On Extensions of supersingular representations of $\text{SL}_2(\mathbb{Q}_p)$.

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October 10, 2018

Abstract

In this note for $p > 5$ we calculate the dimensions of $\text{Ext}^1_{\text{SL}_2(\mathbb{Q}_p)}(\tau, \sigma)$ for any two irreducible supersingular representations $\tau$ and $\sigma$ of $\text{SL}_2(\mathbb{Q}_p)$.

1 Introduction

In this note we calculate the space of extensions of supersingular representations of $\text{SL}_2(\mathbb{Q}_p)$ for $p > 5$. The dimensions of the space of extensions between irreducible supersingular representations of $\text{GL}_2(\mathbb{Q}_p)$ are calculated by Paškūnas in [Paš10]. Understanding extensions between irreducible smooth representations play a crucial role in Paškūnas work on the image of Colmez Montreal functor in (see [Paš13]). We hope that these results have similar application to mod $p$ and $p$-adic local Langlands correspondence for $\text{SL}_2(\mathbb{Q}_p)$.

Let $G$ be the group $\text{GL}_2(\mathbb{Q}_p)$, $K$ be the maximal compact subgroup $\text{GL}_2(\mathbb{Z}_p)$ and $Z$ be the center of $G$. We denote by $I(1)$ the pro-$p$ Iwahori subgroup of $G$. We denote by $G_S$ the special linear group $\text{SL}_2(\mathbb{Q}_p)$. For any subgroup $H$ of $\text{GL}_2(\mathbb{Q}_p)$ we denote by $H_S$ the subgroup $H \cap \text{SL}_2(\mathbb{Q}_p)$. All representations in this note are defined over vector spaces over $\overline{\mathbb{F}}_p$. Let $\sigma$ be an irreducible smooth representation of $K$ and $\sigma$ extends uniquely as a representation of $K_Z$ such that $p \in Z$ acts trivially. The Hecke algebra $\text{End}_G(\text{ind}^G_K Z \sigma)$ is isomorphic to $\overline{\mathbb{F}}_p[T]$. For any constant $\lambda$ in $\overline{\mathbb{F}}_p^*$ let $\mu_\lambda$ be the unramified character of $Z$ such that $\mu_\lambda(p) = \lambda$. Let $\pi(\sigma, \mu_\lambda)$ be the representation

$$\text{ind}_{KZ}^G \sigma \otimes (\mu_\lambda \circ \det).$$

The representations $\pi(\sigma, \mu_\lambda)$ are irreducible (see [Bre03]) and are called supersingular representations in the terminology of Barthel–Livné.

Let $\sigma_r$ be the representation $\text{Sym}^r \overline{\mathbb{F}}_p$ of $\text{GL}_2(\overline{\mathbb{F}}_p)$. We consider $\sigma_r$ as a representation of $K$ by inflation. The $K$-socle of $\pi(\sigma_r, \mu_\lambda)$ is a direct sum of two irreducible smooth representations $\sigma_r$ and $\sigma_{p-1-r}$. Let $\pi_{0,r}$ and $\pi_{1,r}$ be the $G_S$ representations generated by $\sigma_r^{I(1)}$ and $\sigma_{p-1-r}^{I(1)}$. The representations $\pi_{0,r}$ and $\pi_{1,r}$ are irreducible supersingular representations of $G_S$ and

$$\text{res}_{G_S} \pi(\sigma_r, \mu_\lambda) \simeq \pi_{0,r} \oplus \pi_{1,r}.$$

Any irreducible supersingular representation of $G_S$ is isomorphic to $\pi_{i,r}$ for some $r$ such that $0 \leq r \leq p - 1$ and $i \in \{0, 1\}$. Moreover the only isomorphisms between $\pi_{i,r}$ are $\pi_{0,r} \simeq \pi_{1,p-1-r}$ and $\pi_{1,r} \simeq \pi_{0,p-1-r}$ (see [Abd14]). Our main theorem on extensions of supersingular representations of $G_S$ is:
Theorem 1.1. Let \( p \geq 5 \) and \( 0 \leq r \leq (p - 1)/2 \). For any irreducible supersingular representation \( \tau \) of \( G_S \) the space \( \text{Ext}^1_{G_S}(\tau, \pi_{i,r}) \) is non-zero if and only if \( \tau \simeq \pi_{j,r} \) for some \( j \in \{0,1\} \). If \( 0 \leq r < (p - 1)/2 \) then \( \dim_{\mathbb{F}_p} \text{Ext}^1_{G_S}(\pi_{i,r}, \pi_{j,r}) = 2 \) for \( i \neq j \) and \( \dim_{\mathbb{F}_p} \text{Ext}^1_{G_S}(\pi_{i,r}, \pi_{i,r}) = 1 \). For \( r = (p - 1)/2 \) we have \( \dim_{\mathbb{F}_p} \text{Ext}^1_{G_S}(\pi_{0,r}, \pi_{0,r}) = 3 \).

We briefly explain the method of proof. We essentially follow [Pas10]. The functor sending a smooth representation to its \( G \)-invariants induces an equivalence of categories of smooth representations of \( G_S \) generated by \( I(1)_S \)-invariants and the module category of the pro \( p \)-Iwahori Hecke algebra (see [Koz16, Theorem 5.2]). We use the Ext spectral sequence thus obtained by this equivalence of categories to calculate \( \text{Ext}^1_{G_S} \). Extensions of pro \( p \)-Iwahori Hecke algebra modules are calculated from resolutions of Hecke modules due to Schneider and Ollivier. We crucially use results from work of Paskunas [Pas10]. We first obtain lower bounds on the dimensions of \( \text{Ext}^1_{G_S} \) spaces using the spectral sequence and then obtain upper bounds using Paskunas results on \( \text{Ext}^1_K(\sigma, \pi(\sigma, \mu_\lambda)) \).

Acknowledgements I thank Eknath Ghate for showing the fundamental paper [Pas10] and for his interest in this work and discussions on the role of extensions in mod \( p \)-Langlands. I want to thank Radhika Ganapathy for various discussions on mod \( p \) representations and for her mod \( p \) seminar at the Tata Institute.

2 Pro-\( p \) Iwahori Hecke algebra

Let \( B \) be the Borel subgroup consisting of invertible upper triangular matrices, \( U \) be the unipotent radical of \( B \) and \( T \) be the maximal torus consisting of diagonal matrices. We denote by \( \bar{U} \) the unipotent radical of \( \bar{B} \) the Borel subgroup consisting of invertible lower triangular matrices. We denote by \( I \) the standard Iwahori-subgroup of \( G \). Let \( I(1) \) be the pro-\( p \) Iwahori subgroup of \( G \) and \( I(1)_S \) be the pro-\( p \)-Iwahori subgroup of \( G_S \). We note that \( I(1)_S(Z \cap I(1)) \) is equal to \( I(1) \). Let \( \mathcal{H} \) be the pro-\( p \) Iwahori–Hecke algebra \( \text{End}_G(\text{ind}_{I(1)_S}^{G_S} \text{id}) \). Let \( \text{Rep}_{G_S} \) and \( \text{Rep}_{G_S}^{I(1)_S} \) be the category of smooth representations of \( G_S \) and its full subcategory consisting of those smooth representations generated by \( I(1)_S \)-invariant vectors respectively. We denote by \( \text{Mod}_\mathcal{H} \) the category of modules over the ring \( \mathcal{H} \). We have two functors

\[
\mathcal{I} : \text{Rep}_{G_S}^{I(1)_S} \rightarrow \text{Mod}_\mathcal{H} \\
\mathcal{I}(\pi) = \pi^{I(1)_S}
\]

and

\[
\mathcal{T} : \text{Mod}_\mathcal{H} \rightarrow \text{Rep}_{G_S}^{I(1)_S} \\
\mathcal{T}(M) = M \otimes_\mathcal{H} \text{ind}_{I(1)_S}^{G_S} \text{id}.
\]

From [Koz16] Theorem 5.2 the functors \( \mathcal{T} \) and \( \mathcal{I} \) are quasi-inverse to each other. Let \( \sigma \) and \( \tau \) be any two smooth representations of \( G_S \) and \( \sigma_1 \) be the \( G_S \) subrepresentation of \( \sigma \) generated by \( I(1)_S \)-invariants of \( \sigma \). We have

\[
\text{Hom}_G(\tau, \sigma) = \text{Hom}_G(\tau, \sigma_1) = \text{Hom}_H(\mathcal{I}(\tau), \mathcal{I}(\sigma_1)) = \text{Hom}_H(\mathcal{I}(\tau), \mathcal{I}(\sigma)).
\]

We get a Grothendieck spectral sequence with \( E_2^{ij} \) equal to \( \text{Ext}^i(\mathcal{I}(\tau), \mathcal{R}^j\mathcal{I}(\sigma)) \) such that

\[
\text{Ext}^i(\mathcal{I}(\tau), \mathcal{R}^j\mathcal{I}(\sigma)) \Rightarrow \text{Ext}^{i+j}_{G_S}(\tau, \sigma).
\]
The 5-term exact sequence associated to the above spectral sequence gives the following exact sequence:

\[
0 \to \text{Ext}_H^1(\mathcal{I}(\tau), \mathcal{I}(\sigma)) \xrightarrow{i} \text{Ext}_G^1(\tau, \sigma) \xrightarrow{\delta} \\
\text{Hom}_H(\mathcal{I}(\tau), \mathcal{I}(\sigma)) \to \text{Ext}_H^2(\mathcal{I}(\tau), \mathcal{I}(\sigma)) \to \text{Ext}_G^2(\tau, \sigma)
\]

for all \( \tau \) such that \( \tau = < G_S \tau^{(1)} > \). In particular we apply these results when \( \tau \) and \( \sigma \) are irreducible supersingular representations of \( G_S \). We first recall the structure of the ring \( \mathcal{H} \), its modules \( M(i, r) = \pi_{i,r}^{(1)} \) for \( i \) in \( \{0, 1\} \) and \( 0 \leq r \leq p-1 \). The \( \mathcal{H} \) module \( M(i, r) \) is a character and we first calculate the dimensions of the spaces \( \text{Ext}_H^1(M(i, r), M(j, s)) \).

Let \( T_0^S \) and \( T_1^S \) be the maximal compact subgroup of \( T_S \) and its maximal pro-\( p \)-subgroup. We denote by \( s_0, s_1 \) and \( \theta \) the matrices \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), \( \begin{pmatrix} 0 & -p^{-1} \\ p & 0 \end{pmatrix} \) and \( \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \) respectively. Let \( N(T_S) \) be the normaliser of the torus. The extended Weyl group \( W = \theta^\mathbb{Z} \coprod s_0 \theta^\mathbb{Z} \) sits into an exact sequence of the form

\[
0 \to \Omega := \frac{T_0^S}{T_1^S} \to \tilde{W} := \frac{N(T_S)}{T_1^S} \to W = \frac{N(T_S)}{T_0^S} \to 0.
\]

The length function \( l \) on \( W \), given by \( l(\theta^i) = |2i| \) and \( l(s_0 \theta^i) = |1 - 2i| \), extends to a function on \( \tilde{W} \) such that \( l(\Omega) = 0 \). Let \( T_w \) be the element \( \text{Char}_{\mathcal{I}(1)w(1)} \) for all \( w \in \tilde{W} \). We denote by \( e_1 \) the element \( \sum_{w \in \Omega} T_w \). The functions \( T_w \) span \( \mathcal{H} \) and the relations in \( \mathcal{H} \) are given by

\[
T_w T_v = T_{vw} \text{ whenever } l(v) + l(w) = l(vw), \\
T_{s_i}^2 = -e_1 T_{s_i}.
\]

The pro-\( p \)-Iwahori Hecke algebra is generated by \( T_w T_{s_i} \) for \( w \) in \( \Omega \). For any character \( \chi \) of \( \Omega \) let \( e_\chi \) be the element \( \sum_{w \in \Omega} \chi^{-1}(w) T_w \). Let \( \gamma \) be a \( W_0 \) orbit of the characters \( \chi \) and \( e_\gamma \) be the element \( \sum_{\chi \in \gamma} e_\chi \). The elements \( \{e_\gamma; \gamma \in \hat{\Omega}/W_0\} \) are central idempotents in the ring \( \mathcal{H} \) and we have

\[
\mathcal{H} = \bigoplus_{\hat{\Omega}/W_0} \mathcal{H} e_\gamma.
\]

For the group \( G_S \), we know that \( \mathcal{H} \) is the affine Hecke algebra. The characters of affine Hecke algebra are described in a simple manner we recall this for \( G_S \). Let \( I \) be a subset of \( \{s_0, s_1\} \) and \( W_I \) be the subgroup of \( W \) generated by elements of \( I \) and \( W_0 \) is trivial group. The characters of \( \mathcal{H} \) are parametrised by pairs \( (\lambda, I) \) where \( \lambda \) is a character of \( \Omega \) and \( I \subset S_\lambda \). For such a pair \( (\lambda, I) \) the character \( \chi_{\lambda, I} \) associated to it is given by

\[
\chi_{\lambda, I}(T_w t) = 0 \text{ for all } w \in W \setminus W_I \text{ and for all } t \in \Omega, \\
\chi_{\lambda, I}(T_w t) = \lambda(t)(-1)^{l(w)} \text{ for all } w \in W_I \text{ and for all } t \in \Omega.
\]

If \( \lambda \) is nontrivial then we have \( \chi_{\lambda, 0}(T_I) = \lambda(t) \), for all \( t \in \Omega \) and \( \chi_{\lambda, 0}(T_w t) = 0 \) for all \( w \neq \text{id} \) and \( t \in \Omega \).

We denote by \( \chi_{r, \theta} \) the character \( \chi_{r \to r^*, \theta} \). From the above description we get that \( M(0, r) = \chi_{r, 0} \) and \( M(1, r) = \chi_{p-1-r, \theta} \) for \( r \notin \{0, p-1\} \). If \( r \in \{0, p-1\} \) then [OST16] Proposition 3.9 says that \( \chi_{\text{id}, 0} \) and \( \chi_{\text{id}, S} \) are not supersingular characters. This shows that \( M(i, r) \) is either \( \chi_{\text{id}, I} \) or \( \chi_{\text{id}, J} \), for \( r \in \{0, p-1\} \), where \( I = \{s_0\} \) and \( J = \{s_1\} \). Since the element \( T_{s_0} \) belongs to pro-\( p \) Iwahori–Hecke
algebra of $G$ and using the presentation in [BP12, Corollary 6.4] we obtain that $M(1, 0) = \chi_{\text{id}, J}$ and $M(0, 0) = \chi_{\text{id}, J}$. Similarly $M(1, p - 1)$ is given by the character $\chi_{\text{id}, J}$ and $M(0, p - 1)$ is given by the character $\chi_{\text{id}, J}$. Let $0 \leq r, s \leq (p - 1)/2$ then (11) shows that

$$\text{Ext}^1_{\mathcal{H}}(M(i, r), M(j, s)) = 0 \quad (7)$$

for $r \neq s$.

2.1 Resolutions of Hecke modules

In order to calculate extensions between the characters $M(i, r)$, we use resolutions constructed by Schneider and Ollivier for $\mathcal{H}$. Let $X$ be the Bruhat–Tits tree of $G_S$ and let $A(T_S)$ be the standard apartment associated to $T_S$. We fix an edge $E$ and vertices $v_0$ and $v_1$ contained in $E$ such that the $G_S$-stabiliser of $v_0$ is $K_S$. For any facet $F$ of $X$ we denote by $G_F$ the $\mathbb{Z}_p$-group scheme with generic fibre $\text{SL}_2$ and $G_F(\mathbb{Z}_p)$ is the $G$-stabiliser of $F$. We denote by $I_F$ the subgroup of $G_F(\mathbb{Z}_p)$ whose elements under mod-$p$ reduction of $G_F(\mathbb{Z}_p)$ belong to the $\mathbb{F}_p$-points of the unipotent radical of $G_F \times \mathbb{F}_p$. We denote by $\mathcal{H}_F$ the finite subalgebra of $\mathcal{H}$ defined as

$$\mathcal{H}_F := \text{End}_{G_F(\mathbb{Z}_p)}(\text{ind}_{I_F}^{G_F(\mathbb{Z}_p)}(\text{id})).$$

In particular $\mathcal{H}_E$ is a semi-simple algebra.

For any $\mathcal{H}$-module $m$ the construction of Schneider and Ollivier [OS14, Theorem 3.12, (6.4)] gives us a ($\mathcal{H}, \mathcal{H}$)-exact resolution

$$0 \to \mathcal{H} \otimes_{\mathcal{H}_E} m \xrightarrow{\delta_1} (\mathcal{H} \otimes_{\mathcal{H}_{v_0}} m) \oplus (\mathcal{H} \otimes_{\mathcal{H}_{v_1}} m) \xrightarrow{\delta_0} m \to 0. \quad (8)$$

Using the resolution (8) and the observation that $\mathcal{H}_E$ is semi-simple for $p \neq 2$ we get that

$$0 \to \text{Hom}_{\mathcal{H}}(m, n) \to \bigoplus_{v_0, v_1} \text{Hom}_{\mathcal{H}_{v_i}}(m, n) \to \text{Hom}_{\mathcal{H}_E}(m, n) \xrightarrow{\delta} \text{Ext}^1_{\mathcal{H}}(m, n) \to \bigoplus_{v_0, v_1} \text{Ext}^1_{\mathcal{H}_{v_i}}(m, n) \to 0 \quad (9)$$

Note that we have an isomorphism of algebras

$$\mathcal{H}_{v_0} \simeq \mathcal{H}_{v_1} \simeq \text{End}_{\text{SL}_2(\mathbb{F}_p)}(\text{ind}_{N(\mathbb{F}_p)}(\text{id})).$$

The above isomorphism is not a canonical isomorphism. Let $K_0$ and $K_1$ be the compact open subgroups $K \cap G_S$ and $K_{\Pi} \cap G_S$ respectively.

2.2 Extensions of supersingular modules over pro-$p$ Iwahori–Hecke algebra.

The Hecke algebra $\mathcal{H}_{v_i}$ is isomorphic to $\text{End}_{K_i}(\text{ind}_{I_i}^{K_i}(\text{id})$. The Hecke algebra $\mathcal{H}_{v_i}$ is generated by $T_i$ and $T_{s_i}$ for $t \in \Omega$. The relations among them are given by

$$T_{i_1}T_{i_2} = T_{i_1i_2},$$
$$T_iT_{s_i} = T_{i}s_i - T_{s_i}T_{i} = T_{s_i}T_{i} - T_{i}T_{s_i},$$
$$T_{s_i}^2 = -e_1T_{s_i}$$

where $e_1 = \sum_{t \in \Omega} T_t.$
**Lemma 2.1.** Let $0 < r < (p-1)/2$ the space $\text{Ext}^1_{H}(M(i,r), M(j,s))$ is non-zero if and only if $i \neq j$ and has dimension 2 when $i = j$. If $r = (p-1)/2$ then the space $\text{Ext}^1_{H}(M(i,r), M(i,r))$ has dimension 2.

**Proof.** Since $r \neq 0$ the characters $M(0, r)$ and $M(1, r)$ are isomorphic to $\chi_{r,0}$ and $\chi_{p-1-r,0}$ respectively (see [5]). Let $E_{c}$ be a 2-dimensional $\mathbb{F}_p$ module $\mathbb{F}_p e_1 \oplus \mathbb{F}_p e_2$ and $\mathbb{F}_p[\Omega]$ acts on $E$ by $T_ie_0 = t^r e_0$ and $T_ie_1 = t^{p-1-r} e_1$. We set $T_i e_0 = 0$ and $T_s e_1 = c e_0$ for some $c \neq 0$. This makes $E$ a $H_{v_i}$ module and is a non-trivial extension

$$0 \to \chi_{r,0} \to E \to \chi_{p-1-r,0} \to 0.$$ 

Let $E$ be a $H_{v_i}$-extension of $W := \chi_{s,0}$ by $V := \chi_{r,0}$ i.e, we have an exact sequence

$$0 \to V \to E \xrightarrow{f} W \to 0.$$ 

There exists a $\mathbb{F}_p[\Omega]$-equivariant section $s : W \to E$ of $f$. Let $V'$ be the image of this section. Now $E = V \oplus V'$. The action of $T_i$ is trivial on $V$ and observe that $f(T_i(V')) = T_i(f(V')) = 0$. If $E$ is nontrivial then $T_i(V') = V$. This implies that $r + s = p - 1$ and hence $E$ is isomorphic to $E_c$ for some $c \neq 0$. This shows that the space of $H_{v_i}$ extensions of $W$ by $V$ is one dimensional if $r + s = p - 1$ and zero otherwise. Now consider the exact sequence ([6]) when $m$ is $M(i,r)$ and $n$ is $M(j,r)$. For $i = j$ the map $\delta$ in zero ([6]) hence the space $\text{Ext}_H^1(M(i,r), M(i,r))$ is trivial. When $i \neq j$ the Hom spaces in ([6]) are all trivial. This shows that the dimension of the space $\text{Ext}_H^1(M(i,r), M(i,r))$ is 2 from our calculations. \hfill $\Box$

**Lemma 2.2.** The space of extensions $\text{Ext}_H^1(M(i,0), M(j,0))$ is trivial for $i = j$ and has dimension 1 for $i \neq j$.

**Proof.** The algebra $e_1H_{v_i}$ is semi-simple algebra and hence we get that

$$\text{Ext}_{H_{v_i}}^1(\chi_{id,S}, \chi_{id,S'}) = 0$$

for all $i > 0$ and for subsets $S$ and $S'$ of $\{s_0, s_1\}$. Now consider the exact sequence ([7]) when $m$ is $M(i,r)$ and $n$ is $M(j,r)$. For $i = j$ the map $\delta$ in ([7]) is zero hence the space $\text{Ext}_H^1(M(i,r), M(i,r))$ is trivial. When $i \neq j$ the first two Hom spaces in ([7]) are trivial. The space $\text{Hom}_{H_{v_i}}(m,n)$ has dimension one. This shows that the dimension of the space $\text{Ext}_H^1(M(i,r), M(i,r))$ is 1 for $i \neq j$. \hfill $\Box$

### 3 The Hecke module $\mathbb{R}^1T_{\pi_{i,r}}$.

Paškūnas calculated the cohomology groups $\mathbb{R}^1T_{\pi_{i,r}}$ and we now recall his results. Let $\pi_r$ be the supersingular representation $\pi(\sigma_r, \mu_1)$ of $G$. Recall that the $K$-socle of $\pi_r$ is isomorphic to $\sigma_r \oplus \sigma_{p-1-r}$ and the space of $I(1)$ invariants has a basis $(v_0, v_1)$ where $v_0$ and $v_1$ belong to $\sigma_r$ and $\sigma_{p-1-r}$ respectively. Let $I^+$ and $I^-$ be the groups $I \cap U$ and $I \cap \bar{U}$ respectively. Consider the spaces

$$M_0 := I^+ \theta^n v_1; \ n \geq 0 \quad \text{and} \quad M_1 := I^+ \theta^n v_2; \ n \geq 0$$

and let $\Pi$ be the matrix $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ which normalizes $I$ and $I(1)$. We denote by $\pi_0$ and $\pi_1$ the spaces $M_0 + \Pi M_1$ and $M_1 + \Pi M_0$. Let $G^0$ be subgroup of $G$ consisting of elements with integral discriminant. Let $G^+$ be the group $ZG^0$. We denote by $Z_1$ the group $I(1) \cap Z$. 

**Proposition 3.1** (Paškūnas). The spaces $\sigma_0$ and $\sigma_1$ are $G^+$ stable. The space $\tilde{\pi}_r$ is the direct sum of the representations $\pi_1$ and $\pi_0$ as $G^+$ representations and hence $\pi_{i,r}$ is isomorphic to $\pi_i$ as $G_S$ representations for $i \in \{0,1\}$. If $r$ be an integer such that $0 < r < (p-1)/2$ then the Hecke module $\mathbb{R}^1I(\pi_{i,r})$ is isomorphic to $I(\pi_{i,r}) \oplus I(\pi_{i,r})$. In the Iwahori case (i.e, $r = 0$) the Hecke module $\mathbb{R}I(\pi_{0,0}) \oplus \mathbb{R}I(\pi_{1,0})$ is isomorphic to $I(\pi_{0,0}) \oplus I(\pi_{1,0})^2$.

**Proof.** The first part follows from [Paš10, Corollary 6.5]. The second part follows from [Paš10, Proposition 10.5, Theorem 10.7 and equation (49)]. \qed

**Corollary 3.2.** Let $\tau$ be an irreducible supersingular representation of $G_S$. If the space of extensions $\text{Ext}_{G_S}^1(\tau, \pi_{i,r})$ is non-trivial then $\tau \simeq \pi_{j,r}$ for some $j \in \{0,1\}$.

**Proof.** This follows from (3), (7) and Proposition 3.1. \qed

**Corollary 3.3.** Let $0 < r < (p-1)/2$ and $i \neq j$ then the dimensions of the space $\text{Ext}_{G_S}^1(\pi_{i,r}, \pi_{j,r})$ is 2.

**Proof.** Observe that for $0 < r < (p-1)/2$ the modules $M(i,r)$ and $M(j,s)$ are not isomorphic. Now using the exact sequence (3) and Proposition 3.1 we get that

$$\text{Ext}_{G_S}^1(\pi_{i,r}, \pi_{j,r}) \simeq \text{Ext}_R^1(M(i,r), M(j,r)).$$

The corollary follows from the Lemma 2.1. \qed

**Remark 3.4.** The results of Corollary 3.3 remain valid for $r = 0$ but we prove this later. It is interesting to note that for $0 < r < (p-1)/2$ and $i \neq j$ any extension $E$ of $\pi_{i,r}$ by $\pi_{j,r}$ for $i \neq j$ is generated by its $I(1)_S$ invariants, i.e, $E = \langle G_S E^{I(1)_S} \rangle$.

4 Calculation of degree one self extensions.

Let us first consider the case when $0 < r \leq (p-1)/2$. In order to determine the dimensions of $\text{Ext}_{G_S}^1(\pi_{i,r}, \pi_{i,r})$ we first show that the map

$$\text{Ext}_{G_S}^1(\pi_{i,r}, \pi_{i,r}) \to \text{Hom}_R(I(\pi_{i,r}), \mathbb{R}^1I(\pi_{i,r}))$$

is non-zero. Explicitly the above map takes an extension $E$, with $0 \to \pi_{i,r} \to E \to \pi_{i,r} \to 0$, to the delta map in the associated long exact sequence, given by: $I(\pi_{i,r}) \xrightarrow{\delta_E} \mathbb{R}^1I(\pi_{i,r})$. Note that the dimension of $E^{I(1)}$ is one if and only if $\delta_E \neq 0$.

**Lemma 4.1.** For $0 < r \leq (p-1)/2$ then map (11) is non-zero.

**Proof.** For $0 < r \leq (p-1)/2$ there exists a self extension $E$ of $\tilde{\pi}_r$ such that the map $I(\tilde{\pi}_r) \xrightarrow{\delta_E} \mathbb{R}^1I(\tilde{\pi}_r)$ is non-zero. We fix an extension $E$ such that $\delta_E \neq 0$. Since $\delta_E$ is a Hecke-equivariant map and $I(\tilde{\pi}_r)$ is an irreducible Hecke-module of dimension 2 we get that the inclusion map of $I(\tilde{\pi}_r)$ in $I(E)$ is an isomorphism i.e, dim $E^{I(1)} = 2$. Now consider the pullback diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{\pi}_r & \longrightarrow & E_1 & \longrightarrow & \pi_{i,r} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\pi}_r & \longrightarrow & E & \longrightarrow & \tilde{\pi}_r & \longrightarrow & 0.
\end{array}
$$

(12)
The long exact sequences in $I(1)$-group cohomology attached to (12) gives us:

$$
\begin{array}{ccccccc}
0 & \to & \mathcal{I}(\pi_r) & \xrightarrow{f} & \mathcal{I}(E_1) & \to & \mathcal{I}(\pi_{i,r}) & \xrightarrow{\delta_2} & \mathcal{R}^1\mathcal{I}(\tilde{\pi}_r) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{I}(\pi_r) & \to & \mathcal{I}(E) & \to & \mathcal{I}(\tilde{\pi}_r) & \to & \mathcal{R}^1\mathcal{I}(\tilde{\pi}_r).
\end{array}
$$

Since the dimension of $\mathcal{I}(E)$ is 2 we get that $\delta_1$ is injective and hence the map $\delta_2$ is non-zero. The dimension of the space $\mathcal{I}(\pi_{i,r})$ is one hence $f$ is an isomorphism. This shows that the space $\mathcal{I}(E_1)$ has dimension 2. For $r = (p - 1)/2$ the representations $\pi_{1,r} \simeq \pi_{0,r}$. We assume without loss of generality $img\delta_2$ is contained in $\mathcal{R}^1\mathcal{I}(\pi_{i,r})$. For any $r$ such that $0 < r \leq (p - 1)/2$ consider the pushout of $\tilde{\pi}_r$ by $\pi_{i,r}$

$$
\begin{array}{ccccccc}
0 & \to & \pi_{i,r} & \to & E_2 & \to & \pi_{i,r} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \tilde{\pi}_r & \to & E_1 & \to & \pi_{i,r} & \to & 0
\end{array}
$$

(13)

The self extension $E_2$ of $\pi_{i,r}$ is non-split and the induced map $\delta_{E_2}$ is non-zero. To see this consider the long exact sequence in cohomology attached to (13):

$$
\begin{array}{ccccccc}
0 & \to & \mathcal{I}(\pi_{i,r}) & \xrightarrow{g} & \mathcal{I}(E_2) & \to & \mathcal{I}(\pi_{i,r}) & \xrightarrow{\delta_3} & \mathcal{R}^1\mathcal{I}(\pi_{i,r}) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{I}(\tilde{\pi}_r) & \xrightarrow{\sim} & \mathcal{I}(E_1) & \to & \mathcal{I}(\pi_{i,r}) & \xrightarrow{\delta_2} & \mathcal{R}^1\mathcal{I}(\tilde{\pi}_r)
\end{array}
$$

Note that $\mathcal{R}^1\mathcal{I}(\tilde{\pi}_r)$ is isomorphic to $\mathcal{R}^1\mathcal{I}(\pi_{0,r}) \oplus \mathcal{R}^1\mathcal{I}(\pi_{1,r})$ and $\mathcal{R}^1\mathcal{I}(p_2)$ is the projection map. This shows that $\mathcal{R}^1\mathcal{I}(p_2)\delta_2 \neq 0$ and hence $\delta_3 \neq 0$ using which we get that $g$ is an isomorphism. This shows that $E_2$ is a non-split self-extension of $\pi_{i,r}$ by $\pi_{i,r}$. \hfill \Box

**Corollary 4.2.** For any integer $r$ such that $0 < r < (p - 1)/2$ we have $\dim_{\mathbb{F}_p} \text{Ext}^1_{G}(\pi_{i,r}, \pi_{i,r}) \geq 1$.

**Theorem 4.3.** Let $p \geq 5$ and $0 \leq r < (p - 1)/2$ then the dimension of $\text{Ext}^1_{G}(\pi_{i,r}, \pi_{i,r})$ is 1 and dimension of $\text{Ext}^1_{G}(\pi_{i,r}, \pi_{j,r})$ is 2 for $i \neq j$. For $r = (p - 1)/2$ the dimension of $\text{Ext}^1_{G}(\pi_{0,r}, \pi_{0,r})$ is 3.

**Proof.** The subgroup $G_SZ$ is an index 2 subgroup of $G$ and id and $\Pi$ are two double coset representatives for $K \backslash G/G_SZ$. We note that $K \cap G_SZ$ and $K^{\Pi} \cap G_SZ$ are representatives for the two distinct classes of maximal compact subgroups of $G_SZ$ and we denote them by $K_1$ and $K_2$ respectively. Let $\sigma'_r$ be the representation $\sigma^\Pi_r$ of $K^{\Pi}$. Using Mackey-decomposition we get that

$$
\text{res}_{G_SZ} \text{ind}_{K^{\Pi}_SZ}^{G_SZ} \sigma_r = \text{ind}_{K^{\Pi}_SZ}^{G_SZ} \sigma^\Pi_r \oplus \text{ind}_{K^{\Pi}_SZ \cap G_SZ}^{G_SZ} \sigma_r = \text{ind}_{K^{\Pi}_1SZ}^{G_SZ} \sigma_r \oplus \text{ind}_{K^{\Pi}_2SZ}^{G_SZ} \sigma'_r.
$$

(14)

using the long exact sequence of Ext groups for the exact sequence,

$$
0 \to \text{ind}_{ZK}^G \sigma_r \xrightarrow{T} \text{ind}_{ZK}^G \tilde{\pi}_r \to 0
$$

7
we get that an exact sequence
\[ \text{Hom}_G(\text{ind}^G_{ZK} \sigma_r, \bar{\pi}_r) \to \text{Ext}^1_G(\bar{\pi}_r, \bar{\pi}_r) \to \text{Ext}^1_G(\text{ind}^G_{ZK} \sigma_r, \bar{\pi}_r) \xrightarrow{T} \text{Ext}^1_G(\text{ind}^G_{ZK} \sigma_r, \bar{\pi}_r). \] (15)

Now using (14) the exact sequence (15) becomes
\[ 0 \to \text{Hom}_{K_1}(\sigma_r, \bar{\pi}_r) \oplus \text{Hom}_{K_2}(\sigma'_r, \bar{\pi}_r) \to \text{Ext}^1_G(\bar{\pi}_r, \bar{\pi}_r) \to \text{Ext}^1_{K_1}(\sigma_r, \bar{\pi}_r) \oplus \text{Ext}^1_{K_2}(\sigma'_r, \bar{\pi}_r). \] (16)

The groups \( K_1 \) is contained in \( K/Z_1 \). For all \( i \geq 0 \) we note that
\[ \text{Ext}^1_{K_1}(\sigma_r, \bar{\pi}_r) \approx \text{Ext}^1_{K/Z_1}(\text{ind}^{K/Z_1}_{K_1}(\sigma_r), \bar{\pi}_r) \approx \bigoplus_{0 \leq a < p-1} \text{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \bar{\pi}_r). \]

The spaces \( \text{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \bar{\pi}_r) \) can be calculated from the work of Paškūnas. We recall his calculations as needed. There exists a \( G \) smooth representation \( \Omega \) such that \( \text{res}_K \Omega \) is an injective envelope of \( \text{Soc}_K(\pi_r) \) in the category of smooth representations of \( K \). In particular we get that \( \pi_r \) is contained in \( \Omega \). The restriction \( \text{res}_K \Omega \) is isomorphic to \( \text{inj} \sigma_r \oplus \text{inj} \sigma_{p-1-r} \). Now \( \text{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \bar{\pi}_r) \) is isomorphic to \( \text{Hom}_{K/Z_1}(\sigma_r \otimes \det^a, \Omega/\bar{\pi}_r) \).

**We now use the notations from [Pas10, Notations, Section 9].** We make one modification. Paškūnas uses the notation \( \chi \) for the character
\[ \begin{pmatrix} \lambda & 0 \\ 0 & [\mu] \end{pmatrix} \mapsto (\lambda)^r (\mu)^a \]
for all \( \lambda, \mu \in \mathbb{F}_{p^2}^\times \) and \([\cdot]\) is the Teichmuller lift. For convenience we use the notation \( \chi_{a,r} \) instead of \( \chi \). The idempotent \( e_{\chi_{a,r}} \) in [Pas10] Section 9 will be denoted \( e_{\chi_{a,r}} \). The space \( \text{Hom}_{K_1}(\sigma_r \otimes \det^a, \Omega/\bar{\pi}_r) \) is the same as
\[ \ker(\mathcal{I}(\Omega/\bar{\pi}_r)e_{\chi_{a,r}} \xrightarrow{T_n} \mathcal{I}(\Omega/\bar{\pi}_r)e_{\chi_{a,r}}) \] (17)
and from [Pas10] Proposition 10.10 has dimension less than or equal to 2. For \( 0 \leq r \leq (p-1)/2 \) the space \( \text{Hom}_{K/Z_1}(\sigma_r \otimes \det^a, \bar{\pi}_r) \) is non-zero if and only if \( a = 0 \) and has dimension 1 if \( r < (p-1)/2 \) and 2 otherwise. Using (17) for \( 0 \leq r < (p-1)/2 \) the space \( \text{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \bar{\pi}_r) \) is non-zero if and only if \( a = 0 \) and has dimension at most 2 (see [Pas10] Proposition 10.10 for \( 0 < r < (p-1)/2 \) and [Pas13], Corollary 6.13 and Corollary 6.16 for \( r = 0 \)). When \( r = (p-1)/2 \) the space \( \text{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \bar{\pi}_r) \) is non-zero for \( a = 0 \) and \( a = (p-1)/2 \) and in each of these cases the dimension of the space \( \text{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \bar{\pi}_r) \) is less than or equal to 2.

Now using exact sequence (16) the space \( \text{Ext}^1_{GS}(\bar{\pi}_r, \bar{\pi}_r) \) has dimension less than or equal to 6 for \( 0 \leq r < (p-1)/2 \) and its dimension is less than or equal to 12 if \( r = (p-1)/2 \). For \( r \neq 0 \) using this upper bound and the lower bounds from Corollary 12 and Corollary 33 we deduce the theorem in this case. When \( r = 0 \) Paškūnas showed that (see [Pas10] Proposition 6.15) the dimension of \( \text{Ext}^1_{G^+/Z}(\pi_{i,0}, \pi_{j,0}) \) is 0 when \( i \neq j \) and 1 otherwise. Since \( G_S/\{\pm 1\} \) has index a factor of 2 in \( G^+/Z \) and \( G_S \cap Z \) acts trivially on \( \pi_{i,0} \) we get that
\[ \text{Ext}^1_{G^+/Z}(\pi_{i,0}, \pi_{j,0}) \to \text{Ext}^1_{G_S/(\{\pm 1\})}(\pi_{i,0}, \pi_{j,0}) = \text{Ext}^1_{GS}(\pi_{i,0}, \pi_{j,0}). \] (18)

From our upper bounds the inclusions (18) are strict and hence we prove the theorem.

**Corollary 4.4.** The Hecke module \( \mathbb{R}^1 \mathcal{I}(\pi_{i,0}) \) is isomorphic to the module \( \mathcal{I}(\pi_{i,0}) \oplus \mathcal{I}(\pi_{j,0}) \) for \( i \neq j \).

**Proof.** From the Theorem 4.3 exact sequence (3) and (10) we get that dimension of the space \( \text{Hom}_H(\mathcal{I}(\pi_{0,0}), \mathbb{R}^1 \mathcal{I}(\pi_{0,0})) \) is 1. Using the Proposition 3.1 we get that
\[ \mathbb{R}^1 \mathcal{I}(\pi_{i,0}) \simeq \mathcal{I}(\pi_{i,0}) \oplus \mathcal{I}(\pi_{j,0}). \]

\[ \square \]
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