DISCREPANCY SKEW PRODUCTS AND AFFINE RANDOM WALKS

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Abstract. We prove bounded rational ergodicity for some discrepancy skew products whose rotation number has bad rational approximation. This is done by considering the asymptotics of associated affine random walks.

§1 Introduction
Discrepancy skew products. Let \( \mathbb{T} := \mathbb{R}/\mathbb{Z} \cong [0, 1) \) denote the additive circle.

Consider the function \( \varphi : \mathbb{T} \to \mathbb{Z} \) defined by
\[
\varphi := 2 \cdot 1_{[0, \frac{1}{2})} - 1
\]
and the skew products \( T_\alpha : \mathbb{T} \times \mathbb{Z} \to \mathbb{T} \times \mathbb{Z} \) defined for \( \alpha \notin \mathbb{Q} \) by
\[
T_\alpha(x, y) = (x + \alpha, y + \varphi(x)).
\]
These are measure preserving transformations of the \( \sigma \)-finite measure space
\[
(X, B, m) = (\mathbb{T} \times \mathbb{Z}, \mathcal{B}(\mathbb{T} \times \mathbb{Z}), \text{Leb} \times \#).
\]
We have that \( T_\alpha^n(x, y) = (x + n\alpha, y + \varphi_n(x)) \) where
\[
\varphi_n(x) = \sum_{k=0}^{n-1} \varphi(x + k\alpha).
\]
This is related to the discrepancy of the well-distribution of \( \{n\alpha\} \) over \( [0, \frac{1}{2}) \) (see [16]) and accordingly we call the function \( (n, x) \mapsto \varphi_n(x) \) the discrepancy cocycle and \( T_\alpha \) the discrepancy skew product.

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Discrepancy Skew Products and Affine Random Walks.

(aka the “cylinder flow arising from irregularity of distribution” in [16] and “deterministic random walk” as in [3]).

Ergodicity of the discrepancy skew product $T_\alpha$ was established for $\alpha = \frac{\sqrt{5} - 1}{4}$ in [16] and then $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ in [8].

Results.
We call $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ badly approximable if
\[
\inf \{ q^2|\alpha - \frac{p}{q}| : q \in \mathbb{N}, \, p \in \mathbb{Z} \} > 0
\]
and denote
\[
\text{BAD} := \{ \alpha \in \mathbb{R} \setminus \mathbb{Q} : \alpha \text{ badly approximable} \}.
\]

Our main result is that if $\alpha \in \text{BAD}$, then $T_\alpha$ is boundedly rationally ergodic in the sense of [1]. In particular, defining $\Psi_n = \Psi_n^{(\alpha)} : \mathbb{T} \to \mathbb{N}$ by
\[
\Psi_n(x) = S_n(1_{\mathbb{T} \times \{0\}})(x,0)
\]
\[
= \sum_{k=0}^{n-1} 1_{\mathbb{T} \times \{0\}} \circ T_\alpha^k(x,0)
\]
\[
= \#\{0 \leq k \leq n-1 : \varphi_k(x) = 0\}
\]
we prove that $\exists M > 1$ so that
\[
\|\Psi_n^{(\alpha)}\|_{L^\infty(\mathbb{T})} \leq M \int_{\mathbb{T}} \Psi_n^{(\alpha)}(t)dt
\]
and
\[
\int_{\mathbb{T}} \Psi_n^{(\alpha)}(t)dt = M^{s_1} \frac{n}{\sqrt{\log n}}.
\]

Here and throughout, for $a, b > 0, \, M > 1$,
\[
a = M^{s_1} b \text{ means } \frac{1}{M} \leq \frac{a}{b} \leq M.
\]

We’ll also consider more general subsequence versions of (⋆):
\[
\|\Psi_{\ell_n}^{(\alpha)}\|_{L^\infty(\mathbb{T})} \leq M \int_{\mathbb{T}} \Psi_{\ell_n}^{(\alpha)}(t)dt
\]
which implies (as in [11]) that there is a dense hereditary ring $R(T_\alpha)$ of sets of finite measure so that
\[
\sum_{j=0}^{\ell_n-1} m(A \cap T^{-j}B) \sim m(A)m(B) a_{\ell_n} \text{ as } n \to \infty \forall A, \, B \in R(T)
\]
where $a_{\ell_n} := \int_{\mathbb{T}} \Psi_{\ell_n}^{(\alpha)}(t)dt$.

In case $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ quadratic, (⋆) was established in [3] and refined in [4].
The discrepancy skew products occur as good sections (in the sense of [2]) for directional translation flows of the infinite staircase translation surface as in [13] (see [1]). Our result also holds for the corresponding directional, translation flow (with \( \alpha \) badly approximable) as can be seen using lemma 2.1 in [2].

**Proof perspective.**

The proof of (R) relies on a weak, rough local limit theorem for an associated affine random walk arising from sequence of renormalizations related to the orbit of \( 2\alpha \) under a modified continued fraction transformation (see below).

The quadratic case corresponds to a compact (eventually periodic) renormalization sequence, and the \text{BAD} case corresponds to a precompact sequence.

**RATs and ARWs.** A random affine transformation (RAT) on \( \mathbb{R}^d \) is a random variable \( F = (a(F), b(F)) \) taking values in \( M_{d \times d}(\mathbb{R}) \times \mathbb{R}^d \) (associated with the transformation \( x \mapsto ax + b \)).

We call the RAT \( F \) discrete if \( F \in M_{d \times d}(\mathbb{Z}) \times \mathbb{Z}^d \) a.s.

An affine random walk (ARW) is a \( \mathbb{R}^d \)-valued stochastic process \( (X^{(n)})_{n \geq 0} \) defined by

\[
X^{(0)} := 0 \text{ and } X^{(n+1)} := F_{n+1}(X^{(n)}) = a(F_n)X^{(n)} + b(F_n)
\]

where the \( (F_n : n \geq 1) \) is a sequence of independent RATs referred to as the RAT sequence. In this paper, we only have need of a special kind of RAT called flip type (defined in §4). For other works on ARWs (not of flip type), see [10], [14], [17], [11], [9] and references therein.

The rational ergodicity of the discrepancy skew product is governed by the asymptotic behavior of the temporal statistics of the discrepancy cocycle values \( (\varphi_n(0) : n \geq 1) \) (Visit Lemma 2.4 below). These temporal statistics are modeled by certain ARWs (Construction Lemma 4.1 below) and (R) follows from a weak, rough, local limit theorem for the coordinates of these (Theorem 6.2 below).

§2 Visit distributions of the discrepancy cocycle

In this section, as in [3], we show that (R) (as on page 2) follows from certain asymptotic properties of “visit distributions” (to be defined below).

We first note that we may assume without loss of generality that \( 0 < \alpha < \frac{1}{2} \).
This is because $-T_\alpha(-x, -n) = T_{1-\alpha}(x, n)$ and $-\varphi_k^{(\alpha)}(x) = \varphi_k^{(1-\alpha)}(-x)$ whence

$$\Psi_n^{(\alpha)}(x) = \Psi_n^{(1-\alpha)}(-x).$$

Thus (●) (as on page 2) for $\alpha$ and $1 - \alpha$ are equivalent and we only consider the case $0 < \alpha < \frac{1}{2}$.

**Calculation of the jump function orbit.**

We recall from [3] the substitution algorithm to calculate the jump function orbit $(\varphi(\{n\alpha\}) : n \geq 0)$.

We have,

$$\varphi(n\alpha) = (-1)^{\sum_{j=1}^n \psi_j}$$

where

$$\psi_n = \psi_n^{(2\alpha)} := 1_{[1-2\alpha, 1]}(2(n-1)\alpha).$$

The modified continued fraction expansion of $\beta \in (0, 1)$ is

$$\beta = [n_1, n_2, \ldots] = 1/n_1 - 1/n_2 - 1/n_3 - \ldots$$

with each $n_k \in \mathbb{N}_2 := \{a \in \mathbb{N} : a \geq 2\}$. Here (see [15]) $\beta \in \mathbb{Q}$ iff $n_k \to 2$ as $n \to \infty$.

**Theorem 2.1** ([3]) For $\beta = [n_1, n_2, \ldots]$, let $b_0(0) = 0$, $b_0(1) = 1$ and $b_{k+1}(0) = b_k(0)\circ (n_{k+1}-1) \oplus b_k(1)$ and $b_{k+1}(1) = b_k(0)\circ (n_{k+1}-2) \oplus b_k(1)$, then

$$(\psi_1^{(\beta)}, \ldots, \psi_k^{(\beta)}) = b_k(0) \quad (k \geq 1).$$

Here $\oplus$ denotes concatenation, and $\ell_k(i) = |b_k(i)|$ denotes the length of the block $b_k(i)$ ($i = 0, 1$).

For $\beta = 2\alpha = [n_1, n_2, \ldots]$, set for $i = 0, 1$, $B_0(i) := [(-1)^i]$ and for $k \geq 0$, $i = 0, 1$ and $(n_{k+1}, i) \neq (2, 1)$:

$$B_{k+1}(i) = \bigoplus_{j=1}^{n_{k+1}-1-i} (-1)^{(j-1)\epsilon_k(i)} B_k(0) \oplus (-1)^{\epsilon_k(i)\epsilon_k(n_{k+1}-1-i)} B_k(1);$$

where

$$\epsilon_k(i) := \sum_{j=1}^{\ell_k(i)} (b_k(i))_j \mod 2$$

and $B_{k+1}(1) = B_k(1)$ in case $n_{k+1} = 2$.

**Theorem 2.2** ([3])

$$B_k(0) = (\varphi(\{j\alpha\}))_{j=0}^{\ell_k(0)-1} \quad (i = 0, 1).$$
Visit sets.

The visit set to $\nu \in \mathbb{Z}$ is

$$K_\nu := \{ n \geq 1 : \varphi_n(0) = \nu \}$$

and the visit distributions are the measures $U_k^{(i)}$ on $\mathbb{Z}$ defined by

$$U_k^{(i)}(\nu) := \#(K_\nu \cap [1, \ell_k(i)]) \quad (k \geq 1, \ i = 0,1).$$

Lemma 2.3

$$\int_0^1 \Psi_{\ell_k(0)}(x) dx \geq \frac{1}{4\ell_k(0)} \sum_{\nu \in \mathbb{Z}} [U_k^{(1)}(\nu)]^2;$$

$$\int_0^1 \Psi_{\ell_k(1)}(x)^N dx \leq \frac{2N}{\ell_k(1)} \sum_{\nu \in \mathbb{Z}} [U_k^{(0)}(\nu)]^{N+1} \ \forall \ N \geq 1. \tag{4.2}$$

Statement (4.1) is essentially Lemma 4.1 in [3] and proved in the same manner. Statement (4.2) is an upgrade of lemma 4.2 in [3].

Proof of (4.2) As in the proofs of lemmas 4.1 and 4.2 in [3],

$$\ell_{k+r}(0) \int_0^1 \Psi_{\ell_k(1)}(x)^N dx \sim \sum_{r=0}^{\infty} \sum_{\nu \in \mathbb{Z}} \#(K_\nu \cap [j+1, j+\ell_k(1)])^N$$

and for $r \geq 1, \exists J = J_r, r \geq 1, \ 1 = m_1 < \cdots < m_J$ and $\epsilon_1, \ldots, \epsilon_{J-1} = \pm 1, \ i_1, \ldots, i_{J-1} = 0,1$ so that

$$m_{j+1} - m_j = \ell_k(i_j) \ \forall \ j, \ \ [1, \ell_{k+r}(0)] = \bigcup_{j=1}^{J-1} [m_j, m_{j+1}],$$

$$(\varphi(m_j \alpha), \varphi(m_j + 1) \alpha), \ldots, \varphi((m_{j+1} - 1) \alpha)) = \epsilon_j B_k(i_j).$$

For fixed $\nu \in \mathbb{Z},

$$K_\nu \cap [m_j, m_{j+1}) = m_j + K_{\epsilon_j(\nu - \varphi_m(0))} \cap [1, \ell_k(i_j)].$$

Note that

$$(\varphi(m_j \alpha), \varphi((m_j + 1) \alpha), \ldots, \varphi((m_{j+1} + \ell_k(1) - 1) \alpha)) = (\epsilon_j B_k(i_j), \Delta_j B_k(1))$$

for some $\Delta_j = \pm 1$. 
We have as before, for fixed $\nu \in \mathbb{Z}$,

$$\sum_{i \in [1, \ell_k+1(0)] \cap K_\nu} \#(K_\nu \cap [i + 1, i + \ell_k(1)])^N = \sum_{j=1}^{J-1} \sum_{i \in [m_j, m_{j+1}] \cap K_\nu} \#(K_\nu \cap [i + 1, i + \ell_k(1)])^N$$

$$\leq \sum_{j=1}^{J-1} \sum_{i \in [m_j, m_{j+1}] \cap K_\nu} \#(K_\nu \cap [i + 1, m_{j+1} + \ell_k(1)])^N.$$  

Fix $j$. For fixed $i \in [m_j, m_{j+1})$,

$$\#(K_\nu \cap [i + 1, m_{j+1} + \ell_k(1)]) = \#(K_\nu \cap [i + 1, m_{j+1}]) + \#(K_\nu \cap [m_{j+1} + 1, m_{j+1} + \ell_k(1)])$$

$$= \#(K_\nu \cap [i + 1, m_{j+1}]) + U^{(1)}_k(\Delta_j(\nu - \varphi_{m_{j+1}(0)})).$$

Thus

$$\sum_{i \in [m_j, m_{j+1}] \cap K_\nu} \#(K_\nu \cap [i + 1, m_{j+1} + \ell_k(1)])^N = \sum_{i \in [m_j, m_{j+1}] \cap K_\nu} \left( \#(K_\nu \cap [i + 1, m_{j+1}]) + U^{(1)}_k(\Delta_j(\nu - \varphi_{m_{j+1}(0)})) \right)^N$$

$$= \sum_{i \in [m_j, m_{j+1}] \cap K_\nu} \left( \sum_{r=0}^{N} \binom{N}{r} \sum_{i \in [m_j, m_{j+1}] \cap K_\nu} \#(K_\nu \cap [i + 1, m_{j+1}])^r U^{(1)}_k(\Delta_j(\nu - \varphi_{m_{j+1}(0)})) \right)^N$$

$$\leq \sum_{r=0}^{N} \binom{N}{r} U^{(1)}_k(\epsilon_j(\nu - \varphi_{m_j(0)}))^{r+1} U^{(0)}_k(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r}$$

$$\leq \sum_{r=0}^{N} \binom{N}{r} U^{(0)}_k(\epsilon_j(\nu - \varphi_{m_j(0)}))^{r+1} U^{(0)}_k(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r}.$$  

Using this and Hölder’s inequality,

$$\sum_{\nu \in \mathbb{Z}} \sum_{j \in [1, \ell_k+1(0)] \cap K_\nu} \#(K_\nu \cap [j + 1, j + \ell_k(1)])^N \leq \sum_{j=1}^{J} \sum_{r=0}^{N} \binom{N}{r} \sum_{\nu \in \mathbb{Z}} U^{(0)}_k(\epsilon_j(\nu - \varphi_{m_j(0)}))^{r+1} U^{(0)}_k(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r}$$

$$\leq \sum_{j=1}^{J} \sum_{r=0}^{N} \binom{N}{r} \left( \sum_{\nu \in \mathbb{Z}} U^{(0)}_k(\epsilon_j(\nu - \varphi_{m_j(0)}))^{r+1} U^{(0)}_k(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r} \right)^{\frac{N-r}{N}}$$

$$\leq \sum_{r=0}^{N} \binom{N}{r} \left( \sum_{\nu \in \mathbb{Z}} U^{(0)}_k(\nu)^{N-1} \right)^{\frac{r+1}{N-r}} \left( \sum_{\nu \in \mathbb{Z}} U^{(0)}_k(\nu)^{N+1} \right)^{\frac{N-r}{N}}$$

$$= 2^N J \sum_{\nu \in \mathbb{Z}} U^{(0)}_k(\nu)^{N+1}.$$
whence
\[
\int_0^1 \Psi_{\ell_k(1)}(x)x^N dx \sim \frac{1}{\ell_{k+r}(0)} \sum_{\nu \in \mathbb{Z}} \sum_{j \in [1, \ell_{k+r}(0)] \cap K_\nu} \# (K_\nu \cap [j + 1, j + \ell_k(1)])^N \\
\leq \frac{2Nf}{\ell_{k+r}(0)} \sum_{\nu \in \mathbb{Z}} U_k^0(\nu)^{N+1} \\
\leq \frac{2N}{\ell_k(1)} \sum_{\nu \in \mathbb{Z}} U_k^0(\nu)^{N+1}. \quad \Box (4.2)
\]

Visit lemma
This is a Fourier series (or generating function) consequence of Lemma 2.3.

Let
\[
\tilde{U}_k^{(i)}(Z) := \sum_{\nu \in \mathbb{Z}} U_k^{(i)}(\nu)Z^\nu \quad (i = 0, 1, |Z| = 1).
\]

Visit lemma 2.4

(4.1') \quad \int_0^1 \Psi_{\ell_k(0)}(x)x^N dx \geq \frac{1}{4\ell_k(0)} \int_{S^1} |\tilde{U}_k^{(1)}(Z)|^2 dZ

(4.2') \quad \|\Psi_{\ell_k(1)}\|_\infty \leq 2 \int_{S^1} |\tilde{U}_k^{(1)}(Z)|dZ.

Where here and throughout,
\[
\int_{S^1} f(Z) dZ := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.
\]

Proof For fixed $k \geq 1$,

The statement (4.1') follows from (4.1) via the Riesz-Fischer theorem and the statement (4.2'), follows from (4.2) using the Hausdorff-Young theorem as in the proof of theorem 6.1 of [3]. \Box

§3 Visit distribution transitions (as in [3])

Set $s_k(i) := \sum_{j=0}^{\ell_k(i)-1} \varphi\{j\alpha\}$ and define the orbit blocks
\[
\Sigma_k(i) := (\varphi_1(0), \varphi_2(0), \ldots, \varphi_{\ell_k(i)}(0)).
\]

Our goal here is to obtain the transitions of the visit distribution generating functions.

Transitions in terms of blocks

From theorem 2.2 above we see that for $k \geq 1$, $i = 0, 1$, where $(n_{k+1}, i) \neq (2, 1)$:
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- in case $\epsilon_k(0) = 0$,
  \[ \Sigma_{k+1}(i) = \bigotimes_{j=1}^{n_{k+1}-1-i} (\Sigma_k(0) + (j-1)s_k(0)) \odot (\Sigma_k(1) + (n_{k+1}-i-1)s_k(0)) \]

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,
  \[ \Sigma_{k+1}(i) = [\Sigma_k(0), s_k(0) - \Sigma_k(0)]^{\otimes \frac{n_{k+1}-1-i}{2}} \odot \Sigma_k(1) \]

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,
  \[ \Sigma_{k+1}(i) = [\Sigma_k(0), s_k(0) - \Sigma_k(0)]^{\otimes \frac{n_{k+1}-2-i}{2}} \odot \Sigma_k(0) \odot (s_k(0) - \Sigma_k(1)) \]

- and $\Sigma_{k+1}(1) = \Sigma_k(1)$ in case $n_{k+1} = 2$.

**Transitions in terms of visit distributions**

- For $k \geq 1$, $i = 0, 1$ and $\nu \in \mathbb{Z}$, where $(n_{k+1}, i) \neq (2, 1)$:
  - in case $\epsilon_k(0) = 0$,
    \[ U_{k+1}^{(i)}(\nu) = \sum_{j=1}^{n_{k+1}-1-i} U_k^{(0)}(\nu - (j-1)s_k(0)) + U_k^{(1)}(\nu - (n_{k+1}-i-1)s_k(0)) \]
  - in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,
    \[ U_{k+1}^{(i)}(\nu) = \frac{n_{k+1}-1-i}{2} (U_k^{(0)}(\nu) + U_k^{(0)}(s_k(0) - \nu)) + U_k^{(1)}(\nu) \]
  - in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,
    \[ U_{k+1}^{(i)}(\nu) = \frac{n_{k+1}-i}{2} U_k^{(0)}(\nu) + \frac{n_{k+1}-2-i}{2} U_k^{(0)}(s_k(0) - \nu) + U_k^{(1)}(s_k(0) - \nu) \]
  - and $U_{k+1}^{(1)}(\nu) = U_k^{(1)}(\nu)$ in case $n_{k+1} = 2$.

**Transitions in terms of generating functions**

- For $k \geq 1$, $i = 0, 1$ and $Z \in \mathbb{S}^1$, where $(n_{k+1}, i) \neq (2, 1)$:
  - in case $\epsilon_k(0) = 0$,
    \[ \tilde{U}_{k+1}^{(i)}(Z) = \sum_{j=1}^{n_{k+1}-1-i} Z^{(j-1)s_k(0)} \tilde{U}_k^{(0)}(Z) + Z^{(n_{k+1}-1-i)s_k(0)} \tilde{U}_k^{(1)}(Z) \]
  - in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,
    \[ \tilde{U}_{k+1}^{(i)}(Z) = \frac{n_{k+1}-1-i}{2} (\tilde{U}_k^{(0)}(Z) + Z^{s_k(0)} \tilde{U}_k^{(0)}(Z^{-1}) + \tilde{U}_k^{(1)}(Z) \]
in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,

$$\widehat{U}_{k+1}^{(i)}(Z) = \frac{n_{k+1} - i}{2} \widehat{U}_k^{(0)}(Z) + \frac{n_{k+1} - 2 - i}{2} Z^{s_k(0)} \widehat{U}_k^{(0)}(Z^{-1}) + Z^{s_k(0)} \widehat{U}_k^{(1)}(Z^{-1});$$

and $\widehat{U}_{k+1}^{(1)}(Z) = \widehat{U}_k^{(1)}(Z)$ in case $n_{k+1} = 2$.

Simplified visit distributions.

Our next task is to eliminate the dependence of the visit distribution transitions on the positions $s_k(0)$. We’ll do this exploiting the parity sequence $(\epsilon_k : k \geq 1)$.

Parities, block lengths and positions

Let $k \geq 1$. The $k^{th}$:

- parity state vector is $\ell_k := \left( \begin{array}{c} \epsilon_k(0) \\ \epsilon_k(1) \end{array} \right)$;
- block length vector is $\ell_k := \left( \begin{array}{c} \ell_k(0) \\ \ell_k(1) \end{array} \right)$;
- position vector is $s_k := \left( \begin{array}{c} s_k(0) \\ s_k(1) \end{array} \right)$

where $s_k(i) := \sum_{j=0}^{\ell_k(i)-1} \varphi(\{j\alpha\})$.

Next let

$$A(0) := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad A(1) := \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$B(n) := \begin{pmatrix} n-1 & 1 \\ n-2 & 1 \end{pmatrix} \quad \text{and} \quad A(n) := A(n \mod 1),$$

then

$$s_{k+1} = \begin{cases} B(n_{k+1}) s_k & \epsilon_k(0) = 0; \\ A(n_{k+1}) s_k & \epsilon_k(0) = 1. \end{cases}$$

The following diagram gives (as in [3]) the parity transitions $\ell_k \rightarrow \ell_{k+1}$ conditional on $N = n_{k+1}$ together with the corresponding position vector transitions.
Lemma 3.1 (\textit{[3]}) For $k \geq 1$,

\[ s_k(0) = 1, \quad s_k(1) = 1 \quad \text{or} \quad s_k(0) - s_k(1) = 1 \]

according to whether

\[ \epsilon_k := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \]

respectively.

To this end, define $T_k = T_k(\epsilon_k) = T_k(\epsilon_k(0))$ by:

\[ T_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = T_k \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -s_k(0), \quad T_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -s_k(1). \]

Evidently

\[ (\times) \quad (\epsilon_k(0) = 1 \quad \implies \quad T_k = -s_k(0)) \implies \quad T_{k+1} - T_k = T_{k+1} + T_k + 2s_k(0) \]

The following is established by straightforward computation.
Proposition 3.2: Increments of the $T_k$s

\begin{align*}
(1) & \quad T_{k+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - T_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1, \\
(2) & \quad T_{k+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - T_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1, \\
(3) & \quad T_{k+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - T_k \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1, \\
(4) & \quad T_{k+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - T_k \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1, \\
(5) & \quad T_{k+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - T_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -(n_{k+1} - 1). 
\end{align*}

Simplified visit distribution transitions.

Given $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ we define the simplified visit distributions by

\[
V_k^{(i)}(J) := \begin{cases} 
U_k^{(i)}(\nu) & J = 2\nu + T_k, \quad \nu \in \mathbb{Z}, \\
0 & \text{else}.
\end{cases}
\]

where $T_k = T_k(\epsilon_k(0))$.

The generating functions are given by

\[
\widehat{V}_k^{(i)}(Z) = Z^{T_k} \widehat{U}_k^{(i)}(Z^2).
\]

They have the property that

\[
\int_{S^1} |\widehat{V}_k^{(i)}(Z)|^p dZ = \int_{S^1} |\widehat{U}_k^{(i)}(Z)|^p dZ \quad \forall \ p > 0
\]

and satisfy simpler recursions as follows:

For $k \geq 1$, $i = 0, 1$ and $Z \in S^1$, where $(n_{k+1}, i) \neq (2, 1)$:

- in case $\epsilon_k(0) = 0$ we have $s_k(0) = 1$ and $T_{k+1} - T_k = -(n_{k+1} - 1)$ and,

\[
\widehat{V}_{k+1}^{(i)}(Z) = Z^{T_{k+1}} \widehat{U}_{k+1}^{(i)}(Z^2)
= \sum_{j=1}^{n_{k+1}-1-i} Z^{2(j-1)+T_{k+1}} \widehat{U}_k^{(0)}(Z^2) + Z^{2(n_{k+1}-i-1)+T_{k+1}} \widehat{V}_k^{(1)}(Z^2)
= \sum_{j=1}^{n_{k+1}-1-i} Z^{2(j-1)+T_{k+1}} \widehat{V}_k^{(0)}(Z) + Z^{2(n_{k+1}-i-1)+T_{k+1}} \widehat{V}_k^{(1)}(Z)
= \sum_{j=1}^{n_{k+1}-1-i} Z^{2(j-1)-(n_{k+1}-1)} \widehat{V}_k^{(0)}(Z) + Z^{2(n_{k+1}-i-1)-2i} \widehat{V}_k^{(1)}(Z)
\]

In case $\epsilon_k(0) = 1$, we have $T_{k+1} - T_k = T_{k+1} + T_k + 2s_k(0) = 1$. 

in this case, for \( n_{k+1} - 1 - i \in 2\mathbb{Z} \),

\[
\hat{V}_{k+1}^{(i)}(Z) = Z^{T_{k+1}} \hat{U}_{k+1}^{(i)}(Z^2)
\]

\[
= \frac{n_{k+1} - 1 - i}{2} (Z^{T_{k+1}} \hat{U}_{k}^{(0)}(Z^2) + Z^{2s_k(0) + T_{k+1}} \hat{U}_{k}^{(0)}(Z^2)) + Z^{T_{k+1}} \hat{U}_{k}^{(1)}(Z^2)
\]

\[
= \frac{n_{k+1} - 1 - i}{2} (Z^{T_{k+1} - T_k} \hat{V}_{k}^{(0)}(Z) + Z^{2s_k(0) + T_{k+1} + T_k} \hat{V}_{k}^{(0)}(Z^{-1})) + Z^{T_{k+1} - T_k} \hat{V}_{k}^{(1)}(Z)
\]

\[
= \frac{n_{k+1} - 1 - i}{2} (Z \hat{V}_{k}^{(0)}(Z) + Z \hat{V}_{k}^{(0)}(Z^{-1})) + Z \hat{V}_{k}^{(1)}(Z)
\]

in this case, for \( n_{k+1} - 1 - i \in 2\mathbb{Z} + 1 \),

\[
\hat{V}_{k+1}^{(i)}(Z) = Z^{T_{k+1}} \hat{V}_{k+1}^{(i)}(Z^2)
\]

\[
= \frac{n_{k+1} - i}{2} Z^{T_{k+1}} \hat{U}_{k}^{(0)}(Z^2) + \frac{n_{k+1} - 2 - i}{2} Z^{2s_k(0) + T_{k+1}} \hat{U}_{k}^{(0)}(Z^{-2}) + Z^{2s_k(0) + T_{k+1}} \hat{U}_{k}^{(1)}(Z^{-2})
\]

\[
= \frac{n_{k+1} - i}{2} Z^{T_{k+1} - T_k} \hat{V}_{k}^{(0)}(Z) + \frac{n_{k+1} - 2 - i}{2} Z^{2s_k(0) + T_{k+1} + T_k} \hat{V}_{k}^{(0)}(Z^{-2}) + Z^{2s_k(0) + T_{k+1} + T_k} \hat{V}_{k}^{(1)}(Z^{-2})
\]

\[
= \frac{n_{k+1} - i}{2} Z \hat{V}_{k}^{(0)}(Z) + \frac{n_{k+1} - 2 - i}{2} Z \hat{V}_{k}^{(0)}(Z^{-2}) + Z \hat{V}_{k}^{(1)}(Z^{-2})
\]

and \( \hat{V}_{k+1}^{(1)}(Z) = Z \hat{V}_{k}^{(1)}(Z) \) in case \( n_{k+1} = 2 \).

### §4 Extracting the ARW

**Associated sequence of temporal probabilities.**

Now let \( P_{k}^{(i)} \in \mathcal{P}(\mathbb{Z}) \) be defined by

\[
P_{k}^{(i)}(\nu) := \frac{V_{k}^{(i)}(\nu)}{\ell_k(i)}.
\]

We call these “temporal probabilities” because

\[
P_{k}^{(0)}(\nu) = \frac{1}{\ell_k(0)} \# \{ 1 \leq j \leq \ell_k(0) : 2\varphi_j(0) + T_k = \nu \}.
\]

The **generating function** of \( P_{k}^{(i)} \) is \( \Phi_{k}^{(i)} : \mathbb{S}^1 \to \mathbb{C} \) defined by

\[
\Phi_{k}^{(i)}(Z) := \sum_{\nu \in \mathbb{Z}} P_{k}^{(i)}(\nu) Z^\nu = \frac{\hat{V}_{k}^{(i)}(Z)}{\ell_k(i)} \quad (Z \in \mathbb{S}^1).
\]
Set

$$\phi_N(Z) := \frac{1}{N} \sum_{k=0}^{N-1} Z^k \quad (N \geq 1) \text{ and } \phi_0 = \phi_{-1} \equiv 0;$$

$$p_{k+1}(i) := 1 - \frac{\ell_k(1)}{\ell_{k+1}(i)} = \frac{(n_{k+1} - 1 - i)\ell_k(0)}{\ell_{k+1}(i)};$$

$$q_N := 1 - \frac{\lfloor \frac{N}{2} \rfloor}{N}, \quad (N \geq 1) \text{ and } q_0 := 0.$$

**Generating function transitions.**

For $k \geq 1, \ i = 0, 1$ and $Z \in S^1$ we have,

- in case $\epsilon_k(0) = 0,$

$$\Phi_{k+1}^{(i)}(Z) = \frac{\ell_k(0)}{\ell_{k+1}(i)} \sum_{j=1}^{n_{k+1} - 1 - i} Z^{2(j-1) - (n_{k+1} - 1)} \Phi_k^{(0)}(Z) + \frac{1}{\ell_k(i)} Z^{n_{k+1} - 2i - 1} \Phi_k^{(1)}(Z)$$

$$= p_{k+1}(i) Z^{-(n_{k+1} - 1)} \phi_{n_{k+1} - 1 - i}(Z^2) \Phi_k^{(0)}(Z) + (1 - p_{k+1}(i)) Z^{n_{k+1} - 2i - 1} \Phi_k^{(1)}(Z)$$

- in case $\epsilon_k(0) = 1$ and $2 \mid n_{k+1} - 1 - i,$

$$\Phi_{k+1}^{(i)}(Z) = \frac{(n_{k+1} - 1 - i)\ell_k(0)}{2\ell_{k+1}(i)} (Z \Phi_k^{(0)}(Z) + Z \Phi_k^{(0)}(Z^{-1})) + \frac{1}{\ell_k(i)} Z \Phi_k^{(1)}(Z)$$

$$= \frac{p_{k+1}(i)}{2} (Z \Phi_k^{(0)}(Z) + Z \Phi_k^{(0)}(Z^{-1})) + (1 - p_{k+1}(i)) Z \Phi_k^{(1)}(Z)$$

- in case $\epsilon_k(0) = 1$ and $2 \nmid n_{k+1} - 1 - i,$

$$\Phi_{k+1}^{(i)}(Z) = \frac{(n_{k+1} - 1 - i)\ell_k(0)}{2\ell_{k+1}(i)} Z \Phi_k^{(0)}(Z) + \frac{(n_{k+1} - 2 - i)\ell_k(0)}{2\ell_{k+1}(i)} Z \Phi_k^{(0)}(Z^{-1}) + \frac{1}{\ell_k(i)} Z \Phi_k^{(1)}(Z^{-1})$$

$$= p_{k+1}(i) q_{n_{k+1} - 1} Z \Phi_k^{(0)}(Z) + p_{k+1}(i)(1 - q_{n_{k+1} - 1}) Z \Phi_k^{(0)}(Z^{-1}) + (1 - p_{k+1}(i)) Z \Phi_k^{(1)}(Z^{-1})$$

**Smelling the RATs.**

Next, for given $\alpha$ with $2\alpha = [n_1, n_2, \ldots]$ we’ll construct ARWs

$$X^{(k)} = (X^{(k)}(0), X^{(k)}(1)) \quad (k \geq 1)$$

so that

$$P([X^{(k)}(i) = \nu]) = P_k^{(\nu)}(\nu).$$

We call an ARW satisfying (**) an $\alpha$-ARW.
Let
\[ N_N \in \text{RV}(\mathbb{Z}), \quad P(N_N = 2k - (N - 1)) = \frac{1}{N}, \quad 0 \leq k \leq N - 1; \]
\[ x_k(i) \in \text{RV} (\{0, 1\}), \quad P(x_k(i) = 1) = p_k(i); \]
\[ y_N \in \text{RV} (\{0, 1\}), \quad P(y_N = 1) = q_N. \]

**Flip type RATs.**
We call a RAT \( F = (a, b) \in M_{d \times d}(\{-1, 0, 1\}) \times \mathbb{R}^d \), of flip type if
\[ a(k, L) = 1_{\{L(a(k, a) = L)\}} \bar{a}(k, L) \]
where each \( L(k, a) \) is a RV with values in \( \{1, 2, \ldots, d\} \) and each \( \bar{a}(k, L) \) is a RV with values in \( \{-1, 1\} \).

Flip type is preserved under composition. For \( F' = (a', b') \) and \( F = (a, b) \),
\[ F' \circ F = (a', b') \circ (a, b) = (a'a, a'b + b') \]
and
\[ L(k, a'a) = L(L(k, a'), a). \]

The \( \alpha \)-ARWs to be constructed will be generated by RATs of flip type. Indeed, henceforward, we only consider flip type RATs

**Linear recursion for characteristic functions of flip type ARWs.**
Let \( (a^{(n)}, b^{(n)}): n \geq 1 \) be a flip type RAT sequence and consider the generated ARW
\[ X^{(0)} = 0, \quad X^{(n)} = a^{(n)}X^{(n-1)} + b^{(n)}. \]

The characteristic functions of the coordinates of \( (X^{(n)}, -X^{(n)}) \) satisfy a linear recursion.

In the special case where \( P([a_{k,\ell}^{(n)} = -1]) = 0 \ \forall \ n \geq 1, \ 1 \leq k, \ell \leq d, \)
there is a simpler linear recursion for the characteristic functions of the coordinates of \( X^{(n)}. \)

Writing for the \( \mathbb{R}^d \)-valued random variable \( X = (X_1, \ldots, X_d) \):
\[ V_X(\theta) := \begin{pmatrix} \widehat{\Phi}_{X_1}(\theta) \\ \vdots \\ \widehat{\Phi}_{X_d}(\theta) \\ \widehat{\Phi}_{X_1}(-\theta) \\ \vdots \\ \widehat{\Phi}_{X_d}(-\theta) \end{pmatrix}, \]
where \( \widehat{\Phi}_Y(\theta) := E(e^{i\theta Y}) \) denotes the characteristic function of the \( \mathbb{R} \)-valued random variable \( Y \).
and for the flip type RAT $(a, b)$ independent of $X$:

$$X' = aX + b,$$

we have that

$$V_{X'}(\theta) = P(\theta)V_X(\theta).$$

Here $P(\theta) \in M_{2d \times 2d}(\mathbb{C})$ is given by

$$P_{k,L}(\theta) = P_{k,L}(0)\Phi_{C_{k,L}}(\theta)$$

where $P(0)$ is a stochastic matrix and $C_{k,L}$ $(1 \leq k, L \leq 2d)$ are random variables.

Specifically:

$$
\begin{align*}
P_{k,L}(0) &= P([\mathcal{Z}(k, a) = L])P([\overline{a}_{k,L} = 1]) \quad \text{and} \quad C_{k,L} = b_{k,L,1};
P_{k,d+L}(0) &= P([\mathcal{Z}(k, a) = L])P([\overline{a}_{k,L} = -1]) \quad \text{and} \quad C_{d+L,k} = b_{k,L,-1};
P_{d+k,L}(0) &= P([\mathcal{Z}(k, a) = L])P([\overline{a}_{k,L} = -1]) \quad \text{and} \quad C_{d+k,L} = -b_{k,L,-1};
P_{d+k,d+L}(0) &= P([\mathcal{Z}(k, a) = L])P([\overline{a}_{k,L} = 1]) \quad \text{and} \quad C_{d+k,d+L} = -b_{k,L,1}.
\end{align*}
$$

Here, for $1 \leq k, L \leq d$, $J \in \mathbb{Z}$ and for $\epsilon = \pm 1$,

$$P([b_{k,L,\epsilon} = J]) := P([b_k = J|[a_{k,L} = \epsilon]).$$

Equivalently, for $i, j = 0, 1$ and $\epsilon = 1 - 2i, \delta = 1 - 2j$,

$$P_{i\delta+d+k,j\delta+d+L}(\theta) = P([\mathcal{Z}(k, a) = L])P([\overline{a}_{k,L} = \delta])\Phi_{\epsilon\delta b_{k,\ell,\delta}}(\theta).$$

We’ll refer to the function $P : \mathbb{R} \to M_{2d \times 2d}(\mathbb{C})$ as the characteristic function of the flip type RAT: (abbr. RAT-CF).

**Construction procedures.**

Fix a sequence of independent random vectors

$$(x_{k+1}(i), y_{n_{k+1}-1-i}, N_{n_{k+1}-1-i}), \quad i = 0, 1)_{k \geq 1}$$

whose marginals are determined by $2\alpha = [n_1, n_2, \ldots]$ and ($\blacklozenge$).

Define $X^{(k)} = (X^{(k)}(0), X^{(k)}(1)) \in \mathcal{H}(\mathbb{Z}^2)$ by $X^{(0)}(i) = 0, \quad i = 0, 1$ and

- in case $\epsilon_k(0) = 0$,

$$X^{(k+1)}(i) = x_k(i)X^{(k)}(0) + (1-x_k(i))X^{(k)}(1) + x_k(i)N_{n_{k+1}-1-i} + (1-x_k(i))(n_{k+1}-2i-1);$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,

$$X^{(k+1)}(i) = x_k(i)(2y_{n_{k+1}-1-i} - 1)X^{(k)}(0) + (1-x_k(i))X^{(k)}(1) + 1$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,

$$X^{(k+1)}(i) = x_k(i)(2y_{n_{k+1}-1-i} - 1)X^{(k)}(0) - (1-x_k(i))X^{(k)}(1) + 1.$$
The form of the RATs and the independence of the random vectors implies that \((X^{(k)} : k \geq 0)\) is indeed a flip type ARW.

**Construction Lemma 4.1** A flip type ARW is an \(\alpha\)-ARW iff it is defined according to a construction procedure as above.

**Proof** Evidently, \((\star)\) holds if and only if the ARW generating function transitions are the same as those established above for the \((\Phi_k^{(i)}(Z) : k \geq 0, i = 0, 1)\). The latter correspond to linear recursions by RAT-CF of flip type RATs as above. \(\square\)

**Remark.** The plethora of \(\alpha\)-ARWs arises because of the variety of possible joint distributions of the random vectors

\[
\{(x_{k+1}(i), y_{n_{k+1}-1-i}, N_{n_{k+1}-1-i}) : \ i = 0, 1\}
\]

for fixed \(k \geq 1\).

**\(\alpha\)-RAT sequences.**

An \(\alpha\)-RAT sequence is a flip type RAT sequence \((F_k = (a^{(k)}, b^{(k)}): k \geq 1)\) which generates an \(\alpha\)-ARW \((X^{(k)}: k \geq 0)\) by

\[
X^{(0)} = 0, \ X^{(k)} = a^{(k)}X^{(k-1)} + b^{(k)}.
\]

The \(\alpha\)-RAT sequences \((F_k = (a^{(k)}, b^{(k)}): k \geq 1)\) satisfy, for \(n_k \geq 3\),

\[
(0) \quad a^{(k)} = \begin{pmatrix} x_k(0) & 1 - x_k(0) \\ x_k(1) & 1 - x_k(1) \end{pmatrix}, \quad \epsilon_{k-1} = 0;
\]

\[
(1) \quad a^{(k)} = \begin{pmatrix} x_k(0)(2y_{n_k} - 1) & -(1 - x_k(0)) \\ x_k(1)(2y_{n_k-1} - 1) & (1 - x_k(1)) \end{pmatrix}, \quad \epsilon_{k-1} = 1 \text{ and } 2 \mid n_k;
\]

\[
(1) \quad a^{(k)} = \begin{pmatrix} x_k(0)(2y_{n_k} - 1) & (1 - x_k(0)) \\ x_k(1)(2y_{n_k-1} - 1) & -(1 - x_k(1)) \end{pmatrix}, \quad \epsilon_{k-1} = 1 \text{ and } 2 \nmid n_k
\]

\[
(0) \quad b^{(k)} = \begin{pmatrix} x_k(0)N_{n_k-1} + (1 - x_k(0))(n_k - 1) \\ x_k(1)N_{n_k-2} + (1 - x_k(1))(n_k - 2) \end{pmatrix}, \quad \epsilon_{k-1} = 0;
\]

\[
(1) \quad b^{(k)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \epsilon_{k-1} = 1
\]
and for $n_{k+1} = 2$ by

\begin{align}
(0) \quad a^{(k)} &= \begin{pmatrix} x_k(0) & 1 - x_k(0) \\ 0 & 1 \end{pmatrix}, \quad \epsilon_{k-1} = 0; \\
(1) \quad a^{(k)} &= \begin{pmatrix} x_k(0)(2y_{nk} - 1) & -(1 - x_k(0)) \\ 0 & 1 \end{pmatrix}, \quad \epsilon_{k-1} = 1 \\
(0) \quad b^{(k)} &= \begin{pmatrix} 1 - x_k(0) \\ 0 \end{pmatrix}, \quad \epsilon_{k-1} = 0; \\
(1) \quad b^{(k)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \epsilon_{k-1} = 1.
\end{align}

**Special RATs.** We call RATs of the type defined above *special* (spec-RATs).

We define the parity of a spec-RAT $F$ as above to be *even* if it is defined by equations marked “(0)”, and to be *odd* if it is defined by equations marked “(1)”.

The coefficient associated to the spec-RAT $F_k$ as above is $n(F_k) := n_k$.

A spec-RAT with coefficient 2 is called *trivial*.

§5 Asymptotics of ARWs

**Norm of a matrix** Throughout this paper, we use the $L^\infty$-operator norm of matrices. Namely, the norm of the matrix $A \in M_{d \times d}(\mathbb{C}) \cong \mathbb{C}^{d^2}$ is

$$
\|A\| := \sup \{ \|Ax\|_\infty : x \in \mathbb{C}^d, \|x\|_\infty = 1 \} = \max_k \sum_{\ell=1}^{d} |A_k,\ell|
$$

where $\|(x_1, x_2, \ldots, x_d)\|_\infty := \max_{1 \leq k \leq d} |x_k|$.

**Norm of a RAT-CF.**

Let $P = P_F : \mathbb{T} \to M_{2d \times 2d}(\mathbb{C})$ be the characteristic function of the RAT $F = (a, b)$, then for $\theta \in \mathbb{T}$,

$$
(P(\theta)) = \max_{1 \leq j \leq 2d} \sum_{K=1}^{2d} |P_{j,K}(\theta)|
$$

$$
= \max_{1 \leq k \leq d} \sum_{L=1}^{d} \sum_{L=1}^{d} [P(a_{k,L} = 1)|\Phi_{b_k,L,1}(\theta)| + P(a_{k,L} = -1)|\Phi_{b_k,L,-1}(\theta)|]
\leq 1
$$

with equality iff for some $k$, $b_{k,\ell}$ is a constant random variable $\forall \ 1 \leq \ell \leq d$ and $\epsilon = \pm 1$ with $P(a_{k,\ell} = \epsilon) > 0$. 

Irreducibility, mean contractivity and balance.

We call the RAT $F$ irreducible if $P(a_{k,\ell}(F) \neq 0) > 0 \quad \forall \ 1 \leq k, \ell \leq d$ and mean contractive if $\|E(a(F))\| < 1$. Note that $F$ is mean contractive iff for each $1 \leq k \leq d \exists \ 1 \leq \mu \neq \nu \leq d$ so that $P(a_{k,\mu} = 0) > 0$ and $P(a_{k,\nu} = \epsilon) > 0 \quad \forall \ \epsilon = \pm 1$.

For spec-RATs, mean contractivity entails irreducibility.

Nontrivial spec-RATs with odd parity are irreducible and mean contractive.

Nontrivial spec-RATs with even parity are irreducible but not mean contractive.

Trivial spec-RATs are not irreducible.

Call $F$ balanced if $P(a_{k,\ell} = \epsilon) > 0 \quad \forall \ 1 \leq k, \ell \leq d$ and $\epsilon = \pm 1$. Balance entails both irreducibility and mean contractivity.

If either of $F'$, $F$ is irreducible, mean contractive or balanced, then so is $F' \circ F$.

If $F$ is irreducible and $F'$ is mean contractive, then $F' \circ F$ is balanced (but $F \circ F'$ may not be balanced)

Adapted RATs.

We’ll call the RAT $(a, b) \in M_{d \times d}(\mathbb{R}) \times \mathbb{R}^d$

- adapted if $\exists \ \theta \in \mathbb{T}$ so that $\|P(a, b)(\theta)\| < 1$;
- strongly adapted if $\|P(a, b)(\theta)\| < 1 \quad \forall \ \theta \neq 0$;
- partially adapted if $a$ has an adapted row i.e.: $\exists \ 1 \leq k \leq d$ and $\theta \in \mathbb{T}$ so that

$$\sum_{L=1}^{d} [P(a_{k,L} = 1)\hat{\Phi}_{k,L,1}(\theta)] + P(a_{k,L} = -1)[\hat{\Phi}_{k,L,-1}(\theta)] < 1;$$

equivalently $\exists \ 1 \leq \ell \leq d$ and $\epsilon = \pm 1$ with $P(a_{k,\ell} = \epsilon) > 0$, $b_{k,\ell,\epsilon}$ is a non-constant random variable.

Note that the RAT is adapted iff all rows are adapted.

The RAT $(a, b) \in M_{d \times d}(\mathbb{R}) \times \mathbb{R}^d$ with $E(\|b\|^2) < \infty$ is adapted iff

$$\kappa_F := \min_k \left[ -\sum_{\ell=1}^{2d} \frac{d^2}{d\theta^2} P_F(\theta)_{k,\ell} \right]_{\theta=0}$$

$$= \min_k \sum_{L=1}^{d} P(\mathcal{L}(k,a) = L)[P(\bar{a}_{k,L} = 1)\text{Var}(b_{k,L,1}) + P(\bar{a}_{k,L} = -1)\text{Var}(b_{k,L,-1})] > 0.$$
The periodicity group of the RAT $F = (a, b)$ is
\[
\Gamma_F := \{ \theta \in \mathbb{R} : \|P_F(\theta)\| = 1 \}.
\]
It follows from (\ref{eq:per}) that
\[
\Gamma_F = \bigcup_{k=1}^{d} \bigcap_{1 \leq \ell \leq d} \bigcap_{\epsilon = \pm 1, P(a_k, \ell, \epsilon) > 0} \Gamma_{b_k, \ell, \epsilon}
\]
where for a random variable $b$,
\[
\Gamma_b := \{ \theta \in \mathbb{R} : |\Phi_b(\theta)| = 1 \}.
\]
This is a closed subgroup of $\mathbb{R}$ and so, either $\Gamma_F = \mathbb{R}$ or $F$ is adapted and $\Gamma_F = g\mathbb{Z}$ for some $g = g_F > 0$.

Note that $F$ is strongly adapted iff $\Gamma_F = \{0\}$.

This is a closed subgroup of $\mathbb{R}$ and so, either $\Gamma_F = \mathbb{R}$ or $F$ is adapted and $\Gamma_F = g\mathbb{Z}$ for some $g = g_F > 0$.

In case $F$ is discrete, adapted, $g_F = \frac{2\pi}{N}$ for some $N \geq 1$ and $\Gamma_F = \{\frac{2\pi}{N}k : 0 \leq k \leq N - 1\}$ is a finite subgroup of $\mathbb{T}$. Moreover
\[
\|P_F(\theta + \gamma)\| = \|P_F(\theta)\| \quad \forall \ \theta \in \mathbb{T} \text{ and } \gamma \in \Gamma_F
\]
and $\forall \ 0 < \epsilon < \frac{1}{2|\Gamma_F|} \exists \ \Delta > 0$ so that
\[
\|P_F(\theta)\| \leq 1 - \Delta \quad \forall \ \theta \in \mathbb{T} \setminus B(\Gamma_F, \epsilon).
\]
Here, for $(X, d)$ a metric space, $\Gamma \subset X$ and $\epsilon > 0$,
\[
B(\Gamma, \epsilon) := \{ x \in X : \exists \ y \in \Gamma, \ d(x, y) \leq \epsilon \}.
\]

Adapted spec-RATs Any nontrivial spec-RAT $F$ with even parity is:
\begin{itemize}
  \item adapted if $n(F) \geq 4$ and
  \item partially adapted if $n(F) = 3$.
\end{itemize}

No trivial spec-RAT or spec-RAT with odd parity is partially adapted.

Spectral properties of RAT-CFs.

Let $F$ be a RAT. The RAT-CF $P_F : \mathbb{R} \to M_{2d \times 2d}(\mathbb{C})$ is a symmetric function in the sense that $P_F(-\theta) = \overline{P_F(\theta)}$.

Noting that
\[
P_F(\theta) = \begin{pmatrix} Q(\theta) & R(\theta) \\ R(-\theta) & Q(-\theta) \end{pmatrix} = \begin{pmatrix} Q(\theta) & \overline{R(\theta)} \\ \overline{R(\theta)} & \overline{Q(\theta)} \end{pmatrix}
\]
where
\[
Q(\theta) = \sum_{\nu} Q_{\nu} e^{i\nu\theta} \quad \text{and} \quad R(\theta) = \sum_{\nu} R_{\nu} e^{i\nu\theta}
\]
with each $Q_{\nu}$, $R_{\nu} \in M_{d \times d}(\mathbb{C})$.

By applying the row and column permutations
\[
(1, 2, \ldots, 2d) \mapsto (d + 1, \ldots, 2d, 1, 2, \ldots, d)
\]
(as in [3]), we see that for $\lambda \in \mathbb{C}$,

$$\det(P_F(\theta) - \lambda I) = \det(P_F(-\theta) - \lambda I) = \det(P_F(\theta) - \lambda I).$$

It follows that

$$(\mathcal{E}) \quad \det(P_F(\theta) - \lambda I) = \sum_{k=0}^{2d} c_k(\theta)\lambda^k$$

where $c_k : \mathbb{T} \to \mathbb{R}$ is even.

**Spectral theory of RAT-CFs.**

As seen above a RAT-CF is a perturbation of a positive operator. The spectral theory of perturbations (as in chapter III of [12]) is applicable.

Let $F$ be a RAT with $E(\|b(F)\|^2) < \infty$. We call $P_F(0)$ *positive* if $P(0)_{k,\ell} > 0 \forall 1 \leq k, \ell \leq d$ (equivalently, $F$ is balanced), and *aperiodic* if $\exists N \geq 1$ with $P_F(0)^N$ is positive.

Suppose that $P_F(0)$ is aperiodic.

By the Perron-Frobenius theorem, $1 = \|P_F(0)\|$ is a simple, dominant eigenvalue (i.e. its eigenspace $\mathbb{C} \cdot 1$ is one-dimensional and all other eigenvalues are smaller in absolute value).

Since $\theta \mapsto P_F(\theta)$ is $C^2$, by the implicit function theorem, $\exists \epsilon > 0$ so that for $|\theta| < \epsilon$, $P_F(\theta)$ also has a simple, dominant eigenvalue $\lambda_F(\theta) \in \mathbb{C}$ where $\lambda_F : (-\epsilon, \epsilon) \to \mathbb{R}$ is differentiable.

By $(\mathcal{E})$ $\lambda_F : (-\epsilon, \epsilon) \to \mathbb{R}$ is a real valued, even function with $\lambda_F(0) = 0$ and $\gamma_F := -\lambda_F'(0) \geq 0$ with equality iff $\lambda_F \equiv 1$.

If $F$ is adapted, then $\gamma_F > 0$. However, it may be that $\gamma_F > 0$ and $\|P_F\| = 1$.

Fix $\pi = \pi_F \in \mathbb{R}_+^d$ satisfying

$$\langle \pi, 1 \rangle = 1 \text{ and } P_F(0)^t \pi = \pi.$$

There is a unique eigenvalue function $\eta = \eta_F : (-\epsilon, \epsilon) \to \mathbb{C}^{2d}$ so that

$$P_F(\theta)\eta(\theta) = \lambda(\theta)\eta(\theta) \text{ and } \langle \pi, \eta(\theta) \rangle = 1 \forall |\theta| < \epsilon.$$

It follows that $\eta(0) = 1$ and $\eta(-\theta) = \overline{\eta(\theta)}$.

**Theorem 5.1 Coordinate distributional limits for stationary ARWs**

Let $F_n = (a_n, b_n) \in M_{d \times d}(\mathbb{R}) \times \mathbb{R}^d$ ($n \geq 1$) be an iid RAT sequence with each $F_n \overset{\text{dist}}{=} F$ with $F$ adapted and $P_F(0)$ aperiodic and let $(X_1, X_2, \ldots)$ be the associated ARW, then for each $1 \leq k \leq d$

$$(\text{CLT}) \quad \frac{(X_n)_k}{\sqrt{n}} \overset{\text{dist}}{\to} N(0, \gamma_F)$$
If, in addition, $F$ is strongly adapted, then for any bounded interval $J \subset \mathbb{R}$,
\[
\sqrt{n} P\left(\left( X_n \right)_k \in t_n + J \right) \xrightarrow[n \to \infty]{\mathcal{L}} \frac{1}{2 \pi \gamma_F} e^{-\frac{x^2}{2 \pi F}}.
\]

Proof sketch Let $P = P_F$ and let $\pi \in \mathbb{R}^{2d}$ be so that $P_F(0)^t \pi = \pi$.

For each $\theta \in (-\epsilon, \epsilon)$, $\exists! \eta(\theta) \in \mathbb{C}^d$ so that $P_F(\theta) \eta(\theta) = \lambda(\theta) \eta(\theta)$ and $\langle \eta(\theta), \pi \rangle = 1$.

It follows that $\eta : (-\epsilon, \epsilon) \to \mathbb{C}^d$ is a smooth, symmetric function.

Again, for each $\theta \in (-\epsilon, \epsilon)$, $\exists! \pi(\theta) \in \mathbb{C}^d$ so that for $\theta \in (-\epsilon, \epsilon)$,
\[
P_F(\theta) \pi(\theta) = \lambda(\theta) \pi(\theta) \quad \text{and} \quad \langle \pi(\theta), \eta(\theta) \rangle = 1.
\]

It follows that $\pi : (-\epsilon, \epsilon) \to \mathbb{C}^d$ is also a smooth symmetric function.

Define the projection $N(\theta) : \mathbb{C}^d \to \mathbb{C}^d$ by
\[
N(\theta) x := \langle x, \pi(\theta) \rangle \eta(\theta).
\]

We see that
\[
P_F(\theta) N(\theta) = \lambda(\theta) N(\theta),
\]

and that $Q(\theta) := P_F(\theta) - P_F(\theta) N(\theta) : \mathbb{C}^d \to \mathbb{C}^d$ satisfies $QN = NQ = 0$.

The spectral radius of $Q(0)$, $r(Q(0)) < 1$. By continuity of $\theta \mapsto Q(\theta)$, for possibly smaller $\epsilon > 0$,
\[
r(Q(\theta)) \leq \rho < 1 \quad \forall \left| \theta \right| < \epsilon.
\]

Thus
\[
V_{X_n}(\theta) = P_F(\theta)^n \mathbb{1} = \lambda(\theta)^n \eta(\theta) + O(\rho^n).
\]

The first statement follows directly from this.

The second follows also since if $F$ is strongly adapted, $\sup_{\theta \in [-\epsilon, M]} |\lambda(\theta)| < 1 \quad \forall \ 0 < \epsilon < M < \infty$. This is seen via standard proofs of the local limit theorem (see [4]).

Remark: Irreducible, positive RATs

The above does not include the important case of an irreducible RAT $(a, b)$ with $P(a_k, \ell) = 0 \forall \ k, \ \ell$. Here there is possible linear drift, but the $d$-dimensional vectors $(\tilde{\Phi}_X^{(n)}, \tilde{\Phi}_X^{(n)}, \ldots, \tilde{\Phi}_X^{(n)})$ satisfy a linear recursion and an analogous spectral argument applies.

Remark: Quadratic $\alpha$.

The following is a special case of a subsequence version of theorem 1.1 in [5] (see also [4]).
Proposition 5.2 Suppose that $\alpha \in (0, 1)$ is quadratic. There are $K$, $L \in \mathbb{N}$ and constants $c > 0$, $\mu \in \mathbb{R}$ so that for intervals $I \subset \mathbb{R}$

$$\frac{1}{\ell_{K+Ln}(0)} \# \left\{ 1 \leq j \leq \ell_{K+Ln}(0) : \frac{\varphi_j^{(\alpha)}(0) - \mu n}{c\sqrt{n}} \in I \right\} \xrightarrow{n \to \infty} \frac{1}{2\pi} \int_I e^{-t^2} dt.$$ 

Here, $\varphi_j^{(\alpha)}(0) := \sum_{i=0}^{j-1} \varphi \circ r_\alpha(0)$. This does not follow directly from CLT in theorem 5.1 as the $\alpha$-RAT sequence is not eventually periodic (although it converges to an eventually periodic RAT sequence). However, the transitions of the generating functions of the simplified visit distributions are eventually periodic and the result can be deduced from this using arguments in section 5 of [3] as was (R) (see page 2) in this case.

For general $\alpha \in \text{BAD}$ there need be no eventual periodicity of renormalization related sequences. New tools are needed.

§6 The weak, rough local limit theorem

Variance of ARWs.

We call the RAT $(a, b)$ centered if $E(b) = 0$ and the RAT sequence $(F_n = (a^{(n)}, b^{(n)}) : n \geq 1)$ centered if each RAT $F_n = (a^{(n)}, b^{(n)})$ is centered.

From the formula

$$X^{(n)} = F_n(X^{(n-1)}) = a^{(n)} X^{(n-1)} + b^{(n)}$$

we see that

$$E(X^{(n)}) = E(a^{(n)}) E(X^{(n-1)}) + E(b^{(n)})$$

whence the RAT sequence $(F_n = (a^{(n)}, b^{(n)}) : n \geq 1)$ is centered if and only if

$$E(X^{(n)}) = 0 \ \forall \ n \geq 1.$$

6.1 Variance Theorem

Let $(F_n := (a^{(n)}, b^{(n)}) : n \geq 1)$ be a centered, independent RAT sequence and let

$$X^{(n)} := F_1^n(0) = F_n \circ F_{n-1} \cdots \circ F_1(0)$$

be the corresponding ARW, then for $1 \leq k \leq d$ and $n \geq 1$,

$$\sum_{\nu=1}^{n} \min_{1 \leq L \leq d} E([b^{(\nu)}_L]^2) \leq E([X^{(n)}_k]^2) \leq \sum_{\nu=1}^{n} \max_{1 \leq L \leq d} E([b^{(\nu)}_L]^2).$$

Proof

Set
\[ Y^{(n,\nu)} := a_{\nu+1}^{n} b^{(\nu)} \quad (1 \leq \nu \leq n) \]

where

\[
a^N_k := \left\{ \begin{array}{ll} a^{(N)} a^{(N-1)} \ldots a^{(k)} & k \leq N \\ \text{Id} & k > N. \end{array} \right.
\]

so that

\[ X^{(n)} = \sum_{\nu=1}^{n} Y^{(n,\nu)}. \]

As above, \( E(Y^{(n,\nu)} Y^{(n,\mu)}) = 0 \) for each \( 1 \leq K \leq d, \ n \geq 1, \) the random variables \( \{Y^{(n,\nu)} : 1 \leq \nu \leq n\} \) are orthogonal, i.e.

\[ E(Y^{(n,\nu)} Y^{(n,\mu)}) = 0 \quad \forall \ \mu \neq \nu. \]

**Proof of \( \Box \)**

We have for \( 1 \leq K \leq d, \) that

\[ Y^{(n,\nu)} = (a^N_K)_{K,\ell(K,a_{\nu+1}^{n})} b^{(\nu)}_{\ell(K,a_{\nu+1}^{n})} \]

and for \( 1 \leq \nu < \mu \leq n, \) that

\[
\ell(K,a_{\nu+1}^{n}) = \ell(\ell(K,a_{\mu+1}^{n}), a^{\mu}_{\nu+1}) \quad \text{and} \quad
\]

\[
(a_{\nu+1}^{n})_{K,\ell(K,a_{\nu+1}^{n})} = (a_{\nu+1}^{n})_{K,\ell(K,a_{\mu+1}^{n})} (a^{\mu}_{\nu+1})_{\ell(K,a_{\mu+1}^{n}), \ell(K,a_{\mu+1}^{n}), a^{\mu}_{\nu+1}}.
\]

Consequently

\[
Y^{(n,\nu)} Y^{(n,\mu)} =
\]

\[= (a_{\nu+1}^{n})_{K,\ell(K,a_{\nu+1}^{n})} b^{(\nu)}_{\ell(K,a_{\nu+1}^{n})} (a_{\mu+1}^{n})_{K,\ell(K,a_{\mu+1}^{n})} b^{(\mu)}_{\ell(K,a_{\mu+1}^{n})}
\]

\[= (a^{\mu}_{\nu+1})_{\ell(K,a_{\mu+1}^{n}), \ell(K,a_{\mu+1}^{n}), a^{\mu}_{\nu+1}} b^{(\nu)}_{\ell(K,a_{\mu+1}^{n}), a^{\mu}_{\nu+1}) b^{(\mu)}_{\ell(K,a_{\mu+1}^{n})}
\]

\[= \sum_{L=1}^{d} 1[\ell(K,a_{\mu+1}^{n}) = L] b^{(\mu)}_{L} \sum_{M=1}^{d} 1[\ell(L,a_{\nu+1}^{n}) = M] (a_{\nu+1}^{n})_{L,M} b^{(\nu)}_{M}.
\]

whence by independence and centering,

\[
E(Y^{(n,\nu)} Y^{(n,\mu)})
\]

\[= \sum_{L=1}^{d} P[\ell(K,a_{\mu+1}^{n}) = L)] E(b^{(\mu)}_{L}) \sum_{M=1}^{d} E(1[\ell(L,a_{\nu+1}^{n}) = M] (a_{\nu+1}^{n})_{L,M} E(b^{(\nu)}_{M})
\]

\[= 0. \quad \Box
\]

It follows from (\( \ast \)) that
\[ E(Y_{K}^{(n,\nu)2}) = \sum_{L=1}^{d} P([\ell(K, a_{\nu+1}^{n}) = L]) E(b_{L}^{(\nu)2}) \]

and from the above that
\[ E(X_{K}^{(n)2}) = \sum_{\nu=1}^{n} E(Y_{K}^{(n,\nu)2}) \].

The Variance Theorem follows from this. \( \square \)

Compactness properties of RAT sequences.

We’ll say that the RAT sequence
\[(F_{n} = (a^{(n)}, b^{(n)}): n \geq 1) \in \text{RV}(M_{d\times d}(\mathbb{R}) \times \mathbb{R}^{d})^{\mathbb{N}}\]

- is adapted if \( \exists \) a finite subgroup \( \Gamma \leq \mathbb{T} \) (called the adaptivity group) so that
  \[ |\widehat{\Phi}_{b_{k,\ell,\epsilon}}^{(n)}(\theta)| = 1 \implies \theta \in \Gamma, \]
  \[ \forall \epsilon > 0 \exists \delta > 0 \text{ so that } \]
  \[ |\widehat{\Phi}_{b_{k,\ell,\epsilon}}^{(n)}(\theta)| \leq 1 - \delta \forall 1 \leq k, \ell \leq d, \ n \geq 1 \text{ and } \theta \in \mathbb{T} \setminus B(\Gamma, \delta). \]

- is uniform if \( \sup \{|E(b_{k,\ell,\epsilon}^{(n)})| : < \infty \forall 1 \leq k, \ell \leq d, \ n \geq 1, \ \epsilon = \pm 1 \}

  (a) the collection
  \[ \mathcal{F}_{F} := \{ |\widehat{b}_{k,\ell,\epsilon}^{(n)}|^{2} : 1 \leq k, \ell \leq d, \ n \geq 1, \ \epsilon = \pm 1 \} \]
  is uniformly integrable where \( \overline{Y} := Y - E(Y) \) denotes the centering of the integrable random variable \( Y \); and

  (b) \( \exists M > 1 \) so that
  \[ \text{Var} \left(b_{k,\ell,\epsilon}^{(n)}\right) = M^{\pm 1} \sum_{j=1}^{d} \text{Var} \left(b_{j}^{(n)}\right) \]
  \[ \forall n \geq 1, \ 1 \leq k, \ell \leq d, \ \epsilon = \pm 1 \ \text{with} \ P(a_{k,\ell}^{(n)} = \epsilon) > 0. \]

It is standard to show that uniformity of \((a^{(n)}, b^{(n)}): n \geq 1)\) implies that \( \exists M > 1 \) so that \( \text{Var} \left(b_{k,\ell,\epsilon}^{(n)}\right) = M^{\pm 1} \forall n \geq 1, \ 1 \leq k, \ell \leq d, \ \epsilon = \pm 1 \)

with \( P(a_{k,\ell}^{(n)} = \epsilon) > 0. \)

The above properties do not entail discreteness and we’ll need the WRLLT for certain non-discrete ARWs.
Theorem 6.2 (WRLLT) Let \( (F_n = (a^{(n)}, b^{(n)}): n \geq 1) \in (M_{d \times d}(\mathbb{R}) \times \mathbb{R}^d)^\mathbb{N} \) be a RAT sequence.

Suppose that \( (F_n: n \geq 1) \) is adapted, centered and uniform and let \( (X^{(n)})_{n \geq 1} \) be the corresponding ARW.

For each \( 1 \leq k \leq d \) and \( 1 \leq p \leq 2 \),
\[
\int_\mathbb{T} |\overline{\Phi}_{X_k^{(n)}}(\theta)|^p d\theta \lesssim \frac{1}{\sqrt{n}}.
\]

Proof of the WRLLT Since the integral decreases with \( p \), it suffices to show that \( \exists M > 1 \) so that \( \forall n \in \mathbb{N} \) large,
\[
(a) \quad \int_\mathbb{T} |\overline{\Phi}_{X_k^{(n)}}(\theta)|^2 d\theta \geq \frac{1}{M \sqrt{n}} \quad \text{and} \quad (b) \quad \int_\mathbb{T} |\overline{\Phi}_{X_k^{(n)}}(\theta)| d\theta \leq \frac{M}{\sqrt{n}}.
\]

Proof of (a) It follows from adaptedness, centeredness and the Variance theorem that \( \exists \Gamma > 0 \) so that
\[
E((X_k^{(n)})^2) \leq \Gamma n \quad (1 \leq k \leq d).
\]

Next, fix \( M = 2\sqrt{\Gamma} \), then by Chebyshev’s inequality,
\[
P(\{|X_k^{(n)}| \leq M \sqrt{n}\}) \geq \frac{3}{4}.
\]

Now fix \( \Delta > 0 \) so that
\[
|1 - e^{it}| < \frac{1}{4} \quad \forall \quad |t| < \Delta.
\]

We have
\[
\sqrt{n} \int_\mathbb{T} |E(e^{i\theta X_k^{(n)}})|^2 d\theta = \int_{-\pi \sqrt{n}}^{\pi \sqrt{n}} |E(\exp[it \frac{X_k^{(n)}}{\sqrt{n}}])|^2 dt \geq \int_{-\Delta \sqrt{n}}^{\Delta \sqrt{n}} |E(\exp[it \frac{X_k^{(n)}}{\sqrt{n}}])|^2 dt.
\]
For \( |t| < \frac{\Delta}{\sqrt{n}} \), we have
\[
|E(\exp[it \frac{X_k^{(n)}}{\sqrt{n}}])| \geq |E(\exp[it \frac{X_k^{(n)}}{\sqrt{n}}]1_{\{|X_k^{(n)}| < M \sqrt{n}\}})| - P(\{|X_k^{(n)}| \geq M \sqrt{n}\}) \geq \frac{3}{4} \cdot \frac{3}{4} - \frac{1}{4} = \frac{5}{16}
\].
whence

\[
\sqrt{n} \int_{\mathbb{T}} |E(e^{i\theta X_k(n)})|^2 d\theta \geq \int_{\triangle} |E(\exp[it \frac{X_k(n)}{\sqrt{n}}])|^2 dt \\
\geq \frac{2\Delta}{M} \cdot \frac{25}{256} \quad \Box \text{(a)}
\]

**Proof of (b)** Suppose that \( \Gamma = \{ \frac{k}{N} : 0 \leq k \leq N - 1 \} \subset \mathbb{T} \).

We claim first that

- \( \exists \rho \in (0, 1), \ 0 < \Delta < \frac{1}{2N} \) and \( \beta > 0 \) so that for each \( n \geq 1 \) and \( \gamma \in \Gamma_{F_n} \),

  (i) \( \| P_{F_n}(\gamma + \theta) \| \leq 1 - \beta \theta^2 \ \forall \ \| \theta \| < \Delta \); and

  (ii) \( \| P_{F_n}(\gamma + \theta) \| \leq \rho \ \forall \ \theta \notin B(\mathcal{F}, \Delta) \)

  (iii) \( \rho \leq 1 - \beta \Delta^2 \).

**Proof of (i)** Let \( Y \) be a random variable with finite second moment and let \( W = Y - Y' \) be its symmetrization. We have for \( \theta \in \mathbb{T} \simeq (-\frac{1}{2}, \frac{1}{2}) \) that

\[
|\Phi_Y(\theta)|^2 = \Phi_W(\theta) = 1 - 2E(\sin^2(\frac{W\theta}{2}))
\]

Now

\[
E(\sin^2(\frac{W\theta}{2})) \geq E(1[|W| \leq \frac{\pi}{\theta}]) \sin^2(\frac{W\theta}{2}) \\
\geq \frac{2\theta^2}{\pi^2} E(1[|W| \leq \frac{\pi}{\theta}]) W^2 \\
= \frac{2\theta^2}{\pi^2} E(W^2)(1 - \Delta_Y(\theta))
\]

where

\[
\Delta_Y(\theta) := \frac{E(1[|W| \geq \frac{\pi}{\theta}]) W^2)}{E(W^2)}.
\]

It follows from uniformity that \( \exists \Delta > 0, \ \Delta < \frac{1}{2N} \) so that

\[
\Delta_{x(n)} < \frac{1}{2} \ \forall \ \| \theta \| < \Delta, \ 1 \leq k, \ell \leq d, \ n \geq 1
\]
whence for $|\theta| < \Delta$ and $\gamma \in \Gamma_{F_n}$,

$$
\|P_{F_n}(\gamma + \theta)\| = \|P_{F_n}(\theta)\| \\
= \max_{1 \leq k \leq d} \sum_{L=1}^{d} P(\xi(k, a) = L)[P(\bar{a}_{k,L} = 1)\Phi_{b_{k,L,1}}(\theta) + P(\bar{a}_{k,L} = -1)\Phi_{b_{k,L,-1}}(\theta)] \\
\leq 1 - \frac{\theta^2}{\pi^2} \min_{1 \leq k \leq d} \sum_{L=1}^{d} P(\xi(k, a) = L)[P(\bar{a}_{k,L} = 1)\text{Var}(b^{(n)}_{k,k,1}) + P(\bar{a}_{k,L} = -1)\text{Var}(b^{(n)}_{k,k,-1})] \\
= 1 - \frac{\kappa_{F_n} \theta^2}{\pi^2} \\
\leq 1 - \beta \theta^2
$$

for any $0 < \beta \leq \min_{n \geq 1} \frac{\kappa_{F_n}}{\pi^2}$ which latter is positive by uniformity. $\varnothing$(i)

Proof of (ii) and (iii) Statement (ii) follows from adaptedness. Statement (iii) can be obtained by shrinking $\beta$. $\varnothing$(ii), (iii) and $\varnothing$.

To complete the proof of (b), we have

$$
|\Phi_{X_k^{(n)}}(\theta)| = |P_{F_n}(\theta)P_{F_{1-1}}(\theta)\cdots P_{F_1}(\theta)|_k| \\
\leq \prod_{k=1}^{n} \|P_k(\theta)\|.
$$

By $\varnothing$(i), for $n \geq 1$, $\gamma \in \Gamma_{F_n}$ we have

$$
\|P_{F_n}(\gamma + \theta)\| \leq 1 - \beta \theta^2 \quad \forall \quad |\theta| < \Delta
$$

and by $\varnothing$(ii)

$$
\|P_{F_n}(t)\| \leq \rho \quad \forall \quad t \notin B(\Gamma_{F_n}, \Delta).
$$

It follows that for $\gamma \in \Gamma \setminus B(\Gamma_{F_n}, \Delta)$ and $|\theta| < \Delta$, we have

$$
\|P_{F_n}(\gamma + \theta)\| \leq \rho \quad \text{by } \varnothing$(ii) \\
\leq 1 - \beta \theta^2 \quad \text{by } \varnothing$(iii).

Thus for $\gamma \in \Gamma$:

$$
|\Phi_{X_k^{(n)}}(\gamma + \theta)| \leq (1 - \beta \theta^2)^n \quad |\theta| < \Delta
$$

and

$$
|\Phi_{X_k^{(n)}}(t)| \leq \rho^n \quad \forall \quad t \notin B(\Gamma, \Delta),
$$
Discrepancy Skew Products and Affine Random Walks.

whence

\[
\int_{\Gamma} \Phi_{X_k^n}(\theta) d\theta \leq \left( \int_{B(\Gamma, \Delta)} + \int_{T \setminus B(\Gamma, \Delta)} \right) \prod_{k=1}^{n} \|P_k(\theta)\| d\theta
\]

\[
\leq \sum_{\gamma \in \Gamma} \int_{-\Delta}^{\Delta} \prod_{k=1}^{n} \|P_k(\gamma + \theta)\| d\theta + \int_{T \setminus B(\Gamma, \Delta)} \prod_{k=1}^{n} \|P_k(\theta)\| d\theta
\]

\[
\leq N \int_{-\Delta}^{\Delta} (1 - \beta \theta^2)^n d\theta + \rho^n
\]

\[
\leq \frac{N}{\sqrt{n}} \int_{-\Delta}^{\Delta\sqrt{n}} (1 - \frac{\beta t^2}{n})^n dt + \rho^n
\]

\[
\leq \frac{N}{\sqrt{n}} \int_{\mathbb{R}} e^{-\beta t^2} dt + \rho^n
\]

\[
\propto \frac{1}{\sqrt{n}} \varnothing(b) \text{ and WRLLT}
\]

§7 Centering of a RAT sequence

Let \((F_n = (a^{(n)}, b^{(n)}): n \geq 1)\) be an independent RAT sequence and let \(X^{(n)} := F_n \circ F_{n-1} \circ \cdots \circ F_1(0)\) and \(c_n := E(X^{(n)})\). Set \(\tilde{X}^{(n)} = X^{(n)} - c_n\), then

\[
\tilde{X}^{(n)} = X^{(n)} - c_n
\]

\[
= a^{(n)} X^{(n-1)} + b^{(n)} - c_n
\]

\[
= a^{(n)} \tilde{X}^{(n-1)} + b^{(n)} - c_n + a^{(n)} c_{n-1}
\]

\[
= \tilde{F}_n(\tilde{X}^{(n-1)})
\]

\[
= \tilde{F}_n \circ \tilde{F}_{n-1} \circ \cdots \circ \tilde{F}_1(0).
\]

and so \((\tilde{X}^{(n)})_{n \geq 1}\) is a centered ARW with the corresponding centered independent, RAT sequence \((\tilde{F}_n = (a^{(n)}, \tilde{b}^{(n)}): n \geq 1)\) where

\[
\tilde{b}^{(n)} = b^{(n)} - c_n + a^{(n)} c_{n-1} = 0.
\]

The ARW and associated RAT sequence above are unique. We call them the centerings of the ARW and associated RAT sequence respectively.

**Proposition 7.1** Let \((X^{(n)}: n \geq 1)\) be an ARW with centering \((\tilde{X}^{(n)})_{n \geq 1}\), then for each \(n \geq 1\) and \(1 \leq k \leq d\),

\[
\text{Var}(X_k^{(n)}) = E(\tilde{X}_k^{(n)})^2).
\]
Adaptedness preservation.

The centering of a discrete ARW may not be discrete, but centering does not affect adaptedness or any of the other compactness properties.

This is because if \( (F_n : n \geq 1) \) is a RAT with centering \( (\tilde{F}_n : n \geq 1) \), then
\[
\tilde{F}_n = (a(F_n), b(F) - c(n) + a(F)c(n-1))
\]
where \( c(n) \in \mathbb{R}^d \) \( n \geq 1 \) are constant.

Thus, for \( 1 \leq k, \ell \leq d \) and \( \epsilon = \pm 1 \) with \( P(a_k,\ell = \epsilon) > 0 \), the random variable \( b_{k,\ell,\epsilon} \) is a constant translation of \( b_{k,\ell,\epsilon} :\)
\[
\tilde{b}_{k,\ell,\epsilon}(\tilde{F}_n) = b_{k,\ell,\epsilon}(F_n) - c_k + \epsilon c_{\ell}(n-1).
\]
It follows from \( \ast \) that the adaptedness of \( (F_n : n \geq 1) \) is equivalent to that of \( (\tilde{F}_n : n \geq 1) \) and the adaptivity groups are the same.

We’ll need conditions for \( \sup_{n \geq 1} \|b(n) - \tilde{b}(n)\| < \infty \) a.s..

Bounded mean fluctuation.

We say that the ARW \( (X(n) := F_n \circ F_{n-1} \circ \cdots \circ F_1(0) : n \geq 1) \) (and its associated RAT sequence) has bounded mean fluctuations BMF if
\[
\sup_{n \geq 1} \|E(X(n))\| < \infty.
\]

7.2 Bounded Centering Proposition Let \( (F_n = (a(n), b(n)) : n \geq 1) \) be a RAT sequence with centering \( (\tilde{F}_n = (a(n), \tilde{b}(n)) : n \geq 1) \).

If \( (F_n = (a(n), b(n)) : n \geq 1) \) has BMF, then \( \exists M > 1 \) so that
\[
\|\tilde{b}(n) - b(n)\| \leq M \quad a.s. \quad \forall \ n \geq 1.
\]

Proof Since
\[
\tilde{b}(n) - b(n) - c_n + a(n)c_{n-1}
\]
and \( \|a(n)\| \leq 1 \) a.s., the proposition holds with
\[
M = 2 \sup_{n \geq 1} \|E(X(n))\|. \quad \square
\]
The following gives sufficient conditions for BMF of a RAT sequence in terms of uniform mean boundedness and uniform mean contractivity.

7.3 Lemma Let \( (F_n = (a(n), b(n)) : n \geq 1) \) be a RAT sequence. If \( \exists M > 0 \) and \( \rho \in (0, 1) \) so that
\[
\|E(b(k))\| \leq M \quad \text{and} \quad \|E(a(k))\| \leq \rho \quad \forall \ k \geq 1,
\]
then \( (F_n = (a(n), b(n)) : n \geq 1) \) has BMF.
Proof

$$X^{(n)} := F_n \circ F_{n-1} \circ \cdots \circ F_1(0) = \sum_{\nu=1}^{n} a_{\nu+1}^{n} b^{(\nu)}$$

where

$$a_{\nu} := \begin{cases} a^{(N)} a^{(N-1)} \cdots a^{(k)} & \text{if } k \leq N, \\ \text{Id} & \text{if } k > N, \end{cases}$$

we have by independence that

$$E(X^{(n)}) = E\left(\sum_{\nu=1}^{n} a_{\nu+1}^{n} b^{(\nu)}\right) = \prod_{k=1}^{n} E(a^{(k)}) E(b^{(\nu)})$$

whence

$$\|E(X^{(n)})\| \leq \sum_{\nu=1}^{n} \prod_{k=1}^{n} \|E(a^{(k)})\| \|E(b^{(\nu)})\| \leq M \sum_{k=1}^{\infty} \rho^k = \frac{M \rho}{1 - \rho} < \infty.$$

§8 Proof of the main result

Our first step in the proof of (ii) (as on page 2) is to show that for $\alpha \in \text{BAD}$, each $\alpha$-ARW satisfies the WRLLT along a syndetic subsequence.

A canonical subsequence. Define $\nu : \mathbb{N}^2 \to \mathbb{N} \cup \{\infty\}$ by

$$\nu(n_1, n_2, \ldots) := \min \{J \geq 1 : \sum_{k=1}^{J} 1_{[n_k > 2]} \geq 4\}.$$

The number $\beta = 2\alpha = [n_1, n_2, \ldots]$ is irrational iff $\#\{k \geq 1 : n_k > 2\} = \infty$ so $\nu(n_{k+1}, n_{k+2}, \ldots) < \infty \ \forall \ k \geq 1$.

Define

$$\nu_0 = 0, \ \nu_{k+1} := \nu_k + \nu(n_{\nu_k+1}, n_{\nu_k+2}, \ldots),$$

then $\nu_k < \infty \ \forall \ k \geq 1$.

By [15], $\alpha \in \text{BAD}$ iff

(i) $\sup_{k \geq 1} n_k < \infty$ and (ii) $\sup \{J \geq 1 : \exists k, \ n_{k+j} = 2 \ \forall \ 1 \leq j \leq J\} < \infty$.

Thus in case $\alpha \in \text{BAD}$, $\sup_{k \geq 1} (\nu_{k+1} - \nu_k) < \infty$ i.e. $\{\nu_k\}_{k \geq 1}$ is syndetic.

The grouping theorem. Let $(F_n = (a^{(n)}, b^{(n)}) : n \geq 1)$ be a $\alpha$-RAT sequence. The canonical grouping of $\alpha$ (or $(F_n : n \geq 1)$) is the RAT sequence $(G_n : n \geq 1)$ defined by

$$G_0 = \text{Id}, \ G_{k+1} := F_{\nu_{k+1}} \circ F_{\nu_{k+1}-1} \circ \cdots \circ F_{\nu_k+1}.$$
8.1 Grouping theorem  Let $\alpha \in \text{BAD}$, let $(F_k : k \geq 1)$ be the associated \text{RAT}, then the canonical grouping $(G_k : k \geq 1)$ of $(F_k : k \geq 1)$ has \text{BMF} and is adapted.

The proof of the Grouping theorem proceeds via:

Compact \text{RAT} collections.

We’ll call the collection $\mathcal{F} \subset RV(M_{d \times d}(\mathbb{Z}^d) \times \mathbb{Z}^d)$ of flip type \text{RAT}s compact if

\[
\exists M = M_\mathcal{F} > 1 \text{ so that } \forall F := (a, b) \in \mathcal{F} \text{ and } 1 \leq k, \ell \leq d, \epsilon = 0, \pm 1, \nu \in \mathbb{Z} \text{ we have }
\]

(i) $P([a_{k,\ell} = \epsilon] \cap [b_k = \nu]) > 0 \implies P([a_{k,\ell} = \epsilon] \cap [b_k = \nu]) \geq \frac{1}{M}$ and

(ii) $\|b\|_\infty := \sup_{P(b=\beta) > 0} \|\beta\| \leq M$.

We’ll call the \text{RAT} sequence $(F_k : k \geq 1)$ compactly generated if the collection $\{F_k : k \geq 1\}$ is compact.

8.2 Compactness lemma  Suppose that $2\alpha = [n_1, n_2, \ldots] \in \text{BAD}$ with

$\sup_{k \geq 1} n_k =: M$ and $\max \{J \geq 1 : \exists k, n_{k+i} = 2 \forall 1 \leq i \leq J\} =: r$

and let $(F_n : n \geq 1)$ be an $\alpha$-\text{RAT}, then

$\mathcal{F} := \{F_{j_1} \circ F_{j_2} \circ \cdots \circ F_{j_r} : j_1, j_2, \ldots, j_r \in \mathbb{N} \text{ and } 1 \leq r \leq r\}$

is compact.

Before proving the compactness lemma, we need a

8.3 Sublemma  If $\alpha \in \text{BAD}$, then $\exists \Delta > 0$ so that

(i) $r_k := \frac{\ell(1)}{\ell(0)} \geq \Delta \forall k \geq 1$, and

(ii) $\frac{p_k(i)}{p_k(1)} \in \left[n_{k+1} - 1 - i, \frac{n_{k+1} - 1 - i}{\Delta}\right]$.

Proof of (i)

\[
r_{k+1} = \frac{\ell_{k+1}(1)}{\ell_{k+1}(0)} = \frac{(n_{k+1} - 2)\ell_k(0) + \ell_k(1)}{(n_{k+1} - 1)\ell_k(0) + \ell_k(1)} = 1 - \frac{1}{n_{k+1} - 1 + r_k}.
\]

In case $n_{k+1} \geq 3$, we have

\[
r_{k+1} \geq \frac{1}{2}.
\]
When $n_{k+1} = 2$, $r_{k+1} = v(r_k)$ where $v(x) := \frac{x}{1+x}$. Now suppose that $n_k \geq 3$ and $n_{k+1} = n_{k+2} = \cdots = n_{k+R} - 2$, then for $\forall 1 \leq j \leq R$,

$$r_{k+j} = v(r_{k+j-1}) = v^j(r_k) \geq v^j\left(\frac{1}{2}\right).$$

Inspection of this recursion shows that

$$\frac{1}{r_{k+j}} = \frac{1}{r_k} + j \leq 2 + j \quad \forall 1 \leq j \leq R. \tag{1}$$

Let $r \in \mathbb{N}$ be so that $\forall k \exists J \leq r + 1$, $n_{k+J} \geq 3$ and set $\Delta := \frac{1}{2^r r^r}$. It follows that

$$r_k \geq \Delta. \quad \Box \tag{2}$$

**Proof of (ii)**

$$\frac{p_k(i)}{1 - p_k(i)} = \frac{(n_{k+1} - 1 - i)\ell_k(0)}{\ell_k(1)} = \frac{n_{k+1} - 1 - i}{r_k} \in \left[n_{k+1} - 1 - i, \frac{n_{k+1} - 1 - i}{\Delta}\right]. \quad \Box \tag{3}$$

**Proof of the compactness lemma**

We claim first that the RAT collection $\mathcal{F}_0 := \{F_n \mid n \geq 1\}$ is compact.

Condition (ii) is immediate from the boundedness of $\{n_k \mid k \geq 1\}$.

To see (i), note first that

- $q_2 = 1$ and $q_N \in \left[\frac{1}{3}, \frac{2}{3}\right]$ $\forall N \geq 3$,
- $p_k(1) = 0$ when $n_k = 2$ and by the sublemmas
- $\exists R > 0$ so that $p_k(i)$, $1 - p_k(i) \geq R \forall k \geq 1$, $i = 0, 1$, $(n_k, i) \neq (2, 1)$.

Next, for each fixed $k \geq 1$, $i, j = 0, 1$ and $\nu \in \mathbb{Z}$,

$$P([a_{i,j}(F_k) = \epsilon] \cap [b_i(F_k) = \nu])$$

is a polynomial of degree at most 3 in

$$z^{(k)} = (z^{(k)}_1, \ldots, z^{(k)}_9) = (p_k(0), 1-p_k(0), q_{n_k}, 1-q_{n_k}, p_k(1), 1-p_k(1), q_{n_k-1}, 1-q_{n_k-1}, \frac{1}{n_k})$$

whose coefficients are non-negative integers. By sublemmas 8.3, $\exists \epsilon > 0$ so that

$$1 \leq s \leq 9, k \geq 1, z^{(k)}_s > 0 \implies z^{(k)}_s \geq \epsilon.$$

Condition (i) follows because if

$$P([a_{i,j}(F_k) = \epsilon] \cap [b_i(F_k) = \nu]) \leq F(z) = \sum_{1 \leq r, s, t \leq 9} N_{r,s,t} z_r z_s z_t > 0$$

then $1 \leq r, s, t \leq 9$ (maybe not distinct) so that $N_{r,s,t} \geq 1$ and $z_r, z_s, z_t \geq \epsilon$. This shows that $\mathcal{F}_0$ is compact.
For analogous reasons, the compactness persists among concatenations of \( \{F_n : n \geq 1\} \) of length bounded by \( r \).

Let \( F = F_{k_r} \circ F_{k_{r-1}} \circ \cdots \circ F_{k_1} \in \mathfrak{F} \), then

\[
a(F) = a(F_{k_r})a(F_{k_{r-1}}) \cdots a(F_{k_1}) \quad \text{and} \quad b(F) = \sum_{\nu=1}^{r-1} a(F_{\nu+1})b(F_{k_\nu})
\]

where \( F_{\nu+1} := F_{k_r} \circ F_{k_{r-1}} \circ \cdots \circ F_{k_{\nu+1}} \).

Condition (ii) is immediate since

\[
\|b(F)\|_{\infty} \leq \sum_{\nu=1}^{r-1} \|a(F_{\nu+1})b(F_{k_\nu})\|_{\infty} \leq \sum_{\nu=1}^{r-1} \|b(F_{k_\nu})\|_{\infty} \leq r \sup_n \|b(F_n)\|_{\infty}.
\]

To see (i) we note that for fixed \( i, j = 0, 1 \) and \( \nu \in \mathbb{Z} \),

\[
P([a_{i,j}(F) = \epsilon] \cap [b_i(F) = \nu])
\]

is a polynomial of degree at most most \( r \) in the variables

\[
\{P([a_{k,\ell}(F_{k_u}) = \omega] \cap [b_k(F_{k_u}) = \mu]) : u = 1, 2, \ldots, r, \; \omega = 0, \pm 1, \; \mu \in \mathbb{Z}\}
\]

whose coefficients are non-negative integers and so

\[
P([a_{i,j}(F) = \epsilon] \cap [b_i(F) = \nu]) > 0 \implies P([a_{i,j}(F) = \epsilon] \cap [b_i(F) = \nu]) \geq \frac{1}{M_{\delta_0}^r}
\]

where \( M_{\delta_0} \) is as in the definition of compactness of \( \mathcal{F}_0 \). This proves the compactness lemma. \( \square \)

**Proof of the Grouping theorem.**

In order to prove that \( (G_n : n \geq 1) \) has \( \text{BMF} \), it suffices by lemma 7.3 and the compactness lemma, to show that each \( G_n \) is mean contractive.

We claim also that in order to prove adaptedness of the sequence \( (G_n : n \geq 1) \), it suffices to show that each \( \text{RAT} \; G_n \) is adapted.

To prove this latter claim, suppose that each \( G_n \) is individually adapted. Compactness of \( (G_n : n \geq 1) \) entails \( \|b(G_n)\| \leq M \) which implies by discreteness that (possibly increasing \( M \)) \( \forall \; k \geq 1, \; \text{supp} \; b_k(G_n) \subset [-M, M] \) whence

\[
\Gamma_{G_n} \subset \frac{2\pi}{(2M)!} \mathbb{Z}.
\]

Adaptedness of the sequence now indeed follows individual adaptedness by compactness.
The rest of this proof is concerned with establishing these individual properties for the concatenations involved.

**Concatenations of spec-RATs.** Each $G = G_n$ in the canonical grouping is an independent concatenation of four RATs of form

$$H = R \circ T$$

where $R$, $T$ are independent, $R = R(H)$ is a non-trivial spec-RAT and either $T = T(H)$ is a concatenation of finitely many independent, trivial spec-RATs, or $T = \text{Id}$.

**Mean contractivity of spec-RAT concatenations**

Non-trivial, odd spec-RATs are mean contractive. Even spec-RATs are not. Suppose $G = H_1 \circ H_2$ where and $R(H_i)$ $i = 1, 2$ are even. By the parity transition laws, each $R(H_i)$ is followed and preceded by odd (here necessarily trivial) spec-RATs $J$. It can be calculated that for $a = a(J \circ R)$ or $a = a(R \circ J)$, then $P(a_i = \epsilon) > 0$ for $i = 0, 1$ and $\epsilon = \pm 1$. By irreducibility $G$ is mean contractive.

**Adaptedness of spec-RAT concatenations**

If $G = H_1 \circ H_2 \circ H_3 \circ H_4$ where $R(H_i)$ is even for some $i = 2, 3, 4$, then $R(H_i)$ is partially adapted and $H_1 \circ \cdots \circ H_{i-1}$ is irreducible, whence $H_1$, $H_1 \circ \cdots \circ H_{i-1} \circ R(H_2)$ is adapted and so is $G$.

The remaining case is where $G = H_1 \circ \cdots \circ H_4$ where $R(H_i)$ is odd $\forall 2 \leq i \leq 4$.

To show that such $G$ is adapted, we write

$$G = H_1 \circ U \circ T$$

where

$$U := H_2 \circ H_3 \circ R(H_4).$$

Since $H_1$ is irreducible, it suffices to show that row 0 is adapted for either $H_2 \circ H_3$ or for $U$.

Suppose that row 0 is not adapted for $H_2 \circ H_3$, then, in particular the random variable $b_{0,0,1}(H_2 \circ H_3) \equiv c$ is constant.

For each $\alpha = \pm 1$, $P(J_\alpha) > 0$ where

$$J_\alpha = [a_{0,0}(H_2) = 1] \cap [a_{0,0}(H_3) = \alpha] \cap [a_{0,0}(R(H_4)) = \alpha] \subset [a_{0,0}(U) = 1]$$

and, on $J_\alpha$ we have that

$$b_0(U) = b_0(H_2 \circ H_3) + \alpha b_0(R(H_4)) = c_1 + \alpha.$$ 

Thus the random variable $P(b_{0,0,1}(U) = c_1 + \alpha) > 0$ for $\alpha = \pm 1$ and row 0 is adapted for $U$. \(\varnothing\)
This completes the proof of the Grouping theorem.

CONCLUSION OF THE PROOF OF THE MAIN RESULT

By the adaptedness preservation observation, the centering
\((H_k : k \geq 1)\) of \((G_k : k \geq 1)\) is adapted (with the same adaptivity group).

By compactness, BMF and the bounded centering lemma, \((H_k : k \geq 1)\) is uniform.

Thus \((H_k : k \geq 1)\) satisfies WRLLT. Let \((X_n : n \geq 1), (Y_k : k \geq 1)\) and \((Z_k : k \geq 1)\) be the ARWs generated by \((F_n : n \geq 1), (G_k : k \geq 1)\) and \((H_k : k \geq 1)\) respectively, then for each \(i = 0, 1\) and \(1 \leq p \leq 2\),

\[
\int_T |\Phi_{X^{(\nu_k)}(i)}(\theta)|^p d\theta = \int_T |\Phi_{Y^{(k)}(i)}(\theta)|^p d\theta \quad \text{because the processes are identical}
\]

\[
= \int_T |\Phi_{Z^{(k)}(i)}(\theta)|^p d\theta \quad \text{because } Z^{(k)}(i) = Y^{(k)}(i) - E(Y^{(k)}(i))
\]

\[
\asymp \frac{1}{\sqrt{k}} \quad \text{by the WRLLT}.
\]

By the Visit Lemmas 2.4,

\[
\int_T \Psi_{\ell_{\nu_k}(0)}(x) dx \gg \ell_{\nu_k}(1) \int_T |\Phi_{X^{(\nu_k)}(1)}(\theta)|^2 d\theta \asymp \frac{\ell_{\nu_k}(1)}{\sqrt{k}},
\]

and

\[
\|\Psi_{\ell_{\nu_k}(1)}\|_\infty \leq \ell_{\nu_k}(0) \int_T |\Phi_{X^{(\nu_k)}(0)}(\theta)| d\theta \asymp \frac{\ell_{\nu_k}(0)}{\sqrt{k}}.
\]

Next, since \(\alpha \in \text{BAD}\) and \(\sup_{k \geq 1} (\nu_{k+1} - \nu_k) < \infty\), \(\exists 1 < m < M\) so that

\[
m \leq \frac{\ell_{\nu_{k+1}}(i)}{\ell_{\nu_k}(j)} \leq M \quad i, j = 0, 1
\]

whence

\[
\log \ell_{\nu_k}(j) \asymp k.
\]

For \(\ell_{\nu_k}(0) \leq n \leq \ell_{\nu_{k+1}}(0)\),

\[
\int_T \Psi_n(x) dx \geq \int_T \Psi_{\ell_{\nu_k}(0)}(x) dx \gg \frac{\ell_{\nu_k}(1)}{\sqrt{k}} \gg \frac{\ell_{\nu_{k+1}}(1)}{\sqrt{k}} \gg \frac{n}{\sqrt{\log n}}
\]

and for \(\ell_{\nu_k}(1) \leq n \leq \ell_{\nu_{k+1}}(1)\),

\[
\|\Psi_n\|_\infty \ll \frac{\ell_{\nu_{k+1}}(1)}{\sqrt{k}} \ll \frac{\ell_{\nu_k}(0)}{\sqrt{k}} \ll \frac{n}{\sqrt{\log n}}.
\]

[\Box (\bullet) \text{ (as on page 2)}]
Concluding Remarks.

1. It is shown in [7] that for a.e. $\alpha \in \mathbb{T}$, $T_\alpha$ is $\frac{1}{n}$-recurrent in the sense that $\sum_{n=1}^{\infty} \frac{1}{n} f \circ T^n_\alpha = \infty$ a.e. $\forall f \in L^1_+$. It follows easily from $(\Theta)$ (as on page 2) that for $\alpha \in \text{BAD}$ and $\omega_n \downarrow 0$: $T_\alpha$ is $\omega_n$-recurrent ($\sum_{n=1}^{\infty} \omega_n f \circ T^n_\alpha = \infty$ a.e. $\forall f \in L^1_+$) iff

$$\sum_{n \geq 1} n (\omega_n - \omega_{n+1}) \sqrt{\log n} = \infty$$

E.g. $\omega_n = \frac{1}{n \sqrt{\log n}}$.

2. The subsequence condition $(\Theta)$ (as on page 2) is satisfied for some $\alpha \notin \text{BAD}$. For example, if $2\alpha = [2m_1 + 1, 2m_2, 2m_3, \ldots]$ with $m_k \geq 2 \forall k \geq 1$, then by sublemma 8.3(i), $\ell_k(1) \leq \ell_k(0) \leq 2\ell_k(1)$ and it follows as above that $(\Theta)$ is satisfied along $\ell_{\mu_k}(0)$ ($\mu_k \uparrow \infty$) as soon as

$$\int_{\mathbb{T}} |\bar{\Phi}_{X^{\nu_k}}(0)(\theta)|d\theta \ll \int_{\mathbb{T}} |\bar{\Phi}_{X^{\nu_k}}(1)(\theta)|^2d\theta.$$

The parities $\epsilon_k(0) = 1 \forall k \geq 2$, whence the associated RAT collection $\{F_j: j \geq 1\}$ is compact.

The canonical subsequence in §8 is $\nu_k = 1 + 4k$ ($k \geq 1$), the grouping lemma applies and the centering of the canonical grouping RAT sequence satisfies the WRLLT.

Consequently $T_\alpha$ satisfies $(\Theta)$ along $\ell_{\nu_k}(0)$ with $a_{\nu_k} \asymp \frac{\ell_{\nu_k}}{\sqrt{k}}$.

3. It is thus natural to ask if $p(\Theta) = 1$ (or even if $p(\Theta) > 0$) where $p(\Theta) := m(\{\alpha \in (0, 1): T_\alpha \text{ satisfies } (\Theta) \text{ along some subsequence}\})$.

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