Operator-algebraic superrigidity for
\(SL_n(\mathbb{Z}), \ n \geq 3\)

Bachir Bekka

July 6, 2018

Abstract

For \(n \geq 3\), let \(\Gamma = SL_n(\mathbb{Z})\). We prove the following superrigidity result for \(\Gamma\) in the context of operator algebras. Let \(L(\Gamma)\) be the von Neumann algebra generated by the left regular representation of \(\Gamma\). Let \(M\) be a finite factor and let \(U(M)\) be its unitary group. Let \(\pi : \Gamma \to U(M)\) be a group homomorphism such that \(\pi(\Gamma)^{\prime\prime} = M\). Then either

(i) \(M\) is finite dimensional, or

(ii) there exists a subgroup of finite index \(\Lambda\) of \(\Gamma\) such that \(\pi|_{\Lambda}\) extends to a homomorphism \(U(L(\Lambda)) \to U(M)\).

This answers, in the special case of \(SL_n(\mathbb{Z})\), a question of A. Connes discussed in [Jone00, Page 86]. The result is deduced from a complete description of the tracial states on the full \(C^*\)-algebra of \(\Gamma\).

As another application, we show that the full \(C^*\)-algebra of \(\Gamma\) has no faithful tracial state, thus answering a question of E. Kirchberg.

1 Introduction

Two major achievements in the study of discrete subgroups in semi-simple Lie groups are Mostow’s rigidity theorem and Margulis’ superrigidity theorem. A weak version of the latter is as follows. Let \(\Gamma\) be a lattice in a simple real Lie group \(G\) with finite centre and with \(\mathbb{R} - \text{rank}(G) \geq 2\). Let \(H\) be another simple real Lie group with finite centre, and let \(\pi : \Gamma \to H\) be a homomorphism such that \(\pi(\Gamma)\) is Zariski-dense in \(H\). Then, either \(H\) is
compact or there exists a finite index subgroup $\Lambda$ of $\Gamma$ such that $\pi|_\Lambda$ extends to a continuous homomorphism $G \to H$. For more general results, see [Marg91] and [Zimm84]. Moreover, as shown by Corlette, the superrigidity theorem continues to hold for the simple real Lie groups $G$ with $\mathbb{R} - \text{rank}(G) = 1$ which are not locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$.

In the theory of von Neumann algebras, discrete groups (as well as their actions) always played a prominent rôle. To a discrete group $\Gamma$ is associated a distinguished von Neumann algebra $L(\Gamma)$, namely the von Neumann algebra generated by the left regular representation $\lambda_\Gamma$ of $\Gamma$; thus, $L(\Gamma)$ is the closure for the strong operator topology of the linear span of $\{\lambda_\Gamma(\gamma) : \gamma \in \Gamma\}$ in the algebra $\mathcal{L}(l^2(\Gamma))$ of all bounded operators on the Hilbert space $l^2(\Gamma)$.

The first rigidity result in the context of operator algebras is the result by A. Connes [Conn80] showing that, for a group $\Gamma$ with Kazhdan’s Property (T), the group of outer automorphisms of $L(\Gamma)$ is countable. A major problem in this area is whether such a group $\Gamma$ can be reconstructed from its von Neumann algebra $L(\Gamma)$. In recent years, a series of remarkable results concerning this question, with applications to ergodic theory, have been obtained by S. Popa ([Popa06-a], [Popa06-b]; for an account, see [Vae06]). Other relevant work includes [CoHa89] and [Furm99].

The purpose of this paper is to discuss another kind of rigidity, namely the rigidity of a discrete group in the unitary group of its von Neumann algebra. If $\Gamma$ is a discrete group, we view $\Gamma$ as a subgroup of the unitary group $U(L(\Gamma))$ of $L(\Gamma)$, that is, the group of the unitary operators in $L(\Gamma)$. It was suggested by Connes (see [Jone00, Page 86]) that, for $\Gamma$ as in the statement of Margulis’ theorem, a superrigidity result should hold in which $G$ above is replaced by $U(L(\Gamma))$ and $H$ by the unitary group $U(M)$ of a type $II_1$ factor. We prove such a superrigidity result in the case $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$.

Recall that a von Neumann algebra $M$ is a factor if the centre of $M$ is reduced to the scalar operators. The von Neumann algebra $M$ is said to be finite if there exists a finite normal faithful trace on $M$. A finite factor is a type $II_1$ factor which is infinite dimensional. Recall also that $L(\Gamma)$ is a finite von Neumann algebra. Moreover, $L(\Gamma)$ is a factor if and only if $\Gamma$ is an ICC-group, that is, if all its conjugacy classes, except $\{e\}$, are infinite. For an account on the theory of von Neumann algebras, see [Dix-vN].

**Theorem 1** Let $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$. Let $M$ be a finite factor and let $U(M)$ its unitary group. Let $\pi : \Gamma \to U(M)$ be a group homomorphism.
Assume that the linear span of $\pi(\Gamma)$ is dense in $M$ for the strong operator topology. Then either

(i) $M$ is finite dimensional, that is, $M$ is isomorphic to a matrix algebra $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, in which case $\pi$ factorizes to a multiple of an irreducible representation of some congruence quotient $SL_n(\mathbb{Z}/N\mathbb{Z})$ for $N \in \mathbb{N}$, or

(ii) there exists a subgroup of finite index $\Lambda$ of $\Gamma$ such that $\pi|_{\Lambda}$ extends to a normal homomorphism $L(\Lambda) \to M$ of von Neumann algebras. In particular, $\pi|_{\Lambda}$ extends to a group homomorphism $U(L(\Lambda)) \to U(M)$.

Let $C$ be the centre of $SL_n(\mathbb{Z})$. Observe that $C$ is trivial for odd $n$ and $C = \{\pm I\}$ for even $n$. If, in the statement of the theorem above, we take instead $\Gamma = PSL_n(\mathbb{Z}) = SL_n(\mathbb{Z})/C$, then $L(\Gamma)$ is a factor and the conclusion (ii) holds for $\Lambda = \Gamma$.

The method of proof of Theorem 1 can be adapted to establish the same result when $\Gamma$ is the symplectic group $Sp_{2n}(\mathbb{Z})$ for $n \geq 2$; it works presumably for the group of integral points of any Chevalley group of rank $\geq 2$. No such result can be true for the modular group $SL_2(\mathbb{Z})$; see Remark 4 below.

Remark 2 (i) Let $\Gamma$ be a countable ICC–group with Kazhdan’s Property (T). It was shown in [CoJo85] that $L(\Gamma)$ cannot be a subfactor of $L(F_2)$, where $F_2$ is a non-abelian free group. This, combined with Theorem 1, shows that every representation of $PSL_n(\mathbb{Z})$ for $n \geq 3$ into $U(L(F_2))$ decomposes as a direct sum of finite dimensional representations. This is a special case of a result of G. Robertson [Robe93] valid for all groups with Property (T). For a related work, see [Vale97].

(ii) Let $M$ be a finite factor. The unitary group $U(M)$ has as centre a copy of the circle group $S^1$, namely the unitary scalar operators. It was shown in [Harp79] that the projective unitary group $U(M)/S^1$ of $M$ is a simple group.

The result in Theorem 1 amounts to the classification of the characters of $\Gamma$ (see Section 7), that is, the functions $\varphi : \Gamma \to \mathbb{C}$ with the following properties:

- $\varphi$ is central, that is, $\varphi(\gamma x \gamma^{-1}) = \varphi(x)$ for all $\gamma, x \in \Gamma$,
- $\varphi$ is positive definite, that is, $\sum_{i=1}^n \overline{c_j} c_i \varphi(\gamma_j^{-1} \gamma_i) \geq 0$ for all $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$.
\( \varphi \) is normalized, that is, \( \varphi(e) = 1 \),

\( \varphi \) is indecomposable, that is, \( \varphi \) cannot be written in a non-trivial way as a convex combination of two central positive definite normalized functions.

There are two obvious examples of characters of a group \( \Gamma \). First of all, the normalized character (in the usual sense) of an irreducible finite dimensional unitary representations of \( \Gamma \) is a character of \( \Gamma \) in the above sense. For \( \Gamma = SL_n(\mathbb{Z}) \), \( n \geq 3 \), it is well-known that every such representation factorizes through some congruence quotient \( SL_n(\mathbb{Z}/N\mathbb{Z}) \) for an integer \( N \); this a consequence of the solution of the congruence subgroup problem (see [BaMS67], [Menn65]; see also [Stei85]). Much is known about the characters of the finite groups \( SL_n(\mathbb{Z}/N\mathbb{Z}) \); see [Zele81].

Let \( C \) be the centre of the group \( \Gamma \). Assume that all conjugacy classes, except those of the elements from \( C \), are infinite. Then, for every unitary character \( \chi \) of the abelian group \( C \), the trivial extension \( \widetilde{\chi} \) of \( \chi \) to \( \Gamma \), defined by \( \widetilde{\chi} = 0 \) on \( \Gamma \setminus C \), is a character of \( \Gamma \). In particular, if \( \Gamma \) is ICC, then \( \delta_e \), the Dirac function at \( e \), is a character of \( \Gamma \). When \( n \) is even, all conjugacy classes of \( PSL_n(\mathbb{Z}) \), except \( \{I\} \) and \( \{-I\} \), are infinite.

Our main result says that \( SL_n(\mathbb{Z}) \) for \( n \geq 3 \) has no characters other than the obvious ones described above.

**Theorem 3** Let \( \varphi \) be a character of \( SL_n(\mathbb{Z}) \) for \( n \geq 3 \). Then, either

(i) \( \varphi \) is the character of an irreducible finite dimensional representation of some congruence quotient \( SL_n(\mathbb{Z}/N\mathbb{Z}) \) for \( N \geq 1 \), or

(ii) \( \varphi \) is the trivial extension \( \widetilde{\chi} \) of a character \( \chi \) of the centre of \( SL_n(\mathbb{Z}) \).

**Remark 4** No classification of the characters of the modular group \( SL_2(\mathbb{Z}) \) can be expected. Indeed, this group contains the free non-abelian group \( F_2 \) on two generators as normal subgroup. Every character of \( F_2 \) extends to a character on \( SL_2(\mathbb{Z}) \). Now, \( F_2 \) has a huge number of characters: if \( M \) is any finite factor with trace \( \tau \), every pair of unitaries in \( M \) defines a homomorphism \( \pi : F_2 \to U(M) \) and a corresponding character \( \tau \circ \pi \) on \( F_2 \).

The problem of the description of the characters of a discrete group \( \Gamma \) has been considered by several authors. E. Thoma [Thom64b] solved
this problem for the infinite symmetric group $S_\infty$ (see also VerKe81), H - L. Skudlarek [Skud76] for the group $\Gamma = GL(\infty, \mathbb{F})$, where $\mathbb{F}$ is a finite field, and D. Voiculescu [Voic76] for $\Gamma = U(\infty)$; see also StVo75 and Boye83. A.A. Kirillov [Kiri65] described the characters of $\Gamma = GL_n(\mathbb{K})$ or $SL_n(\mathbb{K})$ for $n \geq 2$, where $\mathbb{K}$ is an infinite field (see also Rose89 and Ovci71).

Our proof of Theorem 3 is based on an analysis of the restriction $\varphi|_V$ of a given character $\varphi$ of $SL_n(\mathbb{Z})$ to various copies $V$ of $\mathbb{Z}^{n-1}$. We will see that we have a dichotomy corresponding to the two different types of characters from Theorem 3: either the measure on the torus $\mathbb{T}^{n-1}$ associated to $\varphi|_V$ is atomic or this measure is the Lebesgue measure for every $V$. An important ingredient in our analysis is the solution of the congruence subgroup for $SL_n(\mathbb{Z})$ for $n \geq 3$.

The result of Theorem 3 can be interpreted as a classification of the traces on the full $C^*$-algebra $C^*(\Gamma)$ of $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$ (see Section 2).

E. Kirchberg asked in [Kirc93, Remark 8.2, page 487] whether the full $C^*$-algebra of $SL_4(\mathbb{Z})$ has a faithful trace. He was motivated by the fact that a positive answer to this question would imply a series of outstanding conjectures in the theory of von Neumann algebras (see Section 8). As a consequence of Theorem 3, we will see that the answer to Kirchberg’s question is negative, namely:

**Corollary 5** The full $C^*$-algebra of $SL_n(\mathbb{Z})$ has no faithful tracial state for $n \geq 3$.

In fact, we will prove the stronger result Corollary 19 below.

Recall that the reduced $C^*$-algebra $C^*_r(\Gamma)$ of a group $\Gamma$ is the closure of the linear span of $\{\lambda_\gamma(\gamma) : \gamma \in \Gamma\}$ in $L(\ell^2(\Gamma))$ for the operator norm. Recall also that $\delta_e$ factorizes to a faithful tracial state on $C^*_r(\Gamma)$. The finite dimensional representations of $PSL_n(\mathbb{Z})$ do not factorize through $C^*_r(PSL_n(\mathbb{Z}))$, since $PSL_n(\mathbb{Z})$ is not amenable. As a consequence, Theorem 3 implies that $\delta_e$ is the unique tracial state on $C^*_r(PSL_n(\mathbb{Z}))$. This also follows from BeCH95, where a different method is used.

Theorem 3 leaves open the problem of existence of infinite, semi-finite traces on $C^*(SL_n(\mathbb{Z}))$. We do not know whether such traces exist. Using BeCH95, we can only show that no such trace exists on $C^*_r(PSL_n(\mathbb{Z}))$. In fact, this result is valid for a more general class of groups including $PSL_2(\mathbb{Z})$ (see Proposition 21 below).

This paper is organized as follows. Sections 2 and 3 are devoted to some general facts. The proof of Theorem 3 is spread over three sections: in
Section 4 we show that the proof splits into two cases which are then treated accordingly in Sections 5 and 6. In Section 7 we show that Theorem 1 is a consequence of Theorem 3. Corollary 5 is proved in Section 8 and Section 9 is devoted to a remark on the problem of the existence of infinite traces.

Acknowledgments We are grateful to S. Popa who pointed out to us Connes’ question from [Jone00] and suggested to emphasize the superrigidity result Theorem 1. Thanks are also due to E. Blanchard, E. Kirchberg, and P. de la Harpe, for interesting comments.

2 Factor representations and characters

We review some general facts concerning the relationships between central positive definite functions on groups and factor representations. Details can be found in [Dix-C*] Chapters 6 and 17 or [Thom64a].

Let \( \Gamma \) be a discrete group. We are interested in representations of \( \Gamma \) in the unitary group of a finite von Neumann algebra.

Recall that a finite trace or a tracial state on a \( C^* \)-algebra \( A \) with unit 1 is a linear functional \( \tau \) on \( A \) which has the property

\[
\tau(xy) = \tau(yx) \quad \text{for all } x, y \in A,
\]

which is positive (that is, \( \tau(x^*x) \geq 0 \) for all \( x \in A \)), and which is normalized by \( \tau(1) = 1 \). The trace \( \tau \) is faithful if \( \tau(x^*x) \neq 0 \) for all \( x \neq 0 \).

Let \( M \) be a finite von Neumann algebra, with faithful trace normal \( \tau \). Let \( \pi : \Gamma \to U(M) \) be a group homomorphism. The function \( \varphi = \tau \circ \pi : \Gamma \to \mathbb{C} \) has the following properties:

(i) \( \varphi \) is central;

(ii) \( \varphi \) is positive definite;

(iii) \( \varphi(e) = 1 \).

Let \( CP(\Gamma) \) denote the set of functions \( \varphi : \Gamma \to \mathbb{C} \) with Properties (i), (ii) and (iii) above.

Let \( \varphi \in CP(\Gamma) \). Then there exist a finite von Neumann algebra \( M_\varphi \), with faithful normal trace \( \tau_\varphi \), and a group homomorphism \( \pi_\varphi : \Gamma \to U(M_\varphi) \) such that \( \varphi = \tau_\varphi \circ \pi_\varphi \). Indeed, by GNS-construction, there exists a cyclic unitary
representation \( \pi_\varphi \) of \( \Gamma \) on a Hilbert space \( \mathcal{H}_\varphi \) with a cyclic unit vector \( \xi_\varphi \) such that

\[
\varphi(\gamma) = \langle \pi_\varphi(\gamma) \xi_\varphi, \xi_\varphi \rangle \quad \text{for all } \gamma \in \Gamma.
\]

Since \( \varphi \) is central, there exists another unitary representation \( \rho_\varphi \) of \( \Gamma \) on \( \mathcal{H}_\varphi \) which commutes with \( \pi_\varphi \) (that is, \( \pi_\varphi(\gamma) \rho_\varphi(\gamma') = \rho_\varphi(\gamma') \pi_\varphi(\gamma) \) for all \( \gamma, \gamma' \in \Gamma \)) and with the property that

\[
\rho_\varphi(\gamma) \xi_\varphi = \pi_\varphi(\gamma^{-1}) \xi_\varphi \quad \text{for all } \gamma \in \Gamma.
\]

Let \( M_\varphi = \pi_\varphi(\Gamma)'' \) be the von Neumann subalgebra of \( \mathcal{L}(\mathcal{H}_\varphi) \) generated by \( \pi_\varphi(\Gamma) \), where \( S' = \{ T \in \mathcal{L}(\mathcal{H}_\varphi) : TS = ST \quad \text{for all } S \in S \} \) denotes the commutant of a subset \( S \) of \( \mathcal{L}(\mathcal{H}_\varphi) \). The mapping

\[
T \mapsto \langle T \xi_\varphi, \xi_\varphi \rangle \quad \text{for all } T \in M_\varphi
\]

is a faithful normal trace \( \tau_\varphi \) on \( M_\varphi \) and \( \varphi = \tau_\varphi \circ \pi_\varphi \).

Moreover, if \( N_\varphi = \rho_\varphi(\Gamma)'' \) is the von Neumann subalgebra of \( \mathcal{L}(\mathcal{H}_\varphi) \) generated by \( \rho_\varphi(\Gamma) \), then

\[
M'_\varphi = N_\varphi \quad \text{and} \quad N'_\varphi = M_\varphi.
\]

In particular, the common centre of \( M_\varphi \) and \( N_\varphi \) is \( M_\varphi \cap N_\varphi \).

As an important example, let \( \varphi = \delta_e \) be the Dirac function at the group unit \( e \). Then \( \varphi \in \text{CP}(\Gamma) \). The unitary representations \( \pi_\varphi \) and \( \rho_\varphi \) associated to \( \varphi \) are the left and right regular representations \( \lambda_\Gamma \) and \( \rho_\Gamma \) on \( \ell^2(\Gamma) \). Moreover, \( M_\varphi \) is the von Neumann algebra \( L(\Gamma) \) of \( \Gamma \).

The set \( \text{CP}(\Gamma) \) is a compact and convex subset of the vector space of all bounded functions on \( \Gamma \), equipped with the weak *-topology. The set of extremal points \( E(\Gamma) \) of \( \text{CP}(\Gamma) \) is the set of all indecomposable central positive definite functions on \( \Gamma \). By Choquet theory, every \( \varphi \in \text{CP}(\Gamma) \) may be written as an integral

\[
\varphi = \int_{E(\Gamma)} \psi d\mu(\psi)
\]

for a probability measure \( \mu \) on \( E(\Gamma) \), at least when \( G \) is countable. For \( \varphi \in \text{CP}(\Gamma) \), we have that \( M_\varphi \) is a factor if and only if \( \varphi \in E(\Gamma) \). As an example, the Dirac function \( \delta_e \) belongs to \( E(\Gamma) \) if and only if \( \Gamma \) is an ICC group.

Let \( M \) be a finite von Neumann algebra, with faithful normal trace \( \tau \), and let \( \pi : \Gamma \to U(M) \) be a homomorphism such that \( \pi(\Gamma)'' = M \). Observe
that, if we set $\varphi = \tau \circ \pi \in CP(\Gamma)$, then, with the notation above, the mapping $\pi_\varphi(\gamma) \mapsto \pi(\gamma)$ extends to an isomorphism $M_\varphi \to M$ of von Neumann algebras.

A homomorphism $\pi : \Gamma \to U(M)$ for a finite factor $M$ such that $\pi(\Gamma)'' = M$ will be called a finite factor representation of $\Gamma$. We say that two such representations $\pi_1 : \Gamma \to U(M_1)$ and $\pi_2 : \Gamma \to U(M_2)$ are quasi-equivalent if there exists an isomorphism $\Phi : M_1 \to M_2$ such that $\Phi(\pi_1(\gamma)) = \pi_2(\gamma)$ for all $\gamma \in \Gamma$. Summarizing the discussion above, we see that $E(\Gamma)$ classifies the finite factor representations of $\Gamma$, up to quasi-equivalence.

The set $E(\Gamma)$ parametrizes also the indecomposable traces on the full $C^*$-algebra of $\Gamma$. Recall that the full $C^*$-algebra $C^*(\Gamma)$ is the universal property that every unitary representation of $\Gamma$ on a Hilbert space $\mathcal{H}$ extends to a $*$-homomorphism $C^*(\Gamma) \to \mathcal{L}(\mathcal{H})$. The algebra $C^*(\Gamma)$ can be realized as completion of the group algebra $\mathbb{C}[\Gamma]$ under the norm

$$\left\| \sum_{\gamma \in \Gamma} c_\gamma \gamma \right\| = \sup \left\{ \left\| \sum_{\gamma \in \Gamma} c_\gamma \pi(\gamma) \right\| : \pi \in \text{Rep}(\Gamma) \right\},$$

where $\text{Rep}(\Gamma)$ denotes the set of (equivalence classes of) cyclic unitary representations of $\Gamma$.

We will view $\Gamma$ as a subgroup of the group of unitaries in $C^*(\Gamma)$ by means of the canonical embedding $\Gamma \to C^*(\Gamma)$. Every trace on $C^*(\Gamma)$ defines by restriction to $\Gamma$ an element of $CP(\Gamma)$. Conversely, every $\varphi \in CP(\Gamma)$ extends to a trace on $C^*(\Gamma)$, since, as seen above, $\varphi(\gamma) = \langle \pi_\varphi(\gamma) \xi_\varphi, \xi_\varphi \rangle$ and $\pi_\varphi$ is a unitary representation of $\Gamma$.

### 3 Some subgroups of $SL_n(\mathbb{Z})$

Let $n$ is a fixed integer with $n \geq 2$. For a pair of integers $(i, j)$ with $1 \leq i \neq j \leq n$, denote by $e_{ij}$ the corresponding elementary matrix, that is, the $(n \times n)$-matrix with 1’s on the diagonal, 1 at the $(i, j)$-entry, and 0 elsewhere. It is well-known that $SL_n(\mathbb{Z})$ is generated by

$$\{ e_{ij} : 1 \leq i \neq j \leq n \}.$$ 

Moreover, for $n \geq 3$, any two elementary matrices are conjugate inside $SL_n(\mathbb{Z})$. Indeed, observe that the matrix

$$s_{ij} = e_{ij} e_{ji}^{-1} e_{ij} \in SL_n(\mathbb{Z})$$
permutes the \(i\)-th and the \(j\)-th standard unit vectors of \(\mathbb{Z}^n\), up to a sign. Hence, if \(e_{kl}\) and \(e_{pq}\) are two elementary matrices, conjugation by a suitable product of matrices of the form \(s_{ij}\) will carry \(e_{kl}\) into \(e_{pq}\) or \(e_{pq}^{-1}\). Now, \(e_{pq}\) and \(e_{pq}^{-1}\) are conjugate via a suitable diagonal matrix in \(SL_n(\mathbb{Z})\), when \(n \geq 3\).

The proof of the following two lemmas is by straightforward computation. We will always view an element \(a \in \mathbb{Z}^n\) as column vector. Its transpose \(a^t\) is then a row vector. We denote by \(e_1, \ldots, e_n\) the standard unit vectors in \(\mathbb{Z}^n\).

**Lemma 6** Let \(k\) be a non-zero integer and let \(i, j \in \{1, \ldots, n\}\) with \(1 \leq i \neq j \leq n\). The centralizer of \(e_{ij}^k\) in \(SL_n(\mathbb{Z})\) consists of all matrices with \(\varepsilon e_i\) as \(i\)-th column and \(\varepsilon e_j^t\) as \(j\)-th row for \(\varepsilon \in \{\pm 1\}\). ■

For instance, the centralizer of \(e_{12}^k\) is the subgroup of all matrices of the form

\[
\begin{pmatrix}
\varepsilon & * & * & \cdots & * \\
0 & \varepsilon & 0 & \cdots & 0 \\
0 & * & * & \cdots & * \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & * & * & \cdots & * 
\end{pmatrix},
\]

for \(\varepsilon \in \{\pm 1\}\).

For \(j \in \{1, \ldots, n\}\), let \(V_j \cong \mathbb{Z}^{n-1}\) be the subgroup generated by

\[\{e_{ij} : 1 \leq i \leq n, i \neq j\};\]

for instance, \(V_1\) is the set of matrices of the form

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
* & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots \\
* & 0 & \cdots & 1 
\end{pmatrix}.
\]

**Lemma 7** The normalizer of \(V_j\) in \(SL_n(\mathbb{Z})\) is the subgroup \(G_j\) of all matrices in \(SL_n(\mathbb{Z})\) with \(\varepsilon e_j\) as \(j\)-th row for \(\varepsilon \in \{\pm 1\}\). ■
Thus, for instance, the normalizer $G_1$ of $V_1$ is the group of all matrices

$$
\begin{pmatrix}
\varepsilon & 0 & \cdots & 0 \\
* & & & \\
\vdots & & & A \\
* & & & \\
\end{pmatrix},
$$

where $A \in GL_{n-1}(\mathbb{Z})$ and $\varepsilon = \det A$.

Up to a subgroup of index two, $G_j$ is isomorphic to the semi-direct product $SL_{n-1}(\mathbb{Z}) \rtimes \mathbb{Z}^{n-1}$ for the natural action of $SL_{n-1}(\mathbb{Z})$ on $\mathbb{Z}^{n-1}$.

We will have also to consider the transpose subgroups $V_i^t$ generated by $\{e_{ij} : 1 \leq j \leq n, j \neq i\}$.

Observe that $V_j \cap V_i^t$ is the copy of $\mathbb{Z}$ generated by $e_{ij}$ for $i \neq j$. The normalizer of $V_i^t$ in $SL_n(\mathbb{Z})$ is of course the group $G_i^t$. Observe also that

$$V_j \subset G_i^t \quad \text{and} \quad V_i^t \subset G_j$$

for all $i \neq j$.

We will refer to subgroups of the form $V_j$ and $V_i^t$ as to the copies of $\mathbb{Z}^{n-1}$ inside $SL_n(\mathbb{Z})$.

4 Proof of Theorem 3: A preliminary reduction

The starting point of the proof of Theorem 3 is the following classification from [Burg91, Proposition 9] of the measures on the $n$-dimensional torus $\mathbb{T}^n$ which are invariant under the natural action of $SL_n(\mathbb{Z})$; for a more elementary proof in the case $n = 2$, see [DaKe79].

Lemma 8 ([Bur]) Let $n \geq 2$ be an integer. Let $\mu$ be a $SL_n(\mathbb{Z})$–invariant ergodic probability measure on the Borel subsets of $\mathbb{T}^n$. Then either $\mu$ is concentrated on a finite $SL_n(\mathbb{Z})$–orbit or $\mu$ is the normalized Lebesgue measure on $\mathbb{T}^n$.

Recall that a point $x \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ has a finite $SL_n(\mathbb{Z})$–orbit if and only if $x \in \mathbb{Q}^n/\mathbb{Z}^n$. 
Let \( n \geq 3 \) and let
\[
\varphi : SL_n(\mathbb{Z}) \to \mathbb{C}
\]
be an indecomposable central positive definite function on \( SL_n(\mathbb{Z}) \), fixed throughout the proof.

As in Section 2, let \( \pi \) and \( \rho \) be the corresponding commuting factor representations of \( \Gamma \) on the Hilbert space \( \mathcal{H} \) with cyclic vector \( \xi \) such that
\[
\varphi(\gamma) = \langle \pi(\gamma)\xi, \xi \rangle = \langle \rho(\gamma^{-1})\xi, \xi \rangle, \quad \text{for all } \gamma \in \Gamma.
\]

Fix any copy \( V = V_j \) or \( V = V_j^t \) of \( \mathbb{Z}^{n-1} \) inside \( SL_n(\mathbb{Z}) \) and consider the restriction \( \varphi|_V \) to \( V \).

As \( \varphi \) is central, \( \varphi|_V \) is a \( G \)-invariant positive definite function on \( V \), where
\[
G = G_j \quad \text{or} \quad G = G_j^t
\]
is the normalizer of \( V \) in \( SL_n(\mathbb{Z}) \). Since \( G \) contains a copy of the semi-direct product \( SL_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n-1} \) (for the usual action in case \( V = V_j \) and for the inverse transpose of the usual action in case \( V = V_j^t \)), we have
\[
\varphi(Ax) = \varphi(x) \quad \text{for all } x \in \mathbb{Z}^{n-1}, \ A \in SL_{n-1}(\mathbb{Z}).
\]
Thus, by Bochner’s theorem, \( \varphi|_V \) is the Fourier transform of a \( SL_{n-1}(\mathbb{Z}) \)-invariant probability measure on the torus
\[
\mathbb{T}^{n-1} \cong \hat{V}.
\]

Let \((O_i)_{i \geq 1}\) denote the sequence of finite \( SL_{n-1}(\mathbb{Z}) \)-orbits in \( \mathbb{T}^{n-1} \). For each \( i \geq 1 \), denote by \( \mu_{O_i} \) the uniform distribution on \( O_i \), that is, the probability measure
\[
\mu_{O_i} = \frac{1}{|O_i|} \sum_{\chi \in O_i} \delta_\chi
\]
on \( \mathbb{T}^{n-1} \). Lemma shows that \( \mu \) has a decomposition as a convex combination
\[
\mu = t_\infty \mu_\infty + \sum_{i \geq 1} t_i^{(V)} \mu_{O_i}, \quad \text{with} \quad t_\infty^{(V)} + \sum_{i \geq 1} t_i^{(V)} = 1, \ t_\infty^{(V)} \geq 0, \ t_i^{(V)} \geq 0,
\]
where \( \mu_\infty \) is the normalized Lebesgue measure on \( \mathbb{T}^{n-1} \). Thus, we obtain a corresponding decomposition of \( \varphi|_V \)
\[
\varphi|_V = t_\infty^{(V)} \delta_e + \sum_{i \geq 1} t_i^{(V)} \psi_{O_i}, \quad \text{with} \quad t_\infty^{(V)} + \sum_{i \geq 1} t_i^{(V)} = 1, \ t_\infty^{(V)} \geq 0, \ t_i^{(V)} \geq 0,
\]
where $\psi_{O_i}$ is the Fourier transform of the measure $\mu_{O_i}$.

By general theory, we have a corresponding decomposition of $\mathcal{H}$ into a direct sum of $\pi(V)$–invariant subspaces

$$\mathcal{H} = \mathcal{H}^V_{\infty} \oplus \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}^V_{\chi}$$

where $\mathcal{H}^V_{\chi}$ is the subspace on which $V$ acts according to the character $\chi$, that is,

$$\mathcal{H}^V_{\chi} = \{ \eta \in \mathcal{H} \mid \pi(v)\eta = \chi(v)\eta \quad \text{for all} \quad v \in V \}$$

and where $\mathcal{H}^V_{\infty}$ is a subspace on which $\pi(V)$ is a multiple of the regular representation $\lambda_V$ of $V$. Observe that some of these subspaces may be $\{0\}$.

Observe also that, since the representation $\rho$ commutes with $\pi$, each of the subspaces $\mathcal{H}^V_{\chi}$ and $\mathcal{H}^V_{\infty}$ is invariant under the whole of $\rho(SL_n(\mathbb{Z}))$.

We claim that we have the following dichotomy.

**Lemma 9** We have

- either $\mathcal{H} = \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}^V_{\chi}$ for every copy $V$ of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$, or
- $\mathcal{H} = \mathcal{H}^V_{\infty}$ for every copy $V$ of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$.

**Proof** Let $V$ be a copy of $\mathbb{Z}^{n-1}$ with $\mathcal{H}^V_{\infty} \neq \{0\}$. We will show that

$$\mathcal{H} = \mathcal{H}^W_{\infty}$$

for every copy $W$ of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$.

Clearly, this will prove the lemma.

- **First step:** Let $W$ be a copy of $\mathbb{Z}^{n-1}$ for which we assume that $V \cap W \neq \{0\}$.

We claim that $\mathcal{H}^V_{\infty} = \mathcal{H}^W_{\infty}$.

Indeed, $V \cap W$ is the copy of $\mathbb{Z}$ generated by the appropriate elementary matrix. We have two decompositions of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}^V_{\infty} \oplus \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}^V_{\chi}$$

and

$$\mathcal{H} = \mathcal{H}^W_{\infty} \oplus \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} \mathcal{H}^W_{\chi}.$$
has a decomposition into a direct sum of subspaces under which \( \pi(V \cap W) \) acts according to a character of \( V \cap W \).

On the other hand, the representation \( \pi|_{V \cap W} \) restricted to \( H_{\infty}^V \) or to \( H_{\infty}^W \) is a multiple of the regular representation \( \lambda_{V \cap W} \), since \( \lambda_{V \cap W} \) and \( \lambda_{W \cap W} \) are multiples of \( \lambda_{V \cap W} \). It follows that we necessarily have \( H_{\infty}^V = H_{\infty}^W \).

**Second step:** Let \( W \) be now an arbitrary copy of \( \mathbb{Z}^{n-1} \). We claim that we still have \( H_{\infty}^V = H_{\infty}^W \).

Indeed, as is readily verified, we can find two copies \( W^1 \) and \( W^2 \) of \( \mathbb{Z}^{n-1} \) with

\[
V \cap W^1 \neq \{0\}, \quad W^1 \cap W^2 \neq \{0\}, \quad \text{and} \quad W^2 \cap W \neq \{0\}.
\]

Therefore, by the first step, we have

\[
H_{\infty}^V = H_{\infty}^{W^1}, \quad H_{\infty}^{W^1} = H_{\infty}^{W^2}, \quad H_{\infty}^{W^2} = H_{\infty}^W,
\]

so that \( H_{\infty}^V = H_{\infty}^W \).

**Third step:** We claim that \( H_{\infty}^V = H \).

Indeed, by the second step, we have

\[
H_{\infty}^V = H_{\infty}^W \quad \text{for every copy } W \text{ of } \mathbb{Z}^{n-1} \text{ in } SL_n(\mathbb{Z}).
\]

Since \( H_{\infty}^W \) is invariant under \( \pi(W) \), it follows that \( H_{\infty}^V \) is invariant under \( \pi(SL_n(\mathbb{Z})) \).

On the other hand, \( H_{\infty}^V \) is also invariant under \( \rho(SL_n(\mathbb{Z})) \). Since \( \pi \) is a factor representation and since \( H_{\infty}^V \neq \{0\} \), the claim follows. ■

We have to consider separately the two possible decompositions of \( H \) given by the previous lemma. We will see that the first one corresponds to a character of a congruence quotient, and that the second one to a character induced from the centre.

5 Proof of Theorem 3: First case

With the notation from the last section, we assume in this section that

\[
H = \bigoplus_{\chi \in \mathbb{Q}^{n-1}/\mathbb{Z}^{n-1}} H_{\chi}^V \quad \text{for every copy } V \text{ of } \mathbb{Z}^{n-1} \text{ in } SL_n(\mathbb{Z}).
\]
We claim that there exists some integer \( N \geq 1 \) such that \( \pi \) is trivial on the congruence normal subgroup

\[
\Gamma(N) = \{ \gamma \in SL_n(\mathbb{Z}) : \gamma \equiv I \mod N \}.
\]

Let \( \gamma_0, \gamma_1, \ldots, \gamma_d \) denote the elementary matrices in \( SL_n(\mathbb{Z}) \), where \( d = n(n-1) - 1 \).

For every \( k \in \{0, \ldots, d\} \), we have a decomposition

\[
\mathcal{H} = \bigoplus_{\alpha \in \mathbb{Q}/\mathbb{Z}} \mathcal{H}^\gamma_{\alpha, k}
\]

of \( \mathcal{H} \) under the action of the unitary operator \( \pi(\gamma_k) \), where \( \mathcal{H}^\gamma_{\alpha, k} \) is the eigenspace (possibly equal to \( \{0\} \)) of \( \pi(\gamma_k) \) corresponding to \( \alpha \).

**Lemma 10** There exists an integer \( N \geq 1 \) such that \( \pi(\gamma_0^N), \pi(\gamma_1^N), \ldots, \pi(\gamma_d^N) \) have a non-zero common invariant vector in \( \mathcal{H} \).

**Proof** Let \( M \) be the factor generated by \( \pi(\Gamma) \) and denote by \( \tau \) the trace on \( M \) defined by \( \varphi \).

Write the elements in \( \mathbb{Q}/\mathbb{Z} \) as a sequence \( \{\alpha_i\}_{i \geq 1} \). For every \( i \geq 1 \), let

\[
p_i : \mathcal{H} \to \mathcal{H}^\gamma_{\alpha_i, 0}
\]

denote the orthogonal projection onto \( \mathcal{H}^\gamma_{\alpha_i, 0} \). Observe that \( p_i \in M \) (in fact, \( p_i \) belongs to the abelian von Neumann algebra generated by \( \pi(\gamma_0) \)). We have \( \tau(p_i) \in [0, 1] \) and \( \sum_{i \geq 1} \tau(p_i) = 1 \), since \( \sum_{i \geq 1} p_i = I \).

Let \( \varepsilon \) be a real number with

\[
0 < \varepsilon < 1/2^d.
\]

There exists \( i_0 \geq 1 \) such that

\[
\sum_{i=1}^{i_0} \tau(p_i) \geq 1 - \varepsilon.
\]

Since elements in \( \mathbb{Q}/\mathbb{Z} \) have finite order, we can find an integer \( N \geq 1 \) such that

\[
\alpha_i^N = 1 \quad \text{for all} \quad i \in \{1, \ldots, i_0\}.
\]
Then $\pi(\gamma_0^N)$ acts as the identity on
\[ \bigoplus_{i=1}^{i_0} H_{\alpha_i}^{\gamma_0}. \]

For $l \in \{0, 1, \ldots, d\}$, let $H^{\gamma_l}_N$ be the subspace of $\pi(\gamma_l^N)$–invariant vectors in $H$. We claim that
\[ H^{\gamma_0}_N \cap H^{\gamma_1}_N \cap \cdots \cap H^{\gamma_d}_N \neq \{0\}. \]

For every $k \in \{0, 1, \ldots, d\}$, let $q_k$ denote the orthogonal projection onto $H^{\gamma_0}_N \cap H^{\gamma_1}_N \cap \cdots \cap H^{\gamma_k}_N$. It is clear that $q_k \in M$.

We claim that
\[ (1) \quad \tau(q_k) \geq 1 - 2^k \varepsilon \quad \text{for all} \quad k = 0, 1, \ldots, d. \]

Once proved, this will imply that
\[ \tau(q_d) \geq 1 - 2^d \varepsilon > 0, \]
and hence $q_d \neq 0$ since $\tau$ is faithful on $M$; this will finish the proof of the lemma.

To prove (1), we proceed by induction on $k$. Since
\[ \bigoplus_{i=1}^{i_0} H_{\alpha_i}^{\gamma_0} \subset H^{\gamma_0}_N, \]
we have $q_0 \geq \sum_{i=1}^{i_0} p_i$. Hence,
\[ \tau(q_0) \geq \sum_{i=1}^{i_0} \tau(p_i) \geq 1 - \varepsilon, \]
and this proves (1) in the case $k = 0$.

Let $k \geq 1$ and assume that
\[ (2) \quad \tau(q_{k-1}) \geq 1 - 2^{k-1} \varepsilon. \]
Set
\[ K = \mathcal{H}^N \cap \mathcal{H}^N \cap \cdots \cap \mathcal{H}^{N-1} \]
and set \( q = q_{k-1} \), the orthogonal projection on \( K \).

Since any two elementary matrices are conjugate, we have \( \gamma_k = s\gamma_0s^{-1} \)
for some element \( s \in SL_n(\mathbb{Z}) \). Observe that
\[ \mathcal{H}^N = \pi(s)\mathcal{H}^N. \]

Consider the operator
\[ T = (1 - q)\pi(s^{-1})q \]
on \( \mathcal{H} \). Observe that \( T \in M \). For \( \eta \in \mathcal{H} \), we have
\( T(\eta) = 0 \) if and only if \( \pi(s^{-1})q(\eta) \in K \), that is, if and only \( q(\eta) \in \pi(s)K \). Hence
\[ \text{(3)} \quad \ker T = (K \cap \pi(s)K) \oplus K^\perp. \]

Let
\[ p_{\ker T} : \mathcal{H} \to \ker T \]
be the orthogonal projection on \( \ker T \). Then \( p_{\ker T} \in M \), since \( T \in M \).

Moreover, since the range of \( T \) is contained in \( K^\perp \), we have
\[ \tau(1 - q) \geq \tau(I) - \tau(p_{\ker T}) = 1 - \tau(p_{\ker T}). \]

Hence, by (2),
\[ \text{(4)} \quad \tau(p_{\ker T}) \geq 1 - 2^{k-1}\varepsilon. \]

We have, by (3)
\[ \tau(p_{\ker T}) = \tau(p_{K \cap \pi(s)K}) + \tau(1 - q), \]
where \( p_{K \cap \pi(s)K} \in M \) is the orthogonal projection on \( K \cap \pi(s)K \). Now,
\[ K \cap \pi(s)K \subset K \cap \pi(s)\mathcal{H}^N = K \cap \mathcal{H}^N. \]

Since \( q_k \) is the orthogonal projection on \( K \cap \mathcal{H}^N \), it follows in view of (2) and (4) that
\[ \tau(q_k) \geq \tau(p_{K \cap \pi(s)K}) \]
\[ = \tau(p_{\ker T}) - (1 - \tau(q)) \]
\[ \geq (1 - 2^{k-1}\varepsilon) - 2^{k-1}\varepsilon = 1 - 2^k\varepsilon. \]

This proves the claim (1) and finishes the proof of the lemma. \( \blacksquare \)
Corollary 11  Under the assumption made at the beginning of this section, there exists an irreducible representation \( \pi_0 \) of the congruence quotient
\[
SL_n(\mathbb{Z})/\Gamma(N^2) \cong SL_n(\mathbb{Z}/N^2\mathbb{Z})
\]
such that \( \varphi \) is the (normalized) character of \( \pi_0 \) lifted to \( SL_n(\mathbb{Z}) \), where \( N \) is as in Lemma 10.

Proof  By the previous lemma, the subspace \( K \) of the common invariant vectors under \( \pi(\gamma_0^N), \pi(\gamma_1^N), \ldots, \pi(\gamma_d^N) \) is non-zero. Let \( \Gamma \) be the subgroup of \( SL_n(\mathbb{Z}) \) generated by
\[
\{\gamma_0^N, \gamma_1^N, \ldots, \gamma_d^N\}.
\]
By [Tits76, Proposition 2], \( \Gamma \) contains the congruence normal subgroup \( \Gamma(N^2) \).

Consider the subspace
\[
\mathcal{H}^{\Gamma(N^2)} = \{\eta \in \mathcal{H} : \pi(\gamma)\eta = \eta \quad \text{for all} \quad \gamma \in \Gamma(N^2)\},
\]
of \( \pi(\Gamma(N^2)) \)-invariant vectors. Then \( \mathcal{H}^{\Gamma(N^2)} \neq \{0\} \) since \( K \subseteq \mathcal{H}^{\Gamma(N^2)} \). Moreover, \( \mathcal{H}^{\Gamma(N^2)} \) is invariant under \( \pi(SL_n(\mathbb{Z})) \), as \( \Gamma(N^2) \) is normal in \( SL_n(\mathbb{Z}) \).

On the other hand, \( \mathcal{H}^{\Gamma(N^2)} \) is clearly invariant under \( \rho(SL_n(\mathbb{Z})) \). It follows that
\[
\mathcal{H}^{\Gamma(N^2)} = \mathcal{H}.
\]
Hence, \( \pi \) factorizes through the finite group \( SL_n(\mathbb{Z})/\Gamma(N^2) \). It follows that \( \mathcal{H} \) is finite-dimensional, that \( \pi \) is a multiple \( m\pi_0 \) of an irreducible representation \( \pi_0 \) of \( SL_n(\mathbb{Z})/\Gamma(N^2) \) with \( m = \dim(\pi_0) \), and that \( \varphi \) is the normalized character of \( \pi_0 \).

6  Proof of Theorem 3: Second case

With the notation as in Section 4 we assume now that
\[
\mathcal{H} = \mathcal{H}_V \quad \text{for every copy} \quad V \quad \text{of} \quad \mathbb{Z}^{n-1} \quad \text{in} \quad SL_n(\mathbb{Z}).
\]
This is equivalent to:
\[
\varphi|_V = \delta_e \quad \text{for every copy} \quad V \quad \text{of} \quad \mathbb{Z}^{n-1} \quad \text{in} \quad SL_n(\mathbb{Z}).
\]

Let \( \chi_\varphi \) be the unitary character of the centre \( C = \{\pm I\} \) of \( SL_n(\mathbb{Z}) \) such that
\[
\varphi(z\gamma) = \chi_\varphi(z)\varphi(\gamma) \quad \text{for all} \quad z \in C, \gamma \in SL_n(\mathbb{Z}).
\]
We claim that
\[
\varphi(\gamma) = \begin{cases} 
0 & \text{if } \gamma \in SL_n(\mathbb{Z}) \setminus C \\
\chi_\varphi(\gamma) & \text{if } \gamma \in C.
\end{cases}
\]

The following proposition, which is of independent interest, will play a crucial rôle.

**Proposition 12** Every matrix \( \gamma \in SL_n(\mathbb{Z}) \) is conjugate to the product \( g_1g_2g_3 \) of three matrices of the form

\[
g_1 = \begin{pmatrix}
1 & * & \cdots & * \\
0 & * & \cdots & * \\
\vdots & & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix} \in G_1^t, \\
g_2 = \begin{pmatrix}
* & * & \cdots & * \\
\vdots & & \ddots & \vdots \\
* & * & \cdots & * \\
0 & 0 & 0 & 1
\end{pmatrix} \in G_n
\]

and

\[
g_3 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
* & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & 0 & 0 & \cdots & 1
\end{pmatrix} \in V_1
\]

**Proof**

- **First step:** We first claim that \( \gamma \) is conjugate to a matrix \( \gamma_1 \) with first column of the form \((*,0,*,,0,\ldots,0)^t\). This is Lemma 1 in [Bren60]. The result is proved by conjugating \( \gamma \) by permutation matrices (with sign adjusted) and by elementary matrices of the type \( e_{ij} \) with \( 1 < i \neq j \leq n \).

  So, we can assume that the first column of \( \gamma \) is of the form \((k,0,l,0,\ldots,0)^t\) for \( k,l \in \mathbb{Z} \).

- **Second step:** There exists a matrix \( \gamma_1 \in G_n \) such that the first column of \( \gamma_1\gamma \) is \((k,1,l,0,\ldots,0)^t\). Indeed, since \( \gcd(k,l) = 1 \), there exist \( p,q \in \mathbb{Z} \) such that \( pk + ql = 1 \). We can take

\[
\gamma_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
p & 1 & q & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \in G_n.
\]
• **Third step:** There exists a matrix $\gamma_2 \in G_1^t \cap G_n$ such that the first column of $\gamma_2 \gamma_1 \gamma$ is $(1, 1, l, 0, \ldots, 0)^t$. Indeed, we can take

$$
\gamma_2 = \begin{pmatrix}
1 & 1 - k & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \in G_n.
$$

• **Fourth step:** There exists a matrix $\gamma_3 \in V_1$ such that the first column of $\gamma_3 \gamma_2 \gamma_1 \gamma$ is $(1, 0, 0, 0, \ldots, 0)^t$. Indeed, we can take

$$
\gamma_3 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & \cdots & 0 \\
-l & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \in V_1.
$$

By the last step, $\gamma_4 = \gamma_3 \gamma_2 \gamma_1 \gamma \in G_1^t$. We have

$$
\gamma_4 \gamma_4^{-1} = \gamma_4 (\gamma_1^{-1} \gamma_2^{-1}) \gamma_3^{-1}.
$$

The claim follows, since $\gamma_1^{-1} \gamma_2^{-1} \in G_n$ and $\gamma_3^{-1} \in V_1$. ■

**Remark 13** In the case $n \geq 4$, the previous proposition can be improved: every $\gamma \in SL_n(\mathbb{Z})$ is conjugate to a product $g_1 g_2 \in G_1^t G_n$. Indeed, in this case, the matrix $\gamma_3$ in the fourth step of the proof belongs to $G_n$ and hence

$$
\gamma_4 \gamma_4^{-1} = \gamma_4 (\gamma_1^{-1} \gamma_2^{-1} \gamma_3^{-1}) \in G_1^t G_n.
$$

Returning to the proof of Theorem 3, the previous proposition implies that it suffices to show that

$$
\varphi(\gamma) = 0 \quad \text{for all} \quad \gamma \in G_1^t G_n V_1 \quad \text{with} \quad \gamma \notin C.
$$

For this, several preliminary steps will be needed.

We will use several times the following elementary lemma.
Lemma 14 Let $\Gamma$ be a group and $(\pi, \mathcal{H})$ a unitary representation of $\Gamma$. Let $\psi = \langle \pi(\cdot)\xi, \xi \rangle$ be an associated positive definite function such that $\psi = \delta_e$. Then, for every sequence $(g_k)_{k \in \mathbb{N}}$ of pairwise distinct elements $g_k \in \Gamma$, the sequence $(\pi(g_k)\xi)_{k \in \mathbb{N}}$ converges weakly to 0 in $\mathcal{H}$.

Proof For $k, l \in \mathbb{N}$ with $k \neq l$, we have

$$
\langle \pi(g_k)\xi, \pi(g_l)\xi \rangle = \langle \pi(g_l^{-1}g_k)\xi, \xi \rangle = \psi(g_l^{-1}g_k) = 0.
$$

Therefore, $(\pi(g_k)\xi)_{k \in \mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$ and the claim follows.

The first step in this part of the proof of Theorem 3 is to show that $\varphi(\gamma) = 0$ for all $\gamma \in G_1^t \cup G_n$ with $\gamma \notin C$.

For elements $x, y$ in a group, let $[x, y]$ denote the commutator $x^{-1}y^{-1}xy$.

Lemma 15 Let $V$ be a copy of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$ and let $G$ be the normalizer of $V$. Then $\varphi(\gamma) = 0$ for every $\gamma \in G \setminus C$.

Proof Write

$$V = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n,$$

where $x_1, \ldots, x_n$ are the elementary matrices contained in $V$.

Let $\gamma \in G \setminus C$. We claim that there exists $i \in \{1, \ldots, n\}$ such that

$$x_i^{-k}\gamma x_i^k \neq x_i^{-l}\gamma x_i^l \quad \text{for all} \quad k, l \in \mathbb{Z}, \ k \neq l.
$$

Indeed, otherwise there would exist non-zero integers $k_i$ such that $\gamma$ is in the centralizer of $x_i^{k_i}$ for all $i \in \{1, \ldots, n\}$. This would imply that $\gamma \in C$ (see Lemma 6).

The commutators $[\gamma, x_i^k]$ belong to $V$ and are pairwise distinct. Hence, by Lemma 14, the sequence $(\pi([\gamma, x_i^k])\xi)_{k \in \mathbb{N}}$ is weakly convergent to 0 in $\mathcal{H}$. For $k \in \mathbb{N}$, we have

$$
\varphi(\gamma) = \varphi(x_i^{-k}\gamma x_i^k) = \varphi(\gamma[x, x_i^k]) = \langle \pi([\gamma, x_i^k])\xi, \pi(\gamma^{-1})\xi \rangle.
$$
Hence,

$$\varphi(\gamma) = \lim_{k} \langle \pi([\gamma, x^k])\xi, \pi(\gamma^{-1})\xi \rangle = 0,$$

as claimed. ■

The next step is to show that

$$\varphi(\gamma) = 0 \quad \text{for all} \quad \gamma \in G_1^1 G_n \quad \text{with} \quad \gamma \notin C.$$

**Lemma 16** Let $V, W$ be two copies of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$ with $V \cap W \neq \{0\}$. Let $G, H$ be the normalizers of $V$ and $W$, respectively. Let $\gamma = gh$ with $g \in G$, $h \in H$, and $\gamma \notin C$. Then $\varphi(\gamma) = 0$.

**Proof** If $g \in C$ or $h \in C$, then $\gamma \in G$ or $\gamma \in H$ and then $\varphi(\gamma) = 0$, by Lemma 15. Hence, we can assume that $g \notin C$ and $h \notin C$.

Let $x$ denote the elementary matrix such that $V \cap W = \langle x \rangle$.

It is readily verified that, for $k \in \mathbb{Z} \setminus \{0\}$, the centralizer of $x^k$ is contained in $G \cap H$. Hence, we can assume that $\gamma$ does not belong to this centralizer, that is, that the elements $x^{-k}\gamma x^k$ are pairwise distinct.

We have

$$x^{-k}\gamma x^k = x^{-k}g x^k x^{-k} h x^k = g[g, x^k] x^{-k} h x^k,$$

Set $y_k = [g, x^k] x^{-k} h x^k$. Observe that $V \subset G \cap H$. Since $[g, x^k] \in V$, we have $y_k \in H$. Moreover, the elements $y_k$ are pairwise distinct, since

$$y_k = g^{-1} x^{-k}\gamma x^k.$$

Hence, again by Lemma 14 the sequence $(\pi(y_k)\xi)_{k \in \mathbb{N}}$ is weakly convergent to 0 in $H$. As in the previous lemma, it follows that

$$\varphi(\gamma) = \lim_{k} \varphi(x^{-k}\gamma x^k) = \lim_{k} \langle \pi(y_k)\xi, \pi(g^{-1})\xi \rangle = 0.$$ ■

We will also need the following consequence of Lemma 16.

**Lemma 17** Let $V, W$ two copies of $\mathbb{Z}^{n-1}$ in $SL_n(\mathbb{Z})$ with $V \cap W \neq \{0\}$. Let $G, H$ be the normalizers of $V$ and $W$, respectively. Let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence of pairwise distinct elements in $GH$. Then $(\pi(\gamma_k)\xi)_{k \in \mathbb{N}}$ converges weakly to 0 in $H$. 21
Proof. Observe that \((\pi(\gamma_k)\xi)_{k \in \mathbb{N}}\) is a bounded sequence in \(\mathcal{H}\). Therefore, it suffices to show that every subsequence \((\pi(\gamma_{k_i})\xi)_{i \in \mathbb{N}}\) of \((\pi(\gamma_k)\xi)_{k \in \mathbb{N}}\) has a subsequence which weakly converges to 0.

For \(i \in \mathbb{N}\), write \(\gamma_{k_i} = g_{k_i} h_{k_i}\) for \(g_{k_i} \in G\) and \(h_{k_i} \in H\).

Since \(C\) is finite and since the elements \(\gamma_{k_i}\) are pairwise distinct, we can find a subsequence of \((\gamma_{k_i})_i\), still denoted by \((\gamma_{k_i})_i\), such that \(\gamma_{k_j}^{-1} \gamma_{k_i} \notin C\) for all \(i \neq j\). It follows that

\[
g_{k_j}^{-1} g_{k_i} h_{k_i} h_{k_j}^{-1} \notin C
\]

for all \(i \neq j\).

From Lemma 16, we deduce that, for all \(i \neq j\),

\[
\varphi(\gamma_{k_j}^{-1} \gamma_{k_i}) = \varphi(h_{k_j}^{-1} g_{k_j}^{-1} g_{k_i} h_{k_i}) = \varphi(g_{k_j}^{-1} g_{k_i} h_{k_i} h_{k_j}^{-1}) = 0,
\]

since \(g_{k_j}^{-1} g_{k_i} \in G\) and \(h_{k_i} h_{k_j}^{-1} \in H\). As in the proof of Lemma 14 this shows that \((\pi(\gamma_k)\xi)_i\) weakly converges to 0. ■

We can now conclude the proof of Theorem 3. Let \(\gamma \in SL_n(\mathbb{Z}) \setminus C\). We want to show that \(\varphi(\gamma) = 0\).

By Proposition 12 we can assume that \(\gamma = g_1 g_2 g_3\) for matrices of the form

\[
g_1 = \begin{pmatrix}
1 & \ast & \cdots & \ast \\
0 & \ast & \cdots & \ast \\
\vdots & & & \ast \\
0 & \ast & \cdots & \ast
\end{pmatrix} \in G_1^t, \quad g_2 = \begin{pmatrix}
\ast & \cdots & \ast \\
\vdots & & \\
\ast & \cdots & \ast \\
0 & 0 & 0 & 1
\end{pmatrix} \in G_n
\]

and

\[
g_3 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
a_2 & 1 & 0 & \cdots & 0 \\
\vdots & & & \vdots & \\
a_n & \vdots & \vdots & \cdots & 1
\end{pmatrix} \in V_1.
\]

If \(g_3 \in G_n\), then \(\gamma\) is a non-central element in \(G_1^t G_n\), and it follows from Lemma 16 that \(\varphi(\gamma) = 0\). We can therefore assume that \(g_3 \notin G_n\), that is, \(a_n \neq 0\).
Let $x$ be the elementary matrix $e_{2,n}$, thus

$$x = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.$$

Then $x \in G_1^t \cap G_n$ and the centralizer of every power $x^k$ for $k \neq 0$ is contained in $G_n$. Hence, if $\gamma$ is contained in the centralizer of some power $x^k$ for $k \neq 0$, the claim follows from Lemma 15. We can therefore assume that

$$x^{-k}\gamma x^k \neq x^{-l}\gamma x^l \quad \text{for all } k \neq l.$$

We compute that

$$x^{-k}g_3 x^k = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
a_2 + ka_n & 1 & 0 & \cdots & 0 \\
a_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & 0 & 0 & \cdots & 1
\end{pmatrix}.$$

Hence $x^{-k}g_3 x^k = \alpha_k\beta$, where

$$\alpha_k = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
ka_n & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
a_2 & 1 & 0 & \cdots & 0 \\
a_3 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & 0 & 0 & \cdots & 1
\end{pmatrix}.$$

Observe that $\alpha_k \in G_n$ for every $k$. We have

$$x^{-k}\gamma x^k = x^{-k}g_1g_2g_3x^k = (x^{-k}g_1x^k)(x^{-k}g_2x^k)(x^{-k}g_3x^k) = (x^{-k}g_1x^k)(x^{-k}g_2x^k)\alpha_k\beta.$$

Now, since $x \in G_1^t \cap G_n$, we have $x^{-k}g_1x^k \in G_1^t$ and $x^{-k}g_2x^k\alpha_k \in G_n$. It follows that

$$x^{-k}\gamma x^k\beta^{-1} \in G_1^tG_n \quad \text{for every } k.$$
Set 
\[ \gamma_k = x^{-k} \gamma x^k \beta^{-1}. \]

Since \( \gamma \) is not in the centralizer of \( x^k \), we have \( \gamma_k \neq \gamma_l \) for all \( k \neq l \). Hence, by Lemma 17, the sequence \((\pi(\gamma_k))_{k \in \mathbb{N}}\) converges weakly to 0. It follows that 
\[
\varphi(\gamma) = \lim_k \varphi(\beta x^{-k} \gamma x^k \beta^{-1}) \\
= \lim_k \varphi(\beta \gamma_k) \\
= \lim_k \langle \pi(\beta \gamma_k) \xi, \xi \rangle \\
= \lim_k \langle \pi(\gamma_k) \xi, \pi(\beta^{-1}) \xi \rangle \\
= 0.
\]

This concludes the proof of Theorem 3. ■

7 Deducing Theorem 1 from Theorem 3

Let \( \Gamma = SL_n(\mathbb{Z}) \) for \( n \geq 3 \). Let \( M \) be a finite factor, with trace \( \tau \), and let \( \pi : \Gamma \to U(M) \) be a group homomorphism such that \( \pi(\Gamma)'' = M \). Then \( \varphi = \tau \circ \pi \) is a character of \( \Gamma \).

Assume that \( M \) is finite dimensional. Let \( \pi_\varphi : \Gamma \to U(M_\varphi) \) be the finite factor representation associated to \( \varphi \) (see Section 2). The mapping \( \pi_\varphi(\gamma) \mapsto \pi(\gamma) \) extends to an isomorphism \( M_\varphi \to M \) of von Neumann algebras. Hence \( M_\varphi \) is finite dimensional and, by Theorem 3, \( \varphi \) is the character of an irreducible finite dimensional representation of some congruence quotient \( SL_n(\mathbb{Z}/N\mathbb{Z}) \) for \( N \geq 1 \). It follows that \( \pi \) factorizes through \( SL_n(\mathbb{Z}/N\mathbb{Z}) \).

Assume now that \( M \) is infinite dimensional. By Theorem 3, we have \( \varphi = \tilde{\chi} \) for a character \( \chi \) of the centre \( C \). If \( n \) is odd, let \( \Lambda = \Gamma \) and, if \( n \) is even, let \( \Lambda = \Gamma(N) \) be a congruence subgroup for \( N \geq 3 \). Then \( \Lambda \) has finite index in \( \Gamma \) and \( \Lambda \cap C = \{e\} \). We therefore have \( \varphi|_\Lambda = \delta_e \). The GNS-representation of \( \Lambda \) corresponding to \( \delta_e \) is the regular representation \( \lambda_\Lambda \) which generates the von Neumann algebra \( L(\Lambda) \). The mapping \( \lambda_\Lambda(\gamma) \mapsto \pi(\gamma) \) extends to a normal homomorphism \( L(\Lambda) \to M \).

Remark 18 Observe that the conclusion in (ii) of Theorem 1 is that \( \pi|_\Lambda \) extends to \( L(\Lambda) \) and not just to \( U(L(\Lambda)) \). P. de la Harpe pointed out to me that this is a stronger statement: a homomorphism \( U(M_1) \to U(M_2) \)
between the unitary groups of two finite factors $M_1, M_2$ does not necessarily extend to an algebra homomorphism $M_1 \to M_2$. As a simple example, take $M_1 = M_2(\mathbb{C})$ and $M_2 = M_4(\mathbb{C}) \cong M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. The group homomorphism $\pi : U(2) \to U(4), g \mapsto g \otimes g$ does not extend to an algebra homomorphism $M_2(\mathbb{C}) \to M_4(\mathbb{C})$.

8 A question of Kirchberg

A conjecture of Kirchberg [Kirc93, Section 8, (B4)] is:

The full $C^*$-algebra $C^*(SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}))$ of the direct product $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ has a faithful tracial state.

As shown in [Kirc93], this problem is in fact equivalent to a series of outstanding conjectures, among them the following one which was suggested by Connes in [Conn76, page 105]:

Every factor of type $II_1$ with separable predual is a subfactor of the ultrapower $R_\omega$ of the hyperfinite factor $R$ of type $II_1$.

A positive answer to the following question of Kirchberg [Kirc93, Remark 8.2] would imply the conjecture above:

Does $C^*(SL_4(\mathbb{Z}))$ have a faithful tracial state?

Indeed, $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ embeds as a subgroup of $SL_4(\mathbb{Z})$, for instance, via the mapping

$$SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \ni (\gamma_1, \gamma_2) \mapsto \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \in SL_4(\mathbb{Z}).$$

A faithful tracial state on $C^*(SL_4(\mathbb{Z}))$ would give, by restriction, a faithful tracial state on $C^*(SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}))$.

We proceed to show that the answer to this question is negative. In fact, the following stronger result will be proved. We will consider the copy

$$\Lambda = \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & I \end{pmatrix} : \gamma \in SL_2(\mathbb{Z}) \right\} \cong SL_2(\mathbb{Z})$$

of $SL_2(\mathbb{Z})$ inside $SL_n(\mathbb{Z})$.

**Corollary 19** Let $n \geq 3$ and set $\Gamma = SL_n(\mathbb{Z})$. Let $\varphi$ be a tracial state on $C^*(\Gamma)$. Then $\varphi|_{C^*(\Lambda)}$ is not faithful.
Proof Let \( \pi \) be the cyclic unitary representation of \( \Gamma \) corresponding to \( \varphi \). By Theorem 3, \( \pi \) decomposes as a direct sum

\[
\pi_{\infty} \oplus \bigoplus_{i} \sigma_{i},
\]

where \( \pi_{\infty} \) is a multiple of the regular representation \( \lambda_{\Gamma} \), and where every representation \( \sigma_{i} \) factorizes through some congruence quotient \( \Gamma/\Gamma(N_{i}) \).

Let \( \text{Rep}_{\text{cong}}(\Gamma) \) denote the set of all unitary representations of \( \Gamma \) which factorize through some congruence quotient. In fact, as a consequence of the positive answer to the congruence subgroup problem, \( \text{Rep}_{\text{cong}}(\Gamma) \) coincides with the set of all finite dimensional unitary representations of \( \Gamma \) (see [Bekk99, Proposition 2]). This implies (see, for instance, [Bekk99, Proposition 1]) that

\[
\bigcap_{\sigma \in \text{Rep}_{\text{cong}}(\Gamma)} C^{*} - \text{Ker}\sigma \subset C^{*} - \text{Ker}\lambda_{\Gamma},
\]

where \( C^{*} - \text{Ker}\sigma \) denotes the kernel in \( C^{*}(\Gamma) \) of the extension of a unitary representation \( \sigma \) of \( \Gamma \).

We consider now the restriction \( \pi|_{\Lambda} \) of \( \pi \) to \( \Lambda \). Observe that

\[
\text{Rep}_{\text{cong}}(\Lambda) = \{ \sigma|_{\Lambda} : \sigma \in \text{Rep}_{\text{cong}}(\Gamma) \}.
\]

Since \( C^{*} - \text{Ker}\lambda_{\Lambda} = C^{*} - \text{Ker}(\lambda_{\Gamma}|_{\Lambda}) \), we have

\[
\bigcap_{\sigma \in \text{Rep}_{\text{cong}}(\Lambda)} C^{*} - \text{Ker}\sigma \subset C^{*} - \text{Ker}\lambda_{\Lambda},
\]

It follows from Selberg’s inequality \( \lambda_{1} \geq 3/16 \) (see [Bekk99, Lemma 3]) and from the fact that \( SL_{2}(\mathbb{Z}) \) does not have Kazhdan’s Property (T) that \( \text{Rep}_{\text{cong}}(\Lambda) \) does not separate the points of \( C^{*}(\Lambda) \), that is,

\[
\bigcap_{\sigma \in \text{Rep}_{\text{cong}}(\Lambda)} C^{*} - \text{Ker}\sigma \neq \{0\}.
\]
Hence, we have

\[ C^* - \text{Ker}(\pi|_\Lambda) = C^* - \text{Ker}(\pi_\infty|_\Lambda) \cap \bigcap_i C^* - \text{Ker}(\sigma_i|_\Lambda) \]

\[ = C^* - \text{Ker} \lambda_\Lambda \cap \bigcap_i C^* - \text{Ker}(\sigma_i|_\Lambda) \]

\[ \supset C^* - \text{Ker} \lambda_\Lambda \cap \bigcap_{\sigma \in \text{Rep}_{\text{cong}}(\Lambda)} C^* - \text{Ker} \sigma \]

\[ = \bigcap_{\sigma \in \text{Rep}_{\text{cong}}(\Lambda)} C^* - \text{Ker} \sigma \]

and \( C^* - \text{Ker}(\pi|_\Lambda) \neq \{0\} \). This clearly implies that \( \varphi|_\Lambda \) is not faithful. ■

**Remark 20** The previous result does not hold for \( n = 2 \). Indeed, as was shown in [Choi80, Corollary 9], \( C^*(SL_2(\mathbb{Z})) \) has a faithful trace. In fact a stronger result is proved in [Choi80, Theorem 7]: \( C^*(SL_2(\mathbb{Z})) \) is residually finite dimensional, that is, the finite dimensional representations of \( SL_2(\mathbb{Z}) \) separate the points of \( C^*(SL_2(\mathbb{Z})) \).

It is shown in [LuSh04] that other interesting groups have a residually finite dimensional full \( C^* \)-algebra; this is, for instance, the case for fundamental groups of surfaces.

### 9 A remark on semi-finite traces

As mentioned in the introduction, it is conceivable that semi-finite, infinite traces exist on \( C^*(PSL_n(\mathbb{Z})) \) for \( n \geq 3 \). The following result implies that no such trace factorizes through the reduced \( C^* \)-algebra \( C^*_r(PSL_n(\mathbb{Z})) \) for any integer \( n \geq 2 \).

**Proposition 21** Let \( G \) be a connected real semisimple Lie group without compact factors and with trivial centre. Let \( \Gamma \) be a Zariski-dense subgroup of \( G \). Then the tracial state \( \delta_c \) is, up to a scalar multiple, the unique semi-finite trace on \( C^*_r(\Gamma) \). In particular, \( C^*_r(\Gamma) \) has no normal factor representation of type \( II_\infty \).

**Proof** Let \( \varphi : C^*_r(\Gamma)^+ \to [0, \infty] \) be a semi-finite trace on the set of positive elements of \( C^*_r(\Gamma) \).
We use an observation from [Rose89, page 583]. It is well-known that there exist a non-zero two-sided ideal \( m \), called the ideal of definition of \( \varphi \), and a linear functional on \( m \) which coincides with \( \varphi \) on \( m^+ \) (see [Dix-C* Proposition 6.1.2]). Now, by [BeCH95], \( C^*(\Gamma) \) is simple, that is, \( C^*(\Gamma) \) has no non-trivial two-sided (closed or non-closed) ideals. Hence, \( m = C^*(\Gamma) \) and \( \varphi \) is a finite trace. By [BeCH95], \( \delta_e \) is the unique tracial state on \( C^*(\Gamma) \) and the claim follows. ■

Examples of Zariski dense subgroups \( \Gamma \) of a group \( G \) as in the previous proposition include all lattices in \( G \). So Proposition 21 applies, for instance, when \( \Gamma = \text{PSL}_n(\mathbb{Z}) \) for \( n \geq 2 \) or when \( \Gamma \) is the fundamental group of an oriented compact surface of genus \( \geq 2 \).

References

[BaMS67] H. Bass, M. Lazard and J-P. Serre. Sous-groupes d’indice fini dans \( SL(n, \mathbb{Z}) \). Bull. Amer. Math. Soc. 70, 385-392 (1964).

[BeCH95] B. Bekka, M. Cowling and P. de la Harpe. Some groups whose reduced \( C^* \)-algebra is simple. Publ. Math. IHES 80, 117-134 (1995).

[Bekk99] B. Bekka. On the full \( C^* \)-algebras of arithmetic groups and the congruence subgroup problem. Forum Math. 11, 705-715 (1999).

[Boye83] R.P. Boyer. Infinite traces of AF-algebras and characters of \( U(\infty) \). J. Oper. Theory 9, 205-236 (1983).

[Bren60] J.L. Brenner. The linear homogeneous group, III. Ann. Math. 71, 210-223 (1960).

[Burg91] M. Burger. Kazhdan constants for \( SL(3, \mathbb{Z}) \). J. Reine Angew. Math. 413, 36-67 (1991).

[Choi80] M. D. Choi. The full \( C^* \)-algebra of the free group on two generators. Pac. J. Math. 87, 41-48 (1980).

[Conn76] A. Connes. Classification of injective factors., Ann. Math. 104, 73-115 (1976).

[Conn80] A. Connes. A factor of type II\(_1\) with countable fundamental group. J. Operator Theory, 4: 151–153, 1980.
[CoJo85] A. Connes and V. Jones. Property (T) for von Neumann algebras. *Bull. London Math. Soc.* **17**, 51-62 (1985).

[CoHa89] M. Cowling and U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96:507–542, 1989.

[DaKe79] S.G. Dani and M. Keane. Ergodic invariant measures for action of SL(2, $\mathbb{Z}$). *Ann. Inst. Henri Poincaré* **15**, Nouv. Sér., Sect. B, 79-84 (1979).

[Dix-C\textsuperscript{*}] J. Dixmier. *Les $C^*$-algèbres et leurs représentations*. Gauthier-Villars, 1969.

[Dix-vN] J. Dixmier. *Les algèbres d’opérateurs dans l’espace Hilbertien*. Gauthier-Villars, 1969.

[Furm99] A. Furman. Orbit equivalence rigidity. *Ann. Math.*, 150:1083–1108, 1999.

[Harp79] P. de la Harpe. Simplicity of the projective unitary groups defined by simple factors. *Comment. Math. Helv.* **54**, 334–345 (1979).

[Jone00] V. Jones. Ten problems. In: Mathematics: Frontiers and perspectives. Ed: V. Arnold et al., pages 79-91. American Mathematical Society, 2000.

[Kirc93] E. Kirchberg. On non–split extensions, tensor products and exactness of group $C^*$–algebras. *Invent. Math.* **112**, 449–489 (1993).

[Kiri65] A. A. Kirillov. Positive definite functions on a group of matrices with elements from a discrete field. *Soviet. Math. Dokl.* **6**, 707-709 (1965).

[LuSh04] A. Lubotzky and Y. Shalom. Finite representations in the unitary dual and Ramanujan groups. In: “Discrete geometric analysis”, Proceedings of the 1st JAMS symposium, Sendai, 2002. Contemporary Mathematics **347**, 173-189 (2004).

[Marg91] G.A. Margulis. *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, 1991.
[Menn65] J. L. Mennicke. Finite factor groups of the unimodular group. Ann. Math. 81, 31-37 (1965).

[Popa06-a] S. Popa. Strong rigidity of $II_1$ factors arising from malleable actions of w-rigid groups, I. Inventiones Math. 165, 369-408 (2006).

[Popa06-b] S. Popa. Strong rigidity of $II_1$ factors arising from malleable actions of w-rigid groups, II. Inventiones Math. 165, 409-452 (2006).

[Robe93] G. Robertson. Property (T) for $II_1$ factors and unitary representations of Kazhdan groups. Math. Ann. 296, 547-555 (1993).

[Ovci71] S.V. Ovcinnikov. Positive definite functions on Chevalley groups. Funct. Anal. Appl. 5, 79-80 (1971).

[Rose89] J. Rosenberg. Un complément à un théorème de Kirillov sur les caractères de $GL(n)$ d’un corps infini discret. C. R. Acad.Sc.Paris 309, Série I, 581-586 (1989).

[Skud76] H-L. Skudlarek. Die unzerlegbaren Charaktere einiger diskreter Gruppen. Math. Ann. 223, 213-231 (1976).

[Stein85] R. Steinberg. Some consequences of the elementary relations in $SL_n$. Contemporary Math. 45, 335-350 (1985).

[StVo75] S. Stratila and D. Voiculescu. Representations of AF-algebras and of the group $U(\infty)$. Lecture Notes in Mathematics 486, Springer, 1975.

[Thom64a] E. Thoma. Über unitäre Darstellungen abzählbarer, diskreter Gruppen. Math. Ann. 153, 111-138 (1964).

[Thom64b] E. Thoma. Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. Math. Z. 85, 40-61 (1964).

[Tits76] J. Tits. Systèmes générateurs de groupes de congruence. C. R. Acad. Sci., Paris, Sér. A, 283, 693-695 (1976).

[Vaes06] S. Vaes. Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa). Séminaire Bourbaki, Exposé 961, March 2006
[Vale97] A. Valette. Amenable representations and finite injective von Neumann algebras. *Proc. Amer. Math. Soc.* **125**, 1841-1843 (1997).

[VerKe81] A.M. Vershik and S.V. Kerov. Characters and factor representations of the infinite symmetric group. *Sov. Math. Dokl.* **23**, 389-392 (1981).

[Voic76] D. Voiculescu. Représentations factorielles de type $II_1$ de $U(\infty)$. *J. Math. Pures Appl.* **IX**, Sér. 55, 1-20 (1976).

[Zele81] A. V. Zelevinsky. *Representations of finite classical groups. A Hopf algebra approach*. Lecture Notes in Mathematics **869**, Springer, 1981.

[Zimm84] R.J. Zimmer. *Ergodic theory and semisimple groups*, Birkhäuser, 1984.

**Address**
Bachir Bekka, UFR Mathématique, Université de Rennes 1, Campus Beaulieu, F-35042 Rennes Cedex, France
E-mail: bachir.bekka@univ-rennes1.fr