Explicit Lower Bounds on the Outage Probability of Integer Forcing over $N_r \times 2$ Channels

Elad Domanovitz and Uri Erez
Dept. EE-Systems
Tel Aviv University, Israel

Abstract—The performance of integer-forcing equalization for communication over the compound multiple-input multiple-output channel is investigated. An upper bound on the resulting outage probability as a function of the gap to capacity has been derived previously, assuming a random precoding matrix drawn from the circular unitary ensemble is applied prior to transmission. In the present work a simple and explicit lower bound on this outage probability is derived for the case of a system with two transmit antennas, leveraging the properties of the Jacobi ensemble. The derived lower bound is also extended to random unitary space-time precoding, and may serve as a useful benchmark for assessing the relative merits of various algebraic space-time precoding schemes.

I. INTRODUCTION

This paper addresses communication over a compound multiple-input multiple output (MIMO) channel, where the transmitter only knows the number of transmit antennas and the mutual information. More specifically, the goal of this work is to assess the performance of (randomly precoded) integer-forcing (IF) equalization for such a scenario.

Communication over the compound MIMO channel using an architecture employing space-time linear processing at the transmitter side and IF equalization at the receiver side was proposed in [1]. It was shown that such an architecture achieves capacity up to a constant gap, provided that the precoding matrix corresponds to a linear perfect space-universally random unitary space-time precoding, and may serve as a useful benchmark for assessing the relative merits of various algebraic space-time precoding schemes.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Channel Model

The (complex) MIMO channel is described by

$$y_c = H_c x_c + z_c,$$  

(1)

where $x_c \in \mathbb{C}^{N_t}$ is the channel input vector, $y_c \in \mathbb{C}^{N_r}$ is the channel output vector, $H_c$ is an $N_r \times N_t$ complex channel matrix, and $z_c$ is an additive noise vector of i.i.d. unit-variance circularly symmetric complex Gaussian random variables. We assume that the channel is fixed throughout the transmission period. Further, we may assume without loss of generality that the input vector $x_c$ is subject to the power constraint

$$\mathbb{E}(x_c^H x_c) \leq N_t.$$

Consider the mutual information achievable with a Gaussian isotropic or “white” input (WI)

$$C = \log \det (I + H_c H_c^H).$$  

(2)

We may define the set

$$\mathbb{H}(C) = \{ H_c \in \mathbb{C}^{N_r \times N_t} : \log \det (I + H_c H_c^H) = C \},$$

(3)

of all channel matrices with the same WI mutual information $C$. The corresponding compound channel model is defined by (1) with the channel matrix $H_c$ arbitrarily chosen from the set $\mathbb{H}(C)$. The matrix $H_c$ is known to the receiver, but not to the transmitter. Clearly, the capacity of this compound channel is $C$, and is achieved with an isotropic Gaussian input.

Applying the singular-value decomposition (SVD) to the channel matrix, $H_c = U_c \Sigma_c V_c^H$, we note that the unitary matrices have no impact on the mutual information. Let $D_c$ be defined by

$$(I + H_c^H H_c) = U_c D_c U_c^H,$$  

(4)

and note that $D_c = I + \Sigma_c^H \Sigma_c$. Thus, the compound set $\mathbb{D}(C)$ may equivalently be described by constraining $D_c$ to belong to the set

$$\mathbb{D}(C) = \{ \text{Diagonal } D_c : \sum \log(d_{i,i}) = C \}. $$

(5)

We turn now to the performance of IF. It has been observed that employing the IF receiver allows approaching $C$ for “most” but not all matrices $H_c \in \mathbb{H}(C)$. In the present work, we quantify the measure of the set of bad channel matrices by considering outage events, i.e., those events (channels) where integer forcing fails even though the channel has sufficient mutual information. The probability space here is induced by considering a randomized scheme where a random unitary

\footnote{We denote by $[\cdot]^T$, the transpose of a vector/matrix and by $[\cdot]^H$, the Hermitian transpose.}
precoding matrix $P_c$ is applied prior to transmission over the channel.

More specifically, denoting by $R_{IF}(H_c)$ the rate achievable with IF over a channel $H_c$, the achievable rate of the randomized scheme is $R_{IF}^{W}(H_c)$. As $P_c$ is drawn at random, the latter rate is also random. Following [4], we define the worst-case (WC) outage probability of randomized IF as

$$P_{out}^{WICIF}(C, \Delta C) = \sup_{H_c \in \mathbb{H}(C)} \Pr \left( R_{IF}(H_c \cdot P_c) < C - \Delta C \right),$$

(6)

where the probability is with respect to the ensemble of precoding matrices and $R_{IF}(H_c)$ is the achievable rate of IF as given in [5].

Note that in (6), we take the supremum over the entire compound class rather than taking the average with respect to some putative distribution over $\mathbb{H}(C)$. It follows that $P_{out}^{WICIF}(C, R)$ provides an upper bound on the outage probability that holds for any such distribution. Clearly (6) is not an explicit bound. Nonetheless, by restricting attention to a uniform measure over the (unitary) precoding matrices, we are able to obtain both closed-form upper and lower bounds.

Specifically, we consider precoding matrices $P_c$ drawn from the Circular Unitary Ensemble (CUE), see e.g., [6]. Applying the SVD to the effective channel, we have $H_c P_c = U_c \Sigma_c V_c^T P_c$. From the properties of the CUE it follows that $V_c^T P_c$ has the same (CUE) distribution as $P_c$. Thus, $V_c$ (and of course $U_c$) plays no role in (6) and we may rewrite the latter as

$$P_{out}^{WICIF}(C, \Delta C) = \sup_{D_c \in \mathbb{D}(C)} \Pr \left( R_{IF}(D_c \cdot P_c) < C - \Delta C \right),$$

(7)

and so the analysis for CUE precoding is greatly simplified. Both the upper and lower bounds for (7) developed below heavily rely on the well-studied properties of the CUE and further utilize the Jacobi distribution which gives the eigenvalue distribution of submatrices of such matrices [7].

We similarly denote by $P_{out}^{WICIF-SIC}(C, \Delta C)$ the WC outage probability of IF with successive interference cancellation (SIC), the rate of which we denote by $R_{IF-SIC}(H_c)$ and for which we give an explicit expression next.

**B. Integer-Forcing Equalization: Achievable Rates**

We begin by recalling (only) the achievable rates of the IF equalization scheme, where the reader is referred to [5] and [7] for the derivation, details and proofs. Furthermore, we follow the notation of these works, and in particular we present IF over the reals. We also focus our attention on IF receivers employing successive interference cancellation (SIC).

For a given choice of (invertible) integer matrix $A$, let $L$ be defined by the following Cholesky decomposition

$$A \left( I + H^T H \right)^{-1} A^T = LL^T.$$  

(8)

Denoting by $\ell_{m,m}$ the diagonal entries of $L$, IF-SIC can achieve [7] any rate satisfying $R < R_{IF-SIC}(H)$ where

$$R_{IF-SIC}(H) = 2N_t \frac{1}{2} \max_{A} \min_{m=1,...,2N_t} \log \left( \frac{1}{\ell_{m,m}^2} \right)$$

and the maximization is over all $2N_t \times 2N_t$ full-rank integer matrices.

**C. The Jacobi Ensemble**

In the analysis we carry out, the distribution of the singular values of a submatrix of $P_c$ play a central role. To that end, we recall the Jacobi ensemble that is defined as follows.

**Definition 1. (Jacobi ensemble).** The $J(m_1, m_2, n)$ ensemble, where $m_1, m_2 \geq n$, is an $n \times n$ Hermitian matrix that can be written as $A(A + B)^{-1}$, where $A$ and $B$ belong to the Wishart ensembles $W(m_1, n)$ and $W(m_2, n)$, respectively.

We recall the well-known (see [8] and references therein) joint probability density function of the ordered eigenvalues $0 \leq \lambda_1 \leq \cdots \lambda_n \leq 1$ of the Jacobi ensemble $J(m_1, m_2, n)$. Namely,

$$f(\lambda_1, \cdots, \lambda_n) = \kappa^{-1}(m_1, m_2, n) \prod_{i=1}^{n} \lambda^{m_1-n}(1-\lambda_i)^{m_2-n} \prod_{i<j}(\lambda_i - \lambda_j)^2,$$

(9)

where $\kappa^{-1}(m_1, m_2, n)$ is a normalizing factor (Selberg integral), i.e. (see, e.g., [9]),

$$K_{m_1,m_2,n} = \prod_{j=1}^{n} \frac{\Gamma(m_1+j) \Gamma(m_2+j) \Gamma(1+j)}{\Gamma(2) \Gamma(m_1+m_2+n+j)}.$$  

(10)

As detailed in [8], the singular values of the $\frac{N_t}{2} \times k$ submatrix of the $N_t \times N_t$ unitary matrix $P_c$ have the following Jacobi distribution:

- When $1 \leq k \leq \frac{N_t}{2}$, the singular values of the submatrix have the same distribution as the eigenvalues of the Jacobi ensemble $J(\frac{N_t}{2}, \frac{N_t}{2}, k)$.
- When $\frac{N_t}{2} < k \leq N_t$, using Lemma 1 in [8], we have that the singular values of the submatrix have the same distribution as the eigenvalues of the Jacobi ensemble $J(\frac{N_t}{2}, \frac{N_t}{2}, N_t - k)$.

**III. CLOSED-FORM BOUNDS FOR $N_r \times 2$ CHANNELS**

**A. Space-Only Precoded Integer-Forcing**

1) Upper Bound: We recall known upper bounds for the achievable WC outage probability of CUE-precoded IF-SIC for $N_r \times 2$ channels. The following theorem combines Theorem 2, Lemma 4 and Corollary 2 of [4].

**Theorem 1.** [4] For any $N_r \times 2$ complex channel $H_c$ with white-input mutual information $C > 1$, i.e., $D \in \mathbb{D}(C)$, and for $P_c$ drawn from the CUE (inducing a real-valued precoding matrix $P$), we have

$$P_{out}^{WICIF-SIC}(C, \Delta C) \leq 81 \pi^2 2^{-\Delta C},$$

(9)
for $\Delta C > 1$. A tighter yet less explicit bound is

$$
P_{\text{out,IF-SIC}} (C, \Delta C) \leq \max_{\rho_{\text{max}}} \sum_{a \in \mathbb{H}(\beta, \rho_{\text{max}})} \frac{2^{2\rho_{\text{max}} - \beta/C + \Delta C}}{2 \pi} \sqrt{1 + \rho_{\text{max}}},$$

(12)

where $\rho_{\text{max}} = \max_i \rho_i$ and

$$
\mathbb{H}(\beta, \rho_{\text{max}}) = \{ a : 0 < \|a\| < \sqrt{\beta \rho_{\text{max}}} \text{ and } 0 < c < 1 : ca \in \mathbb{Z}^n \}.
$$

(13)

with $\beta = 2^{-\beta/2(C + \Delta C)}$.

2) Lower Bound on the Outage Probability Via Maximum-Likelihood Decoding: It is natural to compare the performance attained by an IF receiver with that of an optimal maximum likelihood (ML) decoder for the same precoding scheme but where each stream is coded using an independent Gaussian codebook. Since we are confining the encoders to operate in parallel (independent streams), we are in fact considering coding over a MIMO multiple-access channel (MAC).

Thus, a simple upper bound on the achievable rate of integer-forcing is the capacity of the MIMO MAC with independent Gaussian codebooks of equal rates. Specifically, let $H_S$ denote the submatrix of $H_sP_c$ formed by taking the columns with indices in $S \subseteq \{1, 2, \ldots, N_s\}$. For a joint ML decoder, the following is the maximal achievable rate over the considered MIMO multiple-access channel:

$$
R_{\text{JOINT}} = \min_{S \subseteq \{1, 2, \ldots, N_s\}} \frac{N_s}{|S|} \log \det (I_{N_s} + H_S H_S^H).
$$

(14)

Note that since $H_S$ depends on the random precoding matrix $P_c$, $R_{\text{JOINT}}$ is a random variable.

We next derive the exact WC scheme outage for ML decoding when random CUE precoding is applied (with independent Gaussian codebooks).

Using the SVD decomposition, $H_sP_c$ can be written as

$$
H_sP_c = U_c \begin{bmatrix} \sqrt{\rho_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\rho_2} & \cdots & 0 \end{bmatrix} V_c^H P_c,
$$

(15)

where $\rho_i = \Sigma_i^2$. Substituting the latter in (2) yields

$$
C = \log(1 + \rho_1) + \log(1 + \rho_2).
$$

(16)

Theorem 2. For an $N_r \times 2$ compound channel with WI-MI $C$, and CUE precoding matrix, we have

$$
P_{\text{out,JOINT}}(C, \Delta C) = 1 - \sqrt{1 - 2\Delta C}.
$$

(17)

Proof. The capacity of the $N_r \times N_t$ MIMO MAC channel with equal user rates is given by

$$
R_{\text{JOINT}}(H_sP_c) = R_{\text{JOINT}}(\rho_1, \rho_2) = \min_{k \in S_k} \frac{2}{k} \min \log \det (I_{N_s} + H_s H_s^H)
$$

$$
= \min_{k \in S_k} \frac{2}{k} \min \left\{ R_3(s) \right\}
$$

(18)

where $S_k$ is the set of all the subsets of cardinality $k$ from $\{1, 2, \ldots, N_s\}$, where $k = 1, \ldots, N_t$. Hence $H_s$ is a submatrix of $H_sP_c$ formed by taking $k$ columns. Since we assume that $P_c$ is drawn from the CUE, it follows that $P_c$ is equal in distribution to $V_c^H P_c$. Hence, taking $k$ columns from $H_sP_c$ is equivalent to multiplying $H_s$ with $k$ columns of $P_c$. Since, in this theorem, $N_t = 2$, (13) reduces to

$$
R_{\text{JOINT}}(H_sP_c) = \min \{ 2R_3(\{1\}), 2R_3(\{2\}), R_3(\{1, 2\}) \}.
$$

(19)

When $k = 2$, we have $R_3(s = \{1, 2\}) = C$. Plugging it into (19), we get

$$
R_{\text{JOINT}}(H_sP_c) = \min \{ 2R_3(\{1\}), 2R_3(\{2\}), C \}.
$$

(20)

We now turn to study $2R_3(s = \{1\})$. Note that

$$
\log \det (I_{N_r} + H_s H_s^H) = \log (1 + H_s^H H_s),
$$

(21)

so that

$$
2R_3(s = \{1\}) = 2 \log \left( 1 + \left[ \begin{array}{c} \rho_1 \\ 0 \end{array} \right] \left[ \begin{array}{c} 0 & \rho_2 \\ \rho_2 & 0 \end{array} \right] \left[ \begin{array}{c} \rho_1 \\ 0 \end{array} \right] \right)
$$

$$
= 2 \log \left( 1 + \rho_1 H_{1,1}^H P_{1,1} + \rho_2 H_{1,2}^H P_{1,2} \right).
$$

(22)

Also, since $P_{1,1}$ and $P_{1,2}$ form a vector in a unitary matrix,

$$
P_{1,1}^H P_{1,1} + P_{1,2}^H P_{1,2} = 1,
$$

(23)

and hence

$$
2R_3(s = \{1\}) = \log \left( 1 + \rho_1 |P_{1,1}|^2 + \rho_2 (1 - |P_{1,1}|^2) \right)
$$

$$
= 2 \log \left( 1 + \rho_2 + |P_{1,1}|^2 (\rho_1 - \rho_2) \right).
$$

(24)

Without loss of generality we assume that $\rho_2 \leq \rho_1$.

Therefore,

$$
\Pr (2R_3(s = \{1\}) < R) = \Pr \left( 2 \log \left( 1 + \rho_2 + |P_{1,1}|^2 (\rho_1 - \rho_2) \right) < R \right)
$$

$$
= \Pr \left( |P_{1,1}|^2 \frac{2R^2/2 - 1 - \rho_2}{\rho_1 - \rho_2} \right),
$$

(25)

where $0 \leq \rho_2 \leq 2C/2 - 1$.

The probability density function of the squared magnitude of any entry of an $M \times M$ matrix drawn from the circular unitary ensemble is

$$
f_{|P_{1,1}|^2}(\mu) \begin{cases} 
(M - 1)(1 - \mu)^{M-2} & 0 \leq \mu \leq 1 \\
0 & \text{otherwise}
\end{cases},
$$

(26)

where the expression holds for $M \geq 2$. In our case, $M = 2$, and thus $|P_{1,1}|^2 \sim U(0,1)$. Hence,

$$
\Pr (R_3(s = \{1\}) < R) =\max \left( \frac{2R^2/2 - 1 - \rho_2}{\rho_1 - \rho_2} , 0 \right).
$$

(27)

4 It is readily seen that this distribution is a special case of the Jacobi distribution.
As from (16) we have \( \rho_1 = \frac{2^C}{1+\rho_2} - 1 \), it follows that

\[
\Pr( R_3(s = \{1\}) < R ) = \frac{2^{R/2} - 1 - \rho_2}{1 + \rho_2} - 1 - \rho_2.
\]  

(28)

Now, by symmetry, it is clear that

\[
\Pr( R_3(s = \{2\}) < R ) = \Pr( R_3(s = \{1\}) < R ) .
\]  

(29)

Furthermore, it is not difficult to show (a proof appears in Appendix B) that the events \( \{ R_3(s = \{1\}) < R \} \) and \( \{ R_3(s = \{2\}) < R \} \) are disjoint. Therefore by (28) and (20), it follows that for \( 0 \leq R \leq C \), we have

\[
\Pr( R_{\text{JOINT}}(C, \rho_2) < R ) = 2 \cdot \Pr( R_3(s = \{1\}) < R ) = 2 \cdot \frac{2^{R/2} - 1 - \rho_2}{1 + \rho_2} - 1 - \rho_2,
\]  

(30)

which implies that

\[
P_{\text{out,JOINT}}^{\text{WC}}(C, R) = \max_{0 \leq \rho_2 \leq 2^{C/2} - 1} 2 \cdot \frac{2^{R/2} - 1 - \rho_2}{1 + \rho_2}.
\]  

(31)

It is readily verified that the derivative of the expression that is maximized with respect to \( \rho_2 \) is zero for (and only for)

\[
\rho_2^* = 2^{-R-1} \left( 2^{C+1} - 2^{R/2+1} - 2^{2R - 2C - 2C-R} \right),
\]

and moreover, that the second derivative at this point is negative, and hence this is a global maximum. Finally, by plugging \( \rho_2 = \rho_2^* \), we obtain

\[
P_{\text{out,JOINT}}^{\text{WC}}(C, R) = 1 - \sqrt{1 - 2^{-\Delta C}}.
\]  

(32)

3) Comparison of Bounds and Empirical Results: Figure 1 depicts the lower and upper bounds as well as results of an empirical simulation of the scheme. We observe that for \( N_t \times 2 \) channels, the actual performance of randomly precoded IF-SIC is very close to the upper (ML) bound. This suggests that one can expect that the ML bound may serve as a useful design tool for more general cases (\( N_t > 2 \)).

B. Space-Time Precoding

Hitherto the role of random precoding was in facilitating performance evaluation. Namely, applying CUE precoding results in performance being dictated solely by the singular values of the channel, so that one can then consider the worst case performance only with respect to the latter.

In contrast, applying random precoding over space as well as time has operational significance, allowing to improve the guaranteed performance as we quantify next.

1) Background: Combining space-time precoding and integer forcing was suggested in [11], as we next briefly recall.

A block of \( T \) channel uses is processed jointly so that the \( N_t \times N_t \) physical MIMO channel [11] is transformed into a “time-extended” \( N_t T \times N_t T \) MIMO channel. A unitary

\[ \rho_1 = \frac{2^C}{1+\rho_2} - 1, \]

2) Upper Bound: Space-time CUE precoding results in a \( N_t T \times 2T \) MIMO channel. An upper bound on the WC outage probability can be obtained from Theorem 1 in [4], by substituting \( N_t = 2T \).

3) Lower Bound: Define

\[ B_1(T, k, R, p_1, p_2) = \left\{ \Delta : \prod_{i=1}^{k} \left( 1 + p_1 \lambda_i + p_2 (1 - \lambda_i) \right) \leq 2^R \right\} \]

\[ B_2(T, k, R, p_1, p_2) = \left\{ \Delta : \prod_{i=1}^{2T-k} \left( 1 + p_1 \lambda_i + p_2 (1 - \lambda_i) \right) \leq 2^R \right\} \]

\[ K_{m_1, m_2, n} = \prod_{j=1}^{m_1} \frac{\Gamma(m_1 + j) \Gamma(m_2 + j) \Gamma(1 + j) \Gamma(2) \Gamma(m_1 + m_2 + n + j)}{\Gamma(2) \Gamma(m_1 + m_2 + n + j)}. \]

Fig. 1. Lower and upper bound for \( N_t \times 2 \) MIMO channels with \( C = 14 \).
where $\hat{R} = \frac{k}{T} \max(R - (k - T), 0)$.

**Theorem 3.** For an $N_r \times 2$ compound channel with WI-MI equal $C$, and CUE precoding over $T$ time extensions, we have

$$P^{\text{WC}}_{\text{out}}(C, R) \geq \max_{0 \leq P_k \leq 2^C/2} \max_{k} P_{\text{out}}$$

where $P_{\text{out}} = P_{\text{out}}(k, T, R, \rho_1, \rho_2)$ and

- For $1 \leq k \leq T$:
  $$P_{\text{out}} = \kappa_1 \int_{B_1} \prod_{i=1}^{k} \lambda_i^{T-k}(1-\lambda_i)^{T-k} \prod_{i<j}(\lambda_i - \lambda_j)^2 d\Delta$$
- For $T + 1 \leq k \leq T$:
  $$P_{\text{out}} = \kappa_2 \int_{B_2} \prod_{i=1}^{T-k} \lambda_i^{k-T}(1-\lambda_i)^{k-T} \prod_{i<j}(\lambda_i - \lambda_j)^2 d\Delta$$

and where $\kappa_1 = \kappa_{T,T,k}^{-1}$ and $\kappa_2 = \kappa_{T,T,T-k}^{-1}$.

*Proof.* The proof depends on eigenvalue distribution of sub-matrices of $P_c$. As mentioned above, these eigenvalues have Jacobi distribution. The full description of the distribution and proof can be found in Appendix A. \qed

4) Comparison of Bounds and Empirical Results: We compare the obtained upper and lower bounds with the empirical performance results of CUE-precoded IF-SIC. In addition, for a $T = 2$ time-extended $N_r \times 2$ channel, it is natural to also compare performance with that obtained by replacing CUE precoding with algebraic precoding. Specifically, we consider Alamouti and golden code precoding.

To that end, let us define the $\varepsilon$-outage capacity of a scheme $R_{\text{Scheme}}(P_{st,c}; \varepsilon)$ as the rate for which

$$P^{\text{WC, Scheme}}_{\text{out}}(C, R_{\text{Scheme}}(P_{st,c}; \varepsilon)) = \varepsilon.$$  \hspace{1cm} (38)

Further, the guaranteed transmission efficiency of a scheme, at a given outage probability $\varepsilon$ and WI mutual information $C$, is defined as

$$\eta_{\varepsilon}(C, P_{st,c}) = \frac{R_{\text{Scheme}}(P_{st,c}; \varepsilon)}{C}.$$  \hspace{1cm} (39)

Figure 2 depicts the guaranteed efficiency at 1% outage for several precoding options for an $N_r \times 2$ channel and $T = 1, 2$. We plot the empirical efficiency for both IF-SIC and ML receivers. It can be seen that for CUE precoding, the performance of IF-SIC is very close to that of ML.

We also present empirical results for an $N_r \times 4$ channel. Figure 3 depicts the guaranteed efficiency at 1% for several precoding and receiver topologies, where the algebraic codes considered are orthogonal space-time block precoding (rate 3/4), the perfect code 3, the latter punctured to rate 3, and also the MIMO (rate 2) code 12.

*Fig. 2. Guaranteed efficiency at 1% outage probability for the $N_r \times 2$ MIMO channel with various precoding and decoding options, for $T = 1, 2$. *

*Fig. 3. Guaranteed efficiency at 1% outage probability for the $N_r \times 4$ MIMO channel with various precoding and decoding options, for $T = 1, 2, 4$. *

IV. DISCUSSION AND OUTLOOK

For the $N_r \times 2$ compound MIMO channel, using CUE precoding over a time-extend channel offers significant benefit over space-only precoding. However, space-time CUE precoding falls short when compared to algebraic space-time precoding. Specifically, the combination of Alamouti precoding at low rates and golden code precoding (with IF-SIC) at high rates is superior to CUE precoding.

Nonetheless, for the $N_r \times 4$ compound MIMO channel, we observe from the empirical results (Figure 3) that there is a region where using random space-time CUE precoding results in the highest guaranteed efficiency. This provides motivation for searching for fixed precoding matrices that yield better results than perfect codes at the price of a small outage probability.

As a concluding remark, we note that the lower bound we developed holds only for the case of a maximum of two distinct singular value in the SVD decomposition of $H_c$. 

\[\text{When using fixed space-time precoding matrix we apply CUE precoding to the physical channel.}\]
This limitation occurs since we only know the eigenvalue distribution of a single submatrix of unitary matrix drawn from the CUE. In the case of two singular values, we can take the complement as the eigenvalue distribution of the residual matrix (see for example Equation (23)). Nevertheless, the results hold for the important case of an open-loop MAC channel with a single receive antenna, where the transmitters have no channel state information.

APPENDIX A
PROOF OF THEOREM 3

Using \( T \) time extensions we obtain an \( N_c T \times 2T \) equivalent channel. The ML lower bound (14) now takes the form

\[
R_{\text{JOINT,ST}} = \frac{1}{T} \min_{k} \frac{2T}{k} \min_{s \in S_k} \log \det (I_{k} + H_s H_s^H)
\]

(40)

where \( S_k \) is the set of all the subsets of cardinality \( k \) contained in \( \{1, 2, \ldots, 2T\} \). Hence \( H_s \) is a submatrix of \( H \) formed by taking \( k \) columns.

Using (15), and after possibly applying column permutations, the effective channel takes the form

\[
\begin{align*}
E \triangleq \hat{U} e \begin{bmatrix} \sqrt{\rho_1}I_{T \times T} & 0 \\ 0 & \sqrt{\rho_2 I_{T \times T}} \end{bmatrix} P_{st,c},
\end{align*}
\]

(41)

where \( \tilde{A} = I_{T \times T} \otimes A \). Since \( P_{st,c} \) is drawn from the CUE, it follows that \( E \) is equal to \( H_s P_{st,c} \) in distribution and thus we assume that the channel is the former for sake of analysis. In particular, we have

\[
R_J(s \in S_k) = \log \det (I_{N_c T} + E_s E_s^H)
= \log \det (I_{k} + E_s E_s^H).
\]

(42)

Let us use the notation \([ ]_s \) to denote the matrix resulting from a specific selection of \( k \) columns from a matrix, corresponding to the chosen set \( s \). Denoting

\[
E_s = [E]_s = \hat{U} e \begin{bmatrix} \sqrt{\rho_1}I_{T \times T} & 0 \\ 0 & \sqrt{\rho_2 I_{T \times T}} \end{bmatrix} P_{st,c},
\]

we have

\[
R_J(s \in S_k) = \log \det (I_k + E_s E_s^H)
= \log \det (I_k + [P_{st,c}]_s^H \begin{bmatrix} \rho_1 I_{T \times T} & 0 \\ 0 & \rho_2 I_{T \times T} \end{bmatrix} [P_{st,c}]_s)
= \log \det (I_k + \rho_1 P_1^H P_1 + \rho_2 P_2^H P_2),
\]

(43)

where \([P_{st,c}]_s = \begin{bmatrix} P_1^H \gamma_s \end{bmatrix} \).

As described in [8], we note that the singular values of \( P_1 \) (which is a rectangular submatrix of dimensions \( T \times k \) of the \( 2T \times 2T \) unitary matrix \( P_{ST,c} \)) has the following Jacobi distribution

- When \( 1 \leq k \leq T \), the singular values of \( P_1 \) have the same distribution as the eigenvalues of the Jacobi ensemble \( \mathcal{J}(T, T, k) \).

Further, we recall a derivation appearing in Lemma 1 of [8] (which is a corollary of [13]) and note that since \( P_{st,c} \) is unitary, we have

\[
P_1^H P_1 + P_2^H P_2 = I_k.
\]

(44)

Therefore

\[
P_1^H P_1 = I_k - P_2^H P_2
\]

\[
UD_1 V^H = I_k - P_2^H P_2
\]

\[
D_1 = U^H (I_k - P_2^H P_2) V
\]

(45)

Let \( \{\lambda_i(1)\}_{i=1}^k \) and \( \{\lambda_i(2)\}_{i=1}^k \) be the eigenvalues of \( P_1^H P_1 \) and \( P_2^H P_2 \), respectively. It follows that

\[
\lambda_i(2) = 1 - \lambda_i(1),
\]

(46)

and hence

\[
R_J(s \in S_k) = \det (I_m + \rho_1 P_1^H P_1 + \rho_2 P_2^H P_2)
= \prod_{i=1}^m \left( 1 + \rho_1 \lambda_i(1) + \rho_2 (1 - \lambda_i(1)) \right).
\]

(47)

Therefore, for a specific choice of columns \( s \in S_k \), the outage probability may be written as

\[
\begin{align*}
P_{\text{out}} \left( \frac{2}{k} R_J(s \in S_k) \right) &= \Pr \left( \frac{2}{k} R_J(s \in S_k) < R \right)
= \Pr \left( \log \det (I_k + \rho_1 P_1^H P_1 + \rho_2 P_2^H P_2) < R \frac{k}{2} \right)
= \Pr \left( \prod_{i=1}^k \left( 1 + \rho_1 \lambda_i(1) + \rho_2 (1 - \lambda_i(1)) \right) < 2^{R \frac{k}{2}} \right)
= \Pr \left( \prod_{i=1}^k \left( 1 + \rho_2 + \lambda_i(1) (\rho_1 - \rho_2) \right) < 2^{R \frac{k}{2}} \right).
\end{align*}
\]

(48)

Without loss of generality, we assume that \( \rho_2 \leq \rho_1 \) and hence \( 0 \leq \rho_2 \leq 2^{C/2} \). Using the explicit expression for the Jacobi distribution (9) of these singular values, we have

- For \( 1 \leq k \leq T \):

\[
P_{\text{out}}(s \in S_k) = P_{\text{out}}(k, T, R, \rho_1, \rho_2)
= K_{\epsilon} \prod_{i=1}^k \frac{\sqrt{T-k} (1 - \lambda_i)}{\lambda_i} \prod_{i<j}^k (\lambda_j - \lambda_i)^2 d \Delta
\]

(49)

- For \( T \leq k \leq 2T \), by Theorem 3 of [8], we have

\[
P_{\text{out}}(k, T, R, \rho_1, \rho_2) = P_{\text{out}}(2T - k, 2T, R, \rho_1, \rho_2),
\]

(50)

and thus

\[
P_{\text{out}}(s \in S_k) = P_{\text{out}}(2T - k, 2T, R, \rho_1, \rho_2)
\]
necessarily implies that \( R_j(2) > R \), for all \( 0 \leq R \leq C \).

To that end, assume that indeed \( R_j(\{1\}) < R \). By (62), this implies that

\[
1 + \rho_2 + |P_{1,1}|^2(\rho_1 - \rho_2) < 2^{R/2},
\]

or equivalently

\[
-1 - \rho_2 - |P_{1,1}|^2(\rho_1 - \rho_2) > -2^{R/2}.
\]

It follows that

\[1 + \rho_2 + (1 - |P_{1,1}|^2)(\rho_1 - \rho_2) > 2 + \rho_1 + \rho_2 - 2^{R/2}.\]

By (63), we have established that

\[R_j(\{1\}) < R \implies R_j(\{2\}) > 2\log \left(2 + \rho_1 + \rho_2 - 2^{R/2}\right).\]

To show that \( R_j(\{1\}) < R \implies R_j(\{2\}) > R \), it suffices therefore to show that

\[2\log \left(2 + \rho_1 + \rho_2 - 2^{R/2}\right) \geq R,\]

or equivalently,

\[2 + \rho_1 + \rho_2 - 2^{R/2} \geq 2^{R/2}.\]

Using (10), the latter is further equivalent to showing that

\[1 + \frac{2^C}{1 + \rho_2} + \rho_2 - 2^{R/2} \geq 2^{R/2} \]

Finally, denoting \( x = 1 + \rho_2 \), this is equivalent to showing that the following holds.

\[x^2 - 2 \cdot 2^{R/2} \cdot x + 2^C \geq 0.\]

It can be easily verified that for all values of \( \rho_2 \), and for all \( 0 \leq R \leq C \), this inequality indeed holds. Therefore, we conclude that the events \( R_j(\{1\}) < R \) and \( R_j(\{2\}) < R \) are indeed disjoint.

**References**

[1] O. Ordentlich and U. Erez, “Precoded integer-forcing universally achieves the MIMO capacity to within a constant gap,” *Information Theory, IEEE Transactions on*, vol. 61, no. 1, pp. 323–340, Jan 2015.

[2] P. Elia, B. Setharaman, and P. V. Kumar, “Perfect space–time codes for any number of antennas,” *IEEE Transactions on Information Theory*, vol. 53, no. 11, pp. 3853–3868, 2007.

[3] F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, “Perfect space–time block codes,” *IEEE Transactions on Information Theory*, vol. 52, no. 9, p. 3885, 2006.

[4] E. Domanovitz and U. Erez, “Outage behavior of randomly precoded integer forcing over MIMO channels,” *CoRR*, vol. abs/1608.01588, 2016. [Online]. Available: [http://arxiv.org/abs/1608.01588](http://arxiv.org/abs/1608.01588).

[5] J. Zhan, B. Nazer, U. Erez, and M. Gastpar, “Integer-forcing linear receivers,” *Information Theory, IEEE Transactions on*, vol. 60, no. 12, pp. 7661–7685, Dec 2014.

[6] M. Metha, “Random matrices and the statistical theory of energy levels,” *Academic, New York*, 1967.

[7] O. Ordentlich, U. Erez, and B. Nazer, “Successive integer-forcing and its sum-rate optimality,” *CoRR*, vol. abs/1307.2105, 2013. [Online]. Available: [http://arxiv.org/abs/1307.2105](http://arxiv.org/abs/1307.2105).

[8] R. Dar, M. Feder, and M. Shafi, “The Jacobi MIMO channel,” *IEEE Transactions on Information Theory*, vol. 59, pp. 2426–2441, 2013.
[9] A. Selberg, “Remarks on a multiple integral,” *Norsk Mat. Tidsskr.*, vol. 26, pp. 71–78, 1944.

[10] A. Narula, M. Trott, and G. W. Wornell, “Performance limits of coded diversity methods for transmitter antenna arrays,” *Information Theory, IEEE Transactions on*, vol. 45, no. 7, pp. 2418–2433, Nov 1999.

[11] E. Domanovitz and U. Erez, “Combining space-time block modulation with integer forcing receivers,” in *Electrical Electronics Engineers in Israel (IEEE), 2012 IEEE 27th Convention of*, Nov 2012, pp. 1–4.

[12] F. Oggier, C. Hollanti, and R. Vehkalahti, “An algebraic MIMO-MISO code construction,” in *2010 International Conference on Signal Processing and Communications (SPCOM)*, July 2010, pp. 1–5.

[13] C. C. Paige and M. A. Saunders, “Towards a generalized singular value decomposition,” *SIAM Journal on Numerical Analysis*, vol. 18, no. 3, pp. 398–405, 1981.