A Liouville comparison principle for solutions of quasilinear differential inequalities

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Abstract

This work is devoted to the study of a Liouville comparison principle for entire weak solutions of quasilinear differential inequalities of the form

\[ A(u) + |u|^{q-1}u \leq A(v) + |v|^{q-1}v \]

on \( \mathbb{R}^n \), where \( n \geq 1, q > 0 \), and the operator \( A(w) \) belongs to a class of the so-called \( \alpha \)-monotone operators. Typical examples of such operators are the \( p \)-Laplacian and its well-known modifications for \( 1 < p \leq 2 \). The results improve and supplement those in [1].

1 Introduction and Definitions

This work is devoted to the study of a Liouville comparison principle for entire weak solutions of quasilinear differential inequalities of the form

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(1)

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\[ \Delta_p(w) := \text{div} \left( |\nabla w|^{p-2} \nabla w \right) \]

(2)

for \( 1 < p \leq 2 \) and its well-known modification (see, e.g., [2, p. 155]),

\[ \overline{\Delta}_p(w) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |\partial w|^{p-2} \frac{\partial w}{\partial x_i} \right) \]

(3)
for \( n \geq 2 \) and \( 1 < p \leq 2 \). The results obtained in this work improve and supplement those in [1]. To prove these results we further develop the approach that was proposed to solve similar problems in wide classes of partial differential equations and inequalities in [3] and [4].

In what follows, \( n \geq 1 \) is a natural number, \( q > 0 \) and \( \alpha > 1 \) are real numbers. Let \( A(w) \) be a differential operator given formally by the relation

\[
A(w) = \sum_{i=1}^{n} \frac{d}{dx_i} A_i(x, \nabla w). \tag{4}
\]

Assume that the functions \( A_i(x, \xi) \), \( i = 1, \ldots, n \), satisfy the Carathéodory conditions on \( \mathbb{R}^n \times \mathbb{R}^n \). Namely, they are continuous in \( \xi \) at almost all \( x \in \mathbb{R}^n \) and measurable in \( x \) at all \( \xi \in \mathbb{R}^n \).

**Definition 1** Let \( n \geq 1 \) and \( \alpha > 1 \). The operator \( A(w) \), given by (4), is said to be \( \alpha \)-monotone if \( A_i(x, 0) = 0 \), \( i = 1, \ldots, n \), at almost all \( x \in \mathbb{R}^n \), and there exists a positive constant \( K \) such that

\[
0 \leq \sum_{i=1}^{n} (\xi_i^1 - \xi_i^2) (A_i(x, \xi^1) - A_i(x, \xi^2)) \tag{5}
\]

and

\[
\left( \sum_{i=1}^{n} (A_i(x, \xi^1) - A_i(x, \xi^2))^2 \right)^{\alpha/2} \leq K \left( \sum_{i=1}^{n} (\xi_i^1 - \xi_i^2) (A_i(x, \xi^1) - A_i(x, \xi^2)) \right)^{\alpha-1} \tag{6}
\]

hold at all \( \xi^1, \xi^2 \in \mathbb{R}^n \) and almost all \( x \in \mathbb{R}^n \).

Note that the condition (5) is the well-known monotonicity condition in PDE theory, while the condition (6) is the proper \( \alpha \)-monotonicity condition for differential operators, considered first in [4]; see also [5]. Ibidem one can find algebraic inequalities from which it follows immediately that the \( p \)-Laplacian \( \Delta_p \) and its modification \( \tilde{\Delta}_p \), as well as the weighted \( p \)-Laplacian

\[
\nabla_p(w) := \text{div} \left( a(x) |\nabla w|^{p-2} \nabla w \right), \tag{7}
\]
(see, e.g., [6, p. 55]), with any measurable non-negative uniformly bounded function \( a(x) \) on \( \mathbb{R}^n \), satisfy the \( \alpha \)-monotonicity condition for \( \alpha = p \) and \( 1 < p \leq 2 \). Note also that the \( \alpha \)-monotonicity condition (6) in the case \( \xi^2 = 0 \) is in turn a special case of the very general growth condition for quasilinear differential operators, considered first in [7].

**Definition 2** Let \( n \geq 1 \), \( \alpha > 1 \) and \( q > 0 \), and let the operator \( A(w) \) be \( \alpha \)-monotone. By an entire weak solution of the inequality (1) we understand a pair of functions \((u, v)\) measurable on \( \mathbb{R}^n \) which belong to the space \( W^{1,\text{loc}}_\alpha(\mathbb{R}^n) \cap L^{q,\text{loc}}(\mathbb{R}^n) \) and satisfy the integral inequality

\[
\int_{\mathbb{R}^n} \left[ \sum_{i=1}^n \varphi_{x_i} A_i(x, \nabla u) - |u|^{q-1} u \varphi \right] dx \geq \int_{\mathbb{R}^n} \left[ \sum_{i=1}^n \varphi_{x_i} A_i(x, \nabla v) - |v|^{q-1} v \varphi \right] dx
\]

for every non-negative function \( \varphi \in C^{\infty}(\mathbb{R}^n) \) with compact support.

2 Results

**Theorem 1** Let \( n = 1, 2 \geq \alpha > 1 \) and \( q > 0 \). Let the operator \( A(w) \) be \( \alpha \)-monotone, and let \((u, v)\) be an entire weak solution of the inequality (1) on \( \mathbb{R}^n \) such that \( u(x) \geq v(x) \). Then \( u(x) = v(x) \) on \( \mathbb{R}^n \).

**Theorem 2** Let \( n = 2, \alpha = 2 \) and \( q > 0 \). Let the operator \( A(w) \) be \( \alpha \)-monotone, and let \((u, v)\) be an entire weak solution of the inequality (1) on \( \mathbb{R}^n \) such that \( u(x) \geq v(x) \). Then \( u(x) = v(x) \) on \( \mathbb{R}^n \).

**Theorem 3** Let \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, \frac{n(n-1)}{n-\alpha} \geq q > \alpha - 1 \) and \( q \geq 1 \). Let the operator \( A(w) \) be \( \alpha \)-monotone, and let \((u, v)\) be an entire weak solution of the inequality (1) on \( \mathbb{R}^n \) such that \( u(x) \geq v(x) \). Then \( u(x) = v(x) \) on \( \mathbb{R}^n \).

**Theorem 4** Let \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, q > \frac{n(n-1)}{n-\alpha} \) and \( q \geq 1 \), and let the operator \( A(w) \) be \( \alpha \)-monotone. Then there exists no entire weak solution \((u, v)\) of the inequality (1) on \( \mathbb{R}^n \) such that \( u(x) \geq v(x) \) and the relation

\[
\limsup_{R \to +\infty} R^{-\frac{n+\alpha(q-\nu)}{q-\nu+1}} \int_{|x|<R} (u-v)^{q-\nu} dx = +\infty
\]

holds with any given \( \nu \in (0, \alpha - 1] \).
Remark 1 If, in Theorem 4, \( \nu = \alpha - 1 \), then the relation (9) has the form

\[
\limsup_{R \to +\infty} R^{-n+\alpha} \int_{0 < |x| < R} (u - v)^{q-\alpha+1} \, dx = +\infty. \tag{10}
\]

To illustrate the sharpness of Theorem 4 we give Example 1.

Example 1 For \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, q > \frac{n(\alpha-1)}{n-\alpha}, q \geq 1 \) and a suitable constant \( c > 0 \), the pair \((u, v)\) of the functions

\[
u(x) = c(1 + |x|^{\alpha/(\alpha-1)})^{(1-\alpha)/(q-\alpha+1)} \tag{11}\]

and

\[
u(x) = 0 \tag{12}\]

is an entire weak solution of the inequality (1) on \( \mathbb{R}^n \) with \( A(w) = \Delta_{\alpha}(w) \) or \( A(w) = \tilde{\Delta}_{\alpha}(w), \) respectively, such that \( u(x) \geq \nu(x) \) and, for any given \( \nu \in (0, \alpha - 1] \), the relation

\[
\limsup_{R \to +\infty} R^{-n+\frac{\alpha(q-\nu)}{q-\alpha+1}} \int_{0 < |x| < R} (u - v)^{q-\nu} \, dx = C_1, \tag{13}\]

with \( C_1 \) a certain positive constant, holds.

The two following statements are simple corollaries of Theorem 4.

Theorem 5 Let \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, q > \frac{n(\alpha-1)}{n-\alpha} \) and \( q \geq 1 \), and let the operator \( A(w) \) be \( \alpha \)-monotone. Then, for any given constants \( c > 0 \) and \( \delta > 0 \), there exists no entire weak solution \((u, v)\) of the inequality (1) on \( \mathbb{R}^n \) such that \( u(x) \geq \nu(x) + c(1 + |x|^{\alpha/(\alpha-1)})^{(1-\alpha)/(q-\alpha+1)+\delta}. \)

Theorem 6 Let \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, q > \frac{n(\alpha-1)}{n-\alpha} \) and \( q \geq 1 \), and let the operator \( A(w) \) be \( \alpha \)-monotone. Then, for any given constant \( c > 0 \) there exists no entire weak solution \((u, v)\) of the inequality (1) on \( \mathbb{R}^n \) such that \( u(x) \geq \nu(x) + c. \)

Theorem 7 Let \( n \geq 2, 2 \geq \alpha > 1, n > \alpha \) and \( 1 \geq q > 0 \), and let the operator \( A(w) \) be \( \alpha \)-monotone. Then there exists no entire weak solution \((u, v)\) of the inequality (1) on \( \mathbb{R}^n \) such that \( u(x) \geq \nu(x) \) and the relation

\[
\limsup_{R \to +\infty} R^{\alpha-n} \int_{\{0 < |x| < R \} \cap \{x: u(x) \not= \nu(x)\}} (|u|^{q-1} u - |v|^{q-1} v)(u - v)^{1-\alpha} \, dx = +\infty \tag{14}\]

holds.
Remark 2 If, in Theorem 7, \( q = 1 \), then the relation (14) has the form
\[
\limsup_{R \to +\infty} R^{\alpha-n} \int_{\{ |x| < R \} \cap \{ x: u(x) \neq v(x) \}} (u - v)^{2-\alpha} dx = +\infty. \tag{15}
\]

To illustrate the sharpness of Theorem 7 we give three examples.

Example 2 For \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, \alpha - 1 > q > 0, 0 < \mu < \frac{n-\alpha}{n}, \lambda = (\alpha - 1)/(\alpha - 1 - q), \) and a suitable constant \( c > 0 \), the pair \((u, v)\) of the functions
\[
u(x) = c(1 + |x|^\alpha/(\alpha-1))^\lambda + (1 + |x|^\alpha/(\alpha-1))^{-\mu} \tag{16}
\]
and
\[
v(x) = c(1 + |x|^\alpha/(\alpha-1))^\lambda \tag{17}
\]
is an entire weak solution of the inequality (1) on \( \mathbb{R}^n \) with \( A(w) = \Delta_\alpha(w) \) or \( A(w) = \tilde{\Delta}_\alpha(w) \), respectively, such that \( u(x) \geq v(x) \) and the relation
\[
\limsup_{R \to +\infty} R^{\alpha-n} \int_{\{ |x| < R \} \cap \{ x: u(x) \neq v(x) \}} (|u|^{q-1}u - |v|^{q-1}v)(u - v)^{1-\alpha} dx = C_2, \tag{18}
\]
with \( C_2 \) a certain positive constant, holds.

Example 3 For \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, \alpha - 1 > q > 0, 0 < \mu < \frac{n-\alpha}{n}, \lambda > (\alpha - 1)/(\alpha - 1 - q), \) and a suitable constant \( c > 0 \), the pair \((u, v)\) of the functions (16) and (17) is an entire weak solution of the inequality (1) on \( \mathbb{R}^n \) with \( A(w) = \Delta_\alpha(w) \) or \( A(w) = \tilde{\Delta}_\alpha(w) \), respectively, such that \( u(x) \geq v(x) \) and the relation
\[
\limsup_{R \to +\infty} R^{\alpha-n} \int_{\{ |x| < R \} \cap \{ x: u(x) \neq v(x) \}} (|u|^{q-1}u - |v|^{q-1}v)(u - v)^{1-\alpha} dx = 0 \tag{19}
\]
holds.

Example 4 To illustrate the sharpness of Theorem 7 in the case when \( 1 \geq q > 0 \) and \( q > \frac{n(\alpha-1)}{n-\alpha} \), we note that for \( n \geq 2, 2 \geq \alpha > 1, n > \alpha, 1 \geq q > 0, q > \frac{n(\alpha-1)}{n-\alpha} \), and a suitable constant \( c > 0 \), the pair \((u, v)\) of the functions (11) and (12) is an entire weak solution of the inequality (1) on \( \mathbb{R}^n \) with
\[ A(w) = \Delta_\alpha(w) \text{ or } A(w) = \tilde{\Delta}_\alpha(w), \] respectively, such that \( u(x) \geq v(x) \) and the relation

\[
\limsup_{R \to +\infty} \int_{\{ |x| < R \} \cap \{ x: u(x) \neq v(x) \}} (|u|^{q-1}u - |v|^{q-1}v)(u - v)^{1-\alpha} dx = C_3, \quad (20)
\]

with \( C_3 \) a certain positive constant, holds.

**Remark 3** If, in Example 4, \( q = 1 \), then the relation \( (20) \) has the form

\[
\limsup_{R \to +\infty} \int_{\{ |x| < R \}} (u - v)^{2-\alpha} dx = C_3. \quad (21)
\]

The following statement, which is an immediate corollary of Theorem 7 and Remark 2, supplements the results of Theorem 3 for the case \( q = \alpha - 1 \), which is possible in that theorem only if \( \alpha = 2 \) and \( q = 1 \).

**Theorem 8** Let \( n \geq 3 \), \( \alpha = 2 \) and \( q = 1 \), and let the operator \( A(w) \) be \( \alpha \)-monotone. Then there exists no entire weak solution \((u, v)\) of the inequality \( (1) \) on \( \mathbb{R}^n \) such that \( u(x) > v(x) \).

**References**

[1] V. V. Kurta, *On a Liouville-type comparison principle for solutions of quasilinear elliptic inequalities*, C. R. Math. Acad. Sci. Paris, 336 (2003), no. 11, 897–900.

[2] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[3] V. A. Kondrat'ev and E. M. Landis, *Semilinear second-order equations with nonnegative characteristic form*, Mat. Zametki 44 (1988), 457–468.

[4] V. V. Kurta, *Some problems of qualitative theory for nonlinear second-order equations*, Doctoral Dissert., Steklov Math. Inst., Moscow, 1994.
[5] V. V. Kurta, *Comparison principle for solutions of parabolic inequalities*, C. R. Acad. Sci. Paris, Série I, 322 (1996), 1175–1180.

[6] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, The Clarendon Press, Oxford University Press, New York, 1993.

[7] V. M. Miklyukov, *Capacity and a generalized maximum principle for quasilinear equations of elliptic type*, Dokl. Akad. Nauk SSSR 250 (1980), 1318–1320.

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