Some $B$-Difference Sequence Spaces Derived by Using Generalized Means and Compact Operators

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Abstract

This paper presents new sequence spaces $X(r, s, t, p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ defined by using generalized means and difference operator. It is shown that these spaces are complete paranormed spaces and the spaces $X(r, s, t, p; B)$ for $X \in \{c(p), c_0(p), l(p)\}$ have Schauder basis. Furthermore, the $\alpha$-, $\beta$-, $\gamma$- duals of these sequence spaces are computed and also obtained necessary and sufficient conditions for some matrix transformations from $X(r, s, t, p; B)$ to $X$. Finally, some classes of compact operators on the space $l_p(r, s, t; B)$ are characterized by using the Hausdorff measure of noncompactness.

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1 Introduction

The study of sequence spaces has importance in the several branches of analysis, namely, the structural theory of topological vector spaces, summability theory, Schauder basis theory etc. Besides this, the theory of sequence spaces is a powerful tool for obtaining some topological and geometrical results by using Schauder basis.

Let $w$ be the space of all real or complex sequences $x = (x_n), n \in \mathbb{N}_0$. For an infinite matrix $A$ and a sequence space $\lambda$, the matrix domain of $A$, denoted by $\lambda_A$, is defined as $\lambda_A = \{x \in w : Ax \in \lambda\}$ [25]. Some basic methods, which are used to determine the topologies, matrix transformations and inclusion relations on sequence spaces can also be applied to study the matrix domain $\lambda_A$. In recent times, there is an approach of forming a new sequence space by using a matrix domain of a suitable matrix and characterize the matrix mappings between these sequence spaces.

Let $(p_k)$ be a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ and $M =$

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The linear spaces \( c(p), c_0(p), l_\infty(p) \) and \( l(p) \) are studied by Maddox \cite{16}, where

\[
c(p) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} |x_k - l|^p_k = 0 \text{ for some } l \in \mathbb{C} \right\},
\]

\[
c_0(p) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} |x_k|^p_k = 0 \right\},
\]

\[
l_\infty(p) = \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}_0} |x_k|^p_k < \infty \right\}
\]

and

\[
l(p) = \left\{ x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p_k < \infty \right\}.
\]

The linear spaces \( c(p), c_0(p), l_\infty(p) \) are complete with the paranorm \( g(x) = \sup_k |x_k|^\frac{p_k}{k} \) if and only if \( \inf p_k > 0 \) for all \( k \) while \( l(p) \) is complete with the paranorm \( \tilde{g}(x) = \left( \sum_k |x_k|^p_k \right)^{\frac{1}{p}} \). Recently, several authors introduced new sequence spaces by using matrix domain. For example, Başar et al. \cite{4} studied the space \( bs(p) = [l_\infty(p)]_S \), where \( S \) is the summation matrix. Altay and Başar \cite{2} studied the sequence spaces \( r'(p) \) and \( r'_\infty(p) \), which consist of all sequences whose Riesz transform are in the spaces \( l_0 \) and \( l_\infty(p) \) respectively, i.e., \( r'(p) = [l_0(p)]_{R'} \) and \( r'_\infty(p) = [l_\infty(p)]_{R'} \). Altay and Başar also studied the sequence spaces \( r'_0(p) = [c(p)]_{R'} \) and \( r'_0(p) = [c_0(p)]_{R'} \) in \cite{3}.

Kizmaz first introduced and studied the difference sequence space in \cite{13}. Later on, many authors including Ahmad and Mursaleen \cite{1}, Çolak and Et \cite{6}, Başar and Altay \cite{3}, etc. studied new sequence spaces defined by using difference operator. Using Euler mean of order \( \alpha \), \( 0 < \alpha < 1 \) and difference operator, Karakaya and Polat introduced the paranormed sequence spaces \( e_0^\alpha(p; \Delta) \), \( e_0^\alpha(p; \Delta) \), \( e_0^\alpha(p; \Delta) \) in \cite{14}. Mursaleen and Noman \cite{22} introduced a sequence space of generalized means, which includes most of the earlier known sequence spaces. But till 2011, there was no such literature available in which a sequence space is generated by combining both the weighted mean and the difference operator. This was first initiated by Polat et al. \cite{24}. Later on, Demiriz and Çakan \cite{8} introduced the new paranormed difference sequence spaces \( \lambda(r', s', p; \Delta) \) for \( \lambda \in \{l_\infty(p), c(p), c_0(p), l(p)\} \).

Quite recently, Başarir and Kara \cite{5} introduced and studied the \( B \)-difference sequence space \( l(r', s', p; B) \) defined as

\[
l(r', s', p; B) = \left\{ x \in w : (G(r', s').B)x \in l(p) \right\},
\]

where \( r' = (r'_n), s' = (s'_n) \) are non zero sequences and the matrices \( G(r', s') = (g_{nk}) \), \( B = B(u, v) = (b_{nk}), u, v \neq 0 \) are defined by

\[
g_{nk} = \begin{cases} r'_k s'_k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n \end{cases} \quad \text{and} \quad b_{nk} = \begin{cases} 0 & \text{if } 0 \leq k < n - 1, \\ v & \text{if } k = n - 1 \\ u & \text{if } k = n \\ 0 & \text{if } k > n. \end{cases}
\]

By using matrix domain, one can write \( l(r', s', p; B) = [l(p)]_{G(r', s', B)} \).

The aim of this present paper is to introduce the sequence spaces \( X(r, s, t, p; B) \) for \( X \in \{l_\infty(p), c(p), c_0(p), l(p)\} \). We have shown that these sequence spaces are complete paranormed sequence spaces under some suitable paranorm. Some topological results and the \( \alpha-, \beta-, \gamma- \) duals of these spaces are obtained. A characterization of some matrix transformations between these new sequence spaces is established. We also give a characterization of some classes of compact operators on the space \( l_p(r, s, t; B) \) by using
the Hausdorff measure of noncompactness.

2 Preliminaries

Let $l_\infty, c$ and $c_0$ be the spaces of all bounded, convergent and null sequences $x = (x_n)$ respectively, with the norm $\|x\|_\infty = \sup_n |x_n|$. Let $bs$ and $cs$ be the sequence spaces of all bounded and convergent series respectively. We denote by $e = (1, 1, \cdots)$ and $e_n$ for the sequence whose $n$-th term is 1 and others are zero and $N_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all natural numbers. A sequence $(b_n)$ in a normed linear space $(X, \|\|)$ is called a Schauder basis for $X$ if for every $x \in X$ there is a unique sequence of scalars $(\mu_n)$ such that

$$\|x - \sum_{n=0}^{k} \mu_n b_n\| \to 0 \text{ as } k \to \infty,$$

i.e., $x = \sum_{n=0}^{\infty} \mu_n b_n$ [25].

For any subsets $U$ and $V$ of $w$, the multiplier space $M(U, V)$ of $U$ and $V$ is defined as

$$M(U, V) = \{a = (a_n) \in w : au = (a_n u_n) \in V \text{ for all } u \in U\}.$$ 

In particular,

$$U^\alpha = M(U, l_1), U^\beta = M(U, cs) \text{ and } U^\gamma = M(U, bs)$$

are called the $\alpha$, $\beta$- and $\gamma$- duals of $U$ respectively [25].

Let $A = (a_{nk})_{n,k}$ be an infinite matrix with real or complex entries $a_{nk}$. We write $A_n$ as the sequence of the $n$-th row of $A$, i.e., $A_n = (a_{nk})_k$ for every $n$. For $x = (x_n) \in w$, the $A$-transform of $x$ is defined as the sequence $Ax = ((Ax)_n)$, where $A_n (x) = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$, provided the series on the right side converges for each $n$. For any two sequence spaces $U$ and $V$, we denote by $(U, V)$, the class of all infinite matrices $A$ that map from $U$ into $V$. Therefore $A \in (U, V)$ if and only if $Ax = ((Ax)_n) \in V$ for all $x \in U$. In other words, $A \in (U, V)$ if and only if $A_n \in U^\beta$ for all $n$ [25].

The theory of $BK$ spaces is the most powerful tool in the characterization of matrix transformations between sequence spaces. A sequence space $X$ is called a $BK$ space if it is a Banach space with continuous coordinates $p_n : X \to \mathbb{K}$, where $\mathbb{K}$ denotes the real or complex field and $p_n (x) = x_n$ for all $x = (x_n) \in X$ and each $n \in \mathbb{N}_0$. The space $l_1$ is a $BK$ space with the usual norm defined by $\|x\|_{l_1} = \sum_{k=0}^{\infty} |x_k|$. An infinite matrix $T = (t_{nk})_{n,k}$ is called a triangle if $t_{nn} \neq 0$ and $t_{nk} = 0$ for all $k > n$. Let $T$ be a triangle and $X$ be a $BK$ space. Then $X_T$ is also a $BK$ space with the norm given by $\|x\|_{X_T} = \|Tx\|_X$ for all $x \in X_T$ [25].

3 Sequence spaces $X(r, s, t, p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$

In this section, we first begin with the notion of generalized means given by Mursaleen et al. [22]. We denote the sets $U$ and $U_0$ as

$$U = \{u = (u_n)_{n=0}^{\infty} \in w : u_n \neq 0 \text{ for all } n\} \text{ and } U_0 = \{u = (u_n)_{n=0}^{\infty} \in w : u_0 \neq 0\}.$$
Let \( r, t \in \mathcal{H} \) and \( s \in \mathcal{U}_0 \). The sequence \( y = (y_n) \) of generalized means of a sequence \( x = (x_n) \) is defined by
\[
y_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} t_k x_k \quad (n \in \mathbb{N}_0).
\]
The infinite matrix \( A(r, s, t) \) of generalized means is defined by
\[
(A(r, s, t))_{nk} = \begin{cases} \frac{s_{n-k} t_k}{r_n} & 0 \leq k \leq n, \\ 0 & k > n. \end{cases}
\]

Since \( A(r, s, t) \) is a triangle, it has a unique inverse and the inverse is also a triangle\(^5\). Take \( D_0^{(s)} = \frac{1}{s_0} \) and
\[
D_n^{(s)} = \begin{bmatrix} 1 & s_1 & 0 & \cdots & 0 \\ s_1 & 1 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_1 \end{bmatrix}
\]
Then the inverse of \( A(r, s, t) \) is the triangle \( \tilde{B} = (\tilde{b}_{nk})_{n,k} \), which is defined as
\[
\tilde{b}_{nk} = \begin{cases} (-1)^{n-k} \frac{D_n^{(s)}}{t_n} \frac{r_n^k}{t_n^k} & 0 \leq k \leq n, \\ 0 & k > n. \end{cases}
\]

Throughout this paper, we consider \( p = (p_k) \) be a bounded sequence of strictly positive real numbers such that \( H = \sup_k p_k \) and \( M = \max \{1, H\} \).

We now introduce the sequence spaces \( X(r, s, t, p; B) \) for \( X \in \{l_\infty(p), c(p), c_0(p), l(p)\} \), combining both the generalized means and the matrix \( B(u, v) \) as
\[
X(r, s, t, p; B) = \left\{ x \in w : y = A(r, s, t; B)x \in X \right\},
\]
where \( y = (y_k) \) is \( A(r, s; t; B) \)-transform of a sequence \( x = (x_k) \), i.e.,
\[
y_n = \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k} t_k u + s_{n-k-1} t_{k+1} v)x_k + s_0 t_n u x_n \right), \quad n \in \mathbb{N}_0,
\]
where we mean \( \sum_{n}^{m} = 0 \) for \( n > m \). By using matrix domain, we can write \( X(r, s, t, p; B) = X_{A(r, s, t, p; B)} = \{ x \in w : A(r, s, t; B)x \in X \} \), where \( A(r, s, t; B) = A(r, s, t).B \), product of two triangles \( A(r, s, t) \) and \( B(u, v) \). For \( X = l_p, p \geq 1 \), we write \( X(r, s, t; p; B) \) as \( l_p(r, s, t; B) \).

These sequence spaces include many known sequence spaces studied by several authors. For examples,

I. if \( r_n = \frac{1}{k}, t_n = s'_n, s_n = 1 \) \( \forall n \), then the sequence spaces \( l(r, s, t, p; B) \) reduce to \( l(r', s', p; B) \) studied by Başarir and Kara\(^5\).

II. if \( r_n = \frac{1}{r_n}, t_n = s'_n, s_n = 1 \) \( \forall n, u = 1 \) and \( v = -1 \) then the sequence spaces \( X(r, s, t, p; B) \) reduce to \( X(r', s', p; \Delta) \) for \( X \in \{l_\infty(p), c(p), c_0(p), l(p)\} \) studied by Demiriz and Çakan\(^8\).

III. if \( r_n = \frac{1}{n^\alpha}, t_n = s_n = \frac{(1-\alpha)n^\alpha}{n^\alpha}, \) where \( 0 < \alpha < 1, u = 1, v = -1, \) then the sequence spaces
Theorem 4.1. The sequence space $l(r, s, t; p; B)$ is a complete linear metric space paranormed by $\tilde{h}$ defined as

$$\tilde{h}(x) = \left( \sum_{n=0}^{\infty} \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k} t_k u + s_{n-k-1} t_k v) x_k + s_0 t_n u x_n \right)^p \right)^{\frac{1}{p}}.$$

(b) The sequence spaces $X(r, s, t; p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p)\}$ are complete linear metric spaces paranormed by $h$ defined as

$$h(x) = \sup_{n \in \mathbb{N}} \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k} t_k u + s_{n-k-1} t_k v) x_k + s_0 t_n u x_n \right)^{\frac{1}{p}}.$$

(c) The sequence space $\ell_p(r, s, t; B)$, $1 \leq p < \infty$ is a BK space with the norm given by

$$\|x\|_{\ell_p(r, s, t; B)} = \|y\|_{\ell_p},$$

where $y = (y_k)$ is defined in (3.1).

Proof. We prove only the part (a) of this theorem. In a similar way, we can prove the other parts.

Let $x, y \in l(r, s, t; p; B)$. Using Minkowski’s inequality

$$\left( \sum_{n=0}^{\infty} \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k} t_k u + s_{n-k-1} t_k v) (x_k + y_k) + s_0 t_n u (x_n + y_n) \right)^p \right)^{\frac{1}{p}} \leq \left( \sum_{n=0}^{\infty} \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k} t_k u + s_{n-k-1} t_k v) x_k + s_0 t_n u x_n \right)^p \right)^{\frac{1}{p}}$$

$$+ \left( \sum_{n=0}^{\infty} \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k} t_k u + s_{n-k-1} t_k v) y_k + s_0 t_n u y_n \right)^p \right)^{\frac{1}{p}}.$$ (4.1)

So, we have $x + y \in l(r, s, t; p; B)$. Let $\alpha$ be any scalar. Since $|\alpha|^p \leq \max\{1, |\alpha|^M\}$, we have $\tilde{h}(\alpha x) \leq \max\{1, |\alpha|\} \tilde{h}(x)$. Hence $\alpha x \in l(r, s, t; p; B).$ It is trivial to show that $\tilde{h}(\theta) = 0$, $\tilde{h}(-x) = \tilde{h}(x)$ for all $x \in l(r, s, t; p; B)$ and subadditivity of $\tilde{h}$, i.e., $\tilde{h}(x + y) \leq \tilde{h}(x) + \tilde{h}(y)$ follows from (4.1).

Next we show that the scalar multiplication is continuous. Let $(x^m)$ be a sequence in $l(r, s, t; p; B)$, where
For each fixed $x^m = (x^m_k) = (x^m_0, x^m_1, x^m_2, \ldots) \in l(r, s, t; p; B)$ for each $m \in \mathbb{N}_0$ such that $\tilde{h}(x^m - x) \to 0$ as $m \to \infty$ and $(\alpha_m)$ be a sequence of scalars such that $\alpha_m \to \alpha$ as $m \to \infty$. Then $\tilde{h}(x^m)$ is bounded that follows from the following inequality

$$\tilde{h}(x^m) \leq \tilde{h}(x) + \tilde{h}(x - x^m).$$

Now consider

$$\tilde{h}(\alpha_m x^m - \alpha x) = \left( \sum_{n=0}^{\infty} \frac{1}{r^n} \left( \sum_{k=0}^{n-1} (s_{n-k}t_ku + s_{n-k-1}t_{k+1}v)(\alpha_m x^m_k - \alpha x_k) + s_0 t_n u(\alpha_m x^m_n - \alpha x_n) \right)^{p_n} \right)^{\frac{1}{p_n}} \leq |\alpha_m - \alpha| \tilde{h}(x^m) + |\alpha| \tilde{h}(x - x) \to 0 \text{ as } m \to \infty.$$

This shows that the scalar multiplication is continuous. Hence $\tilde{h}$ is a paranorm on the space $l(r, s, t; p; B)$. Now we prove the completeness of the space $l(r, s, t; p; B)$ with respect to the paranorm $\tilde{h}$. Let $(x^m)$ be a Cauchy sequence in $l(r, s, t; p; B)$. So for every $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that

$$\tilde{h}(x^m - x^l) < \frac{\varepsilon}{2} \text{ for all } m, l \geq n_0.$$

Then by definition for each $n \in \mathbb{N}_0$, we have

$$\left| (A(r, s, t; B)x^m)_n - (A(r, s, t; B)x^l)_n \right| \leq \left( \sum_{n=0}^{\infty} \left| (A(r, s, t; B)x^m)_n - (A(r, s, t; B)x^l)_n \right|^{p_n} \right)^{\frac{1}{p_n}} < \frac{\varepsilon}{2} \quad (4.2)$$

for all $m, l \geq n_0$, which implies that the sequence $((A(r, s, t; B)x^m)_n)$ is a Cauchy sequence of scalars for each fixed $n \in \mathbb{N}_0$ and hence converges for each $n$. We write

$$\lim_{m \to \infty} (A(r, s, t; B)x^m)_n = (A(r, s, t; B)x)_n \quad (n \in \mathbb{N}_0).$$

Now taking $l \to \infty$ in (4.2), we obtain

$$\left( \sum_{n=0}^{\infty} \left| (A(r, s, t; B)x^m)_n - (A(r, s, t; B)x)_n \right|^{p_n} \right)^{\frac{1}{p_n}} < \varepsilon$$

for all $m \geq n_0$ and each fixed $n \in \mathbb{N}_0$. Thus $(x^m)$ converges to $x$ in $l(r, s, t; p; B)$ with respect to $\tilde{h}$.

To show $x \in l(r, s, t; p; B)$, we take

$$\left( \sum_{n=0}^{\infty} \frac{1}{r^n} \left( \sum_{k=0}^{n-1} (s_{n-k}t_ku + s_{n-k-1}t_{k+1}v)x_k + s_0 t_n u(x_n - x^m_n + x^m_n) \right)^{p_n} \right)^{\frac{1}{p_n}} \leq \tilde{h}(x - x^m) + \tilde{h}(x^m),$$

which is finite for all $m \geq n_0$. Therefore $x \in l(r, s, t; p; B)$. This completes the proof.

\begin{theorem}
\textbf{Theorem 4.2.} The sequence spaces $X(r, s, t; p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ are linearly isomorphic to the spaces $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ respectively, i.e., $l_\infty(r, s, t; p; B) \cong l_\infty(p), c(r, s, t; p; B) \cong c(p), c_0(r, s, t; p; B) \cong c_0(p)$ and $l(r, s, t; p; B) \cong l(p)$.
\end{theorem}
Proof. We prove the theorem only for the case when $X = l(p)$. To prove this we need to show that there exists a bijective linear map from $l(r, s, t, p; B)$ to $l(p)$. Now we define a map $T : l(r, s, t, p; B) \to l(p)$ by $x \mapsto Tx = y = (y_n)$, where

$y_n = \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k}t_ku + s_{n-k-1}t_{k+1}v)x_k + s_0t_nux_n \right), \quad n \geq 0.$

The linearity of $T$ is trivial. It is easy to see that $Tx = 0$ implies $x = 0$. Thus $T$ is injective. To prove $T$ is surjective, let $y \in l(p)$. Since $y = (A(r, s, t, B)x), \ i.e., x = (A(r, s, t, B)^{-1}y = B^{-1}A(r, s, t)^{-1}y$, so we can get a sequence $x = (x_n)$ as

$x_n = \sum_{j=0}^{n} \bigg( \sum_{k=j}^{n} (-1)^{k-j} \frac{(-v)^{n-k} D_{k-j}^{(s)}}{u^{n-k+1} l_k} r_j y_j \bigg), \quad n \in \mathbb{N}_0.$ (4.3)

Then

$\tilde{g}(x) = \left( \sum_{n=0}^{\infty} \frac{1}{r_n} \left( \sum_{k=0}^{n-1} (s_{n-k}t_ku + s_{n-k-1}t_{k+1}v)x_k + s_0t_nux_n \right) \right)^{\frac{1}{p_n}} = \left( \sum_{n=0}^{\infty} |y_n|^{p_n} \right)^{\frac{1}{p}} = g(y) < \infty.$

Thus $x \in l(r, s, t, p; B)$ and this shows that $T$ is surjective. Hence $T$ is a linear bijection from $l(r, s, t, p; B)$ to $l(p)$. Also $T$ is paranorm preserving. This completes the proof.

Since $X(r, s, t, p; B) \cong X$ for $X \in \{c_0(p), c(p), l(p)\}$, the Schauder bases of the sequence spaces $X(r, s, t, p; B)$ are the inverse image of the bases of $X$ for $X \in \{c_0(p), c(p), l(p)\}$. So, we have the following theorem without proof.

**Theorem 4.3.** Let $\nu_k = (A(r, s, t; B)x)_k, \ k \in \mathbb{N}_0$. For each $j \in \mathbb{N}_0$, define the sequence $b^{(j)} = (b^{(j)}_n)_{n \in \mathbb{N}_0}$ of the elements of the space $c_0(r, s, t, p; B)$ as

$b^{(j)}_n = \sum_{k=j}^{n} (-1)^{k-j} \frac{(-v)^{n-k} D_{k-j}^{(s)}}{u^{n-k+1} l_k} r_j,$

and

$b^{(j)}_{n-1} = \sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^{k-j} \frac{(-v)^{n-k} D_{k-j}^{(s)}}{u^{n-k+1} l_k} r_j.$

Then the followings are true:

(i) The sequence $(b^{(j)})_{j=0}^{\infty}$ is a basis for the space $X(r, s, t, p; B)$ for $X \in \{c_0(p), l(p)\}$ and any $x \in X(r, s, t, p; B)$ has a unique representation of the form

$x = \sum_{j=0}^{\infty} \nu_j b^{(j)}.$

(ii) The set $(b^{(j)})_{j=0}^{\infty}$ is a basis for the space $c(r, s, t, p; B)$ and any $x \in c(r, s, t, p; B)$ has a unique representation of the form

$x = \ell b^{(-1)} + \sum_{j=0}^{\infty} (\nu_j - \ell) b^{(j)},$

where $\ell = \lim_{n \to \infty} (A(r, s, t; B)x)_n.$
Remark 4.1. In particular, if we choose \( r_n = \frac{1}{p_n}, t_n = s_n', s_n = 1 \forall n, \) then the sequence space \( l(r, s, t, p; B) \) reduces to \( l(r', s', p; B) \) \cite{23}. With this choice of \( s_n \), we have \( D_0^{(s)} = D_1^{(s)} = 1 \) and \( D_{n_0}^{(s)} = 0 \) for \( n \geq 2 \). Thus the sequences \( b^{(j)} = (b_n^{(j)})_{n \in \mathbb{N}_0} \) for \( j = 0, 1, \ldots \) reduce to

\[
b_n^{(j)} = \begin{cases} \frac{(-1)^{n-k}}{r_n} \left( \frac{a^{n-j}}{n^{j+1}} \frac{1}{r_j} + \frac{a^{n-j-1}}{n^{j+1}} \frac{1}{r_{j+1}} \right) & \text{if } 0 \leq j < n \\ \frac{1}{a_{n-k} s_n} & \text{if } j = n \\ 0 & \text{if } j > n. \end{cases}
\]

The sequence \( (b^{(j)}) \) is a Schauder basis for the space \( l(r', s', p; B) \) studied in \cite{23}.

4.1 The \( \alpha-, \beta-, \gamma-\)duals of \( X(r, s, t, p; B) \) for \( X \in \{ \ell_\infty(p), c(p), c_0(p), l(p) \} \)

In 1999, K. G. Grosse-Erdmann \cite{11} has characterized the matrix transformations between the sequence spaces of Maddox, namely, \( \ell_\infty(p), c(p), c_0(p) \) and \( l(p) \). To compute the \( \alpha-, \beta-, \gamma-\)duals of \( X(r, s, t, p; B) \) for \( X \in \{ \ell_\infty(p), c(p), c_0(p), l(p) \} \) and to characterize the classes of some matrix mappings between these spaces, we list the following conditions.

Let \( L \) denotes a natural number, \( F \) be a nonempty finite subset of \( \mathbb{N} \) and \( A = (a_{nk})_{n,k} \) be an infinite matrix. We consider \( p'_k = \frac{p_k}{p_k - 1} \) for \( 1 < p_k < \infty \).

\[
\sup_F \sup_k \left| \sum_{n \in F} a_{nk} \right|_{p_k}^k < \infty \tag{4.4}
\]

\[
\sup_F \left( \sum_k \left| \sum_{n \in F} a_{nk} L^{-1} \right|_{p'_k}^k < \infty \right. \quad \text{for some } L \tag{4.5}
\]

\[
\lim_n a_{nk} = 0 \quad \text{for every } k \tag{4.6}
\]

\[
\sup_n \sup_k |a_{nk} L|_{p_k} < \infty \quad \text{for all } L \tag{4.7}
\]

\[
\sup_n \left( \sum_k |a_{nk} L|_{p'_k} < \infty \right. \quad \text{for all } L \tag{4.8}
\]

\[
\sup_n \sup_k |a_{nk}|_{p_k} < \infty \tag{4.9}
\]

\[
\exists (\alpha_k) \lim_{n \to \infty} a_{nk} = \alpha_k \quad \text{for all } k \tag{4.10}
\]

\[
\exists (\alpha_k) \sup_n \left( \left| a_{nk} - \alpha_k L \right|_{p_k} < \infty \right. \quad \text{for all } L \tag{4.11}
\]

\[
\exists (\alpha_k) \sup_n \left( \sum_k \left| a_{nk} - \alpha_k L \right|_{p'_k} < \infty \right. \quad \text{for all } L \tag{4.12}
\]

\[
\sup_n \sup_k |a_{nk} L^{-1}|_{p_k} < \infty \quad \text{for some } L \tag{4.13}
\]

\[
\sup_F \left( \sum_n \left| \sum_{k \in F} a_{nk} L^{-1} \right|_{p'_k} < \infty \right. \quad \text{for some } L \tag{4.14}
\]

\[
\sum_n \left| \sum_k a_{nk} \right| < \infty \tag{4.15}
\]

\[
\sup_F \left( \sum_n \left| \sum_{k \in F} a_{nk} L^{-1} \right|_{p'_k} < \infty \right. \quad \text{for all } L \tag{4.16}
\]

\[
\sup_n \sum_k |a_{nk} L^{-1}|_{p'_k} < \infty \quad \text{for some } L \tag{4.17}
\]
Lemma 4.1. \[\text{(}\ref{lem:4.1}\text{)}\] If \(1 < p_k \leq H < \infty\). Then we have

(i) \(A \in (l(p), l_1)\) if and only if (4.5) holds.

(ii) \(A \in (l(p), c_0)\) if and only if (4.6) and (4.8) hold.

(iii) \(A \in (l(p), c)\) if and only if (4.10), (4.12) and (4.24) hold.

(iv) \(A \in (l(p), l_\infty)\) if and only if (4.24) holds.

(b) if \(0 < p_k \leq 1\). Then we have

(i) \(A \in (l(p), l_1)\) if and only if (4.4) holds.

(ii) \(A \in (l(p), c_0)\) if and only if (4.6) and (4.7) hold.

(iii) \(A \in (l(p), c)\) if and only if (4.9), (4.10) and (4.11) hold.

(iv) \(A \in (l(p), l_\infty)\) if and only if (4.13) holds.

Lemma 4.2. \[\text{(}\ref{lem:4.2}\text{)}\] For \(0 < p_k \leq H < \infty\). Then we have

(i) \(A \in (c_0(p), l_1)\) if and only if (4.14) holds.

(ii) \(A \in (c(p), l_1)\) if and only if (4.14) and (4.15) hold.

(iii) \(A \in (l_\infty(p), l_1)\) if and only if (4.16) holds.

Lemma 4.3. \[\text{(}\ref{lem:4.3}\text{)}\] For \(0 < p_k \leq H < \infty\). Then we have

(i) \(A \in (c_0(p), l_\infty)\) if and only if (4.17) holds.

(ii) \(A \in (c(p), l_\infty)\) if and only if (4.17) and (4.18) hold.

(iii) \(A \in (l_\infty(p), l_\infty)\) if and only if (4.19) holds.

Lemma 4.4. \[\text{(}\ref{lem:4.4}\text{)}\] For \(0 < p_k \leq H < \infty\), we have

(i) \(A \in (c_0(p), c)\) if and only if (4.10), (4.17) and (4.20) hold.

(ii) \(A \in (c(p), c)\) if and only if (4.10), (4.17), (4.20), (4.21) hold.

(iii) \(A \in (l_\infty(p), c)\) if and only if (4.22), (4.23) hold.

We now define the following sets to obtain the \(\alpha\)-dual of the spaces \(X(r, s, t, p; B)\):

\[
H_1(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_F \sum_n \left| \sum_{k \in F} (-v)^{n-j} D^{(s)}_{j-k} \frac{1}{t_j} p_{n-k} a_n L^\frac{1}{p_k} \right| < \infty \right\}
\]
\[ H_2(p) = \left\{ a = (a_n) \in w : \sum_n \sum_{k=0}^n \sum_{j=k}^n (-1)^{j-k} \frac{(-v)^{n-j} D^{(s)}_{j-k}}{u^{n-j+1}} r_j a_n < \infty \right\} \]

\[ H_3(p) = \bigcap_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_{F_k} \left| \sum_{n \in F_k \cap j=k} \left( \sum_{j=k}^n (-1)^{j-k} \frac{(-v)^{n-j} D^{(s)}_{j-k}}{u^{n-j+1}} r_j a_n \right) \right| < \infty \right\} \]

\[ H_4(p) = \left\{ a = (a_n) \in w : \sup_{F_k} \left| \sum_{n \in F_k \cap j=k} (-1)^{j-k} \frac{(-v)^{n-j} D^{(s)}_{j-k}}{u^{n-j+1}} r_j a_n \right| < \infty \right\} \]

\[ H_5(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_{F_k} \left| \sum_{n \in F_k \cap j=k} (-1)^{j-k} \frac{(-v)^{n-j} D^{(s)}_{j-k}}{u^{n-j+1}} r_j a_n, L^{-1} \right| < \infty \right\} \].

**Theorem 4.4.** (a) If \( p_k > 1 \), then \( |l(r, s, t, p; B)|^\alpha = H_5(p) \) and \( |l(r, s, t, p; B)|^\alpha = H_4(p) \) for \( 0 < p_k \leq 1 \).

(b) For \( 0 < p_k \leq H < \infty \), then

(i) \([c_0(r, s, t, p; B)]^\alpha = H_1(p)\).

(ii) \([c(r, s, t, p; B)]^\alpha = H_1(p) \cap H_2(p)\).

(iii) \([l_\infty(r, s, t, p; B)]^\alpha = H_3(p)\).

Proof. (a) Let \( p_k > 1 \forall k, a = (a_n) \in w, x \in l(r, s, t, p; B) \) and \( y \in l(p) \). Then for each \( n \), we have

\[ a_n x_n = \sum_{k=0}^n \sum_{j=k}^n (-1)^{j-k} \frac{(-v)^{n-j} D^{(s)}_{j-k}}{u^{n-j+1}} r_j a_n y_k = (C y)_n, \]

where the matrix \( C = (c_{nk})_{n,k} \) is defined as

\[ c_{nk} = \begin{cases} \sum_{j=k}^n (-1)^{j-k} \frac{(-v)^{n-j} D^{(s)}_{j-k}}{u^{n-j+1}} r_j a_n & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n, \end{cases} \]

and \( x_n \) is given in (4.3). Thus for each \( x \in l(r, s, t, p; B), (a_n x_n)_n \in l_1 \) if and only if \((C y)_n \in l_1 \) where \( y \in l(p) \). Therefore \( a = (a_n) \in |l(r, s, t, p; B)|^\alpha \) if and only if \( C \in (l(p), l_1) \). By using Lemma [4.1] (a), we have

\[ |l(r, s, t, p; B)|^\alpha = H_5(p). \]

If \( 0 < p_k \leq 1 \forall k \), then using Lemma [4.1] (b), we have \([l(r, s, t, p; B)]^\alpha = H_4(p)\).

(b) In a similar way, using Lemma [4.2], it can be derived that \([c_0(r, s, t, p; B)]^\alpha = H_1(p), [c(r, s, t, p; B)]^\alpha = H_1(p) \cap H_2(p) \) and \([l_\infty(r, s, t, p; B)]^\alpha = H_3(p)\). \( \square \)

To find the \( \gamma \)-dual of the spaces \( X(r, s, t, p; B) \) for \( X \in \{ l_\infty(p), c(p), c_0(p), l(p) \} \), we consider the following sets:

\[ \Gamma_1(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \left| \sum_k c_{nk} \right| L^{-\frac{1}{\gamma}} < \infty \right\} \]

\[ \Gamma_2(p) = \left\{ a = (a_k) \in w : \sup_k \left| \sum_k c_{nk} \right| < \infty \right\} \]

\[ \Gamma_3(p) = \bigcap_{L \in \mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \left| \sum_k c_{nk} \right| L^{-\frac{1}{\gamma}} < \infty \right\} \]

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\[ \Gamma_4(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \left( \sum_k |e_{nk}L^{-1}|^{p_k} \right) < \infty \right\} \]
\[ \Gamma_5(p) = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \left( \sum_k |e_{nk}L^{-1}|^{p'_k} \right) < \infty \right\}, \]
where the matrix \( E = (e_{nk}) \) is defined as
\[
e_{nk} = \begin{cases} 
\frac{1}{u} \sum_{j=0}^{k+1} (-1)^{j-k} \frac{D^{(s)}_{j-k}}{t_j} \left( \sum_{l=j+1}^{n} \frac{(-v)^{l-j} a_{lj}}{u^{l-j+1}} \right) + \sum_{j=k+2}^{n} (-1)^{j-k} \frac{D^{(s)}_{j-k}}{t_j} \left( \sum_{l=j}^{n} \frac{(-v)^{l-j} a_{lj}}{u^{l-j+1}} \right) & 0 \leq k \leq n, \\
0 & k > n.
\end{cases}
\] (4.25)

**Theorem 4.5.** (a) If \( p_k > 1 \), then \([l(r,s,t,p;B)]^\gamma = \Gamma_5(p)\) and \([l(r,s,t,p;B)]^\gamma = \Gamma_4(p)\) for \( 0 < p_k \leq 1 \).
(b) For \( 0 < p_k \leq H < \infty \) then
(i) \([e_0(r,s,t,p;B)]^\gamma = \Gamma_1(p)\),
(ii) \([e(r,s,t,p;B)]^\gamma = \Gamma_1(p) \cap \Gamma_2(p)\),
(iii) \([l_\infty(r,s,t,p;B)]^\gamma = \Gamma_3(p)\).

**Proof.** (a) Let \( p_k > 1 \) for all \( k \), \( a = (a_n) \in w, x \in l(r,s,t,p;B) \) and \( y \in l(p) \). Then using (4.3), we have
\[
\sum_k a_k x_k = \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{j=0}^{k} (-1)^{j-l} \frac{D^{(s)}_{l-j}}{t_j} a_{lj} y_n a_k \\
= \sum_{k=0}^{n-1} \sum_{l=0}^{k} \sum_{j=0}^{k} (-1)^{j-l} \frac{D^{(s)}_{l-j}}{t_j} a_{lj} x_k y_n + \sum_{l=0}^{n} \sum_{j=0}^{n} (-1)^{j-l} \frac{D^{(s)}_{l-j}}{t_j} a_{lj} y_n x_k \\
\left[ \frac{1}{u} \sum_{j=0}^{k+1} (-1)^{j-k} \frac{D^{(s)}_{j-k}}{t_j} \left( \sum_{l=j+1}^{n} \frac{(-v)^{l-j} a_{lj}}{u^{l-j+1}} \right) + \sum_{j=k+2}^{n} (-1)^{j-k} \frac{D^{(s)}_{j-k}}{t_j} \left( \sum_{l=j}^{n} \frac{(-v)^{l-j} a_{lj}}{u^{l-j+1}} \right) \right] y_n x_k \\
\left[ \frac{1}{u} \sum_{j=0}^{k+1} (-1)^{j-k} \frac{D^{(s)}_{j-k}}{t_j} \left( \sum_{l=j+1}^{n} \frac{(-v)^{l-j} a_{lj}}{u^{l-j+1}} \right) + \sum_{j=k+2}^{n} (-1)^{j-k} \frac{D^{(s)}_{j-k}}{t_j} \left( \sum_{l=j}^{n} \frac{(-v)^{l-j} a_{lj}}{u^{l-j+1}} \right) \right] y_n x_k \\
= \left( E y \right)_n
\]
where \( E \) is the matrix defined in (4.25).

Thus \( a \in [l(r,s,t,p;B)]^\gamma \) if and only if \( ax = (a_k x_k) \in bs \) for \( x \in l(r,s,t,p;B) \) if and only if \( \left( \sum_k a_k x_k \right)_n \in l_\infty \), i.e., \( (Ey)_n \in l_\infty \), for \( y \in l(p) \). Hence using Lemma 3.11(a), we have
\[
[l(r,s,t,p;B)]^\gamma = \Gamma_5(p).
\]
If \( 0 < p_k \leq 1 \forall k \), then using Lemma 3.11(b), we have \([l(r,s,t,p;\Delta)]^\gamma = \Gamma_4(p)\).
(b) In a similar way, using Lemma 4.23, we can obtain \([c_0(r, s, t, p; B)]^\gamma = \Gamma_1(p), \ [c(r, s, t, p; B)]^\gamma = \Gamma_1(p) \cap \Gamma_2(p)\) and \([l_\infty(r, s, t, p; B)]^\gamma = \Gamma_3(p)\). □

To obtain \(\beta\)-duals of \(X(r, s, t, p; B)\), we define the following sets:

\[ B_1 = \{ a = (a_n) \in w : \sum_{l=k+1}^{\infty} \frac{(-v)^{l-j}}{u^{l-j+1}} a_l \text{ exists for all } k \} \]

\[ B_2 = \{ a = (a_n) \in w : \sum_{j=k+1}^{\infty} (-1)^{j-k} \frac{D_{1-k}^{(a)}}{t_j} \left( \sum_{l=j}^{\infty} \frac{(-v)^{l-j}}{u^{l-j+1}} a_l \right) \text{ exists for all } k \} \]

\[ B_3 = \{ a = (a_n) \in w : \left( \frac{r_k a_k}{t_k} \right) \in l_\infty(p) \} \]

\[ B_4 = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_n \sum_k |e_{nk}| L^{-1-p_k} < \infty \right\} \]

\[ B_5 = \{ a = (a_n) \in w : \sup_{n,k} |e_{nk}| p_k < \infty \} \]

\[ B_6 = \{ a = (a_n) \in w : \exists (\alpha_k) \lim_{n \to \infty} e_{nk} = \alpha_k \forall k \} \]

\[ B_7 = \bigcap_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \exists (\alpha_k) \sup_{n,k} \left( |e_{nk} - \alpha_k| L \right)^{pk} < \infty \right\} \]

\[ B_8 = \bigcap_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \exists (\alpha_k) \sup_{n,k} \left( |e_{nk} - \alpha_k| L \right)^{\rho_k} < \infty \right\} \]

\[ B_9 = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \exists (\alpha_k) \sup_{n,k} \left( |e_{nk} - \alpha_k| L \right)^{\rho_k} < \infty \right\} \]

\[ B_{10} = \bigcup_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_n \sum_{k} |e_{nk}| L^{1-p_k} < \infty \right\} \]

\[ B_{11} = \{ a = (a_n) \in w : \exists \lim_{n} \sum_{k} e_{nk} - \alpha = 0 \} \]

\[ B_{12} = \bigcap_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \sup_n \sum_{k} |e_{nk}| L^{\rho_k} < \infty \right\} \]

\[ B_{13} = \bigcap_{L \in \mathbb{N}} \left\{ a = (a_n) \in w : \exists (\alpha_k) \lim_{n} \sum_{k} |e_{nk} - \alpha_k| L^{\rho_k} = 0 \right\} \]

**Theorem 4.6.** (a) If \(p_k > 1\) for all \(k\), then \([l(r, s, t, p; B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_6 \cap B_8\) and if \(0 < p_k \leq 1\) for all \(k\), then \([l(r, s, t, p; B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_5 \cap B_6 \cap B_7\).

(b) Let \(p_k > 0\) for all \(k\). Then

(i) \([c_0(r, s, t, p; B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_6 \cap B_9 \cap B_{10},\)

(ii) \([c(r, s, t, p; B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_5 \cap B_9 \cap B_{11},\)

(iii) \([l_\infty(r, s, t, p; B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_{12} \cap B_{13}.\)

**Proof.** (a) Let \(p_k > 1\) for all \(k\). By Theorem 4.5, we have

\[ \sum_{k=0}^{n} a_k x_k = (Ey)_n, \]

where the matrix \(E\) is defined in (4.23). Thus \(a \in [l(r, s, t, p; B)]^\beta\) if and only if \(ax = (a_k x_k) \in cs\), where \(x \in l(r, s, t, p; B)\) if and only if \((Ey)_n \in c\), where \(y \in l(p)\), i.e., \(E \in (l(p), c)\). Hence by Lemma 4.1(a), we
Therefore \(|l(r, s, t, p; B)|^\beta = B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5 \cap B_6 \cap B_7\).

If \(0 < p_k \leq 1 \forall k\), then using Lemma 4.4 (b), we have

\[
\sup_{n, k \in N_0} |e_{nk}|^{p_k} < \infty,
\]
\[
\exists (\alpha_k) \lim_{n \to \infty} e_{nk} = \alpha_k \text{ for all } k,
\]
\[
\exists (\alpha_k) \sup_{n, k \in N_0} \left( |e_{nk} - \alpha_k| L \right)^{p_k} < \infty \text{ for all } L \in N.
\]

Thus \(|l(r, s, t, p; B)|^\beta = B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5 \cap B_6 \cap B_7\).

(b) In a similar way, using Lemma 4.4 we can obtain the \(\beta\)-duals of \(c_0(r, s, t, p; B)\), \(c(r, s, t, p; B)\) and \(l_\infty(r, s, t, p; B)\). \(\square\)

### 4.2 Matrix mappings

**Theorem 4.7.** Let \(\tilde{E} = (\tilde{e}_{nk})\) be the matrix which is same as the matrix \(E = (e_{nk})\) defined in (4.25), where \(a_k\) and \(a_n\) is replaced by \(a_{nk}\) and \(a_{nl}\) respectively.

(a) Let \(1 < p_k \leq H < \infty\) for \(k \in N_0\), then \(A \in (l(r, s, t, p; B), l_\infty)\) if and only if there exists \(L \in N\) such that

\[
\sup_n \sum_{k} |\tilde{e}_{nk} L^{-1}|^{p_k} < \infty \quad \text{and} \quad (a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5 \cap B_6 \cap B_8.
\]

(b) Let \(0 < p_k \leq 1\) for \(k \in N\). Then \(A \in (l(r, s, t, p; B), l_\infty)\) if and only if there exists \(L \in N\) such that

\[
\sup_n \sum_{k} |\tilde{e}_{nk} L^{-1}|^{p_k} < \infty \quad \text{and} \quad (a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_5 \cap B_6 \cap B_7.
\]

**Proof.** (a) Let \(p_k > 1\) for all \(k\). Since \((a_{nk})_k \in \{l(r, s, t, p; B)\}^\beta\) for each fixed \(n\), \(Ax\) exists for all \(x \in l(r, s, t, p; B)\). Now for each \(n\), we have

\[
\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} r_k \left[ \frac{1}{u} \frac{a_{nk}}{s_0} + \sum_{j=k+1}^{k+1} (-1)^{j-k} \frac{D^{(s)}}{l_j} \left( \sum_{l=k+1}^{n} \frac{(-v)^{l-j} u^{l-j+1} a_{nl}}{l_j} \right) + \sum_{j=k+2}^{n} (-1)^{j-k} \frac{D^{(s)}}{l_j} \left( \sum_{l=j}^{n} \frac{(-v)^{l-j} u^{l-j+1} a_{nl}}{l_j} \right) \right] y_k
\]

\[
= \sum_{k=0}^{m} \tilde{e}_{nk} y_k,
\]
Taking $m \to \infty$, we have
\[
\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty} \tilde{e}_{nk}y_k \quad \text{for all } n \in \mathbb{N}_0.
\]
We know that for any $T > 0$ and any complex numbers $a, b$
\[
|ab| \leq T(|aT^{-1}|^{p'} + |b|^p)
\]
where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Now one can easily find that
\[
\sup_n \left| \sum_k a_{nk}x_k \right| \leq \sup_n \left| \sum_k \tilde{e}_{nk} \right| \leq T \left[ \sup_n \left| \sum_k \tilde{e}_{nk} T^{-1} |x_k|^{p'} + \sum_k |y_k|^p \right| \right] < \infty.
\]
Conversely, assume that $A \in (l(r, s, t, p; B), l_{\infty})$ and $1 < p_k \leq H < \infty$ for all $k$. Then $Ax$ exists for each $x \in l(r, s, t, p; B)$, which implies that $(a_{nk})_k \in [l(r, s, t, p; B)]^\beta$ for each $n$. Thus $(a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5 \cap B_6 \cap B_7$. Since $\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty} \tilde{e}_{nk}y_k$, we have $\tilde{E} = (\tilde{e}_{nk}) \in (l(p), l_{\infty})$. Now using Lemma 4.1(a), we have $\sup_n \sum_k \left| \tilde{e}_{nk} L^{-1} \right|^{p_k} < \infty$ for some $L \in \mathbb{N}$. This completes the proof.

(b) We omit the proof of this part as it is similar to the previous part.

**Theorem 4.8.** (a) Let $1 < p_k \leq H < \infty$ for $k \in \mathbb{N}$, then $A \in (l(r, s, t, p; B), l_{1})$ iff there exists $L \in \mathbb{N}$ such that
\[
\sup_{F} \sum_{k \in F} \left| \sum_{n \in F} \tilde{e}_{nk} L^{-1} \right|^{p_k} < \infty \quad \text{for some } L \in \mathbb{N} \quad \text{and} \quad (a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5 \cap B_6 \cap B_7.
\]
(b) Let $0 < p_k \leq 1$ for $k \in \mathbb{N}$. Then $A \in (l(r, s, t, p; B), l_{1})$ iff
\[
\sup \sup_{F} \sum_{k \in F} \left| \sum_{n \in F} \tilde{e}_{nk} \right|^{p_k} < \infty \quad \text{and} \quad (a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_5 \cap B_6 \cap B_7.
\]

**Proof.** We omit the proof as it follows as the same way.

## 5 Measure of noncompactness and compact operators on the space $l_p(r, s, t; B)$

In this section, we concentrate on $l_p(r, s, t; B)$, $p \geq 1$, which is a $BK$ space and establish some identities or estimates for the Hausdorff measure of noncompactness of certain matrix operators on the space $l_p(r, s, t; B)$. Moreover, we characterize some classes of compact operators on this space.

The Hausdorff measure of noncompactness was first introduced and studied by Goldenstein, Gohberg and Markus in 1957 and later on studied by Istrătescu in 1972 [12]. It is quite natural to find necessary and sufficient conditions for a matrix mapping between $BK$ spaces to define a compact operator as the matrix transformations between $BK$ spaces are continuous. This can be achieved with the help of Hausdorff measure of noncompactness. Recently several authors, namely, Malkowsky and Rakóčević [18], Dojolović et al. [10], Dojolović [9], Mursaleen and Noman (20, 21), Başarır and Kara [5] etc. have established some identities or estimates for the operator norms and the Hausdorff measure of noncompactness of matrix operators from an arbitrary $BK$ space to arbitrary $BK$ space. Let us recall some definitions and well-known results.
Let $X$, $Y$ be two Banach spaces and $S_X$ denotes the unit sphere in $X$, i.e., $S_X = \{ x \in X : \|x\| = 1 \}$.

We denote by $B(X,Y)$, the set of all bounded (continuous) linear operators $L : X \to Y$, which is a Banach space with the operator norm $\| L \| = \sup_{x \in S_X} \| L(x) \|_Y$ for all $L \in B(X,Y)$. A linear operator $L : X \to Y$ is said to be compact if the domain of $L$ is all of $X$ and for every bounded sequence $(x_n) \in X$, the sequence $(L(x_n))$ has a subsequence which is convergent in $Y$ and we denote by $C(X,Y)$, the class of all compact operators in $B(X,Y)$. An operator $L \in B(X,Y)$ is said to be finite rank if $\dim R(L) < \infty$, where $R(L)$ is the range space of $L$. If $X$ is a $BK$ space and $a = (a_k) \in w$, then we consider

$$
\|a\|_X = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|,
$$

(5.1)

provided the expression on the right side exists and is finite which is the case whenever $a \in X^\beta$. 

Let $(X,d)$ be a metric space and $\mathcal{M}_X$ be the class of all bounded subsets of $X$. Let $B(x,r) = \{ y \in X : d(x,y) < r \}$ denotes the open ball of radius $r > 0$ with centre at $x$. The Hausdorff measure of noncompactness of a set $Q \in \mathcal{M}_X$, denoted by $\chi(Q)$, is defined as

$$
\chi(Q) = \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{i=0}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon, n \in \mathbb{N}_0 \right\}.
$$

The function $\chi : \mathcal{M}_X \to [0,\infty)$ is called the Hausdorff measure of noncompactness. The basic properties of the Hausdorff measure of noncompactness can be found in ([18], [10], [17], [19]). For example, if $Q, Q_1$ and $Q_2$ are bounded subsets of a metric space $(X,d)$ then

$$
\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded} \quad \text{and}
$$

if $Q_1 \subset Q_2$ then $\chi(Q_1) \leq \chi(Q_2)$.

Also if $X$ is a normed space, the function $\chi$ has some additional properties due to linear structure, namely,

$$
\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),
$$

$$
\chi(\alpha Q) = |\alpha| \chi(Q) \text{ for all } \alpha \in \mathbb{K}.
$$

Let $\phi$ denotes the set of all finite sequences, i.e., of sequences that terminate in zeros. Throughout we denote $p'$ as the conjugate of $p$ for $1 \leq p < \infty$, i.e., $p' = \frac{p}{p-1}$ for $p > 1$ and $p' = \infty$ for $p = 1$. The following known results are fundamental for our investigation.

**Lemma 5.1.** ([21]) Let $1 \leq p < \infty$ and $A \in (l_p, \ell)$. Then the followings hold:

(i) $\alpha_k = \lim_{n \to \infty} a_{nk}$ exists for all $k \in \mathbb{N}_0$,

(ii) $\alpha = (\alpha_k) \in l_{p'}$,

(iii) $\sup_n \| A_n - \alpha \|_{l_{p'}} < \infty$,

(iv) $\lim_{n \to \infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k$ for all $x = (x_k) \in l_p$.
Lemma 5.2. (15, Theorem 1.29) Let $1 \leq p < \infty$. Then we have $l_p^2 = l_{p'}$ and $\|a\|_{l_p'} = \|a\|_{l_p}$ for all $a \in l_p'$.

Lemma 5.3. (21) Let $X \supset \phi$ and $Y$ be BK spaces. Then we have $(X, Y) \subset B(X, Y)$, i.e., every matrix $A \in (X, Y)$ defines an operator $L_A \in B(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$.

Lemma 5.4. (9) Let $X \supset \phi$ be a BK space and $Y$ be any of the spaces $c_0$, $c$ or $l_\infty$. If $A \in (X, Y)$, then we have

$$\|L_A\| = \|A\|_{(X, l_\infty)} = \sup_n \|A_n\|_{X} < \infty.$$ 

Lemma 5.5. (15) Let $Q \in M_X$, where $X = l_p$ for $1 \leq p < \infty$ or $c_0$. If $P_m : c_0 \rightarrow c_0$ ($m \in \mathbb{N}_0$) be the operator defined by $P_m(x) = (x_0, x_1, \ldots, x_m, 0, 0, \cdots)$ for all $x = (x_k) \in X$. Then we have

$$\chi(Q) = \lim_{m \rightarrow \infty} \left( \sup_{x \in Q} \|(I - P_m)(x)\| \right),$$

where $I$ is the identity operator on $X$.

Let $z = (z_n) \in c$. Then $z$ has a unique representation $z = \ell e + \sum_{n=0}^{\infty} (z_n - \ell)e_n$, where $\ell = \lim_{n \rightarrow \infty} z_n$. We now define the projections $P_m$ ($m \in \mathbb{N}_0$) from $c$ onto the linear span of $\{e, e_0, e_1, \cdots, e_m\}$ as

$$P_m(z) = \ell e + \sum_{n=0}^{m} (z_n - \ell)e_n,$$

for all $z \in c$ and $\ell = \lim_{n \rightarrow \infty} z_n$. Then the following result gives an estimate for the Hausdorff measure of noncompactness in the BK space $c$.

Lemma 5.6. (15) Let $Q \in M_c$ and $P_m : c \rightarrow c$ be the projector from $c$ onto the linear span of $\{e, e_0, e_1, \cdots, e_m\}$. Then we have

$$\frac{1}{2} \lim_{m \rightarrow \infty} \left( \sup_{x \in Q} \|(I - P_m)(x)\|_{l_\infty} \right) \leq \chi(Q) \leq \lim_{m \rightarrow \infty} \left( \sup_{x \in Q} \|(I - P_m)(x)\|_{l_\infty} \right),$$

where $I$ is the identity operator on $c$.

Lemma 5.7. (15) Let $X, Y$ be two Banach spaces and $L \in B(X, Y)$. Then

$$\|L\|_X = \chi(L(S_X))$$

and

$$L \in C(X, Y) \text{ if and only if } \|L\|_X = 0.$$

Let $\mathcal{F}_m = \{F \in \mathcal{F} : n > m, \ \forall n \in F\}$, $m \in \mathbb{N}$ and $\mathcal{F}$ is the collection of nonempty and finite subsets of $\mathbb{N}$.

Lemma 5.8. (20) Let $X \supset \phi$ be a BK space.

(a) If $A \in (X, c_0)$, then

$$\|L_A\|_X = \lim_{n \rightarrow \infty} \sup_{n} \|A_n\|_X$$

and
Lemma 5.9. Let $X$ be a Banach space. If $A \in (X, l_\infty)$, then

$$L_A \text{ is compact if and only if } \lim_{n \to \infty} \|A_n\|_X = 0.$$  

(b) If $A \in (X, l_\infty)$, then

$$0 \leq \|L_A\|_X \leq \limsup_{n \to \infty} \|A_n\|_X$$

and

$$L_A \text{ is compact if and only if } \lim_{n \to \infty} \|A_n\|_X = 0.$$  

(c) If $A \in (X, l_1)$, then

$$\lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left\| \sum_{n \in \mathcal{F}} A_n \right\|_X^\ast \right) \leq \|L_A\|_X \leq 4 \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left\| \sum_{n \in \mathcal{F}} A_n \right\|_X^\ast \right)$$

and

$$L_A \text{ is compact if and only if } \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left\| \sum_{n \in \mathcal{F}} A_n \right\|_X^\ast \right) = 0.$$  

We establish the following lemmas which are required to characterize the classes of compact operators with the help of Hausdorff measure of noncompactness.

Lemma 5.9. If $a = (a_k) \in [l_p(r, s, t; B)]^\beta$, then $\tilde{a} = (\tilde{a}_k) \in l_{p'}$ and the equality

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k y_k$$

holds for every $x = (x_k) \in l_p(r, s, t; B)$ and $y = (y_k) \in l_p$, where $y = (A(r, s, t)B)x$. In addition

$$\tilde{a}_k = r_k \left[ \frac{a_k}{s_0 l_k} u \sum_{i=k}^{k+1} (-1)^{i-k} D_i(s) \frac{D_{i-k}}{l_i} \left( \sum_{j=k+1}^{\infty} \frac{(-v)^{j-i}}{u^{j-i+1} a_j} \right) + \sum_{i=k+2}^{\infty} (-1)^{i-k} D_i(s) \frac{D_{i-k}}{l_i} \left( \sum_{j=1}^{\infty} \frac{(-v)^{j-i}}{u^{j-i+1} a_j} \right) \right]. \quad (5.2)$$

Proof. Let $a = (a_k) \in [l_p(r, s, t; B)]^\beta$. Then by ([9], Theorem 2.3, Remark 2.4), we have $R(a) = (R_k(a)) \in l_p^\beta = l_{p'}$ and also

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k(a) T_k(x) \quad \forall x \in l_p(r, s, t; B),$$

where

$$R_k(a) = r_k \left[ \frac{a_k}{s_0 l_k} u \sum_{i=k}^{k+1} (-1)^{i-k} D_i(s) \frac{D_{i-k}}{l_i} \left( \sum_{j=k+1}^{\infty} \frac{(-v)^{j-i}}{u^{j-i+1} a_j} \right) + \sum_{i=k+2}^{\infty} (-1)^{i-k} D_i(s) \frac{D_{i-k}}{l_i} \left( \sum_{j=1}^{\infty} \frac{(-v)^{j-i}}{u^{j-i+1} a_j} \right) \right] = \tilde{a}_k.$$  

and $y = T(x) = (A(r, s, t)B)x$. This completes the proof. \hfill \Box

Lemma 5.10. Let $1 \leq p < \infty$. Then we have

$$\|a\|_{[l_p(r, s, t; B)]^\beta} = \|\tilde{a}\|_{l_{p'}} = \left\{ \begin{array}{ll}
\left( \sum_{k=0}^{\infty} |\tilde{a}_k|^{p'} \right)^{\frac{1}{p'}} & 1 < p < \infty, \\
\sup_k |\tilde{a}_k| & p = 1.
\end{array} \right.$$  

for all $a = (a_k) \in [l_p(r, s, t; B)]^\beta$, where $\tilde{a} = (\tilde{a}_k)$ is defined in (5.2).
Proof. Let $a = (a_k) \in \{l_p(r, s, t; B)\}^\beta$. Then from Lemma 5.9 we have $\tilde{a} = (\tilde{a}_k) \in l'_p$. Also $x \in S_{l_p(r, s, t; B)}$ if and only if $y = T(x) \in S_{l'_p}$ as $\|x\|_{l_p(r, s, t; B)} = \|y\|_{l'_p}$. From (5.1), we have

$$\|a\|_{l_p(r, s, t; B)}^* = \sup_{x \in S_{l_p(r, s, t; B)}} \left( \sum_{k=0}^{\infty} a_k x_k \right) = \sup_{y \in S_{l'_p}} \left( \sum_{k=0}^{\infty} \tilde{a}_k y_k \right) = \|\tilde{a}\|_{l'_p}^*.$$  

Using Lemma 5.2 we have $\|a\|_{l_p(r, s, t; B)}^* = \|\tilde{a}\|_{l'_p}^* = \|\tilde{a}\|_{l'_p}$, which is finite as $\tilde{a} \in l'_p$. This completes the proof.

Lemma 5.11. Let $1 \leq p < \infty$, $Y$ be any sequence space and $A = (a_{nk})_{n,k}$ be an infinite matrix. If $A \in \{l_p(r, s, t; B), Y\}$ then $\tilde{A} \in \{l_p, Y\}$ such that $Ax = \tilde{A}y$ holds for all $x \in l_p(r, s, t; B)$ and $y \in l_p$, which are connected by the relation $y = (A(r, s, t), B)x$ and $\tilde{A} = (\tilde{a}_{nk})_{n,k}$ is given by

$$\tilde{a}_{nk} = r_k \left[ \frac{a_{nk}}{n!} k! \right] + \sum_{i=k}^{k+1} \frac{1}{i!} \left( \sum_{j=k+1}^{\infty} \frac{1}{u^j} (a_{n,j}) \right) + \sum_{i=k}^{\infty} \frac{(1)_{i-k}}{i!} \left( \sum_{j=k+1}^{\infty} \frac{1}{u^j} (a_{n,j}) \right),$$

provided the series on the right side converges for all $n,k$.

Proof. We assume that $A \in \{l_p(r, s, t; B), Y\}$, then $A_n \in \{l_p(r, s, t; B)\}^\beta$ for all $n$. Thus it follows from Lemma 5.9 we have $\tilde{A}_n \in l'_p = l'_p$ for all $n$ and $Ax = \tilde{A}y$ holds for every $x \in l_p(r, s, t; B)$ and $y \in l_p$, which are connected by the relation $y = (A(r, s, t), B)x$. Hence $\tilde{A}y \in Y$. Since $x = B^{-1}.A(r, s, t)^{-1}y$, for every $y \in l_p$, we get some $x \in l_p(r, s, t; B)$ and hence $\tilde{A} \in \{l_p, Y\}$. This completes the proof.

Theorem 5.1. Let $1 < p < \infty$.

(a) If $A \in \{l_p(r, s, t; B), c_0\}$ then

$$\|L_A\|_\chi = \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'} \right)^{1/p'} (5.3)$$

and $L_A$ is compact if and only if $\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'} \right)^{1/p'} = 0$.

(b) If $A \in \{l_p(r, s, t; B), l_\infty\}$ then

$$0 \leq \|L_A\|_\chi \leq \limsup_{n \to \infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk}| (5.4)$$

and $L_A$ is compact if and only if $\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'} \right)^{1/p'} = 0$.

Proof. Let $1 < p < \infty$ and $A \in \{l_p(r, s, t; B), c_0\}$, then $A_n \in \{l_p(r, s, t; B)\}^\beta$ for all $n$ and hence $\tilde{A}_n \in l'_p$ by Lemma 5.9. Again using Lemma 5.11 we have

$$\|A_n\|_{l'_p(r, s, t; B)}^* = \|\tilde{A}_n\|_{l'_p}^* = \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'} \right)^{1/p}.$$  

Now by Lemma 5.8 we have $\|L_A\|_\chi = \limsup_{n \to \infty} \|A_n\|_{l'_p(r, s, t; B)}^* = \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'} \right)^{1/p}.$

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Using Lemma [5.7], we have $L$ is compact if and only if \( \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} = 0. \)

Similarly, we can prove the part(b).

**Theorem 5.2.** If $A \in (l_p(r, s, t; B), c)$ then

\[
\frac{1}{2} \limsup_{n \to \infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k| \leq \|L_A\|_\chi \leq \limsup_{n \to \infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k|, \tag{5.5}
\]

where $\tilde{a}_k = \lim_{n \to \infty} \tilde{a}_{nk}$ for all $k$.

**Proof.** Let $A \in (l_p(r, s, t; B), c)$. Then by using Lemma [5.11] & 5.1 we can deduce that the expression in \((5.5)\) exists. We write $S = S_{l_p(r, s, t; B)}$ in short. Then by Lemma [5.7] we have $\|L_A\|_\chi = \chi(AS)$. Since $l_p(r, s, t; B)$ and $c$ are BK spaces, $A$ induces continuous map $L_A$ from $l_p(r, s, t; B)$ to $c$ by Lemma [5.3]. Thus $AS$ is bounded in $c$, i.e., $AS \in M_c$. Let $P_m : c \to c$, $(m \in \mathbb{N}_0)$ be the projection from $c$ onto the span of \(\{e, e_0, e_1, \cdots, e_m\}\) defined by

\[
P_m(z) = \ell e + \sum_{k=0}^{m} (z_k - \ell)e_k,
\]

where $\ell = \lim_{k \to \infty} z_k$. Thus for every $m$, we have

\[
(I - P_m)(z) = \sum_{k=m+1}^{\infty} (z_k - \ell)e_k,
\]

where $I$ is the identity operator. Therefore $\| (I - P_m)(z) \|_\infty = \sup_{k=m+1} |z_k - \ell|$ for all $z = (z_k) \in c$. So by applying Lemma [5.6] we have

\[
\frac{1}{2} \lim_{m \to \infty} \left( \sup_{x \in S} \| (I - P_m)(Ax) \|_{l_\infty} \right) \leq \|L_A\|_\chi \leq \lim_{m \to \infty} \left( \sup_{x \in S} \| (I - P_m)(Ax) \|_{l_\infty} \right). \tag{5.6}
\]

Since $A \in (l_p(r, s, t; B), c)$, we have by Lemma [5.11] $\tilde{A} \in (l_p, c)$ and $Ax = \tilde{A}y$ for every $x \in l_p(r, s, t; B)$ and $y \in l_p$ which are connected by the relation $y = (A(r, s, t).B)x$. Again applying Lemma [5.1] we have $\tilde{a}_k = \lim_{n \to \infty} \tilde{a}_{nk}$ exists for all $k$, $\tilde{a} = (\tilde{a}_k) \in X^\beta = l_{p'}$ and $\tilde{A}_n(y) = \sum_{k=0}^{\infty} \tilde{a}_k y_k$. Since $\| (I - P_m)(z) \|_{l_\infty} = \sup_{k>m} |z_k - \ell|$, we have

\[
\| (I - P_m)(Ax) \|_{l_\infty} = \| (I - P_m)(\tilde{A}y) \|_{l_\infty}
\]

\[
= \sup_{n>m} |\tilde{A}_n(y) - \sum_{k=0}^{\infty} \tilde{a}_k y_k|
\]

\[
= \sup_{n>m} |\sum_{k=0}^{\infty} (\tilde{a}_{nk} - \tilde{a}_k) y_k|.
\]

Also we know that $x \in S = S_{l_p(r, s, t; B)}$ if and only if $y \in S_{l_{p'}}$. From [5.7] and Lemma [5.2] we deduce that
\[
\sup_{x \in S} \| (I - P_m)(Ax) \|_\infty = \sup_{n > m} \left( \sup_{y \in S_p} \left( \sum_{k=0}^{\infty} (\bar{a}_{nk} - \tilde{\alpha}_k)y_k \right) \right) \\
= \sup_{n > m} \| \bar{A}_n - \tilde{\alpha} \|_{l_p} \\
= \sup_{n > m} \| \bar{A}_n - \tilde{\alpha} \|_{l_{p'}}
\]

Hence from (5.10), we have
\[
\frac{1}{p} \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} |\bar{a}_{nk} - \tilde{\alpha}_k|^p \right)^{\frac{1}{p}} \leq \| L_A \|_\chi \leq \limsup_{n \to \infty} \left( \sum_{k=0}^{\infty} |\bar{a}_{nk} - \tilde{\alpha}_k|^p \right)^{\frac{1}{p}}.
\]
This completes the proof. \(\square\)

**Theorem 5.3.** Let \(1 \leq p < \infty\). If \(A \in (l_1(r, s, t; B), l_p)\), then

\[
\| L_A \|_\chi = \lim_{m \to \infty} \left( \sup_{k} \left( \sum_{n=m+1}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \right).
\]

**Proof.** Let \(A \in (l_1(r, s, t; B), l_p)\). We write \(S = S_{l_1(r, s, t; B)}\) in short. Since \(l_1(r, s, t; B)\) and \(l_p\) are BK spaces, by Lemma 5.3 we have \(AS \in M_{l_p}\). Also by Lemma 5.7 we have \(\| L_A \|_\chi = \chi(AS)\). Now by Lemma 5.5

\[
\chi(AS) = \lim_{m \to \infty} \left( \sup_{x \in S} \| (I - P_m)(Ax) \|_{l_p} \right),
\]
where \(P_m : l_p \to l_p\) is the operator defined by \(P_m(x) = (x_0, x_1, \ldots, x_m, 0, 0, \ldots)\) for all \(x = (x_k) \in l_p\) and \(m \in \mathbb{N}_0\). Since \(A \in (l_1(r, s, t; B), l_p)\), we have by Lemma 5.11 \(\tilde{A} \in (l_1, l_p)\) and \(Ax = \tilde{A}y\) holds for every \(x \in l_1(r, s, t; B)\) and \(y \in l_p\) which are connected by the relation \(y = (A(r, s, t,B)x)\). Now for each \(m \in \mathbb{N}_0\),

\[
\| (I - P_m)(Ax) \|_{l_p} = \| (I - P_m)\tilde{A}y \|_{l_p} = \left( \sum_{n=m+1}^{\infty} |\tilde{A}_n(y)|^p \right)^{\frac{1}{p}} \\
= \left( \sum_{n=m+1}^{\infty} \left( \sum_{k=0}^{\infty} |\bar{a}_{nk}|y_k \right)^p \right)^{\frac{1}{p}} \\
\leq \sum_{k=0}^{\infty} \left( \sum_{n=m+1}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \\
\leq \| y \|_{l_1} \left( \sup_{k} \left( \sum_{n=m+1}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \right) \\
= \| x \|_{l_1(r, s, t; B)} \left( \sup_{k} \left( \sum_{n=m+1}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \right)
\]

Thus
\[
\sup_{x \in S} \| (I - P_m)(Ax) \|_{l_p} \leq \left( \sup_{k} \left( \sum_{n=m+1}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \right).
\]
Hence
\[ \|L_A\|_\chi \leq \lim_{m \to \infty} \left( \sup_k \left( \sum_{n=m+1}^\infty |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right). \]

Conversely, let \( b^{(j)} \in l_1(r,s,t;B) \) such that \( (A(r,s,t;B)b^{(j)})_j = e_j \), where the sequence \( (b^{(j)})_{j=0}^\infty \) is a basis in \( l_1(r,s,t;B) \) defined in Theorem 4.3. Thus \( A(b^{(j)}) = \tilde{A}e_j \) for each \( j \). Let \( D = \{b^{(j)} : j \in \mathbb{N}_0\} \). Then \( AD \subset AS \) and by the property of \( \chi \), we have \( \chi(AD) \leq \chi(AS) = \|L_A\|_\chi \). Further, by Lemma 5.5

\[ \chi(AD) = \lim_{m \to \infty} \left( \sup_k \left( \sum_{n=m+1}^\infty |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right) \leq \|L_A\|_\chi. \]

Thus we have \( \|L_A\|_\chi = \lim_{m \to \infty} \left( \sup_k \left( \sum_{n=m+1}^\infty |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right). \)

**Theorem 5.4.** Let \( 1 < p < \infty \). If \( A \in (l_p(r,s,t;B), l_1) \), then

\[ \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right) \leq \|L_A\|_\chi \leq 4 \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right) \]

and

\[ L_A \text{ is compact if and only if } \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right) = 0. \]

**Proof.** Let \( A \in (l_p(r,s,t;B), l_1) \). Then \( A_n \in [(l_p(r,s,t;B)]^\beta \) for all \( n \) and hence \( \tilde{A}_n \in l_{p'} \) by Lemma 5.10. Using Lemma 5.10 we have

\[ \left\| \sum_{n \in F} A_n \right\|_{l_p(r,s,t;B)} = \left\| \sum_{n \in F} \tilde{A}_n \right\|_{l_{p'}} = \left( \sum_{k=0}^\infty \sum_{n \in F} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}}. \]

Now using Lemma 5.10(c), we obtain

\[ \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right) \leq \|L_A\|_\chi \leq 4 \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right). \]

Thus \( L_A \) is compact if and only if \( \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} |\tilde{a}_{nk}|^{p'} \right)^{\frac{1}{p'}} \right) = 0. \)

**Theorem 5.5.** Let \( 1 < p < \infty \). If \( A \in (l_p(r,s,t;B), bv) \), then

\[ \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} (\tilde{a}_{nk} - \tilde{a}_{n-1,k})^{p'} \right)^{\frac{1}{p'}} \right) \leq \|L_A\|_\chi \leq 4 \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} (\tilde{a}_{nk} - \tilde{a}_{n-1,k})^{p'} \right)^{\frac{1}{p'}} \right) \]

and

\[ L_A \text{ is compact if and only if } \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^\infty \sum_{n \in F} (\tilde{a}_{nk} - \tilde{a}_{n-1,k})^{p'} \right)^{\frac{1}{p'}} \right) = 0. \]

**Proof.** Using matrix domain, the sequence space \( bv \) of bounded variation sequences can be written as matrix domain of triangle \( \Delta \), i.e., \( bv = (l_1)_{\Delta} \). Let \( A \in (l_p(r,s,t;B), bv) \). Then for each \( x \in l_p(r,s,t;B) \),
we get \((\Delta A)x = \Delta (Ax)\in l_1\). So by Theorem 5.4, we have

\[
\lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^{\infty} \left| \sum_{n \in F} (\tilde{a}_{nk} - \tilde{a}_{n-1,k}) \right|^p \right) \right) \leq \|L_A\| \left\| X \right\| \leq 4 \lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^{\infty} \left| \sum_{n \in F} (\tilde{a}_{nk} - \tilde{a}_{n-1,k}) \right|^p \right) \right)
\]

and \(L_A\) is compact if and only if \(\lim_{m \to \infty} \left( \sup_{F \in \mathcal{F}_m} \left( \sum_{k=0}^{\infty} \left| \sum_{n \in F} (\tilde{a}_{nk} - \tilde{a}_{n-1,k}) \right|^p \right) \right) = 0\). This proves the theorem.

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