Integrable $SU(m|n)$ supersymmetric electronic models
of strong correlations

James T. Liu
Department of Physics
The Rockefeller University
1230 York Avenue
New York, NY 10021-6399, USA

D. F. Wang
Institut de Physique Théorique
Ecole Polytechnique Fédérale de Lausanne
PHB-Ecublens, CH-1015 Lausanne, Switzerland

We generalize the $SU(2|2)$ symmetric extended Hubbard model of $1/r^2$ interaction to the $SU(m|n)$ supersymmetric case. Integrable models may be defined on both uniform and non-uniform one dimensional lattices. We study both cases in detail and present the ground state wavefunctions and energy spectra of these models.

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I. INTRODUCTION

Much progress has been made in the study of low dimensional systems of strongly correlated electrons. A notable example is the one dimensional Hubbard model, which was solved by Lieb and Wu [1] using the Bethe-ansatz. This model exhibits a hidden $SO(4)$ symmetry [2,3] which is the basis of its properties. The suggestion of Anderson that the two dimensional Hubbard model contains the essential physics of high temperature superconductivity and that its normal state may share the Luttinger-liquid-like feature of one dimensional interacting electron systems [4,5] has induced renewed activity in the study of such strongly correlated one dimensional models.

Recently, an extended Hubbard model with nearest neighbor interactions was introduced [6,7], and is solvable by Bethe-ansatz in one dimension [8]. The spin $SU(2)$ along with the $\eta$-pairing $SU(2)$ of this model are combined and extended to form a $SU(2|2)$ supersymmetry. We have introduced the Calogero-Sutherland [9,10] version of the extended Hubbard model, defined both on a uniform lattice and on a non-uniform lattice in one dimension. The long range versions of the extended Hubbard model are exactly solvable, and the ground states, excitations and integrabilities have been studied in considerable detail [11,12].

In this paper, we generalize the previous results by extending the supersymmetry to $SU(m|n)$. Thus we introduce the $SU(m|n)$ strongly correlated electron system of $1/r^2$ long range interaction. This general model encompasses the $SU(0|2)$ Haldane-Shastry spin chain [13,14] and the $SU(1|2)$ supersymmetric $t$-$J$ model [15] as well as the $SU(2|2)$ extended Hubbard model.

In the next section we present the general features of the $SU(m|n)$ supersymmetric model. Following this, we study the system, first defined on a non-uniform lattice given by the roots of the Hermite polynomials, and then defined on a uniform lattice. In both cases, the $SU(m|n)$ models are completely solvable. We explicitly construct the ground state wavefunctions and provide the ground state energies. We also examine the excitation spectra of these systems, making use of their integrability properties.
II. THE $SU(M|N)$ SUPERSYMMETRIC ELECTRONIC MODEL

Consider a system defined on a lattice with a total of $L$ sites. At each site there is exactly one particle, which may either be a fermion with $n$ flavors (internal degrees of freedom) or a boson with $m$ flavors. Since there are a total of $m + n$ states at each site, $n$ fermionic and $m$ bosonic, this is the basis of the $SU(m|n)$ supersymmetry.

In order to describe these degrees of freedom, we use a representation in terms of fermion and boson operators, $f$ and $b$, satisfying the usual (anti-)commutation relations

$$\{ f_{i\sigma}, f_{j\sigma'}^\dagger \} = \delta_{ij} \delta_{\sigma\sigma'} \quad \{ f_{i\sigma}, f_{j\sigma'} \} = 0$$
$$[ b_{i\alpha}, b_{j\alpha'}^\dagger ] = \delta_{ij} \delta_{\alpha\alpha'} \quad [ b_{i\alpha}, b_{j\alpha'} ] = 0 . \quad (1)$$

The indices $i$ and $j$ label the site, $\sigma = 1, \ldots, n$ labels the fermion flavor and $\alpha = 1, \ldots, m$ labels the boson flavor. These operators may be represented in a supersymmetric manner by $c_{i\nu} = \{ f_{i\sigma} \}, \{ b_{i\alpha} \}$ where $\nu = 1, \ldots, m + n$ labels the (either fermionic or bosonic) species. The $c$’s then satisfy the (anti-)commutation relation

$$[ c_{i\nu}, c_{j\nu'}^\dagger ]_\pm = \delta_{ij} \delta_{\nu\nu'} , \quad (2)$$

where $[\ldots ]_\pm$ indicates either a commutator or an anti-commutator as appropriate. In general, this representation allows for multiple occupancy at each site. In order to project onto single occupancy we use the Gutzwiller operator, $P_G$, defined by

$$P_G = \prod_{i=1}^{L} \delta_1 \sum_{\nu} c_{i\nu}^\dagger c_{i\nu}$$
$$= \prod_{i=1}^{L} \delta_1 \sum_{\sigma} f_{i\sigma}^\dagger f_{i\sigma} + \sum_{\alpha} b_{i\alpha}^\dagger b_{i\alpha} . \quad (3)$$

The Hamiltonian for the $SU(m|n)$ supersymmetric model takes the form

$$H = -\frac{1}{2} P_G \sum_{1 \leq i \neq j \leq L} J_{ij} \Pi_{ij} P_G , \quad (4)$$

where the coupling parameter $J_{ij} = 1/d(i,j)^2$ takes an inverse square form. The distance function, $d(i,j)$, depends on the lattice and is what distinguishes between the models on
uniform and non-uniform lattices. In this section, we concentrate on the features of $H$ that are independent of the specifics of $d(i,j)$. The key features of this model, especially its integrability, are derived from the graded permutation operator, $\Pi_{ij}$, which exchanges particle species at sites $i$ and $j$. In the above representation, the permutation operator takes the form

$$\Pi_{ij} = \sum_{\nu=1}^{m+n} \sum_{\nu'=1}^{m+n} c^\dagger_i \nu' c_j \nu' c^\dagger_j \nu c_i \nu.$$  

(5)

Since the Hamiltonian only permutes particles, the number of bosons and fermions of each flavor are all conserved quantities. It is also easy to verify that $H$ commutes with both the $SU(n)$ fermion and $SU(m)$ boson flavor symmetries generated by

$$S^a_i = \sum_{\alpha=1}^{L} S^a_i, \quad S^a_i = \sum_{\sigma, \sigma'=1}^{n} \langle n \rangle f_i^\dagger(1) f_i \sigma' T^a_{\sigma \sigma'}(n) f_i \sigma,$$  

(6)

$$\eta^a_i = \sum_{\alpha=1}^{L} \eta^a_i, \quad \eta^a_i = \sum_{\alpha, \alpha'=1}^{m} \langle m \rangle b_i^\dagger(1) b_i \alpha' T^a_{\alpha \alpha'}(m) b_i \alpha,$$  

(7)

respectively (here $T(n)$ and $T(m)$ are hermitian $SU(n)$ and $SU(m)$ generators in the fundamental representations). In fact, ignoring the grading, since $\Pi_{ij}$ acts symmetrically on all $m+n$ species [as is evident from the first line of (5)], we would expect a $SU(m+n)$ symmetry of this model. Taking statistics into account, this is modified into a $SU(m|n)$ supersymmetry of which the bosonic subgroup is given by $SU(m) \times SU(n) \times U(1)$ (see e.g. Ref. [16]). The $U(1)$ symmetry is related to the boson and fermion number currents, and is given by

$$J = \sum_{i=1}^{L} J_i, \quad J_i = \frac{1}{n} \sum_{\sigma=1}^{n} f_i^\dagger(1) f_i \sigma + \frac{1}{m} \sum_{\alpha=1}^{m} b_i^\dagger(1) b_i \alpha.$$  

(8)

Although it appears there is another $U(1)$ symmetry given by the remaining orthogonal combination of boson and fermion currents, this is actually not the case since we restrict
ourselves to single occupancy, and thus total particle number is fixed. This means that the second $U(1)$ symmetry is trivial (except in the special case when $m = n$).

Complementing the above bosonic generators of $SU(m|n)$ are the fermionic generators

$$Q_\sigma = \sum_{i=1}^{L} Q_{i\sigma}, \quad Q^\dagger_{i\sigma} = f^\dagger_{i\sigma} b_{i\sigma},$$
$$Q^\dagger_{\sigma} = \sum_{i=1}^{L} Q^\dagger_{i\sigma}, \quad Q^\dagger_{\sigma} = b^\dagger_{i\sigma} f_{i\sigma},$$

transforming as the $(\overline{m}, n)$ and $(m, \overline{n})$ of $SU(m) \times SU(n)$ respectively. The complete $SU(m|n)$ superalgebra is given by these $2mn$ fermionic generators as well as the $m^2 + n^2 - 1$ bosonic generators of Eqns. (3), (4) and (8).

In general, since the Hamiltonian $H$ is $SU(m|n)$ invariant, all states in the system fall into complete representations of the $SU(m|n)$ supersymmetry algebra. These supermultiplets have a very rich structure, and span states with different numbers of fermions and bosons. Representations of $SU(m|n)$ may be divided into two categories, “typical” and “atypical” \[10\]. The typical representations group together states in sectors with boson and fermion numbers given by $(Q, M) = (b, f), (b + 1, f - 1), (b + 2, f - 2), \ldots, (b + mn, f - mn)$, while the atypical ones terminate early and hence span fewer than the $mn + 1$ sectors of typical ones. In both cases, a full $SU(m|n)$ representation may be decomposed in terms of a direct sum (over sectors of decreasing fermion number) of representations of the bosonic subgroup $SU(m) \times SU(n) \times U(1) \subset SU(m|n)$. For an irreducible representation, we refer to the state with greatest fermion number (i.e. the $(b, f)$ sector) as the highest member of a supermultiplet and the state with the fewest fermions (the $(b + mn, f - mn)$ sector for typical representations) as the lowest member.

While the important properties of the system may be understood through this supermultiplet structure, we generally wish to have greater control over the individual species populating the lattice. In particular, we note that since $SU(m|n)$ has a maximal Cartan subalgebra given by $U(1)^{m+n-1}$, there are a set of $m + n - 1$ conserved currents which are equivalent to the fermion and boson number currents of the $m + n$ species (along with the
single occupancy constraint). As a result, the number of bosons of each flavor and the number of the fermions of each flavor are all conserved quantities, and hence we may work in a subspace where all these quantities are fixed. This is easily accomplished by adding to $H$ a set of chemical potentials for the different species:

$$
H = H + \sum_{\nu=1}^{m+n} \mu^{(\nu)} \sum_{i=1}^{L} c_{i\nu}^\dagger c_{i\nu},
$$

(10)

(one of the $\mu$’s is redundant). Each subspace of fixed occupancies may be labeled by the $m+n$ numbers $\{Q_\alpha, M_\sigma\}$ given by

$$
M_\sigma = \sum_{i=1}^{L} f_{i\sigma}^\dagger f_{i\sigma}, \quad \sigma = 1, 2, \cdots, n
$$

$$
Q_\alpha = \sum_{i=1}^{L} b_{i\alpha}^\dagger b_{i\alpha}, \quad \alpha = 1, 2, \cdots, m.
$$

(11)

Since each site is singly occupied, these quantities satisfy the relation that $\sum_\sigma M_\sigma + \sum_\alpha Q_\alpha = L$. Working in such a subspace, we have broken the $SU(m|n)$ symmetry down to its maximal Cartan subalgebra. Alternatively, we often consider the case when $SU(m|n)$ is not broken completely down to $U(1)^{m+n-1}$, but instead to its bosonic subgroup $SU(m) \times SU(n) \times U(1)$.

This is equivalent to working in a subspace of fixed $(Q, M)$ where $Q = \sum_\alpha Q_\alpha$ is the total number of bosons and $M = \sum_\sigma M_\sigma$ is the total number of fermions, and corresponds to using the $U(1)$ current $J$ to break supersymmetry.

We now define the wavefunctions in this Hilbert subspace of fixed $(\{Q_\alpha\}, \{M_\sigma\})$ by writing the state vectors of the system in the following form:

$$
|\phi\rangle = \sum_{\{x\sigma\},\{y\alpha\}} \phi(x\sigma; y\alpha) \prod_{i=1}^{M} f_{x_i\sigma_i}^\dagger \prod_{k=1}^{Q} b_{y_k\alpha_k}^\dagger |0\rangle.
$$

(12)

The amplitude $\phi(x\sigma, y\alpha)$ is antisymmetric in the fermion positions and spins, $\{x\sigma\} = (x_1 \sigma_1, \cdots, x_M \sigma_M)$, while symmetric in the boson positions and spins, $\{y\alpha\} = (y_1 \alpha_1, \cdots, y_Q \alpha_Q)$. Because of the supersymmetry, it is convenient to combine the boson and fermion positions by defining $(q_1, q_2, \cdots, q_L) = (x_1, \cdots, x_M | y_1, \cdots, y_Q)$ and $\phi(x\sigma, y\alpha) = \phi(\{q\}, \{\sigma\}, \{\alpha\})$. Due to single occupancy, the set $\{q\}$ spans the lattice.
With the state vector written in the above manner, the fermionic and bosonic nature of the particles is encoded in the symmetry properties of the wavefunction. Thus the graded permutation operator, $[\mathcal{F}]$, takes a particularly simple form independent of the particle statistics. The resulting Hamiltonian, acting on wavefunctions $\phi(q, \sigma, \alpha)$, is

$$H = -\frac{1}{2} \sum_{1 \leq i \neq j \leq L} \frac{M_{ij}}{d(q_i, q_j)^2}, \quad (13)$$

where $M_{ij}$ interchanges particles $i$ and $j$, $M_{ij}\phi(q, \sigma, \alpha) = \phi(q', \sigma, \alpha)$, with $q' = (q_1', q_2', \ldots, q_L') = (q_1, \ldots, q_j, \ldots, q_i, \ldots, q_L)$. The resulting eigenenergy equation then takes the form

$$-\frac{1}{2} \sum_{1 \leq i \neq j \leq L} d(q_i, q_j)^{-2} M_{ij}\phi(q, \sigma, \alpha) = E\phi(q, \sigma, \alpha) \quad . \quad (14)$$

We immediately see that the lowest energy state in the full Hilbert space corresponds to constant $\phi$. Since this is only compatible with bosonic symmetry, we conclude that the $SU(m|n)$ ground state is in the $M = 0$ sector and is described by the wavefunction

$$\phi_G(y\alpha) = 1 . \quad (15)$$

Because this is a good wavefunction independent of boson flavors $\alpha$, the ground state is in general degenerate, and is given by the $L$-fold symmetric combination of the fundamental representation of $SU(m)$. In terms of the Young tableau, this is the $([L^1], [\cdot])$ of $SU(m) \times SU(n)$. Since the wavefunction is constant, the ground state energy is given by

$$E_G = -\frac{1}{2} \sum_{1 \leq i \neq j \leq L} d(i, j)^{-2} , \quad (16)$$

and may easily be summed. All excitations, whether fermionic or bosonic, are built on top of this.

To be more precise about the nature of the ground state, the purely bosonic $(Q, M) = (L, 0)$ state is just the lowest member of a $SU(m|n)$ supermultiplet, and therefore its degeneracy in the full Hilbert space is larger than indicated by the $M = 0$ sector alone. We find that the ground state representation is atypical, and spans the sectors
\((L-n,n), (L-n+1,n-1), \ldots, (L,0)\) with corresponding representations \(([L-n]^1, [1^n])\), \(([L-n+1]^1, [1^{n-1}]), \ldots, ([L^1], [\cdot])\). Alternatively, this result follows by noting that fermionic wavefunctions which are symmetric in the coordinates \(\{x\}\) may be constructed by taking antisymmetric combinations of their flavors \(\sigma\).

The highest energy state of the \(SU(m|n)\) model is given by a completely antisymmetric wavefunction, corresponding to the fully fermionic sector, \(M = L\). In the absence of periodic boundary conditions, the wavefunction is

\[
\phi_M(x\sigma) = \prod_{i<j}(q_i - q_j),
\]

with energy \(E_M = -E_G\). Since \(\phi_M\) is independent of the fermion flavor \(\sigma\), the highest energy state corresponds to the \(([\cdot], [L^1])\) representation (this time the highest member of a supermultiplet). As a result, all states in the spectrum lie in the energy range \([E_G, -E_G]\). Furthermore, any eigenstate of \(H\), when multiplied by \(\phi_M\), remains a good eigenstate, but with opposite eigenenergy. Since this also interchanges bosonic and fermionic boundary conditions, it corresponds to the interchange of the \(SU(m|n)\) and \(SU(n|m)\) theories. Therefore this indicates that \(H\) for the \(SU(m|n)\) model corresponds to \(-H\) for the \(SU(n|m)\) model \([17]\). We will subsequently make use of this symmetry to obtain the upper bound of the energy levels of the \(SU(m|n)\) permutation model defined on the Hermite lattice.

In general, we wish to work, not in the full Hilbert space, but rather in a given subspace of fixed \((Q,M)\). In this case, wavefunctions have mixed symmetry properties. We will consider the specifics of these wavefunctions in the next two sections. Here we mention that, since the ground state in this sector contains a symmetric combination of \(Q\) bosons, it is again degenerate \([18]\), and corresponds to the \(Q\)-fold symmetric combination of the \(m\) of \(SU(m)\). As a general rule, in order to lower the energy, the wavefunction should be as symmetric as possible. This may be accomplished by distributing the \(M\) fermions as equally as possible into the \(n\) flavors, resulting in the ground state representation \(([Q^1], [1^r])\) where \(r = M \mod n\). This will be made clear subsequently for both the non-uniform and the uniform lattice.
III. $SU(M|N)$ ELECTRONIC MODEL ON A NON-UNIFORM LATTICE

As we have seen above, many properties of the $SU(m|n)$ model are quite general, and result from the permutation nature of the system. In order to proceed further with a detailed study of the excitation spectrum, we now fix the lattice, as specified by the distance function $d(i,j)$. In this section we discuss the $SU(m|n)$ model defined on a non-uniform lattice in one dimension where the sites of the chain are given by the roots of the Hermite polynomial $H_L(x)$. It is well known that this Hermite polynomial has $L$ roots, $r_1, r_2, \ldots, r_L$, which are all real and distinct. Thus the distance function for this non-uniform lattice is defined by

$$d(i,j) = |r_i - r_j|.$$ (18)

This one dimensional chain is well defined and yields an exactly integrable system. The $SU(m|n)$ model on a non-uniform lattice encompasses the $SU(0|n)$ spin chain with inverse square exchange [19] and the $SU(1|n)$ supersymmetric $t$-$J$ model [20,21] as well as the $SU(2|2)$ extended Hubbard model [11].

Since the Hamiltonian, (13), is written solely in terms of permutations on the particle positions and is independent of statistics, many results for the $SU(0|2)$ spin chain on a non-uniform lattice [19] are applicable to the general $SU(m|n)$ system as well. In particular, the Hamiltonian commutes with a set of conserved quantities $\{I_A\}$, namely

$$[H, I_A] = 0, \quad [I_A, I_B] = 0, \quad A, B = 0, 1, 2, \ldots, \infty,$$ (19)

where

$$I_A = \sum_{i=1}^{L} (a_i^\dagger a_i)^A,$$

$$a_k^\dagger = i \sum_{j(\neq k)=1}^{L} (q_k - q_j)^{-1} M_{kj} + iq_k, \quad a_k = (a_k^\dagger)^\dagger.$$ (20)

These relations yield the integrability conditions of the $SU(m|n)$ model. The quantities $I_A$ as written act only on the coordinates of wavefunctions $\phi(\{q\}, \{\sigma\}, \{\alpha\})$ and thus commute with the $SU(m|n)$ algebra. One can use the permutation symmetry property of the wavefunction
\( \phi \) defined by Eqn. (12) in order to write explicit representations for \( I_A \) in terms of the \( b \) and \( f \) operators.

As shown in the previous section, the ground state of this model in the complete Hilbert space has its lowest component in the \((Q, M) = (L, 0)\) subspace, and has an energy

\[
E_G = -\frac{1}{2} \sum_{1 \leq i < j \leq L} \frac{1}{(r_i - r_j)^2} = -\frac{1}{4} L(L - 1) .
\]

While this is the ground state of the system in the complete Hilbert space, we now wish to investigate the model in a given subspace of fixed occupation numbers \(\{Q_\alpha\}, \{M_\sigma\}\). Based on our previous experience with the supersymmetric \(t\)-\(J\) model and the extended Hubbard model, we anticipate the ground state wavefunction in this sector to take the following form:

\[
\phi_0(x_\sigma, y_\alpha) = \prod_{1 \leq i < j \leq M} (x_i - x_j)^{\delta_{\sigma_i, \sigma_j}} e^{i\pi \text{sgn}(\sigma_i - \sigma_j)} .
\]

This is essentially the minimal possible wavefunction that satisfies the appropriate anti-symmetries under fermion exchange. Using the techniques of [20,21], we may prove that this Jastrow wavefunction is an eigenstate of the Hamiltonian (13) with eigenenergy

\[
E_0 = -\frac{1}{4} L(L - 1) + \frac{1}{2} \sum_{\sigma=1}^{n} M_\sigma (M_\sigma - 1) ,
\]

independent of bosonic species, as anticipated in the previous section. Minimizing this energy subject to the constraint \(\sum_\sigma M_\sigma = M\) gives

\[
E(Q, M) = -\frac{1}{4} L(L - 1) + \frac{1}{2n} [M(M - n) + r(n - r)] ,
\]

where \(r = M \text{ mod } n\). The ground state wavefunction has the fermions distributed as evenly as possible among the \(n\) distinct flavors. For \(M \geq n\), this is the highest component of the supermultiplet.

We now turn to the question of excitations above the ground state. As in the supersymmetric \(t\)-\(J\) model and the supersymmetric extended Hubbard model, one can show that there are several ways to create excitations from this ground state. The first way is to excite the \(M\) fermions, the second way is to excite the \(Q\) bosons. In fact, \(a_k^\dagger\) and \(a_k\) are
raising and lowering operators of the Hamiltonian $H$ \cite{20,21}. To construct excited state wavefunctions, we first use the convention implicit in \cite{12} that the fermions are ordered before the bosons. Thus the positions of the fermions are $(q_1, q_2, \cdots, q_M)$ and the positions of the bosons are $(q_{M+1}, \cdots, q_L)$. One way to excite those $f$ fermions is to construct the following wavefunctions:

$$|K_1, K_2, \cdots, K_M \rangle = \sum_{P_1, P_2, \cdots, P_M} \prod_{i=1}^{M} (a_i^\dagger \sigma_i)^{K_{P_i}} |\phi_0 \rangle , \quad (25)$$

where the summation $P$ is over all possible permutations, $K_i$ are nonzero integers, and $|\phi_0 \rangle$ is the state vector corresponding to the amplitude of Eqn. \cite{23}. These states, if not vanishing, will be eigenstates of the Hamiltonian, with eigenenergies given by

$$E(K_1, \cdots, K_M) = E_0 + \sum_{i=1}^{M} K_i . \quad (26)$$

To excite the bosons, one way is to construct the state vectors:

$$|K_1, K_2, \cdots, K_Q \rangle = \sum_{P} \prod_{i=1}^{Q} (a_{i+M}^\dagger \alpha_i)^{K_{P_i}} |\phi_0 \rangle , \quad (27)$$

with eigenenergies given by

$$E = E_0 + \sum_{i=1}^{Q} K_i , \quad (28)$$

if the state vectors constructed this way do not vanish. There are also some other ways to excite the fermions and bosons, such as

$$|K_1, K_2, \cdots, K_Q \rangle = \sum_{P} \prod_{i=1}^{Q} (a_{i+M}^\dagger)^{K_{P_i}} |\phi_0 \rangle . \quad (29)$$

Now, let us return to the question of why $\phi_0$ is anticipated to be the ground state of the system. Heuristically, we note that, by construction, $\phi_0$ is as symmetric as possible while still compatible with fermion statistics. Thus it is as close as possible to the maximally symmetric state $\phi_G$. More rigorously, we can prove explicitly that

$$\left( \sum_{i=1}^{Q} a_{i+M} \alpha_i \right) |\phi_0 \rangle = 0, \quad \left( \sum_{i=1}^{Q} a_{i+M} \right) |\phi_0 \rangle = 0 . \quad (30)$$
Furthermore, one can show that
\[ \sum_{i=1}^{M} a_i \sigma_i |\phi_0\rangle = 0, \quad \sum_{i=1}^{M} a_i |\phi_0\rangle = 0, \]  
for the fermions’ degrees of freedom. These relations yield the impossibility of constructing non-vanishing states of energy \( E_0 - 1 \) by using the lowering operators \( a_k \). We therefore can regard these identities as partial confirmation that \( \phi_0 \) is the ground state. A more complete proof requires further work.

In the subspace of fixed \( (\{Q_\alpha\}, \{M_\sigma\}) \), the full energy spectrum of the \( SU(m|n) \) model on this non-uniform lattice is expected to consist of equally-spaced energy levels:
\[ E_S = E_0 + S, \]  
where \( S = 0, 1, 2, \ldots, S_{max} \). There is an upper bound on the values of \( S \) due to the finite size of the Hilbert space. Using the boson–fermion interchange symmetry, \( H \rightarrow -H \) for \( SU(m|n) \rightarrow SU(n|m) \), and Eqn. (23), we find that \( E_{S_{max}} = \frac{1}{4} L (L - 1) - \frac{1}{2} \sum_\alpha Q_\alpha (Q_\alpha - 1) \). Presently we are unable to characterize the pattern of the energy spectrum by some systematic rule, nor are we able to explain the degeneracies of the energy levels by the underlying symmetries of the system (presumably Yangian symmetry).

### IV. \( SU(M|N) \) ELECTRONIC MODEL ON A UNIFORM LATTICE

We now turn to the \( SU(m|n) \) supersymmetric electronic models on a uniform lattice in one dimension. This uniform lattice is characterized by the trigonometric interaction
\[ d(i, j) = \frac{L}{\pi} \sin \left( \frac{\pi |i - j|}{L} \right). \]  
The general \( SU(m|n) \) model reduces to the Haldane-Shastry spin chain \( [13, 14] \) for the case of \( SU(0|2) \), the supersymmetric \( t-J \) model introduced by Kuramoto and Yokoyama \( [15, 22, 23] \) for \( SU(1|2) \), and the extended Hubbard model studied by us previously \( [12] \) for \( SU(2|2) \).

For the Hamiltonian \( [13] \) on a uniform lattice, we may use the results of Fowler and Minahan for the Haldane-Shastry spin chain \( [24] \), except that one has to use the permutation
properties of the amplitude $\phi$ as defined by Eqn. (12). The result is a set of conserved quantities $I_A$,

\[ [H, I_A] = 0, \quad [I_A, I_B] = 0, \quad A, B = 0, 1, 2, \ldots, \infty, \quad (34) \]

where

\[
I_A = \sum_{i=1}^{L} \pi_i^A, \\
\pi_j = \sum_{k(\neq j)=1}^{L} \frac{Z_k}{(Z_k - Z_j)} M_{jk}, \quad Z_k = \exp\left(\frac{2\pi i q_k}{L}\right), \quad (35)
\]

which yield the integrability of the $SU(m|n)$ electronic model.

In the full Hilbert space, the ground state (as the lowest component of a supermultiplet) is obtained when $M = 0$, with trivial ground state wavefunction \cite{15}. The corresponding ground state energy on the uniform lattice is

\[
E_G/(\pi/L)^2 = -\frac{1}{2} \sum_{1 \leq i < j \leq L} \sin^2(\pi(i-j)/L) = -\frac{1}{6} L(L^2 - 1), \quad (36)
\]

with degeneracy of the lowest component given by the ($[L^1], [\cdot]$) representation of $SU(m) \times SU(n)$.

Once again, we turn to the examination of the ground state in a given subsector specified by $(\{Q_\alpha\}, \{M_\sigma\})$. As before, when fermions are present, we expect the ground state wavefunction to have the minimum antisymmetry possible consistent with fermi statistics. This time, however, due to the uniform lattice, we also require the wavefunctions to have the appropriate periodicity under $q_i \to q_i + L$. The results for the $SU(1|n)$ case \cite{25} motivate us to write the following Jastrow wavefunctions for the $SU(m|n)$ case:

\[
\phi_0(x_\sigma, y_\alpha) = \prod_{i=1}^{M} X_i^{J_{\sigma_i}} \prod_{1 \leq i < j \leq M} (X_i - X_j)^{\delta_{\sigma_i, \sigma_j}} e^{i\frac{\pi}{L} \text{sgn}(\sigma_i - \sigma_j)}, \quad (37)
\]

where $X_k = \exp\left(\frac{2\pi i x_k}{L}\right)$, and $M = \sum_{\sigma} M_\sigma$ is the total number of fermions. These wavefunctions are eigenstates of the Hamiltonian provided the quantum numbers $J_1, J_2, \ldots, J_n$ (uniform lattice momenta for each fermion flavor) obey the constraints $-M_\sigma \leq J_\sigma \leq 0$. 


The eigenenergies corresponding to the above wavefunctions were derived in [25], and take on a more concise form when expressed as a function of the shifted momenta, $K_\sigma = J_\sigma + \frac{1}{2}(M_\sigma - 1)$. Without loss of generality, we assume the occupation numbers, $\{M_\sigma\}$, are ordered according to $M_1 \geq M_2 \geq \cdots \geq M_n$. In this case, we find

$$E_0 = -\frac{1}{6}L(L^2 - 1) + \frac{1}{6n}\sum_{\sigma=1}^{n}(3L + (\sigma - 2)M_\sigma)(M_\sigma^2 - 1) - \frac{1}{2}\sum_{\sigma' < \sigma} M_\sigma(M_{\sigma'}^2 - 1)$$

$$+ 2(L - M)\sum_{\sigma=1}^{n} K_\sigma^2 + 2\sum_{\sigma' < \sigma} M_{\sigma'}(K_{\sigma'} - K_\sigma)^2,$$

(38)
provided the (either integer or half integer) $K_\sigma$'s satisfy

$$|K_\sigma| \leq \frac{1}{2}(M_\sigma - 1)$$

$$|K_\sigma - K_{\sigma'}| \leq \frac{1}{2}(M_\sigma - M_{\sigma'}) \quad \text{for } \sigma' > \sigma.$$

(39)

The lattice momentum carried by this state (over the bosonic background) is simply $P_0 = \sum_\sigma M_\sigma K_\sigma$.

Examination of Eqn. (38) indicates that the lowest energy state in the above set of Jastrow wavefunctions is reached when the $K_\sigma$'s are all as close to 0 as possible. This is accomplished by taking $K_\sigma = 0$ for odd $M_\sigma$ or 1/2 for even $M_\sigma$. Note that whenever some of the $M_\sigma$'s are even, we could equally well have taken $K_\sigma = -1/2$, leading to a two-fold degeneracy of the ground state arising from the reflection symmetry of the lattice (in addition to the degeneracy arising from the bosonic species). As in the case of the non-uniform lattice, the ground state in a given $(Q, M)$ sector has the $M$ fermions distributed as evenly as possible. The ground state energy is then given by

$$E(Q, M)/(\pi/L)^2 = -\frac{1}{6}L(L^2 - 1) + \frac{1}{6n}(3L - M)(M^2 - n^2)$$

$$+ \frac{1}{6n}r(n - r)[3(L + M) + 2(n - 2r)]$$

$$+ \frac{1}{2}(L - M) \begin{cases} n - r, & \text{if } \text{Int}(M/n) \text{ is even;} \\ r, & \text{otherwise.} \end{cases}$$

(40)

As before, $r$ is given by $r = M \mod n$. Only the first line is important in the thermodynamic limit. This ground state (and its reflected pair for $P \neq 0$ or $L/2$) transforms as the $([Q^1], [r^1])$
of $SU(m) \times SU(n)$ (once again the highest component of a supermultiplet for $M \geq n$). We note that when all $M_\sigma$ are odd, this particular wavefunction can also be obtained by taking the strong interaction limit of the corresponding continuum quantum system \cite{26}.

The full energy spectrum for this long range permutation model may be derived using the asymptotic Bethe-ansatz (ABA) \cite{10}. The motivation for this ABA lies in the fact that the scattering is essentially two-body in nature, even in the presence of long range interactions, as indicated by the Jastrow form of the wavefunction. The ABA has been used successfully in the $SU(1|2)$ supersymmetric $t$-$J$ model \cite{27} as well as its $SU(1|n)$ generalization \cite{22}, and gives exact results, even in the non-asymptotic regime, as proven to be true for the $SU(0|2)$ Haldane-Shastry spin chain \cite{28} and for the $SU(1|2)$ $t$-$J$ model \cite{23} — a fact that is related to the integrability of the system.

For the $SU(m|n)$ generalization of the ABA, we need to treat multiple flavors of both bosons and fermions on the lattice. In the case of nearest-neighbor interactions, Lai first discussed such lattice permutation models with mixtures of bosons and fermions in 1974 \cite{29}. This was later generalized by various authors \cite{18,30,31,8}. For the multi-species case, the Bethe-ansatz takes on a nested form, where one species is removed at each step. For the long range $SU(m|n)$ model, this is accomplished by starting with one of the $m+n$ (either bosonic or fermionic) species as the background, and then peeling off the remaining $m+n-1$ species one at a time, resulting in $m+n-1$ sets of Bethe-ansatz equations. There are a total of $(m+n)!$ ways to perform this nesting. However this number may be reduced by the obvious $n!$ fermion permutations and $m!$ boson permutations, yielding $(m+n)!/(m!n!)$ independent ways of writing the nested Bethe-ansatz. A particular choice of nesting may be denoted by a string of $m+n$ $B$’s or $F$’s, indicating the order in which the bosonic and fermionic species are removed, working from right to left (the rightmost character denotes the choice of background). Of course all such choices of the nesting should yield equivalent results.

The ABA obtained previously for the $SU(1|n)$ supersymmetric $t$-$J$ model \cite{22} corresponds to the $BF^n$ choice of nesting.

For a general $\nu (= m+n)$ component system, we encode the nesting of the ABA according
to the statistics $s_i$ of species $i$ where $s_i = 0$ for bosons and 1 for fermions. Since we take the first species as the background, the $B^m F^n$ nesting corresponds to $\{s\} = (1, \ldots, 1, 0, \ldots, 0)$. We denote the occupancies of the species by $M_i$ (here $i$ runs from 1 to $\nu$) and define the quantities

$$N_i = \sum_{j > i}^\nu M_j ,$$  \hspace{1cm} (41)

where $N_0 = L$ and $N_\nu = 0$. For the nested ABA, we introduce $\nu - 1$ sets of pseudomomenta, $\{p^{(1)}\}, \{p^{(2)}\}, \ldots, \{p^{(\nu-1)}\}$ where each set $\{p^{(a)}\}$ consists of the $N_a$ quantities

$$p^{(a)}_i : \quad i = 1, 2, \ldots, N_a ,$$  \hspace{1cm} (42)

all of which are within the range $[-\pi, \pi]$. The energy and the total lattice momentum of the system depend only on the $p^{(1)}_i$'s, and are given by

$$E = (-1)^{s_1+1} \left[ \frac{\pi^2}{6} L (1 - \frac{1}{L^2}) + \frac{1}{2} \sum_{i=1}^{N_1} (p^{(1)}_i)^2 - \pi^2 \right] ,$$

$$P = \left[ s_1 (L - 1) \pi + \sum_{i=1}^{N_1} (p^{(1)}_i - \pi) \right] \mod 2\pi .$$  \hspace{1cm} (43)

The pseudomomenta, $\{p^{(a)}\}$ for $a = 1, \ldots, \nu - 1$, are obtained by solving the $\nu - 1$ set of equations of the nested ABA, expressed concisely in the following form:

$$\sum_{k=1}^{N_a-1} \theta(p^{(a)}_i - p^{(a-1)}_k) = 2\pi I^{(a)}_i + \delta_{s_a,s_{a+1}} \sum_{j=1}^{N_a} \theta(p^{(a)}_i - p^{(a)}_j) + (-1)^{\delta_{s_a,s_{a+1}}} \sum_{l=1}^{N_{a+1}} \theta(p^{(a)}_i - p^{(a+1)}_l) ,$$  \hspace{1cm} (44)

where $\theta(x) = \pi \text{sgn}(x)$ is a step function. For $a = 1$ the left hand side of (44) is replaced by $p^{(1)}_i L$, and for $a = \nu - 1$ the last term on the right hand side is dropped. A given state in the ABA spectrum is thus specified by the set of quantum numbers $\{I^{(a)}\}$. At a given level $a$ of the nesting, there are $N_a$ non-overlapping $I^{(a)}_i$'s (all integers or half-integers as appropriate) which are required to lie in the range

$$|I^{(a)}_i| \leq \frac{1}{2} (N_{a-1} - N_a + N_{a+1} - 1) ,$$  \hspace{1cm} (45)

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for $s_a = s_{a+1}$ or

$$|I_i^{(a)}| \leq \frac{1}{2}(N_{a-1} - N_{a+1} - 2) ,$$

(46)

for $s_a \neq s_{a+1}$. These conditions lead to the restrictions on the occupation numbers, either $M_a \geq M_{a+1}$ or $M_a \geq N_{a+1} + 1$ respectively. An exception happens when $N_a = 0$, in which case the nested ABA terminates early, giving no further restriction on $M_a$.

We remark that in general the ABA still provides exact eigenenergies when conditions (45) and (46) are relaxed by one unit on either end. However we have found the above restrictions to be necessary in order to ensure that the ABA generates only highest weight states of the $SU(m|n)$ superalgebra. It is important to realize that the choice of nesting plays a crucial role in defining what is meant by the highest weight representation. In particular, for $SU(m|n)$, the Cartan subalgebra lies in the bosonic subgroup $SU(m) \times SU(n) \times U(1)$ and has rank $m+n-1$. Thus highest weight representations of $SU(m|n)$ are annihilated by a set of $m+n-1$ simple positive roots. It is this set of simple roots which is determined by the nesting of the ABA, with identical statistics in the nesting ($s_a = s_{a+1}$) corresponding to bosonic roots and $s_a \neq s_{a+1}$ corresponding to fermionic roots. For this reason the most straightforward nesting is given by $B^m F^n \ (or \ \ F^n B^m)$, in which case there are $m+n-2$ bosonic roots corresponding to the simple positive roots of $SU(m) \times SU(n)$ and a single fermionic root taking a state from the $(Q, M)$ sector to $(Q-1, M+1)$. This gives the conventional definition of the highest weight state of $SU(m|n)$ residing in the highest component of the supermultiplet.

In order to be more specific, we now pick the natural $B^m F^n$ nesting and the original notation of $M_\sigma$ and $Q_\alpha$ for fermionic and bosonic occupation numbers respectively. The nested ABA may be written in terms of the $n-1$ sets of “fermionic” pseudomomenta

$$p_i^{(a)} : \quad i = 1, 2, \ldots, N_a \quad \text{where} \quad N_a = Q + \sum_{\sigma=a+1}^{n} M_\sigma ,$$

(47)

and $m$ sets of “bosonic” pseudomomenta

$$q_\mu^{(b)} : \quad \mu = 1, 2, \ldots, N_b' \quad \text{where} \quad N_b' = \sum_{\alpha=b}^{m} Q_\alpha .$$

(48)
The ABA may now be written explicitly as

\[ p_i^{(1)} L = 2\pi I_i^{(1)} + \sum_{i' \neq i} \theta(p_i^{(1)} - p_i') - \sum_j \theta(p_i^{(1)} - p_j^{(2)}) , \]

\[ \sum_i \theta(p_i^{(2)} - p_i^{(1)}) = 2\pi I_j^{(2)} + \sum_{j' \neq j} \theta(p_j^{(2)} - p_j') - \sum_k \theta(p_j^{(2)} - p_k^{(3)}) , \]

\[ \vdots \]

\[ \sum_j \theta(p_k^{(n-1)} - p_j^{(n-2)}) = 2\pi I_k^{(n-1)} + \sum_{k' \neq k} \theta(p_k^{(n-1)} - p_k') - \sum_\mu \theta(p_k^{(n-1)} - q_\mu^{(1)}) , \]

\[ \sum_k \theta(q_\mu^{(1)} - p_k^{(n-1)}) = 2\pi J_\mu^{(1)} + \sum_\nu \theta(q_\mu^{(1)} - q_\nu^{(2)}) , \]

\[ \sum_\mu \theta(q_\nu^{(2)} - q_\mu^{(1)}) = 2\pi J_\nu^{(2)} + \sum_{\nu' \neq \nu} \theta(q_\nu^{(2)} - q_\nu') - \sum_\gamma \theta(q_\nu^{(2)} - q_\gamma^{(3)}) , \]

\[ \vdots \]

\[ \sum_\nu \theta(q_\gamma^{(m-1)} - q_\nu^{(m-2)}) = 2\pi J_\gamma^{(m-1)} + \sum_{\gamma' \neq \gamma} \theta(q_\gamma^{(m-1)} - q_\gamma^{(m-1)}) - \sum_\eta \theta(q_\gamma^{(m-1)} - q_\eta^{(m)}) , \]

\[ \sum_\gamma \theta(q_\eta^{(m)} - q_\gamma^{(m-1)}) = 2\pi J_\eta^{(m)} + \sum_{\eta' \neq \eta} \theta(q_\eta^{(m)} - q_\eta') , \]

with quantum numbers \( \{I^{(a)}\} \) and \( \{J^{(b)}\} \). It is anticipated that the ABA span the full set of energy levels when both the highest weight property of the ABA states and the supermultiplet structure connecting states in different \((Q, M)\) sectors are taken into account.

We have verified that this is the case numerically for the \( SU(2|2) \) model on small lattices. The numerical results also indicate that, while the ABA accounts for the full energy spectrum, it does not give all the proper degeneracies of the states. The missing states presumably arise from localized bound states which are not described by the asymptotic Bethe-ansatz. Thus, in order to properly count the degeneracies and derive the full \( SU(m|n) \) supermultiplet structure of the spectrum, one must generalize the “squeezed string” picture of the \( SU(n) \) Haldane-Shastry spin chain [32].

V. CONCLUSION

In this work, we have introduced the \( SU(m|n) \) electronic model of long range interaction, which is integrable on both uniform and non-uniform one dimensional lattices. The ground
state, excitation spectrum, and integrability properties of this model have been considered in detail. Since the $SU(m|n)$ model includes many previously well-known models as special cases, we have shown how a uniform treatment of all such models may be accommodated.

For the integrable $SU(m|n)$ model on a non-uniform lattice, perhaps the most interesting open problem is to explain the high degeneracy of the equally-spaced energy levels using the underlying symmetries of the system. Presently it is still unclear how to construct the thermodynamics of the long range $SU(m|n)$ permutation model on the Hermite lattice.

For the uniform lattice, since the energies of the ABA states are given by the essentially non-interacting formula, we note the interesting result that for a given lattice of length $L$, the set of allowed energy levels are completely determined, and is independent of the number of fermionic and bosonic species, $n$ and $m$. The differences between the $SU(m|n)$ models on a uniform lattice hence lie only in the degeneracies and supermultiplet structures of the models, with larger $n$ and $m$ leading to higher degeneracies.

Since the full $SU(m|n)$ symmetry is broken in most physical problems, we generally wish to work in a given subspace of fixed $(Q, M)$, corresponding to the breaking of supersymmetry. It is clear that, in this case, a proper understanding of both the supermultiplet structure (including degeneracies) and $SU(m|n)$ representation theory is required. We note that the latter is quite intricate as atypical representations are clearly involved.

Working with fixed $(Q, M)$, one only sees the bosonic $SU(m) \times SU(n)$ subgroup of the full $SU(m|n)$ symmetry group. For both the non-uniform and uniform lattice, we have explicitly found the ground state wavefunctions and their corresponding eigenenergies. Because of the symmetry properties of the wavefunction, the ground state wavefunction is independent of the bosons. Hence the ground state in any $Q > 0$ sector is degenerate whenever $m > 1$, and transforms in the $([Q^1], [r^1])$ representation where $r = M \mod n$.

The ground state structure and ABA results for the excited spectrum of states on the uniform chain presented above opens up the possibility of studying the properties of elementary excitations of all the general $SU(m|n)$ models in a consistent manner. While we have in particular focused on lattice models with inverse square exchange, our results carry over
to many other graded permutation models as well.

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