Challenging Common Assumptions in Convex Reinforcement Learning

Mirco Mutti∗
Politecnico di Milano
Universit`a di Bologna
mirco.mutti@polimi.it

Riccardo De Santi∗
ETH Zurich
rdesanti@ethz.ch

Piersilvio De Bartolomeis
ETH Zurich
pdebartol@ethz.ch

Marcello Restelli
Politecnico di Milano
marcello.restelli@polimi.it

Abstract

The classic Reinforcement Learning (RL) formulation concerns the maximization of a scalar reward function. More recently, convex RL has been introduced to extend the RL formulation to all the objectives that are convex functions of the state distribution induced by a policy. Notably, convex RL covers several relevant applications that do not fall into the scalar formulation, including imitation learning, risk-averse RL, and pure exploration. In classic RL, it is common to optimize an infinite trials objective, which accounts for the state distribution instead of the empirical state visitation frequencies, even though the actual number of trajectories is always finite in practice. This is theoretically sound since the infinite trials and finite trials objectives are equivalent and thus lead to the same optimal policy. In this paper, we show that this hidden assumption does not hold in convex RL. In particular, we prove that erroneously optimizing the infinite trials objective in place of the actual finite trials one, as it is usually done, can lead to a significant approximation error. Since the finite trials setting is the default in both simulated and real-world RL, we believe shedding light on this issue will lead to better approaches and methodologies for convex RL, impacting relevant research areas such as imitation learning, risk-averse RL, and pure exploration among others.

1 Introduction

Standard Reinforcement Learning (RL) [50] is concerned with sequential decision-making problems in which the utility can be expressed through a linear combination of scalar reward terms. The coefficients of this linear combination are given by the state visitation distribution induced by the agent’s policy. Thus, the objective function can be equivalently written as the inner product between the mentioned state distribution and a reward vector. However, not all the relevant objectives can be encoded through this linear representation [2]. Several works have thus extended the standard RL formulation to address non-linear objectives of practical interest. These include imitation learning [30, 42], or the problem of finding a policy that minimizes the distance between the induced state distribution and the state distribution provided by experts’ interactions [1, 29, 32, 33, 22, 17]. risk-averse RL [20], in which the objective is sensitive to the tail behavior of the agent’s policy [52, 43, 51, 16, 15, 8, 61], pure exploration [27], where the goal is to find a policy that maximizes the entropy of the induced state distribution [53, 33, 41, 40, 59, 24, 86, 47, 55, 59, 38], diverse skills

∗Equal contribution

36th Conference on Neural Information Processing Systems (NeurIPS 2022).
However, we note that the usual convex RL formulation makes an implicit infinite trials assumption which is rarely met in practice. Indeed, the objective is written as a function of the state distribution, which is an expectation over the empirical state distributions that are actually obtained by running the policy in a given episode. In practice, we always run our policy for a finite number of episodes (or trials), which in general prevents the empirical state distribution from converging to its expectation. This has never been a problem in standard RL: due to the scalar objective, optimizing the policy over infinite trials or finite trials is equivalent, as it leads to the same optimal policy. Crucially, in this paper, we show that this property does not hold for the convex RL formulation: a policy optimized over infinite trials can be significantly sub-optimal when deployed over finite trials (Figure 1). In light of this observation, we reformulate the convex RL problem from a finite trials perspective, developing insights that can be used to partially rethink the way convex objectives have been previously addressed in RL, with potential ripple effects to research areas of significant interest, such as imitation learning, risk-averse RL, pure exploration, and others.

In this paper, we formalize the notion of a finite trials RL problem, in which the objective is a function of the empirical state distribution induced by the agent’s policy over \( n \) trials rather than its expectation over infinite trials. As an illustrative example, consider a financial application, in which we aim to optimize a trading strategy. In the real world, we can only deploy the strategy over a single trial. Thus, we are only interested in the performance of the strategy in the real-world realization, rather than the performance of the strategy when averaging different realizations. Similar considerations apply to other relevant real-world applications, such as autonomous driving or treatment optimization in a healthcare domain. Following this intuition, we first define the (linear) finite trial RL formulation (Section 3), for which it is trivial to prove the equivalence with standard RL. In Section 4, we provide the finite trial convex RL formulation, for which we prove an upper bound on the approximation error made by optimizing the infinite trials as a proxy of the finite trials objective. In light of this finding, we challenge the hidden assumption that (1) convex RL can be equivalently addressed with an infinite trials formulation, even if the setting is finite-trial. We corroborate this result with an additional numerical analysis showing that the approximation bound is non-vacuous for relevant applications (Section 6). Finally, in Section 5, we include an in-depth analysis of the single trial convex RL, which suggests that other common assumptions in the convex RL literature, i.e., that (2) the problem is always computationally tractable and that (3) stationary policies are in general sufficient, should be reconsidered as well. The proofs of the reported results can be found in the Appendix.

## 2 Preliminaries

In this section, we report the notation and background useful to understand the paper. We denote with \( \{1, \ldots, T\} \) a set of integers, and with a lower case letter \( a \) a scalar or a vector, according to the context. For two vectors \( a, b \in \mathbb{R}^d \), we denote with \( a \cdot b = \sum_{i=1}^d a_i b_i \) the inner product between \( a, b \).

### 2.1 Probabilities and Percentiles

Let \( \mathcal{X} \) denote a measurable space, we will denote with \( \Delta(\mathcal{X}) \) the probability simplex over \( \mathcal{X} \), and with \( p \in \Delta(\mathcal{X}) \) a probability measure over \( \mathcal{X} \). For two probability measures \( p, q \) over \( \mathcal{X} \), we define their \( \ell^p \)-distance as \( \|p - q\|_p := \left( \sum_{x \in \mathcal{X}} |p(x) - q(x)|^p \right)^{1/p} \), and their Kullback-Leibler (KL) divergence
as \( \text{KL}(p||q) := \sum_{x \in X} p(x) \log \left( \frac{p(x)}{q(x)} \right) \). Let \( X \) be a random variable distributed according to \( p \), having a cumulative density function \( F_X(x) = Pr(X \leq x) \). We denote with \( \mathbb{E}[X] \) its expected value, and its \( \alpha \)-percentile is denoted as \( \text{VaR}_\alpha(X) = \inf \{ x \mid F_X(x) \geq \alpha \} = F_X^{-1}(\alpha) \), where \( \alpha \in (0, 1) \) is a confidence level, and \( \text{VaR}_\alpha \) stands for Value at Risk (VaR) at level \( \alpha \). We denote the expected value of \( X \) within its \( \alpha \)-percentile as \( \text{CVaR}_\alpha(X) = \mathbb{E} \left[ X \mid X \leq \text{VaR}_\alpha(X) \right] \), where \( \text{CVaR}_\alpha \) stands for Conditional Value at Risk (CVaR) at level \( \alpha \).

### 2.2 Markov Decision Processes

A tabular Markov Decision Process (MDP) \(^{44}\) is defined as \( \mathcal{M} := (S, A, P, T, \mu, r) \), where \( S \) is a state space of size \( |S| \), \( A \) is an action space of size \( |A| \), \( P \) is a Markovian transition model \( P : S \times A \rightarrow \Delta(S) \), such that \( P(s'|s, a) \) denotes the conditional probability of the next state \( s' \) given the current state \( s \) and action \( a \), \( T \) is the episode horizon, \( \mu \) is an initial state distribution \( \mu : \Delta(S) \), and \( r \) is a scalar reward function \( r : S \rightarrow \mathbb{R} \), such that \( r(s) \) is the reward collected in the state \( s \).

In the typical interaction episode, an agent first observes the initial state \( s_0 \sim \mu \) of the MDP. Then, the agent selects an action \( a_0 \), so that the MDP transitions to the next state \( s_1 \sim P(\cdot|s_0, a_0) \), and the agent collects the reward \( r(s_1) \). Having observed \( s_1 \), the agent then selects an action \( a_1 \) triggering a subsequent transition to \( s_2 \sim P(\cdot|s_1, a_1) \). This process carries on repeatedly until the episode ends.

A policy \( \pi \) defines the behavior of an agent interacting with an MDP, i.e., the strategy for which an action is selected at any step of the episode. It consists of a sequence of decision rules \( (\pi_t)_{t=0}^\infty \) that maps the current trajectory \( h_t = (s_t, a_t)_{t=0}^{t-1} \in \mathcal{H}_t \) with a distribution over actions \( \pi_t : \mathcal{H}_t \rightarrow \Delta(A) \), where \( \mathcal{H}_t \) denotes the set of trajectories of length \( t \). A non-stationary policy is a sequence of decision rules \( \pi_t : S \rightarrow \Delta(A) \). A stationary (Markovian) policy is a time-consistent decision rule \( \pi : S \rightarrow \Delta(A) \), such that \( \pi(a|s) \) denotes the conditional probability of taking action \( a \) in state \( s \).

A trajectory \( h \), obtained from an interaction episode, induces an empirical distribution \( d \) over the states of the MDP \( \mathcal{M} \), such that \( d(s) = \frac{1}{|h|} \sum_{s_t \in h} 1 \left( s_t = s \right) \). We denote with \( p^\pi \) the probability of drawing \( d \) by following the policy \( \pi \). For \( n \in \mathbb{N} \), we denote with \( d_n \) the empirical distribution \( d_n(s) = \frac{1}{n} \sum_{t=1}^n 1 \left( s_t = s \right) \), and with \( p^\pi_n \) the probability of drawing \( d_n \) by following the policy \( \pi \) for \( n \) episodes. Finally, we can call the expectation \( d^\pi = \mathbb{E}_{d \sim p^\pi} [d] \) the state distribution induced by \( \pi \).

### 3 Reinforcement Learning in Finite Trials

In the standard RL formulation \(^{50}\), an agent aims to learn an optimal policy by interacting with an unknown MDP \( \mathcal{M} \). An optimal policy is a decision strategy that maximizes the expected sum of rewards collected during an episode. Especially, we can represent the value of a policy \( \pi \) through the value function \( V^\pi(s) := \mathbb{E}_\pi \left[ \sum_{t=0}^T r(s_t) \mid s_0 = s \right] \). The value function allows us to write the RL objective as \( \max_{\pi \in \Pi} \mathbb{E}_{s_0 \sim \mu} [V^\pi(s_1)] \), where \( \Pi \) is the set of all the stationary policies. Equivalently, we can rewrite the RL objective into its dual formulation \(^{44}\), i.e.,

\[
\max_{\pi \in \Pi} \left( r \cdot d^\pi \right) =: J_\infty(\pi) \tag{1}
\]

where we denote with \( r \in \mathbb{R}^S \) a reward vector, and with \( d^\pi \) the state distribution induced by \( \pi \). We call the problem \(^1\) the infinite trials RL formulation. Indeed, the objective \( J_\infty(\pi) \) considers the sum of the rewards collected during an episode, i.e., \( r \cdot d^\pi \), that we can achieve on the average of an infinite number of episodes drawn with \( \pi \). This is due to the state distribution \( d^\pi \) being an expectation of empirical distributions \( d^\pi = \mathbb{E}_{d \sim p^\pi} [d] \). However, in practice, we can never draw infinitely many episodes following a policy \( \pi \). Instead, we draw a small batch of episodes \( d_n \sim p^\pi_n \). Thus, we can instead conceive a finite trials RL formulation that is closer to what is optimized in practice.

---

\(^1\) We will call a sequence of states and actions a trajectory or a history indifferently.
Table 1: Various convex RL objectives and applications. The last column states the equivalence between infinite trials and finite trials settings, as derived in Proposition[7](Appendix).

| OBJECTIVE $\mathcal{F}$ | APPLICATION | INFINITE $\equiv$ FINITE |
|--------------------------|-------------|--------------------------|
| $r \cdot d$              | RL          | ✓                        |
| $||d - d_E||_p^p$         | RL          | ✓                        |
| $\text{KL}(d||d_E)$      | IMITATION LEARNING | ×                        |
| $-d \cdot \text{log} (d)$| PURE EXPLORATION     | ×                        |
| $\text{CVaR}_\alpha[r \cdot d]$ | RISK-averse RL | ×                        |
| $r \cdot d - \max[r \cdot d]$ | LINEARLY CONSTRANGED RL | ✓                        |
| $-\mathbb{E}_z \text{KL}(d_z||d_k)$ | DIVERSE SKILL DISCOVERY | ×                        |

Finite Trials RL

$$\max_{\pi \in \Pi} \left( \mathbb{E}_{d_n \sim p^n_\pi} [r \cdot d_n] \right) =: J_n(\pi)$$

One could then wonder whether optimizing the finite trials objective $J_n$ leads to results that differ from the infinite trials one $J_\infty$. To this point, it is straightforward to see that the two objective functions are actually equivalent

$$J_n(\pi) = \mathbb{E}_{d_n \sim p^n_\pi} [r \cdot d_n] = r \cdot \mathbb{E}_{d_n \sim p^n_\pi} [d_n] = r \cdot d^n = J_\infty(\pi),$$

since $r$ is a constant vector and the expectation is a linear operator. It follows that the infinite trials and the finite trials RL formulations admit the same optimal policies. Hence, one can enjoy both the computational tractability of the infinite trials formulation and, at the same time, optimize the objective function that is employed in practice. In the next section, we will show that a similar result does not hold true for the convex RL formulation.

4 Convex Reinforcement Learning in Finite Trials

Even though the RL formulation covers a wide range of sequential decision-making problems, several relevant applications cannot be expressed by means of the inner product between a linear reward vector $r$ and a state distribution $d^n$. These include imitation learning, pure exploration, constrained problems, and risk-sensitive objectives, among others. Recently, a convex RL formulation has been proposed to unify these applications in a unique general framework, which is

Convex RL

$$\max_{\pi \in \Pi} \left( \mathcal{F}(d^n) \right) =: \zeta_\infty(\pi)$$

where $\mathcal{F} : \Delta(S) \rightarrow \mathbb{R}$ is a function of the state distribution $d^n$. In Table 1 we recap some of the most relevant problems that fall under the convex RL formulation, along with their specific $\mathcal{F}$ function. As it happens for linear RL, in any practical simulated or real-world scenario, we can only draw finite number of episodes with a policy $\pi$. From these episodes, we obtain an empirical distribution $d_n \sim p^n_\pi$ rather than the actual state distribution $d^n$, where $n$ is the number of episodes. This can cause a mismatch from the objective that is typically considered in convex RL [60, 57], and what can be optimized in practice. To overcome this mismatch, we define a finite trials formulation of the convex RL objective, as we did in the previous section for the linear RL formulation.

Finite Trials Convex RL

$$\max_{\pi \in \Pi} \left( \mathbb{E}_{d_n \sim p^n_\pi} [\mathcal{F}(d_n)] \right) =: \zeta_n(\pi)$$

3In this context, we use the term convex to distinguish it from the standard linear RL objective. However, in the following we will consider functions $\mathcal{F}$ that are either convex, concave or even non-convex. In general, problem (2) takes the form of a max problem for concave $\mathcal{F}$, or a min problem for convex $\mathcal{F}$. 

4
Comparing objectives \(3\) and \(4\), one can notice that both of them include an expectation over the episodes, being \(d^\pi = \mathbb{E}_{d_n \sim p_n}[d]\). Especially, we can write
\[
ζ_\infty(π) = \mathcal{F}(d^\pi) = \mathcal{F}(\mathbb{E}_{d_n \sim p_n}[d_n]) \leq \mathbb{E}_{d_n \sim p_n}[\mathcal{F}(d_n)] = ζ_n(π)
\]
through the Jensen’s inequality. As a consequence, optimizing the infinite trials objective \(ζ_\infty(π)\) does not necessarily lead to an optimal behavior for the finite trials objective \(ζ_n(π)\). From a mathematical perspective, this is due to the fact that the empirical distributions \(d_n\) induced by the policy \(π\) are averaged by the expectation \(d^\pi\) before computing the \(\mathcal{F}\) function into objective (3), thus losing a measure of the performance \(\mathcal{F}\) for each batch of episodes, which we instead keep in the objective (4).

### 4.1 Approximating the Finite Trials Objective with Infinite Trials

Despite the evident mismatch between the finite trials and the infinite trials formulation of the convex RL problem, most existing works consider (3) as the standard objective, even if only a finite number of episodes can be drawn in practice. Thus, it is worth investigating how much we can lose by approximating a finite trials objective with an infinite trials one. First, we report a useful assumption on the structure of the function \(\mathcal{F}\).

**Assumption 4.1 (Lipschitz).** A function \(\mathcal{F} : \mathcal{X} \rightarrow \mathbb{R}\) is Lipschitz-continuous if it holds for some constant \(L\)
\[
|\mathcal{F}(x) - \mathcal{F}(y)| \leq L\|x - y\|_1, \quad \forall (x, y) \in \mathcal{X}^2.
\]

Then, we provide an upper bound on the approximation error in the following theorem.

**Theorem 4.1 (Approximation Error).** Let \(n \in \mathbb{N}\) be a number of trials, let \(δ \in (0, 1]\) be a confidence level, let \(\pi^\dagger \in \arg \max_{\pi \in \Pi} ζ_n(π)\) and \(\pi^* \in \arg \max_{\pi \in \Pi} ζ_\infty(π)\). Then, it holds with probability at least \(1 - δ\)
\[
err := |ζ_n(\pi^\dagger) - ζ_\infty(π^*)| \leq 4LT \sqrt{\frac{2S \log(4T/δ)}{n}}
\]

The previous result establishes an approximation error rate \(err = O(LT \sqrt{S/n})\) that is polynomial in the number of episodes \(n\). Unsurprisingly, the guarantees over the approximation error scales with \(O(1/\sqrt{n})\), as one can expect the empirical distribution \(d_n\) to concentrate around its expected value for large \(n\) [54]. This implies that approximating the finite trials objective \(ζ_n(π)\) with the infinite trials \(ζ_\infty(π)\) can be particularly harmful in those settings in which \(n\) is necessarily small. As an example, consider training a robot through a simulator and deploying the obtained policy in the real world, where the performance measures are often based on a single episode (\(n = 1\)). The performance that we experience from the deployment can be much lower than the expected \(ζ_\infty(π)\), which might result in undesirable or unsafe behaviors. However, Theorem 4.1 only reports an instance-agnostic upper bound, and it does not necessarily imply that there would be a significant approximation error in a specific instance, i.e., a specific pairing of an MDP \(M\) and a function \(\mathcal{F}\). Nevertheless, in this paper we argue that the upper bound of the approximation error is not vacuous in several relevant applications, and we provide an illustrative numerical corroborating of this claim in Section 6.

**Challenged assumption 1.** The convex RL problem can be equivalently addressed with an infinite trials formulation.

Finally, in Figure 2 we report a visual representation\(^4\) of the approximation error defined in Theorem 4.1. Notice that the finite trials objective \(ζ_n\) converges uniformly to the infinite trials objective \(ζ_\infty\) as a trivial consequence of Theorem 4.1. This is particularly interesting as it results in \(π^\dagger\) converging to \(π^*\) in the limit of large \(n\) as shown Figure 2.

### 5 In-Depth Analysis of Single Trial Convex Reinforcement Learning Setting

Having established a significant mismatch between the infinite trials convex RL setting that is usually considered in previous works, i.e., \(ζ_\infty(π)\), and the finite trials formulation that is actually targeted in

---

\(^4\)Note that it is not possible to represent the objective functions in two dimensions in general. Nevertheless, we provide an abstract one-dimensional representation of the policy space to bring the intuition.
Whereas M

We can show that any finite-horizon convex MDP where we denoted with d

Taking inspiration from [57], we can cast the problem (5) defined over an MDP

Notably, the problem of finding an optimal policy for the extended MDP M

Figure 2: The two illustrations report an abstract visualization of ζ

practice, i.e., ζ

Indeed, to avoid the approximation error that can occur by optimizing (4) through the infinite trials formulation (Theorem 4.1), one could instead directly address the optimization of (4). Especially, how does the finite trials convex RL problem compare to its infinite trials formulation and the linear RL problem? What kind of policies do we need to optimize the finite trials objective? Is the underlying learning process statistically harder than infinite trials convex RL? In this section, we investigate the answers to these relevant questions. To this purpose, we will focus on a single trial setting, i.e., ζ

5.1 Extended MDP Formulation of the Single Trial Convex RL Setting

We can show that any finite-horizon convex MDP CM can be actually translated into an equivalent MDP \( M_\ell = (S_\ell, A_\ell, P_\ell, \mu_\ell, r_\ell) \), which we call an extended MDP. The main idea is to temporally-extend CM so that each state contains the information of the full trajectory leading to it, so that the convex objective can be cast into a linear reward. To do this, we define the extended state space \( S_\ell \) to be the set of all the possible histories up to length \( T \), so that \( s_\ell \in S_\ell \) now represents a history. Then, we can keep \( A_\ell, P_\ell, \mu_\ell \) equivalent to \( A, P, \mu \) of the original CM, where for the extended transition model \( P_\ell(s_\ell'|s_\ell, a) \) we solely consider the last state in the history \( s_\ell \) to define the conditional probability to the next history \( s_\ell' \). Finally, we just need to define a scalar reward function \( r_\ell : S_\ell \to \mathbb{R} \) such that \( r_\ell(s_\ell) = \mathcal{F}(d_\ell) \) for all the histories \( s_\ell \) of length \( T \) and \( r_\ell(s_\ell) = 0 \) otherwise, where we denoted with \( d_\ell \) the empirical state distribution induced by \( s_\ell \).

Notably, the problem of finding an optimal policy for the extended MDP \( M_\ell \), i.e., \( \pi^\ast \in \arg\max_{\pi \in \Pi} r_\ell \cdot d^\pi \), is equivalent to solve the problem (5). Indeed, we have

\[
 r_\ell \cdot d^\pi = \sum_{s_\ell \in S_\ell} r_\ell(s_\ell)d^\pi(s_\ell) = \sum_{s_\ell \in S_\ell} \mathcal{F}(d_\ell) \mathbb{1}(|s_\ell| = T)p^\pi(d_\ell) = \mathbb{E}_{d_\ell \sim p^\pi}[\mathcal{F}(d_\ell)].
\]

Whereas \( M_\ell \) can be solved with classical MDP methods [44], the size of the policy \( \pi : S_\ell \to \Delta(A_\ell) \) to be learned does scale with the size of \( M_\ell \), which grows exponentially in the episode horizon as we have \( |S_\ell| > ST \). Thus, the extended MDP formulation and the resulting insight cast some doubts on the notion that the convex RL is not significantly harder than standard RL [60, 57].

Challenged assumption 2. Convex RL is only slightly harder than the standard RL formulation.
5.2 Partially Observable MDP Formulation of the Single Trial Convex RL Setting

Instead of temporally extending the convex MDP $CM$ as in the previous section, which causes the policy space to grow exponentially with the episode horizon $T$, we can alternatively formulate $CM$ as a Partially Observable MDP (POMDP) \[5 \ddagger 31\] $P_M = (S_t, A_t, P_t, \mu_t, r_t, \Omega, O)$, in which $\Omega$ denotes an observation space, and $O : S_t \rightarrow \Delta(\Omega)$ is an observation function. The process to build $P_M$ is rather similar to the one we employed for the extended MDP $M_t$, and the components $S_t, A_t, P_t, \mu_t, r_t$ remain indeed unchanged. However, in a POMDP the agent does not directly access a state $s_t \in S_t$, but just a partial observation $o_t \in \Omega$ that is given by the observation function $O$. Here $O(s_t) = o$ is a deterministic function such that the given observation $o$ is the last state in the history $s_t$. Since the agent only observes $o$, a stationary policy can be defined as a function $\pi : \Omega \rightarrow \Delta(A)$, for which the size depends on the number of states $S$ of the convex MDP $CM$, being $\Omega = S$. However, it is well known \[31\] that history-dependent policies should be considered for the problem of optimizing a POMDP. This is in sharp contrast with the current convex MDP literature, which only considers stationary policies due to the infinite trials formulation \[60\].

Challenged assumption 3. The set of stationary randomized policies is sufficient for convex RL.

5.3 Online Learning in Single Trial Convex RL

Let us assume to have access to a planning oracle that returns an optimal policy $\pi^*$ for a given $CM$, so that we can sidestep the concerns on the computational feasibility reported in previous sections. It is worth investigating the complexity of learning $\pi^*$ from online interactions with an unknown $CM$. A typical measure of this complexity is the online learning regret $R(N)$, which is defined as

$$R(N) := \sum_{t=1}^{N} V^* - V^t,$$

where $N$ is the number of learning episodes, $V^* = V^*_{\pi^*}(s_1)$ is the value of the optimal policy, $V^t = V^*_{\pi_t}$ is the value of the policy $\pi_t$ deployed at the episode $t$. We now aim to assess whether there exists a principled algorithm that can achieve a sub-linear regret $R(N)$ in the worst case. To this purpose, we can cast our learning problem in the Once-Per-Per-EPISODE (OPE) RL formulation \[12\]. In the latter setting, the agent interacts with the MDP for $T$ steps, receiving a 0/1 feedback at the end of the episode, where the feedback is computed according to a logistic model that is function of the history. To translate our objective $\xi_1(\pi) = \mathbb{E}_{d \sim p^n}[F(d)]$ into the OPE framework \[12\], we have to encode $F$ into a linear representation. With the following assumption, we state the existence of such representation.

Assumption 5.1 (Linear Realizability). The function $F$ is linearly-realizable if it holds

$$F(d) = w^\top \phi(d),$$

where $w \in \mathbb{R}^{d_w}$ is a vector of parameters such that $\| w \|_2 \leq B$ for some known $B > 0$, and $\phi(d) = (\phi_j(d))_{j=1}^{d_w}$ is a known vector of basis functions such that $\| \phi(d) \|_2 \leq 1, \forall d \in \Delta(S)$.

With the Assumption 5.1 and other minor changes that are detailed in the Appendix, we can invoke the analysis of OPE-UCBVI in \[12\] to provide an upper bound to the regret $R(N)$ in our setting.

Theorem 5.1 (Regret). For any confidence $\delta \in (0, 1]$ and unknown convex MDP $CM$, the regret of the OPE-UCBVI algorithm is upper bounded as

$$R(N) \leq O \left( \left[ \frac{T^{1/2}B^{3/2}S^{1/2}A^{1/2}}{N} \right] \sqrt{N} \right)$$

with probability $1 - \delta$.

The latter result states that the problem of learning an optimal policy in a unknown convex MDP is at least statistically efficient assuming linear realizability and the access to a planning oracle. Those are fairly strong assumptions, but principled approximate solvers may be designed to overcome the planning oracle assumption, whereas in several convex RL settings the function $F$ is assumed to be known, and thus trivially realizable.
In this section, we evaluate the performance over the finite trials objective (4) achieved by a policy \( \pi^* \in \arg \max_{\pi \in \Pi} \zeta_{H}(\pi) \) maximizing the same finite trials objective (4) against a policy \( \pi^* \in \arg \max_{\pi \in \Pi} \zeta_{H}(\pi) \) maximizing the infinite trials objective (5) instead. The latter infinite trials \( \pi^* \) can be obtained by solving a dual optimization on the convex MDP (see Sec. 6.2 in [41]),

\[
\max_{\omega \in \Delta(S \times A)} F(\omega), \quad \text{subject to } \sum_{a \in A} \omega(s, a) = \sum_{s' \in S, a' \in A} P(s|s', a')\omega(s', a'), \forall s \in S,
\]

To get the finite trials \( \pi^* \), we first recover the extended MDP as explained in Section 5.1, and then we apply standard dynamic programming [7]. In the experiments, we show that optimizing the infinite trials objective can lead to sub-optimal policies across a wide range of applications. In particular, we cover examples from pure exploration, risk-averse RL, and imitation learning. We carefully selected MDPs that are as simple as possible in order to stress the generality of our results. For the sake of clarity, we restrict the discussion to the single trial setting (\( n = 1 \)).

**Pure Exploration** For the pure exploration setting, we consider the state entropy objective [27], i.e., \( F(d) = H(d) = -d \cdot \log d \), and the convex MDP in Figure 3a. In this example, the agent aims to maximize the state entropy over finite trajectories of \( T \) steps. Notice that this happens when a policy induces an empirical state distribution that is close to uniform. In Figure 4a, we compare the average entropy induced by the optimal finite trials policy \( \pi^* \) and the optimal infinite trials policy \( \pi^* \). An agent following the policy \( \pi^* \) always achieves a uniform empirical state distribution leading to the maximum entropy. Moreover, \( \pi^* \) is a non-Markovian deterministic policy. In contrast, the policy \( \pi^* \) is randomized in all the three states. As a result, this policy induces sub-optimal empirical state distributions with strictly positive probability, as shown in Figure 4d.

**Risk-Averse RL** For the risk-averse RL setting, we consider a Conditional Value-at-Risk (CVaR) objective [36] given by \( F(d) = \text{CVaR}_{\alpha}[d \cdot a] \), where \( r \in [0, 1]^S \) is a reward vector, and the convex MDP in Figure 3b, in which the agent aims to maximize the CVaR over a finite-length trajectory of \( T \) steps. First, notice that a financial semantics can be attributed to the given MDP. An agent, starting in state 2, can decide whether to invest in risky assets, e.g., crypto-currencies, or in safe ones, e.g., treasury bills. Because of the stochastic transitions, a policy would need to be reactive to the realizations of the transition model in order to maximize the single trial objective [5]. This kind of behavior is achieved by an optimal finite trials policy \( \pi^* \). Indeed, \( \pi^* \) is a non-Markovian deterministic policy, which can take decisions as a function of history, and thus takes into account the current realization. On the other hand, an optimal infinite trials policy \( \pi^* \) is a Markovian policy, and it cannot take into account the current history. As a result, the policy \( \pi^* \) induces sub-optimal trajectories with strictly positive probability (see Figure 4c). Finally, in Figure 4b, we compare the single trial performance induced by the optimal single trial policy \( \pi^* \) and the optimal infinite trials policy \( \pi^* \). Overall, \( \pi^* \) performs significantly better than \( \pi^* \).

**Imitation Learning** For the imitation learning setting, we consider the distribution matching objective [32], i.e., \( F(d) = \text{KL}(d||d_E) \), and the convex MDP in Figure 3c. The agent aims to learn a policy \( \pi \) inducing an empirical state distribution \( d \) close to the empirical state distribution \( d_E \) demonstrated by an expert. In Figure 4d, we compare single trial performance induced by the optimal single trial policy \( \pi^* \) and the optimal infinite trials policy \( \pi^* \). An agent following \( \pi^* \) induces an empirical state distribution that perfectly matches the expert. In contrast, an agent following \( \pi^* \) induces sub-optimal realizations with strictly positive probability (see Figure 4d).
7 Related Work

To the best of our knowledge, [27] were the first to introduce the convex RL problem, as a generalization of the standard RL formulation to non-linear utilities, especially the entropy of the state distribution. They show that the convex RL objective, while being concave (convex) in the state distribution, can be non-concave (non-convex) in the policy parameters. Anyway, they provide a provably efficient algorithm that overcomes the non-convexity through a Frank-Wolfe approach.

[60] study the convex RL problem under the name of RL with general utilities. Especially, they investigated a hidden convexity of the convex RL objective that allows for statistically efficient policy optimization in the infinite-trials setting. Recently, the infinite trials convex RL formulation has been reinterpreted from game-theoretic perspectives [57, 21]. The former [57] notes that the convex RL problem can be seen as a min-max game between the policy player and a cost player. The latter [21] shows that the convex RL problem is a subclass of mean-field games.

Another relevant branch of literature is the one investigating the expressivity of scalar (Markovian) rewards [2, 49]. Especially, [2] shows that not all the notions of task, such as inducing a set of admissible policies, a (partial) policy ordering, a trajectory ordering, can be naturally encoded with a scalar reward function. Whereas the convex RL formulation extends the expressivity of scalar RL w.r.t. all these three notions of task, it is still not sufficient to cover any instance. Convex RL is powerful in terms of the policy ordering it can induce, but it is inherently limited on the trajectory ordering as it only accounts for the infinite trials state distribution. Instead, the finite trials convex RL setting that we presented in this paper is naturally expressive in terms of trajectory orderings, at the expense of a diminished expressivity on the policy orderings w.r.t. infinite trials convex RL.

Previous works concerning RL in the presence of trajectory feedback are also related to this work. Most of this literature assumes an underlying scalar reward model [e.g., 18] which only delays the feedback at the end of the episode. One notable exception is the once-per-episode formulation in [12], which we have already commented on in Section 5.

Finally, the work in [13, 14] considers infinite-horizon MDPs with vectorial rewards as a mean to encode convex objectives in RL with a multi-objective flavor. They show that stationary policies are in general sub-optimal for the introduced online learning setting, where non-stationarity becomes
essential. In this setting, they provide principled procedures to learn an optimal policy with sub-linear regret. Their work essentially complement our analysis in the infinite-horizon problem formulation, where the difference between finite trials and infinite trials fades away.

8 Conclusion and Future Directions

While in classic RL the optimization of an infinite trials objective leads to the optimal policy for the finite trials counterpart, we have shown that true convex RL does not have this property. First, we have formalized the concept of finite trials convex RL, which captures a problem that until now has been cast into an unfounded optimization problem. Then, we have given an upper bound on the approximation error obtained by erroneously optimizing the infinite trials objective, as it is currently done in practice. Finally, we have presented intuitive, yet general, experimental examples to show that the approximation error can be significant in relevant applications. Since the finite trials setting is the standard in both simulated and real-world RL, we believe that shedding light on the above mentioned performance gap will lead to better approaches for convex RL and related areas. Future work could target approximate solutions to the finite trials objective rather than the infinite trials one, which can cause sub-optimality even when solved exactly. Methods in POMDPs [26, 34] or optimistic planning algorithms [10] could provide useful inspiration.

Acknowledgements

Riccardo De Santi and Piersilvio De Bartolomeis thank professor Niao He for offering graduate students at ETH the opportunity to work in exciting research areas within the “Foundations of Reinforcement Learning” course. Further, we thank Ali Batuhan Yardim for his generous feedback on an early version of this work.

References

[1] Pieter Abbeel and Andrew Y Ng. Apprenticeship learning via inverse reinforcement learning. In International Conference on Machine learning, 2004.

[2] David Abel, Will Dabney, Anna Harutyunyan, Mark K Ho, Michael Littman, Doina Precup, and Satinder Singh. On the expressivity of Markov reward. In Advances in Neural Information Processing Systems, 2021.

[3] Joshua Achiam, David Held, Aviv Tamar, and Pieter Abbeel. Constrained policy optimization. In International Conference on Machine Learning, 2017.

[4] Eitan Altman. Constrained Markov decision processes, volume 7. CRC Press, 1999.

[5] Karl Johan Åström. Optimal control of Markov processes with incomplete state information. Journal of Mathematical Analysis and Applications, 10:174–205, 1965.

[6] Qinbo Bai, Amrit Singh Bedi, Mridul Agarwal, Alec Koppel, and Vaneet Aggarwal. Achieving zero constraint violation for constrained reinforcement learning via primal-dual approach. In AAAI Conference on Artificial Intelligence, 2022.

[7] Richard Bellman. Dynamic programming. Princeton University Press, 1957.

[8] L Bisi, L Sabbioni, E Vittori, M Papini, and M Restelli. Risk-averse trust region optimization for reward-volatility reduction. In International Joint Conference on Artificial Intelligence, 2020.

[9] Kiante Brantley, Miro Dudik, Thodoris Lykouris, Sobhan Miryoosefi, Max Simchowitz, Aleksandrs Slivkins, and Wen Sun. Constrained episodic reinforcement learning in concave-convex and knapsack settings. In Advances in Neural Information Processing Systems, 2020.

[10] Sébastien Bubeck and Rémi Munos. Open loop optimistic planning. In Conference on Learning Theory, 2010.
[11] Víctor Campos, Alexander Trott, Caiming Xiong, Richard Socher, Xavier Giró-i Nieto, and Jordi Torres. Explore, discover and learn: Unsupervised discovery of state-covering skills. In International Conference on Machine Learning, 2020.

[12] Niladri Chatterji, Aldo Pacchiano, Peter Bartlett, and Michael Jordan. On the theory of reinforcement learning with once-per-episode feedback. In Advances in Neural Information Processing Systems, 2021.

[13] Wang Chi Cheung. Exploration-exploitation trade-off in reinforcement learning on online markov decision processes with global concave rewards. arXiv preprint arXiv:1905.06466, 2019.

[14] Wang Chi Cheung. Regret minimization for reinforcement learning with vectorial feedback and complex objectives. In Advances in Neural Information Processing Systems, 2019.

[15] Yinlam Chow, Mohammad Ghavamzadeh, Lucas Janson, and Marco Pavone. Risk-constrained reinforcement learning with percentile risk criteria. Journal of Machine Learning Research, 18(1):6070–6120, 2017.

[16] Yinlam Chow, Aviv Tamar, Shie Mannor, and Marco Pavone. Risk-sensitive and robust decision-making: a cvaR optimization approach. In Advances in Neural Information Processing Systems, 2015.

[17] Robert Dadashi, Leonard Hussenot, Matthieu Geist, and Olivier Pietquin. Primal wasserstein imitation learning. In International Conference on Learning Representations, 2020.

[18] Yonathan Efroni, Nadav Merlis, and Shie Mannor. Reinforcement learning with trajectory feedback. In AAAI Conference on Artificial Intelligence, 2021.

[19] Benjamin Eysenbach, Abhishek Gupta, Julian Ibarz, and Sergey Levine. Diversity is all you need: Learning skills without a reward function. In International Conference on Learning Representations, 2018.

[20] Javier García and Fernando Fernández. A comprehensive survey on safe reinforcement learning. Journal of Machine Learning Research, 16(1):1437–1480, 2015.

[21] Matthieu Geist, Julien Pérolat, Mathieu Laurière, Romuald Elie, Sarah Perrin, Oliver Bachem, Rémi Munos, and Olivier Pietquin. Concave utility reinforcement learning: The mean-field game viewpoint. In International Conference on Autonomous Agents and Multiagent Systems, 2022.

[22] Seyed Kamyar Seyed Ghasemipour, Richard Zemel, and Shixiang Gu. A divergence minimization perspective on imitation learning methods. In Conference on Robot Learning, 2020.

[23] Karol Gregor, Danilo Rezende, and Daan Wierstra. Variational intrinsic control. International Conference on Learning Representations, Workshop Track, 2017.

[24] Zhaohan Daniel Guo, Mohammad Gheshlaghi Azar, Alaa Saade, Shantanu Thakoor, Bilal Piot, Bernardo Avila Pires, Michal Valko, Thomas Mesnard, Tor Lattimore, and Rémi Munos. Geometric entropic exploration. arXiv preprint arXiv:2101.02055, 2021.

[25] Steven Hansen, Will Dabney, Andre Barreto, David Warde-Farley, Tom Van de Wiele, and Volodymyr Mnih. Fast task inference with variational intrinsic successor features. In International Conference on Learning Representations, 2019.

[26] Milos Hauskrecht. Value-function approximations for partially observable markov decision processes. Journal of Artificial Intelligence Research, 13:33–94, 2000.

[27] Elad Hazan, Sham Kakade, Karan Singh, and Abby Van Soest. Provably efficient maximum entropy exploration. In International Conference on Machine Learning, 2019.

[28] Shuncheng He, Yuhang Jiang, Hongchang Zhang, Jianzhun Shao, and Xiangyang Ji. Wasserstein unsupervised reinforcement learning. In AAAI Conference on Artificial Intelligence, 2022.
[29] Jonathan Ho and Stefano Ermon. Generative adversarial imitation learning. In *Advances in Neural Information Processing Systems*, 2016.

[30] Ahmed Hussein, Mohamed Medhat Gaber, Eyad Elyan, and Chrisina Jayne. Imitation learning: A survey of learning methods. *ACM Computing Surveys (CSUR)*, 50(2):1–35, 2017.

[31] Leslie Pack Kaelbling, Michael L Littman, and Anthony R Cassandra. Planning and acting in partially observable stochastic domains. *Artificial Intelligence*, 1998.

[32] Ilya Kostrikov, Ofir Nachum, and Jonathan Tompson. Imitation learning via off-policy distribution matching. In *International Conference on Learning Representations*, 2019.

[33] Lisa Lee, Benjamin Eysenbach, Emilio Parisotto, Eric Xing, Sergey Levine, and Ruslan Salakhutdinov. Efficient exploration via state marginal matching. *arXiv preprint arXiv:1906.05274*, 2019.

[34] Michael L Littman, Anthony R Cassandra, and Leslie Pack Kaelbling. Learning policies for partially observable environments: Scaling up. In *Machine Learning*, pages 362–370. Elsevier, 1995.

[35] Hao Liu and Pieter Abbeel. APS: Active pretraining with successor features. In *International Conference on Machine Learning*, 2021.

[36] Hao Liu and Pieter Abbeel. Behavior from the void: Unsupervised active pre-training. In *Advances in Neural Information Processing Systems*, 2021.

[37] Sobhan Miryoosefi, Kianté Brantley, Hal Daume III, Miro Dudik, and Robert E Schapire. Reinforcement learning with convex constraints. In *Advances in Neural Information Processing Systems*, 2019.

[38] Mirco Mutti, Riccardo De Santi, and Marcello Restelli. The importance of non-markovianity in maximum state entropy exploration. In *International Conference on Machine Learning*, 2022.

[39] Mirco Mutti, Mattia Mancassola, and Marcello Restelli. Unsupervised reinforcement learning in multiple environments. In *AAAI Conference on Artificial Intelligence*, 2022.

[40] Mirco Mutti, Lorenzo Pratissoli, and Marcello Restelli. Task-agnostic exploration via policy gradient of a non-parametric state entropy estimate. In *AAAI Conference on Artificial Intelligence*, 2021.

[41] Mirco Mutti and Marcello Restelli. An intrinsically-motivated approach for learning highly exploring and fast mixing policies. In *AAAI Conference on Artificial Intelligence*, 2020.

[42] Takayuki Osa, Joni Pajarinen, Gerhard Neumann, J Andrew Bagnell, Pieter Abbeel, Jan Peters, et al. An algorithmic perspective on imitation learning. *Foundations and Trends® in Robotics*, 7(1-2):1–179, 2018.

[43] LA Prashanth and Mohammad Ghavamzadeh. Actor-critic algorithms for risk-sensitive mdps. In *Advances in Neural Information Processing Systems*, 2013.

[44] Martin L Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

[45] Zengyi Qin, Yuxiao Chen, and Chuchu Fan. Density constrained reinforcement learning. In *International Conference on Machine Learning*, 2021.

[46] R Tyrrell Rockafellar and Stanislav Uryasev. Optimization of conditional value-at-risk. *Journal of Risk*, 2(3):21–41, 2000.

[47] Younggyo Seo, Lili Chen, Jinwoo Shin, Honglak Lee, Pieter Abbeel, and Kimin Lee. State entropy maximization with random encoders for efficient exploration. In *International Conference on Machine Learning*, 2021.
[48] Archit Sharma, Shixiang Gu, Sergey Levine, Vikash Kumar, and Karol Hausman. Dynamics-aware unsupervised discovery of skills. In *International Conference on Learning Representations*, 2020.

[49] David Silver, Satinder Singh, Doina Precup, and Richard S Sutton. Reward is enough. *Artificial Intelligence*, 299:103535, 2021.

[50] Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

[51] Aviv Tamar, Yinlam Chow, Mohammad Ghavamzadeh, and Shie Mannor. Policy gradient for coherent risk measures. In *Advances in Neural Information Processing Systems*, 2015.

[52] Aviv Tamar and Shie Mannor. Variance adjusted actor critic algorithms. *arXiv preprint arXiv:1310.3697*, 2013.

[53] Jean Tarbouriech and Alessandro Lazaric. Active exploration in Markov decision processes. In *International Conference on Artificial Intelligence and Statistics*, 2019.

[54] Tsachy Weissman, Erik Ordentlich, Gadiel Seroussi, Sergio Verdu, and Marcelo J Weinberger. Inequalities for the L1 deviation of the empirical distribution. *Hewlett-Packard Labs, Tech. Rep.*, 2003.

[55] Denis Yarats, Rob Fergus, Alessandro Lazaric, and Lerrel Pinto. Reinforcement learning with prototypical representations. In *International Conference on Machine Learning*, 2021.

[56] Tiancheng Yu, Yi Tian, Jingzhao Zhang, and Suvrit Sra. Provably efficient algorithms for multi-objective competitive rl. In *International Conference on Machine Learning*, 2021.

[57] Tom Zahavy, Brendan O’Donoghue, Guillaume Desjardins, and Satinder Singh. Reward is enough for convex mdps. In *Advances in Neural Information Processing Systems*, 2021.

[58] Tom Zahavy, Yannick Schroecker, Feryal Behbahani, Kate Baumli, Sebastian Flenerhag, Shaobo Hou, and Satinder Singh. Discovering policies with domino: Diversity optimization maintaining near optimality. *arXiv preprint arXiv:2205.13521*, 2022.

[59] Chuheng Zhang, Yuanying Cai, Longbo Huang, and Jian Li. Exploration by maximizing Rényi entropy for reward-free RL framework. In *AAAI Conference on Artificial Intelligence*, 2021.

[60] Junyu Zhang, Alec Koppel, Amrit Singh Bedi, Csaba Szepesvari, and Mengdi Wang. Variational policy gradient method for reinforcement learning with general utilities. In *Advances in Neural Information Processing Systems*, 2020.

[61] Shangtong Zhang, Bo Liu, and Shimon Whiteson. Mean-variance policy iteration for risk-averse reinforcement learning. In *AAAI Conference on Artificial Intelligence*, 2021.

**Checklist**

1. For all authors...
   - (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes] Lines 57-73 details the main contributions and where to find them.
   - (b) Did you describe the limitations of your work? [No] We did not explicitly comment on the limitations in the paper. However, we note that this is an analytical paper that does not include an algorithmic contribution. This could be seen as a potential limitation, but we believe that developing methodologies for finite trials convex RL settings can be a matter for future works.
   - (c) Did you discuss any potential negative societal impacts of your work? [No] This is a mainly theoretical paper, we cannot foresee any potential societal impact beyond speculation.
   - (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Assumptions 4.1, 5.1, and the Appendix for additional details.
   (b) Did you include complete proofs of all theoretical results? [Yes] See the Appendix.

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No] Our paper is mainly theoretical. While we reported a brief numerical evaluation, the experiments are straightforward to reproduce given the description provided in Section 6.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 6.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] We reported 95% c.i. over 1000 runs, as specified in the caption of Figure 4.
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] The needed computation is negligible.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]