EXISTENCE AND UNIQUENESS OF MEASURES OF MAXIMAL ENTROPY FOR PARTIALLY HYPERBOLIC ENDO MORPHISMS

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Abstract. We prove the existence of measures of maximal entropy for partially hyperbolic endomorphisms with one-dimensional center bundle. We also address the uniqueness of such measures for certain endomorphisms defined on the $n$-torus. More precisely, we obtain a unique measure of maximal entropy — locally in a $C^1$ neighborhood of a linear Anosov endomorphism, and globally with additional hypotheses.

1. Introduction

Ergodic theory is the branch of mathematics that studies dynamical systems equipped with an invariant probability measure. This theory was motivated by statistical mechanics, aiming, for instance, to understand and solve problems connected with the kinetic theory of gases, such as whether a Hamiltonian system is ergodic or not. A system is called ergodic when it is dynamically indivisible from a measure theoretical view, meaning that each invariant set has either zero or total measure.

There are two kinds of entropy used to describe the complexity of a dynamical system: metric entropy and topological entropy. The metric entropy provides the maximum amount of average information one can obtain from a system with respect to an invariant measure, and the topological entropy describes the exponential growth rate of the number of orbits.

A measure of maximal entropy is an invariant measure with its metric entropy coinciding with the topological entropy of the system. Such measures “perceive” the whole complexity of the system, and they are part of the study of thermodynamic formalism, see [24, Chapter 10] and the references therein.

Describing the existence and uniqueness/finiteness of measures of maximal entropy is an active research topic in dynamical systems. When a system has a unique measure of maximal entropy, it is called intrinsically ergodic. Uniformly hyperbolic diffeomorphisms are intrinsically ergodic [9, 10], while partially hyperbolic diffeomorphisms can be intrinsically ergodic [5] or not [21]. For non-invertible systems with some kind of hyperbolicity, this problem has been much less explored.

A uniformly hyperbolic endomorphism (also called Anosov endomorphism) is a generalization of the concepts of uniformly hyperbolic diffeomorphism and expanding map. Anosov endomorphisms in general are not structurally stable, since nearby maps may have different quantities of unstable manifolds for corresponding points, which obstructs the existence of a conjugacy [13, 19]. Nonetheless, their dynamics can be studied in spaces of orbits where a kind of stability holds, and it can be shown in this way that they are intrinsically ergodic, see Section 2 for more details.
A partially hyperbolic endomorphism is the non-invertible generalization of partial hyperbolicity, as we define on Section 2. There are examples of partially hyperbolic endomorphisms that are not intrinsically ergodic, having an expanding map as its linearization. But the intrinsic ergodicity for partially hyperbolic endomorphisms with hyperbolic linearization presenting contracting directions was not yet explored, to the best of our knowledge.

Our first effort is to guarantee a general result of existence in the case that the central direction is one-dimensional, as the one by W. Cowieson and L.-S. Young in [8].

**Theorem A.** For $M$ a closed Riemannian manifold, if $f : M \to M$ is a $C^1$ partially hyperbolic endomorphism with one-dimensional center bundle, then there is a measure of maximal entropy for $f$.

We then explore some cases in the torus for which uniqueness can be established, with an approach inspired by [5, 23]. We use a semiconjugacy with an expansive map on the space of orbits to obtain a local result in the form of Theorem B.

**Theorem B.** If $f : T^n \to T^n$ is a dynamically coherent $C^1$ absolutely partially hyperbolic endomorphism with one-dimensional center bundle, its linearization $A$ is hyperbolic and $d_{C^1}(f, A)$ is sufficiently small, then $f$ has a unique measure of maximal entropy $\mu$ and $(f, \mu)$ is ergodically equivalent to $(A, m)$, where $m$ is the volume measure on $T^n$.

On the other hand, if there is a semiconjugacy between $f$ and a hyperbolic linear endomorphism $A$ on $T^n$, we get a global result with further hypotheses, which we state for simplicity as Theorem C on $T^2$.

**Theorem C.** Let $f : T^2 \to T^2$ be an absolutely partially hyperbolic endomorphism with hyperbolic linearization $A$. If $A$ is a factor of $f$, then $f$ has a unique measure of maximal entropy $\mu$, and $(f, \mu)$ is ergodically equivalent to $(A, m)$.

The previous result holds on $T^n$ if the center direction is one-dimensional, $f$ is dynamically coherent and it has quasi-isometric leaves $W^\sigma_f$, where $\sigma \in \{u, c, s\}$, as we see in Theorem D on Section 5.

If $f : T^n \to T^n$ is a partially hyperbolic endomorphism, a lift $F : \mathbb{R}^n \to \mathbb{R}^n$ of $f$ to the universal cover has $A : \mathbb{R}^n \to \mathbb{R}^n$ as a factor — where $A$ is the unique linear map that projects to $A : T^n \to T^n$, and we use the same notation for both maps. In other words, there is a continuous and surjective map $H : \mathbb{R}^n \to \mathbb{R}^n$ such that $H \circ F = A \circ H$. It remains to explore conditions for this semiconjugacy to descend to $T^n$. In the uniformly hyperbolic case, $H$ is a conjugacy, and it projects to $T^n$ if and only if $f$ has a unique unstable manifold for each point $\mathbb{R}^n$. But there is no similar result for the partial hyperbolic case.

Further results can be explored by the investigation of conditions under which we can obtain dynamical coherence and quasi-isometry of foliations for general partially hyperbolic endomorphisms. Another possibility would be to consider the case in which $\dim(E^c) > 1$.

**Organization of the paper.** Section 2 is dedicated to summarize concepts that give context and tools for our results. Here $M$ is an closed smooth manifold (compact, connected and without boundary), and $(X, d)$ is a compact metric space.

## 2. Preliminary concepts

In this section we introduce some necessary concepts and results for our proofs. Here $M$ is an closed smooth manifold (compact, connected and without boundary), and $(X, d)$ is a compact metric space.
2.1. Measures of maximal entropy. We now review a few facts on measures of maximal entropy. We start this section with an alternative definition of metric entropy.

Let $f : X \to X$ be a continuous map and $\mu$ an ergodic $f$-invariant probability measure. We define the $d_n$ metric over $X$ as

$$d_n(x, y) := \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)).$$

For $\delta \in (0, 1)$, $n \in \mathbb{N}$ and $\epsilon > 0$, a finite set $E \subset X$ is called $(n, \epsilon, \delta)$-spanning if the union of the $\epsilon$-balls $B^n_\epsilon(x) = \{y \in X : d_n(x, y) < \epsilon\}$, centered at points $x \in E$, has $\mu$-measure greater than $1 - \delta$. The metric entropy is defined by

$$h_\mu(f) = \limsup_{n \to \infty} \frac{1}{n} \log \min\{\#E : E \subset X \text{ is } (n, \epsilon, \delta) - \text{spanning}\}.$$ 

Let $K \subset X$ be a non-empty compact set. A set $E \subset K$ is said to be $(n, \epsilon)$-spanning if $K$ is covered by the union of the $\epsilon$-balls centered at points of $E$. The topological entropy of $f$ on $K$ is defined by

$$h(f, K) = \limsup_{n \to \infty} \frac{1}{n} \log \min\{\#E : E \subset K \text{ is } (n, \epsilon) - \text{spanning}\}.$$ 

We denote $h_{\text{top}}(f) := h(f, X)$.

The classical definition of metric entropy can be seen, for instance, on [24, Chapter 9], and the above characterization is for ergodic measures on compact metric spaces.

Dinaburg’s variational principle establishes a relation between metric entropy and topological entropy. It states that

$$\sup\{h_\mu(f) : \mu \in \mathcal{M}(X, f)\} = \sup\{h_\mu(f) : \mu \in \mathcal{M}_e(X, f)\} = h_{\text{top}}(f),$$

where $\mathcal{M}(X, f)$ denotes the set of $f$-invariant Borel probability measures on $X$ and $\mathcal{M}_e(X, f)$ denotes the set of ergodic measures on $X$.

Definition 1. A measure of maximal entropy is a probability measure $\mu \in \mathcal{M}(X, f)$ such that $h_\mu(f) = h_{\text{top}}(f)$.

In general, establishing the existence of measures of maximal entropy is a non-trivial task, but in certain contexts there are results that facilitate this task. For instance, when the entropy function $h : \mathcal{M}(X, f) \to [0, \infty)$ defined by $h(\mu) := h_\mu(f)$ is upper semi-continuous, $f$ admits a measure of maximal entropy, see [24, Lemma 10.5.8].

Let $f : X \to X$ be a homeomorphism. The bi-infinite Bowen ball around $x \in X$ of size $\epsilon > 0$ is the set

$$\Gamma_\epsilon(x) := \{y \in X : d(f^n(x), f^n(y)) < \epsilon \text{ for all } n \in \mathbb{Z}\}.$$ 

We say that $f$ is expansive if there is a constant $\epsilon > 0$ such that $\Gamma_\epsilon(x) = \{x\}$ for all $x \in X$.

To generalize the concept of expansiveness, consider

$$\phi_\epsilon(x) := \bigcap_{n=1}^{\infty} B^n_\epsilon(x).$$

Definition 2. A continuous map $f : X \to X$ is called:

1. $h$-expansive if $h^*_f(\epsilon) := \sup_{x \in \mathcal{M}} h(f, \phi_\epsilon(x)) = 0$;
2. asymptotically $h$-expansive if $\lim_{\epsilon \to 0} h^*_f(\epsilon) = 0$.

Of course asymptotic $h$-expansiveness implies $h$-expansiveness. It is also easy to check that, if $f$ is a homeomorphism, then $\Gamma_\epsilon(x) \subseteq \phi_\epsilon(x)$ and by [2 Corollary 2.3], $h^*_f(\epsilon) = h^*_{f, \text{homeo}}(\epsilon) := \sup_{x \in \mathcal{M}} h(f, \Gamma_\epsilon(x))$. In particular, if $f$ is expansive, then it is $h$-expansive.
Theorem 1 ([17]). If \( f : X \to X \) is asymptotically h-expansive, then the entropy function is upper semi-continuous. In particular, \( f \) admits a measure of maximal entropy.

Uniqueness of measures of maximal entropy is a much more delicate problem, but it also brings interesting properties. For instance, when such measures are unique, they are ergodic. Some classes of maps are known to have a unique measure of maximal entropy. For expansive homeomorphisms with specification property, for instance, this problem was solved by R. Bowen in [3].

Following B. Weiss [25], we call a system intrinsically ergodic if there is a unique measure of maximal entropy. Our main tool to prove intrinsic ergodicity is the following theorem, known as Ledrappier–Walters' formula.

\[
\sup \{ h(\mu(T)) : \mu \in \mathcal{M}(X,T) \text{ and } \mu \circ T^{-1} = \nu \} = \int_Y P(T, f, \pi^{-1}(y)) d\nu(y).
\]

In particular, if \( f = 0 \), we have that
\[
\sup \{ h(\mu(T)) : \mu \in \mathcal{M}(X,T) \text{ and } \mu \circ T^{-1} = \nu \} = h(\nu(S)) + \int_Y h(T, \pi^{-1}(y)) d\nu(y).
\]

The function \( P \) that appears on the above result is the topological pressure, and we will not address it in this work. For more details, see [23].

2.2. Natural extensions. For certain aspects, we need to analyze the past orbit of a point. If the map is not invertible, every point has more than one preimage, and there are several “choices of past”. We can make each one of these choices a point on a new space, defined as follows.

Definition 3. Given \( f : X \to X \) continuous, the natural extension (or inverse limit space) associated to the triple \( X, d \) and \( f \) is

- \( X_f = \{ \tilde{x} = (x_k) \in X^\mathbb{Z} : x_{k+1} = f(x_k), \ \forall k \in \mathbb{Z} \} \),
- \( (f(\tilde{x}))_k = x_{k+1} \ \forall k \in \mathbb{Z} \) and \( \forall \tilde{x} \in X_f \),
- \( \tilde{d}(\tilde{x}, \tilde{y}) = \sum_k \frac{d(x_k, y_k)}{2^{|k|}} \).

We also denote \((X_f, \tilde{f})\) as \( \lim(X, f) \). We have that \((X_f, \tilde{d})\) is a compact metric space and the shift map \( \tilde{f} \) is continuous and invertible. Let \( \pi : X_f \to X \) be the projection on the 0th coordinate, \( \pi(\tilde{x}) = x_0 \), then \( \pi \) is a continuous surjection and \( f \circ \pi = \pi \circ \tilde{f} \).

Therefore, every non-invertible topological dynamical system on a compact metric space is a topological factor of an invertible topological dynamical system on a compact metric space.

With the metric \( \tilde{d} \) over \( X_f \), we can define precisely the continuity of objects that depend on the orbit of a point, such as the invariant manifolds given by the invariant splittings in the definitions of the next subsection. By making use of the inverse limit space, it is possible to better comprehend non-invertible systems, especially with respect to invariant measures.

Unless stated otherwise, all measures on this work are over the Borel \( \sigma \)-algebra on the given space. For any \( \tilde{f} \)-invariant probability measure \( \tilde{\mu} \) on \( X_f \), we obtain \( \pi_* \tilde{\mu} \) as a probability measure on \( X \) that is \( f \)-invariant. What makes the natural extension the natural construction to study ergodic theory of non-invertible systems is the fact that \( \pi_* \)
is actually a bijection between the invariant probabilities for \((X_f, \tilde{f})\) and \((X, f)\), as stated in [20 Proposition I.3.1].

**Proposition 1** ([20]). Let \((X, d)\) be a compact metric space and \(f : X \to X\) continuous. For any \(f\)-invariant probability measure \(\mu\) on \(X\), there is a unique \(\tilde{f}\)-invariant probability measure \(\tilde{\mu}\) on \(X_f\) such that \(\pi_* \tilde{\mu} = \mu\).

Moreover, we have that the metric entropies for corresponding measures are the same [20 Proposition I.3.4], that is, \(h_{\mu}(f) = h_{\tilde{\mu}}(\tilde{f})\). Thus, by the variational principle, the topological entropy of \((X_f, \tilde{f})\) and \((X, f)\) is the same.

An Anosov endomorphism is topologically conjugate to its linearization at the natural extension level [19 Theorem 1.20], so their natural extensions have the same topological entropy and corresponding invariant Borel probability measures. Thus, by the invariance of entropy between a system and its natural extension, Anosov endomorphisms are intrinsically ergodic. We investigate this property for partially hyperbolic endomorphisms.

### 2.3. Partially hyperbolic endomorphisms

Partially hyperbolicity means that the dynamics presents directions with hyperbolicity dominating a central direction. More precisely, a diffeomorphism \(f : M \to M\) is a (weakly) *partially hyperbolic diffeomorphism* if, for every \(x \in M\), \(T_xM = E^s_x \oplus E^c_x \oplus E^u_x\), with this splitting being \(Df\)-invariant \((Df_x E^s_x = E^s_{f(x)}), \sigma \in \{u, c, s\}\), and such that

\[
\|Df_x v^s\| < \|Df_x v^c\| < \|Df_x v^u\|
\]

for unit \(v^\sigma \in E^\sigma_x\), \(\sigma \in \{u, c, s\}\), and \(\|Df|_{E^c_x}\| < 1, \|Df|_{E^u_x}\| > 1\).

A more restrict type of partial hyperbolicity is given if we have the same inequalities, but permitting the vectors \(v^\sigma\) to be from different tangent spaces, that is, if \(x, y, z \in M\), and \(v^s \in E^s_x, v^c \in E^c_y\) and \(v^u \in E^u_z\) are unit vectors, then

\[
\|Df_x v^s\| < \|Df_y v^c\| < \|Df_z v^u\|
\]

and \(\|Df|_{E^c_x}\| < 1, \|Df|_{E^u_z}\| > 1\). In this case, we call \(f\) *absolutely partially hyperbolic diffeomorphism*. Along this work, we deal only with this stronger kind of partial hyperbolicity.

Passing to the endomorphism case, we have \(f : M \to M\) a local diffeomorphism, and we do not have an invariant splitting in general. Indeed, even if \(E^c_x\) is trivial (\(f\) is an Anosov endomorphism), the case in which there is an invariant splitting — which we call *special* — is not robust [19]. So the definition cannot be made by using a global invariant splitting. We can define partially hyperbolicity for splittings along given \(f\)-orbits or using invariant cones.

**Definition 4.** We say that a \(C^1\) local diffeomorphism \(f : M \to M\) is *absolutely partially hyperbolic* if, for any \(f\)-orbit \(\tilde{x} \in M_f\), we have a splitting \(T_{\tilde{x}}M = E^s_{\tilde{x}_i} \oplus E^c_{\tilde{x}_i} \oplus E^u_{\tilde{x}_i}, i \in \mathbb{Z}\), such that

1. it is \(Df\)-invariant: \(Df_{\tilde{x}_i} E^\sigma_{\tilde{x}_{i+1}} = E^\sigma_{\tilde{x}_{i+1}}, \sigma \in \{u, c, s\}\), for all \(i \in \mathbb{Z}\);
2. if \(\tilde{x}, \tilde{y}, \tilde{z} \in M_f\), and \(v^s \in E^s_{\tilde{x}_i}\), \(v^c \in E^c_{\tilde{y}_j}\) and \(v^u \in E^u_{\tilde{z}_k}\) are unit vectors, \(i, j, k \in \mathbb{Z}\), then

\[
\|Df_{\tilde{x}_i} v^s\| < \|Df_{\tilde{y}_j} v^c\| < \|Df_{\tilde{z}_k} v^u\|
\]

Moreover, \(\|Df|_{E^c_{\tilde{x}_i}}\| < 1, \|Df|_{E^u_{\tilde{x}_i}}\| > 1\) for any \(\tilde{x} \in M_f\) and \(i \in \mathbb{Z}\).

For endomorphisms, we may have that: \(E^c\) is trivial, obtaining an Anosov endomorphism with contracting direction; \(E^s\) is trivial; or \(E^s\) and \(E^u\) are both trivial, obtaining an expanding map. Absolutely partially hyperbolic endomorphisms then generalize Anosov endomorphisms (which in turn generalize expanding maps and Anosov diffeomorphisms) and absolutely partially hyperbolic diffeomorphisms (the invertible case).
To see that the class of partially hyperbolic endomorphisms is open in $C^1(M,M)$, for instance, it is convenient to work with an alternative definition. For this, we consider a cone family as $\mathcal{C} = \{\mathcal{C}(x)\}_{x \in M} \subseteq TM$ with $\mathcal{C}(x) \subseteq T_xM$, and we say that it is $Df$-invariant if $Df_x(\mathcal{C}(x)) \subseteq \text{Int} \mathcal{C}(f(x))$.

**Definition 5.** We say that a $C^1$ local diffeomorphism $f : M \to M$ is absolutely partially hyperbolic if there exists a $Df$-invariant cone family $\mathcal{C}$ and $\lambda > 1$ such that $\|Df_xv\| > \lambda \|v\|$ for all $v \in \mathcal{C}u(x)$.

If $M$ is an orientable surface, then the existence of $f : M \to M$ a partially hyperbolic endomorphism implies that $M = \mathbb{T}^2$, since there is a $Df$-invariant line field $E^c \subseteq TM$ transversal to the cone field $[10]$. This direction is described generically with $E^c$, but it could also be contracting or expanding, with less expansion than the cones. A classification of partially hyperbolic endomorphisms on surfaces is given in [11].

We recall that there are local and global unstable/stable manifolds for $f$ in $M$ [19, Theorem 2.1], with the unstable manifolds depending on the whole orbit $\tilde{x}$. In general, a partially hyperbolic endomorphism $f : M \to M$ does not have a global invariant unstable or central bundle, and these manifolds do not form a foliation. However, if each point has only one unstable/central direction, we say that $f$ is u/c-special, and we say that $f$ is special if it is both u- and c-special, as it is the case for linear toral endomorphisms.

As the natural extension is an important tool to understand the ergodic properties of endomorphisms, we have that the lift of the endomorphism to the universal cover is the natural way to explore its geometric and differential properties, due to the following result.

**Proposition 2 ([7]).** Let $\overline{M}$ be the universal cover of $M$ and $F : \overline{M} \to \overline{M}$ a lift for $f$. Then $F$ is a partially hyperbolic endomorphism if and only if $F : \overline{M} \to \overline{M}$ is a partially hyperbolic diffeomorphism.

Thus, at the universal cover level, we do have global unstable and central bundles. This implies that there is an unstable foliation at the universal cover. If $F : \overline{M} \to \overline{M}$ is dynamically coherent, then it also has a center foliation.

**Definition 6.** A partial hyperbolic diffeomorphism $f : M \to M$ is said to be dynamically coherent if there are invariant foliations $W^{uc}$ and $W^{cs}$ tangent to $E^{uc} = E^u \oplus E^c$ and $E^{cs} = E^c \oplus E^s$ respectively.

Since partial hyperbolic endomorphisms do not have unstable or central foliations in general, unless they are special, dynamical coherence is defined as follows.

**Definition 7.** A partial hyperbolic endomorphism $f : M \to M$ is said to be dynamically coherent if there are unique invariant leaves $W^{uc}(\tilde{x})$ and $W^{cs}(\tilde{x})$ tangent to $E^{uc}(\tilde{x}) = E^u(\tilde{x}) \oplus E^c(\tilde{x})$ and $E^{cs} = E^c(\tilde{x}) \oplus E^s(x)$ respectively.

If $f$ is a dynamically coherent partial hyperbolic endomorphism, its lift $F : \overline{M} \to \overline{M}$ is dynamically coherent.

In the case that $M = \mathbb{T}^n$, another property that we verify for the lift $F$ is a general geometric property for foliations on $\mathbb{R}^n$, called quasi-isometry. It means that the metric given by the induced distance between to points along a leaf of the foliation is equivalent to the Euclidean distance between them.

**Definition 8.** Given a foliation $\mathcal{F}$ of $\mathbb{R}^n$, with $d_{\mathcal{F}}$ the distance along the leaves, we say that $\mathcal{F}$ is quasi-isometric if there are constants $a, b > 0$ such that, for every $y \in \mathcal{F}(x)$,

$$d_{\mathcal{F}}(x, y) \leq a\|x - y\| + b.$$
If the foliation $\mathcal{F}$ is uniformly continuous, we can take $b = 0$ in the above definition.

For special Anosov endomorphisms, quasi-isometry is guaranteed by the existence and the properties of a conjugacy between the map $f$ and a linear one, and we cover such conjugacy in the next subsection. For systems with partial hyperbolicity, quasi-isometry is the main ingredient to guarantee dynamical coherence [4], and it is important as well to our proofs in this work.

2.4. Hyperbolic linearization and semiconjugacy. Let $f : T^n \to T^n$ be an endomorphism and $A$ its linearization, that is, $A$ is the unique linear map that induces the same homomorphism of $\mathbb{Z}^n \cong \pi_1(T^n)$ as $f$. Consider $F : \mathbb{R}^n \to \mathbb{R}^n$ a lift of $f$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ the linear lift of $A$ to $\mathbb{R}^n$. If $A$ is hyperbolic, then by [1, Theorem 8.2.1] and its proof, there is a unique continuous surjection $H : \mathbb{R}^n \to \mathbb{R}^n$ on the universal cover with

- $A \circ H = H \circ F$;
- $d(H, Id) < K$ and $K$ goes to 0 as $d_{C^1}(f, A)$ tends to 0;
- $H$ is uniformly continuous.

This semiconjugacy is not necessarily preserved under deck transformations, that is, it does not necessarily projects to a semiconjugacy in $T^n$. If $f$ is an Anosov endomorphism ($E^c$ is trivial), $H$ projects to $T^n$ if and only if $f$ is special [6, Proposition 7]. But we can use $H$ to induce a semiconjugacy $\tilde{h} : T^n_f \to T^n_A$ between the natural extensions of $f$ and $A$. This is a consequence of [1, Propositions 7.2.4 and 8.3.1], which we describe briefly.

Firstly, we need to understand the structure of the natural extension of a toral covering map $f : T^n \to T^n$ as a topological group. For a specific finite covering $\tilde{T}^n$, the natural extension $(\hat{S}, \hat{F}) = \lim \leftarrow (\tilde{T}^n, F')$ of the lift $F'$ is constructed and proved to be a solenoidal group, that is, a compact connected abelian group with finite topological dimension [1 §7.2], that obeys the following the commutative diagram.
The lift of \( f \) and \( F' \) to the universal cover, \( F : \mathbb{R}^n \to \mathbb{R}^n \) is homeomorphic to its image under an injective function \( \psi \). The image \( \psi(\mathbb{R}^n) \) is dense on \( \mathbb{R}^p + S_q \), where \( p + q = n \) and \( S_q \) is a solenoidal group. Thus \( \tilde{F} = \psi^{-1} \circ F \circ \psi \) can be extended to \( \mathbb{R}^p + S_q \).

The map \( p_1 : \mathbb{R}^p + S_q \to \mathbb{T}^p + S_q \) is a projection and \( \tilde{F} \) can be projected to \( \bar{F} \). Then \([1 \text{ Theorem 7.2.4}] \) gives us that \( (\bar{S}, \bar{F}) = \lim \mathbb{T}^n, F' \). Finally, we have an isomorphism \( \beta \) between \( (\mathbb{T}^n, \bar{f}) \) and \( (S, \bar{F}) \) \([1 \text{ Lemma 7.2.5}] \).

The same constructions can be made for the linearization \( A : \mathbb{T}^n \to \mathbb{T}^n \). If \( A \) is hyperbolic, the semiconjugacy \( H : \mathbb{R}^n \to \mathbb{R}^n \) on the universal cover is carried to \( \mathbb{R}^p + S_q \) as \( \mathcal{H} = \psi^{-1} \circ H \circ \psi \), and is shown in \([1 \text{ Theorem 8.3.1}] \) to project to \( S \). Therefore, there is a semiconjugacy \( \tilde{h} : \mathbb{T}^n \to \mathbb{T}^n \) between \( (\mathbb{T}^n, \bar{f}) \) and \( (\mathbb{T}^n, \bar{A}) \).

The existence of this semiconjugacy has consequences to the metric and topological entropies of \( f \). Indeed, if \( f : X \to X \) and \( g : Y \to Y \) are continuous maps, there is a continuous surjection such that \( \phi \circ f = g \circ \phi \), and the measures \( \mu \) and \( \nu = h_*\mu \) are \( f \) and \( g \)-invariant, respectively, then

\[
\begin{align*}
(1) \quad & h_\nu(g) \leq h_\mu(f) \\
(2) \quad & h_{\text{top}}(g) \leq h_{\text{top}}(f).
\end{align*}
\]

3. Proof of Theorem \( A \)

The main step to prove Theorem \( A \) is in demonstrating that the inverse limit \( \tilde{f} : M_f \to M_f \) of \( f \) is \( h \)-expansive. The proof is similar to the one for the invertible case, as proven by \([8]\), and we refer to \([9]\) for a presentation closer to ours.

**Theorem 3.** If \( f : M \to M \) is a \( C^1 \) partially hyperbolic endomorphism with one-dimensional center bundle, then \( \tilde{f} : M_f \to M_f \) is \( h \)-expansive.

**Proof.** In order to prove that the topological entropy of \( \tilde{f} \) restricted to \( \Gamma_\varepsilon(\tilde{x}) \) is equal to 0 for each \( \tilde{x} \in M_f \), we show that \( \Gamma_\varepsilon(\tilde{x}) \) is contained on a local center-stable disk and the exponential growth of its spanning sets is given on a local center curve. This curve is one-dimensional, and its iterates have bounded length, so the entropy along it is 0.

Since \( M \) is a closed manifold, we have that the natural extension is a fiber bundle \( (\tilde{X}, X, \pi, \mathcal{C}) \), where the fiber \( \mathcal{C} \) is a Cantor set \([1 \text{ Theorem 6.5.1}] \).

Let \( \beta > 0 \) be sufficiently small such that

- for each \( x \in M \), \( \pi^{-1}(B(x, \beta)) \simeq B(x, \beta) \times \pi^{-1}(\{x\}) \);
- there is \( \delta \in (0, \beta) \) such that, for each \( \tilde{x} \in M_f \) and \( \tilde{y} \in \tilde{B}(\tilde{x}, \beta) \simeq B(x, \beta) \times \{\tilde{x}\} \), if \( d(\tilde{x}, \tilde{y}) < \delta \) then
  \[ W_\beta^s(\tilde{x}) \cap \gamma_\beta^e(\tilde{y}) \text{ is unitary, where } \gamma_\beta^e(\tilde{y}) \text{ is a curve with center } y = \pi(\tilde{y}) \text{ and radius lesser than } \beta \text{ that is tangent to the central bundle}; \]
  \[ W_\beta^u(\tilde{x}) \cap D_\beta^u(\tilde{y}) \text{ is unitary, where } \]
  \[ D_\beta^u(\tilde{y}) = \bigcup_{z \in \gamma_\beta^e(\tilde{y})} W_\beta^u(z), \]

is a disk tangent to \( E^c \oplus E^s \).

In other words, we are taking \( \beta \) small enough to have inside the ball \( B(x, \beta) \) a local product structure that depends only on \( \tilde{x} \), so that it can be seen either in \( B(x, \beta) \subseteq M \) or in \( \tilde{B}(\tilde{x}, \beta) \), the connected component of \( \tilde{x} \) in \( \pi^{-1}(B(x, \beta)) \subseteq M_f \). See Figure \( \mathbb{I} \) Note that we are not assuming dynamical coherence.

Let \( \alpha > 0 \) be such that \( \lambda \alpha < \delta \), where \( \lambda = \max_{x \in M} \|Df|_{E^u}\| \). Then we can construct foliated boxes \( V(\tilde{x}) \subseteq B(x, \beta) \) with uniform size \( \alpha \) (not depending on \( x \in M \)) such that
they have local product structure and their images \( f(V(\tilde{x})) \subseteq B(f(x), \beta) \) also have local product structure. More precisely,

\[
V(\tilde{x}) := \bigcup_{y \in D_{\alpha}^{cs}(\tilde{y})} W_{\alpha}^{u}(\tilde{y}),
\]

where \( \tilde{y} = \pi^{-1}(\{y\}) \cap \tilde{B}(\tilde{x}, \beta) \).

By construction, these boxes are small enough so that we can lift then to the natural extension inside \( \tilde{B}(\tilde{x}, \beta) \), and we denote this lift by \( \tilde{V}(\tilde{x}) := \pi^{-1}(V(\tilde{x})) \cap \tilde{B}(\tilde{x}, \beta) \), see Figure 2. We use \( \tilde{A} \) to denote the lift of any subset \( A \) of \( V(\tilde{x}) \) to \( \tilde{V}(\tilde{x}) \). Then we can use the inverse \( \tilde{f}^{-1} \) on each box, and the proof is concluded exactly as in [9, Theorem 1.2], which we include here for completeness.

Since the sizes of the boxes \( \tilde{V}(\tilde{x}) \) are uniform, there is \( \epsilon > 0 \) such that \( B(\tilde{x}, \epsilon) \cap \tilde{B}(\tilde{x}, \beta) \subseteq \tilde{V}(\tilde{x}) \) for any \( \tilde{x} \in Mf \).

Fixed \( \tilde{x} \in Mf \), we consider \( \tilde{x}^n := \tilde{f}^n(\tilde{x}) \) and \( \gamma_n := \gamma_{\alpha}^{c}(\tilde{x}^n) \) chosen in such way that \( f(\gamma_{n-1}) \cap \gamma_n \) contains a open interval around \( x_n = \pi(\tilde{x}^n) \). This is possible since the central direction is invariant under \( f \). So we are fixing each central curve coherently along the forward orbit of \( \tilde{x} \), and each foliated box \( \tilde{V} \) henceforth is with respect to this choice.

**Claim 1:** \( \Gamma_{\epsilon}(\tilde{x}) \subseteq D_{\alpha}^{cs}(\tilde{x}) \).

Indeed, for each \( \tilde{z} \in \Gamma_{\epsilon}(\tilde{x}) \), we have that \( \tilde{z}^k := \tilde{f}^k(\tilde{z}) \in B(\tilde{f}^k(\tilde{x}), \epsilon) \) for each \( k \in \mathbb{Z} \) by the definition of \( \Gamma_{\epsilon}(\tilde{x}) \), thus \( \tilde{z}^k \in \tilde{V}(\tilde{x}^k) \).

Consider the projections

\[
\rho^a : \tilde{V}(\tilde{x}) \rightarrow \tilde{D}_{\alpha}^{cs}(\tilde{x}), \\
\tilde{z} \mapsto \tilde{W}_{\alpha}^{u}(\tilde{z}) \cap \tilde{D}_{\alpha}^{cs}(\tilde{x})
\]
Figure 2. The box $\tilde{V}(\tilde{x})$ contained in the connected component of $\tilde{x}$ on $\pi^{-1}(B(x, \beta))$, represented here for the two-dimensional case.

and

$$p^c : \hat{D}_{\alpha}^cs(\tilde{x}) \rightarrow \tilde{\gamma}(\tilde{x})$$

that are well defined by the local product structure. Define $\tilde{y}^n := p^c(\tilde{z}^n)$ and $\tilde{w}^n := p^c(\tilde{y}^n)$ for each $n \in \mathbb{N}$.

Suppose that $\tilde{z}^n \notin \hat{D}_\alpha^cs(\tilde{x}^n)$, that is, $\tilde{z}^n \neq \tilde{y}^n$. Then, since $\tilde{y}^n \in W^u(\tilde{z}^n)$, there is $m \in \mathbb{N}$ such that $d(\tilde{z}^n + m, \tilde{y}^n + m) > \alpha$, which implies that $\tilde{z}^{n+m} \notin \tilde{V}(\tilde{x}^{n+m})$, a contradiction. □

Define

$$\Gamma^c := \bigcap_{n \geq 0} \tilde{f}^{-n}(\tilde{\gamma}_n)$$

as the set of points in $\tilde{\gamma}(\tilde{x})$ such that their images under $\tilde{f}$ are in $\tilde{\gamma}_n$. The length of $\tilde{f}^n(\Gamma^c)$ is lesser than $2\alpha$ for all $n \in \mathbb{N}$. In particular we have that $h(\tilde{f}, \Gamma^c) = 0$, see [5, Lemma 4.2], for instance.

Claim 2: $h(\tilde{f}, \hat{D}_\alpha^cs(\tilde{x})) = h(\tilde{f}, \Gamma^c)$.

Indeed, if $S$ is an $(n, \epsilon/2)$-spanning set for $\Gamma^c$, then it is an $(n, \epsilon)$-spanning set for $B(\Gamma^c, \epsilon/2) = \{ \tilde{x} \in M_f : \exists \tilde{y} \in \Gamma^c \text{ with } d(\tilde{x}, \tilde{y}) < \epsilon/2 \}$.

If $n \in \mathbb{N}$ is sufficiently large, then $\tilde{f}^n(\hat{D}_\alpha^cs(\tilde{x})) \subseteq B(\tilde{f}^n(\Gamma^c), \epsilon/2)$, and $S$ is an $(n, \epsilon)$-spanning set for $\hat{D}_\alpha^cs(\tilde{x})$. Thus

$$h(\tilde{f}, \Gamma^c) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \min \{ \#S : S \subseteq \Gamma^c \text{ is an } (n, \epsilon)-\text{spanning set} \}$$

$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \min \{ \#S : S \subseteq \hat{D}_\alpha^cs(\tilde{x}) \text{ is an } (n, \epsilon/2)-\text{spanning set} \}$$

$$= h(\tilde{f}, \hat{D}_\alpha^cs(\tilde{x})).$$

Since $h(\tilde{f}, \Gamma^c) = 0$ and $\Gamma_{\epsilon}(\tilde{x}) \subseteq \hat{D}_\alpha^cs(\tilde{x})$, then Claim 2 implies that $h(\tilde{f}, \Gamma_{\epsilon}(\tilde{x})) = 0$. □

The previous theorem implies that $\tilde{f}$ is $h$-expansive, and from Theorem 1 we have that $\tilde{f}$ has a measure of maximal entropy $\tilde{\mu}$. Thus, it projects to a measure $\mu = \pi_*\tilde{\mu}$ of maximal entropy for $f$ in $M$ by [20, Proposition I.3.4].
4. Proof of Theorem B

We know, by Theorem A, that \( \tilde{f} \) has measures of maximal entropy. We first use Ledrappier–Walters’ formula to prove that any \( \tilde{\mu} \) measure of maximal entropy for \( \tilde{f} \) satisfies \( h_{\tilde{\mu}} = m \), where \( \tilde{h} \) is the semiconjugacy between \( \tilde{f} \) and \( \tilde{A} \), and \( m \) is the volume on \( T^n \).

If \( \tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y}) \), then

\[
\tilde{A}^k(\tilde{h}(\tilde{x})) = \tilde{A}^k(\tilde{h}(\tilde{y})) \implies \tilde{h}(\tilde{f}^k(\tilde{x})) = \tilde{h}(\tilde{f}^k(\tilde{y})),
\]

for any \( k \in \mathbb{Z} \). Thus

\[
\tilde{d}(\tilde{f}^k(\tilde{x}), \tilde{f}^k(\tilde{y})) \leq \tilde{d}(\tilde{f}^k(\tilde{x}), \tilde{h}(\tilde{f}^k(\tilde{x}))) + \tilde{d}(\tilde{h}(\tilde{f}^k(\tilde{x})), \tilde{h}(\tilde{f}^k(\tilde{y}))) + \tilde{d}(\tilde{h}(\tilde{f}^k(\tilde{y})), \tilde{f}^k(\tilde{y})) \leq 2\delta_f,
\]

where \( \delta_f = \tilde{d}_q(Id_{T^n}, \tilde{h}) = \sup_{x \in T^n} \tilde{d}(\tilde{x}, \tilde{h}(\tilde{x})) \), which goes to 0 as \( d_{C^1}(f,A) \) tends to 0 by the proof of [1] Theorem 8.2.1 and the metrics on the diagram of Section 2.4 defined on [1] §7.2.

So we will fix \( d_{C^1}(f,A) \) small such that \( \delta_f < \min \left\{ \frac{\varepsilon}{2}, \frac{a}{4c} \right\} \), where: \( \varepsilon \) and \( a \) are as in the proof of Theorem A, \( a \) is an expansiveness constant to \( A \); and \( c = c(\alpha) > 1 \) is such that any local unstable curve inside \( V_\alpha(\tilde{x}) \) with endpoints \( \tilde{y} \) and \( \tilde{w} \) has length lesser than \( c \tilde{d}(\tilde{y}, \tilde{w}) \). Such \( c \) exists because any unstable curve can be seen in \( T^n \) as a \( C^1 \) curve, and they vary continuously inside \( V_\alpha(\tilde{x}) \).

Therefore, since \( \tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y}) \) implies \( \tilde{d}(\tilde{f}^k(\tilde{x}), \tilde{f}^k(\tilde{y})) \leq 2\delta_f < \varepsilon \), we have that \( \tilde{h}^{-1}(\tilde{z}) \subseteq \phi_{c}(\tilde{x}) \) for \( \tilde{x} \in \tilde{h}^{-1}(\tilde{z}) \). So we have that \( h(f, \tilde{h}^{-1}(\tilde{z})) = 0 \) for each \( \tilde{z} \in T^n_A \).

Then Ledrappier–Walters’ formula (Theorem 2) implies

\[
\sup_{\nu: \tilde{h}_{\nu} = \tilde{h}_{\tilde{\mu}}} h_{\nu}(\tilde{f}) = h_{\tilde{h}_{\tilde{\mu}}}(A),
\]

for any \( \tilde{f} \)-invariant \( \tilde{\mu} \), and the sup is achieved if \( \tilde{\mu} \) is a measure of maximal entropy for \( \tilde{f} \), that is, \( h_{\tilde{h}}(\tilde{f}) = h_{\tilde{h}_{\tilde{\mu}}}(A) \).

On one hand, \( m \) is the unique measure of maximal entropy for \( \tilde{A} \), then \( h_{\tilde{h}_{\tilde{\mu}}}(\tilde{A}) \leq h_{\tilde{h}}(\tilde{A}) = h_{top}(\tilde{A}) \). On the other hand, since \( \tilde{A} \) is a factor of \( \tilde{f} \), then \( h_{top}(\tilde{A}) \leq h_{top}(\tilde{f}) = h_{\tilde{h}}(\tilde{f}) \). Thus, these entropies are equal and \( h_{\tilde{h}_{\tilde{\mu}}} = m \).

It remains to prove that there is only one \( \tilde{f} \)-invariant measure that is pushed to \( \tilde{m} \) under \( \tilde{h} \). To do so, it suffices to prove that \( \tilde{\Gamma} := \{ \tilde{z} \in T^n_A : \#h^{-1}(\tilde{z}) > 1 \} \) has \( \tilde{m} \)-measure zero.

**Claim 3:** \( \tilde{m}(\tilde{\Gamma}) = 0 \).

We saw that \( \tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y}) \implies \tilde{d}(\tilde{f}^k(\tilde{x}), \tilde{f}^k(\tilde{y})) \leq 2\delta_f \), but a reverse implication holds for our choice of \( \delta_f \). Indeed, if \( \tilde{d}(\tilde{f}^k(\tilde{x}), \tilde{f}^k(\tilde{y})) \leq 2\delta_f \) c then

\[
\tilde{d}(\tilde{A}^k(\tilde{h}(\tilde{x})), \tilde{A}^k(\tilde{h}(\tilde{y}))) = \tilde{d}(\tilde{h}(\tilde{f}^k(\tilde{x})), \tilde{h}(\tilde{f}^k(\tilde{y}))) \leq \tilde{d}(\tilde{h}(\tilde{f}^k(\tilde{x})), \tilde{f}^k(\tilde{x})) + \tilde{d}(\tilde{f}^k(\tilde{x}), \tilde{f}^k(\tilde{y})) + \tilde{d}(\tilde{f}^k(\tilde{y}), \tilde{h}(\tilde{f}^k(\tilde{y}))) \leq 2\delta_f + 2\delta_f c \leq 4\delta_f c < a,
\]

and, by expansiveness of \( \tilde{A} \), \( \tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y}) \). This implies that \( \tilde{h}^{-1}(\tilde{z}) \) is contained on \( \hat{W}_{f,\alpha}^c(\tilde{x}) \) for \( \tilde{x} \in \tilde{h}^{-1}(\tilde{z}) \), using the fact that \( f \) is dynamically coherent.
So $\hat{h}^{-1}(\tilde{z})$ is a closed and bounded subset of $\hat{W}_{f,\alpha}^c(\tilde{x})$, being thus compact. We see in the following that it is also connected. If $\tilde{x}, \tilde{y} \in \hat{h}^{-1}(\tilde{z})$, consider $\tilde{w} \in [\tilde{x}, \tilde{y}]_c$, that is, on the curve from $\tilde{x}$ to $\tilde{y}$ inside $\hat{W}_{f,\alpha}^c(\tilde{x})$. Then

$$d(\hat{f}^k(\tilde{x}), \hat{f}^k(\tilde{w})) \leq \tilde{d}_c(\hat{f}^k(\tilde{x}), \hat{f}^k(\tilde{w})) \leq \tilde{d}_c(\hat{f}^k(\tilde{x}), \hat{f}^k(\tilde{y})) < 2\delta_f,$$

for each $k \in \mathbb{Z}$, thus $\tilde{w} \in \hat{h}^{-1}(\tilde{z})$, and $\hat{h}^{-1}(\tilde{z})$ is connected.

Then for each $\tilde{z} \in \mathbb{T}_{A}^n$, the local linear leaf $\hat{W}_{A,\alpha}^c(\tilde{z})$ has at most countable points in $\tilde{G}$, since for $\tilde{w} \in \hat{W}_{A,\alpha}^c(\tilde{z})$, $\hat{h}^{-1}(\tilde{w}) \subseteq \hat{W}_{f}^c(\tilde{x})$ for $\tilde{x} \in \hat{h}^{-1}(\tilde{z})$, then $\tilde{w} \in \hat{W}_{A,\alpha}^c(\tilde{z}) \cap \tilde{G}$ has an non-trivial arc as pre-image, and there can be at most countable such disjoint arcs on $\hat{W}_{f}^c(\tilde{x})$.

We can cover $\mathbb{T}_{A}^n$ with boxes $\tilde{V}$, inside each $\hat{W}_{A,\alpha}^c(\tilde{z})$ forms a linear foliation, and by Fubini’s Theorem, $\tilde{m}(\tilde{G}) = 0$.

5. Proof of Theorem $\text{C}$

L. Hall and A. Hammerlindl have proved that partially hyperbolic surface endomorphisms with hyperbolic linearization are dynamically coherent and their foliations by unstable and central leaves in the universal cover are quasi-isometric $\text{[10]}$. Then, Theorem $\text{C}$ is a particular case of the following more general result. The proof is similar to the one of Theorem $\text{B}$ but the quasi-isometry and the semiconjugacy with a linear model on the ambient manifold allows a global argument.

**Theorem D.** Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an absolutely partially hyperbolic endomorphism with $\dim(E^c) = 1$, dynamically coherent and with $W_{f}^c$, $\sigma \in \{u, c, s\}$, quasi-isometric, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a lift of $f$. If $A = f_s$ is hyperbolic and $A$ is a factor of $f$, then $A$ has a unique measure of maximal entropy $\mu$. Moreover, $(f, \mu)$ and $(A, m)$ are ergodically equivalent.

**Proof of Theorem $\text{D}$** Even without an unstable foliation for $f$ in $\mathbb{T}^n$, an unstable foliation does exist for $F$ in $\mathbb{R}^n$, and all computations in the universal cover work as in the invertible case. Thus, as in $\text{[23, Lemma 3.4]}$, we prove the following, where $H : \mathbb{R}^n \to \mathbb{R}^n$ is a lift for the semiconjugacy $h : \mathbb{T}^n \to \mathbb{T}^n$.

**Lemma 1.** If $\dim(E^c_F) = 1$, $f$ is dynamically coherent and $A = f_s$ is hyperbolic, then $A$ admits a partially hyperbolic splitting $\mathbb{R}^n = E^u_A \oplus E^c_A \oplus E^s_A$ with $\dim(E^c_A) = 1$. Additionally, $H(W^c_F(x)) = W^c_A(H(x))$ for all $x \in \mathbb{R}^n$.

If $A$ is hyperbolic, we also have that there is $\alpha > 0$ such that, for all $x, y \in \mathbb{R}^n$, $H(x) = H(y)$ if and only if $d(F^k(x), F^k(y)) < \alpha$ for all $k \in \mathbb{Z}$. Indeed, if $H(x) = H(y)$, then

$$d(A^k(H(x)), A^k(H(y))) \implies H(F^k(x)) = H(F^k(y)) \text{ for all } k \in \mathbb{Z}.$$

Since $d(H(z), z) < K$ for all $z \in \mathbb{R}^n$, then

$$d(F^k(x), F^k(y)) \leq d(F^k(x), H \circ F^k(x)) + d(H \circ F^k(x), H \circ F^k(y)) + d(H \circ F^k(y), F^k(y)),$$

which is lesser than $2K$ for all $k \in \mathbb{Z}$.

For the reciprocal implication, we use the expansiveness of $A$ and the fact that it is linear to guarantee that it is expansive with any expansiveness constant (see, for instance, $\text{[11 Lemma 8.2.3]}$). Since

$$d(A^k \circ H(x), A^k \circ H(y)) = d(H \circ F^k(x), H \circ F^k(y)) \leq d(H \circ F^k(x), F^k(x)) + d(F^k(x), F^k(y)) + d(F^k(y), H \circ F^k(y)) < 2K + \alpha$$

...
for all \( k \in \mathbb{Z} \), then \( H(x) = H(y) \).

**Lemma 2.** If \( A \) is hyperbolic, \( F \) is dynamically coherent and \( W^\sigma_F \) is quasi-isometric, then \( H(x) = H(y) \) implies that \( y \in W^F_F(x) \).

**Proof.** Suppose that \( y \notin W^F_F(x) = W^\sigma_F(x) \cap W^\sigma_F(x) \). Then \( y \notin W^\sigma_F(x) \) or \( y \notin W^\sigma_F(x) \). Let us see that the first case is absurd, and the second case is analogous.

If \( y \notin W^\sigma_F(x) \), then there is \( z \neq y \) such that \( z = W^\sigma_F(x) \cap W^\sigma_F(y) \). Indeed, the global product structure follows from the quasi-isometry, as proved in \([12]\) Theorem 1.1\) for partially hyperbolic diffeomorphisms on (not necessary compact) manifolds. It then applies to \( F : \mathbb{R}^n \to \mathbb{R}^n \).

Consider \( D_{cs} = d_{cs}(x, z) \) and \( D_a = d_a(y, z) \). There are \( 1 < \lambda_c < \lambda_u \) such that

\[
d(F^k(x), F^k(z)) \leq \lambda_c^k D_{cs} \quad \text{and} \quad d(F^k(y), F^k(z)) \geq \lambda_u^k D_a
\]

for all \( k \in \mathbb{Z} \). Since \( W^\sigma_F \) is quasi-isometric, we have \( d_a(F^k(y), F^k(z)) \leq ad(F^k(y), F^k(z)) + b \) and thus

\[
d(F^k(y), F^k(z)) \geq \frac{\lambda_u^k D_a - b}{a}.
\]

Therefore

\[
d(F^k(x), F^k(y)) \geq d(F^k(x), F^k(z)) - d(F^k(x), F^k(z)) > \frac{\lambda_u^k D_a - b}{a} - \lambda_c^n D_{cs} \xrightarrow{n \to \infty} \infty,
\]

which implies that \( H(x) \neq H(y) \), a contradiction. \( \square \)

**Remark 1.** The above lemma requires that \( n = \dim(\mathbb{T}^n) \geq 3 \), for we need \( \dim(E^\sigma_F) \neq 0 \), \( \sigma \in \{ u, c, s \} \). If \( n = 2 \), we have the same result just by requiring that \( A \) is hyperbolic. Indeed, in this case, \( \dim(E^c) = \dim(E^u) = 1 \) and we have dynamical coherence, quasi-isometry and global product structure by \([10]\). Supposing that \( y \notin W^F_F(x) \), there is a unique \( z \in W^\sigma_F(y) \cap W^\sigma_F(x) \), \( z \neq y \), and the proof by absurd is the same.

**Remark 2.** By lemmas 1 and 2 for all \( z \in \mathbb{R}^n \) we have that \( H^{-1}(W^\sigma_F(z)) = W^F_F(x) \) for any \( x \in H^{-1}(z) \).

**Lemma 3.** If \( A \) is hyperbolic, \( \dim(E^\sigma_F) = 1 \), \( F \) is dynamically coherent and \( W^\sigma_F \) is quasi-isometric for \( \sigma \in \{ u, c, s \} \), then for all \( z \in \mathbb{R}^n \) we have that \( H^{-1}(z) \) is a compact and connected subset of \( W^F_F(x) \) for \( x \in H^{-1}(z) \).

**Proof.** We have that \( H^{-1}(z) \) is closed, and it is bounded since \( d(H, Id) < K \). For \( x \in H^{-1}(z) \), \( H^{-1}(z) \) it is a subset of \( W^F_F(x) \) by Lemma 2. It remains to prove connectedness. Fixing \( x, y \in H^{-1}(z) \) and given \( w \in [x, y] \), we have by the quasi-isometry of \( W^F_F \) that

\[
d(F^k(x), F^k(w)) \leq d_c(F^k(x), F^k(w)) \leq d_c(F^k(x), F^k(y)) \leq ad(F^k(x), F^k(y)) + b \leq 2aK + b,
\]

which implies that \( z = H(x) = H(w) \), as previously shown using the expansiveness of \( A \). Thus, \( [x, y] \subseteq H^{-1}(z) \) for all \( x, y \in H^{-1}(z) \), and \( H^{-1}(z) \) is connected. \( \square \)

Let \( \Gamma = \{ z \in \mathbb{R}^n : \# H^{-1}(z) > 1 \} \) be the set of points for which \( H \) fails to be invertible. Consider \( p(\Gamma) = \Gamma \), with \( p : \mathbb{R}^n \to \mathbb{T}^n \) the canonical projection.

**Lemma 4.** Under the hypotheses of Lemma 3, \( m(\Gamma) = 0 \).
Proof. For all \( z \in \mathbb{R}^n \) there is \( x \in \mathbb{R}^n \) such that \( H^{-1}(W^c_A(z)) = W^c_k(x) \). Consider \( \Gamma^e_z = W^c_A(z) \cap \Gamma \). Then \( \{H^{-1}(y) : y \in \Gamma^e_z\} \) is a family of disjoint nontrivial intervals in \( W^c_A(z) \), we obtain that \( \Gamma^e_z \) is countable for all \( z \in \mathbb{R}^n \). Therefore, since \( W^c_A \) is a linear foliation, Fubini’s theorem provides that \( m(\Gamma) = 0 \).

Thus, \( \Gamma = p(\Gamma) \) also has zero volume on \( \mathbb{T}^n \), and it satisfies \( \Gamma = \{z \in \mathbb{T}^n : \#h^{-1}(z) > 1\} \) since \( H \) is a lift for \( h \). Moreover, \( h^{-1}(\Gamma) = \Gamma \).

Lemma [3] implies that \( H^{-1}(\overline{z}) \) is a compact and connected one-dimensional central disk, with its length bounded with a constant that does not depend on \( z \). But if \( \overline{x}, \overline{y} \in \mathbb{R}^n \) are such that \( H(\overline{x}) = H(\overline{y}) \), then we have that \( d(F^k(\overline{x}), F^k(\overline{y})) < 2K \) for any \( k \in \mathbb{Z} \). Thus \( \overline{x} \in H^{-1}(\overline{z}) \) if and only if \( F^n(\overline{x}) \in H^{-1}(A^n(\overline{z})) \) for any \( n \in \mathbb{N} \). Therefore, the length of \( F^n(H^{-1}(\overline{z})) \) is also bounded with the same constant than \( H^{-1}(\overline{z}) \). These bounds are the same for the projections of the stable manifolds, and this implies \( h(f, h^{-1}(z)) = 0 \) for all \( z \in \mathbb{T}^n \).

By Theorem [A], there exists a measure of maximal entropy \( \mu \). Then, using Ledrappier–Walters’ formula, we have that \( h_{\ast}\mu = m \). Now, we want to prove that \( \mu \) is the unique measure of maximal entropy that project into \( m \). By contradiction, suppose that there exists a measure of maximal entropy \( \eta \neq \mu \), then \( h_{\ast}\eta = m = h_{\ast}\mu \). From Lemma [4] it follows that \( \mu(\Gamma) = \eta(\Gamma) = 0 \) and for every continuous function \( \psi : \mathbb{T}^n \to \mathbb{R} \) we have that

\[
\int_{\mathbb{T}^n} \psi d\mu = \int_{\mathbb{T}^n \setminus \Gamma} \psi d\mu = \int_{\mathbb{T}^n \setminus \Gamma} \psi \circ h^{-1} \circ h d\mu = \int_{\mathbb{T}^n \setminus \Gamma} \psi \circ h^{-1} d\mu \eta = \int_{\mathbb{T}^n \setminus \Gamma} \psi d\eta.
\]

Hence, \( \mu = \eta \), a contradiction. Therefore, there exists a unique measure of maximal entropy.

\[\square\]

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References

[1] Nobuo Aoki and Koichi Hiraide. Topological theory of dynamical systems, volume 52 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, 1994. Recent advances.

[2] Rufus Bowen. Expansive entropy-expansive maps. Trans. Amer. Math. Soc., 164:323–331, 1972.

[3] Rufus Bowen. Some systems with unique equilibrium states. Math. Systems Theory, 8(3):193–202, 1974/75.

[4] Michael Brin. On dynamical coherence. Ergodic theory and dynamical systems, 23(2):395–401, 2003.

[5] Jérôme Buzzi, Todd Fisher, Martín Sambarino, and Carlos H. Vásquez. Maximal entropy measures for certain partially hyperbolic, derived from Anosov systems. Ergodic Theory Dynam. Systems, 32(1):63–79, 2012.

[6] Marisa Cantarino and Régis Varão. Anosov endomorphisms on the 2-torus: Regularity of foliations and rigidity. arXiv preprint arXiv:2104.01693, 2021.

[7] José S. C. Costa and Fernando Micena. Some generic properties of partially hyperbolic endomorphisms, 2021.

[8] William Cowieson and Lai-Sang Young. SRB measures as zero-noise limits. Ergodic Theory Dynam. Systems, 25(4):1115–1138, 2005.

[9] Lorenzo J. Díaz and Todd Fisher. Symbolic extensions and partially hyperbolic diffeomorphisms. Discrete Contin. Dyn. Syst., 29(4):1419–1441, 2011.
Layne Hall and Andy Hammerlindl. Partially hyperbolic surface endomorphisms. *Ergodic Theory and Dynamical Systems*, 41(1):272–282, 2021.

Layne Hall and Andy Hammerlindl. Classification of partially hyperbolic surface endomorphisms. *Geom. Dedicata*, 216(3):Paper No. 29, 19, 2022.

Andy Hammerlindl. Dynamics of quasi-isometric foliations. *Nonlinearity*, 25(6):1585, 2012.

Ittai Kan. Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin. *Bull. Amer. Math. Soc. (N.S.)*, 31(1):68–74, 1994.

François Ledrappier and Peter Walters. A relativised variational principle for continuous transformations. *Journal of the London Mathematical Society*, 2(3):568–576, 1977.

Ricardo Mañé and Charles Pugh. Stability of endomorphisms. In *Dynamical Systems—Warwick 1974*, pages 175–184. Springer, 1975.

Grigoriy A. Margulis. *On some aspects of the theory of Anosov systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows. Translated from the Russian by Valentina Vladimirovna Szulikowska.

Michal Misiurewicz. Diffeomorphism without any measure with maximal entropy. *Bulletin de l’Académie polonaise des sciences. Série des sciences mathématiques, astronomiques et physiques*, 21(10):903–910, 1973.

Bárbara Núñez Madariaga, Sebastián A. Ramírez, and Carlos H. Vásquez. Measures maximizing the entropy for Kan endomorphisms. *Nonlinearity*, 34(10):7255–7302, 2021.

Feliks Przytycki. Anosov endomorphisms. *Studia Math.*, 58(3):249–285, 1976.

Min Qian, Jian-Sheng Xie, and Shu Zhu. *Smooth ergodic theory for endomorphisms*, volume 1978 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

Joas Elias Rocha and Ali Tahzibi. On the number of ergodic measures of maximal entropy for partially hyperbolic diffeomorphisms with compact center leaves. *Math. Z.*, 301(1):471–484, 2022.

Naoya Sumi. Linearization of expansive maps of tori. In *Proc. of International Conference on Dynamical Systems and Chaos*, volume 1, pages 243–248, 1994.

Raúl Ures. Intrinsic ergodicity of partially hyperbolic diffeomorphisms with a hyperbolic linear part. *Proceedings of the American Mathematical Society*, 140(6):1973–1985, 2012.

Marcelo Viana and Krerley Oliveira. *Foundations of Ergodic Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.

Benjamin Weiss. Intrinsically ergodic systems. *Bull. Amer. Math. Soc.*, 76:1266–1269, 1970.

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