Classical and quantum duality in jordanian quantizations

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Abstract

The limiting transitions between different types of quantizations are studied by the deformation theory methods. We prove that for the first order coboundary deformation of a Lie bialgebra \((g, g^*_1) \rightarrow (g, g^*_1 + \xi g^*_2)\) one can always get the quantized Lie bialgebra \(\mathcal{A}(g, g^*_2)\) as a limit of the sequence of quantizations of the type \(\mathcal{A}(g, g^*_1)\). The obtained results are illustrated by some low-dimensional examples of quantized Lie algebras and superalgebras.

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1 Introduction

The Drinfeld and Jimbo deformations of universal enveloping of simple Lie algebras [1, 2] correspond to Lie bialgebras with classical $r$-matrix

$$r_{DJ} = \sum_{i=1}^{k} t_{ij} H_i \otimes H_j + \sum_{\alpha \in \Delta_+} E_\alpha \otimes E_{-\alpha},$$

where $k$ is the rank, $t_{ij}$ is the inverse Cartan matrix, and $\Delta_+$ is the set of positive roots. The triangular quantum groups and twistings [4], for example, the jordanian quantization of $\mathfrak{sl}(2)$ or of its Borel subalgebra $B_+$ ($\{h, x | [h, x] = 2x\}$) with $r = h \otimes x - x \otimes h = h \wedge x$ [1], have the triangular $R$-matrix $R = F_{21} F^{-1}$ defined by the twisting element [5, 6]

$$F = \exp\left\{ \frac{1}{2} h \otimes \ln(1 + 2\xi x) \right\}. \quad (1)$$

The extension of this twist [7] implies the existence of a subalgebra $L$ with special properties. This is a solvable subalgebra with at least four generators. All simple Lie algebras except $\mathfrak{sl}(2)$ contain such $L$ and in any of them a deformation induced by twist of $L$ can be performed. In particular for $\mathcal{U}(\mathfrak{sl}(N))$ the following twisting element $F \in \mathcal{U}(\mathfrak{sl}(N)) \otimes 2$ can be applied,

$$F = \exp\left\{ 2\xi \sum_{i=2}^{N-1} E_{1i} \otimes E_{1i} e^{-\sigma} \right\} \exp\{H \otimes \sigma\}, \quad (2)$$

where $x = E_{1N}$, $H = E_{11} - E_{NN}$, $\sigma = \frac{1}{2} \ln(1 + 2\xi x)$.

In this work the cohomological interpretation of the interrelations between the Drinfeld-Jimbo (or standard) quantum algebra $\mathcal{U}_q(\mathfrak{sl}(N))$ and the jordanian (or non-standard) $\mathcal{U}_\xi(\mathfrak{sl}(N))$ are discussed. The existence of such interrelations was already pointed out in [5]. The operator $\exp(\xi \text{ad} E_{1N})$ (with the highest root generator $E_{1N}$) turns $r_{DJ}$ into the sum $r_{DJ} + \xi r_j$. Hence,\n
$$r_j = -\xi \left( H_{1N} \wedge E_{1N} + 2 \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN} \right), \quad (3)$$

is a classical $r$-matrix too.

Here the decomposition of $r$-matrix can be treated as a consequence of a specific similarity transformation. We shall show that the effect is in fact connected with the quite general Lie bialgebra deformation properties.
2 Deformed coboundary Lie bialgebras

Let us consider a coboundary Lie bialgebra \((g, g^\ast)\) and its first order deformation which we shall write in a most general form: \((g_h, g^\ast_\xi)\). We can consider it as a four dimensional variety:

\[
(\alpha_1 g_1 + \alpha_2 g_2, \beta_1 g^*_1 + \beta_2 g^*_2) = (g_{\alpha_1, \alpha_2}, g^*_{\beta_1, \beta_2}).
\]

Being the first order deformation this pair contains in fact four Lie bialgebras:

\[
(g^i, g^*_k)_{i, k = 1, 2},
\]

with compositions \(\mu_i, \mu^*_k\). As a coboundary Lie bialgebra the pair \((g_{\alpha_1, \alpha_2}, g^*_{\beta_1, \beta_2})\) corresponds to the classical \(r\)-matrix satisfying the MCYBE, its symmetric part being \(\text{ad}^{\otimes 2}\)-invariant. Hence \(g^*_{\beta_1, \beta_2}\) defines a Lie coalgebra (on the space \(V_g\) of the Lie algebra \(g_{\alpha_1, \alpha_2}\)),

\[
\delta_{\beta_1, \beta_2}(x) = \alpha_1 \mu_1(x \otimes 1 + 1 \otimes x, r) + \alpha_2 \mu_2(x \otimes 1 + 1 \otimes x, r) = (\beta_1 \delta_1(x)) + \beta_2 \delta_2(x).
\]

When the parameters \(\alpha_i, \beta_k\) are independent no solutions of (4) smoothly depending on parameters can be found.

We shall confine ourselves to the following two types of restrictions on parameters \(\{\alpha_i, \beta_k\}\):

I. \(\alpha_i = \beta_i\),  
II. \(\alpha_2 = 0\).

In the first case the constant \(r\)-matrix can serve as a solution of (4). This situation is characteristic for the quantum double case [10]. The equation (4) factorizes:

\[
\delta_i(x) = \mu_i(x \otimes 1 + 1 \otimes x, r).
\]

Thus for a quantum double of two Borel algebras:

\[
[H, X_\pm] = \pm \alpha_1 X_\pm, \quad \Delta X_+ = X_+ \otimes 1 + e^{-\theta \alpha_2 H} \otimes X_+,
\]

\[
[H', X_\pm] = \pm \alpha_2 X_\pm, \quad \Delta X_- = X_- \otimes e^{\theta \alpha_1 H'} + 1 \otimes X_-,
\]

\[
[X_+, X_-] = \frac{1}{2} \left( e^{\theta \alpha_1 H'} - e^{-\theta \alpha_2 H} \right);
\]

the Lie bialgebra has the form \((\alpha_1 g_1 + \alpha_2 g_2, \alpha_1 g^*_1 + \alpha_2 g^*_2)\), where

\[
g_1 = \left\{ \begin{array}{c}
[H, X_\pm] = \pm X_\pm; \\
[X_+, X_-] = H'; 
\end{array} \right\}, \quad g_2 = \left\{ \begin{array}{c}
[H', X_\pm] = \pm X_\pm; \\
[X_+, X_-] = H; 
\end{array} \right\},
\]

\[
g^*_1 = \left\{ \begin{array}{c}
[H', X_-] = -\theta X_+; \\
\end{array} \right\}, \quad g^*_2 = \left\{ \begin{array}{c}
[H, X_-] = -\theta X_+; \\
\end{array} \right\}.
\]
and \( \{ \hat{H}, \hat{H}', \hat{X}_\pm \} \) is the dual base. One can easily see that in this situation to be a solution of the equation (4) the classical \( r \)-matrix must be constant:

\[
r = \theta (X_+ \otimes X_+ + H \otimes H').
\]

In the second case the equation can be solved only with \( r \)-matrix depending on \( \beta_i \) and, moreover, the latter can be decomposed into the sum of two constituent \( r \)-matrices for \( (g, \beta_1 g_1^* ) \) and \( (g, \beta_2 g_2^* ) \) correspondingly

\[
r = \beta_1 r_1 + \beta_2 r_2.
\]

This latter case of the Lie bialgebras first order deformations appear in the studies of jordanian quantizations. Below we shall treat them in details.

### 3 Drinfeld-Jimbo and jordanian deformations as mutually first order ones

It is well known that some sorts of jordanian deformations can be treated as limiting structures for certain sequences of standard quantizations [5, 6, 8]. These properties are more transparent for quantum groups.

Let the \( N \times N \)-matrix \( T \) formed by the generators of the standard (FRT-deformed) quantum group \( \text{Fun}_h(SL(N)) \) be subject to the similarity transformation with the matrix

\[
M = 1 + \frac{\xi}{q - 1} \rho(E_{1N}) \quad (q = e^h)
\]  

(5)

(the coproduct \( \Delta T = T \otimes T \) is conserved). As far as \( q \neq 1 \) the transformed group \( \text{Fun}_{h; \xi}(SL(N)) \) is equivalent to the original one. Compare the corresponding Lie bialgebras: \( (g, g_{h;0}^*) = (sl(N), (sl(N))^*) \) and \( (g, g_{h;\xi}^*) \). Here \( g = sl(N) \) is fixed and only the second Lie multiplication law \( (\mu_{h;0}^* : V_{g^*} \wedge V_{g^*} \rightarrow V_{g^*}) \) changes:

\[
\mu_{h;0}^* \rightarrow \mu_{h;\xi}^*.
\]

One can check that the new Lie product decomposes as:

\[
\mu_{h;\xi}^* = \mu_{h;0}^* + \xi \mu'.
\]  

(6)
\(\mu'\) is defined by the transformed \(RTT = TTR\) equations. Change the coordinate functions of \(SL(N)\) arranged in \(T\) for the exponential ones \(T = \exp(\epsilon Y)\) and the parameters \(h \mapsto \epsilon h, \xi \mapsto \epsilon \xi\). Tending \(\epsilon\) to zero one gets both summands in (3). In the canonical \(gl(N)\)-basis the second one is:

\[
\begin{align*}
\mu'(Y_{1k}, Y_{ij}) &= 2\delta_{ik}Y_{Nj}, \quad \text{for } k, j < N; \quad i > 1, \\
\mu'(Y_{ij}, Y_{1N}) &= -2\delta_{ij}Y_{Nj}, \quad \text{for } j < N; \quad i, l > 1, \\
\mu'(Y_{ij}, Y_{1N}) &= -\delta_{ij}Y_{i1} - \delta_{iN}Y_{Nj}, \quad \text{for } j < N; \quad i > 1, \\
\mu'(Y_{1i}, Y_{i1N}) &= -Y_{1i}, \quad \text{for } N > i > 1, \\
\mu'(Y_{1N}, Y_{kN}) &= Y_{kN}, \quad \text{for } k < N < 1, \\
\mu'(Y_{11}, Y_{1N}) &= \mu'(Y_{1N}, Y_{NN}) = -(Y_{11} - Y_{NN}), \\
\mu'(Y_{11}, Y_{1k}) &= \delta_{i1}Y_{Nk}, \quad \text{for } k, i < N; \quad k > 1, \\
\mu'(Y_{iN}, Y_{1N}) &= -\delta_{kN}Y_{i1}, \quad \text{for } k, i > 1; \quad i < N, \\
\mu'(Y_{11}, Y_{kN}) &= \delta_{11}Y_{k1} - \delta_{kN}Y_{Ni} - 2\delta_{ik}(Y_{11} - Y_{NN}), \quad \text{for } i < N; \quad k > 1.
\end{align*}
\]

The map \(\mu'\) not only defines the infinitesimal deformation of \(\mu_{h,0}^*\) but is itself a Lie multiplication. The full map \(\mu_{h,\xi}^*\) is a deformation of \(\mu_{h,0}^*\) and \(\mu'\) does not depend on \(h\) or \(\xi\):

\[
\mu_{h,\xi}^* = \mu_{h,0}^* + \mu_{0,\xi}^*.
\]

Thus the composition \(\mu_{h,\xi}^*\) is a Lie multiplication deformed in the first order.

Both summands are the Lie maps and at the same time can be considered as deforming functions of each other:

\[
\mu_{h,0}^* \in Z^2 \left( g_{0,\xi}^*, g_{0,\xi}^* \right), \quad \mu_{0,\xi}^* \in Z^2 \left( g_{h,0}^*, g_{h,0}^* \right).
\]

The equivalence \(Fun_{h,\xi}(SL(N)) \approx Fun_h(SL(N))\) (for \(h \neq 0\)) signifies that \(\mu_{0,\xi}^*\) is in fact a coboundary,

\[
\mu_{0,\xi}^* \in B^2 \left( g_{h,0}^*, g_{h,0}^* \right).
\]

On the contrary, the composition \(\mu_{h,0}^*\) corresponds to a nontrivial cohomology class

\[
\mu_{h,0}^* \in H^2 \left( g_{h,\xi}^*, g_{h,\xi}^* \right),
\]

the deformation of \(\mu_{0,\xi}^*\) by \(\mu_{h,0}^*\) is essential [9].

With respect to the cochain complex \(C^n: \wedge^n V_g \to V_g \wedge V_g\), all the Lie algebras \(g_{h,\xi}^*\) are dual to one and the same \(g = sl(N)\):

\[
\mu_{h,0}^*, \mu_{0,\xi}^* \in B^1 \left( g, g \wedge g \right).
\]
Thus (see Section 1) the classical r-matrix of $U_{h;\xi} (sl(N)) \approx (Fun_{h;\xi}(SL(N)))^*$ must exhibit the decomposition property:

$$r_{h;\xi} = r_{h;0} + r_{0;\xi} = \frac{\hbar}{N} \left( \sum_{k=1}^{N-1} k (N - k) H_{k,k+1} \otimes H_{k,k+1} \right. \nonumber$$

$$+ \sum_{k<l} (N - 1) k \left( H_{k,k+1} \otimes H_{1,l+1} + H_{1,l+1} \otimes H_{k,k+1} \right) \nonumber$$

$$\left. + 2\hbar \sum_{k<l} (E_{lk} \otimes E_{kl}) \right) \nonumber$$

$$- \xi H_{1N} \wedge E_{1N} - 2\xi \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN}. \quad (10)$$

In the limit $\hbar \to 0$ one gets the element

$$\lim_{\hbar \to 0} r_{h;\xi} = r_{0;\xi} = -\xi \left( H_{1N} \wedge E_{1N} + 2 \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN}. \right), \quad (11)$$

that coincides with the r-matrix that can be obtained from $\mathcal{R}$

$$\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1} \nonumber$$

$$= \prod_j \exp \left( 2\xi E_{jN} e^{-\sigma} \otimes E_{1j} \right) \exp \left( \sigma \otimes H_{1N} \right) \exp \left( -H_{1N} \otimes \sigma \right) \cdot \prod_j \exp \left( -2\xi E_{1j} \otimes E_{jN} e^{-\sigma} \right). \quad (12)$$

of the twisted algebra $\mathcal{U}_\mathcal{F}(sl(N))$ [9].

To clarify the contraction properties of $Fun_{h;\xi}(SL(N))$ let us consider the 1-parameter subvariety $\{ g_{h;1-h}^* \}$ of Lie algebras $g_{h;\xi}^*$ (putting $\xi = 1 - h$ in [8]). Each dual pair $(sl(N), g_{h;1-h}^*)$ is a Lie bialgebra and thus is quantizable [3]. The result is the set $\mathcal{A}_{s,h}$. This set is smooth (in the formal series topology). The 1-dimensional boundaries $\mathcal{A}_{0,h}$ and $\mathcal{A}_{s;0}$ of $\mathcal{A}_{s,h}$ are the quantizations of $(sl(N), g_{0;1}^*)$ and of $(sl(N), g_{0;0}^*)$ respectively. Each internal point in $\mathcal{A}_{s,h}$ can be connected with a boundary by a smooth parametric curve $a(u)$ starting in $\mathcal{A}_{0,h}$ and ending in $\mathcal{A}_{s;0}$. This does not necessarily mean that this limit is a faithful contraction – it may not be in orbit. This is just what happens in our case. For every positive $h$ fixed the subset $\{ Fun_{h;\xi}(SL(N)) \}$ is in the $GL(N^2)$ -orbit of the corresponding $Fun_{h,0}(SL(N))$. To attain the points $Fun_{0;\xi}(SL(N))$ one must tend $h$ to zero by crossing the set of orbits.

One of the principle conclusions is that the possibility to obtain the jordanian deformation $Fun_{0;\xi}(SL(N))$ as a limiting transformation of the standard quantum group – $Fun_{h,0}(SL(N))$ (and on the dual list to get the twisted q-algebra $\mathcal{U}_\mathcal{F}(sl(N))$ as a limit of the variety of standard quantized algebras $\mathcal{U}_q(sl(N))$) is provided by the fact that the 1-cocycle $\mu_{0;\xi}^* \in$
\[ Z^1(sl(N), sl(N) \wedge sl(N)) \] (that characterizes the Lie bialgebra for \( U(sl(N)) \)) is at the same time the 2-coboundary \( \mu_{0,\xi}^* \in B^2(g_{h,0}^*, g_{h,0}^*) \) the Lie algebra \( g_{h,0}^* \) being the standard dual of \( sl(N) \).

We want to stress that these interrelations of Lie maps are not specific only to the factorizable twists such as (2). Consider the \( R \)-matrix depending on two parameters \( (h, \xi) \) proposed in [11] for the superalgebra \( U(osp(1|2)) \). It leads to a smooth sequence of deformations whose limit (for \( h \to 0 \)) was proved to be a twist. The twisting element consists of two factors \( \mathcal{F}(s_j) = \mathcal{F}(s) \mathcal{F} \), where the first operator \( \mathcal{F} \) is the jordanian twist (1) while the second is defined as

\[
\mathcal{F}(s) = \exp(-2\xi(v_+ \otimes v_+)\phi(\sigma \otimes 1, 1 \otimes \sigma))
\]

where all the terms of the expansion for the symmetric function \( \phi \) can be written down explicitly [11]. This provides the possibility to extract the corresponding Lie bialgebra and find that it reveals the decomposition property (9): \( g_{osp(1|2)}, hg_1^* + \xi g_2^* \). For the defining relations of \( U(osp(1|2)) \) as

\[
[h, v_{\pm}] = \pm v_{\pm}, \quad \{v_+, v_-\} = -h/4, \quad X_{\pm} = \pm 4v_{\pm}v_{\pm}
\]

the corresponding Lie maps (in terms of dual basis) are

\[
\begin{align*}
\mu_1^*(\hat{h}, \hat{X}_{\pm}) &= -2\hat{X}_{\pm}, & \mu_2^*(\hat{X}_{\pm}, \hat{h}) &= 2\hat{h}, \\
\mu_1^*(\hat{h}, \hat{v}_{\pm}) &= -\hat{v}_{\pm}, & \mu_2^*(\hat{X}_{\pm}, \hat{X}_{\pm}) &= 2\hat{X}_{\pm}, \\
\mu_1^*(\hat{v}_{\pm}, \hat{v}_{\pm}) &= 4\hat{X}_{\pm}, & \mu_2^*(\hat{v}_{\pm}, \hat{v}_{\pm}) &= \hat{v}_{\pm},
\end{align*}
\]

Both maps, \( \mu_1^* \) and \( \mu_2^* \), are 2-cocycles of each other and the second of them is a 2-coboundary \( \mu_2^* \in B^2(g_1^*, g_1^*) \):

\[
\mu_2^* = \delta \psi,
\]

the 1-cochain \( \psi \) on the basic elements looks like

\[
\psi : (\hat{h}, \hat{X}_{\pm}, \hat{v}_{\pm}, \hat{v}_{\pm}) \to (-\hat{X}_{\pm}, -\hat{h}, -\hat{X}_{\pm}, \hat{v}_{\pm}, \hat{v}_{\pm}).
\]

The cohomological properties of the involved Lie bialgebras permit to explain the connection of the Drinfeld-Jimbo (standard) quantization with
the twisting. This is a special case of the general dependence: Having the first order coboundary deformation of a Lie bialgebra \((g, g_1^*)\)

\[(g, g_1^*) \rightarrow (g, g_1^* + \xi g_2^*)\]

with \(\mu_2^* \in B^2(\mu_1^*, \mu_1^*)\) one can always get the quantized Lie bialgebra \(A(g, g_2^*)\) as a limit of the sequence of quantizations of the type \(A(g, g_1^*)\).

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