Resilient Consensus Through Asynchronous Event-based Communication
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Abstract—We consider resilient versions of discrete-time multi-agent consensus in the presence of faulty or even malicious agents in the network. In particular, we develop event-triggered update rules which can mitigate the influence of the malicious agents and at the same time reduce the necessary communication. Each regular agent updates its state based on a given rule using its neighbors’ information. Only when the triggering condition is satisfied, the regular agents send their current states to their neighbors. Otherwise, the neighbors will continue to use the state received the last time. Assuming that a bound on the number of malicious nodes is known, we propose two update rules with event-triggered communication. They follow the so-called mean subsequence reduced (MSR) type algorithms and ignore values received from potentially malicious neighbors. We provide full characterizations for the necessary connectivity in the network for the algorithms to perform correctly, which are stated in terms of the notion of graph robustness. A numerical example is provided to demonstrate the effectiveness of the proposed approach.

Index Terms—resilient consensus, distributed event-based control, multi-agent systems, discrete-time systems

I. INTRODUCTION

The study of distributed coordination in multi-agent systems has received much attention in a wide range of areas including control, communications, complex networks, and computer science. More recently, it has been recognized that cyber security for such systems is a critical issue since the extensive use of networks for the interactions among agents creates numerous vulnerabilities for potential attacks (e.g., [24]). Different from cyber attacks limited to information technology applications, as multi-agent systems often have physical aspects, attacks may lead to damages in equipments or even accidents.

In large-scale multi-agent systems, consensus problems form one of the fundamental problems (e.g., [21]). There, agents interact locally and exchange their information with each other in order to arrive at the global objective of sharing a common value. In an uncertain environment where faults or even adversarial attacks can be present, it is of great importance to secure consensus algorithms by raising their resilience levels so as to avoid being influenced by such uncertainties in their decision makings. In this context, adversarial agents are those that do not follow the given algorithms and might even attempt to keep the nonfaulty, regular agents from reaching consensus.

In the area of distributed algorithms in computer science, resilient versions of consensus algorithms have long been studied (see, e.g., [2], [16]), and our work follows this line of research. For each regular agent, a simple but effective approach to reduce the influence of potentially misleading information is to ignore the agents whose states are the most different from its own. It is assumed that the nodes know a priori the maximum number $F$ of adversarial agents in the network. Hence, it is useful to remove the $F$ largest values as well as the $F$ smallest values among those received from the neighbors. This class of algorithms are sometimes called the mean subsequence reduced (MSR) algorithms and has been employed in computer science (e.g., [18], [29]), control theory (e.g., [3], [15], [32]), and robotics (e.g., [7], [23]). An important recent progress lies in the characterization of the necessary requirement on the topology of the agent networks. This was initiated by [15], [29], where the relevant notion of robust graphs was proposed. It is also remarked that, as a different class of cyber attacks, the effects of jamming and denial-of-service (DoS) attacks on multi-agent consensus have recently been analyzed in [12], [25].

In this paper, we develop distributed protocols for resilient consensus with a particular emphasis on the communication loads for node interactions. We reduce the transmissions through the so-called event-triggered protocols (e.g., [9]). Under this method, nodes make transmissions only when necessary in the sense that their values sufficiently changed since their last transmissions. The advantage is that the communication can be greatly reduced in frequency, while the achievable level of consensus may leave some gap in the values of the agents. More concretely, we develop two protocols for resilient consensus under event-based communication. Their convergence properties are analyzed, and the requirement for the network topology is fully characterized in terms of robust graphs. We will show through a numerical example how the two protocols differ in the amounts of communication needed for achieving consensus. Event-based protocols have been developed for conventional consensus without malicious agents in, e.g., [6], [8], [11], [17], [19], [20], [26]. Related results can be found in [10], where event-based consensus-type algorithms are developed for the synchronization of clocks possessed by the nodes in wireless sensor networks (WSNs).

The difficulty in applying event-based communication in the context of resilient consensus problems is due to the handling of the errors between the current values and their last transmitted ones. In our approach, we treat such errors...
neighboring nodes. Furthermore, in [22], resilient distributed consensus is studied, where another class of resilience is developed for agents having second-order dynamics, which simply do not use the values that are the most different. Efforts have been made to develop algorithms to detect the adversaries while performing the given task, for example, in [11].

This paper is organized as follows. In Section II, we introduce some preliminaries and then formulate the event-based resilient consensus problem. We propose two event-based resilient consensus algorithms and study their convergence and necessary network structures in Sections III and IV. A numerical example is given in Section V to demonstrate the effectiveness of the proposed algorithms. We provide concluding remarks in Section VI. This paper is an extended version of [30] with full proofs of the results and further discussions.

II. EVENT-BASED RESILIENT CONSENSUS PROBLEM

A. Preliminaries on graphs

Here, some basic notations related to graphs are introduced for the analysis in this paper.

Consider the directed graph \( G = (V, E) \) consisting of \( n \) nodes. Here the set of nodes is denoted by \( V = \{1, 2, \ldots, n\} \) and the edge set by \( E \subseteq V \times V \). The edge \((j, i) \in E\) indicates that node \( j \) can send a message to node \( i \) and is called an incoming edge of node \( i \). Let \( \mathcal{N}_i = \{j : (j, i) \in E\} \) be the set of neighbors of node \( i \). The number of neighbors of node \( i \) is called its degree and is denoted as \( d_i = |\mathcal{N}_i| \). The path from node \( i_1 \) to node \( i_p \) is denoted as the sequence \((i_1, i_2, \ldots, i_p)\), where \((i_j, i_{j+1}) \in E\) for \( j = 1, 2, \ldots, p-1 \). The graph \( G \) is said to have a spanning tree if there exists a node from which there is a path to all other nodes of this graph.

To establish resilient consensus results, an important topological notion is that of robustness of graphs [15].

Definition 1. The graph \( G = (V, E) \) is called \((r, s)\)-robust \((r, s < n)\) if for any two nonempty disjoint subsets \( V_1, V_2 \subseteq V \), one of the following conditions is satisfied:

1) \( |\mathcal{X}_{V_1}^r| = V_1 \),
2) \( |\mathcal{X}_{V_2}^s| = V_2 \),
3) \(|\mathcal{X}_{V_1}^r| + |\mathcal{X}_{V_2}^s| \geq s \),

where \( \mathcal{X}_{V_i}^r \) is the set of all nodes in \( V_i \) which have at least \( r \) neighbors outside \( V_i \) for \( i = 1, 2 \).

In Fig. 1 we display an example graph with five nodes. It can be checked to have just enough connectivity to be \((2, 2)\)-robust. If any of the edges are removed, this level of robustness will be lost.
Several basic properties of robust graphs in [15] are summarized below.

**Lemma 1.** An \((r, s)\)-robust graph \(G\) satisfies the following:
1. \(G\) is \((r', s')\)-robust, where \(0 \leq r' \leq r\), \(1 \leq s' \leq s\), and in particular, it is \(r\)-robust.
2. \(G\) has a directed spanning tree. Moreover, it is 1-robust if and only if it has a directed spanning tree.
3. \(r \leq \lfloor n/2 \rfloor\). Furthermore, if \(r = \lfloor n/2 \rfloor\), \(G\) is a complete graph, where the ceil function \(\lceil y \rceil\) gives the smallest integer value greater than or equal to \(y\).
4. The degree \(d_i\) for \(i \in V\) is lower bounded as \(d_i \geq r + s - 1\) if \(s < r\) and \(d_i \geq 2r - 2\) if \(s \geq r\).

Moreover, a graph \(G\) is \((r, s)\)-robust if it is \((r + s - 1)\)-robust.

In consensus problems, the property 2) in the lemma above is of interest. Robust graphs may not be strongly connected in general, but this property indicates that the notion of robust graphs is a generalization of graphs containing directed spanning trees, which are of great relevance in the literature of consensus [21].

As we will see, robust graphs play a key role in characterizing the necessary network structure for achieving resilient consensus. It should however be noted that checking the robustness of a given graph involves combinatorial computation and is thus difficult in general [28], [32], [33].

### B. Event-based consensus protocol

We consider the directed graph \(G\) of \(n\) nodes. The nodes in \(V\) are partitioned into two sets: \(R\) denotes the set of regular nodes and \(A = V \setminus R\) represents the set of adversarial nodes. The regular nodes will follow the designed algorithm exactly while the adversarial nodes can have different update rules from that of the regular nodes. The attacker is allowed to know the states of the regular nodes and the graph topology, and to choose any nodes as members of \(A\).

We introduce the event-based protocol for the regular nodes to achieve consensus. It can be outlined as follows: At each discrete-time instant \(k \in \mathbb{Z}_+\), the nodes make updates, but whether they transmit their current values to neighbors depends on the triggering function. More concretely, each node \(i\) has an auxiliary variable which is its state value communicated the last time and compares it with its own current state. If the current state has changed sufficiently, then it will be sent to its neighbors and the auxiliary variable will be replaced.

We employ the following event-based update rule for the agent system:

\[
x_i(k+1) = x_i(k) + \sum_{j \in N_i} a_{ij}(k) (\hat{x}_j(k) - x_i(k)),
\]

where \(x_i(k) \in \mathbb{R}\) is the state of node \(i\), and \(\hat{x}_j(k) \in \mathbb{R}\) is the last communicated state of node \(j\) at time \(k\). The latter is defined as \(\hat{x}_j(k) = x_j(t^*_j)\), \(k \in [t^*_j, t^*_{j+1}]\), where \(t^*_0, t^*_1, \ldots\) denote the transmission times of node \(j\) determined by the triggering function to be given below. The initial values \(x_i(0), \hat{x}_j(0)\) are given, \(a_{ij}(k)\) is the weight for the edge \((j, i)\), which satisfies \(\gamma \leq a_{ij}(k) < 1\), and \(\gamma\) is the lower bound of the weights with \(0 < \gamma \leq 1/2\). The update rule above can be seen as a discrete-time counterpart of the event-based consensus algorithms in, e.g., [8], [17], [26].

Denote the error between the updated state \(x_i(k+1)\) and the last communicated state \(\hat{x}_i(k)\) by \(e_i(k+1) = \hat{x}_i(k) - x_i(k+1)\) with \(e_i(0) = 0\). Then we introduce the triggering function as

\[
f_i(k+1) = |e_i(k+1)| - (c_0 + c_1 e^{-\alpha(k+1)}),
\]

where \(c_0, c_1, \alpha > 0\) are positive constants.

#### C. Adversary model and resilient consensus

For the adversarial nodes in the set \(A\), we use the model introduced in [15]. The classification is based on their number, locations, and behaviors.

**Definition 2. (Malicious nodes):** We say that an adversarial node is malicious if it sends the same value to all of its neighbors at each transmission.

Adversarial nodes more difficult to deal with are those that can transmit different values to different neighbors in an arbitrary way. Such nodes are referred to as being Byzantine [29]. The motivation for considering malicious nodes as defined above comes, for example, from the applications of WSNs, where sensor nodes communicate to their neighbors by broadcasting their data.

We also set a bound on the number of malicious nodes in the network. In this paper, we will deal with networks of the so-called \(F\)-total model as defined below.

**Definition 3.** \((F\)-total and \(F\)-local models): For \(F \in \mathbb{N}\), we say that the adversarial set \(A\) follows an \(F\)-total model if \(|A| \leq F\), and an \(F\)-local model if \(|N_i \cap A| \leq F\) for each node \(i \in R\).

Let the number of malicious agents be denoted by \(N_m = |A|\). Then, let \(N = |V| - N_m\) be the number of regular agents.

Now, we introduce the notion of resilient consensus for multi-agent systems.

**Definition 4.** (Resilient consensus): If for any possible sets and behaviors of the malicious agents and any initial state values of the regular nodes, the following conditions are satisfied, then the multi-agent system is said to reach resilient consensus:

1. Safety condition: There exists an interval \(S \subset \mathbb{R}\) such that \(x_i(k) \in S\) for all \(i \in R, k \in \mathbb{Z}_+\).
2. Consensus condition: There exists a nonnegative number \(c\) such that \(\limsup_{k \to \infty} |x_i(k) - x_j(k)| \leq c\) for all \(i, j \in R\).

In this paper, we would like to design event-based update rules for the regular agents to reach resilient consensus under the \(F\)-total model by using only local information obtained from their neighbors.

### III. ROBUST PROTOCOLS FOR EVENT-BASED CONSENSUS

#### A. E-MSR algorithm

In this section, we outline a resilient consensus protocol to solve the resilient consensus problem. As discussed above, every node makes an update at every time step in a synchronous manner, but only when an event happens, the auxiliary values
will be updated and then sent to neighbors. The basis of the algorithm follows those in the works of, e.g., [3], [4], [15]. The algorithm in this paper is called the event-based mean subsequence reduced (E-MSR) algorithm.

The E-MSR algorithm has four steps as follows:

1. (Collecting neighbors’ information) At each time step \( k \), every regular node \( i \in \mathcal{R} \) uses the values \( \hat{x}_j(k), j \in \mathcal{N}_i \), most recently communicated from the neighbors as well as its own value \( x_i(k) \) and sorts them from the largest to the smallest.

2. (Deleting suspicious values) Comparing with \( x_i(k) \), node \( i \) removes the \( F \) largest and \( F \) smallest values from its neighbors. If the number of values larger or smaller than \( x_i(k) \) is less than \( F \), then all of them are removed. The removed data is considered as suspicious and will not be used in the update. The set of the node indices of the remaining values is written as \( \mathcal{M}_i(k) \subset \mathcal{N}_i \).

3. (Local update) Node \( i \) updates its state by

\[
x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k) (\hat{x}_j(k) - x_i(k)).
\]

4. (Communication update) Node \( i \) checks if its own triggering function \( f_i(k+1) \) in (1) is positive or not. Then, it sets \( \hat{x}_i(k+1) \) as

\[
\hat{x}_i(k+1) = \begin{cases} x_i(k+1) & \text{if } f_i(k+1) > 0, \\ \hat{x}_i(k) & \text{otherwise.} \end{cases}
\]

The communication rule in this algorithm shows that only when the current value has varied enough to exceed a threshold, then the auxiliary variable will be updated, and only at this time the node sends its value to its neighbors. This event triggering scheme can significantly reduce the communication burden as we will see in the numerical example in Section V.

B. Protocol 1

The first protocol of this paper is the E-MSR algorithm as stated above, which will be referred to as Protocol 1. We are now ready to present our main result for this protocol.

We introduce two kinds of minima and maxima of the states of the regular agents: The first involves only the states as \( \overline{x}(k) = \max_{i \in \mathcal{R}} x_i(k) \) and \( \underline{x}(k) = \min_{i \in \mathcal{R}} x_i(k) \) while the second uses also the auxiliary variables as \( \overline{\hat{x}}(k) = \min_{i \in \mathcal{R}} \{x_i(k), \hat{x}_i(k)\} \) and \( \underline{\hat{x}}(k) = \max_{i \in \mathcal{R}} \{x_i(k), \hat{x}_i(k)\} \). The safety interval \( S \) is chosen as \( S = [\underline{x}(0), \overline{x}(0)] \).

Theorem 1. Under the \( F \)-total model, the regular agents with E-MSR using (2) and (3) reach resilient consensus if and only if the underlying graph is \((F+1, F+1)\)-robust. The safety interval is given by \( S = [\underline{x}(0), \overline{x}(0)] \), and the upper bound on consensus error is

\[
\limsup_{k \to \infty} |x_i(k) - x_j(k)| \leq \min \left\{ \frac{4c_0 N}{\gamma N}, |S| \right\} \text{ for } i, j \in \mathcal{R},
\]

where \( |S| = \overline{x}(0) - \underline{x}(0) \).

Proof. (Necessity) This essentially follows from [15], which considers the special case without the triggering function, that is, \( c_0 = c_1 = 0 \).

(Sufficiency) We first show that the interval \( S = [\hat{x}(0), \overline{x}(0)] \) satisfies the safety condition by induction. Note that the update rule (2) can be rewritten as

\[
x_i(k+1) = a_{ii}(k)x_i(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k)\hat{x}_j(k)
\]

where

\[
a_{ii}(k) = 1 - \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k). \]

At time \( k = 0 \), it is clear by definition that \( x_i(0), \hat{x}_i(0) \in S, i \in \mathcal{R} \). Suppose that for each regular agent \( i \), \( x_i(k), \hat{x}_i(k) \in S \). Then, for agent \( i \), its neighbors in \( \mathcal{M}_i(k) \) take values only in \( S \), since there are agents with values outside \( S \) at most \( F \), and they are ignored in step 2 of the E-MSR. From (4), we have \( x_i(k+1) \in S \). Moreover, by (3), it follows that \( \hat{x}_i(k+1) \in S \). Thus, \( S \) is the safety interval.

We next establish the consensus condition. Note that for time \( k \in [t_f, t_{f+1}) \) between two triggering instants, we have \( f_i(k) \leq 0 \). Moreover, for the neighbor node \( j \in \mathcal{N}_i \), if \( f_j(k) > 0 \), then we have \( \hat{x}_j(k) = x_j(k) \). If \( f_j(k) \leq 0 \), then \( \hat{x}_j(k) = \hat{x}_j(k-1) = x_j(k) + e_j(k) \). We define

\[
\hat{e}_j(k) = \begin{cases} e_j(k) & \text{if } f_j(k) \leq 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Note that

\[
|\hat{e}_j(k)| \leq c_0 + c_1 e^{-\alpha k}, \ \forall k \geq 0.
\]

Then, we can write (4) as

\[
x_i(k+1) = a_{ii}(k)x_i(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k) (x_j(k) + \hat{e}_j(k)).
\]

This can be bounded by using the maximum state \( \overline{x}(k) \) as

\[
x_i(k+1) \leq a_{ii}(k)\overline{x}(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k) (\overline{x}(k) + \hat{e}_j(k))
\]

\[
= \overline{x}(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k)\hat{e}_j(k)
\]

\[
\leq \overline{x}(k) + \max_{j \in \mathcal{M}_i(k)} |\hat{e}_j(k)|.
\]

Thus, by (7) it follows

\[
x_i(k+1) \leq \overline{x}(k) + c_0 + c_1 e^{-\alpha k}.
\]

Let \( V(k) = \overline{x}(k) - \underline{x}(k) \). Then we introduce two sequences given by

\[
\overline{x}_0(k+1) = \overline{x}_0(k) + c_0 + c_1 e^{-\alpha k},
\]

\[
\underline{x}_0(k+1) = \underline{x}_0(k) - c_0 - c_1 e^{-\alpha k},
\]

where \( \overline{x}_0(0) = \overline{x}(0) - \sigma_0 \), and \( \underline{x}_0(0) = \underline{x}(0) + \sigma_0 \) with \( \sigma_0 = \sigma V(0) \). We next introduce another sequence \( \varepsilon_0(k) \) defined by

\[
\varepsilon_0(k+1) = \gamma \varepsilon_0(k) - (1 - \gamma) \sigma_0,
\]

where \( \varepsilon_0(0) = \varepsilon V(0) \). Take the parameters \( \varepsilon \) and \( \sigma \) so that

\[
\varepsilon + \sigma = 1, \ \ 0 < \sigma < \frac{\gamma N}{1 - \gamma N} \varepsilon.
\]
For the sequence $\varepsilon_0(k)$, let
\[
\overline{X}_0(k, \varepsilon_0(k)) = \{ j \in V : x_j(0) > \overline{X}_0(k) - \varepsilon_0(k) \},
\]
\[
\underline{X}_0(k, \varepsilon_0(k)) = \{ j \in V : x_j(0) < \underline{X}_0(k) + \varepsilon_0(k) \}.
\]
These two sets are both nonempty at time $k = 0$ and, in particular, each contains at least one regular node; this is because by definition, $\pi_0(0) > \overline{X}_0(0) - \varepsilon_0(0)$ and $\underline{X}_0(0) + \varepsilon_0(0) > \underline{X}_0(0) + \varepsilon_0(0)$.

In the following, we show that $\overline{X}_0(k, \varepsilon_0(k))$ and $\underline{X}_0(k, \varepsilon_0(k))$ are disjoint sets. To this end, we must show
\[
\pi_0(k) - \varepsilon_0(k) \geq \underline{X}_0(k) + \varepsilon_0(k).
\]
By (10) and (11) for $\pi_0(k)$ and $\underline{X}_0(k)$, we have
\[
(\pi_0(k) - \varepsilon_0(k)) - (\underline{X}_0(k) + \varepsilon_0(k)) = (\pi_0(k) - \underline{X}_0) + c_0k + c_1 \frac{1 - e^{-\alpha k}}{1 - e^{-\alpha}} - 2\varepsilon_0(k).
\]
Then by substituting $\pi_0(0) = \pi(0) - \sigma_0$ and $\underline{X}_0(0) = \underline{X}(0) + \sigma_0$ into the right-hand side of (14), we obtain
\[
(\pi_0(k) - \varepsilon_0(k)) - (\underline{X}_0(k) + \varepsilon_0(k)) = (\pi(0) - \underline{X}(0)) - 2\sigma_0 + 2c_0k + 2c_1 \frac{1 - e^{-\alpha k}}{1 - e^{-\alpha}} - 2\varepsilon_0(k) = V(0) - 2\sigma V(0) + 2c_0k + 2c_1 \frac{1 - e^{-\alpha k}}{1 - e^{-\alpha}} - 2\varepsilon_0(k).
\]
By (12) and $0 < \gamma \leq 1/2$, we easily have that $\varepsilon_0(k + 1) < \varepsilon_0(k)$, and hence $\varepsilon_0(k) < \varepsilon_0(0) = \varepsilon V(0)$. We thus obtain
\[
(\pi_0(k) - \varepsilon_0(k)) - (\underline{X}_0(k) + \varepsilon_0(k)) > (1 - 2\varepsilon - 2\varepsilon) V(0) + 2c_0k + 2c_1 \frac{1 - e^{-\alpha k}}{1 - e^{-\alpha}} > 0,
\]
where the last inequality holds since $\sigma + \varepsilon = 1/2$ from (13).

Consequently, it follows that $\overline{X}_0(k, \varepsilon_0(k))$ and $\underline{X}_0(k, \varepsilon_0(k))$ are disjoint sets.

From the above, we have that the two sets $\overline{X}_0(k, \varepsilon_0(0))$ and $\underline{X}_0(0, \varepsilon(0))$ are nonempty with at least one regular node in each and moreover disjoint. Therefore, by the assumption of $(F + 1, F + 1)$-robustness, there are three cases:

1) All nodes in $\overline{X}_0(0, \varepsilon(0))$ have $F + 1$ neighbors or more from outside.

2) All nodes in $\underline{X}_0(0, \varepsilon(0))$ have $F + 1$ neighbors or more from outside.

3) The total number of nodes in $\overline{X}_0(0, \varepsilon(0))$ and $\underline{X}_0(0, \varepsilon(0))$ having $F + 1$ neighbors or more from outside of its own set is no smaller than $F + 1$.

Notice that in any of the three cases, there exists at least one regular agent $i \in R$ in either $\overline{X}_0(0, \varepsilon(0))$ or $\underline{X}_0(0, \varepsilon(0))$ that has $F + 1$ neighbors or more from outside of its own set. In the following, we suppose that this node $i$ belongs to $\overline{X}_0(0, \varepsilon(0))$. A similar argument holds for the case when it is in $\underline{X}_0(0, \varepsilon(0))$.

Now, we go back to (3) and rewrite it by partitioning the neighbor node set $M_i(k)$ of node $i$ into two parts: The nodes which belong to $\overline{X}_0(k, \varepsilon_0(k))$ and those that do not. Since node $i$ has at least $F + 1$ neighbors outside $\overline{X}_0(k, \varepsilon_0(k))$, the latter set is nonempty. Hence, we obtain
\[
x_i(k + 1) = a_{ii}(k)x_i(k) + \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)x_j(k)
\]
\[
+ \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)x_j(k) + \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\hat{e}_j(k),
\]
where we use the shorthand notation $\overline{X}_0$ for $\overline{X}_0(0, \varepsilon_0(k))$, and $a_{ii}(k)$ is given in (5). Then, we can bound this from above as
\[
x_i(k + 1)
\]
\[
\leq a_{ii}(k)\overline{X}_0(k) + \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\overline{X}_0(k) + \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\hat{e}_j(k)
\]
\[
= \left(1 - \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\right)\overline{X}_0(k) + \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\hat{e}_j(k).
\]

We next show that $\overline{X}_0(k) \leq \overline{X}_0(k) + \sigma_0$ (and similarly, $\underline{X}_0(k) \geq \underline{X}_0(k) - \sigma_0$) by induction. For $k = 0$, by definition, we have $\overline{X}_0(0) = \overline{X}_0(0) + \sigma_0$. Suppose that $\overline{X}_0(k) \leq \overline{X}_0(k) + \sigma_0$. Then from (5) and (10), we have
\[
\overline{X}_0(k + 1) \leq \overline{X}_0(k) + \max_j |\hat{e}_j(k)| \leq \overline{X}_0(k) + \sigma_0 + c_1 e^{-\alpha k}
\]
\[
\leq \overline{X}_0(k) + \sigma_0 + c_1 e^{-\alpha k} = \overline{X}_0(k + 1) + \sigma_0.
\]

Then (16) can be further bounded as
\[
x_i(k + 1)
\]
\[
\leq \left(1 - \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\right)\overline{X}_0(k) + \sigma_0
\]
\[
+ \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\overline{X}_0(k) - \varepsilon_0(k) + \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\hat{e}_j(k)
\]
\[
\leq \overline{X}_0(k) + \left(1 - \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\right)\sigma_0
\]
\[
- \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\varepsilon_0(k) + \sum_{j \in M_i(k) \cap \overline{X}_0} a_{ij}(k)\hat{e}_j(k)\right].
\]

We also show that $\varepsilon_0(k) > 0$ holds for $k = 0, 1, \ldots, N$. It is clear from (12) that $\varepsilon_0(k + 1) < \varepsilon_0(k)$. Thus we only need to guarantee $\varepsilon_0(N) > 0$. By (12), $\varepsilon_0(N)$ can be written as
\[
\varepsilon_0(N) = \gamma^N \varepsilon_0(0) - \sum_{i = 0}^{N - 1} \gamma^i (1 - \gamma) \sigma_0
\]
\[
= \gamma^N \varepsilon V(0) - \frac{1 - \gamma^N}{1 - \gamma} (1 - \gamma) \sigma V(0)
\]
\[
= \left(\gamma^N \varepsilon - (1 - \gamma^N) \sigma\right) V(0).
\]
This is positive because we have chosen $\sigma$ as in (14).
Hence, (17) can be written as
\[ x_i(k + 1) \leq \pi_0(k) + (1 - \gamma)\sigma_0 - \gamma \varepsilon_0(k) + c_0 + c_1 e^{-\alpha k} \]
where in the inequality, we used the fact that there always exists \( j \) not in \( \mathcal{K}_0(k, \varepsilon_0(k)) \). This relation shows that if an update happens at node \( i \), then this node will move out of \( \mathcal{K}_0(k + 1, \varepsilon_0(k + 1)) \). We note that inequality (18) also holds for the regular nodes that are not inside \( \mathcal{K}_0(k, \varepsilon_0(k)) \) at time \( k \). This means that such nodes cannot move in \( \mathcal{K}_0(k + 1, \varepsilon_0(k + 1)) \). It is also similar with \( \mathcal{K}_0(k + 1, \varepsilon_0(k + 1)) \).

Thus, after \( N \) time steps, all regular nodes will be out of at least one of the two sets \( \mathcal{K}_0(N, \varepsilon_0(N)) \) and \( \mathcal{K}_0(N, \varepsilon_0(N)) \). We suppose that \( \mathcal{K}_0(N, \varepsilon_0(N)) \cap \mathcal{R} \) is empty. Then we have \( \pi_0(N) - \varepsilon_0(N) \). It hence follows that
\[ V(N) = \pi(N) - \pi_0(N) \]
\[ \leq \pi_0(0) - \varepsilon_0(0) + 2c_0N + 2 \sum_{i=0}^{N-1} c_1 e^{-\alpha i} - \varepsilon_0(N) + \sigma_0 \]
\[ = (\pi(0) - \sigma_0) - (\pi(0) + \sigma_0) + 2c_0N + 2c_1 \frac{1 - e^{-\alpha N}}{1 - e^{-\alpha}} - \varepsilon_0(N) + \sigma_0 \]
\[ = V(0) + 2c_0N + 2c_1 \frac{1 - e^{-\alpha N}}{1 - e^{-\alpha}} - \sigma V(0) \]
\[ - (\gamma N \varepsilon - (1 - \gamma N) \sigma) V(0) \]
\[ = (1 - \gamma N (\varepsilon + \sigma)) V(0) + 2c_0N + 2c_1 \frac{1 - e^{-\alpha N}}{1 - e^{-\alpha}}. \]

By \( \varepsilon + \sigma = 1/2 \) in (13), we have
\[ V(N) \leq \left(1 - \frac{\gamma N}{2}\right) V(0) + 2c_0N + 2c_1 \frac{1 - e^{-\alpha N}}{1 - e^{-\alpha}}. \]

If there are more updates by node \( i \) after time \( k = N \), this argument can be extended further as
\[ V(IN) \leq \left(1 - \frac{\gamma N}{2}\right) V((l-1)N) \]
\[ + 2c_0N + 2c_1 \frac{1 - e^{-\alpha N}}{1 - e^{-\alpha}} \]
\[ \leq \left(1 - \frac{\gamma N}{2}\right)^l V(0) + \sum_{t=0}^{l-1} \left(1 - \frac{\gamma N}{2}\right)^{l-1-t} \]
\[ \times \left(2c_0N + 2c_1 \frac{1 - e^{-\alpha N}}{1 - e^{-\alpha}} \right) \]
\[ \leq \left(1 - \frac{\gamma N}{2}\right)^l V(0) + \frac{1 - \left(1 - \frac{\gamma N}{2}\right)^t}{1 - \left(1 - \frac{\gamma N}{2}\right)^t} 2c_0N \]
\[ + 2c_1 \frac{1 - e^{-\alpha N}}{1 - e^{-\alpha}} \left(1 - \frac{\gamma N}{2}\right)^t \frac{1 - \left(1 - \frac{\gamma N}{2}\right)^t}{1 - \left(1 - \frac{\gamma N}{2}\right)^t} e^{-\alpha N}. \]

From (19) we can easily obtain
\[ \lim_{l \to \infty} V(IN) \leq \frac{2c_0N}{1 - \left(1 - \frac{\gamma N}{2}\right)^t} = \frac{4c_0N}{\gamma N} \]

Now we show the dynamics of \( V(IN + t) \) for \( t = 0, 1, \ldots, N - 1 \). The analysis is similar and we can obtain an inequality like (19), where the only difference is that in the derivation, \( V(0) \) is replaced with \( V(t) \). From the safety condition, we know that \( V(k) \leq |S| \) for all \( k \). Therefore, we finally arrive at
\[ \lim_{l \to \infty} V(IN + t) \leq \min \left\{ \frac{4c_0N}{\gamma N}, |S| \right\}. \]

This completes the proof of the consensus condition.

The above result shows that the multi-agent system is guaranteed to reach resilient consensus despite the presence of \( F \)-total malicious agents. First, the width of the safety interval \( S \) is determined by the initial states of the regular agents. Second, the error that may remain after achieving resilient consensus is also proportional to \( c_0 \). The parameters can be set to any value by the designer and, clearly, by taking \( c_0 = 0 \), exact consensus can be achieved at the expense of having more communications. The role of \( c_1 \) and \( \alpha \) is to reduce the communication in the early stage by making the threshold in the triggering function large. We note that the exponential decaying bound by \( c_1 \) and \( \alpha \) can also decrease the communication in the long run.

Compared with the results of conventional event-based consensus in [26], our upper bound for the consensus condition is larger and thus more conservative. The reason is that due to the malicious agents, the analysis methods of previous works are not applicable to this problem. Our analysis follows those in resilient consensus problems such as [4]. The differences between these two methods lead to the gap in the bounds. We will see the effects of the parameters of the event-triggering function through a numerical example in Section V. A related result for the case of \( F \)-local model can be found in [13] with a particular application to clock synchronization in WSNs. It studies a resilient consensus problem with decaying noise that arises in the system due to the interactions among clock states.

**Remark 1.** We should highlight that in the discrete-time domain, event-based consensus algorithms must be carefully designed especially in the resilient case. We can construct another resilient consensus algorithm motivated by the structures found in [26], [31], which deal with continuous-time multi-agent systems, as
\[ x_i(k + 1) = x_i(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k) (\hat{x}_j(k) - \hat{x}_i(k)). \]

The modification may appear minor as the only difference is that \( \hat{x}_i(k) \) is used instead of \( x_i(k) \) in the second term of the right-hand side.

We however found that for this consensus protocol, the E-MSR algorithm cannot guarantee a safety interval that is bounded if \( c_0 \neq 0 \). We can show that under this update rule, the states of the regular agents remain in the time-varying interval of the form \( [\pi(0) - g(k), \pi(0) + g(k)] \), where \( g(k) = 2c_0k + 2c_1(1 - e^{-\alpha k})/(1 - e^{-\alpha}) \). Notice that the upper and lower boundary points are linear functions of time \( k \) and hence unbounded. Moreover, compared with Protocol 1, the upper bound for the error in the consensus condition is double
as $8c_0N/\gamma^N$. These results can be obtained by following a proof similar to that of Theorem 1.

The reason for the unbounded safety interval of this alternative update rule is that both the current value $x_i(k)$ and the communicated value $\hat{x}_i(k)$ of node $i$ are used in (20). The error between these two values will accumulate if there is a fixed triggering bound $c_0$.

In the next section, we present yet another protocol by further changing the terms in the update rule slightly. It will be found to have an impact especially in the behaviors of state convergence and communication event times.

IV. PROTOCOL 2

In this section, we provide our second resilient consensus algorithm, referred to as Protocol 2.

To this end, we modify the update rule (2) in a way different from the protocol (20) discussed in Remark 1. It is pointed out that in Protocol 1, for obtaining the new state $x_i(k+1)$ of agent $i$, its own data appears only through the current state $x_i(k)$. On the one hand, this means that even when the new state is not communicated, it still needs to be stored at every time step. On the other, as the current state $x_i(k)$ is newer than $\hat{x}_i(k)$, it seems desirable for speeding up the convergence. We will however show that it may be better to use only $\hat{x}_i(k)$ for both storage and convergence reasons. The protocol introduced below is motivated by those in [10], [31].

In the local update, for $k \in \mathbb{Z}_+$, every regular node $i \in \mathcal{R}$ updates its current state by

$$x_i(k+1) = \hat{x}_i(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k) (\hat{x}_j(k) - \hat{x}_i(k)). \quad (21)$$

Note that the new state $x_i(k+1)$ need not be stored until the next time step, but is merely used for checking the condition of the triggering function $f_i(k+1)$ in (1). Accordingly, in the E-MSR, steps 1 and 2 should be adjusted so that agent $i$ uses $\hat{x}_i(k)$ instead of $x_i(k)$ in determining the neighbor set $\mathcal{M}_i(k)$.

Then we are ready to present our second main result of this paper, which is regarding Protocol 2.

**Theorem 2.** Under the $F$-total malicious model, the normal agents with E-MSR using (21) and (3) reach resilient consensus if and only if the underlying graph is $(F+1, F+1)$-robust. The safety interval is given by $\mathcal{S} = [\hat{x}(0), \bar{x}(0)]$, and the upper bound of consensus error for $i, j \in \mathcal{R}$ is

$$\limsup_{k \to \infty} |x_i(k) - x_j(k)| \leq \min \left\{ c_0, \frac{1}{\gamma^{N-1} - (1 - \gamma)}, |\mathcal{S}| \right\}. \quad (22)$$

**Proof.** The necessity part follows similar lines as those in the proof of Theorem 1. In the following, we thus give the sufficiency part.

First, we establish the safety condition in the sense of $x_i(k), \hat{x}_i(k) \in \mathcal{S}$ for regular nodes $i$. This is done by induction. At $k = 0$, for each $i \in \mathcal{R}$, it holds $x_i(0), \hat{x}_i(0) \in \mathcal{S}$ by definition. Next, assume that at time $k$, we have $x_i(k), \hat{x}_i(k) \in \mathcal{S}$ for $i \in \mathcal{R}$. Then, for agent $i$, its neighbors $j \in \mathcal{M}_i(k)$ satisfy $\hat{x}_j(k) \in \mathcal{S}$ since there are at most $F$ agents with values outside $\mathcal{S}$, and they are ignored in step 2 of the E-MSR. From the update rule (21), we have

$$x_i(k+1) = a_{ii}(k)\hat{x}_i(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k)\hat{x}_j(k) \leq a_{ii}(k)\bar{x}(k) + \sum_{j \in \mathcal{M}_i(k)} a_{ij}(k)\bar{x}(k) = \bar{x}(k), \quad (23)$$

where $a_{ii}(k)$ is given in (5). The inequality (23) means that the upper bound of every regular node is nonincreasing. Similarly, we have $x_i(k+1) \geq \hat{x}_i(k)$, so we obtain $x_i(k) \in \mathcal{S}$ for $k \geq 0$. Furthermore, by (3), it holds that $\hat{x}_i(k+1) \in \mathcal{S}$. Hence, we have $\mathcal{S}$ as the safety interval.

For the consensus condition part, we first sort the regular communicated values $\hat{x}_i(k), i \in \mathcal{R}$, at time $k$ in the entire graph from the smallest to the largest. Denote by $s_i(k)$ the index of the agent taking the $i$th value from the smallest. Hence, the values are sorted as $\hat{x}_{s_1}(k) \leq \hat{x}_{s_2}(k) \leq \cdots \leq \hat{x}_{s_N}(k)$.

Introduce two sequences of conditions for the relation of each gap between two nodes. The first is given from below as

$$A_1: \hat{x}_{s_1}(k) - \hat{x}_{s_1}(k) \leq (c_0 + c_1e^{-\alpha k})/\gamma,$$

$$A_2: \hat{x}_{s_3}(k) - \hat{x}_{s_2}(k) \leq (c_0 + c_1e^{-\alpha k})/\gamma^2,$$

\[ \cdots \]

$$A_{N-1}: \hat{x}_{s_N}(k) - \hat{x}_{s_{N-1}}(k) \leq (c_0 + c_1e^{-\alpha k})/\gamma^{N-1}. $$

The other sequence is from above as

$$B_N: \hat{x}_{s_N}(k) - \hat{x}_{s_{N-1}}(k) \leq (c_0 + c_1e^{-\alpha k})/\gamma,$$

$$B_{N-1}: \hat{x}_{s_{N-1}}(k) - \hat{x}_{s_{N-2}}(k) \leq (c_0 + c_1e^{-\alpha k})/\gamma^2,$$

\[ \cdots \]

$$B_2: \hat{x}_{s_3}(k) - \hat{x}_{s_2}(k) \leq (c_0 + c_1e^{-\alpha k})/\gamma^{N-1}. $$

Denote by $j_A$ the minimum $j$, $1 \leq j \leq N - 1$, such that the condition $A_j$ is not satisfied. Also, denote by $j_B$ the maximum $j$, $2 \leq j \leq N$, such that the condition $B_j$ is not satisfied. Thus we have

$$\hat{x}_{s_{j_A}}(k) - \hat{x}_{s_{j_A}}(k) > \frac{c_0 + c_1e^{-\alpha k}}{\gamma^{j_A}}. \quad (24)$$

Moreover, the conditions $A_1$ to $A_{j_A-1}$ and $B_{j_B+1}$ to $B_N$ are satisfied. Then we introduce two sets

$$X_1(k, k') = \left\{ j \in \mathcal{V} : \hat{x}_j(k') < \hat{x}_{s_{j_A}}(k) + c_0 + c_1e^{-\alpha k} \right\},$$

$$X_2(k, k') = \left\{ j \in \mathcal{V} : \hat{x}_j(k') > \hat{x}_{s_{j_B}}(k) - c_0 - c_1e^{-\alpha k} \right\},$$

where $0 \leq k \leq k'$.

There are two cases concerning the relationship between $j_A$ and $j_B$. We study them separately below.

Case 1: $j_A < j_B$. Let the two subsets of the regular nodes be $V_1 = \{ s_1(k), s_2(k), \ldots, s_{j_A}(k) \}$ and $V_2 = \{ s_{j_B}(k), \ldots, s_N(k) \}$. Note that all nodes in $V_1$ are inside $X_1(k, k)$, and those in $V_2$ are inside $X_2(k, k)$. Hence, $X_1(k, k)$ and $X_2(k, k)$ are nonempty. They are moreover disjoint. This is because by using the two inequalities in (24), from $1 \leq j_A < j_B \leq N$ and $0 < \gamma \leq 1/2$, it follows that

$$\hat{x}_{s_{j_B}}(k) - \hat{x}_{s_{j_A}}(k) > \max \left\{ \frac{1}{\gamma^{j_A}}, \frac{1}{\gamma^{N-j_B+1}} \right\} (c_0 + c_1e^{-\alpha k}) \geq 2(c_0 + c_1e^{-\alpha k}).$$
Thus, the $(F + 1, F + 1)$-robust graph guarantees that some regular node $i$ in $X_1(k, k)$ or $X_2(k, k)$ has at least $F + 1$ neighbors outside. We suppose that $i \in X_1(k, k)$. By (21),

$$x_i(k + 1) = a_{ii}(k)\hat{x}_i(k) + \sum_{j \in M_i(k) \cap X_i} a_{ij}(k)\hat{x}_j(k) + \sum_{j \in M_i(k) \cap X_i} a_{ij}(k)\hat{x}_j(k),$$

where the simplified notation $X_1$ is used for $X_1(k, k)$. Since $M_i(k) \setminus X_1(k, k)$ is not empty, we have

$$x_i(k + 1) \geq (1 - \gamma)\hat{x}_i(k) + \gamma\hat{x}_{s_j + 1}(k). \quad (25)$$

Using the conditions $A_1$ to $A_{j_A - 1}$, we can bound $\hat{x}_{s_j}(k)$ from below as

$$\hat{x}_{s_j}(k) \geq \hat{x}_i(k) - \frac{c_0 + c_1e^{-\alpha k}}{\gamma} \geq \hat{x}_{s_j}(k) - \frac{1}{\gamma} \left( c_0 + c_1e^{-\alpha k} \right) \geq \cdots \geq \hat{x}_{s_{j_A}}(k) = \frac{1}{\gamma^{j_A - 1}} \left( c_0 + c_1e^{-\alpha k} \right).$$

Substitute this into (25) and obtain

$$x_i(k + 1) \geq \hat{x}_{s_{j_A}}(k) + \gamma \left( \hat{x}_{s_{j_A + 1}}(k) - \hat{x}_{s_{j_A}}(k) \right) - \frac{1}{\gamma^{j_A - 1}} (c_0 + c_1e^{-\alpha k}) \geq \hat{x}_{s_{j_A}}(k) + \gamma \frac{c_0 + c_1e^{-\alpha k}}{\gamma^j_A} - \frac{1}{\gamma^{j_A - 1}} (c_0 + c_1e^{-\alpha k}) = \hat{x}_{s_{j_A}}(k) + \frac{c_0 + c_1e^{-\alpha k}}{\gamma^j_A} \quad (26)$$

where the second inequality follows by (24). Thus, this node $i$ is moved out of set $X_1(k, k + 1)$ at time $k + 1$.

We next show that the regular nodes not in $X_1(k, k)$ at time $k$ will not move in $X_1(k, k + 1)$ at time $k + 1$. If node $j$ has some neighbors inside $X_1(k, k)$, then (25) and (26) hold and we know that the node does not move in $X_1(k, k + 1)$. If node $j$ has neighbors only in $X \setminus X_1(k, k)$, then we have

$$x_j(k + 1) \geq \hat{x}_{s_{j_A}}(k) > \hat{x}_{s_{j_A}}(k) + \frac{c_0 + c_1e^{-\alpha k}}{\gamma^j_A}.$$

Clearly, node $j$ does not move in $X_1(k, k + 1)$ in this case.

Therefore, the regular nodes in $X_1(k, k + 1)$ decrease in number as

$$X_1(k, k + 1) \cap R \subseteq X_1(k, k) \cap R.$$

Similar results also hold if $i \in X_2(k, k)$, and we have $\hat{x}_i(k + 1)$ decreases more than $c_0 + c_1e^{-\alpha k}$ compared with $\hat{x}_{s_B}(k)$.

As a result, if the conditions $A_{j_A}$ and $B_{j_B}$ with $j_A < j_B$ are not satisfied, after $N$ steps, the set $X_1(k, k + N)$ or $X_2(k, k + N)$ becomes empty in regular nodes. It then follows that $\hat{\bar{x}}(k + N)$ increases more than $c_0 + c_1e^{-\alpha k}$ or $\overline{\bar{x}}(k + N)$ decreases more than $c_0 + c_1e^{-\alpha N}$.

Case 2: $j_A \geq j_B$. This case is impossible. We can show this by contradiction as follows. Since $j_A \geq j_B$, we know that $A_{j_B - 1}$ and $B_{j_A + 1}$ are both satisfied. Combined with $A_{j_A}$ and $B_{j_B}$ not being satisfied, we have

$$d_0 + c_1e^{-\alpha k} \gamma^{N - j_B + 1} < \hat{x}_{s_{j_B}}(k) - \hat{x}_{s_{j_B - 1}}(k) \leq d_0 + c_1e^{-\alpha k} \gamma^{j_B - 1}, \quad (27)$$

$$c_0 + c_1e^{-\alpha k} \gamma^{j_A} < \hat{x}_{s_{j_A + 1}}(k) - \hat{x}_{s_{j_A}}(k) \leq c_0 + c_1e^{-\alpha k} \gamma^{N - j_A}. \quad (28)$$

The inequalities in the first relations in (27) indicate that it must hold $j_B > (N + 1)/2$. The second set of inequalities in (28) also implies $j_A < N/2$. Consequently, we have $j_A < j_B$, which is in contradiction with $j_A \geq j_B$.

We can now conclude that after a finite number of time steps, all conditions from $A_1$ to $A_m$ and $B_{m+2}$ to $B_N$, where $0 \leq m \leq N - 1$, must be satisfied. Otherwise the difference between $\bar{x}(k)$ and $\hat{x}(k)$ will decrease more than $c_0$ by an update induced by an event. From the analysis for the safety condition, we know that $\bar{x}(k)$ is nonincreasing and $\hat{x}(k)$ is nondecreasing. Hence, if the events continuously occur, $\bar{x}(k) - \hat{x}(k)$ will become smaller and eventually negative, which cannot happen. This completes the proof.

Protocol 2 enables us to achieve resilient consensus only with data communicated via event-based protocols. Compared with Protocol 1 considered in Theorem 1 it is more difficult to trigger an event when the regular nodes become close to each other. This property usually means a lower communication rate. The disadvantage, in turn, is of having less sensitivity to deal with small errors between neighboring nodes. That is, if the triggering rules are the same in Protocols 1 and 2, then for this protocol, larger errors may remain in consensus.

The upper bound obtained in Theorem 2 can in general be conservative, but is tight for some graphs in certain situations as we see below. It is also noted that the proof method used for Theorem 1 could be employed here; this will however result in the same error bound as in Theorem 1 which is more conservative than the one we have in (23).

Remark 2. We present an example of a multi-agent system whose error in consensus among the agents is equal to the bound obtained in Theorem 2. Such a graph may be called a worst-case graph. Consider the network in Fig. 2 with four nodes which are all regular and thus $F = 0$. Note that the graph contains a directed spanning tree. The initial values $x_i(0)$ of the nodes and the (constant) weights $a_{ij}(k)$ on the edges are indicated in the figure. Since the weights are all $1/2$ (and thus $\gamma = 1/2$), for nodes having two neighbors, their own values are not used in the update rule (21). Moreover, for the node in the far left, a self-loop is shown to indicate that this node uses its own value. The node in the far right has no incoming edge, and thus its value will not change over time. By setting the parameters for the triggering function as $c_0 = 1$ and $c_1 = 0$, it follows that there will be no event at any time. Note that the difference in their values is 14, which is equal to the upper bound in Theorem 2.
in general, leading the nodes to achieve better performance.

hand, with Protocol 1, more communications will take place

large as 0.3, which is what we observe in Fig. 6. On the other

case of line graph with four nodes, the overall error can be as

consensus for this case.

observe that both protocols managed to achieve a certain level

error in Protocol 2 for the malicious case can be explained as

error level does not decrease further. The reason for the lar

ger error compared to Protocol 1 for the same parameters used in

no transmission after time 6. This however results in the lar

ger number of communications required in Protocol 2 is much smaller, with

neighbors are within 0.1 due to the choice if

c

stopped communicating after the differences in the values with

node 5 results in the line graph. In Protocol 2, the nodes

follow: Due to the oscillatory behavior of node 5, its value

remains in the updates of others after some time. However,

amount of communication required in Protocol 2 is much smaller, with

no transmission after time 6. This however results in the larger

error compared to Protocol 1 for the same parameters used in

the triggering functions.

In Protocol 1, the errors among the agents decay as fast

as those in Protocol 2. However, the agents keep making

updates, resulting in further changes in their values though the

error level does not decrease further. The reason for the large

error in Protocol 2 for the malicious case can be explained as

follows: Due to the oscillatory behavior of node 5, its value

is removed in the updates of others after some time. However,

note that in the network in Fig. 1, the removal of edges from

node 5 results in the line graph. In Protocol 2, the nodes

stopped communicating after the differences in the values with

neighbors are within 0.1 due to the choice if $c_0 = 0.1$. In the

case of line graph with four nodes, the overall error can be as

large as 0.3, which is what we observe in Fig. 5. On the other

hand, with Protocol 1, more communications will take place

in general, leading the nodes to achieve better performance in

consensus for this case.

V. NUMERICAL EXAMPLE

In this section, we illustrate the proposed resilient consensus

approaches via a numerical example.

We consider the multi-agent system with five nodes whose

connectivity graph is shown in Fig. 1. As already mentioned,

this graph is $(2,2)$-robust. We compare the performance of the

two proposed algorithms: Protocol 1 using the update rules in

and $(3)$ and Protocol 2 using $(21)$ and $(3)$. The parameters

of the triggering function are chosen to be the same with $c_0 =

0.1, c_1 = 1, \alpha = 2$. The initial state was chosen as $x(0) =

[1 2 3 5 4]^T$ for all simulations.

First, we computed the time responses of Protocols 1 and 2

under the normal condition when all agents follow the event-

based algorithms. They are depicted in Figs. 3 and 4. In the

plots, the $x$-axis represents the sampling instants $k$ and the $y$-

axis the values of agents. Moreover, the time instants when

each node broadcasts are shown by the markers * in the color

corresponding to that of its time response curve. We observe

that both protocols managed to achieve a certain level of

consensus with about the same sizes of errors based on

event-triggered communication. Protocol 2 required a slightly

smaller number of events.

Next, we demonstrate the case with a malicious agent. Here,

node 5 is set to behave in a malicious manner by continuously

oscillating its value. The time responses of the two protocols

are shown in Figs. 5 and 6. It is easy to see that under both

protocols, the normal nodes reach consensus in safe regions.

In fact, there is very little sign of being influenced by the

behavior of the malicious node. Moreover, again, the amount

of communication required in Protocol 2 is much smaller, with

no transmission after time 6. This however results in the larger

error compared to Protocol 1 for the same parameters used in

the triggering functions.

In Protocol 1, the errors among the agents decay as fast

as those in Protocol 2. However, the agents keep making

updates, resulting in further changes in their values though the

error level does not decrease further. The reason for the large

error in Protocol 2 for the malicious case can be explained as

follows: Due to the oscillatory behavior of node 5, its value

is removed in the updates of others after some time. However,

note that in the network in Fig. 1, the removal of edges from

node 5 results in the line graph. In Protocol 2, the nodes

stopped communicating after the differences in the values with

neighbors are within 0.1 due to the choice if $c_0 = 0.1$. In the

case of line graph with four nodes, the overall error can be as

large as 0.3, which is what we observe in Fig. 5. On the other

hand, with Protocol 1, more communications will take place

in general, leading the nodes to achieve better performance in

consensus for this case.

VI. CONCLUSION

In this paper, we considered a resilient approach for the

multi-agent consensus problem to mitigate the influence of

misbehaving agents due to faults and cyber-attacks. Two

protocols for the updates of the regular nodes have been pro-

posed, and their convergence properties as well as necessary

network structures have been characterized. In both cases,

resilient consensus can be achieved with reduced frequencies

in communication among agents through event triggering.

This is possible at the expense of certain errors in consensus
determined by the parameters in the triggering function. Future

studies will focus on resilient consensus algorithms with time

delays in communications for the event-triggered case and also

those based on model predictive control.

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Fig. 5. Protocol 1 with malicious node 5

Fig. 6. Protocol 2 with malicious node 5

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