Twist Quantization of String and Hopf Algebraic Symmetry

Tsuguhiko Asakawa

Department of Physics, Tohoku University

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“Hopf Algebra Symmetry and String Theory,” arXiv:0805.2203,
“Twist Quantization of String and B Field Background,” arXiv:0811.1638.
(cf)“Twist Quantization of String and Hopf Algebraic Symmetry,” arXiv:0912.xxxx.
Introduction
Abstract

We propose a new description of string worldsheet theory based on a Hopf algebra.

- Quantization = Drinfeld Twist.
- Unifying quantization, covariance and spacetime symmetry
- Suitable framework for general covariance at quantum level.
Motivation 1: String theory as a quantum gravity

Superstring theory

A promising candidate for the unification of elementary particles and quantum gravity. But formulated only as perturbation theory around a fixed background.

**Hint:** To formulate a theory, symmetries (geometries) would play a key role, but

- Concepts in classical general relativity are not manifest.
- What is stringy “geometry”?
Quantization in Minkowski background

- Worldsheets theory is defined by fixing a b.g. Riem. metric $\eta_{\mu \nu}$. Poincaré covariant quantization $\Rightarrow$ 3 massless spin 2 graviton $h_{\mu \nu}$

- Changing to spacetime viewpoint $\Rightarrow$ existence of spacetime gauge symmetry, although it is not a symmetry at the worldsheets theory.
  - Related to infinitesimal diffeomorphism via identification $g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu}$.
  - String on-shell amplitudes $=$ General relativity amplitudes

Difficulties:

- Indirect. No efficient method to deal with diffeo. within the worldsheets theory.
- Another b.g. needs another covariance and quantization.
Our strategy

- Using background independent (classical) language.
- Background-dependence enters minimal way.
- Making an implicit close connection among quantization, covariance and spacetime symmetry explicit.

⇒ Hopf algebra.
Motivation 2: Twisted Poincaré symmetry in NC field theories

The use of Hopf algebras is motivated by

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**NC field theory**

= field theory on a noncommutative space.
Moyal-Weyl product $\phi *_M \phi$ or $[x^\mu, x^\nu] = i\theta^{\mu\nu}$

- **An issue**: explicit breaking of Poincaré symmetry group
- **Breakthrough**: be a kind of symmetries as a Hopf algebra
  = Twisted Poincaré symmetry
  [Chaichian, Kulish, Nishijima, Tureanu]
- **Generalization** to a twisted diffeomorphism
  [Aschieri, Blohmann, Dimitrijevic, Meyer, Schupp, Wess]
The point

a Drinfeld twist governs both

\[ \text{twisted product} = \text{noncommutativity}, \]
\[ \text{twisted Hopf algebra} = \text{twisted Poincaré symmetry}. \]

String theory realization?

NC field theory appears as an effective theory on D-branes in a $B$-field b.g. [Seiberg,Witten]

$\Rightarrow$ Is the twisted Poincaré symmetry derived from string theory?

We will show that

- a similar twisted Hopf algebra structure in string theory even $B = 0$.
- If $B \neq 0$, further twist is needed.
  $\Rightarrow$ twisted Poincaré symmetry on D-branes is derived.
Hopf and Module algebras
A Hopf algebra $H$ over a field $k$ is a $k$-vector space $H$ equipped with following linear maps

- **algebra** $\mu : H \otimes H \rightarrow H$ multiplication, $\iota : k \rightarrow H$ unit,
- **coalgebra** $\Delta : H \rightarrow H \otimes H$ coproduct, $\epsilon : H \rightarrow k$ counit,
- $S : H \rightarrow H$ antipode,

with compatibility conditions (where $\mu(h \otimes g) = hg$ and $\iota(k) = k1_H$.)

\[
(fg)h = f(gh) \quad \text{associativity}, \quad h1_H = h = 1_Hh,
\]
\[
(\Delta \otimes \text{id}) \circ \Delta(h) = (\text{id} \otimes \Delta) \circ \Delta(h) \quad \text{coassociativity},
\]
\[
(\epsilon \otimes \text{id}) \circ \Delta(h) = h = (\text{id} \otimes \epsilon) \circ \Delta(h),
\]
\[
\mu \circ (S \otimes \text{id}) \circ \Delta(h) = \epsilon(h) = \mu \circ (\text{id} \otimes S) \circ \Delta(h),
\]
\[
\Delta(gh) = \Delta(g)\Delta(h), \quad \epsilon(gh) = \epsilon(g)\epsilon(h), \quad S(gh) = S(h)S(g).
\]
Examples

Hopf algebras appear in Lie groups/Lie algebras.

**Function on a Lie group** \( k(G) \)

As manifold, \( k(G) \) is a \( k \)-algebra. Group structure of \( G \) gives

\[
\Delta(f)(g_1, g_2) = f(g_1 g_2), \quad \text{product in } G
\]

\[
\epsilon(f) = f(e), \quad \text{unit in } G
\]

\[
S(f)(g) = f(g^{-1}), \quad \text{inverse in } G
\]
Universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$

A tensor alg. $k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$ generated by elements of $\mathfrak{g}$

$$U(\mathfrak{g}) \ni u = u_0 + u_i g^i + u_{ij} g^i \cdot g^j + \cdots ,$$

modulo the Lie algebra relation $g_1 \cdot g_2 - g_2 \cdot g_1 = [g_1, g_2]$.

For $g \in \mathfrak{g}$, the Hopf algebra structure is defined by

$$\Delta(g) = g \otimes 1 + 1 \otimes g, \quad \epsilon(g) = 0, \quad S(g) = -g,$$

and extended for arbitrary elements $u \in U(\mathfrak{g})$. 
Hopf algebras $H = “symmetry”$

$H$-module $A$ = representation space.

**$H$-module $A$ of a Hopf algebra $H$**

is a $k$-vector space $A$ with a map ($H$-action) $\alpha : H \otimes A \rightarrow A$ denoted $\alpha(h \otimes a) = h \triangleright a$, s.t. $(gh) \triangleright a = g \triangleright (h \triangleright a)$. $A$ is called a $H$-module algebra if $A$ is in addition a unital algebra with a multiplication $m : A \otimes A \rightarrow A$ s.t.

$$h \triangleright m(a \otimes b) = m \circ \Delta(h) \triangleright (a \otimes b).$$

Note: This Hopf/module algebra structure is merely a axiomatic reformulation of standard representation theory of a Lie algebra.

If we start from a representation, we recover the axiom automatically.
Diffeomorphism $H = U(\Gamma(TM))$: [Aschieri et al.]

Let $M$: a manifold, $A = C^\infty(M)$: algebra of functions, $\Gamma(TM)$: vector fields generating diffeo. with the action

$$(\xi \triangleright f)(x) = \xi^\mu(x) \partial_\mu f(x).$$

Two actions $\eta \triangleright (\xi \triangleright f) \equiv (\eta \cdot \xi) \triangleright f$ defines a product in $H$:

$$\eta \cdot \xi = \eta^\nu(x) \partial_\nu \xi^\mu(x) \partial_\mu + \eta^\nu(x) \xi^\mu(x) \partial_\nu \partial_\mu$$

$\Gamma(TM)$ is a Lie algebra with

$$[\eta, \xi] = \eta \cdot \xi - \xi \cdot \eta = (\eta^\nu \partial_\nu \xi^\mu - \xi^\nu \partial_\nu \eta^\mu)(x) \partial_\mu \in \mathfrak{X}$$

Actions on the product defines a coproduct $\Delta(\xi)$:

$$\xi \triangleright (fg) = (\xi \triangleright f)g + f(\xi \triangleright g) \text{ Leibniz rule}$$

$$\Rightarrow \Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi \text{ primitive}$$
Method to obtain a new Hopf algebra (quantum group) from known one.

**Twist element $\mathcal{F}$**

is an invertible element $\mathcal{F} \in H \otimes H$ s.t.

\[
(\mathcal{F} \otimes 1)(\Delta \otimes \text{id})\mathcal{F} = (1 \otimes \mathcal{F})(\text{id} \otimes \Delta)\mathcal{F} \quad \text{cyclocycle condition}
\]

\[
\mu(\text{id} \otimes \epsilon)\mathcal{F} = 1 = \mu(\epsilon \otimes \text{id})\mathcal{F} \quad \text{counital condition}
\]

**Twisted Hopf algebra $H_\mathcal{F}$**

is defined from $H$ by replacing $\Delta$ and $S$ with

\[
\Delta_\mathcal{F}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \quad \text{twisted coproduct}
\]

\[
S_\mathcal{F}(h) = US(h)U^{-1}, \quad \text{where} \ U := \mu \circ (\text{id} \otimes S)(\mathcal{F})
\]

Note: algebra structure (Lie bracket for $U(\mathfrak{g})$) is kept invariant.
$H_{\mathcal{F}}$-module algebra $A_{\mathcal{F}}$

- Same as $A$ as a vector space (same repr.).
- The product should be twisted to $m_{\mathcal{F}}$ (also denoted as $*_{\mathcal{F}}$)

\[ a *_{\mathcal{F}} b = m_{\mathcal{F}}(a \otimes b) := m \circ \mathcal{F}^{-1} \triangleright (a \otimes b). \]

so that $m_{\mathcal{F}}$ is compatible with $\Delta_{\mathcal{F}}$ as

\[ h \triangleright m_{\mathcal{F}}(a \otimes b) = m \circ \Delta(h)\mathcal{F}^{-1} \triangleright (a \otimes b) \]
\[ = m_{\mathcal{F}} \circ \Delta_{\mathcal{F}}(h) \triangleright (a \otimes b). \]
Example: Moyal product and twisted diffeomorphism

Moyal twist $\mathcal{F}_M = e^{-\frac{i}{2} \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu} \in U(\Gamma(TM)) \otimes U(\Gamma(TM))$.

- A module algebra $\mathcal{A} = C^\infty(M)$ is twisted $\mathcal{A}_{\mathcal{F}_M}$ with Moyal product $f \ast_{\mathcal{F}_M} g$. [Oeckl][Watts]

- twisted diffeomorphism $U_{\mathcal{F}_M}(\Gamma(TM))$ [Aschieri et al.]

with a coproduct

$$\Delta(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i}{2} \right)^n \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n}$$

$$\times [\partial_{\mu_1}, \cdots [\partial_{\mu_n}, \xi] \cdots ] \otimes \partial_{\nu_1} \cdots \partial_{\nu_n}$$

$$+ \partial_{\mu_1} \cdots \partial_{\mu_n} \otimes [\partial_{\nu_1}, \cdots [\partial_{\nu_n}, \xi] \cdots ]$$
String Theory: Preparation
∽ Formulas from Polchinski’s Textbook
**String Theory**

Dynamical variables $X^\mu(z)$ ($\mu = 0, \cdots, d - 1$) are functions defining a map $X : \Sigma \to \mathbb{R}^d$ from the worldsheet (2 dim) to the target space ($d$ dim).

**Classical action**

$\sigma$-model of the bosonic string in flat $d$-dimensional Minkowski space as the target. In conformal gauge, it is a 2-dim. scalar field theory:

$$S_0[X] = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu,$$

$\Sigma$: worldsheet (any Riemann surface with boundaries), $X^\mu(z) = X^\mu(z, \bar{z})$: worldsheet field (not holomorphic field), $\eta^{\mu\nu}$: Minkowski metric in the target space $\mathbb{R}^d$. 
Correlation functions

defined by the path integral vacuum expectation value (VEV):

$$\langle V_1(z_1) \cdots V_n(z_n) \rangle_0 = \frac{\int \mathcal{D}X V_1(z_1) \cdots V_n(z_n) e^{-S_0[X]}}{\int \mathcal{D}X e^{-S_0[X]}}$$

Vertex operators

defined by a product of “operators” $X^\mu(z, \bar{z})$ and their derivatives of the form $F[X(z)] = \partial_\alpha X^\mu \cdots \partial_\beta X^\nu(z) e^{ikX(z)}$.

But divergences at the coincidence point are removed by the (conformal) normal ordering

$$V =: F[X] := e^{-\frac{1}{2} \int d^2 z \int d^2 w G_{0}^{\mu\nu}(z,w) \frac{\delta}{\delta X^\mu(z)} \frac{\delta}{\delta X^\nu(w)} F[X]}$$

$G_{0}^{\mu\nu}(z,w)$: propagator (Green’s function on $\Sigma$).

(e.g.) For $\Sigma = \mathbb{C}$, $G_{0}^{\mu\nu} = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z - w|^2$. 
## Operator product

The (time ordered) product of two vertex operators are written as

\[
: F[X(z)] : : G[X(w)] :
\]

\[
= e^{\int d^2 z' \int d^2 w' \, G_{0}^{\mu \nu}(z', w')} \frac{\delta F}{\delta X^\mu(z')} \frac{\delta G}{\delta X^\nu(w')} \, F[X(z)] G[X(w)] :
\]

Using the Taylor expansion around \( w \) it reduces to usual OPE.

## Wick’s Theorem

Because of \( \langle : F[X] : \rangle_0 = F[X] |_{X=0} \), the VEV of the product is written as

\[
\langle : F[X] :: G[X] : \rangle_0
\]

\[
= e^{\int d^2 z \int d^2 w \, G_{0}^{\mu \nu}(z, w)} \frac{\delta F}{\delta X^\mu(z)} \frac{\delta G}{\delta X^\nu(w)} \, F[X] G[X] : \bigg|_{X=0}
\]
Symmetry

If both the action $S_0[X]$ and the measure $dX$ are invariant under a variation $X^\mu(z) \rightarrow X^\mu(z) + \delta X^\mu(z)$, there is a Ward identity

$$0 = \sum_{i=1}^{n} \langle V_1(z_1) \cdots \delta V_i(z_i) \cdots V_n(z_n) \rangle_0 \quad \text{← sum!}$$

Here $\delta V$ is given by either $\delta V = [Q, V]$ w/ symmetry charge or equivalently

$$\delta V(z) = - : \int d^2 w \, \delta X^\mu(w) \frac{\delta F[X]}{\delta X^\mu(w)} :$$

(e.g.) Minkowski case, the Poincaré trf. is a symmetry:

$$P^\mu = -i \int d^2 z \, \eta^{\mu\lambda} \frac{\delta}{\delta X^\lambda(z)}, \quad L^{\mu\nu} = -i \int d^2 z X^{[\mu}(z) \eta^{\nu]\lambda} \frac{\delta}{\delta X^\lambda(z)}$$

We would like to reformulate these formulas in terms of a Hopf/module algebra.
Classical Hopf/Module Algebras of Functionals

We define for a string worldsheet

\[ \mathcal{H}: \text{Hopf algebra of classical diffeomorphism + worldsheet variations} \]
\[ \sim \text{functional derivatives} \]
\[ \mathcal{A}: \text{module algebra of classical functionals} \]

Note: they depend on a target space \( \mathbb{R}^d \)
but are background-independent. (Riem. metric, \( B \) field, \( \cdots \)).
Consider a space of maps \( \{ X : \Sigma \to \mathbb{R}^d \} \).

Let \( \mathcal{A} \): space of functionals \( X \mapsto I[X] \in \mathbb{C} \) of the form

\[
I[X] = \int d^2 z \, \rho(z) F[X(z)] .
\]

\( F[X(z)] = (X^* F)(z) \): pull-back of components for a covariant tensor field \( \Gamma(\bigotimes^n T^* \mathbb{R}^d) \) in the target space, and \( \rho(z) \): a weight function (distribution).

In particular, a **local functional at** \( z_i \) is given by

\[
F[X(z_i)] = \int d^2 z \, \delta^{(2)}(z - z_i) F[X(z)] .
\]

They become an integrated/local vertex operators after quantization.
Remarks:

- For a fixed $X$, $I[X]$ is regarded as a map $I : \Gamma(\otimes^n T^* \mathbb{R}^d) \to \mathcal{A}$. (e.g.) $1$-form $\omega = \omega_\mu(x) dx^\mu \in \Gamma(T^* M)$

$$I_\omega[X] = \int d^2 z \rho^a(z) \partial_\alpha X^\mu(z) \omega_\mu[X(z)].$$

- $\mathcal{A}$ is an commutative algebra with the multiplication

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \quad IJ[X] = I[X]J[X].$$

It corresponds to products in target space, only for local vertex operators at the same insertion point:

$$\omega_1[X(z)]\omega_2[X(z)] = (\omega_1 \otimes \omega_2)[X(z)].$$
Let $\mathcal{X}$ be the space of all functional vector fields of the form

$$\xi = \int d^2 w \xi^\mu(w) \frac{\delta}{\delta X^\mu(w)}.$$ 

$\xi^\mu(w)$: weight functions (distributions) including the following two classes.

i) $\xi^\mu(w)$: function of $w$ but independent of $X(w)$.

$\Leftrightarrow$ change of the embedding $X^\mu(z) \rightarrow X^\mu(z) + \xi^\mu(z)$,

- Used to derive the equation of motion.
- Abelian Lie subalgebra $\mathfrak{c}$ with $[\xi, \eta] = 0$.

ii) $\xi^\mu(w) = \xi^\mu[X(w)]$ : pullback of a target space function.

$\Leftrightarrow$ diffeomorphism $X^\mu(z) \rightarrow X^\mu(z) + \xi^\mu[X(z)]$.

- $\Gamma(T_{\mathbb{R}^d})$-action (Lie derivative) is compatible with $\mathcal{X}$-action as

$$I_{\mathcal{L}_\xi \omega}[X] = (\xi \triangleright I_\omega)[X].$$

- Lie algebra str.: $[\xi, \eta] = \int d^2 w \left( \xi^\mu \frac{\delta \eta^\nu}{\delta X^\mu} - \eta^\mu \frac{\delta \xi^\nu}{\delta X^\mu} \right)(w) \frac{\delta}{\delta X^\nu(w)}.$
\( \mathcal{H} = U(\mathfrak{X}) \): classical Hopf algebra

As usual, a Lie algebra \( \mathfrak{X} \) is extended to its universal enveloping algebra \( U(\mathfrak{X}) \).

Remarks:

- Hopf subalgebras
  - Abelian subalgebra \( U(\mathfrak{C}) \) (class i)).
  - subalgebra \( U(\mathfrak{P}) \) of Poincaré-Lie algebra \( \mathfrak{P} \) generated by

\[
P^\mu = -i \int d^2z \eta^{\mu\lambda} \frac{\delta}{\delta X^\lambda(z)},
\]
\[
L^{\mu\nu} = -i \int d^2z X^{[\mu(z)} \eta^{\nu]\lambda} \frac{\delta}{\delta X^\lambda(z)},
\]

- The algebra \( \mathcal{A} \) of functionals is now considered as a \( \mathcal{H} \)-module algebra.
Quantization as Drinfeld Twist

We propose that

\[
\text{Quantization} = \text{Drinfeld twist of Hopf/module algebras}
\]

We first give our definition of the quantization and then see the equivalence to ordinary path integral quantization.
(Abelian) twist element

We assume \( \mathcal{F} \in U(\mathcal{C}) \otimes U(\mathcal{C}) \) of the form

\[
\mathcal{F} = \exp \left\{ - \int d^2z \int d^2w \ G^{\mu\nu}(z, w) \left( \frac{\delta}{\delta X^\mu(z)} \otimes \frac{\delta}{\delta X^\nu(w)} \right) \right\},
\]

specified by a propagator \( G^{\mu\nu}(z, w) \) on the worldsheet.

It satisfies all the conditions (invertible, unital, 2-cocycle).

Twisting

Given \( \mathcal{F} \), the twisted Hopf/module algebras are defined as usual.
In particular, the product in $\mathcal{A}_\mathcal{F}$ and the $\mathcal{H}_\mathcal{F}$-action on it are

$$ F *_{\mathcal{F}} G = m_\mathcal{F}(F \otimes G) = m \circ \mathcal{F}^{-1} \triangleright (F \otimes G), $$

$$ h \triangleright (F *_{\mathcal{F}} G) = m \circ \Delta(h) \mathcal{F}^{-1} \triangleright (F \otimes G) $$

$$ = m_\mathcal{F} \Delta_\mathcal{F}(h) \triangleright (F \otimes G). $$

**VEV for $\mathcal{A}_\mathcal{F}$**

is defined by the map $\tau : \mathcal{A}_\mathcal{F} \rightarrow \mathbb{C}$ with the evaluation at $X = 0$:

$$ \tau (I[X]) := I[X]|_{X=0}. $$

- For $I = F *_{\mathcal{F}} G$, the correlation function is defined by

  $$ \tau(F[X(z)] *_{\mathcal{F}} G[X(w)]) = \tau \circ m \circ \mathcal{F}^{-1} \triangleright (F \otimes G). $$

- The action of $h \in \mathcal{H}_\mathcal{F}$ inside the VEV is

  $$ \tau (h \triangleright I[X]). $$
Introduction
Hopf
String
Classical Hopf
Twist Quantization
Twisted Sym
B-field
Conclusion

Equivalence to Path Integral

Coboundary relation

Our twist element can be written as

\[ F = (\mathcal{N}^{-1} \otimes \mathcal{N}^{-1}) \Delta(\mathcal{N}), \]

where \( \mathcal{N} \in U(\mathcal{C}) \) (normal ordering) is defined by

\[ \mathcal{N} = \exp \left\{ -\frac{1}{2} \int d^2 z \int d^2 w \ G^{\mu \nu}(z, w) \frac{\delta}{\delta X^\mu(z)} \frac{\delta}{\delta X^\nu(w)} \right\}. \]

This shows that the twist element \( F \) is cohomologically trivial.

\[ \Rightarrow \text{there are isomorphisms to untwisted } \mathcal{H} (\mathcal{A}), \text{ but we denote } \hat{\mathcal{H}} (\hat{\mathcal{A}}), \]

the normal ordered Hopf (module) algebra, to distinguish from them.

\[ \mathcal{H} \xrightarrow{\text{twist}} \mathcal{H}_F \xrightarrow{\sim} \hat{\mathcal{H}} \quad h \mapsto \mathcal{N} h \mathcal{N}^{-1} \equiv \tilde{h} \]

\[ \nabla \rightarrow \nabla \rightarrow \nabla \]

\[ \mathcal{A} \xrightarrow{\text{twist}} \mathcal{A}_F \xrightarrow{\sim} \hat{\mathcal{A}} \quad F \mapsto \mathcal{N} \triangleright F \equiv :F: \]
Normal ordered module algebra $\hat{\mathcal{A}} = \text{vertex operator algebra}$

- Formally $\mathcal{A} = \hat{\mathcal{A}}$, but different on divergences, because its elements are in normal ordered form.

$$\mathcal{N} \triangleright I[X] \equiv :I[X]: \quad \text{vertex operators}$$

- It has the same multiplication as the classical functional $\mathcal{A}$:
  $$\mathcal{N} \triangleright m \circ \mathcal{F}^{-1} \triangleright (F \otimes G) = m \circ (\mathcal{N} \otimes \mathcal{N}) \triangleright (F \otimes G),$$
  An equivalent but more familiar expression
  $$:(F *_{\mathcal{F}} G): =:F::G: \quad \text{OPE}$$

$\text{VEV of } \hat{\mathcal{A}} = \text{VEV } \langle \cdots \rangle \text{ in path integral}$

defined by $\tau \circ \mathcal{N}^{-1} : \hat{\mathcal{A}} \rightarrow \mathbb{C}$ to give the same VEV as $\mathcal{A}_\mathcal{F}$.

For instance, the correlation of two local functionals is

$$\tau \circ \mathcal{N}^{-1} \triangleright (:F[X]::G[X]:) = \tau(F *_{\mathcal{F}} G) \quad \text{Wick’s theorem}$$

Formulas in Polchiski recovered. The equivalence is shown.
Remarks:

\[ \mathcal{F} \leftrightarrow G_{\mu\nu}(z, w) \leftrightarrow S[X] \]
A propagator \( G_{\mu\nu}(z, w) \) corresponds to a quadratic action
\[ S[X] = \frac{1}{2} \int d^2 z X_{\mu} D_{\mu\nu} X_{\nu}, \]
where \( D_{\mu\rho} G^{\rho\nu}(z, w) = \delta_{\mu}^{\nu} \delta^{(2)}(z - w). \)
\[ \Rightarrow \text{the equivalence holds for an arbitrary free theory.} \]

\[ \mathcal{F} \text{ (Propagator) is background dependent.} \]
(e.g.) \( G_{0}^{\mu\nu} = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z - w|^2 \)
A choice of \( \mathcal{F} \leftrightarrow \text{a quantization w.r.t. that background.} \)

\[ \hat{A} \text{ corresponds directly with path integral (operator formulation),} \]
but highly background dependent, since both elements \( :F: \in \hat{A} \) and
the VEV \( \tau \circ \mathcal{N}^{-1} \) contain \( \mathcal{N} \), which depends on the background.
On the other hand, \( \mathcal{A}_F \text{ is minimally background dependent}. \)
\[ \Rightarrow \text{better when considering various background.} \]
Twisted Hopf Algebraic Symmetry

We now focus on the quantized symmetry $\mathcal{H}_F$. 
Unification by the Twisted Hopf algebra

In the following, we choose the twist element to $\mathcal{F}_0$ for the Minkowski background $G_0^{\mu \nu}(z, w) \propto \eta^{\mu \nu}$. Note that the quantization needs $U(\mathcal{C})$. However, the twist $\mathcal{F}_0$ affects the whole Hopf algebra $\mathcal{H} = U(\mathcal{X})$.

Classification

The twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_0}$ consists of (unifies)

\[
\begin{cases}
  U(\mathcal{C}) & : \text{abelian (twist inv.)} \quad \rightarrow \text{used for quantization} \\
  U(\mathcal{P}) & : \text{Poincaré (twist inv.)} \quad \rightarrow \text{unbroken sym. (covariance)} \\
  \text{others} & : \text{twisted} \quad \rightarrow (\text{quantum}) \text{ diffeo.}
\end{cases}
\]
Unbroken Symmetry = Twist invariant Hopf subalgebra

The universal envelope $U(\mathcal{P})$ for the Poincaré-Lie algebra $\mathcal{P}$ is special:
$\Delta_{\mathcal{F}_0}(u) = \Delta(u), \quad S_{\mathcal{F}_0}(u) = S(u)$. 

- In terms of $\hat{H}$, it means $\tilde{u} = \mathcal{N}_0 u \mathcal{N}_0^{-1} = u$. 
  Poincaré transformation is unchanged at the quantum level (covariant quantization).

$$\hat{H} \triangleright \hat{A} \quad \quad \mathcal{H}_{\mathcal{F}_0} \triangleright \mathcal{A}_{\mathcal{F}_0}$$
operator $\xi \triangleright : F : = : (\xi \triangleright F) :$ classical

- Ward identity holds: for $\xi = P_\mu$ or $L_{\mu\nu}$,

$$0 = \sum_{i=1}^{n} \langle V_1(z_1) \cdots (\xi \triangleright V_i(z_i)) \cdots V_n(z_n) \rangle_0 \quad \text{sum (primitive)}$$

We reproduced the conventional description of symmetries.
Twisted diffeomorphisms

Our understanding: Worldsheet description of the target space diffeomorphism.

Not fully understood yet but evidences

- Twist does not change trf. law for a single functional
  \[ I_{\mathcal{L}_\xi} I_\omega = \xi \triangleright I_\omega. \]  
  target \(\Leftrightarrow\) worldsheet

- The twisted action on a \(\ast \mathcal{F}\) product seems to give the correct trf. law in the target. Using \(\Delta_\mathcal{F}(\xi)\) we can show

  \[
  \xi \triangleright (F \ast \mathcal{F} G) = \int d^2 z \xi^\mu [X(z)] \left( \frac{\delta F}{\delta X^\mu(z)} \ast \mathcal{F} G + F \ast \mathcal{F} \frac{\delta G}{\delta X^\mu(z)} \right)
  \]

  i.e., \(\ast \mathcal{F}\) does not act on \(\xi^\mu\)

  better to see this in the effective gravity theory.

- Broken sym. but \(\exists\) Ward-like identities.
Another viewpoint: Let $u = e^\xi$ and define

$$Y^\mu = u \triangleright X^\mu = X^\mu + \xi^\mu + O(\xi^2).$$

Note $(u \triangleright F)[X] = F[Y]$.

The $u$-action is absorbed in $Y^\mu$:

$$u \triangleright (F * \mathcal{F}_0 \ G)[X] = m \circ \mathcal{F}_0^{-1} \Delta \mathcal{F}_0(u) \triangleright (F \otimes G)[X]$$

$$= m \circ \mathcal{F}_1^{-1} (u \otimes u) \triangleright (F \otimes G)[X]$$

$$= F[Y] * \mathcal{F}_1 \ G[Y]$$

$$\mathcal{F}_1 := (u \otimes u) \mathcal{F}_0 \Delta (u^{-1}) = e^{-\int d^2 z d^2 w G_0^{\mu \nu}(z,w) \frac{\delta}{\delta Y^\mu(z)} \otimes \frac{\delta}{\delta Y^\nu(w)}}$$

and

$$\frac{\delta}{\delta Y^\mu} = u \frac{\delta}{\delta X^\mu(z)} u^{-1}.$$  

Equivalent to regard a coordinate transformation

$$F[X] \rightarrow F'[X] = F[Y], \quad \eta^{\mu \nu} \rightarrow \eta^{\alpha \beta} \left( \frac{\partial x'_{\mu}}{\partial x_{\alpha}} \right) \left( \frac{\partial x'_{\nu}}{\partial x_{\beta}} \right)$$

$$\Rightarrow$$ Any b.g. metric diffeomorphic to $\eta_{\mu \nu}$ gives a twist element.
B-field Background

Let

- Target space: $\mathbb{R}^d$
  - with Minkowski metric $\eta_{\mu\nu}$ and a constant 2-form $B_{\mu\nu}$.
- Worldsheet: $\Sigma = \text{upper half plane (UHP)}$. 
String in a B-field background

- A constant $B$-field is turned on through

$$S_1 = S_0 + S_B$$

$$S_B = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z B_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu.$$ 

Neumann b.c. $\rightarrow$ mixed b.c.

- Propagator on the upper half plane (UHP):

$$G_1^{\mu\nu}(z, w) = -\alpha' \left[ \eta^{\mu\nu} \ln |z - w| - \eta^{\mu\nu} \ln |z - \bar{w}| ight.$$  

$$+ G^{\mu\nu} \ln |z - \bar{w}|^2 + \Theta^{\mu\nu} \ln \frac{z - \bar{w}}{\bar{z} - w} \right]$$

$$G^{\mu\nu} = \left( \frac{1}{\eta + B} \eta \frac{1}{\eta - B} \right)^{\mu\nu}, \quad \Theta^{\mu\nu} = \frac{\Theta^{\mu\nu}}{2\pi\alpha'} = - \left( \frac{1}{\eta + B} B \frac{1}{\eta - B} \right)^{\mu\nu}.$$
Twist quantization with B-field

Let $\mathcal{H} = U(\mathcal{X})/\mathcal{A}$ be classical Hopf/Module algebras as before. Only the twist element (normal ordering) is changed to

$$\mathcal{F}_1 = \exp \left\{ - \int d^2 z \int d^2 w \ G_1^{\mu \nu} (z, w) \frac{\delta}{\delta X^\mu(z)} \otimes \frac{\delta}{\delta X^\nu(w)} \right\},$$

$$\mathcal{N}_1 = \exp \left\{ - \frac{1}{2} \int d^2 z d^2 w \ G_1^{\mu \nu} (z, w) \frac{\delta}{\delta X^\mu(z)} \frac{\delta}{\delta X^\nu(w)} \right\}.$$

The whole procedure of the twist is exactly the same as before, to give $\mathcal{H}_{\mathcal{F}_1}$ and $\mathcal{A}_{\mathcal{F}_1}$.

- denote the normal ordering as $\circ \cdots \circ$ and the star product as $F \ast_{\mathcal{F}_1}^G = m_{\mathcal{F}_1}^{-1} \triangleright (F \otimes G)$, then we have

$$\circ F \ast_{\mathcal{F}_1}^G \circ = \circ F \circ G \circ$$

- Poincaré $U(\mathcal{P})$ is no more twist invariant. $\Delta_{\mathcal{F}_1}(u) \neq \Delta(u)$. 

$\Rightarrow$ should be twisted (broken).
Decomposition of the twist

In order to understand this new quantization, we give a method to compare it with conventional quantization with $B = 0$.

- Decompose the propagator as
  
  \[ G_1^{\mu\nu}(z, w) = G_0^{\mu\nu}(z, w) + G_B^{\mu\nu}(z, w), \]
  \[ G_0^{\mu\nu}(z, w) = -\alpha' \eta^{\mu\nu} (\ln |z - w| + \ln |z - \bar{w}|) \]
  \[ G_B^{\mu\nu}(z, w) = -\alpha' \left[ (G - \eta)^{\mu\nu} \ln |z - \bar{w}|^2 + \Theta^{\mu\nu} \ln \frac{z - \bar{w}}{\bar{z} - w} \right]. \]

- The twist element is also divided into $\mathcal{F}_1 = \mathcal{F}_B \mathcal{F}_0$.

- This defines two successive twists of module algebras

\[
\begin{align*}
\mathcal{A} \xrightarrow{\text{twist by } \mathcal{F}_0} & \quad \mathcal{A}_{\mathcal{F}_0} \xrightarrow{\text{twist by } \mathcal{F}_B} (\mathcal{A}_{\mathcal{F}_0})_{\mathcal{F}_B} = \mathcal{A}_{\mathcal{F}_1} \\
\mathcal{A} \xrightarrow{\text{twist by } \mathcal{F}_B} & \quad \hat{\mathcal{A}}_{\mathcal{F}_B}
\end{align*}
\]

2nd twist is also regarded as a twist of normal ordered module algebra. This enables us to compare $B \neq 0$ with $B = 0$. 
An interpretation

1st twist $\mathcal{F}_0$: standard quantization ($B = 0$)
gives standard vertex operators $\hat{A}$.

2nd twist $\mathcal{F}_B$: deformation of OPE

In fact, a correlation function is

$$\langle \circ F \circ G \circ \rangle_1 = \langle :F:*_BF:*G: \rangle_0.$$ 

Note: deformation affects both closed/open strings.

New twisted Poincaré Hopf algebra

$U(\mathcal{P})$ is invariant under 1st twist $\mathcal{F}_0$ but twisted under 2nd twist $\mathcal{F}_B$. The coproduct is modified as

$$\Delta_{\mathcal{F}_B}(L_{\mu\nu}) = \Delta(L_{\mu\nu}) - 2 \int d^2zd^2w \eta[\mu\alpha]G_B^{\alpha\beta}(z,w) \frac{\delta}{\delta X^\beta(z)} \otimes \frac{\delta}{\delta X^\nu(z)}$$

How this is described in the effective theory is an open question.
Another decomposition: If restricting open strings (boundary vertex operators) only, the decomposition into “sym/antisym parts” is also useful.

\[
G_{1}^{\mu\nu}(s, t) = G_{S}^{\mu\nu}(s, t) + G_{A}^{\mu\nu}(s, t),
\]
\[
G_{S}^{\mu\nu}(s, t) = -\alpha' G^{\mu\nu} \ln(s - t)^2, \quad G_{A}^{\mu\nu}(s, t) = \frac{i}{2} \theta^{\mu\nu} \epsilon(s - t).
\]

**Twist by** \(\mathcal{F}_1 = \mathcal{F}_A \mathcal{F}_S\)

1st twist \(\mathcal{F}_S\): quantization w.r.t open string metric \(G_{\mu\nu}\)

2nd twist \(\mathcal{F}_A\): Moyal deformation of OPE

This is equivalent to the argument of [Seiberg-Witten].
Now we have also the same relations for Hopf algebras:

\[ \mathcal{H}_{\mathcal{F}_1} \xrightarrow{\text{field theory}} U_{\mathcal{F}_M}(\mathcal{P}’) \]

\[ \nabla \]

\[ \mathcal{A}_{\mathcal{F}_1} \xrightarrow{\text{field theory}} \mathcal{A}_{\mathcal{F}_M}. \]

where \( \mathcal{P}’ \): Poincaré wrt \( G_{\mu\nu} \) and \( \mathcal{F}_M = e^{i\frac{1}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu} \).

The twisted Poincaré-Lie algebra structure \( U_{\mathcal{F}_M}(\mathcal{P}’) \) [Chaichian et.al.] on the Moyal noncommutative space is derived from worldsheet theory.
Conclusion

- Twist $\mathcal{F}$ unifies quantization, covariance and diffeomorphism.

  \begin{align*}
  \text{Hopf algebraic} & \quad \text{Conventional} \\
  \text{twisted module algebra } \mathcal{A}_\mathcal{F} & \iff \text{path integral quantization} \\
  \text{twist invariant part of } \mathcal{H}_\mathcal{F} & \iff \text{unbroken sym. (covariance)} \\
  \text{twisted part of } \mathcal{H}_\mathcal{F} & \iff \text{quantum diffeo. (bonus!)}
  \end{align*}

- Twist $\mathcal{F}$ is minimally background dependent.
  A suitable to consider various backgrounds in a unified way.
  (e.g.) B-field background $\Rightarrow$ relation to NC field theory.

We are at the starting point to investigate seriously what stringy geometry is.
Discussion

- extending the Hopf-algebraic formalism to include other symmetries.
  - local symmetries (conformal sym.) on the worldsheet

\[ \xi = \int d^2 z \, \epsilon(z) \partial X^\mu (z, \bar{z}) \frac{\delta}{\delta X^\mu (z, \bar{z})} \]

- stringy diffeo. \( \alpha' \) (mode) -dependent
- T-duality

- extension to quantization in other backgrounds.
  (e.g) curved backgrounds (interacting theory)

\[ S[X] = \int d^2 z G_{\mu \nu} (X) \partial X^\mu \bar{\partial} X^\nu \]
Remarks on the noncommutativity

- In fact we should fix the ordering of the insertion points in order to compare with the field theory product.

- String theory twist acting on $V(t) \otimes V(s)$ is related to the Moyal twist as

$$\mathcal{F}^{-1}_A = \mathcal{F}^{-1}_M \left[ \theta(t - s) + \theta(s - t)\mathcal{R}^{-1}_M \right],$$

where $\mathcal{R}_M = \mathcal{F}_{M21}\mathcal{F}^{-1}_M$ is the universal R-matrix (triangular). On the other hand $\mathcal{R}_A = 1 \otimes 1$ because of cocommutativity (trivial).

- To derive the Moyal-twisted Poincaré symmetry, we should fix the ordering, too.
Remarks on Deformation Quantization

Twist quantization is applicable to QM and QFT in any dimension.

- Forgetting about target space ⇒ scalar field theory.
  - related works [Dito][Brouder,Fauser,Frabetti,Oeckl]
- Reducing to 1 dim (QM), it is similar to deformation quantization.
  (e.g.) \((\mathbb{R}^{2n}, \omega^{ij})\), Moyal product given by \(\mathcal{F} = e^{-\frac{i\hbar}{2} \omega^{ij} \partial_i \otimes \partial_j}\).
  similar to the propagator \(G^{\mu\nu}(t, t') \propto \alpha' \eta^{\mu\nu}\).

Differences:

- Our twist is time dependent (Heisenberg picture)
- Symmetric \(G^{\mu\nu}(t, t') = G^{\nu\mu}(t't) \Rightarrow \) twist is trivial.