A hodgepodge of sets of reals

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Abstract

We open up a grab bag of miscellaneous results and remarks about sets of reals. The reader will be treated to a cornucopia of delightful, delectable, and delicious ideas to pound and shake his head at in consternation while muttering “Who would have thought of that?” and “Why didn’t they keep it to themselves?”.

Results concern: Kysiak and Laver-null sets, Kočinac and $\gamma_k$-sets, Fleissner and square $Q$-sets, Alikhani-Koopaei and minimal $Q$-like-sets, Rubin and $\sigma$-sets, and Zapletal and the Souslin number.

See the survey papers Brown, Cox [1], and Miller [10, 18].

1 $\sigma$-sets are Laver null

A subtree $T \subseteq \omega^{<\omega}$ of the finite sequences of elements of $\omega = \{0, 1, 2, \ldots\}$ is called a Laver tree [13] iff there exists $s \in T$ (called the root node of $T$) with the property that for every $t \in T$ with $s \subseteq t$ there are infinitely many $n \in \omega$ with $tn$ in $T$. Here $tn$ is the sequence of length exactly one more than $t$ and ending in $n$. We use $[T]$ to denote the infinite branches of $T$, i.e.,

$$[T] = \{ x \in \omega^\omega : \forall n \in \omega \ x \upharpoonright n \in T \}$$

A set $X \subseteq \omega^\omega$ is Laver-null iff for every Laver tree $T$ there exists a Laver subtree $T' \subseteq T$ such that

$$[T'] \cap X = \emptyset$$

This is analogous to the ideal of Marczewski null sets, $(s)_0$. For some background on this topic, see Kysiak and Weiss [11] and Brown [2].

\[1\] Thanks to the conference organizers: Cosimo Guido, Ljubisa Kočinac, Boaz Tsaban, Liljana Babinkostova, and Marion Scheepers for their generosity in inviting me to speak at the Workshop on Coverings, Selections and Games in Topology, December 2005, University of Lecce, Italy.

Mathematics Subject Classification 2000: 03E17; 54D20; 03E50

Keywords: $\gamma$-set, $\sigma$-sets, Laver forcing, $Q$-sets, universal $G_\delta$-sets, retractive boolean algebras, Souslin numbers, Borel hierarchies, meager ideal.
A separable metric space \( X \) is a \( \sigma \)-set iff every \( G_\delta \) in \( X \) is also \( F_\sigma \). It is known to be relatively consistent (Miller [15]) with the usual axioms of set theory that every \( \sigma \)-set is countable.

At the Lecce conference, Kysiak\(^2\) asked if it is consistent to have a \( \sigma \)-set which is not Laver-null. The answer is no.

**Theorem 1.1** Every \( \sigma \)-set is Laver-null. In fact, the Borel hierarchy of a non-Laver-null set must have \( \omega_1 \) levels.

**Proof**

Here we use a result of Reclaw that appears in Miller [18]. Reclaw proved that if \( X \) is a set of reals and there exists a continuous onto map \( f : X \to 2^\omega \), then the Borel hierarchy on \( X \) has \( \omega_1 \) levels, in particular, \( X \) is not a \( \sigma \)-set.

So let \( X \subseteq \omega^\omega \) be a set which is not Laver-null. Hence there exists a Laver tree \( T \) such that for every Laver subtree \( T' \subseteq T \) we have that \([T']\) meets \( X \).

To simplify our notation assume that \( T = \omega^{<\omega} \). Define the following continuous function \( f : \omega^\omega \to 2^\omega \). \( f \) is the parity function, i.e., for any \( x \in \omega^\omega \) we define \( f(x) = y \in 2^\omega \) by the rule that \( x(n) \) is even iff \( y(n) = 0 \). But note that \( f \) maps \( X \) continuously onto \( 2^\omega \). This is because for any \( y \in 2^\omega \) there is a Laver-tree \( T \) such that \( f[T] = \{y\} \). But since \([T]\) meets \( X \) there is some \( x \in X \) with \( f(x) = y \).

Now in the more general case \( T \) is an arbitrary Laver-tree. In this case note that there is a natural map from \( \omega^{<\omega} \) to \( T \) and by using essentially the same proof as above it is easy to see that the result holds.

QED

2 \( \gamma_k \)-sets

In Kočinak [10] the notion of a \( \gamma_k \)-set is defined. A \( k \)-cover of topological space \( X \) is a family of open subsets with the property that every compact subset of \( X \) is subset of an element of the family. \( X \) is called \( \gamma_k \)-set iff for every \( k \)-cover \( U \) of \( X \) there exists a sequence \((U_n \in U : n \in \omega)\) such that for every compact \( C \subseteq X \) we have that \( C \subseteq U_n \) for all but finitely many \( n \).

This is a generalization of \( \gamma \)-sets first considered by Gerlits-Nagy [7] and studied in many papers.

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\(^2\)As I was writing this I learned from Jack Brown that M.Kysiak, A.Nowik, and T.Weiss [12], also solved this problem at about the same time. In fact, their solution is a little better as it also solves the analogous problem for Ramsey null sets.
A theorem of Galvin and Todorčević (see Galvin and Miller [6]) shows that it is consistent that the union of two \( \gamma \)-sets need not be a \( \gamma \)-set. Kočinak asked at the Lecce conference if such a counterexample exists for \( \gamma_k \)-sets. We show that it does.

**Example 2.1** There exist disjoint subsets of the plane \( X \) and \( Y \) such that both \( X \) and \( Y \) are \( \gamma_k \)-sets but \( X \cup Y \) is not.

Let \( X \) be the open disk of radius one, i.e., \( X = \{(x, y) : x^2 + y^2 < 1\} \), and \( Y \) be any singleton on the boundary of \( X \), e.g., \( Y = \{(1, 0)\} \). The result follows easily from the following:

**Lemma 2.2** Suppose that \( Z \) is a metric space. Then \( Z \) is a \( \gamma_k \)-set iff \( Z \) is locally compact and separable.

Proof
First suppose that \( Z \) is locally compact and separable. Then we can write \( Z \) as an increasing union of compact subsets \( C_n \) whose interiors cover \( Z \). Given a \( k \)-cover \( \mathcal{U} \) we simply choose \( U_n \subseteq \mathcal{U} \) so that \( C_n \subseteq U_n \). This works because for every compact set \( C \) there exists \( n \) with \( C \subseteq C_n \).

Conversely, suppose that \( Z \) is not locally compact. This means that for some \( x \in Z \) we have that \( x \) is not in the interior of any compact set. Define a sequence of \( \mathcal{U}_n \) as follows: Let \( \mathcal{U}_n \) be the set of all open subsets of \( Z \) such that \( U \) does not contain the open ball of radius \( 1/2^n \) around \( x \), i.e., there exists \( y \notin U \) such that \( d(x, y) < 1/2^n \).

Note that each \( \mathcal{U}_n \) is a \( k \)-cover of \( Z \). To see this, suppose \( C \) is a compact subset of \( Z \). Since \( x \) is not in the interior of \( C \), the set \( C \) cannot contain an open ball centered at \( x \). Choose \( y \notin C \) with \( d(x, y) < 1/2^n \). Now cover \( C \) with (finitely) many open balls not containing \( y \). The union of this cover is in \( \mathcal{U}_n \).

We can use the trick of Gerlits and Nagy to get a single \( k \)-cover from the sequence of \( k \)-covers, \( (\mathcal{U}_n : n \in \omega) \). Since \( Z \) cannot be compact there must exist a sequence \( (x_n : n \in \omega) \) with no limit point. Define

\[
\mathcal{U} = \{U \setminus \{x_n\} : n < \omega, \ U \in \mathcal{U}_n\}.
\]

Since any compact set can contain at most finitely many of the \( x_n \), we see that \( \mathcal{U} \) is a \( k \)-cover of \( Z \).
For contradiction, suppose $Z$ is $\gamma_k$-set and $(U_n \in \mathcal{U} : n \in \omega)$ eventually contains each compact set. Without loss, we may assume that $U_n \in \mathcal{U}$, with $U_n$ distinct. This is because at most finitely many $U_n$ can be “from” any $\mathcal{U}$, since they eventually must include $x_l$. Choose $y_n \notin U_n$ with $d(x, y_n) < 1/2^l_n$. Then 

$$\{ y_n : n \in \omega \} \cup \{ x \}$$

is a convergent sequence, hence compact. But it is not a subset of any $U_n$.

It is easy to see that $Z$ must be separable as we can take $\mathcal{U}$ to be the family of finite unions of open balls of radius less than $1/2^n$, then apply the Gerlits Nagy trick as above to obtain a countable basis for $Z$.

QED

In Example 2.1 each of $X$ and $Y$ are locally compact metric spaces but $X \cup Y$ is not locally compact at the point $(0, 1)$, so the result follows.

Kočniač also asked if $X \times Y$ is $\gamma_k$-set if both $X$ and $Y$ are. For metric spaces, this must be true by the Lemma, since the product of locally compact separable metric spaces is a locally compact separable metric space.

### 3 Q-sets

A Q-set is a separable metric space $X$ such that every subset of $X$ is a (relative) $G_\delta$-set. It is easy to see that $|X| = 2^\omega$, hence, if there is an uncountable Q-set, then $2^{\aleph_1} = 2^{\aleph_0}$. So uncountable Q-sets might not exist. Martin’s axiom (MA) implies that every separable metric space of size less than the continuum is a Q-set (see Martin and Solovay [14]).

The Rothberger cardinal, $b$, is defined to be the cardinality of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is some $f \in \mathcal{F}$ with $f(n) \geq g(n)$ for infinitely many $n$. That is to say, $b$ is the size of the smallest unbounded family in the quasi-ordering $(\omega^\omega, \leq^*)$. Martin’s Axiom implies that $b$ is the continuum.

**Theorem 3.1** Suppose $\kappa < b$. Then the following are equivalent:

1. There exists a Q-set $X \subseteq 2^\omega$ with $|X| = \kappa$.
2. There exists $(f_\alpha : \omega^\omega \to 2^\omega :\alpha \kappa)$ continuous functions such that given any $(y_\alpha \in 2^\omega :\alpha \kappa)$ there exists $x \in \omega^\omega$ with the property that $f_\alpha(x) =^* y_\alpha$ for every $\alpha \kappa$. 

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3. There exists a sequence \((U_\alpha \subseteq 2^\omega \times 2^\omega : \alpha < \kappa)\) of \(G_\delta\)-sets which is universal for \(\kappa\) sequences of \(G_\delta\)-sets, i.e., for every sequence \((V_\alpha \subseteq 2^\omega : \alpha < \kappa)\) of \(G_\delta\)-sets there exists \(x \in 2^\omega\) such that for every \(\alpha < \kappa\)

\[ V_\alpha = U_\alpha(x) = \{ y : (x, y) \in U_\alpha \}. \]

Proof

We will need the following lemma and the details of its proof.

**Lemma 3.2** There exists \(U \subseteq 2^\omega \times 2^\omega\) which is a universal \(G_\delta\)-set such that for every \(x_1, x_2 \in 2^\omega\) if \(x_1 =^* x_2\), then \(U(x_1) = U(x_2)\).

Proof

A set \(U\) is a universal \(G_\delta\)-set, if it is \(G_\delta\) and for every \(G_\delta\)-set \(V \subseteq 2^\omega\) there exists \(x \in 2^\omega\) such that

\[ U(x) = \{ y \in 2^\omega : (x, y) \in U \} = V. \]

Define

\[ U = \{(A, y) \in P(2^{<\omega}) \times 2^\omega : \exists \infty n \ y \upharpoonright n \in A\} \]

where \(\exists \infty\) stands for “there exists infinitely many”. It is easy to see that \(U\) is \(G_\delta\). To see that it is universal, suppose that \(V = \cap_{n < \omega} V_n\) where the \(V_n \subseteq 2^\omega\) are open and descending, i.e., \(V_{n+1} \subseteq V_n\) for each \(n\). For \(\sigma \in 2^{<\omega}\) nontrivial let \(\sigma^* \subseteq \sigma\) be the initial segment of \(\sigma\) of length exactly one less than \(\sigma\), i.e., \(|\sigma^*| = |\sigma| - 1\). Define

\[ A = \{ \sigma : [\sigma] \subseteq V \text{ or } \exists n \ [\sigma] \subseteq V_n \text{ and } [\sigma^*] \not\subseteq V_n \} \]

Then \(U(A) = V\). To see this, suppose \(x \in U(A)\). If for some \(n\) we have that \(x \upharpoonright n \in A\) because \([x \upharpoonright n] \subseteq V\) then clearly \(x \in V\). On the other hand, if there are infinitely many \(k\) such that for some \(n\), \([x \upharpoonright k] \subseteq V_n\) but \([x \upharpoonright (k-1)] \not\subseteq V_n\), then these \(n\)’s must all be distinct and since the \(V_n\) were descending \(x \in V\).

Conversely, if \(x \in V\) then either \(x\) is in the interior of \(V\) and so \(x \upharpoonright k \in A\) for all but finitely many \(k\) or it isn’t in the interior of \(V\) and there are thus infinitely many \(n\) with \(x \upharpoonright n \in A\). Hence \(x \in U(A)\).
QED

2→3 This follows immediately from the Lemma. Just define

\[ (x, y) \in U_\alpha \text{ iff } (f_\alpha(x), y) \in U \]

identify \( \omega^\omega \) with a \( G_\delta \) subset of \( 2^\omega \).

3→1 By the proof of Lemma 3.2 there exists \( A_\alpha \subseteq 2^{<\omega} \times 2^{<\omega} \) such that for any \( (x, y) \) we have that \( (x, y) \in U_\alpha \) iff \( \exists n (x \upharpoonright n, y \upharpoonright n) \in A_\alpha \). We claim that

\[ \{ A_\alpha : \alpha < \kappa \} \]

is a \( Q \)-set. Fix \( y \in 2^\omega \) arbitrary. Consider any \( \Gamma \subseteq \kappa \) and define the sequence of \( G_\delta \) sets \( (V_\alpha : \alpha < \kappa) \) by

\[ V_\alpha = \begin{cases} \{ y \} & \text{if } \alpha \in \Gamma \\ \emptyset & \text{if } \alpha \notin \Gamma \end{cases} \]

By assumption there exists \( x \in 2^\omega \) such that \( U(\alpha) = V_\alpha \) for every \( \alpha < \kappa \). But then

\[ \alpha \in \Gamma \text{ iff } y \in U_\alpha(x) \text{ iff } \exists n (x \upharpoonright n, y \upharpoonright n) \in A_\alpha \text{ iff } A_\alpha \in \{ A : \exists n (x \upharpoonright n, y \upharpoonright n) \in A \} \]

But this last set is \( G_\delta \). It follows that \( \{ A_\alpha : \alpha \in \Gamma \} \) is relatively \( G_\delta \) in the set \( \{ A_\alpha : \alpha \in \kappa \} \).

1→2 Let \( \{ v_\alpha^n \in 2^\omega : n < \omega, \ \alpha < \kappa \} \) be a \( Q \)-set. Define the function \( f_\alpha \) as follows: Suppose \( x = (A, (I_n : n < \omega)) \) where \( A \subseteq 2^{<\omega} \) and each \( I_n \subseteq 2^{<\omega} \) is finite. (We can easily identify the set of such \( x \) with \( \omega^\omega \).) Now for each \( \alpha < \kappa \) define a continuous map \( f_\alpha(x) \in 2^\omega \) as follows. Define

\[ f_\alpha((A, (I_n : n < \omega)))(n) = \begin{cases} 1 & \text{if } \exists k v_\alpha^k \upharpoonright k \in I_n \cap A \\ 0 & \text{otherwise} \end{cases} \]

Since the \( I_n \) are finite, the function \( f_\alpha \) is continuous. We verify that it has the property required. Let \( x_\alpha \in 2^\omega \) for \( \alpha < \kappa \) be arbitrary. Since \( \{ v_\alpha^n \in 2^\omega : n < \omega, \ \alpha < \kappa \} \) is a \( Q \)-set, there is a \( G_\delta \)-set \( U \subseteq 2^\omega \) with the property that for every \( \alpha < \kappa \) and \( n < \omega \) we have that \( v_\alpha^n \in U \) iff \( x_\alpha(n) = 1 \). By the proof of Lemma 3.2 there exists \( A \subseteq 2^{<\omega} \) such that for all \( \alpha, n \)
\( v^n_\alpha \in U \text{ iff } A \cap \{ v^n_\alpha \mid k : k < \omega \} \) is infinite.

Since \( b > \kappa \) there exists a partition \((I_l : l < \omega)\) of \( 2^{<\omega} \) into finite sets such that for every \( \alpha < \kappa \) and \( n < \omega \) the set \( A \cap \{ v^n_\alpha \mid k : k < \omega \} \) is infinite iff \( I_l \cap A \cap \{ v^n_\alpha \mid k : k < \omega \} \neq \emptyset \) for all but finitely many \( l < \omega \). But this implies that for \( f_\alpha((A, (I_l : l < \omega)) = x_\alpha \text{ for each } \alpha. \)

QED

Condition 3 is a kind of uncountable version of Luzin’s doubly universal sets, see Kechris [9] page 171 22.15 iv. Luzin used a doubly universal set to prove that the classical properties of separation and reduction cannot hold on the same side of a reasonable point-class.

In condition 2, \( u =^* v \) means that \( u(n) = v(n) \) except for finitely many \( n \). It is impossible to have the stronger condition with \( "=\)" in place of \( "=^*\) at least when \( \kappa \) is uncountable. To see this, fix \( y_0 \in 2^\omega \) and define \( E_\alpha = f_\alpha^{-1}(y_0) \) for \( \alpha < \omega_1 \). It is not hard to see that the \( F_\alpha = \cap_{\beta < \alpha} E_\beta \) would have to be a strictly decreasing sequence of closed sets, which is impossible in a separable metric space.

We do not know iff the condition \( \kappa < b \) is needed for this result. There are several models of set theory where there is a \( Q \)-set and \( b = \omega_1 \), Fleissner and Miller [3], Judah and Shelah [5], and Miller [19].

We obtained this result while working on the square \( Q \)-set problem, see Fleissner [4]. Unfortunately, Fleissner’s proof that it is consistent there is a \( Q \)-set whose square is not a \( Q \)-set contains a gap. In his paper: he claims to show that in his model of set theory:

1. there is a \( Q \)-set \( Y \subseteq 2^\omega \) of size \( \omega_2 \), and
2. for any set of \( Z = \{ z_\alpha : \alpha < \omega_2 \} \subseteq 2^\omega \) the set

\[ \{(z_\alpha, z_\beta) : \alpha < \beta < \omega_2 \} \]

is not \( G_\delta \) in \( Z \times Z \).

But we have a fairly easy proof that (1) implies the negation of (2).

**Theorem 3.3** If there exists a \( Q \)-set \( Y \subseteq 2^\omega \) with \( |Y| = \omega_2 \) then there exists \( Z = \{ z_\alpha : \alpha < \omega_2 \} \subseteq 2^\omega \) such that

\[ \{(z_\alpha, z_\beta) : \alpha < \beta < \omega_2 \} \]

is (relatively) \( G_\delta \) in \( Z^2 \).
Proof
Let \( Y = \{ y_\alpha : \alpha < \omega_2 \} \) and let \( U \subseteq 2^\omega \times 2^\omega \) be a universal \( G_\delta \)-set. Choose for each \( \beta < \omega_2 \) a \( u_\beta \in 2^\omega \) such that for every \( \alpha < \omega_2 \)
\[
y_\alpha \in U_{u_\beta} \text{ iff } \alpha < \beta.
\]
Since \( U \) is \( G_\delta \) there are clopen \( C_{n,m}, D_{n,m} \subseteq 2^\omega \) with
\[
U = \bigcap_{n<\omega} \bigcup_{m<\omega} (C_{n,m} \times D_{n,m})
\]
Now let \( z_\alpha = (y_\alpha, u_\alpha) \) and identify \( 2^\omega \times 2^\omega \) with \( 2^\omega \). Then for any \( \alpha, \beta < \omega_2 \) we have that
\[
\alpha < \beta \iff (y_\alpha, u_\beta) \in U
\]
\[
\alpha < \beta \iff (y_\alpha, u_\beta) \in \bigcap_{n<\omega} \bigcup_{m<\omega} (C_{n,m} \times D_{n,m})
\]
\[
\alpha < \beta \iff (z_\alpha, z_\beta) = ((y_\alpha, u_\alpha), (y_\beta, u_\beta)) \in \bigcap_{n<\omega} \bigcup_{m<\omega} ((C_{n,m} \times 2^\omega) \times (2^\omega \times D_{n,m}))
\]
QED

As far as we know, the problem of the consistency of a \( Q \)-set whose square is not a \( Q \)-set, is open. One way to connect this problem with Theorem 3.1 is the following:

**Corollary 3.4** Suppose there is a \( Q \)-set of size \( \omega_2 \) and \( b > \omega_2 \). Then given any family \( \Gamma \subseteq P(\omega_2 \times \omega_2) \) with \( |\Gamma| = \omega_2 \) there is a \( Q \)-set
\[
Z = \{ z_\alpha : \alpha < \omega_2 \}
\]
such that for every \( A \in \Gamma \) the set \( \{ (z_\alpha, z_\beta) : (\alpha, \beta) \in A \} \) is \( G_\delta \) in \( Z \).

## 4 Minimal \( Q \)-like-sets

At the Slippery-Rock conference in June 2004, Ali A. Alikhani-Koopaei asked me if the following \( Q \)-like example was possible. We show that it is.

**Example 4.1** There exist a \( T_0 \) space \( Y \) such that \( Y \) is not a \( Q \)-set but for every \( A \subseteq Y \) there is a minimal \( G_\delta \) set \( Q \) with \( A \subseteq Q \). By minimal we mean that for any \( G_\delta \) set \( Q' \) if \( A \subseteq Q' \), then \( Q \subseteq Q' \).
Proof
Let $X$ be any $Q$-set, i.e., every subset of $X$ is $G_δ$ and $X$ at least $T_0$. For example, a discrete space. Now let $X'$ be a disjoint copy of $X$ and let $p \mapsto p'$ a bijection from $X$ to $X'$. For each $A \subseteq X$ let $A' = \{p' : p \in A\}$. Define the topology on $Y = X \cup X'$ by letting the open sets of $Y$ be exactly those of the form $U \cup V'$ where $U, V \subseteq X$ are open in $X$ and $U \subseteq V$. Then $Y$ is $T_0$, e.g. $X'$ is open in $Y$ and separates any $p$ and $p'$.

Claim: For $A, B \subseteq X$ the set $A \cup B'$ is $G_δ$ in $Y$ iff $A \subseteq B$. Furthermore, given any $A, B \subseteq X$ the set $A \cup (A \cup B)'$ is the minimal $G_δ$ in $Y$ containing $A \cup B'$.

Proof
Suppose that $A$ is not a subset of $B$ and let $p \in A \setminus B$. Then any open set in $Y$ which contains $A$ must also contain $p'$. The same is true for any $G_δ$ and hence $A \cup B'$ is not $G_δ$.

On the other hand, suppose $A \subseteq B$. Let $A = \cap_{n<ω} U_n$ and $B = \cap_{n<ω} V_n$ where the $U_n$ and $V_n$ are open in $X$. Now since $A \subseteq B$ we may assume that $U_n \subseteq V_n$ (if not just replace $U_n$ by $U_n \cap V_n$). But then in

$$A \cup B' = \cap_{n<ω} (U_n \cup V_n').$$

For the furthermore, note that if $C \cup D'$ is $G_δ$ and contains $A \cup B'$, then $A \subseteq C$ and $C \cup B \subseteq D$ and so $A \cup (A \cup B)' \subseteq C \cup D'$

QED

Question. Can we get an example which is uncountable but contains no uncountable $Q$-set?

Yes. Let $X = ω_1$ have the topology with $U \subseteq X$ is open iff $U = \emptyset$ or there exists $α$ with

$$U = [α, ω_1) = \{β : α \leq β < ω_1\}$$

Given any $A \subseteq X$ the smallest $G_δ$ containing $A$ is $[\text{min}(A), ω_1)$.

5 $σ$-sets and retractive boolean algebras

The definition of thin set of reals is due to Rubin [21] who showed it equivalent to a certain construction yielding a retractive boolean algebra which is not
the subalgebra of any interval algebra. Rubin asked whether or not there is always an uncountable thin set of reals. We show that every thin set is a \(\sigma\)-set and so by the results of Miller \[15\] that it is consistent there are no uncountable \(\sigma\)-sets, it is also consistent there are no uncountable thin sets.

A thin set of reals is defined as follows. An OIT (ordered interval tree) is a family of \((G_n : n \in \omega)\) such that each \(G_n\) is a family of pairwise disjoint open intervals such that for \(n\) and \(I \in G_{n+1}\) there exists \(J \in G_n\) with \(I \subseteq J\). A set of reals \(Y\) is \((G_n : n \in \omega)\)-small iff there exists \((F_n \in [G_n]^{<\omega} : n \in \omega)\) such that for every \(x \in Y\) and \(n \in \omega\) if \(x \in \bigcup G_n\), then \(x \in \bigcup F_n\). A set of reals \(X\) is thin iff for every OIT \((G_n : n \in \omega)\) the set \(X\) is a countable union of \((G_n : n \in \omega)\)-small sets.

**Proposition 5.1** If \(X \subseteq \mathbb{R}\) is thin, then \(X\) is a \(\sigma\)-set.

**Proof**

A thin set cannot contain an interval (see Rubin \[21\]) so without loss of generality we may suppose that \(X\) is disjoint from the rationals \(\mathbb{Q}\). Let \(B\) be the family of nonempty open intervals with end points from \(\mathbb{Q}\). The following claim is easy to prove and left to the reader.

**Claim.** Given any open set \(U \subseteq I\) where \(I \in B\) we can construct a family of pairwise disjoint intervals \(G \subseteq B\) so that

1. \(cl(J) \subseteq I\) for each \(J \in G\) and
2. \(\bigcup G \subseteq U \subseteq \bigcup G \cup \mathbb{Q}\).

Now suppose that \(\cap_{n<\omega} U_n\) is an arbitrary \(G_\delta\) set of reals where the \(U_n\) are open sets. Using the claim it is easy to construct a sequence \(G_n \subseteq B\) of pairwise disjoint rational intervals such that:

1. if \(I \in G_{n+1}\), then for some \(J \in G_n\) we have \(cl(I) \subseteq J\) and
2. \(\bigcup G_n \subseteq U_n \subseteq \bigcup G_n \cup \mathbb{Q}\).

Since \(X\) is thin, we have that \(X = \bigcup_{m<\omega} X_m\) where each \(X_m\) is \({G_n : n < \omega}\)-small. Fix \(m\). There exists \(F_{n,m} \in [G_n]^{<\omega}\) for \(n < \omega\) which witness the smallness of \(X_m\). Let

\[
C_m = \cap_{n<\omega} (\bigcup F_{n,m})
\]
Note that we may assume that for each \( n \) and \( I \in F_{n+1,m} \) there is a \( J \in F_{n,m} \) with \( cl(I) \subseteq J \). Hence

\[
\cap_{n<\omega}(\cup F_{n,m}) = \cap_{n<\omega}(\cup_{I \in F_{n,m}} cl(I))
\]

and since each \( F_{n,m} \) is finite, \( C_m \) is closed. Since \( X \) is disjoint from \( Q \) we have that

\[
X \cap (\cap_{n<\omega} U_n) = X \cap (\cup_{m<\omega} C_m).
\]

Since we started with an arbitrary \( G_\delta \) set we have that \( X \) is a \( \sigma \)-set.

QED

It is not hard to see that a set of reals is thin iff it is hereditarily Hurewicz. See Miller and Fremlin [17] for the definition of the Hurewicz property.

### 6 Souslin number and nonmeager sets

We obtained these results in March 2004. First we define the following small cardinal number:

\[
\text{non}(\mathcal{M}) = \min\{|X| : X \subseteq 2^{\omega} \text{ nonmeager} \}
\]

For \( X \subseteq 2^{\omega} \) we define \( \text{ord}(X) \) (the Borel order of \( X \)) to be the smallest \( \alpha < \omega_1 \) such that every Borel subset \( A \) of \( 2^{\omega} \) there exist a \( \Sigma^0_\alpha \) subset \( B \) of \( 2^{\omega} \) such that \( A \cap X = B \cap X \), if there is no such \( \alpha < \omega_1 \), we define \( \text{ord}(X) = \omega_1 \).

To prove our main result (Theorem 6.2) we will use the following theorem:

**Theorem 6.1** There exists \( X \subseteq 2^{\omega} \) with \( |X| \leq \text{non}(\mathcal{M}) \) and \( \text{ord}(X) = \omega_1 \).

**Proof**

This is similar to the proof of Miller [15] Theorem 18. Notice that it is enough to show that for each \( \alpha < \omega_1 \) there exists an \( X_\alpha \subseteq 2^{\omega} \) with

\[
|X_\alpha| \leq \text{non}(\mathcal{M})
\]

and \( \text{ord}(X_\alpha) \geq \alpha \), since the \( \omega_1 \) union of these sets would be the \( X \) we need.

So fix \( \alpha_0 < \omega_1 \) with \( \alpha_0 > 1 \). According to Miller [15] Theorem 13, there exists a countable subalgebra \( \mathcal{G} \subseteq \mathcal{B} \) where \( \mathcal{B} \) is the complete boolean algebra:

\[
\mathcal{B} = \frac{\text{Borel}(2^{\omega})}{\text{meager}(2^{\omega})}
\]
such that $\mathcal{G}$ countably generates $\mathbb{B}$ in exactly $\alpha_0$ steps. This last statement means the following:

Define $\mathcal{G}_0 = \mathcal{G}$. For $\alpha > 0$ an even ordinal define $\mathcal{G}_\alpha$ to be the family of countable disjuncts of elements from $\bigcup_{\beta < \alpha} \mathcal{G}_\beta$ and for $\alpha$ an odd ordinal define $\mathcal{G}_\alpha$ to be the family of countable conjuncts of elements from $\bigcup_{\beta < \alpha} \mathcal{G}_\beta$. These classes are analogous to the $\Sigma^0_\alpha$ and $\Pi^0_\alpha$ families of Borel sets. Then $\mathcal{G}$ has the property that $\mathcal{G}_{\alpha_0} = \mathbb{B}$ but for each $\beta < \alpha_0$, $\mathcal{G}_\beta \neq \mathbb{B}$.

Now let $Y \subseteq 2^\omega$ be such that $|Y| = \text{non}(\mathcal{M})$ and $Y \cap U$ is nonmeager for every nonempty open subset $U$ of $2^\omega$. Note that $Y$ has the property that for any Borel subsets $A$ and $B$ of $2^\omega$, if $A \cap Y = B \cap Y$, then the symmetric difference, $A \Delta B$ is meager.

Let $\mathcal{F} \subseteq \text{Borel}(2^\omega)$ be a family of representatives for $\mathcal{G}$, i.e.,

$$\mathcal{G} = \{ [A] : A \in \mathcal{F} \}$$

where $[A] \in \mathbb{B}$ is the equivalence class of $A$ modulo the meager ideal in $2^\omega$. Assume $\mathcal{F}$ is chosen so that the map $A \mapsto [A]$ is one-to-one and $2^\omega$ and $\emptyset$ are the representatives of 1 and 0. By throwing out a meager subset of $Y$ we may assume that for any $A, B, C \in \mathcal{F}$

1. $[A] \lor [B] = [C]$ iff $(A \cap B) \cap Y = C \cap Y$
2. $[A] \land [B] = [C]$ iff $(A \cup B) \cap Y = C \cap Y$

Define $\mathcal{F}^Y = \{ Y \cap A : A \in \mathcal{F} \}$. Then we have that $\mathcal{G}$ and $\mathcal{F}^Y$ are isomorphic as boolean algebras:

$$(\mathcal{G}, \lor, \land, 0, 1) \simeq (\mathcal{F}^Y, \cup, \cap, \emptyset, Y)$$

Define $\mathcal{F}_\beta$ and $\mathcal{F}^Y_\beta$ exactly as we did $\mathcal{G}_\beta$ but using countable unions and intersections instead of disjuncts and conjuncts as we do in a boolean algebra.

**Claim.**

1. By induction on $\beta$
   
   (a) $\mathcal{G}_\beta = \{ [B] : B \in \mathcal{F}_\beta \}$ and
   
   (b) $\mathcal{F}^Y_\beta = \{ B \cap Y : B \in \mathcal{F}_\beta \}$.

2. If $\beta < \alpha_0$, then $\mathcal{F}^Y_\beta \neq \mathcal{F}^Y_{\alpha_0}$. 

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Proof

Item (1) is an easy induction. To see (2) suppose that \([B] \in G_{\alpha_0} \setminus \cup_{\beta < \alpha_0} G_\beta\). Without loss \(B \in F_{\alpha_0}\) and we claim that \(B \cap Y \in F_Y^{\alpha_0} \cup \cap_{\beta < \alpha_0} F_Y^\beta\). Suppose for contradiction that \(B \cap Y \in F_Y^\beta\) for some \(\beta < \alpha_0\). Then there would exist \(C \in F_\beta\) with \(B \cap Y = C \cap Y\). But this would imply that \([B] = [C] \in G_\beta\) which is a contradiction. This proves the claim.

Now let \(F_Y = \{C_n : n < \omega\}\) and let \(i : Y \to 2^\omega\) be the Marczewski characteristic function of the sequence, which is defined by

\[
i(a)(n) = \begin{cases} 1 & \text{if } a \in C_n \\ 0 & \text{if } a \notin C_n\end{cases}
\]

Let \(X = i(Y)\). \(i\) need not be one-to-one but by definition, it is onto \(X\), so \(|X| \leq |Y| = \text{non}(\mathcal{M})\). Note that

\[
\{C \cap X : C \text{ is clopen in } 2^\omega\} = \{i(C) : C \in F_Y\}
\]

Hence, since the Borel order of \(F_Y\) is at least \(\alpha_0\) we have that \(\text{ord}(X) \geq \alpha_0\). This proves Theorem 6.1.

QED

We define the Souslin number \(\text{s}n\):

\[
\text{s}n = \min\{|X| : X \subseteq 2^\omega, \exists A \in \Sigma_1^1 \forall B \in \Pi_1^1 A \cap X \neq B \cap X\}
\]

In Zapletal [22] Appendix C, it is shown that \(\text{s}n \geq b\), where \(b\) is the smallest cardinality of an unbounded family in \(\omega^\omega\). In Miller [20] it is shown to be consistent to have \(\text{s}n > b\).

Define the following variant of the Souslin number \(\text{s}n^*\):

\[
\text{s}n^* = \min\{|X| : X \subseteq 2^\omega, \exists A \in \Sigma_1^1 \forall B \in \text{Borel } A \cap X \neq B \cap X\}
\]

The following theorem partially confirms a conjecture of Zapletal that \(\text{s}n \leq \text{non}(\mathcal{M})\), since \(\text{s}n^* \leq \text{s}n\). Zapletal was motivated by results in [22] Appendix C and [23], which roughly speaking show that it is impossible to force \(\text{s}n > \text{non}(\mathcal{M})\) using a countable support iteration of definable real forcing in the presence of suitable large cardinal axioms. Zapletal’s conjecture remains open.

**Theorem 6.2** \(\text{s}n^* \leq \text{non}(\mathcal{M})\)
Proof

Let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\Sigma_1^1$ set and consider the set of reals $X$ from Theorem 6.1. For each $\alpha < \omega_1$ let $B_\alpha \subseteq 2^\omega$ be a $\Sigma_0^0$ such that for every $C$ which is $\Pi_0^0$ we have that

$$B_\alpha \cap X \neq C \cap X$$

Since $U$ is universal there exists $a_\alpha \in 2^\omega$ such that the cross section $U_{a_\alpha} = B_\alpha$. Let $Z$ be defined by

$$Z = \{ a_\alpha : \alpha < \omega_1 \} \times X \subseteq 2^\omega \times 2^\omega$$

Then $|Z| \leq \text{non}(\mathcal{M})$ and there is no Borel set $B \subseteq 2^\omega \times 2^\omega$ such that $Z \cap U = Z \cap B$. This is because if $B$ is say $\Pi_0^0$, then every cross section of $B$ is $\Pi_0^0$, but then

$$B_\alpha \cap X = U_{a_\alpha} \cap X = B_{a_\alpha} \cap X$$

which contradicts our choice of $B_\alpha$.

QED

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