_B$. It is not hard to see that if we replace $f$ with another representative of the class $[f]$, then we get the same $\alpha$.

**Proposition 3.14.** Let $f : B \to C$ be a premorphism in $\text{OBD}_{po}$ and $\alpha : X_C \to X_B$ be its associated map as defined above. Then $\mathcal{V}([f]) = \alpha$.

**Proof.** With the above notation, $\alpha(U(s_1, s_2, \ldots, s_{f_n})) \subseteq U(e_1, e_2, \ldots, e_n)$. This shows that $\alpha$ is continuous. Moreover, $\alpha \circ \lambda_C = \lambda_B \circ \alpha$ and $\alpha$ maps the unique path in $S_{\text{max}}$ to the unique path in $E_{\text{max}}$. Thus, $\alpha : \mathcal{V}(C) \to \mathcal{V}(B)$ is a morphism in $\text{SDS}$. Also, $\mathcal{P}(\alpha) \tau_B = \tau_A([f])$. Hence, $\mathcal{P}(\alpha) = \mathcal{P}(\mathcal{V}([f]))$, and so by Theorem 3.8, $\mathcal{V}([f]) = \alpha$. □

We remark that the proof of Lemma 3.13 gives a second method for computing $\mathcal{V}([f])$ above. This turns out to be easier to follow—at least in some cases—as it requires less computation. In fact, let $x = (s_1, s_2, \ldots)$ be in $X_C$. Define the path $\alpha(x) = (e_1, e_2, \ldots)$ in $X_B$ as follows. First consider the following commutative diagram:

$$
\begin{array}{ccc}
V_0 & \xrightarrow{E_1} & V_1 \\
W_0 & \xrightarrow{F_0} & W_1 \\
\downarrow S_{0,f_1} & & \downarrow S_{0,f_1} \\
V_1 & \xrightarrow{E_1} & V_2 \\
W_1 & \xrightarrow{F_1} & W_2
\end{array}
$$

Then there is a unique path $(e_1, d_1)$ in $E_1 \circ F_1$, corresponding to the path $(s_0, s_1, \ldots, s_{f_1})$ in $F_0 \circ S_{0,f_1}$, where $s_0$ is the unique element of $F_0$. Now consider the following commutative diagram:

$$
\begin{array}{ccc}
V_1 & \xrightarrow{E_2} & V_2 \\
W_1 & \xrightarrow{F_2} & W_2 \\
\downarrow S_{f_1,f_2} & & \downarrow S_{f_1,f_2} \\
V_2 & \xrightarrow{E_2} & V_3 \\
W_2 & \xrightarrow{F_2} & W_3
\end{array}
$$

Then there is a unique path $(e_2, d_2)$ in $E_2 \circ F_2$, corresponding to the path $(d_1, s_{f_1+1}, \ldots, s_{f_2})$ in $F_1 \circ S_{f_1,f_2}$. Continuing this procedure, we obtain a path $(e_1, e_2, \ldots)$ in $X_B$ which is the same as $\alpha(x)$ obtained by the previous method (by the proof of Lemma 3.13).
As stated in the proof of Theorem 3.12, the correspondence (natural transformation) \( \tau \), defined above, gives \( 1_{\text{OBD}} \cong P V \). Using this, a standard categorical procedure gives a correspondence \( \sigma \) which implements \( 1_{\text{SDS}} \cong V P \). In fact, let \( \mathcal{X} \) be in \( \text{SDS} \). Then \( \tau P(\mathcal{X}) : P(\mathcal{X}) \to P(V(P(\mathcal{X}))) \) is an isomorphism in \( \text{OBD} \). By Theorem 3.8, there is a unique isomorphism \( \sigma_{\mathcal{X}} : \mathcal{X} \to V(P(\mathcal{X})) \) such that \( P(\sigma_{\mathcal{X}}) = \tau P^{-1}(\mathcal{X}) \). Moreover, \( \sigma : 1_{\text{SDS}} \cong VP \). Let us summarize the results of this section as follows.

**Theorem 3.15.** The contravariant functors \( P : \text{SDS} \to \text{OBD} \) and \( V : \text{OBD} \to \text{SDS} \) are equivalences of categories which are inverse to each other, with respect to the natural isomorphisms \( \tau : PV \cong 1_{\text{OBD}} \) and \( \sigma : VP \cong 1_{\text{SDS}} \).

4. Cantor Minimal Systems

In this section we shall apply the results of the previous section to Cantor minimal dynamical systems and their factor maps (thus obtaining new results as an application of the categorical methods).

4.1. Cantor Systems. Recall that a dynamical system \((X, \varphi)\) is called a Cantor minimal system if \(X\) is homeomorphic to the Cantor set and \(\varphi\) is a minimal homeomorphism of \(X\). These systems are of great importance in symbolic dynamics. Every Cantor minimal system has a Bratteli–Vershik model (see, e.g., the definition of the functor \(P\) on objects in Section 3).

Recall the definition of a simple Bratteli diagram from Definition 2.2. The following well-known fact follows from the results of [20].

**Proposition 4.1.** Let \(B = (V, E, \geq)\) be a properly ordered Bratteli diagram. Then the following statements are equivalent:

1. The system \((X_B, \lambda_B)\) is minimal;
2. \((V, E)\) is a simple Bratteli diagram.

The preceding proposition and Theorem 3.15 imply that the full subcategory of \(\text{OBD}\) consisting of simple properly ordered Bratteli diagrams is equivalent to the full subcategory of \(\text{SDS}\) (Definition 3.5) consisting of scaled minimal dynamical systems on metrizable, compact, totally disconnected spaces.

Recall that for a Bratteli diagram \(B = (V, E)\), the Bratteli compactum \(X_B\) is a metrizable, compact, totally disconnected space. Thus, to obtain a Cantor set, we need to translate the property of having no isolated point into the language of diagrams. This is not hard and is done in the next lemma. By the definition of the topology on \(X_B\) (Subsection 3.2), (2) is equivalent to having no isolated points. Thus, (1) is equivalent to (2). Also, (3) is just a reformulation of (2).

**Lemma 4.2.** Let \(B = (V, E)\) be a Bratteli diagram. The following statements are equivalent:

1. \(X_B\) is homeomorphic to the Cantor set;
2. For each infinite path \(x = (e_1, e_2, \ldots)\) in \(X_B\) and each \(n \geq 1\) there is an infinite path \(y = (f_1, f_2, \ldots)\) with \(x \neq y\) and \(e_k = f_k, 1 \leq k \leq n\);
3. For each \(n \geq 0\) and each \(v \in V_n\) there is \(m \geq n\) and \(w \in V_m\) such that there is a path from \(v\) to \(w\) and \(|s^{-1}(\{w\})| \geq 2\).

The next result follows immediately from the lemma above.

**Proposition 4.3.** Let \(B = (V, E)\) be a simple Bratteli diagram. Then the following statements are equivalent:
(1) $X_B$ is homeomorphic to the Cantor set;
(2) $X_B$ is infinite;
(3) the set \( \{ n \in \mathbb{N} \mid |E_n| \geq 2 \} \) is infinite.

In this context, Theorem 3.15 restricts as follows.

**Corollary 4.4.** The full subcategory of $\text{OBD}_{po}$ consisting of simple properly ordered Bratteli diagrams $B$ with infinite $X_B$, is equivalent to the full subcategory of $\text{SDS}$ consisting of scaled minimal dynamical systems on Cantor sets.

### 4.2. Factor Maps

In this subsection we use the idea of ordered premorphism to construct factor maps between Cantor minimal systems. For simplicity, we only consider factor maps on odometers. However, an example of an extension of the Chacon system is considered briefly at the end of this subsection (Example 4.15). An objective of this subsection is to illustrate some of our ideas concerning diagrams. In particular, we reprove some facts on extensions of odometers by using premorphisms (though there are also some new results—such as Proposition 4.14). More examples and results in this direction can be found in [19].

Consider Example 2.6. Applying the functor $\mathcal{V}$ to the class of the ordered premorphism $f$ in that example, we get a factor map $\mathcal{V}(f) : \mathcal{V}(C) \to \mathcal{V}(B)$ as defined before Lemma 3.13. In fact, $\mathcal{V}(B)$ is the maximal rational equicontinuous factor of $\mathcal{V}(C)$ (see Theorem 4.12 and [15]). The idea of the this example can be used to reprove the fact that every Cantor minimal system has a maximal rational equicontinuous factor (possibly trivial). Before showing this, we prove that factor maps are in one-to-one correspondence to ordered morphisms. First, we recall the following notion.

**Definition 4.5.** Let $(X, \varphi)$ be a Cantor minimal system. By a Bratteli–Vershik model for $(X, \varphi)$ we mean a properly ordered Bratteli diagram $B$ such that the associated system $(X_B, \lambda_B)$ is conjugate to $(X, \varphi)$. Let $x_0 \in X$. By a Bratteli–Vershik model for $(X, \varphi, x_0)$ we mean a properly ordered Bratteli diagram $B$ such that $(X, \varphi, x_0)$ is pointed topological conjugate to $(X_B, \lambda_B, x_{\max})$, i.e., there is a homeomorphism \( \alpha : X \to X_B \) such that $\alpha \circ x = \lambda_B \circ x$ and $\alpha(x_0) = x_{\max}$ (an isomorphism in $\text{DS}$). Note that in the latter case, $B$ is unique (up to equivalence).

**Proposition 4.6.** Let $(X, \varphi)$ and $(Y, \psi)$ be Cantor minimal systems, and let $x \in X$ and $y \in Y$. Let $C$ and $B$ be Bratteli–Vershik models for $(X, \varphi, x)$ and $(Y, \psi, y)$. The following statements are equivalent:

1. there is a factor map $\alpha : (X, \varphi) \to (Y, \psi)$ with $\alpha(x) = y$;
2. there is an (ordered) premorphism $f$ from $B$ to $C$ (see Definition 2.5).

More precisely, there is a natural one-to-one correspondence between the set of factor maps $\alpha$ as in (1) and the set of equivalence classes of ordered premorphisms $f$ from $B$ to $C$ given by $\alpha = \mathcal{V}(f)$.

**Proof.** This follows from the fact that the functor $\mathcal{V}$ is full and faithful (by Theorem 3.12). More precisely, consider the mapping $[f] \mapsto \mathcal{V}(f)$, from the set of ordered morphism from $B$ to $C$, into the set of factor maps $\alpha : (X, \varphi) \to (Y, \psi)$ with $\alpha(x) = y$. The fullness and faithfulness of $\mathcal{V}$ imply respectively that this mapping is surjective and injective. This completes the proof. □

**Theorem 4.7.** Let $(X, \varphi)$ and $(Y, \psi)$ be Cantor minimal systems. The following statements are equivalent:

1. there is a factor map from $(X, \varphi)$ to $(Y, \psi)$;
(2) there are Bratteli–Vershik models \( C \) and \( B \) for \( (X, \varphi) \) and \( (Y, \psi) \), respectively, such that there is an ordered premorphism from \( B \) to \( C \).

**Proof.** (1)⇒(2): Suppose that \( \alpha : (X, \varphi) \rightarrow (Y, \psi) \) is a factor map. Choose \( x \in X \) and set \( y = \alpha(x) \). Let \( C \) and \( B \) be Bratteli–Vershik models for \( (X, \varphi, x) \) and \( (Y, \psi, y) \), respectively (see Definition 4.5). Applying Proposition 4.6, we get an ordered premorphism from \( B \) to \( C \).

(2)⇒(1): Let \( C \) and \( B \) be Bratteli–Vershik models for \( (X, \varphi) \) and \( (Y, \psi) \), respectively, and let \( f : B \rightarrow C \) be an ordered premorphism. Applying the functor \( V \), we get a factor map \( V(f) : V(C) \rightarrow V(B) \) (see Subsection 3.2). Since \( (X, \varphi) \) and \( (Y, \psi) \) are respectively conjugate to \( V(C) \) and \( V(B) \), we obtain a factor map from \( (X, \varphi) \) to \( (Y, \psi) \).

Let us recall the definition of an odometer, including the trivial odometers (cf. [7]).

**Definition 4.8.** Let \( (k_n)_{n=1}^{\infty} \) be a sequence in \( \mathbb{N} \). By an *odometer of type* \((k_n)_{n=1}^{\infty}\) we mean a minimal system \( (X, \varphi) \) where

\[
X = \prod_{n=1}^{\infty} \{0, 1, \ldots, k_n - 1\}
\]

and the homeomorphism \( \varphi : X \rightarrow X \) is addition of \((1, 0, 0, \ldots)\), with carrying.

It is known that if \( X \) is infinite (i.e., \( k_n \geq 2 \) for infinitely many \( n \)), then \( (X, \varphi) \) is a Cantor minimal system. When \( X \) is finite, \( (X, \varphi) \) is minimal.

The following well-known result follows from Proposition 4.6.

**Lemma 4.9** ([19]). Let \( (X, \varphi) \) and \( (Y, \psi) \) be odometers of types \((k_n)_{n=1}^{\infty}\) and \((l_n)_{n=1}^{\infty}\), respectively. The following statements are equivalent:

1. \( (X, \varphi) \) is a factor of \( (Y, \psi) \);
2. for each \( n \geq 1 \) there is an \( m \geq 1 \) such that \( k_1 \cdots k_n | l_1 \cdots l_m \).

The following proposition is part of the literature.

**Proposition 4.10.** Let \( (X, \varphi) \) and \( (Y, \psi) \) be odometers of types \((k_n)_{n=1}^{\infty}\) and \((l_n)_{n=1}^{\infty}\), respectively. The following statements are equivalent:

1. \( (X, \varphi) \) and \( (Y, \psi) \) are conjugate;
2. \( (X, \varphi) \) and \( (Y, \psi) \) are orbit equivalent;
3. \( (X, \varphi) \) is a factor of \( (Y, \psi) \), and also \( (Y, \psi) \) is a factor of \( (X, \varphi) \).

**Proof.** For the equivalence of (1) and (2) see, e.g., [26]. The equivalence of (1) and (3) follows from Lemma 4.9. \( \square \)

In the following definition we associate an odometer \( O(B) \) to an ordered Bratteli diagram \( B \) and construct an (ordered) premorphism \( f_B : O(B) \rightarrow B \).

**Definition 4.11.** Let \( B = (V, E, \geq) \) be an ordered Bratteli diagram. We associate to \( B \) an odometer \( O(B) = (W, R, \geq) \) of type \((r_n)_{n=1}^{\infty}\) and an ordered premorphism \( f_B : O(B) \rightarrow B \) as follows. Let \( h_n \) be the greatest common divisor of the heights of the towers at level \( n, n \geq 0 \), and set \( r_n = h_n / h_{n-1}, n \geq 1 \). More precisely, write \( V = \bigcup_{n=0}^{\infty} V_n \) and \( E = \bigcup_{n=1}^{\infty} E_n \) as in Definition 2.1. Let \( M(E_n) \) denote the multiplicity matrix of \( E_n \). Then \( E_{0,n} = E_1 \circ E_2 \circ \cdots \circ E_n \) (the edge set from \( V_0 \) to \( V_n \)) is the set of towers at level \( n \), and the column matrix

\[
M(E_{0,n}) = M(E_n) \cdots M(E_{n-1}) M(E_1) = \begin{pmatrix}
h_{n,1} & h_{n,2} \\
& \vdots \\
& h_{n,k_n}
\end{pmatrix},
\]
where the $h_{n,i}$ are non-zero positive integers and $k_n = |V_n|$, consists of the heights of these towers. Thus, $h_n = \gcd(h_{n,1}, h_{n,2}, \ldots, h_{n,k_n})$. Note that $1 = h_0 \mid h_1 \mid h_2 \cdots$ and so the definition of $r_n = h_n/h_{n-1}$ makes sense. Let $O(B) = (W, R, \geq)$ be the odometer of type $(r_n)_{n=1}^\infty$. Thus $R = \bigcup_{n=1}^\infty R_n$ and $|R_n| = r_n$, $n \geq 1$. Now define $f_B : O(B) \to B$ as follows. Set $f_B = (F, (n)_{n=0}^\infty, \geq)$ (see Definition 2.5), where $F = \bigcup_{n=0}^\infty F_n$ is defined as follows. Let $W = \bigcup_{n=0}^\infty W_n$ denote the set of vertices of $O(B)$ and write $W_n = \{w_n\}$, $n \geq 0$. Also, write $V_n = \{v^n_1, v^n_2, \ldots, v^n_{k_n}\}$. Set $F_0 = \{(w_0, v^n_0)\}$. Thus, $F_0$ has only one edge going from $w_0$ to $v_0^n$. For $n \geq 1$ set

$$F_n = \{(w_n, v^n_i, j) \mid 1 \leq i \leq k_n, 1 \leq j \leq \frac{h_{n-1}}{h_n}\}.$$ 

Thus, $F_n$ has $\frac{h_{n-1}}{h_n}$ edges from $w_n$ to $v^n_i$. Put an arbitrary linear order $\geq$ on these edges. Note that the order on $F$ is not important here as any two orders on $F$ give equivalent (ordered) premorphisms, since $O(B)$ has only one vertex at each level. Put $f_B = (F, (n)_{n=0}^\infty, \geq)$.

Observe that $f_B = (F, (n)_{n=0}^\infty, \geq)$ is an ordered premorphism. In fact, we have $M(F_n) = \frac{1}{h_n}M(E_{0,n})$ and $\gcd(M(F_n)) = 1$, $n \geq 0$. The commutativity condition in Definition 2.5 amounts to commutativity of the following diagram, $n = 1, 2, \ldots$:

$$\begin{array}{ccc}
W_{n-1} & \xrightarrow{F_{n-1}} & V_{n-1} \\
R_n \downarrow & & \downarrow E_n \\
W_n & \xrightarrow{F_n} & V_n.
\end{array}$$

To see this, first note that ordered commutativity and unordered commutativity of this diagram coincide as $W_{n-1}$ has only one vertex. We have

$$M(E_n)M(F_{n-1}) = \frac{1}{h_{n-1}}M(E_{n-1})M(E_{0,n-1}) = \frac{1}{h_{n-1}}M(E_{0,n})$$

$$= \frac{h_n}{h_{n-1}}M(F_n) = r_nM(F_n)$$

$$= M(F_n)M(R_n).$$

Now, we give an alternative proof for the existence and uniqueness of the maximal (rational) equicontinuous factor of a Cantor minimal system (cf. [2, Chapter 9]).

**Theorem 4.12.** For any Cantor minimal system $(X, \varphi)$ there is a unique (up to conjugacy) odometer $(Y, \psi)$ with the following properties:

1. $(Y, \psi)$ is a factor of $(X, \varphi)$;
2. every odometer which is a factor of $(X, \varphi)$ is also a factor of $(Y, \psi)$.

Moreover, there is a factor map $\alpha : (X, \varphi) \to (Y, \psi)$ such that, if $\beta : (X, \varphi) \to (Z, \eta)$ is a factor map onto an odometer $(Z, \eta)$, then there is a (necessarily unique) factor map $\gamma : (Y, \psi) \to (Z, \eta)$ such that $\beta = \gamma \circ \alpha$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
(X, \varphi) & \xrightarrow{\alpha} & (Y, \psi) \\
\beta \downarrow & & \downarrow \gamma \\
& (Z, \eta) & \end{array}$$

\]
Proof. First note that uniqueness of \((Y, \psi)\) follows from Proposition 4.10 and (2). Let \(O(B) = (W, R, \geq)\) be the ordered Bratteli diagram of the odometer of type \((r_n)_{n=1}^{\infty}\) associated to \(B\) as in Definition 4.11. Put \((Y, \psi) = \mathcal{V}(O(B))\). Also, consider the ordered premorphism \(f_B : O(B) \to B\) as in Definition 4.11 and set \(\alpha = \mathcal{V}(f_B)\). Thus, \(\alpha : (X, \varphi) \to (Y, \psi)\) is a factor map.

Suppose that \(\beta : (X, \varphi) \to (Z, \eta)\) is a factor map onto an odometer \((Z, \eta)\) of type \((s_n)_{n=1}^{\infty}\). We may assume that \((Z, \eta) = \mathcal{V}(C)\) for some properly ordered Bratteli diagram \(C = (U, S, \geq)\), where \(U = \bigcup_{n=0}^{\infty} U_n, S = \bigcup_{n=1}^{\infty} S_n, |U_n| = 1\) for all \(n \geq 0\), and \(|S_n| = s_n\) for all \(n \geq 1\). Since \((Z, \eta)\) is dynamically homogeneous (i.e., for any \(z_1, z_2 \in Z\) there is a conjugacy from \((Z, \eta)\) to itself mapping \(z_1\) to \(z_2\); see [19]), there is a conjugacy \(\delta : (Z, \eta) \to (Z, \eta)\) such that \(\delta(\beta(x_{\text{min}})) = z_{\text{min}}\) where \(x_{\text{min}} \in X\) and \(z_{\text{min}} \in Z\) are the unique minimal paths. Consider the map \(\tilde{\beta} = \delta \circ \beta : (X, \varphi) \to (Z, \eta)\), a factor map with \(\tilde{\beta}(x_{\text{min}}) = z_{\text{min}}\). Since the contravariant functor \(\mathcal{V}\) is full (by Theorem 3.12—see also Proposition 4.6), there is a considered premorphism \(g : C \to B\) such that \(\mathcal{V}(g) = \tilde{\beta}\).

Let us construct an ordered premorphism \(h : C \to O(B)\) such that \(f_B h \sim g\). Define \(h = (H, (m_n)_{n=0}^{\infty}, \geq)\) where \(H = \bigcup_{n=0}^{\infty} H_n\) is follows. Note that, since \(H_n\) needs to be an edge set from \(U_n\) to \(W_{m_n}\) and both these sets have only one vertex, we need only to determine \(t_n := |H_n|\), and the order on \(H_n\) is not important. Put \(t_0 = 0\). Fix \(n \geq 1\). Since \(G_0 \circ E_{0, m_n} \cong S_1 \circ \cdots \circ S_n \circ G_n\), we get \(M(E_{0, m_n}) = M(G_n)M(S_1 \circ \cdots \circ S_n) = s_1 \cdots s_n M(G_n)\). Taking the gcd of both sides we get \(h_{m_n} = s_1 \cdots s_n \gcd(M(G_n))\), where \(h_{m_n}\) is as in Definition 4.11. (Note that \(M(G_n)\) is a column matrix.) Put \(t_n = h_{m_n}/s_1 \cdots s_n\). Observe that \(h = (H, (m_{n})_{n=0}^{\infty}, \geq)\) thus defined is an ordered premorphism, i.e., the following diagram of \(h\) commutes:

\[
\begin{array}{ccccccccc}
U_0 & \to & U_1 & \to & U_2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
W_0 & \to & W_1 & \to & W_2 & \to & \cdots \\
\end{array}
\]

In fact, for any \(n \geq 1\) we have

\[
M(R_{m_{n-1}, m_n})M(H_{n-1}) = \frac{h_{m_n}}{h_{m_{n-1}}} \cdot t_{n-1} = \frac{h_{m_n}}{s_1 \cdots s_n} \cdot s_n = M(H_n)M(S_n).
\]

Now the premorphisms \(f_B h = ((H_n \circ F_{m_n})_{n=0}^{\infty}, (m_n)_{n=0}^{\infty}, \geq)\) and \(g = (G, (m_n)_{n=0}^{\infty}, \geq)\) are isomorphic. In fact, for any \(n \geq 1\) we have

\[
M(H_n \circ F_{m_n}) = M(F_{m_n})M(H_n) = \frac{1}{h_{m_n}} \cdot h_{m_n} = 1 = M(E_{0, m_n}) = M(G_n).
\]

Thus \(f_B h \cong g\) and so \(f_B h \sim g\) (see Definition 2.7). Therefore, \(\mathcal{V}(f_B h) = \mathcal{V}(g)\) and so \(\mathcal{V}(h) \circ \alpha = \beta = \delta \circ \beta\). Put \(\gamma = \delta^{-1} \circ \mathcal{V}(h)\). Then \(\gamma : (Y, \psi) \to (Z, \eta)\) is a factor map with \(\beta = \gamma \circ \alpha\). \(\square\)
The following (new) result gives a criterion for the existence of a factor map from a Cantor minimal system to an odometer.

**Corollary 4.13.** Let \((X, \varphi)\) be a Cantor minimal system and let \((Y, \psi)\) be the unique odometer of type \((r_n)_{n=1}^{\infty}\) associated with \((X, \varphi)\) as in Theorem 4.12. For an odometer \((Z, \eta)\) of type \((s_n)_{n=1}^{\infty}\), the following statements are equivalent:

1. \((Z, \eta)\) is a factor of \((X, \varphi)\);
2. \((Z, \eta)\) is a factor of \((Y, \psi)\);
3. for each \(n \geq 1\) there is an \(m \geq 1\) such that \(s_1 \cdots s_n \mid r_1 \cdots r_m\).

**Proof.** Note that the sequence \((r_n)_{n=1}^{\infty}\) is defined as in Definition 4.11 in which \(B\) is a Bratteli–Vershik model (unique up to conjugacy) for \((X, \varphi)\). The equivalence of (1) and (2) follows from Theorem 4.12. Also, the equivalence of (2) and (3) follows from Lemma 4.9.

Finally, we obtain the following new uniqueness result.

**Proposition 4.14.** Let \((X, \varphi)\) be a Cantor minimal system and let \((Z, \eta)\) be an odometer. Then there is at most one factor map (up to conjugacy) from \((X, \varphi)\) to \((Z, \eta)\); that is, if \(\beta_1, \beta_2 : (X, \varphi) \to (Z, \eta)\) are factor maps then there is a conjugacy \(\gamma : (Z, \eta) \to (Z, \eta)\) such that \(\beta_1 = \gamma \circ \beta_2\).

**Proof.** Let \(\beta_1, \beta_2 : (X, \varphi) \to (Z, \eta)\) be two factor maps. Let \((Y, \psi)\) be the unique odometer associated with \((X, \varphi)\) as in Theorem 4.12, and let \(\alpha : (X, \varphi) \to (Y, \psi)\) be as in that theorem. By Theorem 4.12, there are factor maps \(\gamma_1, \gamma_2 : (Y, \psi) \to (Z, \eta)\) such that \(\beta_i = \gamma_i \circ \alpha\) for \(i = 1, 2\). If we show that there is a conjugacy \(\gamma : (Z, \eta) \to (Z, \eta)\) such that \(\gamma_1 = \gamma \circ \gamma_2\), then it will follow that \(\beta_1 = \gamma \circ \beta_2\).

Let \((Y, \psi)\) and \((Z, \eta)\) be of type \((r_n)_{n=1}^{\infty}\) and \((s_n)_{n=1}^{\infty}\), respectively. We may assume that \((Y, \psi) = \mathcal{V}(D)\) and \((Z, \eta) = \mathcal{V}(C)\) for some properly ordered Bratteli diagrams \(D = (W, R, \geq)\) and \(C = (U, S, \geq)\), where \(W = \bigcup_{n=0}^{\infty} W_n\), \(U = \bigcup_{n=0}^{\infty} U_n\), \(R = \bigcup_{n=1}^{\infty} R_n\), \(S = \bigcup_{n=1}^{\infty} S_n\), \(|W_n| = |U_n| = 1\) for all \(n \geq 0\), and \(|R_n| = r_n\) and \(|S_n| = s_n\) for all \(n \geq 1\). Since \((Z, \eta)\) is dynamically homogeneous (see the proof of Theorem 4.12), there are conjugacies \(\delta_1, \delta_2 : (Z, \eta) \to (Z, \eta)\) such that \(\delta_i(\gamma_i(y_{\min})) = y_{\min}\) for \(i = 1, 2\), where \(y_{\min} \in Y\) and \(y_{\min} \in Z\) are the unique minimal paths. Since the contravariant functor \(\mathcal{V}\) is full (by Theorem 3.12), there are ordered premorphisms \(f, g : C \to D\) such that \(\mathcal{V}(f) = \delta_1 \circ \gamma_1\) and \(\mathcal{V}(g) = \delta_2 \circ \gamma_2\). We claim that \(f\) is equivalent to \(g\).

Write \(f = (\bigcup_{n=0}^{\infty} F_n, (k_n)_{n=0}^{\infty}\geq)\) and \(g = (\bigcup_{n=0}^{\infty} G_n, (m_n)_{n=0}^{\infty}\geq)\). Fix \(n \geq 1\). By symmetry, we may assume that \(m_n \geq k_n\). Consider the following (a priori non-commutative) diagram:

\[
\begin{array}{c}
U_0 \xrightarrow{S_{0,n}} U_n \\
F_0=G_0 \xrightarrow{F_n} G_n \\
W_0 \xrightarrow{R_{0,k_n}} W_{k_n} \xrightarrow{R_{k_n,m_n}} W_{m_n}.
\end{array}
\]

Let us show that the triangle in this diagram commutes. (The square clearly commutes.) Since \(f\) is a premorphism, we have \(M(F_n)M(S_{0,n}) = M(R_{0,k_n})\). Thus, \(M(F_n) = r_1 \cdots r_{k_n} / s_1 \cdots s_n\) (as a \(1 \times 1\) matrix). Similarly, since \(g\) is a premorphism, we get \(M(G_n) = r_1 \cdots r_{m_n} / s_1 \cdots s_n\). Hence,

\[
M(R_{k_n,m_n})M(F_n) = \frac{r_1 \cdots r_{m_n}}{r_1 \cdots r_{k_n}}, \quad M(F_n) = \frac{r_1 \cdots r_{m_n}}{s_1 \cdots s_n} = M(G_n).
\]
Using Definition 2.10, \( f \) is equivalent to \( g \). Hence, \( \delta_1 \circ \gamma_1 = \mathcal{V}(f) = \mathcal{V}(g) = \delta_2 \circ \gamma_2 \). Put \( \gamma = \delta_1^{-1} \circ \delta_2 \) which is a conjugacy from \((Z, \eta)\) to itself. Then \( \gamma_1 = \gamma \circ \gamma_2 \), and the proof is complete. \( \square \)

We note that Theorem 4.12, Corollary 4.13, and Proposition 4.14 hold also for essentially minimal totally disconnected dynamical systems (Definition 3.2) which are slightly more general than Cantor minimal systems (the same proofs work).

Next let us give an illustrative example of two ordered premorphisms which are inverses of each other. This shows that ordered premorphisms can also be used to verify conjugacy between Cantor minimal systems. This also gives an alternative proof of the fact that the diagrams in Figure 3 are equivalent (as already mentioned in [15, Section 4.2].)

**Example 4.15.** Let \((X, \varphi)\) (the Chacon system) and \(C = (W, S, \geq)\) be as in Example 2.11. The diagram \(C\) is drawn on the left in Figure 4. Let \(B = (V, E, \geq)\) be the properly ordered Bratteli diagram drawn on the left in Figure 3 and let \(B' = (V', E', \geq')\) be the telescoping to the sequence 0, 3, 4, 5... of \(B\). The diagram \(B'\) is drawn on the right in Figure 4. It can be checked easily that \(g : C \to B'\) in Figure 4 is an ordered premorphism, i.e., the ordered commutativity required in Definition 2.5 holds. Write \(g = (G, (u_n)_{n=0}^\infty, \geq)\). Then the multiplicity matrices are the following:

\[
M(E'_0) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad M(E'_n) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M(S_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad M(S_n) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{for } n \geq 2;
\]

\[
M(F_0) = (1), \quad M(F_n) = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}, \quad \text{for } n \geq 1.
\]

Note that \(g\) can also be considered as an ordered premorphism from \(C\) to \(B\), in which case we have \(g = (G, (g_n)_{n=0}^\infty, \geq)\) where \(g_0 = 0\) and \(g_n = n + 2\) for \(n \geq 1\). Also, \(f\) in Example 2.11 can considered as an ordered premorphism from \(B\) to \(C\) and in this case we can write \(f = (F, (f_n)_{n=0}^\infty, \geq)\), where \(f_0 = 0\) and \(f_n = n + 1\) for \(n \geq 1\). With this in mind, the compositions \(fg : C \to C\) and \(gf : B \to B\) make sense. It is easy to check that \(fg \sim id_C\) using Definition 2.10 for equivalence of ordered premorphisms in the second sense. Applying the functor \(\mathcal{V}\) we get \(\mathcal{V}([g]) \circ \mathcal{V}([f]) = id_X\). Note that \(\mathcal{V}([f])\) is surjective (since factor maps between minimal systems are surjective). It follows that \(\mathcal{V}([f])\) is a conjugacy and that \(gf \sim id_B\). In particular, \(B\) and \(C\) are equivalent. A dynamical argument for this can be obtained by using the fact that the Chacon system \((X, \varphi)\) is topologically prime—i.e., it has no non-trivial factor.

5. Weak Orbit Equivalence and C*-Algebras

In this section we give an equivalent condition in terms of C*-algebras for weak orbit equivalence.

Let \((X, \varphi)\) and \((Y, \psi)\) be Cantor minimal systems. Recall from [17] that these systems are *weakly orbit equivalent* if there exists a homeomorphism \(\alpha\) in \([\varphi]\) such that the system \((X, \alpha)\) admits \((Y, \psi)\) as a factor, and there exists a homeomorphism \(\beta\) in \([\psi]\) such that \((Y, \beta)\) admits \((X, \varphi)\) as a factor. (Here, \([\varphi]\) denotes the full group of \((X, \varphi)\); see [17].)

Two simple dimension groups with order unit, \(G\) and \(H\), are called *weakly isomorphic* if there exist order and order unit preserving group homomorphisms from \(G\) into \(H\) and from \(H\) to \(G\).

For a C*-algebra \(A\) let us denote by \(T(A)\) the set of tracial states on \(A\). When \(T(A) \neq \emptyset\), there is a natural pairing \(\rho_A : K_0(A) \to \text{Aff}(T(A))\) defined by \(\rho_A([p]) (\tau) = \tau(p)\) for all \([p] \in K_0(A)\) and \(\tau \in T(A)\). In the next result we have used the notion of UCT class. We refer the reader to [27, Definition 2.4.5] for the definition and details.
Figure 4. An ordered premorphism $g$ from $C$ to $B'$. See Example 4.15.

**Theorem 5.1.** Let $(X, \varphi)$ and $(Y, \psi)$ be Cantor minimal systems and set $C(X) \ltimes_\varphi Z = A$ and $C(Y) \ltimes_\psi Z = B$. The following statements are equivalent:

1. $(X, \varphi)$ and $(Y, \psi)$ are weakly orbit equivalent;
2. there exists a positive homomorphism from $\rho_A(K_0(A))$ to $\rho_B(K_0(B))$, mapping $\rho_A([1_A])$ to $\rho_B([1_B])$, and one from $\rho_B(K_0(B))$ to $\rho_A(K_0(A))$, mapping $\rho_B([1_B])$ to $\rho_A([1_A])$;
(3) there are simple unital AT algebras $C, D$ of real rank zero with $K_1$ equal to $Z$ and $K_0$ not equal to $Z$ which are tracially equivalent to $A$ and $B$, respectively, and there are unital $*$-homomorphisms from $C$ to $D$ and from $D$ to $C$;

(4) there are separable simple unital C*-algebras $C, D$ with tracial rank zero which are tracially equivalent to $A$ and $B$, respectively, and there are unital $*$-homomorphisms from $C$ to $D$ and from $D$ to $C$.

Moreover, in (3) (and in (4), if we further assume that $C, D$ are in the UCT class) we can replace the existence of $*$-homomorphisms with the existence of positive homomorphisms $\alpha : K_0(C) \to K_0(D)$ and $\beta : K_0(D) \to K_0(C)$ such that $\alpha([1_A]) = [1_B]$ and $\beta([1_B]) = [1_A]$ and that $\alpha, \beta$ preserve the infinitesimal subgroups. Also, we can choose $C, D$ in (3) in such a way that the infinitesimal subgroups of $K_0(C)$ and $K_0(D)$ are trivial.

Proof. First note that for any unital exact C*-algebra $A$ we have $\ker \rho_A = \text{Inf}(K_0(A))$, where $\rho_A : K_0(A) \to \text{Aff}(T(A))$ is the natural pairing.

$(1) \Leftrightarrow (2)$: This follows from [17, Theorem 2.3] and the relation between the K-theory of a Cantor minimal system and of the associated crossed product. In fact, let $A$ be as in the statement. Then $K_0(A)/\text{Inf}(K_0(A)) \cong \rho_A(K_0(A))$ as dimension groups with order unit, where the latter group is considered with the positive cone $\rho_A(K_0(A))^\dagger$ and order unit $\rho_A([1_A])$. On the other hand, $K^0(X, \phi) \cong K_0(X)$ as dimension groups with order unit. Thus,

$$\rho_A(K_0(A)) \cong \frac{K^0(X, \phi)}{\text{Inf}(K^0(X, \phi))}$$

as dimension groups with order unit. An analogous result holds for $B$. Now, [17, Theorem 2.3] implies that (1) and (2) are equivalent.

$(1) \Leftrightarrow (3)$: There is a Cantor minimal system $(Z, \phi)$ such that

$$K^0(Z, \phi) \cong \frac{K^0(X, \phi)}{\text{Inf}(K^0(X, \phi))}$$

as dimension groups with order unit (see [25, 20, 14]). We may assume that $Z = X$. Indeed, let $h : X \to Z$ be a homeomorphism. Then $T = h^{-1}\phi h$ is a homeomorphism of $X$ and $h : (X, T) \to (Z, \phi)$ is a conjugacy. So $K^0(X, T) \cong K^0(Z, \phi) \cong K^0(X, \phi)/\text{Inf}(K^0(X, \phi))$ as dimension groups with order unit. Note that $\text{Inf}(K^0(X, T)) = 0$. Then by [17, Theorem 2.3], the systems $(X, \phi)$ and $(X, T)$ are weakly orbit equivalent. Set $C = C(X) \rtimes_T Z$. Then by [21, Theorem 4.2], $A$ and $C$ are tracially equivalent. Similarly, there is a minimal homeomorphism $S$ of $Y$ such that

$$K^0(Y, S) \cong \frac{K^0(Y, \psi)}{\text{Inf}(K^0(Y, \psi))}.$$  

Set $D = C(Y) \rtimes_S Z$. Thus, $B$ and $D$ are tracially equivalent. Note that $C$ and $D$ are simple unital AT algebras of real rank zero with $K_1$ equal to $Z$ and $K_0$ not equal to $Z$. Since $(X, T)$ and $(Y, S)$ are weakly orbit equivalent, by [17, Theorem 2.3] there exist positive unital homomorphisms (i.e., morphisms in the category $\text{DG}_1$) $\alpha : K_0(C) \to K_0(D)$ and $\beta : K_0(D) \to K_0(C)$. Note that $C$ and $D$ are TAF algebras and so by [6], there are unital $*$-homomorphisms $f : C \to D$ and $g : D \to C$ such that $K_0(f) = \alpha$ and $K_0(g) = \beta$.

$(3) \Leftrightarrow (4)$: This follows from the fact that if $C$ is a simple unital AT algebra of real rank zero with $K_1$ equal to $Z$ and $K_0$ not equal to $Z$ then $C$ is a TAF algebra. In fact, by [14, Theorem 1.15], there is a Cantor minimal system $(Z, \phi)$ such that $C \cong C(Z) \rtimes_\phi Z$. By [21], such an algebra is a TAF algebra.
(4)⇔(2): Since $A$ is tracially equivalent to $C$, by [21, Theorem 3.4] there is an order isomorphism from $\rho_A(K_0(A))$ onto $\rho_C(K_0(C))$ which maps $\rho_A([1_A])$ to $\rho_C([1_C])$. Similarly, there is an order isomorphism from $\rho_B(K_0(B))$ onto $\rho_D(K_0(D))$ which maps $\rho_B([1_B])$ to $\rho_D([1_D])$. Now let $f : C \to D$ and $g : D \to C$ be unital $\ast$-homomorphisms as in (4). Then we get ordered group homomorphisms $K_0(f) : K_0(C) \to K_0(D)$ and $K_0(g) : K_0(D) \to K_0(C)$ which induce ordered group homomorphisms from $\rho_C(K_0(C))$ to $\rho_D(K_0(D))$ mapping $\rho_C([1_C])$ to $\rho_D([1_D])$ and from $\rho_D(K_0(D))$ to $\rho_C(K_0(C))$ mapping $\rho_D([1_D])$ to $\rho_C([1_C])$. By composing the appropriate maps we obtain ordered group homomorphisms from $\rho_A(K_0(A))$ to $\rho_B(K_0(B))$ mapping $\rho_A([1_A])$ to $\rho_B([1_B])$, and from $\rho_B(K_0(B))$ to $\rho_A(K_0(A))$ mapping $\rho_B([1_B])$ to $\rho_A([1_A])$. Thus (2) holds.

Observe that in (3) and (4) we may replace the existence of unital $\ast$-homomorphisms with (unital) maps between the $K_0$-groups. This is because the $C^\ast$-algebras in question are separable simple unital TAF algebras in the UCT class and (by [6]) one can lift unital positive homomorphisms between the $K_0$-groups to unital $\ast$-homomorphisms between the corresponding $C^\ast$-algebras.

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