An Exact Solution for Static Scalar Fields Coupled to Gravity in (2 + 1)-Dimensions

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Abstract. We obtain an exact solution for the Einstein’s equations with cosmological constant coupled to a scalar, static particle in static, ”spherically” symmetric background in 2 + 1 dimensions.

KEY WORDS: black holes; Choptuik formation; singularity.

1. Introduction

In reference [1], Choptuik studied numerically a massless scalar field $\phi$ minimally coupled to the gravitational metric in 3 + 1 dimensions and found a scaling relation $M = C(p - p_*)^\gamma$ for the black hole mass $M$ in the limit $p$ close to critical value $p_*$ where $\gamma$ is approximately equal to 0.374 for all 1-parameter families of scalar data. A similar behavior was also obtained for the cylindrically symmetric case in 3 + 1 dimensions studied by Abrahams and Evans [2].

In references 3-6, similar systems in 2 + 1 dimensions were studied numerically and analytically. In reference [7], Birkandan and Hortacsu studied the BTZ system along the lines of Pretorius and Choptuik [3] work. They dropped the time dependency and first studied the static case with no scalar field and then, added the scalar field perturbatively.
In this note we study the Einstein Klein-Gordon system in AdS space-time for the static spherically symmetric background. The original equations studied perturbatively in [7] are reduced to a second order system and then, the exact solution are obtained in a new coordinate frame. Explicit solutions in terms of the first coordinate frame involve hypergeometric functions they were omitted. We used the further coordinate transformations to obtain the exact solution in the form generalizing the AdS metric as given by Matschull [10].

2. Einstein’s Field Equations

Let $M$ be a 3-dimensional Lorentzian manifold with local coordinates $(t, r, \theta)$ and $g_{\mu \nu}$ be a metric on $M$. We start with the corresponding line element

$$ds^2 = -e^{2X(r)} dt^2 + e^{2X(r)} dr^2 + e^{-2Y(r)} d\theta^2,$$

We shall use the notation $X' = \frac{dX}{dr}$ and $Y' = \frac{dY}{dr}$. We give below the non-zero components of the Christoffel symbols of the second kind

$$\Gamma^t_{tt} = X', \quad \Gamma^t_{tr} = X', \quad \Gamma^r_{rr} = X', \quad \Gamma^\theta_{r\theta} = -Y', \quad \Gamma^\theta_{\theta} = Y' e^{-2(X+Y)},$$

of the curvature tensor

$$R_{trtr} = e^{2X} X'', \quad R_{t\theta\theta} = e^{-2Y} (-X'Y'), \quad R_{r\theta r\theta} = e^{-2Y} (Y'' - X'Y' - Y'^2),$$

of the Ricci tensor

$$R_{tt} = X'' - X'Y', \quad R_{rr} = X'' + Y'' - X'Y' - Y'^2, \quad R_{\theta\theta} = e^{-2(X+Y)} (Y'' - Y'^2),$$

and the Ricci scalar $R$

$$R = 2e^{-2X} (-X'' + Y'' - Y'^2).$$

The Einstein’s equations with cosmological constant coupled to a scalar, static particle are [8]

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} = \kappa T_{\mu \nu},$$

where

$$T_{\mu \nu} = \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g_{\mu \nu} (\partial_{\lambda} \phi \partial^{\lambda} \phi),$$

and $\phi$ satisfies the wave equation

$$g^{\mu \nu} \nabla_\mu \nabla_\nu \phi = (-g)^{-1} \partial_\mu [( -g) \frac{1}{2} g^{\mu \nu} \partial_\nu \phi].$$
with $\det |g_{\mu\nu}| = g$ [9]. From Eqn.(7), we obtain

$$T_{tt} = \frac{1}{2} \phi'^2, \quad T_{rr} = \frac{1}{2} \phi'^2, \quad T_{\theta\theta} = -\frac{1}{2} e^{-2(X+Y)} \phi'^2.$$  \hfill (9)

The wave equation for the scalar field is reduced to

$$\phi'' - Y' \phi' = 0$$  \hfill (10)

which can be integrated as

$$\phi' = \frac{\lambda}{\sqrt{2\pi}} e^{Y},$$  \hfill (11)

where $\lambda$ is an integration constant.

Following the conventions in [3] we choose $\kappa = 4\pi$ and after relevant substitutions, (6) reduces to the following system of ODE’s for $X$, $Y$ and $\phi$,

$$X'' + \Lambda e^{2X} + \lambda^2 e^{2Y} = 0,$$
$$X'Y' - \Lambda e^{2X} + \lambda^2 e^{2Y} = 0,$$
$$Y'' - X'Y' - Y'^2 - \lambda^2 e^{2Y} = 0.$$  \hfill (12)

**Remark 1.** If we substitute $Y = X$ in the system (12), it can be seen that after algebraic eliminations the third equation gives $-4 \Lambda e^{2X} = 0$, hence $\Lambda$ should be zero for compatibility. On the other hand for $Y \neq X$, (12) is a consistent system. In the following we shall assume that $Y \neq X$.

We now proceed with the solution of the system (12). We shall first reduce it to a second order system then obtain an analytic solution in a suitable coordinate system.

**Proposition 2.** The system of ODE’s (12) can be reduced to the following system of ODE’s for $X$, $Y$

$$X' = \frac{1}{2} c e^{Y} + \frac{1}{2} [c^2 e^{2Y} + 4(-\Lambda e^{2X} + \lambda^2 e^{2Y})]^{1/2},$$
$$Y' = \frac{1}{2} c e^{Y} + \frac{1}{2} [c^2 e^{2Y} + 4(-\Lambda e^{2X} + \lambda^2 e^{2Y})]^{1/2}$$  \hfill (13)

where $c$ and $\lambda$ are integration constants and $\Lambda$ is the cosmological constant.

**Proof.** After algebraic eliminations, the system (12) is reduced to

$$X'' + \Lambda e^{2X} + \lambda^2 e^{2Y} = 0,$$
$$X'Y' - \Lambda e^{2X} + \lambda^2 e^{2Y} = 0,$$
$$Y'' = Y'^2 + 2\Lambda e^{2X}.$$  \hfill (14)
Let \( Z = X + Y \) and differentiate twice with respect to \( r \) to get \( Z'' = X'' + Y'' \). Substituting the expressions of \( X'' \) and \( Y'' \) we obtain

\[
Z'' = \Lambda e^{2X} - \lambda^2 e^{2Y} + Y'' = X'Y' + Y'' = Z'Y'. \tag{15}
\]

Note that \( Z' = 0 \) is a special solution. For \( Z' \neq 0 \), we can integrate Eqn.(15) as \( Z' = ce ^ Y \) where \( c \) is an integration constant. Hence the system (14) gives

\[
X' + Y' = ce ^ Y, \\
X'Y' - \Lambda e^{2X} + \lambda^2 e^{2Y} = 0. \tag{16}
\]

Solving \( X' \) from the first and substituting in the second, we obtain

\[
(c e ^ Y - Y')Y' - \Lambda e^{2X} + \lambda^2 e^{2Y} = 0.
\]

Rearranging this equation and completing the square, we have

\[
(Y' - \frac{1}{2}(c e ^ Y))^2 = -\Lambda e^{2X} + \lambda^2 e^{2Y} + \frac{1}{4}c^2 e^{2Y}.
\]

Taking the square root and rearranging again we obtain \( Y' \) and \( X' \) as in (13) hence the proof is complete.

We note that the special solution \( Z' = X' + Y' = 0 \) corresponds to the value \( c = 0 \) of the integration constant.

Remark 3. For \( X' + Y' = 0, \ c = 0 \) and the metric reduces to the form

\[
ds^2 = e^{2X} \left( -dt^2 + dr^2 + e^{-2Y_0} d\theta^2 \right),
\]

where \( Y_0 \) is an integration constant. Unless otherwise indicated, we shall assume that \( c \neq 0 \). We note that the solutions corresponding to different choices of the sign of the square root in (13) are related by the sign changes of \( c \) and \( r \), hence we take

\[
X' = \frac{1}{2}(ce ^ Y - \frac{1}{2}[c^2 e^{2Y} + 4(-\Lambda e^{2X} + \lambda^2 e^{2Y})]^\frac{1}{2}}, \\
Y' = \frac{1}{2}(ce ^ Y + \frac{1}{2}[c^2 e^{2Y} + 4(-\Lambda e^{2X} + \lambda^2 e^{2Y})]^\frac{1}{2}), \quad c \neq 0, \tag{17}
\]

allowing \( c \) to have both positive and negative signatures.

The metric (1) admits obviously the abelian \( g_2 \) symmetry algebra generated by \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \theta} \). The computation of the Killing’s equation

\[
\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0
\]

is straightforward. It can be seen that for \( c \neq 0 \) there are only two Killing vectors, i.e.,
\[ X' + Y' \neq 0 : \quad \xi^{(1)} = \partial_t, \quad \xi^{(2)} = \partial_\theta. \quad (18) \]

For \( c = 0 \) there is third Killing vector and the symmetry algebra has the following generators.

\[ X' + Y' = 0 : \quad \xi^{(1)} = \partial_t, \quad \xi^{(2)} = \partial_\theta, \quad \xi^{(3)} = \theta \partial_t + t \partial_\theta \quad (19) \]

Note that as the Ricci tensor is diagonal, using (4-5) and (14) we can obtain the curvature invariants as
\[ R_{tt} = 2\Lambda, \quad R_{rr} = 2\Lambda + 2\lambda^2 e^{-2X + 2Y}, \quad R_{\theta\theta} = 2\Lambda, \quad R = 6\Lambda + 2\lambda^2 e^{-2X + 2Y}. \]

These expressions show that curvature singularities are related either to the cosmological constant \( \Lambda \) or to the particle coupling manifested by \( \lambda \). In the following we shall assume that \( \Lambda = -1 \) and use a series of coordinate transformations to obtain an exact analytic solution.

**Proposition 4.** The system of ODE's (17) is equivalent to the system
\[ \rho' = \frac{1}{2} \rho^2 [\mu \sin \varphi - \cos 2\varphi], \quad (20) \]
\[ \varphi' = \rho \sin \varphi \cos \varphi, \quad (21) \]
where \( \rho \) and \( \varphi \) are defined by
\[ e^X = \frac{1}{2} \rho \cos \varphi, \quad e^Y = \frac{1}{\sqrt{c^2 + 4\lambda^2}} \rho \sin \varphi, \]
and \( \mu = \frac{c}{\sqrt{c^2 + 4\lambda^2}}. \)

**Proof.** Taking derivatives of \( e^X \) and \( e^Y \) and substituting in (17), we obtain
\[ \rho' \cos \varphi - \varphi' \rho \sin \varphi = \frac{1}{2} \rho^2 [\mu \sin \varphi \cos \varphi - \cos \varphi], \]
\[ \rho' \sin \varphi + \varphi' \rho \cos \varphi = \frac{1}{2} \rho^2 [\mu \sin^2 \varphi + \sin \varphi]. \]

By multiplying the first equation with \( \cos \varphi \) and the second one with \( \sin \varphi \) and adding, we obtain \( \rho' \) as in (20). By a similar elimination we obtain \( \varphi' \), hence the proof is complete. \( \bullet \)

We will now obtain \( \rho \) as a function of \( \varphi \), as it will be discussed below, (20) can be seen as a parameter transformation hence we have an analytic solution in the coordinate frame \((t, \rho, \varphi)\). Explicit solutions in terms of \( r \) involve hypergeometric functions and shall be omitted.
Proposition 5. The solution of system of ODE’s (12) in the coordinate frame \((t, \rho, \varphi)\), with \(0 < \varphi < \pi/2\) is

\[
\text{ds}^2 = -\left[\frac{1}{2} \rho \cos \varphi\right]^2 dt^2 + \left[\frac{1}{2} \rho \sin \varphi\right]^2 d\varphi^2 + \left[\frac{\sqrt{c^2 + 4\lambda^2}}{\rho \sin \varphi}\right]^2 d\theta^2, \tag{22}
\]

where

\[
\rho(\varphi) = \frac{\rho_0}{\sqrt{2}} [1 + \sin \varphi]\frac{\mu}{\sin \varphi} - \frac{1}{2} \frac{1}{\cos \varphi} - \frac{1}{2} - \mu. \tag{23}
\]

Proof. From the Eqn.(20) and (21), we obtain

\[
\frac{d\rho}{d\varphi} = \rho \left[\frac{1}{2} \mu \cos \varphi - \cot 2\varphi\right]
\]

which leads to

\[
\frac{d\rho}{\rho} = \left[\frac{1}{2} \mu \cos \varphi - \cot 2\varphi\right] d\varphi.
\]

Integrating both sides, we obtain \(\rho\) as a function of \(\varphi\) as given Eqn.(23). As \(e^X\) and \(e^Y\) are positive, the coordinate transformation used in Proposition 4 implies that \(0 < \varphi < \pi/2\).

Substituting \(\rho\) in \(e^X\) and \(e^Y\) and using \(dr = \frac{d\rho}{d\varphi} d\varphi\), \(dr = (\frac{d\rho}{d\varphi})^{-1} d\varphi\), we obtain

\[
\text{dr}^2 = \left[\frac{1}{\rho \sin \varphi \cos \varphi}\right]^2 d\varphi^2
\]

and reach Eqn.(22) after rearrangements.

The expression of the curvature invariants are

\[
R_t^t = -2, \quad R_\varphi^\varphi = -2 + \frac{8\lambda^2}{(c^2 + 4\lambda^2) \tan^2 \varphi}, \quad R_\theta^\theta = -2,
\]

\[
R = -6 + \frac{8\lambda^2}{(c^2 + 4\lambda^2) \tan^2 \varphi}.
\]

Thus there is a curvature singularity at \(\varphi = \frac{\pi}{2}\). Comparing with the previous expressions of the curvature invariants, it can be seen that the singularity at \(\varphi = \frac{\pi}{2}\) is related to the scalar particle.

We shall now use further coordinate transformations to obtain the exact solution (22) in the form generalizing the AdS metric as given in [10]. Substituting for \(\rho\), Eqn.(22) reduces to

\[
\text{ds}^2 = -\frac{a^2}{\mu} [\sec \varphi + \tan \varphi]^\mu \cot \varphi dt^2 + \left[\frac{1}{2} \frac{1}{\sin \varphi}\right]^2 d\varphi^2 + \frac{2(c^2 + 4\lambda^2)}{\rho_0} [\sec \varphi + \tan \varphi]^{-\mu} \cot \varphi d\theta^2.
\]
First let \( \frac{d\varphi}{\sin \varphi} = dv \).

Then \( v = \ln(\tan \frac{\varphi}{2}) \) or \( e^v = \tan \frac{\varphi}{2} \). Using double angle formulas for the tangent function we can see that

\[
\tan \varphi = \frac{2e^v}{1 - e^{2v}} = -\frac{1}{\sinh v}.
\]

As \( \frac{\pi}{2} \) varies from zero to \( \frac{\pi}{4} \), then \( 0 < \varphi < \frac{\pi}{2} \); \( v \) is negative \( (-\infty < v < 0) \), hence

\[
\sec \varphi = -\frac{\cosh v}{\sinh v}.
\]

Finally setting \( u = -v \) we obtain the metric

\[
ds^2 = -\rho_0^2 \left[ \frac{1 + \cosh u}{\sinh u} \right] \sinh u dt^2 + \frac{1}{4} du^2 + \frac{2(c^2 + 4\lambda^2)}{\rho_0^2} \left[ \frac{1 + \cosh u}{\sinh u} \right]^{-\mu} \sinh u d\theta^2,
\]

where \( 0 < u < \infty \). At the last step we set \( u = 2\chi \) and use double angle formulas to obtain

\[
ds^2 = -\frac{\rho_0^2}{4} \left[ \frac{\cosh \chi}{\sinh \chi} \right] \cosh \chi \sinh \chi dt^2 + d\chi^2 + 4 \left[ \frac{\cosh \chi}{\sinh \chi} \right]^{-\mu} \cosh \chi \sinh \chi d\theta^2.
\]

The coefficients \( \frac{\rho_0^2}{4} \) and \( \frac{4}{\rho_0^2} \) can be eliminated by appropriate scalings of \( t \) and \( \theta \) as

\[
\tilde{t} = \frac{\rho_0}{2} t, \quad \tilde{\theta} = \frac{2}{\rho_0} \theta
\]

and the metric is

\[
ds^2 = - \left[ \frac{\cosh \chi}{\sinh \chi} \right] \cosh \chi \sinh \chi dt^2 + d\chi^2 + (c^2 + 4\lambda^2) \left[ \frac{\cosh \chi}{\sinh \chi} \right]^{-\mu} \cosh \chi \sinh \chi d\tilde{\theta}^2.
\]

Note that \( \sqrt{c^2 + 4\lambda^2} = \frac{2}{\rho_0} \) induces a scaling of \( \tilde{\theta} \) due to the effect of the coupled scalar field which can be interpreted as inducing a topological defect. For \( \lambda = 0 \), \( \mu = \frac{2}{|c|} = \pm 1 \). Taking \( c = 1 \) we have AdS solution

\[
ds^2 = -\cosh^2 \chi dt^2 + d\chi^2 + \sinh^2 \chi d\tilde{\theta}^2
\]

as given by Eqn.(1.1) in [10].

3. Results and Further Discussions

We found an exact solution in \( 2 + 1 \) dimensions for the Einstein’s equations with cosmological constant coupled to a scalar, static particle in static,
spherically symmetric background which was studied perturbatively in reference [7]. Our solution involves a parameter \( \mu \), \( 0 < \mu < 1 \) related to strength of the coupling field and AdS limit is reached for \( \mu = 1 \).

Furthermore, in the final metric (25) the scaling \( \tilde{\theta} \rightarrow \frac{c}{\mu} \tilde{\theta} \) where \( c \) is an integration constant can be interpreted as a topological defect. Although, the absence of \( t \) dependency rules out the existence of trapped surfaces [11] page 435, the metric given in (25) provides an exact module for the problem studied in Matschull [10] with cutting and gluing procedure.

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Note added in proof

The solution for a special case of the metric (1), corresponding to choice of \( \lambda = e^{-Y} \) and \( \nu(\lambda) = X \) is obtained in reference 12. The analytic solution to the complete problem investigated here has been previously obtained in reference 13. With the signature \((+, -, -)\) our metric (25) coincides with equation (3.7) in [13] as below. The equation (3.7) is

\[
\begin{align*}
\text{ds}^2 &= A \left| \rho - \rho_+ \right|^{\frac{1}{2}+a} \left| \rho - \rho_- \right|^{\frac{1}{2}-a} \text{d}t^2 \\
&\quad - \frac{4|A|}{A} \left| \rho - \rho_+ \right|^{\frac{1}{2}-a} \left| \rho - \rho_- \right|^{\frac{1}{2}+a} \text{d}\theta^2 + \frac{1}{4\Lambda(\rho_+ - \rho_-)(\rho_+ + \rho_-)} \text{d}\rho^2.
\end{align*}
\]

For \( \Lambda = -1 \), \( a = -\frac{1}{2} \) and \( \rho = \frac{1}{2}[(\rho_+ - \rho_-)\cosh2\chi + (\rho_+ + \rho_-)] \) it reduces to

\[
\begin{align*}
\text{ds}^2 &= A \left| \rho_+ - \rho_- \right| \cosh \chi \left|^{1+\mu} \right| \sinh \chi \left|^{1-\mu} \right| \text{d}t^2 \\
&\quad - \frac{4|\rho_+ - \rho_-|}{A} \cosh \chi \left|^{1-\mu} \right| \sinh \chi \left|^{1+\mu} \right| \text{d}\theta^2 - \text{d}\chi^2
\end{align*}
\]

where \( A \) and \( |\rho_+ - \rho_-| \) are constants. Thus with appropriate scalings, we get equation (25).

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