Axions: Bose Einstein Condensate or Classical Field?

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Abstract

The axion is a motivated dark matter candidate, so it would be interesting to find features in Large Scale Structures specific to axion dark matter. Such features were proposed for a Bose Einstein condensate of axions, leading to confusion in the literature (to which I contributed) about whether axions condense due to their gravitational interactions. This note argues that the Bose Einstein condensation of axions is a red herring: the axion dark matter produced by the misalignment mechanism is already a classical field, which has the distinctive features attributed to the axion condensate (BE condensates are described as classical fields). This note also estimates that the rate at which axion particles condense to the field, or the field evaporates to particles, is negligible.

1 Introduction

The axion\cite{1,2,3,4,5} is a light pseudo-goldstone boson ($m_a \lesssim m_\nu$), introduced\cite{2} in a solution of the strong CP problem of QCD. It can constitute the Cold Dark Matter (CDM) of the Universe. If the phase transition at which axions appear occurs after inflation, then axions are unconstrained by observations of inflationary density fluctuations (e.g. PLANCK, BICEP2), and there are two axion contributions to CDM: oscillations of the classical axion field produced by the “misalignment mechanism”\cite{6,7}, and a population of non-relativistic modes radiated by strings\cite{9,10}. These two contributions can provide\cite{9} the observed CDM.

Sikivie has raised the interesting question of whether axion dark matter could be distinguished from Weakly Interacting Massive Particles (WIMPs)\cite{11}. This is pursued by various experiments which search for axions\cite{12,13} and/or WIMPs\cite{15,16}. Sikivie and collaborators\cite{17,18} noticed that the stress-energy tensors for axions and WIMPs are different, and proposed that axion dark matter could observably differ from WIMPs in non-linear structures. For example, if the dark matter halo of a rotating galaxy were composed of a Bose Einstein (BE) condensate of axions, then vortices could form, leading to observable caustics in the dark matter distribution. This interesting scenario has generated discussion, both of the rate at which the Bose Einstein condensate could form\cite{19,20,21,22}, and of the behaviour of a halo of condensate\cite{23,24}.

The aim of this note is to argue that the issue of whether axions are a Bose Einstein condensate is a red herring. The Path Integral should allow to compute anything, and in the Path Integral, CDM axions can be described by the “classical field”, and the two-point function (or equivalently, the number density of particles). The dynamics should be controlled by the Lagrangian of a scalar field coupled to gravity, which is simple and well-known. I claim that practically, the classical misalignment axion field is always a Bose Einstein condensate (if one wishes to use those words), and its evolution under gravity can be understood by solving Einstein’s Equations for a classical field\cite{2}. This is known to a segment of the community: Peebles studied classical scalar field dark matter\cite{25}, and in a beautiful series of papers, Rindler-Daller and Shapiro\cite{23} study whether it is energetically favourable for a galactic halo made of classical field to form vortices, and find that the $\phi^4$ coupling of the QCD axion is of the wrong sign.

The outline of this paper is as follows. The next section gives a brief review of axions and the BE condensate literature, and proposes a Path-Integral-motivated translation dictionary among the various vocabularies used to describe cold dark matter axions. In particular, BE condensate = classical field. Section 3 reviews the stress-energy tensor $T^{\mu\nu}$ for the (axion) field and for particles, because $T^{\mu\nu}$ determines the evolution of dark matter, in the classical approximation. Section 4 gives some rough estimates of the $O(G_N^2)$ rate at which gravity might transfer axions between the particle bath and the classical field. This section, estimating the rate at which axion particles could condense to the classical field, is the only new part of this note.

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\footnote{1 See also\cite{13}.}

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\footnote{2This means that discussions of the rate at which the misalignment field forms a BE condensate are irrelevant — despite the considerable confusion (to which I contributed) in the literature about the condensation of the misalignment field.}
2 Notation and notions

This section contains a brief review of axion cosmology, an outline of some literature on the BE condensation of axions, and proposes that axion CDM can be described as a classical field and a distribution of cold particles. More complete references can be found in [21], a more thorough cosmological discussion in [11][26], astrophysical bounds in [3], and up-to-date numbers in [27]. This paper focuses on the QCD axion; the more generic case of Weakly Interacting Slim Particles, and Axion-Like Particles, are reviewed in [28].

2.1 A brief review of axion cosmology

In “invisible” axion models [4][5], the Standard Model is extended by a global “Peccei Quinn” U(1) symmetry, and various new fields. This Peccei Quinn symmetry breaks spontaneously at some high scale, $f_{\text{PQ}} \sim 10^{11}$ GeV for concreteness, where all the new fields become massive except the goldstone, who will become the axion. I suppose that this phase transition occurs after inflation. This is consistent with the BICEP2 [29] value of the inflationary expansion rate $H \sim 10^{14}$ GeV, and avoids “isocurvature bounds” [30][31] on axion CDM. After the Peccei-Quinn phase transition, the phase of the symmetry-breaking field, which is the axion, takes an arbitrary value between $-\pi$ and $\pi$ in each horizon volume, and there is on average a string per horizon (I suppose a potential which does not allow more dangerous defects[8], such as domain walls). Then the Universe expands until the QCD phase transition. During this period, the coherence length of the (massless) axion field grows with the horizon [30], and there remains on average a string per horizon [9]. As the QCD phase transition occurs, the massive pion appears, and mixes with goldstone. This “tilts the mexican hat”, smoothly turning on the axion mass. Once $m_\pi$ reaches a constant, the axion field $\phi$ has a potential [1]

$$V(\phi) \approx f_{\text{PQ}}^2 m^2 [1 - \cos(\phi/f_{\text{PQ}})] \approx \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} f_{\text{PQ}}^2 \phi^4 + \frac{1}{6!} f_{\text{PQ}}^2 \phi^6 + ...$$

(1)

where the axion mass is

$$m \simeq \frac{m_\pi f_{\pi}}{f_{\text{PQ}}} \sqrt{m_u m_d} \simeq 6 \times 10^{-5} \text{eV} \frac{10^{11}\text{GeV}}{f_{\text{PQ}}}$$

(2)

This potential has two implications for CDM axions: the strings go away, via a complicated process studied numerically in [9], who obtain that the energy in the string network is transferred to a population of modes with momenta $\sim H_{\text{QCD}} \approx 2 \times 10^{-20}$ GeV. Once the axion mass reaches its current value, these incoherent non-relativistic modes contribute $\Omega_\alpha \sim 0.2 \times (f_{\text{PQ}}/10^{11} \text{GeV})^{6/5}$. The other effect of the potential [1], is to cause the axion field $\phi$, randomly located in $(-\pi, \pi)$ in each horizon volume, to roll towards its minimum, and oscillate. The QCD horizon scale is $H_{\text{QCD}}^{-1}$, so the oscillations of this “misaligned” classical axion field are non-relativistic, and redshift like CDM [32]. They also grow large-scale density fluctuations like CDM [33][34][35][36]. The fluctuations in the density of the axion field are $O(1)$ on the comoving scale of $H_{\text{QCD}}^{-1}$ ($\sim 10^{-5}$ the distance to the galactic centre today); the interesting behaviour of these “axion miniclusters” is discussed in [37].

2.2 The scenario of axion condensation

The “occupation number” [5] of the misalignment field is very high. In a seminal paper, Sikivie and Yang [17] proposed that “gravitational thermalisation” of the misalignment axions could cause them to Bose Einstein condense. They estimated the gravitational interaction rate, or graviton exchange rate,

$$\Gamma_{\text{int}} \sim G_N m^2 n_\phi \frac{n_{\phi}}{H^2}$$

(3)

where $n_\phi$ is the axion number density. This is faster than the Hubble expansion rate $H$ for photon temperatures of $\lesssim$ keV. This estimate, for the gravitational interaction rate of the misalignment axions, was confirmed in [18][19][20][21]. Notice that the rate is linear in $G_N$, because the misalignment axions are in a coherent state (classical field). Sikivie and Yang then go on to study the evolution of the axion condensate using the non-relativistic Schrodinger equation (or Gross-Pitaevskii equation), as used later by Rindler-Daller and Shapiro [23].

The rigorous analyses of Saikawa et al [19][20] focus on the gravitational interactions of misalignment axions, and obtain equations of motion for the axion number operator in second quantised field theory, both in flat Newtonian space-time, and in a perturbed expanding Friedmann-Robertson-Walker Universe. They show that $dn_k/dt \simeq \Gamma_{\text{int}}$, where $n_k$ is the number of axions of momentum $k$ in the coherent state representing the misalignment axions.

In a previous paper with Elmer, we doubted that (3) was a thermalisation rate, because there is no entropy generation (no fluctuations are averaged over, the evolution is coherent). Using the equations of motion for a classical field in an expanding Universe with small density fluctuations, we reproduced eqn (3). We also recalled that, with a

3Recall that the classical field and particle limits of a Lagrangian require a different distribution of $\hbar$ [35], so that $\hbar$ is explicitly required, to obtain the particle number in a classical field configuration.
different parametrisation, these equations are linear and can be solved: if one studies the fourier modes of the energy density, rather than those of the axion field, then one obtains the familiar equations for density fluctuations in the early Universe. We interpreted that the gravitons exchanged in eqn (3) are coherently growing density fluctuations in the early Universe.

Recently, Berges and Jaeckel\(^4\) recall that thermalisation is not required for Bose Einstein condensation, and make analogies to results obtained in \(\lambda \phi^4\) theories. They recall that an initially highly occupied distribution of low-momentum particles develops an infrared cascade, and that a condensate can form before the high energy tail of the distribution has reached thermal form. Applied to CDM axions, the observation of \(^22\) could suggest that the cold axion particles produced by strings, might join in the axions in the misalignment field. I return to this in section 4.

## 2.3 Proposals from the path integral

Whether axion-CDM behaves differently from WIMP-CDM, is an interesting question. To identify variables and equations with which to address it, one can consult the path integral, which in principle knows everything.

The Path Integral allows to compute \(n\)-point functions, in particular, the expectation value of the field \(\langle \phi(x,t) \rangle = \phi_{cl}(x,t)\), and of the two point function. The “classical field” \(\phi_{cl}(x,t)\) is familiar from the 1PI effective action. In closed time path formalism, the two point function encodes a statistical number (or phase space) density, as well as the propagator. Since axions are feebly interacting, the higher point functions can often be neglected. The classical field and the density of incoherent modes are convenient variables for the discussion of cold Dark Matter axions, precisely they can be identified as the misalignment field, and the modes radiated by strings.

The classical field is distinguished from a distribution of particles by its macroscopic coherence. It can be represented in second-quantised Field Theory as a coherent state \(^{39}\), which is constructed by acting on the vacuum with the exponential of the creation operator (see eqn (10)). Such a state is therefore \textit{not} an eigenstate of the number operator, instead, the expectation value of the field operator in the coherent state gives the classical field. It follows that the classical field is something like an amplitude, and its equations of motion can be linear in the coupling. This differs from particles, where only forward scattering is linear in the coupling, because it interferes with doing nothing.

The statistical part of the two-point function describes an incoherent distribution of modes or particles. This paper assumes that axion strings decay into such a distribution \(^3\). Since the time evolution of a statistical distribution of classical modes, or particles, is approximately the same \(^{10}\), for concreteness, it is supposed here that the strings decay into cold axion particles.

The path integral can also provide equations of motion for the classical field and number density, although more familiar is the recipe to obtain in-out vacuum \(n\)-point functions, with which to calculate cross-sections. However, initial value problems \(^{11}\) can be posed in the Path Integral, using the Schwinger-Keldysh or Closed Time Path formalism \(^{42}\). Furthermore, in the 2PI formalism\(^{43}\), the classical field and the two point function appear as variables. So the indicated formalism for studying axion CDM would be the Closed Time Path 2PI action for axions, preferably in curved space-time. An \(O(N) \lambda \phi^4\) theory was studied in this formalism in \(^{44}\). However, this note uses more simple and familiar formalism. Einsteins equations and \(T^{\mu\nu} = 0\) are used as equations of motion, and the stress-energy tensor of the axion field and particles is evaluated as the expectation value of an operator in the usual way (“in-out vacua”). The classical equations of motion should be acceptable because gravity is a classical theory and the axion is feebly coupled, and equating in and out vacua should be acceptable again because the axion is feebly coupled.

## 2.4 What is a Bose Einstein condensate of axions?

The axion literature uses diverse vocabulary and calculational techniques. I propose to assume the following translation dictionary

\[
\text{classical field} = \text{condensed regime} = \text{Bose Einstein condensate}.
\]

That is, the misalignment axions are a Bose Einstein condensate. And eqn (3) is irrelevant, because it is the gravitational interaction rate of the axions in the condensate.

What is a Bose Einstein condensate? Bogoliubov \(^{45}\) long ago identified the Bose Einstein condensate as the macroscopic occupation of the lowest energy mode. In particular, starting from a second-quantised formalism, he treated as numbers the creation and annihilation operators of the zero mode\(^{4}\), such that the field operator could be written as a classical field in the zero mode plus creation and annihilation operators for the remainder of excitations. This formalism was used by Nambu and Sasaki to describe density fluctuations in the axion misalignment field\(^{36}\).

Important characteristics of a BE condensate seem to be

1. a classical field,
2. carrying a conserved charge.

\(^{4}\) We plan to address this question in more detail in a subsequent publication.

\(^{5}\) This is related to the coherent state, eigenstate of the annihilation operator.
3. whose fourier modes are concentrated at a particular value — that is, most of the “particles” who condense to the BE condensate, should coherently be doing the same thing. However, they do not need to be in the zero-momentum mode.

This is consistent with Bose Einstein condensation in equilibrium statistical mechanics and finite temperature field theory [46, 47] (where in homogeneous systems, the bosons condense in the zero-momentum mode), as well as with the experimental studies of BE condensation in alkali gases [48, 49], where the condensed atoms have similar velocities.

Are the misalignment axions a BE condensate? The axion is a real (pseudo)scalar (so formally has no conserved charge), but the number changing interactions are sufficiently slow that axion number is approximately conserved. If the PQ phase transition is after inflation, the fourier modes have an approximately white noise spectrum, so the classical axion field is not peaked at a particular momentum mode. However, readers with a predilection for BE condensates, who are attached to the third criteria, could then view the axion field as a superposition of BE condensates, which are coupled via gravity [21].

In any case, the question of whether the axion field is a BE condensate seems more about vocabulary than dynamics. Axion cold dark matter should evolve according to its equations of motion, which are approximately those of a (free) non-relativistic scalar interacting with gravity. These are the equations studied in [17, 23]. In my opinion, the BE condensate analogy is not useful for axions, because the familiar BE condensates (4He, etc) have stronger short range interactions than axions.

3 Formalism to calculate with: stress energy tensors

The matter current which couples to gravity is the stress-energy tensor, so this section reviews the stress-energy tensor of a (non-relativistic) classical scalar field (the misalignment axion field), and of a distribution of particles (the cold axions from strings). The equations which govern the formation of galaxies and large structures can then be respectively obtained from $T^{0\mu}_{\mu} = 0$ and $T^{\mu\nu}_{\mu\nu} = 0$.

Recall that the stress-energy tensor for dust, or non-relativistic non-interacting particles with $U^\mu = (1, \vec{v})$, is

$$T_{\mu\nu} = \rho U_{\mu} U_{\nu} = \begin{pmatrix} \rho & -\rho \vec{v} \\ -\rho \vec{v} & \rho v^i v^j \end{pmatrix} \quad \text{(dust)}. \quad (5)$$

With a metric $ds^2 = (1 + 2\psi)dt^2 - (1 - 2\psi)\delta_{ij}dx^i dx^j$, which describes Minkowski space with a Newtonian potential $\psi$ (satisfying $\nabla^2 \psi = 4\pi G_N \rho$ by Einsteins Equations), then $T^{\mu\nu}_{\mu\nu} = 0$ gives

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{continuity} \quad (7)$$

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla \psi \quad \text{Euler} \quad (8)$$

which should describe the evolution of a galactic halo made of cold non-interacting particles [50].

To obtain the stress-energy tensor for the axion field and particles, I work in second-quantised field theory, in the Heisenberg picture (time dependent operators). The axion is real and non-relativistic, however, it is convenient to use covariant (relativistic) notation $x^\alpha = (t, \vec{x})$ and a complex field $\phi$. The relativistic notation is because $T^{\mu\nu}$ is covariant. Then, the almost-non-interacting non-relativistic axion field has an effectively conserved quantum number (particle number). Indeed, a real relativistic field $\varphi = \varphi^\dagger$ can be described in the non-relativistic limit by a complex field $\phi$:

$$\varphi = \phi e^{-imt} + \phi^\dagger e^{imt}. \quad \text{So to have a conserved number in relativistic notation, I use a complex scalar field, which has a conserved current. It is straightforward to check that the stress-energy tensors obtained for the field and the cold particles (the anti-particle modes are neglected) will be the same. In this note, the non-relativistic limit is taken by neglecting $\partial^2_\theta$ and $(\partial_\theta)^2$, and by neglecting $\partial_t \theta$ (see eqn (11)) with respect to $m$.}

The field operator can be fourier-expanded

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left( \hat{a}_k e^{-ik \cdot x} + \hat{a}_k^\dagger e^{ik \cdot x} \right) \quad (9)$$

on particle annihilation and anti-particle creation operators satisfying $[\hat{a}_k, \hat{a}_l^\dagger] = \delta^3(\vec{k} - \vec{p})(2\pi)^3$.

$^6$Alternatively, the gravitational interaction can be put “by hand” into the Euler equation in Minkowski space.
The axion misalignment field can be represented as a coherent state\(^7\):

\[
|\phi\rangle = \frac{1}{N} \exp \left\{ \int \frac{d^3q}{(2\pi)^3} \phi(\vec{q})\hat{a}_i \right\} |0\rangle
\]  

(10)

where \(N\) is a normalisation factor such that \(\langle \phi|\phi\rangle = 1\). The coherent state is constructed such that the expectation value of the field operator is the classical field \(\phi(x)\):

\[
\langle \phi|\hat{\phi}(x)|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2T_{kk}}} \tilde{\phi}(\vec{k})e^{-i\vec{k} \cdot \vec{x}} = \phi(x) = \eta(x)e^{-i\theta(x)} = \eta(x)e^{-i(mt+S(x))}.
\]  

(11)

The Fourier mode expansion of \(\phi\) is convenient for linear perturbations in the early Universe\(^21\), and for evaluating expectation values in a coherent state. However, the parametrisation \(\phi = n e^{-i\theta}\) (with \(n\) and \(\theta\) real) is more appropriate for a classical field with a conserved number (a Bose Einstein condensate\?), and \(\theta = mt + S\) will be useful in the non-relativistic limit. A similar parametrisation is used by Daller-Rindler and Shapiro\(^23\).

### 3.1 The classical field

The stress-energy tensor for a complex scalar \(\phi = n e^{-i\theta}\), with potential \(V(\phi^\dagger \phi) = m^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2\) is

\[
T_{\mu\nu} = \partial_\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial_\mu \phi - g_{\mu\nu} (\partial^\alpha \phi^\dagger \partial_\alpha \phi - V(\phi^\dagger \phi)) \quad \text{(classical field)}
\]

\[
= 2\partial_\mu \eta \partial_\nu \eta + 2\eta^2 \partial_\nu \partial_\alpha \theta \partial^\alpha \eta - g_{\mu\nu} (\partial^\alpha \eta \partial_\alpha \eta + \eta^2 \partial_\alpha \theta \partial^\alpha \theta - V(\eta^2))
\]  

(12)

where \(T_{00} = \rho\), and \(T_{ij} = P\) (no sum on \(i\)).

The flat space equations of motion (gravity will be included by hand) are

\[
0 = \partial^\mu T_{\mu0} = \partial_\eta \left\{ 2\partial_\eta \partial^\mu \eta - 2\eta \partial_\alpha \theta \partial^\alpha \eta + \frac{\partial V}{\partial \eta} \right\} + \partial_\eta \left\{ \partial^\mu (\eta^2 \partial_\mu \theta) \right\}
\]  

(13)

where in curly brackets are the equations of motion that would be obtained from the Lagrangian. In particular, writing \(\theta = mt + S\) in the non-relativistic limit, the current conservation equation

\[
\partial^\mu (\eta^2 \partial_\mu \theta) = \partial_t \left( m\eta^2 \right) - \partial_j \left( \eta^2 \partial_j S \right)
\]  

(14)

becomes the continuity equation \(\partial^\mu T_{\mu0} = 0\), with the approximations

\[
T_{00} = \rho = 2m^2 \eta^2 + ...
\]

(15)

\[
T_{j0} = 2m\eta^2 \partial_j S + ...
\]  

(16)

where “...” can represent derivatives of \(\eta\) and \(S\) and the \(\lambda\eta^4\) interactions. With the identification \(v^j = -\partial_j S/m\), \(\partial^\mu T_{\mu0} = 0\) for the field is identical to eqn (7) for cold, non-interacting particles.

As is well-known\(^{25,17,23}\), a classical field has additional contributions to \(T_{ij}\) with respect to (6):

\[
T_{ij} = 2\partial_i \eta \partial_j \eta + \rho v_i v_j + \delta_{ij} \left( -\nabla \eta \cdot \nabla \eta - \rho |\vec{v}|^2 + 2m\eta^2 \partial_i S - \lambda \eta^4 \right),
\]  

(17)

where \(\lambda = -m^2/12f_Q^2\) from eq. 11. However, the equations of motion, given in\(^23\), are more useful for understanding the modified dynamics. From the first equation in curly brackets of (13)\(^3\) an Euler-like equation can be obtained:

\[
\rho \partial_t \vec{v} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\rho \nabla \psi - \rho \nabla Q - \nabla P_{SI}
\]  

(18)

where \(Q = -\frac{m^2}{2m\eta} \nabla^2 \eta\) describes the “quantum kinetic energy” or “quantum pressure”\(^{19}\) of the classical field, and \(P_{SI} = 2\lambda \eta^4\) is proportional to the pressure arising from the Self Interactions of the field. Comparing to (5), shows that the classical scalar field has extra forces\(^{23,19,23}\) compared to dust.

As noted by Rindler-Daller and Shapiro, the sign of \(P_{SI}\) is important: negative pressures and potential energies (such as the gravitational \(\psi\)) induce attractive forces, whereas positive \(P_{SI}\) would give a repulsive force, countering gravity. The axion potential of eqn (11) gives a negative \(\lambda \eta^4\) coupling, so the self-interactions of axions induce an attractive force which encourages the field to clump. This is a well-known behaviour for BE condensates in atomic traps\(^{19}\). Rindler-Daller and Shapiro look for stationary rotating solutions to eqn (18), which contain a vortex. They find that the “quantum pressure” of a scalar field as massive as the axion is insufficient for a vortex to be energetically favourable. And since the \(\lambda \eta^4\) coupling of axions is negative, the resulting force attracts axions towards the centre of the halo, which also discourages vortices\(^23\).

\(^7\)Alternatively, \(\partial^\mu T_{\mu j} = 0\), combined with current conservation, gives an equation for \(\partial_t S\). Taking its gradient gives the Euler-like eqn (18).


\section*{3.2 Cold particles}

First, it's worth to check that the usual WIMP-CDM eqns apply for the cold axion particles. This requires obtaining the “classical particle limit” from second-quantised field theory, which should be possible using Wigner transforms.

The stress-energy tensor can be introduced as a function of two space-time points $T_{\mu\nu}(x_1, x_2)$, separated by a small distance $\delta$. Then performing a Wigner transform\cite{51},

$$T_{\mu\nu}(X, Q) = \int \frac{d^4\delta}{(2\pi)^4} e^{iQ\cdot\delta} T_{\mu\nu}(X - \delta/2, X + \delta/2) , \quad (19)$$

allows to obtain a (classical) distribution of particles of three-momentum $\vec{Q}$, at point $X$. This classical approximation is expected to be acceptable because there is a separation of scales between the (galactic) distances parametrised by $\vec{X}$, and the inverse-axion-momentum scale $|\vec{\delta}|$ (the inverse-momentum of an axion of $10^{-4}$ eV with $v = 300\text{km/s}$ is a few metres). Intuitively, the coordinate $\vec{X}$ can be imagined discrete, so the galaxy is parametrised by a grid, with a cube in which particles are quantised at each point of the grid. Therefore, the creation and annihilation operators are also labelled by $X$, and the multi-particle state in which $\langle T_{\mu\nu}(X, Q) \rangle$ is evaluated can have a different number distribution of particles at each point.

The quantum field theory operator representing the density can be written (in the $\lambda \to 0$ limit)

$$\rho(X, Q) = \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2k_0p_0}} (k_0 p_0 + \vec{p} \cdot \vec{k} + m^2) \hat{a}^\dagger_k(X) \hat{a}_p(X) \frac{1}{(2\pi)^4} \delta^4(Q - \frac{p + k}{2}) e^{i(p - k) \cdot X} . \quad (20)$$

Evaluating $\rho(X, Q)$ in a multi-particle state $|n\rangle$, defined such that $\langle n| a_k(X)^\dagger a_p(X) |n\rangle = f(X, k) \delta^3(\vec{k} - \vec{p})/(2\pi)^3$, where $f(X, k)$ is the distribution in phase space of axion particles, and integrating over $Q$ (this reestablishes the correct mass dimensions for $T_{\mu\nu}$, and removes a leftover $\delta(p^2 - m^2)$), gives

$$\rho(X) = \int \frac{d^3q}{(2\pi)^3} q_0 f(X, q) , \quad (21)$$

which is the expected “classical particle” result. In the non-relativistic limit, $q_0 \approx m$, and $\rho(X) = mn(X)$.

The energy flux $T_{0i}(X - \delta/2, X + \delta/2)$ can be manipulated in a similar manner. If I neglect $\partial_i \hat{a}_k(X)$ with respect to $m \hat{a}_k(X)$ (the non-relativistic approximation), and $\partial_i \hat{a}_k(X)$ with respect to $k_i \hat{a}_k(X)$ (the separation of scales $\delta \ll X$ discussed after eqn (19)), then

$$T^{0i}(X) = \int \frac{d^3q}{(2\pi)^3} q^i f(X, q) , \quad (22)$$

(as given for instance in \cite{52}). Defining $v^i = T^{0i}/\rho$, this reproduces the top row of eqn [5].

\section*{3.3 What to do with this formalism?}

More interesting than the stress-energy tensors for axions, or their equations of motion, would be the solutions of the equations of motion. These are well-known in the period of linear structure growth, where the axion particles and field grow fluctuations as do WIMPs, because pressure is neglected in the equations of motion on the relevant scales.\cite{20,21} It is interesting to wonder whether the extra pressures of the classical axion field, as compared to axion particles or WIMPs, could be relevant during non-linear structure formation. Structure growth in the non-linear epoch can be studied numerically, so this could be addressed by developing a code where Dark Matter is evolved as a fluid. The pressure and viscosity of the axion field could then be included, allowing to identify observable consequences of axion field dynamics, distinguishing the misalignment axions from WIMPs. For such a code, it would be important to know the rate at which axions pass between the field and the cold particle bath; this is estimated in the next section.

\section*{4 Gravitational evaporation of the field?}

This section estimates the rate at which axions can transfer between the field and the bath of particles. This is interesting because the axion particles should behave like WIMPs (both have the stress-energy tensor of “dust”), whereas the field could cause structure to grow differently, due to its additional $T_{ij}$.

First, the rate mediated by graviton exchange is estimated, because it is infrared divergent, so could be large. Then I compare to the axion self-interaction rate. There are various issues to address for such an estimate: the order in $G_N$, the cutoff for the infrared divergent graviton exchange, the Bose enhancement factors from the high occupation number of axions, and how to include the axions from the classical field. They will be addressed in that order.

The estimate will turn out to be $O(G^2_N \lambda^2)$, corresponding to a cross-section representing “quantum” graviton exchange. I imagine that it is reasonable to estimate quantum gravity corrections using field theory at scales where field
theory is experimentally verified, despite that possible destabilisations of the axion potential from the Planck scale \[53\] are not discussed.

I assume that at \(O(G_N)\), the interactions between gravity and matter are given by Einstein’s Equations, where the stress-energy tensor is the matter-current to which gravity couples. As seen in the previous section, evaluating the expectation value of the operator corresponding to the stress-energy, in a state composed of a bath of axion particles and a classical field, does not give any “cross-terms”, in which appear the field and the particles. This is because any interaction between the field and the particles would change the number of particles, so the expectation value is zero in a particular state (corresponding to a distribution of particles and a classical field). Therefore at \(O(G_N)\), the misalignment field and the particles feel each others stress-energy, but do not exchange axions.

At \(O(G_N^2)\), the cross-section for the gravitational scattering of non-relativistic bosons (see figure 1) is given by DeWitt \[54\] as:

\[
\frac{d\sigma}{d\Omega} = \frac{G_N^2 m^2}{16} \left( \frac{1}{v^2 \sin^2(\theta/2)} + \frac{1}{v^2 \cos^2(\theta/2)} \right)^2
\]

Figure 1: Feynman diagrams for the gravitational scattering of an axion from the condensate \(\langle \phi \rangle\) with an axion from the bath, resulting in two axions in the bath.

where \(v\) is the three-velocity of an incident axion in the centre-of-mass frame, and I removed the annihilation contribution included in \[54\]. The cross-section is clearly infra-red divergent for small angle scattering, corresponding to soft graviton exchange. If one of the axion legs is in the coherent state representing the misalignment field, then the cross-section describes the passage of axions between the particle bath and the field. That is, it could be involved in the condensation of the particles to the field, or in the evaporation of the field to particles.

To determine the rate at which gravitational scattering moves axions between the field and the bath, it is clear that one must identify an infrared cutoff for \[23\]. I claim that it should be \(|\vec{\delta}|^{-1}\), of order the axion three-momentum, because \(|\vec{\delta}|\) was the spatial scale in the Wigner transform, below which there were particles. That is, gravitons couple to stress-energy (in an almost-flat space), because the stress-energy is the variation of the action with respect to the metric. Concretely, the matrix element for gravitational scattering of scalars given by DeWitt \[54\] is:

\[
\mathcal{M}[\phi(p_1) + \phi(p_2) \rightarrow \phi(p_3) + \phi(p_4)] = 16\pi G_N T^{\mu\nu}(p_1, p_3) \frac{g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma}}{(p_1 - p_3)^2} T^{\sigma\tau}(p_2, p_4) + \text{u channel},
\]

where \(T^{\mu\nu}(p_1, p_3) = \sqrt{\frac{2\pi E}{4m^2 E^2}}[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2 + m^2)]\) reduces to \(m^2 \delta^{\mu\nu}\delta^{\rho\sigma}\) in the non-relativistic limit. I claim that gravitons interact with individual axions, if the graviton momentum allows to see inside the box of volume \(|\vec{\delta}|^3\). On longer wavelengths, the graviton sees the stress-energy tensor of the particle distribution discussed in the previous section, and cannot be taken to have incoherent interactions with individual axions. That is, the longer-wavelength graviton interacts coherently with many axions, so these interactions must be summed in the amplitude, giving rise to graviton interactions with the stress-energy. Therefore the infrared cutoff of the \(O(G_N)^2\) cross-section, which represents incoherent graviton-axion interactions, should be the axion 3-momentum \(\sim m \times 10^{-3} c\), which gives

\[
\sigma_G \sim 10^{10} \frac{m^2}{m_{pl}^4}
\]

This cutoff is intuitive, because we expect a graviton of wavelength the size of the earth, to interact coherently with the earth, and not incoherently with the individual gluons and quarks which make it up. This also agrees with the scattering of an MeV photon on a proton, where the photon sees a point particle of charge +1, and not the quarks inside.

This cross-section can be compared to the rate at which an axion particle could scatter an axion out of the condensate via its \(\lambda\phi^4\) coupling.

\[
\sigma_\lambda \sim \frac{\lambda^2}{4\pi m^2} \sim \frac{m^2}{4\pi(4!)^2 f_{PQ}^4}
\]

\(^8\)Preskill, Wise and Wilck\[7\] estimate the rate at which a condensate evaporates via the six-axion coupling, in the process \{four condensate axions\} \rightarrow \{two axion particles\}. This is the lowest order kinematically allowed diagram for the case they consider, of a condensate made of zero-momentum axions in vacuum. If there are axion particles in the initial state, as considered here, they can scatter axions out of the condensate via the \(\lambda\phi^4\) coupling.
which exceeds eqn (28) for $f_{PQ} \lesssim 10^{-2} m_{pl}$. If the PQ phase transition is after inflation (as supposed here), then $\sigma_\lambda > \sigma_G$.

The high occupation numbers of the axion field (coherent state) and particles must be taken into account. In the familiar Boltzmann equation which would describe scatterings among axion particles, the rate at which the number density $n$ of axion particles changes, includes axions being scattered in minus out of each mode, so can be written

$$\frac{\partial n}{\partial t} = \int \frac{d^3 p_1}{2E_1(2\pi)^3} \frac{d^3 p_2}{2E_2(2\pi)^3} \frac{d^3 p_3}{2E_3(2\pi)^3} \frac{d^3 p_4}{2E_4(2\pi)^3} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times |\mathcal{M}|(\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4))^2 \left[f_1 f_2 (1 + f_3) (1 + f_4) - f_3 f_4 (1 + f_1) (1 + f_2)\right].$$

(27)

It is clear that $f_1 f_2 (1 + f_3) (1 + f_4) - f_3 f_4 (1 + f_1) (1 + f_2) \sim f^3$. So the rate at which axions move between the bath and the condensate due to gravity is proportional to the density of targets, multiplied by one Bose enhancement factor $f$, and can be estimated as

$$\Gamma \sim n_\phi \sigma_G f \sim 10^{13} \left(\frac{\rho_{DM}}{\rho_c}\right)^2 \left(\frac{H_0}{m}\right)^3 H_0$$

(28)

where $n_\phi \sim m n_f^2$ is the number density of axions in the field, and $f \sim n_\phi / (m^3 v^3)$ is the occupation number of axion particles of non-relativistic velocity $\lesssim v$ (this estimate applies when the field and particles make similar contributions to the dark matter density). The second estimate in eqn (28) is for the galaxy today, using $\rho_{DM} \sim m n_\phi \sim 0.3$ GeV/cm$^3$, $v \sim 10^{-5} c$, and that the Hubble rate today is $H_0 \sim \sqrt{\rho_c / m_{pl}}$ where $\rho_c$ is the critical density. With the amusing numerical coincidence that $H_0 \sim m^2 / m_{pl}$, one sees that this rate is negligible, compared to the expansion rate of our Universe today, because $H_0 / m \sim m / m_{pl} \lesssim 10^{-24}$. Therefore gravity does not move axions between the field and particle bath, within the age of the Universe.

A similar estimate, using the cross-section for axion self-interactions given in eqn (26), would give a rate amplified by a factor $\sim (\frac{m_{pl}^3}{\pi^2 f_{PQ}})^3$. This clearly cannot compensate the $\sim (m/m_{pl})^3$ factor, so it appears than neither gravity nor self-interactions can move axions between the field and bath within the age of the Universe.

Finally, in the above estimates, the axions in the coherent state were treated “like particles”, except with a different formula for the number density. This should be reasonable, as can be seen from the more correct analysis of [55].

5 Summary

The “classical field”, and the “density of cold particles”, are understandable phrases with which to describe the two contributions which axions can make to dark matter. Avoiding the issue of “Bose Einstein Condensation” allows to focus on the interesting question of how to distinguish axions from WIMPs. Various studies [25] suggest that during non-linear structure formation, cold particles and a classical field could grow galaxies differently, due to the extra pressures and viscosities of the field. For instance, the analytic analysis of Rindler-Daller and Shapiro (RDS) [24], includes a scalar field with the parameters of the QCD axion. RDS did not confirm that vortices in the halo are a signature of an axion field, but other observable differences could be identified by following the dynamics of galaxy formation (RDS look for stable solutions representing the galaxy today). So it would be interesting to model galaxy formation in the presence of scalar field dark matter, or more generally, to study non-linear structure formation with a code evolving dark matter as a fluid with pressure and viscosities.

For the case of axions, which can contribute two components to dark matter, it is relevant to estimate the rate at which axions could move between the field (distinguishable from WIMPs), and the cold particle bath. This note made simple estimates for the evaporation/condensation rate of the field in the presence of a bath of axion particles, due to gravitational interactions, or the $\lambda \phi^4$ self-interactions. Both rates were found to be negligible compared to the Hubble expansion rate, suggesting that the axion field remains coherent during the violent process of galaxy formation.

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