Derived equivalences between matrix subrings and their applications

Yiping Chen

School of Mathematical Sciences, Beijing Normal University
100875 Beijing, People’s Republic of China
E-mail: ypchen@mail.bnu.edu.cn

Abstract

In this paper, we construct derived equivalences between matrix subrings. As applications, we calculate the global dimensions and the finitistic dimensions of some matrix subrings. And we show that the finitistic dimension conjecture holds for a class of Harada algebras and a class of tiled triangular algebras.

1 Introduction

Derived equivalences preserve many homological properties of algebras such as the number of simple modules, the finiteness of global dimension and finitistic dimension, the algebraic K-theory and Hochschild (co)homological groups (see [15, 6, 11, 19, 20, 18]). Thus, in order to study some homological properties of a given algebra, we can turn to the one which is derived equivalent to it.

Recently, Hu and Xi have exhibited derived equivalent endomorphism rings induced by $D$-split sequences. We find that $D$-split sequences give a way to construct derived equivalences between matrix subrings. In this paper, we will study the derived equivalences having a characteristic that one of two rings has relatively simple structure.

As applications, we first investigate the global dimension of a matrix subring. By the definition of global dimension, Kirkman and Kuzmanovich in [13] have calculated the global dimensions of some matrix subrings. Cowley extended some of their results by triangular decomposition [5]. As never before, we investigate some cases by the method of derived equivalences.

Second, we study the finitistic dimension of a matrix subring. For a ring $A$, the finitistic dimensions are defined as follows: $l$.Fin.dim$(A)$ is the supremum of the projective dimensions of left $A$-modules of finite projective dimensions, and fin.dim$(A)$ is the supremum of the projective dimensions of finitely generated left $A$-modules of finite projective dimensions. Kirkman and Kuzmanovich compute $l$.Fin.dim$(A)$ for a noncommutative Noetherian ring $A$ in [13]. By derived equivalences, we calculate $l$.Fin.dim$(A)$ for a matrix subring $A$. This result is helpful to study the finitistic dimension conjecture which states that for an Artin algebra $A$, fin.dim$(A)$ is finite. This conjecture is still open. We refer the reader to [21] on some new advances on this conjecture.

Little is known about whether the finitistic dimension conjecture holds for matrix subalgebras. Note that the Artin algebra $A$ and the matrix algebra $M_n(A)$ are Morita equivalent. Thus, in order to prove that fin.dim$(A)$ is finite, it is equivalent to prove that fin.dim$(M_n(A))$ is finite. Our ideal in this direction is to investigate the finitistic dimension of a matrix subalgebra. If the finitistic dimension of $A$ is finite, what could we say about the finitistic dimension of a matrix subalgebra?

In order to describe the main result precisely, we fix some notations.

Let $A$ be a ring with identity. Let $A_i (2 \leq i \leq n)$ be a family of subrings of $A$ with the same identity with $A$, and let $I_i, I_{ij}, 2 \leq i \leq n, 2 \leq j \leq n − 1$ be ideals of $A$ satisfying that $I_n \subseteq I_{n−1} \subseteq \cdots \subseteq I_2, I_i \subseteq \cdots
In this way, we can construct two rings

\[
\Lambda = \begin{pmatrix}
A_{i_2} & I_2 & I_3 & \cdots & I_{n-1} & I_n \\
A_{j_2} & A_2 & I_3 & \cdots & I_{n-1} & I_n \\
A_{i_3} & I_2 & A_3 & \cdots & I_{n-1} & I_n \\
A_{j_2} & I_2 & I_3 & A_4 & \cdots & I_n \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
A_{i_n} & I_{n-2} & \cdots & I_{n-1} & A_n \\
\end{pmatrix}
\quad \text{and} \quad
\Sigma = \begin{pmatrix}
A_2/I_2 & 0 & \cdots & 0 \\
I_{2}/I_2 & A_3/I_3 & \cdots & \vdots \\
I_{3}/I_2 & I_{3}/I_3 & A_4/I_4 & \cdots \\
I_{4}/I_2 & I_{3}/I_3 & I_{4}/I_4 & \cdots & A_n/I_n \\
I_{n}/I_2 & I_{n-1}/I_3 & I_{n-1}/I_4 & \cdots & A_n/I_n \\
A/I_2 & A/I_3 & A/I_4 & \cdots & A/I_n & A \\
\end{pmatrix}
\]

with identities. Throughout this paper, \(\Lambda\) and \(\Sigma\) are rings of this forms.

The main result in this paper is the following:

**Theorem 1.1.** The two rings \(\Lambda\) and \(\Sigma\) are derived equivalent.

As a direct consequence of Theorem 1.1, we have the following corollary.

**Corollary 1.2.** (1) Let \(\Lambda\) be as in Theorem 1.1. Then

\[l.\text{Fin.dim}(A) - 1 \leq l.\text{Fin.dim}(\Lambda) \leq n + \sum_{i=2}^{n} l.\text{Fin.dim}(A_i/I_i) + l.\text{Fin.dim}(A).
\]

(2) Let \(\Lambda\) as in Theorem 1.1. Then

\[
\max\{l.\text{gl.dim}(A_i/I_i), l.\text{gl.dim}(A), 2 \leq i \leq n\} - 1 \leq l.\text{gl.dim}(\Lambda) \leq \sum_{i=2}^{n} l.\text{gl.dim}(A_i/I_i)
\]

\[+ l.\text{gl.dim}(A) + n.
\]

We define a class of algebras which are called general block extensions of rings with respect to the decomposition of identity. And we calculate their global dimensions and finitistic dimensions. We also get a class of Harada algebras and a class of tiled triangular rings which satisfy the finitistic dimension conjecture.

This paper is arranged as follows. In section 2, we fix some notations and recall some definitions and lemmas needed in this paper. In section 3, the proof of the main result is given. In section 4, we give some applications of the main result. The definition of general block extensions of rings with respect to the decompositon of identity is proposed. We calculate their global dimensions and finitistic dimensions. And we also get some classes of algebras which satisfy the finitistic dimension conjecture. In section 5, we display some examples to illustrate the applications.

## 2 Preliminaries

In this section, we shall recall some basic definitions and results needed in this paper.

Let \(A\) be a ring with identity. We denote \(A\)-\text{Mod} the category of left \(A\)-modules and by \(A\)-\text{mod} the category of all finitely generated left \(A\)-modules. \(\text{Mod}-A\) means the category of right \(A\)-modules. Given an \(A\)-module \(M\), we denote by \(\text{proj.dim}(M)\) the projective dimension of \(M\). The left global dimension of \(M\), denoted by \(l.\text{gl.dim}(A)\), is the supremum of all \(\text{proj.dim}(M)\) with \(M \in \text{A}-\text{Mod}\). By \(\text{add}(M)\), we shall mean the full subcategory of \(A\)-\text{Mod}, whose objects are summands of finite sums of \(M\). For two morphisms \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\) in \(A\)-\text{Mod}, the composition of \(f\) and \(g\) is written as \(fg\), which is a morphism form \(X\) to \(Z\).

Let \(A\) be an Artin algebra. A complex \(X^* = (X^i, d^i_X)\) of \(A\)-modules is a sequence of \(A\)-modules and \(A\)-module homomorphisms \(d^i_X : X^i \rightarrow X^{i+1}\) such that \(d^i_X d^{i+1}_X = 0\) for all \(i \in \mathbb{Z}\). A morphism \(f^* : X^* \rightarrow Y^*\) between two complexes \(X^*\) and \(Y^*\) is a collection of homomorphisms \(f^i : X^i \rightarrow Y^i\) of \(A\)-modules such that \(f^i d^i_X = d^i_Y f^{i+1}\). The morphism \(f^*\) is said to be null homotopic if there exists a homomorphism \(h^i : X^i \rightarrow Y^{i-1}\) such that \(f^i = d^i_Y h^{i+1} + h^i d^{i-1}_X\) for all \(i \in \mathbb{Z}\). A complex \(X^*\) is called bounded below if \(X^i = 0\) for all but finitely many \(i < 0\), bounded above if \(X^i = 0\) for all but finitely many \(i > 0\), and bounded if \(X^*\) is bounded below and above. We denote by \(\mathcal{C}(A)\) (resp., \(\mathcal{C}(A\text{-Mod})\))
the category of complexes of finitely generated (resp., all) \( A \)-modules. The homotopic category \( \mathcal{X}(A) \) is the derived category of \( A \)-mod by \( \mathcal{D}(A) \) which is the quotient category of \( \mathcal{X}(A) \) with respect to the subcategory of \( \mathcal{X}(A) \) consisting of all the acyclic complexes. The full subcategory of \( \mathcal{X}(A) \) consisting of bounded complexes over \( A \)-mod is denoted by \( \mathcal{X}^b(A) \) and \( \mathcal{D}^b(A) \), respectively. We denote \( \mathcal{X}^b(A) \) by the category of complexes of bounded below, and by \( \mathcal{X}^+(A) \) the homotopic category of \( \mathcal{C}^+(A) \). The full subcategory of \( \mathcal{D}(A) \) consisting of bounded below complexes is denoted by \( \mathcal{D}^+(A) \). Similarly, we have the category \( \mathcal{C}^-(A) \) of complexes bounded above, the homotopic category \( \mathcal{X}^-(A) \) of \( \mathcal{C}^-(A) \) and the derived category \( \mathcal{D}^-(A) \) of \( \mathcal{C}^-(A) \).

The two rings \( A \) and \( B \) are called derived equivalent if \( \mathcal{D}^b(A) \) and \( \mathcal{D}^b(B) \) are equivalent as triangulated categories. It is equivalent to say that \( B \) is isomorphic to \( \text{End}_{\mathcal{D}^b(A)}(T^*) \), where \( T^* \) is a complex in \( \mathcal{X}^b(A \text{-proj}) \) satisfying: (a) \( T^* \) is self-orthogonal, that is, \( \text{Hom}_{\mathcal{X}^b(A \text{-proj})}(T^*, T^*[i]) = 0 \) for all \( i \neq 0 \), (b) \( \text{add}(T^*) \) generates \( \mathcal{X}^b(A \text{-proj}) \) as a triangulated category.

In [10], Hu and Xi define the \( \mathcal{D} \)-split sequences which occur in many situation, for instance, the Auslander-Reiten sequence.

**Definition 2.1.** [10] Let \( C \) be an additive category and \( D \) a full subcategory of \( C \). A sequence

\[
X \xrightarrow{f} M \xrightarrow{g} Y
\]

in \( C \) is called a \( D \)-split sequence if

1. \( M \in D \);
2. \( f \) is a left \( D \)-approximation of \( X \), and \( g \) is a right \( D \)-approximation of \( Y \);
3. \( f \) is a kernel of \( g \), and \( g \) is a cokernel of \( f \).

\( D \)-split sequences implies the derived equivalence between the endomorphism algebras. The following theorem reveals how to construct derived equivalence from \( D \)-split sequences.

**Lemma 2.2.** [10] Let \( C \) be an additive category and \( M \) an object in \( C \). Suppose

\[
X \xrightarrow{f} M' \xrightarrow{g} Y
\]

is an almost \( \text{add}(M) \)-split sequence in \( C \). Then the endomorphism ring \( \text{End}_C(X \oplus M) \) of \( X \oplus M \) and the endomorphism ring \( \text{End}_C(Y \oplus M) \) of \( Y \oplus M \) are derived equivalent.

## 3 Results and proofs

To prove our results, we first establish a fact.

**Lemma 3.1.** Let \( R \) be a ring with identity.

1. Let \( M \) be a noetherian left \( R \)-module, and let \( f : M \rightarrow M \) be a surjective homomorphism, then \( f \) is injective.
2. Let \( M \) be an artinian left \( R \)-module, and let \( f : M \rightarrow M \) be an injective homomorphism, then \( f \) is surjective.

**Proof.** We only prove the first part of the lemma. The second part of the lemma is similar. Set \( f^k = f \cdots f \). Note that \( \text{Ker} f^k \) are submodules of \( M \) and \( \text{Ker} f^i \) is the submodule of \( \text{Ker} f^{i+1} \) for any \( i \geq 1 \). Since \( M \) is a noetherian module, there exists \( i_0 \geq 1 \), satisfying \( \text{Ker} f^{i_0} = \text{Ker} f^{i_0+1} \). Then we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker} f^{i_0} & \rightarrow & M & \xrightarrow{f^{i_0}} & M & \rightarrow & 0 \\
& & \downarrow{id} & & \downarrow{id} & & \downarrow{f^{i_0}} & & \\
0 & \rightarrow & \text{Ker} f^{i_0+1} & \rightarrow & M & \xrightarrow{f^{i_0+1}} & M & \rightarrow & 0
\end{array}
\]
By the snake lemma, we can get Ker$f = 0$. So $f$ is injective. □

Now, let us prove the main result in this paper.

**Proof of Theorem 1.1:** Set $\Gamma = M_n(A)$, the $n \times n$ matrix over $A$. Denote by $e_i$ the matrix which has $I_A$ in the $(i, i)$-th position and zeros elsewhere for $1 \leq i \leq n$. So $e_1, e_2, \ldots, e_n$ are piecewise orthogonal idempotents in $A$, such that $I_A = e_1 + e_2 + \cdots + e_n$.

Since $\Lambda$ is a subring of $\Gamma$ with the same identity, the ring $\Gamma$ can be considered as a $\Lambda$-module just by restriction of the scalars of $\Gamma$ to $\Lambda$.

Now, we consider the exact sequence

$$0 \rightarrow \Lambda \xrightarrow{\lambda} \Gamma \xrightarrow{\pi} L \rightarrow 0$$

in $\Lambda$-Mod, where $\lambda$ is the inclusion map and $L$ is the cokernel of $\lambda$. To show the Theorem, we prove the following statements.

(1) The sequence

$$0 \rightarrow \Lambda \xrightarrow{\lambda} \Gamma \xrightarrow{\pi} L \rightarrow 0$$

is an almost add$(\Lambda e_1)$-split sequence in $\Lambda$-Mod.

In fact, we shall check that all conditions in Definition 2.1 are satisfied.

Since the left $\Lambda$-module $\Lambda \Gamma$ is a direct sum of some copies of $\Lambda e_1$, we have $\Gamma \in \text{add}(\Lambda e_1)$. Clearly, $\Lambda e_1$ is projective as a left $\Lambda$-module, then we have an exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(D, \Lambda) \xrightarrow{(-, \lambda)} \text{Hom}_\Lambda(D, \Gamma) \xrightarrow{(-, \pi)} \text{Hom}_\Lambda(D, L) \rightarrow 0$$

for any $D \in \text{add}(\Lambda e_1)$.

This means that the homomorphism $\pi : \Lambda \rightarrow \Lambda \Gamma$ is a right add$(\Lambda e_1)$-approximation of $\Lambda \Lambda L$. Now, we prove that the homomorphism $\lambda : \Lambda \rightarrow \Lambda \Gamma$ is a left add$(\Lambda e_1)$-approximation of $\Lambda$. In fact, every left $\Lambda$-module homomorphism $g : \Lambda \rightarrow \Lambda e_1$ is determined by $g(1)$, the image of 1 under $g$. Similarly, every left $\Gamma$-module homomorphism $h : \Gamma \rightarrow \Gamma e_1$ is determined by $h(1)$, the image of 1 under $h$. Note that $\Gamma e_1$ and $\Lambda e_1$ are isomorphic as left $\Lambda$-module, and any left $\Gamma$-module homomorphism is also left $\Lambda$-module homomorphism. So we assume that $g_1 : \Gamma \rightarrow \Lambda e_1$ is a left $\Lambda$-module homomorphism which sends 1 to $g(1)$. Then the homomorphism $g_1$ satisfies $g = ig_1$. Thus we have proved that the homomorphism $\lambda : \Lambda \Lambda \rightarrow \Lambda \Gamma$ is a left add$(\Lambda e_1)$-approximation. Hence (1) is proved.

Note the sequence

$$0 \rightarrow \Lambda e_i \xrightarrow{\lambda_i} \Lambda e_1 \xrightarrow{\pi_i} L_i \rightarrow 0 \quad (*)$$

where $\lambda_i$ is inclusion map and $L_i$ is the cokernel of $\lambda_i$, are almost add$(\Lambda e_1)$-split sequences for $2 \leq i \leq n$.

By Lemma 2.2, the ring $\Lambda$ and the endomorphism ring $\text{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1)$ are derived equivalent via a tilting module $L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1$.

(2) The ring $\Sigma$ and the endomorphism ring $\text{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1)$ are isomorphic as rings.

Indeed, we note that

$$\text{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1) \cong \begin{pmatrix}
(L_2, L_2) & (L_2, L_3) & \cdots & (L_2, \Lambda e_1) \\
(L_3, L_2) & (L_3, L_3) & \cdots & (L_3, \Lambda e_1) \\
\vdots & \ddots & \ddots & \vdots \\
(\Lambda e_1, L_2) & (\Lambda e_1, L_3) & \cdots & (\Lambda e_1, \Lambda e_1)
\end{pmatrix}$$

as rings.

In the following, we calculate the endomorphism ring $\text{End}_\Lambda(L_2 \oplus L_3 \oplus \cdots \oplus L_n \oplus \Lambda e_1)$.

The morphism set $\text{Hom}_\Lambda(L_i, \Lambda e_1) = 0$ for $2 \leq i \leq n$. Applying the functor $\text{Hom}_\Lambda(-, \Lambda e_1)$ to the exact sequence $0 \rightarrow \Lambda e_i \rightarrow \Lambda e_1 \rightarrow L_i \rightarrow 0$, we can get the following exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(L_i, \Lambda e_1) \rightarrow \text{Hom}_\Lambda(\Lambda e_1, \Lambda e_1) \rightarrow \text{Hom}_\Lambda(\Lambda e_1, \Lambda e_1) \rightarrow 0$$
in $\mathbb{Z}$-$\text{Mod}$ for $2 \leq i \leq n$.

Note that both of $\text{Hom}_A(\Lambda e_1, \Lambda e_1)$ and $\text{Hom}_A(\Lambda e_i, \Lambda e_1)$ are isomorphic to $A$ in $\mathbb{Z}$-$\text{Mod}$. By Lemma 3.1, we have $\text{Hom}_A(\Lambda_1, \Lambda e_1) = 0$ for $2 \leq i \leq n$.

For simplicity, we denote the set $e_i \Lambda e_j$ by $\Lambda_{ij}$ for $1 \leq i, j \leq n$. The morphism $\mu_t$ denote the right multiplication by $x$.

Let $b$ be an element in $\Lambda_{ij}$ for $2 \leq i, j \leq n$. Since $\lambda_i$ is the left $\Lambda e_1$-approximation of $\Lambda e_i$, we have a morphism $\mu_b : \Lambda e_1 \to \Lambda e_i$ such that $\lambda_i \mu_a = \mu_b \lambda_j$ where $a$ is an element in $A$. Thus we can get an element $\alpha_b$ in $\text{Hom}(\Lambda_0, \Lambda e_j)$ such that $\pi_0 \alpha_b = \mu_a \pi_j$. It follows from the commutativity of the left square that $a = b$. For a given morphism $\mu_b$, there is a unique $\alpha_b$ satisfying $\pi_0 \alpha_b = \mu_b \pi_j$. Note that $\pi_0 \mu_b = \mu_b \pi_j$, we can get $\alpha_b = \mu_b$.

\[
\begin{array}{ccc}
0 & \longrightarrow & \Lambda e_i \\
\mu_b \downarrow & & \downarrow \pi_0 \\
0 & \longrightarrow & \Lambda e_j
\end{array}
\]

(\ast) The map $\phi_{ij}$ is well-defined.

Thus, we can define a set of maps from $\Lambda_{ij}$ to $\text{Hom}_A(\Lambda_0, \Lambda e_j)$ for $2 \leq i, j \leq n$.

Define

$$
\phi_{ij} : \Lambda_{ij} \to \text{Hom}_A(\Lambda_0, \Lambda e_j) \\
b \mapsto \alpha_b,$$

for $2 \leq i, j \leq n$.

(a) The map $\phi_{ij}$ is well-defined.

(b) The morphism $\phi_{ij}$ is surjective.

Let $\alpha$ be an element in $\text{Hom}_A(\Lambda_0, \Lambda e_j)$. Note that $\Lambda e_1$ is projective module over $\Lambda$, thus there exists a morphism $\mu_a : \Lambda e_1 \to \Lambda e_1$ such that $\mu_a \pi_j = \pi_0 \alpha$ where $a$ is an element in $\Lambda_{11}$. Thus there is a unique morphism $\mu_b : \Lambda e_i \to \Lambda e_j$ such that $\lambda_i \mu_b = \mu_b \lambda_j$ for $b \in \Lambda_{ij}$. So $\phi_{ij}$ is surjective.

(c) The description of $\text{Ker}\phi_{ij}$, i.e., $\text{Ker}\phi_{ij} = I_j$ for $2 \leq i, j \leq n$.

Suppose that $b = 0$, we have $\lambda_i \mu_b = 0$. It follows that there is a morphism $\pi_j : \Lambda_0 \to \Lambda e_1$ such that $\mu_b = \pi_0 \alpha$. Thus we have $\pi_0 \alpha_b = \mu_b \pi_j = \pi_0 \pi_j$. Since $\pi$ is surjective, we have $\alpha_b = \pi_j$. Note that $\text{Hom}_A(\Lambda_1, \Lambda e_1) = 0$ for $2 \leq i \leq n$, we obtain $\alpha_b = 0$. Hence $\phi_{ij}$ is well-defined.

(d) $\phi_{ij}$ preserves addition and multiplication.

It is easy to prove that $\phi_{ij}$ preserves addition.
Now, we turn to prove that $\phi_{ij}$ preserves multiplication, i.e., $\phi_{ij}(b)\phi_{jk}(b') = \phi_{ik}(bb')$ for $2 \leq i, j \leq n$ where $b$ and $b'$ are elements of $\Lambda_{ij}$ and $\Lambda_{jk}$ respectively.

It suffices to prove that $\alpha_{bb'} = \alpha_b\alpha_{b'}$. Since $\mu_{b'b'} = \mu_{bb'}$, we have $\lambda_{b'b'} = \lambda_b\mu_c$ where $\mu_c$ is a morphism induced by $\mu_{bb'}$. Thus $\mu_{bb'} - \mu_c$ factorizes through $\pi_c$. Note that $\text{Hom}_\Lambda(L_i, \Lambda e_1) = 0$ for $2 \leq i \leq n$, we have $\mu_{bb'} = \mu_c$. Hence we get $\alpha_{bb'} = \alpha_b\alpha_{b'}$.

Now, we can define a map

$$\phi = (\phi_{ij}) : (e_2 + \cdots + e_n)\Lambda (e_2 + \cdots + e_n) \rightarrow \begin{pmatrix}
(L_2, L_2) & (L_2, L_3) & \cdots & (L_2, L_n) \\
(L_3, L_2) & (L_3, L_3) & \cdots & (L_3, L_n) \\
\vdots & \vdots & \ddots & \vdots \\
(L_n, L_2) & (L_n, L_3) & \cdots & (L_n, L_n)
\end{pmatrix}$$

for $2 \leq i, j \leq n$.

The map $\phi_{ij}$ is well-defined and surjective, so is the map $\phi$. It follows from that $\phi_{ij}$ preserves addition and multiplication for $2 \leq i, j \leq n$ that $\phi$ is a ring homomorphism. The kernel of $\phi$ is

$$\begin{pmatrix}
I_2 & I_3 & \cdots & I_n \\
I_2 & I_3 & \cdots & I_n \\
\vdots & \vdots & \ddots & \vdots \\
I_2 & I_3 & \cdots & I_n
\end{pmatrix}$$

Thus, we have a ring isomorphism

$$\tilde{\phi} : \begin{pmatrix}
A_2/I_2 & 0 & \cdots & 0 \\
l_{22}/I_2 & A_{3}/I_3 & \cdots & \vdots \\
l_{32}/I_2 & I_{33}/I_3 & A_{4}/I_4 & \vdots \\
l_{n2}/I_2 & I_{n3}/I_3 & I_{nk}/I_4 & \cdots & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
(L_2, L_2) & (L_2, L_3) & \cdots & (L_2, L_n) \\
(L_3, L_2) & (L_3, L_3) & \cdots & (L_3, L_n) \\
\vdots & \vdots & \ddots & \vdots \\
(L_n, L_2) & (L_n, L_3) & \cdots & (L_n, L_n)
\end{pmatrix}$$

Meanwhile, we have algebra isomorphism $\varphi_2$

$$\varphi_2 : A \rightarrow \text{End}_\Lambda(\Lambda e_1)$$

$$a \mapsto (f_a : \lambda e_1 \mapsto \lambda ae_1)$$

and isomorphism of abelian groups $\varphi_3$

$$\varphi_3 : (A/I_2, A/I_3, \cdots, A/I_n) \rightarrow (\text{Hom}_\Lambda(\Lambda e_1, L_2), \text{Hom}_\Lambda(\Lambda e_1, L_3), \cdots, \text{Hom}_\Lambda(\Lambda e_1, L_n))$$

$$(m_1, m_2, \cdots, m_n) \mapsto (f_{m_2}, f_{m_3}, \cdots, f_{m_n})$$

where $f_{m_i} : \lambda e_1 \mapsto \lambda m_i e_1$ for $2 \leq i \leq n$.

Now, Set

$$\varphi = \begin{pmatrix}
\tilde{\phi} & 0 \\
\varphi_3 & \varphi_2
\end{pmatrix} : \Sigma \rightarrow \begin{pmatrix}
\text{End}_\Lambda(L_2) & \text{Hom}_\Lambda(L_2, L_3) & \cdots & \text{Hom}_\Lambda(L_2, L_n) \\
\text{Hom}_\Lambda(L_3, L_2) & \text{End}_\Lambda(L_3) & \cdots & \text{Hom}_\Lambda(L_3, L_n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Hom}_\Lambda(\Lambda e_1, L_2) & \text{Hom}_\Lambda(\Lambda e_1, L_3) & \cdots & \text{End}_\Lambda(\Lambda e_1)
\end{pmatrix}$$

\[
\left(\begin{array}{cccccccc}
r_{22} & r_{23} & \cdots & r_{2n} & 0 \\
r_{32} & r_{33} & \cdots & r_{3n} & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
r_{n2} & r_{n3} & \cdots & r_{nm} & 0 \\
m_2 & m_3 & \cdots & m_n & a
\end{array}\right) \mapsto \left(\begin{array}{cccccccc}
\Phi_{22}(r_{22}) & \Phi_{23}(r_{23}) & \cdots & \Phi_{2n}(r_{2n}) & 0 \\
\Phi_{32}(r_{32}) & \Phi_{33}(r_{33}) & \cdots & \Phi_{3n}(r_{3n}) & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_{n2}(r_{n2}) & \Phi_{n3}(r_{n3}) & \cdots & \Phi_{nm}(r_{nm}) & 0 \\
\Phi_3(m_2) & \Phi_3(m_3) & \cdots & \Phi_3(m_n) & \Phi_2(a)
\end{array}\right)
\]
Clearly, the map $\varphi$ is an isomorphism of abelian groups. And it easy to check that $\varphi$ is a ring isomorphism. The proof is completed. □

As a direct consequence of Theorem 1.1, we have the following corollary.

**Corollary 3.2.** Let $A$ be a ring with identity, and suppose that $I_2, I_3, \ldots, I_n$ are ideals of $A$.

1. The two rings

$$
\begin{pmatrix}
A & I_2 & I_3 & \cdots & I_n \\
A & A & I_2 & \cdots & I_n \\
A & I_2 & A & \cdots & I_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A & I_2 & I_3 & \cdots & I_{n-1} \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
A/I_2 & 0 & \cdots & 0 \\
0 & A/I_2 & \cdots & \vdots \\
\vdots & \vdots & \ddots & A/I_4 \\
0 & \cdots & 0 & A/I_n \\
A/I_2 & A/I_3 & A/I_4 & \cdots & A/I_n & A \\
\end{pmatrix}
$$

are derived equivalent.

2. The two rings

$$
\begin{pmatrix}
A & I_2 & I_3 & \cdots & I_{n-1} & I_n \\
A & A & I_2 & \cdots & I_{n-1} & I_n \\
A & A & A & \cdots & I_{n-1} & I_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A & A & A & A & \cdots & A \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
A/I_2 & 0 & \cdots & 0 \\
A/I_2 & A/I_3 & \cdots & \vdots \\
A/I_2 & A/I_3 & A/I_4 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & A/I_n \\
A/I_2 & A/I_3 & A/I_4 & \cdots & A/I_n & A \\
\end{pmatrix}
$$

are derived equivalent.

4 Applications

In section 3, we have constructed derived equivalences of matrix subrings. In this section, we will give some applications of the main result. At first, we define a class of rings which are called general block extensions of rings with respect to the decomposition of identity. Then we calculate the finitistic dimension of a general block extension of ring with respect to the decomposition of identity. At last, we will consider the finiteness of finitistic dimension of a tiled triangular ring. Proposition 4.14, will give a condition under which the finitistic dimensions of tiled triangular algebras are finite.

The following lemmas, which are taken from [18, 11], are useful for this section.

**Lemma 4.1.** [18] If two left coherent rings $A$ and $B$ are derived equivalent, and if $T^*$ is a tilting complex over $A$ with $n + 1$ non-zero terms such that $B \cong \text{End}(T^*)$, then $\text{fin.dim}(A) - n \leq \text{fin.dim}(B) \leq \text{fin.dim}(A) + n$.

**Remark.** In Lemma 4.1, if we replace $A$ and $B$ by arbitrary ring with identity, then $l.\text{Fin.dim}(A) - n \leq l.\text{Fin.dim}(B) \leq l.\text{Fin.dim}(A) + n$. The proof is similar.

**Lemma 4.2.** [11] Let $A$ be an ring with identity, and let $T^*$ be a tilting complex over $A$ with $\text{End}(T^*) \cong B$. If $T^*$ has $n + 1$ non-zero terms, where $n \geq 0$, then the following statements hold.

(a) $l.\text{gl.dim}(A) - n \leq l.\text{gl.dim}(B) \leq l.\text{gl.dim}(A) + n$;
(b) $\text{inj.dim}(A) - n \leq \text{inj.dim}(B) \leq \text{inj.dim}(A) + n$.

The following lemma about the estimation of global dimension and finitistic dimension can be found in [7, Corollary 4.21, p.70].

**Lemma 4.3.** [7] Let $R$ and $S$ be rings. Let $M$ be an $S-R$ bimodule and $\Lambda := \begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$. Then the following inequalities hold:

1. $l.\text{Fin.dim}(S) \leq l.\text{Fin.dim}(\Lambda) \leq 1 + l.\text{Fin.dim}(R) + l.\text{Fin.dim}(S)$. 
4.1 General block extensions of rings with respect to the decomposition of identity

In this part, we will define a class of rings which contains hereditary orders, block extensions of basic algebras.

**Definition 4.4.** Let $A$ be a ring with identity $1_A$. And $1_A = e_1 + e_2 + \cdots + e_m$ is a decomposition of the identity where $e_i$ is idempotent. Then $A$ can be represented as the following matrix form

$$A = \begin{pmatrix}
e_1 e_1 & e_1 e_2 & \cdots & e_1 e_m \\
e_2 e_1 & e_2 e_2 & \cdots & e_2 e_m \\
\vdots & \vdots & \ddots & \vdots \\
e_m e_1 & e_m e_2 & \cdots & e_m e_m
\end{pmatrix}.$$ 

Set $A_i = e_i A e_i$ and $A_{ij} = e_i A e_j$. Then $A_i$ is the subring of $A$ with identity element $e_i$, and $A_{ij}$ is a $(A_i, A_j)$-bimodule.

Let $n_1, n_2, \cdots, n_m \in \mathbb{N}$. For $1 \leq i, s \leq m$, $1 \leq j \leq n_i$ and $1 \leq t \leq n_s$, we define

$$P = A(n_1, n_2, \cdots, n_m) = \begin{pmatrix}
P(1,1) & P(1,2) & \cdots & P(1,m) \\
P(2,1) & P(2,2) & \cdots & P(2,m) \\
\vdots & \vdots & \ddots & \vdots \\
P(m,1) & P(m,2) & \cdots & P(m,m)
\end{pmatrix}$$

which is contained in the ring $\text{End}_A((Ae_1)^{n_1} \oplus \cdots \oplus (Ae_m)^{n_m})$ with the restrictions of the binary operations of addition and multiplication of $\text{End}_A((Ae_1)^{n_1} \oplus \cdots \oplus (Ae_m)^{n_m})$.

$$P(i,s) = \begin{pmatrix}
P_{1,s_1} & P_{1,s_2} & \cdots & P_{1,s_n} \\
P_{2,s_1} & P_{2,s_2} & \cdots & P_{2,s_n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{m,s_1} & P_{m,s_2} & \cdots & P_{m,s_n}
\end{pmatrix}_{n_i \times n_s}$$

satisfies that $P_{i,p,q}$ is a $(A_i, A_p)$-bimodule.

For $P(i,s)$, there are three cases:

**Case I**: $i = s$.

$$P(i,s) := \begin{pmatrix}
A_i & I_2 & I_3 & \cdots & I_{i(n-1)} & I_n \\
A_i & B_2 & I_3 & \cdots & I_{i(n-1)} & I_n \\
A_i & I_3 & B_3 & \cdots & I_{i(n-1)} & I_n \\
A_i & I_3 & I_4 & B_4 & \cdots & I_n \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
A_i & I_{n-2} & \cdots & I_{n(n-1)} & B_{n-1}
\end{pmatrix}$$

where $B_l$ is the subring of $A_i$ with the same identity, $I_l$ and $I_{l,p,q}$ are ideals of $A_i$ satisfying $I_{i_{n_l}} \subseteq I_{i_{(n_l-1)}} \subseteq \cdots \subseteq I_2, I_2 \subseteq I_{pq}, I_l \subseteq B_l$ for $2 \leq l \leq n_i, 3 \leq p \leq n_i, 2 \leq q \leq n_i - 1$.

**Case II**: $i < s$.

$$P(i,s) := \begin{pmatrix}
A_{is} & P_{1,s_2} & \cdots & P_{1,s_{n_1}} \\
A_{is} & P_{1,s_2} & \cdots & P_{1,s_{n_1}} \\
\vdots & \vdots & \ddots & \vdots \\
A_{is} & P_{1,s_2} & \cdots & P_{1,s_{n_1}}
\end{pmatrix}$$
Case III: $i > s$.

$$P(i,s) := \begin{pmatrix}
A_{is} & P_{1,i,2} & \cdots & P_{1,m_i}
A_{is} & P_{2,i,2} & \cdots & P_{2,m_i}
\vdots & \vdots & \ddots & \vdots \\
A_{is} & P_{m_i,i,2} & \cdots & P_{m_i,m_i}
\end{pmatrix}.$$

Suppose that $\sum_{l=1}^{m} P(i,l)P(l,j) \subseteq P(i,j)$. Then $P$ is a ring and called general block extension of $A$ with respect to the decomposition of identity.

General block extension of $A$ with respect to the decomposition of identity contains many classes of subrings of $M_n(A)$. In the following, we will give some examples.

**Example.** (1) In Definition 4.4, we assume that $m = 1$. Then

$$P = \begin{pmatrix}
A & I_2 & I_3 & \cdots & I_{n-1} & I_n \\
A & A_2 & I_3 & \cdots & I_{n-1} & I_n \\
A & I_{32} & A_3 & \cdots & I_{n-1} & I_n \\
A & I_{42} & I_{43} & A_4 & \cdots & I_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A & I_{n_2} & \cdots & \cdots & I_{n,n-1} & A_n
\end{pmatrix},$$

where $A$ is a ring with identity, $A_i$ are subrings of $A$ with the same identity for $2 \leq i \leq n$. $I_i, I_{ij}$ are ideals of $A$ for $2 \leq i \leq n, 2 \leq j \leq n - 1$. In particular, set $I_i = a\Omega, A_i = \Omega$ for $2 \leq i \leq n$ and $I_{ij} = \Omega$ for $2 \leq j < i \leq n$, where $\Omega$ is a local $R$-order and $a$ is a regular element in $\Omega$. Then $\Omega/a \cdot \Omega$ is local and $P$ is a QF-order with associated ideal $J = \omega \cdot P$, where

$$\omega = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
a & 0 & \cdots & 0 & 0
\end{pmatrix}_{n \times n}.$$

(2) In Definition 4.4, let $A$ be a basic algebra, and let $\{e_1, \cdots, e_m\}$ be a complete set of orthogonal primitive idempotents of $A$. set

$$P(i,s) = \begin{pmatrix}
P_{1,s,1} & P_{1,s,2} & \cdots & P_{1,s,m_i} \\
P_{2,s,1} & P_{2,s,2} & \cdots & P_{2,s,m_i} \\
\vdots & \vdots & \ddots & \vdots \\
P_{m_i,s,1} & P_{m_i,s,2} \cdots & P_{m_i,s,m_i}
\end{pmatrix} = \begin{pmatrix}
A_i & \cdots & A_i \\
\vdots & \ddots & \vdots \\
\text{rad}(A_i) & \cdots & A_i \\
A_{is} & \cdots & A_{is} \\
\vdots & \vdots & \ddots \\
A_{is} & \cdots & A_{is}
\end{pmatrix}_{(i = s)}$$

Then $P$ is called the block extension of $A$ which can be found in [17]. In particular, if $A$ is a basic QF-algebra, then $P$ is a basic Harada algebra (see [17]).

**Theorem 4.5.** Let $A$ be a ring with identity. Let $P$ be a general block extension of $A$ with respect to the decomposition of identity. Then

$$l.\dim(A) - 1 \leq l.\dim(P) \leq l.\dim(A) + \sum_{j=1}^{m} \sum_{i=2}^{n_j} l.\dim(A_j/I_{ji}) + \sum_{i=1}^{m} n_i - m.$$  

**Proof.** Denote by $e_{\sum_{l=1}^{i} n_l + j}$ the matrix which has $1_{A_{i+1}}$ in the $(\sum_{l=1}^{i} n_l + j)$-th position and zeros elsewhere for $1 \leq j \leq n_{i+1}, \ 0 \leq i \leq m - 1$. Thus $e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_{n_1+n_2}, \cdots, e_{\sum_{l=1}^{i} n_l + 1}, \cdots, e_{\sum_{l=1}^{m} n_l}$ are piecewise orthogonal idempotents in $P$ such that $1_p = e_1 + \cdots + e_{\sum_{l=1}^{m} n_l}.$
Set $\gamma = \text{End}_P(\bigoplus_{j=1}^{m} (\bigoplus_{i=1}^{n} L_{ij}) + \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{m} P_{ij}))$. Since $P$ is a subring of $\gamma$ with the same identity, $\gamma$ can be viewed as a $P$-module by restriction of the scalars of $\gamma$ to $P$. There is an exact sequence

$$0 \to P \xrightarrow{\lambda} \gamma \xrightarrow{\pi} L \to 0$$

in $\gamma$-Mod, where $\lambda$ is the inclusion map and $L$ is the cokernel of $\lambda$. By Theorem 2.2, two rings $P$ and $\text{End}_P((L_{ij})_{1 \leq i \leq n, 1 \leq j \leq m})$ are derived equivalent.

Note that $\text{End}_P((L_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}) \cong \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{m} P_{ij})$. Thus, we get an exact sequence

$$0 \to \bigoplus_{i=1}^{n} P_{i}^p \oplus \bigoplus_{j=1}^{m} L_{ij} \to \bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij} \to \bigoplus_{i=1}^{n} P_{i} \oplus \bigoplus_{j=1}^{m} L_{ij} \to 0.$$  

(1) $\text{Hom}_P(L_{ij}, L_{ij}) = 0$ for $2 \leq i \leq n_p, 2 \leq j \leq n_q, 1 \leq p < q \leq m$.

There are exact sequences

$$0 \to \bigoplus_{i=1}^{n} P_{i}^p \oplus \bigoplus_{j=1}^{m} L_{ij} \to \bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij} \to \bigoplus_{i=1}^{n} P_{i} \oplus \bigoplus_{j=1}^{m} L_{ij} \to 0. \quad (**)$$

Applying the functor $\text{Hom}_P(-, L_{ij})$ to $(**)$, we get an exact sequence

$$0 \to \text{Hom}_P(L_{ij}, L_{ij}) \to \text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij}) \to \text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij}) \to 0.$$  

By calculation, we have $\text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij}) = \text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij}) = 0$. Thus, $\text{Hom}_P(L_{ij}, L_{ij}) = 0$ for $2 \leq i \leq n_p, 2 \leq j \leq n_q, 1 \leq p < q \leq m$.

(2) $\text{Hom}_P(L_{ij}, L_{ij}) = 0$ for $2 \leq i \leq n_q, 1 \leq k \leq m$.

Apply the functor $\text{Hom}_P(-, L_{ij})$ to the exact sequence

$$0 \to \bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij} \to \bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij} \to \bigoplus_{i=1}^{n} P_{i} \oplus \bigoplus_{j=1}^{m} L_{ij} \to 0.$$  

We have an exact sequence

$$0 \to \text{Hom}_P(L_{ij}, L_{ij}) \to \text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij}) \to \text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij}) \to 0.$$  

Note that $\text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^p \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij})$ and $\text{Hom}_P(\bigoplus_{i=1}^{n} P_{i}^{p+1} \oplus \bigoplus_{j=1}^{m} L_{ij}, L_{ij})$ are both isomorphic to $A_k/I_k$ in $\mathbb{Z}$-Mod. It follows from Lemma 3.1 that $\text{Hom}_P(L_{ij}, L_{ij}) = 0$ for $2 \leq i < j < n_q, 1 \leq k \leq m$.

Thus $\text{End}_P((L_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}) \cong \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{m} P_{ij})$.

Let $\delta \in \text{End}_A(Ae_1 \oplus \cdots \oplus Ae_m)$.

By Lemma 4.3 and Lemma 4.1, we can get the conclusion.
Corollary 4.6. Let $\Lambda$ be as in Theorem 1.1. Then
\[ l\text{-Fin} \dim(A) - 1 \leq l\text{-Fin} \dim(A) \leq n + \sum_{i=2}^{n} l\text{-Fin} \dim(A_i/I_i) + l\text{-Fin} \dim(A). \]

**Proof.** It follows from Theorem 1.1 and Lemma 4.1. □

As a consequence of Theorem 4.5, we can get the following corollary. By the corollary, we can get a class of algebras which have finite finitistic dimension.

Corollary 4.7. Let $\Lambda$ be as in Theorem 1.1 and suppose that $A$ is an Artin algebra. Then
1. If $\text{fin} \dim(A) < \infty$ and $\text{fin} \dim(A_j/I_{ij}) < \infty$ for $2 \leq i \leq n, 1 \leq j \leq m$, then $\text{fin} \dim(P) < \infty$.
2. If $\text{fin} \dim(P) < \infty$, then $\text{fin} \dim(A) < \infty$.

In [17], K. Yamaura proved that any block extension of a basic QF-algebra is a basic left Harada algebra. And for any basic left Harada algebra $T$, there exists a basic QF-algebra $R$ such that $T$ is isomorphic to an upper staircase factor algebra of a block extension of $R$. By Theorem 4.5, we can get that the finitistic dimension is finite for the block extension of a QF-algebra. Thus, the finitistic dimension conjecture holds for the class of left Harada algebras.

Corollary 4.8. Suppose that $R$ is a QF-algebra and $P$ is the block extension of $R$. Then
\[ \text{fin} \dim(P) \leq \sum_{i=1}^{m} n_i - m < \infty. \]

**Proof.** Note that $\text{fin} \dim(R) = 0, \text{fin} \dim(A_j/\text{rad}A_j) = 0$ for $2 \leq i \leq n, 1 \leq j \leq m$. □

Proposition 4.9. Let $\Lambda$ as in Theorem 1.1. Then
\[ \max\{l\text{-gl} \dim(A_i/I_i), l\text{-gl} \dim(A), 2 \leq i \leq n\} - 1 \leq l\text{-gl} \dim(\Lambda) \leq \sum_{i=2,3,\ldots,n} l\text{-gl} \dim(A_i/I_i) + l\text{-gl} \dim(A) + n. \]

**Proof.** By Theorem 1.1, we can get that the two rings $\Lambda$ and $\Sigma$ are derived equivalent via a tilting module whose projective dimension is less or equal 1. It follows from 4.2 and Lemma 4.3. □

Corollary 4.10. Let $A$ be a ring with identity, $I_2, I_3, \ldots, I_n$ ideals of $A$. Set
\[ \Gamma = \begin{pmatrix}
A & I_2 & I_3 & \cdots & I_n \\
A & A & I_3 & \cdots & I_n \\
A & I_2 & A & I_4 & \cdots & I_n \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
A & I_2 & I_3 & A & \cdots & I_{n-1} \\
A & I_2 & I_3 & \cdots & I_{n-1} & A
\end{pmatrix} \]

Then \[ \max\{l\text{-gl} \dim(A_i/I_i) - 1, l\text{-gl} \dim(A) - 1, \text{proj} \dim(A_i), 2 \leq i \leq n\} \leq l\text{-gl} \dim(\Gamma) \leq \max\{l\text{-gl} \dim(A_i/I_i) + \text{proj} \dim(A_i) + 3, l\text{-gl} \dim(A) + 1, 2 \leq i, j \leq n\}. \]

**Proof.** By Corollary 3.3(2) and Lemma 4.3, we can get the conclusion. □

### 4.2 Tiled triangular rings

Before we turn to the second topic, we recall the definition of recollement, given by Beilinson, Bernstein and Deligne in their work on perverse sheaves.

**Definition 4.11.** Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be triangulated categories. Then a recollement of $\mathcal{D}$ relative to $\mathcal{D}'$ and $\mathcal{D}''$, diagrammatically expressed by
\[ \mathcal{D}' \longleftarrow \mathcal{D} \rightarrow \mathcal{D}'' \]
Lemma 4.12. [9] Let $A$ be a finite-dimensional algebra and assume that
provide a class of algebras which have finite finitistic dimension.

Proposition 4.14. isomorphism

Lemma 4.13. [16] Let $A$

Now, we turn to consider “tiled triangular ring,” i.e., rings of the form

The following lemma, showing how to construct a recollement, is useful in our proof.

Lemma 4.13. [16] Let $A, B$ and $C$ be algebras. The following assertions are equivalent:

1. $\mathcal{D}^+(A)$ is a recollement of $\mathcal{D}^-(C)$ and $\mathcal{D}^-(B)$.
2. There are two objects $P, Q \in \mathcal{D}^-(A)$ satisfying the following properties:
   a. There are isomorphism of algebras $C \cong \text{Hom}_{\mathcal{D}^-(A)}(P, P)$ and $B \cong \text{Hom}_{\mathcal{D}^-(A)}(Q, Q)$.
   b. $P$ is exceptional and isomorphic in $\mathcal{D}(A)$ to a bounded complex of finitely generated pro-
      jective $A$-modules.
   c. For every set $\Lambda$ and every non-zero integer $i$ we have $\text{Hom}_{\mathcal{D}^+(A)}(Q, Q^{|\Lambda|}[i]) = 0$, the canonical
      isomorphism $\text{Hom}_{\mathcal{D}^+(A)}(Q, Q^{|\Lambda|}) \to \text{Hom}_{\mathcal{D}^+(A)}(Q, Q^{|\Lambda|})$ is an isomorphism, and $Q$ is isomorphic in
      $\mathcal{D}(A)$ to a bounded complex of projective $A$-modules.
   d. $\text{Hom}_{\mathcal{D}^+(A)}(P, Q[i]) = 0$ for all $i \in \mathbb{Z}$.
   e. $P \oplus Q$ generates $\mathcal{D}(A)$.

Now, we turn to consider “tiled triangular ring,” i.e., rings of the form

$\Delta = \begin{pmatrix} A_{1,2} & \cdots & A_{1,n} \\ A_{2,2} & \cdots & A_{2,n} \\ \vdots & \ddots & \vdots \\ A_{n,n} & \cdots & A_{n,n} \end{pmatrix}$

for $I_{ij}$ ideals of $A$.

Now, let us prove the last result in this paper.

Proposition 4.14. Set

$\Phi = \begin{pmatrix} A_{1,2} & I_{1,3} & \cdots & I_{1,n-1} & I_{1,n} \\ A_{2,2} & I_{2,3} & \cdots & I_{2,n-1} & I_{2,n} \\ A_{3,2} & \cdots & I_{3,n-1} & I_{3,n} \\ \vdots & \ddots & \ddots & \vdots \\ A_{n,2} & \cdots & A_{n,n} \end{pmatrix}$

Suppose that $\Phi$ is an Artin algebra, $I_{ij}$ are ideals of $A$ for $1 \leq i < j \leq n$, $\text{proj.dim}(A/I_{i,j+1}/I_{i,j+1}/I_{i,j+1}) < \infty$ for $1 \leq i < j \leq n - 1$ and $\text{fin.dim}(A/I_{i,j+1}) < \infty$, $\text{fin.dim}(A) < \infty$
for $1 \leq i \leq n - 1$. Then $\text{fin.dim}(\Phi) < \infty$. 

12
Proof. Set $\Gamma = M_n(A)$. The exact sequence

$$0 \rightarrow \Phi \xrightarrow{\lambda} \Gamma \xrightarrow{\pi} L \rightarrow 0$$

in $\Phi$-$\text{Mod}$ is an add($\Gamma$)-split sequence. By the method which is similar to the ones in Theorem 1.1, we can prove that the two rings

$$\Phi = \begin{pmatrix} A & I_{12} & I_{13} & \cdots & I_{1,n} \\ A & A & I_{23} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ A & A & A & A & A \end{pmatrix} \quad \text{and} \quad \Sigma_1 = \begin{pmatrix} A/I_{12} & I_{23}/I_{13} & \cdots & I_{2n}/I_{1n} & 0 \\ A/I_{12} & A/I_{13} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ A/I_{12} & A/I_{13} & \cdots & A/I_{1n} & 0 \end{pmatrix}$$

are derived equivalent.

Set

$$\Phi_{n-1} = \begin{pmatrix} A/I_{12} & I_{23}/I_{13} & \cdots & I_{2n}/I_{1n} \\ A/I_{12} & A/I_{13} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A/I_{12} & A/I_{13} & \cdots & A/I_{1n} \end{pmatrix}$$

For simplicity, we denote $\Phi_n$ by

$$\Gamma = \begin{pmatrix} A_1 & J_{12} & J_{13} & \cdots & J_{1,n} \\ \vdots & A_2 & J_{23} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & J_{n-1,n} \\ A_1 & A_2 & A_3 & \cdots & A_n \end{pmatrix}$$

Claim: Suppose that $\text{proj.dim}(A_i,J_{ik}) < \infty, 2 \leq k \leq n$ and $\text{proj.dim}(A_i/J_{i-1,j},J_{ij}/J_{i-1,j}) < \infty, 2 \leq i < j \leq n - 1$. Then $\text{fin.dim}(\Gamma) < \infty$ if and only if $\text{fin.dim}(A_i) \leq \infty$ and $\text{fin.dim}(A_i/J_{i-1,j}) < \infty$, for $2 \leq i \leq n$.

Proof of Claim: Let $e$ be an idempotent of $\Gamma$ which has 1 in the $(1,1)$-th position and zeros elsewhere. By easy computation,

$$\Gamma e \Gamma = \begin{pmatrix} A_1 & J_{12} & J_{13} & \cdots & J_{1n} \\ A_1 & J_{12} & J_{13} & \cdots & J_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 & J_{12} & J_{13} & \cdots & J_{1n} \end{pmatrix}$$

Since $\Gamma e \Gamma$ is projective as right $\Gamma$-module, we have $\text{Tor}_i^\Gamma(\Gamma/\Gamma e \Gamma, \Gamma/\Gamma e \Gamma) = 0$ for $i > 0$. Then $\lambda : \Gamma \rightarrow \Gamma/\Gamma e \Gamma$ is a homological ring homomorphism. Then there is recollement:

$$\begin{array}{ccc}
\mathcal{D}(\text{Mod} - \Gamma/\Gamma e \Gamma) & \xrightarrow{i_*} & \mathcal{D}(\text{Mod} - \Gamma) \\
R\text{Hom}_\Gamma(\Gamma/\Gamma e \Gamma, -) & \xrightarrow{-\otimes_{\Gamma}^{\mathbb{L}} \Gamma e \Gamma} & \text{Tria}_{\mathcal{D}(\Gamma)}(\Gamma e \Gamma) \\
\end{array}$$

where $\text{Tria}_{\mathcal{D}(\Gamma)}(\Gamma e \Gamma)$ is the smallest full triangulated subcategory of $\mathcal{D}(\Gamma)$ containing $\Gamma e \Gamma$ and closed under small coproducts, $i_*$ is the inclusion functor, $- \otimes_{\Gamma}^{\mathbb{L}} \Gamma e \Gamma$ is the left derived functor of $- \otimes_{\Gamma} \Gamma e \Gamma$, $- \otimes_{\Gamma}^{\mathbb{L}} \Gamma e \Gamma$ is the left derived functor of $- \otimes_{\Gamma} \Gamma e \Gamma$ and $R\text{Hom}_\Gamma(\Gamma/\Gamma e \Gamma, -)$ is the right derived functor of $\text{Hom}_\Gamma(\Gamma/\Gamma e \Gamma, -)$. 
Note that $\Gamma/\text{Te}\Gamma$ has finite projective dimension as right $\Gamma$-module. Then, by Lemma 4.13 and [16, Corollary 3 and Example 6], there is a recollement:

$$
\begin{array}{ccc}
\mathcal{D}^- (\text{Mod} - \Gamma/\text{Te}\Gamma) & \xrightarrow{\mathbb{L}} & \mathcal{D}^- (\text{Mod} - \Gamma) \\
\mathbf{R}\text{Hom}_{\Gamma}(\Gamma/\text{Te}\Gamma, -) & \xrightarrow{\mathcal{L}} & \mathcal{D}^- (\text{C})
\end{array}
$$

where $C$ is the dg algebra $C_{dg}(\Gamma/\text{Te}\Gamma, \partial_e \Gamma)$ and $\partial_e \Gamma$ is an injective resolution of the right $\Gamma$-module $\Gamma e$. Note that $\Gamma e$ is isomorphic to $(e \Gamma)^n$ as right $\Gamma$-module. By [12, Theorem 9.2], there is a triangle equivalence between $\mathcal{D}^- (\text{C})$ and $\mathcal{D}^- (\text{Mod} - H^0(\text{C})) = \mathcal{D}^- (\text{Mod} - e \Gamma e)$. Hence there is a recollement:

$$
\begin{array}{ccc}
\mathcal{D}^- (\text{Mod} - \Gamma/\text{Te}\Gamma) & \xrightarrow{\mathbb{L}} & \mathcal{D}^- (\text{Mod} - \Gamma) \\
\mathbf{R}\text{Hom}_{\Gamma}(\Gamma/\text{Te}\Gamma, -) & \xrightarrow{\mathcal{L}} & \mathcal{D}^- (\text{Mod} - e \Gamma e)
\end{array}
$$

where $\iota_e$ is inclusion functor, $j^! = \mathbf{R}\text{Hom}_{\Gamma}(e \Gamma, -)$ is the right derived functor of $\text{Hom}_{\Gamma}(e \Gamma, -)$, $j_* = \mathbf{R}\text{Hom}_{\Gamma}(e \Gamma, -)$ is the right derived functor of $\text{Hom}_{\Gamma}(e \Gamma, -)$, $-\otimes^L_{\Gamma} \Gamma/\text{Te}\Gamma$ is the left derived functor of $-\otimes \Gamma/\text{Te}\Gamma$, $-\otimes_{\Gamma e} e \Gamma$ is the left derived functor of $-\otimes_{\Gamma e} e \Gamma$. Since $\text{proj.dim}(A, J_{ik}) < \infty, 2 \leq k \leq n$, we have that $\Gamma e$ and $\Gamma/\text{Te}\Gamma$ have finite projective dimension as left $\Gamma$-module. Thus, the functors $-\otimes^L_{\Gamma} \Gamma/\text{Te}\Gamma$ and $-\otimes^L_{\Gamma e} e \Gamma$ send complexes of bounded homology to complexes of bounded homology. Note that $e \Gamma$ and $\Gamma/\text{Te}\Gamma$ have finite projective dimension as right $\Gamma$-module. Then the functors $\mathbf{R}\text{Hom}_{\Gamma}(e \Gamma, -)$ and $\mathbf{R}\text{Hom}_{\Gamma}(\Gamma/\text{Te}\Gamma, -)$ restrict to the functors $\mathcal{D}^b(\Gamma/\text{Te}\Gamma) \to \mathcal{D}^b(e \Gamma e - \Gamma)$ and $\mathcal{D}^b(\Gamma/\text{Te}\Gamma) \to \mathcal{D}^b(e \Gamma e - \Gamma/\text{Te}\Gamma)$ respectively. Then, we can get a recollement:

$$
\begin{array}{ccc}
\mathcal{D}^b(\text{Mod} - \Gamma/\text{Te}\Gamma) & \xrightarrow{\mathbb{L}} & \mathcal{D}^b(\text{Mod} - \Gamma) \\
\mathbf{R}\text{Hom}_{\Gamma}(\Gamma/\text{Te}\Gamma, -) & \xrightarrow{\mathcal{L}} & \mathcal{D}^b(e \Gamma e - \Gamma)
\end{array}
$$

Since $\Gamma$ is Artin algebra and the functors, appearing in $(*)$, take finitely generated modules to finite generated modules, it follows that the following diagram is a recollement.

$$
\begin{array}{ccc}
\mathcal{D}^b(\Gamma/\text{Te}\Gamma) & \xrightarrow{\mathbb{L}} & \mathcal{D}^b(\Gamma) \\
\mathbf{R}\text{Hom}_{\Gamma}(\Gamma/\text{Te}\Gamma, -) & \xrightarrow{\mathcal{L}} & \mathcal{D}^b(e \Gamma e)
\end{array}
$$

By Lemma 4.12, $\text{fin.dim}(\Gamma) < \infty$ if and only if $\text{fin.dim}(A_1) < \infty$ and $\text{fin.dim}(\Gamma/\text{Te}\Gamma) < \infty$. Let $\Gamma_{n-1}$ denote $\Gamma/\text{Te}\Gamma$. By similar discussion, we can get $\text{fin.dim}(\Gamma) < \infty$ if and only if $\text{fin.dim}(A_1) < \infty$ and $\text{fin.dim}(A_1/A_{i-1}) < \infty$ for $2 \leq i \leq n$. By Claim, $\text{fin.dim}(\Phi_{n-1}) < \infty$ if and only if $\text{fin.dim}(A_i/A_{i+1}) < \infty$ for $1 \leq i \leq n - 1$. By Lemma 4.3, $\text{fin.dim}(A) \leq \text{fin.dim}(\Sigma_1) \leq \text{fin.dim}(A) + \text{fin.dim}(\Phi_{n-1}) + 1$. By Claim, $\text{fin.dim}(A_i/A_{i+1}) < \infty, 1 \leq i \leq n - 1$ imply $\text{fin.dim}(\Sigma_1) < \infty$. By Lemma 4.1, we can get that $\text{fin.dim}(\Sigma_1) < \infty$ implies $\text{fin.dim}(\Phi) < \infty$. If $\text{fin.dim}(A) < \infty$, then $\text{fin.dim}(\Phi) < \infty$.

The following is a typical case of Theorem 4.14.

**Corollary 4.15.** Set

$$
\Phi = \begin{pmatrix}
A & \text{rad}A & I_{1,3} & \cdots & I_{1,n} \\
A & A & \text{rad}A & \ddots & \vdots \\
A & A & A & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & I_{n-2,n} \\
A & \vdots & \ddots & \ddots & \text{rad}A \\
A & A & \cdots & A & A
\end{pmatrix}
$$

Suppose that $\Phi$ is Artin algebra, $I_{ij}$ are ideals of $A$ for $i, j = 1, 2, \cdots, n$. If $\text{fin.dim}(A) < \infty$, then $\text{fin.dim}(\Phi) < \infty$.

**Proof.** Let $I_{i,i+1} = \text{rad}A$ for $i = 1, 2, \cdots, n - 1$. □
5 Examples

In this section, we display several examples to illustrate our theorem.

**Example 1.** Let $A$ be a ring with ideal $I$. Let $\Gamma$ be the ring

$$
\Gamma = \begin{pmatrix}
A & I & I^2 & I^3 \\
A & A & I^2 & I^3 \\
A & I & A & I^3 \\
A & I & I & A
\end{pmatrix}
$$

By Proposition 4.9, we can get $\max\{0, l.\text{Fin.dim}(A) - 1\} \leq l.\text{Fin.dim}\Gamma \leq l.\text{Fin.dim}(A) + l.\text{Fin.dim}(A/I) + l.\text{Fin.dim}(A/I^2) + l.\text{Fin.dim}(A/I^3) + 4$.

**Example 2.** Let $A = k[x]/(x^n)$ for $n \geq 1$, and $I = \text{rad}(A)$, Let

$$
\Lambda = \begin{pmatrix}
A & I & I^2 & I^3 \\
A & A & I & I^3 \\
A & A & A & I \\
A & A & A & A
\end{pmatrix}
$$

Since $k[x]/(x^n)$ is representation-finite, the finitistic dimension of $k[x]/(x^n)$ is finite. By Corollary 4.15, $\text{fin.dim}(\Lambda) < \infty$.

**Example 3.** Let $A$ be a $k$-algebra given by the following quiver.

\[
\begin{array}{ccc}
\bullet & \overset{\alpha}{\rightarrow} & \bullet \\
\gamma & \overset{\beta}{\rightarrow} & \delta \\
\end{array}
\]

with relations $\{\alpha^3 = \beta\delta, \alpha\beta = 0, \delta\alpha = 0\}$.

Then $A$ can be represented as the following matrix form.

$$
A = \begin{pmatrix}
 k[\alpha]/(\alpha)^4 & k\beta \\
 k\delta & k[\delta\beta]/(\delta\beta)^2
\end{pmatrix}
$$

Suppose that $P(3,2)$ is the block extension of $A$. $P(3,2) =$

\[
\begin{pmatrix}
 k[\alpha]/(\alpha)^4 & (\alpha)/\alpha^4 & k[\alpha]/(\alpha)^4 & k[\alpha]/(\alpha)^4 & k\beta & k\beta \\
 (\alpha)/\alpha^4 & (\alpha)/\alpha^4 & (\alpha)/\alpha^4 & (\alpha)/\alpha^4 & k\beta & k\beta \\
 k\delta & k\delta & k\delta & k[\delta\beta]/(\delta\beta)^2 & k[\delta\beta]/(\delta\beta)^2 \\
 k\delta & k\delta & k\delta & (\delta\beta)/(\delta\beta)^2 & k[\delta\beta]/(\delta\beta)^2 \\
 k[\delta\beta]/(\delta\beta)^2 & k[\delta\beta]/(\delta\beta)^2 \\
\end{pmatrix}
\]

$P(3,2)$ can be described by given the quiver

\[
\begin{array}{ccc}
\bullet & \overset{\delta_{11}}{\rightarrow} & \bullet \\
\beta_{21} & \overset{\beta_{11}}{\rightarrow} & \beta_{12} \\
\end{array}
\]

with relations $\{e(\alpha^3) = \beta_{11}\delta_{11}\delta_{12}\beta_{11}\delta_{11}\delta_{12}\beta_{21} = \beta_{12}\delta_{21}\beta_{21} = e(\beta\delta), e(\alpha\beta) = \beta_{11}\delta_{11}\delta_{12}\beta_{12} = 0, e(\delta\alpha) = \beta_{21}\delta_{11}\delta_{12}\beta_{11} = 0\}$, where $e$ is the extension map defined in [22]. By Corollary 4.8, we have $\text{fin.dim}(P(3,2)) \leq 3$.

**Acknowledgement.** I want to express my gratitude to my supervisor professor Changchang Xi for encouragement and useful suggestions.
References

[1] M. AUSSLANDER AND I. REITEN AND S. O. SMALØ, *Representation Theory of Artin Algebras*. Cambridge University Press. 1995.

[2] A. A. BEILINSON, J. BERNSTEIN AND P. DELIGNE, Faisceaux pervers, in Analyse et topologie sur les espaces singuliers. Astérisque 100 (1982), 1-172.

[3] Y. P. CHEN, Derived equivalences in n-angulated categories. arXiv: 1107.2985.

[4] Y. P. CHEN, Constructions of derived equivalences between subrings. Preprint.

[5] K. N. COWLEY, On the global dimension of matrix subrings. Comm. Algebra 24 (1996), 857-871.

[6] D. DUGGER AND B. SHIPLEY, K-theory and derived equivalences. Duke Math. J. 124 (3)(2004), 587-617.

[7] R. M. FOSSUM, P. A. GRIFFITH AND I. REITEN, *Trivial Extensions of Abelian Categories*. Springer-Verlag, Berlin, 1975.

[8] D. HAPPEL, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*. Cambridge Univ. Press, Cambridge, 1988.

[9] D. HAPPEL, Reduction techniques for homological conjecture. Tsukuba J. Math 17 (1993), 115-130.

[10] W. HU AND C. C. XI, D-split sequences and derived equivalences. Adv. Math. 227 (2011), 292-318.

[11] Y. KATO, On derived equivalent coherent rings. Comm. Algebra 30 (2002), 4437-4454.

[12] B. KELLER, Deriving DG categories. Ann. Sci. École Norm. Sup. (4)27(1994), no. 1, 63-102.

[13] E. KIRKMAN AND J. KUZMANOVICH, Matrix subrings having finite global dimension. J. Algebra 109 (1987), 74-92.

[14] E. KIRKMAN, J. KUZMANOVICH AND L. SMALL, Finitistic dimension of noetherian rings. J. Algebra 147 (1992), 350-364.

[15] J. C. MCCONNEL AND J. C. BOBSON, *Noncommutative Noetherian Rings*. Wiley, Chichester, England, 1987.

[16] P. NICOLÁS AND M. SAORÍN, Lifting and restricting recollement data. Appl. categor. struct., DOI 10.1007/s10485-009-9198-z.

[17] K. OSHIRO, On Harada rings III. Math. J. Okayama Univ. 32 (1990), 111-118.

[18] S. Y. PAN AND C. C. XI, Finiteness of finitistic dimension is invariant of derived equivalences. J. Algebra 322 (2009), 21-24.

[19] J. RICKARD, Morita theory for derived categories. J. London Math. Soc. 39(1989), 436-456.

[20] J. RICKARD, Derived categories and stable equivalences. J. Pure Appl. Algebra 64(1989), 303-317.

[21] C. C. XI AND D. M. XU, The finitistic dimension conjecture and relatively projective modules. Preprint is available at: http://math.bnu.edu.cn/~ccxi/.

[22] K. YAMAURA, Quivers with relations of Harada algebras. Proc. Amer. Math. Soc. 138 (2010), 47-59.