Hardy-type “proofs” or paradoxes as true-implies-false gadgets

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Hardy-type arguments are presented uniformly by enumerating the orthogonality hypergraphs that underly their structure. The resulting collection of observables, if interpreted classically, induce a true-implies-false relation to the respective observable terminal points of the hypergraph. Such relations have already been used by Kochen and Specker, Stairs and Clifton, but for a single quantum in dimension three and higher, and not among entangled quanta. They can be extended to true-implies-true gadgets and even to propositional structures which are very special in that they still allow classical predictions but do no longer support any faithful classical embeddability.

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In 1992 and 1993 Lucien Hardy suggested [1, 2] what is nowadays often synonymously referred to as “Hardy’s theorem” [3, 4], “Hardy’s proof” [5–7], or “Hardy’s paradox” [8, 9]. “Hardy’s wonderful trick” [10], also called “Hardy’s beautiful example” [11, Section 23.5, p. 589f], has received a lot of attention, and attempts to make it accessible to a wider audience abound [12, 13]. Nevertheless, it might be useful to add another discussion, with an emphasis both on the structure of the argument, as well as on similar historic suggestions.

Thereby we shall employ hypergraphs introduced by Greechie, depicting contexts as smooth lines. In what follows we shall use the following terms synonymously: context, block, (Boolean) subalgebra, (maximal) clique, complete graph. In particular, Greechie has suggested to (amendments are indicated by square brackets “[…”])

... present [...] lattices as unions of [contexts] intertwined or pasted together in some fashion [...] by replacing, for example, the $2^n$ elements in the Hasse diagram of the power set of an $n$-element set with the [context aka] complete graph $K_n$ on $n$ elements. The reduction in numbers of elements is considerable but the number of remaining “links” or “lines” is still too cumbersome for our purposes. We replace the [context aka] complete graph on $n$ elements by a single smooth curve (usually a straight line) containing $n$ distinguished points. Thus we replace $n(n+1)/2$ “links” with a single smooth curve. This representation is propitious and uncomplicated provided that the intersection of any pair of blocks contains at most one atom. [14, p. 120]

In what follows we shall refer to such a general representation of observables as (orthogonality) hypergraph [15]. The term should be understood in the broadest possible consistent sense. (That is we shall not restrict our attention to three dimensions and thereby exclude loops of order two, three and four; the latter condition is equivalent to the requirement that the corresponding orthomodular poset is a lattice.) Most of our arguments will be in four-dimensional state space. An exception will be our mentioning the Specker “Käfer” bug gadget [16–18] in Figure 1(a) which has been introduced in 1965 for other purposes and used in 1967 for the first time serving as a true-implies-false construction in three-dimensional state space analogous to the Hardy gadget in four-dimensional state space.

We shall concentrate on orthogonality hypergraphs which are pasting [19] constructions [20, Chapter 2] of a homogeneous single type of contexts $K_n$ where the (maximal) clique number $n$ is fixed. In all our examples those hypergraphs have a faithful orthogonal representation [21–24], and the (maximal) clique number $n$ equals the dimension of the Hilbert space. Note that other authors use similar definitions for Greechie diagrams [25] and McKay-Megill-Pavicic diagrams (MMP) [26].

Furthermore, atomic propositions will be omitted (or only drawn lightly) if they are not essential to the argument. In particular, in three and four dimensions, given two orthogonal (in general non-collinear) vectors it is always possible to “complete” this partially defined context by a Gram-Schmidt process [27, 28]. Indeed, given two (orthogonal) non-collinear vectors, then in three dimensions the span of the “missing” vector is uniquely determined by the span of the cross product of those two vectors. (A generalized cross product of $n-1$ vectors in $n$-dimensional space can be written as a determinant; that is, in the form of a Levi-Civita symbol.) This “lack of freedom” in one dimension may result in the unfeasibility to complete an incomplete hypergraph with the present vector encoding, or even with any vector encoding in this dimensionality – in particular, whenever the missing vector is collinear to some vector occurring in the faithful orthogonal representation of the incomplete hypergraph one is attempting to complete. The easiest such counterexample is a hypergraph with three cyclic contexts $\{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}\}$ and any incomplete faithful orthogonal of its intertwining atoms such as (in what follows column vectors will be represented by the respective transposed row vectors) $1 = (0, 0, 1), 3 = (0, 1, 0), 5 = (1, 0, 0)$; any conceivable completion fails because the

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missing vectors would result in duplicities in the faithful orthogonal representation, that is, in \( 2 = 5, 4 = 1 \), and \( 6 = 3 \).

Nevertheless, in four dimensions, given at least two (orthogonal) non-collinear vectors, the two-dimensional orthogonal subspace is spanned by a continuity of bases. Therefore, in such a case there is always “enough room for breathing”; that is, for accommodating the basis vectors for properly completing any hypergraph without duplicities. I encourage the reader to try to find a faithful orthogonal representation of the cyclic triangular shaped hypergraph \( \{(1,...,4), (4,...,7), (7,...,1)\} \) in four dimensions.

In general and for arbitrary dimensions, as long as there are two or more “free” (without any strings and intertwining contexts attached) vectors per context missing from a faithful orthogonal representation of a hypergraph, its completion is always possible. Stated differently, any faithful orthogonal representation of an incomplete hypergraph can be directly extended (without reshuffling of vector components) to a faithful orthogonal representation in a completed hypergraph (eg, by a Gram-Schmidt process) if coordinatization of at least two or more non-intertwining vectors per context in that hypergraph are missing. Indeed, one may even drop an already existing coordinatization of a vector “blocking” a faithful orthogonal representation of an entire (hyper)graph if the associated atom is not intertwining in two or more contexts, and if the new freedom facilitates continuous bases instead of a single vector whose addition may result in duplicities through collinear vectors.

Because in the case of two or more “free” atoms, any completion involves or “lives in” a two- or higher-dimensional subspace; and any such subspace \( \mathbb{R}^{k \geq n-2} \) or \( \mathbb{C}^{k \geq n-2} \) of the \( n \)-dimensional continua \( \mathbb{R}^n \) or \( \mathbb{C}^n \) is spanned by a continuity of bases. A typical example is an incomplete faithful orthogonal representation of a basis of \( \mathbb{R}^4 \) rotated into a form \( \{(1,0,0,0), (0,1,0,0)\} \). Its completion is then given by the continuity of bases \( \{(0,0,\cos \theta, \sin \theta), (0,0,-\sin \theta, \cos \theta)\} \), with \( 0 \leq \theta < \pi \).

A completion should even be possible if one merely allows sets of bases which are denumerable – or even finitely but “sufficiently” many bases with respect to the hypergraph encoded. From this viewpoint four dimensions offer a much wider variety of completions if compared to the threedimensional case – indeed the difference is a continuum of subspaces versus a single subspace, a fact which is very convenient for all kinds of constructions. The completion of hypergraphs associated consisting of (non-)decomposable tensors – in particular, if one desires to maintain (non-)decomposability – is an altogether different issue which will be elaborated elsewhere.

This possibility to complete incomplete contexts is also the reason why practically all papers introducing and reviewing Hardy’s configuration operate not with the complete eight contexts including 21 atomic vertices, but merely with the nine vectors/vertices in which those eight contexts intertwine. Nevertheless, for tasks such as determining whether or not a particular configuration of observables supports or does not allow a classical two-valued state, as well as for determining the set of two-valued states and their properties (eg, separable, unital), the non-intertwining atomic propositions matter.

For the sake of being able to delineate Hardy’s rather involved original derivation \([2]\) let us stick to his nomenclature as much as possible. We shall, however, drop the particle index as it is redundant; so, for instance, Hardy’s \(|+\rangle_1 |+\rangle_2 \) will be written as \(|+\rangle |+\rangle \). We shall be later very explicit and identify the respective entities in terms of Hardy’s Ansatz, but let us study Hardy’s schematics in some generality first:

(i) Hardy starts out with a specific entangled state of two two-state particles \(|\Psi\rangle\).

(ii) He then suggests measuring two dichotomic (ie, two-valued) observables \( \hat{U} \) (exclusive) or \( \hat{D} \) on each one of the two particles. This results in four measurement configurations \( \hat{U} \otimes \hat{U}, \hat{U} \otimes \hat{D}, \hat{D} \otimes \hat{U}, \hat{D} \otimes \hat{D} \) – that is, effectively, the two-particle observable \( \hat{U} \otimes \hat{D} \) is measured “in Einstein-Podolsky-Rosen (EPR) terms of” \( \hat{U} \otimes \mathbb{1}_2 \) and \( \mathbb{1}_2 \otimes \hat{D} \).

(iii) As both of these dichotomic observables \( \hat{U} \) and \( \hat{D} \) have two possible outcomes called \( u \) and \( v \) for \( \hat{U} \) and \( c \) and \( d \) for \( \hat{D} \), respectively, there are \( 2^2 \times 2^2 = 2^4 = 16 \) different outcomes that denoted by the ordered pairs \( uu, uv, uc, cd, du, dv, dc, dd \).

(iv) From these 16 outcomes one can form 5 groups of (incomplete if not all atoms or vertices are specified; yet as earlier discussed a completion is straightforward if desired) contexts which consist of simultaneously measurable and mutually exclusive observables, namely \( \{dd, ..., cv\}, \{dd, ..., vc\}, \{cv, vu, uu, dv\}, \{vc, uv, uu, vd\}, \{vu, ..., uv\} \).

(v) Finally, one “ties together” this collection of five contexts with the (projection) observable corresponding to the original entangled state \(|\Psi\rangle\) introduced in (i) by the three (incomplete) contexts \( \{dd, ..., \Psi\}, \{uu, ..., \Psi\}, \{dv, ..., \Psi\} \).

As a result these (incomplete) contexts, when pasted \([19]\) together at their respective intertwines result in a collection of eight (incomplete) contexts

\[
\begin{align*}
\{dd, ..., cv\} &= \{dd, 8, 9, cv\}, \\
\{dd, ..., vc\} &= \{dd, 11, 12, vc\}, \\
\{cv, vu, uu, dv\} &= \{cv, vu, uu, dv\}, \\
\{vc, uv, uu, vd\} &= \{vc, uv, uu, vd\}, \\
\{vu, ..., uv\} &= \{vu, 18, 19, uv\}, \\
\{vd, ..., \Psi\} &= \{vd, 2, 3, \Psi\}, \\
\{uu, ..., \Psi\} &= \{uu, 20, 21, \Psi\}, \\
\{dv, ..., \Psi\} &= \{dv, 16, 17, \Psi\}
\end{align*}
\]

whose orthogonality hypergraph is depicted in Figure 1(a).

In what follows we shall prove that:

(i) Hardy’s configuration \((1)\) allows a classical interpretation as it supports a separable set of two-valued states. A “canonical” classical representation will be explicitly enumerated.
(ii) All classical interpretations of Hardy’s configuration (1) enumerated in (i) predict that, if the system is prepared in state \(\Psi\), then the observable \(dd\) never occurs. That is, Hardy’s setup is a gadget graph [16–18] with a “true-implies-false (classical) set of two-valued states” (TIFS). Indeed, it is one out of three minimal non-isomorphic true-implies-false configurations in four dimensions [6, Figure 4(a)].

First, Hardy’s configuration (1) allows a classical interpretation because the set of all 186 two-valued states it supports is separating. Therefore, by Kochen and Specker’s Theorem 0 [29], the structure of observables underlying it can be embedded in some Boolean algebra, which indicates classical representability.

An explicit construction of a classical model of a propositional structure corresponding to Hardy’s 1993 configuration [2] is enumerated in Table I. Its realization is in terms of 8 partitions (corresponding to the 8 contexts) of the index set \(\{1, 2, \ldots, 185, 186\}\) of 186 two-valued states. The elements of the partitions corresponding to the 21 atomic propositions which are obtained from “completing” the context as enumerated in Equation 1 are the index sets of all two-valued states which obtain the value “1” on the respective atoms. A detailed description of this construction can be found in Refs. [30–32].

Next, we shall elaborate on a classical prediction which is violated by quantum predictions: If \(\Psi\) is assumed to be true – that is, if a classical system is prepared (aka pre-selected) in the state corresponding to observable \(\Psi\) – then the outcome corresponding to the observable \(dd\) cannot occur.

For a proof by contradiction depicted in Figure 2 suppose wrongly that both \(\Psi\) as well as \(dd\) were both true simultaneously. Then by the standard admissibility criteria for two-valued states [33, 34] (also denoted as completeness and exclusivity [7, 35, 36]), \(cv = vc = vd = dv = uu = 0\), enforcing \(vu = uv = 1\) which contradicts admissibility (completeness and exclusivity).

The only remaining possibility is that \(\Psi\) and \(dd\) have opposite values if one of them is true (they still may both be 0). Therefore, any two-valued state for which \(\Psi\) is 1 – that is, in which the classical observable corresponding to \(\Psi\) occurs – must classically result in non-occurrence of the outcome corresponding to the observable \(dd\); and vice versa. Such relational properties between an input and an output ports of gadget graphs [37] have been called 1-0-property [38] or true-implies-false set of two-valued states (TIFS) [6].

Historically, the first true-implies-false gadget seems to have been introduced by Kochen and Specker [39, Figure 1, p. 182] and used by them as a subgraph of \(\Gamma_1\) [29, p. 68] in three dimensions. Its orthogonality hypergraph is depicted in Figure 1(b). Pitowsky called this gadget “cat’s cradle” [40, 41]. See also Figure 1 in [33, p. 123] (reprinted in Ref. [42]), a subgraph in Figure 21 in [43, pp. 126-127], Figure B.1 in [44, p. 64], [45, pp. 588-589], Figure 2 in [46, p. 446], and Figure 2.4.6 in [47, p. 39] for early discussions of the true-implies-false prediction.

The full nuances of predictions are revealed when the classical probabilities are computed. As the classical probability distributions are just the convex combinations of all two-valued states [48, Chapter 2], it is easy to read them off from the canonical partition logic enumerated in Table I. In particular, the true-implies-false gadget behavior at the terminals \(\Psi\) and \(dd\) can be directly read off from

\[
P_{\Psi} = \sum_{i \in \Psi} \lambda_i = \sum_{i=1}^6 \lambda_i, \quad \text{and} \quad P_{dd} = \sum_{i \in dd} \lambda_i
\]

with \(\lambda_i \geq 0\), and \(\sum_{i=1}^{186} \lambda_i = 1\).

Since the intersection of the index sets \(\Psi\) and \(dd\) are empty, \(P_{dd} = 0\) whenever \(P_{\Psi} = 1\), and vice versa. For the sake of the example all six two-valued measures assigning 1 to \(\Psi\) are depicted in Figure 3.

One equivalent alternative way to characterize the classical probabilities completely would be to exploit the Minkowski-Weyl “main” representation theorem [49–55] and consider the classical convex polytope spanned by the 186 21-dimensional vectors whose components are the values in \(\{0, 1\}\) of the two-valued states on the atomic propositions of the Hardy gadget and, from these vertices (V-representation), compute the 35 half-spaces that are the bounds of the polytope (H-representation) [48, 56]. But due to space restrictions we omit this discussion, although it might reveal quantum violations of Boole’s (classical) “conditions of experience” [57].

Let us now turn to the quantum realization in terms of some faithful orthogonal representations of the Hardy gadget. Hardy’s original computation is rather involved, but for the sake of delineating it we shall mostly stick to the nomenclature of the 1993 paper [2]. There will be two entangled two-state particles involved. Per particle we shall consider three orthonormal bases of two-dimensional Hilbert space (corresponding to the two orthogonal (ie exclusive) states of each

![Orthogonality hypergraphs of (a) the Hardy gadget with 8 contexts and 21 atoms \{\{dd, 8, 9, cv\}, \{dd, 11, 12, vc\}, \{cv, vu, uu, dv\}, \{vc, vv, uu, vd\}, \{vu, 18, 19, uv\}, \{vv, 2, 3, v\}, \{vu, 20, 21, \Psi\}, \{dv, 16, 17, \Psi\}\}; (b) rendition of the true-implies-false Specker bug/cat’s cradle gadget with 7 contexts and 13 atoms \{\{a_8, a_9\}, \{a_8, a_7\}, \{a_6, a_4, a_3\}, \{a_7, a_5, a_2\}, \{a_4, a_3\}, \{a_2, a_1\}, \{a_3, a_7\}\}.](image-url)
TABLE I. Partition logic representing classical probabilities of the Hardy configuration [2], whose intertwined contexts are enumerated in Equation 1, obtained from the separating set of all 186 two-valued states it supports. Note that the intersection of $\Psi \cap dd = \{1, 2, 3, 4, 5, 6\} \cap \{11, 16, 21, 26, 55, 60, 73, 78, 83, 88, 117, 122, 135, 140, 145, 150, 155, 164, 173, 182\} = \emptyset$ is empty, yielding true-implies-false relations among $\Psi$ and $dd$ and vice versa, respectively.

| Particle if isolated and not entangled | $B_1 = \{\{\uparrow\}, \{\downarrow\}\}$ by $[58]$ |
|---------------------------------------|---------------------------------|
| $B_2 = \{\{|u\rangle, \{|v\rangle\}$ by $[58]$ |
| $B_3 = \{\{|c\rangle, \{|d\rangle\}$ by $[58]$ |

The components of the respective unitary transformations “rotating” these orthonormal bases into each other are defined by

$$B_1 \leftrightarrow B_2 : \quad f_j = U_{jk}^{12}e_i, \quad \text{and} \quad e_j = (U_{jk}^{12})^\dagger f_i$$

$$B_2 \leftrightarrow B_3 : \quad g_k = \sum_{i=1}^2 U_{ki}^{23}f_j, \quad \text{and} \quad f_k = \sum_{i=1}^2 (U_{ki}^{23})^\dagger g_j$$

$$B_1 \leftrightarrow B_3 : \quad g_k = \sum_{i=1}^2 U_{ki}^{23}e_i, \quad \text{and} \quad e_k = \sum_{i,j=1} U_{ki}^{12} (U_{kj}^{23})^\dagger g_j.$$
that, in addition to (vice versa), one needs to define those transformations such that, in order to obtain a contradiction with the classical prediction “if \( \Psi \) is true then \( dd \) must be false” or “if an system is prepared/(pre)selected in state \( \Psi \) then any event/outcome associated with \( dd \) must not occur” (and vice versa), one needs to define those transformations such that, in addition to (6),

\[
\langle \Psi | dd \rangle \neq 0 \text{ and “as great as possible”}. \tag{7}
\]

Hardy did exactly that; that is, he defined

\[
(U^{12})^\dagger = -\frac{i}{\sqrt{\alpha + \beta}} \left( \frac{\sqrt{\beta}}{\alpha} - \frac{\sqrt{\alpha}}{\beta} \right),
\]

\[
(U^{23})^\dagger = \frac{1}{\sqrt{1 - \alpha \beta}} \left( \frac{\sqrt{\alpha \beta} - \alpha + \beta}{\alpha - \beta} \right), \tag{8}
\]

\[
(U^{13})^\dagger = (U^{12})^\dagger \cdot (U^{23})^\dagger.
\]

Suppose, without loss of generality, that the first basis \( B_1 \) in (3) is identified with the Cartesian basis; that is, by \(| + \rangle = (1, 0)\) and \(| - \rangle = (0, 1)\). Consequently, the vectors of the other bases \( B_2 \) and \( B_3 \) are obtained by applying the respective transformations (4) and (8):

\[
\begin{align*}
|u\rangle &= i \frac{(\sqrt{1 - \alpha^2}, \sqrt{\alpha})}{\sqrt{1 - \alpha^2 + \alpha}}, \\
|v\rangle &= i \frac{(\sqrt{\alpha} - \sqrt{1 - \alpha^2})}{\sqrt{1 - \alpha^2 + \alpha}}, \\
|c\rangle &= i \frac{(\alpha^{3/2}, (1 - \alpha^2)^{3/4})}{\sqrt{\alpha^3 - \sqrt{1 - \alpha^2}^2 \alpha^2 + \sqrt{1 - \alpha^2}^2}}, \\
|d\rangle &= i \frac{(1 - \alpha^2)^{3/2}}{\sqrt{\alpha^3 - \sqrt{1 - \alpha^2}^2 \alpha^2 + \sqrt{1 - \alpha^2}^2}}.
\end{align*}
\]

Since we are only dealing with pure states and the associated observables we shall just represent the states as well as the atomic propositional observables as vectors which are the sum of the “delineated” Kronecker products [60, Chapter 1]; eg., \(|\Psi\rangle = \alpha (1, 0) \otimes (1, 0) - \sqrt{1 - \alpha^2} (0, 1) \otimes (0, 1) = (\alpha, 0, 0, -\sqrt{1 - \alpha^2})\).

By applying the transformations (4) \(|\Psi\rangle\) can be rewritten in terms of (i) the second basis \( B_2 \) for the first particle, and the second basis \( B_2 \) for the second particle (ii) the second basis \( B_2 \) for the first particle, and the third basis \( B_3 \) for the second particle, (iii) the third basis \( B_3 \) for the first particle, and the second basis \( B_2 \) for the second particle, and (iv) the third basis
As can be readily read off from these presentations of $|\Psi\rangle$ the conditions (6) and desideratum (7) are satisfied: (10) has no term proportional to $|uu\rangle$, (11) has no term proportional to $|vd\rangle$, (12) has no term proportional to $|dv\rangle$, and (13) has a term proportional to $|dd\rangle$.

To complete Hardy’s original argument we compare the classical prediction of “zero outcome” (non-occurrence) for observable $dd$ to the quantum prediction probability

$$|\langle dd|\Psi\rangle|^2 = \left\{ \frac{\alpha (\sqrt{1 - \alpha^2} + \alpha)}{\alpha \sqrt{1 - \alpha^2} - 1} \right\}^2$$

obtained from preparing (aka preselecting) two entangled particles in state $|\Psi\rangle$ and measuring (eg by postselection) the non-vanishing probability to find them in state $|dd\rangle$ (thus contradicting aforementioned classical predictions). $|\langle dd|\Psi\rangle|^2$ acquires its maximal value $\frac{1}{2}(5\sqrt{5} - 11) \approx 0.09$ at $\alpha_\pm = \sqrt{1 \pm 6\sqrt{5} - 13}/2$, slightly more than the maximal violation for the non-entangled pure three-dimensional “minimal” true-implies-false case (the Specker bug [6, 32, 39] depicted in Figure 1b) performance of $1/9 \approx 0.1$ [27, 43, 44, 61, 62].

The parametrization of Hardy’s (minimal in four dimensions [6]) true-implies-false gadget depicted in Figure 1a in terms of four-dimensional vectors appears rather ad hoc and mainly motivated by what is sometimes referred to as “demonstrations of non-local contextuality”; that is, the “spread” of the relational information [63] among (hopefully spatially [64]) separated pairs of particles. Indeed, presently no general analytic construction for finding even a single faithful orthogonal representation of a (hyper)graph (if any) exists, let alone a method for finding all such representations. Nevertheless, other such faithful orthogonal representations of the Hardy gadget have been suggested and can be generated in extenso with automated searches.

In what follows we shall, therefore, enumerate a few alternative faithful orthogonal representations of the Hardy gadget. The first type can almost directly be read off from the orthogonality hypergraph of the Hardy gadget depicted in Figure 1a.

Note that the two “central full contexts” $\{|cv\}, |vu\rangle, |uu\rangle, |dv\rangle\}$ and $\{|vc\}, |uv\rangle, |uu\rangle, |vd\rangle\}$ intertwined at one common element $|uu\rangle$ are actually “generated” by the flattened tensor products of two non-identical two-dimensional contexts representable by the two orthonormal bases $\{|c\rangle, |d\rangle\}$ and $\{|u\rangle, |v\rangle\}$. So all that is necessary is to make sure that $|\Psi\rangle$ is orthogonal to three vectors $|vd\rangle, |uu\rangle$, and $|dv\rangle$ of four-dimensional space (and no multiplicities occur), as already encoded in Eqs. (6):

$$|\Psi\rangle \propto (d_2u_2^2v_1 - 2d_2u_1u_2v_2 + d_1u_2^2v_2, d_2u_1^2v_2 - d_1u_2^2v_1, d_2u_1^2v_2 - d_1u_2^2v_1, 2d_1u_1u_2v_1 - d_2u_1^2v_1 - d_1u_2^2v_2),$$

where $y_i$ stands for the $i$th component of the vector $y$ with respect to some basis.

In order to be able to claim non-locality additional constraints can be required from the components of $|\Psi\rangle$. Suppose one desires $|\Psi\rangle$ to be entangled then the product of its outer components should not be equal to the product of its inner components [60, p. 18]; that is, $\Psi_1\Psi_4 \neq \Psi_2\Psi_3$ because every non-entangled decomposable product state of two vectors $(a, b)$ and $(c, d)$ is of the (delineated) form $(x_1 = ac, x_2 = ad, x_3 = bc, x_4 = bd)$, so that, because of commutativity of scalars, $x_1x_4 = (ac)(bd) = abcd = (ad)(bc) = x_2x_3$. If one prefers the tensor product in matrix notation then $x_1 = ac, x_2 = ad, x_3 = bc, x_4 = bd$ and the criterion for non-entanglement (ie, factorizability, decomposability) is a vanishing determinant $x_1x_4 - x_2x_3 = 0$. By applying this constraint to Eq. (15) results in

$$(d_2u_1 - d_1u_2)(u_1v_2 - u_2v_1) \neq 0.\quad (16)$$

The first two rows of Table II present two ad hoc configurations satisfying this “inseparability” constraint.

Conversely, it might be desirable to keep $|\Psi\rangle$ separable; that is, all entities should be in a product state. In this case, the product of the outer components of $|\Psi\rangle$ should be equal to the product of its inner components [60, p. 18]; that is, $\Psi_1\Psi_4 = \Psi_2\Psi_3$. This results in the constraint from Eq. (15):

$$d_1 = d_2u_1/u_2, \text{ with } u_2 \neq 0.\quad (17)$$
The seventh row of Table II presents an ad hoc configuration \( u = (1, 2), v = (2, -1), c = (3, -\frac{1}{2}) \), and \( d = (\frac{1}{2}, 3) \) satisfying this “separability” constraint.

The last row of Table II contains a faithful orthogonal representation of the Hardy gadget in which all intertwine vectors appear entangled because for any vector the number of components with imaginary units \( i \) and \(-i\) is odd (that is, either one or three). It has been obtained with a more general, heuristic algorithm developed by McKay, Megill and Pavličič [65] than the ones previously described. So, in summary, with regards to decomposibility, the Hardy gadget allows all types of faithful orthogonal representations of its intertwining atoms: ones which have entangled or non-entangles states at their endpoints; and ones which use entangled or non-entangled states corresponding to their intertwining atomic propositions. From now on the observables need not be formed by some sort of composition, and therefore two symbols such as “uv” should only be understood as a label.

What remains to be mentioned are extensions of the Hardy gadget which have a classical true-implies-true structure, as already employed in Kochen and Specker’s \( \Gamma_2 \) [29] and discussed in Ref. [6]. A further escalation is a combo of these true-implies-true gadgets, similar to Kochen and Specker’s \( \Gamma_3 \), which delivers a truly non-classical performance on the algebraic level of the propositional observables (and not just probabilistic predictions based upon classical probabilities) – because, unlike the Hardy and its extended true-implies-true gadgets, those observables can no longer be embedded into any Boolean algebra [29, Theorem 0].

Figure 4 depicts the extension of the Hardy gadget which delivers a classical true-implies-true prediction at its terminal points \( \Psi \) and \( N \). A faithful orthogonal representation of the extended Hardy gadget can be obtained ad hoc by the heuristic algorithm VECFIN [65] in the coordinate basis \( \{0, \pm1, \pm2, \pm3\} \) and the \( \neg n \) option, which is capable of finding “almost all” vectors, including the true-implies-true terminal points \( \Psi \) and \( N \) ex machina, and (for this coordinate basis) needs a little helping hand (or the additional component basis elements \( \{-3, 5, 7, \sin \theta, \cos \theta\} \) with \( \theta \neq n\pi/4 \), \( n \in \mathbb{Z} \) to find the complete set, given by \[ \Psi = (0, 1, 1, -1), \quad 2 = (2, 2, -1, 1), \quad 3 = (3, -2, 1, -1), \quad vd = (0, 0, 1, 1), \quad uu = (1, 0, 0, 1), \quad \text{cv} = (0, 1, 1, 0), \quad 8 = (3, 1, -1, 2), \quad 9 = (-2, 1, -1, 2), \quad dd = (0, -1, 1, 1), \quad 11 = (3, -2, -1, -1), \quad 12 = (2, 2, 1, 1), \quad \text{vc} = (0, 0, 1, -1), \quad uv = (0, 1, 0, 0), \quad dv = (0, 1, -1, 0), \quad 16 = (-2, 1, 1, 2), \quad 17 = (3, 1, 1, 2), \quad 18 = (\cos \theta, 0, \sin \theta, 0), \quad 19 = (-\sin \theta, 0, \cos \theta, 0), \quad 20 = (0, 4, -3, 1), \quad 21 = (0, 2, 5, 7), \quad M = (0, 1, 0, 1), \quad N = (0, 1, 2, -1), \quad O = (2, -1, 2, 1), \quad P = (3, 1, -2, -1), \quad \text{where } \theta \neq n\pi/4, \quad n \in \mathbb{Z} \). (Actually, the original coordinatization suggested for atom 20 was \( 0, 1, -1, 0 \) but a completion would have resulted in duplicities, namely \[ 21 = (0, 1, 1, 2) = dv; \] and therefore the original suggestion had to be dropped.) Although we do not concentrate on maximal violations of classical predictions by quantum probabilities for reasons that will be mentioned later, it is worth noting that, as \( |\langle \psi | N \rangle|^2 = 8/9 \), the quantum violation of the classical predictions will, in this particular configuration, occur in one out of nine times; that is, with probability 0.1 (proper normalization is always assumed). Fortunately, if one concentrates on the quantum signal for observable \( |dd \rangle \langle dd | \) then one obtains the same quantum prediction \( |\langle \psi | dd \rangle|^2 = 1/9 \) that the outcome occurs although classically it should never occur.

One way to proceed would be what Kochen and Specker did with their true-implies-true gadget \( \Gamma_4 \), and serially compose them at their respective (properly parametrized) terminal points often enough to obtain \( \Gamma_2 \), which renders a complete contradiction with exclusivity [29]. Instead of this head-on strategy for obtaining complete contradictions with classical non-contextual hidden variable models we shall use a more subtle approach and consider a hypergraph which, again
TABLE II. Tabellation of some faithful orthogonal representations of the Hardy gadget. Two missing vectors per context as well as normalizations can be completed with a little effort. Labels of the form “ab” should not be understood as product states but have been used merely to conform to Hardy’s original nomenclature.

| ψ   | dv  | vd  | uu  | uv  | vu  | cv  | vc  | dd  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| CEG-A 1996 [4] | (1,−1,−1) | (1,0,0) | (1,1,0) | (0,0,1) | (0,0,1) | (0,1,0) | (0,1,0) | (1,−1,0) |
| Cabello 1997 [5] | AB | β+ | β− | α | δ+ | δ− | γ+ | γ− | ab |
| BBCGL 2011 [8] | Ψ   | a1b1 | a1b2 | a2b1 | a2b2 | a1b2 | a2b1 | a1b1 |
| VECFIND [65] | (0.1,−2,√2) | (1,−2,√2,0) | (1,−1,−1,0) | (1,0,1,0) | (1,0,0,1) | (0,1,0,0) | (0,0,1,0) | (1,−1,−1) |

in analogy with Kochen and Specker’s Σ3 in three dimensions, cannot be classically embedded in a Boolean algebra. The construction uses two true-implies-true extended Hardy gadgets to construct two pairs of observable propositions which cannot be differentiated by classical two-valued measures – and thus by any classical probability distributions – although “plenty” such two-valued states still exist (but their set is “too meagre” to allow mutual separability of all pairs atomic propositions).

I have been unable to find a faithful orthogonal representation of an extension of the “original” version of the Hardy gadget, as depicted in Figure 4(a). Nevertheless, as it turns out this task can be performed with a slight modification of Hardy’s gadget introduced in Figure 4(b) of Ref. [6], in which the original context {ψ,...,uu} is “relocated” or “reshuffled” into the context {uu,...,dd}. The resulting gadget not only has less atoms but, most importantly, has a less tight “orthogonality backbone” structure, depicted in Figure 6(b), of just two contexts intertwined in a single atom M, namely \{uu.dd1.M,Ψ\}, \{uu.dd2,M,Ψ\}, as compared to the tight configuration resulting from the composition of two of Hardy’s original gadgets \{Ps1,...,uu\}, \{uu.dd1,M,Ψ\}, \{uu.dd2,M,Ψ\}, \{Ps2,...,uu\}, depicted in Figure 6(a).

I have not been able to find faithful orthogonal representations of the latter but VECFIND [65] with the component basis \{0,±1,2,−3,4,5\} yields an ad hoc coordinatization of the intertwine atoms \ψ = (1,0,0,0), \uu = (1,−1,1,−1), \vv = (−3,−1,−1,−1), \dd = (1,−3,0,0), \vv = (−3,−1,−1,−1), \uu = (1,−1,1,−1), \vd = (0,2,−1,1), \uu = (0,0,1,1), \vv = (1,−1,1,−1), \dd = (1,−3,0,0), \vv = (1,−1,1,−1), \dd = (1,−3,0,0), \uu = (0,2,−1,1), \uu = (1,−1,1,−1), \dd = (1,−3,0,0), \uu = (0,2,−1,1), \uu = (1,−1,1,−1), \vd = (0,2,−1,1), \uu = (0,0,1,1), \vv = (1,−1,1,−1), \dd = (1,−3,0,0), \uu = (0,2,−1,1), \uu = (1,−1,1,−1), \vd = (0,2,−1,1), \uu = (0,0,1,1), \vv = (1,−1,1,−1), \dd = (1,−3,0,0), \uu = (0,2,−1,1), \uu = (1,−1,1,−1), \vd = (0,2,−1,1), \uu = (0,0,1,1), \vv = (1,−1,1,−1), \dd = (1,−3,0,0), \uu = (0,2,−1,1), which can be readily completed into a faithful orthogonal representation of the hypergraph depicted in Figure 5(b). Note that in this particular configuration, because of inseparability, the classical prediction to find a particle prepared in a state ψ in the state Ψ is one (certainty), whereas quantum mechanics predicts non-occurrence of the elementary propositional observable ℙψ ℙΨ given a preselected, prepared state ℙΨ with probability \|ℙΨ ℙΨ\|^2 = 9/10: that is, the violation of the classical prediction by quantum mechanics occurs in this case in one out of ten experimental runs.

There are two reasons why not much emphasis has been laid, and efforts dedicated to “optimize” or even “maximize” this sort of performance: (i) because there exist already true-implies-{true,false} gadgets which yield high performance (in terms of disagreements with classical predictions) in three dimensions [18, 37, 66, 67]; and (ii) because suppose any vectors corresponding to pre-and postselected states are fixed, then it is always possible to find any kind of conforming or disagreeing classical-versus-quantum behavior. As I have pointed out, these kind of statements are contingent of the gadget consisting of mostly counterfactual observables “in the mind” of the observer [68, 69]. Nevertheless, any such considerations raise fascinating, challenging issues in a variety of fields which might have been perceived unrelated so far: graph theory, (linear) algebra, functional analysis, geometry, automated theorem proving and – last but not least – quantum physics and quantum information (processing) technology.

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[1] Lucien Hardy, “Quantum mechanics, local realistic theories, and lorentz-invariant realistic theories,” Physical Review Letters 68, 2981–2984 (1992).
[2] Lucien Hardy, “Nonlocality for two particles without inequal-
FIG. 6. Hypergraphs of the “orthogonality backbones” of (a) Figure 5(a), and (b) Figure 5(b) supporting the two-valued states depicted in Figure 5(c) and (d), respectively.
