COHEN-MACAULAY DU BOIS SINGULARITIES
WITH A TORUS ACTION OF COMPLEXITY ONE

ANTONIO LAFACE, ALVARO LIENDO, AND JOAQUÍN MORAGA

ABSTRACT. Using Altmann-Hausen-Süß description of $T$-varieties via divisorial fans and Kovacs-Schwede-Smith characterization of Du Bois singularities, we study Cohen-Macaulay Du Bois $T$-singularities of complexity one. We exhibit cohomological criteria for a $T$-variety to be Cohen-Macaulay and Du Bois in terms of polyhedral divisors. We give an example of a Cohen-Macaulay Du Bois singularity of complexity one which does not have rational singularities.

CONTENTS

Introduction 1
1. $T$-Varieties via polyhedral divisors 3
2. Singularities of normal varieties 5
3. Preliminary results 6
4. Proof of the main theorems 9
References 13

INTRODUCTION

We study the singularities of $T$-varieties, i.e. normal algebraic varieties endowed with an effective action of an algebraic torus $T = (K^*)^k$, where $K$ is an algebraically closed field of characteristic zero. Given a $T$-variety $X$ we define its complexity to be $\dim(X) - \dim(T)$. $T$-varieties of complexity zero are called toric varieties and they admit a well-known combinatorial description in terms of fans of polyhedral cones [9, 12]. Such combinatorial description can be generalized to higher complexity. The complexity one case was considered in [10, 12, 26]. Altmann, Hausen and Süß generalize this combinatorial description for arbitrary complexity using the language of polyhedral divisors and divisorial fans on normal projective algebraic varieties [4, 5]. We will use the language of polyhedral divisors in this paper. In this context, an affine $T$-variety $X$ is determined by a pair $(Y, D)$, where $Y$ is the Chow quotient for the $T$-action on $X$ and $D$ is a $p$-divisor on $Y$, i.e., a formal finite sum of prime divisors on $Y$ whose coefficients are convex polyhedra. Recall that the Chow quotient of $X$ by the action of $T$ is the closure of the set of general $T$-orbit closures seen as points in the Chow variety (see Section 1 for the details).
Using toric resolution of singularities is it possible to prove that every normal toric variety $X$ has log terminal singularities [9, Proposition 11.4.24]. Therefore, normal toric varieties have rational singularities and then are Cohen-Macaulay and Du Bois, see [13]. However, in higher complexity none of this statements is true if we do not impose conditions on the defining combinatorial data. For instance, a simple computation shows that the affine cone $X_C$ over a plane curve $C$ of genus $g$ is a normal $\mathbb{T}$-surface of complexity one which is log canonical if and only if $g$ equals 0 or 1. Furthermore, in the case that $C$ is a curve of genus $g \geq 1$, the singularity at the vertex is not rational [19] and so by virtue of [14], $X$ is an example of a $\mathbb{T}$-variety of complexity one which is Cohen-Macaulay and Du Bois but has a singularity that is not rational.

Therefore it is worthwhile to study characterizations of this kind of singularities for $\mathbb{T}$-varieties of higher complexity in terms of the defining $p$-divisor. There are such characterizations of rational singularities and $\mathbb{Q}$-Gorenstein singularities and partial characterizations of Cohen-Macaulayness in terms of the defining $p$-divisors (see [19]). In the case of positive characteristic, there are also characterizations of $F$-split and $F$-regular $\mathbb{T}$-varieties of complexity one in terms of $p$-divisors (see [3]).

Cohen-Macaulay singularities are the most natural singularities which have the same cohomological behavior as smooth varieties, for instance vanishing theorems still hold for varieties with Cohen-Macaulay singularities (see [13]). Du Bois singularities also play an important role in algebraic geometry since these are the singularities appearing in the minimal model program and moduli theory (see [23, 24]).

In this article, we study $\mathbb{T}$-varieties with Cohen-Macaulay and Du Bois singularities.

Throughout the introduction, we restrict ourselves to the case of rational $\mathbb{T}$-varieties of complexity one. We briefly recall the data defining a rational complexity one affine $\mathbb{T}$-variety (we refer the reader to Section 1 and for the details). These varieties are determined by a finite set of points $p_1,\ldots,p_r \in \mathbb{P}^1$ and a finite set of polyhedra $\Delta_{p_1},\ldots,\Delta_{p_r} \subset N_{\mathbb{Q}} := N \otimes \mathbb{Q} \simeq \mathbb{Q}^r$ with a common recession cone $\sigma \subset N_{\mathbb{Q}}$. Let $M := \text{Hom}(N,\mathbb{Z})$. This data induces a function $D : M \to \text{CaDiv}_{\mathbb{Q}}(\mathbb{P}^1)$ defined by

$$D(u) := \sum_{i=1}^r \min_{v \in V_i} \langle u, v \rangle p_i$$

where $V_i$ is the set of vertices of $\Delta_{p_i}$. We say that $D$ is a $p$-divisor over $\mathbb{P}^1$ if $\sum_{i=1}^r \Delta_{p_i} \subset \sigma$; this is a particular case of Definition 1.1. There exists a rational $\mathbb{T}$-variety of complexity one $X(D)$ associated to the $p$-divisor $D$ (see Construction 1.3 and Definition 1.4). Furthermore, every rational affine $\mathbb{T}$-variety of complexity one is isomorphic to such a $X(D)$ [4]. We say that a ray $\rho \in \sigma(1)$ is big if $\sum_{i=1}^r \Delta_{p_i} \cap \rho = \emptyset$. Otherwise, we say that $\rho \in \sigma(1)$ is not big. Our first theorem, is an application of Kempf criteria for rational singularities [22] in the context of rational $\mathbb{T}$-varieties of complexity one.

**Theorem 1.** Let $D$ be a $p$-divisor on $(\mathbb{P}^1,N)$. Assume that $X(D)$ is Cohen-Macaulay. Then, $X(D)$ is rational if and only if $\deg(D(u)) \geq 1$ for every $u$ in the set

$$\{ u \in M \mid \langle u, \rho \rangle \leq -1 \text{ for all } \rho \text{ big and } \langle u, \rho \rangle \geq 0 \text{ for at least one } \rho \text{ not big} \}.$$

In a similar vein, using Kovács-Schwede-Smith characterization of Du Bois singularities [16, Theorem 1], we prove the following result:
**Theorem 2.** Let $D$ be a $p$-divisor on $(\mathbb{P}^1, N)$. Assume that $X(D)$ is Cohen-Macaulay. Then, $X(D)$ has Du Bois singularities if and only if $\deg\lfloor D(u) \rfloor \geq -1$ for every $u$ in the set

$$\{ u \in M \mid \langle u, \rho \rangle \leq -1 \text{ for all } \rho \text{ big and } \langle u, \rho \rangle \geq 1 \text{ for at least one } \rho \text{ not big } \}.$$ 

The paper is organized as follows: in Section 1, we introduce the basic notation of $T$-varieties via p-divisors. In Section 2, we introduce the classes of singularities needed in the article, i.e., rational, Cohen-Macaulay, Du Bois, and log canonical. In Section 3, we prove some preliminary results regarding the canonical divisor of $T$-varieties and sections of invariant line bundles. In Section 4 we prove Theorem 4.2 and Theorem 4.3 which are generalizations of Theorem 1 and Theorem 2 to higher complexity.

**Acknowledgements.** We would like to thank Karl Schwede and Linquan Ma for useful comments about Du Bois singularities. We wish to thank the anonymous referees of an early version of this work for useful comments and suggestions that helped the authors improve and correct the results.

1. **T-Varieties via polyhedral divisors**

We work over an algebraically closed field $\mathbb{k}$ of characteristic zero. In this section, we briefly introduce the language of p-divisors introduced in [4] and [5]. We begin by recalling the definition of polyhedral divisors, p-divisors and their connection with affine $T$-varieties.

Let $\mathbb{N}$ be a finitely generated free abelian group of rank $k$ and denote by $M$ its dual. We also let $\mathbb{N}_Q$ and $M_Q$ be the corresponding $\mathbb{Q}$-vector spaces obtained by tensoring $\mathbb{N}$ and $M$ with $\mathbb{Q}$ over $\mathbb{Z}$, respectively. We denote by $T := \text{Spec}(\mathbb{k}[M])$ the $k$-dimensional algebraic torus. Given a polyhedron $\Delta \subset \mathbb{N}_Q$ we will denote its recession cone by

$$\text{rec}(\Delta) := \{ v \in \mathbb{N}_Q \mid v + \Delta \subset \Delta \},$$

where $+$ denotes the Minkowski sum, which is defined by

$$\Delta_1 + \Delta_2 := \{ w \in \mathbb{N}_Q \mid w = v_1 + v_2, v_1 \in \Delta_1 \text{ and } v_2 \in \Delta_2 \}.$$ 

Given a pointed polyhedral cone $\sigma \subset \mathbb{N}_Q$ we can define a semigroup with underlying set

$$\text{Pol}_Q(\mathbb{N}, \sigma) := \{ \Delta \subset \mathbb{N}_Q \mid \Delta \text{ is a polyhedron with } \text{rec}(\Delta) = \sigma \}$$

and addition being the Minkowski sum. The neutral element in the semigroup is the cone $\sigma$. The elements in $\text{Pol}_Q(\mathbb{N}, \sigma)$ are called $\sigma$-polyhedra. In what follows we consider the semigroup $\text{Pol}^+_Q(\mathbb{N}, \sigma)$ which is the extension of the above semigroup obtained by including the element $\emptyset$ which is an absorbing element with respect to the addition, meaning that:

$$\emptyset + \Delta := \emptyset \text{ for all } \Delta \in \text{Pol}^+_Q(\mathbb{N}, \sigma).$$

Given a normal projective variety $Y$ we denote by $\text{CaDiv}^\geq_0(Y)$ the semigroup of effective Cartier divisors on $Y$. We define a *polyhedral divisor* on $(Y, N)$ with recession cone $\sigma$ to be an element of the semigroup

$$\text{Pol}^+_Q(\mathbb{N}, \sigma) \otimes_{\mathbb{Z}^\geq_0} \text{CaDiv}^\geq_0(Y).$$
Observe that any polyhedral divisor can be written as
\[ D = \sum Z \Delta Z \cdot Z, \]
where the sum runs over prime divisors of \( Y \), the coefficients \( \Delta Z \) are either \( \sigma \)-polyhedra or the empty set and all but finitely many \( \Delta Z \) are the neutral element \( \sigma \). The recession cone of a polyhedral divisor \( D \) is defined as the recession cone of any non-empty coefficient and is denoted by \( \sigma(D) \). The locus of \( D \) is the open set
\[ \text{loc}(D) := Y \setminus \bigcup_{\Delta Z \neq \emptyset} Z \]
and we say that \( D \) has complete locus if the equality \( \text{loc}(D) = Y \) holds, meaning that all the coefficients \( \Delta Z \) are non-empty \( \sigma(D) \)-polyhedra. A polyhedral divisor \( D \) on \( (Y,N) \) with recession cone \( \sigma \) defines a homomorphism of semigroups as follows
\[ D : \sigma^\vee \to \text{CaDiv}_\mathbb{Q}(\text{loc}(D)) \]
\[ u \mapsto \sum_{\Delta Z \neq \emptyset} \min(\Delta Z, u) Z|_{\text{loc}(D)}, \]
where by abuse of notation we denote by \( D \) both the polyhedral divisor and the homomorphism of semigroups. Observe that this homomorphism is well-defined since all the polyhedra \( \Delta Z \) have recession cone \( \sigma(D) \) and then the minimum appearing in the definition always exists. Moreover, for any \( u \in \sigma^\vee \) we have that \( D(u) \) is a \( \mathbb{Q} \)-divisor in \( \text{loc}(D) \) whose support contained in
\[ \text{supp}(D) := \text{loc}(D) \cap \bigcup_{\Delta Z \neq \emptyset} Z. \]

Recall that a variety \( Y \) is called semiprojective if the natural map \( Y \to H^0(Y, \mathcal{O}_Y) \) is projective. In particular, affine and projective varieties are semiprojective. Furthermore, blow-ups of semiprojective varieties are also semiprojective.

**Definition 1.1.** Let \( D \) be a polyhedral divisor on \( (Y,N) \) with recession cone \( \sigma \). We say that \( D \) is a proper polyhedral divisor, or \( p \)-divisor for short, if \( \text{loc}(D) \) is semiprojective, \( D(u) \) is semiample for \( u \in \sigma^\vee \) and \( D(u) \) is big for \( u \in \text{relint}(\sigma^\vee) \). Note that this condition holds whenever the locus of the polyhedral divisor is affine.

**Remark 1.2.** In what follows, given a \( \mathbb{Q} \)-divisor \( D \) on an algebraic variety \( X \) we will write \( \mathcal{O}_X(D) \) for \( \mathcal{O}_X(\lfloor D \rfloor) \).

**Construction 1.3.** Given a \( p \)-divisor \( D \) on \( (Y,N) \) with recession cone \( \sigma \) we have an induced sheaf of \( \mathcal{O}_{\text{loc}(D)} \)-algebras
\[ \mathcal{A}(D) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\text{loc}(D)}(D(u)) \chi^u. \]
Regard that \( \mathcal{A}(D) \) is indeed a sheaf of \( \mathcal{O}_{\text{loc}(D)} \)-algebras since the inequality
\[ D(u) + D(u') \leq D(u + u') \]
holds for every \( u, u' \in \sigma^\vee \) by convexity of the polyhedral coefficient. We denote by \( \tilde{X}(D) \) the relative spectrum of \( \mathcal{A}(D) \) and by \( X(D) \) the spectrum of the ring of global sections, both \( X(D) \) and \( \tilde{X}(D) \) comes with a \( T \)-action induced by the \( M \)-grading. The main theorem of [4] states that every \( n \)-dimensional normal affine variety \( X \) with an effective action of a \( k \)-dimensional torus \( T \) is \( T \)-equivariantly isomorphic
to $X(D)$ for some $p$-divisor $D$ on $(Y, N)$, where $Y$ is a projective normal variety of dimension $n - k$ and rank$(N) = k$. In this situation we have two natural morphisms

$$\text{loc}(D) \xrightarrow{\pi} \tilde{X}(D) \xrightarrow{r} X(D),$$

where $\pi$ is the good quotient induced by the inclusion of sheaves $O_{\text{loc}(D)} \hookrightarrow A(D)$ and $r$ is the $T$-equivariant birational map induced by taking global sections of $A(D)$. Moreover, every affine $T$-variety can be recovered from a $p$-divisor on the Chow quotient by the $T$-action (see [4, Definition 8.7]).

**Definition 1.4.** Let $D$ be a $p$-divisor on $(Y, N)$. The affine $T$-variety $X(D)$ is called the $T$-variety associated to the $p$-divisor $D$.

Let $D$ be a $p$-divisor. A ray $\rho \in \sigma(D)$ is called big if $D(u)$ is big for $u$ in the relative interior of the cone $\sigma(D)^\vee \cap \rho^\perp$. A vertex $v$ in $\Delta_Z$ is called big if $D(u)|_Z$ is big for every $u$ in the relative interior of the cone $\{u \mid \langle u, w - v \rangle > 0 \text{ for all } w \in \Delta_Z\}$.

2. **Singularities of normal varieties**

In this section, we define the main classes of singularities that we will study and we recall the inclusions between these classes. We will proceed by introducing the smaller classes of singularities first to then move to the wider ones.

**Definition 2.1.** Let $X$ be a $\mathbb{Q}$-Gorenstein normal algebraic variety and $\phi: Y \to X$ be a log resolution, i.e. a resolution such that the exceptional locus with reduced scheme structure is purely divisorial and simple normal crossing. Then we can write

$$K_Y = \phi^*(K_X) + \sum_i a_i E_i,$$

where $E_i$ are pairwise different exceptional divisors over $X$. We say that $X$ is log terminal (resp. log canonical) if $a_i > -1$ (resp. $a_i \geq -1$) for every $i \in I$.

**Definition 2.2.** Let $X$ be a normal algebraic variety and let $\phi: Y \to X$ be any resolution of singularities of $X$. We say that $X$ has rational singularities if the higher direct image sheaves $R^i \phi_* O_Y$ vanish for all $i > 0$.

Log terminal singularities are rational [13, Theorem 5.22] but in general log canonical singularities are not rational. Indeed, let $X$ to be the affine cone over a planar elliptic curve $C$ in $\mathbb{P}^2$. We can produce a log resolution $\phi: Y \to X$ with only one exceptional divisor $E$ such that $K_Y = \phi^*(K_X) - E$, so $X$ is log canonical but the stalk of $R^1 \phi_* O_X$ at the vertex of the cone is isomorphic to $H^1(C, O_C) \cong K$ which is non trivial and so $X$ is not rational.

**Definition 2.3.** We say that a commutative local Noetherian ring $R$ is Cohen-Macaulay if its depth is equal to its dimension. An algebraic variety $X$ is Cohen-Macaulay if the local ring $O_{X,x}$ is Cohen-Macaulay for all $x \in X$.

It is known that rational singularities in characteristic zero are Cohen-Macaulay (see, e.g., [13, Theorem 5.10]), however there are many examples of log canonical singularities which are not Cohen-Macaulay.

We will use the following characterization of Du Bois singularities which are Cohen-Macaulay due to Kóvacs-Schwede-Smith.
Theorem 2.4 ([16, Theorem 1]). Let $X$ be a normal Cohen-Macaulay algebraic variety, let $\phi: Y \to X$ be a log resolution and denote by $E$ the exceptional divisor of $\phi$ with reduced scheme structure. Then $X$ has Du Bois singularities if and only if we have an isomorphism of sheaves $\phi_\ast\omega_Y(E) \simeq \omega_X$.

There is an analogous characterization of rational singularities in terms of canonical sheaves: a Cohen-Macaulay normal algebraic variety $X$ has rational singularities if and only if the natural inclusion of sheaves $\phi_\ast\omega_Y \to \omega_X$ is an isomorphism, where $\phi: Y \to X$ is a resolution of singularities [12]. From Theorem 2.4 and this characterization of rational singularities it follows that rational singularities are Du Bois. Indeed, by [16, Lemma 3.14] we have two natural injections $\phi_\ast\omega_Y \to \phi_\ast\omega_Y(E) \to \omega_X$, and Theorem 2.4 states that $X$ is Du Bois whenever the inclusion of sheaves $\phi_\ast\omega_Y(E) \to \omega_X$ is an isomorphism. Moreover, it is known that log canonical singularities are Du Bois [14].

It is known that toric singularities are log terminal [9, Proposition 11.4.24] and therefore they belong to all the above classes of singularities. In higher complexity the situation becomes more complicated: rational singularities can be characterized in terms of divisorial fans [19, Theorem 3.4], but there are not complete characterizations for the other classes of singularities defined above. Partial characterizations of Cohen-Macaulay and log terminal singularities with torus action are given in [19].

Remark 2.5. The above definitions are independent of the chosen resolution (or log resolution) of the normal variety $X$. By [2] we may assume that the resolution of singularities is $\mathbb{T}$-equivariant.

3. Preliminary results

In this section, we collect all the ingredients we need for the proof of our main theorem. Some are borrowed from the literature on $\mathbb{T}$-varieties while some other are proven here.

3.1. $\mathbb{T}$-invariant divisors. We recall the description of the $\mathbb{T}$-invariant prime divisors of the $\mathbb{T}$-varieties $\tilde{X}(\mathcal{D})$ and $X(\mathcal{D})$ following [20, Proposition 3.13]. Let $\mathcal{D}$ be a $\mathbb{P}$-divisor on $(Y,N)$, where $Y$ is a normal algebraic variety, then any fiber of the good quotient $\pi$ over a point of $Y$ which is not contained in $\text{supp}(\mathcal{D})$ is $\mathbb{T}$-equivariantly isomorphic to the toric variety $X(\sigma(\mathcal{D}))$, therefore the $\mathbb{T}$-variety $\tilde{X}(\mathcal{D})$ admits an open subset which is isomorphic to the product $X(\sigma(\mathcal{D})) \times (Y - \text{supp}(\mathcal{D}))$.

An horizontal $\mathbb{T}$-divisor is a $\mathbb{T}$-invariant divisor of $\tilde{X}(\mathcal{D})$ which dominates $Y$, these divisors are in one to one correspondence with the rays $\rho$ of the cone $\sigma(\mathcal{D})$ and they can be realized as the closure in $\tilde{X}(\mathcal{D})$ of the subvariety $V(\rho) \times (Y - \text{supp}(\mathcal{D}))$, where $V(\rho)$ is the toric invariant divisor of $X(\sigma(\mathcal{D}))$ corresponding to the ray $\rho$ of $\sigma(\mathcal{D})$. On the other hand, we define a vertical $\mathbb{T}$-divisor to be a $\mathbb{T}$-invariant divisor of $\tilde{X}(\mathcal{D})$ which does not dominate $Y$, these divisors are in one-to-one correspondence with pairs $(Z,v)$ where $Z$ is a prime divisor of $Y$ and $v$ is a vertex of the polyhedron $\Delta_Z$. We often write $\mathcal{V}(\Delta_Z)$ for the set of vertices of the polyhedron $\Delta_Z$. Any $\mathbb{T}$-invariant divisor of $\tilde{X}(\mathcal{D})$ is either horizontal or vertical.

Definition 3.1. Now we turn to describe the horizontal and vertical $\mathbb{T}$-divisors which are not contained in the exceptional locus of the morphism $r: \tilde{X}(\mathcal{D}) \to X(\mathcal{D})$.

- A ray $\rho$ of $\sigma(\mathcal{D})$ is big if the $\mathbb{Q}$-divisor $\mathcal{D}(u)$ is big for every $u \in \text{relint}(\sigma^\vee \cap \rho^\perp)$, the corresponding horizontal $\mathbb{T}$-divisor will be called big as well.


• Given a prime divisor \( Z \subset Y \) and a vertex \( v \in \Delta_Z \) we say that the vertex \( v \) is a **big** if the \( \mathbb{Q} \)-divisor \( D(u)|_Z \) is a big divisor for every \( u \) in the interior of the normal cone

\[
\mathcal{N}(\Delta_Z, v) := \{ u \mid (u, w - v) > 0 \text{ for every } w \in \Delta_Z \}.
\]

As before, we say that the corresponding vertical \( T \)-divisor is big.

Then we can say that the codimension one exceptional set of \( r \) is the union of all the \( T \)-divisors which are not big.

Finally we turn to introduce some notation for the horizontal and vertical \( T \)-divisors on \( \bar{X}(D) \) and \( X(D) \). Given a ray \( \rho \) in \( \sigma(D) \) we may denote by \( \bar{D}_\rho \) the corresponding horizontal divisor of \( \bar{X}(D) \) and whenever \( \rho \) is big we denote by \( D_\rho \) its image \( r(\bar{D}_\rho) \). Given a prime divisor \( Z \subset Y \) and a vertex \( v \in \mathcal{V}(\Delta_Z) \) we write \( \bar{D}(Z,v) \) for the corresponding vertical divisor and whenever \( v \) is big we denote by \( D(Z,v) \) its image in \( X(D) \).

3.2. **\( T \)-orbit decomposition.** Now we turn to describe the \( T \)-orbits of the \( T \)-varieties \( \bar{X}(D) \) and \( X(D) \) following [4, Section 10]. In order to do so, we begin by defining the toric bouquet associated to a polytope \( \Delta \subset \mathbb{Q} \).

**Definition 3.2.** Given a \( \sigma \)-polyhedron \( \Delta \subset \mathbb{Q} \) we denote by \( \mathcal{N}(\Delta) \) the **normal fan** of \( \Delta \), which is the fan in \( \mathbb{M}_\mathbb{Q} \) with support \( \sigma^\vee \) and whose cones corresponds to linearity regions of the function \( \min(\Delta, -) : \sigma^\vee \to \mathbb{Q} \). Observe that the cones of \( \mathcal{N}(\Delta) \) are in one-to-one dimension-reversing correspondence with the faces of \( \Delta \).

Given a face \( F \) of \( \Delta \), we will denote by \( \mathcal{N}(F) \) the corresponding cone of \( \mathcal{N}(\Delta) \). We define the **toric bouquet** of \( \Delta \) to be

\[
X(\Delta) := \text{Spec}(k[\mathcal{N}(\Delta) \cap M]),
\]

where \( k[\mathcal{N}(\Delta) \cap M] \) equals \( k[\sigma^\vee \cap M] \) as \( k \)-vector spaces, but the multiplication rule in \( k[\mathcal{N}(\Delta) \cap M] \) is given by

\[
\chi^u \cdot \chi^{u'} := \begin{cases} 
\chi^{u+u'} & \text{if } u,u' \text{ belong to a common cone of } \mathcal{N}, \\
0 & \text{otherwise.}
\end{cases}
\]

Given a point \( y \in Y \), we define the **fiber polyhedron** to be

\[
\Delta_y := \begin{cases} 
\sum_{g \in \mathcal{Z}} \Delta_Z & \text{if } y \in \text{supp}(D), \\
\sigma(D) & \text{otherwise.}
\end{cases}
\]

In the previous section we stated that the fiber \( \pi^{-1}(y) \) over a point \( y \in Y \) which is not contained in the support of \( D \) is \( T \)-equivariantly isomorphic to the toric variety \( X(\sigma(D)) \), in general the special fibers of \( \pi \) are not irreducible, but they can still be interpreted as toric bouquets as follows. Given a point \( y \in \text{supp}(D) \), the fiber \( \pi^{-1}(y) \) is \( T \)-equivariantly isomorphic to the toric bouquet \( X(\Delta_y) \). Therefore, we have a dimension-reversing bijection between between the orbits of \( \pi^{-1}(y) \) and the faces of \( \Delta_y \). Thus, the orbits of \( \bar{X}(D) \) are in one-to-one dimension-reversing correspondence with the pairs \((y,F)\) where \( y \in Y \) and \( F \) is a face of \( \Delta_y \).

Finally, we describe the \( T \)-orbits of \( \bar{X}(D) \) which are identified by the contraction morphism \( r \). Given an element \( u \in \sigma(D)^\vee \cap M \) we have an induced morphism

\[
\phi_u : Y \to Y_u, \text{ where } Y_u = \text{Proj} \left( \bigoplus_{n \in \mathbb{Z}_{\geq u}} H^0(Y, \mathcal{O}_Y(D(nu))) \right).
\]
From [4, Theorem 10.1] we know that two orbits of $\tilde{X}(D)$ corresponding to the pairs $(y, F)$ and $(y', F')$ are identified via the $T$-equivariant contraction $r$ if and only if

$$\mathcal{N}(F) = \mathcal{N}(F')$$

and $\phi_u(y) = \phi_u(y')$ for some $u \in \text{relint}(\mathcal{N}(F))$.

Observe that in the case that $Y$ is $\mathbb{P}^1$, then either $\phi_u$ is an isomorphism or is the constant morphism, depending whether the divisor $D(nu)$ has positive or trivial degree, respectively.

### 3.3. Sections of $T$-invariant sheaves.

In this subsection we describe the sections of the sheaf induced by a $T$-invariant Weil divisor on $X(D)$ (see, e.g., [20, Section 3.3]). Recall that the prime $T$-invariant divisors of $X(D)$ are of the form $D_{\rho}$ or $D_{Z,v}$ for $\rho$ a big ray of $\sigma(D)$ or $v \in \mathcal{V}(\Delta_Z)$ a big vertex. Therefore, any $T$-invariant Weil divisor $D$ of $X(D)$ can be written as

$$D = \sum_{\rho \text{ big}} a_\rho D_{\rho} + \sum_{(Z,v) \text{ big}} b_{Z,v} D_{Z,v}. $$

Observe that the field of rational functions of $\tilde{X}(D)$ and the field of rational functions of $X(D)$ are isomorphic to the fraction field of $\mathbb{K}(Y)[M]$, so any quasi-homogeneous rational function of $\tilde{X}(D)$ or $X(D)$ can be written as $f\chi^u$ where $u \in M$ and $f \in \mathbb{K}(Y)$. By [20, Proposition 3.14] we know that the principal divisor associated to $f\chi^u$ on $\tilde{X}(D)$ is given by

$$\text{div}(f\chi^u) = \sum_{\rho} \langle \rho, u \rangle D_{\rho} + \sum_{(Z,v)} \mu(v)(\langle v, u \rangle + \text{ord}_Z f)D_{Z,v},$$

and the associated principal divisor on $X(D)$ is given by

$$\text{div}(f\chi^u) = \sum_{\rho \text{ big}} \langle \rho, u \rangle D_{\rho} + \sum_{(Z,v) \text{ big}} \mu(v)(\langle v, u \rangle + \text{ord}_Z f)D_{Z,v},$$

where $\mu(v)$ denotes the smaller positive integer such that $\mu(v)v$ is an integral point of $N_\mathbb{Q}$. Therefore, the divisor $\text{div}(f\chi^u) + D$ is effective in $X(D)$ if and only if

$$\langle \rho, u \rangle + a_\rho \geq 0, \text{ for every } \rho \text{ big, and}$$

$$\langle v, u \rangle + \frac{b_{Z,v}}{\mu(v)} + \text{ord}_Z f \geq 0, \text{ for every pair } (Z,v) \text{ with } v \text{ big.}$$

We denote by $\square(D)$ the integer points of the polyhedron defined by

$$\{ u \in M_\mathbb{Q} \mid \langle \rho, u \rangle \geq -a_\rho, \text{ for each } \rho \text{ big } \}. $$

Given an element $u \in \square(D) \cap M$ we denote by $\psi_D(u)$ the divisor of $Y$ given by

$$\sum_{Z \subseteq Y} \left( \min_{v \text{ big}} \langle v, u \rangle + \frac{b_{Z,v}}{\mu(v)} \right)Z.$$

Therefore, we have that

$$H^0 \left( X(D), \mathcal{O}_{X(D)}(D) \right) \simeq \bigoplus_{u \in \square(D) \cap M} H^0 \left( X(D), \mathcal{O}_{X(D)}(D) \right)_u,$$

and

$$H^0 \left( X(D), \mathcal{O}_{X(D)}(D) \right)_u \simeq H^0 \left( Y, \mathcal{O}_Y(\psi_D(u))\chi^u \right).$$
3.4. Canonical sheaf. Let $K_Y$ be a canonical divisor of $Y$. Given a subvariety $Z \subset Y$ we denote by $\text{coef}_Z K_Y$ the coefficient of $K_Y$ along $Z$. Then, we can describe the canonical divisor of $\tilde{X}(D)$ as

\[(3.3) \quad K_{\tilde{X}(D)} = \sum_{(Z,v)} (\mu(v) \text{coef}_Z K_Y + \mu(v) - 1) \tilde{D}_{Z,v} - \sum_{\rho} \tilde{D}_\rho,
\]

and the canonical divisor of $X(D)$ as

\[(3.4) \quad K_X(D) = \sum_{(Z,v) \big} (\mu(v) \text{coef}_Z K_Y + \mu(v) - 1) D_{Z,v} - \sum_{\rho \big} D_\rho.
\]

4. Proof of the main theorems

In this section, we prove the main theorems of the article. We start by stating the theorems for rational singularities and Du Bois singularities of $T$-varieties. In order to state the theorems, we will use the following notation on $p$-divisors:

Notation 4.1. Let $D$ be a $p$-divisor on $(Y,N)$. In what follows, when we write $Z \subset Y$, we mean a prime divisor on $Y$. Let $V_Z$ be the set of vertices of $\Delta_Z$. Here, $\Delta_Z$ is the polyhedral divisor of $\Delta$ at the prime divisor $Z \subset Y$. Unless otherwise stated, $\rho$ is the primitive generator of an extremal ray of the cone $\sigma(D)$.

Let $D$ be a $p$-divisor on $(Y,N)$. Then, we have a function $D : \sigma^Y \to \text{CaDiv}_Q(Y)$. This function can be extended to the whole group $M$ as follows:

\[D(u) := \sum_{Z \subset Y} \min_{v \in V_Z} \langle u, v \rangle Z.
\]

Note that the previous sum is finite as $\Delta_Z = \rho(\Delta)$ for all but finitely many prime divisors $Z$. We define $Y^b \subset Y$ to be the union of the prime divisors $Z \subset Y$ for which no $v \in V_Z$ is big. Now, we turn to introduce some variations of $D$. We define the function

\[D_\omega : M \to \text{CaDiv}_Q(Y) \quad u \mapsto \sum_{Z \subset Y} \min_{v \in V_Z} \langle v, u \rangle Z.
\]

Note that the previous divisor has support in $Y^b$. On the other hand, we define:

\[\langle v, u \rangle := \begin{cases} [\langle v, u \rangle] & \text{if } v \text{ is big} \\ [\langle v, u \rangle] + 1 & \text{if } v \text{ is not big} \end{cases}
\]

Then, we can define the function

\[D_{\omega,E} : M \to \text{CaDiv}_Q(Y) \quad u \mapsto \sum_{Z \subset Y} \min_{v \in V_Z} \langle \langle v, u \rangle \rangle Z.
\]

Theorem 4.2. Let $D$ be a $p$-divisor on $(Y,N)$ with smooth support. Assume that $X(D)$ is Cohen-Macaulay. Then $X(D)$ has rational singularities if and only if the following conditions hold:

1. For every $u \in M$ with $\langle \rho, u \rangle \geq 1$ for all $\rho$, the isomorphism

\[H^0(Y^b, \omega_{Y^b}([D_\omega(u)])) \simeq H^0(Y, \omega_Y([D(u)]))
\]

holds, and
Lemma 4.4. The following conditions hold:

1. For every \( u \in M \) with \( \langle \rho, u \rangle \geq 1 \) for all \( \rho \) big and \( \langle \rho, u \rangle \leq 0 \) for at least one \( \rho \) not big, the equality

\[
h^0(Y^b, \omega_Y(\mathcal{D}_\omega(u))) = 0
\]

holds.

Theorem 4.3. Let \( D \) be a \( p \)-divisor on \((Y, N)\) with smooth support. Assume that \( X(D) \) is Cohen-Macaulay. Then, \( X(D) \) has Du Bois singularities if and only if the following conditions hold:

1. For every \( u \in M \) with \( \langle \rho, u \rangle \geq 1 \) for all \( \rho \) big and \( \langle \rho, u \rangle \geq 0 \) for all \( \rho \) not big, the isomorphism

\[
H^0(Y^b, \omega_Y(\mathcal{D}_\omega(u))) \simeq H^0(Y, \omega_Y(D_{\mathbb{C}, E}(u)))
\]

holds, and

2. For every \( u \in M \) with \( \langle \rho, u \rangle \geq 1 \) for all \( \rho \) big and \( \langle \rho, u \rangle \leq -1 \) for at least one \( \rho \) not big, the equality

\[
h^0(Y^b, \omega_Y(\mathcal{D}_\omega(u))) = 0
\]

holds.

Lemma 4.4. Let \( D \) be a simple normal crossing proper polyhedral divisor on a projective variety \( Y \). We denote by \( r: \tilde{X}(D) \to X(D) \) the associated \( \mathbb{T} \)-equivariant birational contraction. Let \( E \) be the reduced exceptional divisor of \( r \). Assume \( X(D) \) is Cohen-Macaulay. Then, the following statements hold:

1. \( X(D) \) has rational singularities if and only if \( r_*\omega_{\tilde{X}(D)} \simeq \omega_{X(D)} \), and
2. \( X(D) \) has Du Bois singularities if and only if \( r_*\omega_{\tilde{X}(D)}(E) \simeq \omega_{X(D)} \).

Proof. The proof of the first statement is analogous to the proof of the second statement. For simplicity, we denote \( X(D) \) by \( X \) and \( \tilde{X}(D) \) by \( \tilde{X} \).

Since the polyhedral divisor \( D \) has simple normal crossing support, then the variety \( \tilde{X} \) has toroidal singularities (see, e.g., [19, Example 2.5]). Let \( E^+ \) be the reduced sum of all the torus invariant vertical divisors mapping to \( \text{supp}(D) \) plus the reduced sum of all the torus invariant horizontal divisors. Again by [19, Example 2.5], the pair \((\tilde{X}, E^+)\) is toroidal. In particular, the pair \((\tilde{X}, E^+)\) has log canonical singularities (see, e.g. [9, Corollary 11.4.25]). Let \( E \) be the reduced exceptional divisor of \( \tilde{X} \to X \). Note that \( E \leq E^+ \). Let \( \phi: X' \to X \) be a log resolution of \( \tilde{X} \) and \( \tilde{\phi}: X' \to \tilde{X} \) be the induced projective birational morphism. Let \( E' \) be the reduced exceptional divisor of \( X' \to X \). By [16, Theorem 1], we have that \( X \) is Du Bois if and only if

\[
\phi_*\omega_{X'}(E') \simeq \omega_X.
\]

Hence, it suffices to prove that

\[
(4.1) \quad \tilde{\phi}_*\omega_{X'}(E') \simeq \omega_{\tilde{X}}(E).
\]

The isomorphism (4.1) follows from [16, Lemma 3.15].

Proof of Theorem 4.3. By Lemma 4.4, it suffices to check that \( r_*\omega_{\tilde{X}(D)}(E) \simeq \omega_{X(D)} \). Observe that both sheaves \( r_*\omega_{\tilde{X}(D)}(E) \) and \( \omega_{X(D)} \) are \( \mathbb{T} \)-invariant sheaves on an affine \( \mathbb{T} \)-variety. Hence, it suffices to prove that the \( M \)-graded pieces of

\[
(4.2) \quad H^0(\tilde{X}(D), \omega_{\tilde{X}(D)}(E))
\]
agree with the $M$-graded pieces of
\begin{equation}
H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})}).
\end{equation}

First, we compute the $M$-graded pieces of the group $H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})})$. Let $f \chi \in K(Y)[M]$. By (3.2) and (3.3), we know that $f \chi \in H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})})$ if and only if the following conditions hold:

\begin{align}
\langle \rho, u \rangle &\geq 1 \text{ for all } \rho \text{ big} \quad (4.4) \\
\langle \rho, u \rangle &\geq 0 \text{ for all } \rho \text{ not big} \quad (4.5)
\end{align}

By (3.2) and (3.3), we know that $f \chi \in H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})})$ if and only if the following conditions hold:

\begin{align}
\langle \rho, u \rangle &\geq 1 \text{ for all } \rho \text{ big} \quad (4.4) \\
\langle \rho, u \rangle &\geq 0 \text{ for all } \rho \text{ not big}
\end{align}

where $\varepsilon(v) = 0$ if $v$ is big and $\varepsilon(v) = 1$ otherwise. Hence, $f \chi \in H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})})$ if and only if (4.4), (4.5) are satisfied and (4.6) is replaced by

\begin{align}
\langle \rho, u \rangle &\geq 0 \forall \rho \text{ not big}
\end{align}

On the other hand

\begin{align}
\left\lceil \frac{\mu(v)}{\mu(v)} + \langle v, u \rangle \right\rceil = \begin{cases} 
\left\lceil \langle v, u \rangle \right\rceil & \text{if } v \text{ is big} \\
\left\lceil \langle v, u \rangle \right\rceil + 1 & \text{if } v \text{ is not big}
\end{cases}
\end{align}

we conclude that

\begin{align}
H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})}(E)) &\simeq \bigoplus_{\langle \rho, u \rangle \geq 1 \forall \rho \text{ big}} H^0(Y, \omega_Y(\mathcal{D}_{\mathcal{X}(\mathcal{D})}(u))).
\end{align}

Here, the function $\mathcal{D}_{\mathcal{X}(\mathcal{D})}(E)$ is defined as in Notation 4.1. Now, we turn to compute the $M$-graded pieces of the group (4.3). By (3.1) and (3.4), we know that $f \chi \in H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})})$ if and only if (4.4) holds and (4.6) holds for any $v$ big. The latter condition is equivalent to

\begin{align}
\langle \rho, u \rangle &\geq 1 \forall \rho \not\text{ big}
\end{align}

Hence, we conclude that

\begin{align}
H^0(\mathcal{X}(\mathcal{D}), \omega_{\mathcal{X}(\mathcal{D})}) &\simeq \bigoplus_{\langle \rho, u \rangle \geq 1 \forall \rho \not\text{ big}} H^0(Y^b, \omega_Y(\mathcal{D}_{\mathcal{X}(\mathcal{D})}(u))).
\end{align}

Thus, the affine variety $\mathcal{X}(\mathcal{D})$ has Du Bois singularities if and only if the following two conditions are satisfied:

- For every $u \in M$ with $\langle \rho, u \rangle \geq 1$ for all $\rho$ big and $\langle \rho, u \rangle \geq 0$ for all $\rho$ not big, the isomorphism

  \begin{align}
  H^0(Y^b, \omega_Y(\mathcal{D}_{\mathcal{X}(\mathcal{D})}(u))) &\simeq H^0(Y, \omega_Y(\mathcal{D}_{\mathcal{X}(\mathcal{D})}(u)))
  \end{align}

  holds.

- For every $u \in M$ with $\langle \rho, u \rangle \geq 1$ for all $\rho$ big and $\langle \rho, u \rangle \leq -1$ for at least one $\rho$ not big, the equality

  \begin{align}
  H^0(Y^b, \omega_Y(\mathcal{D}_{\mathcal{X}(\mathcal{D})}(u))) = 0
  \end{align}

holds.

This finishes the proof of the theorem. \qed
Proof of Theorem 4.2. By Lemma 4.4, it suffices to check that $r_{\ast}\omega_{\tilde{X}(D)} \simeq \omega_{X(D)}$. Observe that both sheaves $r_{\ast}\omega_{\tilde{X}(D)}$ and $\omega_{X(D)}$ are $\mathbb{T}$-invariant sheaves on an affine $\mathbb{T}$-variety. Hence, it suffices to prove that the $M$-graded pieces of

\begin{equation}
H^0(\tilde{X}(D), \omega_{\tilde{X}(D)})
\end{equation}

agree with the $M$-graded pieces of

\begin{equation}
H^0(X(D), \omega_{X(D)}).
\end{equation}

In view of the proof of Theorem 4.3, it suffices to find the $M$-graded pieces of (4.7). By (3.2) and (3.3), we know that $f^\rho_u$ belongs to $H^0(\tilde{X}(D), \omega_{\tilde{X}(D)}(E))$ if and only if the following conditions hold:

\begin{equation}
\langle \rho, u \rangle \geq 1 \quad \text{for all } \rho
\end{equation}

\begin{equation}
\text{coeff}_Z(K_Y) + \frac{\mu(v) - 1}{\mu(v)} + \langle v, u \rangle + \text{ord}_Z(f) \geq 0 \quad \text{for all } (Z, v).
\end{equation}

Hence, $f^\rho_u$ belongs to (4.7) if and only if (4.8) holds and (4.9) is replaced by

\begin{equation}
\text{ord}_Z(f) + \text{coeff}_Z(K_Y) + \min_{v \in V_Z} \left\{ \frac{\mu(v) - 1}{\mu(v)} + \langle u, v \rangle \right\} \geq 0 \quad \text{for all } Z.
\end{equation}

We conclude that

\begin{equation}
H^0(\tilde{X}(D), \omega_{\tilde{X}(D)}) \simeq \bigoplus_{\langle \rho, u \rangle \geq 1 \forall \rho} H^0(Y, \omega_Y([D(u)])).
\end{equation}

Thus, the affine variety $X(D)$ has rational singularities if and only if the following two conditions are satisfied:

- For every $u \in M$ with $\langle \rho, u \rangle \geq 1$ for all $\rho$, the isomorphism
  \[ H^0(Y^b, \omega_{Y^b}([D_\omega(u)]) \simeq H^0(Y, \omega_Y([D(u)])) \]
  holds.
- For every $u \in M$ with $\langle \rho, u \rangle \geq 1$ for all $\rho$ big and $\langle \rho, u \rangle \leq 0$ for at least one $\rho$ not big, the equality
  \[ H^0(Y^b, \omega_{Y^b}([D_\omega(u)]) = 0 \]
  holds.

This finishes the proof of the theorem. \hfill \Box

Now, we are ready to prove the theorems in the case of complexity one.

Proof of Theorem 1. If $D$ is a $p$-divisor on $(\mathbb{P}^1, N)$, then all the vertices $v \in V_p$, are big. We conclude that $\mathbb{P}^{1,h} = \mathbb{P}^1$ and $D_\omega = D$. Hence, the condition Theorem 4.2.1 is vacuous. On the other hand, the condition of Theorem 4.2.2 is equivalent to: for every $u \in M$ with $\langle \rho, u \rangle \geq 1$ for all $\rho$ big and $\langle \rho, u \rangle \leq 0$ for at least one $\rho$ not big, we have $\deg [D(u)] \leq 1$. This is equivalent to the statement of the theorem after replacing $u$ with $-u$. \hfill \Box

Proof of Theorem 2. If $D$ is a $p$-divisor on $(\mathbb{P}^1, N)$, then all the vertices $v \in V_p$, are big. We conclude that $\mathbb{P}^{1,h} = \mathbb{P}^1$ and $D_{\omega,E} = [D]$. Hence, the condition Theorem 4.3.1 is vacuous. On the other hand, the condition of Theorem 4.3.2 is equivalent to: for every $u \in M$ with $\langle \rho, u \rangle \geq 1$ for all $\rho$ big and $\langle \rho, u \rangle \leq -1$ for at least one $\rho$ not big, we have $\deg [D(u)] \leq 1$. This is equivalent to the statement of the theorem after replacing $u$ with $-u$. \hfill \Box
Example 4.5. Let $M = \mathbb{Z}^2$ so that $N = \mathbb{Z}^2$ and $M_Q = N_Q = \mathbb{Q}^2$. We also let $\sigma$ be the first quadrant, i.e., the cone spanned by $(1,0)$ and $(0,1)$. Taking $Y = \mathbb{P}^1$, we let $\mathcal{D}$ be the $p$-divisor on $Y$ given by $\mathcal{D} = \Delta_1 \cdot z_1 + \Delta_2 \cdot z_2 + \Delta_3 \cdot z_3 + \Delta_4 \cdot z_4$, where $\Delta_1 = \Delta_2 = (1/2, 1)$ and $\Delta_3 = \Delta_4 = (-1/2, 1)$. Since $\deg \mathcal{D} = \sum_{i=1}^{4} \Delta_i = (0,4) + \sigma \subseteq \sigma$ it follows that $\mathcal{D}$ is a $p$-divisor.

There are two rays in $\sigma$. We denote the ray spanned by $(1,0)$ by $\rho_1$ and the ray spanned by $(0,1)$ by $\rho_2$. We have that $\rho_1$ is big since it does not intersect $\deg \mathcal{D}$ while $\rho_2$ is not big since it intersects $\deg \mathcal{D}$.

By [21, Theorem 5] we have that $X(\mathcal{D})$ is a $\mathbb{Z}/2\mathbb{Z}$ quotient of $X(\mathcal{D}')$, where $\mathcal{D}' = \Delta_1' \cdot z_1 + \Delta_2' \cdot z_2 + \Delta_3' \cdot z_3 + \Delta_4' \cdot z_4$, where $\Delta_1 = \Delta_2 = (1,1)$ and $\Delta_3 = \Delta_4 = (-1,1)$ which is a toric variety since $\mathcal{D}'$ is equivalent to a $p$-divisor with at most two non-trivial coefficients. This yields $X(\mathcal{D}')$ is Cohen-Macaulay and so is $X(\mathcal{D})$ by [11, Proposition 13]. Furthermore, the set

\begin{equation}
\{ u \in M \mid \langle u, \rho \rangle \leq -1 \text{ for all } \rho \text{ big and } \langle u, \rho \rangle \geq 0 \text{ for at least one } \rho \text{ not big} \}
\end{equation}

in Theorem 1 is $\{(u_1, u_2) \in M \mid u_1 \leq -1 \text{ and } u_2 \geq 0 \}$. For instance, $u = (-1,0)$ is contained in this set and $\deg [\mathcal{D}(u)] = -2$. Theorem 1 implies that $X(\mathcal{D})$ does not have rational singularities. This agrees with [19, Proposition 5.1]

On the other hand, the set

\begin{equation}
\{ u \in M \mid \langle u, \rho \rangle \leq -1 \text{ for all } \rho \text{ big and } \langle u, \rho \rangle \geq 1 \text{ at least one } \rho \text{ not big} \}
\end{equation}

in Theorem 2 is $\{(u_1, u_2) \in M \mid u_1 \leq -1 \text{ and } u_2 \geq 1 \}$. Now, for every $u = (u_1, u_2)$ in this set, we have

\[ \deg [\mathcal{D}(u)] = 2 \left( \left\lfloor \frac{1}{2} u_1 \right\rfloor + \left\lfloor -\frac{1}{2} u_1 \right\rfloor \right) + 4u_2 \geq -2 + 4u_2 \geq 2 \geq -1. \]

We conclude from Theorem 2 that $X(\mathcal{D})$ has Du Bois singularities.

References

[1] Klaus Altmann, Nathan Owen Ilten, Lars Petersen, Hendrik Süß, and Robert Vollmert, The geometry of $T$-varieties, Contributions to algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 17–69. MR2975658

[2] Dan Abramovich and Jianhua Wang, Equivariant resolution of singularities in characteristic 0, Math. Res. Lett. 4 (1997), no. 2-3, 427–433.

[3] Piotr Achinger, Nathan Ilten, and Hendrik Süß, $F$-Split and $F$-Regular Varieties with a Diagonalizable Group Action, arXiv:1503.03116.

[4] Klaus Altmann and Jürgen Hausen, Polyhedral divisors and algebraic torus actions, Math. Ann. 334 (2006), no. 3, 557–607.

[5] Klaus Altmann, Jürgen Hausen, and Hendrik Süß, Gluing affine torus actions via divisorial fans, Transform. Groups 13 (2008), no. 2, 215–242.

[6] Ivan Arzhantsev, Jürgen Hausen, Elaine Herppich, and Alvaro Liendo, The automorphism group of a variety with torus action of complexity one, Mosc. Math. J. 14 (2014), no. 3, 429–471, 641.

[7] Benjamin Bechtold, Jürgen Hausen, Elaine Huggenberger, and Michele Nicolussi, On terminal Fano 3-folds with 2-torus action, Int. Math. Res. Not. IMRN 5 (2016), 1563–1602.

[8] Jean-François Boutot, Singularités rationnelles et quotients par les groupes réductifs, Invent. Math. 88 (1987), no. 1, 65–68.

[9] David A. Cox, John B. Little, and Henry K. Schenck, Toric varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[10] Hubert Flenner and Mikhail Zaidenberg, Normal affine surfaces with $\mathbb{C}^*$-actions, Osaka J. Math. 40 (2003), no. 4, 981–1009.
[11] M. and Eagon Hochster John A., *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

[12] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973.

[13] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original.

[14] János Kollár and Sándor J. Kovács, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813.

[15] Sándor J. Kovács, *Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink*, Compositio Math. **118** (1999), no. 2, 123–133. MR1713307

[16] Sándor J. Kovács, Karl Schwede, and Karen E. Smith, *The canonical sheaf of Du Bois singularities*, Adv. Math. **224** (2010), no. 4, 1618–1640.

[17] Kevin Langlois, *Singularités canoniques et actions horosphériques*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 4, 365–369, DOI 10.1016/j.crma.2017.03.004 (French, with English and French summaries). MR3634672

[18] Kevin Langlois and Alvaro Liendo, *Additive group actions on affine T-varieties of complexity one in arbitrary characteristic*, J. Algebra **449** (2016), 730–773.

[19] Alvaro Liendo and Hendrik Süß, *Normal singularities with torus actions*, Tohoku Math. J. (2) **65** (2013), no. 1, 105–130.

[20] Lars Petersen and Hendrik Süß, *Torus invariant divisors*, Israel J. Math. **182** (2011), 481–504.

[21] Charlie Petitjean, *Cyclic covers of affine T-varieties*, J. Pure Appl. Algebra **219** (2015), no. 9, 4265–4277.

[22] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973. MR0335518

[23] Sándor J. Kovács and Karl E. Schwede, *Hodge theory meets the minimal model program: a survey of log canonical and Du Bois singularities*, Topology of stratified spaces, Math. Sci. Res. Inst. Publ., vol. 58, Cambridge Univ. Press, Cambridge, 2011, pp. 51–94.

[24] Karl Schwede, *A simple characterization of Du Bois singularities*, Compos. Math. **143** (2007), no. 4, 813–828.

[25] Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28. MR0337963

[26] Dmitri A. Timashev, *Torus actions of complexity one*, Toric topology, 2008, pp. 349–364.

[27] Robert Vollmert, *Toroidal embeddings and polyhedral divisors*, Int. J. Algebra **4** (2010), no. 5-8, 383–388.

[28] Keiichi Watanabe, *Some remarks concerning Demazure’s construction of normal graded rings*, Nagoya Math. J. **83** (1981), 203–211.