Inverse Conductivity Problem for a Parabolic Equation using a Carleman Estimate with One Observation

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Abstract

For the heat equation in a bounded domain we give a stability result for a smooth diffusion coefficient. The key ingredients are a global Carleman-type estimate, a Poincaré-type estimate and an energy estimate with a single observation acting on a part of the boundary.

1 Introduction

This paper is devoted to the identification of the diffusion coefficient in the heat equation using the least number of observations as possible.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ with $n \leq 3$, (the assumption $n \leq 3$ is necessary in order to obtain the appropriate regularity for the solution using classical Sobolev embedding, see Brezis [3]). We denote $\Gamma = \partial \Omega$ assumed to be of class $C^1$. We denote by $\nu$ the outward unit normal to $\Omega$ on $\Gamma = \partial \Omega$. Let $T > 0$ and $t_0 \in (0, T)$. We shall use the following notations $Q_0 = \Omega \times (0, T)$, $Q = \Omega \times (t_0, T)$, $\Sigma = \Gamma \times (t_0, T)$ and $\Sigma_0 = \Gamma \times (0, T)$. We consider the following heat equation:

\begin{equation}
\begin{cases}
\partial_t q = \nabla \cdot (c(x) \nabla q) & \text{in } Q_0, \\
q(t, x) = g(t, x) & \text{on } \Sigma_0, \\
q(0, x) = q_0 & \text{in } \Omega.
\end{cases}
\end{equation}

Our problem can be stated as follows:

**Inverse Problem**

Is it possible to determine the coefficient $c(x)$ from the following measurements:

$$
\partial_t (\partial_t q)|_{(t_0, T) \times \Gamma_0} \text{ and } \nabla(\Delta q(T', \cdot)), \Delta q(T', \cdot), q(T', \cdot) \text{ in } \Omega \text{ for } T' = \frac{t_0 + T}{2},
$$

where $\Gamma_0$ is a part of the boundary $\Gamma$ of $\Omega$ ?

Let $q$ (resp. $\tilde{q}$) be solution of (1.1) associated to $(c, g, q_0)$ (resp. $(\tilde{c}, g, q_0)$), we assume

**Assumption 1.1.**

- $q_0$ belongs to $H^4(\Omega)$ and $g$ is sufficiently regular (e.g. $\exists \epsilon > 0$ such that $g \in H^1(0, T, H^{3/2+\epsilon}(\partial \Omega)) \cap H^2(0, T, H^5/2+\epsilon(\partial \Omega)$))
• $c, \tilde{c} \in C^4(\Omega),$

• There exist a constant $r > 0$, such that $q_0 \geq r$ and $g \geq r$.

Note that the first item of the previous assumptions implies that $[1.1]$ admits a solution in $H^1(t_0, T, H^2(\Omega))$ (see Lions [12]). We will later use this regularity result. The two last items allow us to state that the function $u$ satisfies $|\Delta q(x, T')| \geq r > 0$ and $|\nabla q(x, T')| \geq r > 0$ in $\Omega$ (see Pazy [3]). Benabdallah, Gaitan and Le Rousseau [4].

We assume we can measure both the normal flux $\partial_n(q)$ on $\Gamma_0 \subset \partial \Omega$ in the time interval $(t_0, T)$ for some $t_0 \in (0, T)$ and $\nabla(\Delta q)$, $\Delta q$ and $\nabla q$ at time $T' \in (t_0, T)$.

Our main result is a stability result for the coefficient $c(x)$:

For $q_0$ in $H^2(\Omega)$ there exists a constant $C = C(\Omega, \Gamma, t_0, T, r) > 0$ such that

$$|c - \tilde{c}|^2_{H^1(\Omega)} \leq C|\partial_n(\partial_t q) - \partial_n(\partial_t \tilde{q})|^2_{L^2((t_0, T) \times \Gamma_0)} + C|\nabla(\Delta q(T', \cdot)) - \nabla(\Delta \tilde{q}(T', \cdot))|^2_{L^2(\Omega)} + C|\Delta q(T', \cdot) - \Delta \tilde{q}(T', \cdot)|^2_{L^2(\Omega)} + C|\nabla q(T', \cdot) - \nabla \tilde{q}(T', \cdot)|^2_{L^2(\Omega)}.$$

The key ingredients to this stability result are a global Carleman estimate, a Poincaré-type estimate and an energy estimate. We use the classical Carleman estimate with one observation on the boundary for the heat equation obtained in Fernández-Cara and Guerrero [3], Fursikov and Imanuvilov [8]. Following the method developed by Imanuvilov, Isakov and Yamamoto for the Lamé system in Imanuvilov, Isakov and Yamamoto [11], we give a Poincaré-type estimate. Then, we prove an energy estimate. Such energy estimate has been proved in Lasiecka, Triggiani and Zhang [13] for the Schrödinger operator in a bounded domain in order to obtain a controllability result and in Cristofol, Cardoulis and Gaitan [2] for the Schrödinger operator in an unbounded domain in order to obtain a stability result. Then using these estimates, we give a stability and uniqueness result for the diffusion coefficient $c(x)$. In the perspective of numerical reconstruction, such problems are ill-posed and stability results are thus of importance.

In the stationary case, the inverse conductivity problem has been studied by several authors. There are different approaches. For the two dimensional case, Nachman [14] proved an uniqueness result for the diffusion coefficient $c \in C^2(\Omega)$ and Astala and Paivärinta [1] for $c \in L^\infty(\Omega)$ with many measurements from the whole boundary. In the three dimensional case, with the use of complex exponentially solutions, Faddeev [1], Calderon [1], Sylvester and Uhlmann [10] showed uniqueness for the diffusion coefficient.

There are few results on Lipschitz stability for parabolic equations, we can cite Imanuvilov and Yamamoto [17], Benabdallah, Gaitan and Le Rousseau [4]. In [4], the authors prove a Lipschitz stability result for the determination of a piecewise-constant diffusion coefficient. For smooth coefficients in the principal part of a parabolic equation, Yuan and Yamamoto [17] give a Lipschitz stability result with multiple observations. This paper is an improvement of the simple case in [17] where we consider that the diffusion coefficient is a real valued function and not a $n \times n$-matrix. Indeed, in this case, with the method developed by [17], they need two observations in order to obtain an estimation of the $H^1$-norm of the diffusion coefficient. In this case, we need only one observation.

Our paper is organized as follows. In Section 2, we recall the global Carleman estimate for [13] with one observation on the boundary. Then we prove a Poincaré-type estimate for the coefficient $c(x)$ and an energy estimate. In Section 3, using the previous results, we establish a stability estimate for the coefficient $c(x)$ when one of the solutions $\tilde{q}$ is in a particular class of solutions with some regularity and "positivity" properties.
2 Some Useful Estimates

2.1 Global Carleman Estimate

We recall here a Carleman-type estimate with a single observation acting on a part $\Gamma_0$ of the boundary $\Gamma$ of $\Omega$ in the right-hand side of the estimate (see [8], [9]). Let us introduce the following notations:

Let $\tilde{\beta}$ be a $C^4(\Omega)$ positive function such that there exists a positive constant $C_0$ which satisfies

$$|\nabla \tilde{\beta}| \geq C_0 > 0$$

in $\Omega$, $\partial_{\nu} \tilde{\beta} \leq 0$ on $\Gamma \setminus \Gamma_0$.

Then, we define

$$\beta = \tilde{\beta} + K$$

with $K = m\|\tilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (t_0, T)$, we define the weight functions

$$\varphi(x, t) = \frac{e^{\lambda \beta(x)}}{(t - t_0)(T - t)}$$

$$\eta(x, t) = \frac{e^{2\lambda K} - e^{\lambda \beta(x)}}{(t - t_0)(T - t)}.$$

If we set $\psi = e^{-s\eta}q$, we also introduce the following operators

$$M_1 \psi = \nabla \cdot (c \nabla \psi) + s^2\lambda^2 \psi |\nabla \beta|^2 \varphi^2 \psi + s(\partial_t \eta) \psi,$$

$$M_2 \psi = \partial_t \psi - 2s\lambda \varphi \nabla \beta \cdot \nabla \psi - 2s\lambda^2 \varphi |\nabla \beta|^2 \psi.$$

Then the following result holds (see [8], [9]):

**Theorem 2.2.** There exist $\lambda_0 = \lambda_0(\Omega, \Gamma_0) \geq 1$, $s_0 = s_0(\lambda_0, T) > 1$ and a positive constant $C = C(\Omega, \Gamma_0, T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the following inequality holds:

$$\|M_1(e^{-s\eta}q)\|_{L^2(Q)}^2 + \|M_2(e^{-s\eta}q)\|_{L^2(Q)}^2$$

$$+ s\lambda^2 \int_{t_0}^T \int_Q e^{-2s\eta} \varphi |\nabla q|^2 dx \, dt + s^3 \lambda^4 \int_{t_0}^T \int_Q e^{-2s\eta} \varphi^3 |q|^2 dx \, dt$$

$$\leq C \left[ s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} |\partial_t q|^2 dx \, dt + \int_{t_0}^T \int_Q e^{-2s\eta} |\partial_t q - \nabla \cdot (c \nabla q)|^2 dx \, dt \right],$$

for all $q \in H^1(t_0, T, H^2(\Omega))$ with $q = 0$ on $\Sigma$.

2.2 Poincaré-type estimate

We consider the solutions $q$ and $\tilde{q}$ to the following systems

$$\partial_t q = \nabla \cdot (c(x) \nabla q)$$

in $Q_0$, $q(t, x) = g(t, x)$ on $\Sigma_0$, $q(0, x) = q_0$ in $\Omega$.

and

$$\partial_t \tilde{q} = \nabla \cdot (\tilde{c}(x) \nabla \tilde{q})$$

in $Q_0$, $\tilde{q}(t, x) = g(t, x)$ on $\Sigma_0$, $\tilde{q}(0, x) = \tilde{q}_0$ in $\Omega$. 

3
We set \( u = q - \tilde{q}, \ y = \partial_t u \) and \( \gamma = c - \tilde{c}. \) Then \( y \) is solution to the following problem

\[
\begin{aligned}
\frac{\partial y}{\partial t} &= \nabla \cdot (c(x) \nabla y) + \nabla \cdot (\gamma(x) \nabla (\partial_t \tilde{q})) \quad \text{in} \quad Q_0, \\
y(t, x) &= 0 \quad \text{on} \quad \Sigma_0, \\
y(0, x) &= \nabla \cdot (\gamma(x) \nabla (q_0(x))) \quad \text{in} \quad \Omega.
\end{aligned}
\]

(2.5)

Note that with (2.3) and (2.4) we can determine \( y(T', x) \) and we obtain

\[
y(T', x) = \nabla \cdot (\gamma(x) \nabla (q(T', x))) + \nabla \cdot (c(x) \nabla (u(T', x))).
\]

(2.6)

We use a lemma proved in [11] for Lamé system in bounded domains:

**Lemma 2.3.** We consider the first order partial differential operator

\[
P_0 g := \nabla q_0 \cdot \nabla g
\]

where \( q_0 \) satisfies

\[
|\nabla \beta \cdot \nabla \tilde{q}(T')| \neq 0.
\]

Then there exists positive constant, \( s_1 > 0 \) and \( C = C(\lambda, T') \) such that for all \( s \geq s_1 \)

\[
s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') |g|^2 \, dx \, dy \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') |P_0g|^2 \, dx \, dy
\]

with \( T' = \frac{\kappa + T}{T}, \ \eta(T') := \eta(x, T'), \ \varphi(T') := \varphi(x, T') \) and for \( g \in H_0^1(\Omega). \)

We assume

**Assumption 2.4.** \( |\nabla \beta \cdot \nabla \tilde{q}(T')| \neq 0. \)

**Proposition 2.5.** Let \( \tilde{q} \) be solution of (2.4). We assume that Assumption 2.4 are satisfied. Then there exists a positive constant \( C = C(T', \lambda) \) such that for \( s \) large enough \(( s \geq s_1)\), the following estimate hold true

\[
s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') \langle |\nabla \gamma|^2 + |\gamma|^2 \rangle \, dx \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') \langle |\nabla y(T')|^2 + |y(T')|^2 \rangle \, dx
\]

\[
+ C \int_{\Omega} e^{-2s\eta(T')} \langle |\nabla (\Delta u(T'))|^2 + |\Delta u(T')|^2 + \nabla u(T')^2 \rangle \, dx
\]

for \( \gamma \in H_0^1(\Omega). \)

**Proof.** We are dealing with the following first order partial differential operators given by the equation (2.6)

\[
P_0(\gamma) := \sum_{i=1}^n \partial_{x_i} \tilde{q}(T') \partial_{x_i} \gamma = y(T') - \gamma \Delta \tilde{q}(T') - \nabla (c \nabla u(T')).
\]

We apply the lemma 2.3 for this operator and we can write:

\[
s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') |g|^2 \, dx \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') \langle |y(T')|^2 + |\gamma|^2 \rangle \, dx
\]

\[
+ C \int_{\Omega} e^{-2s\eta(T')} \langle |\Delta u(T')|^2 + |\nabla u(T')|^2 \rangle \, dx
\]

(2.7)
In the other hand, we use the \( x_j \)-derivative of the previous equation (2.6). So, for each \( j \) we deal with the following first order partial differential operator:

\[
P_0(\partial_{x_j} \gamma) = \partial_{x_j} (T') - \partial_{x_j} \gamma \Delta \tilde{q}(T') - \gamma \Delta (\partial_{x_j} \tilde{q})(T') - \partial_{x_j} (\nabla (c \nabla u))(T').
\]

Then under assumption (2.4):

\[
s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T')|\partial_{x_j} \gamma|^2 \, dx \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') \partial_{x_j} y(T')^2 \, dx + C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') \left( |\partial_{x_j} \gamma|^2 + |\gamma|^2 + |\nabla \gamma|^2 + |\partial_{x_j} F|^2 \right) \, dx
\]

So, adding for all \( j \), we can write

\[
(2.8)
\]

\[
s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T')|\nabla \gamma|^2 \, dx \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') \left( |\nabla \gamma|^2 + |\gamma|^2 + |\nabla u(T')|^2 + |\Delta u(T')|^2 \right) \, dx
\]

Taking into account (2.7) and (2.8) and for \( s \) large enough, we can conclude.

\[\square\]

### 2.3 Estimation of \( \int_{\Omega} e^{-2s\eta(T')} |y(T')|^2 \, dx \)

Let \( T' = \frac{1}{2}(T + t_0) \) the point for which \( \Phi(t) = \frac{1}{(t - t_0)(T - t)} \) has its minimum value.

We set \( \psi = e^{-s\eta} y \). With the operator

\[
(2.9)
\]

\[
M_2 \psi = \partial_t \psi - +2s\lambda \varphi \nabla \beta \nabla \psi - 2s\lambda^2 \varphi c |\nabla \beta|^2 \psi,
\]

we introduce, following [2],

\[
\mathcal{I} = \int_{t_0}^{T'} \int_{\Omega} M_2 \psi \, dx dt
\]

We have the following estimates.

**Lemma 2.6.** Let \( \lambda \geq \lambda_1 \), \( s \geq s_1 \) and let \( a, b, c, d \in L^\infty(\Omega) \). Furthermore, we assume that \( u_0, v_0 \) in \( H^2(\Omega) \) and the assumption \([7]\) is satisfied. Then there exists a constant \( C = C(\Omega, \omega, T) \) such that

\[
(2.10)
\]

\[
\int_{\Omega} e^{-2s\eta(T',x)} |y(T', x)|^2 \, dx \leq C \left[ \lambda^{1/2} \int_{t_0}^{T} \int_{\Gamma_0} e^{-2s\eta \varphi |\partial_{\nu} y|^2} \, dx \, dt + s^{-1/2} \lambda^{-1/2} \int_{t_0}^{T} \int_{\Omega} e^{-2s\eta} \left( |\gamma|^2 + |\nabla \gamma|^2 \right) \, dx \, dt \right].
\]
Proof. If we compute $I$, we obtain:

$$\int_{\Omega} e^{-2s\eta(T',x)} |y(T',x)|^2 \, dx = -2I$$

$$-4s\lambda \int_{t_0}^{T'} \int_{\Omega} \varphi \, e^{-\gamma \varphi} \, \nabla \cdot \nabla \varphi \, \psi \, dx \, dt - 4s\lambda \int_{t_0}^{T'} \int_{\Omega} \varphi \, e^{-\gamma \varphi} \, |\psi|^2 \, dx \, dt.$$

Then with the Carleman estimate (2.2), we can estimate all the terms in the right hand side of the previous equality and we have

$$\int_{\Omega} e^{-2s\eta(T',x)} |y(T',x)|^2 \, dx \leq Cs^{-3/2}\lambda^{-2} \left( \|M_2\psi\|^2 + s^3\lambda^4 \int_{\Omega} e^{-2s\eta}\varphi^3 |y|^2 \, dx \, dt \right) + Cs^{-1}\lambda^{-1/2} \left( s\lambda \int_{\Omega} e^{-2s\eta}\varphi \, |\nabla y|^2 \, dx \, dt + s^3\lambda^4 \int_{\Omega} e^{-2s\eta}\varphi^3 |y|^2 \, dx \, dt \right) + Cs^{-2}\lambda^{-2} \left( s^3\lambda^4 \int_{\Omega} e^{-2s\eta}\varphi^3 |y|^2 \, dx \, dt \right).$$

Finally, we obtain

$$\int_{\Omega} e^{-2s\eta(T',x)} |y(T',x)|^2 \, dx \leq C\lambda^{1/2} \int_{t_0}^{T} \int_{\Gamma_0} e^{-2s\eta}\varphi \, |\partial_\nu y|^2 \, d\sigma \, dt + Cs^{-1}\lambda^{-1/2} \int_{Q} e^{-2s\eta}|f|^2 \, dx \, dt,$$

where $f = \nabla \cdot (\gamma \nabla \partial_t \tilde{q})$. We assume that $\tilde{q}$ is sufficiently smooth in order to have $\nabla \partial_t \tilde{q}$ and $\Delta \partial_t \tilde{q}$ in $L^2(\Omega, T, L^\infty(\Omega))$.

Moreover taking into account that $e^{-2s\eta(x)} \leq e^{-2s\eta(T')}$, the proof of Lemma 2.6 is complete. 

### 2.4 Estimation of $\int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') |\nabla y(T')|^2 \, dx$

We introduce

$$E(t) = \int_{\Omega} e^{-\gamma(t,x)} e^{-2s\eta(x,t)} |\nabla y(x,t)|^2 \, dx.$$ 

In this section, we give an estimation for the energy $E(t)$ at $T'$.

**Theorem 2.7.** We assume that Assumptions I-4 are checked, then there exist $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $s_1 = s_1(\lambda_1, T) > 1$ and a positive constant $C = C(\Omega, \Gamma_0, C_0, r, T)$ such that, for any $\lambda \geq \lambda_1$ and any $s \geq s_1$, the following inequality holds:

$$E(T') \leq C \left[ s\lambda \int_{t_0}^{t} \int_{\Gamma_0} e^{-2s\eta} |\partial_\nu y|^2 \, dx \, dt + \int_{Q} e^{-2s\eta} |\gamma|^2 + |\nabla \gamma|^2 \, dx \, dt \right].$$

**Proof.** We note $f = \nabla \cdot (\gamma(x) \nabla \partial_t \tilde{q})$.

We multiply the first equation of (2.3) by $e^{-2s\eta} \nabla \cdot (e\nabla y) \varphi^{-1}$ and integrate over $(t_0, T) \times \Omega$, we have:

$$\int_{t_0}^{T} \int_{\Omega} \varphi^{-1} e^{-2s\eta} \nabla \cdot (e\nabla y) \partial_t y \, dx \, dt = \int_{t_0}^{T} \int_{\Omega} \varphi^{-1} e^{-2s\eta} \nabla \cdot (e\nabla y)^2 \, dx \, dt.$$
Using the fact that \( \int_{\Omega} e^{-2s\eta} \partial_t (c \nabla y) \partial_t y \, dx \, dt \).

We denote \( A := \int_{t_0}^{T'} \int_{\Omega} e^{-2s\eta} \partial_t (c \nabla y) \, dx \, dt \).

Integrating by parts \( A \) with respect to the space variable, we obtain

\[
(2.14) \quad A = \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t (c \nabla y) \, dx \, dt + 2s \lambda \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \nabla y \partial_t y \nabla \beta \, dx \, dt
\]

\[
- \lambda \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t (c \nabla y) \nabla \beta \, dx \, dt.
\]

Observe that

\[
e^{-s\eta} \partial_t (c \nabla y) = \partial_t (e^{-s\eta} \partial_t (c \nabla y)) + s e^{-s\eta} \partial_t \eta \nabla y + \frac{1}{2} e^{-s\eta} \partial_t \nabla y.
\]

Hence, the first integral of the right-hand side of (2.14) can be written as

\[
\int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t (c \nabla y) \, dx \, dt = \int_{t_0}^{T'} \int_{\Omega} c \ e^{-s\eta} \partial_t (c \nabla y) e^{-s\eta} \partial_t \nabla y \, dx \, dt
\]

\[
= \int_{t_0}^{T'} \int_{\Omega} c \ e^{-s\eta} \partial_t (c \nabla y) e^{-s\eta} \partial_t \nabla y \, dx \, dt + s \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t (c \nabla y)^2 \partial_t \nabla y \, dx \, dt
\]

\[
+ \frac{1}{2} \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t ^2 (c \nabla y)^2 \partial_t \nabla y \, dx \, dt.
\]

Using an integration by parts with respect the time variable, the first term of (2.14) is exactly equal to \( \frac{1}{2} E(T') \), since \( E(t_0) = 0 \). Therefore, the equations (2.13), (2.14) and (2.15) yield

\[
E(T') = -2s \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t (c \nabla y)^2 \partial_t \nabla y \, dx \, dt - \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t ^2 (c \nabla y)^2 \partial_t \nabla y \, dx \, dt
\]

\[
- 4s \lambda \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \nabla y \partial_t y \nabla \beta \, dx \, dt + 2 \lambda \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \nabla y \partial_t y \nabla \beta \, dx \, dt
\]

\[
+ 6 \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t ^2 (c \nabla y)^2 \nabla \beta \, dx \, dt + 2 \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t (c \nabla y) f \, dx \, dt
\]

(2.16)

\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

Now, in order to obtain an estimation to \( E(T') \), we must estimate all the integrals \( I_i \), \( 1 \leq i \leq 6 \).

Using the fact that \( |\partial_t \eta| \leq C(\Omega, \omega) T^2 \), we obtain, in first step, for the integral \( I_1 \), the following estimation

\[
|I_1| \leq Cs \int_{t_0}^{T'} \int_{\Omega} c \ e^{-2s\eta} \partial_t (c \nabla y)^2 \, dx \, dt
\]

\[
\leq C \lambda^{-2} \left[ s \lambda^2 \int_{Q} c \ e^{-2s\eta} \partial_t (c \nabla y)^2 \, dx \, dt \right].
\]
In a second step, the Carleman estimate yields
\[ |I_1| \leq C\lambda^{-2} \left[ s\lambda \int_{t_0}^{T} e^{-2s\eta |\partial_y|^2} \, dx \, dt + \int_{Q} e^{-2s\eta |f|^2} \, dx \, dt \right], \]
where \( C \) is a generic constant which depends on \( \Omega, \Gamma_0, c_{max} \) and \( T \).

As the same way, we have, for \( I_2 \), the following estimate
\[ |I_2| \leq Cs^{-1}\lambda^{-2} \left[ s\lambda \int_{t_0}^{T} e^{-2s\eta |\partial_y|^2} \, dx \, dt + \int_{Q} e^{-2s\eta |f|^2} \, dx \, dt \right]. \]

The last inequality holds through the Carleman estimate and the following inequality
\[ |\partial_t \varphi| \leq C(\Omega, \Gamma_0)T^3\phi^{-1}. \]

Using Young inequality, we estimate \( I_3 \).

We have
\[ |I_3| \leq Cs \left[ \lambda^2 \int_{Q} e^{-2s\eta |\nabla y|^2} \, dx \, dt + s^{-1} \int_{Q} e^{-2s\eta \varphi^{-1}|\partial_t y|^2} \, dx \, dt \right] \]
\[ \leq Cs \left[ \lambda \int_{t_0}^{T} e^{-2s\eta |\partial_y|^2} \, dx \, dt + \int_{Q} e^{-2s\eta |f|^2} \, dx \, dt \right], \]

For the integral \( I_4 \), we have
\[ |I_4| \leq C \left[ \lambda^2 \int_{Q} e^{-2s\eta \varphi^{-1}|\nabla y|^2} \, dx \, dt + s^{-1} \int_{Q} e^{-2s\eta |\partial_t y|^2} \, dx \, dt \right] \]
\[ \leq C \left[ \lambda \int_{t_0}^{T} e^{-2s\eta |\partial_y|^2} \, dx \, dt + \int_{Q} e^{-2s\eta |f|^2} \, dx \, dt \right], \]

where we have used, for the term containing \( |\nabla y|^2 \), the following estimate
\[ \varphi^{-1} \leq C(\Omega, \omega)T^4 \phi^{-1}/16. \]

We have immediately the following estimate for \( I_5 \)
\[ |I_5| \leq Cs \left[ \lambda \int_{t_0}^{T} e^{-2s\eta |\partial_y|^2} \, dx \, dt + \int_{Q} e^{-2s\eta |f|^2} \, dx \, dt \right]. \]

Finally, for the last term \( I_6 \), we have
\[ |I_6| \leq C \left[ s^{-1} \int_{Q} e^{-2s\eta \varphi^{-2}|\nabla \cdot (e\nabla y)|^2} \, dx \, dt + s \int_{Q} e^{-2s\eta |f|^2} \, dx \, dt \right] \]
\[ \leq C \left[ \lambda \int_{t_0}^{T} e^{-2s\eta |\partial_y|^2} \, dx \, dt + s \int_{Q} e^{-2s\eta |f|^2} \, dx \, dt \right]. \]

The last inequality holds using the following estimate
\[ \varphi^{-2} \leq C(\Omega, \omega)T^2\phi^{-1} \]

If we come back to (2.16), using the estimations of \( I_i, 1 \leq i \leq 6 \) and expanding the term \( f \), this concludes the proof of Theorem 2.7.
3 Stability Result

Theorem 3.1. Let $q$ and $\tilde{q}$ be solutions of (2.3) and (2.4) such that $c - \tilde{c} \in H^2_0(\Omega)$. We assume that Assumptions (1.4) are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma_0, T)$ such that for $s$ and $\lambda$ large enough,

$$\int_{\Omega} \varphi(T') e^{-2s\eta(T')} ([\nabla(c - \tilde{c})]^2 + [\nabla(c - \tilde{c})]^2) \, dx \, dy \leq C \int_{\Omega} e^{2s\eta(T')} (|\nabla u(T')|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) \, dx$$

$$+ C \int_{\Omega} e^{2s\eta(T')} (|\nabla u(T')|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) \, dx$$

Proof. Using the estimates (2.12), (2.10) and Proposition (2.5), we obtain

$$s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T')([\nabla \gamma]^2 + |\gamma|^2) \, dx \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') (|\nabla y(T')|^2 + |y(T')|^2) \, dx$$

$$+ C \int_{\Omega} e^{-2s\eta(T')} (|\nabla u(T')|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) \, dx$$

$$\leq C \left[ s \lambda \int_{\Omega} \int_{\Gamma_0} e^{-2s\eta(T')} |\zeta_y|^2 \, d\sigma \, dt + s \int_{Q} e^{-2s\eta(T')} \, dx \, dt \right]$$

$$+ C \left[ \lambda^{1/2} \int_{\Omega} \int_{\Gamma_0} e^{-2s\eta(T')} |\zeta_y|^2 \, d\sigma \, dt + s^{-1/2} \lambda^{-1/2} \int_{\Omega} \int_{Q} e^{-2s\eta(T')} \, dx \, dt \right]$$

$$+ C \int_{\Omega} e^{-2s\eta(T')} (|\nabla u(T')|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) \, dx.$$

So we get for $s$ sufficiently large

$$s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T')([\nabla \gamma]^2 + |\gamma|^2) \, dx \leq C s \lambda \int_{\Omega} \int_{\Gamma_0} e^{-2s\eta(T')} |\zeta_y|^2 \, d\sigma \, dt$$

$$+ C \int_{\Omega} e^{-2s\eta(T')} (|\nabla u(T')|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) \, dx,$$

and the theorem is proved. \hfill \Box

Remark

- All the previous results are available for $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ with $n \geq 3$ if we adapt the regularity properties of the initial and boundary data.
- We give a stability result for two linked coefficient ($c$ and $\nabla c$) with one observation. Note that for two independent coefficients, there is no result in the literature with only one observation.

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