Exact Traveling Wave Solutions of One-Dimensional Models of Cancer Invasion

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Abstract—In this paper, we obtain exact analytical solutions of equations of continuous mathematical models of tumor growth and invasion based on the model introduced by Chaplain and Lolas for the case of one spatial dimension. The models consist of a system of three nonlinear reaction–diffusion–taxis partial differential equations describing the interactions between cancer cells, the matrix degrading enzyme and the tissue. The obtained solutions are smooth nonnegative functions depending on the traveling wave variable with certain conditions imposed on model parameters.

Keywords: partial differential equation, exact solution, traveling wave solutions, cancer invasion, chemotaxis, haptotaxis

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INTRODUCTION

A comprehensive presentation of the biological and medical aspects underlying the construction of mathematical models in oncology can be found in the cited literature [1–36] and references therein.

In the present paper, we obtain exact analytical solutions of a system of three nonlinear second-order partial differential equations of the reaction–diffusion–taxis type. This system is based on a continuous mathematical model of the growth and invasion of solid tumors proposed in [4], being a modification of it that allows one to exactly solve the equations of the system in terms of the traveling wave variable.

The original model [4] describes the space–time behavior and evolution of tumor cells with density \( c(t, \vec{r}) \), extracellular matrix (ECM) density \( v(t, \vec{r}) \), and concentration of protease (plasminogen activator, uPA) \( u(t, \vec{r}) \). It is assumed that cell number density changes due to dispersion resulting from random locomotion and the directional migratory response of tumor cells to gradients of diffusible (uPA) and nondiffusible (ECM) macromolecules [4]. It is also assumed that in the absence of any extracellular matrix, the proliferation of cancer cells obeys the logistic growth law and the extracellular matrix does not move and changes solely due to its degradation by uPA protease and its remodeling by cancer and other cells; the terms responsible for proliferation and re-establishment are the last terms in the first two equations below. The complete system of dimensionless equations has the form [4]

\[
\begin{align*}
    c_t &= D_c \nabla^2 c - \chi_c \nabla (c \nabla u) - \xi_c \nabla (c \nabla v) + \mu_1 c(1 - c - v), \\
    v_t &= -\delta uv + \mu_2 v(1 - c - v), \\
    u_t &= D_u \nabla^2 u + \alpha c - \beta u,
\end{align*}
\]

(1)

where the constant positive model parameters \( D_c \) and \( D_u \) are the coefficients of diffusion of cells and uPA, respectively; \( \chi_c \) and \( \xi_c \) are the chemotaxis and haptotaxis coefficients, \( \delta \) is the rate of ECM degradation by uPA; and \( \alpha \) and \( \beta \) are the rates of production and decay of uPA. The transformation of variables and parameters in Eqs. (1) into dimensionless quantities is the same as in [2–4],

\[
\begin{align*}
    t &\to t/\tau, \quad x \to x/L, \quad c \to c/c_0, \quad v \to v/v_0, \quad u \to u/u_0; \\
    D_c &\to D_c/D, \quad D_u \to D_u/D, \quad \chi \to \chi_c u_0/D, \quad \xi \to \xi_c v_0/D, \quad \alpha \to \alpha \tau c_0/u_0;
\end{align*}
\]
where $c_0$, $v_0$, and $u_0$ are the corresponding reference tumor cell density, the extracellular matrix density, and the reference uPA concentration, respectively, $\tau = L^2/D$, $L = 0.1 - 1$ cm, is the maximum invasion distance of cancer cells at the early stage of invasion, and $D = 10^{-6}$ cm$^2$/s is the reference coefficient of chemical diffusion [4]. Since tumor cells and matrix-degrading enzymes are assumed to remain within the tissue region under consideration [3], the new variable $x \in [0; 1]$. New parameters are denoted by the same symbols. Therefore, the parameter values are: $D_c \sim 10^{-5} - 10^{-3}$, $D_u \sim 10^{-3} - 1$, $\chi_c \sim \xi_c \sim 10^{-3} - 1$, $\alpha \sim 0.05 - 1$, $\beta \sim 0.13 - 0.95$, $\delta \sim 1 - 20$, $\mu_1 \sim 0.05 - 2$, and $\mu_2 \sim 0.15 - 2.5$.

This model and its extensions and generalizations are being fruitfully studied. However, in most works the analysis of such systems and their solutions is carried out numerically. The existence of solutions in terms of the traveling wave variable for various models of tumor invasion with haptotaxis was established as a result of numerical calculations in [4], as well as in [37–40], where a detailed study of the behavior of the traveling wave was carried out. However, to the best of our knowledge, the models discussed in this paper and the solutions presented here are new.

It becomes interesting whether it is possible to solve this system analytically and obtain exact solutions, including those capable of satisfactory biological interpretation. It seems that the answer to this question is negative and, unfortunately, the model (1) cannot be solved analytically even in the case of one spatial variable.

The purpose of this article is to consider models that, in our opinion, are as close as possible to the model presented above and that can be solved exactly; we will consider models in one spatial dimension. Following the example of the authors of [4], we study systems under the assumption that they are dimensionless. For certain values of the model parameters, we obtain exact analytical solutions in terms of the traveling wave variable at a speed that depends on these parameters. The presented solutions contain both biologically acceptable and solutions that are not suitable for biological analysis. These latter solutions may be of interest as exact solutions of systems of nonlinear partial differential equations.

1. MODELS CONSIDERED AND CONSTRUCTION OF EXACT SOLUTIONS

1.1. Model without Proliferation

We would like to start with a modification of the model proposed in [3]. We examine this model with logarithmic chemotactic and haptotactic sensitivity functions, without the proliferation and re-establishment terms, and with a slightly modified ECM equation. It can be seen that the model (1) is more complicated and contains a term describing proliferation. However, it initially seems interesting to solve exactly a system in which “cell proliferation was not included in order to focus solely on the role of cancer cell migration in invasion” [5].

So, the model we are considering in the general case has the form

\[
\begin{align*}
\chi c_t &= D_c c_{xx} - \chi c \left( \frac{u_x}{u} \right)_x - \xi c \left( \frac{v_x}{v} \right)_x, \\
v_t &= -\delta uv^p, \\
u_t &= D_u u_{xx} + \alpha c - \beta u, \\
\end{align*}
\]

(2)

where the variables and model parameters have been defined above. The second equation in (2) differs from [3, 4] by the presence of degree $p$ in the function $v$, and we assume that $0 < p < 1$. For $p = 1$ we do not obtain biologically acceptable solutions, however, as we will see below, $p$ can be taken very close to unity, for example, $p = 0.95$. Here the transformation of variables and parameters into dimensionless quantities is the same as above, with the exception of $\delta$, $\chi_c$, and $\xi_c$: $\delta \rightarrow \delta \tau u_0 v_0^{p-1}$, so $\delta \sim 10^{-5(p-1)} \times 10^{-6(p-1)}$, as for $\chi_c$ and $\xi_c$, we take dimensionless quantities as in [4], i.e., $\chi_c \sim \xi_c \sim 10^{-3}$.1
In terms of a traveling wave variable of the form \( y = x - \nu t, \nu = \text{const}, \) this system has the form

\[
\begin{align*}
\nu c + D_c y - \chi_c c(\ln u)_y - \xi_c c(\ln v)_y &= \lambda, \\
\nu v_y - \delta uv^p &= 0, \\
\nu u_y + D_u u_{yy} + \alpha c - \beta u &= 0, \\
\end{align*}
\]

(2')

where \( c = c(y), v = v(y), u = u(y), \) and \( \lambda \) is the integration constant. Further, we put \( \lambda = 0. \) If we introduce the function

\[
F = \frac{v^{1-p}}{1-p},
\]

(3)

the first two equations in (2') yield

\[
\begin{align*}
c &= C_c (e^{-\nu y} \xi_c u \chi_c)^{1/D_c} \\
u u &= \frac{\nu}{\delta} F_y,
\end{align*}
\]

(4)

where the constant \( C_c > 0. \) Substituting (3) and (4) into the third equation in system (2'), we obtain

\[
D_u F_y + \nu F_y - \beta F_y + \frac{C_1}{D_u} e^{-\frac{\xi_c}{D_c} y} F_y e^{-\frac{\xi_c}{D_c} y (1-p)} = 0,
\]

(5)

\[
C_1 = C_c \alpha (\nu/\delta) \frac{\chi_c}{D_c} (1-p) \frac{\chi_c}{D_c}, \quad \text{and further we will study this equation.}
\]

It seems to us that for arbitrary values of the system parameters it is impossible to obtain an exact solution in explicit form. Therefore, we impose a number of restrictions on these parameters. Thus, let

\[
\chi_c/D_c = 1,
\]

(6)

i.e., \( \chi_c = D_c \sim 10^{-3}. \) Then there is a “selected” value of the speed of traveling waves for which we obtain two classes of different solutions. Let

\[
\nu^2 = \frac{\beta D_c^2}{D_u - D_c}.
\]

(7)

This can be done because \( D_u \geq D_c \) and we are not considering the case of \( D_u = D_c. \) Then it can be shown that Eq. (5) reduces to the form

\[
F_{yy} - \frac{\nu(D_u - D_c)}{D_u D_c} F_y + \frac{C_1 D_c (1-p)}{D_u (\xi_c + D_c (1-p))} e^{-\frac{\xi_c}{D_c} y} F_y e^{-\frac{\xi_c}{D_c} y (1-p)+1} = 0
\]

(8)

with an integration constant equal to zero. To integrate this equation, we use the method of Lie groups of infinitesimal transformations [41]. We find the group invariant of the second prolongation of the vector of the one-parameter symmetry group of (8) and transform Eq. (8) with its help into a first-order equation. It turns out that there exist two nontrivial symmetry groups depending on the ratio of parameters that produce two different types of solutions. Let us consider the first of them.

1.2. Exact Solution

The possibility to reduce Eq. (8) to a first-order equation and solve it requires the following condition to be met:

\[
1 - p = \frac{\xi_c (D_u - D_c)}{2D_u D_c}.
\]

(9)

Let us introduce a new variable \( z \) and a new function \( w, \)

\[
\begin{align*}
z &= F e^{-\frac{\nu (1-p)}{\xi_c} y} \\
w &= -\frac{D_c}{\nu} F_y e^{-\frac{\nu (1-p)}{\xi_c} y}.
\end{align*}
\]

(10)
then, after elementary integration, Eq. (8) turns into a quadratic equation for $w(z)$,

$$w^2 + \frac{2(1 - p)D_c}{\xi_c} zw + C_2 z^{\frac{\xi_c}{p-1}} + 2 = 0,$$

where the integration constant is zero and $C_2 = \frac{2C_c D_c (1 - p) {\xi_c}^{p-1}}{v^2(\xi_c + D_c (1 - p)) (\xi_c + 2D_c (1 - p))}$. Returning to the initial function $F$ and the variable $y$ and integrating (11), we obtain a solution for $F$,

$$F = C_3 (e^{-\frac{\nu}{\xi_c} y} + C_F) \frac{2D_c (1 - p)}{\xi_c},$$

where $C_F$ is a positive constant and

$$C_3 = \left( C_F \frac{2v^2 D_u(\xi_c + D_c (1 - p)) (\xi_c + 2D_c (1 - p))}{C_c \alpha D_c^2 \xi_c} \right)^{D_u/\xi_c}.$$  

Substituting the latter into Eqs. (3) and (4), we obtain the first type of solutions of system (2) in the form

$$c(y) = C_3 e^{-\frac{\nu}{\xi_c} y} (e^{-\frac{\nu}{\xi_c} y} + C_F) \frac{D_u (\xi_c + 2D_c (1 - p))}{\xi_c (\alpha D_c \xi_c (1 - p))},$$

$$v(y) = C_3 (e^{-\frac{\nu}{\xi_c} y} + C_F) \frac{2D_c (1 - p)}{\xi_c},$$

$$u(y) = C_3 e^{-\frac{\nu}{\xi_c} y} (e^{-\frac{\nu}{\xi_c} y} + C_F) \frac{D_u (\xi_c + 2D_c (1 - p))}{\xi_c (\alpha D_c \xi_c (1 - p))},$$

where the constants have the form

$$C_v = \left( \frac{C_F \beta (\xi_c + D_c (1 - p)) (\xi_c + 2D_c (1 - p))}{\alpha D_c \xi_c (1 - p)} \right)^{D_u/\xi_c},$$

$$C_u = C_v \frac{\beta D_c}{\alpha D_c (1 - p)},$$

$$C_C = C_c C_u C_v^{\xi_c/\xi_c}.$$

As can be seen from Eqs. (13), these solutions are positive functions defined for all values of $y$. Despite the fact that, due to the biological context, we are interested in solutions in a limited space-time domain, it is easy to see that for $D_u > D_c$, the functions $c(y)$ and $u(y)$ vanish as $y \to \pm \infty$; the function $v(y) \to C_v C_F^{2D_c/\xi_c}$ as $\nu y \to +\infty$ and $v(y) \to 0$ as $\nu y \to -\infty$. It is also clear that $c(y)$ and $u(y)$ have a single maximum; its values, as well as the asymptotic value of $v(y)$ as $\nu y \to +\infty$ depend on the parameters chosen. These functions are presented in Fig. 1 for $\nu > 0$ and various parameter values.

1.3. Model with Proliferation

Now consider a model with terms describing proliferation and re-establishment, modified as follows:

$$c_t = D_c c_{xx} - \chi_c(c(u_x + \lambda_u u)_x) - \xi_c \left( \frac{v_x}{v^p} \right)_x + \mu_1 c(1 - c),$$

$$v_t = -\delta u v^p + \mu_2 v^p (1 - \lambda_c c - v^{1-p}),$$

$$u_t = D_u u_{xx} + \alpha c - \beta u,$$

where we again introduced the degree $p$ of the function $v$, $0 < p < 1$, and the constant $\lambda_c > 0$; other variables and parameters of the model are defined above, with the exception of $\mu_2$: $\mu_2 \to \mu_2 T v_0^{p-1}$, therefore, $\mu_2 \sim 0.15 \times 10^{-5(p-1)} - 2.5 \times 10^{-6(p-1)}$. However, the main difference between system (1) and this one is the presence of an additional term $\lambda_C (cu)_x$, $\lambda_u$ is a constant. This term was originally added to the first equation (15) so that the system could be solved exactly. However, the graphs of the obtained solutions are very close to the graphs of the numerical solutions presented in Figs. 4.
Fig. 1. Graphs of the functions $u(y)$, $v(y)$, and $c(y)$ for $\nu > 0$ with various parameter values: (a) $p = 0.9$, $D_c = \chi_c = 8 \times 10^{-3}$, $D_u = 10^{-2}$, $\xi_c = 8 \times 10^{-3}$, $\alpha = 0.25$, $\beta = 0.3$, $\delta = 10^{0.7}$, $C_F = 15.84$, $C_c = 1$; (b) $p = 0.95$, $D_c = \chi_c = 5 \times 10^{-3}$, $D_u = 10^{-2}$, $\xi_c = 10^{-3}$, $\alpha = 0.1$, $\beta = 0.95$, $\delta = 10^{1.25}$, $C_F = 71.25$, $C_c = 1$.

and 5 in [4] for traveling waves propagating into tissue. This suggests that adding the above term does not distort the original model much, although the coefficient $\lambda_u$ cannot be made arbitrarily small; as can be seen from the following presentation, $\lambda_u \geq D_c$.

In terms of the traveling wave variable $y = x - \nu t$, $\nu = \text{const}$, this system takes the form

$$
D_c c_{yy} + \nu c_y - \chi_c \left( c(u_y + \lambda_u u) \right)_y = \xi_c \left( \frac{v_y}{v_p} \right)_y + \mu_1 c(1 - c) = 0,
$$

$$
\nu v_y - \delta u v^p + \mu_2 v^p \left( 1 - \lambda_c c - v^{1-p} \right) = 0,
$$

$$
D_u u_{yy} + \nu u_y + \alpha c - \beta u = 0,
$$

where, this time, $c = c(y)$, $v = v(y)$, and $u = u(y)$. Everywhere below we assume $\nu > 0$.

As in the case of the previous model (2'), we must impose a number of conditions on the model parameters in order to obtain exact solutions in explicit form. Thus, the degree $p$ and the constant $\lambda_u$
are expressed via other parameters of the model by the relations

\[
1 - p = \frac{\xi_c (\delta \alpha + \beta \mu_2 \lambda_c)}{\mu_2 (\chi_c \alpha + \xi_c \mu_2 \lambda_c)},
\]

\[
\lambda_u = \frac{\beta \sqrt{\xi_c \mu_2 \lambda_c - D_c \mu_1}}{\chi_c \alpha},
\]

(16)

where we are restricted by the condition \( \xi_c \mu_2 \lambda_c - D_c \mu_1 > 0 \). The “selected” value of the speed of traveling waves is equal to

\[
\nu = \frac{\xi_c \mu_2 \lambda_c}{\sqrt{\xi_c \mu_2 \lambda_c - D_c \mu_1}}.
\]

(17)

One more necessary condition that must be imposed on the model constants to obtain exact solutions has the form

\[
D_u (\delta \alpha + \beta \mu_2 \lambda_c)(\xi_c \mu_2 \lambda_c - D_c \mu_1) = \chi_c \alpha \mu_2 \lambda_c (\chi_c \alpha + \xi_c \mu_2 \lambda_c).
\]

(18)

Then the second equation in (15') can be integrated, and we obtain

\[
v(y) = \left(1 - \frac{\mu_2 \lambda_c D_u (1 - p)}{\alpha \nu} u - \frac{\delta \alpha + \beta \mu_2 \lambda_c}{\alpha \mu_2} u \right)^{\frac{1}{1-p}}.
\]

(19)

Further, we introduce a function

\[
F = c_y + K (c - c^2), \quad K = \frac{\xi_c \mu_2 \lambda_c}{D_c \nu}.
\]

(20)

By direct substitution we can check that the first equation in (15') can be reduced to the form

\[
F_y + \frac{\mu_1 \nu}{\xi_c \mu_2 \lambda_c} F = 0.
\]

(21)

1.4. Exact Solution

The first-order linear equation (21) has the obvious solution

\[
F = C_F e^{-\frac{\mu_1 \nu}{\xi_c \mu_2 \lambda_c} y}.
\]

(22)

Consider the case of \( C_F = 0 \) or \( F = 0 \). Then the function \( c(y) \) must be a solution of the Riccati equation

\[
c_y + K (c - c^2) = 0,
\]

(23)

where the left-hand side of Eq. (23) simultaneously represents the Burgers equation integrated in the traveling wave variable. Thus, the function \( c(y) \) is a well-known solution of the Burgers equation, and we are interested in the bounded “shock wave” solution

\[
c = \frac{1}{1 + C_c e^{K y}},
\]

(24)

where \( C_c \) is a positive constant. Substituting this expression into the third equation in system (15') and choosing the integration constant so that the function \( u(y) \) is bounded, we obtain

\[
u(y) = \frac{\alpha}{D_u \Delta k k_+} \left( C_{c^+}^{k_+} \Gamma \left( 1 - \frac{k_+}{K} \right) \Gamma \left( 1 + \frac{k_+}{K} \right) e^{k_+ y} - 2 F_1 \left( - \frac{k_+}{K}, 1, 1 - \frac{k_+}{K}, -C_c e^{K y} \right) \right) + \frac{\alpha}{D_u \Delta k k_-} \left( 2 F_1 \left( - \frac{k_-}{K}, 1, 1 - \frac{k_-}{K}, -C_c e^{K y} \right) \right),
\]

(25)
Fig. 2. Graphs of the functions $u(y)$, $v(y)$, and $c(y)$ for various values of the constant $Cc$: $p = 0.868$, $D_c = 10^{-5}$, $D_u = 1$, $\chi_c = 0.475$, $\xi_c = 3.4 \times 10^{-2}$, $\alpha = 0.5$, $\beta = 0.95$, $\delta = 4.6$, $\mu_1 = 0.05$, $\mu_2 = 2.5$.

where

$$k_{\pm} = \frac{1}{2D_u} \left( -\nu \pm \sqrt{\nu^2 + 4\beta D_u} \right), \quad \Delta k = k_- - k_+; \quad (26)$$

$\Gamma$ and $_2F_1$ are the gamma function and the Gaussian hypergeometric function, respectively. Finally, substituting Eq. (24) into (19), we obtain an explicit expression for the function $v(y)$.

It is easy to see from Eq. (24) that $c(y) \to 0$ as $y \to \infty$ and $c(y) \to 1$ as $y \to -\infty$. Also, for a certain choice of model constants, the function $u(y)$ in (25) is a smooth positive definite function; $u(y) \to 0$ as $y \to \infty$ and $u(y) \to \alpha/\beta$ as $y \to -\infty$. As for the function $v(y)$, from Eq. (19) one can see that $v(y) \to 1$ as $y \to \infty$ and $v(y) \to 1 - \delta \alpha + \beta \mu_2 \lambda \frac{\lambda}{\delta \mu_2}$ as $y \to -\infty$, and the choice of model parameters should ensure the condition $v(y) \geq 0$. These functions are presented in Fig. 2.

It can be noticed that the obtained graphs are very close to the graphs in Figs. 4 and 5 in [4], showing the evolution of $c(x, t)$, $u(x, t)$, and $v(x, t)$.

2. OTHER SOLUTIONS

It seems to us that there are no other exact solutions acceptable from a biological point of view. Let us now present solutions that may be of interest as exact solutions of systems of nonlinear partial differential equations.

2.1. Solutions of the Model without Proliferation

Let us return to Eq. (8) and consider a relation similar to (9),

$$1 - p = \frac{\xi_c(D_u - D_c)}{D_c(2D_c - D_u)}. \quad (27)$$

It should be noted that conditions (9) and (27) do not coincide for any parameter values, since $D_u > D_c$. Further, as can be seen from Eq. (27), the constraint on $D_u$ and $D_c$ becomes stricter, $D_c < D_u < 2D_c$. As in the case of the first class of solutions, we introduce a new variable $z$ and a new function $w$,

$$z = Fe^{-\frac{D_u - D_c}{D_c}y}, \quad w = F - \frac{D_u D_c}{\nu(D_u - D_c)} F_y. \quad (28)$$

Then Eq. (8) turns into a quadratic equation for $w(z)$

$$w^2 + C_4 z \frac{D_u}{\nu(D_u - D_c)} - C_w = 0. \quad (29)$$
where \( C_w > 0 \) is the integration constant and \( C_4 = \frac{2C_w\alpha(D_u-D_c)(1-p)}{\beta D_c}. \) Then we find solutions of Eq. (29) in parametric form with parameter \( \tau \),

\[
\tau^2 + 1 = \frac{C_w}{C_4} z^{-\frac{D_u}{D_u-D_c}}. \tag{30}
\]

Analysis of the asymptotic behavior of solutions for \( \pm \infty \) (see [42]) and the requirement that the functions \( c(y), v(y) \) and \( u(y) \) be positive determine one of the integration constants. The resulting formulas are quite complex, so we introduce the notation

\[
\Theta(\tau) = -\tau \frac{\Gamma(1-D_u/D_c)}{2\Gamma(3/2-D_u/D_c)}
\]

and express our solutions via \( \Theta(\tau) \). We also provide solutions for \( \nu > 0 \). This gives the following expressions for the second type of solutions:

\[
y(\tau) = -\frac{D_u}{\sqrt{\beta(D_u-D_c)}} \ln \left( \frac{C_w \frac{\beta}{D_u} 2(D_u-D_c)}{C_4 \frac{\beta}{D_u}} \Theta(\tau) \right), \tag{32}
\]

\[
v(\tau) = \left( \frac{C_w \frac{\beta}{D_u}(1-p)}{2(D_u-D_c)} \right)^{-\frac{1}{2}} (\tau^2 + 1) \frac{D_u-D_c}{\nu} (\Theta(\tau))^{-\frac{1}{2}}, \tag{33}
\]

\[
u(\tau) = -\frac{C_w}{\delta D_u} \frac{\beta D_c}{(\tau^2 + 1)^{-2+D_u/D_u}} \frac{\beta D_c}{(\tau^2 + 1)^{-1}} \frac{D_u-D_c}{\nu} (\Theta(\tau))^{-1} \tag{34}
\]

Considering formula (4), the expression for the function \( c(\tau) \) has the form

\[
c(\tau) = \frac{2\beta D_c(D_u-D_c)}{\alpha D_u^2} (\tau^2 + 1)^{-\frac{2\beta D_c}{\alpha D_u^2}} (\Theta(\tau))^2 u(\tau). \tag{35}
\]

Here \(_2F_1\) is the Gaussian hypergeometric function, and \( \Gamma \) is the gamma function. It is clear from the expressions (31), (33)–(35) that the functions \( v(\tau), u(\tau) \), and \( c(\tau) \) are smooth positive definite functions for all \( \tau \). The function \( c(\tau) \rightarrow 0 \) as \( \tau \rightarrow \pm \infty \), the function \( u(\tau) \rightarrow u_0 \) as \( \tau \rightarrow -\infty \), and \( u(\tau) \) vanishes as \( \tau \rightarrow +\infty \); the function \( v(\tau) \rightarrow 0 \) as \( \tau \rightarrow -\infty \) and \( v(\tau) \rightarrow v_0 \) as \( \tau \rightarrow +\infty \), where the values \( u_0 \) and \( v_0 \) can be obtained from expressions (33), (34). However, as can be seen from (31), (32), \( y \rightarrow y_0 \) as \( \tau \rightarrow -\infty \), where \( y_0 \) is the final value that can be made \( < 0 \) (or \( \geq 0 \)) by choosing integration constants. This leads to the fact that solutions obtained as functions of \( y \) can be considered only on a limited time interval. In other words, since \( x \in [0;1] \) and \( t \geq 0 \), formally \( y \in [-\nu t;1-\nu t] \) (for \( \nu > 0 \)). However, since \( y \) is bounded on the left by \( y_0 \), for each \( y_0 \) there exists a time value \( t_0 \) after which no solutions are defined. The graphs of these solutions are presented in Fig. 3.

2.2. Solutions of the Model with Proliferation

Consider again Eq. (22) and let \( C_F \neq 0 \),

\[
c_y + K(c-c^2) = C_F e^{-My}, \quad M = \frac{\mu_1 \nu}{\xi_2 \mu_2 \lambda_c}. \tag{36}
\]
As expected, the Cole–Hopf transform linearizes this equation. Introducing a new function $\tilde{c}$ and a new variable $\xi$ as

$$c = -\frac{1}{K} \tilde{c}_y,$$
$$\xi = 2\sqrt{C_F K} e^{-\frac{\nu}{M}} y,$$

we obtain the Bessel equation for $C_F > 0$ and the modified Bessel equation for $C_F < 0$ with order $\tilde{\nu}^2 = (K/M)^2$. We are only interested in a smooth solution, and we consider the Infeld $I_\nu$ and McDonald $K_\nu$ functions. As a result, the solution for the function $c(y)$ has the form

$$c(y) = \frac{1}{2} \left( 1 + \frac{|\tilde{\nu}|}{\nu} \right) + \sqrt{\frac{|C_F|}{K}} e^{-\frac{\nu}{M}} y \frac{C_1 I_{\tilde{\nu}+1}(\xi) - C_K K_{\tilde{\nu}+1}(\xi)}{C_1 I_\nu(\xi) + C_K K_\nu(\xi)}.$$  

(39)

It is easy to see that $c \to \infty$ as $y \to -\infty$. It cannot be said that for arbitrary values of $\tilde{\nu}$ the function $c(y)$ is always nonnegative and does not diverge as $y \to \infty$, but, apparently, this can be corrected by suitable choice of $C_I$ and $C_K$. Nevertheless, exponential growth as $y \to -\infty$ is a general property of the obtained solutions. Based on (15'), we can expect that the function $u(y)$ also grows exponentially as $y \to -\infty$. Solving the third equation in (15'), we obtain integrals that can be calculated exactly only for half-integer $\tilde{\nu}$. In particular, for $\tilde{\nu} = 1/2$ or $\xi_c \mu_2 \lambda_c = 1.5 D_c \mu_1$ with $C_I = \pi C_K$ we obtain very simple expressions for $c(y)$ and $u(y)$,

$$c(y) = \sqrt{|C_F|/K} e^{-K y},$$
$$u(y) = C_+ e^{k_1 y} + C_- e^{k_2 y} - \sqrt{|C_F|/K} \frac{\alpha}{D_u(k_+ + K)(k_- + K)} e^{-K y},$$

(40)

JOURNAL OF APPLIED AND INDUSTRIAL MATHEMATICS  Vol. 17  No. 3  2023
where $C_{\pm}$ are constants; one can set one or both of them equal to zero. It is not obvious whether the requirement $u \geq 0$ and conditions (16)–(18) can be satisfied simultaneously. However, this is not the main problem with this solution. The most dubious thing about it is the expression for the function $v(y)$; as can be seen from expression (19), it either diverges as $y \to -\infty$ or is equal to one for all $y$. All this gives reasons to believe that these solutions do not meet reasonable requirements.

The next solution is even less realistic. The expression for $c(y)$ can be easily obtained by the substitution $c = e^{-Ky}\tilde{c}$, $\xi_c\mu_2\lambda_c = 1.5D_c\mu_1$, but the function $u(y)$ cannot be completely exactly integrated. So,

$$c(y) = e^{-Ky} \sqrt{\frac{|C|}{K}} \frac{1 - \tilde{C}_ce^{-2\sqrt{|C|F}/K}e^{-K_y}}{1 + \tilde{C}_ce^{-2\sqrt{|C|F}/K}e^{-K_y}}.$$  (41)

Like the solution above, this function obviously diverges as $y \to -\infty$. However, the expression for $u(y)$ contains integrals of the form

$$\int \frac{(\ln(t - \tilde{C}_c))^{k_{\pm}/K}}{t} dt,$$

which are expressed in terms of a polylogarithm for a limited set of $k_+$ and cannot be calculated exactly for $k_-$. In addition, we assume that $u(y) \to$ exponentially as $y \to -\infty$, causing $v(y)$ to diverge as in the solution above.

CONCLUSIONS

In the present paper, we consider a system of three nonlinear second-order partial differential equations of the reaction–diffusion–taxis type with one spatial dimension. We obtain exact analytical solutions of this system depending on the traveling wave variable. Due to the fact that the solutions are expressed in terms of known functions and have a fairly simple form, they are easy to analyze. The system under consideration is a modification of a continuous mathematical model of tumor growth and invasion, proposed in its original form in the papers [3, 4]. The authors were solving the system numerically, and, as it seems to us, it cannot be solved analytically. Therefore, we slightly modified the model; this made it possible to obtain solutions in explicit form exactly. The graphs of some of the solutions we have presented are close to the graphs obtained by numerical methods within the framework of the original system. This allows us to conclude that the modifications we introduced did not greatly affect the essence of the original model.

REFERENCES

1. J. Folkman and M. Klagsbrun, “Angiogenic factors,” Science 235 (4787), 442–447 (1987).
2. A. R. A. Anderson and M. A. J. Chaplain, “Continuous and discrete mathematical models of tumour-induced angiogenesis,” Bull. Math. Biol. 60, 857–899 (1998).
3. A. R. A. Anderson, M. A. J. Chaplain, E. L. Newman, R. J. C. Steele, and A. M. Thompson, “Mathematical modelling of tumour invasion and metastasis,” J. Theor. Med. 2 (2), 129–154 (2000).
4. M. A. J. Chaplain and G. Lolas, “Mathematical modelling of cancer invasion of tissue: Dynamic heterogeneity,” Am. Inst. Math. Sci. 1 (3), 399–439 (2006).
5. H. Enderling and M. A. J. Chaplain, “Mathematical modeling of tumor growth and treatment,” Curr. Pharm. Des. 20 (30), 4934–4940 (2014).
6. J. A. Adam and N. Bellomo, A Survey of Models for Tumour-Immune System Dynamics (Birkhäuser, Boston, 1996).
7. L. Preziosi, Cancer Modelling and Simulation (Chapman Hall/CRC Press, Boca Raton, 2003).
8. N. Bellomo, M. A. J. Chaplain, and E. De Angelis, Selected Topics in Cancer Modeling: Genesis, Evolution, Immune Competition, and Therapy (Birkhäuser, Boston, 2008) (Modeling and Simulation in Science, Engineering and Technology).
9. R. P. Araujo and D. L. S. McElwain, “A history of the study of solid tumour growth: The contribution of mathematical modelling,” Bull. Math. Biol. 66 (5), 1039–1091 (2004).

10. J. S. Lowengrub, H. B. Frieboes, F. Jin, Y.-L. Chuang, X. Li, P. Macklin, S. M. Wise, and V. Cristini, “Nonlinear modelling of cancer: Bridging the gap between cells and tumours,” Nonlinearity 23, R1–R91 (2010).

11. R. A. Gatenby and E. T. Gawlinski, “A reaction–diffusion model of cancer invasion,” Cancer Res. 56 (24), 5745–5753 (1996).

12. A. J. Perumpanani, J. A. Sherratt, J. Norbury, and H. M. Byrne, “Biological inferences from a mathematical model for malignant invasion,” Invasion Metastasis 16 (4–5), 209–221 (1996).

13. C. S. Patlak, “Random walk with persistence and external bias,” Bull. Math. Biophys. 15 (3), 311–338 (1953).

14. E. F. Keller and L. A. Segel, “Initiation of slime mold aggregation viewed as an instability,” J. Theor. Biol. 26 (3), 399–415 (1970).

15. E. F. Keller and L. A. Segel, “Model for chemotaxis,” J. Theor. Biol. 30 (2), 225–234 (1971).

16. E. F. Keller and L. A. Segel, “Traveling bands of chemotactic bacteria: A theoretical analysis,” J. Theor. Biol. 30 (2), 235–248 (1971).

17. K. J. Painter, “Mathematical models for chemotaxis and their applications in self-organisation phenomena,” J. Theor. Biol. 481, 162–182 (2019).

18. A. R. A. Anderson, “A hybrid mathematical model of solid tumour invasion: The importance of cell adhesion,” Math. Med. Biol. 22 (2), 163–186 (2005).

19. M. A. J. Chaplain and G. Lolas, “Mathematical modelling of cancer cell invasion of tissue: The role of the urokinase plasminogen activation system,” Math. Models Methods Appl. Sci. 15, 1685–1734 (2005).

20. H. Enderling, A. R. A. Anderson, M. A. J. Chaplain, A. J. Munro, and J. S. Vaidya, “Mathematical modelling of radiotherapy strategies for early breast cancer,” J. Theor. Biol. 241 (1), 158–171 (2006).

21. V. Andasari, A. Gerisch, G. Lolas, A. P. South, and M. A. J. Chaplain, “Mathematical modeling of cancer cell invasion of tissue: Biological insight from mathematical analysis and computational simulation,” J. Math. Biol. 63 (1), 141–171 (2010).

22. A. Gerisch and M. A. J. Chaplain, “Mathematical modelling of cancer cell invasion of tissue: Local and non-local models and the effect of adhesion,” J. Theor. Biol. 250 (4), 684–704 (2008).

23. H. B. Frieboes, X. Zheng, C. H. Sun, B. Tromberg, R. Gatenby, and V. Cristini, “An integrated computational/experimental model of tumor invasion,” Cancer Res. 66, 1597–1604 (2006).

24. K. J. Painter, “Modelling cell migration strategies in the extracellular matrix,” J. Math. Biol. 58 (4–5), 511–543 (2009).

25. I. Ramis-Conde, M. A. J. Chaplain, and A. R. A. Anderson, “Mathematical modelling of cancer cell invasion of tissue,” Math. Comput. Model. 47 (5–6), 533–545 (2008).

26. K. J. Painter, N. A. Armstrong, and J. A. Sherratt, “The impact of adhesion on cellular invasion processes in cancer and development,” J. Theor. Biol. 264, 1057–1067 (2010).

27. L. Peng, D. Trucu, P. Lin, A. Thompson, and M. A. J. Chaplain, “A multiscale mathematical model of tumour invasive growth,” Bull. Math. Biol. 79 (3), 389–429 (2017).

28. P. Domchke, D. Trucu, A. Gerisch, and M. A. J. Chaplain, “Structured models of cell migration incorporating molecular binding processes,” J. Math. Biol. 75 (5–6), 1517–1561 (2017).

29. V. Bitsouni, M. A. J. Chaplain, and R. Eftimie, “Mathematical modelling of cancer invasion: The multiple roles of TGF-β pathway on tumour proliferation and cell adhesion,” Math. Models Methods Appl. Sci. 27 (10), 1929 (2017).

30. V. Bitsouni, D. Trucu, M. A. J. Chaplain, and R. Eftimie, “Aggregation and travelling wave dynamics in a two-population model of cancer cell growth and invasion,” Math. Med. Biol. 35 (4), 541–577 (2018).

31. Z. Szymanska, M. Cytowski, E. Mitchell, C. K. Macnamara, and M. A. J. Chaplain, “Computational modelling of cancer development and growth: Modelling at multiple scales and multiscale modelling,” Bull. Math. Biol. 80 (5), 1366–1403 (2017).
32. P. Y. H. Pang and Y. Wang, “Global existence of a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant,” J. Differ. Equat. 263, 1269–1292 (2017).
33. Y. Ke and J. Zheng, “A note for global existence of a two-dimensional chemotaxis–haptotaxis model with remodeling of non-diffusible attractant,” Nonlinearity 31 (10), 4602 (2018).
34. F. Bubba, C. Pouchol, N. Ferrand, G. Vidal, L. Almeida, B. Perthame, and M. Sabbah, “A chemotaxis-based explanation of spheroid formation in 3d cultures of breast cancer cells,” J. Theor. Biol. 479, 73–80 (2019).
35. T. Xiang and J. Zheng, “A new result for 2D boundedness of solutions to a chemotaxis–haptotaxis model with/without sub-logistic source,” Nonlinearity 32, 4890 (2019).
36. Y. Tao and M. Winkler, “Global classical solutions to a doubly haptotactic cross-diffusion system modeling oncolytic virotherapy,” J. Differ. Equat. 268 (9), 4973 (2020).
37. A. J. Perumpanani, J. A. Sherratt, J. Norbury, and H. Byrne, “A two parameter family of travelling waves with a singular barrier arising from the modelling of matrix mediated malignant invasion,” Physica. Ser. D: Nonlinear Phenom. 126, 145–159 (1999).
38. B. P. Marchant, J. Norbury, and J. A. Sherratt, “Travelling wave solutions to a haptotaxis-dominated model of malignant invasion,” Nonlinearity 14 (6), 1653–1671 (2001).
39. J. Sherratt, “On the form of smooth-front travelling waves in a reaction–diffusion equation with degenerate nonlinear diffusion,” Math. Model. Nat. Phenom. 5 (5), 64–79 (2010).
40. K. Harley, P. Van Heijster, R. Marangell, G. J. Pettet, and M. Wechselberger, “Existence of traveling wave solutions for a model of tumor invasion,” J. Appl. Dyn. Syst. 13 (1), 366–396 (2014).
41. P. J. Olver, Applications of Lie Groups to Differential Equations (Springer-Verlag, 1986).
42. H. Bateman and A. Erdélyi, Higher Transcendental Functions. Vol. 2 (McGraw-Hill, New York–Toronto–London, 1953).