Generalized uncertainty principles and black hole temperatures in rainbow gravity

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Abstract

In this paper, we have obtained modified black hole temperatures in rainbow gravity by employing both the modified dispersion relation (MDR) and the three different types of generalized uncertainty principles (GUPs) including the extended uncertainty principle (EUP). We also investigate their thermodynamic stabilities of the modified Schwarzschild black hole according to the different GUPs.

Keywords: generalized uncertainty principles, modified gravity, quantum black holes

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1 Introduction

Hawking’s semiclassical treatment [1] breaks down as the size of the event horizon of a black hole approaches the Planck length and quantum gravitational effects become large [2, 3]. One common feature of quantum gravity effects is the existence of a minimal length scale, which causes an alteration of the Heisenberg uncertainty principle (HUP) to a generalized uncertainty principle (GUP) [4–33]. A GUP again significantly modifies the Hawking radiation and leaves a remnant at the final stage of the evaporation [34]. Moreover, the existence of a minimal length scale makes us puzzled since there is no length invariant under the special relativity. This contemplation leads to doubly special relativity (DSR), or special relativity with two observer independent scales [35–38]. Of course, it is assumed that the DSR becomes the standard special relativity in the limit of vanishing the minimal length. It is naturally expected in the DSR that the usual energy-momentum dispersion relation should be modified by some nonlinear mass-shell relations.

On the other hand, Magueijo and Smolin (MS) [39] have extended the DSR to general relativity by proposing that quanta of different energies see different background geometry, referred to as rainbow gravity. Since then many efforts have been devoted to rainbow gravity related to extending gravity and other stimulated work at the Planck scale [40–76]. Recently, Ali [77] calculated thermodynamic quantities in rainbow Schwarzschild black hole with a particular choice of rainbow functions and studied corresponding thermodynamics. Here, he obtained the black hole temperature from the definition of the surface gravity in the rainbow Schwarzschild black hole and then by invoking the HUP and the ordinary dispersion relation the energy dependence in the temperature was eliminated. Later, Gim and Kim [78] improved the black hole temperature whose energy dependence was eliminated by employing both the HUP and the MDR. Very recently, we have extended the previous study of the thermodynamics and phase transition of the rainbow Schwarzschild black hole to the Schwarzschild-AdS black hole [79] where metric depends on the energy of a probe by employing both the HUP and the MDR.

In this paper, we will further investigate modified temperatures of black holes in rainbow gravity by fully employing both the MDR and the various GUPs, not just restricted by the HUP, and study their corresponding thermodynamic stabilities. In section 2, we first briefly review the method of Adler-Chen-Santiago (ACS) heuristically to find black hole’s temperatures related
to the HUP. Then, we obtain the desired black hole’s temperature considering three different types of the GUPs including the extended uncertainty principle (EUP) and the generalized extended uncertainty principle (GEUP). In section 3, we study the GUPs effects in black hole temperatures making full use of the MDR in rainbow gravity. In section 4, we investigate thermodynamic stabilities of both the GUP and the MDR corrected Schwarzschild black holes in rainbow gravity. Finally, a conclusion and discussion is given in section 5.

2 Black hole temperatures with various GUPs

According to ACS [34], let us first briefly recapitulate the way of getting the well-known Hawking temperature by considering the HUP for a spherically symmetric black hole.

In the vicinity of a black hole surface, there is intrinsic position uncertainty of the Hawking radiated photons of order of the event horizon $r_+$ as

$$\Delta x \sim r_+. \quad (2.1)$$

On the other hand, according to the HUP, it leads to momentum uncertainty of order $p$ as

$$p = \Delta p \sim \frac{\hbar}{\Delta x} \sim \frac{\hbar}{r_+}. \quad (2.2)$$

Let us plug this momentum uncertainty into the standard dispersion relation

$$E^2 - p^2 c^2 = m^2. \quad (2.3)$$

Then, for a massless particle, the energy uncertainty will be

$$E = pc \sim \frac{\hbar c}{r_+}, \quad (2.4)$$

which can be identified with the characteristic energy of the emitted photon as $E = k_B T$. As a result, it concludes that a black hole’s temperature be

$$T_{HUP} = \frac{1}{4\pi r_+} \quad (2.5)$$

with a calibration factor of $1/4\pi$ and $k_B = \hbar = c = 1$, which is in Fig. 1(a). The figure shows that the temperature diverges as $r_+ \to 0$, while it

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Hereafter, we will use the natural units of $\hbar = 1$ and $c = 1$ except in the uncertainty principles themselves for their clarity.
vanishes as $r_+ \to \infty$. Note that for the case of the Schwarzschild black hole with $r_+ = 2GM$, it gives exactly the same Hawking temperature as

$$T^{HUP}_H = \frac{1}{8\pi GM}. \quad (2.6)$$

### 2.1 GUP case

In order to extend the method of ACS to various GUPs, let us first consider the GUP case which is susceptible when the momentum is close to the Planck scale, given by the inequality

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha \ell_p^2 \frac{\Delta p}{\hbar}, \quad (2.7)$$

where $\alpha$ is a dimensionless constant and $\ell_p^2 = G\hbar/c^3$ is the Planck length [2–31, 33]. This gives us the momentum uncertainty of

$$\frac{\Delta x}{2\alpha \ell_p^2} \left(1 - \sqrt{1 - \frac{4\alpha \ell_p^2}{(\Delta x)^2}}\right) \leq \frac{\Delta p}{\hbar} \leq \frac{\Delta x}{2\alpha \ell_p^2} \left(1 + \sqrt{1 - \frac{4\alpha \ell_p^2}{(\Delta x)^2}}\right). \quad (2.8)$$

It explicitly shows that there exists an absolute minimum in the position uncertainty as

$$\Delta x \geq \Delta x_{\text{min}} \equiv 2\sqrt{\alpha} \ell_p. \quad (2.9)$$

The existence of $\Delta x_{\text{min}}$ implies that the position uncertainty cannot be arbitrarily small, while the momentum uncertainty cannot be arbitrarily large. Note that as $\alpha \to 0$, or, equivalently, when $\Delta x \gg \Delta x_{\text{min}}$ in the semiclassical regime, the HUP can be recovered from the left inequality in Eq. (2.8).

Now, according to ACS [34] as before, the momentum uncertainty of order $p$ can be obtained from the position uncertainty of an emitted photon of order $\Delta x \sim r_+$ as

$$p = \Delta p \sim \frac{r_+}{2\alpha \ell_p^2} \left(1 - \sqrt{1 - \frac{4\alpha \ell_p^2}{r_+^2}}\right). \quad (2.10)$$

Plugging this into the standard dispersion relation for a massless particle, the energy uncertainty gives a GUP–corrected black hole temperature as

$$T^{GUP} = \frac{r_+ \omega_p^2}{8\pi \alpha} \left(1 - \sqrt{1 - \frac{4\alpha}{r_+^2 \omega_p^2}}\right). \quad (2.11)$$
with the calibration factor of $1/4\pi$. This GUP-modified temperature is plotted in Fig. 1(b), which shows that the temperature vanishes as $r_+ \to \infty$. On the other hand, since $r_+$ cannot go through the minimum position uncertainty $\Delta x_{\text{min}}$ as like in Eq. (2.9), the evaporation of the GUP-modified temperature stops at $r_{+0} = 2\sqrt{\alpha}/\omega_p$ where $T_{0}^{\text{GUP}} = \omega_p/4\pi\sqrt{\alpha}$, leaving a remnant. Note that here we have used the left inequality in Eq. (2.8) for the momentum uncertainty since it recovers correctly the HUP limit in the semiclassical regime, and $\omega_p = \ell_p^{-1}$. And for the Schwarzschild black hole with $r_+ = 2GM$, the GUP-modified Hawking temperature is reduced to

$$T_{H}^{\text{GUP}} = \frac{M}{4\pi} \left( 1 - \sqrt{1 - \frac{\omega_p^2}{M^2}} \right)$$

(2.12)

with $G = \omega_p^{-2}$, which shows that the total Hawking evaporation of the black hole is prevented by the effect of the minimal length in the GUP.

Figure 1: Hawking temperatures using (a) the HUP, (b) the GUP, (c) the EUP, and (d) the GEUP.
2.2 EUP case

Second, the extended uncertainty principle (EUP) is given by

\[ \Delta p \geq \frac{\hbar}{\Delta x} + \frac{\hbar \beta^2}{\ell^2} \Delta x, \quad (2.13) \]

where \( \beta \) is a dimensionless constant and \( \ell \) the curvature radius of the AdS spacetime [31–33,47]. This EUP shows that the momentum uncertainty cannot be arbitrarily small but \( \Delta p \geq \Delta p_{\text{min}} = 2\hbar \beta^2 / \ell^2 \).

According to ACS [34] as before, the momentum uncertainty of order \( p \) can also be obtained from a position uncertainty of an emitted photon of order of \( \Delta x \sim r_+ \) as

\[ p = \Delta p \sim \frac{1}{r_+} \left( 1 + \frac{\beta^2}{\ell^2} r_+^2 \right). \quad (2.14) \]

This momentum uncertainty with the standard dispersion relation for a massless particle gives a EUP-corrected black hole temperature as

\[ T^{\text{EUP}} = \frac{1}{4\pi r_+} \left( 1 + \frac{\beta^2}{\ell^2} r_+^2 \right) \quad (2.15) \]

with the same calibration factor as before. The modified temperature obtained using the EUP was plotted in Fig. 1(c), which shows that it behaves as like the case of the Schwarzschild-AdS black hole; it is divergent not only as \( r_+ \to \infty \) but also as \( r_+ \to 0 \). A global minimum temperature, \( T^{\text{EUP}}_{\text{min}} = \beta/2\pi \ell \), is achieved at \( r_{+, \text{min}} = \ell/\beta \). In addition, for \( \beta^2 = 3 \), it is exactly the same with the Hawking temperature of the Schwarzschild-AdS black hole so that one sees that in the presence of the cosmological constant the EUP leads to the Hawking temperature for the Schwarzschild-AdS black hole. Note also that replacing \( \ell^2 \) with \(-\ell^2\) one can obtain the Hawking temperature of the Schwarzschild–dS black hole.

2.3 GEUP case

Finally, the generalized extended uncertainty principle (GEUP) [28–33] is described by

\[ \Delta x \Delta p \geq \hbar + \alpha \ell_p^2 \frac{(\Delta p)^2}{\hbar} + \frac{\hbar \beta^2}{\ell^2} (\Delta x)^2. \quad (2.16) \]
This gives us the momentum uncertainty of

\[
\frac{\Delta x}{2 \alpha \ell_p^2} \left[ 1 - \sqrt{1 - \frac{4 \alpha \ell_p^2}{(\Delta x)^2} \left( 1 + \frac{\beta^2}{\ell^2} (\Delta x)^2 \right)} \right] \leq \frac{\Delta p}{\hbar} \leq \frac{\Delta x}{2 \alpha \ell_p^2} \left[ 1 + \sqrt{1 - \frac{4 \alpha \ell_p^2}{(\Delta x)^2} \left( 1 + \frac{\beta^2}{\ell^2} (\Delta x)^2 \right)} \right].
\]

(2.17)

It also says that the position uncertainty is bounded from below by

\[
\Delta x \geq \Delta x_{\text{min}} \equiv \frac{2 \sqrt{\alpha \ell_p}}{\sqrt{1 - 4 \alpha \beta^2 \ell_p^2 / \ell^2}}.
\]

(2.18)

Now, according to ACS [34] as before, the momentum uncertainty of order \( p \) can be obtained from a position uncertainty of an emitted photon of order of \( \Delta x \sim r_+ \) as

\[
p = \Delta p \sim \frac{r_+}{2 \alpha \ell_p^2} \left[ 1 - \sqrt{1 - \frac{4 \alpha \ell_p^2}{r_+^2} \left( 1 + \frac{\beta^2}{\ell^2} r_+^2 \right)} \right].
\]

(2.19)

Plugging this momentum uncertainty into the standard dispersion relation for a massless particle, the energy uncertainty gives a GEUP-corrected black hole temperature as

\[
T_{\text{GEUP}} = \frac{r_+ \omega_p^2}{8 \pi \alpha} \left[ 1 - \sqrt{1 - \frac{4 \alpha \ell_p^2}{r_+^2} \left( 1 + \frac{\beta^2}{\ell^2} r_+^2 \right)} \right]
\]

(2.20)

with the calibration factor of 1/4\( \pi \). In Fig. 1(d), we have plotted the GEUP-corrected black hole temperature which shows a mixture of the results of the GUP-corrected and the EUP-corrected black hole temperatures; the evaporation of the GEUP-corrected black hole temperature stops at the minimum position uncertainty \( \Delta x_{\text{min}} = r_{+,0} \) as

\[
T^*_{\text{GEUP}} = \frac{\omega_p}{4 \pi \sqrt{\alpha}} \frac{1}{\sqrt{1 - 4 \alpha \beta^2 \ell_p^2 / \omega_p^2}}
\]

(2.21)

where

\[
r_{+,0} = \frac{2 \sqrt{\alpha}}{\omega_p} \frac{1}{\sqrt{1 - 4 \alpha \beta^2 \ell_p^2 / \omega_p^2}}.
\]

(2.22)
while the GEUP-corrected black hole temperature diverges as \( r_+ \to \infty \).

Note that in the GUP limit of \( \beta \to 0 \), the GEUP-corrected black hole temperature becomes

\[
T^{GEUP} \approx T^{GUP} + \beta \left( \frac{r_+}{4\pi \ell \sqrt{1 - 4\alpha/r_+^2 \omega_p^2}} \right) + O(\beta^2). \tag{2.23}
\]

On the other hand, in the EUP limit of \( \alpha \to 0 \), it also well recovers the EUP-corrected black hole temperature as

\[
T^{GEUP} \approx T^{EUP} + \alpha \left( \frac{(1 + \beta^2 r_+^2/\ell^2)^2}{4\pi r_+^2 \omega_p^2} \right) + O(\alpha^2). \tag{2.24}
\]

3 GUPs Effects in rainbow gravity

One of the drastic changes near the Planck scale one may think is the modification of the standard dispersion relation (2.3) which was suggested in rainbow gravity as

\[
\omega^2 f^2(\omega/\omega_p) - p^2 c^2 g^2(\omega/\omega_p) = m^2, \tag{3.1}
\]

where \( f(\omega/\omega_p) \), \( g(\omega/\omega_p) \) are called the rainbow functions which satisfy with the conditions of \( \lim_{\omega \to 0} f(\omega/\omega_p) = 1 \) and \( \lim_{\omega \to 0} g(\omega/\omega_p) = 1 \) at low energies. The MDR (3.1), which is invariant under nonlinear Lorentz transformation in the momentum space, can be realized in the dual position space as one parameter energy dependent family of metric [39] as

\[
d^2 = -\frac{F(r)}{f^2(\omega/\omega_p)} dt^2 + \frac{G(r)}{g^2(\omega/\omega_p)} dr^2 + \frac{r^2}{g^2(\omega/\omega_p)} d\Omega^2 \tag{3.2}
\]

for a spherically symmetric spacetime. One of the most interesting rainbow functions are described by

\[
f(\omega/\omega_p) = 1, \quad g(\omega/\omega_p) = \sqrt{1 - \eta (\omega/\omega_p)^n}, \tag{3.3}
\]

which forms have been studied much in Ref. [57, 58, 60–62, 64, 66, 67, 69–71, 73, 74, 77–79] including the quantum gravity phenomenology [80] and the \( \kappa \)-Minkowski noncommutative spacetime [81]. Note that the choice of \( f(\omega/\omega_p) = 1 \) in the rainbow functions makes a time-like Killing vector in the rainbow gravity as usual and so the local thermodynamic energy independent of a
test particle’s energy. Moreover, we have recently shown that a freely falling observer sees his local temperature dependent only on \( g(\omega/\omega_p) \), needless to think about \( f(\omega/\omega_p) \) \cite{61}. In the subsequent analysis, we will particularly choose \( n = 2 \) in the rainbow function of \( g(\omega/\omega_p) \) without loss of generality.

Now, having seen the efficient way of ACS with the standard dispersion relation in deriving Hawking temperatures in the previous section, we will generalize the method to incorporate the MDR (3.1) in the rainbow gravity to derive Hawking temperatures in this section.

Let us first briefly summarize the HUP result in the rainbow gravity. As like in the section 2, one can relate the momentum uncertainty of order \( p \) with the position uncertainty of an emitted photon of order of \( \Delta x \sim r_+ \) as in Eq. (2.2) using the HUP. Then, plugging this momentum uncertainty into the MDR for a massless particle, one can have

\[
E = pc \frac{g(\omega/\omega_p)}{f(\omega/\omega_p)} \sim \frac{\hbar c g(\omega/\omega_p)}{r_+ f(\omega/\omega_p)}.
\]

Therefore, one can directly obtain a black hole’s temperature in the rainbow gravity as

\[
\tilde{T}_{HUP} = \frac{1}{4\pi r_+} \frac{g(\omega/\omega_p)}{f(\omega/\omega_p)} = T_{HUP} \frac{g(\omega/\omega_p)}{f(\omega/\omega_p)}.
\]

with the calibration factor of \( 1/4\pi \). This shows the effect of the rainbow gravity permeates the black hole’s temperature through the rainbow functions.

When used the rainbow functions (3.3) with \( n = 2 \), one has the black hole’s temperature explicitly as

\[
\tilde{T}_{HUP} = \frac{1}{4\pi r_+} \sqrt{1 - \eta \left( \frac{\omega}{\omega_p} \right)^2} = T_{HUP} \sqrt{1 - \eta \left( \frac{\omega}{\omega_p} \right)^2}.
\]

Now, by making use of the HUP (2.2) and MDR (3.4), one can eliminate the energy \( \omega \) dependence and obtain the HUP-corrected black hole temperature.
Figure 2: Hawking temperatures in the rainbow gravity using (a) the HUP, (b) the GUP, (c) the EUP, and (d) the GEUP.

in the rainbow gravity as

$$\tilde{T}^{HUP} = \frac{1}{4\pi r_+} \frac{r_+ \omega_p}{\sqrt{r_+^2 \omega_p^2 + \eta}}, \quad (3.7)$$

which becomes the usual Hawking temperature as $\omega_p \to \infty$. In Fig. 2(a), we have plotted the HUP-corrected black hole temperature in the rainbow gravity. This extends the previous results [77, 78] where the modified temperature is obtained from the HUP and the standard dispersion relation to the one using the HUP and the MDR. As a result, by considering the rainbow gravity, one can see that the modified black hole temperature has no divergence at $r_+ = 0$, but finite, $\tilde{T}_0^{HUP} = \omega_p / 4\pi \sqrt{\eta}$, so that leaves a remnant.

3.1 GUP in rainbow gravity

Now, we can extend the previous analysis finding a black hole’s temperature to include the various GUPs.

First, by combining the momentum uncertainty (2.10) obtained from a position uncertainty of an emitted photon of order of $\Delta x \sim r_+$ in the GUP case
with the MDR (3.1) and the rainbow functions (3.3) for a massless particle, one can obtain the GUP-corrected black hole temperature as

$$\tilde{T}^{\text{GUP}} = T^{\text{GUP}} \frac{g(\omega/\omega_p)}{f(\omega/\omega_p)} = T^{\text{GUP}} \sqrt{1 - \eta \left(\frac{\omega}{\omega_p}\right)^2}. \quad (3.8)$$

Next, by making use of the momentum uncertainty (2.10) in the GUP case and MDR (3.4), one can easily find the ratio of

$$\frac{\omega^2}{\omega_p^2} = \frac{\frac{r_+^2 \omega_p^2}{2} \left(1 - \sqrt{1 - \frac{4\alpha}{r_+^2 \omega_p^2}}\right) - \alpha}{\alpha^2 + \eta \left(\frac{r_+^2 \omega_p^2}{2} \left(1 - \sqrt{1 - \frac{4\alpha}{r_+^2 \omega_p^2}}\right) - \alpha\right)}. \quad (3.9)$$

This can be used to eliminate the energy \( \omega \) dependence in the GUP-corrected black hole’s temperature (3.8) in the rainbow gravity, and one can finally have

$$\tilde{T}^{\text{GUP}} = T^{\text{GUP}} \left\{1 - \eta \left[\frac{\frac{r_+^2 \omega_p^2}{2} \left(1 - \sqrt{1 - \frac{4\alpha}{r_+^2 \omega_p^2}}\right) - \alpha}{\alpha^2 + \eta \left(\frac{r_+^2 \omega_p^2}{2} \left(1 - \sqrt{1 - \frac{4\alpha}{r_+^2 \omega_p^2}}\right) - \alpha\right)}\right]\right\}^{\frac{1}{2}}. \quad (3.10)$$

In Fig. 2(b), we have plotted the GUP-corrected black hole’s temperature in the rainbow gravity which as like the temperature \( T_0^{\text{GUP}} \) in Eq. (2.11) without considering the rainbow gravity, it stops to evaporate at \( r_{+,0} = \Delta x_{\text{min}} = 2\sqrt{\alpha/\omega_p} \) where \( T_0^{\text{GUP}} = \omega_p/4\pi \sqrt{\alpha + \eta} \), which is lower than \( T_0^{\text{GUP}} \). And as \( r_+ \to \infty \), the temperature \( \tilde{T}^{\text{GUP}} \) approaches the usual Hawking temperature of \( T^{\text{HUP}} \) disappearing the strong gravity effect. Moreover, when the rainbow effect is turned off, i.e., \( \eta \to 0 \), the modified temperature becomes the GUP-corrected black hole temperature (2.11), and when the GUP effect is turned off, i.e., \( \alpha \to 0 \), it becomes the HUP-corrected black hole temperature (3.7).

It is appropriate to comment that the GUP-corrected black hole’s temperature (3.10) is obtained from the consideration of both the MDR and the GUP along with the definition of the temperature in the rainbow gravity as like in Eq. (3.8). On the other hand, the similar result can be found in Ref. [78]
in the conclusion. However, the modified temperature discussed in there was obtained from the consideration of the MDR and the GUP, but not considered the rainbow effect in the temperature, which means that the authors have used the standard dispersion relation to define the temperature.

### 3.2 EUP in rainbow gravity

Second, by combining the momentum uncertainty (2.14) obtained from a position uncertainty of an emitted photon of order of $\Delta x \sim r_+$ in the EUP case with the MDR (3.4) and the rainbow functions (3.3) for a massless particle, one can obtain the EUP-corrected black hole temperature in the rainbow gravity as

$$\tilde{T}^{EUP} = T^{EUP} \frac{g(\omega / \omega_p)}{f(\omega / \omega_p)}$$

$$= T^{EUP} \sqrt{1 - \eta \left( \frac{\omega}{\omega_p} \right)^2}.$$  (3.11)

Moreover, one can determine the ratio of the energy $\omega^2 / \omega_p^2$ in the square root by making use of the momentum uncertainty (2.14) and the MDR (3.4) for the massless particle, and finally obtain the EUP-corrected black hole temperature in the rainbow gravity as

$$\tilde{T}^{EUP} = T^{EUP} \left\{ 1 - \eta \left[ \frac{B(\beta)^2}{r_+^2 \omega_p^2 + \eta B(\beta)^2} \right] \right\}^{\frac{1}{2}},$$  (3.12)

where $B(\beta) \equiv 1 + \frac{\beta^2 r_+^2}{1}$. We have plotted the EUP-corrected black hole temperature in the rainbow gravity in Fig. 2(c). Comparing with the temperature (2.15) without the rainbow effect, the EUP-corrected black hole temperature in the rainbow gravity is finite both at $r_+ = 0$ and as $r_+ \to \infty$, having and approaching $\tilde{T}^{EUP}_0 = \omega_p / 4\pi \sqrt{\eta}$. It also has a global minimum temperature, $\tilde{T}^{EUP}_m = \omega_p / 4\pi \sqrt{\eta + \ell^2 \omega_p^2 / 4\beta^2}$, at $r_+ = r_m = \ell / \beta$.

On the other hand, when the rainbow effect disappears with $\eta \to 0$, the modified temperature becomes the EUP-corrected black hole temperature (2.15), and when the EUP effect is turned off with $\beta \to 0$, it reproduces the HUP-corrected black hole temperature (3.7).
3.3 GEUP in rainbow gravity

Finally, by combining the momentum uncertainty (2.19) obtained from a position uncertainty of an emitted photon of order of $\Delta x \sim r_+$ in the GEUP case with the MDR (3.4) and the rainbow functions (3.3) for a massless particle, one can also easily obtain the GEUP-corrected black hole temperature in the rainbow gravity as

$$\tilde{T}_{\text{GEUP}} = T_{\text{GEUP}} \frac{g(\omega/\omega_p)}{f(\omega/\omega_p)^2}.$$ (3.13)

As like in the previous subsections, one can also find the energy ratio $\omega^2/\omega_p^2$ in the square root by making use of the momentum uncertainty (2.19) and the MDR (3.4) for the massless particle, and finally obtain the GEUP-corrected black hole temperature in the rainbow gravity as

$$\tilde{T}_{\text{GEUP}} = T_{\text{GEUP}} \left\{ 1 - \eta \left[ \frac{r^2 \omega_p^2}{2} \left( 1 - \sqrt{1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2}} \right) - \alpha B(\beta) \right] \right\}^{1/2}.$$ (3.14)

Note that as $\beta \to 0$ (i.e., $B(\beta) \to 1$), it becomes the GUP-corrected black hole temperature (3.10), while as $\alpha \to 0$ it goes to the EUP-corrected black hole temperature (3.12). The GEUP-corrected black hole temperature in the rainbow gravity is plotted in Fig. 2(d), which shows a kind of the mixed result of the GUP-corrected and the EUP-corrected black hole temperatures in Fig. 2(b) and Fig. 2(c).

As like in the GUP-corrected case, the GEUP-corrected black hole temperature in the rainbow gravity stops to evaporate at

$$\tilde{T}_{0,\text{GEUP}} = \frac{\omega_p}{4\pi \sqrt{\alpha}} \frac{1}{\sqrt{1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2} + \frac{\eta}{\alpha}}},$$ (3.15)

when

$$r_{+,0} = \frac{2\sqrt{\alpha}}{\omega_p} \frac{1}{\sqrt{1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2}}}.$$ (3.16)
Moreover, when $r_+ \to \infty$, it approaches the value of the EUP-corrected black hole temperature as

$$\tilde{T}_0^{GEUP} = \frac{\omega_p}{4\pi \sqrt{\eta}}.$$  \hfill (3.17)

It also has a local minimum at $r_+ = r_m$ as shown in Fig. 2(d), even though it is not easy to find an analytic solution. In this respect, one can easily understand that the GEUP-corrected black hole temperature shares the properties of the GUP-corrected and the EUP-corrected black hole temperature: in the rainbow gravity, the former shows that the black hole evaporation stops at some finite $r_+$, while the latter shows that the black hole temperature has a global minimum at some $r_+$ and approaches to a finite value as $r_+ \to 0$ and $r_+ \to \infty$.

4 Thermodynamic stability

4.1 Stability without rainbow gravity

Local thermodynamic stability usually requires the positivity condition on the specific heat. So in this subsection, let us first study it without the rainbow gravity, and in the next subsection, with the rainbow gravity.

To begin with, the specific heat can be simply written as

$$C = \frac{dE}{dT} = \frac{dr_+}{dT} \frac{dM}{dr_+} = \frac{\omega_p^2}{2} \left( \frac{dT}{dr_+} \right)^{-1},$$  \hfill (4.1)

where we have used the Schwarzschild black hole’s mass $M = r_+/2G = r_+ \omega_p^2/2$ for simplicity. Then, the known specific heat obtained from using the HUP is given by

$$C^{HUP} = -2\pi \omega_p^2 r_+^2,$$  \hfill (4.2)

whose figure is drawn in Fig. 3(a) showing that in the whole range of $r_+$ it is negative so that it is unstable thermodynamically.

On the other hand, the specific heat obtained from using the GUP can be obtained as

$$C^{GUP} = -\frac{4\pi \alpha}{1 - \sqrt{1 - \frac{4\alpha r_+^2 \omega_p^2}{r_+^2 \omega_p^2}}},$$  \hfill (4.3)
which is also unstable as shown in Fig. 3(b). Note that the specific heat is only defined with $r_{+,0} \geq \Delta x_{\text{min}} = 2\sqrt{\alpha / \omega_p}$.

Compared with these two, the specific heat obtained from using the EUP given by

$$C^{EUP} = -\frac{2\pi r_+^2 \omega_p^2}{1 - \frac{\beta^2 r_+^2}{\ell^2}}$$

(4.4)

is divided by two regions at $r_+ = \ell / \beta$; when $r_+ > \ell / \beta$, it is stable, while $r_+ < \ell / \beta$, it is unstable, which is plotted in Fig. 3(c). Interestingly, the thermodynamic behavior looks like the one of the Schwarzschild-AdS black hole. And as $r_+ \to \infty$, it goes to a constant value of $C^{EUP} \to 2\pi \omega_p^2 \ell^2 / \beta^2$.

Finally, the specific heat obtained from using the GEUP is given by

$$C^{GEUP} = -\frac{4\pi \alpha \sqrt{1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2}}}{1 - \frac{4\alpha \beta^2}{C^2 \omega_p^2} - \sqrt{1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2}}}.$$  

(4.5)

The figure 3(d) is depicted the GEUP-corrected specific heat which is similar to the EUP-corrected specific heat $C^{EUP}$ except it starts from $r_{+,0} = \Delta x_{\text{min}} = 2\sqrt{\alpha / \omega_p}$. 

Figure 3: Specific heats using (a) the HUP, (b) the GUP, (c) the EUP, and (d) the GEUP.
Now, let us study the thermodynamic stability by considering specific heats with the rainbow gravity.

First, the specific heat with the HUP in the rainbow gravity is described by
\[
\tilde{C}_{HUP} = -\left(\frac{2\pi(\omega_p^2 r_+^2 + \eta)^{3/2}}{r_+\omega_p}\right),
\]
which is plotted in Fig. 4(a) showing that it is unstable and at large \( r_+ \) it is reduced to the specific heat, \( C_{HUP} \), of the HUP case. Near \( r_+ = 0 \), it is a little bit different but still remains negative, so unstable.

The GUP-corrected specific heat with the rainbow gravity can be found as
\[
\tilde{C}_{GUP} = \frac{\pi}{\sqrt{2\alpha}} \left[ 2\alpha(\alpha - \eta) + \eta r_+^2 \omega_p^2 \left( 1 - \frac{1}{\sqrt{1 - \frac{4\alpha}{\omega_p^2}}} \right) \right]^{5/2},
\]
which figure is drawn in Fig. 4(b). It shows that it starts with \( r_{+,0} = \Delta x_{\text{min}} = 2\sqrt{\alpha}/\omega_p \) as like in \( C_{GUP} \) and as \( r_+ \) becomes large, it behaves as like \( C_{GUP} \).

On the other hand, the EUP-corrected specific heat can be written as
\[
\tilde{C}_{EUP} = -\frac{2\pi \omega_p^2}{\sqrt{1 - \frac{\beta^2}{\ell^2}}} \left[ 1 + \frac{2\beta^2}{\omega_p^2} + \frac{\eta \beta^2}{\ell \omega_p^2} + \frac{\eta}{\omega_p^2} \right]^{3/2},
\]
which has a blow-up point at \( r_+ = \ell/\beta \) as shown in Fig. 4(c). It is stable when \( r_+ > \ell/\beta \), but unstable when \( r_+ < \ell/\beta \). Comparing with \( C_{EUP} \), it also blows up as \( r_+ \to \infty \).

Finally, the GEUP-corrected specific heat is given by a little complicated form as
\[
\tilde{C}_{GEUP} = -\frac{A(r_+, \omega_p, \eta, \alpha, \beta)}{B(r_+, \omega_p, \eta, \alpha, \beta)},
\]

\( 15 \)
Figure 4: Specific heats in the rainbow gravity using (a) the HUP, (b) the GUP, (c) the EUP, and (d) the GEUP.

where

\[
A(r_+, \omega_p, \eta, \alpha, \beta) = \sqrt{2\pi} \left\{ \frac{1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2}}{2\alpha (\alpha - \eta B(\beta)) + \eta r_+^2 \omega_p^2 \left( 1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2} \right)} \right\}^{\frac{1}{2}} \\
\times \left\{ \frac{2\eta \alpha \beta^2 r_+^2}{\ell^2} - \left[ 2\alpha (\alpha - \eta) + \eta r_+^2 \omega_p^2 \left( 1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2} \right) \right] \right\}^3,
\]

\text{(4.10)}

\[
B(r_+, \omega_p, \eta, \alpha, \beta) = \alpha^2 \left\{ 2 \left[ \alpha (\alpha - \eta) + \eta r_+^2 \omega_p^2 \left( 1 - \frac{3\alpha \beta^2}{\ell^2 \omega_p^2} \right) \left( 1 - \frac{4\alpha B(\beta)}{r_+^2 \omega_p^2} \right) \right] \right\}^{\frac{1}{2}} \\
- 2\alpha \left[ \alpha \left( 1 - \frac{4\alpha \beta^2}{\ell^2 \omega_p^2} \right) + \eta \left( 3 - \frac{4\alpha \beta^2}{\ell^2 \omega_p^2} \right) \right] \\
- 2\eta \omega_p^2 r_+^2 \left( 1 - \frac{\alpha \beta^2}{\ell^2 \omega_p^2} \right) \left( 1 - \frac{4\alpha \beta^2}{r_+^2 \omega_p^2} \right),
\]

\text{(4.11)}

with \( B(\beta) = 1 + \frac{\beta^2}{\ell^2} r_+^2 \), which was drawn in Fig. 4(d). It has similar struc-
ture with the EUP-corrected specific heat $\tilde{C}^{EUP}$ except it starts from $r_{+0} = \Delta x_{\min} = 2\sqrt{\alpha}/(\omega_p\sqrt{1 - 4\alpha \beta^2 \ell^2/\omega_p})$ as before.

5 Discussion

In this paper, according to the method of Adler-Chen-Santiago, we have studied modified temperatures of black holes in the rainbow gravity. By fully considering both the MDR and the three different types of the GUPs including the extended uncertainty principle (EUP) and the generalized extended uncertainty principle (GEUP), we have eliminated the energy dependence of a particle in the modified black hole temperature in the rainbow gravity. As a result, we have shown that the rainbow gravity makes the black hole temperature finite at $r_+ = 0$ in the HUP-corrected and the EUP-corrected case, while in the GUP-corrected and the GEUP-corrected case the black hole temperatures are also finite not at $r_{+0} = 0$, but at $r_{+0} = 2\sqrt{\alpha}/\omega_p$ and at $r_{+0} = 2\sqrt{\alpha}/(\omega_p\sqrt{1 - 4\alpha \beta^2 \ell^2/\omega_p})$ where they stop the black hole evaporation.

We have also investigated their corresponding thermodynamic stabilities of the various GUPs-corrected modified Schwarzschild black holes in the rainbow gravity. The HUP-corrected and the GUP-corrected black holes are globally unstable. On the other hand, the EUP-corrected and the GEUP-corrected black holes are stable in a certain region of satisfying with $r_+ > r_m$, while they are unstable when $r_+ < r_m$.

Through a further investigation, it will be interesting to analyze thermodynamic phase transitions in the rainbow gravity by fully implementing the MDR and the various GUPs.

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