An extension of S-artinian rings and modules to a hereditary torsion theory setting

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ABSTRACT
For any commutative ring \( A \), we introduce a generalization of S-artinian rings using a hereditary torsion theory \( r \) instead of a multiplicative closed subset \( S \subseteq A \). It is proved that if \( A \) is a totally \( \sigma \)-artinian ring, then \( \sigma \) must be of finite type, and \( A \) is totally \( \sigma \)-noetherian.

0. Introduction

In [6], the authors study the problem of determining the structure of the polynomial ring \( D[X] \) over an integral domain \( D \) with field of fractions \( K \), looking for the structure of the Euclidean domain \( K[X] \). In particular, an ideal \( a \subseteq D[X] \) is said to be almost principal whenever there exist a polynomial \( F \in \alpha \), of positive degree, and an element \( 0 \neq s \in D \) such that \( \alpha s \subseteq FD[X] \subseteq \alpha \). The integral domain \( D \) is an almost principal domain whenever every ideal \( a \subseteq D[X] \), which extends properly to \( K[X] \), is almost principal. Noetherian and integrally closed domains are examples of almost principal domains.

Later, in [2], the authors extend this notion to non-necessarily integral domains in defining, for a given multiplicatively closed subset \( S \subseteq A \) of a ring \( A \), an ideal \( a \subseteq A \) to be S-finite if there exist a finitely generated ideal \( \alpha' \subseteq a \) and an element \( s \in S \) such that \( \alpha s \subseteq \alpha' \), and define a ring \( A \) to be S-noetherian whenever every ideal \( a \subseteq A \) is S-finite. Many authors have worked on S-noetherian rings and related notions, and shown relevant results on its structure. See for instance [3, 7, 10–12, 14].

In [12], the author study S-artinian rings, dualizing the former notion of S-noetherian ring, and give some characterization of S-artinian rings in terms of finite cogeneration with respect to \( S \). Our aim is to show that this theory is part of a more general theory involving hereditary torsion theories. In particular, we show that if \( A \) is totally \( \sigma \)-artinian, then the hereditary torsion theory \( \sigma \) is of finite type, and, in addition, \( A \) it is totally \( \sigma \)-noetherian.

The background will be the hereditary torsion theories on a commutative (and unitary) ring \( A \), see [4, 13], and \( \text{Mod} - A \) denotes the category of \( A \)-modules. Thus, a hereditary torsion theory \( \sigma \) in \( \text{Mod} - A \) is given by one of the following objects:
(1) a torsion class \( T \), a class of modules which is closed under submodules, homomorphic images, direct sums, and group extensions,

(2) a torsionfree class \( \mathcal{T} \), a class of modules which is closed under submodules, essential extensions, direct products, and group extensions,

(3) a Gabriel filter of ideals \( \mathcal{L}(\sigma) \), a non-empty filter of ideals satisfying that every \( b \subseteq A \), for which there exists an ideal \( a \in \mathcal{L}(\sigma) \) such that \( (b : a) \in \mathcal{L}(\sigma) \), for every \( a \in \mathcal{L}(\sigma) \).

(4) a left exact kernel functor \( \sigma : \text{Mod}-A \rightarrow \text{Mod}-A \).

The relationships between these notions are the following. If \( \sigma \) is the left exact kernel functor, then

\[
\begin{align*}
\mathcal{T}_\sigma &= \{ M \in \text{Mod}-A \mid \sigma M = M \}, \\
\mathcal{F}_\sigma &= \{ M \in \text{Mod}-A \mid \sigma M = 0 \}, \\
\mathcal{L}(\sigma) &= \{ a \subseteq A \mid A/a \in \mathcal{T}_\sigma \}.
\end{align*}
\]

If \( \mathcal{L} \) is the Gabriel filter of \( \sigma \), and \( \mathcal{T} \) the torsion class, for any \( A \)-module \( M \) we have the following:

\[
\sigma M = \{ m \in M \mid (0 : m) \in \mathcal{L} \} = \sum \{ N \subseteq M \mid N \in \mathcal{T} \}.
\]

**Example 0.1.**

(1) Let \( \Sigma \subseteq A \) be a multiplicatively closed subset, there exists a hereditary torsion theory, \( \sigma_\Sigma \), defined by

\[
\mathcal{L}(\sigma_\Sigma) = \{ a \subseteq A \mid a \cap \Sigma \neq \emptyset \}.
\]

Observe that \( \sigma_\Sigma \) has a filter basis constituted by principal ideals. Every hereditary torsion theory \( \sigma \) such that \( \mathcal{L}(\sigma) \) has a filter basis of principal ideals is called a principal hereditary torsion theory. We can show that there is a correspondence between principal hereditary torsion theories in \( \text{Mod}-A \), and saturated multiplicatively closed subsets in \( A \).

(2) Let \( \mathcal{A} \) be a set of finitely generated ideals of a ring \( A \), then

\[
\mathcal{L} = \{ b \subseteq A \mid \text{there exists } a_1, \ldots, a_t \in \mathcal{A} \text{ such that } a_1 \cdots a_t \subseteq b \}
\]

is a Gabriel filter.

This paper is organized in sections. In the first one, we introduce the main subject: totally \( \sigma \)-artinian rings and modules, and show examples, their first properties, and the decisive fact: If \( A \) is a totally \( \sigma \)-artinian ring, then \( \sigma \) is a finite-type hereditary torsion theory. In section two we deal with scalar extensions, which will be useful for studying local properties. In section three we give an extra characterization of totally \( \sigma \)-artinian rings and modules with the minimal conditions we found out. In the fourth section, we study the behavior of prime ideals in relation to totally \( \sigma \)-artinian modules. Sections five and six are devoted to establish the necessary background to show that every totally \( \sigma \)-artinian rings is also totally \( \sigma \)-noetherian.

**1. Totally \( \sigma \)-artinian rings and modules**

For any \( \sigma \)-torsion finitely generated \( A \)-module \( M \), if \( M = m_1A + \cdots + m_tA \), since \( (0 : m_i) \in \mathcal{L}(\sigma) \), for any \( i = 1, \ldots, t \), then \( b := \cap_{i=1}^t (0 : m_i) \in \mathcal{L}(\sigma) \), and satisfies \( Mb = 0 \). In general, this result does not hold for \( \sigma \)-torsion non-finitely generated \( A \)-modules. Therefore, we shall define an \( A \)-module \( M \) to be **totally \( \sigma \)-torsion** whenever there exists \( b \in \mathcal{L}(\sigma) \) such that \( Mb = 0 \). The notion of totally torsion appears, for instance, in [9, page 462].

For any ideal \( a \subseteq A \), we have two different notions of finitely generated ideals relative to \( \sigma \):
(1) \( a \subseteq A \) is \( \sigma \)-finitely generated whenever there exists a finitely generated ideal \( a' \subseteq a \) such that \( a/a' \) is \( \sigma \)-torsion.

(2) \( a \subseteq A \) is \textbf{totally} \( \sigma \)-finitely generated whenever there exists a finitely generated ideal \( a' \subseteq a \) such that \( a/a' \) is totally \( \sigma \)-torsion.

In the same way, for any ring \( A \) we have two different notions of noetherian ring relative to \( \sigma \):

(1) \( A \) is \( \sigma \)-noetherian if every ideal is \( \sigma \)-finitely generated.

(2) \( A \) is \textbf{totally} \( \sigma \)-noetherian whenever every ideal is totally \( \sigma \)-finitely generated.

**Example 1.1.**

(1) Every finitely generated ideal is totally \( \sigma \)-finitely generated and every totally \( \sigma \)-finitely generated ideal is \( \sigma \)-finitely generated.

(2) Let \( S \subseteq A \) be a multiplicatively closed subset, an ideal \( a \subseteq A \) is \( S \)-finite if, and only if, it is totally \( \sigma_S \)-finitely generated. The ring \( A \) is \( S \)-noetherian if and only if \( A \) is totally \( \sigma_S \)-noetherian.

We may dualize this notions, thus, if \( A \) is a ring and \( \sigma \) a hereditary torsion theory in \( \text{Mod} \)-\( A \),

(1) \( A \) is \( \sigma \)-artinian if every decreasing chain of ideals is \( \sigma \)-stable.

(2) \( A \) is \textbf{totally} \( \sigma \)-artinian if every decreasing chain of ideals is totally \( \sigma \)-stable.

Being a decreasing chain of ideals \( a_1 \supseteq a_2 \supseteq \ldots \) \( \sigma \)-stable whenever there exists an index \( m \) such that \( a_i \subseteq a_m \), for every \( s \geq m \), i.e. every \( a_i \) is \( \sigma \)-dense in \( a_m \), or equivalently, for every \( x \in a_m \), there exists \( h \in \mathcal{L}(\sigma) \) such that \( xh \subseteq a_i \) (observe that \( h \) depends of \( x \) and \( s \)). Otherwise, the decreasing chain of ideals is \textbf{totally} \( \sigma \)-stable whenever there exist an index \( m \), and \( h \in \mathcal{L}(\sigma) \) such that \( a_m h \subseteq a_i \), for every \( s \geq m \).

**Lemma 1.2.** For any ring \( A \) we have:

\[ \text{A is artinian } \Rightarrow \text{A is totally } \sigma \text{-artinian } \Rightarrow \text{A is } \sigma \text{-artinian.} \]

The notions of \( \sigma \)-artinian (resp. totally \( \sigma \)-artinian) and \( \sigma \)-noetherian (resp. totally \( \sigma \)-noetherian) ring can be extended to \( A \)-modules in an easy way.

Trivial examples of totally \( \sigma \)-artinian modules are the totally \( \sigma \)-torsion modules. Also, every \( \sigma \)-artinian module is totally \( \sigma \)-artinian for every hereditary torsion theory \( \sigma \).

These two notions of torsion, and the derived notions from them, are completely different in its behavior and its categorical properties. For instance, due to the definition, for any \( A \)-module \( M \) there exists a maximum submodule belonging to \( \mathcal{F}_\sigma \), the submodule: \( \sigma M \), and it satisfies \( M/\sigma M \in \mathcal{F}_\sigma \). In the totally \( \sigma \)-torsion case, we cannot assure the existence of a maximal totally \( \sigma \)-torsion submodule. The existence of a maximum \( \sigma \)-torsion submodule allows us to build new concepts relative to \( \sigma \) as lattices, closure operator, and localization, concepts that we do not have in the totally \( \sigma \)-torsion case. For instance, the ring \( A \) is \( \sigma \)-artinian if, and only if, the lattice \( C(A, \sigma) = \{ a \subseteq A \mid A/a \in \mathcal{F}_\sigma \} \) is an artinian lattice. Nevertheless, the totally \( \sigma \)-torsion case allows us to study arithmetic properties of rings and modules which are hidden with that use of \( \sigma \)-torsion, and these properties are those which we are interested in studying.

As we point out before, the \( \sigma \)-torsion allows us, for any \( A \)-module \( M \), to define a lattice \( C(M, \sigma) = \{ N \subseteq M \mid M/N \in \mathcal{F}_\sigma \} \), and in \( \mathcal{L}(M) \), the lattice of all submodules of \( M \), a closure operator \( \text{Cl}_\sigma^M(-) : \mathcal{L}(M) \to C(M, \sigma) \subseteq \mathcal{L}(M) \), defined by the equation \( \text{Cl}_\sigma^M(N)/N = \sigma(M/N) \). The elements in \( C(M, \sigma) \) are
called the \(\sigma\)-closed submodules of \(M\), and the lattice operations in \(C(M, \sigma)\), for any \(N_1, N_2 \in C(M, \sigma)\), are defined by

\[
N_1 \land N_2 = N_1 \cap N_2, \\
N_1 \lor N_2 = C_{\sigma}^M(N_1 + N_2).
\]

Dually, the submodules \(N \subseteq M\) such that \(M/N \in \mathcal{S}_\sigma\) are called \(\sigma\)-dense submodules. The set of all \(\sigma\)-dense submodules of \(M\) is represented by \(\mathcal{L}(M, \sigma)\), \(\mathcal{L}(\sigma)\) in the case in which \(M = A\).

In the following, we assume \(A\) is a ring, \(\text{Mod}\text{--}A\) is the category of \(A\)-modules and \(\sigma\) is a hereditary torsion theory on \(\text{Mod}\text{--}A\). Modules are represented by Latin letters: \(M, N, N_1,\ldots\), and ideals by Gothics letters: \(a, b, b_1,\ldots\). Different hereditary torsion theories will be represented by Greek letters: \(\sigma, \tau, \sigma_1,\ldots\), and induced hereditary torsion theories by adorned Greek letters: \(\sigma', \tau',\ldots\).

In order to establish equivalent condition to (totally) \(\sigma\)-artinian modules, we introduce the definition of finitely cogenerated \(A\)-module.

(1) An \(A\)-module \(M\) is \textbf{finitely cogenerated} if for any family of submodules \(\{N_i \mid i \in I\}\) such that \(\cap_{i \in I} N_i = 0\), there exists a finite subset \(J \subseteq I\) such that \(\cap_{j \in J} N_j = 0\).

(2) In the same way, in [4] the author uses the notion of \(\sigma\)-finitely cogenerated modules; an \(A\)-module \(M\) is \(\sigma\)-finitely cogenerated if for any family of submodules \(\{N_i \mid i \in I\}\) such that \(\cap_{i \in I} N_i\) is \(\sigma\)-torsion there exists a finite subset \(J \subseteq I\) such that \(\cap_{j \in J} N_j\) is \(\sigma\)-torsion.

(3) In our case for totally \(\sigma\)-torsion, we define an \(A\)-module to be \textbf{totally \(\sigma\)-finitely cogenerated} whenever for every family of submodules \(\{N_i \mid i \in I\}\) such that \(\cap_{i \in I} N_i\) is totally \(\sigma\)-torsion there exists a finite subset \(J \subseteq I\) such that \(\cap_{j \in J} N_j\) is totally \(\sigma\)-torsion, i.e. there exists \(b \in \mathcal{L}(\sigma)\) such that \((\cap_{j \in J} N_j)b = 0\).

\textbf{Theorem 1.3.} Let \(A\) be a ring and \(\sigma\) a hereditary torsion theory in \(\text{Mod}\text{--}A\), for any \(A\)-module \(M\) the following statements are equivalent:

(a) \(M\) is totally \(\sigma\)-artinian.
(b) Every quotient \(M/N\) of \(M\) is \(\sigma\)-finitely cogenerated.

\textbf{Proof.} (a) \(\Rightarrow\) (b). Let \(\{N_i/N \mid i \in I\}\) be a family of submodules of \(M/N\) such that \(\cap_{i \in I} (N_i/N)\) is totally \(\sigma\)-torsion. If \(H/N = (\cap_{i \in I} N_i)/N = \cap_{i \in I} (N_i/N)\), then \(H/N\) is totally \(\sigma\)-torsion and \(\cap_{i \in I} (N_i/H) = 0\). We have a family \(\{H_i = N_i/H \mid i \in I\}\) of submodules of \(M/H\) such that \(\cap_{i \in I} H_i = 0\). By the hypothesis, \(M/H\) is \(\sigma\)-artinian, so there are maximal elements in the set

\[
\Gamma = \{\cap_{j \in J} H_j \mid J \subseteq I \text{ is finite}\}.
\]

Let \(\cap_{j \in J} H_j \in \Gamma\) be a minimal element in \(\Gamma\). There exists \(b \in \mathcal{L}(\sigma)\) such that for any \(i \in I \setminus J\), we have

\[
(\cap_{j \in J} H_j)b \subseteq (\cap_{j \in J} H_j) \cap N_i \subseteq \cap_{j \in J} H_j.
\]

In particular, \((\cap_{j \in J} H_j)b \subseteq \cap_{j \in J} H_j = 0\), and \(\cap_{j \in J} H_j\) is totally \(\sigma\)-torsion.

(b) \(\Rightarrow\) (a). Let \(N_0 \supseteq N_1 \supseteq \cdots\) be a decreasing chain of submodules of \(M\), and define \(N = \cap_{n \in \mathbb{N}} N_n\). In \(M/N\), the family \(\{N_n/N \mid n \in \mathbb{N}\}\) satisfies \(\cap_{n \in \mathbb{N}} (N_n/N) = 0\); hence, there exists \(J \subseteq \mathbb{N}\), finite, and \(b \in \mathcal{L}(\sigma)\) such that \((\cap_{j \in J} (N_j/N))b = 0\), hence \((N_k/N)b = 0\), being \(k = \max(J)\), and satisfies \(N_kb \subseteq N\). Therefore, the decreasing chain \(\sigma\)-stabilizes.

Properties about the behavior of totally \(\sigma\)-finitely cogenerated and \(\sigma\)-noetherian modules are collected in the following result.

\textbf{Proposition 1.4.}

(1) Every submodule of a totally \(\sigma\)-finitely cogenerated \(A\)-module also is.
(2) For every submodule $N \subseteq M$, we have: $M$ is totally $\sigma$-artinian if, and only if, $N$ and $M/N$ are totally $\sigma$-artinian.

(3) Finite direct sums of totally $\sigma$-artinian modules also are.

Also, we can build up examples of totally $\sigma$-artinian rings in considering hereditary torsion theories $\sigma_1 \leq \sigma_2$. Thus, we have the following lemma, whose proof is straightforward.

**Lemma 1.5.** Let $\sigma_1 \leq \sigma_2$ be hereditary torsion theories in $\text{Mod} - A$, and $M$ an $A$-module. If $M$ is totally $\sigma_1$-artinian then $M$ is totally $\sigma_2$-artinian.

Regular elements have a particular behavior with respect to totally $\sigma$-artinian rings.

**Lemma 1.6.** If $A$ is a totally $\sigma$-artinian ring, for any regular element $a \in A$, we have $aA \in \mathcal{L}(\sigma)$.

**Proof.** If $a \in A$ is regular, we consider the decreasing chain $(a) \supseteq (a^2) \supseteq \cdots$. By the hypothesis, there exist an index $m$ and $h \in \mathcal{L}(\sigma)$ such that $(a^m)h \subseteq (a^s)$ for every $s \geq m$. Thus, for every $h \in \mathcal{L}(\sigma)$ there exists $x \in A$ such that $a^m h = a^{m+1} x$, hence $h = ax \in aA$, which means that $\mathcal{L}(\sigma)$.

As a consequence, the case of an integral domain is well understood. See [12, Corollary 2.2].

**Corollary 1.7.** Let $A = D$ be an integral domain, if $D$ is totally $\sigma$-artinian, then $\sigma = \sigma_{D\setminus\{0\}}$, the usual torsion theory on $D$.

In particular, we have the following conclusions:

1. If $p \subseteq D$ is a non-zero prime ideal of an integral domain $D$, and we consider the hereditary torsion theory $\sigma_{D\setminus p}$, then $D$ is never totally $\sigma_{D\setminus p}$-artinian.
2. For every integral domain $D$, the hereditary torsion theory $\sigma = \sigma_{D\setminus\{0\}}$ satisfies that $D$ is $\sigma$-artinian, but non-necessarily $D$ is totally $\sigma$-artinian. Indeed, $D$ is $\sigma$-artinian because $D_\sigma$, the field of fractions of $D$ is artinian. Otherwise, the following example shows that the converse non-necessarily holds. Let $D = \mathbb{Q}[X_n \mid n \in \mathbb{N}]$, and $a_n = (X_0 \cdots X_n)$, for every $n \in \mathbb{N}$; the decreasing chain $a_0 \supseteq a_1 \supseteq \cdots$ satisfies that there is not $s \in D \setminus \{0\}$ neither $m \in \mathbb{N}$ such that $a_m s \subseteq a_s$, for every $s \geq m$.

This example raises the following problems:

**Problem 1.8.** Which properties are necessary to add to a $\sigma$-artinian ring to be a totally $\sigma$-artinian ring?

We refer to Theorem (4.4) below.

**Corollary 1.9.** Let $A$ be a totally $\sigma$-artinian ring, and $\text{Reg}(A)$ be the set of all regular elements of $A$, then $\sigma_{\text{Reg}(A)} \leq \sigma$.

**Proof.** It is a consequence of the well-known fact that $\text{Reg}(A)$ is a saturated multiplicatively closed set.

Let $A$ be a ring and $T = T(A)$ the **total ring of fractions** of $A$, i.e. the localization of $A$ at $\text{Reg}(A)$, the multiplicatively closed set of all regular elements, i.e. $T = A_{\sigma_{\text{Reg}(A)}}$. The above example in (2) shows that non-necessarily $A$ must be totally $\sigma_{\text{Reg}(A)}$-artinian, although it is $\sigma_{\text{Reg}(A)}$-artinian.

We said that an ideal $a \subseteq A$ is **invertible** whenever $a(A : a) = A$, being $(A : a) = \{x \in T \mid ax \subseteq A\}$.

**Corollary 1.10.** If $A$ is a totally $\sigma$-artinian ring, then every invertible ideal $a$ belongs to $\mathcal{L}(\sigma)$.
Proof. Let \( a \subseteq A \) be an invertible ideal, we consider the decreasing chain \( a \supseteq a^2 \supseteq \cdots \). By the hypothesis, there exist an index \( m \) and \( b \in \mathcal{L}(\sigma) \) such that \( a^m b \subseteq a^s \) for every \( s \geq m \). In particular, \( a^m b \subseteq a^{m+1} \), hence \( b \subseteq a \), and \( a \in \mathcal{L}(\sigma) \).

Example 1.11. Since invertible ideals are finitely generated ideals, they generate a hereditary torsion theory, that we name \( \sigma_{inv} \), see (2) in Example (0.1). If \( A \) is a totally \( \sigma \)-artinian ring, non-necessarily \( A \) is totally \( \sigma_{inv} \)-artinian.

Indeed, we can consider the ring \( A = \prod_{n \geq 1} \mathbb{Z}_2 \), and the prime ideal \( p = \prod_{n \geq 1} \mathbb{Z}_2 \). We know that \( A \) is totally \( \sigma_{A \text{ - } p} \)-artinian. Since \( A \) is a total ring of fractions, every non-regular element is invertible, hence \( T = T(A) = A \), and the only invertible ideal is the proper \( A \), hence \( \sigma_{\text{Reg}(A)} = 0 \). If \( A \) were totally \( \sigma_{\text{Reg}(A)} \)-artinian, then \( A \) must be exactly artinian, but obviously \( A \) is not artinian.

In general, if \( A \) is a totally \( \sigma \)-artinian ring, we have one more property of the hereditary torsion theory \( \sigma \).

Proposition 1.12. If \( A \) is a totally \( \sigma \)-artinian ring, then the hereditary torsion theory \( \sigma \) is of finite type.

Proof. Since \( A \) is totally \( \sigma \)-artinian, it is \( \sigma \)-artinian and, by Hopkins’ Theorem, \( \sigma \)-noetherian, hence \( \sigma \) is of finite type. We are interested in proving stronger results: if \( A \) is totally \( \sigma \)-artinian, then \( A \) is totally \( \sigma \)-noetherian.

2. Scalar extensions

Let \( f : A \to B \) be a ring map. For every hereditary torsion theory \( \sigma \) in \( \text{Mod-}A \), we may define a new hereditary torsion theory \( f(\sigma) \) in \( \text{Mod-}B \) being

- \( \mathcal{L}(f(\sigma)) = \{ b \subseteq B \mid f^{-1}(b) \in \mathcal{L}(\sigma) \} \),
- \( \mathcal{T}_{f(\sigma)} = \{ M_B \mid M_A \in \mathcal{T}_{\sigma} \} \),
- \( \mathcal{T}_{f(\sigma)} = \{ M_B \mid M_A \in \mathcal{T}_{\sigma} \} \).

In addition, sometimes, we shall impose the condition that every ideal of \( B \) is an extension of an ideal of \( A \), i.e. for every ideal \( b \subseteq B \), there exists an ideal \( a \subseteq A \) such that \( b = f(a)B \). With this condition, we have that the Gabriel filter \( \mathcal{L}(f(\sigma)) \) can be described also as

\[ \mathcal{L}(f(\sigma)) = \{ f(a)B \mid a \in \mathcal{L}(\sigma) \} \]

Lemma 2.1. Let \( f : A \to B \) be a ring map such that every ideal of \( B \) is an extension of an ideal of \( A \), and let \( \sigma \) be a hereditary torsion theory in \( \text{Mod-}A \) such that \( A \) is totally \( \sigma \)-artinian, then \( B \) is totally \( \sigma \)-artinian.

Proof. Let \( b_1 \supseteq b_2 \supseteq \cdots \) be a decreasing chain of ideals of \( B \), and let \( a_i \subseteq A \) be an ideal such that \( b_i = f(a_i)B \); we can obtain a decreasing chain \( a_1 \supseteq a_2 \supseteq \cdots \) of ideals \( A \). By the hypothesis, there exist an index \( m \) and \( b \in \mathcal{L}(\sigma) \) such that \( a_m b \subseteq a_s \) for every \( s \geq m \). In consequence, \( b_m B \subseteq b_s \), for every \( s \geq m \), and \( B \) is totally \( f(\sigma) \)-artinian.

Corollary 2.2. Let \( A \) be a totally \( \sigma \)-artinian ring, then we have:

1. If for any ideal \( a \subseteq A \) we consider the canonical projection \( p : A \to A/a \), then \( A/a \) is totally \( p(\sigma) \)-artinian.
If for any multiplicatively closed subset $\Sigma \subseteq A$ we consider the canonical map $f : A \to \Sigma^{-1}A$, then $\Sigma^{-1}A$ is totally $f(\sigma)$-artinian.

**Corollary 2.3.** Let $\alpha \subseteq A$ be an ideal, and $p : A \to A/\alpha$ the canonical projection. The following statements are equivalent:

1. $A$ is totally $\sigma$-artinian.
2. $\alpha$ is totally $\sigma$-artinian and $A/\alpha$ is totally $\sigma$-artinian (equivalently, totally $p(\sigma)$-artinian).

And, as a consequence of Proposition (1.4), we have:

**Corollary 2.4.** Let $A$ be a totally $\sigma$-artinian ring, then every finitely generated $A$-module is totally $\sigma$-artinian.

### 3. The minimal condition

Let $M$ be an $A$-module, after [12], we establish the following definitions:

1. Let $\mathcal{N} \subseteq \mathcal{L}(M)$ be a family of submodules of $M$. An element $N \in \mathcal{N}$ is $\sigma$-minimal if there exists $h \in \mathcal{L}(\sigma)$ such that for every $H \in \mathcal{N}$ such that $H \subseteq N$ we have $Nh \subseteq H$.
2. The $A$-module $M$ satisfies the $\sigma$-MIN condition if every nonempty family of submodules of $M$ has $\sigma$-minimal elements.
3. A family $\mathcal{N}$ of submodules of $M$ is $\sigma$-lower closed if for every submodule $H \subseteq M$ such that there exist $N \in \mathcal{N}$ and $h \in \mathcal{L}(\sigma)$ satisfying $Nh \subseteq H$, either equivalently $N \subseteq (H : h)$ or equivalently $(H : N) \in \mathcal{L}(\sigma)$, we have $H \in \mathcal{N}$.

We have the following characterization of totally $\sigma$-artinian modules.

**Proposition 3.1.** Let $M$ be an $A$-module, the following statements are equivalent:

(a) $M$ is totally $\sigma$-artinian.
(b) Every nonempty $\sigma$-lower closed family of submodules of $M$ has minimal elements.
(c) Every nonempty family of submodules of $M$ has $\sigma$-minimal elements.

If we have $\Sigma \subseteq A$ a multiplicatively closed subset of $A$ and $\sigma = \sigma_\Sigma$, this proposition is Theorem 2.1 in [12].

Let $\sigma$ be a hereditary torsion theory in $\text{Mod} - A$; an $A$-module $M$ is

1. $\sigma$-finitely cogenerated if for any family of submodules $\{N_i \mid i \in I\}$ of $M$ such that $\bigcap_{i \in I} N_i$ is $\sigma$-torsion, there exists a finite subset $J \subseteq I$ such that $\bigcap_{j \in J} N_j$ is $\sigma$-torsion.
2. totally $\sigma$-finitely cogenerated if for any family of submodules $\{N_i \mid i \in I\}$ of $M$ such that $\bigcap_{i \in I} N_i$ is totally $\sigma$-torsion, there exists a finite subset $J \subseteq I$ such that $\bigcap_{j \in J} N_j$ is totally $\sigma$-torsion.
3. strongly totally $\sigma$-finitely cogenerated if for any family of submodules $\{N_i \mid i \in I\}$ of $M$ such that $\bigcap_{i \in I} N_i = 0$, there exists a finite subset $J \subseteq I$ such that $\bigcap_{j \in J} N_j$ is totally $\sigma$-torsion.

We are mainly interested in modules $M$ such that every quotient $M/N$ is $\sigma$-finitely cogenerated (resp. totally $\sigma$-finitely cogenerated). For that reason, we weaken the condition $\bigcap_{i \in I} N_i$ is $\sigma$-torsion (resp. totally $\sigma$-torsion) to simply consider that $\bigcap_{i \in I} N_i = 0$, obtaining in this way the strongly
Every quotient $M/N$ of $M$ is strongly totally co-$\sigma$; in particular, $M/T$ is totally co-$\sigma$. Let $\mathcal{R}$ be a family of submodules of $M/N$ such that $\bigcap_{i \in I} (N_i/N) = 0$, hence $\bigcap_{i \in I} N_i = N$. By the hypothesis, $M/N$ is totally co-$\sigma$, so there are maximal elements in the set $\Gamma = \{\bigcap_{j \in J} N_j \mid J \subseteq I \text{ is finite}\}$.

Let $\bigcap_{j \in J} N_j \in \Gamma$ be a minimal element in $\Gamma$. There exists $b \in \mathcal{L}(\sigma)$ such that for any $i \in I \setminus J$, we have $(\bigcap_{j \in J} N_j)b \subseteq (\bigcap_{j \in J} N_j) \cap N_i \subseteq \bigcap_{j \in J} N_j$.

In particular, $(\bigcap_{j \in J} N_j)b \subseteq \bigcap_{i \in I} N_i = 0$, and $\bigcap_{j \in J} H_j$ is totally co-$\sigma$-torsion.

(a) $\Rightarrow$ (c). Let $\{N_i/N \mid i \in I\}$ be a family of submodules of $M/N$ such that $\bigcap_{i \in I} (N_i/N)$ is totally co-$\sigma$-torsion. If $H/N = (\bigcap_{i \in I} N_i)/N = \bigcap_{i \in I} (N_i/N)$, then $H/N$ is totally co-$\sigma$-torsion and $\bigcap_{i \in I} (N_i/H) = 0$. We have a family $\{H_i = N_i/H \mid i \in I\}$ of submodules of $M/H$ such that $\bigcap_{i \in I} H_i = 0$. By the hypothesis, $M/H$ is co-$\sigma$-artinian, so there are maximal elements in the set $\Gamma = \{\bigcap_{j \in J} H_j \mid J \subseteq I \text{ is finite}\}$.

Let $\bigcap_{j \in J} H_j \in \Gamma$ be a minimal element in $\Gamma$. There exists $b \in \mathcal{L}(\sigma)$ such that for any $i \in I \setminus J$, we have $(\bigcap_{j \in J} H_j)b \subseteq (\bigcap_{j \in J} H_j) \cap H_i \subseteq \bigcap_{j \in J} H_j$.

In particular, $(\bigcap_{j \in J} H_j)b \subseteq \bigcap_{i \in I} H_i = 0$, and $\bigcap_{j \in J} H_j$ is totally co-$\sigma$-torsion. Therefore, $\bigcap_{j \in J} (N_j/H)$ and $\bigcap_{j \in J} (N_j/N)$ are totally co-$\sigma$-torsion.

(b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a). Let $N_0 \supseteq N_1 \supseteq \cdots$ be a decreasing chain of submodules of $M$, and define $N = \bigcap_{n \in \mathbb{N}} N_n$. In $M/N$, the family $\{N_n/N \mid n \in \mathbb{N} \setminus \{0\}\}$ satisfies $\bigcap_{n \in \mathbb{N}} (N_n/N) = 0$, hence there exist $J \subseteq \mathbb{N}$, finite, and $b \in \mathcal{L}(\sigma)$ such that $(\bigcap_{j \in J} (N_j/N))b = 0$, hence $(N_k/N)b = 0$, being $k = \max(J)$, and satisfies $N_kb \subseteq N$. Therefore, the decreasing chain co-$\sigma$-stabilizes.

**Lemma 3.3.** Let $M$ be an $A$-module and $T \subseteq M$ be a totally co-$\sigma$-torsion submodule, the following statements are equivalent:

(a) $M$ is totally co-$\sigma$-artinian.
(b) $M/T$ is totally co-$\sigma$-artinian.

It is a direct consequence of Proposition (1.4), since every totally co-$\sigma$-torsion module is totally co-$\sigma$-artinian.

In the same line, we find that finitely cogenerated modules have their own characterization. The following is Theorem 3.4 in [12].

**Theorem 3.4.** Let $M$ be an $A$-module, the following statements are equivalent:
(a) $M$ is finitely cogenerated.
(b) $M$ is strongly totally $\sigma_{A/p}$-finitely cogenerated, for every $p \in \text{Spec}(A)$.
(c) $M$ is strongly totally $\sigma_{A/m}$-finitely cogenerated, for every $m \in \text{Max}(A)$.

**Proof.** (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c). They are obvious.

(c) \(\Rightarrow\) (a). Let \(\{N_i \mid i \in I\}\) be a family of submodules such that \(\bigcap_{i \in I} N_i = 0\), for every maximal ideal $m \subseteq A$ there exist a finite subset $J_m \subseteq I$ and $s_m \in A \setminus m$ such that $(\bigcap_{i \in J_m} N_i)s_m = 0$. If $b = \{s_m \mid m \in \text{Max}(A)\} \neq A$, there exists a maximal ideal $m$ such that $b \subseteq m$, which is a contradiction. Therefore, $b = A$, and there are maximal ideals $m_1, \ldots, m_t$ such that $\langle s_{m_1}, \ldots, s_{m_t} \rangle = A$. We define $J = \bigcup_{i=1}^t m_i \subseteq I$, which is finite, and satisfies $(\bigcap_{i \in J} N_i)s_m = 0$, for every $i = 1, \ldots, t$. Hence, $\bigcap_{i \in J} N_i = \bigcap_{i \in J} \langle s_{m_1}, \ldots, s_{m_t} \rangle = 0$, and $M$ is finitely cogenerated.

Also, if $A = D$ is an integral domain strongly totally $\sigma$-finitely cogenerated then $D$ is a field, whenever $\sigma \neq 0$.

**Proposition 3.5.** If $D$ is a $\sigma$-torsionfree strongly totally $\sigma$-finitely cogenerated integral domain, then $D$ is a field. The converse always holds.

**Proof.** We claim $0 \subseteq D$ is strongly prime, see [5]. Indeed, let \(\{a_i \mid i \in I\}\) be a family of ideals such that $\bigcap_{i \in I} a_i = 0$. By the hypothesis, there exist $I \subseteq I$, finite, and $b \in \mathcal{P}(\sigma)$ such that $(\bigcap_{i \in I} a_i)b = 0$. Since $A$ is $\sigma$-torsion-free, then $\bigcap_{i \in I} a_i = 0$. Otherwise, since $0 \subseteq D$ is prime, there exists $j \in I$ such that $a_0 = 0$.

In consequence, the intersection of all non-zero ideals is non-zero, and $D$ contains a minimum nonzero ideal, say $a = aD$. For any $0 \neq x \in D$ we have $ax \neq 0$, and $aD \subseteq axD$, this means that there exists $y \in D$ such that $a = axy$, and $1 = xy$, hence $x$ is invertible.

We put the condition that $D$ is $\sigma$-torsionfree only to avoid the trivial case in which $\sigma = 0$.

### 4. Study through prime ideals

Let $p \subseteq A$ be a prime ideal, we consider $\sigma_{A/p}$, the hereditary torsion theory cogenerated by $A/p$, or equivalently, the hereditary torsion theory generated by the multiplicatively subset $A \setminus p$. For every torsion theory $\sigma$, we associate the following sets of ideals:

1. $\mathcal{P}(\sigma)$, the Gabriel filter of $\sigma$.
2. $\mathcal{D}(\sigma) = \mathcal{P}(\sigma) \cap \text{Spec}(A)$. In particular, if $p \subseteq q$ are prime ideals and $p \in \mathcal{D}(\sigma)$, then $q \in \mathcal{D}(\sigma)$.
3. $\mathcal{C}(A, \sigma) = \{a \mid A/a \in \mathcal{P}(\sigma)\}$.
4. $\mathcal{F}(\sigma) = \mathcal{C}(A, \sigma) \cap \text{Spec}(A)$, it is the complement of $\mathcal{D}(\sigma)$ in $\text{Spec}(A)$. In particular, if $p \subseteq q$ are prime ideals and $q \in \mathcal{D}(\sigma)$, then $p \in \mathcal{D}(\sigma)$.
5. $\mathcal{G}(\sigma) = \text{Max} \mathcal{F}(\sigma)$.

If $\sigma$ is of finite type, then $\sigma = \wedge \{\sigma_{A/p} \mid p \in \mathcal{F}(\sigma)\}$. Otherwise, $\sigma = \wedge \{\sigma_{A/p} \mid p \in \mathcal{G}(\sigma)\}$ whenever $A$ is $\sigma$-noetherian, because $\sigma_{A/q} \leq \sigma_{A/p}$ whenever $p \subseteq q$, for any prime ideals $p, q$.

**Proposition 4.1.** Let $p \subseteq A$ be a prime ideal. If $A$ is totally $\sigma_{A/p}$-artinian, then $p \subseteq A$ is a minimal prime ideal.

**Proof.** Let $q \subseteq p$ be prime ideals such that $A$ is totally $\sigma_{A/p}$-artinian. Taking the quotient by the ideal $q$, $p : A \to A/q$, we have that $A/q$ is a totally $p(\sigma_{A/p})$-artinian domain, hence $p(\sigma_{A/p})$ is the usual hereditary torsion theory in a domain, i.e. $\mathcal{P}(p(\sigma_{A/p}))$ contains only the non-zero ideals of $A/q$. Therefore, $q \not\in \mathcal{D}(\sigma_{A/p})$, which is a contradiction. \qed
Corollary 4.2. Let $A$ be a totally $\sigma$-artinian ring, every prime ideal $\mathfrak{p} \in \mathcal{K}(\sigma)$ is a minimal prime ideal. In consequence, $\mathcal{K}(\sigma) = \mathcal{C}(\sigma)$, i.e., every prime ideal in $\mathcal{K}(\sigma)$ is maximal in $\mathcal{K}(\sigma)$.

Proof. Let $\mathfrak{p} \in \mathcal{K}(\sigma)$, then $\mathfrak{p} \leq \sigma_A \setminus \mathfrak{p}$, and $A$ is $\sigma_A \setminus \mathfrak{p}$-artinian. Therefore, $\mathfrak{p}$ is a minimal prime ideal, hence maximal in $\mathcal{K}(\sigma)$.

For any multiplicatively closed subset $\Sigma \subseteq A$, and $\sigma = \sigma_\Sigma$, we obtain Proposition 2.5 in [12].

Since every totally $\sigma$-artinian ring is $\sigma$-artinian, we establish the next result for $\sigma$-artinian rings.

Corollary 4.3. Let $A$ be a $\sigma$-artinian ring, the following statements hold:

1. $C(A, \sigma)$ is artinian, and the converse also holds.
2. $A$ is $\sigma$-noetherian. In particular, $\sigma$ is of finite type.
3. $\mathcal{K}(\sigma)$ is a finite set, say $\mathcal{K}(\sigma) = \{p_1, ..., p_t\}$.
4. There exists a multiplicatively closed subset $\Sigma \subseteq A$ such that $\mathcal{L}(\sigma) = \{a \mid a \cap \Sigma \neq \emptyset\}$ and $A_\Sigma$ is artinian. The converse also holds.

Proof. (1). It is just the definition.
(2). It is Hopkins’ Theorem. See [4].
(3). If $\mathcal{K}(\sigma)$ is not finite, there exists a numerable family of prime ideals $\{p_n \mid n \in \mathbb{N}\} \subseteq \mathcal{K}(\sigma)$.
Then, we may build a decreasing chain of ideals $p_1 \supseteq p_1 p_2 \supseteq p_1 p_2 p_3 \supseteq \cdots$. By the hypothesis, this chain stabilizes, and there exists an index $m$ such that $Cl^A_\sigma(p_1 \cdots p_m) = Cl^A_\sigma(p_1 \cdots p_m p_{m+1}) = \cdots$. Therefore, for any $x \in p_1 \cdots p_m$ there exists $\mathfrak{h} \in \mathcal{L}(\sigma)$ such that $x\mathfrak{h} \subseteq p_1 \cdots p_m p_{m+1} \subseteq p_{m+1}$. Since $\mathfrak{h} \not\subseteq p_{m+1}$, we have $x \not\in p_{m+1}$. In consequence, $p_1 \cdots p_m \subseteq p_{m+1}$, and there exists an index $i$ such that $p_i \subseteq p_{m+1}$, which is a contradiction.
(4). See [1, Corollary 7.24].

The following result appears as Theorem 2.2 in [12].

Theorem 4.4. Let $A$ be a ring, the following statements are equivalent:

(a) $A$ is artinian.
(b) $A$ is totally $\sigma_A \setminus \mathfrak{p}$-artinian, for every $\mathfrak{p} \in \text{Spec } A$.
(c) $A$ is totally $\sigma_A \setminus \mathfrak{m}$-artinian, for every $\mathfrak{m} \in \text{Max}(A)$.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c). It is evident.
(c) $\Rightarrow$ (a). Let $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ be a decreasing chain of ideals. For every maximal ideal $\mathfrak{m}$, there exist $m_\mathfrak{m}$ and $s_\mathfrak{m} \in A \setminus \mathfrak{m}$ such that $a_{m_\mathfrak{m}} \subseteq \mathfrak{a}_i$ for every $s \geq m_\mathfrak{m}$. Let $G = \{s_\mathfrak{m} \mid \mathfrak{m} \in \text{Max}(A)\}$. If $(G) \neq A$, there exists a maximal ideal $\mathfrak{m}$ such that $(G) \subseteq \mathfrak{m}$, which is a contradiction. Thus, we have $(G) = A$, and there exist $s_1, ..., s_t \in G$ such that $\sum_{i=1}^t s_i A = A$. Let $s_i = s_{m_i}$ and $m_i = m_{s_i}$, for every $i = 1, ..., t$. If $m = \max\{m_1, ..., m_t\}$ we have $a_m \subseteq a_{m_i}$, then
$$a_m s_i \subseteq a_m s_{m_i} \subseteq a_i,$$
for every $s \geq m$. In consequence, $a_m = a_m \sum_{i=1}^t s_i A \subseteq a_i$ for every $s \geq m$, and the chain stabilizes. □

We may study this result in order to characterize totally $\sigma$-artinian rings.

Theorem 4.5. Let $A$ be a ring and $\sigma$ be a finite type hereditary torsion theory. The following statements are equivalent:

(a) $A$ is totally $\sigma$-artinian.
(b) $\mathcal{K}(\sigma) = \mathcal{C}(\sigma)$ is finite and $A$ is totally $\sigma_A \setminus \mathfrak{p}$-artinian for every prime ideal $\mathfrak{p} \in \mathcal{C}(\sigma)$. 
5. Simple and maximal modules

In order to show that every totally $\sigma$-artinian ring is totally $\sigma$-noetherian, we need to study simple modules and maximal ideal relative to the hereditary torsion theory $\sigma$. We shall use the preceding studies of minimal and maximal elements as appears in sections 3 and 6, respectively.

The example of $\sigma$-torsion. The classical theory

Let $M$ be an $A$-module. We have that $M$ is $\sigma$-artinian if, and only if, the family $C(M, \sigma)$ of $\sigma$-closed submodules $\mathcal{M}$ satisfies the decreasing chain condition, or equivalently the minimal condition, which are also equivalent to the condition that for every decreasing chain $N_1 \supseteq N_2 \supseteq \cdots$ of submodules, there exists $m \in \mathbb{N}$ such that $C_{\sigma}^M(N_m) = C_{\sigma}^M(N_s)$, for every $s \geq m$.

A submodule $N \subseteq M$ is $\sigma$-minimal if $C_{\sigma}^M(N) \subseteq M$ is a minimal element in $C(M, \sigma) \setminus \{\sigma M\}$, or equivalently if $C(N, \sigma) = \{\sigma N, N\}$. This means that $N$ is not $\sigma$-torsion, and for every submodule $L \subseteq N$, we have either $L$ is $\sigma$-torsion or $L$ is not $\sigma$-torsion, and in this case $L \subseteq \sigma N$, it is $\sigma$-dense. Observe that if $M$ is not $\sigma$-torsion, a submodule $N \subseteq M$ is $\sigma$-minimal if, and only if, $N$ is a minimal in the family $\mathcal{M} = \{N \subseteq M \mid N \text{ is not } \sigma\text{-torsion}\}$.

An $A$-module $M$ is $\sigma$-simple if $C(M, \sigma) = \{\sigma M, M\}$. If, in addition, $M$ is $\sigma$-torsionfree, we name $M$ a $\sigma$-cocritical $A$-module.

We may dualize $\sigma$-artinian to obtain $\sigma$-noetherian modules. In the case of $\sigma$-maximal submodules, we have that $N \subseteq M$ is $\sigma$-maximal if $M/N$ is not $\sigma$-torsion, and for every submodule $N \subseteq L \subseteq M$ such that $M/L$ is not $\sigma$-torsion we have $N \subseteq L$, which is equivalent to say that $M/N$ is a $\sigma$-simple $A$-module. Observe that if $M$ is not $\sigma$-torsion, a submodule $N \subseteq M$ is $\sigma$-maximal if, and only if, $N$ is a maximal the family $\mathcal{M} = \{N \subseteq M \mid M/N \text{ is not } \sigma\text{-torsion}\}$.

A submodule $N \subseteq M$ is called $\sigma$-critical if it is $\sigma$-maximal and $M/N$ is $\sigma$-torsionfree.

The example of totally $\sigma$-torsion. Simple modules

When we study modules and the totally $\sigma$-torsion we need to change the paradigm. Thus, let $M$ be a totally $\sigma$-artinian $A$-module, and consider the family of submodules

$$\mathcal{M} = \{N \subseteq M \mid N \text{ is not totally } \sigma\text{-torsion}\}.$$ 

$\mathcal{M}$ is not empty whenever $M$ is not totally $\sigma$-torsion. If $M$ is totally $\sigma$-artinian, there exists a $\sigma$-minimal element in $\mathcal{M}$, say $N$, that satisfies:

1. $N$ is not totally $\sigma$-torsion,
2. There exists $\mathfrak{q} \in \mathcal{L}(\sigma)$ such that for every $H \subseteq N$, which is not totally $\sigma$-torsion, we have $N\mathfrak{q} \subseteq H$.

In general, for any $A$-module $M$, a submodule $N$ satisfying (1) and (2) is called a totally $\sigma$-minimal submodule of $M$. An $A$-module $M$ is called totally $\sigma$-simple whenever $M$ is a $\sigma$-minimal element of $\mathcal{M}$, i.e.

1. $M$ is not totally $\sigma$-torsion and
2. there exists $\mathfrak{q} \in \mathcal{L}(\sigma)$ such that for every not totally $\sigma$-torsion submodule $H \subseteq M$ we have $M\mathfrak{q} \subseteq H$. 

Proof. (a) $\Rightarrow$ (b). See Corollaries (4.2) and (4.3).

(b) $\Rightarrow$ (a). Let $a_1 \supseteq a_2 \supseteq \cdots$ be a chain of ideals. For every $p \in \mathcal{C}(\sigma)$, there exists $m_p \in \mathbb{N}$ and $b_p \in \mathcal{L}(\sigma_{A/p})$ such that $a_{m_p}b_p \subseteq a_1$ for every $s \geq m_p$. If we take $m = \max\{m_p \mid p \in \mathcal{C}(\sigma)\}$, and $b = \prod\{b_p \mid p \in \mathcal{C}(\sigma)\}$, then $mb \subseteq a_1$ for every $s \geq m$, which shows that $A$ is totally $\sigma$-artinian. □
Let \( M \) be a totally \( \sigma \)-simple \( A \)-module with companion ideal \( \mathfrak{b} \in \mathcal{L}(\sigma) \), i.e. \( \mathfrak{b} \) satisfies that for every \( H \subseteq M \) which is non-totally \( \sigma \)-torsion we have \( M\mathfrak{b} \subseteq H \).

Observe that we have:

**Proposition 5.1.** Let \( M \) be a totally \( \sigma \)-simple \( A \)-module with companion ideal \( \mathfrak{b} \in \mathcal{L}(\sigma) \), the following statements hold:

1. \( M\mathfrak{b} \) is not totally \( \sigma \)-torsion, hence \( M\mathfrak{b} \subseteq M \) is the minimum of all not totally \( \sigma \)-torsion submodules of \( M \). In particular, every not totally \( \sigma \)-torsion of \( M \) is totally \( \sigma \)-simple.
2. \( M\mathfrak{b} \) is also totally \( \sigma \)-simple, and every proper submodule if totally \( \sigma \)-torsion.
3. For any ideal \( \mathfrak{b}' \in \mathcal{L}(\sigma) \) we always have \( M\mathfrak{b} \subseteq M\mathfrak{b}' \). In particular, if \( \mathfrak{b}' \subseteq \mathfrak{b} \) then \( M\mathfrak{b} = M\mathfrak{b}' \).
4. If \( \mathfrak{b}' \in \mathcal{L}(\sigma) \) is another ideal companion to \( M \), then \( M\mathfrak{b}' = M\mathfrak{b} \).
5. Let \( M' \) be a totally \( \sigma \)-simple \( A \)-module and \( f : M \to M' \) be a surjective map with kernel \( K \), then \( K \) is totally \( \sigma \)-torsion.

**Proof.** (1) to (4) are straightforward.

(5). If \( K \subseteq M \) is not totally \( \sigma \)-torsion, and \( \mathfrak{b} \in \mathcal{L}(\sigma) \) is the companion ideal of \( M \), then \( M\mathfrak{b} \subseteq K \), and we have \( M\mathfrak{b}' = 0 \), which is a contradiction.

If \( M \) is totally \( \sigma \)-simple with companion ideal \( \mathfrak{b} \), we call \( M\mathfrak{b} \) the core submodule of \( M \).

**Proposition 5.2.** If \( M \subseteq M' \) satisfies that there exists \( \mathfrak{b}_0 \in \mathcal{L}(\sigma) \) such that \( M\mathfrak{b}_0 \subseteq M \), then \( M \) is totally \( \sigma \)-simple if, and only if, \( M' \) is. In addition, the core of \( M \) and \( M' \) are equal.

**Proof.** If \( M \) is totally \( \sigma \)-simple, \( \mathfrak{b} \in \mathcal{L}(\sigma) \) is a companion ideal, and \( H \subseteq M' \) is not totally \( \sigma \)-torsion, then \( H\mathfrak{b}_0 \subseteq M \) is not totally \( \sigma \)-torsion, hence \( M\mathfrak{b}_0 \subseteq H \), and \( M'\mathfrak{b}_0 \subseteq H \). Otherwise, if \( M' \) is totally \( \sigma \)-simple, \( \mathfrak{b}' \in \mathcal{L}(\sigma) \) is a companion ideal, and \( H \subseteq M \) is not totally \( \sigma \)-torsion, since \( H \subseteq M' \), we have \( M\mathfrak{b}' \subseteq M\mathfrak{b} \subseteq H \), and \( M \) is totally \( \sigma \)-simple.

If \( M \subseteq M' \) are totally \( \sigma \)-simple modules we may assume \( \mathfrak{b} \) is the same companion ideal to \( M \) and \( M' \), then we have \( M\mathfrak{b} = M'\mathfrak{b} \). Indeed, \( M'\mathfrak{b} \subseteq M \), and \( M'\mathfrak{b} = M'\mathfrak{b}' \subseteq M\mathfrak{b} \subseteq M'\mathfrak{b} \). The core of \( M/L \) is a quotient of the core of \( M \).

**Proposition 5.3.** Let \( L \subseteq M \) be a totally \( \sigma \)-torsion submodule, then \( M \) is totally \( \sigma \)-simple if, and only if, \( M/L \) is. In this case, the core of \( M/L \) is a quotient of the core of \( M \).

**Proof.** If \( M \) is totally \( \sigma \)-simple with companion ideal \( \mathfrak{b} \in \mathcal{L}(\sigma) \), and \( H/L \subseteq M/L \) is a not totally \( \sigma \)-torsion submodule, then \( H \subseteq M \) is not totally \( \sigma \)-torsion, hence \( M\mathfrak{b} \subseteq H \), and \( (M/L)\mathfrak{b} = (M\mathfrak{b} + L)/L \subseteq H/L \). Otherwise, if \( M/L \) is totally \( \sigma \)-simple with companion ideal \( \mathfrak{b} \in \mathcal{L}(\sigma) \), and \( H \subseteq M \) is not totally \( \sigma \)-torsion, then \( (H + L)/L \) is not totally \( \sigma \)-torsion, hence \( (M/L)\mathfrak{b} \subseteq (H + L)/L \), and \( M\mathfrak{b} + L \subseteq H + L \). Since \( L \) is totally \( \sigma \)-torsion, there exists \( \mathfrak{b}' \in \mathcal{L}(\sigma) \) such that \( \mathfrak{b}' \mathfrak{b} \). Therefore, \( M\mathfrak{b}' \subseteq M\mathfrak{b} \subseteq H \), and \( M \) is totally \( \sigma \)-simple.

If \( M\mathfrak{b} \) is the core of \( M \), its image is \( M\mathfrak{b}/(M\mathfrak{b} \cap L) \), which is totally \( \sigma \)-simple and every proper submodule is totally \( \sigma \)-torsion. Indeed, if \( H/(M\mathfrak{b} \cap L) \subseteq M\mathfrak{b}/(M\mathfrak{b} \cap L) \), then \( H \subseteq M \), so it is totally \( \sigma \)-torsion, hence \( H/(M\mathfrak{b} \cap L) \).

An \( A \)-module \( M \) is core totally \( \sigma \)-simple whenever \( M \) is the core of a totally \( \sigma \)-simple module, i.e. whenever \( M \) is not totally \( \sigma \)-torsion and every proper submodule is totally \( \sigma \)-torsion.

Observe that if \( M \) is core totally \( \sigma \)-simple, then \( M\mathfrak{b} = M \) for every \( \mathfrak{b} \in \mathcal{L}(\sigma) \). Indeed, if \( \mathfrak{b}_0 \in \mathcal{L}(\sigma) \) is the companion ideal of \( M \), then \( M\mathfrak{b}_0 = M \), and for every \( \mathfrak{b} \in \mathcal{L}(\sigma) \) we have: \( M\mathfrak{b} = M\mathfrak{b}_0 = M \).

If \( M \) is a core totally \( \sigma \)-simple \( A \)-module, we have two different cases:

1. \( \sigma M \neq M \). In this case, \( \sigma M \) is totally \( \sigma \)-torsion and \( M/\sigma M \) is \( \sigma \)-torsionfree and core totally \( \sigma \)-simple, hence it is simple. Indeed, every proper submodule is totally \( \sigma \)-torsion, hence zero.
(2) \( \sigma M = M \). In this case, since every proper quotient is \( \sigma \)-torsion, it follows that \( M \) has no simple quotients. In particular, \( M \) is not finitely generated.

Let \( M \) be a core totally \( \sigma \)-simple \( A \)-module, we have the following two possibilities:

(1) \( M \) is not cyclic. Since every proper submodule of \( M \) is totally \( \sigma \)-torsion, we have \( M \) is \( \sigma \)-torsion. Hence, \( M \) is not finitely generated because it is not totally \( \sigma \)-torsion.

(2) \( M \) is cyclic. Since it is not totally \( \sigma \)-torsion, then \( \sigma M \neq M \), and \( M \) has simple \( \sigma \)-torsionfree quotients. Otherwise, every simple quotient of \( M \) is \( \sigma \)-torsionfree. Consequences of this fact are \( \sigma(M) \) is totally \( \sigma \)-torsion, and \( \sigma M \) is contained in the intersection of all maximal submodules of \( M \).

**Lemma 5.4.** Let \( A \) be a ring and \( a \subseteq A \) be a proper ideal, then \( M = A/a \) is core totally \( \sigma \)-simple if, and only if, it satisfies:

1. \( a \notin \mathcal{L}(\sigma) \),
2. \( a + b = A \), for every \( b \in \mathcal{L}(\sigma) \), and
3. \( (a : b) \subseteq \mathcal{L}(\sigma) \), for every proper ideal \( b \supseteq a \).

**Proof.** If \( A/a \) is core totally \( \sigma \)-simple, then it is not totally \( \sigma \)-torsion, i.e. \( a \notin \mathcal{L}(\sigma) \). For any \( b \in \mathcal{L}(\sigma) \), we have \((A/a)b = A/a\), i.e. \( a + b = A \). Since every proper submodule of \( A/a \) is totally \( \sigma \)-torsion, then for every \( b \supseteq a \) we have \((a : b) \in \mathcal{L}(\sigma) \).

In particular, for any maximal ideal \( m \) the simple \( A \)-module \( A/m \) is totally \( \sigma \)-simple if, and only if, \( m \notin \mathcal{L}(\sigma) \).

**Lemma 5.5.** The class \( \mathcal{M} \) is \( \sigma \)-lower closed.

**Proof.** Indeed, if \( H \subseteq M \) and there are \( N \in \mathcal{M} \) and \( b \in \mathcal{L}(\sigma) \) such that \( Nb \subseteq H \), then \( H \) is not totally \( \sigma \)-torsion. On the contrary there exists \( b' \in \mathcal{L}(\sigma) \) such that \( Hb' = 0 \), hence \( Hb'b' = 0 \), and \( N \) is totally \( \sigma \)-torsion.

If \( M \) is totally \( \sigma \)-artinian, there are minimal elements in \( \mathcal{M} \); every minimal element \( N \) of \( \mathcal{M} \) is a core totally \( \sigma \)-simple module, i.e. it satisfies:

1. \( N \) is not totally \( \sigma \)-torsion, and
2. Every proper submodule of \( N \) is totally \( \sigma \)-torsion.

**Lemma 5.6.** Let \( M \) be a totally \( \sigma \)-simple module, for any submodule \( N \subseteq M \) we have:

1. If \( N \) is not totally \( \sigma \)-torsion, then \( N \) is totally \( \sigma \)-simple. In addition, \( N \) and \( M \) have the same core.
2. The quotient \( M/N \) is either totally \( \sigma \)-torsion whenever \( N \) is non totally \( \sigma \)-torsion, or totally \( \sigma \)-simple whenever \( N \) is totally \( \sigma \)-torsion.

**Proof.** (1) is straightforward.

(2) If \( N \) is non-totally \( \sigma \)-torsion, there exists \( b \in \mathcal{L}(\sigma) \) such that \( Mb \subseteq N \), hence \( M/N \) is totally \( \sigma \)-torsion. Otherwise, if \( N \) is totally \( \sigma \)-torsion, then \( M/N \) is non-totally \( \sigma \)-torsion; on the other hand, for any non-totally \( \sigma \)-torsion submodule \( L/N \subseteq M/N \), since \( L \subseteq M \) is non-totally \( \sigma \)-torsion, there exists \( b \in \mathcal{L}(\sigma) \) such that \( Mb \subseteq L \); hence \((M/N)b \subseteq (L/N)\), and \( M/N \) is totally \( \sigma \)-simple. □
Lemma 5.7. If $N_1$, $N_2$ are totally $\sigma$-simple modules, and $f : N_1 \rightarrow N_2$ is a module map, then $f$ is either zero or surjective.

6. Maximal submodules

In this section, we assume the reader knows about totally $\sigma$-noetherian rings and modules as it was exposed in [8].

Let $M$ be a totally $\sigma$-noetherian $A$-module, and consider the family of submodules

$$\mathcal{M}' = \{N \subseteq M \mid M/N \text{ is not totally } \sigma\text{-torsion}\}.$$  

We have that $\mathcal{M}'$ is nonempty whenever $M$ is not totally $\sigma$-torsion.

Lemma 6.1. Let $M$ be a non totally $\sigma$-torsion module, the class $\mathcal{M}'$ is $\sigma$-upper closed.

Proof. If $H \subseteq M$ and there are $N \in \mathcal{M}'$ and $h \in \mathcal{L}(\sigma)$ such that $H \cdot h \subseteq N$, then $H \in \mathcal{M}'$. We show that $M/H$ is not totally $\sigma$-torsion. On the contrary, there exists $h_1 \in \mathcal{L}(\sigma)$ such that $M \cdot h_1 \subseteq H$; hence, $M \cdot h_1 \cdot h \subseteq H \cdot h \subseteq N$, and $M/N$ is totally $\sigma$-torsion, which is a contradiction. 

Since $M$ is totally $\sigma$-noetherian, there are maximal elements in $\mathcal{M}$; a maximal element of $\mathcal{M}'$ is called a totally $\sigma$-maximal submodule of $M$. We define a submodule $N$ of $M$ to be a core totally $\sigma$-maximal submodule whenever it satisfies:

1. $M/N$ is not totally $\sigma$-torsion.
2. $N$ is maximal in the $\sigma$-upper closed family $\mathcal{M}' = \{L \subseteq M \mid N \subseteq L, M/L \text{ is not totally } \sigma\text{-torsion}\}.$

In the same way, we can define a totally $\sigma$-maximal submodule of an $A$-module $M$ whenever

1. $M/N$ is not totally $\sigma$-torsion, and
2. $N$ is $\sigma$-maximal in $\mathcal{M}'$, i.e. there exists $h \in \mathcal{L}(\sigma)$ such that for every $H \in \mathcal{M}'$ such that $N \subseteq H$ we have $H \cdot h \subseteq N$.

Even, we can dualize the notion of totally $\sigma$-simple submodule, in defining a submodule $N \subseteq M$ to be totally $\sigma$-cosimple if it satisfies:

1. $M/N$ is not totally $\sigma$-torsion, and
2. There exists $h \in \mathcal{L}(\sigma)$ such that for every $N \subseteq H \not\subseteq M$ satisfying that $H/N$ is not totally $\sigma$-torsion we have $M \cdot h \subseteq H$.

and defining core totally $\sigma$-cosimple if, in addition, for every $N \subseteq H \not\subseteq M$ we have that $H/N$ is totally $\sigma$-torsion.

Lemma 6.2. Let $M$ be a core totally $\sigma$-simple $A$-module, then $\text{Ann}(M) \subseteq A$ is a core totally $\sigma$-cosimple ideal.

Proof. Since $M$ is core totally $\sigma$-simple we have two possibilities for $M$:

1. $\sigma M = M$, and
2. $\sigma M \neq M$.

In case (2), there exists $x \in M \setminus \sigma M$, such that $xA = M$. Indeed, since $xA \subseteq M$ and it is not totally $\sigma$-torsion, then $xa = M$. Hence, $\text{Ann}(M) = \text{Ann}(x)$, and $A/\text{Ann}(x)$ is not totally $\sigma$-torsion. The rest is obvious.

In case (1), we have that $M$ is cyclic and we can proceed in the same way. 

$\square$
Proposition 6.3. Let $M$ be a totally $\sigma$-simple $A$-module, then $\text{Ann}(M) \subseteq A$ is totally $\sigma$-cosimple.

Proof. Let $M\hat{\ell} \subseteq M$ the core of $M$, we have $\text{Ann}(M\hat{\ell}) = \text{Ann}(M) : \hat{\ell}$, hence we can build a short exact sequence $0 \to \frac{\text{Ann}(M\hat{\ell})}{\text{Ann}(M)} \to \frac{A}{\text{Ann}(M)} \to \frac{A}{\text{Ann}(M\hat{\ell})} \to 0$. Since $\frac{\text{Ann}(M\hat{\ell})}{\text{Ann}(M)}$ is totally $\sigma$-torsion, we have the result.

If $M = A$, we may determine more precisely the core totally $\sigma$-cosimple ideals.

Proposition 6.4. Let $a \subseteq A$ be an ideal. If $a \subseteq A$ is core totally $\sigma$-cosimple then:

1. $a$ is prime.
2. $a$ is maximal in $\mathcal{H}(\sigma)$, i.e., $a \in \mathcal{C}(\sigma)$.

In conclusion, core totally $\sigma$-cosimple ideals are exactly the ideals in $\mathcal{C}(\sigma)$.

Proof. Since $a \subseteq A$ is core totally $\sigma$-cosimple, it is not totally $\sigma$-torsion, hence $a \not\subseteq \mathcal{L}(\sigma)$.

1. Let $a_1, a_2 \subseteq A$ be proper ideals properly containing $a$ such that $a_1 a_2 \subseteq a$. Since $a \subseteq a_1 \subseteq A$, then $A/a_1$ is totally $\sigma$-torsion, hence $a_1 \in \mathcal{L}(\sigma)$. Similar result holds for $a_2$. Therefore, $a_1 a_2 \in \mathcal{L}(\sigma)$, and $a \in \mathcal{L}(\sigma)$, which is a contradiction.

2. Since $a \not\subseteq \mathcal{L}(\sigma)$ is prime, then $a \in \mathcal{H}(\sigma)$. Let $a \subseteq p \subseteq \mathcal{H}(\sigma)$, since $A/p$ is $\sigma$-torsionfree and non-zero, it is not totally $\sigma$-torsion, hence $p = a$. Therefore, $a \in \mathcal{H}(\sigma)$ is maximal and $a \not\in \mathcal{C}(\sigma)$.

The converse is obvious because for any $p \in \mathcal{C}(\sigma)$ we have that $A/p$ is $\sigma$-cocritical.

The core totally $\sigma$-Jacobson radical of an $A$-module $M$ is defined as the intersection of all core totally $\sigma$-cosimple submodule, and we represent it by $t\text{Jac}(M)$.

Lemma 6.5. Let $M$ be an $A$-module, then $t\text{Jac}(M/t\text{Jac}(M)) = 0$.

Proof. We have $N \subseteq M/t\text{Jac}(M)$ is core totally $\sigma$-cosimple if, and only if, $N \subseteq M$ is core totally $\sigma$-cosimple.

Proposition 6.6.

1. Every totally $\sigma$-simple $A$-module is totally $\sigma$-artinian.
2. Every core totally $\sigma$-simple $A$-module is totally $\sigma$-noetherian.
3. Every totally $\sigma$-simple $A$-module is totally $\sigma$-noetherian.

Proof. (1). Let $M$ be totally $\sigma$-simple and $N_1 \supseteq N_2 \supseteq \cdots$ be a decreasing chain of submodules of $M$. If for every index $i$, we have that $N_i$ is not totally $\sigma$-torsion, and $\hat{\ell} \in \mathcal{L}(\sigma)$ is the companion ideal of $M$, then $M\hat{\ell} \subseteq N_i$, hence $N_i \hat{\ell} \subseteq M\hat{\ell} \subseteq N_i$, for every index $i$. If there exists an index $m$ such that $N_m$ is totally $\sigma$-torsion, there exists $\hat{\ell}' \in \mathcal{L}(\sigma)$ such that $N_m \hat{\ell}' = 0 \subseteq N_s$, for every $s \geq m$.

(2). If $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of $M$, and $M$ is core totally $\sigma$-simple we have studied the two cases.

2.1. If $M$ is $\sigma$-torsion, then $M$ is cyclic, hence $\cup_{n} N_n \subseteq M$ is either totally $\sigma$-torsion or $\cup_{n} N_n = N$. In the second case, there exists an index $m$ such that $M = N_m$. In both cases, the chain is $\sigma$-stable.

2.2. If $M$ is not $\sigma$-torsion, then $\sigma M$ is totally $\sigma$-torsion. If $\cup_{n} N_n \subseteq \sigma M$, then the chain $\sigma$-stabilizes. If there exists an index $m$ such that $N_m \subseteq \sigma M$, and $\hat{\ell} \in \mathcal{L}(\sigma)$ is the companion ideal of $M$, then $M\hat{\ell} \subseteq N_m$, and in this case, the chain stabilizes.
Proof. If $M$ is totally $\sigma$-simple, with companion ideal $h \in \mathcal{L}(\sigma)$, then $Mh$ is core totally $\sigma$-simple, hence it is totally $\sigma$-noetherian. Otherwise, $M/Mh$ is totally $\sigma$-torsion, hence totally $\sigma$-noetherian. Therefore, $M$ is totally $\sigma$-noetherian because it is an extension of $Mh$ by $M/Mh$. □

**Lemma 6.7.** Let $M$ be an artinian $A$-module such that $\imath\text{Jac}(M) = 0$, then there exists $T \subseteq M$, totally $\sigma$-torsion such that $M/T \subseteq \bigoplus_{j=1}^{r} S_{j}$, for a finite family of core totally $\sigma$-simple $A$-modules. In particular, $M$ is totally $\sigma$-noetherian.

**Proof.** If $\imath\text{Jac}(M) = 0$, the intersection of all core totally $\sigma$-cosimple submodules is zero, and since $M$ is $\sigma$-finitely cogenerated, there exists a finite family of core totally $\sigma$-cosimple submodules, $\{N_{1}, ..., N_{r}\}$ such that $\bigcap_{j=1}^{r} N_{j}$ is totally $\sigma$-torsion. If we call $T = \bigcap_{j=1}^{r} N_{j}$, then $M/T$ is a submodule of $\bigoplus_{j=1}^{r} (M/N_{j})$. Finally, since $\bigoplus_{j=1}^{r} (M/N_{j})$ is totally $\sigma$-noetherian, then $M/T$ is totally $\sigma$-noetherian, and we have $M$ is totally $\sigma$-noetherian. □

**Theorem 6.8.** If $A$ is a totally $\sigma$-artinian ring, then $A$ is totally $\sigma$-noetherian.

**Proof.** Let $J = \imath\text{Jac}(A)$. If $A$ is totally $\sigma$-artinian, then $A/J$ is totally $\sigma$-artinian and totally $\sigma$-noetherian, by Lemma (6.7). We consider $A/J^{2}$, because for every ideal $a \subseteq A$ we have that $a \subseteq A$ is a core totally $\sigma$-cosimple ideal if, and only if, $a/J^{2} \subseteq A/J^{2}$ is core totally $\sigma$-cosimple, then $\imath\text{Jac}(A/J^{2}) = J/J^{2}$, and the same holds for every $m \in \mathbb{N}$, i.e. $\imath\text{Jac}(A/J^{m}) = J/J^{m}$.

The decreasing chain $J = J^{1} \supseteq J^{2} \supseteq \cdots$ is $\sigma$-stable, hence there exist $m \in \mathbb{N}$ and $h \in \mathcal{L}(\sigma)$ such that $J^{m}h \subseteq J^{s}$ for every $s \geq m$. We do induction on $m$. Let us assume $m = 1$, then $J/J^{2}$ is totally $\sigma$-torsion, and we have a short exact sequence

$$0 \rightarrow J/J^{2} \rightarrow A/J^{2} \rightarrow A/J \rightarrow 0$$

Since $J^{1}/J^{2}$ and $A/J$ are totally $\sigma$-artinian and totally $\sigma$-noetherian, then $A/J^{2}$ is. We assume the result holds for any positive integral number smallest than $m$ and that $J^{m}h \subseteq J^{s}$ for every $s \geq m$. Consider the short exact sequence

$$0 \rightarrow J^{m}/J^{m+1} \rightarrow A/J^{m+1} \rightarrow A/J^{m} \rightarrow 0$$

Since $J^{m}/J^{m+1}$ is totally $\sigma$-torsion and $A/J^{m}$ is totally $\sigma$-artinian and totally $\sigma$-noetherian, then $A/J^{m+1}$ is.

For any ring $A$, the totally Jacobson $\sigma$-radical of $A$ which is the intersection of all core totally $\sigma$-cosimple ideals is the intersection of all elements in $\mathcal{E}(\sigma)$, see Proposition (6.4), which coincides with the Jacobson $\sigma$-radical of $A$, i.e.

$$\text{Jac}(A) = \text{Jac}_{\sigma}(A) = \bigcap\{p \mid p \in \mathcal{E}(\sigma)\}.$$

We show that there exist enough core totally $\sigma$-cosimple submodules in the following sense.

**Lemma 6.9.** Let $\sigma$ be a finite type hereditary torsion theory, for any totally $\sigma$-finitely generated module $M$ and any proper submodule $N \subseteq M$ such that $M/N$ is not totally $\sigma$-torsion, there exists $a$ core totally $\sigma$-cosimple submodule $N \subseteq H \subseteq M$.

**Proof.** Let $\Gamma = \{H \subseteq M \mid N \subseteq H \subseteq M \text{ and } M/H \text{ is not totally } \sigma\text{-torsion}\}$. If $\Gamma = \{N\}$, then $N$ is maximal among those submodules which are not totally $\sigma$-torsion, hence it is core totally $\sigma$-cosimple. Otherwise, for any increasing chain $N_{1} \subseteq N_{2} \subseteq \cdots$ in $\Gamma$, we consider $\bigcup_{n \geq 1} N_{n}$. If $M/\bigcup_{n \geq 1} N_{n}$ is totally $\sigma$-torsion, there exists $h \in \mathcal{L}(\sigma)$, finitely generated, such that $Mh \subseteq M/\bigcup_{n \geq 1} N_{n}$, and there is an index $m$ such that $Mh \subseteq N_{m}$. Otherwise, there exist $N' \subseteq M$, finitely generated, and $h' \in \mathcal{L}(\sigma)$, finitely generated, such that $Mh' \subseteq N'$. In consequence, $Mh' \subseteq N_{m}$, and there exists an index $m$ such that $Mh' \subseteq N_{m}$, which is a contradiction.
Thus, $\Gamma$ is an inductive set of submodules, and by Zorn’s lemma, we have that $\Gamma$ has maximal submodules. A maximal submodule $N \subseteq M$ in $\Gamma$ is a core totally $\sigma$-cosimple submodule. 

Of particular interest is the case in which $M = A$; in this case for every ideal $\alpha \subseteq A$ such that $\alpha \not\subset L(\sigma)$, there exist $\mathfrak{p} \in \mathfrak{C}(\sigma)$ such that $\alpha \subseteq \mathfrak{p}$.

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