Interaction of solitons with a qubit in an anisotropic Heisenberg spin chain with first and second-neighbor interactions

S Varbev, I Boradjiev, R Kamburova, H Chamati
Georgi Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences,
72 Tzarigradsko chaussée Blvd., BG-1784 Sofia, Bulgaria
E-mail: stanislavvarbev@issp.bas.bg

Abstract. Solitons provide a promising means to control the state of a spin-qubit in quantum information technology. In this paper, we consider the anisotropic interaction between a soliton, propagating in a ferromagnetic Heisenberg spin chain and a spin-$\frac{1}{2}$ particle (qubit). The spin chain exhibits an on-site anisotropy and spin-spin nearest-neighbor and next-nearest-neighbor interactions. The Bloch sphere picture is employed to investigate the spin-$\frac{1}{2}$ dynamics. We found that increasing the anisotropic and the second neighbor interactions leads to an increase of the deviation of the spin-$\frac{1}{2}$ particle from its initial state.

1. Introduction
The main quantum information unit is the qubit – a quantum mechanical two-level system. A thorough introduction into the physics of the two-level system can be found in R. P. Feynman’s lectures on physics [1]. Experimentally, this system can be realized in various manners, such as an electron spin [2] or a composite particle (molecular cluster) with a magnetic moment and spin [3]. Therefore, coherent control of the spin dynamics plays a crucial role in quantum information science. Formally, quantum computing and quantum communications can be viewed as a central set of techniques for coherent manipulation of spins in one and two dimensional lattices [4].

The dynamics of one- and two-dimensional spin magnetic systems, with different kinds of anisotropy and under the effect of external magnetic field, has attracted a lot of interest in recent years [5]. One-dimensional quantum systems permit the existence, and thus propagation and interaction of solitons between each other and with external systems [6]. Soliton-like spin magnetic excitations propagating along spin chains have also been proposed as a way for manipulating the spin state of a spin-$\frac{1}{2}$ localized particle (qubit) [7–9]. The idea is based on the time evolution of a specific class of quantum spin systems determined by a corresponding Hamiltonian dynamics of classical spins in the limits of large-spin and mean-field [10]. An interesting possible realization of an effective qubit is by using spin solitons pinned by the local breaking of translational symmetry in isotropic Heisenberg chains [11]. Using this scheme one should, in principle, be able to control the state of the qubit by a soliton. These considerations may pave the way for the implementation of a new type of quantum computers.

In a recent paper [12], we studied the possibility to manipulate the qubit (realized as a spin) using a soliton propagating through a Heisenberg spin chain with nearest neighbor interaction and on-site anisotropy. We showed that for large values of the interaction between the soliton and the qubit in the $x−y$ plane the creation of qubit’s state superpositions is possible. Here, we extend our investigation by...
including next-nearest-neighbor spin-spin interactions in our model to examine their potential effect on the qubit dynamics.

2. The model

The problem of interest consists of an anisotropic coupling between a qubit (quantum spin-\(1/2\)) and an arbitrary spin on a ferromagnetic Heisenberg chain with nearest- and next-nearest-neighbor exchange interactions and on-site spin anisotropy. The investigated system is described by the following Hamiltonian

\[
\hat{H} = -J_1 \sum_{n=1}^{N} \hat{S}_n \cdot \hat{S}_{n+1} - J_2 \sum_{n=1}^{N} \hat{S}_n \cdot \hat{S}_{n+2} - A \sum_{n=1}^{N} (\hat{S}_n^z)^2 \\
- \mu H_0 \sum_{n=1}^{N} \hat{S}_n^z - v H_0 \hat{S}_j^z + d_{xy}(\hat{S}_j^x \hat{S}_j^x + \hat{S}_j^y \hat{S}_j^y) + d_z \hat{S}_j^z \hat{S}_j^z,
\]

(1)

where \(\hat{S}_n\) are the spin vector operators, \(n = 1, \ldots, N\). \(N\) is the number of spins in the chain, \(A > 0\) is the on-site anisotropy, \(J_1 > 0\) and \(J_2 \geq 0\) are the exchange integrals between the nearest and the next-nearest neighbors, respectively [13]. \(\hat{S}_j\) is the spin-\(1/2\) vector operator, \(H_0\) is the external magnetic field oriented in z-direction, \(\mu\) and \(v\) are magnetic moments per spin in the chain’s sites and the qubit, respectively. \(d_{xy}, d_z\) are constants describing the anisotropic \((d_{xy} \neq d_z)\) coupling of the chain with the spin-\(1/2\) particle [12].

The equations of motion for spin vectors \(\hat{S}_n\) in our model (1) read \((h = 1)\) [12]

\[
\pm i \frac{\partial \hat{S}_n^z}{\partial t} = A (\hat{S}_n^x \hat{S}_{n+1}^x + \hat{S}_n^y \hat{S}_{n+1}^y) + d_{xy}(\hat{S}_j^x \hat{S}_j^x + \hat{S}_j^y \hat{S}_j^y) - d_z \hat{S}_j^z \hat{S}_j^z + \mu H_0 \hat{S}_j^z

- J_1 [(\hat{S}_n^x (\hat{S}_{n-1}^x + \hat{S}_{n+1}^x) - \hat{S}_n^x (\hat{S}_{n-1}^x + \hat{S}_{n+1}^x)] - J_2 [(\hat{S}_n^x (\hat{S}_{n-2}^x + \hat{S}_{n+2}^x) - \hat{S}_n^x (\hat{S}_{n-2}^x + \hat{S}_{n+2}^x)].
\]

(2)

For large magnitude \(S = |\hat{S}_n|\) of the spin, we may adopt the approximations

\[
\hat{S}_n^+ = S \alpha_n, \quad \hat{S}_n^- = S \alpha_n^*, \quad \hat{S}_n^z = S \sqrt{1 - |\alpha_n|^2},
\]

(3)

where \(\alpha_n\) is a complex function and \(\alpha_n^*\) its complex conjugate. Substituting (3) in (2), we obtain

\[
i \frac{\partial \alpha_n}{\partial t} = 2AS \sqrt{1 - |\alpha_n|^2} - J_1 S \left((\alpha_{n+1} + \alpha_{n-1}) \sqrt{1 - |\alpha_n|^2} - \alpha_n \left(\sqrt{1 - |\alpha_{n+1}|^2} + \sqrt{1 - |\alpha_{n-1}|^2}\right) \right)

- J_2 S \left((\alpha_{n+2} + \alpha_{n-2}) \sqrt{1 - |\alpha_{n+2}|^2} - \alpha_n \left(\sqrt{1 - |\alpha_{n+2}|^2} + \sqrt{1 - |\alpha_{n-2}|^2}\right) \right) + \mu H_0 \alpha_n.
\]

(4)

Further, to seek solutions in the form of amplitude-modulated waves we make use of the anzats

\[
\alpha_n(t) = \varphi_n(t)e^{i(kn\Delta x - \omega_0 t)},
\]

(5)

where the envelope functions \(\varphi_n(t)\) are slowly varying against the position and time, \(k\) and \(\omega_0\) are, respectively, the wave number and the frequency of the carrying wave and \(\Delta x = 1\) is the lattice constant, which sets the length scale. After application of (5), and in the continuum limit \(\varphi_n(t) \rightarrow \varphi(x, t)\), equations (4) transform into the modified nonlinear Schrödinger equation for \(\varphi = \varphi(x, t)\) (for details see e.g. [13,14])

\[
i \left(\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x}\right) = (\omega_0 - \omega)\varphi - b_k S \frac{\partial^2 \varphi}{\partial x^2} + g_k S |\varphi|^2 \varphi
\]

(6)

with \(\omega_0 = \mu H_0 - 2g_k S\) — the characteristic frequency, \(v = 2S(J_1 \sin k + J_2 \sin 2k)\) — the group velocity of the carrier wave, \(b_k = J_1 \cos k + 4J_2 \cos 2k\) — the dispersion coefficient and \(g_k = J_1 (\cos k - 1) + J_2 (\cos 2k - 1) - A\). Remark that the group velocity, the dispersion coefficient and \(g_k\) depend on \(J_1\) and \(J_2\).
The bright soliton solution, \( g_k \cos k < 0 \), of equation (6) that is fulfilled under the condition

\[ \omega = \omega_0 - \frac{b_k S}{L^2} \]

reads

\[ \varphi(x, t) = \varphi_0 \text{sech} \frac{x - vt}{L}, \quad (7) \]

where

\[ \varphi_0^2 = -\frac{2b_k}{g_k L^2}, \]

and the parameters \( L \gg \Delta x \) and \( v \) are the width and the velocity of the soliton, respectively, where \( v \) is determined by the imaginary part of (6). Note that the soliton given by (7) satisfies the boundary conditions \( \lim_{x \to \pm \infty} |\varphi(x, t)| \to 0 \).

3. Qubit (spin \( \frac{1}{2} \)) dynamics

The equations for the evolution of the components of the Bloch vector for the spin-\( \frac{1}{2} \) particle are [12]

\[ i\partial_t (a^-) = \left( d_z S \sqrt{1 - |\varphi|^2} - vH_0 \right) a^- - d_{xy} S \varphi^* e^{-i(kj - \omega t)} a^z, \quad (8a) \]

\[ i\partial_t (a^+) = -\left( d_z S \sqrt{1 - |\varphi|^2} - vH_0 \right) a^+ + d_{xy} S \varphi e^{i(kj - \omega t)} a^z, \quad (8b) \]

\[ i\partial_t (a^z) = \frac{d_{xy}}{2} \left( S \varphi^* e^{-i(kj - \omega t)} a^+ - S \varphi e^{i(kj - \omega t)} a^- \right). \quad (8c) \]

With the transformation

\[ a^\pm(t) = a^x \pm ia^y, \quad (9) \]

the system of equations (8) takes the final form

\[ i\partial_t (a^x) = -i \left( d_z S \sqrt{1 - |\varphi|^2} - vH_0 \right) a^y + id_{xy} S \varphi \sin(kj - \omega t) a^z, \quad (10a) \]

\[ i\partial_t (a^y) = i \left( d_z S \sqrt{1 - |\varphi|^2} - vH_0 \right) a^x - id_{xy} S \varphi \cos(kj - \omega t) a^z, \quad (10b) \]

\[ i\partial_t (a^z) = -id_{xy} S \varphi \sin(kj - \omega t) a^x + id_{xy} S \varphi \cos(kj - \omega t) a^y. \quad (10c) \]

4. Numerical results

We solve numerically (10) to gain insights into the dynamics of the Bloch vector \( \mathbf{a} \), assuming that at the initial moment \( \mathbf{a}_{t=0} = (0, 0, 1) \), i.e. the spin \( \frac{1}{2} \) is collinear to the unit vector of the z-axis. Fix the energy scale \( J_1 = A = 1 \), the spin in units of \( \hbar \), i.e. \( S = \hbar \), and the length scale in units of the lattice constant. Therefore, we may express all other quantities in dimensionless units. The interaction between the external magnetic field and a single spin in the chain is \( \mu H_0 = 0.5 \) and the one for the qubit is \( \nu H_0 = 1.5 \). These interacting terms are taken to be of the order of the exchange interaction in the spin chain and satisfy the inequality \( \mu < \nu \). The site \( j \) where the qubit-spin chain interaction takes place is arbitrary and does not affect the results in our simulations we set \( j = 10 \). The peak of the soliton is attained at \( x = vt \). For \( x = 0 \) and \( t = 0 \), the peak of the soliton is located at the site labeled by \( j \). Let us consider the situation when the qubit-soliton interaction occurs exactly at \( t = 0 \)

\[ \begin{cases} d_{xy} = d_z = 0, & \text{for } t < 0, \\ d_{xy} > 0 \text{ and } d_z \geq 0, & \text{for } t \geq 0. \end{cases} \quad (11) \]
In the numerical investigations the soliton parameters are chosen to be $k = \frac{\pi}{30}$ and $L = 6$. The profile for fixed $x$ ($t \geq 0$) is depicted in figure 1. Here, we are interested in the long-time behavior of the qubit, when the soliton peak is far away from the site $j$ and the $z$ coordinate of the qubit has already reached its stationary value.

![Figure 1](image1.png)

**Figure 1.** Soliton profile $\phi(0, t)$ for $t \geq 0$, $k = \frac{\pi}{30}$, $L = 6$, $S = 1$, $J_1 = A = 1$ and $J_2 = 0.2$.

Figure 2. Qubit dynamics for $d_{xy} = 20$, $d_z = 0$ and different values of next-nearest-neighbor interaction $J_2$. Panels (a) - (c): the $z$-component of the Bloch vector. Panels (d) - (f): the qubit’s state evolution on the Bloch sphere. $J_2 = 0$ for (a) and (d); $J_2 = 0.1$, for (b) and (e); $J_2 = 0.2$ for (c) and (f).

![Figure 2](image2.png)

To study the effect of the interaction of second neighbors on the qubit dynamics we consider a few limiting cases. Let us first consider the case of strong anisotropy in the soliton-qubit interaction ($d_{xy} = 20$, $d_z = 0$). The time evolution of the $z$-component of the Bloch vector and its trajectory on the surface of
the Bloch sphere are plotted in figure 2 (a) – (c) and (d) – (f), respectively. It can be seen from figure 2 (a) - (c), that when $a_z(0) = 1$, then $a_z(t)$ oscillates very fast, and for large enough time these oscillations fade away around some asymptotic value. We choose to determine the asymptotic values for the Bloch vector at $t = 200$. The path on the Bloch sphere (figure 2 (d) - (f)) covers a large area on surface of the sphere, while the asymptotic value is seen at the meridian where the intensity of the curves has the highest density. In figure 2 (a), (d), we consider only the nearest-neighbor interaction, then $a_z(t)$ has an asymptotic value of 0.20 for $t = 200$ and the soliton amplitude $q_0 = 0.23$. Switching on the interaction with second neighbors $J_2$ in the ferromagnetic chain increases the soliton amplitude $q_0 = 0.28$ and we find smaller asymptotic value of $a_z(200) = 0.17$, i.e. $a_z(t)$ is asymptotically closer to zero, corresponding to the state of equal superposition (figure 2 (b), (e)). Further increase of $J_2$ leads to a higher value of the soliton amplitude $q_0 = 0.31$ and to the decrease of $a_z(200)$ towards 0.15 as seen from figures 2 (c) and (f). All values of $a_z(200)$ and $q_0$ are given with an accuracy of two decimal digits and they show that the effect of the next-nearest neighbors is relatively small, with an average decrease of several percentages.

![Figure 3. Qubit dynamics for $d_{xy} = 20$, $d_z = 5$ and different values of next-nearest-neighbor interaction $J_2$. Panels (a) - (c): the $z$-component of the Bloch vector. Panels (d) - (f): the qubit's state evolution on the Bloch sphere. $J_2 = 0$ for (a) and (d); $J_2 = 0.1$, for (b) and (e); $J_2 = 0.2$ for (c) and (f). The asymptotic values for $a_z(t = 200)$ are 0.78 for (a) and (d); 0.72 for (b) and (e); and 0.67 for (c) and (f).](image)

For relatively moderate anisotropy of the qubit-soliton interaction ($d_{xy} = 20$, $d_z = 5$) (figure 3 (a)-(f)), the ferromagnetic second neighbor interaction again reduces the asymptotic value of the qubit’s spin $\frac{1}{2}$ projections on the $z$-axis closer to zero. The amplitudes of the soliton remain unchanged and for this reason (figure 3 is quite similar to figure 2 for strong anisotropy, except that the values for $a_z(200)$ are very close to one. This leads us to the conclusion that smaller anisotropy of the qubit-soliton interaction is associated to higher asymptotic value of $a_z$.

5. Conclusions
We have studied the effect of an anisotropic interaction between a propagating in a spin chain soliton and an external spin $\frac{1}{2}$ that plays the role of a qubit. The spin chain is described by a ferromagnetic Heisenberg
model Possessing an on-site spin anisotropy in conjunction with nearest- and next-nearest-neighbor spin-spin interactions. The equations governing the qubit dynamics are derived and the behavior of the $z$-projection of the associated Bloch vector is analyzed. It is shown that an increase of the magnitudes of the next-nearest-neighbor interaction and the anisotropy in the qubit-spin chain interaction is associated to an increasing probability for the spin $\frac{1}{2}$ flip.

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