Hardy inequalities with double singular weights

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Abstract

The aim of this paper is to obtain new Hardy inequalities with double singular weights – at an interior point and on the boundary of the domain. These inequalities give us the possibility to derive estimates from below of the first eigenvalue of the p-Laplacian with Dirichlet boundary conditions.

Keywords: Hardy inequality; Double singular weights; Sharp estimates; p-Laplacian; First eigenvalue.

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1 Introduction

The paper is devoted to the classical Hardy inequality, its generalizations and applications for estimates from below of the first eigenvalue \( \lambda_{p,n}(\Omega) \) of the p-Laplacian, \( p > 1 \), in a bounded domain \( \Omega \subset R^n, n \geq 2 \). Only the multidimensional case \( n \geq 2 \) is considered because for \( n = 1 \) there exist detailed literature and satisfactory results, see for instance Hardy [51, 52, 53], Nečas [79], Maz’ja [76], Opic and Kufner [80], Hoffmann-Ostenhof et al. [55]. Unlike the one-dimensional case, the theory for \( n \geq 2 \) is far from being completely solved.

The paper can be regarded as a work in the series of works of Maz’ja [76], Opic and Kufner [80], Ghousoub and Moradifam [47], Balinsky et al. [12], Kufner et al. [63]. We focus on the optimality of the Hardy constant and on the sharpness of Hardy inequality. Further on in the paper we say that the Hardy constant is optimal if for a greater one the corresponding Hardy inequality fails for all functions of the admissible class. Sharpness of the Hardy inequality means that an equality is achieved for some admissible function. For Hardy inequality with singular weights at an interior point of \( \Omega \), or on the boundary \( \partial \Omega \), we always prove the optimality of Hardy constant. As for the sharpness of the Hardy inequality, we show that an equality is achieved only for Hardy inequality with additional ‘nonlinear’ term. It is well-known that for inequalities with optimal constant and additional ‘linear’ term, an equality is not achieved. Only in the case when Hardy constant is greater than the optimal one, the sharpness of the inequality is proved by variational technique, see for example Pinchover and Tintarev [83] and the references therein. In fact, in this way the optimality of Hardy constant is shown.

In the literature mainly inequalities with singular weights at a point, or on the boundary, \( \partial \Omega \), or on some \( k \)-dimensional manifold, \( 1 \leq k \leq n - 1 \), have been studied. The subject of the investigations in the paper is Hardy type inequalities in bounded domains \( \Omega \in R^n, n \geq 2 \) with double singular weights: in an interior point of \( \Omega \) and on the boundary \( \partial \Omega \).

Our aim is to derive new Hardy inequalities, which are with an optimal constant and with suitable additional terms that become sharp. For example, Hardy constant is optimal for convex and star-shaped domains and the inequality is sharp as a result of a ‘nonlinear’ additional term.

The background of the theory of Hardy inequalities is mathematical and functional analysis and differential equations. Among many different applications we choose one – the estimate of the first eigenvalue of the p-Laplacian from below, which motivates the study of Hardy inequalities with double singular weights.
There are estimates for $\lambda_{p,n}(\Omega)$ by means of the Cheeger’s constant, Cheeger [27], Lefton and Wei [67], Kawohl and Fridman [59], by the Picone’s identity Benedikt and Drábek [18, 19], with the Sobolev inequality Maz’ja [76], Ludwig et al. [75], with estimates in parallelepiped Lindqvist [74] and others. However, Hardy inequality with double singular weights allows one to get better analytical estimates for $\lambda_{p,n}(\Omega)$. For completeness, we prove estimates from below for $\lambda_{p,n}(\Omega)$ as well as by the well-known Hardy inequalities with singular weights only at a point or on the boundary $\partial \Omega$. The comparison of the results definitely shows that the inequalities with double singular weights produce better analytical estimates for $\lambda_{p,n}(\Omega)$ in comparison with those obtained from other Hardy inequalities or other methods.

In the rest of this section, without claims of completeness, we present results of Hardy inequalities which from our point of view are decisive for the development of the subject in the recent years, as the books of Maz’ja [76], Opic and Kufner [80], Ghoussoub and Moradifam [47], Balinsky et al. [12], Kufner et al. [63], and the papers of Brezis and Marcus [25], Brezis and Vazquez [26], Davies [28], Barbatis et al. [13, 14], Hoffmann-Ostenhof et al. [55], Tidblom [87], Edmunds and Hurri-Syrjänen [31], Kinnunen and Korte [60], Lehrbäck [69] and others. In section 2 we derive a new Hardy inequality with weights in abstract form. Particular cases are presented to demonstrate the applicability of the method and to show some generalizations of the existing results. In section 3 we prove a general Hardy inequality with singular weights at zero and on the boundary $\partial \Omega$ of star-shaped domains and an optimal Hardy constant. In section 4 we propose Hardy inequality with weight singular at $0 \in \Omega$ in the class of functions which are not zero on the boundary $\partial \Omega$. The Hardy constant is optimal and the inequality is sharp due to the additional boundary term. In section 5 we derive improved Hardy inequality with double singular weights in bounded, star-shaped domains $\Omega \subset R^n$, $n \geq 2$ where the singularity is at an interior point and on the boundary of the domain. The Hardy constant is optimal and the inequality is sharp due to the additional term. In section 6 we apply Hardy inequalities from the previous sections and we derive estimates from below of the first eigenvalue of the p-Laplacian.

1.1 Preliminary remarks on Hardy inequalities

In this section classical Hardy inequalities are shown together with some definitions used in the paper. The analysis of the existing in the literature results is far from its completeness. The aim of this section is to recall the well known results in the frame of the study in the paper.

The classical Hardy inequality in $R^1_+ = (0, \infty)$ states
\[
\int_0^\infty |u'(x)|^p x^\alpha dx \geq \left( \frac{p - 1 - \alpha}{p} \right)^p \int_0^\infty x^{-p+\alpha} |u(x)|^p dx, \tag{1.1}
\]
where $1 < p < \infty$, $\alpha < p - 1$ and $u(x)$ is an absolutely continuous function on $[0, \infty)$ with $u(0) = 0$, see Hardy [52, 53] for $\alpha = 0$ and Hardy et al. [54], Sect. 9.8 for $\alpha < p - 1$.

The constant $\left( \frac{p - 1 - \alpha}{p} \right)^p$ is the best possible one, i.e., it can not be replaced with a greater one, but the equality in (1.1) is not achieved.

In the last 20 years the generalizations of (1.1) for the multidimensional case are mainly oriented in two directions with respect to the structure of the singular weight:
• singularities on the boundary: when the prototype inequality is
\[ \int_{\Omega} |\nabla u(x)|^p d^\alpha(x) dx \geq C_{\Omega} \int_{\Omega} d^{-p+\alpha}(x)|u(x)|^p dx, \quad u \in W^{1,p}_0(\Omega) \quad (1.2) \]
with \( d(x) = \text{dist}(x, \partial \Omega) \) and \( \alpha < p-1, \, p > 1, \, n \geq 2, \, p \neq n \);

• singularity at a point: when the inequality
\[ \int_{\Omega} |\nabla u(x)|^p dx \geq C_{p,n} \int_{\Omega} \frac{|u(x)|^p}{|x|^p} dx, \quad u \in W^{1,p}_0(\Omega) \quad (1.3) \]
holds, \( \Omega \subset \mathbb{R}^n \), \( 0 \in \Omega \), \( n \geq 2 \) and \( p > 1, \, p \neq n \). The constant \( C_{p,n} = \left| \frac{n-p}{p} \right|^p \) is optimal one.

We will present some of the results on Hardy inequalities underlining their optimality and sharpness, see Definition 1.1

Next we recall several useful definitions and notions.

**Definition 1.1.** The constants \( C_{\Omega} \) in Hardy inequality
\[ \int_{\Omega} V(x)|\nabla u|^p dx \geq C_{\Omega} \int_{\Omega} W(x)|u|^p dx + A(u) \quad (1.4) \]
with positive weights \( V(x), W(x) \) and additional nonnegative term \( A(u) \) is optimal if for every \( \varepsilon > 0 \) there exists \( u_\varepsilon \) from the admissible class of functions for which the inverse Hardy inequality holds if we replace \( C_{\Omega} \) with \( C_{\Omega} + \varepsilon \). The Hardy inequality (1.4) is sharp if there exists a function from the admissible class of functions for which (1.2) is an equality and both sides of (1.2) are finite.

Let us recall that sharp Hardy inequalities are proved by means of variational technique, see Pinchover and Tintarev [83] and the references therein.

**Inequalities with general weights**

There are generalizations of (1.1) for the \( n \)-dimensional case, \( n \geq 2 \), for bounded domains and for different weights (integral kernels), see for more details Davies [28], Opic and Kufner [80], Ghoussoub and Moradifam [47], Balinsky et al. [12].

For example, in Opic and Kufner [80], Theorems 14.1, 14.2, the following Hardy inequality with general weights is proposed
\[ \sum_{i=1}^{n} \int_{\Omega} v_i(x) \left| \frac{\partial u}{\partial x_i} \right|^p dx \geq \int_{\Omega} w(x)|u(x)|^p dx. \quad (1.5) \]
Here \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( u \in C_0^\infty(\Omega) \), \( 1 < p < \infty \), functions \( v_i \), \( i = 1, \ldots, n \) and \( w \) are measurable, positive and finite for a.e. \( x \in \Omega \). It is proved that under the existence of a solution \( y(x) \) of the equation
\[ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ v_i \left| \frac{\partial y}{\partial x_i} \right|^{p-2} \frac{\partial y}{\partial x_i} \right] + w(x)|y|^{p-2} y = 0 \quad \text{in} \ \Omega, \quad (1.6) \]
where \( y(x) \neq 0 \) and \( \frac{\partial y}{\partial x_i} \neq 0 \) for a.e. \( x \in \Omega \), and some regularity conditions on \( y, v_i \) and \( w \) inequality (1.5) hold.

A necessary and sufficient condition on \( v_i \) and \( w \) for the validity of (1.5) is proved in Maz’ja [76] in terms of capacities, see also Opic and Kufner [80], Theorem 16.3 and the discussion therein.

In Ghoussoub and Moradifam [47], Theorem 4.1.1, Hardy inequality with general positive weights \( V(|x|) \) and \( W(|x|) \) is proposed in the ball \( B_R = \{ x \in \mathbb{R}^n, |x| < R \} \), \( n \geq 2 \), i.e.,

\[
\int_{B_R} V(|x|)|\nabla u|^2 dx \geq c \int_{B_R} W(|x|)u^2 dx, \quad u \in C_0^\infty(B_R).
\] (1.7)

The necessary and sufficient condition for the validity of (1.7) given in the same book is that the couple \((V, W)\) forms \( n \)-dimensional Bessel pair on the interval \((0, R)\), i.e., the equation

\[
y''(r) + \left( \frac{n-1}{r} + \frac{V'(r)}{V(r)} \right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0,
\]

has a positive solution in \((0, R)\).

Another Hardy inequality is proved in Shen and Chen [85], Lemma 1.1,

\[
\int_{\Omega} \phi(r)|\nabla u|^2 dx \geq \int_{\Omega} \phi(r) \left| \frac{h'(r)}{h(r)} \right|^2 u^2 dx, \quad r = |x|,
\] (1.8)
in a bounded domain \( \Omega \subset \mathbb{R}^n \), \( 0 \in \Omega \subset B_R \), \( R > \sup_{x \in \Omega} |x| \). Here \( \phi(r) \in C^1(0, R) \), \( \phi(r) > 0 \) and \( h \in C^1(0, R) \) is a positive solution of the equation

\[
r^{n-1} \phi(r)(h^2)'(r) = c = \text{const},
\] (1.9)
and \( u \in C_0^\infty(\Omega) \) if \( h^{-1}(0) = 0 \) while \( u \in C_0^\infty(\Omega \setminus \{0\}) \) if \( h^{-1}(0) \neq 0 \).

In Davies and Hinz [29], see also Balinsky et al. [12], Theorem 1.2.8, the following Hardy inequality is shown

\[
\int_{\Omega} \frac{|\nabla V|^p}{|\Delta V|^{p-1}} |\nabla u|^p dx \geq \left( \frac{1}{p} \right)^p \int_{\Omega} |\Delta V||u|^p dx,
\]
for all \( u \in C_0^\infty(\Omega) \) and any domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \). The real valued weight \( V(x) \) has derivatives up to order \( 2 \) in \( L_{loc}^1(\Omega) \) and \( \Delta V \) is of one sign for a.e. \( x \in \Omega \).

**Inequalities with weights singular on the boundary**

Another research direction on Hardy inequalities concerns the study of the geometric properties of the domain \( \Omega \subset \mathbb{R}^n \), on which the Hardy’s inequality (1.2) holds.

The inequality (1.2) was proved by Nečas [79] for bounded domains \( \Omega \) with Lipschitz boundary \( \partial \Omega \) and \( u \in C_0^\infty(\Omega) \). Next generalizations of (1.2) were made by Kufner [62] for Hölder \( \partial \Omega \) and by Wannebo [93] for \( \partial \Omega \) under generalized Hölder conditions. Detailed description of these results can be found in Opic and Kufner [80], Maz’ja [76], Hajlasz [50].

Another geometric condition on \( \partial \Omega \) for the validity of (1.2) is suggested in Edmunds and Hurri-Syrjänenn [31]. More precisely, if \( R^\alpha \setminus \Omega \) is \( b \)-plump for some \( b \), then (1.2) holds for \( \alpha \leq 0 \). If additionally \( \Omega \) also satisfies the Whitney cube counting condition then (1.2)
is valid for $\alpha \in (0, \alpha_0)$, where $\alpha_0 > 0$ is given explicitly, see Theorem 3.1 in Edmunds and Hurri-Syrjänen [31] for more details.

Further generalizations were made in Ancona [7], Lewis [71], Wannebo [92], Hajlasz [50], Lehrbäck [68, 69], Koskela and Lehrbäck [61]. They are based on the investigation of the pointwise Hardy inequalities with capacity methods, see the review of Koskela and Lehrbäck [61]. Note that in [50] inequality (1.2) is proved in the domain $\Omega_t = \{ x \in \Omega, d(x) < t \}$ for $u \in C_0^\infty(\Omega)$ and without zero conditions for $u$ on the set $\{ x \in \Omega, d(x) = t \}$. A more generally sufficient condition result is proved in Koskela and Lehrbäck [61].

Another way to describe the properties of $\Omega$ is to connect the validity of inequality (1.2) with the existence of solutions of certain boundary value problems for second order elliptic equations with a singular weight. In Ancona [7] it is proved that a necessary and sufficient condition for the validity of (1.2) when $p = 2$ and $\alpha = 0$ is the existence of a positive super-harmonic function $v$ in $\Omega$ and a positive number $\delta$ such that $\Delta v + \frac{\delta}{d(x)^2} v \leq 0$.

Moreover, in [7] it is shown that $\max \delta = C_{\Omega}$. For more general results for p-Laplace operator see Allegretto and Huang [5].

For arbitrary $p > 1$ and $\alpha = 0$ in Kinnunen and Korte [60], Theorem 5.2, it is proved that (1.2) holds if and only if there exists a positive constant $\lambda_p(\Omega)$ and a positive supersolution $\varphi \in W^{1,p}_0(\Omega)$ of the problem

$$\text{div} (|\nabla \varphi|^{p-2} \nabla \varphi) + \lambda_p |\varphi|^{p-2} \frac{\varphi}{d^p(x)} \leq 0 \quad \text{in } \Omega, \varphi = 0 \text{ on } \partial \Omega.$$ 

The constant $C_{\Omega} = \left( \frac{p-1-\alpha}{p} \right)^p$ is optimal for (1.2) in the case $n = 1$ and for convex domains for $n \geq 2$. For non-convex domains when $n > 1$ the optimal constant $C_{\Omega}$ in (1.2) is unknown. There are only partial results, for example, when $\Omega$ is non-convex and $p = 2$ and $\alpha = 0$, then Ancona [7] proved that $C_{\Omega} \geq \frac{1}{16}$ by means of the Koebe quarter theorem. Later on in Laptev and Sobolev [66] better estimate for $C_{\Omega}$ using a stronger version of Koebe quarter theorem was obtained.

Moreover, in Avkhadiev [10] a sufficient condition on non-convex domain $\Omega$ is given so that (1.2) with $p = 2, \alpha = 0$ holds with optimal constant $C_{\Omega} = \frac{1}{4}$ for $n \geq 2$.

For arbitrary open domain $\Omega$ with $C^2$ smooth boundary $\partial \Omega$ with non-negative mean curvature $H(x)$ inequality (1.2) is proved in Lewis et al. [72], see Theorem 1.2, for $\alpha = 0$ with optimal constant $C_{\Omega} = \left( \frac{p-1}{p} \right)^p$. The curvature condition $H(x) \geq 0$ is optimal because for arbitrary $\varepsilon > 0$ and $H(x) \geq -\varepsilon$ on $\partial \Omega$, the Hardy’s inequality (1.2) with $C_{\Omega} = \left( \frac{p-1}{p} \right)^p$ fails for some $u \in W^{1,p}_0(\Omega)$.

Recently, in Zsuppán [94], Theorem 2.1, an explicit estimate from below for $C_{\Omega}$ in (1.2) for star-shaped domains $\Omega$ was shown. This estimate coincides with $\left( \frac{p-1-\alpha}{p} \right)^p$ for convex domains $\Omega$.

For the annular domain $B_R \setminus B_r = \{ 0 < r < |x| < R < \infty \} \subset \mathbb{R}^n$ the following Hardy
inequality is proved in Avkhadiev and Laptev [9], see Theorem 1 and Corollary 1,
\[
\int_{B_R \setminus B_r} |\nabla u|^2 \, dx \\
\geq \frac{1}{4} \int_{B_R \setminus B_r} \left( \frac{n-2}{2} + \frac{1}{|x-r|^2} + \frac{1}{|x-R|^2} + \frac{2}{|x-R||x-r|} \right) u^2 \, dx,
\]
for every \(u \in H^1_0(B_R \setminus B_r)\). The weights on the right-hand side are singular on the boundary of \(B_R \setminus B_r\) and at the origin, which however does not belong to the domain \(B_R \setminus B_r\).

**Inequalities with weights singular at a point**

Another generalization of (1.1) is an inequality with a weight, singular at an interior point of \(\Omega \subset R^n\), i.e., namely of type (1.3). The optimal constant \(C_{2,n} = \left(\frac{n-2}{2}\right)^2\) is obtained in Leray [70] for \(\Omega = R^3, p = 2\) and in Hardy et al. [54] for \(\Omega = R^n, n > 3, p = 2\), see also Peral and Vazquez [82] and Opic and Kufner [80].

The case \(p = n\) in (1.3) is considered in Ioku and Ishiwata [57], see Theorem 1.1, where Hardy’s inequality
\[
\int_{B_1} \left| \frac{x}{|x|} \nabla u \right|^n \, dx \geq \left( \frac{n-1}{n} \right)^n \int_{B_1} \frac{|u|^n}{|x|^{n(n-1)}} \, dx,
\]
is proved with optimal constant \(\left( \frac{n-1}{n} \right)^n\) for every \(u \in W^{1,n}_0(B_1)\).

For function \(u \in C_0^\infty(\Omega)\), the constant \(C_{p,n}\) in (1.3) is independent on \(\Omega\). However, when \(u \in C^\infty(\Omega)\) the boundary term on \(\partial \Omega\) is taken into account because \(u\) does not necessary vanish on the boundary and the geometry of \(\Omega\) is important. In Wang and Zhu [91], for \(p = 2\), the following Hardy inequality with weights and an additional boundary term was proposed.
\[
\int_{B_1} |x|^{-2\alpha} |\nabla u|^2 \, dx \geq \left( \frac{n-2}{2} - \alpha \right)^2 \int_{B_1} |x|^{-2(\alpha+1)} u^2 \, dx - \frac{n-2\alpha-2}{2} \int_{\partial B_1} u^2 \, dx,
\]
for \(B_1 \subset R^n, n \geq 3, \alpha < \frac{n-2}{2}\).

In Kufner [62] Hardy inequality for functions which are not zero on the boundary (or part on the boundary) is considered. In this case a natural boundary term is added to the right-hand side.

**1.2 Hardy inequalities with an additional term**

When Hardy’s constant in (1.2) is optimal there is no non-trivial function of the admissible class of functions for which the Hardy inequality becomes an equality.

That is why in Brezis and Marcus [25] the question on the existence of an additional positive term \(A(u)\) to the right-hand side of (1.2) is stated such that the improved inequality
\[
\int_{\Omega} |\nabla u(x)|^p d^\alpha(x) \, dx \geq C_{\Omega} \int_{\Omega} |u(x)|^p d^{\alpha-p}(x) \, dx + A(u), \quad (1.10)
\]
still holds in bounded domains for the optimal constant $C_\Omega = \frac{1}{4}$ when $p = 2$, $\alpha = 0$, $n \geq 2$.

For bounded convex domains Brezis and Marcus [25], Theorem II, proved the following Hardy inequality for all $u \in C^\infty_0(\Omega)$

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4} \int_\Omega \frac{u^2}{d^2(x)} \, dx \geq b(\Omega) \int_\Omega u^2 \, dx,$$

where

$$b(\Omega) \geq \frac{1}{4 \text{diam}^2(\Omega)} \text{ and diam}(\Omega) = \max_{x,y \in \Omega} |x - y|.$$

In Hoffmann-Ostenhof et al. [55], Theorem 3.2, for $p = 2$, $n \geq 2$, $C_\Omega = \frac{1}{4}$ and a convex domain $\Omega$, inequality (1.11) is improved by showing that

$$b(\Omega) \geq b_1^{-\frac{n}{2}}, \quad b_1 = \frac{n}{4} \left[ \frac{2\pi^\frac{2}{n}}{n\Gamma(n/2)} \right]^\frac{n}{n},$$

while in Evans and Lewis [33], Theorem 3.2, the estimate (1.12) is improved for bounded convex domain $\Omega \subset \mathbb{R}^n$ to $b(\Omega) \geq \frac{3}{2} b_1 |\Omega|^{-\frac{n}{2}}$.

Later on in Tidblom [87], Theorem 2.2, for $p > 1$, $n \geq 2$ and $\alpha = 0$ an optimal constant $C_\Omega = \left( \frac{p-1}{p} \right)^p$ is shown for inequality (1.10), which holds with $A(u) = b(\Omega) \int_\Omega |u|^p \, dx$ where

$$b(\Omega) = \frac{(p-1)^{p+1}}{pp^n} \left( \frac{\omega_n}{n|\Omega|} \right)^{p/n} \frac{\sqrt{\pi} \Gamma \left( \frac{n+p}{2} \right)}{\Gamma \left( \frac{p+1}{2} \right) \Gamma \left( \frac{n}{2} \right)}.$$

In Filippas et al. [43], Theorem 1.1, the authors proved that in a convex, bounded domain $\Omega \subset \mathbb{R}^n$ inequality (1.11) holds with

$$b(\Omega) \geq 3(D_{\text{int}}(\Omega))^{-2}, \quad D_{\text{int}}(\Omega) = 2 \sup_{x \in \Omega} d(x).$$

Moreover, for $1 < p < n$, $\alpha = 0$, and $C_\Omega = \left( \frac{p-1}{p} \right)^p$, the estimate (1.10) holds with

$$A(u) = b(\Omega) \int_\Omega |u|^p \, dx \quad \text{and} \quad C_1(p,n)D_{\text{int}}^{-p} \geq b(\Omega) \geq C_2(p,n)D_{\text{int}}^{-p},$$

for some positive constants $C_1(p,n), C_2(p,n)$.

A different estimate for $b(\Omega)$ in (1.11) is given in Avkhadiev and Wirths [11], Theorem 1, where

$$b(\Omega) = \frac{b_0^2}{D_{\text{int}}^2},$$

and $b_0 \approx 0.940 \ldots$ is the Lamb constant defined as the first positive zero of the function $J_0(x) + 2xJ'_0(x)$, where $J_0$ is the Bessel function of order 0.

Another additional term for (1.10) is proposed in Brezis and Marcus [25], Theorem 5.1, for convex domains

$$A(u) = \frac{1}{4} \int_\Omega \frac{u^2}{d^2(x) (1 - \log(d(x)/L))^2} \, dx, \quad L = \text{diam} \Omega.$$
For \( p > 1, p \neq n \) and in domain \( \Omega \) satisfying some geometric condition, the result of Brezis and Marcus [25] is generalized in Barbatis et al. [14], Theorem A, to Hardy inequality

\[
\int_{\Omega} |\nabla u|^p \, dx \\
\geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{d^p(x)} \, dx + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \int_{\Omega} \frac{|u|^p}{d^p(x)} \left( \log(d(x)/D) \right)^{-2} \, dx,
\]  

(1.13)

for \( u \in W^{1,p}_0(\Omega) \) and \( D \geq \sup_{x \in \Omega} d(x) \).

In Filippas et al. [43], Theorem 3.1, for a convex domain \( \Omega \subset \mathbb{R}^n \), \( 1 < p < n \), \( \alpha > -p \) and \( C_\Omega = \left( \frac{p-1}{p} \right)^p \) the following Hardy inequality with additional term is proved

\[
\int_{\Omega} |\nabla u|^p \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{d^p(x)} \, dx + C(p, n, \alpha) D^{-\alpha - p} \int_{\Omega} d^\alpha(x) |u|^p \, dx.
\]

The constant \( C(p, n, \alpha) \) is independent of \( \Omega \) and for \( p = 2, \alpha > -2 \) the constant \( C(2, n, \alpha) = C_\alpha \) is given explicitly: \( C_\alpha = 2^\alpha (\alpha + 2)^2 \) when \(-2 < \alpha < -1\), while \( C_\alpha = 2^\alpha (2\alpha + 3)^2 \) when \( \alpha \geq -1 \).

A different generalization of Hoffmann-Ostenhof et al. [55], Tidblom [87], Barbatis et al. [14] is obtained in Edmunds and Hurri-Syrjänen [31], Lewis et al. [72] and in Nasibullin and Tukhvatullina [77].

For example in Lewis et al. [72], Theorem 1.3, for an arbitrary domain \( \Omega \subset \mathbb{R}^{n+1} \) with \( C^2 \) smooth boundary \( \partial \Omega \) with nonnegative mean curvature \( H(x) \geq 0 \) the improved Hardy–Brezis–Marcus inequality

\[
\int_{\Omega} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d^2(x)} \, dx + b(n, \Omega) \int_{\Omega} u^2 \, dx,
\]

is proved where \( b(n, \Omega) \geq \frac{2}{n} \left( \inf_{\partial \Omega} H(x) \right)^2 \).

In Edmunds and Hurri-Syrjänen [31], Theorem 5.1, for the domain \( \Omega \subset \mathbb{R}^n \) which is \( b \)-plump for some \( b \in (0, 1] \), the Hardy inequality

\[
\int_{\Omega} |\nabla u|^p \, dx \geq c(p, n) b^p \left( \int_{\Omega} \frac{|u|^p}{d^p(x)} \, dx + |\Omega|^{-p/n} \int_{\Omega} |u|^p \, dx \right),
\]

is satisfied for every \( u \in W^{1,p}_0(\Omega), 1 < p < \infty \). The constant \( c(p, n) \) is given explicitly.

Analogously to the paper of Brezis and Marcus [25] in Brezis and Vazquez [26] the question about the existence of an additional positive term \( A(u) \) such that the improved inequality

\[
\int_{\Omega} |\nabla u(x)|^2 \, dx \geq \left( \frac{n-2}{2} \right)^2 \int_{\Omega} \frac{u^2(x)}{|x|^2} \, dx + A(u),
\]

(1.14)

with optimal constant still holds for every \( u \in H^1_0(\Omega) \) is addressed. In the same paper, Theorem 4.1, the authors find

\[
A(u) = \lambda(\Omega) \int_{\Omega} u^2 \, dx, \quad \lambda(\Omega) = z_0^2 \left( \frac{\omega_n}{|\Omega|} \right)^{2/n},
\]
where \( z_0 \approx 2.4048 \) is the first zero of the Bessel’s function \( J_0(z) \) whereas \( \omega_n \) and \( |\Omega| \) are the volume of the unit ball and resp. \( \Omega \).

The following generalization of (1.14) for \( 1 < p < n \) is proved in Gazzola et al. [45], Theorem 1,

\[
\int_\Omega |\nabla u|^p dx \geq \left( \frac{n-p}{p} \right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx + C(p,n) \left( \frac{\omega_n}{|\Omega|} \right)^{\frac{n}{n-p}} \int_\Omega |u|^p dx,
\]

where \( \Omega \) is a bounded domain, \( 0 \in \Omega \), the constant \( \left( \frac{n-p}{p} \right)^p \) is optimal and \( C(p,n) \) is given explicitly for \( p \geq 2 \).

In Filippas and Tertikas [41], Theorem D, the additional term in (1.14) is with singular weight, i.e.

\[
\int_\Omega |\nabla u|^2 dx \geq \left( \frac{n-2}{2} \right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^\infty \int_\Omega \frac{u^2}{|x|^2} X_1^2 \left( \frac{|x|}{R} \right) \ldots X_i^2 \left( \frac{|x|}{R} \right) dx,
\]

for every \( u \in H^1_0(\Omega) \). Here \( R \geq \sup_{x \in \Omega} |x| \), \( X_1(t) = (1 - \ln t)^{-1} \), \( X_k(t) = X_1(X_{k-1}(t)) \) for \( k = 2, \ldots, p = 2, n \geq 3 \) and \( \Omega \subset R^n \) is a bounded domain, \( 0 \in \Omega \).

When \( u \in W^{1,p}(\Omega) \) is not zero on \( \partial \Omega \), the result in Filippas and Tertikas [41] is extended in Adimurthi and Esteban [2], Theorem 1.1, to the inequalities

\[
\int_\Omega |\nabla u|^p dx \geq \left( \frac{n-p}{p} \right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx + C(p,n) \int_\Omega \sum_{j=1}^k \left( \log^j(R/|x|) \right) -p \frac{|u|^p}{|x|^p} dx + b(\Omega, p, R) \int_{\partial \Omega} |u|^p ds, \quad \text{for } 1 < p < n,
\]

and

\[
\int_\Omega |\nabla u|^n dx \geq \left( \frac{n-1}{n} \right)^n \int_\Omega \frac{|u|^n}{(|x| \log R/|x|)^n} dx + C(n) \int_\Omega \sum_{j=2}^k \left( \log^j(R/|x|) \right) -n \frac{|u|^n}{|x|^n} dx + b(\Omega, n, R) \int_{\partial \Omega} |u|^n ds, \quad \text{for } p = n.
\]

Here \( \Omega \subset R^n \) is a bounded domain, \( 0 \in \Omega \), \( \log(1) a = \log a \), \( \log(k) a = \log(\log(k-1) a) \) with \( a > e^{(k-1)} \), \( k \geq 2 \), \( \log(k) a = \prod_{j=1}^k \log(j) a \) for \( a > e^{(k-1)} \), \( e^{(1)} = e \), \( e^{(k+1)} = e^{e^k} \) and \( R > e^{(k-1)} \sup_{x \in \Omega} |x| \).

Let us mention the result of Wang and Willem [90], Theorem 2, where the weights in both sides of Hardy inequality (1.14) are singular

\[
\int_\Omega |x|^\alpha |\nabla u|^2 dx \geq \left( \frac{n-2+\alpha}{2} \right)^2 \int_\Omega \frac{u^2}{|x|^{2-\alpha}} dx + \frac{1}{4} \int_\Omega \frac{u^2}{|x|^{2-\alpha}} \ln^{-2} \frac{R}{|x|} dx,
\]

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and \( R > \sup_{x \in \Omega} |x|, \alpha > 2 - n \). The constants \( \left( \frac{n - 2 + \alpha}{2} \right)^2 \) and \( \frac{1}{4} \) are optimal.

Another direction proposed in Vázquez and Zuazua [88], Theorem 2.2, is the improved Hardy–Poincare inequality

\[
\int_{\Omega} |\nabla u|^2 \, dx \geq \left( \frac{n - 2}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx + C(q, \Omega) \left( \int_{\Omega} |u|^q \, dx \right)^{2/q},
\]

for \( 1 \leq q < 2, \Omega \subset R^n \) is a bounded domain, \( 0 \in \Omega \) and \( u \in H^1_k(\Omega) \).

In Filippas and Tertikas [41], Theorem A, and in Adimurthi et al. [4], Theorem A, the following result for the improved Hardy–Sobolev inequality in the unit ball \( B_1 \subset R^n, n \geq 3 \) is obtained

\[
\int_{B_1} |\nabla u|^2 \, dx \geq \left( \frac{n - 2}{2} \right)^2 \int_{B_1} \frac{|u|^2}{|x|^2} \, dx + C_n(a) \left( \int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a, |x|)|u|^\frac{2n}{n-2} \, dx \right)^{\frac{n-2}{n}},
\]

for every \( u \in C^\infty_0(B_1) \), where \( X_1(a, s) = (a - \log s)^{-1}, a > 0, 0 < s < 1. \) Here \( \left( \frac{n - 2}{2} \right)^2 \) and \( C_n(a) \) are optimal constants, where \( C_n(a) \) is given by

\[
C_n(a) = \begin{cases} 
(n - 2)^{\frac{2(n-1)}{n}} S_n, & a \geq \frac{1}{\frac{n}{2} - 2}, \\
 a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{\frac{n}{2} - 2},
\end{cases}
\]

and \( S_n = \pi n(n-2) \left( \Gamma \left( \frac{n}{2} \right) / \Gamma(n) \right)^{\frac{2}{n}} \) is the best constant in the classical Sobolev inequality

\[
\int_{R^n} |\nabla u|^2 \, dx \geq S_n \left( \int_{R^n} |u|^\frac{2n}{n-2} \, dx \right)^{\frac{n-2}{n}}.
\]

In Barbatis et al. [14], Theorem A, the improved Hardy inequality

\[
\int_{\Omega} |\nabla u|^p \, dx \geq \left| \frac{n-p}{p} \right|^p \int_{\Omega} \frac{|u|^p}{|x|^p} \, dx + \frac{p-1}{2p} \left| \frac{n-p}{p} \right|^{p-2} \int_{\Omega} \frac{|u|^p}{|x|^p} X^2 \left( \frac{|x|}{D} \right) \, dx,
\]

is proved in a bounded domain \( \Omega \subset R^n \) with \( 0 \in \Omega \) and \( u \in W_0^{1,p}(\Omega \setminus \{0\}) \).

Here \( X(t) = -1/\log t, t \in (0, 1) \) and \( D \geq D_0 \) where \( D_0(n, p) \geq \sup_{x \in \Omega} |x| \) is some positive constant. The constants in (1.16) are optimal. In fact, (1.13) and (1.16) are a consequence of a more general result in Barbatis et al. [14], Theorem A, when the distance function \( d(x) \) is the distance of \( x \in \Omega \) to a piecewise smooth surface \( K \) of the co-dimension \( k, 1 \leq k \leq n \).

Unfortunately, in all papers mentioned above, inequality (1.14) with optimal constant \( \left( \frac{n - 2}{2} \right)^2 \) or (1.15) with optimal constant \( \left( \frac{n-p}{p} \right)^p \) are not sharp, see Definition 1.1.

In fact, in Pinchover and Tintarev [83] (see also the references therein) it was shown by variational technique that Hardy inequality (1.3) with \( p = 2 \) is not sharp for the optimal constant \( \left( \frac{n - 2}{2} \right)^2 \).
Finally, we will briefly refer to the case of a point singularity of the weights on the boundary $\partial \Omega$, i.e., $0 \in \partial \Omega$, see Filippas and Tertikas [42], Fall and Musina [40], Barbatis et al. [16], Devyver et al. [30] and the references therein. In this case the optimal constant $\left(\frac{n-2}{2}\right)^2$ in (1.14) for the weight singular at an interior point is replaced with greater constant $\frac{n^2}{4}$ when $\Omega$ satisfies some geometric conditions, for example, when $\Omega$ is a convex domain.

2 Hardy inequalities in abstract form

In this section we derive a new Hardy inequalities with weights in abstract form. Examples are presented to demonstrate the applicability of the method and to show generalizations of existing results. The sharpness of the inequalities is proved and the results are illustrated by several examples. The section is based on Fabricant et al. [36, 37].

2.1 General weights

Let $\Omega$ be a bounded domain, $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with a boundary $\partial \Omega \in C^1$. Suppose that $f$ is a vector function defined in $\Omega$, $|f| \neq 0$ with components $f_i \in C^1(\Omega) \cap C(\bar{\Omega})$, $i = 1, \cdots, n$. Let $p > 1$ and assume that in $\Omega$ there exist measurable functions $v$, $w$, $v^1 - p \in L^1(\Omega)$ such that

$$-\text{div} f - (p - 1)v|f|^{p'} \geq w,$$

for a.e. $x \in \Omega$, (2.1)

where $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\partial \Omega$ be divided into two parts $\partial \Omega = \Gamma_- \cup \Gamma_+$, where

$$\Gamma_- = \{x \in \partial \Omega : \langle f, \eta \rangle < 0\}, \quad \Gamma_+ = \{x \in \partial \Omega : \langle f, \eta \rangle \geq 0\}.$$ (2.2)

Here $\eta$ is the unit outward to $\Omega$ normal vector on $\partial \Omega$ and $\langle .., \rangle$ is the scalar product in $\mathbb{R}^n$.

We consider the functions $u \in C^\infty(\Gamma_- \setminus (\bar{\Omega} \cup \Gamma_-))$, where $C^\infty(\Gamma_- \setminus (\bar{\Omega} \cup \Gamma_-)) = \{u \in C^\infty, u = 0 \text{ in a neighbourhood of } \Gamma_-\}$.

Let us introduce the notations

$$L(u) = \int_{\Omega} v^{1-p} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx, \quad K_0(u) = \int_{\Gamma_+} \langle f, \eta \rangle |u|^p dS,$$

$$K(u) = \int_{\Omega} v|f|^{p'} |u|^p dx, \quad N(u) = \int_{\Omega} w|u|^p dx,$$ (2.3)

where $dS$ is the $(n-1)$-dimensional surface measure and $u \in C^\infty(\Gamma_- \setminus (\bar{\Omega} \cup \Gamma_-))$.

In this section our main result is the following theorem.

**Theorem 2.1.** Under condition (2.1) for every $u \in C^\infty(\Gamma_- \setminus (\bar{\Omega} \cup \Gamma_-)), u \neq 0$, and $v > 0$, $w \geq 0$, the following inequality holds

$$L(u) \geq \left(\frac{1}{p}\right)^p \frac{(K_0(u) + (p - 1)K(u) + N(u))^p}{K^{p-1}(u)}.$$ (2.4)
Proof. Since
\[
\int_\Omega \langle f, \nabla|u|^p \rangle dx = p \int_\Omega |u|^{p-2} u \langle f, \nabla u \rangle dx,
\]
applying the H"older inequality on the right-hand side with
\[
v^{-1/p'} \frac{\langle f, \nabla u \rangle}{|f|} \quad \text{and} \quad v^{1/p'} |f| |u|^{p-2} u,
\]
as factor of the integrand we get
\[
\left| \int_\Omega \langle f, \nabla|u|^p \rangle dx \right| \leq p \left( \int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx \right)^{1/p} \left( \int_\Omega v |f|' |u|^p dx \right)^{1/p'}.
\] (2.5)
Rising both sides of (2.5) to power \( p \) it follows that
\[
\int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx \geq \frac{1}{p} \int_\Omega \langle f, \nabla|u|^p \rangle dx \left( \int_\Omega v |f|' |u|^p dx \right)^{p-1}. \] (2.6)
Integrating by parts the numerator of the right-hand side of (2.6), from (2.1), we get
\[
\left| \frac{1}{p} \int_\Omega \langle f, \nabla u \rangle dx \right| = \left| \frac{1}{p} \int_{\partial \Omega} \langle f, \eta \rangle |u|^p dS - \frac{1}{p} \int_\Omega \text{div} |u|^p dx \right|
\]
\[= \left| \frac{1}{p} \int_{\partial \Omega} \langle f, \eta \rangle |u|^p dS - \frac{1}{p} \int_\Omega (\text{div} f + (p - 1)v |f|'|u|^p) dS \right|
\]
\[+ \left( \frac{p-1}{p} \right) \int_\Omega v |f|' |u|^p dx \geq \frac{1}{p} \left| \int_{\partial \Omega} \langle f, \eta \rangle |u|^p dS \right|
\]
\[+ \left| \int_\Omega w |u|^p dx + (p - 1) \int_\Omega v |f|'|u|^p dx \right|
\]
\[\geq \frac{1}{p} \left( \int_{\Gamma_+} \langle f, \eta \rangle |u|^p dS + \int_\Omega w |u|^p dx + (p - 1) \int_\Omega v |f|'|u|^p dx \right). \]
The last equality follows from \( u |_{\Gamma_-} = 0 \).
From (2.6) we obtain (2.4).

Remark 2.1. The idea of the proof of Theorem 2.1 comes from Boggio [24], for \( p = 2 \); Flekinger et al. [44], Theorem II.1 and Barbatis et al. [14], Theorem 4.1. In our case, in contrast to these works, we consider functions not necessarily zero on the whole boundary \( \partial \Omega \) and due to this there is an additional boundary term \( K_0 \) in (2.4). In \( L \) and \( K \) there is also a weight \( v \), which is 1 in the above mentioned papers.

The careful analysis of the proof of Theorem 2.1 shows that we have a more general result than (2.4) without any sign conditions of the boundary term.
**Corollary 2.1.** Suppose that $p > 1$ and there exist in $\Omega$ measurable functions $v > 0$, $w, v^{1-p} \in L^1(\Omega)$ such that condition (2.1) holds. Then for every $u \in C^\infty(\overline{\Omega})$ the following inequality holds

$$L(u) \geq \left(\frac{1}{p}\right)^p \frac{|K_3(u) + (p - 1)K(u) + N(u)|^p}{K^{p-1}(u)},$$

(2.7)

where $K_3(u) = \int_{\partial\Omega} \langle f, \eta \rangle |u|^p dS$.

**Corollary 2.2.** As a consequence of Theorem 2.1 we get under condition (2.1) for $v > 0$, $w \geq 0, v^{1-p} \in L^1(\Omega)$ and $u \in C^\infty(\Gamma^-)$ the following Hardy inequalities:

i) $$L^{\frac{1}{p}}(u) \geq \left(\frac{1}{p}\right)^{\frac{1}{p}} K^{\frac{1}{p}}(u) + \left(\frac{1}{p}\right) K_0(u) K^{-\frac{1}{p}}(u),$$

(2.8)

ii) $$L(u) \geq K_0(u) + N(u) \geq N(u),$$

(2.9)

iii) $$L(u) \geq \left(\frac{1}{p}\right)^p K(u) + \left(\frac{1}{p}\right)^{p-1} (K_0(u) + N(u)) \geq \left(\frac{1}{p}\right)^p K(u),$$

(2.10)

for every $u \in C^\infty(\Gamma^-)$.

**Proof.** i) Rising both sides of (2.4) to power $\frac{1}{p}$ and neglecting $N(u) \geq 0$ we obtain (2.8).

ii) Applying the Young inequality

$$\frac{Q^p}{H^{p-1}} \geq p s^{p-1} Q - (p - 1) s^p H,$$

with $H > 0, Q \geq 0$ and constant $s \geq 0$ to the right-hand side of (2.4) for

$$Q = \frac{1}{p} (K_0(u) + N(u) + (p - 1)K(u)) \text{ and } H = K(u)$$

we get

$$L(u) \geq s^{p-1}(K_0(u) + N(u)) + (p - 1)s^{p-1}(1 - s) K(u).$$

(2.11)

For $s = 1$ in (2.11) we get (2.9) and neglecting $K_0(u)$ since $K_0(u) \geq 0$ we obtain the last inequality in (2.9).

iii) Inequality (2.10) is a consequence of (2.11) for $s = \frac{1}{p'} = \frac{p - 1}{p}$ and neglecting $K_0(u) \geq 0$ and $N(u) \geq 0$ we obtain the last inequality in (2.10).

The form of Hardy inequality (2.8) is not the usual one. It depends on the derivative of $u$ in the direction of the unit vector $\frac{f}{|f|}$, on two functions $v, w$ satisfying (2.1) and on additional term including boundary integral.

Since $\langle f, \eta \rangle \geq 0$ on $\Gamma_+$ and $||\nabla u||^p \geq \left|\frac{\langle f, \nabla u \rangle}{|f|}\right|^p$, in (2.8)–(2.10) we can replace their left-hand sides correspondingly with

$$\int_{\Omega} v^{1-p} ||\nabla u||^p dx \quad \text{and} \quad \left(\int_{\Omega} v^{1-p} ||\nabla u||^p dx\right)^{1/p}.$$
The careful analysis of the proof of Theorem 2.1 shows that (2.4) is an equality if and only if
\[ \left| \int_{\Omega} \langle f, \nabla |u|^p \rangle dx \right| = \int_{\Omega} \langle f, \nabla |u|^p \rangle dx, \] (2.12)
Hölder inequality becomes an equality, i.e.,
\[ \left| v \frac{1}{p^*} \frac{\langle f, \nabla u \rangle}{|f|} \right|^p = k_1^p \left| v \frac{1}{p} |f||u|^{p-2}u \right|^p, \] (2.13)
for a.e. \( x \in \Omega \) and some constant \( k_1 > 0 \) and
\[ -\text{div}f - (p - 1)v|f|^p = w, \quad \text{in } \Omega. \] (2.14)
However, (2.12) and (2.13) are satisfied if
\[ u\langle f, \nabla u \rangle = |u\langle f, \nabla u \rangle|, \] (2.15)
\[ \langle f, \nabla u \rangle = k_1 v|f|^p u, \quad k_1 > 0, \] (2.16)
for a.e. \( x \in \Omega \). Thus we get the following result for sharpness of Hardy inequality (2.4).

**Theorem 2.2.** Suppose \( p > 1, n \geq 2 \), \( \Omega \) is a bounded domain with \( C^1 \) smooth boundary \( \partial \Omega \), \( v > 0 \), \( v \frac{1}{p^*} - \frac{p-2}{p} \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_i} \) \( \in L^1(\Omega) \) and \( w \geq 0 \) for a.e. \( x \in \Omega \). Then Hardy inequality (2.4) becomes a non-trivial equality if \( f, v, w, u \) satisfy (2.14)–(2.16) and \( u \in C^\infty_0(\Omega), u \not\equiv 0 \).

Let us note that the possibility to use a vector function \( f \) and two functions \( v \) and \( w \) in inequalities (2.8)–(2.10) serves for many new Hardy inequalities.

### 2.2 Comparison with some existing results

We will compare the result in Theorem 2.1 with results in Opic and Kufner [80] and Balinsky et al. [12].

**Example 2.1.** Let \( y(x) \) be a solution of the equation (1.6) with properties listed in Sect. 1.1, i.e., \( v_i, w \) are positive measurable functions, finite a.e. in \( \Omega \), so that inequality (1.5) holds, see Opic and Kufner [80], Theorems 14.1 and 14.2. Under these conditions we will prove new Hardy inequalities by means of Theorem 2.1.

Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 2 \). For \( 1 < p < n \) we define vector function \( f = (f_1, \ldots, f_n) \) with
\[ f_i = v_i \left| \frac{\partial y}{\partial x_i} \right|^{p-2} \frac{\partial y}{\partial x_i} (|y|^{p-2} y)^{-1}, \quad i = 1, \ldots, n, \]
and
\[ v(x) = \inf_{|\xi|=1} \left( \sum_{i=1}^n v_i \sum_{i=1}^n v_i \left| \xi_i \right|^{2(p-1)} \right)^{-\frac{p}{2(p-1)}}. \]
For every \( u(x) \in C_0^\infty(\Omega) \) the following Hardy inequalities hold

\[
\left( \int_\Omega v^{1-p} \left| \frac{\langle \nabla y, \nabla u \rangle}{|y|} \right|^p \, dx \right)^{\frac{1}{p}} \geq \frac{p-1}{p} \left[ \int_\Omega \frac{v}{|y|^p} \left( \sum_{i=1}^n v_i^2 \left| \frac{\partial y}{\partial x_i} \right|^{2(p-1)} \right)^{\frac{p}{2(p-1)}} |u|^p \, dx \right]^{\frac{1}{p}} + \frac{1}{p} \int_\Omega w|u|^p \, dx
\]

(2.17)

and

\[
\int_\Omega v^{1-p} \left| \frac{\langle \nabla y, \nabla u \rangle}{|y|} \right|^p \, dx \geq \int_\Omega w|u|^p \, dx.
\]

(2.18)

The proof of (2.17) and (2.18) follows from (2.4) and (2.9). Indeed, the vector function \( f(x) \) satisfies inequality (2.1), i.e.,

\[
- \text{div} \, f - (p-1)v|f|^p \geq w + (p-1) \sum_{i=1}^n \frac{v_i}{|y|^p} \left| \frac{\partial y}{\partial x_i} \right|^p
\]

\[
- (p-1) \frac{v}{|y|^p} \left( \sum_{i=1}^n v_i^2 \left| \frac{\partial y}{\partial x_i} \right|^{2(p-1)} \right)^{\frac{p}{2(p-1)}} \geq w(x) \quad \text{in } \Omega,
\]

because

\[
\sum_{i=1}^n v_i \left| \frac{\partial y}{\partial x_i} \right|^p \left( \sum_{i=1}^n v_i^2 \left| \frac{\partial y}{\partial x_i} \right|^{2(p-1)} \right)^{\frac{p}{2(p-1)}} \geq \inf_{|\xi|=1} \sum_{i=1}^n v_i |\xi|^p \left( \sum_{i=1}^n v_i^2 |\xi|^{2(p-1)} \right)^{\frac{p}{2(p-1)}} = v(x).
\]

Since

\[
|f|^p = \left( \sum_{i=1}^n v_i^2 \left| \frac{\partial y}{\partial x_i} \right|^{2(p-1)} \right)^{\frac{p}{2(p-1)}}
\]

we get

\[
L(u) = \int_\Omega v^{1-p} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p \, dx, \quad K_0(u) = 0,
\]

\[
K(u) = \int_\Omega \frac{v}{|y|^p} \left( \sum_{i=1}^n v_i^2 \left| \frac{\partial y}{\partial x_i} \right|^{2(p-1)} \right)^{\frac{p}{2(p-1)}} |u|^p \, dx, \quad N(u) = \int_\Omega w|u|^p \, dx,
\]

because \( u = 0 \) on \( \partial \Omega \). Applying (2.4) and (2.3) we obtain (2.17) and (2.18).
Example 2.2. Let \( z \) be a real-valued function \( z \in W^{2,1}_{loc}(\Omega) \) in a bounded domain \( \Omega \subset \mathbb{R}^n \), \( \partial \Omega \in C^1 \), \( n \geq 2 \) and \( \Delta z \) is of one sign a.e. in \( \Omega \). Then the following inequalities hold for \( p > 1 \), see Davies and Hinz [29] and Balinsky et al. [12], Theorem 1.2.8,

\[
\int_{\Omega} \frac{1}{|\Delta z|^{p-1}} |(\nabla z, \nabla u)|^p dx \geq \left( \frac{1}{p} \right)^p \int_{\Omega} |\Delta z||u|^p dx \tag{2.19}
\]

for every \( u \in C^\infty_0(\Omega) \).

If additional \( z \in C^1(\Omega) \cap W^{2,1}(\Omega) \) then

\[
\left( \int_{\Omega} \frac{1}{|\Delta z|^{p-1}} |(\nabla z, \nabla u)|^p dx \right)^{\frac{1}{p}} \geq \frac{1}{p} \left( \int_{\Omega} |\Delta z||u|^p dx \right)^{\frac{1}{p}}
\]

\[
- \frac{1}{p} \text{sgn} \Delta z \int_{\Gamma_+} \langle \nabla z, \eta \rangle |u|^p dS \left( \int_{\Omega} |\Delta z||u|^p dx \right)^{\frac{p-1}{p}},
\]

for every \( u \in C^\infty_0(\Omega) \). Here \( \eta \) is the unit outward to \( \Omega \) normal vector on \( \partial \Omega \), where \( \Gamma_-, \Gamma_+ \) are defined in (2.2).

Inequalities (2.19) and (2.20) follow from (2.8) for \( f = - \left( \frac{1}{p-1} \right)^{p-1} (\text{sgn} \Delta z) \nabla z, \)
\( v = |\Delta z||\nabla z|^{-p} \) and \( w = 0 \). Indeed, we get

\[
-\text{div} f - (p-1)v f^{p'} = \left( \frac{1}{p-1} \right)^{p-1} |\Delta z| - \left( \frac{1}{p-1} \right)^{p-1} |\Delta z| = 0,
\]

for a.e. in \( \Omega \).

With the computations

\[
L(u) = \int_{\Omega} v^{1-p} \left| \frac{f}{|f|} \right|^p dx = \int_{\Omega} \frac{1}{|\Delta z|^{p-1}} |(\nabla z, \nabla u)|^p dx,
\]

\[
K_0(u) = - \int_{\Gamma_+} \langle f, \eta \rangle |u|^p dS = \left( \frac{1}{p-1} \right)^{p-1} (\text{sgn} \Delta z) \int_{\Gamma_+} \langle \nabla z, \eta \rangle |u|^p dS,
\]

\[
K(u) = \int_{\Omega} v |f|^{p'} |u|^p dx = \left( \frac{1}{p-1} \right)^p \int_{\Omega} |\Delta z||u|^p dx,
\]

\[
N(u) = 0.
\]

applying (2.8) and (2.10) we obtain (2.19) and (2.20), respectively.

2.3 Sharp Hardy inequalities

We illustrate below the possibility to choose a vector function \( f \) in order to obtain sharp Hardy inequality.
Inequality with weight the first eigenfunction of the p-Laplacian

Let $\varphi$ be the first eigenfunction of the p-Laplacian in a bounded domain $\Omega \subset \mathbb{R}^n$, $p > 1$, $n \geq 2$ with the first eigenvalue $\lambda > 0$

$$-\Delta_p \varphi = \lambda |\varphi|^{p-2}\varphi, \quad \text{in } \Omega,$$

$$\varphi|_{\partial \Omega} = 0.$$

Let us define the vector function $f = \frac{\nabla \varphi}{|\varphi|^{p-2}\varphi}$ in every domain $\Omega_0$, $\Omega_0 \subset \Omega$ such that for $f = f_1 \ldots f_n$ we have $f_i \in C^1(\Omega_0)$ and

$$-\text{div}\, f = -\frac{\Delta_p \varphi}{|\varphi|^{p-2}\varphi} + (p-1)\frac{|\nabla \varphi|^p}{|\varphi|^p} = \lambda + (p-1)|f|^p \quad x \in \Omega_0,$$

i.e., $v = 1$ and $\omega = \lambda$ in (2.1).

If we fix $u \in C^0_0(\Omega)$, then supp $u \subset \Omega$ so that we can apply Theorem 2.1 and obtain the inequality

$$L(u) \geq \left(\frac{1}{p}\right)^p \frac{[(p-1)K(u) + N(u)]^p}{K^{p-1}(u)}, \quad (2.21)$$

where

$$L(u) = \int_\Omega \left|\frac{\nabla \varphi}{|\varphi|^p}\nabla u\right|^p dx, \quad K(u) = \int_\Omega \frac{|\nabla \varphi|^p}{|\varphi|^p} |u|^p dx, \quad N(u) = \lambda \int_\Omega |u|^p dx.$$

Note that under arguments of completeness the inequality (2.21) holds for every $u \in W^{1,p}_0(\Omega)$, moreover simple computation gives us that inequality (2.21) is sharp, i.e., becomes an equality for $u(x) = \varphi(x)$.

We consider the case $p = 2$, $n = 3$ and $\Omega = B_1$. In this case the first eigenfunction is $\varphi = \sqrt{2} \sin \pi r$, $r = |x|$, and the eigenvalue is $\lambda = \pi^2$, see Vladimirov [89], Sect. 28.1.

Now we have

$$f = \frac{\nabla \varphi}{|\varphi|^p} = \left(\pi \cot \pi r - 1\right)\frac{x}{r}, \quad |f|^2 = \left(\pi \cot \pi r - 1\right)^2 \quad \text{and} \quad -\text{div}\, f = |f|^2 + \pi^2.$$

Applying Theorem 2.1 to

$$L(u) = \int_{B_1} |\nabla u|^2 dx, \quad K(u) = \int_{B_1} |f|^2 u^2 dx \quad \text{and} \quad N(u) = \pi^2 \int_{B_1} u^2 dx$$

we get

$$L(u) \geq \frac{1}{4} \left(K(u) + 2N(u) + N^2(u)K^{-1}(u)\right) \quad \text{for } u \in C^0(\Omega). \quad \text{(2.22)}$$

Using the series expansion for the function $\cot(z)$, see Remmert [84], we obtain

$$\left(\pi \cot \pi r - 1\right)^2 = \left(\sum_{k=1}^\infty \frac{2r}{r^2 - k^2}\right)^2$$

$$= \frac{4r^2}{(r^2 - 1)^2} \left[1 + \sum_{k=2}^\infty \frac{r^2 - 1}{r^2 - k^2}\right] = \frac{r^2}{(r - 1)^2} \left[\frac{2}{r + 1} + 2 \sum_{k=2}^\infty \frac{r - 1}{r^2 - k^2}\right], \quad \text{(2.23)}$$

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for the kernel of \( K(u) \) and \( r \in (0,1) \)

Using (2.23) for (2.22), the following Hardy inequality holds

\[
\int_{B_1} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{(1 - |x|)^2} \left( \frac{2}{|x| + 1} + 2 \sum_{k=2}^{\infty} \frac{|x| - 1}{|x|^2 - k^2} \right)^2 \, dx + 2\pi^2 \int_{B_1} u^2 \, dx + \frac{\pi^4 \left( \int_{B_1} u^2 \, dx \right)^2}{\int_{B_1} \frac{u^2}{(1 - |x|)^2} \left( \frac{2}{|x| + 1} + 2 \sum_{k=2}^{\infty} \frac{|x| - 1}{|x|^2 - k^2} \right)^2 \, dx}.
\]

(2.24)

Since the last term in (2.24) is positive, \( \frac{2}{|x| + 1} \geq 1 \) and \( 1 - |x| = d(x) = \text{dist}(x, \partial B_1) \), we can rewrite (2.24) as

\[
\int_{B_1} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{B_1} \frac{u^2}{d(x)^2} \, dx + A(u) \quad u \in W^{1,p}_0(B_1),
\]

(2.25)

where

\[
A(u) = \frac{1}{4} \int_{B_1} \left[ \left( \frac{1 - |x|}{1 + |x|} + 2 \sum_{k=2}^{\infty} \frac{|x| - 1}{|x|^2 - k^2} \right)^2 - 1 \right] u^2 \, dx
\]

\[
+ \frac{1}{2} \pi^2 \int_{B_1} u^2 \, dx + \frac{\pi^4 \left( \int_{B_1} u^2 \, dx \right)^2}{\int_{B_1} \frac{u^2}{(1 - |x|)^2} \left( \frac{2}{|x| + 1} + 2 \sum_{k=2}^{\infty} \frac{|x| - 1}{|x|^2 - k^2} \right)^2 \, dx} > 0.
\]

Inequality (2.25) has an optimal constant \( \frac{1}{4} \) and moreover it is sharp, i.e., for function \( u(x) = \sqrt{2} \sin \frac{\pi}{|x|} \) it becomes an equality. The above example shows that sharp inequality (1.11) with optimal constant \( \frac{1}{4} \) see Definition 1.1, is possible but for more complicated additional term \( A(u) \). Thus, in this special case we give a positive answer to the question of Brezis and Marcus [25].

### Hardy inequalities in an annulus and in a ball

Let us define for \( p > 1 \), \( p' = \frac{p}{p-1} \), \( n \geq 2 \), \( m = \frac{p-n}{p-1} \) and \( 0 \leq r < R \) the sets of functions:

\[
M(r, R) = \begin{cases} 
\{ u : \int_{B_R \setminus B_r} \frac{|\langle x, \nabla u \rangle|^p}{|x|} \, dx < \infty, \ 0 \leq r < R, \\
\quad \quad \text{and} \ |R^m - \hat{R}^m|^{1-p} \int_{\partial B_{\hat{R}}} |u|^p dS \to 0, \ \hat{R} \to R - 0, \ m \neq 0, \\
\quad \quad \ln \frac{R}{\hat{R}} \int_{\partial B_{\hat{R}}} |u|^p dS \to 0, \ \hat{R} \to R - 0, \ m = 0. \end{cases}
\]

(2.26)
Proposition 2.1. For $u \in M(r, R)$ and $0 < r < R$ the following Hardy inequalities hold:

- for $m \neq 0$, i.e., $p \neq n$
  \[
  \left( \int_{B_R \setminus B_r} \frac{\langle x, \nabla u \rangle^p}{|x|^p} \, dx \right)^{1/p} \geq \frac{n-p}{p} \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} |R^m - |x|^m|^p} \, dx \right)^{1/p} 
  \]
  \[+ \frac{1}{p} r^{1-n} |R^m - r^m|^{1-p} \int_{\partial B_r} |u|^p dS \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} |R^m - |x|^m|^p} \, dx \right)^{-\frac{1}{p}}. \tag{2.27} \]

For function $u_k(x) = \left( \frac{R^m - |x|^m}{m} \right)^k$, $k > \frac{1}{p}$ inequality (2.27) becomes an equality;

- for $m = 0$, i.e., $p = n$
  \[
  \left( \int_{B_R \setminus B_r} \frac{\langle x, \nabla u \rangle^n}{|x|^n} \, dx \right)^{1/n} \geq \frac{n-1}{n} \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x|^n \ln \frac{R}{|x|}} \, dx \right)^{1/n} 
  \]
  \[+ \frac{1}{n} \left( \frac{r \ln R}{r} \right)^{1-n} \int_{\partial B_r} |u|^n dS \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x|^n \ln \frac{R}{|x|}} \, dx \right)^{-\frac{1}{n}}. \tag{2.28} \]

For function $u_s(x) = \left( \ln \frac{R}{|x|} \right)^s$, $s > \frac{1}{n'}$ inequality (2.28) becomes an equality.

Proof. Let the function $\psi(x)$ be a solution of the problem:

\[ \begin{cases} 
  \Delta_p \psi = 0, & \text{in } B_R \setminus B_r, \\
  \psi |_{\partial B_R} = 0, & \psi |_{\partial B_r} = 1,
\end{cases} \]

then

\[ \psi(x) = \begin{cases} 
  \frac{R^m - |x|^m}{R^m - r^m}, & m \neq 0, \\
  \ln \frac{r}{\ln \frac{R}{r}}, & m = 0.
\end{cases} \tag{2.29} \]

Indeed
for \( m \neq 0 \) we have
\[
\nabla \psi = -m|x|^{m-1} \frac{x}{|x|} \frac{1}{R^m - |x|^m},
\]
\[
|\nabla \psi|^{p-2} \nabla \psi = -m|p-2|m|x|^{(m-1)(p-1)} \frac{x}{|x|} \left( \frac{1}{R^m - |x|^m} \right)^{p-1},
\]
\[
\Delta_p \psi = \text{div} \left( |\nabla \psi|^{p-2} \nabla \psi \right)
\]
\[
= -m|p-2|m|x|^{(m-1)(p-1)-1} \left( \frac{1}{R^m - |x|^m} \right)^{p-1} [(m-1)(p-1) + n - 1] = 0,
\]

because \((m-1)(p-1) + n - 1 = \left( \frac{p-n}{p-1} - 1 \right)(p-1) + n - 1 = 0.\)

for \( m = 0 \) we have
\[
\nabla \psi = - \frac{1}{|x|} \int \frac{1}{|x|} \frac{x}{\ln R \ |x|} \ |\nabla \psi|^{n-2} \nabla \psi = - \frac{1}{|x|} (\ln R)^{n-1} \ |x|,
\]
\[
\Delta_n \psi = \text{div} \left( |\nabla \psi|^{n-2} \nabla \psi \right) = -|x|^{-n} (\ln R)^{n-1} [-(n-1) + n - 1] = 0.
\]

Using the function \( \psi \) in (2.29) we define the vector function \( f(x) \) in \( B_R \setminus B_r \) as \( f = \frac{|\nabla \psi|^{p-2} \nabla \psi}{|\psi|^{p-2}} \) and let us check that
\[
f(x) = \begin{cases} 
-|x|^{-n}x \left( \frac{R^m - |x|^m}{m} \right)^{1-p}, & m \neq 0, \\
-|x|^{-n}x \left( \ln \frac{R}{|x|} \right)^{1-n}, & m = 0.
\end{cases}
\]

Indeed

• for \( m \neq 0 \) we have
\[
f = -|m|^{p-2}m|x|^{(m-1)(p-1)-1}x (R^m - |x|^m)^{1-1} \left( R^m - |x|^m \right)^{2-p}
\]
\[
= -|x|^{-n}x \left( \frac{R^m - |x|^m}{m} \right)^{1-p},
\]

because \((m-1)(p-1) - 1 = \left( \frac{p-n}{p-1} - 1 \right)(p-1) - 1 = -n;\)

• for \( m = 0 \) we have
\[
f = -|x|^{-(n-1)} \left( \ln \frac{R}{|x|} \right)^{1-n} \frac{x}{|x|} = -|x|^{-n}x \left( \ln \frac{R}{|x|} \right)^{1-n}.
\]
Note that the outward normal $\eta$ to $B_\hat{R} \setminus B_r$, $r < \hat{R} < R$ is defined as

$$\eta|_{\partial B_\hat{R}} = \frac{x}{|x|}|_{\partial B_\hat{R}}, \quad \eta|_{\partial B_r} = -\frac{x}{|x|}|_{\partial B_r}. $$

Moreover, we get for $u \in M(r,R)$ and

- for $m \neq 0$

$$|f|^{p'} = |x|^{(1-n)p'} \left( \frac{R^n - |x|^m}{m} \right)^{-p}, \quad \int_{\partial(B_\hat{R} \setminus B_r)} (f, \eta)|u|^p dS \geq 0,$$

and

$$-\text{div} f = -\text{div} \left( \left| \frac{\nabla \psi}{\psi} \right|^{p-2} \frac{\nabla \psi}{\psi} \right)$$

$$= -\frac{\Delta \psi}{\psi^{p-1}} + |x|^{-n} \left< x, \nabla \left( \frac{R^n - |x|^m}{m} \right)^{1-p} \right>$$

$$= (p-1)|x|^{-n} \left( \frac{R^n - |x|^m}{m} \right)^{-p} |x|^m$$

$$= (p-1)|x|^{m-n} \left( \frac{R^n - |x|^m}{m} \right)^{-p} = (p-1)|f|^p;$$

- for $m = 0$

$$|f|^n = |x|^{-n} \left( \ln \frac{R}{|x|} \right)^{-n}, \quad \int_{\partial(B_\hat{R} \setminus B_r)} (f, \eta)|u|^n dS \geq 0,$$

and

$$-\text{div} f = -\text{div} \left( \left| \frac{\nabla \psi}{\psi} \right|^{n-2} \frac{\nabla \psi}{\psi} \right)$$

$$= -\frac{\Delta \psi}{\psi^{n-1}} + |x|^{-n} \left< x, \nabla \left( \ln \frac{R}{|x|} \right)^{1-n} \right>$$

$$= -(1-n)|x|^{-n} \left| \ln \frac{R}{|x|} \right|^{-n} \left< x, \frac{x}{|x|^2} \right>$$

$$= (n-1)|x|^{-n} \left( \ln \frac{R}{|x|} \right)^{-n} = (n-1)|f|^n.$$

Since the vector function $f(x)$ satisfies (2.1) with $v = 1, w \equiv 0$ then using (2.30), (2.31) and applying Theorem 2.1 in $B_\hat{R} \setminus B_r$ we obtain inequalities (2.27), (2.28) after the limit $\hat{R} \to R$.

We will prove the sharpness of (2.27) only for $m > 0$ because in the case $m < 0$ the proof is similar and we omit it.
First, let us evaluate for $k > \frac{1}{p}$ the integral

$$I_m = \int_{B_R \setminus B_r} \frac{dx}{|x|^{(n-1)p'} (R^m - |x|^m)^{p(1-k)}}.$$  

With a change of variables $y = \frac{x}{R}$ and $\rho = |y|$ we get

$$I_m = R^{m(1-p+kp)} \int_{B_R \setminus B_r} \frac{dy}{|y|^{(n-1)p'} (1 - |y|^m)^{p(1-k)}} = R^{m(1-p+kp)} \omega_n \int_{r/R}^{1} \frac{\rho^{m-1} d\rho}{(1 - \rho^m)^{p(1-k)}}$$

$$= \omega_n m^{-1} (R^m - r^m)^{1-p+kp} (1 - p + kp)^{-1},$$

where we use $(n-1)p' = n - m$ and $1 - p + kp > 0$ because $k > \frac{1}{p'} = \frac{p - 1}{p}$.

With $u_k(x) = \left(\frac{R^m - |x|^m}{m}\right)^{k}$ for the left-hand side of (2.27) we get

$$(lhs) = \left(\int_{B_R \setminus B_r} \left|\frac{\langle x, \nabla u_k\rangle}{|x|}\right|^p dx\right)^{\frac{1}{p}}$$

$$= k \left(\int_{B_R \setminus B_r} |x|^{(m-1)p} \left(\frac{R^m - |x|^m}{m}\right)^{(k-1)p} dx\right)^{\frac{1}{p}}$$

$$= km^{1-k} \left(\int_{B_R \setminus B_r} \frac{dx}{|x|^{(n-1)p'} (R^m - |x|^m)^{p(1-k)}}\right)^{\frac{1}{p}} = km^{1-k} I_m^{\frac{1}{p}}$$

$$= k \omega_n m^{-\frac{1-p+kp}{p}} (R^m - r^m)^{\frac{1-p+kp}{p}} (1 - p + kp)^{-\frac{1}{p}},$$

where we use $(m-1)p = \left(\frac{p-n}{p-1}\right) p = -(n-1)p'$.

For the terms in the right-hand side of (2.27) using the expression (2.32) for $I_m$ we get

$$(rhs)_1 = \frac{p - n}{p} \left(\int_{B_R \setminus B_r} \frac{|u_k|^p dx}{|x|^{(n-1)p'} (R^m - |x|^m)^p}\right)^{\frac{1}{p}}$$

$$= \frac{p - n}{p} m^{-k} \left(\int_{B_R \setminus B_r} \frac{dx}{|x|^{(n-1)p'} (R^m - |x|^m)^{p(1-k)}}\right)^{\frac{1}{p}}$$

$$= \frac{p - n}{p} m^{-1} m^{-k+1} I_m^{\frac{1}{p}} = \frac{p - 1}{p} m^{-k+1} I_m^{\frac{1}{p}}$$

$$= \frac{p - 1}{p} \omega_n m^{-\frac{1-p+kp}{p}} (R^m - r^m)^{\frac{1-p+kp}{p}} (1 - p + kp)^{-\frac{1}{p}}.$$
\[(rhs)_2 = \frac{1}{p} r^{1-n} (R^m - r^m)^{1-p} \int_{\partial B_r} |u_k|^p dS \]
\[\times \left( \int_{B_R \setminus B_r} \frac{|u_k|^p dx}{|x|^{(n-1)p} (R^m - |x|^m)^p} \right)^{-\frac{1}{p'}} \]
\[= \frac{1}{p} \omega_m r^{1-n} \left. \frac{1}{p'} \right| u_k \right|^{1/2} \frac{-p}{(R^m - r^m)^{1-p}} \int_{\partial B_r} \frac{|u_k|^p}{|x|^{(n-1)p}} dS = \frac{1}{p} \omega_m r^{1-n} \left. \frac{1}{p'} \right| u_k \right|^{1/2} \frac{-p}{(R^m - r^m)^{1-p}} \int_{\partial B_r} \frac{|u_k|^p}{|x|^{(n-1)p}} dS \]
\[= \frac{1}{p} \omega_m r^{1-n} \left. \frac{1}{p'} \right| u_k \right|^{1/2} \frac{-p}{(R^m - r^m)^{1-p}} \int_{\partial B_r} \frac{|u_k|^p}{|x|^{(n-1)p}} dS \]
\[(2.34)\]

Adding (2.33) and (2.34) we obtain
\[(rhs)_1 + (rhs)_2 = \frac{1}{p} \omega_m r^{1-n} \left. \frac{1}{p'} \right| u_k \right|^{1/2} \frac{-p}{(R^m - r^m)^{1-p}} \int_{\partial B_r} \frac{|u_k|^p}{|x|^{(n-1)p}} dS \]
\[= \left( \int_{B_R \setminus B_r} \frac{|x, \nabla u|^p}{|x|^n} dx \right)^{1/2} = (lhs) \]

which proves the sharpness of (2.27) for \( m > 0 \).

Analogously, for the sharpness of (2.28), first we evaluate the integral
\[I_0 = \int_{B_R \setminus B_r} |y|^{-n} \left( \ln \frac{1}{|y|} \right)^{n(s-1)} dy = \omega_n \int_{r/R}^1 \left( \ln \frac{1}{\rho} \right)^{n(s-1)} \rho^{-1} d\rho \]
\[= \omega_n \left( \ln \frac{R}{r} \right)^{1-n+ns} (1 - n + ns)^{-1} \]

where we use \( 1 - n + ns > 0 \) because \( s > \frac{1}{n'} \).

With \( u_s(x) = \left( \ln \frac{R}{|x|} \right)^s \) for the left-hand side of (2.28) we get
\[(lhs) = \left( \int_{B_R \setminus B_r} \frac{|x|^{n} \left( \nabla u_s \right)^{|n|}}{|x|^{n}} \right)^{1/2} = s \left( \int_{B_R \setminus B_r} |x|^{-n} \left( \ln \frac{R}{|x|} \right)^{(s-1)n} dx \right)^{1/2} \]
\[= s I_0^{1/2} = s \omega_n \left( \ln \frac{R}{r} \right)^{1-n+sn} (1 - n + sn)^{-1/2} \]

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For the terms in the right-hand side we get

\[
\text{(rhs)}_1 = \frac{n-1}{n} \left( \frac{1}{\int_{B_R \setminus B_r} |u_s|^n dx} \right)^{\frac{1}{n}}
\]

\[
= \frac{n-1}{n} \left( \int_{B_R \setminus B_r} |x|^{-n} \left( \ln \frac{R}{|x|} \right)^{n(s-1)} dx \right)^{\frac{1}{n}}
\]

\[
= \frac{n-1}{n} \omega^{\frac{1}{n}} = \frac{n-1}{n} \omega^{\frac{1}{n}} (1-n + sn)^{-\frac{1}{n}} \left( \ln \frac{R}{r} \right)^{-\frac{1-n+sn}{n}},
\]

and

\[
\text{(rhs)}_2 = \frac{1}{n} \left( \ln \frac{R}{r} \right)^{1-n} \int_{\partial B_r} |u_s|^n dS \left( \int_{B_R \setminus B_r} \frac{|u_s|^n dx}{|x|^n \left( \ln \frac{R}{|x|} \right)^{n(s-1)}} \right)^{-\frac{1}{n}}
\]

\[
= \frac{1}{n} \omega^{\frac{1}{n}} \left( \ln \frac{R}{r} \right)^{1-n+sn} \left( \int_{B_R \setminus B_r} \frac{dx}{|x|^n \left( \ln \frac{R}{|x|} \right)^{n(s-1)}} \right)^{-\frac{1}{n}}
\]

\[
\text{(rhs)}_2 = \frac{1}{n} \omega^{\frac{1}{n}} \left( \ln \frac{R}{r} \right)^{1-n+sn} \left( \ln \frac{R}{r} \right)^{-(1-n+sn)\frac{1}{n}} (1-n + sn)^{\frac{1}{n}}
\]

\[
= \frac{1}{n} \omega^{\frac{1}{n}} \left( \ln \frac{R}{r} \right)^{\frac{1}{n}} (1-n + sn)^{\frac{n-1}{n}}.
\]

Adding (2.35) and (2.36) we obtain

\[
\text{(rhs)}_1 + \text{(rhs)}_2 = \frac{1}{\omega^{\frac{1}{n}}} \left( \ln \frac{R}{r} \right)^{\frac{1-n+sn}{n}} \left[ \frac{1}{(1-n+sn)^{\frac{1}{n}}} \left( \frac{n-1}{n} + \frac{(1-n + sn)^{\frac{n-1}{n}}}{n} \right) \right]
\]

\[
= s \omega^{\frac{1}{n}} \left( \ln \frac{R}{r} \right)^{\frac{1-n+sn}{n}} (1-n + sn)^{-\frac{1}{n}}
\]

\[
\left( \int_{B_R \setminus B_r} \frac{|x \cdot \nabla u_s|^n}{|x|^n} dx \right)^{\frac{1}{n}} = (lhs),
\]

which proves the sharpness of (2.28). \(\square\)

Using Proposition 2.1 we will obtain Hardy inequalities in a ball which are sharp for \(p > n\) and optimal for \(p > 1\).
Proposition 2.2. For functions $u \in M(0,R)$ defined in (2.26) the following inequalities hold:

i) for $m > 0$, i.e., $p > n$

$$
\left( \int_{B_R} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx \right)^{\frac{1}{p}}
\geq \frac{p-n}{p} \left( \int_{B_R} \frac{|u|^p}{|x|^{(n+1)p'}} \frac{|x|^p}{|Rm - |x||^p} dx \right)^{\frac{1}{p}}
+ \frac{1}{p} R^{m-p} \limsup_{r \to 0} \left[ r^{n-p} \int_{\partial B_r} |u|^p dS \right]
\times \left( \int_{B_R} \frac{|u|^p}{|x|^{(n+1)p'}} \frac{|x|^p}{|Rm - |x||^p} dx \right)^{-\frac{1}{p'}}.
$$

(2.37)

For the functions $u_k(x) = \left( \frac{Rm - |x|^m}{m} \right)^k$, $k > \frac{1}{p'}$, inequality (2.37) becomes an equality and the constant $\frac{p-n}{p}$ in (2.37) is optimal.

ii) for $m < 0$, i.e., $p < n$

$$
\left( \int_{B_R} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx \right)^{\frac{1}{p}}
\geq \left| \frac{p-n}{p} \right| \left( \int_{B_R} \frac{|u|^p}{|x|^{(n+1)p'}} \frac{|x|^p}{|Rm - |x||^p} dx \right)^{\frac{1}{p}}
+ \frac{1}{p} R^{m-p} \limsup_{r \to 0} \left[ r^{1-p} \int_{\partial B_r} |u|^p dS \right]
\times \left( \int_{B_R} \frac{|u|^p}{|x|^{(n+1)p'}} \frac{|x|^p}{|Rm - |x||^p} dx \right)^{-\frac{1}{p'}}.
$$

(2.38)

The constant $\left| \frac{p-n}{p} \right|$ in (2.38) is optimal.
iii) for $m = 0$, i.e., $p = n$

\[
\left( \int_{B_R} \frac{\langle x, \nabla u \rangle^n}{|x|^n} \, dx \right)^{\frac{1}{n}} \geq \frac{n - 1}{n} \left( \int_{B_R} \frac{|u|^n}{|x|^n \ln \frac{R}{|x|}} \, dx \right)^{\frac{1}{n}}
\]

\[+ \frac{1}{n} \limsup_{r \to 0} \left( r \ln \frac{R}{r} \right)^{1-n} \int_{\partial B_r} |u|^p \, dS \left( \int_{B_R} \frac{|u|^n}{|x|^n \ln \frac{R}{|x|}} \, dx \right)^{-\frac{1}{n'}}. \tag{2.39}\]

The constant $\frac{n - 1}{n}$ in (2.39) is optimal.

**Proof.** Let us apply (2.27) and (2.28) in Proposition 2.1. Then after the limit $r \to 0$ we obtain (2.37), (2.38) and (2.39) for functions $u \in M(0, R)$. More precisely, 

i) Inequality (2.37) becomes equality for $u_k(x) = \left( \frac{R^m - |x|^m}{m} \right)^k$, $k > \frac{1}{p'}$ and hence the constant $\frac{p-n}{n}$ is optimal. Indeed, the sharpness of (2.37) is a consequence of the sharpness of (2.27) and the limit $r \to 0$ since $u_k \in M(0, R)$ for $k > \frac{1}{p'}$.

ii) Note that for $m < 0$ in the proof of (2.38) we use the identities

\[
\limsup_{r \to 0} \frac{1}{p} r^{1-n} |R^m - r^m|^{1-p} \int_{\partial B_r} |u|^p \, dS
\]

\[= \limsup_{r \to 0} \frac{1}{p} r^{1-n+m(1-p)} |R^m r^m - 1|^{1-p} \int_{\partial B_r} |u|^p \, dS
\]

\[= \limsup_{r \to 0} \frac{1}{p} r^{1-p} |R^m r^m - 1|^{1-p} \int_{\partial B_r} |u|^p \, dS = \limsup_{r \to 0} \frac{1}{p} r^{1-p} \int_{\partial B_r} |u|^p \, dS.
\]

We will prove that the constant $\left\lfloor \frac{p-n}{p} \right\rfloor$ in (2.38) is optimal using the function $u_\varepsilon(x) = |x|^{-\frac{|m|}{p'}(1-\varepsilon)} \left( R^{|m|} - |x|^{|m|} \right)^{\frac{1}{p'}(1+\varepsilon)}$ for $0 < \varepsilon < 1$. Note that $u_\varepsilon(x) \in M(0, R)$ because for the power of $|x|$ we have

\[
\frac{-|m|}{p'}(1-\varepsilon) = \varepsilon \frac{|m|}{p'} + \frac{p-n}{p} = \varepsilon \frac{|m|}{p'} + \frac{n(p-1)}{p} + 1 - n > 1 - n,
\]

hence $u_\varepsilon(x)$ is integrable at 0.

Ignoring the boundary term in (2.38) and rising both sides to $p$-th power for the left-
For the right-hand side we get

\[ \text{(lhs)} = \int_{B_R} \frac{|x, \nabla u_\epsilon|}{|x|}^p \, dx \]

\[ = \int_{B_R} |x|^{-m(p-1)(1-\varepsilon)-p} \left( R^{|m|} - |x|^{|m|} \right)^{(p-1)(1+\varepsilon)-p} \]

\[ \times \left| \frac{m}{p'} \right| [(1-\varepsilon)R^{|m|} + 2\varepsilon |x|^{|m|}]^p \, dx \]

\[ \leq \left| \frac{p-n}{p} \right|^p (1+\varepsilon)^p R^{|m|p} \]

\[ \int_{B_R} |x|^{(p-n)(1-\varepsilon)-p} \left( R^{|m|} - |x|^{|m|} \right)^{(p-1)(1+\varepsilon)-p} \, dx. \]

For the right-hand side we get

\[ \text{(rhs)} = \frac{p-n}{p} \int_{B_R} \frac{|u_\epsilon|^p}{|x|^{(n-1)p'}} \left( R^{|m|} - |x|^{|m|} \right)^p \, dx \]

\[ = \frac{p-n}{p} \int_{B_R} \frac{|x|^{-m(p-1)(1-\varepsilon)p} \left( R^{|m|} - |x|^{|m|} \right)^p R^{|m|p} x^{|m|p}}{|x|^{(n-1)p'}} \, dx \]

\[ = \frac{p-n}{p} R^{|m|p} \int_{B_R} |x|^{-m(p-1)(1-\varepsilon)-p} \left( R^{|m|} - |x|^{|m|} \right)^{(p-1)(1+\varepsilon)-p} \, dx, \]

and hence \( 1 < \frac{\text{(lhs)}}{\text{(rhs)}} < (1+\varepsilon)^p \) because \( \left( \frac{m}{p'} \right)^p = \left( \frac{p-n}{p} \right)^p = \left( \frac{p-n}{p} \right)^p \). For \( \varepsilon \to 0 \) it follows that the constant \( \left| \frac{p-n}{p} \right|^p \) is optimal.

(iii) We will prove that the constant \( \frac{n-1}{n} \) in (2.39) is optimal using the function

\[ u_s(x) = \begin{cases} \left( \frac{R}{|x|} \right)^s, & \text{for } r_0 < |x| < R, \\ \left( \frac{R}{r_0} \right)^s, & \text{for } 0 \leq |x| \leq r_0, \end{cases} \]

with \( s = \frac{n-1}{n} (1+\varepsilon), \, 0 < \varepsilon. \)

Ignoring the boundary term in (2.39) and rising both sides to \( n \)-th power we obtain the inequality

\[ \int_{B_R} \left| \left( x, \nabla u_s \right) \right|^n \, dx \geq \left( \frac{n-1}{n} \right)^n \int_{B_R} \frac{|u_s|^n}{|x|^n} \, dx. \]
For the left-hand side and for the right-hand side of (2.40) as in Proposition 2.1 we get

\[(lhs) = \int_{B_R} \left| \frac{x, \nabla u_s}{|x|} \right|^n dx = s^n \int_{B_{R\setminus B_{r_0}}} |x|^{-n} \left( \ln \frac{R}{|x|} \right)^{n(s-1)} dx \]

\[= s^n \omega_n \int_{r_0/R}^1 \rho^{-1} \left( \ln \frac{1}{\rho} \right)^{n(s-1)} = s^n \omega_n \left( \ln \frac{R}{r_0} \right)^{1-n+sn} (1 - n + sn)^{-1} \]

\[(rhs) = \left( \frac{n-1}{n} \right) \int_{B_R} \frac{|u_s|^n}{|x|^n} \ln \frac{R}{|x|}^n dx \]

\[= \left( \frac{n-1}{n} \right) \omega_n \left( \ln \frac{R}{r_0} \right)^{1-n+sn} (1 - n + sn)^{-1} \]

\[+ \left( \frac{n-1}{n} \right) \omega_n \left( \ln \frac{R}{r_0} \right)^{sn} \int_{r_0/R} \rho^{-1} \left( \ln \frac{1}{\rho} \right)^{-n} d\rho \]

\[= \left( \frac{n-1}{n} \right) \omega_n \left( \ln \frac{R}{r_0} \right)^{1-n+sn} \left[ \frac{1}{1-n+sn} + \frac{1}{n-1} \right] \]

\[= \left( \frac{n-1}{n} \right) \omega_n \left( \ln \frac{R}{r_0} \right)^{1-n+sn} \frac{sn}{(1-n+sn)(n-1)}. \]

Since \( s = (1 + \varepsilon)\frac{n-1}{n} > \frac{1}{n'}, \) then \( 1 \leq \frac{(lhs)}{(rhs)} = (1 + \varepsilon)^{n-1} \) and the sharpness of (2.40) is proved.

\[\square\]

### 2.4 Hardy inequalities with additional logarithmic term

The aim of this section is to prove Hardy inequality in a ball \( B_R \) centered at zero with radius \( 0 < R < \infty, B_R \subset R^n, n \geq 2 \) with double singular weights on the boundary \( \partial B_R \) and at the origin and with an additional logarithmic term. We generalize the results in Barbatis et al. [13] where the weights are singular only on the boundary of the domain.

Let \( p > 1, \) \( p' = \frac{p}{p-1}, n \geq 2, m = \frac{p-n}{p-1}. \) In order to formulate the new Hardy inequality, let us prove Lemma 2.1 following the result of Barbatis et al. [13], Lemma 3.1.

**Lemma 2.1.** For every \( p \geq 2, \) there exists \( a = a(p) < 0 \) and \( \tau_0 > 0 \) such that for every \( \tau > \tau_0 \) the function

\[ Z(s) = \left( \frac{1}{p'} \right)^{p-1} \left( 1 - \frac{1}{1 + \ln \tau - s} + \frac{a}{(1 + \ln \tau - s)^2} \right), \]

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satisfies
\[ Z(s) \in C^1(-\infty, 0), \quad Z > 0, \quad Z' < 0, \quad Z(-\infty) = \left( \frac{1}{p'} \right)^{p-1}, \]  
and is a solution of the inequality
\[ -Z' + (p-1)Z - (p-1)Zp' \geq H(s), \]  
where
\[ H(s) = \left( \frac{1}{p'} \right)^p \left( 1 + \frac{p}{2(p-1)} \frac{1}{1 + \ln \tau - s} \right). \]

Proof. Let us denote for simplicity \( y(s) = \frac{1}{1 + \ln \tau - s} \), so that
\[ Z(s) = \left( \frac{1}{p'} \right)^{p-1} (1 - y + ay^2), \quad Z'(s) = \left( \frac{1}{p'} \right)^{p-1} (-y^2 + 2ay^3) \]

Expanding \( Zp'(y) \) for a small \( y \) near \( y = 0 \) in a Taylor polynomial up to third order we obtain
\[ Zp' = \left( \frac{1}{p'} \right)^p \left\{ 1 - \frac{p}{p-1} y + \frac{p}{p-1} \left( 2a + \frac{1}{p-1} \right) y^2 \right\} \]
\[ + \frac{p}{p-1} \left[ -\frac{6a}{p-1} - \frac{p-2}{(p-1)^2} \right] \frac{y^3}{6} + o(y^3) \}

Then if \( a < -\frac{p-2}{6(p-1)} \) we get
\[ -Z' + (p-1)Z - (p-1)Zp' \]
\[ = \left( \frac{1}{p'} \right)^p \left[ 1 + \frac{p}{2(p-1)} y^2 + \frac{p}{p-1} \left( -a + \frac{p-2}{6(p-1)} \right) y^3 + o(y^3) \right] \]
\[ \geq \left( \frac{1}{p'} \right)^p \left( 1 + \frac{p}{2(p-1)} y^2 \right). \]

With this choice of \( a \) inequalities (2.42) and \( Z'(s) < 0 \) hold. In order to satisfy the rest of the conditions in (2.41) we choose \( \tau \) such that \( Z(s) > 0 \), i.e., \( 1 - y - |a|y^2 > 0 \). This means that
\[ 0 < y < y_0 = \frac{\sqrt{1+4|a|} - 2|a|}{2|a|}. \]  
Let \( \tau_0 = e^{\frac{1}{y_0}} \) then for every \( \tau > \tau_0 \) we get \( Z(s) > 0 \). \( \square \)

First, we will obtain an inequality in an annulus. Let us define the vector function \( f \),
\[ f = \left| \nabla \psi \right|^{p-2} \frac{\nabla \psi}{\psi} Z(\ln \psi) \quad \text{in} \ B_R \setminus B_r, \]  
where \( \psi(x) \) is defined in (2.29) and \( Z \) is given in Lemma 2.1.
Proposition 2.3. The vector function $f = \{f_1, \ldots, f_n\}$ in (2.44) satisfies $f_j \in C^1(B_R \setminus B_r)$ and

$$-\text{div } f - (p - 1)|f|^{p'} \geq w, \quad \text{in } B_R \setminus B_r,$$

where $w = \left| \frac{\nabla \psi}{\psi} \right|^p H(\ln \psi)$ and $H(s)$ is defined in Lemma 2.1.

Moreover, for every $u \in W^{1,p}_0(B_R)$, the following inequality holds

$$L(u) \geq N(u),$$

where

$$L(u) = \int_{B_R \setminus B_r} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx,$$

and

- for $m \neq 0$, i.e., $p \neq n$
  
  $$N(u) = \int_{B_R \setminus B_r} u|u|^p dx$$
  
  $$= \left| \frac{p - n}{p} \right| \int_{B_R \setminus B_r} \left[ 1 + \frac{p}{2(p - 1) \ln^2 \frac{x}{R}} \right] |x|^{(n - 1)p'} \left| \frac{u}{|R - |x|^{m}|} \right|^p dx,$$

- for $m = 0$, i.e., $p = n$
  
  $$N(u) = \int_{B_R \setminus B_r} u|u|^n dx$$
  
  $$= \left( \frac{n - 1}{n} \right)^n \int_{B_R \setminus B_r} \left[ 1 + \frac{n}{2(n - 1) \ln^2 \frac{x}{R}} \right] |x|^{n} |\ln \frac{R}{|x|}| dx,$$

where $\tau > \tau_0 = e^{\frac{1}{m - 1}}$ and $y_0$ is defined in (2.43).

Proof. Let us check that the function $f$ satisfies (2.45). Indeed,

$$-\text{div } f = - \left( \frac{\Delta \psi}{|\psi|^{p-1}} - (p - 1) \frac{\nabla \psi}{\psi} \right) \frac{\nabla \psi}{\psi} |\nabla \psi|^{p'} Z(\ln \psi) - \frac{\nabla \psi}{\psi} |\nabla \psi|^{p'} Z'(\ln \psi)$$

$$= \frac{\nabla \psi}{\psi} \left[ -(p - 1)Z - (p - 1)Z' \right] + (p - 1) \frac{\nabla \psi}{\psi} |\nabla \psi|^{p'} Z'$$

$$\geq (p - 1)|f|^{p'} + \frac{\nabla \psi}{\psi} |\nabla \psi|^{p'} H(\ln \psi).$$

Since $f$ as a function of $x$ has the form

$$f(x) = \begin{cases} 
-x|x|^{-n} \left( \frac{R^m - |x|^m}{m} \right)^{1-p} Z(\ln \psi), & m \neq 0, \\
-x|x|^{-n} \left( \frac{\ln \frac{R}{|x|}}{n} \right)^{1-n} Z(\ln \psi), & m = 0,
\end{cases}$$
then $\langle f, \eta \rangle_{\partial B_r} > 0$. Ignoring the boundary term over $\partial B_r$ we can apply (2.9) in the domain $B_R \setminus B_r, r < \hat{R} < R$ for $v = 1$ and $w = \left| \frac{\nabla \psi}{\psi} \right|^p H(\ln \psi)$ to obtain (2.46) after the limit $\hat{R} \to R$. \hfill \Box

The inequality with additional logarithmic weight in a ball can be obtained only for $p > n$, i.e., $m > 0$. In the case $m \leq 0$ the function $\psi(x)$ defined in (2.29) satisfies

$$\psi(x) = R^m - |x|^m \to R^m - |x|^m$$

for $r \to 0$, and the inequalities with $N(u)$ defined in (2.47), (2.48) are the same as the inequalities (2.38) and (2.39) without boundary and logarithmic terms. When $m > 0$, i.e., $p > n$, the function $\psi(x) = \frac{R^m - |x|^m}{R^m - r^m}$ for $r \to 0$ so we obtain a new Hardy inequality with additional logarithmic term using the expression (2.47) for $N(u)$.

**Proposition 2.4.** For $m > 0$, i.e., $p > n$, the following inequality holds for every $u \in W_0^{1,p}(B_R)$

$$\int_{B_R} |\nabla u|^p \, dx \geq \int_{B_R} \left| \frac{x \cdot \nabla u}{|x|} \right|^p \, dx \geq \left( \frac{p - n}{p} \right)^p \int_{B_R} \left[ 1 + \frac{p}{2(p-1)} \ln^2 \frac{R^m - |x|^m}{|x|^n} \right] \frac{|u|^p}{|x|^{(n-1)p'} |R^m - |x|^m|^p} \, dx,$$

where $\tau_0 = e^{\frac{1}{y_0} - 1}, y_0$ is defined in (2.43) with $a = -\frac{p - 2}{6(p-1)}$.

**Proof.** Since the function $\psi(x) = \frac{R^m - |x|^m}{R^m - r^m}$ for $r \to 0$ and after the limit $r \to 0$ in $N(u)$ from the expression (2.47) we obtain (2.49). By continuity we get $\tau = \tau_0$ in (2.47) \hfill \Box

### 2.5 One-parametric family of Hardy inequalities

For $n \geq 2, p > 1$ we consider the Poisson problem in $B_R$

$$\begin{cases}
-\text{div}(|\nabla \phi|^{p-2} \nabla \phi) = w(|x|) & \text{in } B_R, \\
\phi = 0 & \text{on } \partial B_R,
\end{cases}$$

where

$$\int_0^R s^{n-1}w(s)\,ds < \infty. \quad (2.51)$$

We choose a function $w(|x|)$ such that the function $\phi$ has a simple form.

For this purpose let us apply the result in Biezuner et al. [23], where it is shown that the solution of (2.50), (2.51) is given by

$$\phi(|x|) = \int_{|x|}^R \theta^{\frac{1-n}{p-1}} \left( \int_0^\theta s^{n-1}w(s)\,ds \right)^{\frac{1}{p-1}} \, d\theta. \quad (2.52)$$
Indeed, from the invariance of (2.50) under rotation, problem (2.50) is equivalent to the boundary value problem for ordinary differential equation

\[
\begin{align*}
-(r^{n-1}|\phi'|^{p-2}\phi')' &= r^{n-1}w(r), \quad 0 < r < R, \\
\phi(R) &= 0, \phi'(0) = 0.
\end{align*}
\] (2.53)

Integrating twice the equation in (2.53) and applying boundary conditions we obtain (2.52).

Let us chose the function \(w(|x|) = |x|^{-\delta}, \delta \in (0, n),\) then (2.51) holds and from (2.52) we have

\[
\phi(|x|) = \begin{cases} 
\frac{p-1}{p-\delta}(n-\delta)^{1}\left(R^{\frac{p}{p-1}} - |x|^{\frac{p-\delta}{p-1}}\right) & \text{for } \delta \neq p, \\
(n-p)^{-\frac{1}{p-1}} \ln \frac{R}{|x|} & \text{for } \delta = p.
\end{cases}
\]

For \(\varepsilon \in (0, R)\) we define the vector function \(f = \frac{\nabla \phi}{|\phi|^{p-2} \phi} \). Then \(f \in C^1(B_R \setminus B_{\varepsilon}), \varepsilon < \hat{R} < R\) and \(f\) satisfies the equation

\[-\text{div} f = -\frac{\Delta \phi}{|\phi|^{p-2} \phi} + (p-1)\left|\frac{\nabla \phi}{|\phi|^{p-2} \phi}\right| = \frac{w(|x|)}{|\phi|^{p-1}} + (p-1)|f|^{\frac{p}{p-1}} \text{ in } B_R \setminus B_{\varepsilon}.
\]

According to Corollary 2.1, the following Hardy inequality holds for \(u \in C_0^\infty(B_R)\), with \(\text{supp } u \subset B_{\hat{R}}\)

\[
L_\varepsilon(u) \geq \left(\frac{1}{p}\right)^p \frac{|(p-1)K_\varepsilon(u) + K_{3\varepsilon}(u) + N_\varepsilon(u)|^p}{K_{p-1}^{-1}(u)}, \quad u \in C_0^\infty(B_R),
\] (2.54)

where

\[
K_{3\varepsilon}(u) = \int_{S_\varepsilon} \langle f, \eta \rangle |u|^p dS + \int_{B_{\hat{R}}} \langle f, \eta \rangle |u|^p dS = \int_{S_\varepsilon} \langle f, \eta \rangle |u|^p dS.
\]

Here \(\eta\) is the unit outward normal vector to \(\partial(B_{\hat{R}} \setminus B_{\varepsilon})\). The expressions for \(L_\varepsilon(u), K_\varepsilon(u)\) and \(N_\varepsilon(u)\) in (2.54) are correspondingly:

- in the case \(\delta \neq p\)

\[
\int_{B_{\hat{R}} \setminus B_{\varepsilon}} |\nabla u|^p dx \geq L_\varepsilon(u) = \int_{B_{\hat{R}} \setminus B_{\varepsilon}} \left|\frac{\nabla \phi}{|\phi|^{p-2} \phi}\right|^p dS = \int_{B_{\hat{R}} \setminus B_{\varepsilon}} \left|\frac{x, \nabla u}{|x|}\right|^p dx,
\]

\[
K_\varepsilon(u) = \int_{B_{\hat{R}} \setminus B_{\varepsilon}} \left|\frac{\nabla \phi}{|\phi|^{p-2} \phi}\right|^p |u|^p dx = \left|\frac{p-\delta}{p-1}\right| \int_{B_{\hat{R}} \setminus B_{\varepsilon}} \left|\frac{|u|^p}{|x|^\delta R^{\frac{p-\delta}{p-1}} - |x|^{\frac{p-\delta}{p-1}}}\right| dx,
\]

\[
N_\varepsilon(u) = \int_{B_{\hat{R}} \setminus B_{\varepsilon}} w(|x|) |u|^p dx
\]

\[
= (n-\delta) \left|\frac{p-\delta}{p-1}\right| \int_{B_{\hat{R}} \setminus B_{\varepsilon}} \left|\frac{|u|^p}{|x|^\delta R^{\frac{p-\delta}{p-1}} - |x|^{\frac{p-\delta}{p-1}}}\right|^{p-1} dx,
\] (2.55)
in the case $\delta = p$

$$K_\varepsilon(u) = \int_{B_R \setminus B_{\varepsilon r}} \frac{|u|^p}{|x|^p \ln \frac{R}{|x|}} \, dx, \quad N_\varepsilon(u) = |n - p| \int_{B_R \setminus B_{\varepsilon r}} \frac{|u|^p}{|x|^p \ln \frac{R}{|x|}} \, dx. \quad (2.56)$$

Since

$$\nabla \phi = -(n - \delta) \frac{x}{|x|^2} \frac{x}{|x|}, \quad \eta \big|_{S_\varepsilon} = -\frac{x}{\varepsilon},$$

and

$$\langle \nabla \phi, \eta \rangle = (n - \delta) \frac{1}{p - 1} |\varepsilon|^{\frac{p - 1}{p - 1}} \geq 0, \quad \text{for } x \in S_\varepsilon,$$

we get

$$\int_{S_\varepsilon} \langle f, \eta \rangle |u|^p \, dS = (n - \delta) \frac{1}{p - 1} \varepsilon^{\frac{p - 1}{p - 1}} \int_{S_\varepsilon} |u|^p \, dS \geq 0.$$

Hence, neglecting $K_{3\varepsilon}$ (2.54) becomes

$$L_\varepsilon(u) \geq \left( \frac{1}{p} \right)^p \frac{|(p - 1)K_\varepsilon(u) + N_\varepsilon(u)|^p}{K^p(u)}, \quad u \in C_0^\infty(B_R).$$

After the limit $\varepsilon \to 0$ inequality

$$L(u) \geq \left( \frac{1}{p} \right)^p \frac{|(p - 1)K(u) + N(u)|^p}{K^p(u)}, \quad u \in C_0^\infty(B_R). \quad (2.57)$$

holds in $B_R$ with $L(u)$, $K(u)$ and $N(u)$ defined in (2.55) and (2.56) for $\varepsilon = 0$, i.e.,

- in the case $\delta \neq p$

$$L(u) = \int_{B_R} \frac{|\langle x, \nabla u \rangle|^p}{|x|^p} \, dx;$$

$$K(u) = \left| \frac{p - \delta}{p - 1} \right| \int_{B_R} \left| \frac{x}{|x|} \right|^{\frac{(p - 1)u}{p - 1}} \left| \frac{|u|^p}{R^{\frac{p - 1}{p - 1}} - |x|^{\frac{p - 1}{p - 1}}} \right|^p \, dx, \quad (2.58)$$

$$N(u) = (n - \delta) \left| \frac{p - \delta}{p - 1} \right|^{p - 1} \int_{B_R} \left| \frac{x}{|x|} \right|^{\delta} \left| \frac{|u|^p}{R^{\frac{p - 1}{p - 1}} - |x|^{\frac{p - 1}{p - 1}}} \right|^{p - 1} \, dx;$$

- in the case $\delta = p$

$$K(u) = \int_{B_R} \frac{|u|^p}{|x|^p \ln \frac{R}{|x|}} \, dx, \quad N(u) = |n - p| \int_{B_R} \frac{|u|^p}{|x|^p \ln \frac{R}{|x|}} \, dx. \quad (2.59)$$

From (2.11) and (2.57) we get

$$L(u) \geq (p - 1)s^{p - 1}(1 - s)K(u) + s^{p - 1}N(u). \quad (2.60)$$

In particular, for $s = \frac{p - 1}{p}$ in (2.60) a ‘linear’ form of Hardy inequality holds.
Lemma 2.2. The inequalities $K(u) < \infty$ and $N(u) < \infty$ for $K(u)$ and $N(u)$ defined in (2.58) and (2.59) hold for every $u \in C_0^\infty(B_R)$.

Proof. For $u \in C_0^\infty(B_R)$,

$$d_1 = \text{dist}(\text{supp } u, \partial B_R) > 0, \quad d_2 = R^{\frac{p-\delta}{p-1}} - (R - d_1)^{\frac{p-\delta}{p-1}} > 0,$$

we obtain these statements from the following estimates

- for $0 < \delta < p$, $\delta < n$

$$\int_{B_R} \frac{|u|^p}{|x|^{\frac{(\delta-1)p}{p-1}}} \frac{|x|^{\frac{\delta-1}{p-1}}}{|R^{\frac{\delta-\delta}{p-1}} - |x|^{\frac{\delta-1}{p-1}}|} \ dx \leq \left( \frac{\sup |u|}{d_2} \right)^p \int_{S_1} \int_0^{R-d_1} \rho^{n-1} \frac{\rho^{\frac{(\delta-1)p}{p-1}}}{\rho} \ d\rho d\theta$$

$$= \omega_n \left( \frac{\sup |u|}{d_2} \right)^p \left( n - \delta + \frac{p - \delta}{p - 1} \right)^{-1} (R - d_1)^{n - \delta + \frac{p - \delta}{p - 1}} < \infty,$$

where $\omega_n = \text{meas } S_1$,

$$\int_{B_R} \frac{|u|^p}{|x|^{\delta}} \frac{|x|^{\frac{\delta-1}{p-1}}}{|R^{\frac{\delta-\delta}{p-1}} - |x|^{\frac{\delta-1}{p-1}}|} \ dx \leq \left( \frac{\sup |u|}{d_2} \right)^p \int_{S_1} \int_0^{R-d_1} \rho^{n-1} \frac{\rho^{\frac{(\delta-1)p}{p-1}}}{\rho^p} \ d\rho d\theta$$

$$= \omega_n \left( \frac{\sup |u|}{d_2} \right)^p (n - \delta)^{-1} (R - d_1)^{n - \delta} < \infty;$$

- for $1 < p < \delta < n$

$$\int_{B_R} \frac{|u|^p}{|x|^{\frac{(\delta-1)p}{p-1}}} \frac{|x|^{\frac{\delta-1}{p-1}}}{|R^{\frac{\delta-\delta}{p-1}} - |x|^{\frac{\delta-1}{p-1}}|} \ dx = R^{\frac{(\delta-n)p}{p-1}} \int_{B_R} \frac{|u|^p}{|x|^{\frac{\delta-n}{p-1}}} \frac{|x|^{\frac{\delta-n}{p-1}}}{|R^{\frac{\delta-n}{p-1}} - |x|^{\frac{\delta-n}{p-1}}|} \ dx$$

$$\leq \frac{\omega_n}{n-p} \left( \frac{\sup |u|}{d_2} \right)^p (R - d_1)^{n-p} R^{\frac{(\delta-n)p}{p-1}} < \infty,$$

$$\int_{B_R} \frac{|u|^p}{|x|^{\delta}} \frac{|x|^{\frac{\delta-1}{p-1}}}{|R^{\frac{\delta-\delta}{p-1}} - |x|^{\frac{\delta-1}{p-1}}|} \ dx = R^{\frac{(\delta-n)p}{p-1}} \int_{B_R} \frac{|u|^p}{|x|^{\frac{\delta-n}{p-1}}} \frac{|x|^{\frac{\delta-n}{p-1}}}{|R^{\frac{\delta-n}{p-1}} - |x|^{\frac{\delta-n}{p-1}}|} \ dx$$

$$\leq \frac{\omega_n}{n-p} \left( \frac{\sup |u|}{d_2} \right)^p (R - d_1)^{n-p} R^{\frac{(\delta-n)p}{p-1}} < \infty;$$

- for $p = \delta < n$

$$\int_{B_R} \frac{|u|^p}{|x|^p} \frac{\ln \frac{R}{|x|}}{\ln \frac{R}{d_1}} \ dx \leq \left( \sup |u| \right)^p \frac{\omega_n}{(\ln \frac{R}{d_1})^p} \int_0^{R-d_1} \rho^{n-p-1} d\rho$$

$$= \left( \sup |u| \right)^p \frac{\omega_n}{(\ln \frac{R}{d_1})^p} R^{n-p} < \infty,$$

$$35.$$
\[
\int_{B_R} \frac{|u|^p}{|x|^p \ln \frac{R}{|x|^p}} \leq (\sup |u|)^p \frac{\omega_n}{(\ln \frac{R}{R-d_1})^{p-1}} \int_0^{R-d_1} \rho^{n-p-1} d\rho
\]
\[
= (\sup |u|)^p \frac{\omega_n}{(\ln \frac{R}{R-d_1})^{p-1}} \frac{(R-d_1)^{n-p}}{n-p} < \infty.
\]

Proposition 2.5. For \( \delta \in (0, n) \) and all functions \( u \in W_0^{1,p}(B_R) \) the following Hardy inequalities hold:

(i) for \( \delta \neq p, p < n \)
\[
\int_{B_R} |\nabla u|^p dx \geq \frac{p-\delta}{p} \int_{B_R} \frac{|u|^p}{|x|^{(\delta-1)p}} \left( R^{\frac{p-\delta}{p-1}} - |x|^{\frac{p-\delta}{p-1}} \right) dx
\]
\[
+ (n-\delta) \frac{p-\delta}{p} \int_{B_R} \frac{|u|^p}{|x|^{\delta}} \left( R^{\frac{p-\delta}{p-1}} - |x|^{\frac{p-\delta}{p-1}} \right)^{p-1} dx.
\]

(ii) for \( \delta = p \)
\[
\int_{B_R} |\nabla u|^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^p \ln \frac{R}{|x|^p}} dx
\]
\[
+ \left( \frac{p-1}{p} \right)^{p-1} \left| n-p \right| \int_{B_R} \frac{|u|^p}{|x|^p \ln \frac{R}{|x|^p}}^{p-1} dx.
\]

Proof. With the expressions (2.58), (2.59), applying (2.57) we obtain (i) and (ii). □

By arguments of continuity inequality (2.61) is also true in the case of \( \delta = n \) and \( \delta = 0 \).

It is important to mention that in the new Hardy inequalities (2.61) and (2.62) for \( \delta \in (1, n) \) the constants \( \left( \frac{p-\delta}{p} \right)^p \) and \( \left( \frac{p-1}{p} \right)^p \), correspondingly, at their leading terms in the right-hand side are optimal. This follows from Proposition 2.6 below.

Proposition 2.6. If \( p > 1, n \geq 2, 1 < \delta < n \) then for \( 0 < |x| < R \) the following inequalities hold:

(i) for \( \delta \neq p \)
\[
\left( \frac{p-\delta}{p} \right)^p |x|^{(\delta-1)p} \left( R^{\frac{p-\delta}{p-1}} - |x|^{\frac{p-\delta}{p-1}} \right)^{-p} \geq \left( \frac{p-1}{p} \right)^p (R-|x|)^{-p};
\]

(ii) for \( \delta = p \)
\[
\left( \frac{p-1}{p} \right)^p |x|^{-p} \ln \frac{R}{|x|} \geq \left( \frac{p-1}{p} \right)^p (R-|x|)^{-p}.
\]

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Proof. (i) For $p > \delta$ inequality (2.63) is equivalent to the estimate

$$g(r) = (p - \delta)(R - r) - (p - 1)r^{\frac{\delta - 1}{p - 1}} \left(R^{\frac{\delta}{p - 1}} - r^{\frac{\delta}{p - 1}}\right) \geq 0, \quad \text{for } 0 \leq r \leq R. \quad (2.65)$$

Since $g(R) = 0$ and

$$g'(r) = -(\delta - 1)r^{\frac{\delta - p}{p - 1}} \left(R^{\frac{\delta}{p - 1}} - r^{\frac{\delta}{p - 1}}\right) \leq 0$$

inequality (2.65) follows from the monotonicity of $g(r)$.

For $1 < p < \delta < n$ estimate (2.63) is equivalent to

$$\left(\delta - p\right)^p R^{\frac{(1-\delta)p}{p-1}} r^{\frac{(\delta-p)p}{p-1}} R^{\frac{\delta-p}{p-1}} \left(R^{\frac{\delta-p}{p-1}} - r^{\frac{\delta-p}{p-1}}\right)^{-p} \geq \left(\frac{p-1}{p}\right)^p (R - r)^{-p},$$

or

$$h(r) = (\delta - p)R^{\frac{\delta-p}{p-1}} (R - r) - (p - 1)r \left(R^{\frac{\delta-p}{p-1}} - r^{\frac{\delta-p}{p-1}}\right) \geq 0, \quad \text{for } 0 \leq r \leq R.$$

Since $h(R) = 0$ and

$$h'(r) = -(\delta - 1) \left(R^{\frac{\delta-p}{p-1}} - r^{\frac{\delta-p}{p-1}}\right) \leq 0,$$

inequality (2.63) follows from the monotonicity of $h(r)$.

(ii) Inequality (2.64) is equivalent to

$$z(r) = R - r - r \ln \frac{R}{r} \geq 0, \quad \text{for } 0 < r < R. \quad (2.66)$$

Since $z(R) = 0$ and $z'(r) = -\ln \frac{R}{r} < 0$, for $r \in (0, R)$, the inequality (2.66) is a consequence of the monotonicity of $z(r)$.

However, for Hardy inequality

$$\int_{B_R} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{B_R} \frac{|u|^p}{(R - |x|)^p} dx,$$

the constant $\left(\frac{p-1}{p}\right)^p$ is optimal, see (1.2) with $\alpha = 0$ and $\Omega = B_R$. The same optimality is also true for (2.61) and (2.62).

3 General Hardy inequalities with optimal constant

In this section we prove general Hardy inequalities with singular at zero and on the boundary $\partial \Omega$ weights is proved. The Hardy constant is optimal when $\Omega$ is a ball. The section is based on Fabricant et al. [34].

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain. Suppose that

$$\Omega \subset \Omega^* \text{ and there exists a positive function } \lambda \in C^{0,1}(\Omega^*) \setminus \lambda(x) > 0, \quad \text{such that } |x| < \lambda(x) \text{ and } \langle x, \nabla \lambda \rangle \leq 0 \text{ for a.e. } x \in \Omega^*, \quad (3.1)$$

where $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^n$. If $\Omega$ is convex or star-shaped domain with respect to some interior ball centered at zero, then $\Omega$ satisfies (3.1), see Section 1.1.8 in
Maz’ja [76], so one can take $\Omega^* = \Omega$. When $\Omega$ is an arbitrary domain, then its star-shaped envelope with respect to some fixed interior ball (i.e., the intersection of all star-shaped domains with respect to the fixed ball containing $\Omega$) can be taken as $\Omega^*$. Further on, we suppose that $0 \in \Omega$.

For $\alpha, \beta \in \mathbb{R}$, $p > 1$, $\beta p > 1$, $\alpha p \leq n + \beta p - 1$ and $k = \frac{n - \alpha p}{\beta p - 1}$, we define the function

$$g(s) = \begin{cases} 
\frac{1 - s^k}{k} & \text{for } k \neq 0 \\
\ln 1 & \text{for } k = 0
\end{cases}$$

(3.2)

and the constant $\gamma = \frac{\beta p - 1}{p}$. Note that $\gamma > 0$ and $k \geq -1$. For $s(x) = \frac{|x|}{\lambda(x)}$ let us consider the non-negative weights

$$v(x) = |x|^{1-\alpha}|g(s(x))|^{1-\beta}, \quad w(x) = |x|^{-\alpha}|g(s(x))|^{-\beta}.$$  

(3.3)

The function $v$ is singular at the origin when $k \geq 0$ for $\alpha > 1$ and when $k \in [-1, 0)$ for $\alpha > k(\beta - 1) + 1$. The function $w(x)$ is singular at 0 when $k \geq 0$ for $\alpha > 0$ and when $k \in [-1, 0)$ for $\alpha > -\beta k$. Moreover, $v$ and $w$ are singular on the whole boundary if $\partial \Omega^* = \partial \Omega = \{x : |x| = \lambda(x)\}$. Otherwise, if $\partial \Omega^* \neq \partial \Omega$, the functions $v$ and $w$ are singular only on a part of the boundary.

Let the space $W^{1,p}_{0,v} (\Omega)$ be the completion of $C_0^\infty(\Omega)$ functions with respect to the norm

$$\left( \int_{\Omega} v^p |\nabla u|^p dx \right)^{1/p} < \infty,$$

see Lemma 3.1.

The aim of this section is to prove the following new Hardy inequality with double singular weights.

**Theorem 3.1.** For every $u \in W^{1,p}_{0,v} (\Omega)$ the following inequality holds

$$\int_\Omega v^p |\nabla u|^p dx \geq \gamma^p \int_\Omega w^p |u|^p dx,$$

(3.4)

where $\gamma = \frac{\beta p - 1}{p}$.

At the beginning, let us analyze condition (3.1) and simplify it in polar coordinates. If $S_1 = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere and $\rho = |x|$, $\theta = \frac{x}{|x|} \in S_1$, then the property of $\lambda(x)$, $\langle x, \nabla \lambda \rangle \leq 0$ in (3.1) becomes $\rho \lambda_\rho \leq 0$, where $\lambda = \lambda(\rho, \theta)$.

Indeed, $\langle x, \nabla \lambda \rangle = \rho \lambda_\rho$ because

$$\frac{\partial \lambda}{\partial x_j} = \frac{x_j}{\rho} \lambda_\rho + \frac{\partial \lambda}{\partial \theta_k} \left( \frac{\delta_{jk}}{\rho} - \frac{x_j x_k}{\rho^3} \right),$$

and

$$\sum_{i,j=1}^n \frac{\partial \lambda}{\partial \theta_i} x_j \left( \frac{\delta_{ji}}{\rho} - \frac{x_j x_i}{\rho^3} \right) = 0.$$  

The proof of the Theorem is based on Theorem 2.1 and Lemma 3.1 which clarifies the properties of the weights $w$ and $v$ and the definition of the weighted Sobolev space $W^{1,p}_{0,v} (\Omega)$.  

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Lemma 3.1. Functions $w(x)$ and $v(x)$ belong to $L^p_{\text{loc}}(\Omega)$.

Proof. Since the functions $v$ and $w$ are singular at 0 and on $\partial \Omega$, it is enough to prove Lemma 3.1 only in a small ball containing 0. For this purpose let us fix $\varepsilon \in (0,1)$ and we will prove that $w(x)$ is an $L^p$ function in a small ball $B_\sigma$ with radius $\sigma \in (0,1)$ centered at zero, $B_\sigma \subset \Omega$, so that

$$\frac{\rho}{\lambda} < 1 - \varepsilon \quad (3.5)$$

for $\rho < \sigma$.

In polar coordinates, for $k \neq 0$ we have

$$\int_{B_\sigma} w^p dx = \int_{S_1} \int_0^\sigma H(\rho)d\rho d\theta,$$

where $H(\rho) = \rho^{n-1-\alpha p} \frac{1 - (\frac{\rho}{\lambda})^k}{k}^{-\beta p}$.

We define the constant $C_{k,\varepsilon} = \left(1 - (\frac{1 - \varepsilon}{k})^k\right)^{\beta p} \omega_n$, where $\omega_n$ is the measure of the unit sphere $S_1$ in $\mathbb{R}^n$, and, let us consider all possibilities for the sign of $k$:

(a) $k > 0$, i.e., $n > \alpha p$. Then from

$$\frac{\partial}{\partial \rho} \left(1 - \left(\frac{\rho}{\lambda}\right)^k\right)^{-\beta p} > 0$$

we get

$$\int_{S_1} \int_0^\sigma H(\rho)d\rho d\theta \leq \int_{S_1} \int_0^\sigma \frac{C_{k,\varepsilon}}{\omega_n} \rho^{n-1-\alpha p} d\rho d\theta = C_{k,\varepsilon} \frac{\sigma^{n-\alpha p}}{n-\alpha p} < \infty.$$  

(b) $k < 0$, i.e., $n < \alpha p$. In this case

$$H(\rho) = \left(\frac{\rho}{\lambda}\right)^{-k\beta p} \rho^{n-1-\alpha p} \frac{1 - (\frac{\rho}{\lambda})^k}{|k|}^{-\beta p}$$

$$= \lambda^{k\beta p} \rho^{\frac{\alpha p - n}{\alpha p - 1}} \left(\frac{1 - (\frac{\rho}{\lambda})^k}{|k|}\right)^{-\beta p},$$

and then

$$\int_{S_1} \int_0^\sigma H(\rho)d\rho d\omega \leq C_{|k|,\varepsilon} \frac{\beta p - 1}{\alpha p - n} \sup_{\theta \in S_1} \lambda^{k\beta p}(\sigma, \theta) < \infty,$$

because $\lambda$ is a decreasing function on $\rho$ and $k\beta p < 0$.

(c) $k = 0$, i.e., $n = \alpha p$. Then the simple computations taking into account the monotonicity of $\lambda(\rho, \theta)$ and (3.5) give the following chain of inequalities

$$\frac{\lambda(\rho, \theta)}{\rho} \geq \frac{\lambda(\sigma, \theta)}{\rho} > \frac{1}{1 - \varepsilon} > 1,$$
for every $\rho < \sigma$, and

$$0 < \left( \frac{\ln \lambda(\rho, \theta)}{\rho} \right)^{-\beta p} \leq \left( \frac{\ln \lambda(\sigma, \theta)}{\rho} \right)^{-\beta p}.$$ 

Hence from (3.2) and the choice of $H(\rho)$ we get

$$\int_{S_1} \int_0^\sigma H(\rho) d\rho d\theta = \int_{S_1} \int_0^\sigma \left( \frac{\ln \lambda(\sigma, \theta)}{\rho} \right)^{-\beta p} \frac{d\rho}{\rho} d\theta \leq \int_{S_1} \int_0^\sigma \left( \frac{\ln \lambda(\rho, \theta)}{\rho} \right)^{-\beta p} \frac{d\rho}{\rho} d\theta = \frac{1}{\beta p - 1} \int_{S_1} \left( \frac{\ln \lambda(\rho, \theta)}{\rho} \right)^{1-\beta p} d\theta < \infty,$$

since $\beta p > 1$.

The proof of $v(x) \in L^p_{\text{loc}}(\Omega)$ is analogous for $k \geq 0$ because the singularity of $v(x)$ at the origin is weaker than the singularity of $w(x)$.

As for $k < 0$, i.e., $n < \alpha p$, we obtain

$$\int_{B_\sigma} v^p dx = \int_{S_1} \int_0^\sigma H_1(\rho) d\rho d\theta,$$

and

$$H_1(\rho) = \rho^{n-1+(1-\alpha)p} \left( \frac{1 - (\frac{\rho}{\lambda})^k}{k} \right)^{(1-\beta)p} = \rho^{n-1+(1-\alpha)p+k(1-\beta)p} \left( \frac{1 - (\frac{\rho}{\lambda})^k}{k} \right)^{\chi k(\beta-1)} \left( \frac{1 - (\frac{\rho}{\lambda})^k}{|k|} \right)^{(1-\beta)p}.$$

Hence

$$\int_{B_\sigma} v^p dx \leq \frac{\sigma^{n+(1-\alpha)p+k(1-\beta)p}}{n + (1-\alpha)p + k(1-\beta)p} \left( \frac{1 - (\frac{\rho}{\lambda})^k}{k} \right)^{\chi k(\beta-1)} \omega_n < \infty,$$

because $\lambda < \infty$ and

$$\left( \frac{1 - (\frac{\rho}{\lambda})^k}{|k|} \right)^{(1-\beta)p} \leq \left( \frac{1 - (\frac{\rho}{\lambda})^k}{|k|} \right)^{(1-\beta)p} \quad \text{for } \beta > 1,$$

$$\left( \frac{1 - (\frac{\rho}{\lambda})^k}{|k|} \right)^{(1-\beta)p} \leq 1 \quad \text{for } \beta \in \left[ \frac{1}{p}, 1 \right],$$

$$n + (1-\alpha)p + k(1-\beta)p > \frac{p}{\beta p - 1} (\beta p - 1 + n - \alpha p) \geq 0,$$

due to the condition $\frac{n}{p} < \alpha \leq \frac{n + \beta p - 1}{p}$, as well as $(1-\beta)p < p - 1$. \qed
Proof of Theorem 3.1. Without loss of generality we can suppose that \( u \in C_0^\infty(\Omega) \cap W_0^{1,p}(\Omega) \).

First, let us consider the case \( \alpha \neq \frac{n}{p} \), i.e., \( k \neq 0 \). By (3.1) it holds that

\[
\langle x, \nabla g(s(x)) \rangle = -s(x)^{k-1} \left( \frac{|x|}{\lambda(x)} - \frac{|x|}{\lambda(x)} \frac{\langle x, \nabla \lambda(x) \rangle}{\lambda(x)} \right)
\leq -s(x)^k = kg(s(x)) - 1. \tag{3.6}
\]

For the vector function \( f(x) = \frac{x}{|x|^{1+\alpha(p-1)}} |g(s(x))|^{-\beta(p-1)-1} g(s(x)) \), we have

\[
f(x)v(x) = \frac{x}{|x|^\alpha} |g(s(x))|^{-\beta}|g(s(x))|^{-\beta(p-1)} \text{ and } |f(x)|^{p'} = |x|^{-\alpha p} |g(s(x))|^{-\beta p} \equiv w(x)^p. \tag{3.7}
\]

Since \( \beta p > 1 \) by (3.6) we get

\[
\text{div}(f(x)v(x)) = \frac{n - \alpha p}{|x|^{\alpha p}} |g(s(x))|^{-\beta p} g(s(x)) + \frac{1 - \beta p}{|x|^\alpha} |g(s(x))|^{-\beta(p-1)} \langle x, \nabla g(s(x)) \rangle
\geq \frac{1}{|x|^{\alpha p} |g(s(x))|^{-\beta p}}[(n - \alpha p)g(s(x)) + (1 - \beta p)(kg(s(x)) - 1)]
= (\beta p - 1)w^p. \tag{3.8}
\]

The last equality in (3.8) holds because \((n - \alpha p) + k(1 - \beta p) = 0\).

Thus for the functions

\[
f_0(x) = -f(x)v(x), \quad v_0(x) = v^{-p'}(x), \quad w_0(x) = p(\beta - 1)w^p(x),
\]

from (3.7) and (3.8) we obtain

\[
-\text{div}f_0 - (p - 1)v_0|f_0|^{p'} = \text{div}(fv) - (p - 1)|v|^{-p'}|f|^{p'}|v|^{p'}
\geq (\beta p - 1)w^p - (p - 1)w^p = p(\beta - 1)w^p = p(\beta - 1)w_0. \tag{3.9}
\]

Moreover, the following identities hold in \( \Omega_\varepsilon = \Omega_1 \setminus B_\varepsilon, \) \( B_\varepsilon = \{|x| \leq \varepsilon\}, \) \supp \( u \subset \Omega_1 \subset \Omega, \)
\( B_\varepsilon \subset \Omega_1 \)

\[
L(u) = \int_{\Omega_\varepsilon} v_0^{1-p} \left( \frac{\langle f_0, \nabla u \rangle}{|f_0|} \right)^p dx \leq \int_{\Omega_\varepsilon} v^p |\nabla u|^p dx;
\]

\[
K(u) = \int_{\Omega_\varepsilon} v_0 |f_0|^p |u|^p dx = \int_{\Omega_\varepsilon} |f|^p |u|^p dx = \int_{\Omega_\varepsilon} w^p |u|^p dx;
\]

\[
N(u) = \int_{\Omega_\varepsilon} w_0 |u|^p dx = p(\beta - 1) \int_{\Omega_\varepsilon} w^p |u|^p dx;
\]

\[
K_0(u) = \int_{\Gamma_+} \langle f_0, \eta \rangle |u|^p dS;
\]

where \( \eta \) is outward normal vector to \( \partial\Omega_\varepsilon \). Note that

\[
\int_{\partial\Omega_\varepsilon} \langle f_0, \eta \rangle |u|^p dS = 0\), and \( \int_{\partial B_\varepsilon} \langle f_0, \eta \rangle |u|^p dS \geq 0\),

because \( \langle f_0, \eta \rangle = |x|^{-\alpha(p-1)+1}|g(s(x))|^{-\beta(p-1)-1}g(s(x))\eta(x) > 0 \) on \( \partial B_\varepsilon \).

From (3.9), (3.10) and Corollary 2.1, we obtain

\[
\int_{\Omega_\varepsilon} v^p |\nabla u|^p dx \geq L(u) \geq \left( \frac{1}{p} \right)^p \frac{|K_0(u) + (p-1)K(u) + N(u)|^p}{K^{p-1}(u)}
\]

\[
\geq \left( \frac{1}{p} \right)^p \frac{|(p-1)K(u) + N(u)|^p}{K^{p-1}(u)} = \left( \frac{1}{p} \right)^p \frac{(p-1+p(\beta-1)) \int_{\Omega_\varepsilon} w^p |u|^p dx |^p}{(\int_{\Omega_\varepsilon} w^p |u|^p dx)^{p-1}}
\]

\[
\geq \left( \frac{p\beta-1}{p} \right)^p \int_{\Omega_\varepsilon} w^p |u|^p dx.
\]

After the limit \( \varepsilon \to 0 \) we get (3.4) in \( \Omega \).

Second, for the case \( \alpha = \frac{n}{p} \), i.e., \( k = 0 \), it is enough to replace inequality (3.6) with inequality

\[
\langle x, \nabla g(s(x)) \rangle = -\frac{\lambda(x)}{|x|} \left\langle x, \nabla \left( \frac{|x|}{\lambda(x)} \right) \right\rangle
\]

\[
= \frac{\lambda(x)}{|x|} \left( \frac{|x|}{\lambda(x)} \langle x, \nabla \lambda(x) \rangle \right) - \frac{|x|}{\lambda(x)} \leq -1,
\]

from (3.1). The rest of the proof is similar to the case \( \alpha \neq \frac{n}{p} \).

3.1 Optimality of the Hardy constant \( \gamma^p \)

The optimality of \( \gamma^p \) for \( \Omega^* = \Omega = \{ x : |x| < \lambda \}, \lambda = const \), is guaranteed by the following theorem. However, the question whether the inequality (3.4) is sharp in \( W^{1,p}_0(\Omega) \) is still open.

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**Theorem 3.2.** For every $\varepsilon, 0 < \varepsilon < 1$ there exists $u_\varepsilon \in W^{1,p}_0(\Omega)$ such that for $L(u_\varepsilon) = \int_{\Omega} v^p|\nabla u_\varepsilon|^p \, dx$ and $R(u_\varepsilon) = \int_{\Omega} w^p|u_\varepsilon|^p \, dx$, we have

$$\gamma^p \leq \frac{L(u_\varepsilon)}{R(u_\varepsilon)} \leq (1 + \varepsilon)^p \gamma^p. \quad (3.11)$$

In the proof of the optimality of the constant $\gamma^p$ in inequality (3.4) we will choose the function $u_\varepsilon$ so that the ratio of the left-hand side to the right-hand side of (3.4) is in the interval $[\gamma^p, (1 + \varepsilon)^p \gamma^p]$. In addition, one have to show that both sides of (3.4) are finite.

**Proof of Theorem 3.2.** For simplicity we suppose that $\lambda \equiv 1$ so that $\Omega = B_1$. In polar coordinates $(\rho, \theta)$ with the functions $v$ and $w$ defined in (3.3) and for functions $u$ depending only on $\rho$, the inequality (3.4) becomes

$$L(u) = \omega_n \int_0^1 \rho^{n-1-p(\alpha-1)} \left| \frac{1 - \rho^k}{k} \right| |u_\rho|^p \, d\rho$$

$$\geq \gamma^p \omega_n \int_0^1 \rho^{n-1-\beta p} \left| \frac{1 - \rho^k}{k} \right| |u|^p \, d\rho = \gamma^p R(u), \quad (3.12)$$

where $\omega$ is the measure of the unite sphere $S_1$ in $R^n$.

Let us fix $0 < \varepsilon < 1$. In order to prove (3.11), we choose $u_\varepsilon$ for different cases of $k$ as follows:

(a) Let $k > 0$, i.e., $n > \alpha p$. For $u_\varepsilon = (1 - \rho^k)^{\gamma(1+\varepsilon)} \rho^{-\gamma k(1-\varepsilon)}$, we have

$$(u_\varepsilon)_\rho = -\gamma k(1+\varepsilon)(1 - \rho^k)^{\gamma(1+\varepsilon)-1} \rho^{-\gamma k(1-\varepsilon)+k-1}$$

$$-\gamma k(1-\varepsilon)(1 - \rho^k)^{\gamma(1+\varepsilon)} \rho^{-\gamma k(1-\varepsilon)-1}$$

$$= -\gamma k(1 - \rho^k)^{\gamma(1+\varepsilon)-1} \rho^{-\gamma k(1-\varepsilon)-1}[(1+\varepsilon)^{\rho^k} + (1-\varepsilon)(1 - \rho^k)].$$

Now comparing the functions in the left-hand and right-hand sides in (3.12) with $u = u_\varepsilon$ we obtain

$$\frac{v(\rho)(u_\varepsilon)_\rho(\rho)}{w(\rho)u_\varepsilon(\rho)} = k \gamma \left[ \frac{1 - \rho^k}{k} \right] (1 - \rho^k)^{-1} (1 - \rho^k) \rho^{-1} (2 \varepsilon \rho^k + 1 - \varepsilon)$$

$$\leq (1 + \varepsilon) \gamma k \left[ \frac{1 - \rho^k}{k} \right] \left( \frac{1}{\rho(1 - \rho^k)} \right) \leq (1 + \varepsilon) \gamma,$$

and hence

$$\frac{L(u_\varepsilon)}{R(u_\varepsilon)} \leq (1 + \varepsilon)^p \gamma^p. \quad (3.13)$$

Note that

$$R(u_\varepsilon) = \omega_n k^{\beta p} \int_0^1 \rho^{n-\alpha p - 1 - \gamma k p(1-\varepsilon)} (1 - \rho^k)^{\gamma p(1+\varepsilon)} \rho^{-\beta p} \, d\rho < \infty,$$

since $n - \alpha p - 1 - \gamma k p(1-\varepsilon) = \varepsilon(n - \alpha p) - 1 > -1$
and
\[ \gamma p(1 + \varepsilon) - \beta p = \varepsilon(\beta p - 1) - 1 > -1. \]
Inequalities (3.12) and (3.13) give the estimate (3.11) for \( k > 0 \).

(b) Let \( k < 0 \), i.e., \( n < \alpha p \). In this case we define \( u_\varepsilon = (1 - \rho^{-k})^{(1+\varepsilon)} \). Similar calculations as in (a) give us
\[
(u_\varepsilon)_\rho = k\gamma(1 + \varepsilon)\rho^{-k-1}(1 - \rho^{-k})^{(1+\varepsilon)-1},
\]
and
\[
\frac{v(\rho)(u_\varepsilon)_\rho}{w(\rho)u_\varepsilon(\rho)} = (1 + \varepsilon)\gamma k \left[ \frac{1 - \rho^k}{k} \right] [\rho^{-k-1}(1 - \rho^{-k})^{-1}] = (1 + \varepsilon)\gamma.
\]
Hence
\[
\frac{L(u_\varepsilon)}{R(u_\varepsilon)} = (1 + \varepsilon)p\gamma p,
\]
and with (3.4) we obtain inequality (3.11).

It remains to check that \( R(u_\varepsilon) < \infty \). Indeed,
\[
R(u_\varepsilon) = \omega n \int_0^1 \rho^{n-\alpha p-1}(1 - \rho^{-k})^{\gamma p(1+\varepsilon)} \left( \frac{\rho^k - 1}{|k|} \right)^{-\beta p} d\rho
\]
\[= \omega_n |k|^\beta \int_0^1 \rho^{n-\alpha p-\beta pk-1}(1 - \rho^{|k|})^{\gamma p(1+\varepsilon)-\beta pd} d\rho < \infty,
\]
because
\[ n - \alpha p - \beta pk - 1 = \frac{\alpha p - n}{\beta p - 1} - 1 > -1,
\]
and
\[ \gamma p(1 + \varepsilon) - \beta p = \varepsilon(\beta p - 1) - 1 > -1. \]

(c) Let \( k = 0 \), i.e., \( n = \alpha p \). We define for a fixed \( \mu \) the function \( 0 < \mu < 1 \)
\[
u_\varepsilon = \begin{cases} (\ln \frac{1}{\rho})^{(1+\varepsilon)} & \text{for } \mu < \rho < 1, \\ (\ln \frac{1}{\mu})^{(1+\varepsilon)} & \text{for } 0 \leq \rho \leq \mu. \end{cases}
\]
Then
\[
(u_\varepsilon)_\rho = \begin{cases} -\gamma(1 + \varepsilon)\frac{1}{\rho} \left( \ln \frac{1}{\rho} \right)^{(1+\varepsilon)-1} & \text{for } \mu < \rho < 1, \\ 0 & \text{for } 0 \leq \rho \leq \mu, \end{cases}
\]
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and for
\[ v(\rho) = \rho^{1-\alpha} \left( \ln \frac{1}{\rho} \right)^{1-\beta}, \quad w(\rho) = \rho^{-\alpha} \left( \ln \frac{1}{\rho} \right)^{-\beta}, \]
it follows that
\[ \frac{v(\rho)}{w(\rho)} = \begin{cases} 
\gamma(1+\varepsilon) \left[ \rho \ln \frac{1}{\rho} \right] \left[ \frac{1}{\rho} \left( \ln \frac{1}{\rho} \right)^{-1} \right] = \gamma(1+\varepsilon) \text{ for } \mu < \rho < 1, \\
0, \text{ for } 0 \leq \rho \leq \mu.
\end{cases} \]

Hence we get
\[ \frac{L(u_\varepsilon)}{R(u_\varepsilon)} \leq (1 + \varepsilon)^p \gamma^p, \]
and combining with (3.4) we obtain inequality (3.11).

Let us check that \( R(u_\varepsilon) \) is finite. Simple computations give us
\[ R(u_\varepsilon) = \omega_n \left( \ln \frac{1}{\mu} \right) \frac{p \gamma (1+\varepsilon)}{\beta p - 1} \int_0^\mu \left( \ln \frac{1}{\rho} \right)^{1-\beta} d\rho + \omega_n \int_\mu^1 \left( \ln \frac{1}{\rho} \right)^{p \gamma (1+\varepsilon) - \beta p} d\rho \]
\[ = \omega_n \left( \ln \frac{1}{\mu} \right) \frac{p \gamma (1+\varepsilon)}{\beta p - 1} \left[ \frac{1}{\beta p - 1} + \frac{1}{p \gamma (1+\varepsilon) - \beta p + 1} \right] \]
\[ = \omega_n \left( \frac{1}{\beta p - 1} \right) \left( \frac{1 + \varepsilon}{\varepsilon} \right) \left( \ln \frac{1}{\mu} \right)^{p \gamma (1+\varepsilon) - \beta p + 1} < \infty, \]
because
\[ p \gamma (1+\varepsilon) - \beta p + 1 = \varepsilon(\beta p - 1) > 0 \text{ and } 1 - \beta p < 0. \]

\[ \square \]

### 3.2 Examples and comments

The examples below illustrate that Theorem 3.1 provides a new correction term in Hardy inequalities for weights with one type of singularity either at 0 or on \( \partial \Omega \). Let us recall that the classical Hardy inequality for \( p = 2, n = 3 \) does not attain equality on functions from \( H_0^1(B_1) \), \( B_1 \) is the unit ball, it allows the so-called correction term \( A(u) = \int_{B_1} Q(x) u^2 \, dx, \) see (1.14), i.e.,
\[ \int_{B_1} |\nabla u|^2 \, dx - \frac{1}{4} \int_{B_1} \frac{|u|^2}{|x|^2} \, dx \geq \frac{1}{4} \int_{B_1} Q(x) |u|^2 \, dx, \text{ for } u \in H_0^1(B_1). \]

In Adimurthi et al. [3], Alvino et al. [6], Brezis and Vazquez [26], Filippas and Tertikas [41], inequality (3.14) is proved for different radial symmetric weights \( Q(x) = q(r), r = |x| \):

- \( q(r) = \text{const} \) in Brezis and Vazquez [26];
General characteristics of the possible \( Q(x) \) were given in Ghoussoub and Moradifam [46], Theorem 1. It was shown that (3.14) is valid for \( Q(x) \) with decreasing \( q(r) \) if and only if the ordinary differential equation

\[
y''(r) + \frac{y'(r)}{r} + q(r)y(r) = 0, \tag{3.15}
\]

has a positive solution on \((0, 1)\). Note that the results in Brezis and Vazquez [26], Filippas and Tertikas [41], Adimurthi et al. [3], mentioned above, follow from Ghoussoub and Moradifam [46] with a special choice of \( Q(x) \).

**Example 3.1.** Let \( \Omega = B_1 \), \( \lambda(x) = 1 \), \( \alpha = \beta = 1 \), \( n = 3 \), \( p = 2 \). Then \( \Omega^* = B_1 \), \( k = 1 \), \( \gamma = \frac{1}{2} \), \( v = 1 \), \( w = |x|^{-1}(1 - |x|)^{-1} \) and (3.4) takes the form of

\[
\int_{B_1} |\nabla u|^2 dx - \frac{1}{4} \int_{B_1} \frac{|u|^2}{|x|^2(1 - |x|)^2} dx, \quad \text{for } u \in H_0^1(B_1). \tag{3.16}
\]

The weight of the right-hand side of (3.16) has singularities at zero and on \( \partial B_1 \) in contrast to the weight in the papers of Adimurthi et al. [3], Brezis and Vazquez [26], Filippas and Tertikas [41].

Factorize

\[
\frac{1}{|x|^2(1 - |x|)^2} = \frac{1}{|x|^2} + \frac{2 - |x|}{|x|(1 - |x|)^2}
\]

then inequality (3.16) takes the form of (3.14) with a kernel \( Q(x) = \frac{2 - |x|}{|x|(1 - |x|)^2} \), i.e.,

\[
\int_{B_1} |\nabla u|^2 dx - \frac{1}{4} \int_{B_1} \frac{|u|^2}{|x|^2} dx \geq \frac{1}{4} \int_{B_1} \frac{(2 - |x|)|u|^2}{|x|(1 - |x|)^2} dx, \quad \text{for } u \in H_0^1(B_1). \tag{3.17}
\]

Here \( Q(x) = q(r) \), where \( q(r) \) is radially symmetric for \( r = |x| \in (0, 1) \), convex and \( q(r) \to \infty \) for \( r \to 0 \) or \( r \to 1 \). The inequality (3.17) is not included in Ghoussoub and Moradifam [46] since the function \( q(r) \) in (3.15) is not decreasing on \((0, 1)\). Moreover, the general solution of equation (3.15) for \( q(r) = \frac{2 - r}{r(1 - r)^2} \) is a linear combination of Hypergeometric functions and has no positive solution in the whole interval \((0, 1)\), see Gradsteyn and Ryzhik [48].
Let us compare Hardy inequality \( (3.16) \) in Example 3.1 with two known results. In Shen and Chen [85] under a sufficient condition \((1.9)\), Hardy inequality \((1.8)\) is proved, see Sect. 1.1.

Actually the condition \((1.9)\) is sufficient but not necessary for the validity of \((1.8)\). Indeed, in the special case of \( \Omega = B_1 \) and \( n = 3 \) for \( \phi(r) = 1 \), \( h(r) = \left( \frac{r}{1-r} \right)^{1/2} \), inequality \((1.8)\) follows from Example 3.1. However,

\[
r\phi(r)(h^2)'(r) = \frac{r}{(1-r)^2} \neq \text{const},
\]

and \((1.9)\) fails. Moreover, a simple computation shows that for \( \phi(r) \equiv 1 \), the weight \( \left| \frac{h'(r)}{h(r)} \right|^2 \) of the right-hand side of \((1.8)\) cannot be singular both at 0 and at 1 if condition \((1.9)\) is satisfied.

In Brezis and Marcus [25] the following Hardy inequality

\[
\int_{B_1} |\nabla u|^2 dx - \frac{1}{4} \int_{B_1} \frac{|u|^2}{(1 - |x|^2)} dx \geq \frac{1}{4} \int_{B_1} Q(x)|u|^2 dx, \text{ for } u \in H^1_0(B_1), \tag{3.18}
\]

was proved for \( Q(x) = \text{const} \), see \((1.11)\). If we factorize

\[
\frac{1}{|x|^2(1 - |x|^2)^2} = \frac{1}{(1 - |x|)^2} + \frac{1 + |x|}{|x|^2(1 - |x|)},
\]

inequality \((3.16)\) transforms into \((3.18)\) with \( Q(x) = \frac{(1 + |x|)}{|x|^2(1 - |x|)} \).

In all inequalities obtained in Adimurthi et al. [3], Brezis and Vazquez [26], Brezis and Marcus [25], Filippas and Tertikas [41], the constants are optimal. From Theorem 3.2, it follows that for Hardy inequality \((3.4)\) the constant \( \gamma^p \) is optimal as well.

Finally, in Filippas and Tertikas [41], the authors show the validity of \((3.4)\) under the restriction \( 2 = p < n \). In the present section there are no restrictions on \( p \) except \( p > 1 \).

Let us comment the geometry of the domain \( \Omega^* \) in \((3.1)\). It is well known that there are no conditions on \( \Omega \) for Hardy inequality with singularity at \( 0 \in \Omega \). However, when the singularity is on \( \partial \Omega \) then the restrictions about the convexity of the domain or its generalization are always considered. We will give three simple examples for \( \lambda(x) \) and \( \Omega^* \) when condition \((3.1)\) holds.

**Example 3.2.** Let \( \Omega = B_{\lambda_0} = \{ x, |x| < \lambda_0 \}, \lambda_0 = \text{const} > 0 \) and \( \Omega^* = \Omega \). In this case the weights \( v(x), w(x) \) are singular at 0 and on the whole boundary \( \partial \Omega = \partial B_{\lambda_0} \). If for simplicity \( \alpha = \beta = 1, \gamma = \frac{p-1}{p} \) and \( k = \frac{n-p}{p-1} \neq 0 \), i.e., \( 1 < p \neq n \), then Hardy inequality

\((3.4)\) becomes

\[
\int_{B_{\lambda_0}} |\nabla u|^p dx \geq \left( \frac{n-p|\lambda_0^k}{p} \right)^p \int_{B_{\lambda_0}} \frac{|u|^p}{|x|^{p(k - |x|^k)} dx}.
\]

**Example 3.3.** Let \( \Omega \) be a star-shaped domain with respect to an interior ball centered at zero, so that \( \Omega^* = \Omega \). In this case we can choose \( \lambda = \lambda(\theta) \) where \( \theta \) is the angular variable of \( x \) and \( \partial \Omega = \{ x, |x| = \lambda(\theta) \} \). According to Lemma in section 1.1.8 of Maz’ja [76], \( \lambda(\theta) \in C^{0,1}(\Omega) \) and condition \((3.1)\) and respectively Hardy inequality \((3.4)\) holds. Note that in this case the weights \( v(x) \) and \( w(x) \) are singular at zero and on the whole boundary \( \partial \Omega \).

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Example 3.4. Let \( n > p, \alpha = 1, \beta = 1, \gamma = \frac{p-1}{p} \) so that \( k = \frac{n-p}{p-1} > 0 \) and let (3.1) hold for \( \Omega \). For a domain \( D \subset \mathbb{R}^n \) we define

\[
H(D) = \int_D |\nabla u|^p \, dx, \quad H_1(D) = C_1 \int_D \frac{|u|^p}{|x|^p} \, dx,
\]

\[
H_2(D) = C_2 \int_D \frac{|u|^p}{(\lambda(x) - |x|)^p} \, dx.
\]

We will define two domains \( \Omega_1, \Omega_2 \) such that \( \bar{\Omega}_1 \subset \Omega, \Omega_2 = \Omega \setminus \bar{\Omega}_1, 0 \in \Omega_1 \) and we will show that \( H(\Omega) \geq H_1(\Omega_1) + H_2(\Omega_2) \), i.e.,

\[
\int_\Omega |\nabla u|^p \, dx \geq C_1 \int_{\Omega_1} \frac{|u|^p}{|x|^p} \, dx + C_2 \int_{\Omega_2} \frac{|u|^p}{(\lambda(x) - |x|)^p} \, dx, \tag{3.19}
\]

with \( C_1 = \left( \frac{n-p}{p} \right)^p, \quad C_2 = \left( \frac{1}{p'} \right)^p \) for \( u \in W^{1,p}_0(\Omega) \).

In order to prove (3.19) let us mention that \( \left( \frac{|x|}{\lambda(x)} \right)^k < 1 \) due to (3.1) and \( k > 0 \).

Now we consider the function \( h(s) = 1 - s^k - s^{-1} + 1 \), where \( s = \frac{|x|}{\lambda(x)} \). Since \( h'(s) > 0 \), this function is increasing on \((0, 1]\) and its maximum is attained for \( s = 1 \). Hence \( h(s) \leq 0 \) and the following inequality is true

\[
\frac{|x|}{k} \left( 1 - \left( \frac{|x|}{\lambda(x)} \right)^k \right) \leq \lambda(x) - |x|. \tag{3.20}
\]

From \( k > 0 \) the trivial inequality

\[
\frac{|x|}{k} \left( 1 - \left( \frac{|x|}{\lambda(x)} \right)^k \right) \leq \frac{|x|}{k},
\]

holds and combining with (3.20) we get

\[
\frac{|x|}{k} \left( 1 - \left( \frac{|x|}{\lambda(x)} \right)^k \right) \leq \min \left( \frac{|x|}{k}, \lambda(x) - |x| \right), \quad x \in \Omega. \tag{3.21}
\]

For

\[
v(x) = 1, \quad w(x) = |x|^{-1} \left( 1 - \left( \frac{|x|}{\lambda(x)} \right)^k \right)^{-1},
\]

applying (3.21) in \( \Omega_1 = \{ x \in \Omega : \min \left( \frac{|x|}{k}, \lambda(x) - |x| \right) = \frac{|x|}{k} \}, 0 \in \Omega_1 \) and \( \Omega_2 = \Omega \setminus \bar{\Omega}_1 \) correspondingly, we get from (3.4) the chain of inequalities

\[
H(\Omega) = \int_\Omega |\nabla u|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \int_\Omega \frac{|u|^p}{|x|^p} \left( 1 - \left( \frac{|x|}{\lambda(x)} \right)^k \right)^p \, dx
\]

\[
\geq \left( \frac{p-1}{p} \right)^p (k)^p \int_{\Omega_1} \frac{|u|^p}{|x|^p} \, dx + \left( \frac{1}{p'} \right)^p \int_{\Omega_2} \frac{|u|^p}{(\lambda(x) - |x|)^p} \, dx
\]

\[
= H_1(\Omega_1) + H_2(\Omega_2).
\]
Note that in (3.19) the constants $C_1$ and $C_2$ are optimal for the corresponding classical cases with single singular weights. However, (3.19) cannot be obtained by summing the classical Hardy inequalities $H(\Omega) \geq H_1(\Omega_1)$ and $H(\Omega) \geq H_2(\Omega_2)$ because they are valid only for $u \in W^{1,p}_0(\Omega_1) \cap W^{1,p}_0(\Omega_2)$.

Let us recall that $0 \in \Omega_1$ in Example 3.4 and this is essential for the optimality of the constant $C_1$ in (3.19). Remark that in the case when $0 \in \partial \Omega_1$ there exists a constant $C'_1 > C_1$, see Nazarov [78], Pinchover and Tintarev [83].

### 4 Sharp Hardy inequalities with weights singular at an interior point

In this section we prove Hardy inequality with weight singular at $0 \in \Omega \subset \mathbb{R}^n$, $n \geq 2$ in the class of functions which are not zero on the boundary $\partial \Omega$. Hardy’s constant is optimal and the inequality is sharp due to the additional boundary term. The section is based on Fabricant et al. [35].

In order to formulate our main results we recall the definition of the trace operator, see Adams [1], Evans [32], Ch. 5.5 and Maz’ja [76], Ch. 1.4.5.  

**Definition 4.1.** For a bounded $C^1$ smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and $p > 1$ the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial \Omega)$ is a bounded linear operator, $Tu = u|_{\partial \Omega}$ for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $\|Tu\|_{L^p(\partial \Omega)} \leq C(p,\Omega)\|u\|_{W^{1,p}(\Omega)}$ for $u \in W^{1,p}(\Omega)$.

Let us consider the following inequality

$$
\int_{\Omega} |x|^l \frac{\langle x, \nabla u(x) \rangle}{|x|^n} |x|^{p-l-n} dx \geq |x|^l \int_{\Omega} |\nabla u(x)|^p dx, \quad u \in W^{1,p}_{l,0}(\Omega),
$$

(4.1)

for the constant $l \neq p-n$, where $p > 1$, $n \geq 2$ and $C^1$ is smooth, bounded domain $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$. Here $W^{1,p}_{l,0}(\Omega)$ is the completion of $C^\infty_0(\Omega)$ functions with respect to the norm

$$
\left( \int_{\Omega} |x|^l |\nabla u(x)|^p dx \right)^{1/p} < \infty,
$$

(4.2)

satisfying the condition

$$
\lim_{\varepsilon \to 0} \varepsilon^{l-p+1} \int_{S_\varepsilon} |Tu|^p dS = 0, \quad S_\varepsilon = \{ x \in \Omega ; |x| = \varepsilon \}.
$$

(4.3)

The constant $|x|^l \int_{\Omega} |\nabla u(x)|^p dx$ in (4.1) is optimal but inequality (4.1) is not sharp in $W^{1,p}_{l,0}(\Omega)$, see Hardy et al. [54] for $n = 1$. That is why we introduce a more general class of functions $W^{1,p}_l(\Omega)$, without any restrictions of $u$ on $\partial \Omega$, and define an additional term depending on the trace of $u$ on $\partial \Omega$.

We denote $\partial \Omega_- = \{ x \in \partial \Omega : \text{sgn}(p-l-n)\langle x, \eta \rangle < 0 \}$, where $\eta$ is the unit outward normal vector to $\partial \Omega$, and consider the norm

$$
\left( \int_{\Omega} |x|^l |\nabla u(x)|^p dx \right)^{1/p} + \left( \frac{p-1}{p} \right) \int_{\partial \Omega_-} |\langle x, \eta \rangle||x|^{l-p}u(x)|^p dS < \infty,
$$

(4.4)
see Maz’ja [76], Ch. 1.1.15 in the case $|\partial \Omega_-| \neq 0$ and Ch. 1.1.16 in the case $|\partial \Omega_-| = 0$.

Let us mention that $|\partial \Omega_-| = 0$ if and only if $p > l + n$ and $\Omega$ is a star-shaped domain with respect to the origin, according to Definition 4.2 below.

We define $\hat{W}^{1,p}_l(\Omega)$ as the completion of $C^\infty(\Omega) \cap C(\bar{\Omega})$ functions in the norm (4.4) which satisfy (4.3). Note that for $p - l - n < 0$ condition (4.3) is fulfilled for every $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$, while for $p - l - n > 0$, condition (4.3) requires $u(0) = 0$. Actually, $\hat{W}^{1,p}_l(\Omega)$ for $p - l - n > 0$ is the completion of $C^\infty(\Omega) \cap C(\bar{\Omega})$ functions in the norm (4.4) which are equal to zero near the origin (see Remark 4.1 below).

**Theorem 4.1.** Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^1$ smooth boundary and $0 \in \Omega$. Then for every constant $l \neq p - n$, $p > 1$, $n \geq 2$ and for every $u \in \hat{W}^{1,p}_l(\Omega)$ the following inequality holds

\[
\left( \int_\Omega |x|^l \left| \frac{\langle x, \nabla u(x) \rangle}{|x|} \right|^p dx \right)^{1/p} \geq \left| \frac{p - l - n}{p} \right| \left( \int_\Omega |x|^{l-p} |u(x)|^p dx \right)^{1/p}
\]

\[+ \frac{1}{p} \text{sgn}(p - l - n) \int_{\partial \Omega} |x|^{l-p} \langle x, \eta \rangle |Tu|^p dS \left( \int_\Omega |x|^{l-p} |u(x)|^p dx \right)^{-1/p'},
\]

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\eta$ is the unit outward normal vector to $\partial \Omega$.

**Remark 4.1.** The standard definition of the space of functions in (4.5) for $p - l - n > 0$ is the completion of $C^\infty(\Omega) \cap C(\bar{\Omega})$ functions, with respect to the norm (4.4), which are zero near the origin. However, applying Hardy inequality (4.5) to the ball $B_\varepsilon = \{ x \in \Omega : \ |x| < \varepsilon \}$, after the limit $\varepsilon \to 0$, we get from $\int_{B_\varepsilon} |x|^l |\nabla u|^p dx \to 0$ that

\[
\int_{B_\varepsilon} |x|^{l-p} |u|^p dx \to 0.
\]

Hence from (4.5) and (4.6) it follows that

\[
\int_{S_\varepsilon} |x|^{l-p} \langle x, \eta \rangle |u|^p dS = \varepsilon^{l+p+1} \int_{S_\varepsilon} |u|^p dS \to 0,
\]

i.e., (4.3) is satisfied. In this way we get the same space $\hat{W}^{1,p}_l(\Omega)$.

**Proof of Theorem 4.1.** In order to prove Theorem 4.1, let us introduce the notations

\[
f(x) = \text{sgn}(p - l - n) |x|^{l-p} x, \quad v(x) = \frac{|p - l - n|}{p - 1} |x|^{l/(1-p)}
\]

and

\[
L(u) = \int_\Omega u^{l-1} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx = \left( \frac{|p - l - n|}{p - 1} \right)^{l-1} \int_\Omega |x|^l \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx;
\]

\[
K(u) = \int_\Omega |f|^p |u|^p dx = \frac{|p - l - n|}{p - 1} \int_\Omega |x|^{l-p} |u|^p dx;
\]

\[
K_3(u) = \int_{\partial \Omega} \langle f, \eta \rangle |Tu|^p dS = \text{sgn}(p - l - n) \int_{\partial \Omega} |x|^{l-p} \langle x, \eta \rangle |Tu|^p dS.
\]
Simple computations show that $f$ and $v$ satisfy the equality
\[- \text{div } f - (p - 1)v|f|^{p'} = 0, \quad \text{in } \Omega \setminus \{0\},\] (4.9)
because \[-(p - 1)v|f|^{p'} = -|p - l - n||x|^{l-p}\]
and
\[-\text{div } f = \text{sgn}(p - l - n) \left[ n|x|^{l-p} + (l - p)|x|^{l-p} \right] = |p - l - n||x|^{l-p}.\]
Without loss of generality we will prove (4.5) for every $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ satisfying (4.3), (4.4).

For every small positive constant $\varepsilon$ we apply Corollary 2.1 in $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$, $B_\varepsilon = \{|x| \leq \varepsilon\}$, for $w(x) \equiv 0$.

After the limit $\varepsilon \to 0$ in (2.7), since $N(u) = 0$, in notations (4.8) we obtain
\[L(u) \geq \left( \frac{1}{p} \right)^p \frac{|K_3(u) + (p - 1)K(u)|^p}{K^{p-1}(u)}.\]

Now using the choice of $f$ in (4.7) we obtain (4.5) for $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$.

By standard approximation argument (see Maz’ja [76], Ch. 1.1.15 and Ch. 1.1.6), we get (4.5) for every $u \in \hat{W}^{1,p}_l(\Omega)$.

\[\text{Remark 4.2.}\] By means of the simple inequality $|1 + z|^p \geq 1 + pz$ for every $z$ and $p > 1$, we get
\[L(u) \geq \left( \frac{p - 1}{p} \right)^p K(u) \left| 1 + \frac{1}{p - 1}K_3(u)K^{-1} \right|^p \geq \left( \frac{p - 1}{p} \right)^p K(u) + \left( \frac{p - 1}{p} \right)^{p-1}(K_{3,+}(u) - K_{3,-}(u)),\]
where
\[K_{3,+} = \int_{\partial\Omega \setminus \partial\Omega_-} |x|^{l-p}|\langle x, \eta \rangle||Tu|^pdS, \quad K_{3,-} = \int_{\partial\Omega_-} |x|^{l-p}|\langle x, \eta \rangle||Tu|^pdS.\]

So, for $u \in \hat{W}^{1,p}_l(\Omega)$ we obtain the ‘linear’ form of Hardy inequality (4.5)
\[L(u) + \left( \frac{p - 1}{p} \right)^{p-1}K_{3,-}(u) \geq \left( \frac{p - 1}{p} \right)^p K(u) + \left( \frac{p - 1}{p} \right)^{p-1}K_{3,+}(u).\]

**Theorem 4.2.** Under the assumptions of Theorem 4.1 inequality (4.5) is an equality for
\[u_k = |x|^k \Phi \left( \frac{x}{|x|} \right)\] (4.10)
for every smooth function $\Phi$ and every constant $k > \frac{p - l - n}{p}$, such that $u_k \in \hat{W}^{1,p}_l(\Omega)$, i.e., (4.5) is sharp in $\hat{W}^{1,p}_l(\Omega)$. 51
Proof. From Theorem 2.2, it follows that (4.9) is an equality if (2.14)–(2.16) hold, i.e., if (4.9)
and
\[ u(f, \nabla u) = |u(f, \nabla u)|, \quad (4.11) \]
are fulfilled for a.e. \( x \in \Omega \) and for some constant \( k_1 \geq 0 \). From the choice of \( f \) and \( v \) in (4.7), equation (4.11) is equivalent to
\[ \text{sgn}(p - l - n)u(x, \nabla u) = |u(x, \nabla u)|, \quad \langle x, \nabla u \rangle = ku, \quad (4.12) \]
where \( k = k_1 \rho - l - n \rho - 1 \). We are looking for solution of the second equation in (4.12) in the form \( u = |x|^k z(x) \in \dot{W}_1^{1,p}(\Omega) \). Simple computations give us that \( z(x) \) is a solution of the homogeneous, first order partial differential equation
\[ \sum_{i=1}^{n} x_i z_{x_i} = 0 \text{ in } \Omega \setminus \{0\}. \quad (4.13) \]
The system of characteristic equations of (4.13) becomes
\[ \dot{x}_1(t) = x_1, \ldots, \dot{x}_n(t) = x_n, \quad (4.14) \]
and the functions \( \varphi_1(x) = \frac{x_1}{|x|}, \ldots, \varphi_n(x) = \frac{x_n}{|x|} \) are constants along the trajectories of (4.14), i.e., \( \varphi_1, \ldots, \varphi_n \) are the first integrals of (4.13). Note that only \( n - 1 \) of them are independent first integrals, so that the general solution of (4.13) is given by
\[ z(x) = \Phi \left( \frac{x}{|x|} \right). \]
Hence function \( u_k(x) = |x|^k \Phi \left( \frac{x}{|x|} \right) \) is a general solution of (4.12) for arbitrary smooth function \( \Phi \) and constant \( k \), such that \( |x|^k \Phi \in \dot{W}_1^{1,p}(\Omega) \), i.e., \( u_k \) satisfies (4.2) and (4.3).

Let us check up when condition (4.3) holds. With the change of the variables \( x = \varepsilon y \) we get
\[ \varepsilon^{l-p+1} \int_{S_\varepsilon} |u_k|^p dS = \varepsilon^{l-p+1} \int_{S_\varepsilon} |x|^kp \left| \Phi \left( \frac{x}{|x|} \right) \right|^p dS \]
\[ = \varepsilon^{l-p+1+kp+n-1} \int_{|y|=1} |\Phi(y)|^p dS. \]
Therefore, (4.3) is satisfied if and only if \( k > \frac{p - l - n}{p} \) and \( \int_{|y|=1} |\Phi(y)|^p dS < \infty \).

We will prove that both sides of (4.5) are finite for all functions \( u_k \) defined in (4.10). For this purpose it is enough to check that \( K(u_k) < \infty \). In fact, for some fixed small constant, \( a \in (0, 1) \) and for the ball \( B_a = \{|x| < a\} \), we get the chain of inequalities
\[ \int_{\Omega} |x|^{l-p+kp} \Phi \left( \frac{x}{|x|} \right) dS \leq C_1 \int_{B_a} |x|^l dS + C_2 < \infty, \]
where \( C_1 \) and \( C_2 \) are positive constants.
for \( \lambda = l - p + kp > -n \) and some constants \( 0 < C_1, C_2 < \infty \). The above inequality follows from \( k > \frac{p-l-n}{p} \).

In the following remarks we will compare our result in Theorem 4.1, i.e., inequality (4.5) with the corresponding results about Hardy inequalities with additional boundary term in Wang and Zhu [91], Barbatis et al. [15], Berchio et al. [20].

**Remark 4.3.** Consider the special case of the constants : \( p = 2, \ l = -2a, \ a > 0, \ n - 2a - 2 > 0 \) and \( \Omega \) is the unit ball \( B_1 \subset R^n \). Since \( \frac{|\langle x, \nabla u \rangle|}{|x|} < |\nabla u|^p \), from (4.5), rising both sides of this inequality to second power and since \( \eta = x \) on \( \partial B_1 \) we get

\[
\int_{B_1} |x|^{-2a} |\nabla u|^2 \, dx \geq \left( \frac{n-2-2a}{2} \right)^2 \left( \int_{B_1} |x|^{-2a-2} |u|^2 \, dx \right), \quad u \in \hat{W}^{1,2}_{-2a}(B_1).
\] (4.15)

In Wang and Zhu [91], the following Hardy inequality with additional boundary term was proved

\[
\int_{B_1} |x|^{-2a} |\nabla u(x)|^2 \, dx > \left( \frac{n-2-2a}{2} \right)^2 \int_{B_1} |x|^{-2a-2} |u(x)|^2 \, dx
\]

\[
- \frac{n-2-2a}{2} \int_{\partial B_1} |Tu|^2 \, dS, \quad u \in \hat{W}^{1,2}_{-2a}(B_1).
\] (4.16)

The constant \( \left( \frac{n-2-2a}{2} \right)^2 \) in both inequalities (4.15) and (4.16) is optimal. The difference between (4.16) and (4.15) is the additional positive term. Moreover, (4.16) is not sharp, i.e., it is a strict inequality, while the additional term in (4.15) guarantees its sharpness, i.e., there exists a class of functions \( \hat{W}^{1,2}_{-2a}(B_1) \) for which (4.15) is an equality.

**Remark 4.4.** In Barbatis et al. [15] new Hardy inequalities in bounded domains \( \Omega \) for functions \( H^1(\Omega) \), see eq. (2.4) in Barbatis et al. [15] are obtained

\[
\int_{\Omega} |\nabla u|^2 \, dx + c \int_{\partial \Omega} |x|^{-2} \langle x, \eta \rangle |Tu|^2 \, dS \geq c(n-2-c) \int_{\Omega} |x|^{-2} |u|^2 \, dx,
\] (4.17)

for \( u \in H^1(\Omega) \), where \( 0 < c \leq \frac{n-2}{2} \).

The inequality (4.5) for the case \( p = 2, l = 0, n > p \) with a similar transformation as
Remark 4.3 reads
\[
\int_{\Omega} |\nabla u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} |x|^{-2} |u(x)|^2 dx
\]
\[ - \frac{n-2}{2} \int_{\partial\Omega} |x|^{-2}(x,\eta)|Tu|^2 dS \]
\[ + \frac{1}{4} \left( \int_{\partial\Omega} |x|^{-2}(x,\eta)|Tu|^2 dS \right)^2 \left( \int_{\Omega} |x|^{-2} |u(x)|^2 dx \right)^{-1}, \quad u \in H^1(\Omega). \tag{4.18}
\]

The comparison of (4.18) and (4.17) for the optimal constant \(c = \frac{n-2}{2}\) shows that in (4.18) there exists an additional positive term on the right-hand side. Moreover, with this additional term inequality (4.18) is sharp, i.e., it is an equality for the class of functions \(u = u_k \in H^1(\Omega)\) defined in (4.10).

Remark 4.5. In Berchio et al. [20], see (13), the following Hardy inequality is studied
\[
\int_{\Omega} |\nabla u|^2 dx + c \int_{\partial\Omega} |Tu|^2 dS \geq h(c) \int_{\Omega} |x|^{-2} |u|^2 dx, \quad u \in H^1(\Omega), \tag{4.19}
\]
where \(c \in [0, C_n], C_n \geq \frac{n-2}{2}, (C_n = \frac{n-2}{2} \text{ for } \Omega = B_1)\) and \(h(c) \in \left[0, \left(\frac{n-2}{2}\right)^2\right]\), are defined as
\[
h(c) = \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx + c \int_{\partial\Omega} |Tu|^2 dS}{\int_{\Omega} |x|^{-2} |u|^2 dx}. \tag{4.20}
\]

By means of positive solutions of the eigenvalue problem under Steklov boundary conditions
\[
- \Delta u = h(c) \frac{|u|^2}{|x|^2}, \quad \text{in } \Omega
\]
\[
u x + cu = 0, \quad \text{on } \partial\Omega,
\]
see (15) in Berchio et al. [20], it is shown (see Theorem 8 in Berchio et al. [20]) that for the value of \(c = \frac{n-2}{2}\) the infimum in (4.20), i.e.,
\[
h\left(\frac{n-2}{2}\right) = \left(\frac{n-2}{2}\right)^2
\]
is not achieved. This means that for the optimal constant \(\left(\frac{n-2}{2}\right)^2\) Hardy inequality (4.19) is not sharp.

The inequality (4.5) for the case \(p = 2, l = 0, n > p\) and \(\Omega = B_1(0)\) becomes (4.15) in Remark 4.3 for \(a = 0\) and in comparison with (4.19) has an additional positive term in the
right-hand side. Moreover, the inequality (4.15) is an equality for the functions defined in (4.10).

As a consequence of Theorem 4.1 we get an extension of the classical Hardy inequality (4.1) for functions \( u \) in the largest class \( \hat{W}^1_{1,p}(\Omega) \), i.e., when \( u \) is not necessary zero on the whole boundary \( \partial \Omega \).

### 4.1 Star-shaped domains

We consider the case of domains \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), \( 0 \in \Omega \) which are star-shaped with respect to the origin. Let us recall the definitions of a star-shaped and strictly star-shaped \( C^1 \) smooth domains.

**Definition 4.2.** The domain \( \Omega, \partial \Omega \in C^1 \) is:

1. **star-shaped domains with respect to the origin** if
   
   \[
   \langle x, \eta(x) \rangle \geq 0, \quad \text{for every } x \in \partial \Omega, \tag{4.21}
   \]

   where \( \eta(x) \) is the unit outward normal vector to \( \partial \Omega \) at the point \( x \in \partial \Omega \).

2. **strictly star-shaped domains with respect to the origin** if inequality (4.21) is strict, i.e.,

   \[
   \langle x, \eta(x) \rangle > 0, \quad \text{for every } x \in \partial \Omega, \tag{4.22}
   \]

Let us note that for star-shaped domains the sign of the additional term in (4.5) depends only on the sign of the constant \( p - l - n \).

**Theorem 4.3.** Suppose \( \Omega \) is a bounded, star-shaped domain with respect to the origin in \( \mathbb{R}^n \), \( n \geq 2 \) with \( C^1 \) smooth boundary \( \partial \Omega \) and \( 0 \in \Omega \). Then for every \( p > 1 \) we have:

1. If \( p - l - n > 0 \), inequality (4.1) is satisfied for every \( u(x) \in \hat{W}^1_{1,p}(\Omega) \) and the constant \( \frac{p - l - n}{p} \) in (4.1) is optimal.
2. If additionally \( \Omega \) is a strictly star-shaped domain with respect to the origin, then (4.1) is not sharp in \( \hat{W}^1_{1,p}(\Omega) \).

3. If \( p - l - n < 0 \), then (4.1) in general does not hold, for example, for functions \( u_k \in \hat{W}^1_{1,p}(\Omega) \) defined in (4.10).

**Proof.** (i) From (4.21) inequality (4.1) holds from (4.5).

It is easy to prove that when \( p - l - n > 0 \) the constant \( \frac{p - l - n}{p} \) is optimal in (4.1) and (4.5). For this purpose, we will use function \( u_k \), defined in (4.10) with \( k > \frac{p - l - n}{p} \) and \( \Phi \equiv 1 \). Since (4.5) is an equality for every \( u_k \) we get

\[
\left( \frac{L(|x|^k)}{K(|x|^k)} \right)^{1/p} = \frac{p - l - n}{p} + \frac{1}{p} \int_{\partial \Omega} |x|^{l-p+k} < x, \eta > dS
\]

\[
\times \left( \int_{\Omega} |x|^{-p+k} dx \right)^{-1} \to \frac{p - l - n}{p} \quad \text{for } k \to \frac{p - l - n}{p} + 0.
\]
The above limit follows from the inequalities
\[ \int_{\Omega} |x|^{l-p+kp} dx \geq \int_{B_a} |x|^{l-p+kp} dx = \omega_n \int_{0}^{a} r^{l-p+kp+n-1} dr \]
\[ = \omega_n \frac{a^{l-p+kp+n}}{l-p+kp+n} \rightarrow +\infty, \quad \text{for } k \rightarrow \frac{p-l-n}{p} + 0, \]
and
\[ \frac{1}{p} \int_{\partial \Omega} |x|^{l-p+kp}(x, \eta) dS \rightarrow \frac{1}{p} \int_{\partial \Omega} |x|^{-n}(x, \eta) dS < \infty \]
for \( k \rightarrow \frac{p-l-n}{p} + 0, \)
where \( \omega_n \) is the measure of the unit sphere in \( \mathbb{R}^n \) and \( B_a \subset \Omega, \partial B_a \cap \partial \Omega = \emptyset. \)

If we suppose that (4.1) is sharp for some function \( w(x) \in W^{1,p}_{l}(\Omega) \) in a strictly star-shaped domain \( \Omega \) then from (4.1) and (4.5) we have
\[ \int_{\partial \Omega} |x|^{l-p}(x, \eta)|Tw|^p dS = 0. \]

Hence due to (4.22) it follows that
\[ Tw = 0 \text{ for a.e. } x \in \partial \Omega. \]

This means that (4.1) is also sharp in \( W^{1,p}_{l,0}(\Omega) \) which proves Theorem 4.3 (i).

(ii) If \( p-l-n < 0 \), then for \( u_k(x) = |x|^k \) we get from Theorem 4.2 and (4.21)
\[ \int_{\Omega} |x|^l \left( \frac{\langle x, \nabla u_k(x) \rangle}{|x|} \right)^p dx = -\frac{1}{p} \int_{\partial \Omega} |x|^{l-p}(x, \eta)|u_k(x)|^p dS \]
\[ \times \left( \int_{\Omega} |x|^{l-p}|u_k(x)|^p dx \right)^{-1/p} + \frac{|p-l-n|}{p} \left( \int_{\Omega} |x|^{l-p}|u_k(x)|^p dx \right)^{1/p} \]
\[ < \frac{|p-l-n|}{p} \left( \int_{\Omega} |x|^{l-p}|u_k(x)|^p dx \right)^{1/p}. \]

Hence (4.1) is not satisfied for \( u = u_k(x) = |x|^k \in W^{1,p}_{l}(\Omega) \) and \( k > \frac{p-l-n}{p} \) which proves Theorem 4.3 (ii).

4.2 General domains

In order to prove (4.1) without geometry conditions (4.21) or (4.22) as in Sect. 4.1 we specify the class of functions. Let us introduce the spaces \( W^{1,p}_{l,0}(\Omega) \), resp. \( W^{1,p}_{l,-}(\Omega) \), which are the completion of \( C^\infty(\Omega) \cap C(\Omega) \) functions with respect to the norm (4.4), satisfying in addition (4.3) and (4.23), resp. (4.3) and (4.24):
\[ \int_{\partial \Omega} |x|^{l-p}(x, \eta)|Tu|^p dS \geq 0, \quad (4.23) \]
Lemma 4.1. \(\int_{\partial \Omega} |x|^{1-p} (x, \eta) |Tu|^p dS \leq 0.\) (4.24)

Note that obviously the following inclusions hold:

\[ W_{l,0}^{1,p}(\Omega) \subset W_{l,1}^{1,p}(\Omega) \subset \hat{W}_{l}^{1,p}(\Omega); \quad W_{l,+}^{1,p}(\Omega) \cup W_{l,-}^{1,p}(\Omega) = \hat{W}_{l}^{1,p}(\Omega). \]

By means of conditions (4.23) or (4.24) one can control the sign of the additional term in inequality (4.5), and, we have the following result for general domains.

**Theorem 4.4.** Suppose \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(n \geq 2\) with \(C^1\) smooth boundary, \(0 \in \Omega\) and \(p > 1\). Then:

(i) Inequality (4.1) holds for every \(u \in W_{l,0}^{1,p}(\Omega)\);

(ii) If \(p - l - n < 0\), then inequality (4.1) holds for all functions \(u \in W_{l,-}^{1,p}(\Omega)\);

(iii) If \(p - l - n > 0\), then inequality (4.1) holds for all functions \(u \in W_{l,+}^{1,p}(\Omega)\). The constant \(\frac{p - l - n}{p}\) is optimal but inequality (4.1) is not a sharp one in \(W_{l,+}^{1,p}(\Omega)\). However, an inequality with additional term (4.5) is sharp in \(W_{l,+}^{1,p}(\Omega)\).

In order to prove Theorem 4.4 we need the following Lemma:

**Lemma 4.1.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\), \(n \geq 2\) with \(C^1\) smooth boundary \(\partial \Omega\), \(0 \in \Omega\) and \(p > 1\). Then identity

\[
\int_{\partial \Omega} |x|^{1-p+kp}(x, \eta) \left| T \Phi \left( \frac{x}{|x|} \right) \right|^p dS
= (l - p + kp + n) \int_{\Omega} |x|^{l-p+kp} \left| \Phi \left( \frac{x}{|x|} \right) \right|^p dx > 0
\]

(4.25)

holds for every \(k > \frac{p - l - n}{p}\) and every nontrivial function \(|x|^k \Phi \left( \frac{x}{|x|} \right) \in \hat{W}_{l}^{1,p}(\Omega)\).

**Proof.** If \(\varepsilon > 0\) is a sufficiently small constant such that \(B_{\varepsilon} \subset \Omega\), where \(B_{\varepsilon} = \{ x : |x| < \varepsilon \}\) then

\[
\text{div} \left( |x|^{l-p+kp} \Phi \left( \frac{x}{|x|} \right) \right)
= (l - p + kp + n) |x|^{l-p+kp} \left| \Phi \left( \frac{x}{|x|} \right) \right|^p + |x|^{l-p+kp} |x| \left| \nabla \Phi \right|^p
\]

\[
= (l - p + kp + n) |x|^{l-p+kp} \left| \Phi \left( \frac{x}{|x|} \right) \right|^p, \quad \text{for a.e. } x \in \Omega \setminus B_{\varepsilon}.
\]

With integration by parts of the above equality in \(\Omega \setminus B_{\varepsilon}\), equality (4.25) follows after the limit \(\varepsilon \to 0\) from (4.3) because

\[
\int_{S_{\varepsilon}} |x|^{l-p+kp}(x, \eta) \left| T \Phi \left( \frac{x}{|x|} \right) \right|^p ds
= -\varepsilon^{l-p+kp+1} \int_{S_{\varepsilon}} T \Phi \left( \frac{x}{|x|} \right) \left| \frac{x}{|x|} \right|^p ds \to 0, \quad \text{as } \varepsilon \to 0.
\]

\[\square\]
Proof of Theorem 4.4. (i) The inequality (4.1) is a direct consequence of (4.5). The optimality of the constant $\frac{|p-l-n|}{p}$ for (4.1) follows in the same way as in Theorem 4.3 (i) for (4.5) with $p-l-n>0$.

(ii) For $p-l-n<0$ inequality (4.1) follows from (4.5) and (4.24). If (4.1) is sharp for some function $z(x) \in W_{l}^{1,p}(\Omega)$, then from (4.1) and (4.5) we get

$$G(z) = \int_{\partial\Omega} |x|^{1-p}(x,\eta)|Tz|^p dS \geq 0.$$  \hspace{1cm} (4.26)

Since $z(x) \in W_{l}^{1,p}(\Omega)$ from (4.24) it follows that $G(z) = 0$ and (4.1) is sharp in $\hat{W}_{l,0}^{1,p}(\Omega)$ which is impossible.

(iii) For $p-l-n>0$ inequality (4.1) follows from (4.5) and (4.23). From Theorem 4.2 for functions $u_k(x) = |x|^k$, $k > \frac{p-l-n}{p}$ inequality (4.5) becomes an equality. Since $u_k(x) \in \hat{W}_{l}^{1,p}(\Omega)$ it is enough to show that $u_k(x)$ satisfies (4.23), i.e., $u_k(x) \in W_{l+}^{1,p}(\Omega)$. This follows from Lemma 4.1 for $\Phi \equiv 1$.

If we suppose that (4.1) is sharp for some function $w(x) \in W_{l+}^{1,p}(\Omega)$, then from (4.1) and (4.5) we have $G(w) \leq 0$ where the function $G$ is defined in (4.26). Since $w(x) \in W_{l+}^{1,p}(\Omega)$, i.e., $G(w) \geq 0$, then from (4.23) it follows that $G(w) = 0$. This means that (4.5) is sharp for the function $w(x) \in W_{l,0}^{1,p}(\Omega)$ which is impossible.

The optimality of the constant $\frac{p-l-n}{p}$ follows in the same way as in the proof of Theorem 4.3 (i). \hfill $\square$

5 Sharp Hardy inequalities in star-shaped domains with double singular weights

In the present section we prove Hardy inequalities with double singular weights in bounded, star-shaped domains $\Omega \subset R^n$, $n \geq 2$. The weights are singular at an interior point and on the boundary of the domain. Hardy’s constant is optimal and the inequality is sharp due to the additional term, i.e., there exists a non-trivial function for which the inequality becomes equality, see Definition 1.1.

In section 4.1, star-shaped domain and a strictly star-shaped domain with respect to $0 \in \Omega$ are defined, where $\partial\Omega \in C^1$, see Definition 4.1. Here we use more general Definitions 5.1 and 5.2, when $\partial\Omega \in C^0$.

**Definition 5.1.** The bounded domain $\Omega \subset R^n$, $n \geq 2$ with $C^0$ boundary $\partial\Omega$ is star-shaped domain with respect to a point $x_0 \in \Omega$ if every ray starting from $x_0$ intersects the boundary $\partial\Omega$ only at one point.

**Definition 5.2.** The bounded domain $\Omega$, where $\partial\Omega \in C^0$ is a star-shaped with respect to an interior ball $B_\varepsilon = \{|x| < \varepsilon\} \subset \Omega$ if $\Omega$ is star-shaped with respect to every point of the ball $B_\varepsilon$, see Definition 5.1 and Ch. 1.1.6 in Maz’ja [76].

Let $\Omega = \{|x| < \varphi(x)\} \subset R^n$ be a star-shaped domain with respect to a small ball. Here $0 \in \Omega$, $n \geq 2$, $p > 1$, and $\varphi$ is a homogeneous function of the 0-th order. Note that, according to Ch. 1.1.8 in Maz’ja [76], in this case $\varphi(x)$ is Lipschitz function on the unit sphere $S_1$ in $R^n$.
We denote by $W^{1,p}_0(|x|^{l(1-p)},\Omega)$, $l \leq \frac{n-1}{p-1}, l \in \mathbb{R}$, the completion of $C^\infty_0(\Omega)$ functions with respect to the norm

$$
\|u\|_{W^{1,p}_0(|x|^{l(1-p)},\Omega)} = \left( \int_{\Omega} |x|^{l(1-p)}|\nabla u|^p dx \right)^{\frac{1}{p}} < \infty, \quad (5.1)
$$

(see Maz‘ja [76], Ch.1.1.6).

In Theorem 3.1 the following Hardy inequality with double singular weights (3.4) is proved

$$
\int_{\Omega} |x|^{l(1-p)} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx \geq \frac{|m+l|^p}{p'} \int_{\Omega} |u|^p \left| |x|^{n-m-l} \right| 1 - \left( \frac{|x|}{\varphi} \right)^{m+l} dx, \quad u \in W^{1,p}_0(|x|^{l(1-p)},\Omega), \quad (5.2)
$$

where we use the notations in Sect. 3:

$$
\alpha = 1 + \frac{l}{p'}, \quad \beta = 1, \quad \gamma = \frac{p-1}{p},
$$

$$
v = |x|^{-\frac{p}{p'}}, \quad w = |m+l||x|^{-1-\frac{p}{p'}} \left| 1 - \left( \frac{|x|}{\varphi} \right)^{m+l} \right|^{-1},
$$

$$
g(s(x)) = \frac{1 - s(x)^{m+l}}{m+l}, \quad s(x) = \frac{|x|}{\varphi(x)} \text{ and } m+l \neq 0, \ l \leq 1 - m.
$$

Here $m = \frac{p-n}{p-1} \neq 0, \frac{1}{p'} + \frac{1}{p} = 1$, $\langle ., . \rangle$ is the scalar product in $\mathbb{R}^n$ and the constant $\frac{|m+l|^p}{p'}$ is optimal.

In this section we generalize (5.2) and prove a sharp Hardy inequality with additional term and an optimal constant for star-shaped domains and $m+l > 0$.

### 5.1 Hardy inequalities with additional boundary term

We start with the following theorem:

**Theorem 5.1.** Suppose $\Omega = \{|x| < \varphi(x)\} \subset \mathbb{R}^n$, $n \geq 2$ is a star-shaped domain with respect to a small ball centered at the origin, $p > 1$, $m = \frac{p-n}{p-1}, -m < l \leq 1 - m$. Then
for every \( u \in W^{1,p}_0(|x|^{(1-p)}, \Omega) \), the improved Hardy inequality

\[
\left( \int_\Omega |x|^{(1-p)} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p \, dx \right)^{\frac{1}{p}} \geq \left( \frac{m+l}{p} \right) \left( \int_\Omega |x|^{n-m-l} |\varphi^{m+l}(x) - |x|^{m+l}|^p \, dx \right)^{\frac{1}{p}} \\
+ \frac{1}{p} \limsup_{\varepsilon \to 0} \varepsilon^{1-n} \int_{S_\varepsilon} \frac{|u(x)|^p dS}{\varphi^{(m+l)(p-1)}(x)} \times \left( \int_\Omega |x|^{n-m-l} |\varphi^{m+l}(x) - |x|^{m+l}|^p \, dx \right)^{-\frac{1}{p}},
\]

(5.3)

holds, where \( S_\varepsilon = \{|x| = \varepsilon\} \).

In inequality (5.3) instead of the distance to zero in the denominator there is the distance to the boundary on the ray, and the constant is optimal.

**Remark 5.1.** For \( m > 0 \), i.e., \( p > n \), the choice \( l = 0 \) is possible in (5.3) and in this case Hardy inequality (5.3) is true for every \( u \in W^{1,p}_0(\Omega) \).

In order to prove Theorem 5.1 we need some auxiliary results.

Fix \( r < \inf \varphi(x) \) and for the annulus \( A[r, \varphi) = \{r \leq |x| < \varphi(x)\} \) we introduce the space \( W^{1,p}_0(A[r, \varphi)) \) which is the completion in the norm (5.1) for \( A[r, \varphi) = \Omega \setminus \bar{B}_r \) of the \( C^\infty(A[r, \varphi)) \) functions which are zero in a neighborhood of the boundary \( S_\varphi = \{|x| = \varphi(x)\} \) (see Maz'ja [76], Ch. 1.1.15 and Ch. 1.1.6), i.e., functions in \( C^\infty_0(\Omega) \). The main element of the proof of Theorem 5.1 is the following Proposition 5.1 and Corollary 2.2.

**Proposition 5.1.** Suppose \( n \geq 2 \), \( p > 1 \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), \( -m < l \leq 1 - m \), \( m = \frac{p-n}{p-1} \) and

\[
f(x) = (f_1, \ldots, f_n) = -x|x|^{-n} \left( \frac{\varphi^{m+l}(x) - |x|^{m+l}}{m+l} \right)^{1-p},
\]

(5.4)

where \( f \neq 0 \), \( f_j \in C^1(A[r, \varphi)) \). Then \( f \) satisfies the identity

\[-\text{div} f - (p-1)|f'| |x|^l = 0, \quad \text{in } A[r, \varphi),
\]

(5.5)

and for every \( u \in W^{1,p}_0(|x|^{(1-p)}, A[r, \varphi)) \) the inequality

\[
\left( \int_{A[r, \varphi]} |x|^{(1-p)} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p \, dx \right)^{1/p} \geq \frac{1}{p'} \left( \int_{A[r, \varphi]} |x|^l |f'|^p |u|^p \, dx \right)^{1/p},
\]

(5.6)

\[\frac{1}{p} \int_{S_r} \frac{\langle f, x \rangle}{|x|} |u|^p dS \left( \int_{A[r, \varphi]} |x|^l |f'|^p |u|^p \, dx \right)^{-\frac{1}{p}},\]

holds.
Proof. Without loss of generality we suppose that $u \in C^\infty_0(\Omega)$. Simple computations give us

$$|f'|^p = |x|^{m-n} \left( \frac{\varphi^{m+l}(x) - |x|^{m+l}}{m+l} \right)^{-p},$$

$$- \text{div} f = |x|^{-n} \left( x, \nabla \left( \frac{\varphi^{m+l}(x) - |x|^{m+l}}{m+l} \right)^{1-p} \right)$$

$$= -(p-1)|x|^{-n} \left( \frac{\varphi^{m+l}(x) - |x|^{m+l}}{m+l} \right)^{-p} [\varphi^{m+l-1}(x) \langle x, \nabla \varphi(x) \rangle - |x|^{m+l}]$$

$$= (p-1)|x|^{m+l-n} \left( \frac{\varphi^{m+l}(x) - |x|^{m+l}}{m+l} \right)^{-p} = (p-1)|f'|^p |x|^l,$$

because $\langle x, \nabla \varphi(x) \rangle = 0$.

Thus we have that (5.5) is satisfied. Inequality (5.6) follows from (2.8) in Corollary 2.2 for $v = |x|^l$ and $w = 0$. \hfill \Box

Proposition 5.2. Suppose $n \geq 2$, $p > 1$, and $m = \frac{p-n}{p-1}$, $-m < l \leq 1 - m$. Then for every $u \in W_0^{1,p}(|x|^{l(1-p)}, A[r, \varphi])$ the following inequalities hold:

$$\left( \int_{A[r, \varphi]} |x|^{l(1-p)} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p \right)^{\frac{1}{p}} \geq \frac{m+l}{p'} \left( \int_{A[r, \varphi]} \frac{|u|^p}{|x|^{n-m-l}(\varphi^{m+l}(x) - |x|^{m+l})} \right)^{\frac{1}{p'}}$$

$$+ \frac{r^{1-n}}{p} \int_{S_r} \frac{|u|^p}{(\varphi^{m+l}(x) - r^{m+l})^{p-1}} \, dS$$

$$\times \left( \int_{A[r, \varphi]} \frac{|u|^p}{|x|^{n-m-l}(\varphi^{m+l}(x) - |x|^{m+l})} \right)^{-\frac{1}{p'}},$$

and

$$\int_{A[r, \varphi]} |x|^{l(1-p)} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p \, dx \geq \left( \frac{m+l}{p'} \right)^p \int_{A[r, \varphi]} \frac{|u|^p}{|x|^{n-m-l}(\varphi^{m+l}(x) - |x|^{m+l})} \, dx$$

$$+ \frac{r^{1-n}}{p} \left( \frac{1}{p'} \right)^{p-1} \int_{S_r} \frac{|u|^p}{(\varphi^{m+l}(x) - r^{m+l})^{p-1}} \, dS.$$
where \( \Phi(x) \) is a homogeneous function of the 0-th order, inequality (5.7) becomes an equality.

**Proof.** Without loss of generality we suppose that \( u(x) \) is \( C^\infty \) function which is zero near the boundary \( S_\varphi = \{ |x| = \varphi(x) \} \). Let us choose \( f \) as in Proposition 5.1, see (5.4). The proof of (5.7) follows from (5.6) with the special choice (5.4) of \( f \) since

\[
K_0(u) = - \int_{S_r} \frac{\langle f, x \rangle}{|x|} |u|^p dS = (m + l)^{p-1}r^{1-n} \int_{S_r} \frac{|u|^p}{|\varphi^{m+l}(x) - r^{m+l}|^{p-1}} dx \geq 0.
\]

For the proof of (5.8) we use (2.10).

For function \( u_k(x) \) in (5.9) we get an equality in (5.7). The proof is similar to the proof of Theorem 5.2 and we omit it. \( \square \)

**Proof of Theorem 5.1.** Let \( 0 < \varepsilon < \inf_{|x|=1} \varphi(x) \) be a small positive number and \( u \in W_0^{1,p}(|x|^{1-p}, \Omega) \). Then \( u \in W_0^{1,p}(|x|^{1-p}, A[\varepsilon, \varphi]) \), where \( A[\varepsilon, \varphi] = \{ \varepsilon \leq |x| < \varphi(x) \} \). From Corollary 2.2 we get the following Hardy inequality in the annulus \( A[\varepsilon, \varphi] \) for \( -m < l \leq 1 - m \), \( v = |x|^l \), \( w = 0 \) and \( L(u), K_0(u), K(u), N(u) \) defined in (2.3) for \( \Omega = A[\varepsilon, \varphi] \), i.e.,

\[
L_{\varphi}(u) = \left( \int_{A[\varepsilon, \varphi]} |x|^{l(1-p)} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx \right)^{\frac{1}{p}} \geq \frac{1}{p'} K_{\varphi}(u) + \frac{1}{p} K_0(u) \frac{1-p}{p} \tag{5.10}
\]

\[
= \frac{m + l}{p'} \left( \int_{A[\varepsilon, \varphi]} \frac{|u|^p}{|x|^{n-m-l}|\varphi^{m+l}(x) - |x|^{m+l}|^p dx} \right)^{\frac{1}{p'}} + \frac{\varepsilon^{1-n} - l}{p} \int_{S_\varepsilon} \frac{|u|^p}{|\varphi^{m+l}(x) - \varepsilon^{m+l}|^{p-1}} dS
\]

\[
\times \left( \int_{A[\varepsilon, \varphi]} \frac{|u|^p}{|x|^{n-m-l}|\varphi^{m+l}(x) - |x|^{m+l}|^p dx} \right)^{\frac{1-p}{p}}.
\]

After the limit \( \varepsilon \to 0 \) in (5.10) and from the inequality

\[
\int_{S_\varepsilon} \frac{|u|^p}{|\varphi^{m+l}(x) - \varepsilon^{m+l}|^{p-1}} dS \geq \int_{S_\varepsilon} \frac{|u|^p}{\varphi^{(m+l)(p-1)}(x)} dS,
\]

we get (5.3). \( \square \)

### 5.2 Sharpness of the Hardy inequalities

The inequality (5.3) becomes an equality for a class of functions defined in Theorem 5.2. Moreover, in Corollary 5.2 we show that in the special case of a ball, inequality (5.3) transforms into (5.21) with the distance function in the denominator and the optimal constant.
Theorem 5.2. Suppose $\Omega = \{ |x| < \varphi(x) \} \subset R^n$, $n \geq 2$ is a star-shaped domain with respect to a small ball centered at the origin $p > 1$, $m = \frac{p - n}{p - 1}$, $-m < l \leq 1 - m$. Then Hardy inequality (5.3) is an equality if $u(x) = u_k(x)$,

$$u_k(x) = (\varphi^{m+l}(x) - |x|^{m+l}) \Phi(x), \quad k > \frac{1}{p'},$$

where $\Phi$ is a homogeneous function of the 0-th order. Moreover, the constant $\left( \frac{m + l}{p'} \right)^p$ is optimal for (5.2) and (5.3).

Proof. From Theorem 2.2 it follows that (5.3) is an equality if and only if for a.e. $x \in \Omega$ and some constant $k_1 \geq 0$ identities (2.14)–(2.16) are satisfied, i.e., (5.3) becomes an equality if (5.11) holds for a.e. $x \in \Omega$.

$$u(f, \nabla u) = |u(f, \nabla u)|,$$

$$\langle f, \nabla u \rangle = k_1 v|f|^{p'} u. \quad (5.11)$$

From the choice of $f$ in (5.4) equalities (5.11) are equivalent to

$$-u(x, \nabla u) = |u(x, \nabla u)|,$$

$$\langle x, \nabla u \rangle = -k_1 |x|^{m+l} \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{-1} u, \quad (5.12)$$

for a.e. $x \in \Omega$.

We are looking for solution of the second equation in (5.12) of the form

$$u = \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{\frac{k_1}{m+l}} z.$$

Together with (5.12) a simple computation gives us

$$-k_1 |x|^{m+l} \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{-1} u = \langle x, \nabla u \rangle$$

$$= k_1 \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{\frac{k_1}{m+l}} \left( \varphi^{m+l-1}(x) \langle x, \nabla \varphi(x) \rangle - |x|^{m+l-2} \langle x, x \rangle \right) z$$

$$+ \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{\frac{k_1}{m+l}} \langle x, \nabla z \rangle$$

$$= -k_1 |x|^{m+l} \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{\frac{k_1}{m+l}} z + \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{\frac{k_1}{m+l}} \langle x, \nabla z \rangle$$

$$= -k_1 |x|^{m+l} \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{-1} u + \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^{\frac{k_1}{m+l}} \langle x, \nabla z \rangle.$$

Hence $z$ is a solution of the homogeneous first order partial differential equation

$$\langle x, \nabla z \rangle = 0, \quad \text{in } \Omega \setminus \{0\}. \quad (5.13)$$

It is well known, see Evans [32], that all solutions of (5.13) are in the form of $z(x) = \Phi(x)$, where $\Phi$ is a homogeneous function of the 0-th order, i.e., all solutions of (5.12) are $u_k = \left( \varphi^{m+l}(x) - |x|^{m+l} \right)^k \Phi(x)$ for $k \geq 0$. When $k > \frac{1}{p'}$, then $u_k \in W_0^{1,p}(|x|^{(1-p)}, \Omega)$. 

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Let us check up that for \( u_k \) inequality (5.3) becomes an equality. Since \( \varphi(x) \) and \( \Phi(x) \) are homogeneous functions of the 0-th order we have \( \langle x, \nabla \varphi(x) \rangle = 0, \langle x, \nabla \Phi(x) \rangle = 0 \). We make a polar change of the variables and for simplicity we use the same notations \( \langle \cdot \rangle \) are homogeneous functions of the 0-th order we have

\[
L(u_k) = \int_\Omega |x|^{l(1-p)} \left| \frac{\langle x, \nabla u_k \rangle}{|x|} \right|^p dx
\]

\[
= [k(m + l)]^p \int_\Omega |\Phi(x)|^p |x|^{l(1-p)+p(m+l-1)} \left| \varphi^{m+l}(x) - |x|^{m+l} \right|^{p(k-1)} dx
\]

\[
= [k(m + l)]^p \int_{S_1} \int_0^{\varphi(x)} |\Phi(x)|^p \rho^{m+l-1} \left| \varphi^{m+l}(x) - \rho^{m+l} \right|^{p(k-1)} d\rho dS
\]

\[
= k^p(m + l)^{p-1} \int_{S_1} \int_0^{\varphi(x)} |\Phi(x)|^p \varphi^{m+l}(x) - \rho^{m+l} \right|^{p(p-k-1)} d\rho dS
\]

\[
= k^p(m + l)^{p-1} \frac{p-k-p+1}{p} \int_{S_1} |\Phi(x)|^p \varphi^{(m+l)(pk-p+1)}(x) dS < \infty,
\]

because \( pk - p + 1 > 0 \) for \( k > \frac{1}{p} \) and \( m + l > 0 \).

Analogously, for the terms \( K_1(u_k), K_2(u_k) \) in the right-hand side of (5.3) we obtain

\[
K_1(u_k) = \int_\Omega \frac{|u_k|^p}{|x|^{n-m-l} |\varphi^{m+l}(x) - |x|^{m+l}|^p} dx
\]

\[
= \int_{S_1} \int_0^{\varphi(x)} \frac{|\Phi(x)|^p \rho^{m+l-1}}{|\varphi^{m+l}(x) - \rho^{m+l}|^{p-pk}} d\rho dS
\]

\[
= \frac{1}{(m + l)(pk - p + 1)} \int_{S_1} |\Phi(x)|^p \varphi^{(m+l)(pk-p+1)}(x) dS,
\]

and

\[
K_2(u_k) = \frac{1}{p} \lim_{\varepsilon \to 0} \varepsilon^{1-n} \int_{S_\varepsilon} \frac{|u_k|^p}{|\varphi^{m+l}(x) - \varepsilon^{m+l}|^{p-1}} dS
\]

\[
= \frac{1}{p} \int_{S_1} |\Phi(x)|^p \varphi^{(m+l)(pk-p+1)} dS.
\]

Thus for the right-hand side of (5.3) we get finally

\[
K_{12}(u_k) = \left( \frac{m + l}{p} K_1(u_k) + K_2(u_k) \right) K_1^{-\frac{1}{p}}(u_k)
\]

\[
= \left[ \frac{(m + l)(p - 1)}{(m + l)(pk - p + 1)p} + \frac{1}{p} \right] \left[ (m + l)(pk - p + 1) \right]^{\frac{1}{p}}
\]

\[
\times \left[ \int_{S_1} |\Phi(x)|^p \varphi^{(m+l)(pk-p+1)} dS \right]^{\frac{1}{p}}
\]

\[
= \frac{(m + l)^{\frac{1}{p}} k}{(pk - p + 1)^{\frac{1}{p}}} \left[ \int_{S_1} |\Phi(x)|^p \varphi^{(m+l)(pk-p+1)} dS \right]^{\frac{1}{p}}.
\]
So the left-hand side \((L(u_k))^{1/p}\) of (5.3) coincides with the right-hand side \(K_{12}(u_k)\) of (5.3). Thus (5.3) is an equality for \(u_k(x)\).

Let us check now that the constant \(\left(\frac{m + l}{p'}\right)^p\) is optimal for (5.3). From (5.3), (5.14) and (5.15) we have

\[
\left(\frac{m + l}{p'}\right)^p \leq \int \frac{|\langle x, \nabla u_k \rangle|}{|x|}^p \, dx \\
\times \left(\int \frac{|u_k|^p}{|x|^{n-m-l} |\varphi^{n+l}(x) - |x|^{m+l}|^p} \, dx\right)^{-1} \\
= \frac{(m + l)(pk - p + 1)k^p(m + l)^{p-1}}{(pk - p + 1)} \\
= \left(\frac{m + l}{p'}\right)^p (p'k)^p \to \left(\frac{m + l}{p'}\right)^p + 0, \quad \text{when } k \to \frac{1}{p'} + 0.
\]

Let us illustrate Theorem 5.1 and Theorem 5.2 for the ball \(B_R = \{|x| < R\}\) and \(l = 0\).

**Corollary 5.1.** Suppose \(B_R = \{|x| < R\} \subset \mathbb{R}^n\), \(p > n \geq 2\), \(m = \frac{p - n}{p - 1}\). Then for every \(u \in W^{1,p}_0(B_R)\) Hardy inequality

\[
\left(\int_{B_R} \frac{|\langle x, \nabla u \rangle|^p}{|x|} \, dx\right)^{\frac{1}{p}} \geq \frac{p - n}{p} \left(\int_{B_R} \frac{|u|^p}{|x|^{n-m} |R^m - |x|^{m}|^p} \, dx\right)^{\frac{1}{p}} \\
+ \frac{1}{p} R^{n-p} \omega_n |u(0)|^p \left(\int_{B_R} \frac{|u|^p}{|x|^{n-m} |R^m - |x|^{m}|^p} \, dx\right)^{-\frac{1}{p}},
\]

holds, where \(\omega_n\) is the \((n-1)\)-dimensional measure of the unite sphere \(S_1\).

For \(u_k(x) = (R^m - |x|^m)^k \Phi(x), \ k > \frac{1}{p'}\), where \(\Phi\) is a homogeneous function of the 0-th order, (5.16) becomes an equality, i.e., (5.16) is sharp and the constant \(\frac{p - n}{p}\) is optimal.

**Proof.** Corollary 5.1 follows from Theorem 5.1 for \(\varphi(x) = R\), the constant \(\frac{p - n}{p}\) in (5.16) is optimal and inequality (5.16) is sharp due to Theorem 5.2.

As it is shown in Sect. 1.2, different forms of Hardy inequalities with additional term and optimal constant are considered in Brezis and Marcus [25], Hoffmann-Ostenhof et al. [55], Tidblom [87], Filippas et al. [43], Avkhadiev and Wirths [11], etc. In Hardy inequality (5.16), the weights in the leading term of the right-hand side have singularities at 0 and on the boundary of the ball \(B_R\). This inequality is not in the ‘linear’ form and it is sharp. Moreover, in Corollary 5.2 we obtain the inequality (5.21) where the leading term in the
right-hand side is written with the distance function \( d(x) \) and an additional term which depends on the value of the function at 0.

Inequalities (5.3) and (5.16) are sharp, but they are not in a ‘linear’ form. Using Young inequality, we can get a ‘linear’ form of these inequalities.

**Theorem 5.3.** Suppose \( \Omega = \{ |x| < \varphi(x) \} \subset \mathbb{R}^n, \ n \geq 2 \) is a star-shaped domain with respect to a small ball centered at the origin, \( p > 1, \ m = \frac{p-n}{p-1}, -m < l \leq 1 - m \). Then the following Hardy inequality holds in \( \Omega \):

\[
\int_{\Omega} |x|^{l(1-p)} \left( \frac{\langle x, \nabla u \rangle}{|x|} \right)^p \, dx \geq \left( \frac{m+l}{p'} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^{n-m-l} |\varphi^{m+l}(x)| - |x|^{m+l}} \, dx \\
+ \left( \frac{m+l}{p'} \right)^{p-1} \limsup_{\varepsilon \to 0} \varepsilon^{1-n} \int_{S_{\varepsilon} \varphi^{m+l}(p-1)(x)} |u(x)|^p dS, \ u \in W^{1,p}_0(|x|^{l(1-p)}, \Omega),
\]

When \( p > n, l = 0 \) then in \( B_R \) the inequality

\[
\int_{B_R} \left( \frac{\langle x, \nabla u \rangle}{|x|} \right)^p \, dx \geq \left( \frac{p-a}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^{n-m} |R^m - |x||^p} \, dx \\
+ \left( \frac{p-n}{p} \right)^{p-1} R^{n-p} \frac{\omega_n}{p} |u(0)|^p, \ u \in W^{1,p}_0(B_R),
\]

holds.

**Proof.** Since \( K_0(u) \geq 0 \), we have from (2.10) that

\[
L(u) \geq \left( \frac{1}{p'} \right)^p K(u) + \left( \frac{1}{p'} \right)^{p-1} K_0(u).
\]

The rest of the proof follows from (5.18), Theorem 5.1 and Corollary 5.1. \( \square \)

As a corollary of Theorems 5.1 and 5.3 we get

**Corollary 5.2.** If \( m = \frac{p-n}{p-1} > 0 \), then

\[
\left( \frac{p-n}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^{n-m} |R^m - |x||^p} \, dx \geq \left( \frac{1}{p'} \right)^p \int_{B_R} \frac{|u|^p}{dp'(x)} \, dx
\]

and correspondingly

\[
\left( \int_{B_R} \left( \frac{\langle x, \nabla u \rangle}{|x|} \right)^p \, dx \right)^\frac{1}{p} \geq \frac{1}{p'} \left( \int_{B_R} \frac{|u|^p}{dp'(x)} \, dx \right)^\frac{1}{p'}
\]

\[
+ \frac{R^{n-p}}{p} \omega_n |u(0)|^p \left( \int_{B_R} \frac{|u|^p}{|x|^{n-m} |R^m - |x||^p} \, dx \right)^\frac{1}{p'},
\]

\[
\int_{B_R} \left( \frac{\langle x, \nabla u \rangle}{|x|} \right)^p \, dx \geq \left( \frac{1}{p'} \right)^p \int_{B_R} \frac{|u|^p}{dp'(x)} \, dx + \left( \frac{p-n}{p} \right)^{p-1} R^{n-p} \frac{\omega_n}{p} |u(0)|^p,
\]

hold for every \( u \in W^{1,p}_0(B_R) \). Moreover, the constant \( \left( \frac{1}{p'} \right)^p \) in (5.20) is optimal.
Proof. It is enough to prove the inequality
\[
\left( \frac{p - n}{p} \right)^p |x|^{m-n} (R^m - |x|^m)^{-p} \geq \left( \frac{p - 1}{p} \right)^p (R - |x|)^{-p},
\]
or equivalently
\[
(p - n)(R - \rho) \geq (p - 1)\rho^{\frac{n-1}{p-1}}(R^m - \rho^m),
\]
for $|x| = \rho$ and $n - m = \frac{(n-1)p}{p-1}$. The function
\[
h(\rho) = (p - n)(R - \rho) - (p - 1)\rho^{\frac{n-1}{p-1}}(R^m - \rho^m)
\]
is a decreasing one for $\rho \in [0, R]$ because
\[
h'(\rho) = (n - p) - (n - 1)R^m \rho^{\frac{n-1}{p-1}} - 1 = (n - 1) \left[ 1 - \left( \frac{R}{\rho} \right)^m \right] \leq 0,
\]
and $m > 0$. Since $h(R) = 0$ inequality (5.22) is satisfied.

To obtain inequalities (5.20) and (5.21) we replace (5.19) in (5.16) and (5.17). The optimality of the constants $\frac{1}{p}$ in (5.20) and $\left( \frac{1}{p} \right)^p$ in (5.21) follows from Corollary 5.1.

6 Estimates from below for the first eigenvalue of the p-Laplacian

In this section we give an application of Hardy inequalities for the estimate from below of the first eigenvalue $\lambda_{p,n}$ of the p–Laplacian $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$ in a bounded simply connected domain $\Omega \subset R^n$, $n \geq 2$ with smooth boundary $\partial \Omega$

\[
\begin{cases}
-\Delta_p u = \lambda_{p,n} |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

6.1 Existing analytical estimates from below

Here we listed some estimates of $\lambda_{p,n}$. The first eigenvalue $\lambda_{p,n}(\Omega)$ can be characterized through Reyleigh quotient, see Cheeger [27], Lindqvist [73]

\[
\lambda_{p,n}(\Omega) = \inf_{u \in W^{1,p}_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx},
\]
and $\lambda_{p,n}(\Omega)$ is simple, i.e., the first eigenfunction $u_{p,n}(x)$ is unique up to multiplication with non-zero constant $C$. Moreover, $u_{p,n}(x)$ is positive in $\Omega$, $u_{p,n}(x) \in W^{1,p}_0(\Omega) \cap C^{1,\delta}(\Omega)$ for some $\delta \in (0, 1)$ (see for example Belloni and Kawohl [17] and the references therein).
Analytical values of $\lambda_{p,n}$ are known only for $p > 1$ and $n = 1$ or $p = 2$ and $n \geq 2$. For $p > 1$ and $n = 1$, $\Omega = (a, b)$, see Ōtani [81], the analytical value of $\lambda_{p,1}(\Omega)$ is

$$\lambda_{p,1}(\Omega) = (p-1) \left( \frac{\pi}{(b-a)p\sin\frac{\pi}{p}} \right)^p.$$  

For $n \geq 2$ in the case $p = 2$, i.e., for the Laplace operator, the value of $\lambda_{2,n}(\Omega)$ is known by analytical formulae for domains $\Omega$ with simple geometry like a ball, a spherical shell, a parallelepiped etc. Numerical approximations have been done for more general domains, see Vladimirov [89] and the review by Grebenkov and Nguyen [49]. For example, if $\Omega$ is a ball centered at zero, $B_R \subset R^n$ then

$$\lambda_{2,n}(B_R) = \left( \frac{\mu_1^{(\alpha)}}{R^2} \right)^2, \quad \alpha = \frac{n}{2} - 1,$$

where $\mu_1^{(\alpha)}$ is the first positive zero of the Bessel function $J_\alpha$.

If $p \neq 2$, the explicit value of $\lambda_{p,n}(\Omega)$ is not known even for domains $\Omega$ like a ball or a cube. That is why an explicit lower bound for $\lambda_{p,n}(\Omega)$ is an important task.

For this purpose the Faber–Krahn theorem simplifies the estimate of the first eigenvalue for arbitrary domain to the estimate in a ball. Let us recall the Faber–Krahn inequality which gives an estimate from below of $\lambda_{p,n}(\Omega)$ for arbitrary bounded domain $\Omega \subset R^n$ with $\lambda_{p,n}(\Omega^*)$, where $\Omega^*$ is the n-dimensional ball of the same volume as $\Omega$, see Lindqvist [73], Bhattacharia [21], Huang [56], Kawohl and Fridman [59]. In Kawohl and Fridman [59] is proved that among all domains $\Omega$ of a given n-dimensional volume the ball $\Omega^*$ with the same volume as $\Omega$ minimizes $\lambda_{p,n}(\Omega)$, in other words

$$\lambda_{p,n}(\Omega) \geq \lambda_{p,n}(\Omega^*).$$

**Estimate with a Cheeger’s constant**

One of the first lower bounds for $\lambda_{p,n}(\Omega)$ is based on Cheeger’s constant

$$h(\Omega) = \inf_{D \subset \Omega} \frac{|\partial D|}{|D|}.$$  

Here $D$ varies over all smooth sub-domains of $\Omega$ whose boundary $\partial D$ does not touch $\partial \Omega$, where $|\partial D|$ and $|D|$ are the (n-1)- and n-dimensional Lebesgue measure of $\partial D$ and $D$ respectively.

In Cheeger [27] for $p = 2$ and in Lefton and Wei [67] for $p > 1$ it was proved that the first eigenvalue of (6.1) can be estimated from below via

$$\lambda_{p,n}(\Omega) \geq \left( \frac{h(\Omega)}{p} \right)^p. \quad (6.3)$$

Inequality (6.3) is sharp for $p \to 1$, because $\lambda_{p,n}(\Omega)$ converges to the Cheeger’s constant $h(\Omega)$, see Kawohl and Fridman [59], Corollary 6.

The Cheeger’s constant $h(\Omega)$ is known only for special domains. For example, if $\Omega$ is a ball $B_R \subset R^n$, then $h(\Omega) = \frac{n}{R}$ and (6.3) gives the following lower bound for $\lambda_{p,n}(B_R)$, see Kawohl and Fridman [59],

$$\lambda_{p,n}(B_R) \geq \Lambda_{p,n}^{(1)}(B_R) = \left( \frac{n}{pR} \right)^p, \text{ for } p > 1, \ n \geq 2. \quad (6.4)$$
Thus combining the above results the following inequality holds for $p \to 1$, see Kawohl and Fridman [59], Remark 5,

$$
\lambda_{1,n}(\Omega) \geq n \left( \frac{\omega_n}{|\Omega|} \right)^{1/n} = \Lambda_{1,n}^{(1)}(\Omega), \quad n \geq 2, \quad (6.5)
$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. If $\Omega$ is a ball, then (6.5) becomes an equality.

In the other limit case $p \to \infty$ the result in Juutinen et al. [58] says that

$$
\lambda_{\infty,n}(\Omega) = \lim_{p \to \infty} \left( \lambda_{p,n}(\Omega) \right)^{1/p} = \max \{ \text{dist}(x, \partial \Omega), x \in \Omega \}^{-1}.
$$

In particular for $\Omega = B_R$

$$
\lambda_{\infty,n}(B_R) = \lim_{p \to \infty} \left( \Lambda_{p,n}^{(1)}(B_R) \right)^{1/p} = \frac{1}{R}.
$$

**Estimate with Picone’s identity**

In Benedikt and Drábek [18], Theorem 2, and in Benedikt and Drábek [19], Theorem 2, by Picone’s identity the following estimate for $p > 1$ was proved

$$
\lambda_{p,n}(B_R) \geq \begin{cases} 
\Lambda_{p,n}^{(2,1)}(B_R) = \frac{n}{R^p} \left( \frac{p}{p - 1} \right)^{p-1}, \\
\Lambda_{p,n}^{(2,2)}(B_R) = \frac{np}{R^p},
\end{cases}
$$

Since

$$
\max \left\{ \Lambda_{p,n}^{(2,1)}(B_R), \Lambda_{p,n}^{(2,2)}(B_R) \right\} = \begin{cases} 
\Lambda_{p,n}^{(2,1)}(B_R), & \text{for } 1 < p < 2, \\
\Lambda_{p,n}^{(2,2)}(B_R), & \text{for } p \geq 2,
\end{cases}
$$

see Proposition 6.4, the estimate

$$
\lambda_{p,n}(B_R) \geq \Lambda_{p,n}^{(2)}(B_R) = \begin{cases} 
\Lambda_{p,n}^{(2,1)}(B_R) = \frac{n}{R^p} \left( \frac{p}{p - 1} \right)^{p-1}, & \text{for } 1 < p < 2, \\
\Lambda_{p,n}^{(2,2)}(B_R) = \frac{np}{R^p}, & \text{for } p \geq 2,
\end{cases}
$$

holds.

**Estimate with Sobolev constant**

It is not difficult to estimate $\lambda_{p,n}(\Omega)$ from below in a bounded domain for $1 < p < n$ by the well-known Sobolev and Hölder inequalities

$$
\| \nabla u \|_p \geq C_{n,p} \| u \|_{\frac{np}{n-p}} \geq C_{n,p} \| u \|_{p,\Omega}^{-1/n}. \quad (6.7)
$$

The best Sobolev’s constant $C_{n,p}$ is obtained in Aubin [8] and Talenti [86]. For more details see Maz’ja [76] and Ludwig et al. [75]

$$
C_{n,p} = n^{1/p} \omega_n^{1/n} \left( \frac{n - p}{p - 1} \right)^{\frac{n-1}{p}} \left[ \frac{\Gamma \left( \frac{n}{2} \right) \Gamma \left( n + 1 - \frac{n}{p} \right)}{\Gamma(n)} \right]^{1/n}.
$$
From (6.7) the estimate from below of the first eigenvalue for $\Omega = B_R$ becomes

$$
\lambda_{p,n}(B_R) \geq \frac{n}{R^p} \left( \frac{n-p}{p-1} \right)^{p-1} \left[ \frac{\Gamma \left( \frac{n}{p} \right) \Gamma \left( n + 1 - \frac{n}{p} \right)}{\Gamma(n)} \right]^{n/p}
= \Lambda_{p,n}^{(S)}(B_R), \text{ for } 1 < p < n.
$$

**Lindqvist’s estimate in parallelepiped**

In the parallelepiped

$$
P = \{x \in \mathbb{R}^n, 0 < x_j < a_j, j = 1, 2, \ldots, n\} \text{ with } a_{min} = \min_{i \leq i \leq n} a_i
$$

for $p > n$ we have the estimate

$$
\lambda_{p,n}(P) \geq \frac{p}{a_{min}^p},
$$

by Lindqvist [74]. If $\Omega$ is an arbitrary bounded domain in $\mathbb{R}^n$ and $R$ is the radius of the largest ball inscribed in the smallest parallelepiped (with minimal $a_{min}$) containing $\Omega$, then

$$
\lambda_{p,n}(\Omega) \geq \frac{p}{R^p} = \Lambda_{p,n}^{(L)}(B_R), \text{ for } p > n.
$$

**Numerical estimates**

In Biezuner et al. [22, 23], different numerical methods for computing $\lambda_{p,n}(\Omega)$ inspired by the inverse power method in finite dimensional algebra are developed. By means of iterative technique the authors define two sequences of functions. One of the sequences is monotone decreasing, the other one is monotone increasing. The first eigenvalue $\lambda_{p,n}(\Omega)$ is between the limits of these sequences. In the case of a ball the two limits are equal and $\lambda_{p,n}(\Omega)$ coincides with them.

In Lefton and Wei [67] a finite element technique for numerical approximation of the first eigenfunction and the first eigenvalue of (6.1) is used.

### 6.2 Estimates from below of $\lambda_{p,n}$ using Hardy inequalities

We prove several analytical bounds from below of the first eigenvalue of the $p$–Laplacian in bounded domains using different Hardy inequalities with weights derived in section 2. The section is based on the results in Fabricant et al. [36, 38, 39], Kutev and Rangelov [64, 65].

For this purpose we use the following Faber–Krahn theorem.

**Theorem 6.1** (Kawohl and Fridman [59]). *Among all domains of a given $n$-dimensional volume the ball $\Omega^*$ with the same volume as $\Omega$ minimizes every $\lambda_{p,n}(\Omega)$, in other words

$$
\lambda_{p,n}(\Omega) \geq \lambda_{p,n}(\Omega^*).
$$

Thus, from (6.9) it is enough for us to find an estimate for $\lambda_{p,n}$ only in the ball $\Omega^* = B_R$. Hardy inequalities are with weights singular either at some interior point of $\Omega$, usually at the origin, or with weights singular on the boundary or combined double singularities at 0 and at $\partial \Omega$. We will apply these three types of Hardy inequalities in order to estimate
from below the first eigenvalue \( \lambda_{p,n}(\Omega) \). We are concentrating only on those Hardy’s inequalities (among the large number of results in the literature) which are with explicitly given constants.

Let us note that from the classical Hardy inequality, see (1.3) using the Reyleigh quotient (6.2), we get immediately the estimate

\[
\lambda_{p,n}(B_R) \geq \left| \frac{n-p}{pR} \right|^p = \Lambda_{p,n}^{(R)}(B_R), \quad \text{for } n \geq 2, p > 1, n \neq p. \quad (6.10)
\]

**Estimates by means of Hardy inequalities with double singular weights**

From (2.37), (2.38) and (2.39) for \( u \in W_0^{1,p}(B_R) \) ignoring the boundary terms we have Hardy inequalities

\[
\int_{B_R} |\nabla u|^p \, dx \geq \left| \frac{p-n}{p} \right|^p \int_{B_R} \frac{|u|^p}{|x|^n |R^m - |x|^m|^p} \, dx, \quad \text{for } p \neq n, \quad (6.11)
\]

\[
\int_{B_R} |\nabla u|^n \, dx \geq \left( \frac{n-1}{n} \right)^n \int_{B_R} \frac{|u|^n}{|x|^n |\ln R/|x||} \, dx, \quad \text{for } p = n, \quad (6.12)
\]

where \( p > 1, n \geq 2, m = \frac{p-n}{p-1} \).

With the estimates (6.11) and (6.12) we obtain the following estimate for \( \lambda_{p,n}(B_R) \).

**Theorem 6.2.** For every \( n \geq 2, p > 1 \) the following estimates hold:

(i) If \( p \neq n \), then

\[
\lambda_{p,n}(B_R) \geq \left( \frac{1}{pR} \right)^p \left[ \frac{(n-1)^{n-1}}{(p-1)^{p-1}} \right]^\frac{n-p}{p-n}. \quad (6.13)
\]

(ii) If \( p = n \), then

\[
\lambda_{n,n}(B_R) \geq \left( \frac{n-1}{nR} \right)^n e^n. \quad (6.14)
\]

**Proof.** (i) If \( |x| = \rho \in [0, R] \), then for every \( x \in B_R \) and \( p \neq n \) we get for the right-hand side of (6.11) the estimate

\[
\int_{B_R} \frac{|u|^p}{|x|^n |R^m - |x|^m|^p} \, dx \geq \inf_{\rho \in (0,R)} \left( \rho^{n-m} |R^m - \rho^m|^p \right)^{-1} \int_{B_R} |u|^p \, dx.
\]

Further on we will use the identities

\[
m - n = (m-1)p = (1-n) \frac{p}{p-1} < 0, \quad 1 - m = \frac{n-1}{p-1} > 0, \quad (6.15)
\]

\[
\frac{n-m}{m} = \frac{(1-m)p}{m} = \frac{(n-1)p}{p-n}.
\]

Now applying the definition of \( \lambda_{p,n}(B_R) \) with Reyleigh quotient from (6.2) and (6.15) we have

\[
\lambda_{p,n}(B_R) \geq \left| \frac{p-n}{p} \right|^p \inf_{\rho \in (0,R)} \left[ \rho^{1-m} |R^m - \rho^m| \right]^{-p}
\]

\[
= \left| \frac{p-n}{p} \right|^p \left( \sup_{\rho \in (0,R)} \left( \rho^{1-m} |R^m - \rho^m| \right) \right)^{-p}. \quad (6.16)
\]
For the function $z(\rho) = \rho^{1-m}|R^m - \rho^m|$ in the interval $(0, R)$ we have
\[
z'(\rho) = \left[ (1-m)R^m \rho^{-m} - 1 \right] \text{sgn}(m),
\]
and $z'(\rho) = 0$ only at the point $\rho_0 = R(1-m)^{1/m} = R\left(\frac{n-1}{p-1}\right)^{1/m}$. For $m > 0$, i.e., $p > n$ we have $\frac{n-1}{p-1} < 1$ and hence \(\left(\frac{n-1}{p-1}\right)^{1/m} < 1\) while for $m < 0$, i.e., $p < n$ the inequality $\frac{n-1}{p-1} > 1$ holds and hence from $m < 0$ we get \(\left(\frac{n-1}{p-1}\right)^{1/m} < 1\).

Since $0 < \left(\frac{n-1}{p-1}\right)^{1/m} < 1$ for every $m \neq 0$, then $0 < \rho_0 < R$ and from $z''(\rho_0) = -|m|(1-m)R^m\rho_0^{-m-1} < 0$ it follows that the function $z(\rho)$ has a maximum at the point $\rho_0$ and
\[
z(\rho_0) = R\left(\frac{n-1}{p-1}\right)^{\frac{n-1}{m}} \left| \frac{p-n}{p-1} \right|.
\] (6.17)

Hence from (6.16) and (6.17) we get
\[
\lambda_p(B_R) \geq \left(\frac{p-1}{p}\right)^p \left(\frac{n-1}{p-1}\right)^{-\frac{(n-1)p}{p-n}} R^{-p} = \left(\frac{1}{R_p}\right)^p \left(\frac{n-1}{p-1}\right)^{-\frac{p-1}{n}}. \tag{6.18}
\]

(ii) As in the proof of (6.13) for the right-hand side of (6.12) we get the estimate
\[
\int_{B_R} \frac{|u|^n}{|x|^n \ln \frac{R}{|x|}} dx \geq \inf_{\rho \in (0,R)} \left(\rho \ln \frac{R}{\rho}\right)^{-n} \int_{B_R} |u|^n dx.
\]
From (6.12) and (6.2) we obtain
\[
\lambda_{n,n}(B_R) \geq \left(\frac{n-1}{n}\right)^n \inf_{\rho \in (0,R)} \left(\rho \ln \frac{R}{\rho}\right)^{-n} \tag{6.18}
= \left(\frac{n-1}{n}\right)^n \left(\sup_{\rho \in (0,R)} \rho \ln \frac{R}{\rho}\right)^{-n}.
\]

For the function $y(\rho) = \rho \ln \frac{R}{\rho}$ in the interval $(0, R)$ we have
\[
y'(\rho) = \ln \frac{R}{\rho} - 1 \text{ and } y'(\rho) = 0 \text{ only at the point } \rho_1 = Re^{-1} \in (0, R). \text{ Since } y''(\rho_1) = -\frac{1}{\rho_1} < 0, \text{ the function } y(\rho) \text{ has a maximum at } \rho_1 \text{ and } y(\rho_1) = Re^{-1}.
\]

Hence from (6.18) we obtain
\[
\lambda_{n,n}(B_R) \geq \left(\frac{n-1}{n}\right)^n (Re^{-1})^{-n} = \left(\frac{n-1}{nRn}\right)^n e^n.
\]
Estimates by means of Hardy inequalities with additional logarithmic term

In this section we estimate from below the first eigenvalue of the $p$–Laplacian in $B_R \subset \mathbb{R}^n$, $n \geq 2$, $p > n$, $m = \frac{p-n}{p-1} > 0$ using Hardy inequality (2.49), i.e.,

\[
\int_{B_R} |\nabla u|^p \, dx \geq \int_{B_R} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p \, dx \geq \left( \frac{p-n}{p} \right)^p \int_{B_R} \left[ 1 + \frac{p}{2(p-1)} \ln \frac{1}{2 R^m - |x|^m} \right] \frac{|u|^p}{|x|^{(n-1)p'-|x|^m}} \, dx
\]

where $\tau = e^{\frac{1}{y_0}}$ and

\[
y_0 = \frac{1 - \sqrt{1 + 4|a|}}{-2|a|} = 2 \left( 1 + \sqrt{\frac{5p-7}{3(p-1)}} \right)^{-1} \quad \text{for } a = -\frac{p-2}{6(p-1)}. \tag{6.20}
\]

**Theorem 6.3.** For every ball $B_R \in \mathbb{R}^n$, $n \geq 2$, $p > n$, $m = \frac{p-n}{p-1}$ the estimate

\[
\lambda_{p,n}(B_R) \geq \left( \frac{1}{pR} \right)^p \left[ \frac{(p-1)^{p-1}}{(n-1)^{n-1}} \right]^{\frac{m}{p-n}} \times \left\{ 1 + \frac{p}{4(p-1)} \left[ 1 + \sqrt{\frac{5p-7}{3(p-1)} - 2 \ln m - 2 \ln \tau} \right]^{-2} \right\}, \tag{6.21}
\]

holds, where

\[
\tau = 1 - 4(1-m) \left[ p \left( 1 + \sqrt{\frac{5p-7}{3(p-1)} - 2 \ln m} \right) - 4m \right]^{-1} \in (0,1). \tag{6.22}
\]

**Proof.** Suppose that $\int_{B_R} |u|^p \, dx = 1$ and with the notation $\varepsilon = \frac{1}{e\tau_0}$, i.e.,

\[
\ln \varepsilon = -\frac{1}{y_0} = -\frac{1}{2} \left( 1 + \sqrt{\frac{5p-7}{3(p-1)}} \right), \tag{6.23}
\]

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from (6.20) and from (6.19) we obtain the estimate

\[
\lambda_{p,n}(BR) \geq \left( \frac{p-n}{pR} \right)^p \inf_{\rho \in [0,R]} \left\{ \frac{1}{\left( \frac{\rho}{R} \right)^{n-m} \left( 1 - \left( \frac{\rho}{R} \right)^m \right)^p} \frac{1}{\left( 1 + \frac{p}{2(p-1) \ln^2 \varepsilon} \right) \rho^p \left( 1 - \frac{1}{z} \right)^{n-m}} \right\}^{\frac{1}{p}} \times \left[ 1 + \frac{p}{2(p-1) \ln^2 \varepsilon} \right]^{\frac{1}{p}} \left( \frac{p-n}{pR} \right)^p \inf_{\rho \in [0,R]} \left\{ \frac{1}{\left( \frac{\rho}{R} \right)^{n-m} \left( 1 - \left( \frac{\rho}{R} \right)^m \right)^p} \frac{1}{\left( 1 + \frac{p}{2(p-1) \ln^2 \varepsilon} \right) \rho^p \left( 1 - \frac{1}{z} \right)^{n-m}} \right\}^{\frac{1}{p}} \geq \left( \frac{p-n}{pR} \right)^p \inf_{\rho \in [0,R]} \left\{ \frac{1}{\left( \frac{\rho}{R} \right)^{n-m} \left( 1 - \left( \frac{\rho}{R} \right)^m \right)^p} \frac{1}{\left( 1 + \frac{p}{2(p-1) \ln^2 \varepsilon} \right) \rho^p \left( 1 - \frac{1}{z} \right)^{n-m}} \right\}^{\frac{1}{p}} \geq \left( \frac{p-n}{pR} \right)^p \inf_{\rho \in [0,R]} \left\{ \frac{1}{\left( \frac{\rho}{R} \right)^{n-m} \left( 1 - \left( \frac{\rho}{R} \right)^m \right)^p} \frac{1}{\left( 1 + \frac{p}{2(p-1) \ln^2 \varepsilon} \right) \rho^p \left( 1 - \frac{1}{z} \right)^{n-m}} \right\}^{\frac{1}{p}} \geq \left( \frac{p-n}{pR} \right)^p \left( I_1 + \frac{p}{2(p-1)} I_2 \right),
\]

where

\[
I_1 = \inf_{z \in [0,1]} \frac{1}{\rho^p (1 - z)^{\frac{n-m}{m}}} = \left[ \frac{1}{\rho^p (1 - z)^{\frac{n-m}{m}}} \right]^{-1}
\]

\[
I_2 = \inf_{z \in [0,1]} \frac{1}{\rho^p (1 - z)^{\frac{n-m}{m}}} = \frac{1}{\rho^p (1 - z)^{\frac{n-m}{m}}} \left[ \frac{1}{\rho^p (1 - z)^{\frac{n-m}{m}}} \right]^{-2}.
\]

Since \( \frac{n-m}{m} > 0 \) from (6.15), the function \( h(z) = z^p (1 - z)^{\frac{n-m}{m}} \) satisfies the conditions \( h(0) = h(1) = 0 \) and

\[
h'(z) = p z^{p-1} (1 - z)^{\frac{n-m}{m}} - \frac{n-m}{m} z^p (1 - z)^{\frac{n-m}{m}-1}
\]

\[
= p z^{p-1} (1 - z)^{\frac{n-m}{m}-1} \left( 1 - \frac{z}{m} \right),
\]

so it follows that \( h(z) \) has a maximum at the point \( m \in (0,1) \). Hence

\[
\sup_{z \in [0,1]} h(z) = h(m) = m^p (1 - m)^{\frac{n-m}{m}} = \left( \frac{p-n}{p-1} \right)^p \left( \frac{n-1}{p-1} \right)^{\frac{p(n-1)}{p-n}},
\]

and we get

\[
\left( \frac{p-n}{pR} \right)^p I_1 = \left( \frac{p-n}{pR} \right)^p \left( \frac{p-n}{p-1} \right)^{-p} \left( \frac{n-1}{p-1} \right)^{\frac{p(n-1)}{p-n}} = \left( \frac{1}{pR} \right)^p \left( \frac{(n-1)^{n-1}}{(p-1)^{p-1}} \right)^{\frac{p}{n-p}}.
\]
which gives the same estimate from below for \( \lambda_{p,n}(B_R) \) as in (6.13).

Let us estimate \( I_2 \). For the function 
\[ G(z) = z^{\frac{p}{2}}(1 - z)^{-\frac{n-m}{2m}}(- \ln \varepsilon z) \]
we have 
\[ G'(z) = z^{\frac{p}{2}-1}(1 - z)^{-\frac{n-m}{2m}-1}g(z), \]
where 
\[ g(z) = \frac{p}{2}(1 - z)(- \ln \varepsilon z) - 1 + z - \frac{n-m}{2m}z(- \ln \varepsilon z) \]
\[ = \frac{1}{2} \left[ p - \left( p + \frac{p(n-1)}{p-n} \right) \right] (- \ln \varepsilon z) - 1 + z \]
\[ = \frac{p}{2} \left( 1 - \frac{z}{m} \right) (- \ln \varepsilon z) - 1 + z. \]

Simple computations give us 
\[ g'(z) = \frac{p}{2} \left( \frac{1}{m} - \frac{1}{z} + \frac{1}{m} \ln \varepsilon z \right) + 1, \]
\[ g''(z) = \frac{p}{2} \left( \frac{1}{mz} + \frac{1}{z^2} \right) > 0, \]
for \( z \in [0,1] \), and hence from the monotonicity of \( g'(z) \), (6.15) and (6.23) we get the chain of equalities
\[
\sup_{z \in [0,1]} g'(z) = g'(1) = \frac{p}{2m} (1 - m + \ln \varepsilon) + 1 \\
= \frac{p}{2m} \left( \frac{n-1}{p-1} - \frac{1}{2} \frac{5p-7}{3(p-1)} \right) + 1 \\
= \frac{p}{4(p-n)} \left[ -2(p-n) - \sqrt{p-1} \left( \frac{\sqrt{5p-7}}{3} - \sqrt{p-1} \right) \right] + 1 \\
< - \frac{2p(p-n)}{4(p-n)} + 1 < - \frac{p}{2} + 1 < 0, \quad \text{for } p > n \geq 2.
\]

Since \( g'(z) < 0 \) for \( z \in [0,1] \) and \( \lim_{z \to 0} g(z) = \infty, g(m) = m - 1 < 0 \), it follows that there exists a unique point \( z_* \in (0,m) \) such that \( g(z_*) = 0 \), i.e., \( G'(z_*) > 0 \) for \( z \in [0,z_*) \), \( G'(z) < 0 \) for \( z \in (z_*,1]) \) and
\[
\sup_{z \in [0,1]} G(z) = \sup_{z \in (0,m]} G(z) = G(z_*). \quad (6.26)
\]

In order to localize better the maximum point \( z_* \) we look for \( z = \tau m, \tau \in (0,1) \) such that
\[ G'(\tau m) > 0. \] From (6.23) we get the chain of equalities

\[ g(\tau m) = \frac{p}{2} (1 - \tau)(-\ln(\varepsilon \tau m)) - 1 + \tau m \]

\[ = \frac{p}{2} (1 - \tau)(-\ln(\tau)) + \frac{p}{4} \left( 1 + \sqrt{\frac{5p - 7}{3(p - 1)}} - 2 \ln m \right)(1 - \tau) - 1 + \tau m \]

\[ = \frac{p}{2} (1 - \tau)(-\ln(\tau)) + \frac{p}{4} \left( 1 + \sqrt{\frac{5p - 7}{3(p - 1)}} - 2 \ln m \right) - 1 \]

\[ - \tau \left[ -m + \frac{p}{4} \left( 1 + \sqrt{\frac{5p - 7}{3(p - 1)}} - 2 \ln m \right) \right] \]

\[ = \frac{p}{2} (1 - \tau)(-\ln \tau). \]

Since \( m = \frac{p - n}{p - 4} < 1 \) we get from (6.15) that \( \ln m < 0 \) and

\[ p \left( 1 + \sqrt{\frac{5p - 7}{3(p - 1)}} - 2 \ln m \right) \]

\[ = p \left( 1 + \sqrt{\frac{1 + 2(p - 2)}{3(p - 1)}} - 2 \ln m \right) > 2p > 4 > 4m, \]

\[ 1 - 4(1 - m) \left[ p \left( 1 + \sqrt{\frac{5p - 7}{3(p - 1)}} - 2 \ln m \right) - 4m \right]^{-1} \]

\[ > 1 - 4(1 - m)(4 - 4m)^{-1} = 0. \]

Hence

\[ 0 < \tau = 1 - 4(1 - m) \left[ p \left( 1 + \sqrt{\frac{5p - 7}{3(p - 1)}} - 2 \ln m \right) - 4m \right]^{-1} < 1, \]

and from (6.27) the inequality

\[ G'(\tau m) > 0, \]

is satisfied. Thus \( z_\ast \in (\tau m, m) \), where \( \tau \) is given in (6.22).

From (6.25), (6.26), (6.28) and (6.29) it follows that \(-\ln(\varepsilon z) \leq -\ln(\varepsilon \tau m)\) for every
\[ z \in [\tau m, m] \text{ and} \]
\[ \sup_{z \in [\tau m, m]} \left[ (-\ln(\varepsilon z)) z^\frac{p}{2} (1 - z)^{\frac{n-m}{2m}} \right] \]
\[ \leq (-\ln(\varepsilon m)) \sup_{z \in [\tau m, m]} z^\frac{p}{2} (1 - z)^{\frac{n-m}{2m}} \]
\[ = (-\ln(\varepsilon m)) \sup_{z \in [\tau m, m]} h^\frac{1}{2}(z) = (-\ln(\varepsilon m)) h^\frac{1}{2}(m) \]
\[ = (-\ln \varepsilon m) m^\frac{p}{2} (1 - m)^{\frac{n-m}{2m}} \]
\[ = \left[ \frac{1}{2} \left( 1 + \sqrt{\frac{5p - 7}{3(p-1)}} \right) - \ln \tau - \ln m \right] I_1^{-\frac{1}{2}} \]

from the considerations for \(I_1\). Thus we have the following estimate for \(I_2\)
\[ I_2 \geq \frac{1}{2} I_1 (-\ln \varepsilon m)^{-2} = \frac{1}{2} \left[ 1 + \sqrt{\frac{5p - 7}{3(p-1)}} - 2 \ln m - 2 \ln \tau \right]^{-2} \]
and hence from (6.24) we obtain (6.21). \(\square\)

**Estimates by means of one-parametric family of Hardy inequalities**

We will obtain a new analytical estimate for \(\lambda_{p,n}(B_R)\) from below using one-parametric family of Hardy inequalities developed in section 2.5.

For this purpose we introduce the notations:
\[ A(p,n,\delta) = (p-1)[p-\delta-p(n-\delta)]; \]
\[ B(p,n,\delta) = (p-1)(n-\delta)(p+\delta)-(\delta-1)(p-\delta); \]
\[ C(p,n,\delta) = -\delta(n-\delta)(p-1), D = B^2 - 4AC; \]
\[ A_0(p,n) = -p^2(n-p), B_0(p,n) = p(p-1)(n-p-1); \]
\[ C_0(p,n) = p(p-1), D_0 = B_0^2 - 4A_0C_0. \]

Consider the quadratic equations
\[ Az^2 + Bz + C = 0 \text{ and } Cy^2 + By + A = 0, \]
\[ A_0z^2 + B_0z + C_0 = 0, \]
and note that their discriminants \(D\) and \(D_0\) are correspondingly
\[ D = B^2 - 4AC = [(p-1)(n-\delta)(p+\delta)-(\delta-1)(p-\delta)]^2 \]
\[ + 4(p-1)[p-\delta-p(n-\delta)]\delta(n-\delta)(p-1) \]
\[ = (p-\delta)^2 \left\{ [(p-1)(n-\delta)+1-\delta]^2 + 4\delta(p-1)(n-\delta) \right\} \]
\[ = (p-\delta)^2 D_1 > 0, \]

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for $p > 1, \delta \in (0, n), n \geq 2$ and $\delta \neq p$, 

$$D_0 = B_0^2 - 4A_0C_0 = p^2(p - 1)[(p - 1)(n - p - 1)^2 + 4p(n - p)] > 0.$$ 

Let us define some of the roots of (6.31), (6.32) 

$$z_+(p, n, \delta) = \frac{-B(p, n, \delta) + \sqrt{D(p, n, \delta)}}{2A(p, n, \delta)},$$ 

$$y_+(p, n, \delta) = \frac{-B(p, n, \delta) + \sqrt{D(p, n, \delta)}}{2C(p, n, \delta)},$$ 

$$z^0_-(p, n) = \frac{-B_0(p, n) - \sqrt{D_0(p, n)}}{2A_0(p, n)}.$$ 

In this section our main result is:

**Theorem 6.4.** For $n \geq 2, p > 1$, if $\Sigma = \left\{ \delta \in (0, n), \delta \neq \frac{n - 1}{p} \right\}$ then the estimate 

$$\lambda_{p,n}(B_R) \geq \Lambda_{p,n}^{(3)}(B_R) = \frac{1}{R^p} \sup_{\delta \in \Sigma} H(p, n, \delta),$$ 

(6.35)

holds where 

$$H(p, n, \delta) = \begin{cases} 
\left[ \frac{p - \delta}{p(1 - z_+(p, n, \delta))z_+(p, n, \delta)^{1-p}} \right]^{p-1} \left[ \frac{(p - \delta)z_+(p, n, \delta)}{p(1 - z_+(p, n, \delta))} + n - \delta \right], & 0 < \delta < p, \\
\left[ \frac{p - 1}{p - \delta - p} \right]^{p-1} \left[ \frac{p(1 - z_+(p, n, \delta)) + n - p}{pe^{-z^0_0(p, n)}} + \frac{n - p}{e^{-z^0_0(p, n)}} \right], & \delta = p, \\
\left[ \frac{\delta - p}{p(1 - y_+(p, n, \delta))y_+(p, n, \delta)^{1-p}} \right]^{p-1} \left[ \frac{(\delta - p)}{p(1 - y_+(p, n, \delta))} + n - \delta \right], & p < \delta. 
\end{cases}$$

Let us consider some special cases for $\delta$: 

- Suppose $\delta \to 0$, so that from (6.35) and $A(p, n, 0) = -p(p - 1)(n - 1), B(p, n, 0) = p[(p - 1)n + 1], C(p, n, 0) = 0, D(p, n, 0) = p^2[(p - 1)n + 1]^2$, we obtain $z_+(p, n, 0) = 0$. 

Applying the L'Hospital rule we get 

$$\lim_{\delta \to 0} H(p, n, \delta) = n \lim_{\delta \to 0} z_+(p, n, \delta) \frac{\delta(p - 1)}{p - \delta} = n \lim_{\delta \to 0} e^{-\frac{\delta(p - 1)}{p - \delta}} = n,$$

so that 

$$\lambda_{p,n}^{(3)}(B_R) \geq \Lambda_{p,n}^{(3,1)}(B_R) = \frac{n}{R^p}.$$ 

- We let $\delta \to n$ in (6.35) and from
\[ A(p, n, n) = (p-1)(p-n), \quad B(p, n, n) = (n-p)(n-1), \quad C(p, n, n) = 0, \quad D(p, n, n) = (p-n)^2(n-1)^2, \quad A_0(n, n) = 0, \quad B_0(n, n) = -n(n-1), \quad C_0(n, n) = n(n-1), \quad D_0(n, n) = n^2(n-1)^2 \]

we get

\[ \lim_{\delta \to n} z_+(p, n, \delta) = \frac{n-1}{p-1}, \quad 1 - \lim_{\delta \to n} z_+(p, n, \delta) = \frac{p-n}{p-1}, \quad \text{for } n < p, \]

\[ \lim_{\delta \to n} y_+(p, n, \delta) = \frac{p-1}{n-1}, \quad 1 - \lim_{\delta \to n} y_+(p, n, \delta) = \frac{n-p}{n-1}, \quad \text{for } p < n. \]

\[ \lim_{p \to n} z_0^+(p, n, \delta) = 1, \quad \text{for } p = n. \]

So

\[ \Lambda^{(3)}_{p, n}(B_R) \geq \Lambda^{(3,0)}_{p, n}(B_R) \]

\[ \geq \frac{1}{R^p} \lim_{\delta \to n} H(p, n, \delta) = \left( \frac{1}{pR} \right)^p \left[ \frac{(n-1)^{n-1}}{(p-1)^{p-1}} \right]^{\frac{p}{p-1}}, \quad \text{for } p \neq n. \]  

(6.36)

and

\[ \Lambda^{(3)}_{n, n}(B_R) \geq \Lambda^{(3,0)}_{n, n}(B_R) = \frac{1}{R^n} \lim_{\delta \to n} H(\delta, n, \delta) = \left( \frac{n-1}{nR} \right)^n e^n, \quad \text{for } p = n. \]  

(6.37)

Estimates (6.36), (6.37) coincide with the result in Theorem 6.2, estimates (6.13) and (6.14).

**Proof of Theorem 6.4.** The proof follows by means of the estimate from below of the kernels of the integrals in the right-hand side of (2.61) and (2.62).

We will consider cases \( \delta \neq p \) and \( \delta = p \) separately.

(i) For the case \( \delta \neq p, \delta \in (0, n) \) we have the following estimate from Hardy inequality (2.61)

\[ \int_{B_R} |\nabla u|^p dx \geq \int_{B_R} g(|x|)|u|^p dx, \]  

(6.38)

where

\[ g(|x|) = \left| \frac{p-\delta}{p} \right|^p \frac{1}{|x|^{\frac{(p-1)p}{p-1}} R^{\frac{p-\delta}{p-1}} - |x|^{\frac{\delta}{p-1}}} \]

\[ + \quad (n-\delta) \left| \frac{p-\delta}{p} \right|^{p-1} \frac{1}{|x|^{\delta} R^{\frac{p-\delta}{p-1}} - |x|^{\frac{\delta}{p-1}}}^{p-1}. \]

After the change of the variable \( |x| = Rr, r \in (0, 1) \) we get

\[ g(|x|) = g(Rr) = R^{-p} G(r), \]

where

\[ G(r) = \left| \frac{p-\delta}{p} \right|^{p-1} \left[ \left| \frac{p-\delta}{p} \right|^{\frac{(p-1)p}{p-1}} R^{\frac{p-\delta}{p-1}} - |x|^{\frac{\delta}{p-1}} \right]^{p-1} + \left| (n-\delta)r^\delta \right| R^{\frac{\delta}{p-1}} \left| 1 - r^{\frac{p-\delta}{p-1}} \right|^{1-p}. \]

Since
for $0 < \delta < p$

$$G(r) = g_1(r) = \left(\frac{p - \delta}{p}\right)^{p-1} \left[ \frac{p - \delta}{p} \left(1 - \frac{r - \frac{\delta - 1}{r-1}}{r - \frac{\delta - 1}{r-1}}\right)^{-p} \right]^{p-1}$$

+ \( (n - \delta)r^{-\delta} \left(1 - \frac{p - \delta}{p}\right)^{-1}\). \hspace{1cm} (6.39)

for $1 < p < \delta < n$

$$G(r) = g_2(r) = \left(\frac{\delta - p}{p}\right)^{p-1} \left[ \frac{\delta - p}{p} \left(1 - \frac{r - \frac{\delta - 1}{r-1}}{r - \frac{\delta - 1}{r-1}}\right)^{-p} \right]^{p-1}$$

+ \( (n - \delta)r^{-\delta} \left(1 - \frac{p - \delta}{p}\right)^{-1}\).

It is clear that \( \lim_{r \to 0} G(r) = \lim_{r \to 1} G(r) = \infty \) and the positive function \( G(r) \) has a positive minimum in \((0, 1)\). Our aim is to find the critical point of \( G(r) \) for \( r \in (0, 1) \).

Let us simplify the derivative of \( G(r) \) and

**for $0 < \delta < p$**

we denote \( r^{\frac{p - \delta}{p - 1}} = z, z \in (0, 1) \) we get

$$\frac{\partial G(r)}{\partial r} = \frac{\partial g_1(r)}{\partial r}$$

$$= \left(\frac{p - \delta}{p}\right)^{p-1} \left[ -\frac{p - \delta}{p - 1} (\delta - 1)r^{\frac{1 - \delta p}{p - 1}} \left(1 - \frac{r - \frac{\delta - 1}{r-1}}{r - \frac{\delta - 1}{r-1}}\right)^{-p} \right] \hspace{1cm}$$

+ \( \frac{(p - \delta)^2}{p - 1} r^{\frac{(1 - \delta)(p + 1)}{p - 1}} \left(1 - \frac{r - \frac{\delta - 1}{r-1}}{r - \frac{\delta - 1}{r-1}}\right)^{-1} - \delta(n - \delta)r^{-\delta - 1} \left(1 - \frac{p - \delta}{p - 1}\right)^{-1} \)

+ \( (p - \delta)(n - \delta)r^{-\delta - 1} \left(1 - \frac{p - \delta}{p - 1}\right)^{-p} \]

$$= \left(\frac{p - \delta}{p}\right)^{p-1} \left(1 - \frac{r - \frac{\delta - 1}{r-1}}{r - \frac{\delta - 1}{r-1}}\right)^{-p - 1} \left[ -\frac{p - \delta}{p - 1} (\delta - 1)z(1 - z) \right]$$

+ \( \frac{(p - \delta)^2}{p - 1} z^2 - \delta(n - \delta)(1 - z)^2 + (p - \delta)(n - \delta)z(1 - z) \]

$$= \left(\frac{p - \delta}{p}\right)^{p-1} \frac{1}{p - 1} \left[\frac{(\delta + 1)(p - 1)}{p - 1} - 1\right] (1 - z)^{p-1} (Az^2 + Bz + C).$$
• for $1 < p < \delta < n$ we denote $r_p = y, y \in (0, 1)$ and we get

$$\frac{\partial G(r)}{\partial r} = \frac{\partial g_2(r)}{\partial r} = \left(\frac{\delta - p}{p}\right)^{p-1} \left[ - (\delta - p)r^{p-1} \left(1 - \frac{\delta - p}{p}r\right)^{-p} + \left(\frac{\delta - p}{p}\right)^2 r^{p-1 + \frac{\delta - p}{p}} \left(1 - \frac{\delta - p}{p}r\right)^{-p-1}\right]$$

$$- p(n - \delta)r^{p-1} \left(1 - \frac{\delta - p}{p}r\right)^{1-p} + (\delta - p)(n - \delta)r^{p-1 + \frac{\delta - p}{p}} \left(1 - \frac{\delta - p}{p}r\right)^{-p}$$

$$= \frac{1}{p-1} \left(\frac{\delta - p}{p}\right)^{p-1} r^{p-1} \left(1 - \frac{\delta - p}{p}r\right)^{-p-1} \left[- (p-1)(\delta - p)(1 - z) + (\delta - p)^2 z\right]$$

$$- p(p-1)(n - \delta)(1 - z)^2 + (p-1)(\delta - p)(n - \delta)z(1 - z)$$

$$= \left(\frac{\delta - p}{p}\right)^{p-1} \frac{1}{p-1} y^{\frac{(p-1)(p-1)}{p-1}}(1 - y)^{-p-1} (Cy^2 + By + A),$$

where $A, B, C$ are defined in (6.30).

Suppose that $A \neq 0$, i.e., $\delta \neq \frac{pn - 1}{p - 1}$, then in order to find the critical point of $G(r)$, we have to solve the quadratic equations in (6.31).

The discriminant of the equations in (6.31) for $\frac{\partial G(r)}{\partial r} = 0$ is $D$ given in (6.33).

Since $D > 0$ and $A \neq 0$ the equation

$$P_1(z) = Az^2 + Bz + C = 0$$

(6.40)

has two real roots

$$z_{\pm} = \frac{-B \pm |p - \delta|\sqrt{D_1}}{2A} = \frac{2C}{-B \mp |p - \delta|\sqrt{D_1}}, \text{ for } 0 < \delta < p.$$  

Analogously, from $C > 0$ and $D > 0$ the equation

$$P_2(y) = Cy^2 + By + A = 0$$

has two real roots

$$y_{\pm} = \frac{-B \pm |p - \delta|\sqrt{D_1}}{2C} = \frac{2A}{-B \mp |p - \delta|\sqrt{D_1}} \text{ for } 1 < p < \delta < n.$$  

Later on we will use only the roots of $P_1(z) = 0$ and $P_2(y) = 0$ which are in the interval $(0, 1)$. In order to find which roots satisfy this condition we prove the following proposition.

**Proposition 6.1.** Let $n \geq 2$, $p > 1$, $\delta \in (0, n)$, $\delta \neq p$, then the following statements hold

i) If $p > n$, then
i1) \( A(p, n, \delta) > 0 \) if and only if \( \frac{p(n-1)}{p-1} < \delta < n \) so that \( z_- < 0 < z_+ \) and

\[
\inf_{r \in (0, 1)} G(r) = g_1 \left( \frac{p-1}{z_+^p} \right);
\]

i2) \( A(p, n, \delta) < 0 \) if and only if \( 0 < \delta < \frac{p(n-1)}{p-1} \) so that \( 0 < z_+ < z_- \) and

\[
\inf_{r \in (0, 1)} G(r) = g_1 \left( \frac{p-1}{z_-^p} \right);
\]

i3) \( A(p, n, \delta) = 0 \) if and only if \( 0 = \delta = \frac{p(n-1)}{p-1} \) so that (6.40) has an unique positive root

\[
z_+ \left( p, n, \frac{p(n-1)}{p-1} \right) = -\frac{C(p, n, \frac{p(n-1)}{p-1})}{B(p, n, \frac{p(n-1)}{p-1})}
\]

and

\[
\inf_{r \in (0, 1)} G(r) = g_1 \left( \frac{(p-1)^2}{z_+^p} \left( p, n, \frac{p(n-1)}{p-1} \right) \right);
\]

ii) If \( 1 < p < n \) then \( A(p, n, \delta) < 0 \) for \( \delta \in (0, n) \) and

ii1) for \( 0 < \delta < p \) we have \( 0 < z_+ < z_- \) and

\[
\inf_{r \in (0, 1)} G(r) = g_1 \left( \frac{p-1}{z_+^p} \right);
\]

ii1) for \( 1 < p < \delta < n \) we have \( 0 < y_+ < y_- \) and

\[
\inf_{r \in (0, 1)} G(r) = g_2 \left( \frac{p-1}{y_+^p} \right);
\]

**Proof.** i) Since for \( n > p \) we have \( n > \frac{p(n-1)}{p-1} \) and for \( 1 < p < n \) the inequality \( \frac{p(n-1)}{p-1} > n \) holds, the statements for the sign of \( A(p, n, \delta) \) follow immediately after (6.30)

i1) From \( A > 0, C < 0 \) it follows that \( B^2 - 4AC > B^2 \) and \( z_- = \frac{-B - \sqrt{D}}{2A} < \frac{-B - |B|}{2A} \leq 0 \), \( z_+ = \frac{-B + |B|}{2A} = 0 \). Thus the minimum of \( G(r) \) in the interval \((0, 1)\) is attained at the point \( \frac{p-1}{z_+^p} \);

i2) From \( P_1(0) = C < 0 \) we get that \( z_+ > 0, z_- > 0 \) and

\[
z_- = \frac{-B + \sqrt{D}}{2A} > \frac{-B - \sqrt{D}}{2A} = z_+ \cdot \frac{p-1}{z_+^p}.
\]

Since \( P_1(z) < 0 \) for \( z \in (0, z_+) \) and \( P_1(z) > 0 \) for \( z \in (z_+, z_-) \), it follows that \( G(r) \) has a minimum at the point \( \frac{p-1}{z_+^p} \);

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i3) The proof is trivial.

ii) ii1) The proof is identical with the proof of i2);

ii2) Since $P_2(0) = A < 0$, \(\lim_{z \to \infty} P_1(z) = -\infty\) it follows that \(y_+ > 0, y_- > 0\) and

\[
y_- = -\frac{B + \sqrt{D}}{2C} > -\frac{B - \sqrt{D}}{2C} = y_+.
\]

From the sign of \(P_2(z)\) we get that \(g_2(r)\) attains its minimum at the point \(y_+^{-1/p} \).

\[\square\]

(ii) For the case \(\delta = p < n\), from (2.62) we get

\[
\int_{B_R} |\nabla u|^p dx \geq \int_{B_R} g(|x|)|u|^p dx,
\]

where

\[
g(|x|) = \left(\frac{p-1}{p}\right)^p |x|^{-p} \left|\ln \frac{R}{|x|}\right|^{-p} + \left(\frac{p-1}{p}\right)^{p-1} (n-p)|x|^{-p} \left|\ln \frac{R}{|x|}\right|^{1-p}.
\]

For \(|x| = Rr, r \in (0, 1)\) we obtain

\[
g(|x|) = g(Rr) = R^{-p}G(r),
\]

where

\[
G(r) = \left(\frac{p-1}{p}\right)^{p-1} \left\{\frac{p-1}{p} r^{-p} (-\ln r)^{-p} + (n-p)r^{-p}(-\ln r)^{1-p}\right\}.
\]

Tedious calculations give us

\[
\frac{\partial G}{\partial r} = \left(\frac{p-1}{p}\right)^{p-1} \frac{1}{p} z^{-p-1} e^{-(p+1)z} [A_0 z^2 + B_0 z + C_0], \quad \text{for } z = -\ln r > 0.
\]

Critical points of \(\frac{\partial G}{\partial r}\) are the solutions of the quadratic equation (6.32).

Since \(D_0(p, n) > 0\) and \(A_0(p, n) \neq 0\) equation (6.32) has two real roots

\[
z_0^0 = -\frac{B_0 \pm \sqrt{D_0}}{2A_0}, \quad z_+^0 < z_0^0.
\]

From \(P_0(0) = p(p-1) > 0\) and \(\lim_{z \to \infty} P_0(z) = -\infty\) it follows that \(z_+^0 < 0 < z_0^0\).

Thus \(G(r)\) attains its minimum in \((0, 1)\) at the point \(r_0 = e^{-z_0^0}\), where from (6.34)

\[
z_0^0 = 2C_0 \left[\sqrt{D_0} - B_0\right]^{-1}.
\]

Theorem 6.4 is proved. \[\square\]

### 6.3 Comparison between different analytical estimates of \(\lambda_{p,n}(B_R)\)

In this section we will compare analytical estimates from below of \(\lambda_{p,n}(B_R)\) defined in Theorem 6.4 \(\Lambda_{p,n}^{(3)}(B_R)\) with \(\Lambda_{p,n}^{(1)}(B_R)\), defined in (6.4) with \(\Lambda_{p,n}^{(2)}(B_R)\), defined in (6.6) and with \(\Lambda_{p,n}^{(H)}(B_R)\), \(\Lambda_{p,n}^{(L)}(B_R)\), defined in (6.10) and (6.8). We compare only those estimates of \(\lambda_{p,n}(B_R)\) that are given with analytical formulas for every \(p > 1\), \(n \geq 2\). The estimate in (6.21), Sect. 6.2 is valid only for \(p > n \geq 2\) and that is why we will not use it for the comparison, no matter, it is clear that the right-hand side of (6.21) is greater than \(\Lambda_{p,n}^{(3,0)}(B_R)\)
Comparison of $\Lambda^{(H)}_{p,n}(B_R)$ and $\Lambda^{(L)}_{p,n}(B_R)$ with $\Lambda^{(1)}_{p,n}(B_R)$, $\Lambda^{(2)}_{p,n}(B_R)$ and $\Lambda^{(3)}_{p,n}(B_R)$

Let us compare $\Lambda^{(H)}_{p,n}(B_R) = \left| \frac{n-p}{pR} \right|^p$ for $n \geq 2$, $p > 1$, $n \neq p$ with other lower bounds for $\lambda_{p,n}(B_R)$.

- For $p > n \geq 2$ we get the estimate
  \[ \Lambda^{(H)}_{p,n}(B_R) = \left( \frac{p-n}{pR} \right)^p = \frac{1}{R^p} \left( 1 - \frac{n}{p} \right)^p < \frac{1}{R^p} < \frac{n}{R^p} = \Lambda^{(3,1)}_{p,n}(B_R); \]

- For $1 < p < n$ the inequality
  \[ \Lambda^{(H)}_{p,n}(B_R) = \left( \frac{n-p}{pR} \right)^p < \left( \frac{n}{pR} \right)^p = \Lambda^{(1)}_{p,n}(B_R) \]
  holds.

- As for the Lindqvist’s constant $\Lambda^{(L)}_{p,n}(B_R) = \frac{p}{R^p}$ for $p > n \geq 2$ given in (6.8) we get the estimate
  \[ \Lambda^{(L)}_{p,n}(B_R) = \frac{p}{R^p} < \frac{np}{R^p} = \Lambda^{(2,2)}_{p,n}(B_R). \]

Comparison of $\Lambda^{(3)}_{p,n}(B_R)$ with $\Lambda^{(1)}_{p,n}(B_R)$

We will use the estimate (6.36), which coincides with the estimate (6.13).

Proposition 6.2. For every $n \geq 2$ there exists $p_0_n$, $1 < p_0_n < 2$ such that
\[ \Lambda^{(1)}_{p,n} < \Lambda^{(3,0)}_{p,n} \leq \Lambda^{(3)}_{p,n}, \text{ for } p_0_n < p. \]

Proof. We define the function
\[ f_n(p) = \frac{1}{n-p} [(n-1) \ln(n-1) - (p-1) \ln(p-1)] - \ln n. \]

The inequality $\Lambda^{(3,0)}_{p,n}(B_R) > \Lambda^{(1)}_{p,n}(B_R)$ holds if and only if $f_n(p) > 0$. We will show that for every fixed $n \geq 2$ the function $f_n(p)$ is strictly increasing one for $p > 1$ and $\lim_{p \to 1} f_n(p) < 0$, $f_n(2) > 0$ for $n \geq 2$, $\lim_{n \to 2} f_n(2) = 1 - \ln 2 > 0$. Thus, there exists $p_0_n \in (1,2)$ such that $f_n(p) < 0$ for $1 < p < p_0_n$,
\[ f_n(p_0_n) = (n-1) \ln(n-1) - (p_0_n - 1) \ln(p_0_n - 1) - (n - p_0_n) \ln n = 0, \quad (6.41) \]
and $f_n(p) > 0$ for $p_0_n < p$.

For the first derivative of $f_n(p)$ we have
\[ f'_n(p) = \frac{1}{(n-p)^2} \left[ (n-1) \ln(n-1) - (n-1) + (p-1) - (n-1) \ln(p-1) \right] = \frac{g_n(p)}{(n-p)^2}. \]
Since \( g'_n(p) = \frac{p-n}{p-1} \), \( g''_n(p) = \frac{n-1}{(p-1)^2} > 0 \) then \( g_n(p) \) has a minimum at the point \( p = n \) and \( g_n(n) = 0 \). Using L’Hospital rule we obtain \( \lim_{p \to n} f'_n(p) = \frac{1}{2(n-1)} > 0 \) and hence \( f'_n(p) > 0 \) for every \( p > 1 \). Moreover, \( \lim_{p \to 1} f_n(p) = \ln(n-1) - \ln n < 0 \), and

\[
f_n(2) = \frac{1}{n-2} [ (n-1) \ln(n-1) - (n-2) \ln n ] > 0.
\]

The inequality (6.42) holds because for the function \( z(n) = (n-1) \ln(n-1) - (n-2) \ln n \) we have \( z' = \frac{2}{n} + \ln(n-1) - \ln n \), \( z'' = \frac{2-n}{n^2(n-1)} \leq 0 \), i.e., \( z' \) is a decreasing function, \( z'(n) > \lim_{n \to \infty} z'(n) > 0 \). Hence \( z(n) \) is a strictly increasing function and \( z(n) > z(2) = 0. \]

\[\square\]

**Comparison of \( \Lambda^{(3)}_{p,n}(B_R) \) with \( \Lambda^{(2)}_{p,n}(B_R) \)**

**Proposition 6.3.** For integer \( n \geq 2 \) and \( p \geq p_n = \frac{27}{8} \left( \frac{2n}{2n-3} \right)^2 \) the estimate

\[
\Lambda^{(2)}_{p,n}(B_R) < \Lambda^{(3)}_{p,n}(B_R),
\]

holds.

**Proof.** From (6.38) and (6.39) for \( \delta < p \), \( \delta \in (0, n) \) it follows that

\[
\Lambda^{(3)}_{p,n}(B_R) = R^{-p} \sup_{\delta \in (0, n)} \inf_{r \in (0, 1)} g_1(r) \geq R^{-p} \sup_{\delta \in (0, n)} \inf_{r \in (0, 1)} H_2(r),
\]

where

\[
H_2(R) = (n-\delta) \left( \frac{p-\delta}{p} \right)^{p-1} r^{-\delta} \left( 1 - r^{\frac{p-\delta}{p}} \right)^{1-p}.
\]

The positive function

\[
h_2(r) = \frac{1}{p-\delta} r^{\frac{\delta}{p-\delta}} \left( 1 - r^{\frac{p-\delta}{p-\delta}} \right)
\]

attains its maximum for \( 0 \leq r \leq 1 \) at the point \( r_2 = \left( \frac{\delta}{p} \right)^{\frac{p-1}{p-\delta}} < 1 \), because

\[
h'_2(r) = \frac{\delta^{\frac{p-1}{p-\delta}}}{(p-1)(p-\delta)} \left[ \delta - pr^{\frac{\delta}{p-\delta}} \right],
\]

\[h'_2(r_2) = 0, \; h_2(0) = h_2(1) = 0.\]

Thus from the equalities

\[
\inf_{r \in (0, 1)} H_2(r) = (n-\delta) \inf_{r \in (0, 1)} (ph_2(r))^{1-p} = (n-\delta) \sup_{r \in (0, 1)} (ph_2(r))^{p-1}
\]

\[= (n-\delta) (ph_2(r_2))^{p-1} = (n-\delta) \left( \frac{p}{\delta} \right)^{\frac{p(p-1)}{p-\delta}}.
\]

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we get the estimate
\[
\Lambda_{p,n}^{(3)}(B_R) \geq \Lambda_{p,n}^{(3,2)}(B_R) = R^{-p} \sup_{\delta \in (0,n)} (n - \delta) \left( \frac{p}{\delta} \right)^{\frac{\delta(p-1)}{p-\delta}}. \tag{6.44}
\]
Thus from (6.44), the estimate (6.43) holds if
\[
\Lambda_{p,n}^{(2)}(B_R) < \Lambda_{p,n}^{(3,2)}(B_R), \quad n \geq 2, p \geq p_n. \tag{6.45}
\]
Note that \( p_n \) is a decreasing function for \( n \in [2, \infty) \), so \( 3.375 < p_n < 54 \).

Since \( p_n > 3 \) then for \( n \geq 2, \delta < p \) the estimate (6.45) is equivalent to the inequality
\[
\sup_{\delta \in (0,n),\delta < p} (n - \delta) \left( \frac{p}{\delta} \right)^{\frac{\delta(p-1)}{p-\delta}} > np
\]
for \( p \geq p_n, p > \delta \).

For \( \delta = \frac{3}{2} \) and \( p \geq p_n \) a simple computation gives us
\[
\sup_{\delta \in (0,n),\delta < p} (n - \delta) \left( \frac{p}{\delta} \right)^{\frac{\delta(p-1)}{p-\delta}} > \left( n - \frac{3}{2} \right) \left( \frac{2p}{3} \right)^{\frac{3(p-1)}{p-3}}
\]
\[
= \frac{1}{2} (2n - 3) \left( \frac{2p}{3} \right)^{\frac{3}{2}} \left( \frac{2p}{3} \right)^{\frac{3(p-1)}{p-3}} \geq \frac{1}{3} (2n - 3) \left( \frac{2p}{3} \right)^{\frac{1}{2}} p
\]
\[
\geq \frac{1}{3} (2n - 3) \left( \frac{2p_0}{3} \right)^{\frac{1}{2}} p = \frac{1}{3} (2n - 3) \frac{3}{2} \frac{2n - 3}{p} = np.
\]

\[\square\]

Comparison of \( \Lambda_{p,n}^{(2,1)}(B_R) \) with \( \Lambda_{p,n}^{(2,2)}(B_R) \)

Proposition 6.4. For every \( n \geq 2 \) the estimates
\[
\Lambda_{p,n}^{(2,1)}(B_R) > \Lambda_{p,n}^{(2,2)}(B_R) \quad \text{for } p \in (1, 2), \tag{6.46}
\]
\[
\Lambda_{p,n}^{(2,1)}(B_R) < \Lambda_{p,n}^{(2,2)}(B_R) \quad \text{for } p > 2, \tag{6.47}
\]
hold.

Proof. The inequality (6.46) is equivalent to \( h(p) = (p-1) \ln \frac{p}{p-1} - \ln p > 0 \) for \( p \in (1, 2) \),
while (6.47) holds when \( h(p) < 0 \) for \( p > 2 \).

A simple computation gives us
\[
h'(p) = \ln \frac{p}{p-1} - \frac{2}{p}, \quad h''(p) = \frac{p-2}{p^2(p-1)}, \quad h'(2) = \ln \frac{2}{e} < 0, \quad \tag{6.48}
\]
and from the L'Hospital rule
\[
\lim_{p \to \infty} h'(p) = \lim_{p \to \infty} \frac{p \ln \frac{p}{p-1} - 2}{p} = \lim_{p \to \infty} \left( \ln \frac{p}{p-1} - \frac{1}{p-1} \right) = 0.
\]
Thus we have from (6.48) that \( h'(p) < 0 \) for \( p > 2 \). From \( h(2) = 0 \) it follows that \( h(p) < 0 \) for \( p > 2 \).

From \( \lim_{p \to 1} h(p) = 0 \) and the concavity of \( h(p) \) for \( p \in (1, 2) \) we get \( h(p) > 0 \) for \( p \in (1, 2) \).
Comparison of $\Lambda_{p,n}^{(1)}(B_R)$ with $\Lambda_{p,n}^{(2)}(B_R)$

According to Proposition 6.4 we will compare $\Lambda_{p,n}^{(1)}(B_R)$ with $\Lambda_{p,n}^{(2)}(B_R)$ for $p \in (1, 2)$, and $\Lambda_{p,n}^{(1)}(B_R)$ with $\Lambda_{p,n}^{(2)}(B_R)$ for $p \geq 2$.

**Proposition 6.5.**

- If $n \in [2, 8]$, then
  
  $$\Lambda_{p,n}^{(2)}(B_R) > \Lambda_{p,n}^{(1)}(B_R) \quad \text{for } p > 1, p \neq 2 \text{and } \Lambda_{p,n}^{(2)}(B_R) > \Lambda_{p,n}^{(1)}(B_R).$$

- If $n \geq 9$, then there exist constants $p_1, n_1 \in (1, 2)$ and $p_3, n_3 \geq 2$ such that
  
  $$\Lambda_{p,n}^{(2)}(B_R) > \Lambda_{p,n}^{(1)}(B_R) \quad \text{for } p \in (1, p_1) \cup (p_3, \infty),$$

  $$\Lambda_{p,n}^{(2)}(B_R) < \Lambda_{p,n}^{(1)}(B_R) \quad \text{for } p \in (p_1, p_3, n_3). \quad (6.49)$$

*Proof.* Case 1: $p \in (1, 2)$.

For every fixed $n$, inequality $\Lambda_{p,n}^{(1)}(B_R) > \Lambda_{p,n}^{(2,1)}(B_R)$ is equivalent to

$$h(p) = (p - 1) \ln n - (2p - 1) \ln p + (p - 1) \ln(p - 1) > 0, \quad \text{for } p \in (1, 2).$$

Simple computations give us $h'(p) = \ln n - 2 \ln p - 1 + \frac{1}{p} + \ln(p - 1)$, $h''(p) = -\frac{p^2 - p - 1}{p^2(p - 1)}$

and $h''(p) > 0$ for $p \in \left(1, \frac{\sqrt{5} + 1}{2}\right)$, $h''(p) < 0$ for $p \in \left(\frac{\sqrt{5} + 1}{2}, 2\right)$. Hence $h'(p)$ has a maximum at the point $\frac{\sqrt{5} + 1}{2}$ and

$$h'\left(\frac{\sqrt{5} + 1}{2}\right) = \ln n - 2 \ln \frac{\sqrt{5} + 1}{2} - 1 + \frac{2}{\sqrt{5} + 1} + \ln \frac{\sqrt{5} - 1}{2},$$

i.e.,

$$h'\left(\frac{\sqrt{5} + 1}{2}\right) < 0 \quad \text{for } n < \left(\frac{\sqrt{5} + 1}{2}\right)^3 e^{\frac{\sqrt{7} - 1}{2}} \approx 6.2065$$

and

$$h'\left(\frac{\sqrt{5} + 1}{2}\right) > 0 \quad \text{for } n > 6.2065.$$ 

Since $\lim_{p \to 1} h'(p) = -\infty$, $h'(2) = \frac{1}{2} \ln \frac{n^2}{16e} < 0$ for $n \in [2, 6]$ and $h'(2) > 0$ for $n \geq 7$, it follows that

$$h'(p) < 0, \quad \text{for } n \in [2, 6] \quad \text{and } p \in (1, 2).$$

For $n \geq 7$ there exists $q_n \in \left(1, \frac{\sqrt{5} + 1}{2}\right)$ such that

$$h'(p) < 0, \quad \text{for } p \in (1, q_n),$$

$$h'(p) > 0, \quad \text{for } p \in (q_n, 2).$$

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Since \( \lim_{p \to 1} h(p) = 0, h(2) = \ln \frac{n}{8} \) and \( h(2) < 0 \) for \( n \in [2, 7] \), \( h(2) = 0 \) for \( n = 8 \), \( h(2) > 0 \) for \( n > 8 \) it follows that \( h(p) < 0 \) for \( p \in (1, 2) \), \( n \in [2, 8] \).

Since \( h(2) = \ln \frac{n}{8} > 0 \) for \( n \geq 9 \), there exists \( p_{1,n} \in (1, 2) \) such that

\[
h(p_{1,n}) = (p_{1,n} - 1) \ln n - (2p_{1,n} - 1) \ln p_{1,n} + (p_{1,n} - 1) \ln(p_{1,n} - 1) = 0, \tag{6.50}
\]

and \( h(p) < 0 \) for \( p \in (1, p_{1,n}) \) and \( h(p) > 0 \) for \( p \in (p_{1,n}, 2) \). Thus Proposition 6.5 for \( p \in (1, 2) \) is proved.

Case 2: \( p \geq 2 \).

The inequality (6.49) for \( p \geq 2 \) is equivalent for every fixed \( n \geq 2 \) to

\[
h_1(p) = (p - 1) \ln n - (p + 1) \ln p > 0, \quad \text{for } p \geq 2.
\]

A simple computation gives us \( h'_1(p) = \ln n - p - \frac{p + 1}{p} \), \( h''_1(p) = \frac{1 - p}{p^2} < 0 \) for \( p \geq 2 \), \( h_1(2) = \ln \frac{n}{8} < 0 \) for \( n \in [2, 7] \), \( h_1(2) = 0 \), for \( n = 8 \).

Since \( h'_{1}(2) = \ln \frac{n}{2e^{3/2}} \leq \ln \frac{8}{2e^{3/2}} \approx -0.1137 \) for \( n \in [2, 8] \) it follows that \( h_1(p) < 0 \) for \( p > 2 \) and \( n \in [2, 8] \).

For \( n \geq 9 \) we get \( h_1(2) = \ln \frac{n}{8} > 0 \), \( \lim_{p \to \infty} h_1(p) = -\infty \) and consequently there exists a constant \( p_{3,n} > 2 \),

\[
h_1(p_{3,n}) = (p_{3,n} - 1) \ln n - (p_{3,n} + 1) \ln p_{3,n} = 0,
\]

such that \( h_1(p) > 0 \) for \( p \in [2, p_{3,n}) \), \( h_1(p) < 0 \) for \( p > p_{3,n} \). For example, \( p_{3,n} = 3 \) for \( n = 9 \). \( \square \)

Finally, we summarize the analytical results in Propositions 6.2–6.5.

Suppose \( n \geq 9 \), then from Proposition 6.2 we get \( \Lambda_{p,n}^{(3)}(B_R) > \Lambda_{p,n}^{(1)}(B_R) \) for \( p > p_{0,n} \), where \( p_{0,n} \in (1, 2) \) is a solution of equation (6.41).

Analogously, from Proposition 6.5 we obtain the estimates \( \Lambda_{p,n}^{(1)}(B_R) > \Lambda_{p,n}^{(2)}(B_R) \) for \( p > p_{1,n} \) and \( \Lambda_{p,n}^{(1)}(B_R) < \Lambda_{p,n}^{(2)}(B_R) \) for \( p \in (1, p_{1,n}) \), where \( p_{1,n} \in (1, 2) \) is a solution of the equation (6.50). Thus for \( n \geq 9 \) we identify the following cases:

(i) If \( p_{1,n} < p_{0,n} \) then

\[
\Lambda_{p,n}^{(2)}(B_R) > \max \left\{ \Lambda_{p,n}^{(1)}(B_R), \Lambda_{p,n}^{(3)}(B_R) \right\}, \quad \text{for } p \in (1, p_{1,n}),
\]

\[
\Lambda_{p,n}^{(1)}(B_R) > \Lambda_{p,n}^{(2)}(B_R), \quad \text{for } p \in (p_{1,n}, p_{0,n}),
\]

\[
\Lambda_{p,n}^{(3)}(B_R) > \max \left\{ \Lambda_{p,n}^{(1)}(B_R), \Lambda_{p,n}^{(2)}(B_R) \right\}, \quad \text{for } p > p_{0,n}.
\]

(ii) If \( p_{0,n} < p_{1,n} \) then there exists \( p_{2,n} \in [p_{0,n}, p_{1,n}) \subset (1, 2) \) such that

\[
\Lambda_{p,n}^{(2)}(B_R) > \max \left\{ \Lambda_{p,n}^{(1)}(B_R), \Lambda_{p,n}^{(3)}(B_R) \right\}, \quad \text{for } p \in (1, p_{2,n}),
\]

\[
\Lambda_{p,n}^{(3)}(B_R) > \max \left\{ \Lambda_{p,n}^{(1)}(B_R), \Lambda_{p,n}^{(2)}(B_R) \right\}, \quad \text{for } p > p_{2,n}.
\]

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For $1 < p < p_n$, $n \in [2, 8]$ the comparison is by means of numerical calculations.

**Remark 6.1.** For sufficiently large values of $n$ we get $p_{1,n} > p_{0,n}$. Indeed, after the limit $n \to \infty$ in (6.50) it follows that $\lim_{n \to \infty} p_{1,n} = 1$. From the definitions of $p_{0,n}$ and $p_{1,n}$ we have that $p = p_{0,n}$ satisfying the equation

$$(n - p)f_n(p) = (n - 1)\ln(n - 1) - (p - 1)\ln(p - 1) - (n - p)\ln n = 0,$$

while $y = p_{1,n}$ satisfies the equation

$$h(p) = (y - 1)\ln n - (2y - 1)\ln y + (y - 1)\ln(y - 1) = 0.$$

Hence for $y$ we obtain

$$(n - p)f_n(y) - h(y) = (n - 1)\ln \frac{n - 1}{n} + (2y + 1)\ln y - 2(y - 1)\ln(y - 1) \to_{n \to \infty} -1,$$

because

$$(n-1)\ln \frac{n - 1}{n} = \left(\frac{n - 1}{n - 1}\right) \ln \left(\frac{n - 1}{n} - 1\right) \to_{n \to \infty} \ln e^{-1} = -1, \quad \text{and} \quad \lim_{y \to 1}(y - 1)\ln(y - 1) = 0.$$

From the inequality $f_n(y) = f_n(p_{1,n}) < 0$ for $n$ sufficiently large it follows that $p_{1,n} < p_{0,n}$ for $n \gg 1$.

**Numerical comparison of $\Lambda^{(3)}_{p,n}(B_R)$ and $\Lambda^{(2)}_{p,n}(B_R)$**

Using the formulas (6.35) for $\Lambda^{(3)}_{p,n}(B_R)$ and (6.6) for $\Lambda^{(2)}_{p,n}(B_R)$ we listed below in Table 1 for $R = 1$ and fixed $n \in [2, 9]$ the intervals of $p$ where $\Lambda^{(3)}_{p,n}(B_1) \geq \Lambda^{(2)}_{p,n}(B_1)$ and where $\Lambda^{(2)}_{p,n}(B_1) \geq \Lambda^{(3)}_{p,n}(B_1)$. Numerical calculations are made by Mathematica 6. For example, for $n = 3$ and $p > 4.25$ we have $\Lambda^{(3)}_{p,n} > \Lambda^{(2)}_{p,n}$, while for $p < 4.25$ we have $\Lambda^{(3)}_{p,n} < \Lambda^{(2)}_{p,n}$.

| $n$ | $p \approx 1.32$ | $p \approx 1.33$ | $p \approx 1.35$ | $p \approx 1.38$ | $p \approx 1.43$ | $p \approx 1.64$ | $p \approx 4.25$ | $p \approx 38.68$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2   | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ | $\Lambda^{(3)}_{0,2} \leftarrow \Lambda^{(2)}_{0,2}$ |
| 3   | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ | $\Lambda^{(3)}_{0,3} \leftarrow \Lambda^{(2)}_{0,3}$ |
| 4   | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ | $\Lambda^{(3)}_{0,4} \leftarrow \Lambda^{(2)}_{0,4}$ |
| 5   | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ | $\Lambda^{(3)}_{0,5} \leftarrow \Lambda^{(2)}_{0,5}$ |
| 6   | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ | $\Lambda^{(3)}_{0,6} \leftarrow \Lambda^{(2)}_{0,6}$ |
| 7   | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ | $\Lambda^{(3)}_{0,7} \leftarrow \Lambda^{(2)}_{0,7}$ |
| 8   | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ | $\Lambda^{(3)}_{0,8} \leftarrow \Lambda^{(2)}_{0,8}$ |
| 9   | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ | $\Lambda^{(3)}_{0,9} \leftarrow \Lambda^{(2)}_{0,9}$ |

**Comparison of $\Lambda^{(3,0)}_{p,n}(B_R)$ with numerical values**

As is mention in Sect. 1 iterative numerical method for evaluating the first eigenvalue was developed in Biezuner et al. [23] where the approximate values, denoted here as $\Lambda^{(num)}_{p,n}(B_1)$ of the first eigenvalue $\Lambda_{p,n}(B_1)$ are given for $p \in (1, 4]$ and $n = 2, 3, 4$.

Table 2 shows that the difference between the calculated numerical values of $\lambda_{p,n}(B_1)$ and the estimates from below $\Lambda_{p,n}(B_1)$ is about 2.5 times more. Nevertheless the presented method for estimates of $\Lambda_{p,n}(B_1)$ from below using Hardy inequality with double singular weights gives analytical estimates for every $p > 1$ and $n \geq 2$. 89
Table 2: Numerical comparison of $\Lambda_{p,n}^{(3,0)}$ and numerical values $\Lambda_{p,n}^{(num)}$ in Table 1, Biezuner et al. [23]

| $p$ | $n = 2$ | $n = 3$ | $n = 4$ |
|-----|--------|--------|--------|
|     | $\Lambda_{p,n}^{(3,0)}$ | $\Lambda_{p,n}^{(num)}$ | $\Lambda_{p,n}^{(3,0)}$ | $\Lambda_{p,n}^{(num)}$ | $\Lambda_{p,n}^{(3,0)}$ | $\Lambda_{p,n}^{(num)}$ |
| 1.2 | 1.3021 | 2.9601 | 2.5093 | 4.5026 | 3.7873 | 6.0797 |
| 1.4 | 1.4683 | 3.6637 | 2.8940 | 5.7188 | 4.4860 | 7.8947 |
| 1.6 | 1.6063 | 4.3477 | 3.2628 | 6.9849 | 5.2046 | 9.8786 |
| 1.8 | 1.7308 | 5.0434 | 3.6298 | 8.3443 | 5.9574 | 12.0940 |
| 2.0 | 1.8472 | 5.7616 | 4.0000 | 9.8144 | 6.7500 | 14.5735 |
| 2.2 | 1.9582 | 6.5071 | 4.3755 | 11.405 | 7.5854 | 17.3421 |
| 2.4 | 2.0652 | 7.2823 | 4.7579 | 13.1232 | 8.4658 | 20.4220 |
| 2.6 | 2.1621 | 8.0885 | 5.1476 | 14.9747 | 9.3926 | 23.8345 |
| 2.8 | 2.2707 | 8.9265 | 5.5453 | 16.9646 | 10.3672 | 27.6004 |
| 3.0 | 2.3703 | 9.7967 | 5.9512 | 19.0977 | 11.3906 | 31.7409 |
| 3.2 | 2.4683 | 10.6994 | 6.3655 | 21.3785 | 12.4639 | 36.2769 |
| 3.4 | 2.5648 | 11.6347 | 6.7884 | 23.8111 | 13.5881 | 41.2298 |
| 3.6 | 2.6601 | 12.6027 | 7.2199 | 26.3977 | 14.7642 | 46.6213 |
| 3.8 | 2.7543 | 13.6034 | 7.6601 | 29.1486 | 15.9929 | 52.4734 |
| 4.0 | 2.8476 | 14.6369 | 8.1091 | 32.0618 | 17.2752 | 58.8085 |

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