Möbius automorphisms of surfaces with many circles

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Abstract

We classify real two-dimensional orbits of conformal subgroups such that the orbits contain two circular arcs through a point. Such surfaces must be toric and admit a Möbius automorphism group of dimension at least two. Our theorem generalizes the classical classification of Dupin cyclides.

Keywords: surface automorphisms, weak del Pezzo surfaces, Möbius geometry, circles, Lie algebras, toric geometry, lattice geometry

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1 Introduction

Our main result is Theorem 1, which states a classification of algebraic surfaces that admit many conformal automorphisms. Let us consider some known examples in order to motivate and explain this result. Suppose that a surface $Z \subseteq \mathbb{R}^n$ is a $G$-orbit for some conformal subgroup $G$ and that $Z$ is not contained in a hyperplane or hypersphere. It follows from Liouville’s theorem that the conformal transformations of $\mathbb{R}^n$ for $n \geq 3$ are exactly the Möbius transformations. If $n = 3$, then $Z \subset \mathbb{R}^3$ is Möbius equivalent to either a circular cone, a circular cylinder or a ring torus, and thus a Dupin cyclide. If $\dim G > 2$, then either $Z = \mathbb{R}^2$ or $Z \subset \mathbb{R}^4$ is a stereographic projection of a Veronese surface in the unit-sphere $S^4$. The considered examples of $G$-orbits contain at least two circles through each point and motivate us to address the following problem about surfaces that are in a sense “generalized Dupin cyclides”:

**Problem.** Classify, up to Möbius equivalence, real surfaces that are the orbit of a Möbius subgroup and that contain at least two circles through a point.

We see in Figure 1 a linear projection of an orbit of a Möbius subgroup in $\mathbb{R}^5$ that contains three circles through each point. This surface is characterized by the third row of Theorem 1.

![Figure 1](image)

*Figure 1:* A projection of a smooth surface of degree six in $\mathbb{R}^5$ that is an $\text{SO}(2) \times \text{SO}(2)$-orbit and that contains three circles through each point. The family of Möbius equivalence classes of such surfaces is two-dimensional.

There has been recent interest in the classification of surfaces that contain at least two circles through each point [12, 15]. Surfaces that contain infinitely many circles through a general point are classified in [7] (see Theorem D). In this article we consider the Möbius automorphism group as Möbius invariant and we use methods from [3] (see Theorem B), a classification from [4] (see Theorem C) and results from [9] (see Theorem A).

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For our classification result we require Möbius invariants that capture geometric aspects at complex infinity. To uncover this hidden structure we define a real variety $X$ to be a complex variety together with an antiholomorphic involution $\sigma: X \to X$, which is called its real structure [14, Section I.1]. We denote the real points of $X$ by $X(\mathbb{R}) := \{ p \in X \mid \sigma(p) = p \}$. Real varieties can always be defined by polynomials with real coefficients [13, Section 6.1]. Curves, surfaces and projective spaces $\mathbb{P}^n$ are real algebraic varieties and maps between such varieties are compatible with the real structures unless explicitly stated otherwise. Instead of $\mathbb{R}^n$, it is more natural to use the Möbius quadric for our space:

$$S^n := \{ x \in \mathbb{P}^{n+1} \mid -x_0^2 + x_1^2 + \ldots + x_{n+1}^2 = 0 \},$$

where $\sigma: \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ sends $x$ to $(x_0 : \ldots : x_{n+1})$. The Möbius transformations of $S^n$ are the biregular automorphisms $\text{Aut}(S^n)$ and they are linear so that $\text{Aut}(S^n) \subset \text{Aut}(\mathbb{P}^{n+1})$. We denote a stereographic projection from the unit-sphere $S^n \subset \mathbb{R}^{n+1}$ by $\pi: S^n \to \mathbb{R}^n$. Notice that $S^n(\mathbb{R}) \cong S^n$ and that the inverse stereographic projection $\pi^{-1}(\mathbb{R}^n) \subset S^n$ is an isomorphic copy of $\mathbb{R}^n$ such that the Möbius transformations of $S^n$ restrict to Möbius transformations of $S^n$ and $\pi^{-1}(\mathbb{R}^n)$. In particular, the Möbius transformations that preserve the projection center of $\pi$, correspond to the Euclidean similarities of $\mathbb{R}^n$.

A conic $C \subset S^n$ is called a circle if $C(\mathbb{R})$ defines a circle in $S^n \subset \mathbb{R}^{n+1}$. We call a surface in $S^n$ $\lambda$-circled if it contains exactly $\lambda$ circles through a general point. A celestial surface is a $\lambda$-circled surface $X \subseteq S^n$ such that $\lambda \geq 2$ and such that $X$ is not contained in a hyperplane section of $S^n$. If in addition $X$ is of degree $d$, then its celestial type is defined as $T(X) := (\lambda,d,n)$. If the biregular automorphism group $\text{Aut}(X)$ is a Lie group, then its identity component is denoted by $\text{Aut}_0(X)$ and the Möbius automorphism group of $X$ is defined as $\text{M}(X) := \text{Aut}_0(X) \cap \text{Aut}_0(S^n)$. We denote the singular locus of $X$ by $S(X)$. A complex node, real node, complex tacnode and real tacnode is denoted by $A_1$, $A_1$, $A_3$ and $A_3$, respectively, and the union of such nodes is written as a formal sum. We write $S(X) = \emptyset$, if $X$ is smooth.

The Möbius moduli dimension $D(X)$ is defined as the dimension of the space of Möbius equivalence classes of celestial surfaces $Y \subset S^n$ such that $T(Y) = T(X)$, $S(Y) \cong S(X)$ as algebraic sets and $\text{M}(Y) \cong \text{M}(X)$ as groups.
We use the following notation for subgroups of Aut$_o$(P$^1$). Let the real structure
$\sigma: \mathbb{P}^1 \to \mathbb{P}^1$ be defined by $(x : y) \mapsto (\bar{x} : \bar{y})$ so that Aut$_o$(P$^1$) $\cong$ PSL(2, $\mathbb{R}$). If $p, q, r \in \mathbb{P}^1$ such that $p \neq \sigma(p)$, $q = \sigma(q)$, $r = \sigma(r)$ and $q \neq r$, then we denote

\[
\begin{align*}
\text{PSO}(2) &:= \{ \varphi \in \text{Aut}_o(\mathbb{P}^1) \mid \varphi(p) = p, \ \varphi(\sigma(p)) = \sigma(p) \}, \\
\text{PSX}(1) &:= \{ \varphi \in \text{Aut}_o(\mathbb{P}^1) \mid \varphi(q) = q, \ \varphi(r) = r \}, \\
\text{PSE}(1) &:= \{ \varphi \in \text{Aut}_o(\mathbb{P}^1) \mid \varphi(x : y) = (x + \alpha y : y), \ \alpha \in \mathbb{R} \}, \text{ and} \\
\text{PSA}(1) &:= \{ \varphi \in \text{Aut}_o(\mathbb{P}^1) \mid \varphi(r) = r \}.
\end{align*}
\]

Notice that $\varphi$ in PSO(2) or PSX(1) maps $(x : y)$ up to choice of coordinates to $(\cos(\alpha) x - \sin(\alpha) y : \sin(\alpha) x + \cos(\alpha) y)$ and $(\alpha x : y)$, respectively, for some $\alpha \in \mathbb{R} \setminus \{0\}$. The elements in PSA(1) are combinations of elements in PSX(1) and PSE(1) (see also Remark 29).

**Theorem 1.** (Möbius automorphisms of celestial surfaces)
If $X \subseteq \mathbb{S}^n$ is a celestial surface such that $\dim M(X) \geq 2$, then $X$ is toric and its Möbius invariants $T(X), S(X), M(X)$ and $D(X)$ are characterized by a row in the following table (see Definition 3 for the names):

| $T(X)$ | $S(X)$ | $M(X)$ | $D(X)$ | name |
|--------|--------|--------|--------|------|
| (2, 8, 7) | 0 | PSO(2) $\times$ PSO(2) | 3 | double Segre surface |
| (2, 8, 5) | 0 | PSO(2) $\times$ PSO(2) | 2 | projected dS |
| (3, 6, 5) | 0 | PSO(2) $\times$ PSO(2) | 2 | dP6 (see Figure 1) |
| ($\infty$, 4, 4) | 0 | PSO(3) | 0 | Veronese surface |
| (4, 4, 3) | $A_1 + A_1 + A_1 + A_1$ | PSO(2) $\times$ PSO(2) | 1 | ring cyclide |
| (2, 4, 3) | $\underline{A_1} + \underline{A_1} + A_1 + A_1$ | PSO(2) $\times$ PSX(1) | 0 | spindle cyclide |
| (2, 4, 3) | $\underline{A_3} + A_1 + A_1$ | PSO(2) $\times$ PSE(1) | 0 | horn cyclide |
| ($\infty$, 2, 2) | 0 | PSO(3, 1) | 0 | 2-sphere |

Moreover, if $T(X) \notin \{(2, 8, 7), (2, 8, 5), (\infty, 4, 4)\}$, then $M(X) = \text{Aut}_o(X)$ and if $D(X) = 0$, then $X$ is unique up to Möbius equivalence.

If we replace $\mathbb{S}^n$ with $\mathbb{S}^n \cong \mathbb{S}^n(\mathbb{R})$, then Theorem 1 holds if we replace PSO(3, 1) by SO(3) and remove the remaining P’s in the $M(X)$ column. The case $T(X) = (\infty, 4, 4)$ was already known and is revisited in §8 (see also [7, Theorem 23] and [1, Section 2.4.3]).
A smooth model of a surface $X \subset \mathbb{P}^{n+1}$ is a birational morphism $\tilde{X} \to X$ from a nonsingular surface $\tilde{X}$, such that this morphism does not contract $(-1)$-curves. If $\lambda < \infty$, then the smooth models of the $\lambda$-circled surfaces in Theorem 1 are isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ blown up in either 0, 2 or 4 complex conjugate points (see Notation 10). A smooth model of a Veronese surface is isomorphic to $\mathbb{P}^2$ such that $\sigma: \mathbb{P}^2 \to \mathbb{P}^2$ sends $x$ to $(\overline{x_0} : \overline{x_1} : \overline{x_2})$ (see Lemma 11c).

Instead of $\mathbb{S}^n$ one could also consider hyperquadrics of different signature. Although we do not pursue this, we cannot resist to mention the following result, which will come almost for free during our investigations:

**Corollary 2.** If $Q \subset \mathbb{P}^8$ is a quadric hypersurface of signature $(4, 5)$ or $(3, 6)$, then there exists a unique double Segre surface $X \subset Q$ such that $\text{Aut}_o(X) \subset \text{Aut}_o(Q)$ and $X$ is isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^2$, respectively.

Our methods are constructive and allow for explicit coordinate description of the moduli space of the celestial surfaces. See [8, moebius_aut] for an implementation using [16, Sage].

**Definition 3.** (names of surfaces)

A surface $X \subset \mathbb{S}^3$ is called a spindle cyclide, horn cyclide or ring cyclide if there exists a stereographic projection $\pi: \mathbb{S}^3 \to \mathbb{R}^3$ such that $\pi(X(\mathbb{R})) \subset \mathbb{R}^3$ is a circular cone, circular cylinder and ring torus, respectively. We call $X \subset \mathbb{P}^5$ a Veronese surface if there exists a biregular isomorphism $\mathbb{P}^2 \to X$ whose components form a basis of the vector space of degree 2 forms. We call $X \subset \mathbb{P}^8$ a double Segre surface (or dS for short) if there exists a biregular isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to X$ whose components form a basis of the vector space of bidegree $(2,2)$ forms. A projected dS is a surface $X$ that is a degree preserving linear projection of a dS. We call $X \subset \mathbb{P}^6$ a sextic del Pezzo surface (or dP6 for short) if $X$ is an anticanonical model of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in two general complex points. ◄
Remark 4. (overview)
Suppose that $X$ is a celestial surface of type $T(X) = (\lambda, d, n)$ such that $\dim M(X) \geq 2$ and $\lambda < \infty$.

In §2 we classify $X$ under the additional assumption that $X$ is toric.

In §3 we give coordinates for a double Segre surface $Y_* \subset \mathbb{P}^8$ and we investigate actions of real structures on $Y_*$. This will be needed for finding quadratic forms of signature $(1, n + 1)$ in the ideal of $Y_*$. We establish in §4 that $X$ must be toric. Moreover, there exists a birational linear projection $\rho: Y_* \rightarrow X$ and $M(X)$ is isomorphic to a subgroup of $\text{Aut}_o(Y_*)$ that leaves the center of $\rho$ invariant. We characterize the possible configurations for the center of $\rho$ in $Y_*$ and for each such configuration we restrict the possible values for $T(X)$ and $M(X)$.

In §5 we encode, up to Möbius equivalence, $X$ as an $M(X)$-invariant quadratic form in the ideal of $Y_*$. From the Lie algebra of $M(X)$, we recover the subspace of $M(X)$-invariant quadratic forms and each invariant form of signature $(1, n + 1)$ in this space encodes a possible Möbius equivalence class for $X$.

In §6 we show how toric real structures act on the Lie algebra of $\text{Aut}_o(Y_*)$ and we recall the classification of Lie algebras of complex subgroups of $\text{Aut}_o(Y_*)$.

In §7 we make a case distinction on the established configurations for the center of $\rho$ and Lie algebras of $M(X)$. If $X$ is not a spindle or horn cyclide, then $M(X) \cong \text{SO}(2) \times \text{SO}(2)$ and we obtain coordinates for the $D(X) + 1$ generators of all $M(X)$-invariant quadratic forms of signature $(1, n + 1)$ in the ideal of $Y_*$. This enables us to conclude §7 with a proof for Theorem 1 and Corollary 2.

Finally, we present in §8 an alternative proof for the known $\lambda = \infty$ case by applying the same methods as before, but with $Y_*$ replaced with the Veronese surface $Y_0 \subset \mathbb{P}^5$.

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2 Toric celestial surfaces

In this section we classify toric celestial surfaces and their real structures.

Suppose that $X \subset \mathbb{P}^n$ is a surface that is not contained in a hyperplane section. The linear normalization $X_N \subset \mathbb{P}^m$ of $X$ is defined as the image of its smooth
model $\tilde{X}$ via the map associated to the complete linear series of hyperplane sections of $X$. Thus $m \geq n$, $X$ is a linear projection of $X_N$ and $X_N$ is unique up to $\text{Aut}(\mathbb{P}^m)$.

Let $\mathbb{T}^1 := (\mathbb{C}^*, 1)$ denote the algebraic torus. Recall that $X$ is toric if there exists an embedding $i : \mathbb{T}^2 \hookrightarrow X$ such that $i(\mathbb{T}^2)$ is dense in $X$ and such that the action of $\mathbb{T}^2$ on itself extends to an action on $X$.

If $X$ is a toric surface, then there exists, up to projective equivalence, a monomial parametrization $\xi : \mathbb{T}^2 \to X_N$. The lattice polygon of $X$ is defined as the convex hull of the points in the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, whose coordinates are defined by the exponents of the components of $\xi$. The antiholomorphic involution $\sigma : X \to X$ induces an involution $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$. Consequently, $\sigma$ induces a unimodular involution $\mathbb{Z}^2 \to \mathbb{Z}^2$ that leaves the lattice polygon of $X$ invariant.

**Notation 5.** By abuse of notation we denote involutions on algebraic structures, that correspond functorially with the real structure $\sigma : X \to X$, by $\sigma$ as well.

A lattice projection $\mathbb{Z}^2 \subset \mathbb{R}^2 \to \mathbb{Z}^1 \subset \mathbb{R}^1$ induces a toric map $X_N \to \mathbb{P}^1$. We call a family of curves on $X_N$ toric if the family can be defined by the fibers of a toric map. A family of circles on $X$ is called toric if it corresponds to a toric family on $X_N$ via a linear projection $X_N \to X$. The toric families of circles that cover a toric surface $X$ that is not covered by complex lines, correspond to the projections of the lattice polygon of $X$ to a line segment that is of minimal width among all such projections [10, Proposition 31].

**Figure 2:** Width of lattice polygon along directions $\searrow$ and $\nearrow$.

In Figure 2 we see two examples of lattice projections of a lattice polygon. The width of the polygon in the $\nearrow$ direction is 4. The lattice polygon attains
its minimal width of 2 in the directions →, ↓ and ↘. Notice that the lines through the origin, along these three directions, are invariant under lattice involution defined by 180° rotation around the central lattice point.

The lattice type \( L(X) \) of a toric surface \( X \) consists of the following data

1. The lattice polygon \( \Lambda \subset \mathbb{R}^2 \) of \( X \).

2. The unimodular involution \( \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \) that is induced by the real structure \( \sigma : X \rightarrow X \).

3. The lattice projections that correspond to toric families of circles. We will represent such projections by arrows (↓, →, ↘, ↗) pointing in the corresponding direction.

Lattice types \( L(X) \) and \( L(X') \) are equivalent if there exists a unimodular isomorphism between their lattice polygons that is compatible with the unimodular involution. Data 3 is uniquely determined by data 1 and data 2. The unimodular involutions \( \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \), defined by \( (x, y) \mapsto (x, y) \), \( (x, y) \mapsto (-x, y) \), \( (x, y) \mapsto (-x, -y) \) and \( (x, y) \mapsto (y, x) \), are represented by their symmetry axes in the lattice polygons.

**Proposition 6.** (classification of toric celestial surfaces)

If \( X \subseteq S^n \) is a toric surface that is covered by at least two toric families of circles, then its lattice type \( L(X) \), together with \( T(X) \) and the name of \( X \), is up to equivalence characterized by one of the eight cases in Table 8.

**Corollary 7.** (classification of toric celestial surfaces)

a) The antiholomorphic involutions of the double Segre surface — that act as unimodular involutions as in Table 8a, Table 9a and Table 9b — are inner automorphic via \( \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \).

b) The antiholomorphic involutions of the Veronese surface — that act as unimodular involutions as in Table 8d and Table 9c — are inner automorphic via \( \text{Aut}(\mathbb{P}^2) \).

Before we prove Proposition 6 we state in Lemma 11 and Lemma 12 the known classification of real structures of \( \mathbb{P}^1 \), \( \mathbb{P}^1 \times \mathbb{P}^1 \), \( \mathbb{P}^2 \) and \( \mathbb{T}^2 \). We include proofs in case we could not find a suitable reference. Theorem A collects results from [9] that we need for Proposition 6, Lemma 14 and Proposition 20.
Table 8: See Proposition 6. Lattice types of toric celestial surfaces up to equivalence, together with the corresponding name and possible celestial types. For the celestial types we have $3 \leq n \leq 7$ and $4 \leq m \leq 5$. The directions correspond to the toric families of circles.

| Celestial Type | Real Structure | Corresponding Name |
|----------------|----------------|---------------------|
| $dS (2, 8, n)$ | $\text{id}$  | ring cyclide $(4, 4, 3)$ |
| $dP6 (3, 6, m)$ | $180^\circ$ | spindle cyclide $(2, 4, 3)$ |
| weak $dP6 (2, 6, m)$ | $\text{id}$  | horn cyclide $(2, 4, 3)$ |
| Veronese $(\infty, 4, 4)$ | $\text{id}$  | 2-sphere $(\infty, 2, 2)$ |

Table 9: See Corollary 7 and the proof of Proposition 6.

It will follow from Proposition 20 in §4 that Proposition 6 also holds with the following hypothesis: “If $X \subseteq \mathbb{S}^n$ is a toric celestial surface, then…”.

Notation 10. We consider the following normal forms for real structures:

\[ \sigma_+: \mathbb{P}^1 \to \mathbb{P}^1, \quad (x : y) \mapsto (\overline{x} : \overline{y}), \]
\[ \sigma_-: \mathbb{P}^1 \to \mathbb{P}^1, \quad (x : y) \mapsto (-\overline{y} : \overline{x}), \]
\[ \sigma_s: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad (s : t; u : w) \mapsto (\overline{u} : \overline{w}; \overline{s} : \overline{t}). \]

Notice that $\mathbb{P}^1 \times \mathbb{P}^1$ with real structure $\sigma_+ \times \sigma_+$ or $\sigma_s$ is isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^2$, respectively. \(\triangleright\)
Lemma 11. (real structures for \( \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \mathbb{P}^2 \))

a) If \( \sigma : \mathbb{P}^1 \to \mathbb{P}^1 \) is an antiholomorphic involution, then there exists \( \gamma \in \text{Aut}(\mathbb{P}^1_C) \) such that \( (\gamma^{-1} \circ \sigma \circ \gamma) \) is equal to either \( \sigma_+ \) or \( \sigma_- \).

b) If \( \sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \) is an antiholomorphic involution, then there exists \( \gamma \in \text{Aut}(\mathbb{P}^1_C \times \mathbb{P}^1_C) \) such that \( (\gamma^{-1} \circ \sigma \circ \gamma) \) is equal to either \( \sigma_+ \times \sigma_+ \), \( \sigma_+ \times \sigma_- \), \( \sigma_- \times \sigma_- \) or \( \sigma_s \).

c) If \( \sigma : \mathbb{P}^2 \to \mathbb{P}^2 \) is an antiholomorphic involution, then there exists \( \gamma \in \text{Aut}(\mathbb{P}^2_C) \) such that \( (\gamma^{-1} \circ \sigma \circ \gamma) \) is equal to \( \sigma_0 : (s : t : u) \mapsto (\bar{s} : \bar{t} : \bar{u}) \).

Proof. Claim 1. If \( X \) is a variety with antiholomorphic involution \( \sigma : X \to X \) and very ample anticanonical class \( -k \), then the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_{-k}} & Y \\
\downarrow \sigma & & \downarrow \sigma_0 \\
X & \xrightarrow{\varphi_{-k}} & Y \\
\end{array}
\]

where \( \sigma_0 : (x_0 : \ldots : x_n) \mapsto (\bar{x}_0 : \ldots : \bar{x}_n) \) and \( Y \) is the image of \( X \) under the birational morphism \( \varphi_{-k} \) associated to \( -k \). This claim is a straightforward consequence of [14, I.(1.2) and I.(1.4)].

a) We apply claim 1 with \( X = \mathbb{P}^1 \) so that \( Y \subset \mathbb{P}^2 \) is a real conic. We know that \( Y \) has signature either \((3, 0)\) or \((2, 1)\). Thus there are, up to inner automorphism, two antiholomorphic involutions of \( \mathbb{P}^1 \). Moreover we have that \(|\{p \in \mathbb{P}^1 \mid \sigma(p) = p\}| \in \{0, \infty\}\). This concludes the proof, since the \( \sigma \) must be inner automorphic to either \( \sigma_+ \) or \( \sigma_- \).

b) If \( \sigma \) does not flip the components of \( \mathbb{P}^1 \times \mathbb{P}^1 \), then it follows from a) that \( \sigma \) is inner automorphic to either \( \sigma_+ \times \sigma_+ \), \( \sigma_+ \times \sigma_- \) or \( \sigma_- \times \sigma_- \). Now suppose that \( \sigma \) flips the components of \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( \pi_1 \) and \( \pi_2 \) be the complex first and second projections of \( \mathbb{P}^1 \times \mathbb{P}^1 \) to \( \mathbb{P}^1_C \), respectively. The composition \( \pi_2 \circ \sigma \circ \pi_1^{-1} \) defines an antiholomorphic isomorphism \( \tau : \mathbb{P}^1_C \to \mathbb{P}^1_C \). If \( \sigma \) and \( \sigma' \) are inner automorphic, then there exist complex \( \alpha, \beta \in \text{Aut}(\mathbb{P}^1_C) \) such that \( \pi_2 \circ \sigma \circ \pi_1^{-1} \) is equal to \( \beta \circ \pi_2 \circ \sigma' \circ \pi_1^{-1} \circ \alpha \). Conversely, an antiholomorphic isomorphism \( \tau : \mathbb{P}^1_C \to \mathbb{P}^1_C \) defines an antiholomorphic involution \((p; q) \mapsto (\tau^{-1}(q); \tau(p))\) that flips the components of \( \mathbb{P}^1 \times \mathbb{P}^1 \). There exists \( \alpha, \beta \in \text{Aut}(\mathbb{P}^1_C) \) such that \( \beta \circ \tau \circ \alpha \) is defined by \((s : t) \mapsto (\bar{s} : \bar{t})\). We conclude that \( \sigma \) is unique up to inner automorphisms and thus without loss of generality inner automorphic to
the real structure $\sigma_s$.

c) We apply claim 1 with $X = \mathbb{P}^2$ so that $Y \subset \mathbb{P}^9$ is a surface of degree nine. Since the degree is odd, we obtain infinitely many real points on $Y$ and thus also infinitely many real points on $\mathbb{P}^2$. We know that $-k = 3h$, where $-k$ is the anticanonical class and $h$ is the divisor class of lines in $\mathbb{P}^2$. We can construct two different real lines in $\mathbb{P}^2$, since a line through two real points is real. The linear subspecies of $|−k|$ that consists of all cubics that contain these real two lines, is $|−k − 2h| = |h|$. Notice that choosing a real subsystem of $|−k|$ is geometrically a real linear projection of $Y$. It follows that the map $\varphi_h: \mathbb{P}^2 \to \mathbb{P}^2$ associated to $h$ is real such that $\sigma_0 \circ \varphi_h = \varphi_h \circ \sigma$. We conclude that $\sigma$ is inner automorphic to $\sigma_0$ as was claimed.

\begin{proof}
proof.
\end{proof}

\begin{lemma}
Lemma 12. (real structures for $\mathbb{T}^2$)
\end{lemma}

If $\sigma: \mathbb{T}^2 \to \mathbb{T}^2$ is a toric antiholomorphic involution, then there exists $\gamma \in \text{Aut}(\mathbb{T}^2)$ such that $(\gamma^{-1} \circ \sigma \circ \gamma)$ is equal to either one of the following:

\[
\begin{align*}
\sigma_0 : (s, u) &\mapsto (\overline{s}, \overline{u}), \\
\sigma_1 : (s, u) &\mapsto (\frac{1}{2}, \overline{u}), \\
\sigma_2 : (s, u) &\mapsto (\frac{1}{2}, \frac{1}{2}), \\
\sigma_3 : (s, u) &\mapsto (\overline{u}, \overline{s}),
\end{align*}
\]

and $\sigma_i: \mathbb{T}^2 \to \mathbb{T}^2$ induces, up to unimodular equivalence, the following unimodular involution $\sigma_i: \mathbb{Z}^2 \to \mathbb{Z}^2$:

\[
\begin{align*}
\sigma_0 : (x, y) &\mapsto (x, y), \\
\sigma_1 : (x, y) &\mapsto (−x, y), \\
\sigma_2 : (x, y) &\mapsto (−x, −y), \\
\sigma_3 : (x, y) &\mapsto (y, x).
\end{align*}
\]

The corresponding real points $\Gamma_{\sigma_i} := \{(s, u) \in \mathbb{T}^2 \mid \sigma_i(s, u) = (s, u)\}$ are:

\[
\begin{align*}
\Gamma_{\sigma_0} &= \{(s, u) \in \mathbb{T}^2 \mid s = \overline{s}, \ u = \overline{u} \} \cong (\mathbb{R}^*)^2, \\
\Gamma_{\sigma_1} &= \{(s, u) \in \mathbb{T}^2 \mid s\overline{s} = 1, \ u\overline{u} = 1 \} \cong S^1 \times \mathbb{R}^*, \\
\Gamma_{\sigma_2} &= \{(s, u) \in \mathbb{T}^2 \mid s\overline{s} = 1, \ u\overline{u} = 1 \} \cong S^1 \times S^1, \\
\Gamma_{\sigma_3} &= \{(s, u) \in \mathbb{T}^2 \mid s = \overline{u} \} \cong \mathbb{R}^2 \setminus \{(0, 0)\}.
\end{align*}
\]

\begin{proof}
Proof. Since $\sigma: \mathbb{T}^2 \to \mathbb{T}^2$ extends to an antiholomorphic involution of an algebraic surface we may assume that $\sigma$ is defined by $(s, u) \mapsto (\overline{f(s, u)})$ where $f$ is some bivariate rational function in $\mathbb{C}(s, u)$. From $\sigma(1, 1) = (1, 1)$ it follows that $f(s, u) = (s^a u^b, s^c u^d)$ with $a, b, c, d \in \mathbb{Z}$. From $(\sigma \circ \sigma)(s, u) = (s, u)$ it follows that $ad − bc = ±1$ and thus the induced unimodular involution $(x, y) \mapsto (ax + by, cx + dy)$ is unimodular equivalent to $\sigma_i: \mathbb{Z}^2 \to \mathbb{Z}^2$ for some $i \in \{0, 1, 2, 3\}$ as asserted. For $\sigma_1$ we find that $f(s, u) = (\frac{1}{a}, u)$, and thus
\( \overline{f(s,u)} = (s,u) \) if and only if \( ss = 1 \) and \( u = \overline{u} \) so that \( \Gamma_{\sigma_1} \cong S^1 \times \mathbb{R}^* \). The proofs for \( \Gamma_{\sigma_0}, \Gamma_{\sigma_2} \) and \( \Gamma_{\sigma_3} \) are similar. \( \square \)

For convenience of the reader, Theorem A below extracts result from [9, Theorem 1, Theorem 3, Theorem 4, Corollary 5, Lemma 1 and Lemma 3] that are needed for Proposition 6, Lemma 14 and Proposition 20.

**Notation 13.** The *Neron-Severi lattice* of a surface \( X \subset \mathbb{P}^{n+1} \) consists of a unimodular lattice \( N(X) \) that is defined by the divisor classes on the smooth model \( \tilde{X} \) up to numerical equivalence. In this article, \( N(X) \) will be a sublattice of \( \langle \ell_0, \ell_1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle_{\mathbb{Z}} \), where \( \ell_0 \cdot \ell_1 = 1, \varepsilon_i^2 = -1 \) and \( \ell_0^2 = \ell_1^2 = \ell_0 \cdot \varepsilon_1 = \ell_1 \cdot \varepsilon_i = 0 \) for \( 1 \leq i \leq 4 \). The real structure \( \sigma \) induces a unimodular involution \( \sigma: N(X) \to N(X) \) such that \( \sigma(\ell_0) = \ell_0, \sigma(\ell_1) = \ell_1, \sigma(\varepsilon_1) = \varepsilon_2 \) and \( \sigma(\varepsilon_3) = \varepsilon_4 \) (see Notation 5). The function \( h^0: N(X) \to \mathbb{Z}_{\geq 0} \) assigns to a divisor class the dimension of the vector space of its associated global sections. The two distinguished elements \( h, k \in N(X) \) correspond to the *class of hyperplane sections* and the *canonical class*, respectively. We call a divisor class \( c \in N(X) \) *indecomposable* if \( h^0(c) > 0 \) and if there do not exist nonzero \( a, b \in N(X) \) such that \( c = a + b, h^0(a) > 0 \) and \( h^0(b) > 0 \). The subset of *indecomposable* \(-2\)-*classes* in \( N(X) \) are defined as

\[
B(X) := \{ c \in N(X) \mid -k \cdot c = 0, c^2 = -2 \text{ and } c \text{ is indecomposable} \}.
\]

We use the following shorthand notation for elements in \( B(X) \):

\[
b_{ij} := \ell_0 - \varepsilon_i - \varepsilon_j, \quad b_{ij}' := \ell_1 - \varepsilon_i - \varepsilon_j, \quad b_1 := \varepsilon_1 - \varepsilon_3 \quad \text{and} \quad b_2 := \varepsilon_2 - \varepsilon_4,
\]

and we underline the classes in \( \{ b \in B(X) \mid \sigma(b) = b \} \). A (projected) weak 
\( dP6 \) is (a degree preserving linear projection of) an anticanonical model of \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up in two complex points that lie in a fiber of a projection \( \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \). We call \( X \subset S^3 \) a *CH1 cyclide*, if \( X(\mathbb{R}) \subset S^3 \) is an inverse stereographic projection of a circular hyperboloid of 1 sheet. \( \diamondsuit \)

**Theorem A. [2019]** We use Notation 13.

Suppose that \( X \) is a celestial surface of type \( T(X) = (\lambda, d, n) \).

a) If either \( \lambda = \infty \), \( d > 4 \), \( |S(X_N)| > 2 \) or \( |B(X)| > 3 \), then \( T(X), S(X_N), B(X) \) and the name of \( X \) is characterized by a row in the following table, where \( 3 \leq n \leq 7 \) and \( 4 \leq m \leq 5 \):

| λ | d | n | m |
|---|---|---|---|
| ∞ | 5 | 3 | 4 |
| 4 | 6 | 4 | 5 |

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\begin{tabular}{|c|c|c|c|}
\hline
T(X) & S(X_N) & B(X) & name \\
\hline
(\infty, 4, 4) & 0 & 0 & Veronese surface \\
(\infty, 2, 2) & 0 & 0 & 2-sphere \\
(2, 8, n) & 0 & 0 & (projected) dS \\
(3, 6, m) & 0 & 0 & (projected) dP6 \\
(2, 6, m) & A_1 & \{b_{12}\} & (projected) weak dP6 \\
(4, 4, 3) & 4A_1 & \{b_{13}, b_{24}, b'_{12}, b'_{23}\} & ring cyclide \\
(2, 4, 3) & 2A_1 + 2A_1 & \{b_{13}, b_{24}, b'_{12}, b'_{14}\} & spindle cyclide \\
(2, 4, 3) & A_3 + 2A_1 & \{b_{13}, b_{24}, b'_{12}, b_1, b_2\} & horn cyclide \\
(3, 4, 3) & A_1 + 2A_1 & \{b_{13}, b_{24}, b'_{12}\} & CH1 cyclide \\
\hline
\end{tabular}

b) If \( \lambda < \infty \), then the class \( h \) of hyperplane sections of \( X \) is equal to the anticanonical class \( -k \) and without loss of generality equal to either
\[ 2\ell_0 + 2\ell_1, \ 2\ell_0 + 2\ell_1 - \varepsilon_1 - \varepsilon_2, \text{ or } 2\ell_0 + 2\ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4. \]
If \( \lambda = \infty \), then either \( h = -\frac{3}{2}k \) and \( k^2 = 9 \), or \( h = -\frac{1}{2}k \) and \( k^2 = 8 \).

c) If \( \lambda < \infty \), then the smooth model \( \tilde{X} \) is isomorphic to the blowup of \( \mathbb{P}^1 \times \mathbb{P}^1 \) in either 0, 2 or 4 nonreal complex conjugate points. These points may be infinitely near, but at most two of the non-infinitely near points lie in the same fiber of a projection from \( \mathbb{P}^1 \times \mathbb{P}^1 \) to \( \mathbb{P}^1 \). The pullback into \( \tilde{X} \) of a fiber that contains two points is contracted to an isolated singularity of the linear normalization \( X_N \).

Proof of Proposition 6 and Corollary 7. Let \( X \subseteq S^n \) be a \( \lambda \)-circled toric celestial surface of degree \( d \) that is covered by at least two toric families of circles.

The lattice polygon of \( X \) contains \( i \) interior and \( b \) boundary lattice points.

Claim 1: \( (i, b, d) \in \{(0, 4, 2), (0, 6, 4), (1, 4, 4), (1, 6, 6), (1, 8, 8)\} \) and \( \lambda = \infty \) if \( i = 0 \). We know from [2, Propositions 10.5.6 and 10.5.8] that \( i \) and \( b \) are equal to the sectional genus \( p_a(h) = \frac{1}{2}(h^2 + k \cdot h) + 1 \) and anticanonical degree \( -k \cdot h \), respectively (see also [5]). This claim now follows from Theorem Ab.

Claim 2: The lattice polygon of \( X \) is, up to unimodular equivalence, preserved by the unimodular involution \( \sigma: \mathbb{Z}^2 \to \mathbb{Z}^2 \) that is defined by either \( \sigma_0, \sigma_1, \sigma_2 \) or \( \sigma_3 \). This claim follows from Lemma 12.
Claim 3: A boundary line segment of the lattice polygon of $X$, that contains no more and no less than two lattice points, is not left invariant by the unimodular involution $\sigma : \mathbb{Z}^2 \to \mathbb{Z}^2$. Up to unimodular equivalence, we may assume that the two lattice points on a boundary line segment have coordinates $(0,0)$ and $(0,1)$, and that the remaining lattice points of the polygon lie strictly on the right side of these two points. Without loss of generality the two lattice points correspond to the first two components $s^0u^0$ and $s^0u^1$ of a monomial parametrization $\xi : \mathbb{T}^2 \to X_N$ so that $\xi(0,u) = (1 : u : 0 : \ldots : 0)$ parametrizes a line in $X_N$. This line is either linear equivalent or linearly projected to a line in $X$. We conclude that claim 3 holds, since $X(\mathbb{R}) \subseteq S^n$ does not contain real lines.

Claim 4: If $d \neq 2$, then the lattice polygon of $X$ attains its minimal width along at least two directions that are left invariant by the unimodular involution $\sigma : \mathbb{Z}^2 \to \mathbb{Z}^2$. Indeed, recall that such a direction corresponds to a toric family of circles.

Claim 5: If the lattice polygon of $X$ is contained in a $3 \times 3$ grid centered at the origin, then $L(X)$ together with $T(X)$ and the name of $X$ is in Table 8 or Table 9. For each real structure $\sigma$ and pair $(i,b)$ listed at claims 1 and 2, we list up to equivalence all lattice polygons in the $3 \times 3$ grid that are left invariant by $\sigma$ and that have $i$ interior and $b$ boundary lattice points. Of these polygons we discard those that contradict claim 3 or claim 4. For example, we exclude the lattice types in Table 9[c,d] as they contradict claim 4 and the lattice polygon of Table 9d together with unimodular involution $\sigma_0$ would contradict claim 3. We find that a candidate for $L(X)$ is equivalent to one of Table 8 or Table 9[a,b]. We recover $|S(X_N)|$ from the monomial parametrization associated to the lattice polygon. For each lattice type in Table 8 and Table 9[a,b], we apply claim 1 and find that either $\lambda = \infty$, $d > 4$ or $|S(X_N)| > 2$. It follows that the name and celestial type of $X$ correspond to a row of Theorem Aa. If $L(X)$ is Table 8d or Table 9c, then $X$ is a Veronese surface by Definition 3 as the monomials associated to its lattice polygon span a basis for vector space of degree 2 forms on $\mathbb{P}^2$. If $L(X)$ is Table 8h, then $X$ is a 2-sphere by claim 1. If $L(X)$ is Table 8e, then there are 4 directions and thus $4 \leq \lambda < \infty$ so that $X$ must be a ring cyclide. If $L(X)$ is Table 8f or Table 8g, then $X$ is a spindle cy-
clide and horn cyclide, respectively (see forward Example 33). This concludes the proof of claim 5.

Claim 6: The lattice polygon of $X$ is contained in a $3 \times 3$ grid centered at the origin. If $\lambda = \infty$ or $d > 4$, then it follows from Theorem Aa, that $X_N$ is unique up to projective equivalence and thus its lattice type is already realized in Table 8 or Table 9 at claim 5. If $\lambda < \infty$, then the lattice polygon of $X$ must be one of [2, Theorem 8.3.7] such that $(i, b)$ is as in claim 1. It follows that claim 6 holds.

Claim 7: Corollary 7 holds. Notice that $X$ is covered by two families of conics that contain real points. Therefore the real structure of a celestial double Segre surface must be inner automorphic to $\sigma_+ \times \sigma_+$ by Lemma 11b so that Corollary 7a holds. Corollary 7b is a consequence of Lemma 11c.

We concluded the proof of Proposition 6, since it follows from claims 5, 6 and 7 that $L(X)$, $T(X)$ and the name of $X$ is up to equivalence characterized by one of the eight cases in Table 8.

3 Embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we give explicit coordinates for double Segre surfaces, which are embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^8$. We describe how real structures and projective automorphisms act on these embeddings.

We denote the vector space of quadratic forms in the ideal $I(X)$ of a surface $X$ as

$$I_2(X) := \langle q \in I(X) \mid \deg q = 2 \rangle_{\mathbb{C}}.$$

Lemma 14. If $X$ is a toric celestial surface, then $\dim I_2(X_N)$ for the linear normalization $X_N$ is as follows:

| Table 8 | a | b | c | d | e | f | g | h |
|---------|---|---|---|---|---|---|---|---|
| $\dim I_2(X_N)$ | 20 | 9 | 9 | 6 | 2 | 2 | 2 | 1 |

Proof. The dimension of the space $U$ of quadratic forms vanishing on $X_N \subset \mathbb{P}^m$ is equal to $h^0(2h)$, where $h$ is the class of hyperplane sections. The dimension of the space $W$ of quadratic forms in $\mathbb{P}^m$ is equal to $\binom{2+m}{2}$. Thus we find that

$$\dim I_2(X_N) = \dim W/U = \dim W - \dim U = \binom{2+m}{2} - h^0(2h).$$
We obtain $h^0(2h) = \frac{1}{2}(4h^2 - 2h \cdot k) + 1$ as a straightforward consequence of Theorem Ac, Riemann-Roch theorem and Kawamata-Viehweg vanishing theorem. The main assertion now follows from Theorem Ab.

**Table 15**: Coordinates for lattice points.

| double Segre surface | Veronese surface |
|----------------------|------------------|
| $(-1, 1)$            | $(2, 0)$         |
| $y_8$                | $y_5$            |
| $y_3$                | $y_6$            |
| $y_5$                | $y_7$            |
| $(0, 1)$             | $(2, 1)$         |
| $(1, 1)$             | $(2, 2)$         |
| $(1, 0)$             | $(1, 0)$         |
| $y_2$                | $y_3$            |
| $y_6$                | $y_4$            |
| $y_0$                | $y_0$            |
| $y_7$                | $y_1$            |
| $(0, -1)$            | $(0, 1)$         |
| $(1, -1)$            | $(0, 2)$         |

Let $Y_\ast \subset \mathbb{P}^8$ denote the linear normalization of the double Segre surface with lattice polygon as in Table 8a. We consider the left coordinates in Table 15 so that we obtain the parametric map

$$\xi: \mathbb{T}^2 \to Y_\ast \subset \mathbb{P}^8, \quad (s, u) \mapsto$$

$$(1 : s : s^{-1} : u : u^{-1} : su : s^{-1}u^{-1} : su^{-1} : s^{-1}u) =$$

$$(y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8).$$

Using $\xi$ we find the following 20 generators for the vector space of quadratic forms on $Y_\ast$ and it follows from Lemma 14 that these form a basis:

$$I_2(Y_\ast) = \langle y_0^2 - y_1y_2, y_0^2 - y_3y_4, y_0^2 - y_5y_6, y_0^2 - y_7y_8, y_1^2 - y_5y_7, y_2^2 - y_6y_8, y_3^2 - y_5y_8, y_4^2 - y_6y_7, y_0y_1 - y_4y_5, y_0y_2 - y_3y_6, y_0y_3 - y_2y_5, y_0y_4 - y_1y_6, y_0y_6 - y_2y_4, y_0y_7 - y_1y_4, y_0y_8 - y_2y_3 \rangle.$$

**Lemma 16.** (real structures for $\mathbb{P}^8$)

Let $i$ denote the imaginary unit. The maps $\sigma_i: \mathbb{P}^8 \to \mathbb{P}^8$ and $\mu_i: \mathbb{P}^8 \to \mathbb{P}^8$ which are defined by

$\sigma_0: y \mapsto (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8),$

$\sigma_1: y \mapsto (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8),$

$\sigma_2: y \mapsto (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8),$

$\sigma_3: y \mapsto (y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8),$

$\mu_0: x \mapsto x,$

$\mu_1: x \mapsto (x_0 : x_1 + ix_2 : x_1 - ix_2 : x_3 : x_4 : x_5 + ix_8 : x_7 - ix_6 : x_7 + ix_6 : x_5 - ix_8),$

$\mu_2: x \mapsto (x_0 : x_1 + ix_2 : x_1 - ix_2 : x_3 : x_4 : x_5 + ix_8 : x_7 - ix_6 : x_7 + ix_6 : x_5 - ix_8 : x_7 + ix_8),$

$\mu_3: x \mapsto (x_0 : x_3 - ix_1 : x_2 + ix_4 : x_3 + ix_4 : x_2 - ix_4 : x_5 : x_6 : x_8 - ix_7 : x_8 + ix_7),$
make the following diagram commute for all $0 \leq i \leq 3$:

$$\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{\xi} & Y_\ast \\
\downarrow{\sigma_i} & & \downarrow{\sigma_i} \\
\mathbb{T}^2 & \xrightarrow{\xi} & Y_\ast
\end{array}$$

where $X_i := \mu_i^{-1}(Y_\ast)$ and real structure $\sigma_i : \mathbb{T}^2 \to \mathbb{T}^2$ is defined in Lemma 12.

**Proof.** The specification of $\sigma_i : Y_\ast \to Y_\ast$, for $0 \leq i \leq 3$, follows from the action on the lattice coordinates in Table 15 (recall Notation 5). It is straightforward to verify that $\mu_i^{-1}$ makes the diagram commute.

**Remark 17.** Recall from Corollary 7, Notation 10 and Lemma 11 that the real structures $\sigma_i : Y_\ast \to Y_\ast$, for $0 \leq i \leq 2$, are inner automorphic to $\sigma_+ \times \sigma_+$ via $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ so that $X_i \cong \mathbb{S}^1 \times \mathbb{S}^1$ in these cases. Notice that $\sigma_3 : Y_\ast \to Y_\ast$ is via $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ inner automorphic to $\sigma_+$ so that $X_3 \cong \mathbb{S}^2$. □

The surface $X_2$ from Lemma 16 is contained in $\mathbb{S}^7$. Indeed, if we compose the first four generators of $I_2(Y_\ast)$ with $\mu_2$, then

$$(y_0^2 - y_1 y_2) \circ \mu_2 = \frac{1}{4} x_0^2 - x_1^2 - x_2^2, \quad (y_0^2 - y_5 y_6) \circ \mu_2 = \frac{1}{4} x_0^2 - x_5^2 - x_6^2,$$

$$(y_0^2 - y_3 y_4) \circ \mu_2 = \frac{1}{4} x_0^2 - x_3^2 - x_4^2, \quad (y_0^2 - y_7 y_8) \circ \mu_2 = \frac{1}{4} x_0^2 - x_7^2 - x_8^2,$$

and their sum is the equation of $\mathbb{S}^7$.

We extend $\xi : \mathbb{T}^2 \to Y_\ast$ such that we obtain the biregular isomorphism

$$\tilde{\xi} : \mathbb{P}^1 \times \mathbb{P}^1 \to Y_\ast, \quad (s : t; u : w) \mapsto$$

$$(stuw; s^2uw; t^2uw; stu^2; stw^2; s^2w^2; t^2w^2; t^2u^2) =$$

$$(y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8),$$

and thus $\text{Aut}_o(Y_\ast) \cong \text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$.

**Lemma 18.** $\text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$.

**Proof.** Automorphisms in the identity component $\text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$ act trivially on the Neron-Severi lattice $N(\mathbb{P}^1 \times \mathbb{P}^1) = \langle \ell_0, \ell_1 \rangle_{\mathbb{Z}}$, where generators $\ell_0$ and $\ell_1$ are the classes of the fibers of the first and second projection of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$.

Thus a fiber of $\pi_i$ is mapped by $\varphi \in \text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$ as a whole to a fiber of $\pi_i$ for all $1 \leq i \leq 2$, so that the main assertion is concluded. We remark that $\text{Aut}_o(\mathbb{P}^1) \cong \text{PSL}(2, \mathbb{R}) \cong \text{PGL}(2, \mathbb{R}) \cong \text{Aut}(\mathbb{P}^1)$. □
We associate to $\varphi = (\varphi_1, \varphi_2) \in \text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$ an automorphism

$$S(\varphi) := \text{Sym}_2(\varphi_1) \otimes \text{Sym}_2(\varphi_2) \in \text{Aut}_o(Y_*) \subset \text{Aut}_o(\mathbb{P}^8).$$

We can compute $S(\varphi)$ via the following specification:

$$S(\varphi) : Y_* \to Y_*, \quad \tilde{\xi}(p) \mapsto (\tilde{\xi} \circ \varphi)(p), \quad \text{for all } p \in \mathbb{P}^1 \times \mathbb{P}^1 \cong Y_*.$$

**Example 19. (toric automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$)**

Since $\varphi \circ \xi : \mathbb{T}^2 \hookrightarrow Y_*$ defines an embedding of the algebraic torus $\mathbb{T}^2$ for all automorphisms $\varphi \in \text{Aut}(Y_*)$, the double Segre surface $Y_*$ does not have a unique toric structure. Let $\text{Aut}^\mathbb{T}_0(Y_*)$ denote the identity component of the toric automorphisms with respect to $\xi$. We have the following parametrization:

$$\phi : \mathbb{T}^2 \overset{\sim}{\to} \text{Aut}^\mathbb{T}_0(Y_*), \quad (s, u) \mapsto B(u) \circ A(s), \quad \text{where}$$

$$A(\alpha) := S \left( \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad B(\alpha) := S \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \right).$$

Suppose that the real structure of $Y_*$ is defined by $\sigma_2$ in Lemma 16. It follows from Lemma 12 that $\{ p \in \mathbb{T}^2 \mid \sigma_2(p) = p \} \cong S^1 \times S^1$ and thus

$$\phi : S^1 \times S^1 \overset{\sim}{\to} \text{Aut}^\mathbb{T}_0(Y_*), \quad \left( (\cos(\alpha), \sin(\alpha)), (\cos(\beta), \sin(\beta)) \right) \mapsto S \left( \begin{bmatrix} \cos(\alpha) + i \sin(\alpha) & 0 \\ 0 & \cos(\alpha) - i \sin(\alpha) \end{bmatrix}, \begin{bmatrix} \cos(\beta) + i \sin(\beta) & 0 \\ 0 & \cos(\beta) - i \sin(\beta) \end{bmatrix} \right).$$

Let $\mu_2 : \mathbb{P}^8 \to \mathbb{P}^8$ be as defined in Lemma 16. From the composition of $\phi$ with the pullback $\mu_2^* : \text{Aut}^\mathbb{T}_0(Y_*) \to \text{Aut}^\mathbb{T}_0(X_2)$ we obtain

$$\mu_2^* \circ \phi : S^1 \times S^1 \to \text{Aut}^\mathbb{T}_0(X_2), \left( (\cos(\alpha), \sin(\alpha)), (\cos(\beta), \sin(\beta)) \right) \mapsto S \left( \begin{bmatrix} \cos(\alpha) - \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}, \begin{bmatrix} \cos(\beta) - \sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \right).$$

Notice that the real structure of $X_2 \subset S^7$ is defined by $\sigma_0$ in Lemma 16 and that $\text{Aut}^\mathbb{T}_0(X_2) \cong \text{PSO}(2) \times \text{PSO}(2)$. \hfill \qed

## 4 Blowups of $\mathbb{P}^1 \times \mathbb{P}^1$

The smooth model of a celestial surface that is not $\infty$-circled is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in two or four points. Such a blowup is realized by a linear projection of the double Segre surface $Y_*$ in $\mathbb{P}^8$. The automorphisms of the image surface induce automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ that leave the center
of blowup invariant. This allows us to formulate restrictions on the possible Möbius automorphism groups of celestial surfaces. In particular, we find that celestial surfaces with many symmetries must be toric.

**Proposition 20. (blowups of $\mathbb{P}^1 \times \mathbb{P}^1$)**

If a celestial surface $X \subset \mathbb{S}^n$ is not \( \infty \)-circled and \( \dim \text{Aut}_o(X) \geq 2 \), then its linear normalization $X_N$ is a toric surface, each family of circles on $X$ is toric, and $\text{Aut}_o(X)$ embeds as a subgroup into $\text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$. Moreover, there exists a birational linear projection $\rho: Y \hookrightarrow \mathbb{P}^n \to X \subset \mathbb{P}^n$ whose center of projection is characterized by a row in Table 21 together with $T(X), S(X_N)$ and the projections of $\text{Aut}_o(X)$ to a subgroup of $\text{Aut}_o(\mathbb{P}^1)$. 

**Proof.** By Theorem A[c,b], the smooth model $\tilde{X}$ is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in a center $\Lambda$ such that $|\Lambda| \in \{0, 2, 4\}$ and $X_N$ is its anticanonical model. Hence $\text{Aut}_o(X_N) \cong \text{Aut}_o(\tilde{X})$ and thus $\text{Aut}_o(X)$ defines a subgroup of $\text{Aut}_o(\tilde{X})$. Moreover, $\text{Aut}_o(\tilde{X})$ is isomorphic to a subgroup of $\text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$ whose action leaves the blowup center $\Lambda$ invariant. It now follows from Lemma 18 that $\text{Aut}_o(X)$ is isomorphic to a subgroup of

\[ \{ \varphi \in \text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1) \mid \varphi(\pi_1(\Lambda), \pi_2(\Lambda)) = (\pi_1(\Lambda), \pi_2(\Lambda)) \}, \]

where $\pi_1$ and $\pi_2$ denote the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to its $\mathbb{P}^1$ factors. We will denote the projections of $\text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$ to its $\text{Aut}_o(\mathbb{P}^1)$ factors, by $\pi_1$ and $\pi_2$ as well.

We claim that $|\pi_1(\Lambda)| \leq 2$ and $|\pi_2(\Lambda)| \leq 2$. Suppose by contradiction that $|\pi_1(\Lambda)| > 2$ so that $|\Lambda| = 4$. Let $\pi_1(\text{Aut}_o(X))$ and $\pi_2(\text{Aut}_o(X))$ denote the subgroups of $\text{Aut}_o(\mathbb{P}^1)$ that preserve $\pi_1(\Lambda)$ and $\pi_2(\Lambda)$, respectively. Recall that $\text{Aut}_o(\mathbb{P}^1)$ is 3-transitive and thus $\dim \pi_1(\text{Aut}_o(X)) = 0$. By assumption $\dim \text{Aut}_o(X) \geq 2$, hence $|\pi_2(\Lambda)| = 1$ and $\dim \text{Aut}_o(X) = \dim \pi_2(\text{Aut}_o(X)) = 2$. By Theorem Ac at most two non-infinitely near points lie in the same fiber and these points are nonreal. Thus $\Lambda$ consist of two complex conjugate points and two infinitely near points. We arrived at a contradiction as $\pi_2(\text{Aut}_o(X))$ must be a proper subgroup of $\text{PSA}(1)$ so that $\pi_2(\text{Aut}_o(X)) \cong \text{PSE}(1)$.
See Proposition 20. The possible configurations of the center of blowup $\Lambda \subset \mathbb{P}^1 \times \mathbb{P}^1$ realized by the birational linear projection $\rho: Y_* \to X$ via the isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \cong Y_\ast$. At the entries for $T(X)$ we have $3 \leq n \leq 7$ and $4 \leq m \leq 5$. Since $\text{Aut}_o(X)$ embeds into $\text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$, we find that the projection $\pi_i(\text{Aut}_o(X))$ is a subgroup of $\text{Aut}_o(\mathbb{P}^1)$ for $i \in \{1, 2\}$. An entry for $\pi_1(\text{Aut}_o(X))$ and $\pi_2(\text{Aut}_o(X))$ denotes the maximal possible subgroup. The vertical and horizontal line segments in the diagrams represent fibers of the projections $\pi_i: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for $i \in \{1, 2\}$. The complex conjugate points $q$ and $\bar{q}$ in diagram (f) are infinitely near to $p$ and $\bar{p}$, respectively. A fiber that contains two centers of blowup is contracted by $\rho$ to an isolated singularity of $X$.

| $\Lambda$ | $T(X)$ | $S(X_\ast)$ | $\pi_1(\text{Aut}_o(X))$ | $\pi_2(\text{Aut}_o(X))$ | name |
|---|---|---|---|---|---|
| (a) | $(2, 8, n)$ | $\emptyset$ | PSL(2) | PSL(2) | (projected) dS |
| (b) | $(3, 6, m)$ | $\emptyset$ | PSO(2) | PSO(2) | (projected) dP6 |
| (c) | $(2, 6, m)$ | $A_1$ | PSO(2) | PSA(1) | (projected) weak dP6 |
| (d) | $(4, 4, 3)$ | $4A_1$ | PSO(2) | PSO(2) | ring cyclide |
| (e) | $(2, 4, 3)$ | $2A_1 + 2A_1$ | PSO(2) | PSX(1) | spindle cyclide |
| (f) | $(2, 4, 3)$ | $A_3 + 2A_1$ | PSO(2) | PSE(1) | horn cyclide |

![Diagram](image-url)
As $|\pi_1(\Lambda)| \leq 2$, $|\pi_2(\Lambda)| \leq 2$ and $\Lambda$ does not contain real points, it follows that all possible configurations of $\Lambda$ together with $\pi_1(Aut_o(X))$ and $\pi_2(Aut_o(X))$ are listed in Table 21. Moreover, since the algebraic torus $T^1$ embeds into $\mathbb{P}^1 \setminus \pi_i(\Lambda)$ such that $Aut_o(T^1)$ extends to a subgroup of $\pi_i(Aut_o(X))$ for $i \in \{1, 2\}$, we deduce that $X_N$ must be toric.

The bidegree $(2, 2)$ forms define an isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow Y_* \subset \mathbb{P}^8$. Since $X_N$ is an anticanonical model, the bidegree $(2, 2)$ forms that pass through the blowup center $\Lambda$, define a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow X_N \subset \mathbb{P}^r$ for some $n + 1 \leq r \leq 8$. Assigning linear conditions to the forms, so that they pass through $\Lambda$, corresponds to a linear projection $f: Y_* \subset \mathbb{P}^8 \rightarrow X_N \subset \mathbb{P}^r$. It follows from the definition of linear normalization that there exists a degree preserving linear projection $g: X_N \rightarrow X$. We now define $\rho: Y_* \rightarrow X$ as the composition of $f$ with $g$. Notice that $Y_* \cong \mathbb{P}^1 \times \mathbb{P}^1$ and thus we may interpret the blowup center $\Lambda$ instead as the center of projection of $\rho$. It follows from Theorem Ac that the fibers that contain two points in $\Lambda$ are contracted via $\rho$ to an isolated singularity of $X$.

We assume without loss of generality that the generators $\ell_0$ and $\ell_1$ as defined at Notation 13 are the classes of the pullbacks to $\tilde{X}$ of the fibers of $\pi_1$ and $\pi_2$, respectively. The generators $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ and $\varepsilon_4$ are the classes of the pullbacks of $(-1)$-curves that contract to the points $p$, $\overline{p}$, $q$ and $\overline{q}$, respectively. For each configuration of $\Lambda$ we obtain an explicit description of $B(X)$. For example, if $\Lambda$ is as in Table 21f, then $b_1 \in B(X)$ since $q$ is infinitely near to $p$, and $b_{12} \in B(X)$ since $p$ and $\overline{p}$ lie in a real fiber of $\pi_2$, and so on. Notice that $S(X_N)$ corresponds to the Dynkin diagram with vertex set $B(X)$ and edge set $\{(a, b) \mid a \cdot b > 0\}$. We find that $|B(X)| > 3$ in each case and thus the values at the $T(X)$, $S(X_N)$ and name columns are a direct consequence of Theorem Aa.

If any two choices for the blowup center $\Lambda$ are characterized by the same row of Table 21, then these choices are equivalent up to $Aut(\mathbb{P}^1 \times \mathbb{P}^1)$. It follows that the linear normalization $X_N \subset \mathbb{P}^r$ is up to $Aut(\mathbb{P}^r)$ uniquely determined by the name of the celestial surface $X$ and thus $L(X)$ is up to equivalence determined by this name as well. We now apply Proposition 6 and find by comparing Table 21 with Table 8, that each family of circles on $X$ is realized by some toric family. 

\[\square\]
**Remark 22. (toric projections of the double Segre surface)**

Recall that a lattice polygon in Table 8 defines, up to projective isomorphism, a monomial parametrization of the linear normalization of a toric celestial surface. The inclusion of lattice polygons with the same unimodular involution, defines an arrow reversing projection between the corresponding toric models. The corresponding *toric projection* is defined by omitting components of the monomial parametrization associated to the bigger lattice polygon such that the exponents of the remaining components define the lattice points of the smaller lattice polygon. Thus toric surfaces with lattice types b, c, e, f, g in Table 8 are toric projections of b, a, b, a, a in Table 9, respectively. We will use this concept in Example 23, Example 33 and in the proof of Lemma 35.

**Example 23. (dP6 as the image of a toric projection)**

Suppose that $Y \subset \mathbb{P}^8$ and $Z \subset \mathbb{P}^6$ have lattice types as in Table 9b and Table 8b, respectively. Thus the real structure of $Y$ is defined by $\sigma_2$ in Lemma 16.

We use the left coordinates of Table 15 and omit the monomial components corresponding to $y_5$ and $y_6$ coordinates such that

$$
\xi_b: \mathbb{T}^2 \to Z \subset \mathbb{P}^6, \quad (s,u) \mapsto (1 : s : s^{-1} : u : u^{-1} : su^{-1} : s^{-1}u) = (y_0 : y_1 : y_2 : y_3 : y_4 : y_7 : y_8).
$$

Let the projective isomorphism $\mu_2: \mathbb{P}^6 \to \mathbb{P}^6$ be a restriction of $\mu_2$ as defined in Lemma 16. We find that $X := \mu_2(Z)$ is contained in $\mathbb{S}^5$ and has celestial type $(3,6,5)$ (see Figure 1). The center of the linear projection $\rho: \mathbb{P}^8 \to \mathbb{P}^6$ is a line that intersects $Y$ transversally in $p = (0 : 0 : 0 : 0 : 1 : 0 : 0 : 0)$ and its complex conjugate $\overline{p} = \sigma_2(p) = (0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0)$. We remark that $\sigma_2: Y \to Y$ is inner automorphic to $\sigma_0$ via $\text{Aut}(Y)$ by Corollary 7a.

The projection $\rho$ realizes a blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \cong Y$ with centers $p$ and $\overline{p}$ as in Table 21b. Notice that $\text{Aut}_o(Z) \cong \{ \varphi \in \text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1) \mid \varphi(p) = p, \ \varphi(\overline{p}) = \overline{p} \}$, since $\rho$ is an isomorphism almost everywhere. Recall that $\text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$ by Lemma 18 and thus $\pi_1(\text{Aut}_o(Z)) \cong \pi_2(\text{Aut}_o(Z)) \cong \text{PSO}(2)$ as it is stated in Table 21b.

5 Invariant quadratic forms on $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we reformulate the problem of classifying Möbius automorphism groups of celestial surfaces, into the problem of finding invariant quadratic
forms of given signature in a vector space.

Suppose that $Y \subset \mathbb{P}^m$ is a surface such that $\text{Aut}(Y) \subset \text{Aut}(\mathbb{P}^m)$. For example, $Y \subset \mathbb{P}^8$ is the double Segre surface or $Y \subset \mathbb{P}^5$ is the Veronese surface. Suppose that we have a birational linear projection with $m \geq n + 1 \geq 3$:

$$\rho: Y \subset \mathbb{P}^m \dashrightarrow X \subset \mathbb{S}^n \subset \mathbb{P}^{n+1}.$$ 

The Möbius pair of $X$ with respect to $\rho$ is defined as

$$(Y, Q) \text{ where } Q \subset \mathbb{P}^m \text{ is the Zariski closure of } \rho^{-1}(\mathbb{S}^n).$$

Notice that $Q$ is a hyperquadric of signature $(1, n + 1)$ such that $Y \subset Q$ and such that the singular locus of $Q$ coincides with the center of the linear projection $\rho$. We define the following equivalence relation on Möbius pairs:

$$(Y, Q) \sim (Y, Q') :\iff \exists \varphi \in \text{Aut}(\mathbb{P}^m): \varphi(Y) = Y \text{ and } \varphi(Q) = Q'.$$

Suppose that $G \subseteq \text{Aut}_o(Y)$ is a subgroup. The vector space of $G$-invariant quadratic forms in the ideal $I(Y)$ of $Y$ is defined as

$$I^G_2(Y) := \langle q \in I_2(Y) \mid q \circ \varphi = q \text{ for all } \varphi \in G \rangle_{\mathbb{C}},$$

where we assume that $\varphi \in G \subset \text{PSL}(m+1)$ is normalized to have determinant one. Notice that the real structure $\sigma: Y \to Y$ induces an antiholomorphic involution on $I^G_2(Y)$. We denote the zeroset of a form $q \in I(Y)$ by $V(q)$.

**Proposition 24. (properties of Möbius pairs)**

Let $(Y, Q)$ and $(Y, Q')$ be the Möbius pairs of surfaces $X \subset \mathbb{S}^n$ and $X' \subset \mathbb{S}^n$, respectively.

a) There exists $\alpha \in \text{Aut}(\mathbb{S}^n)$ with $\alpha(X) = X'$ if and only if $(Y, Q)$ and $(Y, Q')$ are equivalent. In particular, we have that

$$M(X) \cong \{ \varphi \in \text{Aut}_o(\mathbb{P}^m) \mid \varphi(Y) = Y \text{ and } \varphi(Q) = Q \}.$$ 

b) The subgroup $G \subseteq \text{Aut}_o(Y)$ is isomorphic to a subgroup of $M(X)$ if and only if $Q = V(q)$ for some $q \in I^G_2(Y)$.

c) If $G, G' \subset \text{Aut}_o(Y)$ are inner automorphic subgroups and $q \in I^G_2(Y)$, then there exists $q' \in I^{G'}_2(Y)$ such that $(Y, V(q))$ and $(Y, V(q'))$ are equivalent as Möbius pairs.
Proof. a) Let $\rho: \mathbb{P}^m \to \mathbb{P}^{m+1}$ be a birational linear projection such that $\rho(Q) = S^n$ and $\rho(Y) = X$. Similarly, let $\rho': \mathbb{P}^m \to \mathbb{P}^{m+1}$ be such that $\rho'(Q') = S^n$ and $\rho'(Y) = X'$. 

$\Rightarrow$: We show that there exists $\varphi \in \text{Aut}(\mathbb{P}^m)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & Q' \\
\rho & \downarrow & \downarrow \rho' \\
X & \xleftarrow{\alpha} & S^n \\
\end{array}
$$

If $m = n + 1$, then $\rho$ and $\rho'$ are projective isomorphisms and the claim follows immediately. If $m > n + 1$, then the centers of the linear projections $\rho$ and $\rho'$ coincide with the singular loci $S(Q)$ and $S(Q')$ of $Q$ and $Q'$, respectively.

Let $\Lambda, \Lambda' \subset Y$ be the centers of the projections $\rho|_Y$ and $\rho'|_Y$, respectively. The linear isomorphism $\alpha$ induces via the projections $\rho$ and $\rho'$ the algebraic isomorphisms $\varphi|_Q: Q \setminus S(Q) \to Q' \setminus S(Q')$ and $\varphi|_Y: Y \setminus \Lambda \to Y \setminus \Lambda'$. The automorphism $\alpha$ leaves the union of exceptional curves that contract to points in $\Lambda$ invariant and thus we can extend $\varphi|_Y$ so that $\varphi \in \text{Aut}_o(Y)$ and $\varphi(\Lambda) = \Lambda'$. Since $\text{Aut}(Y) \subset \text{Aut}(\mathbb{P}^m)$ and since $Q$ contains $\rho^{-1}(S^n)$ by assumption, we find that $\varphi \in \text{Aut}(\mathbb{P}^m)$ such that $\varphi(Q) = Q'$ and $\varphi(Y) = Y$ as was to be shown.

$\Leftarrow$: For the converse, we need to show that for given $\varphi \in \text{Aut}(\mathbb{P}^m)$ there exists $\alpha \in \text{Aut}(S^n)$ such that the above diagram commutes. This is immediate, since we define $\alpha$ as the composition $\rho'|_{Q'} \circ \varphi \circ (\rho|_Q)^{-1}$.

The remaining assertion follows if we set $Q' := Q$ and $X' := X$ in the above diagram.

b) We first show the $\Leftarrow$ direction. By assumption $q \circ \varphi = q$ for all $\varphi \in G$. Since $\varphi^{-1}(V(q)) = \{ \varphi^{-1}(x) \in \mathbb{P}^m \mid q(x) = 0 \} = V(q \circ \varphi)$ we find that $G \subseteq \{ \varphi \in \text{Aut}_o(\mathbb{P}^m) \mid \varphi(Y) = Y \text{ and } \varphi(Q) = Q \}$. It now follows from a) that $G$ embeds as a subgroup into $\mathcal{M}(X)$. For the $\Rightarrow$ direction we again apply the characterization of $\mathcal{M}(X)$ in a) and find that $\varphi^{-1}(Q) = Q$ and thus $q \circ \varphi = q$ for all $\varphi \in G$ so that $q \in I^G_2(Y)$.

c) Since $q \in I^G_2(Y)$ the following holds for all $\varphi \in G$ and $\alpha \in \text{Aut}(Y)$:

$$
q \circ \varphi = q \iff q \circ \varphi \circ \alpha = q \circ \alpha \iff q \circ \alpha \circ \alpha^{-1} \circ \varphi \circ \alpha = q \circ \alpha.
$$

By assumption $G' = \alpha^{-1} \circ G \circ \alpha$ for some $\alpha \in \text{Aut}(Y)$ and thus for all $\varphi' \in G'$
there exists \( \varphi \in G \) such that \( \varphi' = \alpha^{-1} \circ \varphi \circ \alpha \). It follows that \( q' \circ \varphi' = q' \) for all \( \varphi' \in G' \), where \( q' := q \circ \alpha \) so that \( q' \in I^G_q(Y) \). Thus
\[
\alpha^{-1}(V(q)) = \{ \alpha^{-1}(x) \in \mathbb{P}^m \mid q(x) = 0 \} = V(q \circ \alpha) = V(q'),
\]
so that \( (Y, V(q)) \) is equivalent to \( (Y, V(q')) \). \( \square \)

The following theorem is essentially [3, Theorem 2.5] and allows us to compute \( G \)-invariant quadratic forms via the Lie algebra \( \text{Lie}(G) \) of \( G \).

**Theorem B. [DeGraaf-Pilniková-Schicho, 2009]**

Suppose that \( Y \subset \mathbb{P}^{m-1} \) is a variety such that \( \text{Aut}_c(Y) \subset \text{Aut}_c(\mathbb{P}^{m-1}) \). Let the 1-parameter subgroup \( H \subset \text{Aut}_c(Y) \) be represented by an \( m \times m \) matrix whose entries are smooth functions in the parameter \( \alpha \) such that \( \det H(\alpha) = 1 \) for all \( \alpha \) and such that \( H(0) \) is the identity matrix. Let the \( m \times m \) matrix \( D \) in \( \text{Lie}(H) \) be the tangent vector \( (\partial_\alpha H)(0) \) of \( H \) at the identity. Then the \( H \)-invariant quadratic forms are
\[
I^H_2(Y) = \langle q_A \in I_2(Y) \mid D^T \cdot A + A \cdot D = 0 \rangle_C, \tag{2}
\]
where \( q_A \) denotes the quadratic form \( x^T \cdot A \cdot x \) associated to the symmetric \( m \times m \) matrix \( A \).

**Proof.** We observe that \( I^H_2(Y) = \langle q_A \in I_2(Y) \mid H^T \cdot A \cdot H = A \rangle_C \). Let us first assume that \( H^T \cdot A \cdot H = A \). We differentiate both sides of the equation with respect to \( \alpha \) and evaluate at 0 so that we obtain the necessary condition \( D^T \cdot A + A \cdot D = 0 \). For the converse we assume that \( D^T \cdot A + A \cdot D = 0 \). Thus \( G^T \cdot A \cdot G = B \) for some matrix \( B \) where \( G = \exp(\alpha D) \) is a one-parameter subgroup. We differentiate both sides of the equivalent equation \( G^T \cdot A \cdot G \cdot \exp(\alpha) = B \cdot \exp(\alpha) \) with respect to \( \alpha \) and evaluate at 0 so that we obtain \( D^T \cdot A + A \cdot D + A = B \). It follows that \( A = B \) and we know from Lie theory that \( H = G \) so that \( H^T \cdot A \cdot H = A \) as is required. \( \square \)

**Remark 25. (goal)** Our goal is to classify subgroups \( G \subseteq \text{Aut}_c(Y) \) up to inner automorphism such that \( \dim G \geq 2 \) and \( I^G_q(Y) \) contains quadratic forms \( q \) of signature \( (1, n+1) \) with \( n \geq 3 \). It follows from Proposition 24 that the Möbius pairs \( (Y, V(q)) \) for such \( q \), correspond to the celestial surfaces \( X \subset \mathbb{S}^n \) such that \( G \) is isomorphic to a subgroup of \( \mathbf{M}(X) \). \( \triangleright \)
6 Automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

Motivated by Remark 25 with $Y$ the double Segre surface $Y_\ast \cong \mathbb{P}^1 \times \mathbb{P}^1$, we would like to classify Lie subgroups of $\text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$ up to inner automorphism. By Theorem B, it is sufficient to classify Lie subalgebras of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Let us first investigate real structures of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Consider the toric involutions $\sigma_i : T^2 \to T^2$ in Lemma 12 with $0 \leq i \leq 3$. By Lemma 16, these toric involutions induce involutions on related algebraic structures (recall Notation 5): $\sigma_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, $\sigma_i : \text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1) \to \text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\sigma_i : \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

**Lemma 26. (real structures for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$)**

If $m := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, then

\[
\sigma_0(m) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad \sigma_1(m) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \\
\sigma_2(m) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \begin{pmatrix} f & e \\ h & g \end{pmatrix}, \quad \sigma_3(m) = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \begin{pmatrix} d & c \\ b & a \end{pmatrix},
\]

where $\sigma_0, \sigma_1, \sigma_2$ and $\sigma_3$ are real structures $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ induced by the corresponding involutions in Lemma 12 and Lemma 16.

**Proof.** Suppose that $M \subset \text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$ is a 1-parameter subgroup such that $m$ is the tangent vector of $M$ at the identity. The 1-parameter subgroup $\sigma_i(M)$ has tangent vector $\sigma_i(m)$ for all $0 \leq i \leq 3$. We compute the representation $S(M) \in \text{Aut}(\mathbb{P}^8)$ using (1), where the entries of $M$ are set as indeterminates. Let $L_i$ denote the $9 \times 9$ permutation matrix corresponding to the induced antiholomorphic involution $\sigma_i : \mathbb{P}^8 \to \mathbb{P}^8$ as stated in Lemma 16. It is immediate to verify that $L_i^{-1} \circ S(M) \circ L_i = S(\sigma_i(M))$, where $\sigma_i$ acts on $\text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1)$ as an involution and $\overline{\cdot}$ denotes complex conjugation. We conclude that the lemma holds, since the action of $\sigma_i$ on $m$ is the same as the action of $\sigma_i$ on $M$. \qed
Remark 27. The real structure $\sigma_2 : \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is inner automorphic to $\sigma_H := \alpha \circ \sigma_2 \circ \alpha^{-1}$, where
\[
\alpha := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}
\]
so that $\sigma_H \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) = \begin{bmatrix} \bar{d} & -\bar{e} \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} \bar{h} & -\bar{g} \\ -\bar{f} & \bar{e} \end{bmatrix} \right) \). The Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ with real structure $\sigma_H$ is usually denoted by $\mathfrak{su}_2 \oplus \mathfrak{su}_2$. The real elements in $\mathfrak{su}_2$ are skew Hermitian matrices. Similarly, $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ with real structure $\sigma_1$ can be identified with $\mathfrak{su}_2 \oplus \mathfrak{sl}_2 (\mathbb{R})$.

Notation 28. We consider the following elements in $\mathfrak{sl}_2$:
\[ t := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad q := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad s := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad r := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]
Recall that $\mathfrak{sl}_2$ over the complex numbers is generated by $\langle t, q, s \rangle$, where the Lie brackets of the generators are $[t, q] = s$, $[t, s] = -2t$ and $[q, s] = 2q$. We shall denote $(g, e) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ by $g_1$ and $(e, g) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ by $g_2$ for all $g \in \mathfrak{sl}_2$. Notice that $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = \langle t_1, q_1, s_1, t_2, q_2, s_2 \rangle$ where the Lie bracket acts componentwise.

Remark 29. Suppose that the real structure of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is defined by $\sigma_0$ in Lemma 26. In this case,
\[ \text{Lie}(\text{PSE}(1)) = \langle t \rangle, \quad \text{Lie}(\text{PSX}(1)) = \langle s \rangle, \quad \text{Lie}(\text{PSO}(2)) = \langle r \rangle, \quad \text{Lie}(\text{PSA}(1)) = \langle t, s \rangle \quad \text{and} \quad \text{Lie}(\text{PSL}(2)) = \langle t, q, s \rangle. \]
These groups correspond to translations, scalings, rotations, affine transformations, and projective transformations, respectively. Indeed the generators are the tangent vectors at the identity of the following 1-parameter subgroups:
\[ t \mapsto \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \quad s \mapsto \begin{bmatrix} \alpha + 1 & 0 \\ 0 & (\alpha + 1)^{-1} \end{bmatrix}, \quad r \mapsto \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}. \]
Now suppose that the real structure of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is defined by $\sigma_2$ in Lemma 26. In this case $\text{Lie}(\text{PSO}(2)) = \langle \imath s \rangle$, since
\[ \imath s = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \mapsto \begin{bmatrix} \cos(\alpha) + \imath \sin(\alpha) & 0 \\ 0 & \cos(\alpha) - \imath \sin(\alpha) \end{bmatrix}. \]
See also Example 19.

Suppose that $F$ is a Lie group. We call two Lie subalgebras $\mathfrak{g}, \mathfrak{h} \subset \text{Lie}(F)$ (complex) inner automorphic if there exists (complex) $M \in F$ such that $\mathfrak{g} = M^{-1} \cdot \mathfrak{h} \cdot M$. Theorem C and thus Corollary 30 follow from [4].

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Theorem C. [Douglas-Repka, 2016]

A Lie subalgebra \( 0 \subseteq g \subseteq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \) is, up to flipping the left and right factor, complex inner automorphic to either one of the following with \( \alpha \in \mathbb{C}^* \):

\[
\langle t_1 \rangle, \langle s_1 \rangle, \langle t_1 + t_2 \rangle, \langle t_1 + s_2 \rangle, \langle s_1 + \alpha s_2 \rangle, \langle t_1, s_1 \rangle, \langle t_1, t_2 \rangle, \langle t_1, s_2 \rangle, \langle s_1, s_2 \rangle, \\
\langle s_1 + t_2, t_1 \rangle, \langle t_1 + t_2, s_1 + s_2 \rangle, \langle s_1 + \alpha s_2, t_1 \rangle, \langle t_1, q_1, s_1 \rangle, \langle t_1, s_1, t_2 \rangle, \langle t_1, s_1, s_2 \rangle, \\
\langle s_1 + \alpha s_2, t_1, t_2 \rangle, \langle t_1 + t_2, q_1 + q_2, s_1 + s_2 \rangle, \langle t_1, s_1, t_2, s_2 \rangle, \langle t_1, q_1, s_1, t_2 \rangle, \langle t_1, q_1, s_1, s_2 \rangle, \\
\langle t_1, q_1, s_1, t_2, q_2, s_2 \rangle.
\]

Corollary 30. If \( g \subseteq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \) is a Lie subalgebra such that \( \dim g \geq 2 \) and \( g \) is not complex inner automorphic to \( \langle s_1, s_2 \rangle \), then \( g \) contains a subalgebra that is complex inner automorphic to either \( \langle t_1 \rangle, \langle t_2 \rangle \) or \( \langle t_1 + t_2 \rangle \).

7 The classification of \( \mathbb{P}^1 \times \mathbb{P}^1 \)

In a perfect world we directly use Theorem B to compute for each Lie subalgebra \( \text{Lie}(G) \subseteq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \), the vector space \( I_2^G(Y_*) \) generated by \( G \)-invariant quadratic forms on the double Segre surface \( Y_* \cong \mathbb{P}^1 \times \mathbb{P}^1 \). We would then proceed by classifying quadratic forms in \( I_2^G(Y_*) \) of signature \( (1, n + 1) \) as was suggested in Remark 25.

Unfortunately, there are two problems. Theorem C only provides the classification of subalgebras of \( \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \) up to complex inner automorphisms and thus the real structure is not preserved. The second problem is that it is in general difficult to classify quadratic forms in \( I_2^G(Y_*) \) of fixed signature. For example, for what \( n \geq 3 \) do there exist quadratic forms of signature \( (1, n + 1) \) in the vector space \( I_2^G(X_1) \) at Lemma 31c?

Lemma 34 plays a crucial role in circumventing these two problems using geometric arguments. We are able to prove Lemma 35, since the invariant quadratic forms in Lemma 31a have a particularly nice basis. This section will end with a proof for Theorem 1 and Corollary 2. In particular, we will see that the answer to the question in the previous paragraph is \( n \in \{3\} \).

Lemma 31. (invariant quadratic forms for \( \mathbb{P}^1 \times \mathbb{P}^1 \))

Let the real structures \( \sigma_0, \sigma_1, \sigma_2 \) and \( \sigma_3 \) for Lie algebras be as defined in Lemma 26. Let \( Y_*, X_0, X_1, X_2 \) and \( X_3 \) be the double Segre surfaces in \( \mathbb{P}^8 \) as defined in Lemma 16. We suppose that \( G \) is a Lie subgroup of \( \text{Aut}_o(Y_*) \).

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a) If Lie(G) = ⟨is₁, is₂⟩ with real structure σ₂, then G ≅ PSO(2) × PSO(2),

\[ I^G_2(Y_*) = \langle y_0^2 - y_1y_2, y_0^2 - y_3y_4, y_0^2 - y_5y_6, y_0^2 - y_7y_8 \rangle_C, \quad \text{and} \]

\[ I^G_2(X_2) = \langle \frac{1}{4}x_0^2 - x_1^2 - x_2^2, \frac{1}{4}x_0^2 - x_3^2 - x_4^2, \frac{1}{4}x_0^2 - x_5^2 - x_6^2, \frac{1}{4}x_0^2 - x_7^2 - x_8^2 \rangle_C. \]

b) If Lie(G) = ⟨is₁, s₂⟩ with real structure σ₁, then G ≅ PSO(2) × PSX(1),

\[ I^G_2(Y_*) = \langle y_0^2 - y_1y_2, y_0^2 - y_3y_4, y_0^2 - y_5y_6, y_0^2 - y_7y_8 \rangle_C, \quad \text{and} \]

\[ I^G_2(X_1) = \langle x_0^2 - x_1^2 - x_2^2, x_0^2 - x_3^2 - x_4^2, x_0^2 - x_5^2 - x_6^2, x_0^2 - x_7^2 - x_8^2 \rangle_C. \]

c) If Lie(G) = ⟨is₁, t₂⟩ with real structure σ₁, then G ≅ PSE(2) × PSL(2),

\[ I^G_2(Y_*) = \langle y_0^2 - y_3y_4, y_0^2 - y_6y_7, y_0^2 - y_9y_10, y_0^2 - y_7y_8 \rangle_C, \quad \text{and} \]

\[ I^G_2(X_1) = \langle x_0^2 - x_3x_4, x_0^2 - x_5x_6, x_0^2 - x_7x_8, x_0^2 - x_9x_10 \rangle_C. \]

d) If Lie(G) = ⟨t₁, q₁, s₁, t₂, q₂, s₂⟩ with real structure either σ₀ or σ₃, then G ≅ PSL(2) × PSL(2),

\[ I^G_2(Y_*) = \langle 2y_0^2 - 2y_1y_2 - 2y_3y_4 + y_5y_6 + y_7y_8 \rangle_C, \]

\[ I^G_2(X_0) = \langle 2x_0^2 - 2x_1x_2 - 2x_3x_4 + x_5x_6 + x_7x_8 \rangle_C, \quad \text{and} \]

\[ I^G_2(X_3) = \langle 2x_0^2 - 4x_2x_3 - 4x_1x_4 + x_5x_6 + x_7^2 + x_8^2 \rangle_C. \]

Proof. a) We know from Remark 29 that G ≅ PSO(2) × PSO(2), since is₁ and is₂ are the tangent vectors of the following two 1-parameter subgroups of Aut₀(ℙ¹ × ℙ¹):

\[
\begin{pmatrix}
\cos(\alpha) + i\sin(\alpha) & 0 \\
0 & \cos(\alpha) - i\sin(\alpha)
\end{pmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\cos(\alpha) + i\sin(\alpha) & 0 \\
0 & \cos(\alpha) - i\sin(\alpha)
\end{pmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Via the map φ: S¹ × S¹ → Aut₀(Y∗) ⊂ Aut(ℙ₈) from Example 19, we obtain 1-parameter subgroups H₁ and H₂ of Aut₀(ℙ₈). We use Theorem B to compute the vector spaces I^G₁(H₁) and I^G₂(H₂) of invariant quadratic forms. Since Lie(G) = ⟨is₁, is₂⟩ we have I^G₁(Y∗) = I^H₁(Y∗) \cap I^H₂(Y∗). In order to compute I^G₂(X₂) we compose the generators of I^G₂(Y∗) with μ₂ from Lemma 16. The proofs of b), c) and d) are similar. The invariant quadratic forms can be computed automatically with [8, moebius-aut].

Remark 32. It follows from Remark 17 that X₃ is a 2-uple embedding of S² into ℙ₈. Hence, X₃ cannot be projectively equivalent to a celestial surface. We nevertheless included the real structure σ₃: Y∗ → Y at Lemma 31d in order to prove Corollary 2. If G ≅ PSL(2) × PSL(2), then σ₃: sl₂ ⊕ sl₂ → sl₂ ⊕ sl₂, and thus σ₃: G → G, is specified in Lemma 26. Notice that in this case \{ϕ ∈ G | σ₃(ϕ) = ϕ\} is three-dimensional and not six-dimensional.
Example 33. (spindle cyclide and horn cyclide)

Let \( Z_s \subseteq \mathbb{P}^4 \) be the image of the monomial parametrization defined by Table 8f (spindle cyclide) using the left coordinates in Table 15. Let \( Z_h \subseteq \mathbb{P}^4 \) denote the image of the monomial parametrization defined by Table 8g (horn cyclide).

Recall from Remark 22, that both \( Z_s \) and \( Z_h \) are toric projections of \( Y_s \). It follows from Lemma 14 that \( I_2(Z_s) \) is generated by the generators of \( I_2(Y_s) \) that do not contain \( y_i \) for \( i \in \{5, 6, 7, 8\} \). Similarly, \( I_2(Z_h) \) is generated by generators of \( I_2(Y_h) \) that do not contain \( y_i \) for \( i \in \{1, 2, 5, 8\} \). Thus

\[
I_2(Z_s) = \langle y_0^2 - y_1y_2, y_0^2 - y_3y_4 \rangle_C \quad \text{and} \quad I_2(Z_h) = \langle y_0^2 - y_3y_4, y_1^2 - y_6y_7 \rangle_C.
\]

Let \( \mu_s \) and \( \mu_h \) be restrictions of \( \mu_1 : \mathbb{P}^8 \to \mathbb{P}^8 \) in Lemma 16 as follows:

\[
\mu_s : \mathbb{P}^4 \to \mathbb{P}^4, \quad (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 : x_1 + ix_2 : x_1 - ix_2 : x_3 : x_4),
\]

\[
\mu_h : \mathbb{P}^4 \to \mathbb{P}^4, \quad (x_0 : x_3 : x_4 : x_6 : x_7) \mapsto (x_0 : x_3 : x_4 : x_7 - ix_6 : x_7 + ix_6).
\]

We set \( X_s := (\alpha_s \circ \mu_s)^{-1}(Z_s) \) and \( X_h := (\alpha_h \circ \mu_h)^{-1}(Z_h) \), where

\[
\alpha_s : \mathbb{P}^4 \to \mathbb{P}^4, (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_4 : x_1 : x_2 : \frac{1}{\sqrt{2}}(x_0 - x_3) : \frac{1}{\sqrt{2}}(x_0 + x_3)),
\]

\[
\alpha_h : \mathbb{P}^4 \to \mathbb{P}^4, (x_0 : x_3 : x_4 : x_6 : x_7) \mapsto (\frac{1}{\sqrt{2}}x_4 : x_3 : -x_0 - x_3 : x_6 : x_7).
\]

Notice that \( X_s, X_h \subseteq \mathbb{S}^3 \), since \( I_2(X_s) = \langle x_1^2 + x_2^2 - x_4^2, x_0^2 - x_3^2 - 2x_7^2 \rangle_C \) and

\[
I_2(X_h) = \langle x_4^2 + 2x_0x_3 + 2x_0^2, x_0^2 + 2x_0x_3 + x_3^2 - x_6^2 - x_7^2 \rangle_C.
\]

Let \( \pi^h, \pi^s : \mathbb{S}^3 \to \mathbb{R}^3 \) be stereographic projections so that the projective closures of these projections with the above coordinates are

\[
\bar{\pi}^h : \mathbb{S}^3 \to \mathbb{P}^3, \quad (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 - x_3 : x_1 : x_2 : x_4),
\]

\[
\bar{\pi}^s : \mathbb{S}^3 \to \mathbb{P}^3, \quad (x_0 : x_3 : x_4 : x_6 : x_7) \mapsto (x_0 + x_3 : x_4 : x_7 : x_6).
\]

We verify that \( \bar{\pi}^h(X_s(\mathbb{R})) \) is a circular cone and that \( \bar{\pi}^s(X_h) \) is a circular cylinder so that, by Definition 3, \( X_s \) and \( X_h \) are indeed a spindle cyclide and horn cyclide, respectively. Notice that both \( \pi^h \) and \( \pi^s \) have a real isolated singularity as center of projection. Since these isolated singular points have to be preserved by the Möbius automorphisms, it follows from §1 that \( \mathbf{M}(X_s) \) and \( \mathbf{M}(X_h) \) are subgroups of Euclidean similarities. The circular cone and the circular cylinder are unique up to Euclidean similarities and thus \( \mathbf{D}(X_s) = \mathbf{D}(X_h) = 0 \). By Lemma 31b the generators of the vector space \( I_2(Z_s) \) are \( \text{PSO}(2) \times \text{PSX}(1) \) invariant. Similarly, by Lemma 31c, the generators of the vector space \( I_2(Z_h) \) are \( \text{PSO}(2) \times \text{PSE}(1) \) invariant. Proposition 20
characterizes the projections from $\text{Aut}_o(X_s)$ and $\text{Aut}_o(X_h)$ to $\text{Aut}_o(\mathbb{P}^1)$. We conclude that $\mathbf{M}(X_s) \cong \text{PSO}(2) \times \text{PSX}(1)$ and $\mathbf{M}(X_h) \cong \text{PSO}(2) \times \text{PSE}(1)$ so that $\text{Aut}_o(X_s) = \mathbf{M}(X_s)$ and $\text{Aut}_o(X_h) = \mathbf{M}(X_h)$.

**Lemma 34. (Möbius automorphism groups)**

If $X \subset \mathbb{S}^n$ is a $\lambda$-circled celestial surface such that $\lambda < \infty$ and $\dim \mathbf{M}(X) \geq 2$, then either

1. $\mathbf{M}(X) \cong \text{PSO}(2) \times \text{PSO}(2)$ and $\text{Lie}(\mathbf{M}(X)) \subset \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is, up to inner automorphism, equal to $(i\mathfrak{s}_1, i\mathfrak{s}_2)$ with real structure $\sigma_2$ in Lemma 26,
2. $\mathbf{M}(X) \cong \text{PSO}(2) \times \text{PSX}(1)$, $\mathbf{T}(X) = (2,4,3)$, $\mathbf{S}(X) = 2A_1 + 2A_1$, $\mathbf{D}(X) = 0$, $\mathbf{M}(X) = \text{Aut}_o(X)$ and $X$ is a spindle cyclide, or
3. $\mathbf{M}(X) \cong \text{PSO}(2) \times \text{PSE}(1)$, $\mathbf{T}(X) = (2,4,3)$, $\mathbf{S}(X) = A_3 + 2A_1$, $\mathbf{D}(X) = 0$, $\mathbf{M}(X) = \text{Aut}_o(X)$ and $X$ is a horn cyclide.

**Proof.** In Proposition 20 we related $\mathbb{P}^1 \times \mathbb{P}^1$ to $X$, via a birational linear projection $\rho: Y_* \rightarrow X$, where $Y_* \cong \mathbb{P}^1 \times \mathbb{P}^1$. Recall from Remark 32 that the real structure of $Y_*$, that is via $\rho$ compatible with the real structure of $X$, cannot be $\sigma_3$. We know from Lemma 18 that $\text{Aut}_o(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$. The first and second projection are denoted by $\pi_i: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with $i \in \{1, 2\}$ and we denote the projections of $\text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)$ to $\text{Aut}_o(\mathbb{P}^1)$ by $\pi_1$ and $\pi_2$ as well. The automorphisms of $X$ in the identity component factor via $\rho$ through automorphisms of $Y_*$ that leave the center of projection $\Lambda$ invariant.

We make a case distinction on the configurations of $\Lambda$ in Table 21 were we identified $Y_*$ with $\mathbb{P}^1 \times \mathbb{P}^1$. By Proposition 20 these are all possible configurations for $\Lambda$.

We first suppose that $\Lambda$ is the empty-set as in Table 21a.

We consider the action of subgroups of the Möbius automorphism group $\mathbf{M}(X)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We start by showing that either Lemma 34.1 holds or there exists a one-dimensional subgroup of $\mathbf{M}(X)$ whose action on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves a real fiber $L$ of $\pi_2$ invariant and leaves a real point $\hat{c} \in L$ on this fiber invariant. We write $\mathfrak{g} \sim_C \mathfrak{h}$ and $\mathfrak{g} \sim \mathfrak{h}$ if Lie subalgebras $\mathfrak{g}, \mathfrak{h} \subset \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ are complex and real inner automorphic, respectively. Recall from Corollary 30 that if $\text{Lie}(\mathbf{M}(X)) \sim_C \langle s_1, s_2 \rangle$, then there exists a one-dimensional Lie subgroup $H \subset \mathbf{M}(X)$ such that without loss of generality either $\text{Lie}(H) \sim_C \langle t_2 \rangle$ or $\text{Lie}(H) \sim_C \langle t_1 + t_2 \rangle$. 

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Suppose that $\text{Lie}(M(X)) \sim \mathbb{C}\langle s_1, s_2 \rangle$ such that both $\pi_1(M(X))$ and $\pi_2(M(X))$ leave complex conjugate basepoints invariant while acting on $\mathbb{P}^1$. By Corollary 7a we may assume without loss of generality that the real structure of $Y_*$ is defined by $\sigma_2$ in Lemma 16. The induced real structure on $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is as in Lemma 26. It follows from Remark 29 that $\text{Lie}(M(X)) \sim \mathbb{C}\langle is_1, is_2 \rangle$, so that $\pi_i(M(X))$ consists of all automorphisms in $\text{Aut}(\mathbb{P}^1)$ that preserve two complex conjugate basepoints for $1 \leq i \leq 2$. Hence, $M(X) \subseteq \text{PSO}(2) \times \text{PSO}(2)$ and since $\dim M(X) \geq 2$ by assumption this must be an inclusion of connected Lie groups of the same dimension so that $M(X) \cong \text{PSO}(2) \times \text{PSO}(2)$.

We conclude that Lemma 34.1 holds in this case.

Suppose that $\text{Lie}(M(X)) \sim \mathbb{C}\langle s_1, s_2 \rangle$ such that $\pi_2(M(X))$ leaves real points invariant while acting on $\mathbb{P}^1$. Thus there exists a subgroup $H \subset M(X)$ such that $\text{Lie}(H) \sim \mathbb{C}\langle s_2 \rangle$ and the action of $H$ on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves two real fibers $L$ and $L'$ of $\pi_2$ pointwise invariant.

Suppose that $\text{Lie}(H) \sim \mathbb{C}\langle t_2 \rangle$. It follows from Remark 29 that the action of $H$ on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves exactly one fiber $L := \pi_2^{-1}(u)$ pointwise invariant for some point $u \in \mathbb{P}^1$. The number of fibers that are preserved are invariant under complex inner automorphisms and thus this fiber must be real so that $\text{Lie}(H) \sim \mathbb{C}\langle t_2 \rangle$.

Suppose that $\text{Lie}(H) \sim \mathbb{C}\langle t_1 + t_2 \rangle$. Analogously as before we find that $\text{Lie}(H) \sim \mathbb{C}\langle t_1 + t_2 \rangle$, since the action of $H$ on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves $M$ and $L$ invariant, where $M$ and $L$ are real fibers of $\pi_1$ and $\pi_2$, respectively. Moreover, the action leaves the real point $\hat{c} \in L$ invariant such that $\{\hat{c}\} = M \cap L$.

Now suppose by contradiction that Lemma 34.1 does not hold. Notice that we are still in the case where $\Lambda = \emptyset$ as in Table 21a. We showed that there exists a one-dimensional subgroup $H \subset M(X)$ such that either $\text{Lie}(H) \sim \mathbb{C}\langle s_2 \rangle$, $\text{Lie}(H) \sim \mathbb{C}\langle t_2 \rangle$ or $\text{Lie}(H) \sim \mathbb{C}\langle t_1 + t_2 \rangle$. Moreover, the action of $H$ on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves a real fiber $L$ of $\pi_2$ invariant as a whole and leaves a real point $\hat{c} \in L$ invariant. The point $\hat{c}$ corresponds via $\rho$ to a point $c \in X$ such that $c \in X(\mathbb{R}) \subset S^n$. We assume without loss of generality that $c$ is the center of a stereographic projection $\pi : S^n \rightarrow \mathbb{R}^n$. Recall from §1 that $H$ induces a one-dimensional subgroup of the Euclidean similarities of $\mathbb{R}^n$ that leaves $\pi(X(\mathbb{R}))$ invariant as a whole. We call fibers of $\pi_1$ horizontal and fibers of
π_2 \text{ vertical}, since they correspond to horizontal and vertical line segments in Table 21, respectively. A horizontal/vertical fiber that meets \( \hat{c} \) correspond via \( \rho \) and \( \pi \) to a horizontal/vertical line in \( \pi(X(\mathbb{R})) \). The horizontal/vertical fibers that do not meet \( \hat{c} \) correspond to horizontal/vertical circles in \( \pi(X(\mathbb{R})) \). Let \( \mathcal{L} \subset \pi(X(\mathbb{R})) \) be the vertical line corresponding to the vertical fiber \( L \). Thus \( \mathcal{L} \) is the stereographic projection of the set of real points in \( \rho(L) \) and the action of \( H \) on \( \pi(X(\mathbb{R})) \) leaves the line \( \mathcal{L} \) invariant as a whole. The \( H \)-orbits of a general point on \( L \) and a general point on a horizontal fiber corresponds to the \( H \)-orbits of points on \( \mathcal{L} \) and some horizontal circle, respectively. The directions of such orbits in a small neighborhood are illustrated in Figure 3a if \( \text{Lie}(H) \sim \langle s_2 \rangle \) or \( \text{Lie}(H) \sim \langle t_2 \rangle \), and in Figure 3b if \( \text{Lie}(H) \sim \langle t_1 + t_2 \rangle \). Suppose that \( \varphi \) is a general Euclidean similarity in \( H \). Recall that an Euclidean similarity of \( \mathbb{R}^n \) factors as a rotation, translation and/or scaling. Notice that \( \pi(X(\mathbb{R})) \) is not covered by lines and thus the scaling component of \( \varphi \) is trivial. It follows from Figure 3[a,b] that \( \varphi \) has a nontrivial rotational component. We arrived at a contradiction, since the line \( \mathcal{L} \) meets the horizontal circles and thus cannot be left invariant by the action of \( H \). We established that if \( \Lambda = \emptyset \) as in Table 21a, then Lemma 34.1 holds.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure3.png}
\caption{Euclidean similarities acting on a stereographic projection of \( X(\mathbb{R}) \). A dotted arrow depicts the direction of the orbit of the point at the tail.}
\end{figure}

For the next case we suppose that \( \Lambda \) is as in Table 21b or Table 21d. It follows from Proposition 20 that \( M(X) \subseteq \text{PSO}(2) \times \text{PSO}(2) \) and since \( \dim M(X) \geq 2 \) we find as before that \( M(X) \cong \text{PSO}(2) \times \text{PSO}(2) \). We know from Proposition 6 and Table 8[b,e] that \( X \) has real structure \( \sigma_2 \). Hence Lemma 34.1 holds as well for these cases.

If \( \Lambda \) is as in Table 21e or Table 21f, then \( X \subseteq S^3 \) is either the spindle cyclide or the horn cyclide. We showed in Example 33 that \( M(X) \) and \( D(X) \) are as asserted in Lemma 34.2 and Lemma 34.3, respectively. The assertions for \( T(X) \)
and $\mathbf{S}(X)$ follow from Proposition 20. We remark that the fiber corresponding to $L$ as considered for the case $\Lambda = \emptyset$, is in this case contracted to an isolated singularity $c \in X$ so that $\pi(X(\mathbb{R}))$ is covered by lines.

Finally, we suppose by contradiction that $\Lambda$ is as in Table 21c.

Let $L$ be the real vertical fiber of $\pi_2$ spanned by the complex conjugate points $p$ and $\overline{p}$ as depicted in Table 21c. We first show that there exists a subgroup $H \subset \mathbf{M}(X)$ whose action on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves $L$ and the horizontal fibers invariant.

If $\mathrm{Lie}(\mathbf{M}(X)) \sim_{\mathbb{C}} \langle s_1, s_2 \rangle$, then the action of $\pi_2(\mathbf{M}(X))$ on $\mathbb{P}^1$ leaves $\pi_2(p)$ and some other point $r \in \mathbb{P}^1$ invariant. Thus in this case there exists a subgroup $H \subset \mathbf{M}(X)$ whose action on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves the vertical fibers $L' := \pi_2^{-1}(r)$ and $L$ pointwise invariant, and leaves each horizontal fiber invariant as a whole. Now suppose that $\mathrm{Lie}(\mathbf{M}(X)) \sim_{\mathbb{C}} \langle s_1, s_2 \rangle$. It follows from Corollary 30 that there exists a subgroup $H \subset \mathbf{M}(X)$ such that $\mathrm{Lie}(H) \sim_{\mathbb{C}} \langle t_1 + t_2 \rangle$ or $\mathrm{Lie}(H) \sim_{\mathbb{C}} \langle t_i \rangle$ for $i \in \{1, 2\}$. Since automorphisms of $\mathbb{P}^1$ are 3-transitive and $|\pi_1(\Lambda)| = 2$, it follows that $\dim \pi_1(\mathbf{M}(X)) \leq 1$ and $\mathrm{Lie}(H) \sim_{\mathbb{C}} \langle t_1 \rangle$. Since $\dim \mathbf{M}(X) \geq 2$ by assumption, we find that $\dim \pi_2(\mathbf{M}(X)) \geq 1$. Therefore there exists a subgroup $H \subset \mathbf{M}(X)$ such that $\mathrm{Lie}(H) \sim_{\mathbb{C}} \langle t_2 \rangle$. In this case, the action of $H$ on $\mathbb{P}^1 \times \mathbb{P}^1$ leaves $L$ and the horizontal fibers invariant.

Since $|\Lambda \cap L| = 2$ it follows that $\rho(L)$ is an isolated singularity of $X$. We assume without loss of generality that this isolated singularity $c \in X(\mathbb{R})$ is the center of stereographic projection $\pi: S^n \to \mathbb{R}^n$. We use the same notation as before and find that, except for $L$, the horizontal fibers and vertical fibers correspond via $\rho$ to horizontal lines and vertical circles in $\pi(X(\mathbb{R})) \subset \mathbb{R}^n$, respectively. We showed that there exists a subgroup $H \subset \mathbf{M}(X)$ of Euclidean similarities whose action on $\pi(X(\mathbb{R}))$ leaves the horizontal lines invariant and sends vertical circles to vertical circles as in Figure 3c. Thus the orbit of a point in a vertical circle is a horizontal line. If we let the subgroup of scalings or translations act on the spanning plane of a circle contained in $\pi(X(\mathbb{R}))$, then we obtain $\mathbb{R}^3$ so that $X \subset S^3$. We arrived at a contradiction as $T(X)$ is equal to $(2, 6, m)$, where $m > 3$ by Proposition 20.

We concluded the proof, as we considered all cases for $\Lambda$ in Table 21. \qed
Lemma 35. (rotational Möbius automorphism group)

If \( X \subset \mathbb{S}^n \) is a \( \lambda \)-circled celestial surface such that \( \lambda < \infty \) and such that \( M(X) \cong \text{PSO}(2) \times \text{PSO}(2) \), then Theorem 1 holds for \( X \).

Proof. Let \((Y_*, Q_c)\) denote the Möbius pair of \( X \), where \( Q_c \) is a hyperquadric of signature \((1, n+1)\). The existence of this pair follows from Proposition 20 and we denote the corresponding birational linear projection by \( \rho: Y_* \to X \). By Corollary 7, we may assume without loss of generality that the real structure of \( Y_* \) is defined by \( \sigma_2 \) in Lemma 16. We know from Proposition 24 that we may assume up to Möbius equivalence that \( Q_c = V(q) \) for some invariant quadratic form \( q \in I^G_2(Y_*) \), where \( G \) is isomorphic to \( \text{PSO}(2) \times \text{PSO}(2) \). Thus it follows from Lemma 34 and Lemma 31a that

\[
Q_c = \left\{ y \in \mathbb{P}^8 \mid \sum_{i \in \{1,3,5,7\}} c_i \left( y_i^2 - y_i y_{i+1} \right) = 0 \right\},
\]

for some coefficient vector \( c = (c_1 : c_3 : c_5 : c_7) \in \mathbb{P}^3 \). The singular locus of \( Q_c \) is defined by

\[
S(Q_c) = \bigcap_{i \in I} \{ y \in \mathbb{P}^8 \mid y_i = y_{i+1} = 0 \} \text{ with } I := \{ i \in \{1, 3, 5, 7\} \mid c_i \neq 0 \}.
\]

It follows from Lemma 16 that \( \rho \) factors as \( \mu_\ell^{-1} \circ \rho_\ell \) and \( \rho_r \circ \mu_\ell^{-1} \) as in the following commutative diagram

\[
\begin{array}{ccc}
Y_* & \xrightarrow{\mu_\ell^{-1}} & X_2 \\
\rho_\ell \downarrow & & \downarrow \rho_r \\
Z & \xrightarrow{\mu_\ell^{-1}} & X
\end{array}
\]

Thus \( \rho_\ell \) and \( \rho_r \) are birational linear projections so that the real structures of \( Z \) and \( X \) are induced by \( \sigma_2: Y_* \to Y_* \) and \( \sigma_0: X_2 \to X_2 \), respectively. The center of \( \rho_\ell \) coincides with the singular locus of \( Q_c \) by the definition of Möbius pair and thus \( \rho_\ell \) is a toric projection (see Remark 22). The vector space \( I^G_2(Z) \) is generated by the generators of \( I^G_2(Y_*) \) that do not contain an element in \( \{ y_i \}_{i \in I} \cup \{ y_{i+1} \}_{i \in I} \) as a variable. We obtain the lattice type \( L(Z) \) by taking the convex hull of the lattice polygon that is obtained by removing the lattice points of the polygon in Table 9b that are indexed by \( \{ y_i \}_{i \in I} \cup \{ y_{i+1} \}_{i \in I} \) in Table 15.

We first want to determine the possible values for \( T(X), S(X), \dim \mathbb{P}(I^G_2(X)) \) and whether \( M(X) \) is equal to \( \text{Aut}_o(X) \). We make a case distinction on \( I \subset \)
\{1,3,5,7\}. Notice that \(|I| \leq 2\), otherwise the resulting lattice polygon is 1-dimensional.

- If \(I = \emptyset\), then \(T(X) = (2,8,7)\), \(S(X) = \emptyset\), \(\dim \mathbb{P}(I^G_2(X)) = 3\) and \(M(X) \subsetneq \text{Aut}_o(X)\) as a direct consequence of the definitions.

- If \(I \in \{\{1\}, \{3\}\}\), then \(L(Z)\) is as in Table 9b, \(T(X) = (2,8,5)\) and \(M(X) \subsetneq \text{Aut}_o(X)\). Notice that if \(I = \{3\}\), then the surface \(Z\) is projectively isomorphic to the surface obtained with \(I = \{1\}\). If \(I = \{1\}\), then, as discussed before, we omit the generators of \(I^G_2(Y)\) that contain \(y_1\) or \(y_2\) as variable and find that \(I^G_2(Z) = \langle y_0^2 - y_3y_4, y_0^2 - y_5y_6, y_0^2 - y_7y_8 \rangle\) so that \(\dim \mathbb{P}(I^G_2(X)) = 2\). We conclude from the monomial parametrization \(\rho \circ \xi: \mathbb{T}^2 \rightarrow Z\) that \(S(Z) = \emptyset\) and thus \(S(X) = \emptyset\).

- If \(I \in \{\{5\}, \{7\}\}\), then \(T(X) = (3,6,5)\) and \(L(Z)\) is equivalent to Table 8b. It follows from Proposition 20 that \(M(X) = \text{Aut}_o(X)\) and \(S(X) = \emptyset\). As before we verify that \(\dim \mathbb{P}(I^G_2(X)) = 2\).

- If \(I \in \{\{1,5\}, \{1,7\}, \{3,5\}, \{3,7\}, \{5,7\}\}\), then \(T(X) = (4,4,3)\) and \(L(Z)\) is equivalent to Table 8e. It follows from Proposition 20 that \(M(X) = \text{Aut}_o(X)\) and \(S(X) = 4A_1\). We verify that \(\dim \mathbb{P}(I^G_2(X)) = 1\) as before.

- If \(I = \{1,3\}\), then the lattice points corresponding to \(y_0\), \(y_5\), \(y_6\), \(y_7\) and \(y_8\) in Table 15, correspond after the unimodular transformation \((x, y) \mapsto (x - y, y + x)\) to a 2:1 monomial map \(\xi(s^2, t^2)\) such that the lattice type of the monomial parametrization \(\xi(s, t)\) is as in Table 8e. Thus \(L(Z)\) is equivalent to Table 8e and we may assume without loss of generality that \(I = \{5,7\}\) which we already considered.

We verify that \(X\) is in all five cases biregular isomorphic to its linear normalization \(X_N\). We know from Proposition 20 that \(X_N\) is toric and thus \(X\) is toric as well.

It remains to show that \(D(X) = \dim \mathbb{P}(I^G_2(X))\). It follows from Proposition 24a that \((Y*, Q_c)\) and \((Y*, Q_c')\) correspond to Möbius equivalent celestial surfaces if and only if there exists \(\alpha \in \text{Aut}(Y)\) such that \(\alpha(Q_c) = Q_c'\). Let \(\varphi = (\varphi_1, \varphi_2)\) be an indeterminate element of \(\text{Aut}_o(\mathbb{P}^1) \times \text{Aut}_o(\mathbb{P}^1)\). Thus \(\varphi_1\) and \(\varphi_2\) are nonsingular 2×2-matrices in eight indeterminates \(\vec{a} = (a_1, \ldots, a_8)\).
Recall from (1) that there exist a value for $\vec{a}$ such that $\alpha$ is defined by the $9 \times 9$-matrix $S(\varphi)$. We compose, for all $i \in I$, the polynomials $y_0^2 - y_i y_{i+1}$ with the map defined by $S(\varphi)$ so that we obtain quadratic polynomials in $y_i$ and coefficients in $\mathbb{Q}[a_1, \ldots, a_8]$. Since $\alpha(Q_c) = Q_{c'}$, we require that coefficients of monomials, that are not of the form $y_0^2$ or $y_j y_{j+1}$ for some $j > 0$, vanish. We verify with a computer algebra system that the only possible value for $\vec{a}$ such that $\varphi_1$ and $\varphi_2$ have nonzero determinant, is when $\vec{a}$ defines the identity automorphism. Therefore $(Y_\ast, Q_c)$ and $(Y_\ast, Q_{c'})$ are equivalent if and only if $c = c'$. We conclude that $D(X) = \dim \mathbb{P}(I^2_2(X))$ as was left to be shown.

**Proof of Theorem 1.** Suppose that the celestial surface $X \subset \mathbb{S}^n$ is $\lambda$-circled. If $\lambda < \infty$, then Theorem 1 follows from Lemma 34 and Lemma 35. If $\lambda = \infty$, then Theorem 1 follows from [7, Section 1] and [1, Section 2.4.3]; we will give an alternative proof at Theorem D.

**Proof of Corollary 2.** Our goal is as in Remark 25, but with signature $(4, 5)$ or $(3, 6)$ instead of $(1, n + 1)$. Notice that everything in §5 works if we replace $\mathbb{S}^n$ with a hyperquadric $Q$ of different signature. It follows from Lemma 11b that the real structure of $\mathbb{P}^1 \times \mathbb{P}^1$ with real points is either $\sigma_+ \times \sigma_+$ or $\sigma_s$. These real structures are compatible with $\sigma_0: Y_\ast \to Y_\ast$ and $\sigma_3: Y_\ast \to Y_\ast$ in Lemma 16, respectively. This corollary is now a direct consequence of Proposition 24 and Lemma 31d. We remark that if $Q$ has signature $(3, 6)$, then the unique double Segre surface in $Q$ is not covered by real conics.

## 8 The classification of $\mathbb{P}^2$ revisited

If $X \subseteq \mathbb{S}^n$ is $\infty$-circled, then $T(X)$ is either $(\infty, 4, 4)$ or $(\infty, 2, 2)$. We know from [7, Section 1] and [1, Section 2.4.3] that $M(X)$ is either $\text{PSO}(3)$ or $\text{PSO}(3, 1)$. Moreover, $X \subseteq \mathbb{S}^n$ is in both cases unique up to Möbius equivalence. We believe it might be instructive to give an alternative proof by using the methods of §5. We hope that this convinces the reader that our methods have the potential to be used outside the scope of this paper.

Suppose that $Y_\circ \subset \mathbb{P}^5$ is the Veronese surface with lattice type $L(Y_\circ)$ as in Table 8d. Indeed, by Corollary 7b, we may assume without loss of generality that the antiholomorphic involution acting on $Y_\circ$ is complex conjugation. We
proceed analogously as in §3 with the coordinates in Table 15 (right side) so that we obtain the following parametric map

$$\xi_d: \mathbb{T}^2 \to Y_0 \subset \mathbb{P}^5, \quad (s, t) \mapsto (1 : st : s : t : s^2 : t^2) = (y_0 : \ldots : y_5),$$

which extends to $$\tilde{\xi}_d: \mathbb{P}^2 \to Y_0 \subset \mathbb{P}^5, \quad (s : t : u) \mapsto (u^2 : st : su : tu : s^2 : t^2).$$

Since $$Y_0$$ is isomorphic to $$\mathbb{P}^2$$ via $$\tilde{\xi}_d$$, we have $$\text{Aut}_o(Y_0) \cong \text{PSL}(3)$$. Using $$\xi_d$$ we find the following 6 generators for the vector space of quadratic forms on $$Y_0$$ and it follows from Lemma 14 that these form a basis:

$$I_2(Y_0) = \langle y_1y_1 - y_4y_5, \ y_0y_1 - y_2y_3, \ y_2y_2 - y_0y_4, \ y_3y_3 - y_0y_5, \ y_1y_2 - y_3y_4, \ y_1y_3 - y_2y_5 \rangle_\mathbb{C}.$$

Our goal is as in Remark 25 with $$Y_0$$ as $$Y$$. Notice that the real structure of $$Y_0$$ acts as complex conjugation on the Lie algebra $$\mathfrak{sl}_3$$. We consider the following elements in $$\mathfrak{sl}_3$$:

$$a_1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad c_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$b_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad c_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

**Lemma 36. (invariant quadratic forms for $$\mathbb{P}^2$$)**

Suppose that $$G \subseteq \text{Aut}_o(Y_0)$$ is a Lie subgroup, where $$Y_0 \subset \mathbb{P}^5$$ is the Veronese surface. If $$\text{Lie}(G) = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$, then $$G \cong \text{PSO}(3)$$ and

$$I_2^G(Y_0) = \langle x_1^2 + x_2^2 + x_3^2 - x_0x_4 - x_0x_5 - x_4x_5 \rangle_\mathbb{C}.$$

If $$\text{Lie}(G) = \langle a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2 \rangle$$, then $$G \cong \text{PSL}(3)$$ and $$I_2^G(Y_0) = \langle 1 \rangle_\mathbb{C}.$$

**Proof.** The subgroup $$\text{PSO}(3) \subset \text{PSL}(3)$$ is generated by the three $$3 \times 3$$ rotation matrices and thus $$\mathfrak{so}_3 = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$. The generators for the Lie algebra $$\mathfrak{sl}_3$$ can be found for example in [6, Section 6.2]. For the remaining assertions we apply Theorem B as in the proof of Lemma 31. □

**Lemma 37.** If $$(Y_0, Q)$$ and $$(Y_0, Q')$$ are Möbius pairs of celestial surfaces in $$\mathbb{S}^n$$ for $$n \geq 3$$, then these pairs are equivalent.

**Proof.** We consider the following group actions

$$\mathcal{A}: \text{PSL}(3) \times \mathbb{P}^2 \to \mathbb{P}^2 \quad \text{and} \quad \mathcal{B}: \text{PSL}(3) \times Y_0 \to Y_0.$$

As in (1), the group action $$\mathcal{B}$$ is defined via $$\text{Sym}_2(\cdot)$$ and can be computed via the isomorphism $$\tilde{\xi}_d: \mathbb{P}^2 \to Y_0 \subset \mathbb{P}^5$$. These group actions induce group actions
on the spaces of quadratic forms $V := \mathbb{P}(I_2(\mathbb{P}^2))$ and $W := \mathbb{P}(I_2(Y_0))$:

$$\mathcal{A}_* : \text{PSL}(3) \times V \to V \quad \text{and} \quad \mathcal{B}_* : \text{PSL}(3) \times W \to W.$$  

Recall that a quadratic form in $V$ is equivalent via $\mathcal{A}_*$ to a diagonal form of signature $(1, 0)$, $(2, 0)$, $(1, 1)$, $(3, 0)$ or $(2, 1)$. It is left to the reader to verify that $W$ contains quadratic forms of signatures $(1, 2)$, $(1, 3)$, $(1, 5)$, $(2, 2)$ and $(3, 3)$. We can also define a group action $\mathcal{C}_* : \text{PSL}(3) \times W \to W$ via the action $\mathcal{A}_*$ and an isomorphism $V \to W$. Irreducible representations $\text{PSL}(3) \to \text{Aut}(\mathbb{P}^5)$ are isomorphic and thus $\mathcal{B}_*$ and $\mathcal{C}_*$ must be isomorphic group actions. Hence we can match the orbits of $\mathcal{A}_*$ with the orbits of $\mathcal{B}_*$ and thus we identified all possible signatures of quadratic forms in $W$. Thus $Q = V(q)$ and $Q' = V(q')$, where the quadratic forms $q$ and $q'$ in $W$ have both signature $(1, 5)$. The group action $\mathcal{A}_*$, and thus also the group action $\mathcal{B}_*$, acts transitively on quadratic forms of the same signature. It follows that $q' = q \circ \varphi^{-1}$ for some $\varphi \in \text{Aut}_*(Y_0)$. Therefore $\varphi(Q) = Q'$ so that $(Y_0, Q)$ and $(Y_0, Q')$ are equivalent as Möbius pairs. 

The following theorem is a consequence of [7, Theorem 23]. We give an alternative proof by applying the methods in §5.

**Theorem D. [Kollár, 2018]**

*If $X \subset S^n$ is an $\infty$-circled celestial surface, then Theorem 1 holds for $X$.***

**Proof.** If $n \leq 2$, then $T(X) = (\infty, 2, 2)$, $S(X) = \emptyset$, $M(X) \cong \text{PSO}(3, 1)$ and $D(X) = 0$ so that Theorem 1 holds. If $n > 2$, then we know from Theorem Aa that $T(X) = (\infty, 4, 4)$, $S(X) = \emptyset$ and $X$ is projectively equivalent to the Veronese surface $Y_0$. By assumption there exists a subgroup $G \subseteq M(X)$ such that $\dim G \geq 2$. We know from Proposition 24b that $X$ has Möbius pair $(Y_0, V(q))$ for some invariant quadratic form $q \in I_2^G(Y_0)$ of signature $(1, 5)$. It follows from Lemma 36 and Proposition 24c that $G \not\cong \text{PSL}(3)$ and that if $G \cong \text{PSO}(3)$, then $q \circ \varphi = x_1^2 + x_2^2 + x_3^2 - x_0 x_4 - x_0 x_5 - x_4 x_5$ for some $\varphi \in \text{Aut}(Y_0)$. It follows from Lemma 37 and Proposition 24a that $X$ is unique up to Möbius equivalence. Therefore $\text{PSO}(3) \subseteq M(X)$ and $D(X) = 0$. There exists no subgroup $G$ such that $\text{PSO}(3) \not\subseteq G \subseteq \text{PSL}(3)$ and thus we conclude that $M(X) \cong \text{PSO}(3)$. 

\[\square\]
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10 References

[1] J. Berndt, S. Console, and C. E. Olmos. Submanifolds and holonomy. CRC Press, second edition, 2016. ISBN 978-1-4822-4515-8.

[2] D.A. Cox, J.B. Little, and H.K. Schenck. Toric varieties, volume 124. American Mathematical Society, 2011. ISBN 978-0-8218-4819-7.

[3] W. A. de Graaf, J. Pílniková, and J. Schicho. Parametrizing del Pezzo surfaces of degree 8 using Lie algebras. J. Symbolic Comput., 44(1):1–14, 2009.

[4] A. Douglas and J. Repka. The subalgebras of $\mathfrak{so}(4, \mathbb{C})$. Comm. Algebra, 44(12):5269–5286, 2016.

[5] C. Haase and J. Schicho. An inequality for adjoint rational surfaces. 2013. arXiv:1306.3389.

[6] B. Hall. Lie groups, Lie algebras, and representations, volume 222. Springer, 2015. ISBN 978-3-319-13466-6.

[7] J. Kollár. Quadratic solutions of quadratic forms. In Local and global methods in algebraic geometry, volume 712 of Contemp. Math., pages 211–249. Amer. Math. Soc., Providence, RI, 2018.

[8] N. Lubbes. Software library for computing Möbius automorphisms of surfaces, 2019. github.com/niels-lubbes/moebius_aut.

[9] N. Lubbes. Surfaces that are covered by two families of circles. 2020. arXiv:1302.6710.
[10] N. Lubbes and J. Schicho. Lattice polygons and families of curves on rational surfaces. *J. Algebraic Combin.*, 34(2):213–236, 2011.

[11] *Povray*. Persistence of Vision Pty. Ltd., 2004. [www.povray.org](http://www.povray.org).

[12] H. Pottmann, L. Shi, and M. Skopenkov. Darboux cyclides and webs from circles. *Comput. Aided Geom. Design*, 29(1):77–97, 2012.

[13] J-P. Serre. *Topics in Galois theory*. Jones and Bartlett Publishers, 1992. ISBN 0-86720-210-6.

[14] R. Silhol. *Real algebraic surfaces*. Springer, 1989. ISBN 3-540-51563-1.

[15] M. Skopenkov and R. Krasauskas. Surfaces containing two circles through each point. *Math. Ann.*, 2018. doi: 10.1007/s00208-018-1739-z.

[16] The Sage Developers. *SageMath, the Sage Mathematics Software System*, 2018. [www.sagemath.org](http://www.sagemath.org).

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