Fluctuation Induced Non-Fermi Liquid Behavior near a Quantum
Phase Transition in Itinerant Electron Systems.

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The signature for a non-Fermi liquid behavior near a quantum phase transition has been observed in thermal and transport properties of many metallic systems at low temperatures. In the present work we consider specific examples of itinerant ferromagnet as well as antiferromagnet in the limit of vanishing transition temperature. The temperature variation of spin susceptibility, electrical resistivity, specific heat, and NMR relaxation rates at low temperatures is calculated in the limit of infinite exchange enhancement within the framework of a self consistent spin fluctuation theory. The resulting non-Fermi liquid behavior is due to the presence of the low lying critically damped spin fluctuations in these systems. The theory presented here gives the leading low temperature behavior, as it turns out that the fluctuation correlation term is always smaller than the mean fluctuation field term in three as well as in two space dimensions. A comparison with illustrative experimental results of these properties in some typical systems has been done. Finally we make some remarks on the effect of disorder in these systems.

I. INTRODUCTION

The description of electronic contribution to the low temperature behavior of metals in terms of the Fermi liquids has been highly successful. The low lying excitations of the Fermi liquid manifest themselves in various thermodynamic and transport properties, such as the specific heat varying as $C_v = \gamma T$, a temperature independent (Pauli) spin susceptibility, $\chi = 2\mu_B N(\epsilon_F)$, where $N(\epsilon_F)$ is the density of states at the Fermi energy, a temperature dependent electrical resistivity varying as $\Delta \rho \sim AT^2$, and a linearly temperature dependent NMR relaxation rate. $T_1^{-1} \sim T$ (Korringa). The values of these coefficients, such as $\gamma$,...
A, etc., however, are material dependent. For some transition metals these are about one order of magnitude larger than in normal metals, and in some compounds containing a large concentration of rare-earth or actinide elements such as Ce, Yb or U, these values are about a thousand times larger, particularly the value of $\gamma$ and the zero temperature susceptibility.

The normal Fermi liquid behavior as mentioned above is understood within the Landau phenomenological theory, where the effect of interaction in a Fermi system is expressed in terms of a few parameters which renormalize the physical quantities with respect to their free Fermi gas values. For example, the modifications in specific heat, spin susceptibility and isothermal compressibility are given by,

\[
\frac{C_v}{C_v^0} = 1 + \left(\frac{F_s}{3}\right), \quad \frac{\chi}{\chi^0} = \left(\frac{m^*/m}{1 + F_s^0}\right),
\]

and \[\frac{\kappa_s}{\kappa_s^0} = \left(\frac{m^*/m}{1 + F_s^0}\right)\] respectively. (The superscript 0 denotes the free Fermi gas values, other notations are standard). The basic reason for the success of the Landau theory is the largeness of quasi-particle life time near the Fermi surface, i.e., \(\tau^{-1} \sim |\epsilon|^2 \ll \epsilon\), where \(\epsilon = (E - E_F)/E_F\). From these relations it is clear that for a certain value of the Landau parameters (i.e. \(F_0\) and \(F_1\)), the corresponding quantities become very large, which in turn may indicate a neighborhood of certain phase transition. For example, \(F_0^a \rightarrow -1\) implies magnetic instability, or \(F_s^0 \rightarrow \infty\), a condensation. In the present work we consider the Fermi system in the vicinity of such a transition and seek an explanation of the non-Fermi liquid behavior of certain substances in this regime. It seems the Fermi liquid theory gives indication of the incoming electronic phase transition as the coupling constant is changed, but it does not consider the effect of incipient fluctuations in a self-consistent manner.

There are many examples of electronic phase transitions where the coupling constant tunes the transition. These are known as the quantum phase transition. For example \(1 - UN(\epsilon_F) > 0\) gives instability towards ferromagnetism, \(1 - U\chi(Q) > 0\), gives antiferromagnetic instability corresponding to a wave vector \(Q\), and \(n^{1/3}a_H > 0.26\) describes the metal insulator transition due to Coulomb correlation as suggested by Mott. These are essentially zero temperature transitions, however, in general, \(T_c \ll T_F\), where \(T_F\) is the Fermi temperature of the system. In contrast the classical phase transition occurs at finite temperature and is described by balance in the energy needed (loss) to create disorder with gain
in entropy due to disorder such that the Free Energy, \( F = U - TS \), is reduced. One more difference is that the statics and dynamics become correlated in quantum phase transition. \([4,5]\) This is principally due to non-commutativity of various terms in the Hamiltonian. For example, consider the Hubbard model,

\[
H = \sum_k \epsilon_k n_k + U \sum_i n_{i\uparrow} n_{i\downarrow},
\]

for correlated electrons. Here the kinetic energy and the \( U \) terms do not commute. (Otherwise the model will be trivial to solve). Technically, this means that one should introduce “time” and the Feynmann time ordering in the functional integral for the partition function. The order parameter field becomes “time dependent”. The time variable thus acts as an extra dimension. This leads to a change in the critical behavior. \([4]\) At a first glance it seems that the critical behavior would be the same as that of a \( D + 1 \) dimensional classical system. However, detailed analysis shows that the critical behavior (or the Upper Critical Dimension) depends on the dispersion and damping of the order parameter fluctuations. The reason is that the spin susceptibility for a ferromagnet above \( T_c \) is given by, \([6]\)

\[
\chi(q, \omega^+) \approx \frac{N(\epsilon_F)}{(1 - U N(\epsilon_F)) + \delta q^2 - \frac{i \gamma \omega}{q}}.
\]

For free electron gas \( \gamma = 1/12 \) and \( \delta = 1/2 \). At \( T_c, 1 - U N(\epsilon_F) \to 0 \), and therefore \( \omega \approx q^3 \), gives the order parameter dispersion. In case of antiferromagnetism, the staggered spin susceptibility is given by, \([7]\)

\[
\chi(Q + q, \omega^+) \approx \frac{\chi^0(Q)}{1 - U \chi^0(Q) + \delta q^2 - i \gamma \omega}.
\]

In this case, \( \omega \approx q^2 \) at the critical point. A dynamical exponent \( z \) is introduced, which reflects the change in the static critical behavior. In particular the scaling dimension of the quartic interaction is given by \( \epsilon = 4 - (d + z) \) with \( z = 3 \) for ferromagnets and \( z = 2 \) for antiferromagnets. \([8]\) In field theory \( z = 1 \), since \( \omega \) and \( q \) are linearly related, and have the same scaling form. At present the application of the renormalization group to quantum critical phenomenon in particular the correlation of the static and dynamic behavior is a subject of intense activity. We refer the reader to \([4,5,8,9]\) for detailed discussion. To summarize, the vicinity to the phase transition point and the fermionic nature of correlated electronic
system undergoing a phase transition change the nature of the phase transition itself as well as the Fermi liquid behavior expected in this system. The reason for this behavior is the smallness of the transition temperature, $T_c$, compared with the Fermi temperature $T_F$. This aspect gets revealed more clearly as $T_c \to 0$.

To calculate various physical properties, we take specific examples of ferromagnetic and antiferromagnetic transition in itinerant electron system in two as well as three dimensions in the limit of vanishing transition temperature near the transition temperature. These two examples represent two different types of quantum critical behavior. The basic reason is that in ferromagnet the order parameter is a conserved quantity, while in the antiferromagnet it is not. This difference is reflected in the dispersion of their respective order parameter fluctuation as shown in Eqns. (2) and (3). The microscopic calculation is done within the self consistent spin fluctuation theory developed earlier by Ramakrishnan and one of us [10–13] among many others [14–16]. For details of the spin fluctuation theory we refer the reader to the monograph by Moriya [17]. A brief review is given in [18]. We first briefly review the spin fluctuation theory and then write expressions for spin susceptibility, resistivity, specific heat and the nuclear magnetic relaxation rate. Similar expressions for staggered susceptibility and other quantities in antiferromagnets are also written. These quantities are then calculated in the limit of large exchange enhancement (i.e. in the limit of $\chi_P(T = 0)^{-1} \equiv \alpha(0) \to 0$). Though $T_c = 0$, fluctuation effects are observable well above $T_c$. The temperature dependence need not be Fermi-Liquid-like because of the low lying fluctuation (Bosonic) degrees of freedom.

II. SPIN FLUCTUATION THEORY

The basic motivation for constructing the spin fluctuation theory is the largeness of the susceptibility (Stoner) enhancement factor $1/\alpha(0)$. In such case a highly paramagnetic system at low temperature can be considered to be in the vicinity of a magnetic transition. The temperature variation of various physical quantities is therefore governed by transverse and longitudinal spin fluctuations. Though the order parameter vanishes above the transition, the effect of fluctuations is observable well above the transition. There are many equivalent formulations available on this idea [17,18]. A recent one is due to McMullan, [19] which is
based on functional integration using Grassmann variables and collective coordinate transformation. We briefly summarize our approach and then compile results on some physical properties.

Consider the Landau expansion for the free energy \( F(M, T) \), in power of the order parameter \( M \), viz.

\[
F(M, T) = F(0, T) + \frac{1}{2} A(T)M^2 + \frac{1}{4} BM^4 - HM
\]  

(4)

where \( H \) is the field conjugate to \( M \). The temperature dependence of various quantities in this theory arises due to \( A(T) \) and \( B \). For example, the spin susceptibility for the paramagnetic phase is given by,

\[
\chi^{-1}(T) = A(T).
\]  

(5)

\( (A(T) \) and \( \alpha(T) \) have qualitatively the same temperature dependence and differ only by some numerical factors, e.g. \( A(T) = \alpha(T)/2N(\epsilon_F) \) for ferromagnets, which we ignore and identify \( A(T) \) with \( \alpha(T) \) now onwards.) Similarly the magnetization in the ordered phase is,

\[
M^2(T) = -\frac{\alpha(T)}{B},
\]  

(6)

and the equation of state is given by,

\[
\frac{H}{M} = \alpha(T) + BM^2.
\]  

(7)

The expansion coefficients \( \alpha(T) \) and \( B \) have been calculated in various approximation schemes. In the Ginzburg Landau theory for classical phase transition, \( \alpha(T) \) is taken as \( (T - T_c) \) and \( B \) as independent of temperature. This leads to the Curie Weiss law for the susceptibility and the well known mean field critical exponents. In the mean field theory of itinerant ferromagnet,

\[
\alpha_{MF}(T) = \left(1 - UN(\epsilon_F)\right),
\]  

(8)

and \( B \) is again a constant. In this case the temperature dependence of physical quantities near \( T_c \) comes from that of integral over density of states through a Sommerfeld expansion.
It is weak, of the order of $T^2/T_F^2$, and therefore it does not give a Curie-Weiss form for the spin susceptibility. This issue is tackled in the spin fluctuation theory, where $\alpha(T)$ is given by, \[\alpha^{\text{SF}}(T) = \alpha(0) + u_4(2D^T + 3D^L).\] Here $\alpha(0)$ is the susceptibility enhancement factor at $T = 0$. This includes the mean field part $\alpha^{\text{MF}}(T)$ and the zero temperature part of the fluctuation self energy whose finite temperature part comprises the second term. Here $D^T$ and $D^L$ are transverse and longitudinal spin fluctuation amplitudes obtained by the internal frequency summation in the diagrams shown in Fig. (1) (a,b,c). The main contribution to the temperature variation of various physical quantities is governed by these amplitudes. The factor $u_4$ in the second term is is a dimensionless short range four fluctuation coupling constant obtained after integration over fast fermionic degree of freedom.

The above result has been derived microscopically, within the functional integral scheme on a model of interacting electrons. We consider Hubbard model as applied to itinerant ferromagnets and for brevity consider only spin degrees of freedom. Applying the Stratanovich-Hubbard functional integral transformation the partition function can be written as

\[ Z = \text{tr} \int \prod_{q,m} \frac{d\xi_{q,m}}{\pi} \exp[-\sum_{q,m} |\xi_{q,m}|^2 - \int_0^\beta du \left[ \sum_k \epsilon_k n_{k,\sigma,u} - \frac{U}{\beta} \right]^{1/2} \sum_{q,m} (\xi^*_{q,m} S_{q,m} \exp(z_m u) + \text{h.c.})] \]

where $\xi_{q,m}$ is the spin fluctuation field of wave vector $q$ and frequency $z_m$ ($= 2\pi im/\beta$). Also, $\epsilon_k$ is the kinetic energy of the electrons and $U$ denotes a short range inter atomic repulsion. Integrating over the electronic degrees of freedom, we have free energy functional $F(\xi_{q,m})$ for interacting spin fluctuations; that is

\[ Z = \int \prod_{q,m} \frac{d\xi_{q,m}}{\pi} \exp[-\beta F(\xi_{q,m})]. \]  

Parameters of this model, e.g. fluctuation spectrum, fluctuation coupling vertices are determined by properties of the underlying fermion system. Since these parameters (e.g. the Stoner enhancement factor for ferromagnets or the staggered susceptibility for antiferromagnets) are such that spin fluctuations are low lying excitations, this transformation is specially
helpful for an analysis of temperature dependent properties of weak itinerant electron ferromagnets and antiferromagnets. The free energy functional $F(\xi_{q,m})$ then expanded in powers of these fluctuation fields up to a quartic term and a self consistent mean fluctuation field approximation (quasi harmonic approximation, or the self consistent renormalization scheme of Moriya) can be generated. The mean fluctuation field approximation corresponds to the diagram (a), (b) and (c) in Fig. 1, and shown in a compact manner in Fig. 2 where the double wiggle represents the dressed propagator $D(q)$. The details are given in earlier papers [11,12]. One can also estimate corrections due to higher order fluctuation terms. Figs. 1 (d) and (e) and (f) represent typical higher order fluctuation correlated terms.

III. PHYSICAL PROPERTIES NEAR QUANTUM CRITICAL POINT

A. Spin Susceptibility

The self consistent equation for the temperature dependence of $\alpha(T)$ is given by Eq. (3) which is written explicitly as

$$\alpha(T) = \alpha(0) + \lambda \sum_q \int d\omega n(\omega) \text{Im} \chi(q, \omega^+)$$

(12)

where, $\lambda$ is related to $U_4$, $n(\omega) = (e^\omega/T - 1)^{-1}$, is the Bose distribution function, and

$$\chi(q, \omega) = \frac{N(\epsilon_F)}{\alpha(T) + \delta q^2 - i \frac{\omega}{2q}}$$

(13)

is spin susceptibility for the ferromagnetic case. ($\omega$ and $T$ are written in units of $\epsilon_F$ and $q$ in units of $k_F$. We have taken $\hbar = 1$ and $k_B = 1$). Performing the frequency integral,

$$\alpha(T) = \alpha(0) + \frac{\lambda}{\pi} \sum_q q \{ \ln(y) - \frac{1}{2y} - \psi(y) \}$$

(14)

where,

$$y = \frac{q}{\pi^2 \gamma T} (\alpha(T) + \delta q^2).$$

(15)

An interpolation formula for,

$$\phi(y) \equiv \{ \ln(y) - \frac{1}{2y} - \psi(y) \}$$

$$\simeq \frac{1}{2y + 12y^2}$$

(16)
which is valid for small as well as large $y$ is useful in calculating the momentum integral. For three dimension,

$$\alpha(T) = \alpha(0) + \frac{\lambda}{2\pi^3} \int \frac{q^3 dq}{(2y + 12y^2)}. \quad (17)$$

A finite $\alpha(0)$ introduces two regions of temperatures. If $T < \alpha(0)$ one gets the standard paramagnon theory results; and for $\alpha(0) < T < 1$ one gets the classical Curie Weiss susceptibility,

$$\chi = \chi_P/\alpha(T) \simeq \mu_B^2/(a\alpha(0) + T). \quad (18)$$

That is like susceptibility of a collection of classical spins. This feature gets revealed more clearly if we put $\alpha(0) = 0$ in the expression for $\chi(T)$ and solve the equation self consistently. In this case the paramagnon regime ($T \leq \alpha(0)$) shrinks to zero and a classical behavior is expected down to $T = 0$. One is then essentially calculating susceptibility of a ferromagnet with $T_c = 0$. Since $\alpha(0)$ is taken to be zero and there is only one region of temperature $T < 1$. In this case, typical $y \leq 1$, the limiting form is obtained using the form $\phi(y) \approx \frac{1}{2y}$ (valid for $y << 1$). We then find that,

$$\alpha(T) = \frac{T}{\delta} \left[ q_T - \left( \frac{\alpha(T)}{\delta} \right)^{1/2} \arctan \left( \frac{q_T \delta}{\alpha(T)} \right)^{1/2} \right] \quad (19)$$

where, $q_T$ is a thermal cutoff such that $y_{q_T} \approx 1$. For the form of $y$ given by Eq. (15), the estimate of the cutoff is $q_T^3 \approx T \gamma/\delta$ or $q_T \approx T^{1/3}$. The dominating contribution to $\alpha(T)$ comes from the first term which is given by, $T^{4/3}$. However, since $\delta$ is small, the thermal cutoff $q_T$ is high $\approx q_c$ (the spin fluctuation energy rises only slowly with $q$). Thus, $\alpha(T)$ rises nearly linearly with $T$. This is the classical spin fluctuation behavior, first pointed out for itinerant ferromagnets by Murata and Doniach. Note that we have assumed $T < 1$, i.e. the system is degenerate. Even so, since the characteristic fluctuation energy $\alpha(0)$ is zero, the system behaves classically with regard to spin fluctuations. An estimate of the size of the second term is obtained by putting $\alpha(T) \approx T^{4/3}$. We then find it to be of the order $T^{1/3}$ relative to the first term. Since $T^{1/3}$ is not very small, it is essential to do a self consistent calculation, particularly in the classical regime which is of interest in the present calculation.
We have calculated $\alpha(T)$ and other properties in two dimension also. For this we consider the same approximate form of the spin susceptibility or the fluctuation propagator as in three dimension, the effect of dimensionality is considered only through the phase space in the momentum integration. The assumption regarding the form of susceptibility function in two dimension is in doubt. It is well known [20] that the Lindhardt function from where this functional form has been derived has a different analytic form in two dimension. As far as the low momentum behavior is concerned the assumption is closer to the reality if $\delta$ is considered to be far smaller than its value in three dimension. For the sake of comparison we assume the same value of $\delta$ in three as well as in two dimension.

Following the same procedure as in three dimensions we find in two dimensions a logarithmic temperature dependence,

$$\alpha(T) = \frac{T}{2\delta} \ln \left( \frac{\delta q^2}{\alpha(T)} \right)$$

(20)

Because of the Bose factor $1/(\exp(\omega/T) - 1)$, the number of thermal (classical) fluctuations becomes smaller and smaller as $T \to 0$ (i.e. as the $T_c$ approaches). This reduces the phase space for the fluctuation correlations. In the RNG analysis of Hertz [4] and others [5] this requires the introduction of a suitably scaled ‘energy’ variable as a degree of freedom additional to the three momentum variables. In effect the dimensionality increases, and the behavior becomes mean-field. We see this explicitly in our procedure of calculating the fluctuation correlation correction perturbatively. The terms involving two or more internal thermal spin fluctuations are shown in Fig. (2). These have been calculated in detail earlier. [11] It turns out that apart from a numerical factor the two internal thermal spin fluctuation term has the same temperature dependence as the mean fluctuation field term. However, the three internal thermal spin fluctuation term, $\approx T^2 \ln \left( 1/3\alpha(T) \right)$ in 3D ferromagnet. We see that this term is of the order of $T \ln T$ relative to the simplest non-vanishing contribution. The perturbation expansion therefore converges.

For a finite $T_c$ ferromagnet the mean fluctuation field theory is valid outside the critical regime. As the critical regime approaches higher order fluctuation correlations become comparable to the mean fluctuation term. In the present case, in contrast the mean fluctuation field term gives the leading critical behavior. The reason is the following. Suppose for $\alpha(0) = 0$, $\alpha(T) \sim T^{\lambda}$. Then the quantum region $T \ll \alpha(T)$ means
$T^{1-\lambda} \ll 1$ and it occurs only if $\lambda \leq 1$. This is not possible, and so one always has the other classical (Curie - Weiss) region. Here the fluctuation correlation term is of the form $T^2 \ln(1/\alpha(T)) \sim T^2 \ln(1/T)^{\lambda} \ll T^{\lambda}$. If $\lambda \sim 1$, the correlation term never becomes more important than the mean fluctuation field term.

In case of Antiferromagnets, the formalism is identical. One replaces the Pauli susceptibility with the staggered susceptibility for non interacting electron system $\chi_0(Q)$, for brevity we retain the same notation for the enhancement factor which is defined in the present case as $\alpha(0) = \chi_0(Q)/\chi(Q)$. The expansion of the dynamic staggered susceptibility, $\chi_0(Q+q, \omega)$ for small $q$ and small $\omega$ around the static staggered susceptibility is also written in the form,

$$\chi(Q+q, \omega) = \frac{\chi_0^0(Q)}{\alpha(T) + \delta q^2 - i\gamma \omega}. \quad (21)$$

Making similar transformations as for the ferromagnetic case, we get,

$$\alpha(T) = \alpha(0) + \frac{\lambda}{2} \sum_q \frac{1}{2z + 12z^2} \quad (22)$$

where

$$z = \frac{(\alpha(T) + \delta q^2)}{2\pi \gamma T} \quad (23)$$

Thus, for 3D antiferromagnets,

$$\alpha(T) = \alpha(0) + \frac{\lambda}{2\pi} \int \frac{q^2 dq}{(2z + 12z^2)}. \quad (24)$$

The result turns out to be identical to the ferromagnetic case once we consider only the $z < 1$, where the corresponding momentum cut off turns out to be $T^{1/2}$,

$$\alpha(T) = \frac{T}{\delta} \left[ q_c - \sqrt{\frac{\alpha(T)}{\delta}} \arctan \left( \sqrt{\frac{\delta}{\alpha(T)} q_c} \right) \right]. \quad (25)$$

and in two dimension there is again a logarithmic behavior,

$$\alpha(T) = \frac{T}{2\delta} \ln \left( \frac{\delta q_c^2}{\alpha(T)} \right). \quad (26)$$
B. Resistivity

The electrical resistivity for pure transition and rare earth metals is usually calculated within a two band model, \[21\] where the ‘conducting’ electrons come from s-band while the d-electrons contribute to magnetism. The d-band is assumed to be narrow and the d-electrons heavy. The conducting s-electrons scatter from the spin fluctuations corresponding to d-electrons. The temperature dependent part of the resistivity due to this mechanism for a 3d-ferromagnet is given by, \[22–24\]

\[
\rho(T) \propto \frac{1}{T} \int q^3 dq \int \text{Im}\chi(q, \omega^+)\omega n(\omega)(1 + n(\omega))d\omega
\]  

(27)

The frequency integral can be performed by first using the identity,

\[
\frac{1}{T^2} \omega n(\omega)(1 + n(\omega)) = \frac{\partial}{\partial T} n(\omega)
\]  

(28)

leading to,

\[
\rho(T) \approx T \int dq q^3 \int d\omega \frac{dn(\omega)}{dT} \frac{q\omega}{q^2(\alpha(T) + \delta q^2)^2 + \omega^2}
\]

\[
\approx \int dq q^4 y \phi'(y)
\]  

(29)

where, \(\phi'(y) = \frac{d\phi(y)}{dy}\), and \(\phi(y)\) and \(y\) are given by Eqns. \[16\] and \[15\] respectively. In the limit \(y < 1\) the momentum integral gives,

\[
\rho(T) = \frac{T}{2\delta} \left[ q_c^2 - \frac{\alpha(T)}{\delta} \ln\left( \frac{\alpha(T) + \delta q_c^2}{\alpha(T)} \right) \right].
\]  

(30)

With \(q_c \sim T^{1/3}\) we recover the well known result \(\Delta \rho \sim T^{5/3}\). \[23\] However, the self consistency correction changes the power of temperature. Similarly for two dimensions,

\[
\rho(T) = \frac{T}{\delta} \left[ q_c - \sqrt{\frac{\alpha(T)}{\delta}} \arctan\left( \sqrt{\frac{\delta}{\alpha(T)}} q_c \right) \right].
\]  

(31)

The case of 3D antiferromagnets formalism is similar except for the power of \(q\) in the momentum integral. This is due to the fact that the small momentum expansion is not done around \(q = 0\) but around \(q = Q\), the antiferromagnetic wave vector. The result is, \[7\]

\[
\rho(T) \propto \frac{1}{T} \int q^2 dq \int \text{Im}\chi(Q + q, \omega^+)\omega n(\omega)(1 + n(\omega))d\omega.
\]  

(32)
Following the same steps as for the ferromagnetic case, we get,

\[ \rho(T) \propto T \int q^2 dq \frac{(1 + 12z)}{2z(1 + 6z)^2} \]  \tag{33}

where \( z \) is given by Eq. (23). The result in the limit of \( z < 1 \) is,

\[ \rho(T) = T \delta \left[ q_c - \sqrt{\frac{\alpha(T)}{\delta}} \arctan \left( \sqrt{\frac{\delta}{\alpha(T)}} q_c \right) \right]. \]  \tag{34}

Similarly for two dimensions, \( \rho(T) = \frac{T}{23} \ln \left( \frac{\delta q^2}{\alpha(T)} \right) \).

C. Specific heat

The spin fluctuation contribution to the free energy within the mean fluctuation field approximation (or quasi harmonic approximation) is given by, \[13\]

\[ \Delta \Omega = \frac{3T}{2} \sum_{q,m} \ln \{1 - U \chi^0_{qm} + \lambda T \sum_{q',m'} D_{q'm'} \}. \]  \tag{35}

Where \( D_{q,m} \) is the fluctuation propagator which is related to inverse dynamical susceptibility, and \( \chi^0_{qm} \) is the free Fermi gas (Lindhardt) response function. The argument of the logarithm is related to inverse dynamic susceptibility. Considering only the thermal part of the integral and ignoring the zero point part, we perform the frequency summation and obtain,

\[ \Delta \Omega_{\text{thermal}} = \frac{3}{2} \sum_q \int_0^\infty \frac{d \omega}{e^{\omega/T} - 1} \arctan \{ \frac{\pi \omega/4q}{\alpha(T) + \delta q^2} \}, \]  \tag{36}

Integrating over frequency, we get,

\[ \Delta \Omega_{\text{thermal}} = 3T \sum_q \left( \ln \Gamma(y) - (y - \frac{1}{2}) \ln(y) + y - \frac{1}{2} \ln(2\pi) \right). \]  \tag{37}

where, \( y \) is given by Eq. \[15\]. Once the free energy correction is known, the specific heat correction is given by

\[ \frac{\Delta C_v}{k_B} = -T \frac{\partial^2 \Delta \Omega}{\partial T^2} \]

\[ = -3T^2 \sum_q \left[ \frac{2}{T} \frac{\partial y}{\partial T} + \left( \frac{\partial^2 y}{\partial T^2} \right) \phi(y) + \left( \frac{\partial y}{\partial T} \right)^2 \frac{\partial \phi(y)}{\partial y} \right] \]

\[ = 6 \int q^2 dq \left\{ \phi'(y) \left( \frac{q}{\pi^2 \gamma} \frac{\partial \alpha(T)}{\partial T} - y \right)^2 + T \phi(y) \frac{q}{\pi^2 \gamma} \frac{\partial^2 \alpha(T)}{\partial T^2} \right\}. \]  \tag{38}
Making the small $y$ approximation and introducing the appropriate cutoff,

$$
\frac{\Delta C_v}{k_B} \approx \frac{1}{2\pi^2} \left[ T^2 \frac{\partial^2 \alpha(T)}{\partial T^2} + 2T \frac{\partial \alpha(T)}{\partial T} \right] \int_0^{q_c} dq \frac{q^2}{\alpha(T) + \delta q^2} - \int_0^{q_c} dq \frac{q^2}{2} \left[ \frac{\partial \alpha(T)}{\partial T} \right]^2 - \frac{1}{\delta} \left( T^2 \frac{\partial^2 \alpha(T)}{\partial T^2} + 2T \frac{\partial \alpha(T)}{\partial T} \right) q_c - \frac{1}{\sqrt{\alpha(T)\delta}} \arctan \left( \sqrt{\frac{\delta}{\alpha(T)q_c}} \right) - \frac{1}{\alpha(T) + \delta q_c^2} + \frac{q_c}{3}.
$$

The last result is obtained after the momentum integration. Approximately the terms can be arranged as,

$$
C_v \approx \frac{1}{2\pi^2} \left[ q_c^3 + T^2 \left( \frac{\partial \alpha(T)}{\partial T} \right)^2 \frac{1}{\sqrt{\alpha(T)}} - \left( T^2 \frac{\partial^2 \alpha(T)}{\partial T^2} + 2T \frac{\partial \alpha(T)}{\partial T} \right) q_c \right].
$$

The first term gives the classical result (for constant cutoff), the second dominant term gives leading temperature correction, the last term is about two order of magnitude small at the temperature range of interest. Similarly in two dimensions,

$$
C_v = -\frac{1}{2\delta} \left( T^2 \frac{\partial^2 \alpha(T)}{\partial T^2} + 2T \frac{\partial \alpha(T)}{\partial T} \right) \ln \left( \frac{\delta q_c^2}{\alpha(T)} \right) + \frac{T^2}{2\delta \alpha(T)} \left( \frac{\partial \alpha(T)}{\partial T} \right)^2 \left( 1 + \frac{\alpha(T)}{\delta q_c^2} \right)^{-1} + \frac{q_c^2}{2}
$$

The calculation for antiferromagnet is identical except that $y$ is replaced by $z$ in Eq. (40) and the final equation in terms of $\alpha(T)$ turns out to be identical except that the temperature dependence of $\alpha(T)$ is different in an antiferromagnet.

### D. Nuclear Spin Relaxation Rate

The nuclear spin relaxation rate in metals is given by the Korringa relation, [25]

$$
\frac{1}{T_1 T} \approx \left( \frac{\Delta H}{H} \right)^2
$$

which essentially tells that $1/T_1$ is proportional to the square of the static spin susceptibility of metals, which in turn, is independent of temperature for most normal metals. However, it has been pointed out by Moriya [26] long ago that this relation gets modified in presence of electron correlations. The nuclear spin-lattice relaxation rate in metals is given by,
\[
\frac{1}{T_1 T} \sim \sum_q \frac{\text{Im} \chi^{-+}(q, \omega_0^+)}{\omega_0} \]

(43)

where \(\omega_0\) is the nuclear magnetic resonance frequency which is taken to be very small \((\to 0)\) in the problem of nuclear spin relaxation rate. Substituting the expression for \(\chi^{-+}(q, \omega_0^+)\) and taking the limit, we have,

\[
\frac{1}{T_1 T} \sim \sum_q \frac{1}{q(\alpha(T) + \delta q^2)^2}, \quad (44)
\]

for a ferromagnet in three dimension. After the momentum integration, the result is \((T_1 T)^{-1} \sim \alpha(T)^{-1}\). For normal Fermi liquid \(\alpha(T)\) is constant but in the present case it varies as \(T^{4/3}\). This leads to a non-Fermi liquid behavior again. Similar calculation is done for antiferromagnets.

**E. Effect of Disorder**

The effect disorder can be included in the above mentioned formalism by modifying the propagators and vertices in diagrams for the spin fluctuation self energy. This has been done in our earlier papers. [27][28] In presence of disorder the electron moves in random way getting scattered from impurities repeatedly. This introduces a finite mean free path for electron, and a finite life time \(\tau\) in the electron propagator, which also modifies the free particle-hole propagator (diffuson), free particle-particle propagator (Cooperon) and electron-spin fluctuation vertex. The correction to \(\alpha(T)\) to leading order is given by,

\[
\alpha(T) = \alpha_{SF}(T) - \alpha_d(T), \quad (45)
\]

where \(\alpha_d(T)\) is correction due to diffusive modes. It is given to the leading order in \(1/\epsilon_F \tau\) as

\[
\alpha_d(T) \sim [1 - \sqrt{(2\pi \tau T)})]/(\epsilon_F \tau)^2, \quad (46)
\]

for \(T \ll 1/\tau\) and it vanishes otherwise. Clearly the disorder introduces a new energy scale \((1/\tau)\), in the lowest temperature range as shown schematically in Fig. (3). In the case of non-vanishing \(\alpha(0)\), in the case of ferromagnet in three dimension, the susceptibility inverse \(\alpha(T)\) behaves as \(\sim T^{1/2}\) for \(T < 1/\tau\), as \(T^2/\alpha(0)\) for \(1/\tau < T < \alpha(0)\), and as \(T^{4/3}\) for
\( \alpha(0) < T < 1 \). Similarly the resistivity correction \( \Delta \rho(T) \) behaves as \( \sim T^{1/2} \) for \( T < 1/\tau \), as \( T^2/\sqrt{\alpha(0)} \) for \( 1/\tau < T < \alpha(0) \), and as \( T^{5/3} \) for \( \alpha(0) < T < 1 \). In the case of a zero-\( T_c \) system the quantum fluctuation regime \( (T < \alpha(0)) \) vanishes and the other two regimes merge. At lowest temperature the effect of diffusive mode seems to give the dominant contribution, i.e. \( \alpha(T) \sim T^{1/2} \) and \( \Delta \rho(T) \sim T^{1/2} \) but a more detailed analysis is needed.

**F. Summary**

The following table summarizes our results. In the first column Fermi liquid theory results are written, other columns compile the fluctuation theory results. These results are presented in three rows for each property. The first row gives results from a non-self-consistent calculation, i.e., for example, when only the first term in Eq. (19) for \( \alpha(T) \) is considered but with a proper momentum cut off. This behavior is expected in the extreme low temperature range. These results are in general known, but presented in the coherent form for the first time. The second row gives these results with the temperature dependence of \( \alpha(T) \) taken in account and the integration performed with the functional form for \( \phi(y) \) valid for all \( y \) but approximated by Eq. (16). The power of temperatures so obtained depends slightly on the temperature regime considered (i.e. whether \( T \) is in \( 10^{-3} - 10^{-2} \) or otherwise). The third row gives the classical spin fluctuation results, where the Bose factor \( n(\omega) \) is approximated as \( T/\omega \), (effectively the first row with a constant cutoff). The experimental results are expected to lie between those given in rows one and three.
| $\chi(T)^{-1}$ | Fermi Liquid | Ferro (3D) | Antiferro (3D) | Ferro (2D) | Antiferro (2D) |
|----------------|--------------|------------|---------------|------------|---------------|
| $\rho(T)$      | $T^2$        | $T^{5/3}$  | $T^{3/2}$     | $T^{4/3}$  | $T\ln T$     |
|                | $T^{1.20}$   | $T^{1.56}$ | $T^{1.45}$    | $T^{1.24}$ | $T\ln T$     |
|                | $T$          | $T$        | $T$           | $T$        | $T$           |
| $C_v(T)$       | $T$          | $T$        | $T^{3/2}$     | $T^{2/3}$  | $T$           |
|                | $T^{0.74}$   | $T^{0.99}$ | $T^{0.52}$    | $T^{0.52}$ | $T^{0.86}$    |
|                | Const.       | Const.     | Const.        | Const.     | Const         |
| $(T_1 T)^{-1}$ | Const.       | $T^{-4/3}$ | $T^{-3/4}$    | $(T\ln T)^{-3/2}$ | $(T\ln T)^{-1}$ |
|                | $T^{-1.284}$ | $T^{-0.72}$| $T^{-1.305}$  | $T^{-1}$   | $T^{-1}$      |
|                | $T^{-1}$     | $T^{-1/2}$ | $T^{-3/2}$    | $T^{-1}$   | $T^{-1}$      |
IV. EXPERIMENTAL RESULTS:

In this section we give examples of materials exhibiting non-Fermi liquid behavior at low temperatures and also compare some results with a theory presented above. The most popular example of system showing non-Fermi liquid behavior is, of course, the high temperature superconductors [29]. It seems, however, the effective low dimensionality, the specific nature of the density of states and structural aspect of the Fermi surface (nesting etc.) play important role in this system. We therefore want to consider examples from three dimensional correlated electronic system in the neighborhood of electronic phase transition.

The next example is that of phosphorus-doped silicon (Si:P). [30] This material goes through a Mott insulator to metal transition as the doping by P increases. At $n_c \sim 3.7 \times 10^{18}$ P per cm$^3$ there is a metallic state. The spin susceptibility of Si:P gets enhanced and becomes strongly temperature dependent as the metal-insulator transition is approached from the metallic side. The T-dependence observed does not fit the $1/T$ behavior expected for weakly interacting localized spins either. Moreover, the spin-lattice relaxation times in barely metallic Si:P is strongly temperature dependent, $T_1^{-1} \sim T^{-1/2}$, similarly the correction to the zero-T conductivity, $\sigma \sim T^{1/2}$, in this material. There are theories which associate these anomalies to spin fluctuations induced due to incipient localization. There is a subtle interplay of disorder and correlation effect in this material. Only spin fluctuation kind of theory will not work.

Some transition metal (and also some actinide) inter-metallic compounds show a low saturation moment per transition metal atom and a low magnetic transition temperature $T_c$ compared with conventional ferromagnetic materials like Fe, Co and Ni. These compounds are known as weak itinerant electron ferromagnets. The prototype examples are, ZrZn$_2$, Ni$_3$Al, and Sc$_3$In. [18] Their low temperature properties have been discussed within spin fluctuation theories for a long time. [17] Here we compare the specific heat behavior of Sc$_3$In above $T_c$ with our present calculation (Fig. (4)). The experimental curves are due to Ikeda [31] and show a good fit to the theory with $\Delta \rho(T) \sim T$, and $C_v(T)/T \sim (T - T_c)^{-0.25}$.

The example of MnSi is interesting from the perspective of the present work. The material has a transition temperature around 30$K$. As the hydrostatic pressure is applied the $T_c$ decreases continuously and collapses towards absolute zero at $p_c = 14.6 kbar$. This
is an example where an approach to a zero temperature quantum phase transition can be observed as a function of pressure. This has been done by the Cambridge group. The deviation of the resistivity curve from the $T^2$ behavior gets pronounced as $p_c$ is approached. In Fig. (5) we have compared $\Delta \rho/T^2$ with a power law temperature dependence as suggested in our calculation.

We have compared the nuclear spin relaxation rate of $^{27}Al$ in $Ni_3Al$ as a function of temperature. This material has a transition temperature about 41 K and shows all other characteristic properties of weak itinerant ferromagnet. The low field data fit to power law $T^{-0.89}$ for $(T_1T)^{-1}$ as shown in Fig. (6).

The heavy fermion material CeCu$_6$ is non-magnetic. On alloying with Au the lattice expands and an antiferromagnetic order is observed in CeCu$_{6-x}$Au$_x$ above a critical concentration $x_c \approx 0.1$. The Néel temperature of the anti-ferromagnetic heavy-fermion alloy CeCu$_{5.7}$Au$_{0.3}$ can be continuously tuned to zero with increasing hydrostatic pressure. At the critical pressure the specific heat has been fitted to $C/T \sim \ln T_0/T$ curve. We analyze the data again and fit the curve to our prediction ($T^{0.58}$ corresponding to the temperature range of interest) in the Fig. (7).

V. CONCLUSION

We have calculated the temperature dependence of various physical properties near the quantum phase transition point. The results hold for electronic phase transitions with a finite $T_c$ also. This is clear, as the results from the first rows in Table (1) match with some well known results in literature. However, they are pronounced and a clear non-Fermi liquid behavior is obtained when $T_c \to 0$. Our results are perturbative, but as discussed in the text the fluctuation correlation term is always smaller than the mean fluctuation field term. The behavior of these quantities is different in ferromagnet from the antiferromagnetic system. This is a reflection of the fact that the order parameter fluctuations have different form of dispersion in these systems. Finally we made some remarks about the inclusion of the effect of disorder near quantum critical point within the spin fluctuation formalism. The present approach can be applied to other systems also. One only needs an appropriate form of the order parameter correlation function to calculate various quantities. For example this
approach can be applied to systems with a pseudo gap \[36\] in the excitation spectrum and also with phonon like dispersion as it happens in short coherence length superconductors. For example in 2D short coherence length superconductors it has been shown through Monte Carlo simulations that the relaxation rate varies as the spin susceptibility, \[37\] \((\alpha(T))^{-1}\) of the system, which matches with our result on 2D antiferromagnets.
FIGURE CAPTIONS

FIG. 1. Self energy diagrams for the spin fluctuation propagator.

FIG. 2. Self energy in the mean fluctuation field approximation

FIG. 3. Schematic diagram of the various temperature scales involved in the disordered material

FIG. 4. Plot of $C_v/T$ as a function of $T - T_c$ for Sc$_3$In. The experimental points are represented by circles and the solid line represents the theoretical fit. $\delta = 1/12, \gamma = 1/2, T_F \approx 1000K$.

FIG. 5. Plot of $\rho(T)$ as a function of $T$ for MnSi. The experimental points are represented by circles and the solid line represents the theoretical fit. $\delta = 1/12, \gamma = 1/2, T_F \approx 1000K$.

FIG. 6. Plot of $(T_1T)^{-1}$ as a function of $T$ for Ni$_3$Al. The experimental points are represented by circles and the solid line represents the theoretical fit. $\delta = 1/12, \gamma = 1/2, T_F \approx 1000K$.

FIG. 7. Plot of $C_v$ as a function of $T$ for CeCu$_{5.7}$Au$_{0.3}$ at a pressure of 8.2 kbar. The experimental points are represented by circles and the solid line represents the theoretical fit. $\delta = 1/2\pi, \gamma = 1, T_F \approx 5K$.

TABLE CAPTIONS

TABLE I. Summary of the temperature dependence of various thermal and transport properties near a quantum phase transition point. The first row gives the non-selfconsistent calculation scheme. For ferromagnets the upper cutoff for $q$ is $T^{1/3}$ while for antiferromagnets it is $T^{1/2}$. The second row gives the selfconsistent calculation scheme results. The upper cutoff for $q$ has been taken to be 1. The range of temperatures in which these exponents have been calculated is $T = 10^{-3}$ to $10^{-2}$. The third row gives the classical spin fluctuation results. (i.e. the first row with $q_T$ as a constant.)
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