Solving the heat equation with variable thermal conductivity

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Abstract

We consider the heat equation with spatially variable thermal conductivity and homogeneous Dirichlet boundary conditions. Using the Method of Fokas or Unified Transform Method, we derive solution representations as the limit of solutions of constant-coefficient interface problems where the number of subdomains and interfaces becomes unbounded. This produces an explicit representation of the solution, from which we can compute the solution and determine its properties. Using this solution expression, we can find the eigenvalues of the corresponding variable-coefficient eigenvalue problem as roots of a transcendental function. We can write the eigenfunctions explicitly in terms of the eigenvalues. The heat equation is the first example of more general variable-coefficient second-order initial-boundary value problems that can be solved using this approach.

1 Introduction

The Method of Fokas or Unified Transform Method (UTM) can be used to solve constant-coefficient Initial-Boundary Value Problems (IBVPs) [3, 6]. The purpose of this paper is to demonstrate a method to generalize the UTM to solve variable-coefficient IBVPs. In [5, 12], Fokas and Treharne use a Lax-Pair approach to analyze some variable-coefficient IBVPs. This method reduces the problem of solving a partial differential equation (PDE) to that of solving an ordinary differential equation (ODE) by writing the solution of the PDE as an integral over the solutions to a non-autonomous ODE, but it does not provide an explicit representation of the solution. This approach, along with separation of variables, is useful if the associated ODE is a second-order, self-adjoint problem on a finite domain, for which we have standard Sturm-Liouville theory. However, it does not generalize well to problems that are not self adjoint, of higher order, or are posed on an infinite domain.

In our approach to variable-coefficient IBVPs, we break the domain into \( N \) subdomains. The variable coefficients are approximated on each subdomain by constants, resulting in a constant-coefficient interface problem. We solve this problem using the UTM as shown in [1, 2, 8, 9, 10, 11]. Cramer’s rule gives the solution in each part as a ratio of determinants. Using an explicit expression for these determinants, we take the limit as \( N \) goes to infinity. Obtaining the explicit expressions for the determinants and calculating the limit are both non-trivial steps. Finally, we obtain an explicit (albeit complicated) solution to the original variable-coefficient IBVP, useful for computing the solution, for instance. Further, our solution representation characterizes the eigenvalues of the spectral problem obtained after separation of variables and it gives the eigenfunctions explicitly in terms of these eigenvalues.

Since the UTM is applicable to non-constant boundary conditions, higher-order and non-self-adjoint problems, we expect our method to generalize similarly. Indeed, we have found explicit solutions for general, second-order IBVPs with spatially-variable coefficients and with general boundary conditions in terms of sums and integrals over known quantities. This will be reported in [4]. Although our approach is entirely different, some of our notation has been inspired by [7].

2 The heat equation with homogeneous, Dirichlet boundary conditions

Consider the heat equation on the finite interval, \( x \in (0,1) \), with spatially-variable thermal conductivity \( \sigma^2(x) \), without forcing and with homogeneous, Dirichlet boundary conditions:
\[ q_t = (\sigma^2(x)q_x)_x, \quad x \in (0,1), \quad t > 0, \quad (1a) \]
\[ q(x,0) = q_0(x), \quad x \in (0,1), \quad (1b) \]
\[ q(0,t) = q(1,t) = 0, \quad t > 0, \quad (1c) \]

where the \( x \) index denotes partial differentiation.

**Theorem 1.** If \( \sigma(x) > 0 \) is absolutely continuous and if \( \sigma'(x)/\sigma(x) \) and \( q_0(x) \) are absolutely integrable, then the IBVP (1) has the solution

\[ q(x,t) = \frac{1}{i\pi} \int_{\partial \Omega} \frac{\Phi(k,x)}{\Delta(k)} e^{-k^2 t} dk, \quad (2) \]

where \( \Omega = \{ k \in \mathbb{C} : \pi/4 < \text{Arg}(k) < 3\pi/4 \text{ and } |k| > r \} \) for some \( r > 0 \) as shown in Figure 1a, with

\[ \Delta(k) = \sum_{n=0}^{\infty} S_n^{(0,1)}(k) \quad \text{and} \quad \Phi(k,x) = \int_0^1 \frac{\Psi(k,x,y)q_0(y)}{\sqrt{\sigma(x)\sigma(y)}} dy, \quad (3) \]

where, for \( 0 < y < x < 1, \)

\[ \Psi(k,x,y) = \left( \sum_{n=0}^{\infty} S_n^{(0,y)}(k) \right) \left( \sum_{n=0}^{\infty} S_n^{(x,1)}(k) \right) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} S_n^{(0,y)}(k)S_{\ell}^{(x,1)}(k), \quad (4) \]

and \( \Psi(k,x,y) = \Psi(k,y,x) \) for \( 0 < x < y < 1. \) Here

\[ S_n^{(a,b)}(k) = \frac{1}{2^n} \int_{a \leq y_0 \leq y_1 \leq \cdots \leq y_n \leq b} \left( \prod_{p=1}^{n} \frac{\sigma'(y_p)}{\sigma(y_p)} \right) \sin \left( k \sum_{p=0}^{n} (-1)^p \int_{y_p}^{y_{p+1}} \frac{d\xi}{\sigma'(\xi)} \right) dy_1 \cdots dy_n. \quad (5) \]

Note that this reduces to the solution given in [6] for constant \( \sigma(x). \)

## 3 Derivation

We form a partition \( x_j \) of the interval \([0,1]\), such that \( x_0 = 0 \) and \( x_N = 1 \), see Figure 1b. On each subdomain, we replace (1a) with a constant-coefficient PDE with \( \sigma(x) \) replaced by \( \sigma_j \) such that \( \sigma_j \to \sigma(x_j) \) as \( N \to \infty \), with the corresponding initial condition. At each interface, we require continuity of the solution and a jump discontinuity in the derivative of the solution, consistent with the evolution equation, i.e.,

\[ q_1^{(j)} = \sigma_j^2 x_j q_x, \quad q^{(j)}(x,0) = q_0(x), \quad x \in (x_{j-1},x_j), \quad t > 0, \quad j = 1, \cdots, N, \quad (6a) \]
\[ q^{(j)}(x_j,t) = q^{(j+1)}(x_j,t), \quad \sigma_j^2 x_j q_x(t) = \sigma_{j+1}^2 x_j q_x(t), \quad t > 0, \quad j = 1, \cdots, N - 1, \quad (6b) \]
\[ q^{(1)}(0,t) = 0, \quad q^{(N)}(1,t) = 0, \quad t > 0. \quad (6c) \]
Note that the derivative jump (6b) can be derived by integrating (1a) over a small interval containing \( x_j \).

We follow [1, 2, 8, 9, 10, 11]. Defining for \( j = 1, \ldots, N \) and \( m = 0, 1 \),

\[
\begin{align*}
\hat{q}_0^{(j)}(k) &= \int_{x_{j-1}}^{x_j} e^{-iky}q_0(y)dy, \quad \hat{q}^{(j)}(k, t) = \int_{x_{j-1}}^{x_j} e^{-iky}q^{(j)}(y, t)dy, \quad g_m^{(j)}(W, t) = \int_0^t e^{W} q_m^{(j)}(x_j, s)ds, \\
g_m^{(0)}(W, t) &= \int_0^t e^{W} q_m^{(0)}(0, s)ds, \quad g_0^{(j)}(W, t) = g_0^{(N)}(W, t) = 0,
\end{align*}
\]

(7) where the last equations originate from the boundary conditions (6c). Introducing the dispersion relation \( w_j = \sigma_j^2 \kappa^2 \), we obtain the local relations corresponding to (6a) defined in each subdomain \( D_j = (x_{j-1}, x_j) \times (0, T), j = 1, \ldots, N \). Integrating over the boundary of each subdomain and using Green’s theorem, we find the global relations. Changing variables \( \kappa = \nu_j(k) = k/\sigma_j \), these are

\[
\begin{align*}
e^{k^2t}\hat{q}^{(j)}(\nu_j) &= q_0^{(j)}(\nu_j) + e^{-i\nu_j x_j} \left( \sigma_j^2 g_1^{(j)}(k^2, t) + i\sigma_j k g_0^{(j)}(k^2, t) \right) \\
&\quad - e^{-i\nu_j x_{j-1}} \left( \sigma_j^2 g_1^{(j-1)}(k^2, t) + i\sigma_j k g_0^{(j-1)}(k^2, t) \right), \quad j = 1, \ldots, N.
\end{align*}
\]

(9) These relations are valid for \( k \in \mathbb{C} \), since all integrals are over bounded domains. Letting \( k \mapsto -k \), and \( \nu_j \mapsto -\nu_j \), results in a total of \( 2N \) linear equations for the \( 2N \) unknowns \( (g_0^{(1)}(k^2, t), \ldots, g_0^{(N-1)}(k^2, t), g_1^{(0)}(k^2, t), \ldots, g_1^{(N)}(k^2, t)) \).

We write this system of equations in matrix form as

\[
\mathcal{A}(k)X(k^2, t) = Y(k, t) - e^{k^2t} \mathcal{Y}(k, t),
\]

(10) where \( \mathcal{A}(k) \) is the coefficient matrix corresponding to the global relations (9), and

\[
\begin{align*}
X(k^2, t) &= \left( ik g_0^{(1)}(k^2, t), \ldots, ik g_0^{(N-1)}(k^2, t), \sigma_0^2 g_1^{(0)}(k^2, t), \ldots, \sigma_N^2 g_1^{(N)}(k^2, t) \right)^T, \\
Y(k) &= \left( q_0^{(j)}(\nu_1), \ldots, q_0^{(j)}(\nu_N), q_0^{(1)}(-\nu_1), \ldots, q_0^{(1)}(-\nu_N) \right)^T, \\
\mathcal{Y}(k, t) &= \left( \hat{q}^{(1)}(\nu_1, t), \ldots, \hat{q}^{(N)}(\nu_N, t), \hat{q}^{(1)}(-\nu_1, t), \ldots, \hat{q}^{(N)}(-\nu_N, t) \right)^T.
\end{align*}
\]

(11) (12) (13) Following [11], we can show that the contribution to the solution of \( \mathcal{Y} \) vanishes, so that, in effect, we may solve \( \mathcal{A}X = Y \), for the unknown functions \( g_m^{(j)} \). Using Cramer’s rule,

\[
X_j = ik g_0^{(j)}(k^2, t) = \frac{\det(\mathcal{A}_j(k))}{\det(\mathcal{A}(k))}, \quad j = 1, \ldots, N-1,
\]

(14) where the matrix \( \mathcal{A}_j(k) \) is \( \mathcal{A}(k) \) with the \( j \)th column replaced by \( Y \). If we multiply this equation by \( e^{-k^2t} \) and integrate over \( \partial \Omega \), shown in Figure 1a, we recover the solution at the interfaces \( q_j^{(j)}(x_j, t), j = 1, \ldots, N-1, \) [6] obtaining

\[
q^{(j)}(x_j, t) = -\frac{1}{\pi} \int_{\partial \Omega} \frac{\det(\mathcal{A}_j(k))}{\det(\mathcal{A}(k))} e^{-k^2t} dk \Rightarrow q(x, t) = \lim_{N \to \infty} q^{(j)}(x_j, t). \]

(15) It is possible to compute the solution of the full interface problem as in [1, 9, 10, 11], and obtain the same limit from there. To obtain (2), we proceed as follows.

We introduce \( \Lambda_p^\pm = \sigma_{p+1} + (-1)^{\ell_p+1} \sigma_p \), with \( \ell_p, \ell_{p+1} \in \{0, 1\} \), using \( \Lambda_p^+ \) when \( \ell_p = \ell_{p+1} \) and \( \Lambda_p^- \) when \( \ell_p \neq \ell_{p+1} \). We define

\[
D_N(k) = \frac{i}{2} \det(\mathcal{A}(k)) \left( \prod_{p=1}^{N-1} \frac{1}{\Lambda_p^\pm} \right) = \sum_{\ell \in \{0, 1\}^N} \left( \prod_{p=1}^{N-1} \frac{\Lambda_p^\pm}{\Lambda_p} \right) \sin \left( k \sum_{p=1}^{N} (-1)^{\ell_p} \Delta x_p / \sigma_p \right),
\]

(16) where we have used the explicit form of \( \mathcal{A}(k) \), see [11] with slight modifications. We can show that
\[
\prod_{p=1}^{m-1} \frac{\Lambda_p^+}{2\sigma_p} = \sqrt{\frac{\sigma_m}{\sigma_1}} + O(L), \quad \text{and} \quad \frac{\Lambda_p^-}{\Lambda_p^+} = \frac{\sigma_{p+1} - \sigma_p}{\sigma_{p+1} + \sigma_p} = \frac{\sigma'(x_p) \Delta x_p}{2\sigma_p} + O(L^2),
\]

as \( N \to \infty \) and \( \Delta x_p \to 0^+ \), with \( L = \max_p \Delta x_p \). Next, we show

\[
\Delta(k) = \lim_{N \to \infty} D_N(k) = \sum_{n=0}^{\infty} S_n^{(0,1)}(k),
\]

with \( S_n^{(a,b)}(k) \) defined in (5). Let \( s_1, \ldots, s_n \) be the \( n \) locations where \( N \)-dimensional vector \( \ell \) has its entry switch values (from 0 to 1 or from 1 to 0), with \( s_0 = 0 \) and \( s_{n+1} = N \), for convenience. Since \( \Lambda_p^+ = \Lambda_p^- \) except where the switches occur,

\[
D_N(k) = \sum_{n=0}^{N-1} \sum_{0=s_0 \leq s_1 \leq \cdots \leq s_{n+1} = N} \left( \prod_{p=1}^{n} \frac{\Lambda_{s_p}^-}{\Lambda_{s_p}^+} \right) \sin \left( k \sum_{p=0}^{n} (-1)^{s_p+1} \sum_{r=1}^{s_p+1} \frac{\Delta x_r}{\sigma_r} \right).
\]

Using (17), (18) follows.

Turning to the numerator, we define

\[
E_N(k,j) = \frac{1}{2} \det(\mathcal{A}_j(k)) \left( \prod_{p=1}^{N-1} \frac{1}{\Lambda_p^+} \right) = \sum_{m=1}^{N} \left[ Y_m C_{m,j} + Y_{m+N} \tilde{C}_{m,j} \right],
\]

after doing a cofactor expansion using the \( j \)th column of \( \mathcal{A}_j \). Here \( C_{m,j} \) and \( \tilde{C}_{m,j} \) are the relevant scaled cofactors of \( \mathcal{A}_j \). Using

\[
Y_m = e^{-i\nu_m x_m} q_0(x_m) \Delta x_m + O(L^2) \quad \text{and} \quad Y_{m+N} = e^{i\nu_m x_m} q_0(x_m) \Delta x_m + O(L^2),
\]

as \( N \to \infty \), we have

\[
E_N(k,j) = \sum_{m=1}^{N} q_0(x_m) \left( e^{-i\nu_m x_m} C_{m,j} + e^{i\nu_m x_m} \tilde{C}_{m,j} \right) \Delta x_m + O(L).
\]

Defining

\[
\Psi_{j,m} = \sqrt{\sigma_j / \sigma_m} \left( e^{-i\nu_m x_m} C_{m,j} + e^{i\nu_m x_m} \tilde{C}_{m,j} \right),
\]

then

\[
\Phi(k,x) = \lim_{N \to \infty} E_N(k,j) = \lim_{N \to \infty} \sum_{m=1}^{N} \frac{\Psi_{j,m}}{\sqrt{\sigma_j / \sigma_m}} \Delta x_m = \int_{0}^{1} \frac{\Psi(k,x,y)q_0(y)}{\sqrt{\sigma(x) \sigma(y)}} \, dy,
\]

where \( x_j \to x \) and \( x_m \to y \), and \( \Psi(k,x,y) = \lim_{N \to \infty} \Psi_{j,m} \). For \( 1 \leq m < j \leq N \),

\[
\Psi_{j,m} = \sqrt{\frac{\sigma_j}{\sigma_m}} \left( \prod_{p=m}^{j-1} \frac{2\sigma_p}{\Lambda_p^+} \right) \left[ \sum_{\ell=\{0,1\}^m}^{m-1} \prod_{p=1}^{m} \frac{\Lambda_{p}^{\pm}}{\Lambda_{p}^{\mp}} \right] \sin \left( k \sum_{p=1}^{m} \frac{(-1)^p \Delta x_p}{\sigma_p} \right) \times \left[ \sum_{\ell=\{0,1\}^{N-j}}^{N-1} \prod_{p=j+1}^{N} \frac{\Lambda_{p}^+}{\Lambda_{p}^+} \right] \sin \left( k \sum_{p=j+1}^{N} \frac{(-1)^p \Delta x_p}{\sigma_p} \right),
\]

Again using the explicit form of \( \mathcal{A}(k) \) [11]. Taking the limit as before in (18), we find for \( 0 < y < x < 1 \),

\[
\Psi(k,x,y) = \lim_{N \to \infty} \Psi_{j,m} = \left( \sum_{n=0}^{\infty} S_n^{(0,y)}(k) \right) \left( \sum_{n=0}^{\infty} S_n^{(x,1)}(k) \right) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} S_n^{(0,y)}(k) S_{\ell}^{(x,1)}(k),
\]

and similarly, for \( 0 < x < y < 1 \), \( \Psi(k,x,y) = \Psi(k,y,x) \).

We can prove that the solution is well defined and solves the IBVP (1), see [4].
Figure 2: (a) The exact solution \( q(x, t) = x(1 - x)e^{-t} \) and its successive approximations (29) for \( N = 0, 1, 2 \). (b) The error of the approximations.

4 Eigenvalue problem

Consider the Sturm-Liouville problem

\[
\left( \sigma^2(x)y' \right)' = \lambda y, \quad y(0) = 0 = y(1). \tag{27}
\]

The eigenvalues \( \lambda_m = -\kappa_m^2 \) of this problem are related to the zeros \( \kappa_m \) \((m = 1, 2, \ldots)\) of \( \Delta(k) \). Since these eigenvalues are negative, it follows that these zeros are real. Since \( \Delta(k) \) is odd, it suffices to only consider the positive zeros.

**Theorem 2.** The problem (27) has the eigenfunctions

\[
X_m(x) = \frac{1}{\sqrt{\sigma(x)}} \sum_{n=0}^{\infty} \Phi_n^{(0, x)}(\kappa_n), \quad m = 1, 2, \ldots \tag{28}
\]

**Proof.** The proof is straightforward differentiation, noting the absolute convergence of the sums. \( \square \)

5 Numerical example

With \( q_0(x) = x(1 - x) \) and \( \sigma^2(x) = (3 - (2x - 1)^2)/24 \), we have the exact solution \( q(x, t) = x(1 - x)e^{-t} \). We construct an approximation to the solution (2) such that \( q_N(x, t) \to q(x, t) \) as \( N \to \infty \) (the index \( N \) does not denote differentiation):

\[
q_N(x, t) = \frac{1}{i\pi} \int_{\gamma} \frac{dk}{\sigma(x)} \frac{\Phi_N(k, x)}{\exp(ik \int_0^1 \frac{d\xi}{\sigma^{(2)}})} \Delta_N(k) e^{-k^2 t} dk, \tag{29}
\]

where we multiply denominator and numerator by the exponential so that both are decaying in the upper-half complex \( k \) plane, and where we truncate each series up to \( n = N \). The contour \( \gamma \) is used instead of \( \partial \Omega \) to aid convergence as the factor \( \exp(-k^2 t) \) decays along it. The results are shown in Figure 2.

To demonstrate the computation of the eigenvalues and eigenfunctions, we consider (27) with the same \( \sigma(x) \). To find the eigenvalues, we use Mathematica’s FindRoot command on \( \Delta_N(k) \). The results are shown in Table 1. We see that our method converges to the eigenvalues and outperforms Mathematica’s built-in NDEigenvalues command for \( n = 4 \). Furthermore, we are able to provide explicit bounds on the eigenvalue approximations [4]. Lastly, we denote the order-\( N \) truncated eigenfunctions (28) as \( X_m^{(N)}(x) \). These are shown in Figure 3. For the simple \( \sigma(x) \), given above, the order-0 truncation is quite accurate. For a more complicated \( \sigma(x) \), the order-1 truncation gives an accurate representation.

References

[1] B. Deconinck, B. Pelloni, and N. E. Sheils. Non-steady-state heat conduction in composite walls. *Proceedings of the Royal Society. A, Mathematical, Physical, and Engineering Sciences*, 470:1–22, 2014.
Table 1: Eigenvalues of the system (27) calculated using MATLAB’s chebfun package compared to using Mathematica’s NDEigenvalues and Mathematica’s FindRoot on $\Delta_N(k)$ for $N = 0, 1, 2$.

| Method          | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|-----------------|--------------|--------------|--------------|--------------|
| chebfun         | $-1.0000$    | $-4.2540$    | $-9.6812$    | $-17.2800$   |
| NDEigenvalues   | $-1.0000$    | $-4.2540$    | $-9.6818$    | $-17.2834$   |
| FindRoot: $\Delta_0(k)$ | $-1.0856$    | $-4.3423$    | $-9.7702$    | $-17.3692$   |
| FindRoot: $\Delta_1(k)$ | $-0.9917$    | $-4.2474$    | $-9.6749$    | $-17.2737$   |
| FindRoot: $\Delta_2(k)$ | $-1.0006$    | $-4.2542$    | $-9.6814$    | $-17.2801$   |

Figure 3: (a) The successive approximations to the eigenfunctions (28) shown with the numerically computed eigenfunctions using MATLAB’s chebfun package. (b) The same, using $\sigma^2(x) = \frac{6337 - 252\sqrt{111} - 4500x^2(11 + x(-14 + 5x))}{9000(11 + 6x(-7 + 5x))}$. 

[2] B. Deconinck, N. E. Sheils, and D. A. Smith. The linear KdV equation with an interface. Communications in Mathematical Physics, 347:489–509, 2016.

[3] B. Deconinck, T. Trogdon, and V. Vasan. The method of Fokas for solving linear partial differential equations. SIAM Review, 56:159–186, 2014.

[4] M. Farkas and B. Deconinck. Solving linear, second-order evolution equations with spatially varying coefficients. In Preparation, 2022.

[5] A. S. Fokas. Boundary-value problems for linear PDEs with variable coefficients. Proceedings of the Royal Society. A, Mathematical, Physical, and Engineering Sciences, 460:1131–1151, 2004.

[6] A. S. Fokas. A unified approach to boundary value problems, volume 78 of CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, Philadelphia, PA, 2008.

[7] J. Pöschel and E. Trubowitz. Inverse Spectral Theory. Academic Press Inc., 1987.

[8] N. Sheils and B. Deconinck. Interface problems for dispersive equations. Studies in Applied Mathematics (Cambridge), 134:253–275, 2015.

[9] N. Sheils and B. Deconinck. The time-dependent Schrödinger equation with piecewise constant potentials. European Journal of Applied Mathematics, 31:57–83, 2020.

[10] N. E. Sheils and B. Deconinck. Heat conduction on the ring: Interface problems with periodic boundary conditions. Applied Mathematics Letters, 37:107–111, 2014.

[11] N. E. Sheils and B. Deconinck. Initial-to-interface maps for the heat equation on composite domains. Studies in Applied Mathematics (Cambridge), 137:140–154, 2016.

[12] P. A. Treharne and A. S. Fokas. Initial-boundary value problems for linear PDEs with variable coefficients. Mathematical Proceedings of the Cambridge Philosophical Society, 143:221–242, 2007.