Dynamic mechanical property analysis of 3D membrane-imbedded acoustic metamaterials

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Abstract. Membrane-imbedded metamaterials have great potential applications in noise reduction in a low frequency range, and anisotropic mechanic properties of the materials can be easily achieved by artificially adjusting membrane parameters. Effects of the membrane parameters on equivalent elastic moduli of the 3D embedded membrane acoustic metamaterials are investigated firstly. And then, in order to demonstrate how to manipulating acoustic waves by the membrane-imbedded metamaterials, the transformation acoustics method is employed to design the equivalent material parameters according to given wave propagation paths. Finally, several numerical examples are presented to illustrate abilities of the 3D embedded membrane acoustic metamaterial to control wave propagation.

1. Introduction

Low-frequency noises and vibrations have become annoying engineering problems due to their superior ability to penetrate and a difficulty to absorb. Generally, methods of controlling vibrations and noises have two types, one is controlling vibration sources, mainly developing low vibration and mute products, and the other is controlling propagation paths, including isolation, attenuation and absorption. With continuous advancements and developments of science and technology, product performances and structures have become more and more optimized, and controlling vibration sources has become increasingly difficult [2]. However, the transformation acoustics theory proposed by A.N. Norris [1] and unique characteristics of embedded membrane acoustic metamaterials, provide us a new way of controlling vibrations and noises on their propagation path.

In 2006, Pendry et al. proposed the transformation optics theory based on the covariance of Maxwell's equations under coordinate transformation [3]. Because acoustic wave equations also have the covariance of coordinate transformation, the coordinate transformation method has also been applied to acoustics fields and the theory of transformation acoustics is formed [4]. The transformation acoustics theory is mainly used to design material parameters to control propagation directions and paths of acoustic waves. Therefore, it is widely used in designs and developments of new acoustic devices, for example, an acoustic cloak [5].

To realize controlling propagation paths of acoustic waves, it is necessary that the density and the elastic modulus of a material have the non-uniformity. But an anisotropic density is difficult to be achieved by manual designing and constructing an acoustical metamaterial with a wide frequency band. According to Norris's transformation acoustics theory, a designed material with an isotropic density and an orthogonal anisotropic elastic modulus can be achieved in case of the choice of special coordinate transformation, for instance, a purely tensile and non-rotating coordinate transformation. At this point, acoustic waves can propagate according to preset paths and maintain their propagation natures.
In 2017, Mei et al. designed a kind of 3D embedded membrane acoustic metamaterial, where the membrane parameters have simple relationships with equivalent elastic moduli of the metamaterial in the in-plane direction of membrane, which make it different from previous membrane-type metamaterials[6]. The membrane-imbedded acoustic metamaterials have good application prospects in the acoustics control, especially for the propagation direction control of acoustic waves.

This paper firstly briefly introduces the transformation acoustic theory and investigates relationships between equivalent anisotropic elastic moduli and imbedded membrane parameters of the 3D embedded membrane acoustic metamaterials. And then, the transformation acoustics method is employed to design the equivalent material parameters according to given wave propagation paths. Finally, several numerical examples are presented to illustrate abilities of the 3D embedded membrane acoustic metamaterial to control wave propagations.

2. Coordinate Transformation and the Transformation Acoustic Theory

The coordinate transformation theory is based on the invariance of coordinate transformation of acoustic wave equations. By introducing the concept of coordinate transformation, a field line distortion is effectively equivalent to a spatial deformation, and a corresponding relationship between equivalent physical parameters of materials and the spatial deformation is established [7]. With this relationship, when designing a metamaterial device, if the device's control effect on field lines is known, the equivalent physical parameters of the material can be determined by constructing mapping equations between position coordinates of the field lines in a virtual space, as shown in Fig.1(a), and position coordinates of the field lines in a physical space, as shown in Fig.1(b), which provides a solid theoretical basis for controlling and guiding waves, enabling waves to propagate along paths determined by the coordinate transformation[8]. The traditional transformation acoustics theory requires that the material density is anisotropic, and the elastic modulus also changes gradually with the position change, which is hard to achieve. However, the transformation acoustics theory proposed by A.N. Norris has improved, and the material with isotropic density and orthogonal stiffness can also control wave propagations.

![Figure 1. Coordinate transformation diagram](image)

For changes of two spaces, the coordinate \((x, y, z)\) of a point in the virtual space transforms to the coordinate \((u, v, w)\) of the corresponding point in the physical space, satisfying \(u = u(x, y, z)\), \(v = v(x, y, z)\) and \(w = w(x, y, z)\). A deformation gradient matrix \(F\) can be expressed as

\[
F = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{bmatrix}
\] (1)

According to Norris's transformation acoustics theory, letting \(J = \det F\), \(V^2 = FF^t\), \(F = VR\) \((RR^t = I, \ \det R = 1)\), and then, the elastic modulus tensor \(C\) and the material density \(\rho\) satisfy
Equation (2) can be rewritten as

\[ \rho = J^{-1} \quad K = J \quad C = KS \otimes S \quad S = J^{-1}V = J^{-1}F \]

The acoustic wave equation \( \nabla^2_p \rho p - \ddot{p} = 0 \) in the virtual space transforms into \( \nabla^2_x V \cdot (Vp) = 0 \) in the physical space. The material has an isotropic density and an orthogonally transformed elastic modulus, and the wave propagation path is determined by the wave propagation speed in all directions. The wave velocity satisfies

\[ v^2 = Kn \cdot S \rho^{-1} \mathbf{n} \]

where, \( \mathbf{n} \) is the directional vector. According to Eq.(4), the change can be controlled only by the elastic modulus, and the material density can keep constant before and after transformation, which not only guarantees waves propagating along preset paths, but also reduces difficulties of controlling acoustic waves by designing and fabricating metamaterials in practical applications.

3. Mechanical Properties of the Membrane-imbedded Acoustic Metamaterials

A material model of the membrane-imbedded acoustic metamaterial is established, and its unit cell is shown in Fig.2. Where, the matrix material is chosen as rubber with the elastic modulus of \( E = 7.8 \text{ MPa} \), the Poisson's ratio of \( \mu = 0.47 \) and the density of \( \rho = 980 \text{ kg/m}^3 \). In order to study influences of membrane's elastic moduli on the membrane-imbedded acoustic metamaterials and ensure the membrane volume fraction does not change, the membrane's thickness is set as the same, and membrane's elastic moduli are different. Assuming \( a = 1 \text{ m} \), \( h_m = 0.2 \text{ m} \), the homogenization method is used to calculate equivalent elastic moduli of the material model in all directions, including \( E_x, E_y \) and \( E_z \). Fig.3 and Fig.4 show changes of equivalent elastic moduli of the model with membrane's elastic moduli.

![Figure 2. Structure and parameters of the unit cell](image)
Figure 3. Change of $E_z$ with membrane's elastic moduli

Figure 4. Change of $E_x, E_y$ with membrane's elastic moduli

Since the membrane does not have a perpendicular stiffness when the membrane has no prestress, the membrane doesn’t influence the elastic modulus in the Z direction. If the membrane does not been tailored, the same effect is exerted on the elastic moduli in the X and Y directions, so the simple membrane-imbedded acoustic metamaterial has equal $E_x$ and $E_y$. After fitting, the function of $E_x$ or $E_y$ is yielded

$$E_x = E_y = 0.2569 \times E_m + 46.8 \text{MPa}$$  \hspace{1cm} (5)$$

$$E_z = 46.8 \text{MPa}$$  \hspace{1cm} (6)

Wave propagation paths in a material are determined by wave velocities, so the wave propagation can be controlled by designing wave velocities in different directions. According to Eq.(4), changing the density and elastic modulus of the material can control the wave velocities. Due to the small thickness of the membrane, the density variation of the membrane-imbedded acoustic metamaterial, caused by embeding membrane, may be ignored. The membrane material has a good control over the elastic modulus of the acoustic metamaterial in the direction parallel to the membrane, which can be used to control the wave propagation path in the metamaterials.

4. Numerical Examples of Transformation Acoustics

4.1. Cubic Acoustic Transformation

A cube deformation is shown in Fig.5, where Fig.5(a) is the initial shape and Fig.5(b) is the shape after deformation. In order to keep wave propagation properties in the model unchanged before and after deformation, a plane wave is considered in the example. The propagation path of the plane wave in the initial model is shown in Fig. 6(a), and the propagation path and direction in the deformed model is shown in Fig.6(b).
Considering the model’s symmetry, only the mapping relationship of the first quadrant is studied. For the model’s deformation, the coordinate transformation relationship is expressed as

\[ x' = \left(\frac{4.5 - z}{3}\right)x \quad y' = \left(\frac{4.5 - z}{3}\right)y \quad z' = z \quad (7) \]

Then the transformation matrix \( F \) and its corresponding \( J \) are as follows

\[
F = \begin{bmatrix}
\frac{4.5 - z}{3} & 0 & -\frac{x}{3} \\
0 & \frac{4.5 - z}{3} & -\frac{y}{3} \\
0 & 0 & 1
\end{bmatrix} \quad J = \left(\frac{4.5 - z}{3}\right)^2 \quad (8)
\]

To apply Norris's transformation acoustics theory to this model, the transformation matrix \( F \) should meet \( F = F' \), so a new coordinate transformation relationship is described as

\[ x' = \left(\frac{4.5 - z}{3}\right)x \quad y' = \left(\frac{4.5 - z}{3}\right)y \quad z' = z - \frac{x^2}{6} - \frac{y^2}{6} + \text{sgn}(xy) \frac{xy}{3} \quad (9) \]

At this point, the transformation matrix satisfies \( F \approx F' \), and \( \left| \frac{x^2}{6} - \frac{y^2}{6} - \frac{xy}{3} \right| \) is small, so \( F \) and \( J \) are rewritten as
Through Norris's transformation acoustic theory, the coefficients $C_{1111}, C_{2222}$ and $C_{3333}$ in the elastic stiffness matrix can be obtained:

\[
F = \begin{bmatrix}
\frac{4.5 - z}{3} & 0 & -\frac{x}{3} \\
0 & \frac{4.5 - z}{3} & -\frac{y}{3} \\
-\frac{x}{3} & +\text{sgn}(xy)\frac{y}{3} & 1 \\
\end{bmatrix}
\]

(10)

\[
J = \left(\frac{4.5 - z}{3}\right)^2 - \left(\frac{4.5 - z}{3}\right)\left(\frac{x^2 + y^2 - \text{sgn}(xy)xy}{9}\right)
\]

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\]

\[
C_{1111} = KS_{11}S_{11} = J^{-1}F_{11}^2 = J^{-1}\left(\frac{4.5 - z}{3}\right)^2
\]

\[
C_{2222} = KS_{22}S_{22} = J^{-1}F_{22}^2 = J^{-1}\left(\frac{4.5 - z}{3}\right)^2
\]

\[
C_{3333} = KS_{33}S_{33} = J^{-1}F_{33}^2 = J^{-1}
\]

(11)

At this point, the density $\rho' = J^{-1}\rho$, in combination with the wave velocity formula (4), all the variables can be controlled by the elastic modulus, and the density can remain unchanged before and after deformation. The resulting equivalent elastic modulus and density are expressed as

\[
Ex' = \left(\frac{4.5 - z}{3}\right)^2 Ex \quad Ey' = \left(\frac{4.5 - z}{3}\right)^2 Ey \quad Ez' = Ez \quad \rho' = \rho
\]

(12)

According to the formula, the equivalent modulus of the required metamaterial is orthogonal gradient. This material properties can be approximated by embedding membrane metamaterials. Basing on the effective material parameter model of the membrane-imbedded acoustic metamaterials, it can be seen that in the absence of prestress, membrane's different elastic moduli will only change the equivalent elastic modulus in the direction parallel to the membrane, but has no effect on the equivalent elastic modulus in the direction perpendicular to the membrane. So the membrane-imbedded acoustic metamaterials can be used as transformation acoustics materials set by Eq.(12). The total model is equally divided into 15 parts in the Z direction, and membranes with different elastic moduli are embedded in each part, as shown in Fig.7.
Figure 7. Deformed model embedded membranes

The finite element method is used to analyze the model, as illustrated in Fig.8. Solid elements and membrane elements are used as the matrix model and the membrane model, respectively. In order to compare results, the finite element analysis are separately performed on the initial model shown in Fig.8(a), the deformed model without metamaterials shown in Fig.8(b), and the membrane-imbedded acoustic metamaterial model shown in Fig.8(c). The transient analysis method is adopted. The incident wave is a sinusoidal plane wave of 1000 Hz from the bottom surface along the Z positive direction. A perfect absorption layer (PML) is added on the top of the Z-direction to prevent reflection interference. To facilitate the observation of the wave propagation throughout the model, the wave from the symmetrical section of the model are illustrated.

Figure 8. Plane wave propagation in models

From simulation results, it can be seen that the wave propagates in the initial model can keep plane wave characteristics. In the deformed model, propagation paths of the plane wave are disturbed, and plane wave propagation characteristics are no longer maintained. In the model of membrane-imbedded acoustic metamaterial, the plane wave is still propagating forward in the form of a plane wave and maintains its propagation characteristics. However, at bottom corners of the model, the propagation properties cannot be well maintained.
4.2. Cylindrical Acoustic Transformation

![Initial model](image1)

![Deformed model](image2)

![Deformed model embedded membrane](image3)

**Figure 9.** Cylindrical model and circular truncated cone embedded membrane

An initial model, a deformed model and a deformed model embedded membranes are shown in Fig.9. In Fig.9(c), the material parameters of membrane-imbedded acoustic metamaterials are designed and set according to the following expression:

\[
Er' = \left(\frac{4.5 - z}{3}\right)^2 Er \quad E\theta' = \left(\frac{4.5 - z}{3}\right)^2 E\theta \quad Ez' = Ez \quad \rho' = \rho
\]  

(13)

For the initial cylinder in Fig.9(a) and the deformed circular truncated cone in Fig.9(c), the full wave analysis is performed with a plane wave of 1000 Hz from the bottom circle. By observing the propagation of the plane wave in the model at a certain time, it can be seen that the initial cylinder can keep plane wave propagation characteristics, as shown in Fig.10(a), and the circular truncated cone model can’t maintain the propagation characteristics well at corner points, as shown in Fig.10(b).

![Plane wave propagation in the model](image4)

**Figure 10.** Plane wave propagation in the model

4.3. Error Analysis and Improvement

Both of the cubic model and the circular truncated cone model cannot maintain plane wave propagation characteristics at the corners, although the wave propagation at corner points of the circular truncated cone is a little better than that at the same location in the cubic model. Because the model of the membrane-imbedded acoustic metamaterial has the same effective material parameters at the same coordinate of \(z\), but the material parameters given by the coordinate transformation Eq.(3) have not the feature, there will be errors in the material design procedure, and the maximum error will occur at the model bottom.
Table 1. Equivalent elastic moduli errors at corners

| Model type                  | Elastic moduli | Calculated value \([10^7 \text{ Pa}]\) | Ideal value \([10^7 \text{ Pa}]\) | Errors  |
|-----------------------------|----------------|---------------------------------------|----------------------------------|---------|
| Prism                       | \(E_x\)        | 10.02                                 | 10.58                            | 5.30%   |
| Circular truncated cone     | \(E_x\)        | 7.14                                  | 7.52                             | 5.18%   |

Table 1 shows that errors between calculated values and ideal values in the circular truncated cone model are smaller than those in the cubic model. The two models have the same mapping, the only difference is the boundary shape. Obviously, the boundary shape affects elastic modulus errors at corners in models. In order to improve the wave propagation at corners of the prism, side borders of the prism need to be optimized and adjusted.

In Fig.11, original side borders of the prism are replaced with convex and concave arc surfaces, respectively; meanwhile, boundary parameters, the radii \(R_1\) and \(R_2\), are adjusted and optimized to reduce elastic modulus errors at corner points.

In Fig.12, the relationship curves between relative errors of model elastic modulus and boundary parameters are given. The relative error is defined as the relative error between the elastic moduli calculated by the transformation acoustics at the bottom corner points and the bottom central points.

![Model with the arc-like curved surfaces](image)

(a) Convex arc surface model  
(b) Concave arc surface model

**Figure 11.** Model with the arc-like curved surfaces

![Relationship curves](image)

(a) Relationship between relative errors and \(R_1\)  
(b) Relationship between relative errors and \(R_2\)

**Figure 12.** Relationship between relative errors of model elastic modulus and boundary parameters

For the model with the convex arc surface, the relative errors of model elastic modulus becomes small with the decrease of \(R_1\). Due to the model size limitation, the minimum arc radius is \(R_1 = 4.75m\), the error of the elastic modulus is minimized when \(R_1 = 4.75m\). For the model with the concave arc surface, the relative errors of model elastic modulus becomes small with the increase of \(R_2\). However, when \(R_2\) reaches a certain value, the errors remain stable, and there is still big errors of model elastic modulus at corners.
From calculation data, it can be seen that the model with the convex circular arc surface boundary has less errors at corners. In order to fully illustrate the plane wave propagation improved by adjusting boundaries, for the models with different lateral boundaries, finite element transient analysis is performed, as shown in Fig. 13.

In Fig. 13, it can be seen that the wave propagation at corners in the models with the convex circular arc surface boundary is well improved, and the plane wave characteristics can be well maintained. However, for the models with the concave arc surface boundary, although the wave propagation has been improved at corners, the waveform still has certain disturbance.

5. Summary
In this paper, according to Norris's transformation acoustics theory and characteristics of the membrane-imbedded acoustic metamaterials, the propagation path control of acoustic waves in 3D models is realized, and the following conclusions are drawn:

For the membrane-imbedded acoustic metamaterials, material properties of orthogonal gradient stiffness and isotropic density can be easily designed and achieved by adjusting parameter settings of embedded membranes. Combining the transformation acoustics theory, the propagation path control of acoustic waves can be realized conveniently.

The wave at some area in the transformed models cannot propagate strictly according to preset paths, that is, there are errors in equivalent elastic moduli at the area due to the restriction of the material properties. However, the wave propagation can be improved by adjusting model boundaries, so that the propagation wave can better maintain incident plane wave characteristics.

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