THE VARIATIONAL THEORY OF PERFECT FLUID WITH INTRINSIC HYPERMOMENTUM IN SPACE-TIME WITH NONMETRICITY

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Abstract

The variational theory of the perfect fluid with an intrinsic hypermomentum is developed. The Lagrangian density of such fluid is stated and the equations of motion of the fluid and the evolution equation of the hypermomentum tensor are derived. The expressions of the matter currents of the fluid (the metric stress-energy 4-form, the canonical energy-momentum 3-form and the hypermomentum 3-form) are obtained.

1 Introduction

The perfect fluid with an intrinsic hypermomentum as a new type of matter was announced in [1], [2]. The variational theory of such fluid in a metric-affine space-time \((L^4, g)\) [3] was developed in [4]-[8], in the paper [7] the exterior form language being used. The used variational method generalizes the variational theory of the Weyssenhoff-Raabe perfect spin fluid based on accounting the constraints in the Lagrangian density of the fluid with the help of Lagrange multipliers, which has been developed in case of a Riemann-Cartan space-time in [9]-[17] and in case of a metric-affine space-time in [2], [18]. On the other variational methods of the perfect spin fluid in a Riemann-Cartan space-time see [13], [14].

The theory of the perfect fluid with internal degrees of freedom being developed, the additional internal degrees of freedom of a fluid element are described by the four vectors \(\bar{l}_p (p = 1, 2, 3, 4)\), called directors, adjoined with the each element of the fluid. Three of the directors \((p = 1, 2, 3)\) are space-like and the fourth \((p = 4)\) is time-like and is chosen to be equal to 4-velocity of the fluid element.

The distinctions in the various variational approaches consist in the different properties the directors to be endowed. In [8], [9], [11] the orthonormalization of the four directors is maintained while a fluid element moving. In [7] the three space-like directors are elastic in the sense that they can undergo arbitrary deformations during the motion of the fluid and the orthogonality of each of them to the 4-velocity is maintained. In [8] none of the orthogonality conditions of the four directors is maintained and all directors are elastic.

The second distinction of the variational machinery consists in using the generalized Frenkel condition \(J^{\alpha \beta} u_\beta = J^{\alpha \beta} u_\alpha = 0\) [1]-[7] or the Frenkel condition in its classical form \(S^{\alpha \beta} u_\beta = 0\) [8], where \(J^{\alpha \beta}\) and \(S^{\alpha \beta} = J^{[\alpha |\beta]}\) are the specific intrinsic hypermomentum tensor and the specific spin tensor of a fluid element, respectively.

In this paper we use the exterior form variational method according to Trautman [19] (see also [7], [20]). In our approach it is essential that none of the orthogonality
conditions of the four directors is maintained during the motion of the fluid and the usual Frenkel condition is valid in its usual form as in [8].

2 The dynamical variables and constraints

In the exterior form language the directors turn into the fields of 3-form $\tilde{l}_q$ and 1-form $\tilde{l}^p$ ($p = 1, 2, 3, 4$) representing the material frame and coframe, respectively, adjoined with a fluid element, while the constraint

$$\tilde{l}^p \wedge \tilde{l}_q = \delta^p_q \tilde{\omega}$$

(2.1)

being fulfilled, where $\tilde{\omega}$ is the volume 4-form. We shall consider the 1-form $\tilde{l}^p$ as an independent variable and the 3-form $\tilde{l}_q$ as a function of $\tilde{l}^p$ by means of (2.1). In the component representation one has

$$\tilde{l}_p = l_p^\alpha \tilde{\theta}_\alpha , \quad \tilde{l}_q = l_q^\beta \omega_\beta , \quad l_p^p l_p^q = \delta_\alpha^\beta ,$$

(2.2)

where $\tilde{\theta}_\alpha$ is a basis 1-form and $\tilde{\omega}_\beta$ is a 3-form defined as [19]

$$\tilde{\omega}_\beta = *\tilde{e}_\beta , \quad \tilde{\theta}_\alpha \wedge \tilde{\omega}_\beta = \delta^\beta_\alpha \tilde{\omega} .$$

(2.3)

Here $*$ is the Hodge dual operator and $\tilde{e}_\beta$ is a basis vector, a coordinate system being nonholonomic in general.

Each fluid element possesses a 4-velocity vector $\tilde{u}$ which is corresponded to a velocity 1-form $\tilde{u} = \dot{g}(\tilde{u}, \cdots)$ and a flow 3-form $*\tilde{u}$ [21] with

$$*\tilde{u} \wedge \tilde{u} = c^2 \tilde{\omega} ,$$

(2.4)

that means the usual condition $\dot{g}(\tilde{u}, \tilde{u}) = -c^2$.

The director field and the 4-velocity field are compatible in the sense that the director $\tilde{l}_4$ is oriented along the velocity: $\tilde{l}_4 = c^{-1} \tilde{u}$. This condition yields the constraint

$$*\tilde{u} \wedge \tilde{l}^p = -c \tilde{\omega} .$$

(2.5)

A fluid element moving, the mass and entropy conservation laws are fulfilled,

$$d(\mu *\tilde{u}) = 0 ,$$

(2.6)

$$d(\mu s *\tilde{u}) = 0 ,$$

(2.7)

where $\mu$ and $s$ are the mass density and the specific entropy of the fluid in the rest frame of reference, respectively.

An element of the fluid with intrinsic hypermomentum possesses the additional kinetic energy 4-form

$$E = \frac{1}{2} \mu J^p_q \Omega^p_q \tilde{\omega} ,$$

(2.8)

where $J^p_q$ is the specific intrinsic hypermomentum tensor representing the new dynamical quantity which generalizes the spin density of the Weyssenhoff fluid. The
quantity $\Omega_q^p$ is the measure of ability of a fluid element to perform the intrinsic motion and generalizes the fluid element angular velocity of the Weyssenhoff spin fluid theory. It has the form

$$\Omega_q^p \tilde{\omega} = \tilde{u} \wedge l^p_{\alpha} D l_{\alpha}^p ,$$  \hspace{1cm} (2.9)

where $D$ means the exterior covariant derivative,

$$D l_{p}^\alpha = d l_{p}^\alpha + \tilde{\Gamma}_{\beta p}^\alpha l_{p}^\beta .$$  \hspace{1cm} (2.10)

The specific intrinsic hypermomentum tensor $J_q^p$ can be decomposed into irreducible parts

$$J_q^p = S_q^p + \mathcal{T}_q^p + \frac{1}{4} \delta_q^p J , \hspace{0.5cm} S_q^p := J_{[q]}^p , \hspace{0.5cm} \mathcal{T}_q^p = 0 .$$  \hspace{1cm} (2.11)

Here $S_q^p$ is the specific spin tensor of a fluid element and $J$ is the specific dilaton charge of a fluid element, respectively. The former one obeys the Frenkel condition $S_q^p u^\alpha l_{\alpha}^q = 0$ which can be represented in the form

$$J_{[q]}^p \ast \tilde{u} \wedge \tilde{l}^q = 0 .$$  \hspace{1cm} (2.12)

The perfect fluid Lagrangian density 4-form should be chosen as the remainder after subtraction the internal energy density of the fluid $\varepsilon$ from the kinetic energy (2.8) with regard to the constraints (2.4)-(2.7), (2.9), (2.12) which should be introduced into the Lagrangian density by means of the Lagrange multipliers $\lambda, \nu, \varphi, \tau, \kappa_q^p, \chi_p$, respectively.

The internal energy density of the fluid $\varepsilon$ depends on the extensive (additive) thermodynamic parameters $\mu, s, J_q^p$ and obeys to the first thermodynamic principle

$$d \varepsilon(\mu, s, J_q^p) = \frac{\varepsilon + p}{\mu} d \mu + \mu T d s + \frac{\partial \varepsilon}{\partial J_q^p} d J_q^p .$$  \hspace{1cm} (2.13)

We need the following variation of the dependent variables which can be derived as a result of the resolution of the constrants (2.1), (2.2) with the help of the relations (2.3),

$$\tilde{\omega} \delta l_{p}^\alpha = l_{p}^\alpha \tilde{\omega}_{\alpha} \wedge \delta \tilde{l}^p - \tilde{\omega}_{\alpha} \wedge \delta \tilde{l}^p ,$$  \hspace{1cm} (2.14)

$$\tilde{\lambda} \delta l_{p}^\alpha = - l_{p}^\alpha \wedge \delta \tilde{l}^\alpha + l_{p}^\alpha \tilde{l}^\alpha \wedge \delta \tilde{l}^p .$$  \hspace{1cm} (2.15)

As a result of the relation $s \tilde{u} \wedge \tilde{\theta}^\alpha = - u^\alpha \tilde{\omega}$ one also has

$$\tilde{\omega} \delta u^\alpha = - \delta (s \tilde{u}) \wedge \tilde{\theta}^\alpha = s \tilde{u} \wedge \delta \tilde{\theta}^\alpha - u^\alpha \delta \tilde{\omega} ,$$  \hspace{1cm} (2.16)

$$\delta \tilde{\omega} = - \frac{1}{2} g^{\sigma \rho} \delta g_{\sigma \rho} + \delta \tilde{\theta}^\sigma \wedge \tilde{\omega}_{\sigma} .$$  \hspace{1cm} (2.17)

3 The Lagrangian density and the equations of motion of the fluid

As a result of the previous section the Lagrangian density 4-form of the perfect fluid with an intrinsic hypermomentum has the form

$$\tilde{L}_m = L_m \tilde{\omega} = - \varepsilon(\mu, s, J_q^p) \tilde{\omega} + \frac{1}{2} \mu J_q^p \Omega_q^p \tilde{\omega} + \mu \ast \tilde{u} \wedge d \varphi + \mu \tau \ast \tilde{u} \wedge d s + \mu \lambda(s \tilde{u} \wedge \tilde{\omega} - c^2 \tilde{\omega}) + \mu \nu_p(s \tilde{u} \wedge \tilde{l}^p + c \delta_q^p \tilde{\omega}) + \mu \chi_p J_{[q]}^p \ast \tilde{u} \wedge \tilde{l}^q + \mu \kappa_q^p (s \tilde{u} \wedge l_{q}^\alpha D l_{\alpha}^p - \Omega_q^p \tilde{\omega}) .$$  \hspace{1cm} (3.1)
The fluid motion equations and the evolution equations of the hypermomentum tensor are derived by the variation of (3.1) with respect to the independent variables \( \mu, s, J^p_q, \Omega^q_p, \bar{u}, \bar{l}^p \) and the Lagrange multipliers. As a result of such variational machinery one gets the constraints (2.4)-(2.7), (2.9), (2.12) and the following variational equations,

\[
\begin{align*}
\delta \mu : & \quad -(\varepsilon + p)\omega + \frac{1}{2} \mu J^p_q \Omega^q_p \omega + \mu \bar{u} \land d\varphi = 0, \\
\delta s : & \quad T\bar{\omega} + \bar{u} \land d\tau = 0, \\
\delta J^p_q : & \quad \frac{\partial \varepsilon}{\partial J^p_q} = \frac{1}{2} \mu \Omega^q_p - \mu c\chi[p\delta^q_4], \\
\delta \Omega^q_p : & \quad \kappa^q_p = \frac{1}{2} J^p_q, \\
\delta \bar{u} : & \quad d\varphi + \tau ds + 2\lambda \bar{u} + \nu_p \bar{l}^p + \chi_p J^p_q \bar{l}^q + \kappa^p_\alpha D\bar{l}^\alpha = 0, \\
\delta \bar{l}^q : & \quad \nu_q \bar{u} + \chi_p J^p_q \bar{u} + \frac{1}{2} \dot{J}^\alpha_\beta \bar{l}^\beta_q \omega_\alpha = 0.
\end{align*}
\]

In the equation (3.7) the notation

\[ \dot{J}^\alpha_\beta = *(\bar{u} \land D J^\alpha_\beta) \]  

was introduced.

Let us derive some consequences of these equations. Multiplying the equation (3.6) by \( \bar{u} \) from the left externally and using (3.5) and (3.2) one derives the expression for the Lagrange multiplier \( \lambda \):

\[ 2\mu c^2 \lambda = -(\varepsilon + p) + \mu c\nu(4). \]  

Multiplying the equation (3.7) by \( \bar{u} \) from the right externally one gets:

\[ \nu_q + \chi_p S^p_q = \frac{1}{2c^2} \dot{J}^\alpha_\beta \bar{l}^\beta_q u_\alpha. \]  

This equation with regard of the Frenkel condition (2.12) has the consequence

\[ c\nu(4) = \frac{1}{2c^2} \dot{J}^\alpha_\beta u_\alpha u^\beta. \]  

The Lagrange multiplier \( \chi_p \) can be found as a consequence of the correspondence principle of the theory under consideration to the Weyssenhoff spin fluid theory. Namely, the quantity canonically conjugated to the spin tensor should be the spatial angular velocity of the directors,

\[ \frac{\partial \varepsilon}{S^q_p} = \frac{\partial \varepsilon}{J^p_q} = \frac{1}{2} \mu \Pi^q_r \Pi^s_p \Omega^r_s, \quad \Pi^\alpha_\gamma = \delta^\alpha_\gamma + \frac{1}{c^2} u^\alpha u_\gamma. \]  

Comparing (3.12) with (3.4) one can derive

\[ \chi_q \Pi^q_p = \frac{1}{c^2} \Omega[pq] u^q. \]
Obviously that the Lagragian density (3.1) determines only the spatial part of the Lagrange multiplier $\chi_p$ and therfore the condition $c\chi_4 = \chi_p u^p = 0$ can be imposed without loss of generality. Thus one has

$$\chi_p = \frac{1}{c^2} \Omega_{[pq]} u^q .$$

(3.14)

As a consequence of the fluid motion equations it is easy to verify that the Lagrangian density 4-form (3.1) is proportional to the hydrodynamic fluid pressure,

$$\tilde{L}_m = p\tilde{\omega} .$$

(3.15)

Substituting (3.11) into (3.7) one finds the evolution equation of the hypermomentum tensor,

$$\Pi^\alpha_\gamma \dot{j}_\beta = 0 .$$

(3.16)

This equation has the consequense

$$\dot{j} + \frac{1}{c^2} \dot{j}_\alpha u_\alpha u^\beta = 0 , \quad J = \frac{1}{2}J_\alpha .$$

(3.17)

4 The energy-momentum tensor of the perfect fluid with an intrinsic hypermomentum

The matter Lagrangian density makes possible to derive the external currents of a matter field which are the sources of the gravitational field. In case of the perfect fluid with an intrinsic hypermomentum one has as the matter currents: the metric stress-energy 4-form $\tilde{T}^{\sigma\rho}$, the canonical energy-momentum 3-form $\tilde{t}_\sigma$ and the hypermomentum 3-form $\tilde{J}_\alpha^\beta$, which are determined as variational derivatives [20].

By virtue of the explicit form the Lagrangian density (3.1) the metric stress-energy 4-form reads

$$\tilde{T}^{\sigma\rho} = 2\frac{\delta \tilde{L}_m}{\delta g_{\sigma\rho}} = T^{\sigma\rho}\tilde{\omega} , \quad T^{\sigma\rho} = g^{\sigma\rho}p + \frac{1}{c^2}(\epsilon + p + \frac{1}{2}\mu j)u^\sigma u^\rho .$$

(4.1)

With regard to (3.9) and (3.11) the variational derivative of (3.1) with respect to $\tilde{\theta}^\sigma$ yields

$$\tilde{t}_\sigma = \frac{\delta \tilde{L}_m}{\delta \tilde{\theta}^\sigma} = p\tilde{\omega}_\sigma + \frac{1}{c^2}(\epsilon + p + \frac{1}{2}j)u_\sigma *\tilde{u} + \frac{1}{2}\mu j^\alpha \tilde{\omega}_\alpha .$$

(4.2)

On the basis of the evolution equation of the hypermomentum tensor (3.16) the expression of the canonical energy-momentum 3-form takes the form [22]

$$\tilde{t}_\sigma = p\tilde{\omega}_\sigma + \frac{1}{c^2}(\epsilon + p + \frac{1}{2}j)u_\sigma *\tilde{u} - \frac{1}{2c^2}\mu j^\alpha \tilde{\omega}_\alpha *\tilde{u}$$

$$= p\tilde{\omega}_\sigma + \frac{1}{c^2}(\epsilon + p)u_\sigma *\tilde{u} - \frac{1}{2c^2}\mu j^\beta \Pi^\alpha_\beta u_\alpha *\tilde{u} .$$

(4.3)

For the hypermomentum 3-form expression one finds

$$\tilde{J}_\alpha^\beta = -\frac{\delta \tilde{L}_m}{\delta \tilde{\Gamma}_\alpha^\beta} = \frac{1}{2}\mu J^\alpha_\beta *\tilde{u} .$$

(4.4)
The expressions of the metric stress-energy 4-form (4.1), the canonical energy-momentum 3-form (4.3) and the hypermomentum 3-form (4.4) are compatible in the sense that they satisfy to the Noether identity

\[ T^\sigma_\rho \tilde{\omega} + \frac{1}{2} \mu \ast \bar{u} \wedge D J^\sigma_\rho = \tilde{\theta}^\sigma \wedge \bar{t}_\rho , \]

(4.5)

that corresponds to the \( GL(n, R) \)-invariance of the Lagrangian density (3.1) [20].

5 Conclusion

The essential feature of the constructed variational theory of the perfect fluid with an intrinsic hypermomentum is the assumption that the frame realized by the all four directors is elastic and can be deformed during the motion of the fluid element according to nonmetricity of the space-time. Therefore the Lagrangian density (3.1) does not contain the term maintaining the orthogonality of the directors. The second distinction from the other variational approaches is the using of the Frenkel condition in its classical form (2.12).

As a result we have obtained the new expression for the energy-momentum tensor of the fluid which is the source of the gravitational field in the metric-affine space-time. It should be important to investigate the consequences of the employing of this energy-momentum tensor to cosmological and astrophysical problems. For example, it is interesting to clarify whether the corresponding field equations have the regular solution with the upper limit for \( \epsilon \) (the limiting energy density of the fluid). These questions are under consideration now.

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