Hierarchical organization of cities and nations

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(10 May 2000)

Universality in the behavior of complex systems often reveals itself in the form of scale-invariant distributions that are essentially independent of the details of the microscopic dynamics. A representative paradigm of complex behavior in nature is cooperative evolution. The interaction of individuals gives rise to a wide variety of collective phenomena that strongly differ from individual dynamics—such as demographical evolution, cultural and technological development, and economic activity. A striking example of such cooperative phenomena is the formation of urban aggregates which, in turn, can be considered to cooperate in giving rise to nations. We find that population and area distributions of nations follow an inverse power-law behavior, as is known for cities. The exponents, however, are radically different in the two cases ($\mu \approx 1$ for nations, $\mu \approx 2$ for cities). We interpret these findings by developing growth models for cities and for nations related to basic properties of partition of the plane. These models allow one to understand the empirical findings without resort to the introduction of complex socio-economic factors.

The way in which urban aggregates are distributed was first investigated by Zipf who, half a century ago, observed that the population distribution of cities follows a power-law behavior with exponent $\mu \approx 2$. This Zipf law has a “universal” character since it holds at the world level as well as within a single nation, and the exponent is essentially independent of the area of the nation and its socio-economical conditions. More recently it has been observed that the area distribution of satellite cities, towns and villages around huge urban centers also obeys a power-law with exponent $\mu \approx 2$.

The remarkable universal result $\mu \approx 2$ has very recently attracted the attention of a number of physicists who model urban growth processes. Makse et al. using a correlated percolation model in the presence of a density gradient, reproduced the observed morphology of cities and the area distributions of subclusters in an urban system. Zanette and Manrubia proposed a stochastic model which generates intermittent spatiotemporal structures, and predicts a population distribution in agreement with that observed empirically. Very recently Marsili and Zhang proposed a model based on a master equation approach which is able to give a population distribution close to that obeyed by cities. The transition probabilities which enter the master equation were related to some estimate of the interactions among individuals living in the same city. Though the concept of interaction among human beings is not so clearly defined as that among particles, it is nevertheless true that individuals living in the same city are related to each other by a number of “links” which in the end define the very concept of city.

People living within the same nation are also related to each other—e.g., they share language and cultural heritage. Since the average “interactions” among the inhabitants of a nation may differ from those among people living in the same city, one could ask if the distribution of nations obeys the same law as that of cities. In order to answer to this question we analysed the population and area distributions of the world’s nations. The log-log plot of the population distribution $f(P)$ is shown in Fig. a for all nations of the world. In the same figure we show also the population distribution for the 140 largest city agglomerates of the USA. Both distributions obey a power law dependence, $f(P) \sim P^{-\mu_P}$. Regression fits give $\mu_P = 0.97$ for nations, and $\mu_P = 1.94$ for cities. Figure b shows a log-log plot of the area distribution $f(A)$ for all nations of the world as well as for nations belonging to a subset of all nations (Europe). In both cases $f(A)$ obeys a power law dependence, $f(A) \sim A^{-\mu_A}$ with exponent close to unity.

The strikingly different values of the exponent for cities and nations suggest fundamental differences in the historical and social processes that lead to such distributions. Here we propose a simple explanation of the distributions of nations and cities based only on geometrical considerations. We observe that, at least in principle, there is no restriction to the land accessible to a nation except that of the total existing land: a sufficiently powerful nation could expand to absorb all other nations. Cities, being the result of spontaneous aggregation of individuals around sites having attractive features, can form away.
from existing ones. This separates the plane region into land basins, and each new city spans a single basin. The resulting distribution of areas is not destroyed when a city expands to absorb nearby cities and gives rise to a compact urban aggregate. In fact, unlike nations, cities usually do not lose their land to neighbours: small towns and villages retain their identity and usually become administrative districts of the bigger aggregate (obeying, as shown in Ref. [13], the same area distribution that holds for separate cities). Thus a major difference in the way according to which land is occupied by cities and nations is that for cities the accessible land is fragmented into basins while it is not for nations.

Next we model the land occupation processes of nations and cities. We first note that the above considerations suggest that geometric properties may be behind the differences between city and nation distributions. Hence we analyse these processes as random partitions of the plane. Since different nations (or cities) do not all form at the same time, we consider partition processes in which the different portions are sequentially selected [9,10].

One of the simplest ways of partitioning a plane is to divide it using straight lines that are randomly oriented and positioned. Each line divides the region into two portions, of which the smaller is selected and the larger is further partitioned. This process is close in spirit to the way in which land is occupied by nations. Because this partition model resembles the positioning of fences, we refer to it as fence model.

This new model can be solved analytically. After $n$ partitions, the land “available” for further division is $A_n$, and $A_n = r_n A_{n-1}$, where $r_n$ is a random factor uniformly distributed between $1/2$ and 1. The area $T_n$ of the “taken” portion is $T_n = (1-r_n) A_{n-1}$, and its logarithm can be written as $\ln T_n = \sum_{i=1}^n \ln r_i + \ln(1-r_n) + \ln A_0$. If we plot on the $x$ axis the values of $\ln T_n$, we get random points, the average distance between which is equal to $-\langle \ln r_i \rangle = 1 - \ln 2 \approx 0.3$. The largest nation corresponds to almost an entire continent $A_0/2\ell$, and the smallest one corresponds to $A_0(1/2\ell)e^{-0.3n}$, where $n$ is the number of nations on the continent, and the factor $2\ell$ corresponds to the average $(\ln(1-r_d)) \approx -1.7$. Thus, the distribution of the logarithm of a nation’s area in each continent is a flat distribution between $\ln A_0 - 0.3n^{-1.7}$ and $\ln A_0^{-1.7}$ described by the probability density $P(\ln S) = 1/0.3n$.

The world has 5 continents; some, such as Australia and North America, have very few nations and some, such as Europe and Africa, have many. Because there are, on average, approximately $n = 50$ nations per continent, we can expect the approximate distribution of the logarithms of the number of nations to be uniformly distributed between the average largest country $\sim 10^5\text{km}^2$ and $10^2 \cdot e^{-15} \approx 3\text{km}^2$, which is consistent with a spread observed in the distribution of real nations. The flat distribution of the logarithms $\text{const} d(\log S)$ corresponds to the distribution $(\text{const} / S) \text{d}S$ of the areas, which is close to what we observe in Fig. 1b. Note, however, that, due to the few nations in each bin, a particular realization of the partition process described above may significantly deviate from the flat distribution we would expect.

The above model could be, however, oversimplified. This concern can be alleviated by incorporating into it the possibility that nations can evolve, growing or shrinking. Consider, e. g., the variant of the fence model (called, in the following, evolution model) in which with probability $1/2$ a nation can grow or shrink by some given amount. The $P(A)$ histogram, remarkably, is not affected. To see this, we first note that such change in area corresponds to the variable $\log(A)$ increasing or decreasing by a constant number, i.e., a simple random walk in the variable $\log(A)$. As we mention above, $P(A) \sim 1/A$ is equivalent to $P[\log(A)] \sim \text{const}$.

In the present case, the random walk is confined by reflecting boundaries: $A_{\text{max}} = A_0$ is some minimal size $A_{\text{min}}$ below which the nation is not stable. The distribution of a random walk confined in an interval with reflecting boundaries converges to a uniform distribution [3]. Thus $P[\log(A)]$ converges to a constant and hence $P(A) \sim 1/A$. The the $P(A)$ distribution is immune to the “noise” of growth and shrinking. The distribution of nation areas, simulated using the evolution model, is shown in Fig. 1c. The distribution of the logarithms of the nation areas are shown in Fig. 1d.

To model city distributions, we shall consider a radically different way of partitioning the plane. As suggested by the way urban geographers have thought about “central place” theory and the hierarchy of towns [11,12], we assume that cities have on the average a circular shape and thus can be approximately represented as circles. Land occupation by cities can thus be modeled through the partition of the plane in nonoverlapping circles. Each new portion is a circle (with radius chosen from a uniform distribution) at a randomly-chosen position, but outside of previously-selected circles. The maximal area of each new circle is limited by the distance from the closest existing circle. The resulting fragmentation of accessible land reduces the space available for the next circle more rapidly with respect to what happens for the portions generated through the fence model, thus making small portions much more probable. Hence we expect $\mu_A$ to be larger for the present model.

We simulate the circle model and find that the distribution of the circle areas follows a power-law behavior which is close to the empirical data of Fig.1a. A linear fit of the distribution (with the exclusion of the region affected by finite size effects) gives an exponent $\mu \approx 1.94$ (Fig.1e). One advantage of the circle model, compared to the previous models [13,14], is that it is based only on the geometrical features of the land occupation process.

Another advantage of this model is that it can be
solved analytically in the limit when the area of the newly-formed circle is a small number proportional to the area of the unoccupied land closest to the center of this circle, with proportionality coefficient $k \ll 1$. Suppose that there are $N$ circles in a region of total area $A_0$. Then the probability that a randomly chosen point is surrounded by an empty space of area larger than $X$ is governed by a Poisson distribution $F(X, N) = \exp(-XN/A_0)$. The number of circles with area between $A$ and $A+dA$ is $p(A, N) dA = N/(A_0k) \exp(-AN/A_0k)dA$, the derivative of the Poisson distribution function. The distribution of circles, $P(A)$, at some given instant when there are $N_c$ circles is given by the integral $\int_0^{N_c} p(A, N) dN = \frac{kA_0}{A^2}[1 - \exp(-AN_c/A_0k)]$. For large $N_c$, we recover $P(A) \sim A^{-2}$, so $\mu_A = 2$, in agreement with empirical findings. It is interesting to note that the exponent $\mu_A = 2$ is robust since, as our simulations show, it holds even in the case when $k$ is not a small number, but is any random number constrained only by the fact that two circles cannot overlap (as for urban agglomerates that consist of tightly-packed villages and towns).

In summary, we show, through the analysis of accurate geographical and demographical data, that the nation population and nation area distributions obey power-laws. The exponent is, however, surprising and completely unexpected. Moreover, its value (1) is quite different from the known exponent for cities (2). In addition to our empirical discovery, we propose an explanation for why the population distribution exponent of cities differs so strikingly from that of nations.

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We thank J. S. Andrade Jr., M. Batty, R. S. Dokholyan, I. Grosse, P. L. Krapivsky, H. A. Makse, M. Morrissey, and S. Redner for interesting and stimulating discussions. We thank NSF for support. N. V. D. is supported by NIH postdoctoral fellowship (GM20251-01).
FIG.1. (a) Double-logarithmic plot of the histogram of the population, $P$, of the 255 world nations and 140 largest urban agglomerates in the USA in mid-1997. For nations, the slope gives $\mu_P = 0.97$ and the linear regression coefficient is $R = 0.97$; for cities $\mu_P = 1.94$ and $R = 0.99$. For visual convenience, city data are multiplied by $10^2$. (b) Double-logarithmic plot of the histogram of areas, $A$, of the 255 nations of the world and 49 European nations; the corresponding coefficients are $R = 0.99$ and 0.99 respectively. European data are divided by $10^2$. The source for both plots is [http://www.stats.demon.nl](http://www.stats.demon.nl). (c) Double-logarithmic plot of the histogram of areas for the evolution model. (d) Distributions of logarithms of nations areas produced by (i) 255 nations and (ii) by the evolution model. (e) Double-logarithmic plot of the histogram of areas for the circle model; the linear regression coefficient is $R = 0.96$. 