Fast charging of quantum battery assisted by noise

Srijon Ghosh,1 Titas Chanda,2 Shiladitya Mal,1 and Aditi Sen(De)1

1Harish-Chandra Research Institute and HBNI, Chhatnag Road, Jhunsi, Allahabad - 211019, India
2Instytut Fizyki Teoretycznej, Uniwersytet Jagielloński, Łojasiewicza 11, 30-348 Kraków, Poland

We investigate the performance of a quantum battery exposed to local Markovian and non-Markovian dephasing noises. The battery is initially prepared as the ground state of a one-dimensional transverse XY model with open boundary condition and is charged (discharged) via interactions with local bosonic reservoirs. We show that in the transient regime, quantum battery (QB) can store energy faster and has a higher maximum extractable work, quantified via ergotropy, when it is affected by local phase-flip or bit-flip Markovian noise compared to the case when there is no noise in the system. In both the charging and discharging processes, we report the enhancement in work-output as well as in ergotropy when all the spins are affected by non-Markovian Ohmic bath both in the transient and the steady-state regimes, thereby showing a counter-intuitive advantage of decoherence in QB. Both in Markovian and non-Markovian cases, we identify the system parameters and the corresponding noise models which lead to maximum enhancement of work-output and ergotropy. Moreover, we show that the benefit due to noise persists even with the initial state being prepared at a moderate temperature.

I. INTRODUCTION

In the last century, quantum mechanical description of nature was shown to improve the performance of the devices compared to the analogous classical devices and at the same time, fulfilled the effort to miniaturize them [1]. It has also been established that quantum properties, like coherence [2], entanglement [3, 4], quantum discord [5] are responsible to enhance the capability in communication [6], computation [7], optimal control [8] etc. Moreover, many of these discoveries have successfully been realized in laboratories [9, 10].

Towards this aim, recently a quantum-version of an energy-storage device, a battery, was proposed [11, 12]. Such a consideration is also a part of an active field of research, called quantum thermodynamics [13, 14], which for example, deals with whether thermodynamical principles maintain their validity or get modified in the quantum mechanical domain or not. The idea of quantum batteries has been introduced with the aim of obtaining advantages in the process of charging and discharging. It was shown [11] that the entangling operations are required to enhance extractable work stored in a quantum battery (QB) which consists of an arbitrary number of independent and identical quantum systems. Interestingly, it was observed that the required entangling operations do not necessarily generate entanglement in the system to maximize the storage of energy [15–17]. Moreover, there can be a model of QB from which a high amount of power-extraction is possible without generating entanglement in the system [18]. Although resources responsible for showing quantum advantages in QB is still not properly understood [19] – characterizing it via the various figures of merit is essential.

Towards implementing quantum batteries in realizable systems, solid-state QB employing Dicke state has been proposed [20], where extensive quantum advantage has been obtained due to the interaction between the system and the common reservoir, induced by collective charging compared to parallel charging of the qubits. In a similar spirit, ordered quantum spin chain which can be charged via a local magnetic field has been shown to be a potential candidate for QB [21–23] (see also [24]). Recently, some of us have shown that disordered interactions, instead of having detrimental effects, can enhance the extraction of power from the QB [25] (see also [26]).

During the charging-discharging process of QB in laboratories, one of the possible obstacles is the decoherence [27] which can, in general, decrease the performance of the device. In very recent times, studying the dynamics of QB-system in presence of an environment [28–37] has created lots of attention and has revealed several interesting characteristics including the effect on ergotropy due to the interplay between a coherent and an incoherent source of the charger, i.e., between the unitary and the non-unitary processes, the stability of the battery, the superextensive capacity of the battery by using dark states [34], and enhanced charging and preserving of energy via non-Markovian dynamics [31]. Note here that states evolved in the presence of non-Markovian noisy environment was shown to be useful for generating (or enhancing) certain quantum features, such as the periodic revival of quantum entanglement, quantum coherence etc., necessary for building quantum technologies [38–50].

In the present work, we consider a one-dimensional XY spin model with a transverse magnetic field in open boundary condition as a quantum battery. Every individual spin of it is connected to two sets of local bosonic reservoirs – one acts as a charger while the other one is an energy extractor responsible for the discharging process. In addition, we consider dephasing noise which acts locally on some of the spins or on all the spins of the model (see Fig. 1 for the schematic of the model). In the Markovian domain, we find that during the storing of energy and in terms of maximal gained work, i.e., the ergotropy, there exists a crossover time up to which noise helps to pump more energy (work-gained) in QB compared to that of the noiseless case, while non-Markovian further enhances the capacity of the charging process in the entire time period. In the later case, the work-gained as well as the amount of extractable work by the QB also increase when more number of spins are exposed to noise.

We analytically show the advantage obtained for
Markovian bit-flip and phase-flip noises when the system is prepared in the ground state of the transverse Ising spin chain consisting of two spin-1/2 systems. We also find the range of parameters for which bit-flip noise leads to a higher amount of work than that of the phase-flip ones. We then illustrate that such a hierarchy remains valid also for the ergotropy in the intermediate time of the dynamics. We observe that the bit-flip noise performs better in terms of the work-output if the initial state is chosen from the paramagnetic phase compared to that of the antiferromagnetic one, both in the presence of Markovian and non-Markovian noises while we demonstrate that such a phase dependence does not appear in the computation of ergotropy to show the advantage of the bit-flip noise over the phase-flip ones. In both the noisy scenarios, the enhancement per spin is independent of the system size. During discharging, non-Markovian noise turns out to be more efficient than that of the Markovian dynamics as well as in the absence of noise. Moreover, we notice that the advantage due to noise can be obtained in terms of work-output, as well as ergotropy, both for the ground and the canonical equilibrium states of the transverse Ising model having moderate temperatures as the initial states.

The paper is organized as follows. In Sec. II, we discuss the formalism of quantum battery and its dynamics considered here. The consequence of Markovian noise on QB based on transverse Ising model consisting of two spins is studied analytically in Subsec. III A, while we present the results about the effects of other parameters and Markovian noise on work and ergotropy of QB having more number of spins in Subsec. III B. Sec. IV deals with non-Markovian noise and its advantage in the work and ergotropy generation as well as their extraction in QB. Finally, we end with concluding remarks in Sec. V.

II. QUANTUM BATTERY EXPOSED TO NOISY ENVIRONMENT: PRELIMINARIES

A quantum battery is a collection of a $N$-body system, used to store energy, which can be extracted in a suitable later time. Specifically, the ground or the canonical equilibrium state of an interacting Hamiltonian, $H_B$, can be chosen as the initial state ($\rho_0$) of the battery. If we assume the QB and the charging device, which can be a local magnetic field, are isolated from the environment, the charging-discharging process of QB can be well-described by the unitary dynamics [21, 22, 25], and after a certain time, $t$, total work-gained by the QB can be defined as

$$W(t) = \text{Tr}(H_B\rho_t) - \text{Tr}(H_B\rho_0),$$

where $H_B$ is the Hamiltonian of the QB, and $\rho_t = U\rho_0U^\dagger$ is the time-evolved state of the battery with $\rho_0$ being the initial state. Note that extracting energy from the QB in this scenario is the reverse process of the charging one which will not be the case for open quantum dynamics.

In this paper, we mainly focus on the effect of the environment on the generation (and extraction) of energy in the QB. The Hamiltonian of an interacting quantum spin model considered as a battery can be described as

$$H_B = \frac{h}{2} \sum_{j=1}^{N} \sigma_j^z + J \sum_{j=1}^{N} \{(1 + \gamma)\sigma_j^x\sigma_{j+1}^x + (1 - \gamma)\sigma_j^y\sigma_{j+1}^y\},$$

where $\sigma_i^s$, $i = x, y, z$ are the Pauli spin matrices, $N$ is the total number of spins, $h$ and $J$ respectively represent the strength of the magnetic field and nearest-neighbor coupling constant, and $\gamma \geq 0$ is the anisotropy parameter [51]. In the thermodynamic limit, the model undergoes a quantum phase transition at $\lambda \equiv J/h = 1$ — it is in a paramagnetic phase when $|\lambda| \leq 1$ while in an antiferromagnetic phase with $\lambda > 1$ and a ferromagnetic phase with $\lambda < -1$.

During the charging process, the QB is attached to a set of local bosonic reservoirs, which acts as an absorption channel (an effective dynamical map after integrating-out the bosonic degrees of freedom) that is responsible to supply energy to the system, as depicted in Fig. 1. After the completion of the charging process, i.e., when the energy in QB does not change with time, the discharging process starts, when the spins of the QB are attached to a different set of local bosonic reservoirs, which extracts the energy by acting as a dissipation channel (which can again be an effective dynamical map for the spins after integrating-out the bosonic degrees of freedom). Moreover, during these two processes, each spin of QB can be exposed to dephasing noise — (a) phase-flip and (b) bit-flip noise. Therefore, the reduced dynamics of QB can be described by Lindblad-Gorini-Kossakowski-Sudarshan [27, 52, 53] master equation as

$$\frac{d\rho(t)}{dt} = -i[H_B, \rho(t)] + \sum_k \gamma(k) (L_{k,j}\rho(t)L_{k,j}^\dagger - \frac{1}{2} (L_{k,j}^\dagger L_{k,j}, \rho(t))).$$

Figure 1. (Color online.) Schematic model of the quantum battery. An interacting quantum spin model with open boundary condition acts as a battery, while charging and discharging processes take place by using local bosonic reservoir. Moreover, the local noise (Markovian as well as non-Markovian) acts on each spins (grey clouds).
mensionless. There are three kinds of $L_{k,j}$ applied at the $j$-th spin, given by

$$L_{1,j} = \sigma_j^+, \quad L_{2,j} = \sigma_j^-, \quad L_{3,j} = \sigma_j^i, \quad i = z, x,$$

(4)

for absorption (during charging process), dissipation (during discharging process) and dephasing channels respectively, and the corresponding coefficients read as

$$\gamma(1) = \Gamma_{abs}, \quad \gamma(2) = \Gamma_{diss}, \quad \gamma(3) = \Gamma_{dph}^i \quad i = z, x,$$

(5)

where $\sigma^+ = \frac{\sigma_z+i\sigma_x}{\sqrt{2}}, \quad \sigma^- = \frac{\sigma_z-i\sigma_x}{\sqrt{2}}$ and $\Gamma_{abs}, \Gamma_{diss}$ and $\Gamma_{dph}^i, i = x, z$ denote respectively the rate of absorption when QB consumes energy from the charger, the rate of dissipation during discharging, and the strength of the dephasing noise.

We consider two scenarios –

Case 1. Time-independent noise. Each spin of QB interacts with the bosonic reservoirs, and at the same time, with local dephasing noise, either in the $x$ or in the $z$-direction. All the $\gamma(k)$s are time independent. The dynamics of QB in this case is Markovian.

Case 2. Time-dependent noise. $\gamma(k), k = 1, 2$ are considered to be time independent but the strength of the dephasing noise, i.e., $\gamma(3)$ is taken as a time-dependent variable, leading to non-Markovian effects in dynamics which will be discussed in details in the succeeding section.

To quantify the maximal amount of work that one can gain (extract) from the battery at any time instant $t$, we calculate ergotropy, denoted by $Erg(t)$, which is defined as

$$Erg(t) = E_B(t) - \min_{U_B} \text{Tr}[H_B U_B \rho(t) U_B^d].$$

Here $\rho(t)$ and $E_B(t)$ is the time evolved state of the system and energy of the battery at time instant $t$ respectively. $U_B$’s are the unitaries over which minimization occurs to get the extractable work from the system [11]. It can be computed by using spectral decompositions of $\rho(t)$ and $H_B$ [28].

### III. NOISE-INDUCED ENHANCEMENT IN A QUANTUM BATTERY BASED ON A TRANSVERSE ISING CHAIN WITHOUT MEMORY

We will now show that Markovian noise can help to give certain improvement in the charging-discharging duo, when the battery is in the ground state or the thermal state of the transverse Ising model. In Subsec. III A, we will analytically show that QB consists of two spins, both phase-flip and bit-flip noise acting on QB can give increment in total work-output and ergotropy during the initial periods of the charging process compared to the scenario where the noise is absent. We will then report the improvement of noisy QB composed of the transverse Ising model consisting of higher number of spins even in presence of Markovian noise in Subsec. III B.

#### A. Noise-assisted battery constructed via transverse Ising chain of two spins: Markovian regime

Let us suppose that quantum battery consists of two spin-1/2 particles, described by the Hamiltonian in Eq. (2) with unit anisotropy, $\gamma = 1$, i.e., the transverse Ising model. The battery is initially prepared as the ground state of the model. Let us first concentrate on the QB when phase-flip channel acts on each spin of the battery independently. In this case, we obtain the following result:

**Proposition 1.** In the transient regime, the work-gained by the transverse Ising model-based QB with the help of local bosonic reservoirs is higher in presence of local phase-flip noise than the one without the influence of noise.

**Proof.** The master equation in Eq. (3) for the noise in the $z$-direction can explicitly be written as

$$\frac{d\rho}{dt} = -i[H_B, \rho(t)]$$

$$+ \Gamma_{abs}[(\sigma^+ \otimes I)\rho(t)(\sigma^- \otimes I) - \frac{1}{2}((\sigma^- \otimes I)(\sigma^+ \otimes I), \rho(t))$$

$$+ (I \otimes \sigma^+ )\rho(t)(I \otimes \sigma^-) - \frac{1}{2}((I \otimes \sigma^-)(I \otimes \sigma^+), \rho(t))]$$

$$+ \Gamma_{dph}^z[(\sigma_z \otimes I)\rho(t)(\sigma_z \otimes I) + (I \otimes \sigma_z)\rho(t)(I \otimes \sigma_z) - 2\rho(t)],$$

(7)

where the charger and the noise act locally. Let us consider the density matrix of the initial state, $\rho_0$, of the battery as the ground state of the transverse Ising Hamiltonian $H_B$, given by

$$\rho_0 = \begin{pmatrix}
\frac{p^2}{1+p^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{p}{1+p^2} & 0 & 0 & 1
\end{pmatrix},$$

(8)

where $p = \frac{4(h+e_0)}{J}$ and $e_0 = -\sqrt{h^2 + \frac{4J^2}{h}}$ is the ground state energy of $H_B$. Solving Eq. (7) (see Appendix A), we can obtain the time-evolved state of the battery, $\rho_t$. Since the two-body Ising Hamiltonian of the QB in the computational basis takes the form of a “X”-matrix, given by

$$H_B = \begin{pmatrix}
h & 0 & 0 & J \\
0 & 0 & \frac{J}{2} & 0 \\
0 & \frac{J}{2} & 0 & 0 \\
J & 0 & 0 & -h
\end{pmatrix},$$

(9)

a compact form of extracted work in presence of phase-flip noise can be computed analytically. Since we want to calculate the effects of noise on the dynamics of $W(t)$, we denote the quantity as $W(\Gamma_{dph}^z)$, omitting its functional dependence on $t$ from notation. The straightforward calculation leads us to

$$W(\Gamma_{dph}^z) = \frac{e^{-t\Gamma_{abs}}}{4\Gamma_{dph}^z A} \times [4JpB\Gamma_{dph}^z + h(2Jp(B-1) + 4\Gamma_{dph}^z(e^{t\Gamma_{abs}}A - 2))],$$

(10)
with $A = (1 + p^2)$ and $B = e^{-4t\Gamma_{ph}}$. On the other hand, in the noiseless case, i.e., for $\Gamma_{ph} = 0$, the extracted work, $W(\Gamma_{ph}^z = 0)$, is given by

$$W(\Gamma_{ph}^z = 0) = h + \frac{e^{-4t\Gamma_{ph}}[2Jp - 2h(2 + Jpt)]}{2A}.$$  (11)

By subtracting Eq. (11) from Eq. (10), and expanding them around $t = 0$, we get

$$\delta_{\text{adv}} \equiv W(\Gamma_{ph}^z) - W(0) = -\frac{8\Gamma_{ph}(1 + e_0)t}{(1 + p^2)} + O(t^2).$$  (12)

Keeping the first-order term in $t$ (and ignoring the higher order ones), we find that $\delta_{\text{adv}}$ is a positive number (since $e_0$ is a negative number and its absolute value is greater than $J$), thereby implying the advantage in a noisy scenario than the noiseless one in a transient regime. ■

The above Proposition shows that after the evolution starts, noise in the $z$-direction helps to store more energy in the QB of the transverse Ising model than that in the noiseless situation. Instead of phase-flip noise, if both the spins are affected by the bit-flip noise, the noisy situation still remains advantageous with the restriction in the parameter-space of the transverse Ising chain as shown in the Proposition below.

**Proposition 2.** When local bit-flip noise acts on both the spins of QB which is initially in the ground state of a transverse Ising chain, the work-output, in the transient regime, gives higher value in the noisy case than that of the noiseless one, provided the initial state is prepared with $\lambda < 0.89$.

**Proof.** For bit-flip noise, we can rewrite the master equation as in Eq. (7) by replacing $\sigma_z$ by $\sigma_x$ and the strength of the noise by $\Gamma_{ph}^z$ (see Appendix A for details). If we take the ground state as the initial state of $H_B$, given in Eq. (8), and compute the work-output during charging process, we obtain

$$W(\Gamma_{ph}^x) = \frac{1}{4\Gamma_{ph}^z A(\Gamma_{abs} + 2\Gamma_{ph}^z)} \times e^{-t[I_{abs} + 4I_{ph}]A'} - B'e^{2\Gamma_{phi}^z} + e^{4\Gamma_{phi}^z}(C'e^{I_{abs} + D'}),$$  (13)

where $A' = 2Jhp(\Gamma_{abs} + 2\Gamma_{ph}^z)$, $B' = 8h\Gamma_{ph}^z(\Gamma_{abs} + \Gamma_{ph}^z - p^2\Gamma_{ph}^z)$, $C' = 4\Gamma_{abs}h\Gamma_{ph}^z(1 + p^2)$ and $D' = Jhp(\Gamma_{abs} + 2\Gamma_{ph}^z)(2\Gamma_{ph}^z - h)$ while the $W(\Gamma_{ph}^z = 0)$ is same as in Eq. (11). In the transient regime, expansion of the exponential at $t = 0$ gives

$$\delta_{\text{adv}} \equiv W(\Gamma_{ph}^z) - W(0) = 2\Gamma_{ph}^z(1 - p^2) + O(t^2).$$  (14)

As before, we only keep the above series up to the first order in $t$. To get the improvement in energy stored in presence of noise along the $x$-direction, $p^2$ should be less than 1 which holds when $0 < \lambda \lesssim 0.89$. Unlike phase-flip noise, we find here that in the transient regime, the work can be enhanced in presence of noise when the initial state is the ground state of the transverse Ising model with suitably chosen parameters. ■

The above two Propositions also lead to a comparison between the work-outputs when the amount of noise applied to the $x$- and the $z$-direction respectively. Suppose the amount of noise in both the cases are same, i.e $\Gamma_{ph}^x = \Gamma_{ph}^z = \Gamma_{ph}$.

**Proposition 3.** The amount of stored energy in the transient region is more in case of bit-flip noise than that of the scenario with the phase-flip noise, provided the initial state is prepared from the paramagnetic phase of the transverse Ising chain, far away from the critical point.

**Proof.** From Eqs. (10) and (13), we immediately find (up to the first-order approximation of $t$)

$$\delta_{x > z} \equiv W(\Gamma_{ph}^z) - W(\Gamma_{ph}^x) = 2\Gamma_{ph}^z(1 + \lambda p - p^2) + \frac{1}{1 + p^2},$$  (15)

which is positive only when $(1 + \lambda p) > p^2$. It implies that more amount of work is generated in the bit-flip noise than that of the phase-flip ones if $\lambda \lesssim 0.62$. When $\lambda > 0.62$, noise in the $z$ direction turns out to be more beneficial than that of the bit-flip one as we will also see from the numerical simulations with more number of spins in the next section. ■

Let us now perform the similar analysis for ergotropy and find out whether noise really helps to produce more ergotropy, when the initial state of the battery is the ground state of the transverse Ising model. In this case, we notice that the ergotropy shows noise-induced enhancement in the intermediate time-interval, and so the expansion at $t = 0$ performed for work-output is not valid in this situation, thereby making the analytical treatment difficult. However, we find that for the noiseless situation, $Erg(t = 1.5) = 0.398$, while when dephasing noise along the $x$-direction acts on the first spin as well as on both the spins, the numerical values of ergotropy turn out to be 0.417 and 0.435 respectively, showing clear increment over that of the noiseless case for $\lambda = 0.5$. However, when we apply phase-flip noise on a single spin and both the spins, the values of ergotropy become 0.403 and 0.407 respectively, thereby showing that the effects of bit-flip noise on the performance is more pronounced than that of the phase-flip ones in certain parameter values and time-window. Moreover, Fig. 2 depicts such advantage in ergotropy in case of bit-flip noise for the initial period of the dynamics, when the battery is initially prepared as the ground state of the transverse Ising model (see the next subsection for more details).

**B. Battery made up of transverse Ising model with arbitrary spins: Markovian dynamics**

Let us now move to the QB, built up with $N(> 2)$ number of interacting spin-1/2 particles, governed by the transverse Ising Hamiltonian, $H_B$ with $\gamma = 1$. For a better comparison between different scenarios, we normalize the spectrum of the Hamiltonian in the rest of the paper as

$$\langle H_B - e_0 F \rangle / (\epsilon_{\text{max}} - e_0) \rightarrow H_B, \quad (16)$$

where $e_0$ and $\epsilon_{\text{max}}$ are the lowest and highest energy levels of the former Hamiltonian. Due to this normalization,
$W$ is restricted between 0 and 1, fixing the unit of energy as well as of time, irrespective of the system parameters.

In this subsection, our aim is to confirm whether the results obtained in the previous subsection with two spins also hold when the number of spins in the QB designed via transverse Ising model increases. To find the optimal work-gained or ergotropy of the battery, we have to set all the parameters involved in the charging-discharging process in a proper manner. From the set-up of the problem, we find that $W$ ($E_{\text{rg}}$) is a function of $\lambda, \gamma, \Gamma_{\text{abs}}, \Gamma_{\text{dph}}, (i = x, z)$, apart from its dependence on time. To choose parameters suitably, we now carefully analyze the dependence of $E_{\text{rg}}$ on the parameter space. Qualitatively, similar dependence can also be found for $W$.

1. Noise-absorption (dissipation) ratio.

Let us first identify the role of $\Gamma_{\text{dph}}/\Gamma_{\text{abs}}$ in the charging process. By fixing the initial state as the ground state of the transverse Ising chain and by choosing $\lambda = 0.5$ from the paramagnetic phase, we investigate the enhancement of ergotropy due to the introduction of bit-flip noise, $\Delta_{\text{erg-adv}} = E_{\text{rg}}(\Gamma_{\text{dph}}) - E_{\text{rg}}(0)$, with the variation of time and $\Gamma_{\text{dph}}/\Gamma_{\text{abs}}$, as depicted in Fig. 2. The number of spins in the Ising chain is taken to be four and all the spins are affected by local bit-flip noise. As shown in Proposition 2 for the amount of stored energy in the QB at the initial period of time, ergotropy mimics the similar behavior. We also observe that the amount of increment remains sufficiently high for sufficiently long time when $0.2 \lesssim \Gamma_{\text{dph}}/\Gamma_{\text{abs}} \lesssim 0.4$ which can be referred to as the optimal operating region for the QB. When the noise is applied in the $z$-direction, we also notice that with $0.2 \lesssim \Gamma_{\text{dph}}^z/\Gamma_{\text{abs}} \lesssim 0.4$, $\Delta_{\text{erg-adv}} > 0$ although unlike the noise in the $x$-direction, the increment is not prominent in this case. Motivated by these observations, we have chosen the ratio to be 0.3, for both the noisy cases, throughout the paper.

Let us now consider other parameter regimes involved in this problem so that we can identify the exact conditions which lead to the gain of noise in the performance of the QB.

2. Role of noise on charging-discharging.

We first demonstrate the effects of local bit-flip noise on QB. For analyzing this, the ground state of the transverse Ising chain with $N = 4$ and $\lambda = 0.5$ is taken as the initial state of the QB. To systematically probe the situation, we consider the cases, when noise acts on $i$, $i = 1, \ldots, 4$, number of spins (marked as $(i)$, $i = 1, \ldots, 4$ in

Figure 2. (Color online.) Role of noise-absorption ratio. Contour of the advantage in ergotropy, $\Delta_{\text{erg-adv}} = E_{\text{rg}}(\Gamma_{\text{dph}}) - E_{\text{rg}}(0)$, with respect to time, $t$, (abscissa) and $\Gamma_{\text{dph}}/\Gamma_{\text{abs}}$ (ordinate). Different contours represent the fixed value of $\Delta_{\text{erg-adv}}$. Here $N = 4$ and the initial state is in the ground state of the transverse Ising model with $\lambda \equiv J/h = 0.5$. Both the axes are dimensionless.

Figure 3. (Color online.) Variation of $W$ (ordinate) against $t$ (abscissa). The initial state is prepared as the ground state of the transverse Ising chain. Left panel is for the charging process while the right one is for the discharging case. Plots are for different number of spins that are exposed to bit-flip noise. $W_0$ indicates the noiseless case while $W_i$, $i = 1, \ldots, 4$ represents $i$ number of spins effected through depolarizing channels. To show the advantage obtained by noise in the transient regime, we plot up to $t = 4$. Here $\lambda = 0.5$, and $N = 4$. $\Gamma_{\text{dph}}/\Gamma_{\text{abs}} = 0.3$ (left panel) and $\Gamma_{\text{dph}}/\Gamma_{\text{dis}} = 0.3$ (right panel). Inset: Behavior of $W$ with the variation of $t$. $t$ is chosen up to the point where all of them goes to a steady state. Both the axes are dimensionless.

Figure 4. (Color online.) Ergotropy vs. time, $t$. All the other parameters and conditions are same as in Fig. 3. Both the axes are dimensionless.
Figs. 3 and 4). We then compare them with the noiseless scenario which is indicated as \( W_0 \) and \( Erg_0 \) in Figs. 3 and 4 respectively. Let us first consider the initial period of time of the dynamics during charging, i.e., in the transient part of the dynamics. In this regime, we find a crossover time, \( t_c \), up to which we observe (see Fig. 3)

\[
W_4 > W_3 > W_2 > W_1 > W_0,
\]

where subscripts denotes the number of spins in the QB being influenced with noise. In Fig. 4, we notice that almost the similar patterns for ergotropy emerge although in this case, the hierarchy is prominent in the intermediate time interval instead of initial time-period of the evolution. In this case also, there exists \( t < t_c \) up to which presence of noise actually helps to extract higher amount of work from the battery than in the noiseless scenario (see Fig. 4). Both the results for work-output and ergotropy indicate that for storing energy quickly in the battery described by the transverse Ising model, decoherence outperforms over the noiseless case. Exactly opposite orderings among optimal work-gained appear when \( t > t_c \) which include the region where the system reaches its steady state. In particular, in this region, no crossover occurs for Erg between different noisy scenarios. In the entire duration of time, before reaching to the steady state, noise actually helps rapid decay of the extractable work. If we design a device for which quick extraction of energy from the battery is required, dephasing noise turns out to be beneficial as shown in the right panel of Fig. 4. Note, however, that in absence of noise, total extraction is possible only in the asymptotic case due to a very slow rate of dissipation, i.e., the choice of a low value of \( \Gamma_{dis} = 0.5 \).

3. Relation of quantum phases with power of QB.

All the above discussions were restricted to local bit-flip noise. As shown in Proposition 3 for the battery with two spins, there exists a range of \( \lambda \) in the paramagnetic regime, where bit-flip noise in all the spins can produce a higher work-output than that of the phase-flip ones. Let us now see if we increase the number of spins whether the picture remains same or not. We find that this is indeed the case.

To make the comparison more effective, we investigate the trends of \( \delta_{z,z}^{adv} = W(\Gamma_{dph}) - W(\Gamma_{adv}) \) and \( \Delta_{z,z}^{adv} = Erg(\Gamma_{dph}) - Erg(\Gamma_{adv}) \) with time and the coupling constant \( \lambda \) of the QB. Note that since Markovian noise can give advantage only in the transient regime, we are interested only in that period. Fig. 5 also depicts that the dependence of time and \( \lambda \) in \( \delta_{z,z}^{adv} \) is complementary in nature. On the other hand, intermediate time of the evolution is clearly beneficial as far as ergotropy is concerned as depicted in Fig. 6. Let us summarize all the observations for \( \delta_{z,z}^{adv} \) and \( \Delta_{z,z}^{adv} \) in this respect.

**Observation 1.** For small values of \( \lambda \), i.e., when the transverse Ising model is deep into the paramagnetic phase, the bit-flip noise is always better than the phase-flip one with respect to storing of energy.

**Observation 2.** With the increase of \( \lambda \), the time up to which \( \delta_{z,z}^{adv} \) remains positive decreases. For example, with \( \lambda = 0.5 \), it is positive up to \( t \approx 2.01 \) while \( \lambda = 0.9 \), the time reduces to \( \approx 1.0 \).

**Observation 3.** If QB is prepared in the antiferromagnetic phase at \( t = 0 \), the phase-flip noise can give more work-output during charging than the bit-flip one in the transient regime.

However, by considering the figure of merit as ergotropy, we find out the following:

**Observation 1.** Fig. 6 clearly demonstrates that in the
intermediate time of the dynamics, bit flip noise is helpful to obtain high ergotropy in comparison with phase-flip one.

**Observation 2.** The behavior of $\Delta_{<2>}$ depends almost negligibly on $\lambda$ as seen from the almost parallel lines with $\lambda$ in Fig. 6, especially for high values of $t$, i.e., the trends of ergotropy do not crucially depend on the phase of the transverse Ising model. This behavior is in sharp contrast with stored work difference $\delta_{<2>}$ in Fig. 5.

4. **Dependence on anisotropy.**

Let us justify here the reason to choose $\gamma = 1$ i.e., transverse Ising model for presenting all the results. When there is no noise in the system, we can find that the storage of work can increase with the decrease of the anisotropy in the $XY$ model and similarly more extraction of energy is possible for low values of $\gamma$. Such a difference wipes out when the noise acts on the system, i.e., there is almost no difference found in the trends of $W$ for different values of $\gamma$ in presence of noise, thereby confirming that the results reported here also hold for other values of $\gamma$.

5. **Scale invariance.**

We can check how the results scales with the increase of the system-size of the battery. We notice that after normalizing the Hamiltonian, as in Eq. (16), the work-gained (or -extracted) or the ergotropy has no qualitative as well as quantitative change with the increase in the number of spins in the battery. For example, $W$ for $N = 2$ and $N = 8$ are essentially same up to the numerical accuracy of $10^{-4}$. Moreover, a good agreement of the above results for $N = 4$ with the Propositions in Sec. III A also confirms the scale-invariance, thereby indicating the validity of all the results for arbitrary number of spins of the battery.

6. **Finite temperature tolerance of QB under dephasing noise.**

Up to now, we prepare the ground state of the transverse Ising Hamiltonian as the initial state of the battery. From an experimental point of view, it is not possible to reach the absolute zero temperature. Therefore, it is interesting to check whether the noise-induced performance of QB constructed via transverse Ising model persists even in a finite temperature or not. To investigate it, let us take the initial state as the thermal state of the transverse Ising spin chain which can be represented as $\rho_\beta = \exp(-\beta H_B)/Z$, with $\beta = 1/k_B T$, $T$ being the temperature, and $k_B$ the Boltzmann constant. Here $Z = \text{Tr}(\exp(-\beta H_B))$ is the partition function of the system. Keeping all other parameters unchanged, and by applying dephasing channels in all the spins in the $x$-direction, we study the variation of ergotropy with respect to time for different values of inverse temperature $\beta$. In Fig. 7, we show that again at the intermediate time period in the transient regime, noisy states are beneficial in terms of ergotropy, even at finite temperature. Specifically, in this noisy situation, the value of ergotropy increases with the decrease of $\beta$, i.e., with the increase of temperature which establishes the robustness of the quantum battery based on quantum spin chain. Moreover, in that period of time, the ground state as the initial state without noise performs worst in terms of ergotropy production than that of any other thermal states with noise.

7. **Entanglement in dynamics.**

We compute the nearest neighbor entanglement, quantified by logarithmic negativity [54], between spin 1 and 2 after tracing out the rest of the spins. We find that although the initial state of the transverse Ising model acted as QB is entangled [57, 58], the entanglement of the evolved state decreases with the increases of time and finally vanishes after a short period of time, both in the noiseless and in the noisy scenarios. Moreover, in the transient regime, entanglement of the noisy state is less than that of the noiseless case. Hence in this model, generation of entanglement in the charging process is not important to achieve better performances of QB (cf. [34]).

IV. **Non-Markovian Noise Leads to Better Efficiency in Quantum Battery**

In the preceding section, we restrict to the time-independent strength of the Lindblad operators that leads to the Markovian dynamics of the reduced battery-system. If this condition is relaxed, and we now want to
see the memory effect in the charging-discharging cycle of the battery, we consider the evolution of the battery to be non-Markovian which was shown to increase several resources like entanglement, coherence, of quantum states, thereby ensuring the possibility to increase the efficiency in quantum information processing tasks [39–50]. These results motivate us to study the dynamics of QB in this context.

1. The QB model in non-Markovian domain

The system still remains the quantum XY model with a transverse magnetic field. Again each spin of the battery is connected to bosonic reservoirs for storing and for extracting energy. Each spin of the battery is further attached to the dephasing channel which can be either in the x- or in the z-direction. Unlike Markovian channel, we assume that the strength of the dephasing channel, \( \gamma(3) \), is time-dependent, i.e., \( \Gamma_{dph}(t) = \Gamma_{dph}(t) \) can be characterized by the Ohmic spectral density \( G(\omega) \) [47], given by

\[
G(\omega) = \frac{\omega^s}{\omega_c^s} \exp\left(-\frac{\omega}{\omega_c}\right),
\]

where \( \omega \) and \( \omega_c \) being respectively the frequency and the cut-off frequency of the reservoir and \( s \) is the Ohmicity parameter. The noise strength reads as

\[
\Gamma_{dph}(t, s) = (1 + (\omega_c t)^2)^{-\frac{1}{2}} \Gamma(s) \sin[s \tan^{-1}(\omega_c t)],
\]

where \( \Gamma(s) \) is the Euler gamma function. It can be checked that when \( s > 2 \), the above mentioned dephasing rate can become temporarily negative and memory effect assists the back-flow of information from the environment to the QB, thereby representing a non-Markovian domain, while \( s \leq 2 \) is the Markovian regime [47]. Importantly, such an environment was shown to be realized by ultracold atoms [59].

A. Non-Markovian dynamics always enhances the performance of QB

1. Charging-discharging process.

Like Markovian case, the system is initially prepared in the ground of the transverse Ising model, which is attached through a bosonic reservoir with \( \Gamma_{abs} = 0.5 \). In case of decoherence, we again compare five situations – when there is no noise in the system, and when a single spin, or two, or three, or all the spins are affected by bit-flip noise. With a sharp contrast to the Markovian noise, we report here that in the charging process, the amount of work stored increases with the increase of noise in the system, both in the transient as well as the steady-state regime and interestingly, the same feature persists in case of ergotropy as depicted in Figs. 8 and 9. Specifically, we find that for extractable work, there is no existence of crossover time, as obtained in the Markovian scenario and in the entire evolution-time, \( t \), we have

\[
W(4) > W(3) > W(2) > W(1) > W(0), \quad \forall t
\]

where the notations are the same as in Eq. (17) as depicted in the left panels of Fig. 8 while the similar hierarchy is also observed for ergotropy after an initial time interval as shown in Fig. 9. It clearly shows that the non-Markovian noise in all the four spins can lead to higher storing capacity of energy (ergotropy) in QB of transverse Ising model compared to the low noise and noiseless cases, thereby showing the usefulness to build noise-induced QB.

The discharging process also shows a remarkable improvement in presence of non-Markovian noise – extracted amount of energy and ergotropy increases with the increase of noise as seen in the right panels of Figs. 8 and 9 with \( \Gamma_{dis} = 0.5 \). Moreover, strong effect of back-flow of information can be perceived from the patterns of work-gained and extracted (Fig. 8). Unlike Markovian regime, complete extraction of energy from the battery is possible when all the spins are under local bit-flip noise.
2. **Markovian vs. non-Markovian dynamics.**

We have already seen that there is a clear difference between Markovian and non-Markovian noise. To perform the comparison more concrete, we now investigate the patterns of work-output as well as ergotropy by changing the Ohmicity parameter, $s$, from 0.5 to 4, i.e., by sweeping the system from Markovian to non-Markovian domain, when all the spins which are initially prepared as the ground state of the transverse Ising chain are affected by local noise. Firstly, in the charging scenario, $E_{rg}$ as well as $W$ increases faster with the increase of $s$ and they reach their maxima with $s = 4$. Quantitative analysis reveals that the difference between $E_{rg}$ with $s = 0.5$ and $s = 4.0$ is of the order of $\approx 0.537$ when both of them saturates (see Fig. 10). The second observation is the nonmonotonic nature of $E_{rg}$ ($W$) in the non-Markovian case which is absent in the Markovian domain (Fig. 10).

In case of discharging, noise induces the fast and high amount of extraction of energy as well as ergotropy from the battery. We make the other observations below by fixing the Ohmicity parameter to be 4.

3. **Dephasing noise vs. interaction strength.**

To study the role of the direction of noise on the work-stored and -extracted, we plot $\delta_{x > z} = W(\Gamma_{x z}^{dph}) - W(\Gamma_{x z}^{\lambda})$ in Fig. 11 with the variation of time and $\lambda$ (cf. Fig. 5) during charging of the battery. Comparing Fig. 5 with Fig. 11, we find a difference between Markovian and non-Markovian cases due to the nonmonotonic nature of work-output in the later. Surely, if the initial state belongs to the paramagnetic regime, unlike the Markovian case, the advantage of local bit-flip noise over the phase-flip one, quantified by $\delta_{x > z}^{dph}$, can be of the order of $\approx 0.05 - 0.3$. On the other hand, in Fig. 12, we observe a striking difference in ergotropy compared to that obtained in Fig. 6 for Markovian noise. In the non-Markovian scenario, we report $\Delta_{x > z}^{adv} \geq 0$ after the transient regime, i.e., the bit-flip noise is always better than the phase-flip one except for small values of $t$. Specifically, a significant amount of advantage is observed for intermediate time, $t \approx (2, 3)$ and for small values of $\lambda$.

Like the Markovian case, we also observe that system-size does not play a role (after normalizing the Hamiltonian) in the storing-extracting process of QB in non-Markovian regime, thereby again confirming the scale-invariance feature. Moreover, we notice that bipartite entanglement between spin 1 and 2 in presence of non-Markovian noise collapses, then again revives during charging as well as discharging and the amount of entanglement increases with the increase of Ohmicity parameter. However, the pattern of entanglement cannot explain the behavior of work-output as well as ergotropy. In particular, when the bipartite state is separable, we can still obtain stored energy or extracted energy (maximal extractable work) to be greater in case of non-Markovian noise than the noiseless and Markovian ones.
V. DISCUSSION

Preparing an isolated system is essentially impossible in laboratories and hence decoherence is one of the main obstructions which reduce the performance of the devices. The main reason behind such observations is that all the quantum properties like entanglement, coherence responsible for quantum advantage are fragile in the presence of noise. Therefore, designing quantum machinery which is robust against noise has immense importance in quantum technologies. We address this question in the context of a storage device, especially for the quantum battery.

We considered the ground or the canonical equilibrium state of an interacting spin-1/2 chain, specifically a quantum XY model with a transverse magnetic field as the initial state of the quantum battery (QB) which is exposed to an environment. For storing energy, each spin-1/2 particle of the battery is connected to bosonic reservoirs which act as a charger. During the extraction of energy from the battery, another set of bosonic reservoirs are attached to the system, which helps to dissipate the energy. Moreover, some or all the spins are exposed to a local noisy environment which is modelled as bit- and phase-flip Markovian as well as non-Markovian noise. Our aim was to find the consequence of noise on the performance of the battery.

In general, we know that decoherence decreases the physical property of the system, thereby reducing the efficiency of the devices. We usually see some exceptions when the noise is of non-Markovian kind due to the memory effects. We reported here that even in presence of Markovian, phase- and bit-flip noise, storing of energy in the battery as well as maximum work-gained quantified via ergotropy can be enhanced in the transient regime while during the extraction of work from the battery, the process can be made faster in presence of noise. The exact condition for obtaining such improvement in case of work-output is analytically found when QB consists of two spins and was argued to be true for ergotropy. We then showed that such increment persists even with more number of spins in the battery. We found that any initial canonical equilibrium state under decoherence can outperform over the noiseless scenario in the transient regime, although the ground state of the transverse Ising model as QB gives the maximum advantage in the stead state. From numerical simulations, we identified the proper range in the parameter-space where such noise-induced work and ergotropy from the QB can be obtained. We observed that the entire charging-discharging process can be upgraded in presence of non-Markovian Ohmic noise for the entire time period. Therefore, both in the transient and the steady-state regime, non-Markovian noise can perform better than the Markovian as well as noiseless cases. Since both the noise models considered here can be realized in cold-atomic systems, we believe that such improvements in quantum battery can be achieved in the laboratory.

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Appendix A: Master equation: Dephasing Noise acted on two spins of QB

Let us first consider the case when phase-flip noise acts on both the spins independently. Starting with the initial state as the ground state in Eq. (8), we have to solve Eq. (7) to get the time-evolved two-qubit state, \( \rho_t \) at any time instant \( t \). To obtain the solution, we work with the following coupled linear differential equations, given by

\[
\begin{align*}
\dot{\rho}_{11} &= \Gamma_{abs}(\rho_{22} + \rho_{33}) - \frac{J}{2} \rho_{14}, \\
\dot{\rho}_{22} &= \Gamma_{abs}(-\rho_{22} + \rho_{44}) + \frac{J}{2} \rho_{23}, \\
\dot{\rho}_{33} &= \Gamma_{abs}(-\rho_{33} + \rho_{44}) + \frac{J}{2} \rho_{23}, \\
\dot{\rho}_{44} &= -2\Gamma_{abs}\rho_{44} + \frac{J}{2} \rho_{14}. \\
\end{align*}
\]

(A1)

\[
\begin{align*}
\dot{\rho}_{12} &= -2\Gamma_{dph}\rho_{12} + \Gamma_{abs}(-\frac{\rho_{12}}{2} + \rho_{34}) + \text{Im}, \\
\dot{\rho}_{13} &= -2\Gamma_{dph}\rho_{13} + \Gamma_{abs}(-\frac{\rho_{13}}{2} + \rho_{24}) + \text{Im}, \\
\dot{\rho}_{14} &= (-\Gamma_{abs} - 4\Gamma_{dph})\rho_{14} + \text{Im}, \\
\dot{\rho}_{23} &= (-\Gamma_{abs} - 4\Gamma_{dph})\rho_{23} + \text{Im}, \\
\dot{\rho}_{24} &= (-3\Gamma_{abs} - 2\Gamma_{dph})\rho_{24} + \text{Im}, \\
\dot{\rho}_{34} &= (-3\Gamma_{abs} - 2\Gamma_{dph})\rho_{34} + \text{Im}. \\
\end{align*}
\]

(A2)

Here \( \rho_{ij} \), \( i, j = 1 \ldots 4 \) are the matrix elements of the \( 4 \times 4 \) matrix of \( \rho_t \) and Im is the imaginary part of the matrix elements. Interestingly, after solving differential equations, all imaginary parts vanish. Multiplying \( \rho_t \) with \( H_B \) in Eq. (9), only a few nonvanishing terms survive and hence we can obtain \( W(\Gamma_{dph}) \).

When dephasing noise in the the \( x \)-direction acts individually on each spins, the master equation can be written as

\[
\frac{d\rho}{dt} = -i[H_B, \rho(t)] + \Gamma_{abs}[(\sigma^+ \otimes I)\rho(t)(\sigma^- \otimes I) - \frac{1}{2}(\sigma^- \otimes I)(\sigma^+ \otimes I), \rho(t)] + (I \otimes \sigma^+)(\rho(t)(I \otimes \sigma^-) - \frac{1}{2}(I \otimes \sigma^-)(I \otimes \sigma^+), \rho(t))] + \Gamma_{dph}[(\sigma_x \otimes I)\rho(t)(\sigma_x \otimes I) + (I \otimes \sigma_x)\rho(t)(I \otimes \sigma_x) - 2\rho(t)].
\]

(A3)

Taking ground state as the initial state, the following coupled differential equations can be obtained for diagonal
matrix elements,

\[
\rho_{11} = \Gamma_{\text{abs}}(\rho_{22} + \rho_{33}) + \Gamma_{\text{deph}}^x(-2\rho_{11} + \rho_{22} + \rho_{33}) - \frac{J}{2} \rho_{14}, \\
\rho_{22} = \Gamma_{\text{abs}}(-\rho_{22} + \rho_{44}) + \Gamma_{\text{deph}}^x(\rho_{11} - 2\rho_{22} + \rho_{44}) - \frac{J}{2} \rho_{23}, \\
\rho_{33} = \Gamma_{\text{abs}}(-\rho_{33} + \rho_{44}) + \Gamma_{\text{deph}}^x(\rho_{11} - 2\rho_{33} + \rho_{44}) + \frac{J}{2} \rho_{23}, \\
\rho_{44} = -2\Gamma_{\text{abs}}\rho_{44} + \Gamma_{\text{deph}}^x(\rho_{22} + \rho_{33} - 2\rho_{44}) + \frac{J}{2} \rho_{14}.
\]  

(A4)

and for off-diagonal ones,

\[
\begin{align*}
\rho_{12} &= \Gamma_{\text{abs}}(\rho_{44} - \frac{\rho_{12}}{2}) + \Gamma_{\text{deph}}^x(\rho_{34} - \rho_{12}) + \text{Im}, \\
\rho_{13} &= \Gamma_{\text{abs}}(\rho_{24} - \frac{\rho_{12}}{2}) + \Gamma_{\text{deph}}^x(\rho_{24} - \rho_{13}) + \text{Im}, \\
\rho_{14} &= -\Gamma_{\text{abs}}\rho_{14} + 2\Gamma_{\text{deph}}^x(-\rho_{14} + \rho_{23}) + \text{Im}, \\
\rho_{23} &= -\Gamma_{\text{abs}}\rho_{23} + 2\Gamma_{\text{deph}}^x(-\rho_{23} + \rho_{14}) + \text{Im}, \\
\rho_{24} &= (-3\Gamma_{\text{abs}}/2 - \Gamma_{\text{deph}}^x)\rho_{24} + \Gamma_{\text{deph}}^x\rho_{13} + \text{Im}, \\
\rho_{34} &= (-3\Gamma_{\text{abs}}/2 - \Gamma_{\text{deph}}^x)\rho_{34} + \Gamma_{\text{deph}}^x\rho_{12} + \text{Im}. \\
\end{align*}
\]  

(A5)

The notations are similar to those used in the phase-flip noise. Again similar simplification leads to \(W(\Gamma_{\text{deph}}^x)\).

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