Anisotropic $q$-Gaussian 3D velocity distributions in $\Lambda$CDM halos

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ABSTRACT

The knowledge of the velocity distribution function (VDF) of dark matter halos is required for calibrating the direct dark matter detection experiments and useful for recovering the mass profile from the observed distribution of tracers in projected phase space when the VDF has unknown anisotropy. Unfortunately, the VDF of halos in $\Lambda$CDM dissipationless cosmological simulations is still poorly known. We consider the $q$-Gaussian (also called Tsallis) family of VDFs, among which the Gaussian is a special case. We extend the $q$-Gaussian to anisotropic VDFs by considering the isotropic set of dimensionless spherical velocity components normalized by the velocity dispersion along that component. We test our anisotropic VDF on 90 cluster-mass halos of a dissipationless cosmological simulation. While our anisotropic $q$-Gaussian model adequately reproduces the VDF averaged in spherical shells with radii greater than 2 virial radii, no $q$-Gaussian model can adequately represent the VDF in spherical shells of radius smaller than 2 virial radii, for which the distribution of dimensionless normalized velocity moduli is flatter than any $q$-Gaussian can allow. Nevertheless, the anisotropic $q$-Gaussian fits significantly better the simulated VDF than the commonly used Maxwellian (Gaussian) distribution, at all radii except near one-tenth of the virial radius. Beyond one-tenth of a virial radius, the radial variation of non-Gaussianity diverges between the mean fits and the fit on a random stack of halos, a feature that appears to be the consequence of the triaxiality of individual halos. We provide a parametrization of the modulation of the non-gaussianity parameter with radius for both the mean fits and the fit of the stacked halo. At a radii of a few percent of the virial radius, corresponding to the Solar position in the Milky Way, the best-fit $q$-Gaussian model, although fitting better the VDF than the Gaussian, overproduces significantly the fraction of high velocity objects, indicating that one should not blindly use these $q$-Gaussian fits to make predictions on the direct detection rate of dark matter particles.

1 INTRODUCTION

While dark matter appears to constitute 85% of the mass of the Universe, much work is being performed to detect dark matter particles and to quantify its distribution in astronomical systems. In particular, experiments have been developed in order to detect the passage of dark matter particles through terrestrial detectors: DAMA (Bernabei et al. 2013), CoGeNT (Aalseth et al. 2013), CRESST-II (Petricca et al. 2012), CDMS-Si (CDMS Collaboration et al. 2013), and Xenon100 (Aprile et al. 2012). The knowledge of the precise high-end part of the distribution of space (3D) velocities (hereafter velocity distribution function or VDF) in the inner halo, corresponding to the Solar position in the Milky Way galaxy, is required to quantify the expected event rate in direct dark matter detection experiments. Indeed, these experiments involve a detection threshold in kinetic energy, which for light (e.g. $\approx 10$ GeV in mass) dark matter particles corresponds to velocities of order of $300 \, \text{km s}^{-1}$, i.e. somewhat higher than the expected velocity dispersion of halo dark matter particles in the Solar neighborhood. With this goal in mind, the VDF in $\Lambda$CDM halos has drawn attention during the last few years. The fits of Fairbairn & Schwetz (2009), Vogelsberger et al. (2009), Ling et al. (2010), Kuhlen et al. (2010), Lisanti et al. (2011), Mao et al. (2013), Pato et al. (2013).

The knowledge of the VDF is also important for modeling the radial profiles of mass (including dark matter) and velocity anisotropy of quasi-spherical systems from the distribution of their tracers (stars in galaxies; galaxies in clusters) in projected phase space (PPS: projected radius and line-of-sight velocity). The cleanest way to perform this mass / anisotropy analysis is to model the distribution of tracers in PPS, but this requires a triple integral of the six-dimensional distribution function (DF) expressed in terms of energy and angular momentum, $f(E, J)$ (Dejonghe & Merritt 1992). For example, the method of Wojtak et al. (2009) that starts from the $\Lambda$CDM halo DF of Wojtak et al. (2008) is very slow (requiring a day to run for a 500-tracer system with full error sampling from Markov Chain Monte-Carlo [MCMC] methods). Orbit modeling (Schwarzschild 1979; Richstone & Tremaine 1984; Syer & Tremaine 1996) is much slower, thus preventing proper error sampling by MCMC. Recently, Mamon, Biviano, & Boué...
have developed an algorithm called MAMPOSSSt, in which the distribution of tracers in PPS is expressed as a single integral: $f(E, J)$ is replaced by the distribution of line-of-sight velocities at a given (3D) position, which in turn depends on the combination of the radial profiles of the total mass and the velocity anisotropy $1 - (\sigma_r^2 + \sigma_t^2)/(2\sigma_v^2)$, as well as a suitably simple form for the VDF. So far, MAMPOSSSt has only been used with a Maxwellian (or Gaussian) VDF \citep{Mamon2013, Biviano2013, Munari2013, Guennou2013}. However, N-body simulations (both cosmological and academic ones) indicate that VDFs show departures from Gaussianity \citep[e.g.,][]{Wojtak2003, Hansen2006}. Moreover, forcing Gaussianity in the VDF of isotropic systems built from the Jeans equation of local dynamical equilibrium leads to unstable density profiles, whereas analogous systems built from distribution functions are not \citep{Kazantzidis2004}.

The appropriate statistical mechanical description of the 6D structure of self-gravitating spherical systems is an old and still open problem. A generalization of the Maxwellian VDF has been proposed in the context of the non-extensive thermodynamics developed by \cite{Tsalidis1988}. The Tsallis VDF, alternatively called $q$-Gaussian, has been applied to describe phenomena of diverse fields of physics, particularly self-gravitating systems. The Tsallis distribution is equivalent to the polytropic gas model \citep{Plastino1993}, of which the isothermal sphere is a particular case.

To the best of our knowledge, all analyses of non-Maxwellian VDFs assume either velocity isotropy \citep{Vogelsberger2006, Lisanti2011, Mao2013} or a VDF that is separable into its radial and tangential components \citep{Hansen2006, Fairbairn2009, Kuhlen2010} or tried both \citep{Ling2010}. Interestingly, the radial and tangential components of the VDF of structures in cosmological and academic N-body simulations are well fit, separately, by the $q$-Gaussian formula \citep{Hansen2006}, although other modifications to the Gaussian have been shown to also fit well the VDFs of halos \citep[at the solar radius,][]{Fairbairn2009, Kuhlen2010, Lisanti2011, Mao2013}.

In fact, were the dynamical evolution of these systems just determined by two-body interactions, as is the case for ideal gases, i.e., if the two-body relaxation time were short, then the system would rapidly evolve to isotropic velocities in a short time scale, the distribution function would then depend solely on energy, $f = f(E)$, and could be obtained from the density profile \citep{Eddington1916}, and finally, the velocity modulus distribution function at radius $r$ would then simply be $f(v) \propto v^2 / [2 + \Phi (r)]$.

However, in most large-scale astronomical systems (galaxies and clusters), the two-body relaxation time of the dark matter component is longer than the age of the Universe. One might still expect that violent relaxation, caused by a rapidly varying gravitational potential \citep{Lynden-Bell1967}, will redistribute energies and lead to a possibly stationary configuration. However, violent relaxation is not thought to be long-term, hence it is not clear that energies will be completely redistributed.

On the other hand, simulations and observational modeling suggest that the VDF in elliptical galaxies and galaxy clusters is most likely anisotropic. Indeed, CDM halos of cluster-mass halos show radial velocities at outer \citep[e.g.,][]{Lemze2012} or all \citep{Wojtak2013} radii. Moreover, dynamical studies of galaxies

\textsuperscript{1} We will refer to the Maxwellian VDF in the physical context, and to the Gaussian VDF in the mathematical context.

\citep{Wojtak2013} and clusters \citep{Biviano2004, Wojtak2010, Biviano2013, Munari2013} point to radial outer velocity anisotropy.

In principle, the VDF can be deduced from the DF. For anisotropic spherical systems, since $\int dv f(E, J) = r$, the VDF at radius $r$ will be

$$f_\nu (v_r, v_t | r) = \frac{2\pi}{\rho (r)} v_\nu \left( \frac{v_r^2 + v_t^2}{2 + \Phi (r)} \right) .$$

For example, one could use the separable form of the DF that \cite{Wojtak2008} measured for $\Lambda$CDM halos. However, it involves a total of 8 parameters, so, although interesting, this approach is left for future work. This separable form of the DF leads to a non-separable VDF. Indeed, if $f(E, J) = f_E (E) f_J (J)$, then according to equation (1), $v_\nu$ cannot be separated from $v_t$ within $f_E (E)$, unless $f_E (E) = c \exp (-a E)$, where $a$ is a constant, but this is not the case for $\Lambda$CDM halos \citep{Wojtak2008}. It thus seems preferable to consider a VDF that is a non-separable function of radial and tangential velocities. Moreover, it is straightforward to compute the distribution of line-of-sight velocities from the non-separable $q$-Gaussian VDF that we will present in Sect.\textsuperscript{3} while the distribution of line-of-sight velocities for a separable $q$-Gaussian VDF cannot be expressed in analytical form in a single quadrature.

Thanks to the interest in direct dark matter detection, most work on the VDF has been restricted to radii of $\approx 3$ percent of the virial radius, $r_{\text{vir}}$, i.e., the position of the Earth in the halo of the Milky Way. On the other hand, as in many other mass / velocity anisotropy modeling methods, MAMPOSSSt involves integrals along the line-of-sight, corresponding to physical radii extending from $r = R$ to infinity, in principle. In practice, the Hubble flow stretches the velocity vs. distance-to-halo-center relation so that beyond $r_{\text{max}} \approx 13 r_{\text{vir}}$, the line-of-sight velocities extend beyond $3 \sigma_{\text{los}}$ (e.g., \cite{Mamon2010}). Thus, the knowledge of the VDF is required at all radii from the halo center to $\approx 13 r_{\text{vir}}$.

In this work, we propose an extension of the $q$-Gaussian VDF to spherical systems with anisotropic velocities. We do not advocate any fundamental basis for the $q$-Gaussian velocity distribution. Instead, we treat it as a powerful parametrization that allows us to phenomenologically describe systems whose 3D velocity distributions depart from the Gaussian distribution. We then fit our anisotropic $q$-Gaussian model to the VDF of simulated $\Lambda$CDM halos, between 0.03 and 13 $r_{\text{vir}}$, to check if it provides a significantly better representation of the VDF than does the Gaussian model with one parameter less.

In Sect.\textsuperscript{4} we review the classical and simplest case, of the Gaussian velocity distribution from the Maxwellian approach. In Sect.\textsuperscript{5} we present the $q$-Gaussian velocity distribution and briefly describe its extensions to thermodynamics. Then, in Sect.\textsuperscript{6} we generalize the $q$-Gaussian velocity distribution to spherical systems with anisotropic velocities. In Sect.\textsuperscript{7} we describe the simulated data that we use to analyze in Sect.\textsuperscript{6} the properties of the 3D velocity distribution of cluster-mass $\Lambda$CDM halos as a function of radial distance to the halo center. We discuss our results in Sect.\textsuperscript{7}.

2 GAUSSIAN VELOCITY DISTRIBUTION

The velocity distribution of an ideal gas in equilibrium was first determined by \cite{Maxwell1860}, based on two symmetry hypotheses (see \cite{Sommerfeld1993, Silva1998, Diu2007}).

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Together, these hypotheses imply that
\[ F(v) = f(v_x) f(v_y) f(v_z) . \] (2)

Following standard steps, from eq. (2) we write
\[ \ln F(v) = \ln f(v_x) + \ln f(v_y) + \ln f(v_z) \] (3)

and differentiate both sides relative to \( v_x \), obtaining
\[ \frac{1}{v} \frac{d}{dv} F(v) = \frac{1}{v_x} \frac{d}{dv_x} f(v_x) . \] (4)

In equation (4), the left-hand side is only a function of \( v \), while the right-hand side is only a function of \( v_x \), which implies that both are equal to some constant \(-k\). This leads to
\[ f(v_x) \propto \exp \left( -\frac{k \cdot \sigma}{2} \right) . \] (5)

Equation (5) also holds for \( v_y \) and \( v_z \). Therefore, the joint velocity distribution is
\[ F(v) = F(v) = A \exp \left( -\frac{k \cdot \sigma}{2} \right) , \] (6)

where \( A = [k/(2\pi)]^{3/2} \) is determined by the normalization condition
\[ \int F(v) d^3v = \int_0^{\infty} F(v) 4\pi v^2 \, dv = 1 , \] (7)

while \( k = 1/\sigma^2 \), with \( \sigma \) the one-dimensional velocity dispersion determined by the 2nd velocity moment condition
\[ \int F(v) v^2 d^3v = \int_0^{\infty} F(v) 4\pi v^4 \, dv = 3 \sigma^2 . \] (8)

Boltzmann (1872) showed that the velocity distribution of equation (6) is not changed by molecular collisions and obtained the expression for the entropy that, maximized, gives the velocity distribution previously derived by Maxwell.

3 Tsallis (or q-Gaussian) Velocity Distribution

The DF of isotropic spherical systems implied by equation (6) is then \( f(E) \propto \exp(-E/\sigma^2) \). Since the joint assumptions of isotropy (i) and separability of the velocity components (ii) lead to purely exponential energy distributions, then non-exponential energy distributions of isotropic systems will necessarily lead to the non-separability of the VDF. For example, if the DF is truncated because of escaping particles (e.g., King 1966), the VDF will be non-separable. This provides a natural motivation to explore non-Maxwellian VDFs such as the Tsallis distribution.

Historically speaking, the q-Gaussian velocity distribution was derived in the opposite order. Firstly, a generalized version of the Boltzmann entropy was proposed by Tsallis (1988), and then the VDF was obtained by maximizing this Tsallis entropy (Plastino & Plastino 1993). Finally, the same velocity distribution was obtained (Silva et al. 1998) following symmetry arguments similar to that of Maxwell.

Anisotropic q-Gaussian Velocity Distributions

In fact, assuming the velocity isotropy hypothesis (i) above — which allows us to write \( F(v) = F(v) \) but abandoning the coordinate-independence hypothesis (ii) — Silva & Alcaniz (1998) proposed, as a generalization of the joint Maxwellian distribution (eq. [6]), the expression
\[ F(v) = \exp_q \left[ \sum_{i=1}^{n} f^{q-1}(v_i) \ln_q f(v_i) \right] , \] (9)

where \( \exp_q(f) = [1 + (1 - q)f]^{1/(1 - q)} \) and \( \ln_q(f) = \frac{f^{1-q} - 1}{1 - q} \) is called the q-log function, and follows \( \exp_q(f) \to e^f \) as \( q \to 1 \), while
\[ \ln_q(f) \to f^{1-q} - 1 \]

is called the q-exp function, and follows \( \exp_q(\ln_q(f)) \to f \) as \( q \to 1 \). One can easily check that \( \exp_q(\ln_q(f)) = f \). Then, in the limit \( q \to 1 \), the joint velocity distribution of equation (9) tends to the Maxwellian distribution. Following the same steps as for the Maxwellian VDF, we arrive at
\[ F(v) = F(v) = B_q \left[ 1 - (1 - q)k \right]^{1/(1 - q)} , \] (10)

where the constants \( B_q \) and \( k \) are obtained using the normalization equations (7) and (8) (see Silva & Alcaniz 2003),
\[ B_q = \left( \frac{k}{2\pi^2} \right)^{3/2} \]
\[ \begin{cases} \Gamma[1/(1-q) + 3/2]/\Gamma[1/(1-q)] & (0 < q < 1) , \\ \Gamma[1/(q-1)]/\Gamma[1/(q-1) - 3/2] & (1 < q < 5/3) , \end{cases} \] (11)

where
\[ k = \frac{2}{7 - 5q} \frac{1}{\sigma^2} . \] (12)

This value of \( k \) imposes the tighter restriction \( q < 7/5 \). Note the velocity limit of \( \sqrt{2/[k(1 - q)]} \) when \( q < 1 \).

4 Anisotropic Tsallis (q-Gaussian) Velocity Distribution

The velocity distribution of equation (10) depends only on the modulus of the velocity, hence is isotropic, as expected by construction. Since simulated astrophysical systems have anisotropic velocities, we now extend the q-Gaussian velocity distribution to anisotropic velocities.

One possible approach to extend the q-Gaussian VDF to anisotropic velocities would be to maximize the Tsallis entropy, as done by Plastino & Plastino (1993), but with additional constraints (see Stiavelli & Bertin 1987). Instead, inspired by eq. (10), and concerned with spherically symmetric self-gravitating systems, and correcting for streaming motions (e.g., streaming radial motions beyond the virial radius), we introduce the anisotropic q-Gaussian
VDF as
\[ F(v) = \frac{C_q}{(1-\beta)\sigma_i^2} \times \left[ 1 - (1-q)\frac{D_q}{2} \times \left( \frac{(v_r - \bar{v}_r)^2}{\sigma_r^2} + \frac{(v_t - \bar{v}_t)^2}{\sigma_t^2} + \frac{(v_\phi - \bar{v}_\phi)^2}{\sigma_\phi^2} \right)^{1/(1-q)} \right]^{(q)}, \]
where the $\bar{v}_r$, $\sigma_r$, $\bar{v}_t$, $\sigma_t$, and $\bar{v}_\phi$, $\sigma_\phi$ are respectively the mean streaming velocities and velocity dispersions in the direction $i$ of the spherical coordinate system, while $C_q$ and $D_q$ are constants (dependent on $q$) that we shall determine below.

Defining the dimensionless normalized velocities as
\[ u_i = \frac{v_i - \bar{v}_i}{\sigma_i}, \] (14)
and noting that the Jacobian relating the $v_i$ to the $u_i$ is $(1-\beta)\sigma_i^3$, equation (13) can equivalently be written
\[ F(u) = C_q \left[ 1 - (1-q)\frac{D_q}{2} \left( u_r^2 + u_t^2 + u_\phi^2 \right) \right]^{1/(1-q)}. \] (15)
One notices that the vector field $u$ is isotropic by construction. Equations (17) and (S) become
\[ \int F(u) u_r^2 u_r = \int_{0}^{\infty} F(u) 4\pi u^2 du = 1, \] (16)
\[ \int F(u) u^2 u^4 = \int_{0}^{1} F(u) 4\pi u^4 du = 3. \] (17)
Equation (15) is identical to equation (10), once one sets $\sigma_r$ in the latter equation to unity. Therefore,
\[ D_q = \frac{2}{7-5q}, \] (18)
and
\[ C_q = \left( \frac{D_q}{2\pi} \right)^{3/2} \times \begin{cases} (1-q)^{3/2} \Gamma[1/(1-q) + 5/2] \Gamma[1/(1-q) + 1] & (0 < q < 1), \\ (q-1)^{3/2} \Gamma[1/(q-1) - 3/2] \Gamma[1/(q-1) - 1] & (1 < q < 7/5). \end{cases} \] (19)

Considering the radial and tangential dimensionless normalized velocities
\[ u_r = \frac{v_r - \bar{v}_r}{\sigma_r}, \] (20)
\[ u_t = \sqrt{u_t^2 + u_\phi^2} = \sqrt{\frac{(v_t - \bar{v}_t)^2}{\sigma_t^2} + \frac{(v_\phi - \bar{v}_\phi)^2}{\sigma_\phi^2}}, \] (21)
the probability distribution function of $(u_r, u_t)$ is then
\[ F(u_r, u_t | q) = 2\pi u_t F(u_r, u_t, u_\phi) = 2\pi C_q u_t \left[ 1 - (1-q)\frac{D_q}{2} \left( u_r^2 + u_t^2 \right) \right]^{1/(1-q)}. \] (22)
The VDF expressed in dimensionless normalized velocities in equation (22) is clearly not separable into two terms respectively depending on $u_r$ and on $u_t$.

### 4.1 Velocity modulus (or speed) distribution

As for the Maxwellian VDF, it is interesting to define the distribution of the modulus of the velocity, i.e. the speed distribution function (SDF) of the $q$-Gaussian VDF. Defining the dimensionless normalized speed as
\[ u = \sqrt{u_r^2 + u_t^2}, \] (23)
the SDF is
\[ G(u | q) = 4\pi u^2 F(u_r, u_t, u_\phi) = 4\pi C_q u^2 \left[ 1 - (1-q)\frac{D_q}{2} u^2 \right]^{1/(1-q)}. \] (24)
Again, we have a maximum velocity for $q < 1$, which is now $\sqrt{2/[D_q(1-q)]} = \sqrt{(7-5q)/(1-q)}$.

### 5 SIMULATIONS

To test the performance of the VDF of equation (13), we have analyzed a cosmological dark matter $N$-body simulation performed with Gadget-2 [Springel 2005]. The simulation was run with 512$^3$ particles in a periodic box of comoving size $100\ h^{-1}\ Mpc$, using a WMAP7 cosmology: $\Omega_m = 0.272, \Omega_\Lambda = 0.728, h = 0.704, \sigma_8 = 0.807$. The particle mass is $5.62 \times 10^8\ h^{-1}\ M_{\odot}$. The Plummer-equivalent force softening is 5% of the mean interparticle distance and kept constant in comoving units. Initial conditions have been generated using the MPgrafic code [Prunet 2008].

Halos were extracted with the HaloMaker 2.0 using a Friends-of-Friends technique [Davis et al. 1985], with linking length $b = 0.2$ (in units of the mean interparticle separation).

We selected 90 halos from the $z = 0$ output of the cosmological simulation, divided in 3 subsamples of comparable mass: the first subsample contains the 30 most massive halos, while the other two subsamples contain halos with geometric mean differing by 0.5 and by 1.0 dex from the geometric mean of the first subsample (i.e., the halos of mass rank 53 – 82 and 221 – 250). The mean mass in each subsample is $(M) = 2.44 \times 10^{14} M_{\odot}, 6.15 \times 10^{13} M_{\odot}$ and $1.92 \times 10^{13} M_{\odot}$, respectively.

We analyzed the halos as follows. First, we defined the center of each halo using an iterative median center scheme, starting on the halo particles returned by the halo finder, computing the median halo coordinates, and restricting to the particles within one-fifth of the initial (virial) radius around the new center, iterating 3 times. Once we had the center, we reestimated the virial radius by finding the radius, $r_{\text{vir}}$, where the mean density within is 100 times the critical density of the Universe at $z = 0$. To avoid assigning particles outside of halo virial spheres to two or more halos, we reassigned all particles to the nearest halos in units of their virial radii.

We determine the maximum likelihood estimate of $q$ from the distribution of $(u_r, u_t)$, by minimizing
\[ -\ln \mathcal{L}(q) = -\sum \ln F(u_r, u_t | q), \] (25)
where $F(u_r, u_t | q)$ is given in equation (24) and depends on $q$, or equivalently by minimizing
\[ -\ln \mathcal{L}(q) = -\sum \ln G(u | q) = -\ln \mathcal{L}(q) + \text{extra term}, \] (26)
where $G(u | q)$ is given in equation (24), and the extra term is independent of $q$.

We performed two types of analyses: on one hand, we built a

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3 For the cosmology of our simulation the mean density within the virial radius is 97 according to the approximation of Bryan & Norman [1992].
and error bars

Figure 1. Best-fit value of $q$ obtained with maximum-likelihood (eq. 26) fits of equation (24) to the distribution of the dimensionless normalized speed $u$ (eqs. 20, 21, and 22), as function of distance from halo center (in virial units), for the 90 individual halos (black lines). The orange line and error bars are the medians and their uncertainties ($1.25 \sigma / \sqrt{N}$, where $N \leq 90$ corresponds to the number of halos used for the median). The green dashed horizontal line shows the (anisotropic) Gaussian.

stacked halo, normalized by $r_{100}$, with all particles of our 90 halos, and on the other hand we fit all halos individually and then determined the mean and median value of $q$ in each radial bin. To fit the stacked halo, the radial bins were defined requiring $3.0 \times 10^5$ particles in each bin. For the fits in individual halos, we required 5000 particles in each radial bin and to calculate the mean and median, we defined 15 radial bins equally spaced in log. The mean was computed by only considering those halos whose radii were in one of our 15 final bins. For the median, we first performed linear interpolation (without extrapolation) of $q(r)$ and then we took the median of the $N \leq 90$ interpolated values of $q$.

We compute $q$ in this manner starting from 0.03 $r_{100}$ (within which the definition of the center may affect the results) to 13 $r_{100}$ (see Sect. 4).

6 RESULTS

6.1 Radial profiles of non-Gaussianity

Fig 1 shows the best-fit $q$ of our anisotropic model (eq. 24) versus distance from the halo center for the 90 individual halos. No halo exhibits a Gaussian behaviour at all radii. The non-Gaussianity index starts above unity, decreases to unity at typically $r_{100}/10$, keep decreasing to $q \approx 0.85$ at $r \approx r_{100}$, then rises rapidly to $q \approx 0.94$ at $2 \sim 2.5 r_{100}$, where it reaches a plateau.

Fig. 2 shows the best fit values of $q$ obtained in the stacked halo as well as the mean and median $q(r)$ of individual halos (see Sect. 5). We can see how $q$ changes in respect to the Gaussian case of $q = 1$ plotted as a dashed horizontal line. We note that $q(r) \propto -\log (r/r_{100})$ in the inner region, while it rises approximately as some power of $(r/r_{100})$ at larger radii, until it reaches a plateau.

This behavior of $q(r)$ can be described with the following 5-parameter analytical function:

$$q(r) = q_{\text{low}} - a \left( \frac{1 - y^b}{b \ln 10} + \log y \right),$$

$$y = \frac{\text{Min}(r/r_{100}, x_{\text{flat}})}{x_{\text{low}}}. \quad (27)$$

In equation (27), $a$ is the limit of $dq/\log r/r_{100}$ when $r \rightarrow 0$, $b$ is close to the power of $r/r_{100}$ in the rising portion of $q(r)$, while in equation (28) $x_{\text{low}} = r_{\text{low}}/r_{100}$ is where $q(r)$ is minimized at $q(r_{\text{low}}) = q_{\text{low}}$, and $x_{\text{flat}} = r_{\text{flat}}/r_{100}$, such that $q(r)$ reaches its plateau at $r_{\text{flat}}$.

The continuous lines in Fig. 2 show the result of this fit. Table 1 shows the values of the parameters obtained in the “stack”, “mean”, and “median” cases.

The striking feature of Fig. 2 is that the $q(r)$ profile for the stacked halo is very different from that of the mean and median $q(r)$ profiles of the individual halos. The mean and median profiles are computed with quite different methods (see Sect. 5), and yet are

![Figure 1](image1.png)

![Figure 2](image2.png)

Table 1. Parameters of the best-fit $q(r)$ function of equations 27 and 28 to the data of the 90 simulated halos

| Method | $a$  | $b$  | $x_{\text{low}}$ | $q_{\text{low}}$ | $x_{\text{flat}}$ |
|--------|------|------|------------------|------------------|------------------|
| Stack  | 0.247| 1.031| 0.552            | 0.924            | 3.010            |
| Mean   | 0.347| 0.970| 0.791            | 0.816            | 2.347            |
| Median | 0.221| 1.631| 1.028            | 0.835            | 2.512            |

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We now ask whether the best-fit \( q \)-Gaussian model provides an adequate representation of the simulated data. Fig. 3 displays the distribution of dimensionless normalized SDFs measured in the simulated halos in the stacked case as well as the best fit Gaussian and \( q \)-Gaussian SDFs. The dimensionless normalized SDF reaches its mode at low normalized velocity modulus near the virial radius, similar to what Hansen et al. (2006) found for the distribution function of the total speed. We see that within \( r \lesssim 2r_{100} \) (upper and middle panels of Fig. 3), the \( q \)-Gaussian does not describe well the data (nor does the Gaussian, since it is a special case of the \( q \)-Gaussian). For example, at \( r = 0.03r_{100} \), while the \( q \)-Gaussian with \( q = 1.17 \) fits the data better than the Gaussian (mostly in the low-end tail of \( G(u) \)), its predicted velocity distribution is too peaked, and it presents an important excess of very high \( u \) velocities. These two characteristics are present at all radii \( r \lesssim 2r_{100} \). On the other hand, at large radii \( r \gtrsim 2r_{100} \), the \( q \)-Gaussian model is an adequate representation of the data is rejected at over 95% confidence) at all radii. Similarly, the KS test fails at all radii for the mean sample.

Nevertheless, given that we have roughly equal numbers of particles per radial bin in both cases, we can illustrate the result of the KS test by plotting the maximum absolute difference in the CDFs. Figs. 4 and 5 show this maximum absolute difference of CDFs for the stacked halo and mean cases respectively. We see that, for both cases, the maximum difference between the CDF of the best-fit \( q \)-Gaussian and the data is less than 5%, and is smaller than the difference associated with the Gaussian distribution, except at some specific radii. In the stacked case, Fig. 4 shows that, for \( r/r_{100} \gtrsim 2 \), the maximum discrepancies in the CDFs of the simulation data and the best-fit \( q \)-Gaussian are \( \approx 10 \) times smaller than between the Gaussian and the data. For the mean of the individual 90 halos, Fig. 5 indicates that the discrepancy in CDFs at large radii is only a factor of 1.6 worse for the Gaussian CDF relative to the \( q \)-Gaussian one. We will see in Sect. 6.3 below that this smaller discrepancy for the \( q \)-Gaussian CDF (with the simulated data) in comparison with that of the Gaussian CDF is statistically significant. This shows that the \( q \)-Gaussian provides a better representation of the simulated data than is the Gaussian distribution.

Another way to qualitatively evaluate the goodness-of-fit is to observe the contours for 2-dimensional velocity distribution defined by the radial and tangential components. This is shown in Fig. 6. While the basic shapes of the contours of the models match...
fairly well the contours of the simulated data, there are differences, in the shapes, and, more strikingly, in the extent of the contours. At \( r = 0.03 \, r_{100} \), the contours of the best-fit \( q \)-Gaussian extend to much greater combinations of \( u_r \) and \( u_t \), i.e. to greater dimensionless normalized speeds. This is another sign that the \( q \)-Gaussian predicts much more very high velocity objects than is seen in the simulation. The better fit of the \( q = 1.17 \) \( q \)-Gaussian in comparison with the Gaussian is caused by the difference in the low-end tail (\( u < 0.8 \)) of \( G(u) \) for these two cases (see upper left panel of Fig. 5), which is difficult to distinguish in the contours. At \( r = 0.1 \, r_{100} \), the best-fit value of \( q \) is unity, so the contours of the Gaussian and \( q \)-Gaussian are identical. Moreover, they are quite similar to the contours extracted from the simulated data, indicating that the high-end of the VDFs are similar (see upper-right panel of Fig. 5). At \( r = 0.32 \) and \( 1 \, r_{100} \), the data contours are more extended in \( u_r \) than both the best-fit \( q \)-Gaussian and Gaussian cases. Finally, at \( r = 3.2 \) and \( 10 \, r_{100} \), despite the good fit to \( G(u) \) (lower panels of Fig. 5), the best-fit \( q \)-Gaussian shows an excess of very high values of \( u_r \).

### 6.3 Does the \( q \)-Gaussian model reproduce the data significantly better than the Gaussian?

We now ask whether the \( q \)-Gaussian provides a significantly better fit to the simulation data than does the Gaussian, taking into account the extra parameter involved in the former. For this, we evaluate the so-called Bayes factor between the two models, defined by the ratio of the evidences which, in turn, are defined as the integral of the likelihood \( L \), weighted by the prior, over the parameter space. In symbols, the Bayes factor between models \( q \) (for \( q \)-Gaussian) and \( G \) (for Gaussian) is

\[
B_{q:G} = \frac{\int \pi(q) L_q \, dq}{\int \pi(q) L_G \, dq},
\]

where \( \pi(q) \) is the prior associated to the parameter \( q \). As the Gaussian case does not involve any free parameter, the evidence is equal to the likelihood, as if its prior was a Dirac’s delta function.

According to Trotta (2008), values of \( \ln B_{12} \) smaller than 1.0 represent inconclusiveness about the preference of model 2 relative to the simpler model 1. If \( \ln B_{12} \approx 1.0 \), it represents weak evidence in favor of model 2. For \( \ln B_{12} \approx 2.5 \), we have moderate evidence, and for \( \ln B_{12} \gtrsim 5.0 \) we have strong evidence in favor of model 2. With these values in mind, we show in Fig. 7 the Bayes factor for the fit done in each radial bin, calculated with the approximation (proposed by Kass & Raftery 1995)

\[
\ln B_{12} \approx -\frac{1}{2} \Delta \text{BIC} = \ln L_2 - \ln L_1 - \frac{1}{2} (d_1 - d_2) \ln N,
\]

where \( d_i \) is the number of parameters of model \( i \) and \( N \) is the number of data points. BIC is the Bayes Information Criterion introduced by Schwarz (1978).

Fig. 7 shows that there is strong evidence that the \( q \)-Gaussian is preferable to the Gaussian distribution for the stacked halo (red line), except, of course, at the points where \( q \approx 1 \), where the addition of the parameter \( q \) is not necessary. For the mean of halos (smoother black line), we have strong evidence in favor of the \( q \)-Gaussian at all radii. However, with a finer grid we would necessarily find no evidence in favor of the \( q \)-Gaussian at \( r \approx 0.1 \, r_{100} \), where \( q \approx 1 \).
Bayes factor (Eq. [30]), obtained by fitting the non-Gaussianity parameter \( q \) to the stacked halo (red broken line) and to the mean of halos (continuous black smoother line). The y axis follows an arc sinh scaling. For the stack, there is strong evidence that the \( q \)-Gaussian model is a better representation than the Gaussian at all radii except \( r = 0.1 \) and 1.5 virial radii. For the mean of the fits, the evidence that the \( q \)-Gaussian is a better representation than the Gaussian is strong at all radii, but the coarse grid is missing \( r \approx 0.1 \), where \( q \simeq 1 \), hence there should be negative evidence (as in the case of the stacked halo).

Table 2. Parameters of the best-fit \( q(r) \) function of equations (27) and (28) to the stacked data split in 3 bins of halo mass

| \( (M)/M_\odot \) | \( a \) | \( b \) | \( x_{\text{low}} \) | \( q_{\text{low}} \) | \( x_{\text{flat}} \) |
|-----------------|-----|-----|-------------|---------|---------|
| \( 2.44 \times 10^{14} \) | 0.348 | 1.115 | 0.642 | 0.839 | 2.709 |
| \( 6.15 \times 10^{13} \) | 0.078 | 2.519 | 0.831 | 0.983 | 2.712 |
| \( 1.92 \times 10^{13} \) | 12.356 | 0.027 | 0.394 | 0.903 | 2.681 |

Notes: Values are for the 3 mass subsamples, whose mean is indicated in the first column.

6.4 Mass dependence

We now investigate a possible dependence of the parameter \( q \) on the mass of the halos. To do this, we divide our sample in the 3 mass subsamples already defined.

For each of these 3 subsamples, we performed the same fit procedure as before, for the stack and mean cases. Fig. 8 shows that the \( q(r) \) profile of the 3 stacked halos show reasonable differences at intermediate radii, but no trivial dependence on mass (like mass ordering in the height of curves) can be inferred. In the mean case (Fig. 9), the 3 subsamples produce very similar mean \( q(r) \) profiles.

7 CONCLUSIONS AND DISCUSSION

In this work, we propose an anisotropic version of the \( q \)-Gaussian velocity distribution, given by equation (13), or equivalently equations (15), (22) or (24), which is built on an an isotropic dimensionless normalized velocity field \( u \). Our VDF involves a single dimensionless normalized velocity, the modulus of \( u \), which can be written (for negligible streaming motions expected with the virial radius)

\[
u = \frac{\sqrt{v^2 + v^2_t/(1 - \beta)}}{\sigma_r},
\]

(31)

(see eq. [23]), and a single value of non-Gaussianity. In other words, at a given radius \( r \), the velocity distribution function is a function of \( \{2[E - \Phi(r)] + \beta(r)J^2/r^2\}/\sigma_r^2(r) \) instead of \( 2[E - \Phi(r)] \) for the isotropic distribution. We find this parametriza-
Table 3. Parameters of the mean best-fit $q(r)$ function of equations (27) and (28) to the data of individual halos, where the mean is in 3 bins of halo mass

| $\langle M \rangle / M_\odot$ | $a$     | $b$    | $x_{\text{low}}$ | $q_{\text{low}}$ | $x_{\text{flat}}$ |
|-----------------------------|---------|--------|------------------|------------------|------------------|
| $2.44 \times 10^{14}$      | 0.370   | 0.946  | 0.757            | 0.811            | 2.284            |
| $6.15 \times 10^{13}$      | 0.253   | 1.479  | 0.968            | 0.820            | 2.417            |
| $1.92 \times 10^{13}$      | 0.160   | 2.253  | 1.138            | 0.853            | 2.559            |

Notes: Values are for the 3 mass subsamples, whose mean is indicated in the first column.

Anisotropic $q$-Gaussian velocity distributions

mass much earlier and are more relaxed at $z = 0$. Could more relaxed regions lead to $q$ closer to unity? While there is considerable scatter ($\sigma_q \approx 0.1$) between the $q(r)$ profiles of individual halos (Fig. 1), there is little modulation with mass (Fig. 5), in the 1 dex cluster-mass range studied here.

One should not over-interpret the results of our $q$-Gaussian fits to small radii, such as $r = 0.03\ r_{100}$, close to the expected position of the Sun in the Milky Way’s halo. Indeed, despite the good fit, the best-fit $q$-Gaussian model strongly over-predicts the fraction of objects with velocities greater than $4\ \sigma$. If the threshold for direct dark matter detection is that high, then the best-fit $q$-Gaussian model will strongly over-predict the dark matter detection rate.

It is interesting to compare our results at $r = 0.03\ r_{100}$ to those obtained by other workers. Ling et al. (2010) found that the SDF of the dark matter in their hydrodynamical cosmological simulation was well fit by a $q$-Gaussian with $q = 0.70$ (after translating from their formula with exponent $q/(1 - q)$ to our exponent of $(1/(1 - q))$. Hence, their SDF is more truncated than the Gaussian, while our SDF with $q = 1.17$ (stack) or $1.14$ (mean) is less truncated than a Gaussian. Lisanti et al. (2011) also find SDFs that are more truncated at large velocities than the best-fit Gaussian (see their Fig. 3). In comparison, the top left panel of Fig. 5 also shows that the simulated SDF is flatter than the best-fit Gaussian model, and falls off faster at large $u$. The preference for the $q = 1.17$ Tsallis model comes from the low-end half of the SDF. Our result is robust, as none of our 90 halos has $q < 1$ at $r = 0.03\ r_{100}$ (Fig. 1) although a half-dozen halos have too few particles for reliable determinations at this radius.

The differences between our results and those of previous authors might be explained by the different behavior of the radial and tangential VDFs: at the small radius corresponding to the Solar radius in the Milky Way halo, Fairbairn & Schwetz (2009) and Kuhlen et al. (2010) find more truncated than Gaussian radial VDFs but more extended than Gaussian tangential VDFs. Our anisotropic VDF (and our SDF) involve some averaging between the radial and tangential components: (see eq. 31). It would be worthwhile to redo the analysis presented here using equation (1) on the separable DF that Wojtak et al. (2008) measured on simulated CDM halos.

Finally, the analysis presented here indicates that it is not optimal to assume Gaussian 3D velocities, as currently implemented in MAMPOSS. The inclusion of our anisotropic $q$-Gaussian VDF (eq. 13) into MAMPOSS would improve the mass / anisotropy modeling of this algorithm, for example using the shape of $q(r)$ of equations (27) and (29), and possibly forcing its parameters.

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