Robust Pareto Set Identification with Contaminated Bandit Feedback
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Abstract—We consider the Pareto set identification (PSI) problem in multi-objective multi-armed bandits (MO-MAB) with contaminated reward observations. At each arm pull, with some probability, the true reward samples are replaced with the samples from an arbitrary contamination distribution chosen by the adversary. We propose a median-based MO-MAB algorithm for robust PSI that abides by the accuracy requirements set by the user via an accuracy parameter. We prove that the sample complexity of this algorithm depends on the accuracy parameter inverse squarely. We compare the proposed algorithm with a mean-based method from MO-MAB literature on Gaussian reward distributions. Our numerical results verify our theoretical expectations and show the necessity for robust algorithm design in the adversarial setting.

Index Terms—Multi-objective multi-armed bandits (MOMABs), robust optimization, adversarial attack, median based optimization, Pareto set identification

I. INTRODUCTION

Multi-armed bandit (MAB) problem involves decision making under uncertainty in which a finite amount of resources are allocated between a limited number of options (arms) in order to optimize gain over time (or equally minimize regret). In the classical setting, each arm is associated with a reward distribution that is unknown or only partially known at the time of allocation and the information on distributions increase as more observations are made over time [11]–[6].

Over the last decades, MAB algorithms have been used in a broad range of applications such as medical treatment allocation [7], financial portfolio design [8], [9], adaptive routing [10], cellular coverage optimization [11], news article recommendation [12], and online advertising [13]. Due to the security concerns in these applications, adversarial MABs have attracted considerable attention [14], [15]. A variety of adversary models are considered that come with different restrictions on the adversary. One of the widely studied attack model is the attack that has a bounded attack value. A notable attack model is the adversarial attack model. In this model, an attack can occur at every round with fixed probability. Unlike the attack models mentioned before, this model does not put any restrictions on the attack value. This attack model is considered in [14], [23], [24], and also in this study.

Multi-objective MABs (MO-MABs) are another significant extension of the MAB setting where multiple, possibly conflicting objectives are optimized simultaneously. Unlike the single objective optimization, in multi-objective optimization (MOO) problems, it is not possible to identify a single optimal arm in most of the cases. Therefore, in MOO, the aim is to identify the Pareto optimal set of arms which is the generalization of single objective optimality to the multi-objective case (see Section III for definition of the Pareto optimality).

MO-MABs are extensively studied in the non-adversarial, stochastic settings for regret minimization and the Pareto set identification (PSI) problems. PSI problem in the MO-MAB setting is considered in [25], where an elimination-based adaptive arm sampling algorithm is proposed. Upper and lower bounds on the sample complexity are analyzed. In [26], MO-MAB is investigated from the regret minimization perspective using scalarization based methods. These methods turn the multi-objective problem into a single-objective problem, which can be solved efficiently via well-known single-objective bandit algorithms. Multi-objective variants of Thompson sampling and Knowledge Gradient (KG) algorithms are investigated in [27] and [28]. Regret minimization in multi-objective contextual bandit problems is studied in [29] and [30]. In [26], MO-MAB problem is considered with correlated objectives. The authors of this work design a variant of the well-known UCB policy with the aim of minimizing the cumulative regret.

Another line of work [31]–[36] focuses on PSI with Gaussian process priors and propose acquisition strategies to utilize prior induced dependencies between mean arm rewards.

A. Contribution and Comparison with Related Works

In the literature, single objective MAB problem is studied under various attack models. In most of the MAB literature, the goal is to identify the arm that corresponds to the reward distribution with the highest first order statistic (mean) [37]. This is only justified when the attack model is assumed to...
TABLE I
Comparison with the related work.

| Work | Setting | Goal | Adversary | Bound |
|------|---------|------|-----------|-------|
| [23] | MO-MAB | PSI  | Adv. free | Samp. com. |
| [23] | MAB     | Best arm Id. | Oobl.,Presc.,Mal. | Samp. com. |
| [16] | MAB     | Regret min. | Bounded attack | Cum. reg. |
| [25] | MO-GP   | Regret min. | Adv. free | Cum. reg. |
| [26] | MAB     | PSI    | Adv. free | Samp. com. |
| [14] | MO-GP   | PSI    | Adv. free | Samp. com. |
| [16] | MAB     | Regret min. | Prescient like | Cum. reg. |
| [21] | MAB     | Regret min. | Prescient like | Cum. reg. |

Ours MO-MAB PSI Oobl.,Presc.,Mal. Samp. com.

be bounded in value since mean cannot be estimated from samples contaminated with an attack that has unbounded value. However in many applications, it is more plausible to restrict the probability of occurrence of an attack instead of the attack value. For instance, consider the bit corruptions that might occur on a digitally stored dataset that arbitrarily alters the value of one or multiple data points. In this case, the bounded attack probability model is more suitable to capture the effect of the adversary compared to the bounded attack value model.

As shown in the single objective adversarial MAB studies that consider bounded probability attack model, the median is a robust measure against the unbounded attacks [24]. In parallel, we establish the median statistic as a robust measure in the multi-objective case and propose a method to solve the PSI in the adversarial setting. The detailed comparison of our work with the prior work from the literature is provided in Table I.

Our contribution can be summarized as follows: (i) We propose a robust algorithm that can approximate the set of Pareto optimal arms under adversarial attacks. (ii) We provide sample complexity bounds for our algorithm that inverse squarely depend on the accuracy parameter $\alpha$. We show that when the reward distributions are subgaussian, our sample complexity bound has the same dependence on $\alpha$ as Algorithm 1 of [23] up to logarithmic factors. (iii) We conduct experiments on synthetic and real world data that verify the robustness of our algorithm in the adversarial setting.

### B. Organization

In Section II we introduce the necessary notation. In Section II we formulate the adversarial MO-MAB problem. In Section IV we describe our median-based Pareto selection algorithm. In Section V we prove that the proposed method satisfy the accuracy and coverage requirements defined in Section III-D and prove a sample complexity bound. In Section VI we give the experimental results. Conclusions of the research and future directions are highlighted in Section VII.

### II. Notation

We denote the Bernoulli distribution with parameter $\rho \in [0,1] \text{ by } \text{Ber}(\rho)$. We denote the set of positive integers by $\mathbb{N}_+$ and the set $\{1, ..., n\}$ by $[n]$ for $n \in \mathbb{N}$. We use the short hand notation $[a \pm b]$ to denote the interval $[a - b, a + b]$. We use the abbreviation w.h.p. to denote with high probability and cdf to denote cumulative distribution function.

Let $F$ represent a cdf and $X$ be a random variable such that $X \sim F$. We denote the right and left quantile functions of $X$ by $Q_RF(p) := \inf\{x \in \mathbb{R} : F(x) > p\}$ and $Q_LF(p) := \inf\{x \in \mathbb{R} : F(x) \geq p\}$ for $p \in [0,1]$, respectively. The following notations are borrowed from [24]. The set of medians is denoted by $m_1(F) := [Q_{L}F(\frac{1}{2}), Q_{R}F(\frac{1}{2})]$. We also use the shorthand $m_1(X)$ to denote $m_1(F)$. In the case where median is unique, we use $m_1(F)$ to denote the median instead of the singleton set containing this value. Note that $m_1(F)$ can be considered to be the robust analogue to mean. We denote the empirical median of a sequence of samples $x_1, \ldots, x_n \in \mathbb{R}$ as $\hat{m}_1(x_1, \ldots, x_n)$. If $n$ is odd, this corresponds to the middle value in the sequence. If $n$ is even, it corresponds to the average of two middle values.

Suppose $x$ is an $M$-dimensional vector. We denote the $i$th element of $x$ by $x^i$. Consider another $M$-dimensional vector $y$. We use the notation $x \preceq y$ to denote that the vector $x$ is weakly dominated by vector $y$, or equivalently, $\forall i \in [M] \colon x^i \leq y^i$. Also we use the notation $x \not\preceq y$ to denote that $x$ is not weakly dominated by $y$, or equivalently, $\exists i \in [M] : x^i > y^i$.

Suppose $a$ is a scalar and $x$ is a vector. We use the notation $x + a$ and $x - a$ to denote the summation of each element of $x$ with $a$ and the subtraction of each element of $x$ by $a$ respectively. We also define the ordering relations between scalars and vectors similar to the ones defined between the vectors above: $x \preceq a$ denotes that $\forall i : x^i \leq a^i$; $a \preceq x$ denotes that $\forall i : a^i \leq x^i$; $a \not\preceq x$ denotes that $\exists i : a^i > x^i$ and $x \not\preceq a$ denotes that $\exists i : x^i > a$.

### III. Problem Formulation

We consider the multi-objective optimization problem with $M$ objectives and a finite number of arms indexed by $i = 1, \ldots, K$. The cdf of reward distributions are denoted by $\{F_{i,n}^{d}\}_{i \in [K], d \in [M]}$ and the cdf of contamination distributions by $\{G_{i,n}^{d}\}_{i \in [K], d \in [M], n \in \mathbb{N}}$. The corresponding random variables are denoted by $Y_{i,n}^{d} \sim F_{i,n}^{d}$ and $Z_{i,n}^{d} \sim G_{i,n}^{d}$ respectively. Note that the contamination distributions depend on sampling step $n$ since the adversary is free to pick a new distribution at every sampling step. The contamination probability is fixed across all arms and objectives and denoted by $\epsilon \in (0, \frac{1}{2})$. Bernoulli random variable corresponding to arm $i$ and objective $d$ that determines whether a contamination occurs at sampling step $n$ is denoted by $B_{i,n}^{d} \sim \text{Ber}(\epsilon)$. The contaminated reward distribution at sampling step $n$ that belongs to arm $i$ and objective $d$ is denoted by $F_{i,n}^{d}$ and the corresponding random variable by $Y_{i,n}^{d}$. If a contamination occurs at an arm pull, the observed reward is sampled from the contamination distribution instead of the true reward distribution. Formally,

$$Y_{i,n}^{d} = \begin{cases} Y_{i,n}^{d} & \text{if } B_{i,n}^{d} = 0 \\ Z_{i,n}^{d} & \text{if } B_{i,n}^{d} = 1 \end{cases}$$

Equivalently, $Y_{i,n}^{d} = (1 - B_{i,n}^{d})Y_{i,n}^{d} + B_{i,n}^{d}Z_{i,n}^{d}$. We define median of interest $m_1^{d}$, corresponding to cdf $F_{i,n}^{d}$, as the mean.
of right and left $\frac{1}{2}$-quantiles of $F_i^d$, i.e.,

$$m_{i,d} := \frac{Q_{R,F_i^d}(\frac{1}{2}) + Q_{L,F_i^d}(\frac{1}{2})}{2}.$$ 

Note that if $F_i^d$ has a unique median, $m_{i,d}$ is equivalent to this median. We also define median of interest vector of arm $i$ which we denote by $m_i$, as the $M$-dimensional vector whose elements are the medians of interest that is associated with arm $i$. Next, we define the Pareto optimal set of arms according to median of interest.

**Definition 1.** $P^* := \{i \in [K] \mid \forall j \in [K] \setminus \{i\}: m_i \geq m_j\}.$

A PSI algorithm stops after conducting a series of sequential evaluations of arms in $[K]$ with the aim of returning a predicted Pareto set $P$ that approximates $P^*$ up to a given level of accuracy (formally defined in Section III-C). A desirable property is to stop after as few samplings as possible.

**A. Adversarial Attack Models**

We consider three attack models, which we give in the order from the weakest to the strongest below in terms of the adversarial power. Our attack models are extensions of the attack models in [24] to MO-MAB.

**Oblivious adversary:** Chooses all the contamination distributions apriori without the knowledge of the arm outcomes or the rounds in which the samples are corrupted. Formally, for any given $i \in [K]$ and all $d \in [M]$, $\{Y_{i,n}^d, Z_{i,n}^d, B_{i,n}^d\}_{n \geq 1}$ triples are independent. Furthermore, for any given $n$, $Y_{i,n}^d$ and $B_{i,n}^d$ are independent for all $i$ and $d$. Therefore, $F_{i,n}^d$ is equivalent to $(1 - \epsilon)F_{i,n} + \epsilon C_{i,n}$ in this model. In practice, a motivating example for oblivious adversary can be randomly occurring measurement errors due to the environmental effects.

**Prescient adversary:** Can choose contamination distributions based on all the past and future true arm outcomes and the outcome of Bernoulli random variable that determines if a contamination occurs. Formally, for any given $i \in [K]$ and $d \in [M]$, the pairs $\{Y_{i,n}^d, B_{i,n}^d\}_{n \geq 1}$ are independent. Furthermore, for any given $n$, $Y_{i,n}^d$ and $B_{i,n}^d$ are independent for all $i$ and $d$ and $Z_{i,n}^d$ may depend on all the realizations $\{Y_{j,s}^d, Z_{j,s}^d, B_{j,s}^d\}_{j \in [K], d \in [M], s \geq 1}$. The prescient adversary model can be a good fit for an adversary that tries to corrupt the samples in a readily available dataset.

**Malicious adversary:** Same as the prescient except that malicious adversary can also couple the random variable that determines if the contamination occurs with the true reward outcomes. Formally, for any given $i \in [K]$ and all $d \in [M]$, $\{Y_{i,n}^d, B_{i,n}^d\}_{n \geq 1}$ pairs are independent. $Z_{i,n}^d$ may depend on all $\{Y_{j,s}^d, Z_{j,s}^d, B_{j,s}^d\}_{j \in [K], d \in [M], s \geq 1}$. For malicious adversary, it is not required for $Y_{i,n}^d, B_{i,n}^d$ pairs to be independent for a fixed $n$ since they can be coupled by the malicious adversary. For instance, a hacker attack on a dataset that target the data points based on their value could be a good fit for malicious adversary model.

In all three attack models, the random variables corresponding to the true arm reward distributions can be correlated with each other, i.e., for all $i, j \in [K]$ and $a, b \in [M]$, the random variables $Y_{i,n}^a$ and $Y_{j,n}^b$ can be correlated.

**B. Unavoidable Bias and Median Concentration**

Because our adversarial attack model allows for an arbitrary contamination distribution, mean statistics cannot be predicted from the contaminated samples. Furthermore, median statistic is subject to an unavoidable bias which makes the median identifiable only up to a certain interval. Below we will review results from [24], which allows us to determine the amount of unavoidable bias.

**Definition 2.** [24 Definition 5]. For any $t \in (0, \frac{1}{2})$ and positive, non-decreasing function $R$ defined on range $[0, t]$, we define $C_{R,t}$ to be the family of all distributions $F$ that satisfy the following:

$$R(t) \geq \max\left\{Q_{R,F}(\frac{1}{2}t + m) - m, m - Q_{L,F}(\frac{1}{2}t - t)\right\} \quad (1)$$

for all $t \in [0, t]$ and $m \in m_1(F)$.

In words, $R$ bounds the maximum quantile deviation that can occur from the median. It will play a key role in our sample complexity analysis. We will choose a common $R$ for all $\{F_i^d\}_{i,d}$ in order to facilitate our analysis. In the example below, we show how $R$ can be defined for $\sigma$-subgaussian distributions, which is a very common distribution considered in MAB problems [1].

**Example 1.** All $\sigma$-subgaussian distributions are members of the family $C_{R,t}$, where $t < 1/2$ and

$$R(t) = \sigma \sqrt{2 \left(\log \left(\frac{1}{1/2 - t}\right) + \sqrt{\log(2)}\right)} \quad (2)$$

**Proof.** Let any $X$ such that $X - \mathbb{E}[X]$ is $\sigma$-subgaussian distributed, be called a $\sigma$-subgaussian variable. Then, for any $q > 0$, we have:

$$\mathbb{P}(X - \mathbb{E}[X] \geq q) \leq e^{\frac{q^2}{2\sigma^2}} \quad (3)$$

$$\mathbb{P}(X - \mathbb{E}[X] \leq -q) \leq e^{\frac{-q^2}{2\sigma^2}} \quad (4)$$

By definition:

$$\mathbb{P}(X < Q_{R,F}(1/2 + t)) \leq 1/2 + t, \forall t \in \left(0, \frac{1}{2}\right) \quad (5)$$

Next, we will bound $Q_{R,F}(1/2 + t) - m$ for the three cases given below. Let $q_t := Q_{R,F}(1/2 + t)$. Case 1: $\mathbb{E}[X] \leq m \leq q_t$. Case 2: $m \leq \mathbb{E}[X] \leq q_t$. Case 3: $m \leq q_t \leq \mathbb{E}[X]$.

**Case 1:** Since $q_t - \mathbb{E}[X] > 0$, then, by (3 and 5):

$$1 - (1/2 + t) \leq \mathbb{P}(X \geq q_t) = \mathbb{P}(X - \mathbb{E}[X] \geq q_t - \mathbb{E}[X]) \leq e^{\frac{q_t - q_t}{2\sigma^2}}.$$

Simplifying above gives:

$$q_t \leq \mathbb{E}[X] + 2\sigma^2 \log \left(\frac{1}{1/2 - t}\right) \quad (6)$$

Since $m \geq \mathbb{E}[X]$:

$$q_t - m \leq q_t - \mathbb{E}[X] \leq 2\sigma^2 \log \left(\frac{1}{1/2 - t}\right).$$
Case 2: In this case, (6) is valid since \( q_i \geq \mathbb{E}[X] \). Also, by (4), we can bound \( \mathbb{E}[X] - m \):

\[
1/2 \leq \mathbb{P}(X \leq m) = \mathbb{P}(X - \mathbb{E}[X] \leq m - \mathbb{E}[X]) \leq e^{-\frac{D(X,m)^2}{2m^2}}.
\]

Therefore, we have:

\[
\mathbb{E}[X] - m \leq \sqrt{2\sigma^2 \log 2}.
\]  (7)

Combining (7) with (6), we obtain:

\[
 q_i - m = (q_i - \mathbb{E}[X]) + (\mathbb{E}[X] - m) \leq \sqrt{2\sigma^2 \log \left(\frac{1}{2} - \frac{1}{t}\right)} + \sqrt{2\sigma^2 \log 2}. 
\]

Case 3: By (1) and by the fact that \( q_i \leq \mathbb{E}(X) \):

\[
 q_i - m \leq \mathbb{E}[X] - m \leq \sqrt{2\sigma^2 \log 2}.
\]

Combining the results for all three cases, we obtain:

\[
 q_i - m \leq \sqrt{2\sigma^2 \log \left(\frac{1}{2} - \frac{1}{t}\right)} + \sqrt{2\sigma^2 \log 2}.
\]

Following a similar argument, one can obtain the same bound on \( m - q_i \), from which the result follows.

Below, we state results on concentration of the empirical median. These will be used in our sample complexity analysis.

Lemma 1. (Upper bound on empirical median deviation for prescient and oblivious adversaries) [22] Lemma 7). Let \( t \in (0, \frac{1}{2}) \), \( \epsilon \in (0, \frac{1}{2t}) \), \( \delta \in (0, 1) \) and \( F \in C_{R} \). Let \( Y_i \sim F \) and \( B_i \sim \text{Ber}(\epsilon) \) all be independently drawn for \( i \in [n] \). Let \( \{Z_i\}_{i \in [n]} \) be arbitrary random variables possibly depending on \( \{Y_i, B_i\}_{i \in [n]} \), and \( Y_i = (1 - B_i)Y_i + B_iZ_i \). Then for \( n \geq 2(t - \frac{1}{1 - \epsilon})^2 \log (\frac{2}{\delta}) \):

\[
 \mathbb{P}\left( \sup_{m \in m_1(F)} |\hat{m}(Y_1, \ldots, Y_n) - m| \leq R\left( 2 \left(1 - \epsilon\right) + \sqrt{\frac{2 \log(2/\delta)}{n}} \right) \right) \geq 1 - \delta.
\]

Lemma 2. (Upper bound on empirical median deviation for malicious adversary) [22] Lemma 8). Let \( t \in (0, \frac{1}{2}) \), \( \epsilon \in (0, t) \), \( \delta \in (0, 1) \) and \( F \in C_{R} \). Let \( (Y_i, B_i) \) pairs be independently drawn for \( i \in [n] \) with marginals \( Y_i \sim F \) and \( B_i \sim \text{Ber}(\epsilon) \). Let \( \{Z_i\}_{i \in [n]} \) be arbitrary random variables possibly depending on \( \{Y_i, B_i\}_{i \in [n]} \), and \( Y_i = (1 - B_i)Y_i + B_iZ_i \). Then for \( n \geq 2(t - \epsilon)^{-2} \log (\frac{2}{\delta}) \):

\[
 \mathbb{P}\left( \sup_{m \in m_1(F)} |\hat{m}(Y_1, \ldots, Y_n) - m| \leq R\left( \epsilon + \sqrt{\frac{2 \log(3/\delta)}{n}} \right) \right) \geq 1 - \delta.
\]

By taking the limit \( n \to \infty \) in above, we determine the unavoidable bias term as \( D = R\left( \frac{1}{2(1 - \epsilon)} \right) \) for oblivious and prescient adversaries and \( D = R(\epsilon) \) for malicious adversary.

Note that the above lemmas can be used for bounding the deviation of empirical median from the median of interest since median of interest is also a median of the given distribution. In the rest of the paper, we will simply refer to median of interest as the median and the median of interest vector as the median vector.

C. Multi-objective Suboptimality Gap

The number of samples required to distinguish an arm \( i \notin P^* \) from an arm \( j \in P^* \) depends on distance between arms \( i \) and \( j \). We quantify this distance by the notion of suboptimality gap.

Definition 3. We define \( \Delta_{i,j} := \max\{0, \min_d(m_i^d - m_j^d)\} \) as the suboptimality gap of an arm \( i \) with respect to arm \( j \) and \( \Delta_i := \max_{j \in P^*} \Delta_{i,j} \) as the suboptimality gap of arm \( i \).

\( \Delta_i \) measures how much arm \( i \) is dominated by the Pareto set. Given a positive real number \( \alpha \), we call an arm \( i \) \( \alpha \)-suboptimal if its suboptimality gap is smaller than \( \alpha \), i.e., \( \Delta_i \leq \alpha \). Note that in the rest of the paper, we refer to the arms that are not Pareto optimal as suboptimal which should not be confused with the \( \alpha \)-suboptimality defined above. Also we note that all the Pareto optimal arms are \( \alpha \)-suboptimal for any positive real number \( \alpha \) since \( \Delta_j = 0 \) for a Pareto optimal arm \( j \).

Due to the unavoidable bias, in general, it is not possible to detect Pareto optimal arms with more than \( 2D \) accuracy, as proven in the following remark.

Lemma 3. Suppose that an adversary can alter a reward sample as much as \( D \) so that either \( \lim_{n \to \infty} \bar{m}(Y_{1,1}, \ldots, Y_{1,n}) = m + D \) or \( \lim_{n \to \infty} \bar{m}(Y_{1,1}, \ldots, Y_{1,n}) = m - D \) holds for \( i \in [K] \). Then, given any \( \epsilon > 0 \), there are bandit environments in which it is impossible to distinguish \( (2D - \epsilon) \)-suboptimal arms that are not Pareto optimal from the Pareto optimal ones.

Proof. Consider a simple environment with only 2 arms. Suppose that arm 1 is Pareto optimal and arm 2 is such that

\[
\forall d \in [M] : m_1^d - m_2^d = 2D - \epsilon.
\]

Adversary can manipulate the experiment so that \( \forall d \in [M] \), \( \lim_{n \to \infty} \bar{m}(Y_{1,1}^d, \ldots, Y_{1,n}^d) = m - D \) and \( \lim_{n \to \infty} \bar{m}(Y_{2,1}^d, \ldots, Y_{2,n}^d) = m + D \). This implies that:

\[
\forall n \geq 0 : \forall N > N_0, \forall d \in [M], m_1^d - D - \epsilon/4 \leq \bar{m}(Y_{1,1}^d, \ldots, Y_{1,N}^d) \leq m_1^d - D + \epsilon/4 \text{ and } m_2^d + D - \epsilon/4 \leq \bar{m}(Y_{2,1}^d, \ldots, Y_{2,N}^d) \leq m_2^d + D + \epsilon/4.
\]

Therefore, \( \forall N \geq N_0 \):

\[
\bar{m}(Y_{2,1}^d, \ldots, Y_{2,N}^d) - \bar{m}(Y_{1,1}^d, \ldots, Y_{1,N}^d) \geq \epsilon/2 > 0.
\]

Hence, we conclude that, even if infinitely many samples are collected from both arms, it is not possible to decide on their optimality based on the empirical results.

D. Pareto Accuracy

In the following, we define the class of algorithms that is of interest to us in the adversarial MO-MAB setting.

Definition 4. (Pareto accurate algorithm) Suppose that the reward distributions belong to \( C_{R,t} \) for some \( t \in (0, \frac{1}{2}) \). Then, given the accuracy parameter \( \alpha \geq 0 \), the confidence probability \( 0 < \delta < 1 \) and the adversarial attack probability \( 0 \leq \epsilon < t \), we call an algorithm Pareto accurate in the adversarial MO-MAB setting, if the set of arms \( P \) that
the algorithm returns at termination satisfies the following conditions:

1) Accuracy: All the returned arms are at least \(2D + \alpha\) suboptimal:

\[
\forall i \in P, \Delta_i \leq 2D + \alpha.
\]

2) Coverage: If a Pareto optimal arm is not returned in the predicted set, then, there exists at least one arm in the predicted set that \((2D)\)-covers the eliminated Pareto optimal arm:

\[
\forall j \in P^*, \exists i \in P : m_j - m_i \leq 2D.
\]

The Pareto accuracy can be considered as the generalization of the probably approximately correct (PAC) learning concept from the single objective MAB setting to the adversarial MO-MAB setting. However, in the adversarial MO-MAB setting, since the Pareto optimal arms cannot be distinguished from other \(2D\)-suboptimal arms as shown in Lemma 3, it is not possible to approximate the optimal solution arbitrarily well. This reflects itself in the accuracy and the coverage conditions defined above. We also note that, from the algorithms that are in the class defined above, the ones that have smaller sample complexity would be favorable since in many practical settings, taking samples induce some type of cost, e.g., monetary and time that we would want to minimize.

IV. A ROBUST LEARNING ALGORITHM

We call our algorithm Robust Pareto Set Identification (RPSI) whose pseudocode is given in Algorithm 1. As will be shown in Section V, this algorithm is Pareto accurate. The procedure of RPSI is described as follows: At the beginning, all the arms are assigned to the undecided set \(S\). Then each arm is sampled \(n_0\) times which is defined as:

\[
n_0 = \left\lfloor \frac{2\hat{\beta}_{i,e} \log \left( \frac{\pi^2MK}{6\delta} \right) }{\epsilon^2} \right\rfloor
\]

(8)

where

\[
\hat{\beta}_{i,e} = \left( \frac{\bar{t} - \epsilon}{2(1 - \epsilon)} \right)^{-2}, \quad \hat{\delta} = \frac{\delta}{2}
\]

(9)

for prescient and oblivious adversaries, and

\[
\beta_{i,e} = (\bar{t} - \epsilon)^{-2}, \quad \hat{\delta} = \frac{\delta}{3}
\]

(10)

for malicious adversary.

At time steps \(t \geq 1\), algorithm selects the arm with the largest statistical uncertainty which we denote by \(i^*\). The selected arm is successively sampled \(n_{\tau^*_i}\) times where \(n_{\tau^*_i}\) is given by the following expression:

\[
n_{\tau^*_i} = 1 + 4\tau_i \cdot \hat{\beta}_{i,e} \log \left( \frac{\tau^*_i \cdot \tau_i - 1}{\tau_i^* - 1} \right) + 2\hat{\beta}_{i,e} \log \left( \frac{(\tau^*_i - 1)^2MK\pi^2}{6\hat{\delta}} \right)
\]

where \(\tau_i^*\) denotes the number of sampling rounds for arm \(i^*\). This should not be confused with the total number of samples taken from arm \(i\) since in each sampling round there are multiple number of samplings made from the same arm.

As will be proven in Lemma 4 \(n_{\tau^*_i}\) is chosen in such a way that the sample complexity requirements of Lemma 1 and Lemma 2 are met and the resulting empirical median deviation bound of Lemma 1 and Lemma 2 depends on \(\tau_i^*\) through the expression \(R(h_e + \sqrt{\frac{1}{2\hat{\beta}_{i,e}}})\) where \(h_e = \frac{1}{\alpha - \epsilon}\) for oblivious and malicious adversary and \(h_e = \epsilon\) for malicious adversary.

Given a sampling round \(\tau_i\), we define the statistical bias term \(U_{\tau}\) as follows:

\[
U_{\tau} = R(h_e + 1/\sqrt{\hat{\beta}_{i,e}}) - R(h_e).
\]

(11)

Remark 1. The difference between the sampling round numbers of any two arms in \(S\) cannot be larger than 1 since the algorithm selects the arm with the largest statistical bias at the sampling step.

\[
\forall i, j \in S, |\tau_i - \tau_j| \leq 1.
\]

After the sampling step, algorithm enters the elimination step where the arms that are guaranteed to be worse than \((2D + \alpha)\)-suboptimal are eliminated. As shown in Lemma 5, none of the Pareto optimal arms can be eliminated at this step which is crucial for our theoretical analysis to hold since the “additive property of suboptimality” is not satisfied in the adversarial setting as shown in Remark 4 of [24].

After the elimination step, algorithm enters the identification step where the arms that are guaranteed to satisfy the accuracy requirements are collected in \(O_1\). Among these arms in \(O_1\), the ones that can potentially eliminate an arm in \(S\) at future rounds are dropped back to \(S\). The rest of the arms in \(O_1\) are collected in \(O_2\). This prevents arms that are not \((2D + \alpha)\)-suboptimal to potentially end up in \(P\). Note that we have an if statement in the identification step that checks whether \(U_k > \alpha/4\) and when \(U_k \leq \alpha/4\), the algorithm terminates by moving all the arms in \(O_1\) to \(P\). This is to guarantee the termination of the algorithm as there might be some arms left in \(S\) with suboptimality gaps between \((4D + \alpha)\) and \((2D + \alpha)\) that might cause algorithm to stuck in an infinite loop in the absence of this step.

V. SAMPLE COMPLEXITY ANALYSIS

In this section, we give our accuracy and sample complexity analysis.

A. Good Event

We start by showing that the good event in which the sample median concentrates sufficiently around the true median occurs with high probability. The rest of our analysis is based on this good event.

Lemma 4. Define \(E\) as the event in which for all \(i \in [K], d \in [M]\) and \(\tau_i \geq 1\), the following is satisfied:

\[
m_i^d + D + U_{\tau_i} \geq \tilde{m}_i^d \geq m_i^d - D - U_{\tau_i}
\]

or equivalently, \(|\tilde{m}_i^d - m_i^d| \leq D + U_{\tau_i}\). Then:

\[
P(E) \geq 1 - \delta.
\]

Proof. When the sampling round of arm \(i\) is \(\tau_i\), the total number of samples \(N_{\tau_i}\) taken from arm \(i\) is given by:
Algorithm 1 R-PSI

Input: $\alpha$, $\delta$, $\epsilon$, $t$

Initialize: $S = [K]$, $P = 0$, $\tau_i = 0 \forall i \in [K]$, $t = 0$.

Sample each arm $n_0$ times as in (8).

Update $U_i, \tau_i$ according to (11) \( \forall i \in S \)

Update $\hat{m}_i \forall i \in S$.

while $S \neq \emptyset$ do

if $t > 0$ then

Sampling:

Choose arm $i^* = \arg\max_{k \in [K]} U_{k, \tau_k}$

$\tau_{i^*} \leftarrow \tau_{i^*} + 1$

Sample $i^*$ successively $n_{i^*, \tau_{i^*}}$ times.

Update $U_{i^*, \tau_{i^*}}$ according to (11).

Update $\hat{m}_i$.

end if

Elimination:

$S \leftarrow S \setminus \{i \in S | \exists j \in S \setminus \{i\} : \hat{m}_i - D + U_i \leq \hat{m}_j - D - U_j\}$

Identification:

$O_1 \leftarrow \{i \in S | \exists j \in S \setminus \{i\} : \hat{m}_i - U_i + \alpha \leq \hat{m}_j + U_i\}$

if $\exists k \in S : U_k > \alpha/4$ then

$O_2 \leftarrow \{i \in O_1 | \exists j \in S \setminus \{i\} : \hat{m}_j - U_j + \alpha \leq \hat{m}_i + U_i\}$

$S \leftarrow S \setminus O_2$

$P \leftarrow P \cup O_2$

else:

$P \leftarrow P \cup O_1$

return $P$

end if

$t \leftarrow t + 1$

end while

return $P$

---

\[ N_{\tau_i} = \sum_{\tau=1}^{\tau_i} n_\tau = [2\beta_{t, \epsilon} \log\left(\frac{\pi^2 MK}{6\delta}\right)] + \sum_{\tau=2}^{\tau_i} \left[4\tau \beta_{t, \epsilon} \log\left(\frac{\tau}{\tau - 1}\right) + 2\beta_{t, \epsilon} \log\left(\frac{\tau - 1}{6\delta}\right) \right] \]

\[ \geq 2\beta_{t, \epsilon} \log\left(\frac{\pi^2 MK}{6\delta}\right) + \sum_{\tau=2}^{\tau_i} \left[4\tau \beta_{t, \epsilon} \log\left(\frac{\tau}{\tau - 1}\right) + 2\beta_{t, \epsilon} \log\left(\frac{\tau - 1}{6\delta}\right) \right] \]

Simplifying R.H.S., we obtain:

\[ N_{\tau_i} \leq 2\tau_i \beta_{t, \epsilon} \log\left(\frac{\tau_i^2 MK \pi^2}{6\delta}\right) = 2\tau_i \beta_{t, \epsilon} \log\left(\frac{\delta}{\delta\delta_{\tau_i}}\right) \]

\[ = 2\tau_i \beta_{t, \epsilon} \log\left(\frac{1}{\delta_{\tau_i}}\right) \]

---

Where $\delta_{\tau_i} = 6\delta_{\Delta_i}$, $\hat{\delta}_{\tau_i} = (\delta_{\tau_i} \delta_{\bar{\tau}})$ and $\bar{\delta}$ is given in Equations (9) and (10). Hence, by Lemmas (1) and (2) and noting that $\hat{\delta}_{\tau_i} = \delta_{\tau_i}/2$ for oblivious and prescient adversary and $\bar{\delta}_{\tau_i} = \delta_{\tau_i}/3$ for malicious adversary, the following holds for $\forall i \in [K]$ and $\forall d \in [M]$:

\[ \mathbb{P}\left(\sup_{m_d^i} |\hat{m}_d^i - m_d^i| \geq R(h_e + \sqrt{\frac{2 \log(1/\delta_{\tau_i})}{N_{\tau_i}}})\right) \leq \delta_{\tau_i}. \]

Since $N_{\tau_i} \geq 2\tau_i \beta_{t, \epsilon} \log\left(\frac{1}{\delta_{\tau_i}}\right)$ and $R$ is a non-decreasing function:

\[ R\left(h_e + \sqrt{\frac{1}{\beta_{t, \epsilon}\tau_i}}\right) \leq \mathbb{P}\left(\sup_{m_d^i} |\hat{m}_d^i - m_d^i| \geq R(h_e + \sqrt{\frac{2 \log(1/\delta_{\tau_i})}{N_{\tau_i}}})\right) \]

Hence:

\[ \mathbb{P}\left(\sup_{m_d^i} |\hat{m}_d^i - m_d^i| \geq R(h_e + \sqrt{\frac{2 \log(1/\delta_{\tau_i})}{N_{\tau_i}}})\right) \leq \delta_{\tau_i}. \]

Inserting $U_{\tau_i}$ and $D$ in the left hand side of the inequality above we obtain:

\[ \mathbb{P}\left(\sup_{m_d^i} |\hat{m}_d^i - m_d^i| \geq U_{\tau_i} + D\right) \leq \delta_{\tau_i}. \]

The result follows by the union bound:

\[ \mathbb{P}(E) \geq 1 - \sum_{i \in [K]} \sum_{j \in [M]} \sum_{\tau \geq 1} \delta_{\tau_i} = 1 - \sum_{i \in [K]} \sum_{j \in [M]} \sum_{\tau \geq 1} \frac{6\delta}{\tau^2 MK \pi^2} \]

\[ = 1 - MK \frac{6}{MK \pi^2} \sum_{\tau \geq 1} \frac{1}{\tau_i} = 1 - \delta. \]

\[ \square \]

B. Main Results

Next, we state our first main result which gives an accuracy guarantee for R-PSI and establishes a high probability upper bound on its sample complexity.

Theorem 1. Assume that the reward distributions belong to $C_{R,i}$ given in Definition (2) and the event $E$ defined in Lemma (2) holds. Then, given any $\alpha \in \mathbb{R}_+$, and $\epsilon \leq \frac{2}{\tau_1 \pi^2}$ for the oblivious and prescient adversary ($\epsilon \leq 1$ in the case of the malicious adversary), R-PSI is Pareto accurate with sample complexity $N$ bounded by:

\[ N \leq \sum_{i : \Delta_i > 4D + \alpha} 2\beta_{t, \epsilon} \log\left(\frac{\tau_i^2 MK \pi^2}{6\delta}\right) + 1 + \sum_{i : \Delta_i \leq 4D + \alpha} 2\beta_{t, \epsilon} \log\left(\frac{\tau_i^2 MK \pi^2}{6\delta}\right) \]

\[ \leq K \tau_{\alpha}(2\beta_{t, \epsilon} \log\left(\frac{\tau_{\alpha}^2 MK \pi^2}{6\delta}\right) + 2) \]

where $\bar{\Delta}_i := \Delta_i - 4D$, and $\tau_{\alpha} := \inf\{\tau : U_{\tau} \leq \alpha/4\}$ for $\alpha \in \mathbb{R}_+$. 

\[ \square \]
The above theorem gives the most general expression for the sample complexity without making any further assumptions on the reward distributions. If a suitable $R$ can be determined, for the given reward distributions, then it is possible to derive an explicit gap-dependent bound on the sample complexity. Below, we provide such a result for subgaussian reward distributions.

**Corollary 1.** Suppose that the reward distributions are subgaussian with parameter $\sigma$. Then, the asymptotic sample complexity is given by:

$$
O\left( \frac{1}{(1/2 - \epsilon)^2} K \right) = \frac{1}{\alpha^2} \log \left( \frac{MK}{\alpha} \right).
$$

**Proof.** Let $\tau_{\epsilon} := \tau(\epsilon) - 1$. By definition of $\tau(\epsilon)$, we have:

$$
U_{\tau_{\epsilon}} = R(h_{\epsilon}) + 1/\beta\tau_{\epsilon} - R(h_{\epsilon}) > \alpha/4.
$$

Using $R(t)$ from Example 1, we have:

$$
\frac{\sqrt{2}}{\sigma^2} \left( \frac{1}{1 - 2 - h_{\epsilon}} \right)^2 \frac{1}{\sqrt{h_{\epsilon} e}} - \frac{1}{\sqrt{h_{\epsilon} e \tau_{\epsilon}}} > \alpha/4.
$$

Arranging the terms:

$$
\tau_{\epsilon} < \frac{1}{(1/2 - h_{\epsilon})^2 \beta \tau_{\epsilon}} \left( 1 - e^{-c_1 \alpha^2 + c_2 \alpha} \right),
$$

where $c_1 = \frac{1}{32\sigma^2} > 0$ and $c_2 = \frac{1}{2\sigma^2} \log \left( \frac{1/2 - h_{\epsilon}}{1/2 - \epsilon} \right) > 0$.

Note that $\frac{1}{1 - e^{-\epsilon}} = 1 + \frac{1}{e - 1} \leq 1 + \frac{1}{\epsilon}$, from which we obtain:

$$
\tau(\epsilon) < 1 + \frac{1}{(1/2 - h_{\epsilon})^2 \beta \tau_{\epsilon}} \left( \frac{1}{c_1 \alpha^2 + c_2 \alpha} \right)^2.
$$

Note that as $\alpha \to 0$, the dominant term in the denominator of $\frac{1}{c_1 \alpha^2 + c_2 \alpha}$ becomes $c_2 \alpha$. Putting the bound on $\tau(\epsilon)$ in (13):

$$
N \geq O\left( \frac{1}{(1/2 - h_{\epsilon})^2 \alpha^2} \frac{K}{\log \left( \frac{MK}{\alpha} \right)} \right).
$$

The sample complexity bound derived for subgaussian distributions in (14) has a worst case asymptotic matching to the bounds derived in Theorem 4 of [24] and Theorem 3 of [24] in terms of $\alpha$-dependence. Also, this bound nearly matches, in the worst case, the adversary-free lower bound in Theorem 17 of [25]. We also note that unlike these studies, our bound contains an $\epsilon$-dependent factor that comes from the adversarial attack. As expected, this term increases the sample complexity as $\epsilon$ increases and goes to infinity at $\epsilon = 1/2$.

**C. Proof of Theorem [7]**

In the proof, we assume that event $E$ holds. First, we prove that the Pareto optimal arms are not eliminated in the ‘Elimination’ step.

**Lemma 5.** R-PSI does not eliminate Pareto optimal arms in the ‘Elimination’ step.

**Proof.** At elimination step, an arm $i$ is eliminated if $\exists j \in S \setminus \{i\} : \hat{m}_i + D + U_i \leq \hat{m}_j - D - U_j$. By Lemma 4 this implies that $m_i = (m_i - D - U_i) + D + U_i \leq \hat{m}_i + D + U_i \leq \hat{m}_j - D - U_j \leq (m_j + D + U_j) - D - U_j$. Hence, $m_i \leq m_j$. By definition of Pareto optimality, this is not possible if $i$ is a Pareto optimal arm. Hence, Pareto optimal arms cannot be discarded at the elimination step of R-PSI.

Next, we prove that R-PSI has a suboptimality guarantee of $(2D + \alpha)$, i.e., any arm in $P$ is $(2D + \alpha)$ suboptimal. We first state a technical result needed in the proof in the following lemma.

**Lemma 6.** Suppose an arm $i$ is moved to $O_2$ at some round $t_1$. Then, the following is satisfied for all $t \geq t_1$:

$$
\forall j \in S, \exists d_j \in [M] : m_j^{d_j} + D + \alpha > m_i^{d_i} - D.
$$

**Proof.** Since $i$ is moved to $O_2$ at round $t_1$, the condition for $O_2$ requires:

$$
\forall j \in S \setminus \{i\}, \exists d_j \in [M] : m_j^{d_j} - U_j + \alpha > \hat{m}_i - U_i.
$$

This implies:

$$
\forall j \in S \setminus \{i\}, \exists d_j \in [M] : m_j^{d_j} - U_j + \alpha > \hat{m}_i - U_i.
$$

Applying Lemma 4 gives:

$$
\forall j \in S \setminus \{i\}, \exists d_j \in [M] : (m_j^{d_j} + D + U_j) - U_j + \alpha \geq m_j^{d_j} - U_j + \alpha > m_i^{d_i} - U_i \geq (m_i^{d_i} - D - U_i) + U_i.
$$

Hence:

$$
\forall j \in S \setminus \{i\}, \exists d_j \in [M] : m_j^{d_j} + D + \alpha > m_i^{d_i} - D.
$$

The result follows by considering that $S$ does not admit any new arms over time.

We are now ready to prove the suboptimality guarantee for the predicted arms.

**Lemma 7.** $P$ can only contain arms that are $(2D + \alpha)$-suboptimal.

**Proof.** Consider $j \in S$ that is moved to $P$ in round $t_2$. Note that an arm in $S$ needs to be first moved to $O_1$ to end up in $P$. Denote the Pareto optimal arms in $S$ by $S^{(p)}$. $j \in S$ moved to $O_1$ implies that:

$$
\forall i \in S^{(p)} \setminus \{j\} : \hat{m}_i - U_j + \alpha \leq \hat{m}_i + U_i
$$
or equivalently:

$$
\forall i \in S^{(p)} \setminus \{j\}, \exists d_i \in [M] : m_i^{d_i} - U_j + \alpha > \hat{m}_i^{d_i} + U_i.
$$

By Lemma 4

$$
\forall i \in S^{(p)} \setminus \{j\}, \exists d_i \in [M] : (m_i^{d_i} + D + U_j) - U_j + \alpha \geq \hat{m}_i^{d_i} - U_j + \alpha > \hat{m}_i^{d_i} + U_i \geq (m_i^{d_i} - D - U_i) + U_i.
$$

Hence:

$$
\forall i \in S^{(p)} \setminus \{j\}, \exists d_i \in [M] : m_i^{d_i} + D + \alpha > m_i^{d_i} - D.
$$

(15)
Next, consider the set that consists of Pareto optimal arms in $P$, which we denote by $P^*(\alpha)$. By Lemma 6

$$\forall i \in P^*(\alpha), \exists d_i \in [M]: m_{d_i}^i + D + \alpha > m_{i}^i - D .$$

Note that at any round, $S^*(\alpha) \cup P^*(\alpha) = P^*$ since Lemma 5 implies that a Pareto optimal arm is either in $S$ or $P$. Hence, by (15) and (16):

$$\forall i \in P^* \setminus \{j\}, \exists d_i \in [M]: m_{d_i}^i + D + \alpha > m_{i}^i - D .$$

By Definition 3, this implies that $\Delta_j < 2D + \alpha$. Hence, we conclude that if arm $j$ is moved to $P$, it needs to be $(2D + \alpha)$-suboptimal.

Next, we show that for any optimal arm that is not returned in $P$, there exists an arm returned in $P$ which is not worse than the optimal arm more than $2D$ in any objective.

**Lemma 8.** If a Pareto optimal arm $p^*$ is not returned in $P$, then there exists an arm $j$ returned in $P$ such that $m_{p^*} - m_j \leq 2D$.

**Proof.** First, note that, if any of the optimal arms is not returned in $P$, this implies that at the final round before termination the following is satisfied: $\forall i \in S, U_i \leq \alpha/4$. Since this is the only possible way to enter the if statement in the identification step where it is possible for a Pareto optimal arm to not end up in $P$ in the end. Otherwise, all the Pareto optimal arms are returned in $P$ because of Lemma 5. Also, considering that $p^*$ is not included in $O_1$ at the final round, we have an arm $l_1 \in S$ such that:

$$m_{p^*} - U_{p^*} + \alpha \leq m_{l_1} + U_{\tau_1}.$$  \hspace{1cm} (17)

Let $p^* \neq l_1$ denote the above relation. Also, considering that $U_{\tau_1} \leq \alpha/4$, the above inequality implies:

$$m_{p^*} + U_{p^*} < m_{l_1} - U_{\tau_1} + \alpha .$$

Hence:

$$m_{l_1} - U_{\tau_1} + \alpha \leq m_{p^*} + U_{p^*} .$$  \hspace{1cm} (18)

We use the notation $l_1 \neq p^*$ to denote above. Now, let $\{l_i\}_{i=1}^n$ denote a set of arms in $S$ that satisfies the following:

1. $p^* \neq l_1 \cdots \neq l_n$.
2. There are no repeating arms in the set.

Now, we will prove by induction that:

$$\forall z \in \{p^*\} \cup [l_i]_{i=1}^{k-2}, l_k \neq z$$  \hspace{1cm} (19)

for all $k \in [n]$. We have already proven in (18) that when $k = 1$, (19) holds. Now, assuming that (19) holds for $k - 1$, we show that it also holds for $k$. First, observe that, since $l_{k-1} \neq l_k$ and since $\alpha > 2U_{\tau_{k-1}}$ and $\alpha > 2U_{\tau_k}$:

$$m_{l_{k-1}} + U_{\tau_{k-1}} < m_{l_{k-1}} + U_{\tau_{k-1}} + (\alpha - 2U_{\tau_{k-1}}) = m_{l_{k-1}} - U_{\tau_{k-1}} + \alpha \leq m_{l_k} + U_{\tau_k} .$$

$$< m_{l_k} + U_{\tau_k} + (\alpha - 2U_{\tau_k}) = m_{l_k} - U_{\tau_k} + \alpha .$$  \hspace{1cm} (20)

Therefore, $m_{l_{k-1}} + U_{\tau_{k-1}} < m_{l_{k-1}} - U_{\tau_{k-1}} + \alpha$, which implies that $m_{l_k} - U_{\tau_k} + \alpha \leq m_{l_{k-1}} + U_{\tau_{k-1}}$, or $l_k \neq l_{k-1}$. Also, by assumption, $l_{k-1} \neq z$, $\forall z \in \{p^*\} \cup [l_i]_{i=1}^{k-2}$, or equivalently:

$$m_{l_{k-1}} - U_{\tau_{k-1}} + \alpha \leq m_{l_k} + U_{\tau_k} + \alpha .$$  \hspace{1cm} (21)

Also it is shown in (20) that:

$$m_{l_{k-1}} - U_{\tau_{k-1}} + \alpha \leq m_{l_k} - U_{\tau_k} + \alpha .$$  \hspace{1cm} (22)

Hence, by (21), and (22):

$$m_{l_k} - U_{\tau_k} + \alpha \leq m_{l_{k-1}} + U_{\tau_{k-1}} .$$

Also, we had proved above that $l_k \neq l_{k-1}$. From this and above, we conclude:

$$\forall z \in \{p^*\} \cup [l_i]_{i=1}^{k-2} \cup \{l_{k-1}\}, l_k \neq z$$

which proves the induction. Now, suppose that we can find another arm $l_{n+1} \in L$ that we can add to this set. Since the induction proven above applies to this arm as well, we can deduce that $l_{n+1}$ is not prevented by any other arm in this set to enter $O_1$. Now, suppose that we sequentially keep adding other arms to this set. Considering that we have a finite amount of arms in $S$, we will eventually reach a point where we will not be able to add any more arms. Denote by $L$ a set that is constructed through such a procedure and denote the last element of this set by $j$. Then, $j$ satisfies the following:

$$p^* \leq l_1 \cdots \leq j .$$

Also, since we cannot keep adding any more arms to this set, we must have:

$$\forall k \in S \setminus \{L \cup \{p^*\}\}, j \neq k .$$

Considering that $j$ satisfies the above proven induction:

$$\forall k \in L \cup \{p^*\}, j \neq k .$$

Hence, combining two inequalities above yields:

$$\forall k \in S, j \neq k .$$

The above inequality means that arm $j$ is returned in $P$. Next, we prove by another induction that $p^* \leq l_k$ for any $l_k \in L$. This is already given for the first element, i.e., $p^* \leq l_1$. Now assume that this is true for arm $l_{k-1}$. Then, $p^* \leq l_{k-1}$ or equivalently $m_{p^*} - U_{p^*} + \alpha \leq m_{l_{k-1}} + U_{\tau_{k-1}}$. As shown in (20), we also have that:

$$m_{l_{k-1}} + U_{\tau_{k-1}} < m_{l_k} + U_{\tau_k} .$$

Hence:

$$m_{p^*} - U_{p^*} + \alpha \leq m_{l_k} + U_{\tau_k} .$$

or $p^* \leq l_k$ and the induction is proven. Since arm $j$ described above is part of this set, $p^* \leq j$ or equivalently:

$$m_{p^*} - U_{p^*} + \alpha \leq m_j + U_{\tau_j} .$$

Applying Lemma 4 above, we get:

$$m_{p^*} - D + U_{p^*} + \alpha \leq m_{p^*} - U_{p^*} + \alpha \leq m_j + U_{\tau_j} \leq m_j + D + 2U_{\tau_j} .$$

Hence, $m_{p^*} - D + U_{p^*} + \alpha \leq m_j + D + 2U_{\tau_j} .$. Since
$\alpha \geq 2U_{r_s} + 2U_{r_f}$, this implies that $m_{r_s} - D \leq m_j + D$. \hfill $\square$

Next, we state the termination condition for R-PSI in the remark below. We obtain this condition based on the observation that if the algorithm enters the “else” statement inside the identification step at some round $t$, then it terminates after performing the operation inside this “else”.

**Remark 2.** R-PSI terminates at the latest when $\forall i \in S, U_i \leq \alpha/4$.

Next, we give a relaxed elimination condition for the arms that are not $(4D + \alpha)$-suboptimal in the lemma below. The proof technique of this lemma is similar to the previous lemmas and therefore deferred to the appendix.

**Lemma 9.** An arm $i$ that is not $(4D + \alpha)$-suboptimal is guaranteed to be eliminated at the latest when $\forall k \in S, U_k \leq \Delta_i/4$ where $\Delta_i = \Delta_i - 4D$.

With this, we are ready to complete the proof of Theorem 1.

**Final Steps in the Proof of Theorem 1** By Remark 1 and 2,

$\tau_i \leq \inf\{\tau : U_\tau \leq \alpha/4\}, \forall i \in [K].$

By Lemma 2, we can obtain tighter bounds on the number of sampling rounds of arms that are not $(4D + \alpha)$-suboptimal:

$\tau_i \leq \inf\{\tau : U_\tau \leq \Delta_i/4\}.$

The total number of samples taken from an arm that has a sampling round number $\tau$, can be bounded by:

$N_\tau = n_0 + \sum_{\tau = 2}^{\tau} n_\tau = \left[2\beta_{t,\epsilon} \log\left(\frac{\pi^2MK}{6\delta}\right)\right] + \sum_{\tau = 2}^{\tau} \left[4\tau\beta_{t,\epsilon} \log\left(\frac{\tau}{\tau - 1}\right) + 2\beta_{t,\epsilon} \log\left(\frac{\tau - 1)^2MK\pi^2}{6\delta}\right)\right]

\leq 1 + 2\beta_{t,\epsilon} \log\left(\frac{\pi^2MK}{6\delta}\right) + \sum_{\tau = 2}^{\tau} \left(2 + 4\tau\beta_{t,\epsilon} \log\left(\frac{\tau}{\tau - 1}\right) + \right)

\leq 2\beta_{t,\epsilon} \log\left(\frac{\pi^2MK\pi^2}{6\delta}\right) + 2\beta_{t,\epsilon} \log\left(\frac{\tau - 1)^2MK\pi^2}{6\delta}\right)\right].$

Simplifying R.H.S., we obtain:

$N_\tau \leq 2\tau(\beta_{t,\epsilon} \log\left(\frac{\tau^2MK\pi^2}{6\delta}\right) + 1) - 1.$

Hence, the sample complexity $N$ can be bounded as:

$N \leq K\tau(\alpha)(2\beta_{t,\epsilon} \log\left(\frac{\tau^2(\alpha)MK\pi^2}{6\delta}\right) + 2).$

Lastly, the Pareto accuracy of R-PSI follows from Lemma 7 and 8.

**VI. Numerical Results**

In this section, we compare our algorithm with the Algorithm 1 from [25] which considers Pareto set identification problem for adversary free MO-MAB setting. We name this algorithm *Auer*. Auer algorithm is a Pareto accurate algorithm in the adversary free setting (so that $D = 0$) and its ranking of the arms is based on the mean of the distributions instead of the median like our work. For both algorithms to be comparable, we consider the Gaussian reward distributions which have the same median and mean values. We conduct experiments on synthetic Gaussian reward distributions, SW-LLVM dataset, and the data obtained from the UVA/PADOVA Type 1 Diabetes Simulator [38]. Note that in [25], the success condition differs from the accuracy and coverage requirements we use to define the Pareto accurate algorithms in adversarial MO-MAB setting. In particular, their success condition requires the predicted arms to be at least $\alpha$-accurate and all the Pareto optimal arms to be returned in the predicted set. In terms of our accuracy and coverage arguments, this success condition is equivalent to an $\alpha$-accuracy and 0 margin coverage requirement. For a fair comparison, while evaluating Algorithm 1, we relax this success condition to $(2D + \alpha)$ accuracy and $(2D)$ coverage requirement, which are equivalent to the requirements set for R-PSI.

**A. Experiments on Synthetic Gaussian Distributions**

First, we conduct experiments on the synthetic Gaussian reward distributions. For each $\epsilon$, we run both algorithms for 10 independent iterations. The reward samples are generated from Gaussian reward distributions with 0.1 standard deviation. We create 10 arms with 2 objectives with reward medians (or equivalently means) that are generated randomly from the interval $[0, 10]$ using a uniform distribution. We create a synthetic adversarial attack that contaminates with a fixed contamination value of 1 when the arms are suboptimal and -1 when the arms are Pareto optimal. This adversarial attack fits the prescient type of attack model and therefore we use the parameters that corresponds to the prescient model. We choose $\bar{\epsilon} = 0.49$ and maximum $\epsilon$ value of 0.4 so that $\epsilon \leq \frac{\bar{\epsilon}}{1 + 2\epsilon}$ is satisfied. We choose $\alpha = 0.1$ and $\delta = 0.1$. We report the results in Table 1. From left to right, columns display the attack probability, the ratio of successful runs, the total number of samples taken by algorithm averaged over 10 runs, ratio of optimal arms returned in $P$ to the total number of optimal arms and the average number of arms returned in $P$ that does not satisfy the $(2D + \alpha)$ accuracy requirement. When $\epsilon = 0$, the results are consistent with the theoretical expectations. The higher sample complexity of Algorithm 1 from [25] is partly due to their construction of confidence intervals and more stringent Pareto accuracy requirements (e.g., requiring all arms in $P^*$ to be returned). When $\epsilon$ exceeds 0.3, it can be seen that Algorithm 1 success ratio drops drastically while R-PSI consistently makes successful predictions for all $\epsilon$ values.

**B. Experiments on SW-LLVM Dataset**

Next, we use SW-LLVM dataset from [39] which consists of 1023 compiler settings characterized by 11 binary features. The objectives are performance and memory footprint of some software when compiled with these settings. Similar to [25], to obtain a stochastic-like data for our algorithm, we use the combinations of 4 of the binary features to form 16 arms. By ignoring all the other features we end up with 64 datapoints for each arm. When an arm is pulled, one of the 64 data-points pertaining to that arm is randomly selected.
and the corresponding objectives are returned as reward. We normalize the data to obtain a similar range for both objectives. We assume that the reward distributions are Gaussian. Note that this assumption is for the purpose of determining the parameters of the algorithm and does not affect the rewards obtained from arm pulls in any way. We take the mean of 64 data-points assigned to an arm and use this as the mean (median) of the corresponding Gaussian reward distribution. We pick a $\sigma$ value of 0.2 in (2). We use the same type of adversarial model as in the synthetic Gaussian setting with a contamination value of ±10. We choose $\delta = 0.1$ and $\alpha = 0.1$. We use $\ell = 0.49$ and maximum $\epsilon$ value of 0.4 as in the previous settings. We report the average results over 10 runs in Table II. It can be seen that the results are similar to the results of the synthetic Gaussian setting.

### C. Experiments on UVA/PADOVA Diabetes Simulator

Next, we run experiments on the UVA/PADOVA Type 1 Diabetes Simulator [38] which simulates the blood glucose (BG) levels of type 1 diabetes patients over time. We select an adult patient and use the parameters of this patient when running the simulator. We aim to determine the insulin doses that keep the selected patient’s measured BG levels close to the ideal BG level as much as possible. Specifically, we try to optimize the BG levels of the patient measured after 90 and 180 minutes following an insulin injection. For this patient, we choose the ideal BG level to be 140 mg/dL. We pick 11 different insulin doses and assign each of them to an arm to test the effect of each dose on the BG levels of the patient. We compute the objective values for each arm by taking the minus difference between the measured and the ideal BG levels so that an insulin dose that results in a BG measurement that is close to the ideal level takes a large objective value. Note that for this experiment, we normalize the objectives. Given an insulin dose level and the time that passes after the injection, the simulator computes a deterministic BG level based on various patient-dependent parameters [38]. For a more realistic simulation model, we add a Gaussian noise that has zero mean and 0.1 standard deviation to the result computed by the simulator to simulate the errors in the measurement device. We also model the effect of other occasionally occurring factors such as stress, insomnia, and heavy physical activity with an oblivious adversary that interferes with the simulation results. We assume a fixed uniform contamination distribution for this adversary. The choice for the parameters $\alpha$, $\delta$, and $\ell$ are the same as in the previous settings. We report the average results over 10 runs in Table II. It can be seen that the results are similar to the results of the synthetic Gaussian setting.

### VII. CONCLUSION

We investigated the Pareto set identification problem under adversarial attacks. We proposed a sample-efficient algorithm that returns a predicted set that abides by the accuracy and coverage requirements. We also proved a sample complexity upper bound that matches the adversary-free case lower bound proved in previous studies in terms of accuracy parameter dependence. We further proved a tighter gap-dependent sample complexity bound. Experiment results proved that our theoretical results hold as expected and the multi-objective methods developed for the adversary-free setting are unable to cope with a strong adversary. To the best of our knowledge, this is the first study to propose an algorithm with theoretical guarantees that is capable of approximating the Pareto optimal set when the reward samples are corrupted by adversarial attacks with arbitrary distributions. An interesting future research direction is to investigate the success and the sample complexity of the Pareto set identification for large-scale MO-MAB problems with correlated arms.

### TABLE II

| $\epsilon$ | Synthetic | SW-LLVM | UVA/PADOVA |
|---|---|---|---|
| | RSR | AS | RO | VC | RSR | AS | RO | VC | RSR | AS | RO | VC |
| 0.0 | 1.0 | 654.3 | 1.0 | 0.0 | 1.0 | 1611.1 | 1.0 | 0.0 | 1.0 | 5629.6 | 1.0 | 0.0 |
| 0.05 | 1.0 | 2564.7 | 1.0 | 0.0 | 1.0 | 54901.0 | 1.0 | 0.0 | 1.0 | 5780.4 | 1.0 | 0.0 |
| 0.2 | 1.0 | 3580.0 | 1.0 | 0.0 | 1.0 | 67216.2 | 1.0 | 0.0 | 1.0 | 7844.6 | 1.0 | 0.0 |
| 0.3 | 1.0 | 5186.6 | 1.0 | 0.0 | 1.0 | 9836.1 | 1.0 | 0.0 | 1.0 | 11432.4 | 1.0 | 0.0 |
| 0.4 | 1.0 | 13553.6 | 1.0 | 0.0 | 1.0 | 20797.5 | 1.0 | 0.0 | 1.0 | 26772.0 | 1.0 | 0.0 |
| | | | | | | | | | | | | |
| 0.0 | 1.0 | 8189.4 | 1.0 | 0.0 | 1.0 | 7198.2 | 1.0 | 0.0 | 1.0 | 9585.4 | 1.0 | 0.0 |
| 0.05 | 1.0 | 28045.6 | 0.96 | 0.0 | 1.0 | 284796.0 | 0.96 | 0.0 | 0.7 | 4629.1 | 0.75 | 0.3 |
| 0.1 | 1.0 | 28239.7 | 0.96 | 0.0 | 1.0 | 182361.0 | 0.96 | 0.0 | 0.6 | 612.1 | 0.5 | 0.5 |
| 0.2 | 1.0 | 13898.6 | 0.89 | 0.0 | 0.7 | 100986.0 | 0.9 | 0.1 | 0.3 | 122.6 | 0.2 | 0.4 |
| 0.3 | 0.9 | 25633.7 | 0.84 | 0.1 | 0.9 | 54480.8 | 0.9 | 0.3 | 0.1 | 66.2 | 0.3 | 1.2 |
| 0.4 | 0.1 | 25947.1 | 0.71 | 1.3 | 0.9 | 32020.5 | 0.9 | 0.2 | 0.1 | 29.3 | 0.3 | 1.1 |
if statement in the identification step. Then, the algorithm is guaranteed to be eliminated (if it is not already eliminated) as soon as does not enter the if statement in the identification step. Then, since is eliminated. Now suppose that the earliest round where . Hence, by the elimination rule of R-PSI, when . Now, suppose that so that . Therefore, it is not possible for arm to be in at . Similarly, cannot be in at because of Lemma These together imply that it was eliminated at a round .

Next we proceed proving Lemma By Lemma if is already eliminated since is not be in . Now, suppose that is in . Consider the optimal arm . Note that . Therefore, it is not possible for arm to be in in . Similarly, is eliminated at . Therefore, it is not possible for arm to be in in . Hence, by the elimination rule of R-PSI, when . Now, suppose that so that . Therefore, it is not possible for arm to be in in . Similarly, is eliminated at . Therefore, it is not possible for arm to be in in . Hence, by the elimination rule of R-PSI, when .

To summarize, in all possible scenarios, we showed that is guaranteed to be eliminated (if it is not already eliminated) as soon as .

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APPENDIX

Proof of Lemma 9

First, we give the following lemma needed in the proof.

Lemma 10. Consider an arm that is not suboptimal and suppose that the Pareto optimal arm . Then, the algorithm is guaranteed to be eliminated (if it is not already eliminated) as soon as .

Proof. By Lemma ∀ ∈ , . Therefore, it is not possible for arm to be in at . Similarly, cannot be in at because of Lemma These together imply that it was eliminated at a round .

Now suppose that , then is already eliminated since is not in . Now, suppose that is in . Consider the optimal arm . Note that . Therefore, it is not possible for arm to be in in . Similarly, is eliminated at . Therefore, it is not possible for arm to be in in . Hence, by the elimination rule of R-PSI, when . Now, suppose that so that . Therefore, it is not possible for arm to be in in . Similarly, is eliminated at . Therefore, it is not possible for arm to be in in . Hence, by the elimination rule of R-PSI, when .

To summarize, in all possible scenarios, we showed that is guaranteed to be eliminated (if it is not already eliminated) as soon as ∀ ∈ , }.
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