Boundary Regularity for p-Harmonic Functions and Solutions of Obstacle Problems on Unbounded Sets in Metric Spaces

Abstract: The theory of boundary regularity for $p$-harmonic functions is extended to unbounded open sets in complete metric spaces with a doubling measure supporting a $p$-Poincaré inequality, $1 < p < \infty$. The barrier classification of regular boundary points is established, and it is shown that regularity is a local property of the boundary. We also obtain boundary regularity results for solutions of the obstacle problem on open sets, and characterize regularity further in several other ways.

Keywords: barrier, boundary regularity, Kellogg property, metric space, obstacle problem, $p$-harmonic function

MSC: Primary: 31E05; Secondary: 30L99, 35J66, 35J92, 49Q20

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set and let $f \in C(\partial \Omega)$. The Perron method (introduced on $\mathbb{R}^2$ in 1923 by Perron [47] and independently by Remak [48]) provides a unique function $Pf$ that is harmonic in $\Omega$ and takes the boundary values $f$ in a weak sense, i.e., $Pf$ is a solution of the Dirichlet problem for the Laplace equation. A point $x_0 \in \partial \Omega$ is regular if
$$\lim_{\Omega \ni y \to x_0} Pf(y) = f(x_0)$$
for all $f \in C(\partial \Omega)$. Regular boundary points were characterized in 1924 by the so-called Wiener criterion and in terms of barriers, by Wiener [51] and Lebesgue [42], respectively.

A nonlinear analogue is to consider the Dirichlet problem for $p$-harmonic functions, which are solutions of the $p$-Laplace equation $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) = 0$, $1 < p < \infty$. This leads to a nonlinear potential theory, which has been studied since the 1960s, initially for $\mathbb{R}^n$, and later generalized to weighted $\mathbb{R}^n$, Riemannian manifolds, and other settings. For an extensive treatment in weighted $\mathbb{R}^n$, the reader may consult the monograph Heinonen–Kilpeläinen–Martio [33].

More recently, nonlinear potential theory has been developed on complete metric spaces equipped with a doubling measure supporting a $p$-Poincaré inequality, $1 < p < \infty$, see, e.g., the monograph Björn–Björn [11] and the references therein. The Perron method was extended to this setting by Björn–Björn–Shanmugalingam [17] for bounded open sets and Hansevi [30] for unbounded open sets. Note that when $\mathbb{R}^n$ is equipped with a measure $d\mu = w \, dx$, our assumptions on $\mu$ are equivalent to assuming that $w$ is $p$-admissible as in [33], and our definition of $p$-harmonic functions is equivalent to the one in [33], see Appendix A.2 in [11].

Boundary regularity for $p$-harmonic functions on metric spaces was first studied by Björn [22] and Björn–MacManus–Shanmugalingam [26]. Björn–Björn–Shanmugalingam [16] obtained the Kellogg property saying
that the set of irregular boundary points has capacity zero. Björn–Björn [9] obtained the barrier characterization, showed that regularity is a local property, and also studied boundary regularity for obstacle problems showing that they have essentially the same regular boundary points as the Dirichlet problem. These studies were pursued on bounded open sets.

In this paper we study boundary regularity for $p$-harmonic functions on unbounded sets $\Omega$ in metric spaces $X$ (satisfying the assumptions above). The boundary $\partial \Omega$ is considered within the one-point compactification $X^* = X \cup \{\infty\}$ of $X$, and is in particular always compact. We also impose the condition that the capacity $C_p(X \setminus \Omega) > 0$.

In this generality it is not known if continuous functions $f$ are resolutive, i.e., whether the upper and lower Perron solutions $\overline{P}_f$ and $\underline{P}_f$ coincide. We therefore make the following definition.

**Definition 1.1.** A boundary point $x_0 \in \partial \Omega$ is regular if

$$\lim_{\Omega \ni y \to x_0} \overline{P}_f(y) = f(x_0) \quad \text{for all } f \in C(\partial \Omega).$$

With a few exceptions, we limit ourselves to studying regularity at finite boundary points.

Our main results can be summarized as follows.

**Theorem 1.2.** Let $x_0 \in \partial \Omega \setminus \{\infty\}$ and let $B = B(x_0, r)$ for some $r > 0$.

(a) The Kellogg property holds, i.e., $C_p(I \setminus \{\infty\}) = 0$, where $I$ is the set of irregular boundary points.

(b) $x_0$ is regular if and only if there is a barrier at $x_0$.

(c) Regularity is a local property, i.e., $x_0$ is regular with respect to $\Omega$ if and only if it is regular with respect to $B \cap \Omega$.

Once the barrier characterization (b) has been shown, the locality (c) follows easily. Our proofs of these facts are however intertwined, and even though we use that these facts are already known to hold for bounded open sets, our proof is significantly longer than the proof in Björn–Björn [9] (or [11]). On the other hand, once (c) has been deduced, (a) follows from its version for bounded domains. Several other characterizations of regularity are also given, see Sections 5 and 9.

We also study the associated (one-sided) obstacle problem with prescribed boundary values $f$ and an obstacle $\psi$, where the solution is required to be greater than or equal to $\psi$ q.e. in $\Omega$ (i.e., up to a set of capacity zero). This problem obviously reduces to the Dirichlet problem for $p$-harmonic functions when $\psi \equiv -\infty$. In Section 8, we show that if $x_0 \in \partial \Omega \setminus \{\infty\}$ is a regular boundary point and $f$ is continuous at $x_0$, then the solution $u$ of the obstacle problem attains the boundary value at $x_0$ in the limit, i.e.,

$$\lim_{\Omega \ni y \to x_0} u(y) = f(x_0)$$

if and only if $C_p^* \overline{\text{ess \ sup}}_{\Omega \ni y \to x_0} \psi(y) \leq f(x_0)$. The results in Section 8 generalize the corresponding results in Björn–Björn [9] to unbounded sets, with some improvements also for bounded sets. These results are new even on unweighted $\mathbb{R}^n$.

Boundary regularity for $p$-harmonic functions on $\mathbb{R}^n$ was first studied by Maz’ya [45] who obtained the sufficiency part of the Wiener criterion in 1970. Later on the full Wiener criterion has been obtained in various situations including weighted $\mathbb{R}^n$ and for Cheeger $p$-harmonic functions on metric spaces, see [37], [43], [46], and [23]. The full Wiener criterion for $p$-harmonic functions defined using upper gradients remains open even for bounded open sets in metric spaces (satisfying the assumptions above), but the sufficiency has been obtained, see [26] and [24], and a weaker necessity condition, see [25]. An important consequence of Theorem 1.2(c) is that the sufficiency part of the Wiener criterion holds for unbounded open sets. (Hence also the porosity-type conditions in Corollary 11.25 in [11] imply regularity for unbounded open sets.)

In nonlinear potential theory, the Kellogg property was first obtained by Hedberg [31] and Hedberg–Wolff [32] on $\mathbb{R}^n$ (see also Kilpeläinen [36]). It was extended to homogeneous spaces by Vodop’yanov [50], to weighted $\mathbb{R}^n$ by Heinonen–Kilpeläinen–Martio [33], to subelliptic equations by Markina–Vodop’yanov [44], and to bounded open sets in metric spaces by Björn–Björn–Shanmugalingam [16]. In some of these papers
boundary regularity was defined in a different way than through Perron solutions, but these definitions are now known to be equivalent. See also [1] and [41] for the Kellogg property for \( p(\cdot) \)-harmonic functions on \( \mathbb{R}^n \).

Granlund–Lindqvist–Martio [28] were the first to define boundary regularity using Perron solutions for \( p \)-harmonic functions, \( p \neq 2 \). They studied the case \( p = n \) in \( \mathbb{R}^n \) and obtained the barrier characterization in this case for bounded open sets. Kilpeläinen [36] generalized the barrier characterization to \( p > 1 \) and also deduced resolutivity for continuous functions. The results in [36] covered both bounded and unbounded open sets in unweighted \( \mathbb{R}^n \), and were extended to weighted \( \mathbb{R}^n \) (with a \( p \)-admissible measure) in Heinonen–Kilpeläinen–Martio [33, Chapter 9].

As already mentioned, the Perron method for \( p \)-harmonic functions was extended to metric spaces in Björn–Björn–Shanmugalingam [17] and Hansevi [30]. It has also been extended to other types of boundaries in [19], [20], [27], and [7]. Various aspects of boundary regularity for \( p \)-harmonic functions on bounded open sets in metric spaces have also been studied in [2], [4]–[10] and [13].

Very recently, Björn–Björn–Li [14] studied Perron solutions and boundary regular for \( p \)-harmonic functions on unbounded open sets in Ahlfors regular metric spaces. There is some overlap with the results in this paper, but it is not substantial and here we consider more general metric spaces than in [14].

2 Notation and preliminaries

We assume that \((X, d, \mu)\) is a metric measure space (which we simply refer to as \( X \)) equipped with a metric \( d \) and a positive complete Borel measure \( \mu \) such that \( 0 < \mu(B) < \infty \) for every ball \( B \subset X \). It follows that \( X \) is separable, second countable, and Lindelöf (these properties are equivalent for metric spaces). For balls \( B(x_0, r) := \{ x \in X : d(x, x_0) < r \} \), we let \( \lambda B = \lambda B(x_0, r) := B(x_0, \lambda r) \) for \( \lambda > 0 \). The \( \sigma \)-algebra on which \( \mu \) is defined is the completion of the Borel \( \sigma \)-algebra. We also assume that \( 1 < p < \infty \). Later we will impose further requirements on the space and on the measure. We will keep the discussion short, see the monographs Björn–Björn [11] and Heinonen–Koskela–Shanmugalingam–Tyson [35] for proofs, further discussion, and references on the topics in this section.

The measure \( \mu \) is doubling if there exists a constant \( C \geq 1 \) such that

\[
0 < \mu(2B) \leq C\mu(B) < \infty
\]

for every ball \( B \subset X \). A metric space is proper if all bounded closed subsets are compact, and this is in particular true if the metric space is complete and the measure is doubling.

We use the standard notation \( f_+ = \max\{f, 0\} \) and \( f_- = \max\{-f, 0\} \), and let \( \chi_E \) denote the characteristic function of the set \( E \). Semicontinuous functions are allowed to take values in \( \mathbb{R} := [-\infty, \infty] \), whereas continuous functions will be assumed to be real-valued unless otherwise stated. For us, a curve in \( X \) is a rectifiable nonconstant continuous mapping from a compact interval into \( X \), and it can thus be parametrized by its arclength \( ds \).

By saying that a property holds for \( p \)-almost every curve, we mean that it fails only for a curve family \( \Gamma \) with zero \( p \)-modulus, i.e., there exists a nonnegative \( \rho \in L^p(X) \) such that \( \int_{\Gamma} \rho \, ds = \infty \) for every curve \( y \in \Gamma \).

Following Koskela–MacManus [40] we make the following definition, see also Heinonen–Koskela [34].

**Definition 2.1.** A measurable function \( g : X \to [0, \infty] \) is a \( p \)-weak upper gradient of the function \( f : X \to \mathbb{R} \) if

\[
|f(y(0)) - f(y(l_y))| \leq \int_y g \, ds
\]

for \( p \)-almost every curve \( y : [0, l_y] \to X \), where we use the convention that the left-hand side is \( \infty \) when at least one of the terms on the left-hand side is infinite.

Shanmugalingam [49] used \( p \)-weak upper gradients to define so-called Newtonian spaces.
Definition 2.2. The \( \text{Newtonian space} \) on \( X \), denoted \( N^{1,p}(X) \), is the space of all extended real-valued functions \( f \in L^p(X) \) such that
\[
\|f\|_{N^{1,p}(X)} := \left( \int_X |f|^p \, d\mu + \inf_{g} \int_X g^p \, d\mu \right)^{1/p} < \infty,
\]
where the infimum is taken over all \( p \)-weak upper gradients \( g \) of \( f \).

Shanmugalingam [49] proved that the associated quotient space \( N^{1,p}(X) / \sim \) is a Banach space, where \( f \sim h \) if and only if \( \|f - h\|_{N^{1,p}(X)} = 0 \). In this paper we assume that functions in \( N^{1,p}(X) \) are defined everywhere (with values in \( \mathbb{R} \)), not just up to an equivalence class. This is important, in particular for the definition of \( p \)-weak upper gradients to make sense.

Definition 2.3. An everywhere defined, measurable, extended real-valued function on \( X \) belongs to the \( \text{Dirichlet space} \) \( D^p(X) \) if it has a \( p \)-weak upper gradient in \( L^p(X) \).

A measurable set \( A \subset X \) can be considered to be a metric space in its own right (with the restriction of \( d \) and \( \mu \) to \( A \)). Thus the Newtonian space \( N^{1,p}(A) \) and the Dirichlet space \( D^p(A) \) are also given by Definitions 2.2 and 2.3, respectively. If \( X \) is proper and \( \Omega \subset X \) is open, then \( f \in N^{1,p}_{\text{loc}}(\Omega) \) if and only if \( f \in N^{1,p}(V) \) for every open \( V \) such that \( V \) is a compact subset of \( \Omega \), and similarly for \( D^p_{\text{loc}}(\Omega) \). If \( f \in D^p_{\text{loc}}(X) \), then \( f \) has a \( \text{minimal} \) \( p \)-weak upper gradient \( g_f \in L^p_{\text{loc}}(X) \) in the sense that \( g_f \leq g \) a.e. for all \( p \)-weak upper gradients \( g \in L^p_{\text{loc}}(X) \) of \( f \).

Definition 2.4. The \( \text{(Sobolev) capacity} \) of a set \( E \subset X \) is the number
\[
C_p(E) := \inf_{f} \|f\|_{N^{1,p}(X)}^p,
\]
where the infimum is taken over all \( f \in N^{1,p}(X) \) such that \( f \geq 1 \) on \( E \).

Whenever a property holds for all points except for those in a set of capacity zero, it is said to hold \( \text{quasieverywhere} \) (q.e.).

The capacity is countably subadditive, and it is the correct gauge for distinguishing between two Newtonian functions: If \( f \in N^{1,p}(X) \), then \( f \sim h \) if and only if \( f = h \) q.e. Moreover, if \( f, h \in N^{1,p}_{\text{loc}}(X) \) and \( f = h \) a.e., then \( f = h \) q.e.

There is a subtle, but important, difference to the standard theory on \( \mathbb{R}^n \) where the equivalence classes in the Sobolev space are (usually) up to sets of measure zero, while here the equivalence classes in \( N^{1,p}(X) \) are up to sets of capacity zero. Moreover, under the assumptions from the beginning of Section 3, the functions in \( N^{1,p}_{\text{loc}}(X) \) and \( N^{1,p}_{\text{loc}}(\Omega) \) are quasicontinuous. On weighted \( \mathbb{R}^n \), the Newtonian space \( N^{1,p}(X) \) therefore corresponds to the refined Sobolev space mentioned on p. 96 in Heinonen–Kilpeläinen–Martio [33].

In order to be able to compare boundary values of Dirichlet and Newtonian functions, we need the following spaces.

Definition 2.5. For subsets \( E \) and \( A \) of \( X \), where \( A \) is measurable, the \( \text{Dirichlet space with zero boundary values} \) in \( A \setminus E \), is
\[
D^p_0(E; A) := \{ f \mid_{E \cap A} : f \in D^p(A) \text{ and } f = 0 \text{ in } A \setminus E \}.
\]

The \( \text{Newtonian space with zero boundary values} \) \( N^{1,p}_0(E; A) \) is defined analogously. We let \( D^p_0(E) \) and \( N^{1,p}_0(E) \) denote \( D^p_0(E; X) \) and \( N^{1,p}_0(E; X) \), respectively.

The condition “\( f = 0 \) in \( A \setminus E \)” can in fact be replaced by “\( f = 0 \) q.e. in \( A \setminus E \)” without changing the obtained spaces.

Definition 2.6. We say that \( X \) supports a \( p \)-\( \text{Poincaré inequality} \) if there exist constants, \( C > 0 \) and \( \lambda \geq 1 \) (the dilation constant), such that
\[
\int_B |f - f_B| \, d\mu \leq C \text{diam}(B) \left( \int_{\lambda B} g^p \, d\mu \right)^{1/p}
\]
for all balls \( B \subset X \), all integrable functions \( f \) on \( X \), and all \( p \)-weak upper gradients \( g \) of \( f \).
In (2.1), we have used the convenient notation $f_B := \int_B f \, d\mu := \frac{1}{|B|} \int_B f \, d\mu$. Requiring a Poincaré inequality to hold is one way of making it possible to control functions by their $p$-weak upper gradients.

### 3 The obstacle problem and $p$-harmonic functions

We assume from now on that $1 < p < \infty$, that $X$ is a complete metric measure space supporting a $p$-Poincaré inequality, that $\mu$ is doubling, and that $\Omega \subset X$ is a nonempty (possibly unbounded) open subset such that $C_p(X \setminus \Omega) > 0$.

One of our fundamental tools is the following obstacle problem, which in this generality was first considered by Hansevi [29].

**Definition 3.1.** Let $V \subset X$ be a nonempty open subset with $C_p(X \setminus V) > 0$. For $\psi : V \to \mathbb{R}$ and $f \in D^p(V)$, let

$$\mathcal{X}_{\psi,f}(V) = \{ v \in D^p(V) : v - f \in D^p_0(V) \text{ and } v \geq \psi \text{ q.e. in } V \}. $$

We say that $u \in \mathcal{X}_{\psi,f}(V)$ is a solution of the $\mathcal{X}_{\psi,f}(V)$-obstacle problem (with obstacle $\psi$ and boundary values $f$) if

$$\int_V g_v^p \, d\mu \leq \int_V g_f^p \, d\mu \quad \text{for all } v \in \mathcal{X}_{\psi,f}(V).$$

When $V = \Omega$, we usually denote $\mathcal{X}_{\psi,f}(\Omega)$ by $\mathcal{X}_{\psi,f}$.

It was proved in Hansevi [29, Theorem 3.4] that the $\mathcal{X}_{\psi,f}$-obstacle problem has a unique (up to sets of capacity zero) solution whenever $\mathcal{X}_{\psi,f}$ is nonempty. Furthermore, in this case, there is a unique lsc-regularized solution of the $\mathcal{X}_{\psi,f}$-obstacle problem, by Theorem 4.1 in [29]. A function $u$ is lsc-regularized if $u = u^*$, where the lsc-regularization $u^*$ of $u$ is defined by

$$u^*(x) = \text{ess lim inf}_{y \to x} u(y) := \lim_{r \to 0} \text{ess inf}_{B(x,r)} u.$$

**Definition 3.2.** A function $u \in N^{1,p}_{\text{loc}}(\Omega)$ is a minimizer in $\Omega$ if

$$\int_{\varphi \neq 0} g_u^p \, d\mu \leq \int_{\varphi \neq 0} g_{u^*+\varphi} \, d\mu \quad \text{for all } \varphi \in N^{1,p}_{\text{loc}}(\Omega).$$

If (3.1) is only required to hold for all nonnegative $\varphi \in N^{1,p}_{0}(\Omega)$, then $u$ is a superminimizer.

Moreover, a function is $p$-harmonic if it is a continuous minimizer.

Kinnunen–Shanmugalingam [39, Proposition 3.3 and Theorem 5.2] used De Giorgi’s method to show that every minimizer $u$ has a Hölder continuous representative $\tilde{u}$ such that $\tilde{u} = u$ q.e. They also obtained the strong maximum principle [39, Corollary 6.4] for $p$-harmonic functions. Björn–Marola [21, p. 362] obtained the same conclusions using Moser iterations. See alternatively Theorems 8.13 and 8.14 in [11]. Note that $N^{1,p}_{\text{loc}}(\Omega) = D^p_{\text{loc}}(\Omega)$ (under our assumptions), by Proposition 4.14 in [11].

If $\psi : \Omega \to [-\infty, \infty]$ is continuous as an extended real-valued function, and $\mathcal{X}_{\psi,f} \neq \emptyset$, then the lsc-regularized solution of the $\mathcal{X}_{\psi,f}$-obstacle problem is continuous, by Theorem 4.4 in Hansevi [29]. Thus the following definition makes sense. It was first used in this generality by Hansevi [29, Definition 4.6].

**Definition 3.3.** Let $V \subset X$ be a nonempty open set with $C_p(X \setminus V) > 0$. The $p$-harmonic extension $H_{\Omega} f$ of $f \in D^p(V)$ to $V$ is the continuous solution of the $\mathcal{X}_{-\infty,f}(V)$-obstacle problem. When $V = \Omega$, we usually write $H f$ instead of $H_{\Omega} f$.

**Definition 3.4.** A function $u : \Omega \to (-\infty, \infty]$ is superharmonic in $\Omega$ if

(i) $u$ is lower semicontinuous;
(ii) $u$ is not identically $\infty$ in any component of $\Omega$;}
(iii) for every nonempty open set $V$ such that $\overline{V}$ is a compact subset of $\Omega$ and all $v \in \text{Lip}(\overline{V})$, we have $H_{\gamma}v \leq u$ in $V$ whenever $v \leq u$ on $\partial V$.

A function $u : \Omega \to [-\infty, \infty]$ is subharmonic if $-u$ is superharmonic.

There are several other equivalent definitions of superharmonic functions, see, e.g., Theorem 6.1 in Björn [3] (or Theorem 9.24 and Propositions 9.25 and 9.26 in [11]).

An lsc-regularized solution of the obstacle problem is always superharmonic, by Proposition 3.9 in Hansevi [29] together with Proposition 7.15 in Kinnunen–Martio [38] (or Theorem 9.24 and Propositions 9.25 and 9.26 in [11]).

When proving Theorem 9.2 we will need the following generalization of Proposition 7.15 in [11], which may be of independent interest.

**Lemma 3.5.** Let $u$ be superharmonic in $\Omega$ and let $V \subset \Omega$ be a bounded nonempty open subset such that $C_p(X \setminus V) > 0$ and $u \in D^p(V)$. Then $u$ is the lsc-regularized solution of the $\mathcal{K}_{u,v}(V)$-obstacle problem.

The boundedness assumption cannot be dropped. To see this, let $1 < p < n$ and $\Omega = V = \mathbb{R}^n \setminus B(0,1)$ in unweighted $\mathbb{R}^n$. Then $u(x) = |x|^{(p-n)/(p-1)}$ is superharmonic in $\Omega$ and belongs to $D^p(V)$. However, $\omega \equiv 1$ is the lsc-regularized solution of the $\mathcal{K}_{u,v}(V)$-obstacle problem.

**Proof.** Corollary 9.10 in [11] implies that $u$ is superharmonic in $\Omega$, and hence it follows from Corollary 7.9 and Theorem 7.14 in Kinnunen–Martio [38] (or Corollary 9.6 and Theorem 9.12 in [11]) that $u$ is an lsc-regularized superminimizer in $\Omega$. Because $u \in D^p(V)$, it is clear that $u \in \mathcal{K}_{a,u}(V)$. Let $v \in \mathcal{K}_{a,u}(V)$ and let $w = \max\{u, v\}$. Then $\varphi := w - u = (v - u)_+ \in D^p(V)$, and since $X$ supports a $p$-Friedrichs inequality (Definition 2.6 in Björn–Björn [12]) and $V$ is bounded, we see that $\varphi \in N^1_{\gamma}(V)$, by Proposition 2.7 in [12]. Because $v = w$ q.e. in $V$, it follows from Definition 3.2 that

$$
\int_V g_v^p \, d\mu \leq \int_V g_w^{p+\varphi} \, d\mu = \int_V g_w^p \, d\mu = \int_V g_v^p \, d\mu.
$$

Hence $u$ is the lsc-regularized solution of the $\mathcal{K}_{a,u}(V)$-obstacle problem. \qed

## 4 Perron solutions

In addition to the assumptions given at the beginning of Section 3, from now on we make the convention that if $\Omega$ is unbounded, then the point at infinity, $\infty$, belongs to the boundary $\partial \Omega$. Topological notions should therefore be understood with respect to the one-point compactification $\Omega^* := \Omega \cup \{\infty\}$.

Note that this convention does not affect any of the definitions in Sections 2 or 3, as $\infty$ is not added to $\Omega$ (it is added solely to $\partial \Omega$).

Since continuous functions are assumed to be real-valued, every function in $C(\partial \Omega)$ is bounded even if $\Omega$ is unbounded.

**Definition 4.1.** Given a function $f : \partial \Omega \to \mathbb{R}$, let $\mathcal{F}(\Omega)$ be the collection of all functions $u$ that are superharmonic in $\Omega$, bounded from below, and such that

$$
\liminf_{\Omega \ni y \to x} u(y) \geq f(x) \quad \text{for all } x \in \partial \Omega.
$$

The upper Perron solution of $f$ is defined by

$$
P_{\partial \Omega} f(x) = \inf_{u \in \mathcal{F}(\Omega)} u(x), \quad x \in \Omega.
$$

Let $\mathcal{L}(\Omega)$ be the collection of all functions $\nu$ that are subharmonic in $\Omega$, bounded from above, and such that

$$
\limsup_{\Omega \ni y \to x} \nu(y) \leq f(x) \quad \text{for all } x \in \partial \Omega.
$$

The lower Perron solution of $f$ is defined by

$$
L_{\partial \Omega} f(x) = \sup_{\nu \in \mathcal{L}(\Omega)} \nu(x), \quad x \in \Omega.
$$

Note that $P_{\partial \Omega} f$ and $L_{\partial \Omega} f$ are both well-defined.
The lower Perron solution of $f$ is defined by

$$\underline{P}_\Omega f(x) = \sup_{v \in \mathcal{L}_f(\Omega)} v(x), \quad x \in \Omega.$$  

If $\overline{P}_\Omega = \underline{P}_\Omega f$, then we denote the common value by $P_\Omega f$. Moreover, if $P_\Omega f$ is real-valued, then $f$ is said to be resolutive (with respect to $\Omega$). We often write $Pf$ instead of $P_\Omega f$, and similarly for $\overline{P}$ and $\underline{P}$.

Immediate consequences of the definition are: $\overline{P}f = -\underline{P}(-f)$ and $\overline{P}f \preceq \overline{P}h$ whenever $f \leq h$ on $\partial \Omega$. If $a \in \mathbb{R}$ and $\beta \geq 0$, then $\overline{P}(\alpha + \beta f) = \alpha + \beta \overline{P}f$. Corollary 6.3 in Hansevi [30] shows that $\overline{P}f \preceq \overline{P}f$. In each component of $\Omega$, $\overline{P}f$ is either $p$-harmonic or identically $\pm \infty$, by Theorem 4.1 in Björn–Björn–Shanmugalingam [17] (or Theorem 10.10 in [11]); the proof is local and applies also to unbounded $\Omega$.

**Definition 4.2.** Assume that $\Omega$ is unbounded. Then $\Omega$ is $p$-parabolic if for every compact $K \subset \Omega$, there exist functions $u_j \in N^{1,p}(\Omega)$ such that $u_j \geq 1$ on $K$ for all $j = 1, 2, \ldots$, and

$$\int_{\Omega} \phi_{u_j} d\mu \to 0 \quad \text{as} \quad j \to \infty.$$  

Otherwise, $\Omega$ is $p$-hyperbolic.

For examples of $p$-parabolic sets, see, e.g., Hansevi [30]. The main reason for introducing $p$-parabolic sets in [30] was to be able to obtain resolutivity results. We formulate this in a special case, which will be sufficient for us.

**Theorem 4.3.** ([17, Theorem 6.1] and [30, Theorems 75 and 78]) Assume that $\Omega$ is bounded or $p$-parabolic. If $f \in C(\partial \Omega)$, then $f$ is resolutive.

If $f \in D^p(\Omega)$ and $f(\infty)$ is defined (with a value in $\mathbb{E}$), then $f$ is resolutive and $Pf = Hf$.

Recall from Section 2 that under our standing assumptions, the equivalence classes in $D^p(\Omega)$ only contain quasicontinuous representatives. This fact is crucial for the validity of the second part of Theorem 4.3.

## 5 Boundary regularity

For unbounded $p$-hyperbolic sets resolutivity of continuous functions is not known, which will be an obstacle to overcome in some of our proofs below. This explains why regularity was defined using upper Perron solutions in Definition 1.1. In our definition it is not required that $\Omega$ is bounded, but if it is, then it follows from Theorem 4.3 that it coincides with the definitions of regularity in Björn–Björn–Shanmugalingam [16], [17], and Björn–Björn [9], [11], where regularity is defined using $Pf$ or $Hf$. Thus we can use the boundary regularity results from these papers when considering bounded sets.

Since $\overline{P}f = -\underline{P}(-f)$, the same concept of regularity is obtained if we replace the upper Perron solution by the lower Perron solution in Definition 1.1.

**Theorem 5.1.** Let $x_0 \in \partial \Omega$. Fix $x_1 \in X$ and define $d_{x_0} : X^* \to [0, 1]$ by

$$d_{x_0}(x) = \begin{cases} \min \{d(x, x_0), 1\} & \text{when } x \neq \infty, \\ 1 & \text{when } x = \infty, \end{cases}$$  

if $x_0 \neq \infty$,  

and

$$d_{\infty}(x) = \begin{cases} \exp(-d(x, x_1)) & \text{when } x \neq \infty, \\ 0 & \text{when } x = \infty. \end{cases}$$

Then the following are equivalent:

(a) The point $x_0$ is regular.
(b) **It is true that**
\[
\lim_{\Omega \ni y \to x_0} Pf(y) = 0.
\]
(c) **It is true that**
\[
\limsup_{\Omega \ni y \to x_0} Pf(y) \leq f(x_0)
\]
for all \( f : \partial \Omega \to [-\infty, \infty) \) that are bounded from above on \( \partial \Omega \) and upper semicontinuous at \( x_0 \).

(d) **It is true that**
\[
\lim_{\Omega \ni y \to x_0} Pf(y) = f(x_0)
\]
for all \( f : \partial \Omega \to \mathbb{R} \) that are bounded on \( \partial \Omega \) and continuous at \( x_0 \).

(e) **It is true that**
\[
\limsup_{\Omega \ni y \to x_0} Pf(y) \leq f(x_0)
\]
for all \( f \in C(\partial \Omega) \).

The particular form of \( d_{x_0} \) is not important. The same characterization holds for any nonnegative continuous function \( d : X^* \to [0, \infty) \) which is zero at and only at \( x_0 \). For the later applications in this paper it will also be important that \( d \in D^p(X) \), which is true for \( d_{x_0} \).

**Proof.** (a) \(\Rightarrow\) (b) This is trivial.

(b) \(\Rightarrow\) (c) Fix \( \alpha > f(x_0) \). Since \( f \) is upper semicontinuous at \( x_0 \), there exists an open set \( U \subset X^* \) such that \( x_0 \in U \) and \( f(x) < \alpha \) for all \( x \in U \cap \partial \Omega \). Let \( \beta = \sup_{\partial \Omega \setminus U} f \), and \( \delta := \inf_{\partial \Omega \setminus U} d_{x_0} > 0 \). (Note that \( \delta = \infty \) if \( \partial \Omega \setminus U = \emptyset \)) Then \( \beta < \infty \) and \( f \leq \alpha + \beta d_{x_0} / \delta \) on \( \partial \Omega \), and hence it follows that
\[
\limsup_{\Omega \ni y \to x_0} Pf(y) \leq \lim_{\Omega \ni y \to x_0} Pf(y) = \alpha.
\]

Letting \( \alpha \to f(x_0) \) yields the desired result.

(c) \(\Rightarrow\) (d) Let \( f \) be bounded on \( \partial \Omega \) and continuous at \( x_0 \). Both \( f \) and \( -f \) satisfy the hypothesis in (c). The conclusion in (d) follows as
\[
\limsup_{\Omega \ni y \to x_0} Pf(y) \leq f(x_0) \leq \limsup_{\Omega \ni y \to x_0} Pf(-f(y)) = \liminf_{\Omega \ni y \to x_0} Pf(y) \leq \liminf_{\Omega \ni y \to x_0} Pf(y).
\]

(d) \(\Rightarrow\) (e) This is trivial.

(e) \(\Rightarrow\) (a) This is analogous to the proof of (c) \(\Rightarrow\) (d).

We will mainly concentrate on the regularity of finite points in the rest of the paper.

## 6 Barrier characterization of regular points

**Definition 6.1.** A function \( u \) is a barrier (with respect to \( \Omega \)) at \( x_0 \in \partial \Omega \) if

(i) \( u \) is superharmonic in \( \Omega \);
(ii) \( \lim_{\Omega \ni y \to x_0} u(y) = 0 \);
(iii) \( \liminf_{\Omega \ni y \to x} u(y) > 0 \) for every \( x \in \partial \Omega \setminus \{x_0\} \).

Superharmonic functions satisfy the strong minimum principle, i.e., if \( u \) is superharmonic and attains its minimum in some component \( G \) of \( \Omega \), then \( u|_G \) is constant (see Theorem 9.13 in [11]). This implies that a barrier is always nonnegative, and furthermore, that a barrier is positive if every component \( G \subset \Omega \) has a boundary point in \( \partial G \setminus \{x_0\} \).

**Theorem 6.2.** If \( x_0 \in \partial \Omega \setminus \{\infty\} \) and \( B \) is a ball such that \( x_0 \in B \), then the following are equivalent:

- \( u \) is superharmonic on \( B \);
- \( \lim_{z \to x_0} u(z) = \inf_{\partial B} u \) (the infimum is taken over \( \partial B \));
- \( u \) satisfies the strong minimum principle in \( B \).
Letting $h \in C(\partial \Omega)$ and fix $\alpha > f(x_0)$. Then the set $U := \{ x \in \partial \Omega : f(x) < \alpha \}$ is open relative to $\partial \Omega$, and $\beta := \sup_{\partial \Omega}(f - \alpha) < \infty$. Assume that $u$ is a barrier at $x_0$, and extend $u$ lower semicontinuously to the boundary by letting

$$u(x) = \liminf_{\Omega \ni \gamma \to x} u(y), \quad x \in \partial \Omega.$$ 

Because $u$ is lower semicontinuous and satisfies condition (iii) in Definition 6.1, we have $\delta := \inf_{\partial \Omega \setminus U} u > 0$. (Note that $\delta = \infty$ if $\partial \Omega \setminus U = \emptyset$.) It follows that

$$f \leq \alpha + \frac{\beta u}{\delta} =: h \quad \text{on} \ \partial \Omega.$$

Since $h$ is bounded from below and superharmonic, we see that $h \in \mathcal{H}$, and hence $\mathcal{P}f \leq h$ in $\Omega$. As $u$ is a barrier, it follows that

$$\limsup_{\Omega \ni \gamma \to x_0} \mathcal{P}f(y) \leq \alpha + \frac{\beta}{\delta} \lim_{\Omega \ni \gamma \to x_0} u(y) = \alpha.$$ 

Letting $\alpha \to f(x_0)$, and appealing to Theorem 5.1 shows that $x_0$ is regular.

(a) $\Rightarrow$ (c) Assume that $x_0$ is regular. We begin with the case when $C_p(\{x_0\}) > 0$. Let $d_{x_0} \in D^p(\Omega)$ be given by (5.1). We let $u$ be the continuous solution of the $\mathcal{X}_{d_{x_0}, -d_{x_0}}$ obstacle problem, which is superharmonic (see...
Section 3) and hence satisfies condition (i) in Definition 6.1. We also extend \( u \) to \( X \) by letting \( u = d_{x_0} \) outside \( \Omega \) so that \( u \in D^p(X) \). Then \( 0 \leq u \leq 1 \) (as \( 0 \leq d_{x_0} \leq 1 \)), and thus \( U := \{ x \in \Omega : u(x) > d_{x_0}(x) \} \subset B(x_0, 1) \). Since \( u \) and \( d_{x_0} \) are continuous, we see that \( U \) is open and \( u = d_{x_0} \) on \( \partial U \).

Suppose that \( x_0 \in \partial U \). Proposition 3.7 in Hansevi [29] implies that \( u \) is the continuous solution of the \( \mathcal{K}_{d_{x_0}, d_{x_0}}(U) \)-obstacle problem. Since \( u > d_{x_0} \) in \( U \), we see that \( u|_U = H_U d_{x_0} \), and hence, by Theorem 4.3,

\[
u u|_U = H_U d_{x_0} = \mathcal{P}_U d_{x_0}.
\]

(6.1)

The Kellogg property for bounded sets (Theorem 3.9 in Björn–Björn–Shanmugalingam [16] or Theorem 10.5 in [11]) implies that \( x_0 \) is regular with respect to \( U \) as \( C_p((x_0)) > 0 \). It thus follows that

\[
\lim_{U \ni y \to x_0} u(y) = \lim_{U \ni y \to x_0} \mathcal{P}_U d_{x_0}(y) = 0.
\]

On the other hand, if \( x_0 \in \partial(\Omega \setminus U) \), then

\[
\lim_{\Omega \ni y \to x_0} u(y) = \lim_{\Omega \ni y \to x_0} d_{x_0}(y) = 0,
\]

and hence \( u(y) \to 0 \) as \( \Omega \ni y \to x_0 \) regardless of the position of \( x_0 \) on \( \partial \Omega \). (Note that it is possible that \( x_0 \) belongs to both \( \partial U \) and \( \partial(\Omega \setminus U) \).) Thus \( u \) satisfies condition (ii) in Definition 6.1.

Furthermore, \( u \) also satisfies condition (iii) in Definition 6.1, as

\[
\lim_{U \ni y \to x} \inf u(y) \geq \lim_{U \ni y \to x} \inf d_{x_0}(y) = d_{x_0}(x) > 0 \quad \text{for all} \ x \in \partial \Omega \setminus \{x_0\}.
\]

Thus \( u \) is a positive continuous barrier at \( x_0 \).

Now we consider the case when \( C_p((x_0)) = 0 \). As the capacity \( C_p \) is an outer capacity, by Corollary 1.3 in Björn–Björn–Shanmugalingam [18] (or [11, Theorem 5.31]), \( \lim_{r \to 0} C_p(B(x_0, r)) = 0 \). This, together with the fact that \( C_p(X \setminus \Omega) > 0 \), shows that we can find a ball \( B := B(x_0, r) \) with sufficiently small radius \( r > 0 \) so that \( C_p(X \setminus (\Omega \cup 2B)) > 0 \). Let \( h : X \to [-r, 0] \) be defined by

\[
h(x) = -\min\{d(x, x_0), r\}.
\]

Let \( v \) be the continuous solution of the \( \mathcal{K}_{h, h}(\Omega \cup 2B) \)-obstacle problem, and extend \( v \) to \( X \) by letting \( v = h \) outside \( \Omega \cup 2B \). Then \( -r \leq h \leq v \leq v(x_0) = 0 \) in \( \Omega \cup 2B \). Let \( u = \mathcal{P}_\Omega w \), where

\[
w(x) := \begin{cases} -v(x), & x \in \Omega, \\ -\liminf_{U \ni y \to x} v(y), & x \in \partial \Omega. \end{cases}
\] (6.2)

Then \( u \) is \( p \)-harmonic, see Section 4, and in particular continuous. Thus \( u \) satisfies condition (i) in Definition 6.1.

Because \( x_0 \) is regular and \( w \) is continuous at \( x_0 \) and bounded, it follows from Theorem 5.1 that \( u \) satisfies condition (ii) in Definition 6.1, as

\[
\lim_{U \ni y \to x_0} u(y) = \lim_{U \ni y \to x_0} \mathcal{P}_U w(y) = -\lim_{U \ni y \to x_0} \mathcal{P}_U (-w)(y) = w(x_0) = 0.
\]

Let \( V = \{ x \in \Omega \cup 2B : v(x) > h(x) \} \). Clearly, \( v = h < 0 \) in \( (\Omega \cup 2B) \setminus \{x_0\} \setminus V \). Suppose that \( V \neq \emptyset \) and let \( G \) be a component of \( V \). Then

\[
C_p(X \setminus G) \geq C_p(X \setminus V) \geq C_p(X \setminus (\Omega \cup 2B)) > 0,
\]

and hence Lemma 4.3 in Björn–Björn [9] (or Lemma 4.5 in [11]) implies that \( C_p(\partial G) > 0 \). Let \( B' \) be a sufficiently large ball so that \( C_p(B' \cap \partial G) > 0 \). Since \( C_p((x_0)) = 0 \), it follows from the Kellogg property for bounded sets (Theorem 3.9 in Björn–Björn–Shanmugalingam [16] or Theorem 10.5 in [11]) that there is a point \( x_1 \in (B' \cap \partial G) \setminus \{x_0\} \) that is regular with respect to \( G' := G \cap B' \). As in (6.1) for \( U \), we see that \( v|_{G'} = \mathcal{P}_{G'} v \), and it follows that

\[
\lim_{G' \ni y \to x_1} v(y) = \lim_{G' \ni y \to x_1} \mathcal{P}_{G'} v(y) = v(x_1) = h(x_1) < 0.
\]
Thus \( v \neq 0 \) in \( G \). As \( v \leq 0 \) is \( p \)-harmonic in \( G \) (by Theorem 4.4 in Hansevi [29]), it follows from the strong maximum principle (see Corollary 6.4 in Kinnunen–Shanmugalingam [39] or [11, Theorem 8.13]), that \( v < 0 \) in \( G \) (and thus also in \( V \)). We conclude that \( v < 0 \) in \( (\Omega \cup 2B) \setminus \{ x_0 \} \).

Let \( m = \sup_{\partial B} v \). By compactness, we get that \( -r \leq m < 0 \). Since \( v_{(\Omega \cup 2B) \setminus \overline{B}} \) is the continuous solution of the \( \mathcal{X}_{\partial \Omega}((\Omega \cup 2B) \setminus \overline{B}) \)-obstacle problem (by Proposition 3.7 in [29]) and \( h = -r \) in \( (\Omega \cup 2B) \setminus \overline{B} \), we see that \( \sup_{(\Omega \cup 2B) \setminus \overline{B}} v = m \). It follows that

\[
\lim \sup_{\partial \Omega \to x} v(y) \leq m < 0 \quad \text{for all } x \in \partial \Omega \setminus \overline{B}.
\]

Moreover, as \( v \) is continuous in \( 2B \), it follows that

\[
\lim \sup_{\partial \Omega \to x} v(y) = v(x) < 0 \quad \text{for all } x \in (\partial \Omega \cap \overline{B}) \setminus \{ x_0 \},
\]

and hence

\[
\lim \sup_{\partial \Omega \to x} v(y) < 0 \quad \text{for all } x \in \partial \Omega \setminus \{ x_0 \}.
\]

Since \( v \) is bounded and superharmonic in \( \Omega \), defining \( w \) in the particular way on \( \partial \Omega \) as we did in (6.2) makes sure that \( w \in \mathcal{X}_w \), and hence \( u \geq w \) in \( \Omega \). It follows that \( u \) is positive and satisfies condition (iii) in Definition 6.1, as

\[
\lim \inf_{\partial \Omega \to x} u(y) \geq \lim \inf_{\partial \Omega \to x} (-v(y)) = -\lim \sup_{\partial \Omega \to x} v(y) > 0 \quad \text{for all } x \in \partial \Omega \setminus \{ x_0 \}.
\]

Thus \( u \) is a positive continuous barrier at \( x_0 \).

\[\square\]

7 The Kellogg property

**Theorem 7.1.** (The Kellogg property) If \( I \) is the set of irregular points in \( \partial \Omega \setminus \{ \infty \} \), then \( C_p(I) = 0 \).

**Proof.** Cover \( \partial \Omega \setminus \{ \infty \} \) by a countable set of balls \( \{ B_i \}_{i=1}^\infty \) and let \( I_j = I \cap B_j \). Furthermore, let \( I_j' \) be the set of irregular boundary points of \( \Omega \cap B_j \), \( j = 1, 2, \ldots \). Theorem 6.2 (using that \( -a \Rightarrow -d \)) implies that \( I_j \subset I_j' \). Moreover, \( C_p(I_j') = 0 \), by the Kellogg property for bounded sets (Theorem 3.9 in Björn–Björn–Shanmugalingam [16] or Theorem 10.5 in [11]). Hence \( C_p(I_j) = 0 \) for all \( j \), and thus by the subadditivity of the capacity, \( C_p(I) = 0 \).

As a consequence of Theorem 7.1 we obtain the following result, which in the bounded case is a direct consequence of the results in Björn–Björn–Shanmugalingam [16], [17].

**Theorem 7.2.** If \( f \in C(\partial \Omega) \), then there exists a bounded \( p \)-harmonic function \( u \) on \( \Omega \) such that there is a set \( E \subset \partial \Omega \setminus \{ \infty \} \) with \( C_p(E) = 0 \) so that

\[
\lim_{\partial \Omega \to x} u(y) = f(x) \quad \text{for } x \in \partial \Omega \setminus (E \cup \{ \infty \}). \tag{7.1}
\]

If moreover, \( \Omega \) is bounded or \( p \)-parabolic, then \( u \) is unique and \( u = Pf \).

Existence holds also for \( p \)-hyperbolic sets, which follows from the proof below, but uniqueness can fail. To see this, let \( 1 < p < n \) and \( \Omega = \mathbb{R}^n \setminus \overline{B}(0, 1) \) in unweighted \( \mathbb{R}^n \). Then both \( u(x) = |x|^{(p-n)/(p-1)} \) and \( v \equiv 1 \) are functions that are \( p \)-harmonic in \( \Omega \) and satisfy (7.1) when \( f \equiv 1 \), with \( E = \emptyset \).

**Proof.** Let \( u = Pf \) and let \( E \) be the set of irregular boundary points in \( \partial \Omega \setminus \{ \infty \} \). Then \( C_p(E) = 0 \) by the Kellogg property (Theorem 7.1), and \( u \) is bounded, \( p \)-harmonic, and satisfies (7.1), which shows the existence.

For uniqueness, suppose that \( \Omega \) is bounded or \( p \)-parabolic, and that \( u \) is a bounded \( p \)-harmonic function that satisfies (7.1). By Lemma 5.2 in Björn–Björn–Shanmugalingam [19], \( C_p(E, \Omega) \leq C_p(E) \) (the proof is valid also if \( \Omega \) is unbounded), and hence Corollary 7.9 in Hansevi [30] implies that \( u = Pf \).

\[\square\]
Another consequence of the barrier characterization is the following restriction result.

**Proposition 7.3.** Let \( x_0 \in \partial \Omega \setminus \{ \infty \} \) be regular, and let \( V \subset \Omega \) be open and such that \( x_0 \in \partial V \). Then \( x_0 \) is regular also with respect to \( V \).

**Proof.** Using the barrier characterization the proof of this fact is almost identical to the proof of the implication (c) \( \Rightarrow \) (e) in Theorem 6.2. We leave the details to the reader. \( \square \)

# 8 Boundary regularity for obstacle problems

**Theorem 8.1.** Let \( \psi : \Omega \to \mathbb{R} \) and \( f \in D^p(\Omega) \) be functions such that \( \mathcal{X}_{\phi,f} \neq \emptyset \), and let \( u \) be the lsc-regularized solution of the \( \mathcal{X}_{\phi,f} \)-obstacle problem. If \( x_0 \in \partial \Omega \setminus \{ \infty \} \) is regular, then

\[
m = \liminf_{\Omega \ni y \to x_0} u(y) \leq \limsup_{\Omega \ni y \to x_0} u(y) = M,
\]

where

\[
m := \sup \{ k \in \mathbb{R} : (f - k)_- \in D^p_0(\Omega; B) \text{ for some ball } B \ni x_0 \},
\]

\[
M := \max \left\{ M', C_p \cdot \text{ ess lim sup } \psi(y) \right\},
\]

\[
M' := \inf \{ k \in \mathbb{R} : (f - k)_+ \in D^p_0(\Omega; B) \text{ for some ball } B \ni x_0 \}.
\]

Roughly speaking, \( m \) is the lim inf of \( f \) at \( x_0 \) in the Sobolev sense and \( M' \) is the corresponding lim sup.

Observe that it is not possible to replace \( M \) by \( M' \), as it can happen that \( C_p \cdot \text{ ess lim sup}_{\Omega \ni y \to x_0} \psi(y) > M' \), see Example 5.7 in Björn–Björn [9] (or Example 11.10 in [11]).

In the case when \( \Omega \) is bounded, this improves upon Theorem 5.6 in [9] (and Theorem 11.6 in [11]) in two ways: By allowing for \( f \in D^p(\Omega) \) and by having (two) equalities in (8.1), instead of inequalities.

**Lemma 8.2.** Assume that \( 0 < \tau < 1 \). If \( h \in D^p_0(\Omega; B) \) for some ball \( B \), then \( h \in N^{1,p}_0(\Omega; \tau B) \).

**Proof.** Let \( h \in D^p_0(\Omega; B) \) for some ball \( B \). Extend \( h \) to \( B \) by letting \( h \) be equal to zero in \( B \setminus \Omega \) so that \( h \in D^p(B) \). Theorem 4.14 in [11] implies that \( h \in N^{1,p}_{\text{loc}}(B) \), and hence \( h \in N^{1,p}(\tau B) \). As \( h = 0 \) in \( \tau B \setminus \Omega \), it follows that \( h \in N^{1,p}_0(\Omega; \tau B) \). \( \square \)

It follows from Lemma 8.2 that the space \( D^p_0(\Omega; B) \) in the expressions for \( m \) and \( M' \) in Theorem 8.1 can in fact be replaced by the space \( N^{1,p}_0(\Omega; B) \) without changing the values of \( m \) and \( M' \).

**Proof of Theorem 8.1.** Let \( k > M \) be real and, using Lemma 8.2, find a ball \( B = B(x_0, r) \), with \( r < \frac{1}{4} \text{ diam } X \), so that \( (f - k)_+ \in N^{1,p}_0(\Omega; B) \) and \( k \geq C_p \cdot \text{ ess sup}_{B \cap \Omega} \psi \). Let \( V = B \cap \Omega \) and let

\[
v = \begin{cases} 
\max \{ u, k \} & \text{in } V, \\
k & \text{in } B \setminus V.
\end{cases}
\]

Since \( 0 \leq (u - k)_+ \leq (u - f)_+ + (f - k)_+ \), Lemma 5.3 in Björn–Björn [9] (or Lemma 2.37 in [11]) shows that \( (u - k)_+ \in N^{1,p}_0(\Omega; B) \). Because \( \max \{ u, k \} = k + (u - k)_+ \), we see that \( (v - k)_+ \in N^{1,p}(V; B) \) and \( v \in N^{1,p}(B) \). Let \( U = \Omega \cap \frac{1}{4} B \). The boundary weak Harnack inequality (Lemma 5.5 in [9] or Lemma 11.4 in [11]) implies that \( H_{V \cap U} \) is bounded from above on \( U \).

By Lemma 4.7 in Hansevi [29], it follows that

\[
H_{V \cap U} v \geq H_{V \cap U} k = k \geq C_p \cdot \text{ ess sup } \psi \quad \text{in } V,
\]

and hence \( H_{V \cap U} v \) is a solution of the \( \mathcal{X}_{\phi,V}(V) \)-obstacle problem. Furthermore, Proposition 3.7 in [29] shows that \( u \) is a solution of the \( \mathcal{X}_{\phi,U}(V) \)-obstacle problem, and thus \( u \leq H_{V \cap U} v \) in \( V \), by Lemma 4.2 in [29]. Hence \( u \) is bounded from above on \( U \), and thus \( v \) is bounded on \( U \).
By replacing $V$ by $U$ in the previous paragraph, we see that $u \leq H_U v$ in $U$. It follows from Theorem 4.3 (after multiplication by a suitable cutoff function) that $H_U v = \bar{P}_U v$. Theorem 6.2 asserts that $x_0$ is regular also with respect to $U$. Hence, as $v \equiv k$ on $\frac{1}{2} B \cap \partial U$, Theorem 5.1 shows that

$$\limsup_{\Omega \ni y \to x_0} u(y) = \limsup_{\Omega \ni y \to x_0} u(y) \leq \lim_{\Omega \ni y \to x_0} \bar{P}_U v(y) = v(x_0) = k.$$ 

Taking infimum over all $k > M$ shows that

$$\limsup_{\Omega \ni y \to x_0} u(y) \leq M.$$  

(8.2)

Now let $k > \limsup_{\Omega \ni y \to x_0} u(y)$ be real. Then there is a ball $B \ni x_0$ such that $u \leq k$ in $B \cap \Omega$, and hence $(u - k)_+ \equiv 0$ in $B \cap \Omega$. It follows that

$$0 \leq (f - k)_+ \leq (f - u)_+ + (u - k)_+ \in D^p_0(\Omega; B),$$

and thus $(f - k)_+ \in D^p_0(\Omega; B)$, by Lemma 2.8 in Hansevi [29]. This implies that $k \geq M'$, and hence taking infimum over all $k > \limsup_{\Omega \ni y \to x_0} u(y)$ shows that

$$\limsup_{\Omega \ni y \to x_0} u(y) \geq M'.$$

(8.3)

We also know that $u \geq \psi$ q.e., so that

$$\limsup_{\Omega \ni y \to x_0} u(y) \geq C_{p'} \limsup_{\Omega \ni y \to x_0} u(y) \geq C_{p'} \limsup_{\Omega \ni y \to x_0} \psi(y),$$

which combined with (8.2) and (8.3) shows that

$$\limsup_{\Omega \ni y \to x_0} u(y) = M,$$

and thus we have shown the last equality in (8.1).

To prove the other equality, let $k < \liminf_{\Omega \ni y \to x_0} u(y)$. Then there is a ball $B \ni x_0$ such that $k < u$ in $B \cap \Omega$, and hence $(k - u)_+ \equiv 0$ in $B \cap \Omega$. Lemma 2.8 in Hansevi [29] implies that $(f - k)_- \in D^p_0(\Omega; B)$, since

$$0 \leq (k - f)_- \leq (k - u)_+ + (u - f)_+ \in D^p_0(\Omega; B).$$

Thus $k \leq m$, and hence taking supremum over all $k < \liminf_{\Omega \ni y \to x_0} u(y)$ shows that

$$\liminf_{\Omega \ni y \to x_0} u(y) \leq m.$$

We complete the proof by applying the first part of the proof to $h := -f$ and $\psi \equiv -\infty$. Note that $Hh$ is the lsc-regularized solution of the $\mathcal{X}_{-\infty, -f}$-obstacle problem, and that $u \equiv Hf = -Hh$, by Lemma 4.2 in Hansevi [29]. Let

$$M'' = \inf \{ k \in \mathbb{R} : (h - k)_+ \in D^p_0(\Omega; B) \text{ for some ball } B \ni x_0 \}.$$

Then, as

$$\max \{ M'', -\infty \} = \inf \{ k \in \mathbb{R} : (f + k)_- \in D^p_0(\Omega; B) \text{ for some ball } B \ni x_0 \} = -\sup \{ k \in \mathbb{R} : (f - k)_- \in D^p_0(\Omega; B) \text{ for some ball } B \ni x_0 \} = -m,$$

it follows that

$$\liminf_{\Omega \ni y \to x_0} u(y) = -\limsup_{\Omega \ni y \to x_0} (-u)(y) \geq -\limsup_{\Omega \ni y \to x_0} Hh(y) = m. \quad \square$$

**Theorem 8.3.** Let $\psi : \Omega \to \mathbb{R}$ and $f \in D^p(\Omega)$ be functions such that $\mathcal{X}_{\psi, f} \neq \emptyset$, and let $u$ be the lsc-regularized solution of the $\mathcal{X}_{\psi, f}$-obstacle problem. Assume that $x_0 \in \partial \Omega \setminus \{ \infty \}$ is regular and that either
(a) \( f(x_0) := \lim_{\Omega \ni y \to x_0} f(y) \) exists, or
(b) \( f \in D^p(\mathbb{T}) \) for some ball \( B \ni x_0 \), and \( f|_{\partial \Omega \cap B} \) is continuous at \( x_0 \).

Then \( \lim_{\Omega \ni y \to x_0} u(y) = f(x_0) \) if and only if \( f(x_0) \geq C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} \psi(y) \).

In both cases we allow \( f(x_0) \) to be \( \pm \infty \).

Note that it is possible to have \( f(x_0) < C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} \psi(y) \) and still have a solvable obstacle problem, see Example 5.7 in Björn–Björn [9] (or Example 11.10 in [11]).

The proof of Theorem 8.3 is similar to the proof of Theorem 5.1 in Björn–Björn [9] (or Theorem 11.8 in [11]), but appealing to Theorem 8.1 above instead of Theorem 5.6 in [9] (or Theorem 11.6 in [11]). That one can allow for \( f(x_0) = \pm \infty \) seems not to have been noticed before.

**Proof.** Let \( m, M \), and \( M' \) be defined as in Theorem 8.1. We first show that \( M' \leq f(x_0) \). If \( f(x_0) = \infty \) there is nothing to prove, so assume that \( f(x_0) \in [-\infty, \infty) \) and let \( \alpha > f(x_0) \) be real. Also let \( B' = B(x_0, r) \) be chosen so that

\[
\lim_{x \in B' \cap \Omega} f(x) < \alpha
\]

for \( x \in B' \cap \Omega \) in case (a),
\[
\lim_{x \in B' \cap \partial \Omega} f(x) < \alpha
\]

in case (b),

with the additional requirement that \( B' \subset B \) in case (b). Then \( (f - \alpha)_+ \in D^0(\Omega; B') \) and thus \( M' \leq \alpha \). Letting \( \alpha \to f(x_0) \) shows that \( M' \leq f(x_0) \). Applying this to \( -f \) yields \( f(x_0) \leq m \).

If \( f(x_0) \geq C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} \psi(y) \), then by Theorem 8.1,

\[
f(x_0) \leq m = \lim \inf_{\Omega \ni y \to x_0} u(y) \leq \lim \sup_{\Omega \ni y \to x_0} u(y) = M \leq f(x_0),
\]

and hence \( \lim_{\Omega \ni y \to x_0} u(y) = f(x_0) \).

Conversely, if \( f(x_0) < C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} \psi(y) \), then, as \( u \geq \psi \) q.e., we see that

\[
f(x_0) < C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} \psi(y) \leq C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} u(y).
\]

\( \square \)

The following corollary is a special case of Theorem 8.3. (For the existence of a continuous solution see Section 3.)

**Corollary 8.4.** Let \( f \in D^p(\Omega) \cap C(\mathbb{T}) \) and let \( u \) be the continuous solution of the \( \mathcal{K}_{f,f} \)-obstacle problem. If \( x_0 \in \partial \Omega \setminus \{\infty\} \) is regular, then \( \lim_{\Omega \ni y \to x_0} u(y) = f(x_0) \).

### 9 Additional regularity characterizations

**Theorem 9.1.** Let \( x_0 \in \partial \Omega \setminus \{\infty\} \) and let \( B \) be a ball such that \( x_0 \in B \). Then the following are equivalent:

(a) The point \( x_0 \) is regular.
(b) For all \( f \in D^p(\Omega) \) and all \( \psi : \Omega \to \mathbb{R} \) such that \( \mathcal{K}_{\psi, f} \neq \emptyset \) and

\[
f(x_0) := \lim_{\Omega \ni y \to x_0} f(y) \geq C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} \psi(y)
\]

(where the limit in the middle is assumed to exist in \( \mathbb{R} \)), the lsc-regularized solution \( u \) of the \( \mathcal{K}_{\psi, f} \)-obstacle problem satisfies

\[
\lim_{\Omega \ni y \to x_0} u(y) = f(x_0).
\]

(c) For all \( f \in D^p(\Omega \cup (B \cap \mathbb{T})) \) and all \( \psi : \Omega \to \mathbb{R} \) such that \( \mathcal{K}_{\psi, f} \neq \emptyset \), \( f|_{\partial \Omega \cap B} \) is continuous at \( x_0 \) (with \( f(x_0) \in \mathbb{R} \)), and

\[
f(x_0) \geq C_p \cdot \text{ess lim sup}_{\Omega \ni y \to x_0} \psi(y),
\]

the lsc-regularized solution \( u \) of the \( \mathcal{K}_{\psi, f} \)-obstacle problem satisfies

\[
\lim_{\Omega \ni y \to x_0} u(y) = f(x_0).
\]
(d) The continuous solution \( u \) of the \( \mathcal{X}_{d, x_0} \)-obstacle problem, where \( d_{x_0} \) is defined by (5.1), satisfies

\[
\lim_{\Omega \ni y \to x_0} u(y) = 0.
\]

Moreover, \( u \) is a positive continuous barrier at \( x_0 \).

**Proof.** (a) \( \Rightarrow \) (b) and (a) \( \Rightarrow \) (c) These implications follow from Theorem 8.3.

(b) \( \Rightarrow \) (d) and (c) \( \Rightarrow \) (d) That (9.1) holds follows directly since (b) or (c) holds. Moreover, as \( u \geq d_{x_0} \) everywhere in \( \Omega \), we see that

\[
\lim_{\Omega \ni y \to x_0} u(y) \geq d_{x_0}(x) > 0 \quad \text{for all } x \in \partial \Omega \setminus \{x_0\}.
\]

As \( u \) is superharmonic (see Section 3), it is a positive continuous barrier at \( x_0 \).

(d) \( \Rightarrow \) (a) Since \( u \) is a barrier at \( x_0 \), Theorem 6.2 implies that \( x_0 \) is regular. \( \square \)

**Theorem 9.2.** Let \( x_0 \in \partial \Omega \setminus \{\infty\} \) and let \( B \) be a ball such that \( x_0 \in B \). Then (a) implies parts (b)–(d) below. Moreover, if \( \Omega \) is bounded or \( p \)-parabolic, then parts (a)–(d) are equivalent.

(a) The point \( x_0 \) is regular.

(b) It is true that

\[
\lim_{\Omega \ni y \to x_0} Hf(y) = f(x_0)
\]

for all \( f \in D^p(\Omega) \) such that \( f(x_0) := \lim_{\Omega \ni y \to x_0} f(y) \) exists.

(c) It is true that

\[
\lim_{\Omega \ni y \to x_0} Hf(y) = f(x_0)
\]

for all \( f \in D^p(\Omega \cup (B \cap \overline{\Omega})) \) such that \( f|_{\partial \Omega \cap B} \) is continuous at \( x_0 \).

(d) It is true that

\[
\lim_{\Omega \ni y \to x_0} f(y) \geq f(x_0)
\]

for all \( f \in D^p(\Omega \cup (B \cap \overline{\Omega})) \) that are superharmonic in \( \Omega \) and such that \( f|_{\partial \Omega} \) is lower semicontinuous at \( x_0 \).

As in Theorems 8.3 and 9.1 we allow for \( f(x_0) = \infty \) in (b)–(d). We do not know if (a)–(d) are equivalent when \( \Omega \) is \( p \)-hyperbolic.

**Proof.** (a) \( \Rightarrow \) (b) and (a) \( \Rightarrow \) (c) Apply Theorem 9.1 to \( f \) (with \( \psi \equiv \infty \)). Then these implications are immediate as \( Hf \) is the continuous solution of the \( \mathcal{X}_{\infty, f} \)-obstacle problem.

(a) \( \Rightarrow \) (d) Theorem 6.2 asserts that the point \( x_0 \) is regular with respect to \( V := \Omega \cap B \). If \( f(x_0) = \infty \) there is nothing to prove, so assume that \( f(x_0) \in (-\infty, \infty] \) and let \( a < f(x_0) \) be real.

As \( f|_{\partial \Omega} \) is lower semicontinuous at \( x_0 \), there is \( r \) such that \( 0 < r < \text{dist}(x_0, \partial B) \) and \( f \geq a \) in \( B(x_0, r) \cap \partial V \).

Let \( h = \min(f, a) \), which is also superharmonic in \( \Omega \), by Lemma 9.3 in [11]. It thus follows from Lemma 3.5 that \( h \) is the lsc-regularized solution of the \( \mathcal{X}_{h, h}(V) \)-obstacle problem. Since \( h - a = 0 \) in \( B(x_0, r) \cap \partial V \), we get that

\[
h - a \in D^p_0(V; B(x_0, r)).
\]

By applying Theorem 8.1 with \( h \) and \( V \) in the place of \( f = \psi \) and \( \Omega \), respectively, we see that \( m \geq a \), where \( m \) is as in Theorem 8.1, and hence

\[
\lim_{\Omega \ni y \to x_0} f(y) = \lim_{\Omega \ni y \to x_0} f(y) \geq \lim_{\Omega \ni y \to x_0} h(y) = m \geq a.
\]

Letting \( a \to f(x_0) \) yields the desired result.

We now assume that \( \Omega \) is bounded or \( p \)-parabolic.

(b) \( \Rightarrow \) (a) and (c) \( \Rightarrow \) (a) Observe that the function \( d_{x_0} \) in Theorem 5.1 satisfies the conditions for \( f \) in both (b) and (c). Theorem 4.3 applies to \( d_{x_0} \), and hence it follows that \( x_0 \) is regular, by Theorem 5.1, as

\[
\lim_{\Omega \ni y \to x_0} \mathcal{F}d_{x_0}(y) = \lim_{\Omega \ni y \to x_0} Hd_{x_0}(y) = d_{x_0}(x_0) = 0.
\]
(d) ⇒ (a) Let
\[ f = \begin{cases} 
Hd_{x_0} & \text{in } \Omega, \\
d_{x_0} & \text{on } \partial \Omega.
\end{cases} \]

Because both \( f \) and \(-f\) satisfy the hypothesis in (d), we see that
\[ 0 = f(x_0) = \liminf_{y \to x_0} f(y) = \liminf_{y \to x_0} Hd_{x_0}(y) \]
and
\[ \limsup_{y \to x_0} Hd_{x_0}(y) = -\liminf_{y \to x_0} (-f(y)) \leq f(x_0) = 0. \]

Theorem 4.3 implies that \( Hd_{x_0} = \overline{Pd}_{x_0} \), and hence
\[ 0 = \liminf_{y \to x_0} \overline{Pd}_{x_0}(y) \leq \limsup_{y \to x_0} \overline{Pd}_{x_0}(y) \leq 0, \]
which shows that \( \lim_{y \to x_0} \overline{Pd}_{x_0}(y) = 0 \). Thus \( x_0 \) is regular by Theorem 5.1.

The following two results remove the assumption of bounded sets from the \( p \)-harmonic versions of Lemma 7.4 and Theorem 7.5 in Björn [6] (or Theorem 11.27 and Lemma 11.32 in [11]).

**Theorem 9.3.** If \( x_0 \in \partial \Omega \setminus \{ \infty \} \) is irregular with respect to \( \Omega \), then there is exactly one component \( G \) of \( \Omega \) with \( x_0 \in \partial G \) such that \( x_0 \) is irregular with respect to \( G \).

**Lemma 9.4.** Suppose that \( \Omega_1 \) and \( \Omega_2 \) are nonempty disjoint open subsets of \( X \). If \( x_0 \in (\partial \Omega_1 \cap \partial \Omega_2) \setminus \{ \infty \} \), then \( x_0 \) is regular with respect to at least one of these sets.

The lemma follows directly from the sufficiency part of the Wiener criterion, see [6] or [11]. With straightforward modifications of the proof of Theorem 7.5 in [6] (or Theorem 11.27 in [11]), we obtain a proof for Theorem 9.3. For the reader’s convenience, we give the proof here.

**Proof of Theorem 9.3.** Suppose that \( x_0 \in \partial \Omega \setminus \{ \infty \} \) is irregular. Then Theorem 5.1 implies that
\[ \limsup_{y \to x_0} \overline{Pd}_{x_0}(y) > 0. \]

Let \( \{y_j\}_{j=1}^\infty \) be a sequence in \( \Omega \) such that
\[ \lim_{j \to \infty} y_j = x_0 \quad \text{and} \quad \lim_{j \to \infty} \overline{Pd}_{x_0}(y_j) = \limsup_{y \to x_0} \overline{Pd}_{x_0}(y). \]

Assume that there are infinitely many components of \( \Omega \) containing points from the sequence \( \{y_j\}_{j=1}^\infty \).

Then we can find a subsequence \( \{y_{j_k}\}_{k=1}^\infty \) such that each component of \( \Omega \) contains at most one point from the sequence \( \{y_{j_k}\}_{k=1}^\infty \). Let \( G_k \) be the component of \( \Omega \) containing \( y_{j_k} \), \( k = 1, 2, \ldots \). Then
\[ \lim_{k \to \infty} \overline{Pd}_{x_0}(y_{j_k}) = \lim_{k \to \infty} \overline{Pd}_{x_0}(y_{j_k,1}) > 0, \]
and thus \( x_0 \) is irregular both with respect to \( \Omega_1 := \bigcup_{k=1}^\infty G_{2k} \) and with respect to \( \Omega_2 := \bigcup_{k=1}^\infty G_{2k+1} \), by Theorem 5.1. Since \( \Omega_1 \) and \( \Omega_2 \) are disjoint, this contradicts Lemma 9.4. We conclude that there are only finitely many components of \( \Omega \) containing points from the sequence \( \{y_j\}_{j=1}^\infty \).

Thus there is a component \( G \) that contains infinitely many of the points from the sequence \( \{y_j\}_{j=1}^\infty \). So there is a subsequence \( \{y_{j_k}\}_{k=1}^\infty \) such that \( y_{j_k} \in G \) for every \( k = 1, 2, \ldots \). It follows that \( x_0 \in \partial G \) and as
\[ \lim_{k \to \infty} \overline{Pd}_{x_0}(y_{j_k}) > 0, \]
x_0 must be irregular with respect to \( G \).

Finally, if \( G' \) is any other component of \( \Omega \) with \( x_0 \in \partial G' \), then, by Lemma 9.4, \( x_0 \) is regular with respect to \( G' \).
Acknowledgement: The first author was supported by the Swedish Research Council, grant 2016-03424.

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