DIMENSION RESULTS AND KPZ FORMULA FOR TWO DIMENSIONAL MULTIPLICATIVE CASCADE PROCESS

XIONG JIN

Abstract. We prove a Hausdorff dimension result for the image of two-dimensional multiplicative cascade process, and we obtain from this result a KPZ-like formula that normally has one point of phase transition. We also prove that this Hausdorff dimension result holds almost surely for all Borel sets if and only if the process is fractional with unique parameter. The “only if” part relies on the multifractal analysis of the process and a level set dimension result.

1. Introduction

In a recent paper [5] Benjamini and Schramm proved a formula relating the Hausdorff dimension of a Borel subset of the unit interval to the Hausdorff dimension of the same set with respect to a random metric obtained from multiplicative cascade measures. Their goal, inspired by [6], is to give a Hausdorff dimension version of the well-known KPZ formula of Knizhnik, Polyakov and Zamolodchikov from quantum gravity [15]. To be more precise, let $W$ be a positive random variable of expectation 1 and let $\{W(w) : w \in \bigcup_{n \geq 1} \{0, 1\}^n\}$ be a sequence of independent copies of $W$ encoded by the dyadic words. The multiplicative cascade measure $\mu$ on $[0, 1]$ generated by $W$ is defined as the weak limit of

$$\left( d\mu_n(x) = W(x_{|1})W(x_{|2}) \cdots W(x_{|n}) \, dx \right)_{n \geq 1},$$

where for $i = 1, 2, \cdots$ and $x \in [0, 1]$, $x_{|i}$ stands for the first $i$ letters of the dyadic expansion of $x$. From [13, 12] one knows that if $\mathbb{E}(W \log_2 W) < 1$, then $\mu$ is almost surely non-degenerate and without atom, so it induces a random metric $\rho_\mu$ on $[0, 1]$ given by $\rho_\mu(x, y) = \mu([x, y])$ for $0 \leq x < y \leq 1$ (such a metric was previously considered in [1]). Denote by $\dim_H$ the Hausdorff dimension with respect to the Euclidean metric and by $\dim_H^\mu$ the Hausdorff dimension with respect to $\rho_\mu$, it is shown in [5] that if $\mathbb{E}(W^{-s}) < \infty$ for all $s \in (0, 1)$, then for any Borel set $K \subset [0, 1]$ with $\dim_H K = \xi_0$, almost surely $\dim_H^\mu K$ is equal to a constant $\xi \in [0, 1]$ satisfying

$$2^{-\xi_0} = 2^{-\xi} \cdot \mathbb{E}(W^\xi).$$

In the special case when $W = e^{\sigma \cdot Y - \sigma^2/2}$, where $Y = \mathcal{N}(0, 1)$ is a normal random variable and $\sigma \geq 0$ (the condition $\mathbb{E}(W \log_2 W) < 1$ is equivalent to $\sigma^2 < 2 \ln 2$), they obtain the KPZ formula for the random metric $\rho_\mu$:

$$\xi_0 - \xi = \frac{\sigma^2}{2 \ln 2} \cdot (1 - \xi).$$

\begin{flushright}
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Notice that if we consider the indefinite integral of \( \mu \), that is \( F_\mu(x) = \mu([0,x]) \) for \( x \in [0,1] \), then by definition one directly gets
\[
\dim_{\mu}^H K = \dim_{\mu} F_\mu(K).
\]
So Benjamini and Schramm’s result can be also understood as a Hausdorff dimension result for the image of the increasing process \( F_\mu \).

The main goal of this paper is to extend Benjamini and Schramm’s result to the signed multiplicative cascade processes, a class of random multifractal functions recently constructed in [2] as a natural generalization of \( F_\mu \). We will consider the more general two dimensional case (see Remark 1.1 for the reason) and our results are the following:

(i) We prove a Hausdorff dimension result for the image of the two dimensional multiplicative cascade process (Theorem 1.1), and we obtain from this result a KPZ-like formula which normally has one point of phase transition (Example 1 and 2).

(ii) By using the so-called restricted singularity spectra (Theorem 1.2) and a level set dimension result (Theorem 1.3), we show that the Hausdorff dimension result in Theorem 1.1 cannot hold almost surely for all Borel sets if the process is multifractal, or one dimensional fractional, or two dimensional fractional with different parameters.

(iii) When the process is two dimensional fractional with unique parameter, we prove that the Hausdorff dimension result in Theorem 1.1 holds almost surely for all Borel sets (Theorem 1.4).

Before stating in more detail the results we need to recall the definition of two dimensional multiplicative cascade processes. Let us begin with some notations on the coding space.

**Coding space.** Let \( b \geq 2 \) be an integer and let \( \mathcal{A} = \{0, \cdots, b-1\} \) be the alphabet. Let \( \mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n \) (by convention \( \mathcal{A}^0 = \{\emptyset\} \) the set of empty word) and \( \mathcal{A}^{\mathbb{N}+} = \{0, \cdots, b-1\}^{\mathbb{N}+} \). The length of a word \( w \) is denoted by \( |w| = n \) if \( w \in \mathcal{A}^n \), \( n \geq 0 \) and \( |w| = \infty \) if \( w \in \mathcal{A}^{\mathbb{N}+} \).

The word obtained by concatenation of \( w \in \mathcal{A}^* \) and \( t \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}+} \) is denoted by \( w \cdot t \) and sometimes \( wt \). If \( n \geq 1 \) and \( w = w_1 \cdots w_n \in \mathcal{A}^n \), then for every \( 1 \leq i \leq n \), the word \( w_1 \cdots w_i \) is denoted by \( w[i] \), and if \( i = 0 \) then \( w[0] \) stands for \( \emptyset \). Also, for any infinite word \( t = t_1 t_2 \cdots \in \mathcal{A}^{\mathbb{N}+} \) and \( n \geq 1 \), \( t[n] \) denotes the word \( t_1 \cdots t_n \) and \( t[0] \) the empty word.

Let \( \pi : t \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}+} \mapsto \sum_{i=1}^{[t]} t_i b^{-i} \) be the canonical projection from \( \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}+} \) onto the interval \([0,1)\). For \( w \in \mathcal{A}^* \) denote by \( I_w = [\pi(w), \pi(w) + b^{-|w|}] \) the b-adic interval encoded by \( w \). For \( x \in [0,1) \) and \( n \geq 1 \), we define \( x|_n = x_1 \cdots x_n \) the unique element of \( \mathcal{A}^n \) such that \( x \in I_{x_1 \cdots x_n} \), as well as \( 1|_n = b-1 \cdots b-1 \).

Sometimes we will use the convention \( I_n(x) = I_{x_1 \cdots x_n} \).

**Two dimensional multiplicative cascade processes.** Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be the probability space and let \( W = (W_1, W_2) \) be a random vector satisfying

(A0) \( \mathbb{E}(W_1) = \mathbb{E}(W_2) = b^{-1} \);

(A1) \( \exists q \in (1, 2] \) such that \( \mathbb{E}(|W_1|^q) \vee \mathbb{E}(|W_2|^q) < b^{-1} \);

(A2) \( \exists s > 2 \) such that \( \mathbb{E}(|W_1|^{-s}) \vee \mathbb{E}(|W_2|^{-s}) < \infty \).
Let \( \{W(w) : w \in \mathcal{A}^*\} \) be a sequence of independent copies of \( W \). For \( k = 1, 2 \), \( x \in [0, 1] \) and \( n \geq 1 \) define the product
\[
Q_k(x|_n) = Q_k(I_x|_n) = W_k(x|_1) \cdot W_k(x|_2) \cdots W_k(x|_n).
\]

For \( n \geq 1 \) define the random piecewise linear function
\[
F_{k,n} : t \in [0,1] \mapsto \int_0^t b^n \cdot Q_k(x|_n) \, dx.
\]

From [2] one has almost surely \( F_{k,n} \) converges uniformly to a limit \( F_k \). Then the two dimensional multiplicative cascade process considered in this paper is defined as
\[
F = (F_1, F_2) : t \in [0,1] \mapsto (F_1(t), F_2(t)) \in \mathbb{R}^2.
\]

Now we are ready to present the results. Our first result is the following:

**Hausdorff dimension result for the image.** Given \( \xi_0 \in [0,1] \), denote by \( \xi \) the smallest solution of the equation
\[
b^{-\xi_0} = \mathbb{E}(|W_1|^\xi) \vee \mathbb{E}(|W_2|^\xi)
\]
and \( \zeta \) the smallest solution of the equation
\[
b^{-\zeta_0} = \mathbb{E}(|W_1|^\zeta^{-1} \cdot |W_2|) \vee \mathbb{E}(|W_1| \cdot |W_2|^\zeta^{-1}).
\]

Also denote by
\[
\xi_* = -\log_b \left( \mathbb{E}(|W_1|) \vee \mathbb{E}(|W_2|) \right) \in (1/2, 1).
\]

**Theorem 1.1.** Let \( K \subset [0,1] \) be any Borel set with \( \dim_H K = \xi_0 \).

(i) If \( \mathbb{P}(W_1 = W_2) < 1 \), then almost surely
\[
\dim_H F(K) = \xi \wedge \zeta = \begin{cases} 
\xi, & \text{if } \xi_0 \in [0, \xi_*] ; \\
\zeta, & \text{if } \xi_0 \in (\xi_*, 1]. 
\end{cases}
\]

(ii) If \( \mathbb{P}(W_1 = W_2) = 1 \), then almost surely
\[
\dim_H F(K) = \xi \wedge 1 = \begin{cases} 
\xi, & \text{if } \xi_0 \in [0, \xi_*] ; \\
1, & \text{if } \xi_0 \in (\xi_*, 1]. 
\end{cases}
\]

Let us give two examples to help understanding the result.

**Example 1.** Let \( X_1 \) and \( X_2 \) be two random variables both taking values \( b^{-\alpha} \) and \( -b^{-\alpha} \) with respective probabilities \( (1 + b^{\alpha - 1})/2 \) and \( (1 - b^{\alpha - 1})/2 \) for some \( \alpha \leq 1 \). Suppose that \( \mathbb{P}(X_1 = X_2) < 1 \). Let \( \sigma \geq 0 \) and let \( Y = \mathcal{N}(0, 1) \) be a normal random variable independent of \( X_1 \) and \( X_2 \). Define
\[
W_1 = X_1 \cdot e^{\sigma Y - \sigma^2/2} \quad \text{and} \quad W_2 = X_2 \cdot e^{\sigma Y - \sigma^2/2}.
\]

By simple calculation one has for \( \{k, l\} = \{1, 2\} \),
\[
\mathbb{E}(|W_k|^\xi) = \mathbb{E}(|W_l|^\xi) = \mathbb{E}(|W_k|^{\xi^{-1}} \cdot |W_l|) = b^{-\alpha \xi} \cdot e^{\sigma^2(\xi^2 - \xi)/2}.
\]

Let \( \beta = \sigma^2/(2 \ln b) \). Then assumption (A1) ((A0) and (A2) are automatically satisfied) is equivalent to requiring
\[
\beta < 1 \quad \text{and} \quad \begin{cases} 
2\beta^{1/2} - \beta < \alpha \leq 1, & \text{if } \beta \geq 1/4 ; \\
\beta + 1/2 < \alpha \leq 1, & \text{if } \beta < 1/4.
\end{cases}
\]
In such a case, Theorem 1.1 says that for any Borel set \( K \subseteq [0, 1] \) with \( \dim H K = \xi_0 \), almost surely \( \dim_H F(K) \) is equal to a constant \( \xi \in [0, 2) \) satisfying
\[
\xi_0 - \alpha \cdot \xi = \beta \cdot (1 - \xi).
\]
Comparing to (2), this formula has a new parameter \( \alpha \) varying in the region given by (5), and when \( \alpha < 1 \), the maximal dimension \( \dim_H F([0, 1]) \) is equal to
\[
\frac{\beta + \alpha - \sqrt{(\beta + \alpha)^2 - 4\beta}}{2\beta} \in (1, 2)
\]
if \( \beta > 0 \) and is equal to \( 1/\alpha \in (1, 2) \) if \( \beta = 0 \).

**Example 2.** Now let
\[
W_1 = X_1 \cdot e^{\alpha Y - \sigma^2/2} \quad \text{and} \quad W_2 = b^{-1} \cdot e^{\alpha Y - \sigma^2/2},
\]
so \( W_2 \) is almost surely positive. For \( \xi \geq 0 \) one has
\[
\mathbb{E}(|W_1|^{\xi}) \vee \mathbb{E}(|W_2|^{\xi}) = b^{-\alpha \xi} \cdot e^{\sigma^2(\xi^2 - \xi)/2},
\]
and \( \zeta \geq 1 \) one has
\[
\mathbb{E}(|W_1|^{\zeta - 1}|W_2|) \vee \mathbb{E}(|W_2|^{\zeta - 1}|W_1|) = b^{-(\zeta - 1 + \alpha)} \cdot e^{\sigma^2(\xi^2 - \xi)/2},
\]
as well as \( \xi_* = \alpha \). We need the same condition as in (5). In this case, since \( F_2 \) is almost surely increasing, one can deduce a random metric \( \rho_F \) from \( F \) on \([0, 1] \) given by \( \rho_F(x, y) = |F(x) - F(y)| \) for \( x, y \in [0, 1] \). Then Theorem 1.1 says that for any Borel set \( K \subseteq [0, 1] \) with \( \dim H K = \xi_0 \), almost surely \( \dim_H^\rho F K \) is equal to a constant \( \xi \in [0, 2) \) satisfying
\[
\left\{ \begin{array}{ll}
\xi_0 - \alpha \cdot \xi = \beta \cdot (1 - \xi), & \text{if } \xi_0 \in [0, \alpha]; \\
\xi_0 - \xi = \beta \cdot (1 - \xi) + \alpha - 1, & \text{if } \xi_0 \in (\alpha, 1].
\end{array} \right.
\]
If \( \alpha = 1 \) then we go back to (2). If \( \alpha < 1 \), then this KPZ-like formula for \( \rho_F \) has a phase transition at \( \alpha \), and the maximal dimension \( \dim_H^\rho F [0, 1] \) is equal to
\[
\frac{\beta + 1 - \sqrt{(\beta + 1)^2 - 4\beta(2 - \alpha)}}{2\beta} \in (1, 2)
\]
if \( \beta > 0 \) and is equal to \( 2 - \alpha \in (1, 3/2) \) if \( \beta = 0 \).

**Remark 1.1.** The reason why we consider the two dimensional case can be easily seen from Theorem 1.1 and Example 1, 2. If we only consider one dimensional case, as shown in Theorem 1.1(ii), the formula will always have a phase transition at \( \xi_* \), and such phase transition is indeed caused by the limitation of the image space. Also it is necessary to consider the two or higher dimensional case to obtain the uniform dimension result given later.

**Remark 1.2.** Example 1 and 2 are special cases of Theorem 1.1, in general the theorem could provide us with more colorful formulas. In principle, the formula can have as many points of phase transition as we want.

**Remark 1.3.** To finding the Hausdorff dimension of the image of a stochastic process restricted to any Borel subset of its domain is a classical problem in probability theory. The first work on this subject can be traced back to Lévy [16] and Taylor [20], regarding the Hausdorff dimension and Hausdorff measure of the image of Brownian motion. Since then much progress has been made for fractional Brownian motion, stable Lévy process and many other processes. We refer to the survey paper [23] and the references therein for more information on this subject.
Our second result is a “multifractal singularity spectra” type result. The starting point of such a result is the following natural question: Could the dimension result in Theorem 1.1 hold almost surely for all Borel sets? The answer to this question is negative, and we can see it quickly from the case \( F = F_\mu \): From [13] we know that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), the multiplicative cascade measure \( \mu_\omega \) is carried by a random Borel set \( E_\omega \) of Hausdorff dimension \( 1 \) if \( \mathbb{E}(W \log_2 W) \), which is strictly less than \( 1 \) unless \( W = 1 \) and \( \mu \) is Lebesgue. On the other hand, from [1] one can deduce that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), the random Borel set \( E_\omega \) that carries \( \mu_\omega \) has Hausdorff dimension \( 1 \) with respect to \( \rho_{\mu_\omega} \). So unless we are in the trivial case that \( \mu \) is Lebesgue, (1) cannot hold almost surely for all Borel sets.

This observation leads us to the next question: Given a Borel subset \( K \), of which random subset \( E_{\omega,K} \) of \( K \) one has \( \dim_H F(E_{\omega,K}) = \dim_H F(K) \) and what is Hausdorff dimension of \( E_{\omega,K} \) (for \( \mathbb{P} \)-almost every \( \omega \in \Omega \))? To answer this question it is natural to consider the Hölder level sets of \( F \), and inspired by [10] we get the following “multifractal singularity spectra” type result:

**Restricted singularity spectra.** Let \( K \subset [0,1] \) be any Borel set such that
\[
\dim_H K = \dim_P K = \xi_0 > 0,
\]
where \( \dim_P \) stands for packing dimension.

For \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_+^2 \) define the Hölder level set:
\[
K(\alpha) = \{ x \in K : h_F(x) = \alpha \},
\]
where \( h_F = (h_{F_1}, h_{F_2}) \) is the Hölder vector of \( F \) defined as (whenever exists)
\[
h_{F_k}(x) = \lim_{n \to \infty} \frac{\log \sup_{s,t \in I_n(x)} |F_k(s) - F_k(t)|}{\log |I_n(x)|}, \quad k = 1, 2.
\]

For \( q = (q_1, q_2) \in \mathbb{R}^2 \) define
\[
\Phi(q) = \Phi(q_1, q_2) = -\log_3 \mathbb{E}(|W_1|^{q_1} |W_2|^{q_2}).
\]

Let
\[
J(\xi_0) = \left\{ q \in \mathbb{R}^2 : \Phi(q) > -\infty \text{ and } q \cdot \nabla \Phi(q) - \Phi(q) > -\xi_0 \right\},
\]
and
\[
\Lambda(\xi_0) = \left\{ \nabla \Phi(q) : q \in J(\xi_0) \right\}.
\]

**Theorem 1.2.** If \( \mathcal{H}^{\xi_0}(K) > 0 \), then for any \( q \in J(\xi_0) \) and \( \alpha = \nabla \Phi(q) \) one has almost surely
\[
\dim_H K(\alpha) = \xi_0 + q \cdot \alpha - \Phi(q).
\]

Moreover, denote by \( \alpha_* = \alpha_1 \land \alpha_2 \) and \( \alpha^* = \alpha_1 \lor \alpha_2 \). One has:

(i) If \( \mathbb{P}(W_1 = W_2) < 1 \), then almost surely
\[
\dim_H F(K(\alpha)) = \frac{\dim_H K(\alpha)}{\alpha_*} \land \left( 1 + \frac{\dim_H K(\alpha) - \alpha_*}{\alpha^*} \right);
\]

(ii) If \( \mathbb{P}(W_1 = W_2) = 1 \), then almost surely
\[
\dim_H F(K(\alpha)) = \frac{\dim_H K(\alpha)}{\alpha_*} \land 1.
\]
Remark 1.4. The image dimension result in Theorem 1.2 looks very like that obtained in [22] for Gaussian vector fields. In fact from the Hölder regularity point of view, when restricted to the Hölder level set $K(\alpha)$, the process $F$ behaves exactly like fractional Brownian motion of index-$\alpha$.

Remark 1.5. The condition $\mathcal{H}^0(K) > 0$ in Theorem 1.2 can be replaced by $\mathcal{H}^q(K) > 0$ for any gauge function $g$ of the form $g(r) = r^q |\log r|^n$ for some $n \geq 1$. These conditions together with $\dim_H K = \dim_P K$ are only used to provide a finer Frostman measure carried by $K$ and having Hausdorff dimension $\xi_0$ in a stronger sense (see Lemma 3.1).

Remark 1.6. Following [3] one can construct a family of multiplicative cascade measures carried by $K(\nabla \Phi(q))$ simultaneously for all $q \in J(\xi_0)$. Then using the methods in [3, 10] we can prove that the results in Theorem 1.2 hold almost surely for all $q \in J(\xi_0)$. But such results would rely on heavy calculations and need a great effort on the parameters estimates. So in order to simplify the purpose of this paper, we only give the results in the present form.

Now we are ready to answer the question asked before:

For $\xi_0 \in (0, 1]$ recall the definition of $\xi$ and $\zeta$ in (3) and (4). Suppose that $\Pr(W_1 = W_2) < 1$. By simple calculation one can get that for any $\xi_0 \in (0, \xi_1]$, the maximal dimension of $F(K(\alpha))$ over $\alpha \in \Lambda(\xi_0)$ is equal to $\xi = \dim_H F(K)$ and it is reached at

\begin{equation}
\alpha = \begin{cases}
\nabla \Phi(\xi, 0), & \text{if } \dim_H F(K(\nabla \Phi(\xi, 0))) \geq \dim_H F(K(\nabla \Phi(0, \xi))); \\
\nabla \Phi(0, \xi), & \text{otherwise},
\end{cases}
\end{equation}

and for any $\xi_0 \in (\xi_1, 1]$, the maximal dimension of $F(K(\alpha))$ over $\alpha \in \Lambda(\xi_0)$ is equal to $\zeta = \dim_H F(K)$ and it is reached at

\begin{equation}
\alpha = \begin{cases}
\nabla \Phi(\zeta - 1, 1), & \text{if } \dim_H F(K(\nabla \Phi(\zeta - 1, 1))) \geq \dim_H F(K(\nabla \Phi(1, \zeta - 1))); \\
\nabla \Phi(1, \zeta - 1), & \text{otherwise}.
\end{cases}
\end{equation}

In both cases we have $\dim_H K(\alpha) = \xi_0 + q \cdot \alpha - \Phi(q)$ for the corresponding $q \in J(\xi_0)$. So if $q \cdot \nabla \Phi(q) - \Phi(q) < 0$ for one of such $q$, then the dimension result in Theorem 1.1 cannot hold almost surely for all Borel sets. To the contrary, if for any $q \in \mathbb{R}^2$ of the type $(\xi, 0)$, $(0, \xi)$, $(\zeta - 1, 1)$ or $(1, \zeta - 1)$ such that

$$\dim_H F(K(\nabla \Phi(q))) = \dim_H F(K)$$

we have $q \cdot \nabla \Phi(q) - \Phi(q) = 0$, then from the analyticity of $\Phi$ and (6) we know that there exists $k \in \{1, 2\}$ such that $\mathbb{R} \ni p \mapsto -\log_b \mathbb{E}(|W_k|^p)$ is an affine map, thus there exists $\alpha_k \in (1/2, 1]$ such that $|W_k| = b^{-\alpha_k}$. Let us assume $k = 1$, then

$$\Phi(q_1, q_2) = q_1 \cdot \alpha_1 - \log_b \mathbb{E}(|W_2|^{q_2})$$

and

$$q \cdot \nabla \Phi(q) - \Phi(q) = q_2 \cdot \phi_2(q_2) - \phi_2(q_2),$$

where $\phi_2 : q_2 \mapsto -\log_b \mathbb{E}(|W_2|^{q_2})$. From (7) we know that either we have $\phi'_2(1) - \phi(1) = 0$ or there exists an open interval such that $q_2 \cdot \phi'_2(q_2) - \phi_2(q_2) = 0$ holds for all $q_2$ belonging to the interval. Either of the two cases will imply that there exists $\alpha_2 \in (1/2, 1]$ such that $|W_2| = b^{-\alpha_2}$. 
Fractional case. Now we are in the situation that for \( k = 1, 2 \), \( W_k \) is a random variable taking value \( b^{-\alpha_k} \) and \( -b^{-\alpha_k} \) with respective probabilities \((1 + b^{\alpha_k})/2\) and \((1 - b^{\alpha_k})/2\) for some \( \alpha_k \in (1/2, 1] \), which is indeed the fractional case considered in [4]. In the following we will show that in this case the dimension result in Theorem 1.1 holds almost surely for all Borel sets if and only if \( \mathbb{P}(W_1 = W_2) < 1 \) and \( \alpha_1 = \alpha_2 \).

First let us state the following level set dimension result:

**Theorem 1.3.** For \( k = 1, 2 \) almost surely for Lebesgue almost every \( y \in F_k([0, 1]) \),

\[
\dim_H L_k(y) = 1 - \alpha_k,
\]

where \( L_k(y) = \{ x \in [0, 1] : F_k(x) = y \} \) is the level set of \( F_k \) at level \( y \).

There is nothing to prove when \( \alpha_1 = \alpha_2 = 1 \), and we have the following three other cases:

(i) If \( \mathbb{P}(W_1 = W_2) = 1 \) and \( \alpha_1 = \alpha_2 < 1 \), then one can deduce from Theorem 1.3 that the dimension result in Theorem 1.1 cannot hold almost surely for all Borel set, since almost surely for Lebesgue almost every \( y \in F_1([0, 1]) \) one has

\[
\dim_H L_1(y) = 1 - \alpha_1 > 0
\]

and

\[
\dim_H F(L_1(y)) = \dim_H \{ (y, y) \} = 0.
\]

(ii) If \( \mathbb{P}(W_1 = W_2) < 1 \) and \( \alpha_1 > \alpha_2 \), then from Theorem 1.3 one has almost surely \( \dim_H L_2(y) = 1 - \alpha_2 > 0 \) for Lebesgue almost every \( y \in F_2([0, 1]) \). On the other hand, since \( F_1 \) is \( \alpha \)-Hölder continuous for any \( \alpha \in (0, \alpha_1) \) (see [4]), and \( F(L_2(y)) = F_1(L_2(y)) \times \{ y \} \), so

\[
\dim_H F(L_2(y)) = \dim_H F_1(L_2(y)) \leq \frac{\dim_H L_2(y)}{\alpha_1} < \frac{\dim_H L_2(y)}{\alpha_2} < 1.
\]

This shows that the dimension result in Theorem 1.1 cannot hold almost surely for all Borel sets.

(iii) If \( \mathbb{P}(W_1 = W_2) < 1 \) and \( \alpha_1 = \alpha_2 = \alpha \in (1/2, 1) \), we have

**Theorem 1.4.** Almost surely

\[
\dim_H F(K) = \frac{\dim_H K}{\alpha}
\]

for all Borel set \( K \subset [0, 1] \).

**Remark 1.7.** In the literature, results like Theorem 1.4 are often mentioned as uniform dimension results. The first result of this kind was given by Kaufman in [14] for planar Brownian motion, and was extended to strictly stable Lévy processes by Hawkes and Pruitt in [9] and to fractional Brownian motion by Monrad and Pitt in [19]. We refer again to the survey paper [23] and the references therein for more information.

The rest of the paper consists of four sections that individually present the proofs of Theorems 1.1 to 1.4. We end up this section with some preliminaries.

**Hausdorff dimension.** If \( (X, \rho) \) is a locally compact metric space, for \( d \geq 0 \), \( \delta > 0 \) and \( K \subset X \) let

\[
\mathcal{H}_\delta^d(K) = \inf \left\{ \sum_{i \in I} |U_i|_\rho^d \right\}
\]
where the infimum is taken over the set of all the at most countable coverings $\bigcup_{i=1}^{\infty} U_i$ of $K$ such that $0 \leq |U_i|_\rho \leq \delta$, where $|U_i|_\rho$ stands for the diameter of $U_i$ with respect to $\rho$. Define
\[
H^{\rho,d}(K) = \lim_{\delta \to 0} H^{\rho,d}_{\delta}(K).
\]
Then $H^{\rho,d}(K)$ is called the $d$-dimensional Hausdorff measure of $K$ with respect to $\rho$, and the Hausdorff dimension of $K$ with respect to $\rho$ is the number
\[
\dim H^\rho E = \inf\{d : H^{\rho,d}(K) < \infty\}.
\]
For any positive Borel measure $\nu$ defined on $(X, \rho)$, the lower Hausdorff dimension of $\nu$ with respect to $\rho$ is defined as
\[
\dim H^\rho(\nu) = \inf\{\dim H^\rho E : E \subset X \text{ and } \nu(E) > 0\}.
\]
When $\rho$ is the standard Euclidean metric, we often omit the index $\rho$.

**Stationary self-similarity of multiplicative cascade processes.** For $k = 1, 2, w \in \mathcal{A}^*$ and $n \geq 1$ define
\[
F^{[w]}_{k,n}(t) = \int_0^t b^n \cdot W_k(w \cdot x_{1}) \cdots W_k(w \cdot x_{n}) \, dx.
\]
Since $\mathcal{A}^*$ is countable, so almost surely for all $w \in \mathcal{A}^*$, $F^{[w]}_{k,n}$ converges uniformly to a limit $F^{[w]}_k$ which has the same law as $F_k$.

By construction one has for any $w \in \mathcal{A}^*$ and $t \in [0, 1]$,
\[
F_k(\pi(w) + t \cdot b^{-|w|}) - F_k(\pi(w)) = Q_k(w) \cdot F^{[w]}(t)
\]
(8)

For $w \in \mathcal{A}^*$ define
\[
Z_k(w) = F^{[w]}_k(1) \text{ and } X_k(w) = \sup_{s,t \in [0,1]} |F^{[w]}_k(s) - F^{[w]}_k(t)|.
\]
Then from (8) one has
\[
F_k(\pi(w) + b^{-|w|}) - F_k(\pi(w)) = Q_k(w) \cdot Z_k(w)
\]
and
\[
O_k(w) = O_k(I_w) := \sup_{s,t \in I_w} |F_k(s) - F_k(t)| = |Q_k(w)| \cdot X_k(w),
\]
where $Q_k(w)$ is independent of $Z_k(w)$ and $X_k(w)$. We will often use the convention that $Z_k = Z_k(\emptyset)$ and $X_k = X_k(\emptyset)$. By direct calculation, one has for any $q_1, q_2 \in \mathbb{R}$,
\[
\mathbb{E}(O_1(w)^{q_1} O_2(w)^{q_2}) = \mathbb{E}(X_1^{q_1} X_2^{q_2}) \cdot \mathbb{E}([W_1]^{q_1} [W_2]^{q_2})^{[w]} = \mathbb{E}(X_1^{q_1} X_2^{q_2} \cdot b^{-|w|q(q_1,q_2)}),
\]
whenever the expectation exists.

**Moments control.** It is proved in [2] that for $k = 1, 2$,
\[
(9) \quad \begin{align*}
\text{(i)} \ & \mathbb{E}(|W_k|^q) < b^{-1} \text{ for some } q > 1, \text{ then } \mathbb{E}(X_k^q) < \infty; \\
\text{(ii)} \ & \mathbb{E}(|W_k|^{-s}) < \infty \text{ for some } s > 0, \text{ then } \mathbb{E}(X_k^{-s}) < \infty.
\end{align*}
\]
2. Proof of Theorem 1.1

2.1. Upper bound.

Proof. For \( p \geq 0 \) let

\[
\phi(p) = \mathbb{E}(|W_1|^p) \vee \mathbb{E}(|W_2|^p);
\]

\[
\tilde{\phi}(p) = \mathbb{E}(|W_1|^{p-1} \cdot |W_2|) \vee \mathbb{E}(|W_1| \cdot |W_2|^{p-1}).
\]

We have the following lemma:

Lemma 2.1. One has \( \phi(p) \leq \tilde{\phi}(p) \) if \( p \in [0,1] \) and \( \phi(p) \geq \tilde{\phi}(p) \) if \( p \geq 1 \).

Proof. Obviously \( \phi(1) = \tilde{\phi}(1) \). Since \( \frac{|W_1|}{|W_2|} + \frac{|W_2|}{|W_1|} \geq 2 \), so \( \tilde{\phi}(0) \geq 1 = \phi(0) \).

Let \( \{k,l\} = \{1,2\} \). For \( p > 1 \) from Hölder inequality one gets

\[
\mathbb{E}(|W_k|^{p-1}|W_l|) \leq \mathbb{E}(|W_k|^{p-1})^{\frac{p-1}{p}} \cdot \mathbb{E}(|W_l|^p)^{\frac{1}{p}} = \mathbb{E}(|W_k|^{p-1})^{\frac{p-1}{p}} \cdot \mathbb{E}(|W_l|^p)^{\frac{1}{p}} \leq \phi(p).
\]

This implies \( \tilde{\phi}(p) \leq \phi(p) \). For \( p \in (0,1) \) from Hölder inequality one gets

\[
\mathbb{E}(|W_1|^p) = \mathbb{E}(|W_1|^p \cdot |W_2|^{-p(1-p)} \cdot |W_2|^{p(1-p)}) \leq \mathbb{E}(|W_1|^p \cdot |W_2|^{-p(1-p)}) \cdot \mathbb{E}(|W_2|^{p(1-p)}) \leq \phi(p)^p \cdot \mathbb{E}(|W_2|^p)^{1-p} \leq \tilde{\phi}(p)^p \cdot \mathbb{E}(|W_1|^p)^{1-p}.
\]

In an analogous way one can also obtain

\[
\mathbb{E}(|W_2|^p) \leq \tilde{\phi}(p)^p \cdot \mathbb{E}(|W_1|^p)^{1-p}.
\]

Then

\[
\mathbb{E}(|W_1|^p) \leq \tilde{\phi}(p)^p \cdot \tilde{\phi}(p)^{p(1-p)} \cdot \mathbb{E}(|W_1|^p)^{(1-p)(1-p)} \Rightarrow \mathbb{E}(|W_1|^p) \leq \tilde{\phi}(p)^p \cdot \mathbb{E}(|W_2|^p)^{1-p} \leq \phi(p),
\]

which also implies \( \mathbb{E}(|W_2|^p) \leq \tilde{\phi}(p) \), thus \( \phi(p) \leq \tilde{\phi}(p) \).

Given \( \xi_0 \in [0,1] \) recall the definition of \( \xi \) and \( \zeta \) in (3) and (4). Lemma 2.1 tells us that \( \xi \leq \zeta \leq 1 \) if \( \xi_0 \in [0,\xi_*] \) and \( \xi \geq \zeta > 1 \) if \( \xi_0 \in (\xi_*,1] \). Due to the convexity of \( \phi \) and \( \tilde{\phi} \) we also have \( \phi'(\xi^+) < 0 \) and \( \tilde{\phi}'(\zeta^+) < 0 \). Thus given any \( \epsilon > 0 \) smaller enough, one can find an \( \eta > 0 \) such that

\[
\phi(\xi + \eta) \leq b^{-(\xi_0+\epsilon)} \quad \text{and} \quad \tilde{\phi}(\zeta + \eta) \leq b^{-(\xi_0+\epsilon)}.
\]

From the moments control (9) and the definition of \( \xi \) and \( \zeta \) it is easy to deduce that \( \mathbb{E}(X_{1}^{\xi + \eta}) \vee \mathbb{E}(X_{2}^{\xi + \eta}) < \infty \) and

\[
\mathbb{E}(X_{k}^{\xi + \eta - 1} X_{l}) \leq \mathbb{E}(X_{k}^{\xi + \eta})^{\frac{k-1}{k}} \cdot \mathbb{E}(X_{l}^{\xi + \eta})^{\frac{l-1}{l}} < \infty, \quad \{k,l\} = \{1,2\}.
\]

For each \( n \geq 1 \) one can find a sequence \( \mathcal{I}_n \) of \( b \)-adic intervals such that

\[
K \subset \bigcup_{I \in \mathcal{I}_n} I \quad \text{and} \quad \sum_{I \in \mathcal{I}_n} |I|^{\xi_0+\epsilon} \leq 2^{-n}.
\]

Let \( \delta_n = \sup_{I \in \mathcal{I}_n} |F(I)| \). Since \( F \) is almost surely continuous, so \( \delta_n \to 0 \) almost surely. For any interval \( I \in \mathcal{I}_n \) denote by

\[
O_+(I) = \min\{O_1(I), O_2(I)\} \quad \text{and} \quad O^+(I) = \max\{O_1(I), O_2(I)\}.
\]

Then we can obtain the desired upper bounds from the following two facts:
(i) If \( \xi_0 \in (0, \xi_*] \): For each \( I \subset \mathcal{I}_n \) one can use a single square of side length \( 2O^\ast(I) \) to cover \( F(I) \), thus

\[
\mathbb{E}\left( \mathcal{H}^{\xi_0+\eta}_n(F(K)) \right) \leq 2^{\xi+\eta} \mathbb{E}\left( \sum_{I \in \mathcal{I}_n} O_1(I)^{\xi+\eta} \vee O_2(I)^{\xi+\eta} \right)
\leq 2^{\xi+\eta} \sum_{I \in \mathcal{I}_n} \mathbb{E}(O_1(I)^{\xi+\eta}) + \mathbb{E}(O_2(I)^{\xi+\eta})
\leq C \cdot \sum_{I \in \mathcal{I}_n} |I|^\xi_0+\epsilon \leq C \cdot 2^{-n},
\]

where \( C = 2^{\xi+\eta+1} \mathbb{E}(X_1^{\xi_0+\eta}) \vee \mathbb{E}(X_2^{\xi_0+\eta}) \).

(ii) If \( \xi_0 \in (\xi_*, 1] \): For each \( I \subset \mathcal{I}_n \) one can use no more than \( |O^\ast(I)/O_\ast(I)| \)-many squares of side length \( 2O_\ast(I) \) to cover \( F(I) \), thus

\[
\mathbb{E}\left( \mathcal{H}^{\xi_0+\eta}_n(F(K)) \right) \leq 2^{\xi+\eta+1} \mathbb{E}\left( \sum_{I \in \mathcal{I}_n} \left( \frac{O_2(I)}{O_1(I)} \cdot O_1(I)^{\xi+\eta} \right) \vee \left( \frac{O_1(I)}{O_2(I)} \cdot O_2(I)^{\xi+\eta} \right) \right)
\leq 2^{\xi+\eta+1} \mathbb{E}(O_2(I)O_1(I)^{\xi+\eta-1} + O_1(I)O_2(I)^{\xi+\eta-1})
\leq C' \cdot \sum_{I \in \mathcal{I}_n} |I|^\xi_0+\epsilon \leq C' \cdot 2^{-n},
\]

where \( C' = 2^{\xi+\eta+1} \mathbb{E}(X_1^{\xi_0+\eta-1}X_2) \vee \mathbb{E}(X_2^{\xi_0+\eta-1}X_1) \).

\[\square\]

2.2. Lower bound.

Proof. There is nothing to prove when \( \dim_H K = 0 \), since \( F(K) \) is always non-empty. Let \( \dim_H K = \xi_0 > 0 \). We will use a similar method as in [5] to estimate the lower bound.

Given any \( \delta \in (0, \xi_0) \), due to Frostman lemma there exists a Borel probability measure \( \mu_0 \) supported by \( K \) such that

\[
\int_{s,t \in [0,1]} \frac{d\mu_0(s)}{|s-t|^{\xi_0-\delta}} < \infty.
\]

First we consider the case \( \mathbb{P}(W_1 = W_2) < 1 \).

Let \( \{k, t\} = \{1, 2\} \) and \( \gamma \in (0, 2) \) be the unique number such that

\[
\begin{cases}
\mathbb{E}(|W_k|^{\gamma}) = b^{-(\xi_0-\delta)}, & \text{if } \xi_0 \in (0, \xi_*]; \\
\mathbb{E}(|W_k| \cdot |W_1|^{\gamma-1}) = b^{-(\xi_0-\delta)}, & \text{if } \xi_0 \in (\xi_*, 1].
\end{cases}
\]

We may assume that \( \delta \) is smaller enough so that \( \gamma > 1 \) when \( \xi_0 \in (\xi_*, 1] \), and we always have \( \gamma \in (0, 1) \) when \( \xi_0 \in (0, \xi_*] \). For \( w \in \mathcal{A}^* \) denote

\[
\widetilde{W}(w) = \begin{cases}
\; b^{\xi_0-\delta} \cdot |W_k(w)|^{\gamma}, & \text{if } \xi_0 \in (0, \xi_*]; \\
\; b^{\xi_0-\delta} \cdot |W_k(w)| \cdot |W_1(w)|^{\gamma-1}, & \text{if } \xi_0 \in (\xi_*, 1].
\end{cases}
\]

and

\[
Q(w) = \widetilde{W}(w_1)\widetilde{W}(w_2)\cdots \widetilde{W}(w).
\]

For \( n \geq 1 \) define the random measure \( \mu_n \):

\[
d\mu_n(x) = Q(x_1^n) d\mu_0(x).
\]
By construction, \((\mu_n)_{n \geq 1}\) is a measure-valued martingale thus yields a weak limit \(\mu\), and the support of \(\mu\) is a subset of \(K\).

For \(s, t \in [0, 1]\) define

\[
K_n^\gamma(s, t) = |F_k(s) - F_k(t)|^\gamma \vee O_k(s|_n)^\gamma
\]

if \(\gamma \in (0, 1]\) and

\[
K_n^\gamma(s, t) = (|F_k(s) - F_k(t)|^2 + |F_i(s) - F_i(t)|^2)^{\frac{\gamma}{2}} \vee (O_k(s|_n)^2 + O_l(s|_n)^2)^{\frac{\gamma}{2}}
\]

if \(\gamma > 1\). Due to the continuity of \(F\), one has almost surely \(K_n^\gamma\) converges uniformly to

\[
K^\gamma(s, t) = \begin{cases} |F_k(s) - F_k(t)|^\gamma, & \text{if } \gamma \in (0, 1]; \\ (|F_k(s) - F_k(t)|^2 + |F_i(s) - F_i(t)|^2)^{\frac{\gamma}{2}}, & \text{if } \gamma > 1. \end{cases}
\]

We have the following proposition:

**Proposition 2.1.** There exists a constant \(C\) such that for any \(0 \leq s < t \leq 1\) and \(n \geq 1\),

\[
\mathbb{E}\left(\frac{d\mu_n(s)d\mu_n(t)}{K_n^\gamma(s, t)}\right) \leq C \cdot \frac{d\mu_0(s)d\mu_0(t)}{|s - t|^{5\gamma - \delta}}.
\]

By using Fubini’s Theorem, Proposition 2.1 yields that for any \(n \geq 1\),

\[
\mathbb{E}\left(\int_{[0, 1]} \frac{d\mu_n(s)d\mu_n(t)}{K_n^\gamma(s, t)}\right) \leq 2C \int_{[0, 1]} \frac{d\mu_0(s)d\mu_0(t)}{|s - t|^{5\gamma - \delta}} < \infty.
\]

Since for any \(s, t \in [0, 1]\) one has

\[
K_n^\gamma(s, t) \leq \sup_{s, t \in [0, 1]} |F_k(s) - F_k(t)|^\gamma = X_k^\gamma,
\]

so (12) implies

\[
\sup_{n \geq 1} \mathbb{E}(X_k^{-\gamma} \cdot \mu_n([0, 1])^2) < \infty.
\]

Notice that for \(\gamma \in (0, 1)\) we have

\[
\mathbb{E}(\mu_n([0, 1])^{\frac{1}{1+\gamma}}) = \mathbb{E}(X_k^{\frac{1}{1+\gamma}} \cdot X_k^{-\frac{\gamma}{1+\gamma}} \cdot \mu_n([0, 1])^{\frac{\gamma}{1+\gamma}}) \leq \mathbb{E}(X_k^{\frac{1}{1+\gamma}}) \cdot \mathbb{E}(X_k^{-\gamma} \cdot \mu_n([0, 1])^2)^{\frac{\gamma}{1+\gamma}},
\]

and for \(\gamma \in (1, 2)\) we have

\[
\mathbb{E}(\mu_n([0, 1])) = \mathbb{E}(X_k^{\frac{1}{\gamma}} \cdot X_k^{-\frac{\gamma}{\gamma}} \cdot \mu_n([0, 1])) \leq \mathbb{E}(X_k^{\frac{1}{\gamma}}) \cdot \mathbb{E}(X_k^{-\gamma} \cdot \mu_n([0, 1])^2)^{\frac{1}{\gamma}}.
\]

Thus by using the corresponding martingale convergence theorem we get from (13) that \(\mathbb{E}(\mu([0, 1])) = 1\). Also by using the same tail event argument as in [5] we can get \(\mathbb{P}(\mu([0, 1]) > 0) = 1\).

Due to the fact that almost surely \(\mu_n\) converges weakly to \(\mu\) and \(K_n^\gamma\) converges uniformly to \(K^\gamma\), we get from (12) that

\[
\mathbb{E}\left(\int_{[0, 1]} \frac{d\mu(s)d\mu(t)}{K^\gamma(s, t)}\right) \leq \liminf_{n \to \infty} \mathbb{E}\left(\int_{[0, 1]} \frac{d\mu_n(s)d\mu_n(t)}{K_n^\gamma(s, t)}\right) < \infty.
\]

By using the mass distribution principle we get the desired lower bound

For the case \(\mathbb{P}(W_1 = W_2) = 1\), it is the same proof as above when \(\xi_0 \in (0, \xi_*]\). When \(\xi_0 \in (\xi_*, 1]\), we just take \(\gamma \in (0, 1)\) such that \(\mathbb{E}(|W_k|^\gamma) = b^{-\xi_* - \delta}\) and let

\[
\hat{W}(w) = b^{\xi_* - \delta} \cdot |W_k(w)|^\gamma.
\]
Then the same proceeding as we did for the case $\xi_0 \in (0, \xi_*)$ will yield the lower bound $\gamma$, which can be arbitrary close to 1, thus the conclusion. \hfill \Box

2.3. Proof of Proposition 2.1.

Proof. Recall that $Z_k = F_k(1)$. We will frequently use the following lemma, whose proof will be given in the next section.

Lemma 2.2.

(i) For any $\gamma \in (0, 1)$ there exists a constant $C_\gamma$ such that for any constants $A, B \in \mathbb{R}$ with $A \neq 0$, one has

$$
E\left(|AZ_k + B|^{-\gamma}\right) \leq C_\gamma \cdot |A|^{-\gamma}.
$$

(ii) If $\mathbb{P}(W_1 = W_2) < 1$, then for any $\gamma \in (1, 2)$ there exists a constant $C_\gamma$ such that for any constants $A_1, A_2, B_1, B_2 \in \mathbb{R}$ with $A_1A_2 \neq 0$, one has

$$
E\left((|A_1Z_k + B_1|^2 + |A_2Z_i + B_2|^2)^{-\gamma/2}\right) \leq C_\gamma \cdot (|A_1| \cdot |A_2|^{-1})^{-1}.
$$

For $n \geq 1$ and $w \in \mathcal{A}^n \setminus \{b - 1 \cdots b - 1\}$ denote by $w^+$ the unique word in $\mathcal{A}^n$ such that $\pi(w^+) = \pi(w) + b^+$. Since $s < t$, there exists a unique $j \geq 0$ such that $s^+|_j = t^+|_j$ and $s^+|_{j+1} \neq t^+|_{j+1}$. This implies $\pi(s^+|_j) + b^+-j-1 \leq t$ and

$$
b^-(j+1) \leq |s - t| \leq 2b^{-j} \leq b^{-(j-1)}.
$$

Notice that one has either $(s^+|_{j+1})|_j = s^+|_j$ or $(s^+|_{j+1})|_j = t^+|_j$. Without loss of generality we may assume $s^+|_{j+1} = s^+|_j \cdot r$ for $r = s_{j+1} + 1 \in \{1, \cdots, b - 1\}$.

Recall the definition of $K_n^+$ in (10) and (11). We have the following two situations.

2.3.1. When $\gamma < 1$.

(i) If $j \geq n$ then

$$
\frac{d\mu_n(s) d\mu_n(t)}{K_n^+(s, t)} \leq O_k(s|_n)^{-\gamma} \cdot Q(s|_n) \cdot Q(t|_n) \cdot d\mu_0(s) d\mu_0(t)
= b^{n(\xi_0 - \delta)} \cdot X_k(s|_n)^{-\gamma} \cdot |Q(t|_n)| \cdot d\mu_0(s) d\mu_0(t).
$$

Since $X_k(s|_n)$ and $Q(t|_n)$ are independent, so we get

$$
E\left(\frac{d\mu_n(s) d\mu_n(t)}{K_n^+(s, t)}\right) \leq b^{n(\xi_0 - \delta)} \cdot E(X_k^{-\gamma}) \cdot d\mu_0(s) d\mu_0(t)
\leq b^{\xi_0 - \delta} \cdot E(X_k^{-\gamma}) \cdot b^{(j-1)(\xi_0 - \delta)} \cdot d\mu_0(s) d\mu_0(t)
\leq b^{\xi_0 - \delta} \cdot E(X_k^{-\gamma}) \cdot \frac{d\mu_0(s) d\mu_0(t)}{|s - t|^\xi_0 - \delta}.
$$

(ii) If $j \leq n - 1$ then

$$
K_n^+(s, t)^{-1} \leq |F_k(s) - F_k(t)|^{-\gamma} = |Q_k(s^+|_{j+1}) \cdot Z_k(s^+|_{j+1}) + \Delta_k|^{-\gamma},
$$

where $\Delta_k = F_k(t) - F_k(\pi(s^+|_{j+1}) + b^+-j-1) + F_k(s^+|_{j+1}) - F_k(s)$. Notice that $Z_k(s^+|_{j+1})$ is independent of $Q(s|_n)$, $Q(t|_n)$, $Q_k(s^+|_{j+1})$ and $\Delta_k$. Let

$$
F(s^+|_{j+1}) = \sigma(W(w) : |w| \leq j + 1 \text{ or } w|_{j+1} \neq s^+|_{j+1}).
$$
From Lemma 2.2(i) we get
\[
E\left( \frac{d\mu_n(s)d\mu_n(t)}{K_n(s,t)} \middle| \mathcal{F}(s_{j+1}) \right)
\leq C \cdot |Q_k(s \cdot r)|^{-\gamma} \cdot Q(s_n) \cdot Q(t_n) \cdot d\mu_0(s)d\mu_0(t)
\]
\[
= C \cdot |W_k(s \cdot r)|^{-\gamma} \cdot b^{(j+1)(\xi_0-\delta)} \cdot \prod_{i=j+1}^{n} \hat{W}(s_i) \cdot Q(t_n) \cdot d\mu_0(s)d\mu_0(t).
\]
Since all the random variables in the above products are independent, we have
\[
E\left( \frac{d\mu_n(s)d\mu_n(t)}{K_n(s,t)} \right) \leq C \cdot \mathbb{E}(|W_k|^{-\gamma}) \cdot b^{(j+1)(\xi_0-\delta)} \cdot d\mu_0(s)d\mu_0(t)
\leq C \cdot b^{2(\xi_0-\delta)} \cdot \mathbb{E}(|W_k|^{-\gamma}) \cdot \frac{d\mu_0(s)d\mu_0(t)}{|s-t|^{\xi_0-\delta}}.
\]

### 2.3.2. When \( \gamma > 1 \).

(i) If \( j \geq n \) then
\[
\frac{d\mu_n(s)d\mu_n(t)}{K_n(s,t)} \leq \frac{Q(s_n) \cdot Q(t_n) \cdot d\mu_0(s)d\mu_0(t)}{(O_k(s_n))^2 + O_l(s_n)^2)^{\gamma}}
\leq \frac{Q(s_n) \cdot Q(t_n) \cdot d\mu_0(s)d\mu_0(t)}{((Q_k(s_n) \cdot Z_k(s_n))^2 + (Q_l(s_n) \cdot Z_l(s_n))^2)^{\gamma}}.
\]
Let \( F_n = \sigma(W(w) : w \in \mathcal{A}^n) \). From Lemma 2.2(ii) we get
\[
E\left( \frac{d\mu_n(s)d\mu_n(t)}{K_n(s,t)} \middle| F_n \right)
\leq C \cdot \mathbb{E}(|Q_k(s_n)| \cdot |Q_l(s_n)|^{-\gamma-1}) \cdot Q(s_n) \cdot Q(t_n) \cdot d\mu_0(s)d\mu_0(t)
\]
\[
= C \cdot b^{n(\xi_0-\delta)} \cdot Q(t_n) \cdot d\mu_0(s)d\mu_0(t).
\]
This implies
\[
E\left( \frac{d\mu_n(s)d\mu_n(t)}{K_n(s,t)} \right) \leq C \cdot b^{n(\xi_0-\delta)} \cdot d\mu_0(s)d\mu_0(t)
\leq C \cdot b^{\xi_0-\delta} \cdot \frac{d\mu_0(s)d\mu_0(t)}{|s-t|^{\xi_0-\delta}}.
\]

(ii) If \( j \leq n - 1 \), like in Section 2.3.1(ii) one has
\[
K_n(s,t)^{-1}
\leq (|F_k(s) - F_k(t)|^2 + |F_l(s) - F_l(t)|^2)^{-\frac{\gamma}{2}}
\]
\[
= (|Q_k(s_{j+1}^+ \cdot Z_k(s_{j+1}^+) + \Delta_k|^2 + |Q_l(s_{j+1}^+ \cdot Z_l(s_{j+1}^+) + \Delta_l|^2)^{-\frac{\gamma}{2}}.
\]
By using Lemma 2.2(ii) we get

\[ \mathbb{E}\left( \frac{d\mu_n(s)}{K_n(s,t)} \left| F(s|_{j+1}^+) \right| \right) \]

\[ \leq C \cdot \left( |Q_k(s)\cdot r| \cdot |Q(t)|^{-1} \cdot Q(s) \cdot Q(t) \cdot d\mu_0(s) \cdot d\mu_0(t) \right) \]

\[ = C \cdot Q \cdot (s) \cdot r^{-1} \cdot Q(s) \cdot Q(t) \cdot d\mu_0(s) \cdot d\mu_0(t) \]

\[ = C \cdot \bar{W}(s) \cdot r^{-1} \cdot b^{(j+1)(\xi_0 - \delta)} \cdot \prod_{i=j+1}^{n} W(s) \cdot Q(t). \]

Thus

\[ \mathbb{E}\left( \frac{d\mu_n(s)}{K_n(s,t)} \right) \leq C \cdot \mathbb{E}(|W_k|^{-1}|W_i|^{-1}) \cdot b^{(j+1)(\xi_0 - \delta)} \cdot d\mu_0(s) \cdot d\mu_0(t) \]

\[ \leq C \cdot \mathbb{E}(|W_k|^{-1}|W_i|^{-1}) \cdot b^{2(\xi_0 - \delta)} \cdot \sum_{s=|s|}^{\gamma} \left| \mathbb{E}(\nu_0(s) \cdot d\mu_0(t)) \right|. \]

2.3.3. Conclusion. Let

\[ C = \left\{ \begin{array}{ll}
\max \left\{ b^{\xi_0 - \delta} \cdot \mathbb{E}(X^{-\gamma}), C \cdot b^{2(\xi_0 - \delta)} \cdot \mathbb{E}(|W_k|^{-\gamma}) \right\}, & \text{if } \xi_0 \in (0, \xi_*]; \\
\max \left\{ C \cdot b^{\xi_0 - \delta}, C \cdot b^{2(\xi_0 - \delta)} \cdot \mathbb{E}(|W_k|^{-1}|W_i|^{-1}) \right\}, & \text{if } \xi_0 \in (\xi_*, 1].
\end{array} \right. \]

Then we get the conclusion from Section 2.3.1 and 2.3.2

2.4. Proof of Lemma 2.2.

Proof. (i) Let \( \varphi_k(x) = \mathbb{E}(e^{ixZ_k}) \) be the characteristic function of \( Z_k \). From (8) we have the following functional equation:

\[ Z_k = \sum_{j=0}^{b-1} W_k(j) \cdot Z_k(j). \]

This implies \( \varphi_k(x) = \mathbb{E}\left( \prod_{j=0}^{b-1} \varphi_k(x \cdot W_k(j)) \right) \) and

\[ |\varphi_k(x)| \leq \mathbb{E}\left( |\varphi_k(x \cdot W_k)| \right)^b. \]

Starting from (16) and following the proof of Theorem 2.1 in [17] one can prove the following result:

If \( \mathbb{E}(W_k^{-s}) < \infty \) for some \( s > 0 \), then \( |\varphi(x)| = O(|x|^{-s}) \) when \( x \to \infty \).

Under assumption (A2) this result will imply that \( \varphi_k \in L^1(\mathbb{R}) \), thus \( Z_k \) has a bounded density function \( f_k \) with \( \|f_k\|_{\infty} \leq C_k := \int_{\mathbb{R}} |\varphi_k(x)| \, dx < \infty \). Then

\[ \mathbb{E}\left( |AZ_k + B|^{-\gamma} \right) = \int_{|A|^{-\gamma}} \frac{f_k(x)}{|A_x + B|^{-\gamma}} \, dx \leq \left( 1 + C_k \int_{|u| \leq 1} \frac{1}{|u|^{-\gamma}} \, du \right) \cdot |A|^{-\gamma}. \]
(ii) First we assume that \( Z_k \) and \( Z_l \) have a bounded joint density function \( f \) with \( \|f\|_\infty = C < \infty \), then

\[
\begin{align*}
\mathbb{E}\left( \left( |A_1 Z_k + B_1|^2 + |A_2 Z_l + B_2|^2 \right)^{-\gamma/2} \right) \\
= \iint f(x, y) \, dx \, dy \\
= |A_2|^{-\gamma} \iint \frac{f(x, y)}{\left( |A_2|^{-\gamma} |A_2 x + B_1|^2 + |y + B_2|^{\gamma/2} \right)} \\
= \frac{|A_2|^{-\gamma} A_2}{|A_1|} \iint \frac{f\left( \frac{A_1}{A_2} u - \frac{B_1}{A_2}, v - \frac{B_2}{A_2} \right)}{(u^2 + \nu^2)^{\gamma/2}} \, du \, dv \\
\leq |A_1|^{-1} |A_2|^{-\gamma+1} \left( 1 + C \int \int_{\sqrt{u^2 + \nu^2} \leq 1} \frac{dv \, du}{(u^2 + \nu^2)^{\gamma/2}} \right),
\end{align*}
\]

which gives us the conclusion. So it is enough to show that the characteristic function

\[
\varphi : (x, y) \in \mathbb{R}^2 \mapsto \varphi(x, y) = \mathbb{E}(e^{i(xZ_k + yZ_l)})
\]

is in \( L^1(\mathbb{R}^2) \). For we consider the polar coordinates: for \( r \in \mathbb{R}_+ \) and \( \theta \in [0, 2\pi) \) define

\[
\bar{\varphi}(r, \theta) = \varphi(r \cos \theta, r \sin \theta) = \mathbb{E}(e^{i(r \cos \theta Z_k + r \sin \theta Z_l)}).
\]

Let \( \psi(r) = \sup_{\theta \in [0, 2\pi]} |\bar{\varphi}(r, \theta)| \). Clearly \( \psi(r) \leq 1 \), so it is enough to show that \( \psi(r) = O(r^{-s}) \) for some \( s > 2 \) when \( r \to \infty \). This can be done by using a similar argument as in (i): From (17) and (15) one has

\[
\bar{\varphi}(r, \theta) = \mathbb{E}\left( \prod_{j=0}^{b-1} \bar{\varphi}(r \cdot \bar{W}(j), \theta + \bar{j}(j)) \right),
\]

where \( \bar{W}(j) = \sqrt{|W_k(j)|^2 + |W_l(j)|^2} \) and \( \bar{j}_j = \arccos(W_k(j)/\bar{W}(j)) \). Due to Fatou’s lemma this implies

\[
\psi(r) \leq \mathbb{E}\left( \psi(r \cdot \bar{W}) \right)^b, \text{ where } \bar{W} = \sqrt{|W_k|^2 + |W_l|^2}.
\]

Again, starting from inequality (18) and following the proof of Theorem 2.1 in [17] (with a non-trivial modification which we will present later), one can prove the following result:

\[
(19) \quad \text{If } \mathbb{E}(\bar{W}^{-s}) < \infty \text{ for some } s > 0, \text{ then } \psi(r) = O(r^{-s}) \text{ when } r \to \infty.
\]

Then we can get the conclusion due to assumption (A2).

The non-trivial modification for proving (19) is the part that proves \( \psi(r) < 1 \) holds for all \( r > 0 \), the rest of the proof will follow easily from the proof of Theorem 2.1 in [17]. In order to prove \( \psi(r) < 1 \) holds for all \( r > 0 \), first we show that \( \psi(r) < 1 \) holds for all \( r \) smaller enough.

Suppose that it is not the case, then we can find sequences \( r_n \to 0 \) and \( \theta_n \in [0, 2\pi) \) such that \( |\bar{\varphi}(r_n, \theta_n)| = 1 \), thus there exists a subset \( \Omega' \subset \Omega \) with \( \mathbb{P}(\Omega') = 1 \) such that there exists a sequence \( \zeta_n \in [0, 2\pi) \) such that

\[
r_n \cos \theta_n Z_k(\omega) + r_n \sin \theta_n Z_l(\omega) \in \zeta_n + 2\pi \mathbb{Z}
\]

holds for all \( n \geq 1 \) and \( \omega \in \Omega' \). In other words, for any \( \omega, \omega' \in \Omega' \) one has

\[
r_n \cos \theta_n (Z_k(\omega) - Z_k(\omega')) + r_n \sin \theta_n (Z_l(\omega) - Z_l(\omega')) \in 2\pi \mathbb{Z}, \quad \forall \, n \geq 1.
\]
From $r_n \to 0$ one gets
\[
\cos \theta_n(Z_k(\omega) - Z_k(\omega')) + \sin \theta_n(Z_l(\omega) - Z_l(\omega')) = 0
\]
for all $n$ large enough. Since $\cos \theta_n$ and $\sin \theta_n$ cannot be equal to 0 at the same time and $Z_k, Z_l$ are not almost surely a constant, so there exist a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ and a constant $c \neq 0$ such that
\[
Z_k(\omega) - Z_k(\omega') = c(Z_l(\omega) - Z_l(\omega'))
\]
holds for all $\omega, \omega' \in \Omega'$. This implies that $Z_k - cZ_l$ is a constant on $\Omega'$. In other words, $\sum_{j=0}^b W_j(j)Z_k(j) - cW_j(j)Z_l(j)$ is almost surely a constant. But this could happen only if $W_j(j)Z_k(j) - cW_j(j)Z_l(j)$ is almost surely equal to 0 for each $j = 0, \cdots, b - 1$ (since they are i.i.d. random variables). So we get $c = 1$ and $W_k = W_l$ almost surely, which is a contradiction to the assumption $\mathbb{P}(W_1 = W_2) < 1$.

Now suppose there exists a $h > 0$ such that $\psi(h) = 1$, and we may assume that $\psi(r) < 1$ holds for all $0 < r < h$. From (18) we get
\[
1 = \psi(h) \leq \mathbb{E}(\psi(h \cdot \tilde{W}))^b \leq 1.
\]
This implies that almost surely $\psi(h \cdot \tilde{W}) = 1$. Due to (A1) there exists $q \in (1, 2]$ such that $\mathbb{E}(|W_1|^q) \vee \mathbb{E}(|W_2|^q) < b^{-1}$, so by using Chebyshev’s inequality we get
\[
\mathbb{P}(|W_i| \geq 1) \leq \mathbb{E}(|W_i|^q) < b^{-1}, \quad i = 1, 2.
\]
This implies that
\[
\mathbb{P}(|W_i| < 1 \& |W_i| < 1) \geq 1 - \mathbb{P}(|W_i| \geq 1) - \mathbb{P}(|W_i| \geq 1) > 1 - 2b^{-1} \geq 0.
\]
Thus there exists $\delta < 1$ such that $\psi(h \cdot \delta) = 1$, which is a contradiction. \qed

3. Proof of Theorem 1.2

3.1. Upper bound.

Proof. For the upper bound we only need $\dim_H K = \xi_0$.

Given $\epsilon > 0$, for each $n \geq 1$ one can find a sequence $\mathcal{I}_n$ of $b$-adic intervals such that
\[
K \subset \bigcup_{I \in \mathcal{I}_n} I \quad \text{and} \quad \sum_{I \in \mathcal{I}_n} |I|^s \leq 2^{-n}.
\]
Fix $q = (q_1, q_2) \in J(\xi_0)$, denote by $\nabla \Phi(q) = \alpha = (\alpha_1, \alpha_2) \in \Lambda(\xi_0)$. For $n \geq 1$ define
\[
\mathcal{I}_n(\alpha) = \left\{ I \in \mathcal{I}_n : |I|^{\alpha_k + \epsilon} \leq |O_k(I)| \leq |I|^{\alpha_k - \epsilon} \text{ for } k = 1, 2 \right\}.
\]
It is easy to see that for each $N \geq 1$, $\bigcup_{n \geq N} \mathcal{I}_n(\alpha)$ is a covering of $K(\alpha)$.

Let $C = \mathbb{E}(X_1^n \cdot X_2^m) < \infty$. Then like in Section 2.1, we can obtain the desired upper bounds from the following three estimates:

(i) For $s = \xi_0 + q \cdot \alpha - \Phi(q) + (|q_1| + |q_2| + 1)\epsilon$,
\[
\mathbb{E} \left( \sum_{I \in \mathcal{I}_n(\alpha)} |I|^s \right) \leq \mathbb{E} \left( \sum_{I \in \mathcal{I}_n(\alpha)} |O_1(I)|^{q_1} |O_2(I)|^{q_2} |I|^{-q \cdot \alpha - (|q_1| + |q_2|)\epsilon} \cdot |I|^s \right)
\]
\[
\leq 2C \sum_{I \in \mathcal{I}_n} |I|^\xi_0 \cdot 2^{-n},
\]

(ii) For $s = \xi_0 + q \cdot \alpha - \Phi(q) + (|q_1| + |q_2| + 1)\epsilon$,
Proof. Since \( \mu \) is of full measure, for \( q \cdot \alpha - \Phi(q) + (|q_1| + |q_2| + 1)\epsilon \),
\[
\mathbb{E} \left( \sum_{I \in \mathcal{I}_n(\alpha)} O_1(I)^s \vee O_2(I)^s \right)
\leq \mathbb{E} \left( \sum_{I \in \mathcal{I}_n(\alpha)} |I|^{(\alpha_1 - \epsilon)s} + |I|^{(\alpha_2 - \epsilon)s} \right)
\leq 2\mathbb{E} \left( \sum_{I \in \mathcal{I}_n} |I|^{(\alpha_1 - \epsilon)s} \cdot |O_1(I)|q_1 |O_2(I)|q_2 |I|^{-q \alpha - (|q_1| + |q_2|)\epsilon} \right)
\leq 2C \sum_{I \in \mathcal{I}_n} |I|^{\epsilon_0 + \epsilon} \leq 2C \cdot 2^{-n};
\]
(iii) For \( s = 1 + (\alpha^* - \epsilon^{-1}) \cdot (\zeta_0 + q \cdot \alpha - \Phi(q) + (|q_1| + |q_2| + 2)\epsilon - \alpha^*) \),
\[
\mathbb{E} \left( \sum_{I \in \mathcal{I}_n(\alpha)} \left( \frac{O_2(I)}{O_1(I)} \cdot O_1(I)^s \right) \vee \left( \frac{O_1(I)}{O_2(I)} \cdot O_2(I)^s \right) \right)
\leq \mathbb{E} \left( \sum_{I \in \mathcal{I}(\alpha)} O_2(I)O_1(I)^{s-1} + O_1(I)O_2(I)^{s-1} \right)
\leq \mathbb{E} \left( \sum_{I \in \mathcal{I}(\alpha)} |I|^{\alpha_2+\alpha_1} |I|^{-1} + |I|^{\alpha_1+\alpha_2} |I|^{-s-1} \right)
\leq 2\mathbb{E} \left( \sum_{I \in \mathcal{I}_n} |I|^{\alpha_2+\alpha_1} |I|^{-s-1} \cdot |O_1(I)|q_1 |O_2(I)|q_2 |I|^{-q \alpha - (|q_1| + |q_2|)\epsilon} \right)
\leq 2C \sum_{I \in \mathcal{I}_n} |I|^{\epsilon_0 + \epsilon} \leq 2C \cdot 2^{-n}.
\]
\[\square\]

3.2. Lower bound (part I).

Proof. We will use the following modified Frostman’s lemma: 

**Lemma 3.1.** If \( \dim_H K = \dim_P K = \zeta_0 \) and \( \mathcal{H}^{\zeta_0}(K) > 0 \), then there exist a Borel probability measure \( \mu_0 \) supported by \( K \) and a non-increasing sequence \( (\epsilon_n)_{n \geq 1} \) tending to 0 such that 

(i) \( \mu_0(I) \leq |I|^{\zeta_0} \) holds for all \( b \)-adic intervals;

(ii) \( N_n(w) := \# \{ u \in \mathbb{Z}^n : \mu_0(I_{wu}) \neq 0 \} \leq \mu_0(I_w) \cdot b(|w| + n)(\zeta_0 + \epsilon_{|w| + n}) \) holds for all \( n \geq 1 \) and \( w \in \mathbb{Z}^n \).

Proof. Since \( \mathcal{H}^{\zeta_0}(K) > 0 \), due to Frostman’s lemma (see Theorem 8.8 in [18] for example) there exists a Borel probability measure \( \mu \) supported by \( K \) such that \( \mu(I) \leq |I|^{\zeta_0} \) holds for any \( b \)-adic interval \( I \). Since we also have \( \dim_P K = \zeta_0 \), so the set 
\[
K' = \left\{ x \in K : \limsup_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leq \zeta_0 \right\}
\]
is of full \( \mu \)-measure. For \( n \geq 1 \) define
\[
\epsilon_n = \max \left\{ \frac{\log \mu(I_n(x))}{\log |I_n(x)|} - \zeta_0 : x \in K' \right\}.
\]
First we have $\limsup_{n \to \infty} \epsilon_n \leq 0$. Otherwise we can find a positive number $\epsilon > 0$, a sequence $\{n_j\}_{j \geq 1}$ of integers and a sequence $\{w_j \in \mathcal{A}^{n_j}\}_{j \geq 1}$ of finite words such that

$$I_{w_j} \cap K' \neq \emptyset \quad \text{and} \quad \frac{\log \mu(I_{w_j})}{\log |I_{w_j}|} \geq \xi_0 + \epsilon$$

holds for all $j \geq 1$. Clearly one can find $x_1 \in \{0, \cdots, b-1\}$ such that $I_{x_1} \cap I_{w_j} \neq \emptyset$ holds for infinitely many $j \geq 1$, and for the same reason, for any $k \geq 2$ there exists $x_k \in \{0, \cdots, b-1\}$ such that $I_{x_1 \cdots x_{k-1} x_k} \cap I_{w_j} \neq \emptyset$ holds for infinitely many $j \geq 1$. So $x = \sum_{k \geq 1} x_k b^{-k} \in K'$ and $\limsup_{n \to \infty} \frac{\log \mu(I_{\theta^{(n)}(x)})}{\log |I_{\theta^{(n)}(x)}|} \geq \xi_0 + \epsilon$, which is a contradiction.

By definition for any $n \geq 1$ and $w \in \mathcal{A}^n$ such that $\mu(I_w) \neq 0$ one has

$$|I_w|^\xi_0 + \epsilon \leq \mu(I_w) \leq |I_w|^\xi_0,$

thus $\epsilon_n \geq 0$. This implies that $\lim_{n \to \infty} \epsilon_n = 0$. Moreover, for any $w \in \mathcal{A}^* \cap \omega$ and $n \geq 1$ one has

$$\#\{u \in \mathcal{A}^n : \mu(I_{wu}) \neq 0\} \leq \sum_{u \in \mathcal{A}^n, \mu(I_{wu}) \neq 0} b^{(|w|+n)(\xi_0 + \epsilon_{|w|+n})} \mu(I_{wu}) = \sum_{u \in \mathcal{A}^n, \mu(I_{wu}) \neq 0} b^{(|w|+n)(\xi_0 + \epsilon_{|w|+n})} \mu(I_{wu}).$$

Then we get the conclusion by redefining $\epsilon_n = \sup_{k \geq n} \epsilon_k$. \hfill \qed

Let $\mu_0$ be the Frostman measure given by Lemma 3.1. Recall that $\Phi(q_1, q_2) = -\log_b \mathbb{E}(|W_1(w)|^{q_1} |W_2(w)|^{q_2})$.

For $w \in \mathcal{A}^*$ define

$$\tilde{W}(w) = b^{\Phi(q_1, q_2)} \cdot |W_1(w)|^{q_1} |W_2(w)|^{q_2},$$

and

$$Q(w) = \tilde{W}(w|_1) \cdot \tilde{W}(w|_2) \cdots \tilde{W}(w|_n).$$

For $n \geq 1$ we define the random measure

$$d\mu_n(x) = Q(x|_n) \, d\mu_0(x).$$

By construction, $(\mu_n)_{n \geq 1}$ forms a measure-valued martingale thus yields a weak limit $\mu$.

The following results about the convergence of $\mu_n$ and the lower Hausdorff dimension of $\mu$ can be deduced from [3]. For readers’ convenient we present the proofs here. Since we are dealing with the ideal case of multiplicative chaos and only asking for non-simultaneously results, the proofs are much easier.

3.2.1. Almost surely $b$-adic decomposition. Let $d = \xi_0 + q \cdot \alpha - \Phi(q) > 0$.

For $p > 1$ define

$$\theta(p) = p \Phi(q) - \Phi(pq_1, pq_2).$$

Since $\Phi$ is analytic around $q$, there exist constants $\epsilon_0 > 0$ and $C_0 > 1$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$\theta(1 + \epsilon) \leq \epsilon (\Phi(q) - q \cdot \alpha) + C_0 \epsilon^2 = \epsilon (\xi_0 - d) + C_0 \epsilon^2.$$

For $w, u \in \mathcal{A}^*$ let $Q[w](u) = Q(wu)/Q(w)$. For any $w \in \mathcal{A}^*$ such that $\mu_0(I_w) \neq 0$ and $n \geq 1$ define

$$Y_n(w) = \mu_0(I_w)^{-1} \sum_{u \in \mathcal{A}^n} Q[w](u) \cdot \mu_0(I_{wu}).$$
By construction \((Y_n(w))_{n \geq 1}\) is a positive martingale and of expectation 1.

We need the following moments inequality given by Von Bahr and Esseen in [21]: for any \(p \in (1, 2]\), there exists a constant \(c_p\) such that for any independent centered random variables \(V_1, \ldots, V_n\), we have

\[
\mathbb{E}\left( \left| \sum_{i=1}^{n} V_i \right|^p \right) \leq c_p \cdot \mathbb{E}(|V|^p).
\]

Then for any \(\epsilon \in (0, \epsilon_0)\) one has

\[
\mathbb{E}\left(|Y_{n+1}(w) - Y_n(w)|^{1+\epsilon}\right) \leq c_{1+\epsilon} \cdot \mu_0(I_w)^{-1-\epsilon} \cdot \\
\sum_{w \in \mathcal{A}^n} \mathbb{E}(Q[w]^{1+\epsilon}) \cdot \mathbb{E}\left( \left| \sum_{j=0}^{b-1} \widetilde{W}(wu_j) \mu_0(I_{wu_j}) - \mu_0(I_{wu}) \right|^{1+\epsilon}\right) \leq 2c_{1+\epsilon} \cdot \mu_0(I_w)^{-1-\epsilon} \cdot b(n+1)(1+\epsilon) \cdot \\
\sum_{w \in \mathcal{A}^{n+1}, \mu_0(I_{wu}) \neq 0} \mu_0(I_{wu})^{1+\epsilon} \leq 2c_{1+\epsilon} \cdot \mu_0(I_w)^{-\epsilon} \cdot b^{n+\epsilon}(1+\epsilon) \cdot N_{n+1}(w) \cdot b^{-(|w|+n+1)\epsilon_0(1+\epsilon)} \leq 2c_{1+\epsilon} \cdot \mu_0(I_w)^{-\epsilon} \cdot b^{n+\epsilon}(1+\epsilon) \cdot b^{(|w|+n+1)(\epsilon_0+c_{|w|+n+1} - \epsilon_1(1+\epsilon))},
\]

(20)

\[
\text{Since } d > 0 \text{ and } \lim_{n \to \infty} \epsilon_n = 0, \text{ inequality (20) shows that for any } \epsilon \in (0, \epsilon_0/C_0) \text{ and for any } w \in \mathcal{A}^n \text{ with } \mu_0(I_w) \neq 0, \text{ the martingale } (Y_n(w))_{n \geq 1} \text{ converges almost surely and in } L^{1+\epsilon} \text{ to a limit } Y(w), \text{ and there exists a constant } C_\epsilon \text{ independent of } w \text{ such that}
\]

\[
\mathbb{E}(Y(w)^{1+\epsilon}) \leq C_\epsilon \cdot \mu_0(I_w)^{-\epsilon} \cdot b^{n(\epsilon_1+c_{|w|})}.
\]

Thus almost surely for all \(w \in \mathcal{A}^n\) such that \(\mu_0(I_w) \neq 0\) we have the following decomposition:

\[
\mu(I_w) = Q(w) \cdot Y(w) \cdot \mu_0(I_w),
\]

where \(Q(w)\) and \(Y(w)\) are independent and of expectation 1. As a direct consequence, \(\mu\) is supported by \(K\) and \(\mu(K) > 0\) almost surely.

3.2.2. Lower bound of \(\dim_H(\mu)\). For \(n \geq 1\) and \(\delta > 0\) define the set

\[
E_{n, \delta} = \left\{ x \in K : \mu(I_n(x)) \geq b^{-n(d-\delta)} \right\}.
\]

Then by using (21) one has for any \(\epsilon \in (0, \epsilon_0/C_0)\),

\[
\mathbb{E}(\mu(E_{n, \delta})) \leq \mathbb{E} \left( \sum_{w \in \mathcal{A}^n, \mu_0(I_w) \neq 0} [Q(w) \cdot Y(w) \cdot \mu_0(I_w)]^{1+\epsilon} \cdot b^{n(\epsilon(d-\delta))} \right) \leq C_\epsilon \cdot b^{n(\epsilon_0+c_{|w|})} \cdot b^{n(-\epsilon_1+c_{|w|})} \cdot b^{n(\epsilon(d-\delta))} \cdot \\
\sum_{w \in \mathcal{A}^n, \mu_0(I_w) \neq 0} \mu_0(I_w) = C_\epsilon \cdot b^{n(-\epsilon_0+c_{|w|}^2+c_{|w|})}.
\]

This implies that \(\sum_{n \geq 1} \mathbb{E}(\mu(E_{n, \delta})) < \infty\) and we get from Borel-Cantelli lemma that almost surely for \(\mu\)-almost every \(x \in K\):

\[
\lim \inf_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \geq d - \delta.
\]
Thus yields that almost surely $\dim_H(\mu) \geq d$.

3.2.3. Asymptotic behavior of $|Q_k(x|n)|$ with respect to $\mu$. For $\delta > 0$, $k = 1, 2$, $\lambda = \pm 1$ and $n \geq 1$ define the set

$$E_{k,n,\delta}^\lambda = \left\{ x \in K : |Q_k(x|n)| \cdot b^{n\delta} \geq b^{n\delta} \right\}.$$ 

One has for any $\eta > 0$

$$\mu(E_{k,n,\delta}^\lambda) \leq \sum_{w \in \mathcal{X}^n} \left[ |Q_k(w)| \cdot b^{n\alpha_k} \cdot b^{-n\delta} \right]^\eta \cdot Q(w) \cdot Y(w) \cdot \mu_0(I_w).$$

Taking the expectation from both side we get

$$\mathbb{E}(\mu(E_{k,n,\delta}^\lambda)) \leq b^{n(\Phi(q_1, q_2) - \Phi(q_1 + \lambda \eta, q_2) + \lambda \eta \alpha_1 - \delta \eta)},$$

$$\mathbb{E}(\mu(E_{k,n,\delta}^\lambda)) \leq b^{n(\Phi(q_1, q_2) - \Phi(q_1, q_2 + \lambda \eta) + \lambda \eta \alpha_2 - \delta \eta)}.$$ 

For $\eta$ smaller enough one has

$$\Phi(q_1, q_2) - \Phi(q_1 + \lambda \eta, q_2) + \lambda \eta \alpha_1 - \delta \eta = O(\eta^2) - \delta \eta < 0,$$

$$\Phi(q_1, q_2) - \Phi(q_1, q_2 + \lambda \eta) + \lambda \eta \alpha_2 - \delta \eta = O(\eta^2) - \delta \eta < 0.$$ 

Thus $\sum_{n \geq 1} \mathbb{E}(\mu(E_{k,n,\delta}^\lambda)) < \infty$. From Borel-Cantelli lemma we get that almost surely for $\mu$-almost every $x \in K$,

$$\alpha_k - \delta \leq \liminf_{n \to \infty} \frac{\log |Q_k(x|n)|}{\log b^{-n}} \leq \limsup_{n \to \infty} \frac{\log |Q_k(x|n)|}{\log b^{-n}} \leq \alpha_k + \delta, \ k = 1, 2.$$ 

By taking a sequence $\delta_n$ tending to 0 this implies that almost surely for $\mu$-almost every $x \in K$,

$$\limsup_{n \to \infty} \frac{\log |Q_1(x|n)|}{\log |Q_2(x|n)|} = \alpha.$$ 

3.2.4. Asymptotic behavior of $X_k(x|n)$ with respect to $\mu$. For $\delta > 0$, $k = 1, 2$, $\lambda = 1, -1$ and $n \geq 1$ define the set

$$E_{k,n,\delta}^\lambda = \left\{ x \in K : X_k(x|n) \cdot b^{n\delta} \right\}.$$ 

One has for any $\epsilon \in (0, \epsilon_0/C_0)$,

$$\mathbb{E}(\mu(E_{k,n,\delta}^\lambda)) \leq \mathbb{E}
\left( \sum_{w \in \mathcal{X}^n, \mu_0(I_w) \neq 0} 1_{\left\{ X_k(w) \geq b^{n\delta} \right\}} \cdot Q(w) \cdot Y(w) \cdot \mu_0(I_w) \right)$$

$$\leq \mathbb{E}
\left( \sum_{w \in \mathcal{X}^n, \mu_0(I_w) \neq 0} 1_{\left\{ X_k(w) \geq b^{n\delta} \right\}} \cdot \mu_0(I_w) \right) \cdot \mathbb{E}(Y^\alpha) \cdot \mathbb{E}(Y^{1+\epsilon}) \cdot \mu_0(I_w)$$

$$\leq C_{\frac{1}{1+\epsilon}} \cdot \mathbb{E}(X_k^\alpha) \cdot b^{-n\delta} \cdot \mathbb{E}(Y^{1+\epsilon}) \cdot \sum_{w \in \mathcal{X}^n, \mu_0(I_w) \neq 0} \mu_0(I_w)$$

$$\leq C_{\frac{1}{1+\epsilon}} \cdot \mathbb{E}(X_k^\alpha) \cdot b^{-n\delta} \cdot \mathbb{E}(Y^{1+\epsilon}) \cdot b^{-\alpha \xi_0} \cdot \mathbb{E}(Y^{1+\epsilon}) \cdot b^{-\alpha \xi_0}$$

$$= C_{\frac{1}{1+\epsilon}} \cdot \mathbb{E}(X_k^\alpha) \cdot b^{-n((\delta - (2+\epsilon)\epsilon_0)/(1+\epsilon)}. $$
Thus implies \( \sum_{n \geq 1} \mathbb{E}(\mu(\tilde{E}_{1,n}^\lambda)) < \infty \). From Borel-Cantelli lemma we get that for \( k = 1, 2 \), almost surely for \( \mu \)-almost every \( x \in K \),

\[-\delta \leq \liminf_{n \to \infty} \frac{\log X_k(x|_n)}{\log |I_n(x)|} \leq \limsup_{n \to \infty} \frac{\log X_k(x|_n)}{\log |I_n(x)|} \leq \delta.\]

By taking a sequence \( \delta_n \) tending to 0 this implies that almost surely for \( \mu \)-almost every \( x \in K \),

\[
\lim_{n \to \infty} \frac{\log X_1(x|_n)}{\log |I_n(x)|} = \lim_{n \to \infty} \frac{\log X_2(x|_n)}{\log |I_n(x)|} = 0.
\]

Together with (22) we proved that almost surely \( \mu \) is carried by \( K(\alpha) \), thus almost surely \( \dim_H K(\alpha) \geq \dim_H \mu \geq d. \)

3.3. Lower bound (part II).

**Proof.** We will keep using the same notation as in Section 3.2.

3.3.1. The asymptotic behavior of \( |Q_k^\prime(x|_n^+)\| \) with respect to \( \mu \). Recall that for \( n \geq 1 \) and \( w \in \mathcal{A}^n \setminus \{b \cdot 1 \cdots b \cdot 1\} \), \( w^+ \) is the unique word in \( \mathcal{A}^n \) such that \( \pi(w^+) = \pi(w) + b^{-n} \). For \( \delta > 0 \) and \( n \geq 1 \) define the set

\[ E_{n,\delta}' = \{ x \in K : |Q_1(x|_n^+) \| \leq b^{-n(\alpha_1 + \delta)} \}. \]

One has for any \( \eta > 0 \),

\[ \mu(E_{n,\delta}') \leq \sum_{w \in \mathcal{A}^n} \left[ |Q_1(w^+) \| \cdot b^{n(\alpha_1 + \delta)} \right]^{-\eta} \cdot Q(w) \cdot Y(w) \cdot \mu_0(I_w). \]

For \( n \geq 1 \) we have

\[ \bigcup_{w \in \mathcal{A}^n} (w, w^+) = \bigcup_{m=0}^{n-1} \bigcup_{u \in \mathcal{A}^m} \bigcup_{i=0}^{b-2} (u \cdot i \cdot g_{n-1-m}, u \cdot (i+1) \cdot d_{n-1-m}), \]

where \( g_n \) (resp. \( d_n \)) is the word consisting of \( n \) times the letter \( b \cdot 1 \) (resp. 0). For \( w = u \cdot i \cdot g_{n-1-m} \) and \( w^+ = u \cdot (i+1) \cdot d_{n-1-m} \) with \( u \in \mathcal{A}^m \) one has

\[
Q(w)|Q_1(w^+)|^{-\eta} = Q(u)|Q_1(u)|^{-\eta} \cdot \tilde{W}(ui)|W_1(u(i+1))|^{-\eta} \cdot \prod_{k=1}^{n-1-m} \tilde{W}(uig_k)|W_1(u(i+1)d_k)|^{-\eta}.
\]

This implies that

\[ \mathbb{E}(Q(w)|Q_1(w^+)|^{-\eta}) = b^{m(\Phi(q_1,q_2) - \Phi(q_1-q_2))} \cdot \mathbb{E}(|W_1|^{-\eta}) \cdot n-m. \]

Thus

\[ \mathbb{E}(\mu(E_{n,\delta}')) \leq b^{-n\eta(\alpha_1 + \delta)} \cdot \sum_{m=0}^{n-1} b^{m(\Phi(q_1,q_2) - \Phi(q_1-q_2))} \cdot \mathbb{E}(|W_1|^{-\eta}) \cdot n-m \cdot \sum_{u \in \mathcal{A}^m} \sum_{i=0}^{b-2} \mu_0(I_{uig_{n-1-m}}). \]

Notice that when \( \mu_0(I_{uid_{n-1-m}}) \neq 0 \) one has

\[ \mu_0(I_{uid_{n-1-m}}) \leq b^{-n\xi_0} \leq \mu_0(I_{ui}) \cdot b^{-(n-m-1)\xi_0} \cdot b^{(m+1)\xi_{m+1}}, \]
so from (24) we get

$$
\mathbb{E}(\mu(E_{n, \delta})) \leq b^{\delta_0} \cdot b^{n(\Phi(q_1, q_2) - \Phi(q_1 - \eta, q_2)) - \eta \alpha_1 - \delta \eta)} \cdot \sum_{m=0}^{n-1} A(\eta)^m \cdot b^{(m+1)\epsilon_{m+1}},
$$

where $A(\eta) = b^{\Phi(q_1, q_2) - \Phi(q_1 - \eta, q_2)} \cdot \mathbb{E}(|W_1| - \eta) \cdot b^{-\delta_0}$. Since $A(0) = b^{-\delta_0} < 1$, so $A(\eta) < 1$ for $\eta$ smaller enough, as well as

$$
\Phi(q_1, q_2) - \Phi(q_1 - \eta, q_2) - \eta \alpha_1 - \delta \eta = O(\eta^2) - \delta \eta < 0.
$$

This implies that $\sum_{n \geq 1} \mathbb{E}(\mu(E_n(\delta))) < \infty$. From Borel-Cantelli lemma we get that almost surely for $\mu$-almost every $x \in K$, $\liminf_{n \to \infty} \frac{\log |Q_k(x)^+|}{\log b - n} \geq \alpha_1 - \delta$. In an analogous way we can also prove that almost surely for $\mu$-almost every $x \in K$, $\liminf_{n \to \infty} \frac{\log |Q_k(x)^+|}{\log b - n} \geq \alpha_2 - \delta$. Then by taking a sequence $\delta_n$ tending to 0 we get that almost surely for $\mu$-almost every $x \in K$,

$$
\liminf_{n \to \infty} \frac{\log |Q_k(x)^+|}{\log |I_n(x)|} \geq \alpha_k, \quad k = 1, 2.
$$

3.3.2. Lower bound. We will only give the proof for the case $\mathbb{P}(W_1 = W_2) < 1$, the case $\mathbb{P}(W_1 = W_2) = 1$ would follow easily from the same argument as in the end of Section 2.2.

Recall that $d = \xi_0 + q \cdot \alpha - \Phi(q) > 0$ and fix $\delta \in (0, d)$.

Let $\{k, l\} = \{1, 2\}$ be such that $\alpha_k \leq \alpha_l$ and let

$$
\gamma = \begin{cases}
(\alpha_k + \delta)^{-1} \cdot (d - (|q_1| + |q_2| + 1) \delta), & \text{if } d \leq \alpha_k, \\
1 + (\alpha_l + \delta)^{-1} \cdot (d - \alpha_k - (|q_1| + |q_2| + 2) \delta), & \text{if } d > \alpha_k.
\end{cases}
$$

We may assume that $\delta$ is smaller enough so that $\gamma > 1$ if $d > \alpha_k$, and $\gamma \in (0, 1)$ if $d \leq \alpha_k$.

For $w \in \mathcal{A}^*$ we define the indicator function

$$
\chi(w) = \begin{cases}
1, & \text{if } b^{-|w|(|\alpha_1 + \delta)} \leq |Q_i(w)| \leq b^{-|w|(|\alpha_i - \delta)} \text{ and } |Q_i(w)^+| \geq b^{-|w|(|\alpha_i + \delta)} \text{ holds for } i = 1, 2, \\
0, & \text{otherwise}.
\end{cases}
$$

For $m \geq 1$ define the set

$$
E_m = \{x \in K : \chi(x|_n) = 1 \text{ for all } n \geq m\}.
$$

Recall the definition of $K_n^\alpha$ in (10) and (11). We have the following proposition:

**Proposition 3.1.** There exists a constant $C$ such that for any $s < t$, $|s - t| \leq b^{-m}$ and $n \geq m$,

$$
\mathbb{E}\left(1_{E_m \times E_m(s, t)} \cdot \frac{d\mu_n(s) d\mu_n(t)}{K_n(s, t)} \right) \leq C \cdot \frac{d\mu_0(s) d\mu_0(t)}{|s - t|^{\xi_0 - \delta}}.
$$

For $n \geq 1$ define

$$
\mathcal{E}_{m, n} = \int\int_{|s - t| \leq b^{-m}} 1_{E_m \times E_m(s, t)} \cdot \frac{d\mu_n(s) d\mu_n(t)}{K_n(s, t)}.
$$
By using Fatou’s lemma and Fubini’s theorem, Proposition 3.1 yields that
\[
\mathbb{E}\left(\int_{s \in E_m \mid |s-t| < b^{-m}} \frac{d\mu(s) d\mu(t)}{K^n(s, t)}\right) \leq \liminf_{n \to \infty} \mathbb{E}(\mathcal{E}_{m, n})
\]
\[
\leq 2C \cdot \int_{s, t \in [0,1]} \frac{d\mu_0(s) d\mu_0(t)}{|s-t|^{\xi_0 - \delta}}.
\]
Since \(\mu_0(I) \leq |I|^{\xi_0}\) holds for any \(b\)-adic interval \(I\), we have
\[
\int_{s, t \in [0,1]} \frac{d\mu_0(s) d\mu_0(t)}{|s-t|^{\xi_0 - \delta}} < \infty
\]
holds for any \(\delta > 0\). This implies that almost surely for any \(m \geq 1\),
\[
\int_{s, t \in E_m \mid |s-t| < b^{-m}} \frac{d\mu(s) d\mu(t)}{K^n(s, t)} < \infty.
\]
From Section 3.2 and 3.3.1 we know that almost surely \(\mu(\lim_{m \to \infty} E_m) > 0\), thus for \(\mathbb{P}\)-almost every \(\omega \in \Omega\) there exists a smallest integer \(N = N(\omega)\) such that \(\mu(E_N) > 0\). Then from mass distribution principle we get that almost surely \(\dim_H F(K(\alpha)) \geq \gamma\). Then by taking a sequence \(\delta_n \to 0\) we get the conclusion. \(\square\)

3.4. **Proof of Proposition 3.1.**

**Proof.** The proof of Proposition 3.1 is very like that of Proposition 2.1. The main difference is that here we need to choose certain indicator function that remains fine information after taking the conditional expectation.

Recall that \(s < t\) and there exists a unique \(j \geq 0\) such that \(s_j^+ = t_j\) and \(s_{j+1}^+ \neq t_{j+1}\), as well as \(\pi(s_{j+1}^+) + b^{-j-1} \leq t\) and
\[
b^{-j} \leq |s-t| \leq 2b^{-j} \leq b^{-j-1}.
\]
We may also assume that \(s_{j+1}^+ = s_j^+ \cdot r\), where \(r = s_j+1 + 1 \in \{1, \ldots, b-1\}\).

3.4.1. **When \(\gamma < 1\).**

(i) If \(j \geq n\): Since \(n \geq m\), so \(1_{E_m \times E_m}(s, t) \leq \chi(s|n)\). Notice that under \(\chi(s|n) = 1\) one has
\[
|Q_k(s|n)| \geq b^{-n(\alpha_k + \delta)}\) and \(Q(s|n) \leq b^{-n(q\cdot\alpha - \Phi(q) - (|q_1| + |q_2|)\delta)}\),
\]
thus
\[
1_{E_m \times E_m}(s, t) \cdot \frac{d\mu_n(s) d\mu_n(t)}{K^n(s, t)} \leq \chi(s|n) \cdot O_k(s|n)^{-\gamma} \cdot Q(s|n) \cdot Q(t|n) d\mu_0(s) d\mu_0(t)
\]
\[
\leq (b^{-n(\alpha_k + \delta)} X_k(s|n))^{-\gamma} \cdot b^{-n(q\cdot\alpha - \Phi(q) - (|q_1| + |q_2|)\delta)} \cdot Q(t|n) d\mu_0(s) d\mu_0(t).
\]
Recall that \(\gamma = (\alpha_k + \delta)^{-1}(\xi_0 + q\cdot\alpha - \Phi(q) - (|q_1| + |q_2| + 1)\delta)\). Since \(X_k(s|n)\) and \(Q(t|n)\) are independent, by taking expectation from both side we get
\[
\mathbb{E}\left(1_{E_m \times E_m}(s, t) \cdot \frac{d\mu_n(s) d\mu_n(t)}{K^n(s, t)}\right) \leq \mathbb{E}(X_k^{-\gamma}) \cdot b^{n(\xi_0 - \delta)} d\mu_0(s) d\mu_0(t)
\]
\[
\leq b^{\xi_0 - \delta} \mathbb{E}(X_k^{-\gamma}) \cdot \frac{d\mu_0(s) d\mu_0(t)}{b^{-j-1}(\xi_0 - \delta)}
\]
\[
\leq b^{\xi_0 - \delta} \mathbb{E}(X_k^{-\gamma}) \cdot \frac{d\mu_0(s) d\mu_0(t)}{|s-t|^{\xi_0 - \delta}}.
\]
Recall that 

\[ 1 \mathbb{E}_m \times E_m (s, t) \leq \chi (s | j + 1). \]

Notice that under \( \chi (s | j + 1) = 1 \) one has

\[ |Q_k (s | j + 1)| \geq b^{-(j + 1)(\alpha_k + \delta)} \] and \( Q(s | j + 1) \leq b^{-(j + 1)(q \alpha - \Phi(q) + (|q_1| + |q_2|) \delta)} \).

Recall (14). Since \( \chi (s | j + 1) \) is measurable with respect to \( \mathcal{F}(s | j + 1) \), thus by using Lemma 2.2(i) we get

\[
\mathbb{E} \left( 1_{E_m \times E_m} (s, t) \cdot \frac{d\mu_n (s) d\mu_n (t)}{K_n (s, t)} \bigg| \mathcal{F}(s | j + 1) \right) \\
\leq C \gamma \cdot \chi (s | j + 1) \cdot |Q_k (s | j + 1)|^{- \gamma} \cdot Q(s | n) \cdot Q(t | n) d\mu_0 (s) d\mu_0 (t) \\
\leq C \gamma \cdot (b^{-(j + 1)(\alpha_k + \delta)})^{- \gamma} \cdot b^{-(j + 1)(q \alpha - \Phi(q) + (|q_1| + |q_2|) \delta)} \\
\cdot \prod_{i=j+1}^n \mathbb{E} \left( W(s_i) \cdot Q(t | n) d\mu_0 (s) d\mu_0 (t). \right)
\]

Since \( \prod_{i=j+1}^n \mathbb{E} \left( W(s_i) \right) \) and \( Q(t | n) \) are independent, by taking the expectation from both side we get

\[
\mathbb{E} \left( 1_{E_m \times E_m} (s, t) \cdot \frac{d\mu_n (s) d\mu_n (t)}{K_n (s, t)} \right) \leq C \gamma \cdot b^{2(\xi_0 - \delta)} \cdot \frac{d\mu_0 (s) d\mu_0 (t)}{b^{-(j + 1)(\xi_0 - \delta)}} \\
\leq C \gamma \cdot b^{2(\xi_0 - \delta)} \cdot \frac{d\mu_0 (s) d\mu_0 (t)}{|s - t|^{\xi_0 - \delta}}.
\]

### 3.4.2. When \( \gamma > 1 \).

(i) If \( j \geq n \): We have \( 1_{E_m \times E_m} (s, t) \leq \chi (s | n) \). Notice that under \( \chi (s | n) = 1 \) one has

\[ |Q_k (s | n)| \geq b^{-n(\alpha_k + \delta)}, \quad |Q(t | n)| \geq b^{-n(\alpha_1 + \delta)} \] and

\[ Q(s | n) \leq b^{-n(q \alpha - \Phi(q) - (|q_1| + |q_2|) \delta)}. \]

Since \( \chi (s | n) \) is measurable with respect to \( \mathcal{F}_n = \sigma (W(w) : |w| \leq n) \), so we get from Lemma 2.2(ii) that

\[
\mathbb{E} \left( 1_{E_m \times E_m} (s, t) \cdot \frac{d\mu_n (s) d\mu_n (t)}{K_n (s, t)} \bigg| \mathcal{F}_n \right) \\
\leq C \gamma \cdot \chi (s | n) \cdot \frac{Q(s | n) \cdot Q(t | n) d\mu_0 (s) d\mu_0 (t)}{|Q_k (s | n)| \cdot |Q(t | n)|^{- \gamma}} \\
\leq C \gamma \cdot (b^{-n(\alpha_k + \delta)})^{- \gamma} \cdot (b^{-n(\alpha_1 + \delta)})^{1 - \gamma} \cdot b^{-n(q \alpha - \Phi(q) - (|q_1| + |q_2|) \delta)} \\
\cdot Q(t | n) d\mu_0 (s) d\mu_0 (t). \]

Recall that \( \gamma = 1 + (\alpha_1 + \delta)^{-1} \cdot (\xi_0 + q \cdot \Phi(q) - \alpha_k - (|q_1| + |q_2| + 2) \delta) \), then by taking expectation from both side we get

\[
\mathbb{E} \left( 1_{E_m \times E_m} (s, t) \cdot \frac{d\mu_n (s) d\mu_n (t)}{K_n (s, t)} \right) \leq C \gamma \cdot b^{n(\xi_0 - \delta)} d\mu_0 (s) d\mu_0 (t) \\
\leq C \gamma \cdot b^{\xi_0 - \delta} \cdot \frac{d\mu_0 (s) d\mu_0 (t)}{|s - t|^{\xi_0 - \delta}}.
\]
(ii) If \( j \leq n - 1 \): Like in Section 2.3.2(ii), we have
\[
\mathbb{E}\left( 1_{E_{m} \times E_{m}}(s, t) \cdot \frac{d\mu_{n}(s) d\mu_{n}(t)}{K_{n}^{s}(s, t)} | \mathcal{F}(s_{j+1})^{+} \right) \\
\leq C_{\gamma} \cdot \chi(s_{j+1}) \cdot \frac{Q(s | n)Q(t | n) d\mu_{0}(s) d\mu_{0}(t)}{|Q_{k}(s_{j+1})| \cdot |Q_{l}(s_{j+1})|^{\gamma - 1}} \\
\leq C_{\gamma} \cdot b^{(j+1)(\xi_{0} - \delta)} \cdot \prod_{i=j+1}^{n} \tilde{W}(s_{i}) \cdot Q(t | n) d\mu_{0}(s) d\mu_{0}(t).
\]

Since \( \prod_{i=j+1}^{n} \tilde{W}(s_{i}) \) and \( Q(t | n) \) are independent, we get
\[
\mathbb{E}\left( 1_{E_{m} \times E_{m}}(s, t) \cdot \frac{d\mu_{n}(s) d\mu_{n}(t)}{K_{n}^{s}(s, t)} \right) \leq C_{\gamma} \cdot b^{2(\xi_{0} - \delta)} \cdot \frac{d\mu_{0}(s) d\mu_{0}(t)}{b^{-j(j+1)(\xi_{0} - \delta)}} \\
\leq C_{\gamma} \cdot b^{2(\xi_{0} - \delta)} \cdot \frac{d\mu_{0}(s) d\mu_{0}(t)}{|s - t|^{\xi_{0} - \delta}}.
\]

3.4.3. Conclusion. From Section 3.4.1 and 3.4.2 we get the conclusion by letting
\( C = \max \{ b^{\xi_{0} - \delta} \mathbb{E}(X_{k}^{\gamma}), C_{\gamma} \cdot b^{2(\xi_{0} - \delta)} \}. \)

\[ \square \]

4. PROOF OF THEOREM 1.3

Proof. It is enough to prove the result for \( F_{1} \). We will use the same method as one constructs the local time of fractional Brownian motion to compute the Hausdorff dimension of its level sets, see [11] for example.

Let \( \nu \) be the occupation measure of \( F_{1} \) with respect to the Lebesgue measure on \([0, 1]\), that is the Borel measure defined as
\[
\nu(B) = \int_{0}^{1} \mathbf{1}_{\{F_{1}(t) \in B\}} dt \quad \text{for any Borel set } B \subset \mathbb{R}.
\]

First we show that almost surely \( \nu \) is absolutely continuous with respect to the Lebesgue measure. For we consider the Fourier transform of \( \nu \):
\[
\hat{\nu}(u) = \int_{0}^{1} e^{iuF_{1}(t)} dt.
\]

We will show that
\[
\mathbb{E}\left( \int_{\mathbb{R}} |\hat{\nu}(u)|^{2} du \right) < \infty.
\]

This will imply that almost surely \( \hat{\nu} \) is in \( L^{2}(\mathbb{R}) \). Therefore almost surely \( \nu \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \) and its density function belongs to \( L^{2}(\mathbb{R}) \).

By using Fubini’s theorem one has
\[
\mathbb{E}\left( \int_{\mathbb{R}} |\hat{\nu}(u)|^{2} du \right) = \mathbb{E}\left( \int_{s, t \in [0, 1]} \int_{\mathbb{R}} e^{iu(F_{1}(t) - F_{1}(s))} du ds dt \right).
\]

Fix \( 0 \leq s < t \leq 1 \). Recall that there exists a unique \( j \geq 0 \) such that \( s_{j+1}^{+} = t_{j} \), \( s_{j+1}^{+} \neq t_{j+1} \) and
\[
F_{1}(t) - F_{1}(s) = Q_{1}(s_{j+1}^{+}) \cdot Z_{1}(s_{j+1}^{+}) + \Delta_{1},
\]
where \( \Delta_{1} = F_{1}(t) - F_{1}(\pi(s_{j+1}^{+}) + b^{-j-1}) + F_{1}(s_{j+1}^{+}) - F_{1}(s) \). By construction \( Z_{1}(s_{j+1}^{+}) \) is independent of \( Q_{1}(s_{j+1}^{+}) \) and \( \Delta_{1} \).
Recall (14) and \( \varphi_1(u) = \mathbb{E}(e^{iuZ_1}) \). We have

\[
\mathbb{E}\left( \int_{\mathbb{R}} e^{iu(F_1(t) - F_1(s))} \, du \mid \mathcal{F}(s)_{j+1}^+ \right) = \int_{\mathbb{R}} e^{iu\Delta_1} \cdot \varphi_1(Q_1(s)_{j+1}^+ \cdot u) \, du.
\]

We know from the proof of Lemma 2.2 that \( \varphi_1 \) is in \( L^1(\mathbb{R}) \), thus

\[
\left| \mathbb{E}\left( \int_{\mathbb{R}} e^{iu(F_1(t) - F_1(s))} \, du \right) \right| \leq \mathbb{E}\left( \int_{\mathbb{R}} |\varphi_1(Q_1(s)_{j+1}^+ \cdot u)| \, du \right)
\]

\[
= \mathbb{E}\left( \int_{\mathbb{R}} |\varphi_1(Q_1(s)_{j+1}^+)|^{-1} \right) \cdot \int_{\mathbb{R}} |\varphi_1(u)| \, du
\]

\[
= b^{(j+1)\alpha_1} \cdot \int_{\mathbb{R}} |\varphi_1(u)| \, du
\]

(26)

\[
\leq b^{2\alpha_1} \cdot |s - t|^{-\alpha_1} \cdot \int_{\mathbb{R}} |\varphi_1(u)| \, du.
\]

This implies that

\[
\mathbb{E}\left( \int_{\mathbb{R}} |\tilde{v}(u)|^2 \, du \right) \leq b^{2\alpha_1} \cdot \left( \int_{\mathbb{R}} |\varphi_1(u)| \, du \right) \cdot \int_{s,t \in [0,1]} |s - t|^{-\alpha_1} \, ds \, dt < \infty.
\]

We have proved that almost surely \( \nu \) is absolutely continuous with respect to Lebesgue measure. This implies that almost surely for Lebesgue almost every \( y \in F_1([0,1]) \) the following limit

\[
\lim_{r \to 0} \frac{1}{r} \int_{0}^{1} 1\{ |F_1(t) - y| \leq r \} \, dt
\]

exists and belongs to \((0, \infty)\), thus yields a positive finite Borel measure \( \nu_y \) carried by \( L_1(y) = \{ t \in [0,1] : F_1(t) = y \} \), defined as

\[
\int_{0}^{1} g(t) \, d\nu_y(t) = \lim_{r \to 0^+} \frac{1}{r} \int_{0}^{1} 1\{ |F_1(t) - y| \leq r \} g(t) \, dt, \quad \forall \ g \in \mathcal{C}([0,1]).
\]

Moreover, for any Borel measurable function \( G : [0,1] \times \mathbb{R} \to \mathbb{R}_+ \) one has

\[
\int_{y \in F_1([0,1])} \int_{[0,1]} G(t, y) \, d\nu_y(t) \, dy = \int_{0}^{1} G(t, F_1(t)) \, dt.
\]

Let \( \gamma > 0 \). Due to Fatou’s lemma and Fubini’s theorem we have

\[
\int_{y \in F_1([0,1])} \int_{0}^{1} \int_{0}^{1} |s - t|^{-\gamma} \, d\nu_y(s) \, d\nu_y(t) \, dy
\]

\[
= \int_{y \in F_1([0,1])} \int_{0}^{1} \left[ \lim_{r \to 0^+} \frac{1}{r} \int_{0}^{1} 1\{ |F_1(s) - y| \leq r \} |s - t|^{-\gamma} \, ds \right] \, d\nu_y(t) \, dy
\]

\[
\leq \lim_{r \to 0^+} \frac{1}{r} \int_{0}^{1} \int_{y \in F_1([0,1])} \int_{0}^{1} 1\{ |F_1(s) - y| \leq r \} |s - t|^{-\gamma} \, d\nu_y(t) \, ds \, dy
\]

(27)

\[
= \lim_{r \to 0^+} \frac{1}{r} \int_{0}^{1} \int_{0}^{1} 1\{ |F_1(s) - F_1(t)| \leq r \} |s - t|^{-\gamma} \, dt \, ds.
\]
Fix $0 \leq s < t \leq 1$. Recall (25) and the fact that $Z_1(s|_{j+1})$ has a bounded density function $f_1$ with $\|f_1\|_\infty = C_1 < \infty$, so
\[
E\left(1\{f_1(s) - f_1(t) \leq r\} | \mathcal{F}(s|_{j+1})\right)
\]
\[
= \int_{\mathbb{R}} 1\{|x|\leq \frac{r}{Q_1(s|_{j+1})}\} f_1(x) \, dx
\]
\[
= \int_{\mathbb{R}} 1\{|z|\leq \frac{r}{Q_1(s|_{j+1})}\} f_1(z - \frac{\Delta_1}{Q_1(s|_{j+1})}) \, dz
\]
(28)
\[
\leq C_1 \cdot \frac{2r}{|Q_1(s|_{j+1})|} = 2C_1 \cdot r \cdot \beta(j+1)\alpha_1.
\]
By using again Fatou’s lemma and Fubini’s theorem we get from (27) and (28) that
\[
E\left(\int_{y \in F_1([0,1])} \int_0^1 \int_0^1 |s - t|^{-\gamma} \, d\nu_\gamma(s) \, d\nu_\gamma(t) \, dy \right) \leq C \int_0^1 \int_0^1 |s - t|^{-(\gamma + \alpha_1)} \, ds \, dt,
\]
where $C = 2C_1 \beta^{2\alpha_1}$. Due to mass distribution principle we get that for any $\gamma < 1 - \alpha_1$, almost surely for Lebesgue almost every $y \in F_1([0,1])$,
\[
\dim_H L_1(y) \geq \gamma.
\]
This gives us the desired lower bound.

For the upper bound, we use the fact that almost surely the Hausdorff dimension of the graph of $F_1$, defined as $\{(t, F_1(t)) : t \in [0,1]\}$, is equal to $2 - \alpha_1$ (see [4]). Then from Corollary 7.12 in [7] we know that there can’t exist a subset $E \subset F_1([0,1])$ with positive Lebesgue measure such that for every $y \in E$, $\dim_H L_1(y) > 1 - \alpha_1$.

5. PROOF OF THEOREM 1.4

Proof. The proof is inspired by [8], but different.

For $n \geq 1$ denote by $\mathcal{T}_n = \{\pi(w) : w \in \mathcal{A}^n\} \cup \{1\}$ the set of $b$-adic numbers of generation $n$ in $[0,1]$. We also denote by $\mathcal{S}_n$ the collection of all $b$-adic squares in $\mathbb{R}^2$ with side length $b^{-n}$.

Fix $p \geq 2$, $n \geq 1$ and $S \in \mathcal{S}_n$, denote by $m = \lfloor n(1-1/p) \rfloor$ and
\[
\mathcal{T}_m = \{0 = t_0 < t_1 < \cdots < t_m = 1\}.
\]
Let
\[
N(S) := \#\{j = 0, \cdots, b^m : F(t_j) \in S\}.
\]
For $k \geq 1$ one has $P(N(S) \geq k) \leq E(A_k(S))$, where
\[
A_k(S) := \sum_{s_1 < s_2 < \cdots < s_k \in \mathcal{T}_m} \prod_{l=1}^k 1\{F(s_l) \in S\}.
\]
We have
\[
A_{k+1}(S)
\]
\[
= \sum_{s_1 < s_2 < \cdots < s_k \in \mathcal{T}_m, \ s_k \neq 1} \left(\prod_{l=1}^k 1\{F(s_l) \in S\}\right) \cdot \sum_{s_k < t \in \mathcal{T}_m} 1\{F(t) \in S\}
\]
\[
\leq \sum_{s_1 < s_2 < \cdots < s_k \in \mathcal{T}_m, \ s_k \neq 1} \left(\prod_{l=1}^k 1\{F(s_l) \in S\}\right) \cdot \sum_{s_k < t \in \mathcal{T}_m} 1\{|F(t) - F(s_k)| \leq \sqrt{2}|S|\}.
\]
Notice that for any \( t \in \mathcal{T}_m \) with \( t > s_k \), one has

\[
\mathbb{1}_{\{|F(t) - F(s_k)| \leq \sqrt{2}|S|\}} \leq |F(t) - F(s_k)|^{\frac{1+1/p}{\alpha}} \cdot (\sqrt{2}|S|)^{\frac{1+1/p}{\alpha}}.
\]

Fix \( t > s_k \), then there exists a unique \( j_t \geq m \) such that \( s_k|_{j_t} = t|_{j_t} \) and \( s_k|_{j_t+1} \neq t|_{j_t+1} \). Recall (14) and (25). Like in Section 2.3.2(ii), by using Lemma 2.2(ii) we can find a constant \( C \) such that

\[
\mathbb{E}\left( |F(t) - F(s_k)|^{\frac{1+1/p}{\alpha}} \left| \mathcal{F}(s_k|_{j_t+1}) \right. \right) \leq C : |t - s_k|^{(1+1/p)}.
\]

This implies

\[
\sum_{s_k < t \in \mathcal{T}_m} \mathbb{E}\left( \mathbb{1}_{\{|F(t) - F(s_k)| \leq \sqrt{2}|S|\}} \left| \mathcal{F}(s_k|_{j_t+1}) \right. \right) \leq 2^{\frac{1+1/p}{2\alpha}} C \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot \sum_{s_k < t \in \mathcal{T}_m} |t - s_k|^{(1+1/p)}
\]

\[
\leq 2^{\frac{1+1/p}{2\alpha}} C \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot b^m (1+1/p) \cdot \sum_{l=1}^{\infty} l^{-(1+1/p)}
\]

\[
\leq C' \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot b^{m \frac{1+1/p}{1-p}} < \infty,
\]

where \( C' = 2^{\frac{1+1/p}{2\alpha}} C \cdot \sum_{l=1}^{\infty} l^{-(1+1/p)} < \infty \). Then let

\[
B_k(S) = \sum_{s_1 < s_2 < \cdots < s_k \in \mathcal{T}_m, s_k \neq 1} \left( \prod_{l=1}^{k} \mathbb{1}_{\{F(s_l) \in S\}} \right).
\]

Obviously \( B_k(S) \leq A_k(S) \). Notice that for all \( s_k < t \in \mathcal{T}_m \), \( \mathcal{F}(s_k|_{j_t+1}) \) is independent of \( B_k(S) \), thus we get from (29) that

\[
\mathbb{E}(A_{k+1}(S)) \leq C' \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot \mathbb{E}(B_k(S))
\]

\[
\leq C' \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot \mathbb{E}(A_k(S))
\]

\[
\leq (C')^k \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot \mathbb{E}(A_1(S)).
\]

This gives us

\[
\mathbb{P}( \sup_{S \in \mathcal{S}_n} N(S) \geq k + 1) \leq (C')^k \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot \mathbb{E}\left( \sum_{j=0}^{b^m} \mathbb{1}_{\{F(t_j) \in S\}} \right)
\]

\[
= (C')^k \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot \mathbb{E}\left( \sum_{j=0}^{b^m} \mathbb{1}_{\{F(t_j) \in S\}} \right)
\]

\[
= (C')^k \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot (b^m + 1)
\]

\[
\leq 2(C')^k \cdot b^{-n \frac{1+1/p}{\alpha}} \cdot b^{m \frac{1-1/p}{\alpha}}
\]

\[
= 2(C')^k \cdot b^{-n \frac{1}{\alpha}} \cdot (b-1).
\]

Let \( k = p + 1 \), then from Borel-Cantelli lemma one gets for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) there exists a integer \( n_p(\omega) \) such that for any \( n \geq n_p(\omega) \),

\[
\sup_{S \in \mathcal{S}_n} N(S) \leq p + 2.
\]
For \( w \in \mathcal{A}^* \) let
\[
X(w) = \sup_{s,t \in [0,1]} |F^w(s) - F^w(t)|.
\]
From the fact that \( \mathbb{E}(W_1^q) \leq \mathbb{E}(W_2^q) = b^{-q\alpha} < b^{-1} \) holds for all \( q > 1/\alpha \) we get \( \mathbb{E}(X') < \infty \) holds for all \( q > 0 \). Then for any \( n \geq 1 \) one has
\[
\mathbb{P}( \sup_{w \in \mathcal{A}^n} X(w) \geq b^{n/p} ) \leq \sum_{w \in \mathcal{A}^n} \mathbb{P}(X(w) \geq b^{n/p}) \leq b^n \cdot b^{-n(p+1)/p} \cdot \mathbb{E}(X_{p+1}).
\]
Due to Borel-Cantelli lemma, this implies that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) there exists an integer \( n_p(\omega) \) such that for any \( n \geq n_p(\omega) \) and \( w \in \mathcal{A}^n \),
\[
X(w) \leq b^{n/p}.
\]

Let \( \Omega' = \bigcap_{p \geq 2} \{ \omega \in \Omega : n_p(\omega) \wedge n_p(\omega) < \infty \} \).
So \( \mathbb{P}(\Omega') = 1 \). Now fix \( \omega \in \Omega' \), \( p \geq 2 \) and \( n \geq n_p(\omega) \wedge n_p(\omega) \).

For any \( S \in \mathcal{S}_n \) there are at most \( p + 2 \) many \( w \in \mathcal{A}^{(n(1-1/p)/\alpha)} \) such that \( F(\pi(w)) \in S \). Also for any \( w \in \mathcal{A}^{(n(1-1/p)/\alpha)} \) one has
\[
\sup_{s,t \in I_w} |F'(s) - F'(t)| = b^{-\alpha|w|} \cdot X(w) \leq b^{-\varsigma n(1-1/p)/\alpha}(\alpha^{-1/p}).
\]

By comparing the area, this implies that there are at most
\[
b^2n \cdot \pi^2 \cdot \left( b^{-\varsigma n(1-1/p)/\alpha}(\alpha^{-1/p}) + \sqrt{2}b^{-\gamma} \right)^2 \cdot (p + 2)
\]
many \( w \in \mathcal{A}^{(n(1-1/p)/\alpha)} \) such that \( F(I_w) \cap S \neq \emptyset \), where \( C = (p + 2) \cdot \pi^2 \cdot (1 + \sqrt{2})^2 \).

**Lower bound.** For any Borel set \( K \subseteq [0,1] \), let \( C_N \) be any \( b \)-adic square covering of \( F(K) \) such that
\[
\sum_{S \in C_N} |S|^{\dim_H F(K) + 1/p} \leq 2^{-N}.
\]
Denote by \( n(S) = \left\lfloor \frac{-\log_b |S|^{(1-1/p)}}{\alpha} \right\rfloor \). For \( N \) large enough one has
\[
\bigcup_{S \in C_N} = \{ I_w : w \in \mathcal{A}^{n(S)}, F(I_w) \cap S \neq \emptyset \}
\]
forms a covering of \( K \). Moreover, due to (30), for
\[
s = (\dim_H F(K) + 1/p) \cdot \frac{\alpha}{1 - 1/p} + (2 + \frac{2\alpha}{1 - 1/p})/p
\]
one has
\[
\sum_{S \in C_N} \sum_{w \in \mathcal{A}^{n(S)}, F(I_w) \cap S \neq \emptyset} |I_w|^s \leq C \sum_{S \in C_N} |S|^{\dim_H F(K) + 1/p} \leq C \cdot 2^{-N},
\]
which implies
\[
\dim_H K \leq (\dim_H F(K) + 1/p) \cdot \frac{\alpha}{1 - 1/p} + (2 + \frac{2\alpha}{1 - 1/p})/p.
\]
Since this works for all \( p \geq 2 \), we get \( \dim_H K \leq \alpha \dim_H F(K) \).
Upper bound. Now consider any \( b \)-adic interval covering \( I_N \) of \( K \) such that
\[
\sum_{I \in I_N} |I|^\dim_H K + 1/p \leq 2^{-N}.
\]
For \( N \) large enough one has \( \sup_{s,t \in I} |F(s) - F(t)| \leq |I|^{|\alpha| - 1/p} \) holds for any \( I \in I_N \), thus for each \( I \) one can use a square of side length \( 2|I|^{|\alpha| - 1/p} \) to cover \( F(I) \). From
\[
\sum_{I \in I_N} (2|I|^{|\alpha| - 1/p})^{(\dim_H K + 1/p)/(|\alpha| - 1/p)} = \sum_{I \in I_N} |I|^\dim_H K + 1/p \leq 2^{-N}
\]
we get
\[
\dim_H F(K) \leq \frac{\dim_H K + 1/p}{\alpha - 1/p}.
\]
Since it works for all \( p \geq 2 \), we get \( \dim_H K \geq \dim_H F(K) \).

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LAGA, Institut Galilée, Université Paris 13, 93430 Villetaneuse, France

*E-mail address: xiongjin82@gmail.com*