KINEMATIC CONSTRAINTS TO THE KEY INFLATIONARY OBSERVABLES

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ABSTRACT

The observables $T/S$ and $n-1$ are key to testing and understanding inflation. ($T$, $S$, and $n-1$ respectively quantify the gravity-wave and density-perturbation contributions to CMB anisotropy and the deviation of the density perturbations from the scale-invariant form.) Absent a standard model, there is no definite prediction for, or relation between, $T/S$ and $n-1$. By reformulating the equations governing inflation we show that models generally predict $T/S \approx -5(n-1)$ or 0, and in particular, if $n > 0.85$, $T/S$ is expected to be $> 10^{-3}$. 


**Introduction.** Cosmic microwave background (CMB) anisotropy measurements have begun to test inflation, the leading paradigm to extend the standard big-bang cosmology. Within a decade they should test inflation decisively and even probe the underlying physics \([1, 2, 3]\). Recent results from the BOOMERanG and MAXIMA CMB experiments \([4, 5]\) (as well as results from earlier experiments \([6]\)) confirm the flat Universe predicted by inflation and are beginning to address its second basic prediction: almost scale-invariant adiabatic, Gaussian density perturbations produced by quantum fluctuations during inflation \([7]\). The third prediction, a nearly scale-invariant spectrum of gravity waves, will be more difficult to confirm, but is a critical probe of inflation \([8]\).

The key inflationary observables are: the level of anisotropy arising from density (scalar) perturbations (quantified by the contribution to the CMB quadrupole anisotropy, \(S\)), the level of anisotropy arising from gravity-wave (tensor) perturbations (\(T\)), and the power-law index \(n\) that characterizes the density perturbations (scale invariance refers to equal amplitude fluctuations in the gravitational potential on all length scales and corresponds to \(n = 1\)). If \(T\), \(S\) and \(n-1\) can be measured, then the scalar-field potential that drove inflation can be reconstructed \([10]\). The most promising means of measuring \(T\) is its unique signature in the polarization of CMB anisotropy \([9]\) (however, direct detection by a future space-based experiment should not be dismissed).

While there is no standard model of inflation, all known models can be cast in terms of the classical evolution of a new scalar field \(\phi\) (dubbed the inflaton) \([13]\). Predictions for \(S\), \(T\) and \(n-1\) can be expressed in terms of the scalar-field potential \(V(\phi)\) and its first two derivatives. While there is a model-independent relation between \(T/S\) and the power-law index \(n_T\) that characterizes the gravity-wave spectrum, \(T/S = -5n_T\) \([11, 12]\), no such relation for \(n\) and \(T/S\) exists \([14]\).

This is unfortunate because \(n_T\) is very difficult to measure, and \(n\) will be measured to a precision of better than 1% by the MAP and PLANCK experiments (BOOMERanG and MAXIMA have already determined that \(n = 1 \pm 0.06\)). Even an approximate or generic relation between \((n - 1)\) and \(T/S\) would be valuable, both as a test of inflation and as a guide for the expected level of gravity waves when \(n\) is measured.

The formation of large-scale structure and CMB measurements already indicate that a significant part of CMB anisotropy arises from scalar perturbations (\(T/S\) cannot be \(>1\)). On the other hand, nothing precludes \(T/S \ll 1\), and if \(T/S\) is much less than \(10^{-3}\), the prospects for measuring \(T\) are poor \([9]\) (one inflation theorist has opined that \(T/S \ll 1\) for all reasonable models \([15]\)).

The goal of this work is to provide objective theoretical guidance. By casting the equations governing inflation in a form that is essentially independent of the inflaton potential (“flow equations” for \(T/S\) and \(n-1\)), we show that the \(T/S - (n-1)\) plane is not uniformly populated by models of inflation: The lines \(T/S \approx 0\) and \(T/S \approx -5(n-1)\) act as attractors for models that are consistent with the equations governing inflation. For \(n < 1\), there is an excluded region between these two attractors; for \(n > 1\), other values for \(T/S\) and \(n-1\) are possible, but at the expense of a spectrum of density perturbations that is poorly represented by a power law. (the CMB will be able to test how well a power law describes the density perturbations.)
Flow equations. The kinematic equations that govern inflation are well known \[16\]
\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0 \tag{1}
\]
\[
H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_{\text{Pl}}^2} \left[V(\phi) + \frac{1}{2} \dot{\phi}^2\right] \tag{2}
\]
where $a(t)$ is the cosmic scale factor, prime denotes $d/d\phi$, and overdot denotes $d/dt$. During inflation $\phi$ rolls slowly and the $\ddot{\phi}$ term in its equation of motion and its kinetic term in the Friedmann equation can be neglected \[16, 17\], so that
\[
\dot{\phi} \simeq -\frac{V'}{3H} \tag{3}
\]
\[
N(\phi) \equiv \int_{\phi}^{\phi_{\text{end}}} H dt \simeq -\frac{8\pi}{m_{\text{Pl}}^2} \int_{\phi}^{\phi_{\text{end}}} d\phi \frac{1}{x(\phi)} \tag{4}
\]
where $x(\phi) \equiv V'(\phi)/V(\phi)$ measures the steepness of the potential and $N(\phi)$, the number of e-folds before the end of inflation, is the natural time variable. Inflation ends when the slow-roll conditions,
\[
m_{\text{Pl}} V'/V = m_{\text{Pl}} x < \sqrt{48\pi} \tag{5}
\]
\[
m_{\text{Pl}}^2 V''/V = m_{\text{Pl}}^2 (x' + x^2) < 24\pi \tag{6}
\]
are violated (at $\phi = \phi_{\text{end}}$) \[16, 17\].

The inflationary observables are related to the same quantities that govern the kinematics of inflation \[12\]
\[
(n - 1) = \frac{m_{\text{Pl}}^2}{8\pi} \left[2x' - x^2\right] \tag{7}
\]
\[
T/S = \frac{5m_{\text{Pl}}^2}{8\pi} x^2 \tag{8}
\]
\[
T = 0.6V/m_{\text{Pl}}^4 \tag{9}
\]
These expressions are given to lowest order in $x^2$ and $x'$ (see Ref. \[18\] for higher-order corrections). Note, $n - 1$ is only equal to $n_T = -5(T/S)$ if $x' = 0$.

By combining the slow-roll equations with those governing $(n - 1)$ and $T/S$, we can write equations that govern the inflationary observables (almost) without reference to a model,
\[
\frac{d(T/S)}{dN} = (n - 1) \frac{T}{S} + \frac{1}{5} \left(\frac{T}{S}\right)^2 \tag{10}
\]
\[
\frac{d(n - 1)}{dN} = -\frac{1}{5} (n - 1) \frac{T}{S} - \frac{1}{25} \left(\frac{T}{S}\right)^2 \pm \frac{m_{\text{Pl}}^3}{16\pi^2} \sqrt{\frac{2\pi T}{5}} S x'' \tag{11}
\]
where the sign of the last term matches that of $V'$.

We call these “flow equations” as they describe the trajectory in the $T/S - (n - 1)$ plane during inflation. Because of the $x''$ term they are not completely independent of the potential.
To “close” the flow equations we will assume that the potential is smooth enough so that we can treat $x''$ as being approximately constant. For sufficiently smooth and featureless potentials $x''$ will also be small.

Finally, one might wonder what happened to the most stringent constraint on inflation: achieving density perturbations of amplitude $10^{-5}$ or so ($S \sim 10^{-10}$). The flow equations involve the quantities $T/S$, $(n - 1)$ and $dN/d\phi$ which are unaffected by a rescaling of the potential, $V \rightarrow aV$. This rescaling changes $S$: $S \sim aS$. Thus, any potential can be rescaled to give proper size density perturbations without affecting the flow equations.

**Trajectories and attractors.** The scales relevant for structure formation (1 Mpc to $10^4$ Mpc) crossed outside the horizon roughly $50 e$-folds before the end of inflation (i.e., when $N = 50$), and so it is $T/S$ and $(n - 1)$ at this time that can be measured by CMB experiments. We find them by evolving $T/S$ and $(n - 1)$ until inflation ends and counting back 50 e-folds.

To determine when inflation ends, we recast the slow-roll conditions (5,6):

$$T/S < 30$$

$$|n - 1| + \frac{3 T}{5 S} < 6$$

To be specific, we pick “initial” values in the range, $0 < T/S < 10$ and $-0.5 < (n - 1) < 0.5$, and then integrate with fixed $x''$ until one of the slow-roll conditions is violated, signaling the end of inflation. We then count back 50 e-folds to find $(T/S)_{50}$ and $(n - 1)_{50}$. Some trajectories are shown in Fig. 1.

Figs. 2 and 3 summarize the $(T/S)_{50}$ – $(n - 1)_{50}$ phase space generated from the range of initial conditions considered. The $(T/S)_{50}$ – $(n - 1)_{50}$ plane is not uniformly populated. For $x'' < O(1)$, solutions cluster around two attractors, $(T/S)_{50} \approx 0$ and $(T/S)_{50} \approx -5(n - 1)_{50}$, and for $(n - 1)_{50} < 0$, there is an excluded region between the two attractor lines, which cannot be reached for any value of $x''$. We call the region between the excluded area and $x'' = 3$ as the favored region for the inflationary observables $n - 1$ and $T/S$.

Taking $x'' = 0$ it is simple to show how the attractors arise. In this limit, the flow equations are: $s \equiv (n - 1) - \frac{1 T}{5 S} = \text{const}$ and $r \propto \exp(sN)$, where $r = T/S$. Unless $r$ and/or $s$ are small, corresponding to the attractor solutions, $r$ grows very rapidly and inflation does not last 50 e-folds.

For $n > 1$, values of $(T/S)$ and $(n - 1)$ outside the favored region are possible at the expense of large $x''$. Models with a large $x''$ have density-perturbation spectra which are not well represented by a power law: the running of the spectral index \[20\], $dn/d\ln k = -dn/dN$ includes the term

$$\frac{dn}{d\ln k} = \cdots \pm \frac{m_{Pl}^2}{16 \pi^2} \sqrt{\frac{2 \pi T}{5 S}} x''$$

which becomes large for large $x''$. This explains the results of a recent paper [21] in which models with $n$ as large as 2 were constructed. In particular, for the model with $n = 2$, $x'' \approx 2000$, $T/S \approx 3 \times 10^{-3}$ and $dn/d\ln k \approx 0.3$.

Fig. 3 shows the $T/S \approx 0$ attractor region in more detail. As $n$ increases toward unity, values of $T/S$ in the favored region grow, making the prospects for measuring $T$ better; in particular, for $n > 0.85$, $T/S > 10^{-3}$. This figure also confirms that almost, but not exactly,
Figure 1: Trajectories in the $T/S - (n-1)$ plane. Squares indicate the initial choices for $T/S$ and $(n-1)$; circles indicate the values 50 e-folds before the end of inflation. A trajectory ends when $T/S$ and/or $|n-1|$ become large; most of inflation occurs when $T/S$ and $|n-1|$ are small. The upper left panel shows a complete trajectory, with ticks indicating e-folds before the end of inflation (from the circle, 50, 49, ⋯, 1). The other three panels show trajectories in more detail. Note how $T/S$ and $(n-1)$ outside the attractor region are “pulled in” (the attractors are shown as broken lines and the boundary of the excluded region is a solid curve).
Figure 2: Summary of our model search using the flow equations. The lines $T/S = -5(n-1)$ and $T/S = 0$ act as attractors; the dotted curves correspond to $x'' = 0, 1, 2, 5$ (from left to right). We found no model in the excluded region, and we call the region between it and the curve $x'' = 3$ the favored region. Models outside the favored region (upper right part) have large $dn/d\ln k$ and density perturbations that are not well represented by a power law. Diamonds indicate various known inflationary models: chaotic, $V(\phi) = \lambda \phi^n$ for $n = 2, 3, \cdots$ (diamonds on the diagonal); new inflation ($n = 0.94$) and natural inflation (with $n = 0.84$). The ellipse is the $2\sigma$ error ellipse “forecasted” for the PLANCK satellite [19].
scale-invariant density perturbations are a generic prediction of inflation [7, 17]: the favored
region just touches $n = 1$ (for $x'' \simeq 1.5$). Finally, in the disfavored region where $T/S \ll 1$, large $x''$ does not imply a poor power law because $dn/d\ln k$ is proportional to $T/S$. Indeed one of the models in this region is new inflation.

So far, we have only considered one-field inflation. There are potentials that are so flat and smooth that the slow-roll conditions never break down; the most well known of these is power-law inflation, $V(\phi) \propto \exp(-\beta\phi/m_{Pl})$. In a “never ending” model, another field causes the slow-roll conditions to break down (e.g., by classical evolution in hybrid inflation or a phase transition in extended inflation). The flow equations can also be applied to such models.

In our framework two-field models are models that would inflate forever on their own. We find such models when the right hand sides of Eqs. (10,11) vanish prior to violating the slow-roll conditions. When this happens, we obtain fixed points in the $T/S - (n - 1)$ plane, which are the most likely values for $T/S$ and $(n - 1)$ 50 e-folds prior to when the second field ends inflation. These points, shown in Fig. 3, populate the region $T/S \approx 0$ and for $n > 1$ along with the attractor line, $T/S = -5(n - 1)$.

It is also possible that a self-ending model has an auxiliary field that ends inflation “early”. We treat this possibility by populating the $(T/S)_{50} - (n - 1)_{50}$ plane with the values of $T/S$ and $(n - 1)$ at $N > 50$ for all one-field models. We find that the two-field models behave similarly to the one-field models. The only significant difference is that two-field models extend the $(T/S)_{50} = 0$ attractor to $(n - 1)_{50} > 0$ (see Fig. 4).

Finally, what about our taking $x'' \simeq$ constant? It can affect the relationship between the initial and final values of $n - 1$ and $T/S$ if $x''$ is large, since $x''$ need not be constant (as is the case in some known models). Since we have covered a wide range of initial values we would expect that this fact would only slightly modify the $(n - 1)_{50} - (T/S)_{50}$ phase space; indeed, we have also formulated the flow equations assuming $V''/V = constant$ and obtain similar results. Further, in the favored part of the $(n - 1)_{50} - (T/S)_{50}$ plane, $x''$ is small.

Discussion. Prior to this work there was one guiding relation for the inflationary observables: $T/S = -5nT$. It has the virtue of exactitude and can test the consistency of the scalar-field inflationary framework, but it involves the power-law index of the gravity wave perturbations, the most difficult inflationary observable to measure. By reformulating the equations governing inflation, we have found generic relations between $T/S$ and $(n - 1)$: Inflationary kinematics constrain models to cluster along the lines $T/S = -5(n - 1)$ and $T/S = 0$, with a forbidden region between these two lines for $n < 1$. Large $n - 1$ is possible, but at the expense of a poor power-law for the density perturbations (i.e., large $dn/d\ln k$), unless $T/S$ is very small. Further, our results support the view that inflation generically predicts almost, but not exactly scale-invariant density perturbations [17].

These results provide practical guidance to CMB experimenters and additional tests for inflation. For example, if $n$ is found to be significantly greater than 1, then a poor power-law is also expected unless $T/S \ll 1$. If $n$ is found to be $> 0.85$, then $T/S > 10^{-3}$ is likely, which would makes prospects for detecting the gravity-wave of inflation signature more favorable.
As $n \to 1$, $T/S$ increases in the favored region. Below and to the left of the broken line ($dn/d\ln k = 10^{-2}$), a poor power law does not occur for large $x''$ because $dn/d\ln k \propto \sqrt{T/S} x''$ and $T/S$ is small.

Figure 3: Same as Fig. 2, except with a logarithmic scale for $T/S$ to show more detail. As $n \to 1$, $T/S$ increases in the favored region.
Figure 4: Summary of two-field models. The filled circles represent the values of \((T/S)_{50}\) and \(n_{50}\) for the corresponding one-field models, and the attached curves are the values obtained for inflation ending early due to an auxiliary field. The dashed lines represent fixed points in the \((T/S) - n\) plane that result from models that do not end without an auxiliary field. In general, two-field models populate the same region as one-field models and extend the \(T/S \approx 0\) attractor to \(n > 1\).


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