Approximation of integration over finite groups, difference sets and association schemes

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Abstract
Let $G$ be a finite group and $f : G \to \mathbb{C}$ be a function. For a non-empty finite subset $Y \subset G$, let $I_Y(f)$ denote the average of $f$ over $Y$. Then, $I_G(f)$ is the average of $f$ over $G$. Using the decomposition of $f$ into irreducible components of $\mathbb{C}^G$ as a representation of $G \times G$, we define non-negative real numbers $V(f)$ and $D(Y)$, each depending only on $f$, $Y$, respectively, such that an inequality of the form $|I_G(f) - I_Y(f)| \leq V(f) \cdot D(Y)$ holds. We give a lower bound of $D(Y)$ depending only on $\#Y$ and $\#G$. We show that the lower bound is achieved if and only if $\#\{(x, y) \in Y^2 \mid x^{-1}y \in [a]\}/\#[a]$ is independent of the choice of the conjugacy class $[a] \subset G$ for $a \neq 1$. We call such a $Y \subset G$ as a pre-difference set in $G$, since the condition is satisfied if $Y$ is a difference set. If $G$ is abelian, the condition is equivalent to that $Y$ is a difference set. We found a non-trivial pre-difference set in the dihedral group of order 16, where no non-trivial difference set exists. The pre-difference sets in non-abelian groups of order 16 are classified. A generalization to commutative association schemes is also given.

Keywords Difference set · Association scheme · Group · Quasi-Monte carlo method · Pre-difference set

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1 Introduction and main results

Let \( X \) be a non-empty finite set, and \( Y \) be a non-empty subset of \( X \). We denote by \( \mathbb{C}^X \) the space of functions from \( X \) to \( \mathbb{C} \). For \( f \in \mathbb{C}^X \), its integration \( I(f) \) over \( X \) is defined as \( \frac{1}{\#X} \sum_{x \in X} f(x) \). We use the term “integration” because of a similarity to the theory of quasi-Monte Carlo integration, stated later. Similarly, the integration \( I_Y(f) \) over \( Y \) is defined as \( \frac{1}{\#Y} \sum_{x \in Y} f(x) \). We would like to find a finite subset so that the absolute integration error \( |I(f) - I_Y(f)| \) is small, for \( f \) being in some function space \( F \subset \mathbb{C}^X \). This is an analogue of quasi-Monte Carlo (QMC) methods in approximating the integration, where \( X \) is a hyper cube \([0, 1]^s\) and the integration of \( f : X \to \mathbb{R} \) is with respect to the Lebesgue measure. A large amount of studies exist, see for example [7, 8, 14]. Some recent researches would be found in the conference books from international conferences titled “Monte Carlo and quasi-Monte Carlo methods” held every other year. For the readers’ convenience, we briefly explain a typical QMC and its relation with group characters in Appendix A.

In this manuscript, we consider the case where \( X \) is a finite group \( G \), and in Sect. 5 a generalization where \( X \) has a structure of commutative association scheme.

Here, \( \mathbb{C}^X \) is equipped with a standard inner product and a norm:

\[
\langle f, g \rangle := \sum_{x \in X} f(x)\overline{g(x)}, \quad ||f|| := \sqrt{\langle f, f \rangle}.
\]

When \( G \) is a finite group, \( \mathbb{C}^G \) is a left \( G \)-module by defining the action \( g \in G \) on \( f(-) \in \mathbb{C}^G \) by \( g(f(-)) = f(g^{-1}(-)) \). Then \( \mathbb{C}^G \) has an orthogonal decomposition \( \mathbb{C}^G = \bigoplus_{\rho \in \hat{G}} V_{\rho} \), where \( \hat{G} \) is the set of isomorphism classes of irreducible characters and \( V_{\rho} \) is the submodule isomorphic to the direct sum of the \( \dim \rho \) copies of the representation \( \rho \). Hence, any \( f \in \mathbb{C}^G \) is decomposed as \( f = \bigoplus_{\rho \in \hat{G}} f_\rho \), and \( f_\rho \) is called the \( \rho \)-component of \( f \).

We define non-negative real numbers \( \partial_\rho(Y) \) (which is \( ||2^\rho|| \) defined in Proposition 2.6) such that \( |I(f_\rho) - I_Y(f_\rho)| \leq ||f_\rho|| \partial_\rho(Y) \) holds, which is sharp since there is an \( f \) with equality holds for all \( \rho \) (see Remark 2.7 below).

**Theorem 1** Let \( G \) be a finite group, \( Y \) a nonempty subset of \( G \), and \( f : G \to \mathbb{C} \) a function. Let \( I(f) \) be the average of \( f \) over \( G \), \( I_Y(f) \) the average of \( f \) over \( Y \). Let \( f_\rho \) be the \( \rho \)-component of \( f \), where \( \rho \) is an irreducible representation of \( G \), namely, \( \rho \in \hat{G} \). Define

\[
\partial_\rho(Y) := \sqrt{\frac{\dim \rho}{\#Y^2 \#G} \sum_{x, y \in Y} \chi_\rho(x^{-1}y)},
\]
where $\chi_\rho$ is the character of $\rho$. Then we have $|I(f_\rho) - I_Y(f_\rho)| \leq ||f_\rho|| \partial_\rho(Y)$. For the trivial representation $\rho = 1_G$, we have $|I(f_{1_G}) - I_Y(f_{1_G})| = 0$.

A proof is given in Sect. 2.

**Corollary 2** We have

$$|I(f) - I_Y(f)| \leq \sum_{\rho \in \hat{G}_{\{1_G\}}} ||f_\rho|| \partial_\rho(Y) \leq V(f) D(Y),$$

where $V(f) := \sum_{\rho \in \hat{G}_{\{1_G\}}} (||f_\rho|| \dim \rho)$ and $D(Y) := \max_{\rho \in \hat{G}_{\{1_G\}}} \frac{\partial_\rho(Y)}{\dim \rho}$.

**Proof** We have $I(f) = \sum_{\rho \in \hat{G}} I(f_\rho)$. The above theorem implies the second inequality in

$$|I(f) - I_Y(f)| \leq \sum_{\rho \neq 1_G} |I(f_\rho) - I_Y(f_\rho)| \leq \sum_{\rho \neq 1_G} ||f_\rho|| \partial_\rho(Y)$$

$$= \sum_{\rho \neq 1_G} ||f_\rho|| \dim \rho \cdot \frac{\partial_\rho(Y)}{\dim \rho} \leq \left( \sum_{\rho \neq 1_G} ||f_\rho|| \dim \rho \right) D(Y).$$

$\square$

Note that $V(f)$ depends only on $f$, and $D(Y)$ depends only on $Y$. Corollary 2 implies

$$|I(f) - I_Y(f)| \leq V(f) D(Y).$$

Such type of error bounds on QMC-integration appears in many researches; a famous example is Koksma-Hlawka inequality [14].

It is easy to show that $D(G) = 0$, so this bound is tight in this sense. Because of this inequality, we are interested in finding a $Y$ with fixed cardinality which minimizes $D(Y)$.

**Proposition 1.1** Let $D(Y)$ be the value defined in Corollary 2. Under the assumptions in Theorem 1, a lower bound

$$D(Y) \geq \sqrt{\frac{1/\#G - 1/\#Y}{\#G - 1}}$$

holds. The equality holds if and only if $\frac{\partial_\rho(Y)}{\dim \rho} = \sqrt{\frac{1/\#Y - 1/\#G}{\#G - 1}}$ holds for any $\rho \neq 1_G$.

A proof is given in Sect. 3. Thus, $D(Y)$ is bounded below by a formula depending only on $\#G$ and $\#Y$. We are interested in the case where the equality holds. If $G$ is an abelian group, then the equality holds if and only if $Y$ is a difference set[2], as shown in Theorem 4.

**Definition 1.2** Let $G$ be a finite group, and $Y$ its subset. Define $\lambda_a := \# \{(x, y) \in Y \times Y \mid x^{-1} y = a\}$, where $a \in G$. A non-empty proper subset $Y$ of $G$ is a $(v, k, \lambda)$-difference set if $v = \#G$, $k = \#Y$, and $\lambda_a = \lambda$ for any $a \in G$ except $a = 1$. Note that $\lambda_1 = \#Y$. A difference set $Y$ is said to be trivial if $\#(Y) = 1$, $v - 1$. 

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We define the notion of pre-difference set. We could not find this notion in the literatures.

**Definition 1.3** (pre-difference set) Let $G$ be a finite group, and $Y$ its non-empty subset. Define $\lambda[a] := \#\{(x, y) \in Y \times Y \mid x^{-1}y \in [a]\}$, for the conjugacy class $[a] \subset G$ of $a \in G$. Then $Y$ is a $(v, k, \lambda)$ pre-difference set if $v = \#G$, $k = \#Y$, and $\lambda[a] = \lambda\#[a]$ for any $a \in G$ except $a = 1$. Note that $\lambda[1] = \#Y$. A pre-difference set $Y$ is said to be trivial if $\#(Y) = 1$, $v = 1$, $v - 1$, $v$.

If $Y$ is a difference set, then it is a pre-difference set. The converse is not true, because there is a non-trivial pre-difference set in the dihedral group of order 16, see Sect. 4. The parameter $\lambda$ for pre-difference set appears to be a rational number, but we shall show that it is an integer in Sect. 4.

**Remark 1.4** The set $Y = G$ is not considered as a difference set, but is considered as a pre-difference set. This is because in the context of the integration, $Y = G$ is the best choice and we don’t want to exclude.

**Theorem 3** Let $G$ be a finite group and $Y$ its non-empty subset. Then, the following conditions are equivalent.

1. $Y$ is a pre-difference set, i.e., $\lambda[a]/\#[a]$ is independent of the choice of $a \neq 1$.
2. $\lambda[a]/\#[a] = (\#Y^2 - \#Y)/(\#G - 1)$ holds for any $a \neq 1$, and $\lambda[1] = \#Y$ holds.
3. $\partial_\rho(Y)/(\dim \rho) = \sqrt{1/\#Y - 1/\#G}/\#G - 1$ holds for any $\rho \neq 1_G$, and $\partial_{1_G}(Y) = \sqrt{1/\#G}$ holds.
4. $\partial_\rho(Y)/(\dim \rho)$ is independent of the choice of $\rho \in \hat{G} \setminus \{1_G\}$.
5. The equality holds in the inequality

$$D(Y) \geq \sqrt{1/\#Y - 1/\#G}/\#G - 1$$

in Proposition 1.1.

A proof is given in Sect. 3. Thus, when the cardinality of $Y$ is fixed, $D(Y)$ attains the lower bound above if and only if $Y$ is a pre-difference set. It seems interesting that the optimal choice for the quasi-Monte Carlo in the above setting is equivalent to a natural generalization of the notion of classical difference set.

If $G$ is abelian, then $[a] = \{a\}$, $\lambda[a] = \lambda_a$, and $\dim \rho = 1$. This shows the following proposition.

**Proposition 1.5** Let $G$ be an abelian group. A subset $Y \subset G$ is a difference set if and only if $Y \neq G$ and $Y$ is a pre-difference set.

Thus, the above theorem implies the following theorem.

**Theorem 4** Let $G$ be a finite abelian group, and $Y$ its non-empty subset. Then the following conditions are equivalent.

1. $Y$ is a difference set or $Y = G$.
2. $\partial_\rho(Y)$ is independent of the choice of $\rho \in \hat{G} \setminus \{1_G\}$.
The equality holds in the inequality

$$D(Y) \geq \sqrt{\frac{1/\#Y - 1/\#G}{\#G - 1}}$$

in Proposition 1.1.

In Sect. 5, we prove a theorem generalizing Theorem 3 in the context of commutative association schemes.

## 2 Preliminary for the proofs and a proof of theorem 1

### Definition 2.1
For a finite set $X$, $\mathbb{C}^X$ denotes the set of functions from $X$ to $\mathbb{C}$, and $\mathbb{C}[X]$ denotes the linear vector space with basis $X$. We identify $\mathbb{C}^X = \mathbb{C}[X]$ by $f \mapsto \sum_{x \in X} f(x)x$. A standard Hermitian inner product and the norm are defined as in Sect. 1.

### Definition 2.2
For $x \in X$, an element $x \in \mathbb{C}[X]$ corresponds to the delta function $\delta_x \in \mathbb{C}^X$, defined by $\delta_x(y) = 0$ if $x \neq y$ and $\delta_x(y) = 1$ if $x = y$, for $y \in X$. For a subset $Y$, we define $\delta_Y = \sum_{y \in Y} \delta_y \in \mathbb{C}^X$. Note that $\langle \delta_Y, \delta_Z \rangle = \#(Y \cap Z)$. In Sect. 1, we defined an operator taking the average over $Y$: $I_Y : \mathbb{C}^G \to \mathbb{C}$. We define $I_Y := \frac{1}{\#Y} \delta_Y \in \mathbb{C}^X$ representing the operator $I_Y$, i.e.,

$$I_Y(f) = \langle f, I_Y \rangle$$

holds. By definition, we have

$$\langle I_Y, I_Z \rangle = \frac{1}{\#Y \#Z} \#(Y \cap Z).$$

If $X$ is a group $G$, then for $\rho \in \hat{G}$ (see below), we denote the $\rho$-component of $\delta_y, \delta_Y, I_Y$ by $\delta^\rho_y, \delta^\rho_Y, I^\rho_Y$, respectively.

### Definition 2.3
In the above definition, suppose that $G = X$ is a group. Then, $\mathbb{C}[G]$ is a group ring. By left multiplication, $\mathbb{C}[G]$ is a left $G$-module. Let $\hat{G}$ denote the set of isomorphism classes of irreducible representations of $G$. The character of $\rho$ is denoted by $\chi_\rho$. Normalized inner product is defined for $\mathbb{C}^G$ (and hence for $\mathbb{C}[G]$) by

$$(f, g)_G := \frac{1}{\#G} \sum_{x \in G} f(x)\overline{g(x)}.$$  

(This inner product is used only in Proposition 2.4 for stating orthonormality of the characters.) Let $C(G)$ be the set of conjugacy class, and for $a \in G$, $[a] \in C(G)$ denotes the class that contains $a$. The linear subspace $\mathbb{C}^{C(G)}$ of $\mathbb{C}^G = \mathbb{C}[G]$ is the center of $\mathbb{C}[G]$.  

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The following are well-known, see for example Serre [16, I, §2].

**Proposition 2.4** (1) \( \{ \chi_\rho \mid \rho \in \hat{G} \} \) is an orthonormal basis of the space of class functions \( \mathbb{C}^{\mathbb{C}(G)} \) with respect to the normalized inner product.

(2) As a left \( G \)-module, \( \mathbb{C}[G] = \bigoplus_{\rho \in \hat{G}} H_\rho \), where \( H_\rho \) is the isotypical component to the irreducible representation \( \rho \). The decomposition is orthogonal. Thus, \( f \in \mathbb{C}[G] \) is decomposed as \( f = \bigoplus_{\rho \in \hat{G}} f_\rho \), and \( f_\rho \in H_\rho \) is called the \( \rho \)-component of \( f \). The multiplicity of \( \rho \) in \( H_\rho \) is \( \dim \rho \). Hence, \( \sum_{\rho \in \hat{G}} (\dim \rho)^2 = \#G \).

(3) Let

\[
E_\rho := \frac{\dim \rho}{\#G} \sum_{g \in G} \chi_\rho(g) g \in \mathbb{C}[G].
\]

Then, the left multiplication of \( E_\rho \) to \( \mathbb{C}[G] \) is the projection \( \mathbb{C}[G] \to H_\rho \). Thus, for \( f \in \mathbb{C}^X \), its \( \rho \)-component \( f_\rho \) is \( E_\rho f \) (through the identification \( \mathbb{C}^G = \mathbb{C}[G] \) as left \( G \)-modules given in Definition 2.1).

The above proposition implies the following:

**Proposition 2.5** For \( x \in G \), the \( \rho \)-component \( \delta^\rho_x = E_\rho \delta_x \) of \( \delta_x \) is given in \( \mathbb{C}[G] \) by

\[
\delta^\rho_x = \frac{\dim \rho}{\#G} \sum_{g \in G} \chi_\rho(g) x g = \frac{\dim \rho}{\#G} \sum_{g \in G} \chi_\rho(g x^{-1}) g.
\]

Through the identification \( \mathbb{C}[G] = \mathbb{C}^G \),

\[
\delta^\rho_x(g) = \frac{\dim \rho}{\#G} \chi_\rho(g x^{-1}) = \frac{\dim \rho}{\#G} \chi_\rho(x g^{-1}) = \frac{\dim \rho}{\#G} \chi_\rho(g^{-1} x).
\]

**Proposition 2.6** Let \( Y \) be a subset of \( G \). Recall Definition 2.2 for various elements in \( \mathbb{C}^G \) related with \( Y \). For \( f \in \mathbb{C}^G \),

\[
I_Y(f) = \langle f, I_Y \rangle = \sum_{\rho \in \hat{G}} \langle f, I^\rho_Y \rangle = \sum_{\rho \in \hat{G}} \langle f_\rho, I^\rho_Y \rangle.
\]

**Proof** The first equality is shown in Definition 2.2. The second follows from \( I_Y = \sum_\rho I^\rho_Y \). The third follows from the orthogonality of \( H_\rho \). \( \square \)

**Proof of Theorem 1** Let \( 1_G \) denote the trivial representation. Note that for any \( f \in \mathbb{C}^G \), \( I(f) = I_G(f) \), and \( f^{1_G} = E_{1_G}(f) = I(f) \) (where \( I(f) \) denotes the constant function with value \( I(f) \) in \( \mathbb{C}^G \)).

Since \( G \) acts on \( I_G \) trivially, \( I^\rho_G = 0 \) for \( \rho \neq 1_G \), and \( I^{1_G}_G = I_G = 1/\#G \) (i.e. the constant function with value \( 1/\#G \) in \( \mathbb{C}^G \)). By Proposition 2.6, we have

\[
I_G(f) - I_Y(f) = \langle f, I_G - I_Y \rangle = \sum_\rho \langle f_\rho, I^\rho_G - I^\rho_Y \rangle.
\]
\[
\langle f_1, \mathcal{I}_G - \mathcal{I}_Y \rangle + \sum_{\rho \neq 1_G} \langle f_\rho, -\mathcal{I}_Y^\rho \rangle \\
= \sum_{\rho \neq 1_G} \langle f_\rho, -\mathcal{I}_Y^\rho \rangle.
\]
(The last equality follows from \( I_G(f) = I_Y(f) \) for a constant function \( f \), and that for any \( f \), \( f_1 \) is a constant function.) Thus
\[
|I_G(f) - I_Y(f)| \leq \sum_{\rho \neq 1_G} |\langle f_\rho, -\mathcal{I}_Y^\rho \rangle| \\
\leq \sum_{\rho \neq 1_G} ||f_\rho|| \cdot ||\mathcal{I}_Y^\rho||.
\]

Then, by \( \mathcal{I}_Y^\rho = \frac{1}{\#Y} \delta_y^\rho \) and using Propositions 2.5 and 2.6
\[
||\mathcal{I}_Y^\rho||^2 = \langle \mathcal{I}_Y^\rho, \mathcal{I}_Y^\rho \rangle = \langle \mathcal{I}_Y^\rho, \mathcal{I}_Y \rangle \\
= I_Y(\mathcal{I}_Y^\rho) = \frac{1}{\#Y} \sum_{y \in Y} \mathcal{I}_Y^\rho(y) \\
= \frac{1}{\#Y^2} \sum_{y \in Y} \sum_{x \in Y} \delta_x^\rho(y) \\
= \frac{\dim \rho}{\#Y^2 G} \sum_{y,x \in Y} \chi_\rho(x y^{-1}) \\
= \frac{\dim \rho}{\#Y^2 G} \sum_{x,y \in Y} \chi_\rho(x^{-1} y).
\]

Hence, by putting
\[
\partial_\rho(Y) := ||\mathcal{I}_Y^\rho|| = \sqrt{\frac{\dim \rho}{\#Y^2 G} \sum_{x,y \in Y} \chi_\rho(x^{-1} y)},
\]
we prove Theorem 1.

**Remark 2.7** If we put \( f := \mathcal{I}_Y \), then the equalities hold for the two inequalities in the proof. This is a worst function for the quasi-Monte Carlo integration by \( Y \). Moreover, if \( Y \) is a pre-difference set, then (4) of Theorem 3 implies that the equalities hold in the inequalities in Corollary 2. Thus the bound \( V(f) D(Y) \) is tight in this sense.

### 3 Proofs of proposition 1.1 and Theorem 3

To obtain a lower bound of \( D(Y) \), we begin with a relation among \( \partial_\rho(Y) \):
Proposition 3.1 For $\partial_\rho(Y)$ defined in Theorem 1,
\[
\sum_\rho \partial_\rho(Y)^2 = \frac{1}{\#Y}
\]
and $\partial_{1_G}(Y)^2 = \frac{1}{\#G}$ for the trivial representation $1_G$. Thus,
\[
\sum_{\rho \neq 1_G} \partial_\rho(Y)^2 = \frac{1}{\#Y} - \frac{1}{\#G}.
\]

Proof Since $\partial_\rho(Y) = ||I_\rho Y||$, 
\[
\sum_\rho \partial_\rho(Y)^2 = \sum_\rho ||I_\rho Y||^2 = ||\sum_\rho I_\rho Y||^2 = ||I_Y||^2
\]
holds by a formula in Definition 2.2. For $\rho = 1_G$, $I_Y^{1_G}$ is a constant function with value $I_G(I_Y) = I_G(\frac{1}{\#Y}Y) = 1/\#G$. Thus 
\[
||I_Y^{1_G}||^2 = \sum_{g \in G}(1/\#G)^2 = \frac{1}{\#G}.
\]

Lemma 3.2 Let $y_1, \ldots, y_n$ be non negative real numbers with $\sum_i y_i = a$. Let $d_1, \ldots, d_n$ be positive real numbers. Let $M$ be $\max_{i=1}^n y_i/d_i$. Then $M \geq a/(\sum_i d_i)$, and the equality follows if and only if $y_i/d_i = a/(\sum_i d_i)$ for each $i = 1, \ldots, n$.

Proof Since $M \geq y_i/d_i$, $\sum_i (M d_i) \geq \sum_i y_i = a$ and the inequality holds. If the equality holds, then $M = y_i/d_i$ holds for each $i$, hence the result.

Now we prove Proposition 1.1. By Proposition 3.1, it follows that 
\[
\sum_{\rho \neq 1_G} \partial_\rho(Y)^2 = \frac{1}{\#Y} - \frac{1}{\#G}.
\]

By the lemma above with $y_i$ being $\partial_\rho(Y)^2$ and $d_i$ being $(\dim \rho)^2$, it follows that 
\[
D(Y)^2 = \max_{\rho \neq 1_G} (\partial_\rho(Y)/\dim \rho)^2 \geq \left( \frac{1}{\#Y} - \frac{1}{\#G} \right) / \left( \sum_{\rho \neq 1_G} (\dim \rho)^2 \right),
\]
where $\sum_{\rho \neq 1_G} (\dim \rho)^2 = \#G - 1$ by Proposition 2.4, and the equality holds if and only if $\partial_\rho(Y)/\dim \rho = \sqrt{(\frac{1}{\#Y} - \frac{1}{\#G})/(\#G - 1)}$ for any $\rho \neq 1_G$. This proves Proposition 1.1.
**Proof of Theorem 3** From definition of \( \partial_\rho(Y) \) and Proposition 2.5 we have

\[
\left( \frac{\partial_\rho(Y)}{\text{dim} \rho} \right)^2 = \frac{1}{\text{dim} \rho^2} \sum_{x,y \in Y} \text{dim} \rho \#Y^2 \#G \chi_\rho(x^{-1}y)
\]

\[
= \frac{1}{\# Y^2 \# G} \sum_{x,y \in Y} \chi_\rho(x^{-1}y) \text{dim} \rho
\]

\[
= \frac{1}{\# Y^2 \# G} \sum_{[a] \in C(G)} \lambda_{[a]} \chi_\rho(a) \text{dim} \rho
\]

\[
= \frac{1}{\# Y^2 \# G} \sum_{[a] \in C(G)} \frac{\lambda_{[a]} \chi_\rho(a)}{\# [a] \text{dim} [a]}
\]

Summarizing the above computation, we have:

**Lemma 3.3** Let \((P(\rho,[a])) \) \( (\rho \in \hat{G}, [a] \in C(G)) \) be the matrix defined by \( P(\rho,[a]) = \frac{1}{\# Y^2 \# G} \chi_\rho(a) \# [a] \). By the orthogonality of the characters, this matrix is invertible. The above formula shows that the vector \( \left( \frac{\lambda_{[a]}}{\# [a]} \right) \) \( ([a] \in C(G)) \) is mapped to the vector \( \left( \frac{\partial_\rho(Y)}{\text{dim} \rho} \right)^2 \) \( (\rho \in \hat{G}) \) by \( P \).

We shall prove the equivalence between (1)-(5) in Theorem 3. □

Proof of (1) \( \Leftrightarrow \) (2). Now we assume (1). Then \( \lambda_{[a]}/\# [a] = \lambda \) for \( a \neq 1 \). Since \( \sum_{[a] \in C(G)} \lambda_{[a]} = \# Y^2 \), \( \sum_{[a] \in C(G) \setminus \{[1]\}} \lambda_{[a]} = \# Y^2 - \# Y \) follows. By \( \lambda_{[a]} = \# [a] \lambda \), \( \sum_{[a] \in C(G) \setminus \{[1]\}} \lambda_{[a]} = \lambda \sum_{[a] \in C(G) \setminus \{[1]\}} \# [a] = \lambda (\# G - 1) \). Thus, \( \lambda = \frac{\# Y^2 - \# Y}{\# G - 1} \), and (2) follows since \( \lambda_{[1]} = \# Y \) always hold. Clearly (2) implies (1).

Proof of (2) \( \Leftrightarrow \) (3). We show that (2) implies (3). Put \( \lambda := \frac{\# Y^2 - \# Y}{\# G - 1} \). Lemma 3.3 gives for \( \rho \neq 1_G \)

\[
\left( \frac{\partial_\rho(Y)}{\text{dim} \rho} \right)^2 = \frac{1}{\# Y^2 \# G} \sum_{[a] \in C(G)} \frac{\lambda_{[a]} \chi_\rho(a)}{\# [a] \text{dim} [a]}
\]

\[
= \frac{1}{\# Y^2 \# G} \left( \lambda_{[1]} \chi_\rho(1) \# [1] + \sum_{[a] \neq [1]} \lambda \frac{\chi_\rho(a)}{\text{dim} \rho} \# [a] \right)
\]

\[
= \frac{1}{\# Y^2 \# G} \left( \# Y - \lambda \frac{\chi_\rho(1)}{\text{dim} \rho} \right)
\]

\[
= \frac{1}{\# Y^2 \# G} (\# Y - \lambda)
\]

\[
= \frac{1}{\# Y^2 \# G} \left( \# Y - \frac{\# Y^2 - \# Y}{\# G - 1} \right)
\]

\[
= \frac{1}{\# Y^2 \# G} \frac{\# Y \# G - \# Y^2}{\# G - 1}
\]

\[
= \frac{1}{\# Y - 1/\# G}.
\]
The third equality used the orthogonality of the characters

$$\sum_{a \in G} \chi_{1_G}(a) \chi_{\rho}(a) = 0$$

for $\rho \neq 1_G$ ([16, 1, §2, Proposition 7]). For $\rho = 1_G$, a similar computation using Lemma 3.3 gives $\left( \frac{\partial_{\rho}(Y) \dim \rho}{\dim \rho} \right)^2 = 1/\#G$. This implies (3), and that $P$ maps the vector $(v_{[a]} \mid [a] \in C(G))$ with components $v_{[a]} = (\#Y^2 - \#Y)/(\#G - 1)$ ($[a] \neq [1]$) and $v_{[1]} = \#Y$ to the vector $(w_\rho \mid \rho \in \hat{G})$ with $w_\rho = \frac{1/\#Y - 1/\#G}{\#G - 1}$ ($\rho \neq 1_G$) and $w_{1_G} = 1/\#G$. Now assume (3). Since $P$ is a regular matrix, the above fact shows that (3) implies (2).

Proof of (3) $\iff$ (4). Clearly (3) implies (4). We assume (4), namely, $\left( \frac{\partial_{\rho}(Y) \dim \rho}{\dim \rho} \right)^2 = C$ for $\rho \neq 1_G$. For $\rho = 1_G$, $\frac{\partial_{1_G}(Y)^2}{\dim 1_G} = \frac{1}{\#G}$ is proved in Proposition 3.1, and since $\partial_{\rho}(Y)^2 = C(\dim \rho)^2$, their sum over $\rho \neq 1_G$ is

$$C \sum_{\rho \neq 1_G} (\dim \rho)^2 = C(\#G - 1).$$

By Proposition 3.1, this value is $\frac{1}{\#Y} - \frac{1}{\#G}$, and hence $C = (\frac{1}{\#Y} - \frac{1}{\#G})/(\#G - 1)$. This implies (3).

Proof of (3) $\iff$ (5). By Proposition 1.1, the equality of (5) holds if and only if (3) holds, since $\partial_{1_G}(Y) = \sqrt{\frac{1}{\#G}}$ always holds.

These prove Theorem 3 (and thus Theorem 4).

### 4 Pre-difference sets

#### 4.1 A pre-difference set which is not a difference set

**Proposition 4.1** Let $D_{16}$ be the dihedral group of order 16 presented by

$$\left\{ s, r \mid s^2 = r^8 = (sr)^2 = 1 \right\}.$$  

Then, the subset

$$Y = \{ 1, r, s, sr^3, sr^5, sr^7 \}$$

is a $(16, 6, 2)$ pre-difference set. This is not a difference set.

A proof is done by a program using GAP[10]. The element $sr$ is not a difference of two elements in $Y$, and hence $Y$ is not a difference set.

**Remark 4.2** It is conjectured that any dihedral group $D_{2n}$ has only trivial difference sets[9, Remark 4.4]. The conjecture is true for $n$ being a prime power ([5, Theorem 1.3]
for the prime 2, [9, Theorem 4.5] for odd primes). Thus, $D_{16}$ has no non-trivial difference set, but has non-trivial pre-difference set.

### 4.2 Groups of order 16

In this section, we consider only non-trivial pre-difference sets and non-trivial difference sets. There are 14 isomorphism classes among groups of order 16. A complete list of these classes and all the difference sets there (up to equivalence stated below) is given in [12]. Let $G$ be a group and $Y \subset G$. Then, $(\sigma, g) \in \text{Aut}(G) \times G$ acts on $Y$ by $g \cdot \sigma(Y)$. If $Y$ is a pre-difference set (a difference set, respectively), then so is $g \cdot \sigma(Y)$, respectively. Such a (pre-)difference set is said to be equivalent to $Y$. Using GAP, we obtained a complete list of pre-difference sets for these groups, as follows. Among the 14 classes, the cyclic group and the dihedral group have no difference set. Accordingly in [12], the rest 12 classes are tagged by (A) to (L). Among them, (A) to (D) are abelian, so we omit them since pre-difference sets in these groups are difference sets classified there. We add the dihedral group as the class (M). We list all the pre-difference sets in each class up to equivalence. If the pre-difference sets are difference sets, we noted so. For each class, only the relators are described. For example, in (E), the described group is generated by $a, b, c$ and the relations are $1 = a^4 = b^2 = c^2 = (ab)^2 = aca^{-1}c^{-1} = bcb^{-1}c^{-1}$.

(E) $a^4, b^2, c^2, (ab)^2, aca^{-1}c^{-1}, bcb^{-1}c^{-1}$ ($G \cong D_8 \times \mathbb{Z}/2\mathbb{Z}$)

1. $1, a, b, c, a^2, abc,$
2. (difference set) $1, a, b, c, a^{-1}, a^2bc,$
3. (difference set) $1, a, b, c, a^2b, a^{-1}c.$

(F) $a^4, c^2, a^2b^{-2}, bab^{-1}a^{-3}, aca^{-1}c^{-1}, bcb^{-1}c^{-1}$ ($G \cong \text{Central product of } Q_8 \text{ and } \mathbb{Z}/4\mathbb{Z}$)

1. (difference set) $1, a, b, c, a^2, abc$
2. (difference set) $1, a, b, c, a^{-1}, b^{-1}c$

(G) $a^4, a^2b^{-2}, a^2c^{-2}, bab^{-1}a^{-3}, aca^{-1}c^{-1}, bcb^{-1}c^{-1}$ ($G \cong \text{Central product of } D_8 \text{ and } \mathbb{Z}/4\mathbb{Z}$)

1. $1, a, b, c, a^2, abc,$
2. $1, a, b, c, a^{-1}, bc^{-1},$
3. (difference set) $1, a, b, c, ab, c^{-1},$
4. (difference set) $1, a, b, c, ac, b^{-1}.$

(H) $a^4, b^2, aba^{-1}b^{-1}, c^2b^{-1}, cac^{-1}b^{-1}a^{-3}$ ($G \cong \text{SmallGroup}(16, 3)$)

1. $1, a, c, a^2b, a^2, ac,$
2. $1, a, c, a^2b, a^2, a^{-1}c^{-1},$
3. (difference set) $1, a, c, a^2b, a^{-1}, c^{-1},$
4. (difference set) $1, a, c, a^2b, a^2c, ab,$
5. (difference set) $1, a, c, a^2, a^{-1}b, c^{-1},$
6. (difference set) $1, a, c, a^2, a^2c^{-1}, ab.$
(I) $a^4, b^4, bab^{-1} a^{-3}$ ($G \cong \text{Nontrivial semidirect product of } \mathbb{Z}/4\mathbb{Z} \text{ and } \mathbb{Z}/4\mathbb{Z}$)

1. $1, a, b, a^2, b^2, ab^{-1},$
2. (difference set) $1, a, b, a^2, ab^2, a^2b^{-1},$
3. (difference set) $1, a, b, a^2, b^{-1}, a^{-1}b^2,$
4. (difference set) $1, a, b, b^2, a^2b, a^{-1}b^2.$

(J) $a^8, a^2b^{-2}, bab^{-1}a^{-5}$ ($G \cong M_{16}$)

1. $1, a, b, a^2, a^4, ab,$
2. $1, a, b, a^2, a^3, aba^{-1},$
3. $1, a, b, a^4, a^3, b^{-1},$
4. $1, a, b, ab, a^3, a^{-3},$
5. $1, a, b, ab, a^{-3}, a^{-1},$
6. $1, a, b, a^3, aba^{-1}, a^{-2},$
7. (difference set) $1, a, b, a^4, a^3, a^2b,$
8. (difference set) $1, a, b, a^{-1}, b^{-1}.$

(K) $a^8, a^4b^{-2}, bab^{-1}a^{-3}$ ($G \cong SD_{16}$)

1. $1, a, b, a^{-2}, b^2, ab,$
2. $1, a, b, a^{-2}, b^2, a^{-1}b^{-1},$
3. $1, a, b, a^{-2}, b^2, ab^{-1},$
4. $1, a, b, a^{-2}, b^2, a^{-1}b,$
5. $1, a, b, a^{-2}, a^{-1}, b^{-1},$
6. $1, a, b, b^2, a^{-1}, a^2b,$
7. $1, a, b, b^2, a^2b, a^3,$
8. (difference set) $1, a, b, a^{-2}, ab^2, a^2b,$
9. (difference set) $1, a, b, ab^2, a^2b, a^2.$

(L) $a^8, a^4b^{-2}, bab^{-1}a^{-7}$ ($G \cong Q_{16}$)

1. $1, a, b, a^2, b^2, ab^{-1},$
2. (difference set) $1, a, b, a^2, ab^2, a^2b^{-1},$
3. (difference set) $1, a, b, b^2, a^3, a^2b^{-1}.$

(M) $a^8, b^2, abab$ ($G = D_{16}$)

1. $1, a, b, a^2, a^4, a^{-3}b,$
2. $1, a, b, a^2, a^{-3}, a^{-2}b,$
3. $1, a, b, a^4, a^3, a^{-2}b.$

### 4.3 Properties of pre-difference set

The following theorem shows that pre-difference sets share many properties of difference sets.

**Theorem 4.3** Let $G$ be a finite group and $Y$ be a $(v, k, \lambda)$ pre-difference set. Then we have the following:

(I) $\lambda$ is a positive integer.
(2) \( \lambda(v - 1) = (k^2 - k) \) holds.
(3) If \( Y \neq G \), then the complement \( Y^c \) is a \((v, v - k, v - 2k + \lambda)\) pre-difference set.

**Proof** (1). For any \( a \in G \) with \( a \neq 1 \), \( \lambda[a] \) is a positive integer by definition. Now the conjugacy class formula tells that

\[
\#G = 1 + \sum_{[a] \in C(G) \setminus \{1\}} \#[a].
\]

Multiply the both sides by \( \lambda \). Since \( \#G \) is a multiple of any \( \#[a] \), \( \#G \lambda \) is an integer. The latter term of the right hand side, when multiplied by \( \lambda \), is also an integer. Hence, \( 1 \times \lambda \) is an integer.

(2) This is equivalent to the condition (2) of Theorem 3.

(3) For a subset \( Y \subset G \), by abuse of language, let \( Y \) denote \( \sum_{y \in Y} y \in \mathbb{C}[G] \), and \( Y^{-1} \) denote \( \sum_{y \in Y} y^{-1} \). For any subset \( H \subset G \), \( HG = (#H)G \) holds in \( \mathbb{C}[G] \). Let \( C \) be \( C(G) \). Let \( f : \mathbb{C}[G] \to \mathbb{C}[C] \) be the induced linear map from \( G \to C \), \( a \mapsto [a] \). Then, \( Y \) being a pre-difference set is equivalent to that \( f(YY^{-1}) = f((k - \lambda)e + \lambda G) \), where \( e \) is the unit of \( G \). Put \( Z = G - Y \). Then, in \( \mathbb{C}[G] \), we have

\[
ZZ^{-1} = (G - Y)(G - Y^{-1}) = G^2 - GY - GY^{-1} + YY^{-1}.
\]

Thus

\[
f(ZZ^{-1}) = f(#G - 2#Y) f(G) + f((k - \lambda)e + \lambda G)
\]

\[
= f((v - 2k + \lambda)G + (k - \lambda)e) = f(\lambda' G + (k' - \lambda')e)
\]

holds for \( \lambda' = v - 2k + \lambda \) and \( k' = v - k \). This shows that \( Y^c \) is a \((v, v - k, v - 2k + \lambda)\) pre-difference set. \( \square \)

## 5 Association schemes and Delsarte theory

### 5.1 Basic facts

Let us recall the notion of commutative association schemes briefly. See [1, 3] for details. Let \( X \) be a finite set. By \( M(X; \mathbb{C}) \) we denote the matrix algebra over \( \mathbb{C} \), where the rows and columns are indexed by \( X \). Let \( C \) be a finite set with a specified element \( i_0 \in C \). Let \( R : X \times X \to C \) be a surjective function. For \( i \in C \), \( R^{-1}(i) \subset X \times X \) gives a square matrix \( A_i \in M(X; \mathbb{C}) \), where \( A_i(x, y) = 1 \) if \( R(x, y) = i \) and 0 otherwise. We assume that \( A_{i_0} \) is the identity matrix. For any \( i \), we assume that there is an \( i' \) such that \( A_i = A_{i'} \). The tuple \( (R, X, C) \) is called an association scheme if the linear span of \( A_i(x, y) (i \in C) \) over \( \mathbb{C} \) in the matrix algebra \( M(X; \mathbb{C}) \) is closed under matrix multiplication, and hence a subalgebra of \( M(X; \mathbb{C}) \). This subalgebra is called the Bose-Mesner algebra \( A_X \) of the association scheme. If it is commutative, then the
association scheme is said to be commutative. In this subsection, we deal with only commutative association schemes.

The \( \{A_i\} \) is a linear basis of \( A_X \). Hadamard product of \( A, B \in M(X; \mathbb{C}) \) is defined by the component-wise product \( (A \circ B)(x, y) = (A(x, y)B(x, y)) \). The set \( \{A_i \mid i \in C\} \) consists of the primitive idempotents of \( A_X \) with respect to the Hadamard product. It is known that \( A_X \) is closed under transpose and complex conjugate. Since \( M(X; \mathbb{C}) \) acts on \( \mathbb{C}[X] \), so does \( A_X \). Since \( A_X \) is commutative, we may simultaneously diagonalize all elements of \( A_X \). That is, we have a set of common eigen vectors \( e_k \in \mathbb{C}[X] \) consisting a basis. The action of \( A_X \) on \( e_k \) gives a ring homomorphism \( A_X \to \mathbb{C} \). Different \( e_k \) may give the same ring homomorphism, so let \( \hat{X} \) be the set of different ring homomorphisms \( \rho : A_X \to \mathbb{C} \) obtained in this way. Then, \( A_X \to \mathbb{C}^{\hat{X}} \) is an isomorphism (where the multiplication of \( A_X \) is given by the matrix multiplication). Thus, there is a set of primitive idempotents \( E_\rho \in A_X, \rho \in \hat{X} \). There is a special primitive idempotent \( E_{j_0} := \frac{1}{|X|} I, \) where \( I \) denotes the matrix with all components being 1. The corresponding representation \( j_0 : A_X \to \mathbb{C} \) is given by \( A_i E_{j_0} = j_0(A_i)E_{j_0} \). Thus \( j_0(A_i) \) is the number of ones in a column in \( A_i \) (independent of the choice of the column), called the \( i \)-th valency and denoted by \( k_i \). We call \( j_0 \) the trivial representation.

This shows that

\[
\mathbb{C}^C \to A_X \to \mathbb{C}^{\hat{X}}
\]

are isomorphisms of \( \mathbb{C} \)-vector spaces, where the left map is an isomorphism as a ring (\( \mathbb{C}^C \) the direct product and \( A_X \) the Hadamard product), and the right map is an isomorphism as a ring (\( A_X \) the matrix product and \( \mathbb{C}^{\hat{X}} \) the direct product).

We have orthogonal decomposition of \( \mathbb{C}[X] = \bigoplus_{\rho \in \hat{X}} V_\rho, \) where \( V_\rho \) is the largest subspace such that \( A_X \) acts on \( V_\rho \) via character \( \rho \) of \( A_X \). The one dimensional subspace spanned by \( \sum_{x \in X} x \in \mathbb{C}[X] \) is \( V_{j_0} \).

A typical example of commutative association schemes is a group association scheme associated to a finite group \( G \). In this case, \( X = G, C = C(G) \), and \( R : G \times G \to C(G) \) is given by \( x, y \mapsto [x^{-1}y] \). The group ring \( \mathbb{C}[G] \) acts from left on \( \mathbb{C}[G] \), and thus \( \mathbb{C}[G] \subset M(G; \mathbb{C}) \). It is known that the Bose-Mesner algebra \( A_X \) is the center of \( \mathbb{C}[G] \) \( \mathbb{C}[C(G)] \), \( A_{[a]} \) is the matrix representation of \( [a] \in \mathbb{C}[G] \). The set \( \hat{X} \) may be taken as \( \hat{G} \), and \( E_\rho \) is the projection from \( \mathbb{C}[G] \) to the \( \rho \)-component of \( \mathbb{C}[G] \) (as a left \( G \)-module).

### 5.2 Quasi-Monte Carlo in an association scheme

Let \( Y \subset X \) be a non-empty finite set. Objects defined in Definition 2.1 such as \( \delta_Y := \sum_{y \in Y} y \in \mathbb{C}[X] \) are available for any set. Take \( f \in \mathbb{C}[X] \), and let \( I_Y \) be \( \frac{1}{|Y|} \delta_Y \). Then \( \langle f, I_Y \rangle = \frac{1}{|Y|} \sum_{y \in Y} f(y) \). Any \( f \) decomposes to the sum of \( f_\rho \in V_\rho \) uniquely, and \( f_\rho \) is called the \( \rho \)-component of \( f \). (For \( f = I_Y \), its \( \rho \)-component is denoted by \( I_Y^\rho \).) Because of the orthogonality of \( V_\rho, ||f||^2 = \sum_\rho ||f_\rho||^2 \). Because \( \mathbb{C} \cdot I_X \subset \mathbb{C}[X] \) is the \( j_0 \) component of \( \mathbb{C}[X] \), \( I_X^\rho = 0 \) for \( \rho \neq j_0 \). Generally, \( f_{j_0} = \sum_{x \in X} \langle f, I_X \rangle x \) holds.
By defining $\partial_{\rho}(Y) := ||I^\rho_Y||$, we obtain the same inequalities as in Theorem 1.

**Theorem 5.1** Let $(R, X, C)$ be a commutative association scheme, $Y$ a nonempty subset of $X$, and $f : X \to \mathbb{C}$ a function. This function is identified with $\sum_{x \in X} f(x)x \in \mathbb{C}[X]$. Let $I(f)$ be the average of $f$ over $X$, $I_Y(f)$ the average of $f$ over $Y$. Let $f_\rho$ be the $\rho$-component of $f$ for each $\rho \in \hat{X}$. Define $\partial_{\rho}(Y) := ||I^\rho_Y||$. Then we have $|I(f_\rho) - I_Y(f_\rho)| \leq ||f_\rho||\partial_{\rho}(Y)$. For the trivial representation $\rho = j_0$, we have $|I(f_{j_0}) - I_Y(f_{j_0})| = 0$.

**Proof** This is because

$$I(f_\rho) - I_Y(f_\rho) = \langle f_\rho, I_X - I_Y \rangle,$$

$(I^{j_0}_X - I^{j_0}_Y) = 0$ and $I^\rho_X = 0$ for $\rho \in \hat{X}$, $\rho \neq j_0$. The right hand side $\langle f_\rho, I_X - I_Y \rangle$ of the equation for $\rho \neq j_0$ is

$$\langle f_\rho, I^\rho_X - I^\rho_Y \rangle = \langle f_\rho, -I^\rho_Y \rangle$$

and its absolute value is bounded by

$$||f_\rho|| \cdot ||I^\rho_Y|| = ||f_\rho|| \cdot \partial_{\rho}(Y).$$

$\square$

**Corollary 5.2** We have

$$|I(f) - I_Y(f)| \leq \sum_{\rho \in \hat{X}\backslash\{j_0\}} ||f_\rho||\partial_{\rho}(Y) \leq V(f)D(Y),$$

where $V(f) := \left(\sum_{\rho \in \hat{X}\backslash\{j_0\}} ||f_\rho||\sqrt{\dim V_\rho}\right)$ and $D(Y) := \max_{\rho \in \hat{X}\backslash\{j_0\}} \frac{\partial_{\rho}(Y)}{\sqrt{\dim V_\rho}}$.

The proof is the same as that of Corollary 2.

**Proposition 5.3** For $\partial_{\rho}(Y)$ defined in Theorem 5.1,

$$\sum_{\rho} \partial_{\rho}(Y)^2 = \frac{1}{\#Y},$$

and $\partial_{j_0}(Y)^2 = \frac{1}{\#X}$. Thus,

$$\sum_{\rho \neq j_0} \partial_{\rho}(Y)^2 = \frac{1}{\#Y} - \frac{1}{\#X}.$$
Proof The first equality follows from
\[ \sum_{\rho} \partial_{\rho}(Y)^2 = \sum_{\rho} || I^{\rho}_Y ||^2 = \langle I_Y, I_Y \rangle = 1/\#Y. \]
(See the formula in Definition 2.2.) We have
\[ \partial_{j_0}(Y)^2 = \langle I^{j_0}_Y, I^{j_0}_Y \rangle, \]
but this is $1/\#X$ since $I^{j_0}_Y(x) = \langle I_Y, I_X \rangle = \frac{1}{\#X} \frac{1}{\#Y} (\#(Y \cap X)) = \frac{1}{\#X}$ and hence
\[ \langle I^{j_0}_Y, I^{j_0}_Y \rangle = \#X \cdot \frac{1}{\#X^2} = \frac{1}{\#X}. \]

Proposition 5.4 Let $D(Y)$ be the value defined in Corollary 5.2. Under the assumptions in Theorem 5.1, a lower bound
\[ D(Y) \geq \sqrt{\frac{1/\#Y - 1/\#X}{\#X - 1}} \]
holds. The equality holds if and only if $\frac{\partial_{\rho}(Y)}{\sqrt{\dim V_{\rho}}} = \sqrt{\frac{1/\#Y-1/\#G}{\#G-1}}$ holds for any $\rho \neq j_0$.

The proof is the same as that of Proposition 1.1.

5.3 Delsarte theory and difference sets

Definition 5.5 The $\mathbb{C}$-vector space $M(X; \mathbb{C})$ is equipped with the standard Hermitian inner product
\[ \langle A, B \rangle = \text{trace}(AB^*) = \sum_{x, y \in X} a_{x, y} \overline{b_{x, y}}. \]

Definition 5.6 For $Y \subset X$, we define $\Delta_Y \in M(X; \mathbb{C})$ by $(\Delta_Y)(x, y) = 1$ for $x, y \in Y$ and 0 otherwise.

Lemma 5.7 We define
\[ \lambda_i(Y) := \langle A_i, \Delta_Y \rangle. \]
Then $\lambda_i(Y) = \#\{(x, y) \in Y^2 \mid A_i(x, y) = 1\}$. (This is the inner distribution in [3] multiplied by a scalar $\#Y$. If $X$ is a group association scheme of $G$, then $\lambda_{[a]}(Y) = \lambda_{[a]}$ defined in Theorem 3.)
We have
\[ ||\delta^\rho_{Y}||^2 = \langle E_{\rho}, \Delta_Y \rangle. \]
Proof For any $A \in M(X, \mathbb{C})$, we have $\langle A, \Delta_Y \rangle = \langle A \delta_Y, \delta_Y \rangle$ where the second inner product is that for $\mathbb{C}[X]$. The first statement follows immediately, and the second statement follows from $\delta_Y^c = E_\rho \delta_Y$ and the orthogonality of $V_\rho$. \hfill $\square$

We prepare for a generalization of Theorem 3.

Proposition 5.8 Let $X, R : X \times X \to \mathbb{C}$ be a commutative association scheme. The Hadamard idempotents $A_i$ ($i \in \mathbb{C}$) form a (not necessary normalized) orthogonal basis of $A_X$, and $\langle A_i, A_i \rangle = k_i \# X$.

The ordinal idempotents $E_\rho$ ($\rho \in \hat{X}$) form also a (not necessary normalized) orthogonal basis of $A_X$, and $\langle E_\rho, E_\rho \rangle = \dim V_\rho$.

Proof Orthogonality of $A_i$ comes from that $\langle A, B \rangle$ is the sum of all components of the Hadamard product $A \circ B$ and $A_i$’s are primitive idempotents. The value of inner product is an easy counting. Orthogonality of $E_\rho$ comes from $\langle A, B \rangle = \text{trace} AB^*$, $E_\rho^* = E_\rho$, and that $E_\rho$’s are primitive idempotents. Since $E_\rho$ is the projector to $V_\rho$, its trace is $\dim V_\rho$, which is $\langle E_\rho, E_\rho \rangle = \text{trace} E_\rho^2 = \text{trace} E_\rho$. \hfill $\square$

The following lemma is obvious.

Lemma 5.9 Let $V$ be a $\mathbb{C}$-vector space with Hermitian inner product $\langle \cdot, \cdot \rangle$. Let $W \subset V$ be a subspace, with an orthogonal basis $w_1, \ldots, w_n$. Then the orthogonal projection $V \to W$, $v \mapsto v_W$ is given by

$$v_W = \sum_{i=1}^n \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i.$$ 

Proof For a unique $h \in W^\perp$, we have

$$v = v_W \oplus h = \sum_i a_i w_i \oplus h.$$ 

Since

$$\langle v, w_i \rangle = a_i \langle w_i, w_i \rangle,$$ 

it follows that

$$v_W = \sum_i a_i w_i = \sum_{i=1}^n \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i.$$ 

\hfill $\square$

Corollary 5.10 Let $X$ be a commutative association scheme. For any $M \in M(X, \mathbb{C})$, its orthogonal projection $M_A$ to $A_X$ is

$$M_A = \sum_{i \in \mathbb{C}} \frac{\langle M, A_i \rangle}{\langle A_i, A_i \rangle} A_i$$
\[ \sum_{\rho \in \hat{X}} \left( \frac{M, E_{\rho}}{E_{\rho}, E_{\rho}} \right) E_{\rho}. \]

Putting \( M = \Delta_Y \), we proved

**Corollary 5.11**

\[
(\Delta_Y)_A = \sum_{i \in C} \frac{\lambda_i(Y)}{k_i \#X} A_i
= \sum_{\rho \in \hat{X}} \frac{||\delta_{\rho}^Y||^2}{(\dim V_{\rho})} E_{\rho}.
\]

**Remark 5.12** The vector \( \lambda_i(Y) \) \((i \in C)\) is called the inner distribution vector of \( Y \) (times \( \#Y \)) in \([3, \S 3.1 (3.1)]\). The above result is proved implicitly in \([3, \S 3.1]\), but we give a proof here for simplicity and the self-containedness. Note that the definition of inner distribution adopted in \([4, \text{Definition 4.1}]\) coincides with \( \lambda_i(Y)/k_i \). The above result is also deduced from the arguments in Sect. 3 there.

**Lemma 5.13**

Let \( V \) be the linear subspace of \( A_X \) spanned by the identity matrix \( I \) and \( J \). Then, \( \sum_{i \in C} a_i A_i \in V \) if and only if the value \( a_i \) is independent of the choice of \( i \in C \backslash \{i_0\} \). Similarly, \( \sum_{\rho \in \hat{X}} a_{\rho} E_{\rho} \in V \) if and only if the value \( a_{\rho} \) is independent of the choice of \( \rho \in \hat{X} \backslash \{j_0\} \).

**Proof** It is known that \( A_{i_0} = I \) and \( \sum_{i \in C} A_i = J \). The first statement follows from the linear independence of \( A_i \). Also, it is known that \( E_{j_0} = \frac{1}{\#X} J \) and \( \sum_{\rho \in \hat{X}} E_{\rho} = I \). The second statement follows from the linear independence of \( E_{\rho} \).

The next theorem is a direct consequence of the above lemma and Corollary 5.11.

**Theorem 5.14** Let \( X \) be a commutative association scheme, and \( Y \) its subset. Then, \( \frac{\lambda_i(Y)}{k_i \#X} \) is independent of the choice of \( i \neq i_0 \) if and only if \( \frac{||\delta_{\rho}^Y||^2}{(\dim V_{\rho})} \) is independent of the choice of \( \rho \neq j_0 \).

This theorem and Proposition 5.4 show the following theorem, which generalizes Theorem 3.

**Theorem 5.15** Let \((X, R, C)\) be a commutative association scheme and \( Y \) a non-empty subset of \( X \). Then, the following conditions are equivalent.

1. \( \lambda_i(Y)/k_i \) is independent of choice of \( i \in C \) except \( i = i_0 \), where \( k_i \) is the number of 1 in a column of \( A_i \).
2. \( \lambda_i(Y)/k_i = (\#Y^2 - \#Y)/(\#X - 1) \) holds for any \( i \neq i_0 \), and \( \lambda_{i_0} = \#Y \) holds.
3. \( \partial_{\rho}(Y)/\sqrt{\dim V_{\rho}} = \sqrt{\frac{1}{\#Y - 1/\#X}} \) holds for any \( \rho \neq j_0 \), and \( \partial_{j_0}(Y) = \sqrt{\frac{1}{\#X}} \) holds.
4. \( \partial_{\rho}(Y)/\sqrt{\dim V_{\rho}} \) is independent of the choice of \( \rho \in \hat{X} \backslash \{j_0\} \).

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(5) The equality holds in the inequality
\[ D(Y) \geq \sqrt{\frac{1/\#Y - 1/\#X}{\#X - 1}} \]
in Proposition 5.4.

Proof Recall that \( \partial_\rho(Y) = ||I^\rho_Y|| \).

- Equivalence (1) \( \iff \) (4): Theorem 5.14 implies the equivalence between (1) and (4).
- Equivalence (3) \( \iff \) (5): Proposition 5.4 implies the equivalence between (3) and (5), since \( \partial_{j_0}(Y) = \sqrt{1/\#X} \) always hold.
- Equivalence (3) \( \iff \) (4): (3) implies (4). Assume (4), and put \( K := \partial_\rho(Y)^2/(\dim V_\rho) \). Proposition 5.3 implies \( \partial_{j_0}(Y)^2 = K(\dim V_\rho) \) in the second equality in Proposition 5.3, we have \( \sum_{\rho \neq j_0} K(\dim V_\rho) = 1/\#Y - 1/\#X \). Since \( \sum_{\rho \neq j_0} \dim V_\rho = \#X - 1 \), we have \( K = (1/\#Y - 1/\#X)/(\#X - 1) \), which implies (3).
- Equivalence (1) \( \iff \) (2): (2) implies (1). Assume (1), and put \( K := \lambda_i(Y)/k_i \). From Lemma 5.7, it follows that \( \lambda_{i_0}(Y) = \#Y \), and \( \lambda_i(Y) = (\#Y^2 \cap R^{-1}(i)) \). Thus, \( \sum_{i \in C} \lambda_i(Y) = \#Y^2 \). Putting \( \lambda_i(Y) = k_i K \) for \( i \neq i_0 \) and using \( \sum_{i \neq i_0} k_i = \#X - 1 \) (since \( \sum_i A_i = J \)), we have \( K(\#X - 1) + \#Y = \#Y^2 \), and thus \( K = (\#Y^2 - \#Y)/(\#X - 1) \), which implies (2). Note that this proof does not use the commutativity of the association scheme.

We give a second proof of Theorem 3.

Proof Let \( G \) be a finite group, and consider the associated commutative group association scheme \( (G, R, C(G)) \). Let \( Y \) be a non-empty subset of \( G \). Then \( \lambda_{|a|}(Y) = \lambda_{|a|} \) and \( ||\delta_\rho^a||^2/(\dim V_\rho) = ||\delta_\rho^a||^2/(\dim \rho)^2 \), since \( H_\rho \) has dimension \((\dim \rho)^2 \). Now the conditions (1)-(5) in Theorem 3 are the same as those in Theorem 5.15.

We close this paper by proposing a notion of a difference set in an association scheme, which is equivalent to the condition (1) in Theorem 5.15 and unifies the notions of difference set and pre-difference set.

Definition 5.16 Let \((X, R, C)\) be an association scheme, which may be non-commutative. Let \( i_0 \in C \) be the element with \( A_{i_0} = I \). A non-empty subset \( Y \) of \( X \) is said to be a difference set in the association scheme \((X, R, C)\), if there is a constant \( \lambda \in \mathbb{Q} \) such that \( \lambda = (\#(Y^2 \cap R^{-1}(i)))/k_i \) holds for any \( i \in C \) except \( i = i_0 \), where \( k_i \) is the valency of \( A_i \). Let \( v := \#X, k := \#Y \). Then \( Y \) is called a \((v, k, \lambda)\) difference set in \((X, R, C)\).

It is known that for a finite group \( G, R : G \times G \to G \) given by \( R(g, h) = g^{-1}h \) is an association scheme \((G, R, G)\) (which is commutative if and only if \( G \) is commutative). A difference set \( Y \subset G \) (in a usual sense) is a difference set in the association scheme \((G, R, G)\) with \( Y \neq G \). A pre-difference set is a difference set in the group association scheme \((G, R, C(G))\).
Proposition 5.17 Let \((X, R, C)\) be an association scheme and \(Y\) a difference set as in Definition 5.16. Then,

1. \(\lambda(v - 1) = k(k - 1)\) holds.
2. The complement \(Y^c\) is a \((v, v - k, v - 2k + \lambda)\) difference set in \((X, R, C)\).

It is clear that if \(\#Y = 1\) or \(v, Y\) is a difference set. By (2), so is \(Y\) if \(\#Y = v - 1\). We call these difference sets trivial.

Proof Theorem 5.15 (1) \(\Rightarrow\) (2) (whose proof does not use the commutativity) shows that \(\lambda(v - 1) = k(k - 1)\) holds.

We use notations as in the proof of Lemma 5.7. Then \(\lambda_i(Y) = \langle A_i \delta_Y, \delta_Y \rangle\). We have \(\delta_{Y^c} = \delta_X - \delta_Y\), and

\[
\lambda_i(Y^c) = \langle A_i(\delta_X - \delta_Y), \delta_X - \delta_Y \rangle
= \langle A_i \delta_X, \delta_X \rangle - \langle A_i \delta_X, \delta_Y \rangle - \langle A_i \delta_Y, \delta_X \rangle + \langle A_i \delta_Y, \delta_Y \rangle
= k_i \#X - k_i \#Y - \langle \delta_Y, A_i \delta_X \rangle + \lambda_i(Y)
= k_i v - 2k_i k + \lambda_i(Y).
\]

By dividing by \(k_i\), we have

\[
\frac{\lambda_i(Y^c)}{k_i} = v - 2k + \lambda
\]

for every \(i \neq i_0\), which proves the second statement. \(\Box\)

Remark 5.18 Using a table of all association schemes of order 16 [11], we find examples of difference sets in association schemes with non-integer value of \(\lambda = 2/5, 4/5, 4/3, 14/5\), etc., as well as usual integer values \(\lambda = 2, 6\), by using GAP.

Example 5.19 Let \(k \leq n\) positive integers. Let \([n]\) be \(\{1, 2, \ldots, n\}\), and \(X\) the set of all subsets of \([n]\) with cardinality \(s\). For \(B_1, B_2 \in X\), we define \(R(B_1, B_2) := s - \#(B_1 \cap B_2)\). Then, \((X, R, C)\) with \(R : X \times X \rightarrow C := \{0, 1, \ldots, s\}\) is a symmetric (hence commutative) association scheme called a Johnson scheme, denoted by \(J(n, s)\).

Difference sets \(Y\) and \(Y'\) in \(J(n, s)\) is said to be equivalent if there is a bijection \(\phi : X \rightarrow X\) with \(\phi(Y) = \phi(Y')\) (note that \(\phi\) acts on the set of subsets of \(X\)).

In \(J(5, 2)\), we show that only the following difference sets exist up to equivalence, with an aid of a computer. Let \(k\) denote the cardinality of \(Y\). If \(k = 1, 9\) or 10, then every \(Y\) is a trivial difference set. If \(k = 2, 5, 8\), there are no difference sets. If \(k = 3,\) \([\{1, 2\}, \{1, 3\}, \{3, 4\}\] is a difference set with \(\lambda = 2/3\). If \(k = 4\), two difference sets \([\{1, 2\}, \{1, 3\}, \{3, 4\}, \{2, 4\}\] and \([\{1, 2\}, \{1, 4\}, \{3, 4\}, \{4, 5\}\] exist with \(\lambda = 4/3\). If \(k = 6\), there are two difference sets with \(\lambda = 10/3\), which are the complement sets of the \(k = 4\) cases. If \(k = 7\), there is one difference set with \(\lambda = 14/3\), which is the complement of the \(k = 3\) case. These examples show that \(\lambda\) may not be integers.

Appendix A. quasi-Monte Carlo integration and characters

Here we explain a typical example of QMC integration. Put \(X := (\mathbb{R}/\mathbb{Z})^s\). We consider a periodic real valued function of \(s\)-variables \(f : X \rightarrow \mathbb{R}\). Let \(\alpha\) be a positive integer,
and \( r = (r_1, \ldots, r_s) \). We write \( r \leq \alpha \) if \( r_i \leq \alpha \) holds for each \( 1 \leq i \leq s \). Let \( x = (x_1, \ldots, x_s) \in (\mathbb{R}/\mathbb{Z})^s \). For \( f(x) \), we define
\[
D^r(f) := \frac{\partial^{r_1 + \cdots + r_s}}{\partial x_1^{r_1} \cdots \partial x_s^{r_s}}(f)
\]
if it exists. We assume that \( D^r(f) \) exists and is continuous for \( r \leq \alpha \). We define a norm
\[
||f||_\alpha := \sum_{0 \leq r \leq \alpha} ||D^r(f)||_{L^\infty}.
\]
The set \( \hat{X} \) of characters of \( X \) is
\[
\hat{X} = \{ E_h(x) := \exp(2\pi i h \cdot x) \mid h = (h_1, \ldots, h_s) \in \mathbb{Z}^s \}.
\]
Let
\[
f(x) = \sum_{h \in \mathbb{Z}^s} \hat{f}(h) E_h(x)
\]
be the Fourier-expansion of \( f \). Note that \( \hat{f}(0) = I(f) := \int_X f(x) \). Let \( P \) be a finite subset in \( X \). The QMC integration of \( f \) by \( P \) is the average
\[
I(f; P) := \frac{1}{\# P} \sum_{x \in P} f(x),
\]
and the integration error is
\[
\text{Err}(f; P) = |I(f) - I(f; P)| = \left| \hat{f}(0) - \sum_{h \in \mathbb{Z}^s} \hat{f}(h) I(E_h; P) \right|
\]
\[
\leq \sum_{h \in \mathbb{Z}^s - \{0\}} \left| \hat{f}(h) \right| |I(E_h; P)|.
\]
Let
\[
\rho(h) := \max\{1, |h_1| \} \times \cdots \times \max\{1, |h_s| \}.
\]
It is not difficult to show that the inequalities on the Fourier coefficients
\[
|\hat{f}(h)| \leq C_s,\alpha ||f||_\alpha \rho(h)^{-\alpha}
\]
hold for a constant \( C_{s, \alpha} \) depending only on \( s, \alpha \) (cf. [6, §2.2] [18, §5.2.2]), and we have a Koksma-Hlawka type inequality on the error bound:

\[
\text{Err}(f; P) \leq C_{s, \alpha} \| f \|_\alpha \times \sum_{\mathbf{h} \in \mathbb{Z}^n - \{0\}} \left| \rho(\mathbf{h})^{-\alpha} I(E_\mathbf{h}; P) \right|.
\]

Thus, we want a point set \( P \) that makes \( \sum_{\mathbf{h} \in \mathbb{Z}^n - \{0\}} \left| \rho(\mathbf{h})^{-\alpha} I(E_\mathbf{h}; P) \right| \) small. This is attained if \( |I(E_\mathbf{h}; P)| \) is small (or ideally 0) for \( \mathbf{h} \) with small \( \rho(\mathbf{h}) \).

From now on, we assume that \( P \subset X \) is a finite cyclic subgroup of order \( N \). Such a point set is called a rank-1 lattice and well-studied; see a nice introduction [17]. Let \( P^\perp \subset \mathbb{Z}^s \cong X \) be the subgroup defined by

\[
P^\perp := \{ \mathbf{h} \in \mathbb{Z}^s \mid E_\mathbf{h}(\mathbf{x}) = 1 \text{ for all } \mathbf{x} \in P \}.
\]

It is easy to show that \( I(E_\mathbf{h}; P) = 0 \) if \( \mathbf{h} \notin P^\perp \), and \( I(E_\mathbf{h}; P) = 1 \) if \( \mathbf{h} \in P^\perp \). Thus, we obtain the error-bound

\[
\text{Err}(f; P) \leq C_{s, \alpha} \| f \|_\alpha \times \sum_{\mathbf{h} \in P^\perp - \{0\}} \left| \rho(\mathbf{h})^{-\alpha} \right|.
\]

It is known that for any \( N \) there are \( P \) such that the summation in the right hand side is bounded by \( C_{s, \alpha}^* N^{-\alpha} (\log N)^{\alpha} \) [14, Chapter 5], which implies that when \( N \) is increased, the error-bound decreases with order \( O(N^{-\alpha} (\log N)^{\alpha}) \).

When compared with our study, the above error analysis (1) is essentially obtained from the left inequality in Corollary 2 (if we neglect that we treat only finite groups), where \( \rho \in \hat{G} \) corresponds to \( E_\mathbf{h} \in X \), \( f_\rho \) corresponds to \( \hat{f}(\mathbf{h}) \), \( Y \) corresponds to \( P \), and \( \partial_\rho(Y) = ||Z_\rho^Y|| = |\{E_\rho, I_Y\}| = |I(E_\rho; Y)| \) corresponds to \( |I(E_\mathbf{h}; P)| \). A big difference is that as in (2), \( \hat{f}(\mathbf{h}) \) (the \( \mathbf{h} \)-component of \( f \)) decays when \( \rho(\mathbf{h}) \) gets large, and to make the error bound smaller, it is better to choose \( P \) such that \( |I(E_\mathbf{h}; P)| = 0 \) holds for \( \mathbf{h} \) with small \( \rho(\mathbf{h}) \) since \( I(E_\mathbf{h}; P) \) has a large “weight” in the bound (3) (this condition is close to the idea of Delsarte’s “designs” in [3, §3.4]), while in our study, as in Theorem 3, we have no reasonable “weight” on characters (i.e. degree of importance of each character), and we need to treat them with equal importance, which leads to the notion of pre-difference sets. At present, we think that our results treating general groups and association schemes are rather wide and abstract (say, compared with \((\mathbb{R}/\mathbb{Z})^s\)) and that a practical application to QMC is a future work.

We here briefly explain the notion of digital nets in the theory of QMC [7, 14], which are widely used and closely related with character theory. The unit hypercube \([0, 1]^s\) is approximated by \((\mathbb{F}_2)^s\) via two-adic expansion up to \( n \) digits. A digital net \( P \) is a subgroup of \((\mathbb{F}_2)^s\), identified as a subset of the hypercube. An important figure-of-merit of \( P \) is its \( t \)-value ([14, Chapter 4]), which is obtained from the minimum weight of \( P^\perp \) with respect to Niederreiter-Rosenbloom-Tsfasman(NRT) metric [15], which is a generalization of the Hamming weight. \( P \) has a good (large) \( t \)-value if and only if \( I(E_\rho; P) = 0 \) holds for every \( \rho \in (\mathbb{F}_2)^s - \{0\} \) with small NRT metric, which can be formalized by the notion of designs by Delsarte, as mentioned above. See [13].
for analysis of the digital nets via association schemes. Our study is different in that we treat the cases where the $\frac{\partial \rho(Y)}{\dim \rho} = |I(E_\rho; Y)|/\dim \rho$ are independent of $\rho \neq 1_G$, while the above studies treat the cases where $|I(E_\rho; Y)| = 0$ holds for some “important” characters $\rho$.

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