Parabolic - hyperbolic boundary layer

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Abstract

A boundary value problem related to a parabolic higher order operator with a small parameter $\varepsilon$ is analyzed. For $\varepsilon$ tends to zero, the reduced operator is hyperbolic. When $t \to \infty$ and $\varepsilon \to 0$ a parabolic hyperbolic boundary layer appears. In this paper a rigorous asymptotic approximation uniformly valid for all $t$ is established.

1 Introduction

The parabolic operator

$$L_\varepsilon = \partial_{xx}(\varepsilon \partial_t + c^2) - \partial_{tt}$$

(1.1)

is related to the well known Kelvin -Voigt viscoelastic model. Further, it characterizes also the principal part of numerous models with non linear dissipation, such as

$$L_\varepsilon = \beta(u, u_x, u_t).$$

(1.2)

Typical example is the perturbed Sine Gordon equation. Moreover, by

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means of (1.2), wave equations with non linear terms are regularized obtaining a priori estimates and considering the \( \varepsilon \) parameter vanishing.\cite{1}. Further third operators are also considered to value the Cauchy problem for a second order hyperbolic equation \cite{2} or to regularize parabolic forward-backward equations.\cite{3}

Singular perturbation problem related to equations like (1.2) have interest also to evaluate the influence of the dissipative causes on the wave propagation.\cite{4}. In particular, in the linear case \( \beta = f(x,t) \), it’s interesting to compare the effects of the diffusion with the pure waves which occur when \( \varepsilon = 0 \). In this case one has a parabolic-hyperbolic boundary layer with the unique singularity for \( t \to \infty \).

In this paper, we consider the strip problem for equation (1.2) and analyze the singular perturbation problem when \( \beta = f(x,t) \) is linear. The Green function related to this problem has been already determined in term of a rapidly decreasing Fourier series.\cite{5}.

An appropriate analysis of this series when \( \varepsilon \to 0 \) allows to obtain a rigorous asymptotic estimate of the solution, uniformly valid even \( t \to \infty \).

2 Statement of the problem

If \( v(x,t) \) is a function defined in

\[
\Omega = \{(x,t) : 0 < x < l, \ t \geq 0 \},
\]

with \( l \) arbitrary positive constant, let IBC the following system of initial-boundary conditions:

\[
\begin{aligned}
&v(x,0) = f_0(x), \ \ v_t(x,0) = f_1(x), \ x \in [0,l], \\
&v(0,t) = \psi_0, \ \ v(l,t) = \psi_1, \ t \geq 0,
\end{aligned}
\]

with \( f_i, \psi_i \ (i = 0,1) \) regular data.

Consider the operators:

\[
\begin{aligned}
\mathcal{L}_0 &= c^2 \partial_{xx} - \partial_{tt}; \quad \mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon \partial_{xx}t
\end{aligned}
\]
and denote by $u_0$ and $u_\varepsilon$ the solutions of the problems:

\begin{align*}
\text{Problem } P_0 : & \quad \mathcal{L}_0 u_0 = -f \text{ with IBC } 21) \\
\text{Problem } P_\varepsilon : & \quad \mathcal{L}_\varepsilon u_\varepsilon = -f \text{ with IBC } 21),
\end{align*}

where $f(x,t)$ is a prefixed source term.

To obtain a rigorous approximation of $u_\varepsilon$ when $\varepsilon \to 0$, we put

\begin{equation}
(2.3) \quad u(x,t,\varepsilon) = u_0(x,t) + \varepsilon r(x,t,\varepsilon)
\end{equation}

where $u_0$ is the well-known solution of the classical problem $P_0$, while the error term represent the solution of the $\text{Problem } P_r$:

\begin{equation}
(2.4) \quad \left\{ \begin{array}{l}
\mathcal{L}_r \varepsilon x r = -F(x,t) \quad (x,t) \in \Omega \\
v(x,0) = f_0(x), \quad v_t(x,0) = f_1(x), \quad x \in [0,l], \\
v(0,t) = \psi_0, \quad v(l,t) = \psi_1, \quad t \geq 0,
\end{array} \right.
\end{equation}

with $F(x,t) = \partial x xt u_0$. Therefore, following results in ??, one has:

\begin{equation}
(2.5) \quad r(x,t,\varepsilon) = -\int_0^t d\xi \int_0^t F(\xi,\tau,\varepsilon) G(x,\xi,t - \tau) d\tau
\end{equation}

where $G(x,\xi,t)$ is the Green function related to $\mathcal{L}_\varepsilon$ operator.

In particular, for all integer $n \geq 1$, letting:

\begin{equation}
(2.6) \quad \gamma_n = \pi l n \quad a_n = \frac{\varepsilon}{2} \gamma_n k = \frac{2c l}{\pi \varepsilon} \\
\quad b_n = \gamma_n c \sqrt{1 - (n/k)^2} \quad H_n = e^{a_n \frac{\sin(b_n t)}{b_n}},
\end{equation}

\begin{equation}
(2.7) \quad G(x,\xi,t) = \frac{2}{l} \sum_{n=1}^\infty H_n(t) \sin \gamma_n x \sin \gamma_n \xi
\end{equation}

with
\( H_n(t) = \frac{e^{-bn^2t}}{bn^2\sqrt{1 - (k/n)^2}} \sinh(bn^2t\sqrt{1 - (k/n)^2}) \) 

and

\( b = \frac{\pi^2}{2l^2} = q\varepsilon, \quad k = \frac{2cl}{\pi\varepsilon}, \quad \gamma_n = \frac{\pi}{l} n. \)

Now, denote with \( u(x, t) \) the solution of the reduced problem obtained by (2.1) with \( \varepsilon = 0 \). To obtain an asymptotic approximation for \( w(x, t) \) when \( \varepsilon \to 0 \), we put:

\( w(x, t, \varepsilon) = e^{-\varepsilon t}u(x, t) + r(x, t, \varepsilon) \)

where the error \( r(x, t, \varepsilon) \) must be evaluated.

By means of standard computations one verifies that \( r(x, t, \varepsilon) \) is the solution of the problem:

\[
\begin{cases}
\partial_{xx}(\varepsilon r_t + c^2 r) - \partial_{tt} r = f(x, t, \varepsilon), & (x, t) \in D, \\
r(x, 0) = 0, & r_t(x, 0) = 0, \quad x \in [0, l], \\
r(0, t) = 0, & r(l, t) = 0, \quad 0 < t < T,
\end{cases}
\]

where the source term \( f(x, t, \varepsilon) \) is:

\( f(x, t, \varepsilon) = F'(x, t)(1 - e^{-\varepsilon t}) + e^{-\varepsilon t}[-\varepsilon \lambda_t + \varepsilon^2(u + u_{xx})] \)

with \( \lambda = 2u + u_{xx} \).

The problem (2.11) has already been solved in [5] and the solution is given by:

\( r(x, t, \varepsilon) = -\int_0^l d\xi \int_0^t f(\xi, \tau, \varepsilon)G(x, \xi, t - \tau) \, d\tau \)

where \( G(x, \xi, t) \) is:
(2.14) \[ G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} H_n(t) \sin \gamma_n x \sin \gamma_n \xi \]

with

(2.15) \[ H_n(t) = \frac{e^{-bn^2t}}{bn^2 \sqrt{1 - (k/n)^2}} \sinh(bn^2t \sqrt{1 - (k/n)^2}) \]

and

(2.16) \[ b = \frac{\pi^2}{2l^2\varepsilon} = q\varepsilon, \quad k = \frac{2cl}{\pi\varepsilon} \quad \gamma_n = \frac{\pi}{l} n. \]

3 Analysis of \( G(x,t,\xi,\varepsilon) \) when \( \varepsilon \) tends to zero.

In order to investigate the behaviour of the Green function \( G \) when parameter \( \varepsilon \to 0 \), referring to the function \( G \) defined in (2.14), let:

(3.1) \[ H_n^1(t) = \frac{e^{-bn^2t}}{bn^2 \sqrt{(k/n)^2 - 1}} \sin bn^2t \sqrt{(k/n)^2 - 1} \]

and

(3.2) \[ G(x, \xi, t) = \frac{2}{l} \left( \sum_{n=1}^{[k]} H_n^1(t) + \sum_{[k]+1}^{\infty} H_n(t) \right) \sin \gamma_n x \sin \gamma_n \xi = G_1 + G_2. \]

If \( \alpha \) is an arbitrary constant such that:

(3.3) \[ 1/2 < \alpha < 1, \quad \bar{n} = \frac{2cl}{\pi\varepsilon^\alpha}, \]

the term \( G_1 \) of \( G \) can be given the forms:
\[
G_1(x, \xi, t) = \frac{2}{l} \left\{ \sum_{n=1}^{[\bar{n}]} H_n(t) + \sum_{[\bar{n}]+1}^{[k]} H_n(t) \right\} \sin \gamma_n x \sin \gamma_n \xi.
\]

It is easy to prove that if \(1 \leq n \leq [\bar{n}]\) it holds:

\[
\sqrt{(k/n)^2 - 1} \geq \frac{1 - \varepsilon^{2(1-\alpha)}}{\varepsilon^{1-\alpha}}; \quad e^{-bn^2t} \leq e^{-qt\varepsilon}.
\]

Otherwise, if \([\bar{n}] + 1 \leq n \leq [k]\):

\[
\sqrt{(k/n)^2 - 1} \geq \frac{\sqrt{\pi \varepsilon \beta} \sqrt{4cl - \beta \pi \varepsilon}}{2cl - \pi \varepsilon \beta}; \quad e^{-bn^2t} \leq e^{-2c^2t/\varepsilon^{2n-1}},
\]

where \(0 < \beta < 1\). In particular, if \(k\) is an integer we will assume \(\beta = 1\) and we will explicitly consider the term with \(n = k\), having \(te^{-2c^2t/\varepsilon}\).

Since (3.5) and (3.6), the following inequality holds:

\[
|G_1(x, \xi, t)| \leq N(\varepsilon)e^{-\alpha}e^{-qt\varepsilon} + N_1(\varepsilon) e^{-3/2} e^{-c^2t/\varepsilon^{2n-1}}
\]

where

\[
N(\varepsilon) = \frac{2\zeta(2)}{ql}[1 - \varepsilon^{2(1-\alpha)}]^{-1/2}; \quad N_1(\varepsilon) = \frac{2\zeta(2)(2cl - \pi \varepsilon \beta)}{ql\sqrt{\pi \beta} \sqrt{4cl - \beta \pi \varepsilon}}
\]

and \(\zeta(2)\) is the Riemann zeta function.

There remains to determine an upper bound for hyperbolic terms. This may be done using inequalities proved in [5]. So, being \(\forall n \geq [k] + 1\):

\[
b n^2t(1 \pm \sqrt{1 - (k/n)^2}) \geq c^2/\varepsilon,
\]

and since

\[
\sqrt{1 - (k/n)^2} \geq \frac{\pi \varepsilon (1 - \beta)[4cl + \pi \varepsilon (1 - \beta)]}{2cl + \pi \varepsilon (1 - \beta)},
\]
with \( \beta \equiv 0 \) if \( k \) is an integer, we can write:

\[
|G_2(x, \xi, t)| \leq C_1(\varepsilon) \varepsilon^{-2} e^{-c^2t/\varepsilon} \tag{3.11}
\]

where

\[
C_1(\varepsilon) = \frac{2\zeta(2)[cl + \pi\varepsilon(1 - \beta)]}{ql\pi(1 - \beta)[4cl + \pi\varepsilon(1 - \beta)]}. \tag{3.12}
\]

The previous results lead to prove the following

**Theorem 3.1** - The Green function \( G(x, \xi, t) \) defined in (2.14) converges absolutely for all \((x, t) \in D\). Moreover, indicating by \( M(\varepsilon) = \max\{N_1(\varepsilon) \varepsilon^{-3/2}, C_1(\varepsilon) \varepsilon^{-2}\} \), it holds:

\[
|G(x, \xi, t)| \leq N(\varepsilon)\varepsilon^{-\alpha} e^{-qt\varepsilon} + M(\varepsilon)e^{-c^2t/\varepsilon^{2\alpha-1}}. \tag{3.13}
\]

4 **Asymptotic approximation**

Now, we are able to estimate function \( r(x, t, \varepsilon) \) i.e. it is possible to have an upper bound for the solution of problem (2.11). In fact, recalling expression (2.12)-(2.13), it holds:

\[
|r(x, t, \varepsilon)| \leq l\varepsilon \int_0^t e^{-\varepsilon\tau}\{|\lambda_t(x, \tau)| + \varepsilon|\lambda - u|\}|G(x, \xi, t - \tau)|d\tau +
\]

\[
+l\int_0^t |F(x, \tau)||1 - e^{-\varepsilon\tau}|G(x, \xi, t - \tau)|d\tau.
\]

So, choosing:

\[
3/4 < \alpha < 1 \quad \text{and} \quad 2(2\alpha - 1)^{-1} < \delta < 1,
\]
let:

\[ \beta = \delta (2\alpha - 1) - 1/2, \quad 0 < \gamma < 1. \]  

(4.3)

So, if

\[ \eta = \min\{\beta, \gamma, 1 - \alpha, 1/2\}; \]  

(4.4)

and

\[ A = \max\{\sup_D |F|, \sup_D |\lambda - u|, \sup_D |\lambda_t|\} \]  

(4.5)

the following lemma holds:

**Lemma 4.1** - If the function \( f(x, t, \varepsilon) \) defined in (2.12) is a continuous function in \( D \) with continuous derivative with respect to \( x \), then the function \( r(x, t, \varepsilon) \) satisfies the inequality:

\[ |r| \leq A \varepsilon^\eta \left\{ t^2 Z(\varepsilon) + tY(\varepsilon) + \{t^{2-\delta} + t^{1-\delta}\} W(\varepsilon) + t^{1-\gamma} V(\varepsilon) \right\} + 
\]

\[ + A \{ U(\varepsilon) e^{-c^2 t/\varepsilon} + S(\varepsilon) \} \]

with

\[ Z(\varepsilon) = N(\varepsilon)/2; \quad Y(\varepsilon) = \max\{2N(\varepsilon), N_1(\varepsilon)\} \]

\[ W(\varepsilon) = N_1(\varepsilon)(\delta/\varepsilon) \delta; \quad V(\varepsilon) = C(\varepsilon)[(1 + \gamma)/\varepsilon]^{1+\gamma(1-\gamma)^{-1}} \]

\[ U(\varepsilon) = 2q\varepsilon/c^2 \zeta(2) + C(\varepsilon)/c^2 + \varepsilon/c^2; \quad S(\varepsilon) = 2q\varepsilon\zeta(2)/c^2 + \varepsilon/c^2. \]

**Proof** - Since the well known inequality [7]:

\[ e^{-x} \leq [a/(ex)]^a \quad \forall a > 0, \forall x > 0 \]  

(4.8)

and (4.1), it holds:
\[ |r| \leq Al[\varepsilon^{1-\alpha}(t^2/2 + t + \varepsilon t)N + N_1[\varepsilon^\beta(\delta/e)^\delta(t^{2-\delta} + t^{1-\delta}) + t\sqrt{\varepsilon}] + \]

\[ +C_1(\varepsilon)[(1 + \gamma)/e^{1+\gamma}(1 - \gamma)^{-1}t^{1-\gamma} + \varepsilon/c^2 + e^{\varepsilon t/\varepsilon}(\varepsilon/c^2 + c^{-2}) + \]

\[ +2\varepsilon\zeta(2)q/c^2e^{-c^2t/\varepsilon} + 2q\varepsilon/c^2\zeta(2), \]

from which, taking into account (4.2), (4.3), lemma follows.

In this way, if we consider the set

\[ Q_\varepsilon = \{(x, t) : 0 \leq x \leq l, 0 < t < \varepsilon^{-\eta/2}\} \]

the following theorem holds:

**Theorem 4.1** - When \( \varepsilon \to 0 \), the solution of the parabolic problem (2.1) verifies the following asymptotic estimate

\[ w(x, t, \varepsilon) = e^{-\varepsilon t}u(x, t) + r(x, t, \varepsilon) \]

where the error \( r(x, t, \varepsilon) \) is uniformly bounded every where in \( Q_\varepsilon \).

**References**

[1] A.I. Kozhanov N. A. Lar’kin, *Wave equation with nonlinear dissipation in noncylindrical Domains*, Dokl. Math 62, 2, 2000 17-19

[2] V.P. Maslov, P. P. Mosolov *Non linear wave equations perturbed by viscous terms* Walter deGruyher Berlin N. Y. 2000 pp 329

[3] G. I. Barenblatt, M. Bertsch, R. Del Passo M. Ughi, it Adegenerate pseudoparabolic regularization of a nonlinear forward- backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow. Siam J. Math Anal 24, no 6 1414-1439 (1993).
[4] Ali Nayfey *A comparison of perturbation methods for nonlinear hyperbolic waves* in Proc. Adv. sem. Wisconsin no 45 (1980).

[5] M. De Angelis *Asymptotic analysis for the strip problem related to a parabolic third-order operator*, Appl. Math. Lett. 14,4 pp 425-430 (2001)

[6] B. D’Acunto, M. De Angelis, P. Renno, *Fundamental solution of a dissipative operator*, Rend. Acc. Sc. Fis. Mat. (1997)

[7] D.S. Mitrinovic *Analytic Inequalities* Springer 1970