On homoclinic solutions for a second order difference equation with $p-$Laplacian

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Abstract

In this paper, we obtain conditions under which the difference equation

$$-\Delta (a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda f(k, u(k)), \quad k \in \mathbb{Z},$$

has infinitely many homoclinic solutions. A variant of the fountain theorem is utilized in the proof of our theorem. It improves the results in [L. Kong, Homoclinic solutions for a second order difference equation with $p-$Laplacian, Appl. Math. Comput., 247 (2014), 1113–1121], where the set of conditions imposed on nonlinearity is inconsistent.

Math Subject Classifications: 39A10, 47J30, 35B38

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1 Introduction

In the present paper we deal with the following nonlinear second-order difference equation:

$$\begin{cases} -\Delta (a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda f(k, u(k)) & \text{for all } k \in \mathbb{Z}, \\ u(k) \to 0 \text{ as } |k| \to \infty. \end{cases}$$

Here $p > 1$ is a real number, $\lambda$ is a positive real parameter, $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, $a, b : \mathbb{Z} \to (0, +\infty)$, while $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, the forward difference operator is defined as $\Delta u(k-1) = u(k) - u(k-1)$. We say that a solution $u = \{u(k)\}$ of (1) is homoclinic if $\lim_{|k| \to \infty} u(k) = 0$.

In this paper, similarly to [3], our goal is to apply the variational method and a variant of the fountain theorem to find a sequence of homoclinic solutions for the problem (1). Our theorem improves the results in [3], where the set of conditions imposed on nonlinearity is inconsistent. We not only show that one of the assumptions is in fact superfluous, but also that others can be relaxed. The problem (1) has been studied recently in several papers. Infinitely many solutions were obtained in [8] by employing Nehari manifold methods, in [6] by use of the Ricceri’s theorem (see [1], [5]), and in [7] directly applying the variational method.

We assume that potential $b(k)$ and the nonlinearity $f(k, t)$ satisfies the following conditions:

(B) $b(k) \geq b_0 > 0$ for all $k \in \mathbb{Z}$, $b(k) \to +\infty$ as $|k| \to +\infty$;

(H$_1$) $f(k, -t) = -f(k, t)$ for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$;

(H$_2$) there exist $d > 0$ and $q > p$ such that $|F(k, t)| \leq d(|t|^p + |t|^q)$ for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$;

(H$_3$) $\lim_{t \to 0} \frac{f(k, t)}{|t|^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;
\((H_4)\) \(\lim_{|t| \to +\infty} \frac{f(k,t)t}{|t|^p} = +\infty\) for all \(k \in \mathbb{Z}\);

\((H_5)\) there exists \(\sigma \geq 1\) such that \(\sigma F(k,t) \geq F(k,st)\) for \(k \in \mathbb{Z}, t \in \mathbb{R}\), and \(s \in [0,1]\),

where \(F(k,t)\) is the primitive function of \(f(k,t)\), that is \(F(k,t) = \int_0^t f(k,s)ds\) for every \(t \in \mathbb{R}\) and \(k \in \mathbb{Z}\), and \(F(k,t) = f(k,t)t - pF(k,t)\).

Kong \([3]\) gave conditions for existence of a sequence of solutions of the problem \([1]\). In additions to hypotheses \((B), (H_1), (H_5)\), he offered also the following conditions:

\((H'_1)\) there exist \(d > 0\) and \(q > p\) such that \(|F(k,t)| \leq dt^q\) for all \(k \in \mathbb{Z}\) and \(t \in \mathbb{R}\);

\((H'_3)\) \(\sup_{|t| \leq T} |F(\cdot,t)| \in l_1\) for all \(T > 0\);

\((H'_4)\) \(\lim_{|t| \to +\infty} \frac{f(k,t)t}{|t|^p} = +\infty\) uniformly for all \(k \in \mathbb{Z}\).

Obviously, \((H'_1)\) is stronger than \((H_1)\), and \((H'_2)\) is stronger than both \((H_2)\) and \((H_5)\). In \([3]\), as an example of function, which satisfied conditions \((H_1), (H_2), (H'_3), (H'_4), (H_5)\) is given the function

\[
f(k,t) = \frac{1}{k^{\nu}} |t|^{\mu - 2} t \ln (1 + |t|^p), \quad (k,t) \in \mathbb{Z} \times \mathbb{R}
\]

with \(\mu > 1\) and \(\nu \geq 1\). But this does not satisfy the condition \((H'_1)\). Moreover, the conditions \((H'_3)\) and \((H'_4)\) are contradictory. Indeed, since \(p > 1\) the hypothesis \((H'_3)\) does give us \(T_1 > 0\) such that \(|f(k,t)| \geq 1\) for all \(|t| \geq T_1\) and \(k \in \mathbb{Z}\). Put \(\alpha_k = F(k,T_1)\) for all \(k \in \mathbb{Z}\). Then \(\{\alpha_k\} \in l_1\), by \((H'_3)\). As \(f\) is continuous we have for \(T > T_1\) and \(k \in \mathbb{Z}\)

\[
|F(k,T)| = \left| \int_0^T f(k,t)dt \right| = \left| \int_0^{T_1} f(k,t)dt + \int_{T_1}^T f(k,t)dt \right| = \left| \alpha_k + \int_{T_1}^T f(k,t)dt \right| \\
\geq \left| \int_{T_1}^T f(k,t)dt \right| - |\alpha_k| = \int_{T_1}^T |f(k,t)| dt - |\alpha_k| \geq (T - T_1) - |\alpha_k|,
\]

and so \(|F(\cdot,T)| \notin l_1\), contrary to \((H'_3)\).

It is easy to verify that the function \([2]\) does satisfy conditions \((H_1) - (H_5)\). Note also that it does not satisfy the standard Ambrosetti-Rabinowitz condition.

2 Preliminaries

We repeat the relevant for us material from \([3]\). We begin by defining some Banach spaces. For all \(1 \leq p < +\infty\), we denote \(\ell^p\) the set of all functions \(u : \mathbb{Z} \to \mathbb{R}\) such that

\[
\|u\|_p = \sum_{k \in \mathbb{Z}} |u(k)|^p < +\infty.
\]

Moreover, we denote \(\ell^\infty\) the set of all functions \(u : \mathbb{Z} \to \mathbb{R}\) such that

\[
\|u\|_\infty = \sup_{k \in \mathbb{Z}} |u(k)| < +\infty
\]

We set

\[
X = \left\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{k \in \mathbb{Z}} [\alpha(k) |\Delta u(k-1)|^p + b(k)|u(k)|^p] < \infty \right\}
\]
\[ \|u\| = \left( \sum_{k \in \mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] \right)^{\frac{1}{p}}. \]

Clearly we have
\[ \|u\|_{\infty} \leq \|u\|_p \leq b_0^{-\frac{1}{p}} \|u\| \quad \text{for all} \quad u \in X. \]  

(4)

Moreover, \((X, \| \cdot \|)\) is a reflexive and separable Banach space and the embedding \(X \hookrightarrow l^p\) is compact (see Lemma 2.2 in [3]).

**Lemma 1** If \(S\) is a compact subset of \(l^p\), then, for every \(\delta > 0\), there exists \(h > 0\) such that
\[ |u(k)| \leq \left( \sum_{|k| > h} |u(k)|^p \right)^{\frac{1}{p}} < \delta \]
for all \(u \in S\) and \(|k| > h\).

This is Lemma 3.3 in [3].

Let
\[ \Phi(u) := \frac{1}{p} \sum_{k \in \mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] \quad \text{for all} \quad u \in X \]
and
\[ \Psi(u) := \sum_{k \in \mathbb{Z}} F(k, u(k)) \quad \text{for all} \quad u \in l^p \]
where \(F(k, s) = \int_0^s f(k, t)dt\) for \(s \in \mathbb{R}\) and \(k \in \mathbb{Z}\). Let \(J : X \to \mathbb{R}\) be the functional associated to problem (1) defined by
\[ J_\lambda(u) = \Phi(u) - \lambda \Psi(u). \]

**Lemma 2** Assume that (B) and (H\(\lambda\)) are satisfied. Then

(a) \( \Phi \in C^1(X)\);
(b) \( \Psi \in C^1(l^p) \) and \( \Psi \in C^1(X)\);
(c) \( J_\lambda \in C^1(X) \) and every critical point \( u \in X \) of \( J_\lambda \) is a homoclinic solution of problem (1).

This version of the lemma can be proved essentially by the same way as Propositions 5,6 and 7 in [2], where \(a(k) \equiv 1\) on \(\mathbb{Z}\) and the norm on \(X\) is slightly different. See also Lemma 2.3 in [3].

**3 Main results**

Now we are ready to state our result.

**Theorem 3** Suppose that the conditions (B), (H\(\lambda\)) - (H\(\lambda\)) hold. Then, for any \(\lambda > 0\), the problem (1) has a sequence \(\{u_n(k)\}\) of solutions such that \(J_\lambda(u_n) \to \infty\) as \(n \to \infty\).
Our main tool is the following version of the fountain theorem with Cerami’s condition (see [4]). We say that \( I, a C^1 \)-functional defined on a Banach space \( X \), satisfies the Cerami condition if any sequence \( \{u_n\} \subset X \) such that \( \{I(u_n)\} \) is bounded and \( (1 + \|u_n\|) \|I'(u_n)\| \to 0 \) has a convergent subsequence; such a sequence is then called a Cerami sequence. Now, let \( X \) be a reflexive and separable Banach space. It is well known that there exists \( e_i \in X \) and \( e^*_i \in X^* \) such that
\[
X = \text{span}\{e_i : i \in \mathbb{N}\}, \quad X^* = \text{span}\{e^*_i : i \in \mathbb{N}\}^{w^*}
\]
and
\[
\langle e^*_i, e_j \rangle = \delta_{ij}, \text{ where } \delta_{ij} = 1 \text{ for } i = j \text{ and } \delta_{ij} = 0 \text{ for } i \neq j.
\]
Put
\[
X_i = \text{span}\{e_i\}, \quad Y_n = \bigoplus_{i=1}^n X_i \quad \text{and} \quad Z_n = \bigoplus_{i=n}^\infty X_i.
\]

**Theorem 4** Assume that \( I \in C^1(X, \mathbb{R}) \) satisfies the Cerami condition and \( I(-u) = I(u) \). If for almost every \( n \in \mathbb{N} \), there exist \( \rho_n > r_n > 0 \) such that
\[
(i) \ a_n = \inf_{u \in Z_n, \|u\| = r_n} I(u) \to +\infty \text{ as } n \to \infty;
\]
\[
(ii) \ b_n = \max_{u \in Y_n, \|u\| = \rho_n} I(u) \leq 0,
\]
then I has a sequence of critical points \( \{u_n\} \) such that \( I(u_n) \to +\infty \).

In the remainder of this paper, let \( X \) be defined by (3), and \( Y_n \) and \( Z_n \) be given in (5). To prove Theorem 1, we will also need the following.

**Lemma 5** Let \( q \geq p \). For \( n \in \mathbb{N} \), define
\[
\beta_{q,n} = \sup_{u \in Z_n, \|u\| = 1} \|u\|_q.
\]
Then, \( \lim_{n \to \infty} \beta_{q,n} = 0 \).

For \( q > p \) this is Lemma 3.2 in [3]. As proof shows, the instance \( q = p \) is also true.

**Lemma 6** Suppose that \((H_5)\) holds, then
\[
F(k, t) \geq 0 \quad \text{and} \quad f(k, t) t \geq 0
\]
for all \((k, t) \in \mathbb{Z} \times \mathbb{R}\).

**Proof.** By \((H_5)\),
\[
f(k, t) t - pF(k, t) \geq 0
\]
for all \((k, t) \in \mathbb{Z} \times \mathbb{R}\). From this, for \( t > 0 \) and \( k \in \mathbb{Z} \), we have
\[
\frac{\partial}{\partial t} \left( \frac{F(k, t)}{t^p} \right) = \frac{tp^p f(k, t) - pt^{p-1} F(k, t)}{t^{2p}} \geq 0.
\]
Since \( F(k, 0) = 0 \), we obtain \( F(k, t) \geq 0 \) and \( f(k, t) t \geq 0 \) for all \( k \in \mathbb{Z} \) and \( t \geq 0 \). Arguing similarly for the case \( t \leq 0 \), we complete the proof. \( \blacksquare \)
Lemma 7 Assume that (B) and \((H_1), (H_4), (H_5)\) hold. Then, for any \(\lambda > 0\), \(J_\lambda\) satisfies Cerami’s condition.

Proof. Let \(\lambda > 0\) be fixed. Let \(\{u_n\}\) be a Cerami sequence of \(J_\lambda\). Firstly, we assume that \(\{u_n\}\) is bounded. Up to considering a subsequence, we may assume that for some \(c \in \mathbb{R}\), \(J_\lambda(u_n) \to c\) and \(J'_\lambda(u_n) \to 0\). Then the proof proceeds along the same lines as the first part of the proof of Lemma 3.4 in [3], giving the desired conclusion. This proof uses assumptions (B) and \((H_3)\).

Now, let we suppose that \(\{u_n\}\) is unbounded. Then, up to a subsequence, we may assume that for some \(c \in \mathbb{R}\),
\[
J_\lambda(u_n) \to c, \quad \|u_n\| \to \infty, \quad \|u_n\| \|J'_\lambda(u_n)\| \to 0.
\]

And again, proceeding as in the second part of the proof of Lemma 3.4 in [3], we obtain a contradiction. We must only use Lemma 3 and assumption \((H_4)\) instead of using condition \((H'_4)\).

Proof of Theorem 1. Let \(\lambda > 0\) be fixed. By \((H_1)\) and Lemma 4, \(J_\lambda\) is even and satisfies Cerami’s condition.

In the following, we show that, for almost every \(n \in \mathbb{N}\), there exist \(\rho_n > r_n > 0\) such that conditions (i) and (ii) of Theorem 4 with \(I = J_\lambda\) are satisfied.

By Lemma 4 there exists \(n_0 \in \mathbb{N}\) such that \(\beta_{p,n} < \left(\frac{1}{2^{\frac{1}{p}}}\right)^{\frac{1}{p}}\) for all \(n \geq n_0\). Let \(r_n = \left(\frac{\lambda d - \lambda d_{p,n}}{\lambda d_{p,n}}\right)^{\frac{1}{p}}\) for all \(n \geq n_0\). Here \(d > 0\) and \(q > p\) are given in \((H_2)\). Then, one has \(r_n > 0\) and \(\lim_{n \to \infty} r_n = \infty\). For \(u \in Z_n\) we have \(\|u\|_p \leq \beta_{p,n} \|u\|\) and \(\|u\|_q \leq \beta_{q,n} \|u\|\), by Lemma 5. Let \(u \in Z_n\) with \(\|u\| = r_n\). Then
\[
J_\lambda(u) = \frac{1}{p} \sum_{k \in \mathbb{Z}} \left|a(k)\Delta u(k-1) + b(k)|u(k)|^p\right| - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k)) \geq \frac{1}{p} \|u\|_p^p - \lambda d \left(\left\|u\right\|_p^p + \left\|u\right\|_q^q\right)
\geq \frac{1}{p} \|u\|_p^p - \lambda d \left(\beta_{p,n} \|u\|_p^p + \beta_{q,n} \|u\|_q^q\right) = \frac{1}{2p} \left(\frac{\lambda d_{p,n} - \lambda d}{\lambda d_{q,n}}\right)^{\frac{1}{p}} = \frac{1}{2p} r_n^p.
\]
Therefore
\[
a_n = \inf_{u \in Z_n \cap \|u\| = r_n} J_\lambda(u) \to +\infty \quad \text{as} \quad n \to \infty
\]
and condition (i) holds.

Since \(\dim Y_n < \infty\), there exists \(C_n > 0\) such that
\[
\frac{1}{p} \|u\|_p^p \leq \lambda C_n \|u\|_\infty^p \quad \text{for all} \quad u \in Y_n. \tag{6}
\]
As \(S_n = \{u \in Y_n : \|u\| = 1\}\) is compact in \(L^p\), there exists \(h_n > 0\) such that \(|u(k)| < 1\) for all \(u \in S_n\) and \(|k| > h_n\), by Lemma 6. Consequently, for every \(u \in Y_n\), there exists \(k_0 \in \mathbb{Z}\) with \(|k_0| \leq h_n\) such that \(\|u\|_\infty = |u(k_0)|\). By \((H_4)\), there exists \(T > 0\) such that
\[
F(k, t) \geq 2C_n |t|^p \quad \text{for} \quad (k, t) \in \mathbb{Z} \times \mathbb{R} \quad \text{with} \quad |k| \leq h_n \quad \text{and} \quad |t| > T. \tag{7}
\]
Choose \(\rho_n > \max\{\lambda p C_n\}^{1/p} T, r_n\}\) for all \(n \geq n_0\). Then \(\rho_n > r_n > 0\). Taking \(u \in Y_n\) with \(\|u\| = \rho_n\) we have \(\|u\|_\infty > T\) and \(\|u\|_\infty = |u(k_0)|\) for some \(k_0 \in \mathbb{Z}\) with \(|k_0| \leq h_n\). Thus, from (6), (7) and Lemma 6 it follows that
\[
J_\lambda(u) = \frac{1}{p} \|u\|_p^p - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k)) \leq \frac{1}{p} \|u\|_p^p - \lambda F(k_0, u(k_0)) \leq \frac{1}{p} \|u\|_p^p - \lambda 2C_n |u(k_0)|^p
\]
\[
= \frac{1}{p} \|u\|_p^p - \lambda 2C_n \|u\|_\infty^p \leq -\frac{1}{p} \|u\|_p^p.
\]
Therefore
\[
b_n = \max_{u \in Y_n \cap \|u\| = \rho_n} J_\lambda(u) \leq 0
\]
and condition (ii) holds. \qed
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