Triangular \( C^* \)-bialgebra defined as the direct sum of matrix algebras

Katsunori Kawamura*

College of Science and Engineering, Ritsumeikan University,
1-1-1 Noji Higashi, Kusatsu, Shiga 525-8577, Japan

Abstract

Let \( M_*(C) \) denote the \( C^* \)-algebra defined as the direct sum of all matrix algebras \( \{ M_n(C) : n \geq 1 \} \). It is known that \( M_*(C) \) has a non-cocommutative comultiplication \( \Delta_\varphi \). We show that the \( C^* \)-bialgebra \( (M_*(C), \Delta_\varphi) \) has a universal \( R \)-matrix \( R \) such that the quasi-cocommutative \( C^* \)-bialgebra \( (M_*(C), \Delta_\varphi, R) \) is triangular.

Mathematics Subject Classifications (2000). 16W35, 81R50, 46K10.

Key words. universal \( R \)-matrix, triangular \( C^* \)-bialgebra.

1 Introduction

The purpose of this paper is to construct a new triangular \( C^* \)-bialgebra such that its universal \( R \)-matrix is defined by using a certain set of arithmetic transformations. In this section, we show our motivation, definitions and our main theorem.

1.1 Motivation

In this subsection, we roughly explain our motivation and the background of this study. Explicit mathematical definitions will be shown after §1.2.

For \( n \geq 2 \), let \( M_n(C) \) denote the \( C^* \)-algebra of all \( n \times n \) matrices and we define \( M_1(C) = C \) for convenience. Define the \( C^* \)-algebra \( M_*(C) \) as the direct sum of \( \{ M_n(C) : n \geq 1 \} \):

\[
M_*(C) = M_1(C) \oplus M_2(C) \oplus M_3(C) \oplus \cdots .
\]  

\[ (1.1) \]

*e-mail: kawamura@kurims.kyoto-u.ac.jp.
In § 6.3 of [9], we constructed a non-cocommutative comultiplication $\Delta_{\varphi}$ of $M_s(C)$ such that $(M_s(C), \Delta_{\varphi})$ is a C*-subbialgebra of a certain C*-bialgebra. As a C*-algebra, $M_s(C)$ is almost trivial and there is no new property, but the bialgebra structure is new, which is not a deformation of a known cocommutative bialgebra.

On the other hand, in the theory of quantum groups, a universal $R$-matrix for a quasi-cocommutative bialgebras is important for applications to mathematical physics and low-dimensional topology [5, 6, 7, 8]. Especially, quasi-triangular (or braided) bialgebras generate solutions of Yang-Baxter equation. As a stronger property, a triangular bialgebra was introduced by Drinfel’d [5]. In this case, the tensor category of all representations of the bialgebra is symmetric ([8], XIII.6 Exercises 1). See also [4, 6, 14].

Our interest is to find a universal $R$-matrix of $(M_s(C), \Delta_{\varphi})$ in (1.1) if there exists. In this paper, we construct a universal $R$-matrix $R$ of $(M_s(C), \Delta_{\varphi})$ defined as a double infinite sequence of permutation matrices arising from certain arithmetic transformations of quotients and residues of positive integers. Furthermore, we show that the quasi-cocommutative C*-bialgebra $(M_s(C), \Delta_{\varphi}, R)$ is triangular.

1.2 Definitions

In this subsection, we recall definitions of C*-bialgebra and universal $R$-matrix [11]. At first, we prepare terminologies about C*-bialgebra according to [12, 13].

1.2.1 C*-bialgebra

For a C*-algebra $A$, let $A''$ denote the enveloping von Neumann algebra of $A$. The multiplier algebra $\mathcal{M}(A)$ of $A$ is defined by

$$\mathcal{M}(A) \equiv \{ a \in A'' : aA \subset A, Aa \subset A \}. \quad (1.2)$$

Then $\mathcal{M}(A)$ is a unital C*-subalgebra of $A''$. Especially, $A = \mathcal{M}(A)$ if and only if $A$ is unital. The algebra $\mathcal{M}(A)$ is the completion of $A$ with respect to the strict topology.

For two C*-algebras $A$ and $B$, let $\text{Hom}(A, B)$ and $A \otimes B$ denote the set of all *-homomorphisms from $A$ to $B$ and the minimal C*-tensor product of $A$ and $B$, respectively. A *-homomorphism from $A$ to $B$ is not always extended to the map from $\mathcal{M}(A)$ to $\mathcal{M}(B)$. If $f \in \text{Hom}(A, B)$ is surjective and both $A$ and $B$ are separable, then $f$ is extended to a surjective *-homomorphism of $\mathcal{M}(A)$ onto $\mathcal{M}(B)$. We state that $f \in \text{Hom}(A, \mathcal{M}(B))$
is nondegenerate if \( f(A)B \) is dense in \( B \). If both \( A \) and \( B \) are unital and \( f \) is unital, then \( f \) is nondegenerate. For \( f \in \text{Hom}(A, \mathcal{M}(B)) \), if \( f \) is nondegenerate, then \( f \) is called a morphism from \( A \) to \( B \) [15]. If \( f \) is a nondegenerate \(*\)-homomorphism from \( A \) to \( B \), then we can regard \( f \) as a morphism from \( A \) to \( B \) by using the canonical embedding of \( B \) into \( \mathcal{M}(B) \). Each morphism \( f \) from \( A \) to \( B \) can be extended uniquely to a homomorphism \( \tilde{f} \) from \( \mathcal{M}(A) \) to \( \mathcal{M}(B) \) such that \( \tilde{f}(m)f(b)a = f(mb)a \) for \( m \in \mathcal{M}(B), b \in B, \) and \( a \in A \). If \( f \) is injective, then so is \( \tilde{f} \).

A pair \((A, \Delta)\) is a \( C^*\)-bialgebra if \( A \) is a \( C^*\)-algebra with \( \Delta \in \text{Hom}(A, \mathcal{M}(A \otimes A)) \) such that \( \Delta \) is nondegenerate and the following holds:

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta.
\]  

(1.3)

We call \( \Delta \) the comultiplication of \( A \). A \( C^*\)-bialgebra \((A, \Delta)\) is counital if there exists \( \varepsilon \in \text{Hom}(A, \mathbb{C}) \) such that

\[
(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta.
\]  

(1.4)

We call \( \varepsilon \) the counit of \( A \) and write \((A, \Delta, \varepsilon)\) as the counital \( C^*\)-bialgebra \((A, \Delta)\) with the counit \( \varepsilon \). Remark that we do not assume \( \Delta(A) \subset A \otimes A \). Furthermore, \( A \) has no unit for a \( C^*\)-bialgebra \((A, \Delta)\) in general.

1.2.2 Universal \( R \)-matrix

We recall a unitary universal \( R \)-matrix and the quasi-cocommutativity for a \( C^*\)-bialgebra [11].

**Definition 1.1** Let \((A, \Delta)\) be a \( C^*\)-bialgebra.

(i) The map \( \tilde{\tau}_{A,A} \) from \( \mathcal{M}(A \otimes A) \) to \( \mathcal{M}(A \otimes A) \) is the extended flip defined as

\[
\tilde{\tau}_{A,A}(X)(x \otimes y) = \tau_{A,A}(X(y \otimes x)) \quad (X \in \mathcal{M}(A \otimes A), x, y \in A) \tag{1.5}
\]

where \( \tau_{A,A} \) denotes the flip of \( A \otimes A \).

(ii) The map \( \Delta^{\text{op}} \) from \( A \) to \( \mathcal{M}(A \otimes A) \) defined as

\[
\Delta^{\text{op}}(x) = \tilde{\tau}_{A,A}(\Delta(x)) \quad (x \in A) \tag{1.6}
\]

is called the opposite comultiplication of \( \Delta \).

(iii) A \( C^*\)-bialgebra \((A, \Delta)\) is cocommutative if \( \Delta = \Delta^{\text{op}} \).
An element \( R \) in \( \mathcal{M}(A \otimes A) \) is called a (unitary) universal \( R \)-matrix of \( (A, \Delta) \) if \( R \) is a unitary and
\[
R \Delta(x) R^* = \Delta^{op}(x) \quad (x \in A).
\]
(1.7)

In this case, we state that \( (A, \Delta) \) is quasi-cocommutative (or almost cocommutative [3]).

We write a quasi-cocommutative \( C^\ast \)-bialgebra \( (A, \Delta) \) with a universal \( R \)-matrix \( R \) as \( (A, \Delta, R) \). If \( A \) is unital, then \( \mathcal{M}(A \otimes A) = A \otimes A \) and \( \tilde{\tau}_{A,A} = \tau_{A,A} \). In addition, if \( (A, \Delta) \) is quasi-cocommutative with a universal \( R \)-matrix \( R \), then \( R \in A \otimes A \).

Next, we introduce quasi-triangular and triangular \( C^\ast \)-bialgebra according to [5].

**Definition 1.2** Let \( (A, \Delta, R) \) be a quasi-cocommutative \( C^\ast \)-bialgebra.

(i) \( (A, \Delta, R) \) is quasi-triangular (or braided [8]) if the following holds:
\[
(\Delta \otimes \text{id})(R) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13}R_{12}
\]
where we use the leg numbering notation [1].

(ii) \( (A, \Delta, R) \) is triangular if \( (A, \Delta, R) \) is quasi-triangular and the following holds:
\[
R \tilde{\tau}_{A,A}(R) = I
\]
where \( \tilde{\tau}_{A,A} \) is as in (1.5) and \( I \) denotes the unit of \( \mathcal{M}(A \otimes A) \).

Since both \( \Delta \otimes \text{id} \) and \( \text{id} \otimes \Delta \) are nondegenerate, (1.8) makes sense. The equation (1.9) is written as “\( R_{12}R_{21} = 1 \)” in [5]. In Appendix A we will show basic facts about quasi-triangular \( C^\ast \)-bialgebras.

### 1.2.3 Direct product and direct sum of \( C^\ast \)-algebras

For an infinite set \( \{A_i : i \in \Omega\} \) of \( C^\ast \)-algebras, there are separate notions of direct sum and product which do not coincide with the algebraic ones [2]. We define two \( C^\ast \)-algebras \( \prod_{i \in \Omega} A_i \) and \( \bigoplus_{i \in \Omega} A_i \) as follows:
\[
\prod_{i \in \Omega} A_i \equiv \{ (a_i) : \|(a_i)\| \equiv \sup_{i} \|a_i\| < \infty \},
\]
(1.10)
\[
\bigoplus_{i \in \Omega} A_i \equiv \{ (a_i) : \|(a_i)\| \to 0 \text{ as } i \to \infty \}
\]
(1.11)
where I

In fact,

Then (\ref{eq:main_theorem}), we show our main theorem. For each \( i \in \Omega \), we have the direct product and the direct sum of \( A_i \)'s, respectively. The algebra \( \bigoplus_{i \in \Omega} A_i \) is a closed two-sided ideal of \( \prod_{i \in \Omega} A_i \). The algebraic direct sum \( \bigoplus_{alg} \{ A_i : i \in \Omega \} \) is a dense *-subalgebra of \( \bigoplus \{ A_i : i \in \Omega \} \). Since \( \mathcal{M}(\bigoplus_{i \in \Omega} A_i) \cong \prod_{i \in \Omega} \mathcal{M}(A_i) \) \((\ref{II.8.1.3}), \ref{PP}, \ref{A.1})\), if \( A_i \) is unital for each \( i \), then

\[
\mathcal{M}\left( \bigoplus_{i \in \Omega} A_i \right) \cong \prod_{i \in \Omega} A_i. \tag{1.12}
\]

1.3 C*-bialgebra (\( M_\ast(C), \Delta_\varphi \))

In this subsection, we recall the C*-bialgebra \( (M_\ast(C), \Delta_\varphi) \) \((\ref{P})\). Let \( M_\ast(C) \) be as in \((\ref{P})\) and let \( \{ E_{ij}^{(n)} \} \) denote the set of standard matrix units of \( M_n(C) \). For \( n, m \geq 1 \), define \( \varphi_{n,m} \in \text{Hom}(M_{nm}(C), M_n(C) \otimes M_m(C)) \) by

\[
\varphi_{n,m}(E_{m(i-1)+j,m(i'-1)+j'}) = E_{i,i'}^{(n)} \otimes E_{j,j'}^{(m)} \tag{1.13}
\]

for \( i, i' \in \{1, \ldots, n\} \) and \( j, j' \in \{1, \ldots, m\} \). By using \( \{ \varphi_{n,m} \}_{n,m \geq 1} \), define two maps \( \Delta_\varphi \in \text{Hom}(M_\ast(C), M_\ast(C) \otimes M_\ast(C)) \) and \( \varepsilon \in \text{Hom}(M_\ast(C), C) \) by

\[
\Delta_\varphi(x) = \sum_{m,l,m+l=n} \varphi_{m,l}(x) \quad \text{when } x \in M_n(C), \tag{1.14}
\]

\[
\varepsilon(x) \equiv 0 \quad \text{when } x \in \bigoplus \{ M_n(C) : n \geq 2 \}, \quad \varepsilon(x) \equiv x \quad \text{when } x \in M_1(C). \tag{1.15}
\]

Then \( (M_\ast(C), \Delta_\varphi, \varepsilon) \) is a counital C*-bialgebra, which is non-cocommutative. In fact,

\[
\Delta_\varphi(E_{2,2}^{(6)}) = I_1 \otimes E_{2,2}^{(6)} + E_{1,1}^{(2)} \otimes E_{2,2}^{(3)} + E_{1,1}^{(3)} \otimes E_{2,2}^{(2)} + E_{2,2}^{(6)} \otimes I_1 \tag{1.16}
\]

where \( I_1 \) denotes the unit of \( M_1(C) = C \). The second and third terms show \( \Delta_\varphi \neq \Delta_\varphi^\varepsilon \). Remark \( \Delta_\varphi(M_\ast(C)) \subset M_\ast(C) \otimes M_\ast(C) \). The C*-bialgebra \( (M_\ast(C), \Delta_\varphi, \varepsilon) \) satisfies the cancellation law \((\ref{P}), \ref{A.1}\), and it never has an antipode \((\ref{P}), \ref{A.2}\).

1.4 Main theorem

In this subsection, we show our main theorem. For \( n \geq 1 \), let \( \{ e^{(n)}_i \}_{i=1}^n \) denote the standard basis of the finite dimensional Hilbert space \( C^n \).
Definition 1.3 Define the unitary transformation $R^{(n,m)}$ on $\mathbb{C}^n \otimes \mathbb{C}^m$ by

$$R^{(n,m)}(e^{(n)}_i \otimes e^{(m)}_j) \equiv e^{(n)}_{\hat{i}} \otimes e^{(m)}_{\hat{j}} \quad (1.17)$$

for $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ where the pair $(\hat{i}, \hat{j}) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ is uniquely defined as the following integer equation:

$$m(i - 1) + j = n(j - 1) + \hat{i}. \quad (1.18)$$

For example, $m(i - 1) + j$ divided by $n$ equals $j - 1$ with a remainder of $\hat{i}$ when $1 \leq \hat{i} \leq n - 1$.

By the natural identification $\text{End}_\mathbb{C}(\mathbb{C}^n \otimes \mathbb{C}^m) \cong M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$, $R^{(n,m)}$ is regarded as a unitary element in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ for each $n, m \geq 1$.

From (1.12), $M(M_*(\mathbb{C}) \otimes M_*(\mathbb{C})) = \prod_{n,m \geq 1} M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. Hence the set \{R^{(n,m)}\}_{n,m \geq 1} in (1.17) defines a unitary element $R$ in $\mathcal{M}(M_*(\mathbb{C}) \otimes M_*(\mathbb{C}))$:

$$R \equiv (R^{(n,m)})_{n,m \geq 1} \in \mathcal{M}(M_*(\mathbb{C}) \otimes M_*(\mathbb{C})). \quad (1.19)$$

Then the main theorem is stated as follows.

Theorem 1.4 Let $(M_*(\mathbb{C}), \Delta_\varphi)$ be as in §1.3.

(i) The unitary $R$ in (1.19) is a universal $R$-matrix of $(M_*(\mathbb{C}), \Delta_\varphi)$.

(ii) In addition to (i), the quasi-cocommutative $\ast$-bialgebra $(M_*(\mathbb{C}), \Delta_\varphi, R)$ is triangular.

We discuss the meaning of $R$ in (1.19) as follows.

Remark 1.5 From (1.18), the operator $R^{(n,m)}$ in (1.17) is induced from the arithmetic transformation $\chi_{n,m}$ defined as

$$(i, j) \mapsto \chi_{n,m}(i, j) \equiv (\hat{i}, \hat{j}) \quad (1.20)$$

The map $\chi_{n,m}$ is a permutation of the set $\{1, \ldots, n\} \times \{1, \ldots, m\}$. For a given integer $N$ in $\{1, \ldots, nm\}$, $(i, j), (\hat{i}, \hat{j}) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ are uniquely determined by

$$N = m(i - 1) + j = n(j - 1) + \hat{i}. \quad (1.21)$$

Hence both $(i, j)$ and $(\hat{i}, \hat{j})$ are modifications of quotients and residues of $N$. From this, $\chi_{n,m}$ means a transformation between quotients and residues of a given integer with respect to a pair of fixed integers $n$ and $m$. For example,

$$\chi_{2,3}(1, 2) = (2, 1), \quad \chi_{2,3}(2, 1) = (2, 2). \quad (1.22)$$

From this, $\chi_{2,3} \neq (\chi_{2,3})^{-1}$. This implies $R^2 \neq id$. It is interesting that the triangular structure of a bialgebra is induced from such arithmetic transformations.
In §2 we will introduce locally triangular \( \mathcal{C}^* \)-weakly coassociative system as a generalization of \( \{(M_n(\mathbb{C}), \varphi_{n,m}, R^{(n,m)}) : n, m \geq 1\} \). By using general statements in §2 we will prove Theorem 1.4 in §3.

2 \( \mathcal{C}^* \)-weakly coassociative system

In this section, we consider a general method of construction of \( \mathcal{C}^* \)-bialgebras in order to prove Theorem 1.4.

2.1 Definitions

According to §3 in [9], we recall a general method to construct a \( \mathcal{C}^* \)-bialgebra from a set of \( \mathcal{C}^* \)-algebras and \( * \)-homomorphisms among them. We call \( M \) a monoid if \( M \) is a semigroup with unit.

**Definition 2.1** Let \( M \) be a monoid with the unit \( e \). A data \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \) is a \( \mathcal{C}^* \)-weakly coassociative system (= \( \mathcal{C}^* \)-WCS) over \( M \) if \( A_a \) is a unital \( \mathcal{C}^* \)-algebra for \( a \in M \) and \( \varphi_{a,b} \) is a unital \( * \)-homomorphism from \( A_{ab} \) to \( A_a \otimes A_b \) for \( a, b \in M \) such that

(i) for all \( a, b, c \in M \), the following holds:

\[
(id_a \otimes \varphi_{b,c}) \circ \varphi_{a,bc} = (\varphi_{a,b} \otimes id_c) \circ \varphi_{ab,c}
\]

where \( id_x \) denotes the identity map on \( A_x \) for \( x = a, c \),

(ii) there exists a counit \( \varepsilon_e \) of \( A_e \) such that \( (A_e, \varphi_{e,e}, \varepsilon_e) \) is a counital \( \mathcal{C}^* \)-bialgebra,

(iii) \( \varphi_{e,a}(x) = I_e \otimes x \) and \( \varphi_{a,e}(x) = x \otimes I_e \) for \( x \in A_a \) and \( a \in M \).

From this definition, the following holds.

**Theorem 2.2** ([9], Theorem 3.1) Let \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \) be a \( \mathcal{C}^* \)-WCS over a monoid \( M \). Assume that \( M \) satisfies

\[
\#N_a < \infty \text{ for each } a \in M
\]

where \( N_a \equiv \{(b, c) \in M \times M : bc = a\} \). Define \( \mathcal{C}^* \)-algebras

\[
A_\star \equiv \oplus \{A_a : a \in M\}, \quad C_\star \equiv \oplus \{A_b \otimes A_c : (b, c) \in N_a\} \quad (a \in M),
\]

where \( \mathcal{C}^* \)-weakly coassociative system as a generalization of \( \{(M_n(\mathbb{C}), \varphi_{n,m}, R^{(n,m)}) : n, m \geq 1\} \). By using general statements in §2 we will prove Theorem 1.4 in §3.

2 \( \mathcal{C}^* \)-weakly coassociative system

In this section, we consider a general method of construction of \( \mathcal{C}^* \)-bialgebras in order to prove Theorem 1.4.

2.1 Definitions

According to §3 in [9], we recall a general method to construct a \( \mathcal{C}^* \)-bialgebra from a set of \( \mathcal{C}^* \)-algebras and \( * \)-homomorphisms among them. We call \( M \) a monoid if \( M \) is a semigroup with unit.

**Definition 2.1** Let \( M \) be a monoid with the unit \( e \). A data \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \) is a \( \mathcal{C}^* \)-weakly coassociative system (= \( \mathcal{C}^* \)-WCS) over \( M \) if \( A_a \) is a unital \( \mathcal{C}^* \)-algebra for \( a \in M \) and \( \varphi_{a,b} \) is a unital \( * \)-homomorphism from \( A_{ab} \) to \( A_a \otimes A_b \) for \( a, b \in M \) such that

(i) for all \( a, b, c \in M \), the following holds:

\[
(id_a \otimes \varphi_{b,c}) \circ \varphi_{a,bc} = (\varphi_{a,b} \otimes id_c) \circ \varphi_{ab,c}
\]

where \( id_x \) denotes the identity map on \( A_x \) for \( x = a, c \),

(ii) there exists a counit \( \varepsilon_e \) of \( A_e \) such that \( (A_e, \varphi_{e,e}, \varepsilon_e) \) is a counital \( \mathcal{C}^* \)-bialgebra,

(iii) \( \varphi_{e,a}(x) = I_e \otimes x \) and \( \varphi_{a,e}(x) = x \otimes I_e \) for \( x \in A_a \) and \( a \in M \).

From this definition, the following holds.

**Theorem 2.2** ([9], Theorem 3.1) Let \( \{(A_a, \varphi_{a,b}) : a, b \in M\} \) be a \( \mathcal{C}^* \)-WCS over a monoid \( M \). Assume that \( M \) satisfies

\[
\#N_a < \infty \text{ for each } a \in M
\]

where \( N_a \equiv \{(b, c) \in M \times M : bc = a\} \). Define \( \mathcal{C}^* \)-algebras

\[
A_\star \equiv \oplus \{A_a : a \in M\}, \quad C_\star \equiv \oplus \{A_b \otimes A_c : (b, c) \in N_a\} \quad (a \in M),
\]
and define ⋆-homomorphisms \( \Delta^{(a)} \in \text{Hom}(A_a, C) \), \( \Delta \in \text{Hom}(A, A \otimes A) \) and \( \varepsilon \in \text{Hom}(A, C) \) by

\[
\Delta^{(a)}(x) \equiv \sum_{(b,c) \in \mathbb{N}} \varphi_{b,c}(x) \quad (x \in A_a), \quad \Delta \equiv \oplus \{ \Delta^{(a)} : a \in M \},
\]

\[
\varepsilon(x) \equiv \begin{cases} 
0 & \text{when } x \in \oplus \{ A_a : a \in M \setminus \{ e \} \}, \\
\varepsilon_e(x) & \text{when } x \in A_e.
\end{cases}
\]

Then \( (A^*, \Delta, \varepsilon) \) is a counital \( C^\ast \)-bialgebra.

We call \( (A^*, \Delta, \varepsilon) \) in Theorem 2.2 by a (counital) \( C^\ast \)-bialgebra associated with \( \{ (A_a, \varphi_{a,b}) : a, b \in M \} \). In this paper, we always assume the condition (2.2).

In this subsection, we do not assume that \( M \) is abelian. For example, we constructed a \( C^\ast \)-WCS over a non-abelian monoid in [10].

### 2.2 Locally triangular \( C^\ast \)-weakly coassociative system

In addition to §2.1, we introduce locally triangular \( C^\ast \)-weakly coassociative system (=\( C^\ast \)-WCS) in this subsection.

**Definition 2.3** Let \( \{ (A_a, \varphi_{a,b}) : a, b \in M \} \) be a \( C^\ast \)-WCS.

(i) For \( a, b \in M \), define \( \varphi_{a,b}^{op} \in \text{Hom}(A_{ab}, A_b \otimes A_a) \) by

\[
\varphi_{a,b}^{op} \equiv \tau_{a,b} \circ \varphi_{a,b}
\]

where \( \tau_{a,b} \) denotes the flip from \( A_a \otimes A_b \) to \( A_b \otimes A_a \).

(ii) \( \{ (A_a, \varphi_{a,b}) : a, b \in M \} \) is locally quasi-cocommutative if there exists \( \{ R^{(a,b)} : a, b \in M \} \) such that \( R^{(a,b)} \) is a unitary in \( A_a \otimes A_b \) and

\[
R^{(a,b)} \varphi_{a,b}(x)(R^{(a,b)})^* = \varphi_{b,a}^{op}(x) \quad (x \in A_{ab})
\]

for each \( a, b \in M \). In this case, we call \( \{ (A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M \} \) a locally quasi-cocommutative \( C^\ast \)-WCS.

(iii) A locally quasi-cocommutative \( C^\ast \)-WCS \( \{ (A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M \} \) is locally quasi-triangular if the following holds:

\[
(\varphi_{a,b} \otimes \text{id}_c)(R^{(ab,c)}) = R^{(a,c)}_{13} R^{(b,c)}_{23},
\]

\[
(\text{id}_a \otimes \varphi_{b,c})(R^{(a,b,c)}) = R^{(a,c)}_{13} R^{(a,b)}_{12}
\]

for each \( a, b, c \in M \).
A locally quasi-cocommutative $C^*$-WCS $\{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M\}$ is locally triangular if $\{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M\}$ is locally quasi-triangular and the following holds:

$$R^{(a,b)}\tau_{b,a}(R^{(b,a)}) = I_a \otimes I_b \quad (a, b \in M) \quad (2.10)$$

where $I_x$ denotes the unit of $A_x$ for $x = a, b$.

For a $C^*$-WCS $\{(A_a, \varphi_{a,b}) : a, b \in M\}$, the following holds from (1.12):

$$\mathcal{M}(A_\ast \otimes A_\ast) \cong \prod_{a,b \in M} A_a \otimes A_b. \quad (2.11)$$

Hence we identify an element in $\mathcal{M}(A_\ast \otimes A_\ast)$ with that in $\prod_{a,b \in M} A_a \otimes A_b$.

By Definition 2.3, the following holds.

**Lemma 2.4** Assume that a monoid $M$ is abelian.

(i) If a $C^*$-WCS $\{(A_a, \varphi_{a,b}) : a, b \in M\}$ is locally quasi-cocommutative with respect to $\{R^{(a,b)} : a, b \in M\}$ in (2.7), then the unitary $R \in \mathcal{M}(A_\ast \otimes A_\ast)$ defined by

$$R = (R^{(a,b)})_{a,b \in M} \quad (2.12)$$

is a universal $R$-matrix of $(A_\ast, \Delta_{\varphi})$.

(ii) If a locally quasi-cocommutative $C^*$-WCS $\{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M\}$ is locally quasi-triangular, then $(A_\ast, \Delta_{\varphi}, R)$ is quasi-triangular for $R$ in (2.12).

(iii) If a locally quasi-cocommutative $C^*$-WCS $\{(A_a, \varphi_{a,b}, R^{(a,b)}) : a, b \in M\}$ is locally triangular, then $(A_\ast, \Delta_{\varphi}, R)$ is triangular for $R$ in (2.12).

**Proof.** (i) Let $a \in M$ and $x \in A_a$. From (2.3),

$$R\Delta_{\varphi}(x)R^\ast = R\Delta_{\varphi}^{(a)}(x)R^\ast = \sum_{b,c; bc = a} R\varphi_{b,c}(x)R^\ast. \quad (2.13)$$

From (2.7),

$$R\varphi_{b,c}(x)R^\ast = R^{(b,c)}\varphi_{b,c}(x)(R^{(b,c)})^\ast = \varphi_{c,b}^{op}(x). \quad (2.14)$$

From these and the assumption that $M$ is abelian,

$$R\Delta_{\varphi}(x)R^\ast = \sum_{b,c; bc = a} \varphi_{c,b}^{op}(x) = \sum_{b,c; cb = a} \varphi_{c,b}^{op}(x). \quad (2.15)$$
On the other hand,
\[ \Delta_{\varphi}^\sigma(x) = \tilde{\tau}_{A_*, A_*}(\Delta_{\varphi}^{(a)}(x)) = \sum_{b, c; cb = a} \tau_{b, c}(\varphi_{c, b}(x)) = \sum_{b, c; cb = a} \varphi_{c, b}^{\text{op}}(x). \tag{2.16} \]

Hence \( R\Delta_{\varphi}(x)R^* = \Delta_{\varphi}^\sigma(x) \) for each \( a \in M \) and \( x \in A_a \). Therefore the statement holds.

(ii) Let \( a, b, c \in M \) and \( z \in A_a \otimes A_b \otimes A_c \). By (2.5),
\[ (\Delta_{\varphi} \otimes \text{id})(R)z = (\varphi_{a, b} \otimes \text{id}_c)(R^{(ab, c)})z = R_{13}^{(a, c)}R_{23}^{(b, c)}z = R_{13}R_{23}z. \tag{2.17} \]

From this, \( (\Delta_{\varphi} \otimes \text{id})(R) = R_{13}R_{23} \). By the same token, we can verify that
\[ (\text{id} \otimes \Delta_{\varphi})(R) = R_{13}R_{12}. \] Hence the statement holds.

(iii) Let \( a, b \in M \) and \( z \in A_a \otimes A_b \). From (2.10),
\[ R\tilde{\tau}_{A_*, A_*}(R)z = R^{(a, b)}(\tilde{\tau}{b, a}(R^{(b, a)})z) = z. \tag{2.18} \]

This holds for each \( a, b \in M \) and \( z \in A_a \otimes A_b \). Therefore \( R\tilde{\tau}_{A_*, A_*}(R) = I \). Hence the statement holds.

We use the assumption that \( M \) is abelian in the proof of Lemma 2.4(i).

3 Proof of Theorem 1.4

We prove Theorem 1.4 in this section. We regard the set \( N = \{1, 2, 3, \ldots\} \) of all positive integers as a monoid with respect to the multiplication. Then we see that \( \{(M_n(C), \varphi_{n, m}) : n, m \in N\} \) in (1.13) is a C*-WCS over the abelian monoid \( N \). From Lemma 2.4, it is sufficient for the proof of Theorem 1.4 to show the following equations for \( \{R^{(n, m)} : n, m \in N\} \) in (1.17):
\[ R^{(n, m)}\varphi_{n, m}(x)(R^{(n, m)})^* = \varphi_{m, n}^{\text{op}}(x) \quad (x \in M_{nm}(C)), \tag{3.1} \]
\[ (\varphi_{n, m} \otimes \text{id}_l)(R^{(nm, l)}) = R_{13}^{(n, l)}R_{23}^{(m, l)}; \tag{3.2} \]
\[ (\text{id}_n \otimes \varphi_{m, l})(R^{(n, ml)}) = R_{13}^{(n, l)}R_{12}^{(n, m)}; \tag{3.3} \]
\[ R^{(n, m)}\tau_{m, n}(R^{(m, n)}) = I_n \otimes I_m \tag{3.4} \]
for each \( n, m, l \in N \).
3.1 Proof of Theorem 1.4(i)

In this subsection, we show (3.1) in order to prove Theorem 1.4(i). We introduce several new symbols for convenience as follows: Let $F_n \equiv \{1, \ldots, n\}$ and define the bijective map $\phi_{n,m}$ from $F_n \times F_m$ to $F_{nm}$ by

$$\phi_{n,m}(i, j) \equiv m(i - 1) + j \quad ((i, j) \in F_n \times F_m).$$

(3.5)

Let $\{E_{i,j}^{(n)} : i, j \in F_n\}$ be as in §1.3. For $i, j \in F_n$ and $k, l \in F_m$,

$$E_{(i,k),(j,l)}^{(n,m)} \equiv E_{i,j}^{(n)} \otimes E_{k,l}^{(m)}. \quad (3.6)$$

Lemma 3.1 For $k, l \in F_{nm}$, the following holds:

$$R^{(n,m)} \varphi_{n,m}(E_{k,l}^{(nm)})(R^{(n,m)})^* = E_{\chi_{n,m} \circ \phi_{n,m}^{-1}, \chi_{n,m} \circ \phi_{n,m}^{-1}}^{(n,m)}(k), \chi_{n,m} \circ \phi_{n,m}^{-1}(l),$$

(3.7)

$$\varphi_{m,n}^{op}(E_{k,l}^{(nm)}) = E_{\theta_{m,n} \circ \phi_{m,n}^{-1}, \theta_{m,n} \circ \phi_{m,n}^{-1}}^{(n,m)}(k), \theta_{m,n} \circ \phi_{m,n}^{-1}(l) \quad (3.8)$$

where $\chi_{n,m}$ is as in (1.20) and $\theta_{m,n}$ denotes the flip from $F_m \times F_n$ to $F_n \times F_m$.

Proof. From (3.5) and (1.13),

$$\varphi_{n,m}(E_{k,l}^{(nm)}) = E_{\phi_{n,m}^{-1}(k), \phi_{n,m}^{-1}(l)}^{(n,m)}(k, l \in F_{nm}).$$

(3.9)

By definition, $R^{(n,m)}$ is written as follows:

$$R^{(n,m)} = \sum_{(i,j) \in F_n \times F_m} E_{\chi_{n,m} \circ \phi_{n,m}^{-1}, \chi_{n,m} \circ \phi_{n,m}^{-1}}^{(n,m)}(i, j).$$

(3.10)

From (3.9) and (3.10), (3.7) holds.

We see that

$$\tau_{n,m}(E_{(i,k),(j,l)}^{(n,m)}) = E_{(i,k),(j,l)}^{(m,n)} \otimes E_{i,j}^{(m,n)} = E_{(i,k),(j,l)}^{(m,n)}.$$  

(3.11)

By (3.9), $\varphi_{m,n}(E_{k,l}^{(nm)}) = E_{\phi_{m,n}^{-1}(k), \phi_{m,n}^{-1}(l)}^{(m,n)}$. From this and (3.11), (3.8) holds.

The equation (3.10) shows a representation of $R$ restricted on the vector subspace $M_n(C) \otimes M_m(C)$ of $M_n(C) \otimes M_m(C)$.
**Proof of Theorem 1.4(i).** We prove (3.1) for each \( n, m \in \mathbb{N} \). If it is done, then Theorem 1.4(i) holds from Lemma 2.4(i).

For \( \chi_{n,m} \) in (1.20), we see that

\[
\chi_{n,m} = \theta_{m,n} \circ \phi_{m,n}^{-1} \circ \phi_{n,m} \quad (3.12)
\]

where \( \theta_{m,n} \) is as in Lemma 3.1. From this, we see that (3.7) equals (3.8). Hence (3.1) is verified. Therefore the statement holds.

### 3.2 Proof of Theorem 1.4(ii)

In order to prove Theorem 1.4(ii), we prove (3.2-3.4). Especially, we show (3.2) in § 3.2.1 and § 3.2.2 step by step.

#### 3.2.1 Proof of (3.2) — Step 1

In this subsubsection, we reduce (3.2) to equations of maps on integers.

**Lemma 3.2** Let \( F_n \) and \( \{E_{i,j}^{(n)}\} \) be as in § 3.2.1. For \( i, a \in F_n, j, b \in F_m \) and \( k, c \in F_l \), let

\[
E_{(i,j,k),(a,b,c)}^{(n,m,l)} = E_{i,a}^{(n)} \otimes E_{j,b}^{(m)} \otimes E_{k,c}^{(l)}. \quad (3.13)
\]

Then the following holds:

\[
(\varphi_{n,m} \otimes id_l)(R_{nm,l}) = \sum_{(a,b,c) \in F_n \times F_m \times F_l} E_{P(a,b,c),(a,b,c)}^{(n,m,l)}, \quad (3.14)
\]

\[
R_{13}^{(n,l)} R_{23}^{(m,l)} = \sum_{(a,b,c) \in F_n \times F_m \times F_l} E_{Q(a,b,c),(a,b,c)}^{(n,m,l)} \quad (3.15)
\]

where \( P \) and \( Q \) are maps on \( F_n \times F_m \times F_l \) defined by

\[
P \equiv (\phi_{n,m}^{-1} \times id_l) \circ \chi_{nm,l} \circ (\phi_{n,m} \times id_l), \quad (3.16)
\]

\[
Q \equiv (id_n \times \theta_{l,m}) \circ (\chi_{n,l} \times id_m) \circ (id_n \times \theta_{m,l}) \circ (id_n \times \chi_{m,l}) \quad (3.17)
\]

where \( id_x \) denotes the identity map on \( F_x \) for \( x = n, m, l \).

**Proof.** From (3.10),...
\((\varphi_{n,m} \otimes \text{id}_l)(R^{(nm,l)})\)

\[= \sum_{(t,k) \in F_{nm} \times F_l} (\varphi_{n,m} \otimes \text{id}_l)(E^{(nm,l)}_{\chi_{nm,l}(t,k)}, (t,k)) \]

\[= \sum_{(i,j,k) \in F_n \times F_m \times F_l} (\varphi_{n,m} \otimes \text{id}_l)(E^{(nm,l)}_{\chi_{nm,l}(\phi_{n,m}(i,j),k), (\phi_{n,m}(i,j),k)}). \]

When \(t = \phi_{n,m}(i, j)\),

\[(\varphi_{n,m} \otimes \text{id}_l)(E^{(nm,l)}_{\chi_{nm,l}(t,k)}) = \varphi_{n,m}(E^{(nm)}_{t,t}) \otimes E^{(l)}_{k,k} \quad \text{(by (3.9))} \]

where \((t, k) = \chi_{nm,l}(t, k)\). We see that

\[(\phi^{-1}_{n,m}(t, k) = \{(\phi^{-1}_{n,m} \times \text{id}_l) \circ \chi_{nm,l}(t, k) \}

\[= \{(\phi^{-1}_{n,m} \times \text{id}_l) \circ \chi_{nm,l} \circ (\phi_{n,m} \times \text{id}_l)\}(i, j, k) = P(i, j, k). \]

Hence (3.14) holds.

From (3.10),

\[R^{(n,l)}_{13} R^{(m,l)}_{23} = \{\text{id}_n \otimes \tau_{l,m}\}(R^{(n,l)} \otimes \text{id}_m)\{\text{id}_n \otimes \tau_{m,l}\}(\text{id}_n \otimes R^{(m,l)}) \]

\[= \sum_{(i,t) \in F_n \times F_l} \sum_{(j,k) \in F_m \times F_l} Y^{(n,m,l)}_{i,t,j,k} \]

where

\[Y^{(n,m,l)}_{i,t,j,k} \equiv \{\text{id}_n \otimes \tau_{l,m}\}(E^{(n,l)}_{\chi_{n,l}(i,t), (i,t)} \otimes \text{id}_m)\{\text{id}_n \otimes \tau_{m,l}\}(\text{id}_n \otimes E^{(m,l)}_{\chi_{m,l}(j,k), (j,k)}). \]

Then we see that

\[Y^{(n,m,l)}_{i,t,j,k} = E^{(n,l)}_{\xi_{i,t}} \otimes E^{(m,l)}_{\xi_{j,k}} \otimes E^{(l,k)}_{\varphi_{n,m}} = \delta_{t,k} E^{(n,m,l)}_{\xi_{i,t}, \xi_{j,k}} \]

(3.19)

where \((\xi_{i,t}) = \chi_{n,l}(i, t)\) and \((\xi_{j,k}) = \chi_{m,l}(j, k)\). From this,

\[R^{(n,l)}_{13} R^{(m,l)}_{23} = \sum_{(i,j,k) \in F_n \times F_m \times F_l} E^{(n,m,l)}_{\xi_{i,j,k}, (i,j,k)} \quad \text{(3.20)} \]

13
When \( t = k \),

\[
(i, j, t) = \{ (id_n \times \theta_{l,m}) \circ (\chi_{n,l} \times id_m) \} \{ i, t, j \} = \{ (id_n \times \theta_{l,m}) \circ (\chi_{n,l} \times id_m) \} \{ i, k, j \} = Q(i, j, k).
\]

Therefore (3.15) holds.

From Lemma 3.2, it is sufficient for the proof of (3.2) to show the equality \( P = Q \) for two maps \( P \) and \( Q \) in (3.16) and (3.17).

3.2.2 Proof of (3.2) — Step 2

In this subsubsection, we prove equations of maps on integers in Lemma 3.2.

Lemma 3.3 For \( P \) and \( Q \) in (3.16) and (3.17), \( P = Q \), that is, the following diagram is commutative:

\[
\begin{array}{ccc}
\phi_{n,m} \times id_l & F_n \times F_m \times F_l & id_n \times \chi_{m,l} \\
F_{nm} \times F_l & \downarrow & F_n \times F_m \times F_l \\
\chi_{nm,l} & & id_n \times \theta_{m,l} \\
F_{nm} \times F_l & \phi_{n,m}^{-1} \times id_l & F_n \times F_m \times F_l \\
& & id_n \times \theta_{l,m}
\end{array}
\]

Proof. Here we omit the symbol “\( \circ \)” for simplicity of description. For \( \{ \phi_{n,m} : n, m \in \mathbb{N} \} \) in (3.5), the following holds:

\[
\phi_{nm,l}(\phi_{n,m} \times id_l) = \phi_{n,ml}(id_n \times \phi_{m,l}) \quad (n, m, l \in \mathbb{N}).
\] (3.21)

From (3.21), we obtain

\[
\phi_{nm,l} = \phi_{n,ml}(id_n \times \phi_{m,l})(\phi_{n,m} \times id_l)^{-1}.
\] (3.22)

By the same token, we see that

\[
\phi_{n,ml} = \phi_{nl,m}(id_l \times \phi_{n,m})(\phi_{l,n} \times id_m)^{-1},
\] (3.23)

\[
\phi_{nl,m} = \phi_{tn,m}(id_l \times \phi_{n,m})(\phi_{l,n} \times id_m)^{-1}.
\] (3.24)
Substituting (3.24) into (3.23), and substituting it into (3.22),

\[
\phi_{nm,l} = \phi_{l,nm} (\text{id}_l \times \phi_{n,m})^{-1} \times (\phi_{n,l} \times \text{id}_m) (\text{id}_n \times \phi_{l,m})^{-1} (\phi_{n,m} \times \text{id}_l)^{-1}.
\]

Hence

\[
\theta_{nm,l} = (\text{id}_l \times \phi_{n,m}) (\theta_{n,l} \times \text{id}_m) (\phi_{n,m} \times \text{id}_l)^{-1}.
\] (3.25)

From this,

\[
(id_l \times \phi_{n,m})^{-1} (\theta_{nm,l} \times \text{id}_m) = (\theta_{n,l} \times \text{id}_m) (\phi_{n,m} \times \text{id}_l). \tag{3.26}
\]

By multiplying \((id_n \times \theta_{l,m})(\text{id}_n \times \phi_{n,m})^{-1} \theta_{nm,l} \times \text{id}_m)\) at both sides of (3.26) from the left,

\[
(id_n \times \theta_{l,m})(\theta_{n,l} \times \text{id}_m)(id_l \times \phi_{n,m})^{-1} (\theta_{nm,l} \times \text{id}_m) = (id_n \times \theta_{l,m})(\beta_{n,m} \times \text{id}_l) (\phi_{n,m} \times \text{id}_l)^{-1}.
\] (3.27)

The R.H.S. of (3.28) is \(Q\). On the other hand, the L.H.S. of (3.28) is

\[
(id_n \times \theta_{l,m})(\theta_{n,l} \times \text{id}_m)(\phi_{n,m} \times \text{id}_l)^{-1} = (id_n \times \theta_{l,m})(\beta_{n,m} \times \text{id}_l)(\phi_{n,m} \times \text{id}_l)^{-1}.
\] (3.29)

where \(\beta_{n,m,l}(i,j,k) \equiv (k,i,j)\). Since \((id_n \times \theta_{l,m})(\theta_{n,l} \times \text{id}_m) \eta_{n,m,l} = id_n \times id_m \times id_l\), the L.H.S. of (3.28) is \(P\). Hence the statement holds.

During initial phases of this study, Lemma 3.3 was forecasted by a computer experiment. Essential parts of the proof of Lemma 3.3 are equations in (3.21).

### 3.2.3 Proof of Theorem 1.4(ii)

From Lemma 3.2 and Lemma 3.3 (3.2) holds. By the same token, (3.3) can be verified. From Lemma 2.4(ii), the quasi-cocommutative \(C^*-\)bialgebra \((M_s(C), \Delta_\varphi, R)\) is quasi-triangular.

From (3.11) and (3.10),

\[
\tau_{nm}(R^{(n,m)}) = \sum_{(i,j) \in \text{F}_n \times \text{F}_m} \tau_{nm}(E_{\chi_{n,m}(i,j), (i,j)})
\]

\[
= \sum_{(i,j) \in \text{F}_n \times \text{F}_m} E_{\theta_{n,m}, \chi_{n,m}}(i,j), \theta_{n,m}(i,j)
\]

\[
= \sum_{(a,b) \in \text{F}_n \times \text{F}_m} E_{\theta_{n,m}, \chi_{n,m}}(b,a), (b,a).
\]
From this and (3.10),
\[
R^{(n,m)} \tau_{m,n}(R^{(m,n)}) = \sum_{(i,j),(b,a) \in F_n \times F_m} E^{(n,m)}_{\chi_{n,m}(i,j),(i,j)} E^{(n,m)}_{\theta_{m,n} \chi_{m,n} \theta_{n,m}}(b,a), (b,a)
\]
\[
= \sum_{(b,a) \in F_n \times F_m} E^{(n,m)}_{\chi_{n,m} \theta_{m,n} \chi_{m,n} \theta_{n,m}}(b,a), (b,a)
\]

On the other hand, \(\chi_{n,m} \theta_{m,n} \chi_{m,n} \theta_{n,m} = id_n \times id_m\) from (3.12). Hence
\[
R^{(n,m)} \tau_{m,n}(R^{(m,n)}) = \sum_{(b,a) \in F_n \times F_m} E^{(n,m)}_{(b,a), (b,a)} = I_n \otimes I_m. \tag{3.30}
\]

Hence, (3.31) holds. From this and Lemma 2.4(iii), the quasi-triangular \(C^\ast\)-bialgebra \((M_\ast(C), \Delta_\varphi, R)\) is triangular.

\section*{Appendix}

\section{Basic facts about quasi-triangular \(C^\ast\)-bialgebras}

In this section, we show basic facts about quasi-triangular \(C^\ast\)-bialgebras.

\textbf{Fact A.1} Let \((A, \Delta, R)\) be a quasi-triangular \(C^\ast\)-bialgebra. Then the following holds:

(i) \(R\) satisfies the Yang-Baxter equation
\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{A.1}
\]

(ii) If \((A, \Delta)\) has a counit \(\varepsilon\), then
\[
(\varepsilon \otimes id)(R) = id = (id \otimes \varepsilon)(R) \tag{A.2}
\]
where \(id\) denotes the unit of \(M(A)\).

(iii) Let \((\mathcal{H}, \pi)\) be a nondegenerate representation of the \(C^\ast\)-algebra \(A\). Let \(\Pi\) denote the extension of \(\pi \otimes \pi\) on \(M(A \otimes A)\) and let \(T\) denote the flip on \(\mathcal{H} \otimes \mathcal{H}\). Define the unitary operator \(C\) on \(\mathcal{H} \otimes \mathcal{H}\) by
\[
C = T\Pi(R). \tag{A.3}
\]
For \( n \geq 3 \), let \( \mathcal{H} \otimes^n \) denote the \( n \)-times tensor power of \( \mathcal{H} \). For \( 1 \leq i \leq n-1 \), let \( C_i \equiv I_{\mathcal{H}} \otimes (i-1) \otimes C \otimes I_{\mathcal{H}} \otimes (n-i) \) where \( I_{\mathcal{H}} \) denotes the identity map on \( \mathcal{H} \). Then

\[
C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1}.
\]

(A.4)

In addition, if \( (A, \Delta, R) \) is triangular, then \( C^2 = I \).

Proof. Proofs of (i) and (ii) are given along with the proof of Theorem VIII.2.4 of \[8\] which is modified to a \( C^* \)-bialgebra as follows:

(i)

\[
R_{12} R_{13} R_{23} = R_{12} (\Delta \otimes id)(R) \quad \text{(by (1.8))}
\]
\[
= (\Delta \otimes id)(R) R_{12} \quad \text{(by (1.7))}
\]
\[
= (\tilde{\tau}_{A,A} \otimes id)(\Delta \otimes id)(R) R_{12} \quad \text{(by (1.8))}
\]
\[
= (\tilde{\tau}_{A,A} \otimes id)(R_{13} R_{23}) R_{12} \quad \text{(by (1.7))}
\]
\[
= (\tilde{\tau}_{A,A} \otimes id)(R_{12}) \cdot (\tilde{\tau}_{A,A} \otimes id)(R_{23}) R_{12}
\]
\[
= R_{23} R_{13} R_{12}.
\]

(ii) Since \( \varepsilon \) is nondegenerate, it can be extended to the \( * \)-homomorphism \( \tilde{\varepsilon} \) from \( \mathcal{M}(A) \) to \( \mathbb{C} \) such that \( \tilde{\varepsilon}(I) = 1 \). We write \( \tilde{\varepsilon} \) as \( \varepsilon \) here. Since \( (\varepsilon \otimes id) \circ \Delta = id \),

\[
R = \{ (\varepsilon \otimes id) \circ (\Delta \otimes id) \}(R)
\]
\[
= (\varepsilon \otimes id \otimes id)(R_{13} R_{23}) \quad \text{(by (1.8))}
\]
\[
= (\varepsilon \otimes id)(R_{12}) \cdot (\varepsilon \otimes id)(R_{23})
\]
\[
= (\varepsilon \otimes id)(R) \cdot \varepsilon(I) R.
\]

From this, we obtain \( (\varepsilon \otimes id)(R) = id \) because \( \varepsilon(I) = 1 \) and \( R \) is invertible.

By the same token, we obtain \( (id \otimes \varepsilon)(R) = id \).

(iii) Assume \( n = 3 \) and \( i = 1 \). Let \( U \equiv \Pi(R) \). Then

\[
C_1 C_2 C_1 = T_{12} U_{12} T_{23} U_{23} T_{12} U_{12}
\]
\[
= T_{12} T_{23} U_{12} U_{13} U_{12}
\]
\[
= T_{23} T_{12} U_{12} U_{13} U_{23} \quad \text{(by (A.1))}
\]
\[
= T_{23} U_{23} T_{12} U_{12} T_{23} U_{23}
\]
\[
= C_2 C_1 C_2
\]

where we use the leg numbering notations \( T_{ij} \) and \( U_{ij} \) on \( \mathcal{H}^{\otimes 3} \) and \( T_{12} T_{23} T_{12} = T_{23} T_{12} T_{23} \). This implies (A.4).

Assume that \( (A, \Delta, R) \) is triangular. For \( a, b \in A \), we see that \( T\{(\pi \otimes \pi)(a \otimes b)\} = (\pi \otimes \pi)(b \otimes a) \). From this,

\[
T \Pi(R) T = \Pi(\tilde{\tau}_{A,A}(R)).
\]

(A.5)
From (A.5) and (1.9),
\[ C^2 = T\Pi(R)T\Pi(R) = \Pi(\tilde{\tau}_{A,A}(R)R) = \Pi(I_{M(A\otimes A)}) = I. \quad \text{(A.6)} \]

In addition to Fact (A.1(iii)), it is clear that \( \{ C_i \}_{i=1}^{n-1} \) satisfies \( C_i C_j = C_j C_i \) for \( i, j = 1, \ldots, n-1 \) when \( |i-j| \geq 2 \). Therefore a nondegenerate representation of a quasi-triangular (resp. triangular) \( C^* \)-bialgebra gives a unitary representation of the braid group \( B_n \) ([8], Lemma X.6.4) (resp. the symmetric group \( \mathfrak{S}_n \) ([8], § X.6.3)).

References

[1] S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de \( C^* \)-algèbres, *Ann. Scient. Ec. Norm. Sup.*, 4e série 26 (1993), 425–488.

[2] B. Blackadar, Operator algebras. Theory of \( C^* \)-algebras and von Neumann algebras, Springer-Verlag Berlin Heidelberg New York, 2006.

[3] V. Chari and A. Pressley, A guide to quantum groups, Cambridge University Press, 1994.

[4] M. Cohen, D. Fischman and S. Westreich, Schur’s double centralizer theorem for triangular Hopf algebras, *Proc. Amer. Math. Soc.* 122(1) (1994), 19–29.

[5] V. G. Drinfel’d, Quantum groups, Proceedings of the international congress of mathematicians, Berkeley, California, 1987, 798–820.

[6] C. Gómez, M. Ruiz-Altaba and G. Sierra, Quantum groups in two-dimensional physics, Cambridge University Press, 1996.

[7] M. Jimbo, A \( q \)-difference analogue of \( U(g) \) and Yang-Baxter equation, *Lett. Math. Phys.* 10 (1985), 63–69.

[8] C. Kassel, Quantum groups, Springer-Verlag, 1995.

[9] K. Kawamura, \( C^* \)-bialgebra defined by the direct sum of Cuntz algebras, *J. Algebra* 319 (2008), 3935–3959.

[10] K. Kawamura, \( C^* \)-bialgebra defined as the direct sum of Cuntz-Krieger algebras, *Comm. Algebra* 37 (2009), 4065–4078.
[11] K. Kawamura, Non-existence of universal $R$-matrix for some $C^*$-bialgebras, math.OA/0912.3578v1.

[12] J. Kustermans and S. Vaes, The operator algebra approach to quantum groups, *Proc. Natl. Acad. Sci. USA* **97**(2) (2000), 547–552.

[13] T. Masuda, Y. Nakagami and S. L. Woronowicz, A $C^*$-algebraic framework for quantum groups, *Int. J. Math.* **14** (2003), 903–1001.

[14] A. Van Daele and S. Van Keer, The Yang-Baxter and pentagon equation, *Composit. Math.* **91**(2) (1994), 201–221.

[15] S. L. Woronowicz, $C^*$-algebras generated by unbounded elements, *Rev. Math. Phys.* **7** (1995), 481–521.