Toeplitz Quantization of an Algebra with Conjugation

Stephen Bruce Sontz
Centro de Investigación en Matemáticas, A. C.
(CIMAT)
Guanajuato, Mexico

Abstract

Toeplitz quantization is defined in a general setting in which the symbols are the elements of a possibly non-commutative algebra with a conjugation and a possibly degenerate inner product. There need not be a measure space in this theory, though the inner product could arise from a measure. This theory is based on the mathematical structures in recent work of the author which dealt with specific examples of this Toeplitz quantization in the non-commutative case. Creation and annihilation operators are defined as certain Toeplitz operators, and their commutation relations are discussed. Due to the possibility of non-commuting symbols, there are now definitions for two types of anti-Wick quantization, and these two definitions are equivalent in the commutative case. This Toeplitz quantization satisfies one of these definitions, but not necessarily the other. This can be considered as a type of second quantization.

Keywords: Toeplitz quantization, anti-Wick quantization, creation and annihilation operators, canonical commutation relations

MSC 2010: 47B35, 81S99

1 Introduction

The history of Toeplitz operators covers a bit over one hundred years and includes many major works, far too numerous to mention here. For a recent reference that will give the reader some first links to that extensive literature, see Section 3.5 in [17]. Speaking for myself, the papers [6], [7] and [11] have been rather influential. But the study of Toeplitz operators with symbols coming from a non-commutative algebra seems to be limited mostly to cases where $\mathcal{A}$ is a matrix algebra or is some other quite specific non-commutative algebra such as in two recent works, [21] and [22], of the author. The papers [21] and [22] can be considered as two rather elaborated examples of the
theory presented here. That study is continued in this paper, but in a much more general setting intended to clarify the mathematical structures at play in those two examples. We will discuss the relation of our theory to the other non-commutative cases in the last section. One goal of this paper is to encourage more research of the theory of Toeplitz quantization of not necessarily commutative algebras, which of course includes non-commutative geometry as a special case.

The paper is organized as follows. After presenting the foundations of this theory in the next section we define and analyze in Section 3 the Toeplitz quantization. In particular, Toeplitz operators are defined. In Section 4 we introduce two special cases of Toeplitz operators; they are the creation and annihilation operators. Then in Section 5 we study the relation of this Toeplitz quantization with two definitions of anti-Wick quantization, each of which arises naturally in a non-commutative setting. (These two definitions are equivalent in the commutative case.) This Toeplitz quantization always satisfies one of these definitions, which is of course then taken to be the ‘correct’ generalization of anti-Wick quantization for this setting. The notion of canonical commutation relations in this abstract context is discussed in Section 6. This last topic is treated in a very rudimentary manner, and it deserves further investigation. In the last section we conclude with a few remarks about our approach to Toeplitz quantization as contrasted with other approaches.

2 The Setting

We will study an algebra $\mathcal{A}$ over the complex numbers $\mathbb{C}$ with unit 1 and with an involutive, anti-linear conjugation (also called a $*$-operation), denoted by $f^*$ for $f \in \mathcal{A}$, together with a unital sub-algebra $\mathcal{P}$ of $\mathcal{A}$ (that is, with $1 \in \mathcal{P}$). We assume that $1^* = 1$. The algebra $\mathcal{A}$ will be the space of symbols for the Toeplitz operators which we will define later. So $\mathcal{A}$ is the ‘classical’ space we wish to quantize. For example, it could be an algebra of ‘functions’ on a non-commutative phase space, but in this paper we do not impose a Poisson structure on $\mathcal{A}$. The typical case that we have in mind is that $\mathcal{P}$ is not closed under the conjugation and that in fact the intersection $\mathcal{P} \cap \mathcal{P}^*$ is as small as it possibly could be, namely $\mathbb{C}1$. Here $\mathcal{P}^* := \{g^* \mid g \in \mathcal{P}\}$. However, we will make no hypothesis about $\mathcal{P} \cap \mathcal{P}^*$.

We do not assume that $\mathcal{A}$ is a $*$-algebra, namely that $(fg)^* = g^*f^*$ holds
for all \( f, g \in \mathcal{A} \). Also, we do not put any restriction on the dimensions of these vector spaces. The existence of a unit in \( \mathcal{A} \) is not an essential element of this theory and many of the results go through without assuming that it exists. The fact that we are using an algebra of symbols allows us to include non-commutative geometry as a special case of this theory. So \( \mathcal{A} \) could be any algebra that is considered to be a non-commutative space. The possibility of such a theory was raised, but not realized, in a remark in [2].

We suppose there is a sesquilinear, complex symmetric form (or inner product) \( \mathcal{A} \times \mathcal{A} \to \mathbb{C} \), which is denoted by \( \langle \cdot , \cdot \rangle _{\mathcal{A}} \). Our convention is that this form is anti-linear in the first entry and linear in the second. We allow the possibility that this inner product could be degenerate. However, we impose the requirement that when this inner product is restricted to \( \mathcal{P} \) it is positive definite. Therefore \( \mathcal{P} \) is a pre-Hilbert space. We let \( \mathcal{H} \) denote the completion of \( \mathcal{P} \). Therefore \( \mathcal{P} \) can be realized as a dense subspace in the Hilbert space \( \mathcal{H} \) with no loss of generality. We assume from now on that this is the case.

We suppose there exists an orthonormal indexed set \( \Phi = \{ \varphi _{j} \mid j \in J \} \subset \mathcal{P} \) that satisfies the following three conditions:

1. \( \Phi \) is a Hamel basis of \( \mathcal{P} \). (A Hamel basis of a vector space is a maximal, linearly independent subset of that vector space.) So,

\[
\mathcal{P} = \left\{ \sum _{k} a_{k} \varphi _{k} \mid a_{k} = 0 \text{ for all but finitely many } k \right\}.
\]

2. \( \Phi \) is an orthonormal basis of \( \mathcal{H} \). So,

\[
\mathcal{H} = \left\{ \sum _{k} a_{k} \varphi _{k} \mid \sum _{k} |a_{k}|^{2} < \infty \right\}.
\]

This is actually a consequence of Condition 1, but for the sake of clarity we state it separately.

3. For every \( f \in \mathcal{A} \), the set defined by \( \Phi _{f} := \{ \varphi _{j} \in \Phi \mid \langle \varphi _{j} , f \rangle _{\mathcal{A}} \neq 0 \} \) is finite. However, the cardinality of the set \( \Phi _{f} \) can depend on \( f \).

If \( \mathcal{P} \) is finite dimensional, as in [20], then \( \mathcal{P} = \mathcal{H} \) and such a subset \( \Phi \) exists. Notice that neither the \( * \)-operation nor the unit \( 1 \) is mentioned in these three conditions and so some of this theory can be developed without those structures, though the more interesting results do use those structures. (Cp. Proposition [23]) We now fix such a set \( \Phi \) and continue developing this
theory further. We address later the question if the subsequent theory really depends on the choice of $\Phi$. These conditions are technical in nature and could be modified to give similar theories. The first two conditions concern relations among $\Phi$, the pre-Hilbert space $\mathcal{P}$ and its completion $\mathcal{H}$.

The third condition has a quite different character, since it relates the sub-algebra $\mathcal{P}$ with the larger algebra $\mathcal{A}$. (Recall $\mathcal{A}$ will be the symbol space.) Intuitively, this condition says that $\mathcal{A}$ is not "too" big or equivalently that $\mathcal{P}$ is not "too" small. In particular, using Condition 3 the Toeplitz operators which we will define presently leave their common domain invariant and so the composition of two of them is completely straightforward. An alternative condition, weaker than Condition 3, would be to require:

$$3'. \text{ The series } \sum_j |\langle \varphi_j, f \rangle_{\mathcal{A}}|^2 \text{ converges for all } f \in \mathcal{A}. \tag{2.1}$$

With this weaker condition the common domain of our Toeplitz operators need not be invariant and so the composition of them becomes problematic. For example, the discussion of the canonical commutation relations becomes more complicated in this case.

This paragraph is a non-rigorous discussion which is only meant to serve as motivation. We first consider the formal sum

$$K := \sum_{j \in J} \varphi^*_j \otimes \varphi_j. \tag{2.1}$$

We emphasize that the cardinality of the index set $J$ is completely arbitrary. If we restrict $j$ in the previous sum to lie in some finite subset of $J$, this gives a well-defined element in $\mathcal{P}^* \otimes \mathcal{P}$, which in turn can be identified (essentially by thinking of Dirac’s bra-ket notation) as a finite rank projection operator mapping $\mathcal{P}$ to itself. If $\mathcal{H}$ has finite dimension, then (2.1) itself immediately identifies $K$ as the kernel of the identity operator of $\mathcal{P} = \mathcal{H}$. If $\mathcal{H}$ has infinite dimension, then (2.1) also identifies $K$ as the kernel of the identity operator of $\mathcal{P}$ provided that we interpret the infinite sum in the topology corresponding to the strong operator topology of bounded operators. So (2.1) is basically a resolution of the identity of $\mathcal{P}$. It seems reasonable to suppose that (2.1) could be replaced with a resolution of the identity of $\mathcal{P}$ by coherent states without changing this theory dramatically. However, we must emphasize that the Toeplitz quantization to be defined below is not the coherent state quantization (see [10]) associated to (2.1). The latter quantization in this setting maps the function $\alpha : J \rightarrow \mathbb{C}$ to the operator
associated to \( \sum_{j \in J} \alpha(j) \varphi_j^* \otimes \varphi_j \), modulo the usual technical details about convergence of the sum. Moreover, the set of all such \( \alpha \)'s forms a commutative algebra, while we will quantize the possibly non-commutative algebra \( A \). A formal computation now gives for all \( f \in P \) that

\[
\langle K, f \rangle_A = \langle \sum_j \varphi_j^* \otimes \varphi_j, f \rangle_A = \sum_j \langle \varphi_j^* \otimes \varphi_j, f \rangle_A \varphi_j.
\]

These remarks motivate this formal definition for all \( f \in P \):

\[
\langle K, f \rangle_A := \sum_j \langle \varphi_j, f \rangle_A \varphi_j.
\]

Even though the conjugation was used to motivate this definition, note that the conjugation does not appear in the definition. By Condition 3 the sum on the right side has only finitely many non-zero terms. It gives us an element in \( P \), since each \( \varphi_j \in P \). Moreover, since \( f \in P \) we have that

\[
\sum_j \langle \varphi_j, f \rangle_A \varphi_j = f,
\]

which is an elementary result. So \( \langle K, f \rangle_A = f \) and thus \( K \) can also be viewed formally as a generalized reproducing kernel for \( P \). While all the material of this paragraph can be developed rigorously (for example, following the presentations in [20] or [22]), for the moment we merely wished to give an idea of what the set \( \Phi \) is good for.

It is natural to require that the inner product in \( A \) has this relation with the conjugation in \( A \):

\[
\langle f, g \rangle_A^* = \langle f^*, g^* \rangle_A
\]

(2.2)

for all \( f, g \in A \). This simply means that \( f \mapsto f^* \) is an anti-unitary map of \( A \) to itself. This condition is satisfied by the paragrassmann algebras (see [20]) and by the complex quantum plane (see [22]). Here is an immediate consequence of this requirement.

**Proposition 2.1** \( P^* \) is a pre-Hilbert space with respect to the restriction of the inner product \( \langle \cdot, \cdot \rangle_A \) to it.

**Sketch of Proof:** The set \( \Phi^* = \{ \varphi_j^* \mid j \in J \} \) is an orthonormal set in \( P^* \) by (2.2). It is left to the reader to prove that \( \Phi^* \) is a Hamel basis of \( P^* \) as
well. Then it follows that the inner product $\langle \cdot, \cdot \rangle_A$ restricted to $P^*$ is positive definite, and so $P^*$ is a pre-Hilbert space. ■

We denote the completion of the pre-Hilbert space $P^*$ by $H^*$. There is an identification as inner product spaces between the pair of spaces $(P, H)$ and the pair of spaces $(P^*, H^*)$ induced by $P \ni f \mapsto f^* \in P^*$. We sometimes refer to $H$ as the holomorphic space (or Segal-Bargmann space) and to $H^*$ as the anti-holomorphic space (or anti-Segal-Bargmann space). It turns out that these designations are completely arbitrary and can be reversed with absolutely no loss of rigor nor (if one is savvy enough) of intuition. Since $P \cap P^*$ consists either way of elements which, according to this classification, are both holomorphic and anti-holomorphic, we see the intuition behind the condition $P \cap P^* = \mathbb{C}1$. But we continue with $P \cap P^*$ being completely arbitrary.

Curiously, the statement “$P^*$ is a sub-algebra” could be false, though it is true for the examples in [21] and [22] (even when $A$ is not a $*$-algebra) or whenever $A$ is a $*$-algebra. In [21] $P$ is the holomorphic Hilbert space, while in [22] the sub-algebra $Pre(\theta)$ plays the role of $P$.

**Proposition 2.2** If $(A, P, \Phi, \langle \cdot, \cdot \rangle_A)$ satisfy Conditions 1–3, then it follows that $(A, P^*, \Phi^*, \langle \cdot, \cdot \rangle_A)$ satisfy Conditions 1–3.

**Sketch of Proof:** We have already commented that $\Phi^*$ satisfies Condition 1. And, as also noted, Condition 2 readily follows from Condition 1. That Condition 3 holds we leave to the reader as a quick exercise using (2.2). ■

A reasonable condition relating the inner product, the conjugation and the multiplication in $A$ turns out to be

$$\langle f_1 f_2^*, g_1 g_2^* \rangle_A = \langle f_1 g_2, f_2 g_1 \rangle_A$$  \hspace{1cm} (2.3)

for all $f_1, f_2, g_1, g_2 \in A$. However, in general this condition is not satisfied by the paragrassmann algebras (see [20]) nor by the complex quantum plane (see [22]). Nonetheless we will assume (2.3) for the time being.

If we take $A$ to be an algebra of (complex valued) functions contained in $L^2(\Omega, \mu)$, where $(\Omega, \mu)$ is a measure space, with the multiplication and conjugation defined pointwise, and the inner product on $A$ is the associated $L^2$ inner product

$$\langle f, g \rangle = \int_{\Omega} d\mu \, f^* g$$  \hspace{1cm} (2.4)
for \( f, g \in \mathcal{A} \), then (2.3) is trivially satisfied. The point is that the condition (2.3) provides some structure to the new, more general setting described here, while still including the classical case (2.4). Of course, to carry out the Toeplitz quantization in this case one would still have to choose a sub-algebra \( \mathcal{P} \) of \( \mathcal{A} \) as well as a set \( \Phi \) as above. (In many particular cases \( \Omega \) is a domain in \( \mathbb{C}^n \), and \( \mathcal{P} \) is the sub-algebra of holomorphic polynomials. So the symbol \( \mathcal{P} \) can be thought of as coming from ‘polynomial’ as well as from ‘pre-Hilbert’.) This example shows that we can base this Toeplitz quantization on a measure which also is an essential ingredient of coherent state quantization. (See [10].) However, we wish to emphasize again that the Toeplitz quantization presented in this paper does not require a measure.

The identities in the next proposition also serve to motivate (2.3).

**Proposition 2.3** The identity (2.3) is equivalent to the first identity in the following list. Also each identity in this list implies the next. Here \( f, g, h, f_1, f_2, g_1, g_2 \in \mathcal{A} \).

1. \( \langle f_1 f_2^*, g_1 g_2^* \rangle_A = \langle f_1 g_1^*, g_2^* f_2 \rangle_A \).
2. \( \langle f, gh \rangle_A = \langle fg^*, h \rangle_A = \langle fh^*, g \rangle_A \).
3. \( \langle f, gh \rangle_A = \langle h^*, f^* g \rangle_A = \langle g^*, hf^* \rangle_A \).
4. \( \langle f, g \rangle_A^* = \langle f^*, g^* \rangle_A \).

Notice that the last identity is (2.2).

Moreover, if \( \mathcal{A} \) is a \(*\)-algebra, then Part 3 implies one of the identities in Part 2. (Details in the proof.)

**Remark:** I conjecture that none of the statements in this list implies the previous statement, since it seems (to me!) that each statement says less than the previous one. But I have not constructed counterexamples.

**Proof:** Part 1: Applying (2.3) twice yields

\[
\langle f_1 f_2^*, g_1 g_2^* \rangle_A = \langle f_1 g_2^*, f_2^* g_1 \rangle_A = \langle f_1 g_1^*, g_2^* f_2 \rangle_A.
\]

Similarly, applying Part 1 twice gives us (2.3).

Part 2: Using Part 1 we have that

\[
\langle f, gh \rangle_A = \langle f, gh \rangle_A = \langle fg^*, h^* \rangle_A = \langle fg^*, h \rangle_A.
\]
For the second equality we use (2.3), which is equivalent to Part 1, to get
\[ \langle f, gh \rangle_A = \langle f 1, gh \rangle_A = \langle fh^*, 1^* g \rangle_A = \langle fh^*, g \rangle_A. \]

Part 3: We note that
\[ \langle f, gh \rangle_A = \langle fg^*, h \rangle_A = \langle h, fg^* \rangle_A^* = \langle g^*, hf^* \rangle_A, \tag{2.5} \]
where we have used Part 2 twice. We now apply the identity just proved to the expression \( \langle g^*, hf^* \rangle_A \) itself to get \( \langle g^*, hf^* \rangle_A = \langle h^*, f^* g \rangle_A \).

Part 4: Taking \( h = 1 \) in equation (2.5) of Part 3 gives
\[ \langle f, g \rangle_A = \langle g^*, f^* \rangle_A = \langle f^*, g^* \rangle_A^*, \]
which implies Part 4 by taking the complex conjugate.

Finally, suppose that \( A \) is a \( \ast \)-algebra and the identities in Part 3 hold. Then we have
\[ \langle f, gh \rangle_A = \langle g^*, hf^* \rangle_A = \langle hf^*, g^* \rangle_A^* = \langle (hf^*)^*, g^* \rangle_A = \langle fh^*, g \rangle_A \]
for all \( f, g, h \in A \). Here we have used Part 3 in the first equality, Part 4 (which is a consequence of Part 3) in the third equality and the hypothesis that \( A \) is a \( \ast \)-algebra in the fourth equality. This proves one of the identities in Part 2. ■

A mnemonic for the identities in Parts 1 to 3 as well as (2.3) is that one cyclically permutes three factors and introduces a conjugation on each factor that moves from one side of the comma to the other side. But be careful, since one of those factors is 1 in Part 2.

**Proposition 2.4** For all \( f, g, h \in A \) we have the identity
\[ \langle fg, h \rangle_A = \langle (f^* g^*)^*, h \rangle_A \]
and so all of the elements \( fg - (f^* g^*)^* \) are in \( A^\perp \). In particular if \( A \) is a \( \ast \)-algebra, then all of the commutators \( fg - gf \) are in \( A^\perp \) and, provided that \( A \) is also not commutative, then the inner product is degenerate.

**Remark:** We will not use this property in this paper, but we feel it merits mention. Anyway it says degenerate inner products are not to be shunned.
Proof: By Parts 2, 3 and 4 of Proposition 2.3 we get

\[
\langle fg, h \rangle_A = \langle f, g^* h \rangle_A = \langle h^*, f^* g^* \rangle_A = \langle f^* g^*, h^* \rangle_A = \langle (f^* g^*)^*, h \rangle_A.
\]

Before quantizing the ‘classical’ algebra \(A\), we note that it need not be commutative and so in some sense already may have a ‘quantum’ nature. So we wish to make some remarks about the probabilistic interpretation of \(A\) as a quantum object. The following comments are fairly well known, but merit repetition because of their relevance in the current context. In the spirit of the rest of this paper we will use linear functionals, which will be interpreted as expectation values, since we are not using measures. First, we say that an element \(\phi \in A\) is a state if \(\langle \phi, \phi \rangle_A = 1\). Then given such a state we define the expectation with respect to \(\phi\),

\[
E_\phi : A \to \mathbb{C},
\]

to be \(E_\phi(g) := \langle \phi, g\phi \rangle_A\) for all \(g \in A\). Then \(E_\phi(1) = 1\) immediately follows. This looks like standard material in the theory of \(C^*\)-algebras. But note for example that since the inner product \(\langle \cdot, \cdot \rangle_A\) is not assumed to be positive definite, there may be no states \(\phi \in \mathcal{P}^1\). Moreover, when we consider the self-adjoint case \(g = g^*\), the situation is complicated due to the possible lack of commutativity. Using Part 2 of Proposition 2.3 we see that for \(g = g^* \in A\) we have

\[
E_\phi(g) = \langle \phi, g\phi \rangle_A
= \langle \phi g^*, \phi \rangle_A
= \langle \phi g, \phi \rangle_A
= \langle \phi, \phi g \rangle_A^*.
\]

Now, if the state \(\phi\) is in the center of \(A\), we can continue getting

\[
E_\phi(g) = \langle \phi, \phi g \rangle_A^*
= \langle \phi g, \phi \rangle_A^*
= E_\phi(g)^*.
\]

which is to say that \(E_\phi(g)\) is real. But without this extra hypothesis on the state \(\phi\) it seems that we are in the doldrums. And this hypothesis is extremely strong, since the center of \(A\) could be simply \(\mathbb{C}1\). Moreover, the functional \(E_\phi\) need not be positive (that is, \(E_\phi(gg^*) \geq 0\) for all \(g \in A\), and
so is not necessarily a state in the usual sense of that term. We leave the
details of this to the reader.

Since this theory is still in its preliminary stages, we have presented an
extended discussion about the interrelations among the product, conjugation
and inner product. While (2.3) seems to be a reasonable starting point,
it does not include the motivating examples in [21] and [22]. In the rest
of this paper we will only use the identities in the following corollary of
Proposition 2.3. Note that these identities only differ from two identities
in Proposition 2.3 by the restriction that the elements $f_1$ and $f_2$ lie in the
sub-algebra $\mathcal{P}$.

**Corollary 2.1** For $f_1, f_2 \in \mathcal{P}$ and $g \in \mathcal{A}$ we have

1. $\langle f_1, f_2 g \rangle_{\mathcal{A}} = \langle f_1 g^*, f_2 \rangle_{\mathcal{A}}$.
2. $\langle f_1, f_2 g \rangle_{\mathcal{A}} = \langle f_1 f_2^*, g \rangle_{\mathcal{A}}$.

The point in presenting this Corollary separately is that the identities in it
hold for the examples in [21] and [22]. So these two examples are special
cases of the theory in the rest of this paper.

### 3 Toeplitz Quantization

First we take any $g \in \mathcal{A}$ and use it to define a linear map

$$M_g : \mathcal{P} \to \mathcal{A}$$

by $M_g \psi := \psi g$ for all $\psi \in \mathcal{P}$. Notice that $\psi g \in \mathcal{A}$, since it is a product
of two elements in the algebra $\mathcal{A}$. In this paper this is the main use of
the multiplication of $\mathcal{A}$. So a bilinear map $\mathcal{P} \times \mathcal{A} \to \mathcal{A}$ could be used instead of
the multiplication $(\psi, g) \mapsto \psi g$. This map would have to satisfy some other
conditions as well to make the theory work out. We will not go into further
details about this more general approach.

We next wish to use the kernel $K$ to extend the identity map on $\mathcal{P}$ to
a projection map $P_K : \mathcal{A} \to \mathcal{A}$. The technique is standard in analysis.
We simply use the same formula to define a different operator, where the
difference consists in using a different domain of definition. So we define for
$f \in \mathcal{A}$:

$$P_K f := \sum_j \langle \varphi_j, f \rangle_{\mathcal{A}} \varphi_j.$$

(3.1)
Of course, by our previous discussion we have \( P_K f = f \) provided that \( f \in \mathcal{P} \). Now for \( f \in \mathcal{A} \) we have assumed that only finitely many of the coefficients \( \langle \varphi_j, f \rangle_{\mathcal{A}} \) are non-zero. So the sum on the right side of (3.1) is effectively over a finite number of terms and so \( P_K f \in \mathcal{P} \) for all \( f \in \mathcal{A} \), that is, \( P_K : \mathcal{A} \to \mathcal{P} \).

**Theorem 3.1** \( P_K \) is a projection, that is \( P_K^2 = P_K \), and is symmetric with respect to the inner product on \( \mathcal{A} \), that is

\[
\langle P_K f, g \rangle_{\mathcal{A}} = \langle f, P_K g \rangle_{\mathcal{A}}
\]

(3.2)

for all \( f, g \in \mathcal{A} \). If the inner product is non-degenerate, then we can write (3.2) as \( P_K^* = P_K \), where \( P_K^* \) is the unique adjoint operator of \( P_K \).

**Proof:** First we note that \( P_K f \in \mathcal{P} \) for all \( f \in \mathcal{A} \) and that \( P_K \) acts as the identity on \( \mathcal{P} \). So \( P_K(P_K f) = P_K f \) for all \( f \in \mathcal{A} \), thereby proving that \( P_K^2 = P_K \). Next, one readily calculates that each side of (3.2) is equal to \( \sum_j \langle f, \varphi_j \rangle_{\mathcal{A}} \langle \varphi_j, g \rangle_{\mathcal{A}} \). And this is a sum with only finitely many non-zero terms, and so there is no problem with the convergence of this sum. \( \blacksquare \)

We now return to the question whether this theory depends on the choice of orthonormal set \( \Phi = \{ \varphi_j | j \in J \} \). The point is that this set is used to define \( P_K \). But suppose that \( \Psi = \{ \psi_j | j \in J \} \) is another orthonormal set in \( \mathcal{P} \) that is also a Hamel basis of \( \mathcal{P} \). And let \( P_K \) be the projection operator defined above using the set \( \Phi \). We temporarily denote \( P_K \) by \( P_K^\Phi \) to indicate its dependence on \( \Phi \). Suppose that \( f \in \mathcal{A} \). Then as we have seen \( P_K^\Phi f \in \mathcal{P} \). So we can expand \( P_K^\Phi f \) uniquely in the Hamel basis \( \Psi \) of \( \mathcal{P} \) to get

\[
P_K^\Phi f = \sum_k a_k \psi_k
\]

(3.3)

with all but finitely many \( a_k = 0 \). Taking the inner product of this with \( \psi_j \) yields

\[
a_j = \langle \psi_j, P_K^\Phi f \rangle_{\mathcal{A}} = \langle P_K^\Phi \psi_k, f \rangle_{\mathcal{A}} = \langle \psi_k, f \rangle_{\mathcal{A}}
\]

(3.4)

for all \( j \in J \), since \( \Psi \) is orthonormal and \( P_K^\Phi \) acts as the identity of \( \mathcal{P} \). So the set \( \Psi_f := \{ \psi_j | \langle \psi_j, f \rangle_{\mathcal{A}} \neq 0 \} \) is finite.

Substituting (3.4) back into (3.3) we see for all \( f \in \mathcal{A} \) that

\[
P_K^\Phi f = \sum_k \langle \psi_k, f \rangle_{\mathcal{A}} \psi_k = P_K^\Psi f
\]

11
using the (obvious but unstated) definition of the projection operator $P_K^\Phi$ defined by the set $\Psi$. In short, the definition of $P_K^\Phi$ does not depend on the choice of the set $\Phi$. Since the only essential use of the set $\Phi$ is exactly to define $P_K^\Phi$, we now have shown that the subsequent theory does not depend on the particular choice $\Phi$. And so we revert to our original notation: $P_K$.

Since $\Phi$ will not make another appearance in this paper and it was used only to define $P_K$, an alternative approach to this theory is to start with $\mathcal{A}$, $\mathcal{P}$ and the inner product, all as above. But instead of $\Phi$, one introduces an operator $P : \mathcal{A} \to \mathcal{P}$ straightaway with the properties in Theorem 3.1.

**Definition 3.1** For any $g \in \mathcal{A}$ we can form the composition of linear maps

$$
\mathcal{P} \xrightarrow{M_g} \mathcal{A} \xrightarrow{P_K} \mathcal{P}
$$

which we define to be the Toeplitz operator associated with the symbol $g \in \mathcal{A}$, denoted by $T_g := P_K M_g$.

Notice that $T_g$ is defined in the dense domain $\text{Dom}(T_g) := \mathcal{P}$, which does not depend on $g$. Furthermore, $\mathcal{P}$ is invariant under the action of $T_g$ and so we can always compose any finite number of Toeplitz operators. This is not a usual situation in Toeplitz operator theory in function spaces, where the domain typically depends on the symbol and where that domain is not necessarily invariant.

The symbol $g$ in $T_g$ is known as the upper symbol in Lieb’s paper [16] and as the contravariant symbol in Berezin’s paper [5]. The corresponding lower or covariant symbol of those papers does not seem to have an exact analogue in this general non-commutative setting. For example, see [4] where lower symbols are introduced in a non-commutative setting that includes coherent states. So it may well be a worthwhile avenue for future research to modify the present theory so as to include coherent states as well.

The linear map $T : g \mapsto T_g$ for $g \in \mathcal{A}$ is called the **Toeplitz quantization**. Since $\mathcal{A}$ can be a non-commutative algebra, we do include the possibility that the symbols of the Toeplitz quantization do not commute among themselves. This is in rather sharp contrast to the theories of Toeplitz operators in classical analysis where the symbols are real or complex valued functions with multiplication defined pointwise. This is a strictly mathematical point of view of what is being done here.

However, from a physics point of view, we are quantizing the (possibly non-commutative, i.e., quantum) space $\mathcal{A}$ by densely defined operators acting
in the quantum Hilbert space $\mathcal{H}$. The space $\mathcal{A}$ could be a configuration space or a phase space or just about anything else. When $\mathcal{A}$ is non-commutative this can be considered as a type of second quantization (that is, it is the quantization of something that is already quantum), though the result of the quantization is not a quantum field theory by any means. However with a little bit more work, this Toeplitz quantization can be realized as a functor and so is in accord with Nelson’s maxim that second quantization is a functor. (See Section X.7 in [19].)

Before stating the next theorem, we recall the standard definition

$$ S^\perp := \{ f \in \mathcal{A} \mid \langle f, g \rangle_{\mathcal{A}} = 0 \quad \forall g \in S \}, $$

where $S$ is any subset of $\mathcal{A}$.

**Theorem 3.2** The kernel of the Toeplitz quantization map $T$ is given by

$$ \ker T = (\mathcal{PP}^*)^\perp. $$

Therefore $T$ is a monomorphism if and only if $(\mathcal{PP}^*)^\perp = 0$.

**Proof:** The following statements are equivalent:

- $g \in \ker T$.
- $T_g = 0$.
- $T_g \phi = 0$ for all $\phi \in \mathcal{P}$. (Recall that $T_g : \mathcal{P} \to \mathcal{P}$.)
- $\langle \psi, T_g \phi \rangle_{\mathcal{A}} = 0$ for all $\psi, \phi \in \mathcal{P}$. (Since $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is positive definite on $\mathcal{P}$.)
- $\langle \psi, P_K M_g \phi \rangle_{\mathcal{A}} = 0$ for all $\psi, \phi \in \mathcal{P}$.
- $\langle P_K \psi, \phi g \rangle_{\mathcal{A}} = 0$ for all $\psi, \phi \in \mathcal{P}$.
- $\langle \psi, \phi g \rangle_{\mathcal{A}} = 0$ for all $\psi, \phi \in \mathcal{P}$.
- $\langle \psi \phi^*, g \rangle_{\mathcal{A}} = 0$ for all $\psi, \phi \in \mathcal{P}$.
- $g \in (\mathcal{PP}^*)^\perp$.

Here we have used Part 2 of Corollary 2.1 in the next to last equivalence. So we have shown that $\ker T = (\mathcal{PP}^*)^\perp$. And then the second statement immediately follows. ■

**Corollary 3.1** If $\mathcal{A} = \mathcal{PP}^*$ and the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is non-degenerate, then the Toeplitz quantization map $T$ is a monomorphism.
Remark: The condition $A = PP^*$ is another way of saying that $A$ is not ‘too’ big. Given this condition, we definitely would not wish to impose as well the condition $P = P^*$, since then $A$ would be too small.

We can take the co-domain of the Toeplitz quantization $T$ to be the vector space

$$\mathcal{L} \equiv \mathcal{L}(\mathcal{H} : \mathcal{P}) := \{ S : \mathcal{P} \to \mathcal{P} \mid S \text{ is linear} \}$$

of densely defined linear operators in $\mathcal{H}$ with common invariant domain $\mathcal{P}$. So, $\mathcal{L}(\mathcal{H} : \mathcal{P})$ is an algebra under composition, though it does not have a natural norm putting it on the same footing as the algebra $A$. Then the Toeplitz quantization

$$T : A \to \mathcal{L}(\mathcal{H} : \mathcal{P})$$

is a linear map between algebras. However, it is not expected to be an algebra morphism in any reasonable set-up. Nonetheless, it does have some properties related to the multiplication as well as some other nice properties. Parts 2 and 3 below can be false without the hypothesis that $\mathcal{P}$ is a sub-algebra.

Theorem 3.3 The following hold:

1. $T_1 = I_P$ where $I_P$ is the identity map of $\mathcal{P}$.

2. If $g \in \mathcal{P}$, then $T_g = M_g$.

3. If $g \in A$ and $h \in \mathcal{P}$, then $T_gT_h = T_{hg}$.

4. Suppose that $f_1, f_2 \in \mathcal{P}$ and $g \in A$. Then

$$\langle T_gf_1, f_2 \rangle_A = \langle f_1, T_g^*f_2 \rangle_A. \quad (3.5)$$

This can also be expressed by saying that $T_g^* \subset (T_g)^*$ or equivalently that $T_g \subset (T_g^*)^*$. This shows a compatibility between the $*$-operation in the algebra $A$ and the adjoint operation of densely defined operators.

5. If $g = g^*$, then $T_g$ is a symmetric operator.

Proof: For Part 1, we note that $M_1$ is just the inclusion map of $\mathcal{P}$ into $A$. Since $P_K$ acts as the identity on $\mathcal{P}$, we get $T_1 = P_KM_1 = I_P$.

For Part 2 we remark that the range of $M_g$ is contained in $\mathcal{P}$ for $g \in \mathcal{P}$, since $M_g\psi = \psi g \in \mathcal{P}$ for all $\psi \in \mathcal{P}$. Here we are using the hypothesis that $\mathcal{P}$ is a sub-algebra of $A$. But $P_K$ acts as the identity on $\mathcal{P}$. So, $T_g = P_KM_g = M_g$. 

14
For Part 3 we first note for all \( g \in \mathcal{A} \) and all \( h, \phi \in \mathcal{P} \) that
\[
M_g M_h \phi = (M_h \phi)g = (\phi h)g = \phi (hg) = M_{hg} \phi.
\]
Note that \( h \in \mathcal{P} \) was used here to guarantee that the composition \( M_g M_h \) makes sense. So, using this and Part 2 we see that
\[
T_g T_h = P_K M_g P_K M_h = P_K M_g M_h = P_K M_{hg} = T_{hg}.
\]
For Part 4 we suppose \( f_1, f_2 \in \mathcal{P} \) and \( g \in \mathcal{A} \). Then we calculate
\[
\langle T_g f_1, f_2 \rangle_A = \langle P_K M_g f_1, f_2 \rangle_A = \langle f_1 g, P_K f_2 \rangle_A = \langle f_1 g, f_2 \rangle_A,
\]
where the last equality follows from \( f_2 \in \mathcal{P} \). Similarly we see that
\[
\langle f_1, T_g^* f_2 \rangle_A = \langle f_1, P_K M_g^* f_2 \rangle_A = \langle P_K f_1, f_2 g^* \rangle_A = \langle f_1, f_2 g^* \rangle_A,
\]
where now the last equality follows from \( f_1 \in \mathcal{P} \). Finally, we use the identity \( \langle f_1, f_2 g^* \rangle_A = \langle f_1 g, f_2 \rangle_A \) from Part 1 of Corollary 2.1 to get the result.

Next, the two relations \( T_g^* \subset (T_g)^* \) and \( T_g \subset (T_g^*)^* \) follow immediately from (3.5). These relations are equivalent using the substitution \( g \mapsto g^* \).

Part 5 is an immediate consequence of Part 4 and the definition (see [18]) of a symmetric operator. ■

We now are presented with a classical problem in the functional analysis of densely defined operators, namely, in the case of a self-adjoint symbol \( g = g^* \) we have the symmetric, densely defined operator \( T_g \). One would like to know whether this operator has self-adjoint extensions and, if it does, how to explicitly classify them. In particular, it could be that \( T_g \) is self-adjoint or essentially self-adjoint for particular choices of self-adjoint \( g \). For example, \( T_1 = I_\mathcal{P} \) is essentially self-adjoint. Of course, the probabilistic interpretation of any self-adjoint extension of \( T_g \) would be based on its projection valued measure, just as is done in [18]. This last remark complements our previous discussion of the probabilistic interpretation of the algebra \( \mathcal{A} \). We will not discuss these questions further here.

Another mathematical possibility is that this Toeplitz quantization \( T \) could be iterated. Since \( \mathcal{L} \), the codomain of \( T \), is an algebra, we have one ingredient. But what will be the inner product on \( \mathcal{L} \) and what sub-algebra of \( \mathcal{L} \) would be the pre-Hilbert space with this inner product? And after addressing these questions, what would play the role of the set \( \Phi \)? So far I
have not been able to find adequate answers to these questions in this general setting. For example, a natural choice for an inner product of operators is the Hilbert-Schmidt inner product, which gives rise to the Hilbert space of Hilbert-Schmidt operators. This space could play the role of $\mathcal{H}$, while the role of the pre-Hilbert space could be played by the space $\mathcal{FR}$ of finite rank operators mapping $\mathcal{P}$ to itself. Then $\mathcal{FR}$ is also a sub-algebra of $\mathcal{L}$ but it does not contain the unit element $I_{\mathcal{P}}$ if $\mathcal{P}$ has infinite dimension. Also, letting $\ast$ denote the adjoint we have $\mathcal{FR} = \mathcal{FR}^\ast$, which is somewhat of a pathological situation since it intuitively means that every finite rank operator is both holomorphic and anti-holomorphic. The role of the set $\Phi$ could be played by the set of rank one operators $\{ |\varphi_j\rangle\langle\varphi_k| \mid j, k \geq 0 \}$. But here the train runs out of steam, since there is seemingly no natural way to put $\mathcal{L}$ into play. However, from a physics point of view there is little, if any, motivation for iterating a quantization.

We continue with some technical, but important, mathematical details.

**Theorem 3.4** For any $g \in \mathcal{P}$, the Toeplitz operator $T_g$ is closable and its closure $\overline{T_g}$ satisfies

$$\overline{T_g} = (T_g)^{**} \subset (T_{g^\ast})^\ast.$$  

**Proof:** All of this follows from basic functional analysis. (See [18].) For example, an operator $R$ is closable if and only if $Dom R^\ast$ is dense. But $Dom(T_g)^\ast \supset Dom T_{g^\ast} = \mathcal{P}$ and $\mathcal{P}$ is dense in $\mathcal{H}$. So, $Dom(T_g)^\ast$ is also dense and therefore $T_g$ is closable. Next $\overline{T_g} = (T_g)^{**}$ comes directly from [18]. Finally, $T_{g^\ast} \subset (T_g)^\ast$ implies that $(T_g)^{**} \subset (T_{g^\ast})^\ast$. \vspace{1mm} \hfill \blacksquare 

Since $T_g$ is closable, we would expect that in this setting there is a more explicit description of its closure $\overline{T_g} = (T_g)^{**}$. We leave this as a problem for future consideration.

Many other problems that are considered in the usual, classical Toeplitz quantization of functions also arise in this non-commutative context. These include finding necessary conditions as well as sufficient conditions for a Toeplitz operator to be bounded. Then given that a Toeplitz operator is bounded, there are open problems remaining to find necessary conditions as well as sufficient conditions for it to be compact, to be in a Schatten class, to be normal, to be unitary and so forth. However, these questions are known to depend on the particular properties of $\mathcal{P}$ and $\mathcal{A}$ in the case of classical Toeplitz operators and so may not be amenable to much more analysis in this general setting.
The material in this section deals with one of eight (at least!) possible Toeplitz quantizations that can be defined in this setting. For starters, one could change the definition of the operators $M_g$ to be multiplication on the left (instead of on the right) by $g$. This would give us a different, but very similar theory. Another possibility is to consider the Toeplitz quantization given by Toeplitz operators acting in the anti-holomorphic space $\mathcal{H}^\ast$ together with the two options for how the multiplication operators $M_g$ act, namely, on the right or on the left. This gives us two more Toeplitz quantizations provided that $\mathcal{P}^\ast$ is a sub-algebra of $\mathcal{A}$. Again, these are quite similar to the theory developed here. And yet another variation is to replace $M_g$ with $M_g^\ast$ in each of the previous four cases, thereby resulting in anti-linear Toeplitz quantizations. But these are all minor variations on the same theme and will not be discussed further.

4 Creation and Annihilation Operators

**Definition 4.1** Let $g \in \mathcal{P}$ be given. Then the creation operator associated to $g$ is defined to be

$$A^\dagger(g) := T_g$$

and the annihilation operator associated to $g$ is defined to be

$$A(g) := T_g^\ast.$$ 

These are reasonable definitions given that they are in accord with the usual meaning of these terms as exemplified in [11] and [22]. However, there are other normalizations used as well for these operators. One of these entails putting a factor of $\hbar^{-1/2}$ on the right sides of these definitions, where $\hbar$ denotes Planck’s constant. But we will postpone the introduction of Planck’s constant to a bit later. Notice that $g \mapsto A^\dagger(g)$ is linear (as already remarked) and that $g \mapsto A(g)$ is anti-linear. Also $A^\dagger(g) = T_g = M_g$ holds, since $g \in \mathcal{P}$. Since $A^\dagger(1) = A(1) = T_1 = I_\mathcal{P}$, we see that $I_\mathcal{P}$ is both a creation and an annihilation operator. In fact for any $g \in \mathcal{P} \cap \mathcal{P}^\ast$, one has $T_g = A^\dagger(g) = A(g^\ast)$ and so $T_g$ is both a creation and an annihilation operator.

One of the important contributions of Bargmann’s seminal paper [3] is that it realizes the creation and annihilation operators introduced by Fock as adjoints of each other with respect to the inner product on the Hilbert space which is nowadays called the Segal-Bargmann space. The creation operator
\(A^\dagger(g)\) and the annihilation operator \(A(g)\) also have this relation, modulo domain considerations, as we have already seen in Theorem 3.3, Part 4. Whether each is exactly the adjoint of the other as in [3] is an open question if \(\mathcal{P}\) has infinite dimension, but is trivially so for finite dimensional \(\mathcal{P}\).

5 Anti-Wick Quantizations

We now have the language needed to discuss whether this is an anti-Wick quantization, as is expected from a Toeplitz quantization. First recall that we have shown

\[T_{gh} = T_h T_g\]  \hspace{1cm} (5.1)

provided that \(g \in \mathcal{P}\) but with \(h \in \mathcal{A}\) being arbitrary. Because we are allowing non-commutative algebras, we are led to two definitions for ‘anti-Wick’ in this theory. These are clearly equivalent conditions if \(\mathcal{A}\) is commutative as the reader will soon appreciate.

**Definition 5.1** We say that \(T\) is an anti-Wick quantization if

\[T_{hg^\ast} = T_{g^\ast} T_h\]

for all \(g, h \in \mathcal{P}\). We say that \(T\) is an alternative anti-Wick quantization if

\[T_{g^\ast h} = T_{g^\ast} T_h\]

for all \(g, h \in \mathcal{P}\).

Notice that on the right side in both of these definitions we have the product of an annihilation operator \(T_{g^\ast}\) to the left of a creation operator \(T_h\). And so the right side is in anti-Wick order for each of these definitions. The naming of these two properties was determined only after proving the following results. What we deem to call the anti-Wick quantization turns out to be the ‘correct’ generalization of this notion to the present setting as the next result shows.

**Theorem 5.1** The Toeplitz quantization \(T\) is an anti-Wick quantization.

**Proof:** Take \(g, h \in \mathcal{P}\). Then \(T_{hg^\ast} = T_{g^\ast} T_h\), where we have used (5.1). ■

This clarifies why the examples in [21] and [22] are anti-Wick quantizations even though they arise in a non-commutative context. The longer, explicit computations given in those references are not needed as we can now see.

This theorem has several immediate consequences:

18
Corollary 5.1 If $\mathcal{A} = \mathcal{P}\mathcal{P}^*$, then one can write any Toeplitz operator as a sum of terms in anti-Wick order.

Corollary 5.2 If $\mathcal{A}$ is commutative, then the Toeplitz quantization $T$ is an alternative anti-Wick quantization.

The examples in [21] and [22] for $q \neq 1$ (which is the non-commutative case) are not alternative anti-Wick quantizations.

The last corollary has a partial converse.

Theorem 5.2 $T$ is not an alternative anti-Wick quantization if and only if there exist elements $g, h \in \mathcal{P}$ such that $T_{g^*h} \neq T_{hg^*}$.

Proof: As already shown $T_{g^*h} = T_{hg^*}$ is an identity for all $g, h \in \mathcal{P}$. Now by definition $T$ is not an alternative anti-Wick quantization if and only if

$$T_{g^*h} \neq T_{g^*}T_h$$

for some $g, h \in \mathcal{P}$. These two statements give the result. ■

Corollary 5.3 Suppose that there exists an element in $\mathcal{P}$ which does not commute with some element in $\mathcal{P}^*$ and that $T$ is a monomorphism. Then $T$ is not an alternative anti-Wick quantization.

Proof: By hypothesis there exist elements $g, h \in \mathcal{P}$ such that $g^*h \neq hg^*$. Since $T$ is a monomorphism, this implies that $T_{g^*h} \neq T_{hg^*}$. And now the previous theorem applies. ■

My hunch is that the Toeplitz quantization of some non-commutative algebra can be an alternative anti-Wick quantization, but I have not constructed an example. The results just presented indicate where not to look for such an example. This remains an open, though relatively minor, problem.

Just for completeness we read into the record two more related definitions.

Definition 5.2 We say that $T$ is a Wick quantization if

$$T_{hg^*} = T_h T_{g^*}$$

for all $g, h \in \mathcal{P}$. We say that $T$ is an alternative Wick quantization if

$$T_{g^*h} = T_h T_{g^*}$$

for all $g, h \in \mathcal{P}$.
These definitions are not expected in any way at all to describe a typical Toeplitz quantization. Their value lies in the possibility that some other types of quantizations of non-commutative algebras may have these properties. If the range of the Toeplitz quantization $T$ consists of operators which commute among themselves, then $T$ is trivially a Wick quantization. Of course, this condition on $T$ is not what one wants in a quantum theory and should be considered as a pathological condition.

To show more clearly that our definition of anti-Wick ordering compares well with the discussion of this topic in Theorem 8.2 in [11] we prove the next result. We say that the sub-algebra $P$ is $\ast$-friendly if $(p_1 \cdots p_n)^\ast = p_n^\ast \cdots p_1^\ast$ for all $p_1, \ldots, p_n \in P$. Returning to an earlier remark, we can see now that $P^\ast$ is a sub-algebra if $P$ is $\ast$-friendly, and moreover in this case the algebras $P$ and $P^\ast$ are anti-isomorphic via the conjugation. Clearly, if $A$ is a $\ast$-algebra, then $P$ is $\ast$-friendly.

**Theorem 5.3** Suppose that $g_1, \ldots, g_n, h_1, \ldots, h_m \in P$. Then

1. $T_{g_1 \cdots g_n} = T_{g_n} \cdots T_{g_1}$,
2. $T_{h_1^\ast \cdots h_m^\ast} = T_{h_m^\ast} \cdots T_{h_1^\ast}$ if $P$ is a $\ast$-friendly.
3. $T_{g_1 \cdots g_nh_1^\ast \cdots h_m^\ast} = T_{h_m^\ast} \cdots T_{h_1^\ast}T_{g_n} \cdots T_{g_1}$ if $P$ is $\ast$-friendly.

**Remark:** The extra hypothesis on $P$ in Parts 2 and 3 is not used elsewhere in this paper. In [21] and [22] the sub-algebra $P$ is $\ast$-friendly for all allowed values of the parameter $q$, namely, $q \in \mathbb{C} \setminus \{0\}$. But $A$ is a $\ast$-algebra if and only if $q \in \mathbb{R} \setminus \{0\}$.

**Proof:** For Part 1 we use induction. The case $n = 1$ is trivial, while the case $n = 2$ follows from (5.1). For $n \geq 3$ we have that

$$T_{g_1g_2\cdots g_n} = T_{g_1(g_2\cdots g_n)} = T_{g_2\cdots g_n}T_{g_1} = T_{g_n} \cdots T_{g_2}T_{g_1},$$

where we used (5.1) for the second equality and the induction hypothesis for $n - 1$ for the third equality.

For the proof of Part 2 we take the notation $T_f^\ast$ for any $f \in A$ to mean the restriction of the adjoint $(T_f)^\ast$ of $T_f$ to the sub-algebra $P$. So, $T_f^\ast = T_{f^\ast}$.
has already been proved. We then note that

\[ T_{h_m} \cdots T_{h_1}^* = T_{h_m}^* \cdots T_{h_1}^* = (T_{h_1} \cdots T_{h_m})^* = (T_{h_m \cdots h_1})^* = T_{(h_m \cdots h_1)^*} = T_{h_1^* \cdots h_m^*}, \]

where we used Part 1 in the third equality and that \( \mathcal{P} \) is a \(*\)-friendly in the last equality. We leave the second equality as an exercise for the reader.

For Part 3 we have

\[ T_{h_m}^* \cdots T_{h_1}^* T_{g_n} \cdots T_{g_1} = T_{h_1^* \cdots h_m^*} T_{g_1 \cdots g_n} = T_{g_1 \cdots g_n h_1^* \cdots h_m^*} \]

by applying Parts 1 and 2 in the first equality and applying (3.1) in the second equality, since \( g_1 \cdots g_n \in \mathcal{P} \) follows from \( \mathcal{P} \) being a sub-algebra. □

6 Canonical Commutation Relations

We next want to consider the canonical commutation relations satisfied by these creation and annihilation operators. But our approach here is quite the opposite of the usual approach in which one starts with some generalization or modification of the standard canonical commutation relations (considered as formal relations to be satisfied), and then one looks for realizations (namely, representations) of them as actual operators in some Hilbert space. Here we would like to find the appropriate canonical commutation relations that arise from a given Toeplitz quantization, that is, the operators are given first.

Our first observation is that the creation and annihilation operators all sit inside the algebra \( \mathcal{L} \). So they generate a sub-algebra of \( \mathcal{L} \), which is an object well known in mathematical physics.

**Definition 6.1** The sub-algebra of \( \mathcal{L} \) generated by all the creation operators \( T_g \), where \( g \in \mathcal{P} \), and all the annihilation operators \( T_h \), where \( h \in \mathcal{P}^* \), is called the algebra of canonical commutation relations (CCR) and is denoted by \( \text{CCR}(A, \mathcal{P}) \).

Alternatively, we will write \( \text{CCR} \) if context resolves the ambiguity in this notation. This may be a good time to point out that the inner product has...
been suppressed from our notation of Toeplitz operators. Therefore even the notation \( \mathcal{CCR}(A, \mathcal{P}) \) is ambiguous.

We also define the *Toeplitz algebra*, denoted \( \mathcal{T} \), to be the sub-algebra of \( \mathcal{L} \) generated of all the Toeplitz operators \( T_g \) for arbitrary symbols \( g \in A \). Clearly, \( \mathcal{CCR} \subset \mathcal{T} \). An explicit description of either of the algebras \( \mathcal{CCR} \) or \( \mathcal{T} \) seems to be in no way trivial in general.

Notice that in this abstract approach we first define the algebra of CCR before defining the canonical commutation relations themselves. Typically in studies in physics and mathematical physics, one defines the algebra of CCR in terms of a presentation of generators and relations, where the relations are exactly the canonical commutation relations. In the present abstract approach this corresponds to writing \( \mathcal{CCR} \) as the quotient of some other free algebra \( \mathcal{F} \) and then identifying the kernel of the quotient map \( \pi : \mathcal{F} \to \mathcal{CCR} \) as the ideal of relations. Then we could pick a minimal set of generators of this ideal of relations as the CCR of this theory. However, the trick is to do this (or at least some of it) in a functorial way, because otherwise we will not have a general theory.

We propose the following construction. We define \( \mathcal{F} \) to be the free algebra over \( \mathbb{C} \) generated by the set \( \mathcal{P} \cup \mathcal{P}^* \). Since \( \mathcal{P} \cup \mathcal{P}^* \subset A \), we distinguish the product in \( \mathcal{F} \) from that in \( A \) by writing the algebra generators of \( \mathcal{F} \) as \( G_f \) for \( f \in \mathcal{P} \cup \mathcal{P}^* \). So \( \mathcal{F} \) is the complex vector space with a basis given by all monomials \( G_{f_1}G_{f_2}\cdots G_{f_n} \) with each \( f_j \in \mathcal{P} \cup \mathcal{P}^* \). The algebra morphism \( \pi : \mathcal{F} \to \mathcal{CCR} \) is defined on the algebra generators of \( \mathcal{F} \) by \( \pi : G_f \mapsto T_f \) for all \( f \in \mathcal{P} \cup \mathcal{P}^* \). Since the algebra \( \mathcal{F} \) is free on these generators, this defines \( \pi \) uniquely. Also since the elements \( T_f \) for \( f \in \mathcal{P} \cup \mathcal{P}^* \) are algebra generators for the algebra \( \mathcal{CCR} \), it follows that \( \pi \) is surjective. Moreover, \( \pi(G_{f_1}G_{f_2}\cdots G_{f_n}) = T_{f_1}T_{f_2}\cdots T_{f_n} \) gives the map \( \pi \) on a basis of \( \mathcal{F} \). We define the ideal of canonical commutation relations in \( \mathcal{F} \) to be \( \mathcal{R} := \ker \pi \).

This seems to be as far as one can go before getting down to the details of picking ideal generators of \( \mathcal{R} \). It appears to be impossible to do this step in a functorial way. But it is reasonable to say that any minimal set of algebra generators of \( \mathcal{R} \) is a set of CCR. Notice that such a set need not be unique in general. So we are still some ways from having the typical situation found in most studies of CCRs.

We now discuss the well known, standard CCRs of quantum mechanics in \( \mathbb{R}^n \) in this setting. These are given by the generators \( A_1, \ldots A_n, A_1^\dagger, \ldots, A_n^\dagger \).
together with the relations (the standard CCR):

\begin{align*}
A_j A_k - A_k A_j & \quad (6.1) \\
A_j^\dagger A_k^\dagger - A_k^\dagger A_j^\dagger & \quad (6.2) \\
A_j A_k^\dagger - A_k^\dagger A_j - \delta_{j,k} \hbar 1 & \quad \text{(Kronecker delta)} (6.3)
\end{align*}

for \( j, k \in \{1, 2, \ldots, n\} \). Here \( \hbar > 0 \) is Planck’s constant. Notice that we have deliberately written these as relations to be quotiented out and that we have not used the standard notation for commutators. One important point, often sluffed over, is that (6.1) and (6.2) are non-trivial relations mathematically, since they impose the commutativity of certain pairs of generators in the quotient algebra of CCR. The corresponding generators of the free algebra do not commute, of course. However, in physics the intuition is that commuting operators are like objects in classical mechanics and so are deemed to be trivial in the quantum setting. Using this physics intuition, the only non-trivial case is (6.3) when \( j = k \), while from a mathematical perspective all the cases of (6.3) as well as (6.1) and (6.2) are non-trivial.

Instead of concerning ourselves with what is trivial and what is not (and from whose point of view), let us simply note that the relations (6.1), (6.2) and, when \( j \neq k \) (6.3), are homogeneous elements (in this case of degree 2) in the free algebra, while (6.3) for \( j = k \) is not a homogeneous element. Also, Planck’s constant only plays a role in (6.3) for \( j = k \) and then only in a lower order term. The free algebra \( \mathcal{F} \) introduced above is also a graded algebra, where the linear span of all of the basis elements \( G_{f_1} G_{f_2} \cdots G_{f_n} \) for some fixed integer \( n \geq 0 \) is by definition the subspace of homogeneous elements of degree \( n \). Then a homogeneous element in \( \mathcal{R} \subset \mathcal{F} \) is called a classical relation while any non-homogeneous element in \( \mathcal{R} \) is called a quantum relation. This dichotomy is important more for ideal generators of \( \mathcal{R} \) rather than for arbitrary elements in \( \mathcal{R} \) itself. For example, using this dichotomy, one sees that the \( q \)-commutation relation \( AA^\dagger - q A^\dagger A \) (usually written as \( xy - qyx \)) for \( q \in \mathbb{C} \) is a classical relation, while the relation \( AA^\dagger - q A^\dagger A - \hbar 1 \) is a quantum relation. Notice that both of these relations, classical and quantum, arise in the study of Toeplitz operators associated with the quantum plane. See [22].

The next definition is motivated by the examples discussed above.

**Definition 6.2** Let \( R \in \mathcal{R} \) be a non-zero relation. Then we can write \( R \) uniquely as \( R = R_0 + R_1 + \cdots + R_n \), where \( \deg R_j = j \) for each \( j = 0, 1, \ldots, n \) and \( R_n \neq 0 \). Then we say that \( R_n \) is the classical relation associated to \( R \).
Notice that $R_n$ is indeed a classical relation. Both of the cases $R_n \in \mathcal{R}$ and $R_n \notin \mathcal{R}$ can occur. Intuitively, to get the classical relation $R_n$ from $R$ we throw away the ‘quantum corrections’ $R_0, R_1, \ldots, R_{n-1}$ in $R$. We let

$$\mathcal{R}_{cl} := \langle R_n \mid R_n \text{ is the classical relation associated to some } R \in \mathcal{R} \rangle,$$

where the brackets $\langle \cdot \rangle$ indicate that we are taking the two-sided ideal in $\mathcal{F}$ generated by the elements inside the brackets.

**Definition 6.3** The dequantized algebra associated to $\mathcal{A}$ is defined to be

$$\mathcal{DQ} := \mathcal{F}/\mathcal{R}_{cl}.$$ 

Note that $\mathcal{DQ}$ need not be commutative.

We can realize $\mathcal{DQ}$ as the case $\hbar = 0$ of a family of algebras defined for all $\hbar \in \mathbb{C}$ and with $\hbar = 1$ corresponding to $\mathcal{CCR}$. Of course, when $\hbar > 0$ we interpret $\hbar$ as Planck’s constant. To achieve this we define the $\hbar$-deformed relations to be

$$\mathcal{R}_\hbar := \langle \hbar^{n/2}R_0 + \hbar^{(n-1)/2}R_1 + \cdots + \hbar^{1/2}R_{n-1} + R_n \mid R \in \mathcal{R} \rangle \quad (6.4)$$

$$= \langle R_0 + \hbar^{-1/2}R_1 + \cdots + \hbar^{-(a-1)/2}R_{n-1} + \hbar^{-n/2}R_n \mid R \in \mathcal{R} \rangle, \quad (6.5)$$

using the above notation $R = R_0 + R_1 + \cdots + R_n$. And then we define

$$\mathcal{CCR}_\hbar := \mathcal{F}/\mathcal{R}_\hbar.$$ 

In the second expression (6.5) the powers of $\hbar^{-1/2}$ correspond to the degree of homogeneity of each of the terms, while in the first expression (6.4) each of the homogeneous terms has been given its intuitively correct degree of ‘quantumness’. The first expression (6.4) also clarifies in a formal way what happens when we take the limit when $\hbar \to 0$. For $\hbar \neq 0$ the two expressions (6.4) and (6.5) are clearly equivalent, but for $\hbar = 0$ only the definition (6.4) makes sense. In physics one considers $\hbar > 0$, but here the discussion is valid for $\hbar \in \mathbb{C}$.

We have included Planck’s constant $\hbar$ to emphasize that this theory has semi-classical behavior (more precisely, what happens when $\hbar$ tends to zero) as well as a classical counterpart $\mathcal{DQ}$ (that is, what happens when we put $\hbar$ equal to zero). However, the semi-classical theory as well as the general relation between the algebras $\mathcal{A}$ and $\mathcal{DQ}$ remain as open problems. Each of the algebras $\mathcal{A}$ and $\mathcal{DQ}$ is a ‘classical’ algebra (though possibly in different
senses of the word ‘classical’) with \( \text{CCR} \) being an intermediate quantum algebra of interest. The Toeplitz algebra \( \mathcal{T} \) is also a quantum algebra with its own intrinsic interest.

This section is nothing but a preliminary discussion of what generalized canonical commutation relations might look like in this theory. It is basically a call for further research.

7 Concluding Remarks

There are many papers, especially in the physics literature, dedicated to the study of a given deformation of the canonical commutation relations (CCRs). Realizations of these deformed CCRs can be a non-trivial problem. Often the solution is given not by using a Toeplitz quantization but rather some other approach. Also the approach in those papers typically involves the definition of the algebra under study in terms of generators and relations. This leads to highly specific studies of rather concrete mathematical structures. A ‘slight’ change of the presentation in terms of generators and relations can entail a rather different theory. Also, one is faced with the usually intractable problem of identifying when two presentations in terms of generators and relations define isomorphic objects.

As mentioned earlier the approach in this paper is quite the opposite. Here we start with the Toeplitz quantization of an algebra and then look for the corresponding generalized CCR’s. And we have not imposed many restrictions on the algebra \( \mathcal{A} \) besides the quite standard ones of associativity and existence of a unit. Also we require a \( \ast \)-operation and an inner product. We have avoided the use of generators and relations as a starting point. Of course, one can generalize or modify any theory, and for this theory one could drop the associativity condition or the existence of the unit. Or these could be replaced by other conditions. Similar comments apply to the \( \ast \)-operation and the inner product. While these are possibilities for further research, we think that the theory as presented here is still quite rudimentary and merits further study. For example, we look forward to an understanding of how to find the (best?) generators of the generalized CCR’s associated with a given Toeplitz quantization.

Finally, here are some comments on other Toeplitz quantizations which use non-commuting symbols. First there is the impressive monograph \cite{BottcherSilbermann} by Böttcher and Silbermann. These authors, and the researchers associated
with them, have produced a significant body of work on Toeplitz operators whose symbols are matrices with entries in various function spaces or in an algebra. In this regard also see the papers [13] and [14] by Karlovich. These works can hardly be described in a few words, but it seems that they always use measures. Their works include the study of Toeplitz operators in Banach spaces, such as $L^p$ and $H^p$. Of course, in the present paper we do not use measures but we do use an inner product. And our Toeplitz operators are only defined in a Hilbert space. A major difference in emphasis is that the Böttcher-Silbermann school takes an operator theory approach, whereas we are also treating topics because of their interest in physics as well as in analysis and operator theory.

The papers [1] and [2] by Ali and Englis use matrix valued symbols. So again, these symbols are functions but with values in a non-commutative algebra. Their results are in the setting of $L^2$ spaces, so there is a measure being used. Their papers are concerned with Berezin-Toeplitz quantization, where one has quantum Hilbert spaces $\mathcal{H}_h$ indexed by Planck’s constant $h > 0$. These two papers are concerned with the asymptotics as $h \to 0$. This is a mathematical-physics approach, but treats themes complementary of those of this paper. The paper [15] of Kerr is similar to the work of Ali and Englis, but now the symbols are matrices with entries in a scalar valued Bergman space. So this is based on a measure, and it also has more of a flavor of functional analysis and operator theory.

The papers [8] by Borthwick et al. and [12] by Iuliu-Lazaroiu et al. study super-Toeplitz operators, that is, those that arise naturally in super-manifold theory. The symbols are super-functions, meaning they have commuting and anti-commuting parts. This theory arose from Berezin’s work in quantum physics and has become an area in and of itself in geometry. However, we find it to be rather complementary to the current approach.

None of these prior works was known to me until I was finishing up this paper. They have many superficial similarities to this paper, but are not sources for it. A major, important feature of this paper is that it provides a quantization scheme without using a measure, or some sort of generalization of a measure as is done in [14]. And this is a significant difference of this paper from those mentioned above. Also, we are presenting a theory intended to be simultaneously applicable both in operator theory and in mathematical physics.
Acknowledgments: I thank S.T. Ali, J. Cruz Sampedro, M. Đurđevich, J.-P. Gazeau and R.A. Martínez-Avendaño for clarifying remarks.

References

[1] S.T. Ali, M. Englis, Berezin-Toeplitz quantization over matrix domains, in: Contributions in Mathematical Physics: A Tribute to Gerard G. Emch, Eds. S.T. Ali and K.B. Sinha, Hindustan Book Agency, New Delhi, India (2007). arXiv: math-ph/0602015.

[2] S.T. Ali, M. Englis, Matrix-valued Berezin-Toeplitz quantization, J. Math. Phys. 48, (2007), 053504, (14 pages). arXiv: math-ph/0611082.

[3] V. Bargmann, On a Hilbert space of analytic functions and its associated integral transform. I, Commun. Pure Appl. Math. 14 (1961) 187–214.

[4] M. El Baz, R. Fresneda, J.-P. Gazeau and Y. Hassouni, Coherent state quantization of paragrassmann algebras, J. Phys. A: Math. Theor. 43 (2010) 385202 (15pp). Also see the Erratum for this article in arXiv:1004.4706v3.

[5] F.A. Berezin, General Concept of Quantization, Commun. Math. Phys., 40 (1975) 153–174.

[6] C.A. Berger and L.A. Coburn, Toeplitz operators and quantum mechanics, J. Funct. Anal., 68 (1986), 273–299.

[7] C.A. Berger and L.A. Coburn, Toeplitz operators on the Segal-Bargmann space, Trans. Am. Math. Soc., 301 (1987), 813–829.

[8] D. Borthwick et al., Matrix Cartan superdomains, super Toeplitz operators, and quantization, J. Funct. Anal. 127 (1995), 456–510. arXiv: hep-th/9406050

[9] A. Böttcher and B. Silbermann, Analysis of Toeplitz Operators, Springer, first edition 1990, second edition 2006.

[10] J.-P. Gazeau, Coherent States in Quantum Physics, Wiley-VCH, 2009.

[11] B.C. Hall, Holomorphic methods in analysis and mathematical physics, in: “First Summer School in Analysis and Mathematical Physics” (S. Perez-Esteva and C. Villegas-Blas, Eds.), 1–59, Contemp. Math., 260, Am. Math. Soc., 2000.
[12] C. Iuliu-Lazaroiu, D. McNamee and C. Sämann, Generalized Berezin-Toeplitz quantization of Kähler supermanifolds, J. High Energy Phys., (2009), 055. arXiv: 0811.4743v2

[13] A. Yu. Karlovich, Higher order asymptotic formulas for Toeplitz matrices with symbols in generalized Hölder spaces in: Operator Algebras, Operator Theory and Applications, Eds. Maria Amélia Bastos (2008) et al., Birkhäuser, pp. 207–228. arXiv: 0705.0432.

[14] A. Yu. Karlovich, Asymptotics of Toeplitz Matrices with Symbols in Some Generalized Krein Algebras, Modern Anal. Appl. (2009) 341–359. arXiv: 0803.3767.

[15] R. Kerr, Products of Toeplitz Operators on a Vector Valued Bergman Space, Integral Equations Operator Theory 66 (2010), 571–584. arXiv:0804.4234

[16] E.H. Lieb, The classical limit of quantum spin systems, Commun. Math. Phys., 31 (1973) 327–340.

[17] R.A. Martínez-Avendaño and P. Rosenthal, An Introduction to Operators on the Hardy-Hilbert space, Springer, 2007.

[18] M. Reed and B. Simon, Mathematical Methods of Modern Physics, Vol. I: Functional Analysis, Academic Press, 1972.

[19] M. Reed and B. Simon, Mathematical Methods of Modern Physics, Vol. II: Fourier Analysis, Self-Adjointness, Academic Press, 1975.

[20] S.B. Sontz, Paragrassmann Algebras as Quantum Spaces, Part I: Reproducing Kernels, in: Geometric Methods in Physics. XXXI Workshop 2012. Trends in Mathematics, Eds. P. Kielanowski et al., (2013) Birkhäuser, pp. 47–63. arXiv:1204.1033v3.

[21] S.B. Sontz, Paragrassmann Algebras as Quantum Spaces, Part II: Toeplitz Operators, Journal of Operator Theory, to appear. arXiv:1205.5493

[22] S.B. Sontz, A Reproducing Kernel and Toeplitz Operators in the Quantum Plane, arXiv:1305.0986