SHARP \((H_p, L_p)\) TYPE INEQUALITIES OF MAXIMAL OPERATORS OF \(T\) MEANS WITH RESPECT TO VILENKIN SYSTEMS WITH MONOTONE COEFFICIENTS

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Abstract. In this paper we prove and discuss some new \((H_p, L_p)\) type inequalities of maximal operators of \(T\) means with respect to the Vilenkin systems with monotone coefficients. We also apply these inequalities to prove strong convergence theorems of such \(T\) means. We also show that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

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1. Introduction

The definitions and notations used in this introduction can be found in our next Section.

It is well-known that Vilenkin systems do not form bases in the space \(L_1\). Moreover, there is a function in the Hardy space \(H_p\), such that the partial sums of \(f\) are not bounded in \(L_p\)-norm, for \(0 < p \leq 1\). Approximation properties of Vilenkin-Fourier series with respect to one- and two-dimensional case can be found in [17] and [32]. Simon [24] proved that there exists an absolute constant \(c_p\), depending only on \(p\), such that the inequality

\[
\frac{1}{\log^{|p|} n} \sum_{k=1}^{n} \frac{\|S_kf\|^p}{k^{2-p}} \leq c_p \|f\|^p_{H_p} \quad (0 < p \leq 1)
\]

holds for all \(f \in H_p\) and \(n \in \mathbb{N}_+\), where \(|p|\) denotes the integer part of \(p\). For \(p = 1\) analogous results with respect to more general systems were proved in Blahota [2] and Gát [4] and for \(0 < p < 1\) simpler proof was given in Tephnadze [31]. Some new strong convergence result for partial sums with respect to Vilenkin system was considered in Tutberidze [33].

In the one-dimensional case the weak \((1,1)\)-type inequality for the maximal operator of Fejér means \(\sigma^nf := \sup_{n\in\mathbb{N}} |\sigma_nf|\) can be found in Schipp [21] for Walsh series and in Pál, Simon [15] for bounded Vilenkin series. Fujii [8] and Simon [23] verified that \(\sigma^*\) is bounded from \(H_1\) to \(L_1\). Weisz [38] generalized this result and proved boundedness of \(\sigma^*\) from the martingale space \(H_p\) to the space \(L_p\), for \(p > 1/2\). Simon [22] gave a counterexample, which shows that boundedness does not hold for \(0 < p < 1/2\). A counterexample for \(p = 1/2\) was given by Goginava [6] (see also Tephnadze [25]). Moreover, Weisz [40] proved that the

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maximal operator of the Fejér means \( \sigma^* \) is bounded from the Hardy space \( H_{1/2} \) to the space \( \text{weak} - L_{1/2} \). In [26] and [27] the following result was proved:

**Theorem T1:** Let \( 0 < p \leq 1/2 \). Then the following weighted maximal operator of Fejér means

\[
\sigma^*_p f := \sup_{n \in \mathbb{N}_+} \frac{|\sigma_n f|}{(n + 1)^{1/p-2} \log^{2[1/2+p]}(n + 1)}
\]

is bounded from the martingale Hardy space \( H_p \) to the Lebesgue space \( L_p \).

Moreover, the rate of the weights \( \left\{ 1/ (n + 1)^{1/p-2} \log^{2[p+1/2]}(n + 1) \right\}_{n=1}^{\infty} \) in \( n \)-th Fejér mean was given exactly.

Similar results with respect to Walsh-Kaczmarz systems were considered in [7] for \( p = 1/2 \) and in [28] for \( 0 < p < 1/2 \). Approximation properties of Fejér means with respect to Vilenkin and Kaczmarz systems can be found in Tephnadze [29], Tutberidze [34], Persson, Tephnadze and Tutberidze [19].

In [3] it was proved that there exists an absolute constant \( c_p \), depending only on \( p \), such that the inequality

\[
\frac{1}{\log^{1/2+p} n} \sum_{k=1}^{n} \left\| \sigma_k f \right\|_p^p \leq c_p \left\| f \right\|_p^p \quad (0 < p \leq 1/2, \quad n = 2, 3, \ldots).
\]

holds. Some new strong convergence result for Vilenkin-Fejér means was considered [20].

Móricz and Siddiqi [11] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of \( L_p \) function in norm. In the two-dimensional case approximation properties of Nörlund means was considered by Nagy [12, 13, 14]. In [16] it was proved that the maximal operators of Nörlund means \( T^* f := \sup_{n \in \mathbb{N}} |T_n f| \) either with non-decreasing coefficients, or non-increasing coefficients, satisfying condition

\[
\frac{1}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty
\]

are bounded from the Hardy space \( H_{1/2} \) to the space \( \text{weak} - L_{1/2} \) and are not bounded from the Hardy space \( H_p \) to the space \( L_p \), when \( 0 < p \leq 1/2 \).

In [18] it was proved that for \( 0 < p < 1/2, f \in H_p \) and non-decreasing sequence \( \{q_k : k \geq 0\} \) there exists an absolute constant \( c_p \), depending only on \( p \), such that the inequality holds

\[
\sum_{k=1}^{\infty} \left\| t_k f \right\|_p^p \leq c_p \left\| f \right\|_p^p
\]

Moreover, if \( f \in H_{1/2} \) and \( \{q_k : k \geq 0\} \) be a sequence of non-decreasing numbers, satisfying the condition

\[
\frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty,
\]

then there exists an absolute constant \( c \), such that the inequality holds

\[
\frac{1}{\log n} \sum_{k=1}^{n} \left\| t_k f \right\|_{1/2}^{1/2} \leq c \left\| f \right\|_{H_{1/2}}^{1/2}
\]

In [35] was proved that the maximal operators \( T^* f := \sup_{n \in \mathbb{N}} |T_n f| \) of \( T \) means either with non-increasing coefficients, or non-decreasing sequence satisfying condition [3] are bounded.
from the Hardy space $H_{1/2}$ to the space weak $- L_{1/2}$. Moreover, there exists a martingale and such $T$ means for which boundedness from the Hardy space $H_p$ to the space $L_p$ do not hold when $0 < p \leq 1/2$.

One of the most well-known mean of $T$ means is Riesz summability. In [30] it was proved that the maximal operator $R^*$ of Riesz means is bounded from the Hardy space $H_{1/2}$ to the space weak $- L_{1/2}$ and is not bounded from $H_p$ to the space $L_p$, for $0 < p \leq 1/2$. There also was proved that Riesz summability has better properties than Fejér means. In particular, the following weighted maximal operators

$$\frac{\log n |R_n f|}{(n + 1)^{1/p - 2} \log^{2(1/2 + p)} (n + 1)}$$

are bounded from $H_p$ to the space $L_p$, for $0 < p \leq 1/2$ and the rate of weights are sharp. Moreover, in [9] was also proved that if $0 < p < 1/2$ and $f \in H_p(G_m)$, then there exists an absolute constant $c_p$, depending only on $p$, such that the inequality holds:

$$(4) \sum_{n=1}^{\infty} \frac{\log^p n \|R_n f\|_{H_p}^p}{n^{2 - 2p}} \leq c_p \|f\|_{H_p}^p$$

If we compare strong convergence results, given by (1) and (4), we obtain that Riesz means has better properties than Fejér means, for $0 < p < 1/2$, but in the case $p = 1/2$ is was not possible to show even similar result for Riesz means as it is proved for Fejér means given by inequality (1).

In this paper we prove and discuss some new $(H_p, L_p)$ type inequalities of maximal operators of $T$ means with respect to the Vilenkin systems with monotone coefficients. Moreover, we apply these inequalities to prove strong convergence theorems of such $T$ means. In particular, we also study strong convergence theorems of $T$ means with non-increasing sequences in the case $p = 1/2$, but under the condition (2). For example, this condition is fulfilled for Fejér means but does not hold for Riesz means. We also show that these inequalities are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary Lemmas, some of them are new and of independent interest. These results are presented in Section 4. The detailed proofs are given in Section 5.

2. Definitions and Notation

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, ...)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, ..., m_k - 1\}$ the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the groups $Z_{m_i}$ with the product of the discrete topologies of $Z_{m_j}$’s.

The direct product $\mu$ of the measures $\mu_k (\{j\}) := 1/m_k$ ($j \in Z_{m_k}$) is the Haar measure on $G_m$ with $\mu (G_m) = 1$. 
In this paper we discuss bounded Vilenkin groups, i.e. the case when sup \( m_n < \infty \).

The elements of \( G_m \) are represented by sequences
\[
x := (x_0, x_1, ..., x_j, ...) \ , \ (x_j \in \mathbb{Z}_{m_j}).
\]

Set \( e_n := (0, ..., 0, 1, 0, ...) \in G \), the \( n \)-th coordinate of which is 1 and the rest are zeros \( (n \in \mathbb{N}) \). It is easy to give a basis for the neighborhoods of \( G_m \):
\[
I_0 (x) := G_m, \ I_n (x) := \{ y \in G_m \mid y_0 = x_0, ..., y_{n-1} = x_{n-1} \},
\]
where \( x \in G_m, n \in \mathbb{N} \).

If we define \( I_n := I_n (0) \), for \( n \in \mathbb{N} \) and \( \overline{I}_n := G_m \setminus I_n \), then
\[
(5) \quad \overline{I}_N = \left( \bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_{k,l}^N \right) \bigcup \left( \bigcup_{k=1}^{N-1} I_{k,N}^N \right),
\]
where
\[
I_{k,l}^N := \left\{ \begin{array}{ll}
I_N (0, ..., 0, x_k \neq 0, 0, ..., 0, x_{l+1} \neq 0, x_{l+1}, ..., x_{N-1}, ...) \quad \text{for } k < l < N, \\
I_N (0, ..., 0, x_k \neq 0, 0, ..., 0, x_{N-1} = 0, x_N, ...) \quad \text{for } l = N.
\end{array} \right.
\]

If we define the so-called generalized number system based on \( m \) in the following way:
\[
M_0 := 1, \ M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),
\]
then every \( n \in \mathbb{N} \) can be uniquely expressed as \( n = \sum_{j=0}^{\infty} m_j \) where \( n_j \in \mathbb{Z}_{m_j} \) \( (j \in \mathbb{N}_+) \) and only a finite number of \( n_j \)'s differ from zero.

We introduce on \( G_m \) an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function \( r_k (x) : G_m \to \mathbb{C} \), the generalized Rademacher functions, by
\[
r_k (x) := \exp (2\pi i x_k / m_k), \quad (i^2 = -1, x \in G_m, \ k \in \mathbb{N}).
\]

Next, we define the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) by:
\[
\psi_n (x) := \prod_{k=0}^{\infty} r_k^n (x), \quad (n \in \mathbb{N}).
\]

Specifically, we call this system the Walsh-Paley system when \( m \equiv 2 \).

The norms (or quasi-norms) of the spaces \( L_p (G_m) \) and \( \text{weak} - L_p (G_m) \) \( (0 < p < \infty) \) are respectively defined by
\[
\| f \|_p := \int_{G_m} |f|^p \, d\mu, \quad \| f \|_{\text{weak} - L_p} := \sup_{\lambda > 0} \int_{G_m} f(x) \, d\mu (f > \lambda) < +\infty.
\]

The Vilenkin system is orthonormal and complete in \( L_2 (G_m) \) (see [36]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If \( f \in L_1 (G_m) \) we can define Fourier coefficients, partial sums and Dirichlet kernels with respect to the Vilenkin system in the usual manner:
\[
\hat{f} (n) := \int_{G_m} f \overline{\psi}_n \, d\mu, \quad S_n f := \sum_{k=0}^{n-1} \hat{f} (k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).
\]
It is well known that if $n \in \mathbb{N}$, then

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases}$$

Moreover, if $n = \sum_{i=0}^{\infty} n_i M_i$, and $1 \leq s_n \leq m_n - 1$, then we have the following identity:

$$D_n = \psi_n \left( \sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j-n_j}^{m_j-1} r_j^k \right),$$

The $\sigma$-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $F_n(n \in \mathbb{N})$. Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to $F_n(n \in \mathbb{N})$. (for details see e.g. [37]). The maximal function of a martingale $f$ is defined by $f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|$. For $0 < p < \infty$ the Hardy martingale spaces $H_p$ consist of all martingales $f$ for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

A bounded measurable function $a$ is called a $p$-atom, if there exists an interval $I$, such that

$$\int_I a d\mu = 0, \quad \|a\|_{\infty} \leq \mu(I)^{-1/p}, \quad \text{supp} (a) \subset I.$$

If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \overline{\psi_i} d\mu.$$

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. The $n$-th $T$ means for a Fourier series of $f$ are respectively defined by

$$T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f,$$

where $Q_n := \sum_{k=0}^{n-1} q_k$.

It is obvious that $T_n f(x) = \int_{G_m} f(t) F_n(x-t) d\mu(t)$, where $F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_k D_k$ is called the $T$ kernel.

We always assume that $\{q_k : k \geq 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then the summability method (8) generated by $\{q_k : k \geq 0\}$ is regular if and only if $\lim_{n \to \infty} Q_n = \infty$.

If we invoke Abel transformation we get the following identities, which are very important for the investigations of $T$ summability:

$$Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=0}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n - 1),$$

$$F_n = \frac{1}{Q_n} \left( \sum_{j=0}^{n-2} (q_j - q_{j+1}) j K_j + q_{n-1}(n - 1) K_n \right).$$
\[(11) \quad T_n f = \frac{1}{Q_n} \left( \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \sigma_j f + q_{n-1}(n-1)\sigma_{n-1} f \right). \]

Let \( \{q_k : k \geq 0\} \) be a sequence of nonnegative numbers. The \( n \)-th Nörlund mean \( t_n \) for a Fourier series of \( f \) is defined by
\[(12) \quad t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,\]
where \( Q_n := \sum_{k=0}^{n-1} q_k \).

If \( q_k \equiv 1 \), we respectively define the Fejér means \( \sigma_n \) and Kernels \( K_n \) as follows:
\[ \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k. \]

It is well-known that (for details see [1])
\[(13) \quad n \left| K_n \right| \leq c \sum_{l=0}^{\lfloor n \rfloor} M_l \left| K_{M_l} \right| \]
and
\[(14) \quad \|K_n\|_1 \leq c < \infty. \]

The well-known example of Nörlund summability is the so-called \((C, \alpha)\)-mean (Cesàro means) for \( 0 < \alpha < 1 \), which are defined by
\[ \sigma_\alpha^n f := \frac{1}{A_\alpha^n} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f, \]
where
\[ A_\alpha^0 := 0, \quad A_\alpha^n := \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}. \]

We also consider the "inverse" \((C, \alpha)\)-means, which is an example of a \( T \)-means:
\[ U_\alpha^n f := \frac{1}{A_\alpha^n} \sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \quad 0 < \alpha < 1. \]

Let \( V_\alpha^n \) denote the \( T \) mean, where \( \{q_0 = 0, \ q_k = k^{\alpha-1} : k \in \mathbb{N}_+\} \), that is
\[ V_\alpha^n f := \frac{1}{Q_n} \sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \quad 0 < \alpha < 1. \]

The \( n \)-th Riesz logarithmic mean \( R_n \) and the Nörlund logarithmic mean \( L_n \) are defined by
\[ R_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{k} \quad \text{and} \quad L_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}, \]
respectively, where \( l_n := \sum_{k=1}^{n} 1/k \).
Up to now we have considered $T$ means in the case when the sequence $\{q_k : k \in \mathbb{N}\}$ is bounded but now we consider $T$ summabilities with unbounded sequence $\{q_k : k \in \mathbb{N}\}$.

Let $\alpha \in \mathbb{R}_+, \beta \in \mathbb{N}_+$ and $\log^{(\beta)}(x) := \log \ldots \log x$. If we define the sequence $\{q_k : k \in \mathbb{N}\}$ by $\{q_0 = 0, q_k = \log^{(\beta)}(k^\alpha) : k \in \mathbb{N}_+\}$, then we get the class of $T$ means with non-decreasing coefficients:

$$B_n^{\alpha,\beta} f := \frac{1}{Q_n} \sum_{k=1}^{n-1} \log^{(\beta)}(k^\alpha) S_k f.$$ 

We note that $B_n^{\alpha,\beta}$ are well-defined for every $n \in \mathbb{N}$

$$B_n^{\alpha,\beta} f = \sum_{k=1}^{n-1} \frac{\log^{(\beta)}(k^\alpha)}{Q_n} S_k f.$$ 

It is obvious that $\frac{n}{2} \log^{(\beta)}(\frac{n^\alpha}{2}) \leq Q_n \leq n \log^{(\beta)}(n^\alpha)$. It follows that

$$\frac{q_{n-1}}{Q_n} \leq \frac{c \log^{(\beta)}(n^\alpha)}{n \log^{(\beta)}(n^\alpha)} = O\left(\frac{1}{n}\right) \to 0, \text{ as } n \to \infty.$$ 

We also define the maximal operator of $T$ and Nörlund means by

$$T^* f := \sup_{n \in \mathbb{N}} |T_n f|, \quad t^* f := \sup_{n \in \mathbb{N}} |t_n f|.$$ 

Some well-known examples of maximal operators of $T$ means are the maximal operator of Fejér $\sigma^*$ and Riesz $R^*$ logarithmic means, which are defined by:

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|, \quad R^* f := \sup_{n \in \mathbb{N}} |R_n f|.$$ 

3. THE MAIN RESULTS AND APPLICATIONS

Our first main result reads:

**Theorem 1.** Let $0 < p \leq 1/2$, $f \in H_p$ and $\{q_k : k \geq 0\}$ be a sequence of non-increasing numbers. Then the maximal operator

$$\tilde{T}_p^* f := \sup_{n \in \mathbb{N}_+} \frac{|T_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space $H_p$ to the space $L_p$.

**Corollary 1.** Let $0 < p \leq 1/2$ and $f \in H_p$. Then the maximal operator

$$\tilde{R}_p^* f := \sup_{n \in \mathbb{N}_+} \frac{|R_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space $H_p$ to the space $L_p$.

**Corollary 2.** Let $0 < p \leq 1/2$ and $f \in H_p$. Then the maximal operator

$$\tilde{U}_p^{\alpha,*} f := \sup_{n \in \mathbb{N}_+} \frac{|U_n^\alpha f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space $H_p$ to the space $L_p$. 
Corollary 3. Let $0 < p \leq 1/2$ and $f \in H_p$. Then the maximal operator
\[
\tilde{V}^{\alpha, \ast} f := \sup_{n \in \mathbb{N}^+} \frac{|V_n^\alpha f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}
\]
is bounded from the Hardy space $H_p$ to the space $L_p$.

Next, we consider maximal operators of $T$ means with non-decreasing sequence:

Theorem 2. Let $0 < p \leq 1/2$, $f \in H_p$ and \{q_k : k \geq 0\} be a sequence of non-decreasing numbers, satisfying the condition
\[
\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty.
\]

Then the maximal operator
\[
\tilde{T}^*_p f := \sup_{n \in \mathbb{N}^+} \frac{|T_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}
\]
is bounded from the martingale Hardy space $H_p$ to the space $L_p$.

Corollary 4. Let $0 < p \leq 1/2$, $f \in H_p$ and \{q_k : k \geq 0\} be a sequence of non-decreasing numbers, such that
\[
\sup_{n \in \mathbb{N}} q_n < c < \infty.
\]

Then
\[
\frac{q_{n-1}}{Q_n} \leq \frac{c}{Q_n} \leq \frac{c}{q_0} = \frac{c_1}{n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to 0,
\]
and weighted maximal operators of such $T$ means, given by (18) are bounded from the Hardy space $H_p$ to the space $L_p$.

Corollary 5. Let $0 < p \leq 1/2$ and $f \in H_p$. Then the maximal operator
\[
\tilde{T}^*_p f := \sup_{n \in \mathbb{N}^+} \frac{|B_n^{\alpha, \beta} f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}
\]
is bounded from the martingale Hardy space $H_p$ to the space $L_p$.

Remark 1. According to Theorem T1 we obtain that weights in (16) and (18) are sharp.

Theorem 3. a) Let $0 < p < 1/2$, $f \in H_p$ and \{q_k : k \geq 0\} be a sequence of non-increasing numbers. Then there exists an absolute constant $c_p$, depending only on $p$, such that the inequality holds:
\[
\sum_{k=1}^{\infty} \frac{\|T_k f\|^p}{k^{2-2p}} \leq c_p \|f\|^p_{H_p}
\]

b) Let $f \in H_{1/2}$ and \{q_k : k \geq 0\} be a sequence of non-increasing numbers, satisfying the condition
\[
\frac{1}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty.
\]

Then there exists an absolute constant $c$, such that the inequality holds:
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|T_k f\|_{1/2}}{k} \leq c \|f\|_{H_{1/2}}^{1/2}
\]
Corollary 6. Let $0 < p \leq 1/2$ and $f \in H_p$. Then there exists absolute constant $c_p$, depending only on $p$, such that the following inequality holds:

$$\frac{1}{\log^{1/2+p}} \sum_{k=1}^{n} \left\| \sigma_k f \right\|_p^p \leq c_p \left\| f \right\|_{H_p}^p.$$  

Corollary 7. Let $0 < p \leq 1/2$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that the following inequalities hold:

$$\sum_{k=1}^{\infty} \left\| U_k f \right\|_{k^2-2p}^p \leq c_p \left\| f \right\|_{H_p}^p,$$

$$\sum_{k=1}^{\infty} \left\| V_k f \right\|_{k^2-2p}^p \leq c_p \left\| f \right\|_{H_p}^p,$$

$$\sum_{k=1}^{\infty} \left\| R_k f \right\|_{k^2-2p}^p \leq c_p \left\| f \right\|_{H_p}^p.$$  

Theorem 4. a) Let $0 < p < 1/2$, $f \in H_p$ and $\{q_k : k \geq 0\}$ be a sequence of non-decreasing numbers. Then there exists an absolute constant $c_p$, depending only on $p$, such that the inequality holds:

$$\sum_{k=1}^{\infty} \left\| T_k f \right\|_{k^2-2p}^p \leq c_p \left\| f \right\|_{H_p}^p.$$  

b) Let $f \in H_{1/2}$ and $\{q_k : k \geq 0\}$ be a sequence of non-increasing numbers, satisfying the condition (17). Then there exists an absolute constant $c$, such that the inequality holds:

$$\frac{1}{\log n} \sum_{k=1}^{n} \left\| T_k f \right\|_{1/2}^{1/2} \leq c \left\| f \right\|_{H_{1/2}}^{1/2}.$$  

Corollary 8. Let $0 < p \leq 1/2$, $f \in H_p$ and $\{q_k : k \geq 0\}$ be a sequence of non-decreasing numbers, such that $\sup_{n \in \mathbb{N}} q_n < c < \infty$. Then condition (17) is satisfied and for all such $T$ means there exists an absolute constant $c$, such that the inequality (22) holds.

We have already considered the case when the sequence $\{q_k : k \geq 0\}$ is bounded. Now, we consider some Nörlund means, which are generated by an unbounded sequence $\{q_k : k \geq 0\}$.

Corollary 9. Let $0 < p \leq 1/2$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that the following inequality holds:

$$\frac{1}{\log^{1/2+p}} \sum_{k=1}^{n} \left\| B_k^{\alpha, \beta} f \right\|_{k^2-2p}^p \leq c_p \left\| f \right\|_{H_p}^p.$$  

4. Auxiliary Lemmas

We need the following auxiliary Lemmas:

Lemma 1 (see e.g. [39]). A martingale $f = (f(n), n \in \mathbb{N})$ is in $H_p (0 < p \leq 1)$ if and only if there exists a sequence $(\alpha_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{\alpha_k} a_k = f(n), \quad a.e., \quad \text{where} \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$  

Moreover,

$$\left\| f \right\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.$$
where the infimum is taken over all decompositions of \( f \) of the form (23).

**Lemma 2** (see e.g. [39]). Suppose that an operator \( T \) is \( \sigma \)-sublinear and for some \( 0 < p \leq 1 \)
\[
\int I |Ta|^p d\mu \leq c_p < \infty,
\]
for every \( p \)-atom \( a \), where \( I \) denotes the support of the atom. If \( T \) is bounded from \( L_{\infty} \) to \( L_{\infty} \), then
\[
\|Tf\|_p \leq c_p \|f\|_{H_p}, \quad 0 < p \leq 1.
\]

**Lemma 3** (see [5]). Let \( n > t, t, n \in \mathbb{N} \). Then
\[
K_{M_n}(x) = \begin{cases} 
\frac{M_n}{M_{n-1}}, & x \in I_t \setminus I_{t+1}, \ x - x_t e_t \in I_n, \\
0, & \text{otherwise}.
\end{cases}
\]

For the proof of our main results we also need the following new Lemmas of independent interest:

**Lemma 4.** Let \( n \in \mathbb{N} \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence either of non-increasing numbers, or non-decreasing numbers satisfying condition (17). Then
(24) \( \|F_n\|_1 < c < \infty \).

**Proof:** Let \( n \in \mathbb{N} \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. By combining (9) and (11) with (14) we can conclude that
\[
\|T_n\|_1 \leq \frac{1}{Q_n} \left( \sum_{j=0}^{n-2} |q_j - q_{j+1}| j \|\sigma_j\|_1 + q_{n-1}(n-1)\|\sigma_{n-1}\|_1 \right)
\leq \frac{c}{Q_n} \left( \sum_{j=0}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1) \right) \leq c < \infty.
\]

Let \( n \in \mathbb{N} \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence non-decreasing sequence satisfying condition (17). Then By using again (9) and (11) with (14) we find that
\[
\|T_n\|_1 \leq \frac{1}{Q_n} \left( \sum_{j=0}^{n-2} |q_j - q_{j+1}| j \|\sigma_j\|_1 + q_{n-1}(n-1)\|\sigma_{n-1}\|_1 \right)
\leq \frac{c}{Q_n} \left( \sum_{j=0}^{n-2} (q_{j+1} - q_j) j + q_{n-1}(n-1) \right)
= \frac{c}{Q_n} \left( 2q_{n-1}(n-1) - \sum_{j=0}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1) \right)
= \frac{c}{Q_n} (2q_{n-1}(n-1) - Q_n) \leq c < \infty.
\]

The proof is complete.
Lemma 5. Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers and \( n > M_N \). Then

\[
\left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x) \right| \leq \frac{c}{M_N} \left\{ \sum_{j=0}^{\lfloor n/2 \rfloor} M_j |K_M| \right\},
\]

Proof. Since sequence is non-increasing number we get that

\[
\frac{1}{Q_n} \left( q_{M_N} + \sum_{j=M_N}^{n-2} |q_j - q_{j+1}| + q_{n-1} \right) \leq \frac{1}{Q_n} \left( q_{M_N} + \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) + q_{n-1} \right) \leq \frac{2q_{M_N}}{Q_n} \leq \frac{2q_{M_N}}{Q_{M_N+1}} \leq \frac{c}{M_N}.
\]

If we apply Abel transformation and \([13]\) we immediately get that

\[
\left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x) \right| = \frac{1}{Q_n} \left( q_{M_n} K_{M_n-1} + \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) K_j + q_{n-1} K_{n-1} \right) \leq \frac{1}{Q_n} \left( q_{M_n} + \sum_{j=M_N}^{n-2} |q_j - q_{j+1}| + q_{n-1} \right) \sum_{i=0}^{\lfloor n/2 \rfloor} M_i |K_M| \leq \frac{c}{M_N} \sum_{i=0}^{\lfloor n/2 \rfloor} M_i |K_M|.
\]

The proof is complete. \(\square\)

Lemma 6. Let \( x \in I_{N}^{k,l} \), \( k = 0, \ldots, N-1 \), \( l = k+1, \ldots, N \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then there exists an absolute constant, such that

\[
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x - t) \right| d\mu (t) \leq \frac{c M_k M_l}{M_N^2}.
\]

Proof: Let \( x \in I_{N}^{k,l} \), for \( 0 \leq k < l \leq N-1 \) and \( t \in I_N \). First, we observe that \( x - t \in I_{N}^{k,l} \). Next, we apply Lemmas \([3]\) and \([5]\) to obtain that

\[
(25) \qquad \int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x - t) \right| d\mu (t) \leq \frac{c M_k M_l}{M_N^2}.
\]
and the first estimate is proved.

Now, let $x \in I_{N}^{k,N}$. Since $x - t \in I_{N}^{k,N}$ for $t \in I_{N}$, by combining (6) and (7) we have that

$$|D_{i}(x - t)| \leq M_{k}$$

and

$$\int_{I_{N}} \left| \frac{1}{Q_{n}} \sum_{j=M_{N}}^{n-1} q_{j} D_{j}(x - t) \right| d\mu(t) \leq \frac{c}{Q_{n}} \sum_{i=0}^{[n]} q_{i} \int_{I_{N}} |D_{i}(x - t)| d\mu(t) \leq \frac{c}{Q_{n}} \sum_{i=0}^{[n]-1} M_{k} d\mu(t) \leq \frac{cM_{k}}{M_{N}}.$$  

According to (25) and (26) the proof is complete.

**Lemma 7.** Let $n > M_{N}$ and $\{q_{k} : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers, satisfying condition (20). Then

$$\left| \frac{1}{Q_{n}} \sum_{j=M_{N}}^{n-1} q_{j} D_{j} \right| \leq \frac{c}{n} \left\{ \sum_{j=0}^{[n]} M_{j} |K_{M_{j}}| \right\},$$

where $c$ is an absolute constant.

**Proof.** Since sequence is non-increasing number satisfying condition (20), we get that

$$\frac{1}{Q_{n}} \left( q_{M_{n}} + \sum_{j=M_{N}}^{n-2} |q_{j} - q_{j+1}| + q_{n-1} \right) \leq \frac{2q_{M_N}}{Q_{n}} \leq \frac{c}{n} \leq \frac{c}{n}.$$ 

If we apply (13) we immediately get that

$$\left| \frac{1}{Q_{n}} \sum_{j=M_{N}}^{n-1} q_{j} D_{j} \right| \leq \left( \frac{1}{Q_{n}} \left( q_{M_{n}} + \sum_{j=M_{N}+1}^{n-2} |q_{j} - q_{j+1}| + q_{n-1} \right) \right) \sum_{i=0}^{[n]} M_{i} |K_{M_{i}}| \leq \frac{c}{n} \sum_{i=0}^{[n]} M_{i} |K_{M_{i}}|.$$ 

The proof is complete. □
Lemma 8. Let \( x \in I_N^{k,l} \), \( k = 0, \ldots, N-2 \), \( l = k + 1, \ldots, N-1 \) and \( \{ q_k : k \in \mathbb{N} \} \) be a sequence of non-increasing numbers, satisfying condition (20). Then

\[
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M}^{n-1} q_j D_j (x - t) \right| d\mu(t) \leq \frac{cM_l M_k}{nM_N}.
\]

Let \( x \in I_N^{k,N} \), \( k = 0, \ldots, N-1 \). Then

\[
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M}^{n-1} q_j D_j (x - t) \right| d\mu(t) \leq \frac{cM_k}{M_N}.
\]

Here \( c \) is an absolute constant.

**Proof**: Let \( x \in I_N^{k,l} \), for \( 0 \leq k < l \leq N-1 \) and \( t \in I_N \). First, we observe that \( x - t \in I_N^{k,l} \).

Next, we apply Lemmas 3 and 7 to obtain that

\[
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M}^{n-1} q_j D_j (x - t) \right| d\mu(t) \leq \frac{c}{n} |n| \sum_{i=0}^{M} M_i \int_{I_N} |K_{M_i} (x - t)| d\mu(t)
\]

\[
\leq \frac{c}{n} \sum_{i=0}^{M} M_i M_k d\mu(t) \leq \frac{cM_k M_l}{nM_N}
\]

and the first estimate is proved.

Now, let \( x \in I_N^{k,N} \). Since \( x - t \in I_N^{k,N} \) for \( t \in I_N \), by combining again Lemmas 3 and 7 we have that

\[
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M}^{n-1} q_j D_j (x - t) \right| d\mu(t) \leq \frac{c}{n} |n| \sum_{i=0}^{M} M_i \int_{I_N} |K_{M_i} (x - t)| d\mu(t)
\]

\[
\leq \frac{c}{n} \sum_{i=0}^{M} M_i M_k d\mu(t) \leq \frac{cM_k}{M_N}.
\]

By combining (27) and (28) we complete the proof.

Lemma 9. Let \( n \geq M_N \), \( x \in I_N^{k,l} \), \( k = 0, \ldots, N-1 \), \( l = k + 1, \ldots, N \) and \( \{ q_k : k \in \mathbb{N} \} \) be a sequence of non-increasing sequence, satisfying condition (20). Then

\[
\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M}^{n-1} q_j D_j (x - t) \right| d\mu(t) \leq \frac{cM_l M_k}{M_N^2},
\]

where \( c \) is an absolute constant.

**Proof**: Since \( n \geq M_N \) if we apply Lemma 8 we immediately get the proof.
Lemma 10. Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers satisfying (17). Then
\[
|F_n| \leq \frac{c}{n} \left\{ \sum_{j=0}^{n} M_j |K_{M_j}| \right\},
\]
where \( c \) is an absolute constant.

Proof. Since sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing if we apply condition (17) we can conclude that
\[
\frac{1}{Q_n} \left( \sum_{j=0}^{n-2} |q_j - q_{j+1}| + q_{n-1} \right) \leq \frac{1}{Q_n} \left( \sum_{j=0}^{n-2} (q_{j+1} - q_j) + q_{n-1} \right) \leq \frac{2q_{n-1} - q_0}{Q_n} \leq \frac{q_{n-1}}{Q_n} \leq \frac{c}{n}.
\]
If we apply (10) and (13) we immediately get that
\[
|F_n| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_j - q_{j+1}| + q_0 \right) \right) \sum_{i=0}^{n} M_i |K_{M_i}| = \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_j - q_{j+1}) + q_0 \right) \right) \sum_{i=0}^{n} M_i |K_{M_i}| \leq \frac{q_{n-1}}{Q_n} \sum_{i=0}^{n} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{n} M_i |K_{M_i}|.
\]
The proof is complete. \( \square \)

Lemma 11. Let \( x \in I_{N}^{k,l}, \ k = 0, \ldots, N-2, \ l = k+1, \ldots, N-1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers, satisfying condition (17). Then
\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{cM_l M_k}{nM_N}.
\]
Let \( x \in I_{N}^{k,N}, \ k = 0, \ldots, N-1 \). Then
\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{cM_k}{M_N}.
\]
Here \( c \) is an absolute constant.

Proof: The proof is quite analogously to Lemma 8. So we leave out the details.

Lemma 12. Let \( n \geq M_N, \ x \in I_{N}^{k,l}, \ k = 0, \ldots, N-1, \ l = k+1, \ldots, N \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing sequence, satisfying condition (17). Then
\[
\int_{I_N} |F_n(x-t)| \, d\mu(t) \leq \frac{cM_l M_k}{nM_N^2}.
\]
Proof: Since \( n \geq M_N \) if we apply Lemma 11 we immediately get the proof.
5. PROOFS OF THE THEOREMS

Proof of Theorem 1 Let $0 < p \leq 1/2$ and sequence $\{q_k : k \geq 0\}$ be non-increasing. By combining (9) and (11) we get that

$$\tilde{T}^*_p f := \frac{|T_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \leq \frac{1}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \left| \frac{1}{Q_n} \sum_{j=1}^{n-1} q_j S_j f \right| \leq \frac{1}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \left| \sum_{j=1}^{n-2} (q_j - q_{j+1}) j |\sigma_j f| + q_{n-1}(n-1) |\sigma_n f| \right| \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1) \right) \sup_{n \in \mathbb{N}^+} \frac{|\sigma_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \leq \sup_{n \in \mathbb{N}^+} \frac{|\sigma_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} := \tilde{\sigma}^*_p f,$$

so that $\tilde{T}^*_p f \leq \tilde{\sigma}^*_p f$. Hence, if we apply Theorem T1 we can conclude that the maximal operators $\tilde{T}^*_p$ of $T$ means with non-increasing sequence $\{q_k : k \geq 0\}$ are bounded from the Hardy space $H_p$ to the space $L_p$ for $0 < p \leq 1/2$. The proof is complete. □

Proof of Theorem 2 Let $0 < p \leq 1/2$ and sequence $\{q_k : k \geq 0\}$ be non-decreasing satisfying the condition (17). By combining (9) and (11) we find that

$$\tilde{T}^*_p f := \frac{|T_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \leq \frac{1}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \left| \frac{1}{Q_n} \sum_{j=1}^{n-1} q_j S_j f \right| \leq \frac{1}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \left| \sum_{j=1}^{n-2} (q_j - q_{j+1}) j |\sigma_j f| + q_{n-1}(n-1) |\sigma_n f| \right| \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (j + 1)^{1/p - 2} \log^{2[1/2+p]} (j + 1) \right) \sup_{n \in \mathbb{N}^+} \frac{|\sigma_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \leq \frac{2q_{n-1}(n-1) - q_n}{Q_n} \sup_{n \in \mathbb{N}^+} \frac{|\sigma_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} \leq \sup_{n \in \mathbb{N}^+} \frac{|\sigma_n f|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)} = \tilde{\sigma}^*_p f.$$

□
so that

\[(29)\quad \tilde{T}_p^* f \leq \tilde{\sigma}_p^* f\]

If we apply (29) and Theorem T1 we can conclude that the maximal operators \(\tilde{T}_p^*\) of \(T\) means with non-decreasing sequence \(\{q_k : k \geq 0\}\), are bounded from the Hardy space \(H_p\) to the space \(L_p\) for \(0 < p \leq 1/2\). The proof is complete. \(\Box\)

\textbf{Proof of Theorem 3.} Let \(0 < p < 1/2\) and sequence \(\{q_k : k \geq 0\}\) be non-increasing. By Lemma \(1\), the proof of part a) will be complete, if we show that

\[
\sum_{m=1}^{\infty} \frac{\|T_m a\|_{p_{H_p}}^p}{m^{2-2p}} \leq c_p,
\]

for every \(p\)-atom \(a\), with support \(I, \mu (I) = M_N^{-1}\). We may assume that \(I = I_N\). It is easy to see that \(S_n (a) = T_n (a) = 0\), when \(n \leq M_N\). Therefore, we can suppose that \(n > M_N\).

Let \(x \in I_N\). Since \(T_n\) is bounded from \(L_\infty\) to \(L_\infty\) (boundedness follows Lemma \(4\) and \(\|a\|_\infty \leq M_N^{1/p}\), we obtain that

\[
\int_{I_N} |T_m a|^p d\mu \leq \frac{\|a\|_\infty^p}{M_N} \leq c < \infty.
\]

Hence,

\[(30)\quad \sum_{m=1}^{\infty} \int_{I_N} |T_m a|^p d\mu \leq \sum_{k=1}^{\infty} \frac{1}{m^{2-2p}} \leq c < \infty, \quad 0 < p < 1/2.
\]

It is easy to see that

\[(31)\quad |T_m a (x)| = \left| \int_{I_N} a (t) F_n (x - t) d\mu (t) \right|
\]

\[
= \left| \int_{I_N} a (t) \frac{1}{Q^n} \sum_{j=M_N}^{n} q_j D_j (x - t) d\mu (t) \right|
\]

\[
\leq \|a\|_\infty \int_{I_N} \frac{1}{Q^n} \sum_{j=M_N}^{n} q_j D_j (x - t) d\mu (t)
\]

\[
\leq M_N^{1/p} \int_{I_N} \frac{1}{Q^n} \sum_{j=M_N}^{n} q_j D_j (x - t) d\mu (t)
\]

Let \(T_n\) be \(T\) means, with non-decreasing coefficients \(\{q_k : k \geq 0\}\) and \(x \in I_{N_l}^{k,l}, 0 \leq k < l \leq N\). Then, in the view of Lemma \(6\), we get that

\[(32)\quad |T_m a (x)| \leq c M_l M_k M_N^{1/p-2}, \quad \text{for} \quad 0 < p < 1/2.
\]
Let $0 < p < 1/2$. By using (5), (31) and (32) we find that

$$
\int_{I_N} |T_m a|^p d\mu = \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{j=0}^{N-1} \int_{I_N} |T_m a|^p d\mu + \sum_{k=0}^{N-1} \int_{I_N} |T_m a|^p d\mu
$$

$$
\leq \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \cdots m_{N-1}}{M_N} (M_l M_k)^p M_N^{1-2p} + \sum_{k=0}^{N-1} \frac{1}{M_N} M_k M_N^{1-p}
$$

Moreover, according to (33) we get that

$$
\frac{\int_{I_N} |T_m a|^p d\mu}{m^{2-2p}} \leq \sum_{m=M_N+1}^{\infty} \frac{c M_N^{1-2p}}{m^{2-2p}} < c < \infty, \quad (0 < p < 1/2).
$$

The proof of part a) is complete by just combining (30) and (34).

Let $p = 1/2$ and $T_n$ be $T$ means, with non-increasing coefficients $\{q_k : k \geq 0\}$, satisfying condition (20). By Lemma 1, the proof of part b) will be complete, if we show that

$$
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|T_m a\|_{H_1/2}}{m} \leq c,
$$

for every $1/2$-atom $a$, with support $I$, $\mu (I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $S_n (a) = T_n (a) = 0$, when $n \leq M_N$. Therefore, we can suppose that $n > M_N$.

Let $x \in I_N$. Since $T_n$ is bounded from $L_\infty$ to $L_\infty$ (boundedness follows from Lemma 1) and $\|a\|_\infty \leq M_N^2$, we obtain that

$$
\int_{I_N} |T_m a|^{1/2} d\mu \leq \frac{\|a\|_\infty^{1/2}}{M_N} \leq c < \infty.
$$

Hence,

$$
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\int_{I_N} |T_m a|^{1/2} d\mu}{m} \leq \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{m} \leq c < \infty.
$$

Analogously to (31) we find that

$$
|T_m a (x)| = \left| \int_{I_N} a (t) \frac{1}{Q_n} \sum_{j=M_N}^{n} q_j D_j (x - t) d\mu (t) \right|
$$

$$
\leq \|a\|_\infty \int_{I_N} |F_m (x - t)| d\mu (t) \leq M_N^2 \int_{I_N} |F_m (x - t)| d\mu (t).
$$

Let $x \in I_N^{kl}$, $0 \leq k < l < N$. Then, in the view of Lemma 8 we get that

$$
|T_m a (x)| \leq c M_l M_k M_N.
$$

Let $x \in I_N^{kN}$. Then, according to Lemma 8 we obtain that

$$
|T_m a (x)| \leq c M_k M_N.
$$
By combining (5), (36), (37) and (38) we obtain that
\[
\int_{I_N} \left| T_m a(x) \right|^{1/2} d\mu(x)
\]
\[
\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \cdots m_{N-1}}{M_N} (M_l M_k)^{1/2} M_N^{1/2} m^{1/2} + \sum_{k=0}^{N-1} \frac{1}{M_N} M_k^{1/2} M_N^{1/2}
\]
\[
\leq M_N^{1/2} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^{1/2}}{m^{1/2} M_l} + \sum_{k=0}^{N-1} \frac{M_k^{1/2}}{M_N^{1/2}} \leq \frac{c M_N^{1/2} N}{m^{1/2}} + c.
\]

It follows that
\[
(39) \quad \frac{1}{\log n} \sum_{m=M_N+1}^{n} \frac{\int_{I_N} \left| T_m a(x) \right|^{1/2} d\mu(x)}{m} \leq \frac{1}{\log n} \sum_{m=M_N+1}^{n} \left( \frac{c M_N^{1/2} N}{m^{3/2}} + \frac{c}{m} \right) < c < \infty.
\]

The proof of part b) is completed by just combining (35) and (39). □

**Proof of Theorem 4.** If we use Lemmas [11] and [12] and follows analogical steps of proof of Theorem 3 we immediately get the proof of Theorem 4. So, we leave out the details. □

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