On Dependence of Sets of Functions on the Mean Value of their Elements

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Abstract. The paper considers, for a given closed bounded set $M \subset \mathbb{R}^m$ and $K = (0,1)^n \subset \mathbb{R}^n$, the set $M = \{ h \in L_2(K;\mathbb{R}^m) \mid h(x) \in M \text{ a.e. } x \in K \}$ and its subsets

$$M(\hat{h}) = \left\{ h \in M \mid \int_K h(x)dx = \hat{h} \right\}.$$ 

It is shown that, if a sequence $\{ \hat{h}_k \} \subset \text{co}M$ converges to an element $h_k \in M(\hat{h}_k)$ there is $h_k' \in M(\hat{h}_0)$ such that $h_k' - h_k \to 0$ as $k \to \infty$. If, in addition, the set $M$ is finite or $M$ is the convex hull of a finite set of elements, then the multivalued mapping $\hat{h} \mapsto M(\hat{h})$ is lower semicontinuous on $\text{co}M$.

Key words: multivalued mapping, subsets of functions with fixed mean value, continuous dependence.

1 Introduction

Most sets of admissible control functions in the theory of optimal control are given as sets of measurable functions with values from a given set: for a given reference domain $Q \subset \mathbb{R}^n$ and a given set $M \subset \mathbb{R}^m$ the set of admissible controls is defined as

$$M = \left\{ h \text{ measurable in } Q \mid h(x) \in M \text{ a.e. } x \in Q \right\}.$$ 

Here $n$ and $m$ are arbitrary fixed positive integers.

Provided that $Q$ is a bounded domain and $M$ is a bounded and closed set, the set $M$ can be split as $M = \bigcup_{\hat{h} \in \text{co}M} M(\hat{h})$, where

$$M(\hat{h}) := \left\{ h \text{ measurable}, h(x) \in M \text{ a.e. } x \in Q, \quad \frac{1}{|Q|} \int_Q h(x)dx = \hat{h} \right\}.$$ 

Here by $\text{co}A$ we denote the convex hull of the set $A$ and by $|Q|$ we denote the Lebesgue measure of the set $Q \subset \mathbb{R}^n$. 

Such a representation of $M$ is useful when weak limits of sequences of control functions are involved, especially in procedures of relaxation via convexification, see, for instance, Warga [4]. Analogous splitting is used in the homogenization theory defining the so-called $G_0$-closures, see, for instance, Milton [2]. The corresponding relaxation procedures often involve the evaluation of integrals (over the periodicity cell $K = (0,1)^n$) of the kind

$$I(\hat{h}) = \inf_{h \in \mathcal{M}(\hat{h})} \int_K f(x, h(x)) \, dx$$  \hspace{1cm} (1.1)

and the investigation of continuity properties of the function $\hat{h} \to I(\hat{h})$. To do that, obviously, one needs to know certain properties of the dependence of sets $\mathcal{M}(\hat{h})$ on $\hat{h}$.

In Sections 2 and 3 we shall show the following results.

**Theorem 1.** Let $Q \subset \mathbb{R}^n$ be bounded Lipschitz domain and let the set $M \subset \mathbb{R}^m$ is bounded and closed. Then for every given sequences $\{\hat{h}_k\}$ and $\{h_k\}$ such that

(i) $\{\hat{h}_k\} \subset \text{co} M$ and $\hat{h}_k \to \hat{h}_0$ in $\mathbb{R}^m$ as $k \to \infty$;

(ii) $h_k \in \mathcal{M}(\hat{h}_k)$, $k = 1,\ldots,$

there exists a sequence $\{h_{0k}\} \subset \mathcal{M}(\hat{h}_0)$ such that

$$h_k - h_{0k} \to 0 \text{ strongly in } L_2(Q;\mathbb{R}^m) \text{ as } k \to \infty.$$

**Theorem 2.** Let $Q \subset \mathbb{R}^n$ be bounded Lipschitz domain and let the set $M \subset \mathbb{R}^m$ is finite or $M$ is the closed convex hull of a finite set of elements. Then for every fixed sequence $\{\hat{h}_k\} \subset \text{co} M$ that converges in $\mathbb{R}^m$ to an element $\hat{h}_0$ and for every given element $h_0 \in \mathcal{M}(\hat{h}_0)$ there exists a sequence $\{h_k\}$, $h_k \in \mathcal{M}(\hat{h}_k)$, $k = 1,2,\ldots,$ such that

$$h_k - h_0 \to 0 \text{ strongly in } L_2(Q;\mathbb{R}^m) \text{ as } k \to \infty.$$

**Remark 1.** Obviously, from Theorem 2 it follows immediately that the multivalued mapping $\hat{h} \to \mathcal{M}(\hat{h})$ is lower semicontinuous on $\text{co} M$ (for the definition and properties of multivalued mappings we refer to Kuratowski [1]).

**Remark 2.** It is easy to see that under hypotheses of Theorem 1 the function $\hat{h} \to I(\hat{h})$ defined by (1.1) is lower semicontinuous provided that $f$ is Caratheodory function and that $f$ has a majorant $f_0 \in L_1(Q)$ (we recall that the set $M$ is bounded). If, in addition, the hypotheses of Theorem 2 are satisfied, then the function $\hat{h} \to I(\hat{h})$ is continuous on $\text{co} M$.

# 2 Proof of Theorem 1

In this Section, we give the proof of Theorem 1. Since all reasonings below do not depend on concrete properties of the reference domain $Q$, then, without loosing a generality, all proofs are given for the standard case $Q = K := (0,1)^n$. Since the set $M$ is bounded and closed, then the convex hull $\text{co} M$ of $M$ is
closed too and all sets $\mathcal{M}(\hat{h})$ with $\hat{h} \in coM$ are nonempty closed sets. In what follows, we shall use the notion of the relative interior $riA$ for convex sets $A$ from Euclidean spaces, for instance $ricoM$ stands for the relative interior of the convex hull of $M$. For the definition of $riA$ and other notations and properties for convex sets we refer to Rockafellar [3]. Let $r_0$ be dimension of $coM$.

Step 1. Let $\hat{h}_0 \in ricoM$. Then there exists $d > 0$ such that $\hat{h} \in ricoM$ whenever $\hat{h} \in coM$ and $|\hat{h} - \hat{h}_0| \leq d$. Let us fix $\varepsilon > 0$, $0 < \varepsilon < d/4$, and let $|\hat{h} - \hat{h}_0| \leq \varepsilon$. Then the element

$$\hat{h}_* = \hat{h} + \frac{d}{\varepsilon}(\hat{h}_0 - \hat{h}) \in ricoM.$$ 

Let $h \in \mathcal{M}(\hat{h})$, $h_* \in \mathcal{M}(\hat{h}_*)$ be arbitrary chosen elements. By virtue of Lyapunov's theorem on the range of vectorial measures for every $\lambda \in [0, 1]$ there exists a measurable set $E_\lambda \subset K$ such that

$$|E_\lambda| = \lambda, \quad \int_{E_\lambda} h(y) \, dy + \int_{K \setminus E_\lambda} h_*(y) \, dy = \lambda \hat{h} + (1 - \lambda) \hat{h}_*.$$ 

For a special choice $\lambda = \lambda_0 = 1 - \varepsilon/d$ we define $h_0$ as

$$h_0(\cdot) = \chi_{E_\lambda_0}(\cdot)h(\cdot) + (1 - \chi_{E_\lambda_0}(\cdot))h_*(\cdot),$$

where $\chi_E$ denotes the characteristic function of the set $E$. By construction, $h_0 \in \mathcal{M}(\hat{h}_0)$ and

$$\int_K (h(y) - h_0(y))^2 \, dy = \int_{K \setminus E_{\lambda_0}} (h(y) - h_0(y))^2 \, dy \leq 4c(M)\varepsilon/d,$$

where $c(M)$ depends only on $M$. Thus, the assertion of Theorem 1 holds whenever $\hat{h}_0 \in ricoM$.

Step 2. Let $\hat{h}_0$ does not belong to $ricoM$. Because $ricoM$ is not empty (provided that $M$ consists of more than one element), then there exist a vector $a \in \mathbb{R}^m$ and a constant $c$ such that

$$|a| = 1, \quad \langle a, \hat{h}_0 \rangle = c < \langle a, \hat{h} \rangle \quad \text{for all } \hat{h} \in ricoM.$$ 

Without loosing generality, we can assume that $c = 0$, otherwise we can use the transform $\hat{h} \mapsto \hat{h} - \hat{h}_0$.

Let $M_1 = \{h \in M | \langle a, h \rangle = 0 \}$. Because the sets $M$ and $M_1$ are compact, then there exists a continuous function $\gamma$, $\gamma(t) = 0$ if $t \leq 0$, $\gamma(t) > 0$ if $t > 0$, such that

$$\langle a, h - \hat{h}_0 \rangle \geq \gamma(\text{dist}\{h; M_1\}) \quad \text{for all } h \in M. \quad (2.1)$$

Without loosing generality, we can assume that the function $\gamma$ is convex, otherwise we can pass to the bipolar $\gamma^{**}$, which has the desired properties. By construction, for nonnegative $\tau$ there exists the inverse function $\tau \mapsto \gamma^{-1}(\tau)$, $\gamma^{-1}(\gamma(t)) = t$ for $t \geq 0$, which is continuous and strictly increasing on $\{\tau \in
Now, from (2.1) and convexity of $\gamma$ it follows that for every chosen $h \in \mathcal{M}$ there exists an element $h_*$,

$$h_* \in \mathcal{M}_1 = \left\{ h \in L_2(K; \mathbb{R}^r) \| h(y) \in \mathcal{M}_1 \text{ a.e. } y \in K \right\},$$

such that

$$\| h - h_* \|_{L_2(K; \mathbb{R}^r)}^2 \leq c(m, M) \int_K |h(y) - h_*(y)| \, dy \leq c(m, M) \gamma^{-1} \left( \gamma \left( \int_K |h(y) - h_*(y)| \, dy \right) \right) \leq c(m, M) \gamma^{-1} \left( \int_K \gamma(|h_*(y)|) \, dy \right) \leq c(m, M) \gamma^{-1} \left( \int_K h(y) \, dy - \int_K h_*(y) \, dy \right),$$

where $c(m, M)$ depends only on $m$ and $M$. This way, for our situation with a fixed $\hat{h}_0 \in \text{co}\mathcal{M}_1$, for every $\hat{h} \in \text{co}\mathcal{M}$ and arbitrary chosen $h \in \mathcal{M}(\hat{h})$ there exists a corresponding $h_* \in \mathcal{M}_1$ such that

$$\| h - h_* \|_{L_2(K; \mathbb{R}^r)}^2 \leq c(m, M) \gamma^{-1} (|\hat{h} - \hat{h}_0|).$$

By construction,

$$\int_K h_*(y) \, dy = \hat{h}_* \in \text{co}\mathcal{M}_1,$$

$\mathcal{M}(\hat{h}_0) \subset \mathcal{M}_1$ and the dimension of $\text{co}\mathcal{M}_1$ is less than $r_0$. From now on, we have to approximate the element $h_* \in \mathcal{M}(\hat{h}_*) \subset \mathcal{M}_1$ by elements from $\mathcal{M}(\hat{h}_0) \subset \mathcal{M}_1$, i.e. we have reduced the dimension $r_0$ of our problem to the problem with dimension less than or equal to $r_0 - 1$.

**Step 3.** To conclude our reasoning by induction over the dimension $r_0$ we have to prove our assertion for the case $r_0 = 1$. If $\hat{h}_0 \in \text{rico}\mathcal{M}$, then we apply reasoning from Step 1. If $\hat{h}_0$ does not belong to $\text{rico}\mathcal{M}$, then the set $\mathcal{M}_1$ from the Step 2 consists of only one element $h_0$ and the set $\mathcal{M}_1$ consists of one constant function $h_0(y) = h_0$ a.e. $y \in K$. For this case we can apply the same reasoning as in Step 2, what gives the statement of Theorem for $r_0 = 1$.

### 3 Proof of Theorem 2

In this Section, we give the proof of Theorem 2. Let $M = \{\overline{h}_1, \ldots, \overline{h}_N\} \subset \mathbb{R}^m$. Let $H$ be $m \times N$ matrix with columns $\overline{p}_1, \ldots, \overline{p}_N$ respectively and let

$$\Lambda := \left\{ \overline{\lambda} \in \mathbb{R}^N \mid \overline{\lambda} = (\lambda_1, \ldots, \lambda_N), \lambda_j \geq 0, j = 1, \ldots, N; \lambda_1 + \cdots + \lambda_N = 1 \right\}.$$

To a given vector-function $h \in \mathcal{M}$ (it has only $N$ admissible values from $M$) we can appoint an element $\overline{\lambda}$ whose components $\lambda_j$ represent the volume fractions...
in \( K \) of the sets where the vector-function \( h \) has the value \( \mathbf{1} \), \( j = 1, \ldots, N \), respectively. Let \( E := \{ \mathbf{r} \in \mathbb{R}^N \mid H \mathbf{r} = 0 \} \).

In these notations the statement of Theorem 2 is a straight consequence of:

\[
\begin{cases}
  \text{if } \lambda_0 \in \Lambda, \{ \mathbf{r}_k \} \subset \mathbb{R}^N, \mathbf{r}_k \to 0 \text{ as } k \to \infty \\
  \text{and } \{ \lambda_0 + \mathbf{r}_k + E \} \cap \Lambda \neq \emptyset, k = 1, 2, \ldots, \\
  \text{then there exists a sequence } \{ \lambda_k \} \text{ such that} \\
  \mathbf{r}_k \to \lambda_0 \text{ as } k \to \infty, \\
  \{ \lambda_k + \mathbf{r}_k + E \} \cap \Lambda, k = 1, 2, \ldots.
\end{cases}
\]

Indeed, first of all we have to take care only about volume fractions of sets where the functions under consideration take the corresponding values \( \{ \overline{h}_1, \ldots, \overline{h}_N \} \) (we always can rearrange the corresponding sets preserving their measures).

Further, for \( h \in \cal{M} \) with corresponding volume fractions \( (\lambda_1, \ldots, \lambda_N) = \overline{h} \) we have that \( h \in \cal{M}(\overline{h}) \), and every \( h \in \text{co} \cal{M} \) has the representation \( h = H(\lambda + E) \) with some \( \lambda \in \Lambda \). The convergence \( \mathbf{r}_k \to 0 \) as \( k \to \infty \) in (3.1) implies the corresponding convergence \( \overline{h}_k \to h_0 \) in Theorem 2, and the convergence \( \lambda_k \to \lambda_0 \) implies the corresponding convergence \( h_k \to h_0 \) in Theorem 2.

Let us denote \( \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N \) and let us represent \( E \) as the direct sum \( E = E_0 + E_1 \) where

\[
E_0 = \{ \mathbf{r} \in E \mid \langle \mathbf{r}, \mathbf{1} \rangle = 0 \}.
\]

Here the subspace \( E_1 \) can be equal to \{0\} if \( \mathbf{1} \) is orthogonal to \( E \). From assumptions on \( \mathbf{r}_k \) we have the existence of \( \mathbf{r}_{0k} \in E_0 \) and \( \mathbf{r}_{1k} \in E_1 \) such that

\[
\begin{align*}
\lambda_0 + \mathbf{r}_k &= \mathbf{r}_{0k} + \mathbf{r}_{1k} \\
\langle \mathbf{r}_k + \mathbf{r}_{1k}, \mathbf{1} \rangle &= 0, \quad \mathbf{r}_{1k} \to 0 \quad \text{as} \quad k \to \infty.
\end{align*}
\]

So, if necessary, using the transform \( \mathbf{r}_k \to \mathbf{r}_k + \mathbf{r}_{1k} \) and replacing \( E \) by \( E_0 \), without loosing generality, we can assume that

(i) the vector \( \mathbf{1} \) is orthogonal to \( E \);

(ii) \( \langle \mathbf{r}_k, \mathbf{1} \rangle = 0, \quad k = 1, 2, \ldots. \)

That means (since \( \langle \lambda_0, \mathbf{1} \rangle = 1 \)) our further reasoning concerns only the hyperplane \( \{ \mathbf{r} \in \mathbb{R}^N \mid \langle \mathbf{r}, \mathbf{1} \rangle = 1 \} \). There are two possibilities:

(a) \( (\lambda_0 + E) \cap riA \neq \emptyset \);

(b) \( \lambda_0 \) belongs to a façade \( A_s \) of \( A \) with the dimension \( s \), \( 0 \leq s \leq N - 2 \), and \( \lambda_0 + E \) can be separated from \( riA \).

For the case (a) there exists a \( \lambda_* \in (\lambda_0 + E) \cap riA \) and the elements

\[
\lambda_k = \lambda_0 + \mathbf{r}_k + \tau_k(\lambda_* - \lambda_0), \quad k = 1, 2, \ldots,
\]

with appropriate \( \tau_k > 0, k = 1, 2, \ldots \), solve the problem for \( k \geq k_0 \) with some \( k_0 \).

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For the case \((b)\), without loosing generality, we can assume that \(A_s\) is the façade with the minimal dimension \(s\) compared to all façades, which contain \(\hat{\lambda}_0\). Hence, after relabeling indexes we obtain

\[
A_s = \{ \overline{x} \in A \mid \overline{x} = (\lambda_1, \ldots, \lambda_N), \lambda_{s+2} = \cdots = \lambda_N = 0 \},
\]

\[
\hat{\lambda}_0 = (\lambda_0^1, \ldots, \lambda_0^N), \quad 0 < \lambda_0^1, \ldots, 0 < \lambda_0^{s+1}, \quad \lambda_0^{s+2} = \cdots = \lambda_0^N = 0.
\]

If

\[
\hat{\lambda}_0 + \overline{a}_k + \overline{z}_k \in A \quad \& \quad \overline{z}_k \in E, \quad k = 1, 2, \ldots,
\]

then from

\[
\overline{a}_k \to 0 \text{ as } k \to \infty, \quad \text{and} \quad \langle \overline{z}_k, \overline{1} \rangle = 0, k = 1, 2, \ldots,
\]

it follows immediately that the sequence \(\{\overline{z}_k\}\) is bounded.

Let us assume that the sequence \(\{\overline{z}_k\}\) converges to an element \(\overline{z}_0\). If \(\overline{z}_0 = 0\), then the sequence

\[
\overline{\lambda}_k := \hat{\lambda}_0 + \overline{a}_k + \overline{z}_k, \quad k = 1, 2, \ldots,
\]

solves the problem.

If \(\overline{z}_0 \neq 0\), then those entries of \(\overline{z}_0 = (z_{01}, \ldots, z_{0N})\), which are different from zero, are positive for \(j \geq s + 2\) and negative for those indexes \(j^{**}\), for which \(\lambda_{j^{**}}^0 = 1\) (if any). Therefore, there exists \(d_0 > 0\) such that \(\hat{\lambda}_0 + \tau \overline{z}_0 \in A\) provided

\[
0 \leq \tau \leq d_0.
\]

Since \(\overline{z}_k - \overline{z}_0 \to 0\) as \(k \to \infty\), then the elements

\[
\overline{\lambda}_k := \hat{\lambda}_0 + \tau \overline{a}_k + \overline{z}_k + \overline{a}_k + \overline{z}_0 + \overline{z}_k
\]

for \(k \geq k_0\) and with appropriate \(\tau_k, \tau_k \to 0\) as \(k \to \infty\), belong to \(A\). Indeed, since

\[
\langle \overline{z}_0, \overline{1} \rangle = 0, \langle \overline{a}_k, \overline{1} \rangle = 0, \langle \overline{z}_k, \overline{1} \rangle = 0, k = 1, 2, \ldots,
\]

we have to check only inequalities

\[
\lambda_{kj} \geq 0, \quad j = 1, \ldots, N; \quad k = k_0, k_0 + 1, \ldots
\]

(Obviously, \(\overline{\lambda}_k \to \hat{\lambda}_0\) as \(k \to \infty\)). For those indexes \(\{j^{'}\}\), for which entries of \(\overline{z}_0\) are equal to zero,

\[
\lambda_{k^{j^{'}}} + a_{k^{j^{'}}} + (z_{k^{j^{'}}} - z_{0^{j^{'}}}) \geq 0, \quad k = 1, 2, \ldots,
\]

(by the initial assumptions on the sequence \(\{\overline{z}_k\}\)), but for the rest of indexes \(\{j^{**}\}\) either

\[
1 > \lambda_{j^{**}}^0 > 0
\]

or

\[
\lambda_{j^{**}}^0 = 1 \quad \& \quad z_{0^{j^{**}}} < 0
\]

what is sufficient for the existence of \(\tau_k\) with desired properties.

The general case of an arbitrary sequence \(\{\overline{z}_k\}\) is treated by standard reasoning by contradiction, i.e., we assume the contrary that there exist \(d > 0\) and a sequence of indexes \(\{k^{'}\}\) such that the distance from \(\hat{\lambda}_0\) to \(\{\hat{\lambda}_0 + \overline{a}_{k^{'}}, E\} \cap A\)
is greater than \( d \). After that we take an arbitrary subsequence of \( \{ \tilde{h}_k \} \), for which the corresponding sequence \( \{ \tilde{z}_k \} \) converges. The proof of the first part of Theorem 2 is completed.

Now, let \( M \) be closed convex hull of a finite number of elements \( \{ h_1, \ldots, h_N \} \) and set

\[
S := \{ \sigma \in L_2(K; \mathbb{R}^N) \mid \sigma = (\sigma_1, \ldots, \sigma_N), 0 \leq \sigma_j(x) \leq 1, j = 1, \ldots, N; \\
\sum_{j=1}^N \sigma_j(x) = 1 \text{ a.e. } x \in K \}.
\]

Since the function

\[
(\sigma, x) \rightarrow (h(x) - \sum_{j=1}^N \sigma_j h_j)^2
\]

is a normal integrand on \( \Lambda \times K \) (for every fixed \( h \in \mathcal{M} \) ), then every \( h \in \mathcal{M} \) has the representation

\[
h(x) = \sum_{j=1}^N \sigma_j(x) h_j \text{ a.e. } x \in K
\]

with some \( \sigma \in S \). In turn, a subset of piecewise constant elements is dense in \( S \) and sets \( \mathcal{M}(\tilde{h}) \) have the same property.

This way, by using Cantor’s diagonal process, we have that it is sufficient to show the existence of the approximating sequence \( \{ h_k \} \) for the case of a piecewise element \( h_0 \in \mathcal{M}(\tilde{h}_0) \). Let \( Q_i \subset K, i = 1, \ldots, s \), are the sets where the function \( h_0 \) is constant and takes values \( g_1, \ldots, g_s \) from \( M \) respectively.

Now, we define the set \( \hat{M} := \{ h_1, \ldots, h_N, g_1, \ldots, g_s \} \) and sets

\[
\hat{\mathcal{M}}(\hat{h}) := \{ h \text{ measurable in } K \mid h(x) \in \hat{M} \text{ a.e. } x \in K, \int_K h(x) dx = \hat{h} \}.
\]

By construction, \( \text{co} \hat{M} = M \) and \( \hat{\mathcal{M}}(\hat{h}) \subset \mathcal{M}(\hat{h}) \) for every \( \hat{h} \in \mathcal{M} \).

If \( \{ h_k \} \subset M \) and \( h_k \rightarrow \hat{h} \) as \( k \rightarrow \infty \) then also \( \{ \hat{h}_k \} \subset \text{co} \hat{M}, \hat{h}_0 \in \text{co} \hat{M} \) and \( h_0 \in \mathcal{M}(\hat{h}_0) \). This way, the existence of the desired approximating sequence \( \{ h_k \} \) follows immediately from the proof of the first part of Theorem 2. The proof of Theorem 2 is completed.

We conclude with a simple example, which shows that the statement of Theorem 2 is not, in general, true under hypotheses of Theorem 1. Let

\[ M = \left\{ (-1,0,0), (1,0,0), (0,t,t^2), \ 0 \leq t \leq 1 \right\} \subset \mathbb{R}^3. \]

By construction,

\[
\mathcal{M}((0,t,t^2)) = \left\{ (h_1,h_2,h_3) \in L_2(K;\mathbb{R}^3) \mid h_1(x) = 0, h_2(x) = t, h_3(x) = t^2; x \in K \right\} \text{ for } 0 < t < 1,
\]

\[
\mathcal{M}((0,0,0)) = \left\{ (h_1,h_2,h_3) \in L_2(K;\mathbb{R}^3) \mid h_1(x) = -1 \text{ or } 0 \text{ or } 1, \int_K h_1(x) dx = 0; h_2(x) = 0, h_3(x) = 0; x \in K \right\},
\]

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and the statement of Theorem 2 does not hold.

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