ALMOST KÄHLER 4-DIMENSIONAL LIE GROUPS WITH J-INVARIANT RICCI TENSOR

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Abstract. The aim of this paper is to determine left-invariant strictly almost Kähler structures on 4-dimensional Lie groups \((g, J, \Omega)\) such that the Ricci tensor is \(J\)-invariant.

1. Introduction

An almost Kähler structure on a manifold \(M\) of real dimension \(n = 2m\) consists of a symplectic form \(\Omega\), an almost complex structure \(J\) and a Riemannian metric \(g\), satisfying the compatibility condition

\[
\Omega(X, Y) = g(JX, Y),
\]

for any vector fields \(X, Y\) on \(M\). If \(J\) is integrable, then the triple \((g, J, \Omega)\) is a Kähler structure on \(M\). In real dimension 2, the notions of almost Kähler and Kähler structure coincide, but this does not hold in higher dimensions. Through the paper, “strictly almost Kähler” will mean that the corresponding almost complex structure is non-integrable (or, equivalently, that the almost Kähler structure is non-Kähler).

Given a symplectic manifold \((M, \Omega)\), there are many \(\Omega\)-compatible almost Kähler structures on \(M\), i.e. many pairs \((g, J)\) which satisfy the relation (1). The space \(\mathcal{AK}(M, \Omega)\) of all \(\Omega\)-compatible metrics (or almost complex structures) is an infinite dimensional, contractible Fréchet space. The problem of finding distinguished Riemannian metrics in \(\mathcal{AK}(M, \Omega)\) has been intensively studied (for a survey, see [3]). In particular, such a kind of metric is given by an Einstein one. A still open conjecture by Goldberg affirms that there exist no Einstein, strictly almost Kähler metrics on a compact symplectic manifold [10]. This conjecture is true if the scalar curvature is non-negative [21] and there many positive results in dimension four under different additional assumptions on the curvature [1, 4, 5, 19]. Moreover, compactness is important in the above conjecture since in [18] an example of a non-compact Ricci flat strictly almost Kähler manifold is given.

On a compact manifold \(M\), the Einstein condition agrees with the Euler-Lagrange equation of the Hilbert functional (integral of the scalar curvature) acting on the space

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of all Riemannian metrics on $M$ of a given volume $[7]$. Blair and Ianus $[8]$ restricted such a functional to the space $AK(M, \Omega)$ and proved that its critical points are the almost Kähler metrics $(g, J)$ whose Ricci tensor $\rho$ is $J$-invariant, i.e. satisfies
\begin{equation}
\rho(JX, JY) = \rho(X, Y),
\end{equation}
for any vector fields $X, Y$. Therefore, the previous condition is weaker than the Einstein and Kähler condition. Almost Kähler metrics $(g, J)$ satisfying (2) have been recently studied in $[14]$ and have been called “harmonic almost Kähler structures”.

In this paper we study such harmonic strictly almost Kähler structures on simply-connected 4-dimensional Riemannian homogeneous spaces, in particular on 4-dimensional real Lie groups. In general, any simply connected 4-dimensional Riemannian homogeneous space is either symmetric or isometric to a Lie group with a left-invariant metric $[6]$. By $[17]$ there exists no a compact strictly almost Kähler locally symmetric space.

The problem of existence of special left-invariant metrics, like Einstein and anti-self-dual metrics, have been studied respectively in $[12]$ and $[9]$, where a classification is obtained.

The almost Kähler structures $(g, J, \Omega)$ that we consider are invariant, in the sense that $g, J, \Omega$ are left-invariant tensors. We prove that a simply-connected 4-dimensional real Lie group with an invariant harmonic strictly almost Kähler structure is necessarily solvable and we give a general description of its Lie algebra. Moreover, we prove that all these Lie groups are isometric (up to homothety) to the (unique) 4-dimensional proper 3-symmetric space (described by Kowalski $[13]$) and satisfy the condition $[11]$
\begin{equation}
(G2) \quad R_{XYZW} = R_{JXJYZW} + R_{JXJYZW} + R_{JXYJZW}.
\end{equation}
This space is given by $\mathbb{R}^4$ endowed with the Riemannian metric
\[
g = \left( -x + \sqrt{x^2 + y^2 + 1} \right) du^2 + \left( x + \sqrt{x^2 + y^2 + 1} \right) dv^2 - 2y du dv + \lambda^2(1 + x^2 + y^2)^{-1}[(1 + y^2) dx^2 + (1 + x^2) dy - 2xy dx dy], \quad \lambda > 0.
\]
The underlying homogeneous space $M$ is $\mathbb{R}^2 \ltimes SL(2, \mathbb{R})/SO(2)$
\[
\left\{ \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{with} \quad \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1
\]
and admits a simply transitive action of a 4-dimensional (real) Lie group $\mathbb{R}^2 \ltimes \text{Sol}_2$ $[3]$, where $\text{Sol}_2$ consists of the upper triangular matrices with positive diagonal entries, $[3, 24]$. Choosing a basis of the Lie algebra of $\mathbb{R}^2 \ltimes \text{Sol}_2$ consisting of $\{e_1, e_2\}$ in $\mathbb{R}^2$ and
\[
e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
in $\mathfrak{so}_2 \subset \mathfrak{sl}_2(\mathbb{R})$, the following are the non zero Lie brackets
\[ [e_1, e_3] = -e_1 = [e_2, e_4], \quad [e_2, e_3] = e_2, \quad [e_3, e_4] = 2e_4. \]

2. Structure equations

Let $G$ be a real 4-dimensional Lie group with a left-invariant Riemannian metric $g$ (which corresponds to an inner product $g$ on the Lie algebra $\mathfrak{g}$ of $G$). The curvature tensor $R$ of a left-invariant Riemannian metric is completely determined by its value at the identity element of $G$. Hence $R$ is an element of the vector space
\[ \mathcal{R} = S^2(\Lambda^2 \mathfrak{g}^*) \oplus \Lambda^4 \mathfrak{g}^*, \]
where $\mathfrak{g}$ is the Lie algebra of $G$, and the Ricci tensor $\rho$ can be identified as the component of $R$ on $S^2(\mathfrak{g}^*)$.

Then, the curvature tensor of $g$ and its Ricci tensor are completely determined by the structure constants of the Lie algebra $\mathfrak{g}$ and its inner product. Therefore the problem is purely algebraic.

**Theorem 2.1.** If $G$ is a simply-connected real 4-dimensional Lie group with an invariant strictly almost Kähler structure $(g, J, \Omega)$ such that the Ricci tensor is $J$-invariant then there exist an orthonormal basis $\{e^1, e^2, e^3, e^4\}$ of left-invariant 1-forms such that
\[
\begin{align*}
\Omega &= e^1 \wedge e^2 + e^3 \wedge e^4, \\
J e^1 &= e^2, \quad J e^3 = e^4,
\end{align*}
\]
\[
\begin{align*}
d e^1 &= -s(e^1 \wedge e^3 - e^2 \wedge e^4) + \frac{(-s^2 + t^2)}{2t} e^1 \wedge e^4 + te^2 \wedge e^3, \\
d e^2 &= -s^2 e^1 \wedge e^3 + s(e^1 \wedge e^4 + e^2 \wedge e^3) + \frac{(s^2 - t^2)}{2t} e^2 \wedge e^4, \\
d e^3 &= \frac{(t^2 + s^2)}{t} e^3 \wedge e^4, \quad d e^4 = 0,
\end{align*}
\]
where $t \neq 0$ and $s$ are real numbers. Relative to the above orthonormal basis, the Ricci tensor has the diagonal form
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -3(t^2 + s^2)^2 & 0 \\
0 & 0 & 0 & -3(t^2 + s^2)^2
\end{pmatrix}.
\]

**Proof.** Since $(G, \Omega)$ is a 4-dimensional Lie group with a left-invariant symplectic structure $\Omega$, $G$ has to be solvable [15]. Then the dimension of the commutator $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is less than equal to 3. One can suppose that there exists an orthonormal basis of
left-invariant 1-forms \( \{e^1, e^2, e^3, e^4\} \) on \( G \) such that

\[
\begin{align*}
\Omega &= e^1 \wedge e^2 + e^3 \wedge e^4, \\
J e^1 &= e^2, J e^3 = e^4, \\
de^1 &= a_1 e^1 \wedge e^2 + a_2 e^1 \wedge e^3 + a_3 e^1 \wedge e^4 + a_4 e^2 \wedge e^3 + a_5 e^2 \wedge e^4 + a_6 e^3 \wedge e^4, \\
de^2 &= b_1 e^1 \wedge e^2 + b_2 e^1 \wedge e^3 + b_3 e^1 \wedge e^4 + b_4 e^2 \wedge e^3 + b_5 e^2 \wedge e^4 + b_6 e^3 \wedge e^4, \\
de^3 &= c_1 e^1 \wedge e^2 + c_2 e^1 \wedge e^3 + c_3 e^1 \wedge e^4 + c_4 e^2 \wedge e^3 + c_5 e^2 \wedge e^4 + c_6 e^3 \wedge e^4, \\
de^4 &= 0,
\end{align*}
\]

where \( a_i, b_i, c_i, i = 1, \ldots, 6 \), are arbitrary real numbers. Thus, to describe the 4-dimensional Lie groups with a strictly almost Kähler structure such that the Ricci tensor is \( J \)-invariant is equivalent to determine the real numbers \( a_i, b_i, c_i, i = 1, \ldots, 6 \), such that

\[
\begin{cases}
\frac{d^2 e^i}{dt^2} = 0, & i = 1, 2, 3, \\
\frac{d\Omega}{dt} = 0, \\
\rho(e_1, e_1) = \rho(e_2, e_2), & \rho(e_1, e_2) = 0, \\
\rho(e_1, e_3) = \rho(e_2, e_4), & \rho(e_1, e_4) = -\rho(e_2, e_3), \\
\rho(e_3, e_3) = \rho(e_4, e_4), & \rho(e_3, e_4) = 0,
\end{cases}
\]

(3)

(where \( \{e_1, \ldots, e_4\} \) is the dual basis of \( \{e^1, \ldots, e^4\} \) ) and \( J \) is non integrable.

The condition \( d\Omega = 0 \) is equivalent to

\[
\begin{cases}
c_1 = a_3 + b_5, \\
a_2 + b_4 = 0, \\
b_6 - c_2 = 0, \\
a_6 + c_4 = 0
\end{cases}
\]

(4)

and the conditions \( \frac{d^2 e^i}{dt^2} = 0, i = 1, 2, 3 \), are:

\[
\begin{cases}
-a_1 b_6 - c_6 a_2 - b_3 a_4 + a_5 b_2 + b_6 a_6 = 0, \\
-a_1 a_2 - b_1 a_4 - b_6 a_4 - a_6 a_2 = 0, \\
a_1 b_5 + a_2 c_5 - a_3 a_6 - c_3 a_4 - b_1 a_5 - b_5 a_6 = 0, \\
a_1 a_6 - 2a_2 a_5 + a_3 a_4 - b_3 a_4 - c_6 a_4 - a_6^2 = 0, \\
b_1 b_6 - a_3 b_2 - c_6 b_2 + 2a_2 b_3 + b_5 b_2 + b_6^2 = 0, \\
b_1 a_2 - a_1 b_2 - a_6 b_2 + b_6 a_2 = 0, \\
b_1 a_3 + b_2 c_5 - a_1 b_3 + a_2 c_3 - b_5 b_6 - a_3 b_6 = 0, \\
b_1 a_6 - a_5 b_2 + b_3 a_4 + c_6 a_2 - b_6 a_6 = 0, \\
-2a_3 b_6 - b_5 b_6 + a_6 b_3 + c_3 a_2 + b_2 c_5 = 0, \\
2a_3 b_5 + b_2^2 + a_3^2 + b_6 c_5 + c_3 a_6 - a_1 c_3 - c_6 a_3 - c_6 b_5 - b_1 c_5 = 0, \\
a_3 a_6 + 2b_5 a_6 - b_6 a_5 + c_3 a_4 - a_2 c_5 = 0, \\
a_1 b_6 - a_6 b_1 = 0.
\end{cases}
\]

(5)
The Ricci tensor $\rho$ of the metric $g$ is $J$-invariant if and only if

$$
\begin{align*}
\rho(e_1, e_1) - \rho(e_2, e_2) &= \frac{1}{2}c_5^2 - c_6a_3 - b_5a_4 + a_1^2 - b_2^2 + a_5^2 - b_3^2 + c_6b_5 \\
-\frac{3}{2}b_6^2 + \frac{3}{2}a_6^2 + b_6^2 - a_3b_2 - a_3a_4 - a_2^2 - b_5b_2 - \frac{1}{2}c_3^2 - a_1a_6 + b_1b_6 = 0, \\
\rho(e_1, e_2) &= -b_1a_6 - a_2a_4 + \frac{3}{2}b_6a_6 - \frac{1}{2}c_3c_5 - b_3b_5 + a_2b_5 \\
a_2a_3 + a_2b_2 - a_5a_3 - \frac{1}{2}c_6a_5 - \frac{1}{2}c_6b_3 = 0, \\
\rho(e_1, e_3) - \rho(e_2, e_4) &= a_6c_6 + \frac{3}{2}b_6a_2 - \frac{1}{2}b_6a_5 \\
-\frac{1}{2}a_4a_6 - c_6c_3 + \frac{1}{2}a_5c_5 + \frac{1}{2}a_3a_6 - a_6b_5 \\
+\frac{1}{2}a_2c_5 - \frac{3}{2}a_6a_3 - a_6b_2 \\
\frac{1}{2}c_3a_3 - \frac{1}{2}c_3a_4 + a_1a_3 + a_1a_4 + b_1b_3 - b_1a_2 = 0, \\
\rho(e_1, e_3) + \rho(e_2, e_4) &= -\frac{1}{2}a_6b_3 + \frac{3}{2}a_6a_2 \\
+\frac{1}{2}a_4b_6 + \frac{1}{2}c_5b_5 + \frac{1}{2}c_5b_2 + \frac{1}{2}c_3b_3 + \frac{1}{2}c_3a_2 - a_3b_6 - \frac{3}{2}b_6b_5 + b_6c_6 \\
-a_1a_2 + a_1a_5 - b_1b_2 + b_1b_5 - c_6c_5 + \frac{1}{2}b_2b_6 = 0, \\
\rho(e_3, e_3) - \rho(e_4, e_4) &= -2a_2^2 + c_5^2 - c_6a_3 - \frac{1}{2}a_4^2 - a_4b_2 \\
-\frac{3}{2}b_6^2 + a_5b_3 + \frac{1}{2}a_2^2 + \frac{1}{2}b_2^2 - c_6b_5 + b_6c_5 + c_3a_6 - a_6^2 \\
-\frac{3}{2}b_6^2 + \frac{5}{2}b_2^2 + a_3b_5 + \frac{5}{2}a_2^2 - b_1b_6 - a_1a_6 + c_3^2 = 0, \\
\rho(e_3, e_4) &= a_6c_5 - \frac{1}{2}b_1c_3 + \frac{1}{2}b_1a_6 - a_2a_3 + a_6b_6 - b_6c_3 \\
-b_6a_6 + \frac{1}{2}a_1c_5 - \frac{1}{2}a_1b_6 - \frac{1}{2}b_3a_4 - \frac{1}{2}b_3b_2 \\
a_2b_5 - \frac{1}{2}b_2a_4 - \frac{1}{2}b_2b_5 = 0.
\end{align*}
$$

Moreover, $J$ is integrable if and only if

$$
\begin{align*}
c_4 &= -c_3, c_5 = c_2, \\
b_4 &= -b_3 + a_2 - a_5, \\
b_5 &= b_2 + a_3 + a_4.
\end{align*}
$$

Then, by imposing (4), (5), and (6), we get the following cases for the relations between the non-zero parameters:

**case 1)** $b_3 = -a_5, c_6$ any real number;

**case 2)** $c_6 = 2b_5, b_3 = -a_5, a_3 = b_5$;

**case 3)** $c_4 = -a_6, a_1 = \frac{a_3^2 + a_4a_6 + a_5^2}{a_6}, a_3 = -a_4, c_3 = a_6$;

**case 4)** $c_6 = -a_5, a_3 = -a_4, a_4$ any real value;

**case 5)** $c_6 = 2b_5, a_6 = -b_5, b_2 = -2b_5$;

**case 6)** $c_6 = 2b_5, a_3 = -b_5, b_2 = 2b_5$;

**case 7)** $a_4 = -b_2, a_5 = b_3$;

**case 8)** $a_2 = -b_4, a_3 = \frac{b_4}{b_5}, a_4 = -\frac{b_2}{b_5}, a_5 = -b_4, b_2 = b_5, b_3 = -b_4, c_6 = \frac{(b_4^2 + b_2^2)}{b_5}$;

**case 9)** $a_1, b_1, c_6$ any real value;

**case 10)** $a_3 = -\frac{1}{2}b_2, b_5 = \frac{1}{2}b_2, c_6 = b_2$;
case 11) \( a_3 = \frac{1}{2}b_2, b_5 = -\frac{1}{2}b_2, c_6 = -b_2; \)

case 12) \( b_1 = \frac{1}{2}c_6, b_4 = \frac{1}{2}b_5, c_5 = b_5, c_6 = b_6; \)

case 13) \( a_1 = a_6, b_1 = b_6, c_2 = b_6, c_3 = a_6, c_5 = b_6, c_6 \) any real value;

case 14) \( a_3 = \frac{1}{2}c_6, a_4 = c_6, b_5 = -\frac{1}{2}c_6; \)

case 15) \( a_3 = \frac{1}{2}c_6, a_4 = -c_6, b_5 = -\frac{1}{2}c_6; \)

case 16) \( a_2 = -a_5, a_3 = \left(\frac{a_5-a_4^2}{a_4}\right), b_2 = -\frac{a_5}{a_4}, b_3 = a_5, b_4 = a_5, b_5 = \left(\frac{a_5-a_4^2}{a_4}\right), \)
\( c_6 = \left(\frac{a_5-a_4^2}{a_4}\right), \)

Using (11) we can exclude the cases 1), 2), 3), 4), 6), 7), 8), 9), 10), 12), 13), 15), since in all these cases \( J \) is integrable. In the remaining cases the corresponding Lie group has the following structure equations:

\[
5) \begin{cases} 
\text{de}^1 = -b_5 e^1 \wedge e^4, \text{de}^2 = b_5 (e^2 \wedge e^4 - 2e^1 \wedge e^3), \\
\text{de}^3 = 2b_5 e^3 \wedge e^4, \text{de}^4 = 0,
\end{cases}
\]

(with \( \rho(e_1, e_1) = 0 \) and \( \rho(e_3, e_3) = -6b_2^2 \))

11) \[
\begin{cases} 
\text{de}^1 = \frac{1}{2}b_2 e^1 \wedge e^4, \text{de}^2 = b_2 e^1 \wedge e^3 - \frac{1}{2}b_2 e^2 \wedge e^4, \\
\text{de}^3 = -b_2 e^3 \wedge e^4, \text{de}^4 = 0,
\end{cases}
\]

(with \( \rho(e_1, e_1) = 0 \) and \( \rho(e_3, e_3) = -\frac{3}{2}b_2^2 \))

14) \[
\begin{cases} 
\text{de}^1 = \frac{1}{2}c_6 e^1 \wedge e^4 + c_6 e^2 \wedge e^3, \text{de}^2 = -\frac{1}{2}c_6 e^2 \wedge e^4, \\
\text{de}^3 = c_6 e^3 \wedge e^4, \text{de}^4 = 0,
\end{cases}
\]

(with \( \rho(e_1, e_1) = 0 \) and \( \rho(e_3, e_3) = -\frac{3}{2}c_6^2 \))

16) \[
\begin{cases} 
\text{de}^1 = -a_5 (e^1 \wedge e^3 - e^2 \wedge e^4) + \left(\frac{a_5-a_4^2}{a_4}\right) e^1 \wedge e^4 + a_4 e^2 \wedge e^3, \\
\text{de}^2 = \frac{a_5}{a_4} e^1 \wedge e^3 + a_5 (e^1 \wedge e^4 + e^2 \wedge e^3) + \left(\frac{a_5-a_4^2}{a_4}\right) e^2 \wedge e^4, \\
\text{de}^3 = \left(\frac{a_5-a_4^2}{a_4}\right) e^3 \wedge e^4, \text{de}^4 = 0,
\end{cases}
\]

(with \( \rho(e_1, e_1) = 0 \) and \( \rho(e_3, e_3) = -\frac{3}{2}(a_4^2 + a_4^2)^2 a_4^2 \))

17) \[
\begin{cases} 
\text{de}^1 = a_5 (e^1 \wedge e^3 + e^2 \wedge e^4) + \left(\frac{a_5-a_4^2}{a_4}\right) e^1 \wedge e^4 + a_4 e^2 \wedge e^3, \\
\text{de}^2 = -\frac{a_5}{a_4} e^1 \wedge e^3 + a_5 (e^1 \wedge e^4 - e^2 \wedge e^3) + \left(\frac{a_5-a_4^2}{a_4}\right) e^2 \wedge e^4, \\
\text{de}^3 = \left(\frac{-a_4^2-a_4^2}{a_4}\right) e^3 \wedge e^4, \text{de}^4 = 0,
\end{cases}
\]

(with \( \rho(e_1, e_1) = 0 \) and \( \rho(e_3, e_3) = -\frac{3}{2}(a_4^2 + a_4^2)^2 a_4^2 \)).

If one puts \( b_5 = -\frac{1}{2}b_2 \) in the structure equations 5) one gets the structure equations 11). Moreover, the Lie group with structure equations 14) is isomorphic to 5)
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considering the new basis
\[ f_1 = -e^2, f_2 = e^1, f_3 = e^3, f_4 = e^4. \]

If one put \( a_5 = 0 \) in the structure equations 16) one gets 5). Then 5) is a particular case of 16). Finally 17) is isomorphic to 16) considering the new basis
\[ f_1 = e^1, f_2 = -e^2, f_3 = -e^3, f_4 = -e^4. \]

and the respective metrics are isometric. Thus \( G \) must have structure equations 16).

Remark 2.1. One has that
\[ \text{trace}(ad_{e_4}) = \frac{t^2 + s^2}{t} \neq 0. \]

Therefore \( G \) is not unimodular and it does not admit any compact quotient [16].

Remark 2.2. For any \( t \neq 0 \) and \( s \in \mathbb{R} \) the metric \( g \) of Theorem 2.1 is never Einstein. Then, there exist no left-invariant Einstein, strictly almost Kähler metrics on real 4-dimensional Lie groups.

If one investigates left-invariant (non flat) Kähler-Einstein metrics on 4-dimensional Lie groups, by using the proof of the above theorem and then by imposing the conditions (4), (5), (6), (7) and
\[ \rho(e_1, e_1) = \rho(e_3, e_3), \]

one gets the following cases for the relations between the non zero parameters:

\text{case 2)} with \( b_5 \neq 0 \): the corresponding Lie group has structure equations:
\[ \begin{align*}
  de_1 &= b_5 e_1 \land e^4 + a_5 e^2 \land e^4, \\
  de_2 &= -a_5 e_1 \land e^4 + b_5 e^2 \land e^4, \\
  de_3 &= 2b_5(e_1 \land e^2 + e^3 \land e^4), \\
  de_4 &= 0,
\end{align*} \]

and \( \rho(e_1, e_1) = -6b_5^2 \).

\text{case 3)} with \( c_6 = \frac{a_1^2 + 3a_0^2a_4}{a_6^2 - a_4^2} \): the corresponding Lie group has structure equations:
\[ \begin{align*}
  de_1 &= \frac{a_1^2 + 3a_0^2a_4}{a_6^2 - a_4^2} e_1 \land e^2 - a_4 e_1 \land e^4 + a_4 e^2 \land e^3 + a_6 e^3 \land e^4, \\
  de_2 &= 0, \\
  de_3 &= -a_4 e_1 \land e^2 + a_6(e_1 \land e^4 - e^2 \land e^3) + \frac{a_1^2 + 3a_0^2a_4}{a_6^2 - a_4^2} e_3 \land e^4, \\
  de_4 &= 0,
\end{align*} \]

and \( \rho(e_1, e_1) = -2\frac{(a_1^2 + a_0^2)^3}{(a_1^2 - a_6^2)^3} \).

\text{case 4)} with \( a_1 = 0 \): the corresponding Lie group has structure equations:
\[ \begin{align*}
  de_1 &= a_4(-e_1 \land e^4 + e^2 \land e^3), \\
  de_2 &= 0, \\
  de_3 &= -a_4(e_1 \land e^2 + e^3 \land e^4), \\
  de_4 &= 0,
\end{align*} \]
and \( \rho(e_1, e_1) = -2a^2_t \). This case is equivalent to the case 3) imposing \( a_6 = 0 \).

**case 8):** the corresponding Lie group has structure equations:

\[
\begin{align*}
d e^1 &= -b_4(e^1 \wedge e^3 + e^2 \wedge e^4) + \frac{b_0^2}{b_5}(e^1 \wedge e^4 - e^2 \wedge e^3), \\
d e^2 &= b_5(e^1 \wedge e^3 + e^2 \wedge e^4) - b_4(e^1 \wedge e^4 - e^2 \wedge e^3), \\
d e^3 &= \frac{b_1 + b_2}{b_5}(e^1 \wedge e^2 + e^3 \wedge e^4),
\end{align*}
\]

and \( \rho(e_1, e_1) = -\frac{2(b_1^2 + b_2^2)^2}{b_5} \).

**case 9) with** \( a_1^2 + b_1^2 = c_6^2 \): the corresponding Lie group has structure equations:

\[
\begin{align*}
d e^1 &= a_1 e^1 \wedge e^2, \\
d e^2 &= b_1 e^1 \wedge e^2, \\
d e^3 &= c_6 e^3 \wedge e^4, \\
d e^4 &= 0
\end{align*}
\]

and \( \rho(e_1, e_1) = -a_1^2 - b_1^2 \).

**case 12) with** \( c_6 = \frac{b_1^2 + b_2^2}{b_5^2 - b_6^2} \): the corresponding Lie group has structure equations:

\[
\begin{align*}
d e^1 &= 0, \\
d e^2 &= \left(\frac{-b_0^2 - 3b_2^2 b_6}{b_5^2 - b_6^2}\right) e^1 \wedge e^2 + b_2(e^1 \wedge e^3 + e^2 \wedge e^4) + b_6 e^3 \wedge e^4, \\
d e^3 &= b_2 e^1 \wedge e^2 + b_6(e^1 \wedge e^3 + e^2 \wedge e^4) + \frac{b_1^2 + 3b_2^2 b_6}{b_5^2 - b_6^2} e^3 \wedge e^4, \\
d e^4 &= 0,
\end{align*}
\]

and \( \rho(e_1, e_1) = -2(b_1^2 + b_2^3)^3 \). This case is equivalent to the case 8) taking the new basis \( \{ f^1 = -e^2, f^2 = e^1, f^3 = e^3, f^4 = e^4 \} \) and \( b_2 = -a_4, b_6 = a_6 \).

**case 13) with** \( c_6 = 0 \): the corresponding Lie group has structure equations:

\[
\begin{align*}
d e^1 &= a_6(e^1 \wedge e^2 + e^3 \wedge e^4), \\
d e^2 &= b_6(e^1 \wedge e^2 + e^3 \wedge e^4), \\
d e^3 &= b_6(e^1 \wedge e^3 + e^2 \wedge e^4) + a_6(e^1 \wedge e^4 - e^2 \wedge e^3), \\
d e^4 &= 0,
\end{align*}
\]

and \( \rho(e_1, e_1) = -2a_6^2 - 2b_6^2 \).

Comparing all the above possibilities with the list of non-isomorphic Lie algebras given in [12] in the case 2) one gets the family of non isomorphic Lie algebras, for distinct values of \( t \), with structure equations

\[
\begin{align*}
d f^1 &= 0, \\
d f^2 &= f^1 \wedge f^2 - tf^1 \wedge f^3, \\
d f^3 &= tf^1 \wedge f^2 - f^1 \wedge f^3, \\
d f^4 &= df^2 = -2f^1 \wedge f^4 + 2f^2 \wedge f^3, \quad 0 \leq t < \infty,
\end{align*}
\]

and as Riemannian space each of these is the hermitian hyperbolic space. In the remaining cases one gets the direct sum of a 2-dimensional Lie algebra with itself and as Riemannian space the direct product of a 2-dimensional solvable group manifold, of curvature \(-1\), with itself.

A classification of 4-dimensional solvable Lie algebras appears for example in [23] and a description of all simply-connected, 4-dimensional solvable real Lie groups which have commutators of dimension 3 is also given in [20], where a classification of such Lie groups admitting a left-invariant complex structure is given. We can prove that
all the Lie algebras with structure equations 16) are isomorphic to the Lie algebra of type \( g_{4,9}(\alpha) \) with structure equations

\[
\begin{align*}
df^1 &= 0, \\
df^2 &= (1 - \alpha)f^1 \wedge f^2, \\
df^3 &= -f^1 \wedge f^3, \\
df^4 &= -\alpha f^1 \wedge f^4 - f^2 \wedge f^3.
\end{align*}
\]

and \( \alpha = \frac{1}{2} \). The Lie algebra \( g_{4,9}(2) \) is one of the Lie algebras with anti-self-dual non conformally flat inner product \([9]\).

**Theorem 2.2.** Let \( G \) be a 4-dimensional Lie group admitting an invariant strictly almost Kähler structure \( (g, J, \Omega) \) such that the Ricci tensor is \( J \)-invariant then its Lie algebra \( g \) is isomorphic to the Lie algebra \( g_{4,9}(\frac{1}{2}) \) (in the notation of [23]) with structure equations

\[
\begin{align*}
df^1 &= 0, \\
df^2 &= \frac{1}{2}f^1 \wedge f^2, \\
df^3 &= -f^1 \wedge f^3, \\
df^4 &= -\frac{1}{2}f^1 \wedge f^4 - f^2 \wedge f^3.
\end{align*}
\]

and \( g \) is homothetic to the metric

\[ h = (f^1)^2 + (f^2)^2 + (f^3)^2 + (f^4)^2. \]

**Proof.** By the previous Theorem, \( g \) has structure equations 16) and thus it is an abelian extension of the Lie algebra of the real 3-dimensional Heisenberg Lie algebra. Indeed, one has that \( G \) is 3-step solvable with the commutator \( g^1 = [g, g] \) the real 3-dimensional Heisenberg Lie algebra, spanned by \( \{Z, X, Y\} \) with \( Z \) central and \( [X, Y] = Z \), where

\[ Z = a_4 e_1 + a_5 e_2, \quad X = e_3, \quad Y = e_2. \]

One can remark that \( ad_{e_4} \) is a derivation of \( g^1 \) such that

\[
\begin{align*}
ad_{e_4}(Z) &= \eta Z, \\
ad_{e_4}(X) &= aZ + \mu X + \delta Y, \\
ad_{e_4}(Y) &= bZ + kX + \nu Y,
\end{align*}
\]

with

\[
\eta = \frac{(a_4^2 + a_5^2)}{2a_4} = \frac{1}{2}\mu = -\nu, \quad \delta = a = k = 0, \quad b = \frac{a_5}{a_4}.
\]

Then if one consider on \( G \) the new basis \( \{A, Z, X, Y\} \) (where \( A = \frac{1}{\mu} e_4 \)), one has the following non zero Lie brackets

\[ [A, Z] = \frac{1}{2} Z, \quad [A, X] = X, \quad [A, Y] = \frac{b}{\mu} Z - \frac{1}{2} Y, \quad [X, Y] = Z. \]
If one considers the new basis
\[
\begin{align*}
 f_1 &= A, f_2 = \frac{1}{\sqrt{a_4^2 + a_5^2}}(-Y + \frac{b}{\mu}Z), f_3 = \frac{1}{\mu}X, f_4 = \frac{a_4}{(a_4^2 + a_5^2)^{3/2}}Z
\end{align*}
\]
(which is orthogonal with respect to \(g\)) one has
\[
[f_1, f_4] = \frac{1}{2}f_4, \quad [f_1, f_2] = -\frac{1}{2}f_2, \quad [f_1, f_3] = f_3, \quad [f_2, f_3] = f_4
\]
and then one obtains the Lie algebra \(\mathfrak{g}_{4,9}(\frac{1}{2})\) with structure equations \([8]\). Moreover, \((\mathfrak{g}, g)\) is isometric to \(\mathfrak{g}_{4,9}(\frac{1}{2})\) endowed with the metric
\[
\frac{(a_4^2 + a_5^2)^2}{a_4^2}((f^1)^2 + (f^2)^2 + (f^3)^2 + (f^4)^2),
\]
since \(||f_i||^2 = \frac{1}{\mu^2} = \frac{a_4^2}{(a_4^2 + a_5^2)^2}\), for any \(i = 1, 2, 3, 4\).

**Remark 2.3.** It is easy to check that the Lie algebra \(\mathfrak{g}_{4,9}(\frac{1}{2})\) is isomorphic to the Lie algebra of \(\mathbb{R}^2 \ltimes \text{Sol}_2\).

### 3. Curvature

If \((M, g)\) is an oriented 4-dimensional Riemannian manifold, the action of the Hodge \(*\) operator on the bundle of the 2-forms \(\Lambda^2 M\) induces the decomposition
\[
\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,
\]
into the subbundles of self-dual and anti-self-dual 2-forms. If one considers the Riemannian curvature tensor \(R\) as a symmetric endomorphism of \(\Lambda^2 M\) one has the following \(SO(4)\)-splitting
\[
R = \frac{s}{12}Id|_{\Lambda^2 M} + \tilde{\rho}_0 + W^+ + W^-,
\]
where \(s\) is the scalar curvature, \(\tilde{\rho}_0\) is the Kulkarni-Nomizu extension of the traceless Ricci tensor \(\rho_0\) to an endomorphism of \(\Lambda^2 M\) (anti-commuting with \(*\)) and \(W^\pm\) are the self-dual and anti-self-dual parts of the Weyl tensor \(W\).

If \((M, g, J)\) is an almost Hermitian 4-manifold and \(\Omega\) the corresponding Kähler form, the action of \(J\) gives rise to the following orthogonal splitting
\[
\Lambda^+ M = \mathbb{R}\Omega \oplus [[\Lambda^{0,2} M]],
\]
where \([[\Lambda^{0,2} M]]\) denotes the bundle of the bundle of the \(J\)-anti-invariant real 2-forms. Then, the vector bundle \(W^+ = S_0^2(\Lambda^+ M)\) of the symmetric traceless endomorphisms of \(\Lambda^+ M\) decomposes into the sum of the three sub-bundles \(W^+_i, i = 1, 2, 3\) [22]. More precisely:
\(W^+_1\) is the trivial line bundle generated by the element \(\frac{1}{8}\Omega \otimes \Omega - \frac{1}{12}Id|_{\Lambda^+ M};\)
\(W^+_2 = [[\Lambda^{0,2} M]]\) is the subbundle of elements which exchange the two factor in \([9]\);
$W^+_3 = S^3_0(\Lambda^{0,2}M)$ is the subbundle of elements preserving the splitting and acting trivially on the first factor $\mathbb{R} \Omega$.

Moreover
\[
\tilde{\rho}_0 = \tilde{\rho}_0^{\text{inv}} + \tilde{\rho}_0^{\text{anti}},
\]
where $\tilde{\rho}_0^{\text{inv}}$ and $\tilde{\rho}_0^{\text{anti}}$ are the Kulkarni-Nomizu extensions of the $J$-invariant and $J$-anti-invariant parts of the traceless Ricci tensor.

By [2] any strictly almost Kähler 4-manifold whose curvature satisfies the three conditions
(i) $\tilde{\rho}_0^{\text{anti}} = 0$,  
(ii) $W^+_2 = 0$,  
(iii) $W^+_3 = 0$

is locally isometric to the (unique) 4-dimensional proper 3-symmetric space.

The relations (i), (ii), (iii) are closely related to the conditions defined by A. Gray [11]

\begin{align*}
(G1) & \quad R_{XYZW} = R_{XYJZW}, \\
(G2) & \quad R_{XYZW} = R_{JXJYW} + R_{JXYJZW} + R_{JXYZW}, \\
(G3) & \quad R_{XYZW} = R_{JXJYJZW}.
\end{align*}

Indeed, by [2] an almost Hermitian 4-manifold $(M, g, J)$ satisfies the property (G3) if and only if the Ricci tensor is $J$-invariant and $W^+_2 = 0$. It satisfies (G2) if, in addition, $W^+_3 = 0$. By [2], a complete, simply connected strictly almost Kähler 4-manifold which satisfies the condition (G2) is isometric to the proper 3-symmetric space.

By Theorem 2.1 if $G$ is a simply connected real 4-dimensional Lie group with an invariant strictly almost Kähler structure $(g, J, \Omega)$ such that the Ricci tensor is $J$-invariant, then the non zero components of the Riemannian curvature $R$ in terms of the orthonormal basis $\{e_1, \ldots, e_4\}$ are given by
\begin{align*}
R_{1212} &= R_{1234} = R_{3412} = \frac{1}{2} \frac{(t^2+s^2)^2}{t^2}, \\
R_{1313} &= -R_{1324} = R_{1414} = R_{1423} = R_{2314} = R_{2323} = -R_{2413} = R_{2424} = -\frac{1}{4} \frac{(t^2+s^2)^2}{t^2}, \\
R_{3434} &= -\frac{(t^2+s^2)^2}{t^2}.
\end{align*}

By direct computation, one has that $W^+_2 = 0$, since $\rho^*(R - L_3 R) = 0$, where
\begin{align*}
\rho^*(R)(x, y) &= \sum_{i=1}^4 R(x, e_i, Jy, Je_i), \\
(L_3 R)(x, y, z, w) &= R(Jx, Jy, Jz, Jw)
\end{align*}

and $W^+_3 = 0$ since $\frac{1}{4}(I - L_2)(I + L_3)(R) = 0$, where
\begin{align*}
(L_2 R)(x, y, z, w) &= \frac{1}{2} [R(x, y, z, w) + R(Jx, Jy, z, w) + R(Jx, y, Jz, w) + R(Jx, y, z, Jw)].
\end{align*}

Then the Lie group $G$ satisfies the condition (G2). As a consequence,

**Corollary 3.1.** If $G$ is a simply connected real 4-dimensional Lie group with an invariant strictly almost Kähler structure $(g, J, \Omega)$ such that the Ricci tensor is $J$-invariant, then $G$ satisfies the condition (G2) and (up to homothety) is isometric to the (unique) 4-dimensional proper 3-symmetric space.
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