O(D, D)-constraint on D-dimensional effective actions

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Abstract

Double Field Theory is a manifestly T-duality invariant formulation of string theory in which the effective theory at any order of α' is invariant under global O(D, D) transformations and ought to be invariant under gauge transformations which receive α'-corrections. On the other hand, the effective theory in the usual D-dimensional formulation of string theory is manifestly gauge invariant and ought to be invariant under T-duality transformations which receive α'-corrections. We speculate that the combination of these two constraints may fix both the 2D-dimensional and the D-dimensional effective actions without knowledge of the α'-corrections of the gauge and the T-duality transformations.

In this paper, using generalized fluxes, we construct arbitrary O(D, D)-invariant actions at orders α'^0 and α', and then dimensionally reduce them to the D-dimensional spacetime. On the other hand, at these orders, we construct arbitrary covariant D-dimensional actions. Constraining the two D-dimensional actions to be equal up to non-covariant field redefinitions, we find that both actions are fixed up to overall factors and up to field redefinitions.

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1 Introduction

One of the most exciting discoveries in string theory is T-duality [1, 2]. This duality may be used to construct the effective field theory at low energy. One approach for constructing this effective action is the Double Field Theory (DFT) approach [3, 4, 5, 6, 7]. This approach doubles the spacetime coordinates, i.e., adds to the usual $D$-dimensional spacetime coordinates which correspond to the momentum excitations, another $D$-dimensional coordinates which correspond to the winding excitations. The level-matching condition in its strong form, however, constrains the $2D$-dimensional dynamical fields to be independent of the winding coordinates. The T-duality is manifested in this approach as the effective action is $O(D, D)$-invariant by constructions. The effective action is also constrained to satisfy some gauge transformations. The appropriate gauge transformations at the leading order of $\alpha'$ are the generalized diffeomorphisms and double-Lorentz transformations [3, 7], however, one of them receive $\alpha'$-corrections at the higher orders of $\alpha'$ [9, 10]. The form of these corrections at order $\alpha'$ have been found in [9, 10], however, it is hard to find them at higher orders of $\alpha'$. Using the $2D$-dimensional field redefinitions freedom, the effective action may appear in different schemes. The DFT effective action at order $\alpha'$ in one particular scheme has been constructed in [10, 11].

Another T-duality based approach for constructing the effective action at higher orders of $\alpha'$, is to use the constraint that the dimensional reduction of the effective action on a circle must be invariant under the T-duality transformations [12]. In this approach, one begins with the most general gauge invariant action in the $D$-dimensional spacetime. The dimensional reduction of this action on the circle must be invariant under the T-duality transformations. The gauge transformations in this approach are the standard coordinate transformations, the B-field gauge transformations and the non-standard Lorentz transformation of the B-field which is required for anomaly cancellations. The T-duality transformations at the leading order of $\alpha'$ are the Buscher rules [13, 14], however, they receive $\alpha'$-corrections at the higher orders of $\alpha'$ [16, 17]. The form of these corrections at order $\alpha'$ have been found in [15, 16, 17], however, it is hard to find them at higher orders of $\alpha'$. Using the $D$-dimensional field redefinitions freedom, the effective action may appear in different schemes. Using the T-duality approach, the gravity and dilaton couplings in the effective actions at orders $\alpha', \alpha'^2, \alpha'^3$ in one particular scheme have been found in [19, 20].

Since the higher derivative corrections to the gauge transformations in the DFT approach and the $\alpha'$-corrections to the Buscher rules in the T-duality approach are hard to find in general, it is desirable to find constraints which do not receive $\alpha'$-corrections. Merging the above two approaches, one may finds such constraints as follows: Using the strong constraint, one can reduce the $2D$-dimensional effective action in the DFT approach to the $D$-dimensional effective action. This action should then be the same as the $D$-dimensional effective action in the T-duality approach. The field variables in the two approaches, however, are not the same. Non-covariant field redefinitions are required to relate the two field variables [21]. The T-duality transformations in the DFT approach are the standard Buscher rules [7] whereas

\footnote{It has been observed in [18] that the renormalization group flows is covariant under the T-duality transformations at order $\alpha'$.}
the gauge transformations are not the standard gauge transformations. On the other hand, the gauge transformations in the T-duality approach are the standard gauge transformations whereas the T-duality transformations are not the standard Buscher rules. As a result, the two $D$-dimensional actions must be the same up to non-covariant field redefinitions. Therefore, the effective actions should satisfy the following constraints:

1-The 2$D$-dimensional action is constrained to be invariant under the $O(D, D)$ transformations and under the generalized diffeomorphisms which do not receive $\alpha'$-corrections. However, it is not constrained to be invariant under the double-Lorentz transformations which receive $\alpha'$-corrections.

2-After reducing it to the $D$-dimensional spacetime and using non-covariant field redefinitions, the action is constrained to be the same as a $D$-dimensional action which is invariant under the standard coordinate transformations, the B-field gauge transformations and the non-standard Lorentz transformation of the B-field, however, it is not invariant under the T-duality transformations which receive $\alpha'$-corrections. We speculate that the above two constraints can fix both the 2$D$-dimensional and $D$-dimensional effective actions. We confirm this idea in this paper by explicit calculations at orders $\alpha'^0$ and $\alpha'$.

The outline of the paper is as follows: In section 2, we perform the calculations at order $\alpha'^0$. In particular, in subsection 2.1, we use generalized metric and dilaton as dynamical fields which are invariant under the double-Lorentz transformations, to construct the most general $O(D, D)$-invariant action at order $\alpha'^0$. Using the strong constraint, we then reduce it to the $D$-dimensional action. Then, using the $D$-dimensional metric, B-field and dilaton, we construct the most general covariant action at order $\alpha'^0$. Constraining the two actions to be identical, we fix both effective actions. Up to an overall factor, they are exactly the known effective actions in the literature. In section 2.2, we use the generalized frame and dilaton as the dynamical fields. Using the generalized fluxes, which are invariant under the generalized diffeomorphisms, we construct the most general $O(D, D)$-invariant action at order $\alpha'^0$, and then reduce it to the $D$-dimensional action. Comparing it with the most general covariant $D$-dimensional action, we fix both the effective actions. The 2$D$-dimensional effective action is the same as the action in the literature.

In section 3, we extend the calculations to the order $\alpha'$. In particular, using the generalized fluxes, we first construct the most general $O(D, D)$-invariant action at order $\alpha'$ without fixing its field redefinitions freedom, and then reduce it to the $D$-dimensional action. To convert the non-covariant field variables in the resulting action to the covariant variables, we use the most general non-covariant field redefinitions. We then construct the most general covariant action at order $\alpha'$ up to covariant field redefinitions. The constraint that the two $D$-dimensional actions must be identical, fixes both actions. Up to an overall factor, the $D$-dimensional action is exactly the same as the action in the literature. Since the field redefinitions freedom is not fixed in the 2$D$-dimensional action, we have found the 2$D$-dimensional action with some arbitrary parameters. In one particular scheme in which dilaton appears as an overall factor, we write the effective action.
2 Effective action at order $\alpha'^0$

Using the strong constraint in the DFT formalism, the effective action of string theory at order $\alpha'^0$ can be written as $O(D, D)$-invariant and invariant under 2D-dimensional gauge transformations which are generalized diffeomorphisms and local double-Lorentz transformations. If one uses the generalized metric and dilaton as dynamical fields which are invariant under the double-Lorentz transformations, then the gauge transformations are the generalized diffeomorphisms $[8]$. On the other hand, if one uses the generalized frame and dilaton as the dynamical fields, then the action can be written in terms of generalized fluxes which are invariant under the generalized diffeomorphisms $[7]$. Hence, the nontrivial gauge transformations in this case is the double-Lorentz transformations. Using these gauge transformations, the 2D-dimensional effective actions have been found in $[8, 22]$.

In this section we are going to find these actions by comparing the most general $O(D, D)$-invariant action with the most general $D$-dimensional covariant action.

2.1 Generalized metric formulation

We begin with the case that the generalized metric $\mathcal{H}_{\mu\nu}$ and dilaton $d$ are the dynamical fields. They are invariant under the double-Lorentz transformations as they carry no index in this space, however, the generalized metric $\mathcal{H}_{\mu\nu}$ is a matrix that transforms under the $O(D, D)$ transformations as $^3$

$$\mathcal{H} \rightarrow O \mathcal{H} O^T$$

(1)

The $D$-dimensional coordinate $x^a$ conjugated to the momentum excitations and the $D$-dimensional coordinate $\tilde{x}_a$ conjugated to the winding excitations, transforms as vector, i.e.,

$$x^\mu \equiv \left( \tilde{x}_a \atop x^a \right) \rightarrow O \left( \tilde{x}_a \atop x^a \right) ; \quad \partial_\mu \equiv \left( \tilde{\partial}^a \atop \partial_a \right) \rightarrow O \left( \tilde{\partial}^a \atop \partial_a \right)$$

(2)

And the 2D-dimensional dilaton $d$ is invariant under the $O(D, D)$ transformations $^3$. Using these $O(D, D)$ tensors, one can write the most general $O(D, D)$-invariant action at two-derivative level as $^3$

$$S_0 = -\frac{2}{\kappa^2 V} \int dx \tilde{x} e^{-2d} \left( c_0 \mathcal{H}^{\alpha\beta} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_3 \mathcal{H}^{\alpha\beta} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_1 \mathcal{H}^{\alpha\gamma} \mathcal{H}^{\beta\delta} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_2 \mathcal{H}^{\alpha\gamma} \mathcal{H}^{\beta\delta} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_5 \mathcal{H}^{\alpha\delta} \mathcal{H}^{\beta\gamma} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_6 \mathcal{H}^{\alpha\delta} \mathcal{H}^{\beta\gamma} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_8 \mathcal{H}^{\alpha\delta} \mathcal{H}^{\beta\gamma} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_9 \mathcal{H}^{\alpha\delta} \mathcal{H}^{\beta\gamma} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} + c_{10} \mathcal{H}^{\alpha\delta} \mathcal{H}^{\beta\gamma} \partial_\alpha \tilde{x} \partial_\beta \tilde{x} \right)$$

(3)

$^3$Our index conversion is that the Greek letters ($\mu, \nu, \cdots$) are the indices of the curved 2D-dimensional space, the Latin letters ($a, d, c, \cdots$) are the indices of the curved $D$-dimensional spacetime, the letters ($A, B, C, \cdots$) are the indices of flat 2D-dimensional tangent space, and the letters ($i, j, k, \cdots$) are the flat $D$-dimensional tangent space.

$^4$We use the mathematica package ‘xAct’ $^{23}$ for performing the calculations in this paper.
where $\mathcal{H}^{\mu\nu}$ is inverse of the generalized metric and $\tilde{V} = \int d\tilde{x}$. Since there is no double-Lorentz index in the couplings (3), this action is invariant under the local double-Lorentz transformations as well.

There is also a $O(D, D)$-invariant metric

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4)$$

which raises and lowers the 2D-indicies, i.e.,

$$\mathcal{H}^{\mu\nu} = \eta^{\mu\alpha} \mathcal{H}_{\alpha\nu} ; \quad \mathcal{H}_{\mu\nu} = \eta_{\mu\alpha} \eta_{\nu\beta} \mathcal{H}^{\alpha\beta} ; \quad \partial^\alpha = \eta^{\alpha\beta} \partial_\beta \quad (5)$$

The symmetry of the effective action under $B_{ab} \rightarrow -B_{ab}$, requires the couplings in it to have even number of constant metric $\eta$ [8]. The couplings involving odd number of $\eta$ is antisymmetric under $B_{ab} \rightarrow -B_{ab}$. At two derivative level, one can convince oneself that any term which is independent of the winding coordinates and contains two constant metrics, is identical to the couplings in (3). For example, the term $\partial_\alpha \mathcal{H}^{\alpha\beta} \partial_\gamma \mathcal{H}_{\gamma\mu}$ can be written as $\partial_\alpha \mathcal{H}^{\alpha\beta} \partial_\gamma \mathcal{H}_{\gamma\mu} \mathcal{H}^{\beta\mu}$, using the constant metric to raise and lower the indices. The latter coupling can be written as the couplings in (3) using the identity

$$\mathcal{H}^{\alpha\beta} \mathcal{H}_{\beta\mu} = \delta^\alpha_\mu \quad (6)$$

At higher derivative level, however, there are couplings involving the metric $\eta$ which can not be written in terms of only generalized metric, e.g., $\partial_\alpha \mathcal{H}^{\alpha\beta} \partial_\gamma \tilde{H}^{\alpha\beta}$. The couplings in (3) with coefficients $c_4, c_7, c_{10}$ become zero using the above identity. Using this identity and total derivative terms, one can relate the coefficients of some of the above terms. One may either use these relations to write (3) in terms of independent couplings and then compare them with the $D$-dimensional gauge invariant action, or one may fix them after comparing the non-independent couplings with the $D$-dimensional gauge invariant action. In the latter case that we are going to do in this paper, they appear as free parameters which can be chosen arbitrarily.

The reduction of the generalized metric and its inverse in terms of the $D$-dimensional metric and the B-field are [8]

$$\mathcal{H}_{\mu\nu} = \begin{pmatrix} G^{ab} B_{ac} & -G^{ac} B_{cb} \\ B_{ac} G^{cb} & G_{ab} - B_{ac} G^{cd} B_{db} \end{pmatrix} ; \quad \mathcal{H}^{\mu\nu} = \begin{pmatrix} G^{ab} - B_{ac} G^{cd} B_{db} & B_{ac} G^{cb} \\ -G_{ac} B_{cb} & G^{ab} \end{pmatrix} \quad (7)$$

The reduction of the 2D-dimensional dilaton to the $D$-dimensional dilaton and metric is $e^{-2d} = e^{-2\Phi \sqrt{-G}}$. Using the strong constraint that fields do not depend on the coordinate $\tilde{x}$, one can reduce (3) to the following $D$-dimensional action:

$$S_0 = \int dx e^{-2\Phi \sqrt{-G}} \left( c_9 G^{ab} \partial_a \Phi \partial_b \Phi + c_3 G^{ab} \partial_b \partial_a \Phi - \frac{1}{4} G^{ac} G^{bd} \partial_c \partial_a G_{db} + 2(c_1 + c_5) G^{ad} B_{be} \partial_a B_{be} + (c_2 + c_6) G^{ad} G^{be} G^{cf} \partial_a B_{ac} \partial_d B_{ef} \right)$$
\[-\frac{1}{4}c_1 G^{ad} G^{bc} G^{cf} \partial_b G_{fc} \partial_d G_{ae} + \frac{1}{4} (8c_1 + c_3 + 8c_5) G^{ad} G^{be} G^{cf} \partial_a G_{bc} \partial_d G_{ef} \]

\[ + c_6 G^{ad} G^{be} G^{cf} \partial_b G_{ae} \partial_d G_{ef} - \frac{1}{2} c_9 G^{ad} G^{be} \partial_a G_{ca} \partial_d \Phi + c_{11} G^{ad} G^{be} \partial_c G_{ab} \partial_d \Phi \]

\[ + \frac{1}{16} c_9 G^{ae} G^{bf} G^{cd} \partial_a G_{de} \partial_e G_{fb} + (-c_2 - c_3) G^{ad} G^{be} G^{cf} \partial_d B_{ae} \partial_f B_{bc} \]

\[ + c_8 G^{ad} G^{be} G^{cf} \partial_d G_{ae} \partial_f G_{bc} + c_2 G^{ae} G^{bd} \partial_c \partial_d G_{ab} \]  

(8)

Note that the coefficients \(c_4, c_7, c_{10}\) do not appear in above \(D\)-dimensional action, so they appear in the final action as arbitrary parameters which can be set to zero. It is interesting to note that even though the generalized metric contains no derivative on the B-field, the above \(D\)-dimensional action contains only terms which have derivative on it. The above \(O(D, D)\) and double-Lorentz invariant action is not invariant under the generalized diffeomorphism for arbitrary parameters. We do not impose this constraint.

We now construct the most general \(D\)-dimensional action at two-derivative level which is invariant under the coordinate transformations and under the B-field gauge transformations, \textit{i.e.},

\[ S_0^* = -\frac{2}{\kappa^2} \int dx e^{-2\Phi} \sqrt{-G} \left( a_1 R + a_2 \nabla_a \Phi \nabla^a \Phi + \frac{a_3}{12} H^2 \right) \]  

(9)

where \(H_{abc} = \partial_a B_{bc} + \partial_b B_{ac} + \partial_c B_{ba}\) and \(a_1, a_2, a_3\) are three constants. Since the action \(S_{0}^*\) is in terms of metric and B-field, we rewrite the above covariant action in terms of metric and B-field, \textit{i.e.},

\[ S_0^* = -\frac{2}{\kappa^2} \int dx e^{-2\Phi} \sqrt{-G} \left( a_2 G^{ab} \partial_a \Phi \partial_b \Phi - a_1 G^{ac} G^{bd} \partial_c \partial_d G_{ab} + a_1 G^{ac} G^{bd} \partial_c \partial_d G_{ab} \right) \]

\[ - \frac{1}{4} a_3 G^{ad} G^{be} G^{cf} \partial_a B_{bc} \partial_d B_{ef} + \frac{1}{2} a_3 G^{ad} G^{be} G^{cf} \partial_b B_{ae} \partial_d B_{ef} \]

\[ + a_1 G^{ad} G^{be} G^{cf} \partial_b G_{fc} \partial_d G_{ae} + \frac{3}{4} a_1 G^{ad} G^{be} G^{cf} \partial_a G_{be} \partial_d G_{ef} \]

\[ - \frac{1}{2} a_1 G^{ad} G^{be} G^{cf} \partial_b G_{ac} \partial_d G_{ef} - \frac{1}{4} a_1 G^{ae} G^{bf} G^{cd} \partial_a G_{de} \partial_c G_{fb} \]

\[ - a_1 G^{ad} G^{be} G^{cf} \partial_d G_{ae} \partial_f G_{bc} \]  

(10)

The two \(D\)-dimensional Lagrangians \(S_{0}^*\) and \(S_{0}^*\) are not equal for any non-zero parameters. However, to compare the two actions, one should take into account non-covariant total derivative terms as well. There are three total derivative terms, \textit{i.e.},

\[ J = -\frac{2}{\kappa^2} \int dx \partial_a \left( e^{-2\Phi} \sqrt{-G} (f_3 G^{ab} \partial_b \Phi + f_2 G^{ab} G^{da} \partial_c G_{db} + f_1 G^{ca} G^{db} \partial_c G_{db}) \right) \]  

(11)

where \(f_1, f_2, f_3\) are three arbitrary parameters. Now adding these total derivative terms to \(S_{0}^*\), the two actions can be equated, \textit{i.e.}, \(S_0 = S_0^* + J\), for the following constraints on the
parameters:

\[ a_2 = 4a_1, \quad a_3 = a_1, \]

\[ c_3 = 2a_1 - c_{11} - 2c_2, \quad c_5 = -a_1/8 - c_1, \quad c_6 = a_1/2 - c_2, \quad c_8 = -c_2, \quad c_9 = 2c_{11} + 4c_2, \]

\[ f_1 = a_1/2 + c_{11}/4 + c_2/2, \quad f_2 = -a_1 + c_2, \quad f_3 = 2a_1 - c_{11} - 2c_2 \] (12)

The equations in the last line give the coefficients of the total derivative terms that are needed to equate the two actions.

The equations in the first line constrain the \(D\)-dimensional action (9) to be

\[ S_c^0 = \frac{-2a_1}{k^2} \int dx e^{-2d} \sqrt{-G} \left( R + 4\nabla_a \Phi \nabla^a \Phi - \frac{1}{12} H^2 \right). \] (13)

which is the standard effective action at order \(\alpha'^0\), up to an overall factor. The equations in the second line constrain the \(O(D, D)\)-invariant action (3) to be

\[ S_0 = \frac{-2a_1}{k^2 V} \int dx d\bar{x} e^{-2d(2\mathcal{H}^{\alpha\beta} \partial_{\alpha} \partial_{\beta} d + \frac{1}{8} \mathcal{H}^{\alpha\delta} \partial_{\alpha} \mathcal{H}_{\beta\gamma} \partial_{\delta} \mathcal{H}^{\beta\gamma} - \frac{1}{2} \mathcal{H}^{\alpha\delta} \partial_{\alpha} \mathcal{H}_{\beta\gamma} \partial_{\delta} \mathcal{H}^{\beta\gamma})} \] (14)

where we have also used the identity \(\partial_{\alpha} \mathcal{H}^{\mu\nu} = -\mathcal{H}^{\mu\beta} \mathcal{H}^{\nu\gamma} \partial_{\alpha} \mathcal{H}_{\beta\gamma}\). The terms with coefficients \(c_2, c_{11}\) are total derivative terms, and terms with coefficient \(c_1\), i.e.,

\[ c_1 \mathcal{H}^{\alpha\beta} \partial_{\gamma} \partial_{\delta} \mathcal{H}_{\beta\gamma} - c_1 \mathcal{H}^{\delta\alpha} \mathcal{H}^{\epsilon\gamma} \partial_{\alpha} \mathcal{H}_{\beta\gamma} \partial_{\delta} \mathcal{H}_{\epsilon\epsilon} \] (15)

become zero using the identity (6), so we have discarded them. The \(O(D, D)\)-invariant action (14) is the one that has been found in [8]. Therefore, the requirement that the \(O(D, D)\)-invariant action and the covariant \(D\)-dimensional action to be identical, fixes both actions, up to an overall factor.

The action (14) has been found in [8] by requiring the \(O(D, D)\)-invariant couplings (3) to be invariant under the generalized diffeomorphisms which are

\[ \delta(e^{-2d}) = \partial_{\mu} (\zeta^{\mu} e^{-2d}) \]

\[ \delta \mathcal{H}_{\mu\nu} = \zeta^{\rho} \partial_{\rho} \mathcal{H}_{\mu\nu} + 2 (\partial_{(\mu} \zeta^{\rho} - \partial^{\rho} \zeta_{(\mu}) \mathcal{H}_{\nu)}) \] (16)

Unlike the terms in (9) which are invariant under the \(D\)-dimensional diffeomorphisms, none of the terms in (3) is invariant under the above \(2D\)-dimensional generalized diffeomorphisms. Only the combination of terms in (14) is invariant under these transformations [8]. This combination may be defined as the definition of \(2D\)-dimensional scalar curvature [8]. It is hard to extend the couplings (14) to the higher order of \(\alpha'\) because the conventional \(2D\)-dimensional Riemann curvature does not transform covariantly under the generalized diffeomorphisms [4, 24, 25, 26, 27].
2.2 Generalized frame formulation

A convenient frame work for constructing the higher derivative couplings in DFT is the general-
ized frame construction of the DFT \[3, 7\]. The generalized frame \( E_\mu^A \) is defined to relate the
generalized metric \( \mathcal{H}_{\mu\nu} \) to the flat generalized metric \( \mathcal{H}_{AB} \) and the \( O(D, D) \)-metric \( \eta_{\mu\nu} \) to flat
metric \( \eta_{AB} \), i.e.,

\[
\mathcal{H}_{\mu\nu} = E_\mu^A \mathcal{H}_{AB} E_\nu^B \\
\eta_{\mu\nu} = E_\mu^A \eta_{AB} E_\nu^B
\] (17)

In terms of the generalized dilaton \( d \), the generalized frame \( E_\mu^A \) and its transverse \( E_\mu^A \), one can
construct flat space tensors which transform as scalar under the generalized diffeomorphisms. They are

\[
\mathcal{F}_A = 2 \partial_A d - E_\mu^B \partial_B E_\mu^A \\
\mathcal{F}_{ABC} = 3 \partial_\nu E_\mu^B E_\rho^C \eta_{\mu\nu}
\] (18)

where the flat space derivative is \( \partial_A = E_\mu^A \partial_\mu \). These tensors transform as scalar under the
generalized diffeomorphisms \[7\]

\[
\delta \mathcal{F}_A = \zeta^\mu \partial_\mu \mathcal{F}_A \\
\delta \mathcal{F}_{ABC} = \zeta^\mu \partial_\mu \mathcal{F}_{ABC}
\] (19)

The flat space derivatives of these tensors transform as scalar under the generalized diffeo-
morphisms as well \[7\]. However, these tensors do not transform covariantly under local double-
Lorentz transformations. It has been shown in \[10, 11\] that these transformations receive \( \alpha' \)
corrections as well \[8\].

One may construct \( O(D, D) \)-invariant and the generalized diffeomorphism invariant effective
actions by considering all contractions of these tensors with constant metric \( \eta^{AB} \) and \( \mathcal{H}^{AB} \), i.e.,
at two-derivative level they are

\[
S_0 = -\frac{2}{\kappa^2 V} \int dx \tilde{e} e^{-2d} \left( c_3 \mathcal{F}_A \mathcal{F}^A + c_4 \partial_\mu \mathcal{F}_A + c_1 \mathcal{F}^A \mathcal{F}^B \mathcal{H}_{AB} + c_2 \partial_\mu \mathcal{F}^B \mathcal{H}_{AB} + c_7 \mathcal{F}_{ABC} \mathcal{F}^{ABC} \\
+ c_6 \mathcal{H}^{AB} \mathcal{F}_A \mathcal{F}^{CD} \mathcal{F}_{BCD} + c_5 \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{F}_{AC} \mathcal{F}_B \mathcal{F}_{DE} + c_3 \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{F}_{AC} \mathcal{F}_{BDF} \right)
\] (20)

The flat indices are raised by the flat metric \( \eta^{AB} \), i.e., \( \mathcal{F}^A = \eta^{AB} \mathcal{F}_B \). This action is invariant
under the generalized diffeomorphisms for arbitrary parameters \( c_1, \cdots, c_8 \), however, it is not
invariant under the local double-Lorentz transformations. Imposing the invariance under the
double-Lorentz transformations, one can fix these parameters \[22, 11\]. However, we are not
going to fix the parameters in this way.

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5If one uses the generalized metric and dilaton as the dynamical fields which are invariant under the double-
Lorentz transformations, then the \( \alpha' \)-corrections would appear in the generalized diffeomorphisms \[9\].
To fix the parameters $c_1, \cdots, c_8$, instead, we constrain the reduction of this action to be identical with the $D$-dimensional covariant action \( \mathcal{H}^{AB} \). The reduction of the metric $\eta^{AB}$, $\mathcal{H}^{AB}$ and the generalized frame $E^{\mu A}$ are \( \mathbb{1} \):

$$
\eta^{AB} = \begin{pmatrix} 0 & \delta^i_j \\ \delta_j^i & 0 \end{pmatrix} ; \quad \mathcal{H}^{AB} = \begin{pmatrix} \eta_{ij} & 0 \\ 0 & \eta^{ij} \end{pmatrix} ; \quad E^{\mu A} = \begin{pmatrix} e_a^i \\ -e_b^i B_{ba} \end{pmatrix} 
$$

(21)

where $e_a^i$ is the $D$-dimensional frame, i.e., $e_a^i e_j^b \eta_{ij} = G_{ab}$. Using the constraint that fields in the 2D-dimensional action \( \mathcal{S}^f \) do not depend on the coordinate $\tilde{x}$, one can reduce it to the following $D$-dimensional action:

$$
\mathcal{S}^f_0 = -\frac{2}{\kappa^2} \int dx e^{-2\Phi} \sqrt{-g} \left( c_2 G^{ab} \frac{\partial}{\partial x} \Phi \frac{\partial}{\partial x} \Phi + 2 c_2 G^{ab} \Phi \frac{\partial}{\partial x} \Phi + 2 c_2 G^{ab} \frac{\partial}{\partial x} \Phi \frac{\partial}{\partial x} \Phi \right)
$$

(22)

The terms with coefficients $c_3, c_4, c_7$, which have no $\mathcal{H}^{AB}$, vanish when reducing the couplings \( \mathcal{S}^f \) to the $D$-dimensional spacetime. Hence, these terms are zero by the strong constraint. Note that even though the generalized frame contains no derivative on the B-field, the above $D$-dimensional action contains only terms which have derivative on the B-field. The above $D$-dimensional action must be equal to \( \mathcal{S}^f_0 \) plus some $D$-dimensional total derivative terms.

To compare the two actions, one has to rewrite the derivatives of metric in the action \( \mathcal{S}^f_0 \) in terms of derivatives of the frame $e_a^i$. This comparison fixes both the effective actions \( \mathcal{S}^f_0 \) and \( \mathcal{S}^f_0 \) up to an overall factor. The action \( \mathcal{S}^f_0 \) is fixed as in \( \mathcal{S}^f_0 \) and the action \( \mathcal{S}^f_0 \) is fixed as

$$
\mathcal{S}^f_0 = -\frac{2a_1}{\kappa^2 V} \int dx d\tilde{x} e^{-2d} \left( -\mathcal{F}^A \mathcal{F}^B \mathcal{H}^{AB} + 2\partial^A \mathcal{F}^B \mathcal{H}^{AB} + \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{F}_{ACE} \mathcal{F}_{BDF} + \cdots \right)
$$

(23)

where dots represent the terms which vanish after using the strong constraint. The above action is the action has been found in \( \mathcal{S}^f_0 \). In the next section we consider this approach to find both $D$-dimensional and 2D-dimensional effective actions at order $\alpha'$. 

8
3 Effective action at order $\alpha'$

The most general four-derivative action which is $O(D, D)$-invariant and is invariant under generalized diffeomorphisms can be constructed by all possible contractions of the following tensors with constant metric $\eta^{AB}$ and $H^{AB}$:

$$
\mathcal{F}_A \mathcal{F}_B \mathcal{F}_C \mathcal{F}_D + \mathcal{F}_A \mathcal{F}_B \mathcal{F}_C \mathcal{F}_{DEF} + \mathcal{F}_A \mathcal{F}_B \mathcal{F}_{CDE} \mathcal{F}_{FGH} + \mathcal{F}_A \mathcal{F}_{BCD} \mathcal{F}_{EFG} \mathcal{F}_{HIJ}
$$

$$
+ \mathcal{F}_{ABC} \mathcal{F}_{DEF} \mathcal{F}_{GHI} \mathcal{F}_{JKL} + \mathcal{F}_{CDE} \mathcal{F}_{FGH} \partial_A \mathcal{F}_B + \mathcal{F}_{DEF} \partial_A \partial_B \mathcal{F}_C + \partial_A \partial_B \partial_C \mathcal{F}_D
$$

$$
+ \partial_A \partial_B \partial_C \mathcal{F}_{DEF} + \mathcal{F}_A \mathcal{F}_{DEF} \partial_B \mathcal{F}_C + \mathcal{F}_A \partial_B \partial_C \mathcal{F}_D + \mathcal{F}_A \partial_B \partial_C \mathcal{F}_{DEF} + \mathcal{F}_A \mathcal{F}_{BCD} \partial_E \mathcal{F}_{FGH}
$$

$$
+ \partial_A \mathcal{F}_{BCD} \partial_E \mathcal{F}_{FGH} + \mathcal{F}_{ABC} \mathcal{F}_{DEF} \partial_G \mathcal{F}_{HIJ}
$$

(24)

It produces 275 terms, i.e.,

$$
S_1' = -\frac{2}{k^2 V} \int dx \bar{e} e^{-2\Phi} \left( c_1 H^{AB} H^{CD} \mathcal{F}_A \mathcal{F}_B \mathcal{F}_C \mathcal{F}_D + c_2 \mathcal{F}^A \mathcal{F}^B \mathcal{F}^D \mathcal{F}^f + \cdots \right)
$$

(25)

where $c_1, \ldots, c_{275}$ are parameters. The terms which have no tensor $H^{AB}$, e.g., $c_2$-term, are again zero after using the strong constraint. There are 34 such terms and there are 91 terms with other structures that become zero after using the strong constraint. The remaining 150 terms are in five classes. One class includes terms that cancel each others after using the strong constraint, one class includes terms that become zero using appropriate identities, one class includes terms that are reproduced by field redefinitions at order $\alpha'$ of the fields in (23), and the last class includes all other terms in which we are interested. One may examine all terms in details to exclude all terms except the terms in the last class, and then imposes the $D$-dimensional gauge symmetry. Alternatively, one may impose the $D$-dimensional gauge symmetry on all 241 terms. In this case, the terms in the first four classes appear in the final action with free parameters which can be chosen arbitrarily.

Using the relation (18), one can write the couplings (25) in terms of dilaton and the generalized frame. Then using the dimensional reduction (21), one can reduce the $2D$-dimensional action (25) to the $D$-dimensional action, e.g., the reduction of $c_1$-term is

$$
S_1' = -\frac{2}{k^2} \int dx e^{-2\Phi} \sqrt{-G} \left( 16 c_1 D \Phi_a D \Phi^a D \Phi_b D \Phi^b + 32 c_1 D e^{a b i} D \Phi_b D \Phi_c D \Phi^c e_{a i} 
$$

$$
- 32 c_1 D e^{a b i} D \Phi_a D \Phi^a D \Phi^c e_{b i} + 16 c_1 D e^{a b i} D e^{c d j} D \Phi_b D \Phi_d e_{a i} e_{c j}
$$

$$
- 16 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} + 8 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j}
$$

$$
- 32 c_1 D e^{a b i} D e^{c d j} D \Phi_a D \Phi_d e_{a i} e_{c j} + 8 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j}
$$

$$
+ 16 c_1 D e^{a b i} D e^{c d j} D \Phi_a D \Phi_d e_{a i} e_{c j} e_{d j} - 16 c_1 D e^{a b i} D e^{c d j} D \Phi_b D \Phi_d e_{a i} e_{c j} e_{d k}
$$

$$
+ 8 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{d k} + 8 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{d k}
$$

$$
+ 16 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{e k} - 8 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{e k}
$$

$$
- 8 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{e k} + 4 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{e k}
$$

$$
- 4 c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{e k} + c_1 D e^{a b i} D e^{c d j} D \Phi_d e_{a i} e_{c j} e_{e k}
$$

(26)
where $D e_{a i} \equiv \partial_a e^i$ and $D \Phi_a \equiv \partial_a \Phi$. The above action is manifestly invariant under T-duality as its parents (25) are invariant under the $O(D, D)$ transformations, however, it is not invariant under the usual $D$-dimensional transformations for arbitrary parameters. We are going to find these parameters by comparing them with a $D$-dimensional action at order $\alpha'$ which is not invariant under the T-duality, but is invariant under the conventional gauge transformations, i.e., the standard coordinate transformations, the B-field gauge transformations and the non-standard Lorentz transformation of the B-field which is required for anomaly cancellations.

The most general $D$-dimensional action which is invariant under the conventional gauge transformations has four class of terms. One class contains terms that are zero by Bianchi identities, one class contains terms that are total derivative terms, one class contains terms that are reproduced by the field redefinitions at order $\alpha'$ of the fields in (13), and the last class contains all other terms in which we are interested. One may choose the couplings to be (28)

$$
\begin{align*}
S_1 &= \frac{-2}{\kappa^2} \alpha' \int d^{d+1}x e^{-2\Phi} \sqrt{-G} \left( b_1 R_{abcd} R^{abcd} + b_2 R_{abcd} H^{ab} H^{cd e} \\
&\quad + b_3 H_f g_h H_a^{f} b H^b H^c h + b_4 H_f^{ab} h g a b H^{fch} H^{g} c h + b_5 (H^2)^2 + b_6 H_{abcd} H^{cd e} \Phi \partial e \Phi \right) \\
&\quad + b_7 H^2 \partial_a \Phi \partial_a \Phi + b_8 (\partial_a \Phi \partial^a \Phi)^2 - \frac{d_1}{6} H^{abc} \Omega_{abc} \right) \right)
\end{align*}
$$

where $b_1, b_2, \cdots, b_8$ are eight parameters. The field redefinitions freedom allows us to choose the eight arbitrary couplings in (27) in many different schemes. The above is one particular scheme. The last term is zero for the bosonic string theory because the B-field gauge transformation is the standard transformation, i.e., $B_{ab} \rightarrow B_{ab} + \partial [a \lambda_b]$. This term is non-zero for the heterotic string theory and its coefficient is the same as the overall coefficient of (13) in the heterotic string theory. This term is a result of non-standard gauge transformation of B-field which is

$$
B_{ab} \rightarrow B_{ab} + \partial [a \lambda_b] + \alpha' \partial [a \Lambda_i \omega_{bij}^i]
$$

where $\Lambda_i$ is the matrix of the Lorentz transformations. The Chern-Simons three-form $\Omega$ which is defined as

$$
\Omega_{abc} = \omega_{[ai} \partial_b \omega_{cj]}^i + \frac{2}{3} \omega_{[ai} \omega_{bj} \omega_{cj]}^i - \Gamma_{ab}^c \omega_{ci}^j e^b_i
$$

makes $H_{abc} + \alpha' \Omega_{abc}$ to be invariant under the Lorentz transformations, i.e., $H_{abc} + \alpha' \Omega_{abc} \rightarrow H_{abc} + \alpha' \Omega_{abc}$. The action (27) is not invariant under T-duality for arbitrary parameters $b_1, b_2, \cdots, b_8$. However, one expects for some specific values for these parameters, the couplings become invariant under T-duality. We are going to find these parameters by comparing them with the manifestly T-duality invariant action (26). To compare the two actions, one has to rewrite the couplings in (27) in terms of frame $e_a^i$ and B-field. If one compares the above $D$-dimensional action with the reduced action (26), one would find zero value for all parameters.
So the actions are not the same. Let us check their Lagrangians. So we have to include some total derivative terms.

To construct all $D$-dimensional total derivative terms in terms of frame $e_a^i$, $B$-field and dilaton $\Phi$, we have to construct all contractions of the following tensors with metric $G^{ab}$ to produce the current $I^a$:

$$
\partial_a \partial_b \partial_c \partial_d e^e_d + \partial_a \partial_b \partial_c \partial_d e^e_d + \partial_a \partial_b \Phi \partial_c \partial_d \Phi + \partial_a \partial_b \Phi \partial_c \partial_d \Phi + \partial_a \delta G_{ab} \Phi \partial_c \partial_d \Phi
$$

$$
\partial_a B_{bc} \partial_d e_d e^g + \partial_a B_{bc} \partial_d e_d e^g + \partial_a \partial_b \partial_c \partial_d \Phi
$$

where $\partial_a e^c_b \equiv \epsilon^c_i \partial_a e^i_b$, $\partial_a \partial_d e^c_b \equiv \epsilon^c_i \partial_a \partial_d e^i_b$ and $\partial_a \partial_d \partial_f e^c_b \equiv \epsilon^c_i \partial_a \partial_d \partial_f e^i_b$. Then the following expression produces all total derivative terms:

$$
J = -\frac{2}{\kappa^2} \alpha' \int dx \partial_a (e^{-2\Phi} \sqrt{-G} I^a)
$$

We have examined the equality $S_f^i = S_i^c + J$ and again found zero result for all parameters. This is a indication of the observation made in [21] that fields in the conventional $D$-dimensional action (27) are not the same as the fields defined in the reduction of the 2D-dimensional fields. In particular, the fields $B, e_a^i$ in (21) are not the same as the dynamical $B$-field and frame used in (27). So we have to use field redefinitions on the $D$-dimensional fields in (13), (27) and then compare them with (26).

The variation of action (13) under field redefinition $G_{ab} \rightarrow G_{ab} + \delta G_{ab}$, $B_{ab} \rightarrow B_{ab} + \delta B_{ab}$ and $\Phi \rightarrow \Phi + \delta \Phi$ is

$$
\delta S_0^c = -\frac{2a_1}{\kappa^2} \int dx e^{-2\Phi} \sqrt{-G} \left[ -R^{ab} - 2\nabla^a \nabla^b \Phi + \frac{1}{4} H^{abc} H_{bcd} \right] \delta G_{ab}
$$

$$
+ \left( \frac{1}{2} R + 2 \nabla_c \nabla^c \Phi - 2 \nabla_c \Phi \nabla^c \Phi - \frac{1}{24} H^2 \right) \left( G^{ab} \delta G_{ab} - 4 \delta \Phi \right) - \frac{1}{2} H^{abc} \partial_d (\delta B_{bc})
$$

where $\delta G_{ab}$, $\delta \Phi$ and $\delta B_{ab}$ can be constructed by all contractions of the following tensors with metric $G^{ab}$:

$$
\partial_a \partial_b B_{cd} + \partial_a \partial_b e^d_c + \partial_a \partial_b \Phi + \partial_a \partial_b \Phi + \partial_a \partial_b \partial_c \partial_d B_{ef}
$$

$$
+ \partial_a B_{bc} \partial_d e^f_c + \partial_a e^c_b \partial_d e^f_c + \partial_a B_{bc} \partial_d \Phi + \partial_a e^c_b \partial_d \Phi
$$

We then examine the following equality:

$$
S^f_1 = S^c_1 + J + \delta S^c_0
$$

It produces many algebraic equations for the parameters with non-zero result for them.
The most important part of the result is that they fix uniquely all eight parameters in the $D$-dimensional action (27) in terms of $b_1$, i.e.,

$$S_1^c = \frac{-2}{\kappa^2} \alpha' \int d^{d+1}x e^{-2\Phi} \sqrt{-G} \left[ b_1 \left( R_{abcd} R^{abcd} - \frac{1}{2} R_{abcd} H^{ab} H^{cd} \right) - \frac{1}{24} H_{f\alpha\beta} H_{\alpha^f a} H_{\beta^f b} c H_{c^f a} - \frac{1}{8} H_{f ab} H_{\alpha c g} H_{b^c cg} \right] \right]$$  \hspace{1cm} (35)

In the scheme (27), the effective action (35) then has no derivative of dilaton. Up to the overall factor $b_1$, the above couplings are the standard effective action of the bosonic and heterotic string theories which has been found in [28] by the S-matrix calculations. This action now is invariant under T-duality. Since we have used field redefinitions to relate the manifestly T-duality invariant action (26) to the manifestly gauge invariant action (27), the T-duality transformations rules in above action would be the Buscher rules plus their $\alpha'$-corrections. They have be be found directly by using the T-duality approach [16] [17].

The parameters in the 2D-dimensional action (25), however, are not fixed uniquely in terms of $b_1, a_1$. There remain many parameters in (25) that we did not removed them from the list of independent couplings in (25). Choosing different values for the arbitrary parameters correspond to different scheme in which the 2D-dimensional action can be written. As in the D-dimensional action (35), we choose a scheme in which the 2D-dimensional dilaton appears only as an overall factor. Since the derivative of the dilaton appears in the flux $F_A$, we choose the arbitrary parameters to have no terms in (25) which has $F_A$ or its derivatives. Using this constraint, there are still some residual parameters in this scheme. We further constrain the scheme to have no terms with structure $\partial_A \partial_B \partial_C F_{DEF}, \partial_A \partial_B F_{CDE}$, nor $\partial_A F_{BCD} \partial_E F_{EFH}$. Then the effective action in this scheme for the bosonic string theory becomes

$$S_1^{fe} = \frac{-2 b_1}{\kappa^2 V} \alpha' \int d^2x d^2\tilde{x} e^{-2\phi} \left( -\frac{1}{2} H^{AB} H^{CD} H^{EF} H^{GH} H^{IJ} H^{KL} F_{ACE} F_{BDG} F_{FIK} F_{HJI} \\
+ \frac{1}{4} H^{AB} H^{CD} F_{A}^{EF} F_{B}^{GH} F_{CE} F_{F}^{GHI} + \frac{1}{4} H^{AB} H^{CD} H^{EF} H^{GH} F_{ACE} F_{BDG} F_{F}^{IJ} F_{HJI} \\
+ \frac{3}{8} H^{AB} H^{CD} H^{EF} H^{GH} F_{AC} F_{BE} F_{DGJ} F_{FHI} \\
- \frac{1}{2} H^{AB} H^{CD} H^{EF} H^{GH} F_{ACE} F_{B}^{IJ} F_{DH} F_{FHI} \\
+ \frac{1}{24} H^{AB} H^{CD} H^{EF} H^{GH} H^{IJ} H^{KL} F_{ACE} F_{BDG} F_{DHI} F_{JLI} \\
- 2 H^{AB} H^{CD} H^{EF} H^{GH} F_{ACE} F_{B}^{FG} \partial_{[F} F_{E]} F_{DG} + H^{AB} H^{CD} H^{EF} H^{GH} F_{ACE} F_{B}^{IJ} \partial_{(H} F_{I)} F_{DF} + \cdots \right)$$  \hspace{1cm} (36)

where dots represent terms that are zero under the strong constraint. Our notation for anti-symmetrization is $L_{[APB]} = (L_A P_B - L_B P_A)/2$, similarly for symmetrization. This action now is invariant under double-Lorentz transformations. Since we have used field redefinitions
to relate the manifestly T-duality invariant action (26) to the manifestly Lorentz invariant action (27), the double-Lorentz transformations in above action would be the standard Lorentz transformations plus their \( \alpha' \)-corrections. The above action is even under \( B \to -B \). It should be the same as the even part of the action has been found in [11] up to the terms that are zero under the strong constraint and up to 2D-dimensional field redefinitions.

The 2D-dimensional effective action of the heterotic string theory at order \( \alpha' \) contains the above action plus the following terms which are odd under \( B \) under the strong constraint and up to 2D:

\[
S^{\prime o}_1 = \frac{a_1}{3 \kappa^2 V} \alpha' \int dx \tilde{e}^{-2d} \left( \frac{1}{8} \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{F}_{AC}^G \mathcal{F}_{BD}^I \mathcal{F}_{EG}^H \mathcal{F}_{FI}^K \right.
\]

\[
-1 \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{H}^{GI} \mathcal{H}^{HJ} \mathcal{F}_{AC} \mathcal{F}_{BG} \mathcal{F}_{DI} \mathcal{F}_{FK}^K \mathcal{F}_{FI}^K
\]

\[
-1 \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{H}^{GJ} \mathcal{H}^{HI} \mathcal{F}_{AC} \mathcal{F}_{BE} \mathcal{F}_{DF}^H \mathcal{F}_{GI}^H \mathcal{F}_{GH}^H
\]

\[
+ \frac{1}{12} \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{H}^{GJ} \mathcal{H}^{HI} \mathcal{F}_{AC} \mathcal{F}_{BE} \mathcal{F}_{DF}^H \mathcal{F}_{GI}^H \mathcal{F}_{GH}^H
\]

\[
+ \frac{1}{4} \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{H}^{GI} \mathcal{H}^{HJ} \mathcal{F}_{AC} \mathcal{F}_{BG} \mathcal{F}_{DI} \mathcal{F}_{FK}^K \mathcal{F}_{FI}^K
\]

\[
- \frac{1}{4} \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{H}^{GJ} \mathcal{H}^{HI} \mathcal{F}_{AC} \mathcal{F}_{BE} \mathcal{F}_{DF}^H \mathcal{F}_{GI}^H \mathcal{F}_{GH}^H
\]

\[
- \frac{3}{8} \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}^{EF} \mathcal{H}^{GJ} \mathcal{H}^{HI} \mathcal{F}_{AC} \mathcal{F}_{BG} \mathcal{F}_{DI} \mathcal{F}_{FK}^K \mathcal{F}_{FI}^K + \ldots \right)
\]

(37)

where dots represent terms that are zero under the strong constraint. While the action (36) has even number of \( \mathcal{H}^{AB} \), the above action has odd number of \( \mathcal{H}^{AB} \). The above action should be the same as the odd part of the action has been found in [11] up to the terms that are zero under the strong constraint and up to 2D-dimensional field redefinitions.

The algebraic equations fix also the non-covariant field redefinitions and total derivative terms required to relate the two 2D-dimensional actions. The field redefinitions for the bosonic theory are

\[
\delta G^{ab}_B = b_1 \left( 2D e^c_d a_i D e_{bcd} - 4D e^c_d (D e_{ca}) b_i + 2D e^c_d (D e_{ab}) c_i + 8D e^c_d (D \Phi e_{a}) b_i - 8D e^c_d (D \Phi e_{a}) a_i \right)
\]

\[
-4D e^c_d (D e_{ab}) c_i + 4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i + 4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i
\]

\[
+4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i
\]

\[
-2D e^c_d (D e_{ab}) c_i - 2D e^c_d (D e_{ab}) c_i - 2D e^c_d (D e_{ab}) c_i - 2D e^c_d (D e_{ab}) c_i
\]

\[
+2D e^c_d (D e_{ab}) c_i - 2D e^c_d (D e_{ab}) c_i - 2D e^c_d (D e_{ab}) c_i - 2D e^c_d (D e_{ab}) c_i
\]

\[
-4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i - 4D e^c_d (D e_{ab}) c_i
\]

\[
\delta \Phi_B = \frac{1}{4} G^{ab} \delta G_{ab}
\]

\[
\delta B_{ab}^B = b_1 H^{cd}_{(a} (\omega_{b)ij} e^i_d + e^i_d e_{bd})
\]

(38)

where \( D e_{abc}^i \equiv \partial_a \partial_e c^i, D e_{ab}^i \equiv \partial_a e^i_b \) and \( D \Phi_a \equiv \partial_a \Phi \). The curved indices are raised with metric \( G^{ab} \) and flat indices are lowered by \( \eta_{ij} \). The above \( \alpha' \)-corrections must be added to the metric, dilaton and the B-field in the reductions of 2D-dilaton, i.e., \( e^{-2d} = e^{-2b} \sqrt{-G} \)
and the generalized metric (7) in order to make them identical to the corresponding covariant fields in the D-dimensional action (27). The resulting new 2D-dilaton $\tilde{d}$ and generalized metric $\tilde{H}$, then should transform covariantly as (16). In this way one may find $\alpha'$-corrections to the generalized diffeomorphisms for the fields that are transformed by the Buscher rules under T-duality. Alternatively, one may add the above $\alpha'$-corrections, with a minus sign, to the D-dimensional covariant fields $G_{ab}, \Phi, B_{ab}$ in order to make them identical to the corresponding $\alpha'$-corrections to the Buscher rules for the D-dimensional covariant fields.

The field redefinitions for the heterotic theory are

$$\delta G_{ab}^H = \delta G_{ab}^B - \frac{a_1}{6} H^{cd} (a e_{b}) i D e_{cd}^i$$
$$\delta \Phi^H = \frac{1}{4} G^{ab} \delta G_{ab}^H$$
$$\delta B_{ab}^H = \delta B_{ab}^B - \frac{a_1}{6} \left( D e_{[a}^i D e_{c]}^i + D D e_{[a}^c e_{b]}^i - 4 D e_{[a}^i D e_{c]}^i + D D e_{[a} e_{b]}^i e^{ci} + \right.$$  
$$\left. + D e_{[a}^i D e_{c]}^i e_{d]} e_{di} - 2 D e_{[a}^i D e_{c]}^i e_{d]} e_{di} + D e_{[a}^i D e_{c]}^i e_{d]} e_{di} + 2 D e_{[a}^i D e_{c]}^i e_{d]} e_{di} \right) \right)$$

(39)

While the field redefinition terms in the bosonic theory do not change the symmetry under $B \rightarrow -B$, the field redefinition terms corresponding to the Chern-Simons term in the heterotic theory change the symmetry of fields under $B \rightarrow -B$. This is as expected because the field redefinitions should produce odd terms under $B \rightarrow -B$ from the even terms in (32).

The above field redefinition terms encode $\alpha'$-corrections of the double-Lorentz transformations or $\alpha'$-corrections of the Buscher rules. The $\alpha'$-corrections of the double-Lorentz transformations may also be found more easily from the fact that the actions (36) and (26) must be invariant under the double-Lorentz transformations. It would interesting to find these transformations and compare them with the double-Lorentz transformations that have been found in [10, 11].

Using the generalized frame and dilaton as the dynamical fields, we have found the 2D-dimensional actions (36) and (37). One may wish to use the generalized metric and dilaton as the dynamical fields to find the 2D-dimensional action at order $\alpha'$, i.e., extension of the action (14) to the order $\alpha'$. We have done this calculation. We have found that for the case that B-field is zero there are non-zero D-dimensional and 2D-dimensional actions, however, in the presence of the B-field one would find no effective action at order $\alpha'$. In fact, when we write the couplings (27) in terms of B-field and metric, the second coupling in (27) produces, among other things, the following terms:

$$-2G^{fg} \partial_a G_{d} \partial^a B^{bc} \partial_b B^{de} \partial_c G_{ef} + G^{fg} \partial_a B^{de} \partial^a B^{bc} \partial_b G_{d} \partial_c G_{ef}$$

(40)

None of them is reproduced by any 2D-couplings, any non-covariant field redefinition or any total derivative term at order $\alpha'$. So we expect the convenient frame-work for studying the
higher derivative couplings in the $D$-dimensional string theory and in DFT is the frame-like formulation of DFT.

The gravity and dilaton couplings in the effective actions of string theories at orders $\alpha'^2$ and $\alpha'^3$ are known in the literature. Using the T-duality approach, it has been shown in [19, 20] that they are invariant under the T-duality transformations which are the Buscher rules and their $\alpha'$-corrections. However, the B-field couplings at these orders are not known in the literature. It would be interesting to use the method in this paper to find these couplings as well as their corresponding 2D-dimensional actions.

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