GENERIC MEASURES FOR TRANSLATION SURFACE FLOWS

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Abstract. We consider straight line flows on a translation surface that are minimal but not uniquely ergodic. We give bounds for the number of generic invariant probability measures.

1. Introduction and Statement of Theorem

An Abelian differential or holomorphic 1-form \( \omega \) on a Riemann surface \( X \), assigns to each holomorphic coordinate \( z \) on \( X \) a holomorphic function \( f^z(z) \), which in an overlapping coordinate \( w \), transforms by

\[
  f^w(w) \frac{dw}{dz} = f^z(z).
\]

If the Riemann surface is closed of genus \( g \geq 1 \) an Abelian differential \( \omega \) has zeroes of order \((\alpha_1, \ldots, \alpha_s)\) with \( \sum_{i=1}^s \alpha_i = 2g - 2 \). In a more geometric fashion one can also describe \( \omega \) as a union of polygons each embedded in \( \mathbb{C} \) with pairs of sides identified by translations. Since each polygon is embedded in \( \mathbb{C} \), letting \( z \) be the local coordinate on the polygon, one defines the holomorphic 1-form \( \omega \) to be \( dz \) in each polygon away from the vertices. The fact that sides are identified by translations says \( dz \) defines a global 1-form. Every Abelian differential \( \omega \) can be expressed this way. This justifies the term translation surface, and these surfaces are usually denoted by \((X, \omega)\). The 1-form \( \omega \) defines a metric \( |\omega| \) and an area form \( |\omega|^2 \). In the polygon description these are just the Euclidean metric and Lebesgue measure.

For each direction \( 0 \leq \theta < 2\pi \) there is a straight line flow \( \phi^\theta_t : (X, \omega) \to (X, \omega) \) in direction \( \theta \). If \( \omega \) is written locally as \( dz \) then these are the lines that make angle \( \theta \) with the positive real axis. Since translations preserve slopes this gives a flow defined for all times except at the points which encounter a zero. Lebesgue measure is invariant under the flow. On a flat torus the classical Weyl theorem says that the minimality of the flow implies unique ergodicity meaning that Lebesgue measure is the only invariant probability measure for the flow. Equivalently, every orbit is equidistributed on the torus. In genus \( g \geq 2 \), however, there are examples of minimal flows on translation surfaces that are not uniquely ergodic.

The first examples appeared in the papers of Veech [20] and Satayev [13]. In a slightly different context there are interval exchange transformations (IET) that are minimal, but not uniquely ergodic. An example based on the construction of Veech appeared in [12]. Then Keane [11] gave a general method of constructing examples. It is a basic fact that the set of invariant probability measures forms a convex set and the extreme points are ergodic.

The motivation for this paper is the following classical theorem of Katok ([10]).

Theorem 1.1. Suppose \((X, \omega)\) is a translation surface on a closed surface of \( g \geq 2 \). If \( \phi^\theta_t \) is a minimal straight line flow on \((X, \omega)\), then there are at most \( g \) ergodic probability measures.
The goal of this paper is to extend the Katok result to bound the number of generic measures.

**Definition 1.2.** Let \( \phi^t \) be a measure preserving flow on a probability space \((X, \mu)\). A point \( p \) is 2-sided generic for \( \mu \) if for all continuous \( f : X \to \mathbb{R} \),

\[
\lim_{S, T \to \infty} \frac{1}{S + T} \int_{-S}^{T} f(\phi^t(p)) dt = \int_X f d\mu.
\]

The Birkhoff Theorem says that if an invariant measure is ergodic then almost every point is 2-sided generic for the invariant measure. A measure may have generic points without being ergodic. For example for the full shift on \( d \geq 2 \) letters every invariant measure has generic points.

**Definition 1.3.** A measure \( \mu \) is 2-sided generic if it has a 2-sided generic point.

In the rest of the paper we will just use the description generic to refer to a 2-sided measure or 2-sided generic point. If a point satisfies

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\phi^t(p)) dt = \int_X f d\mu,
\]

we will say it is a forward generic point and the measure is forward generic. This is the definition of generic given by V. Cyr and B.Kra in [4], where bounds are given for the number of forward generic measures in quite general situations. Their results will be discussed in the next section. The purpose of this paper is to prove the following result.

**Theorem 1.4.** Let \((X, \omega)\) be a translation surface on a closed surface of genus \( g \geq 2 \) with \( s \) zeroes. Let \( \phi^t \) be a minimal straight line flow on \((X, \omega)\) which is not uniquely ergodic. Then the number of invariant generic probability measures is bounded by \( g + s - 1 \).

We do not know if the bound in Theorem 1.4 is in general sharp. An interesting question is if forward generic implies generic in the case of flows on translation surfaces, or equivalently interval exchange transformations. We do not in general have a bound for the number of forward generic measures that improves on the bounds found in [4]. There is one special case where we can bound the number of forward generic measures that does improve on their bounds.

**Theorem 1.5.** If \((X, \omega)\) has genus \( g \geq 2 \) with 1 or 2 zeroes, and there are \( g \) ergodic measures, then there are no invariant forward (hence 2-sided) generic non-ergodic probability measures.

An example is the case of genus 2 with two zeroes. If the flow is minimal, but not uniquely ergodic, there are two ergodic measures and no other forward generic measures.

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1.1. **Connections to Interval Exchange Transformations and History.** Recall to define an interval exchange one is given \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) where \( x_i > 0 \). Form the \( d \) sub-intervals of the interval \( [0, \sum_i x_i) \):

\[
I_1 = [0, x_1), I_2 = [x_1, x_1 + x_2), \ldots, I_d = [x_1 + \ldots x_{d-1}, x_1 + \ldots + x_{d-1} + x_d).
\]
Given a permutation \( \pi \) on the set \( \{1, 2, \ldots, d\} \), one obtains a \( d \)-\textit{Interval Exchange Transformation} (IET) \( T_{\pi,x} : [0, \sum_{i=1}^{d} x_i) \to [0, \sum_{i=1}^{d} x_i) \) which exchanges the intervals \( I_i \) according to \( \pi \). That is, if \( z \in I_j \) then
\[
T_{\pi,x}(z) = z - \sum_{k<j} x_k + \sum_{\pi(k')<\pi(j)} x_{k'}.
\]

Lebesgue measure is invariant under the action of \( T \).

There is a relationship between straight line flows on translation surfaces \( (X, \omega) \) and IET. The first return map to a transversal of the flow is an IET. Conversely, an IET can be suspended to form a closed translation surface. The genus \( g \) and number of zeroes \( s \) depend not only on the number of intervals but also on the permutation \( \pi \) of the IET. For complete details we refer to \cite{[22]}. For example, if \( \pi \) is the hyperelliptic permutation
\[
\pi(j) = d - j + 1
\]
for \( j = 1, \ldots, d - 1 \) and \( d \) is even, then the suspended \( (X, \omega) \) has genus \( g = \frac{d}{2} \) and a single zero. If \( d \) is odd, then the suspended \( (X, \omega) \) has genus \( g = \frac{d-1}{2} \) and two zeroes. Theorem 1.4 gives a bound of \( \frac{d}{2} \) and \( \frac{d+1}{2} \) in the two cases. In general, we have
\[
d = 2g + s - 1.
\]

For further results on counting ergodic measures under certain conditions see \cite{[5]}.
1.3. Outline. We briefly outline the idea of the proof of Theorem 1.4. We use Teichmüller dynamics and the idea of renormalization. By rotating, we can assume that the straight line flow is in the vertical direction. We assume throughout the paper that it is minimal, but not uniquely ergodic. We then apply the Teichmüller geodesic flow \( g_t \) to the surface \((X, \omega)\). Along the new surfaces \( g_t(X, \omega) \) the vertical lines of the translation surface flow are contracted, and the horizontal lines expanded. It is known, [14], that as \( t \to \infty \), the Riemann surfaces along \( g_t(X, \omega) \) eventually leave every compact set of the moduli space of Riemann surfaces. This means that for large \( t \) there is a maximal collection of disjoint curves that are hyperbolically short or equivalently short in extremal length. The curves depend on \( t \). The complimentary components are either cylinders or thick surfaces which means that they do not have any short essential curves. This defines a thick-thin decomposition of the surface. We show first in Proposition 5.3 that the image under \( g_t \) of generic points on \((X, \omega)\) of different ergodic measures, lie in different connected components of the complement of the short curves and have area bounded below away from 0. These can be either thick surfaces or cylinders. Thus we can associate to each ergodic measure and large time, a subsurface depending on the time, and further that different ergodic measures are associated to disjoint subsurfaces. These ideas are related to work of McMullen in [15]. He found a bound for the number of ergodic measures under a certain no loss of mass assumption in terms of the number of connected components.

If there are generic measures that are not ergodic, they will be associated to subsets of other complements of the short curves. Because the set of generic points of these measures has measure 0 it turns out these subsets may not be an entire complimentary component which leads to complications in the counting of the number of possible measures. To guarantee that distinct generic measures determine disjoint subsets we need them to be 2-sided except in the cases of Theorem 1.5. This analysis is carried out in Proposition 5.7. Counting the maximum number of disjoint subsets of the original surface, will give the theorem.

The arguments necessary for Proposition 5.3 and Proposition 5.7 begin with Proposition 5.2. It says that vertical lines through the images of generic points of different measures cannot bound rectangles on \( g_t(X, \omega) \) for large \( t \). The rectangle argument is used in Proposition 5.3 to show that subsurfaces that have limits of positive area in the Deligne-Mumford compactification or are cylinders with circumferences going to 0 and area bounded away from 0 provide the desired subsurfaces for the ergodic measures.

To use limiting ideas in considering non ergodic measures requires arguments where one renormalizes these complementary surfaces, to account for the fact that areas approach 0. This is carried out in Proposition 5.7 based on work in [7], which in turn used work in [16] and [17]. We will also need a preliminary result, Lemma 2.1 on quantitative genericity.

1.4. Notation. Given quantities \( x, y \) we use the notation \( x \asymp y \) to indicate there is a constant \( C \) depending only on the genus \( g \) so that

\[
\frac{1}{C} x \leq y \leq C x.
\]

2. Translation surfaces, Genericity

2.1. Saddle connection, cylinders. A translation surface \((X, \omega)\) defines a metric \(|\omega(z)dz|\) on \( X \) which is flat except at the singularities, which have concentrated negative curvature. In the polygon version one takes the Euclidean metric \(|dz|\) in each polygon. Translations preserve the metric. Moreover slopes of lines are preserved under the side identifications.
In particular this means that each slope defines a flow by lines of that slope. A line leaving a zero is called a *separatrice*. A line segment joining singularities without singularities in its interior is called a *saddle connection*. The holomorphic 1-form $\omega$ defines the area form $|\omega|^2$. We will assume that our translation surfaces have unit area

$$\int_X |\omega|^2 = 1.$$ 

If the metric $|\omega|$ is understood we will denote the length of a curve by $|\alpha|$. 

Given an oriented line segment $\gamma$ one defines the holonomy of $\gamma$ by $\text{hol}(\gamma) := \int_\gamma \omega \in \mathbb{C}$. The real and imaginary parts of the holonomy are also called the horizontal and vertical components of the holonomy.

If a geodesic $\beta$ joins a nonsingular point to itself without passing through a singularity it is the core curve of a *cylinder* $C(\beta)$, i.e. the isometric image of a Euclidean cylinder $[0, a] \times (0, b)/(0, y) \sim (a, y)$ for some positive real numbers $a$ and $b$. Then $b$ is the height of the cylinder. We also refer to it as the distance across the cylinder. The cylinder is swept out by closed parallel loops homotopic to $\beta$. We will suppose throughout that all cylinders are maximal, i.e. in the notation of the previous sentence that $b$ is as large as possible. The boundary of the cylinder is then composed of saddle connections. There are also saddle connections crossing the cylinder joining zeroes on opposite boundary components.

The angle around a zero of order $\alpha_i$ is $2\pi(\alpha_i + 1)$ and is called a cone angle. For every free homotopy class $\gamma$ of a simple closed curve there is a geodesic in its homotopy class. It is either unique or there is a cylinder. If it is unique, then it is a union of saddle connections. The angle at any zero between incoming and outgoing saddle connections is at least $\pi$.

It is a classical result of Strebel’s [19] that on a closed translation surface (or more generally on any half translation surface or quadratic differential), for any direction, the flow in that direction decomposes into cylinders and minimal domains in which every line that does not hit a zero is dense in that domain. If the minimal domain is not the entire surface then the boundary consists of saddle connections in that direction.

### 2.2. Trapezoids

We also need to discuss translation surfaces with boundary. A complete treatment can be found in [19]. We consider the horizontal direction. We assume that each boundary component consists of a union of saddle connections, but assume none are horizontal. In the horizontal direction there may still be minimal domains and cylinders. There are in addition, horizontal segments leaving points on each boundary component. A horizontal segment leaving the boundary at a nonzero point either hits a zero in the interior or returns to a (possibly the same) boundary component.

If a segment joins a pair of nonzero points on the two boundaries, there is a maximal family of parallel horizontal segments joining nonzero points on the same boundary components, determining a trapezoid. The boundary of the trapezoid has two horizontal sides each of which contains a zero, either on the boundary of the surface or in the interior. If the only zero on a horizontal side is on the boundary, then at the zero there are at least two horizontal separatrices leaving the zero and entering the surface. For if there was only one, the trapezoid could be extended, contrary to the assumption it is maximal. The other two sides of the trapezoid are subsegments of the boundary components. On each horizontal side of the trapezoid there is at least one segment that the trapezoid shares with a different domain, which is another trapezoid, a cylinder or a minimal domain.
2.3. Transverse Measures and Effective Genericity. An invariant measure $\mu$ for the vertical flow $\phi^t$ on $(X, \omega)$ defines a transverse measure $\rho$ on horizontal segments $I$ by assigning to each segment $I$ and sufficiently small $t_0$,
\[
\rho(I) = \frac{\mu(I \times [0, t_0])}{t_0},
\]
where $I \times [0, t_0]$ is the set of points $\phi^t(x)$ where $x \in I$ and $0 \leq t \leq t_0$. The fact that $\mu$ is measure preserving under $\phi^t$ implies $\rho(I)$ is independent of $t_0$ for small $t_0$. Furthermore it is easy to see that the transverse measure is flow invariant. It also follows that if $\mu$ and $\mu'$ are distinct flow invariant probability measures, then the associated transverse measures $\rho$ and $\rho'$ are distinct as well.

**Lemma 2.1.** Given a vertical flow on a translation surface with invariant measure $\mu$ and transverse measure $\rho$, a pair of numbers $0 < \eta < 1$ and $0 < \epsilon < 1$, a horizontal interval $I$, and a (2-sided) generic point $q$ for $\mu$, there exists $L_0$, such that for the vertical line $\gamma$ of length $L > L_0$ in either direction, with one endpoint $q$, any subsegment $\gamma'$ of $\gamma$ of length at least $\eta|\gamma|$ satisfies
\[
|\text{card}(\gamma' \cap I) - \rho(I)|\gamma'| \leq \frac{2\epsilon|\gamma'|}{\eta}.
\]

**Proof.** Choose $L_1$ large enough so that if $\tau$ is a vertical segment in either direction with one endpoint $q$ and $|\tau| \geq L_1$, then
\[
|\text{card}(\tau \cap I) - |\tau|\rho(I)| < \epsilon|\tau|.
\]
Let $M$ be the maximum of the pair of numbers of intersection the vertical segments of length $L_1$ in the two directions with one endpoint $q$ have with $I$. Let
\[
L_0 = \max\left(\frac{L_1}{\eta}, \frac{\rho(I)L_1 + M}{\epsilon}\right).
\]
Now take a segment $\gamma$ of length $L \geq L_0$ and a subsegment $\gamma' \subset \gamma$ of length at least $\eta L \geq L_1$ which starts at some $y$ and ends at some $z$ further away from $q$ than $y$. Let $\tau$ the subsegment of $\gamma$ starting at $q$ and ending at $z$. We have $|\tau| \geq L_1$ so
\[
|\text{card}(\tau \cap I) - \rho(I)|\tau| < \epsilon|\tau|.
\]
Let $\sigma$ be the segment from $q$ to $y$. Then
\[
\text{card}(\gamma' \cap I) = \text{card}(\tau \cap I) - \text{card}(\sigma \cap I)
\]
The first case is if $|\sigma| \geq L_1$. Then (2) and (3) give
\[
|\text{card}(\gamma' \cap I) - \rho(I)|\gamma'| \leq \epsilon(|\tau| + |\sigma|) \leq 2\epsilon|\tau| \leq 2\epsilon L = \frac{2\epsilon}{\eta}L \leq \frac{2\epsilon|\gamma'|}{\eta}.
\]
We are done in this case.

Now suppose $|\sigma| \leq L_1$ and so $|\gamma'| \leq |\tau| \leq |\gamma'| + L_1$. Now the second term on the right in (3) is bounded by $M$. Then
\[
|\gamma'|\rho(I) - \epsilon(2|\gamma'| + L_1) - M \leq \text{card}(\tau \cap I) - M \leq \text{card}(\gamma' \cap I) - \text{card}(\tau \cap I) \leq (|\gamma'| + L_1)\rho(I) + \epsilon(|\gamma'| + L_1).
\]
Together with the choice of $L_0$ in (1) this gives
\[
|\text{card}(\gamma' \cap I) - |\gamma'|\rho(I)| \leq \rho(I)L_1 + M + \epsilon(L_1 + |\gamma'|) \leq 2\epsilon L \leq \frac{2\epsilon}{\eta}|\gamma'|.
\]
\[\square\]
3. Teichmüller geodesic flow, thick-thin decomposition

The translation surfaces of genus \( g \) with fixed orders of zeroes \( \alpha_1, \ldots, \alpha_s \) with \( \sum_{i=1}^{s} \alpha_i = 2g - 2 \) fit together to form a moduli space or stratum \( \mathcal{H}(\alpha_1, \ldots, \alpha_s) \). There is a natural map

\[
\pi : \mathcal{H}(\alpha_1, \ldots, \alpha_s) \to \mathcal{M}_g,
\]

the moduli space of Riemann surfaces which simply records the Riemann surface of \((X, \omega)\).

There is defined on \( \mathcal{H}(\alpha_1, \ldots, \alpha_s) \) an action of the group \( SL(2, \mathbb{R}) \). The action is the linear action on polygons. The diagonal subgroup

\[
g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}
\]

is the Teichmüller geodesic flow. It contracts the vertical lines of \( \omega \) by \( e^{t/2} \) and expands along the horizontal lines by \( e^{t/2} \). We will think of \( g_t \) in two ways. If we represent translation surfaces as polygons, then it is a linear map, called the Teichmüller map from one polygon to another. We can also think of it as a self map from the stratum to itself which takes the point corresponding to one polygon to the point corresponding to its image under the Teichmüller map.

We will denote the underlying Riemann surface of \( g_t(X, \omega) \) by \( X(t) \) and the holomorphic 1-form by \( \omega(t) \).

An excellent short survey of translation surfaces and the \( SL(2, \mathbb{R}) \) action on their moduli spaces can be found in [21].

For the homotopy class \( \alpha \) of a closed curve on a closed Riemann surface \( X \) the extremal length of \( \alpha \) on \( X \) is defined to be

\[
\text{Ext}_X(\alpha) = \sup_{\sigma} \frac{\ell_\sigma^2(\alpha)}{\text{area}(\sigma)}.
\]

Here, \( \sigma \) ranges over all metrics in the conformal class of \( X \) and \( \ell_\sigma(\alpha) \) is the infimum of the \( \sigma \)-length of all representatives of the homotopy class of the curve \( \alpha \).

For constants \( \epsilon_0 > \epsilon_1 > 0 \), the \((\epsilon_0, \epsilon_1)\) thick-thin decomposition of \((X, \omega)\) is the pair \((\mathcal{A}, \mathcal{Y})\) where \( \mathcal{A} \) is the set of geodesic representatives in the metric defined by \((X, \omega)\) of closed curves \( \alpha \) such that \( \text{Ext}_X(\alpha) \leq \epsilon_0 \) and \( \mathcal{Y} \) is the set of the components of \( X \) cut along \( \mathcal{A} \). We assume that \( \mathcal{A} \) is maximal in the sense that any curve not in \( \mathcal{A} \) has extremal length at least \( \epsilon_1 \).

**Remark 3.1.** It may happen that distinct short curves in \( \mathcal{A} \) share a zero. This would be the case, for example, if \( \omega \) has a single zero and there is more than one short curve. If this happens then the translation surface representative of a thick subsurface has strictly fewer boundary components than the number of short curves on its boundary.

We have the following definition of size due to Rafi [16].

**Definition 3.2.** Let \( Y \) be a subsurface of a translation surface \((X, \omega)\) whose boundary consists of a collection of curves with extremal length at most \( \epsilon_0 \). If \( Y \) is not topologically a 3-times punctured sphere or a cylinder, the size \( \lambda(Y) \) is defined to be the infimum of the \( |\omega| \) lengths of the essential closed geodesics in \( Y \). A 3-times punctured sphere does not have essential curves so the size is defined to be the diameter. If \( Y \) is a cylinder \( C(\alpha) \), the size \( \lambda(C(\alpha)) \) is the length of a vertical segment joining the two boundary components.
For the definition in the case of cylinders to make sense it is necessary that the core curve of the cylinder not to be vertical as it will not be for the rest of the paper since the vertical flow is assumed to be minimal.

Let \((A, Y)\) be the thick-thin decomposition of \((X, \omega)\). If \(\alpha \in A\) has a unique geodesic representative as a union of saddle connections it is on the boundary of a pair of surfaces \(Y\) and \(Z\), which may coincide. If there is a cylinder \(C(\alpha)\) then each boundary component of \(C(\alpha)\) is on the boundary of such a \(Y\). We will need the following result of Rafi’s which is (part of) Theorem 3.1 of [17].

**Theorem 3.3.**

\[
\frac{1}{\text{Ext}_X(\alpha)} \approx \log \frac{\lambda(Y)}{|\alpha|} + \log \frac{\lambda(Z)}{|\alpha|} + \frac{\lambda(C(\alpha))}{|\alpha|}
\]

By way of further explanation, the first two terms are the moduli of what is called the expanding annulus in \(Y\) (resp. \(Z\)) that are isotopic to \(\alpha\). The expanding annulus is foliated by closed curves that are equidistant from the loop \(\alpha\). The last term on the right is the modulus \(\text{Mod}(C(\alpha))\) of the cylinder. This is the ratio of the height to the circumference of the cylinder.

We also will use Theorem 4 of [16] which says

\[
\text{diam}(Y) \approx \lambda(Y).
\]

Now suppose \(t_n \to \infty\) is a sequence with the property that for some maximal collection of curves \(\alpha(t_n)\) we have \(\text{Ext}(\alpha(t_n)) \to 0\) on the Riemann surface \(X(t_n)\) of \(g_{t_n}(X, \omega)\). This is equivalent to saying that the hyperbolic lengths of \(\alpha(t_n)\) go to zero as well. In other words we are leaving every compact set in the moduli space \(M_g\). Consider any sequence of components \(Y(t_n)\) of the complement of the flat geodesic representatives of \(\alpha(t_n)\) along \(X(t_n)\). Assume they are not cylinders. We write \(|\alpha(t_n)|\) to be the flat length of \(\alpha(t_n)\) with respect to metric defined by \(\omega(t_n)\).

**Definition 3.4.** We say such a sequence of noncylindrical complementary components \(Y(t_n)\) is a plump sequence if for all boundary components \(\alpha(t_n)\) of \(Y(t_n)\),

\[
\lim_{t_n \to \infty} \frac{|\alpha(t_n)|}{\lambda(Y(t_n))} = 0.
\]

We say it is gaunt if there is some boundary component \(\alpha(t_n)\) of \(Y(t_n)\) with

\[
\liminf_{t_n \to \infty} \frac{|\alpha(t_n)|}{\lambda(Y(t_n))} > 0.
\]

We note that if a sequence is gaunt we can pass to a subsequence so that the above \(\liminf\) is replaced by limit. It may happen that \(Y(t_n)\) has empty interior, (see [16] for an example). We also may have that \(Y(t_n)\) is a disc. This can happen if the short extremal length curves bound a sphere, but their representatives in the flat metric form a connected set.

### 4. Limits of Translation Surfaces

A compactification of a stratum of Abelian differentials and its relation to the Deligne-Mumford compactification of the Riemann moduli space is carried out in the paper [2]. We will not need their entire spectrum of results, but essentially we will use Theorem 10 of [7]
which is a theorem about limits of translation surfaces. We begin with a brief explanation of some ideas from the Deligne Mumford compactification $\overline{\mathcal{M}}_g$ of the moduli space $\mathcal{M}_g$ of closed Riemann surfaces of genus $g$. References are the papers [9], [11], [13]. The points in $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ are possibly disconnected Riemann surfaces $X(\infty)$ with punctures, or in the language of algebraic geometry noded stable algebraic curves. We denote the nodes by $\Sigma(\infty)$. These are points where two distinct surfaces are glued together or two points of the same surface are identified. By separating the pair of points, we will think of them as punctures.

We put a topology on $\overline{\mathcal{M}}_g$ by saying $X_n \to X(\infty)$ if there is an exhaustion of $X(\infty) \setminus \Sigma(\infty)$ by compact sets $K_n$ and for any $\epsilon > 0$, for large $n$, injective $(1 + \epsilon)$ quasiconformal maps $h_n : K_n \to X_n$.

This allows us to define if a sequence $p_n \in X_n$ has a limit $q_\infty \in X(\infty) \setminus \Sigma(\infty)$. Namely, there is a sequence $q_n \in K_n$ such that $p_n = h_n(q_n)$ and $q_n \to q_\infty$.

We can extend all this to subsurfaces $X^b(\infty) \subset X(\infty)$ with boundary and say a sequence $X^b_n \subset X_n$ with boundary converges to $X^b(\infty)$ if there is a nested sequence of neighborhoods $U_n$ of $\partial X^b(\infty)$ converging to the boundary and for any $\epsilon$ there are $1 + \epsilon$ quasiconformal maps of the complement of $U_n$ to $X^b_n$.

We will apply this to sequences $X_j(t_n)$ that are thick components of thick-thin decompositions of $\omega(t_n)$ whose boundary satisfies $\text{Ext}(\alpha(t_n)) \to 0$.

Now let $\omega(t_n)$ a holomorphic 1-form on $X(t_n)$. Let $\omega_j(t_n)$ the restriction of $\omega(t_n)$ to $X_j(t_n)$.

**Proposition 4.1.** Suppose $X_j(t_n)$ is sequence of thick components of the thick-thin decomposition. By passing to a subsequence we can assume that there is a limiting surface $X^b_j(\infty)$, possibly with boundary, together with a 1-form $\omega_j(\infty)$, such that $\frac{\omega_j(t_n)}{\text{Ext}(X_j(t_n))}$ converges to $\omega_j(\infty)$. Moreover for boundary loops $\alpha(t_n)$ of $X_j(t_n)$

- if $\lim_{t_n \to \infty} \frac{\alpha(t_n)}{\text{Ext}(X_j(t_n))} > 0$ then $X_j(\infty)$ has a boundary loop corresponding to $\alpha(t_n)$.
- if $\lim_{t_n \to \infty} \frac{\alpha(t_n)}{\text{Ext}(X_j(t_n))} = 0$ then $X_j(\infty)$ has a puncture corresponding to $\alpha(t_n)$

**Remark 4.2.** This is an example of what is called multi-scaling in [2]. In order to achieve a limit, the 1-forms need to be scaled by their size, which can depend on the component.

**Proof.** Theorem 10 of [7] essentially gives the convergence. (We remark that in that paper if the assumption of the first bullet holds, one acquires a pole if one takes the limit of the renormalized differential on a region in $X(t_n)$ that includes $\alpha(t_n)$ in its interior. In that case the curve $\alpha(t_n)$ itself limits on a geodesic in the homotopy class of a loop surrounding the puncture). Here we just take a limition the region $X_j(t_n)$ itself.

We prove the second bullet. Since the length on the renormalized differential still goes to 0, the surface must develop a puncture denoted $p$. Arguing by contradiction, suppose though $\omega_j(\infty)$ has a pole at $p$. Then there is a positive lower bound $a_0$ for the length of any loop surrounding $p$. Fix a point $z_0 \in X_j(\infty)$ with $z_0 \neq p$. The diameter of $X_j(t_n)$ measured with respect to the normalized metric $\frac{w_j(t_n)}{\text{Ext}(X_j(t_n))}$ is uniformly bounded; say by a constant $C$. Since the distance from $z_0$ to $p$ measured in the metric induced by $|\omega_j(\infty)|$ is infinite, choose $U$ a neighborhood of $p$ such that the distance from $z_0$ to $U$ is greater than 2C. For large $t_n$, $h_n(z_0) \in X_j(t_n)$, where $h_n : X_j(\infty) \setminus U \to X(t_n)$ is the quasiconformal map defining the topology. Since $\frac{w_j(t_n)}{\text{Ext}(X_j(t_n))}$ converges to some $\omega_j(\infty)$ on $X_j(\infty) \setminus U$, we
cannot have \( \alpha_{t_n} \subset h_{t_n}(X_j(\infty) \setminus U) \). This follows from the assumption \( \frac{[\alpha(t_n)]}{\lambda(X_J(t_n))} \to 0 \), while the length of \( h_{t_n}^{-1}(\alpha(t_n)) \) with respect to \( |\omega(\infty)| \) is bounded below by \( a_0 > 0 \).

Thus \( h_{t_n}(\partial U) \) contains points of \( X_j(t_n) \). But the \( |\omega(\infty)| \) distance between \( z_0 \) and \( \partial U \) is at least \( 2C \) while the \( \frac{|\omega(t_n)|}{\lambda(X_j(t_n))} \) distance between their images is at most \( C \). This is a contradiction. \( \square \)

Remark 4.3. As a consequence of Proposition 4.1 if the sequence is plump, the limiting \( \omega(\infty) \) is a finite area holomorphic 1-form on \( Y(\infty) \). In particular we still have vertical and horizontal line flows defined by \( \omega(\infty) \). If the sequence is gaunt, then for a (possibly empty) sequences of boundary curves \( \alpha(t_n) \) satisfying \( \frac{|\alpha(t_n)|}{\lambda(X_J(t_n))} \to 0 \), we still have punctures in the limit, where the 1-form is holomorphic. However by definition there are boundary curves with \( \frac{|\alpha(t_n)|}{\lambda(X_J(t_n))} \) bounded away from 0. (Again note lengths \( |\alpha(t_n)| \) are measured with respect to the 1-form \( \omega(t_n) \)). In the limit we get a surface with boundary corresponding to these curves.

We adopt the notation that if a sequence of loops \( \alpha_J(t_n) \) on the boundary of a sequence of gaunt \( Y_J(t_n) \) satisfy \( \frac{|\alpha(t_n)|}{\lambda(Y_J(t_n))} \to 0 \), we will say it is a short sequence. Otherwise it is long.

We have the following which was essentially proved by Eskin, Mirzakhani, and Rafi in \( \mathbb{S} \).

Lemma 4.4. A gaunt sequence \( Y_J(t_n) \) cannot have two or more short sequences of boundary loops sharing a common zero.

Proof. We consider the sequence of loops \( \alpha_J(t_n) \) surrounding the set of all short boundary curves \( \beta(t_n) \) sharing a zero. The flat length of \( \alpha(t_n) \) goes to 0 on the normalized surface. It cannot be the case that the extremal length of \( \alpha(t_n) \) goes to 0 for then \( Y_J(t_n) \) would be a disc and the surrounded boundary loops \( \beta(t_n) \) would not all be short. Then suppose the extremal length of \( \alpha(t_n) \) is bounded away from 0. Let \( \gamma(t_n) \) be any closed curve on \( Y_J(t_n) \) with bounded flat length. Since the flat length of \( \alpha(t_n) \to 0 \), by the exact same proof as given in Lemma 3.9 of \( \mathbb{S} \) there is a curve \( \tau(t_n) \neq \alpha(t_n) \) separating \( \gamma(t_n) \) from \( \alpha(t_n) \) hence from the \( \beta(t_n) \) with the extremal length of \( \tau(t_n) \) going to 0. But then the boundary loops \( \beta(t_n) \) are not on the boundary of \( Y_J(t_n) \), again a contradiction. \( \square \)

Definition 4.5. Suppose \( Y_J(t_n) \) is gaunt sequence. We define a short boundary segment \( J(t_n) \) to be a connected segment on a short boundary loop of \( Y_J(t_n) \) each of whose endpoints is either a zero or a point lying on a vertical segment that hits a zero. We require that there be no other such points in the interior of \( J(t_n) \).

It is clear from the definition that short boundary segments are in pairs; every vertical segment leaving one hits the other.

5. Main Propositions

This section is devoted to Proposition 5.3 and Proposition 5.7, the main ingredients in the proof of the main theorem. Their proofs are based on ideas from \( [14] \) and \( [15] \). We are supposing the vertical line flow \( \phi^t \) is minimal, but not uniquely ergodic. In \( [14] \) it was shown that as \( t \to \infty \), the Riemann surface \( X(t) \) of \( g_t(X, \omega) \) eventually leaves every compact set in the moduli space \( \mathcal{M}_g \). As discussed previously this means that for all large times \( t \) there is a disjoint collection of simple closed curves \( \alpha(t) \) whose extremal or hyperbolic
lengths approach 0. They define a thick-thin decomposition of $X(t)$. The curves $\alpha(t)$ have representatives as geodesics in the flat metric defined by $\omega(t)$ on $X(t)$. Pulled back by $g_t$ their representatives as geodesics on $X$ have horizontal component of their holonomy going to zero and since the flow $\phi^t$ is minimal, the vertical component of their holonomy on $X$ goes to infinity, so by the definition of the Teichmüller flow, their vertical holonomy, hence lengths, denoted $|\alpha(t)|$ on $X(t)$ satisfy

$$\lim_{t \to \infty} e^{t/2}|\alpha(t)| = \infty.$$  

Definition 5.1. Suppose $\beta, \beta'$ are vertical segments of the same length $|\beta| = |\beta'|$ on a translation surface $(Y, \omega)$. Given $0 < \eta \leq 1$ we say that $\beta$ and $\beta'$ $\eta$-interact, if there are subsegments of $\beta$ and $\beta'$ of length at least $\eta|\beta|$ that are the two vertical sides of an isometrically embedded rectangle on $(Y, \omega)$.

In what follows we have a sequence of times $t_n \to \infty$ and consider the 1 forms $\omega(t_n)$ of $g_{t_n}(X, \omega)$. Again we will think of $g_{t_n}$ as maps of the Riemann surface $X$ to a Riemann surface $X(t_n)$. We will denote by $\omega(t_n)$ the restriction of the 1-form to a subsurface $X(t_n)$ of the thick-thin decomposition.

Proposition 5.2. Suppose $p, p'$ are generic points of distinct (2-sided) generic measures $\mu, \mu'$. Suppose $\beta(t_n)$ and $\beta'(t_n)$ are vertical lines with endpoints $g_{t_n}(p)$ and $g_{t_n}(p')$ of $g_{t_n}(X, \omega)$ of equal length such that as $t_n \to \infty$,

$$e^{t_n/2}|\beta(t_n)| \to \infty.$$ 

Suppose they $\eta(t_n)$ interact. Then $\lim_{n \to \infty} \eta(t_n) = 0$.

Proof. Suppose on the contrary for a subsequence of $t_n \to \infty$ they $\eta$-interact for some fixed $\eta > 0$. That is, there exists subsegments $\hat{\beta}(t_n) \subset \beta(t_n)$ and $\hat{\beta}'(t_n) \subset \beta'(t_n)$ of equal length $\eta|\beta(t_n)|$ which are two vertical sides of a rectangle on $X(t_n)$. Let $I$ be a horizontal interval with respect to the 1-form on the base surface $(X, \omega)$ such that the transverse measures $\rho, \rho'$ corresponding to $\mu, \mu'$ satisfy $\rho(I) \neq \rho'(I)$. Let

$$\epsilon = \frac{\eta|\rho(I) - \rho'(I)|}{8}.$$ 

Then for $\eta, \epsilon$ and interval $I$, choose $L_0$ large enough so that it satisfies the conclusion of Lemma 2.4 and also so that

$$L_0 \geq \frac{2}{\epsilon}.$$ 

Let $\hat{\beta} = g_{t_n}^{-1}(\hat{\beta}(t_n))$ and $\hat{\beta}' = g_{t_n}^{-1}(\hat{\beta}'(t_n))$ the equal length vertical segments pulled back to $(X, \omega)$ by the Teichmüller map $g_{t_n}^{-1}$. Then

$$|\hat{\beta}'|_{\omega} = |\hat{\beta}|_{\omega} = |g_{t_n}^{-1}(\hat{\beta}(t_n))|_{\omega} = \eta e^{t_n/2}|\beta(t_n)| \geq \eta L_0,$$

the last inequality holding for large enough $t_n$. (The lengths on the left are measured with respect to the flat metric on the base surface $(X, \omega)$).

Applying Lemma 2.4 to the subsegments $\hat{\beta}$ and $\hat{\beta}'$ and the interval $I$, we get both

$$|\text{card}(\hat{\beta} \cap I) - \rho(I)||\hat{\beta}|_{\omega} \leq \frac{2\epsilon}{\eta} |\hat{\beta}|_{\omega},$$

and

$$|\text{card}(\hat{\beta}' \cap I) - \rho'(I)||\hat{\beta}'|_{\omega} \leq \frac{2\epsilon}{\eta} |\hat{\beta}'|_{\omega}.$$ 

The triangle inequality then gives
\[
\begin{align*}
|\text{card}(\hat{\beta} \cap I) - \text{card}(\hat{\beta}' \cap I)| & \geq |\hat{\beta}|_\omega \rho(I) - \rho'(I)| - \frac{4\epsilon}{\eta} |\hat{\beta}|_\omega = \frac{8\epsilon}{\eta} |\hat{\beta}|_\omega - \frac{4\epsilon}{\eta} |\hat{\beta}|_\omega \geq 4\epsilon L_0 \geq 2. \\
\end{align*}
\]

Applying \(g_{t_n}\) we see that the horizontal segment \(g_{t_n}(I)\) satisfies
\[
|\text{card}(g_{t_n}(I) \cap \hat{\beta}(t_n)) - \text{card}(g_{t_n}(I) \cap \hat{\beta}'(t_n))| \geq 2,
\]
which means \(\hat{\beta}(t_n), \hat{\beta}'(t_n)\) cannot be two vertical sides of an embedded rectangle.

Now suppose \(\mu_1, \ldots, \mu_k\) are the ergodic probability measures and \(\nu_1, \ldots, \nu_m\) are generic but not ergodic probability measures for the minimal vertical flow \(\phi^t\) of \((X, \omega)\). We have \(k \geq 2\). Our goal in proving Theorem 1.4 is to show
\[
k + m \leq q + s - 1.
\]

Let \(\mu\) be Lebesgue measure on \((X, \omega)\). Since the ergodic measures are the extreme points of the simplex of invariant measures, there exists \(a_i \geq 0\) such that
\[
\mu = \sum_{i=1}^k a_i \mu_i.
\]

For each \(i\) the set of generic points of \(\mu_i\) has Lebesgue measure \(a_i\). By choosing a measure in the interior of the simplex, and taking the vertical line flow with that measure (on a possibly different \((X, \omega)\)) we can assume \(a_i > 0\) for all \(i\). Let
\[
A_0 = \min_i a_i.
\]

**Proposition 5.3.** For any sequence of times \(t_n \to \infty\) there is a subsequence, again denoted \(t_n\), an integer \(\ell \geq k\), a number \(a_0 > 0\) and disjoint open subsurfaces or cylinders \(X_1(t_n), \ldots, X_\ell(t_n) \subset g_{t_n}(X, \omega)\) such that

1. the boundary of each \(X_i(t_n)\) is made up of a union of loops \(\alpha_i(t_n)\) each of which in turn is a union of saddle connections and such that \(\lim_{t_n \to \infty} \text{Ext}(\alpha_i(t_n)) = 0\).
2. for each \(i\), \(\text{area}(X_i(t_n)) \geq a_0\)
3. If \(X_i(t_n)\) are not cylinders and \(\omega_i(t_n)\) is the restriction of \(\omega(t_n)\) to \(X_i(t_n)\) then \(\lim_{t_n \to \infty} \omega_i(t_n)\) exists and is a non-zero finite area \(1\)-form \(\omega_i(\infty)\) on a surface \(X_i(\infty)\) in the Deligne-Mumford compactification,
4. For each sequence \(X_i(t_n)\) there is an ergodic measure \(\mu_j\), a generic point \(p_j\) of \(\mu_j\), independent of \(t_n\) such that \(g_{t_n}(p_j) \in X_i(t_n)\). In addition, if the \(X_i(t_n)\) are not cylinders, then \(g_{t_n}(p_j)\) has a limit in a compact subset of \(X_i(\infty)\).
5. If \(X_i(t_n)\) and \(\mu_j\) are related as in (4) and \(q\) is a generic point for a measure \(\nu \neq \mu_j\), then for large \(t_n\), \(g_{t_n}(q) \notin \overline{X_i(t_n)}\).
6. \(\lim_{t_n \to \infty} \text{area}(g_{t_n}(X, \omega) \setminus \bigcup_i X_i(t_n)) \to 0\).

**Definition 5.4.** We say that \(X_i(t_n)\) given in (4) is associated to the generic point \(p_j\) and generic measure \(\mu_j\).

**Remark 5.5.** Some of the ideas in this proposition are contained in Theorem 1.4 of [15]. One difference is that in that theorem there is the assumption that there is no loss of mass in passing to limits of the \(1\)-form on surfaces in the Deligne-Mumford compactification. Here we allow for the possibility of cylinders with area bounded below.
Proof. By passing to a subsequence we can assume that the Riemann surfaces \(X(t_n)\) converge in the Deligne-Mumford compactification to a possibly disconnected Riemann surface \(X(\infty)\) with punctures. A collection of curves have their extremal length approaching 0 so (1) holds. We consider sequences of those components \(X_i(t_n)\) of the \((\epsilon_0, \epsilon_1)\) thick-thin decomposition of \(g_{t_n}(X, \omega)\) whose area determined by the restriction \(\omega_i(t_n)\) of the 1-form \(\omega(t_n)\) to \(X_i(t_n)\) is bounded away from 0. For these (2) holds. Among those that are not cylinders, the area \(\int_{X_i(t_n)} |\omega_i(t_n)|^2\) is bounded above by the size \(\lambda(X_i(t_n))\) up to a uniform multiplicative constant. This follows from the fact, Lemma 4.2 of [7], that one can triangulate \(X_i(t_n)\) by saddle connections that are comparable in length to the size. Thus the size \(\lambda(X_i(t_n))\) is bounded below away from 0. Theorem 10 of [7] says that by passing to subsequences \(\omega_i(t_n)\) converges to a nonzero finite area holomorphic 1-form \(\omega_i(\infty)\) on the limiting Riemann surface \(X_i(\infty)\). The limiting 1-form does not have poles at the punctures. This proves (3).

Now consider any such sequence \(X_i(t_n)\) which has limit \(X_i(\infty)\) which has positive area. Since the union of the sets of generic points for the ergodic measures has full Lebesgue measure, it follows that any open set \(U \subset X_i(\infty) \setminus \Sigma\), contains limit points of \(g_{t_n}(p_j)\) where \(p_j\) is generic for some ergodic measure \(\mu_j\). (This argument appears in Corollary A.3 in the Appendix of [13]) and gives (4) in the non cylinder case.

We next consider the case that there are components \(X_i(t_n)\) of the \((\epsilon_0, \epsilon_1)\) thick-thin decomposition that form a sequence of cylinders with areas bounded below away from 0. Since the areas are bounded below and the set of generic points for the ergodic measures has full Lebesgue measure we see that any open set \(\subset X_i(\infty)\) cannot lie in the same isometrically embedded open disc of \(X_i(\infty)\). For if they did, then there would be vertical segments \(\gamma\) and \(\gamma_i\) of \(\omega_i(\infty)\), of the same length each containing a limit point and which are two sides of a rectangle \(R(\infty)\). The pair \(\gamma\) and \(\gamma_i\) are limits of vertical segments \(\gamma(t_n)\) and \(\gamma_i(t_n)\) on the approximate \(X_i(t_n)\) through \(g_{t_n}(p)\) and \(g_{t_n}(p_j)\) resp, that bound a rectangle \(R(t_n) \to R(\infty)\). Since \(p_j\) is 2-sided the vertical sides would \(\eta(t_n)\) interact, where \(\eta(t_n) \to 1\). This would violate Proposition 6.2. We finish the proof of (5) in the non-cylindrical case by noting that we can cover \(X_i(\infty)\) with open discs.

We see similarly that there cannot be images of generic points \(q \in X_i(\infty)\) in the same cylinder for a sequence \(t_n \to \infty\). Otherwise there would be two vertical segments of the same length through these image points bounding a rectangle. We can take these vertical segments to be longer than the circumference of the cylinder, so that when pulled back to \((X, \omega)\) under \(g_{t_n}^{-1}\), their lengths go to infinity. This is again a contradiction. Thus (5) holds for cylinders.

Since the set of generic points for ergodic measures has full Lebesgue measure we see that (6) holds. □

The rest of this section deals with generic points of measures that are not ergodic. We wish to associate a subset for each as we did for ergodic measures. This is accomplished in Proposition 5.7. The main difficulty is that the set of generic points for generic but not ergodic measures has Lebesgue measure 0, so we cannot use the method in Proposition 5.8 to deal with them. We continue to use the notation \(X_i(t_n)\) for the components of the \((\epsilon_0, \epsilon_1)\)
thick-thin decomposition of $X(t_n)$ associated to ergodic measures. We will denote by $Y(t_n)$ a component of the complement of $\cup_i X_i(t_n)$.

We next divide each $Y(t_n)$ into $(\epsilon_0, \epsilon_1)$ thick-thin components $Y_j(t_n)$. As in Remark 4.3 we pass to the limit of $\omega_j(\infty)$ on $Y_j(\infty)$ of the normalized $\omega_j(t_n) / \lambda(Y_j(t_n))$. If the sequence is plump, then $Y_j(\infty)$ does not have boundary and has finite area.

**Definition 5.6.** We say a subset $W(t_n)$ of a component $Y_j(t_n)$ is associated to a generic point $q$ for a $(2$-sided) generic non-ergodic measure $\nu$ and vertical segment $\beta(t_n)$ through $g_{t_n}(q)$, if $\beta(t_n) \subset W(t_n)$,

$$e^{\pm |\beta(t_n)|} \rightarrow \infty \text{ and } |\beta(t_n)| \gtrsim \lambda(Y_j(t_n))$$

and such that exactly one of the following holds.

- $W(t_n)$ is a cylinder whose limit on $Y_j(\infty)$ is a cylinder of $\omega_j(\infty)$.
- $W(t_n)$ is a minimal domain of $\omega_j(\infty)$
- $W(t_n)$ is a union of horizontal trapezoids $\text{Trap}_k(t_n)$ each of which has two of its sides on two long boundary components and $\beta(t_n)$ crosses the trapezoids and joins nonzero points on a pair of short boundary loops of $Y_j(t_n)$.
- $W(t_n)$ is a union of horizontal trapezoids $\text{Trap}_k(t_n)$ each of which has two of its sides on two long boundary components of $Y_j(t_n)$ and such that $\cup_k \text{Trap}_k(t_n)$ contains a closed geodesic $\gamma(t_n)$ which is isotopic to the composite of $\beta(t_n)$ and a horizontal arc in one of the trapezoids. The arcs of the trapezoids are not isotopic to a boundary component.

We will refer to these $W(t_n)$ given above as type (I), (II), (III) or (IV). Types (III) and (IV) only can occur in a gaunt sequence.

**Proposition 5.7.** Suppose there are $m$ generic but non-ergodic invariant measures $\nu_i$ with generic points $q_i$ contained in $Y_j(t_n)$. For large $t_n$ there are $m$ disjoint subsets $W_1(t_n), \ldots, W_m(t_n)$ where $W_i(t_n)$ is associated to $q_i$ and vertical $\beta_i(t_n)$ and for each $i$ there is $j$ such that $W_i(t_n) \subset Y_j(t_n)$.

**Proof.** First suppose $Y_j(t_n)$ with corresponding $\omega_j(t_n)$ is a plump sequence. By Proposition 4.1 its limit $\omega_j(\infty)$ does not have poles at the punctures. Suppose there are $m'$ generic points $q_i$ of distinct non-ergodic generic measures such that $g_{t_n}(q_i) \in Y_j(t_n)$. In this case we will prove that there are $m'$ disjoint $W_i(t_n)$ of types (I) and (II) that are subsets of $Y_j(t_n)$ and are associated to these generic points.

We claim there exists $\delta_0 > 0$ such that for any generic point $q$, and large $t_n$, a vertical segment shorter than $\delta_0 \lambda(Y_j(t_n))$ through $g_{t_n}(q) \in Y_j(t_n)$ does not leave $Y_j(t_n)$ in both directions.

Arguing by contradiction, suppose there is a sequence $\delta_n \rightarrow 0$ such that the vertical segment $\beta(t_n)$ in both directions through $g_{t_n}(q)$ of length at most $\delta_n \lambda(Y_j(t_n))$ hits boundary components $\alpha_1(t_n)$ and $\alpha_2(t_n)$ of $Y_j(t_n)$. By definition of plumpness there is $\epsilon_n \rightarrow 0$ such that $|\alpha_i(t_n)| \leq \epsilon_n \lambda(Y_j(t_n))$. If $\alpha_1(t_n) \neq \alpha_2(t_n)$ then the concatenation $\beta(t_n) \ast \alpha_1(t_n) \ast \beta^{-1}(t_n) \ast \alpha_2(t_n)$ of arcs produces a closed essential curve, with length at most $(2\delta_n + 2\epsilon_n) \lambda(Y_j(t_n))$. Since $2\delta_n + 2\epsilon_n \rightarrow 0$, we have a contradiction to the definition of $\lambda(Y_j(t_n))$. If $\alpha_1(t_n) = \alpha_2(t_n)$ there is a similar surgery using a segment of $\alpha_1(t_n)$ joining the endpoints of $\beta(t_n)$. This proves the claim.
Since for all boundary components \(\alpha(t_n)\), \(\frac{\lambda(Y_j(t_n))}{|\alpha(t_n)|}\) is bounded away from 0, the claim, and (10) imply that
\[
e^{\epsilon n/2} |\beta(t_n)| \to \infty.
\]

We now take the horizontal foliation of \(Y_j(\infty)\) defined by \(\omega_j(\infty)\) and the limits \(\beta_i(\infty)\) of the \(\beta_i(t_n)\). Each \(\beta_i(\infty)\) contains a subsegment \(\beta'_i(\infty)\) of length proportional to \(\beta_i(\infty)\) such that \(\beta'_i(\infty)\) is contained in a horizontal domain \(W'_i(\infty)\) either a cylinder or a minimal domain.

Now we claim that \(\beta'_i(\infty)\) and \(\beta'_j(\infty)\) cannot be contained in the same domain for \(i \neq j\). For otherwise in the minimal case choosing them as cross sections for the horizontal line flow, we would find segments of each of length comparable to \(\beta_i(\infty)\) that bound a rectangle, which is impossible by Proposition 5.2. The same argument holds for cylinders. Thus the approximate domain to it just as in the plump case.

Next suppose the sequence \(Y_j(t_n)\) is gaunt. Again take its limit \(Y_j(\infty)\) with 1-form \(\omega_j(\infty)\). Now there are long boundary components so the limiting \(Y_j(\infty)\) has boundary. Let \(B_0\) the sum of the lengths of the boundary components of \(Y_j(\infty)\). Unlike the plump case, for any fixed \(\delta_0\) it is possible as \(t_n \to \infty\) there might be a vertical segment that might cross boundary components in both directions while being shorter than \(\delta_0\lambda(Y_j(t_n))\).

Now consider a vertical segment \(\beta_i(t_n)\) through a limit point of \(g_{n_i}(q_i) \in Y_j(t_n)\) of the generic point \(q_i\) which after renormalization has length \(2B_0\). Suppose it remains in \(Y_j(\infty)\). The first possibility is it enters a cylinder or minimal domain that comes from the horizontal line flow. If that happens we associate the approximate domain to it just as in the plump case.

The next possibility is a subsegment of \(\beta_i(t_n)\) joins two short boundary segments and does not enter a cylinder or minimal domain. Its length is at least comparable to \(\lambda_j(t_n)\). It is entirely contained in a union of trapezoids \(W'_i(t_n)\).

This gives the type (III) possibility. No other \(\beta_k(t_n)\) may have a subsegment contained in any of the same trapezoids intersected by \(\beta_i(t_n)\). Otherwise \(\beta_i(t_n)\) and \(\beta_k(t_n)\) would \(\eta\) interact for some \(\eta > 0\), which again is impossible by Proposition 5.2.

The next possibility is that the limiting \(\beta_i(\infty)\) is still contained in a union of trapezoids, but does not join two short boundary loops. Since the sum of distances across the union of trapezoids is at most \(B_0\) and \(\beta_i(\infty)\) has length \(2B_0\), it must return to some trapezoid. Closing the path up in that trapezoid using a horizontal segment we produce a closed loop, denoted \(\gamma_i(\infty)\). The loop \(\gamma_i(\infty)\) and the trapezoids are approximated by loops \(\gamma_i(t_n)\) and approximating trapezoids \(\text{Trap}(t_n)\). We let \(W'_i(t_n)\) be again the union of trapezoids. This gives a type (IV) \(W'_i(t_n)\). As is the case of type (III) sets \(W\), no other \(\beta_k(t_n)\) may have a subsegment contained in any of the same trapezoids intersected by \(\beta_i(t_n)\).

Now assume none of the previous possibilities hold so the the vertical segment \(\beta_i(t_n)\) through \(g_{n_i}(q)\) of lengths \(2B_0\) hits a boundary loop of \(Y_j(t_n)\) in both directions, at least one of which, denoted \(\alpha(t_n)\) is long. Since the length \(|\alpha(t_n)|\) is comparable to \(\lambda(Y_j(t_n))\), the estimate (11) shows the modulus of the corresponding expanding annulus of \(\alpha(t_n)\) inside \(Y_j(t_n)\) that \(\beta_i(t_n)\) crosses is uniformly bounded. Then there must either be a cylinder \(Y_i(t_n)\) of modulus going to infinity as \(t_n \to \infty\) isotopic to \(\alpha(t_n)\), or an expanding annulus of modulus going to infinity in a neighboring \(Y_i(t_n)\) of the thick-thin decomposition. If there is a cylinder \(Y_i(t_n)\) with modulus going to infinity we can associate to \(\beta_i(t_n)\) that cylinder and denote it by \(W'_i(t_n)\). If there is not such a cylinder, but rather an expanding annulus...
in a neighboring $Y_j(t_n)$ then \(\square\) shows that as $t_n \to \infty$,
$$\frac{\lambda(Y_j(t_n))}{\lambda(Y_i(t_n))} \to \infty.$$ This implies that $\alpha(t_n)$ is a short loop of $Y_i(t_n)$.

We extend $\beta_i(t_n)$ to have length $2B_0\lambda(Y_i(t_n))$. If it remains in $Y_i(t_n)$, then as in the previous discussions, we associate the corresponding subsurface $W_i(t_n) \subset Y_i(t_n)$ to $\beta_i(t_n)$ and then the vertical segment through any other generic point cannot intersect $W_i(t_n)$. If $\beta_i(t_n)$ intersects a short boundary loop $\alpha(t_n)$ of $Y_i(t_n)$, then exactly as before, a subsegment of $\beta_i(t_n)$ of length at least $\delta_0\lambda(Y_i(t_n))$ joins the short boundary components of $Y_i(t_n)$, and is associated to a $W_i(t_n)$. Suppose finally that $\beta_i(t_n)$ hits a long boundary loop of $Y_i(t_n)$. Then exactly as in the previous paragraph it must then enter a cylinder of big modulus or enter a neighboring $Y_k(t_n)$ of the thick-thin decomposition with increasing size $\lambda$ can only occur a bounded number of times, in terms of the genus. Therefore after a bounded number of steps this process must terminate with some $W_i(t_n)$ associated to $\beta_i(t_n)$.

\[\square\]

The following Lemma gives additional restrictions on sets $W$ for different measures. Recall \(\approx\) means up to uniform multiplicative constants independent of $t_n$. They all will follow from Proposition 5.2. First we need a notion of closeness of interior zeroes to the boundary.

**Definition 5.8.** An interior zero $z(t_n)$ of $Y_j(t_n)$ is close to the short loop $\alpha(t_n)$ on the boundary of $Y_j(t_n)$ if a vertical separatrice $\beta(t_n)$ leaving $z(t_n)$ hits $\alpha(t_n)$ and
$$\lim_{t_n \to \infty} \frac{|\beta(t_n)|}{\lambda(Y_j(t_n))} \to 0.$$ **Lemma 5.9.** For large $t_n$

1. a pair of gaunt sequences $Y_k(t_n), Y_i(t_n)$ cannot have vertical segments $\beta_k(t_n), \beta_i(t_n)$ associated to type (III) sets $W_k(t_n)$ and $W_i(t_n)$ that have endpoints in $K_k(t_n) \cap J_i(t_n)$ where these are short boundary segments on a common boundary loop.
2. If $Y(t_n)$ is a component of the complement of the union of domains $X_j(t_n)$ associated to ergodic measures there is no type (III) $W(t_n)$ with associated vertical segment $\beta(t_n)$ with an endpoint on $\partial Y(t_n)$.
3. If $W_1(t_n), W_2(t_n) \subset Y_j(t_n)$ are type (III) sets with associated vertical segments $\beta_1(t_n), \beta_2(t_n)$ with endpoints on the same short boundary loop $\gamma(t_n)$ then either they are separated by a zero on $\gamma(t_n)$ or there is a zero close to the segment bounded by the endpoints.

**Proof.** We argue by contradiction. For (1) without loss of generality assume $\lambda(Y_k(t_n)) \geq \lambda(Y_i(t_n))$. Extend $\beta_i(t_n)$ into $Y_k(t_n)$. For a fixed $\eta > 0$ it will $\eta$ interact with $\beta_i(t_n)$ for large $t_n$, which is a contradiction. For (2), if there were such a vertical segment, extending the vertical segment into $\cup X_j(t_n)$ it would interact with an ergodic vertical segment also contradicting (4) of Proposition 5.3. For the proof of (3), if there is no such zero then the vertical segments would $\eta$ interact for some fixed $\eta > 0$ and large $t_n$. \(\square\)
6. Proof of Theorems

Now for the rest of the paper, by passing to subsequence of $t_n$ we can assume that the topology and combinatorics of the ergodic components $X_j(t_n)$, complementary components $Y_j(t_n)$, thick thin components $Y_j(t_n)$ of $Y(t_n)$ and $W_j(t_n)$ are all independent of $t_n$ in the sense that there are homeomorphisms of the underlying surface that preserve these subsets. We will write $Y_j(\infty)$ for limit of $Y_j(t_n)$.

**Definition 6.1.** For any $Z = Z(t_n)$, each component of which is a union of $Y_j(t_n) \subset Y(t_n)$ in the thick-thin decomposition glued together along boundary loops, let $m(Z)$ be the maximum number of $W$ of type (I)-(IV) that can be contained in $Z$.

We define a quantity $\rho(Z)$ for any such $Z$.

**Definition 6.2.** For any $Z$, let $g(Z)$ be the genus, $s(Z)$ the number of interior zeroes, and $n(Z)$ the number of boundary components. Define

$$\rho(Z) = g(Z) + s(Z) + n(Z) - 1.$$ 

**Remark 6.3.** We can make the same definition for any $\Omega \subset Z$ a domain with boundary which is a union of saddle connections.

The next two propositions together will immediately give the main theorem.

**Proposition 6.4.** Let $Y(t_n)$ be a component of the complement of the union of the ergodic components $X_j(t_n)$. Then $k + \sum_{Y(t_n)} \rho(Y(t_n)) \leq \rho(S) = g + s - 1$, where $k$ as before is the number of ergodic probability measures and $g$ is the genus of the entire surface $S$ and $s$ is the number of zeroes of $\omega$.

**Proposition 6.5.** For $Y(t_n)$ a component of the complement of the ergodic components, $m(Y(t_n)) \leq \rho(Y(t_n))$.

**Proof of Theorem 1.4.** Combine Proposition 6.4 with Proposition 6.5.

**Proof of Proposition 6.4.** We successively remove the domains $X_j(t_n)$. At each stage we maximize the value $\rho$ of the complement if the domain we remove $X_j(t_n)$ is a cylinder. Inductively, given a collection of cylinders $X_j(t_n)$ with complement $Y'$ we remove a cylinder giving a new complement $Y''$. We continue $k$ times until we are left with $Y$. It is enough to show $\rho(Y'') < \rho(Y')$ each time we remove a cylinder.

We have $n(Y'') \geq n(Y')$ but $n(Y'') - n(Y') \leq s(Y') - s(Y'')$ since each new boundary component must have at least one zero on it. If we also decrease the genus, then we see that $\rho(Y'') - \rho(Y') \leq -1$.

Suppose then the genus of $Y''$ equals the genus of $Y'$. Thus $Y''$ has one more connected component than $Y'$. If the complementary connected components are not discs, then the extra $-1$ in the definition of $\rho$ again gives $\rho(Y'') \leq \rho(Y') - 1$. If a complementary component $\hat{Y}$ is a disc then $\rho(\hat{Y}) = 0$ Then ignoring the discs $n(Y'') = n(Y')$, but again since $Y''$ has at least one fewer interior zero so again $\rho(Y'') \leq \rho(Y') - 1$. Iterating this reduction $k$ times until all cylinders $X_j(t_n)$ have been removed we have proved the desired formula

$$k + \sum_{Y} \rho(Y) \leq \rho(S) = g + s - 1.$$ 

$\square$
We will prove Proposition 6.10 in two steps. In the first step, Proposition 6.11, we bound \( m(Y_j(t_n)) \). The main complication are the type (III) sets \( W \) and accounting for zeroes on \( \partial Z(t_n) \). In the second step we will glue the \( Y_j(t_n) \) together to form \( Y \).

We now fix a domain \( Y_j(t_n) \). Suppose we have a collection \( \mathcal{T} \) of type (III) sets \( W \subset Y_j(t_n) \). Each determines a pair of segments along loops of \( \partial Y_j(t_n) \) and a vertical line \( \beta \) with endpoints on those segments.

**Definition 6.6.** Let \( \mathcal{G} \) be a graph with vertices that are the boundary components \( C(t_n) \) of \( Y_j(t_n) \) that contain a vertical segment \( \beta \) associated to a type (III) \( W \). The edges \( \mathcal{E} \) are the \( \beta \).

**Definition 6.7.** Let \( \mathcal{G}_1 \) be a possibly disconnected subgraph of \( \mathcal{G} \) with edges \( \mathcal{E}_1 \) such that each connected component of \( \mathcal{G}_1 \) does not contain any loops and \( \mathcal{G}_1 \) is maximal in that \( \text{card}(\mathcal{E}_1) \) is as large as possible.

**Definition 6.8.** For each component \( B(t_n) \) of \( \partial Y_j(t_n) \), let \( \tau(B(t_n)) \) be the number of edges \( \beta \in \mathcal{G} \setminus \mathcal{G}_1 \) that have an endpoint on \( B(t_n) \) and let

\[
\tau(Y_j(t_n)) = \sum \tau(B(t_n)),
\]

the sum over the boundary components \( B(t_n) \) of \( Y_j(t_n) \).

**Definition 6.9.** Let \( s''(Y_j(t_n)) \) be the number of close zeroes to short loops on \( \partial Y_j(t_n) \) that contain endpoints of type (III) \( \beta \subset W \).

By definition,

\[
\tau(Y_j(t_n)) = \text{card}(\mathcal{T}) - \text{card}(\mathcal{E}_1).
\]

**Remark 6.10.** Since \( \mathcal{G}_1 \) has maximal cardinality, then for any vertex \( v = B(t_n) \) either \( v \) is a vertex of some edge in \( \mathcal{E}_1 \) or the only edges with \( v \) as one endpoint join \( v \) to itself.

**Proposition 6.11.** For large \( t_n \)

\[
m(Y_j(t_n)) \leq \rho(Y_j(t_n)) + \tau(Y_j(t_n)) - s''(Y_j(t_n))
\]

**Proof.** We consider first the case that \( Y_j(t_n) \) is a plump sequence. In this case \( s''(Y(t_n)) = \tau(Y_j(t_n)) = 0 \) since there are no type (III) sets \( W \). Inductively starting with \( Y_j(t_n) \) we remove cylinders and minimal domains, denoted by \( U \), leaving a surface \( \Omega \) with corresponding \( \rho(\Omega) = g(\Omega) + s(\Omega) + n(\Omega) - 1 \). The proposition follows from the claim that when we remove another \( U \) that

\[
\rho(\Omega \setminus U) \leq \rho(\Omega) - 1.
\]

We prove the claim. If \( U \) is a cylinder then the number of additional boundary components of \( \Omega \setminus U \) is offset by the fewer number of interior zeroes. If the genus of \( \Omega \setminus U \) is smaller than the genus of \( \Omega \) we have the claim. If the genus is the same, then the cylinder is separating and \( \Omega \setminus U \) has one more connected component than \( \Omega \) and so there is an additional \(-1\) in the definition of \( \rho \) and again we have the claim. The situation of adding a minimal domain is similar.

Next consider the case of a gaunt sequence. Consider the vertical segments \( \beta(t_n) \) of all type (III) sets \( W \). They limit on vertical saddle connections \( \beta(\infty) \) joining nodes in \( Y_j(\infty) \). We denote the complement of these vertical saddle connections in \( Y_j(\infty) \) by \( Y_j^f(\infty) \). We may define \( \rho(Y_j(\infty)) \) in the same way as before, where in the definition, the nodes or punctures are counted as interior zeroes in \( s(Y_j(\infty)) \).
Lemma 6.12. For large $t_n$, \( \text{card}(T) \leq \rho(Y_j(t_n)) - \rho(Y_j^F(\infty)) + \tau(Y_j(t_n)) - s''(Y_j(t_n)) \).

Proof. We start with $Y_j(t_n)$ and first remove the collection of edges $\beta(t_n) \in \mathcal{E}_1$. Then remove the rest of $\beta(t_n) \in \mathcal{E}$. At each stage we have a set $\Omega(t_n)$ whose boundary consists of segments of $\partial Y_j(t_n)$ and vertical $\beta(t_n)$ of $W$ that have been removed. We then have a new $W$ and vertical $\beta'(t_n)$ and set $\Omega'(t_n) = \Omega(t_n) \setminus \beta'(t_n)$. For edges $\beta(t_n) \in \mathcal{E}_1$, at each stage $\beta(t_n)$ will join distinct boundary components which says that $\Omega'(t_n)$ has fewer boundary components than $\Omega(t_n)$ which implies

\[ \rho(\Omega'(t_n)) < \rho(\Omega(t_n)). \]

For $\beta(t_n) \notin \mathcal{E}_1$ we may increase the number of boundary components, but if so, either we decrease the genus or increase the number of connected components and therefore the number of $-1$ in the definition of $\rho(\Omega')$ compared to $\rho(\Omega)$. In either case we still have

\[ \rho(\Omega'(t_n)) \leq \rho(\Omega(t_n)), \]

but not necessarily strict inequality.

Removing all $\beta(t_n)$ we end with some final $\Omega_j^F(t_n)$. As remarked earlier, the $\beta(t_n) \subset Y_j(t_n)$ limit onto $\beta(\infty) \subset Y_j(\infty)$ joining the nodes. The segments of $\partial Y_j(t_n)$ on the boundary of $\Omega_j(t_n)$ as well as the close zeroes limit to the nodes. We conclude that

\[ \rho(Y_j^F(\infty)) = \rho(\Omega_j^F(t_n)) - s''(t_n). \]

Now \( \text{card}(T) = \text{card}(\mathcal{E}_1) + \tau(Y_j(t_n)). \) By \( \text{card}(\mathcal{E}_1) \leq \rho(Y_j(t_n)) - \rho(\Omega_j^F(t_n)). \) The lemma follows.

Lemma 6.13. $m(Y_j^F(\infty)) \leq \rho(Y_j^F(\infty))$.

Proof. The proof begins the same way as the proof in the plump case. We remove all $W$ that are cylinders and minimal domains decreasing $\rho$ by at least that number of domains until we are left with a possibly disconnected $\Omega(t_n)$ and need to consider type (IV) sets $W \subset \Omega(t_n)$. Each $W$ is a union of trapezoids $\text{Trap}(t_n)$ and contains a closed loop $\gamma(t_n)$. The first possibility is that cutting along $\gamma(t_n)$ reduces the genus of the surface. In addition it reduces the number of interior zeroes. One adds a pair of boundary components. At this stage the value of $\rho$ has not been reduced. However we also remove the arcs in the trapezoid crossing $\gamma(t_n)$. Since the arcs join different boundary components, cutting along the arc reduces their number and hence reduces $\rho$.

The other possibility that is cutting along $\gamma(t_n)$ does not reduce the genus but divides the surface into two components. There must be a zero on $\gamma(t_n)$ and a pair of $-1$ in the definition of $\rho$ as compared to a single $-1$ to begin. There are 2 additional boundary components. Thus the corresponding $\rho$ are the same. But now an arc of the trapezoid joins boundary components of the surface to the two new two components found by cutting along $\gamma(t_n)$. Cutting along this arc reduces the total number of boundary components, reducing the value of $\rho$. Finally the same analysis holds if $\gamma(t_n)$ is isotopic to a boundary component since the trapezoid arc cannot be isotopic to the boundary. We conclude that each time we remove a set $W$ the value of $\rho$ strictly decreases which is the desired result.
We finish the proof of Proposition 6.11. We first note that by construction
\[ m(Y_j(t_n)) = \text{card}(T) + m(Y_j^F(\infty)). \]
Combining this with the inequalities from Lemma 6.12 and Lemma 6.13 gives
\[ m(Y_j(t_n)) \leq \rho(Y_j(t_n)) + \tau(Y_j(t_n)) - s''(t_n). \]

**Proof of Proposition 6.3** We do not need to record dependence on time \( t_n \) in this proof. Consider boundary loops of any \( Y_j \) which is either plump or gaunt but the loops are not subsets of boundary components which are vertices of edges in \( G \setminus G_1 \). We start by considering the case of gluing \( Y_j \) to itself along two such loops to form \( Y_j' \). We have
\[ s(Y_j') \geq s(Y_j) + 1, g(Y_j') = g(Y_j) + 1, n(Y_j') = n(Y_j) - 2. \]
This gives \( \rho(Y_j') \geq \rho(Y_j) \) so \( m(Y_j') = m(Y_j) \) and \( \rho(Y_j') \leq \rho(Y_j) \). If there were terms \( \tau(Y_j) \) or \( s''(Y_j) \) in the bound for \( m(Y_j) \), given in Proposition 6.11 they are not effected by the gluing and so appear in the expression for \( m(Y_j') \), and we conclude using Proposition 6.11 that
\[ m(Y_j') \leq \rho(Y_j') + \tau(Y_j) + s''(Y_j). \]

Next we consider that \( Y_j \) is glued to itself, but a cylinder is inserted. We have again \( g(Y_j') = g(Y_j) + 1 \) but the possibility that a cylinder is associated to a generic measure gives \( m(Y_j') = m(Y_j) + 1 \). If the boundary circles of the cylinder are not joined at a point, then \( s(Y_j') \geq s(Y_j) + 2 \) and \( n(Y_j') = n(Y_j) - 2 \). Putting these facts together we have \( \rho(Y_j) + 1 \leq \rho(Y_j') \). If the two circles are joined at a point, then \( s(Y_j') = s(Y_j) + 1 \) and \( n(Y_j') = n(Y_j) - 1 \), so we again have \( \rho(Y_j) + 1 \leq \rho(Y_j') \). We conclude again using Proposition 6.11 that (9) holds.

Next consider distinct surfaces \( Y_j \) and \( Y_k \) glued together to form a new \( Z \) so \( m(Z) = m(Y_j) + m(Y_k) \). Again assume that the assumptions described in the first paragraph hold. Now \( \rho(Y_j) + \rho(Y_k) \) has a \(-2\) in its definition, while \( \rho(Z) \) contains a single \(-1\) term. We also have
\[ g(Z) = g(Y_k) + g(Y_k), n(Z) = n(Y_j) + n(Y_k) - 2, s(Z) \geq s(Y_k) + s(Y_k) + 1, \]
giving
\[ \rho(Z) \geq \rho(Y_j) + \rho(Y_k). \]
If a cylinder is inserted between \( Y_j \) and \( Y_k \) then \( m(Z) = m(Y_j) + m(Y_k) + 1 \), but now \( s(Z) \geq s(Y_j) + s(Y_k) + 2 \). In either case (9) holds for \( Z \).

We continue this process of gluing. If there are no gaunt \( Y_j \), or there are gaunt \( Y_j \), but \( \tau(Y_j) = 0 \) for all of them, then at the end of the gluing we conclude \( m(Y) \leq \rho(Y) \) and we are done. Notice if a loop of \( Y_j \) is glued to a loop on the boundary of the ergodic domains, it cannot contain the endpoint of \( \beta \) of type (III) by (2) of Lemma 6.9.

Thus assume that \( \tau(Y_j) > 0 \) for some \( Y_j \). We assume we have performed all possible gluings described in the previous paragraphs leaving surfaces, denoted \( Z_k \) composed of a number of \( Y_j \). They have \( n'_k \) boundary components which contribute to \( \tau(Z_k) \) and \( n''_k \) boundary components in common with the boundary of the ergodic components. Consider then a boundary component \( B \) of some \( Y_j \subset Z_k \) and such that \( \tau(B) > 0 \). Suppose \( \gamma_1, \ldots, \gamma_p \subset B \) are the short boundary loops that each contain endpoints of a vertical segment \( \beta \) associated to some \( W \in T \). On each \( \gamma_i \) there is a collection of disjoint segments, each containing an
endpoint of exactly one of these $\beta$. The endpoints of the segments themselves are either zeroes or points on vertical lines that hit a close zero. By Lemma 4.4 the $\gamma_i$ do not share any zeroes, and as a consequence do not share close boundary zeroes with each other. Suppose $Y_j$ is glued along each $\gamma_i$ to some $Y_t \subset Z_m$ for some $Z_m$. By Lemma 5.9 the segments along $\gamma_i$ that correspond to type (III) $W \subset Y_j$ are disjoint from the possible segments from the type (III) $W \subset Y_t$. Let $a_j(\gamma_i)$ be the number of the segments on $\gamma_i$ corresponding to $W \subset Y_j$ and $a_j(B) = \sum_{\gamma_i \subset B} a_j(\gamma_i)$.

Since $\tau(B) > 0$, by Remark 6.10 one of two possibilities holds. The first is that $B$ is a vertex of $G_1$, in which case there is a $W$ of type (III) with vertical $\beta$ having endpoint on $\gamma_i$ but the edge $\beta \in G_1$ and so this $\beta$ does not contribute to the set of $W$ that contribute to counting $\tau(Y_j)$. This then says that $a_j(\gamma_i)$ is larger than the number of $\beta$ that do contribute to $\tau$ with endpoints on $\gamma_i$.

The second possibility is that $B$ is not a vertex of $G_1$. In this case all vertical $\beta$ of type (III) with endpoints on $B$ must join $B$ to itself. Thus there are two segments on $B$ associated to a single $\beta$ and again the number of segments is larger than the number of $\beta$ contributing to $\tau(B)$. We conclude that

$$a_j(B) > \tau(B).$$

Summing the above inequality over the boundary components $B$ of $\partial Y_j$ we see

$$a(Y_j) := \sum_{B \subset \partial Y_j} a_j(B) \geq \tau(Y_j) + n'(Y_j).$$

After all the gluings to form $Y$ we compute the number of interior zeroes $s(Y)$ of $Y$. For each $Y_j$ the number of segments considered above is the same as the number of zeroes on the boundary plus the number of close zeroes. The former become new interior zeroes of $Y$ while the latter were already interior zeroes of $Y_j$. We conclude that

$$s(Y) = \sum_j s(Y_j) + \sum_j (a(Y_j) - s''(Y_j)) \geq \sum_j (s(Y_j) + \tau(Y_j) + n'(Y_j) - s''(Y_j)).$$

This and the definition of $\rho$ implies

$$\sum_j (\rho(Y_j) + \tau(Y_j) - s''(Y_j)) \leq \sum_j (g(Y_j) + s(Y_j) + n'(Y_j) + n''(Y_j) - 1 + \tau(Y_j) - s''(Y_j)) \leq g(Y) + s(Y) + n''(Y) - 1 = \rho(Y).$$

This together with Proposition 6.11 gives

$$m(Y) = \sum_j m(Y_j) \leq \sum_j (\rho(Y_j) + \tau(Y_j) - s''(Y_j)) \leq \rho(Y),$$

and we are done.

Proof of Theorem 1.3. The collection of $X_j(t_n)$ associated to ergodic measures, together with $Y(t_n)$ consists of at least $g+1$ surfaces. We claim that the $X_j(t_n)$ must all be cylinders. The boundary of $X_j(t_n)$ cannot be a single loop joining a zero to itself since it would be trivial in homology, and that is impossible for a translation surface. If $X_j(t_n)$ is not a cylinder, the complement $S \setminus X_j(t_n)$ either has $s' = 0$ interior zeroes, genus $g' \leq g - 1$ and $n' = 1$ boundary components, or genus $g' \leq g - 2$, no interior zeroes, and $n' = 2$ boundary components. In either of the two cases, we have

$$\rho(S \setminus X_j(t_n)) \leq g' + n' + s' - 1 \leq g - 1.$$
But this contradicts that $S \setminus X_j(t_n)$ contains at least $g$ subsurfaces, to account for the $g + 1$ generic measures. This proves the claim. The areas of the cylinders $X_j(t_n)$ are bounded below by $A_0 > 0$.

Now suppose the theorem does not hold so there is a 1-sided generic non-ergodic measure. We have associated to the $g$ ergodic measures $g$ disjoint cylinders $X_j(t_n)$ whose boundaries have lengths going to 0 as $t_n \to \infty$. Since twisting in these cylinders are independent in cohomology, the saddle connections crossing them are $g$ independent elements in the relative homology $H_1(S, \Sigma, \mathbb{Z})$ where $\Sigma$ is the set of either 1 or 2 zeroes. This space has dimension either $g$ or $g + 1$. Consequently the complement of the $g$ cylinders, $Y(t_n) = g_{t_n}(X, \omega) \setminus \cup_{j=1}^{g} X_j(t_n)$ is either a disc in the case of a single zero or an annulus in the case of 2 zeroes. It cannot be a disc, so we have a contradiction in the case of a single zero.

Thus assume there are a pair of zeroes. For large $t_n$ the image of the generic point $q$ for a non-ergodic measure $\nu$ must lie in the annulus $Y(t_n)$. We then have $g + 1$ cylinders with circumferences going to 0 along a sequence $t_n \to \infty$. We will arrive at a contradiction. For fixed cylinder $X_j(t_n)$ with boundary $\alpha_j(t_n)$ and fixed time $t_n$, we have $\lim_{t \to \infty} g_t(\alpha_j(t_n))|_{\omega(t)} = \infty$. That is, for any fixed $\alpha_j(t_n)$, its length goes to infinity as $t \to \infty$. Here we are measuring lengths with respect to $\omega(t)$. Consequently, for each fixed $t_n$ there is some $j$ and smallest $s_n > t_n$ such that $|g_{s_n}(\alpha_j(t_n))|_{\omega(s_n)} = \sqrt{A_0}$. Since the area of $X_j(t_n)$ is at least $A_0$, and the circumference is $\sqrt{A_0}$, the distance across $X_j(t_n)$ is therefore at least $\sqrt{A_0}$. Thus all short circumference cylinders must be contained in the complement of $X_j(t_n)$ at time $s_n$. The complement of $X_j(t_n)$ still must contain $g + 1$ cylinders to account for the cylinders associated to the ergodic measures and the non-ergodic generic measures. This is impossible since the corresponding quantity $g' + n' + s' - 1$ is at most $g$.

\[ \square \]

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