Rescaling approach for a stochastic population dynamics equation perturbed by a linear multiplicative Gaussian noise

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Abstract. We are concerned with a nonlinear nonautonomous model represented by an equation describing the dynamics of an age-structured population diffusing in a space habitat $O$, governed by local Lipschitz vital factors and by a stochastic behavior of the demographic rates possibly representing emigration, immigration and fortuitous mortality. The model is completed by a random initial condition, a flux type boundary conditions on $\partial O$ with a random jump in the population density and a nonlocal nonlinear boundary condition given at age zero. The stochastic influence is expressed by a linear multiplicative Gaussian noise perturbation in the equation. The main result proves that the stochastic model is well-posed, the solution being in the class of path-wise continuous functions and satisfying some particular regularities with respect to the age and space. The approach is based on a rescaling transformation of the stochastic equation into a random deterministic time dependent hyperbolic-parabolic equation with local Lipschitz nonlinearities. The existence and uniqueness of a strong solution to the random deterministic equation is proved by combined semigroup, variational and approximation techniques. The information given by these results is transported back via the rescaling transformation towards the stochastic equation and enables the proof of its well-posedness.

Keywords: stochastic nonlinear equation, noise induced phenomena, multiplicative noise, Brownian motion, population dynamics, semigroup approach

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1 Problem presentation

This paper addresses the well-posedness of a nonlinear stochastic population dynamics equation describing the time and age evolution of a population of density $p$, in a space habitat $O$, governed by nonlinear vital factors, as natality and mortality and environmental particularities and influenced by a linear multiplicative noise perturbation. The equation reads

$$dp(t, a, x) + p_a(t, a, x)dt - \Delta p(t, a, x)dt + \mu_S(t, a, x; U(p))p(t, a, x)dt = p(t, a, x)dW(t, a, x), \text{ in } (0, T) \times (0, a^+) \times O.$$  \hspace{1cm} (1.1)

It is completed by two boundary conditions, the first of Robin type on the boundary of $O$, and the second at $a = 0$, and by the initial condition $\text{(1.3)}$, below:

$$-\nabla p(t, a, x) \cdot \nu = \alpha_0(t, a, x)p(t, a, x) + k_0(t, a, x), \text{ on } (0, T) \times (0, a^+) \times \partial O,$$  \hspace{1cm} (1.2)

$$p(t, 0, x) = \int_0^{a^+} m_0(a, x; U(p))p(t, a, x)da, \text{ in } (0, T) \times O,$$  \hspace{1cm} (1.3)

$$p(0, a, x) = p_0(a, x), \text{ in } (0, a^+) \times O.$$  \hspace{1cm} (1.4)

In these equations, $t$ is the time running in $(0, T), a$ is the age belonging to $(0, a^+), a^+$ is the maximum age life and $x$ is the space variable in $O$ which is an open bounded domain of $\mathbb{R}^d$ ($d = 1, 2, 3$). The Laplacian $\Delta$ and the gradient $\nabla$ refer only to the space variable and $\nu$ is the unit outward normal vector to the boundary of $O$. Moreover, $\mu_S$, called supplementary or additional mortality, is the mortality
rates are allowed to depend nonlinearly on \( p \), by the variable

\[
U(p) = \int_0^t \int_{O_U} \gamma(a, x)p(t, a, x)dxda,
\]

where \( O_U \) is a subset of \( O \). This dependence means that the total population lying in the environment \( O_U \) may produce a perturbation of the vital rates according to the weight factor \( \gamma \) varying with respect to age and space. It is relevant to assume that the dependence of \( \mu \) is locally Lipschitz (see e.g., [4] and [5]).

The boundary condition (1.3) written for \( a = 0 \) is the well-known birth equation in population dynamics. The boundary condition (1.2) expresses a change of population living in the habitat \( O \) with the outer environment, supplemented by a possible jump in the population density on the boundary. Other types of conditions, indicating a hostile boundary or a closed habitat, can be considered by assuming homogeneous Dirichlet or Neumann boundary conditions, respectively.

We note that the population dynamics equation with age-structure (1.1) normally includes also, on the left-hand side, a term \( \mu_0(a)p \), where \( \mu_0(a) \) is the natural mortality due to reaching the maximum life age. Since the natural assumptions for \( \mu_0 \) indicates that this is a \( L^1_{loc} \) function in \((0, a^+)\), a standard treatment is to replace \( p \) by \( p\exp\left(-\int_0^a \mu_0(s)ds\right) \). In this way the term \( \mu_0(a)p \) is cancelled from the equation and so, without loss of generality, the equation reduces to the so-called normalized equation (1.5).

Now, let us pass to the stochastic context. A deterministic model (with zero on the right-hand side of (1.1)) cannot reproduce or explain the effects of random fluctuations which come from the intrinsic stochastic nature of open systems. Random effects may be also induced by the interplay between the behavior of natural systems and random fluctuations generated by the environment. The presence of noise produced by this interaction determines an increase of the complexity of the system evolution which can substantially drift apart from its known deterministic feature. Moreover, demographic events, which can substantially drift apart from its known deterministic feature. Moreover, demographic events can be analyzed by rewriting this system in an operatorial form.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, with the natural filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) and let \( W \) be a stochastic Gaussian process of the form

\[
W(t, a, x) = \sum_{j=1}^N \mu_j(a, x)\beta_j(t),
\]

where \( \{\beta_j\}_{j=1}^N \) is an independent system of real-valued Brownian motions and \( \mu_j \) are regular functions. Thus, in relation with equation (1.5), \( W \) mimics a random fluctuation in the interaction between the population and the environment which can be due to a possible immigration, emigration or incidental stochastic mortality.

Let us note that \( W \) can be taken as well as an infinite series, like e.g., in [1] under certain convergence conditions for the series of the square coefficients. As usually, the argument \( \omega \in \Omega \) will be not explicitly specified in the list of the function arguments.

The deterministic nonlinear model (for \( W = 0 \)) described by equations of type (1.1)-(1.4) made the subject of a large amount of papers in the literature. A synthetic presentation of the most important achievements can be found in the monograph [8] and in the references therein, and in relation with the deterministic nonlinear model with locally Lipschitz nonlinearities for the vital rates, in [4] and [5].

The autonomous stochastic linear model of type (1.1)-(1.4), characterized by \( \gamma = 0 \) and time independent coefficients \( \mu_S, \alpha_0, k_0 \), can be analyzed by rewriting this system in an operatorial form and treating it by a semigroup formulation in the \( L^2 \)-approach, as e.g., in [7], or [10]. A path-wise continuous solution to the linear autonomous stochastic equation can be proved, if \( \mu_S \) is globally Lipschitz continuous. Under a supplementary condition on the operator, the well-posedness may follow in the case of a local Lipschitz term \( \mu_S \) for the stochastic equation with an additive noise (see e.g., [6], chapter 7). We also refer to the paper [2] in which the existence for stochastic equations with a linear multiplicative noise, with a general nonlinear monotone, demicontinuous and coercive time dependent operator between two dual spaces is provided. However, these results are not directly applicable in our case because the problem is not parabolic-like as in [2] and the nonlinearities are not
globally Lipschitz. As far as we know, the stochastic equation \((1.1)\) with \(m_0\) and \(\mu_S\) local Lipschitz has been not addressed in the literature.

The proof we provide begins by applying to our problem a rescaling transformation. More exactly, by a suitable function transformation for \(p\), system \((1.1)-(1.4)\) is transformed into the random deterministic one, in the unknown \(y\), see \((2.10)-(2.19)\) in Section 2. This is a nonlinear time dependent hyperbolic-parabolic system with local Lipschitz nonlinearities and it cannot be fitted in any functional framework for which general existence theorems can be applied. The proof of the solution existence requires a long and technical approach which is split in many intermediate results, beginning with the study of the well-posedness of a generic hyperbolic-parabolic problem with globally Lipschitz nonlinearities, in Section 3, Proposition 3.2. This proof is led by means of combined semigroup, variational and approximation techniques. Relying on this result and by using two types of regularizations, one for the time coefficients and the other for the operator acting in the equation, the existence and uniqueness of the solution to the random system is given in Theorem 4.1, for all \(\omega \in \Omega\). Much effort is done to get estimates for the solution to the intermediate problems in order to ensure the strong convergence in the approximated equations. In addition, some space and age regularity and the existence of a strong solution for the rescaled equation are obtained in Corollary 4.2. All information provided by the solution to the random system is fructified, while going back via the rescaling transformation, into the proof of well-posedness of the stochastic system, in Theorem 4.3.

**Notation.** For \(1 \leq p \leq \infty\), \(L^p(\Omega)\) is the space of all \(p\)-integrable real valued functions with the norm \(\| \cdot \|_{L^p(\Omega)}\) and \(L^q(0,T;L^p(\Omega))\) denotes the space of measurable functions \(u : [0, T] \rightarrow L^p(\Omega)\) such that \(t \rightarrow \|u(t)\|_{L^p(\Omega)}\) belongs to \(L^q(0,T;L^p(\Omega))\) is the continuous \(L^p\)-valued functions with the supremum norm in \(t\). As usually, \(W^{1,p}(\Omega)\) is the classical Sobolev space, i.e., \(W^{1,p}(\Omega) = \{u \in L^p(\Omega) ; \nabla u \in L^p(\Omega)\} \) and \(H^1(\Omega) = W^{1,2}(\Omega)\). The scalar product and the norm in a Hilbert space \(X\) are denoted by \((\cdot, \cdot)_X\) and \(\|\cdot\|_X\), respectively. In particular, \(\|\cdot\|_{\infty}\) indicates the norm of functions belonging to \(L^{\infty}(0,T) \times (0, a^+) \times \partial O\) or \(L^{\infty}(0,T) \times (0, a^+) \times \partial O\).

If no confusion can be done, some function arguments will be not specified in the integrands. \(C, C_i, c_i, i = 0, 1, 2, \ldots\) will stand for several constants that may change in the computations from line to line. Moreover, we shall denote

\[
H = L^2(\Omega), \quad V = H^1(\Omega), \quad V' = (H^1(\Omega))',
\]

\[
\mathcal{H} = L^2(0, a^+;H), \quad \mathcal{V} = L^2(0, a^+;V), \quad \mathcal{V}' = L^2(0, a^+;V').
\]

where \(V'\) is the dual of \(V\), and \(\mathcal{V}'\) is the dual of \(\mathcal{V}\). We note that \(V \subset H \subset V'\) and \(\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'\) with compact injections.

## 2 Preliminaries

We specify the hypotheses which will be in effect in this work (see e.g., [3]).

We assume, as relevant in population dynamics, that \(\mu_S(t,a,x;r)\) and \(m_0(a,x;r)\) are local Lipschitz functions on \(\mathbb{R}\) in the variable \(r\), uniformly with respect to \(t, a, x\), i.e., for any \(R > 0\), there exists \(L_{\mu_S}(R)\) and \(L_{m_0}(R)\) such that

\[
|\mu_S(t,a,x;r) - \mu_S(t,a,x;\overline{r})| \leq L_{\mu_S}(R) |r - \overline{r}|, \tag{2.1}
\]

\[
|m_0(a,x;r) - m_0(a,x;\overline{r})| \leq L_{m_0}(R) |r - \overline{r}|,
\]

whenever \(|r| \leq R\) and \(|\overline{r}| \leq R\). Moreover,

\[
\mu_S(\cdot, \cdot; r) \in L^\infty((0,T) \times (0, a^+) \times O), \text{ for all } r \in \mathbb{R}, \tag{2.2}
\]

\[
0 \leq \mu_S(t,a,x;r) \leq \mu_\infty \text{ a.e. in } (0,T) \times (0, a^+) \times O, \text{ for all } r \in \mathbb{R},
\]

\[
m_0(\cdot; r) \in L^\infty((0, a^+) \times O), \text{ for all } r \in \mathbb{R}, \tag{2.3}
\]

\[
0 \leq m_0 \leq m_0\infty \text{ a.e. in } (0, a^+) \times O,
\]
\[ \gamma \in L^\infty((0, a^+) \times O), \quad 0 \leq \gamma(a, x) \leq \gamma_\infty \text{ a.e. in } (0, a^+) \times O. \]

We still assume that
\[ \alpha_0 \in L^\infty((0, T) \times (0, a^+) \times \partial O), \quad \alpha_0 \geq 0 \text{ a.e. in } (0, T) \times (0, a^+) \times O. \] (2.4)

Moreover, \( k_0 \) and \( p_0 \) are random functions, such that
\[ k_0 \in L^2((0, T) \times (0, a^+) \times \partial O), \quad \mathbb{P}\text{-a.s.,} \] (2.5)
\[ p_0 \in L^2(0, a^+; L^2(O)), \quad \mathbb{P}\text{-a.s.,} \] (2.6)
\[ p_0(\cdot, a, x) \text{ is measurable with respect to } \mathcal{F}_0, \text{ a.a. } (a, x), \] (2.7)
\[ k_0(\cdot, t, a, x) \text{ is } \mathcal{F}_t\text{-adapted, a.a. } (t, a, x). \]

Finally, \( \beta_j \in C([0, T]; \mathbb{R}), \quad \beta_j(0) = 0, j = 1, ..., N, \) and assume that
\[ \mu_j \in C^2([0, a^+] \times \bar{O}), \quad \nabla \mu_j \cdot \nu = 0 \text{ on } (0, a^+) \times \partial O, \quad j = 1, ..., N. \] (2.8)

In particular, for a.a. \( \omega \in \Omega, \) the mapping \((t, a, x) \rightarrow W(t, a, x)(\omega)\) is continuous and the process \( \{W(t, a, x)\}_{t \geq 0} \) is real-valued \( \mathcal{F}_t\) adapted. As usually, we shall not specify the variable \( \omega \) in all random functions that occur.

**Definition 2.1.** A process \( p : [0, T] \times \Omega \rightarrow \mathcal{H} \) is called a solution to (1.1)-(1.4) if it is an \( \mathcal{F}_t\)-adapted process, \( t \geq 0, \)
\[ p \in C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap C([0, a^+]; L^2(0, T; H)), \quad \mathbb{P}\text{-a.s.,} \] (2.9)

and
\[
\begin{align*}
(p(t), \psi)_\mathcal{H} + & \int_0^t \int_O p(\tau, a^+, x)\psi(a^+, x)dx d\tau - \int_0^t \int_0^{a^+} \int_O p_\omega dx d\tau \\
- & \int_0^t \int_0^{a^+} \int_O m_0(a, x; U(p))p\psi(0, x)dx d\tau \\
+ & \int_0^t \int_0^{a^+} \int_O (\nabla p \cdot \nabla \psi + \mu_S(\tau, a, x; U(p))p\psi)dx d\tau + \int_0^t \int_0^{a^+} \int_{\partial O} (\alpha_0 p + k_0)\psi d\sigma d\tau \\
= & (p_0, \psi)_\mathcal{H} + \int_0^t (p(\tau)dW(\tau), \psi)_\mathcal{H}, \quad \mathbb{P}\text{-a.s., for all } \psi \in \mathcal{V}, \text{ with } \psi_a \in \mathcal{V}'.
\end{align*}
\] (2.10)

We specify that since \( p \in C([0, T]; \mathcal{H}) \) \( \mathbb{P}\text{-a.s.,} \) the Itô integral
\[ \int_0^t (p(\tau)dW(\tau), \psi)_\mathcal{H} = \sum_{j=1}^N \int_0^t \left( \int_0^{a^+} \int_O \mu_j(a, x)\psi(a, x)p(\tau, a, x)dx \right) d\beta_j(\tau) \] (2.11)
is well defined.

We begin by transforming equation (1.1), using for this the rescaling formula
\[ p(t, a, x) = e^{W(t, a, x)}y(t, a, x), \quad \text{for } t \geq 0. \] (2.12)

In the calculations implied by this transformation we use the Itô’s relation
\[ de^{W} = e^{W} dW + \mu e^{W} dt \] (2.13)
where,
\[ \mu(a, x) = \frac{1}{2} \sum_{j=1}^N \mu_j^2(a, x). \] (2.14)
Then, (2.12)-(2.14) imply that
\[ dp = e^W dy + e^W y dW + \mu e^W y dt. \] (2.15)

After plugging (2.12) in (1.1)-(1.4) and performing some calculations by expanding the terms \( \Delta(e^W y) \) and \( \nabla(e^W y) \) we deduce the system
\[
\begin{align*}
 y_t + y_a - \Delta y + g_1(t, a, x)y + g_2(t, a, x) \cdot \nabla y + \mu_S(t, a, x; U(e^W y))y &= 0, \quad \text{in } (0, T) \times (0, a^+) \times \mathcal{O}, \quad (2.16) \\
- \nabla y \cdot \nu &= \alpha(t, a, x)y + k(t, a, x), \quad \text{in } (0, T) \times (0, a^+) \times \partial \mathcal{O}, \quad (2.17) \\
y(t, 0, x) &= \int_0^{a^+} m(t, a, x; U(e^W y))y(t, a, x) da, \quad \text{in } (0, T) \times \mathcal{O}, \quad (2.18) \\
y(0, a, x) &= y_0(a, x) = p_0(a, x), \quad \text{in } (0, a^+) \times \mathcal{O}, \quad (2.19)
\end{align*}
\]

where
\[
\begin{align*}
g_1 &= W_a - \Delta W - |\nabla W|^2 + \mu, \quad g_2 = -2 \nabla W, \quad (2.20) \\
\alpha &= \alpha_0 + \nabla W \cdot \nu = \alpha_0, \quad k = k_0 e^{-W}, \\
m(t, a, x; r) &= m_0(a, x; r)e^{W(t, a, x)} - W(t, 0, x).
\end{align*}
\]

The functions \( g_1, g_2, \alpha \) and \( k \) depend on \( t, a, x \), and \( \alpha \geq 0 \) by (2.4) and (2.5) and obviously, the functions \( \mu_S \) and \( m \) are locally Lipschitz continuous with respect to the fourth variable, with the Lipschitz constants \( L_{\mu_S}(R) \) and \( L_m(R) = L_{m_0}(R) c_{W_0} \), where
\[
e_{W_0} = \left\| e^{W(\cdot, \cdot)} - W(\cdot, 0, \cdot) \right\|_{L^\infty}. \quad (2.21)
\]

On behalf of the hypotheses (2.1)-(2.8) we deduce
\[
\begin{align*}
g_1 &\in C([0, T]; C^1([0, a^+] \times C(\overline{\Omega})), \quad g_2 \in C([0, T]; C^2([0, a^+] \times C^1(\overline{\Omega})), \quad (2.22) \\
\alpha &= \alpha_0 \in L^\infty((0, T) \times (0, a^+) \times \partial \mathcal{O})), \quad k \in L^2((0, T) \times (0, a^+) \times \partial \mathcal{O}).
\end{align*}
\]

It is obvious that (2.16)-(2.19) is deterministic but random.

**Definition 2.2.** A solution \( y : [0, T] \times \Omega \rightarrow \mathcal{H} \) to (2.16)-(2.19) is an \( \mathcal{F}_t \)-adapted process such that
\[
y \in C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap C([0, a^+]; L^2(0, T; H)), \quad \mathbb{P}\text{-a.s.}, \quad (2.23)
\]
and
\[
\begin{align*}
- \int_0^T \int_0^{a^+} \int_O y \psi_1 dx da dt &- \int_0^{a^+} \int_O y_0 \psi(0, a, x) da \quad (2.24) \\
+ \int_0^T \int_O y(t, a^+, x) \psi(t, a^+, x) dx dt &- \int_0^T \int_0^{a^+} \int_O y \psi_a dx da dt \\
- \int_0^T \int_O \left( \int_0^{a^+} m(t, a, x; U(e^W y)) da \right) \psi(t, 0, x) dx dt &+ \int_0^T \int_0^{a^+} \int_{\partial \mathcal{O}} (\alpha y \psi + k \psi) d\sigma dt \\
+ \int_0^T \int_0^{a^+} \int_O (\nabla y \cdot \nabla \psi + yg_1 \psi + \psi g_2 \cdot \nabla y + \mu_S(t, a, x; U(e^W y))y \psi) dx da dt &= 0, \quad \mathbb{P}\text{-a.s.},
\end{align*}
\]
for all \( \psi \in W^{1,2}(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \), with \( \psi_a \in L^2(0, T; \mathcal{V}) \) and \( \psi(T, a, x) = 0 \).
### 3 Intermediate results

As we shall see, due to the local Lipschitz properties of \( m \) and \( \mu_S \) the proof of the existence of the solution to the deterministic random system is very long and technical. For making the arguments more understandable we shall split it in many parts. We begin with an intermediate result for a generic deterministic hyperbolic-parabolic time dependent system with global Lipschitz nonlinearities.

We introduce the problem

\[
Y_t + Y_a - \Delta Y + f_1(t, a, x)Y + f_2(t, a, x) \cdot \nabla Y + E_1(t, a, x; Y) = f, \quad \text{in } (0, T) \times (0, a) \times O,
\]

\[
- \nabla Y \cdot \nu = Y f_T + f^0, \quad \text{in } (0, T) \times (0, a) \times \partial O, \tag{3.2}
\]

\[
Y(t, 0, x) = \int_0^{a^+} E_2(t, a, x; Y) da, \quad \text{in } (0, T) \times O, \tag{3.3}
\]

\[
Y(0, a, x) = Y_0(a, x), \quad \text{in } (0, a) \times O, \tag{3.4}
\]

where

\[
f_1 \in C^1([0, T]; C^1([0, a^+) \times C(\overline{O})), \tag{3.5}
\]

\[
f_2 \in C^1([0, T]; C^2([0, a^+) \times C^1(\overline{O})), \quad f_2 \cdot \nu = 0 \text{ on } (0, T) \times (0, a^+) \times \partial O, \tag{3.6}
\]

\[
f_T \in C^1([0, T]; L^\infty((0, a^+) \times \partial O)), \quad f_T(t, a, x) \geq 0, \quad \text{a.e. in } (0, T) \times (0, a^+) \times \partial O, \tag{3.7}
\]

\[
f^0 \in L^2(0, T; L^2((0, a^+) \times \partial O)), \quad f \in L^2(0, T; \mathcal{H}), \quad Y_0 \in \mathcal{H}. \tag{3.8}
\]

Here, \( E_i : (0, T) \times (0, a^+) \times O \times \mathcal{H} \rightarrow \mathcal{H}, \ i = 1, 2 \), and both operators are globally Lipschitz on \( \mathcal{H} \), uniformly for \((t, a, x) \in (0, T) \times (0, a^+) \times O\), i.e., there exist \( L_i > 0, \ i = 1, 2 \), such that for any \( v, \overline{v} \in \mathcal{H} \) we have

\[
\|E_{11}(t, \cdot, v) - E_i(t, \cdot, \cdot; \overline{v})\|_{\mathcal{H}} \leq L_i \|v - \overline{v}\|_{\mathcal{H}},
\]

for any \( t \in (0, T) \). Moreover,

\[
\|E_1(t, \cdot, \cdot; v)\|_{\mathcal{H}} \leq \mu_\infty \|v\|_{\mathcal{H}}, \quad \|E_2(t, \cdot, \cdot; v)\|_{\mathcal{H}} \leq m_\infty \|v\|_{\mathcal{H}}, \quad \text{for all } v \in \mathcal{H},
\]

uniformly with respect to \((t, a, x)\) where \( \mu_\infty \) and \( m_\infty \) are precisely given by

\[
\mu_\infty := \sup_{(t, a, x, r) \in (0, T) \times (0, a^+) \times O \times \mathbb{R}} |\mu_S(t, a, x; r)|, \quad m_\infty := \sup_{(t, a, x, r) \in (0, T) \times (0, a^+) \times O \times \mathbb{R}} |m(t, a, x; r)| = cW_0m_0\infty.
\]

**Definition 3.1.** A solution \( Y \) to (3.1)-(3.3) is a function

\[
Y \in C([0, T]; \mathcal{H}) \cap C([0, a^+]; L^2(0, T; \mathcal{H})) \cap L^2(0, T; \mathcal{V}) \tag{3.9}
\]

which satisfies the equation

\[
- \int_0^T \int_0^{a^+} \int_O Y \psi_t dxdt - \int_0^T \int_0^{a^+} \int_O Y \psi_0 dxda - \int_0^T \int_0^{a^+} \int_O Y \psi_a dxdt \tag{3.10}
\]

\[
\quad + \int_0^T \int_0^{a^+} \int_O Y(t, a^+, x) \psi(t, a^+, x) dxdt - \int_0^T \int_0^{a^+} \left( \int_0^{a^+} E_2(t, a, x; Y) da \right) \psi(t, 0, x) dxdt
\]

\[
\quad + \int_0^T \int_0^{a^+} \int_{\partial O} (\nabla Y \cdot \nu + f_1 Y \psi + \psi f_2 \cdot \nabla Y) dxdt + \int_0^T \int_0^{a^+} \left( \int_0^{a^+} f_1(t, a, x; Y) \psi dxdt \right) + \int_0^T \int_0^{a^+} \left( \int_0^{a^+} f^0(t, a, x; Y) \psi dxdt \right)
\]

\[
\quad = \int_0^T \int_0^{a^+} f(t, a, x; Y) \psi dxdt,
\]

for all \( \psi \in W^{1,2}(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \), with \( \psi_a \in L^2(0, T; \mathcal{V}') \) and \( \psi(T, a, x) = 0 \).
Proposition 3.2. Under the assumptions (3.5), problem (3.1)-(3.4) has a unique solution, which satisfies the estimate

\[
\|Y(t)\|_{\mathcal{H}}^2 + \int_0^t \int_0^a Y^2(\tau, a, x)dx d\tau + \int_0^t \|Y(\tau)\|_{\mathcal{V}}^2 d\tau \leq \mathcal{C} := C_{est} \left( \|Y_0\|_{\mathcal{H}}^2 + \int_0^t \left( \|f_2(\tau)\|_{L^2(0,a+;L^2(\partial O))}^2 + \int_0^\tau \|f(\tau)\|_{\mathcal{H}}^2 d\tau \right) \right),
\]

for all \( t \in [0, T] \) and \( a \in [0, a^+] \), where

\[
C_{est} = c_0 e^{c_1 \left( 1 + \|f_1\|_{L^\infty} + \|f_2\|_{L^\infty} + a^+ m_{b_{\infty}} c_{W_0} + c_{L^2} \right)} T,
\]

\( c_0 \) and \( c_1 \) are positive numbers, \( c_0 \) depending on the domain and space dimension. Moreover, for two solutions \( Y \) and \( V \) corresponding to \( \{Y_0, f_1^0, f_2^0, f_0^{01}, f_1^0\} \) and \( \{Y_0^2, f_1^2, f_2^2, f_2^0, f_2^2, f_2^{02}, f_2^2\} \), respectively, we have

\[
\|Y(t) - Y_0(t)\|_{\mathcal{H}}^2 + \int_0^t \int_0^a (Y_0(t) - Y_2(t)) Y_0(t) dx d\tau + \int_0^t \|Y(t) - Y_2(t)\|_{\mathcal{V}}^2 d\tau \leq \mathcal{C} \left( \|f_1(t) - f_1^0\|_{L^\infty} + \|f_2(t) - f_2^0\|_{L^\infty} + \|f_1(t) - f_1^0\|_{L^\infty} + \|f_2(t) - f_2^0\|_{L^\infty} \right) \left( \int_0^t \|f(t) - f^0\|_{L^2(0,a+;L^2(\partial O))}^2 d\tau \right),
\]

for all \( t \in [0, T] \) and \( a \in [0, a^+] \).

**Proof.** The proof is done in 4 steps.

**Step 1.** Let us consider \( E_1 = E_2 = f_0^0 \). For all \( t \in [0, T] \) we introduce the time dependent operator \( A_0(t) : \mathcal{V} \rightarrow \mathcal{V}' \) by

\[
\langle A_0(t)v, \psi \rangle_{\mathcal{V}', \mathcal{V}} = \langle v_a, \psi \rangle_{\mathcal{V}', \mathcal{V}} + \int_0^a \int_O \nabla v \cdot \nabla \psi dx da + \int_0^a \int_{\partial O} v f_1(t, a, x) \psi d\sigma da + \int_0^a \int_O \int_0^\tau \nabla v \cdot \nabla \psi d\tau da + \int_0^a \int_O (f_1(t, a, x) \psi + \nabla v \cdot f_2(t, a, x)) \psi dx da, \text{ for } \psi \in \mathcal{V}.
\]

We specify that \( \langle \cdot, \cdot \rangle_{\mathcal{V}', \mathcal{V}} \) is the pairing between the dual spaces \( \mathcal{V}' \) and \( \mathcal{V} \), defined as

\[
\langle \phi, \psi \rangle_{\mathcal{V}', \mathcal{V}} = \int_0^a \langle \phi(a), \psi(a) \rangle_{\mathcal{V}', \mathcal{V}} da, \text{ for } \phi \in \mathcal{V}', \psi \in \mathcal{V}.
\]

Next, we define the restriction \( A(t) : D(A(t)) \subset \mathcal{H} \rightarrow \mathcal{H} \), where

\[
D(A(t)) = \{ v \in \mathcal{V}; v_a \in \mathcal{V}', v(0, x) = 0, A(t)v \in \mathcal{H} \},
\]

and \( A(t)v = A_0(t)v \) for all \( v \in D(A(t)) \). Thus, (3.1)-(3.4) can be written as the Cauchy problem

\[
\frac{dY}{dt}(t) + A(t)Y(t) = f(t), \text{ a.e. } t \in (0, T),
\]

\[
Y(0) = Y_0,
\]

and show further that it is well-posed. Since \( A(t) \) is time dependent, the existence proof relies on the result of Kato (see [7]) extended by Crandall and Pazy (see [3]) for nonlinear evolution equations. To this end we proceed to check, according e.g., to [3], the following properties of \( A(t) \):

(i) \( D(A(t)) \) is independent of \( t \);
(ii) \( A(t) \) is quasi \( m \)-accretive on \( \mathcal{H} \) for all \( t \in [0, T] \);
(iii) For each \( u \in \mathcal{H}, t \rightarrow J_\lambda(t)u \) is Lipschitz from \([0, T] \) to \( \mathcal{H} \), where \( J_\lambda(t) \) is the resolvent of \( A(t) \).
At point (i) we assert that $\overline{D(A(t))} = \mathcal{H}$ and this can follow as a particular case of the proof given in [3], Proposition 1, because one can note that $\overline{D(A(t))} = \{ v \in \mathcal{V}; v_a \in \mathcal{V}', v(0,x) = 0, v_a - \Delta v \in \mathcal{H}\}$. Let $t$ be fixed. Let us compute

\[
\langle A(t)v, v \rangle_{\mathcal{V}', \mathcal{V}} = \frac{1}{2} \| v(a^+) \|^2_{\mathcal{H}} + \| \nabla v \|^2_{\mathcal{H}} + \int_0^{a^+} \int_{\partial O} (f_1(t)v^2 + v\nabla v \cdot f_2(t))d\sigma da \\
+ \int_0^{a^+} \int_{\partial O} v^2f_1(t)d\sigma da \geq \| v \|^2_{\mathcal{V}} - \| v \|^2_{\mathcal{H}} \left( \| f_1 \|_{\infty} + \frac{1}{2} \| \nabla \cdot f_2 \|_{\infty} + 1 \right),
\]

which shows that $A(t)$ is quasi accretive for $\lambda > \lambda_0 = \| f_1 \|_{\infty} + \frac{1}{2} \| \nabla \cdot f_2 \|_{\infty} + 1$, where $\| f_i \|_{\infty} = \| f_i \|_{L^\infty((0,T) \times (0,a^+) \times X_i)}$, $X_i = \partial O$, $i = 1,2$. Here, we used the properties of $f_1$, $f_2$ and $f_{\tau}$ by (3.5), and the Gauss-Ostrogradski formula, namely

\[
\int_0^{a^+} \int_O \nabla v \cdot f_2(t)d\sigma da = \frac{1}{2} \int_0^{a^+} \int_O f_2(t) \cdot \nabla v^2 d\sigma da \\
= \frac{1}{2} \int_0^{a^+} \int_O (\nabla \cdot (f_2v^2)) - v^2 \nabla \cdot f_2 d\sigma da \leq \frac{1}{2} \| \nabla \cdot f_2 \|_{\infty} \| v \|^2_{\mathcal{H}}.
\]

The operator is quasi $m$-accretive because the equation

\[
\lambda z + A(t)z = h
\]

has a solution $z \in D(A(t))$, for each $h \in \mathcal{H}$. Indeed, let us introduce the linear Cauchy problem

\[
\frac{dz}{da}(a) + B_0(t,a)z(a) = h(a), \text{ a.e. } a \in (0,a^+),
\]

\[
z(0) = 0,
\]

where $B_0(t,a) : \mathcal{V} \rightarrow \mathcal{V}'$, \[
\langle B_0(t,a)z, \psi \rangle_{\mathcal{V}', \mathcal{V}} = \int_O (\lambda z\psi + \nabla z \cdot \nabla \psi + f_1(t,a,x)z\psi + \psi \nabla z \cdot f_2(t,a,x)) dx + \int_{\partial O} zf_{\tau}(t,a,\sigma)\psi d\sigma,
\]

for all $\psi \in \mathcal{V}$ and $a \in [0,a^+]$.

Recall that $t$ is fixed. The operator $B_0(t,a)$ is bounded and $\langle B_0(t,a)z, \psi \rangle_{\mathcal{V}', \mathcal{V}} \geq \| z \|^2_{\mathcal{V}} - \| z \|^2_{\mathcal{H}} (\lambda - \lambda_0)$, so that, by Lions’ theorem (see [3]), problem (3.15) has a unique solution $z \in L^2(0,a^+; \mathcal{V}) \cap W^{1,2}(0,a^+; \mathcal{V}')$. By (3.15), $A(t)z = h - \lambda z \in \mathcal{H}$, hence $z \in D(A(t))$.

To prove (iii) we start from the resolvent equation (3.15) which has a unique solution, as seen before, denoted further by $z^t = (\lambda I + A(t)^{-1})h$. Writing the difference between two equations (3.15) considered for $A(t)$ and $A(s)$,

\[
\lambda(z^t - z^s) + A(t)z^t - A(s)z^s = 0,
\]

setting $z := z^t - z^s$ and multiplying scalarly in $\mathcal{H}$ by $z$ we get

\[
\lambda \| z \|^2_{\mathcal{H}} + \| \nabla z \|^2_{\mathcal{H}} + \int_0^{a^+} \int_O (f_1(t) - f_1(s)) z^t z^s d\sigma da + \int_0^{a^+} \int_{\partial O} f_{\tau}(t) z^2 d\sigma da \\
+ \int_0^{a^+} \int_O f_1(t) - f_1(s) z^t z^s d\sigma da + \int_0^{a^+} \int_O f_2(t) z^2 d\sigma da \\
+ \int_0^{a^+} \int_O (f_2(t) - f_2(s)) z \cdot \nabla z^t z^s d\sigma da + \int_0^{a^+} \int_O f_2(s) z^t \cdot z \nabla z^s d\sigma da = 0.
\]

By the regularity assumptions (3.5), we have

\[
| f_i(t) - f_i(s) | = \left| \int_{s}^{t} f_{i,\tau}(\tau) d\tau \right| \leq \| f_{i,\tau} \|_{\infty} \left| t - s \right|,
\]

8
\( f_{i, \tau} \) and \( f_{\Gamma, \tau} \) below being the partial derivatives of \( f_i \), \( i = 1, 2 \), and \( f_{\Gamma} \), respectively, with respect to \( t \). Then,

\[
\left| \int_0^{a^+} \int_\partial \Omega f_2(s) \cdot z \nabla z \, dx \, da \right| \leq \frac{1}{2} \| \nabla \cdot f_2 \|_\infty \| z \|_{\mathcal{H}}^2,
\]

\[
\left| \int_0^{a^+} \int_\partial \Omega |f_{\Gamma}(t) - f_{\Gamma}(s)| z^t \, z \, d\sigma da \right| \leq \| f_{\Gamma, \tau} \|_\infty |t - s| \int_0^{a^+} \| z(a) \|_{L^2(\partial \Omega)} \| z^t(a) \|_{L^2(\partial \Omega)} \, da
\]

\[
\leq \| f_{\Gamma, \tau} \|_\infty |t - s| c_{\tau r}^2 \int_0^{a^+} \| z(a) \|_{V} \| z^t(a) \|_{V} \, da \leq \frac{1}{2} \| \nabla z \|_V^2 + \frac{1}{2} \| f_{\Gamma, \tau} \|_\infty^2 c_{\tau r}^2 |t - s|^2 \| z^t \|_{V}^2,
\]

where \( c_{\tau r} \) is the constant in the trace theorem. Performing all calculations we obtain

\[
\lambda \| z \|_{\mathcal{H}}^2 + \frac{1}{2} \| \nabla z \|_V^2 \leq (\lambda_0 + 2) \| z \|_{\mathcal{H}}^2
\]

\[
+ |t - s|^2 \left( \| f_{i, \tau} \|_\infty^2 \| z \|_{\mathcal{H}}^2 + \| f_{\Gamma, \tau} \|_\infty^2 \| \nabla z \|_{\mathcal{H}}^2 + \frac{1}{2} \| f_{\Gamma, \tau} \|_\infty^2 c_{\tau r}^2 \| z^t \|_{V}^2 \right).
\]

Relation

\[
\langle A(t)v, v \rangle_{V', V} = (A(t)v, v)_\mathcal{H} \geq \| v \|_{V'}^2 - \lambda_0 \| v \|_{\mathcal{H}}^2
\]

implies that

\[
\| v \|_{V'}^2 \leq (A(t)v, v)_\mathcal{H} + \lambda_0 \| v \|_{\mathcal{H}}^2 \leq \| A(t)v \|_{\mathcal{H}}^2 + (\lambda_0 + 1) \| v \|_{\mathcal{H}}^2
\]

and so we deduce that

\[
(\lambda - \lambda_1) \| z \|_{\mathcal{H}}^2 \leq C |t - s|^2 (\| z^t \|_{\mathcal{H}}^2 + \| A(t)z^t \|_{\mathcal{H}}^2)
\]

with \( C \) a constant and \( \lambda > \lambda_1 = \lambda_0 + 2 \).

Let us note that \( z^t = J_\lambda(t)h, \ z^s = J_\lambda(s)h \) and \( \| A(t)z \|_{\mathcal{H}} = \| A_{\lambda}(t)h \|_{\mathcal{H}} \leq \| A(t)h \|_{\mathcal{H}} \). Thus, we obtain point (iii), as claimed.

Let \( f \in W^{1,1}(0, T; \mathcal{H}) \) and \( y_0 \in D(A(t)) \). Then, the Cauchy problem (3.13) has a unique strong solution

\[
Y \in C([0, T]; \mathcal{H}) \cap L^\infty(0, T; D(A(t))) \cap W^{1,\infty}(0, T; \mathcal{H}).
\]

We multiply (3.13) by \( \psi \in W^{1,2}(0, T; \mathcal{H}) \cap L^2(0, T; V) \), with \( \psi_a \in L^2(0, T; V') \), integrate over \((0, t) \times (0, a) \times O \), and obtain

\[
\int_0^a \int_0^t \int_O Y(t, s, x)\psi(t, s, x)dxds \, dt - \int_0^t \int_0^a \int_O Y\psi_t dxdsdt \, \tau - \int_0^a \int_O Y_0 \psi(0, s, x)dxds = (3.18)
\]

\[
+ \int_0^t \int_0^a \int_O \nabla Y \cdot \nabla \psi + f_1 Y \psi + f_2 \cdot \nabla Y \, dxdsdt + \int_0^t \int_{\partial O} f_\psi Y \psi d\sigma dsdt
\]

\[
= \int_0^t \int_0^a \int_0^t f \psi dxdsdt,
\]

which, in particular, for \( t = T \), \( a = a^+ \) and \( \psi(T, a, x) = 0 \) yields (3.33) with \( E_1 = E_2 = f_1^0 = 0 \).

Next, by setting \( \psi = Y \) in (3.18), we get,

\[
\| Y(t) \|_{\mathcal{H}}^2 + \int_0^t \int_0^a \int_O Y^2(\tau, a, x)dxdsdt \leq \| Y_0 \|_{\mathcal{H}}^2 + \int_0^t \| f(\tau) \|_{V'}^2 d\tau + 2 \left( 1 + \| f_1 \|_\infty + \frac{1}{2} \| \nabla \cdot f_2 \|_\infty \right) \int_0^t \| Y(\tau) \|_{\mathcal{H}}^2 d\tau
\]

\[
\leq \| a(t) + a_1 \int_0^t Y(\tau) \|_{\mathcal{H}}^2 d\tau,
\]
with $a_1 = 2 \left(1 + \|f_1\|_\infty + \frac{1}{2} \|\nabla \cdot f_2\|_\infty \right)$ and $a(t) = \|Y_0\|^2_H + \int_0^t \|f(\tau)\|^2_{V'} \, d\tau$. By Gronwall’s lemma applied for $a(t)$ non-decreasing we get

$$\|Y(t)\|^2_H + \int_0^t \int_O Y^2(\tau, a, x) \, dx \, d\tau + \int_0^t \|Y(\tau)\|^2_{V'} \, d\tau \leq e^{a_1 T} \left(\|Y_0\|^2_H + \int_0^t \|f(\tau)\|^2_{V'} \, d\tau \right). \quad (3.19)$$

Since the operator is linear we also have an estimate for the difference of two solutions $Y_1$ and $Y_2$ corresponding to two pairs of data \{\(Y_0^1, f^1\)\} and \{\(Y_0^2, f^2\)\},

$$\|Y_1 - Y_2\|^2_H + \int_0^t \int_O (Y_1 - Y_2)^2(\tau, a, x) \, dx \, d\tau + \int_0^t \|Y_1 - Y_2(\tau)\|^2_{V'} \, d\tau \leq \left(\|Y_1^0 - Y_2^0\|^2_H + \int_0^t \|(f^1 - f^2)(\tau)\|^2_{V'} \, d\tau \right) e^{a_1 T}. \quad (3.20)$$

**Step 2.** Let $f \in L^2(0, T; H)$, $Y_0 \in H$, $f_0^0 \neq 0$. Let us define $F_1(t) \in V'$ by

$$\langle F_1(t), \psi \rangle_{V', V} = -\int_0^{a^+} \int_{\partial O} f_0^0(t, s, \sigma) \psi(a, \sigma) \, d\tau \, da, \quad \text{for all } \psi \in V, \ a.e. \ t \in (0, T), \quad (3.21)$$

and note that

$$\|F_1(t)\|_{V'} = \sup_{\psi \in V, \|\psi\|_{V'} \leq 1} \langle F_1(t), \psi \rangle_{V', V} \leq c_{V'} \|f_0^0(t)\|_{L^2(0, a^+: L^2(\partial O))}.$$

Let $F_1^n \in W^{1,1}(0, T; H)$, $f^n \in W^{1,1}(0, T; H)$, $Y_0^n \in D(A(t))$, such that $F_1^n \to F_1$ strongly in $L^2(0, T; V')$, $f^n \to f$ strongly in $L^2(0, T; H)$, and $Y_0^n \to Y_0$ strongly in $H$, as $n \to \infty$. Thus, as $n \to \infty$,

$$\int_0^t \langle F_1^n(\tau), \psi \rangle_{V', V} \, d\tau \to \int_0^t \langle F_1(\tau), \psi \rangle_{V', V} \, d\tau = -\int_0^{a^+} \int_{\partial O} f_0^0(t, s, \sigma) \psi(a, \sigma) \, d\tau \, da, \quad \text{for all } \psi \in V, \quad (3.21)$$

and

$$\int_0^t \|F_1^n(\tau)\|^2_{V'} \, d\tau \to \int_0^t \|F_1(\tau)\|^2_{V'} \, d\tau \leq c_{V'}^2 \int_0^t \|f_0^0(\tau)\|^2_{L^2(0, a^+: L^2(\partial O))} \, d\tau.$$

Let us consider problem (3.13) with $f$ replaced by $f^n + F_1^n$. This has a unique solution $Y^n$ satisfying (3.13), that is

$$\int_0^a \int_O Y^n(t, s, x) \psi(t, s, x) \, dx \, ds = \int_0^a \int_O Y^n(0, s, x) \psi(t, s, x) \, dx \, ds - \int_0^a \int_O Y^n(\tau, s, x) \psi(\tau, s, x) \, dx \, d\tau \quad (3.22)$$

$$+ \int_0^a \int_O \nabla Y^n \cdot \nabla \psi + f_1 Y^n \psi + f_2 \cdot \nabla Y^n \, dx \, ds \, d\tau \quad (3.22)$$

Moreover, the solution satisfies estimate (3.19), with $f$ replaced by $f^n + F_1^n$,

$$\|Y^n(t)\|^2_H + \int_0^t \int_O (Y^n)^2(\tau, a, x) \, dx \, d\tau + \int_0^t \|Y^n(\tau)\|^2_{V'} \, d\tau \leq e^{a_1 T} \left(\|Y_0^n\|^2_H + \int_0^t \|f^n(\tau)\|^2_{V'} \, d\tau \right), \quad (3.23)$$

and (3.20), for the difference $Y^n_1 - Y^n_2$, corresponding to two sets of data, \{\(Y_0^i, f^i, F_1^i\)\}$_{i=1,2}$,

$$\|Y^n_1 - Y^n_2\|^2_H + \int_0^t \int_O (Y^n_1 - Y^n_2)^2(\tau, a, x) \, dx \, d\tau + \int_0^t \|Y^n_1 - Y^n_2(\tau)\|^2_{V'} \, d\tau \leq 2e^{a_1 T} \left(\|Y_0^{i1} - Y_0^{i2}\|^2_H + \int_0^t \|(f_1^{i1} - f_2^{i1})(\tau)\|^2_{V'} \, d\tau + c_{V'}^2 \int_0^t \|(F_1^{i1} - F_2^{i2})(\tau)\|^2_{V'} \, d\tau \right). \quad (3.24)$$

10
This particularized for $Y^n - Y^m$ gives
\[
\|(Y^n - Y^m)(t)\|_{L^2}^2 + \int_0^t \int_O (Y^n - Y^m)^2(\tau, a, x) dx d\tau + \int_0^t \|(Y^n - Y^m)(\tau)\|_{L^2}^2 d\tau \geq 2e^{a_1 T} \left( \|Y_0^n - Y_0^m\|_{L^2}^2 + \int_0^t \|(f^n - f^m)(\tau)\|_{L^2}^2 d\tau + \int_0^t ||(F^n - F^m)(\tau)||_{\mathcal{V}'}^2 d\tau \right) \tag{3.25}
\]
whence it follows that $\{Y^n\}$ is a Cauchy sequence in the spaces indicated in (3.8), therefore tending strongly to $Y$ in these spaces. Moreover, by passing to the limit in (3.22) we get that the solution satisfies (3.18) with the right-hand side
\[
\int_0^t \int_O f\psi dx ds d\tau - \int_0^t \int_O f_0^0 \psi dx ds d\tau.
\]
Next, (3.23) and (3.24) are preserved at limit, and imply
\[
\left( \|Y_0\|_{L^2}^2 + \int_0^t \|f(\tau) + F_1(\tau)\|_H^2 d\tau \right) e^{a_1 T} \leq c_0 e^{a_1 T} \left( \|Y_0\|_{L^2}^2 + \int_0^t \|f(\tau)\|_H^2 d\tau + \int_0^t \|f_0^0(\tau)\|_{L^2(0,a^+;L^2(\partial O))}^2 d\tau \right),
\]
(because $\|f(\tau)\|_{L^2} \leq \|f(\tau)\|_H$) and
\[
\|(Y - Y\bar{})J(\tau)\|_{L^2}^2 + \int_0^t \int_O (Y - Y\bar{})^2(\tau, a, x) dx d\tau + \int_0^t \|(Y - Y\bar{})J(\tau)\|_{L^2}^2 d\tau \leq c_0 e^{a_1 T} \left( \|Y_0^1 - Y_0^2\|_{L^2}^2 + \int_0^t \|(f_0^1 - f_0^2)(\tau)\|_{L^2(0,a^+;L^2(\partial O))}^2 d\tau + \int_0^t \|f^1 - f^2(\tau)\|_{H^1}^2 d\tau \right).
\]
The uniqueness is obvious. Here, $c_0$ is a constant depending on the domain and dimension (via $c_{\epsilon r}$).

**Step 3.** Let $f \in L^2(0,T;H)$, $Y_0 \in H$, $f_0^0 \neq 0$ and let us consider the boundary condition
\[
Y(t,0,x) = F(t,x) \text{ with } F \in L^2(0,T;H). \tag{3.28}
\]
In a similar way as done at Step 2, we regularize all functions $f, Y_0$ and $F$, for the last one choosing a sequence $F^n \in C^0([0,T] \times \bar{O})$, such that $F^n(0,x) = 0$ and $F^n \to F$ strongly in $L^2(0,T;H)$. We have
\[
Y^n_t + Y^n_a - \Delta Y^n + f_1(t,a,x)Y^n + f_2(t,a,x) \cdot \nabla Y^n = f^n, \quad \text{in } (0,T) \times (0,a^+) \times O, \tag{3.29}
\]
\[
- \nabla Y^n \cdot \nu = Y^n f_1 + f_0^1, \quad \text{in } (0,T) \times (0,a^+) \times \partial O, \tag{3.30}
\]
\[
Y^n(t,0,x) = F^n(t,x), \quad \text{in } (0,T) \times O, \tag{3.31}
\]
\[
Y^n(0,a,x) = Y^n_0(a,x), \quad \text{in } (0,a^+) \times O. \tag{3.32}
\]
Homogenizing the boundary condition, by setting $Z := Y - F^n$ we get the system
\[
Z_t + Z_a - \Delta Z + f_1(t,a,x)Z + f_2(t,a,x) \cdot \nabla Z = \tilde{f}^n, \quad \text{in } (0,T) \times (0,a^+) \times O, \tag{3.29}
\]
\[
- \nabla Z \cdot \nu = Z f_1 + f_0^1 + \nabla F^n \cdot \nu + f_1 F^n, \quad \text{in } (0,T) \times (0,a^+) \times \partial O, \tag{3.30}
\]
\[
Z(t,0,x) = 0, \quad \text{in } (0,T) \times O, \tag{3.31}
\]
\[
Z(0,a,x) = Z^n_0(a,x), \quad \text{in } (0,a^+) \times O, \tag{3.32}
\]
where
\[
\tilde{f}^n = f^n - F^n_t - F^n_a + \Delta F^n - f_1 F^n - f_2 \cdot \nabla F^n \in W^{1,1}(0,T;\mathcal{H}),
\]
\[
\tilde{f}_0^0 = f_0^0 + \nabla F^n \cdot \nu + f_1 F^n \in C^1([0,T];L^2((0,a^+) \times \partial O)),
\]
\[
Z^n_0 = Y^n_0 - F^n(0,x) \in D(A(t)).
\]
Denoting \( \tilde{F}^n_1 = F^n_1 + G^n \), where \( F^n_1 \) is the regularization of \( F_1 \) given by (3.21) and

\[
\langle G^n(t), \psi \rangle_{\mathcal{V}', \mathcal{V}} = - \int_0^a \int_{\partial \Omega} (\nabla F^n(t, a, \sigma) \cdot \nu + f_1(t, a, \sigma)F^n(t, a, \sigma))\psi(a, \sigma)d\sigma da,
\]

we can write the Cauchy problem

\[
\frac{dZ}{dt}(t) + A(t)Z(t) = \tilde{f}^n(t) + \tilde{F}^n_1(t), \text{ a.e. } t \in (0, T),
\]

\[
Z(0) = Z_0^n.
\]

Thus, we can apply Step 2 and assert that this new system has a unique solution \( Z^n \), satisfying

\[
\int_0^a \int_O Z^n(t, s, x)\psi(t, s, x)dxd\tau - \int_0^a \int_O Z^n(0, 0, x)dxd\tau = 0
\]

\[
\int_0^a \int_O Z^n(\tau, a, x)\psi(\tau, a, x)dxd\tau - \int_0^a \int_O Z^n(0, \tau, a, x)dxd\tau
\]

\[
\int_0^a \int_O F^n(\tau, x)\psi(\tau, 0, x)dxd\tau + \int_0^a \int_{\partial \Omega} f_1^n \psi(\tau, 0, x)dxd\tau + \int_0^a \int_{\partial \Omega} f_1^n \psi(\tau, a, x)dxd\tau + \int_0^a \int_O f_1^n \psi(\tau, a, x)dxd\tau.
\]

Making some computations for going back to \( Y^n = Z^n + F^n \) we get that it satisfies

\[
\int_0^a \int_O Y^n(t, s, x)\psi(t, s, x)dxd\tau - \int_0^a \int_O Y^n(0, 0, x)dxd\tau = 0
\]

\[
\int_0^a \int_O F^n(\tau, x)\psi(\tau, 0, x)dxd\tau + \int_0^a \int_{\partial \Omega} f_1^n \psi(\tau, 0, x)dxd\tau + \int_0^a \int_{\partial \Omega} f_1^n \psi(\tau, a, x)dxd\tau + \int_0^a \int_O f_1^n \psi(\tau, a, x)dxd\tau.
\]

Therefore, we obtain the estimates

\[
\|Y^n(t)\|_H^2 + \int_0^t \int_O (Y^n)^2(\tau, a, x)dxd\tau + \int_0^t \|Y^n(\tau)\|_{\mathcal{V}}^2 d\tau \leq c_0 e^{a_1 T} \left( \|Y_0^n\|_H^2 + \int_0^t \int_O (F^n)^2dxd\tau + \int_0^t \|f^n(\tau)\|_{\mathcal{V}}^2 d\tau + \int_0^t \|F^n_1(\tau)\|_{\mathcal{V}}^2 d\tau \right)
\]

and

\[
\|(Y^n - \overline{Y^n})(t)\|_H^2 + \int_0^t \int_O (Y^n - \overline{Y^n})^2(\tau, a, x)dxd\tau + \int_0^t \|(Y^n - \overline{Y^n})(\tau)\|_{\mathcal{V}}^2 d\tau \leq c_0 e^{a_1 T} \left( \|Y_0^n - \overline{Y_0^n}\|_H^2 + \int_0^t \int_O (F^n - \overline{F^n})^2dxd\tau + \int_0^t \|(F^n - \overline{F^n})(\tau)\|_{\mathcal{V}}^2 d\tau \right).\]

where \( \overline{Y^n} \) is the solution corresponding to \( \{\overline{Y_0}, f_1, \overline{F}_1, \overline{F}\} \) and \( a_1 \) depends on the problem parameters \( \|f_1\|_{\infty}, \|\nabla \cdot f_2\|_{\infty} \) and \( T \). Here, \( f_1 \) is the same for both solutions.
Arguing as before, we get that \( \{Y^n\}_n \) is Cauchy in \( C([0, T]; \mathcal{H}) \cap C([0, a^+]; L^2(0, T; H)) \cap L^2(0, T; \mathcal{V}) \), hence \( Y^n \rightarrow Y \) strongly in these spaces as \( n \rightarrow \infty \), so that by passing to the limit in (3.33) we obtain

\[
\int_0^a \int_O Y(t, s, x)\psi(t, s, x)dxds - \int_0^t \int_O Y(t, s, x)\psi(t, s, x)dxds - \int_0^a \int_O Y_0\psi(0, s, x)dxds \quad (3.36)
\]

\[
+ \int_0^t \int_O Y(t, a, x)\psi(t, a, x)dxds - \int_0^t \int_O Y(t, a, x)\psi(t, a, x)dxds
\]

\[
- \int_0^t \int_O F(t, x)\psi(t, 0)dxds + \int_0^a \int_O (\nabla Y \cdot \nabla \psi + f_1 Y + f_2 \psi \cdot \nabla Y)dxds + \int_0^t \int_0^a \int_{\partial O} (f_T Y \psi + f_T^0 \psi)dxds d\sigma d\tau.
\]

Setting \( t = T \) and \( a = a^+ \) and taking \( \psi(T, a, x) = 0 \), we get that system (3.1), (3.2), (3.4), (3.28) has a solution. By passing to the limit as \( n \rightarrow \infty \), in estimates (3.31) and (3.33) we get

\[
\|Y(t)\|_{\mathcal{H}}^2 + \int_0^T \int_O Y^2(t, a, x)dxds + \int_0^T \|Y(t)\|_{\mathcal{V}}^2 dt \leq c_0 e^{a T} \left( \|Y_0\|_{\mathcal{H}}^2 + \int_0^T \int_O F^2 dxds + \int_0^T \|f(t)\|_{\mathcal{H}}^2 dt + \int_0^T \|f_T^0(t)\|_{L^2(0, a^+; L^2(\partial O))}^2 dt \right) := \mathcal{C}
\]

and

\[
\|Y(t) - \overline{Y}(t)\|_{\mathcal{H}}^2 + \int_0^T \int_O (Y - \overline{Y})^2(t, a, x)dxds + \int_0^T \|Y(t) - \overline{Y}(t)\|_{\mathcal{V}}^2 dt \leq c_0 e^{a T} \left( \|Y_0 - \overline{Y}_0\|_{\mathcal{H}}^2 + \int_0^T \int_O (F - \overline{F})^2 dxds + \int_0^T \|(f - \overline{f})\|_{\mathcal{V}}^2 dt + \int_0^T \|(f_T^0 - \overline{f}_T^0)\|_{L^2(0, a^+; L^2(\partial O))}^2 dt \right),
\]

corresponding to two sets of data \( \{Y_0, f, f_T^0, F\} \) and \( \{\overline{Y}_0, f, \overline{f}_T^0, \overline{F}\} \) and to the same \( f_1, f_2 \) and \( f_T \). Again (3.35) ensures the uniqueness.

For a later use we deduce the estimate for the difference of two solutions \( Y_1 \) and \( Y_2 \) corresponding to two completely different sets of data \( \{Y_0^1, f_1^1, f_2^1, f_T^1, F^1\} \) and \( \{Y_0^2, f_1^2, f_2^2, f_T^2, F^2\} \), computing first the estimate for the regular solutions and then passing to the limit. For simplicity we do not indicate the superscript \( n \) for the regularized solutions in the following computations. We have

\[
\frac{1}{2} \|(Y_1 - Y_2)(t)\|_{\mathcal{H}}^2 + \int_0^t \int_O (Y_1 - Y_2)^2(t, a, x)dxds + \int_0^t \|Y_1 - Y_2\|_{\mathcal{V}}^2 dt \leq \frac{1}{2} \|Y_0^1 - Y_0^2\|_{\mathcal{H}}^2 + 6 \int_0^t \|(f^1 - f^2)\|_{\mathcal{V}}^2 dt + \frac{1}{6} \int_0^t \|Y_1 - Y_2\|_{\mathcal{V}}^2 dt
\]

\[
+ \int_0^t \int_0^a \int_{\partial O} (|f_1^1 - f_2^1||Y_1| + |Y_1 - Y_2|^2 + |Y_1 - Y_2|^2 |f_1^1|) dxds d\sigma d\tau
\]

\[
+ \int_0^t \int_0^a \int_{\partial O} (|f_2^1 - f_2^2| |\nabla Y_1||Y_1 - Y_2| + \frac{1}{2} |Y_1 - Y_2|^2 |\nabla \cdot f_2^1|) dxds d\sigma d\tau
\]

\[
+ \frac{1}{2} \int_0^t \int_O (F^1 - F^2)^2 dxds + \int_0^t \|(Y_1 - Y_2)(t)\|_{\mathcal{H}}^2 dt
\]

\[
+ \int_0^t \int_0^a \int_{\partial O} |f_1^1 - f_2^1||Y_1||Y_1 - Y_2| d\sigma d\tau + \int_0^t \int_0^a \int_{\partial O} |f_2^1 - f_2^2||Y_1 - Y_2| d\sigma d\tau.
\]
Further we have
\[ \frac{1}{2} \| (Y_1 - Y_2)(t) \|_{\mathcal{H}}^2 + \frac{1}{2} \int_0^t \int_O (Y_1 - Y_2)^2(\tau, a, x) dx d\tau + \int_0^t \| (Y_1 - Y_2)(\tau) \|_V^2 d\tau \]
\[ + \int_0^t \int_0^a \int_{\partial O} |Y_1 - Y_2|^2 |f_2^e| d\sigma ds d\tau \]
\[ \leq \frac{1}{2} \| Y_1^0 - Y_2^0 \|_{\mathcal{H}}^2 + 6 \int_0^t \| (f_1^2 - f_2^2)(\tau) \|_{\mathcal{H}}^2 d\tau + \frac{1}{6} \int_0^t \| (Y_1 - Y_2)(\tau) \|_V^2 d\tau \]
\[ + \frac{1}{2} \int_0^t \int_0^a \int_{\partial O} (f_1^1 - f_2^1)(\tau) \| |f_1^2 - f_2^2| \|_{\mathcal{H}}^2 d\sigma ds d\tau \]
\[ + \frac{1}{6} \int_0^t \| (Y_1 - Y_2)(\tau) \|_V^2 d\tau + 6c_r^e \int_0^t \| (f_1^1 - f_2^1)(\tau) \|_{L^2(0,a^+; L^2(\partial O))}^2 d\tau, \]
which implies by (3.37)
\[ \| (Y_1 - Y_2)(t) \|_{\mathcal{H}} + \int_0^t \int_O (Y_1 - Y_2)^2(\tau, a, x) dx d\tau + \int_0^t \| (Y_1 - Y_2)(\tau) \|_V^2 d\tau \]
\[ \leq c_1 \left\{ \| Y_1^0 - Y_2^0 \|_{\mathcal{H}}^2 + \int_0^t \| (f_1^1 - f_2^1)(\tau) \|_{\mathcal{H}}^2 d\tau + \int_0^t \int_O (F_1^1 - F_2^1)^2 dx d\tau \right\} \]
\[ + c_0 \int_0^t \| (f_1^1 - f_2^1)(\tau) \|_{L^2(0,a^+; L^2(\partial O))}^2 d\tau + c_0 \bar{C}_1^2 \left( \| f_1^1 - f_2^1 \|_{\mathcal{H}}^2 + \| f_1^2 - f_2^2 \|_{\mathcal{H}}^2 + \| f_1^1 - f_2^1 \|_{\mathcal{H}}^2 \right) \]
\[ + c_1 (1 + \| f_1^2 \|_{\infty} + \| \nabla f_2^2 \|_{\infty}) \int_0^t \| (Y_1 - Y_2)(\tau) \|_{\mathcal{H}}^2 d\tau, \]
where $\bar{C}_1$ is $\bar{C}$ given by (3.37) corresponding to the functions indexed by $1$ and $c_0$ is a constant depending on $c_{tr}$.

**Step 4.** Let us consider the complete system (3.1)-(3.3). We shall apply the Banach fixed point theorem in the space $C((0, T; \mathcal{H})$. Let us fix $\zeta \in C((0, T; \mathcal{H})$ and consider the problem
\[ v_t + v_a - \Delta v + f_1(t, a, x)v + f_2(t, a, x) \cdot \nabla v = f^c(t, a, x), \text{ in } (0, T) \times (0, a^+) \times O, \]
\[ - \nabla v \cdot \nu = f_1(t, a, x)v + f_2^0(t, a, x), \text{ in } (0, T) \times (0, a^+) \times \partial O, \]
\[ v(t, 0, x) = F^c(t, x), \text{ in } (0, T) \times O, \]
\[ v(0, a, x) = Y_0(a, x), \text{ in } (0, a^+) \times O, \]
where
\[ f^c(t, a, x) = -E_1(t, a, x; \zeta), \quad F^c(t, x) = \int_0^{a^+} E_2(t, a, x; \zeta) da. \]
Note that $f^c \in L^2(0, T; \mathcal{H})$ and $F^c \in L^2(0, T; H)$ and so we are entitled to apply Step 3 to find that system (3.41)-(3.44) has a unique solution
\[ v^c \in C((0, T; \mathcal{H}) \cap C([0, a^+]; L^2(0, T; H)) \cap L^2(0, T; V)), \]
These solutions reads and actually it can be denoted by \( Y \), which proves, by a suitable choice of (3.36) and estimate (3.37), that is

\[
\|v^\zeta(t)\|_H^2 + \|v^\zeta(a)\|_{L^2(0,T;H)}^2 + \int_0^t \|v^\zeta(t)\|_V^2 \, d\tau \\
\leq c_0 e^{a_1 T} \left( \|y_0\|_H^2 + \int_0^t \int_O (F^\zeta)^2(\tau,x) \, dxd\tau + \int_0^t \|f^0(\tau)\|_{L^2(0,a^+;L^2(\partial\Omega))}^2 \, d\tau + \int_0^t \|E^1(\tau)\|_H^2 \, d\tau \right) \\
\leq c_0 e^{a_1 T} \left( \|y_0\|_H^2 + (a^+ m_\infty^2 + \mu_\infty^2) \int_0^t \|\zeta(\tau)\|_H^2 \, d\tau + \int_0^T \|f^0(\tau)\|_{L^2(0,a^+;L^2(\partial\Omega))}^2 \, d\tau \right).
\]

For the passage to the last line in (3.45) we used the properties (3.7) for \( E_1 \) and \( E_2 \), e.g.,

\[
\int_0^t \int_O (F^\zeta)^2(\tau,x) \, dxd\tau = \int_0^t \int_O \left( \int_0^{a^+} E^2(t,a,x;\zeta) \, da \right)^2 \, d\tau \leq a^+ \int_0^t \int_O \int_0^{a^+} E^2_2(t,a,x;\zeta) \, dxd\tau \\
\leq a^+ m_\infty^2 \int_0^t \|\zeta(\tau)\|_H^2 \, d\tau.
\]

Then, we define \( \Psi : \mathcal{M} \to C([0,T];\mathcal{H}) \) which maps \( \zeta \in \mathcal{M} \) into the solution \( v^\zeta \) to (3.41)-(3.44). Obviously, \( \Psi(\mathcal{M}) \subseteq \mathcal{M} \) and we show that \( \Psi \) is a contraction on \( \mathcal{M} \). Indeed, let \( v^\zeta \) and \( v^{\zeta} \) be two solutions to (3.41)-(3.44) corresponding to \( \zeta \) and \( \zeta \). Then, by (3.38), the estimate of the difference of these solutions reads

\[
\|v^\zeta(t) - v^{\zeta}(t)\|_H^2 + \|v^\zeta(a) - v^{\zeta}(a)\|_{L^2(0,T;H)}^2 + \int_0^t \|v^\zeta(\tau) - v^{\zeta}(\tau)\|_V^2 \, d\tau \\
\leq c_0 e^{a_1 T} \left( \int_0^t \int_O (F^\zeta(\tau,x) - F^{\zeta}(\tau,x))^2 \, dxd\tau + \int_0^t \int_0^{a^+} \left( E^1(\tau,a,x;\zeta) - E^1(\tau,a,x;\zeta) \right)^2 \, dxd\tau \right) \\
\leq c_0 e^{a_1 T} \left( a^+ L_2^2 \int_0^t \|\zeta - \zeta(\tau)\|^2_\mathcal{H} \, d\tau + L_2^2 \int_0^t \|\zeta - \zeta(\tau)\|^2_\mathcal{H} \, d\tau \right) \leq C \int_0^t \|\zeta - \zeta(\tau)\|^2_\mathcal{H} \, d\tau.
\]

Considering now the norm \( \|v^\zeta\|_B = \sup_{t \in [0,T]} (e^{-\gamma_0 t} \|v^\zeta(t)\|_H) \) which is equivalent with the standard norm in \( C([0,T];\mathcal{H}) \), it follows by some calculations that

\[
\|v^\zeta - v^{\zeta}\|_B \leq C e^{-\gamma_0 t} \int_0^t e^{\gamma_0 s} \|\zeta - \zeta\|_B^2 \, ds \leq \frac{C}{2\gamma_0} \left( 1 - e^{-2\gamma_0 t} \right) \|\zeta - \zeta\|_B^2 \\
\leq \frac{C}{2\gamma_0} \|\zeta - \zeta\|_B^2
\]

which proves, by a suitable choice \( 2\gamma_0 > C \), that \( \Psi \) is a contraction on \( \mathcal{M} \). Then, \( \Psi \) has a fixed point, \( \Psi(\zeta) = \zeta = v^\zeta \), which is the unique solution to (3.41)-(3.44). Thus, \( v^\zeta \) turns out to solve (3.41)-(3.44) and actually it can be denoted by \( Y \).

Finally, assuming that on the right-hand side of (3.1) we have \( f - E_1(t,a,x;Y) \), we get by using
\[ \|(Y_1 - Y_2)(t)\|_{H}^2 + \int_{0}^{t} \int_{\Omega} (Y_1 - Y_2)^2(\tau, \alpha, x)dx d\tau + \int_{0}^{t} \|(Y_1 - Y_2)(\tau)\|_{V}^2 d\tau \]

\[ \leq c_1 \left( \|Y_0^1 - Y_0^2\|_{H}^2 + \int_{0}^{t} \|(f^1 - f^2)(\tau)\|_{H}^2 d\tau \right. \]

\[ + \int_{0}^{t} \int_{\Omega} (E_1(\tau, \alpha, x; Y_1) - E_1(\tau, \alpha, x; Y_2))^2 dx d\tau \]

\[ + \int_{0}^{t} \int_{\Omega} \left( \int_{\alpha}^{+} (E_2(\tau, \alpha, x; Y_1) - E_2(\tau, \alpha, x; Y_2))da \right)^2 d\tau \]

\[ + c_0 C_1 \left( \|f_1^1 - f_1^2\|_{\infty} + \|f_2^1 - f_2^2\|_{\infty} + \|f_1^1 - f_1^2\|_{\infty} \right) \]

\[ + c_0 \int_{0}^{t} \|(f_1^{01} - f_1^{02})(\tau)\|_{L^2(0, a^+; L^2(\partial\Omega))}^2 d\tau \]

\[ + c_1 \left( 1 + \|f_1^2\|_{\infty} + \|f_2^2\|_{\infty} \right) \int_{0}^{t} \|(Y_1 - Y_2)(\tau)\|_{H}^2 d\tau, \]

which implies (3.12) as claimed.

If the data are the same, this implies the uniqueness too. This ends the proof.

4 Main results

In this section we shall prove that the random system (2.16) - (2.19) has a unique solution and then we shall go back through the transformation (2.12) to the stochastic system (1.1) - (1.4) proving that it has a unique solution in the sense of Definition 2.1.

**Theorem 4.1.** Under the assumptions (2.22) system (2.16) - (2.19) has, for each fixed \( \omega \in \Omega \), a unique solution \( y \), and the process \( t \to y(t, \omega) \) is \( F_t \)-adapted. The solution satisfies the estimate

\[ \|y(t)\|_{H}^2 + \|y(\alpha)\|_{L^2(0, T; H)}^2 + \int_{0}^{t} \|y(t)\|_{V}^2 d\tau \]

\[ \leq C_{est} \left( \|y_0\|_{H}^2 + \int_{0}^{t} \|k(\tau)\|_{L^2(0, a^+; L^2(\partial\Omega))}^2 d\tau \right), \text{ for all } t \in [0, T]. \]

**Proof.** We shall study first an approximating problem introduced to endow the coefficients with more time regularity and deduce then the necessary estimates in order to pass to the limit. Thus, we consider a mollifier \( \rho_\varepsilon \) and define

\[ W_\varepsilon(t, a, x) = \int_{0}^{T} W(t, a, x) \rho_\varepsilon(t - s) ds, \]

(4.2)
\[ \alpha_\varepsilon(t,a,x) = \int_0^T \alpha_0(t,a,x) \rho_\varepsilon(t-s) ds. \]

Recall that a mollifier is defined by \( \rho_\varepsilon(t) = \frac{1}{\varepsilon} \rho\left(\frac{t}{\varepsilon}\right) \) where \( \rho \in C^\infty(\mathbb{R}^d) \), \( \rho(t) \geq 0 \), \( \rho(t) = \rho(-t) \), \( \int_{\mathbb{R}^d} \rho(t) dt = 1 \). Thus, \( W_\varepsilon \in C^\infty([0,T];C^2([0,a^+] \times \overline{O})) \), \( \alpha_\varepsilon \in C^\infty([0,T];L^\infty((0,a^+) \times \partial O)) \), and as \( \varepsilon \to 0 \) we have

\[
W_\varepsilon \to W \text{ strongly in } C([0,T];C^2([0,a^+] \times \overline{O})),
\]

\[
\alpha_\varepsilon \to \alpha \text{ strongly in } L^\infty((0,T) \times (0,a^+) \times \partial O)).
\]

The approximating system reads

\[
y_t + y_a - \Delta y + g_{1\varepsilon}(t,a,x,y) + g_{2\varepsilon}(t,a,x) \cdot \nabla y + \mu_S(t,a,x;U(e^{W_\varepsilon} y))y = 0, \quad \text{in } (0,T) \times (0,a^+) \times O, \quad (4.3)
\]

\[
- \nabla y \cdot \nu = \alpha_\varepsilon(t,a,x,y) + k(t,a,x), \quad \text{in } (0,T) \times (0,a^+) \times \partial O,
\]

\[
y(t,0,x) = \int_0^a m_\varepsilon(t,a,x;U(e^{W_\varepsilon} y)) y(t,a,x) da, \quad \text{in } (0,T) \times O,
\]

\[
y(0,a,x) = y_0(a,x), \quad \text{in } (0,a^+) \times O,
\]

where \( g_{1\varepsilon}, g_{2\varepsilon} \), are given by \((2.20)\) in which \( W \) is replaced by \( W_\varepsilon \), and \( \alpha \) by \( \alpha_\varepsilon \). Relations \((2.20)\) and \((2.22)\) imply

\[
g_{1\varepsilon} \in C^\infty([0,T];C^1([0,a^+] \times C(\overline{O})), \quad g_{2\varepsilon} \in C^\infty([0,T];C^2([0,a^+] \times C(\overline{O})),
\]

\[
\alpha_\varepsilon \in C^\infty([0,T];L^\infty(0,a^+;L^\infty(\partial O))), \quad k \in L^2(0,T;L^1((0,a^+) \times O)).
\]

Recall that \( \mu_S \) and \( m \) are local Lipschitz continuous with constants, \( L_{\mu_S}(R), L_m(R) \).

A solution to \((4.3)-(4.6)\) is defined by replacing in \((2.24)\) the corresponding previous coefficients.

Let us denote

\[
S_1(t,a,x;u) = \mu_S(t,a,x;U(e^{W_\varepsilon} u))u, \quad S_2(t,a,x;u) = m_0(t,a,x;U(e^{W_\varepsilon} u))u, \quad \text{for } u \in \mathcal{H}.
\]

Under the local Lipschitz conditions on \( m \) and \( \mu_S \) it follows that \( S_1 \) and \( S_2 \) turn out to be only local Lipschitz on \( \mathcal{H} \). Indeed, let us take \( R > 0 \) and \( u, \overline{\pi} \in \mathcal{H} \), such that \( \|u\|_{\mathcal{H}} \leq R \) and \( \|\overline{\pi}\|_{\mathcal{H}} \leq R \) and calculate

\[
|U(e^{W_\varepsilon} u)| \leq \left| \int_0^a \int_{O_\varepsilon} \gamma(a,x)e^{W_\varepsilon} u(a,x) da dx \right| \leq e^{\|W_\varepsilon\|_{\infty}} \|u\|_{\mathcal{H}} \gamma_\infty \sqrt{a^+ \text{meas}(O_U)} R \quad (4.8)
\]

where

\[
c_W = e^{\sup_{t \in [0,T]} \|W_\varepsilon(t)\|_{\infty}}
\]

and

\[
|S_2(t,a,x;u) - S_2(t,a,x;\overline{\pi})| = |m_\varepsilon(t,a,x;U(e^{W_\varepsilon} u))u - m_\varepsilon(t,a,x;U(e^{W_\varepsilon} \overline{\pi}))(u)| \leq c_{W_\varepsilon} \left( |m_0(a,x;U(e^{W_\varepsilon} u))| + \|U(e^{W_\varepsilon} \overline{\pi})\|_{\mathcal{H}} |u| + c_{W_\varepsilon} m_{\alpha}(a,x;U(e^{W_\varepsilon} \overline{\pi})) \right)
\]

whence, denoting \( C_m(R) = c_{W_\varepsilon} c_W L_{m_\varepsilon}(R) \gamma_\infty \sqrt{a^+ \text{meas}(O_U)} R + c_{W_\varepsilon} m_{\alpha} \), we have

\[
\|S_2(t,\cdot,\cdot;u) - S_2(\cdot,\cdot,\cdot;\overline{\pi})\|_{\mathcal{H}} \leq C_m(R) \|u - \overline{\pi}\|_{\mathcal{H}}.
\]

This shows that \( S_2 \) is locally Lipschitz on \( \mathcal{H} \). A similar relation follows for \( S_1 \) with the constant denoted \( C_{\mu_S}(R) = c_W L_{\mu_S}(R) \gamma_\infty \sqrt{a^+ \text{meas}(O_U)} R + m_{\alpha} \).

The proof will be done in two steps, the first for proving the existence of the approximating solution and the second for passing to the limit.
Step 1. Let $N \geq 1$. We approximate $S_1$ and $S_2$ by

$$S_i^N(t, a, x; u) = \begin{cases} S_i(t, a, x; u), & \|u\|_\mathcal{H} \leq N \\ S_i(t, a, x; \frac{Nu}{\|u\|_\mathcal{H}}), & \|u\|_\mathcal{H} > N \end{cases}$$

for $i = 1, 2$. Then, it can be easily checked that $S_i^N(t, a, x; u)$ are Lipschitz continuous on $\mathcal{H}$ with the constants $3C_{\mu_i}(N)$ and $3C_{m_i}(N)$, corresponding to $i = 1, 2$, respectively.

Now, we consider system (4.3)-(4.6) with $S_1^N(t, a, x; y)$ and $S_2^N(t, a, x; y)$ instead of $S_1(t, a, x; y)$ and $S_2(t, a, x; y)$. In fact this is (4.11)-(4.14) with $E_i(t, a, x; y) = S_i^N(t, a, x; y)$, $i = 1, 2$, and

$$f_1 = g_{1\varepsilon}, \quad f_2 = g_{2\varepsilon}, \quad f_\tau = \alpha_\varepsilon, \quad f_0 = k, \quad Y_0 = y_0, \quad f = 0.$$  

According to Proposition 3.2, this system has a unique solution $y_\varepsilon^N \in C([0, T]; \mathcal{H}) \cap C([0, a^+]; L^2(0, T; \mathcal{H})) \cap L^2(0, T; \mathcal{V})$ verifying (3.9).

$$- \int_0^T \int_a^a \int_0^y e^N \psi dx dt - \int_0^y \int_0^y \psi(0, a, x) dx da \tag{4.9}$$

$$+ \int_0^T \int_0^y e^N(t, a^+, x) \psi(t, a^+, x) dx dt - \int_0^T \int_0^y \left( \int_0^y S_2^N(t, a, x; y_\varepsilon) dx \right) \psi(t, 0, x) dx dt$$

$$- \int_0^T \int_0^y \int_0^y \psi dx dt + \int_0^T \int_0^y \int_0^y \int_0^y (\nabla y_\varepsilon \cdot \nabla \psi + g_{1\varepsilon} y_\varepsilon^N \psi + g_{2\varepsilon} \psi \cdot \nabla y_\varepsilon^N) dx dt +$$

$$\int_0^T \int_0^y \int_0^y (\alpha_\varepsilon y_\varepsilon^N + k) \psi dx dt = 0. \tag{4.10}$$

Moreover, the solution satisfies the estimates (3.10),

$$\|y_\varepsilon^N(t)\|_\mathcal{H}^2 + \|y_\varepsilon^N(t)\|_{L^2(0, T; \mathcal{H})}^2 + \int_0^T \|y_\varepsilon^N(t)\|_{L^2(\mathcal{V})}^2 \, dt \tag{4.11}$$

and two solutions corresponding to two sets of data obey the inequality (3.12).

Here, $C_{est} = c_0 e^c_1 (1 + \|g_1\|_\infty + \|g_2\|_\infty + a^{+} m_a^2 + \mu_\infty)$, by (3.11), where $\|g_1\|_\infty \leq \|g_i\|_\infty \leq C_i$ (depending on $\|W_0\|_\infty, \|\Delta W\|_\infty$), $i = 1, 2$, because the functions $g_i$ are continuous.

Now, we set

$$R_0 := c_0 e^c_1 (1 + \|g_1\|_\infty + \|g_2\|_\infty + a^{+} m_a^2 + \mu_\infty) \left( \|y_0\|_\mathcal{H}^2 + \int_0^T \|k(t)\|_{L^2(\mathcal{V})}^2 \, dt \right). \tag{3.12}$$

It follows that for $N \geq \sup \{R_0 + 1: = N_0\}$ we get $\|y_\varepsilon^N(t)\|_\mathcal{H}^2 \leq R_0 < N_0 \leq N$ and so, $S_i^N(t, a, x; y) = S_i(t, a, x; y)$, $i = 1, 2$, meaning that $y_\varepsilon^N$ actually satisfies system (4.3)-(4.6), if $N \geq N_0$. Thus, we deduce that $y_\varepsilon^N(t)$ is in fact a solution to problem (4.3)-(4.6) and we denote it by $y_\varepsilon(t)$. We also note that the Lipschitz constants for $S_i^N$ specified before depend actually on $R_0$, namely

$$L_1 = C_{m_i}(R_0) = c_{W_0} c_{L_{m_0}}(R_0) \gamma \sqrt{a^{+}} \text{meas}(\mathcal{O}_U) R + c_{W_0} m_{0\infty},$$

$$L_2 = C_{\mu_\infty}(R_0) = c_{W_0} L_{\mu_\infty}(R_0) \gamma \sqrt{a^{+}} \text{meas}(\mathcal{O}_U) R + \mu_\infty.$$
Obviously, the solution \( y_{e,0} = y_{e} \) satisfies (1.18), in which \( E_{i} \) are replaced by \( S_{i}(t, a, x; U(e^{W}, y_{e})) \), \( i = 1, 2 \),

\[
\int_{0}^{t} \int_{O} y_{e}(t, s, x)\psi(t, s, x)dxda - \int_{0}^{t} \int_{O} y_{e}\psi_{t}dxds + \int_{0}^{t} \int_{O} y_{e}\psi(0, s, x)dxds \quad (4.12)
\]

\[
+ \int_{0}^{t} \int_{O} y_{e}(t, s, x)\psi(t, s, x)dxds - \int_{0}^{t} \int_{O} y_{e}\psi_{t}dxds + \int_{0}^{t} \int_{O} y_{e}\psi(0, s, x)dxds
\]

\[
- \int_{0}^{t} \int_{O} \left( \int_{0}^{s} m_{e}(t, s, x; U(e^{W}, y_{e}))y_{e}ds \right) \psi(t, 0, x)dx
\]

\[
+ \int_{0}^{t} \int_{O} \left( \nabla y_{e} \cdot \nabla \psi + g_{1e}y_{e}\psi + g_{2e}\psi \cdot \nabla y_{e} \right)dxds + \int_{0}^{t} \int_{O} \int_{0}^{s} \mu_{S}(t, s, x; U(e^{W}, y_{e}))y_{e}\psi dxds \tau = 0,
\]

and inherits estimates (4.11). Moreover, (3.27), written for \( Y_{1} = y_{e}, Y_{2} = y_{e}^{*} \), corresponding to \( W_{e} \) and \( W_{e}^{*} \), respectively, yields

\[
\|(y_{e} - y_{e}^{*})(t)\|_{H_{e}}^{2} + \int_{0}^{t} \int_{O} (y_{e} - y_{e}^{*})^{2}(\tau, a, x)dxds + \int_{0}^{t} \int_{O} (y_{e} - y_{e}^{*})(\tau)\|_{H_{e}}^{2} d\tau \quad (4.13)
\]

\[
\leq c_{0}R_{0}^{2} \left( \|g_{1e} - g_{1e}^{*}\|_{\infty}^{2} + \|g_{2e} - g_{2e}^{*}\|_{\infty}^{2} + \|\alpha_{e} - \alpha_{e}^{*}\|_{\infty}^{2} \right)
\]

\[
+ c_{1} (1 + \|g_{1e}\|_{\infty} + \|\nabla \cdot g_{2e}\|_{\infty}) \int_{0}^{t} \int_{O} (y_{e} - y_{e}^{*})(\tau)\|_{H_{e}}^{2} d\tau
\]

\[
+ c_{1} \int_{0}^{t} \int_{O} \left( S_{2}(\tau, a, x; y_{e}) - S_{2}(\tau, a, x; y_{e}^{*}) \right)ds \int_{0}^{t} \int_{O} \left( S_{1}(\tau, a, x; y_{e}) - S_{1}(\tau, a, x; y_{e}^{*}) \right)^{2}dxds \tau
\]

**Step 2.** The second step is devoting to passing to the limit as \( \varepsilon \to 0 \). To this end, we use (4.13) and detail first some computations.

Recall that by (1.8), \( \|U(e^{W_{e}}y_{e})\| \leq c_{W}\gamma_{\infty}\sqrt{a^{+}\text{meas}(O_{U})}R_{0} \), where \( R_{0} \) is precisely (4.11), and we calculate

\[
|S_{2}(t, a, x; y_{e}) - S_{2}(t, a, x; y_{e}^{*})| = |m_{e}(t, a, x; U(e^{W_{e}}y_{e}))y_{e} - m_{e}(t, a, x; U(e^{W_{e}}y_{e}^{*}))y_{e}^{*}|
\]

\[
= |m_{e}(t, a, x; U(e^{W_{e}}y_{e}) - m_{e}(t, a, x; U(e^{W_{e}}y_{e}^{*}))| + |y_{e} - y_{e}^{*}|m_{e}(t, a, x; U(e^{W_{e}}y_{e}^{*}))|.
\]

Recall that \( \{e^{W_{e}(t,a,x)-W_{e}(0,a,x)}\}_{\varepsilon} \) is a Cauchy sequence and by (2.20) we have

\[
\left|\left(m_{e}(t, a, x; U(e^{W_{e}}y_{e}) - m_{e}(t, a, x; U(e^{W_{e}}y_{e}^{*}))\right)
\right| = \left|\left(m_{0}(a, x; U(e^{W_{e}}y_{e}))e^{W_{e}(t,a,x)-W_{e}(0,a,x)} - m_{0}(a, x; U(e^{W_{e}}y_{e}^{*}))e^{W_{e}(t,a,x)-W_{e}(0,a,x)}\right)\right|
\]

\[
= \left|\left(m_{0}(a, x; U(e^{W_{e}}y_{e}) - m_{0}(a, x; U(e^{W_{e}}y_{e}^{*}))\right)\left|e^{W_{e}(t,a,x)-W_{e}(t,0,x)}\right|\right|
\]

\[
+ \left|\left(e^{W_{e}(t,a,x)-W_{e}(0,a,x)} - e^{W_{e}(t,a,x)-W_{e}(t,0,x)}\right|m_{0}(a, x; U(e^{W_{e}}y_{e}))\right|
\]

\[
\leq c_{W_{e}}L_{m_{0}}(R_{0})\gamma_{\infty} \int_{0}^{a} \int_{O_{U}} \left|e^{W_{e}} - e^{W_{e}^{*}}\right| |y_{e}(t)| + |y_{e} - y_{e}^{*}| |e^{W_{e}}| dxd\tau + m_{0}\delta_{\varepsilon,\varepsilon^{*}}
\]

with \( \delta_{\varepsilon,\varepsilon^{*}} \) arbitrarily small. Then,

\[
|S_{2}(t, a, x; y_{e}(t)) - S_{2}(t, a, x; y_{e}^{*}(t))|
\]

\[
\leq (c_{W_{e}}L_{m_{0}}(R_{0})\gamma_{\infty} \sqrt{a^{+}\text{meas}(O_{U})}) |y_{e}(t)|_{H_{e}} \delta_{\varepsilon,\varepsilon^{*}} + |y_{e} - y_{e}^{*}|_{H_{e}} c_{W} + m_{0}\delta_{\varepsilon,\varepsilon^{*}} |y_{e}(t)|_{H_{e}} + m_{\infty} |y_{e} - y_{e}^{*}|
\]

19
whence
\[ \|S_2(\cdot, \cdot; y_e(t)) - S_2(\cdot, \cdot; y_{e'}(t))\|_H \leq C_m(R_0) \|y_e(t) - y_{e'}(t)\|_H + C_2(R_0) \delta_{e,e'}. \]

For \( S_1 \) we get
\[ \|S_1(\cdot, \cdot; y_e(t)) - S_1(\cdot, \cdot; y_{e'}(t))\|_H \leq C_{\mu_0}(R_0) \|y_e(t) - y_{e'}(t)\|_H + C_3(R_0) \delta_{e,e'}. \]

Then, (4.13) yields
\[
\| (y_e - y_{e'})(t) \|^2_H + \int_0^t \int_O (y_e - y_{e'})^2(\tau, a, x)dx d\tau + \int_0^t \| (y_e - y_{e'})(\tau) \|^2_V d\tau \\
\leq c_0 R_0^2 \left( \|g_{1e} - g_{1e'}\|^2_\infty + \|g_{2e} - g_{2e'}\|^2_\infty + \|\alpha_e - \alpha_{e'}\|^2_\infty + \|v - v_{e'}\|^2_\infty \right) \\
+ c_1 (1 + \|g_{1e'}\|_\infty + \|v \cdot g_{2e'}\|_\infty) \int_0^t \| (y_e - y_{e'})(\tau) \|^2_H d\tau \\
+ c_1 \left( (a + C^2_m(R_0) + C^2_{\mu_0}(R_0)) \int_0^t \| y_e(\tau) - y_{e'}(\tau) \|^2_H d\tau + (C^2_2(R_0) + C^2_3(R_0)) \delta_{e,e'}^2 \right),
\]
and applying the Gronwall’s lemma we get
\[
\| (y_e - y_{e'})(t) \|^2_H + \int_0^t \int_O (y_e - y_{e'})^2(\tau, a, x)dx d\tau + \int_0^t \| (y_e - y_{e'})(\tau) \|^2_V d\tau \\
\leq e^{c_1(1 + \|g_{1e'}\|_\infty + \|v \cdot g_{2e'}\|_\infty + (a + C^2_m(R_0) + C^2_{\mu_0}(R_0))) T} \\
\times c_0 R_0^2 \left( \|g_{1e} - g_{1e'}\|^2_\infty + \|g_{2e} - g_{2e'}\|^2_\infty + \|\alpha_e - \alpha_{e'}\|^2_\infty + c_1 (C^2_2(R_0) + C^2_3(R_0)) \delta_{e,e'}^2 \right).
\]

Taking into account that \( \{W_e, \alpha_e\} \), are Cauchy sequences, we deduce that \( \{y_e\}_{e>0} \) is a Cauchy sequence too, hence
\[ y_e \to y \text{ strongly in } C([0, T]; H) \cap L^2(0, T; V) \cap C([0, a^+]; L^2(0, T; H)) \].

Consequently, since
\[ |U(y_e) - U(y)| = \left| \int_0^{a^+} \int_{O_{t,a}^m} \gamma(a, x)(y_e - y)(t) dx da \right| \leq \gamma_\infty \sqrt{a^+ \text{ meas}(O_U)} \|y_e(t) - y(t)\|_H, \]
it follows that \( U(y_e) \to U(y) \) strongly in \( C([0, T]) \), and a.e. on \( (0, T) \). By Egorov theorem, there exists a measurable subset \( A_0^c \subset (0, T) \), with \( \text{meas}(A_0^c) < \delta^c \), and \( U(y_e) \to U(y) \) uniformly on \( (0, T) \setminus A_0^c \).

Then, since \( m_0 \) is continuous with respect to the fourth variable we have
\[ m_\varepsilon(t, a, x; U(y_e)) = e^{W_{\varepsilon}(t, a, x) - W_{\varepsilon}(0, 0, x)} m_0(a, x; U(y_e)) \to m_0(a, x; U(y)) e^{W_{\varepsilon}(t, a, x) - W(t, 0, x)} \]
on \( (0, T) \setminus A_0^c \times (0, a^+) \times O \) and so it tends strongly in \( L^2(0, T; H) \). This implies
\[ m_\varepsilon(\cdot, \cdot; U(y)) y_e \to m(\cdot, \cdot; U(y)) y \text{ strongly in } L^2(0, T; H). \]

A similar convergence is true for \( \mu_0(\cdot, \cdot; U(y)) y \). Since the coefficients \( g_{1e}, g_{2e}, \alpha_e, k_e \) tend strongly to \( g_1, g_2, \alpha, k \) in their corresponding spaces, it follows by passing to the limit in (4.12) that \( y \) satisfies (2.23). In particular for \( t = T, a = a^+ \psi(T, a, x) = 0 \), it is in conclusion a solution to (2.10)–(2.19).

Relations (4.10) and (3.12) are satisfied at limit by \( y \) and the difference \( y - \overline{y} \), respectively, and imply (4.1) and the solution uniqueness.

Next, we show that \( y(t) \) is a \( F_t \)-adapted process. Let us recall problem (3.13) and assume that \( Y_0 \) is measurable with respect to \( F_0 \). In fact, \( Y_0 \) stands for \( y_0 = p_0 \) which has this property by (2.7). Since \( A(t) \) is quasi \( m \)-accrative, one can consider this equation with \( A(t) \) replaced by its Yosida approximation \( A_1(t) \) which is Lipschitz. The solution to the approximating equation can be obtained by an iterative process and so it is measurable with respect to \( F_t \). Also this property is preserved.
by passing to the limit, then the solution to (3.13), as well as all the other solutions, is \( v_{\varepsilon}(t), y_{\varepsilon}(t) \) and \( y(t) \) in Theorem 3.1 which are deduced as limits of \( F_t \)-adapted sequences, so that they are \( F_t \)-adapted.

The proof is ended. \( \Box \)

In addition to the properties of \( y \) proved in Theorem 3.1 one can add that, for each \( \omega \in \Omega \), there exists the strong derivative of \( y \) and equations (2.16)–(2.19) are satisfied in the sense of distributions.

Let us define

\[ \mathcal{X} = \{ u \in \mathcal{V}; \ u_0 \in \mathcal{V}' \}, \ H^1_T(0,T) = \{ \varphi \in H^1(0,T); \varphi(T) = 0 \} \]

and denote by \( \mathcal{X}' \) and \( (H^1_T(0,T))' \) the dual spaces of \( \mathcal{X} \) and \( H^1_T(0,T) \), respectively.

**Corollary 4.2.** Under the assumptions of Theorem 4.1 it follows that

\[
\frac{dy}{dt}(t) \in L^2(0,T; \mathcal{X}').
\] (4.14)

**Proof.** In (3.9) \( \psi \) can be taken of the form \( \psi(t,a,x) = \varphi(t)\psi_0(a,x) \), with \( \varphi \in H^1_T(0,T) \) and \( \psi_0 \in \mathcal{X} \). Obviously, \( \psi_0 \in C([0,a^+];H) \). Let us define \( \tilde{A}(t) : \mathcal{V} \cap C[0,a^+];H) \rightarrow \mathcal{X}' \) by

\[
\left< \tilde{A}(t)v, \psi_0 \right>_{\mathcal{X}' \times \mathcal{X}} = \int_0^a \int_0^t v(a^+,x)\psi_0(a^+,x)dx - \int_0^a \int_0^t v(\psi_0)_a dx da \\
- \int_0^a \int_0^t m(t,a,x;U(v))vd da + \int_0^a \int_0^t \mu_{\Omega}(t,a,x;U(v))v\psi_0 dx da \\
+ \int_0^a \int_0^t \nabla u \cdot \psi_0 x + v g_1 \psi_0 + v g_2 \cdot \nabla x dx da + \int_0^a \int_0^t (\alpha v + k)\psi_0 d\sigma da,
\] (4.15)

for all \( v \in \mathcal{X} \), where \( \mathcal{X}' \) is the dual of \( \mathcal{X} \), with the pivot space \( \mathcal{H} \).

One can easily calculate that \( \left\| \tilde{A}(t)v \right\|_{\mathcal{X}'} \leq C \left( \|v\|_V + \|v\|_{C([0,a];H)} \right) \), hence \( \tilde{A}(t) \) is well defined.

Moreover, for any \( \varphi \in H^1_T(0,T) \) and \( \psi_0 \in \mathcal{X} \), we define the distributional derivative

\[
\frac{dy}{dt}(t) = -\int_0^T y(t,a,x)\varphi_t(t)dt - y_0(a,x)\varphi(0),
\] (4.16)

and

\[
\left< \frac{dy}{dt}(\varphi), \psi_0 \right>_{\mathcal{X}' \times \mathcal{X}} = -\int_0^T \int_0^t y(t,a,x)\varphi_t(t)\psi_0 dx dt - \int_0^T \int_0^a y_0(a,x)\varphi(0)\psi_0 dx da.
\]

Then, one can write (3.9) as

\[
\left< \frac{dy}{dt}(\varphi), \psi_0 \right>_{\mathcal{X}' \times \mathcal{X}} + \int_0^T \left< \tilde{A}(t)y(t), \varphi(t)\psi_0 \right>_{\mathcal{X}' \times \mathcal{X}} dt = 0,
\] (4.17)

for any \( \varphi \in H^1_T(0,T) \) and \( \psi_0 \in \mathcal{X} \). This implies

\[
\frac{dy}{dt}(\varphi) + \tilde{A}(t)y(\varphi) = 0, \text{ for all } \varphi \in H^1_T(0,T),
\]

which can be still written

\[
\frac{dy}{dt} + \tilde{A}(t)y = 0, \text{ in } \mathcal{D}'(0,T;\mathcal{X}').
\] (4.18)

Moreover, since

\[
\int_0^T \left| \left< \tilde{A}(t)y(t), \varphi(t)\psi_0 \right>_{\mathcal{X}' \times \mathcal{X}} \right| dt \leq \int_0^T |\varphi(t)||A(t)y(t)||_{\mathcal{X}'}||\psi_0||_X dt \\
\leq ||\varphi||_{L^2(0,T)} \left( \int_0^T ||A(t)y(t)||^2_{\mathcal{X}'} dt \right)^{1/2} \cdot ||\psi_0||_X \leq C ||\varphi||_{L^2(0,T)} ||\psi_0||_X \left( ||v||_V + ||v||_{C([0,a];H)} \right)
\]

21
it follows that
\[ \left\| \frac{dy}{dt}(\varphi) \right\| _{X'} \leq \sup_{\| \psi_0 \|_X \leq 1} \int_0^T \left\langle \tilde{A}(t)y(t), \varphi(t)\psi_0 \right\rangle _{X',X} \left\| dt \leq C \| \varphi \| _{L^2(0,T)} \]
implying (4.14), and so (4.18) can be written
\[ \frac{dy}{dt}(t) + \tilde{A}(t)y(t) = 0, \text{ a.e. } t \in (0,T). \]  

**Theorem 4.3.** Under the assumptions 1.1-2.8 the stochastic problem 1.1-1.4 has a unique solution and $e^{-w_0} \in L^2(0,T;X')$.

**Proof.** Recall that (2.10)-(2.19) has a unique solution (2.24), for each $\omega \in \Omega$, given by Theorem 3.1.

We go back to $p$ by the transformation (2.12).

Next, by Itô’s formula we have
\[ \int T e W \frac{dy}{dt}(t) + e W (\rho_x \star \tilde{A}(t)y_x)(t) = 0. \]  

Let us denote $p_x := e W y_x$ and note that $p_x \to e W y := p$ strongly in all spaces indicated in (4.20).

Next, by Itô’s formula we have
\[ e W \frac{dy}{dt} = d(e W y_x) - y_x deW \]
and using (2.13) in (4.21) we get
\[ dp_x - p_x dW - \mu p_x dt + e W (\rho_x \star \tilde{A}(t)y_x)(t)dt = 0. \]

Integrating from 0 to $t$ and taking into account that
\[ \int_0^T \left( \int_0^{a^+} \int_0^T \mu_y(t)dy_0 \right)^2 dt \leq \| \mu_x \|_\infty \int_0^T \| p_x(t) \|_{L^2} dt \leq C, \text{ P-a.s.,} \]
which ensures that the Itô integral makes sense, we have
\[ p_x(t) - p_x(0) - \int_0^t \mu_p d\tau - \int_0^t p_x(\tau)dW(\tau) + \int_0^t e W(\tau)(\rho_x \star \tilde{A}(\tau)y_x)(\tau)d\tau = 0. \]

Then, passing to the limit as $\varepsilon \to 0$ and taking into account the convergence of $p_x$ to $p$ and the definition (4.15) we obtain
\[ p(t) - p(0) - \int_0^t \mu p d\tau - \int_0^t p(\tau)dW(\tau) + \int_0^t e W(\tau,a_x)\tilde{A}y(\tau)d\tau = 0. \]
This equation tested at $\psi_0 \in \mathcal{X}$, yields

\begin{equation}
(p(t), \psi_0)_\mathcal{H} - (p(0), \psi_0)_\mathcal{H} - \int_0^t (\mu p(\tau), \psi_0)_\mathcal{H} \, d\tau - \int_0^t (p(\tau), \psi_0)_\mathcal{H} \, dW(\tau) + \int_0^t \left\langle e^{W(\tau)} \bar{A}(\tau) y(\tau), \psi_0 \right\rangle_{\mathcal{X}', \mathcal{X}} \, d\tau = 0.
\end{equation}

But

\begin{equation}
\left\langle e^{W(\tau)} \bar{A}(\tau) y(\tau), \psi_0 \right\rangle_{\mathcal{X}', \mathcal{X}} = \left\langle \bar{A}(\tau) y(\tau), e^{W(\tau)} \psi_0 \right\rangle_{\mathcal{X}', \mathcal{X}},
\end{equation}

and replacing in (4.22) the definition of $\bar{A}(\tau) y(\tau)$ by (4.13) where the test function is $e^{W(\tau)} \psi_0$ we obtain after performing all necessary calculations the weak form (2.10).

The solution $p$ is constructed as the limit of an $\mathcal{F}_t$-adapted sequence, so that $p$ is a $\mathcal{F}_t$-adapted process.

Finally, let us assume that there are two solutions $p_1$ and $p_2$ satisfying (1.1)-(1.4). By substituting $y_i = e^{-W} p_i$, $i = 1, 2$ and by making all calculations we are led to two systems in $y_i$ with the same coefficients. As we know that the solution to the deterministic random system (2.16)-(2.19) is unique, it follows that the solution $p$ to the stochastic system is unique. This ends the proof. \(\square\)

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