Random Walk Fundamental Tensor and Its Applications to Network Analysis
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Abstract
We first present a comprehensive review of various random walk metrics used in the literature and express them in a consistent framework. We then introduce fundamental tensor – a generalization of the well-known fundamental matrix – and show that classical random walk metrics can be derived from it in a unified manner. We provide a collection of useful relations for random walk metrics that are useful and insightful for network studies. To demonstrate the usefulness and efficacy of the proposed fundamental tensor in network analysis, we present four important applications: 1) unification of network centrality measures, 2) characterization of (generalized) network articulation points, 3) identification of network most influential nodes, and 4) fast computation of network reachability after failures.

1. Introduction

Random walk metrics are powerful tools for network analysis and have been used extensively in many applications domains ranging from communications and computer networks to social network analysis and bioinformatics. Newman introduces a random walk centrality measure [1] to capture the importance of nodes in a network. Chen et al. [2] present a clustering algorithm via (expected) hitting time on directed graphs. Hitting time and commute time have also been employed in studying protein interactions in biological systems [3] and also applied in recommendation systems [4]. Despite their importance and wide usage, few attempts have been made to place various random walk metrics defined and used in the literature in a consistent framework and study the connections between them to provide a unified framework for computing these metrics. In addition, while certain classical random walk metrics such as hitting and commute times have at-
tracted enormous attention\cite{5, 6, 7, 2}, other random walk metrics have been largely ignored — few studies have been conducted to reveal their usefulness or effectiveness for network analysis.

In this paper, we first provide a comprehensive review of various random walk metrics used in the literature and express them in a consistent framework using several different forms. We then introduce fundamental tensor — a generalization of the well-known fundamental matrix — and show that it provides a basis for computing other random walk metrics in a unified manner. We have also reviewed, gathered, and derived many insightful relations between various random walk metrics which can be useful for network studies. To demonstrate the usefulness and efficacy of the proposed fundamental tensor in network analysis, we consider four important and represent applications as case studies. More specifically, we show that: 1) the fundamental tensor provides a unified framework for random walk distance, centrality measures, and topological index as the aggregation over tensor dimensions; 2) the (normalized) fundamental tensor can be exploited to find the cut vertices and articulation points of a network and devise a metric to measure the load balancing in the networks; 3) the (normalized) fundamental tensor can be used in the field of social networks to infer the cascade and spread of a phenomena or an influence in a network and derive a formulation to find the most influential nodes for maximizing the influence spread over the network; and 4) the fundamental tensor yields an efficient and fast method for finding the readabilities in a network, especially after some failures.

The remainder of this paper is organized as follows. A preliminary on network terminology is presented in Section 2. In Section 3 we review and present various random walk metrics in a unified format. In Section 4 we gather and derive useful relations among various random walk metrics. Finally, four applications are presented in Section 5 to demonstrate the usefulness and efficacy of the fundamental tensor in network analysis.

2. Preliminaries

In general, a network can be abstractly modeled as a weighted and directed graph, denoted by $G = (\mathcal{V}, \mathcal{E}, W)$. Here $\mathcal{V}$ is the set of nodes in the network such as routers or switches in a communication network or users in a social network, and its size is assumed to be $n$ throughout the paper $|\mathcal{V}| = n$; $\mathcal{E}$ is the set of (directed) edges representing the (physical or logical) connections between nodes (e.g., a communication link from a node $i$ to a node $j$) or entity
relations (e.g., follower-followee relation between two users). The affinity (or adjacency) matrix \( A = [a_{ij}] \) is assumed to be nonnegative, i.e., \( a_{ij} \geq 0 \), where \( a_{ij} > 0 \) if and only if edge \( e_{ij} \) exists, \( e_{ij} \in E \). Network \( G \) is called strongly connected if all nodes can be reachable from each other via at least one path. In this paper, we focus on strongly connected networks, unless stated otherwise.

A random walk in \( G \) is modeled by a discrete time Markov chain, where the nodes of \( G \) represent the states of the Markov chain. The Markov chain is fully described by its transition probability matrix: \( P = D^{-1}A \), where \( D \) is the diagonal matrix of (out-)degrees, i.e., \( D = \text{diag}[d_i] \) and \( d_i = \sum_j a_{ij} \) (\( d_i \) is often referred to as the (out-)degree of node \( i \)). Throughout the paper, the words “node” and “state”, “network” and “Markov chain” are often used interchangeably depending on the context. If the network \( G \) is strongly connected, the associated Markov chain is irreducible and the stationary probabilities \( \pi \) are strictly positive according to Perron-Frobenius theorem [8]. For an undirected and connected \( G \), the associated Markov chain is reversible and the stationary probabilities are a scalar multiple of node degrees: \( \pi_i = \frac{d_i}{\sum_i d_i} \).

3. Definitions of Random Walk Metrics

We first present and review various random walk metrics in terms of the classical fundamental metrics using three unified forms: 1) matrix form, 2) recursive form, and 3) stochastic form. The matrix form is often the preferred form in this paper and we show how two other forms can be obtained from the matrix form. The stochastic form, however, provides a more intuitive definition of random walk metrics. We then introduce fundamental tensor as a generalization of the fundamental matrix.

3.1. Fundamental Matrix

The expected number of visits is a random walk metric which counts the expected number of visits at a node, when a random walk starts from a source node and before a stopping criterion. The stopping criterion in random walk metrics is often “visiting a target node for the first time” which is referred to as hitting the target node. With respect to a target node, the fundamental matrix \( F \) is formed, where the entries are the expected number of visits at a medial node starting from a source node, for all such pairs. The fundamental matrix is defined formally below using the three different forms.
Matrix form \([9, 10]\): Let \(P\) be an \(n \times n\) transition probability matrix for a strongly connected network \(G\) and node \(t\) be the target node. If the nodes are arranged in a way to assign the last index to the target node, transition probability matrix can be written in the form of

\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{nn}
\end{bmatrix}
\]

and the fundamental matrix is defined as follows:

\[
F = (I - P_{11})^{-1},
\]

where entry \(F_{sm}\) represents the expected number of visits of medial node \(m\), starting from source node \(s\), and before hitting (or absorption by) target node \(t\) \([10]\). Note that it is more precise to denote the fundamental matrix by \(F^{(t)}\) to clarify that it is computed for target node \(t\). This is discussed more in random walk metrics generalization to a set of targets \((3.6)\).

Expanding \(F_{sm}\) as a geometric series, namely, \(F_{sm} = [(I - P_{11})^{-1}]_{sm} = [I]_{sm} + [P_{11}]_{sm} + [P^{2}_{11}]_{sm} + ...\), it is easy to see the probabilistic interpretation of the expected number of visits as a summation over the number of steps required to visit node \(m\).

Recursive form: Each entry of the fundamental matrix, \(F_{sm}\), can be recursively computed in terms of the entries of \(s\)'s outgoing neighbors. Note that if \(s = m\), \(F_{sm}\) is increased by 1 to account for \(X_{0} = m\) (the random walk starts at \(s = m\), thus counting as the first visit at \(m\)).

\[
F_{sm} = 1_{\{s=m\}} + \sum_{j \in N_{\text{out}}(s)} \ p_{sj} F_{jm}
\]

It is easy to see the direct connection between the recursive form and the matrix form: from \(F = I + P_{11}F\), we have \(F = (I - P_{11})^{-1}\).

Stochastic form \([11]\): Let \(G = (X_{k})_{k>0}\) be a discrete-time Markov chain with the transition probability matrix \(P\), where \(X_{k}\) is the state of Markov chain in time step \(k\). The indicator function \(1_{\{X_{k} = m\}}\) is a Bernoulli random variable, equal to 1 if the state of Markov chain is \(m\) at time \(k\), i.e. \(X_{k} = m\), and 0 otherwise. The number of visits of node \(m\), denoted by \(\nu_{m}\), can be written in terms of the indicator function:

\[
\nu_{m} = \sum_{k=0}^{\infty} 1_{\{X_{k} = m\}}.
\]

The stopping criteria is hitting node \(t\) for the first time. In an irreducible chain; this event is guaranteed to occur in
a finite time. Hence \( k < \infty \). \( F_{sm} \) is defined as the expected value of \( \nu_m \) starting from \( s \).

\[
F_{sm} = \mathbb{E}_s(\nu_m) = \mathbb{E}_s \sum_{k=0}^{\infty} 1_{\{X_k=m\}} = \sum_{k=0}^{\infty} \mathbb{E}_s(1_{\{X_k=m\}})
\]

\[
= \sum_{k=0}^{\infty} \mathbb{P}(X_k = m | X_0 = s, X_{<k} \neq t) = \sum_{k=0}^{\infty} [P^k_{11}]_{sm}, \tag{3}
\]

where the expression is simply the expanded version of the matrix form. Note that in order for \( F_{sm} \) to be finite (namely, the infinite summation converges), it is sufficient that node \( t \) be reachable from all other nodes in network. In other words, the irreducibility of the entire network is not necessary.

### 3.2. Fundamental Tensor

We introduce and define the fundamental tensor, \( F \), a generalization of the fundamental matrix \( F^{(t)} \) by stacking up the fundamental matrices constructed for each node \( t \) as the target node in a strongly connected network. In other words, the fundamental tensor exists for any triplets of nodes \((s, m, t)\) in a strongly connected network. The \( t \)-th cross section of the fundamental tensor is computed from the corresponding fundamental matrix \( F^{(t)} \):

\[
F_{smt} = \begin{cases} 
F^{(t)}_{sm} & \text{if } s, m \neq t \\
0 & \text{if } s = t \text{ or } m = t 
\end{cases} \tag{4}
\]

### 3.3. Hitting Time

The (expected) hitting time metric, also known as the first transit time, first passage time, and expected absorption time in the literature, counts the expected number of steps (or time) required to hit a target node for the first time when the random walk starts from a source node. Hitting time is frequently used in the literature as a form of (random walk) distance metric for network analysis. We formally represent it in the three different forms below.

- **Matrix form [10]**: Hitting time can be computed from the fundamental matrix [1] as follows:

\[
h^{(t)} = F^{(t)} 1, \tag{5}
\]
where \( \mathbf{1} \) is a vector of all ones and \( \mathbf{h}^{(t)} \) is a vector of \( H_s^{(t)} \) computed for all \( s \in V \setminus \{ t \} \). \( H_s^{(t)} \) represents the expected number of steps required to hit node \( t \) starting from \( s \) and is obtained from: 
\[
H_s^{(t)} = \sum_m F_{sm}^{(t)}.
\]
The intuition behind this formulation is that enumerating the average number of nodes visited on the way from the source node to the target node yields the number of steps (distance) required to reach to the target node.

- **Recursive form** [12] [11] [4]: The recursive form of \( H_s^{(t)} \) is the most well-known relation presented in the literature for deriving the hitting time:
\[
H_s^{(t)} = 1 + \sum_{m \in N_{out}(s)} p_{sm} H_m^{(t)}
\]
(6)
It is easy to see the direct connection between the recursive form and the matrix form: from \( \mathbf{h} = 1 + \mathbf{P}_{11} \mathbf{h} \), we have \( \mathbf{h} = (I - \mathbf{P}_{11})^{-1} \mathbf{1} \).

- **Stochastic form** [11]: Let \( G = (X_k)_{k>0} \) be a discrete-time Markov chain with the transition probability matrix \( \mathbf{P} \). The hitting time of the target node \( t \) is denoted by a random variable \( \kappa_t : \Omega \to \{ 0, 1, 2, \ldots \} \cup \{ \infty \} \) given by \( \kappa_t = \inf \{ \kappa \geq 0 : X_\kappa = t \} \), where by convention the infimum of the empty set \( \emptyset \) is \( \infty \). Assuming that the target node \( t \) is reachable from all the other nodes in the network, we have \( \kappa_t < \infty \). The (expected) hitting time from \( s \) to \( t \) is then given by
\[
H_s^{(t)} = \mathbb{E}_s[\kappa_t] = \sum_{k=1}^{\infty} k \mathbb{P}(\kappa_t = k | X_0 = s) + \infty \mathbb{P}(\kappa_t = \infty | X_0 = s)
\]
\[
= \sum_{k=1}^{\infty} k \mathbb{P}(X_k = t | X_0 = s, X_{<k} \neq t)
\]
\[
= \sum_{k=1}^{\infty} k \sum_{m \neq t} \mathbb{P}(X_{k-1} = m | X_0 = s, X_{<k-1} \neq t) \mathbb{P}(X_k = t | X_{k-1} = m)
\]
\[
= \sum_{k=1}^{\infty} k \sum_{m \neq t} [P_{11}^{k-1}]_{sm} [p_{12}]_m,
\]
(7)
where \( [P_{11}^0]_{sm} = 1 \) for \( m = s \) and it is 0 otherwise. The connection
between the stochastic form and the matrix form can be found in the appendix.

3.3.1. Commute Time

The commute time between node $i$ and node $j$ is defined as the sum of the hitting time from $i$ to $j$ and the hitting time from $j$ to $i$:

$$C_{ij} = H_{ij}^{(j)} + H_{ji}^{(i)} \tag{8}$$

Clearly, commute time is a symmetric quantity, i.e., $C_{ji} = C_{ij}$. In contrast, hitting time is in general not symmetric, even when the underlying network is undirected.

3.4. Hitting Cost

The (expected) hitting cost, also known as average first-passage cost in the literature, generalizes the (expected) hitting time by assigning a cost to each transition. Hitting cost from $s$ to $t$, denoted by $IH_{s}^{(t)}$, is the average cost incurred by the random walk starting from node $s$ to hit node $t$ for the first time. The cost of transiting edge $e_{ij}$ is given by $w_{ij}$. The hitting cost was first introduced by Fouss et al. [4] and given in in a recursive form. In the following, we first provide a rigorous definition for hitting cost in the stochastic form, and then show how the matrix form and recursive form can be driven from this definition.

- **Stochastic form:** Let $G = (X_k)_{k>0}$ be a discrete-time Markov chain with the transition probability matrix $P$ and cost matrix $W$. The hitting cost of the target node $t$ is a random variable $\eta_t : \Omega \rightarrow C$ which is defined by $\eta_t = \inf \{ \eta \geq 0 : \exists k, X_k = t, \sum_{i=1}^{k} w_{X_i-1,X_i} = \eta \}$. $C$ is a countable set. If we view $w_{ij}$ as the length of edge (link) $e_{ij}$, then the hitting cost $\eta_t$ is the total length of steps that the random walk takes until it hits $t$ for the first time. The expected value of $\eta_t$ when the random walk starts at node $s$ is given by

$$IH_{s}^{(t)} = \mathbb{E}_s[\eta_t] = \sum_{l \in C} l \mathbb{P}(\eta_t = l | X_0 = s) \tag{9}$$

For compactness, we delegate the more detailed derivation of the stochastic form and its connection with the matrix form to the appendix.
• **Matrix form:** Hitting cost can be computed from the following *closed form* formulation:

\[ h^{(t)} = Fr, \]  

(10)

where \( r \) is the vector of expected outgoing costs and \( h^{(t)} \) is a vector of \( H_s^{(t)} \) computed for all \( s \in V \setminus \{t\} \). The expected outgoing cost of node \( s \) is obtained from: \( r_s = \sum_{m \in N_{out}(s)} p_{sm} w_{sm} \). Note that the hitting time matrix \( H \) in Eq. (5) is a special case of the hitting cost matrix \( H \), obtained when \( w_{ij} = 1 \) for all \( e_{ij} \).

• **Recursive form [4]:** The recursive computation of \( H_s^{(t)} \) is given as follows:

\[ H_s^{(t)} = r_s + \sum_{m \in N_{out}(s)} p_{sm} H_m^{(t)}. \]  

(11)

It is easy to see the direct connection between the recursive form and the matrix form: from \[ h = r + P_{11} h, \] we have \[ h = (I - P_{11})^{-1} r. \]

### 3.4.1. Commute Cost

Commute cost \( C_{ij} \) is defined as the expected cost required to hit \( j \) for the first time and get back to \( i \). As in the case of commute time, commute cost is a symmetric metric and is given by

\[ C_{ij} = H_i^{(j)} + H_j^{(i)}. \]  

(12)

### 3.5. Absorption Probability

The absorption probability, also known as hitting probability in the literature, is the probability of hitting or getting absorbed by a target node (or any node in a set of target nodes) in a finite time [11]. For a single target node, this probability is trivially equal to 1 for all nodes in a strongly connected network. We therefore consider more than one target nodes in this paper.

Let indexes \( n - 1 \) and \( n \) be assigned to two target nodes in a strongly connected network. We partition the transition probability matrix \( P \) as follows:

\[ P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{n-1,n-1} & P_{n-1,n} \\ P_{31} & P_{n,n-1} & P_{n,n} \end{bmatrix} \]  

(13)
where \( P_{11} \) is an \((n - 2) \times (n - 2)\) matrix, \( p_{12}, p_{12}, p_{21}, \) and \( p_{31} \) are \((n - 2) \times 1\) vectors, and the rest are scalars. The corresponding absorption probability can be expressed in the three forms as follows:

- **Matrix form [10]**: The absorption probability matrix denoted by \( Q \) is a \((n - 2) \times 2\) matrix whose columns represent the absorption probabilities to target \( n - 1 \) and \( n \) respectively:

  \[
  Q^{(n-1, n)} = F p_{12}, \quad (14)
  \]

  \[
  Q^{(n-1, n)} = F p_{13}, \quad (15)
  \]

  where \( F = (I - P_{11})^{-1} \). The notation \( Q^{(n-1, n)} \) emphasizes that target \( n - 1 \) is hit sooner than target \( n \), and \( Q^{(n-1, n)} \) indicates hitting target \( n \) occurs sooner than target \( n - 1 \). The formulation above states that to obtain the probability of getting absorbed (hit) by a given target when starting a random walk from a source node, we add up the absorption probabilities of starting from the neighbors of the source node, weighted by the number of times we expect to be in those neighboring nodes [10]. For a strongly connected network, these two probabilities are sum up to 1 for each starting node \( s \), i.e., \( Q^{(n-1, n)}_s + Q^{(n-1, n)}_s = 1 \).

- **Recursive form [11]**: For each of the target nodes, the absorption probability starting from any source node can be found from the absorption probabilities starting from its neighbors:

  \[
  Q^{(n-1, n)}_s = \sum_{m \in N_{out}(s)} p_{sm} Q^{(n-1, n)}_m, \quad (16)
  \]

  \[
  Q^{(n-1, n)}_s = \sum_{m \in N_{out}(s)} p_{sm} Q^{(n-1, n)}_m, \quad (17)
  \]

  where \( s, m \in V \setminus \{n - 1, n\} \). Note that the neighbors of a node can also be the target nodes. Thus, the right-hand side of the above equations is decomposed into two parts: \( Q^{(n-1, n)}_s = p_{sn} + \sum_{m \neq n, n-1} p_{sm} Q^{(n-1, n)}_m \), and the same way for \( Q^{(n-1, n)}_s \). Now, it is easy to see how the recursive form is connected to the matrix form: from \( Q^{(n-1, n)} = P_{13} + P_{11} Q^{(n-1, n)} \), we have \( Q^{(n-1, n)} = (I - P_{11})^{-1} p_{13} \).
• **Stochastic form [11]:** Let $G = (X_k)_{k>0}$ be a discrete-time Markov chain with the transition matrix $P$. The hitting time of the target state $n$ before $n-1$ is denoted by a random variable $\kappa_n : \Omega \rightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$ given by $\kappa_n = \inf \{\kappa \geq 0 : X_k = n, X_{k<k} \neq n, n-1\}$. Then the probability of ever hitting $n$ is $\mathbb{P}(\kappa_n < \infty)$ [11]. This can be derived as follows:

$$Q^{(n-1,n)}_s = \sum_{k=1}^{\infty} \mathbb{P}(\kappa_n = k|X_0 = s)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_k = n|X_0 = s, X_{<k} \neq n, n-1)$$

$$= \sum_{k=1}^{\infty} \sum_{m \neq n, n-1} \mathbb{P}(X_{k-1} = m|X_0 = s, X_{<k-1} \neq n, n-1) \cdot \mathbb{P}(X_k = n|X_{k-1} = m)$$

$$= \sum_{k=1}^{\infty} \sum_{m} [P_{11}^{k-1}]_{sm} [P_{13}]_m$$

$$= \sum_{m} [(I - P_{11})^{-1}]_{sm} [P_{13}]_m. \tag{18}$$

The stochastic form for $Q^{(n-1,\pi)}_s$ is derived in a similar vein.

3.6. **Generalization: Random Walk Metrics for a Set of Targets**

Let $A = \{t_1, \ldots, t_{|A|}\}$ be a set of target nodes. Then the transition probability matrix can be written in the following form:

$$P = \begin{bmatrix} P_{TT} & P_{TA} \\ P_{AT} & P_{AA} \end{bmatrix}, \tag{19}$$

where $T = V \setminus A$ is the set of non-target nodes. Note that set of target nodes $A$ can be modeled as the set of absorbing states in a Markov chain, and then $T = V \setminus A$ is the set of transient (non-absorbing) nodes. Since hitting the target nodes is the stopping criterion for all of the random walk metrics we have introduced so far, it does not matter where the random walk can go afterwards and what the outgoing edges of the target nodes are. Therefore, there is no difference between $P = \begin{bmatrix} P_{TT} & P_{TA} \\ P_{AT} & P_{AA} \end{bmatrix}$ and $P' = \begin{bmatrix} P_{TT} & P_{TA} \\ P_{AT} & 0 \end{bmatrix} I$ for computing the random walk metrics.
For a given set of target nodes $\mathcal{A}$, the fundamental matrix $F^A$ is obtained using the following relation:

$$ F^A = I + \sum_{k=1}^{\infty} P_{TT}^k = (I - P_{TT})^{-1}, \quad (20) $$

which is a general form of the fundamental matrix defined for a single target (Eq.(1)). Entry $F^A_{sm}$ represents the expected number of visits to $m$ before hitting any of the target nodes in $\mathcal{A}$ when starting a random walk from $s$.

A hitting time for $\mathcal{A}$ is defined as the expected number of steps to hit the set for the first time which can occur by hitting any of the target nodes in this set. The vector of hitting times with respect to a target set $\mathcal{T}$ can be computed using

$$ h^A = F^A 1 \quad (21) $$

If there exists a matrix of costs $W$ defined for the network, the hitting cost for target set $\mathcal{A}$ is given below

$$ hl^A = F^A r, \quad (22) $$

where $r$ is a vector of expected outgoing cost $r_s$’s: $r_s = \sum_{m \in N_{\text{out}}(s)} p_{sm} w_{sm}$.

The absorption probability of target set $\mathcal{A}$ is a $|\mathcal{T}| \times |\mathcal{A}|$ matrix whose columns represents the absorption probability for each target node if it gets hit sooner than the other target nodes:

$$ Q^A = F^A P_{\mathcal{T},\mathcal{A}}, \quad (23) $$

where $Q^A$ is a row-stochastic matrix for a strongly connected network.

We remark that if the network is not strongly connected (thus the corresponding Markov chain is not irreducible), $I - P_{TT}$ may not be non-singular for every set of $\mathcal{A}$. Hence $F^A$ may not exist. The necessary and sufficient condition for the existence of $F^A$ is that target set $\mathcal{A}$ includes at least one node from each recurrent equivalence class in the network. The recurrent equivalence class is the minimal set of nodes that have no outgoing edge to nodes outside the set. Once a random walk reaches a node in a recurrent equivalence class, it can no longer get out of that set. A recurrent equivalence class can be as small as one single node, which is called an absorbing node.
4. Useful Relations for Random Walk Metrics

In this section, we first establish several important theorems, and then gather and derive a number of useful relations among the random walk metrics. We start by relating the fundamental tensor with the Laplacian matrices of a general network. For an undirected network or graph \( G \), the graph Laplacian \( L = D - A \) (where \( A \) is the adjacent matrix of \( G \) and \( D = \text{diag}[d_i] \) is the diagonal matrix of node degrees) and its normalized version \( \tilde{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}} \) have been widely studied and found many applications (see, e.g., [13] and the references therein). In particular, it is well-known that commute times are closely related to the Penrose-Moore pseudo-inverse of \( L \) (and a variant of Eq. (24) also holds for \( \tilde{L} \)):

\[
C_{ij} = L_{ii}^{u+} + L_{jj}^{u+} - L_{ij}^{u+} - L_{ji}^{u+}. \tag{24}
\]

Li and Zhang [14, 15, 16] were first to introduce the (correct) generalization of graph Laplacians for directed networks/graphs (digraphs) using the stationary distribution \( \{\pi_i\} \) of the transition matrix \( P = D^{-1}A \) for the associated (random walk) Markov chain defined on a directed network \( G \). For a strongly connected network \( G \), its normalized digraph Laplacian is defined as

\[
\tilde{L} = \Pi^{-\frac{1}{2}}(I - P)\Pi^{-\frac{1}{2}},
\]

where \( \Pi = \text{diag}[\pi_i] \) is the diagonal matrix of stationary probabilities. Li and Zhang showed that the Penrose-Moore pseudo-inverse \( \tilde{L}^{+} \) of \( \tilde{L} \) is closely related to the fundamental matrix of the random walk Markov chain \( \tilde{P} \), and proved that the hitting time and commute time can be computed from \( \tilde{L}^{+} \) using the following relations:

\[
H_{i}^{(j)} = \tilde{L}_{ij}^{+} \frac{\tilde{L}_{jj}^{+}}{\pi_j} - \tilde{L}_{jj}^{+} \frac{\tilde{L}_{ij}^{+}}{\sqrt{\pi_i \pi_j}} \tag{25}
\]

and

\[
C_{ij} = H_{i}^{(j)} + H_{j}^{(i)} = \tilde{L}_{ii}^{+} \frac{\tilde{L}_{jj}^{+}}{\pi_i} + \tilde{L}_{jj}^{+} \frac{\tilde{L}_{ij}^{+}}{\pi_j} - \frac{\tilde{L}_{ij}^{+}}{\sqrt{\pi_i \pi_j}} - \frac{\tilde{L}_{ji}^{+}}{\sqrt{\pi_i \pi_j}}. \tag{26}
\]

We define the (unnormalized) digraph Laplacian for a general (directed or undirected) network \( G \) as \( L = \Pi(I - P) \) and the random walk Laplacian as \( L^{p} = I - P \). Clearly, \( \tilde{L} = \Pi^{-\frac{1}{2}}L\Pi^{-\frac{1}{2}} = \Pi^{\frac{1}{2}}L^{p}\Pi^{-\frac{1}{2}} \). Note that for a (connected) undirected graph, as \( \pi_i = \frac{d_i}{\text{vol}(G)} \) where \( \text{vol}(G) = \sum_j d_j \), we see that the classical graph Laplacian \( L^{u} = D - A = \text{vol}(G)L \). Any results which hold for \( L \) also hold for \( L^{u} = D - A \) with a scalar multiple. In the following
we relate the fundamental tensor to the digraph and random walk Laplacians $L$ and $L^p$, and use this relation to establish similar expressions for computing hitting and commute times using $L$, analogous to Eqs. (25) and (26).

**Lemma 1** ([17]). Let \( \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{nn} \end{bmatrix} \) be an $n \times n$ irreducible matrix such that \( \text{nullity}(L) = 1 \). Let $M = L^+$ be the Penrose-Moore pseudo-inverse of $L$ partitioned similarly and $(\mathbf{u}', 1)L = 0$, $L(\mathbf{v}; 1) = 0$, where $\mathbf{u}$ and $\mathbf{v}$ are $(n-1)$-dim column vectors, $\mathbf{u}'$ is the transpose of the column vector $\mathbf{u}$ ($(\mathbf{u}', 1)$ is a $n$-dim row vector and $(\mathbf{v}; 1)$ is a $n$-dim column vector, a la Matlab). Then the inverse of the $(n-1) \times (n-1)$ matrix $L_{11}$ exists and is given by:

$$L_{11}^{-1} = (I + \mathbf{v}\mathbf{v}')M_{11}(I + \mathbf{u}\mathbf{u}'),$$

(27)

where $I$ denotes the $(n-1) \times (n-1)$ identity matrix.

Note that node $n$ in the above lemma can be substituted by any other node (index).

**Lemma 2.** The fundamental tensor can be computed from the Moore-Penrose pseudo-inverse of the digraph Laplacian matrix $L = \Pi(I - P)$ as well as the random walk Laplacian matrix $L^p = I - P$ as follows:

$$F_{\text{smt}} = (L^p_{sm} - L^p_{tm} + L^p_{tt} - L^p_{st})\pi_m, \quad (28)$$

$$F_{\text{smt}} = L^p_{sm} - L^p_{tm} + \frac{\pi_m}{\pi_t}L^p_{tt} - \frac{\pi_m}{\pi_t}L^p_{st}, \quad (29)$$

where $\pi_i$ is the stationary probability of node $i$ and $\Pi$ is a diagonal matrix whose $i$-th diagonal entry is equal to $\pi_i$.

**Proof.** Note that $F = (I - P_{11})^{-1} = L_{11}^{-1}$ as in Lemma 1. The above equations follow from Lemma 1 with $\mathbf{v} = \mathbf{u} = 1$.

**Corollary 1.** The fundamental tensor $F$ of a strongly connected network can be computed in $O(n^3)$ time complexity.

**Proof.** The nullity of matrix $L^p = I - P$ for a strongly connected network is 1. Using Eq. (28), all $n^3$ entries of the fundamental tensor $F$ can be computed from $L^+$ in constant time.
Corollary 2.

\[
\sum_{s,t} F_{smt} = c\pi_m,
\]

(30)

where \(c\) is a constant independent of \(m\).

Proof.

\[
\sum_{s,t} F_{smt} = \sum_{s,t} (L_{sm}^+ - L_{tm}^+ - L_{st}^+ + L_{tt}^+)\pi_m
\]

(31)

\[
= 0 - 0 - 0 + (n \sum_t L_{tt}^+)\pi_m
\]

(32)

\[
= c\pi_m,
\]

(33)

where the second equality follows from the fact that the column sum of \(L^+ = (\Pi(I - P))^+\) is zero. Later in Section 5.1 we will show that \(c = |E|K\), where \(K\) is the Kirchhoff index of a network.

Corollary 3. Hitting time and commute time can also be expressed in terms of entries in the digraph Laplacian matrix \(L = \Pi(I - P)\) [110]:

\[
H_{ij}^{(j)} = \sum_m (L_{im}^+ - L_{jm}^+)\pi_m + L_{jj}^+ - L_{ij}^+,
\]

(34)

\[
C_{ij} = L_{ii}^+ + L_{jj}^+ - L_{ij}^+ - L_{ji}^+,
\]

(35)

Proof. Use Eq. (5) and (28).

Note that we can also write the metrics in terms of the random walk Laplacian matrix \(L^p\) by a simple substitution: \(L_{im}^+ - L_{ij}^+ = \frac{L_{im}^p\pi_m}{\pi_m} - \frac{L_{ij}^p}{\pi_j}\).

Corollary 4. Hitting cost \(HH\) and commute cost \(C\) can be expressed in terms of the digraph Laplacian matrix \(L = \Pi(I - P)\):

\[
HH_{ij} = \sum_m (L_{im}^+ - L_{jm}^+ + L_{jj}^+ - L_{ij}^+)g_m,
\]

(36)

\[
C_{ij} = (L_{im}^+ - L_{jm}^+ + L_{jj}^+ - L_{ij}^+)\sum_m g_m,
\]

(37)

where \(g_m = r_m\pi_m\) and \(r_m = \sum_{k \in N_{out}(m)} P_{mk}w_{mk}\). 

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Proof. Use Eq. (10) and (28). From Eq. (35) and (37), it is also interesting to note that commute cost is a multiple scalar of commute time. □

Theorem 1 (Incremental Computation of the Fundamental Matrix). The fundamental matrix for target set $S_1 \cup S_2$ can be computed from the fundamental matrix for target set $S_1$ as follows,

$$F_{im}^{S_1 \cup S_2} = F_{im}^{S_1} - F_{iS_2}^{S_1} F_{S_2}^{-1} F_{S_2m}^{S_1},$$

(38)

where $F_{iS_2}^{S_1}$ denotes the row corresponding to node $i$ and the columns corresponding to set $S_2$ of the fundamental matrix $F^{S_1}$, and the (sub-)matrices $F_{S_2}^{S_1}$ and $F_{S_2m}^{S_1}$ are similarly defined.

Proof. Consider the matrix $M = I - P_{TT}$, where the absorbing set is $A = S_1$ and the transient set $T = V \setminus S_1$. The inverse of $M$ yields the fundamental matrix $F^{S_1}$, and the inverse of its sub-matrix obtained from removing rows and columns corresponding to set $S_2$ yields the fundamental matrix $F^{S_1 \cup S_2}$. Using the following equations from the Schur complement, we see that the inverse of a sub-matrix can be derived from that of the original matrix.

If $A$ is invertible, we can factor the matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as follows

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

(39)

Inverting both sides of the equation yields

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BS^{-1} \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

(40)

$$= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$$

(41)

$$= \begin{bmatrix} X & Y \\ Z & W \end{bmatrix},$$

(42)

where $S = D - CA^{-1}B$. Therefore, $A^{-1}$ can be computed from $A^{-1} = X - YW^{-1}Z$. □

Corollary 5. The simplified form of Theorem 4 for a single target is given by

$$F_{im}^{\{j,k\}} = F_{im}^{\{j\}} - \frac{F_{ik}^{\{j\}} F_{k,m}^{\{j\}}}{F_{k,k}^{\{j\}}}$$

(43)
Lemma 3.  
\[ P_{TT}F^A = F^A P_{TT} = F^A, \]  
(44)  
where \( \mathcal{T} \cup \mathcal{A} = \mathcal{V} \)

Proof. It follows easily from Eq.(1).

Corollary 6 (Another Recursive Form for the Fundamental Matrix).  
\[ F_{im}^{(j)} = \begin{cases} \sum_{k \in \mathcal{N}_m(m)} F_{ik}^{(j)} p_{km} & \text{if } i \neq m \\ 1 + \sum_{k \in \mathcal{N}_m(m)} F_{ik}^{(j)} p_{km} & \text{if } i = m \end{cases} \]  
(45)  
Proof. It is a special case of Lemma 3. Note that the recursive relation in Eq.(2) is in terms of \( s \)’s outgoing neighbors, while this one is in terms of incoming neighbors of \( m \).

Theorem 2 (Absorption Probability & Normalized Fundamental Matrix).  
The absorption probability of a target node \( j \) in an absorbing set \( \mathcal{A} = \{ j \} \cup \mathcal{S} \) can be written in terms of the normalized fundamental matrix \( F^S \), where the columns are normalized by the diagonal entries:  
\[ Q_{ij}^A = \frac{F_{ij}^S}{F_{jj}^S} \]  
(46)  
Proof.  
\[ Q_{ij}^A = [F^A P_{\mathcal{T} \mathcal{A}}]_{ij} \]  
(47)  
\[ = \sum_{m \in \mathcal{T}} F_{im}^A p_{mj} \]  
\[ = \sum_{m \in \mathcal{T}} (F_{im}^S - \frac{F_{ij}^S F_{jm}^S}{F_{jj}^S}) p_{mj} \]  
\[ = \sum_{m \in \mathcal{T}} F_{im}^S p_{mj} - \frac{F_{ij}^S}{F_{jj}^S} \sum_{m \in \mathcal{T}} F_{jm}^S p_{mj} \]  
\[ = F_{ij}^S - \frac{F_{ij}^S}{F_{jj}^S} (F_{jj}^S - 1) \]  
\[ = \frac{F_{ij}^S}{F_{jj}^S}, \]
where the third and fifth equalities follow directly from of Theorem 1 and Lemma 3 respectively.
We are now in a position to gather and derive a number of useful relations among the random walk metrics.

**Relation 1** (Complementary relation of absorption probabilities).

\[ Q_{ij}^A = 1 - \sum_{k \in \mathcal{A}\setminus\{j\}} Q_{ik}^A, \quad (48) \]

where \( i \in \mathcal{T} \) and \( j \in \mathcal{A} \).

*Proof.* Based on the definition of \( Q \) and the assumption that all the nodes in \( \mathcal{T} \) are transient, the probability that a random walk eventually ends up in set \( \mathcal{A} \) is 1. \( \square \)

**Relation 2** (Relations between the fundamental matrix and commute time).

1. \[ F_{ii}^{\{j\}} = \pi_i C_{ij}, \quad (49) \]
2. \[ \frac{F_{im}^{\{j\}}}{\pi_m} + \frac{F_{mi}^{\{j\}}}{\pi_i} = C_{ij} + C_{jm} - C_{im}, \quad (50) \]
3. \[ \frac{F_{im}^{\{j\}}}{\pi_m} + \frac{F_{ij}^{\{m\}}}{\pi_j} = C_{jm}, \quad (51) \]
4. \[ F_{im}^{\{j\}} + F_{jm}^{\{i\}} = \pi_mC_{ij}, \quad (52) \]

*Proof.* Use (28) and (35). \( \square \)

**Relation 3** (The hitting time detour overhead in terms of other metrics).

1. \[ H_{ij}^{\{j\}} + H_{jk}^{\{m\}} - H_{ijn}^{\{m\}} = F_{im}^{\{j\}} / \pi_m, \quad (53) \]
2. \[ H_{ij}^{\{j\}} + H_{jm}^{\{m\}} - H_{ijn}^{\{m\}} = Q_{ij}^{\{m,j\}} C_{mj}, \quad (54) \]

*Proof.* For the first equation use (28) and (34), and for the second one use the previous equation along with (2) and (49). \( \square \)

**Relation 4** (The hitting time for two target nodes in terms of hitting time for a single target).

\[ H_{ij}^{\{j,k\}} = H_{ij}^{\{k\}} - Q_{ij}^{\{j,k\}} H_{ik}^{\{k\}} = H_{ij}^{\{j\}} - Q_{ij}^{\{k,j\}} H_{ik}^{\{j\}}, \quad (55) \]

which can also be reformulated as: \( H_{ij}^{\{j\}} = H_{ij}^{\{j,k\}} + Q_{ij}^{\{k,j\}} H_{ik}^{\{j\}} \).
Proof. Aggregate two sides of Eq. (1) over $m$ and substitute Eq. (2) in it. □

**Relation 5** (Inequalities for hitting time).

1. $H^{(m)}_i + H^{(j)}_m \geq H^{(j)}_i$ \textit{(triangular inequality)} (56)
2. $H^{(j)}_i \geq H^{(j,m)}_i$ (57)
3. $H^{(m)}_i + H^{(j,k)}_m \geq H^{(j,k)}_i$ (58)

Proof. For the first inequality, use (34) and (62). For the second inequality, use the aggregated form of Eq. (1) over $m$ and the fact that the entries of $F$ are non-negative. The third inequality is a generalization of the first one. □

**Relation 6** (Inequalities for the fundamental matrix).

1. $F^{(j)}_{im} F^{(j)}_{kk} \geq F^{(j)}_{ik} F^{(j)}_{km}$ (59)
2. $F^{(j)}_{kk} \geq F^{(j)}_{ik}$ (60)

Proof. For the first inequality, use Eq. (1) and the fact that $F$ is non-negative. The second one can be derived from Eqs. (49), (53) and (56). Note that these two inequalities hold for any absorbing set $A$, hence we drop the superscripts. □

**Relation 7** (Inequality for absorption probabilities).

$$Q^{(m,j)}_i \geq Q^{(k,j)}_i Q^{(m,j)}_k$$ (61)

Proof. Use (2) and (59). □

**Relation 8** (Inequality for the digraph Laplacian matrix).

$$L^{+}_{im} + L^{+}_{kk} \geq L^{+}_{ik} + L^{+}_{km}$$ (62)

Proof. Use (28) and the fact that $F$’s entries are always non-negative. □

**Relation 9** (Relations for undirected networks (reversible Markov chain)).

1. $\frac{F^{(S)}_{im}}{\pi_m} = \frac{F^{(S)}_{mi}}{\pi_i}$ (63)
2. $Q^{(m,j)}_{im} C^{(j)}_m = Q^{(i,j)}_{m} C^{(j)}_i$ (64)
3. $H^{(m)}_i + H^{(j)}_m + H^{(i)}_j = H^{(i)}_m + H^{(m)}_j + H^{(j)}_i$ (65)

Proof. The first equation follows from Eq. (28) and the fact that $L^+$ is symmetric for undirected networks. The second equation can be derived by using Eqs. (2), (28), (35) and the fact that $L^+$ is symmetric. The third equation follows from Eq. (34) and $L^+$ being symmetric. □
5. Applications of the Fundamental Tensor in Network Analysis

5.1. Unifying Network Centrality/Distance Measures

Many network measures have been proposed in the literature for network analysis purposes [18], such as distance metrics for measuring the similarity (or diversity) between nodes or entities of a network, centrality measures to assess a node’s involvement or importance in the connectivity or communication between network entities, and topological indices to measure the overall robustness of networks. Many of these network measures leverage or can be derived from random walk metrics. In this section we review some of these network measures proposed in the literature, and show that these measures can be unified under the proposed fundamental tensor, which provides a coherent framework for computing them and understanding the relations among them.

Statement 1. Fundamental tensor $F$ unifies various network random-walk measures via summation along one or more dimensions shown in Figure 1.

5.1.1. Random-walk distance measure:

The hitting time metric has been used extensively in different application domains, such as a distance (or dissimilarity) measure for clustering...
and classification purposes [2]. Note that this distance metric satisfies two out of three conditions for the general distance metric: It is positive when two ends are different and zero when two ends are identical. As noted earlier, the hitting time metric is in general not symmetric, but it satisfies the triangle inequality. In Section 3, we have shown that hitting time can be computed from the fundamental tensor by summing over m’s (the middle node dimension, see Figure 1).

\[ \text{Distance}_{rw}(s, t) = H_s^{(t)} = \sum_m F_{smt}. \]  

(66)

With a cost matrix \( W \), the hitting cost distance (10) is obtained by the weighted sum over the medial node dimension of the fundamental tensor: 

\[ IH_s^{(t)} = \sum_m F_{smt} b_m, \]

where \( b_m = \sum_i w_{mi} p_{mi} \) is the expected outgoing cost of node \( m \).

5.1.2. Random-Walk Centrality Measure:

Network centrality measures can be broadly categorized into two main types [18]: i) distance-based and ii) volume-based. The closeness centrality is an example of the distance-based measures, whereas the betweenness centrality is an example of volume-based measures.

- Random-walk closeness measure: Closeness centrality ranks nodes in a network based on their total distance from other nodes of the network. This measure reflects how easily/closely the node is accessible/reachable from the other parts of the network, and in a nutshell how “central” the node is located within a network. The classical closeness centrality metric is defined using the shortest path distances. Noh and Rieger [19] introduces the random walk closeness centrality, which is defined using the hitting time metric: A node is considered to have a high centrality value if and only if its total hitting time distances from other nodes in the network is small. This closeness centrality measure can be easily expressed in terms of the random walk fundamental tensor:

\[ \text{Closeness}_{rw}(t) = \sum_s H_s^{(t)} = \sum_{s,m} F_{smt}, \]  

(67)

or in the reciprocal form to imply lower importance with small closeness value: 

\[ \text{Closeness}_{rw}(t) = \frac{\|V\|}{\sum_{s,m} F_{smt}}. \]
• Random-walk betweenness measure: Betweenness measure quantifies the number of times a node acts as a “bridge” along the paths between different parts of the network. The larger the number of paths crossing that node, the more central the node is. As a special case, the node degree, \( \text{deg}(m) \), can be viewed as a betweenness centrality measure in an undirected network. Clearly, it captures how many paths of length 1 going through node \( m \) (or many 1-hop neighbors it has) [18]. It is also proportional to the total number of (random) walks passing through node \( m \) from any source to any target in the network. This follows from the following more general statement. For a general (strongly connected) network, we define the random walk betweenness of node \( m \) as follows and show that it is proportional to \( \pi_m \), the stationary probability of node \( m \):

\[
\text{Betweenness}_{rw}(m) = \sum_{s,t} F_{smt} \tag{68}
\]

\[
= \sum_{s,t} (L_{sm}^+ - L_{tm}^+ - L_{st}^+ + L_{tt}^+) \pi_m \tag{69}
\]

\[
= |V| \sum_t L_{tt}^+ \pi_m \tag{70}
\]

\[
= |E| K \pi_m, \tag{71}
\]

where \( K \) is the Kirchhoff index (see Section 5.1.3). The third equality follows by using the fact that the column sum of the digraph Laplacian matrix \( L^+ = (\Pi(I - P))^+ \) is zero [16 17]. For a (connected) undirected network, \( \pi_m = \frac{d_m}{2|E|} \), where \( d_m \) is the degree of node \( m \).

For undirected networks, Newman [1] proposes a variation of the random walk betweenness measure defined above, which we denote by \( \text{Betweenness}_{Newman, bidirect}(m) \) (the use of subscript \( \text{bidirect} \) will be clear below): it is defined as the (net) electrical current flow \( I \) through a medial node in an undirected network (which can be viewed as an electrical resistive network with bi-directional links with resistance), when a unit current flow is injected at a source and removes at a target (ground),
aggregated over all such source-target pairs. Formally, we have

\[ \text{Betweenness}_{\text{Newman,bidirect}}(m) = \sum_{s,t} I(s \rightarrow m \rightarrow t) \]

\[ = \sum_{s,t} \sum_{k} \frac{1}{2} |F_{smt}p_{mk} - F_{skt}p_{km}|. \]

We remark that the original definition given by Newman is based on current flows in electrical networks, and is only valid for undirected networks. Define \( f(F_{smt}) = \sum_{k} \frac{1}{2} |F_{smt}p_{mk} - F_{skt}p_{km}| \), then 

\[ \text{Betweenness}_{\text{Newman,directed}}(m) = \sum_{s,t} f(F_{smt}) \]

yields a general definition of Newman’s random walk betweenness measure that also holds for directed networks. In particular, we show that if a network is strictly unidirectional, namely, if \( e_{ij} \in E \) then \( e_{ji} \notin E \), Newman’s random walk betweenness centrality reduces to 

\[ \text{Betweenness}_{\text{rw}}(m) = \frac{|E|}{K \pi_m}. \]

\[ \text{Betweenness}_{\text{Newman,unidirect}}(m) = \sum_{s,t} \sum_{k} |F_{smt}p_{mk}| \]

\[ = \sum_{s,t} F_{smt} \sum_{k} p_{mk} \]

\[ = \sum_{s,t} F_{smt} = |E| K \pi_m, \quad (72) \]

where \( K \) is the Kirchhoff index (see Section 5.1.3) and the last equality follows from Corollary 5.1.3.

5.1.3. Kirchhoff Index

The term topological index is a single metric that characterizes the topology (“connectivity structure”) of a network; it has been widely used in mathematical chemistry to reflect certain structural properties of the underlying molecular graph [20][21]. Perhaps the most known topology index is the Kirchhoff index [22] which has found a variety of applications [23, 24, 25, 6]. The Kirchhoff index is often defined in terms of effective resistances [22], 

\[ K(G) = \frac{1}{2} \sum_{s,t} \Omega_{st}, \]

which is closely related to commute times, as \( \Omega_{st} = \frac{1}{|G|} C_{st} [26] \). Hence we have

\[ K(G) = \frac{1}{2|E|} \sum_{s,t} C_{st} = \frac{|V|}{|E|} \sum_{t} L^+_{tt} = \frac{1}{|E|} \sum_{s,m,t} F_{smt}, \quad (73) \]
where the second equality comes from Eq. (35). In other words, the Kirchhoff index can be computed by summing over all three dimensions in Figure 1 normalized by the total number of edges. The authors in [6] provide three interpretations of $L_{ii}^{+}$ as a topological centrality measure, from effective resistances in an electric network, random walk detour costs, and graph-theoretical topological connectivity via connected bi-partitions, and demonstrate that the Kirchhoff index, as a topological index, captures the overall robustness of a network.

5.2. Characterization of Network Articulation Points and Network Load Distribution

Given a general network, an articulation point is a node whose removal reduces the amount of reachability in the network. For instance, in a (directed) network $G$, if $t$ is previously reachable from $s$, but after removing a node $m$, $t$ is no longer reachable from $s$ (but $s$ may still be reachable from $t$), node $m$ is an articulation point for network $G$. In an undirected network, an articulation point is also a so-called cut vertex; namely, its removal increases the number of connected components in the network. Removal of an articulation point in a directed network, however, does not necessarily increase the number of connected components in the network. As an application of the fundamental tensor, we introduce the normalized fundamental tensor $\tilde{F}$ and show that its entries contain information regarding articulation points in a general (directed or undirected) network. If $F_{smt}$ exists, its normalized version is defined as follow,

$$\tilde{F}_{smt} = \begin{cases} 
\frac{F_{smt}}{F_{mmt}} & \text{if } s, m \neq t \\
0 & \text{if } s = t \text{ or } m = t,
\end{cases}$$

(74)

The normalized fundamental tensor satisfies the following properties: a) $0 \leq \tilde{F}(s, m, t) \leq 1$, and b) $\tilde{F}_{smt} = Q_s^{(m, t)}$. Recall that $Q_s^{(m, t)}$ is the absorption probability that a random walk starting from node $s$ hits (is absorbed by) node $m$ sooner than node $t$. The second property (b) is a result of Theorem 2 and the first property (a) follows from (b). Clearly, $\tilde{F}_{smt} = Q_s^{(m, t)} = 1$ means that with probability 1, any random walk starts from node $s$ always hit node $m$ before node $t$. Hence node $m$ is on any path (thus walk) from $s$ to $t$. Hence it is an articulation point. We therefore have the following statement

**Statement 2.** The normalized fundamental tensor captures the articulation points of a network: if $\tilde{F}_{smt} = 1$, then node $m$ is an articulation point;
namely, node \( m \) is located on all paths from \( s \) to \( t \). On the other extreme, \( \hat{F}_{smt} = 0 \) indicates that \( m \) is not located on any path from \( s \) to \( t \) and thus it plays no role for this reachability.

The figure needs to be revised: based on your definition in Eq. (74), the diagonal entries in the matrices should always be set to zero!!! Figure 2 depicts two simple networks, one undirected and one directed, and displays the corresponding normalized fundamental tensors we have computed for these two networks (the tensors are “unfolded” as a series of matrices, each with fixed \( t \)). Any column that contains an entry with value 1 indicates the corresponding node \( m \), \( 1 \leq m \leq 5 \), is an articulation point for the network. Counting the number of 1’s in each column \( m \) over the entire tensor yields the number of source-target pairs for which node \( m \) is an articulation point. The larger this count is, the more critical node \( m \) is for the overall network reachability. For instance, for both networks, node 3 is the most critical node for network reachability.

More generally, we can view \( \hat{F}_{smt} \) as a measure of how critical a role node \( m \) plays in the reachability from node \( s \) to node \( t \). As a generalization of articulation points, we define the overall load that node \( m \) carries for all source-target pairs in a network as follows:

\[
Load(m) = \frac{1}{(n-1)^2} \sum_{s,t} \hat{F}_{smt},
\]

(75)

It is interesting to constrain Eq. (75) with Eq. (68), where the latter (the unnormalized summation \( \sum_{s,t} F_{smt} \)) is proportional to the stationary probability of node \( m \) (and degree of \( m \) if the network is undirected). The distribution of \( Load(m) \)'s provides a characterization how balanced a network in terms of distributing its load (reachability between pairs of nodes), or how robust it is against targeted attacks. A network with a few high-valued articulation points (e.g., a star network) is more vulnerable to the failure of a few nodes. Using a few synthetic networks with specific topologies as well as real-world networks as examples, Figure 3 plots the distribution of \( Load(m) \) for these networks (sorted based on increasing values of \( Load(m) \)'s. Among the specific-shaped networks, it is interesting to note that comparing to a chain network, the loads on a cycle network are evenly distributed – this shows the huge difference that adding a single edge can make in the structural property of a network. It is not surprising that the complete graph has evenly distributed loads. In contrast, a star graph has the most
Figure 2: Two networks, one undirected and one directed, and the corresponding normalized fundamental tensor.
skewed load distribution, with the center node an articulation point of the network. Comparing the binary tree, the grid network has a less skewed load distribution. It is also interesting to compare the load distribution of the binary with that of a 3-ary “fat tree” network – such a topology is used widely in data center networks [27]. The real networks used in Figure 3(b) include the Arxiv High Energy Physics - Phenomenology collaboration network (CAHepPh) [28], a sampled network of Facebook [29], the coauthorship network of network scientists (netSci) [30], the Italian power grid [31], and a protein-protein interaction network [32]. For comparison, we also include three networks generated via two well-known random network models, the Preferential attachment generative model (PA) [33] and Erdos Renyi (ER) random graph model [34] with two different initial links of 8 (random) and 40 (random2). We see that the two ER random networks yield most balanced load distributions, whereas the PA network exhibits behavior similar to a tree network, with a few nodes bearing much higher loads than others. The real networks exhibit load distributions varying between these types of random networks (with the Italian power grid closer to an ER random network, whereas netSci closer to a PA random network).

5.3. Most Influential Nodes in Social Networks

Online social networks have played a key role as a medium for the spread of information, ideas, or “influence” among their members. The Influence maximization problem in social networks is about finding the most influential persons who can maximize the spread of influence in the network. This
problem has applications in viral marketing, where a company may wish to spread the publicity, and eventually the adoption, of a new product via the most influential persons in popular social networks. A social network is modeled as a (directed or undirected) graph where nodes represent the users, and edges represent relationships and interactions between the users. An influence cascade over a network can be modeled by a diffusion process, and the objective of the influence maximization problem is to find the $k$ most influential persons as the initial adopters who will lead to most number of adoptions.

The heat conduction model \cite{35} is a diffusion process which is inspired by how heat transfers through a medium from the part with higher temperature to the part with lower temperature. In this diffusion process, the probability that a user adopts the new product is a linear function of adoption probabilities of her friends who have influence on her as well as her own independent tendency. The independent tendency, denoted by $o$, is modeled by an exogenous node added to the network and linked to all of the nodes in the network as representing the independent tendency of users for the product adoption. Network $G$ with added node $o$ is called extended $G$, denoted by $G^o$. It is shown that the influence maximization problem for $k=1$, where $k$ = # initial adopters, under the heat conduction diffusion process has the following solution in terms of the normalized fundamental tensor over $G^o$ \cite{35}:

$$t^* = \arg \max_t \sum_{s \in V} \hat{F}_{sto}.$$  \hspace{1cm} (76)

Note that the general influence maximization problem for $k > 1$ has been shown to be NP-hard \cite{35}. However, an efficient greedy solution, called C2Greedy \cite{35}, exists. The algorithm finds a set of initial adopters which produces a provably near-optimal influence spread in the network. It iteratively finds the most influential node using Eq.\,(76); it then removes this from the network and solves the equation to find the next best initiator.

**Statement 3.** For $k = 1$, $\arg \max_t \sum_{s \in V} \hat{F}_{sto}$ finds the most influential node of network $G^o$ as the initial adopter for maximizing the influence spread over the network with heat conduction \cite{35} as the diffusion process. For $k > 1$, the greedy algorithm, C2Greedy \cite{35}, employs this relation to iteratively find the $k$ most influential nodes, which yields a provably near-optimal solution.

We remark that the metric in Eq.\,(76) addresses both the global characteristics of the network by placing the most influential node in the critical and
strategical “center” of the network, and the local characteristics by specifying the highly populated and “neighbor-rich” nodes. In Figure 4(a), we visualize the influence spread using the ESNet network of the two most influential nodes found from C2Greedy. The initiators are colored in black and the green intensity indicates the probability of influence spread over the nodes in the network. In Figure 4(b), we pick two nodes, which are a neighbor of the two most influential nodes identified by C2Greedy, as the initiators, and visualize the probability of influence spread caused by these two nodes over other nodes in the network. The lower green intensity of Figure 4(b), compared to that of Figure 4(a) shows that not any two initiators – even if they are their immediate neighbors and globally located very closely – can cause the same influence spread as the two most influential nodes identified by C2Greedy. Using three social networks, wiki vote, hepPh citation, and Facebook, in Figure 5 we compare the performance of C2Greedy with that of several state-of-art algorithms: picking the $k$ highest degree nodes, the nodes with $k$ highest closeness centrality scores, and the top $k$ nodes selected using the Pagerank method, as well as $k$ nodes that are picked randomly. We see that C2Greedy outperforms all these algorithms.

5.4. Fast Computation of Dynamic Network Reachability Queries

Reachability information between nodes in a network is crucial for a wide range of applications from gene interactions in bioinformatics to XML query processing. Given a network, a reachability query $R(s,t)$ has a
binary answer with 1 indicating that target node $t$ is reachable from source node $s$, and 0 representing that it is not. Several efficient algorithms have been devised to answer reachability queries when the network is static [42, 43, 44, 45]. However, few efficient solutions have been developed to answer reachability query for dynamic networks, e.g., after node or link failures. For example, garbage collection in programming languages requires dynamic (re-)computation of reachability to balance the reclamation of memory, which might be reallocated. The speed of answering reachability queries affect the performance of applications [46].

As a final application of the fundamental tensor, we illustrate how it can be employed to develop an efficient algorithm to answer reachability queries for dynamic networks. Here we do not require the network $G$ under consideration (before or after failures) is strongly connected, otherwise the reachability query problem is trivial. For simplicity of exposition, in the following we only consider node failures. Similar in Section 5.3, we add an exogenous node $o$ to network $G$ and connecting all the nodes to it. We note that this extended network $G^o$ has only one recurrent equivalence class, and $F_{sto}$ exists for any pairs of $s$ and $t$. Moreover, $F_{sto}$ is non-zero if and only $t$ is reachable from $s$ in $G$. This is because with non-zero probability a random walk will visit every node that is reachable from $s$ before hitting $o$. By pre-computing the fundamental matrix $F^{(o)}$ once, we can answer any reachability query $R(s, t)$ in constant time using $F^{(o)}$ by performing a table look-up.

Now suppose a set of nodes, $F$, fail. We claim that we can answer the dynamic reachability query $R(s, t, F)$ (after the nodes in $F$ fail, but without prior knowledge of the node failure set $F$) in $O(|F|)$. In particular, if $|F|$
is of a constant order \(O(1)\) compared to the size of network \(|V|\), then the queries are answered in constant \(O(1)\) time. This is achieved by leveraging Theorem 1 for incremental computation of the fundamental matrix. Let \(S = F \cup \{o\}\) and define \(F_{stS}\)

\[
F_{stS} = F_{sto} - F_{sF_o} F^{-1}_{F_Fo} F_{Fto},
\]

which is the tensor form of \(F^S_{st} = F^{(o)}_{st} - F^{(o)}_{sF} (F^{(o)}_{F_F})^{-1} F^{(o)}_{F_Fo} F_{Fto}\). Note that the sub-matrix \((F^{(o)}_{F_F})^{-1}\) is non-singular. This comes from the fact that \(F^{(o)}\) is an inverse M-matrix (an inverse M-matrix is a matrix whose inverse is an M-matrix, hence each of its principal sub-matrix is also an inverse M-matrix. We have the following statement.

**Statement 4.** In the extended network \(G^o\), \(F_{sto}\) is non-zero if and only if \(t\) is reachable from \(s\) in the original network \(G\). Furthermore, if the nodes in the set \(F\) fail, \(F_{stS}\) is non-zero if and only if \(t\) is still reachable from \(s\) in network \(G\) after the failures.

Using the above statement and Theorem 1, we can answer (static and dynamic) reachability queries both before and after failures in constant times (for a constant size node failure set \(F\)) by pre-computing \(F_{::o} (=F^{(o)})\) and storing the results in a table. The method for answering reachability queries is summarized in Algorithm 1. The function \(1_{\{b\}}\) is an indicator function which is equal to 1 if \(b = \text{True}\) and 0 if \(b = \text{False}\).

**Algorithm 1: Answering a Reachability Query**

1. **query:** \(R(s, t, \sim F)\)
2. **input:** transition matrix \(P\) of the extended network \(G^o\)
3. **precomputed oracle:** \(F_{::o} = (I - P^{(o)})^{-1}\)
4. **output:** answer to reachability queries.
5. **if** \(F = \emptyset\) **then**
6. \(R(s, t) = 1\{F_{sto} > 0\}\)
7. **else**
8. \(R(s, t, \sim F) = 1\{F_{sto} - F_{sF_o} F^{-1}_{F_Fo} F_{Fto} > 0\}\)
9. **end if**
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6. Appendix

In this appendix, we provide the detailed derivations regarding the relations between the stochastic form and matrix form of hitting time and hitting costs, respectively.

- **Relation between the stochastic form and matrix form of hitting time**
  Let \( t \) be the only absorbing node and rest of nodes belong to \( T \), then:

  \[
  H_{\{t\}}^{s} = \sum_{k=1}^{\infty} k \sum_{m \in T} \left[ P_{TT}^{k-1} \right]_{sm} [P_{Tt}]_{mt} = \sum_{k=1}^{\infty} k \sum_{m \in T} \left[ P_{TT}^{k-1} \right]_{sm} (1 - \sum_{j \in T} [P_{TT}]_{mj})
  \]

  \[
  = \sum_{k=1}^{\infty} k \left( \sum_{m \in T} \left[ P_{TT}^{k-1} \right]_{sm} - \sum_{j \in T} [P_{TT}]_{sj} \right) = \sum_{m \in T} \sum_{k=1}^{\infty} k \left( [P_{TT}^{k-1}]_{sm} - [P_{TT}]_{sm} \right)
  \]

  \[
  \sum_{m \in T} \sum_{k=1}^{\infty} \left( [P_{TT}^{k-1}]_{sm} - [P_{TT}]_{sm} \right) = \sum_{m \in T} \sum_{k=1}^{\infty} \left( [P_{TT}^{k-1}]_{sm} - [P_{TT}]_{sm} \right)
  \]

  which is the matrix form of hitting time Eq.(5).

- **Relation between the stochastic form and matrix form of hitting cost**
  Let \( Z_{sm} \) be the set of all possible walks from \( s \) to \( m \) and \( \zeta_j \) be the \( j \)-th walk from this set. We use \( Z_{sm}(l) \) to denote the subset of walks whose total length is \( l \), and \( Z_{sm}(k,l) \) to specify the walks which have total length of \( l \) and total step size of \( k \). Recall that a walk (in contrast to a path) can have repetitive nodes, and the length of a walk is the sum of the edge weights in the walk and its step size is the number of edges. Recall that \( \mathbb{P}(\eta = l | X_0 = s) \) denotes the probability of hitting \( t \) in total length of \( l \) when starting from \( s \), which can be obtained from the probability of walks: \( \mathbb{P}(\eta = l | X_0 = s) = \sum_{\zeta_j \in Z_{sm}(l)} \text{Pr}_{\zeta_j} \).

  Probability of walk \( \zeta_j \) denoted by \( \text{Pr}_{\zeta_j} \) is computed by the production over the probabilities of passing edges: \( \text{Pr}_{\zeta_j} = p_{sv_1}p_{v_1v_2}...p_{v_{k-1}m} \), where \( p_{vu} \) is the edge probability from \( v \) to \( u \). The summation over the walk probabilities is computed using the following relation:

  \[
  \sum_{\zeta_j \in Z_{sm}} \text{Pr}_{\zeta_j} = \begin{cases} 
  [P_{TT}]_{sm} & \text{if } m \in T \\
  [P_{TT}^{k-1}P_{Tt}]_{sm} & \text{if } m \in A
  \end{cases}
  \]

  With this introduction, the derivation of the stochastic form of hitting cost Eq.(9) can proceed as follows:
\[ B(s, t) = \sum_{l \in \mathcal{C}} \sum_{k=1}^{< \infty} \sum_{\zeta_j \in \mathcal{Z}_{st}(k,l)} \Pr_{\zeta_j} \]  

\[ = \sum_{l \in \mathcal{C}} \sum_{k=1}^{< \infty} \sum_{\zeta_j \in \mathcal{Z}_{st}(k,l)} l_{\zeta_j} \Pr_{\zeta_j} \]  

\[ = \sum_{\zeta_j \in \mathcal{Z}_{st}} \Pr_{\zeta_j} \sum_{k=1}^{k_{\zeta_j}} w_{v_{k-1}v_k} \]  

\[ = \sum_{\zeta_j \in \mathcal{Z}_{st}} \Pr_{\zeta_j} \sum_{k=1}^{k_{\zeta_j}} \prod_{i=1}^{k} p_{v_{i-1}v_i}.(p_{v_{k}v_{k+1}}w_{v_{k}v_{k+1}}). \prod_{i=k+2}^{k_{\zeta_j}} p_{v_{i-1}v_i} \]  

\[ = \sum_{e_{xy} \in E} p_{xy}w_{xy} \left( \sum_{\zeta_j \in \mathcal{Z}_{ax}} \Pr_{\zeta_j} \right). \left( \sum_{\zeta_j \in \mathcal{Z}_{yt}} \Pr_{\zeta_j} \right) \]  

\[ = \sum_{e_{xy} \in E} p_{xy}w_{xy} \left( \sum_{k} \sum_{\zeta_j \in \mathcal{Z}_{ax}(k)} \Pr_{\zeta_j} \right). \left( \sum_{k} \sum_{\zeta_j \in \mathcal{Z}_{yt}(k)} \Pr_{\zeta_j} \right) \]  

\[ = \sum_{e_{xy} \in E} p_{xy}w_{xy} \left[ P_{TT}^{k} \right]_{ax}. \left( \sum_{k} \left[ P_{TT}^{k-1} P_{TA} \right]_{yt} \right) \]  

\[ = \sum_{e_{xy} \in E} p_{xy}w_{xy} F_{\{t\}}(s, x)Q_{\{t\}}(y, t) \]  

\[ = \sum_{e_{xy} \in E} p_{xy}w_{xy} F_{\{t\}}(s, x) \]  

\[ = \sum_{x} F_{\{t\}}(s, x) \sum_{y \in N_{\text{out}}(x)} p_{xy}w_{xy} \]  

\[ = \sum_{x} F_{\{t\}}(s, x) r_x, \]  

where \( l_{\zeta_j} \) and \( k_{\zeta_j} \) denote the length and step size of a walk \( \zeta_j \), respectively, and \( r_x = \sum_{y \in N_{\text{out}}(x)} p_{xy}w_{xy} \) is the average outgoing cost of node \( x \). In the above derivation, Eq.(86) comes from Eq.(79), and Eq.(88) follows from the fact that \( Q_{\{t\}}(y, t) = 1 \) when having \( t \) as the only absorbing node in the network and reachable from all the other nodes.