Intrinsic torsion classes of Riemannian structures

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Abstract

This article introduces the problem of finding intrinsic torsion varieties associated to $G$-structures on a fixed parallelizable Riemannian manifold. As an illustration, the intrinsic torsion varieties of orthogonal almost product structures are analysed on the Iwasawa manifold.

Introduction

Determining integrability is a fundamental problem in differential geometry. A suitable formulation relates the equivalence problem for $G$-structures with the theory of $k$-jets. In such a context, the intrinsic torsion of a $G$-structure provides a first-order obstruction to integrability. However, in certain circumstances, vanishing of the intrinsic torsion is not only a necessary but also a sufficient integrability condition. The celebrated Darboux and Newlander–Nirenberg theorems assert exactly this for almost symplectic and almost complex structures respectively. On the other hand, in the Riemannian situation, non-vanishing intrinsic torsion impedes a reduction of the holonomy group.

In this paper, we shall consider reductions where $G$ a proper subgroup of $O(N)$. Symmetry properties of the intrinsic torsion tensor $\tau$ have been exploited in various cases for classifying manifolds with a $G$-structure. The prototype classification is given by Gray and Hervella [7], and regards almost Hermitian manifolds. It is based on criteria whereby $\tau$ belongs to some combination of irreducible $U(n)$-components of the relevant space. In the same article, Gray and Hervella subdivide almost symplectic manifolds into four classes using the irreducible representations of $Sp(n, \mathbb{R})$. Later in [12], Naveira applied a similar approach to the classification of almost product manifolds, Cabrera and Swann studied the case of quaternion-Hermitian manifolds [3] etc.

A typical application of a Gray–Hervella type classification is to consider a fixed $G$-structure on a Riemannian manifold, with $G$ a subgroup of $O(N)$, and then determine the components of the intrinsic torsion relative to the action of $G$. A variation of this approach consists in fixing a subgroup $G$ of $O(N)$, with the purpose of realizing $G$-structures of
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a specific class. Such constructions of $G_2$-structures on 7-manifolds are described in [5], symplectic structures in [6], generalized complex structures in [4], and in other articles by Salamon, Fino, Chiossi and others.

Our approach is somewhat different. Whilst we again consider a Riemannian manifold $M$ with a fixed metric and subgroup $G$ of $O(N)$, our aim is to detect different $G$-structures on $M$ as subsets of a suitable parameter space. The motivation of such a set-up arises in theoretical physics. The current lack of a consistent quantum-gravitational theory often requires the development of dynamical models in gauge theories, string theories and $M$-theory, compatible with a fixed (or slightly varying) gravitational background.

A favorable context for such an analysis is the case in which $M$ is a nilmanifold. Nilmanifolds are parallelizable in a natural way, in fact their geometry reflects properties of the underlying nilpotent algebra in terms of invariant tensors. An invariant $G$-structure becomes a point of the homogeneous space $O(N)/G$, and this space contains our subsets (parametrizing special $G$-structures) as real subvarieties. These are the intrinsic torsion varieties associated to the problem.

In the first two sections, we introduce some general considerations and results for determining intrinsic torsion varieties. In the rest of the article, we analyse a specific case, namely the classification of orthogonal almost product structures on the Iwasawa manifold $\mathcal{I}$. This case is based on the fact that $SO(6)/G$ can be interpreted as the Grassmannian of 2-planes in real six-dimensional space. Inside this space, we identify varieties corresponding to null-torsion classes.

A similar analysis has been described in [1] where the Gray–Hervella classes of invariant Hermitian structures on $\mathcal{I}$ are described as subvarieties of the projective space $\mathbb{CP}^3$. In this context, the conditions defining the null-torsion classes can be formulated purely in terms of the exterior derivative, the Hodge star operator and the wedge product. This is far from true in general: only part of the null-torsion classes of in our situation can be determined by analogous conditions. This observation motivates further the techniques developed in the present article.

1 Intrinsic torsion varieties

A point $p$ of the frame bundle $LM$ over a differentiable manifold $M$ corresponds to a specific identification of the tangent space (at the point $\pi(p)$ of $M$) with $\mathbb{R}^n$. The soldering form $\theta : T_p(LM) \to \mathbb{R}^n$ on $LM$ is canonically defined by

$$\theta(X) = p^{-1}(\pi_*(X)).$$

The restriction of $\theta$ to a $G$-structure $P$ then allows one to view the tangent space in each point of $M$ as a representation $\rho_P$ of $G$ on $\mathbb{R}^n$, determined by the right $G$-action on $P$, 

i.e. $R^*_g \theta = g^{-1}\theta$. We also use $\rho_P$ to denote the representation induced on each associated $G$-module.

The covariant derivative $\nabla_\phi \theta$ associated to an affine connection $\phi$ defines a horizontal torsion form $\Theta$ on $LM$, with values in $\text{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)$. This object can be interpreted as a vector-bundle valued torsion tensor on $M$. The structure function on $P$ represents, at each point, that component of $\Theta$ independent of the choice of connection compatible with the structure. More precisely, it is the image of $\Theta$ in the quotient space

$$\mathcal{W} = \frac{\text{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)}{\alpha(\text{Hom}(\mathbb{R}^n, \mathfrak{g}))}.$$  

Here, $\alpha$ is the skew-symmetrization, given by $\alpha(S)(u)v = S(u)v - S(v)u$ for $S$ in $\text{Hom}(\mathbb{R}^n, \mathfrak{g})$, and $\mathfrak{g}$ is the Lie algebra of $G$. We shall refer to this image as the intrinsic torsion, and denote it by $\tau$. It too can be interpreted as a section over $M$ of the vector bundle with fibre (1).

The variation of $\Theta$ along a fiber of $P$ highlights the representation $\rho_P$ on $\mathcal{W}$:

$$\rho_P(g)\Theta(u, v) = g\Theta(g^{-1}u, g^{-1}v)$$

Since $\alpha(\text{Hom}(\mathbb{R}^n, \mathfrak{g}))$ is a $G$-invariant subspace, the representation passes to the quotient. If the structure function at a point $p \in P$ takes values in some $\rho_P$-submodule $\mathcal{U}$, (2) will keep it in the same submodule. This implies that the value of $\tau$ at $\pi(p)$ belongs to the corresponding subbundle associated to the representation of $G$ on $\mathcal{W}$. Equivalently, we can assert that $\tau$ maps to zero in the quotient $\mathcal{W}/\mathcal{U}$. Following the expression coined in [10], we state

**Definition 1.** A null-torsion structure on a manifold is a $G$-structure for which $\tau$ belongs to a proper $G$-submodule $\mathcal{U}$ of $\mathcal{W}$.

We shall next apply these concepts to the orthogonal case. Consider a fixed Riemannian manifold $M$, so that the initial structure we are starting from corresponds to the orthogonal group $O(N)$. For $G = O(N)$, the space (1) is zero, and the consequent vanishing of the structure function explains the existence of a canonical (Levi-Civita) torsion-free connection $\phi_{LC}$ [16].

Now let $G$ be a closed subgroup of $O(N)$. A reduction to a $G$-structure can be detected by realizing $G$ as the stabilizer of a tensor $\xi$ in a suitable $O(N)$-module $V$. We shall call such a reduction $P_\xi$. The parameter space of such reductions pointwise can be identified with the $O(N)$-orbit $\mathcal{O}_\xi$ of $\xi$ inside $V$. In these terms, a $G$-structure on $M$ is a smooth section of a bundle with fibre $\mathcal{O}$, isomorphic to the coset space $O(N)/G$.

The torsion of the reduced structure is given by the restriction of $\Theta$ to $P_\xi$, and $\tau_\xi$ arises by eliminating the dependence of $\Theta$ on connections compatible with $P_\xi$. Of course, in general, the Levi-Civita connection does not reduce to $P_\xi$, so $\tau_\xi$ does not vanish. This
incompatibility arises from the non-zero projection of a vector horizontal with respect to \( \phi_{LC} \) to \( g_\xi^\perp \) (the orthogonal complement of \( g_\xi \) inside the vertical space of the bundle of orthonormal frames). In other words, \( \tau_\xi \) depends on the mutual positions of \( \phi_{LC} \) and the connections compatible with \( P_\xi \), and we always have an isomorphism

\[
\mathcal{W} \cong \mathbb{R}^n \otimes g_\xi^\perp. \tag{3}
\]

In general, the space (1) decomposes into a direct sum of irreducible subspaces \( \mathcal{W}_i \) under the action of the compact group \( G \). We may therefore speak of classes for which \( \tau \) lies in some proper subspace \( \mathcal{W} = \bigoplus_{i \in I} \mathcal{W}_i \). This motivates the popular classifications of \( G \)-structures exploiting \( \mathcal{W} \), in which \( I \) can be used to label the null torsion class.

Now suppose that \( M \) is a parallelizable manifold, equipped with a global section \( s \) of the frame bundle \( LM \). If we declare this to be orthonormal, we have defined a Riemannian metric on \( M \). The bundle of orthonormal frames is a product \( M \times O(N) \), and a subordinate \( G \)-structure is a function \( \xi : M \rightarrow O(N)/G = \mathcal{O} \). We shall call a \( G \)-structure invariant if (relative to the parallelization \( s \)) \( \xi \) is constant. The invariant \( G \)-structures on \( M \) are therefore parametrized by a single ‘classifying’ orbit \( \mathcal{O} \).

**Definition 2.** An intrinsic torsion variety (ITV) is a subset of \( \mathcal{O} \) defined by invariant structures on a parallelizable Riemannian manifold \( M \) whose intrinsic torsion belongs to some proper submodule of \( \mathcal{W} \).

One may now formulate the following general problem: Analyse the geometry of the intrinsic torsion varieties of invariant structures on parallelizable Riemannian manifolds. In the invariant set-up, each point \( \xi \) in \( \mathcal{O} \) is effectively an invariant reduction \( P_\xi \), and determines

1) the corresponding intrinsic torsion \( \tau_\xi \),
2) a space (1) in which \( \tau_\xi \) is located.

The fact that (1) itself depends on \( \xi \) renders the above problem geometrically interesting.

The action \( \rho_{O(N)} \) moves \( \xi \) around in the homogeneous space \( \mathcal{O}_\xi \), and the corresponding variation of \( \Theta \) is given by (2). Futhermore, we may regard (1) as defining a fibre bundle over \( \mathcal{O} \), and the action of \( O(N) \) lifts to this bundle. We may identify the space (1) containing \( \tau_\xi \) with the orthogonal complement \( \alpha(\text{Hom}(\mathbb{R}^n, g)) \perp \) in \( \text{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n) \). The latter space therefore contains a distinguished \( O(N) \)-orbit generated by the \( \tau_{g_\xi} \) in these orthogonal complements. The stabilizer \( H \) of \( \tau_{\xi} \) in \( \rho_{O(N)} \) can be regarded as a proper subgroup of the stabilizer of \( \xi \) (we shall give an illuminating example below). In these terms an explicit formula for \( \tau_{g_\xi} \) can be constructed. In any case:

**Remark 3.** The intrinsic torsion \( \tau_\xi \) can itself be used to define an \( H \)-structure on \( M \). These structures are parametrized by the \( O(N) \)-orbit \( \mathcal{O}_{\tau_\xi} \cong O(N)/H \). The orbit \( \mathcal{O}_{\tau_\xi} \) fibres over \( \mathcal{O}_\xi \) with fibre \( G/H \).
The action of $O(N)$ on the vector space $V$ containing $\xi$ induces an infinitesimal action $\mathfrak{so}(N) \to V$. Fix a $G$-structure $P_\xi$. The Levi-Civita covariant derivative $\nabla_{LC}\xi$ belongs to $T^*M \otimes \mathfrak{so}(N) \cdot \xi$ and we have an injective map depending on $P_\xi$:

$$T_\xi : \mathbb{R}^n \otimes \mathfrak{g}^\perp \to T_pM \otimes V, \quad T_\xi(v \otimes A) = \theta^{-1}(X) \otimes A \cdot \xi.$$  

The intrinsic torsion of the structure determined by $\xi$ is given by $\tau = T_\xi^{-1}(\nabla_{LC}\xi)$. In these terms, the null-torsion condition translates into

$$\nabla_{LC}\xi \in T_\xi(\mathcal{W}).$$

**Remark 4.** Even without exploiting an explicit variation rule which (given $\tau_\xi$) predicts $\tau_{g\xi}$, we can apply the following heuristic argument. The action of $O(N)$ on $TM$ and $T^*$ can be expressed in terms of polynomials in the components, and the same is true for their tensor powers seen as $O(N)$-modules (including the space of connections, torsion and intrinsic torsion). So given a $G$-structure $P_\xi$ which belongs to some known class, the answer of the question ‘which are the elements $g$ of $O(N)$ such that $P_{g\xi}$ belongs to the same class’ can be expressed in such polynomial terms. This implies that the intrinsic torsion varieties are determined by polynomial functions over suitable algebraic sets and thus they are actually varieties in this more general algebraic sense.

The above observation, involving invariant polynomial functions, suggests that the theory of ITV’s can be included in the more general framework of the Geometric Invariant Theory (GIT). In [10, 11] has already been highlighted the role that natural (for example symplectic or complex) structures on $\mathcal{O}$ are related to the geometry of ITV’s. We expect that establishing more general relations between the geometry of $\mathcal{O}$ and a set of quadratic invariants (analogous to those in [7]) of the $G$-representations on $\mathcal{W}$ will lead to a more general and complete theory of ITV’s.

## 2 The canonical tangent distribution

Observe that the space of intrinsic torsion $\mathcal{W}_p \cong T_p^*M \otimes \mathfrak{g}^\perp$ can be interpreted as:

$$\mathcal{W}_p \cong T_p^*M \otimes T_\xi \mathcal{O}.$$  

So at each point of $M$ $\tau$ can be seen as an equivariant linear map from $T_pM$ to $T\mathcal{O}$. Leaving $\xi$ free to vary in $\mathcal{O}$ we obtain a tangent vector distribution on the orbit. In the following sections we shall provide examples showing that the rank of this distribution varies from point to point. Such an object is known as a generalized or Stefan-Sussmann distribution. We highlight this nature of $\tau$ in the following:

**Proposition 5.** The map $\tau$ induces on $\mathcal{O}$ a generalized tangent distribution $\mathcal{T}$.  

The distribution $\mathcal{F}$ is a canonical object on the classifying orbit $\mathcal{O}$. In certain sense it reflects the geometry of $M$ onto $\mathcal{O}$. Even if it contains less information than the map $\tau$ (it cares no information about the $G$-representation on $\mathcal{V}$) the study of this distribution is interesting on its own right.

A natural attempt to interpret $\mathcal{F}$ consists in regarding it as the expression of an infinitesimal action of the diffeomorphisms of $M$ on the orbit $\mathcal{O}$. The condition that a set of maps implements the action of a set of infinitesimal diffeomorphisms in our case translates in:

$$\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi = 0$$

As $\xi$ is free to vary on $\mathcal{O}$, we conclude that the generalized distribution $\mathcal{F}$ is the expression of an infinitesimal action only in the case in which the curvature of the associated bundle $\mathcal{V}$ vanishes.

Suppose that some components of the tensor $\tau$ vanish invariantly with respect to the representation. In other words the position of $\tau$ in $\mathcal{V}$ is not generic, but confined to some specific subspace. This means that the map $\tau : T_p M \to T\mathcal{O}$ admits a non-trivial kernel.

As explained previously (see also [10]) the intrinsic torsion of a reduction $P$ of a $G$-structure $Q$ can be analysed in terms of mutual positions of the horizontal spaces corresponding to a specific $Q$-connection and the connections compatible with $P\xi$. Varying $P\xi$ the possible mutual positions change. In this context a refinement of the proof of the statement [1] given in [14] leads to the following:

**Proposition 6.** The space $\ker \tau$ is composed by vectors whose vertical lift by a connection compatible with $P$ is horizontal with respect to the Levi-Civita connection.

But the existence of a kernel of $\tau$ implies the existence of a non trivial proper $G$-submodule of $TM$, and thus:

**Proposition 7.** The reducibility of the representation of $G$ on $T_p M$ is a necessary condition for the existence of non-trivial kernel of the map $\tau$.

In conclusion when the required $G$-representation $\rho$ is irreducible, the position of $\tau$ inside $\mathcal{V}$ is generic, and intrinsic torsion class of a $G$-structure depends only on the position of $\tau$ with respect to the representation induced on $\mathcal{V}$.

## 3 Almost product structures

We believe that a suitable starting point for the general analysis of the ITV’s can be the Riemannian $G$-reductions defined by a 2-form. The isomorphism $\Lambda^2 T^* \cong \mathfrak{so}(n)$ (the vertical space of the bundle) facilitates a comprehension of the general features. The theory can be generalized for other $O(n)$-modules.
Consider a smooth section $\tilde{\mathfrak{F}}$ of the tensor bundle $T^*M \otimes TM$ over a differentiable manifold $M$. The value in $p$ of $\tilde{\mathfrak{F}}$ is an endomorphism $\tilde{\mathfrak{F}}_p \in \text{End}(T_pM)$. If $\tilde{\mathfrak{F}}$ is stabilized by a subgroup $G \subset \text{Gl}(n, \mathbb{R})$ we can distinguish a $G$-structure on $M$ (see [8]). The action of the endomorphism $\tilde{\mathfrak{F}}_p$ on $T_pM$ can be analysed in terms of its eigenspaces (in particular its kernel) which give rise to distributions of vector subspaces in $TM$. The integrability properties of such distributions over $M$ characterize the manifold. Many geometrical (almost complex, $f$-structures [2, 18], mixed [10, 11] etc.) structures are relevant examples. This set-up includes the almost product structures (APS) introduced by Naveira [12].

Namely considering at each point of $M$ the splitting of the tangent space in the following horizontal and vertical subspaces:

$$V = \text{Im} \tilde{\mathfrak{F}}, \quad H = \ker \tilde{\mathfrak{F}}$$

An endomorphism $\tilde{\mathfrak{F}}_p$ on a Riemannian manifold $(M, g)$ is is compatible with $g$ if satisfies:

$$g(X, Y) = g(\tilde{\mathfrak{F}}_p X, \tilde{\mathfrak{F}}_p Y), \quad X, Y \in (\text{Im} \tilde{\mathfrak{F}}_p).$$

**Definition 8.** Orthogonal almost product structure is an APS compatible with a metric.

So an orthogonal almost product structure (OPS) corresponds to the splitting of an Euclidian vector space as a direct sum of two complementary orthogonal subspaces. The structure group reduces from $O(N)$ to $O(p) \times O(q)$ where $p + q = N$.

In the case of an (oriented) Riemannian manifold $(M, g)$, the ‘contraction’ of any 2-form with the metric determines at each point a skew-symmetric endomorphism $\tilde{\mathfrak{F}}_p$ on the (co)tangent space in $p$. More precisely, given two tangent vectors $X, Y$, $\tilde{\mathfrak{F}}_p$ is defined by:

$$\omega(X, Y) = g(\tilde{\mathfrak{F}}_p X, Y). \quad (4)$$

We call simple a two form which with suitable choice of basis can be written as $e \wedge f$ with $e$ and $f$ 1-forms. In Riemannian context such a form defines a reduction of the structure group to $U(1) \times O(N - 2)$, or equivalently $SO(2) \times O(N - 2)$. The parameter space of such structures is the Grassmannian of oriented 2-planes (by analogy to the Plücker embedding):

$$\text{Gr}_2(\mathbb{R}^N) = \frac{SO(N)}{S(O(2) \times O(N - 2))}.$$ 

The intrinsic torsion $\tau$ of a $G$-structure defined by a simple 2-form $\omega$ (almost complex structure, $f$-structure or an OPS) is determined by $\nabla_{LC}\omega$. The classes of OPS’s are distinguished by the $O(p) \times O(q)$-irreducible components of the intrinsic torsion space [12].

$$\mathfrak{s}o(N) \cong \Lambda^2(V \oplus H)^* = \Lambda^2V^* \oplus (V^* \wedge H^*) \oplus \Lambda^2H^* \cong \mathfrak{s}o(p) \oplus \mathfrak{s}o(q) \oplus (V^* \otimes H^*),$$
so the theory of the intrinsic torsion tensor $\tau$ tells us that
\[
\tau \in (\mathbb{R}^N)^* \otimes g^\perp \cong (V^* \oplus H^*) \otimes (V^* \otimes H^*) \\
\cong (\mathbb{R} \oplus S_0^2 V^* \oplus \Lambda^2 V^*) \otimes H^* \oplus V^* \otimes (\mathbb{R} \oplus S_0^2 H^* \oplus \Lambda^2 H^*)
\]

The Naveira’s classification exploits the following six basic irreducible components:
\[
\begin{align*}
\mathcal{W}_1 &= \Lambda^2 V^* \otimes H^*, & \mathcal{W}_4 &= \Lambda^2 H^* \otimes V^* \\
\mathcal{W}_2 &= S_0^2 V^* \otimes H^*, & \mathcal{W}_5 &= S_0^2 H^* \otimes V^* \\
\mathcal{W}_3 &= V^* \otimes H^*, & \mathcal{W}_6 &= H^* \otimes V^*.
\end{align*}
\]

On the last line the spaces $V^*$ and $H^*$ are present as the missing trace part of the symmetric tensor on the second line. The null-torsion conditions detect thirty-six distinguished classes of almost product structures obtained taking unions of two or more of this primary classes and exchanging by duality ‘vertical’ with ‘horizontal’.

We are mainly interested in OPS’s modeled as $\mathbb{R}^6 = \mathbb{R}^4 \oplus \mathbb{R}^2$. In such case $\Lambda^2 H^* \otimes V^*$ decomposes into two components $\Lambda^2_+ H^* \otimes V^*$ and $\Lambda^2_- H^* \otimes V^*$. Then observe that:
\[
\Lambda^3(\mathbb{R}^6)^* = \Lambda^3(V^* \oplus H^*) \cong \Lambda^2 V^* \otimes H^* \oplus \Lambda^2 H^* \otimes V^* \\
\cong 2H^* \oplus V^* \Lambda^2_+ H^* \oplus V^* \Lambda^2_- H^*.
\]

where some tensor products are omitted. Given a simple 2-form $\alpha$, the Hodge star operator defines a "complementary" simple 4-form $\beta = *\alpha$. Then
\[
d\beta \in \Lambda^5(\mathbb{R}^6)^* \cong V^* \oplus H^*.
\]

Remark 9. It follows that, between them, $d\alpha$ and $d\beta$ determine only five of the seven irreducible components of intrinsic torsion.

4 The case of the Iwasawa manifold

Consider the Heisenberg group $G_H$ and the following lattice:
\[
G_H = \left\{ \begin{pmatrix} 1 & z^1 & z^2 \\ 0 & 1 & z^3 \\ 0 & 0 & 1 \end{pmatrix} \mid z^k \in \mathbb{C} \right\} \\
\mathbb{Z}^6 = \left\{ \begin{pmatrix} 1 & a^1 & a^2 \\ 0 & 1 & a^3 \\ 0 & 0 & 1 \end{pmatrix} \mid a^k \in \mathbb{Z}[i] \right\}
\]

(7)
The Iwasawa manifold is defined as the set of right cosets $\mathscr{I} = \mathbb{Z}^6 \backslash G_H$. So $\mathscr{I}$ is a complex manifold and admits complex coordinate charts with transition functions given by:

$$(z^1, z^2, z^3) \rightarrow (z^1 + a^1, z^2 + a^2, z^3 + a^3 + a^1 z^2).$$

The 1-forms $\xi_1 = dz^1$, $\xi_2 = dz^2$ and $\xi_3 = -dz^3 + z^1 dz^2$ are left invariant on $G_H$ and the Lie brackets of the corresponding complex Lie algebra are determined by:

$$d\xi_i = 0, \quad i = 1, 2,$$
$$d\xi_3 = \xi_1 \wedge \xi_2.$$

(8)

Observe that $\xi_i$ are holomorphic with respect to the natural complex structure $J_0$ on $\mathscr{I}$ determined by the complex coordinates $z^i$. The complex Heisenberg group can be also treated as a real Lie group. Indeed (9) defines a real basis $e^i$ of the dual space $g^*$ of the associated Lie algebra $g$.

$$\xi^1 = e^1 + ie^2,$$
$$\xi^2 = e^3 + ie^4,$$
$$\xi^3 = e^5 + ie^6,$$

$$\begin{cases}
de^k = 0, & k \leq 4 \\
de^5 = e^{13} + e^{42}, \\
de^6 = e^{14} + e^{23},
\end{cases}$$

(9)

where $e^{ij} = e^i \wedge e^j$. The Heisenberg group, and the corresponding Lie algebra, encoded by these equations are 2-step nilpotent.

The real geometry of $\mathscr{I}$ reflects the natural decomposition of $g^* = \mathbb{K} \oplus \langle e^5, e^6 \rangle$, where:

$$\mathbb{K} = \ker(d|g^*).$$

The map $\pi : G_H \rightarrow \mathbb{C}^2$ defined for $g \in G_H$ by $g \rightarrow (z^1, z^2)$ induces a fibration $N \rightarrow T^4$ with fibre $T^2$. The space tangent to the fibre is $\mathbb{K}^0 = \mathbb{K}^\perp = \langle e_5, e_6 \rangle$. The fact that $\mathscr{I}$ is a 2-torus bundle over a 4-torus reflects a general property of nilmanifolds, namely their geometry can be understood in terms of tower of toric fibrations (see [13]).

An invariant tensor on $\mathscr{I}$ can be expressed in terms of the bases $e^i$ and $e_i$ with constant coefficients. Declaring that the chosen 1-forms are orthonormal, we determine on $\mathscr{I}$ an invariant Riemannian metric. In the same way, any invariant Riemannian almost product structure is determined by a simple 2-form expressed with constant coefficients with respect to this base. The $SO(2) \times SO(4)$-structures in six dimensions are parametrized by the real Grassmannian $G_{T^2}(\mathbb{R}^6)$.

Any invariant OPS $P$ on $\mathscr{I}$ can be characterized in terms of its intersection with $\mathbb{K}$ and $\mathbb{K}^\perp$, denoting by $(m, n)$ the couple $(\dim P \cap \mathbb{K}, \dim P \cap \mathbb{K}^\perp)$ the types of OPS’s are:

$$(0, 0), \ (0, 1), \ (1, 0), \ (1, 1), \ (2, 0), \ (0, 2)$$

(10)
We can also express a generic simple 2-form in a way to put in evidence the component in the subspace \((e^5, e^6)\). Let \(v_1 \perp v_2\) be such a form with \(|v_1| = |v_2| = 1\), obviously we can choose \(v_1\) inside a 6-sphere and we can choose \(v_2 \perp v_1\) in a 5-sphere, but each \(v_1\) and any unitary element in the plane \((v_1, v_2)\) defines the same element in \(G_{v_2}(\mathbb{R}^6)\). So this parametrization is characterized by an \(S^1\)-ambiguity. Denote by \(f, g\) (respectively \(e, d\)) the normalized components of the \(v_i\)'s in \(\mathbb{K}\) (in \(\mathbb{K}^2\)), then:

\[
\omega = v^1 \wedge v^2 = (f \cos \alpha_1 + e \sin \alpha_1) \wedge (g \cos \alpha_2 + d \sin \alpha_2)
\]  

(11)

This parametrization is highly overabundant (\(f\) and \(g\) can vary in a 3-spheres, \(e, d, \alpha_1, \alpha_2\) give other \(S^1\) degrees of freedom) but will be useful for detecting some relevant subsets and their inclusions.

In this context, we can apply in opportune way the Frobenius Theorem (see [17]) for establishing the integrability of the distributions of an almost product structure on \(\mathcal{I}\).

**Proposition 10.** Given on \(\mathcal{I}\) an orthonormal basis of vector fields \(\{h_i\}\) and its dual \(\{h^i\}\),

1) a four-dimensional distribution \(\langle h_1, ..., h_4 \rangle\) is integrable if and only if \(d(h^56) = 0\).

2) a two-dimensional distribution \(\langle h_1, h_2 \rangle\) is integrable if and only if \(d(h^{3456}) = 0\).

**Proof.** The integrability of a tangent distribution \(V \subset TM\) is equivalent to the condition that the annihilator \(V^\circ \subset \Omega^1M\) generates a differential ideal. Suppose that \(V^\circ = \langle v^1, v^2 \rangle\).

\[
dv^1 = v^1 \wedge \alpha + v^2 \wedge \beta, \quad dv^2 = v^1 \wedge \alpha' + v^2 \wedge \beta',
\]  

and so

\[
d(v^1 \wedge v^2) = (v^1 \wedge v^2) \wedge \gamma
\]  

(12)

for some 1-form \(\gamma\). However the nilpotency condition forces \(d(v_1 \wedge v_2) = 0\). In fact, we can re-assemble (11) in the form:

\[
\omega = v^1 \wedge v^2 = f^1 \wedge f^2 + f^3 \wedge e^5 + f^4 \wedge e^6 + ae^{56}
\]  

(13)

with \(f^1, ..., f^4 \in \mathbb{R}\), then (12) becomes:

\[
d\omega = f^{12} \wedge \gamma + f^3 \wedge e^5 \wedge \gamma + f^4 \wedge e^6 \wedge \gamma + ae^{56} \wedge \gamma
\]  

(14)

The absence of an \(f^{12}\) term on the left-hand side forces \(\gamma\) to be linear combination of \(f^1\) and \(f^2\). In the case \(a \neq 0\), the same consideration applied to the \(e^{56}\)-term shows that \(\gamma\) is a linear combination of \(e^5\) and \(e^6\), which is a contradiction. In the case \(a = 0\),

\[
f^3 \wedge (e^{13} + e^{42}) = f^3 \wedge e^5 \wedge \gamma, \quad f^4 \wedge (e^{14} + e^{23}) = f^3 \wedge e^6 \wedge \gamma
\]

which again contradicts the above condition. We conclude that the equality (14) cannot be realized except in the case \(v_1, v_2 \in \mathbb{K}\) when both sides are zero. On a nilmanifold, an analogous argument works for any subspace with annihilator whose generators define a simple \(k\)-form with \(k \geq 2\). This includes the second item. \(\square\)
Obviously from (14) follows that a simple 2-form \( \omega \) is closed if and only if:

\[
\omega = v^{12} \in \Lambda^2 \mathbb{K}.
\] (15)

In this light we shall also determine the subset of \( \mathbb{G}r_2(\mathbb{R}^6) \) characterized by a closed complementary 4-form associated to the 2-form (13). The simple 4-form in question equals

\[
* \omega = g^{34} \wedge e^{56} - g^{234} \wedge e^6 - g^{134} \wedge e^5 + g^{1234},
\] (16)

in terms of suitable 1-forms \( g^i \), where \( g^{ij} \cdots = g^i \wedge g^j \wedge \cdots \) etc.

Computing \( d * \omega \), we notice that the last term gives no contribution, so there is no constraint on the \( e^{56} \)-component of \( \omega \). Furthermore

\[
d(g^{234} \wedge e^6) = g^{234} \wedge (e^{14} + e^{23}) = 0
\]

\[
d(g^{134} \wedge e^5) = g^{134} \wedge (e^{13} + e^{42}) = 0,
\]

these terms being 5-forms on a four-dimensional space. Finally,

\[
d(g^{34} \wedge e^{56}) = g^{34} \wedge (e^{13} + e^{42}) \wedge e^6 - g^{34} \wedge e^5 \wedge (e^{14} + e^{23}).
\]

The vanishing of \( d * \omega \) is now seen to be equivalent to the equations:

\[
g^{34} \wedge \beta_2 = 0, \quad g^{34} \wedge \beta_3 = 0,
\] (17)

where

\[
\beta_1 = e^{12} + e^{34}, \quad \beta_2 = e^{13} + e^{42}, \quad \beta_3 = e^{14} + e^{23}
\] (18)

is a basis of the space \( \Lambda^2 \mathbb{K} \).

Any simple 2-form \( \gamma \in \Lambda^2(\mathbb{R}^4)^* \) can be written as \( \gamma = \sigma + \tau \) with \( \sigma \in \Lambda^2_- \) and \( \tau \in \Lambda^2_+ \) of equal norm. In the case of \( \gamma = g^{34} \), the condition (17) forces \( \sigma = \pm \beta_1 \), so we obtain:

\[
g^{34} = \pm \beta_1 + \tau,
\] (19)

with \( \tau \) any unit element of \( \Lambda^2_+ \). Such 2-forms represent oriented real 2-dimensional subspaces \( \langle v, J_1 v \rangle \) or \( \langle v, -J_1 v \rangle \) that are invariant by the action of the complex structure \( J_1 \) associated to the 2-form \( \beta_1 \). The set of all of them is parametrized by two disjoint 2-spheres, given by the sign in (19) and the choice of \( \tau \).

We are interested in finding a simple form \( \omega \) compatible with the correct choice of \( g^{34} \). Returning to (13) we know that \( * (f^{12}) \) must be \( J_1 \)-invariant. This implies that \( f^{12} \) is itself \( J_1 \) invariant, or \( f^2 = \alpha f^1 + \beta J_1 f^1 \) for some \( \alpha, \beta \in \mathbb{R} \). In a base-independent setting

\[
\alpha \in \mathbb{G}r_2(\mathbb{R}^6) \subset \Lambda^2(\mathbb{R}^6) \longrightarrow \Lambda^2_+(\mathbb{R}^4) \longrightarrow \langle \beta_2, \beta_3 \rangle \cong \mathbb{R}^2.
\] (20)

Denoting by \( \rho \) the orthogonal projection \( \mathbb{R}^{15} \longrightarrow \mathbb{R}^2 \) the above discussion can be summarized saying that \( \omega \) admits a complementary closed 4-form if:

\[
\omega \in \mathbb{G}r_2(\mathbb{R}^6) \cap \ker \rho
\] (21)
Proposition 11. The class $\mathcal{H}$ of OPS’s on $\mathcal{I}$ characterized by $[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}$ is a complex submanifold $Gr_2(\mathbb{R}^4) \cong \mathbb{C}P^1 \times \mathbb{C}P^1 \cong S^2 \times S^2$ of $Gr_2(\mathbb{R}^6)$ determined by (13).

Combining Equations (11) and (16) we obtain:

Proposition 12. The class $\mathcal{V}$ of OPS’s $P \in G$ on $\mathcal{I}$ characterized by $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$ is the real 6-dimensional ‘slice’ defined by (21).

Then by Proposition 11 and equation (19) we can assert:

Proposition 13. $\mathcal{H} \cap \mathcal{V}$ consists of the disjoint union of two 2-spheres.

Remark 14. Note that the integrability of both $\mathcal{H}$ and $\mathcal{V}$ does not imply that the OPS is parallel. The reasons are that $SO(2) \times SO(4)$ is not a group in Berger’s list, and the holonomy of the Iwasawa cannot reduce.

5 ITV’s of invariant OPS’s on $\mathcal{I}$

In this section we analyse the intrinsic torsion of a generic left-invariant OPS on the Iwasawa manifold, exploiting the global orthonormal basis $e^i$ (9). The following result is easily verified using the methods of [15]. Let us denote by $\nabla$ the Levi-Civita derivative and by $e^i \odot e^j = \frac{1}{2} (e^i \otimes e^j + e^j \otimes e^i)$ the symmetric tensor product.

Proposition 15. For the standard Riemannian metric on $\mathcal{I}$:

\[
\begin{align*}
2 \nabla e^1 &= e^3 \odot e^5 + e^4 \odot e^6 \\
2 \nabla e^2 &= e^3 \odot e^6 - e^4 \odot e^5 \\
2 \nabla e^3 &= -e^1 \odot e^5 - e^2 \odot e^6 \\
2 \nabla e^4 &= -e^1 \odot e^6 + e^2 \odot e^5 \\
2 \nabla e^5 &= e^{13} + e^{12} \\
2 \nabla e^6 &= e^{14} + e^{23}
\end{align*}
\]

Observe that $\nabla e^k$ is symmetric for $k = 1, 2, 3, 4$ because $e^k$ is closed, and $\nabla e^k$ is anti-symmetric for $k = 5, 6$ because $e^k$ is dual to a Killing field.

Now $\tau_\omega = \nabla \omega$, where $\omega$ is a generic simple left-invariant 2-form on $\mathcal{I}$. Let us consider for the moment, omitting the coefficients, the following special form of (11). We shall see that such a choice does not reduce the generality of our considerations.

\[
(f - e^6) \wedge (g + e^5)
\]

where

\[
f = ae^1 + be^2 + ce^3 + de^4 \\
g = ke^1 + le^2 + me^3 + ne^4
\]

(22)

So we compute:

\[
\nabla_X [(f - e^6) \wedge (g + e^5)] = [\nabla_X (f - e^6)] \wedge (g + e^5) + (f - e^6) \wedge [\nabla_X (g + e^5)] =
\]
\[ i_X[a(e^3 \circ e^5 + e^4 \circ e^6) + b(e^3 \circ e^6 + e^4 \circ e^5) +
\]
\[ +c(-e^1 \circ e^5 - e^2 \circ e^6) + d(-e^1 \circ e^6 + e^2 \circ e^5) - e^{14} - e^{23}] \land (g + e^5) +
\]
\[ (f - e^6) \land i_X[k(e^3 \circ e^5 + e^4 \circ e^6) + l(e^3 \circ e^6 + e^4 \circ e^5) +
\]
\[ m(-e^1 \circ e^5 - e^2 \circ e^6) + n(-e^1 \circ e^6 + e^2 \circ e^5) + e^{13} + e^{42}) \]

but this expression can be interpreted as:

\[ i_X(f^1 \circ e^5 + f^2 \circ e^6 - e^{14} - e^{23}) \land (g + e^5) + (f - e^6) \land (i_X(g^1 \circ e^5 + g^2 \circ e^6 + e^{13} + e^{42})) \] (23)

where:

\[ f^1 = -ce^1 + de^2 + ae^3 - be^4 \quad g^1 = -me^1 + ne^2 + ke^3 - le^4 \]
\[ f^2 = -de^1 - ce^2 + be^3 + ae^4 \quad g^2 = -ne^1 - me^2 + le^3 + ke^4 \] (24)

Observe that \( f \perp f^1 \), \( f \perp f^2 \), \( f^1 \perp f^2 \). The same for \( g \) and \( g^i \)'s. Furthermore, as \( \nabla \omega \) belongs to the space \( T^*M \otimes \Lambda^2 T^*M \), in the above expression the contraction with the tangent vector \( X \) regards the 1-form on the first position of each term and the wedge product regards the second position. More explicitly:

\[ \nabla_X \omega = i_X[ f^1 \otimes e^5 \otimes (g + e^5) - f^1 \otimes (g + e^5) \otimes e^5 + e^5 \otimes f^1 \otimes (g + e^5) - e^5 \otimes (g + e^5) \otimes f^1 + f^2 \otimes e^6 \otimes (g + e^5) - f^2 \otimes (g + e^5) \otimes e^6 + e^6 \otimes f^2 \otimes (g + e^5) - e^6 \otimes f^2 \otimes e^6 - e^1 \otimes (g + e^5) \otimes e^4 + e^4 \otimes e^1 \otimes (g + e^5) - e^4 \otimes (g + e^5) \otimes e^1 + e^2 \otimes e^3 \otimes (g + e^5) + e^2 \otimes (g + e^5) \otimes e^3 + e^3 \otimes e^2 \otimes (g + e^5) - e^3 \otimes (g + e^5) \otimes e^2 + g^1 \otimes (f - e^6) \otimes e^5 - g^1 \otimes e^5 \otimes (f - e^6) + e^5 \otimes (f - e^6) \otimes g^1 - e^5 \otimes g^1 \otimes (f - e^6) + g^2 \otimes (f - e^6) \otimes e^6 - g^2 \otimes e^6 \otimes (f - e^6) + e^1 \otimes (f - e^6) \otimes e^3 - e^3 \otimes e^1 \otimes (f - e^6) - e^3 \otimes (f - e^6) \otimes e^1 + e^2 \otimes e^1 \otimes (f - e^6) + e^1 \otimes (f - e^6) \otimes e^2 - e^4 \otimes e^2 \otimes (f - e^6) - e^2 \otimes (f - e^6) \otimes e^4 + e^2 \otimes e^4 \otimes (f - e^6) ] \] (25)

and now the contraction with \( X \) involves the first term of the triple tensor product.

The analysis of the intrinsic torsion varieties of OPS in \( \mathbb{G}_{r_2}(\mathbb{R}^6) \) consists of establishing that special forms of (25) belong to a fixed irreducible component. As from Naveira's
viewpoint:
\[ V^* \otimes H^* \otimes V^* \cong V^* \otimes V^* \otimes H^* = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3, \]
we can group the terms in (25) and exploit analogous isomorphisms to interpret them. In view of (10) and (11) we consider the following cases realized by precise values of \( \alpha_1 \) and \( \alpha_2 \) and then omit the coefficients which are irrelevant in this case:

1. OPS’s in the class \( H \) corresponding to type \((2, 0)\), or in other words:
\[ \omega = f \wedge g \quad f, g \in \mathbb{K} \]
Then (25) becomes:
\[ \nabla \omega = f^1 \otimes e^5 \otimes g - f^1 \otimes g \otimes e^5 + e^5 \otimes f^1 \otimes g - e^5 \otimes g \otimes f^1 + f^2 \otimes e^6 \otimes g - f^2 \otimes g \otimes e^6 + e^6 \otimes f^2 \otimes g - e^6 \otimes g \otimes f^2 + g^1 \otimes f \otimes e^5 - g^1 \otimes e^5 \otimes f + e^5 \otimes f \otimes g^1 - e^5 \otimes g^1 \otimes f + g^2 \otimes f \otimes e^6 - g^2 \otimes e^6 \otimes f + g^2 \otimes f \otimes e^2 - g^2 \otimes e^2 \otimes f \]
(26)
The forms \( e^5 \) and \( e^6 \) are certainly horizontal and \( f^i, g^i \) can stay generically in \( V^* \oplus H^* \), so we write \( f^i = f^i_H + f^i_V \). Let us take the first row of (26), evaluate it in \( f^i_H \) and \( g^i_H \), and group together the terms as follows:
\[ (f^1_H \otimes e^5 \otimes g + e^5 \otimes f^1_H \otimes g) - (g \otimes f^1_H \otimes e^5 - g \otimes e^5 \otimes f^1_H) \]
With opportune symmetrized assembling all the other terms in (26) can be interpreted in such a way. As \( f^i \perp e^5 \) and \( g^i \perp e^6 \) no term contains ‘trace part’ \( e \otimes e \) of \( H^* \otimes H^* \). Thus all these terms belong to \( S^2_0 H^* \otimes V^* = \mathcal{W}_5 \).
Now evaluate (26) considering \( f^i_V \)’s and \( g^i_V \)’s. The first row becomes:
\[ f^1_V \otimes e^5 \otimes g - g \otimes f^1_V \otimes e^5 + e^5 \otimes f^1_V \otimes g - g \otimes e^5 \otimes f^1_V \]
The component \( f^1_V \) is necessarily proportional to \( g \), so the last two terms of this expression cancel and the remaining part belongs to the ‘trace part’ of \( V^* \otimes V^* \). In conclusion for an OPS in \( H \):
\[ \nabla \omega \in \mathcal{W}_3 \oplus \mathcal{W}_5 \]
From the point of view of the Naveira’s classification such OPS’s are foliations in \( H \). This set is parametrized by the Grassmannian of 2-planes in \( \mathbb{K} \).

2. OPS’s in the class \( V \). The condition of \( J_1 \)-invariance is trivially fulfilled if the 2-form has no \( f^{12} \) component at all. This happens for obvious values of \( \alpha_1 \) and \( \alpha_2 \) exactly in the following cases:
2.1 OPS of type $(0, 2)$ is the most natural choice of an OPS on $\mathcal{I}$, meaning:

$$\omega = e_{56}$$

(analogously $-e_{56}$). In this case (25) becomes:

$$\nabla \omega_{56} = e_1 \otimes e_3 \otimes e_6 - e_1 \otimes e_6 \otimes e_3 - e_3 \otimes e_1 \otimes e_6 + e_3 \otimes e_6 \otimes e_1 + e_4 \otimes e_2 \otimes e_6 - e_4 \otimes e_6 \otimes e_2 - e_2 \otimes e_4 \otimes e_6 + e_2 \otimes e_6 \otimes e_4 + e_1 \otimes e_5 \otimes e_4 - e_1 \otimes e_4 \otimes e_5 - e_4 \otimes e_5 \otimes e_1 + e_4 \otimes e_1 \otimes e_5 + e_2 \otimes e_5 \otimes e_3 - e_2 \otimes e_3 \otimes e_5 - e_3 \otimes e_5 \otimes e_2 + e_3 \otimes e_2 \otimes e_5$$

(27)

The interpretation of these terms is immediate. As $V = \langle e_5, e_6 \rangle$ and $H = \langle e_1, e_2, e_3, e_4 \rangle$, the expression $e_1 \otimes e_3 \otimes e_6 - e_3 \otimes e_1 \otimes e_6$ belongs to $\Lambda^2 H \otimes V$. Analogous skew-symmetrization can be done by the rest of the terms obtaining:

$$\nabla \omega_{56} = (e_{13} + e_{42}) \otimes e_6 + (e_{14} + e_{23}) \otimes e_5 + (e_{13} + e_{42}) \otimes e_6 - (e_{14} + e_{23}) \otimes e_5$$

(28)

The dot means that the skew-symmetrization in the last terms regards the first and the third position in the triple products and all those terms belong to $\mathcal{W}_4$. More precisely:

**Proposition 16.** The intrinsic torsion $\nabla \omega$ of the standard almost product structure on $\mathcal{I}$ belongs to the irreducible component $V \otimes \Lambda^2_{+} H$. It is completely determined by $d(e_{56})$.

2.2. OPS of type $(1, 1)$ corresponding to

$$\omega = f \wedge e \quad f \in \mathbb{K}, \quad e \in \mathbb{K}^{1}$$

$$\nabla_X \omega = i_X (f^1 \otimes e^5 + f^2 \otimes e^6) \wedge (e^5 + e^6) + f \wedge (i_X (e_{13} + e_{42} + e_{14} + e_{23}))$$

(29)

In this case $f^1, f^2 \in H$ and $e^i \in H \oplus V$, but obviously the vertical projections of $e^1, \ldots, e^4$ are proportional to $f$ and those of $e^5$ and $e^6$ to $e$. Being $\text{dim} V = 2$, the second totally skew-symmetric part of (29) gives contribution only to the $\Lambda^2 H \otimes V$ component. For the same reason the first part of (29) belongs to $S^2_{0} H \otimes V$ and $V^* \otimes H^*$. In conclusion:

$$\nabla \omega \in \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5$$

Which are again horizontal foliations. Recalling (11), this set is parametrized choosing the unitary 1-forms $f$ and $e$, so there are $S^3 \times S^1$ possibilities. This parametrization is not affected by the $S^1$ ambiguity described above. There remains only a $\mathbb{Z}_2$ ambiguity determined by the fact that $f \wedge e$ and $-f \wedge (-e)$ define the same oriented plane.
2.3 Then we take a generic $\omega \in V$. The previous discussion led us to the expression:

$$\omega = (f - e^6) \wedge ((\alpha f + \beta J_1 f) + e^5)$$

where $J_1$ is the almost complex structure associated to $\beta_1$.

$$\nabla_X \omega = i_X (f^1 \circ e^5 + f^2 \circ e^6 - e^{14} - e^{23}) \wedge ((\alpha f + \beta J_1 f) + e^5) + (f - e^6) \wedge (i_X (\alpha f^1 \circ e^5 + f^2 \circ e^6) + \beta (f^1 \circ e^6 + f^2 \circ e^5) + e^{13} + e^{42})$$

As $f$, $f^1$, $f^2$ and $Jf$ are mutually orthogonal, there is no way to express $f^1$ and $f^2$ as a linear combination of the vectors generating $V$, so $f^1$ and $f^2$ are certainly horizontal, whereas $e^i$ stay generically in $H \oplus V$. Substituting $e^i_H$ in (30) we obtain terms in $S^2_H \otimes V$ and $\Lambda^2 H \otimes V$. Substituting $e^i_V$ we obtain terms in $V \otimes V \otimes H$. In conclusion the class $V$ is a vertical foliation with

$$\nabla \omega \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$$

To parametrize this ITV we should assign values to the degrees of freedom in the expression:

$$(f \cos \alpha_1 + e \sin \alpha_1) \wedge ((f \cos \alpha_3 + J_1 f \sin \alpha_3) \cos \alpha_2 + d \sin \alpha_2)$$

which depends on seven parameters. In this case the parametrization is affected by the $S^1$ ambiguity so we have to quotient by the $S^1$ action inside the plane of each OPS to recover the expected 6 dimensions of Proposition 21.

To understand better the geometry of this ITV we remark that an OPS of the type $(0,1)$, realizes in a trivial way the $J_1$ invariance and belongs to the same class. The 2-forms belonging to this set have all the terms in (13) except the critical $f^1 \wedge f^2$ term. The set of such OPS’s $\mathcal{G}_0$ determines a sort of a ‘skeleton’ for the ITV of class $V$. We parametrize $\mathcal{G}_0$ by $(f \cos \alpha_1 + e \sin \alpha_1) \wedge e^\perp$ and this form is again not affected by the $S^1$ ambiguity and is faithful up to a $\mathbb{Z}_2$ choice of a sign. Given a form in the five-dimensional set $\mathcal{G}_0$ we should just add a form in $\Lambda^2 \mathbb{K}$ corresponding to positively or negatively oriented $J_1$-invariant 2-plane. And such planes can be chosen inside a disjoint union of 2-spheres.

3. OPS’s in the class $V \cap H$. Suppose $f^1, f^2, g^1, g^2$ belong to $H^\ast$. This implies $g \perp f^1, g \perp f^2, f \perp g^1, f \perp g^2$. Substituting (24) we see that these four conditions are equivalent to the following one:

$$g \in (f^1, f^2)^\perp$$

Now $f^3 := be^1 - ae^2 + de^3 - ce^4$ belongs to $(f, f^1, f^2)^\perp$, a neat way to express $g$ is:

$$g = \alpha f + \beta f^3$$
but as \( f^3 = J_1 f \), we conclude that \( f^1, f^2, g^1, g^2 \in \mathcal{H}^* \) means \( \omega = f \wedge (\alpha f + \beta J_1 f) \), which is the defining condition of \( \mathcal{H}^* \). Following item 1 we conclude that:

\[
\nabla \omega \in S^2_0 \mathcal{H}^* \otimes \mathcal{V}^* = \mathcal{W}_5.
\]

The Naveira’s classification refers to an OPS in \( \mathcal{W}_5 \) as a horizontal totally geodesic foliation. By Proposition 13 this set is a disjoint union of two 2-spheres.

4. The general case is given by \((0,0)\) and \((1,0)\) type forms. The only term we did not find in the previous cases is the ‘trace part’ inside \( \mathcal{H}^* \otimes \mathcal{H}^* \). In this case we cannot exclude the presence of such a term. In fact \( e^i, f^1 \) and \( g^1 \) stay in \( \mathcal{H} \oplus \mathcal{V} \), giving rise to terms of the form \((\mathcal{H} \oplus \mathcal{V}) \otimes (\mathcal{H} \oplus \mathcal{V}) \otimes \mathcal{V} \) and suppose that \( f^1 \) is proportional to \( g \). Then we can express both \( e^5 \) and \( f^1 \) as \((g + e^5) \pm (g - e^5)\) where \( g - e^5 \) is the coinciding horizontal component.

Let us now compute \( \nabla \omega \) putting on evidence all the degrees of freedom (11):

\[
\nabla X (c_1 f + s_1 e) \wedge (c_2 g + s_2 d) =
\]

\[
\nabla X (c_1 f + s_1 c_3 e^5 + s_1 s_3 e^6) \wedge (c_2 g + s_2 d) + (c_1 f + s_1 e) \wedge \nabla X (c_2 g + s_2 c_4 e^5 + s_2 s_4 e^6)
\]

where we denote by \( c_i = \cos \alpha_i \) and by \( s_i = \sin \alpha_i \). Applying the above procedure, we obtain an element in generic position inside the intrinsic torsion space. The question is if it is possible that terms obtained opening the first bracket cancel with terms of the second for some values of \( \alpha_i \) different from the analysed cases. This is obviously impossible except from being \( v_1 = v_2 \). We conclude that the above list of cases is exhaustive, so we can resume the results of this section in the following:

**Theorem 17.** The intrinsic torsion varieties of OPS’s on the Iwasawa manifold are:

| Class | Type | ITV |
|-------|------|-----|
| \( \mathcal{W}_4 \) | horizontal totally geodesic foliations | \( \mathcal{G}_4 \cong \) two points |
| \( \mathcal{W}_5 \) | totally geodesic foliations | \( \mathcal{G}_5 \cong S^2 \sqcup S^2 \) |
| \( \mathcal{W}_{35} \) | horizontal foliations | \( \mathcal{G}_{35} \cong \text{Gr}_2(\mathbb{R}^4) \) |
| \( \mathcal{W}_{345} \) | vertical foliations | \( \mathcal{G}_{345} \cong S^2 \times S^2 \times U(1) \) |
| \( \mathcal{W}_{12345} \) | vertical foliations | \( \mathcal{G}_{12345} \cong (S^2 \times S^2 \times S^1 \times S^1) / U(1) \) |
| \( \mathcal{W}_{123456} \) | general OPS | generic point in \( \text{Gr}_2(\mathbb{R}^6) \) |
6 Moment mapping

Suppose that there exists some natural group action on the tangent space of a manifold. This action induces one on the module \( V \), on the classifying orbit and on the intrinsic torsion space. An interesting question is whether such overall action preserves the null-torsion classes of \( G \)-structures. Such an invariance can be useful for detecting the ITV’s.

For example the classes of Hermitian structures on six-dimensional nilmanifolds has been described in [1] in terms of elements (faces, edges and more general segments) in a solid tetrahedron. An accurate geometrical interpretation of this fact has been given later in [9, 10, 11]. Namely, the reductions of a Riemannian structure determined by a 2-form are parametrized by a suitable \( SO(N) \) (co)adjoint orbit and the 2-form in question can be seen as the image of a standard moment mapping associated to the Konstant–Kirillov–Souriau (KKS) symplectic structure. Typical constructions in symplectic geometry, as symplectic fibrations, quotients etc. can be interpreted in terms of compatibility of \( G \)-structures, and appear to be efficient tools for their parametrization. The restriction of the structural group \( SO(N) \) to a maximum torus \( T \subset SO(N) \) gives rise to a Hamiltonian toric action with moment map \( \mu_T \) to the dual of the Lie algebra \( t^* \cong \mathbb{R}^3 \). As a consequence of the celebrated Atiyah and Guillemin–Sternberg (AGS) Convexity Theorem, the image of each classifying orbit by \( \mu_T \) is a specific polytope.

The action of \( T \) induces a splitting of the tangent space into real two-dimensional subspaces. We pointed out that \( I \) has a preferred real basis \( \{ e_i \} \) compatible with the standard complex structure coming from the complex Heisenberg group. Therefore a natural invariant splitting is given by the planes \( \langle e_1, e_2 \rangle, \langle e_3, e_4 \rangle, \langle e_5, e_6 \rangle \) and we denote the corresponding maximum torus by \( \bar{T} \). Recalling case by case from the previous section the specific expression of the 2-form defining each ITV, we observe that:

**Proposition 18.** The conditions, detecting the intrinsic torsion classes of OPS’s on \( I \) are invariant under the action of \( T \).

As a consequence, the positions and the geometry of the ITV’s can be efficiently described. The action of \( \bar{T} \) produces moment mapping consisting of the projection onto the subspace \( \Lambda^2(\mathbb{R}^6) \supset \langle e_{12}, e_{34}, e_{56} \rangle \cong t^* \). The image of \( \text{Gr}_2(\mathbb{R}^6) \) is an octahedron \( \Delta_g \) with vertices \( \pm e_{12}, \pm e_{34}, \pm e_{56} \) (see Fig. [1]). As the dimension of \( \text{Gr}_2(\mathbb{R}^6) \) is higher than twice the dimension of \( \bar{T} \), the Hamiltonian \( T \)-action gives rise to a non-trivial symplectic quotient \( Q_P = \mu^{-1}_T(P)/T \) at each point \( P \) of \( \Delta_g \). In our case a generic \( Q_P \) is diffeomorphic to a real 2-sphere (see [11]). In [10] we have also defined the intrinsic quotient \( I Q_P \) of an ITV to be the subset of \( Q_P \) representing the ITV.

**Theorem 19.** The ITV’s of APS’s on \( I \) by \( \mu_T \) are represented by:
| ITV $\mathcal{G}$ | $\mu_T(\mathcal{G}) \subset \Delta_\mathcal{G}$ (see Fig. 1) | $\mathcal{I}_{\mathcal{Q},P}$ of $\mathcal{G}$ |
|-------------------|-------------------------------------------------|---------------------------------|
| $\mathcal{G}_4$   | vertices $C$ and $F$ | single point |
| $\mathcal{G}_5$   | line segments $AB$ and $DE$ | single point |
| $\mathcal{G}_{35}$ | filled square $ABCD$ | single point |
| $\mathcal{G}_{345}$ | the origin | a line (arch) segment |
| $\mathcal{G}_{12345}$ | union of the tetrahedra $ABCF$ and $CDEF$ | a line (arch) segment |

Figure 1: The moment polytope of $\text{Gr}_2(\mathbb{R}^6)$, $A = e^{12}$, $B = e^{34}$, $C = e^{56}$.

**Proof.** The points forming $\mathcal{G}_4$ are fixed by the action of $\bar{T}$. The ITV’s $\mu_T(\mathcal{G}_5)$ and $\mu_T(\mathcal{G}_{35})$ are toric varieties, and as such can be represented as unique $\bar{T}$-orbit fibrations with degeneracy on faces and edges over their images by $\mu_T$, which are well known.

The projection of $\mu_T(\mathcal{G}_{345})$ to the space $\langle e^{12}, e^{34}, e^{56} \rangle$ is obvious. Then we express the $S^3$, determined by the condition that $|f|^2 = 1$ setting $z = ae^1 + be^2$, $w = ce^3 + de^4$ with $|z|^2 + |w|^2 = 1$. If $f$ is equal to $z$ or $w$, the corresponding $\bar{T}$ orbit is an $S^1$. In the generic case it is isomorphic to a 2-torus as represented on Fig. 2.

The image of $\mu_T(\mathcal{G}_{12345})$ can be determined recalling (31) with $f$ given by (22). Actually the angular parameters $\alpha_1$ and $\alpha_2$ can be chosen in $[0, \pi]$. Given a 2-form

$$(f + ke^5 + le^6) \wedge (\alpha f + \beta J_1 f + me^5 + ne^6)$$

which projects to a point $P(x, y, z)$ with:

$$x = \beta(a^2 + b^2), \quad y = \beta(c^2 + d^2), \quad z = kn - lm$$

we see that the coordinates of $x$ and $y$ of the projection have according signs depending only on the orientation of the $J_1$-invariant plane. The coordinates $(x, y, z)$ can be exploited to fix the parameters except the components of $f$ (this is particular easy to observe on the plane $z = 0$ where $\beta = x + y$), so $\mathcal{I}_{\mathcal{Q}}$ depends again only on the $\bar{T}$-orbits of $S^3$. $\square$
In particular, observe that $\mu_T(\mathcal{G}^0)$ is the line segment $CF$ and the inclusions:

$$\mathcal{G}_4 \subset \mathcal{G}^0 \subset \mathcal{G}_{12345}, \quad \mathcal{G}_{345} \subset \mathcal{G}^0 \subset \mathcal{G}_{12345}, \quad \mathcal{G}_5 \subset \mathcal{G}_{35}, \quad \mathcal{G}_5 \subset \mathcal{G}_{12345}$$

are properly represented by the inclusion of their images. The intersection of the images of $\mathcal{G}_{35}$ and $\mathcal{G}_{12345}$ does not represent correctly the image of $\mathcal{G}_5$. For each point of the intersection of their images these ITV’s are represented by disjoint intrinsic quotients, which are forced to coincide on the edges $\mu_T(\mathcal{G}_{35})$.

The specific example we have analysed here highlights some of the characteristic phenomena described in the introductory sections, see Remark 3 and Proposition 5.

First, recall the special expression that the intrinsic torsion $\nabla \omega$ assumes in case 2.1. The stabilizer of $\omega$ is $SO(4) \times SO(2)$, which obviously does not leave $\tau_\omega$ invariant. Proposition 16 implies that $\tau_\omega$ is stabilized by $SU(2)_- \subset SO(4)$. Furthermore we can apply a rotation in the plane $\langle \beta_2, \beta_3 \rangle$ (see (18)) which leaves invariant $\tau$ if the same rotation is applied in the plane $\langle e^5, e^6 \rangle$. In conclusion the stabilizer of $\tau_\omega$ is:

$$(SU(2)_- \times SO(2)) \times SO(2) \subset (SU(2)_- \times SU(2)_+ \times SO(2))$$ (32)

where the action of the two $SO(2)$’s is ‘coupled’ in the way just described.

We can also test the properties of the generalized distribution tangent to the classifying orbit. Its rank equals the dimension of the space generated by the 1-forms located on the first position in the triple tensor products giving $\nabla \omega$ and varies from an ITV to another. In fact if an OPS belongs to $\mathcal{G}_4$ and $\mathcal{G}_4$ (case 2.1 and 3), the rank of the distribution is four. In case 1, it is determined by the linear dependence of $f$’s and the $g$’s. In the generic case the rank is equal to six etc.

The author believes that the techniques developed in this article could lead to a more general theory regarding the geometry of the intrinsic torsion varieties.

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