Exactly solvable extended Hubbard model

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One dimensional chiral Hubbard model reduces to the Haldane-Shastry spin chain at half-filling with large but finite on-site energy $U$. In this talk, we show that the Gutzwiller-Jastrow wavefunctions are the eigen-states of the Hubbard model at $U = +\infty$ at less than half-filling. The full energy spectrum and an infinite set of mutually commuting constants of motion are also given in this limit for the system.

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Low dimensional electron systems of strong correlation have been of considerable recent interests, particularly due to the discoveries of high $T_c$ materials and quantum Hall effects. For quantum systems of many degrees of freedom, usually it is impossible to solve the equations of motion exactly, because of many degrees of freedom and the interactions of the particles. However, in some special cases, where systems have infinite number of simultaneous constants of motion, the systems are completely integrable. Exact solutions have provided us with an interesting nonperturbative approach to strongly correlated electron systems. The field of solvabilities has attracted attention from mathematicians, high energy theorists, as well as from condensed matter theorists. Interesting condensed matter models can be solved with Bethe-ansatz, such as the one dimensional Hubbard model \cite{1,2} (a unique example of Luttinger liquid), the Kondo model \cite{3} (a local Fermi liquid), the Anderson model \cite{4} for moment formation, and so on.

In this talk, we review some recent results on the one dimensional chiral Hubbard model. This system was introduced a few years ago by Gebhard and Ruckenstein \cite{7}, and it was conjectured to be integrable for any on-site energy. One very interesting aspect of this system is that the Mott-Hubbard transition occurs at non-zero on-site energy. At half-filling, the Hubbard model in the large on-site energy limit reduces to the Haldane-Shastry spin chain \cite{5,6}. By far, the complete integrability is still lacking for finite $U$. The special case of strong interaction $U = +\infty$ has been studied rather completely, with explicit construction of wavefunctions, analytical derivation of full energy spectrum, thermodynamics, as well as the proof of integrability \cite{8–10}.

The Hamiltonian for the one-dimensional Hubbard model is given by

$$H = \sum_{i \neq j; \sigma = \uparrow, \downarrow} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow},$$  \hspace{1cm} (1)

where $c_{i\sigma}^\dagger$ and $c_{i\sigma}$ are creation and annihilation operators at site $i$ with spin component $\sigma$. We take $t_{ij} = it(-1)^{(i-j)}/d(i - j)$ where $d(n) = \frac{L}{\pi} \sin(n\pi/L)$ is the chord distance \cite{7}. Here we assume periodic boundary condition for the wavefunctions for odd $L$, or anti-periodic boundary condition for even $L$.  

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In the strong interaction limit $U = \infty$, each site can be occupied by at most one electron. In the following, where-ever in case of the strong interaction, we always implicitly assume no double occupancy. Let us denote the number of holes by $Q$, the number of down-spins by $M$. Following notations used in previous literatures, the state vectors can be represented by creating spin and charge excitations from the fully polarized up-spin state $|P\rangle$,

$$|\Phi\rangle = \sum_{\alpha,j} \Phi(x_\alpha, y_j) \prod_\alpha b_{x_\alpha} \prod_j h_{y_j} |P\rangle,$$

where $b_{x_\alpha} = c_{x_\alpha} \downarrow c_{x_\alpha} \uparrow$ is the operator to create a down-spin at site $\alpha$, and $h_{y_j} = c_{y_j} \uparrow$ creates a hole at site $j$.

To describe uniform motion and magnetization, consider the following generalized Gutzwiller-Jastrow wavefunctions [8],

$$\Phi(x, y; J_s, J_h) = \exp \left( \frac{2\pi i}{L} (J_s \sum_\alpha x_\alpha + J_h \sum_i y_i) \times \Phi_0 \right).$$

where

$$\Phi_0 = \prod_{\alpha<\beta} d(x_\alpha - x_\beta) \cdot \prod_\alpha d(x_\alpha - y_i) \cdot \prod_{i<j} d(y_i - y_j),$$

where the function $d(n)$ is defined as before. The quantum numbers $J_s$ and $J_h$ govern the momenta of the down-spins and holes, respectively. They can be integers or half integers so that we have appropriate periodicities (or anti-periodicities) for the wavefunctions under the translations $x_\alpha \rightarrow x_\alpha + L$, or $y_i \rightarrow y_i + L$ for odd $L$ (or even $L$).

These Jastrow wavefunctions have been shown to be eigenstates of the Hamiltonian, with eigenenergies given by

$$E(J_s, J_h) = (2\pi t/L)[2J_h - J_s + L/2]Q,$$

where the momenta of the holes and down-spins meet the following conditions:

$$|J_h| \leq L/2 - (M + Q)/2,$$

$$|J_h - J_s + L/2| \leq M/2.$$

(5)

For other excitations, one may assume following more generalized Jastrow functions:

$$\phi = \phi_s(X, Y)\phi_h(Y) \times \Phi_0.$$
Here the functions $\phi_s$ and $\phi_h$ are polynomials of $X = \{\exp(2\pi ix_\alpha/L)\}$, $Y = \{\exp(2\pi iy_i/L)\}$. They are totally symmetric in their arguments, respectively. The eigen-energy equation thus reduces to

$$
(2\pi t/L)[\sum_{i=1}^Q \partial_i(\phi_s\phi_h) + \sum_{i=1}^Q \phi_s(\partial_i + L/2)\phi_h] = E\phi_s\phi_h,
$$

where $\partial_i = Y_i\partial/\partial Y_i$. This eigen-value equation can be solved exactly, yielding the spectrum given by

$$
E = (2\pi t/L)[\sum_{i=1}^Q n_i + \sum_{\mu=1}^Q m_{\mu}],
$$

The integers (or half integers) satisfy the conditions $|n_i| \leq M/2$, $|m_{\mu}| \leq L/2 - (M + Q)/2$, where $n_i \leq n_{i+1}$ and $m_{\mu} \leq m_{\mu+1}$. This result shows that the spectrum is invariant when changing the sign of $t$.

In terms of a set of conjugate quantum numbers $K_i = n_i + m_i + (L - Q)/2 + i$, the spectrum can be rewritten as

$$
E = -(2\pi t/L) \sum_{i=1}^Q K_i + (\pi tQ/L)(L + 1),
$$

where $K_i$ takes values from $(1, 2, \cdots, L)$. One regards these quantum numbers as the momenta of quasi-particles “holons”. This gives the full energy spectrum of the system. The unoccupied numbers can be considered as the momenta for the quasi-particles of spin degrees. Our numerical result shows that the degeneracy of each energy level is given by the number of the ways to distribute $L - Q - M$ spins $s = +\frac{1}{2}$ and $M$ spins $s = -\frac{1}{2}$ among the empty values. For fixed number of electrons $N_e = L - Q$ on the lattice, the free energy consists of two parts, $F = F_1 - TN_e\ln 2$, where the second term comes from the decoupled spin degrees of freedom, and $F_1$ is the contribution from the charge degree of freedom, which is that of $Q$ spinless fermions with the relativistic spectrum. We see that the spins and charges are completely decoupled in the free energy, and this system is a simple example of Luttinger liquid.

Finally, I wish to note that the integrability can be proved analytically \[10\]. All the results for the $SU(2)$ case can be generalized to the $SU(N)$ case in this strong interaction
limit \[4\]. Recently, some exact eigenfunctions for this model at finite \(U\) have been provided \[11\]. It is hoped that further analytical results on finite \(U\) will be found.

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