Cosmological inflation with minimal and non-minimal coupling of scalar field from Horndeski theory

G Hikmawan\textsuperscript{1}, A Suroso\textsuperscript{1,2} and F P Zen\textsuperscript{1,2}

\textsuperscript{1}Theoretical Physics Laboratory, THEPI Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
\textsuperscript{2}Indonesia Center for Theoretical and Mathematical Physics (ICTMP), Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia

Email: getbogi@fi.itb.ac.id

Abstract. We consider a cosmological model with combination of minimal and non-minimal coupling of scalar field by considering a set of particular coefficient function from Horndeski theory. We study cosmological inflation from the background solution and get the responsible coupling constant regarding the slow-roll inflation parameters. Then we compare the result with the stability conditions to get the exact value of coupling constant for this cosmological model.

1. Introduction

Inflationary universe [1, 2] firstly developed to find solution for the basic problems in cosmology, such as horizon, flatness and monopole problems. There was several model established to explain this phenomena [3, 4, 5, 6, 7]. However, although the fluctuation of temperature observation of the Cosmological Microwave Background (CMB) by WMAP [8] and COBE [9] give possibility for inflationary universe scenario to be right, it is still become a speculative theory, because of the lack of knowledge about the origin of the generator, the scalar field.

There was a lot of cosmological model developed where scalar field coupled with curvature tensor [10, 11, 12]. However, Horndeski [13] derived the most general scalar-tensor theories, with Lagrangian,

\[ L = \sum_{i=2}^{5} L_i \]

where,

\[ L_2 = \kappa \phi X \]

\[ L_3 = -G_3(\phi, X) \]

\[ L_4 = G_4(\phi, X)R - 2G_{4,\chi}(\phi, X) \left[ \phi_{\mu}^{\phi_{\mu}} - \phi_{\mu}^{\phi_{\mu}} \phi_{\mu}^{\phi_{\mu}} \right] \]

\[ L_5 = G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3} G_{5,\chi}(\phi, X) \left[ \phi_{\mu}^{\phi_{\mu}} \phi_{\mu}^{\phi_{\mu}} \right]^3 - 3(\phi_{\mu}^{\phi_{\mu}} (\phi_{\mu}^{\phi_{\mu}})_{\phi_{\mu}} + 2(\phi_{\mu}^{\phi_{\mu}} \phi_{\mu}^{\phi_{\mu}} \phi_{\mu}^{\phi_{\mu}}) \]

with \( X \equiv -\phi_{\mu}^{\phi_{\mu}} / 2 \). This theory comprises all the models with gravity-coupled scalar field that previously studied by choosing particular coefficient functions (\( \kappa (\phi, X), G_i(\phi, X) \)), such as minimally coupled scalar field [14], Brans-Dicke theory [15], Dilaton gravity [16], f (R) gravity [17], derivative...
couplings [18], Gauss-Bonnet couplings [6, 19] and many others. Although the Horndeski theory is not fully satisfactory, this theory may indeed offer alluring solution for cosmology features with the possibility to obtain a new theory of inflation by adjusting the coefficient functions.

This paper is organized as follows, in Section II we describe and derive the background field equations of the model. In section III we impose the De Sitter expansion and vanishing scalar field to get the range of the coupling constant. After that we use the range to restrict our numerical calculations of the model in section IV. The last section is for Conclusion.

2. Field Equation and Slow-Roll Parameters
To get the field equations describing the background evolution for the Horndeski Lagrangian (2-5), we take the scalar field just as function of time, \( \phi = \phi(t) \) and take metric \( ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij} dx^i dx^j \).

So the Horndeski Lagrangian can be written as,

\[
L_2 = \kappa(\phi, X),
\]

\[
L_3 = \frac{G_s(\phi, X)}{N^2} \left[ \frac{\dot{\phi}}{N} + \left( 3H - \frac{N}{N} \right) \phi \right],
\]

\[
L_4 = \frac{6}{N^2} \left[ G_4(\phi, X) \left( 2H^2 + H - \frac{N}{N} H \right) + \frac{G_{4x}(\phi, X)}{N^2} \left[ H\ddot{\phi} + \left( 3H^2 - \frac{N}{N} H \right) \phi \right] - \frac{G_{5x}(\phi, X)}{N^6} \left[ H - \frac{3N}{N} \right] \dot{\phi} - \frac{3}{N} \phi \right],
\]

with \( H = \dot{a} / a \) is the Hubble parameter and the dot denotes a derivative with respect to time. We can get the field equations from variating the Lagrangian with respect to \( N(t) \) and \( a(t) \),

\[
0 = 2X\kappa_X - \kappa + 6XH\phi G_{3,\chi} - 2XG_{3,\phi} - 6H^2 G_4 + 24H^2 X(4G_{4,\chi} + XG_{4,\chi})
\]

\[
-6H\left( \phi G_{4d} + \phi G_{4d} \right) + 2H^3 \phi G_{5,\chi} (5G_{5,\chi} + 2XG_{5,\chi}) - 6H^2 X(3G_{5,\phi} + 2XG_{5,\phi}),
\]

\[
0 = \kappa - 2X(\dot{\phi} G_{3,\chi} + G_{3,\phi}) + 2 \left( 3H^2 + 2H \right) G_4 - \left( 12H^2 X + 4HX + 8HX \right) G_{4,\chi} - 8HXG_{4,\chi}
\]

\[
+ 2(\dot{\phi} + 2H\phi) G_{4,\phi} + 4XG_{4,\phi} + 4X(\dot{\phi} - 2H\phi) G_{4d} + 2(2H^2 + 2HH\phi + 3H^2 \phi) G_{5,\chi}
\]

\[
- 4H^2 X^2 \phi G_{5,\chi} + 4HX(\dot{H} - HX) G_{5,\phi} + 2 \left( HX + HX \right) + 3H^2 X G_{5,\phi} + 4HX \phi G_{5,\phi},
\]

Then the equation of motion for the scalar field can be obtained from variating the Lagrangian with respect to \( \phi(t) \),

\[
0 = \left( \dot{\phi} + 3H\phi \right) \kappa_X - \kappa_X + 2X(\dot{\phi} k_{XX} - \kappa_{XX}) + 6 \left( 3H^2 + H \right) HXX \right) G_{3,\chi} - 2 \left( \phi + 3H\phi \right) G_{3,\phi}
\]

\[
+ 6HXY G_{3,\chi} + 2X(\dot{\phi} + 3H\phi) G_{4d} + 2X G_{3,\phi} - 6(2H^2 + H) G_{4d}
\]

\[
+ 6H \left[ H(\dot{\phi} + 3H\phi) + 2H\phi \right] G_{4,\chi} + 12HX \left( 3H^2 + 2H \right) \phi + 4H\phi \right] G_{4,\chi}
\]

\[
- 6 \left[ 10H^2 + 2H \right) X + 3HX \right] G_{4d} + 2XX \left( 2X(\dot{\phi} - 12H\phi \right) G_{4,\chi} - 12H\phi X \right) G_{4,\phi},
\]

\[
+ 2HX(3H^2 + 3H + H^2 \phi X G_{5,\chi} + 2HX \left( 6H^3 + 6H + 2H^2 \phi \right) X + 5H^2 \phi \right) G_{5,\chi}
\]

\[
- 6H \left[ 3H^2 + 2H \phi + H\phi \right] + 2HX \left[ 9H\phi - 7H^2 + 6H \right] G_{5,\phi} + 18H^2 X \phi G_{5,\phi},
\]

\[
+ 4H^2 X^2 X G_{5,\chi} + 4H^2 X^2 \left( H\phi + 3H\phi \right) G_{5,\phi} - 12H^2 X^2 G_{5,\phi},
\]

In this work, we consider the coefficient functions as,

\[
\kappa = \omega(\phi) X; \quad G_3 = 0; \quad G_4 = \frac{M_{pl}^2}{2} \cdot \frac{1}{2} \zeta \phi^2; \quad G_5 = \zeta \phi,
\]

(13)
where $M_{pl}^2 = (8\pi G)^{-1}$, and we take $M_{pl}^2 \approx 1$. If we substitute the coefficient functions to the Lagrangian in equation (2-5), one can see that the model extracted is the cosmological model with non-minimally coupling between curvature tensor with scalar field and its derivative, where $\zeta$ and $\xi$ are the coupling constants for the scalar field coupling and the derivative of scalar field coupling. Using equation (10-12), we can obtain the field equation as,

$$\frac{3H^2}{2} = \frac{\omega(\phi)}{2} \dot{\phi} + 3\zeta \left( H^2 \dot{\phi}^2 + 2H\dot{\phi} \right) - 9\xi H^2 \dot{\phi}^2,$$

(14)

$$3H^2 + 2H = -\frac{\omega(\phi)}{2} \dot{\phi}^2 + \zeta \left[ (3H^2 + 2H) \dot{\phi}^2 + 2\phi \ddot{\phi} + 4H\dot{\phi} \right] - 6\xi (2H^2 + 2H) \ddot{\phi} = 0,$$

(15)

$$\left( \omega(\phi) - 6\xi H^2 \right) \ddot{\phi} + 3H\dot{\phi} + \frac{\omega' (\phi)}{2} \dot{\phi}^2 - 12\xi H^2 \dot{\phi} + 6\xi (2H^2 + 2H) \phi = 0,$$

(16)

where prime denotes a derivative with respect to scalar field. If we combine (14) with (15), we can obtain,

$$H + \frac{\omega(\phi)}{2} \dot{\phi}^2 + \zeta \left[ H\phi \dot{\phi} - H\phi^2 - \dot{\phi}^2 \right] + \xi \left[ (H - 3H^2) \dot{\phi}^2 + 2H\phi \ddot{\phi} \right] = 0.$$

(17)

In this inflationary model, we can obtain the slow-roll parameters which are defined as $\epsilon = -\frac{\dot{H}}{H^2}$ and $\eta = -\frac{1}{H} \frac{\ddot{H}}{H \dot{H}} = \frac{e}{2He} \frac{dE}{dt}$,

$$\epsilon = \frac{E}{2H^2 \left( \zeta \phi^2 - \xi \ddot{\phi}^2 - 1 \right)},$$

(18)

$$\eta = \epsilon - \frac{E}{2H^2 \dot{\phi} \left( \zeta \phi^2 - \xi \ddot{\phi}^2 - 1 \right)} + \frac{EH}{H^4 \dot{\phi} \left( \zeta \phi^2 - \xi \ddot{\phi}^2 - 1 \right)} + \frac{E \left( \zeta \phi \dot{\phi}^2 - \xi \ddot{\phi} \right)}{H^3 \left( \zeta \phi^2 - \xi \ddot{\phi}^2 - 1 \right)},$$

(19)

with,

$$E = -\frac{\omega(\phi)}{2} \dot{\phi}^2 + \zeta \left( \phi \dot{\phi}^2 + \phi \ddot{\phi} - H \phi \dot{\phi} \right) + \xi \left[ 3H \phi \dot{\phi}^2 - 2H \phi \ddot{\phi} \right].$$

(20)

When inflation occurs, the conditions $\epsilon << 1$ and $\eta << 1$ must be satisfied because the Hubble parameter evolve slowly in this regime.

3. De Sitter Expansion and Decaying Scalar Field

For this work, consider $\omega(\phi) = 1$ as canonical field case, so that (16) and (17) can be written as,

$$\dot{\phi} + 3H \dot{\phi} + \zeta \left[ (2H^2 + \dot{H}) \dot{\phi} - \dot{\phi} \left[ (3H^2 + 2H) \dot{\phi} + H \dot{\phi} + H^2 \ddot{\phi} \right] \right] = 0,$$

(21)

$$\dot{H} + \frac{\dot{\phi}}{2} \dot{\phi}^2 + \zeta \left( H\phi \dot{\phi} - H\phi^2 - \dot{\phi}^2 \right) + \xi \left[ 2H \phi \dot{\phi} + (H - 2H^2) \dot{\phi}^2 \right] = 0,$$

(22)

For inflation case, we impose de Sitter Expansion,

$$a(t) \propto \exp(H_0 t) \Rightarrow H(t) = \frac{\dot{a}}{a} = H_0,$$

(23)

where $H_0 > 0$ to get positive expansion, so the field equation can be written as,

$$\dot{\phi} + 3H_0 \dot{\phi} + \frac{12\xi H_0^2}{1 - 6\xi H_0^2} \dot{\phi} = 0,$$

(24)

$$\left( H_0 \zeta \right) \dot{\phi} - \zeta \ddot{\phi} - \zeta \phi \ddot{\phi} + 2\xi H_0 \phi \ddot{\phi} = 0.$$

(25)

Combine (24) and (25),

$$\left( \zeta + 9\xi H_0^2 \right) \ddot{\phi} - \left( \frac{4\xi H_0^2 - 4\zeta \xi H_0^2}{1 - 6\xi H_0^2} \right) \dot{\phi} - \frac{12\xi^2 H_0^2}{1 - 6\xi H_0^2} = 0.$$

(26)
For scalar field, we impose that the scalar field in this model is vanish at big \( t \), or decaying,
\[
\phi \propto \exp(\phi t),
\]
where \( \phi_0 > 0 \) to ensure the decaying. So (26) can be written as,
\[
(\zeta + 9 \xi H_0^2) \phi_0^2 - \left( \frac{4 \xi H_0 - 48 \xi H_0^3}{1 - 6 \xi H_0^2} \right) \phi - \frac{2 \zeta \xi H_0^2}{1 - 6 \xi H_0^2} = 0.
\]

We consider \( \zeta = \xi \) case, so the non-minimally coupling between curvature tensor with scalar field and its derivative have same portion, then from (30) we get a quadratic function in \( \phi_0 \),
\[
(1 + 9 H_0^2) \phi_0^2 - \left( \frac{4 H_0 - 48 \xi H_0^3}{1 - 6 \xi H_0^2} \right) \phi_0 - \frac{2 \zeta \xi H_0^2}{1 - 6 \xi H_0^2} = 0.
\]
The solution for \( \phi_0 \) of equation above is,
\[
(\phi_0)_{1,2} = \frac{2 H_0}{1 + 9 H_0^2 - 6 \xi H_0^2} \left[ (1 - 12 \xi H_0^2) \pm \sqrt{(306 H_0^4 + 18 H_0^2) \zeta^2 - (51 H_0^2 + 3) \zeta + 1} \right].
\]

So if we take the condition we impose before, \( \phi_0 < 0 \), we get the value range of \( \zeta \), the coupling constant, in the function of \( H_0 \),
\[
0 < \zeta \leq \frac{51 H_0^2 + 3 - \sqrt{1377 H_0^4 + 234 H_0^2 + 9}}{612 H_0^4 + 36 H_0^2},
\]
\[
\frac{51 H_0^2 + 3 + \sqrt{1377 H_0^4 + 234 H_0^2 + 9}}{612 H_0^4 + 36 H_0^2} \leq \zeta < \frac{27 H_0^2}{162 H_0^4 + 18 H_0^2}.
\]

4. Numerical Calculation
In this section, we do numerical calculation for \( \zeta = \xi \) case of the model consider the value range of \( \zeta \) obtained in previous section, \( 0 < \zeta \leq 0.021 \sim 10^{-2} \). For this case, the field equation can be written as,
\[
\ddot{\phi} + 3 H \dot{\phi} + 6 \zeta \left[ 2 H^2 + \dot{H} \right] \phi - \left( 3 H^2 + 2 \dot{H} \right) H \dot{\phi} - H^2 \phi = 0,
\]
\[
\dot{H} + \frac{\dot{\phi}^2}{2} + \zeta \left[ H \dot{\phi} - \dot{H} \phi^2 - \phi \dot{\phi} + 2 H \dot{\phi} \ddot{\phi} + \left( \dot{H} - 3 H^2 - 1 \right) \phi^2 \right] = 0.
\]
We do numerical calculation for several value of the possible corresponding constants, \( \zeta, \xi \), and then plot the solution in e-fold number, \( N(t) = \log a(t) \). We set \( a(0) = 1, H(0) = 1, \phi(0) = 1 \) and \( \dot{\phi}(0) = 1 \) as boundary condition because to get zero initial e-fold number and the scalar field decreases after inflation generated. Natural units (\( \hbar = c = 1 \)) is used, so each scale in time coordinate corresponds the time scale in natural units. From the scale factor obtained, we plot the slow-roll parameter to ensure that the model with particular coupling constant obtained suffice inflation solution.
Figure 1. Plot of $N(t) = \log a(t)$ of the model for $\zeta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$.

Figure 2. Result for slow-roll parameter $\epsilon$ of the model for $\zeta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$.
Conclusion

We have analyzed a cosmological model with specific coefficient function from Horndeski theory to find the inflation solution. From de Sitter Expansion and vanishing scalar field approach, we get the specific range of coupling constant, \( 0 < \zeta \leq 0.021 - 10^{-2} \), and then we use the result for the numerical calculation for the model considered. As we can see in the plot, the background can give inflationary solution, as the \( N(t) = \log a(t) \) expands rapidly in early time, and both of slow roll parameter turn to less than one as time goes. This result can be used in perturbation analysis in future work.

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