CONTRACTIBLE EXTREMAL RAYS ON $\overline{M}_{0,n}$.

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§1 Introduction and statement of results

One of the richest objects of study in higher dimensional algebraic geometry is the Mori-Kleiman (closed) cone of curves, $\overline{NE}(M)$, defined as the closed convex cone in $H_2(M, \mathbb{R})$ generated by classes of irreducible curves on $M$. A lot of geometric information about $M$ is encoded in the cone of curves. For example the possibilities for maps with connected fibres are determined by the cone’s faces. Not surprisingly, $\overline{NE}(M)$ is difficult to compute. Even when $M$ is well understood, it can be difficult to find generators for the cone, that is, to find all of the “edges”, or to use the technical term, “extremal rays” (“edge” is potentially misleading as portions of the cone may be curved). Indeed, it is not even generally obvious whether or not a given curve spans an extremal ray. One expects better luck understanding rays on which $-c_1(M) = K_M$ (or more generally log terminal $K_M + \Delta$) are negative, as these are described by the powerful cone and contraction theorems of Mori-Kawamata-Shokurov: each is generated by a smooth rational curve, and can be “contracted”, i.e. there is a map (with domain $M$) whose fibral curves are precisely the curves whose homology class lies on the extremal ray. Thus as a first step in computing the cone, it is natural to consider such “negative” extremal rays, or more generally, rays which can be contracted.

Here we consider $\overline{M}_{0,n}$, the moduli space of stable $n$-pointed rational curves, as well as $\widetilde{M}_{0,n}$ the quotient of $\overline{M}_{0,n}$ by the natural symmetric group action, which is (an irreducible component of) the moduli space of log pairs (see [Alexeev94]).

The locus of points in $\overline{M}_{0,n}$ corresponding to a curve with at least $k+1$ components has pure codimension $k$; we call its irreducible components the vital codimension $k$-cycles. Vital divisors, curves, $k$-cycles etc. are analogously defined. By a vital cycle in $\widetilde{M}_{0,n}$ we mean the image of a vital cycle in $\overline{M}_{0,n}$. It is relatively easy to check that the vital cycles generate the Chow group. It is natural to wonder if much more is true:

1.1 Question. Is every effective cycle linearly equivalent to an effective sum of vital cycles?

This was first posed to us by William Fulton. In the interest of drama, we will refer to (1.1) as Fulton’s conjecture. Here we consider only the cases of curves and divisors. As homological
and linear equivalence are the same on $\overline{M}_{0,n}$, the conjecture in these cases is equivalent to the statement that vital cycles generate all extremal rays of $\overline{NE}_1$ and $\overline{NE}_d$, the cones of curves and divisors. We prove this for $\overline{NE}_1(\tilde{M}_{0,n})$ and for contractible extremal rays of $\overline{NE}_1(\overline{M}_{0,n})$.

Let $D \subset \overline{M}_{0,n}$ be the boundary, i.e. the sum of the vital divisors. Let $D = \sum B_i$ be its decomposition into $S_n$ orbits (there are $\lfloor n/2 \rfloor$ such orbits). For a subvariety $Z \subset \overline{M}_{0,n}$, let $\tilde{Z}$ be its image (with reduced structure) in $\tilde{M}_{0,n}$.

Here are precise statements of our results:

1.2 Theorem. Let $R$ be an extremal ray of the cone of curves $\overline{NE}_1(\overline{M}_{0,n})$. Then $R$ is spanned by a vital curve under any of the following conditions

1. There is a morphism $f : \overline{M}_{0,n} \to Y$, contracting $R$, with $\rho(Y) = \rho(\overline{M}_{0,n}) - 1$, and such that the exceptional locus of $f$ is not a curve.
2. $(K_{\overline{M}_{0,n}} + G) \cdot R < 0$, where $G$ is an effective boundary whose support is contained in $D$.
3. $n \leq 7$.

Of course (1.2.3) says Fulton’s conjecture holds for curves, provided $n \leq 7$. We were able to prove much stronger results for $\tilde{M}_{0,n}$ (especially (1.3.1-2)):

1.3 Theorem.

1. The cone of effective divisors $\overline{NE}_1(\tilde{M}_{0,n})$ is simplicial, generated by the $\tilde{B}_i$.
2. An effective divisor on $\tilde{M}_{0,n}$ fails to be big iff its support is a proper subset of $\tilde{D}$. Any non-trivial nef divisor is big.
3. The cone of curves of $\overline{NE}_1(\tilde{M}_{0,n})$ is generated by curves in $\tilde{D}$.

Now suppose $n \leq 11$.

4. $\overline{NE}_1(\tilde{M}_{0,n})$ is a finite rational polyhedron, with edges spanned by images of vital curves,
5. Every proper face is contractible by a log Mori fibre space. In particular every nef divisor is eventually free.
6. The divisor $\sum_{i=2}^{\lfloor n/2 \rfloor} r_i \tilde{B}_i$ is nef (resp. ample) iff

$$r_{a+b} + r_{a+c} + r_{b+c} - r_a - r_b - r_c - r_d$$

is non-negative (resp. strictly positive), for all positive integers $a$, $b$, $c$ and $d$, with $n = a + b + c + d$ (where we define $r_1 = 0$ and $r_i = r_{n-i}$ for $i > \lfloor n/2 \rfloor$).

(By a simplicial cone we mean a cone over a simplex, i.e. a polyhedral cone whose edges are linearly independent)

The spaces $\overline{M}_{0,n}$ and $\tilde{M}_{0,n}$ are interesting from a number of viewpoints. They are closely related to the moduli space of curves, $\overline{M}_g$. A finite quotient of $\overline{M}_{0,n}$ occurs as a locus of
degenerate curves in the boundary of $\overline{M}_g$, while $\overline{M}_{0,n}$ is the base of the complete Hurwitz scheme (see [HarrisMumford82]) which can be used, for example, to prove that $\overline{M}_g$ is irreducible. By [Kapranov93a], $\overline{M}_{0,n}$ parameterises degenerations of rational normal curves. Generalisations of $\overline{M}_{0,n}$ are important for Quantum Cohomology calculations, see [KonsevichManin94]. $\overline{M}_{0,n}$ is useful for studying fibrations with general fibre $\mathbb{P}^1$, as in particular it can sometimes be used in lieu of a minimal model program. Kawamata exploits this in [Kawamata77] to prove additivity of log Kodaira dimension for one dimensional fibres, and in [Kawamata95] to prove a codimension two subadjunction formula.

We note that there is an explicit construction of $\overline{M}_{0,n}$, as a blow up of $\mathbb{P}^{n-3}$ along a sequence of simple centres (see (3.1)). In particular $\overline{M}_{0,5}$ is a del Pezzo of degree five, $\overline{M}_{0,6}$ is log Fano, and $\overline{M}_{0,7}$ is nearly log Fano, in the sense that $-K_{\overline{M}_{0,7}}$ is effective. We do not know of such an explicit construction of $\widetilde{M}_{0,n}$, and we have in general a much weaker grasp on its geometry (though a much stronger grasp on its cones). Note by (1.3.3), $\widetilde{M}_{0,n}$ admits no nontrivial fibrations. See also (3.7).

Despite the fact that the blowup construction gives an easy computation of some invariants of $\overline{M}_{0,n}$, one cannot expect the same for the cone of curves. For example the blow up of $\mathbb{P}^2$ in eight points has a finite polyhedral cone of curves, but one can choose a ninth point in such a way that the blow up has a cone with infinitely many edges. We do not use the blow up description in any significant way in our proof of (1.2-3).

As we note in (3.5) $-K_{\overline{M}_{0,n}}$ and $-K_{\widetilde{M}_{0,n}}$ are not effective for $n \geq 8$. In view of this, the cases of (1.3) for $8 \leq n \leq 11$ are interesting in that they give examples of non log Fano varieties, for which every face of the cone of curves is none the less contractible. From this perspective, (1.1), if true, would really be rather surprising. Each vital curve (indeed every vital cycle) is smooth and rational. The cone of curves of a log Fano is generated by rational curves, but one does not expect this in general, even for a rational variety. For example, let $S$ be the blow up of $\mathbb{P}^2$ in a large number of general points. As observed by Kollár, and independently by Caporaso and Harris, $K_S$ is strictly negative on rational curves, but of course $K^2_S < 0$, so $K_S$ must be positive on some curves (but see the remark after (2.4)).

If (1.1) holds for curves, then one can describe the ample cones of $\overline{M}_{0,n}$ or $\widetilde{M}_{0,n}$ by a series of inequalities analogous to those in (1.3), using (4.3). One can then describe, at least in theory, the cone of curves, since it is dual to the ample cone. As an example of the complexity of these cones, $NE_1(\overline{M}_{0,7})$ is a polyhedral cone of dimension 42 with 350 edges (see (4.3) and (4.6)).

Fulton’s conjecture implies every vital curve spans an extremal ray and each is $K_{\overline{M}_{0,n}} + G$ negative for some $G$ as in (1.2.2) (see (4.6)). So by the contraction theorem [KMM87] each vital curve is contracted by a map of relative Picard number one. For $n \geq 9$ every vital curve
deforms. So if (1.1) holds, then (1.2) contains all the possibilities for extremal rays, and (1.2.1) has all the possibilities for \( n \geq 9 \).

Here is a brief outline of our proofs of (1.2-3). It turns out each component of \( D \) is a product \( \overline{M}_{0,i} \times \overline{M}_{0,j} \), for \( i, j < n \) (see §3). For (1.2) we proceed by induction, the main work is to show the extremal ray \( R \) is in the subcone generated by curves in \( D \). For this our main tool is (2.2). (1.3) follows from (1.2) and some simple intersection calculations: one set to show that \( NE^1(\overline{M}_{0,n}) \) is simplicial, and a second to show that for \( n \leq 11 \) every face of the cone contracts to a log Mori fibre space.

§2 contains some results about the cone spanned by curves lying in a divisor. Most of the results of §2 are of independent interest (in particular (2.3-2.6)), and hopefully have broader applications. For this reason we work in greater generality. §3 contains the necessary ingredients to apply some of the results of §2 to \( D \subset \overline{M}_{0,n} \). Intersection products of various vital cycles are easy to compute, and the pairing between divisors and curves is described in §4. §5 finishes the proof of (1.3).

We would like to say a few words about other seemingly natural approaches to (1.1). For curves, it is enough (in fact equivalent) to show that if a divisor intersects all vital curves non-negatively, then it is nef. By induction it is sufficient to show that such a divisor is linearly equivalent to an effective sum of vital divisors. As the vital divisors generate the Picard group, the intersection conditions give a finite collection of simple inequalities on the coefficients. Unfortunately the combinatorics are intimidating, and we were not able to make any progress in this direction, even for \( n = 6 \).

Throughout we will use the main results of the minimal model program, the contraction theorem, the cone theorem etc., as well as the established notation as set out in [KMM87]. We also use elementary properties and notions of cones from Chapter II.4 of [Kollár96]. In particular, by an extremal ray \( R \) of a closed convex cone \( W \) we mean a one dimensional subcone with the property that if \( x + y \in R \) for \( x, y \in W \) then \( x, y \in R \). We note that every closed convex cone is the convex hull of its extremal rays. All spaces are assume to be of finite type over \( \mathbb{C} \). Unless otherwise stated, by a divisor we mean an \( \mathbb{R} \)-divisor. In [Shokurov96] the main results of the MMP are extended to \( \mathbb{R} \)-divisors. However we only need one such result (see (2.2)).

\section{The cone spanned by curves inside a divisor}

We first introduce some notation and definitions. Let \( D \) be a reduced Weil divisor inside the projective \( \mathbb{Q} \)-factorial klt variety \( M \) of dimension \( n \). Let \( W \) be the closed subcone of \( NE^1(M) \) generated by curves lying in \( D \).

We are interested in extremal rays that lie outside of \( W \) and moreover under what conditions
$W = \overline{NE}_1(M)$.

2.1 Definitions. We say that $D$ has anti-nef normal bundle if for every curve $C \subset D$, $C \cdot D \leq 0$.

We will say an extremal ray $R$ of $\overline{NE}_1(M)$ is log extremal if there exists a klt divisor $K_M + \Delta$ such that $(K_M + \Delta) \cdot R < 0$.

Log extremal rays are very special: By the cone and contraction Theorems they are spanned by rational curves $C$, and there is a morphism $f : M \rightarrow Y$ contracting $C$ such that $f_*(O_M) = O_Y$ and $\rho(Y) = \rho(M) - 1$.

The following recent result of Shokurov will prove useful:

2.2 Lemma (Shokurov). Let $X$ be a projective variety, and let $L \in N^1(X)$ be a nef class (not necessarily rational) with $L^\dim(X) > 0$. Then $L$ is in the interior of $NE_1^1(X)$.

Proof. This is implied by the proof of (6.17) of [Shokurov96]. □

(2.2) has some interesting corollaries:

2.3 Corollary. If the components of $D$ span $\overline{NE}_1^1(M)$ then $W = \overline{NE}_1(M)$.

Proof. Let $D = \sum D_i$ be the decomposition of $D$ into irreducible components.

Let $A$ be an ample divisor with support in $D$, and let $R \subset \overline{NE}_1(M)$ be an extremal ray. Assume $R \not\in W$. Let $L$ be a nef class supporting $R$. $L|D$ is ample. Since $L$ is an effective sum of $D_i$, $L^\dim M > 0$ thus by (2.2), $R$ cannot be numerically effective. Since the $D_i$ generate $\overline{NE}_1^1(M)$, $R \cdot D_i < 0$ for some $i$. But this implies $R \in W$, a contradiction. □

2.4 Proposition. Let $G$ be an effective $\mathbb{Q}$-divisor, with non-empty support $D$.

Let $R$ be an extremal ray of $\overline{NE}_1^1(M)$, which does not lie in $W$. If $(K_M + G) \cdot R \leq 0$ then $R$ is log extremal and $K_M \cdot R \leq 0$.

In particular, if $-(K_M + G)$ is nef then $\overline{NE}_1(M)$ is spanned by $W$ and log extremal rays $R$, such that $K_M \cdot R \leq 0$.

Proof. Let $R$ be an extremal ray of $\overline{NE}_1(M)$, not lying in $W$. In particular $R \cdot D_i \geq 0$ and so $K_M \cdot R \leq 0$.

On the other hand we are done if $K_M \cdot R < 0$. Thus we may assume $K_M \cdot R = 0$. Let $L \in N^1(M)$ be a nef class supporting $R$. Then $L$ is strictly positive on $W \setminus 0$ and so by compactness of a slice of $W$, $L + \epsilon D$ is nef and supports $R$ for $0 < \epsilon \ll 1$. As $L$ is ample on $D$ $L^{n-1} \cdot D > 0$. In particular we can replace $L$ by $L + \epsilon D$ and assume $L^n > 0$. Then by (2.2) $R \cdot V < 0$ for some effective Weil divisor $V$. But $(K_M + \epsilon V) \cdot R < 0$ and $(K_M + \epsilon V)$ is klt for $0 < \epsilon \ll 1$. □
Remark. (2.4) is interesting even in the case of a surface. For example pick a cubic in $\mathbb{P}^2$ and blow up as many points as you like along the cubic. Let $M$ be the resulting surface and $D$ the strict transform of the cubic. (2.4) then says that $D$ union all the $-2$-curves and $-1$-curves generate the cone of curves of $M$.

2.5 Proposition. Let $f : M \to Y$ be a proper surjection from a smooth projective variety $M$ to a normal variety $Y$ with $f_* (\mathcal{O}_M) = \mathcal{O}_Y$, and $\rho(Y) = \rho(M) - 1$. Suppose $D$ has ample support and each irreducible component of $D$ has anti-nef normal bundle. If $f|D$ is finite then $f$ is birational, and its exceptional locus is a curve.

Proof. Suppose on the contrary that there is an irreducible surface $E$ whose image has dimension at most one.

Let $D = \sum_i D_i$ be the decomposition of $D$ into irreducible components. Note the assumptions on Picard number imply that any class in $N^1(M)$ which is zero on some fibral curve, is pulled back from $N^1(Y)$.

Since $D$ has ample support $I = D \cap E$ is non-empty. As $f|D$ is finite, $I$ and each $D_i \cap E = D_i \cap I$, is an effective $\mathbb{Q}$-Cartier divisor of $E$, and in particular, is purely one dimensional. Thus if $I$ meets $D_i$, it has an irreducible component contained in $D_i$. Since $D$ has ample support, and $f|D$ is finite, $E$ contracts to an irreducible curve $C \subset f(D) = f(I)$ and $f|I$ is finite.

Claim. We can find two irreducible components $B_1, B_2$ of $I$ and (after renaming) two divisors $D_1, D_2$ with $B_i \subset D_i$ such that $B_i \cdot D_j \geq 0$ (for $i \neq j$) and at least one inequality is strict:

Choose an irreducible component $B_1$ of $I$ contained in a maximal number of $D_i$. Suppose (after reordering) $D_1, D_2, \ldots, D_k$ are the components of $D$ containing $B_1$. Since the $D_i$ have anti-nef normal bundles, and $D$ has ample support, for some $j > k$ we have $D_j \cdot B_1 > 0$. Let $B_2$ be an irreducible component of $D_j \cap I$. By the choice of $B_1$ we can assume (after reordering) that $B_2 \not\subset D_1$. Now set $D_2 = D_j$.

This establishes the claim.

Since $D_1, D_2$ each meet a fibre, we can choose $\lambda > 0$ such that $D_1 - \lambda D_2$ is pulled back from $Y$. Let $J = (D_1 - \lambda D_2)|E$. Then $J \cdot B_1 \leq 0$ and $J \cdot B_2 \geq 0$, and one inequality is strict. Since $J$ is pulled back from $C$, and the $B_i$ are multi-sections, this is a contradiction. □

Remarks.

(1) The assumption on the relative Picard number in (2.5) is necessary; it cannot be replaced by the weaker assumption that $f$ is the contraction of an extremal ray. For example consider $M = E \times E$ for an elliptic curve $E$, $D = F_1 + F_2$ the sum of the two fibres and $f : M \to E$ the addition map.

(2) The assumption on Picard number holds when $f$ is the contraction of a log extremal ray.
(3) One can not rule out the final possibility. For example: Let $M$ be a del Pezzo surface whose cone of curves is not a simplex (e.g. blow up $\mathbb{P}^2$ at three non-collinear points). Let $D$ be a sum of $\rho(M) - 1$-curves with ample support (any effective class is a sum of at most $\rho(M)$ extremal rays, and all the extremal rays are $-1$-curves). Let $f$ blow down some other $-1$-curve.

2.6 Lemma. Suppose $M$ is smooth of dimension at least three, every component of $D$ has anti-nef normal bundle, and $D$ has ample support. Let $G$ be a nonempty effective $\mathbb{Q}$-divisor whose support lies in $D$.

(1) Let $R$ be an extremal ray of $NE_1(M)$. If either $(K_M + G) \cdot R < 0$, or the dimension of $M$ is at least four and $(K_M + G) \cdot R \leq 0$ then $R \in W$.

(2) If $-(K_M + G)$ is nef, and either the support of $G$ is exactly $D$ or the dimension of $M$ is at least four, then $W = NE_1(M)$.

Proof. Let $R$ be an extremal ray of $NE_1(M)$, and suppose $R \notin W$ but $(K_M + G) \cdot R \leq 0$. Then by (2.4) we know that $R$ is spanned by a contractible rational curve $C$. (1) and (2) now follow easily from (2.5) and the observation that if $K_M \cdot C < 0$ and $M$ is a threefold (resp. $K_M \cdot C \leq 0$ and $M$ has dimension at least four) then $C$ deforms inside $M$ (see II.1.13 of [Kollár96]). □

We will use the following technical result in the next section.

2.7 Lemma. Let $N \subset M$ be a normal divisor and suppose that $N^1(M) \to N^1(N)$ is surjective. Let $f : M \to Y$ be a map to a normal projective variety with $f_*(\mathcal{O}_M) = \mathcal{O}_Y$, and $\rho(Y) = \rho(M) - 1$. Let $g : N \to Z$ be the Stein factorisation of $f|_N$. If $f|_N$ is not finite, then $\rho(Z) = \rho(N) - 1$.

Proof. $f$ contracts an extremal ray $R$. Suppose $f|_N$ is not finite. Then $R \in N_1(N)$. If $L \in N^1$ and $L \cdot R = 0$, then $L|N$ is pulled back from $Z$. Since every class in $N^1(D)$ extends to $M$, the result follows. □

§3 Geometry of $\overline{M}_{0,n}$ and $\tilde{M}_{0,n}$.

We will use (a slight modification of) the notation of, as well as several simple facts from pg. 551–554 of [Keel92]. For the readers convenience we will recall the most important ideas:

A vital divisor is determined by a partition of \{1, 2, \ldots, n\} into disjoint subsets $T$, $T^c$, each containing at least two elements. The generic point of the corresponding vital divisor $D_{T,T^c}$ is a curve with two irreducible components, with the labels of $T$ on one component, and the labels of $T^c$ on the other. There is a canonical isomorphism

$$D_{T,T^c} = M_{T \cup \{b\}} \times M_{T^c \cup \{b\}}$$
where e.g. by $M_{T \cup \{b\}}$ we mean a copy of $\overline{M}_{0,|T|+1}$ with the indices labeled by the elements of $T$, with $b$ an extra index, corresponding to the singular point. We indicate the two projections by $\pi_T$ and $\pi_T^c$.

The vital divisors have normal crossings, and each vital codimension $k$-cycle is uniquely expressible as a complete intersection of vital divisors. Each vital $k$-cycle has an expression as a product of $\overline{M}_{0,i}$ analogous to that for the vital divisors. In particular, under the above decomposition, any vital curve of $D_{T,T^c}$ is a product of a vital curve on one factor, with a vital point on the second.

3.1 Proposition (Kapranov). For each index $i \in \{1, 2, \ldots, n\}$ there is a birational map $q_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ with the following properties:

1. $q_i$ is a composition of blow ups along smooth centres, constructed as follows. Fix $n - 1$ general points, and blow up successively (from lowest to highest dimensional) the (strict transforms) of every linear subspace spanned by any subset of these points.

2. $q_i$ takes vital cycles to to linear spaces spanned by the chosen points.

3. If $i \in T$ then $q_i |_{D_{T,T^c}} = q_i \circ \pi_T$ for $i \in T$.

4. If $F$ is the general fibre of the map $\overline{M}_{0,n} \to \overline{M}_{0,n-1}$ given by dropping the $i^{th}$ point, then $q_i(F)$ is a rational normal curve.

5. $q_i$ is a composition of smooth blow downs, blowing down iteratively the (images of) the divisors $D_{T,T^c}$ with $i \notin T$, and $|T| = 3, 4, \ldots, n-2$.

Proof. See [Kapranov93b]. □

3.2 Lemma. Let $\phi$ be an element of $\text{Aut}(\mathbb{P}^1)$ of finite order $p$ and let $Z$ be the closure of the locus of $n$-tuples of distinct elements of $\mathbb{P}^1$ whose coordinates are permuted by $\phi$. Let $q$ be a general element of $Z$. If $\phi$ fixes $j$ coordinates of $q$ then the dimension of $Z$ at $q$ is $(n - j)/p$.

Proof. Let $G \subset \text{Aut}(\mathbb{P}^1)$ be the subgroup generated by $\phi$. Then $G$ has a non-trivial finite orbit, from which it follows that $G$ has exactly two fixed points, and after changing coordinates (so the fixed points are 0 and $\infty$) $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ is multiplication by a $p^{th}$ root of unity. The coordinates divide into orbits, each of which is either a fixed point, or has exactly $p$ elements. The result follows. □

3.3 Lemma. $S_n$ acts freely in codimension one on $M_n \setminus D$ for $n \geq 7$, and faithfully for $n \geq 5$.

The action of $S_4$ on $M_4$ factors through the action on the set of partitions of $\{1, 2, 3, 4\}$ into disjoint subsets of two elements. Nontrivial elements of the kernel are of form $(ij)(kl)$ for $i, j, k, l$ distinct.

Proof. The claims about the $S_4$ action are easily checked, and left to the reader.
So assume \( n \geq 5 \). We bound the dimension of the locus of points in \( M_n \setminus D \) which are fixed by some element of \( S_n \). Equivalently, we bound the dimension of the locus \( Z \subset M_n \setminus D \) of \( n \)-tuples (modulo automorphisms) whose coordinates are permuted by some automorphism of \( \mathbb{P}^1 \).

Let \( \mathcal{U} \subset \mathbb{P}^1^{\times n} \) be the locus of distinct points, and let

\[ \mathcal{T} = \{(q, \phi, a, b) | q \in \mathcal{U}, \phi \in \text{Aut}(\mathbb{P}^1)a \neq b \in \mathbb{P}^1 \text{ s.t. } \phi \text{ permutes } q \text{ and fixes } (a, b)\} . \]

Replace \( \mathcal{T} \) by any one of its irreducible components.

Let \( \phi \) be a general point of the image of \( pr : \mathcal{T} \to \text{Aut}(\mathbb{P}^1) \), and let \( q \in pr^{-1}(\phi) \). By (3.2) \( pr^{-1}(\phi) \) has dimension \( (n - j)/p \) at \( q \), while the fibre of \( \mathcal{T} \to U \) has dimension three. Thus at the image of \( q \), \( U \) has dimension \( (n - j)/p - 1 \). The result follows. □

Let \( B_i = \sum_{|T|=i} D_{T,T^c} \) for \( 2 \leq i \leq k = \left\lfloor \frac{n}{2} \right\rfloor \). \( B_i \) is the orbit under \( S_n \) of any \( D_{T,T^c} \) with \( |T| = i \).

3.4 Lemma. For \( n \geq 7 \) the quotient map \( q : \overline{M}_{0,n} \to \tilde{M}_{0,n} \) is unramified in codimension one outside of \( B_2 \), and has ramification index two along \( B_2 \).

Proof. Suppose \( \sigma \in S_n \) fixes each point of the irreducible divisor \( G \subset \overline{M}_{0,n} \). By (3.3), \( G = D_{T,T^c} \) for some \( T \) preserved by \( \sigma \). Since the action of \( \sigma \) on \( M_{T \cup \{b\}} \) factors through the subgroup of \( S_{|T|+1} \) which fixes \( b \), it follows from (3.3) that \( T = \{i, j\} \) and \( \sigma = (i, j) \). □

We will use the following formulae, essentially from [Pandhapripande95]:

3.5 Lemma.

\[
K_{\overline{M}_{0,n}} + \sum_{j=2}^{k} \left(2 - \frac{j(n-j)}{n-1}\right) B_j = 0 = K_{\tilde{M}_{0,n}} + \left(\frac{1}{2} + \frac{1}{n-1}\right) \tilde{B}_2 + \sum_{j=3}^{k} \left(2 - \frac{j(n-j)}{n-1}\right) \tilde{B}_j.
\]

In particular \(-K_{\overline{M}_{0,n}}\) (resp. \(-K_{\tilde{M}_{0,n}}\)) is pseudo-effective iff \( n \leq 7 \).

Proof. The first formula is Proposition 1 of [Pandhapripande95], the second follows easily from the first and (3.4) and the last statement then follows from (4.8). In fact we may use (4.3) to prove the first formula in a similar way to the way it is derived in [Pandhapripande95].

However it is possible to prove the first formula in an entirely elementary way, using (3.1). Indeed the image \( D' \) of \( D \) is the union of \( \binom{n-1}{2} \) hyperplanes, and the coefficients of \( B_i \) are easily identified as the discrepancies of the divisor \( K_{\mathbb{P}^{n-3}} + (2/(n-1))D' \). □
3.6 Lemma. \( K_{\tilde{M}_{0,n}} + D \) is ample and is linearly equivalent to an effective divisor with the same support as \( D \).

Proof. We proceed by induction on \( n \). The result is easy for \( n = 4 \).

By (3.5), \( K_{\tilde{M}_{0,n}} + D \) is linearly equivalent to an effective divisor \( \Gamma \) with support \( D \).

Note that \( (K_{\tilde{M}_{0,n}} + D)|D_T \) is the tensor product of the “same expressions” pulled back from the two components in the product description of \( D_T \). Thus by induction \( (K_{\tilde{M}_{0,n}} + D)|D \) is ample.

It is easy to see that \( D \) meets (set theoretically) every curve. Use induction and consider the map \( f : \tilde{M}_{0,n} \rightarrow \tilde{M}_{0,n-1} \), observe that \( D \) meets every fibral curve, and note that \( D \supset f^{-1}(D(\tilde{M}_{0,n-1})) \).

Thus \( (K_{\tilde{M}_{0,n}} + D) \cdot C > 0 \) for all curves \( C \).

It follows that \( K_{\tilde{M}_{0,n}} + D \) is nef, and nef and big by induction. Thus by the base point free theorem (applied to the big and nef klt divisor \( K_{\tilde{M}_{0,n}} + D - \varepsilon \Gamma \)) \( m(K_{\tilde{M}_{0,n}} + D) \) is basepoint free for \( m \gg 0 \). Since it intersects every curve positively, it is thus ample. \( \square \)

The results above have some interesting geometric consequences:

3.7 Remarks.

(1) By (3.1.1) \( \tilde{M}_{0,5} \) is isomorphic to \( \mathbb{P}^2 \) blown up at four points. Thus it is a del Pezzo surface of degree five. It is interesting to note that \( K_{\tilde{M}_{0,5}} + D = -K_{\tilde{M}_{0,5}} \) is very ample and defines the anticanonical embedding of \( \tilde{M}_{0,5} \) inside \( \mathbb{P}^5 \). For any \( n \), if \( C \) is a vital curve, then \( (K_{\tilde{M}_{0,n}} + D) \cdot C = 1 \). Thus it would be nice to know if \( K_{\tilde{M}_{0,n}} + D \) is very ample, for if it were, then under the corresponding map, every vital curve would be embedded as a line.

(2) Note that \( \tilde{M}_{0,5} \) is a log del Pezzo of rank one. It is easy to compute, using (3.2) that \( \tilde{M}_{0,5} \) has two quotient singularities, one of index two and the other of index five. It is then easy, from the classification of log del Pezzos, (see [KeelMcKernan95]) to conclude that \( \tilde{M}_{0,5} \) has one \( A_1 \)-singularity and one singularity of type \((2,3)\).

(3) Note that the map \( D_{T,T^c} \rightarrow \tilde{M}_{0,n} \) factors through \( M|_{T_{+1}} \times S|_{T_{+1}} \times S|_{T^c_{+1}} \), but not through the quotient by \( S|_{T_{+1}} \times S|_{T^c_{+1}} \). Thus an inductive study of \( N_{E_1}(\tilde{M}_{0,n}) \) is problematic. In particular one cannot obtain the analog of (3.9) as below.

(4) By (3.1), \( q_i^*(\mathcal{O}(1)) \) is numerically equivalent to an effective divisor with support exactly \( D \). It follows by (3.6) that for any curve \( C \subset D \) there is some vital divisor which is negative on \( C \).

3.8 Lemma. For any projective variety \( T \), \( N_1(\tilde{M}_{0,n} \times T) = N_1(\tilde{M}_{0,n}) \times N_1(T) \) under the map.
induced by the two projections. The same map induces an isomorphism

\[ \text{NE}_1(\overline{M}_{0,n} \times T) = \text{NE}_1(\overline{M}_{0,n}) \times \text{NE}_1(T) \]

Proof. This follows from Theorem 2 of [Keel92]. □

3.9 Corollary. Fulton's conjecture for \( \text{NE}_1(\overline{M}_{0,n}) \) implies the conjecture for \( \text{NE}_1(\overline{M}_{0,n}) \).

Proof. Immediate from (2.2) and (3.8). □

Proof of (1.2). As we are going to use induction it is actually more convenient to prove a slightly stronger result. Let \( M \) be any product of \( \overline{M}_{0,4} \). We will prove (1.2) for \( M \). By a vital cycle on \( M \) we mean a product of vital cycles on each component. We will continue to use the same notation, so for example by \( D_T \) we mean the inverse image of this divisor from a projection onto one of the components of \( M \).

Let \( m \) be the dimension of \( M \). When \( m \leq 2 \) it is easy to check that vital curves generate the cone (see (3.7.1)).

\( D \) has ample support by (3.6), and each \( D_{T,T_c} \) has anti-nef normal bundle by (4.5).

When \( m \leq 4 \) then we can apply (3.5), and (2.6) inductively, to prove (1.2.3).

Let \( R \) be an extremal ray satisfying either (1.2.1) or (1.2.2). We show \( R \) is spanned by a vital curve by induction on \( m \) which we may assume is at least 5.

Note \( M \) retracts onto any vital cycle, thus if \( Z \) is any vital cycle, the restriction \( N_1(M) \to N_1(Z) \) is surjective. In particular if \( R \) is in the image of (the injection) \( i : \text{NE}_1(Z) \to \text{NE}_1(M) \), then \( R \) spans an extremal ray on \( Z \).

Assume \( R \) satisfies (1.2.2). By (2.6), \( R \) spans an extremal ray on some \( D_T \). Since \( D_T \) has anti-nef normal bundle, we can increase its coefficient in \( G \) to one, restrict to \( D_T \), and apply adjunction and induction.

Assume \( R \) satisfies (1.2.1). By (2.5), \( R \) is spanned by a curve \( C \subset D_T \). By (3.7.4) we may assume \( C \cdot D_T < 0 \), so \( C \) does not deform away from \( D_T \). By (2.7), \( C \subset D_T \) is contracted by a map of relative Picard number one, and so we can apply induction. □

§4 Intersecting vital curves and divisors.

By a marked point of an \( n \)-pointed curve, we either mean one of the singular points of the curve, or one of the labeled points \( p_1, p_2, \ldots, p_n \).

4.1 Notation: Let \( C \) be a vital curve. Let \( G = G(C) \) be the \( n \)-pointed stable curve corresponding to the generic point of \( C \). \( G \) has \( n - 3 \) components, all but one of which contain \( 3 \) marked points, and exactly one of which contains \( 4 \) marked points. We call this last component
\[ Q = Q(C), \text{ the distinguished component of } G. \] Let \( s(C) \) be the number of singular points on \( Q \), \( l(C) \) be the number of labeled points. \( C \) determines a decomposition of \{1, 2, \ldots, n\} into 4 disjoint subsets: \( G \setminus Q \) has exactly \( s(C) \) connected components. We decompose \{1, 2, \ldots, n\} into those labeled points on each of the components. Additionally we take the singleton sets for each of the \( l(C) \) labeled points on \( G \). We call this decomposition \( P_C \).

There are \( n - 4 \) singular points on \( G \) (intersection points of two components). Each singular \( p \in G \) defines a decomposition, by letting \( T_p \) and \( T_p^c \) be the labels on the two connected components of \( G \setminus \{p\} \). \( C \) is the complete intersection \( \bigcap_{p \in \text{Sing}(G)} D_{T_p, T_p^c} \). Let \( A_{T_p} \) and \( A_{T_p^c} \) be the connected components of \( G \setminus \{p\} \).

4.2 Lemma. Let \( C \) be a vital curve.

1. For \( p \in \text{Sing}(G) \)
   \[ \pi_{T_p} : D_{T_p, T_p^c} \longrightarrow M_{T_p \cup \{b\}} \]
   contracts the vital curve \( C \) iff \( A_{T_p^c} \) contains the generic point of \( Q(C) \).
2. \( q_i \) contracts \( C \) iff \( i \) is not one of the labeled points of \( Q(C) \) (in particular any \( C \) with \( l(C) = 0 \) is contracted).

Proof. (1) is immediate and (2) follows from (1) and (3.1.3). \( \square \)

4.3 Lemma. \( P_C \) uniquely determines the numerical class of \( C \). \( K_{\overline{M}_{0,n}} \cdot C = 2 - l(C) \). For any vital divisor \( D_{T,T^c} \) we have:

1. \( D_{T,T^c} \cdot C = -1 \) iff \( T \) or \( T^c \) is one of the equivalence classes of \( P_C \). Equivalently, iff \( T \) or \( T^c \) is \( T_p \) for some singular point \( p \in Q \).
2. \( D_{T,T^c} \cdot C = 1 \) iff \( T \) or \( T^c \) is the union of two equivalence classes.
3. Otherwise \( D_{T,T^c} \cdot C = 0 \).

Proof. Since the vital divisors generate \( \text{Pic}(\overline{M}_{0,n}) \) the description of \( D_{T,T^c} \cdot C \) implies the first statement. The expression for \( K_{\overline{M}_{0,n}} \cdot C \) follows from the expression for \( D_{T,T^c} \cdot C \) using the adjunction formula, since \( C \) is a complete intersection of vital divisors.

Fix \( p \in \text{Sing}(G) \) and let \( S \) be the intersection of the \( D_{T_q, T_q^c} \) for \( q \neq p \). Then \( C \cdot D_{T_p, T_p^c} \) is the self intersection of \( C \) in \( S \). \( S \) is a vital surface, and so it is either \( \overline{M}_{0,5} \) (which is \( \mathbb{P}^2 \) blown up in 4 points) or \( \overline{M}_{0,4} \times \overline{M}_{0,4} \) (which is \( \mathbb{P}^1 \times \mathbb{P}^1 \)), and \( C \) is a vital curve in \( S \). In the first case \( C \) is a \(-1\)-curve, and in the second a fibre of one of the two projections. Let \( \gamma \) be the pointed stable curve corresponding to a generic point of \( S \). In the first case \( \gamma \) has one component with 5 marked points, and in the second case, two components each with 4 marked points. \( G \) is obtained as the limit as two of the marked points (on the same component) come together at
p. It’s clear that the first case occurs iff \( p \in Q \), whence (1). Note the argument shows that if \( C \subset D_{T,T^c} \) then \( C \cdot D_{T,T^c} \) is either 0 or 1.

If \( D_{T,T^c} \cdot C > 0 \) then \( D_{T,T^c} \cap C \) is a vital point of \( C = \overline{M}_{0,4} \), i.e. a reduced point, thus \( D_{T,T^c} \cdot C = 1 \). This occurs if \( T \) or \( T^c \) is a union of two equivalence classes of \( P_C \) and every vital divisor of \( C \) can be obtained in this way. Since each vital cycle is uniquely a complete intersection of vital divisors, this gives (2).

Since the possibilities with \( C \cdot D_{T,T^c} \) nonzero are classified by (1) and (2), (3) follows.

\[4.4\text{ Corollary.} \text{ The numerical class of } \tilde{C} \text{ is determined by the cardinalities of the subsets in } P_C. \text{ If these cardinalities are } a, b, c, d \text{ then} \]

\[C \cdot \sum r_i B_i = -r_a - r_b - r_c - r_d + r_{a+b} + r_{a+c} + r_{a+d}\]

where we define \( r_1 = 0 \) and \( r_i = r_{n-i} \) for \( i > [n/2] \).

\[4.5\text{ Lemma.} \]

\[N_{D_{T,T^c}} \overline{M}_{0,n} = (q_b \circ \pi_T)^* (\mathcal{O}(-1)) \otimes (q_b \circ \pi_{T^e})^* (\mathcal{O}(-1)).\]

**Proof.** Since the vital curves generate \( N_1 \) we only need to check how both sides intersect a vital curve \( C \subset D_{T,T^c} \). By (3.1.2), and (4.3) the possible values of these intersections are 0 and \(-1\), and it is enough to show show \( D_{T,T^c} \cdot C = -1 \) iff one of the two maps \( q_b \circ \pi_T \) or \( q_b \circ \pi_{T^e} \) fails to contract \( C \).

By (4.2.1) we may assume that \( \pi_T \) is finite on \( C \) (otherwise switch \( T \) and \( T^e \)). By (4.2.1) and (4.3.1), \( D_{T,T^c} \cdot C = -1 \) iff \( b \) is a labeled point of \( Q(\pi_T(C)) \), thus by (4.2.2), iff \( q_b \) is finite on \( \pi_T(C) \). \( \square \)

\[4.6\text{ Lemma.} \text{ Let } C \text{ be a vital curve, and let} \]

\[D_C = \sum_{p \in \text{Sing}(G) \cap Q} D_{T_p,T^e_p}.\]

\[(K_{\overline{M}_{0,n} + D_C}) \cdot C = -2. \text{ } K_{\overline{M}_{0,n} + D + 1/sD_C} \text{ intersects vital curves non-negatively, and vanishes on exactly those vital curves numerically equivalent to } C.\]

**Proof.** Immediate from (4.3). \( \square \)

\[4.6.1\text{ Remark.} \text{ (1.1) and the basepoint free theorem imply } K + D + 1/sD_C \text{ is eventually free, and thus } C \text{ spans an extremal ray. Presumably this could be checked directly.} \]

The following is immediate:
4.7 Lemma. Let $T \subset \{1, 2, \ldots, n\}$ with $|T| \geq 3, |T^c| \geq 2$. For $i \in T$,

$$
D_{T\setminus\{i\}, T^c\cup\{i\}}|_{D_{T,T^c}} = D_{ib,T\setminus\{i\}} \times M_{T^c\cup\{b\}}
$$

under the canonical product decomposition. There is no other vital divisor with the same restriction.

4.8 Lemma. Suppose there is a numerical equality

$$
\sum_{i=2}^{k} m_i B_i \sim F
$$

and either $F$ is nef, or both sides are effective and have no divisor common to their supports. Then

$$
rm_{r-1} \geq (r - 2)m_r \quad \text{for} \quad 3 \leq r \leq k \\
(n - r)m_{r+1} \geq (n - r - 2)m_r \quad \text{for} \quad 2 \leq r \leq k - 1.
$$

When the left hand side is effective, it is either trivial, or has support exactly $D$. (1.3.1-4) hold.

Proof. We prove the first inequality, the argument for the second is analogous. The final remarks follow from the inequalities.

Choose $T$ with $|T| = r$. Let $Z_r$ be the general fibre of

$$
M_{T\cup\{b\}} \longrightarrow M_T.
$$

Let $p \in M_{T^c\cup\{b\}}$ be a general point, and let $D_{T,T^c} \supset C_r = Z_r \times \{p\}$. By (3.1), (4.5) and (4.7) we have

$$
C_r \cdot B_i = \begin{cases} 
  r & \text{if } i = r - 1 \\
  -(r - 2) & \text{if } i = r \\
  0 & \text{otherwise}
\end{cases}
$$

The inequality is obtained by intersecting both sides with $C_r$.

Note that (1.3.1-3) follow immediately and that (1.3.4) then follows from (2.3). \qed

§5 $NE_1(\widetilde{M}_{0,n})$ FOR SMALL $n$

Given (4.8), it is natural to hope that every nef divisor on $\widetilde{M}_{0,n}$ is eventually free. The obvious approach is to try to use the basepoint free theorem, and thus to realise some positive multiple of a big nef class $E$ (pulled back from $\widetilde{M}_{0,n}$) as a klt divisor $K_{\widetilde{M}_{0,n}} + \Delta$. 
5.1 Lemma. If $E$ is a big nef class on a normal $\mathbb{Q}$-factorial variety $M$, and there is a divisor $\Delta$ with $K_M + \Delta$ klt and numerically equivalent to a positive multiple of $E$, then the extremal subcone of $NE_1(M)$ supported by $E$ is rational polyhedral, and is contracted by a log Mori fibre space. If $M = \overline{M}_{0,n}$, the subcone supported by $E$ is spanned by vital curves.

Proof. By (2.2) we have $E = A + Z$ where $A$ is ample and $Z$ is effective. If $V \subset NE_1(M)$ is the extremal subcone supported by $E$, then $K_M + \Delta + \epsilon Z$ is negative on $V \setminus 0$. Thus the result follows from the cone and contraction theorems, together with (1.2) \[\square\]

Let $E$ be a nef divisor on $\overline{M}_{0,n}$, pulled back from $\tilde{M}_{0,n}$. By (3.5), for $n \leq 7$ the conditions of (5.1) are satisfied (if $K + \Gamma$ is klt and trivial, let $\Delta = \Gamma + \epsilon E$).

In general, by (3.5) and (4.8), replacing $E$ by a large multiple one has $E = K_{\overline{M}_{0,n}} + \Delta$ for some $\Delta$ supported on $D$. We can try to make $\Delta$ a boundary by subtracting off part of $E$, thus we are lead to consider:

5.2 Definition-Lemma. Let $E$ be a non-trivial nef class on $\overline{M}_{0,n}$, pulled back from $\tilde{M}_{0,n}$ with $n \geq 8$. Then there is a unique effective class $\Delta_E$ with the following properties

1. $\Delta_E$ has support a proper subset of $D$
2. $K_{\overline{M}_{0,n}} + \Delta_E = \lambda E$ for some $\lambda > 0$.

Proof. For any $\lambda$, $-K_{\overline{M}_{0,n}} + \lambda E$ is pulled back from $\tilde{M}_{0,n}$, thus by (4.8), (1) is the requirement that $\Delta_E$ be on the boundary of $NE_1$. Since $E$ is in the interior of $NE_1$, and by (3.5), $-K_{\overline{M}_{0,n}} \not\in NE_1$, the result is clear. \[\square\]

Notation: For the next corollary, define the integer function $f(a, b, c, d)$ to be $2$ minus the number of variables equal to one.

We will say that $P_n$ holds if for a given integer $n$ the following implication holds:

Let $r_1, r_2, \ldots, r_{n-1}$ be a collection of non-negative real numbers, with $r_1 = 0$, $r_i = r_{n-i}$, and $r_j = 0$ for some $2 \leq j \leq k$. If

$$f(a, b, c, d) + r_{a+b} + r_{a+c} + r_{a+d} \geq r_a + r_b + r_c + r_d$$

for every set of positive integers $a, b, c, d$ with $n = a + b + c + d$, then $r_i < 1$ for all $i$.

5.3 Corollary. $\Delta_E$ is a pure boundary for every non-trivial nef class pulled back from $\tilde{M}_{0,n}$ iff $P_n$ holds.

Proof. By (4.8) $P_n$ is equivalent to the statement: If $\sum r_i B_i$ has support a proper subset of $D$ and $(K_{\overline{M}_{0,n}} + \sum r_i B_i) \cdot C \geq 0$ for all vital curves $C$, then $\sum r_i B_i$ is a pure boundary. Thus
the only thing to show is that if $\Delta_E$ is a pure boundary for every non-trivial nef class, then the images of vital curves generate $NE_1(\overline{M}_{0,n})$. This follows from (5.1). □

For a given $n$ it is straightforward to check whether or not $P_n$ holds:

**5.4 Lemma.** $P_n$ holds for $8 \leq n \leq 11$.

*Proof.* We will check $P_9$. The cases $n = 8, 10$ and $11$ are similarly checked.

Let $r_1, r_2, \ldots, r_8$ be a collection of non negative numbers, as in the definition of $P_9$. From the sums

\[
1 + 2 + 3 + 3 = 9 \\
1 + 1 + 1 + 6 = 9 \\
1 + 2 + 2 + 5 = 9
\]

we obtain the inequalities

\[
1 + r_4 \geq r_3 + r_2 \\
2r_3 \geq r_4 \\
3r_2 \geq r_3 + 1
\]

The result follows easily by considering in turn the possibilities $r_4 = 0, r_3 = 0$ and $r_2 = 0$. □

Observe that (1.3) follows from (5.1), (5.3) and (5.4).

We have checked that $P_n$ holds for several $n \geq 12$, and we suspect that (if motivated) one could prove this for all such $n$.

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