Conserved currents for general teleparallel models

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(December 13, 2018)

The obstruction for the existence of an energy-momentum tensor for the gravitational field is connected with differential-geometric features of the Riemannian manifold. It has not to be valid for alternative geometrical structures.

A teleparallel manifold is defined as a parallelizable differentiable 4D-manifold endowed with a class of smooth coframe fields related by global Lorentz, i.e., $SO(1,3)$ transformations. In this article a general 3-parameter class of teleparallel models is considered. It includes a 1-parameter subclass of models with the Schwarzschild coframe solution (generalized teleparallel equivalent of gravity).

A new form of the coframe field equation is derived here from the general teleparallel Lagrangian by introducing the notion of a 3-parameter conjugate field strength $F_a$. The field equation turns out to have a form completely similar to the Maxwell field equation $d \ast F_a = T_a$. By applying the Noether procedure, the source 3-form $T_a$ is shown to be connected with the diffeomorphism invariance of the Lagrangian. Thus the source of the coframe field is interpreted as the total conserved energy-momentum current of the system.

A reduction of the conserved current to the Noether current and the Noether charge for the coframe field is provided. The energy-momentum tensor is defined as a map of the module of current 3-forms into the module of vector fields. Thus an energy-momentum tensor for the coframe field is defined in a diffeomorphism invariant and a translational covariant way. The total energy-momentum current of a system is conserved. Thus a redistribution of the energy-momentum current between material and coframe (gravity) field is possible in principle, unlike as in GR.

The energy-momentum tensor is calculated for various teleparallel models: the pure Yang-Mills type model, the anti-Yang-Mills type model and the generalized teleparallel equivalent of GR. The latter case can serve as a very close alternative to the GR description of gravity.

I. INTRODUCTION

The concept of an energy-momentum tensor for the gravitational field is, undoubtedly, the most puzzling issue in general relativity (GR). This quantity is well defined for other classical fields acting in a fixed geometrical background and has the following properties. It is

(i) local - i.e., constructed only from the fields taken at some point on a manifold and the derivatives of the fields taken at the same point,

(ii) covariant - i.e., transforms as a tensor under diffeomorphisms of the manifold,

(iii) conserved - i.e., it satisfies the covariant divergence equation $T^\mu{}_{\nu;\mu} = 0$,

(iv) “the first integral of the field equation” - it is derivable from the field equations by integration and includes the field derivatives with an order of one less than the order of the field equation.

It is well known that in Einstein’s theory of gravity a quantity satisfying the conditions listed above does not exist. This fact is usually related to the existence of the equivalence principle. It implies that the gravitational field can not be detected at a point as a covariant object. This conclusion can also be viewed as a purely differential-geometric fact. Indeed, the components of the metric tensor are managed by a system of second order partial differential equations. Thus the energy-momentum quantity has to be a local tensor constructed from the metric components and their first order derivatives. The corresponding theorem of (pseudo) Riemannian geometry (due to Weyl) states that every expression of such a type is trivial. Thus the objection for the existence of a gravitational energy-momentum tensor is directly related to the geometric property of the (pseudo) Riemannian manifold.

It is natural to expect that this objection can be lifted in an alternative geometric model of gravity.

In recent time teleparallel structures in the geometry of spacetime has evoked a considerable interest for various reasons. They was considered as a possible physical relevant geometry by itself as well as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory or metric-affine gravity. Another important subject are the various applications of the frame technique in physical theories based on classical (pseudo) Riemannian geometry.
The construction of a manifold endowed with a smooth field of frames (repère, vierbein) originated in the “Repère Mobile” method of Darboux-E. Cartan \cite{2} in differential geometry. Weitzenböck \cite{3} was the first who recognized that this construction can be also viewed as a self-sufficient geometrical structure. The geometrical structure of the teleparallel manifold was studied intensively in the first third of the last century - see \cite{4} and the references therein. The teleparallel description of gravity has been also studied for a long time. The pioneering works of Cartan \cite{2} and Einstein \cite{5} dealt meaningfully with various models of unified (gravitational-electromagnetic) field theory. Investigations in gauge field theory of gravity and in Einstein-Cartan gravity (see \cite{14}, \cite{15} and the references therein), renewed the interest in teleparallel geometry. In the framework of the general geometrical metric-affine theory of gravity (MAG) \cite{16}, the teleparallel structure appears as one of the basic substructures. Accordingly, the general Lagrangian of MAG includes the pure teleparallel terms as a separate part. For investigations in this area see Refs. \cite{17} to \cite{19}.

The dynamics of classical fields is completely determined by an appropriate integral functional of action. One derives the field equation by applying the least action principle. Conserved field currents are the first integrals of the field equation. Hence they can be derived, in principle, from the field equations by algebraic manipulations. However, one cannot describe in this way the origin and the meaning of the conservation law. The Noether theorem (see e.g. \cite{9}, \cite{10}) provides a natural description of conservation laws. The conserved currents are associated with the invariant properties of the action under certain groups of transformations. The Noether theorem also provides an algorithmic procedure for the actual construction of a conserved current. Recently the classical Noether technique was extended and reformulated in the language of modern differential geometric. The investigation into the Noether technique by Wald and his collaborators \cite{7} motivated mostly by the problem of the black hole entropy. Important progress in the variational technique was provided by the free variation bicomplex of Anderson \cite{11}. Results similar to those of Wald were obtained in the framework of this general theory. In both approaches the general diffeomorphic Lagrangians was studied.

In this article we will make use of the covariant Noether procedure for specific Lagrangians with additional (gauge) symmetry. The outline of the paper is as follows:

We start in the first section with Maxwell-scalar system in flat Minkowski space. The field equations, the conserved currents, and the energy-momentum tensor are exhibited by explicitly covariant expressions of differential forms. This is used for the comparison with the teleparallel models. The second section serves as a brief survey of the teleparallel description of gravity.

Our main results are presented in the third section. We consider a coframe-scalar system with the most general odd quadratic coframe Lagrangian. The field equation is derived in a form almost literally similar to the Maxwell equation. By applying the Noether procedure, the conserved current associated with the diffeomorphism invariance of the Lagrangian is derived. It is interpreted as the total energy-momentum current of the system. This is a conserved current. It serves as the source of the coframe field. Consequently, a redistribution of energy between material and gravitational (coframe) fields is possible in principle.

The notion of the Noether current and the Noether charge for the coframe field are introduced. The energy-momentum tensor is defined as a map of the module of current 3-forms into the module of vector fields. Thus an energy-momentum tensor for the coframe field is defined in a diffeomorphism invariant and a translational covariant way.

In the fourth section the energy-momentum tensor is calculated for various teleparallel models: the pure Yang-Mills type model, the anti-Yang-Mills type model, and the generalized teleparallel equivalent of gravity. The latter case can serve as a very close alternative to the GR description of gravity.

II. ELECTROMAGNETIC-SCALAR SYSTEM

Before diving into the consideration of the teleparallel models, let us start with a rather simple example of a system that includes a gauge vector field and a scalar field. We consider this example mostly in order to give a brief account of the Noether procedure and to fix the notations employed in this paper. It should be noted, however, that the teleparallel description of gravity will be formulated in the next section in a form very similar to the familiar description of the electromagnetic field. A coframe field (i.e. a set of four 1-forms) is a basic dynamical variable in teleparallel gravity, while electromagnetic potential 1-form serves as the basic variable in Maxwell theory. As a consequence of that, the field equations, the conserved currents and the energy-momentum tensor in both theories will turn out to be of similar form.
A. Lagrangian

Let the Minkowski space of special relativity be given, i.e., a flat 4D-manifold $M$ with a metric $\eta_{ab} = (-1, +1, +1, +1)$ and with vanishing torsion. Consider a real even 1-form field $A$ which couples minimally to a complex even scalar field $\phi$. The total Lagrangian density for the system is given by an odd differential 4-form

$$L = (e) L + (s) L = \frac{1}{2} F \wedge *F - \frac{1}{2} D\phi \wedge *D\phi,$$

where

$$F = dA, \quad D\phi = (d + ieA)\phi$$

are the even forms of the corresponding field strengths. $\overline{D\phi}$ means the complex conjugate, and $*$ denotes the Hodge dual map defined by the flat metric $\eta_{ab}$. Our conventions for the operations on forms are listed in the Appendix. The Lagrangian density (1) is invariant under the actions of two continuous groups: the Poincaré group of transformations of $M$, and the internal $U(1)$-gauge group of transformations of the field $A$. Hence two conserved currents of a different nature is expected for the system (1). The infinitesimal variation of the Lagrangian is given by the variational relation

$$\delta L = \delta F \wedge *F - \text{Re} \left( \delta(D\phi) \wedge *D\phi \right).$$

(3)

Applying the Leibniz rule to extract the total derivatives and using (2) we obtain an equivalent form of the variation relation (3)

$$\delta L = \delta A \wedge (d *F + e \text{Im}(\phi *D\phi)) + \text{Re}(\delta\phi D *D\phi)$$

$$+d \left( \delta A \wedge *F - \text{Re}(\delta\phi *D\phi) \right).$$

(4)

B. Field equations

The variation (4) of the Lagrangian has to vanish for a class of arbitrary infinitesimal variations of the dynamical variables $A$ and $\phi$, hence it should be zero also for a subclass of independent variations vanishing at infinity. The total derivative term in (4) vanishes for such variations, while the first two terms result in two field equations. The field equation for the scalar field is

$$D *D\phi = 0,$$

(5)

while the field equation for the electromagnetic field is

$$d *F = (s) I, \quad \text{or} \quad d *dA = (s) I.$$  

(6)

The odd 3-form in the right hand side of (6) is the electromagnetic current produced by the complex scalar field

$$(s) I = -e \text{Im}(\phi * D\phi),$$

(7)

which serves as a source for the field strength $F$. Observe the structure of the field equations (6). The left hand side is the exterior derivative of the Hodge dual of the odd strength (2-form) while the right hand side is the odd form of a current (3-form). We will see later that this is a generic structure. An immediate consequence of the field equation (6) is the conservation law for the electromagnetic current

$$d (s) I = 0$$

(8)

This current is known to be associated with the $U(1)$-gauge symmetry of the Lagrangian (4). Let us turn, however, to another conserved current associated with the invariance of the total Lagrangian (1) under the action of Poincaré group of the space-time transformations.
C. Conserved currents

On shell, i.e., for the fields satisfying the field equations (5) and (6), the variation relation (4) reduces to
\[ d\left( (e)\Theta + (s)\Theta \right) - \delta L = 0, \] (9)
where the 3-forms for the electromagnetic and for the scalar fields are
\[ (e)\Theta = \delta A \wedge \ast F, \quad \text{and} \quad (s)\Theta = -\text{Re}(\delta \varphi \ast \overline{D\varphi}). \] (10)

Consider a vector field \( X \) which represents an infinitesimal action of the Poincaré group on \( M \). Let the variations of the fields \( \delta A \) and \( \delta \varphi \) be produced by the Lie derivative operator \( \mathcal{L}_X \) taken with respect to \( X \). Since the Lagrangian (1) is Poincaré invariant, its variation, being generated by the variations of the fields, will also be produced by the Lie derivative \( \mathcal{L}_X \). Thus the equations \( \delta \varphi = \mathcal{L}_X \varphi, \quad \delta A = \mathcal{L}_X A \) result in \( \delta L = \mathcal{L}_X L \). We will make use of a known formula for the Lie derivative of an arbitrary \( p \)-form
\[ \mathcal{L}_X \alpha = d\left( X\alpha \right) + Xd\alpha. \] (11)

Note that the first term in the right hand side of (11) vanishes for scalars. The second term is zero for 4-forms. Hence, the Lie derivative and consequently the variation of an arbitrary Lagrangian 4-form is a total derivative. After substituting the corresponding expressions for the variations into (9) it takes a form of a conservation law for a certain odd 3-form \( J(X) \)
\[ dJ(X) = 0. \] (12)
This 3-form is expressed as the sum of the electromagnetic and the scalar parts
\[ J(X) = (e)J(X) + (s)J(X). \] (13)
This fact is in agreement with the minimal coupling form of the Lagrangian (3). The explicit expressions for the currents are
\[ (e)J(X) = \left( d(XJ\alpha) + (XJ F) \right) \wedge \ast F - XJ^{(e)}L, \] (14)
\[ (s)J(X) = -\text{Re}\left( (XJd\varphi) \ast \overline{D\varphi} \right) - XJ^{(s)}L. \] (15)

The conserved 3-form \( J(X) \) can be considered as a preliminary candidate for the total energy-momentum current for the scalar-electromagnetic system. It is odd, covariant, local, and associated with the transformations of the manifold. Unfortunately, the electromagnetic 3-form \( (e)J(X) \) includes a term (the first one) with a non-algebraic dependence on the vector field \( X \). Consequence will be shown that only a current which depends linearly (algebraic) on an arbitrary vector field admits a reformulation in terms of the energy-momentum tensor. Observe also another problem with the reduction (13). The electromagnetic part \( (e)J(X) \) and the scalar part \( (s)J(X) \) are not separately gauge invariant.

The 3-form \( J(X) \) can be amended in order to avoid the non-algebraic dependence on an arbitrary vector field and also to recover the separate gauge invariance of its pieces. For that, we use the Leibniz rule for extracting the total derivative in the first term of (14). Apply the field equation (6) to obtain a new reduction
\[ J(X) = (e)T(X) + (s)T(X) + d\left( (e)Q(X) \right), \] (16)
where the 3-forms are
\[ (e)T(X) = (XJ F) \wedge \ast F - \frac{1}{2}XJ(F \wedge \ast F), \] (17)
\[ (s)T(X) = -\text{Re}\left( (XJd\varphi) \ast \overline{D\varphi} \right) + \frac{1}{2}XJ(D\varphi \wedge \ast D\varphi). \] (18)
As for the 2-form \( (e)Q(X) \), it is exhibited as
\[ (e)Q(X) = (XJ\alpha) \ast F. \] (19)
The 3-forms \( s \mathcal{T}(X) \) and \( e \mathcal{T}(X) \) are odd, covariant, local and their sum

\[
\mathcal{T}(X) = (e \mathcal{T}(X) + (s \mathcal{T}(X) \tag{20}
\]

is conserved:

\[
d\mathcal{T}(X) = 0. \tag{21}
\]

Thus \( \mathcal{T}(X) \) can be interpreted as the total conserved current of the system. Accordingly, by (17) the 3-forms (17), (18) will be referred to as the electromagnetic and the scalar Noether currents correspondingly. As for the odd 2-form \( Q(X) \) it can be identified with the Noether charge (7). Note that this object is not gauge invariant and depends only on the electromagnetic field variables.

As for the gauge invariance of the reduction (13), it is recovered: the electromagnetic part (17) and the scalar part (18) are both gauge invariant.

Although, the 3-forms \( s \mathcal{T}(X) \) and \( m \mathcal{T}(X) \) depend on an arbitrary vector field \( X \), this dependence is algebraically linear. Thus every one of these currents can be treated as a linear map of the module of vector fields on \( M \) into the module of 3-forms on \( M \), i.e., as a tensor field. A natural representation of a linear map on vector spaces (on modules) is obtained by its action on the basis vectors. In this case the currents are referred to as the canonical currents of the corresponding fields. Take \( e_a = \partial/\partial x_a \) to obtain the canonical energy-momentum current of the electromagnetic field

\[
(e \mathcal{T})(a) := (e \mathcal{T}(e_a) = (e_a \mathcal{J} F) \wedge *F - \frac{1}{2} e_a \mathcal{J} (F \wedge *F), \tag{22}
\]

and the canonical energy-momentum current of the scalar field

\[
(s \mathcal{T})(a) := (s \mathcal{T}(e_a) = -\text{Re} (e_a \mathcal{J} D\varphi \wedge D\varphi) + \frac{1}{2} e_a \mathcal{J} (D\varphi \wedge *D\varphi). \tag{23}
\]

The canonical Noether charge 2-form for the electromagnetic field can also be defined:

\[
(e Q)(a) := (e Q(e_a) = (e_a \mathcal{J} A) \wedge *F. \tag{24}
\]

Recall that this quantity is not gauge invariant.

### D. The energy-momentum tensor

Let us introduce now the notion of the energy-momentum tensor by this differential-form formalism. We are looking for a second rank tensor field of a type \((0,2)\). Such a tensor can always be treated as a bilinear map

\[
T : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{M}),
\]

where \( \mathcal{F}(\mathcal{M}) \) is the algebra of \( C^\infty \)-functions on \( M \) while \( \mathcal{X}(\mathcal{M}) \) is the \( \mathcal{F}(\mathcal{M}) \)-module of vector fields on \( M \). The unique way to construct a scalar from a 3-form and a vector is to take the Hodge dual of the 3-form and to contract the result by the vector. Consequently, we define the energy-momentum tensor as

\[
T(X,Y) := Y \mathcal{J} \ast T(X). \tag{25}
\]

Observe that this quantity is a tensor if and only if the 3-form current \( T \) depends linearly (algebraic) on the vector field \( X \). Certainly, \( T(X,Y) \) is not symmetric in general. The canonical form of the energy-momentum tensor is defined as

\[
T_{ab} := T(e_a,e_b) = e_b \mathcal{J} \ast T_a. \tag{26}
\]

Another useful form of this tensor can be obtained from (26) by applying the rule (A15)

\[
T_{ab} = -\ast (\partial_b \wedge T_a). \tag{27}
\]

The familiar relation \( T^{a}_{\,b} = \eta^{bc}T_{ac} \) defines tensors of a type \((1,1)\)

\[
T_{a}^{\,b} = -\ast (\partial^b \wedge T_a), \tag{28}
\]

1-5
and
\[ T^a_b = -* (\vartheta_b \wedge T^a), \]

which are different for non-symmetric \( T_{ab} \). By applying the rule (29) the relation (26) can be converted into
\[ T_a = T_a^b * \vartheta_b. \]

Thus the components of the energy-momentum tensor are the coefficients of the current \( T_a \) in the dual basis \( *\vartheta^a \) of the vector space \( \Omega^3 \) of odd 3-forms.

In order to justify our definition of the energy-momentum tensor let us first show that the current satisfies the ordinary conservation law. The coframe \( \vartheta^a = 0 \) is close, thus \( d*\vartheta_b = 0 \). From (30) we derive
\[ dT_a = dT_a^b \wedge *\vartheta_b = -T_a^b, b * 1. \]

Hence the differential-form conservation law \( dT_a = 0 \) is equivalent to the tensorial conservation law \( T_a^b, b = 0 \) and conversely.

Let us write down the explicit forms of the energy-momentum tensor.

For the electromagnetic field we obtain by substituting (17) in (24) and using (A15,A17)
\[ (e)T(X,Y) = -* \left( (X \mathcal{J} F) \wedge *(Y \mathcal{J} F) \right) - \langle X, Y \rangle \ast L. \]

By this formula the energy-momentum tensor in an explicitly symmetric form:
\[ (e)T(X,Y) = (e)T(Y,X). \]

The canonical form of the electromagnetic energy-momentum tensor is
\[ T_{ab} = (e)T(e_a,e_b) = -* \left( (e_a \mathcal{J} F) \wedge *(e_b \mathcal{J} F) \right) - \eta_{ab} \ast L. \]

In a specific coordinate chart \( \{x^\mu\} \) we take the coordinate basis vectors \( X = \partial_\alpha \) and \( Y = \partial_\beta \) to obtain the familiar expression for the energy-momentum tensor of the electromagnetic field
\[ (e)T_{\alpha\beta} := (e)T(\partial_\alpha, \partial_\beta) = -F_{\alpha\mu}F_{\beta}^\mu + \frac{1}{4} \eta_{\alpha\beta} F_{\mu\nu}F^{\mu\nu}. \]

This tensor is obviously traceless: \( \eta^{\alpha\beta}T_{\alpha\beta} = 0. \)

For the scalar field we obtain using (18) an explicitly symmetric form of the energy-momentum tensor
\[ (s)T(X,Y) = -\text{Re} \left( (X \mathcal{J} D\varphi) \ast (Y \mathcal{J} \overline{D\varphi}) \right) - \langle X, Y \rangle \ast L. \]

In a specific coordinate chart \( \{x^\mu\} \) we use the covariant derivative notations \( D\varphi = D_\alpha \varphi dx^\alpha \) to derive the familiar expression for the energy-momentum tensor of a complex scalar field.
\[ (s)T_{\alpha\beta} = -\text{Re}(D_{\alpha} \varphi D_{\beta} \overline{\varphi}) + \frac{1}{2} \eta_{\alpha\beta} (D_{\mu} \varphi \overline{D^{\mu} \varphi}). \]

III. TELEPARALLEL GRAVITY

Let us give a brief account of gravity on teleparallel manifolds. Consider a coframe field \( \{\vartheta^a, a = 0, 1, 2, 3\} \) defined on a 4D differential manifold \( M \). The 1-forms \( \vartheta^a \) declared to be pseudo-orthonormal. This determines completely a metric on the manifold \( M \) via the relation
\[ g = \eta_{ab} \vartheta^a \otimes \vartheta^b. \]

Thus it is possible to consider the coframe field \( \vartheta^a \) as a basic dynamical variable and to treat the metric \( g \) as only a secondary structure.

In order to have an isotropic structure on \( M \) (without peculiar directions) the coframe variable have to be defined only up to global pseudo-rotations of the group \( SO(1, 3) \). Thus, the truly dynamical variable is the equivalence class of
The field equation is derived from the Lagrangian \( L \) in the form \([17], [26]\)

\[
\sum_{i=1}^{3} \rho_i (L) = \frac{1}{2} \sum_{i=1}^{3} \rho_i (L)
\]

where

\[
(1) L = d\vartheta^a \wedge \ast d\vartheta_a,
\]

\[
(2) L = \left( d\vartheta_a \wedge \vartheta^a \right) \wedge \ast \left( d\vartheta_b \wedge \vartheta^b \right),
\]

\[
(3) L = \left( d\vartheta_a \wedge \vartheta^b \right) \wedge \ast \left( d\vartheta_b \wedge \vartheta^a \right).
\]

The coefficients \( \rho_i \) are dimensionless free parameters of the theory.

Every term of the Lagrangian \([38]\) is independent of a specific choice of a coordinate system and invariant under the global (rigid) \( SO(1, 3) \) transformations of the coframe. Thus different choices of the free parameters \( \rho_i \) yield different translation invariant classical field models. Some of them are known to be applicable for description of the gravitational field.

The field equation is derived from the Lagrangian \([38]\) in the form \([17], [24]\)

\[
\rho_1 \left( 2d \ast d\vartheta_a + e_a \left( d\vartheta^b \wedge \ast d\vartheta_b \right) - 2(e_a \ast d\vartheta^b) \wedge \ast d\vartheta_b \right) + \\
\rho_2 \left( -2d \ast d\vartheta_a + d \ast (d\vartheta^b \wedge \vartheta_b) + 2d\vartheta_a \wedge \ast (d\vartheta^b \wedge \vartheta_b) + \\
e_a \left( d\vartheta^c \wedge \vartheta_c \wedge \ast (d\vartheta^b \wedge \vartheta_b) \right) - 2(e_a \ast d\vartheta^b) \wedge \vartheta_b \wedge \ast (d\vartheta^c \wedge \vartheta_c) \right) + \\
\rho_3 \left( -2d \ast \vartheta_a + d \ast \vartheta_a \wedge d\vartheta^b + 2d\vartheta_a \wedge \ast \vartheta_a \wedge d\vartheta^b + \\
e_a \left( \vartheta_c \wedge d\vartheta^b \wedge \ast (d\vartheta^c \wedge \vartheta_b) \right) - 2(e_a \ast d\vartheta^b) \wedge \vartheta_c \wedge \ast (d\vartheta^c \wedge \vartheta_b) \right) = 0.
\]

In \([30]\) the class of “diagonal” spherical-symmetric static solution to the field equation \([42]\) is described. It was also shown there that a solution with Newtonian behavior at infinity appears only in the case \( \rho_1 = 0 \). This is the Schwarzschild coframe

\[
\vartheta^0 = \frac{1 - m/2r}{1 + m/2r} dt, \quad \vartheta^i = \left( 1 + \frac{m}{2r} \right)^2 dx^i, \quad i = 1, 2, 3,
\]

which yields via \([37]\) the Schwarzschild metric in isotropic coordinates.

Few remarks on the structure of the field equation \([42]\) are now in order.
On one hand, the coframe field is a complex of 16 independent variables at every point of $M$ while the symmetric metric tensor field has only 10 independent components. An additional requirement of local $SO(1,3)$ invariance, which satisfied in the case

$$\rho_1 = 0, \quad \rho_2 + 2\rho_3 = 0$$

restricts the set of 16 independent variables to a subset of 10 variables. This subset is certainly related to 10 independent components of the metric.

On the second hand for an arbitrary choice of the parameters $\rho_i$, the field equation (42) is a system of 16 independent equations. This system can be reduced to two covariant systems - a symmetric tensorial system of 10 independent equations and an antisymmetric tensorial system of 6 independent equations. In [28] it was shown that, if (44) holds, the antisymmetric equation vanishes identically. Thus in the case of local $SO(1,3)$ invariance the system (42) is restricted to a system of 10 independent equations for 10 independent variables. Therefore the coframe (teleparallel) structure coincides with the metric structure. The local $SO(1,3)$ invariant teleparallel model is referred to as the teleparallel equivalent of gravity [19]-[34].

IV. COFRAME-SCALAR SYSTEM

A. The total Lagrangian

Let a differential 4D-manifold $M$ be given. Consider a system containing two smooth fields defined on $M$: an even coframe field $\vartheta^a$ and an even (real) scalar field $\varphi$. Our goal is to derive a conserved current expression for the coframe field for a set of models parameterized by the constants $\rho_i$. As in the case of the electromagnetic field, the scalar field $\varphi$ will play a role of an indicator of a true current.

The minimal coupling total Lagrangian density of the system is an odd 4-form, which can be written as a sum of the coframe Lagrangians with the scalar Lagrangian

$$L = \left(\begin{array}{c}
\frac{1}{2} \sum_{i=1}^{3} \rho_i \left(\vartheta^a \right) L - \frac{1}{2} d\varphi \wedge * d\varphi,
\end{array}\right)$$

where the coframe Lagrangians $\left(\vartheta^a \right) L$ are as defined in (39-41).

The standard computations of the variation of a Lagrangian define d on a teleparallel manifold are rather complicated. It is because one needs to vary not only the coframe $\vartheta^a$ itself, but also the the dual frame $e_a$ and even the Hodge dual operator, that depends on the coframe implicitly [26].

In order to avoid these technical problems we will rewrite the total Lagrangian (45) in a compact form which will be useful for the variation procedure.

Consider the exterior differentials of the basis 1-forms $d\vartheta^a$ and introduce the $C$-coefficients of their expansion in the basis of even 2-forms $\vartheta_{ab}$ (we use here and later the abbreviation $\vartheta_{a\cdots b\cdots} = \vartheta^a \wedge \vartheta^b \wedge \cdots$)

$$d\vartheta^a = \vartheta^a_{\beta\alpha} dx^\alpha \wedge dx^\beta := \frac{1}{2} C^a_{bc} \vartheta^{bc}. \quad (46)$$

By definition, the coefficients $C^a_{bc}$ are antisymmetric: $C^a_{bc} = -C^a_{cb}$. The explicit expression can be derived from the definition (46)

$$C^a_{bc} := e_c \int_{\mathcal{I}} (e_b \int d\vartheta^a). \quad (47)$$

In terms of the $C$-coefficients the teleparallel parts of the Lagrangian (45) are

$$\begin{align*}
(1) L &= \frac{1}{2} C_{abc} C^{abc} * 1, \\
(2) L &= \frac{1}{2} C_{abc} \left( C^{abc} + C^{bca} + C^{cab} \right) * 1, \\
(3) L &= \frac{1}{2} \left( C_{abc} C^{abc} - 2 C^a_{ac} C^b_{bc} \right) * 1. \quad (48)
\end{align*}$$

The form (48) is useful for a proof of the completeness of the set of quadratic invariants [24]. It is enough to consider all the possible combinations of the indices. Thus a linear combination of the Lagrangians (48) is the most general
quadratic coframe Lagrangian. Using (48) we rewrite the coframe Lagrangian in a compact form

\[ (c) L = \frac{1}{4} C_{abc} C_{ijk} \lambda^{abcijk} * 1, \]  

(49)

where the constant symbols

\[ \lambda^{abcijk} := (\rho_1 + \rho_2 + \rho_3) \eta^a \eta^b \eta^c \eta^d \eta^e \eta^f + \rho_2 (\eta^a \eta^b \eta^c + \eta^d \eta^e \eta^f) - 2 \rho_3 \eta^a \eta^d \eta^f \eta^i \eta^j \eta^k \]  

(50)

are introduced. It can be checked, by straightforward calculation, that these \( \lambda \)-symbols are invariant under a transposition of the triplets of indices

\[ \lambda^{abcijk} = \lambda^{ijkabc}. \]  

(51)

We also introduce an abbreviated notation

\[ F^{abc} := \lambda^{abcijk} C_{ijk}. \]  

(52)

As for the scalar Lagrangian, it can also be rewritten in a product form similar to (49) by introducing the scalar notations for the derivatives

\[ d\varphi = \varphi_a \vartheta^a, \quad \text{or} \quad \varphi_a = e_a \hat{d} \varphi. \]  

(53)

The total Lagrangian (44) reads now as

\[ L = \frac{1}{4} (C_{abc} F^{abc} - 2 \varphi_a \varphi^a) * 1. \]  

(54)

B. Variation of the Lagrangian

Using the symmetry property of the \( \lambda \)-symbols (51) the variation of the total Lagrangian (54) takes the form

\[ \delta L = \frac{1}{2} \left( \delta C_{abc} F^{abc} - 2 \delta \varphi_a \varphi^a \right) * 1 - L * \delta (*1). \]  

(55)

The variation of the volume element is

\[ \delta (*1) = -\delta (\vartheta^{0123}) = -\delta \vartheta^0 \wedge \vartheta^{123} - \cdots = -\delta \vartheta^0 \wedge * \vartheta^0 - \cdots = \delta \vartheta^m \wedge * \vartheta_m. \]

Thus

\[ L * \delta (*1) = (\delta \vartheta^m \wedge * \vartheta_m) * L = -\delta \vartheta^m \wedge (e_m \hat{d} L) \]  

(56)

As for the variation of the \( C \)-coefficients, we calculate them by equating the variations of the two sides of the equation (46)

\[ \delta d\vartheta_a = \frac{1}{2} \delta C_{amn} \vartheta^{mn} + C_{amn} \delta \vartheta^m \wedge \vartheta^n. \]

Using the formulas (A12) and (A15) we derive

\[ \delta d\vartheta_a \wedge * \vartheta_{bc} = \frac{1}{2} \delta C_{amn} \vartheta^{mn} \wedge * \vartheta_{bc} + C_{amn} \delta \vartheta^m \wedge \vartheta^n \wedge * \vartheta_{bc} \]

\[ = \frac{1}{2} \delta C_{amn} \vartheta^{mn} \wedge (e^n \hat{d} \vartheta_{bc}) - C_{amn} \delta \vartheta^m \wedge (e^n \hat{d} \vartheta_{bc}) \]

\[ = \delta C_{abc} * 1 - \delta \vartheta^m \wedge \left( C_{amb} * \vartheta_c - C_{amc} * \vartheta_b \right). \]
Therefore
\[ \delta C_{abc} \ast 1 = \delta (d \vartheta_a) \wedge \ast \vartheta_{bc} + \delta \vartheta^m \wedge (C_{amb} \ast \vartheta_c - C_{amb} \ast \vartheta_b). \]  \hspace{1cm} (57)

The variation of the scalar field coefficients \( \varphi_a \) can also be calculated using their definition (53). We write
\[ \delta (d \varphi) = \delta \varphi_a \vartheta^a + \varphi_a \delta \vartheta^a, \]
consequently,
\[ \delta \varphi^a \ast 1 = \delta (d \varphi) \wedge \ast \vartheta_a - \varphi_m \delta \vartheta^m \wedge \ast \vartheta_a. \]  \hspace{1cm} (58)

After substituting (56–58) into (55) the variation of the Lagrangian takes the form
\[ \delta L = \frac{1}{2} F^{abc} (\delta (d \vartheta_a) \wedge \ast \vartheta_{bc} + \delta \vartheta^m \wedge (C_{amb} \ast \vartheta_c - C_{amb} \ast \vartheta_b)) \]
\[ - \left( \delta (d \varphi) - \varphi_m \delta \vartheta^m \right) \wedge \ast d \varphi + \delta \vartheta^m \wedge (e_m \mathcal{L}). \]  \hspace{1cm} (59)

The variation relation (59) will play a basic role in the sequel. Let us rewrite it in a compact form by introducing the following abbreviated notations.

Define one-indexed 2-forms
\[ C^a := \frac{1}{2} C^{abc} \vartheta_{bc} = d \vartheta^a. \]  \hspace{1cm} (60)

and a conjugate strength 2-form
\[ F^a := \frac{1}{2} F^{abc} \vartheta_{bc} = (\rho_1 + \rho_3) C^a + \rho_2 e^a \mathcal{J} (\vartheta^m \wedge C_m) - \rho_3 \vartheta^a \wedge (e_m \mathcal{L}). \]  \hspace{1cm} (61)

The 2-form \( F^a \) can be also represented via the irreducible (under the Lorentz group) decomposition of the 2-form \( C^a \) (see [26]). Write
\[ C^a = (1) C^a + (2) C^a + (3) C^a, \]  \hspace{1cm} (62)
where
\[ (1) C^a = C^a - (2) C^a - (3) C^a, \]
\[ (2) C^a = \frac{1}{2} \vartheta^a \wedge (e_m \mathcal{L}), \]
\[ (3) C^a = \frac{1}{2} e^a \mathcal{J} (\vartheta^m \wedge C_m). \]  \hspace{1cm} (63)

Substitute (60) into (61) to obtain
\[ F^a = (\rho_1 + \rho_3) (1) C^a + (\rho_1 - 2 \rho_3) (2) C^a + (\rho_1 + 3 \rho_2 + \rho_3) (3) C^a. \]  \hspace{1cm} (64)

The coefficients in (64) coincide with those calculated in [26].

The 2-forms \( C^a \) and \( F^a \) do not depend on a choice of a coordinate system. They change as vectors by \( SO(1,3) \) transformations of the coframe.

Using (60), the coframe Lagrangian can be rewritten as
\[ (c) L = \frac{1}{4} C_{abc} F^{abc} \ast 1 = \frac{1}{2} C^a \wedge \ast F^a \]  \hspace{1cm} (65)
Let us turn now to the variation relation (59). The terms of the form \( F \cdot C \) can be rewritten as

\[
F^{abc}(C_{amb} \ast \partial_c - C_{amc} \ast \partial_b) = (F^{abc} - F^{acb})C_{amb} \ast \partial_c
\]

\[
= 2C_{amb} \ast (e^b \mathcal{J}^a) = -2(e_m \mathcal{J}_a) \wedge \ast \mathcal{F}^a.
\]

Consequently (59) takes the form

\[
\delta L = \delta \vartheta_m \wedge \left( d(\ast F^m) - (e_m \mathcal{J}_a) \wedge \ast \mathcal{F}^a + (e_m \mathcal{J} d\varphi) \wedge \ast d\varphi + e_m \mathcal{J} L \right)
\]

\[
+ \delta \varphi d \ast d\varphi + d(\delta \vartheta_m \wedge F^m - \delta \varphi \ast d\varphi)
\]

(66)

Collect now the quadratic terms in the brackets in two different 3-forms. The scalar field 3-form is defined as

\[
(s)_m = - (e_m \mathcal{J} d\varphi) \wedge \ast d\varphi + \frac{1}{2} e_m \mathcal{J} (d\varphi \wedge \ast \varphi),
\]

(67)

while the coframe field 3-form is defined as

\[
(c)_m = (e_m \mathcal{J}_a) \wedge \ast \mathcal{F}^a - \frac{1}{2} e_m \mathcal{J} (C_a \wedge \ast \mathcal{F}^a).
\]

(68)

In (67) and (68) the explicit expressions for the scalar and the coframe Lagrangians are inserted. We will use also the notation

\[
\mathcal{T}_m = (s)_m + (c)_m
\]

(69)

for the total 3-form of the system.

Using the definitions above the variational relation (59) results in the final form

\[
\delta L = \delta \vartheta_m \wedge \left( d(\ast F^m - \mathcal{T}_m) \right) + \delta \varphi d \ast d\varphi + d(\delta \vartheta_m \wedge F^m - \delta \varphi \ast d\varphi).
\]

(70)

C. The field equations

We are ready now to write down the field equations. For the scalar field it takes the familiar form

\[
d \ast d\varphi = 0
\]

(71)

As for the coframe, the field equation is

\[
d \ast \mathcal{F}^m = \mathcal{T}_m.
\]

(72)

Note that this is the same equation as (42). The equivalence of the forms can be established by straightforward but tedious algebraic manipulations. The form of the coframe field equation (72) is exactly the same as discussed above: exterior derivative of the dual strength in the left hand side and a 3-form in the right hand side. Thus the 3-forms \( \mathcal{T}_m \) serves as a source for the strength \( \mathcal{F}^m \). The field equation (72) yields the conservation law

\[
d \mathcal{T}_m = 0.
\]

(73)

Thus we obtained a conserved total 3-form for the system which is constructed from the first order derivatives of the field variables (coframe). It is local and covariant. Moreover, it is naturally reduced to the sum of a scalar field 3-form and a coframe field 3-form. The 3-form \( \mathcal{T}_m \) is our candidate for the coframe energy-momentum current. We obtained this expression directly from the field equation and it is similar to the electromagnetic current 3-form (7). Note however an important distinction. The source term for electromagnetic field depends only on material field (scalar in our case). The electromagnetic field itself is uncharged. As for the the coframe field its source is the sum of a material (scalar) field current and a coframe field current \( (c)_m \).

In order to identify the conserved 3-form \( \mathcal{T}_m \) with the energy-momentum current we have to answer the question: What symmetry this conserved current can be associated with?
## D. Conserved currents

Return to the variational relation (70). On shell, for fields satisfying the field equations (71) and (72), it takes the form

\[ \delta L = d\Theta(X), \]  

where

\[ \Theta = \delta \vartheta^a \wedge \ast F_a - \delta \phi \ast d\phi. \]  

Consider the variations of the fields produced by the Lie derivative taken relative to a smooth vector field \( X \) on the manifold \( M \). The total Lagrangian (54) is a diffeomorphic invariant, hence its variation is produced by the Lie derivative taken relative to the same vector field \( X \). The Lie derivative of an arbitrary 4-form is a total derivative thus the relation (74) takes a form of a conservation law for a certain 3-form

\[ dJ(X) = 0, \quad J(X) = \Theta(X) - X\llcorner L. \]  

The explicit form of this conserved current is

\[ J(X) = \left( d(X\llcorner \vartheta^a) + X\llcorner C^a \right) \wedge \ast F_a - (X\llcorner d\phi) \ast d\phi - X\llcorner L. \]  

As in the case of electromagnetic field, this quantity includes a term (the first one) which is non-linear (non-algebraic) relative to an arbitrary vector field \( X \). Such a form of the conserved current can not be used for definition of an energy-momentum tensor. Unlike the electromagnetic case this problem can be solved merely by using the canonical form of the current.

Let us take \( X = e_a \). The first term of (77) vanishes identically. Thus

\[ J(e_m) = (e_m\llcorner C^a) \wedge \ast F_a - (e_m\llcorner d\phi) \ast d\phi - e_m\llcorner L. \]  

Observe that this expression coincides with the source of the field equation (72): \( J(e_m) = T_m \). Thus the conserved current \( T_m \) is associated with the diffeomorphic invariance of the Lagrangian and consequently represents the energy-momentum current.

Another way to avoid the first term in (77) is to extract the total derivative.

\[
J(X) = d\left( (X\llcorner \vartheta^a) \wedge \ast F_a \right) - (X\llcorner d\phi) \ast d\phi - X\llcorner L.
\]

Thus, up to the field equation (72), the current \( J(X) \) represents a total derivative of a certain 2-form

\[ J(X) = d \left( (X\llcorner \vartheta^a) \ast F_a \right). \]  

This result is a special case of a general proposition due to Wald [7] for a diffeomorphic Lagrangians. The 2-form

\[ ^{(c)}Q(X) = (X\llcorner \vartheta^a) \ast F_a. \]

can be referred to as the **Noether charge for the coframe field**. The canonical form of the Noether charge for the coframe field coincides with the dual of the conjugate strength \( F^a \).

\[ ^{(c)}Q_a = ^{(c)}Q(e_a) = \ast F_a. \]

Note, that such 2-form plays an important role in Wald’s treatment of the black hole entropy [1].
The coframe Lagrangian is defined by (45). Take the variation of (85) relative to the coframe field

\[ \delta L = \delta \vartheta_a \wedge \left( d * F^a - c T^a - (m) T^a \right), \]  

(86)

where the coframe Lagrangian is defined by (45). Take the variation of (85) relative to the coframe field \( \vartheta^a \)

\[ \delta L = \delta \vartheta_a \wedge \left( d * F^a - c T^a - (m) T^a \right), \]  

(86)

In this section we present the expressions for the energy-momentum tensor for the scalar and coframe fields. Apply the definition (24) to the conserved current (67). Thus, the energy-momentum tensor for the scalar field is

\[ (s) T_{mn} = * \left( (e_m \downarrow d \varphi) \wedge * (e_n \downarrow d \varphi) \right) + \frac{1}{2} \eta_{mn} * (d \varphi \wedge d \varphi). \]  

(82)

Observe that this expression is symmetric. \( (s) T_{mn} = (s) T_{nm} \) and coincides with the familiar coordinate-wise expression.

As for the coframe field, its energy-momentum tensor is derived from the current (68)

\[ (c) T_{mn} = e_n \downarrow * (c) T_m = e_n \downarrow * \left( (e_m \downarrow C_a) \wedge * F^a - \frac{1}{2} e_m \downarrow (C_a \wedge * F^a) \right). \]  

(83)

Using (A13) we rewrite the first term in (83) as

\[ e_n \downarrow \left( (e_m \downarrow C_a) \wedge * F^a \right) = - * \left( \partial_m \wedge * \left( (e_m \downarrow C_a) \wedge * F^a \right) \right) \]

\[ = * \left( (e_m \downarrow C_a) \wedge * (\partial_n \wedge * F^a) \right) = - * \left( (e_m \downarrow C_a) \wedge * (e_n \downarrow F^a) \right). \]

As for the second term in (83) it takes the form

\[ - \frac{1}{2} e_n \downarrow \left( e_m \downarrow (C_a \wedge * F^a) \right) = \frac{1}{2} \left( \partial_m \wedge * (e_m \downarrow (C_a \wedge * F^a)) \right) \]

\[ = - \frac{1}{2} * \left( \partial_n \downarrow \wedge \partial_m \right) \left( C_a \wedge * F^a \right) = \frac{1}{2} \eta_{mn} * (C_a \wedge * F^a). \]

Consequently the energy-momentum tensor for the coframe field is

\[ (c) T_{mn} = - * \left( (e_m \downarrow C_a) \wedge \ast (e_n \downarrow F^a) \right) + \frac{1}{2} \eta_{mn} * (C_a \wedge * F^a). \]  

(84)

Observe that this expression is exactly of the same form as the familiar expression for the energy momentum tensor of the Maxwell electromagnetic field (34). It also satisfies the following

**Proposition** For all teleparallel models described by the Lagrangian (13) the energy-momentum tensor defined by (34) is traceless \( (c) T_m^m = 0 \).

**Proof** Compute the trace of (34):

\[ (c) T_m^m = (c) T_m \eta^{mn} = - * \left( (e_m \downarrow C_a) \wedge \ast (e_n \downarrow F^a) \right) + 2 * (C_a \wedge * F^a) \]

\[ = \ast \left( (e_m \downarrow C_a) \wedge \ast (\partial_n \wedge * F^a) \right) + 2 * (C_a \wedge * F^a) \]

\[ = - \ast \left( \partial_n \wedge (e_m \downarrow C_a) \wedge * F^a \right) + 2 * (C_a \wedge * F^a) = 0 \]

In the latter equality the relation (A9) was used.

**E. Energy-momentum tensor**

**F. The field equation for a general system**
where the material form current 3-form is defined via the variation derivative of the material Lagrangian relative to the coframe field $\vartheta^a$

$$(m)T^a := -\frac{\delta}{\delta \vartheta^a} (m)L.$$  (87)

Consequently, the field equation for the general system takes the form

$$d * F^a = T^a,$$  (88)

where the notion of the total current of the system $T = (c)T^a + (m)T^a$ is introduced. In the tensorial form the equation can be rewritten as

$$e_b \lrcorner d * F^a = T^a_b,$$  (89)

or equivalently

$$\vartheta_b \wedge d * F^a = T^a_b \ast 1.$$  (90)

The conservation law for the total current $dT^a = 0$ is a consequence of the field equation (88). The field equation is similar to the Maxwell field equation for the electromagnetic field. Observe, however, an important difference. The source term in the right hand side of the electromagnetic field equation depends only on external fields. The electromagnetic field itself is not charged and does not produces additional fields. On the other hand the tensorial form (89) of the field equation is similar to the Einstein field equation for the metric tensor

$$R^a_b - \frac{1}{2} \delta_b^a R = (m)T^a_b.$$  (91)

Again, the source terms in the field equations (88) and (91) are different. The source of the Einstein gravity is the energy-momentum tensor only of the materials fields. The conservation of this tensor is a consequence of the field equation. Thus even if some meaningful conserved energy-momentum current for the metric field existed it would have been conserved regardless of the material field current. Consequently, any transmission of the energy-momentum current between the material and gravitational fields is forbidden in the framework of the traditional Einstein gravity. As for the coframe field equation, the total energy-momentum current plays a role of the source of the field. Consequently the coframe field is completely “self-interacted” - the energy-momentum current of the coframe field produces an additional field. The conserved current of the coframe field equation is the total energy-momentum current, not only the material current. Thus the transmission of the current between the material field and the coframe field is, in principle, possible.

V. NON-GRAVITY TELEPARALLEL MODELS

Let us turn now to concrete teleparallel models, which can be constructed by choosing a fixed set of parameters $\rho_i$ in the Lagrangian (45). We start with two models, which have unique static, spherical symmetric solutions. These solutions, however, have not Newtonian limit at infinity. So these models are not viable models of gravity. We continue with the so-called teleparallel equivalent of gravity. This model has a unique static, spherical symmetric solution, which leads to the Schwarzschild metric. The consideration is restricted to the vacuum case.

A. Yang-Mills-type model

We start with a teleparallel model described by a Lagrangian 4-form of a pure Yang-Mills type

$$L = \frac{1}{2} \rho d\vartheta^a \wedge * d\vartheta_a.$$  (92)

Here the choice of the free parameters in the Lagrangian (45) is

$$\rho_1 = \rho, \quad \rho_2 = \rho_3 = 0.$$  (93)
The conjugate strength (61) takes the form
\[ \mathcal{F}^a = \rho \ C^a. \] (94)

Consequently the field equation (88) is
\[ d(\ast C^a) = \frac{1}{\rho} \mathcal{F}^a, \] (95)
where the source term is the energy-momentum current of the coframe field
\[ \mathcal{T}^a = \rho \left( (e^a \mathcal{J} \ C_m) \wedge \ast C^m - \frac{1}{2} e^a \mathcal{J}(C_m \wedge \ast C^m) \right). \] (96)

It was proved in [30] that the field equation (95) has a unique static solution of a ”diagonal” form, which is spherical symmetric (\( r = \sqrt{x^2 + y^2 + z^2} \))
\[ \vartheta^0 = \left( \frac{r_0}{r} \right)^{4/3} dt, \quad \vartheta^\alpha = \left( \frac{r_0}{r} \right)^{2/3} dx^\alpha \quad \alpha = 1, 2, 3. \] (97)

The corresponding metric is
\[ ds^2 = \left( \frac{r_0}{r} \right)^{8/3} dt^2 - \left( \frac{r_0}{r} \right)^{4/3} (dx^2 + dy^2 + dz^2). \] (98)

Substituting the solution (97) in (96) we obtain
\[ T_0 = - \frac{4 \rho}{9 r_0^2} \left( \frac{r_0}{r} \right)^{2/3} \ast \vartheta_0, \] (99)
\[ T_\alpha = \frac{4 \rho}{9 r_0^2} \left( \frac{r_0}{r} \right)^{2/3} \left( -2 \delta_\alpha\beta + 5 \frac{x_\alpha x_\beta}{r^2} \right) \ast \vartheta_\beta, \] (100)
where \( \alpha, \beta = 1, 2, 3 \) and \( \delta_\alpha\beta = \text{diag}\{1, 1, 1\} \).

The energy-momentum tensor for the coframe field takes the form
\[ T_{mn} = - \frac{4 \rho}{9 r_0^2} \left( \frac{r_0}{r} \right)^{2/3} \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -2 + 5 \frac{x^2}{r^2} & 5 \frac{x^2}{r^2} & 5 \frac{x^2}{r^2} \\
0 & 5 \frac{x^2}{r^2} & -2 + 5 \frac{y^2}{r^2} & 5 \frac{y^2}{r^2} \\
0 & 5 \frac{x^2}{r^2} & 5 \frac{y^2}{r^2} & -2 + 5 \frac{z^2}{r^2}
\end{array} \right). \] (101)

Observe that this matrix is traceless and symmetric. The leading coefficient in (97) can be interpreted as follows. Compute the scalar curvature of the metric (98). It is negative at every point of \( M \) and coincides with the leading coefficient of (101)
\[ R = - \frac{4 \rho}{9 r_0^2} \left( \frac{r_0}{r} \right)^{2/3}. \] (102)

Identify the energy of the field (97) with the \{00\} component of the energy-momentum tensor to obtain
\[ E = - \frac{4 \rho}{9 r_0^2} \left( \frac{r_0}{r} \right)^{2/3}. \] (103)

Thus, in order to have positive energy of the field, we have to require that the parameter \( \rho \) should be negative. The choice \( \rho = -1 \) is in accordance with [26].

Observe a remarkable relation between the energy of the coframe field and the scalar curvature
\[ E = \rho R. \] (104)

It should be noted, however, that the right hand side of (104) is invariant under local \( SO(1, 3) \) transformations of the coframe as well as under arbitrary diffeomorphisms of the manifold. The left hand side of (104), however, depends on
a specific coframe and on a choice of the coordinates.

It was proved in [30] that the solution of the type (100) is a generic solution in a wide class of teleparallel models. For almost arbitrary choice of the parameters \( \rho_i \) the unique spherical symmetric static solution has a form

\[ \vartheta^0 = \left( \frac{r_0}{r} \right)^a dt, \quad \vartheta^i = \left( \frac{r_0}{r} \right)^b dx^i, \]

(105)

where the exponents \( a \) and \( b \) are functions of \( \rho_i \)'s

\[ a = \frac{2\rho_1}{2\rho_3 - 3\rho_1}, \quad b = \frac{\rho_1 - 2\rho_3}{2\rho_3 - 3\rho_1}. \]

(106)

The corresponding metrics is

\[ ds^2 = \left( \frac{r_0}{r} \right)^{2a} dt^2 - \left( \frac{r_0}{r} \right)^{2b} (dx^2 + dy^2 + dz^2). \]

(107)

The metric (107) is singular at the origin of the coordinates as well as at infinity and regular for \( 0 < r < \infty \). Thus it does not have the Minkowskian limit and can not be used to describe the gravitational field.

**FIG. 1.** The energy (103) plotted as a function of a distance.

**B. Anti Yang-Mills-type model.**

Consider a teleparallel model with a Lagrangian of the form

\[ L = \rho d^a \vartheta^a \land *d^a \vartheta_a. \]

(108)

This is obtained from the Yang-Mills Lagrangian (92) by replacing the familiar exterior derivative operator with the coderivative operator \( d^a = *d \). The coderivative of the basis 1-forms can be expressed by its exterior derivative [16]

\[ d^a \vartheta^a = *(\vartheta^{am} \land *d\vartheta_m) = e^a \mathcal{J}(e_m \mathcal{J}\vartheta^m). \]

(109)

Consequently, by using the rule (A9), the Lagrangian can be rewritten as

\[ L = -\rho(e_a \mathcal{J}(e_m \mathcal{J}d\vartheta^m)) \land (\vartheta^{am} \land *d\vartheta_n) = \rho(e_m \mathcal{J}d\vartheta^m) \land \vartheta^a \land *d\vartheta_n. \]

(110)

In order to embed the Lagrangian (110) into the general form of the teleparallel Lagrangian (43) we use again the rule (A9) to obtain

\[ L = \rho(e_m \mathcal{J}(d\vartheta^m \land \vartheta^n) - d\vartheta^n) \land *d\vartheta_n = \rho(d\vartheta_n \land \vartheta_m \land *d\vartheta^m \land \vartheta^n) - \rho d\vartheta^n \land *d\vartheta_n \]

Thus the Lagrangian (108) is

\[ L = -\rho^{(1)}L + \rho^{(3)}L, \]

(111)
and the corresponding set of parameters is
\[
\rho_1 = -\rho, \quad \rho_2 = 0, \quad \rho_3 = \rho.
\] (112)

The conjugate strength is
\[
F^a := \rho \vartheta^a \wedge (e_m J^m)
\] (113)

The field equation (72) takes the form
\[
d \ast (\vartheta^a \wedge (e_m J^m)) = \frac{1}{\rho} T^a,
\] (114)

where the energy-momentum current $T^a$ is obtained from (68) by inserting the conjugate strength (113).

A unique spherical symmetric, static, “diagonal” solution for the equation (114) is
\[
\vartheta^0 = dt, \quad \vartheta^i = \left(1 + \frac{\rho}{r}\right) dx^i.
\] (115)

This coframe generates an asymptotically-flat metric
\[
ds^2 = dt^2 - \left(1 + \frac{\rho}{r}\right)^2 (dx^2 + dy^2 + dz^2).
\] (116)

The metric (116) represents a point-like solution if the corresponding ADM-mass is positive. Rewrite the metric (116) in the spherical Schwarzschild-type coordinates. In the spherical coordinates the metric is
\[
ds^2 = dt^2 - \frac{d\tilde{r}^2}{\left(1 - \frac{\rho}{r}\right)^2} - \tilde{r}^2 d\Omega^2.
\] (117)

Using now the translation
\[
\tilde{r} = r + \rho_0
\]
we obtain the asymptotic-flat metric in the Schwarzschild coordinates
\[
ds^2 = dt^2 - \frac{d\tilde{r}^2}{\left(1 - \frac{\rho}{r}\right)^2} - \tilde{r}^2 d\Omega^2.
\] (117)

The ADM mass for the metric (117) takes the form
\[
m := \lim_{\tilde{r} \to \infty} \tilde{r} \left(1 - \frac{\rho}{r}\right)^2 = \rho_0.
\] (118)

Thus by taking the parameter $\rho_0$ to be positive we obtain a particle-type solution with a finite positive ADM-mass.

The metric (116) is singular at the origin $r = 0$ and consequently the metric (117) is singular at $\tilde{r} = \rho_0$. In order to clarify the nature of this singularity compute the scalar curvature of the metric (116) via the formula (124). The result is
\[
R = \frac{\rho_0^2}{(r_0 + r)^4} = \frac{\rho_0^2}{\tilde{r}^4}.
\] (119)

This function is non-zero and regular for all values of $r$ including the origin ($r = 0$).

The proper distance for a radial null geodesic in the metric (117) is equal to the proper time and attach the infinity
\[
l = t = \int_{r_0}^{\tilde{r}_1} \frac{d\tilde{r}}{1 - \frac{\rho}{r}} \to \infty
\]

Thus the point $r = 0$ ($\tilde{r} = \rho_0$) does not belong to any final part of the space-time.

Computing the conjugate strength with the coframe (110) we obtain
\[
F^a = \frac{2m\rho}{r(m + r)^2} \delta_{\mu x^\mu} \vartheta^{\alpha\nu},
\] (120)
where \( a = 0, 1, 2, 3 \), while \( \mu, \nu = 1, 2, 3 \).

Calculating the energy-momentum current for the coframe field (1 10) we obtain

\[
T^0 = \frac{2m^2 \rho}{(m + r)^4} \ast \vartheta^0,
\]

\[
T^\alpha = -\frac{2m^2 \rho}{(m + r)^4} \frac{x^\alpha x^\mu}{r^2} \delta_{\mu\nu} \ast \vartheta^\nu,
\]

Consequently the energy-momentum tensor obtains the form

\[
T_{mn} = \frac{2m^2 \rho}{(m + r)^4} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{x^2}{r^2} & -\frac{xy}{r^2} & -\frac{xz}{r^2} \\
0 & -\frac{xy}{r^2} & -\frac{y^2}{r^2} & -\frac{yz}{r^2} \\
0 & -\frac{xz}{r^2} & -\frac{yz}{r^2} & -\frac{z^2}{r^2}
\end{pmatrix}.
\]

Observe that this matrix is traceless and symmetric. The leading coefficient in (97) is proportional again to the scalar curvature. The free coefficient \( \rho \) have to be chosen to be positive. Taking \( \rho = \frac{1}{2} \) we obtain also in this model the relation \( E = R \).

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**FIG. 2.** The energy (123) plotted as a function of a distance.

VI. TELEPARALLEL MODELS FOR GRAVITY

Let us delete the first (Yang-Mills-type) term in the general Lagrangian (38) by requiring

\( \rho_1 = 0 \).

Thus the Lagrangian under consideration is

\[
L = \rho_2 (d\vartheta_a \wedge \vartheta^a) \wedge \ast (d\vartheta_b \wedge \vartheta^b) + \rho_3 (d\vartheta_a \wedge \vartheta^b) \wedge \ast (d\vartheta_b \wedge \vartheta^a).
\]

The conjugate strength (61) takes the form

\[
\mathcal{F}^\alpha = \rho_2 e^a \mathbf{j} (\vartheta^m \wedge C_m) + \rho_3 e^m \mathbf{j} (\vartheta^a \wedge C_m)
\]

The field equation (72) takes the form

\[
\rho_2 d \left( \ast (\vartheta^m \wedge C_m) \wedge \vartheta^a \right) + \rho_3 d \left( \ast (\vartheta^a \wedge C_m) \wedge \vartheta^m \right) = T^a,
\]

where the energy-momentum current \( T^a \) is obtained from (68) by inserting the conjugate strength (113).

A unique spherical symmetric, static, “diagonal” solution for the equation (126) is
\[ \vartheta^0 = \frac{1 - m/2r}{1 + m/2r} dt, \quad \vartheta^i = \left(1 + \frac{m}{2r}\right)^2 dx^i, \quad i = 1, 2, 3, \]  

(127)

This coframe corresponds to the Schwarzschild metric in isotropic coordinates

\[ ds^2 = \left(\frac{1 - m/2r}{1 + m/2r}\right)^2 dt^2 - \left(1 + \frac{m}{2r}\right)^4 \left(dx^2 + dy^2 + dz^2\right). \]  

(128)

Observe that the first term of the strength (125) as well as the first term of the field equation (126) vanish identically for an arbitrary choice of the “diagonal” form \( \vartheta^a = F dx^a \).

Computing the conjugate strength for the coframe (110) we obtain

\[ \mathcal{T}^0 = \frac{-2m\rho_3}{r^3(1 + m/2r)^3} \delta_{\mu\nu} x^\mu \vartheta^\nu, \]  

(129)

\[ \mathcal{T}^a = \frac{-2m\rho_3 (1 - m/4r)}{r^4(1 - m/2r)(1 + m/2r)^3} \delta_{\mu\nu} x^\mu \vartheta^{a\nu}, \]  

(130)

where \( a = 0, 1, 2, 3 \), while \( \alpha, \mu, \nu = 1, 2, 3 \).

Calculating the energy-momentum current for the coframe field (110) we obtain

\[ T^0 = \frac{-3\rho_3 (m^2/r^4)(1 - m/6r)}{(1 - m/2r)(1 + m/2r)^6} * \vartheta^0, \]  

(131)

\[ T^a = \frac{\rho_3 (m^2/r^4)}{(1 - m/2r)(1 + m/2r)^6} \left( \vartheta^a - \frac{m}{r} \frac{x^a x^\mu}{r^2} \delta_{\mu\nu} \vartheta^\nu \right). \]  

(132)

Consequently the energy-momentum tensor is

\[
T_{mn} = \frac{\rho_3 (m^2/r^4)}{(1 - m/2r)(1 + m/2r)^6} \begin{pmatrix}
3 - \frac{m}{2r} & 0 & 0 & 0 \\
0 & 1 - \frac{m x^2}{2r r^2} & -\frac{m x y}{2r r^2} & -\frac{m x z}{2r r^2} \\
0 & -\frac{m x y}{2r r^2} & 1 - \frac{m y^2}{2r r^2} & -\frac{m y z}{2r r^2} \\
0 & -\frac{m x z}{2r r^2} & -\frac{m y z}{2r r^2} & 1 - \frac{m z^2}{2r r^2}
\end{pmatrix}.
\]  

(133)

Observe that this matrix is traceless and symmetric. The leading coefficient in (17) is proportional again to the scalar curvature. The free coefficient \( \rho \) has to be positive. Taking \( \rho = \frac{1}{2} \) we obtain also in this model the relation \( E = R \).

![FIG. 3. The energy (133) plotted as a function of a distance.](image)

**ACKNOWLEDGMENTS**

I am deeply grateful to F.W. Hehl and to S. Kaniel for constant support, interesting discussions and valuable comments.
APPENDIX A: BASIC NOTATIONS AND DEFINITIONS

Let us list our basic conventions. We consider an $n$-dimensional differential manifold $M$ of signature

$$\eta_{ab} = \text{diag}(-1, +1, \cdots, +1).$$

(A1)

Let the manifold $M$ be endowed with a smooth coframe field (1-forms)

$$\{\vartheta^a(x), \ a = 0, \cdots, n-1\}.$$  \hspace{1cm} (A2)

Note that a smooth non-degenerate frame (coframe) field can be defined on a manifold of a zero second Stiefel-Whitney class. However this topological restriction is not exactly relevant in physics because the solutions of physical field equations can degenerate at a point or on a curve. Moreover, these solutions produce the most important physical models (particles, strings, etc.).

The coframe $\vartheta^a(x)$ represents, at a given point $x \in M$, a basis of the linear space of 1-forms $\Omega^1$. The set of all non-zero exterior products of basis 1-forms

$$\vartheta^{a_1, \cdots, a_p} := \vartheta^{a_1} \wedge \cdots \wedge \vartheta^{a_p}$$

(A3)

represents a basis of the linear space of $p$-forms $\Omega^p$. Note the (anti)commutative rule for arbitrary forms $\alpha \in \Omega^p$ and $\beta \in \Omega^q$

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$ \hspace{1cm} (A4)

The dual set of vector fields

$$\{e_a(x), \ a = 0, \cdots, n-1\}$$

(A5)

forms a basis of the linear space of vector fields at a given point.

The duality of vectors and 1-forms can be expressed by \textit{inter product} operation for which we use the symbol $\downarrow$. Namely,

$$e_a \downarrow \vartheta^b = \delta_a^b.$$

(A6)

The action $X \downarrow w$ of a vector $X$ on a form $w$ of arbitrary degree $p$ is defined by requiring: (i) linearity in $X$ and in $w$, (ii) modified Leibniz rule for the wedge product of $\alpha \in \Omega^p$ and $\beta \in \Omega^q$

$$X \downarrow (\alpha \wedge \beta) = (X \downarrow \alpha) \wedge \beta + (-1)^p \alpha \wedge (X \downarrow \beta).$$

(A7)

These properties together with (A6) guarantee the uniqueness of the map $\downarrow : \Omega^p \to \Omega^{p-1}$.

The following relations involving the inner product operation ($p = \text{deg}(w)$) are useful for actual calculations.

$$X \downarrow (Y \downarrow w) = -Y \downarrow (X \downarrow w),$$

(A8)

$$\vartheta^a \wedge (e_a \downarrow w) = pw,$$

(A9)

$$e_a \downarrow (\vartheta^a \wedge w) = (n-p)w.$$ \hspace{1cm} (A10)

We use also the forms $\vartheta_a := \eta_{ab} \vartheta^b$ with subscript and the corresponding vector fields $e^a := \eta^{ab} e_b$ with superscript. Thus

$$e_a \downarrow \vartheta_b = \eta_{ab}.\hspace{1cm}$$

(A11)

The linear spaces $\Omega^p$ and $\Omega^{n-p}$ have the same dimensions $\binom{n}{p} = \binom{n}{n-p}$. Thus they are isomorphic. This isomorphism $\textit{Hodge dual map}$ is linear. Thus it is enough to define its action on basis forms:

$$\ast(\vartheta^{a_1 \cdots a_p}) = \frac{1}{(n-p)!} \epsilon^{a_1 \cdots a_{p+1} \cdots a_n} \vartheta_{a_{p+1} \cdots a_n}. $$

(A12)

We use here the complete antisymmetric pseudo-tensor $\epsilon^{a_1 \cdots a_{n-1}}$ normalized as $\epsilon^{01 \cdots (n-1)} = 1$. The set of indices $\{a_1, \cdots, a_n\}$ is an even permutation of the standard set $\{0, 1, \cdots, (n-1)\}$.

1-20
Thus $\star \vartheta^{0-(n-1)} = 1$ and $\star 1 = - \vartheta^{0-(n-1)}$.
The consequence of the definition (A12) is $(\deg(\alpha) = \deg(\beta))$
\[\alpha \wedge \star \beta = \beta \wedge \alpha.\] (A13)

For the choice of the signature (A1) we obtain
\[\star^2 w = (-1)^{p(n-p)+1} w.\] (A14)

In the case $n = 4$ the operator $\star^2$ preserves the forms of odd degree and changes the sign of the forms of even degree.

The following equation is useful for actual calculations
\[e_a \wedge w = - \star (\vartheta_a \wedge \star w).\] (A15)

To prove this linear relation it is enough to check it for the basis forms.
The pseudo-orthonormality for the basis forms $\vartheta^a$ yields the metric tensor $g$ on the manifold $M$
\[g = \eta_{ab} \vartheta^a \otimes \vartheta^b.\] (A16)

The formulas (A11) and (A15) can be applied to derive a useful form of a scalar product of two vectors $X$ and $Y$.
We write these vectors in the basis $e_a$ as $X = X^m e_m$ and $Y = Y^m e_m$. Thus the scalar product is
\[\langle X, Y \rangle = X^m Y^n \eta_{mn}.\]

Using (A11) we obtain
\[\langle X, Y \rangle = X^m Y^n (e_m \vartheta^m)\]
Thus
\[\langle X, Y \rangle = X \lrcorner Y = Y \lrcorner X,\] (A17)
where $\lrcorner X$ is the 1-form dual to the vector $X$ which obtained by a canonical map from vectors to 1-forms
\[\underline{\zeta} : X^m e_m \to X^m \vartheta_m.\]

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