FLEXIBILITY OF SECTIONS OF NEARLY INTEGRABLE HAMILTONIAN SYSTEMS

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ABSTRACT. Given any symplectomorphism on $D^{2n}(n \geq 1)$ which is $C^\infty$ close to the identity, and any completely integrable Hamiltonian system $\Phi^t_H$ in the proper dimension, we construct a $C^\infty$ perturbation of $H$ such that the resulting Hamiltonian flow contains a “local Poincaré section” that “realizes” the symplectomorphism. As a (motivating) application, we show that there are arbitrarily small perturbations of any completely integrable Hamiltonian system which are entropy non-expansive (and, in particular, exhibit hyperbolic behavior on a set of positive measure). We use some results in Berger-Turaev [7], though in higher dimensions we could simply apply a construction from [11].

1. Introduction

We presume that a potential reader interested in this paper is familiar with such notions as symplectic manifolds and Lagrangian subspaces, Hamiltonian vector fields and flows, Poisson and Lie brackets, first integrals, Lyapunov exponents, metric and topological entropy, integrable systems, and has a basic idea of the main concept of KAM Theory. We refresh some of these notions below, to set up notations, and briefly discuss a few more technical aspects of KAM Theory. Open questions are at the end of the paper.

We work on a symplectic manifold $(\Omega^{2n}, \omega_0)$ with $n \geq 2$. A Hamiltonian system (flow) with $n$ degrees of freedom is called completely integrable if it enjoys $n$ algebraically independent first integrals which pair-wisely Poisson commute.

In a loose formulation, the main goal of this paper is as follows. We are given a symplectic transformation of a ball and a completely integrable flow (in appropriate dimension). We assume that the transformation is close to the identity. We show that the Hamiltonian admits $C^\infty$ perturbations which are arbitrarily $C^\infty$ small and such that each perturbed flow contains, in a natural sense, a Poincaré section on which the return map is, after a proper rescaling and iterations, the given transformation (the precise statement is a bit technical, see Proposition 4.1 for more details). Moreover, the section is taken near some periodic orbit of the unperturbed flow. When the unperturbed Hamiltonian flow is the geodesic flow on the standard $S^n (n \geq 4)$, a “dual lens map technique” has been recently developed and used in [11] to construct desired perturbations of Finsler metrics. The ideas...
grew from Inverse and Boundary Rigidity Problems. The proof of our main result, Proposition 4.1, is based on a far-extended dual lens map technique. In some sense, in this paper the dual lens map argument is applied to Lagrangian submanifolds in a symplectic manifold rather than to geodesics in a Finsler manifold.

We begin with explaining an application that motivated this work. According to the Liouville-Arnold Theorem (a precise statement can be found in [20], p.227), except for a zero measure set, the phase space of a completely integrable system with compact common level sets of the integrals is foliated by invariant tori, called the Liouville Tori. The motion on each of these tori is conjugate to a linear flow on a standard torus. These invariant tori are in fact common level sets of the angle variables in the action-angle coordinates.

If one perturbs the Hamiltonian function of a completely integrable system, the resulting Hamiltonian flow is called nearly integrable. Once the unperturbed system is nondegenerate in a suitable sense, the celebrated Kolmogorov-Arnold-Moser (KAM) Theorem [3][19][23] shows that under a $C^\infty$ perturbation (though the smoothness can be lower depending on the degree of freedom), a large measure of invariant tori (called KAM tori) survive and the dynamics on these tori is still quasi-periodic. “A large measure” means that the measure of the tori which do not survive goes to zero as the size of the perturbation diminishes (the concrete estimates are of no importance for us here). The tori which survive have rotation vectors whose directions are “sufficiently irrational” (a certain degree of being Diophantine, the precise condition is a bit technical and of no importance for this paper).

The dynamic outside KAM tori draws a lot of attention. Arnold [4] (in a number of papers followed by papers by Douady [16] and others) gave examples of what is now known as the Arnold diffusion: there may be trajectories asymptotic to one invariant torus at one end and then asymptotic to another torus on the other end. Furthermore, there might be trajectories which spend a lot of time near one torus, then leave and spend even longer time very close to another one and so on. This sort of hyperbolicity is, however, very slow. We know this by the double-exponential estimates on the transition time due to Nekhoroshev [24].

An interesting but relatively easy (by modern standards, though quite important at its time) question is whether topological entropy could become positive. This means the presence of some hyperbolic dynamics there. Newhouse [25] proved that a $C^2$ generic Hamiltonian flow contains a hyperbolic set (a horseshoe), hence the flow has positive topological entropy. The Riemannian geodesic flows versions of this result were later established by Knieper and Weiss [21] for surfaces and Contreras [15] for higher dimensions.

Positivity of topological entropy is nowadays not so exciting: it can (and often does) live on a set of zero measure. To get positive topological entropy, it suffices to find one Poincaré-Smale horseshoe (even of zero measure). Little was known about metric entropy, i.e. the measure theoretic entropy with respect to the Liouville measure on a level set (or to the symplectic volume on the entire space). Positive metric entropy implies positive topological entropy, but not vice versa [8]. Despite the strong interest in nearly integrable Hamiltonian systems, what was lacking is understanding whether these systems may admit positive metric entropy.

In this paper we give a positive answer to this question by constructing specific perturbations near any Liouville torus:
Theorem 1.1. Let $\Phi^t_{\tilde{H}}$ be a completely integrable Hamiltonian flow on a symplectic manifold $\Omega = (\Omega^{2n}, \omega)$ with $n \geq 2$. For any Liouville torus $T \subset \Omega$, one can find a $C^\infty$-small perturbation $\tilde{H}$ of $H$ such that the resulting Hamiltonian flow $\Phi^t_{\tilde{H}}$ has positive metric entropy. Furthermore, such perturbation can be made in an arbitrarily small neighborhood of $T$ and such that the flow is entropy non-expansive. If $\Omega$ is compact, there are perturbations with positive metric entropy with respect to $\omega^n$ on the whole $\Omega$.

Remark 1. Here a flow $\Phi^t$ is called entropy non-expansive if the positive metric entropy can be generated in an arbitrarily small tubular neighborhood of an orbit. This issue attracts a lot of interest, see for instance [9][26]. In particular, the first author [10] introduced this notion in 1988 being in mathematical isolation in the former Soviet Union. This situation is a bit counter-intuitive since hyperbolic dynamics tends to expand and occupy all space. In fact, when Bowen defined entropy expansiveness in [9], he confessed he knew no example of entropy non-expansive diffeomorphism of a compact manifold. In our situation, however, the positive metric entropy can be obtained even near any periodic orbit.

Remark 2. The theorem holds for Hamiltonian (Lagrangian) flows that are geodesic flows on the co-tangent bundle of Finsler manifolds. Furthermore, the perturbation can be made in the class of Finsler metrics. We leave to the reader checking this. For the Riemannian metrics, however, this remains an open problem.

Remark 3. Theorem 1.1 almost answers perhaps the most intriguing question in the KAM theory. The question, which is probably due to A. Kolmogorov, asks if arbitrarily small perturbations of a completely integrable system may result in dynamics with positive metric entropy. Our construction, as an additional bonus, gives examples which are entropy non-expansive, thus resolving another well-known problem. We say, however, “almost answers” since the construction is rather special. We do not know what happens for generic perturbations, and this probably is a quite difficult and important question.

By the dual lens map technique, one can make a $C^\infty$ small Lagrangian perturbation of the geodesic flow on the standard $S^n (n \geq 4)$ to get positive (though extremely small due to [26]) metric entropy and the resulting flow is entropy non-expansive [11]. Together with the Maupertuis principle, the second author managed to obtain positive metric entropy by perturbing the geodesic flow on an Euclidean $T^n (n \geq 3)$ [13]. Unlike the case of spheres, the geodesic flows on flat tori are KAM-nondegenerate. Therefore in view of KAM theory, the construction in [13] is an improvement of that in [11]. With this result we know that in some region in the complement of KAM tori, the dynamics of a nearly integrable Hamiltonian flow can be quite stochastic on a set of positive measure. On the other hand, unlike the construction in [11], the perturbed flows in [13] are entropy expansive.

There are drastic differences between examples in [11][13] and the one in this paper. Papers [11] and [13] only deal with very specific systems: the geodesic flows of the standard metrics on $S^n$ and $T^n$; the original flow on $S^n$ is KAM-degenerate (and periodic) and the perturbed flow is entropy non-expansive, while those on $T^n$ are KAM-nondegenerate and entropy expansive, respectively. In contrast to that, here we can work with any completely integrable system, regardless of whether it is KAM-nondegenerate or not, and the perturbed flow is entropy non-expansive.
Also, thanks to the results in Berger-Turaev [7], our result applies to \( n = 2 \). In this case, the 2-dimensional KAM tori separate the 3-dimensional energy level thus no Arnol’d diffusion is possible in such systems. The existence of positive metric entropy between these tori is a little bit surprising. Before [7], we perturbed the identity map by symplectically embedding the geodesic flow in the cotangent bundle of a negatively curved surface into Euclidean space (see [11, Lemma 5.1]), which inevitably required larger dimension. One of the advantages of our approach, however, is that we have more flexibility of perturbations than those in [7]. Of course, there is an obvious infinite-dimensional space of perturbations obtained by conjugating any one by a symplectomorphism; here we can, however, make essential changes by varying the metric of the negatively curved surface we employ.

The above improvements run into serious difficulties and require new techniques and ideas. The first challenge is how to generate positive metric entropy by perturbing the Poincaré map \( R \). The result in [7] allows us to perturb \( R \) to get positive metric entropy on one level set, but it is not enough to guarantee positive metric entropy on neighboring level sets since the metric entropy may vanish under tiny perturbation. To overcome this obstruction, we build up very special symplectic coordinates on each level set and then apply the Morse-Bott Lemma to make sure the perturbations on different level sets behave in the same way. A more serious challenge is to realize the perturbed return map \( \tilde{R} \) as the Poincaré map of some Hamiltonian \( \tilde{H} \) that is \( C^\infty \) close to \( H \). If \( \tilde{R} \) is the time-one map of some flow, one could construct \( \tilde{H} \) using the suspension of the generating vector field (see e.g. [13]). However, neither the map in Berger-Turaev [7], nor the \( \tilde{R} \) in our construction, is generated by a flow. Our key result, Proposition 4.1, uses Lagrangian submanifolds to perturb a Hamiltonian flow so that a given (symplectic) diffeomorphism becomes the Poincaré return map of a section transverse to a periodic trajectory near this trajectory, which is the major novelty in this paper.

In a nutshell, we have three steps: first, we create a “periodic spot” (a disc consisting of points with the same period, see [17][18][21][22] for related results of interest) in the cross-section; second, we insert a positive-entropy symplectomorphism of the disc there; and third, we extend this to a perturbation of the Hamiltonian. The first step is more difficult in higher dimensions and we have to use a seemingly non-trivial construction.

The paper is organized in the following way: in Section 4, we transform the problem of perturbing the Hamiltonian to perturbation of the Poincaré map. In Section 5, we show how to get positive metric entropy in a small invariant set by perturbing a family of symplectic maps (including those we get in Section 4) and give a proof of Theorem 1.1. A more detailed plan of proof can be found in Section 3.3.

2. Acknowledgments

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3. Preliminaries

3.1. Hamiltonian flows. Let \((\Omega^{2n}, \omega_0)\) be a \(2n\)-dimensional symplectic manifold and \(H\) a smooth function on \(T^*M\). The Hamiltonian vector field \(X_H\) is defined as the unique solution to the equation

\[
\omega_0(X_H, V) = dH(V)
\]

for any smooth vector field \(V\) on \(\Omega\). \(X_H\) is well-defined due to non-degeneracy of \(\omega_0\).

The flow \(\Phi^t_H\) on \(\Omega\) defined by \(\Phi^t_H = X_H\) is called the Hamiltonian flow on \(\Omega\) with Hamiltonian \(H\). One can easily verify that \(\Phi^t_H\) preserves \(\omega_0\) and hence the volume form \(\omega^n\). Any Hamiltonian flow is locally integrable. To be more specific, we have the following generalization of Darboux’s theorem [22, Chapter I, Theorem 17.2]:

**Theorem 3.1** (Carathéodory-Jacobi-Lie). Let \((\Omega^{2n}, \omega_0)\) be a symplectic manifold. Let a family \(p_1, ..., p_k\) of \(k\) differentiable functions \((k \leq n)\), which are pairwise Poisson commutative and algebraic independent, be defined in the neighborhood \(V\) of a point \(x \in \Omega\). Then there exists \(2n - k\) other functions \(p_{k+1}, ..., p_n, q_1, ..., q_n\) defined in an open neighborhood \(U\) of \(x\) in \(V\) such that in \(U\) we have

\[
\omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i.
\]

**Corollary 3.2.** For any point \(x \in \Omega\) and any Hamiltonian function \(H\), one can find an open neighborhood \(U\) of \(x\) and symplectic coordinates \((q, p)\) in \(U\) such that \(H|_U = p_n\).

3.2. Sections and Poincaré maps. Throughout the paper \(\Omega = (\Omega^{2n}, \omega)\) denotes a symplectic manifold, \(n \geq 2\), \(H: \Omega \to \mathbb{R}\) a smooth Hamiltonian, \(X = X_H\) the Hamiltonian vector field of \(H\), and \(\{\Phi^t_H\}_{t \in \mathbb{R}}\) the corresponding Hamiltonian flow.

A section of the flow \(\{\Phi^t_H\}\) is a \((2n-1)\)-dimensional smooth submanifold \(\Sigma \subset \Omega\) transverse to the trajectories of the flow. The transversality means that \(X_H\) is nowhere tangent to \(\Sigma\). Note that this implies in particular that \(X_H\) does not vanish on \(\Sigma\). A section \(\Sigma\) determines the Poincaré return map which sends a point \(x \in \Sigma\) to the first intersection point of the trajectory \(\{\Phi^t_H(x)\}_{t > 0}\) with \(\Sigma\). This is a partially defined map from \(\Sigma\) to itself.

We need to consider a more general situation: given two sections \(\Sigma_0\) and \(\Sigma_1\) of \(\{\Phi^t_H\}\), the associated Poincaré map is a partially defined map \(R_H: \Sigma_0 \to \Sigma_1\) defined as follows: for \(x \in \Sigma_0\), \(R_H(x)\) is the first intersection point of the trajectory \(\{\Phi^t_H(x)\}_{t > 0}\) with \(\Sigma_1\). (If the trajectory does not intersect \(\Sigma_1\), then \(R_H(x)\) is undefined.) We denote this map by \(R_H, \Sigma_0, \Sigma_1\) or by \(R_H\) when \(\Sigma_0\) and \(\Sigma_1\) are clear from context.

Since \(\Sigma_0\) and \(\Sigma_1\) are transverse to the trajectories, \(R_H\) is defined on an open subset of \(\Sigma_0\) and it is a diffeomorphism from this subset to its image in \(\Sigma_1\). In this paper we always choose sections \(\Sigma_0\) and \(\Sigma_1\) so that \(R_H, \Sigma_0, \Sigma_1\) is a diffeomorphism between \(\Sigma_0\) and \(\Sigma_1\). This is achieved by replacing \(\Sigma_0\) and \(\Sigma_1\) by suitable small neighborhoods of some \(x \in \Sigma_0\) and \(R_H(x) \in \Sigma_1\).

The Hamiltonian induces a number of structures on sections. Here is a list of structures and their properties that we need in this paper. For a detailed exposition of relations between flows and their sections, see [27, §6.1].
3.2.1. Induced measure on sections. Since the flow $\Phi^t_H$ preserves the canonical symplectic volume on $\Omega$, it naturally induces a measure $\text{Vol}_\Sigma$ on a section $\Sigma$ as follows: for a Borel measurable $A \subset \Sigma$,

$$\text{Vol}_\Sigma(A) = \text{Vol}_\Omega\{\Phi^t_H(x) : x \in A, \ t \in [0,1]\}$$

where $\text{Vol}_\Omega$ in the right-hand side is the symplectic volume counted with multiplicity.

One easily sees that Poincaré maps preserve the induced measure on sections. Furthermore, from Abramov’s formula [1], one sees that the positivity of metric entropy of a Poincaré return map implies that of the flow:

**Proposition 3.3.** Let $\Sigma$ be a section such that the Poincaré return map $R_{H, \Sigma, \Sigma}$ is a diffeomorphism and it has positive metric entropy. Then the flow $\{\Phi^t_H\}$ has positive metric entropy.

3.2.2. Slicing sections by level sets. For a section $\Sigma$ and $h \in \mathbb{R}$ we denote by $\Sigma^h$ the $h$-level set of $H|_\Sigma$:

$$\Sigma^h = \{x \in \Sigma : H(x) = h\}.$$  

Since $\Sigma$ is transverse to the flow, the differential of $H|_\Sigma$ does not vanish. Hence $\Sigma^h$ is a smooth $(2n-2)$-dimensional submanifold. Furthermore, one easily checks the following:

1. The restriction of the symplectic form $\omega$ to $\Sigma^h$ is non-degenerate. Thus $\Sigma^h$ is a symplectic manifold.
2. Let $\Sigma_0$, $\Sigma_1$ be two sections such that the Poincaré map $R_H : \Sigma_0 \to \Sigma_1$ is a diffeomorphism. Since the flow preserves $H$, $R_H$ sends $\Sigma_0^h$ to $\Sigma_1^h$. We denote by $R^h_H$, the restriction $R_H|_{\Sigma_0^h}$. One easily sees that $R^h_H$ preserves the symplectic form, hence it is a symplectomorphism between $\Sigma_0^h$ and $\Sigma_1^h$.
3. Let $\text{Vol}_{\Sigma^h}$ denote the $(2n-2)$-dimensional symplectic volume on $\Sigma^h$. A straightforward computation shows that the $(2n-1)$-dimensional volume $\text{Vol}_\Sigma$ (see above) is determined by the family $\{\text{Vol}_{\Sigma^h}\}_{h \in \mathbb{R}}$ of symplectic volumes of level sets:

$$\text{Vol}_\Sigma(A) = \int_{\mathbb{R}} \text{Vol}_{\Sigma^h}(A \cap \Sigma^h) \, dh$$

The identity (3.1) and Abramov-Rokhlin entropy formula [2] imply that it suffices to obtain positive metric entropy for the Poincaré return map on the slices $\Sigma^h$. Namely the following holds:

**Proposition 3.4.** Let $\Sigma$ be a section such that the Poincaré return map $R_H = R_{H, \Sigma, \Sigma}$ is a self-diffeomorphism of $\Sigma$. Suppose there is a set $\Lambda \subset \mathbb{R}$ of positive Lebesgue measure such that for every $h \in \Lambda$, the symplectomorphism $R^h_H : \Sigma^h \to \Sigma^h$ has positive metric entropy. Then $R_H$ has positive metric entropy.

**Remark 3.5.** The above structures depend on both $\Sigma$ and $H$. The measure $\text{Vol}_\Sigma$ and the level sets $\Sigma^h$ are determined by the restriction of $H$ to an arbitrary neighborhood of $\Sigma$. In the proof of Theorem [1] we fix suitable sections $\Sigma_0$ and $\Sigma_1$ and perturb $H$ in a small region between them. The perturbation does not change $H$ near $\Sigma_0 \cup \Sigma_1$, hence the volume forms $\text{Vol}_{\Sigma_0}$ on the sections and the splitting of $\Sigma_i$ into symplectic submanifolds $\Sigma^h_i$ remain the same.

However the Poincaré map can be affected by such perturbations of $H$. Our plan is to perturb $H$ in such a way that the resulting Poincaré return map on a
suitable section $\Sigma$ has an invariant open set where it satisfies the assumptions of Proposition 3.4.

3.3. Plan of the proof. Let $\Omega$, $H$, $T$ be as in Theorem 1.1. First, by a small perturbation of $H$ preserving integrability and Liouville tori, we make the flow on $T$ nonvanishing and periodic. Then pick a point $y_0 \in T$ and choose a small section $\Sigma$ through $y_0$ such that $y_0$ is a fixed point of the Poincaré return map $R_H = R_{H, y_0}$. Let $\Sigma_0$ be a small neighborhood of $y_0$ in $\Sigma$ such that $R_H|_{\Sigma_0}$ is a diffeomorphism onto its image $\Sigma_1 := R_H(\Sigma_0)$. The remaining perturbations of $H$ occur within a tiny neighborhood of a point $x_0$ lying on the trajectory of $y_0$. This guarantees that the Poincaré map $R_{\tilde{H}} = R_{H, \Sigma_0, \Sigma_1}$ is still a diffeomorphism between $\Sigma_0$ and $\Sigma_1$.

The perturbed Hamiltonian $\tilde{H}$ should satisfy the following conditions: the Poincaré map $R_{\tilde{H}} : \Sigma_0 \to \Sigma_1$ has an invariant open set $U \ni y_0$, and the restriction of $R_{\tilde{H}}$ to this invariant set has a positive metric entropy. Then, by Proposition 3.4 applied to $U$ in place of $\Sigma$, the flow $\{\Phi_{\tilde{H}}^t\}$ has positive metric entropy.

The construction of $\tilde{H}$ is divided into two parts. The first part, summarized in Lemma 5.2, is a construction of a perturbed Poincaré map $\tilde{R} : \Sigma_0 \to \Sigma_1$ with the properties desired from the Poincaré map $R_{\tilde{H}}$. The second part, described in Section 4, is a construction of a perturbed Hamiltonian $\tilde{H}$ which realizes the given $\tilde{R}$ as its Poincaré map: $\tilde{R} = R_{\tilde{H}}$.

In order to be realizable as a Poincaré map, the diffeomorphism $\tilde{R}$ has to satisfy the natural requirements: it should map the slices $\Sigma^0_n$ to the respective slices $\Sigma^1_n$, and it should preserve the symplectic form on the slices. In fact, in Section 4 we show that any sufficiently small compactly supported perturbation of $R_H$ satisfying these requirements is realizable as a Poincaré map of some perturbed Hamiltonian, see Proposition 4.1.

4. Hamiltonian perturbations with prescribed Poincaré maps

In this section we fulfill the last step of the above plan. We use the notation introduced in Section 3.2: $\Omega = (\Omega^2, \omega)$ is a symplectic manifold, $n \geq 2$, $H : \Omega \to R$ is a Hamiltonian and $\{\Phi_{\tilde{H}}^t\}$ is the corresponding flow, $\Sigma_0$ and $\Sigma_1$ are sections such that the Poincaré map $R_H : \Sigma_0 \to \Sigma_1$ is a diffeomorphism. Let $y_0 \in \Sigma_0$ and let $x_0$ be a point on the trajectory $\{\Phi_{\tilde{H}}(y_0)\}$ between $\Sigma_0$ and $\Sigma_1$.

Let $\tilde{R}$ be a perturbation of $R_H$ with the same properties as $R_H$, namely

1. $\tilde{R} : \Sigma_0 \to \Sigma_1$ is a diffeomorphism;
2. $\tilde{R}$ preserves $H$, that is, $H \circ \tilde{R} = H$ on $\Sigma_0$. Equivalently, $\tilde{R}(\Sigma^0_n) = \Sigma^1_n$ for every $h \in R$;
3. the restriction of $\tilde{R}$ to each $\Sigma^0_n$ preserves the symplectic form.

We also assume that $\tilde{R}$ is $C^\infty$-close to $R_H$ and they coincide outside a small neighborhood of our base point $y_0$. Our goal is to realize $\tilde{R}$ as a Poincaré map of some perturbed Hamiltonian $\tilde{H}$. Moreover $\tilde{H}$ can be chosen $C^\infty$-close to $H$ and such that $\tilde{H} - H$ is supported in a small neighborhood of $x_0$. More precisely, we prove the following.

Proposition 4.1. Let $\Omega$, $H$, $\Sigma_0$, $\Sigma_1$, $y_0$ and $x_0$ be as above. Then for every neighborhood $U$ of $x_0$ in $\Omega$ there exists a neighborhood $V$ of $y_0$ in $\Sigma_0$ such that, for
every neighborhood $\mathcal{H}$ of $H$ in $C^\infty(\Omega, \mathbb{R})$ there exists a neighborhood $\mathcal{R}$ of $R_H$ in $C^\infty(\Sigma_0, \Sigma_1)$ such that the following holds.

For every $\tilde{R} \in \mathcal{R}$ satisfying (1)–(3) above and such that $\tilde{R} = R_H$ outside $V$, there exists $\tilde{H} \in \mathcal{H}$ such that $\tilde{H} = H$ outside $U$, and $\tilde{R} = R_{\tilde{H}}$ where $R_{\tilde{H}} : \Sigma_0 \to \Sigma_1$ is the Poincaré map induced by $\tilde{H}$.

In the sequel we assume that the neighborhood $U$ in Proposition 4.1 is so small that $\overline{U} \cap (\Sigma_0 \cup \Sigma_1) = \emptyset$ where $\overline{U}$ denotes the closure of $U$. This guarantees that the Hamiltonian remains the same in a neighborhood of $\Sigma_0 \cup \Sigma_1$ and therefore the induced structures on $\Sigma_0$ and $\Sigma_1$ are preserved by the perturbation, see Remark 5.4.

In order to prove Proposition 4.2 we first prove the following variant where we realize $\tilde{R}$ as a Poincaré map only on one level set $H^{-1}(h)$. Here we denote by $R^h_H$ the restriction of $R_H$ on $\Sigma^h_0$.

**Proposition 4.2.** Let $\Omega$, $H$, $\Sigma_0$, $\Sigma_1$, $y_0$ and $x_0$ be as above, and let $h = H(x_0)$. Then for every neighborhood $U$ of $x_0$ in $\Omega$ there exists a neighborhood $V^h$ of $y_0$ in $\Sigma^h_0$ such that, for every neighborhood $\mathcal{H}$ of $H$ in $C^\infty(\Omega, \mathbb{R})$ there exists a neighborhood $\mathcal{R}^h$ of $R^h_H$ in $C^\infty(\Sigma^h_0, \Sigma^h_1)$ such that the following holds.

For every symplectic $\tilde{R}^h \in \mathcal{R}^h$ such that $\tilde{R}^h = R^h_H$ outside $V^h$, there exists $\tilde{H} \in \mathcal{H}$ such that $\tilde{H} = H$ on $H^{-1}(h) \setminus U$ and $\tilde{R}^h = R^h_{\tilde{H}}$.

**4.1. Proof of Propositions 4.1, 4.2.** The proof of Proposition 4.2 is divided into a number of steps.

**Step 1. Localization.** In this step we show that it suffices to prove the proposition in the canonical case where

- $\Omega = \mathbb{R}^{2n} = \{(q, p) : q, p \in \mathbb{R}^n\}$, with the standard symplectic structure $\omega = \sum_{i=1}^n dq_i \wedge dp_i$.
- $H(q, p) = p_n$.
- $x_0$ is the origin of $\mathbb{R}^{2n}$.
- $\Sigma_0 = \{(q, p) : q_n = -1\}$ and $\Sigma_1 = \{(q, p) : q_n = 1\}$.

Indeed, our assumptions imply that $dH(x_0) \neq 0$. By adding a constant to $H$ we may assume that $H(x_0) = 0$. By Theorem 3.1, there exist a neighborhood $U_0$ of $x_0$ and a symplectic coordinate system $(q, p)$, $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n)$, in $U_0$ such that $p_n = H|_{U_0}$ and $p(x_0) = q(x_0) = 0$. In these coordinates we have $X_H = \frac{\partial}{\partial q_n}$.

Let $\varepsilon_0 > 0$ be so small that the coordinate cube $Q_0 := \{(q, p) : |q_i| \leq \varepsilon_0 \text{ and } |p_i| \leq \varepsilon_0 \text{ for all } i\}$ is contained in both $U_0$ and the union of the trajectories between $\Sigma_0$ and $\Sigma_1$. Let $\Sigma_-$ and $\Sigma_+$ be the opposite open faces of $Q$ defined by

\begin{equation}
\Sigma_i = \{(q, p) : q_n = \pm \varepsilon_0, \ |q_i| < \varepsilon_0 \text{ for all } i < n, \ |p_i| < \varepsilon_0 \text{ for all } i\}.
\end{equation}

It suffices to prove the propositions for $\Sigma_-$ and $\Sigma_+$ in place of $\Sigma_0$ and $\Sigma_1$.

Indeed, the assumptions on $\varepsilon_0$ imply that there are diffeomorphic Poincaré maps $R_- = R_{H, \Sigma^-_0, \Sigma^-_1}$ and $R_+ = R_{H, \Sigma^+_0, \Sigma^+_1}$ where $\Sigma_0$ and $\Sigma_1$ are suitable neighborhoods of $y_0$ and $R_H(y_0)$ in $\Sigma_0$ and $\Sigma_1$, resp. In the statements of the Propositions 4.1 and 4.2 we may assume that $U \subset Q_0$ and require that $V \subset \Sigma_0$ (resp. $V^h \subset \Sigma^h_0$).

Then a perturbation $\tilde{H}$ of $H$ does not change the Poincaré map outside $\Sigma_0'$, and
it induces a Poincaré map $\tilde{R}: \Sigma_0' \to \Sigma_1'$ if and only if it induces a a Poincaré map $R_i^{-1} \circ \tilde{R} \circ R^{-1}$ from $\Sigma_-$ to $\Sigma_+$.

Thus we may replace $\Sigma_0$ and $\Sigma_1$ with $\Sigma_-$ and $\Sigma_+$ and assume that $U \subset Q_0$. Using the coordinates to identify $Q_0$ with a cube in $\mathbb{R}^{2n}$ where $\mathbb{R}^{2n}$ is equipped with the standard symplectic structure and the standard Cartesian coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$. The hypersurfaces $\Sigma_{\pm}$ are now subsets of the affine hyperplanes $\{q_n = \pm \varepsilon_0\}$. By applying the symplectic transformation $(q, p) \mapsto (\varepsilon_0^{-1} q, p \cdot \varepsilon_0)$ and multiplying $H$ by a constant, we make $\Sigma_{\pm}$ subsets of the hyperplanes $\{q_n = \pm 1\}$, while $H$ is still the coordinate function $p_n$. Now we may extend the structures to the whole $\mathbb{R}^{2n}$ and reduce the propositions to the canonical case described above.

Throughout the rest of the proof we work (without loss of generality) in this canonical setting. Recall that the hypersurfaces $\Sigma_i$, $i = 0, 1$, are foliated by level sets $\Sigma_{\pm}^h$ of $H$. In our standardized setting we have $\Sigma_{\pm}^h = \{(q, p) \in \Sigma_i : p_n = h\}$, so $\Sigma_{\pm}^h$ is an $(2n - 2)$-dimensional affine subspace.

**Step 2.** For each $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_n) \in \mathbb{R}^n$, define
\begin{equation}
A_{\hat{p}} = \{(q, p) \in \Sigma_0 : p = \hat{p}\}.
\end{equation}
Each set $A_{\hat{p}}$ is an $(n - 1)$-dimensional affine subspace contained in $\Sigma_0^h$ for $h = \hat{p}_n$. Moreover $A_{\hat{p}}$ is a Lagrangian submanifold of $\Sigma_0^h$. We denote by
\begin{equation}
\mathbb{R}^n_h = \{((\hat{p}_1, \ldots, \hat{p}_n) : \hat{p}_n = h\}.
\end{equation}
A map $\tilde{R}^h$ satisfying the requirements of Proposition 4.2 maps the partition $\{A_{\hat{p}}\}_{\hat{p} \in \mathbb{R}^n_h}$ of $\Sigma_0^h$ to a partition of $\Sigma_1^h$ into Lagrangian submanifolds $\tilde{R}(A_{\hat{p}})$. The next lemma shows that $\tilde{R}^h$ is uniquely determined by the resulting partition of $\Sigma_1^h$.

**Lemma 4.3.** Let $R_2^h: \Sigma_0^h \to \Sigma_1^h$ be symplectomorphisms such that $R_1^h = R_2^h$ outside a compact subset of $\Sigma_0^h$. Suppose that $R_1^h(A_{\hat{p}}) = R_2^h(A_{\hat{p}})$ for every $\hat{p} \in \mathbb{R}^n_h$. Then $R_1^h = R_2^h$.

**Proof.** Let $f = (R_2^h)^{-1} \circ R_1^h$. The map $f$ is a symplectomorphism from $\Sigma_0^h$ to itself and it is the identity outside a compact set. Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$ be the coordinate vectors corresponding to the coordinates $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n)$ in $\mathbb{R}^{2n}$. The affine space $\Sigma_0^h$ is naturally equipped with coordinates $(q_1, \ldots, q_{n-1}, p_1, \ldots, p_{n-1})$. The assumption that $R_1^h(A_{\hat{p}}) = R_2^h(A_{\hat{p}})$ for all $\hat{p} \in \mathbb{R}^n_h$ implies that $f$ preserves the coordinates $p_1, \ldots, p_{n-1}$. Hence for every $(q, p) \in \Sigma_0$ the partial derivatives of $f$ at $(q, p)$ have the form
\begin{equation}
\frac{\partial f}{\partial p_j}(q, p) = e_{n+j} + v_j, \quad j = 1, \ldots, n - 1,
\end{equation}
and
\begin{equation}
\frac{\partial f}{\partial q_i}(q, p) = w_i, \quad i = 1, \ldots, n - 1,
\end{equation}
where $v_j, w_i$ belong to the linear span of $e_1, \ldots, e_{n-1}$.

Since $f$ preserves the symplectic form $\omega$ on every slice $\{p_n = \text{const}\}$ of $\Sigma_0$, the vectors $v_j$ and $w_i$ from (4.3) and (4.4) satisfy
\begin{equation*}
\omega(e_{n+j} + v_j, w_i) = \omega(e_{n+j}, e_i) = \delta_{ij}
\end{equation*}
for all $i, j \in \{1, \ldots, n-1\}$, where $\delta_{ij}$ is the Kronecker delta. Since $v_j$ and $w_i$ are from the linear span of $e_1, \ldots, e_{n-1}$, we have $\omega(v_j, w_i) = 0$. Thus $\omega(e_{n+j}, w_i) = \delta_{ij}$ for all $i, j$. Hence $w_i = e_i$ for all $i$. Now (4.4) takes the form

$$\frac{\partial f}{\partial q_i}(q, p) = e_i, \quad i = 1, \ldots, n-1.$$  

Hence the restriction of $f$ to every subset $\{q = \text{const}\}$ is a parallel translation. Since $f$ is the identity outside a compact set, it follows that $f$ preserves the coordinates $q_1, \ldots, q_{n-1}$ and is the identity everywhere. Hence $\hat{R}_1^h = R_2^h$. \hfill \Box

We may assume that the set $U$ where we are allowed to change the Hamiltonian is a cube $(-\varepsilon, \varepsilon)^{2n}$ where $\varepsilon \in (0, 1)$. We prove the statement of Proposition 4.2 for $V^h \subset \Sigma_0^h$ defined as the projection of $U$ to $\Sigma_0^h$, namely

$$V^h = (-\varepsilon, \varepsilon)^{n-1} \times \{-1\} \times (-\varepsilon, \varepsilon)^{n-1} \times \{h\}. \quad (4.5)$$

We may assume that the neighborhood $\mathcal{H}$ of $H$ (see the formulation of the proposition) is so small that every $\tilde{H} \in \mathcal{H}$ such that $\tilde{H} = H$ on $H^{-1}(h) \setminus U$ induces a smooth bijection Poincaré map $\tilde{R}_H^h : \Sigma_0^h \rightarrow \Sigma_1^h$. Moreover, $\tilde{R}_H^h = R_H^h$ outside $V^h$ for every such $\tilde{H}$.

With Lemma 4.3, Proposition 4.2 boils down to the following statement: Given a sufficiently small perturbation $\tilde{R}_H^h$ of $R_H^h$ such that $\tilde{R}_H^h = R_H^h$ outside $V^h$, we can construct a perturbed Hamiltonian $\tilde{H} \in \mathcal{H}$ such that $\tilde{H} = H$ on $H^{-1}(h) \setminus U$ and

$$R_H^h(A_{\tilde{p}}) = \tilde{R}_H^h(A_{\tilde{p}}) \text{ for all } \tilde{p} \in \mathbb{R}_h^n. \quad (4.6)$$

Step 3. For each $\tilde{p} \in \mathbb{R}^n$, define a Lagrangian affine subspace $L_{\tilde{p}} \subset \mathbb{R}^{2n}$ by

$$L_{\tilde{p}} = \{(q, \tilde{p}) : q \in \mathbb{R}^n\}.$$  

The subspaces $L_{\tilde{p}}$, where $\tilde{p}$ ranges over $\mathbb{R}^n$, form a foliation of $\mathbb{R}^{2n}$. Our plan is to perturb the subfoliation $\{L_{\tilde{p}}\}_{\tilde{p} \in \mathbb{R}_h^n}$ and obtain another foliation by Lagrangian submanifolds $\{\tilde{L}_{\tilde{p}}\}_{\tilde{p} \in \mathbb{R}_h^n}$ such that

$$\tilde{L}_{\tilde{p}} \cap \Sigma_0^h = A_{\tilde{p}} \quad \text{and} \quad (4.7)$$

$$\tilde{L}_{\tilde{p}} \cap \Sigma_1^h = \tilde{R}_H^h(A_{\tilde{p}}) \quad \text{for all } \tilde{p} \in \mathbb{R}_h^n. \quad (4.8)$$

for all $\tilde{p} \in \mathbb{R}_h^n$, and define the perturbed Hamiltonian $\tilde{H}$ so that it is constant on each submanifold $\tilde{L}_{\tilde{p}}$. Since the flow (without fixed points) on a level set of a Hamiltonian is determined by this set up to a time change, once we fix the level set $\tilde{H}^{-1}(h)$, the resulting Poincaré map on $\Sigma_0^h$ is independent of other level sets.

The next lemma says that this construction solves our problem. We say that a line segment $[x, y] \subset \mathbb{R}^{2n}$ is horizontal if it is parallel to the coordinate axis of the $q_n$-coordinate.

Lemma 4.4. Let $\{\tilde{L}_{\tilde{p}}\}_{\tilde{p} \in \mathbb{R}_h^n}$ be a foliation by Lagrangian submanifolds satisfying (4.7) and (4.8). Let $\tilde{H} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a smooth function such that

$$\tilde{H}|_{\tilde{L}_{\tilde{p}}} = h \quad \text{for all } \tilde{p} \in \mathbb{R}_h^n. \quad (4.9)$$

and suppose that $\tilde{H}$ defines a smooth Poincaré map $\tilde{R}_{\tilde{H}}^h : \Sigma_0^h \rightarrow \Sigma_1^h$. 


Suppose in addition that every horizontal segment intersecting \( \Sigma_0^h \cup \Sigma_1^h \) but not intersecting \( U = (−\varepsilon, \varepsilon)^{2n} \) is contained in one of the submanifolds \( \tilde{L}_\tilde{p} \). Then \( R^h_H = \tilde{R}^h \) and \( \tilde{H} = H \) on \( H^{-1}(h) \setminus U \).

**Proof.** The key implication of the assumption (4.9) is that every submanifold \( \tilde{L}_\tilde{p} \) is contained in one level set of \( \tilde{H} \).

First we show that \( \tilde{H} = H \) on \( H^{-1}(h) \setminus U \). Recall that \( H = p_n \) and \( \tilde{R}(\Sigma_0^h) = \Sigma_1^h \).

This and (4.7), (4.8), (4.9) imply that \( \tilde{H} = H \) on \( \Sigma_0^h \cup \Sigma_1^h \). Since \( U \) is a convex set lying between the hyperplanes \( \Sigma_0^h \) and \( \Sigma_1^h \), every point \( x \in H^{-1}(h) \setminus U \) can be connected to a point \( y \in \Sigma_0^h \cup \Sigma_1^h \) by a horizontal segment not intersecting \( U \). By the assumptions of the lemma the segment \([x, y]\) is contained in one level set of \( \tilde{H} \), and it is contained in a level set of \( H \) since \( H = p_n \). Therefore \( \tilde{H}(x) = H(x) \) for all \( x \in H^{-1}(h) \setminus U \).

Now we show that \( R^h_H = \tilde{R}^h \). Fix \( \tilde{p} \in \mathbb{R}^n_h \) and consider the leaf \( \tilde{L}_\tilde{p} \) of our foliation. By elementary linear algebra, the property that \( \tilde{L}_\tilde{p} \) is contained in a level set of \( \tilde{H} \) implies that the Hamiltonian vector field \( \tilde{X}_{\tilde{H}} \) is tangent to \( \tilde{L}_\tilde{p} \). Therefore \( \tilde{L}_\tilde{p} \) is invariant under the flow \( \Phi^\epsilon \). By (4.7) and (4.8) it follows that

\[
(4.10) \quad R^h_H(A_{\tilde{p}}) = L_{\tilde{p}} \cap \Sigma_1^h = \tilde{R}^h(A_{\tilde{p}}).
\]

The fact that \( \tilde{H} = H \) on \( H^{-1}(h) \setminus U \) implies that \( R^h_H = \tilde{R}^h = \tilde{R}^h \) outside a compact set. Now with (4.10) at hand we can apply Lemma 4.3 to \( R^h_H \) and \( \tilde{R}^h \) in place of \( R_1^h \) and \( R_2^h \) and conclude that \( R^h_H = \tilde{R}^h \).

It remains to construct a foliation \( \{\tilde{L}_\tilde{p}\} \) satisfying Lemma 4.3 and such that the resulting Hamiltonian \( \tilde{H} \) is sufficiently close to \( H \) in \( C^\infty \). This is achieved in the next two steps.

**Step 4.** We begin with a construction of \( \tilde{L}_\tilde{p} \) for a fixed \( \tilde{p} \in \mathbb{R}^n_h \). Throughout this step, \( \tilde{p}_i \) denotes a fixed real number (the \( i \)-th coordinate of \( \tilde{p} \)) and \( p_i \), \( q_i \) are still coordinate functions, \( i = 1, \ldots, n \).

We identify \( \mathbb{R}^{2n} \) with the cotangent bundle \( T^*\mathbb{R}^n \) using \( q_i \)'s as spatial coordinates and \( p_i \)'s as coordinates in the fibers of the cotangent bundle. We construct the desired leaf \( \tilde{L}_\tilde{p} \) as a graph of a closed 1-form \( \tilde{\alpha} = \tilde{\alpha}_{\tilde{p}} \) on \( \mathbb{R}^n \). Recall that a graph of a 1-form is a Lagrangian submanifold of the cotangent bundle if and only if the 1-form is closed, see e.g. (12).

Define \( W = (−\varepsilon, \varepsilon)^n \). The last requirement of Lemma 4.3 prescribes \( \{\tilde{L}_\tilde{p}\} \) and hence \( \tilde{\alpha} \) outside \( W \). We cover \( \mathbb{R}^{2n} \) by two open half-spaces \( \{q_n < \varepsilon\} \) and \( \{q_n > −\varepsilon\} \) and consider \( \tilde{\alpha} \) on these half-spaces with \( W \) removed. First consider a 1-form \( \alpha = \sum \tilde{p}_i dq_i \) (with constant coefficients) and define the restriction of \( \tilde{\alpha} \) to \( \{q_n < \varepsilon\} \setminus W \) by

\[
(4.11) \quad \tilde{\alpha} = \alpha \quad \text{on} \quad \{q_n < \varepsilon\} \setminus W.
\]

The graph of \( \alpha \) is the unperturbed leaf \( \tilde{L}_{\tilde{p}} \). It satisfies (4.7) and consists of horizontal segments, fulfilling the respective part of requirements of Lemma 4.3.

Now we define \( \tilde{\alpha} \) on the set \( \{q_n > −\varepsilon\} \setminus W \). In fact, \( \tilde{\alpha} \) on this set is uniquely determined by the map \( \tilde{R} \) and the requirement (4.8). Consider the set \( \tilde{A} := \tilde{R}^h(A_{\tilde{p}}) \), the desired intersection of the graph of \( \tilde{\alpha} \) with \( \Sigma_1^h \). Since \( \tilde{R} \) preserves \( H \) and
the symplectic form in the levels of $H$, $\tilde{A}$ is an $(n-1)$-dimensional Lagrangian submanifold of the affine subspace $\Sigma^h_1$. In the unperturbed case $\tilde{R}^h = R^h_H$, $\tilde{A}$ is an affine subspace of $\Sigma^h_1$. Hence, if $\tilde{R}^h$ is sufficiently close to $R^h_H$ in $C^\infty$, then $\tilde{A}$ is a graph of a closed 1-form $\tilde{\beta}$ defined on the hyperplane $\{q_n = 1\} \subset \mathbb{R}^n$. We define the restriction of $\tilde{\alpha}$ to $\{q_n > -\varepsilon\} \setminus W$ by

$$\tilde{\alpha} = \Pi^* \tilde{\beta} + \tilde{p}_n dq_n \quad \text{on} \quad \{q_n > -\varepsilon\} \setminus W,$$

where $\Pi$ is the orthogonal projection from $\mathbb{R}^n$ to the hyperplane $\{q_n = 1\}$. The graph of the 1-form defined by (4.12) consists of horizontal segments and satisfies (4.8). Since $\tilde{R}^h = R^h_H$ outside $V$ (see (4.5)), the definitions (4.11) and (4.12) agree on the common domain $\{-\varepsilon < q_n < \varepsilon\} \setminus W$.

Thus we have defined the desired 1-form $\tilde{\alpha}$ on $\mathbb{R}^n \setminus W$. Our goal is to extend $\tilde{\alpha}$ to the whole $\mathbb{R}^n$. We need the following lemma.

**Lemma 4.5.** $\tilde{\alpha}$ defined above is exact on $\mathbb{R}^n \setminus W$.

**Proof.** The statement is trivial if $n > 2$, since the 1-form is closed and the set $\mathbb{R}^n \setminus W$ is simply connected.

For $n = 2$, it suffices to check that the integral of $\tilde{\alpha}$ over any one cycle going around the hole $W = (-\varepsilon, \varepsilon)^2$, is zero. We do this for the boundary of the square $[-1, 1]^2$. Let $s$ be the side $[-1, 1] \times \{1\}$ of this square. Since $\tilde{\alpha} = \alpha$ on the remaining three sides of the square, it suffices to verify that $\int_s \tilde{\alpha} = \int_s \alpha$. Each integral is the signed area between the graph of the respective 1-form and the line $\{p_1 = 0\}$ in the plane $\Sigma^h_1$ with coordinates $(q_1, p_1)$. Since one graph is taken to the other by a symplectomorphism $\tilde{R}^h \circ (R^h_H)^{-1}$ which is the identity outside the small square $(-\varepsilon, \varepsilon)^2$, this signed area is preserved. □

The right-hand sides of (4.11) and (4.12) are closed 1-forms defined on the entire $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be their antiderivatives. To ensure that $f$ and $g$ depend smoothly on the parameter $\tilde{p}$, we choose them so that $g(0, \ldots, 0, -1) = f(0, \ldots, 0, -1) = 0$. Observe $f = g$ outside the set $(-\varepsilon, \varepsilon)^n-1 \times \mathbb{R}$ since (4.11) and (4.12) agree there.

We combine $f$ and $g$ using a suitable partition of unity as follows. Fix a smooth function $\mu : \mathbb{R} \to [0, 1]$ such that $\mu(t) = 1$ for all $t < -1/2$ and $\mu(t) = 0$ for all $t > 1/2$ and define a smooth function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}(q) = \mu(q_n/\varepsilon) \cdot f(q) + (1 - \mu(q_n/\varepsilon)) \cdot g(q).$$

Now define

$$\tilde{\alpha} = d\tilde{f}$$

everywhere on $\mathbb{R}^n$. This definition agrees with (4.11) and (4.12) on their respective domains since $f$ and $g$ agree on the set $\{-\varepsilon < q_n < \varepsilon\} \setminus W$.

This finishes the construction of the 1-form $\tilde{\alpha} = \tilde{\alpha}_{\tilde{p}}$ for a fixed $\tilde{p}$. The graph $\tilde{L}_{\tilde{p}}$ of $\tilde{\alpha}_{\tilde{p}}$ is a Lagrangian submanifold satisfying the requirements of Lemma 4.4.

**Step 5.** It remains to show that $\tilde{H}$ can be chosen to be $C^\infty$ close to $H$. In order to prove it we prove the family of Lagrangian submanifolds $\{\tilde{L}_{\tilde{p}}\}_{\tilde{p} \in \mathbb{R}^n}$, constructed in the previous step, is a $C^\infty$ perturbation of the original foliation $\{L_{\tilde{p}}\}_{\tilde{p} \in \mathbb{R}^n}$.

Going through the constructions of Step 4 one sees that $\tilde{\alpha}_{\tilde{p}}$ depends smoothly on $\tilde{p}$. Hence one can define a smooth map $F^h : H^{-1}(h) \to \mathbb{R}^n$ by

$$F^h(q, \tilde{p}) = (q, \alpha_{\tilde{p}}(q)), \quad q \in \mathbb{R}^n, \tilde{p} \in \mathbb{R}^n.$$
This map takes each leaf of the original foliation \( \{ L_p \} \) to the corresponding Lagrangian submanifold \( \tilde{L}_p \). Note that \( F^h \) is determined by \( \tilde{R}^h \) by means of explicit formulae (involving some inverse functions) and in the unperturbed case (when \( \tilde{R}^h = R^h_H \)) the resulting map \( F^h \) is the identity. Therefore \( F \) is \( C^\infty \)-close to the identity on any fixed compact set as long as \( \tilde{R} \) is \( C^\infty \)-close to \( R_H \). We may assume that the neighborhood \( \mathcal{R} \) of \( R^h_H \) from which \( \tilde{R}^h \) is chosen (see the formulation of Proposition \[12\]) is so small that

\[
\| F^h - \text{id} \|_{C^1([-1,1])} < \frac{\varepsilon}{2}
\]

where the norm of the first derivative is understood as the operator norm.

The construction in Step 4 implies that \( F \) is the identity on the set \( \{ q_n < -\varepsilon \} \) and \( F^h - \text{id} \) is constant along any horizontal segment not intersecting \( U \). This implies that the norm estimate in (4.13) holds in \( C^1 \left( H^{-1}(h) \right) \), and this norm estimate implies that \( F^h \) is a diffeomorphism from \( H^{-1}(h) \) to its image.

Thus \( \tilde{H} \) can be chosen to be \( C^\infty \) close to \( H \) and we finish the proof of Proposition \[12\].

Proposition \[12\] can be applied to prove the following fact which is known in folklore but for which the authors could not find a reference.

**Proposition 4.6.** Let \( \varphi_0 : D^{2n} \to D^{2n} \), \( n \geq 1 \), be a symplectomorphism \( C^\infty \)-close to the identity and coinciding with the identity near the boundary. Then there exists a smooth family of symplectomorphisms \( \{ \varphi_t \}_{t \in [0,1]} \) of \( D^{2n} \) fixing a neighborhood of the boundary and such that \( \varphi_t = \varphi_0 \) for all \( t \in [0, \frac{1}{2}] \), \( \varphi_t = \text{id} \) for all \( t \in [\frac{1}{2}, 1] \), and the family \( \{ \varphi_t \} \) is \( C^\infty \)-close to the trivial family (of identity maps).

*Proof.* Consider \( \Omega = \mathbb{R}^{2n+2} = (q_1, \ldots, q_{n+1}, p_1, \ldots, p_{n+1}) \) with the standard symplectic structure and Hamiltonian \( H = p_{n+1} \). Let \( \Sigma_0 \) and \( \Sigma_1 \) be affine hyperplanes defined by the equations \( q_{n+1} = -1 \) and \( q_{n+1} = 1 \), resp.

We introduce notation \( \bar{p} \) and \( \bar{q} \) for the coordinate \( n \)-tuples \( (p_1, \ldots, p_n) \) and \( (q_1, \ldots, q_n) \). The Poincaré map \( R_H : \Sigma_0 \to \Sigma_1 \) is given by

\[
R_H(\bar{q}, -1, \bar{p}, p_{n+1}) = (\bar{q}, 1, \bar{p}, p_{n+1}).
\]

We define a perturbed map \( \tilde{R}^0 : \Sigma_0 \to \Sigma_1 \) by

\[
\tilde{R}^0(\bar{q}, -1, \bar{p}, 0) = (\varphi_{0,q}(\bar{q}, \bar{p}), 1, \varphi_{0,p}(\bar{q}, \bar{p}), 0)
\]

where \( \varphi_{0,q} \) and \( \varphi_{0,p} \) are \( q \)- and \( p \)-coordinates of \( \varphi_0 \). By applying Proposition \[4.2\] to \( h = 0 \) we get \( \tilde{H} \) such that \( \tilde{R}^0 = R^0_H \). Let \( \mathcal{H} \) be the Hamiltonian such that \( \mathcal{H}^{-1}(0) = \tilde{H}^{-1}(0) \) (hence \( \tilde{R}^0 = R^0_H = R^0_{\mathcal{H}} \)), and \( \mathcal{H} - p_{n+1} \) does not depend on \( p_{n+1} \). Then \( \partial \mathcal{H} / \partial p_{n+1} = 1 \) and the return map \( R_{\mathcal{H}} \) is given by

\[
R_{\mathcal{H}}(\bar{q}, -1, \bar{p}, p_{n+1}) = (\varphi_{0,q}(\bar{q}, \bar{p}), 1, \varphi_{0,p}(\bar{q}, \bar{p}), p_{n+1}).
\]

Define a time-dependent Hamiltonian \( H_t \) on \( \mathbb{R}^{2n} \) as follows:

\[
H_t(\bar{q}, \bar{p}) = \mathcal{H}(\bar{q}, t, \bar{p}, 0).
\]

Then the flow on \( \mathbb{R}^{2n} \) generated by \( H_t \) connects the identity to \( \varphi_0 \). \( \square \)
4.2. Proof of Proposition 4.1. We deduce Proposition 4.1 from Proposition 4.2 by applying it to all $h \in \mathbb{R}$ and to the corresponding restrictions $\tilde{R}|_{2 \Sigma_0}$ in place of $\tilde{R}^h$. Let $\tilde{H}^h$ denote the resulting perturbed Hamiltonian for a given $h$. Then the desired $\tilde{H}$ in Proposition 4.1 is constructed from the family $\{\tilde{H}^h\}_{h \in \mathbb{R}}$ in such a way that $\tilde{H}^{-1}(h) = (\tilde{H}^h)^{-1}(h)$ for every $h \in \mathbb{R}$. The resulting Poincaré map is the prescribed $\tilde{R}$ since the Hamiltonian flow is determined by level sets up to a time change.

We have only to show that the family of Lagrangian submanifolds $\tilde{L}_p$, $\tilde{p} \in \mathbb{R}^n$, constructed in the Step 4 in the proof of Proposition 4.2 forms a foliation of $\mathbb{R}^{2n}$. Similar to Step 5, one can define a smooth map $F: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$F(q, \tilde{p}) = (q, \alpha_\tilde{p}(q)), \quad q, \tilde{p} \in \mathbb{R}^n.$$  

$F$ is $C^\infty$-close to the identity on any fixed compact set as long as $\tilde{R}$ is $C^\infty$-close to $R_H$. Furthermore one can slightly adjust the argument in Step 5 to show that $F$ is a diffeomorphism from $\mathbb{R}^{2n}$ to itself. This finishes the proof of Proposition 4.1.

5. Proof of Theorem 1.1

We begin with the following result by Berger-Turaev [7].

Theorem 5.1 ([7]). For any $n \geq 1$, there is a $C^\infty$-small perturbation of the identity map $\text{id}: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that the resulting map is symplectic and coincides with the identity map near the boundary and has positive metric entropy.

Proof. Theorem A in [7] is proven for $n = 1$. Though they did not mention in the theorem whether the perturbed map agrees with the original map near the boundary, Corollary 5 in [7] (see also Corollary 4.8 in [7]) guarantees that they can coincide near $\partial D$.

To extend the result to $n \geq 2$, one can do the following. Let $\varphi: D^2 \to D^2$ be the perturbation of the identity constructed for $n = 1$ and $\{\varphi_t\}$ the family of symplectomorphisms constructed in Proposition 4.1 for $\varphi_0 = \varphi$. Define a diffeomorphism $\Phi: D^2 \times D^{2n-2} \to D^2 \times D^{2n-2}$ by

$$\Phi(x, y) = (\varphi_t(x), y), \quad x \in D^2, y \in D^{2n-2}.$$  

One easily sees that $\Phi$ is a symplectomorphism fixing the neighborhood of the boundary. Since $\Phi(x, y) = \varphi(x)$ for all $y$ with $|y| \leq \frac{x}{3}$ and $\varphi$ has positive metric entropy, so does $\Phi$. □

Let $\Omega = (\Omega^{2n}, \omega)$ be a symplectic manifold, $H: \Omega \to \mathbb{R}$ a Hamiltonian such that the flow $\{\Phi^t_H\}$ is completely integrable, and $\mathcal{T}$ a Liouville torus and $x_0 \in \mathcal{T}$. By the Liouville-Arnold theorem, there exist action-angle coordinates $(\mathbf{q}, \mathbf{p}) = (q_1, ..., q_n, p_1, ..., p_n)$ near $\mathcal{T}$.

These coordinates identify a neighborhood of $\mathcal{T}$ with the product $\mathbb{T}^n \times D$ where $D \subset \mathbb{R}^n$ is a small $n$-dimensional disc and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the standard $n$-torus. The coordinates $p_1, ..., p_n$ parametrize $D$ and $q_1, ..., q_n$ are the standard angle coordinates on $\mathbb{T}^n$ (taking values in $\mathbb{R}/\mathbb{Z}$). The Hamiltonian $H$ depends only on $\mathbf{p}$-coordinates, hence it can be regarded as a function on $D$.

The flow $\Phi^t_H$ in these coordinates is governed by the equations

$$\begin{align*}
q_i &= A_i(p) := \frac{\partial H}{\partial p_i}(p) \\
\dot{p}_i &= 0.
\end{align*}$$  

(5.1)
Thus, along every trajectory the $p$-coordinates are constant and $q$-coordinates vary linearly with velocity $A_i(p)$, $i = 1, \ldots, n$. We may assume that $p = 0$ on $\mathcal{T}$ and $H(0) = 0$.

By a small perturbation of the function $H = H(p)$ near $p = 0$ we can satisfy the following conditions:

- The flow on $\mathcal{T}$ is nonvanishing. (This means that at least one of the numbers $\frac{\partial H}{\partial p_i}(0)$ is nonzero).
- The flow on $\mathcal{T}$ is periodic. (It suffices to perturb $H$ so that all numbers $\frac{\partial H}{\partial p_i}(0)$ are rational).
- The system is KAM-nondegnerate at $\mathcal{T}$. In our notation this condition means that the Hessian of $H$ at $p = 0$ is nondegenerate.

We change the coordinates by an action of some matrix from $SL(n, \mathbb{Z})$ on $\mathbb{T}^n$ to assure that the (periodic) flow on $\mathcal{T}$ is the flow along the $q_n$-coordinate, that is, $A_i(0) = 0$ for $i = 1, \ldots, n - 1$ and $A_n(0) > 0$.

After having made these modifications we abuse notation and use the same letter $H$ for the modified Hamiltonian and $p, q$ for the modified coordinates. It suffices to prove the theorem for Hamiltonians and coordinates satisfying the conditions above.

For $h \in \mathbb{R}$, denote

$$D_h := \{p \in D : H(p) = h\}.$$  

Replacing $D$ by a smaller neighborhood of $0$ if necessary, we apply the implicit function theorem (using the fact that $\frac{\partial H}{\partial p_i}(0) = A_n(0) > 0$) and obtain a smooth family $\{f_h\}_{h \in \mathbb{R}}$ of smooth functions $f_h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$p \in D_h \iff p_n = -f_h(p_1, \ldots, p_{n-1})$$

for every $h \in \mathbb{R}$ and $p \in D$. The minus sign here is introduced to be canceled out later in \cite{1}. We introduce notation $\bar{p}$ and $\bar{q}$ for the coordinate $(n - 1)$-tuples $(p_1, \ldots, p_{n-1})$ and $(q_1, \ldots, q_{n-1})$. With this notation \cite{5.2} implies that

$$H(\bar{p}, -f_h(\bar{p})) = h$$

for all $\bar{p} \in \mathbb{R}^{n-1}$ sufficiently close to the origin and $h \in \mathbb{R}$ sufficiently close to $0$.

Differentiating \cite{5.3} with respect to $p_i$ we obtain that

$$\frac{\partial f_h}{\partial p_i}(\bar{p}) = \frac{A_i}{A_n}(\bar{p}, f_h(\bar{p})), \quad i = 1, \ldots, n - 1.$$  

Since $A_n(0) = 0$ for $i < n$, the origin is a critical point of $f_h$.

Now we cut our invariant tubular neighborhood of $\mathcal{T}$ by a hypersurface

$$\Sigma = \{(q, p) : q_n = 0\}$$

and consider the resulting Poincaré return map $R = R_H : \Sigma \to \Sigma$. The hypersurface $\Sigma$ is naturally identified with $\mathbb{T}^{n-1} \times D$ and parametrized by coordinates $(\bar{q}, \bar{p})$ where $\bar{q} = (q_1, \ldots, q_{n-1})$ and $\bar{p} = (p_1, \ldots, p_n)$. By \cite{5.1}, $R$ is given by

$$R(\bar{q}, \bar{p}) = \left(q_1 + \frac{A_i}{A_n}(\bar{p}), \ldots, q_{n-1} + \frac{A_{n-1}}{A_n}(\bar{p}), \bar{p}\right).$$

Note that the origin of $\Sigma$ is a fixed point of $R$.

The next lemma is one of the key ingredients of the proof.
Lemma 5.2. There exists a diffeomorphism $\tilde{R}: \Sigma \to \Sigma$ arbitrarily close to $R$ in $C^\infty$ and such that $\tilde{R} = R$ outside an arbitrarily small neighborhood of the origin and the following conditions are satisfied:

1. For every $h \in \mathbb{R}$, $\tilde{R}$ maps the level set $\Sigma^h := \{x \in \Sigma : H(x) = h\}$ to itself and preserves the symplectic form on this set.
2. There is a small $\tilde{R}$-invariant neighborhood of the origin and the restriction of $\tilde{R}$ to this neighborhood has positive metric entropy. Moreover, $\tilde{R}$ is entropy non-expansive.

Remark 5.3. In order to speak about metric entropy of $\tilde{R}$, we regard $\Sigma$ with the measure induced by the symplectic volume on $\Omega$ and the original flow $\Phi^t_h$, see (3.1). One easily sees that any map $\tilde{R}$ satisfying the first requirement of the lemma preserves this measure.

Proof. Our nondegeneracy assumption on the Hessian of $H(p)$ implies that $\bar{0} \in \mathbb{R}^{n-1}$ is a nondegenerate critical point of $f_0$. Thus for all $h$ near $0$, the function $f_h$ has a nondegenerate critical point $\bar{c}(h) = (c_1(h), \ldots, c_{n-1}(h))$ depending smoothly on $h$ with $\bar{c}(0) = \bar{0}$.

First we fix $h \in \mathbb{R}$ sufficiently close to 0 and describe the construction within $\Sigma^h$. By (5.2), the intersection of $\Sigma^h$ with a suitable neighborhood of the origin is parametrized by a map

$$\Gamma_h: O_q \times O_p \to \Sigma$$

given by

$$\Gamma_h(q, \bar{p}) = (q, \bar{p} - f_h(\bar{p}))$$

where $O_q$ and $O_p$ are certain neighborhoods of the origin in $\mathbb{R}^{n-1}$. Here in the right-hand side we use the coordinates $(q, p) = (q, \bar{p}, p_n)$ on $\Sigma$. Since $O_p$ comes from the implicit function theorem, it can be chosen uniformly in $h$. Hence we may assume that $\bar{c}(h) \in O_p$. Observe that $\Gamma_h$ preserves the symplectic form, where $O_q \times O_p \subset \mathbb{R}^{2n-2}$ is equipped with the standard symplectic form $dq \wedge dp = \sum_{i=1}^{n-1} dq_i \wedge dp_i$. Therefore the restriction of $R$ to the set $\Gamma_h(O_q \times O_p) \subset \Sigma^h$ is conjugate to a symplectic map $G_h: O_q \times O_p \to \mathbb{R}^{2n-2}$,

$$(5.6) \quad G_h = \Pi \circ R \circ \Gamma_h$$

where $\Pi$ is the standard projection forgetting the last coordinate. For brevity, we define $m = n - 1$.

We are going to perturb $G_h$ so that the resulting map $\tilde{G}_h: O_q \times O_p \to \mathbb{R}^{2m}$ is still symplectic, it coincides with $G_h$ outside a compact subset of $O_q \times O_p$, has an invariant neighborhood of $(\bar{0}, \bar{c}(h))$ and positive metric entropy in this neighborhood.

By (5.5) and (5.4), $G_h$ can be written in the form

$$(5.7) \quad G_h(q, \bar{p}) = (q + \nabla f_h(\bar{p}), \bar{p})$$

where $\nabla f_h$ is the Euclidean gradient of $f_h: \mathbb{R}^m \to \mathbb{R}$. Notice that $G_h$ is the time-1 map of the Hamiltonian flow $\Phi^t_{F_h}$ with the Hamiltonian $F_h$ given by

$$(5.8) \quad F_h(q, \bar{p}) := f_h(\bar{p})$$

Our plan is to perturb $F_h$ and define $\tilde{G}_h$ as the time-1 map of the flow defined by the perturbed Hamiltonian.
Since $\bar{c}(h)$ is a nondegenerate critical point of $f_h$, by Morse Lemma there exist a coordinate chart $\bar{P} = (P_1, \ldots, P_m)$ in $O_p$ such that $\bar{P}$ vanishes at $\bar{c}(h)$ and
\begin{equation}
\tag{5.9}
f_h = f_h(\bar{c}(h)) + P_1^2 + \cdots + P_k^2 - P_{k+1}^2 - \cdots - P_m^2
\end{equation}
in a neighborhood of $\bar{c}(h)$. We regard $P_1, \ldots, P_m$ as functions on $O_q \times O_p$ by setting $P_i(\bar{q}, \bar{p}) = P_i(\bar{p})$. Then $\bar{P}$ is a formula for $F_h$ as well, cf. (3.1). Since $P_1, \ldots, P_m$ depend only on $\bar{p}$-coordinates, they are Poisson commuting. Hence, by Theorem 4.1, we can extend this collection of functions to a symplectic coordinate system $(\bar{Q}, \bar{P}) = (Q_1, \ldots, Q_m, P_1, \ldots, P_m)$ in a neighborhood of the point $(\bar{q}, \bar{p}) = (0, \bar{c}(h))$ in $O_q \times O_p$. We may assume that $Q_1, \ldots, Q_m$ vanish at $(0, \bar{c}(h))$.

We perturb the Hamiltonian $F_h$ in a region $U_\delta := \{ \bar{P}^2 < \delta, Q^2 < \delta \}$ where $\delta$ is a sufficiently small positive number. Let $\xi$ be a smooth function on $[0, 1]$ with $\xi \equiv 1$ on $[0, \delta/2]$ and $\xi \equiv 0$ on $[\delta, 1]$. For any $\varepsilon > 0$, define a perturbed Hamiltonian $F_{h, \varepsilon, \delta}$ by
\begin{equation}
\tag{5.10}
F_{h, \varepsilon, \delta} := F_h + \varepsilon \xi(\bar{P}^2) \xi(\bar{Q}^2) (Q_1^2 + \cdots + Q_k^2 - Q_{k+1}^2 - \cdots - Q_m^2).
\end{equation}
Due to the formula (5.9) for $F_h$, the Hamiltonian flow $\Phi^t_{F_{h, \varepsilon, \delta}}$ within $U_{\delta/2}$ is governed by the following equations in coordinates $(\bar{Q}, \bar{P})$:
\[
\begin{cases}
\dot{Q}_i = 2P_i, & i \leq k, \\
\dot{P}_i = -2\varepsilon Q_i, & i \leq k, \\
\dot{Q}_i = -2P_i, & i > k, \\
\dot{P}_i = 2\varepsilon Q_i, & i > k.
\end{cases}
\]
This defines a periodic flow with period $\pi/\sqrt{\varepsilon}$ and a fixed point at $\bar{P} = \bar{Q} = 0$. Outside $U_\delta$, the flow $\Phi^t_{F_{h, \varepsilon, \delta}}$ coincides with the original flow $\Phi^t_{F_h}$. We assume that $\delta$ is so small that the trajectories of the flow $\Phi^t_{F_h}$ starting in $U_\delta$ stay within the domain of $(\bar{Q}, \bar{P})$ for all $t \in [0, 1]$. Then the same property holds for the flow $\Phi^t_{F_{h, \varepsilon, \delta}}$.

We choose $\varepsilon$ so that $N := \pi/\sqrt{\varepsilon}$ is an integer. Define
\begin{equation}
\tag{5.11}
G_{h, \varepsilon, \delta} := \Phi^1_{F_{h, \varepsilon, \delta}},
\end{equation}
the time-1 map of the flow determined by the Hamiltonian $F_{h, \varepsilon, \delta}$. This map is defined on an open subset of $O_q \times O_p$ containing the closure of $U_\delta$ (provided that $\delta$ is sufficiently small), and it tends to $G_h$ in $C^\infty$ as $\varepsilon \to 0$ (for each fixed $\delta$). The disc $D_{\delta/2}$ is invariant under $G_{h, \varepsilon, \delta}$ and the restriction of $G_{h, \varepsilon, \delta}$ to this disc is $N$-periodic.

Choose a closed disc $B \subset U_{\delta/2}$ such that the sets $B, G_{h, \varepsilon, \delta}(B), \ldots, G_{h, \varepsilon, \delta}^{N-1}(B)$ are disjoint. This disc is just a sufficiently small ball centered at a non-fixed point of $G_{h, \varepsilon, \delta}$. By Theorem 5.1, there exist a symplectomorphism $\theta : B \to B$ arbitrarily $C^\infty$-close to the identity fixing a neighborhood of the boundary and having positive metric entropy. We extend $\theta$ to the whole $\mathbb{R}^{2m}$ by the identity map outside $B$ and use the same letter $\theta$ for its extension to $\mathbb{R}^{2m}$.

Now define
\begin{equation}
\tag{5.12}G_h = G_{h, \varepsilon, \delta} \circ \theta.
\end{equation}
This formula defines $G_h$ in a neighborhood of the closure of $U_\delta$. Outside $U_\delta$ this map coincides with $G_h$ and we extend it by $G_h$ to obtain a map $\tilde{G}_h : O_q \times O_p \to \mathbb{R}^{2m}$. By construction, $U_{\delta/2}$ is invariant under $\tilde{G}_h$, $B$ is invariant under $\tilde{G}_h^N$, and $(\tilde{G}_h^N)|_B = \theta|_B$. Therefore the restriction of $\tilde{G}_h$ to $U_{\delta/2}$ has positive metric entropy.
By choosing ε sufficiently small and θ sufficiently close to the identity, \( \bar{G}_h \) can be made arbitrarily close to \( G_h \) in the \( C^\infty \) topology.

In order to make the perturbed map entropy non-expansive, the construction can be modified as follows. Instead of working with one disc \( B_i \), we choose a sequence of disks \( \{ B_i \} \) tending to the origin, with diameters going to 0, and such that the sets \( G_{h,\varepsilon,\delta}(B_i) \), \( i \in \mathbb{N}, k = 0, \ldots, N-1 \), are disjoint. We perturb the identity map within each \( B_i \) as in Theorem 5.1 so that the composition of these perturbations is a \( C^\infty \) map \( \theta : \mathbb{R}^{2m} \to \mathbb{R}^{2m} \) which is close to the identity. Then the map \( \bar{G}_h \) defined by (5.12) is entropy non-expansive.

By choosing \( \varepsilon \) sufficiently close to the identity, \( \bar{G}_h \) can be made arbitrarily close to \( G_h \) in the \( C^\infty \) topology.

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By the conjugation inverse to (5.6) we transform \( \bar{G}_h \) to a perturbation \( \tilde{R}_h \) of \( R_h = R|_{\Sigma^h} \). Namely

\[
(5.13) \quad \tilde{R}_h = \Gamma_h \circ \bar{G}_h \circ \Pi|_{\Sigma^h}
\]

within the coordinate domain \( \Gamma_h(\mathcal{O}_q \times \mathcal{O}_p) \) and \( \tilde{R}_h \) coincides with \( R_h \) outside this domain. This finishes the description of the construction within one level set.

It remains to show that one can apply the construction simultaneously on all level sets \( \Sigma^h, h \in \mathbb{R} \), so that the union of the resulting maps \( \tilde{R}_h \) is a diffeomorphism \( \tilde{R} : \Sigma \to \Sigma \) satisfying the requirements of the lemma.

In order to do this, we first construct coordinates \( (Q, \bar{P}) = (Q^h, \bar{P}^h) \) as above for all \( h \) from a neighborhood of 0 so that they depend smoothly on \( h \). The \( P \)-coordinates are constructed using the Morse Lemma. In order to make sure that they depend smoothly on \( h \), one can apply the Morse-Bott Lemma (a.k.a. the parametric Morse Lemma), see e.g. [22, Theorem 2]. More precisely, to obtain a smooth family \( \{ f_h \} \) of functions satisfying (5.9), one applies the Morse-Bott Lemma to the function \( \bar{p}, h) \to f_h(\bar{p}) - f_h(\bar{c}(h)) \) defined in a neighborhood of 0 in \( \mathbb{R}^{n-1} \times \mathbb{R} \). The \( Q \)-coordinates are constructed from \( P \)-coordinates by means of Theorem 3.1

By analyzing the proof of Theorem 3.1 in [22], one can see that this construction boils down to explicit formulae involving algebraic computations and solutions of O.D.E.s, hence it can be made smooth in \( h \) in a suitable neighborhood.

Having constructed the \( (Q^h, \bar{P}^h) \)-coordinates for all \( h \in (-h_0, h_0) \), we define \( F_{h,\varepsilon,\delta} \) by (5.10) using a small fixed \( \delta \) and \( \varepsilon = \varepsilon(h) \) such that \( \varepsilon(h) \) is a small constant for \( |h| < h_0/3 \) and \( \varepsilon(h) = 0 \) for \( |h| > 2h_0/3 \). Then define \( \bar{G}_h \) by (5.11) and (5.12) using \( \theta = \theta_h \) depending on \( h \) as follows: \( \{ \theta_h \} \) is a smooth family \( C^\infty \)-close to a constant one, \( \theta_h \) is a fixed map \( \theta \) as above for \( |h| < h_0/3 \), and \( \theta_h = id \) for \( |h| > 2h_0/3 \). The existence of such a family is guaranteed by Proposition 4.1.

Finally, define \( \tilde{R}_h \) by (5.13). The union of maps \( \tilde{R}_h \) forms a self-diffeomorphism of the set \( \{ x \in \Sigma : H(x) \in (-h_0, h_0) \} \). By construction, this diffeomorphism coincides with \( \tilde{R} \) on the set of \( x \) such that \( |H(x)| \in (2h_0/3, h_0) \). We extend it to a diffeomorphism \( \tilde{R} : \Sigma \to \Sigma \) by setting \( \tilde{R} = R \) on the remaining part of \( \Sigma \).

The resulting map \( \tilde{R} \) has an invariant neighborhood \( \{ P^2 < \delta/2, Q^2 < \delta/2, |H| < h_0/3 \} \). Since the maps \( \tilde{R}_h, h \in (-h_0/3, h_0/3) \), have the coordinate representation and they have positive metric entropy and are entropy non-expansive, the restriction of \( \tilde{R} \) to the above neighborhood has a positive metric entropy.

Now Theorem 4.1 follows from Lemma 5.2 and Proposition 4.1

6. Some open problems

Here we briefly discuss a few open problems, some of them are mentioned above.
(1) In case of the geodesic flow on a Riemannian manifold, we do not know how to make the perturbation Riemannian. This seems to be quite an intriguing problem.

(2) How large entropy can be generated depending on the size of perturbation (any estimate would certainly involve some characteristics of the unperturbed system)? Probably some (very non-sharp) lower bounds can be obtained by a careful analysis of the proof. As for the upper bounds, we suspect they should be double-exponential like Nekhoroshev estimates.

(3) Our construction is very specific and non-generic. What about a generic perturbation?

References

[1] L. Abramov, On the entropy of a flow (Russian), Dokl. Akad. Nauk SSSR 128 (1959) 873–875.
[2] L. M. Abramov and V. A. Rohlin, The entropy of a skew product of measure-preserving transformations, Amer. Math. Soc. Transl. Ser. 2, 48 (1966), 255–265.
[3] V. Arnold, Proof of a Theorem by A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian, Russian Math. Survey 18 (1963), 13–40.
[4] V. Arnold, Instabilities in dynamical systems with several degrees of freedom, Sov. Dokl. 5 (1964), 581–585.
[5] A. Avila. On the regularization of conservative maps, Acta Math., 205(1):5–18, 2010.
[6] A. Banyaga, D. E. Hurtubise, A proof of the Morse-Bott Lemma, Expo. Math. 22 (2004), no. 4, 365–373.
[7] P. Berger, D. Turaev, On Herman’s Positive Entropy Conjecture, Adv. Math. 349 (2019), 1234–1288.
[8] A. Bolsinov, I. Taimanov, Integrable geodesic flows on the suspensions of toric automorphisms, Proceedings of the Steklov Institute of Math. 231 (2000), 42–58.
[9] B. Bowden, Entropy-expansive maps, Trans. Amer. Math. Soc. 164 (1972), 323-331.
[10] D. Burago, A new approach to the computation of the entropy of geodesic flow and similar dynamical systems, Soviet Math. Doklady 37 (1988), 1041–1044.
[11] D. Burago, S. Ivanov. Boundary distance, lens maps and entropy of geodesic flows of Finsler metrics, Geometry & Topology, 20 (2016), 469–490.
[12] A. Cannas da Silva, Lectures on symplectic geometry, Lecture Notes in Mathematics, 1764. Springer-Verlag, Berlin, 2001. xii+217 pp.
[13] D. Chen, Positive metric entropy arises in some nondegenerate nearly integrable systems, Journal of Modern Dynamics, Volume 11 (2017), 43-56.
[14] J. Chen, H. Hu, Y. Pesin, K. Zhang, The essential coexistence phenomenon in Hamiltonian dynamics, Ergodic Theory and Dynamical Systems, 1-22. doi:10.1017/etds.2021.13.
[15] C. Contrares, Geodesic flows with positive topological entropy, twist map and hyperbolicity, Annals of Mathematics, Second Series, Vol. 172, No. 2 (2010), 761-808.
[16] R. Douady, Stabilité ou instabilité des points fixes elliptiques, Ann. Sci. École Norm. Sup. (4), 21 (1988), 1–46.
[17] V. Gelfreich, D. Turaev, Universal dynamics in a neighborhood of a generic elliptic periodic point, Regul. Chaotic Dyn. 15 (2010), no. 2-3, 159–164.
[18] V. Gelfreich, D. Turaev, On three types of dynamics and the notion of attractor, Proc. Steklov Inst. Math. 297 (2017), no. 1, 116–137.
[19] A. Kolmogorov, On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian, Dokl. Akad. Nauk SSSR 98 (1954), 525-530.
[20] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Vol. 54. Cambridge University Press (1997).
[21] G. Knieper, H. Weiss, A surface with positive curvature and positive topological entropy, J. Differential Geom. 39 (1994), no. 2, 229-249.
[22] P. Libermann, C.-M. Marle, Symplectic Geometry and Analytical Mechanics, Vol. 35, Springer Science and Business Media (2012).
[23] J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nach. Akad. Wiss. Göttingen, Math. Phys. Kl. II 1 (1962), 1-20.
[24] N. Nekhoroshev, *The behavior of Hamiltonian systems that are close to integrable ones* (Russian), Funkcional. Anal. i Prilozen. 5 (1971), no. 4, 82—83.
[25] S. Newhouse, *Quasi-elliptic periodic points in conservative dynamical systems*, American Journal of Mathematics 99.5 (1977), 1061-1087.
[26] S. Newhouse, *Continuity Properties of Entropy*, Annals of Mathematics, Second Series, Vol. 129, No. 1 (1989), 215-235.
[27] K. Petersen, *Ergodic theory*, Cambridge University Press, Cambridge, 1983.
[28] D. Turaev, *Richness of chaos in the absolute Newhouse domain*, Proceedings of the International Congress of Mathematicians. Volume III, 1804–1815, Hindustan Book Agency, New Delhi, 2010.

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