ON THREE-PARAMETER FAMILIES OF FILIPPOV SYSTEMS – THE FOLD-SADDLE SINGULARITY.

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ABSTRACT. This paper presents results concerning bifurcations of 2D piecewise-smooth vector fields. In particular, the generic unfoldings of codimension three fold-saddle singularities of Filippov systems, where a boundary-saddle and a fold coincide, are considered and the bifurcation diagrams exhibited.

1. Introduction

The general purpose of this paper is to study non-smooth vector fields (NSVF’s for short), also called Filippov systems, represented by the following three-parameter family of differential equations in $\mathbb{R}^2$:

\[
\begin{align*}
Z_{\tau,\lambda,\mu,\beta} = \begin{cases}
X_\lambda = \left( \frac{1}{\alpha_1(\tau)}(x - \lambda) + \alpha_2(\tau)(x - \lambda)^2 \right) & \text{if } y \geq 0, \\
Y_{\mu,\beta} = \left( \frac{\mu x + (\mu - 2)}{2} y + \beta \right) & \text{if } y \leq 0,
\end{cases}
\end{align*}
\]

where $\tau = \{\text{inv, vis}\}$, $\alpha_1(\text{inv}) = -1$, $\alpha_1(\text{vis}) = 1$, $\alpha_2(\text{inv}) = 1$, $\alpha_2(\text{vis}) = 0$ and $(\lambda, \beta, \mu) \in (-1, 1) \times (-\sqrt{3}/2, \sqrt{3}/2) \times (-\varepsilon_0, 1)$ with $\varepsilon_0 > 0$ sufficiently small.

In Figure 1 (respectively, Figure 2) we consider $\lambda = \mu = \beta = 0$ and $\tau = \text{inv}$ (respectively, $\tau = \text{vis}$) in Equation (1).

Call $S_X = \{X^2 = 0\}$ and $S_Y = \{Y^2 = 0\}$ where $X_\lambda = X = (X^1, X^2)$ and $Y_{\mu,\beta} = Y = (Y^1, Y^2)$. Denote $Z_{\tau,\lambda,\mu,\beta} = (X_\lambda, Y_{\mu,\beta}) = (X, Y)$. In short our goal is to study the local dynamics of $Z_{\tau,\lambda,\mu,\beta}$ consisting of two smooth vector fields $X_\lambda$ and $Y_{\mu,\beta}$ in $\mathbb{R}^2$ such that on one side of a smooth surface $\Sigma = \{y = 0\}$ we take $Z_{\tau,\lambda,\mu,\beta} = X_\lambda$ and on the other side $Z_{\tau,\lambda,\mu,\beta} = Y_{\mu,\beta}$. We mention two particular generic situations that occur in our system when $\beta \neq 0$. The first one is the fold-fold singularity (or two-fold singularity – see Definition 2), studied in some detail in [Guardia et al. (2011)] and [Kuznetsov et al. (2003)] (among others), in which the trajectories of both $X$ and $Y$ have a quadratic tangency to $\Sigma$ at a point $p \in \Sigma$. The other

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situation is the occurrence of branches of canard cycles (see Definition 5) which are typical minimal sets that appear in NSVF’s.

It is worth saying that the set $\Omega^r$ of all NSVF’s $Z = (X, Y)$ as below described (with some specified topology) is a differentiable manifold modeled in a Banach space (see Section 3 for details).

We now give a brief and rough overall description of the main features of this work:

- In $\Omega^r$ all NSVF’s presenting a generic fold-saddle singularity form a codimension-three submanifold $W_1$ in such a way that any $Z \in W_1$ is locally equivalent (see Definition 8) to the above described $Z_{\lambda,\mu,\beta}^r$ with $\lambda = \mu = \beta = 0$.
- The bifurcation diagram of $Z_{\lambda,\mu,\beta}^r$ is exhibited for $\tau = inv$ and $\tau = vis$.

We emphasize that this paper is inserted in a larger 2D classification program where are included papers like [Teixeira(1977), Teixeira(1991), Kuznetsov et al.(2003), Guardia et al.(2011)]. One of the goals of this program is to classify (via topological equivalence) typical singularities of NSVF’s. For this purpose it is necessary to present generic unfoldings and give non-degeneracy conditions on the system in order to characterize the codimension of the singularity. The bifurcation diagram of the codimension-three singularity presented here includes that one exhibited in [Guardia et al.(2011)]. The later claim is commented in the sequel. We finish this introduction presenting a physical model where a fold-saddle singularity can be found.

**Example 1.** One of the reasons that second order linear equations with constant coefficients are worth studying is that they serve as mathematical models of some important physical processes. Two important areas of application are in the fields of mechanical and electrical oscillations. For example, the motion of a mass on a vibrating spring, the torsional oscillations of a shaft with a flywheel, the flow of electric current in a simple series circuit, and many other physical problems are all described by the solution of an equation of the form

\[
a \dddot{x} + b \dot{x} + cx = g(t, x).
\]
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Here we consider the external force \( g \) not depending on \( t \) but depending on \( x \). For example consider

\[
g(x) = Ax + 1 - \text{sgn}(x) \quad \text{with} \quad A > \frac{c}{a}.
\]

Now if we call \( \dot{x} = y \) then the equation (2) with \( g \) given by (3) became

\[
\dot{x} = y, \quad \dot{y} = (A - \frac{c}{a}) x - \frac{b}{a} y, \quad \text{if} \quad x > 0
\]

and

\[
\dot{x} = y, \quad \dot{y} = (A - \frac{c}{a}) x - \frac{b}{a} y + 2, \quad \text{if} \quad x < 0.
\]

System (4) has a saddle equilibrium at the origin because \( A > \frac{c}{a} \). And system (5) has an invisible fold (see Definition 1) at the origin. If \( g(x) = Ax - 1 + \text{sgn}(x) \) with \( A > \frac{c}{a} \) then it has a saddle equilibrium and a visible fold. □

The paper is organized as follows: In Section 2 we present the main results of the paper. In Section 3 we give some basic concepts about NSVF’s in order to setting the problem in Section 4. The remaining sections are dedicated to prove the main results of the paper.

2. Statement of the Main Results

In what follows consider

\[
\mu_0(\beta) = 2 - (12\beta/(-3 + 6\beta + \sqrt{9 - 12\beta^2})).
\]

**Theorem 1.** Assume \( \tau = \text{inv} \) and \( \mu = \mu_0(\beta) \) in Equation (1). Then its bifurcation diagram in the \( (\lambda, \beta) \)-plane contains 19 distinct phase portraits (see Figure 19).

First of all observe that if \( \beta > 0 \) and \( \lambda = -1/2 + \sqrt{9 - 12\beta^2}/6 \) in Theorem 1, then the NSVF has a homoclinic loop surrounding a non hyperbolic singularity. So, it is easy to see that the cases covered by Theorem 1 do not represent the full unfolding of the (invisible) fold-saddle singularity and the next two theorems become necessary.

**Theorem 2.** Assume \( \tau = \text{inv} \) and \( \mu_0(\beta) < \mu < 1 \) in Equation (1). Then its bifurcation diagram in the \( (\lambda, \beta) \)-plane contains 21 distinct phase portraits (see Figure 21).

**Theorem 3.** Assume \( \tau = \text{inv} \) and \( -\epsilon_0 < \mu < \mu_0(\beta) \) in Equation (1). Then its bifurcation diagram in the \( (\lambda, \beta) \)-plane contains 21 distinct phase portraits (see Figure 21).
Remark 1. The bifurcation diagrams exhibited in Theorems 2 and 3 present a homoclinic loop surrounding a hyperbolic singularity. Observe that, under the conditions of Theorem 2 (respectively, Theorem 3) this singularity is an attractor (respectively, repellor) as illustrated in Figure 3, cases (a) and (b). Moreover, when the parameter $\mu$ varies from $-\varepsilon_0$ to 1 there is an element $\mu_0(\beta) \in (-\varepsilon_0,1)$, given by Equation (6), such that $Z_{\lambda,\mu(\beta)}^{inv}$ presents a like Hopf bifurcation phenomenon as illustrated in Figure 3. This phenomenon is fully treated in [Guardia et al. (2011)] and [Kuznetsov et al. (2003)] and it is not covered by Theorems 1, 2 and 3.

![Figure 3](image)

**Figure 3.** A like Hopf bifurcation. The singularities in (a) and (c) (that appear in Theorems 2 and 3 respectively) are called $\Sigma$–attractor and $\Sigma$–repeller throughout this paper (see Figure 4, cases (b) and (c)). In (b) we observe a non hyperbolic singularity present in Theorem 1.

Remark 2. In Theorems 2 and 3 beyond the cases (a) and (c) in Figure 3 it also appears another relevant phenomenon that is the presence of a fold-fold singularity. Note that in Theorem 1 it occurs simultaneously the occurrence of a loop (a global phenomenon) and a fold-fold singularity (a local phenomenon).

Theorem 4. Assume $\tau = \text{vis}$ in Equation (11) or equivalently, take $\tau = \text{vis}$ and $\mu = 0$ in Equation (1). Then its bifurcation diagram in the $(\lambda, \beta)$–plane contains 13 distinct phase portraits (see Figure 27).

The cases covered by Theorem 4 do not represent the full unfolding of the (visible) fold-saddle singularity. So, the next two theorems are necessary. Each one of them describes a distinct generic codimension two singularity.

Theorem 5. Assume $\tau = \text{vis}$ and $0 < \mu < 1$ in Equation (I). Then its bifurcation diagram in the $(\lambda, \beta)$–plane contains 13 distinct phase portraits (see Figure 27).
Theorem 6. Assume \( \tau = \text{vis} \) and \( \varepsilon_0 < \mu < 0 \) in Equation (1). Then its bifurcation diagram in the \((\lambda, \beta)\)-plane contains 13 distinct phase portraits (see Figure 27).

Remark 3. In Theorems 5 and 6 one observes the birth of a singularity on \( \Sigma \). This singularity behaves like a saddle (see Figure 27) and is known as \( \Sigma \)-saddle. This phenomenon does not occur under the conditions presented in Theorem 4.

3. Basic Theory about NSVF

Let \( K \subseteq \mathbb{R}^2 \) be a compact set such that \( \partial K \) is a smooth curve. Consider \( \Sigma \subseteq K \) given by \( \Sigma = f^{-1}(0) \), where \( f : K \to \mathbb{R} \) is a smooth function having \( 0 \in \mathbb{R} \) as a regular value (i.e. \( \nabla f(p) \neq 0 \), for any \( p \in f^{-1}(0) \)) such that \( \partial K \cap \Sigma = \emptyset \) and \( \partial K \not\subset\Sigma \). Clearly the switching manifold \( \Sigma \) is the separating boundary of the regions \( \Sigma^+ = \{ q \in K \mid f(q) \geq 0 \} \) and \( \Sigma^- = \{ q \in K \mid f(q) \leq 0 \} \). We can assume that \( \Sigma \) is represented, locally around a point \( q = (x, y) \), by the function \( f(x, y) = y \).

Designate by \( \chi^r \) the space of \( C^r \)-vector fields on \( K \) endowed with the \( C^r \)-topology with \( r \geq 1 \) large enough for our purposes. Call \( \Omega^r = \Omega^r(K, f) \) the space of vector fields \( Z : K \to \mathbb{R}^2 \) such that

\[
Z(x, y) = \begin{cases} 
X(x, y), & \text{for } (x, y) \in \Sigma^+, \\
Y(x, y), & \text{for } (x, y) \in \Sigma^-,
\end{cases}
\]

where \( X = (X^1, X^2) \) and \( Y = (Y^1, Y^2) \) are in \( \chi^r \). We write \( Z = (X, Y) \), which we will accept to be multivalued in points of \( \Sigma \). The trajectories of \( Z \) are solutions of \( \dot{q} = Z(q) \), which has, in general, discontinuous right-hand side. The basic results of differential equations, in this context, were stated by Filippov in [Filippov(1988)]. Related theories can be found in [Kozlova(1984), Teixeira(1991)] among others.

3.1. Orbits, trajectories and singularities of NSVF’s. In what follows we will use the notation

\[
X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad X^i.f(p) = \langle \nabla (X^{i-1}.f)(p), X(p) \rangle, \quad i \geq 2
\]

where \( \langle ., . \rangle \) is the usual inner product in \( \mathbb{R}^2 \).

Following the Filippov rule, we distinguish the following regions on the discontinuity set \( \Sigma \):

- **Crossing region:** \( \Sigma_c = \{ p \in \Sigma \mid (X.f(p))(Y.f(p)) > 0 \} \).
- **Escaping region:** \( \Sigma_e = \{ p \in \Sigma \mid X.f(p) > 0 \} \) and \( Y.f(p) < 0 \).
- **Sliding region:** \( \Sigma_s = \{ p \in \Sigma \mid X.f(p) < 0 \} \) and \( Y.f(p) > 0 \).

Consider \( Z = (X, Y) \in \Omega^r \) and \( p \in \Sigma_c \cup \Sigma_s \). The sliding vector field \( Z^\Sigma \) associated to \( Z \) at \( p \) is the convex combination of \( X(p) \) and \( Y(p) \) tangent to \( \Sigma \) at \( p \) (see Figure 3).
We say that \( q \in \Sigma \) is a \( \Sigma \)-regular point if
- \((X.f(q))(Y.f(q)) > 0 \) or
- \((X.f(q))(Y.f(q)) < 0 \) and \( Z^{\Sigma}(q) \neq 0 \) (that is \( q \in \Sigma_e \cup \Sigma_s \) and it is not an equilibrium point of \( Z^{\Sigma} \)).

The points of \( \Sigma \) which are not \( \Sigma \)-regular are called \( \Sigma \)-singular. We distinguish two subsets in the set of \( \Sigma \)-singular points: \( \Sigma' \) and \( \Sigma^p \). Any \( q \in \Sigma^p \) is called a pseudo equilibrium of \( Z \) and it is characterized by \( Z^{\Sigma}(q) = 0 \). Any \( q \in \Sigma' \) is called a tangential singularity and is characterized by \( Z^{\Sigma}(q) \neq 0 \) and \((X.f(q))(Y.f(q)) = 0 \).

A pseudo equilibrium \( q \in \Sigma^p \) is a \( \Sigma \)-saddle provided that one of the following condition is satisfied: (i) \( q \in \Sigma_e \) and \( q \) is an attractor for \( Z^{\Sigma} \) or (ii) \( q \in \Sigma_s \) and \( q \) is a repeller for \( Z^{\Sigma} \). A pseudo equilibrium \( q \in \Sigma^p \) is a \( \Sigma \)-repeller (resp. \( \Sigma \)-attractor) provided \( q \in \Sigma_e \) (resp. \( q \in \Sigma_s \)) and \( q \) is a repeller (resp. attractor) equilibrium point for \( Z^{\Sigma} \) (see Figure 5).

Definition 1. We say that \( p_0 \in \Sigma^t \) is a \( \Sigma \)-fold point of \( X \in \chi_r \) if \( X.f(p_0) = 0 \) but \( X^2.f(p_0) \neq 0 \). Moreover, \( p_0 \in \Sigma \) is a visible (respectively invisible) \( \Sigma \)-fold point of \( X \) if \( X.f(p_0) = 0 \) and \( X^2.f(p_0) > 0 \) (respectively \( X^2.f(p_0) < 0 \)).
Definition 2. Let $Z = (X, Y) \in \Omega'$. We say that $p \in \Sigma^t$ is a fold-fold singularity of $Z$ if $p$ is a $\Sigma$–fold point for both $X$ and $Y$.

Definition 3. Let $Z = (X, Y) \in \Omega'$. We say that $q \in \Sigma$ is a fold-saddle singularity of $Z$ if $q$ is a $\Sigma$–fold point of $X$ and a saddle equilibrium of $Y$ (in this case $q$ is called a boundary-saddle of $Y$).

The following construction is presented in [Teixeira(2008)]. Let $Z = (X, Y) \in \Omega'$ such that $T$ is an invisible $\Sigma$–fold point of $X$. From Implicit Function Theorem, for each $p \in \Sigma$ in a neighborhood $V_T$ of $T$ we derive that there exists a unique $t(p)$ such that the orbit $-solution$ $t \mapsto \phi_X(t, p)$ of $X$ through $p$ meets $\Sigma$ at a point $\bar{p} = \phi_X(t(p), p)$. Define the map $\gamma_X : V_T \cap \Sigma \to V_T \cap \Sigma$ by $\gamma_X(p) = \bar{p}$. This map is a $C^r$–diffeomorphism and satisfies: $\gamma_X^2 = Id$. Analogously, when $\bar{T}$ is an invisible $\Sigma$–fold point of $Y$ we define the map $\gamma_Y : V_{\bar{T}} \cap \Sigma \to V_{\bar{T}} \cap \Sigma$ associated to $Y$ which satisfies $\gamma_Y^2 = Id$. We define now the first return map associated to $Z = (X, Y)$:

Definition 4. The first return map $\varphi_Z : T \to T$ is defined by the composition $\varphi_Z = \gamma_Y \circ \gamma_X$ when both $\gamma_X$ and $\gamma_Y$ are well defined in $T \subset \Sigma$.

Remark 4. $\varphi_Z$ is an area-preserving map.

Definition 5. A curve $\Gamma$ is a canard cycle of $Z = (X, Y) \in \Omega'$ if it is closed and composed by orbit-arcs of at least two of the vector fields $X|_{\Sigma^+}$, $Y|_{\Sigma^-}$ and $Z|_{\Sigma}$. We say that $\Gamma$ is hyperbolic if $\varphi_Z'(p) \neq 1$, where $\varphi_Z$ is the first return map defined on a segment $T$ with $p \in T \cap \gamma$.

Definition 6. Consider $Z \in \Omega'$. A closed path $\Delta$ is a $\Sigma$–graph if it is a union of equilibria, pseudo equilibria, tangential singularities of $Z$ and orbit-arcs of $Z$ joining these points in such a way that $\Delta \cap \Sigma \neq \emptyset$.

Definition 7. Consider $Z \in \Omega'$. A point $q \in \Sigma$ is a $\Sigma$–center if there is a neighborhood $U$ of $q$ filled up with a one-parameter family $\gamma_U$ of canard cycles of $Z$ in such a way that $\gamma_U \cap \Sigma \subset \Sigma_c$ (see Figure 7).
3.2. Structural Stability on $\Omega^r$. Bifurcation theory describes how continuous variations of parameter values in a dynamical system can, through topological changes, cause the phase portrait to change suddenly. In this paper we focus on certain structurally unstable NSVF’s within a generic context. In [Andronov & Pontryagin(1937)] the concept of $k^{th}$-order structural stability is presented; in a local approach such setting gives rise to the notion of a codimension $k$ singularity. Now we present the concept of equivalence which will guide us for all the paper.

Definition 8. Two NSVF’s $Z, \tilde{Z} \in \Omega^r(K, f)$ defined in open sets $U, \tilde{U} \subset K$ with switching manifolds $\Sigma \subset U$ and $\tilde{\Sigma} \subset \tilde{U}$ respectively are $\Sigma -$equivalent if there exists an orientation preserving homeomorphism $h : U \to \tilde{U}$ which sends $\Sigma$ to $\tilde{\Sigma}$ and sends orbits of $Z$ (respectively $\tilde{Z}$) to orbits of $\tilde{Z}$ (respectively $\tilde{\Sigma}$). From this definition the concept of local structural stability in $\Omega^r$ is naturally obtained.

As we said in Section 1 our paper is a generalization of some papers that unfold typical singularities of NSVF’s. Below we present the program used in the literature and in our paper to exhibit the diagram bifurcation of a singularity of NSVF’s.

Let $Z \in \Omega^r$ and $p \in \Sigma$. Following the approach in [Sotomayor(1974)] (and also exposed in [Guardia et al.(2011)], Subsection 3.1, pg 1982), we get that:

- By Theorem 3.5 of [Guardia et al.(2011)] we already know the characterization of the set $\Phi^0_0 = \{Z \in \Omega^r | Z$ is locally structurally stable at $p\}$. In fact, $\Phi^0_0$ is open and dense in $\Omega^r$. So, $\Phi^0_0$ is the codimension zero local bifurcation set.

- Let $\Omega_1 = \Omega^r \setminus \Phi^0_0$ and $\Phi^1_0 = \{Z \in \Omega_1 | Z$ is locally structurally stable at $p$ relative to $\Omega_1\}$. The set $\Phi^1_0$ is the codimension 1 local bifurcation set in $\Omega^r$. The characterization $\Phi^1_0$ was given in [Kuznetsov et al.(2003)].

In addition, if $Z_0 \in \Phi^1_0$, it is also known (see [Teixeira(1979)]) that there exists a neighborhood $\mathcal{U}$ of $Z_0$ in $\Omega^r$ such that:

- There exists a $C^r-$function $L : \mathcal{U} \to \mathbb{R}$, satisfying $L(Z_0) = 0$ and $DL_{Z_0}$, the differential of $L$ at $Z_0$, is surjective. Moreover, $L^{-1}(0) = \Phi^1_0 \cap \mathcal{U}$.

- Consider now all the embeddings $\Theta : (-\varepsilon, \varepsilon) \times \mathcal{U} \to \mathcal{U}$ transversal to $\Phi^1_0$ at some $Z \in \Phi^1_0$ and such that $\Theta(0, Z_0) = Z$. We refer to such $\Theta$ as an unfolding of $Z_0$.

- We consider now the set $\Omega_2 = \Omega_1 \setminus \Phi^1_0$ and similar objects $\Phi^2_0$ (the set of codimension 2 singularities) and families of objects $L : \mathcal{U} \to \mathbb{R}^2$, with surjective derivative at $Z_0$ and embeddings $\Theta : (\varepsilon, \varepsilon) \times (-\zeta, \zeta) \times \mathcal{U} \to \mathcal{U}$.

- In this way we can get sequences $\Omega_k$ and $\Phi^k_0$ in $\Omega^r$, that establish a program to characterize all codimension $k$ singularities.
Definition 9. Let $V(0, \mathbb{R}^k)$ and $S(0, \mathbb{R}^l)$ be neighborhoods of 0 in $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively and let $U$ be a neighborhood of $Z_0$ in $\Omega'$. We say that two unfoldings $\Theta : V(0, \mathbb{R}^k) \times U \to U$ and $\Xi : S(0, \mathbb{R}^l) \times U \to U$ are equivalent if there is a homomorphism $A : V(0, \mathbb{R}^k) \to S(0, \mathbb{R}^l)$ such that $A(\lambda) = \mu$ and for each $Z \in U$ the vector fields $\Theta(\lambda, Z)$ and $\Xi(A(\lambda), Z)$ are $\Sigma$-equivalent according to Definition 8. Moreover, we say that an unfolding $\Theta(\lambda, .)$ is a generic unfolding if there is a neighborhood $W(\Theta(\lambda, .))$ of $\Theta(\lambda, .)$ such that any unfolding $\Theta(\lambda, .) \in W(\Theta(\lambda, .))$ is equivalent to $\Theta(\lambda, .)$.

Remark 5. In Definition 4 it is important to say that the homomorphism $A$ does not vary, necessarily, continuously with respect to the parameter $\lambda \in V(0, \mathbb{R}^k)$.

3.3. The Direction Function $H$. Here we introduce a function that will be very useful in the sequel.

In $(A, B) \subset \Sigma_e \cup \Sigma_s$, consider the point $C = (C_1, C_2)$, the vectors $X(C) = (D_1, D_2)$ and $Y(C) = (E_1, E_2)$ (as illustrated in Figure 8). The straight segment passing through $C + X(C)$ and $C + Y(C)$ meets $\Sigma$ in a point $p(C)$. We define the $C^r$-map

$$p : (A, B) \to \Sigma \quad z \mapsto p(z).$$

Since $\Sigma$ is the $x$-axis, we have that $C = (C_1, 0)$ and $p(C) \in \mathbb{R} \times \{0\}$ can be identified with points in $\mathbb{R}$. According with this identification, the direction function on $\Sigma$ is defined by

$$H : (A, B) \to \mathbb{R} \quad z \mapsto p(z) - z.$$

![Figure 8. Direction function.](image)

We obtain that $H$ is a $C^r$-map and

- if $H(C) < 0$ then the orientation of $Z^\Sigma$ in a small neighborhood of $C$ is from $B$ to $A$;
- if $H(C) = 0$ then $C \in \Sigma^p$;
- if $H(C) > 0$ then the orientation of $Z^\Sigma$ in a small neighborhood of $C$ is from $A$ to $B$. 
Simple calculations show that \( p(C_1) = \frac{E_2(D_1+C_1)-D_2(E_1+C_1)}{E_2-D_2} \) and consequently,

\[
H(C_1) = \frac{E_2D_1 - D_2E_1}{E_2 - D_2}.
\]

**Remark 6.** If \( X.f(p) = 0 \) and \( Y.f(p) \neq 0 \) then, in a neighborhood \( V_p \) of \( p \) in \( \Sigma \), the direction function \( H \) has the same sign of \( D_1 \), where \( X(p) = (D_1, D_2) \). In fact, since \( S \) and the following generic normal forms exist a neighborhood \( V \) of \( \Sigma \), \( a \) hyperbolic boundary saddle \( \Gamma \) is an open set in \( \Gamma_+ \) having a \( \Sigma \)-fold point. \( \Gamma_+^S \) is an open set in \( \Gamma_+ \) (see [Teixeira(1977)]). We may consider \( f(x, y) = y \) and the following generic normal forms \( X_0(x, y) = (\alpha_1, \beta_1 x) \) with \( \alpha_1 = \pm 1 \) and \( \beta_1 = \pm 1 \) (see [Vishik(1972)], Theorem 2).

Let \( \Gamma_+^S \subset \Gamma_+ \) be the set of all elements \( x \in \chi^r \) having a \( \Sigma \)-fold point. \( \Gamma_+^S \setminus \Gamma_+ \) is an open set in \( \Gamma_+ \) (see [Teixeira(1977)]). We can consider \( f(x, y) = y \) and the following generic normal forms \( X_0(x, y) = (\alpha_1, \beta_1 x) \) with \( \alpha_1 = \pm 1 \) and \( \beta_1 = \pm 1 \) (see [Vishik(1972)], Theorem 2).

Let \( \Gamma_+ \) be the set of all elements \( y \in \Gamma_+ \) presenting a hyperbolic saddle equilibrium \( S_{Y_0} \) on \( \Sigma \) (called boundary saddle in the literature – as a reference, in [Roy & Roy(2008)], are exhibited some border collisions in three-dimension NSVF’s) and such that the eigenspaces of \( DY_0(S_{Y_0}) \) are transverse to \( \Sigma \) at \( S_{Y_0} \). \( \Gamma_+^S \) is a codimension one submanifold of \( \Gamma_+ \). Note that, \( \Gamma_+^S \) has a \( C^r \)-structure (for more details about this construction see [Dumortier(1978)]). In fact, since \( Y_0 \) has a hyperbolic boundary saddle \( S_{Y_0} \) and the eigenspaces of \( DY_0(S_{Y_0}) \) are transverse to \( \Sigma \) at \( S_{Y_0} \), then there exists a neighborhood \( \mathcal{V}(Y_0) \) of \( Y_0 \) in \( \chi^r \) such that all \( y \in \mathcal{V}(Y_0) \) have a hyperbolic boundary saddle \( S_Y = (x^*, y^*) \) with the same properties of the eigenspaces. Moreover, the correspondence \( L : \mathcal{V}(Y_0) \to \mathbb{R}^2 \), given by \( L(Y) = S_Y \), is \( C^r \) at \( Y_0 \). Define \( \Pi : \mathcal{V}(Y_0) \to \mathbb{R} \), given by \( \Pi(Y) = (f \circ L)(Y) \). We say that two vector fields \( Y, \tilde{Y} \in \Gamma_+^S \) defined in open sets \( U \) and \( \tilde{U} \), respectively, are \( C^0 \)-orbitally equivalent if there exists an orientation preserving homeomorphism \( h : U \to \tilde{U} \) that sends orbits of \( Y \) to orbits of \( \tilde{Y} \). From [Teixeira(1977)] we know that any \( y \in \Gamma_+^S \) is generically \( C^0 \)-orbitally equivalent to its linear part by a \( \Sigma \)-preserving homeomorphism. And the linear saddle with eigenspaces transverse to the \( x \)-axis has the generic normal forms \( Y_0(x, y) = (\alpha_2 y, \alpha_2 x) \) with \( \alpha_2 = \pm 1 \). So it is easy to see that the generic unfolding of the singularity is given by \( Y_\beta = (\alpha_2(y + \beta), \alpha_2 x) \) where \( \beta \in \mathbb{R} \).

Moreover:

- There exists a \( C^r \)-function \( \Pi : \mathcal{V}(Y_0) \to \mathbb{R} \), such that \( D\Pi_{Y_0} \) is surjective.
- The correspondence \( Y \mapsto S_Y \) is \( C^r \), where \( S_Y \) is a saddle point of \( Y \).
• If $\Pi(Y) < 0$ then $S_Y \in \Sigma_-$.
• If $\Pi(Y) = 0$ then $S_Y \in \Sigma$.
• If $\Pi(Y) > 0$ then $S_Y \in \Sigma_+$.

In this paper we are concerned with the bifurcation diagram of systems $Z_0 = (X_0, Y_0)$ in $\Omega^r$ such that $X_0 \in \Gamma^{F+}_\Sigma$, $Y_0 \in \Gamma^S_\Sigma$, and $p_0 = S_{Y_0} \in \Sigma$. The fold-saddle singularity $p_0 = S_{Y_0}$ is illustrated in Figures 1 and 2—the dotted lines in these and later figures represent the points where $X.f = 0$ and $Y.f = 0$.

Let $p = (0,0)$ be a fold-saddle singularity of $Z = (X,Y)$. We denote the set of all NSVF $Z = (X,Y)$ such that $X \in \Gamma^{F+}_\Sigma$ and $Y \in \Gamma^S_\Sigma$ by $\Gamma^{F-S}$. We endow $\chi^r \times \chi^r$ (consequently $\Omega^r$ and $\Gamma^{F-S}$) with the product topology. Let $Z_0 = (X_0, Y_0) \in \Gamma^{F-S}$. Observe that 0 is the unique singularity of $X_0$ around a neighborhood $W_0$ of the origin in $\mathbb{R}^2$. So, there exists a neighborhood $U_0$ of $Z_0$ in $\Omega^r$ such that for any $Z = (X,Y) \in U_0$ we may find a $\Sigma$--fold point $p_Z = (k_Z, 0) \in W_0$ such that it is the unique singularity of $X$ in $W_0$.

Moreover the correspondence $Z \mapsto p_Z$ is $C^r$.

In the same way, for any $Z = (X,Y) \in U_0$ we find a $C^r$--correspondence $B : U_0 \rightarrow \mathbb{R}^2$ where $B(Z) = s_Z = (a_Z, b_Z)$ is the (unique) equilibrium (saddle) of $Y$ in $U_0$. We are assuming that the eigenspaces of $DY_{s_Z}(q_Z)$ are transverse to $\Sigma$ at $s_Z$. We have to distinguish the cases: (i) $b_Z < 0$, (ii) $b_Z = 0$ and (iii) $b_Z > 0$. Observe that when $b_Z < 0$ (resp. $b_Z > 0$) there is associated to $Z$ an invisible (resp. visible) $\Sigma$--fold point of $Y$ given by $q_Y = (c_Z, 0) \in W_0$. Moreover $\lim_{b_Z \rightarrow 0} c_Z = a_Z$.

Define $F(Z) = (k_Z - a_Z, b_Z)$. Knowing that $s_Z$ is a hyperbolic equilibrium point of $Z$ and 0 is a regular value of $F$, it is not hard to prove (see Teixeira(1979)) that:

• The derivative $DF : U_0 \rightarrow \mathbb{R}^2$ is surjective and
• $F^{-1}(0) = \Omega_2$ is a codimension two submanifold of $\Omega^r$.

Therefore this fold-saddle singularity occurs generically in two-parameter families of vector fields in $\Omega^r$.

4.1. Normal Form. We start this section with the following model:

\[ Z^\tau = \begin{cases} 
X^\tau = \left( \frac{\rho_1}{\alpha_1(\tau)x} \right) & \text{if } y \geq 0, \\
Y = \left( \frac{k_1y}{k_1x} \right) & \text{if } y \leq 0,
\end{cases} \tag{8} \]

where $\tau = \{inv, vis\}$, $\alpha_1(inv) = -1$, $\alpha_1(vis) = 1$, $\rho_1 = \pm 1$ and $k_1 = \pm 1$.

The next lemma provides explicitly the equivalence between $Z \in \Omega_2$ and the model (8). We present an outline of proof of the previous lemma in Section 5.

**Lemma 1.** If $Z \in \Omega_2$ then $Z$ is $\Sigma$--equivalent to $Z^\tau$ given by (8).
Note that there exists a NSVF $\tilde{Z} \in \Omega^r$ nearby $Z^{\text{inv}}$, given by (8), such that $\tilde{Z}$ presents a $\Sigma-$center (see Figure 7). In fact, this suggests that the unfolding of (8) has infinite codimension. At this point it seems natural to propose the following conjecture.

**Conjecture:** For any neighborhood $\mathcal{W} \subset \Omega^r$ of $Z^{\text{inv}}$ (given by (8)), and for any integer $k > 0$ there exists $\tilde{Z} \in W$ such that the codimension of $\tilde{Z}$ is $k$.

So, based on this conjecture, we have to sharpen our normal normal. In fact, in order to get low codimension bifurcation we have to impose some generic assumptions.

Without loss of generality, throughout the rest of this paper we consider $\rho_1 = 1$ and $k_1 = -1$ at the model (8). The other choices of the parameters are treated analogously. When $\tau = \text{inv}$ we add the extra generic assumption $X^3_0 f(p) \neq 0$ on the $\Sigma-$fold point $p$ of $Z_0 = (X_0, Y_0) \in \Gamma^{F-S}$. By means of Theorem 2 in [Vishik(1972)], we may conclude that around the invisible $\Sigma-$fold point the vector field $X_0$ can be expressed as $X_0 = (1, -x + a_1 x^2)$ and $f(x, y) = y$, where $a_1 \neq 0$. We say that $X_0$ is contractive (respectively, expansive) at $p$ if $a_1 < 0$ (respectively $a_1 > 0$).

According to the previous discussion, we will consider $Z^{\text{inv}}_0, Z^{\text{vis}}_0 \in \Omega^r$ written in the following forms:

\begin{align*}
Z^{\text{inv}}_0 &= \begin{cases} 
X^{\text{inv}}_0 &= \begin{pmatrix} 1 \\ -x + x^2 \end{pmatrix} & \text{if } y \geq 0, \\
Y_0 &= \begin{pmatrix} -y \\ -x \end{pmatrix} & \text{if } y \leq 0, \text{ and}
\end{cases} \\
Z^{\text{vis}}_0 &= \begin{cases} 
X^{\text{vis}}_0 &= \begin{pmatrix} 1 \\ x \end{pmatrix} & \text{if } y \geq 0, \\
Y_0 &= \begin{pmatrix} -y \\ -x \end{pmatrix} & \text{if } y \leq 0.
\end{cases}
\end{align*}

Note that $X^{\text{inv}}_0$ presents an invisible expansive $\Sigma-$fold point in its phase portrait and $X^{\text{vis}}_0$ presents a visible one.

**4.2. Unfoldings.** The main question of this paper is to exhibit the bifurcation diagram of $Z^{\tau}_0$ with either $\tau = \text{inv}$ or $\tau = \text{vis}$. For this reason we consider unfoldings of the normal forms (9) and (10). We obtain that:

I. There is an imbedding $F^{\tau}_0 : \mathbb{R}^2, 0 \to \chi^r, Z^{\tau}_0$ such that $F^{\tau}_0 (\lambda, \beta) = Z^{\tau}_{\lambda,\beta}$ is expressed by:

\begin{align*}
Z^{\tau}_{\lambda,\beta} &= \begin{cases} 
X^{\tau}_{\lambda} &= \begin{pmatrix} \alpha_1(\tau)(x - \lambda) + \alpha_2(\tau)(x - \lambda)^2 \\ -(y + \beta) \\ -x \end{pmatrix} & \text{if } y \geq 0, \\
Y_{\beta} &= \begin{pmatrix} 1 \\ -x \end{pmatrix} & \text{if } y \leq 0.
\end{cases}
\end{align*}
where $\tau = \{\text{inv, vis}\}$, $(\lambda, \beta) \in (-1, 1) \times (-\sqrt{3}/2, \sqrt{3}/2)$, $\alpha_1(\text{inv}) = -1$, $\alpha_1(\text{vis}) = 1$, $\alpha_2(\text{inv}) = 1$ and $\alpha_2(\text{vis}) = 0$. Moreover, the two-parameter family given by (11) is transversal to $\Omega_2$. We stress that in [Guardia et al. (2011)] the unfolding of the case $\tau = \text{inv}$ is done. Nevertheless we observe that there are some typical topological types nearby $Z_{\tau}^0$ that do not appear in the bifurcation diagram of $Z_{\lambda,\beta}^0$. For example, when $\tau = \text{inv}$ the configurations in Figure 9 cases (a) and (b) are excluded and when $\tau = \text{vis}$ the configuration in Figure 9-(c) also is excluded.

II- In order to consider a more general situation which includes the above mentioned cases we add an auxiliary parameter $\mu$. As a result the model in Equation (1) is obtained.

We stress that the configuration illustrated in Figure 9-(a) plays a very important role in our analysis. In this resonant configuration we note, simultaneously, a fold-fold singularity (which is a local phenomenon) and a $\Sigma-$graph (loop) passing through the saddle equilibrium (which is a global phenomenon). Only the bifurcation of these two unstable configurations already represents a relevant development (the fold-fold singularity was studied recently in [Guardia et al. (2011)] and the non-smooth loop bifurcation, as far as we know, was not studied until the present work). In fact, this configuration is reached in (11), taking $\mu = \mu_0(\beta)$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure9}
\caption{Cases do not covered in [Guardia et al. (2011)]. In (a) and (b) (respectively, in (c)) the fold singularity is invisible (respectively, visible).}
\end{figure}
In what follows, in order to simplify the calculations, we take $\mu = \alpha + 1$ in (11) and obtain the following expression:

\[
Z_{\lambda, \alpha, \beta}^T = \begin{cases} 
X_\lambda = \left( \frac{1}{\alpha_1(\tau)(x - \lambda) + \alpha_2(\tau)(x - \lambda)^2} \right) & \text{if } y \geq 0, \\
Y_{\alpha, \beta} = \left( \frac{1}{\alpha_1(\tau)(x - \lambda) + \alpha_2(\tau)(x - \lambda)^2} \right) & \text{if } y \leq 0,
\end{cases}
\]

where $\tau = \{\text{inv}, \text{vis}\}$, $\alpha_1(\text{inv}) = -1$, $\alpha_1(\text{vis}) = 1$, $\alpha_2(\text{vis}) = 0$ and $(\lambda, \beta, \alpha) \in (-1, 1) \times (-\sqrt{3}/2, \sqrt{3}/2) \times (-1 - \varepsilon_0, 1)$ with $\varepsilon_0 > 0$ sufficiently small. Since $\mu_0(\beta)$ is given by (6), we obtain that

\[
\alpha_0(\beta) = 1 - (12\beta/(-3 + 6\beta + \sqrt{9 - 12\beta^2})).
\]

When it does not produce confusion, in order to simplify the notation we use $Z = (X, Y)$ or $Z_{\lambda, \alpha, \beta} = (X, Y)$ instead $Z_{\lambda, \alpha, \beta}^T = (X_\lambda, Y_{\alpha, \beta})$.

4.2.1. Geometrical Analysis of the Normal Form (12). Given $Z = (X, Y)$, we describe some properties of both $X = X_\lambda$ and $Y = Y_{\alpha, \beta}$.

The real number $\lambda$ measures how the $\Sigma$–fold point $d = (d_1, d_2) = (\lambda, 0)$ of $X$ is translated away from the origin. More specifically, if $\lambda < 0$ then $d$ is translated to the left hand side and if $\lambda > 0$ then $d$ is translated to the right hand side.

Some calculations show that the curve $Y, f = 0$ is given by $y = \frac{(1 - \alpha)}{(1 + \alpha)} x - \beta$. So the points of this curve are equidistant from the separatrices when $\alpha = -1$. It becomes closer to the stable separatrix of the saddle equilibrium $S = S_{\alpha, \beta} = (s_1, s_2)$ when $\alpha \in (-1, 0)$. It becomes closer to the unstable separatrix of $S$ when $\alpha \in (-1 + \varepsilon_0, -1)$. Moreover, the smooth vector field $Y$ has distinct types of contact with $\Sigma$ according to the particular deformation considered. In this way, we have to consider the following behaviors:

- $Y^-$: In this case $\beta < 0$. So $S$ is translated to the $y$–direction with $y > 0$ (and $S$ is not visible for $Z$). It has a visible $\Sigma$–fold point $e = e_{\alpha, \beta} = (e_1, e_2) = \left( \frac{(1 + \alpha)}{(1 - \alpha)} \beta, 0 \right) = (e_1, 0)$ (see Figure 10).
- $Y^0$: In this case $\beta = 0$. So $S$ is not translated (see Figure 11).
- $Y^+$: In this case $\beta > 0$. So $S$ is translated to the $y$–direction with $y < 0$. It has an invisible $\Sigma$–fold point $i = i_{\alpha, \beta} = (i_1, i_2) = \left( \frac{(1 + \alpha)}{(1 - \alpha)} \beta, 0 \right)$. Moreover, we distinguish two points: $h = h_\beta = (h_1, h_2) = (-\beta, 0)$ which is the intersection between the unstable separatrix with $\Sigma$ and $j = j_\beta = (j_1, j_2) = (\beta, 0)$ which is the intersection between the stable separatrix with $\Sigma$ (see Figure 11).

In Figure 11 we distinguish the arcs of trajectory $\sigma_1$ joining $S$ to $h$ and $\sigma_2$ joining $j$ to $S$. 
5. Proof of Lemma 1

Now, we show how we can construct the homeomorphism in Lemma 1.

Outline of Proof of Lemma 1. Here we construct a $\Sigma$—preserving homeomorphism $h$ that sends orbits of $Z = (X, Y) \in \Omega_2$, defined in a sufficiently small neighborhood $U_0$ of the fold-cusp singularity of $Z$, to orbits of $\tilde{Z} = (\tilde{X}, \tilde{Y})$, defined in a sufficiently small neighborhood $U_0$ of the fold-cusp singularity of $\tilde{Z}$, where $\tilde{Z} = Z^{inv}$ is given by (8) with $\rho_1 = 1$ and $k_1 = -1$ (the other cases are treated analogously). Consider $A_0$ an arbitrary point of the stable separatrix of the saddle point $S$ of $Y$ (see Figure 12). Let $T_1$ be a transversal section of $Y$ at $A_0$. The section $T_1$ also is transversal to $\tilde{Y}$ and it crosses the stable separatrix of the saddle point $\tilde{S}$ of $\tilde{Y}$ at $B_0$. Let $A_1 \in T_1$ be a point on the left of $A_0$. The trajectory of $Y$ passing through $A_1$ crosses $\Sigma$ at $A_2$. In the same way, the trajectory of $\tilde{Y}$ passing through $\tilde{B}_1$ crosses $\Sigma$ at $\tilde{B}_2$. The trajectory of $X$ that passes through $A_2$ crosses $\Sigma$ in a point $A_3$. The trajectory of $\tilde{X}$ that passes through $\tilde{B}_2$ crosses $\Sigma$ in $B_3$. Consider $A_4$ an arbitrary point of the unstable separatrix of $S$. Let $T_2$ be a transversal section of $Y$ passing through $A_4$. The section $T_2$ also is transversal to $\tilde{Y}$ and it crosses the unstable separatrix of $\tilde{S}$ at the point $B_4$. The trajectory of $Y$ passing through $A_3$ crosses $T_2$ in a point $A_5$. In the same way, the trajectory of $\tilde{Y}$ passing through $\tilde{B}_3$ crosses $T_2$ at $B_5$. Let $A_6 \in T_1$ be a point at the right of $A_0$. The trajectory of $Y$ passing through $A_6$ crosses $T_2$ at $A_7$. The trajectory of $\tilde{Y}$ passing through $A_6$ crosses $T_2$ at $B_7$. The homeomorphism $h$ sends $T_1$ to $T_1$, the arc of trajectory $\gamma_1 = \overline{A_1A_5}$ to the arc of trajectory $\tilde{\gamma}_1 = \overline{\tilde{A}_1\tilde{B}_5}$ and the arc of trajectory $\gamma_2 = \overline{A_6A_7}$ to the arc of trajectory $\tilde{\gamma}_2 = \overline{\tilde{A}_6\tilde{B}_7}$. Now we can extend continuously $h$ to the interior of the region bounded by $T_1 \cup \gamma_1 \cup T_2 \cup \gamma_2$. In this way, there exists a $\Sigma$—preserving homeomorphism $h$ that sends orbits of $Z$ to orbits of $\tilde{Z}$.
When $\tilde{Z} = Z^{vis}$ is given by (8), the first coordinate of $\tilde{X}$ is equal to 1 and $k_1 = -1$. We build the same above construction until the appearance of $A_2$ and $B_2$. Now, consider $C_4$ an arbitrary point of the stable separatrix of the $\Sigma$–fold point $F$ of $X$ (see Figure 13). Let $T_3$ be a transversal section to $X$ at $C_4$. The section $T_3$ is also transversal to $\tilde{X}$ and it crosses the stable separatrix of the $\Sigma$–fold point $\tilde{F}$ of $\tilde{X}$ at the point $D_4$. The trajectory of $X$ passing through $A_2$ crosses $T_3$ at a point $C_3$. In the same way, the trajectory of $\tilde{X}$ passing through $B_2$ crosses $T_3$ at a point $D_3$. Consider $C_5$ an arbitrary point of the unstable separatrix of $F$. Let $T_4$ be a transversal section to $X$ at $C_5$. The section $T_4$ also is transversal to $\tilde{X}$ and it crosses the unstable separatrix of $\tilde{F}$ at $D_5$. Let $C_6 \in T_3$ be a point at the right of $C_4$. The trajectory of $X$ passing through $C_6$ crosses $T_4$ at $C_7$. In the same way, let $D_6 \in T_3$ be a point at the right of $D_4$. The trajectory of $\tilde{X}$ passing through $D_6$ crosses $T_4$ at $D_7$. Let $C_8 \in T_3$ be a point at the right of $C_5$. The trajectory of $X$ passing through $C_8$ crosses $\Sigma$ at $A_3$. In the same way, let $D_8 \in T_4$ be a point at the right of $D_5$. The trajectory of $\tilde{X}$ passing through $D_8$ crosses $\Sigma$ at $B_3$. Now, it is enough to repeat the construction made in the previous case. The homeomorphism $h$ sends $T_1$ to $T_1$, the arc of trajectory $\gamma_1 = A_1C_3$ to the arc of trajectory $\tilde{\gamma}_1 = \tilde{A}_1D_3$, $T_3$ to $T_3$, the arc of trajectory $\gamma_2 = C_6C_7$ to the arc of trajectory $\tilde{\gamma}_2 = \tilde{D}_6\tilde{D}_7$, $T_4$ to $T_4$, the arc of trajectory $\gamma_3 = C_8A_5$ to the arc of trajectory $\tilde{\gamma}_3 = \tilde{D}_8\tilde{B}_5$, $T_2$ to $T_2$ and the arc of trajectory $\gamma_4 = A_6A_7$ to the arc of trajectory $\tilde{\gamma}_4 = A_6B_7$. Now we can extend continuously $h$ to the interior of the region bounded by $T_1 \cup \gamma_1 \cup T_3 \cup \gamma_2 \cup T_4 \cup \gamma_3 \cup T_2 \cup \gamma_4$. In this way, there exists a $\Sigma$–preserving homeomorphism $h$ that sends orbits of $Z$ to orbits of $\tilde{Z}$. □
6. PROOF OF THEOREM 1

Proof of Theorem 1. In Cases 1, 2 and 3 we assume that \( Y \) presents the behavior \( Y^- \). In Cases 4, 5 and 6 we assume that \( Y \) presents the behavior \( Y^0 \). In these cases canard cycles do not arise.

\[ \diamond \text{ Cases } (1_1) d_1 < e_1, (2_1) d_1 = e_1 \text{ and } (3_1) d_1 > e_1: \] The points of \( \Sigma \) outside the interval \((d_1, e_1)\) (or \((e_1, d_1)\)) belong to \( \Sigma_c \). The points inside this interval, when it is not degenerate, belong to \( \Sigma_a \) in Case 1 and to \( \Sigma_s \) in Case 3. In both cases \( H(z) > 0 \) for all \( z \in \Sigma_e \cup \Sigma_s \). See Figure 14.

\[ \lambda < 0 \quad \lambda = 0 \quad \lambda > 0 \]

\[ \diamond \text{ Cases } (4_1) d_1 < s_1, (5_1) d_1 = s_1 \text{ and } (6_1) d_1 > s_1: \] The points of \( \Sigma \) outside the interval \((d_1, s_1)\) (or \((s_1, d_1)\)) belong to \( \Sigma_c \). The points inside this interval, when it is not degenerate, belong to \( \Sigma_a \) in Case 4 and to \( \Sigma_s \) in Case 6. In both cases \( H(z) > 0 \) for all \( z \in \Sigma_e \cup \Sigma_s \). See Figure 15.

In Cases 7−19 we assume that \( Y \) presents the behavior \( Y^+ \).

Observe that as the parameter \( \lambda \) increases − assuming the values \( L_0, L_1 \) and \( L_2 \) below described − it appear orbit-arcs of \( X \) connecting the points \( h \) and \( i \), \( h \) and \( j \) and \( i \) and \( j \) respectively.

Remembering that \( \alpha = 1 - \left( \frac{12\beta}{(-3 + 6\beta + \sqrt{9 - 12\beta^2})} \right) \) the values of \( L_0, L_1 \) and \( L_2 \) are:
\[\begin{align*}
\lambda < 0 & \quad \lambda = 0 & \quad \lambda > 0 \\
\end{align*}\]

**Figure 15.** Cases 4, 5, and 6.

\[L_0 = \left[-9 - 6\beta + \sqrt{9 - 12\beta^2} + \sqrt{2} \sqrt{15 + 9 - 12\beta^2 - 2\beta(-3 + 2\beta + \sqrt{9 - 12\beta^2})}\right]/12\]

\[L_1 = -1/2 + \sqrt{9 - 12\beta^2}/6\]

\[L_2 = \left[-9 + 6\beta + \sqrt{9 - 12\beta^2} + \sqrt{2} \sqrt{15 + 9 - 12\beta^2 + 2\beta(-3 + 2\beta + \sqrt{9 - 12\beta^2})}\right]/12.\]

\[\diamond \text{Cases (7)} \lambda < -\beta, \ (8) \lambda = -\beta, \ (9) -\beta < \lambda < L_0, \ (10) \lambda = L_0 \text{ and (11) } L_0 < \lambda < L_1: \text{ The points of } \Sigma \text{ outside the interval } (d_1, i_1) \text{ belong to } \Sigma_c. \text{ The points inside this interval belong to } \Sigma_s. \text{ The direction function } H \text{ assumes positive values in a neighborhood of } d_1, \text{ negative values in a neighborhood of } i_1 \text{ and there exists only one value } \bar{P}_{\lambda,\alpha,\beta} \text{ such that } H(\bar{P}) = 0. \text{ So, by (7), the } \Sigma-\text{attractor } P = (\bar{P},0), \text{ nearby } (0,0), \text{ is the unique pseudo equilibrium of } Z. \text{ In these cases canard cycles do not arise. See Figure 15.}\]

**Figure 16.** Cases 7, 9, 10, 11.

\[\diamond \text{Case (12) } \lambda = L_1: \text{ Since } \lambda = L_1 \text{ there is an orbit-arc } \gamma^X_1 \text{ of } X \text{ connecting the points } h \text{ and } j. \text{ It generates a } \Sigma-\text{graph } \Gamma = \gamma^X_1 \cup \Sigma_c \cup S \cup \Sigma_c\]
of kind I. Moreover, since $\alpha = \alpha_0$, where $\alpha_0$ is given by (13), there exists a non generic tangential singularity at the point $d = i$. So, the points of $\Sigma/\{d\}$ belong to $\Sigma_e$. As the $\Sigma$-fold point of $X$ is expansive, a direct calculus shows that the First Return Map $\eta : (h, d) \to (h, d)$ has derivative bigger than 1. As consequence, $\Gamma$ is a repeller for the trajectories inside it, $d = i$ behaves itself like an attractor (weak focus) and canard cycles do not arise. See Figure 17.

![Figure 17](image_url)

**Figure 17.** Cases 12, 13 and 14.

- **Case (13)** $L_1 < \lambda < L_3$: The meaning of $L_3$ will be given below in this case. The points of $\Sigma$ outside the interval $(i_1, d_1)$ belong to $\Sigma_e$ and the points inside this interval belong to $\Sigma_c$. The direction function $H$ assumes positive values in a neighborhood of $d_1$, negative values in a neighborhood of $i_1$ and there exists a unique value $\tilde{P} = \tilde{P}_{\lambda, \alpha, \beta}$ such that $H(\tilde{P}) = 0$. So $P = (\tilde{P}, 0)$ is a $\Sigma$-repeller. When $\lambda$ is a bit bigger than $L_1$, the First Return Map $\eta$ has two fixed points, i.e., $Z$ has two canard cycles. One of them, called $\Gamma_1$, is born from the bifurcation of the $\Sigma$-graph $\Gamma$ of the previous case and the other one, called $\Gamma_2$, is born from the bifurcation of the non generic tangential singularity presented in the previous case. Both of them are canard cycles of kind I. Moreover, we obtain that $\Gamma_1$ is a hyperbolic repeller canard cycle and $\Gamma_2$ is a hyperbolic attractor canard cycle. Note that, as $\lambda$ increases, $\Gamma_1$ becomes smaller and $\Gamma_2$ becomes bigger. When $\lambda$ assumes the limit value $L_3$, one of them collides to the other. See Figure 17.

- **Case (14)** $\lambda = L_3$: The distribution of the connected components of $\Sigma$ and the behavior of $H$ are the same as Case 13. Since $\lambda = L_3$, as described in the previous case, there exists a non hyperbolic canard cycle $\Gamma$ of kind I which is an attractor for the trajectories inside it and is a repeller for the trajectories outside it. See Figure 17.

- **Cases (15)** $L_3 < \lambda < L_2$, (16) $\lambda = L_2$, (17) $L_2 < \lambda < \beta$, (18) $\lambda = \beta$ and (19) $\lambda > \beta$: The points of $\Sigma$ outside the interval $(i_1, d_1)$ belong to $\Sigma_e$ and the points inside this interval belong to $\Sigma_e$. The direction function...
$H$ assumes positive values in a neighborhood of $d_1$, negative values in a neighborhood of $i_1$ and there exists a unique value $\tilde{P}$ such that $H(\tilde{P}) = 0$. So, by (7), the $\Sigma$-repeller $P = (\tilde{P}, 0)$, nearby $(0,0)$, is the unique pseudo equilibrium of $Z$. In these cases canard cycles do not arise. See Figure 18.

\[ L_3 < \lambda < L_2 \quad \lambda = L_2 \quad L_2 < \lambda < \beta \quad \lambda = \beta \quad \lambda > \beta \]

**Figure 18.** Cases 15$_1$ – 19$_1$.

The bifurcation diagram is illustrated in Figure 19.

**Remark 7.** In Cases 11$_1$ and 15$_1$ the ST-bifurcations (as described in Guardia et al. (2011)) arise. In fact, note that the trajectory passing through
Proof of Theorem 2. In Cases 1, 2 and 3 we assume that $Y$ presents the behavior $Y^-$. In Cases 4, 5 and 6 we assume that $Y$ presents the behavior $Y^0$. In Cases 7, 8 we assume that $Y$ presents the behavior $Y^+$. 

\begin{itemize}
\item \emph{Cases (12)} $d_1 < e_1$, (22) $d_1 = e_1$, (32) $d_1 > e_1$, (42) $d_1 < s_1$, (52) $d_1 = s_1$ and (62) $d_1 > s_1$: The analysis of these cases are done in a similar way as the cases 1, 2, 3, 4, 5 and 6.
\end{itemize}

Observe that as the parameter $\lambda$ increases — assuming the values $M_0$, $M_1$ and $M_2$ below described — it appear orbit-arcs of $X$ connecting the points $h$ and $i$, $h$ and $j$ and $i$ and $j$ respectively. The values of $M_0$, $M_1$ and $M_2$ are:

\begin{align*}
M_0 &= (-3 - 3\alpha(-2 + \alpha + 2(-1 + \alpha)\beta) + \\
&+ \sqrt{9(-1 + \alpha)^4 - 12(-1 + \alpha)^2\alpha^2\beta^2})/(6(-1 + \alpha)^2), \\
M_1 &= -1/2 + \sqrt{9 - 12\beta^2}/6 \\
M_2 &= (-3 + 6\beta - 3\alpha(-2 + \alpha + 2\beta) + \\
&+ \sqrt{9(-1 + \alpha)^4 - 12(-1 + \alpha)^2\alpha^2\beta^2})/(6(-1 + \alpha)^2).
\end{align*}

\begin{itemize}
\item \emph{Cases (72)} $\lambda < -\beta$, (82) $\lambda = -\beta$, (92) $-\beta < \lambda < M_0$, (102) $\lambda = M_0$ and (112) $M_0 < \lambda < M_1$: Analogous to Cases 7, 11 changing $L_0$ by $M_0$ and $L_1$ by $M_1$.
\end{itemize}

\begin{itemize}
\item \emph{Case (122)} $\lambda = M_1$: The points of $\Sigma$ outside the interval $(d_1, i_1)$ belong to $\Sigma_\epsilon$ and the points inside this interval belong to $\Sigma_\delta$. The direction function $H$ assumes positive values in a neighborhood of $d_1$, negative values in a neighborhood of $i_1$ (see Remark 3) and there exists a unique value $P = P_{\lambda, \alpha, \beta}$ such that $H(P) = 0$. So $P = (P, 0)$ is a $\Sigma$–attractor. Since $\lambda = M_1$, there is an orbit-arc $\gamma^X$ of $X$ connecting the points $h$ and $j$. It generates a $\Sigma$–graph $\Gamma = \gamma^X \cup \Sigma_\epsilon \cup S \cup \Sigma_c$ of kind I. Since $\alpha > \alpha_0$, where $\alpha_0$ is given by (13), it is straightforward to show that the First Return Map defined in the interval $(h_1, d_1) \subset \Sigma$ do not have fixed points. By consequence, $\Gamma$ is a repellor for the trajectories inside it and canard cycles do not arise. See Figure 20.
\end{itemize}

\begin{itemize}
\item \emph{Case (132)} $M_1 < \lambda < i_1$: The distribution of the connected components of $\Sigma$ and the behavior of $H$ are the same as Case 12. Since $M_1 < \lambda < i_1$, there is an orbit-arc $\gamma^X$ of $X$ connecting $j$ to a point $k = (k_1, 0) \in \Sigma$, where $k_1 \in (h_1, d_1)$, for negative time. Also there is an orbit-arc $\gamma^Y$ of $Y$ connecting $k$ to a point $l = (l_1, 0) \in \Sigma$, where $l_1 \in (i_1, j_1)$, for negative time. Repeating this argument, we can find an increasing sequence $(k_i)_{i \in \mathbb{N}}$. We can prove that there is an interval $I \subset (k, d)$ such that $\eta = (\varphi_Y \circ \varphi_X)^\prime < 1$ on $I$. As $P$ is a $\Sigma$–attractor, there is an interval $J \subset (k, d)$ such that $\eta^\prime > 1$ on $J$. Moreover, we can prove that $\eta$ has a unique fixed point $Q \in (k, d)$. As consequence, there passes through $Q$ a repellor canard cycle $\Gamma$ of kind I. See Figure 20. This canard cycle is born from the bifurcation of the $\Sigma$–graph present in Case 12. The expression of $\eta$ is too large, so the
\[ \lambda = M_1, \quad M_1 < \lambda < i_1, \quad \lambda = i_1 \]

**Figure 20.** Cases 12, 13 and 14.

The general case will be omitted. For the particular case when \( \alpha = -1, \beta = 1/2 \) and \( \lambda = -1/2 + 11\sqrt{6}/60 \), the map \( \eta \) is given by

\[
\eta(x) = \frac{3}{4} + \frac{3}{2} \left( -\frac{1}{2} + \frac{11}{10\sqrt{6}} \right) + \frac{x}{2} + \frac{1}{4} \sqrt{3} \left( \left( 1 - 2 \left( -\frac{1}{2} + \frac{11}{10\sqrt{6}} \right) - 2x \right) \left( 3 + 2 \left( -\frac{1}{2} + \frac{11}{10\sqrt{6}} \right) + 2x \right) \right). 
\]

A straightforward calculation shows that the unique fixed point of this particular \( \eta \) occurs when \( x = -\sqrt{29}/2/10 \).

\( \diamond \) **Case (142) \( \lambda = i_1 \):** Every point of \( \Sigma \) belongs to \( \Sigma_c \) except the point \( d = i \). The canard cycle presented in the previous case is persistent for this case (remember that this canard cycle is born from the bifurcation of the \( \Sigma \)-graph of Case 12). So, it radius does not tend to zero when \( \lambda \) tends to \( i_1 \). So the non generic tangential singularity \( d = i \) behaves itself like a weak attractor focus. See Figure 20.

\( \diamond \) **Cases (152) \( i_1 < \lambda < M_3 \) and (162) \( \lambda = M_3 \):** Analogous to Cases 13, 14 replacing \( L_1 \) by \( i_1 \) and \( L_3 \) by \( M_3 \), where \( M_3 \) is the limit value for which \( \Gamma_1 \) collides with \( \Gamma_2 \).

\( \diamond \) **Cases (172) \( M_3 < \lambda < M_2, (182) \lambda = M_2, (192) \lambda = i_1 < \beta, (202) \lambda = \beta \) and (212) \( \lambda > \beta \):** Analogous to Cases 15, 16 replacing \( L_2 \) by \( M_2 \) and \( L_3 \) by \( M_3 \).

The bifurcation diagram is illustrated in Figure 20. \( \square \)

### 8. Proof of Theorem 3

**Proof of Theorem 3.** In Cases 1, 2 and 3 we assume that \( Y \) presents the behavior \( Y^- \). In Cases 4, 5 and 6 we assume that \( Y \) presents the behavior \( Y^0 \). In Cases 7 and 21 we assume that \( Y \) presents the behavior \( Y^+ \).

\( \diamond \) **Cases (13) \( d_1 < e_1, (23) d_1 = e_1, (33) d_1 > e_1, (43) d_1 < s_1, (53) d_1 = s_1 \) and (63) \( d_1 > s_1 \):** Analogous to Cases 1, 2, 3, 4, 5 and 6.
In what follows we consider $M_0$, $M_1$, $M_2$ and $M_3$ as in the previous theorem.

- Cases $(7_3) \lambda < -\beta$, $(8_3) \lambda = -\beta$, $(9_3) -\beta < \lambda < M_0$, $(10_3) \lambda = M_0$ and $(11_3) M_0 < \lambda < i_1$: Analogous to Cases $7_2 - 11_2$ changing $M_1$ by $i_1$.

- Case $(12_3) \lambda = i_1$: Every point of $\Sigma/\{d\}$ belongs to $\Sigma_c$. In a similar way as Case $13_2$, we can construct sequences $(k_i)_{i \in \mathbb{N}}$ and $(l_i)_{i \in \mathbb{N}}$. Since $d = i$ we have that $k_i \to d$ and $l_i \to d$. So $d$ is a non generic tangential singularity that behaves itself like an attractor. See Figure 22.

- Case $(13_3) i_1 < \lambda < M_1$: Analogous to Case $13_2$ except that there is a change of stability on $P = (\tilde{P}, 0)$, which is a $\Sigma-$repeller, and on $\Gamma$, which is an attractor canard cycle of kind I. This canard cycle is born from the

![Figure 21. Bifurcation Diagram of Theorems 2 and 3.](image)

![Figure 22. Cases 12, 13, and 14.](image)
bifurcation of the non-generic tangential singularity of Case 12. See Figure 22

- Case (14) \( \lambda = M_1 \): Analogous to Case 12 except that occurs a change of stability on \( P = (\tilde{P}, 0) \), which is a \( \Sigma \)-repeller. This fact generates a bifurcation like Hopf near \( P \) and there appears a hyperbolic attractor canard cycle \( \Gamma_1 \), of kind I, between \( P \) and the \( \Sigma \)-graph \( \Gamma_2 \). See Figure 22.
- Cases (15) \( M_1 < \lambda < M_3 \) and (16) \( \lambda = M_3 \): Analogous to Cases 15 − 16, replacing \( i_1 \) by \( M_1 \).
- Cases (17) \( M_3 < \lambda < M_2 \), (18) \( \lambda = M_2 \), (19) \( M_2 < \lambda < \beta \), (20) \( \lambda = \beta \) and (21) \( \lambda > \beta \): Analogous to Cases 17 − 21.

The bifurcation diagram is illustrated in Figure 21. □

9. Proof of Theorem 4

Proof of Theorem 4. Since \( X \) has a unique \( \Sigma \)-fold point which is visible we conclude that canard cycles do not arise.

In Cases 1, 2 and 3 we assume that \( Y \) presents the behavior \( Y^- \). In Cases 4, 5 and 6 we assume that \( Y \) presents the behavior \( Y^0 \). In these cases, when it is well defined, the direction function \( H \) assumes positive values.

- Case (1) \( d_1 < e_1 \): The points of \( \Sigma \) inside the interval \((d_1, e_1)\) belong to \( \Sigma_c \). The points on the left of \( d_1 \) belong to \( \Sigma_s \) and the points on the right of \( e_1 \) belong to \( \Sigma_e \). See Figure 23.
- Case (2) \( d_1 = e_1 \): Here \( \Sigma_c = \emptyset \). The vector fields \( X \) and \( Y \) are linearly dependent on \( d_1 = e_1 \) which is a tangential singularity. Moreover, it is an attractor for the trajectories of \( Z \) crossing \( \Sigma_s \) and a repeller for the trajectories of \( Z \) crossing \( \Sigma_e \). See Figure 23.
- Case (3) \( d_1 > e_1 \): The points of \( \Sigma \) inside the interval \((e_1, d_1)\) belong to \( \Sigma_c \). The points on the left of \( e_1 \) belong to \( \Sigma_s \) and the points on the right of \( d_1 \) belong to \( \Sigma_e \). See Figure 23.

![Figure 23. Cases 1, 2 and 3.](image)

- Case (4) \( d_1 < s_1 \): The points of \( \Sigma \) inside the interval \((d_1, s_1)\) belong to \( \Sigma_c \). The points on the left of \( d_1 \) belong to \( \Sigma_s \) and the points on the right of \( s_1 \) belong to \( \Sigma_e \). See Figure 24.
Case $5_4$ $d_1 = s_1$: Here $\Sigma_c = \emptyset$ and $S$ is an attractor for the trajectories of $Z$ crossing $\Sigma_s$ and it is a repeller for the trajectories of $Z$ crossing $\Sigma_e$. See Figure 24.

Case $6_4$ $d_1 > s_1$: The points of $\Sigma$ inside the interval $(d_1, s_1)$ belong to $\Sigma_c$. The points on the left of $s_1$ belong to $\Sigma_s$ and the points on the right of $d_1$ belong to $\Sigma_e$. See Figure 24.

In Cases $7_4 - 13_4$ we assume that $Y$ presents the behavior $Y^+$.

Cases $7_4$ $d_1 < h_1$, $8_4$ $d_1 = h_1$ and $9_4$ $h_1 < d_1 < i_1$: The points of $\Sigma$ inside the interval $(d_1, i_1)$ belong to $\Sigma_c$. The points on the left of $d_1$ belong to $\Sigma_s$ and the points on the right of $i_1$ belong to $\Sigma_e$. The direction function $H$ assumes positive values on $\Sigma_s$ and negative values in a neighborhood of $i_1$. Moreover, $H(\beta\lambda/(-1+\beta)) = 0$ and the $\Sigma$-repeller $P = (\beta\lambda/(-1+\beta), 0)$ is the unique pseudo equilibrium. See Figure 25.

Case $10_4$ $d_1 = i_1$: Here $\Sigma_c = \emptyset$. The vector fields $X$ and $Y$ are linearly dependent on the tangential singularity $d_1 = i_1$. A straightforward calculation shows that $H(z) = (1 - \beta)/2 \neq 0$ for all $z \in \Sigma/\{d\}$. So $d_1 = i_1$ is an attractor for the trajectories of $Z$ crossing $\Sigma_s$ and a repeller for the trajectories of $Z$ crossing $\Sigma_e$. Moreover, $\Delta = \{d\} \cup \overline{dj} \cup \Sigma_e \cup \{S\} \cup \Sigma_c \cup \overline{hd}$ is a $\Sigma$-graph of kind III in such a way that each $Q$ in its interior belongs to another $\Sigma$-graph of kind III passing through $d$. See Figure 25.

\[ \lambda < 0 \quad \lambda = 0 \quad \lambda > 0 \]

\[ \lambda < -\beta \quad -\beta < \lambda < 0 \quad \lambda = 0 \]

Figure 24. Cases $4_4$, $5_4$ and $6_4$.

Figure 25. Cases $7_4 - 10_4$. 
\( \diamond \) Cases \((11_4)\) \(i_1 < d_1 < j_1\), \((12_4)\) \(d_1 = j_1\) and \((13_4)\) \(j_1 < d_1\): The points of \(\Sigma\) inside the interval \((i_1, d_1)\) belong to \(\Sigma_c\). The points on the left of \(i_1\) belong to \(\Sigma_s\) and the points on the right of \(d_1\) belong to \(\Sigma_e\). The direction function \(H\) assumes positive values on \(\Sigma_e\) and negative values in a neighborhood of \(i_1\). Moreover, \(H(\beta \lambda / (-1 + \beta)) = 0\) and the \(\Sigma\)-attractor \(P = (\beta \lambda / (-1 + \beta), 0)\) is the unique pseudo equilibrium. See Figure 26.

![Figure 26. Cases 11_4 – 13_4.](image)

\(0 < \lambda < \beta\) \(\lambda = \beta\) \(\beta < \lambda\)

10. Proof of Theorem 5

Proof of Theorem 5. The direction function \(H\) has a root \(Q = (q, 0)\) where
\[
q = \frac{1}{2(\alpha + 1)}((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha) + \\
+ \sqrt{((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha))^2 + 4\beta(1 + \alpha)(1 + \alpha + \lambda(-1 + \alpha))}).
\]

The bifurcation diagram is illustrated in Figure 27.

![Figure 27. Bifurcation Diagram of Theorems 4, 5 and 6.](image)
Moreover, $H$ assumes positive values on the right of $Q$ and negative values on the left of $Q$. Note that when $\alpha \to -1$ so $Q \to -\infty$ under the line $\{y = 0\}$ and it occurs the configurations showed in Theorem 4.

In Cases $1_5, 2_5$ and $3_5$ we assume that $Y$ presents the behavior $Y^-$. In Cases $4_5, 5_5$ and $6_5$ we assume that $Y$ presents the behavior $Y^0$. In Cases $7_5 - 13_5$ we assume that $Y$ presents the behavior $Y^+$. 

⋄ Cases $1_5$ $d_1 < e_1$, $2_5$ $d_1 = e_1$, $3_5$ $d_1 > e_1$, $4_5$ $d_1 < s_1$, $5_5$ $d_1 = s_1$ and $6_5$ $d_1 > s_1$: Analogous to Cases $1_4, 2_4, 3_4, 4_4, 5_4$ and $6_4$ respectively, except that here it appears the $\Sigma$–saddle $Q$ on the left of $d$ and $e$ or $S$. See Figure 28.

![Figure 28. Cases 1_5, 2_5 and 3_5.](image)

⋄ Cases $7_5$ $d_1 < h_1$, $8_5$ $d_1 = h_1$, $9_5$ $h_1 < d_1 < i_1$: Analogous to Cases $7_4 - 9_4$, except that here the $\Sigma$–saddle $Q$ appears on the left of $d_1$ and $i_1$. So $P = (p, 0)$ where

$$p = \frac{1}{2(\alpha + 1)}((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha)+$$

$$-\sqrt{((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha))^2 + 4\beta(1 + \alpha)(1 + \alpha + \lambda(-1 + \alpha)))}).$$

⋄ Case $10_5$ $d_1 = i_1$: Analogous to Case $10_4$, except that here appear the $\Sigma$–saddle $Q$ on the left of $d_1 = i_1$.

⋄ Cases $11_5$ $i_1 < d_1 < j_1$, $12_5$ $d_1 = j_1$ and $13_5$ $j_1 < d_1$: Analogous to Cases $11_4 - 13_4$, except that here the $\Sigma$–saddle $Q$ appears on the left of $d_1$ and $i_1$.

The bifurcation diagram is illustrated in Figure 27. □

11. Proof of Theorem 6

Proof of Theorem 6. The direction function $H$ has a root $Q = (q, 0)$ where $q$ is given by (14). Moreover, $H$ assumes positive values on the left of $Q$ and negative values on the right of $Q$. Note that when $\alpha \to -1$ so $Q \to \infty$ under the line $\{y = 0\}$ and the configurations shown in Theorem 4 occur.
In Cases 1, 2 and 3 we assume that $Y$ presents the behavior $Y^{-}$. In Cases 4, 5 and 6 we assume that $Y$ presents the behavior $Y^{0}$. In Cases 7 - 13 we assume that $Y$ presents the behavior $Y^{+}$.

D Case (1) $d_1 < e_1$, (2) $d_1 = e_1$, (3) $d_1 > e_1$, (4) $d_1 < s_1$, (5) $d_1 = s_1$ and (6) $d_1 > s_1$, (7) $d_1 < h_1$, (8) $d_1 = h_1$, (9) $h_1 < d_1 < i_1$, (10) $d_1 = i_1$, (11) $i_1 < d_1 < j_1$, (12) $d_1 = j_1$ and (13) $j_1 < d_1$: Analogous to Cases 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 respectively, except that here the $\Sigma$–saddle $Q$ takes place on the right of $d_1$, $e_1$, $s_1$ and $i_1$ when these points appear.

The bifurcation diagram is illustrated in Figure 27.

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