Euclidean 4-simplices and invariants of four-dimensional manifolds: II. An algebraic complex and moves $2 \leftrightarrow 4$

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Abstract
We write out some sequences of linear maps of vector spaces with fixed bases. Each term of a sequence is a linear space of differentials of metric values ascribed to the elements of a simplicial complex — a triangulation of a manifold. If the sequence turns out to be an acyclic complex then one can construct a manifold invariant out of its torsion. We demonstrate this first for three-dimensional manifolds, and then we conduct the part, related to moves $2 \leftrightarrow 4$, of the corresponding work for four-dimensional manifolds.

1 Introduction
The present work is second in a series of papers whose aim is to construct invariants of four-dimensional piecewise-linear manifolds similar to the invariants of three-dimensional manifolds constructed in papers [2] and [3]. The first one was paper [1] where we considered only rebuildings (Pachner moves) of type $3 \rightarrow 3$, that is replacements of three 4-simplices from a manifold triangulation with three different simplices having the same common boundary. Below we often use the Roman numeral I when referring to formulas, theorems, etc. from paper [1] (and we call the paper [1] itself paper I). For instance, the expression for the invariant of moves $3 \rightarrow 3$ is given in formula (I.23).

It turns out that the further construction of our invariants involves in a natural way the algebraic language of acyclic complexes. Moreover, already in the three-dimensional case one of our constructions used in papers [2] and [3] has a natural interpretation as the calculation of the torsion of some properly constructed acyclic complexes.

Below in Section 2 we study the three-dimensional case from this new standpoint. One the one hand, this looks interesting in itself, and on the other hand, the investigation of the three-dimensional case prepares the ground for constructing and studying
our algebraic complex for four-dimensional manifolds. The general form of the complex and general plan of work in the four-dimensional case are given in Section 3. Part of this plan is realized in this paper, and the other part — in the planned paper III of this series.

In Section 4 of the present paper we introduce edge deviations that constitute one of the linear spaces in our complex. In Section 5 we partly justify the name “complex” by proving a theorem which states that the image of one of the mappings lies in the kernel of the next mapping (we plan to do this work in full, for the case of four dimensions, in paper III). In Section 6 we explain the relation between edge deviations and moves 2 ↔ 4, i.e. replacements of a cluster of two 4-simplices with a cluster of four ones and vice versa.

In the concluding Section 7 we sum up the results of paper I and the present paper and designate the aims of the next paper in this series.

2 Revisiting the three-dimensional case: invariant $I$ as the torsion of an algebraic complex

We recall that in papers [2] and [3] we were considering orientable three-dimensional manifolds represented as simplicial complexes (or “pre-complexes”, where a simplex can enter several times in the boundary of a simplex of a greater dimension). We ascribed to every edge a Euclidean length, and to every 3-cell (tetrahedron) — a sign + or − in such way that the following “zero curvature” condition was satisfied: the algebraic sum of dihedral angles abutting at each edge was zero modulo $2\pi$. Here a dihedral angle is determined by the lengths of edges of the relevant tetrahedron and is taken with the sign ascribed to that tetrahedron. Then we considered infinitesimal variations of lengths and edges and infinitesimal “deficit angles” at edges depending on them. A matrix $A$ that expressed these linear dependencies played a key role in the resulting formula for manifold invariants.

It is natural to regard matrix $A$ as a linear mapping of one vector space with a fixed basis in it into another similar space. It turns out that $A$ makes up in a natural way, together with four other linear maps of based linear spaces, an exact sequence, or acyclic complex, that can be written out as follows:

$$0 \leftarrow (\cdots) B^T \xleftarrow{(d\omega)} A \xleftarrow{(dl)} B \xleftarrow{(dx \text{ and } dg)} 0.$$  

Here $(dx)$ means all differentials of coordinates of vertices in the complex, taken up to the motions of the Euclidean space (for example, in Section 3 of paper [3] these were $\varphi$, $\sigma$, $s$ and $\alpha$); $(dg)$ means all differentials of continuous parameters on which a representation of the manifold’s fundamental group in $E_3$ can depend (in the case of an infinite fundamental group — see explanations below); $(dl)$ is the column of differentials of edge lengths; $(d\omega)$ is the column of differentials of deficit angles. Recall that the superscript T means the matrix transposing and that $A = A^T$. The left-hand half of the sequence (1) is obtained from its right-hand half by the transposing
of all matrices, therefore, the exactness in terms "(⋯)" and \( dw \) follows from the exactness in the respective terms in the right-hand half. We do not need to determine the geometric meaning of the quantities denoted as "(⋯)", — it will follow from the sequence exactness that they are simply elements of space \( \text{Im} B^T \) — the image of map \( B^T \).

We will need the following lemma.

**Lemma 1** If edge lengths and tetrahedron signs in the complex are chosen so that all deficit angles are zero then the vertices of its universal cover can be put in a Euclidean space \( \mathbb{R}^3 \) in such way that the lengths will coincide with the distances between vertices, while tetrahedron signs — with the orientations of their images in \( \mathbb{R}^3 \). This can be done in a unique way up to the motions of space \( \mathbb{R}^3 \).

*Proof.* Choose a tetrahedron in the universal cover — we call it the “first” tetrahedron — and associate a Euclidean system of coordinates with it. This system of coordinates can be naturally extended to any adjoining tetrahedron, i.e. one having a common face with the first one (recall that every separate tetrahedron can always be put in \( \mathbb{R}^3 \)). Proceeding further this way, we can introduce coordinates in any chosen tetrahedron by using a sequence of tetrahedra starting at our first tetrahedron, ending at our chosen “last” tetrahedron and such that any tetrahedron in it adjoins the two neighbouring ones. We think of such a sequence in geometric terms as determined by a broken line whose every straight segment links the barycenters of two adjoining tetrahedra and whose beginning and end are in the first and last tetrahedron respectively.

We are going to prove that if there are two such broken lines with the same beginnings and ends then they yield the same system of coordinates in the last tetrahedron. Assume that it is not so. Then we would have a closed path with the beginning and end in the last tetrahedron and such that we obtain a changed system of coordinates on going around that path. This path is contractible (recall that we are in a universal cover). It can be contracted into the barycenter of the last tetrahedron, and we can assume that in the process of that contraction it passes through edges only (not vertices).

The fact that the deficit angle around every edge is zero implies that Euclidean coordinates can be extended uniquely and without a contradiction from any tetrahedron containing an edge to all tetrahedra containing that edge. Thus, the monodromy matrix that describes the change of coordinates corresponding to the way around our closed path does not change when that path traverses an edge. Finally, we get a “path” of just one point and two different systems of coordinates — this contradiction concludes the proof of the lemma.

We will define the entries of sequence (1) and prove its exactness with enough rigour only for manifolds \( M \) with a finite fundamental group \( \pi_1(M) \), and we will only show what we can expect in the case of infinite fundamental group on a simple example. Recall [3] that our invariant depends also on a representation \( f: \pi_1(M) \to E_3 \). We
start from the case considered in paper [3]: \( M = L(p, q) \) with a nontrivial representation of group \( \pi_1(L(p, q)) = \mathbb{Z}_p \) in \( E_3 \) by rotations around the \( z \) axis.

On taking a universal cover, each vertex of the triangulation of \( L(p, q) \) turns into \( p \) copies of itself. We place those \( p \) points, according to paper [3], in the vertices of a regular \( p \)-gon in such way that they turn into one another under rotations through multiples of \( 2\pi/p \) around the \( z \) axis. In contrast with paper [3], we are now considering, however, an arbitrary triangulation — having any number of vertices. Let there be \( m \) vertices, then their positions with respect to each other are determined, first, by \( m \) distances \( \rho_1, \ldots, \rho_m \) between them and the \( z \) axis and, second, by \((m - 1)\) differences for each of two remaining cylindrical coordinates, e.g., \((\varphi_2 - \varphi_1), \ldots, (\varphi_m - \varphi_1)\) and \((z_2 - z_1), \ldots, (z_m - z_1)\). The experience of paper [3] shows that the differentials \( dx \) must be chosen as (cf. formula [3, (3.16)])

\[
(dx) = (\rho_1 d\rho_1, \ldots, \rho_m d\rho_m; d(\varphi_2 - \varphi_1), \ldots, d(\varphi_m - \varphi_1); d(z_2 - z_1), \ldots, d(z_m - z_1)).
\]

As for the differentials \((dg)\), there are none of them in this case: a representation of a finite group cannot be deformed continuously in a non-equivalent one.

Map \( B \) determines the deformations of all edge lengths in the complex for given \((dx)\). Its injectivity for generic \( \rho, \varphi \) and \( z \) is obvious: the correspondence between \( \rho_i, \varphi_i - \varphi_1, z_i - z_1 \), on the one hand, and the edge lengths in the complex yielding zero deficit angles, is locally one-to-one (this follows easily from Lemma 1), and thus the Jacobian determinant of map \( B \) is almost everywhere nonzero. The injectivity of \( B \) means the exactness of sequence (1) in the term \((dx)\).

So, it remains to prove the exactness in the term \((dl)\). The fact that \( \text{Im} \, B \subset \text{Ker} \, A \) is obvious: if length differentials are generated by translations of vertices within the Euclidean space \( \mathbb{R}^3 \), then the deficit angles are zero. Let us prove that \( \text{Ker} \, A \subset \text{Im} \, B \). This means that if infinitesimal deficit angles are zero then our complex — the universal cover of a triangulation of \( L(p, q) \) — can be put in the Euclidean space \( \mathbb{R}^3 \). This is, however, nothing else but an infinitesimal version of the same Lemma 1 (with the same proof).

Thus, the exactness of sequence (1) for \( L(p, q) \) is proven. We would like to generalize this result at once for all (connected closed orientable) manifolds \( M \) with a finite fundamental group \( \pi_1(M) \). A single important difference from the case of \( L(p, q) \) can consist in the choice of variables \((dx)\). As one can see from papers [2, 3], the variables \((dx)\) must possess the following property: on adding a new vertex \( E \) (as a result of a move \( 1 \rightarrow 4 \)) the form \( \bigwedge dx \) must get multiplied by \( dx_E \land dy_E \land dz_E \) — the form of Euclidean volume. Recall that our invariant corresponds to a pair \((M, f)\), where \( f: \pi_1(M) \rightarrow E_3 \). Three situations are possible depending on the image \( \text{Im} \, f \) of the group \( \pi_1(M) \) in \( E_3 \):

1) \( \text{Im} \, f = \{e\} \). This case has been studied in paper [2].

2) \( \text{Im} \, f \) contains only rotations around one axis. In this case, the same reasonings as presented above for \( L(p, q) \) are valid. In particular, \((dx)\) must be chosen according to formula (2).
3) Im \( f \) contains rotations around two or more (nonparallel) axes. The presence of such axes fixes the system of coordinates in \( \mathbb{R}^3 \), and \((dx)\) must simply consist of all differentials of Euclidean coordinates of vertices in the complex:

\[
(dx) = (dx_1, dy_1, dz_1, \ldots, dx_m, dy_m, dz_m).
\]

Having presented an acyclic complex (1) for a manifold \( M \) with a finite group \( \pi_1(M) \), we now consider its torsion. As in papers [2, 3], we choose a maximal subset \( C \) in the set of edges of \( M \)'s triangulation such that the square matrix \( A|_C \) (obtained from \( A \) by removing the rows and columns corresponding to the edges from \( C \) — the complement of \( C \)) is nondegenerate. We also need a matrix \( B|_C \) obtained by removing from matrix \( B \) the rows corresponding to the edges from \( C \). According to usual formulae [4], the torsion \( \tau \) is

\[
\tau = \frac{(\det B|_C)^2}{\det A|_C}.
\]

As is well-known, \( \tau \) does not depend on the choice of \( C \).

We can write in the style of papers [2, 3]:

\[
\det B|_C = \frac{\bigwedge \tau \bigwedge dx}{\bigwedge dl}.
\]

After this, the translation of those works into our new language presents no difficulty. We formulate the result as the following Theorem.

**Theorem 1** The value

\[
\tau = \prod_{\text{over all edges}} \frac{l^2}{\prod_{\text{over all tetrahedra}} 6V}
\]

is an invariant of a three-dimensional connected closed orientable manifold with a finite fundamental group.

Clearly, (5) is the squared invariant \( I \) from [2] or \( I_k \) from [3] in the cases considered in those papers.

**Proof** of Theorem 1 consists in tracing what happens with expression (5) under moves \( 2 \leftrightarrow 3 \) and \( 1 \leftrightarrow 4 \), and this is done in the very same way as in papers [2] and [3]. The theorem is proven.

We show now what we can expect in the case of an infinite \( \pi_1(M) \) on the example of a manifold \( M \) with \( \pi_1(M) = \mathbb{Z} \) (for instance, \( M = S^1 \times S^2 \)). Generically, the generating element of \( \pi_1(M) \) is sent by homomorphism \( f \) in a rotation around some axis — let that be the \( z \) axis — through an angle \( \alpha \) and a translation along the same
axis through a distance $a$. Arguments similar to those given above show that we will get an exact sequence (1) if we add, in the capacity of $(dg)$, the differentials $d\alpha$ and $da$ to the same $(dx)$ as in formula (2). After this, the torsion and invariant are calculated according to the old formulas (4) and (5). We are not giving the details here and just put forward a conjecture that one can choose $(dx)$ and $(dg)$ for any $M$ so that (1) will be an exact sequence.

3 Four-dimensional case: the algebraic complex in its general form

We will now present a sequence of linear maps of based linear spaces which is our candidate for the rôle of acyclic complex, similar to (1), in the four-dimensional case. To be exact, we write down two “conjugate” sequences: the matrices defining the maps in one of them can be obtained by matrix transposing from the matrices in the other sequence. This enables us, while investigating the exactness in various terms of the sequences, to use that sequence which is more suitable for the given case. Moreover, this will allow us to bypass the question about the geometrical meaning of some linear spaces entering in a sequence if the meaning of the corresponding fragment of the other sequence is clear and that fragment lends itself to investigation. We will denote such linear spaces by the marks of omission ($\cdots$). So, we are using this notation for different linear spaces which are not (yet) involved in our reasonings.

The definitions of some linear spaces entering in the first of our sequences will be given in the next sections of the present paper and in the forthcoming papers of this series. Nevertheless, we believe it reasonable to write down the sequences right now, because they will determine the plan of our further work and the interrelations of its individual parts.

So, here are our sequences:

\begin{align*}
0 & \leftarrow (\cdots) \leftarrow (d\Omega_a) \leftarrow (\partial S_i/\partial \Omega_a) \leftarrow (d\vec{v}_a) \leftarrow (d\sigma) \leftarrow 0, \quad (6) \\
0 & \rightarrow (dx \text{ and } dg) \rightarrow (dL_a) \rightarrow (\partial \omega_i/\partial L_a) \rightarrow (d\omega_i) \rightarrow (\cdots) \rightarrow (\cdots) \rightarrow 0. \quad (7)
\end{align*}

We start the explanations with the sequence (6). Its left-hand side to $(d\omega_i)$ inclusive is a direct analogue of the right-hand side of sequence (1). We only recall that $L_a$ is the squared length of the $a$th edge, while $\omega_i$ is the deficit angle around the $i$th two-dimensional face.

The link between (6) and (7) is provided by formula (I.16) which states the mutual conjugacy of matrices $(\partial \Omega_a/\partial S_i)$ and $(\partial \omega_i/\partial L_a)$. Recall that $S_i$ is the area of the $i$th two-dimensional face, while $\Omega_a$ is the deficit angle around the $a$th edge defined according to formula (I.13). The values $d\vec{v}_a$ — we call them edge deviations — are introduced in Section 4, together with the mapping $(d\vec{v}_a) \rightarrow (dS_i)$. As for the values $d\sigma$ — vertex deviations — we leave their definition to the future paper III of this series.
It seems that there exist enough interesting manifolds for which our sequences (6) and (7) are acyclic complexes and hence one can hope to obtain four-dimensional manifold invariants out of their torsion in analogy with formula (5). In the present paper, we concentrate on the information extractable from fragment \((d\Omega_a) \leftarrow (dS_i) \leftarrow (d\vec{v}_a)\) of sequence (6). This fragment turns out to be responsible, in a sense, for moves \(2 \leftrightarrow 4\).

4 Edge deviations and area differentials determined by them

In this Section we introduce the notion of edge deviation and explain how these deviations give rise to differentials of two-dimensional face areas, or, in other words, how the matrix \(\partial S_i/\partial \vec{v}_a\) is constructed that gives the map \((d\vec{v}_a) \to (dS_i)\) in sequence (6).

Let \(AB\) be some edge of our complex. Recall that in Section 3 of paper I we have mapped all cells of the universal cover of the complex in a Euclidean space \(\mathbb{R}^4\). Using the pull-back of edge \(AB\) and adjoining 4-simplices up to the universal cover, we can assume that this edge and all neighbouring elements of the complex are put in \(\mathbb{R}^4\). We call the deviation of edge \(AB\) any infinitesimal vector \(d\vec{v}_{AB}\) orthogonal to \(AB\). We imagine such vector as having its beginning in the middle of edge \(AB\).

The area of any Euclidean triangle is four times area of the triangle with vertices in the middles of its sides. If edge deviations in triangle \(ABC\) are \(d\vec{v}_{AB}\), \(d\vec{v}_{AC}\) and \(d\vec{v}_{BC}\) then we set by definition that the differential \(dS_{ABC}\) generated by them is

\[
 dS_{ABC} = 4S_{A'B'C'} - S_{ABC} \tag{8}
\]

(Figure 1).

Now let there be a 4-simplex \(ABCDE\) lying in Euclidean space \(\mathbb{R}^4\) with coordinates \((x, y, z, t)\). Assume that vertices \(A\) and \(B\) lie on the \(t\) axis. Consider a situation where
only edge $AB$ has a nonzero deviation

$$\tilde d_{\vec v_{AB}} = \tilde d_{\vec v} = (dv_x, dv_y, dv_z, 0).$$

We are going to derive a formula that relates $dv_x \wedge dv_y \wedge dv_z$ to $dS_{ABC} \wedge dS_{ABD} \wedge dS_{ABE}$.

We project simplex $ABCDE$ along the $t$ axis onto a three-dimensional space and draw the result in thick lines in Figure 2, denoting somewhat loosely the projections of points $A, \ldots, E$ by the same letters. The other points in Figure 2 are: $C_{1/2}$ is the common projection of middle points of edges $AC$ and $BC$; similarly $D_{1/2}$ is the projection of middle points of edges $AD$ and $BD$, and $E_{1/2}$ of edges $AE$ and $BE$; $K$ is the projection of the point distant from the middle of $AB$ by the deviation $\tilde d_{\vec v}$.

Forgetting for a moment about the four-dimensional origin of Figure 2, we can write down, using an easy trigonometry (cf. formulas (31) and (32) of paper [2]):

$$|dv_x \wedge dv_y \wedge dv_z| = \frac{l_{AC_{1/2}} dl_{KC_{1/2}} \wedge l_{AD_{1/2}} dl_{KD_{1/2}} \wedge l_{AE_{1/2}} dl_{KE_{1/2}}}{6V_{AC_{1/2}D_{1/2}E_{1/2}}}.$$

Here, of course, $dl_{KC_{1/2}} = l_{KC_{1/2}} - l_{AC_{1/2}}$ and so on; the letter $l$ itself denotes the length of the corresponding edge lying in the three-dimensional space, while $V$ is the three-dimensional volume.

Now we go back into the fourth dimension. Denote the length of edge $AB$ as $h$. It follows from the orthogonality of this edge to the three-dimensional projection depicted in Figure 2 that

$$|6V_{AC_{1/2}D_{1/2}E_{1/2}} \cdot h| = |3V_{ABCDE}|$$

(as usual, we use the absolute value signs in order not to care about the orientation),

$$l_{AC_{1/2}} \cdot h = S_{ABC}, \quad dl_{KC_{1/2}} \cdot h = dS_{ABC}, \quad (9)$$
and two more pairs of formulas of type (9) are got by changes $C \rightarrow D$ and $C \rightarrow E$.

All this together gives

$$L_{AB}^{5/2} |dv_x \wedge dv_y \wedge dv_z| = \left| \frac{d(S_{ABC}^2) \wedge d(S_{ABD}^2) \wedge d(S_{ABE}^2)}{24V_{ABCD}} \right|, \quad (10)$$

where we have returned to the notation $L_{AB} = h^2$ of paper I for a squared edge length.

5 Edge deviations yield zero deficit angles $d\Omega$

The following Theorem gives “half” of the exactness of sequence (5) in the term $(dS_i)$.

**Theorem 2** In sequence (5), $\text{Im}(\partial S_i/\partial \bar{v}_a) \subset \text{Ker}(\partial \Omega_a/\partial S_i)$.

**Proof** constitutes all the remaining part of this Subsection.

We represent the totality of all differentials $dS_i$ of two-dimensional face areas in the complex as a column vector $(dS_i)$. We consider first those $dS_i$ that arise from deviations of one edge $AB$ common for exactly four 4-simplices $ABCDE = \hat{F}$, $ABCFD = -\hat{E}$, $ABCEF = \hat{D}$ and $ABDFE = -\hat{C}$ (written in such form, these simplices have a consistent orientation). As $AB$ is common also for exactly four two-dimensional faces, vector $(dS_i)$ can have in our case only four nonzero components. They turn out to obey the following linear dependence:

$$V_\hat{C} d(S_{ABC}^2) + V_\hat{D} d(S_{ABD}^2) + V_\hat{E} d(S_{ABE}^2) + V_\hat{F} d(S_{ABF}^2) = 0. \quad (11)$$

Indeed, we see from formula (10) that the expression

$$|d(S_{ABC}^2) \wedge d(S_{ABD}^2) \wedge d(S_{ABE}^2)/V_\hat{F}| \quad (12)$$

is invariant with respect to permutations of letters $C, D, E, F$. One can check that (11) is exactly the linear dependence that ensures this invariance. To be exact, the summands in (11) are determined from the invariance of (12) up to their signs, but one can easily fix those signs by considering, e.g., such limit cases when one of points $D, E$ or $F$ comes very closely to point $C$: if $D \rightarrow C$, then the differentials $dS_{ABC}$ and $dS_{ABD}$ must be identical.

We prove that (11) is equivalent to the condition $d\Omega_{AB} = 0$, where $d\Omega_{AB}$ — the deficit angle around edge $AB$ — is considered as a function of $dS_{ABC}$, $dS_{ABD}$, $dS_{ABE}$ and $dS_{ABF}$. Indeed, it follows from the mutual conjugacy of matrices $(\partial \Omega_a/\partial S_i)$ and $(\partial \omega_i/\partial L_a)$ (Theorem I.2) that a vector $(dS_i)$ that yields zero deficit angles around edges must be orthogonal to the image of matrix $(\partial \omega_i/\partial L_a)$, i.e. to all its columns. Considering the column corresponding to edge $a = AB$, and applying formula (I.5) and formulas obtained from it by changes $C \leftrightarrow D$, $C \leftrightarrow E$ and $C \leftrightarrow F$, we get:

$$\frac{\partial \omega_{ABC}}{\partial L_{AB}} : \frac{\partial \omega_{ABD}}{\partial L_{AB}} : \frac{\partial \omega_{ABE}}{\partial L_{AB}} : \frac{\partial \omega_{ABF}}{\partial L_{AB}} = V_\hat{C} S_{ABC} : V_\hat{D} S_{ABD} : V_\hat{E} S_{ABE} : V_\hat{F} S_{ABF}. \quad (13)$$
This leads at once to linear dependence (11), and we have only to note that only one dependence exists between our four $dS$.

The fact that equality (11), on the one hand, holds for area differentials generated by deviations of edge $AB$ and, on the other hand, is equivalent to the condition $d\Omega_{AB} = 0$, shows that we have proved the following lemma.

**Lemma 2** If an edge $AB$ is common for exactly four 4-simplices, then the infinitesimal changes of areas arising from deviations $d\vec{v}_{AB}$ give rise, in their turn, to a zero deficit angle $d\Omega_{AB} = 0$.

We want to prove that deviations of all edges in the complex give rise to such $dS_i$ that all $d\Omega_a = 0$. Let us prove this first for a complex representing a triangulation of sphere $S^4$ consisting of six vertices $A, \ldots, F$ and six 4-simplices $\hat{A}, \ldots, \hat{F}$.

In such complex, first, every edge is common for exactly four 4-simplices. Second, all $d\Omega_a$ are proportional (the rank of matrix $(\partial\Omega_a/\partial S_i)$ is 1). Thus, Lemma 2 gives at once the needed result.

If we now have an arbitrary complex that contains an edge $AB$ common for exactly four 4-simplices, then, again, deviations of all edges yield $d\Omega_{AB} = 0$, because $d\Omega_{AB}$ can depend, in principle, only on the deviations of the edges belonging to the four “nearby” simplices, and this dependence is the same as in the previous case of the triangulation of $S^4$.

It remains to consider only one more case, with more than four 4-simplices situated around the edge $AB$. We can always assume that our manifold triangulation was obtained by means of Pachner moves, i.e. rebuildings $3 \rightarrow 3$, $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$, from another triangulation where there was no edge $AB$. That edge appeared as a result of a move $2 \rightarrow 4$ or $1 \rightarrow 5$, which means that it was “at first” surrounded by four 4-simplices, so that all the derivatives $(\partial\Omega_{AB}/\partial \vec{v}_a)$ were zero. Then, more rebuildings of various types could be performed in a neighbourhood of edge $AB$, but any of them may be interpreted in the following way: a cluster of 4-simplices in our simplicial complex was replaced with another cluster in such way that they both formed together the triangulation of sphere $S^4$ consisting of six 4-simplices (as described above). One may say that six new 4-simplices are added to the complex, and then some of them “cancel out” with those already present, because their orientations are opposite. Together with six added simplices (whose orientations are consistent between themselves for any rebuilding), the deficit angles pertaining to $S^4$ are added to the corresponding deficit angles in the complex. Since all derivatives $\partial\Omega_{AB}/\partial \vec{v}_a$ are zero in both $S^4$ and the “old” complex, they remain zero after the rebuilding as well.

Thus, Theorem 2 is proven.

6 **Moves 2 ↔ 4 and the invariant of moves 3 → 3**

In paper I, we have constructed the invariant (I.23) of moves $3 \rightarrow 3$ only in the form of the product of two differential forms (acting on exterior powers of two different
linear spaces). From the standpoint of our sequences (6) and (7), we were studying only the map \((\partial \Omega_a / \partial S_i)\) and its conjugate \((\partial \omega_i / \partial L_a)\). Expression (I.23) cannot be invariant with respect to moves \(2 \leftrightarrow 4\) because they change the degree of form \(\bigwedge dS\) — three differentials are added or taken away. Still, we are going to show that, assuming the exactness of sequence (8) in the term \((dS_i)\), we can describe the behaviour of expression (I.23) under moves \(2 \leftrightarrow 4\) in quite simple terms.

For concreteness, we speak below about the replacement of two 4-simplices with four ones, i.e. a move \(2 \rightarrow 4\). All our reasonings and formulas admit a reversion so that they can also describe a move \(4 \rightarrow 2\).

So, let two adjoining 4-simplices \(ACDEF = \hat{B}\) and \(BCDFE = -\hat{A}\) be replaced by four of them: \(ABCDE = \hat{F}\), \(ABCFD = -\hat{E}\), \(ABCEF = \hat{D}\) and \(ABDFE = -\hat{C}\). This rebuilding provides the complex with a new edge \(AB\) and four new two-dimensional faces containing that edge. Arguments similar to those in Section 3 of paper [2] show that the rank of matrix \((\partial \omega_i / \partial L_a)\) increases thus by 1 and, if edge \(AB\) is added to the subset \(C\) of the set of all edges (see I, Section 5) and, for instance, face \(ABF\) is added to the subset \(D\) of the set of all faces (ibidem), then the determinant of matrix \(B = D(\partial \omega_i / \partial L_a)_C\) gets multiplied under our rebuilding by \(\pm \partial \omega_{ABF} / \partial L_{AB}\):

\[
\det B \xrightarrow{2 \rightarrow 4} \pm \frac{\partial \omega_{ABF}}{\partial L_{AB}} \det B. \tag{14}
\]

As for the partial derivative in (14), it can be obtained by comparing formula (I.5) for a similar derivative \(\partial \omega_{ABC} / \partial L_{AB}\) with formula (13):

\[
\frac{\partial \omega_{ABF}}{\partial L_{AB}} = \frac{S_{ABF}}{24} \frac{V_A}{V_C} \frac{V_B}{V_D} \frac{V_E}{V_F}. \tag{15}
\]

Using formulas (14) and (15), we find that invariant (I.23) gets multiplied under a move \(2 \rightarrow 4\) by

\[
\pm 3 \frac{d(S^2_{ABC}) \wedge d(S^2_{ABD}) \wedge d(S^2_{ABE})}{V_F}. \tag{16}
\]

(of course, the exterior product of new area differentials by those already existing is taken).

Recall now our assumption about the exactness of sequence (8) in the term \((dS_i)\). We will need the inclusion inverse to that proven in Theorem 2, namely,

\[
\text{Ker} \left( \frac{\partial \Omega_a}{\partial S_i} \right) \subset \text{Im} \left( \frac{\partial S_i}{\partial \vec{v}_a} \right). \tag{17}
\]

It means that any column vector \((dS_i)\) yielding zero \(d\Omega_a\) can be obtained from deviations \(d\vec{v}_a\). Therefore, the whole exterior product \(\bigwedge dS\) in invariant (I.23) can be expressed through edge deviations.

Inclusion (17) is conserved under a move \(2 \rightarrow 4\): one can always choose three components of the new deviation \(d\vec{v}_{AB}\) so as to get any values of three new independent
the conditions $d\Omega_a = 0$) area differentials. The new $dS$ (entering in formula (16)) depend not only on $d\vec{v}_{AB}$ but also on other deviations, but the dependence on those latter plays no rôle when taking the exterior product $\bigwedge dS$ in (I.23), because $\bigwedge dS$ has always the maximal degree, coinciding with the maximal number of linearly independent $dS$ that can be obtained from edge deviations. Thus, the exterior product (16) can be re-written with the help of formula (10). Hence, we get the following Theorem.

**Theorem 3** Assuming that inclusion (17) holds, invariant (I.23) of moves $3 \rightarrow 3$ gets multiplied under a move $2 \rightarrow 4$ by

$$\pm 72 L_{AB}^{5/2} dv_x \wedge dv_y \wedge dv_z,$$

where $AB$ is the added edge, while $dv_x$, $dv_y$ and $dv_z$ are the components of its deviation.

### 7 Discussion

Theorem 3 shows that the invariant of moves $3 \rightarrow 3$ introduced in paper [1] undergoes in fact only very simple changes under moves $2 \leftrightarrow 4$ as well, at least if we assume the inclusion (17). One can check that this inclusion is satisfied for the sphere $S^4$. Theorem 3 shows also that it does make sense to consider at least the part of our sequence (6) from $(d\Omega_a)$ to $(d\vec{v}_a)$ inclusive.

In the next, third paper of this series we plan to construct a linear space of “vertex deviations” $d\sigma$ together with the linear map from it to the space of edge deviations $d\vec{v}$. This map will be intimately connected with moves $1 \leftrightarrow 5$. Thus, we will be able to take into account all Pachner moves and construct the full invariant of a four-dimensional piecewise-linear manifold with the help of the torsion of sequence (6). This requires that the latter should be an acyclic complex — we will prove this fact, in particular, for the sphere $S^4$. We will show also that the exactness property itself is conserved under Pachner moves.

Note that the investigation of sequence (6) looks very interesting even if it fails to be exact for some manifold. According to some arguments (suggested by Wall’s stabilization theorem), this may happen for the product $S^2 \times S^2$ of two-dimensional spheres.

In passing, we have translated some constructions in papers [2, 3] into the language of the torsion of acyclic complexes, and thus clarified their algebraic nature. Schematically, the main characters in our play are, first, solutions to the pentagon equation or its generalizations (in dimensions $> 3$) and, second, acyclic complexes. In this connection, it would be very interesting to construct similar complexes corresponding to the “$SL(2)$ solution” of the pentagon equation presented in paper [5] and its analogues for higher dimensions.

We conclude this paper with a question which looks the most intriguing: what are the quantum objects whose limit yields our constructions, with their so clearly quasiclassical appearance.
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