ON DIMENSIONS OF VISIBLE PARTS OF SELF-SIMILAR SETS WITH FINITE ROTATION GROUPS

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Abstract. We derive an upper bound for the Assouad dimension of visible parts of self-similar sets generated by iterated function systems with finite rotation groups and satisfying the weak separation condition. The bound is valid for all visible parts and it depends on the direction and the penetrable part of the set, which is a concept defined in this paper. As a corollary, we obtain in the planar case that if the projection is a finite or countable union of intervals then the visible part is 1-dimensional. We also prove that the Assouad dimension of a visible part is strictly smaller than the Hausdorff dimension of the set provided the projection contains interior points. Our proof relies on Furstenberg’s dimension conservation principle for self-similar sets.

1. Introduction

The visible part of a compact set $K \subset \mathbb{R}^d$ from a hyperplane $L$ consists of those points of $K$ which one can see from $L$ (for a precise definition, see (2.1)). This definition extends naturally to $k$-dimensional planes $L$ for $k = 0, \ldots, d−1$ but, in this paper, we concentrate on the case $k = d−1$. We are interested in the dimensions of visible parts. From the definition it follows easily that the orthogonal projection of $K$ onto $L$ equals that of the visible part. Therefore, the Hausdorff, packing and box counting dimensions of a visible part are always between the corresponding dimensions of the set $K$ and its projection. In general, this is all what can be said, that is, given any two numbers $t < s$ such that there exists a set with dimension $s$ whose projection has dimension $t$ then, for every $u \in [t, s]$, one can construct a set whose visible part has dimension $u$. In order to obtain some nontrivial results, one has to study several visible parts simultaneously. It follows from the Marstrand-Mattila projection theorem [15][17] (see also [18, Corollary 9.4]) that the Hausdorff dimension of a visible part equals that of the set $K$ for almost all visible parts provided the Hausdorff dimension of $K$ is at most $d−1$ and almost all visible parts are at least $(d−1)$-dimensional if the Hausdorff dimension of $K$ is larger than $d−1$. This was shown in [12], where a more general result concerning $k$-planes was proved.

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According to the well-known visible part conjecture (see [19, Problem 11]) almost all visible parts are \((d-1)\)-dimensional provided the set has Hausdorff dimension at least \(d-1\). An example of Davies and Fast [5] shows that the measure theoretic notation “almost all” cannot be replaced by a topological typicality, that is, there is a compact 2-dimensional set in the plane whose visible part is the whole set from a dense \(G_\delta\)-set of lines \(L\). Orponen [21], in turn, proved in the plane that this phenomenon, where the visible part is the whole set \(K\), may happen only for a set of directions whose Hausdorff dimension is at most \(2 - \dim_H(K)\), where \(\dim_H\) stands for the Hausdorff dimension. In [12], the visible part conjecture was proved for quasicircles, graphs of continuous functions and some special examples of self-similar sets. In [1], it was shown to be almost surely valid for the fractal percolation. Falconer and Fraser [7] proved that if \(K \subset \mathbb{R}^2\) is a self-similar set satisfying the convex open set condition and if the projection of \(K\) is an interval for all lines, then the Hausdorff and box counting dimensions of all visible parts are equal to 1.

In the case when there are no rotations in the defining iterated function system of \(K\), they showed that if the projection of \(K\) onto a line \(L\) is an interval, then the Hausdorff and box counting dimensions of the visible part from \(L\) are equal to 1. Rossi [23] proved the visibility conjecture for self-affine sets satisfying a projection condition and a property called strong cone separation. As far as we know, these are the only cases, where the visible part conjecture has been proved. However, there are several results which show that a typical visible part is strictly smaller than the set itself. A result of [13] states that almost all visible parts of a compact \(s\)-set, that is, a compact set having positive and finite \(s\)-dimensional Hausdorff measure, have zero \(s\)-dimensional Hausdorff measure provided \(s > d-1\). O’Neil [20] studied planar continua and proved an explicit nontrivial upper bound for the Hausdorff dimension of almost all visible parts from points, that is, from 0-planes. Quite recently, Orponen [22] proved a general theorem according to which the Hausdorff dimensions of almost all visible parts are at most \(d - \frac{1}{50d}\) for all compact sets in \(\mathbb{R}^d\). In particular, if the Hausdorff dimension of a set in \(\mathbb{R}^d\) is larger than \(d - \frac{1}{50d}\), then the Hausdorff dimensions of almost all visible parts are strictly smaller than the Hausdorff dimension of the set itself.

In this paper, we study dimensions of visible parts of self-similar sets \(K\) generated by iterated function systems having finite rotation groups and satisfying the weak separation condition. In Theorem 2.4, we prove an upper bound for the Assouad dimension of visible parts, which depends on the direction and on the penetrable part of the set (for definition, see (2.2)). As a corollary, we obtain a criterion which guarantees the validity of the visible part conjecture for a fixed hyperplane \(L\), see Corollary 2.6. In particular, for planar self-similar sets generated by iterated function systems with finite rotation groups and satisfying the weak separation condition, the visible part from a line \(L\) is 1-dimensional provided the projection of \(K\) onto \(L\) is a finite or countable union of intervals. Under the above assumptions, we also show that the Assouad dimension of a visible part is strictly smaller than the Hausdorff dimension of the set provided the projection contains interior points and
the Hausdorff dimension of $K$ is strictly larger than $d - 1$, see Theorem 2.11. In the case of trivial rotation group, we prove that if the projection satisfies the weak separation condition, then the upper box counting dimension of the visible part equals the Hausdorff dimension of the projection, see Proposition 2.12. Finally, in Section 3 we give examples and discuss the role of our assumptions.

2. Results

Fix a finite set $\Lambda$ and an integer $d \in \mathbb{N} \setminus \{1\}$, where $\mathbb{N} := \{1, 2, \ldots \}$. Set $\Lambda^*: = \bigcup_{n=0}^{\infty} \Lambda^n$, where $\Lambda^0$ is the empty word. Let $\mathcal{F} := \{f_i(x) := r_iO_i(x) + t_i\}_{i \in \Lambda}$ be an iterated function system on $\mathbb{R}^d$, where $0 < r_i < 1$, $O_i$ is an element of the orthogonal group $O(d, \mathbb{R})$ and $t_i \in \mathbb{R}^d$ for all $i \in \Lambda$. Denote by $K$ the attractor of $\mathcal{F}$, that is, $K$ is the unique nonempty compact set in $\mathbb{R}^d$ satisfying $K = \bigcup_{i \in \Lambda} f_i(K)$.

Since all maps in $\mathcal{F}$ are similarity transformations, $K$ is said to be a self-similar set.

Equip $\Lambda^\mathbb{N}$ with the metric $d(i,j) := \prod_{n=1}^{|i \land j|} r_{i_n}$, where $i \land j$ is the longest common prefix of the sequences $i$ and $j$ and $|u|$ is the length of a word $u \in \Lambda^*$. Note that this metric generates the product topology on $\Lambda^\mathbb{N}$. For $u \in \Lambda^*$, we write $f_u := f_{u_1} \circ \cdots \circ f_{u_{|u|}}$ and define the cylinder $[u] := \{i \in \Lambda^\mathbb{N} \mid i_{|u|} = u\}$, where $i|_n := i_1 \ldots i_n$ for $n \in \mathbb{N}$. Let $\pi: \Lambda^\mathbb{N} \to K$ be the coding map associated to $\mathcal{F}$, that is, for all $i \in \Lambda^\mathbb{N}$, we define $\pi(i) := \lim_{n \to \infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(0)$.

The rotation group of $\mathcal{F}$, denoted by $G(\mathcal{F})$, is the closure of the subgroup of $O(d, \mathbb{R})$ generated by $\{O_i \mid i \in \Lambda\}$.

**Definition 2.1.** A self-similar set $K$ generated by an iterated function system $\mathcal{F} := \{f_i\}_{i \in \Lambda}$ satisfies the weak separation condition if the identity is not an accumulation point of the set $\{f_u^{-1} \circ f_v \mid u, v \in \Lambda^*, u \neq v\}$ in the space of similarities equipped with the topology of pointwise convergence.

**Remark 2.2.** (a) We note that there are several different versions of the weak separation condition, most of them being equivalent, see [2][14][25].

(b) For all $z \in \mathbb{R}^d$ and all $r > 0$, let $Q(z, r) := r[0, 1]^d + z - \frac{r}{2}(1, \ldots, 1)$ be the closed cube centred at $z$ and with side length $r$. We record here a useful fact [25 Theorem 1 (4a)] concerning iterated function systems satisfying the weak separation condition:
there exist \( M \in \mathbb{N} \) and \( 0 < c < 1 \) such that, for all \( z \in K \) and all \( r > 0 \), there exist \( u_1, u_2, \ldots, u_M \in \Lambda^* \) satisfying \( cr \leq \text{diam}([u_j]) \leq r \) for all \( j = 1, \ldots, M \) and

\[
Q(z, r) \cap K \subset \bigcup_{j=1}^{M} f_{u_j}(K).
\]

For \( z \in \mathbb{R}^d \setminus \{0\} \), let \( \langle z \rangle \) be the line spanned by \( z \). For \( \theta \in S^{d-1} \) and \( y \in \langle \theta \rangle \), let \( L_{\theta} \) be the linear subspace perpendicular to \( \theta \) and set \( L_{\theta,y} := y + L_{\theta} \). Denote by \( P_{\theta,y} \) the projection from \( \mathbb{R}^d \) onto \( L_{\theta,y} \), that is, \( P_{\theta,y}(z) \) is the point of \( L_{\theta,y} \) that is closest to \( z \). In the case \( y = 0 \), we simply write \( P_{\theta} \). Given a compact set \( F \subset \mathbb{R}^d \), a direction \( \theta \in S^{d-1} \) and a point \( y \in \langle \theta \rangle \) such that \( L_{\theta,y} \cap F = \emptyset \), the visible part of \( F \) from \( L_{\theta,y} \) is the set

\[
V_{\theta,y}(F) := \{ z \in F \mid [z, P_{\theta,y}(z)] \cap F = \{z\}\},
\]

where \([z, z']\) is the line segment connecting the points \( z, z' \in \mathbb{R}^d \). Heuristically, \( V_{\theta,y}(F) \) consists of those points of \( F \) which one can see from the hyperplane \( L_{\theta,y} \) while looking out perpendicular to the plane. For a more general definition of visible parts, see [12].

For a set \( F \subset \mathbb{R}^d \), denote its interior by \( \text{Int}(F) \). For every \( \theta \in S^{d-1} \), we define the penetrable part of \( F \) in direction \( \theta \) by setting

\[
\text{Pen}_{\theta}(F) := F \cap P_{\theta}^{-1}(P_{\theta}(F) \setminus \text{Int}(P_{\theta}(F)))).
\]

**Remark 2.3.** In the case \( G(\mathcal{F}) = \{\text{Id}\} \), the projection of \( K \) is a self-similar set. We remark that there are self-similar sets with nonempty interior, whose boundary is a Cantor set (see e.g. [3]). A planar self-similar set with an interesting projectional structure is presented in Example 3.1.

We denote by \( \dim_{\Lambda} \) the Assouad dimension, by \( \dim_B \) the box counting dimension and by \( \overline{\dim}_{\Lambda} \) the upper box counting dimension. Recall that \( \dim_H(F) \leq \overline{\dim}_{\Lambda}(F) \leq \dim_{\Lambda}(F) \) for all bounded sets \( F \subset \mathbb{R}^d \). We are now ready to state our main theorem.

**Theorem 2.4.** Let \( K \subset \mathbb{R}^d \) be a self-similar set generated by an iterated function system \( \mathcal{F} \) with a finite rotation group \( G(\mathcal{F}) \) and satisfying the weak separation condition. Then, for all \( \theta \in S^{d-1} \) and \( y \in \langle \theta \rangle \) with \( L_{\theta,y} \cap K = \emptyset \), we have that

\[
\dim_{\Lambda}(V_{\theta,y}(K)) \leq \max\{d - 1, \max_{g \in G(\mathcal{F})} \dim_H(\text{Pen}_{g(\theta)}(K))\}.
\]

**Remark 2.5.** We note that the statement of Theorem 2.4 is general. However, it gives a useful bound for \( \dim_{\Lambda}(V_{\theta,y}(K)) \) only when \( \text{Int}(P_{g(\theta)}(K)) \neq \emptyset \) for all \( g \in G(\mathcal{F}) \), for if \( \text{Int}(P_{g(\theta)}(K)) = \emptyset \) for some \( g \in G(\mathcal{F}) \), then \( \text{Pen}_{g(\theta)}(K) = K \). Recall that \( \dim_H(K) = \dim_{\Lambda}(K) \) by the weak separation condition (see [9] Theorem 1.4).

We have an immediate corollary of Theorem 2.4.

**Corollary 2.6.** Let \( K \subset \mathbb{R}^d \) be a self-similar set generated by an iterated function system \( \mathcal{F} \) with a finite rotation group \( G(\mathcal{F}) \) and satisfying the weak separation condition.
condition. Fix $\theta \in S^{d-1}$. If $\dim_H(P_{g(\theta)}(\text{Pen}_g(\theta)(K))) \leq d - 2$ for all $g \in G(\mathcal{F})$, then
\[ \dim_A(V_{\theta,y}(K)) \leq d - 1 \]
for all $y \in \langle \theta \rangle$ with $L_{\theta,y} \cap K = \emptyset$. In particular, if $d = 2$ and if $P_{g(\theta)}(K)$ is a finite or countable union of intervals for every $g \in G(\mathcal{F})$, then
\[ \dim_H(V_{\theta,y}(K)) = \dim_A(V_{\theta,y}(K)) = 1. \]

Proof. The claim follows since $\dim_A(A) \leq \dim_H(P_{g(\theta)}(A)) + 1$ for all sets $A \subset \mathbb{R}^d$ (see e.g. [18, Theorem 8.10]). □

Remark 2.7. Recall that Falconer and Fraser [7] proved that all visible parts of a self-similar set have Hausdorff and box counting dimension equal to 1 provided that the defining iterated function system satisfies the convex open set condition and all projections of the self-similar set are intervals. These assumptions play a crucial role in [7]. Corollary 2.6 shows that the assumption of convexity of the open set is not needed, at least in the case of a finite rotation group. Further, the condition that every projection is a single interval may be relaxed to a finite or countable union of intervals. We remind that our method is completely different from that of [7]. Also Rossi’s work [23] on visible parts of self-affine sets relies essentially on the assumption that the projection of the attractor is a single interval for many directions. Furthermore, the strong cone separation condition used in [23] is never valid for self-similar sets.

Before the proof of Theorem 2.4 we introduce the Furstenberg’s dimension conservation principle for compact sets (see [11]), which is a key ingredient in our proof. For a set $E \subset \mathbb{R}^d$ and $\theta \in S^{d-1}$, we say that the projection $P_{\theta} : \mathbb{R}^d \to L_{\theta}$ is dimension conserving for $E$ if for some $\delta \geq 0$,
\[ \delta + \dim_H \left( \{ x \in L_{\theta} \mid \dim_H \left( P_{\theta}^{-1}(x) \cap E \right) \geq \delta \} \right) \geq \dim_H(E). \]

Here we adopt the convention that the dimension of the empty set is $-\infty$. Observe that one could replace the latter $\geq$-sign in (2.4) by equality, since the left hand side can never be strictly larger than the right hand side by a classical theorem of Marstrand [16], (see also [8, Corollary 2.10.27]), but we do not need that information in our paper. Let $E$ be a closed subset of $[0, 1]^d$. A closed set $A \subset [0, 1]^d$ is a mini-set of $E$ if $A \subset (\lambda E + x) \cap [0, 1]^d$ for some scalar $\lambda \geq 1$ and $x \in \mathbb{R}^d$. A closed set $B \subset [0, 1]^d$ is a micro-set of $E$ if there is a sequence $(A_n)$ of mini-sets of $E$ with $\lim_{n \to \infty} A_n = B$ in the Hausdorff metric.

The following result is a special case of [11, Theorem 6.1].

Theorem 2.8. For every compact set $E \subset \mathbb{R}^d$ and for every $\theta \in S^{d-1}$, there is a micro-set $B$ of $E$ with $\dim_H(B) \geq \dim_A(E)$ for which $P_{\theta}$ is dimension conserving.

Remark 2.9. In the statement of [11, Theorem 6.1], there is $\dim_H(E)$ instead of $\dim_A(E)$. However, the proof of [11, Theorem 6.1] works for the Assouad dimension as is evident from the remarks before [11, Proposition 6.1]. Recall also [10, Theorem...
2.4], which states that the Assouad dimension of a set equals the dimension of the gallery of its micro-sets.

In the next lemma, we state some basic properties needed in the proof of Theorem 2.4. Recall that a homothety is a map \( h: \mathbb{R}^d \to \mathbb{R}^d \), where \( h(x) := r(x - t) + t \) with \( r \in \mathbb{R} \setminus \{0\} \) and \( t \in \mathbb{R}^d \).

**Lemma 2.10.** Fix \( \theta \in S^{d-1} \), \( y \in \mathbb{R}^d \), \( g \in O(d, \mathbb{R}) \) and a homothety \( h \). Let \( F \subset \mathbb{R}^d \) be a nonempty compact set. Then

1. \( h(V_{\theta,y}(F)) = V_{\theta,h(y)}(h(F)) \),
2. \( g \circ h(F) \) and \( h \circ g(F) \) differ only by a translation and
3. \( V_{\theta,y}(g(F)) = g(V_{g^{-1}(\theta),g^{-1}(y)}(F)) \).

**Proof.** The claims follow from straightforward calculations. \( \square \)

We are now ready for the proof of Theorem 2.4.

**Proof of Theorem 2.4.** Fix \( \theta \in S^{d-1} \) and \( y \in \{\theta\} \) such that \( \Lambda^\theta_{\theta,y} \cap K = \emptyset \). We will only deal with the case, where \( K \) is included in one of the components of \( \mathbb{R}^d \setminus \Lambda^\theta_{\theta,y} \). The general situation can be deduced from this special case as follows: Since \( \Lambda^\theta_{\theta,y} \cap K = \emptyset \), there is \( k \in \mathbb{N} \) such that, for all \( u \in \Lambda^k \), the set \( f_u(K) \) is included in one of the components of \( \mathbb{R}^d \setminus \Lambda^\theta_{\theta,y} \). Since \( f_u \) is a similarity, we can write \( f_u(K) \) as a composition

\[
(2.5) \quad f_u(K) = g_u \circ h_u(K),
\]

where \( g_u \in G(F) \) is the orthogonal component of \( f_u \) and \( h_u \) is a homothety.

Notice also that \( f_u(K) \) is a self-similar set generated by the iterated function system \( \mathcal{F}_u := \{ f_i \circ f_u^{-1} \mid i \in \Lambda \} \) that also satisfies the weak separation condition. Clearly, \( G(\mathcal{F}_u) = G(F) \) and

\[
V_{\theta,y}(K) \subset \bigcup_{u \in \Lambda^k} V_{\theta,y}(f_u(K)).
\]

Assuming that the special case has been proved, one concludes by Lemma 2.10 that \( \dim_A(V_{\theta,y}(f_u(K))) \) is bounded by the upper bound given in (2.3). Thus, the stability of the Assouad dimension under finite unions implies that (2.3) holds in the general situation.

Let \( E = \overline{V_{\theta,y}(K)} \) be the closure of \( V_{\theta,y}(K) \). By Remark 2.2, for every \( x \in E \) and \( r > 0 \), there exist \( u_1, u_2, \ldots, u_M \in \Lambda^* \) satisfying \( cr \leq \text{diam}([u_j]) \leq r \) for all \( j = 1, \ldots, M \) such that

\[
Q(x, r) \cap E \subset \bigcup_{j=1}^{M} f_{u_j}(K) \cap E.
\]

Observe that \( V_{\theta,y}(K) \cap f_{u_j}(K) \subset V_{\theta,y}(f_{u_j}(K)) \) for every \( j \in \{1, \ldots, M\} \). Therefore, by (2.5),

\[
(2.6) \quad Q(x, r) \cap E \subset \bigcup_{j=1}^{M} V_{\theta,y}(f_{u_j}(K)) = \bigcup_{j=1}^{M} V_{\theta,y}(g_{u_j} \circ h_{u_j}(K)).
\]
If $A$ is a mini-set of $E$, then $A \subset h(Q(x, r) \cap E)$ for some $x \in [0, 1]^d$ and $0 < r \leq 1$, where $h$ is a homothety with ratio $r^{-1}$. Combining (2.6) and Lemma 2.10 we conclude that

$$
(2.7) \quad A \subset \bigcup_{j=1}^{M} V_{g_j}(g_j \circ h_j(K)),
$$

where $g_j \in G(\mathcal{F})$, $y' \in \mathbb{R}^d$ and the ratio of the homothety $h_j$ is between $c$ and 1 for all $j = 1, \ldots, M$. Let $B$ be a micro-set of $E$. Then $B = \lim_{n \to \infty} A_n$ for a sequence $(A_n)$ of mini-sets of $E$. Applying (2.7) to $A_n$ for all $n \in \mathbb{N}$, we find a sequence of $M$-tuples of sets $(g_1^n \circ h_1^n(K), \ldots, g_M^n \circ h_M^n(K))$. Since $G(\mathcal{F})$ is finite, there is a constant subsequence $(g_1^{n_k} \circ h_1^{n_k}(K), \ldots, g_M^{n_k} \circ h_M^{n_k}(K))$. Since the ratios and translations of the homotheties $h_j^{n_k}$ are bounded, we may choose a further subsequence such that $(h_1^{n_k}, \ldots, h_M^{n_k})$ converges. Using Lemma 2.10, we conclude that (2.7) is valid with a mini-set $A$ replaced by a micro-set $B$.

Now we apply Theorem 2.8 to $E$ in order to find a micro-set $B$ of $E$ whose Hausdorff dimension is at least $\dim_A(E)$ and for which $P_\theta$ is dimension conserving. Applying (2.7) to $B$, we conclude that there exist $\delta \geq 0$ and $F := \overline{V_{g_j}(g_j \circ h_j(K))}$ such that

$$
(2.8) \quad \delta + \dim_H \left\{ x \in P_\theta(F) \mid \dim_H \left( P_\theta^{-1}(x) \cap F \right) \geq \delta \right\} \geq \dim_A(E).
$$

Let $\tau := g_j^{-1}(\theta)$. Combining Lemma 2.10 with (2.8), we deduce that

$$
(2.9) \quad \delta + \dim_H \left\{ x \in P_\tau \left( \overline{V_{\tau,z}(K)} \right) \mid \dim_H \left( P_\tau^{-1}(x) \cap \overline{V_{\tau,z}(K)} \right) \geq \delta \right\} \geq \dim_A(E)
$$

for some $z \in \mathbb{R}^d$, which we may choose such that $K$ is included in one of the components of $\mathbb{R}^d \setminus L_{\tau,z}$. Clearly, we may assume that $\dim_A(E) > d - 1$. In view of (2.9), the constant $\delta$ must be strictly positive, since $\dim_H \left( P_\tau \left( \overline{V_{\tau,z}(K)} \right) \right) \leq d - 1$. Let

$$
D := \left\{ x \in P_\tau \left( \overline{V_{\tau,z}(K)} \right) \mid \dim_H \left( P_\tau^{-1}(x) \cap \overline{V_{\tau,z}(K)} \right) \geq \delta \right\}.
$$

Since $\delta > 0$, we can find, for every $\varepsilon > 0$, a nonempty subset $D_\varepsilon \subset D$ and $\gamma > 0$ such that

$$
(2.10) \quad \dim_H(D_\varepsilon) \geq \dim_H(D) - \varepsilon
$$

and, for every $x \in D_\varepsilon$, there are $C_x \subset P_\tau^{-1}(x) \cap \overline{V_{\tau,z}(K)}$ and $w \in P_\tau^{-1}(x) \cap \overline{V_{\tau,z}(K)}$ with

$$
(2.11) \quad \dim_H(C_x) \geq \delta - \varepsilon
$$

and

$$
(2.12) \quad \text{dist}(w, L_{\tau,z}) \geq \text{dist}(w', L_{\tau,z}) + \gamma \text{ for every } w' \in C_x.
$$

Fix such an $n_0 \in \mathbb{N}$ that, for every $u \in \Lambda^{n_0}$, we have that $\text{diam}(f_u(K)) \leq \frac{\delta}{2}$. Let $x \in D_\varepsilon$ and $w' \in C_x$. Since $K$ is closed, $\overline{V_{\tau,z}(K)} \subset K$. Therefore, $w' \in f_u(K)$ for some $u \in \Lambda^{n_0}$. If $P_\tau(w') \in \text{Int}(P_\tau(f_u(K)))$, there is no point
y ∈ \overline{V}_\tau, z(K) \cap P_{2}^{-1}(P_{2}(w')) with \text{dist}(y, L_{\tau, z}) > \text{dist}(w', L_{\tau, z}) + \text{diam}(f_{u}(K)) (recall that K is included in one of the components of \mathbb{R}^{d} \setminus L_{\tau, z}). Inequality (2.12) and the fact that w ∈ \overline{V}_\tau, z(K) imply that w' ∈ \text{Pen}_{\tau}(f_{u}(K)). From this we deduce that

\begin{equation}
\bigcup_{x ∈ D_{x}} \{C_{x}\} ⊆ \bigcup_{u ∈ \Lambda^{n_0}} \text{Pen}_{\tau}(f_{u}(K)).
\end{equation}

(2.13)

Applying the classical Marstrand theorem \cite{16} and using the dimension bounds in (2.10) and (2.11) together with (2.9), we conclude that

\[ \text{dim}_{\text{H}} \left( \bigcup_{x ∈ D_{x}} \{C_{x}\} \right) ≥ \text{dim}_{\text{H}}(D) + \delta - 2\varepsilon ≥ \text{dim}_{\Lambda}(E) - 2\varepsilon. \]

On the other hand, note that the Hausdorff dimension of the set on the right hand side of (2.13) is bounded above by

\[ \max_{g ∈ G(\mathcal{F})} \text{dim}_{\text{H}}(\text{Pen}_{g(\theta)}(K)). \]

Since \varepsilon is arbitrary, we get that \text{dim}_{\Lambda}(E) is bounded by the above quantity, which is the desired conclusion. \flushright{□}

Our second theorem states that if the projection of a self-similar set with a finite rotation group contains interior points, the dimension of the visible part is strictly smaller than the dimension of the set itself.

**Theorem 2.11.** Let K ⊂ \mathbb{R}^{d} be a self-similar set generated by an iterated function system \mathcal{F} with a finite rotation group G(\mathcal{F}) and satisfying the weak separation condition. Assume that \text{dim}_{\text{H}}(K) > d - 1. Fix θ ∈ S^{d-1}. Suppose that \text{Int}(P_{0}(K)) ≠ ∅. Then \text{dim}_{\Lambda}(V_{\theta, y}(K)) < \text{dim}_{\text{H}}(K) for all y ∈ \langle \theta \rangle with L_{\theta, y} \cap K = ∅.

**Proof.** By Theorem 2.14, it is enough to show that

\begin{equation}
\max_{g ∈ G(\mathcal{F})} \text{dim}_{\text{H}}(\text{Pen}_{g(\theta)}(K)) < \text{dim}_{\text{H}}(K).
\end{equation}

(2.14)

We do that using a porosity argument. Recall that a set A in a metric space Z is porous in Z if there exists a constant c > 0 so that, for every ball B(z, r) in Z, there exists a ball B(z_1, cr) ⊂ B(z, r) satisfying B(z_1, cr) ∩ A = ∅. It is well known (see \cite[Lemma 3.12]{14}) that if the space Z is Ahlfors regular, the inequality

\[ \text{dim}_{\Lambda} A ≤ \text{dim}_{\text{H}} Z - \delta \]

is valid for every porous set A in Z, where \delta > 0 is a constant depending only on Z and the parameter c involved in the definition of porosity of A in Z. Since \text{dim}_{\text{H}}(K) > d - 1, the attractor K is not contained in a hyperplane. Further, as \mathcal{F} satisfies the weak separation condition, the self-similar set K is Ahlfors regular by \cite[Theorem 2.1]{4}. Thus, to prove (2.14), we only need to show that, for every \( g ∈ G(\mathcal{F}), \) the set \( \text{Pen}_{g(\theta)}(K) \) is porous in K.

To that end, observe first that, for all \( g ∈ G(\mathcal{F}), \)

\begin{align}
& \text{if } P_{g(\theta)}(x) ∈ \text{Int}(P_{g(\theta)}(K)) \text{ for } x ∈ K, \\
& \text{then } P_{O_{i}, g(\theta)}(f_{i}(x)) ∈ \text{Int}(P_{O_{i}, g(\theta)}(K)) \text{ for all } f_{i} ∈ \mathcal{F}. \tag{2.15}
\end{align}
As $\text{Int}(P_{\theta}(K)) \neq \emptyset$, we have that $\text{Int}(P_{\theta}(\theta)(K)) \neq \emptyset$ for all $g \in G(\mathcal{F})$ by (2.15) and the finiteness of $G(\mathcal{F})$. Further, there are $r > 0$ and points $x_g \in P_{\theta}(\theta)(K)$ for all $g \in G(\mathcal{F})$ such that $B(x_g, r) \subset \text{Int}(P_{\theta}(\theta)(K))$. Thus, for all $g \in G(\mathcal{F})$, one can find a finite word $u_g \in \Lambda^*$ so that $P_{\theta}(\theta)(f_{u_g}(K)) \subset B(x_g, r)$. From this we infer that

$$\text{(2.16)}\quad P_{\theta}(\theta)\left(\text{Conv}(f_{\nu_n^{-1}\theta}(K))\right) \subset \text{Int}(P_{\theta}(\theta)(K))$$

for all $g \in G(\mathcal{F})$ and for all finite words $\nu \in \Lambda^*$, where $f_{\nu}(x) = r, O_{\nu}(x) + t_{\nu}$ is the decomposition of $f_{\nu}$ and $\text{Conv}(A)$ denotes the convex hull of a set $A$. The inclusion (2.16) clearly implies that

$$\text{(2.17)}\quad \text{Conv}(f_{\nu_n^{-1}\theta}(K)) \cap \text{Pen}_{\theta}(\theta)(K) = \emptyset.$$

Since $G(\mathcal{F})$ is finite, $n_0 := \max\{|u_g| \mid g \in G(\mathcal{F})\} < \infty$. Using this fact and (2.17), it is now readily checked that $\text{Pen}_{\theta}(\theta)(K)$ is porous in $K$.

We complete this section with an observation connecting the weak separation condition of projections to the dimensions of visible parts.

**Proposition 2.12.** Let $K \subset \mathbb{R}^d$ be a self-similar set generated by an iterated function system $\mathcal{F} := \{f_i\}_{i \in \Lambda}$ with $G(\mathcal{F}) = \{\text{Id}\}$. Suppose that, for some $\theta \in S^1$, the projection $P_\theta(K)$ satisfies the weak separation condition. Then, for all $y \in \langle \theta \rangle$ with $L_{\theta,y} \cap K = \emptyset$, we have that

$$\text{diam}_{B}(V_{\theta,y}(K)) = \dim_{H}(P_\theta(K)).$$

**Proof.** As in the proof of Theorem 2.4 we may assume that $K$ is included in one of the components of $\mathbb{R}^d \setminus L_{\theta,y}$. Fix $y \in \langle \theta \rangle$ with $L_{\theta,y} \cap K = \emptyset$. Recall that the Hausdorff and box counting dimensions are equal for every self-similar set, see [6, Corollary 3.3]. The fact that $G(\mathcal{F}) = \{\text{Id}\}$ implies that $(P_\theta \circ f)_u = P_\theta \circ f_u$ for all $u \in \Lambda^*$ and $P_\theta(K)$ is a self-similar set generated by the iterated function system $\{P_\theta \circ f_i\}_{i \in \Lambda}$. Thus, $\dim_{H}(P_\theta(K)) = \dim_{B}(P_\theta(K))$. Let $r_{\min} := \min\{r_i \mid i \in \Lambda\}$. If the claim is not true, then $\text{diam}_{B}(V_{\theta,y}(K)) > \text{diam}_{B}(P_\theta(K))$. Therefore, for every $M \geq 1$, there are arbitrarily small numbers $\delta > 0$ and sets $I_{\delta,M} \subset \Lambda^*$ with cardinality at least $M$ such that the following properties hold: for all $u \in I_{\delta,M},$

$$f_u(K) \cap V_{\theta,y}(K) \neq \emptyset \text{ and } \delta \leq \text{diam}(f_u(K)) \leq r_{\min}^{-1}\delta$$

and, moreover, for all $u, v \in I_{\delta,M}$ with $u \neq v$,

$$\text{dist}(f_u(K), f_v(K)) > \text{diam}(f_u(K)) \text{ and } \text{d}_{H}(P_\theta \circ f_u(K), P_\theta \circ f_v(K)) \leq \delta,$$

where the Hausdorff metric is denoted by $d_{H}$.

Now all the sets $P_\theta \circ f_u(K)$ are included in a ball, whose radius is comparable to $\delta$. Further, they are homothetic copies of each other, since $G(\mathcal{F}) = \{\text{Id}\}$, and their diameters are comparable to $\delta$. Therefore, there exist $\varepsilon := \varepsilon(M, r_{\min}, d)$ with $\lim_{M \to \infty} \varepsilon = 0$ and $u, v \in I_{\delta,M}$ such that

$$|P_\theta \circ f_u(z) - P_\theta \circ f_v(z)| \leq \varepsilon \text{diam}(f_u(K)) \text{ for all } z \in K,$$
that is, there exists a sequence \((P_\theta \circ f_{u_n})^{-1} \circ (P_\theta \circ f_{v_n})\) converging to \(\text{Id}\). Under the weak separation condition for \(P_\theta(K)\), there exist \(u \neq v\) such that \(P_\theta \circ f_u = P_\theta \circ f_v\). Since \(\text{dist}(f_u(K), f_v(K)) > \text{diam}(f_u(K))\) and \(f_u(K)\) and \(f_v(K)\) are included in the same component of \(\mathbb{R}^d \setminus L_{\theta,y}\), it follows that either \(f_u(K) \cap V_{\theta,y}(K) = \emptyset\) or \(f_v(K) \cap V_{\theta,y}(K) = \emptyset\), which is a contradiction. \(\square\)

3. Examples

In this section, we present some examples illustrating the role of assumptions in our theorems. One may ask, if it is necessary to take the maximum over the dimensions of the penetrable parts over the orbit of \(\theta\) under the rotation group in Theorem 2.4 or if the dimension is constant along this orbit. We show that the dimension of the penetrable part is not necessarily constant. Recall the open set condition implies the weak separation condition.

**Example 3.1.** Let \(\mathcal{F} := \{f_1, \ldots, f_5\}\) be an iterated function system on \(\mathbb{R}^2\), where \(\{f_1, \ldots, f_4\}\) generates the standard four corner Cantor set with contraction ratio \(\lambda \geq 1/3\) and \(f_5\) contracts by \(\rho < \frac{1-2\lambda}{\sqrt{2}}\), rotates by an angle \(\pi/4\) and has the centre of \([0,1]^2\) as the fixed point, see Figure 1. Then \(\mathcal{F}\) satisfies the open set condition and \(G(\mathcal{F}) = \{O(\frac{k\pi}{4}) \mid k = 0, \ldots, 7\}\), where \(O(\alpha)\) is the rotation by angle \(\alpha\). Let \(K\) be the attractor of \(\mathcal{F}\). Then \(P_{O(\frac{k\pi}{4})}(1,0)(K)\) is an interval for odd \(k\) and, for even \(k\), it is the union of the \(\lambda\)-Cantor set on the line and the countable union of intervals, which are the projections of the squares \(f_u \circ f_5([0,1]^2)\), where \(u \in \{1, 2, 3, 4\}^*\). Observe that the \(\lambda\)-Cantor set consists of the limit points of the end points of the intervals. In particular, \(\text{Pen}_{O(\frac{2k\pi}{4})}(1,0)(K)\) consists of two opposite corner points of \([0,1]^2\), while \(\text{Pen}_{O(\frac{2(2k+1)\pi}{4})}(1,0)(K)\) is the union of the standard four corner Cantor set generated by \(\{f_1, \ldots, f_4\}\) and the left and right endpoints of the squares \(f_u \circ f_5([0,1]^2)\), where \(u \in \{1, 2, 3, 4\}^*\).
Next example shows that there are self-similar sets satisfying the open set condition such that their projections are finite unions of intervals. This property of the following example must be well known, but we could not find a reference.

**Example 3.2.** Consider the standard four corner Cantor set $K$ in $\mathbb{R}^2$ with contraction ratio $\lambda \geq \frac{1}{3}$. Then, for all $\theta \in S^1 \setminus \{\pm(1,0), \pm(0,1)\}$, the projection $P_\theta(K)$ is a finite union of intervals. We describe them in the case $\lambda = \frac{1}{3}$. Denote by $\alpha_\theta$ the angle which $\theta \in S^1$ makes with the positive $x$-axis. By symmetry, it is enough to consider the case $\alpha_\theta \in [0, \frac{\pi}{4}]$. For all $k \in \mathbb{N}$, if $\tan \alpha_\theta \in [3^{-k}, 3^{-k-1}]$, then $P_\theta(K)$ is a union of $2^{k-1}$ intervals. Indeed, denoting by $K_{k-1}$ the approximation of $K$ at construction level $k-1$, the projection $P_\theta(K_{k-1})$ is a union of $2^{k-1}$ intervals. At level $k$, the projection of every level $k-1$ construction square is the union of two intervals, but the gap between these intervals is covered by the projection of another level $k-1$ construction square due to the choice of $\alpha_\theta$ and the fact that $1 - 2\lambda \leq \lambda$, that is, the gap between the intervals is smaller than the length of the intervals, see Figure 2.

The projection of a self-similar set may be a countable union of intervals. This is demonstrated by the following example.

**Example 3.3.** Let $K$ be the attractor generated by an iterated function system $\mathcal{F} := \{f_1, f_2, f_3\}$ on $[0,1]^2$, where $f_1$ and $f_2$ contract by the factor $\frac{1}{3}$, $f_1$ translates by $\left(\frac{1}{3}, 0\right)$, $f_2$ translates by $\left(\frac{1}{3}, \frac{1}{3}\right)$ and $f_3$ contracts by $\frac{1}{3}$, rotates by an angle $\frac{\pi}{2}$ and translates to the upper left corner. For an illustration of the third construction step, see Figure 3. Now the projection of $K$ onto the $y$-axis is the unit interval and onto the $x$-axis it is the union $\bigcup_{k=0}^{\infty} [1 - 2^{-k}, 1 - 2^{-k} + 2^{k-2}] \cup \{1\}$. We remark that the different contraction ratios are not essential in Example 3.3 and, with four maps, it is possible to construct a homogeneous iterated function...
Figure 3. In Example 3.3, the projection of the attractor onto the $x$-axis is a countable union of intervals.

system having a projection which is a countable union of intervals. Indeed, this is achieved by replacing the maps $f_1$ and $f_2$ by three maps with contraction ratio $\frac{1}{3}$ in Example 3.3 and using the same contraction $\frac{1}{3}$ also in the map $f_3$.

Our last example demonstrates the weak separation condition for projections.

**Example 3.4.** Fix $n \in \mathbb{N} \setminus \{1\}$ and let $\mathcal{F}$ be any nonempty subset of the family

$\{(x, y) \mapsto \frac{1}{n}(x, y) + \frac{1}{n}(k, j) \mid k, j \in \mathbb{N} \cup \{0\} \text{ with } 0 \leq k, j \leq n - 1\}$.

Then $\mathcal{F}$ generates a self-similar set $K$ on $[0,1]^2$ which is invariant under the multiplication by $n \mod 1$. Now the projection of $K$ in any rational direction satisfies the weak separation condition, see [24, Remark 1]. This is due to the fact that, for every rational $\theta$, the orbit $\{k\theta \mod 1 \mid k \in \mathbb{N}\}$ is finite. To be more precise about the rational directions, observe that, for any vector $(a, b) \neq (0, 0)$ with $a, b \in \mathbb{Z}$, the map $(x, y) \mapsto ax + by$ defines a rational projection up to scaling (which does not affect the dimension) and every rational projection can be defined in this way. Thus, for this class of "integral" self-similar sets, we can find a dense set of directions for which the visible part problem is well understood due to Proposition 2.12.

**Remark 3.5.** In the proof of Theorem 2.4, the weak separation condition guarantees that the number $M$ in (2.6) is independent of $r$. The finiteness of $G(\mathcal{F})$, in turn, allows us to find a subsequence with a constant rotation part in the argument after (2.7). Both these properties are essential while we prove that the inclusion (2.7) is valid also for micro-sets. In Theorem 2.11 in addition to using Theorem 2.4.
we apply the weak separation condition to conclude that the attractor $K$ is Ahlfors regular. In the proof of Theorem 2.4 we consider the closure of a visible part. Recall that the restriction of the projection to a visible part is injective, but this is not true for the closure. For example, let $F := \{0\} \times [1, 3] \cup \text{graph}(f)$, where $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = \sin \frac{1}{x} + 2$, and $\text{graph}(f)$ is the graph of $f$. Then the visible part of $F$ from the $x$-axis is $\{(0, 1)\} \cup \text{graph}(f)$, but its closure is $F$. So the closure of a visible part may be much bigger than the visible part. However, Assouad dimension is an upper bound for the box counting dimension, and the box counting dimension of a set equals that of its closure. Thus, in this sense we do not lose anything by considering the closure of a visible part. Nevertheless, we point out that the upper bound in Theorem 2.4 is not always optimal. For example, in Example 3.1 the dimension of the penetrable part from the $x$-axis is larger than one, but the visible part and its closure are 1-dimensional.

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