LOCAL MARTINGALES ASSOCIATED WITH SCHRAMM-LOEWNER EVOLUTIONS WITH INTERNAL SYMMETRY

SHINJI KOSHIDA

Abstract. We consider Schramm-Loewner evolutions (SLEs) with internal degrees of freedom that are associated with representations of affine Lie algebras, following group theoretical formulation of SLEs. We reconstruct the SLEs considered by Bettelheim et al. [Phys. Rev. Lett. 95, 251601 (2005)] and Alekseev et al. [Lett. Math. Phys. 97, 243-261 (2011)] in correlation function formulation. We also explicitly formulate stochastic differential equations on internal degrees of freedom for Heisenberg algebras and the affine $sl_2$. Our formulation enables us to find several local martingales associated with SLEs with internal degrees of freedom from computation on a representation of an affine Lie algebra. Indeed, we formulate local martingales associated with SLEs with internal degrees of freedom described by Heisenberg algebras and the affine $sl_2$. We also find an affine $sl_2$ symmetry of a space of SLE local martingales for the affine $sl_2$.

1. Introduction

Growth processes have been proven to give frameworks that describe various equilibrium and non-equilibrium phenomena exhibited in nature. Examples of such growth processes we consider in this paper are variants of Schramm-Loewner evolutions (SLEs), which were introduced by Schramm [Sch00] as the subsequent scaling limit of loop erased random walks and uniform spanning trees. Actually, Schramm defined two types of SLEs, chordal and radial, but in this paper we only treat chordal SLEs and simply call them SLEs. SLE is the solution of the following stochastic Loewner equation

\begin{equation}
\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t},
\end{equation}

on a formal power series $g_t(z) \in \mathbb{C}[[z^{-1}]]$, with the initial condition $g_0(z) = z$. Here $B_t$ is the standard Brownian motion with values in $\mathbb{R}$ starting from the origin and $\kappa$ is a positive number. The SLE specified by this number $\kappa$ is denoted as SLE($\kappa$). Though we have regarded $g_t(z)$ as just a formal power series, it becomes a uniformization map of a simply connected domain. Namely, for each realization of $g_t(z)$, we can take a subset $K_t \subset \mathbb{H}$ called a hull such that $g_t$ becomes a biholomorphic map $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$. Moreover, for each realization, the family $\{K_t\}_{t \geq 0}$ of hulls parametrized by time is increasing, i.e., if $t < s$, $K_t \subset K_s$ holds. When we investigate an evolution of hulls in more detail, we find that it is governed by an evolution of the tip $\gamma_t$ in the upper half plane, which is captured in the following manner. At the initial time $t = 0$, the uniformization map $g_0$ is the identity, which means that the hull $K_0$ is empty. At a small time $t = t_1$, the corresponding hull $K_{t_1}$ is a slit in the upper half plane, one of whose end points is on the origin. Then we name the other end point $\gamma_{t_1}$ and call it the tip at $t = t_1$. For
small time, the hull is simply the trace of the tip, but when time evolves further, the
trace may touch itself or the real axis. If such an event occurs, the area enclosed by
the trace and the real axis is absorbed in the hull. This is a way of identifying the
evolution of hulls with the evolution of the tip. In this manner, SLEs give a probability
measure on the space of curves in the upper half plane, which is called the SLE(κ)-
measure. The SLE(κ)-measure has been shown to describe an interface of clusters in
several critical systems in two dimensions including the critical percolation [Smi01] and
the Ising model at criticality [CDCH+14]. After their introduction, many aspects of
SLEs have been clarified [Law04, RS05, LSW01a, LSW01b, LSW02b, LSW02a, Wer03].

There is another framework to investigate two-dimensional critical systems. It is
two-dimensional conformal field theory (CFT), [BPZ84] which has been one of the most
powerful tools in a wide variety of fields ranging from condensed matter physics to string
theory, and in mathematics. A milestone of CFT prediction on a critical system is
Cardy’s formula [Car92] which gives the crossing probability for the critical percolation
in two dimensions from computation of a correlation function in CFT. Cardy’s formula
was proven by Smirnov [Smi01] to be a theorem, while the derivation by Cardy has not
been verified.

Since SLE and CFT are different frameworks that describe the same phenomena, they
are expected to be connected to each other in some sense. The connection between SLE
and CFT has been studied under the name of SLE/CFT correspondence from various
points of view. Studies by Friedrich, Werner, Kalkkinen, and Kontsevich, [FW03, FK04,
Fri04, Kon03] proposed that the SLE(κ)-measure was constructed as a section of the
determinant bundle over the moduli space of Riemann surfaces based on observation
of transformation of the correlation function of CFT under conditioning. In a more-
recent approach by Dubédat [Dub15b, Dub15a], the SLE(κ)-measure was constructed
using a localization technique, and its partition function was identified with a highest
weight vector of a representation of Virasoro algebra. A significant development was
the group theoretical formulation of SLEs by Bauer and Bernard [BB02, BB03a, BB03b],
which proposed an elegant way of constructing local martingales associated with SLEs
(SLE local martingales for short) from a representation of the Virasoro algebra. We will
review this formulation in Sect.2.

The notion of SLE has been generalized to several directions along the SLE/CFT
correspondence. Examples include the notion of multiple SLEs [BBK05] and SLEs
corresponding to logarithmic CFT, [Ras04a, MARR04] the \( \mathcal{N} = 1 \) superconformal alge-
bra [Ras04b].

We note that there are other directions of generalization of SLEs. An example is the
notion of SLE(κ, ρ), [LSW03] which is obtained by replacing the Brownian motion in
the stochastic Loewner equation by a Bessel process. CFT interpretation of SLE(κ, ρ)
was obtained by Cardy [Car06] and Kytölä [Kyt06]. Several variants of SLEs associated
with representation of the Virasoro algebra were unified by Kytölä [Kyt07].

CFTs that are associated with representation theory of affine Lie algebras are known
as Wess-Zumino-Witten (WZW) theories [WZ71, Wit84, KZ84]. SLEs corresponding to
WZW theories have been considered by Bettelheim et al. [BGL05] and Alekseev et al.
[AB11] in correlation function formulation and by Rasmussen [Ras07] for the \( \mathfrak{sl}_2 \)
and the present author [Kos17] for simple Lie algebras in group theoretical formulation.
Note that the group theoretical formulation of SLEs corresponding to WZW theory first
given by Rasmussen [Ras07] did not contain the original SLE as a part, and the
current author [Kos17] presented an idea for improving it to recover the original SLE as the geometric part and the result given by correlation function formulation. We will now review the approach in correlation function formulation [BGLW05, ABI11] of SLEs corresponding to WZW theory. Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra and \( k \in \mathbb{C} \) be a level. They start from an object

\[
M_t = \frac{\langle \phi_{\lambda}(z) \phi_{\lambda}(z_1) \cdots \phi_{\lambda}(z_N) \phi_{\lambda}(\bar{z}_1) \cdots \phi_{\lambda}(\bar{z}_N) \phi_{\lambda}(\infty) \rangle^g}{\langle \phi_{\lambda}(z) \phi_{\lambda}(\infty) \rangle^g}.
\]

(1.2)

Here \( \phi_{\lambda} \) is the primary field corresponding to a weight \( \lambda \), with the convention that \( \lambda^* \) denotes the dual representation of \( \lambda \). The points \( z_1, \ldots, z_N \) are put on the upper half plane and \( z_i \) is the tip of the SLE slit defined by \( z_i = \rho_i^{-1}(0) \), where \( \rho_i(z) = g_i(z) + B_t \) satisfies \( d\rho_i(z) = \frac{2i}{\kappa} \rho_i(z) - dB_t \) with \( B_t \) being the Brownian motion of variance \( \kappa \). The numerator of Eq. (1.2) takes a value in the \( \mathfrak{g} \)-invariant subspace of \( L(\lambda) \otimes L(\lambda_1) \otimes \cdots \otimes L(\lambda)^* \), where \( L(\lambda) \) is the irreducible representation of \( \mathfrak{g} \) of highest weight \( \lambda \). The denominator of Eq. (1.2) takes a value in the \( \mathfrak{g} \)-invariant subspace of \( L(\lambda) \otimes L(\lambda)^* \), which is one dimensional due to Schur’s Lemma.

Since a primary field of a WZW theory has internal degrees of freedom, random evolution of a primary field involves ones along the internal degrees of freedom. Studies by Ref. [BGLW05] and Ref. [ABI11] proposed the following stochastic differential equation (SDE):

\[
d\phi_{\lambda}(w_i) = \mathcal{G}_i \phi_{\lambda}(w_i),
\]

where \( w_i = \rho_i(z_i) \) and

\[
\mathcal{G}_i = dt \left( \frac{2}{w_i} \partial_{w_i} - \frac{\tau C_i}{2w_i^2} \right) - dB_i \partial_{w_i} + \left( \frac{1}{w_i} \sum_a d\theta^a t^a_i + \frac{\tau}{2w_i^2} \sum_a t^a_i t^a_i dt \right).
\]

Here \( \{\theta^a\} \) is a basis of \( \mathfrak{g} \) and \( \{t^a_i\} \) are their representation matrices on \( L(\lambda_i) \). Random processes \( \theta^a \) are independent Brownian motions of variance \( \tau \). The number \( C_i \) is the value of the Casimir on the representation \( L(\lambda_i) \).

The claim by Ref. [BGLW05] and Ref. [ABI11] is that the random process \( M_t \) is a local martingale for a certain choice of \( \kappa \) and \( \tau \), and Eq. (1.3) is a generalization of the stochastic Loewner equation so as to correspond to a WZW theory. Their formulation has been extended to multiple SLEs [Sak13] and to coset WZW theories [Naz12, Fuk17].

The motivation in the present work is to better understand the previous studies on SLEs corresponding to WZW theory. In their formulations, the SDEs along internal degrees of freedom appear to be \textit{ad hoc}, random processes along internal degrees of freedom are not constructed in a concrete way, and thus local martingales that are associated with SLEs corresponding to WZW theory are hard to formulate. These issues are addressed in this paper. In particular, we will see that SDEs on internal degrees of freedom arise naturally in the group theoretical formulation. We also construct a random process along internal degrees of freedom for Heisenberg algebras and the affine \( sl_2 \), and formulate several local martingales associated with them.

This paper is organized as follows. In Sect. 2 we review the group theoretical formulation of SLEs originally proposed by Bauer and Bernard. In Sect. 3 we recall the notion of affine Lie algebras associated with finite-dimensional Lie algebras that are simple or commutative and their representation theory. In Sect. 4 we introduce an infinite-dimensional Lie group, which becomes the target space of random
processes generating SLEs corresponding to representations of affine Lie algebras. In Sect. 5 we construct a random process on the infinite-dimensional Lie group assuming existence of an annihilating operator of a highest weight vector. We also formulate SDEs on internal degrees of freedom in the case when the underlying Lie algebra is commutative and $\mathfrak{sl}_2$. In Sect. 6 we discuss an annihilating operator of a highest weight vector, the existence of which is assumed in Sect. 3. In Sect. 7 as an application of the construction of SDEs in Sect. 5 we compute several local martingales associated with the solutions. In Sect. 8 we clarify the $\mathfrak{sl}_2$-module structure on a space of SLE local martingales for $\mathfrak{sl}_2$. Then we present some conclusions. In Appendix A we recall the notion of vertex operator algebra (VOA), which is useful in this paper. In Appendix B we review an Itô process on a Lie group. Appendix C contains computational details that are referred to in Sect. 5. In Appendix D we show a detailed derivation of operators that define the action of $\mathfrak{sl}_2$ on a space of local martingales referred to in Sect. 8.

2. Group theoretical formulation of SLEs

In this section, we recall the group theoretical formulation of SLEs corresponding to the Virasoro algebra originally proposed by Bauer and Bernard. [BB02, BB03a, BB03b] The main purpose of this section is to introduce the infinite-dimensional Lie group $\text{Aut}_+\mathcal{O}$ and a random process on it.

2.1. Virasoro algebra and its representations. Virasoro algebra is an infinite-dimensional Lie algebra $\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$ with Lie brackets defined by

\begin{equation}
[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C,
\end{equation}

\begin{equation}
[C, \text{Vir}] = \{0\}.
\end{equation}

We only consider highest weight representations of the Virasoro algebra that are constructed in the following manner. Let us decompose the Virasoro algebra into subalgebras $\text{Vir} = \text{Vir}_{>0} \oplus \text{Vir}_0 \oplus \text{Vir}_{<0}$, where $\text{Vir}_0 = \mathbb{C}L_0 \oplus \mathbb{C}C$ and $\text{Vir}_{<0} = \bigoplus_{n > 0} \mathbb{C}L_n$. We also set $\text{Vir}_{\geq 0} = \text{Vir}_{>0} \oplus \text{Vir}_{=0}$. For a pair $(c, h) \in \mathbb{C}^2$, let $\mathcal{C}_{(c,h)} = \mathbb{C}1_{(c,h)}$ be a one-dimensional representation of $\text{Vir}_{\geq 0}$ on which $C$ and $L_0$ act as multiplication by $c$ and $h$, respectively. The highest weight Verma module $M(c, h)$ of highest weight $(c, h)$ is defined by induction $M(c, h) = U(\text{Vir}) \otimes _{U(\text{Vir}_{<0})} \mathcal{C}_{(c,h)}$, which is isomorphic to $U(\text{Vir}_{\geq 0}) \otimes \mathcal{C}_{(c,h)}$ as a vector space or a $\text{Vir}_{\geq 0}$-module. The numbers $c$ and $h$ in the highest weight are called the central charge and the conformal weight of the highest weight Verma module $M(c, h)$, respectively. Since we will only treat highest weight representations, we call a highest weight Verma module simply a Verma module. The highest weight vector $1 \otimes 1_{(c,h)}$ is denoted by $(c, h)$. It is clear by construction that a Verma module $M(c, h)$ decomposes into the direct sum of eigenspaces of $L_0$ so that $M(c, h) = \bigoplus_{n \in \mathbb{Z}_>0} M(c, h)_{h+n}$, where we have defined $M(c, h)_\lambda = \{ v \in M(c, h) | L_0 v = \lambda v \}$ for $\lambda \in \mathbb{C}$.

For a generic highest weight $(c, h)$, the corresponding Verma module is irreducible, but for a specific highest weight, it is not. Then we denote the irreducible quotient of the Verma module by $L(c, h)$, and call an element in $J(c, h) := \ker(M(c, h) \rightarrow L(c, h))$ a null vector.

Among other irreducible modules, that of the highest weight $(c, 0)$ denoted by $L(c, 0)$ above has a special feature, which is that it carries the structure of a VOA. We simply denote this VOA by $L_c$ and call it the Virasoro VOA of central charge $c$. An exposition
of VOA structure on $L_c$ is presented in Appendix A and we shall sketch the argument here. The vacuum vector is the highest weight vector $|0\rangle = |c,0\rangle$, and it is generated by a conformal vector $L_{-2}|0\rangle$ that is transferred to the Virasoro field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ under the state-field correspondence map. Simple modules over the Virasoro VOA $L_c$ are realized as highest weight irreducible representations of the same central charge. A nondegenerate bilinear form $\langle \cdot | \cdot \rangle$ on an $L_c$-module $M$ is invariant if it satisfies

$$\langle Y(a,z)u|v\rangle = \langle u|Y(e^{zL_1}(-z^{-2})L_0 a, z^{-1})v\rangle$$

for $a \in L_c$ and $u, v \in M$. This condition is rephrased as $\langle L_n u|v\rangle = \langle u|L_{-n}v\rangle$ and $\langle Cu|v\rangle = \langle u|Cv\rangle$ for $u, v \in M$, which specify a bilinear form $\langle \cdot | \cdot \rangle$ on $M$. It is well known that such a bilinear form uniquely exists under the normalization $\langle c, h|c, h \rangle = 1$.

2.2. Conformal transformation. Here we review how to implement a conformal transformation as an operator on a VOA or its module following Frenkel and Ben-Zvi. \cite{FBZ04}

Let $O = \mathbb{C}[[w]] = \lim w^\mathbb{N} \mathbb{C}$ be a complete topological $\mathbb{C}$-algebra and $D = \text{Spec} O$ be the formal disk. A continuous automorphism $\rho$ of $O$ is identified with the image of the topological generator $w \in O$ by the same automorphism $\rho$. Under this identification, the group $\text{Aut} O$ of continuous automorphisms of $O$ is realized as

$$\text{Aut} O \simeq \{ a_1 w + a_2 w^2 + \cdots | a_i \in \mathbb{C}, a_i \in O, i \geq 2 \}.$$  

Indeed, a nonzero constant term is prohibited to preserve the algebra $O$, and $a_1 \neq 0$ is required for the existence of the inverse. The group law is defined by $(\rho \ast \mu)(w) = \mu(\rho(w))$ for $\rho, \mu \in \text{Aut} O$. The purpose of this subsection is to define a representation of this group on a VOA or its modules, which is significant in application to the theory of SLEs.

It is shown that the Lie algebra of $\text{Aut} O$ is one of vector fields $\text{Der}_0 O = w\mathbb{C}[[w]]\partial_w$. The same Lie algebra is also constructed as a completion of a Lie subalgebra $\text{Vir}_{\leq 0} = \bigoplus_{n=1}^{\infty} \mathbb{C} L_n$ of the Virasoro algebra. Since a subalgebra $\text{Vir}_{\geq m} = \bigoplus_{n\geq m} \mathbb{C} L_n$ in $\text{Vir}_{\leq 0}$ is an ideal, the quotient $\text{Vir}_{\geq 0}/\text{Vir}_{\geq m}$ carries a Lie algebra structure; moreover, we have a family of projections $\text{Vir}_{\geq 0}/\text{Vir}_{\geq m} \to \text{Vir}_{\geq 0}/\text{Vir}_{\geq n}$ for $m > n$. The projective limit $\lim \text{Vir}_{\geq 0}/\text{Vir}_{\geq m}$ of this projective system of Lie algebras is the desired Lie algebra $\text{Der}_0 O$. Since, for an arbitrary vector $v$ in a VOA, $V$ or its module $M$, $L_nv = 0$ for $n \gg 0$, so we have a well-defined action of $\text{Der}_0 O$ on $V$ and $M$.

There is a significant subgroup $\text{Aut}_c O$ of $\text{Aut} O$ that is described as $\text{Aut}_c O \simeq \{ w + a_2 w^2 + \cdots | a_i \in \mathbb{C}, i \geq 2 \}$. It is shown that the Lie algebra of this subgroup is $\text{Der}_c O = w^2 \mathbb{C}[[w]]\partial_w$, which is a Lie subalgebra of $\text{Der}_0 O$.

We shall exponentiate the action of the Lie algebra $\text{Der}_0 O$ to the action of the Lie group $\text{Aut} O$. This is possible if the $L_n$ for $n > 1$ act locally nilpotently and $L_0$ is diagonalizable with integer eigenvalues, the former of which automatically holds for a highest weight representation, and the latter of which is true if the conformal weight of the highest weight is an integer. On such a highest weight representation of the Virasoro algebra, we construct the linear operator $R(\rho)$ for $\rho \in \text{Aut} O$ that defines a representation of $\text{Aut} O$. For an automorphism $\rho \in \text{Aut} O$, we uniquely find $v_i, i \geq 0$, such that

$$\rho(w) = \exp \left( \sum_{i>0} v_i w^{i+1} \partial_w \right) v_0 w \partial_w \cdot w.$$  

Here the exponentiation of the Euler vector field is defined by $v_0 w \partial_w \cdot w = v_0$. The above expression of $\rho$ is a specification of its action on $K = \mathbb{C}[[w]]$ defined by $(\rho F)(w) = f(\rho(w))$ for $F(w) \in K$, where the group law of invertible operators on $K$ is defined
by composition. The first few values of \( v_i \) for a given \( \rho \) are computed by comparing coefficients of each powers of \( w \) so that
\[
v_0 = \rho'(0), \quad v_1 = \frac{1}{2} \frac{\rho''(0)}{\rho'(0)}, \quad v_2 = \frac{1}{6} \frac{\rho'''(0)}{\rho'(0)} - \frac{1}{4} \left( \frac{\rho''(0)}{\rho'(0)} \right)^2, \quad \ldots.
\]
Let \( V \) be a VOA. Then for an automorphism \( \rho \in \text{Aut} \mathcal{O} \), the following operator is well defined in \( \text{End}(V) \):
\[
R(\rho) = \exp \left( - \sum_{i \geq 0} v_i L_i \right) v_0^{-L_0},
\]
and satisfies \( R(\rho) R(\mu) = R(\rho * \mu) \). In the case when \( \rho \in \text{Aut} \mathcal{O} \), we have \( v_0 = 1 \), which means that \( R(\rho) \) can also be regarded as an operator on a \( V \)-module.

We investigate the behavior of a field \( Y(A, z) \) on a VOA \( V \) under the adjoint action by \( R(\rho) \). Let \( L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) be the Virasoro field. Then,
\[
[L(z), Y(A, w)] = \sum_{m \geq -1} Y(L_m A, w) \partial_w^{(m+1)} \delta(z-w),
\]
which implies
\[
[L_n, Y(A, w)] = \sum_{m \geq -1} \left( \frac{n+1}{m+1} \right) Y(L_m A, w) w^{-n-m}.
\]
For \( v = -\sum_{n \in \mathbb{Z}} v_n L_n \) such that \( v_n = 0 \) for \( n \ll 0 \), then
\[
[v, Y(A, w)] = - \sum_{m \geq -1} \left( \partial_w^{(m+1)} v(w) \right) Y(L_m A, w),
\]
where \( v(w) = \sum_{n \in \mathbb{Z}} v_n w^{n+1} \).

**Proposition 2.1.** For \( A \in V \) and \( \rho \in \text{Aut} \mathcal{O} \),
\[
Y(A, w) = R(\rho) Y(R(\rho_w)^{-1} A, \rho(w)) R(\rho)^{-1}.
\]
Here \( \rho_w(t) = \rho(w + t) - \rho(w) \).

**Proof.** We denote by \( \text{Fie}(V) \) the space of fields on \( V \). The state field correspondence map \( Y(-, w) \) is regarded as an element in \( \text{Hom}(V, \text{Fie}(V)) \). For an automorphism \( \rho \in \text{Aut} \mathcal{O} \), we define an endomorphism \( T_\rho \) on \( \text{Hom}(V, \text{Fie}(V)) \) by
\[
(T_\rho \cdot X)(A, w) := R(\rho) X(R(\rho_w)^{-1} A, \rho(w)) R(\rho)^{-1}
\]
for \( X \in \text{Hom}(V, \text{Fie}(V)) \) and \( A \in V \). Then this assignment \( \rho \mapsto T_\rho \) is a group homomorphism. Indeed,
\[
(T_\rho \cdot (T_\mu \cdot X))(A, w)
= R(\rho) (T_\mu \cdot X)(R(\rho_w)^{-1} A, \rho(w)) R(\rho)^{-1}
= R(\rho) R(\mu) X(R(\mu_{\rho(w)})^{-1} R(\rho_w)^{-1} A, \mu(\rho(w))) R(\mu)^{-1} R(\rho)^{-1}.
\]
Note that
\[
(\rho_w * \mu_{\rho(w)})(t) = \mu_{\rho(w)}(\rho_w(t)) - \mu(\rho(w) + \rho_w(t)) - \mu(\rho(w))
= \mu(\rho(w) + \rho(w + t) - \rho(w)) - \mu(\rho(w))
= (\rho * \mu)_w(t).
\]
to obtain
\[(T_\rho \cdot (T_\mu \cdot X))(A, w) = (T_{\rho \mu} \cdot X)(A, w) .\]

Since the exponential map $\text{Der}_0 \mathcal{O} \to \text{Aut}\mathcal{O}$ is surjective, we can assume $\rho$ to be infinitesimal. For an infinitesimal transformation $\rho(w) = w + \epsilon v(w) + o(\epsilon)$ with $v(w) = \sum_{n \geq 0} v_n w^{n+1}$,
\[(2.13) \quad R(\rho) = \text{Id} + \epsilon v + o(\epsilon),\]
where $v = -\sum_{n \geq 0} v_n L_n$. The associated transformation $\rho_w(t)$ is approximated up to a linear order of $\epsilon$ by
\[
\rho_w(t) = \rho(w + t) - \rho(w) = w + t + \epsilon v(w + t) - w - \epsilon v(w) + o(\epsilon) = t + \epsilon \sum_{m \geq 0} \partial^{(m+1)} v(w) t^m + o(\epsilon).
\]
Thus $R(\rho_w)^{-1}$ becomes
\[(2.14) \quad R(\rho_w)^{-1} = \text{Id} + \epsilon \sum_{n \geq 0} \partial^{(n+1)} v(w) L_n + o(\epsilon).
\]

We now show that the state-field correspondence map $Y(-, w)$ is fixed under the action of $T_\rho$ up to a linear order of $\epsilon$:
\[(T_\rho \cdot Y)(A, w) = (\text{Id} + \epsilon v) Y \left( \text{Id} + \epsilon \sum_{n \geq 0} \partial^{(n+1)} v(w) L_n \right) A, w + \epsilon v(w) \right) (\text{Id} - \epsilon v) = Y(A, w) + \epsilon \left( [v, Y(A, w)] + v(w) \partial Y(A, w) + \sum_{n \geq 0} \partial^{(n+1)} v(w) Y(L_n A, w) \right) = Y(A, w).
\]

**Corollary 2.2.** Let $A \in V$ be a primary vector of conformal weight $h$, i.e., it satisfies $L_n A = 0$ for $n > 0$ and $L_0 A = hA$. For an automorphism $\rho \in \text{Aut}\mathcal{O}$,
\[(2.15) \quad Y(A, w) = R(\rho) Y(A, \rho(w)) R(\rho)^{-1} (\rho'(w))^h .\]

**Proof.** For a primary vector $A$ of conformal weight $h$, the one-dimensional space $\mathbb{C}A$ is preserved by the operator $R(\rho_w)$, where $R(\rho_w)$ is given by
\[(2.16) \quad R(\rho_w) = \exp \left( - \sum_{j > 0} v_j(w) L_j \right) v_0(w)^{-L_0} .\]
with $v_j(w)$ being chosen so that
\[(2.17) \quad \rho_w(t) = \exp \left( \sum_{j > 0} v_j(w) t^{j+1} \partial_t \right) v_0(w) \partial_t \cdot t .\]

Since $A$ is primary, the nontrivial effect comes from the action by $L_0$, thus we have $R(\rho_w) A = v_0(w)^{-h} A$, where $v_0(w)$ is computed as $v_0(w) = \partial_0 \rho_w(t = 0) = \rho'(w)$, which implies that $R(\rho_w)^{-1} A = (\rho'(w))^h A$.

One important field that is not primary is the Virasoro field $L(w) = Y(L_{-2}|0), w)$, which transforms as follows.
The infinite series in $z\mathbb{C}$ is the formal neighborhood at infinity, and have to reformulate all the components so that they are associated with the coordinate $z$ at 0. While an automorphism $\rho$ sends $\rho(w) = a_1 w + a_2 w^2 + \cdots$, the same automorphism sends $z$ to $1/\rho(1/z)$. If we expand the image in $z\mathbb{C}$, we can also identify the group $\text{Aut}_z$ with

$$L(w) = R(\rho)L(\rho(w))R(\rho)^{-1}(\rho'(w))^2 + \frac{c}{12}(S\rho)(w).$$

Here $c \in \mathbb{C}$ is the central charge and $(S\rho)(w)$ is the Schwarzian derivative defined by

$$(S\rho)(w) = \frac{\rho'''(w)}{\rho'(w)} - \frac{3}{2} \left( \frac{\rho''(w)}{\rho'(w)} \right)^2.$$ 

Proof. It is clear that the space $\mathbb{C}L_{-2}[0] \oplus \mathbb{C}[0]$ is preserved by the operator $R(\rho_w)$, thus we first compute the inverse $R(\rho_w)^{-1}$ on this space. Let $v_j(w) \in \mathbb{C}[[w]]$ be chosen so that

$$\rho_w(t) = \exp \left( \sum_{j>0} v_j(w) t^{j-1} \partial_t \right) v_0(w)^{\partial_t} \cdot t,$$

then $R(\rho_w)$ can be expressed as

$$R(\rho_w) = \exp \left( - \sum_{j>0} v_j(w) L_j \right) v_0(w)^{-L_0}.$$ 

The matrix form of this operator on $\mathbb{C}L_{-2}[0] \oplus \mathbb{C}[0]$ is expressed in this basis

$$R(\rho_z) = \begin{pmatrix} v_0(w)^{-2} & 0 \\ -v_0(w)^{-2}v_2(w) & 1 \end{pmatrix},$$

and its inverse is

$$R(\rho_w)^{-1} = \begin{pmatrix} v_0(w)^2 & 0 \\ \frac{v_2}{2}v_2(w) & 1 \end{pmatrix} = \begin{pmatrix} (\rho'(w))^2 & 0 \\ \frac{1}{\rho'(w)(S\rho)(w)} & 1 \end{pmatrix},$$

which implies the desired result. \qed

In application to the theory of SLE, we regard the formal disk introduced here as the formal neighborhood at infinity, and have to reformulate all the components so that they are associated with the coordinate $z = \frac{1}{w}$ at 0. While an automorphism $\rho$ sends $\rho(w)$ to $a_1 w + a_2 w^2 + \cdots$, the same automorphism sends $z$ to $1/\rho(1/z)$. If we expand the image in $z\mathbb{C}$, we can also identify the group $\text{Aut}_z$ with

$$\text{Aut}_z \cong \{ b_1 z + b_0 + b_{-1} z^{-1} + \cdots | b_i \in \mathbb{C}, i \leq 0 \}$$

The infinite series in $z\mathbb{C}$ that is identified with an automorphism $\rho$ will be denoted by $\rho(z)$. In the following, we regard formal variables $z$ and $w$ as formal coordinates at 0 and infinity, respectively, and $\rho(z)$ and $\rho(w)$ as infinite series identified with an automorphism $\rho$ via Eq. (2.23) and Eq. (2.24), respectively.

Under realization Eq. (2.23) of the group $\text{Aut}_0$, its subgroup $\text{Aut}_z$ consists of formal series $z + b_0 + b_{-1} z^{-1} + \cdots$ with $b_i \in \mathbb{C}$ for $i \leq 0$, and Lie algebras are realized as $\text{Der}_z = \mathbb{C}[[z^{-1}]] \partial_z$ and $\text{Der}_0 = z\mathbb{C}[[z^{-1}]] \partial_z$.

Since the Lie algebra $\text{Der}_0 = z\mathbb{C}[[z^{-1}]] \partial_z$ consists of vector fields, the coefficients of which are Laurent series in $z^{-1}$, it cannot act on a VOA $V$ or its module $M$ by assignment $-z^{n+1} \partial_z \to L_n$ for $n \leq 0$. Nevertheless, we can define well-defined operators that represent the Lie algebra $\text{Der}_0$ on the completion of the vector space. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be the $\mathbb{Z}$-gradation of a $V$-module $M$. Then we define its formal completion by $
\n\nM = \bigoplus_{n \in \mathbb{Z}} M_n$. Recall that $M_n = 0$ for sufficiently small $n$. Moreover this action of
 Derivative $\partial_t \mathcal{O}$ is exponentiated as a representation of $\text{Aut} \mathcal{O}$ on $\mathcal{V}$, and a representation of its subgroup $\text{Aut}_+ \mathcal{O}$ on $\mathcal{M}$.

For a given $\rho \in \text{Aut} \mathcal{O}$, we can uniquely find numbers $v_i$ ($i \leq 0$) that satisfy

$$
\exp \left( \sum_{j=0}^\infty v_j z^{j+1} \partial_z \right) v_0^{-z} z = \rho(z).
$$

Then the operator $Q(\rho)$ defined by

$$
Q(\rho) = \exp \left( - \sum_{j=0}^\infty v_j L_j \right) v_0^{-L_0}
$$

is a well-defined one on $\mathcal{V}$ and a representation of $\text{Aut} \mathcal{O}$ can be defined. Indeed, the part $v_0^{-L_0}$ behaves as multiplication by $v_0^{-n}$ when restricted on $V_n$, and $L_j$ with $j < 0$ strictly raises the degree, while the $\mathbb{Z}$-gradation on $V$ is bounded from below.

We investigate the covariance property of a field $Y(A, z)$ under the adjoint action by $Q(\rho)$. For $v(z) = \sum_{n \in \mathbb{Z}} v_n z^{n+1} \in \mathcal{C}((z^{-1}))$, 

$$
[v, Y(A, z)] = \sum_{m \geq -1} \partial^{(m+1)} v(z) Y(L_m A, z),
$$

with $v = -\sum_{n \in \mathbb{Z}} v_n L_n$, but here the both sides belong to $\text{End} \mathcal{V}[z, z^{-1}]$.

**Proposition 2.4.** For $A \in V$ and $\rho \in \text{Aut} \mathcal{O}$,

$$
Y(A, z) = Q(\rho) Y(R(\rho_z)^{-1} A, \rho(z)) Q(\rho)^{-1}.
$$

On a $V$-module on which eigenvalues of $L_0$ are not integers, the whole group $\text{Aut} \mathcal{O}$ cannot act, while its subgroup $\text{Aut}_+ \mathcal{O}$ can act. In application to SLEs, this subgroup is sufficient since each realization of the SLE is always normalized so that its expansion around infinity begins from $z$ with the coefficient unity.

For an operator $T$ on a VOA $V$, we define its adjoint operator $T^*$ by the property that $(Tu|v) = (u|T^*v)$ for $u, v \in V$. In this terminology, the operator $Q(\rho)$ defined above is the inverse of the adjoint operator of $R(\rho)$, while $Q(\rho)$ is not an operator on a VOA but on its formal completion.

### 2.3. Appearance of SDEs.

A fundamental object in the group theoretical formulation of SLEs is a random process $\rho_t$ on the infinite-dimensional Lie group $\text{Aut}_+ \mathcal{O}$. A random process on a Lie group induces one on the space of operators on a representation space. Let us take $(\gamma, \mathcal{K} = \mathcal{C}((z^{-1})))$ as a representation of $\text{Aut}_+ \mathcal{O}$ defined by $(\gamma(\rho) F)(z) = F(\rho(z))$. Following the description of a random process on a Lie group presented in Appendix E, we assume that the induced random process on $\text{Aut} \mathcal{K}$ satisfies the SDE

$$
\gamma(\rho_t)^{-1} d \gamma(\rho_t) = \left( 2z^{-1} \partial_z + \frac{\kappa}{2} \partial_z^2 \right) dt - \partial_z dB_t
$$

under the initial condition $\gamma(\rho_0) = \text{Id}$. Here $B_t$ is the $\mathbb{R}$-valued Brownian motion of variance $\kappa$ that starts from the origin. Then we observe that $\gamma(\rho_t) z = \rho_t(z)$ satisfies the SDE

$$
d\rho_t(z) = \frac{2}{\rho_t(z)} \, dt - dB_t
$$
under the initial condition $\rho_0(z) = z$. If we introduce $g_t(z) = \rho_t(z) + B_t$, we find that $g_t(z)$ satisfies the stochastic Loewner equation

\begin{equation}
\frac{d}{dt}g_t(z) = \frac{2}{g_t(z)} - B_t.
\end{equation}

Moreover, since $B_0 = 0$, we have $g_0(z) = z$. Thus $g_t(z)$ is identified with the SLE($\kappa$).

We have just derived the stochastic Loewner equation from a random process on the Lie group $\text{Aut}_c \mathcal{O}$. This manner of formulation enables us to obtain several local martingales associated with SLE. Let us consider the object $Q(\rho_t) |c, h\rangle$, which is regarded as a random process on $L(c, h)$, of which the increment is

\begin{equation}
d(Q(\rho_t) |c, h\rangle) = Q(\rho_t) \left( -2L_{-2} + \frac{\kappa}{2}L_{-1}^2 \right) |c, h\rangle dt + L_{-1} |c, h\rangle dB_t.
\end{equation}

Thus if the vector $\chi = \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right) |c, h\rangle$ is a null vector in the Verma module $M(c, h)$, the random process $Q(\rho_t) |c, h\rangle$ is a local martingale. Notice that $\chi$ is a null vector if and only if it is a singular vector, conditions for which are that we have $c = 1 - \frac{3(\kappa-4)^2}{2\kappa}$ and $h = \frac{6-\kappa}{2\kappa}$. Thus for such a choice of $(c, h)$, the random process $Q(\rho_t) |c, h\rangle$ in $L(c, h)$ is a local martingale, and produces several local martingales associated with SLE. An example is given by $\langle c, h | L(z) Q(\rho_t) |c, h\rangle$, where $L(z)$ is the Virasoro field on $L(c, h)$. From Prop. 2.4 and the fact that the dual of the highest weight vector $|c, h\rangle$ is invariant under the right action by $Q(\rho)$, we find that

\begin{equation}
\langle c, h | L(z) Q(\rho_t) |c, h\rangle = h \left( \frac{\rho'_t(z)}{\rho_t(z)} \right)^2 + \frac{c}{12} (S\rho_t)(z)
\end{equation}

is a local martingale. We can show that such a quantity is indeed a local martingale by a standard Ito calculus, but the group theoretical formulation of SLE in the sense of Bauer and Bernard further clarifies its theoretical origin.

Since the solution $g_t$ of the stochastic Loewner equation is also described as $g_t(z) = (\rho_t(z) + B_t)(z)$, the operator $Q(g_t)$ corresponding to $g_t$ is written as $Q(g_t) = Q(\rho_t) e^{-B_tD_{-1}}$. Let $\mathcal{Y}(-, z)$ be an intertwining operator of type $(L_{(c,h)}^{c,h}, L_{c,h})$, then $\mathcal{Y}(|c, h\rangle, z)$ is a primary field, which is applied to the vacuum vector $|0\rangle$ to yield $\mathcal{Y}(|c, h\rangle, z) |0\rangle = e^{zD_{-1}} |c, h\rangle$. If we are allowed to substitute the Brownian motion $B_t$ in the formal variable $z$, then

\begin{equation}
Q(g_t) \mathcal{Y}(|c, h\rangle, B_t) |0\rangle = Q(\rho_t) |c, h\rangle,
\end{equation}

which is a local martingale for a certain choice of $(c, h)$ depending on $\kappa$. The left-hand side is a convenient form of the same local martingale in revealing a Virasoro module structure on a space of SLE local martingales.

The origin of the infinite-dimensional Lie group $\text{Aut}_c \mathcal{O}$ in CFT was a seminal work, in which the group $\text{Aut}_c \mathcal{O}$ appeared as a part of the fiber of the fiber bundle $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$, where $\mathcal{M}_{g,n}$ is the moduli space of Riemann surfaces of genus $g$ and with $n$ punctures, and $\mathcal{M}_{g,n}$ is the moduli space decorated by local coordinates at punctures. One can put on this infinite-dimensional Lie group a line bundle, the sheaf of sections of which admits an action of the Virasoro algebra. This action essentially gives rise to the Virasoro action on the space of SLE local martingales. These subjects on SLE were developed and unified by Friedrich.
3. Affine Lie algebras and their representations

In this section, we recall the notion of affine Lie algebras and their representation theory. Let \( g \) be a finite-dimensional Lie algebra that is simple or commutative and \((·,·) : g \times g \to \mathbb{C}\) be a nondegenerate symmetric invariant bilinear form on \( g \). The affinization \( \widehat{g} \) of \( g \) is defined by \( \widehat{g} = g \otimes \mathbb{C}[\zeta, \zeta^{-1}] \oplus \mathbb{C}K \) with Lie brackets being defined by

\[
[X(m), Y(n)] = [X, Y](m + n) + m(X)Y(m) + nY(m)X + m(nK)X + nX(mK) , \quad [K, \widehat{g}] = \{0\},
\]

where we denote \( X \otimes \zeta^n \) by \( X(n) \) for \( X \in g \) and \( n \in \mathbb{Z} \). Let \( M \) be a finite-dimensional representation of the finite-dimensional Lie algebra \( g \). Then we lift the action of \( g \) to an action of a Lie subalgebra \( g \otimes \mathbb{C}[\zeta] \oplus \mathbb{C}K \) of the affine Lie algebra so that \( g \otimes \zeta^0 \) acts naturally, \( g \otimes \zeta \mathbb{C}[\zeta] \) acts trivially, and \( K \) acts as multiplication by a complex number \( k \). Then we obtain a representation \( \widehat{M}_k \) of the affine Lie algebra \( \widehat{g} \) by

\[
\widehat{M}_k = \text{Ind}_{g \otimes \mathbb{C}[\zeta] \oplus \mathbb{C}K}^{\widehat{g}} M = U(\widehat{g}) \otimes_{U(g \otimes \mathbb{C}[\zeta] \oplus \mathbb{C}K)} M.
\]

Here the introduced complex number \( k \) is called the level of the representation. By the Poincaré–Birkhoff–Witt theorem, \( \widehat{M}_k \) is isomorphic to \( U(g \otimes \zeta^{-1} \mathbb{C}[\zeta^{-1}]) \otimes_{\mathbb{C}} M \) as a vector space or a \( U(g \otimes \zeta^{-1} \mathbb{C}[\zeta^{-1}]) \)-module.

To classify finite-dimensional irreducible representations of \( g \), we assume that \( g \) is simple in this paragraph. We fix a Cartan subalgebra \( h \) of \( g \), and let \( \Pi' = \{\alpha_1', \ldots, \alpha_\ell'\} \subset h \) be the set of simple coroots of \( g \). Then the fundamental weights \( \Lambda_i \in h^* \) for \( i = 1, \ldots, \ell \) are defined by \( \langle \Lambda_i, \alpha_j' \rangle = \delta_{ij} \), and span the weight lattice \( P = \bigoplus_{i=1}^\ell \mathbb{Z} \Lambda_i \). A weight \( \lambda \in P \) is called dominant if \( \langle \lambda, \alpha_i' \rangle \geq 0 \) for all \( i = 1, \ldots, \ell \). We denote the set of dominant weights by \( P_+ \). Finite-dimensional irreducible representations of \( g \) are labeled by \( P_+ \). Namely, for a dominant weight \( \Lambda \in P_+ \), there is a finite-dimensional irreducible representation \( L(\Lambda) \) of \( g \) with highest weight \( \Lambda \), and conversely, the highest weight of a finite-dimensional irreducible representation of \( g \) is dominant. For an irreducible representation \( L(\Lambda) \) of \( g \), we can construct a representation \( \widehat{L(\Lambda)}_k \) of \( \widehat{g} \) in the manner described in the previous paragraph. Note that although \( L(\Lambda) \) is irreducible as a representation of \( g \), \( \widehat{L(\Lambda)}_k \) is not necessarily an irreducible representation of \( \widehat{g} \), then we denote by \( \hat{L}(\Lambda, k) \) the irreducible quotient of \( \widehat{L(\Lambda)}_k \) as a representation of \( \widehat{g} \).

In the case when \( g \) is commutative, the representation theory is simpler: an irreducible representation \( L(\Lambda) \) of \( g \) is one dimensional and labeled by an element \( \Lambda \in g^* \) so that an element \( X \in g \) acts as \( \langle \Lambda, X \rangle \) times the identity operator. The corresponding representation \( \widehat{L(\Lambda)}_k \) of \( \widehat{g} \), which we denote by \( \hat{L}(\Lambda, k) \) is a Fock representation and irreducible. Notice that Fock representations \( \hat{L}(\Lambda, k) \) are all isomorphic if \( k \neq 0 \), thus we think that \( k = 1 \) in \( \hat{L}(\Lambda, k) \) if the finite-dimensional Lie algebra \( g \) is commutative.

On a representation space \( \hat{L}(\Lambda, k) \) of an affine Lie algebra \( \widehat{g} \) constructed above, we can define an action of the Virasoro algebra through the Segal-Sugawara construction. We normalize the bilinear form so that \( \theta' \theta = 2 \) if \( g \) is simple, where \( \theta \) is the highest root of \( g \). We define a number \( h^\vee \) by the dual Coxeter number \( h^\vee \) of \( g \) if \( g \) is simple, and by \( 0 \) if \( g \) is commutative, and assume that \( k \neq -h^\vee \). Let \( \{X_a\}_{a=1}^{\dim g} \) be an orthonormal basis of \( g \) with respect to \( (·,·) \). Then the operators \( L_n \) for \( n \in \mathbb{Z} \) acting on \( \hat{L}(\Lambda, k) \) that are defined by

\[
L_n = \frac{1}{2(k + h^\vee)} \sum_{a=1}^{\dim g} \sum_{k \in \mathbb{Z}} :X_a(n-k)X_a(k):
\]
give an action of the Virasoro algebra of central charge \( c_{0,k} = \frac{k \dim \mathfrak{g}}{k+\frac{h_\mathfrak{g}}{2}} \). Here the normal ordered product \( \mathcal{A}(p) \mathcal{B}(q) \) is defined by \( \mathcal{A}(p) \mathcal{B}(q) \) for \( p < q \) and \( \mathcal{B}(q) \mathcal{A}(p) \) for \( p \geq q \). Moreover a vector \( v_\Lambda \in L(\Lambda) \to L_{\mathfrak{g},k}(\Lambda) \) is an eigenvector of \( L_0 \) corresponding to an eigenvalue \( h_\Lambda = \frac{\Lambda(\Lambda+2)p_0}{2(k+\frac{h_\mathfrak{g}}{2})} \), with \( \rho_\mathfrak{g} = \sum_{i=1}^\ell \Lambda_i \) if \( \mathfrak{g} \) is simple and \( \rho_\mathfrak{g} = 0 \) if \( \mathfrak{g} \) is commutative. The operator \( L_0 \) is diagonalizable on \( L_\mathfrak{g}(\Lambda, k) \), so that \( L_\mathfrak{g}(\Lambda, k) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} L_\mathfrak{g}(\Lambda, k)_{h_\Lambda + n} \) with each \( L_\mathfrak{g}(\Lambda, k)_{h_\Lambda} \) being the eigenspace of \( L_0 \) corresponding to an eigenvalue \( h \). This action of the Virasoro algebra is compatible with the action of \( \widehat{\mathfrak{g}} \) in the sense that

\[
[L_\mathfrak{g}, A \otimes f(\zeta)] = -A \otimes \zeta^{n+1} \frac{df(\zeta)}{dc}.
\]

Among representations \( L_\mathfrak{g}(\Lambda, k) \), we can equip \( L_\mathfrak{g}(0, k) \) with a VOA structure. The vacuum vector is \( |0\rangle = 1 \otimes \mathbf{1} \), where \( \mathbf{1} \) spans a one-dimensional representation \( L(0) \) of \( \mathfrak{g} \). Let \( \{ X_a \}_{a=1}^{\dim \mathfrak{g}} \) be a basis of \( \mathfrak{g} \), then this VOA is strongly generated by vectors \( X_a(-1)|0\rangle \).

In the following, we call this VOA the affine VOA of \( \mathfrak{g} \) with level \( k \) and denote it by \( L_{\mathfrak{g},k} \). Simple modules over \( L_{\mathfrak{g},k} \) are realized as highest weight representations \( L_{\mathfrak{g},k}(\Lambda, k) \) of the same level. For an \( L_{\mathfrak{g},k} \)-module \( M \), the invariance in Eq. (4.3) of a nondegenerate bilinear form \( \langle \cdot | \cdot \rangle : M \times M \to \mathbb{C} \) is rephrased as \( \langle X(n)u|v\rangle = -\langle u|X(-n)v\rangle \) for \( u, v \in M \) and \( n \in \mathbb{Z} \). Such an invariant bilinear form is specified on an irreducible representation \( L_\mathfrak{g}(\Lambda, k) \) by the normalization \( \langle v_\Lambda|v_\Lambda\rangle = 1 \) with \( v_\Lambda \) being the highest weight vector.

4. INTERNAL SYMMETRY

We again assume that \( \mathfrak{g} \) is a finite-dimensional complex Lie algebra that is simple or commutative. Let \( G \) be a finite-dimensional complex Lie group of which the Lie algebra is \( \mathfrak{g} \), i.e., it is a simple Lie group if \( \mathfrak{g} \) is simple and a torus if \( \mathfrak{g} \) is commutative. To construct a generalization of SLE associated with a representation of an affine Lie algebra \( \widehat{\mathfrak{g}} \), we consider the positive loop group \( G(O) = \mathbb{G}[[\zeta^{-1}]] \) of \( G \) as a group of internal symmetry. A significant subgroup \( G_+(O) \) consists of elements that are the unit element modulo \( \mathbb{G}[[\zeta^{-1}]][\zeta^{-1}] \). The Lie algebras of \( G(O) \) and \( G_+(O) \) are \( \mathfrak{g}[[\zeta^{-1}]] \) and \( \mathfrak{g}[[\zeta^{-1}]][\zeta^{-1}] \), respectively. The group of automorphisms \( \text{Aut} O \) acts on \( G(O) \) to define a semi-direct product \( \text{Aut} O \ltimes G(O) \). Moreover, the subgroup \( \text{Aut}_+ O \) normalizes \( G_+(O) \), thus their semi-direct product \( \text{Aut}_+ O \ltimes G_+(O) \) is also defined.

On a representation \( L_\mathfrak{g}(\Lambda, k) \) of the affine Lie algebra \( \widehat{\mathfrak{g}} \), the Lie algebra \( \mathfrak{g} \otimes \mathbb{C}[[\zeta^{-1}]] \) cannot act, but its formal completion \( \overline{L}_\mathfrak{g}(\Lambda, k) = \prod_{n \in \mathbb{Z}_{\geq 0}} L_\mathfrak{g}(\Lambda, k)_{h_\Lambda + n} \) admits an action of \( \mathfrak{g} \otimes \mathbb{C}[[\zeta^{-1}]] \). It is also obvious that the action of \( \mathfrak{g} \otimes \mathbb{C}[[\zeta^{-1}]] \) is exponentiated to define an action of \( G(O) \). Indeed, an element in \( \mathfrak{g} \otimes \zeta^{-1} \mathbb{C}[[\zeta^{-1}]] \) strictly raises the degree, and a zero-mode element \( X \otimes \zeta^0 \) is exponentiated to be an action of \( e^X \in G \) while each homogeneous space is a representation of the finite-dimensional Lie group \( G \). Moreover, this action of \( G(O) \) is compatible with the action of \( \text{Aut} O \) due to the Segal-Sugawara construction. Thus \( \text{Aut}_+ O \ltimes G_+(O) \) acts on \( \overline{L}_\mathfrak{g}(\Lambda, k) \).

We investigate how each field is transformed under the adjoint action of \( e^a \) where \( a = A \otimes a(\zeta) \in \mathfrak{g} \otimes \mathbb{C}[[\zeta^{-1}]] \). We compute the commutator \( [a, Y(B, w)] \) for \( B \in L_{\mathfrak{g},k} \).

From the operator product expansion (OPE) formula

\[
(4.1) \quad [Y(A(-1)|0\rangle, z), Y(B, w)] = \sum_{k \geq 0} Y(A(k)B, w) \frac{\delta(k)}{\delta(z - w)},
\]
we obtain
\begin{equation}
[A(n), Y(B, w)] = \sum_{k \geq 0} \binom{n}{k} w^{n-k} Y(A(k)B, w).
\end{equation}
Thus the desired commutator is computed as
\begin{equation}
[a, Y(B, w)] = Y(a_w B, w),
\end{equation}
where $a_w = \sum_{k \geq 0} \partial^k a(w) A(k)$. This enables us to obtain the following transformation formula:
\begin{equation}
Y(B, w) = e^{a} Y(e^{-a} B, w) e^{-a}
\end{equation}
Now we compute $e^{-a_w} X(-1) |0\rangle$ for some $X \in g$ to investigate the transformation rule of $Y(X(-1) |0\rangle, z)$ under the adjoint action by $e^a$. The action of $a_w$ on $X(-1) |0\rangle$ gives
\begin{equation}
a_w X(-1) |0\rangle = a(w)(\text{ad} A)(X)(-1) |0\rangle + k(A)X \partial a(w) |0\rangle.
\end{equation}
Applying $a_w$ once more:
\begin{equation}
a_w^2 X(-1) |0\rangle = a(w)^2(\text{ad} A)^2(X)(-1) |0\rangle,
\end{equation}
where we have used the invariance of the bilinear form $(A[[A,X]] = ([A,A]]X) = 0$, and inductively,
\begin{equation}
a_w^n X(-1) |0\rangle = a(w)^n(\text{ad} A)^n(X)(-1) |0\rangle
\end{equation}
for $n \geq 2$. Thus we can see that
\begin{equation}
e^{-a_w} X(-1) |0\rangle = (e^{-a} w)^{\text{ad} A} X(-1) |0\rangle - k(A)X \partial a(w) |0\rangle,
\end{equation}
which implies that
\begin{equation}
Y(X(-1) |0\rangle, w) = e^{a} Y\left(\left(e^{-a} w\right) \text{ad} A X(-1) |0\rangle, w\right) e^{-a} - k(A)X \partial a(w).
\end{equation}
It is also convenient to note the formula for the object in the form $e^{-a} X \otimes x(\zeta) e^a$, where $a = A \otimes a(\zeta) \in g \otimes \mathbb{C}\{[\zeta^{-1}]\}$, $x(\zeta) \in \mathbb{C}\{[\zeta^{-1}]\}$ and $X \in g$ are taken as above. This becomes
\begin{equation}
e^{-a} X \otimes x(\zeta) e^a = \text{Res}_{w} \sum_{n \in \mathbb{Z}} (e^{-a} w)^{\text{ad} A} X \otimes \zeta^n w^{-n-1} x(w) - k(A)X \text{Res}_{w} \partial a(w) x(w)
\end{equation}
\begin{equation}
= \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!} (\text{ad} A)^m(X) \otimes a(\zeta)^m x(\zeta) - k(A)X \text{Res}_{w} \partial a(w) x(w).
\end{equation}
We next investigate the transformation rule of the Virasoro field $L(z)$ under the action of $G(\mathcal{O})$. We compute $e^{-a_z} L_{-2} |0\rangle$ where $a_z = A \otimes a(\zeta) \in g \otimes \mathbb{C}\{[\zeta^{-1}]\}$ and correspondingly $a_z = \sum_{k \geq 0} \partial^k a(z) A(k)$. Note that the OPE
\begin{equation}
[L(z), Y(A(-1) |0\rangle, w)] = Y(A(-1) |0\rangle, w) \partial_w \delta(z-w) + \partial Y(A(-1) |0\rangle, w) \delta(z-w)
\end{equation}
is equivalent to
\begin{equation}
[Y(A(-1) |0\rangle, z), L(w)] = Y(A(-1) |0\rangle, w) \partial_w \delta(z-w),
\end{equation}
which implies
\begin{equation}
A(n) L_{-2} |0\rangle = \begin{cases} A(-1) |0\rangle, & n = 1, \\ 0, & n \in \mathbb{Z}_{\geq 0}\setminus\{1\}. \end{cases}
\end{equation}
Thus
\begin{equation}
- a_z L_{-2} |0\rangle = - \partial a(z) A(-1) |0\rangle.
\end{equation}
If we apply $-a_z$ one more time,
\begin{equation}
(-a_z)^2 L_{-2} |0\rangle = k(\partial a(z))^2 (A|A) |0\rangle.
\end{equation}

Then we obtain the following transformation formula:
\begin{equation}
L(z) = e^a L(z)e^{-a} - \partial a(z) e^a Y(A(-1)|0\rangle , z) e^{-a} + \frac{k(A|A)(\partial a(z))^2}{2}.
\end{equation}

4.1. **Formulas in the case of commutative** $g$. Let us note the formulas in Eq.(4.9) and Eq.(4.10) in a more explicit way in the case when $g$ is commutative. Now, we have $[A, X] = 0$ for any $A, X \in g$, which implies that
\begin{equation}
X(z) = e^a X(z)e^{-a} - k(A|X) \partial a(z),
\end{equation}
\begin{equation}
e^{-a}X \otimes x(\zeta)e^a = X \otimes x(\zeta) - k(A|X) \text{Res}_w \partial w x(w).
\end{equation}

4.2. **Formulas in** $g = \mathfrak{sl}_2$ **case.** We now focus our attention on the case of $g = \mathfrak{sl}_2$ and explicitly note the formulas in Eq.(4.9) and Eq.(4.10). We take as a standard basis of $\mathfrak{sl}_2$
\begin{equation}
E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{equation}
and denote $E \otimes e(\zeta), H \otimes h(\zeta)$ and $F \otimes f(\zeta)$ for $e(\zeta), h(\zeta), f(\zeta) \in C[[\zeta^{-1}]]$ simply by $e, h$ and $f$, respectively. We also write a current field $Y(X(1)|0\rangle, z)$ by $X(z)$ for $X \in g$.

(1) $X = A = H.$
\begin{align*}
H(z) &= e^h H(z)e^{-h} - 2k\partial h(z), \\
E^{-h} \otimes x(\zeta)e^h &= H \otimes x(\zeta) - 2k\text{Res}_w \partial h(w)x(w).
\end{align*}

(2) $X = H, A = E.$
\begin{align*}
H(z) &= e^e H(z)e^{-e} + 2e(z) E(z), \\
e^{-e} \otimes x(\zeta)e^e &= H \otimes x(\zeta) + 2E \otimes e(\zeta)x(\zeta).
\end{align*}

(3) $X = H, A = F.$
\begin{align*}
H(z) &= e^f H(z)e^{-f} - 2f(z) F(z), \\
e^{-f} \otimes x(\zeta)e^f &= H \otimes x(\zeta) - 2F \otimes f(\zeta)x(\zeta).
\end{align*}

(4) $X = E, A = H.$
\begin{align*}
E(z) &= e^{-2h(z)} e^h E(z)e^{-h}, \\
e^{-h} E \otimes x(\zeta)e^h &= E \otimes e^{-2h(\zeta)}x(\zeta).
\end{align*}

(5) $X = A = E.$
\begin{align*}
E(z) &= e^e E(z)e^{-e}, \\
e^{-e} E \otimes x(\zeta)e^e &= E \otimes x(\zeta).
\end{align*}

(6) $X = E, A = F.$
\begin{align*}
E(z) &= e^f E(z)e^{-f} + f(z)e^f H(z)e^{-f} - f(z)z e^f F(z)e^{-f} - k\partial f(z), \\
e^{-f} E \otimes x(\zeta)e^f &= E \otimes x(\zeta) + H \otimes f(\zeta)x(\zeta) - F \otimes f(\zeta)^2 x(\zeta) - k\text{Res}_w \partial f(w)x(w).
\end{align*}
(7) $X = F$, $A = H.$

$$F(z) = e^{2h(z)}e^{h}F(z)e^{-h},$$
$$e^{-h}F \otimes x(\zeta)e^{h} = F \otimes e^{2h(\zeta)}x(\zeta).$$

(8) $X = F$, $A = E.$

$$F(z) = e^{e}F(z)e^{-e} - e(z)e^{e}H(z)e^{-e} - e(z)^2E(z) - k\partial e(z),$$
$$e^{-e}F \otimes x(\zeta)e^{e} = F \otimes x(\zeta) - H \otimes e(\zeta)x(\zeta) - E \otimes e(\zeta)^2x(\zeta) - k\text{Res}_w\partial e(w)x(w).$$

(9) $X = A = F.$

$$F(z) = e^{f}F(z)e^{-f},$$
$$e^{-f}F \otimes x(\zeta)e^{f} = F \otimes x(\zeta).$$

5. Construction of a random process

In this section, we construct a random process on the infinite-dimensional Lie group $\text{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$, which was introduced in the previous Sect. [2]. It is a natural generalization of the random process on $\text{Aut}_+\mathcal{O}$, which was the fundamental object in the group theoretical formulation of SLEs in Sect. [2] to a case with internal symmetry.

5.1. General Lie algebras $\mathfrak{g}$. We shall construct a random process that is a generalization of SLEs with internal symmetry described by $G_+(\mathcal{O})$. Such a random process is expected to be induced from a random process on an infinite-dimensional Lie group $\text{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$. To decide a direction for designing a random process on this group, we first make an observation on an annihilator of the vacuum vector in the vacuum representation $L_{\mathfrak{g},k}$. Since we have defined a representation of the Virasoro algebra by the Segal-Sugawara construction, $L_{-2}|0\rangle = \frac{1}{2(k+h)^2} \sum_{r=1}^{\text{dim} \mathfrak{g}} X_r(-1)^2|0\rangle$. Combining the fact that the vacuum vector is translation invariant, we see that the operator

$$-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{1}{k + h^2} \sum_{r=1}^{\text{dim} \mathfrak{g}} X_r(-1)^2$$

annihilates the vacuum vector for arbitrary $\kappa$. We now assume that the highest weight vector $v_\lambda$ of a representation $L_{\mathfrak{g}}(\Lambda, k)$ is annihilated by an operator of the form

$$-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{\tau}{2} \sum_{r=1}^{\text{dim} \mathfrak{g}} X_r(-1)^2$$

with parameters $\kappa$ and $\tau$ being finely tuned positive numbers. The existence of such an annihilator of the above form will be discussed later in Sect. [6].

We consider a random process $\mathfrak{g}_t$ on $\text{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ that satisfies the following SDE

$$\mathfrak{g}_t^{-1}d\mathfrak{g}_t = \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{\tau}{2} \sum_{r=1}^{\text{dim} \mathfrak{g}} X_r(-1)^2\right)dt + L_{-1}dB^{(0)}_t + \sum_{r=1}^{\text{dim} \mathfrak{g}} X_r(-1)dB^{(r)}_t,$$

where $B^{(i)}_t$ for $i = 0, 1, \ldots, \text{dim} \mathfrak{g}$ are mutually independent Brownian motions with variance $\kappa$ for $B^{(0)}_t$ and $\tau$ for $B^{(r)}_t$ with $r = 1, \ldots, \text{dim} \mathfrak{g}$. An idea for considering a random process on such an infinite-dimensional Lie group as $\text{Aut}_+\mathcal{O} \ltimes G_+(\mathcal{O})$ has already appeared in the work by Rasmussen. [Ras07] but it lacks an SDE based on an annihilating
operator, and it does not include the classical SLE in the coordinate transformation part.

**Proposition 5.1.** Assume that the highest weight vector \( v_\Lambda \) of \( L_\Phi(\Lambda, k) \) is annihilated by the operator in Eq. (5.2). Then for a random process \( \mathcal{G}_t \) on \( \text{Aut}_\Phi \mathcal{O} \times G_+ (\mathcal{O}) \) satisfying Eq. (5.3), the random process \( \mathcal{G}_t v_\Lambda \) in \( L_\Phi(\Lambda, k) \) is a local martingale.

We can write the random process \( \mathcal{G}_t \) as \( \mathcal{G}_t = \Theta_t Q (\rho_t) \) where the random process \( \rho_t \) on \( \text{Aut}_\Phi \mathcal{O} \) induces the SLE(\( \kappa \)) and \( \Theta_t \) is a random process on \( G_+ (\mathcal{O}) \).

**Proposition 5.2.** Under the ansatz \( \mathcal{G}_t = \Theta_t Q (\rho_t) \) described above, the random process \( \Theta_t \) on \( G_+ (\mathcal{O}) \) satisfies the SDE

\[
\Theta_t^{-1} d \Theta_t = \frac{t}{2} \sum_{r=1}^{\dim \mathfrak{g}} (X_r \otimes \rho_t (\zeta))^{-1} dB_t^r.
\]

**Proof.** The action of the Virasoro algebra on an affine Lie algebra, which is described by the relation \([L_n, X(m)] = -m X(n + m)\), implies the transformation formula

\[
G(\rho) X \otimes f (\zeta) G(\rho)^{-1} = X \otimes f (\rho (\zeta)) \]

for \( f (\zeta) \in \mathcal{C}(\{ \zeta \}) \) and \( \rho \in \text{Aut}_\Phi \mathcal{O} \). If we apply this formula in the case when \( f (\zeta) = \zeta^{-1} \), we obtain the desired result.

Equation (5.4) has already appeared in an equivalent form in the correlation function formulation of SLEs corresponding to WZW models. \cite{BGLW05, ABI11} Let \( \mathcal{V}(\zeta, v) \) be an intertwining operator of type \( L_\Phi(\Lambda_3, k) \), \( L_\Phi(\Lambda_2, k) \), and \( \zeta \in L(\Lambda_1) \) be a primary vector in the top space of \( L_\Phi(\Lambda_1, k) \). If we take the adjoint of the primary field \( \mathcal{V}(\zeta, v) \) by \( \Theta_t^{-1} \), we obtain

\[
\mathcal{G}_t^{-1} \mathcal{V}(v, z) \mathcal{G}_t = \mathcal{V}(\Theta_t^{-1} (z) v, \rho_t (z)) (\partial \rho_t (z)) h_{\Lambda_1}.
\]

Here the object \( \Theta_t^{-1} (z) \) is a random process on the group of \( \mathbb{C}[\{ \zeta \}] \)-points in \( G \) obtained by substituting \( \zeta = z \) in \( \Theta_t^{-1} \). From the identity \( \Theta_t^{-1} \Theta_t = \text{Id} \), the SDE on \( \Theta_t^{-1} (z) \) becomes

\[
d \Theta_t^{-1} (z) \Theta_t (z) = \frac{t}{2} \sum_{r=1}^{\dim \mathfrak{g}} (\rho_t (z))^{-1} X_r \otimes \rho_t (z)^{-1} X_r dB_t^r.
\]

Apart from the Jacobian part, the right-hand side of Eq. (5.6) is the random transformation of a primary field in Eq. (1.3) considered in the correlation function formulation of SLEs, \cite{BGLW05, ABI11} which seemed to be \textit{ad hoc}, while it naturally appears in the group theoretical formulation presented here.

The SDE in Eq. (5.4) for the random process along the internal symmetry is still not sufficient to compute matrix elements like \( \langle u | \mathcal{G}_t | v_\Lambda \rangle \). In the following two subsections, we construct the random process \( \Theta_t \) in the most explicit way in the cases when \( \mathfrak{g} \) is commutative and \( \mathfrak{g} = \mathfrak{s} \ell_2 \).

### 5.2. Case when \( \mathfrak{g} \) is commutative.

We temporarily denote the dimension of \( \mathfrak{g} \) by \( \ell \). Let \( H_1, \ldots, H_\ell \) be an orthonormal basis of \( \mathfrak{g} \) with respect to the bilinear form \( (\cdot | \cdot) \). We put an ansatz on \( \Theta_t \) as

\[
\Theta_t = e^{H_1 \otimes h_1 (\zeta)} \cdots e^{H_\ell \otimes h_\ell (\zeta)},
\]

where \( h_i (\zeta) \) are \( \mathbb{C}[\{ \zeta \}] \)-valued random processes.
Proposition 5.3. Under the above ansatz on $\Theta_t$, the random processes $h^i_t(\zeta)$ satisfy
\begin{equation}
 dh^i_t(\zeta) = \frac{1}{\rho_i(\zeta)} dB^{(i)}_t
\end{equation}
for $i = 1, \cdots, \ell$.

Thus the random processes $h^i_t(\zeta)$ are completely determined by the solution of SLEs so that $h^i_t(\zeta) = \int_0^t dB^{(i)}_s / \rho_i(\zeta)$.

5.3. Specialization to $\mathfrak{sl}_2$. To construct the random process $\Theta_t$ in a sufficiently explicit way, we make an ansatz that it is written as $\Theta_t = e^{e^h_1} e^{e^h_2} e^{f_3}$, where $e^h_1 = E \otimes h_1(\zeta)$, $h_1 = H \otimes h_1(\zeta)$, and $f_3 = F \otimes f_3(\zeta)$ are random processes on $\mathfrak{g} \otimes \mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$ associated with $\mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$-valued random processes $e_t(\zeta)$, $h_t(\zeta)$, and $f_t(\zeta)$. Then we shall derive SDEs on $e_t(\zeta)$, $h_t(\zeta)$ and $f_t(\zeta)$. We assume the SDEs as
\begin{equation}
 dx_t(\zeta) = \mathfrak{p}_t(\zeta) dt + \sum_{r=1}^3 x^{(r)}_t(\zeta) dB^{(r)}_t, \quad x = e, h, f.
\end{equation}

Since $X(n)$ with $n < 0$ are mutually commutative for a fixed $X \in \mathfrak{sl}_2$, the increment of the random process $\Theta_t$ is computed by the standard Ito calculus, and we can determine data $\mathfrak{p}_t(\zeta)$ and $x^{(r)}_t(\zeta)$ so the increment of $\Theta_t$ is in the desired form in Eq.\,\,(5.4). After the computation that is presented in Appendix \,\,[C] we obtain the following proposition

Proposition 5.4. Under the ansatz $\Theta_t = e^{e^h_1} e^{e^h_2} e^{f_3}$ described above, the SDE in Eq.\,\,(5.4) implies the following set of SDEs:
\begin{align}
(5.11) \quad & dx_t(\zeta) = \frac{e^{2h_1(\zeta)}}{2\rho_1(\zeta)} dB^{(2)}_t - \frac{i e^{2h_1(\zeta)}}{2\rho_1(\zeta)} dB^{(3)}_t, \\
(5.12) \quad & dh_t(\zeta) = \frac{\tau}{2\rho_1(\zeta)^2} dt - \frac{1}{\sqrt{2\rho_1(\zeta)}} dB^{(1)}_t + \frac{f_3(\zeta)}{\sqrt{2\rho_1(\zeta)}} dB^{(2)}_t + \frac{i f_3(\zeta)}{\sqrt{2\rho_1(\zeta)}} dB^{(3)}_t, \\
(5.13) \quad & df_t(\zeta) = \frac{\sqrt{2} f_3(\zeta)}{\rho_1(\zeta)} dB^{(1)}_t - \frac{1 - f_3(\zeta)^2}{\sqrt{2\rho_1(\zeta)}} dB^{(2)}_t + \frac{i(1 + f_3(\zeta)^2)}{\sqrt{2\rho_1(\zeta)}} dB^{(3)}_t.
\end{align}

6. Annihilating operator of a highest weight vector

We have assumed in Sect.\,[5] that the highest weight vector $v_\Lambda$ of $L_\mathfrak{g}(\Lambda, k)$ is annihilated by an operator of the form in Eq.\,(5.2) with finely tuned parameters $\kappa$ and $\tau$. In this section we see examples of such annihilating operators. As we have already seen, the vacuum vector $|0\rangle$ is annihilated by the operator in Eq.\,(5.2) for $\tau = 2\kappa + h_2$ and arbitrary $\kappa$. Thus we shall search for an example acting on a “charged” representation.

6.1. Case when $\mathfrak{g}$ is commutative. We first compute vectors $L_{-2} v_\Lambda$ and $L_{-1} v_\Lambda$. By the expression of $L_n$ via the Segal-Sugawara construction in Eq.\,(3.3), they can be computed as
\begin{align}
(6.1) \quad & L_{-2} v_\Lambda = \left( \frac{1}{2} \sum_{i=1}^{\ell} H_i (-1)^2 + \Lambda (-2) \right) v_\Lambda, \\
(6.2) \quad & L_{-1} v_\Lambda = \left( \Lambda (-1)^2 + \Lambda (-2) \right) v_\Lambda.
\end{align}
Here we have identified \( g^* \) with \( g \) via the nondegenerate bilinear form \( \langle \cdot , \cdot \rangle \). We assume that \( \Lambda \) is proportional to \( H_1 \) with the coefficient being written as \( \lambda : \Lambda = \lambda H_1 \). Under this assumption,

\[
(6.3) \quad \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right) v_\lambda = \left(-(1 - 2\lambda^2)H_1(-1)^2 - \sum_{i=2}^\ell H_i(-1)^2\right) v_\lambda,
\]

for \( \kappa = 4 \). Thus we have found an operator that annihilates \( v_\lambda \) of a suitable form.

**Proposition 6.1.** The following operator annihilates the highest weight vector \( v_\lambda \) for \( \Lambda = \lambda H_1 \):

\[
(6.4) \quad -2L_{-2} + \frac{\kappa}{2}L_{-1}^2 + \frac{1}{2} \sum_{i=1}^\ell \tau_i \Lambda_i(-1)^2,
\]

where \( \kappa = 4 \), \( \tau_1 = 2 - 4\lambda^2 \) and \( \tau_i = 2 \) for \( i \geq 2 \).

**6.2. Case when \( g = sl_2 \).** Here we assume that the level is \( k = 1 \). In this case, the vacuum representation \( L_{sl_2,1} \) is isomorphic as a VOA to the lattice VOA \( V_Q \) associated with the root lattice \( Q = \mathbb{Z}\alpha \), \( (\alpha|\alpha) = 2 \) of \( sl_2 \). The isomorphism is described by

\[
(6.5) \quad E(z) \mapsto \Gamma_\alpha(z), \quad H(z) \mapsto \alpha(z), \quad F(z) \mapsto \Gamma_{-\alpha}(z).
\]

Here \( \alpha(z) \) is the free Bose field and \( \Gamma_{\pm\alpha}(z) \) are the vertex operators associated with \( \pm\alpha \in Q \). This isomorphism of VOAs is called the Frenkel-Kac construction of an affine VOA, [FK80] of which an exposition is also contained in Appendix A.3. The dominant weights of level \( k = 1 \) are exhausted by 0 and the fundamental weight \( \Lambda \) such that \( (\Lambda|\alpha) = 1 \). The spin-\( \frac{1}{2} \) representation \( L_{g}(\Lambda,1) \) corresponding to \( \Lambda \) is also realized as a module of the lattice VOA \( V_Q \) by \( V_{Q+\Lambda} \), which is defined by

\[
(6.6) \quad V_{Q+\Lambda} = \bigoplus_{\beta \in Q} L_{C\otimes \mathbb{Z}Q}(0,1) \otimes e^{\beta+\Lambda}.
\]

Here \( L_{C\otimes \mathbb{Z}Q}(0,1) \) is the vacuum Fock space introduced in Sect.3. Let the top space of \( L_{sl_2}(\Lambda,1) \) be realized as \( L(\Lambda) = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \) so that \( Ev_0 = 0 \). Then the isomorphism \( L_{sl_2}(\Lambda,1) \simeq V_{Q+\Lambda} \) is determined by

\[
(6.7) \quad v_0 \mapsto e^{\Lambda}, \quad v_1 \mapsto e^{-\Lambda}.
\]

We show that both \( v_0 \) and \( v_1 \) are annihilated by an operator of the form in Eq.(5.2). Let \( \mathcal{Y}(\cdot , z) \) be the intertwining operator of type \( (L_{sl_2}(\Lambda,1) \quad L_{sl_2,i}) \). Then we have \( \mathcal{Y}(e^{\pm\Lambda},z) = \Gamma_{\pm\Lambda}(z) \), where \( \Gamma_{\pm\Lambda}(z) \) are generalized vertex operators associated with \( \pm\Lambda \). Such a realization of an intertwining operator allows us to obtain

\[
(6.8) \quad L_{-2}e^{\mp\Lambda} = L_{-1}^2e^{\mp\Lambda} = \left(\frac{1}{4}\alpha(-1)^2 \pm \frac{1}{2}\alpha(-2)\right)e^{\mp\Lambda}
\]

by computation of operator product expansions. In the case of \( g = sl_2 \), then

\[
(6.9) \quad \sum_{r=1}^3 X_r(-1)^2 = \frac{1}{2}H(-1)^2 + E(-1)F(-1) + F(-1)E(-1).
\]

It is obvious that \( E(-1)e^{\Lambda} = 0 \) from \( \Gamma_{\alpha}(w)\Gamma_\Lambda(z) = (z-w)\Gamma_{\alpha,\Lambda}(z,w) \). On the other hand, \( F(-1) \) nontrivially acts on \( e^{\Lambda} \) and further applying \( E(-1) \), we have \( E(-1)F(-1)e^{\Lambda} = \).
\( \alpha(-2)e^\Lambda \). Combining them we can see that

\[
(6.10) \quad \left( -2L_2 + \frac{\kappa}{2} L_{-1} + \frac{\tau}{2} \sum_{r=1}^{3} X_r (-1)^2 \right) e^\Lambda = 0
\]

if the relation \( \kappa + 2\tau - 4 = 0 \) holds. Computation for \( e^\Lambda \) is carried out in an analogous way. We have \( F(-1)e^\Lambda = 0 \), while \( F(-1)E(-1)e^\Lambda = -\alpha(-2)e^\Lambda \), which leads us to

\[
(6.11) \quad \left( -2L_2 + \frac{\kappa}{2} L_{-1} + \frac{\tau}{2} \sum_{r=1}^{3} X_r (-1)^2 \right) e^{-\Lambda} = 0
\]

if the parameters \( \kappa \) and \( \tau \) satisfy the same relation \( \kappa + 2\tau - 4 = 0 \) as in the case of \( e^\Lambda \).

We summarize the above computation as follows.

**Proposition 6.2.** Let \( \Lambda \) be the fundamental weight of \( \mathfrak{sl}_2 \), and let the fundamental representation of \( \mathfrak{sl}_2 \) be described by \( L(\Lambda) = \mathbb{C}v_\Lambda \oplus \mathbb{C}Fv_\Lambda \). Here \( v_\Lambda \) is the highest weight vector of highest weight \( \Lambda \). We also denote the vector \( Fv_\Lambda \) by \( v_{-\Lambda} \). Then we have in \( L_{\mathfrak{sl}_2}(\Lambda, 1) \)

\[
(6.12) \quad \left( -2L_2 + \frac{\kappa}{2} L_{-1} + \frac{\tau}{2} \sum_{r=1}^{3} X_r (-1)^2 \right) v_{\pm\Lambda} = 0
\]

if the relation \( \kappa + 2\tau - 4 = 0 \) holds.

7. Local martingales

As an application of construction of a random process \( \mathcal{G}_t \) on an infinite-dimensional Lie group presented in Sect. 5, we compute several local martingales associated with the solution of SLEs with internal degrees of freedom by taking the inner product \( \langle u | \mathcal{G}_t | v_\Lambda \rangle \).

7.1. Case when \( \mathfrak{g} \) is commutative. The local martingale \( \mathcal{G}_t v_\Lambda \) on \( L_{\mathfrak{g}}(\Lambda, 1) \) generates local martingales when we take the inner product of it with any vectors in \( L_{\mathfrak{g}}(\Lambda, 1) \). To describe them explicitly, we first investigate how a current field \( H(z) \) and the Virasoro field \( L(z) \) are transformed under the adjoint action by \( \mathcal{G}_t \). First a current field \( H(z) \) transforms under the adjoint action by \( e^{-h_1} \) as in Eq. (4.17), which implies

\[
(7.1) \quad \Theta_t^{-1} H(z) \Theta_t = H(z) - \sum_{i=1}^{\ell} (H_i | H) \partial h_i(z).
\]

Since the transformation rule of \( H(z) \) under the adjoint action by \( Q(\rho_t)^{-1} \) has been already obtained,

\[
(7.2) \quad \mathcal{G}_t^{-1} H(z) \mathcal{G}_t = H(\rho_t(z)) \rho_t'(z) - \sum_{i=1}^{\ell} (H_i | H) \partial h_i(z).
\]

This can be used to formulate a local martingale \( \langle v_\Lambda | H(z) \mathcal{G}_t | v_\Lambda \rangle \).

**Theorem 7.1.** Let \( g_t(z) = \rho_t(z) + B_t \) be the SLE\( (\kappa) \) and \( h_i(z) \) be the solutions of Eq. (5.9). Then the following quantity is a local martingale:

\[
(7.3) \quad \langle v_\Lambda | H(z) \mathcal{G}_t | v_\Lambda \rangle = \lambda(H_i | H) \frac{\rho_i'(z)}{\rho_t(z)} - \sum_{i=1}^{\ell} (H_i | H) \partial h_i(z).
\]
We move on to derive the transformation rule for the Virasoro field $L(z)$. The formula in Eq. (4.16) implies

$$e^{-H \otimes h(\xi)} L(z) e^{H \otimes h(\xi)} = L(z) - \partial h(z) H(z) + \frac{1}{2} (H|H) \partial h(z)^2.$$  

(7.4)

Note that $\{H_i\}_{i=1}^\ell$ is an orthonormal basis, thus the corresponding currents $H_i(z)$ are mutually commutative. This enables us to compute the quantity $\Theta_{t-1} L(z) \Theta_t$ so that

$$\Theta_{t-1} L(z) \Theta_t = L(z) - \sum_{i=1}^\ell \partial h_i(z) H_i(z) + \frac{1}{2} \sum_{i=1}^\ell \partial h_i(z)^2.$$  

(7.5)

When we further take the adjoint by $Q(\rho_t)^{-1}$ on it, we obtain

$$\mathcal{G}_t^{-1} L(z) \mathcal{G}_t = L(\rho_t(z)) \partial \rho_t(z)^2 - \sum_{i=1}^\ell \partial h_i(z) \partial \rho_t(z) H_i(\rho_t(z))$$

$$+ \frac{c}{12} (S\rho_t)(z) + \frac{1}{2} \sum_{i=1}^\ell \partial h_i(z)^2.$$  

(7.6)

This relation again helps us to formulate a local martingale $\langle v_A | L(z) \mathcal{G}_t | v_A \rangle$ associated with the random processes $\rho_t(z)$ and $h_i(z)$.

**Theorem 7.2.** Let $g_t(z) = \rho_t(z) + B_t$ be the SLE($\kappa$) and $h_i(z)$ be the solutions of Eq. (5.5). Then the following quantity is a local martingale:

$$\langle v_A | L(z) \mathcal{G}_t | v_A \rangle = h_A \left( \frac{\partial \rho_t(z)}{\rho_t(z)} \right)^2 - \lambda \partial h_t(z) \frac{\partial \rho_t(z)}{\rho_t(z)} + \frac{c}{12} (S\rho_t)(z) + \frac{1}{2} \sum_{i=1}^\ell \partial h_i(z)^2.$$  

(7.7)

Since on our representation space $L_\kappa(\Lambda, 1)$ the Virasoro field is realized by using current fields, the local martingale $\langle v_A | L(z) \mathcal{G}_t | v_A \rangle$ has another description. From the transformation rule of a current field $H(z)$, its positive and negative power parts are transformed as

$$\mathcal{G}_t^{-1} H(z) \mathcal{G}_t = \sum_{m \in \mathbb{Z}} \text{Res}_w \frac{\partial \rho_t(w) \rho_t(w)^{-m-1}}{w - z} H(m) - \text{Res}_w \frac{1}{w - z} \sum_{i=1}^\ell (H_i|H) \partial h_i(w),$$  

(7.8)

$$\mathcal{G}_t^{-1} H(z) \mathcal{G}_t = \sum_{m \in \mathbb{Z}} \text{Res}_w \frac{\partial \rho_t(w) \rho_t(w)^{-m-1}}{z - w} H(m) - \text{Res}_w \frac{1}{z - w} \sum_{i=1}^\ell (H_i|H) \partial h_i(w).$$  

(7.9)

Here rational functions $\frac{1}{z - w}$ and $\frac{1}{w - z}$ are expanded in regions $|z| > |w|$ and $|w| > |z|$, respectively. We will use a similar convention in the following. Thus the local martingale associated with the normal ordered product $:H(z)^2:$ is computed as

$$\langle v_A | :H(z)^2: \mathcal{G}_t | v_A \rangle = (H|H) \text{Res}_w \left[ \frac{\partial \rho_t(w)}{w - z} \frac{1}{\rho_t(w) - \rho_t(z)} - \frac{\partial \rho_t(w)}{z - w} \frac{1}{\rho_t(z) - \rho_t(w)} \left( \frac{\rho_t(z) \rho_t(w)^{-1}}{\rho_t(z) - \rho_t(w)} \right) \right]$$

$$+ (\lambda (H_1|H))^2 \left( \frac{\partial \rho_t(z)}{\rho_t(z)} \right)^2 - 2\lambda (H_1|H) \sum_{i=1}^\ell (H_i|H) \partial h_i(z) \frac{\partial \rho_t(z)}{\rho_t(z)}.$$  

(7.10)
This enables us to derive another form of the local martingale \( \langle v_A | L(z) \mathcal{G} | v_A \rangle \) so that

\[
\langle v_A | L(z) \mathcal{G} | v_A \rangle = \frac{\ell}{2} \text{Res}_w \left[ \frac{\partial \rho_t(w)}{w - z} \partial_z \left( \frac{1}{\rho_t(w) - \rho_t(z)} \right) - \frac{\partial \rho_t(w)}{z - w} \partial_z \left( \frac{\rho_t(z) \rho_t(w)^{-1}}{\rho_t(z) - \rho_t(w)} \right) \right] \\
+ h_A \left( \frac{\partial \rho_t(z)}{\rho_t(z)} \right)^2 - \lambda \frac{\partial \rho_t(z)}{\rho_t(z)} \partial h_A^t(z).
\]

Comparing this with the same quantity, which is seemingly different, derived previously, we obtain an equality among random processes

\[
\frac{\ell}{2} \text{Res}_w \left[ \frac{\partial \rho_t(w)}{w - z} \partial_z \left( \frac{1}{\rho_t(w) - \rho_t(z)} \right) - \frac{\partial \rho_t(w)}{z - w} \partial_z \left( \frac{\rho_t(z) \rho_t(w)^{-1}}{\rho_t(z) - \rho_t(w)} \right) \right] \\
= \frac{c}{12} (S \rho_t(z)) + \frac{1}{2} \sum_{i=1}^{\ell} (\partial h_A^i(z))^2.
\]

7.2. Case when \( g = \mathfrak{sl}_2 \). We formulate in this subsection several local martingales associated with SLEs with affine symmetry that are generated by a local martingale \( \mathcal{G} v_A \) in \( L_{\mathfrak{sl}_2}(\Lambda, k) \). We treat the case when \( \Lambda \) is the fundamental weight of \( \mathfrak{sl}_2 \) and \( k = 1 \) and use the description \( L(\Lambda) = \mathbb{C} v_A \oplus \mathbb{C} v_{-\Lambda} \) of the fundamental weight as in Prop. 6.2.

Firstly we note transformation formulas for current fields \( X(z) \) for \( X = E, H, F \) under the adjoint action by \( \mathcal{G}^{-1}_t \).

**Lemma 7.3.**

\[
\mathcal{G}^{-1}_t E(z) \mathcal{G} = e^{-2h_t(z)} \partial \rho_t(z) E(\rho_t(z)) + e^{-2h_t(z)} f_t(z) \partial \rho_t(z) H(\rho_t(z)) \\
- e^{-2h_t(z)} f_t(z)^2 \partial \rho_t(z) F(\rho_t(z)) - k \partial f_t(z),
\]

\[
\mathcal{G}^{-1}_t H(z) \mathcal{G} = 2e^{-2h_t(z)} e_t(z) \partial \rho_t(z) E(\rho_t(z)) + (1 + 2e^{-2h_t(z)} e_t(z) f_t(z)) \partial \rho_t(z) H(\rho_t(z)) \\
- 2 f_t(z) + 2e^{-2h_t(z)} e_t(z) f_t(z)^2 \partial \rho_t(z) F(\rho_t(z)) \\
- k (2 \partial h_t(z) + 2e^{-2h_t(z)} e_t(z) \partial f_t(z)),
\]

\[
\mathcal{G}^{-1}_t F(z) \mathcal{G} = - e^{-2h_t(z)} e_t(z)^2 \partial \rho_t(z) E(\rho_t(z)) \\
+ (e_t(z) + e^{-2h_t(z)} e_t(z) f_t(z)) \partial \rho_t(z) H(\rho_t(z)) + 2 e_t(z) f_t(z) + e^{-2h_t(z)} e_t(z) f_t(z)^2 \partial \rho_t(z) F(\rho_t(z)) \\
+ k (2 e_t(z) \partial f_t(z) + e^{-2h_t(z)} e_t(z)^2 \partial f_t(z) - \partial e_t(z)).
\]

This will allow us to compute local martingales of the form \( \langle v_{\pm A} | X(z) \mathcal{G} | v_{\pm A} \rangle \) for \( X = E, H, F \).

**Theorem 7.4.** Let \( \Lambda \) be the fundamental weight of \( \mathfrak{sl}_2 \), and the fundamental representation of \( \mathfrak{sl}_2 \) be described by \( L(\Lambda) = \mathbb{C} v_A \oplus \mathbb{C} v_{-\Lambda} \) as in Prop. \( 6.2 \). We assume that \( \kappa \) and \( \tau \) be positive real numbers satisfying the relation \( \kappa + 2\tau - 4 = 0 \). For the SLE(\( \kappa \)) \( g_t(z) = \rho_t(z) + B_t \) and random processes \( e_t(z) \), \( h_t(z) \) and \( f_t(z) \) satisfying the SDEs in Prop. \( 7.4 \), the following quantities are local martingales.
(1) \( X = E \).

\[
\langle v_{\Lambda} | E(z) \mathcal{G}_{t} | v_{\Lambda} \rangle = e^{-2h_{t}(z)} f_{t}(z) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)} - \partial f_{t}(z),
\]

(7.16)

\[
\langle v_{-\Lambda} | E(z) \mathcal{G}_{t} | v_{\Lambda} \rangle = -e^{-2h_{t}(z)} f_{t}(z)^{2} \frac{\partial \rho_{t}(z)}{\rho_{t}(z)};
\]

(7.17)

\[
\langle v_{\Lambda} | E(z) \mathcal{G}_{t} | v_{-\Lambda} \rangle = e^{-2h_{t}(z)} \frac{\partial \rho_{t}(z)}{\rho_{t}(z)};
\]

(7.18)

\[
\langle v_{-\Lambda} | E(z) \mathcal{G}_{t} | v_{-\Lambda} \rangle = -e^{-2h_{t}(z)} f_{t}(z) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)} - \partial f_{t}(z).
\]

(7.19)

\[
\langle v_{\Lambda} | H(z) \mathcal{G}_{t} | v_{\Lambda} \rangle = (1 + 2e^{-2h_{t}(z)} e_{t}(z) f_{t}(z)) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)} - (2\partial h_{t}(z) + 2e^{-2h_{t}(z)} e_{t}(z) \partial f_{t}(z)),
\]

(7.20)

\[
\langle v_{-\Lambda} | H(z) \mathcal{G}_{t} | v_{\Lambda} \rangle = -(2f_{t}(z) + 2e^{-2h_{t}(z)} e_{t}(z) f_{t}(z)^{2}) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)};
\]

(7.21)

\[
\langle v_{\Lambda} | H(z) \mathcal{G}_{t} | v_{-\Lambda} \rangle = 2e^{-2h_{t}(z)} e_{t}(z) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)};
\]

(7.22)

\[
\langle v_{-\Lambda} | H(z) \mathcal{G}_{t} | v_{-\Lambda} \rangle = -(1 + 2e^{-2h_{t}(z)} e_{t}(z) f_{t}(z)) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)} - (2\partial h_{t}(z) + 2e^{-2h_{t}(z)} e_{t}(z) \partial f_{t}(z)).
\]

(7.23)

\[
\langle v_{\Lambda} | F(z) \mathcal{G}_{t} | v_{\Lambda} \rangle = -(e_{t}(z) + e^{-2h_{t}(z)} e_{t}(z)^{2} f_{t}(z)) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)} + (2e_{t}(z) \partial f_{t}(z) + e^{-2h_{t}(z)} e_{t}(z)^{2} \partial f_{t}(z) - \partial e_{t}(z)),
\]

(7.24)

\[
\langle v_{-\Lambda} | F(z) \mathcal{G}_{t} | v_{\Lambda} \rangle = (2e_{t}(z) f_{t}(z) + e^{-2h_{t}(z)} e_{t}(z)^{2} f_{t}(z)^{2}) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)};
\]

(7.25)

\[
\langle v_{\Lambda} | F(z) \mathcal{G}_{t} | v_{-\Lambda} \rangle = -e^{-2h_{t}(z)} e_{t}(z)^{2} \frac{\partial \rho_{t}(z)}{\rho_{t}(z)};
\]

(7.26)

\[
\langle v_{-\Lambda} | F(z) \mathcal{G}_{t} | v_{-\Lambda} \rangle = (e_{t}(z) + e^{-2h_{t}(z)} e_{t}(z)^{2} f_{t}(z)) \frac{\partial \rho_{t}(z)}{\rho_{t}(z)} + (2e_{t}(z) \partial f_{t}(z) + e^{-2h_{t}(z)} e_{t}(z)^{2} \partial f_{t}(z) - \partial e_{t}(z)).
\]

(7.27)

Proof. By assumption, we have that \( \mathcal{G}_{t} | v_{\pm \Lambda} \) are local martingales in \( L^{\mu 0}(\Lambda, 1) \) from Prop. 5.1 and Prop. 6.2. Thus the quantities \( \langle u | \mathcal{G}_{t} | v_{\pm \Lambda} \rangle \) are local martingales. Noticing that \( \langle v_{\pm \Lambda} | \mathcal{G}_{t} = \langle v_{\pm \Lambda} \mid \) and using the formula in Lemma 7.3, we obtain the desired result.

We also compute the local martingales \( \langle v_{\pm \Lambda} | L(z) \mathcal{G}_{t} | v_{\pm \Lambda} \rangle \) for the Virasoro field \( L(z) \). The Virasoro field is found to be transformed under the adjoint action by \( \mathcal{G}_{t}^{-1} \) as follows.
Lemma 7.5.
\[ \mathcal{G}^{-1}_t L(z) \mathcal{G}_t = (\partial \rho_t(z))^2 L(\rho_t(z)) - e^{-2h_t(z)} \partial e_t(z) \partial \rho_t(z) E(\rho_t(z)) - \partial h_t(z) + e^{-2h_t(z)} f_t(z) \partial \rho_t(z) \partial \rho_t(z) H(\rho_t(z)) - (\partial f_t(z) - 2f_t(z) \partial h_t(z) - e^{-2h_t(z)} f_t(z)^2 \partial e_t(z)) \partial \rho_t(z) F(\rho_t(z)) \]
\[ + k((\partial h_t(z))^2 + e^{-2h_t(z)} \partial e_t(z) \partial f_t(z)) + \frac{c}{12}(S \rho_t(z)). \]

Theorem 7.6. Let \( \Lambda \) be the fundamental weight of \( \mathfrak{sl}_2 \), and the fundamental representation of \( \mathfrak{sl}_2 \) be described by \( L(\Lambda) = \mathbb{C} v_\Lambda \oplus \mathbb{C} v_{-\Lambda} \) as in Prop. 7.3. We assume that \( \kappa \) and \( \tau \) be positive real numbers satisfying the relation \( \kappa + 2\tau - 1 = 0 \). For the SLE(\( \kappa \)) \( g_t(z) = \rho_t(z) + B_t \) and random processes \( e_t(z), h_t(z) \) and \( f_t(z) \) satisfying the SDEs in Prop. 7.4, the following quantities are local martingales:

\[ \langle v_{\Lambda} | L(z) \mathcal{G}_t | v_{\Lambda} \rangle = \frac{1}{4} \left( \frac{\partial \rho_t(z)}{\rho_t(z)} \right)^2 - \partial h_t(z) + e^{-2h_t(z)} f_t(z) \partial e_t(z) \partial \rho_t(z) \rho_t(z) \]
\[ + \frac{1}{12}(S \rho_t(z)), \]
\[ \langle v_{-\Lambda} | L(z) \mathcal{G}_t | v_{-\Lambda} \rangle = -\partial f_t(z) - 2f_t(z) \partial h_t(z) - e^{-2h_t(z)} f_t(z)^2 \partial e_t(z) \frac{\partial \rho_t(z)}{\rho_t(z)}, \]

\[ \langle v_{-\Lambda} | L(z) \mathcal{G}_t | v_{-\Lambda} \rangle = -e^{-2h_t(z)} \partial e_t(z) \frac{\partial \rho_t(z)}{\rho_t(z)}, \]
\[ + \frac{1}{12}(S \rho_t(z)). \]

Proof. The proof is analogous to that of Theorem 7.4. We note that on \( L_{\mathfrak{sl}_2}(\Lambda, 1) \), the central charge is \( c = 1 \) and the conformal weight of the highest weight vector \( v_{\Lambda} \) is \( \frac{1}{2} \). \( \square \)

8. Symmetry of the space of local martingales

In the previous section, we saw that a local martingale \( \mathcal{G}_t | v_{\Lambda} \) that takes its value in \( L_{\mathfrak{sl}_2}(\Lambda, k) \) generates several local martingales. We shall describe this phenomenon from a different point of view.

Let \( \mathcal{Y}(-, z) \) be an intertwining operator of type \( \left( L_{\mathfrak{sl}_2}(\Lambda, k), L_{\mathfrak{sl}_2}(\Lambda, k) \right) \). Then for a vector \( v \in L_0(\Lambda, k) \), we have \( \mathcal{Y}(v, z) \mid 0 \rangle = e^{z L_{-1}} v \). This implies that for a vector \( v \in L(\Lambda) \) in the top space of \( L_0(\Lambda, k) \) that is annihilated by an operator of the form of Eq. 5.2,

\[ \mathcal{G}_t v = \Theta_t Q(g_t) \mathcal{Y}(v, B_t) \mid 0 \rangle \]

is a local martingale.

For a generic element in \( \text{Aut}_+ \mathcal{O} \rtimes G_+ \mathcal{O} \) and an intertwining operator \( \mathcal{Y}(-, z) \) of type \( \left( L_{\mathfrak{sl}_2}(\Lambda, k), L_{\mathfrak{sl}_2}(\Lambda, k) \right) \), the quantity

\[ \mathcal{M}_u = \{ u | \mathcal{Y}(-, x) \mid 0 \rangle \} \in L(\Lambda)^* \left[ g_{n+1}, e_n, h_n, f_n | n < 0 \right] [[x]] = \mathcal{F}_{\text{aff}}(\Lambda) \]
symmetry of a space of local martingales in $F_X$ if we find a operator for any vector $u$. Thus we may find the space of local martingales as a subspace of $F_{aff}(\Lambda)$. Since $u$ is arbitrarily taken, the quantity $M_X(t)u$ for $X \in \mathfrak{sl}_2$ and $t \in \mathbb{Z}$ has the same property. Thus if we find a operator $\mathcal{H}_\ell$ such that $M_X(t)u = \mathcal{H}_\ell M_u$, it can describe affine Lie algebra symmetry of a space of local martingales in $F_{aff}(\Lambda)$. The derivation of the operators $\mathcal{H}_\ell$ is presented in Appendix D and we note the results here:

$$
\mathcal{E}_\ell = - \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1} e^{2h(w)} e^{-2h(z) \frac{\ell}{z}} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial e_n} \\
- \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1} e^{-2h(z)} (f(z) - f(w)) z^{-\ell} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial h_n} \\
+ \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1} e^{-2h(z)} (f(z) - f(w))^2 z^{-\ell} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial f_n} \\
+ \text{Res}_z \frac{e^{-2h(z)} x^{-\ell} g'(z)}{g(z) - x} \pi(E) \\
+ \text{Res}_z \frac{e^{-2h(z)} f(z) z^{-\ell} g'(z)}{g(z) - x} \pi(H) \\
- \text{Res}_z \frac{e^{-2h(z)} f(z)^2 z^{-\ell} g'(z)}{g(z) - x} \pi(F) + k \text{Res}_z \partial f(z) e^{-2h(z) \frac{\ell}{z}}.
$$

$$
\mathcal{H}_\ell = - 2 \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1} e^{2h(w)} e^{-2h(z) \frac{\ell}{z}} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial e_n} \\
- \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1}(1 + 2 e^{-2h(z)} (f(z) - f(w))) z^{-\ell} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial h_n} \\
- 2 \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1}(f(w) - f(z) - e^{-2h(z)}) e(z) (f(w) - f(z))^2 z^{-\ell} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial f_n} \\
+ 2 \text{Res}_z \frac{e^{-2h(z)} e(z) z^{-\ell} g'(z)}{g(z) - x} \pi(E) \\
+ \text{Res}_z \frac{(1 + 2 e^{-2h(z)} e(z) f(z)) z^{-\ell} g'(z)}{g(z) - x} \pi(H) \\
- 2 \text{Res}_z \frac{(1 + e^{-2h(z)} e(z) f(z)) f(z) z^{-\ell} g'(z)}{g(z) - x} \pi(F) \\
+ 2k \text{Res}_z \partial h(z) - \partial f(z) e^{-2h(z) \frac{\ell}{z}} \pi(\partial).
$$
\[ F_\ell = \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1} e^{2h(w)} e^{-2h(z)} e(z)^2 z^{-\ell} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial e_n} \]

\[ - \sum_{n \leq -1} \text{Res}_z \text{Res}_w \frac{w^{-n-1} (1 + e^{-2h(z)} e(z) (f(w) - f(z))) e(z)^{-\ell} g'(z)}{g(w) - g(z)} \frac{\partial}{\partial h_n} \]

\[ - \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} \left[ \frac{e^{2h(z)} + 2 e(z)(f(z) - f(w))}{g(w) - g(z)} + \frac{e^{-2h(z)} e(z)^2 (f(z) - f(w))^2}{g(w) - g(z)} \right] z^{-\ell} g'(z) \frac{\partial}{\partial f_n} \]

\[ - \text{Res}_z \frac{e^{-2h(z)} e(z)^2 z^{-\ell} g'(z)}{g(z) - x} \pi(E) \]

\[ - \text{Res}_z \frac{(1 + e^{-2h(z)} e(z) f(z)) e(z)^{-\ell} g'(z)}{g(z) - x} \pi(H) \]

\[ + \text{Res}_z \frac{(e^{2h(z)} + 2 e(z) f(z) + e^{-2h(z)} e(z)^2 f(z)^2) z^{-\ell} g'(z)}{g(z) - x} \pi(F) \]

\[ - \text{Res}_z (2 \partial h(z) e(z) - \partial e(z) + \partial f(z) e^{-2h(z)} e(z)^2) z^{-\ell}. \]

(8.5)

Here the representation \( \pi : \mathfrak{sl}_2 \rightarrow \text{End}(L(\Lambda)^*) \) is defined by \( \pi(X) \phi)(v) = -\phi(Xv) \) for \( X \in \mathfrak{sl}_2, \phi \in L(\Lambda)^* \) and \( v \in L(\Lambda) \).

We can also derive operators \( \mathcal{L}_\ell \) that associate with the action of the Virasoro algebra such that \( M_{L\ell u} = \mathcal{L}_\ell M_u \). The detailed derivation is conducted in Appendix D but it
yields

\[ \mathcal{L} = - \sum_{n \geq 0} \text{Res}_z \text{Res}_w z^{-\ell+1} w^{-n-1} \frac{g'(z)^2}{g(w) - g(z)} \frac{\partial}{\partial g_n} \]

\[ - \sum_{n \geq 1} \text{Res}_z \text{Res}_w z^{-\ell+1} w^{-n-1} e^{2\beta(w)_{-1}} e^{-2\beta(z)_{-1}} \frac{\partial}{\partial e_n} \]

\[ - \sum_{n \geq 1} \text{Res}_z \text{Res}_w z^{-\ell+1} w^{-n-1} \frac{\partial h(z) + e^{2\beta(z)_{-1}} \partial e(z)(f(z) - f(w)) g'(z)_{-1}}{g(w) - g(z)} \frac{\partial}{\partial h_n} \]

\[ - \sum_{n \geq 1} \text{Res}_z \text{Res}_w z^{-\ell+1} w^{-n-1} \left[ \frac{\partial f(z) - 2\partial h(z)(f(z) - f(w))}{g(w) - g(z)} \right] \frac{\partial}{\partial f_n} \]

\[ + \text{Res}_z z^{-\ell+1} \frac{g'(z)^2}{g(z) - x} \left( \frac{h}{(g(z) - x)^2} + \frac{1}{g(z) - x} \frac{\partial}{\partial x} \right) \]

\[ + \text{Res}_z z^{-\ell+1} \frac{g'(z)(z)_{-1}}{g(z) - x} \pi(E) \]

\[ + \text{Res}_z z^{-\ell+1} (\partial h(z) + e^{2\beta(z)_{-1}} f(z) \partial e(z)) g'(z)_{-1} \pi(H) \]

\[ + \text{Res}_z z^{-\ell+1} (\partial f(z) - 2 f(z) \partial h(z) - e^{2\beta(z)_{-1}} f(z)^2 \partial e(z)) g'(z)_{-1} \pi(F) \]

\[ + \text{Res}_z z^{-\ell+1} \left( \frac{c}{12} (Sg)_{-1} + k (\partial h(z)^2 + e^{-2\beta(z)_{-1}} \partial f(z) \partial e(z)) \right). \]

(8.6)

For a vector \( v \in L(\Lambda) \) in the top space of \( L_{\text{aff}}(\Lambda, k) \), the corresponding local martingale \( \mathcal{M}_v \) is a constant function in \( x \) that takes value \( \langle v \mid \cdot \rangle \in L(\Lambda)^* \). Applying the operators \( \mathcal{X}_\ell \) on elements \( \mathcal{M}_v \) for \( v \in L(\Lambda) \), we obtain all local martingales that are generated by a random process \( \mathcal{G}_t \) on \( \text{Aut}_+ \mathbb{O} \ltimes G_+ (\mathbb{O}) \).

**Theorem 8.1.** Assume that we have an operator of the form in Eq. (5.2) that annihilates the highest weight vector of \( L_{\text{aff}}(\Lambda, k) \). Let \( \mathcal{U} \) be the subspace of \( \mathcal{F}_{\text{aff}}(\Lambda) \) that is obtained by applying operators \( \mathcal{X}_\ell \) for \( \mathcal{X} = \mathcal{E}, \mathcal{H}, \mathcal{F} \) and \( \ell \in \mathbb{Z} \) to elements of the form \( \langle u \mid \cdot \rangle \in L(\Lambda)^* \) for \( u \in L(\Lambda) \). Then an element of \( \mathcal{U} \) gives a local martingale when the SLE(\( \kappa \)), the solution of the SDEs in Prop. 5.4, and the Brownian motion of variance \( \kappa \) are substituted. Namely, for an element \( f(g_n, e_n, h_n, f_n, x) \in \mathcal{U} \),

(8.7)

\[ f(g_n(t), e_n(t), h_n(t), f_n(t), B_t)(u) \]

is a local martingale for an arbitrary \( u \in L(\Lambda) \). Here

(8.8)

\[ g_n(z) = z + \sum_{n \geq 0} g_n(t) z^n \]
is the SLE($\kappa$) and

\begin{align}
  e_t(z) &= \sum_{n<0} e_n(t) z^n, \\
  h_t(z) &= \sum_{n<0} h_n(t) z^n, \\
  f_t(z) &= \sum_{n<0} f_n(t) z^n
\end{align}

satisfy the SDEs in Prop. 5.4.

9. Conclusion

In this paper, we have established the group theoretical formulation of SLEs corresponding to affine Lie algebras following previous work by the present author. As illustrated in Sect. 2, SLE/CFT correspondence in the sense of Bauer and Bernard [BB02, BB03a, BB03b] allowed us to compute local martingales associated with SLEs from representations of the Virasoro algebra. Our achievement was to generalize this notion of SLE/CFT correspondence to connection between SDEs and representations of affine Lie algebra. Our strategy was to extend a random process on an infinite-dimensional Lie group Aut, O that was naturally connected to SLEs associated with the Virasoro algebra to a random process on a larger group Aut, O $\ltimes G_+(O)$, which was introduced in Sect. 4. The SDE for a random process on such an infinite-dimensional Lie group was given in Sect. 5 based on consideration of an annihilating operator of a highest weight vector. It is significant that the Virasoro module structure on a representation of an affine Lie algebra was introduced via the Segal-Sugawara construction. Note that the resulting SDEs have already appeared in the correlation function formula [BGLW05, ABI11] of SLEs corresponding to WZW theory in an equivalent form, but we gave another natural derivation of it from a random process on an infinite-dimensional Lie group. We also constructed the random process in the case when the underlying finite-dimensional Lie algebra was commutative and sl_2. A significant achievement in the present paper was the derivation of SDEs in Prop. 5.4 which gives a rigorous formulation of the random process along the internal space. Thus it paves the way for further studies of SLE with internal degrees of freedom in probability theory. Such a construction made it possible in Sect. 7 to formulate several local martingales associated with SLEs from computation on a representation of an affine Lie algebra. We also revealed an affine sl_2 symmetry of a space of local martingales in Sect. 8, which was again possible due to the construction in Sect. 5. It is clear that the content of Sect. 8 can be extended to other affine Lie algebras in principle, but it will be harder to formulate operators defining the action.

Let us discuss other possibility of a random process on Aut, O $\ltimes G_+(O)$. In Sect. 5 we considered a random process $\mathcal{H}_t$ on an infinite-dimensional Lie group Aut, O $\ltimes G_+(O)$, of which the $dt$ term in its increment is an annihilating operator

\begin{equation}
  -2L_{-2} + \frac{\kappa}{2} L_{-1}^2 + \frac{\tau}{2} \sum_{i=1}^{\dim g} X_i(-1)^2
\end{equation}

of the highest weight vector. This annihilator is chosen by the following principle. Firstly, our construction should derive the ordinary SLE in the coordinate transformation part, which forces an annihilator to have the part $-2L_{-2} + \frac{\kappa}{2} L_{-1}^2$. Secondly, the
operator of the above form indeed annihilates the vacuum vector due to the Segal-Sugawara construction of the Virasoro generators. The third term of the annihilator has room for generalization, which we shall discuss. We can allow the variance $\tau$ to depend on $i$; namely, an annihilator of the form

$$-2L_{-2} + \frac{\kappa}{2} L_{-1}^2 + \frac{1}{2} \sum_{i=1}^{\text{dim} g} \tau_i X_i (-1)^2$$

(9.2)

can be considered. We can also deform the annihilator by adding a term like $X (-2)$ for $X \in \mathfrak{g}$. Such a deformation will be inevitable if we twist the Virasoro generators by a derivative of a current field. The problem of whether annihilators generalized in these ways indeed annihilate the highest weight vector, of course, requires individual case-by-case investigation.

A possible application of our construction of SLEs corresponding to affine Lie algebras is to derive a generalization of Cardy’s formula. In the case of Virasoro algebra, SLE/CFT correspondence derives Cardy’s formula. \cite{BB03} We shall discuss the possibility of generalization of Cardy’s formula. This work will be two-fold. One aim is to find an appropriate scaling limit of a model of statistical physics in which a kind of cluster interface is described by the SLE derived in our formulation. An important point to be considered is that our SLE with internal degrees of freedom describes a stochastic deformation of $G$-bundles which requires us to find a scaling limit that captures such internal degrees of freedom as well as a cluster interface itself. The other aim is to relate an object like

$$\langle u| \mathcal{Y}(A_1, z_1) \cdots \mathcal{Y}(A_n, z_n) \mathbb{F} | v_A \rangle$$

(9.3)

to the defining function of an event associated with SLEs with internal degrees of freedom derived in this paper. Here $\mathcal{Y}(-, z)$ is an intertwining operator and $\mathbb{F} | v_A \rangle$ is a representation-space-valued local martingale constructed in this paper. If such a discussion is possible, the probability of the event is computed as the expectation value of the above quantity, which is time independent and thus reduces to a purely algebraic quantity

$$\langle u| \mathcal{Y}(A_1, z_1) \cdots \mathcal{Y}(A_n, z_n) | v_A \rangle$$

(9.4)

and which may be computed.

It is natural to seek other examples of generalization of SLEs involving more general internal symmetry. In a forthcoming paper, \cite{Kos18} we will construct SLEs for which the internal symmetry is described by an affine Lie superalgebra. Since the Segal-Sugawara construction also works for a twisted affine Lie algebra, a parallel construction to ours presented in this paper will be possible for a twisted affine Lie algebra. We consider the case that internal symmetry is encoded in a more complicated Lie algebra to be nontrivial. We can associate with a VOA a Lie algebra called a current Lie algebra. A Lie subalgebra of a current Lie algebra possibly describes an internal symmetry in the terminology of the book by Frenkel and Ben-Zvi. \cite{FBZ04} For example, the current Lie algebra of an affine VOA has the corresponding affine Lie algebra as a subalgebra, and this is the internal symmetry we treated in this paper. However it is not always possible to take such a good Lie subalgebra for a given VOA, and it is nontrivial whether one can construct SLEs with internal degrees of freedom in such situations when we do not know they have a good Lie subalgebra.
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Appendix A. Remarks on VOAs

In this appendix, we recall the notion of vertex (operator) algebras which is useful in the present paper. Detailed expositions of the theory of vertex (operator) algebras can be found in literatures. [Kac98, FBZ04] The appendix of the book by Iohara and Koga [IK11] is also useful.

A.1. Definition of vertex algebras, modules and intertwining operators. Let $V$ be a vector space. A field on $V$ is a series $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$ in a formal variable $z$ with coefficients $a(n)$ being in End($V$) such that for any $v \in V$ we have $a(n)v = 0$ for $n \gg 0$. Equivalently, a field is a linear map from $V$ to $V((z)) = V[[z]][z^{-1}]$. We let the space of fields be denoted by $\text{Fie}(V) := \text{Hom}_\mathbb{C}(V,V((z)))$.

Definition A.1 (Vertex algebra). A vertex algebra is a quadruple $(V,0,T,Y)$ of a vector space $V$, a distinguished vector $|0\rangle \in V$, an operator $T \in \text{End}(V)$, and a linear operator $Y \in \text{Hom}(V,\text{Fie}(V))$, on which the following axioms are imposed:

(A.1) $[T,Y(a,z)] = \partial Y(a,z)$

(A.2) $T|0\rangle = 0$, $Y(|0\rangle , z) = \text{Id}_V$, $Y(a,z)|0\rangle|_{z=0} = a$.

(A.3) $(z-w)^N [Y(a,z),Y(b,w)] = 0$, $N \gg 0$.

Here we have denoted the image of $a \in V$ via $Y$ by $Y(a,z)$.

We often denote a vertex algebra $(V,|0\rangle,T,Y)$ simply by $V$. We also often expand a field $Y(A,z)$ so that $Y(A,z) = \sum_{n \in \mathbb{Z}} A(n) z^{-n-1}$.

Definition A.2. Let $V$ be a vertex algebra and $S \subset V$ be a subset. We say that $V$ is generated by $S$ if $V$ is spanned by vectors of the form $A_{(i_1)}^{1} \cdots A_{(i_n)}^{n}|0\rangle$ for $A^{1}, \cdots, A^{n} \in S$, $i_1, \cdots, i_n \in \mathbb{Z}_{\geq 1}$ and $n \geq 0$.

Definition A.3. A vertex algebra $V$ is said to be $\mathbb{Z}$-graded if it admits a $\mathbb{Z}$-gradation $V = \bigoplus_{n \in \mathbb{Z}} V_n$ such that $|0\rangle \in V_0$, $TV_n \subset V_{n+1}$, and $(V_h)_{(n)}(V_{h'}) \subset V_{h+h'-n-1}$ for any $h, h', n \in \mathbb{Z}$. We say that a vector in $V_h$ has conformal weight $h$.

Definition A.4. A vector $\omega \in V$ is a conformal vector of central charge $c$ if the coefficients of $Y(\omega,z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ define a representation of the Virasoro algebra of central charge $c$, or explicitly satisfy the commutation relation

(A.4) $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n,0}$.
with $L_{-1} = T$, and $L_0$ is diagonalizable on $V$. A vertex algebra endowed with a conformal vector $\omega$ is called a conformal vertex algebra of central charge $c$. The field $Y(\omega, z)$ is called a Virasoro field of the conformal vertex algebra $V$.

**Definition A.5** (Vertex operator algebra). A $\mathbb{Z}$-graded conformal vertex algebra $(V = \bigoplus_{n \in \mathbb{Z}} V_n, \omega)$ is called a VOA if
- $L_0|_{V_n} = n \omega_{n}$ for all $n \in \mathbb{Z}$.
- $\dim V_n < \infty$ for all $n \in \mathbb{Z}$.
- There exists $N \in \mathbb{Z}$ such that $V_n = \{0\}$ for $n < N$.

**Definition A.6.** Let $(V, |0\rangle, T, Y, \omega)$ be a VOA. A weak $V$-module is a pair $(M, Y^M)$ of a vector space $M$ and a linear map $Y^M : V \to \text{End}(M)[[z, z^{-1}]]$ satisfying the following conditions:
- $Y^M(|0\rangle, z) = \text{id}_M$.
- For arbitrary $A \in V$ and $v \in M$,
  \[ Y^M(A, z)v \in M((z)). \]
- For arbitrary $A, B \in V$ and $m, n \in \mathbb{Z}$,
  \[ \text{Res}_{z-w} Y^M(Y(A, z-w)B, w)i_{z-w}z^m(z-w)^n \]
  \[ = \text{Res}_z Y^M(A, z)Y^M(B, w)i_{z-w}z^m(z-w)^n \]
  \[ - \text{Res}_z Y^M(B, w)Y^M(A, z)i_{z-w}z^m(z-w)^n. \]

Here for a rational function $R(z, w)$ in two variables $z$ and $w$ possibly with poles at $z = 0$, $w = 0$, and $z = w$, we denote its expansion in the region $|z| > |w|$ by $i_{z, w}R(z, w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$.

For a weak $V$-module $(M, Y^M)$, the image of $A \in V$ by $Y^M$ is expressed as
\[(A.5) \quad Y^M(A, z) = \sum_{n \in \mathbb{Z}} A^M_n z^{-n-1}\]
with $A^M_n \in \text{End}(M)$.

If $A \in V$ has the conformal weight $\Delta$, it is convenient to expand $Y^M(A, z)$ as
\[(A.6) \quad Y^M(A, z) = \sum_{n \in \mathbb{Z}} A^M_n z^{-n-\Delta}\]
so that $\text{deg} A^M_n = -n$.

**Definition A.7.** Let $V$ be a VOA and $\omega \in V$ be the conformal vector of $V$. An ordinary $V$-module is a weak $V$-module $M$ such that
- $L^M_0$ in the expansion
  \[ Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L^M_n z^{-n-2} \]
is diagonalizable on $M$.
- In the $L^M_0$-eigenspace decomposition
  \[ M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda, \]
  \[ \dim M_\lambda < \infty \text{ for all } \lambda \in \mathbb{C}. \]
  Moreover, for arbitrary $\lambda \in \mathbb{C}$, $M_{\lambda-n} = 0$ for $n \gg 0$. 

Definition A.8. Let $M^1$, $M^2$ and $M^3$ be $V$-modules. An intertwining operator of type $(\frac{M^1}{M^2}, \frac{M^3}{M^3})$ is a linear operator

$$\mathcal{Y}(\cdot, z) : M^1 \to \text{Hom}(M^2, M^3) z^K := \left\{ \sum_{\alpha \in K} v_\alpha z^\alpha \middle| v_\alpha \in \text{Hom}(M^2, M^3) \right\},$$

where $K = \bigcup_i (\alpha_i + \mathbb{Z})$ with finitely many $\alpha_i \in \mathbb{C}$ being chosen associated with $M^1$, $M^2$ and $M^3$ with the following conditions imposed:

- For any $A \in V$, $v \in M^1$ and $m, n \in \mathbb{Z}$ we have
  $$\text{Res}_{z-w} \mathcal{Y}(Y^{M^1}(A, z-w)v, w) i_{w, z-w} z^m (z-w)^n = \text{Res}_z Y^{M^1}(A, z) \mathcal{Y}(v, w) i_{w, z} z^m (z-w)^n - \text{Res}_z \mathcal{Y}(v, w) Y^{M^1}(A, z) i_{w, z} z^m (z-w)^n.$$

- For any $v \in M^1$, we have

$$\mathcal{Y}(L_{-1} v, z) = \frac{d}{dz} \mathcal{Y}(v, z).$$

A.2. Examples.

A.2.1. Virasoro vertex algebra. In Sect. [2] we have introduced two types of representations of the Virasoro algebra, Verma modules and their simple quotients. We can also consider intermediate objects in the theory of VOAs. The Verma module $M(c, 0)$ of highest weight $(c, 0)$ has a submodule generated by $L_{-1} 1_{c, 0}$. Then the universal Virasoro VOA of central charge $c$ is defined by

$$V_c := M(c, 0)/U(Vir_{c, 0})L_{-1} 1_{c, 0}.$$

Now we prepare the components of the vertex algebra structure on $V_c$.

- $|0\rangle = 1_{c, 0}$,
- $T = L_{-1}$,
- A single generator $\omega = L_{-2}|0\rangle$ with the corresponding field $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$.

From these data, we construct a vertex algebra structure on $V_c$ by

$$Y(L_{j_1} \cdots L_{j_k} |0\rangle, z) = i \partial^{(-j_1-2)} L(z) \cdots \partial^{(-j_k-2)} L(z),$$

with $L(z) = Y(\omega, z)$. Moreover, $V$ is $\mathbb{Z}$-graded by

$$\text{deg}(L_{j_1} \cdots L_{j_k} |0\rangle) = -\sum_{i=1}^k j_i.$$

Then $\omega \in V_2$ and $\text{deg} L_n = -n$, implying $V$ is equipped with a $\mathbb{Z}$-graded vertex algebra. We also see $\omega$ is a conformal vector, and $V$ is a VOA. It is obvious that the maximal proper submodule of $V_c$ as a Vir-module is a vertex subalgebra. Thus the irreducible representation $L(c, 0)$ of the Virasoro algebra also carries a vertex algebra structure and we denote this vertex algebra by $L_c$.

Modules over $L_c$ are realized as simple highest weight representations $L(c, h)$ of the same central charge. Note that an arbitrary simple representation of the Virasoro algebra
is not necessarily a module over \( L_c \), since nontrivial relations may be imposed on the VOA \( L_c \). For instance, if the central charge is given by

\[
(A.12) \quad c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}
\]

with coprime integers \( p \) and \( q \) greater than or equal to 2, the corresponding Virasoro VOA is rational and its simple modules are exhausted by \( L(c_{p,q}, h_{p,q,r,s}) \) with

\[
(A.13) \quad h_{p,q,r,s} = \frac{(rp-sq)^2 - (p-q)^2}{4pq}, \quad 0 < r < q, \ 0 < s < p.
\]

A.2.2. **Affine vertex algebra.** Representations \( \widehat{L}(0)_k \) and \( L_{\hat{g},k} \) of an affine Lie algebra \( \hat{g} \) introduced in Sect.3 are also equipped with VOA structure by the following data:

- \( |0\rangle = v_0 \),
- \( T = L_{-1} \),
- Generators \( X_a(-1)|0\rangle \) for \( a = 1, \ldots, \dim \hat{g} \) with the corresponding fields \( X_a(z) = \sum_{n \in \mathbb{Z}} X_a(n) z^{-n-1} \).

Modules over an affine VOA are realized as \( L_{\hat{g}}(\Lambda,k) \) of the same level \( k \), but again all these representations of the affine Lie algebra are not necessarily modules over the simple VOA \( L_{\hat{g},k} \). Indeed, we have the following example.

**Theorem A.9** (Frenkel-Zhu \[FZ92\]). Let \( \hat{g} \) be a finite-dimensional simple Lie algebra and \( k \in \mathbb{Z}_{>0} \). The simple \( L_{\hat{g},k} \)-modules are exhausted by \( L_{\hat{g}}(\Lambda,k) \) with \( \Lambda \in P^k_c \), where \( P^k_c \) is the set of dominant weights of level \( k \) defined by

\[
(A.14) \quad P^k_c = \{ \Lambda \in P_c | (\theta|\Lambda) \leq k \}.
\]

A.2.3. **Lattice vertex algebra.** Let \( L \) be a nondegenerate even lattice of rank \( \ell \); namely, it is a free \( \mathbb{Z} \)-module of rank \( \ell \) endowed with a nondegenerate \( \mathbb{Z} \)-bilinear form \( (\cdot|\cdot) : L \times L \rightarrow \mathbb{Z} \), such that \( (\alpha|\alpha) \in 2\mathbb{Z} \) for \( \alpha \in L \). There uniquely exists a cohomology class \( [\epsilon] \in H^2(L, \mathbb{C}^*) \) satisfying

\[
(A.15) \quad \epsilon(0,0) = \epsilon(0,\alpha) = 1,
\]

\[
(A.16) \quad \epsilon(\alpha,\beta) = (-1)^{(\alpha|\beta)+|\alpha|^2|\beta|^2} \epsilon(\beta,\alpha)
\]

for \( \alpha, \beta \in L \). Here we denote \( |\alpha|^2 = (\alpha|\alpha) \). Notice that conditions Eq. (A.15) and Eq. (A.16) are independent of the choice of a representative \( \epsilon \) of \([\epsilon]\). It can be shown that we can choose a 2-cocycle \( \epsilon \in [\epsilon] \) so that it takes values in \( \{ \pm 1 \} \). (See Remark 5.5a in the booklet by Kac. \[Kac90\]) We let \( \epsilon \) be such a 2-cocycle in the following. Let \( \mathbb{C}_\epsilon[L] \) be the \( \epsilon \)-twisted group algebra of \( L \), which is

\[
(A.17) \quad \mathbb{C}_\epsilon[L] = \bigoplus_{\alpha \in L} \mathbb{C} e^\alpha
\]

as a vector space with multiplication defined by

\[
(A.18) \quad e^\alpha e^\beta = \epsilon(\alpha,\beta) e^{\alpha+\beta}
\]

for \( \alpha, \beta \in L \).

We set \( \mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L \) and extend the symmetric \( \mathbb{Z} \)-bilinear form \((\cdot|\cdot)\) on \( L \) to a symmetric \( \mathbb{C} \)-bilinear form on \( \mathfrak{h} \). Then we obtain the corresponding Heisenberg algebra \( \mathfrak{h} \) and its
The vacuum representation $L_0(0,1)$ of level 1. The lattice vertex algebra $V_L$ associated to $L$ is

$$V_L = L_0(0,1) \otimes \mathbb{C}_\ell[L]$$

as a vector space. We define the action of $\tilde{\mathfrak{h}}$ on $V_L$ by

$$H(m).\left(s \otimes e^\alpha\right) \coloneqq (H(m) + \delta_{m,0}(H(\alpha)))s \otimes e^\alpha$$

for $H \in \mathfrak{h}$, $m \in \mathbb{Z}$, $s \in L_0(0,1)$, and $\alpha \in L$. We also define the action of $\mathbb{C}_\ell[L]$ on $V_L$ by

$$\epsilon^\beta.(s \otimes e^\alpha) \coloneqq \epsilon(\beta,\alpha)s \otimes e^{\alpha + \beta}$$

for $\alpha, \beta \in L$ and $s \in L_0(0,1)$. The lattice vertex algebra is generated by vectors $H(-1)|0\rangle \otimes e^0$ with $H \in \mathfrak{h}$ and $|0\rangle \otimes e^\alpha$ with $\alpha \in L$, of which the corresponding fields are given by

$$H(z) = \sum_{n \in \mathbb{Z}} H(n)z^{-n-1},$$

$$\Gamma_{\omega}(z) = e^\alpha z \omega(0) e^{-\Sigma_{\alpha \alpha} \frac{z^j}{j} \alpha(j)} e^{-\Sigma_{\alpha \alpha} \frac{z^j}{j} \alpha(j)},$$

respectively. Then $V_L$ admits a unique structure of a vertex algebra.

Let $\{H_\ell\}_{\ell=1}^\ell$ be an orthonormal basis of $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle$. Then the vector

$$\omega = \frac{1}{2} \sum_{i=1}^\ell H_{\ell}(1)^2 |0\rangle \otimes e^0$$

is a conformal vector of central charge $\ell$.

The irreducible $V_L$-modules are classified by elements of $L^*/L$. [Don93] Here $L^*$ is the dual lattice of $L$ in $\mathfrak{h}$, then $L$ is naturally a sublattice of $L^*$. For $\varpi \in L^*$, we can construct a $V_L$-module in the following way. Let $\mathbb{C}[L+\varpi]$ be a vector space spanned by elements of $L+\varpi$ so that $\mathbb{C}[L+\varpi] = \bigoplus_{\beta \in \mathbb{Z}L} e^{\beta+\varpi}$, on which a Lie subalgebra $\mathfrak{h} \otimes \mathbb{C}[\varpi] \otimes \mathbb{C}K$ of the Heisenberg algebra acts as $H(m)e^{\beta+\varpi} = 0$ for $m > 0$ and $H(0)e^{\beta+\varpi} = (H(\beta+\varpi)e^{\beta+\varpi}$ for $H \in \mathfrak{h}$ and $\beta \in L$, and $K = \text{Id}$. Then the $V_L$-module $V_{L+\varpi}$ is constructed as

$$V_{L+\varpi} = \text{Ind}_{\mathfrak{h} \otimes \mathbb{C}[\varpi] \otimes \mathbb{C}K}^{\mathbb{C}[L+\varpi]} \mathbb{C}[L+\varpi],$$

on which the action of $V_L$ is defined in an obvious way. It is also clear that $V_{L+\varpi}$ depends only on the equivalence class $[\varpi]$ of $\varpi$ in $L^*/L$.

### A.3. Frenkel-Kac construction.

One of the most significant examples of lattice vertex algebras is one associated with a root lattice of ADE type, which is isomorphic to the irreducible affine vertex algebra associated with the corresponding Lie algebra. We shall explain this example.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of ADE type and fix its Cartan subalgebra $\mathfrak{h}$. Correspondingly we denote the set of roots by $\Delta$, and the root lattice by $Q = \mathbb{Z}\Delta$. Let $\langle \cdot, \cdot \rangle$ be the nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ normalized so that $\langle \theta, \theta \rangle = 2$ for the highest root $\theta$. Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be the set of simple roots, then they form a basis for the root lattice. We also denote the root space decomposition of $\mathfrak{g}$ by $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = CE_\alpha$ is the root space of the root $\alpha \in \Delta$ spanned by normalized vector $E_\alpha$ so that $\langle E_\alpha, E_-\alpha \rangle = 1$, and the set of simple coroots by $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_\ell^\vee\}$.
\textbf{Theorem A.10} (Frenkel-Kac \cite{FK80}). \textit{There is an isomorphism }$L_{\mathfrak{g},k} \to V_Q$ \textit{of vertex algebras such that}

\begin{equation}
\alpha_i^\vee(-1)|0\rangle \mapsto \alpha_i(-1)|0\rangle, \quad E_{\alpha}(-1)|0\rangle \mapsto e^\alpha, \quad \alpha \in \Delta.
\end{equation}

\section*{Appendix B. Ito Process on a Lie Group}

This appendix is devoted to a short description of Ito processes on Lie groups. Detailed expositions on this matter can be found in the literature. \cite{Chi12,App14}

Let $G$ be a finite-dimensional complex Lie group and $\mathfrak{g}$ be its Lie algebra. A strategy to construct an Ito process on the Lie group $G$ may be exponentiating an Ito process on the Lie group $\mathfrak{g}$. For convenience of description, we take a basis $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$. Then an Ito process $X_t$ on $\mathfrak{g}$ expanded in this basis so that $X_t = \sum_{i=1}^{\dim \mathfrak{g}} x_t^i X_i$, where $x_t^i$ are Ito processes that are characterized by SDEs of the form

\begin{equation}
dx_t^i = \bar{x}_t^i dt + \sum_{j \in \mathcal{I}} x_t^{(j)i} dB_t^{(j)}.
\end{equation}

Here $\bar{x}_t^i$ and $x_t^{(j)i}$ are random processes with proper finiteness properties, and $B_t^{(j)}$ are mutually independent Brownian motions labeled by a set $\mathcal{I}_B$. We set the variance of $B_t^{(j)}$ as $\kappa_j$. Then we can obtain a random process $g_t$ on $G$ by exponentiating $X_t$ as $g_t = \exp(X_t)$, but it is not easy to formulate the SDE on $g_t$ due to noncommutativity in the Lie algebra $\mathfrak{g}$.

Instead, we construct a random process on $G$ via the McKean-Gangolli injection. \cite{McK05} In this approach, we identify the value $X_t$ at each time $t$ as a left invariant vector field on $G$, and a random process $g_t$ on $G$ evolves along this random vector field. Then the infinitesimal time evolution of $g_t$ is described by

\begin{equation}
g_{t+dt} = g_t \exp \left( \sum_{i=1}^{\dim \mathfrak{g}} dx_t^i X_i \right).
\end{equation}

To formulate the SDE on such a constructed $g_t$, we take finite-dimensional faithful representation $V$ of $\mathfrak{g}$. Then on the vector space $V$, an action of $G$ is defined by exponentiating the action of $\mathfrak{g}$. In the following, we do not distinguish an element of $\mathfrak{g}$ from its action on $V$. When we expand the exponential function in Eq. (B.2) and notice that quadratic terms in $dx_t^i$ may give contributions proportional to $dt$, we obtain an SDE

\begin{equation}
g_t^{-1} dg_t = \left( \sum_{i=1}^{\dim \mathfrak{g}} \bar{x}_t^i X_i + \frac{1}{2} \sum_{j \in \mathcal{I}_B} \kappa_j \left( \sum_{i=1}^{\dim \mathfrak{g}} x_t^{(j)i} X_i \right)^2 \right) dt + \sum_{i=1}^{\dim \mathfrak{g}} x_t^{(j)i} dB_t^{(j)}.
\end{equation}

We regard this equation as the standard form of an SDE on an Ito processes on a Lie group.

We have to handle a random process on an infinite-dimensional Lie group in application to SLE. The construction above can be naturally extended to an infinite-dimensional setting. Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra and $G$ be the corresponding Lie group. Examples of such infinite-dimensional Lie groups include the group of coordinate transformations $\text{Aut} \mathcal{O}$ on a formal disk, loop groups of finite-dimensional Lie groups and their semi-direct products. In typical cases, a faithful representation $V$ of $\mathfrak{g}$ is infinite dimensional, thus it is, in general, nontrivial whether the action of $\mathfrak{g}$ on $V$ is exponentiated to an action of $G$, but we assume that it is. The validity of this assumption can be
Then by Ito calculus, we obtain
\[ \Theta^{-1}_t d\Theta_t = - \frac{\kappa}{2} L_2 + \frac{\tau}{2} \sum_{r=1}^{3} X_r(-1)^2 \] 
\[ dt + L_1 dB^{(0)}_t - \sum_{r=1}^{3} X_r(-1) dB^{(r)}_t. \]

Here \( \{X_r\}_{r=1}^{3} \) is an orthonormal basis of \( \mathfrak{g} \) defined by
\[ X_1 = \frac{1}{\sqrt{2}} H, \quad X_2 = \frac{1}{\sqrt{2}} (E + F), \quad X_3 = \frac{i}{\sqrt{2}} (E - F), \]
and \( B^{(i)}_t, i = 0, 1, 2, 3 \) are independent Brownian motions with variances given by
\[ dB^{(0)}_t \cdot dB^{(0)}_t = \kappa dt, \quad dB^{(r)}_t \cdot dB^{(r)}_t = \tau dt, \quad r = 1, 2, 3. \]

Since each element \( X \otimes f(\zeta) \) in the affine Lie algebra transforms under adjoint action by \( Q(\rho_t) \) as \( Q(\rho_t)^{-1} X \otimes f(\zeta)(Q(\rho_t) = X \otimes f(\rho_t^{-1}(\zeta)) \), it suffices to derive SDEs so that \( \Theta_t = e^{\omega_t} e^{h_t} e^{f_t} \) satisfies
\[ \Theta^{-1}_t d\Theta_t = - \frac{3}{2} \sum_{r=1}^{3} (X_r \otimes \rho_t(\zeta)^{-1})^2 dt - \sum_{r=1}^{3} X_r \otimes \rho_t(\zeta)^{-1} dB^{(r)}_t. \]

We suppose that \( e_t(\zeta), h_t(\zeta), \) and \( f_t(\zeta) \) satisfy
\[ de_t(\zeta) = \tilde{e}_t(\zeta) dt + \sum_{r=1}^{3} e_t^r(\zeta) dB^{(r)}_t, \]
\[ dh_t(\zeta) = \tilde{h}_t(\zeta) dt + \sum_{r=1}^{3} h_t^r(\zeta) dB^{(r)}_t, \]
\[ df_t(\zeta) = \tilde{f}_t(\zeta) dt + \sum_{r=1}^{3} f_t^r(\zeta) dB^{(r)}_t. \]

Then by Ito calculus, we obtain
\[ de^{\omega_t} = e^{\omega_t} \left( E \otimes \tilde{e}_t(\zeta) + \frac{\tau}{2} (E \otimes e^r_t(\zeta))^2 \right) dt + e^{\omega_t} \sum_{r=1}^{3} E \otimes e^r_t(\zeta) dB^{(r)}_t, \]
\[ dh^{h_t} = e^{h_t} \left( H \otimes \tilde{h}_t(\zeta) + \frac{\tau}{2} (H \otimes h^r_t(\zeta))^2 \right) dt + e^{h_t} \sum_{r=1}^{3} H \otimes h^r_t(\zeta) dB^{(r)}_t, \]
\[ df^{f_t} = e^{f_t} \left( F \otimes \tilde{f}_t(\zeta) + \frac{\tau}{2} (F \otimes f^r_t(\zeta))^2 \right) dt + e^{f_t} \sum_{r=1}^{3} F \otimes f^r_t(\zeta) dB^{(r)}_t. \]

The increment of \( \Theta_t \) is also computed as
\[ d\Theta_t = (de^{\omega_t}) h_t e^{f_t} + e^{\omega_t} (dh^{h_t}) e^{f_t} + e^{\omega_t} h_t (df^{f_t}) + (de^{\omega_t}) e^{h_t} (df^{f_t}) + e^{\omega_t} (h_t e^{f_t}) \]
\[ (de^{\omega_t})(dh^{h_t}) e^{f_t} + (de^{\omega_t}) e^{h_t} (df^{f_t}) + e^{\omega_t} (h_t e^{f_t}) (df^{f_t}). \]
Terms in the increment $d\Theta_t$ proportional to increments of the Brownian motions are

\[
\sum_{r=1}^{3}\left( E \otimes e^{-2h_t(\zeta)} e^t(\zeta) \\
+ H \otimes (e^{-2h_t(\zeta)} f_t(\zeta) e^t(\zeta) + h_t(\zeta)) \\
+ F \otimes (f_t(\zeta) - e^{-2h_t(\zeta)} f_t(\zeta)^2 e^t(\zeta) - 2 f_t(\zeta) h_t(\zeta)) \right) dB_t(r)
\]

(C.12)

Comparing this to $\sum_{r=1}^{3} X_r \otimes \rho_t(\zeta)^{-1} dB_t(r)$, we identify $e^t(\zeta)$, $h^t(\zeta)$ and $f^t(\zeta)$ as

(C.13) $e^1_t(\zeta) = 0, \quad h^1_t(\zeta) = -\frac{1}{\sqrt{2\rho_t(\zeta)}}, \quad f^1_t(\zeta) = -\frac{\sqrt{2} f_t(\zeta)}{\rho_t(\zeta)}$

(C.14) $e^2_t(\zeta) = -\frac{e^{2h_t(\zeta)}}{\sqrt{2\rho_t(\zeta)}}, \quad h^2_t(\zeta) = \frac{f_t(\zeta)}{\sqrt{2\rho_t(\zeta)}}, \quad f^2_t(\zeta) = -\frac{1 - f_t(\zeta)^2}{\sqrt{2\rho_t(\zeta)}}$

(C.15) $e^3_t(\zeta) = -\frac{i e^{2h_t(\zeta)}}{\sqrt{2\rho_t(\zeta)}}, \quad h^3_t(\zeta) = \frac{i f_t(\zeta)}{\sqrt{2\rho_t(\zeta)}}, \quad f^3_t(\zeta) = \frac{i(1 + f_t(\zeta)^2)}{\sqrt{2\rho_t(\zeta)}}$

Then the term in the increment $d\Theta_t$ proportional to $dt$ becomes

\[
E \otimes e^{-2h_t(\zeta)} e^t(\zeta) \\
+ H \otimes \left( h_t(\zeta) + e^{-2h_t(\zeta)} f_t(\zeta) e^t(\zeta) + \frac{\tau}{2\rho_t(\zeta)^2} \right) \\
+ F \otimes \left( f_t(\zeta) - e^{-2h_t(\zeta)} f_t(\zeta)^2 e^t(\zeta) - 2 f_t(\zeta) h_t(\zeta) - \frac{\tau f_t(\zeta)}{\rho_t(\zeta)^2} \right) \\
+ \frac{\tau}{2} \sum_{r=1}^{3} (X_r \otimes \rho(\zeta)^{-1})^2
\]

(C.16)

Comparing this to $\frac{1}{2} \sum_{r=1}^{3} (X_r \otimes \rho(\zeta)^{-1})^2$, we obtain

(C.17) $e_t(\zeta) = 0, \quad h_t(\zeta) = -\frac{\tau}{2\rho(\zeta)^2}, \quad f_t(\zeta) = 0$

We can finally formulate SDEs

(C.18) $de_t(\zeta) = -\frac{e^{2h_t(\zeta)}}{\sqrt{2\rho_t(\zeta)}} dB_t(2) - \frac{i e^{2h_t(\zeta)}}{\sqrt{2\rho_t(\zeta)}} dB_t(3)$

(C.19) $dh_t(\zeta) = -\frac{\tau}{2\rho(\zeta)^2} dB_t(1) - \frac{1}{\sqrt{2\rho_t(\zeta)}} dB_t(1) + \frac{f_t(\zeta)}{\sqrt{2\rho_t(\zeta)}} dB_t(2) + \frac{i f_t(\zeta)}{\sqrt{2\rho_t(\zeta)}} dB_t(3)$

(C.20) $df_t(\zeta) = -\frac{\sqrt{2} f_t(\zeta)}{\rho(\zeta)} dB_t(1) - \frac{1 - f_t(\zeta)^2}{\sqrt{2\rho_t(\zeta)}} dB_t(2) + \frac{i(1 + f_t(\zeta)^2)}{\sqrt{2\rho_t(\zeta)}} dB_t(3)$

**Appendix D. Derivation of operators $\mathcal{X}_t$**

In this appendix, we derive the operators $\mathcal{X}_t$ in Sect.8 that define an action of $\mathcal{H}_2$ on a space of SLE local martingales.

We first derive differential equations satisfied by $\mathcal{G} = e^\epsilon e^h e^f Q(g)$. Here $e = E \otimes e(\zeta)$, $h = H \otimes h(\zeta)$ and $f = F \otimes f(\zeta)$ are elements in $\mathfrak{g} \otimes \mathbb{C}[[\zeta^{-1}]]\zeta^{-1}$ with $e(\zeta) = \sum_{n<0} e_n \zeta^n$, $h(\zeta) = \sum_{n<0} h_n \zeta^n$, $f(\zeta) = \sum_{n<0} f_n \zeta^n$. 


\[ h(\zeta) = \sum_{n<0} h_n \zeta^n, \quad f(\zeta) = \sum_{n<0} f_n \zeta^n. \] The element \( g \in \text{Aut}_+ \mathcal{O} \) is identified with a Laurent series \( g(z) = z + \sum_{n<0} g_n z^n \). By differentiating \( G \) by \( e_n \) we obtain

\begin{equation}
\frac{\partial G}{\partial e_n} = e^a E \otimes \zeta^n e^h f G(g).
\end{equation}

After transferring \( E \otimes \zeta^n \) to the rightest position in the product, we have a differential equation

\begin{equation}
G^{-1} \frac{\partial G}{\partial e_n} = E \otimes e^{-2h(g^{-1}(\zeta))} g^{-1}(\zeta)^n + H \otimes e^{-2h(g^{-1}(\zeta))} f(g^{-1}(\zeta)) g^{-1}(\zeta)^n
\end{equation}

\begin{equation}
- F \otimes e^{-2h(g^{-1}(\zeta))} f(g^{-1}(\zeta))^2 g^{-1}(\zeta)^n
\end{equation}

Similarly, we can compute derivatives of \( G \) in variables \( h_n \) and \( f_n \) as

\begin{equation}
G^{-1} \frac{\partial G}{\partial h_n} = H \otimes g^{-1}(\zeta)^n - 2F \otimes f(g^{-1}(\zeta)) g^{-1}(\zeta)^n,
\end{equation}

\begin{equation}
G^{-1} \frac{\partial G}{\partial f_n} = F \otimes g^{-1}(\zeta)^n.
\end{equation}

We shall invert these relations; namely, we express an object like \( G X \otimes \theta(\zeta) \) for a certain \( \theta(\zeta) \in \mathbb{C}[\zeta^{-1}] \) by a linear combination of derivatives of \( G \).

**Lemma D.1.** Let \( \theta(\zeta) \in \mathbb{C}[\zeta^{-1}] \). Then

\begin{equation}
G F \otimes \theta(\zeta) = \sum_{n \leq 1} \left( \text{Res}_{w} w^{-n-1} \theta(g(w)) \right) \frac{\partial G}{\partial f_n},
\end{equation}

\begin{equation}
G H \otimes \theta(\zeta) = \sum_{n \leq 1} \left( \text{Res}_{w} w^{-n-1} \theta(g(w)) \right) \frac{\partial G}{\partial h_n} + 2 \sum_{n \leq -1} \left( \text{Res}_{w} w^{-n-1} f(w) \theta(g(w)) \right) \frac{\partial G}{\partial f_n},
\end{equation}

\begin{equation}
G E \otimes \theta(\zeta) = \sum_{n \leq -1} \left( \text{Res}_{w} w^{-n-1} e^{2h(w)} \theta(g(w)) \right) \frac{\partial G}{\partial e_n} - \sum_{n \leq -1} \left( \text{Res}_{w} w^{-n-1} f(w) \theta(g(w)) \right) \frac{\partial G}{\partial h_n}
\end{equation}

\begin{equation}
- \sum_{n \leq -1} \left( \text{Res}_{w} w^{-n-1} f(w)^2 \theta(g(w)) \right) \frac{\partial G}{\partial f_n}.
\end{equation}

**Proof.** We have to search for an infinite series \( a(z) = \sum_{n \leq \ell} a_n z^n \) such that \( a(g^{-1}(\zeta)) = \theta(\zeta) \) for a given infinite series \( \theta(\zeta) \in \mathbb{C}[\zeta^{-1}] \). Such an infinite series is indeed obtained by setting \( a_n = \text{Res}_{w} w^{-n-1} \theta(g(w)) \), which enables us to obtain the desired result.

We next prepare formulas to compute \( G^{-1} X(-\ell) G \) for \( X \in \mathfrak{sl}_2 \) and \( \ell \in \mathbb{Z} \), which is straightforward from the formulas in Subsect 4.2.
Lemma D.2. We set $\xi := g^{-1}(\zeta)$:

$$\mathcal{G}^{-1} E \otimes \zeta^{-\ell} \mathcal{G} = E \otimes e^{-2h(\zeta)}\zeta^{-\ell} + H \otimes e^{-2h(\zeta)} f(\zeta)\xi^{-\ell}$$

(D.8)

$$- F \otimes e^{-2h(\zeta)} f(\zeta)^2 \xi^{-\ell} - k \text{Res}_w \partial f(w)e^{-2h(w)}w^{-\ell},$$

$$\mathcal{G}^{-1} H \otimes \zeta^{-\ell} \mathcal{G} = 2E \otimes e^{-2h(\zeta)}e(\xi)\xi^{-\ell} + H \otimes (1 + 2e^{-2h(\zeta)}e(\xi)f(\zeta))\xi^{-\ell}$$

$$- 2F \otimes (f(\xi) + e^{-2h(\zeta)}e(\xi)f(\xi)^2)\xi^{-\ell}$$

(D.9)

$$- 2k \text{Res}_w (\partial h(w) + \partial f(w)e^{-2h(w)}e(w))w^{-\ell},$$

$$\mathcal{G}^{-1} F \otimes \zeta^{-\ell} \mathcal{G} = - E \otimes e^{-2h(\zeta)}e(\xi)^2\xi^{-\ell} - H \otimes (e(\xi) + e^{-2h(\zeta)}e(\xi)^2f(\xi))\xi^{-\ell}$$

$$+ F \otimes (e^{2h(\xi)} + 2e(\xi)f(\xi) + e^{-2h(\xi)}e(\xi)^2f(\xi)^2)\xi^{-\ell}$$

(D.10)

$$+ k \text{Res}_w (2\partial h(w) e(w) - \partial e(w) + \partial f(w)e^{-2h(w)}e(w)^2)w^{-\ell}.$$ 

Next we express the objects like $\mathcal{G} X \otimes \theta(\zeta) \mathcal{Y}(v, x) \mid 0$ for $X \in \mathfrak{sl}_2$, $\theta(\zeta) \in \mathbb{C}(\zeta^{-1})$ and an intertwining operator $\mathcal{Y}(\cdot, x)$ in a convenient form with the help of Lemma D.1.

Lemma D.3. Let $\mathcal{Y}(\cdot, x)$ be an intertwining operator, $v \in L(\Lambda)$ be a primary vector in the top space of $L_{\mathfrak{sl}_2}(\Lambda, k)$, and $\theta(\zeta) \in \mathbb{C}(\zeta^{-1})$. Then

$$\mathcal{G} E \otimes \theta(\zeta) \mathcal{Y}(v, x) \mid 0 = \left( \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} e^{2h(w)} \theta(z) \frac{\partial}{\partial e_n} \right) \mathcal{Y}(v, x) \mid 0$$

$$- \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} f(w) \theta(z) \frac{\partial}{\partial h_n} \left( \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} f(w)^2 \theta(z) \frac{\partial}{\partial f_n} \right) \mathcal{Y}(v, x) \mid 0$$

(D.11)

$$+ \text{Res}_{z-x} \frac{\theta(z)}{z-x} \mathcal{G}(E \otimes \mathcal{Y}(v, x)) \mid 0,$$

$$\mathcal{G} H \otimes \theta(\zeta) \mathcal{Y}(v, x) \mid 0 = \left( \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} \theta(z) \frac{\partial}{\partial h_n} \right) \mathcal{Y}(v, x) \mid 0$$

$$+ 2 \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} f(w) \theta(z) \frac{\partial}{\partial f_n} \left( \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} f(w)^2 \theta(z) \frac{\partial}{\partial f_n} \right) \mathcal{Y}(v, x) \mid 0$$

(D.12)

$$+ \text{Res}_{z-x} \frac{\theta(z)}{z-x} \mathcal{G}(H \otimes \mathcal{Y}(v, x)) \mid 0,$$

$$\mathcal{G} F \otimes \theta(\zeta) \mathcal{Y}(v, x) \mid 0 = \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} \theta(z) \frac{\partial}{\partial h_n} \mathcal{G}(F \otimes \mathcal{Y}(v, x)) \mid 0$$

(D.13)

$$+ \text{Res}_{z-x} \frac{\theta(z)}{z-x} \mathcal{G}(F \otimes \mathcal{Y}(v, x)) \mid 0.$$ 

Proof. As an example, we show Eq. (D.13). The other two equalities are shown in a similar way. We divide a Laurent series $\theta(\zeta) = \sum_{n \in \mathbb{Z}} \theta_n \zeta^n$ into the negative power part...
and the non-negative power part as
\begin{equation}
\theta(\zeta) = \theta(\zeta)_- + \theta(\zeta)_+,
\end{equation}
where \( \theta(\zeta)_- = \sum_{n<0} \theta_n \zeta^n \) and \( \theta(\zeta)_+ = \sum_{n\geq0} \theta_n \zeta^n \). Notice that \( \theta(\zeta)_- \) is expressed as the following:
\begin{equation}
\theta(\zeta)_- = \text{Res}_{\zeta = -\zeta} \frac{\theta(z)}{-z}.
\end{equation}

Together with Lemma \([D.1]\) this implies that
\begin{equation}
\mathcal{G} F \otimes \theta(\zeta)_- = \sum_{n \leq -1} \text{Res}_w \text{Res}_v w^{-n-1} \frac{\theta(z)}{g(w) - z} \frac{\partial}{\partial f_n}.
\end{equation}

Since \( \mathcal{Y}(v, x) \) is a primary field,
\begin{equation}
[F(n), \mathcal{Y}(v, x)] = x^n \mathcal{Y}(Fv, x),
\end{equation}
which implies that
\begin{equation}
[F \otimes \theta(\zeta)_+, \mathcal{Y}(v, x)] = \sum_{n=0}^{\infty} \theta_n x^n \mathcal{Y}(Fv, x) = \text{Res} \frac{\theta(z)}{z - x} \mathcal{Y}(Fv, x).
\end{equation}

Noting that \( F \otimes \theta(\zeta)_+ \) annihilates the vacuum vector \( |0\rangle \), we obtain the desired result. \( \square \)

For an intertwining operator \( \mathcal{Y}(-, z) \) of type \( \left( L_{\mathfrak{sl}_2}(\Lambda, k) \right) \), we regard \( \langle u|\mathcal{Y}(-, x)|0\rangle \) as an element of \( L(\Lambda)^*[g_{n+1}, e_n, h_n, f_n|n < 0[[z]]] \). The dual space \( L(\Lambda)^* \) is equipped with a representation \( \pi \) of \( \mathfrak{sl}_2 \) defined by \( (\pi(X)\phi)(v) = -\phi(Xv) \) for \( X \in \mathfrak{sl}_2 \), \( \phi \in L(\Lambda)^* \) and \( v \in L(\Lambda) \). Combining Lemma \([D.2]\) and \([D.3]\) we derive operators \( \mathcal{X}_\ell \) that satisfy
\( \langle X(\ell)u|\mathcal{Y}(-, x)|0\rangle = \mathcal{X}_\ell \langle u|\mathcal{Y}(-, x)|0\rangle \) for \( X \in \mathfrak{sl}_2 \) and \( \ell \in \mathbb{Z} \).

We begin with the computation of \( \langle E(\ell)u|\mathcal{Y}(v, x)|0\rangle \):
\begin{equation}
\langle E(\ell)u|\mathcal{Y}(v, x)|0\rangle = -\langle u|E(-\ell)|\mathcal{Y}(v, x)|0\rangle = \mathcal{X}_\ell \langle u|\mathcal{Y}(v, x)|0\rangle,
\end{equation}
where
\begin{align*}
\mathcal{X}_\ell &= -\sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} e^{2h(w)} z^{-\ell} e^{-2h(z)} z^{-\ell} g'(z) \frac{\partial}{\partial c_n} \\
&\quad - \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} e^{-2h(z)} (f(z) - f(w)) z^{-\ell} g'(z) \frac{\partial}{\partial h_n} \\
&\quad + \sum_{n \leq -1} \text{Res}_z \text{Res}_w w^{-n-1} e^{-2h(z)} (f(z) - f(w))^2 z^{-\ell} g'(z) \frac{\partial}{\partial f_n} \\
&\quad + \text{Res}_z \frac{e^{-2h(z)} z^{-\ell} g'(z)}{g(z) - x} \pi(E) \\
&\quad + \text{Res}_z \frac{e^{-2h(z)} f(z) z^{-\ell} g'(z)}{g(z) - x} \pi(H) \\
&\quad - \text{Res}_z \frac{e^{-2h(z)} f(z)^2 z^{-\ell} g'(z)}{g(z) - x} \pi(F) \\
&\quad + k \text{Res}_z \partial f(z) e^{-2h(z)} z^{-\ell}.
\end{align*}
We also obtain

\[(D.21) \quad \langle H(\ell)u|\mathcal{V}(v,x)|0\rangle = -\langle u|H(-\ell)\mathcal{V}(v,x)|0\rangle = \mathcal{H}_\ell \langle u|\mathcal{V}(v,x)|0\rangle,\]

where

\[
\mathcal{H}_\ell = -2 \sum_{n \leq 1} \text{Res}_w \text{Res}_u w^{n-1} e^{2h(w)} e^{-2h(z)} e(z) z^{-\ell} g'(z) \frac{\partial}{\partial e_n} \\
- \sum_{n \leq 1} \text{Res}_w \text{Res}_u w^{n-1} (1 + 2e^{-2h(z)}(f(z) - f(w))) z^{-\ell} g'(z) \frac{\partial}{\partial h_n} \\
- 2 \sum_{n \leq 1} \text{Res}_w \text{Res}_u w^{n-1} (f(w) - f(z) - e^{-2h(z)}e(z)(f(w) - f(z))^2) z^{-\ell} g'(z) \frac{\partial}{\partial f_n} \\
+ 2 \text{Res}_z e^{-2h(z)} e(z) z^{-\ell} g'(z) \frac{\partial}{\partial \pi(E)} \\
+ \text{Res}_z \left(1 + 2e^{-2h(z)}(f(z) - f(w))\right) z^{-\ell} g'(z) \frac{\partial}{\partial \pi(H)} \\
- 2 \text{Res}_z \left(1 + e^{-2h(z)}(f(z) - f(w) + e^{-2h(z)}e(z)^2 f(z))^2\right) z^{-\ell} g'(z) \frac{\partial}{\partial \pi(F)}
\]

\[(D.22) + 2k \text{Res}_z (\partial h(z) - \partial f(z) e^{-2h(z)}e(z)) z^{-\ell},\]

and

\[(D.23) \quad \langle F(\ell)u|\mathcal{V}(v,x)|0\rangle = -\langle u|F(-\ell)\mathcal{V}(v,x)|0\rangle = \mathcal{F}_\ell \langle u|\mathcal{V}(v,x)|0\rangle,\]

where

\[
\mathcal{F}_\ell = \sum_{n \leq 1} \text{Res}_w \text{Res}_u w^{n-1} e^{2h(w)} e^{-2h(z)} e(z) z^{-\ell} g'(z) \frac{\partial}{\partial e_n} \\
- \sum_{n \leq 1} \text{Res}_w \text{Res}_u w^{n-1} (1 + e^{-2h(z)}e(z)(f(w) - f(z))) e(z) z^{-\ell} g'(z) \frac{\partial}{\partial h_n} \\
- \sum_{n \leq 1} \text{Res}_w \text{Res}_u w^{n-1} \left[\frac{e^{2h(z)} + 2e(z)(f(z) - f(w))}{g(w) - g(z)} + \frac{e^{-2h(z)}e(z)^2(f(z) - f(w))^2}{g(w) - g(z)}\right] z^{-\ell} g'(z) \frac{\partial}{\partial f_n} \\
- \text{Res}_z e^{-2h(z)} e(z)^2 z^{-\ell} g'(z) \frac{\partial}{\partial \pi(E)} \\
- \text{Res}_z \left(1 + e^{-2h(z)}e(z)f(z)\right) e(z) z^{-\ell} g'(z) \frac{\partial}{\partial \pi(H)} \\
+ \text{Res}_z \left(e^{2h(z)} + 2e(z)f(z) + e^{-2h(z)}e(z)^2 f(z)^2\right) z^{-\ell} g'(z) \frac{\partial}{\partial \pi(F)} \\
- \text{Res}_z (2\partial h(z) e(z) - \partial e(z) + \partial f(z) e^{-2h(z)}e(z)^2) z^{-\ell}.
\]

We look for an operator $$\mathcal{L}_\ell$$ such that $$\langle L_\ell u|\mathcal{V}(v,x)|0\rangle = \mathcal{L}_\ell \langle u|\mathcal{V}(v,x)|0\rangle.$$ We first prepare a lemma.
Lemma D.4. We set $\xi = g^{-1}(\zeta)$:

\[
\mathcal{L}_m \mathcal{L}_n = \sum_{m \leq 0} (\text{Res}_z z^{-\ell+1} g(z)^{-n-1} g'(z)^2) L_m \\
- E \oplus e^{-2h(\xi)} \partial e(\xi) \xi^{-\ell+1} \\
- H \oplus (\partial h(\xi) + e^{-2h(\xi)} f(\xi) \partial e(\xi)) \xi^{-\ell+1} \\
- F \oplus (\partial f(\xi) - 2f(\xi) \partial h(\xi) - e^{-2h(\xi)} f(\xi)^2 \partial e(\xi)) \xi^{-\ell+1} \\
+ \text{Res}_z z^{-\ell+1}(c_{12}(Sg)(z) + k(\partial h(z)^2 + e^{-2h(z)} \partial f(z) \partial e(z)))
\]

(D.25)

Notice that $\mathcal{L}$ satisfies the same differential equation as the one in the case of the Virasoro algebra, thus

\[
\mathcal{L}_m = - \sum_{n \leq 0} (\text{Res}_z z^{-n-1} g(z)^{m+1}) \frac{\partial \mathcal{L}}{\partial g_n}
\]

for $m \leq -1$. Terms of type $\mathcal{L}_n X \otimes x(\zeta)$ for $X \in \mathfrak{sl}_2$ can be also expressed as derivatives of $\mathcal{L}$ as shown previously. Thus the desired operator $\mathcal{L}_m$ is specified as

\[
\mathcal{L}_m = - \sum_{n \leq 0} (\text{Res}_z z^{-n-1} g(z)^{m+1}) \frac{\partial \mathcal{L}}{\partial g_n} \\
- \sum_{n \leq -1} \text{Res}_z \frac{z^{-\ell+1} w^{-n-1} e^{2h(w)} e^{-2h(z)} \partial e(z) g'(z)}{g(w) - g(z)} \frac{\partial}{\partial e_n} \\
- \sum_{n \leq -1} \text{Res}_z \frac{z^{-\ell+1} w^{-n-1} (\partial h(z) + e^{-2h(z)} \partial e(z) (f(z) - f(w))) g'(z)}{g(w) - g(z)} \frac{\partial}{\partial h_n} \\
- \sum_{n \leq -1} \text{Res}_z \frac{z^{-\ell+1} w^{-n-1} \left[ \frac{\partial f(z) - 2\partial h(z)(f(z) - f(w))}{g(w) - g(z)} \\
- \frac{e^{-2h(z)} \partial e(z) (f(z) - f(w))^2}{g(w) - g(z)} \right] g'(z) \frac{\partial}{\partial f_n}}{g(w) - g(z)} \\
+ \text{Res}_z z^{-\ell+1} g'(z)^2 \left( \frac{h}{(g(z) - x)^2} + \frac{1}{g(z) - x} \frac{\partial}{\partial x} \right) \\
+ \text{Res}_z z^{-\ell+1} e^{-2h(z)} \partial e(z) g'(z) \frac{\partial}{\partial x} \\
+ \text{Res}_z z^{-\ell+1} \partial h(z) + e^{-2h(z)} f(z) \partial e(z) g'(z) \frac{\partial}{\partial x} \\
+ \text{Res}_z z^{-\ell+1} \partial f(z) - 2f(z) \partial h(z) - e^{-2h(z)} f(z)^2 \partial e(z) g'(z) \frac{\partial}{\partial x} \\
+ \text{Res}_z z^{-\ell+1}(c_{12}(Sg)(z) + k(\partial h(z)^2 + e^{-2h(z)} \partial f(z) \partial e(z)))
\]

(D.27)

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**Department of Basic Science, The University of Tokyo**

*E-mail address: koshida@vortex.c.u-tokyo.ac.jp*