Non-Abelian discrete symmetries and neutrino masses: two examples

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New Journal of Physics 6 (2004) 104
Received 17 May 2004
Published 9 August 2004
Online at http://www.njp.org/
doi:10.1088/1367-2630/6/1/104

Abstract. Two recent examples of non-Abelian discrete symmetries (S3 and A4) in understanding neutrino masses and mixing are discussed.

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1. Introduction

In the standard model of quark and lepton interactions, quark and charged-lepton masses come from the Yukawa couplings of the left-handed doublets (u, d)L and (ν, l)L with the right-handed singlets uR, dR and lR through the vacuum expectation value of the one scalar Higgs doublet.
The quark mixing matrix $V_{CKM}$ is then obtained from the mismatch in the diagonalization of the up and down quark mass matrices. Remarkably, $V_{CKM}$ turns out to be almost identically the unit matrix, i.e. quark mixing angles are all small (for three families, $V_{CKM}$ has three angles and one phase). On the other hand, the analogous $U_{MNS(P)}$, which is obtained from the mismatch in the diagonalization of the charged-lepton mass matrix and that of the neutrino mass matrix, is far from being the unit matrix. Whereas one angle is indeed small, the other two are definitely large. Indeed, to a good first approximation,

\[
U_{MNS(P)} \simeq \begin{pmatrix}
\sqrt{2/3} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix}.
\] (1)

In the convention

\[
U_{MNS(P)} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & -s_{23} \\
0 & s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13}e^{-i\delta} \\
0 & 1 & 0 \\
-s_{13}e^{i\delta} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & -s_{12} & 0 \\
s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix},
\] (2)

this means that

\[
\theta_{23} \simeq \pi/4, \quad \theta_{12} \simeq \tan^{-1}(-1/\sqrt{2}), \quad \theta_{13} \simeq 0,
\] (3)

which are consistent with the present experimental constraints [1]:

\[
\sin^2 2\theta_{23} > 0.91 \text{ (90\% CL)}, \quad 0.30 < \tan^2 \theta_{12} < 0.52 \text{ (90\% CL)}, \quad \sin^2 \theta_{13} < 0.067 \text{ (3\sigma)}.
\] (4)

There are a number of approaches in trying to understand the origin of quark and lepton mass matrices. Most of these attempts relate mixing angles with mass ratios. Historically, this was motivated by the phenomenologically successful ansatz $\theta_C \simeq \sqrt{m_d/m_s}$ for the Cabibbo angle. One often assumes that there is a symmetry behind this relationship. In addition, a better question to ask is perhaps whether or not there exists a family symmetry, which tells us that $V_{CKM} = 1$ and $U_{MNS(P)} \neq 1$. Obviously, if each family has its own Abelian (continuous or discrete) symmetry, then there is no mixing among families. That works well for $V_{CKM}$ but not for $U_{MNS(P)}$. If neutrino masses are purely Dirac, then the analogous structure of the quark and lepton sectors would definitely rule out the existence of such a family symmetry. However, if neutrino masses are Majorana, then it is indeed possible to have $V_{CKM} = 1$ and $U_{MNS(P)}$ as given by equation (1) as the result of a symmetry, as shown below. To fit the experimental data, small (radiative) corrections are needed from physics beyond the Standard Model.

2. $S_3$ for two families

2.1. Representations of $S_3$

The group of permutations of three objects is $S_3$. It is isomorphic to the group of three-dimensional rotations of an equilateral triangle to itself, i.e. the dihedral group $D_3$. It has six elements and three irreducible representations: $1$, $1'$, and $2$. As such, it is ideal for describing two families.
Since \( 1' \times 1' = 1 \), it is clear that a field transforming as \( 1' \) should have a \( Z_2 \) parity of \(-1\), i.e. \( \phi \rightarrow -\phi \). On the other hand, a doublet \((\phi_1, \phi_2)\) under \( S_3 \) has a choice of representations as long as it satisfies
\[
2 \times 2 = 1 + 1' + 2.
\]
Different representations are simply related by a unitary transformation. The most convenient representation of \( S_3 \) is a complex representation \[2, 3\] such that the products of the doublets \( \phi_1, \phi_2 \) and \( \psi_1, \psi_2 \) are given by
\[
\begin{align*}
\phi_1 \psi_2 + \phi_2 \psi_1 & \sim 1, \\
\phi_1 \psi_2 - \phi_2 \psi_1 & \sim 1', \\
(\phi_2 \psi_2, \phi_1 \psi_1) & \sim 2.
\end{align*}
\]
Note that \( \phi_1 \phi_1^* + \phi_2 \phi_2^* \) is an invariant; hence \( (\phi_2, \phi_1^*) \) is a doublet. Note also that \( S_3 \) has the special property that the symmetric product of three doublets, i.e. \( \phi_1 \psi_1 \chi_1 + \phi_2 \psi_2 \chi_2 \) is a singlet.

Specifically, the six group elements are the identity: \( e \), the cyclic and anti-cyclic permutations of three objects: \( g_c, g_a \), and the three interchanges of two objects leaving the third fixed: \( g_1, g_2, g_3 \). Their representation matrices are \([1, 1, 1, 1, 1, 1]\) in \( 1 \), \([ 1, 1, 1, -1, -1, -1]\) in \( 1' \) and
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\omega & 0 \\
0 & \omega \\
\omega^2 & 0 \\
0 & \omega^2 \\
0 & \omega \\
\omega & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}
\]
in \( 2 \), respectively, for \([e, g_c, g_a, g_1, g_2, g_3]\), where \( \omega = e^{2\pi i/3} \).

### 2.2. Quarks and leptons under \( S_3 \)

Consider a world of only two (i.e. the second and third) families of quarks and leptons. Choosing the convention that all fermions are left-handed, a natural assignment is \([4]\)
\[
\begin{align*}
Q_i &= (u_i, d_i), \\
L_i &= (\nu_i, l_i) \sim 2 & (i = 2, 3)
\end{align*}
\]
\[
u_c^i, d_c^i, l_c^i \sim 1, \quad u_c^i, d_c^i, l_c^i \sim 1'.
\]
To allow \( u, d \) and \( l \) to have Dirac mass terms, two scalar electroweak doublets \((\phi_0^i, \phi_0^{-i})(i = 2, 3)\) transforming as an \( S_3 \) doublet are required. The invariant leptonic Yukawa couplings are then
\[
L_Y = f_2(\phi_2 L_3 + \phi_3 L_2)l_2^c + f_3(\phi_2 L_3 - \phi_3 L_2)l_3^c + \text{h.c.},
\]
resulting in the \( 2 \times 2 \) mass matrix linking \( l_{2,3}^c \) to \( l_{2,3}^c \) below:
\[
\mathcal{M}_{llc} = \begin{pmatrix}
2v_2 & -3v_3 \\
2v_2 & 3v_3
\end{pmatrix},
\]
where \( v_i = \langle \phi_0^i \rangle \). On the other hand, the Majorana neutrino mass matrix depends on the product of \( L_i \) and \( L_j \), i.e. equation (5). Thus, a choice of scalar representations is available. Suppose two
scalar triplets \((\xi_i^{++}, \xi_i^+, \xi_i^0)\) \((i = 2, 3)\) transforming as an \(S_3\) doublet are used, then
\[
\mathcal{L}_Y = h(L_2L_2\xi_2 + L_3L_3\xi_3) + \text{h.c.},
\]
which leads to
\[
\mathcal{M}_v = \begin{pmatrix}
hu_2 & 0 \\
0 & hu_3
\end{pmatrix},
\]
where \(u_i = \langle \xi_i^0 \rangle\). Comparing equations (13) and (15), the lepton mixing matrix is easily obtained for \(v_2 = v_3 = v\), because equation (13) is diagonalized by
\[
U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\]
on the left and the unit matrix on the right, which of course means maximal mixing. The charged-lepton mass eigenvalues are then \(\sqrt{2}f_2v\) and \(\sqrt{2}f_3v\), whereas the Majorana neutrino mass eigenvalues are \(hu_2\) and \(hu_3\). They all can be different and yet maximal mixing is ensured. This depends of course on the condition \(v_2 = v_3\), which can be maintained in the Higgs potential by the interchange symmetry \(\phi_2 \leftrightarrow \phi_3\). The trilinear scalar couplings \(\mu_2\phi_2\phi_3\xi_2\) and \(\mu_3\phi_2\phi_3\xi_3\) break \(S_3\) softly but are invariant under \(\phi_2 \leftrightarrow \phi_3\). Hence, \(\mu_2 \neq \mu_3\) would imply \(u_2 \neq u_3\). Note that since \(m_\tau \gg m_\mu\) means \(f_3 \gg f_2\), the charged-lepton matrix is diagonalized by
\[
U^\dagger = \frac{1}{\sqrt{v_2^2 + v_3^2}} \begin{pmatrix}
v_2 & v_3 \\
-v_3 & v_2
\end{pmatrix}
\]
to a good approximation even if \(v_2 \neq v_3\). In that case, the mixing angle is given by \(\tan^{-1}(v_3/v_2)\), which can differ from \(\pi/4\).

In the quark sector, the \textit{down}-quark mass matrix has the same form as equation (13), i.e.
\[
\mathcal{M}_{d\bar{d}} = \begin{pmatrix}
f_2^d v_3 & -f_3^d v_3 \\
f_2^d v_2 & f_3^d v_2
\end{pmatrix},
\]
but the \textit{up}-quark mass matrix is given by
\[
\mathcal{M}_{u\bar{u}} = \begin{pmatrix}
f_2^u v_3^* & -f_3^u v_3^* \\
f_2^u v_2^* & f_3^u v_2^*
\end{pmatrix}
\]
instead, because \((\phi_3^*, \phi_2^*)\) must be used in place of \((\phi_2, \phi_3)\). This means that there is now a mismatch between the two diagonalized mass matrices and the mixing angle is given by [4]
\[
\theta_q = 2 \left[ \frac{\pi}{4} - \tan^{-1}(v_3/v_2) \right] = \frac{\pi}{2} - 2\theta_l.
\]
In this way, the smallness of the quark mixing between the second and third families is related to the deviation from maximal mixing in the \(\mu-\tau\) sector. To include the first family of quarks and leptons, \(S_3\) singlets must be used. Since either 1 or 1’ must be chosen, there has to be mixing of the first family into the 2–3 sector, but its exact form or magnitude cannot be fixed by \(S_3\) alone; see [4] for a specific successful application.
3. \( A_4 \) for three families

3.1. Representations of \( A_4 \)

The group of even permutations of four objects is \( A_4 \). It is isomorphic to the group of three-dimensional rotations of a regular tetrahedron, one of five perfect geometric solids known to the ancient Greeks and identified by Plato with the element ‘fire’. It is thus a discrete subgroup of \( \text{SO}(3) \). It is also isomorphic to \( \Delta(12) \), which is a discrete subgroup of \( \text{SU}(3) \) \[5\]. It has 12 elements and four irreducible representations: \( 1, 1', 1'' \) and \( 3 \), with the multiplication rule

\[
3 \times 3 = 1 + 1' + 1'' + 3 + 3,
\]

in analogy to equation (5) for \( S_3 \). As such, it is ideal for describing three families. Specifically, the products of the triplets \( \phi_{1,2,3} \) and \( \psi_{1,2,3} \) are given by \[6\]

\[
\begin{align*}
\phi_1 \psi_1 + \phi_2 \psi_2 + \phi_3 \psi_3 & \sim 1, \\
\phi_1 \psi_1 + \omega^2 \phi_2 \psi_2 + \omega \phi_3 \psi_3 & \sim 1', \\
\phi_1 \psi_1 + \omega \phi_2 \psi_2 + \omega^2 \phi_3 \psi_3 & \sim 1'', \\
(\phi_2 \psi_3, \phi_3 \psi_1, \phi_1 \psi_2) & \sim 3, \\
(\phi_3 \psi_2, \phi_1 \psi_3, \phi_2 \psi_1) & \sim 3,
\end{align*}
\]

where \( \omega = e^{2\pi i/3} \). Note that \( A_4 \) also has the special property that the symmetric product of three triplets, i.e.

\[
\phi_1 \psi_2 \chi_3 + \phi_1 \psi_3 \chi_2 + \phi_2 \psi_1 \chi_3 + \phi_2 \psi_3 \chi_1 + \phi_3 \psi_1 \chi_2 + \phi_3 \psi_2 \chi_1
\]

is a singlet.

Specifically, the 12 group elements are divided into four equivalence classes: \( C_1 \) contains only the identity, \( C_2 \) has 4 elements of order 3, \( C_3 \) also has 4 elements of order 3 and \( C_4 \) has 3 elements of order 2. The representation matrices in 3 are given by

\[
C_1: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2: \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}
\]
They are $[1, 1, 1, 1], [1, \omega, \omega^2, 1]$ and $[1, \omega^2, \omega, 1]$ in $\mathbb{1}, \mathbb{1}'$ and $\mathbb{1}''$, respectively.

3.2. Quarks and leptons under $A_4$

In analogy to equations (10) and (11) for $S_3$, a natural assignment for three families of quarks and leptons is [6]–[10]

$$Q_i = (u_i, d_i), \quad L_i = (\nu_i, l_i) \sim 3 \quad (i = 1, 2, 3)$$

$$u_1^c, d_1^c, l_1^c \sim 1, \quad u_2^c, d_2^c, l_2^c \sim 1', \quad u_3^c, d_3^c, l_3^c \sim 1''.$$  

To allow $u$, $d$ and $l$ to have Dirac mass terms, three scalar electroweak doublets $(\phi^0_i, \phi^-_i) \ (i = 1, 2, 3)$ transforming as an $A_4$ triplet are required. The invariant leptonic Yukawa couplings are then

$$\mathcal{L}_Y = f_1 (L_1 \phi_1 + L_2 \phi_2 + L_3 \phi_3) l_1^c + f_2 (L_1 \phi_1 + \omega L_2 \phi_2 + \omega^2 L_3 \phi_3) l_2^c + f_3 (L_1 \phi_1 + \omega^2 L_2 \phi_2 + \omega L_3 \phi_3) l_3^c + \text{h.c.},$$

where $1' \times 1'' = \mathbb{1}$ has been used. The resulting $3 \times 3$ mass matrix linking $l_{1,2,3}$ to $l_{1,2,3}^c$ is

$$\mathcal{M}_{llc} = \begin{pmatrix}
  f_1 v_1 & f_2 v_1 & f_3 v_1 \\
  f_1 v_2 & f_2 \omega v_2 & f_3 \omega^2 v_2 \\
  f_1 v_3 & f_2 \omega^2 v_3 & f_3 \omega v_3
\end{pmatrix},$$

which is diagonalized simply by

$$U_L^T = \frac{1}{\sqrt{3}} \begin{pmatrix}
  1 & 1 & 1 \\
  1 & \omega & \omega^2 \\
  1 & \omega^2 & \omega
\end{pmatrix}$$

on the left and the unit matrix on the right for $v_1 = v_2 = v_3 = v$. The charged-lepton mass eigenvalues are then $\sqrt{3} f_1 v, \sqrt{3} f_2 v$ and $\sqrt{3} f_3 v$, which are of course free to be chosen as $m_e, m_\mu$ and $m_\tau$. Since this matrix also diagonalizes the up and down-quark mass matrices, the resulting quark mixing matrix is just the unit matrix, i.e. $V_{\text{CKM}} = 1.$
3.3. Three degenerate neutrino masses

Consider now the $3 \times 3$ Majorana neutrino mass matrix. Since the product of $L_i$ and $L_j$ is given by equation (21), a choice of scalar representations is available, as for $S_3$ discussed earlier. The simplest choice is to have one scalar triplet $(\xi^+_{i1}, \xi^+_{i1}, \xi^0_{11})$ transforming as $1$ under $A_4$. In that case,

$$L_Y = h_1(L_1 L_1 + L_2 L_2 + L_3 L_3)\xi_1 + \text{h.c.},$$

resulting in three degenerate neutrino masses, i.e.

$$M_\nu = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & m_0 \end{pmatrix},$$

where $m_0 = 2h_1\langle \xi^0_1 \rangle$. In the $(e, \mu, \tau)$ basis, it becomes

$$M_{\nu}^{(e,\mu,\tau)} = U_L^\dagger M_\nu U_L^* = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & 0 & m_0 \\ 0 & m_0 & 0 \end{pmatrix}.$$

From the high scale where $A_4$ is broken to the electroweak scale, one-loop radiative corrections will change equation (39) to

$$\begin{pmatrix} m_0 & 0 & 0 \\ 0 & 0 & m_0 \\ 0 & m_0 & 0 \end{pmatrix} + R \begin{pmatrix} m_0 & 0 & 0 \\ 0 & 0 & m_0 \\ 0 & m_0 & 0 \end{pmatrix} R^T,$$

where the radiative correction matrix is assumed to be of the most general form, i.e.

$$R = \begin{pmatrix} r_{ee} & r_{e\mu} & r_{e\tau} \\ r_{e\mu}^* & r_{\mu\mu} & r_{\mu\tau} \\ r_{e\tau}^* & r_{\mu\tau}^* & r_{\tau\tau} \end{pmatrix}.$$

Thus, the observed neutrino mass matrix is given by

$$M_\nu = m_0 \begin{pmatrix} 1 + 2r_{ee} & r_{e\tau} + r_{e\mu}^* & r_{e\mu} + r_{e\tau}^* \\ r_{e\tau}^* + r_{e\mu} & 2r_{\mu\tau} & 1 + r_{\mu\mu} + r_{\tau\tau} \\ r_{e\tau} + r_{e\mu} & 1 + r_{\mu\mu} + r_{ee} & 2r_{\mu\tau}^* \end{pmatrix}.$$
Then using the redefinitions:

\[ \delta_0 \equiv r_{\mu\mu} + r_{\tau\tau} - r_{\mu\tau} - r_{\mu\tau}^* \tag{43} \]

\[ \delta \equiv 2r_{\mu\tau} \tag{44} \]

\[ \delta' \equiv r_{ee} - \frac{1}{2}r_{\mu\mu} - \frac{1}{2}r_{\tau\tau} - \frac{1}{2}r_{\mu\tau} - \frac{1}{2}r_{\mu\tau}^* \tag{45} \]

\[ \delta'' \equiv r_{e\mu} + r_{e\tau} \tag{46} \]

it becomes

\[ M_\nu = m_0 \begin{pmatrix} 1 + \delta_0 + \delta + \delta^* + 2\delta'' & \delta'' & \delta^* \\ \delta'' & \delta & 1 + \delta_0 + (\delta + \delta^*)/2 \\ \delta^* & 1 + \delta_0 + (\delta + \delta^*)/2 & \delta \end{pmatrix}. \tag{47} \]

Without loss of generality, \( \delta \) may be chosen real by absorbing its phase into \( \nu_\mu \) and \( \nu_\tau \) and \( \delta_0 \) set equal to zero by redefining \( m_0 \) and the other \( \delta \)s. As a result,

\[ \sin^2 2\theta_{atm} \simeq 1, \quad \Delta m_{atm}^2 \simeq 4\delta m_0^2, \quad U_{e3} \simeq \frac{i \text{Im } \delta''}{\sqrt{2}\delta}, \tag{48} \]

\[ \Delta m_{sol}^2 \simeq 4\sqrt{(\delta')^2 + 2(\text{Re } \delta'')^2} m_0^2, \quad \tan \theta_{sol} \simeq \frac{\sqrt{2} \text{Re } \delta''}{\sqrt{(\delta')^2 + 2(\text{Re } \delta'')^2} - \delta'}, \tag{49} \]

where \( \delta' = \delta + (\text{Im } \delta'')^2/2\delta < 0 \).

Thus, this model explains \( \theta_{23} \simeq \pi/4 \) and predicts three nearly degenerate neutrino masses with neutrinoless double beta decay given by \( |m_0| \). Since \( \delta \) is a radiative correction, it cannot be too large. Given that \( \Delta m_{atm}^2 \) is known to be of order \( 10^{-5} \) eV\(^2\), \( m_0 \) cannot be much smaller than about 0.3 eV. Remarkably, this is also the upper limit on neutrino mass from the large-scale structure of the Universe \cite{11} and possibly the value of \( |m_0| \) as measured in neutrinoless double beta decay \cite{12}.

In the Standard Model, there are no flavour-changing leptonic interactions; thus \( \delta = \delta'' = 0 \) and equation (47) does not lead to neutrino oscillations at all. However, if there is some new physics which allows all the \( \delta \)s to be nonzero, then equation (47) can be realistic. A recent detailed example \cite{9} is available in the context of supersymmetry with arbitrary soft supersymmetry breaking terms.
3.4. Arbitrary neutrino masses

In addition to $\xi_1$ transforming as $1$ under $A_4$, consider $\xi_2$, $\xi_3$ and $\xi_{4,5,6}$ transforming as $1'$, $1''$ and $\bar{3}$ as well [10]. In that case, $\mathcal{M}_\nu$ in the original basis is given by

$$\mathcal{M}_\nu = \begin{pmatrix} a + b + c & 0 & 0 \\ 0 & a + \omega b + \omega^2 c & d \\ 0 & d & a + \omega^2 b + \omega c \end{pmatrix},$$

(50)

where $a$ comes from $\langle \xi_1^{(0)} \rangle$, $b$ from $\langle \xi_2^{(0)} \rangle$, $c$ from $\langle \xi_3^{(0)} \rangle$ and $d$ from $\langle \xi_{4,5,6}^{(0)} \rangle$, assuming that $\langle \xi_{5,6}^{(0)} \rangle = 0$.

In the basis where the charged-lepton mass matrix is diagonal, the neutrino mass matrix becomes

$$\mathcal{M}_{\nu}^{(e, \mu, \tau)} = U_L^\dagger \mathcal{M}_\nu U_L^* = \begin{pmatrix} a + (2d/3) & b - (d/3) & c - (d/3) \\ b - (d/3) & c + (2d/3) & a - (d/3) \\ c - (d/3) & a - (d/3) & b + (2d/3) \end{pmatrix}.$$  

(51)

This matrix has one obvious eigenstate, i.e. $\nu_2 = (\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$ with the eigenvalue $m_2 = a + b + c$. Let

$$U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix},$$

(52)

then in the basis defined by this transformation, i.e.

$$\nu_1 = \sqrt{2/3} \nu_e - \frac{1}{\sqrt{6}} (\nu_\mu + \nu_\tau),$$

(53)

$$\nu_2 = \frac{1}{\sqrt{3}} (\nu_e + \nu_\mu + \nu_\tau),$$

(54)

$$\nu_3 = \frac{1}{\sqrt{2}} (-\nu_\mu + \nu_\tau),$$

(55)

the neutrino mass matrix of equation (51) rotates to

$$\mathcal{M}_{\nu}^{(1,2,3)} = U^\dagger \mathcal{M}_{\nu}^{(e, \mu, \tau)} U^* = \begin{pmatrix} m_1 & 0 & m_4 \\ 0 & m_2 & 0 \\ m_4 & 0 & m_3 \end{pmatrix}.$$  

(56)
where
\[
\begin{pmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4
\end{pmatrix} = \begin{pmatrix}
1 & -1/2 & -1/2 & 1 \\
1 & 1 & 1 & 0 \\
-1 & 1/2 & 1/2 & 1 \\
0 & -\sqrt{3}/2 & \sqrt{3}/2 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}.
\] (57)

In the limit \(m_4 = 0\), equation (56) is diagonal and \(U\) becomes the neutrino mixing matrix of equation (1) with the prediction \(\tan^2 \theta_{12} = 1/2\), as well as \(\sin^2 2\theta_{23} = 1\) and \(\theta_{13} = 0\). This is of course a well-known ansatz [13], but has only just been derived from the symmetry of a complete theory, without arbitrary assumptions regarding its charged-lepton sector, in [10].

Note that \(m_1, m_2, m_3\) in the above are all arbitrary. In other words, the mixing angles are determined without regard to the masses, just as in the quark sector. There is however an important difference. Whereas all quark mixing angles are zero, the lepton mixing angles are not. Additional small corrections from physics beyond the Standard Model, such as supersymmetry [9, 14], are of course necessary to modify these predictions to coincide with present data.

Experimentally, \(|U_{e3}|\) is known to be small, i.e. \(m_4 = (\sqrt{3}/2)(c - b)\) may be considered small compared with \(|b + c|\). Now \(\Delta m^2_{\text{atm}} \gg \Delta m^2_{\text{sol}}\) implies that either \(d \simeq (3/2)(b + c)/2\) or \(d \simeq -2a - (b + c)/2\). If \(d \simeq (3/2)(b + c)\), then
\[
m_{1,2} \simeq a + b + c, \quad m_3 \simeq -a + 2(b + c).
\] (58)

If \(d \simeq -2a - (b + c)/2\), then
\[
m_{1,2} \simeq a + b + c, \quad m_3 \simeq -3a.
\] (59)

Either one will allow a normal hierarchy or an inverted hierarchy or nearly degenerate masses.

If \(m_4 \neq 0\), \(v_1\) mixes with \(v_3\), but \(v_2\) remains the same. Let the new mass eigenstates be
\[
v'_1 = v_1 \cos \theta + v_3 e^{i\delta} \sin \theta, \quad v'_3 = -v_1 e^{-i\delta} \sin \theta + v_3 \cos \theta,
\] (60)

then the new mixing matrix \(U\) has elements
\[
U_{e1} = \sqrt{2/3} \cos \theta, \quad U_{e2} = \frac{1}{\sqrt{3}}, \quad U_{e3} = -\sqrt{2 / 3} e^{i\delta} \sin \theta,
\] (61)
\[
U_{\mu 3} = -\frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{6}} e^{i\delta} \sin \theta = -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{3}{2} |U_{e3}|^2} - \frac{1}{2} U_{e3}.
\] (62)

Therefore, the experimental constraint [15]
\[
|U_{e3}| < 0.16
\] (63)
implies
\[
0.61 < |U_{\mu 3}| < 0.77,
\] (64)

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or, using \( \sin^2 2\theta_{\text{atm}} = 4|U_{\mu 3}|^2 (1 - |U_{\mu 3}|^2) \),

\[
0.94 < \sin^2 2\theta_{\text{atm}} < 1. \tag{65}
\]

Similarly, using \( \tan^2 \theta_{\text{sol}} = |U_{e 3}|^2 / |U_{e 1}|^2 \),

\[
0.5 < \tan^2 \theta_{\text{sol}} < 0.52 \tag{66}
\]

is obtained. Whereas equation (65) is well satisfied by the current data, equation (66) is at the high end of the 2\(\sigma\)-allowed range centred at \( \tan^2 \theta_{\text{sol}} \simeq 0.4 \) [16].

If future experimental measurements persist in getting a value of \( \tan^2 \theta_{\text{sol}} \) outside the range predicted by equation (66), one possible explanation within the context of this model is through radiative corrections. Just as equation (39) is radiatively corrected to become equation (47), equation (51) may also get corrected so that \( \nu_1 \) mixes with \( \nu_2 \) in equation (56). For example, if \( b, c, d < a \), then combining equations (47) and (51) with \( b = c \),

\[
\tan 2\theta_{\text{sol}} \simeq -2\sqrt{2} \left[ \frac{b - (d/3) + \delta'' a}{b - (d/3) - 2\delta' a} \right], \tag{67}
\]

where

\[
\delta'' = \delta_{\mu \mu} + \delta_{\tau \tau}, \quad \delta' = \delta_{ee} - \frac{1}{2}(\delta_{\mu \mu} + \delta_{\tau \tau}) - \delta_{\mu \tau}, \tag{68}
\]

and \( \tan^2 \theta_{\text{sol}} \simeq 0.4 \) is obtained if \( [b - (d/3) + \delta'' a]/[b - (d/3) - 2\delta' a] \simeq 0.75 \). Note that, this may occur even if \( \delta_{\alpha \beta} = 0 \) for \( \alpha \neq \beta \), i.e. in the absence of flavour-changing radiative corrections, in contrast with the requirement of [8, 9].

### 4. Conclusion

The non-Abelian discrete symmetry \( S_3 \) is ideal for explaining maximal mixing in the \( \mu-\tau \) sector with a normal hierarchy of neutrino masses. In a specific application [4], it also explains why \( U_{e 3} \) is small but non-zero.

The non-Abelian discrete symmetry \( A_4 \) is a natural candidate for describing three families of quarks and leptons. Whereas Dirac fermion masses come from the decomposition

\[
\bar{3} \times (1 + 1' + 1'') = \bar{3}, \tag{69}
\]

Majorana neutrino masses come from the decomposition

\[
\bar{3} \times \bar{3} = 1 + 1' + 1'' + \bar{3}. \tag{70}
\]

The mismatch between the quark mass matrices is then naturally given by \( V_{\text{CKM}} = 1 \), whereas that between the charged-lepton and neutrino mass matrices is definitely not the unit matrix, but rather \( U_{\text{MNS}}(P) \) of equation (1) in a certain symmetry limit, thus predicting a relationship among \( \theta_{23}, \theta_{12} \) and \( \theta_{13} \). Specifically, if \( \theta_{13} = 0 \), then \( \sin^2 2\theta_{23} = 1 \) and \( \tan^2 \theta_{12} = 0.5 \), independent of the values of the three neutrino masses. (Note that all six quarks and all three charged leptons have names, but the three neutrinos do not, as yet.) To obtain small nonzero quark mixing angles as well as deviations from the lepton-mixing angles constrained by this model, new physics beyond the Standard Model is expected, such as supersymmetry at the TeV scale.

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Acknowledgments

We thank the Institute for Particle Physics Phenomenology, Durham for hospitality. This work was supported in part by the US Department of Energy under grant no. DE-FG03-94ER40837.

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