ZERO DISTRIBUTION OF RANDOM SPARSE POLYNOMIALS

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Abstract. We study asymptotic zero distribution of random Laurent polynomials whose support are contained in dilates of a fixed integral polytope $P$ as their degree grow. We assume that the coefficients are i.i.d. random variables whose distribution law has bounded density with logarithmically decaying tails. Along the way, we develop a pluripotential theory for multi-circled plurisubharmonic functions which grow like the support function of $P$ in the logarithmic coordinates. As a result, we prove a quantitative localized version of Bernstein-Kouchnirenko Theorem.

1. Introduction

Recall that Newton polytope of a Laurent polynomial $f(z_1, \ldots, z_m) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$ is the convex hull (in $\mathbb{R}^m$) of the exponents in $f(z)$. It is well-known that for a system $(f_1, \ldots, f_m)$ of Laurent polynomials in general position the common zeros is a discrete set in $(\mathbb{C}^*)^m$ and the number of simultaneous zeros of such a system is given by the mixed volume of Newton polytopes of $f_i$’s [Ber75, Kou76]. In this work, we study asymptotic behavior of zeros of the systems of random Laurent polynomials with prescribed Newton polytope as their degree grow. More precisely, we consider random linear combinations of Laurent polynomials relative to a suitable basis (eg. orthogonal polynomials) whose support is contained in $NP$ of a fixed integral polytope $P \subset \mathbb{R}^m$ with non-empty interior. We assume that the random coefficients are i.i.d and their distribution law is absolutely continuous with respect to Lebesgue measure and has logarithmically decaying tails. Standard real and complex Gaussians are among the examples of such distributions.

Computation of simultaneous zeros of deterministic as well as Gaussian systems of sparse polynomials has been studied by various authors (see eg. [HS95, Roj96, MR04, DGM13]) by using mostly methods of algebraic and toric geometry. In this work, we employ methods of pluripotential theory (cf. [SZ04, BS07, BL13, Bay13]). Along the way, we develop a pluripotential theory for plurisubharmonic (psh for short) functions which are dominated by the support function of $P$ (up to a constant) in logarithmic coordinates on $(\mathbb{C}^*)^m$. We remark that the class of psh functions that we work with is a generalization of the Lelong class which corresponds here to the special case $P = \Sigma$ where $\Sigma$ is the standard unit simplex in $\mathbb{R}^m$. For a weighted compact set $(K,q)$ i.e. a nonpluripolar compact set $K \subset (\mathbb{C}^*)^m$ and a continuous weight function $q : (\mathbb{C}^*)^m \to \mathbb{R}$, we define a weighted global extremal function $V_{P,K,q}$ on $(\mathbb{C}^*)^m$ for which we prove a Siciak-Zaharyuta type result (Theorem 3.2) that is of independent interest. Then for given integral polytopes $P_i$ with non-empty interior, we show that the mixed complex Monge-Ampère measure $MAC(V_{P_1,K,q}, \ldots, V_{P_m,K,q})$ of the extremal functions $V_{P_i,K,q}$ is well defined on $(\mathbb{C}^*)^m$ and is of total mass equal to the mixed volume of $P_1, \ldots, P_m$. We use Bergman kernel asymptotics to prove that the normalized expected zero current along simultaneous zero set of independent random Laurent polynomials converges weakly to the external product $dd^c V_{P_1,K,q} \wedge \cdots \wedge dd^c V_{P_m,K,q}$ in any codimension (Theorem 4.1). Moreover, if $P \subset \mathbb{R}^m_{\geq 0}$, expected distribution of zeros has a self-averaging property in the sense that almost surely the normalized zero currents are asymptotic to $dd^c V_{P_1,K,q} \wedge \cdots \wedge dd^c V_{P_m,K,q}$. In particular, almost surely number of zeros of $m$ independent Laurent polynomials $(f_1, \ldots, f_m)$ in an open set $U \subset (\mathbb{C}^*)^m$ is asymptotic to $N^m MAC(V_{P_1,K,q}, \ldots, V_{P_m,K,q})(U)$ (Theorem 4.2). As a result, we obtain a quantitative localized
version of Bernstein-Kouchnirenko theorem. In the last section, we obtain a generalization of the above results (Theorem 1.4) for certain unbounded merely closed subsets \( K \subset (\mathbb{C}^*)^m \) and weight functions \( q \). Recall that zero distribution of Gaussian Laurent polynomials is studied in [SZ04]. The setting of [SZ04] corresponds here to the special case \( P \subset p\Sigma \) for some \( p \in \mathbb{Z}_+ \), \( K = (\mathbb{C}^*)^m \) and \( q(z) = \frac{d}{2} \log(1 + \|z\|^2) \).

For a Laurent polynomial \( f \) the amoeba \( \mathcal{A}_f \) is by definition [GKZ94] the image of the zero locus of \( f \) under the map \( \text{Log}(z_1, \ldots , z_m) = (\log|z_1|, \ldots , \log|z_m|) \). Amoebas are useful tools in several areas such as complex analysis, real algebraic geometry and tropical algebra (see eg. [PR04], [FPT00], [Mik04] and references therein). Complex plane curve amoebas were studied by Passare and Rullgård [PR04] in which they proved that area of such amoebas is bounded by a constant times the volume of Newton polytope of \( f \). In certain cases, one can obtain asymptotic distribution of amoebas from our results.

1.1. Statement of results. Recall that a Laurent polynomial on \((\mathbb{C}^*)^m\) is of the form

\[
 f(z) = \sum a_J z^J \in \mathbb{C}[z_1^{±1}, \ldots , z_m^{±1}]
\]

where \( a_J \in \mathbb{C} \) (or \( \mathbb{R} \)) and \( z^J := z_1^{j_1} \cdots z_m^{j_m} \). The set \( S_f := \{ J \in \mathbb{Z}^m : a_J \neq 0 \} \) is called the support of \( f \) and convex hull of \( S_f \) in \( \mathbb{R}^m \) is called Newton polytope of \( f \). For an integral polytope \( P \) (i.e. convex hull of a finite subset of \( \mathbb{Z}^m \)), we denote the space of Laurent polynomials whose Newton polytope is contained in \( P \) by

\[
 \text{Poly}(P) := \{ f \in \mathbb{C}[z_1^{±1}, \ldots , z_m^{±1}] : S_f \subset P \}
\]

Such polynomials are called sparse polynomials in the literature.

We are interested in asymptotic patterns of zero distribution of systems of Laurent polynomials \((f_{1,N}, \ldots , f_{m,N})\) such that \( S_{f_{j,N}} \subset NP_j \) as \( N \to \infty \). It follows from Bernstein-Kouchnirenko theorem [Ber75] [Kou76] that for systems in general position the set of common zeros are isolated points in \((\mathbb{C}^*)^m\) and the number of simultaneous roots of the system counting multiplicities is given by \( DN^m \).

For a weighted compact set \((K,q)\) i.e. a nonpluripolar compact set \( K \subset (\mathbb{C}^*)^m \) and a continuous function \( q : (\mathbb{C}^*)^m \to \mathbb{R} \), we define the weighted global extremal function

\[
 V_{P,K,q} := \sup \{ \psi \in Psh((\mathbb{C}^*)^m) : \psi(z) \leq \max_{f \in P} \log|z^J| + C_\psi \text{ on } (\mathbb{C}^*)^m \text{ and } \psi \leq q \text{ on } K \}.
\]

We remark that in the special case \( P = \Sigma \), the function \( V_{\Sigma,K,q} \) coincides with the upper envelope of Lelong class of psh functions defined in [ST97] Appendix B. It follows that \( V_{P,K,q} \) is a locally bounded psh function on \((\mathbb{C}^*)^m\) and grows like the support function of \( P \) in logarithmic coordinates (see section 2.2.1 for details). By definition, a weighted compact set \((K,q)\) is regular if \( V_{P,K,q} \) is continuous. Throughout this note we assume that \((K,q)\) is a regular weighted compact set. Unit polydisc and round sphere in \( \mathbb{C}^m \) are among the examples of regular compact sets.

For a measure \( \tau \) supported in \( K \), we fix an ONB basis \( \{ F_j \}_{j=1}^{d_N} \) for Poly\((NP)\) with respect to the inner product

\[
 \langle f, g \rangle := \int_K f(z) g(z) e^{-2Nq(z)} d\tau(z).
\]

Then a Laurent polynomial \( f_N \) can be written uniquely as

\[
 f_N = \sum_{j=1}^{d_N} a_j F_j
\]

where \( d_N = \text{dim}(\text{Poly}(NP)) \). We assume that the coefficients \( a_j \) are i.i.d. complex (respectively real) valued random variables such that distribution law of \( a_j \), is of the form \( P = \phi(z) dz \) (respectively...
\(\phi(x)dx\) where \(\phi\) is a bounded function and \(dz\) (respectively \(dx\)) is the Lebesgue measure on \(\mathbb{C}\) (respectively on \(\mathbb{R}\)) and assume that

\[
F(t) := P\{a : \log |a| > t\} = O(t^{-\alpha})
\]

for some \(\alpha > m + 1\). We remark that our setting includes standard real and complex Gaussian distributions of mean zero and variance one. Then we endow \(\text{Poly}(NP)\) with a \(d_N\)-fold product measure \(\text{Prob}_N\).

Throughout this note we assume that the Bergman functions

\[
B(\tau, q)(z) := \sup_{\|f\|_{L^2(\mathbb{C}^n, e^{-\tau|z|^2})} = 1} |f(z)| e^{-q(z)}
\]

associated with \(\text{Poly}(P)\) has sub-exponential growth, that is

\[
\sup_{z \in \mathbb{C}^n} B(\tau, Nq)(z) = O(e^{N\epsilon})
\]

for all \(\epsilon > 0\) and \(N \geq 1\). Such measures \(\tau\) which always exists on regular weighted compact sets \((K, q)\) when \(P \subset \mathbb{R}^{m+1}_+\), are called Bernstein-Markov (BM) measures in the literature (see §3.1 for details). We remark that although the inner product on \(\text{Poly}(NP)\) (and hence the ONB) depends on the choice of BM-measure \(\tau\), asymptotic distribution of zeros does not depend on it.

In what follows we let \(\omega := \frac{1}{2}d\omega^c \log(1 + \|z\|^2)\) denote the restriction of the Fubini-Study form to \((\mathbb{C}^*)^m\) where \(d\omega^c := \frac{1}{2}d\partial\). Since the distribution law of \(f_N^k\)'s are absolutely continuous by Bertini’s theorem for generic systems \((f_N^1, \ldots, f_N^k)\) of i.i.d. random Laurent polynomials, their zero loci are smooth and intersect transversely. In particular,

\[
Z_{f_N^1, \ldots, f_N^k} := \{z \in (\mathbb{C}^*)^m : f_N^1(z) = \cdots = f_N^k(z) = 0\}
\]

is smooth and of codimension \(k\) in \((\mathbb{C}^*)^m\). The zero set \(Z_{f_N^1, \ldots, f_N^k}\) carries a natural \(2m-2k\) dimensional Riemannian volume \(\text{Vol}_{2m-2k}\) obtained by restricting \(\omega^{m-k}\) to \(Z_{f_N^1, \ldots, f_N^k}\). We let \([Z_{f_N^1, \ldots, f_N^k}]\) denote the current of integration along the zero set \(Z_{f_N^1, \ldots, f_N^k}\). For generic systems \((f_N^1, \ldots, f_N^k)\) the current \(N^{-k}[Z_{f_N^1, \ldots, f_N^k}]\) has finite mass on \((\mathbb{C}^*)^m\) bounded by the mixed volume \(\text{MV}_m(P_1, \ldots, P_k, \Sigma_1, \ldots, \Sigma)\) (see Remark 2.7), hence the expected zero current

\[
\langle \mathbb{E}[Z_{f_N^1, \ldots, f_N^k}], \Theta \rangle := \int_{\text{Poly}(NP_1) \times \cdots \times \text{Poly}(NP_k)} ([Z_{f_N^1, \ldots, f_N^k}], \Theta) d\text{Prob}_N(f_N^1) \cdots d\text{Prob}_N(f_N^k)
\]

is well-defined on test forms \(\Theta \in \mathcal{D}_{m-k, m-k}(\mathbb{C}^*)^m\).

**Theorem 1.1.** Let \(P_i \subset \mathbb{R}^m\) be an integral polytope with non-empty interior for each \(i = 1, \ldots, k \leq m\) and \((K, q)\) be a regular weighted compact set. Then

\[
N^{-k}\mathbb{E}[Z_{f_N^1, \ldots, f_N^k}] \to d\omega^c(V_{P_1, K, q}) \wedge \cdots \wedge d\omega^c(V_{P_k, K, q})
\]

weakly on \((\mathbb{C}^*)^m\) as \(N \to \infty\). In particular, expected volume

\[
N^{-k}\mathbb{E}[\text{Vol}_{2m-2k}(Z_{f_N^1, \ldots, f_N^k} \cap U)] \to \int_U d\omega^c(V_{P_1, K, q}) \wedge \cdots \wedge d\omega^c(V_{P_k, K, q}) \wedge \omega^{m-k}
\]

for every relatively compact open set \(U \subset (\mathbb{C}^*)^m\).

In the special case \(P \subset \mathbb{P}^m\) for some \(p \in \mathbb{Z}_+\), we can identify \(\text{Poly}(NP)\) with a subspace \(\Pi_{NP}\) of \(H^0(\mathbb{P}^m, \mathcal{O}(pN))\) where \(\mathcal{O}(1) \to \mathbb{P}^m\) denotes the hyperplane bundle on the complex projective space \(\mathbb{P}^m\). Then we consider the product space \(\mathcal{E} = \prod_{N=1}^{\infty} \Pi_{NP}\) endowed with the product measure. Thus, elements of \(\mathcal{E}\) are random sequences of global holomorphic sections of powers of \(\mathcal{O}(p)\). The following self averaging property of random zero currents is a consequence of Theorem 1.1 and the variance estimate in [Bay13, §5]:
Theorem 1.2. Let \( P_i \subset \mathbb{R}_{\geq 0}^m \) be an integral polytope with non-empty interior for each \( i = 1, \ldots, k \leq m \) and \((K,q)\) be a regular weighted compact set then almost surely
\[
N^{-k}[Z_{f_1^k, \ldots, f_N^k}] \to df^*(V_{P_1,K,q}) \wedge \cdots \wedge df^*(V_{P_m,K,q})
\]
weakly on \((\mathbb{C}^*)^m\) as \( N \to \infty\).

In particular, letting \( k = m \), it follows from Proposition 2.8 that the total mass
\[
\int_{(\mathbb{C}^*)^m} MA_{V}(P_1, \ldots, P_m) = MV_m(P_1, \ldots, P_m)
\]
where \( MA_{V}(V_{P_1,K,q}, \ldots, V_{P_m,K,q}) \) denotes the mixed complex Monge-Ampère of the extremal functions \( V_{P_1,K,q}, \ldots, V_{P_m,K,q} \). Hence, almost surely the number of zeros in an open set \( U \subset (\mathbb{C}^*)^m \) of \( m \) independent random Laurent polynomials is asymptotic to \( N^m MA_{V}(V_{P_1,K,q}, \ldots, V_{P_m,K,q})(U) \). Thus, Theorem 1.2 gives a quantitative localized version of the Bernstein-Kouchnirenko theorem.

1.2. Comparison with the results in the literature. In the special case \( P = \Sigma \), the set \( Poly(\Sigma) \) coincides with the (ordinary) complex polynomials in \( m \)-variables of degree at most \( N \) and the extremal function \( V_{K,\Sigma,q} \) coincides with the weighted global extremal function of psh functions with logarithmic growth at infinity (cf [ST97] Appendix B). In particular, if the random coefficients \( a_j \) in \( f_j \) are i.i.d. standard complex Gaussian then we recover [BS07 Theorem 3.1] (see also [BL13 Theorem 7.3] and [Bay13 Theorem 1.2] for more general distributions).

Theorem 1.1 can be also considered as a global universality result in the sense that it extends earlier known results for the Gaussian distributions to setting of distributions that has logarithmically decaying tails. For instance, letting \( K = (S^1)^m \) the real torus and \( q(z) = 0 \), we see that the monomials \( \{z^j\}_{j \in N P \cap \mathbb{Z}^m} \) form an ONB for \( Poly(NP) \) with respect to the normalized Lebesgue measure on the real torus. Moreover, endowing \( Poly(NP) \) with complex (or real) Gaussian distribution with mean zero and a (positive definite and diagonal) variance matrix \( C \) for each \( 1 \leq k \leq m \) we observe that
\[
N^{-k}[Z_{f_1^k, \ldots, f_N^k}] = \omega_{NP_1} \wedge \cdots \wedge \omega_{NP_k}
\]
where \( \omega_{NP} = \frac{1}{2} dd^c \sum_{J \in NP \cap \mathbb{Z}^m} \log |z|^2 \) is a Kähler form for sufficiently large \( N \) and we obtain [MR04 Theorem 2]. Then Examples 2.6 and 3.3 together with Theorem 1.1 yields
\[
N^{-k}[Z_{f_1^k, \ldots, f_N^k}] \to \frac{MV_m(P_1, \ldots, P_m)}{(2\pi)^m} d\theta_1 \cdots d\theta_m \text{ weakly as } N \to \infty
\]
hence, we recover [DGM13 Theorem 1.8]. Next, we provide the following example to illustrate the impact of the choice of \((P,K)\) on zero distribution:

Example 1.3. Let \( P = Conv((0,0), (0,1), (1,1), (T,0)) \subset \mathbb{R}^2 \)

where \( T \geq 2 \) is an integer and \( K = S^3 \) is the unit sphere in \( \mathbb{C}^2 \). Then taking \( q \equiv 0 \) we see that
\[
e^jz^J := \left( \frac{(j_1 + j_2 + 1)!}{j_1!j_2!} \right)^{\frac{1}{2}} z_1^{j_1} z_2^{j_2} \text{ for } J = (j_1, j_2) \in NP
\]
form an ONB for \( Poly(NP) \) with respect to the inner product induced from \( L^2(\sigma) \) where \( \sigma \) is the probability surface area measure on \( S^3 \). Then a random Laurent polynomial is of the form
\[
f_N(z) = \sum_{J \in NP} a_J e^jz^J.
\]
and by Theorem 1.2 almost surely

\[ N^{-2} \sum_{\zeta \in \mathbb{Z}} \delta \zeta \to MA_C(V_{P,K}). \]

weakly as \( N \to \infty \) where the measure \( MA_C(V_{P,K}) \) is the complex Monge-Ampère of the unweighted (i.e. \( q \equiv 0 \)) global extremal function \( V_{P,K} \). Since \( V_{P,K} \equiv 0 \) on \( S^3 \) by Proposition 2.7 the measure \( MA_C(V_{P,K}) \) is supported in \( S^3 \). However, unlike the case \( P = \Sigma \) the mass of \( MA_C(V_{P,K}) \) is not uniformly distributed on \( S^3 \) (see Figures 1 and 2 below).

Figures 1 and 2 illustrate zero distribution of independent system of two random polynomials of the form (1.2) whose coefficients are complex i.i.d. standard Gaussian respectively Pareto-distributed with \( T = 5 \) and \( N = 10 \).

**Figure 1.** Standard Gaussian

**Figure 2.** Pareto distribution with \( P\{|a| > R\} \sim R^{-3} \)

In the last part of this work, we obtain a generalization of Theorem 1.1 for certain unbounded closed sets \( K \subset (\mathbb{C}^*)^m \) and weight functions \( q \) (see §5 for details):

**Theorem 1.4.** Let \( P_i \subset \mathbb{R}_{\geq 0}^m \) be an integral polytope with non-empty interior for each \( i = 1, \ldots, k \leq m \) and \( (K,q) \) be a regular weighted closed set with \( q: (\mathbb{C}^*)^m \to \mathbb{R} \) be weakly admissible continuous weight function. Then

\[ N^{-k} E[Z_{f_{P_1}, \ldots, f_{P_k}}] \to dd^c(V_{P_1,K,q}) \wedge \cdots \wedge dd^c(V_{P_k,K,q}) \]

weakly as \( N \to \infty \). Moreover, almost surely

\[ N^{-k} [Z_{f_{P_1}, \ldots, f_{P_k}}] \to dd^c(V_{P_1,K,q}) \wedge \cdots \wedge dd^c(V_{P_k,K,q}) \]

weakly on \( (\mathbb{C}^*)^m \) as \( N \to \infty \).

In the special case, \( P_i \subset p \Sigma \) for some \( p \in \mathbb{Z}_+ \) and \( K = (\mathbb{C}^*)^m \) together with \( q(z) = \frac{p}{2} \log(1 + \|z\|^2) \) zero distribution of random Laurent polynomials with i.i.d. standard complex Gaussian coefficients are studied by Shiffman and Zelditch [SZ04, Shi08]. It follows from [SZ04, Theorem 4.1] that \( V_{K,P,q} \) is continuous on \( (\mathbb{C}^*)^m \), in particular \( (K,q) \) is a regular weighted set (see Example 6.4 for details). Hence, Theorem 1.4 applies in this setting and we recover [SZ04, Theorem 1.4] and [Shi08, Theorem 1.5]. Specializing further, if \( P := P_1 = \cdots = P_m \) by Proposition 2.7 we see that asymptotically zeros of random polynomials concentrate in the region \( A_P := \mu_{P}^{-1}(P^3) \) which is called classically allowed region in [SZ04], where

\[ \mu_p: (\mathbb{C}^*)^m \to \mathbb{R}^m \]

\[ \mu_p(z) = ( \frac{p|z_1|^2}{1 + \|z\|^2}, \ldots, \frac{p|z_m|^2}{1 + \|z\|^2} ). \]
Example 1.5. Let \( P = \text{Conv}((0,0), (0,1), (1,1), (1,0)) \subset \mathbb{R}^2 \) be the unit square.

We also let \( K = (\mathbb{C}^*)^2 \) and \( q(z) = \log(1 + \|z\|^2) \) (i.e. \( p = 2 \)). It follows from \cite{SZ04} Example 1 that the classically allowed region is given by

\[ A_P = \{(z_1, z_2) \in (\mathbb{C}^*)^2 : |z_1|^2 - 1 < |z_2|^2 < |z_1|^2 + 1\} \]

and

\[ V_{P,K,q}(z_1, z_2) = \begin{cases} \log(1 + \|z\|^2) & \text{for } z \in A_P \\ \frac{1}{2} \log |z_2|^2 + \frac{1}{2} \log(1 + |z_1|^2) + \log 2 & \text{for } |z_2|^2 \geq |z_1|^2 + 1 \\ \frac{1}{2} \log |z_1|^2 + \frac{1}{2} \log(1 + |z_2|^2) + \log 2 & \text{for } |z_1|^2 \geq |z_2|^2 + 1 \end{cases} \]

Hence, \((K,q)\) is a regular weighted closed set and Theorem 1.2 applies. Moreover,

\[ c_J z^J := \left(\frac{(N+2)!}{2(N+|J|)!|J_1|!|J_2|!}\right)^{\frac{1}{2}} z_1^{j_1} z_2^{j_2} \]

form an ONB for \( \text{Poly}(NP) \) with respect to the inner product

\[ \langle f, g \rangle : = \int_{(\mathbb{C}^*)^2} f(z) \overline{g(z)} e^{-2Nq(z)} \omega_{FS}^2 = \int_{(\mathbb{C}^*)^2} f(z) \overline{g(z)} \frac{2}{\pi^2(1 + \|z\|^2)^{Np+3}} dz. \]

Thus a random polynomial in the present setting is of the form

\[ f_N(z) = \sum_{J \in NP} a_J c_J z^J \]

and by Theorem 1.2 almost surely

\[ N^{-2} \sum_{\zeta \in Z_{N^2} \times Z_{N^2}} \delta_{\zeta} \to 1_A \frac{2}{\pi^2(1 + \|z\|^2)^3} dz. \]

1.3. Connection with toric varieties. Recall that an integral polytope \( P \subset \mathbb{R}^m \) is called Delzant if a neighborhood of any vertex of \( P \) is \( SL(m, \mathbb{Z}) \) equivalent to \( \{x_i \geq 0 : i = 1, \ldots, m\} \subset \mathbb{R}^m \). A theorem of Delzant asserts that if \( P \) is an integral Delzant polytope then one can construct a toric variety \( X_P \) which is a (smooth) projective manifold and an ample line bundle \( L \to X_P \) such that \( \frac{1}{2} dd^c \sum_{J \in NP \subset \mathbb{Z}^m} \log |z|^2 \) is a Kähler metric on \((\mathbb{C}^*)^m\) and it extends to a smooth global Kähler metric on the toric variety \( X_P \) for sufficiently large \( N \). Moreover, the space of global holomorphic sections \( H^0(X_P, L^{\otimes N}) \) can be identified with \( \text{Poly}(NP) \). In this setting, the asymptotic distribution of zeros was obtained in \cite[Theorem 1.1]{Bay13} (see also \cite{SZ99} for the Gaussian setting). We remark that this is the only overlap with \cite{Bay13} and neither of these results can be obtained from each other.
2. Preliminaries

2.1. Lattice points, polytopes and convex analysis. In what follows $\mathbb{R}^m$ (respectively $\mathbb{R}^m_{\geq 0}$) denotes the set of points in the real Euclidean space with positive (respectively non negative) coordinates. By an integral polytope we mean convex hull $\text{Conv}(A)$ in $\mathbb{R}^m$ of a non-empty finite set $A \subset \mathbb{Z}^m$. We let $\Sigma$ denote the standard unit simplex that is $\Sigma = \text{Conv}(0, e_1, \ldots, e_m)$ where $e_i$ denote the standard basis elements in $\mathbb{Z}^m$. For two non-empty convex sets $P_1, P_2$ we denote their Minkowski sum by

$$P_1 + P_2 := \{x_1 + x_2 : x_1 \in P_1, x_2 \in P_2\}.$$ 

In the present section, we let $P \subset \mathbb{R}^m$ be a convex body i.e. a compact convex set with non-empty interior. Let $\text{Vol}_m$ denote the volume of a subset of $\mathbb{R}^m$ with respect to Lebesgue measure which is normalized such that $\text{Vol}_m(\Sigma) = \frac{1}{m}$.

A theorem by Minkowski and Steiner asserts that $\text{Vol}_m(N_1 P_1 + \cdots + N_k P_k)$ is a homogeneous polynomial of degree $m$ in the variables $N_1, \ldots, N_k \in \mathbb{Z}_+$ (see for instance [CLO05] §4 for details). In the special case $k = m$, the coefficient of the monomial $N_1 \cdots N_m$ in the homogenous expansion of $\text{Vol}_m(N_1 P_1 + \cdots + N_m P_m)$ is called mixed volume of $P_1, \ldots, P_m$ and denoted by $\text{MV}_m(P_1, \ldots, P_m)$. One can compute the mixed volume of convex sets $P_1, \ldots, P_m$ by means of polarization formula

$$\text{MV}_m(P_1, \ldots, P_m) = \sum_{k=1}^m \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq m} (-1)^{m-k} \text{Vol}_m(P_{j_1} + \cdots + P_{j_k}).$$

In particular, if $P = P_1 = \cdots = P_m$ then

$$\text{MV}_m(P) := \text{MV}_m(P, \ldots, P) = m! \text{Vol}_m(P).$$

For example, $\text{MV}_m(\Sigma) = 1$.

Recall that for a convex function $\varphi$ its Legendre-Fenchel transform (or conjugate function of $\varphi$) is defined by

$$\varphi^*(p) := \sup_{x \in \mathbb{R}^m} \{ \langle p, x \rangle - \varphi(x) \}$$

which is a convex function in $\mathbb{R}^m$ with values in $(-\infty, \infty]$. In particular, $K_\varphi := \{ \varphi^* < \infty \}$ is a convex set. Moreover, for convex functions $\varphi_1, \varphi_2$ we have $K_{\varphi_1 + \varphi_2} = K_{\varphi_1} + K_{\varphi_2}$. We refer the reader to the manuscript [Roc70] for basic results in convex analysis.

We denote the support function of a convex body $P$ by $\varphi_P : \mathbb{R}^m \to \mathbb{R}$

$$\varphi_P(x) = \sup_{p \in P} \langle x, p \rangle$$

which is a one-homogenous convex function. Since Legendre transform is an involution we have

$$\varphi_P^*(x) := \begin{cases} 0 & \text{if } x \in P \\ \infty & \text{if } x \notin P. \end{cases}$$

We let $d\varphi|_x$ denote the sub-gradient of $\varphi$ of a finite convex function at a point $x \in \mathbb{R}^m$. Recall that $d\varphi|_x$ is a closed convex set in $\mathbb{R}^m$ defined by

$$d\varphi|_x := \{ p \in \mathbb{R}^m : \varphi(y) \geq \varphi(x) + \langle p, y - x \rangle \text{ for every } y \in \mathbb{R}^m \}.$$ 

We remark that if $\varphi$ is differentiable at $x$ then $d\varphi|_x$ is a point and coincides with $\nabla \varphi(x)$. In the sequel we let $d\varphi(E)$ denote the image of $E \subset \mathbb{R}^m$ under the sub-gradient. It is well know that $x \to d\varphi|_x$ is invertible on its image for almost every $x \in \mathbb{R}^m$ (see for instance, [RRT77] Theorem 2.5]). The later property allows us to define a bona fide measure:
2.1.1. **Real Monge-Ampère of a convex function.** Following [RT77], we define real Monge-Ampère (or Monge-Ampère in the sense of Aleksandrov) of a finite convex function $\varphi$ by

$$
(2.1) \quad MA^R_\varphi(E) := m! \int_{d\varphi(E)} dVol_m
$$

where $E \subset \mathbb{R}^m$ is a Borel set. It is worth mentioning that the above definition differs from that of [RT77] by the normalization constant $m!$ and the role of it is explained in (2.2.2). If $\varphi \in C^2(\mathbb{R}^m)$ then its real Monge-Ampère coincides with its Hessian that is

$$
(2.2) \quad MA^R_\varphi(E) = m! \int_{\nabla \varphi(E)} dVol_m

(2.3) \quad = m! \int_E \det(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}) dx.
$$

Moreover, for a convex function $\varphi \in C^2(\mathbb{R}^m)$ one can also define the Monge-Ampère as

$$
MA_R(\varphi) := d(\varphi_{x_1}) \wedge \cdots \wedge d(\varphi_{x_m})
$$

where $\varphi_{x_i} := \frac{\partial \varphi}{\partial x_i}$. In fact, endowing the cone of convex functions with the topology of locally uniform convergence and the space of measures on $\mathbb{R}^m$ by topology of weak converge it follows from [RT77] that the operator $MA_R$ extends as a continuous symmetric multilinear operator from merely convex functions to measures and the equality

$$
MA_R(\varphi) = MA_R(\varphi)
$$

remains valid for merely convex functions $\varphi$. Finally, following [PR04] one can define mixed real Monge-Ampère of convex functions $\varphi_1, \ldots, \varphi_m$ by means of the polarization formula

$$
(2.4) \quad MA_R(\varphi_1, \ldots, \varphi_m) := \frac{1}{m!} \sum_{k=1}^m \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq m} (-1)^{m-k} MA_R(\varphi_{j_1} + \cdots + \varphi_{j_k}).
$$

The following result provides a key link between mixed volume and the (mixed) real Monge-Ampère operator. We refer the reader to [PR04, Proposition 3] and [BB13, Lemma 2.5] for the proof.

**Proposition 2.1.** Let $P_i \subset \mathbb{R}^m$ be a convex body and $\varphi_i$ be a convex function on $\mathbb{R}^m$ such that $\varphi_i - \varphi_{P_i}$ is bounded for each $i = 1, \ldots, m$. Then

$$
\int_{\mathbb{R}^m} MA_R(\varphi_1, \ldots, \varphi_m) = MV_m(P_1, \ldots, P_m).
$$

2.2. **Pluri-potential theory.** We let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\|z\|$ denote the Euclidean norm of $z \in \mathbb{C}^m$. For a convex body $P \subset \mathbb{R}^m$, we denote

$$
H_P(z) := \max_{J \in P} \log |z^J|
$$

where we use the multi-dimensional notation $z^J := z_1^{j_1} \cdots z_m^{j_m}$ and $J = (j_1, \ldots, j_m) \in \mathbb{Z}^m$. Clearly, $H_P$ is a psh function on $(\mathbb{C}^*)^m$. Note that $H_P$ coincides with $\varphi_P$, the support function of $P$ in the logarithmic coordinates on $(\mathbb{C}^*)^m$. Namely, letting

$$
Log : (\mathbb{C}^*)^m \to \mathbb{R}^m

Log(z) = (\log |z_1|, \ldots, \log |z_m|)
$$

we see that $H_P(z) = \varphi_P \circ Log(z)$ for $z \in (\mathbb{C}^*)^m$. For instance, if $P = \Sigma$ then

$$
H_\Sigma(z) = \max_{i=1, \ldots, m} \log^+ |z_i|.
$$
We let $\mathcal{L}({\mathbb{C}}^m)$ (respectively $\mathcal{L}_+({\mathbb{C}}^m)$) denote the Lelong class i.e. the set of psh functions $\psi$ on $\mathbb{C}^m$ such that $\psi(z) \leq \log^+ ||z|| + C_\psi$ (respectively $\psi(z) - \log^+ ||z||$ is bounded). Following [Bert09], we also define the following classes of psh functions:

$$\mathcal{L}_P := \{ \psi \in Psh((\mathbb{C}^*)^m) : \psi \leq H_P + C_\psi \text{ on } (\mathbb{C}^*)^m \}$$

$$\mathcal{L}_{P,+} := \{ \psi \in \mathcal{L}_P : \psi \geq H_P + C_\psi \text{ on } (\mathbb{C}^*)^m \}$$

We say that a function $\psi \in \mathcal{L}_P$ is $m$-circled if $\psi(z) = \psi([|z_1|, \ldots, |z_m|])$, i.e. $\psi$ is invariant under the action of the real torus $(\mathbb{S}^1)^m$. We denote the set of all $m$-circled functions in $\mathcal{L}_P$ by $\mathcal{L}_{P,0}$. The class $\mathcal{L}_P$ is a generalization of the Lelong class $\mathcal{L}({\mathbb{C}}^m)$ which correspond to the case $P = \Sigma$. Indeed, since every $\psi \in \mathcal{L}_{\Sigma}$ is locally bounded from above near points of the set $\{z \in \mathbb{C}^m : z_1 \cdots z_m = 0\}$, it extends to a psh function $\hat{\psi}$ on $\mathbb{C}^m$. Moreover, since

$$\max_{j \in \Sigma} \log |z|^j = \max_{i=1, \ldots, m} \log^+ |z_i| \leq \log^+ ||z||$$

the extension $\hat{\psi} \in \mathcal{L}({\mathbb{C}}^m)$.

The following lemma will be useful in the sequel.

**Lemma 2.2.** Let $P$ be a convex body and $\psi \in \mathcal{L}_{P,+}$. Then for every $p \in P^o$ there exists $\kappa, C_\psi > 0$ such that

$$\psi(z) \geq \kappa \max_{j=1, \ldots, m} \log |z_j| + \log |z^P| - C_\psi \quad \text{for every } z \in (\mathbb{C}^*)^m.$$

**Proof.** Let $\varphi_P(x)$ denote the support function of $P$. Fixing a small ball $B(p, \kappa) \subset P^o$, by definition we have $\varphi_P \equiv 0$ on $B(p, \kappa)$. Since $(\varphi_P^*)^* = \varphi_P$ this implies that

$$\varphi_P(x) \geq \sup_{q \in B(p, \kappa)} \langle q, x \rangle = \sup_{y \in B(0,1)} \langle ky, x \rangle + \langle p, x \rangle = \kappa \|x\| + \langle p, x \rangle$$

hence, using $H_P(z) = \varphi_P(Log(z))$ for $z \in (\mathbb{C}^*)^m$ we obtain

$$H_P(z) \geq \kappa \max_{j=1, \ldots, m} |\log |z_j|| + \log |z^P|$$

which implies the assertion. \qed

2.2.1. **Global extremal function.** In this section, we let $K \subset (\mathbb{C}^*)^m$ be a non-pluripolar compact set and $q : (\mathbb{C}^*)^m \to \mathbb{R}$ be a continuous function. We define the *weighted global extremal function* $V_{p,K,q}^*$ to be the usc regularization of

$$V_{p,K,q}^* := \sup \{ \psi \in \mathcal{L}_P : \psi \leq q \text{ on } K \}.$$

We remark that in the special case $P = \Sigma$ the function $V_{\Sigma,K,q}^*$ coincides with the weighted global extremal function defined in [ST97, Appendix B]. Moreover, specializing further, in the unweighted case (i.e. $q \equiv 0$) $V_{\Sigma,K}^*$ is the pluricomplex Green function of $K$ (cf. [Kli91 §5]). The following example is a consequence of standard arguments (cf. [Kli91 §5]):

**Example 2.3.** For $P = [a, b] \subset \mathbb{R}$, $K = \mathbb{S}^1$ unit circle and $q \equiv 0$ we have

$$V_{P,\mathbb{S}^1}(z) = \max \{a \log |z|, b \log |z|\} = H_P(z) \quad \text{for } z \in \mathbb{C}^*.$$

This implies that (more generally) for a convex body $P \subset \mathbb{R}^m$, $K = (\mathbb{S}^1)^m \subset (\mathbb{C}^*)^m$ is the real torus and $q \equiv 0$ the (unweighted) global extremal function

$$V_{P,(\mathbb{S}^1)^m}(z) = H_P(z) = \max_{j \in P} \log |z^j| \quad \text{for } z \in (\mathbb{C}^*)^m.$$

In particular, $V_{P,(\mathbb{S}^1)^m}$ is continuous.

**Proposition 2.4.** $V_{p,K,q}^* \in \mathcal{L}_{P,+}$.
Proof. Since $K$ is non-pluripolar and $q$ is continuous the defining family for $V_{P,K,q}$ is locally bounded from above and it follows that $V_{P,K,q}^* \in Psh((\mathbb{C}^*)^m)$ (see [Kli91, §5] for details). Then there exists $C > 0$ such that $\psi - C \leq 0$ on the real torus $(S^1)^m$ for every competitor $\psi$ in the defining family of $V_{P,K,q}$. Thus, by Example 2.3 we have

$$V_{P,K,q}^* - C \leq H_P(z)$$

for $z \in (\mathbb{C}^*)^m$. Finally, since

$$H_P - (\text{sup}(H_P - q)) \in L_P$$

is a competitor for $V_{P,K,q}$ we see that $V_{P,K,q}^* \in L_{P,+}$. □

In particular, $V_{P,K,q}^* \in Psh((\mathbb{C}^*)^m) \cap L_{P,\infty}^\infty((\mathbb{C}^*)^m)$. In the special case $P \subset \mathbb{R}_{>0}^m$, one can show that the extremal function $V_{P,K,q}$ is lsc. However, we do not know if this is the case for more general polytopes. In general, the extremal function $V_{P,K,q}$ is not usc. We say that the weighted compact set $(K,q)$ is regular if $V_{P,K,q}$ is continuous. Note that in this case $V_{P,K,q}^* \leq q$ on $K$. This means that there exists a “maximal element” (which is $V_{P,K,q}^*$) in the competing family for $V_{P,K,q}$.

2.2.2. Complex Monge-Ampère versus Real Monge-Ampère. In what follows we denote $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2\pi} (\partial - \bar{\partial})$ so that $dd^c = \frac{1}{\pi} \bar{\partial} \partial$. The relation between complex Monge-Ampère of a m-circled psh function and the real Monge-Ampère of it (in the logarithmic coordinates) has been observed by various authors (see for instance [Ras00, §3] and references therein). We provide some details for convenience of the reader. Recall that for an open set $\Omega \subset (\mathbb{C}^*)^m$ and a $\psi \in C^2(\Omega)$ we have

$$MA_C(\psi) := dd^c \psi \wedge \cdots \wedge dd^c \psi = m! \left( \frac{2}{\pi} \right)^m \det \left[ \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} \right] d\beta$$

where $d\beta := \frac{1}{2} dz_1 \wedge \cdots \wedge \frac{1}{2} dz_m \wedge d\bar{z}_m$. Now, if $\varphi \in C^2(\log(\Omega))$ then by the chain rule

$$4 \zeta_j \bar{\zeta}_k \frac{\partial^2 \varphi \circ \log}{\partial \zeta_j \partial \bar{\zeta}_k} = \left. \frac{\partial \varphi}{\partial x_j} dx_k \right|_{x = \log(z)}$$

where $\zeta_j = e^{x_j+i\theta_j}$ for some $x_j \in \mathbb{R}$ and $\theta_j \in [0, 2\pi]$. This means that for a given m-circled psh function on $(\mathbb{C}^*)^m$ one can define a convex function function $\varphi(x)$ on $\mathbb{R}^m$ by setting $\varphi(x) = \psi(z)$ where $x = \log(z)$. Moreover, by (2.5)

$$4^m |z_1|^2 \cdots |z_m|^2 \det \left[ \frac{\partial^2 \varphi \circ \log}{\partial \zeta_j \partial \bar{\zeta}_k} \right] = \det \left[ \frac{\partial \varphi}{\partial x_j} \right]$$

then using $d\beta = |z_1|^2 \cdots |z_m|^2 dx_1 \theta_1 \cdots dx_m \theta_m$ in the coordinates $z_j = e^{x_j+i\theta_j}$ where $x_j \in \mathbb{R}$ and $\theta_j \in [0, 2\pi]$ we see that

$$\log(MA_C(\psi)) = MA(\varphi).$$

That is for a Borel set $E \subset (\mathbb{C}^*)^m$

$$\int_E MA(\varphi) = \int_{\log^{-1}(E)} MA_C(\psi).$$

Furthermore, by the results of [Kli77, BTS82] the equality (2.6) holds for any locally bounded m-circled psh function $\psi$ on $(\mathbb{C}^*)^m$. Then (2.6) together with polarization formula for complex Monge-Ampère

$$\sum_{i=1}^m dd^c \psi_i = \frac{1}{m!} \sum_{i=1}^m \sum_{1 \leq i_1 \leq \cdots \leq i_m} (-1)^{m-j} MA_C(\psi_{i_1} + \cdots + \psi_{i_j})$$

and (2.4) implies that for locally bounded m-circled psh functions $\psi_1, \ldots, \psi_m$

$$\log(\int_{i=1}^m dd^c \psi_i) = MA(\varphi_1, \ldots, \varphi_m)$$
where $\varphi_i(x) = \psi_i(z)$ is the corresponding convex function defined as above. Thus, the following is an immediate consequence of Corollary 2.1.

**Proposition 2.5.** Let $\psi_i \in \mathcal{L}_{P,+}^m$ for $i = 1, \ldots, m$ then the total mass of the mixed complex Monge-Ampère

$$
\int_{(\mathbb{C}^*)^m} \bigwedge_{i=1}^m dd^c \psi_i = MV_m(P_1, \ldots, P_m).
$$

By Example 2.3 and Proposition 2.5 we obtain:

**Example 2.6.** Let $P_i \subset \mathbb{R}^m$ be convex bodies for $i = 1, \ldots, m$, $K = (S^1)^m$ is the real torus and $q = 0$. Then the mixed complex Monge-Ampère

$$
\bigwedge_{i=1}^m dd^c (V_{P,K}) = \frac{MV_m(P_1, \ldots, P_m)}{(2\pi)^m} d\theta_1 \ldots d\theta_m.
$$

Recall that $V := V_{P,K,q}$ is a locally bounded psh function on $(\mathbb{C}^*)^m$. Thus, by [BT82] its complex Monge-Ampère measure

$$
MA_c(V) := dd^c (V) \wedge \ldots \wedge dd^c (V)
$$

is well defined and does not charge pluripolar subsets of $(\mathbb{C}^*)^m$. We denote the support of complex Monge-Ampère of the extremal function by $supp(MA_c(V))$.

**Proposition 2.7.** Let $P$ be a convex body and $(K, q)$ be a regular weighted compact set. Then

$$
supp(MA_c(V_{P,K,q})) \subset \{z \in K : V_{P,K,q}(z) = q(z)\}.
$$

In particular, if $q \in \mathcal{L}_{P,+}^m \cap C^2((\mathbb{C}^*)^m)$ then

$$
(\ref{2.7}) \quad Log\left(supp(MA_c(V))\right) \subset \nabla \varphi^{-1}(P^o)
$$

where $\varphi$ is the convex function defined by relation $q(z) = \varphi(Log(z))$.

**Proof.** By a Theorem of T. Bloom in [ST97, Appendix B, Lemma 2.3] for the case $P = \Sigma$

$$
supp(MA_c(V_{\Sigma,K,q})) \subset \{z \in K : V_{\Sigma,K,q}(z) = q(z)\}.
$$

The argument in [ST97] is local and carries over to our setting hence we obtain

$$
supp(MA_c(V_{P,K,q})) \subset \{z \in K : V_{P,K,q}(z) = q(z)\}.
$$

On the other hand, since $q \in \mathcal{L}_p$, we can define a convex function $\varphi$ by setting $\varphi(x) = q(z)$ where $x = Log(z)$. Then by [PRH] Proposition 3 and [BB13, Lemma 2.5] the support of $MA_R(\varphi)$ is a subset of $\nabla \varphi^{-1}(P^o)$. Hence, the assertion follows from (2.6). \[ \square \]

A remarkable property of the Lelong class functions $\psi \in \mathcal{L}(\mathbb{C}^m) \cap L^\infty_{loc}(\mathbb{C}^m)$ is that the total mass

$$
\int_{\mathbb{C}^m} MA_c(\psi) \leq 1.
$$

Moreover, if $\psi \in \mathcal{L}_+(\mathbb{C}^m)$

$$
(\ref{2.8}) \quad \int_{\mathbb{C}^m} MA_c(\psi) = \int_{\mathbb{C}^m} MA_c\left(\frac{1}{2} \log(1 + ||z||^2)\right) = 1
$$

which was observed in [Tay83]. The equality (2.8) is a consequence of comparison theorem (see [Kli91] §5 for the details and references). We generalize the result of [Tay83] to our setting: In what follows we denote the product of annuli

$$
A_{\rho,R} := \{z \in (\mathbb{C}^*)^m : \rho < |z_i| < R \text{ for each } i = 1, \ldots, m\} \text{ for } \rho, R > 0.
$$

We also let

$$
\varpi := dd^c H_{\Sigma}(z) = dd^c (\max_{i=1,\ldots,m} \log^+ |z_i|).
$$
Proposition 2.8. Let $P_i \subset \mathbb{R}^m$ be a convex body and $u_i, v_i \in \mathcal{L}_{P_i} \cap L_{\text{loc}}^\infty((\mathbb{C}^*)^m)$ such that

$$u_i(z) \leq v_i(z) + C_i$$

for each $i = 1, \ldots, k$. Then the total masses

$$\int_{(\mathbb{C}^*)^m} \bigwedge_{i=1}^k dd^c u_i \wedge \omega^{m-k} \leq \int_{(\mathbb{C}^*)^m} \bigwedge_{i=1}^k dd^c v_i \wedge \omega^{m-k}.$$ 

In particular, if $u_i \in \mathcal{L}_{P_i,+}$ each $i = 1, \ldots, k$ then the total mass of the mixed Monge-Ampère

$$\int_{(\mathbb{C}^*)^m} dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \omega^{m-k} = MV_m(P_1, \ldots, P_k, \Sigma, \ldots, \Sigma).$$

Proof. Since the complex Monge-Ampère is a symmetric operator by replacing $v_i$ with $u_i$ successively in the $i^{th}$ step, it is enough to prove the assertion for the case $v_i = u_i$ for $2 \leq i \leq k$.

We fix a convex body $Q \subset \mathbb{R}^m$ such that $0 \in Q^\circ$. Then by Lemma 2.2 and replacing $v_1$ by $v'_1 := v_1 + \epsilon H_Q$ for $\epsilon > 0$ if necessary, we may assume that

$$u_1 - v'_1 \to -\infty$$

as $\|z\| \to \infty$ as well as $|z_i| \to 0$ for some $i \in \{1, \ldots, m\}$. Now, we define

$$\psi_N = \max\{u_1, v'_1 - N\}.$$ 

Note that $\psi_N = v'_1 - N$ near the boundary of the set $A_{\rho,R}$ for sufficiently large $R > 0$ and small $\rho > 0$. Thus, applying Stokes theorem we obtain

$$\int_{(\mathbb{C}^*)^m} dd^c v'_1 \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \omega^{m-k} \geq \int_{A_{\rho,R}} dd^c v'_1 \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \omega^{m-k} \geq \int_{A_{\rho,R}} dd^c \psi_N \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \omega^{m-k}.$$ 

Since $\psi_N$ decreases to $u_1$ as $N \to \infty$, by Bedford-Taylor theorem [BTS2] on continuity of Monge-Ampère measures along decreasing sequences we infer that

$$\int_{(\mathbb{C}^*)^m} dd^c v'_1 \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \omega^{m-k} \geq \int_{A_{\rho,R}} dd^c u_1 \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \omega^{m-k}.$$ 

Finally, since $R \gg 1, \rho > 0$ and $\epsilon > 0$ are arbitrary letting $R \to \infty, \rho \to 0$ and $\epsilon \to 0$ in $v'_1 = v_1 + \epsilon H_Q$ we obtain the first assertion.

To prove the second assertion we let $v_i = H_{P_i}$ and apply the first part together with Proposition 2.3. \qed

Remark 2.9. We remark that the condition $u_i \in L^\infty_{\text{loc}}((\mathbb{C}^*)^m)$ in Proposition 2.8 is used to make sure that the mixed complex Monge-Ampère is well defined. Thus, we infer that for $\psi \in \mathcal{L}_P$ the total mass of $\text{MA}_{\mathbb{C}}(\psi)$ is finite as soon as it is well defined on $(\mathbb{C}^*)^m$. Note that by Bertini’s theorem for generic $f_N^i \in \text{Poly}(NP_i)$ their zero sets $Z_{f_N^i}$ are smooth and intersect transversely. It follows from [Dem09] §III, Theorem 4.5] that for systems $(f_N^1, \ldots, f_N^k)$ in general position

$$[Z_{f_N^1}, \ldots, f_N^k] = \frac{1}{N} dd^c \log |f_N^1| \wedge \cdots \wedge \frac{1}{N} dd^c \log |f_N^k|$$

is well defined and has locally finite mass. Thus, by Proposition 2.8 we see that

$$(2.9) \quad \frac{1}{N^k} \int_{(\mathbb{C}^*)^m} [Z_{f_N^1}, \ldots, f_N^k] \wedge \omega^{m-k} \leq MV_m(P_1, \ldots, P_k, \Sigma, \ldots, \Sigma)$$

which was also observed in [Ras03 Cor. 6.1] when $P \subset \mathbb{R}_{\geq 0}^m$.\n
The results in the rest of this section can be considered as the local version of the ones obtained in [BEGZ10, §2]. First, we prove the following version of comparison principle (cf. [BEGZ10] Corollary 2.3] and references therein):

**Proposition 2.10** (Comparison principle). Let $P \subset \mathbb{R}^m$ a convex body and $\varphi, \psi \in L_P \cap L^\infty_{loc}((\mathbb{C}^*)^m)$ then

$$\int_{\{\varphi < \psi\}} MA_C(\psi) \leq \int_{\{\varphi < \psi\}} MA_C(\varphi) + MV_m(P) - \int_{(\mathbb{C}^*)^m} MA_C(\varphi)$$

**Proof.** By Proposition 2.8 for every $\epsilon > 0$

$$MV_m(P) \geq \int_{(\mathbb{C}^*)^m} MA_C(\max\{\varphi, \psi - \epsilon\})$$

$$\geq \int_{\{\varphi < \psi - \epsilon\}} MA_C(\psi) + \int_{\{\varphi \geq \psi - \epsilon\}} MA_C(\varphi)$$

$$\geq \int_{\{\varphi < \psi - \epsilon\}} MA_C(\psi) + \int_{(\mathbb{C}^*)^m} MA_C(\varphi) - \int_{\{\varphi < \psi\}} MA_C(\varphi).$$

Thus, letting $\epsilon \to 0$ and using Monotone convergence theorem the assertion follows. \(\square\)

We remark that in Proposition 2.10 if $\varphi \in L_{P,+}$ then by Proposition 2.8 we obtain

$$\int_{\{\varphi < \psi\}} MA_C(\psi) \leq \int_{\{\varphi < \psi\}} MA_C(\varphi).$$

Next result is an immediate consequence of (2.10):

**Corollary 2.11** (Domination principle). Let $P \subset \mathbb{R}^m$ a convex body, $\psi \in L_P \cap L^\infty_{loc}((\mathbb{C}^*)^m)$ and $\varphi \in L_{P,+}$. If $\psi \leq \varphi$ a.e. with respect to $MA_C(\varphi)$ then $\psi \leq \varphi$ everywhere on $(\mathbb{C}^*)^m$.

**Proof.** We use standard arguments. Fix $\rho(z) := \frac{1}{|N|} \log \sum_{p \in N} |z|^2$ so that $MA_C(\rho)$ dominates Lebesgue measure for sufficiently large $N$ (see [Gro90]). Shifting $\rho$ by a constant if necessary we may assume that $\rho \leq \varphi$. Then by Proposition 2.10

$$\epsilon^m \int_{\{\varphi < (1-\epsilon)\psi + \epsilon \rho\}} MA_C(\rho) \leq \int_{\{\varphi < (1-\epsilon)\psi + \epsilon \rho\}} MA_C((1-\epsilon)\psi + \epsilon \rho)$$

$$\leq \int_{\{\varphi < (1-\epsilon)\psi + \epsilon \rho\}} MA_C(\varphi)$$

$$\leq \int_{\{\varphi < \psi\}} MA_C(\varphi) = 0$$

for every $\epsilon > 0$. Thus, $\varphi \geq (1 - \epsilon)\psi + \epsilon \rho$ a.e. with respect to Lebesgue measure and hence, $\varphi \geq (1 - \epsilon)\psi + \epsilon \rho$ everywhere on $(\mathbb{C}^*)^m$ for every $\epsilon > 0$. Letting $\epsilon \to 0$ assertion follows. \(\square\)

3. A Siciak-Zaharyuta theorem

We start with a basic result which is an easy consequence of Cauchy’s estimates on the product of annuli

$$A_{\rho,R} := \{z \in (\mathbb{C}^*)^m : \rho < |z_i| < R \text{ for each } i = 1, \ldots, m\}$$

for $\rho, R > 0$ together with a Liouville type argument.

**Proposition 3.1.** Let $P \subset \mathbb{R}^m$ be an integral polytope and $f \in O((\mathbb{C}^*)^m)$ such that

$$\int_{(\mathbb{C}^*)^m} |f(z)|^2 e^{-2NH_P(z)}(1 + |z|^2)^{-r} \, dz < \infty$$

for some $0 \leq r \ll 1$ then $f$ is a Laurent polynomial such that its support $S_f \subset NP$. 

Throughout this section we denote $V := V^*_{P,K,q}$ where $K$ and $q$ as in (2.2.1) and $P \subset \mathbb{R}^m$ is an integral polytope with non-empty interior. Next, we define

$$\Phi_N := \sup_{z \in \mathbb{C}^m} \{ |f_N(z)| : f_N \in \text{Poly}(NP) \text{ and } \max_{K} |f_N(z)e^{-Nq(z)}| \leq 1 \}$$

Note that $\Phi_N \leq \Phi_{N+M}$ which implies that $\lim_{N \to \infty} \frac{1}{N} \log \Phi_N \in (C^*)^m$. Observe also that for each $f_N \in \text{Poly}(NP)$ the function $\frac{1}{N} \log |f_N(z)|$ belongs to $L_P$. Hence, $\lim_{N \to \infty} \frac{1}{N} \log \Phi_N \leq V$ on $(C^*)^m$. If $P$ is the unit simplex $\Sigma$ then it follows from seminal works of Siciak and Zaharyuta (see [K] for details) that

$$\lim_{N \to \infty} \frac{1}{N} \log \Phi_N = V$$

point-wise on $\mathbb{C}^m$. We prove a slightly stronger version of this result in the present setting:

**Theorem 3.2.** Let $P \subset \mathbb{R}^m$ be an integral polytope with non-empty interior and $(K,q)$ be a regular weighted compact set. Then

$$V_{P,K,q} = \lim_{N \to \infty} \frac{1}{N} \log \Phi_N$$

locally uniformly on $(C^*)^m$.

**Proof.** For a given compact set $X \subset (C^*)^m$ we need to show that for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$0 \leq V(z) - \frac{1}{N} \log \Phi_N(z) < \epsilon$$

for every $z \in X$ and $N \geq N_0$. To this end, fix $z_0 \in X$, and $B(z_0, \delta) \subset (C^*)^m$ be a small ball centered at $z_0$ such that for every $z \in B(z_0, \delta)$

$$|V(z) - V(z_0)| < \epsilon$$

(3.1)

First, we assume that $V$ is a smooth function on $(C^*)^m$. We also let $\chi$ be a test function with compact support in $B(z_0, \delta)$ such that $\chi \equiv 1$ on $B(z_0, \frac{\delta}{2})$. For a fixed point $p \in P^*$, we define

$$\psi_N(z) := (N - \frac{m}{\kappa})(V(z) - \frac{\epsilon}{2}) + \frac{m}{\kappa} \log |z|^k + m \max_{j=1, \ldots, m} \log |z_j - z_0,j|$$

where $\kappa > 0$ is as in Lemma 2.2 and $\frac{m}{\kappa} \ll N$. Note that $\psi_N$ is psh on $(C^*)^m$, smooth away $z_0$ and its Lelong number $\nu(\psi_N, z_0) = m$. Since $(C^*)^m$ is pseudoconvex by Hörmander’s $L^2$-estimates [Dem99] §VIII for every $r \in (0,1)$ there exists a smooth function $u_N$ on $(C^*)^m$ such that $\partial u_N = \partial \chi$ and

$$\int_{(C^*)^m} |u_N|^2 e^{-2\psi_N} \leq \frac{4}{r^2} \int_{(C^*)^m} |\partial \chi|^2 e^{-2\psi_N} (1 + |z|^2)^{-r} dz.$$

Note that the $(0,1)$ form $\partial \chi$ is supported in $B(z_0, \delta) \backslash B(z_0, \frac{\delta}{2})$ therefore both integrals are finite. Since $\nu(\psi_N, z_0) = m$ this implies that $u_N(z_0) = 0$. Moreover, by Lemma 2.2 we obtain

$$\int_{(C^*)^m} |u_N|^2 e^{-2N(V(z_0) - \epsilon)} \leq C_1 e^{-2N(V(z_0) - \epsilon)}$$

(3.2)

where $C_1 > 0$ does not depend on $N$. Next, we let $f_N := \chi - u_N$. Then $f_N$ is a holomorphic function on $(C^*)^m$ such that $f(z_0) = 1$. Furthermore,

$$\int_{(C^*)^m} |f_N|^2 e^{-2N(V(z_0) - \epsilon)} \leq C_2 e^{-2N(V(z_0) - \epsilon)}$$

(3.3)

where $C_2 > 0$ is independent of $N$. Then using $V \in L_{P,+}$ again we see that

$$\int_{(C^*)^m} |f_N|^2 e^{-2N\rho P} \leq \infty$$

and taking $r > 0$ small, Proposition 3.1 implies that $f_N$ is a polynomial such that $S_{f_N} \subset NP$.

Finally, if $V$ is not smooth on $(C^*)^m$ then we approximate $V$ by smooth psh functions $V_t := \varrho_t * V \geq V$ (where $\varrho_t$ is an approximate identity) on an increasing sequence of pseudoconvex domains
Then applying sub-mean value inequality to subharmonic function $|C|$, where (3.4) 1 and that for every $y \in B(z, \rho)$ and $\delta, \epsilon > 0$ such that $K_\rho := \{z \in (\mathbb{C}^*)^m : \text{dist}(z, K) < \rho\} \subset (\mathbb{C}^*)^m$ and for every $z \in K$

$$|q(y) - q(z)| < \frac{\epsilon}{2}$$

and

$$q(y) > V(y) - \frac{\epsilon}{2}$$

for every $y \in B(z, \rho) \subset (\mathbb{C}^*)^m$. Now, let

$$C_r := \min_{z \in K_\rho} (1 + |z|^2)^{-r}$$

then applying sub-mean value inequality to subharmonic function $|f_N(z)|^2$ on $B(z, \rho)$ we obtain

$$C_r |f_N(z)|^2 e^{-2Nq(z)} \leq C_3 \int_{B(z, \rho)} |f_N(y)|^2 e^{-2N(V(y) - \frac{\epsilon}{2}) + (q(z) - q(y))} (1 + |y|^2)^{-r} dy$$

$$\leq C_3 \int_{(\mathbb{C}^*)^m} |f_N(y)|^2 e^{-2N(V(y) - \epsilon)} (1 + |y|^2)^{-r} dy$$

$$\leq C_4 e^{-2N(V(z_0) - \epsilon)}$$

where $C_4 > 0$ is as above independent of $N$. Thus, replacing $f_N$ by $F_N := \sqrt{\frac{C_r}{C_4}} e^{N(V(z_0) - \epsilon)} f_N$, we see that

$$\max_{z \in K} |F_N(z) e^{-q(z)}| \leq 1.$$
on $B(z_0, \delta)$. Then
\[
V(z) - \frac{1}{N} \log \Phi_N \leq V(z) - \frac{1}{N_0 + \frac{k}{N}} \log \Phi_{N_0} + \epsilon \\
\leq V(z) - \frac{1}{N_0} \log \Phi_{N_0} + \frac{j}{N_0(kN_0 + j)} V(z) + \epsilon \\
\leq V(z) - \frac{1}{N_0} \log \Phi_{N_0} + 2\epsilon.
\]

Now, by (3.4) for $N \geq N_0$
\[
0 \leq V(z) - \frac{1}{N} \log \Phi_N(z) \\
\leq (V(z_0) - \frac{1}{N_0} \log \Phi_{N_0}(z_0)) + (V(z) - V(z_0)) + \frac{1}{N_0} \log \Phi_{N_0}(z_0) - \frac{1}{N_0} \log \Phi_{N_0}(z)) + 2\epsilon \\
\leq 2\epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{2} + 2\epsilon = 5\epsilon
\]
for $z \in B(z_0, \delta)$ and $N \geq N_0^2$. Finally, covering the compact set $X$ with finitely many $B(z_i, \delta_i)$ we see that
\[
0 \leq V - \frac{1}{N} \log \Phi_N \leq 5\epsilon \text{ for every } N \geq \max_{i} N_i^2.
\]
on $X$. This finishes the proof. \hfill \square

3.1. Bernstein-Markov measures. Next we turn our attention to $L^2$ space of weighted polynomials. A measure $\tau$ supported in $K$ is called a Bernstein-Markov measure for the triple $(P, K, q)$ if it satisfies the weighted Bernstein-Markov inequality: there exists constants $M_N > 0$ such that for every $f_N \in Poly(NP)$
\[
\max_{K} |f_N e^{-Nq}| \leq M_N \|f_N e^{-Nq}\|_{L^2(\tau)}
\]
and $\limsup_{N \to \infty} (M_N)^{\frac{1}{N}} = 1$. This roughly means that sup-norm and $L^2(\tau)$-norm on $Poly(NP)$ are asymptotically equivalent. We remark that if $P \subset p\Sigma$ then any BM measure (for polynomials of degree at most $N$) induces a BM measure for our setting. For instance, for $P = \Sigma$ it follows from $[NZ83]$ that the complex Monge-Ampère of the unweighted (i.e. $q \equiv 0$) global extremal function $V_K^*$ of a regular compact set $K$ satisfies BM inequality.

Next, we fix an orthonormal basis $\{F_j\}_{j=1}^{d_N}$ for $Poly(NP)$ with respect the inner product induced from $L^2(e^{-2Nq}\tau)$. Then associated Bergman kernel is given by
\[
S_N(z, w) = \sum_{j=1}^{d_N} F_j(z)\overline{F_j(w)}
\]
where $d_N := \dim Poly(NP)$.

**Example 3.3.** Let $P \subset \mathbb{R}^m$ be an integral polytope with non-empty interior. Then $\tau = \frac{1}{(2\pi)^m} d\theta_1 \ldots d\theta_m$ satisfies BM inequality for $K = (S^1)^m$ and $q \equiv 0$. Indeed, monomials $\{z^j\}_{j \in NP}$ form an ONB for $Poly(NP)$ with respect to the inner product induced from $L^2(\tau)$ and by Schwarz inequality for every $f_N \in Poly(NP)$ and $z \in (S^1)^m$
\[
|f_N(z)| \leq \sup_{w \in (S^1)^m} \sqrt{S_N(w, w)} \|f_N\|_{L^2(\tau)} \\
\leq \sqrt{d_N} \|f_N\|_{L^2(\tau)}
\]
where $d_N = \dim(Poly(NP)) = O(N^m)$ by $[Ehr67]$.

The following result was proved in $[BS07, \text{Lemma 3.4}]$ for the case $P = \Sigma$. Their argument generalizes to our setting mutatis-mutandis.
**Proposition 3.4.** Let $P$ be an integral polytope with non-empty interior, $K \subset (\mathbb{C}^*)^m$ be a compact set and $q : (\mathbb{C}^*)^m \rightarrow \mathbb{R}$ be continuous weight function such that $V := V_{P,K,q}$ is continuous. If $\tau$ be a BM measure supported on $K$ then

$$\frac{1}{2N} \log S_N(z,z) \rightarrow V$$

uniformly on compact subsets of $(\mathbb{C}^*)^m$.

4. Expected distribution of zeros

Recall that if $P \subset \mathbb{R}^m$ is an integral polytope then

$$\#(NP \cap \mathbb{Z}^m) = \dim(Poly(NP)) = Vol(P)N^m + o(N^m)$$

where the later is known as Ehrhart polynomial of $P$.\cite{Ehr67}.

We identify each $f_N \in Poly(NP)$ with a point in $\mathbb{C}^{dN}$ by

$$\Psi_N : Poly(NP) \rightarrow \mathbb{C}^{dN}$$

$$f_N = \sum_{j=1}^{dN} a_j^N F_j \rightarrow a^N := (a_j^N).$$

We endow $\mathbb{C}^{dN}$ with $d_N$-fold product measure $P_N$ induced by the distribution law of $a_j^N$ and denote $Prob_N := P^\star(P_N)$.

For a complex manifold $Y$ we denote the set of bidegree $(m-k,m-k)$ test forms i.e. smooth forms with compact support by $D_{m-k,m-k}(Y)$. Then a bidegree $(k,k)$ current is a continuous linear functional on $D_{m-k,m-k}(Y)$ with respect to the weak topology. We denote the set of bidegree $(k,k)$ currents by $D^{k,k}(Y)$. We refer the reader to the manuscript \cite{Dem09} for detailed information regarding the theory of currents.

For each $f_N \in Poly(NP)$ we let $[Z_{f_N}]$ denote the current of integration along regular points of the zero locus of $f_N$ and denote the action of it on a test form $\Theta \in D_{m-1,m-1}(Y)$ by $\langle [Z_{f_N}], \Theta \rangle$. Then the *expected zero current* of random Laurent polynomials $f_N \in Poly(NP)$ was defined in the introduction by

$$\langle E[Z_{f_N}], \Theta \rangle = \int_{Poly(NP)} \langle [Z_{f_N}], \Theta \rangle dProb_N(f_N).$$

**Lemma 4.1.**

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{C}^{dN}} \log |\langle a, u \rangle| |dP_N(a) = 0$$

uniformly with respect to unit vectors $u \in \mathbb{C}^{dN}$.

**Proof.** We let $u \in \mathbb{C}^{dN}$ be a unit vector and fix $\epsilon > 0$ satisfying

$$P\{a \in \mathbb{C} : \log |a| > R \} = O(R^{-(m+1+\epsilon)}).$$

First, we show that

$$\int_{\{\log |\langle a, u \rangle| > N^{1+\frac{1}{m+\epsilon}}\}} \log |\langle a, u \rangle| |dP_N(a) \leq C_m$$

where $C_m > 0$ does not depend on $N$ and the unit vector $u$. Indeed, it follows from \cite{4.2} and \cite{4.1} for sufficiently large $N$ that

$$r_k(N) := P_N\{a \in \mathbb{C}^{dN} : \log |\langle a, u \rangle| > kN^{1+\frac{1}{m+\epsilon}}\} \leq \sum_{i=1}^{dN} P_N\{a \in \mathbb{C}^{dN} : |a_i| > \frac{1}{\sqrt{d_N}} \exp(kN^{\frac{1}{m+\epsilon}})\} \leq \frac{CN^m}{k^{m+1+\epsilon} N^{\frac{1}{m+\epsilon}} (m+1+\epsilon)}$$
for some $C > 0$ independent of $N$ and $u$. Then we see that
\[
\int_{\{ \log |(a, u)| > N^{-1 - \frac{m}{m+1}} \}} \log |(a, u)| dP_N(a) \leq C_1 \sum_{k=1}^{\infty} (k + 1) N^{1 - \frac{m}{m+1}} (r_k(N) - r_{k+1}(N)) \leq C_2 \sum_{k=1}^{\infty} k^{-(m+1+\epsilon)},
\]
where $C_2 > 0$ does not depend on $N$ and $u$. Next, we show that
\[
(4.4) \quad \int_{\{ \log |(a, u)| < -N^{-1 - \frac{m}{m+1}} \}} \log |(a, u)| dP_N(a) \leq C_m
\]
where $C_m > 0$ does not depend on $N$ and $u$. To this end we observe that
\[
l_k(N) := P_N\{a \in \mathbb{C}^dN : |(a, u)| < e^{-kN^{-1 - \frac{m}{m+1}}} \} \leq C_dN e^{-2kN^{-1 - \frac{m}{m+1}}},
\]
where $C > 0$ is independent of $N$ and $u$. Indeed, since $u$ is a unit vector we may assume that $|u| \geq \frac{1}{\sqrt{dN}}$ where $u = (u_1, u_2, \ldots, u_{dN})$. Then applying the change of variables
\[
\alpha_1 = \sum_{i=1}^{dN} a_i u_i, \quad \alpha_2 = a_2, \ldots, \alpha_{dN} = a_{dN}
\]
we obtain
\[
l_k = \int_{\mathbb{C}^dN-1} \int_{|u| < e^{-kN^{-1 - \frac{m}{m+1}}}} \frac{1}{|u|^2} |\phi(\alpha_1 - \alpha_2 u_2 - \cdots - \alpha_{dN} u_{dN}) \phi(\alpha_2) \ldots \phi(\alpha_{dN}) d\lambda(\alpha_1) \ldots d\lambda(\alpha_{dN})
\]
where $P = \phi d\lambda$ then
\[
\int_{\{ \log |(a, u)| < -N^{-1 - \frac{m}{m+1}} \}} \log |(a, u)| dP_N(a) \leq C_4 \sum_{k=1}^{\infty} (k + 1) N^{m+1 - \frac{m}{m+1}} e^{-2kN^{-1 - \frac{m}{m+1}}} \leq C_5 N^{m+1 - \frac{m}{m+1}} \int_1^{\infty} (x + 1) e^{-2x N^{\frac{m}{m+1}}} dx \leq C_6 e^{-2N^{\frac{m}{m+1}}} N^m
\]
where $C_6 > 0$ does not depend on $N$ and $u$ which proves (4.4).

Combining (4.3) and (4.4) we conclude that
\[
\int_{\{ \log |(a, u)| > N^{-1 - \frac{m}{m+1}} \}} \log |(a, u)| dP_N(a) \leq C_m
\]
since $P_N$ is a probability measure this implies that
\[
(4.5) \quad \frac{1}{N} \int_{\mathbb{C}^dN} \log |(a, u)| dP_N(a) = O(N^{-\frac{m}{m+1}})
\]
which finishes the proof. □

Next lemma provides a link between Bergman kernels and expected distribution of zeros of random sparse polynomials.

**Lemma 4.2.** Let $P \subset \mathbb{R}^m$ be an integral polytope with non-empty interior then there exists a real closed $(1, 1)$ current $T_N \in D^{1,1}((\mathbb{C}^*)^m)$ such that for every test form $\Theta \in D_{m-1,m-1}((\mathbb{C}^*)^m)$
\[
\frac{1}{N} \mathbb{E}[Z_{f_N}], \Theta) = \frac{1}{2N} (dd^c (\log S_N(z, z)), \Theta) + \langle T_N, \Theta \rangle
\]
and the mass $\|T_N\| = O(N^{-\epsilon})$ for some small $\epsilon > 0$. 
Proof. It follows from Poincaré-Lelong formula that

\[ |Z_{f_N}| = dd^c \log |f_N|. \]

Writing \( f_N = \sum_{j=1}^{d_N} a_j F_j \) where \( \{ F_j \} \) is the fixed ONB for \( \text{Poly}(NP) \) and letting \( u_N(z) := \frac{f_N(z)}{\sqrt{S_N(z,z)}} \) for \( z \in (\mathbb{C}^*)^m \), by Fubini’s theorem we see that

\[
\frac{1}{N} \langle \mathbb{E}[Z_{f_N}], \Theta \rangle = \int_{\mathbb{C}^m} \left( \frac{1}{2N} dd^c \log S_N(z,z), \Theta \right) d\mathbb{P}_N(a_N) + \frac{1}{N} \int_{(\mathbb{C}^*)^m} dd^c \Theta \int_{\mathbb{C}^m} \log \langle a_N, u_N(z) \rangle |d\mathbb{P}_N(a_N) |
\]

\[
= \frac{1}{2N} \left( dd^c \log S_N(z,z), \Theta \right) + \langle T_N, \Theta \rangle
\]

Moreover, by (4.5)

\[
|\langle T_N, \Theta \rangle | \leq CN^{-\frac{\epsilon}{m}} \|dd^c \Theta\|_{\infty}
\]

where \( \epsilon > 0 \) as in (4.2) and \( \|dd^c \Theta\|_{\infty} \) denotes the sum of the sup norms of the coefficients of the smooth form \( dd^c \Theta \).

For an algebraic submanifold \( Y \subset (\mathbb{C}^*)^m \) we let \( Z_{f_N} := \{ z \in Y : f(z) = 0 \} \) denote the restriction of the zero locus of \( f \) on \( Y \). It follows from Lemma 4.2 that

\[
\mathbb{E}[Z_{f_N}] = dd^c \left( \frac{1}{2N} \log S_N(z,z) + h_N(z) \right)
\]

for \( z \in Y \) where

\[
h_N(z) := \frac{1}{N} \int_{\mathbb{C}} \log \langle a_N, u_N(z) \rangle |d\mathbb{P}_N(a) |
\]

The following is a well known probabilistic version of Poincaré-Lelong formula [SZ04, §5]:

**Proposition 4.3.** The expected zero current of independent random Laurent polynomials \( f_N^i \in \text{Poly}(NP) \), \( 1 \leq i \leq k \) is given by

\[
\mathbb{E}[Z_{f_N^i}] = \bigwedge_{i=1}^{k} \mathbb{E}[Z_{f_N^i}]
\]

**Proof of Theorem 1.1.** We prove the theorem by induction on bidegrees. Note that for every continuous \( (m-1, m-1) \) form \( \Theta \) with compact support in \( (\mathbb{C}^*)^m \)

\[
|\langle Z_{f_N^i}, \Theta \rangle | \leq \langle Z_{f_N^i}, \omega^{m-1} \rangle \| \Theta \|_\infty \leq MV_m(P_1, \Sigma, \ldots, \Sigma) \| \Theta \|_\infty
\]

by approximating \( \Theta \) with smooth forms it is enough to consider test forms on \( (\mathbb{C}^*)^m \). It follows from Lemma 4.2 that

\[
\frac{1}{N} \langle \mathbb{E}[Z_{f_N^i}], \Theta \rangle = \frac{1}{N} \int_{(\mathbb{C}^*)^m} dd^c \Theta \int_{\mathbb{C}^m} \log \langle a_N, u_N(z) \rangle |d\mathbb{P}_N(a_N) | + \frac{1}{2N} dd^c \log S_N(z,z), \Theta \rangle
\]

\[
= : I_1(N) + I_2(N)
\]

for every test form \( \Theta \) \( \in D_{m-1, m-1}((\mathbb{C}^*)^m) \). Note that the second term is deterministic and by Proposition 3.4

\[
I_2(N) \to \langle dd^c V_{P, K, q}, \Theta \rangle \quad \text{as } N \to \infty
\]

and by Lemma 4.1

\[
I_1(N) \to 0 \quad \text{as } N \to \infty
\]

which proves the case \( k = 1 \).

We denote

\[
\alpha_N^j := \frac{1}{2N} dd^c \log S_N^j(z, z)
\]
where \( S_N^j(z, w) \) is the Bergman kernel for \( \text{Poly}(NP_j) \). We claim that for every test form \( \Theta \in D_{m-k, m-k}(\mathbb{C}^m) \)

\[
\frac{1}{N^k} \langle Z[f_1, \ldots, f_k], \Theta \rangle = \left( \bigwedge_{j=1}^k \alpha_N^j, \Theta \right) + \langle T_N^k, \Theta \rangle
\]

where \( T_N^k \) is a real closed \((k, k)\) current which satisfies the uniform estimate

\[
|\langle T_N^k, \Theta \rangle| \leq C N^{-\epsilon} \|dd^c \Theta\|_{\infty}
\]

where \( \epsilon > 0 \) small as in Lemma 4.2 and \( C > 0 \) is independent of smooth form \( \Theta \) and \( N \). Note that the case \( k = 1 \) was proved in Lemma 4.2. Assume that the the claim holds for \( k - 1 \). By Bertini’s theorem for generic \( f_N^k \in \text{Poly}(NP_k) \) its zero locus \( Z_{f_N^k} \) is smooth and has codimension one in \((\mathbb{C}^*)^m\). Then, using the notation in Proposition 4.3 and by applying induction hypothesis

\[
\frac{1}{N^k} \int_{Z_{f_N^k}} \langle Z[f_1, \ldots, f_{k-1}], \Theta \rangle d\text{Prob}_N(f_N^k) \ldots d\text{Prob}_N(f_N^{k-1}) = \int_{Z_{f_N^k}} \left( \bigwedge_{j=1}^{k-1} \alpha_N^j \wedge \Theta + \langle T_N^{k-1}, \Theta \rangle \right)
\]

where

\[
|\langle \langle T_N^{k-1} \rangle_{z_{f_N^k}}, \Theta \rangle_{z_{f_N^k}} \rangle| \leq C N^{-\epsilon} \|dd^c \Theta\|_{\infty}
\]

\[
\leq C N^{-\epsilon} \|dd^c \Theta\|_{\infty} \int_{Z_{f_N^k}} \omega^{m-1}
\]

\[
\leq C_1 N^{-\epsilon} \|dd^c \Theta\|_{\infty} MV_m(P_k, \Sigma, \ldots, \Sigma)
\]

where \( \omega = \frac{1}{2} dd^c \log(1 + \|z\|^2) \) and \( C_1 > 0 \) independent of \( f_N^k \) and \( \Theta \). Now, taking the average over \( f_N^k \in \text{Poly}(NP_k) \) and using the estimate for the case \( k = 1 \) we obtain

\[
\frac{1}{N^k} \langle Z[f_1, \ldots, f_N], \Theta \rangle = \left( \bigwedge_{j=1}^k \alpha_N^j, \Theta \right) + \langle T_N^k, \Theta \rangle + \int_{\text{Poly}(NP_k)} \langle T_N^{k-1}, \Theta \rangle d\text{Prob}_N(f_N^k)
\]

\[
= \left( \bigwedge_{j=1}^k \alpha_N^j, \Theta \right) + C_{\Theta, N}
\]

where

\[
C_{\Theta, N} = \langle T_N^k, \Theta \rangle + \int_{\text{Poly}(NP_k)} \langle T_N^{k-1}, \Theta \rangle d\text{Prob}_N(f_N^k)
\]

then by Lemma 4.2 we have

\[
|C_{\Theta, N}| \leq |\langle T_N^k, \Theta \rangle| + |\int_{\text{Poly}(NP_k)} \langle T_N^{k-1}, \Theta \rangle d\text{Prob}_N(f_N^k)|
\]

\[
\leq C N^{-\epsilon} \|dd^c \Theta\|_{\infty} MV_m(P_1, \ldots, P_{k-1}, \Sigma, \ldots, \Sigma) + C_1 N^{-\epsilon} \|dd^c \Theta\|_{\infty} MV_m(P_k, \Sigma, \ldots, \Sigma)
\]

where \( C_1, C > 0 \) independent of \( N \) and \( \Theta \). Thus, the assertion follows from the above estimate and the uniform convergence of Bergman kernels to weighted global extremal function (Proposition 3.4) together with a theorem of Bedford and Taylor [BT82] on convergence of Monge-Ampere measures along uniformly convergent sequences of psh functions. \( \square \)
5. Self-averaging

In this section we prove Theorem 1.2. Let \( \mathbb{P}^m \) denote the complex projective space and \( \omega_{FS} \) is the Fubini-Study form. We also let \( dV \) denote the volume form induced by \( \omega_{FS} \). Recall that an usc function \( \varphi \in L^1(\mathbb{P}^m, dV) \) is called \( \omega_{FS} \)-psh if \( \omega_{FS}+dd^c\varphi \geq 0 \) in the sense of currents. It is well known that there is a 1-1 correspondence between Lelong class psh function \( \mathcal{L}(\mathbb{C}^m) \) and the set of \( \omega_{FS} \)-psh functions which is given by the natural identification

\[
(5.1) \quad u \in \mathcal{L}(\mathbb{C}^m) \rightarrow \varphi(z) := \begin{cases} 
 u(z) - \frac{1}{2} \log(1 + \|z\|^2) & \text{for } z \in \mathbb{C}^m \\
 \limsup_{w \in \mathbb{C}^m \to z} u(w) - \frac{1}{2} \log(1 + \|w\|^2) & \text{for } z \in H_\infty
\end{cases}
\]

where \( \mathbb{P}^m = \mathbb{C}^m \cup H_\infty \) and \( H_\infty \) denotes the hyperplane at infinity. Note that since \( \mathbb{P}^m \) is compact there is no global psh functions other than the constant ones. On the other hand, we can associate each \( \omega_{FS} \)-psh function \( \varphi \) to its curvature current \( \omega_{FS} + dd^c\varphi \) which yields compactness properties of \( \omega_{FS} \)-psh functions. We use the later properties quite often in this section. In addition, working in the compact setting makes the usage of integration by parts more simple since there is no boundary.

We denote the hyperplane bundle \( L \rightarrow \mathbb{P}^m \) by \( L := \mathcal{O}(1) \) which is endowed with the Fubini-Study metric \( h_{FS} \). In the sequel, we identify \( \mathbb{C}^m \) with the affine piece in \( \mathbb{P}^m \). Then the elements of \( H^0(\mathbb{P}^m, \mathcal{O}(N)) \) can be identified with the homogenous polynomials in \( m+1 \) variables of degree \( N \). Thus, restricting them to \( \mathbb{C}^m \), we may identify \( H^0(\mathbb{P}^m, \mathcal{O}(N)) \) with the space of polynomials \( \text{Poly}(N\Sigma) \) of total degree at most \( N \) and the smooth metric \( h_{FS} \) can be represented by the weight function \( \frac{1}{N} \log(1 + \|z\|^2) \) on \( \mathbb{C}^m \). For each \( s_N \in H^0(\mathbb{P}^m, \mathcal{O}(N)) \) we let \( \|s_N(z)\|_{Nh_{FS}} \) denote the pointwise norm of \( s_N \) evaluated with respect to the metric \( h_{FS} \). Then by (5.1) for each \( f_N \in \text{Poly}(N\Sigma) \) the function \( \frac{1}{N} \log |f_N| \) can be naturally identified with \( \frac{1}{N} \log \|s_N\|_{Nh_{FS}} \).

For \( P \subset \mathbb{R}^m_{\geq 0} \) denoting \( p = \max\{p_1 + \cdots + p_m : (p_1, \ldots, p_m) \in P\} \) (so that \( P \subset p\Sigma \)), we may identify \( \text{Poly}(NP) \) with a subspace of \( H^0(\mathbb{P}^m, \mathcal{O}(p\Sigma)) \) and denote it by \( \Pi_{NP} \). The BM measure \( \tau \) induce in inner product on the space \( H^0(\mathbb{P}^m, \mathcal{O}(p\Sigma)) \) defined by

\[
\|s_N\|^2 := \int_K \|s_N(z)\|^2_{pNh_{FS}} d\tau(z).
\]

We also let \( S_N(z, w) \) denote the associated Bergman kernel. We remark that the Bergman kernel asymptotics generalize the current setting (cf. [Bay13, Proposition 2.9]). We can endow \( \Pi_{NP} \) with \( d_N \)-fold product measure \( \text{Proh}_N \) and we endow the product space \( \mathcal{P} = \prod_{N=1}^\infty \Pi_{NP} \) with the product measure \( \mathcal{P}_\infty \). Note that the elements of the probability space \( (\mathcal{P}, \mathcal{P}_\infty) \) are sequences of random holomorphic sections. For each \( s_N \in \Pi_{NP} \) denoting its zero divisor by \( Z_{s_N} \), it follows form Poincaré-Lelong formula that

\[
[Z_{s_N}] = pN\omega_{FS} + dd^c \log \|s_N\|_{Nh_{FS}}.
\]

We remark that \( [Z_{s_N}] \) coincides with the (unique) extension of the current of integration \( dd^c \log |f_N| \) through the hyperplane at infinity \( H_\infty \). Finally, by (5.1), the function \( V_{P,K,q} \) also extends to a \( p\omega_{FS} \)-psh function on \( \mathbb{P}^m \) which we denote by \( V_{P,p\omega_{FS}} \) and let

\[
T_{P,K,q} := p\omega_{FS} + dd^c V_{P,p\omega_{FS}}.
\]

**Slicing and regularization of currents:** Let \( Y \) be a complex manifold of dimension \( n \) and \( \pi_Y : Y \times \mathbb{P}^m \rightarrow Y, \pi_{\mathbb{P}^m} : Y \times \mathbb{P}^m \rightarrow \mathbb{P}^m \) denote the projections onto the factors. Given a positive closed \((k,k)\) current \( \mathcal{R} \) on \( Y \times \mathbb{P}^m \) it follows from [Fed63] (see also [DS09]) that the slices \( \mathcal{R}_y := (\mathcal{R}, \pi_{\mathbb{P}^m}, y) \) exist for a.e. \( y \in Y \). The currents \( \mathcal{R}_y \) (if it exists) is a positive closed \((k,k)\) current on \( \{y\} \times \mathbb{P}^m \). For instance, if \( \mathcal{R} \) is a continuous form then \( \mathcal{R}_y \) is just restriction of \( \mathcal{R} \) on \( \{y\} \times \mathbb{P}^m \). We can identify \( \mathcal{R}_y \) with a positive closed \((k,k)\) current on \( \mathbb{P}^m \) whose mass is independent of \( y \) [DS09, Lemma 2.4.1].

Following [DS09], we say that the map \( y \rightarrow \mathcal{R}_y \) defines a structural variety in the set of positive closed \((k,k)\) currents on \( \mathbb{P}^m \). We also say that a structural variety is special if the slice \( \mathcal{R}_y \) exists for every \( y \in Y \) and the map \( y \rightarrow \mathcal{R}_y \) is continuous. In this work, we will focus on the following special structural disc: Consider the holomorphic map

\[
H : \text{Aut}(\mathbb{P}^m) \times \mathbb{P}^m \rightarrow \mathbb{P}^m
\]
defined by $H(\tau, z) = \tau^{-1}(z)$. Given a positive closed $(k, k)$ current $R$ on $\mathbb{P}^m$ we define $\mathcal{R} := H^*(R)$. Then it is easy to see that the slice $\mathcal{R}_\tau = \tau_*(R)$. This in particular implies that $\tau \to R_\tau$ is continuous and $\{R_\tau\}_\tau$ defines a special structural variety [DS09 Proposition 2.5.1].

We let $\Delta \subset \mathbb{C}$ denote the unit disc. We fix a holomorphic chart for $\text{Aut}(\mathbb{P}^m)$ and denote the local holomorphic coordinates by $y$ where $\|y\| < 1$ and $y = 0$ corresponds to the identity map $\text{id} \in \text{Aut}(\mathbb{P}^m)$. We also let $\tau_y \in \text{Aut}(\mathbb{P}^m)$ denote the automorphism that correspond to local coordinate $y$. Next, we fix a smooth positive function $\psi$ with compact support in $\{\|y\| < 1\}$ such that $\int \psi(y)dy = 1$ and define $\psi_\theta(y) := |\theta|^{-2\theta} \psi\left(\frac{\theta}{|\theta|}\right)$ for $\theta \in \Delta$. Note that $\psi_\theta(y)dy$ is an approximate identity for the Dirac mass at 0. Finally, we define the current $\mathcal{R} \wedge \pi_Y^*(\psi_\theta dy)$ by

$$\langle \mathcal{R} \wedge \pi_Y^*(\psi_\theta dy), \Psi \rangle := \int \langle (\mathcal{R}, \pi_Y, y), \Psi \rangle \psi_\theta(y)dy$$

and

$$\langle \mathcal{R} \wedge \pi_Y^*(\psi_\theta dy), \Psi \rangle := \int (\mathcal{R}_{\theta_y}, \Psi) \psi_\theta(y)dy$$

where $\Psi$ is a $(m-k, m-k)$ test form on $Y \times \mathbb{P}^m$. Note that the slice of $\mathcal{R} \wedge \pi_Y^*(\psi_\theta dy)$ can be identified with the current $R_\theta$ whose action on the $(m-k, m-k)$ test form $\Theta$ on $\mathbb{P}^m$ defined by

$$\langle R_\theta, \Theta \rangle := \int (\langle \tau_y, R, \Theta \rangle \psi_\theta(y)dy = \int (\langle \tau_{\theta_y}, R, \Theta \rangle \psi_\theta(y)dy$$

by setting $\Psi = \pi_Y^*(\theta)$.}

**Proposition 5.1.** Let $R$ be a positive closed $(k,k)$ current on $\mathbb{P}^m$ and $\Theta$ is a smooth $(m-k, m-k)$ form on $\mathbb{P}^m$ such that $dd^c\Theta \geq 0$. Then

(i) $R_\theta$ is a smooth positive $(k,k)$ form for $\theta \in \Delta^*$. The current $R_\theta$ depends continuously on $\theta$ and $R_\theta \to R$ weakly as $\theta \to 0$.

(ii) There exists $C > 0$ such that $\|R_\theta, \Theta\| \leq C \|\Theta\| \|R\|$ for every $\theta \in \Delta$.

(iii) $\varphi(\theta) := \langle R_\theta, \Theta \rangle$ is a continuous subharmonic function on $\Delta$.

**Proof.** Part (i) is proved in [DS09 Proposition 2.1.6]. Adding a large multiple of $\omega_{FS}$ to $\Theta$ we may assume that $0 \leq \Theta \leq C \omega_{FS}$ for some $C > 0$. Since each $R_\theta$ is positive closed and its mass is independent of $\theta$, (ii) follows. For part (iii) let $\Psi := \pi_Y^*(\Theta)$ and observe that $\Phi = (\pi_Y)_*(\mathcal{R} \wedge \Psi)$ is of bidegree $(0,0)$ on $Y$ satisfying

$$dd^c\Phi = (\pi_Y)_*(\mathcal{R} \wedge dd^c\Psi) \geq 0.$$  

This implies that $\Phi$ coincides with a psh function on $Y$. Note that for fixed $y \in Y$ we have $\varphi(\theta) = \Phi(\theta y)$ for $\theta \in \Delta$ thus $\varphi$ is subharmonic. Continuity follows from (i). \qed

We continue with the proof of Theorem 1.2. We will need the following probabilistic lemma in the sequel:

**Lemma 5.2.** For a.e. $\{a^N\}_{n \geq 1} \in \prod_{N=1}^\infty \mathbb{C}^{d_N}$

$$\lim_{N \to \infty} \frac{1}{N} \log \|a^N\| = 0.$$  

**Proof.** We fix $\epsilon > 0$ such that $\alpha(1 - \epsilon) > m + 1$ where $\alpha$ as in (1.1). First, we show that with probability one, $\|a^N\| \leq d_N e^{cN^{1-\epsilon}}$ for sufficiently large $N$. Indeed, we have that with probability one, $\|a^N\| \leq d_N e^{cN^{1-\epsilon}}$ for sufficiently large $N$. Indeed,

$$\mathbb{P}_N\{a_j^N \in \mathbb{C}^{d_N} : |a_j^N| > e^{N^{1-\epsilon}}\} \leq \frac{C}{N^{1-\epsilon} \alpha}$$

which implies that

$$\mathbb{P}_N\{a^N \in \mathbb{C}^{d_N} : \|a^N\| > d_N e^{cN^{1-\epsilon}}\} \leq \frac{Cd_N}{N^{1-\epsilon} \alpha}$$

where the later defines a summable sequence. Thus, the claim follows from Borel-Cantelli lemma. Since $d_N = O(n^m)$ we conclude that

$$\limsup_{N \to \infty} \frac{1}{N} \log \|a^N\| \leq 0$$
with probability one.

On the other hand, since $|\phi(z)| \leq C$ by independence of $a_j^N$,s

$$P_N \{ a^N \in \mathbb{C}^{dN} : \|a^N\| < \frac{1}{N} \} \leq P \{ z \in \mathbb{C} : |z| < \frac{1}{N} \} \leq C\lambda \{ z \in \mathbb{C}^m : |z| < \frac{1}{N} \} = \frac{C\pi}{N^2}$$

thus, again, by using Borel-Cantelli lemma we obtain

$$\|a^N\| \geq \frac{1}{N}$$

with probability one. Hence,

$$\liminf_{N \to \infty} \frac{1}{N} \log \|a^N\| \geq 0.$$

□

For a $(m - 1, m - 1)$ smooth form $\Theta$ on $(\mathbb{C}^*)^m$ we define a random variables

$$X_N : \Pi_{NP} \to \mathbb{C}$$

$$X_N(s_N) := \langle \frac{1}{N}[Z_{s_N}], \Theta \rangle$$

and denote by $\text{Var}(X_N)$ variance of $X_N$. The following is a consequence of Theorem 1.1 and Proposition 2.8. Since the proof is identical to that of [Bay13, Lemma 5.2] we omit it.

**Lemma 5.3.** Under the hypothesis of Theorem 1.2 there exists $C > 0$ and $\epsilon > 0$ small

$$\text{Var}(X_N) \leq CN^{-\epsilon}\|dd^c \Theta\|_{\infty}.$$

Then by Kolmogorov’s strong law of large numbers and Theorem 1.1 we obtain that for $\mathcal{P}_\infty$-a.e. $\{s_N\}_{N \geq 1} \in \mathcal{P}$

$$\lim_{N \to \infty} \frac{1}{L} \sum_{N=1}^{L} \langle \frac{1}{N}[Z_{s_N}], \Theta \rangle = \langle TP_{P,K,q}, \Theta \rangle$$

as $L \to \infty$. Then by [Wal82, Theorem 1.20] we conclude that $\mathcal{P}_\infty$-a.e. $\{s_N\}_{N \geq 1} \in \mathcal{P}$ has a subsequence $s_{N_j}$ of relative density one (i.e. $\frac{1}{N_j} \to 1$ as $j \to \infty$) such that

$$\langle \frac{1}{N_j}[Z_{s_{N_j}}], \Theta \rangle = \langle TP_{P,K,q}, \Theta \rangle$$

as $N_j \to \infty$. We will show that in this case the whole sequence converges. First, we assume that $dd^c \Theta \geq 0$ then for $\{s_N\}_{N \geq 1} \in \mathcal{P}$ we define regularizations $R_{N,\theta} := \frac{1}{N}[Z_{s_N}]_{\theta}$ and functions

$$\Psi_{s_N}(\theta) := \langle R_{N,\theta}, \Theta \rangle,$$

$$\Phi(\theta) := \langle (TP_{P,K,q})_{\theta}, \Theta \rangle$$

Note that by Proposition 5.1, $\Psi_{s_N}, \Phi$ are continuous subharmonic functions on $\Delta$.

**Lemma 5.4.** For $\mathcal{P}_\infty$-a.e. $\{s_N\} \in \mathcal{P}$

$$\limsup_{N \to \infty} \Psi_{s_N}(0) \leq \Phi(0) = \langle TP_{P,K,q}, \Theta \rangle$$

**Proof.** We need to show that almost surely

$$\limsup_{N \to \infty} \frac{1}{N} \langle dd^c \log \|s_N\|_{P_{NHFS}}, \Theta \rangle \leq \langle dd^c V_{P_{PHFS}}, \Theta \rangle.$$
Indeed, by Stokes theorem and Schwarz inequality we have
\[
\left\langle \frac{1}{N} \int_{\mathbb{R}^m} \log \|s_N\|_{p N h_{FS}} \, d\Theta, \Theta \right\rangle = \frac{1}{N} \int_{\mathbb{R}^m} \log \|s_N\|_{p N h_{FS}} d\Theta \\
\leq \frac{1}{2N} \log \sum_{j=1}^{d_N} |a_j|^2 \int_{\mathbb{R}^m} d\Theta + \frac{1}{2N} \int_{\mathbb{R}^m} \log S_N(z, z) d\Theta.
\]

Then the first term tends to zero by Lemma 5.2 and the second term converges to \(\langle dd^c V_{p N h_{FS}}, \Theta \rangle\) by uniform convergence of \(\frac{1}{N} \log S_N\) to \(V_{p N h_{FS}}\) (cf. [Bay13 Proposition 2.9]).

Now, by Proposition 5.1 the functions \(\{\Psi_{s_N}(\theta)\}_N\) are uniformly bounded from above. We claim that for \(P\_\infty\)-a.e. \(s_N \in \mathcal{D}\)
\[
\lim_{N \to \infty} \Psi_{s_N}(0) = \Phi(0).
\]

Indeed, by Hartogs' lemma the sequence \(\{\Psi_{s_N}\}_N\) is relatively compact in \(L^1(\Delta)\). Now, assume that \(\Psi_{N_k} \to \Psi\) in \(L^1(\Delta)\) for some subharmonic function \(\Psi\) and \(\Psi(0) < \Phi(0)\). Then by continuity of \(\Phi(\theta)\)
\[
\psi_{N_k}(0) < \Phi(0)
\]
for \(N_k\) large. But this contradicts (5.2). Hence the claim follows.

Finally, we may write the test form \(\Theta \in D_{m-1, m-1}((\mathbb{C}^*)^m)\) as \(\Theta = \Theta^+ - \Theta^-\) where \(\Theta^\pm\) are smooth forms such that \(dd^c\Theta^\pm \geq 0\) and apply above argument to obtain
\[
\lim_{N \to \infty} \frac{1}{N} \langle [Z_{s_N}], \Theta \rangle = \langle T_{P, K, q}, \Theta \rangle.
\]

This finishes the of Theorem 1.2 for the case \(k = 1\).

Assume that the assertion holds for \(k - 1\). We will show that for every \((m - k, m - k)\) test form \(\Theta\) on \((\mathbb{C}^*)^m\) almost surely
\[
\langle [Z_{s_{N,k}}, \ldots, s_{N,k}], \Theta \rangle \to \langle \bigwedge_{i=1}^{k} T_{P_i, K_i}, \Theta \rangle.
\]

An induction argument in [Bay13] Lemma 5.2 gives the variance estimate as in Lemma 5.3. Hence, by Theorem 1.1 and Kolmogorov's law of large numbers we obtain
\[
\frac{L}{L} \sum_{N=1}^{L} \langle N^{-k}[Z_{s_{N,k}}, \ldots, s_{N,k}], \Theta \rangle \to \langle \bigwedge_{i=1}^{k} T_{P_i, K_i}, \Theta \rangle
\]
as \(L \to \infty\). Assuming \(dd^c\Theta \geq 0\) and letting \(R_{N,\theta}^k := [Z_{s_{N,k}}, \ldots, s_{N,k}]_{\theta}\) also \(R^k := \bigwedge_{i=1}^{k} T_{P_i, K_i}\), for each \(S_N^k = (s_{N,1}, \ldots, s_{N,k})\) we define
\[
\Psi_{S_N^k}(\theta) := \langle R_{N,\theta}^k, \Theta \rangle \\
\Phi_k(\theta) := \langle R^k, \Theta \rangle.
\]

Then by Proposition 5.1 \(\Psi_{S_N^k}\) and \(\Phi_k\) are continuous subharmonic functions on \(\Delta\).

**Lemma 5.5.** Almost surely
\[
\limsup_{N \to \infty} \Psi_{S_N^k}(0) \leq \Phi_k(0).
\]

**Proof.** We prove the lemma by induction on \(k\). The case \(k = 1\) was proved in Lemma 5.4. By Bertini's theorem for systems \(S_N^k = (s_{N,1}, \ldots, s_{N,k})\) in general position we have
\[
[Z_{s_{N,1}, \ldots, s_{N,k}}] = [Z_{s_{N,1}, \ldots, s_{N,k-1}}] \land [Z_{s_{N,k}}].
\]

Then by Stokes theorem
\[
N^{-k}(\langle [Z_{s_{N,1}, \ldots, s_{N,k}], \Theta \rangle) = N^{-k}(\langle [Z_{s_{N,1}, \ldots, s_{N,k-1}} \land [Z_{s_{N,k}}], \Theta \rangle) \\
= N^{-k+1} \int_{Z_{s_{N,1}, \ldots, s_{N,k-1}}} \tilde{p}_i \omega_{FS} \land \Theta + N^{-k} \int_{Z_{s_{N,1}, \ldots, s_{N,k-1}}} \log \|s_N\|_{p N h_{FS}} d\Theta
\]
applying Schwarz lemma the second term can be bounded by
\[ \frac{1}{2N} \log \sum_{j=1}^{dN} |a_j^k|^2 \int_{Z_{s_N,\ldots,s_N}} N^{1-k} \omega^S + \int_{Z_{s_N,\ldots,s_N}} \frac{1}{2N} \log S_N^k(z,z) N^{1-k} \omega^S \]
where \( S_N^k \) denotes the Bergman kernel associated with \( \Pi_{NP_k} \). Then by Lemma \ref{lem5.2} and induction hypothesis together with uniform convergence of \( \frac{1}{2N} \log S_N^k(z,z) \) to \( V_{p\omega_{FS}} \) we infer that
\[ \limsup_{N \to \infty} N^{-k} \langle [Z_{s_N,\ldots,s_N}], \Theta \rangle \leq \left( \bigwedge_{i=1}^{k-1} T_{P_i,K,q_i} \land p_i \omega_{FS}, \Theta \right) + \left( \bigwedge_{i=1}^{k-1} T_{P_i,K,q_i} \land dd\bar{c} V_{P_i,p\omega_{FS}}, \Theta \right) \]
and the lemma follows. \( \square \)

Now, we claim that almost surely
\[ \lim_{N \to \infty} \Psi_{S_N^k}(0) = \Phi_k(0) \]
Indeed, by Proposition \ref{prop5.1} \( \Psi_{S_N^k} \) is uniformly from bounded above. Thus, \( \{ \Psi_{S_N^k} \} \) is relatively compact in \( L^1(\Delta) \). Now, assume that \( \Psi_{S_N^j} \to \Psi \) in \( L^1(\Delta) \) for some subharmonic function \( \Psi \). If \( \Psi(0) \neq \Phi_k(0) \) then \( \Psi(0) < \Phi_k(0) \) by Lemma \ref{lem5.3} and by continuity of \( \Phi_k(\theta) \) we have
\[ \psi_{S_N^k}(0) < \Phi_k(0) \]
for large \( N_j \). But this contradicts \( \ref{eq5.3} \). Hence the claim follows. Now, writing the test form \( \Theta \) as \( \Theta = \Theta^+ - \Theta^- \) for some smooth forms \( \Theta^\pm \) where \( dd\bar{c} \Theta^\pm \geq 0 \) and applying above argument to obtain
\[ \lim_{N \to \infty} \langle \frac{1}{N} [Z_{s_N,\ldots,s_N}], \Theta \rangle = \left( \bigwedge_{i=1}^{k} T_{P_i,K,q_i}, \Theta \right) \].

6. Unbounded case

In this section, we provide an extension of Theorem \ref{thm11} for certain unbounded merely closed subsets \( K \subset (\mathbb{C}^*)^m \). Throughout this section we assume that \( P \subset \mathbb{R}^m_\geq0 \) is an integral polytope with non-empty interior. In the sequel we let \( p := \max\{p_1 + \cdots + p_m : (p_1, \ldots, p_m) \in P \} \) so that \( P \subset p\Sigma \). We use the same notation as in section \ref{n5}

A lower semi-continuous function \( q : \mathbb{C}^m \to \mathbb{R} \) for which \( \{ z \in K : q(z) < \infty \} \) is non-pluripolar, is called weakly admissible if there exists \( M \in (-\infty, \infty) \) such that
\[ \liminf_{z \to K, z \to \infty} q(z) - \frac{p}{2} \log(1 + \|z\|^2) = M. \]
We say that \( q \) is a continuous weakly admissible weight function for \( K \) if it is weakly admissible and it extends to a continuous weight function of a continuous metric on \( O(p) \). A weighted closed set \( (K,q) \) is called regular weighted closed set if the global extremal function \( V_{P,K,q} \) extends to a continuous \( \omega_{FS} \)-psh function on \( P^m \). We remark that if \( q \) is a weakly admissible weight function for \( K = (\mathbb{C}^*)^m \) then the set of polynomials \( Poly(NP) \subset L^2(e^{-2Nq}dV) \) where \( dV = h(z)dz \) denotes a probability volume form on \( \mathbb{C}^m \) (eg. \( dV = \omega_{FS}^{m} \)). Then the proof of Theorem \ref{thm3.2} carries over to the present setting and we obtain:

**Theorem 6.1.** Let \( P \subset \mathbb{R}^m_{\geq0} \) be an integral polytope with non-empty interior, \( (K,q) \) be a regular weighted closed set and \( q : \mathbb{C}^m \to \mathbb{R} \) be weakly admissible continuous weight function. Then
\[ V_{P,K,q} = \lim_{N \to \infty} \frac{1}{N} \log \Phi_N \]
locally uniformly on \( (\mathbb{C}^*)^m \).
Next, we fix an ONB \( \{ F^j_N \} \) for \( Poly(NP) \) with respect to the inner product induced from
\[
(f, g) := \int_{(\mathbb{C}^*)^m} f(z)g(z)e^{-2Nq(z)}dV.
\]
We also let \( S_N(z, w) \) denote the associated Bergman kernel (cf. [Bay13 §1.1]). We remark that volume form \( dV \) satisfies the weighted Berstein-Markov inequality on \((\mathbb{C}^*)^m\) and the argument in [BS07] (see also [SZ04 Proposition 4.2]) generalizes to our setting and we obtain:

**Proposition 6.2.** Let \( P \subset \mathbb{R}^m_0 \) be an integral polytope with non-empty interior, \((K, q)\) be a regular weighted closed set and \( q : \mathbb{C}^m \to \mathbb{R} \) be weakly admissible continuous weight function. Then
\[
\frac{1}{2N} \log S_N(z, z) \to V_{P, K, q}
\]
uniformly on compact subsets of \((\mathbb{C}^*)^m\).

Hence, following the arguments in proofs of Theorem 1.1 and Theorem 1.2 we obtain:

**Theorem 6.3.** Let \( P \subset \mathbb{R}^m_0 \) be an integral polytope with non-empty interior, \((K, q)\) be a regular weighted closed set and \( q : \mathbb{C}^m \to \mathbb{R} \) be weakly admissible continuous weight function for each \( 1 \leq i \leq k \). Then
\[
N^{-k}E[Z_{f_{\lambda_1}^1, \ldots, f_{\lambda_k}^k}] \to \ddc^k(V_{P_1, K, q}) \wedge \cdots \wedge \ddc^k(V_{P_k, K, q})
\]
weakly as \( N \to \infty \). Moreover, almost surely
\[
N^{-k}Z_{f_{\lambda_1}^1, \ldots, f_{\lambda_k}^k} \to \ddc^k(V_{P_1, K, q}) \wedge \cdots \wedge \ddc^k(V_{P_k, K, q})
\]
weakly as \( N \to \infty \).

Next, we provide an example (from [SZ04]) which falls in the framework of Theorem 6.3

**Example 6.4.** Let \( K = (\mathbb{C}^*)^m \) and \( q(z) = \frac{2}{N} \log(1 + \|z\|^2) \) and \( P \subset \mathbb{R}^m_0 \) be an integral polytope with non-empty interior. For each \( p \in P \) we denote the normal cone to \( P \) at \( x \) by \( C_x := \{ u \in \mathbb{R}^m : (u, x) = \varphi_P(u) \} \)
where \( \varphi_P \) is the support function of \( P \). Then by [SZ04 Lemma 4.3] for every \( z \in (\mathbb{C}^*)^m \) there exists unique \( \tau_z \in \mathbb{R}^m \) and \( r(z) \in P \) such that
\[
\mu_p(e^{-\frac{x}{2}} \cdot z) = r(z) \text{ and } \tau_z \in C_{r(z)}
\]
where \( x \cdot z := (x_1 z_1, \ldots, x_m z_m) \) denotes \( \mathbb{R}^m_+ \) action on \((\mathbb{C}^*)^m \) and \( \mu_P \) denotes the moment map defined in the introduction. Then by [SZ04 Theorem 4.1]
\[
V_{P, \omega_{FS}} = \begin{cases} 
0 & \text{for } z \in A_P \\
\frac{1}{2} (r(z), \tau_z) - \frac{N}{2} \log \left[ \frac{1 + \|z\|^2}{1 + e^{-\frac{x}{r(z)} \cdot z}} \right] & \text{for } z \in (\mathbb{C}^*)^m \setminus A_P
\end{cases}
\]
extends as a continuous \( \omega_{FS} \)-psh function on \( \mathbb{P}^m \). In particular, the weighted global extremal function is given by
\[
V_{P, q}(z) = \begin{cases} 
\frac{2}{N} \log(1 + \|z\|^2) & \text{for } z \in A_P \\
\frac{1}{2} (r(z), \tau_z) + \frac{N}{2} \log[1 + e^{-\frac{x}{2}} \cdot z, z] & \text{for } z \in (\mathbb{C}^*)^m \setminus A_P.
\end{cases}
\]
Letting
\[
(f, g) := \int_{(\mathbb{C}^*)^m} f(z)g(z)e^{-2Nq(z)}\omega_{FS}^m
\]
we see that
\[
c_{J^j} := \frac{(N + m)!}{m!(N - |J|)!j_1! \cdots j_m!} z_1^{j_1} \cdots z_m^{j_m} \text{ for } J \in NP
\]
where $|J| = j_1 + \cdots + j_m$ form an ONB for $\text{Poly}(NP)$ and a random Laurent polynomial in this context is of the form

$$f_N(z) = \sum_{J \in NP} a_J c_J z^J.$$

Thus, Theorem 6.3 applies and almost surely

$$N^{-m} \sum_{\zeta \in \mathbb{Z}^{m}} \delta_\zeta \rightarrow MAC(V_{P,q})$$

weakly as $N \to \infty$.

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