Linearly Converging Error Compensated SGD

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1. The Problem
\[
\min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right\}
\]
Parameters of the model

\[
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x)
\]

\(n\) workers/devices

\(f_1(x)\), \(f_2(x)\), \(f_n(x)\)

loss on the data accessible by worker \(i\)
The parameters of the model are given by the minimization of a function $f(x)$ over the data accessible by the workers/devices. The loss function is defined as:

$$f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x)$$

where $n$ is the number of workers/devices and $f_i(x)$ is the loss on the data accessible by worker $i$.

For each worker $i$, the loss function is given by:

$$f_i(x) = \mathbb{E}_{\xi_i \sim D_i} \left[ f_{\xi_i}(x) \right]$$

or

$$f_i(x) = \frac{1}{m} \sum_{j=1}^{m} f_{ij}(x)$$

where $m$ is the number of data points per worker.
Assumptions

\[ \| \nabla f_i(x) - \nabla f_i(y) \| \leq L \| x - y \| \]
\[ f_i(x) - f_i(y) \geq \langle \nabla f_i(y), x - y \rangle \]

\[ f_1, f_2, \ldots, f_n \] – L-smooth and convex
Assumptions

\[ \| \nabla f_i(x) - \nabla f_i(y) \| \leq L \| x - y \| \]

\[ f_i(x) - f_i(y) \geq \langle \nabla f_i(y), x - y \rangle \]

- \( f_1, f_2, \ldots, f_n \) - L-smooth and convex

- \( f \) - strongly quasi-convex

\[ f(x^*) \geq f(x) + \langle \nabla f(x), x^* - x \rangle + \frac{\mu}{2} \| x^* - x \|^2 \]

the solution of the problem
2. Parallel SGD
1. Server broadcasts the parameters $x^k$. 

- $x^k$ to a computer
- $x^k$ to a smartphone
- $x^k$ to a laptop
1. Server broadcasts the parameters

2. Devices compute **stochastic gradients** in parallel

\[ x^k \rightarrow g_1^k \quad x^k \rightarrow g_2^k \quad \ldots \quad x^k \rightarrow g_n^k \]
1. Server broadcasts the parameters

2. Devices compute **stochastic gradients** in parallel

\[ \mathbb{E} g_i^k = \nabla f_i(x^k) \]
1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
3. Server gathers stochastic gradients
4. Server updates the parameters
5. Repeat steps 1 – 4

\[ x^k \rightarrow x^{k+1} = x^k - \gamma \cdot \frac{1}{n} \sum_{i=1}^{n} g_i^k \]
1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
3. Server gathers stochastic gradients
4. Server updates the parameters
5. Repeat steps 1 – 4

Good news:
- Very simple algorithm
- Can be much faster than non-parallel SGD

Issues:
- Overload of the server
3. Communication Bottleneck
How to Handle Communication Bottleneck?

- Change the topology of the network ➔ Decentralized optimization
- Do more work on each worker and communicate less ➔ Local-SGD/Federated Averaging
How to Handle Communication Bottleneck?

- Change the topology of the network  
  → Decentralized optimization

- Do more work on each worker and communicate less  
  → Local-SGD/Federated Averaging

- Send less information to reduce the communication cost

Example:

\[ g_i^k = \begin{pmatrix} 1 \\ -15 \\ 0.2 \\ -7 \\ 10 \end{pmatrix} \]  
\[ C(g_i^k) = \begin{pmatrix} 0 \\ -15 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
How to Handle Communication Bottleneck?

- Change the topology of the network ➔ Decentralized optimization
- Do more work on each worker and communicate less ➔ Local-SGD/Federated Averaging
- Send less information to reduce the communication cost

We focus on this approach

Example:

\[
g^k_i = \begin{pmatrix} 1 \\ -15 \\ 0.2 \\ -7 \\ 10 \end{pmatrix}
\]

Compression operator

\[
C(g^k_i) = \begin{pmatrix} 0 \\ -15 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]
How to Handle Communication Bottleneck?

- Change the topology of the network ➔ Decentralized optimization
- Do more work on each worker and communicate less ➔ Local-SGD/Federated Averaging
- Send less information to reduce the communication cost

We focus on this approach.

Example:

Send $g_i^k = \begin{pmatrix} 1 \\ -15 \\ 0.2 \\ -7 \\ 10 \end{pmatrix}$

Compression operator $C(g_i^k) = \begin{pmatrix} 0 \\ -15 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

What are the options for choosing this?
Compression Operators

Unbiased compressors (quantizations)

$$x \rightarrow \mathcal{Q}(x) \quad \mathbb{E}[\mathcal{Q}(x)] = x$$

Biased compressors

$$x \rightarrow \mathcal{C}(x)$$
Compression Operators

Unbiased compressors (quantizations)

\[ x \to Q(x) \quad \mathbb{E}[Q(x)] = x \]

\[ \mathbb{E}\|Q(x) - x\|^2 \leq \omega \|x\|^2 \]

Biased compressors

\[ x \to C(x) \]

\[ \mathbb{E}\|C(x) - x\|^2 \leq (1 - \delta)\|x\|^2 \]
Compression Operators

Unbiased compressors (quantizations)

\[ x \rightarrow Q(x) \quad \mathbb{E}[Q(x)] = x \]

\[ \mathbb{E}\|Q(x) - x\|^2 \leq \omega \|x\|^2 \]

Example: RandK (for K = 2)

\[
\begin{pmatrix}
1 \\
-15 \\
0.2 \\
-7 \\
10
\end{pmatrix}
\xrightarrow{5,2}
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\]

for unbiasedness

Pick K = 2 components uniformly at random

Biased compressors

\[ x \rightarrow C(x) \]

\[ \mathbb{E}\|C(x) - x\|^2 \leq (1 - \delta) \|x\|^2 \]

Example: TopK (for K = 2)

\[
\begin{pmatrix}
1 \\
-15 \\
0.2 \\
-7 \\
10
\end{pmatrix}
\xrightarrow{0,15,0,0,10}
\begin{pmatrix}
0 \\
-15 \\
0 \\
0 \\
10
\end{pmatrix}
\]

Pick K = 2 components with largest absolute value
Methods with Unbiased Compressors

- **QSGD**
  - Alistarh, Dan, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. "QSGD: Communication-efficient SGD via gradient quantization and encoding." *In Advances in Neural Information Processing Systems*, pp. 1709-1720. 2017.

- **TernGrad**
  - Wen, Wei, Cong Xu, Feng Yan, Chunpeng Wu, Yandan Wang, Yiran Chen, and Hai Li. "Terngrad: Ternary gradients to reduce communication in distributed deep learning." *In Advances in neural information processing systems*, pp. 1509-1519. 2017.

- **DQGD**
  - Khirirat, Sarit, Hamid Reza Feyzmahdavian, and Mikael Johansson. "Distributed learning with compressed gradients." *arXiv preprint arXiv:1806.06573* (2018).

- **DIANA**
  - Mishchenko, Konstantin, Eduard Gorbunov, Martin Takáč, and Peter Richtárik. "Distributed learning with compressed gradient differences." *arXiv preprint arXiv:1901.09269* (2019).
  - Horváth, Samuel, Dmitry Kovalev, Konstantin Mishchenko, Sebastian Stich, and Peter Richtárik. "Stochastic distributed learning with gradient quantization and variance reduction." *arXiv preprint arXiv:1904.05115* (2019).

**Sublinear** convergence rates even in the case when workers quantize full gradients.

**Converges linearly** when workers quantize full gradients.
Parallel SGD with Biased Compressor Can Diverge at Exponential Rate

Beznosikov, Aleksandr, Samuel Horváth, Peter Richtárik, and Mher Safaryan. "On Biased Compression for Distributed Learning." arXiv preprint arXiv:2002.12410 (2020).

\[ n = d = 3 \]
\[
\begin{align*}
    f_1(x) &= \langle a, x \rangle^2 + \frac{1}{4}\|x\|^2 \\
    f_2(x) &= \langle b, x \rangle^2 + \frac{1}{4}\|x\|^2 \\
    f_3(x) &= \langle c, x \rangle^2 + \frac{1}{4}\|x\|^2
\end{align*}
\]
\[
\begin{align*}
    a &= (-3, 2, 2)^\top \\
    b &= (2, -3, 2)^\top \\
    c &= (2, 2, -3)^\top
\end{align*}
\]
\[ x^0 = (t, t, t)^\top \]

In this case Parallel SGD with Top1 compression operator satisfies

\[ x^k = \left(1 + \frac{11\gamma}{6}\right)^k x^0 \]
Parallel SGD with Biased Compressor Can Diverge at Exponential Rate

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In this case Parallel SGD with Top1 compression operator satisfies

\[ n = d = 3 \]

\[ f_1(x) = \langle a, x \rangle^2 + \frac{1}{4} \|x\|^2 \quad f_2(x) = \langle b, x \rangle^2 + \frac{1}{4} \|x\|^2 \quad f_3(x) = \langle c, x \rangle^2 + \frac{1}{4} \|x\|^2 \]

\[ a = (-3, 2, 2)^\top \quad b = (2, -3, 2)^\top \quad c = (2, 2, -3)^\top \]

\[ x^0 = (t, t, t)^\top \]

One can fix this using one special trick called error-compensation
4. Error-Compensated SGD
Papers on EC-SGD

Seide, Frank, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. "1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns." In Fifteenth Annual Conference of the International Speech Communication Association. 2014.

Stich, Sebastian U., Jean-Baptiste Cordonnier, and Martin Jaggi. "Sparsified SGD with memory." In Advances in Neural Information Processing Systems, pp. 4447-4458. 2018.

Karimireddy, Sai Praneeth, Quentin Rebjock, Sebastian Stich, and Martin Jaggi. "Error Feedback Fixes SignSGD and other Gradient Compression Schemes." In International Conference on Machine Learning, pp. 3252-3261. 2019.

Stich, Sebastian U., and Sai Praneeth Karimireddy. "The error-feedback framework: Better rates for SGD with delayed gradients and compressed communication." arXiv preprint arXiv:1909.05350 (2019).

Beznosikov, Aleksandr, Samuel Horváth, Peter Richtárik, and Mher Safaryan. "On Biased Compression for Distributed Learning." arXiv preprint arXiv:2002.12410 (2020).
Step 1

Server broadcasts the parameters
1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
1. Server broadcasts the parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send **compressed vectors** and save unsent information
1. Server broadcasts the parameters

2. Devices compute **stochastic gradients** in parallel

3. Compression

4. Devices send compressed vectors and save unsent information

5. Server gathers the information and updates the parameters

\[
x^0 \Rightarrow x^1 = x^0 - \frac{1}{n} \sum_{i=1}^{n} v_i^0
\]

\[v_1^0 = \mathcal{C}(\gamma g_1^0) \quad v_2^0 = \mathcal{C}(\gamma g_2^0) \quad v_n^0 = \mathcal{C}(\gamma g_n^0)
\]

\[e_1^1 = \gamma g_1^0 - v_1^0 \quad e_2^1 = \gamma g_2^0 - v_2^0 \quad e_n^1 = \gamma g_n^0 - v_n^0\]
Step 1

devices keep these vectors for the next iterations to *partially* send them later
Step 2

Server broadcasts new parameters

\( x^1 \)

\[ e^1_1 \]

\[ e^1_2 \]

\[ \ldots \]

\[ e^1_n \]
1. Server broadcasts new parameters

2. Devices compute **stochastic** gradients in parallel
1. Server broadcasts new parameters
2. Devices compute **stochastic** gradients in parallel
3. Compression

\[ v_1 = C(e_1 + \gamma g_1) \]
\[ v_2 = C(e_2 + \gamma g_2) \]
\[ v_n = C(e_n + \gamma g_n) \]
1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression

\[ v_1^1 = C(e_1^1 + \gamma g_1^1) \]
\[ v_2^1 = C(e_2^1 + \gamma g_2^1) \]
\[ v_n^1 = C(e_n^1 + \gamma g_n^1) \]

New information devices want to send
1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression

\[ x^1 \]

\[ v_1^1 = C(e_1^1 + \gamma g_1^1) \]
\[ v_2^1 = C(e_2^1 + \gamma g_2^1) \]
\[ v_n^1 = C(e_n^1 + \gamma g_n^1) \]

Old information devices want to send (memory)
1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information

$$v^1 = C(e^1 + \gamma g^1)$$
$$v^2 = C(e^2 + \gamma g^2)$$
$$v_n^1 = C(e_n^1 + \gamma g_n^1)$$

$$e_1^2 = e_1^1 + \gamma g_1^1 - v_1^1$$
$$e_2^2 = e_2^1 + \gamma g_2^1 - v_2^1$$
$$e_n^2 = e_n^1 + \gamma g_n^1 - v_n^1$$
1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information
5. Server gathers the information and updates the parameters

\[
x^1 \rightarrow x^2 = x^1 - \frac{1}{n} \sum_{i=1}^{n} v_i^1
\]

\[
v_1^1 = C(e_1^1 + \gamma g_1^1)
\]

\[
e_1^2 = e_1^1 + \gamma g_1^1 - v_1^1
\]

\[
v_2^1 = C(e_2^1 + \gamma g_2^1)
\]

\[
e_2^2 = e_2^1 + \gamma g_2^1 - v_2^1
\]

\[
v_n^1 = C(e_n^1 + \gamma g_n^1)
\]

\[
e_n^2 = e_n^1 + \gamma g_n^1 - v_n^1
\]
1. Server broadcasts new parameters
2. Devices compute **stochastic gradients** in parallel
3. Compression
4. Devices send compressed vectors and update unsent information
5. Server gathers the information and updates the parameters
6. Devices update their memory

\[ x^1 \rightarrow x^2 = x^1 - \frac{1}{n} \sum_{i=1}^{n} v_i^1 \]

\[ e^2_1 = e^1_1 + \gamma g^1_1 - v^1_1 \]
\[ e^2_2 = e^1_2 + \gamma g^1_2 - v^1_2 \]
\[ e^2_n = e^1_n + \gamma g^1_n - v^1_n \]
Server broadcasts new parameters

Step $k+1$
1. Server broadcasts new parameters
2. Workers compute **stochastic gradients** in parallel
3. Compression
4. Devices send **compressed vectors** and update unsent information
5. Server gathers the information and updates the parameters
6. Repeat steps 1 – 5

\[
x^k \rightarrow x^{k+1} = x^k - \frac{1}{n} \sum_{i=1}^{n} v_i^k
\]
Error-Compensated SGD

Converges even with biased compression operators

EC-SGD finds such $\hat{x}$ that $E[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

$\mathcal{O}\left(\frac{L}{\delta} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{\sqrt{L(\sigma^2 + \frac{\zeta^2}{\delta})}}{\mu\sqrt{\delta\varepsilon}}\right)$ iterations

$E\|C(x) - x\|^2 \leq (1 - \delta)\|x\|^2$

$E\left[\|g^k_i - \nabla f_i(x^k)\|^2 \mid x^k\right] \leq \sigma^2$

$\zeta^2 = \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x^*)\|^2$
Error-Compensated SGD

- Converges even with biased compression operators
- Fails to converge with **linear rate** even when workers compute full gradients

EC-SGD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

\[
\mathcal{O}\left(\frac{L}{\delta \mu} + \frac{\sigma^2}{n \mu \varepsilon} + \frac{\sqrt{L(\sigma^2 + \zeta^2_*/\delta)}}{\mu \sqrt{\delta \varepsilon}}\right)
\]

\[
\mathbb{E}\|C(x) - x\|^2 \leq (1 - \delta)\|x\|^2 \quad \quad \mathbb{E}\left[\|g_i^k - \nabla f_i(x^k)\|^2 \mid x^k\right] \leq \sigma^2
\]

\[
\zeta^2_* = \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x^*\|)^2
\]
EC-GD and Logistic Regression

\[
\begin{aligned}
\min_{x \in \mathbb{R}^d} \left\{ f(x) = \frac{1}{N} \sum_{i=1}^{N} \log(1 + \exp(-y_i \cdot (Ax)_i)) + \frac{\mu}{2} \|x\|^2 \right\}
\end{aligned}
\]

sub-linear convergence

linear convergence

- a9a, 20 workers
- w8a, 20 workers
Error-Compensated SGD

- Converges even with biased compression operators
- Fails to converge with linear rate even when workers compute full gradients

Questions:
1. Is it possible to design linearly converging SGD with error compensation when workers compute full gradients, i.e., linearly converging EC-GD?
2. Is it possible to design linearly converging SGD with error compensation when the local loss functions have a finite-sum form?

The answer is Yes for both questions
5.1. New method: EC-GDstar
Error-Compensated GD

EC-GD finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

$\mathcal{O} \left( \frac{L}{\delta \mu} + \frac{\sqrt{L} \zeta_*^2}{\mu \delta \sqrt{\varepsilon}} \right)$

Hides logarithmical factors

$\zeta_*^2 = \frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x^*) \|^2$
Error-Compensated GD

EC-GD finds such \( \hat{x} \) that \( \mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon \) after

\[
\tilde{O} \left( \frac{L}{\delta \mu} + \frac{\sqrt{L \zeta_2^2}}{\mu \delta \sqrt{\varepsilon}} \right) \]

Hides logarithmical factors

What if devices know these vectors from the beginning?
Server broadcasts new parameters
1. Server broadcasts new parameters

2. Workers compute **shifted gradients** in parallel

\[ g_i^k = \nabla f_i(x^k) - \nabla f_i(x^*) \]
EC-GDstar

1. Server broadcasts new parameters
2. Workers compute **shifted gradients** in parallel
3. Compression
4. Devices send **compressed vectors** and update unsent information
5. Server gathers the information and updates the parameters
6. Repeat steps 1 – 5

\[ x^k \xrightarrow{} x^{k+1} = x^k - \frac{1}{n} \sum_{i=1}^{n} v_i^k \]
EC-GDstar: Rate of Convergence

EC-GDstar finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

$$O\left(\frac{L}{\delta \mu \ln \frac{1}{\varepsilon}}\right)$$ iterations

- **Linear rate**
- **The method is impractical: it uses the gradients at the solution**

Can we develop a practical analog?
5.2. New method: EC-SGD-DIANA
The same scheme as for EC-SGD

\[ x^k \rightarrow x^{k+1} = x^k - \frac{1}{n} \sum_{i=1}^{n} v_i^k \]
EC-SGD-DIANA

\[ g_i^k = \hat{g}_i^k - h_i^k + h^k \]
The key insight why do we need $\{h_i^k\}_{i=1}^n$:

it reduces the variance coming from compressions via learning the gradients at the solution!

$$g_i^k = \hat{g}_i^k - h_i^k + h^k$$

$$h_i^{k+1} = h_i^k + \alpha Q(\hat{g}_i^k - h_i^k)$$

EC-SGD-DIANA

Works for both cases:

- $f_i(x) = \mathbb{E}_{\xi_i \sim D_i} [f_{\xi_i}(x)]$
  $$\hat{g}_i^k = \nabla f_{\xi_i}(x_i^k)$$

- $f_i(x) = \frac{1}{m} \sum_{j=1}^m f_{ij}(x)$
  $$l \sim [m] \text{ uniformly at random}$$

Workers send these vectors to the server

Server broadcasts this vector to the workers
EC-SGD-DIANA: Rate of Convergence

EC-SGD-DIANA finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

Option I:

$$O \left( \frac{\omega + \frac{L}{\delta \mu} + \frac{\sigma^2}{n \mu \varepsilon} + \sqrt{L\sigma^2}}{\delta \mu \sqrt{\varepsilon}} \right)$$

Hides logarithmical factors

Option II:

$$O \left( \frac{1 + \omega}{\delta} + \frac{L}{\delta \mu} + \frac{\sigma^2}{n \mu \varepsilon} + \frac{\sqrt{L\sigma^2}}{\mu \sqrt{\delta \varepsilon}} \right)$$
EC-SGD-DIANA: Rate of Convergence

EC-SGD-DIANA finds such $\hat{x}$ that $\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon$ after

Option I:

$$\tilde{O}\left(\frac{L}{\delta \mu} + \frac{\sigma^2}{n \mu \varepsilon} + \frac{\sqrt{L \sigma^2}}{\delta \mu \sqrt{\varepsilon}}\right)$$

Hides logarithmical factors

Option II:

$$\tilde{O}\left(\frac{L}{\delta} + \frac{\sigma^2}{\delta \mu} + \frac{\sqrt{L \sigma^2}}{n \mu \varepsilon} + \frac{\sqrt{L \sigma^2}}{\mu \sqrt{\delta \varepsilon}}\right)$$

Moreover, if workers compute full gradients, then the rate of convergence is linear

$$\tilde{O}\left((\omega + \frac{L}{\delta \mu}) \log \frac{1}{\varepsilon}\right)$$
5.3. New method: EC-LSVRG-DIANA
EC-LSVRG-DIANA

\[ g_i^k = \hat{g}_i^k - h_i^k + h^k \]

Works for the case:

\[ f_i(x) = \frac{1}{m} \sum_{j=1}^{m} f_{ij}(x) \]
EC-LSVRG-DIANA

\[ g^k_i = \begin{cases} \hat{g}^k_i \\ \nabla f_{il}(x^k) - \nabla f_{il}(w^k_i) + \nabla f_i(w^k_i) \end{cases} \quad l \sim [m] \text{ uniformly at random} \]

Works for the case:

\[ f_i(x) = \frac{1}{m} \sum_{j=1}^{m} f_{ij}(x) \]
EC-LSVRG-DIANA

\[ g_i^k = \hat{g}_i^k - h_i^k + h^k \]

\[ l \sim [m] \text{ uniformly at random} \]

Reduction of the variance introduced due to the stochasticity of the gradients

\[ w_i^{k+1} = \begin{cases} 
  x_i^k, & \text{with probability } p \\
  w_i^k, & \text{with probability } 1 - p 
\end{cases} \]

Works for the case:

\[ f_i(x) = \frac{1}{m} \sum_{j=1}^{m} f_{ij}(x) \]
EC-LSVRG-DIANA: Rate of Convergence

EC-LSVRG-DIANA finds such \( \hat{x} \) that

\[
\mathbb{E}[f(\hat{x})] - f(x^*) \leq \varepsilon \quad \text{after}
\]

\[
\mathcal{O} \left( \left( \omega + m + \frac{L}{\delta \mu} \right) \log \frac{1}{\epsilon} \right) \text{ iterations}
\]
6. Unified Convergence Analysis of Methods with Error Compensation
Key Assumption

\[ g^k = \frac{1}{n} \sum_{i=1}^{n} g^k_i, \quad \mathbb{E} \left[ g^k \mid x^k \right] = \nabla f(x^k) \quad \tilde{g}^k_i = \mathbb{E} \left[ g^k_i \mid x^k \right] \]

\[ \frac{1}{n} \sum_{i=1}^{n} \left\| \tilde{g}^k_i \right\|^2 \leq 2A \left( f(x^k) - f(x^*) \right) + B_1 \sigma^2_{1,k} + B_2 \sigma^2_{2,k} + D_1 \]

\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| g^k_i - \tilde{g}^k_i \right\|^2 \mid x^k \right] \leq 2\bar{A} \left( f(x^k) - f(x^*) \right) + \bar{B}_1 \sigma^2_{1,k} + \bar{B}_2 \sigma^2_{2,k} + \bar{D}_1 \]

\[ \mathbb{E} \left[ \left\| g^k \right\|^2 \mid x^k \right] \leq 2A' \left( f(x^k) - f(x^*) \right) + B'_1 \sigma^2_{1,k} + B'_2 \sigma^2_{2,k} + D'_1 \]

\[ \mathbb{E} \left[ \sigma^2_{1,k+1} \mid \sigma^2_{1,k}, \sigma^2_{2,k} \right] \leq (1 - \rho_1) \sigma^2_{1,k} + 2C_1 \left( f(x^k) - f(x^*) \right) + G \rho_1 \sigma^2_{2,k} + D_2 \]

\[ \mathbb{E} \left[ \sigma^2_{2,k+1} \mid \sigma^2_{2,k} \right] \leq (1 - \rho_2) \sigma^2_{2,k} + 2C_2 \left( f(x^k) - f(x^*) \right) \]
Key Assumption

\[ g^k = \frac{1}{n} \sum_{i=1}^{n} g_i^k, \quad \mathbb{E} \left[ g^k \mid x^k \right] = \nabla f \left( x^k \right) \quad \bar{g}_i^k = \mathbb{E} \left[ g_i^k \mid x^k \right] \]

\[
\frac{1}{n} \sum_{i=1}^{n} \| \bar{g}_i^k \|^2 \leq 2A \left( f \left( x^k \right) - f \left( x^* \right) \right) + B_1 \sigma_{1,k}^2 + B_2 \sigma_{2,k}^2 + D_1
\]

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| g_i^k - \bar{g}_i^k \|^2 \mid x^k \right] \leq 2\bar{A} \left( f \left( x^k \right) - f \left( x^* \right) \right) + \bar{B}_1 \sigma_{1,k}^2 + \bar{B}_2 \sigma_{2,k}^2 + \bar{D}_1
\]

\[
\mathbb{E} \left[ \| g^k \|^2 \mid x^k \right] \leq 2A' \left( f \left( x^k \right) - f \left( x^* \right) \right) + B'_1 \sigma_{1,k}^2 + B'_2 \sigma_{2,k}^2 + D'_1
\]

\[
\mathbb{E} \left[ \sigma_{1,k+1}^2 \mid \sigma_{1,k}^2, \sigma_{2,k}^2 \right] \leq (1 - \rho_1) \sigma_{1,k}^2 + 2C_1 \left( f \left( x^k \right) - f \left( x^* \right) \right) + G \rho_1 \sigma_{2,k}^2 + D_2
\]

\[
\mathbb{E} \left[ \sigma_{2,k+1}^2 \mid \sigma_{2,k}^2 \right] \leq (1 - \rho_2) \sigma_{2,k}^2 + 2C_2 \left( f \left( x^k \right) - f \left( x^* \right) \right)
\]

Reflects smoothness properties of the problem and noises introduced by compressions and stochastic gradients.
Key Assumption

\[
g^k = \frac{1}{n} \sum_{i=1}^{n} g_i^k, \quad \mathbb{E} \left[ g^k \mid x^k \right] = \nabla f (x^k) \quad \bar{g}_i^k = \mathbb{E} \left[ g_i^k \mid x^k \right]
\]

\[
\frac{1}{n} \sum_{i=1}^{n} \left\| \bar{g}_i^k \right\|^2 \leq 2A \left( f (x^k) - f (x^*) \right) + B_1 \sigma_{1,k}^2 + B_2 \sigma_{2,k}^2 + D_1
\]

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| g_i^k - \bar{g}_i^k \right\|^2 \mid x^k \right] \leq 2\tilde{A} \left( f (x^k) - f (x^*) \right) + \tilde{B}_1 \sigma_{1,k}^2 + \tilde{B}_2 \sigma_{2,k}^2 + \tilde{D}_1
\]

\[
\mathbb{E} \left[ \left\| g^k \right\|^2 \mid x^k \right] \leq 2A' \left( f (x^k) - f (x^*) \right) + B'_1 \sigma_{1,k}^2 + B'_2 \sigma_{2,k}^2 + D'_1
\]

\[
\mathbb{E} \left[ \sigma_{1,k+1}^2 \mid \sigma_{1,k}^2, \sigma_{2,k}^2 \right] \leq (1 - \rho_1) \sigma_{1,k}^2 + 2C_1 \left( f (x^k) - f (x^*) \right) + G \rho_1 \sigma_{2,k}^2 + D_2
\]

\[
\mathbb{E} \left[ \sigma_{2,k+1}^2 \mid \sigma_{2,k}^2 \right] \leq (1 - \rho_2) \sigma_{2,k}^2 + 2C_2 \left( f (x^k) - f (x^*) \right)
\]

Reflects smoothness properties of the problem and noises introduced by compressions and stochastic gradients

Describes the process of variance reduction of the variance coming from compressions
Key Assumption

\[ g^k = \frac{1}{n} \sum_{i=1}^{n} g^k_i, \quad \mathbb{E} \left[ g^k \mid x^k \right] = \nabla f (x^k) \]

\[ \mathbb{E} \left[ \| \mathbf{g}^k \| \mid x^k \right] \leq \frac{1}{n} \sum_{i=1}^{n} \| \mathbf{g}^k_i \|^2 \leq 2A (f (x^k) - f (x^*)) + B_1 \sigma_{1,k}^2 + B_2 \sigma_{2,k}^2 + D_1 \]

\[ \mathbb{E} \left[ \| g^k_i - \bar{g}^k_i \|^2 \mid x^k \right] \leq 2 \tilde{A} (f (x^k) - f (x^*)) + \tilde{B}_1 \sigma_{1,k}^2 + \tilde{B}_2 \sigma_{2,k}^2 + \tilde{D}_1 \]

\[ \mathbb{E} \left[ \| g^k \|^2 \mid x^k \right] \leq 2A' (f (x^k) - f (x^*)) + B'_1 \sigma_{1,k}^2 + B'_2 \sigma_{2,k}^2 + D'_1 \]

\[ \mathbb{E} \left[ \sigma_{1,k+1}^2 \mid \sigma_{1,k}^2, \sigma_{2,k}^2 \right] \leq (1 - \rho_1) \sigma_{1,k}^2 + 2C_1 (f (x^k) - f (x^*)) + G \rho_1 \sigma_{2,k}^2 + D_2 \]

\[ \mathbb{E} \left[ \sigma_{2,k+1}^2 \mid \sigma_{2,k}^2 \right] \leq (1 - \rho_2) \sigma_{2,k}^2 + 2C_2 (f (x^k) - f (x^*)) \]

- Reflects smoothness properties of the problem and noises introduced by compressions and stochastic gradients
- Describes the process of variance reduction of the variance coming from compressions
- Describes the process of variance reduction of the variance coming from stochastic gradients
Main Theorem

Some quantity depending only on the starting point and stepsize

$$\mathbb{E} \left[ f (\bar{x}^K) - f(x^*) \right] \leq (1 - \eta)^K \frac{\Psi(x^0, \gamma)}{\gamma} + \gamma \Phi \left( D_1, \tilde{D}_1, D_1', D_2 \right)$$

$$\eta = \min \left\{ \frac{\gamma \mu}{2}, \frac{\rho_1}{4}, \frac{\rho_2}{4} \right\}$$
### Methods with Error Compensation Covered by Our Framework

| Problem | Method             | Alg # | Citation | Sec # | Rate (constants ignored)                                                                 |
|---------|--------------------|-------|----------|-------|-----------------------------------------------------------------------------------------|
| (1)+(3) | EC-SGDsr           | Alg 3 | new      | H.1   | $\tilde{O}\left(\frac{\mathcal{L}}{\mu} + \sqrt{\frac{L \mathcal{L}}{\delta \mu}} + \frac{\text{Var}}{\eta \mu \varepsilon} + \sqrt{\frac{L(\text{Var} + \frac{\sigma^2}{L})}{\mu \sqrt{\delta \varepsilon}}}\right)$ |
| (1)+(2) | EC-SGD             | Alg 4 | [45]     | H.2   | $\tilde{O}\left(\frac{\kappa}{\delta} + \frac{\text{Var}}{\eta \mu \varepsilon} + \sqrt{\frac{L(\text{Var} + \frac{\sigma^2}{L})}{\delta \mu \sqrt{\varepsilon}}}\right)$ |
| (1)+(3) | EC-GDstar          | Alg 5 | new      | H.3   | $\mathcal{O}\left(\frac{\kappa}{\delta} \log \frac{1}{\varepsilon}\right)$            |
| (1)+(2) | EC-SGD-DIANA       | Alg 6 | new      | H.4   | Option I: $\tilde{O}\left(\omega + \frac{\kappa}{\delta} + \frac{\sigma^2}{\eta \mu \varepsilon} + \sqrt{\frac{L \sigma^2}{\delta \mu \sqrt{\varepsilon}}}\right)$ |
|         |                    |       |          |       | Option II: $\tilde{O}\left(\frac{1+\omega}{\delta} + \frac{\kappa}{\delta} + \frac{\sigma^2}{\eta \mu \varepsilon} + \sqrt{\frac{L \sigma^2}{\mu \sqrt{\delta \varepsilon}}}\right)$ |
| (1)+(3) | EC-SGDsr-DIANA     | Alg 7 | new      | H.5   | Option I: $\tilde{O}\left(\omega + \frac{\mathcal{L}}{\mu} + \sqrt{\frac{L \mathcal{L}}{\delta \mu}} + \frac{\text{Var}}{\eta \mu \varepsilon} + \sqrt{\frac{L \text{Var}}{\delta \mu \sqrt{\varepsilon}}}\right)$ |
|         |                    |       |          |       | Option II: $\tilde{O}\left(\frac{1+\omega}{\delta} + \frac{\mathcal{L}}{\mu} + \sqrt{\frac{L \mathcal{L}}{\delta \mu}} + \frac{\text{Var}}{\eta \mu \varepsilon} + \sqrt{\frac{L \text{Var}}{\mu \delta \varepsilon}}\right)$ |
| (1)+(2) | EC-GD-DIANA†       | Alg 6 | new      | H.4   | $\mathcal{O}\left((\omega + \frac{\kappa}{\delta}) \log \frac{1}{\varepsilon}\right)$           |
| (1)+(3) | EC-LSVRG          | Alg 8 | new      | H.6   | $\tilde{O}\left(m + \frac{\kappa}{\delta} + \sqrt{\frac{L \mathcal{L}}{\delta \mu \sqrt{\varepsilon}}}\right)$ |
| (1)+(3) | EC-LSVRGstar      | Alg 9 | new      | H.7   | $\mathcal{O}\left((m + \frac{\kappa}{\delta}) \log \frac{1}{\varepsilon}\right)$           |
| (1)+(3) | EC-LSVRG-DIANA    | Alg 10| new      | H.8   | $\mathcal{O}\left((\omega + m + \frac{\kappa}{\delta}) \log \frac{1}{\varepsilon}\right)$ |
Methods with Error Compensation Covered by Our Framework

| Problem | Method | Alg # | Citation | Sec # | Rate (constants ignored) |
|---------|--------|-------|----------|-------|--------------------------|
| (1)+(3) | EC-SGDsr | Alg 3 | new      | H.1   | $\tilde{O}\left( \frac{\mathcal{L}}{\mu} + \sqrt{\frac{L}{\delta \mu}} + \frac{\text{Var}}{\eta \mu \varepsilon} + \sqrt{\frac{L(\text{Var} + \frac{\kappa^2}{\delta})}{\mu \sqrt{\varepsilon}}} \right)$ |
| (1)+(2) | EC-SGD  | Alg 4 | [45]     | H.2   | $\tilde{O}\left( \frac{\kappa}{\delta} + \frac{\text{Var}}{\eta \mu \varepsilon} + \sqrt{\frac{L(\text{Var} + \frac{\kappa^2}{\delta})}{\delta \mu \sqrt{\varepsilon}}} \right)$ |
| (1)+(3) | EC-GDstar | Alg 5 | new      | H.3   | $\mathcal{O}\left( \frac{\kappa}{\delta} \log \frac{1}{\varepsilon} \right)$ |
| (1)+(2) | EC-SGD-DIANA | Alg 6 | new      | H.4   | Option I: $\tilde{O}\left( \frac{\omega + \kappa}{\delta} + \frac{\sigma^2}{\eta \mu \varepsilon} + \frac{\sqrt{L \sigma^2}}{\delta \mu \sqrt{\varepsilon}} \right)$  
    Option II: $\tilde{O}\left( \frac{1+\omega}{\delta} + \frac{\omega}{\delta} + \frac{\kappa}{\delta} + \frac{\sigma^2}{\eta \mu \varepsilon} + \frac{\sqrt{L \sigma^2}}{\mu \sqrt{\delta \varepsilon}} \right)$ |
| (1)+(3) | EC-SGDsr-DIANA | Alg 7 | new      | H.5   | Option I: $\tilde{O}\left( \frac{\omega + \kappa}{\delta} + \frac{\mathcal{L}}{\mu} + \frac{\sqrt{L \mathcal{L}}}{\delta \mu} + \frac{\text{Var}}{\eta \mu \varepsilon} + \frac{\sqrt{L \text{Var}}}{\delta \mu \sqrt{\varepsilon}} \right)$  
    Option II: $\tilde{O}\left( \frac{1+\omega}{\delta} + \frac{\omega}{\delta} + \frac{\mathcal{L}}{\mu} + \frac{\sqrt{L \mathcal{L}}}{\delta \mu} + \frac{\text{Var}}{\eta \mu \varepsilon} + \frac{\sqrt{L \text{Var}}}{\mu \sqrt{\delta \varepsilon}} \right)$ |
| (1)+(2) | EC-GD-DIANA† | Alg 6 | new      | H.4   | $\mathcal{O}\left( \left( \frac{\omega + \kappa}{\delta} \right) \log \frac{1}{\varepsilon} \right)$ |
| (1)+(3) | EC-LSVRG | Alg 8 | new      | H.6   | $\tilde{O}\left( \frac{m + \kappa}{\delta} + \frac{\sqrt{L \mathcal{L}^2}}{\delta \mu \sqrt{\varepsilon}} \right)$ |
| (1)+(3) | EC-LSVRGstar | Alg 9 | new      | H.7   | $\mathcal{O}\left( \left( \frac{m + \kappa}{\delta} \right) \log \frac{1}{\varepsilon} \right)$ |
| (1)+(3) | EC-LSVRG-DIANA | Alg 10 | new     | H.8   | $\mathcal{O}\left( \left( \frac{\omega + m + \kappa}{\delta} \right) \log \frac{1}{\varepsilon} \right)$ |
### Methods with Error Compensation Covered by Our Framework

| Problem | Method          | Alg # | Citation | Sec # | Rate (constants ignored)                                                                 |
|---------|-----------------|-------|----------|-------|----------------------------------------------------------------------------------------|
| (1)+(3) | EC-SGDsr        | Alg 3 | new      | H.1   | $\tilde{O}\left(\frac{L}{\mu} + \frac{\sqrt{L \mathcal{L}}}{\delta \mu} + \frac{\text{Var}}{\eta \mu \varepsilon} + \frac{\sqrt{L \text{Var} + \zeta_0^2 / \delta}}{\mu \sqrt{\delta \varepsilon}}\right)$ |
| (1)+(2) | EC-SGD          | Alg 4 | [45]     | H.2   | $\tilde{O}\left(\frac{\kappa}{\delta} + \frac{\text{Var}}{\eta \mu \varepsilon} + \frac{\sqrt{L \text{Var} + \zeta_0^2 / \delta}}{\delta \mu \sqrt{\delta \varepsilon}}\right)$ |
| (1)+(3) | EC-GDstar       | Alg 5 | new      | H.3   | $\mathcal{O}\left(\frac{\kappa}{\delta} \log \frac{1}{\varepsilon}\right)$              |
| (1)+(2) | EC-SGD-DIANA    | Alg 6 | new      | H.4   | Option I: $\tilde{O}\left(\omega + \frac{\kappa}{\delta} + \frac{\sigma_0^2}{\eta \mu \varepsilon} + \frac{\sqrt{L \sigma_0^2}}{\delta \mu \sqrt{\delta \varepsilon}}\right)$  
Option II: $\tilde{O}\left(\frac{1+\omega}{\delta} + \frac{\kappa}{\delta} + \frac{\sigma_0^2}{\eta \mu \varepsilon} + \frac{\sqrt{L \sigma_0^2}}{\mu \sqrt{\delta \varepsilon}}\right)$ |
| (1)+(3) | EC-SGDsr-DIANA  | Alg 7 | new      | H.5   | Option I: $\tilde{O}\left(\omega + \frac{L \mathcal{L}}{\mu \delta \mu} + \frac{\text{Var}}{\eta \mu \varepsilon} + \frac{\sqrt{L \text{Var}}}{\delta \mu \sqrt{\delta \varepsilon}}\right)$  
Option II: $\tilde{O}\left(\frac{1+\omega}{\delta} + \frac{L \mathcal{L}}{\mu \delta \mu} + \frac{\text{Var}}{\eta \mu \varepsilon} + \frac{\sqrt{L \text{Var}}}{\mu \sqrt{\delta \varepsilon}}\right)$ |
| (1)+(2) | EC-GD-DIANA†    | Alg 6 | new      | H.4   | $\mathcal{O}\left((\omega + \frac{\kappa}{\delta}) \log \frac{1}{\varepsilon}\right)$   |
| (1)+(3) | EC-LSVRG       | Alg 8 | new      | H.6   | $\tilde{O}\left(m + \frac{\kappa}{\delta} + \frac{\sqrt{L \mathcal{C}_m^2}}{\delta \mu \sqrt{\delta \varepsilon}}\right)$ |
| (1)+(3) | EC-LSVRGstar   | Alg 9 | new      | H.7   | $\mathcal{O}\left((m + \frac{\kappa}{\delta}) \log \frac{1}{\varepsilon}\right)$ |
| (1)+(3) | EC-LSVRG-DIANA | Alg 10| new      | H.8   | $\mathcal{O}\left((\omega + m + \frac{\kappa}{\delta}) \log \frac{1}{\varepsilon}\right)$ |

Our framework covers even methods without error compensation and methods with delayed updates.
7. Experiments
Logistic Regression with l2-regularization

- Partial variance reduction
- Full variance reduction
8. More Methods
More Methods Fitting our Framework

The generality of our approach helps to obtain convergence guarantees for a big number of different stochastic methods (even without error compensation). Here are some examples.

- **Methods without error feedback:** SGD, SGD-SR (arbitrary sampling), SAGA, SVRG, L-SVRG, QSGD, TernGrad, DQGD, DIANA, **DIANA**sr-DQ, VR-DIANA, JacSketch, SEGA
- **Methods with delayed updates:** D-SGD, **D-SGD-SR** (arbitrary sampling), **D-QSGD**, D-SGD-DIANA, D-LSVRG, D-QLSVRG, D-LSVRG-DIANA

*bold font* = new method
More Methods Fitting our Framework

The generality of our approach helps to obtain convergence guarantees for a big number of different stochastic methods (even without error compensation). Here are some examples.

- Methods without error feedback: SGD, SGD-SR (arbitrary sampling), SAGA, SVRG, L-SVRG, QSGD, TernGrad, DQGD, DIANA, **DIANA**sr-DQ, VR-DIANA, JacSketch, SEGA

- Methods with delayed updates: D-SGD, **D-SGD-SR** (arbitrary sampling), **D-QSGD**, D-SGD-DIANA, D-LSVRG, D-QLSVRG, D-LSVRG-DIANA

- In one theorem, we recover the sharpest rates for all known special cases
- 16 new methods
- Our analysis works for weakly convex objectives as well

Thank you for watching!