Two-dimensional Inflow-wind Solution of Hot Accretion Flow. I. Hydrodynamics

Amin Mosallanezhad1, Fatemeh Zahra Zeraatgari1, Liquan Mei1, and De-Fu Bu2

1 School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an, Shaanxi 710049, People’s Republic of China; fzeraatgari@xjtu.edu.cn
2 Key Laboratory for Research in Galaxies and Cosmology, Shanghai Astronomical Observatory, Chinese Academy of Sciences, 80 Nandan Road, Shanghai 200030, People’s Republic of China

Received 2020 September 23; revised 2021 January 18; accepted 2021 January 19; published 2021 March 12

Abstract

We solve the 2D hydrodynamical equations of hot accretion flow in the presence of the thermal conduction. The flow is assumed to be in steady state and axisymmetric, and a self-similar approximation is adopted in the radial direction. In this hydrodynamical study, we consider the viscous stress tensor to mimic the effects of the magnetorotational instability for driving angular momentum. We impose the physical boundary conditions at both the rotation axis and the equatorial plane and obtain the solutions in the full \( r-\theta \) space. We have found that thermal conduction is an indispensable term for investigating the inflow-wind structure of the hot accretion flows with very low mass accretion rates. One of the most interesting results here is that the disk is convectively stable in hot accretion mode and in the presence of the thermal conduction. Furthermore, the properties of wind and also its driving mechanisms are studied. Our analytical results are consistent with previous numerical simulations of hot accretion flow.

Unified Astronomy Thesaurus concepts: High energy astrophysics (739); Black hole physics (159); Low-luminosity active galactic nuclei (2033); Accretion (14)

1. Introduction

Various black hole (BH) accretion models have been proposed in the past several decades, including standard thin disk (Shakura & Sunyaev 1973), super-Eddington accretion flow (slim disk; Abramowicz et al. 1988), and also hot accretion flow (Narayan & Yi 1994; Abramowicz et al. 1995). Based on the temperature of the accretion flow, these models can be divided into cold and hot modes where the standard thin disk and super-Eddington accretion flow belong to cold mode (Yuan & Narayan 2014).

In recent years, wind has become a fascinating subject in the study of accretion flows in both cold and hot modes. Wind appears to carry significant mass, angular momentum, and energy away from the disk and has the potential for a greater impact on its surroundings. This discernible effect on its environment is persuasive enough for interest in this topic. Further, the elimination of mass and angular momentum from the disk might essentially alter the accretion process (Shields et al. 1986). There has been substantial observational evidence of wind in cold accretion mode via blueshifted absorption lines, in luminous active galactic nuclei (AGNs; Crenshaw et al. 2003; Tombesi et al. 2010, 2014; Liu et al. 2013; King & Pounds 2015) and X-ray binaries in the high/soft state (Neilsen & Homan 2012; Díaz Trigo & Boirin 2016; Homan et al. 2016). Nevertheless, the challenging objects for detection are systems in hot accretion mode, since the accreting gas is virially hot and fully ionized in low-luminous AGNs (LLAGNs; Tombesi et al. 2010, 2014; Crenshaw & Kraemer 2012; Cheung et al. 2016) and the low/hard state of X-ray binaries (Homan et al. 2016; Munoz-Darias et al. 2019).

Wind can be driven by different physical mechanisms such as thermal, radiation, and magnetic pressures. Radiation pressure and magnetic forces would act on smaller scales (Proga et al. 2000; Fukumura et al. 2015), while thermal driving might work in outer regions of the disk and could adjust the mass accretion rate through the disk (Shakura & Sunyaev 1973; Begelman et al. 1983; Shields et al. 1986). A large number of numerical simulations have been done to show that the wind is potentially able to transfer a significant amount of power from the BH accreting system in hot accretion mode (e.g., Igumenshchev & Abramowicz 1999, 2000; Stone et al. 1999; Hawley & Balbus 2002; Pang et al. 2011; Narayan et al. 2012; Yuan et al. 2012a, 2012b; Bu et al. 2013; Li et al. 2013); however, the actual mechanism driving such winds is still a source of much debate.

An analytical method is often invoked to investigate the existence of wind from accretion flow in hydrodynamic (HD) and magnetohydrodynamic (MHD) studies (e.g., Bu et al. 2009; Mosallanezhad et al. 2014; Samadi et al. 2017; Bu & Mosallanezhad 2018; Kumar & Gu 2018; Zeraatgari et al. 2020) and to calculate the physical properties of the wind. In principle, it is very difficult and time-consuming to perform numerical simulations with the most updated physical terms and different input parameters to examine the dependency of the results on the initial set of parameters. Therefore, analytical studies are very powerful tools for better understanding the dependency of the inflow-wind structure of the system on the physical parameters and attempting to find the real mechanism for producing wind in such accreting systems.

The analytical study of hot accretion flow started from height-averaged, radially self-similar solutions presented by Narayan & Yi (1994), which could not show a clear picture of the vertical structure of such a system. Next, in Narayan & Yi (1995, hereafter NY95), they revisited their solutions in the \( r-\theta \) plane in spherical coordinates. They adopted self-similar solutions in the radial direction and, in order to solve the equations in the \( \theta \)-direction, used proper boundary conditions at both equatorial plane and rotation axis. Unfortunately, the solutions did not show an inflow-wind structure, since they considered \( v_\theta = 0 \) and zero radial velocity at the pole, i.e., \( v_r(0) = 0 \). By eliminating \( v_\theta \) and adopting axisymmetric and steady-state assumptions, the radial self-similar solution of the density is followed by \( \rho \propto r^{-3/2} \) (see the first term in Equation (B1)). Their solutions also became singular when...
γ = 5/3, although the numerical simulations of hot accretion flow suggest that in the nonrelativistic cases γ is very close to 5/3 (e.g., Balbus & Hawley 1998; Blandford & Begelman 1999). However, based on their positive value of the Bernoulli parameter, they argued that bipolar outflow must exist near the rotation axis.

After that, Xu & Chen (1997, hereafter XC97) brought up all components of the velocity including \( v_\phi \), adopted Fourier series, and obtained accretion and ejection solutions. Blandford & Begelman (2004) also represented self-similar 2D solutions for radiatively inefficient accretion flows with outflow called adiabatic inflow–outflow solutions (ADIos). They suggest that the mass accretion rate decreases inward owing to the mass loss in the outflow. Based on their solution, the mass inflow and outflow fluxes follow the power law of \( \dot{M} \propto r^s \) with equal and opposite values. They also determined the power index \( s \) to be in the range 0 ≤ \( s \) ≤ 1.

Tanaka & Menou (2006, hereafter TM06) followed NY95 and mainly focused on the effects of the thermal conduction on the global properties of hot accretion flows. Even though they did not include \( v_\phi \), their solutions obtained positive radial velocity near the rotation axis interpreting the outflow. Here we intend to emphasize the requisite role of \( v_\phi \) in wind studies. As an example, XC97 considered \( v_\phi \), but some researchers argued that the solution would not be correct (e.g., Xue & Wang 2005; Jiao & Wu 2011). This is mainly because they believed that the mass flux of the outflow became exactly equal to the mass flux of the inflow at a certain radius. To avoid this knotty issue, for instance, Xue & Wang (2005) truncated the solutions where \( v_r = 0 \) by prescribing the opening angle \( \theta_0 \) and considered this place as the surface of the disk. They set the sound speed on the surface as \( c_s / \Omega_0 \lesssim (\pi / 2 - \theta_0) r \). Therefore, the second boundary is set as an input parameter rather than being calculated in their solution. Actually, their solutions were limited to the inflow region near the equatorial plane and a surface from which the wind would blow out. Also, Jiao & Wu (2011) integrated the equations from the equatorial plane toward the rotation axis and stopped integration where the density or gas pressure became negative. Although an outflow structure could be shown, their solutions did not reach to the pole, \( \theta = 0 \), where some of the physical boundary conditions must be satisfied.

Khajenabi & Shadmehri (2013, hereafter KS13) also solved the HD equations of hot accretion flow in the presence of the thermal conduction, as well as all three components of the velocity. Further, they imposed the same physical constraint as Xue & Wang (2005) for the opening angle and obtained this angle self-consistently from their numerical integration starting from the equatorial plane. Their results showed that the thermal conduction affected the opening angle of the wind, as an increase of that would shrink the size of the wind region.

The main aim of this paper is to find the inflow–wind solution of hot accretion flow in the whole vertical direction by imposing proper boundary conditions at both the rotation axis and the equatorial plane and then compare the results with those in the above-mentioned studies. In the following section, we will find the origin of discrepancy between all those aforementioned analytical solutions by focusing on the continuity and the energy equations. We will illustrate that it would be possible to solve the hydrodynamic equations of hot accretion flow in the whole \( \theta \)-direction using the two-point boundary value problem.

In this paper, to mimic the effects of magnetorotational instability (MRI) in driving angular momentum, we will consider the viscous stress tensor. Another aim of the present paper is to find the dependency of the results on the initial set of parameters such as the conductivity coefficient, density index, and advection parameter. Note here that the gradient of the magnetic pressure is one of the driving mechanisms for producing wind and plays an important role in the dynamics of hot accretion flows. In this regard, we intend to embark on a series of papers to solve HD and MHD\(^8\) equations of hot accretion flow using appropriate boundary conditions at the rotation axis and the equatorial plane.

The remainder of the paper is organized as follows. In Section 2, we will investigate the origin of the discrepancy between all previous analytical solutions and find the reason why their solutions could not reach the rotation axis. The basic HD equations, physical assumptions, self-similar solutions, and also the boundary conditions will be introduced in Section 3. In Section 4, the detailed explanations of numerical results will be presented. Finally, in Section 5 we will provide the summary and discussion.

2. The Origin of Discrepancy between Previous Analytical Solutions

Based on the Reynolds transport theorem, the time rate of change of integrals of physical quantities within material volumes can be calculated as

\[
\frac{dF}{dt} = \oint_V \left( \frac{dF}{dt} + \mathbf{F} \cdot \nabla \mathbf{v} \right) dV,
\]

where \( F(\mathbf{x}, t) \) can be any scalar, vector, or tensor field, with \( F = \oint_V F(\mathbf{x}, t) dV \) (see Equation (18.2) in Mihalas & Mihalas 1984 for more details). According to the conservation law of mass for the fluid, the mass within a material volume must be always the same, i.e.,

\[
\frac{d}{dt} \oint_V \rho \, dV = 0.
\]

By applying the Reynolds transport theorem for \( F = \rho \), we have

\[
\oint_V \left( \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} \right) dV = 0.
\]

In principle, this integral will be vanished only if the integrand vanishes at all points in the flow field. The integrand then is the continuity equation (see Equation (11)). In spherical coordinates, by imposing the axisymmetric \( (\partial / \partial \phi = 0) \) and steady-state \( (\partial / \partial t = 0) \) assumptions into the equation of continuity, it will be reduced to Equation (B1). Integrating the first term of this equation over angle will give us

\[
\dot{M} = -\oint 2\pi r^2 \sin \theta \rho v_r \, d\theta,
\]

where \( \dot{M} \) is the net mass accretion rate. In NY95, since \( v_\phi = 0 \), the second term was spontaneously eliminated from Equation (B1). As a result, the density index of their self-similar solutions became \( n = 3/2 \) in \( \rho \propto r^{-n} \) (see Equations (2.11)–(2.15) of NY95). In the case of \( v_\phi = 0 \), the sum of two terms in Equation (B1) must be always zero at each specific radius to satisfy the continuity equation. Keeping the second term of Equation (B1) would also

\(^8\) The MHD equations will be solved in the second paper of this series, and the HD and MHD results will be compared there.
cause flattening of the density profile, i.e., the density index would be in the range of \(0 < n < 3/2\).

In XC97, they included \(v_\theta\) in their solutions and found two different kinds of solutions, i.e., accretion outflow and ejection outflow for the density index of \(n < 3/2\) and \(n > 3/2\), respectively. However, the ejection outflow with the large value for the density index of \(n > 3/2\) might not be correct. The reason would emanate from the energy equation, which is defined as

\[
Q_{\text{adv}} = Q_+ - Q_-,
\]

where \(Q_{\text{adv}}\), \(Q_+\), and \(Q_-\) are energy advection, total heating, and total cooling terms per unit volume, respectively. In hot accretion flows or radiatively inefficient accretion flows (RIAFs), it is common to omit \(Q_-\) and introduce the advection parameter as \(f \equiv Q_{\text{adv}}/Q_+\) with \(0 < f \leq 1\). Hence, the energy equation of the hot accretion flow in XC97 was written as

\[
Q_{\text{adv}} = f Q_+,
\]

with the total energy heating released by the viscosity. By assuming an axisymmetric, steady-state, and radially self-similar approximation, the advection and the viscous terms of the energy equation were reduced to the dimensionless forms as described in Equations (A14) and (A15), respectively (see Appendix A for more details). Based on the global picture of the accretion flow, inflow happens around the equatorial plane, while the outflow does appear at high latitudes. To show that the range of the density index, only for highly nonrelativistic cases, must be \(n < 3/2\), in what follows we investigate the energy equation in inflow and wind regions in detail.

2.1. Energy Equation in the Inflow Region

As proved by Mihalas & Mihalas (1984), the viscous dissipation term is always positive everywhere, \(Q_{\text{vis}} > 0\). Besides, the symmetry boundary conditions dictate that the latitudinal component of the velocity should be null at the equatorial plane, i.e., \(v_\theta(\pi/2) = 0\) (see Equation (26)), which means that the first term of Equation (A14) is dominated at the midplane. Necessarily, this term must be positive to satisfy the energy equation

\[
\left( n - \frac{1}{\gamma - 1} \right) \rho g v_\theta > 0,
\]

where \(\gamma\) is the adiabatic index. Since the gas pressure is always positive and \(v_\theta < 0\) at the equatorial plane, then the above equation will be satisfied there only if

\[
n - \frac{1}{\gamma - 1} < 0.
\]

As mentioned in the introduction, numerical simulations of the hot accretion flow suggest that in the nonrelativistic cases \(\gamma\) is very close to \(5/3\). Consequently, to satisfy the above equation, with \(\gamma = 5/3\), the density index must be

\[
n < \frac{3}{2}.
\]

This value for \(n\) clearly proves that the bipolar ejection outflow solution of XC97 is not physical where they set \(n = 2.5\) (see Section 3 and also panel (b) of Figure 1 in XC97).

2.2. Energy Equation in the Wind Region

At high latitudes, if wind exists and launches, the radial velocity must be positive (\(v_r > 0\)). At the rotation axis, we have a similar boundary condition for the latitudinal component of the velocity, i.e., \(v_\theta(0) = 0\). Additionally, as proved from the energy equation in the inflow region, the density index must be \(n < 3/2\). Therefore, the advection term at the pole must be negative as

\[
\left( n - \frac{1}{\gamma - 1} \right) \rho g v_\theta < 0.
\]

Since the viscous heating term of the energy equation is always positive, to satisfy the above equation, we need an additional term on the right-hand side of the energy equation that allows for the positive radial velocity near the pole. TM06 and KS13 showed that thermal conduction can be negative enough at high latitudes to overcome the positive sum of the viscous heating terms and generate positive nonzero radial velocity about the rotation axis.

In essence, in hot accretion flows with very low mass accretion rates, the electron mean free path is much larger than the electron gyroradius. Hence, Coulomb collisions are not exceptional. For instance, in our Galactic center, Sgr A*, with mass accretion rate \(M \approx 10^{-7} - 10^{-8} M_\odot\) yr\(^{-1}\), the electron mean free path \(\sim 1.3 \times 10^{17}\) cm and the gyroradius of electrons \(\gamma = v_\text{thermal} / qB \sim 10^4\) cm \((m_e, v_{\text{in}}, q, B)\) are electron mass, thermal speed of electron, speed of light, electron charge, and magnetic field, respectively). Thermal conduction can play a striking role in such a system as it can make a heat flux from the inner hot regions to the outer cold parts of the flow. The gas in the outer region has the possibility to be heated to a temperature higher than the local virial temperature. This increase in the temperature can drive thermal outflow/wind and consequently decrease the mass accretion rate.

In this paper, we will solve the HD equations of the hot accretion flow with thermal conduction in the whole \(\theta\) direction. We will also consider all components of the velocity and the viscous stress tensor and impose the boundary conditions at both the rotation axis and the equatorial plane.

3. Basic Equations and Assumptions

The HD equations of the hot accretion flow with thermal conduction can be written as

\[
\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} = 0,
\]

\[
\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho \nabla \psi - \nabla p + \nabla \cdot \mathbf{\sigma},
\]

\[
Q_{\text{adv}} = Q_{\text{vis}} + Q_c,
\]

where \(\rho\) is the mass density, \(\mathbf{v}\) is the velocity, \(\psi[-= -GM/r]\) is the Newtonian potential (where \(r\) is the distance from the central BH, \(M\) is the BH mass, and \(G\) is the gravitational constant), \(p\) is the gas pressure, \(\mathbf{\sigma}\) is the viscous stress tensor, and \(d/dt \equiv \partial / \partial t + \mathbf{v} \cdot \nabla\) denotes the Lagrangian or comoving flow.
In a one-temperature structure, as in our case, where
\[ \nu = \frac{k}{\rho T}, \]
\[ Q_{\text{adv}} = \rho \frac{d e}{d t} - \frac{p}{\rho} \frac{d \rho}{d t}, \]  
(14)
\[ Q_{\text{vis}} = f \nabla \cdot \mathbf{v}, \]
(15)
\[ Q_{c} = -\nabla \cdot \mathbf{F}_c. \]  
(16)

Here \( e \) is the internal energy of the gas, and \( F_c \) is the heat flux due to the thermal conduction. We adopt the adiabatic equation of state as \( p = (\gamma - 1)\rho e \). In the purely HD limit, \( F_c \) is defined as
\[ F_c = -\chi \nabla T, \]  
(17)
where \( \chi \) is the thermal diffusivity and \( T \) is the gas temperature.

In one-temperature structure, as in our case, \( T \) can be written as
\[ T = \frac{\mu m_p}{k_B} \rho, \]  
(18)
where \( \mu, m_p, \) and \( k_B \) are mean molecular weight, proton mass, and Boltzmann constant, respectively. Numerical simulations of Sharma et al. (2008) and Bu et al. (2016) assumed \( \kappa = \chi T/p = \alpha_c (GMr)^{2/3}, \) with the dimensionless conductivity taking the value of \( \alpha_c \approx 0.2-2 \). Here we follow those numerical simulations by adopting the same definition for thermal conduction and consider the same range for \( \alpha_c, \) except in Section 4.4 we consider a small value of \( \alpha_c \) to check the dependency of the results on this parameter.

The viscous stress tensor is given by
\[ \sigma_{ij} = \nu \left[ \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} (\nabla \cdot \mathbf{v}) \delta_{ij} \right], \]  
(19)
where \( \nu \) is the kinematic viscosity coefficient and \( \delta_{ij} \) is the usual Kronecker delta. Note that the bulk viscosity is neglected here. It is well known that in a real accretion flow the angular momentum is transferred by Maxwell stress associated with MHD turbulence driven by MRI (Balbus & Hawley 1998).

In our current HD case, we approximate the effect of the magnetic stress by adding viscous terms in momentum and energy equations (see, e.g., Yuan et al. 2012b). The kinematic viscosity coefficient is calculated with the \( \alpha \)-prescription (Shakura & SUNyaev 1973) as
\[ \nu = \alpha \frac{\rho}{\rho \Omega K}, \]  
(20)
where \( \Omega K \equiv (GM/r^3)^{1/2} \) is the Keplerian angular velocity and \( \alpha \) is the viscosity parameter. We use spherical coordinates \( (r, \theta, \phi) \) to solve the full set of equations including all viscous terms.

The disk is taken to be axisymmetric and in steady state. By implementing all the above-mentioned assumptions and definitions in Equations (11)–(13), we obtain the partial differential equations (PDEs) presented in Appendix B (see Equations (B1)–(B5) for more details).

The global numerical simulations of hot accretion flows show that the physical variables of the flow can be described by a power-law function of radius far away from the radial boundaries. For instance, the radial profile of the density follows \( \rho(r) \propto r^{-n} \) with \( n < 3/2 \) (see, e.g., Stone et al. 1999; Yuan et al. 2012a, 2012b, 2015). In order to solve Equations (B16)–(B5) by numerical methods, we impose self-similar solutions to remove the radial dependency of the variables. To do so, we introduce the fiducial radius \( r_0 \) with the self-similar solutions as a power-law form of \((r/r_0)\). Accordingly, the physical variables of the hot accretion flow will be written in the following forms:
\[ \rho(r, \theta) = \rho_0 \left( \frac{r}{r_0} \right)^{-n}, \]
(21)
\[ v_r(r, \theta) = v_{0r} \left( \frac{r}{r_0} \right)^{-1/2}, \]
(22)
\[ v_{\theta}(r, \theta) = v_{0\theta} \left( \frac{r}{r_0} \right)^{-1/2}, \]
(23)
\[ v_\phi(r, \theta) = v_{0\phi} \left( \frac{r}{r_0} \right)^{-1/2} \Omega(\theta) \sin(\theta), \]
(24)
\[ p(r, \theta) = p_0 \left( \frac{r}{r_0} \right)^{n-1}, \]
(25)
where \( r_0, \rho_0, v_{0r}, v_{0\theta}, v_{0\phi} \) are the units of length, density, velocity, and gas pressure, respectively. Substituting the above self-similar solutions into Equations (B16)–(B5), the radial dependency will be removed and the system of PDEs will be reduced to a set of ordinary differential equations (ODEs) presented in Appendix C. The ODE Equations (C1)–(C5) consist of five physical variables, \( v_{r}(\theta), v_{\theta}(\theta), \Omega(\theta), \rho(\theta), \) and \( p_\phi(\theta), \) as well as their first and second derivatives. As we mentioned in the introduction, the integration will not stop at some angle near the rotation axis (see, e.g., Jiao & Wu 2011; Mosallanezhad et al. 2016; Samadi & Abbassi 2016). Instead, the computational domain will be extended from the rotation axis, \( \theta = 0, \) to the equatorial plane, \( \theta = \pi/2. \) Following previous analytical solutions of hot accretion flow, e.g., NY95 and TM06, all physical variables are assumed to be even symmetric, continuous, and differentiable at both boundaries. Since we also include the latitudinal component of the velocity, \( v_\phi, \) its value will be null at both the equatorial plane and the rotation axis. Thus, the following boundary conditions at \( \theta = 0 \) and \( \theta = \pi/2 \) will be imposed:
\[ \frac{d \rho}{d \theta} = \frac{d p_\phi}{d \theta} = \frac{d \Omega}{d \theta} = \frac{d v_r}{d \theta} = v_\phi = 0. \]  
(26)

To satisfy all boundary conditions at both ends, the relaxation method will be adopted mainly because the set of ODEs has extraneous solutions, and also there exists singularity at the rotation axis. To have a good resolution at both sides where the boundary conditions are set, we divide the \( \theta \) direction into 5000 grids with a stretch grid as follows: from \( \theta = 0 \) to \( \theta = \pi/4 \) the grid size ratio is set as \( d\theta_{i+1}/d\theta_i = 1.003, \) while from \( \theta = \pi/4 \) to \( \theta = \pi/2 \) the grid size ratio is set as \( d\theta_{i+1}/d\theta_i = 0.997. \) We calculate the variables at the cell center of these grids. The absolute error tolerance is set to \( 10^{-15}. \) The most difficult part of solving the set of equations is providing an appropriate guess for the required solutions. In this study, we use \textit{Fourier cosine series} for the initial guess of all physical
variables except for \(v_\theta(\theta)\), for which we used Fourier sine series. Since our solutions are satisfying the boundary conditions at both the rotation axis and the equatorial plane, it does assure us that well-behaved solutions will be derived in the whole \(\theta\) direction. In the next section we will explain the behaviors of the physical variables in detail.

4. Numerical Results

4.1. The Solutions of the Fiducial Model

To solve the system of ODEs numerically, we integrate Equations (C1)–(C5) from the rotation axis to the equatorial plane to get the latitudinal profiles of all physical variables of our fiducial model shown in Figure 1. The parameters are set as \(\alpha = 0.15, \alpha_5 = 0.2, n = 0.85, f = 1\), and \(\gamma = 5/3\). The velocities are scaled with the Keplerian velocity at \(r = r_0\), i.e., \(v_0 = \sqrt{GM/r_0}\). The density is normalized with \(\rho_0\), and the pressure is also normalized with \(\rho_0v_0^2\) at \(r_0\). We assumed that the radius \(r_0\) is located at the equatorial plane, where the maximum density of the accretion flow is accumulated. In the top left panel of Figure 1, \(v_r = 0\) shows the inclination of the inflow to the wind region, which is around \(\theta \approx 52^\circ\). As is seen, \(v_r\) is negative in the inflow region, where the matter goes toward the BH, while in the wind region it becomes positive and reaches a value slightly above the Keplerian velocity of that radius. In the top middle panel, \(v_\theta\) is negative in the vertical direction from the equatorial plane toward the rotation axis. Furthermore, due to the boundary conditions, it is zero in both boundaries and minimum around \(\theta \approx 20^\circ\). In the top right panel, the angular velocity tendency, as it increases from the inflow region toward the wind region, shows that the angular momentum is transported away from the system by the wind. Moreover, the angular velocity exceeds the Keplerian velocity around \(\theta \approx 45^\circ\) and is quite super-Keplerian in the wind region. In the bottom left panel, the concentration of the density is at the equatorial plane and then drops toward the wind region so that it reaches its minimum value around the rotation axis. The trend of the gas pressure, in the bottom middle panel, is also the same for the density, the maximum at the equator, and the minimum at the rotation axis. From the bottom right panel, \(v_r\) and \(v_\theta\) both remain subsonic in the whole \(r - \theta\) domain. The main reason is that the density drops faster than the pressure from the equatorial plane toward the rotation axis. Therefore, the sound speed, \(c_s = \sqrt{\gamma p/\rho}\), or equivalently the gas temperature, increases as the \(\theta\) angle decreases. This panel clearly shows that both \(|v_r|/c_s\) and \(|v_\theta|/c_s\) do not pass through a sonic point (where the singularity exists) at small angles. The behavior of Mach number for \(v_r\) is that it first drops and then increases going from \(\theta = 52^\circ\) (where the radial velocity is zero, \(v_r = 0\)) toward the rotation axis. Also, \(|v_\theta|/c_s\) first increases until around \(\theta \approx 35^\circ\) and then decreases toward the rotation axis, and this is because we plot the absolute value of Mach numbers. The overall behavior of the physical variables is almost the same as that in XC97, TM06, and KS13. More precisely, the similarity can be seen in the inflow-wind structure, inflow with negative radial velocity around the

---

5 We made a parameter study and found that the inclination of the inflow to the wind region does not change too much with different sets of input parameters. The changes are only in the range of \(\theta \sim 50^\circ-55^\circ\).
absolute value, behavior of super-Keplerian at small angles. Another one is the U-shape and XC97 with KS13 is that the angular velocity becomes with decreasing \( \theta \). However, in Figure 2 of TM06, angular velocity, different assumptions that these studies have made in their

6 Note here that the main differences between our solution and the accretion outflow solution of XC97 are (1) we include thermal conduction and (2) the value of the radial velocity at the equatorial plane is self-consistently determined here, while in XC97 it is fixed as \( v_r(\pi/2) = -0.05v_K \).

equatorial plane, and wind with positive radial velocity around the rotation axis. It is very diagnostic in the density profile that drops from the equatorial plane to the rotation axis in all the above studies showing a disk-shape structure (see also the left panel of Figure 2). Since in the present study we include \( v_\theta \) we can easily compare our results with XC97 and KS13. For instance, the wind radial velocity is much higher than the inflow radial velocity at high latitudes, which is exactly similar to Figure 2 of XC97 and also Figure 1 of KS13.\(^6\) The trend of the angular velocity in the present work increasing toward the rotation axis is similar to that in both XC97 and KS13, which helps the wind to accelerate with high speed and causes the angular momentum to be transferred outward (see Figure 2 of XC97 and also Figure 2 in KS13).\(^6\) However, in Figure 2 of TM06, angular velocity, \( \Omega \), decreases with decreasing \( \theta \). One of the differences of the present work and XC97 with KS13 is that the angular velocity becomes super-Keplerian at small angles. Another one is the U-shape behavior of \( v_K \) it first falls down and then goes up toward the rotation axis, while it keeps increasing continuously toward the opening angle in KS13. The reason can be interpreted as due to different assumptions that these studies have made in their calculations. In addition, we and XC97 both put the second boundary condition at the rotation axis, where the gradient of the physical variables and \( v_\theta \) must be zero. Nevertheless, KS13 started the integration from the equator and went to a certain inclination where the physical constraint was satisfied, and they considered this inclination as the upper boundary of the accretion flow. Overall, due to appropriately imposing physical boundary conditions at the rotation axis, as well as considering thermal conduction in this study, our results show strong wind at high latitudes.

The 2D inflow-wind structure of the flow is shown in Figure 2, in the density and temperature contours. The left panel is overlaid with the poloidal velocity at six different radii in the units of Schwarzschild radius, i.e., \( r = [100, 150, 200, 250, 300, 350] r_s \) to show the strength of the outflow. The white dashed–dotted lines show the location of \( v_r = 0 \). In the right panel, the poloidal velocity is normalized with its absolute value, \( |v_p| = \sqrt{v_r^2 + v_\theta^2} \) to denote the direction of the vectors. Here \( T_\text{vir}(r_0) = GMm_p/(3k_\text{B}r_0) \) is the virial temperature at \( r_0 = 10 r_s \).

Figure 2. 2D distribution of the density (left panel) and the temperature (right panel) based on the self-similar solutions. Both panels are overlaid with the poloidal velocity, \( \vec{v}_p = v_r \hat{\rho} + v_\theta \hat{\theta} \). In the left panel, the arrows are plotted only at six different radii in the units of Schwarzschild radius, i.e., \( r = [100, 150, 200, 250, 300, 350] r_s \) to show the strength of the outflow. The white dashed–dotted lines show the location of \( v_r = 0 \). In the right panel, the poloidal velocity is normalized with its absolute value, \( |v_p| = \sqrt{v_r^2 + v_\theta^2} \) to denote the direction of the vectors. Here \( T_\text{vir}(r_0) = GMm_p/(3k_\text{B}r_0) \) is the virial temperature at \( r_0 = 10 r_s \).
of the density is located at the inner region of the equatorial plane. So, as we go to outer regions and small \( \theta \) angles, the density decreases rapidly and reaches its minimum in the disk. The density contour is identical to that plotted in the top left panel of Figure 2 in the accretion outflow solution. The torus-like shape of the density is also in agreement with the numerical simulations of hot accretion flow (see, e.g., Yuan et al. 2012b). From the right panel, the minimum temperature belongs to the inflow region inside the disk, while the wind region has the highest temperature, which is much less dense than the disk. Indeed, at high latitudes, thermal conduction heats up the flow, and then the temperature goes up and leads to a rapid change of the gradient of the gas pressure, launching the thermal wind (see also the top left panel of Figure 4).

To show the strength of the wind, we also plot a 3D velocity field in Figure 3 in four cylindrical radii, i.e., \( R = [4, 8, 15, 25] r_s \), and also in four different heights, i.e., \( z = [0, 10, 25, 40] r_s \) (left panel). From the 3D figure, it is clear that the flow is purely inflow at the equatorial plane of the disk, \( z = 0 \). In higher \( z \), the velocity vectors are not parallel to the equatorial plane and are deflected away. In addition, arrows become stronger around the rotation axis at small heights and small radii. For the better view of the velocity field, the 2D plots in four different heights in the \( x-y \) plane are shown in the right panel of Figure 3.

It is of interest to know which mechanism is dominant to drive wind in the hot accretion flow in the HD case. For this purpose, in Figure 4 we analyze all forces in the units of the gravitational force (black), the centrifugal force (green), and the sum of the forces (red). It is also worthwhile to find the prominent radial and angular components of the forces in the inflow and wind regions. In the top left panel, the radial components of the forces are shown. Since the gravitational force only changes with radius, in this panel it has a fixed negative value, i.e., \( F_{\text{gravity}} = -1 \), in the whole \( \theta \) direction. In the inflow region the gravity is prevailing, while from \( \theta \simeq 52^\circ \) upward the gradient of the gas pressure becomes dominant (wind region). As can be seen, the sum of the radial components of the gradient of the gas pressure and the centrifugal force cannot dominate the gravity in the inflow region, so the total force remains negative there. This is also clear from the bottom panel of Figure 4, where the forces are plotted at the region near the equatorial plane, i.e., \( \theta = 85^\circ \), since the latitudinal components of the forces are almost negligible. The top right panel of this figure shows that the vertical component of the centrifugal force is always positive and dominant in the inflow region. On the other hand, in the wind region the \( \theta \) component of the gradient of the gas pressure is dominated, and the sum of these forces in the vertical direction becomes negative. In bottom panel of Figure 4, the forces are calculated at two representative locations in wind and inflow regions, \( \theta = 30^\circ \) and \( \theta = 85^\circ \), respectively. The dashed–dotted line represents the barrier between inflow and wind regions corresponding to \( \nu_r = 0 \), and the dotted line is for the radius where the forces are evaluated, i.e., \( r = 70 r_s \). This panel clearly shows that in the inflow region gravity is dominant, so the matter moves inward, and in the wind region...
the sum of the forces is outward owing to the strong value of the gradient of the gas pressure. These results are in agreement with those presented in numerical HD simulations of the hot accretion flow (see Yuan et al. 2012b).

Figure 5 shows the latitudinal profile of three terms of the energy equation, viscous heating (dotted line), conduction (dashed line), and advection (solid line), based on the solutions of the fiducial model. This figure shows that in the inflow region the viscosity and advection terms have positive values while the thermal conduction has a negative value, so the advection can cool the flow in the disk. At the polar region, the thermal conduction is much more negative and the viscous heating term is almost negligible. Consequently, the sum of these two terms permits a negative advection term near the rotation axis, which means that the advection acts to heat the flow and produce wind. Note that the latitudinal profiles of three terms of the energy equation are not identical to the ones presented in TM06 for two main reasons: (1) the thermal conduction term defined here is not similar to that written in TM06 (we follow the numerical simulation of...
Sharma et al. 2008 since the TM06 definition of the thermal conduction is not consistent with the self-similarity adopted here⁷, and (2) the latitudinal component of the velocity, \( v_\theta \), exists in both advection and viscous terms of our equations, while this component of the velocity was ignored in TM06.

4.2. Bernoulli Parameter

In almost all previous analytical solutions of the hot accretion flow, the Bernoulli parameter was calculated to show the existence of the wind/outflow. The Bernoulli parameter is defined as the sum of the kinetic energy, the potential energy, and the enthalpy of the accreting gas. In fact, the Bernoulli parameter has been of significant concern, as it shows whether wind or outflow would probably emanate from the accretion flow (Narayan & Yi 1995). The Bernoulli parameter can be defined as

\[
Be = \frac{1}{2} v^2 + h + \psi, \tag{27}
\]

where \( h = \gamma p/\rho (\gamma - 1) \) is the enthalpy.

Figure 6 shows the Bernoulli parameter (solid line) and its three terms in Equation (27), kinetic energy (dashed line), enthalpy (dashed–dotted line), and gravitational energy (dotted line). From this figure, the Bernoulli parameter is negative around the equatorial plane, and its value becomes larger and positive at the high latitudes. In the region near the equatorial plane, the gravitational energy is the dominant one, and it is followed by the Bernoulli parameter. Additionally, in high latitudes, the Bernoulli parameter follows the enthalpy owing to the fact that the density drops faster than the pressure from the equatorial plane to the rotation axis. Therefore, the square of the sound speed, and equivalently the enthalpy, rises as \( \theta \) angle decreases. The trend of the Bernoulli parameter is similar to the accretion outflow solution in XC97. However, in that solution the Bernoulli parameter is always positive independent of \( \theta \).

4.3. Convective Stability

In this subsection we investigate the convective stability of hot accretion flows based on our self-similar solutions. In this regard, we use the well-known Solberg–Høiland criteria in cylindrical coordinates \((R, \phi, z)\). If the disk is convectively stable, the two following Solberg–Høiland criteria must be positive as

\[
\frac{1}{R^2} \frac{\partial l^2}{\partial R} - \frac{1}{C_P \rho} \nabla P \cdot \nabla S > 0, \tag{28}
\]

\[
- \frac{\partial P}{\partial z} \left( \frac{\partial l^2}{\partial R} \frac{\partial S}{\partial z} - \frac{\partial l^2}{\partial z} \frac{\partial S}{\partial R} \right) > 0, \tag{29}
\]

where \( l = r \sin \theta_0 \) is the specific angular momentum per unit mass; \( C_P \) is the specific heat at constant pressure; \( P \) is the total pressure, which equals the gas pressure in the current study; and \( S \) is the entropy, defined as

\[
dS \propto d \ln \left( \frac{P}{\rho^\gamma} \right). \tag{30}
\]

The first criterion can be reduced as

\[
N_{eff} = \kappa^2 + N_R^2 + N_\phi^2 > 0, \tag{31}
\]

with

\[
\kappa^2 = \frac{1}{R^2} \frac{\partial l^2}{\partial R}, \tag{32}
\]

\[
N_R^2 = - \frac{1}{\gamma \rho} \frac{\partial P}{\partial R} \ln \left( \frac{P}{\rho^\gamma} \right). \tag{33}
\]
The Astrophysical Journal, 909:140 (16pp), 2021 March 10

Mosallanezhad et al.

thermal conduction can convectively stabilize the accretion system in hot accretion mode. In our future MHD study, we will investigate the convective stability of the hot accretion flow in more detail.

4.4. Dependency of the Solutions on Input Parameters

In this section, we investigate the dependency of the solutions on the input parameters, including density index, \( n \), conductivity parameter, \( \alpha_\text{f} \), and advection parameter, \( f \). As was mentioned before, similar to XC97 and TM06, we obtained the solutions in the whole \( \theta \) direction, from the rotation axis to the equatorial plane, showing the strong wind at high latitudes. However, in contrast to XC97, which solved the HD equations of hot accretion flow without thermal conduction, we evinced that the thermal conduction should be inevitably considered in the energy equation (see Section 2.1). Moreover, as introduced in Section 3, the density index, \( n \), of the self-similar solutions shows how the density changes along the radius. In the case of \( n = 3/2 \), the solution is similar to that of TM06, where \( v_\theta \) was eliminated from the system of the equations. In Section 2, we also showed that for highly nonrelativistic cases (\( \gamma = 5/3 \)) the density index must be \( n < 3/2 \). However, the real accretion systems around the BHs, where this conclusion may not be satisfied, happen in the relativistic regime, especially when the thermal energy of the gas becomes comparable to (or exceeds) the rest-mass energy of the electron (Chattopadhyay & Ryu 2009; Kumar et al. 2013).

We have shown the dependency of the results to the index parameter, \( n \), in Figure 8. We consider three different values of the density index, i.e., \( n = 0.55, 0.85 \), and 1.15. From the top left panel of this figure, we can see that the maximum amount of the radial velocity of the wind is for \( n = 1.15 \). The latitudinal component of the velocity is always negative and becomes null at both boundaries for all values of \( n \) (see the top right panel). In addition, from \( n = 0.55 \) to \( n = 1.15 \), \( v_\theta \) decreases, which shows an opposite behavior with respect to \( v_r \). This result can be predictable from the continuity equation (see Equation (B16)). In fact, in our current study this equation has two terms, and the sum of these terms should be always equal to zero in the whole \( \theta \) direction. Therefore, in a fixed density index, any increase in \( v_r \) causes a decrease in \( v_\theta \). The bottom right panel of Figure 8 illustrates that the density profile drops faster with \( n = 0.55 \), rather than \( n = 1.15 \). Also, the flow rotates faster for low-density indices.

The numerical simulations of the hot accretion flow also studied the dependency of the solutions on the conductivity coefficient, \( \alpha_\text{c} \) (e.g., Bu et al. 2016). At a fixed radius, from the time-averaged properties of the flow in the steady state, they found that the density and pressure change slightly with increasing \( \alpha_\text{c} \). To compare our results with numerical simulations, we also plot Figure 9. We pick three values of the conductivity coefficient for this comparison, \( \alpha_\text{c} = [0.02, 0.2, 2.0] \). In the top left panel, it is shown that as the value of \( \alpha_\text{c} \) increases, the radial velocity at the equator becomes slightly larger, which is consistent with the results obtained in KS13 (see Figure 1 of KS13). Moreover, our results show that for the wind region, unlike in KS13, \( v_\theta \) is still rising, which is commensurate with the numerical simulations of Bu et al. (2016). In the top right panel, \( v_\theta \) decreases with increasing \( \alpha_\text{c} \).

---

Note that the numerical MHD simulations (Narayan et al. 2012; Yuan et al. 2012a) show that magnetic field can cause the flow to become convective stable. We believe that our results are totally in agreement with the prediction of the numerical MHD simulations. We adopted viscosity to mimic the effect of MRI in driving angular momentum from the system. In addition, the density index of the accretion outflow solution of XC97 is in the same range as we considered, i.e., \( n < 3/2 \), where they set \( n = 0.5 \).
conductivity coefficient. This trend is consistent with the continuity equation terms discussed above. However, KS13 found that $v_{\theta}$ had an increasing tendency with an enhancement in the thermal conduction (see the top panel of Figure 2 in KS13). The behaviors of $v_{\theta}$ in the present study and also in KS13 are not totally similar since we impose $v_{\theta} = 0$ at the rotation axis as a symmetric boundary condition. From the bottom left panel, we find that the value of $\alpha_c$ affects the angular velocity of the flow, where the flow rotates more slowly with the greater values of the conductivity coefficient. The changes of the angular velocity with thermal conduction are consistent with the result of TM06 (see the top right panel of Figure 2 in TM06). From the density profile, the flow indicates a more spherical structure with a rise in thermal conduction, which is consistent with numerical simulations and also KS13 (see the top panel of Figure 3 in KS13).

We further treated the dependency of the solution on the advection parameter in Figure 10 for two different values: $f = 0.7$ (dotted line) and $f = 1$ (solid line). In the top left panel, the radial velocity, $v_r$, of the wind drops with the increase of the advection parameter. In the top right panel, the minimum of the latitudinal velocity, $v_{\theta}$, moves toward the rotation axis as the advection parameter increases. Moreover, $v_{\theta}$ in comparison with $v_r$ shows inverse behavior as it increases with the growth of the advection parameter. The enhancement of $f$ would raise the angular velocity of the gas particles significantly as can be seen in the bottom left panel of this figure. However, from the bottom right panel, it can be seen that the density profile would not extensively suffer from the changes of the advection parameter.9

In this work, we solved the HD equations considering thermal conduction and viscosity to mimic the effects of the magnetic field. This will motivate us to solve the full set of MHD equations to understand the properties and nature of the hot accretion flow in a more realistic case. Therefore, in our future studies we mostly focus on MHD equations and investigate the dependency of the solutions on different configurations of the magnetic field. We will also compare the HD and MHD analytical solutions to find a unique and accurate solution for the hot accretion flow, which might be useful for numerical simulations.

5. Summary and Discussion

In summary, we have shown that thermal conduction is a crucial term for investigating the inflow-wind structure of hot accretion flows. As argued in Section 2, thermal conduction is only significant in very low accretion rate systems, which are suitable for very low luminosity AGNs, e.g., our Galactic

9 It should be mentioned here that at small $f$ the radiation may not be totally ignored as in this paper.
center Sgr A*, supermassive BHs in early-type galaxies, and a considerable number of quiescent X-ray binaries. On this matter, we have solved the 2D HD equations of hot accretion flow with the inclusion of the thermal conduction and all components of the viscous stress tensor. For simplicity, we adopted steady-state as well as axisymmetric assumptions and used a self-similar approximation in the radial direction. Our integration starts from the rotation axis and stops at the equatorial plane, so the solutions will be obtained in the full $r-\theta$ space. We imposed the physical boundary conditions at both boundaries and used the relaxation method to solve the coupled system of equations.

We have obtained an inflow-wind solution extended over the full range of the $\theta$ direction. The inflow region is around the equatorial plane, while the wind region is located at high latitudes around the rotation axis, i.e., $0^\circ < \theta \leq 52^\circ$ (see Figure 2). The density is torus-like, and its concentration occurs in the equatorial plane and decreases in the wind region. From our results, angular velocity behavior shows that wind is able to transfer angular momentum outward. Moreover, the gradient of the gas pressure is the dominant force in driving wind. Our results also show that there is no sonic point in our calculation domain. Analysis of the energy balance between advection, thermal conduction, and viscous heating (Figure 5) indicates that while viscous dissipation heats the flow everywhere, thermal conduction cools it at the equator and polar region. Therefore, thermal conduction conducts the heat from inner regions to outer regions and proceeds as a mechanism for launching thermal wind. Moreover, to balance the two terms on the right-hand side of the energy equation (viscous heating and thermal conduction), advection cools the gas in the disk region, while in the polar region it acts as a heating mechanism and heats the flow. The Bernoulli parameter is positive in the wind region and negative in the inflow region. Therefore, it could still be a useful value, if not an arguable factor, to show the existence of wind in the hot accretion flows. We compared our results with previous related analytical studies on hot accretion flow with and without thermal conduction.

We also treated the convective stability of the hot accretion flow in the HD case and found that the disk is convectively stable in the presence of the thermal conduction. From a parameter study of the density index, $n$, the radial velocity, $v_r$, increases with growth of $n$ in the wind region, while the latitudinal velocity, $v_\theta$, and angular velocity, $\Omega$, decrease with increasing $n$. In this work, we have explored the role of thermal conduction in hot accretion flows in the HD case with a parameter study (Figure 9). From our results, thermal conduction did not have an effective role in changing $v_r$ and $v_\theta$ in the inflow region. However, in the wind region, an inverse behavior was shown for these two components of the velocity. With an increase in the conductivity coefficient, the radial
velocity enhances while the latitudinal component drops. Moreover, the disk rotates more slowly with growth of the thermal conduction strength. Thermal conduction also slightly enlarges the density in the entirety of the flow. These results are fully consistent with the numerical simulations of the hot accretion flow (e.g., Bu et al. 2016). To tackle the influence of the advection parameter on the physical variables, we did a comparison between two values, \( f = 0.7 \) and \( f = 1 \). A rise in the advection parameter would increase both the latitudinal and the angular velocities while decreasing the radial velocity. Moreover, growth of the advection parameter could not make a substantial change in the density profile of the accretion flow.

There are several caveats in this study that will be improved in our future studies. The first one is that we only solved the HD equations of hot accretion flow. In a real accretion flow, angular momentum is transferred by Maxwell stress associated with MHD turbulence driven by MRI. In addition, the magnetic field is one of the driving mechanisms for wind production. With this aim, we will solve the MHD equations of the accretion flow. In our next papers we will mainly focus on different magnetic field configurations on the structure of the hot accretion flow and compare the results with the HD case. A further simplification here is that we adopted a single one-temperature fluid. In the hot accretion model, the ions are expected to be much hotter than the electrons (see Rees et al. 1982; Yuan & Narayan 2014). Thus, two different energy equations for electrons and ions should be solved.

The authors would like to thank the referee for the thoughtful and constructive comments. F. Z. Z. is supported by the National Natural Science Foundation of China (grant No. 12003021) and also the China Postdoctoral Science Foundation (grant No. 2019M663664). L. M. is supported by the Science Challenge Project of China (grant No. TZ2016002). A. M. is supported by the China Postdoctoral Science Foundation (grant No. 2020M673371). D.-F. B. is supported by the Natural Science Foundation of China (grant No. 11773053). A. M. also acknowledges the support of Dr. X. D. Zhang at the Network Information Center of Xi’an Jiaotong University. The computation has made use of the High Performance Computing (HPC) platform of Xi’an Jiaotong University.

Appendix A

Energy Equation Presented in Xu & Chen (1997)

XC97 solved the HD equations of hot accretion flows, which were very similar to the equations described in this paper. The only difference is the definition of the energy equations. The

![Figure 10. Dependency of the solution on the advection parameter; \( f = 0.7 \) (dotted line), \( f = 1 \) (solid line). Here \( \alpha = 0.2 \), \( \gamma = 5/3 \), and \( n = 0.85 \).]
energy equation described in XC97 can be written as

$$Q_{\text{adv}} = f Q_{\text{vis}},$$

(A1)

with

$$Q_{\text{adv}} = \rho \frac{dE}{dt} - p \frac{dp}{dt},$$

(A2)

$$Q_{\text{vis}} = \nabla \cdot \sigma.$$  

(A3)

By imposing steady-state (\(\partial / \partial t = 0\)) and axisymmetric (\(\partial / \partial N = 0\)) assumptions, the components of the stress tensor, \(\sigma\), in the spherical coordinates are given by

$$\sigma_{rr} = \rho \nu \left[ 2 \frac{\partial v_r}{\partial r} - \frac{2}{3} (\nabla \cdot v) \right],$$

(A4)

$$\sigma_{\theta \theta} = \rho \nu \left[ 2 \frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{v_r}{r} \right] - \frac{2}{3} (\nabla \cdot v),$$

(A5)

$$\sigma_{\phi \phi} = \rho \nu \left[ 2 \left( \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) - \frac{2}{3} (\nabla \cdot v) \right],$$

(A6)

$$\sigma_{r \theta} = \rho \nu \left[ \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right],$$

(A7)

$$\sigma_{r \phi} = \rho \nu \left[ \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \sin \theta \right) + \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} \right],$$

(A8)

$$\sigma_{\theta \phi} = \rho \nu \left[ \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right].$$

(A9)

where \(\nabla \cdot v\) is written as

$$\nabla \cdot v = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( v_\theta \sin \theta \right).$$

(A10)

Substituting the above terms into the advection and the viscous terms of the energy equations, we have

$$Q_{\text{adv}} = \rho \left[ v_r \frac{\partial e}{\partial r} + v_\theta \frac{\partial e}{\partial \theta} + v_\phi \frac{\partial e}{\partial \phi} \right] - \rho \left[ v_r \frac{\partial p}{\partial r} + v_\theta \frac{\partial p}{\partial \theta} + v_\phi \frac{\partial p}{\partial \phi} \right],$$

(A11)

$$Q_{\text{vis}} = f \left[ \frac{\partial v_r}{\partial r} \sigma_{rr} + \frac{\partial v_\theta}{\partial r} \sigma_{r \theta} + \frac{\partial v_\phi}{\partial r} \sigma_{r \phi} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \sigma_{\theta \theta} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} \sigma_{\phi \phi} - \frac{v_\theta}{r} \sigma_{\phi \theta} - \frac{v_\phi}{r} (\sigma_{r \phi} + \sigma_{\phi \phi} \cot \theta) + \frac{\sigma_{\phi \phi}}{r} (v_r + v_\theta \cot \theta) \right].$$

(A12)

They assumed the fully advection case, i.e., \(f = 1\). Substituting a self-similar approximation, Equations (21)–(25), into the above terms, the radial dependency will be removed. Therefore, the dimensionless form of the energy equation described in XC97 can be written as

$$Q_{\text{adv}} = Q_{\text{vis}},$$

(A13)

with

$$Q_{\text{adv}} = \left[ n - \frac{1}{\gamma - 1} \right] \rho \frac{dE}{dt} + \frac{v_\theta}{\gamma - 1} \left[ \frac{dp}{dt} - \frac{p_\theta}{\rho} \frac{dp}{dt} \right],$$

(A14)

$$Q_{\text{vis}} = \alpha \rho \left[ \frac{1}{2} v_r^2 + 2 \left( \frac{dv_r}{dt} \right)^2 + 2 (v_r + v_\theta \cot \theta)^2 \right],$$

(A15)

\[ \times \left( \frac{2}{3} \left( \frac{dv_r}{dt} \right)^2 + \left[ \frac{3}{2} \frac{v_r}{dt} + \frac{v_\theta}{dt} \right]^2 \right) \sin^2 \theta \]

\[ - \frac{2}{3} (v_r + v_\theta \cot \theta)^2 \]

Appendix B

Partial Differential Equations in Spherical Coordinates

In this study, to reach the final PDEs governing the system, we adopt spherical coordinates \((r, \theta, \phi)\). We consider the disk to be axisymmetric and steady state. We further assume that the velocity field consists of all its components as \(v = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi}\). All components of the viscous stress tensor will be included as described in Equations (A4)–(A9). By imposing the assumptions and definitions introduced in Section 3, Equations (11)–(13) will be reduced to the following system of PDEs. Thus, we can rewrite the continuity equation as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho v_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \rho v_\theta \sin \theta \right) = 0.$$

(B1)

Three components of the equation of motion, Equation (12), can be read as

$$\rho \left[ v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \phi} \right]$$

$$= - \frac{GM \rho}{r^2} - \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial r^2} \left( r^2 \sigma_{rr} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \sigma_{\theta \theta} \right)$$

$$- \frac{1}{r} \left( \sigma_{\theta \theta} + \sigma_{\phi \phi} \right),$$

(B2)

$$\rho \left[ v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \frac{\partial v_\theta}{\partial \phi} \right]$$

$$= - \frac{1}{r \sin \theta} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \left( r^2 \sigma_{r \phi} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \sin \theta \sigma_{\phi \phi} \right)$$

$$+ \frac{1}{r} \left( \sigma_{\theta \phi} - \sigma_{\phi \theta} \cot \theta \right).$$

(B3)
\[ \rho \left[ \frac{\partial \phi_0}{\partial r} + \frac{\nu_0}{r} \frac{\partial \phi_0}{\partial \theta} + \frac{v_r}{r} (v_r + \nu_0 \cot \theta) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \sigma_{\phi\phi} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \sigma_{\phi\theta} \right) + \frac{1}{r} (\sigma_{\phi\phi} + \sigma_{\phi\theta} \cot \theta) . \] (B4)

Finally, the energy equation can be written as

\[ \rho \left[ \frac{\partial e}{\partial r} + \frac{\nu_0}{r} \frac{\partial e}{\partial \theta} + \frac{\nu_0}{r} \frac{\partial \rho}{\partial \theta} \right] - \frac{\rho}{r^2} \left[ \frac{\partial \rho}{\partial r} + \frac{\nu_0}{r} \frac{\partial \rho}{\partial \theta} \right] = f \]

\[ \times \left[ \frac{\partial v_r}{\partial r} \sigma_{\phi\theta} + \frac{\partial \phi_0}{\partial r} \sigma_{\phi\phi} + \frac{\partial v_\theta}{\partial r} \sigma_{\phi\theta} + \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} - v_r \right) \sigma_{\phi\theta} + \frac{1}{r} \left( \frac{\partial \phi_0}{\partial \theta} - v_r \right) \sigma_{\phi\phi} + \frac{v_r}{r} (v_r + \nu_0 \cot \theta) \right] \]

\[ + \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \chi \frac{\partial T}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\chi \sin \theta}{r} \frac{\partial T}{\partial \theta} \right) . \] (B5)

### Appendix C

**Ordinary Differential Equations**

By substituting self-similar solutions presented in Section 3 into the partial differential Equations (B16)–(B5), the following coupled ODEs in the \( \theta \) direction will be obtained:

\[ \left[ \left( \frac{3}{2} - n \right) \tilde{v}_r + \frac{d \tilde{v}_0}{d \theta} + \tilde{v}_0 \cot \theta \right] \tilde{\rho} + \tilde{v}_0 \frac{d \tilde{\rho}}{d \theta} = 0 \] (C1)

\[ \tilde{\rho} \left[ - \frac{1}{2} \tilde{v}_r^2 + \tilde{v}_0 \frac{d \tilde{v}_0}{d \theta} - \tilde{v}_0^2 - \tilde{\Omega}^2 \sin^2 \theta \right] \]

\[ = - \tilde{\rho} + (n + 1) \frac{d \tilde{\phi}_g}{d \theta} + \alpha \left( \frac{d \tilde{v}_r}{d \theta} - \frac{3}{2} \tilde{v}_0 \right) \frac{d \tilde{\rho}}{d \theta} \]

\[ + \alpha \frac{1}{2} \left( n + 1 \right) \left[ 3 \tilde{v}_r + \tilde{v}_0 \cot \theta + \frac{d \tilde{v}_0}{d \theta} \right] \]

\[ - \frac{7}{2} \left( \frac{d \tilde{v}_0}{d \theta} + \tilde{v}_0 \cot \theta \right) + \frac{d^2 \tilde{v}_r}{d \theta^2} + \frac{d^2 \tilde{v}_0}{d \theta^2} \cot \theta - 6 \tilde{v}_r , \] (C2)

\[ \tilde{\rho} \left[ \frac{1}{2} \tilde{v}_r \tilde{v}_0 + \tilde{v}_0 \frac{d \tilde{v}_0}{d \theta} - \tilde{\Omega}^2 \sin \theta \cos \theta \right] \]

\[ = - \frac{d \tilde{\phi}_g}{d \theta} + \alpha \left( \tilde{v}_r + \frac{4}{3} \tilde{v}_0 - \frac{2}{3} \tilde{v}_0 \cot \theta \right) \frac{d \tilde{\rho}}{d \theta} \]

\[ + \alpha \frac{1}{2} \left( n + 1 \right) \left[ \frac{3}{2} \tilde{v}_r - \frac{d \tilde{v}_0}{d \theta} + \frac{4}{3} \tilde{v}_0 \right] \]

\[ \times \left( \frac{d^2 \tilde{v}_0}{d \theta^2} + \frac{d \tilde{v}_0}{d \theta} \cot \theta - \tilde{v}_0 \csc^2 \theta \right) - \frac{5}{2} \frac{d \ln \tilde{\rho}}{d \theta} . \] (C3)

**ORCID iDs**

Amin Mosallanezhad [https://orcid.org/0000-0002-4601-7073](https://orcid.org/0000-0002-4601-7073)
Fatemeh Zahra Zeraatgari [https://orcid.org/0000-0003-3345-727X](https://orcid.org/0000-0003-3345-727X)
Liquan Mei [https://orcid.org/0000-0003-3468-8803](https://orcid.org/0000-0003-3468-8803)
De-Fu Bu [https://orcid.org/0000-0002-0427-520X](https://orcid.org/0000-0002-0427-520X)

**References**

Abramowicz, M. A., Chen, X., Kato, S., Lasota, J. P., & Regev, O. 1995, *ApJl*, 438, L37
Abramowicz, M. A., Czerny, B., Lasota, J. P., & Szuszkiewicz, E. 1988, *ApJl*, 332, 646
Balbus, S. A., & Hawley, J. F. 1998, *RvMP*, 70, 1
Begelman, M. C., McKee, C. F., & Shields, G. A. 1983, *ApJl*, 271, 70
Blandford, R. D., & Begelman, M. 1999, *MNRAS*, 303, 1
Blandford, R. D., & Begelman, M. 2004, *MNRAS*, 349, 68
Bu, D. F., & Mosallanezhad, A. 2018, *A&A*, 615, A35
Bu, D. F., Wu, M. C., & Yuan, Y. F. 2016, *MNRAS*, 459, 746

---

\(^{10}\) Note that, for simplicity, in Equations (C1)–(C5) we remove the \( \theta \) dependency of the variables, as well as their derivatives.
