CUP-PRODUCT FOR EQUIVARIANT LEIBNIZ COHOMOLOGY AND ZINBIEL ALGEBRAS

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Abstract. We study finite group actions on Leibniz algebras, define equivariant cohomology groups associated to such actions. We show that there exists a cup-product operation on this graded cohomology groups which makes it a graded zinbiel algebra.

1. Introduction

In [10], J.-L. Loday introduced some new types of algebras along with their (co)homologies and studied the associated operads. Leibniz algebras and their Koszul duals, zinbiel algebras are examples of such algebras. Let \textbf{Leib} be the category of Leibniz algebras over a fixed field \( K \). Given any Leibniz algebra \( \mathfrak{g} \), J.-L. Loday [9] introduced a cup-product operations

\[ \cup : H^{L^p}(\mathfrak{g}; A) \times H^{L^q}(\mathfrak{g}; A) \to H^{L^{p+q}}(\mathfrak{g}; A) \]

on the graded Leibniz cohomology \( H^{L^*}(\mathfrak{g}; A) \) groups with coefficients in a commutative, associative algebra \( A \). This product is neither associative nor commutative, but satisfies the formula

\[ ([a] \cup [b]) \cup [c] = [a] \cup ([b] \cup [c]) + (-1)^{|b||c|} [a] \cup ([c] \cup [b]), \]

which is, the defining relation of a zinbiel algebra. Thus, the author proved that \( H^{L^*}(\mathfrak{g}; A) \) is a graded zinbiel algebra. The aim of this paper is to study finite group actions on Leibniz algebras. Let \( G \) be a finite group and \( \mathfrak{g} \) be a Leibniz algebra equipped with a given action of \( G \). We discuss examples of such actions and introduce equivariant cohomology groups of a Leibniz algebra \( \mathfrak{g} \) equipped with an action of a finite group \( G \), along the line of Bredon cohomology of a \( G \)-space [2]. We introduce a cup-product operation in the equivariant context and prove that for a Leibniz algebra \( \mathfrak{g} \) equipped with an action of \( G \), equivariant graded Leibniz cohomology groups also admit a graded zinbiel algebra structure.

2. Preliminaries

In this section, we recall some definitions, notations and results from [8],[9],[10].

In [9], J.-L. Loday observed that for a Lie algebra \( \mathfrak{g} \) if one replaces the exterior product \( \wedge \) by the tensor product \( \otimes \) in the classical formula for the boundary map \( d \) of the Chevalley-Eilenberg complex and modifies the boundary map \( d \) so as to put the commutator \([x, x_j]\) at the place \( i \) when \( i < j \) (see 10.6.2.1, [8]), then one obtains a new complex \((T\mathfrak{g}, d)\). The only relation that is used to get \( d^2 = 0 \) is

\[ [x, [y, z]] = [[x, y], z] - [[x, z], y] \text{ for } x, y, z \in \mathfrak{g}. \]

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Dualizing this complex one gets the Leibniz cohomology of the Lie algebra \( g \). Thus, Leibniz cohomology is defined for a larger class of algebras: the Leibniz algebras. More explicitly, it is defined as follows.

2.1. Definition. Let \( K \) be a field. A Leibniz algebra is a vector space \( g \) over \( K \), equipped with a bracket operation, which is \( K \)-bilinear and satisfies the Leibniz identity:

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad \text{for } x, y, z \in g.
\]

A graded Leibniz algebra is a graded \( K \)-vector space \( g = \{g_i\}_{i \geq 0} \), equipped with a graded bracket operation of degree 0 which is \( K \)-bilinear and satisfies the graded Leibniz identity:

\[
[x, [y, z]] = [[x, y], z] - (-1)^{|y||z|}[[x, z], y] \quad \text{for homogeneous elements } x, y, z \in g.
\]

Any Lie algebra is automatically a Leibniz algebra, as in the presence of skew symmetry, the Jacobi identity is equivalent to the Leibniz identity.

2.2. Example. Let \((g, d)\) be a differential Lie algebra with the Lie bracket \([, ,]\). Then \( g \) is a Leibniz algebra with the bracket operation \([x, y]_d := [x, dy]\). The new bracket on \( g \) is called the derived bracket.

2.3. Example. Consider a three dimensional vector space \( g \) spanned by \( \{e_1, e_2, e_3\} \) over \( C \). Define a bilinear map \([, ,] : g \times g \rightarrow g\) by \([e_1, e_3] = e_2\) and \([e_3, e_1] = e_1\), all other products of basis elements being 0. Then \((g, [, ,])\) is a Leibniz algebra over \( C \) of dimension 3. The Leibniz algebra \( g \) is nilpotent and is denoted by \( \lambda_6 \) in the classification of three dimensional nilpotent Leibniz algebras, see [17].

2.4. Definition. A morphism \( \phi : (g_1, [, ,]) \rightarrow (g_2, [, ,]) \) of Leibniz algebras is a Linear map which preserves the brackets, that is,

\[
\phi([x, y]_1) = [\phi(x), \phi(y)]_2, \quad x, y \in g_1.
\]

Recall that the homology \( HL_*(g) \) of a Leibniz algebra is defined as follows. To any Leibniz algebra \( g \) there is an associated chain complex

\[
CL_n(g) : \cdots \rightarrow g^{\otimes n} \rightarrow g^{\otimes (n-1)} \rightarrow \cdots \rightarrow g^{\otimes 2} \rightarrow g
\]

where

\[
d(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} (-1)^i (x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, x_n).
\]

The map \( d \) satisfy \( d^2 = 0 \) (16). The homology groups of this complex are denoted by \( HL_n(g) \), \( n \geq 1 \).

Next, recall that the Leibniz cohomology \( HL^*(g; A) \) of a Leibniz algebra \( g \) with coefficients in an associative commutative \( K \)-algebra \( A \) is defined as follows.

Set \( CL^*(g; A) = \text{Hom}_K(g^{\otimes n}, A) \). Then define

\[
\delta : CL^*(g; A) \rightarrow CL^{n+1}(g; A)
\]

by \( \delta(c) = c \circ d, \quad c \in CL^n(g; A) \), where \( d : g^{\otimes (n+1)} \rightarrow g^{\otimes n} \) is the boundary map (21). Explicitly, for any \( c \in CL^n(g; A) \) and \((x_1, \ldots, x_{n+1}) \in g^{\otimes (n+1)}\), \( \delta(c)(x_1, \ldots, x_{n+1}) \) is given by the expression

\[
\sum_{1 \leq i < j \leq n+1} (-1)^i c(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, x_{n+1}).
\]

Clearly, \( \delta^2 = 0 \) as \( d^2 = 0 \), and therefore, \((CL^*(g; A), \delta)\) is a cochain complex. Its homology groups are called Leibniz cohomology groups of \( g \) with coefficients in \( A \) and denoted by \( HL^*(g; A) \).
3. GROUP ACTIONS ON LEIBNIZ ALGEBRAS

The purpose of this section is to introduce finite group actions on Leibniz algebras and provide examples of group actions.

3.1. Definition. Let \( g \) be a Leibniz algebra and \( G \) be a finite group. The group \( G \) is said to act from the left if there exists a function \( \phi : G \times g \rightarrow g, \ (g, x) \mapsto \phi(g, x) = gx \) satisfying the following conditions.

1. For each \( g \in G \) the map \( x \mapsto gx \), denoted by \( \psi_g \), is linear.
2. \( ex = x \) for all \( x \in g \), where \( e \in G \) is the group identity.
3. \( g_1(g_2x) = (g_1g_2)x \) for all \( g_1, g_2 \in G \) and \( x \in g \).
4. \( g[x, y] = [gx, gy] \) for all \( g \in G \) and \( x, y \in g \).

When \( g = \{g_i\}_{i \geq 0} \) is a graded Leibniz algebra, we further assume that for each \( g \in G \) the map \( \psi_g \) is graded linear of degree 0.

The following is an equivalent formulation of the above definition.

3.2. Proposition. Let \( G \) be a finite group and \( g \) be a Leibniz algebra. Then \( G \) acts on \( g \) if and only if there exists a group homomorphism \( \psi : G \rightarrow \text{Iso}_{\text{Leib}}(g, g), g \mapsto \psi(g) = \psi_g \) from the group \( G \) to the group of Leibniz algebra isomorphisms from \( g \) to \( g \), where \( \psi_g(x) = gx \) is the left translation by \( g \).

3.3. Remark. Let \( K[G] \) be the group ring. If a Leibniz algebra \( g \) is equipped with an action of \( G \) then \( g \) may be viewed as a \( K[G] \)-module.

Observe that if \( g \) is a Leibniz algebra equipped with an action of a group \( G \) as above, then for every subgroup \( H \subset G \), the \( H \)-fixed point set \( g^H \) is defined by

\[ g^H = \{ x \in g : hx = x \text{ for all } h \in H \}. \]

Clearly, for every subgroup \( H \subset G \), \( g^H \) is a sub Leibniz algebra of \( g \). Moreover, note that if \( H, K \) are subgroups of \( G \) with \( g^{-1}Hg \subset K \), \( g \in G \), then the Leibniz algebra homomorphism \( \psi_g \) maps \( g^K \) to \( g^H \).

3.4. Example. Let \( V \) be \( K \)-module which is a representation space of a finite group \( G \). On \( \tilde{T}(V) = V \oplus V^2 \oplus \cdots \oplus V^n \oplus \cdots \) there is a unique bracket that makes it into a Leibniz algebra and verifies

\[ v_1 \otimes v_2 \otimes \cdots \otimes v_n = [\cdots[[v_1, v_2], v_3], \ldots, v_n] \text{ for } v_i \in V \text{ and } i = 1, \ldots, n. \]

This is the free Leibniz algebra over the \( K \)-module \( V \). The linear action of \( G \) on \( V \) extends naturally to an action on \( \tilde{T}(V) \) and the bracket defined above satisfies the conditions of Definition 3.1. Thus, \( (G, \tilde{T}(V)) \) is an action of \( G \) on the free Leibniz algebra \( \tilde{T}(V) \).

3.5. Example. Let \( (g, d) \) be a differential Lie algebra with the Lie bracket \([ , ,]\). Assume that a finite group \( G \) acts linearly on \( g \) such that

1. \( [gx, gy] = [g, x]y \) for all \( g \in G \) and \( x, y \in g \),
Then, the group $G$ acts on the Leibniz algebra $(g, [\cdot, \cdot], d)$, where $[x, y]_d := [x, dy]$ is the derived bracket (cf. Example 2.2).

Our next example is based on a specific case of the above Example.

3.6. Example. Let $V$ be a vector space over a field $\mathbb{K}$. Assume that a finite group $G$ acts linearly on $V$. Let

$$C^k(V) = \{ \alpha : V \times \cdots \times V \to V \mid \alpha \text{ is linear in each argument} \}$$

be the set of all $\mathbb{K}$-multilinear maps on $V$. For $\alpha \in C^k(V)$ and $\beta \in C^l(V)$ let $\alpha \circ \beta \in C^{k+l-1}(V)$ be the element defined by

$$\alpha \circ \beta(x_1, \ldots, x_{k+l-1}) = \sum_{i=1}^{k} (-1)^{(i-1)(l-1)} \alpha(x_1, \ldots, x_{i-1}, \beta(x_i, \ldots, x_{i+l-1}), x_{i+1}, \ldots, x_{k+l-1})$$

and let $[\alpha, \beta] \in C^{k+l-1}(V)$ be the Gerstenhaber bracket defined by:

$$[\alpha, \beta] := \alpha \circ \beta - (-1)^{(k-1)(l-1)} \beta \circ \alpha.$$

Let $C^*(V) = \bigoplus_k C^k(V)$. Then, if we declare an element of $C^k(V)$ to have degree $k-1$, the Gerstenhaber bracket defined a structure of graded Lie algebra on $C^*(V) = \bigoplus_k C^k(V)$.

Next observe that the linear action of $G$ on $V$ induces an action on $C^k(V)$ for each $k$ and is given by $(g, \alpha) \mapsto g(\alpha)$, where

$$g(\alpha)(x_1, \ldots, x_k) := g(\alpha(g^{-1}x_1, \ldots, g^{-1}x_k)).$$

Thus, an element $\alpha \in C^k(V)$ is invariant with respect to the above action if and only if $\alpha : V \times \cdots \times V \to V$ is an equivariant multilinear map where $G$ acts component wise on any product of $V$. Moreover, it is straightforward to verify that the above action on the graded vector space $C^*(V)$ satisfies $[g\alpha, g\beta] = g[\alpha, \beta]$.

Assume that there is an element $\alpha \in C^2(V)$ which is invariant with respect to the above action and defines an associative algebra structure on $V$. The last assertion is equivalent to assume that $[\alpha, \alpha] = 0$. For such an element $\alpha \in C^2(V)$ we define a map $d_\alpha : C^*(V) \to C^{*+1}(V)$ by $d_\alpha(\beta) = [\alpha, \beta]$. Then, $d_\alpha : C^*(V) \to C^{*+1}(V)$ is equivariant and by graded Jacobi identity $d_\alpha^2 = 0$. Thus, $(C^*(V), [\cdot, \cdot], d_\alpha)$ is a differential graded Lie algebra equipped with a linear action of $G$ such that $[g\alpha, g\beta] = g[\alpha, \beta]$ for all $g \in G, \alpha \in C^k(V)$ and $\beta \in C^l(V)$. On $C^*(V)$ we consider the derived bracket

$$[\beta, \gamma]_{d_\alpha} := [\beta, d_\alpha \gamma].$$

Then, $(C^*(V), [\cdot, \cdot], d_\alpha)$ is a graded Leibniz algebra equipped with an action of the group $G$.

Finally, we discuss two geometric examples.

3.7. Example. Recall that every Lie algebra, in particular, is a Leibniz algebra as the in the presence of skew symmetry, the Leibniz identity reduces to the Jacobi identity. Let $M$ be a smooth manifold equipped with a smooth action of $G$. For each $g \in G$, let $l_g : M \to M$ denote the left translation by $g$, that is, $l_g(x) = gx, x \in M$. Consider the Lie algebra $(\chi(M), [\cdot, \cdot], d)$ of vector fields on $M$, where for vector fields $X, Y \in \chi(M)$ their Lie bracket $[X, Y]$ is a vector field which acts on smooth functions $f \in C^\infty(M)$ by $[X, Y](f) := X(Yf) - Y(Xf)$. Define $G \times \chi(M) \to \chi(M)$ by $(g, X) \mapsto (l_g)_*(X)$, where $(l_g)_*(X)$ is the push forward by $l_g$. Explicitly, for $f \in C^\infty(M)$, $(l_g)_*(X)(f) = X(f \circ l_g)$. Then,
it is easy to check that \((l_g)_*([X,Y]) = [(l_g)_*(X),(l_g)_*(Y)]\) and hence \((G,\chi(M))\) is an action of \(G\) on the Lie algebra \(\chi(M)\).

The following discussion is a prelude to our next example.

3.8. **Definition.** Let \(G\) be a finite group and \(M\) is a smooth \(G\)-manifold. Then for any \(k \geq 0\), there is an action of \(G\) on the space of \(k\)-forms \(\Omega^k(M)\) on \(M\), given by

\[
G \times \Omega^k(M) \to \Omega^k(M), \quad (g, \alpha)(x) := (dl_{g^{-1}})^* \alpha(g^{-1}x),
\]

for \(\alpha \in \Omega^k(M)\) and \(x \in M\). Moreover, there is an action of \(G\) on the space of vector fields \(\mathcal{X}(M)\) on \(M\), namely,

\[
G \times \mathcal{X}(M) \to \mathcal{X}(M), \quad (g, X)(x) := (dl_g) X(g^{-1}x),
\]

for \(X \in \mathcal{X}(M)\) and \(x \in M\). This action is the same as the action of \(G\) on \(\chi(M)\) by push forward by \(l_g\) as discussed in the above example.

3.9. **Lemma.** The contraction operator and the de Rham differential operator satisfies the following properties. For any \(\alpha \in \Omega^k(M)\), \(X \in \mathcal{X}(M)\) and \(g \in G\),

(i) \(i_{(g,X)}(g,\alpha) = (g,i_X \alpha)\)

(ii) \(d(g,\alpha) = (g,da)\)

(iii) \(\mathcal{L}_{(g,X)}(g,\alpha) = (g,\mathcal{L}_X \alpha)\)

(iv) If \(\Pi\) is a \(G\)-invariant \((k+1)\)-vector field, that is, \(\Pi(gx) = (dl_g) \Pi(x)\), for all \(g \in G\) and \(x \in M\), then \(\Pi^2(g,\alpha) = (g,\Pi \alpha)\).

**Proof.** (i) For any \(x \in M\) and \(X_1, \ldots, X_{k-1} \in T_x M\), we have

\[
(i_{(g,X)}(g,\alpha))(X_1, \ldots, X_{k-1}) = (g,\alpha)(x) \left( (g, X)(x), X_1, \ldots, X_{k-1} \right)
\]

\[
= (dl_{g^{-1}})^* \alpha(g^{-1}x) \left( (dl_g)X(g^{-1}x), X_1, \ldots, X_{k-1} \right)
\]

\[
= \alpha(g^{-1}x) \left( X(g^{-1}x), (dl_{g^{-1}})X_1, \ldots, (dl_{g^{-1}})X_{k-1} \right)
\]

\[
= i_X \alpha(g^{-1}x) \left( (dl_{g^{-1}})X_1, \ldots, (dl_{g^{-1}})X_{k-1} \right)
\]

\[
= ((dl_{g^{-1}})^* (i_X \alpha)(g^{-1}x))(X_1, \ldots, X_{k-1})
\]

\[
= (g, i_X \alpha)(x)(X_1, \ldots, X_{k-1}).
\]

(ii) Note that the action of \(G\) on \(C^\infty(M)\) is given by \((g,f) := (l_{g^{-1}})^* f\). Therefore, for any \(x \in M\),

\[
(g, df)(x) = (dl_{g^{-1}})^* df(g^{-1}x) = d((l_{g^{-1}})^* f)(x) = d(g,f)(x).
\]

The result now follows from the observation that for any \(\alpha, \beta \in \Omega^*(M)\),

\[
(g, \alpha \land \beta)(x) = (dl_{g^{-1}})^* (\alpha \land \beta)(g^{-1}x)
\]

\[
= (dl_{g^{-1}})^* (\alpha \land g^{-1}x) \land (dl_{g^{-1}})^* (\beta \land g^{-1}x)
\]

\[
= (dl_{g^{-1}})^* \alpha(g^{-1}x) \land (dl_{g^{-1}})^* (\beta(g^{-1}x) = ((g, \alpha) \land (g, \beta))(x).
\]

(iii) It follows from part (i) and (ii) and the Cartan magic formula

\[
\mathcal{L}_X = i_X d + d i_X.
\]
3.11. Remark. The Nambu-Poisson tensor corresponding to the given bracket, and is defined by

\[ \langle \Pi^2(g, \alpha)(x), \beta_x \rangle = \langle (i_{(g, \alpha)}(x)) \Pi(x)(\beta_x) = \Pi(x)((dl_{g^{-1}})^* \alpha(g^{-1}x), \beta_x) = (dl_y) \Pi(g^{-1}x)((dl_{g^{-1}})^* \alpha(g^{-1}x), \beta_x) \]

(since \( \Pi \) is \( G \)-invariant)

\[ \Pi(g^{-1}x)(\alpha(g^{-1}x), (dl_y)^* \beta_x) = \langle (dl_y) (i_{\alpha(g^{-1}x)} \Pi(g^{-1}x)), \beta_x \rangle = \langle (dl_y) (\Pi^2 \alpha)(g^{-1}x), \beta_x \rangle = \langle (g, \Pi^2 \alpha)(x), \beta_x \rangle. \]

\[ \square \]

Recall that a Nambu-Poisson manifold is a generalization of the notion of Poisson manifolds and is defined as follows [11], [4].

3.10. Definition. Let \( M \) be a smooth manifold. A Nambu-Poisson bracket of order \( n \) (\( 2 \leq n \leq \dim M \)) on \( M \) is an \( n \)-multilinear mapping

\[ \{,\ldots,\} : C^\infty(M) \times \cdots \times C^\infty(M) \longrightarrow C^\infty(M) \]

satisfying the following conditions:

1. Skew-symmetric: \( \{f_1, \ldots, f_n\} = \text{sign}(\sigma) \{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\} \) for any \( \sigma \in \Sigma_n \);
2. Leibniz rule: \( \{fg, f_2, \ldots, f_n\} = f \{g, f_2, \ldots, f_n\} + g \{f, f_2, \ldots, f_n\} \);
3. Fundamental identity:

\[ \{f_1, \ldots, f_{n-1}, g_1, \ldots, g_n\} = \sum_{i=1}^{n} (g_1, \ldots, g_{i-1}, \{f_1, \ldots, f_{n-1}, g_i\}, \ldots, g_n) \]

for \( f_i, g_j, f, g \in C^\infty(M) \). The pair \( (M, \{,\ldots,\} ) \) is called a Nambu-Poisson manifold of order \( n \).

3.11. Remark. Recall [11] that Poisson manifolds are Nambu-Poisson manifolds of order 2.

Given a Nambu-Poisson bracket on \( M \), there exists an \( n \)-vector field \( P \in \Gamma(\Lambda^n TM) \), called the Nambu-Poisson tensor corresponding to the given bracket, and is defined by \( P(df_1, \ldots, df_n) = \{f_1, \ldots, f_n\} \), for \( f_1, \ldots, f_n \in C^\infty(M) \). Note that \( P \) induces a bundle map \( P^1 : \Lambda^{n-1} T^* M \rightarrow TM \) given by

\[ \langle \beta, P^2(\alpha_1 \wedge \cdots \wedge \alpha_{n-1}) \rangle = P(\alpha_1, \ldots, \alpha_{n-1}, \beta), \]

for all \( \alpha_1, \ldots, \alpha_{n-1}, \beta \in \Omega^1(M) \).

Recall the following definition from [12].

3.12. Definition. A (left) Leibniz algebroid over a smooth manifold \( M \) is a smooth vector bundle \( A \) over \( M \) together with a bracket \([,\]) on the space \( \Gamma A \) of smooth sections of \( A \) and a bundle map \( \rho : A \rightarrow TM \), called the anchor such that the bracket satisfies

1. (left) Leibniz identity: \( [X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]], \quad X, Y, Z \in \Gamma A; \)
2. \( [X, fY] = f[X, Y] + (\rho(X)f)Y; \)
3. \( \rho([X, Y]) = [\rho(X), \rho(Y)], \quad X, Y \in \Gamma A, \quad f \in C^\infty(M). \)

We recall the following result from [12], [4].
Let $M$ be a Nambu-Poisson manifold of order $n$ with the corresponding Nambu tensor $\Pi$. Then the bundle $\bigwedge^{n-1}T^*M$ carries a Leibniz algebroid structure with bracket

$$[\alpha, \beta] := \mathcal{L}_{\Pi \alpha} \beta - i_{\Pi \beta} d\alpha$$

on the space $\Omega^{n-1}(M)$ of $(n-1)$-forms on $M$ and the anchor is given by the bundle map $\Pi^1$. Thus $\Omega^{n-1}(M)$ is a Leibniz algebra with respect this bracket.

3.13. Example. Suppose $G$ is a finite group. Let $M$ be a Nambu-Poisson manifold of order $n$ with the associated Nambu tensor $\Pi$. Assume that $G$ acts smoothly on $M$ and $\Pi$ is $G$-invariant with respect to the action of $G$ as defined in Definition 3.8. Then, it follows from Lemma 3.9 that for any $\alpha, \beta \in \Omega^{n-1}(M)$ and $g \in G$,

$$[(g, \alpha), (g, \beta)] = \mathcal{L}_{\Pi^1(g, \alpha)}(g, \beta) - i_{\Pi^1(g, \beta)}d(g, \alpha)$$

$$= \mathcal{L}_{\Pi^1(g, \alpha)}(g, \beta) - i_{(g, \Pi^1(\beta))}d(g, \alpha) = (g, [\alpha, \beta]).$$

Thus, with the bracket $[,]$ as defined above $\Omega^{n-1}(M)$ is a Leibniz algebra equipped with the action of the given group $G$.

4. Equivariant cohomology of a Leibniz algebra equipped with a group action

In this section we introduce equivariant cohomology groups of a Leibniz algebra equipped with an action of a finite group following [2],[3].

Let $G$ be a finite group. Recall that the category of canonical orbits of $G$, denoted by $O_G$, is a category whose objects are left cosets $G/H$, as $H$ runs over all subgroups of $G$. Note that the group $G$ acts on the set $G/H$ by left translation. A morphism from $G/H$ to $G/K$ is a $G$-map. Recall that such a morphism determines and is determined by a subconjugacy relation $g^{-1}Hg \subseteq K$ and is given by $\hat{g}(eH) = gK$. We denote this morphism by $\hat{g}$ [2].

4.1. Definition. An $O_G$-module is a contravariant functor $M : O_G \rightarrow \text{Mod}$, where $\text{Mod}$ is the category of modules over $\mathbb{k}$. The category whose objects are $O_G$-modules and with morphisms the natural transformations between $O_G$-modules is an abelian category denoted by $\mathcal{C}_G$. Let $\text{Comm}$ be the category of associative commutative algebras over $\mathbb{k}$. An $O_G$-algebra is a contravariant functor $A : O_G \rightarrow \text{Comm}$. Similarly, an $O_G$-Leibniz algebra is a contravariant functor $L : O_G \rightarrow \text{Leib}$. If $g$ is a Leibniz algebra equipped with an action of $G$, then we have contravariant functor $\Phi_g : O_G \rightarrow \text{Leib}$, given by $\Phi_g(G/H) = g^H$ and for a morphism $\hat{g} : G/H \rightarrow G/K$ corresponding to a sub conjugacy relation $g^{-1}Hg \subseteq K$, $\Phi_g(\hat{g}) = \psi_g : g^K \rightarrow g^H$. Thus, $\Phi_g$ is an $O_G$-Leibniz algebra, which will be referred to as the $O_G$-Leibniz algebra associated to $g$.

We now proceed to define the notion of equivariant cohomology of a Leibniz algebra $g$ equipped with an action of a finite group $G$. Let $A : O_G \rightarrow \text{Comm}$ be an $O_G$-algebra. Let the product in $A(G/H)$ is denoted by $\mu_H : A(G/H) \otimes A(G/H) \rightarrow A(G/H)$. Note that for every morphism $\hat{g} : G/H \rightarrow G/K$ in $O_G$ corresponding to a sub conjugacy relation $g^{-1}Hg \subseteq K$, we have

$$\mu_H \circ (A(\hat{g}) \otimes A(\hat{g})) = A(\hat{g}) \circ \mu_K.$$

4.2. Definition. For every $n \geq 1$, let $CL_n(g)$ be the $O_G$-module defined by $CL_n(g)(G/H) := CL_n(g^H) = (g^H)^{\otimes n}$ and $CL_n(g)(\hat{g}) : CL_n(g^K) \rightarrow CL_n(g^H)$ is given by $(\psi_g)^{\otimes n}$. The boundary map $\partial$ induces a natural transformation $d : CL_{n+1}(g) \rightarrow CL_n(g)$, where $d(g(H)) = d_H$ is the boundary map for the Leibniz algebra $g^H$. Clearly, $d \circ d = 0$. This gives a chain complex in the
an isomorphism

Lemma.

The following is an equivalent formulation of the groups $HL_G^n(g; A)$.

Set $S^n(g; A) = \oplus_{H \subset G} CL^n(g^H; A(G/H))$ and define

$$
\delta : S^n(g; A) \to S^{n+1}(g; A)
$$

by $\delta = \oplus_{H \subset G} \delta_H$, where $\delta_H : CL^n(g^H; A(G/H)) \to CL^{n+1}(g^H; A(G/H))$ is the non-equivariant coboundary map (2.2) for the Leibniz algebra $g^H$. Clearly, $\{S^n(g; A), \delta\}$ is a cochain complex. We define a subcomplex of this cochain complex as follows.

4.3. Definition. A cochain $c = \{c_H\} \in S^n(g; A)$ is said to be invariant under the action of $G$ if for every morphism $\hat{g} : G/H \to G/K$, corresponding to a subconjugacy relation $g^{-1} H g \subset K$ following holds:

$$
e H \circ (\psi_g)^{\otimes n} = \hat{A}(\hat{g}) \circ c_K.
$$

4.4. Lemma. The set of all invariant $n$-cochains is a subcomplex $S^n_G(g; A)$ of $S^n(g; A)$. If $c = \{c_H\} \in S^n(g; A)$ is invariant then $\delta(c) = \{\delta_H(c_H)\} \in S^{n+1}(g; A)$ is an invariant $(n+1)$-cochain.

Proof. It is clear that $S^n_G(g; A)$ is a subgroup of $S^n(g; A)$ as $A(\hat{g})$ is a homomorphism for every $\hat{g} : G/H \to G/K$. Let $c = \{c_H\} \in S^n(g; A)$ be invariant and $\hat{g} : G/H \to G/K$ be a morphism in $O_G$ corresponding to a subconjugacy relation $g^{-1} H g \subset K$. Thus, for every $x_1, \ldots, x_n \in (g^K)^{\otimes n}$ we have

$$
(c_H(\psi_g(x_1), \ldots, \psi_g(x_n)) = A(\hat{g})(c_K(x_1, \ldots, x_n)).
$$

Next, recall from (2.2) that

$$
\delta_H(c_H)(\psi_g(x_1), \ldots, \psi_g(x_{n+1}))
= \sum_{1 \leq i < j \leq n+1} (-1)^j c_H(gx_1, \ldots, gx_{i-1}, [gx_i, gx_j], \ldots, \hat{g}x_j, \ldots, gx_{n+1})
= \sum_{1 \leq i < j \leq n+1} (-1)^j A(\hat{g}) c_K(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_{n+1})
= A(\hat{g}) \delta_K(c_K)(x_1, \ldots, x_{n+1}).
$$

Therefore, $\delta_H(c_H) \circ (\psi_g)^{\otimes n+1} = A(\hat{g}) \circ \delta_K(c_K)$ and hence,

$$
\{\delta_H(c_H)\} \in S^{n+1}_G(g; A).
$$

Thus, we have a cochain subcomplex $S^n_G(g; A) = \{S^n_G(g; A), \delta\}$.

4.5. Theorem. Let $g$ be a Leibniz algebra with a given action of $G$. For any $O_G$-algebra $A$ we have an isomorphism

$$
H_n(S^n_G(g; A)) \cong HL^n_G(g; A)
$$

for all $n$. 

The abelian category of $O_G$-modules. Let

$$
CL^n_G(g; A) := \text{Hom}_{O_G}(CL_n(g), A).
$$

We have an induced homomorphism $\delta : CL^n_G(g; A) \to CL^{n+1}_G(g; A)$ given by $\delta(c) := c \circ \delta$ for any natural transformation $c \in CL^n_G(g; A)$. Thus, we have a cochain complex $CL^n_G(g; A) = \{CL^n_G(g; A), \delta\}$. The homology groups of this cochain complex are equivariant Leibniz algebra cohomology groups of $(G, g)$ with coefficients in $A$ and denoted by $HL^n_G(g; A)$.
Proof. Let $F \in CL_G^0(\mathfrak{g}; A)$. Note that the collection $\{F(G/H) : (\mathfrak{g}^H)^{\otimes n} \to A(G/H)\}$ of non-equivariant $n$-cochain of $\mathfrak{g}^H$, as $H$ varies over subgroups of $G$, is invariant. Because, for every morphism $\hat{g} : G/H \to G/K$ of $O_G$, we have $A(\hat{g}) \circ F(G/K) = F(G/H) \circ (\psi_g)^{\otimes n}$ by naturality of $F$. Thus, we have a map

$$\alpha_n : CL_G^0(\mathfrak{g}; A) \to S_G^n(\mathfrak{g}; A)$$

defined by $F \mapsto \alpha_n(F) := \{F(G/H)\}$ for all $n$. We claim that $\alpha = \{\alpha_n\}$ is a cochain map. Let $F \in CL_G^0(\mathfrak{g}; A)$. Then, $(\delta \circ \alpha)(F) = \{\delta_H F(G/H)\}$. On the other hand, $(\alpha \circ \delta)(F) = \alpha(F \circ \delta) = \{F \circ \delta(H/G/H)\} = \{F(G/H) \circ d_H\}$. Since $\delta_H F(G/H) = F(G/H) \circ d_H$ the result follows. Thus, $\alpha$ induces a homomorphism

$$\tilde{\alpha}_n : HL_G^0(\mathfrak{g}; A) \to H_n(S_G^n(\mathfrak{g}; A))$$

for all $n$.

Finally, observe that the cochain map $\alpha = \{\alpha_n\}$ is a cochain isomorphism with inverse $\beta = \{\beta_n\}$ defined as follows:

$$\beta : S_G^n(\mathfrak{g}; A) \to CL_G^0(\mathfrak{g}; A), \{c_H\} \mapsto C,$$

where the natural transformation $C : CL_G(\mathfrak{g}) \to A$ is given by $C(G/H) := c_H$. The naturality of $C$ follows from the invariance of $\{c_H\}$. Thus, $\tilde{\alpha}_n$ is an isomorphism for all $n$.  

5. Equivariant Leibniz cohomology as Zinbiel algebra

In [3], J.-L. Loday introduced zinbiel algebras and proved that for any Leibniz algebra $\mathfrak{g}$ the graded Leibniz cohomology with coefficients in a commutative, associative algebra admits a graded product which makes it a graded zinbiel algebra. The aim of this section is to prove an equivariant version of this result. Explicitly, for a Leibniz algebra $\mathfrak{g}$ equipped with an action of $G$, we prove that equivarient graded Leibniz cohomology $HL_G^*(\mathfrak{g}; A)$ as introduced in the previous section also admits a graded zinbiel algebra structure. To show this, we use the equivalent formulation (Theorem 14.5) of equivariant cohomology to define a cup-product operation which is based on the product defined at the cochain level of the fixed points Leibniz algebras $\{\mathfrak{g}^H\}_{H \leq G}$.

Recall the following definition from [9],[10].

5.1. Definition. A dual Leibniz algebra or a Zinbiel algebra is a $K$-vector space $R$ equipped with a bilinear map

$$(-,-) : R \times R \to R$$

satisfying the relation

$$(rs)t = (r(st)) + (r(ts)), \forall r,s,t \in R.$$  

A graded Zinbiel algebra is a graded $K$-vector space $R$ equipped with a graded bilinear map

$$(-,-) : R \times R \to R$$

satisfying the relation

$$(rs)t = (r(st)) + (-1)^{|r||s|} (r(ts)),$$

for all homogeneous elements $r,s,t \in R$.

Let $S_n$ be the permutation group of $n$ elements $1,\ldots,n$. A permutation $\sigma \in S_n$ is called a $(p,q)$-shuffle if $p + q = n$ and

$$\sigma(1) < \cdots < \sigma(p) \text{ and } \sigma(p + 1) < \cdots < \sigma(p + q).$$
In the group algebra \( \mathbb{K}[S_n] \), let \( sh_{p,q} \) be the element

\[
sh_{p,q} := \sum_{\sigma} \sigma,
\]

where the summation is over all \((p, q)\)-shuffles.

For any vector space \( V \) we let \( \sigma \in S_n \) act on \( V^\otimes n \) by

\[
\sigma(v_1 \ldots v_n) = (v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(n)}),
\]

where the generator \( v_1 \otimes \cdots \otimes v_n \) of \( V^\otimes n \) is denoted by \( v_1 \ldots v_n \). The free Zinbiel algebra over the vector space \( V \) is \( T(V) = \oplus_{n \geq 1} V^\otimes n \) equipped with the following product

\[
(v_0 \ldots v_p)(v_{p+1} \ldots v_{p+q}) = v_0 sh_{p,q}(v_1 \ldots v_{p+q}) = (Id_1 \otimes sh_{p,q})(v_0 \ldots v_{p+q}).
\]

Here \( Id_1 \) is the identity on the first factor.

Note that the linear map from \( \mathbb{K}[S_n] \) to itself induced by \( \sigma \mapsto sgn(\sigma)\sigma^{-1} \) for \( \sigma \in S_n \) is an anti-homomorphism. Let us denote the image of \( \alpha \in \mathbb{K}[S_n] \) under this map by \( \tilde{\alpha} \).

Let \( \mathfrak{g} \) be a Leibniz algebra equipped with a given action of a finite group \( G \). Let \( \Phi_{\mathfrak{g}} \) be the corresponding \( O_G \)-Leibniz algebra. Denote by \( \Phi_{\mathfrak{g}}^{\otimes p+q} \) the \( O_G \)-vector space given by

\[
\Phi_{\mathfrak{g}}^{\otimes p+q}(G/H) = (\Phi_{\mathfrak{g}}(G/H))^{\otimes p+q} = (\mathfrak{g}^H)^{\otimes p+q}
\]

for objects \( G/H \) in \( O_G \) and for a morphism \( \hat{g} : G/H \to G/K \),

\[
\Phi_{\mathfrak{g}}^{\otimes p+q}(\hat{g}) = (\Phi_{\mathfrak{g}}(\hat{g}))^{\otimes p+q}.
\]

For any non-negative integers \( p \) and \( q \) we define a natural transformation

\[
(5.6) \quad \rho_{p,q} := Id_1 \otimes \tilde{sh}_{p-1,q} = \Phi_{\mathfrak{g}}^{\otimes p+q} \to \Phi_{\mathfrak{g}}^{\otimes p+q}.
\]

Explicitly, for every object \( G/H \in O_G \), the linear map

\[
\rho_{p,q}(G/H) : (\mathfrak{g}^H)^{\otimes p+q} \to (\mathfrak{g}^H)^{\otimes p+q}
\]

is given by

\[
(5.7) \quad \rho_{p,q}(G/H)(x_1 \ldots x_{p+q}) = \sum_{\sigma} sgn(\sigma)(x_1 x_{\sigma(2)} \ldots x_{\sigma(p+q)}),
\]

where the above sum is over all \((p-1, q)\)-shuffles \( \sigma \).

Let \( \tau_{p,q} : \Phi_{\mathfrak{g}}^{\otimes p+q} \to \Phi_{\mathfrak{g}}^{\otimes p+q} \) be the natural transformation defined as follows. For any object \( G/H \) in \( O_G \) and for generators \( x = v_1 \ldots v_p \in (\mathfrak{g}^H)^{\otimes p} \) and \( y = v_{p+1} \ldots v_{p+q} \in (\mathfrak{g}^H)^{\otimes q} \), \( \tau_{p,q}(G/H)(xy) = yx \) with the obvious definition on morphisms in \( O_G \). Then, for non-negative integers \( p, q, r \), we have the following equality

\[
(5.8) \quad (\rho_{p,q} \otimes Id_r) \circ \rho_{p+q,r} = (Id_p \otimes \rho_{q,r} + (-1)^q \circ \tau_{r,q} \circ \rho_{r,q}) \circ \rho_{p,q+r}
\]

(cf. [9].)

Let \( A : O_G \to \text{Comm} \) be an \( O_G \)-algebra. Let the product in \( A(G/H) \) is denoted by \( \mu_H : A(G/H) \otimes A(G/H) \to A(G/H) \). Note that for every morphism \( \hat{g} : G/H \to G/K \) in \( O_G \) corresponding to a sub conjugacy relation \( g^{-1}Hg \subseteq K \), \( A(\hat{g}) : A(G/K) \to A(G/H) \) is an algebra map and hence, we have

\[
\mu_H \circ (A(\hat{g}) \otimes A(\hat{g})) = A(\hat{g}) \circ \mu_K.
\]

In other words,

\[
\mu : A \times A \to A, \quad \mu(G/H) = \mu_H
\]
is a natural transformation.

5.2. Definition. For \( c = \{c_H\} \in S^p_G(g; A) \), \( p > 0 \) and \( d = \{d_H\} \in S^q_G(g; A) \), \( q > 0 \), we define

\[
\delta(c \cup d) = \{\delta_H(c_H \cup d_H)\} = \{\delta_H(c_H) \cup d_H + (-1)^{|c|n} c_H \cup \delta_H(d_H)\} \quad \text{by the non-equivariant case}
\]

\[
= \{\delta_H(c_H) \cup d_H\} + (-1)^{|c|n} \{c_H \cup \delta_H(d_H)\}
\]

\[
= \delta(c) \cup d + (-1)^{|c|} c \cup \delta(d) \quad \text{(since \((-1)^{|c|n}\) = \((-1)^{|c|}\)).}
\]

Let \([a] \in HL_G^p(g; A), [b] \in HL_G^q(g; A)\) and \([c] \in HL_G^r(g; A)\). We claim

\[
(A(\hat{g}) (c_K \cup d_K)
\]

\[
= \mu_H \circ (A(\hat{g}) \circ (c_K \circ d_K) \circ \rho_{p,q}(G/K))
\]

\[
= \mu_H \circ (A(\hat{g}) \circ (c_K \circ d_K) \circ \rho_{p,q}(G/K))
\]

\[
= \mu_H \circ (c_H \circ \rho_{p,q}(G/H)) \circ (\psi_g)^{\otimes p+q} (\text{by invariance of } c\text{ and } d)
\]

\[
= \mu_H \circ (c_H \circ d_H) \circ (\psi_g)^{\otimes p+q} \circ \rho_{p,q}(G/K)
\]

\[
= \mu_H \circ (c_H \circ d_H) \circ \rho_{p,q}(G/H) \circ (\psi_g)^{\otimes p+q} \circ \rho_{p,q}(G/K)
\]

\[
= (c_H \cup d_H) \circ (\psi_g)^{\otimes p+q}.
\]

Next, note that the cup-product operation is well-defined. This is because,

\[
\delta(c \cup d) = \{\delta_H(c_H \cup d_H)\}
\]

\[
= \{\delta_H(c_H) \cup d_H\} + (-1)^{|c|n} \{c_H \cup \delta_H(d_H)\}
\]

\[
= \delta(c) \cup d + (-1)^{|c|} c \cup \delta(d) \quad \text{(since \((-1)^{|c|n}\) = \((-1)^{|c|}\)).}
\]

To prove the relation (5.9) we proceed as follows. We choose representative cocycles \( a = \{a_H\} \), \( b = \{b_H\} \) and \( c = \{c_H\} \). By Definition 5.2, \( c \cup b := \{c_H \cup b_H\} = \{\mu_H \circ (c_H \otimes b_H) \circ \rho_{p,q}(G/H)\} \). Since \( A(G/H) \) is commutative for every object \( G/H \) in \( O_G \), we have

\[
\mu_H \circ (c_H \otimes b_H) \circ \rho_{p,q}(G/H) = \mu_H \circ (b_H \otimes c_H) \circ \rho_{p,q}(G/H) \circ \rho_{p,q}(G/H).
\]

We pre-compose \( \mu_H \circ ((a_H \otimes b_H) \otimes c_H) \) on both sides of the relation (5.8) for every object \( G/H \) in \( O_G \) to deduce

\[
\mu_H \circ ((a_H \otimes b_H) \otimes c_H) \circ (\rho_{p,q}(G/H) \otimes Id_r) \circ \rho_{p+q,r}(G/H)
\]

\[
= \mu_H \circ ((a_H \otimes b_H) \otimes c_H) \circ (Id_p \otimes \rho_{q,r}(G/H)) \circ \rho_{p+q,r}(G/H)
\]

\[
+ (-1)^{pq} \mu_H \circ ((a_H \otimes b_H) \otimes c_H) \circ (Id_p \otimes \tau_{r,q}(G/H) \circ \rho_{r,q}(G/H)) \circ \rho_{p,q+r}(G/H).
\]

Thus, for every object \( G/H \) in \( O_G \), we have

\[
(a_H \cup b_H) \cup c_H = a_H(b_H \cup c_H) + (-1)^{|c||b|} a_H(c_H \cup b_H).
\]

Therefore,

\[
(a \cup b) \cup c = a(b \cup c) + (-1)^{|c||b|} a(c \cup b).
\]
5.3. **Theorem.** Given a Leibniz algebra $\mathfrak{g}$ equipped with an action of a finite group $G$ and an $O_G$-algebra $A$, the graded equivariant cohomology $\text{HL}_G^*(\mathfrak{g}; A)$ is a graded zinbiel algebra with respect to the cup-product operation (5.2).

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