Robust structured light in atmospheric turbulence

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Abstract. Structured light is routinely used in free space optical communication channels, both classical and quantum, where information is encoded in the spatial structure of the mode for increased bandwidth. Both real-world and experimentally simulated turbulence conditions have revealed that free-space structured light modes are perturbed in some manner by turbulence, resulting in both amplitude and phase distortions, and consequently, much attention has focused on whether one mode type is more robust than another, but with seemingly inconclusive and contradictory results. Here, we present complex forms of structured light which are invariant under propagation through the atmosphere: the true eigenmodes of atmospheric turbulence. We provide a theoretical procedure for obtaining these eigenmodes and confirm their invariance both numerically and experimentally. Although we have demonstrated the approach on atmospheric turbulence, its generality allows it to be extended to other channels too, such as aberrated paths, underwater and in optical fibre.

Keywords: structured light, turbulence, orbital angular momentum, eigenmodes.

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1 Introduction

Free-space transmission of electromagnetic waves is crucial in many diverse applications, including sensing, detection and ranging, defence and communication, and extends over distances from the long (Earth monitoring) to the short (WiFi and LiFi). Lately there has been a resurgence of interest in free-space optical links,\textsuperscript{1,2} driven in part by the need for increased communication bandwidths,\textsuperscript{3,4} with the potential to bridge the digital divide in a manner that is license-free.\textsuperscript{5} Here the spatial modes of light have come to the fore, for so-called space division multiplexing\textsuperscript{6} and mode division multiplexing,\textsuperscript{7} where the spatial structure of light is used as an encoding degree of freedom. This in turn has fuelled interest in structured light,\textsuperscript{8,9} where light is tailored in all its degrees of freedom, including amplitude, phase and polarisation, enabled by a modern structured light toolkit.\textsuperscript{10}

A commonly used form of structured light is that of beams carrying orbital angular momentum (OAM), where the phase spirals around the path of propagation azimuthally.\textsuperscript{11} These modes
Fig 1 Propagation through turbulence. Most common forms of structured light (such as Laguerre-Gaussian modes) become distorted when propagating through free space due to the effects of atmospheric turbulence, whereas an eigen-mode of atmospheric turbulence will remain unchanged when propagating through the same channel, as shown in the left panels. In contrast, the eigenmodes of turbulence will not be the eigenmodes of pristine free space, as shown on the right panels.

provide a (theoretically) infinite and easily realised alphabet for encoding information\textsuperscript{12,13} and have been used extensively in optical communication (see Refs.\textsuperscript{14,15} for good reviews). Vectorial combinations of such beams create inhomogeneous polarisation structures\textsuperscript{16–18} and too have found applications in free-space links.\textsuperscript{19,20} Although these structured light fields hold tremendous potential for free-space optical communication, they are distorted by atmospheric turbulence as a phase perturbation in the near-field and an amplitude, phase and polarisation perturbation in the far-field.\textsuperscript{21} This corrects the myth that vectorial light is immune to atmospheric turbulence by virtue of its polarisation components - it is not. What is invariant is its vectorness, how inhomogeneous the polarisation structure is (but not how it looks), which can potentially be exploited for error-free optical communication.\textsuperscript{22} This modal scattering induced cross-talk decreases the information capacity of classical atmospheric transmission channels\textsuperscript{23–32} while reducing the degree of entanglement in quantum links.\textsuperscript{33–42} Mitigating this remains an open challenge that is intensely studied.

Arguments have been put forward for one mode family being more robust than another, with studies covering Bessel-Gaussian,\textsuperscript{43–54} Hermite-Gaussian,\textsuperscript{55–58} Laguerre-Gaussian\textsuperscript{59–63} and Ince-
Gaussian beams, with mixed and contradictory results. In the context of OAM, since the atmosphere itself can be thought of giving or taking OAM from the beam, it has been shown theoretically and experimentally that atmospheric turbulence distortions are independent of the original OAM mode, all susceptible to the deleterious effects of atmospheric turbulence, and indeed OAM has been suggested as not the ideal modal carrier through turbulence. Vectorial structured light has been suggested to improve resilience because of the invariance of the polarisation degree of freedom, but numerous studies in turbulence have been inconclusive, with some reporting that the vectorial structure is stable, and others not. Careful inspection of the studies that report vectorial robustness in noisy channels reveal that the distances propagated were short and the strength of perturbation low, mimicking a phase-only near-field effect where indeed little change is expected, and hence these are not true tests for robustness or invariance. Studies that claim enhanced resilience of vector modes over distances comparable to the Rayleigh length have used the variance in the field’s intensity as a measure, a quantity that one would expect to be robust due to the fact that each polarisation component behaves independently and so will have a low covariance. This failing of structured light in turbulence has led to numerous correction techniques, including novel encoding/decoding methods, modal diversity as an effective error-reduction scheme, traditional adaptive optics for pre- and post-correction as well as vectorial adaptive tools, iterative routines and deep learning models.

Here we present a class of structured light whose entire structure in amplitude and phase remains invariant as they propagate through a turbulent free-space channel. We deploy an operator approach to find the eigenmodes of atmospheric turbulence, a significant departure from prior phenomenological approaches. Unlike other spatial modes, these exotically structured eigenmodes need no corrective procedures and are naturally devoid of deleterious effects such as modal
The first turbulent screen is placed at the beginning of the channel, at \( z = 0 \), with subsequent screens placed a distance \( \Delta z = L/N \) away from the prior, where \( N \) is the number of turbulent phase screens used. Each phase screen and distance form a unit cell, the first highlighted in blue, forming \( N \) unit cells over the complete path length of \( z = L \). The operator for each unit cell, \( T \), is identical, so we need only consider the first unit cell. The initial plane is discretised into pixels with side length \( \delta \) and turbulence is simulated with a strength characterised by the ratio \( D/r_0 \), where \( D \) is the aperture of the inscribed circle and \( r_0 \) is the Fried parameter. The operator describes the action of an imprinted turbulent phase on the beam, followed by vacuum propagation over a distance \( \Delta z \).

crosstalk. Moreover, they are valid over long paths and strong aberrations, a regime that is no longer phase-only and thus traditional adaptive solutions for correction fail. We demonstrate this invariance numerically and confirm it experimentally with a laboratory simulated long path comprising weak, medium and strong turbulence, implemented using multiple turbulent phase screens along the propagation path. Our approach offers a new pathway for exploiting structured light in turbulence, and can be easily extended to arbitrary noisy channels.

2 The eigenmodes of turbulence

2.1 Instantaneous eigenmodes
The concept we tackle here is illustrated in left panels of Fig. 1. Some optical field passes through an arbitrary distance of atmospheric turbulence, treated as a continuous medium of potentially strong turbulence, and which we will refer to as our channel. The conventional forms of structured light such as Laguerre-Gaussian beams are typically distorted after propagation through such a channel, but are invariant to unperturbed free-space. In contrast, the eigenmodes of the medium are complex forms of structured light that are invariant to the channel, emerging distortion free, but then are not eigenmodes of unperturbed free-space.

In our approach to the problem, we treat the channel as an operator, \( \hat{U} \), that acts on the input field to return an output field, namely

\[
\hat{U} u_{\text{in}} = u_{\text{out}} .
\]
This is illustrated in right panels of Fig. 1, with the operator and input/output fields shown schematically (note that in the absence of turbulence the operator just describes free-space). The operator is unitary if the transmitting and receiving apertures are sufficiently large to accommodate the number of eigenmodes utilised, which can be approximated by its free space limit of

\[ N_{\text{max}} \approx \frac{A_t A_r}{(\lambda^2 L^2)} \]

where \( A_t \) and \( A_r \) are the transmitter and receiver aperture areas separated by a distance \( L \) for light of wavelength \( \lambda \). Equation 1 can be recast as an eigenvector problem by insisting that \( u_{\text{in}} = u_{\text{out}} = |\gamma\rangle \) so that

\[ \hat{U} = \sum_i s_i |\gamma_i\rangle \langle \gamma_i| . \] (2)

The challenge is to find these eigenvectors, \( |\gamma_i\rangle \), by decomposition of the channel operator as a transmission matrix with elements \( U_{ij} \) that maps the input to the output, i.e., \( u_i = \sum_j U_{ij} u_j \). This is a variant of singular value decomposition (SVD), which solves the eigenvector equation at two planes for two different sets of modes.\(^{84}\) This ensures orthogonality at the receiver by allowing the optical fields to evolve in transmission, say from modes \( |\phi_i\rangle \) at the transmitter to \( |\psi_i\rangle \) at the receiver, but at the expense of invariance. We wish to find the modes that are invariant (robust) to the channel, the true eigenmodes of the channel, so that transmitter and receiver share a common mode set. Because of the unitarity of the problem, the eigenvector equation is numerically stable and can be solved by a variety of standard numerical tools, so the task is to decompose the channel operator as matrix elements. There are a variety of approaches to do this, with successful theoretical demonstrations including using SVD in turbulence with OAM and pixel modes at the transmitter and receiver,\(^ {85,86}\) and experimental demonstrations using point sources for eigenmodes.
of scattering media.\textsuperscript{87,88} In the language of quantum mechanics, the nature of the problem lends itself to a process tomography of the channel,\textsuperscript{89} which by the isomorphism of channel and state means that a quantum state tomography\textsuperscript{90} will completely retrieve the channel matrix, as shown in quantum channels of complex optical fibre\textsuperscript{91} as well as in classical-quantum channels through turbulence.\textsuperscript{92,93} We believe that this is a promising avenue to explore as it may offer benefits over the classical approach, which typically probes the channel one mode at a time. Nevertheless, the point is that standards tools exist to tackle the problem both experimentally and computationally.

In our work we will use the pixel basis to express the eigenmodes, inspired by the form of the paraxial free-space Green’s function. The channel tomography however can be done in any complete and orthogonal basis for sending and receiving modes, to reconstruct a channel matrix $A = PUQ^\dagger$, where $P$ and $Q$ are the unitary matrices that perform the basis transformations, which following the earlier example might be $Q = \sum_p |\phi_p\rangle \langle \gamma_p|$ and $P = \sum_q |\psi_q\rangle \langle \gamma_q|$ from which $U$ can be again deduced.

2.2 Time averaged eigenmodes

The analysis in the previous section assumed that the channel matrix was fixed at some instant in time, or equivalently, that the light transit time is much shorter than the coherence time of the turbulence. It is instructive to consider what might happen if one instead considers a time averaged result. Turbulence is a stochastic process in which the refractive index of the Earth’s atmosphere varies according to well-known statistics, having zero mean and some non-zero variance. To see the impact of averaging over many different instances of turbulence on the robustness of modes, we use the Helmholtz equation in the non-paraxial form through a thick and varying medium
defined by the function $\delta n(r)$

$$ (\nabla^2 + k^2) V = -2k^2 \delta n V, \tag{3} $$

which has the solution

$$ V(r') = 2k^2 \int d^3 r \ G(r, r') V(r) \delta n(r), \tag{4} $$

with $G(r, r') = \exp(ik||r - r'||)/4\pi||r - r'||$ being the free-space Green’s function and $r = (x, z)$. Taking the ensemble average and using the result that $\langle \delta n V \rangle = A \langle V \rangle$,\textsuperscript{94} we find

$$ \langle V(r') \rangle = 2k^2 A \int d^3 r \ G(r, r') \langle V(r) \rangle, \tag{5} $$

where the constant $A$ is related to the covariance of the refractive index fluctuations. We recognize that Eq. 5 is identical to the usual, zero-turbulence Fresnel integral, up to a constant. Therefore, the averaged eigenmodes should be solutions to the free-space, no turbulence, case. In other words, if the channel involves some form of averaging, say at the detector, then the best mode set in this case is identically the traditional free-space modes in various geometries, e.g., the Hermite-Gaussian and Laguerre-Gaussian modes.

3 Numerical simulation: multiple phase screen example

Conceptually one can imagine that the real path through turbulence is subdivided into many units, each containing a single phase-only turbulent screen and a zero turbulence propagation path of length $\Delta z$. We realise that in the language of operators, the action of each unit on some field is given by the product of the operators for turbulence and free-space propagation, which we denote
by $\mathcal{T}$. This picture serves to confirm that the complete channel operator can be treated as unitary since it can be written as a product of unitary operators. We will now use this approach by way of example to build up a turbulence operator for the long/thick medium because (in the absence of a real-world channel) it lends itself directly to numerical testing and experimental verification of the concept in the laboratory.

To begin, we note that the effects of turbulence are mathematically captured in the stochastic refractive index $n = 1 + \delta n$, where $\delta n$ is the random variation in the refractive index of the Earth’s atmosphere. It is assumed that $\delta n$ has a zero mean value, i.e. $\langle \delta n \rangle = 0$, and that the variation is small, so $|\delta n| \ll 1$. The introduction of this varying term produces the stochastic paraxial Helmholtz equation for a field $V(x, y, z)$

$$\left( \nabla_t^2 + 2ik\partial_z + 2k^2\delta n \right) V = 0,$$

where $\nabla_t^2$ is the transverse Laplacian and $k = 2\pi/\lambda$ is the wavenumber for wavelength $\lambda$.

Equation 6 can be solved numerically according to the split-step method,\textsuperscript{95} illustrated in Fig. 2. Multiple random phase screens are placed at various distances along the beam’s propagation path. Importantly, each screen is in the weak turbulence limit and contributes a random phase $\Theta_j$, where $j$ labels the $j$th screen, so that a single screen approximation is valid, but the sum of many such screens can lead to medium or even strong turbulence. In general the screens at each distance are different, but for pedagogical reasons we start with a simplistic example (to illustrate the concept) where we imagine that the path is subdivided into identical units, each containing such a single screen and a zero turbulence propagation path of length $\Delta z$. The action of $\mathcal{T}$ on a field $V$ is given
by the Huygen-Fresnel integral with a turbulent phase factor

\[ T \mathcal{V} \equiv \int d^2 \mathbf{x} \ g(\mathbf{x}, \mathbf{x}'; \Delta z) \exp(i\Theta) \mathcal{V}(\mathbf{x}, z = 0), \]  

(7)

where

\[ g(\mathbf{x}, \mathbf{x}'; \Delta z) = \frac{1}{i\lambda\Delta z} \exp \left( \frac{i\pi}{\lambda\Delta z} \| \mathbf{x} - \mathbf{x}' \|^2 \right) \]  

(8)

is the paraxial free space Green’s function and \( \mathbf{x}, \mathbf{x}' \) are the two-dimensional coordinates of the initial and final planes, respectively. We then discretize \( \mathbf{x} \) and \( \mathbf{x}' \) into grids of \( N \times N \) points. The coordinates are labelled \( \mathbf{x} \equiv (x_\alpha, y_\beta) \) and \( \mathbf{x}' \equiv (x_\mu, y_\nu) \), so that \( T \) is given by

\[ T_{\mu\nu\alpha\beta} = \frac{1}{i\lambda\Delta z} \exp \left( \frac{i\pi}{\lambda\Delta z} (x_\mu - x_\alpha)^2 \right) \times \exp \left( \frac{i\pi}{\lambda\Delta z} (y_\nu - y_\beta)^2 \right) \exp (i\Theta(x_\alpha, y_\beta)). \]  

(9)

An eigenmode \( \mathcal{E} \) is then a solution to the tensor eigenvalue equation

\[ \gamma_n \mathcal{E}^n_{\mu\nu} = T_{\mu\nu\alpha\beta} \mathcal{E}^n_{\alpha\beta}, \]  

(10)

where \( \gamma_n \) is the eigenvalue of the nth eigenmode. Repeated indices are implicitly summed over and \( \mathcal{E}_{\mu\nu} \equiv \mathcal{E}(x_\mu, y_\nu) \). To convert the above tensor equation into the usual matrix-vector form, we specify a mapping \( \rho \) that acts on the indices \( (\alpha, \beta) \) and \( (\mu, \nu) \) and “counts” them, first by columns and then by rows, such that \( \rho(1, 1) = 1, \ldots, \rho(N, 1) = N, \rho(1, 2) = N + 1 \) up to \( \rho(N, N) = N^2 \).
This mapping lets us rewrite Eq. 10 as

\[ \gamma_n E_i^n = T_{ij} E_j^n, \]  

since \( \rho(\alpha, \beta) = j \) and \( \rho(\mu, \nu) = i \). This equation can be routinely solved using numerical methods to find the eigenmodes of the unit cell operator. The action of the full channel is then described by the product \( T_n \ldots T_1 \) of repeated unit cells, and as per the definition of eigenmodes, they remain invariant regardless of the number of operators applied. To simulate more realistic conditions that change from cell to cell, the individual operators can be set appropriately so that in general \( T_n \neq T_{n+1} \), but the product of operators still holds true.

3.1 Numerical results

We follow the split-step approach shown in Fig. 2 to calculate the eigenmodes and numerically propagate them through atmospheric turbulence. For clarity and brevity we show only the low order eigenmodes and use the OAM modes as our point of comparison. We describe the turbulence conditions by Fried’s parameter \( r_0 = (0.423 k^2 C_n^2 L)^{-3/5} \) and the Rytov variance \( \sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6} \) using the plane-wave approximations\(^96 \) over a path of length \( L \). These parameters are given in the captions of all results.

Examples of the intensity and phase of the eigenmodes are shown in Fig. 3 for various examples of turbulence. Here the first five eigenmode solutions are shown in the left column, increasing from left to right, with the rows corresponding to the turbulence strength, increasing from top to bottom. Although these are complex forms of structured light, as eigenmodes of turbulence they should be invariant after propagation through a turbulent atmosphere. To test this, we propagate
Fig 4 Invariance of eigenmodes under numerical propagation through turbulence. The eigenmodes (left) and LG modes (right) after numerical propagation through weak, medium and strong turbulence through a channel equivalent to propagating over a distance of 100 m. The insets shows the modes before experiencing turbulence. The numerical simulations used the split step method with three unit cells each consisting of a turbulence screen with a given $r_0$ followed by 33.33 m of propagation. Weak turbulence was characterised by $\sigma_R^2 = 0.5$ and $r_0 = 1.8$ mm, medium turbulence by $\sigma_R^2 = 1$ and $r_0 = 1.2$ mm and strong turbulence by $\sigma_R^2 = 1.5$ and $r_0 = 0.91$ mm.

OAM carrying Laguerre-Gaussian (LG) modes and the eigenmodes through various scenarios of turbulence over a 100 m path length, with the results shown in Fig. 4. OAM modes were selected as they are very popular forms of structured light used in free-space studies. Their popularity, however, is not commensurate with their robustness in turbulence, as their phase profiles are sensitive to atmospheric distortions. We see that while the OAM modes are distorted (as expected), the eigenmodes are robust. This can be quantified by performing a modal analysis at the end of
Fig 5 **Crosstalk-free transmission.** Simulated crosstalk matrices for OAM modes $\ell \in [0, 4]$ (top) and eigenmodes (bottom) with insets showing the intensity of the beams. The eigenmodes are unchanged and remain orthogonal, whereas the OAM modes scatter into each other. Turbulence results shown for $D/r_0 = 2$ with a total path length of 100 m and a beam waist parameter for the OAM beams of $w_0 = 6.67$ mm.

the turbulent channel, as would be the case in optical communication at the receiver. In Fig. 5 we see that while the crosstalk is substantial for OAM modes when propagated through turbulence, evident from the many off-diagonal terms, the eigenmode crosstalk matrix remains diagonal after the same channel, for minimal crosstalk. It is useful to study the behaviour of the eigenmodes of turbulence as they propagate through a channel with no turbulence and that is absent of any other perturbations. Laguerre-Gaussian and Hermite-Gaussian are eigenmodes of such a channel and
Fig 6 Eigenmodes of turbulence through an unperturbed, uniform medium. The Laguerre-Gaussian modes (top row) propagate through free space unaberrated as they are solutions to the free space paraxial Helmholtz equation. The eigenmodes of weak, medium and strong turbulence (second, third and fourth row respectively), while still recognizable, show noticeable changes in when passing through a channel with no turbulence. The insets show the modes before propagation and the larger images show the modes after propagation through a 100 m free space channel with no turbulence. Weak turbulence was characterised by $\sigma_R^2 = 0.5$ and $r_0 = 1.8$ mm, medium turbulence by $\sigma_R^2 = 1$ and $r_0 = 1.2$ mm and strong turbulence by $\sigma_R^2 = 1.5$ and $r_0 = 0.91$ mm thus will propagate through it unperturbed. This can be seen in the first row of Figure 6 where several LG modes of increasing OAM are numerically propagate through a 100 m free space channel and show no aberrations. The eigenmodes of weak, medium and strong turbulence (in the second, third and fourth rows respectively) are also propagated through this channel. They show very noticeable changes in their intensity distributions thus demonstrating that they are not eigenmodes of free space. However, the change they undergo in free space is not so significant as to make them unrecognizable and they retain many of their original features such as a general pattern and number of lobes. When comparing this to Figure 4, it appears that the changes undergone by eigenmodes of turbulence in free space are less severe than changes undergone by the eigenmodes of free space.
Real-world turbulent channels have differing unit cells, and to illustrate such an example we simulate a slant path from high altitude to the Earth’s surface. In this case, the turbulence strength changes as a function of altitude, and likewise the phase screen in each cell. The starting altitude of the channel was 500 m and the zenith angle was 170° (the zenith angle is greater than 90° as we are propagating downward from a point of high altitude to the surface of the Earth). This corresponds to a path length of 508 m and an angle of 80° below the horizontal. To calculate the turbulence
strength at each altitude, the Tartarski model\textsuperscript{98} was used to calculate the values for the refractive index structure constant

\begin{equation}
C_{n_i}^2(h) = C_{n0}^2 h^{-b},
\end{equation}

where $C_{n0}^2 = 4.16 \times 10^{-13}$ m$^{2/3}$ and $b = 4/3$ are constants selected to most closely fit experimental data and $h$ is the altitude. The slant path length through the atmospheric layer is given by $\Delta l_i^2 = \Delta h_i^2 + \Delta z_i^2$ where $\Delta h_i^2$ is the height of the atmospheric layer and $\Delta z_i^2$ is the horizontal component of the beam’s path through the layer and the channel was broken up into 14 unit cells. The phase screens were then distributed evenly along the path and the values for $C_{n_i}^2$ where calculated for each phase screen at position of each phase screen. The calculated eigenmodes for this example are shown in Figure 7, where we see some striking similarity to low-order superpositions of the free space modes. More importantly, we note that the before and after intensity structures are in good agreement, indicative of an eigenmode.

\section{Experimental Results}

The experiment, shown in Fig. 8, is conceptually divided into three parts. In the generation stage, a He-Ne laser beam (wavelength $\lambda = 633$ nm) was expanded using a 10× objective lens $L_1$ and then collimated by $L_2$ ($f_2 = 150$ mm) before being directed onto a reflective PLUTO-VIS HoloEye spatial light modulator (SLM). The initial field was generated by using the Arrizón type 3 technique\textsuperscript{99} to shape the incident beam into the desired mode by complex amplitude modulation (amplitude and phase control). This field then entered the turbulent section of the setup where it passed through three unit cells, each comprising the same random phase screen and a propagation distance of one metre. The same phase screen was repeated for ease of calculation. In a real-
Fig 8 Experimental setup. Lenses $L_1$ and $L_2$ expand and collimate a laser beam onto an SLM on which a phase-only hologram of the initial beam is displayed, but implementing amplitude and phase control by complex amplitude modulation. The ideal, turbulence-free beam is generated at this plane and subsequently propagates through three turbulent screens which are also displayed on SLMs, each followed by 1 m of free-space propagation. The final aberrated field is captured on a CCD to image its intensity. The panels on the right show examples of the desired eigenmodes (Calculated Eigenmodes), the holograms to create them, and the measured eigenmodes without any turbulence or propagation (Generated Eigenmodes).

world channel, the phase screens would not be correlated, however, the method would still remain unchanged as demonstrated in the previous section. The phase screens were generated using the sub-harmonic random matrix transform method and displayed on the (phase-only) SLMs. The intensity of the perturbed field was then detected and measured on a camera (CCD). The panels on the left of the experimental setup in Fig. 8 show four examples of the desired (calculated) eigenmodes, the holograms to create them by complex amplitude modulation, and the experimental validation that without any turbulence or propagation, that they are created (generated eigenmodes) with high fidelity (bottom panel).

Our setup differs from conventional laboratory simulations of turbulence in that we are able to
mimic a thick path, from weak to strong turbulence, whereas often only a single phase screen is used, allowing only weak turbulence to be tested. Using our setup, we studied an effective real-world channel of \( L = 100 \) m, at our wavelength of \( \lambda = 633 \) nm and with Rytov variances of \( \sigma_R^2 = 1.5, 1 \) and 0.5, corresponding (respectively) to strong, medium and weak turbulence, with Fried parameters \((r_0)\) of 0.47 mm, 0.62 mm and 0.93 mm, respectively. We required three screens for each turbulence strength, separated by a distance of 33.3 m, each with effective Fried parameters \(r_{0,s} = 0.9\) mm, 1.2 mm and 1.8 mm while maintaining a Rytov variance in each slab (segment of the channel) to be smaller than 0.9, 0.6 and 0.3, respectively. This channel was simulated on the setup shown in Fig. 8 using the Fresnel scaling procedure, allowing a long path to be generated within laboratory distances. The scaling factors were chosen to be: \(\alpha_x = \alpha_{x'} = \sqrt{0.03} \approx 0.173\) and \(\alpha_z = 0.03\). This corresponded to a total path length of \(L' = 3\) m and segment Fried parameters of \(r_{0,s} = 0.081\) mm, 0.11 mm and 0.16 mm (see Appendix for details).

The results of OAM and the eigenmodes for weak, medium and strong turbulence are shown in Fig. 9. The collage shows the final measured eigenmodes after the channel, with the insets showing the initial mode as prepared prior to the channel. The robustness of the eigenmodes is clearly evident, in contrast to the highly distorted OAM modes.

5 Discussion

We have outlined how to find the eigenmodes of a turbulent path, and demonstrated it with a laboratory-based split-step example both numerically and experimentally. These modes are eigenmodes in their truest sense, i.e., they are fixed under the action of the channel. Importantly, our approach works even when the medium is long and the aberration strong, a regime where traditional adaptive optics often fails (beyond a Rytov variance of order 1).
Fig 9 Experimental eigenmodes. In the left column we see the measured intensities of the eigenmodes of turbulence after propagating through the experimental setup. The results show eigenmodes of weak, medium and strong turbulence. In the right column we see the measured intensities of OAM modes after propagating through the same experimental setup with the same turbulence phase screens for comparison. The insets show the initial input mode. Weak turbulence was characterised by $\sigma_{R}^2 = 0.5$ and $r_0 = 1.8$ mm, medium turbulence by $\sigma_{R}^2 = 1$ and $r_0 = 1.2$ mm and strong turbulence by $\sigma_{R}^2 = 1.5$ and $r_0 = 0.91$ mm.
A natural feature of the eigenmodes is that they are channel specific. To be useful in a real-world setting, the transmission should be faster than the coherence time of the turbulence. For typical turbulence conditions this is always true, with the atmosphere changing on millisecond time scales (typically $1 - 10$ ms), whereas the transport of light across km length scales is on the order of nanoseconds. The turbulence thus appears frozen, and numerical simulations on singular value modes have shown that they outperform adaptive solutions even in dynamic turbulence.\(^\text{103}\)

The time frames of acquiring the information to deduce the eigenmodes can be very fast. A tomography of the channel can be done by modal decomposition in any basis\(^\text{97}\) with projective holograms, which with digital micro-mirror devices (DMDs) and photodiodes can certainly be done on millisecond time scales. This can be improved down to microseconds (limited by the response of fast photodiodes) if the projective holograms are hardcoded, such as on a diffractive optical element or metasurface, possible because the channel does not have to be probed in the basis of eigenmodes. Further, one can surmise that machine learning would be ideally suited to the task of anticipating the new eigenmodes in a dynamical system given the prior set, certainly appealing if the conditions are slowly changing. We take some confidence from the fact that such channel analysis is needed in MIMO as well as in singular value decomposition approaches, both of which have been applied to non-turbulence studies already with success. Nevertheless, it is a challenge that should not be downplayed, and is certainly deserving of a full experimental study. We point out that turbulence was used here only as an extreme example, whereas the formulation of the idea is such that it will work for any long path in a complex media, for instance, turbid, underwater, optical fibre or thermally aberrated paths, all of which may change more slowly or not at all.

Our analysis only considered scalar modes, whereas it is clear that vectorial superpositions would also be eigenmodes of this channel. In this way the invariance of the inhomogeneity of vectorial
light in complex channels\textsuperscript{21} could be generalised to the invariance of all properties of the field, an appealing notion for energy transport and imaging through complex systems.

6 Conclusion

The search for robust states of structured light in noisy channels is a pressing challenge, promising enhanced channel capacity and reach. Here, using free-space and atmospheric turbulence as our example, we have outlined a theoretical approach to finding the complex forms of structured light which are invariant under propagation through the atmosphere, the true eigenmodes of turbulence, and confirmed its validity both numerically and experimentally. These exotically structured eigen-modes need no corrective procedures, are naturally devoid of deleterious effects and are valid over any path length in the medium. Our approach offers a new pathway for exploiting structured light in turbulence, and can be easily extended to other noisy channels, such as underwater and optical fibre.

Appendix

Some channel parameters, such as path length, are highly restricted in the laboratory setting. This presents an apparent difficulty to experimentally verifying the eigenmodes. However, a scaling procedure exists\textsuperscript{100} which allows us to verify real-world channels in the laboratory. This procedure is presented below.

The Fresnel integral for the full (real-world) channel of length $L$ is

$$U_f(r, L) = \frac{\exp(ikL)}{i\lambda L} \int d^2r' U_i(r') \exp \left( \frac{i\pi}{\lambda L} \| r - r' \|^2 \right).$$

(13)
We then apply the following scaling parameters: $r_{lab} = \alpha_x r$, $r'_{lab} = \alpha_{r'} r'$ and $L' = \alpha_z L$, where $r_{lab}$ and $r'_{lab}$ are the coordinates used in the experiment. The diffraction integral becomes

$$U_f \left( \frac{r_{lab}}{\alpha_r} \right) = \exp \left( \frac{ikL' \alpha_z}{\alpha r} \right) \int \left( \frac{d^2 r'_{lab}}{\alpha_r} \right) U_i \left( \frac{r'_{lab}}{\alpha_{r'}} \right) \times$$

$$\exp \left( \frac{i\pi \alpha_z}{\lambda L'} \left\| \frac{r_{lab}}{\alpha_r} - \frac{r'_{lab}}{\alpha_{r'}} \right\|^2 \right). \quad (14)$$

To keep the diffraction equivalent with these scaled coordinates we require the Fresnel number

$$F = \frac{\pi D_i D_f}{4\lambda L} \quad (15)$$

to be the same in both the full and scaled-down cases, where $D_i$ and $D_f$ are the aperture diameters in the initial and final planes, respectively. This sets $\alpha_r \alpha_{r'} = \alpha_z$ and the diffraction integral becomes

$$\exp \left( \frac{ikL' \left( 1 - \frac{1}{\alpha_r} \right)}{\alpha_r} \right) U_f \left( \frac{r_{lab}}{\alpha_r} \right) = \exp \left( -\frac{i\pi r_{lab}^2}{\lambda f_r} \right) \frac{\exp (ikL')}{i\lambda L'} \times$$

$$\int \left( \frac{d^2 r'_{lab}}{\alpha_{r'}} \right) \exp \left( -\frac{\pi (r'_{lab})^2}{\lambda f_{r'}} \right) \exp \left( \frac{i\pi}{\lambda L'} \left\| \frac{r_{lab}}{\alpha_r} - \frac{r'_{lab}}{\alpha_{r'}} \right\| \right), \quad (16)$$

where

$$f_r = \frac{L'}{1 - \alpha_r / \alpha_{r'}}, \quad (17)$$

$$f_{r'} = \frac{L'}{1 - \alpha_r / \alpha_{r'}}. \quad (18)$$

Setting $\alpha_r = \alpha_{r'}$ means that the final and initial planes have the same size in the laboratory setting and $f_{r,r'} \to \infty$. Ignoring constant phase factors which arise due to scaling, the final Fresnel integral
becomes
\[
U_f \left( \frac{r_{\text{lab}}}{\alpha_r} \right) = \frac{\alpha_r \exp(i k L')}{i \lambda L'} \int d^2 r'_{\text{lab}} \ U_i \left( \frac{r'_{\text{lab}}}{\alpha_{r'}} \right) 
\]
\[
\times \exp \left( \frac{i \pi}{\lambda L'} \| r_{\text{lab}} - r'_{\text{lab}} \| \right) .
\] (19)

**Disclosures**

The authors declare no conflicts of interest.

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**Code, Data, and Materials Availability**

Code, data and materials are available on request from the corresponding author.

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