Folding model analysis of $^{12}C - ^{12}C$ and $^{16}O - ^{16}O$ elastic scattering using the density-dependent LOCV averaged effective interaction

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Abstract

The averaged effective two-body interaction (AEI) which can be generated through the lowest order constrained variational (LOCV) method for symmetric nuclear matter (SNM) with the input Reid68 nucleon-nucleon potential, is used as the effective nucleon-nucleon potential in the folding model to describe the heavy-ion (HI) elastic scattering cross sections. The elastic scattering cross sections of $^{12}C - ^{12}C$ and $^{16}O - ^{16}O$ systems are calculated in the above frameworks. The results are compared with the corresponding calculations coming from the fitting procedures with the input finite range $DDM3Y1-Reid$ potential and the available experimental data at different incident energies. It is shown that a reasonable description of the elastic $^{12}C - ^{12}C$ and $^{16}O - ^{16}O$ scattering data at the low and the medium energies can be obtained by using the above LOCV AEI, without any need to define a parameterize density dependent function in the effective nucleon-nucleon potential, which is formally considered in the typical $DDM3Y1-Reid$ interactions.

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I. INTRODUCTION

In recent years, there has been a growing interest in the heavy-ion (HI) scattering. These collision processes were investigated widely both experimentally and theoretically. One of the goals of studying the HI reactions is to determine the form of the most suitable effective nucleon-nucleon potential, to explain the experimental elastic scattering cross section data [1, 2]. For many years, the use of empirical parametrization of nuclear potential was very common in the HI studies, but it is desirable to relate the nucleus-nucleus (NN) interactions to the nucleon-nucleon (NN) nuclear potential [3]. Many attempts in this direction have been made, and recently, the double-folding (DF) model was extensively used by many groups in describing the HI scattering, since it gives a simple possibility of numerical handling in two nucleus scattering calculations [4].

In the folding model, the potential is usually generated by folding an effective NN interaction over the ground-state density distribution of the two nuclei [1, 2]. In general, we need a well-defined effective NN interaction which reproduces the basic nuclear matter properties (like the saturation energy and density), and, on the other hand, it can be used as a basic input in the description of HI scattering qualitatively with respect to the experimental data [5]. The M3Y interaction [6] and its density dependent versions [7–13], are usually used into the folding model. Recently the G-matrix and extended Hartree–Fock approaches [14–19] with and without the inclusion of the three body force (TBF) and the rearrangement term (RT), were applied for calculating the nucleon-nucleus and the nucleus-nucleus scattering cross-section calculations (but mainly at 70 MeV), as well as obtaining the nuclear matter saturation properties (EOS) [14]. The RT comes out in case of calculating the single particle energy and the corresponding potential. But in the present work, we intend to apply the lowest order constrained variational averaged effective interaction LOCV AEI, which was generated by using the input Reid68 potential in our previous work [20], as the effective NN interaction, into the folding model to test the validity of our interaction in describing the HI elastic scattering. In this paper, we limit ourselves to the elastic scattering of spherical projectile and spherical target nuclei, so we consider the $^{12}\text{C} - ^{12}\text{C}$ and $^{16}\text{O} - ^{16}\text{O}$ elastic scattering.

A brief discussion about the LOCV method is given in the appendix A. Contrary to G-matrix approach, in the LOCV formalism (which is based on the cluster expansion [21]),
the wave functions, e.g. the correlation functions, are calculated through the Euler-Lagrange differential equations, whereas the application of G operator on the plane wave generate the interacting wave functions. Another advantage of the cluster expansion is its expansion in the powers of correlation functions (in the G-matrix language the wound parameter) and the first power of the $NN$ potential. So it converges faster than the G-matrix approaches which is an expansion in the powers of the potential. On the other hand since we directly calculate the $LOCV AEI$, there is no need to calculate the RT in our approach. In the table 1, the results of the $LOCV$ saturation properties of symmetrical nuclear matter ($SNM$) calculation for the $Reid68$ and $\Delta – Reid68$ potentials, (in comparison to the empirical one), are presented. The $LOCV$ method is self-consistently predict the EOS of $SNM$ (for the detail see the appendix and the table A.1). The one-body, ($E_1$ (is simply the Fermi energy)), the two-body cluster, ($E_2$), and the three-body cluster ,($E_3$), terms as well as the convergence parameters are discussed in the appendix.

In some of our $LOCV$ calculations, we have taken into account the effects of $TBF$ such as the $\Delta$ box diagram (see the appendix A). But in the present work since we intend to compare our results with those coming from the $M3Y$ interaction [6] which is based on the $Reid68$ potential, so our results will be limited to this interaction. However we hope in our future works, the other interactions as well as the effects of the $TBF$ on the nucleus-nucleus differential cross sections are evaluated. In the table A.1 it is clearly demonstrated that the $LOCV$ method predicts the $SNM$ saturation properties close to other methods, even with or without $TBF$ [22]. We should point out here that there is no extra parameters and conditions on the $LOCV$ method to predict the saturation properties of $SNM$.

In our recent paper [20] , we derived the averaged effective two-body interactions ($AEI$) through the lowest order constrained variational ($LOCV$) calculations for the $SNM$ with the $Reid68$ [23], the $\Delta-Reid68$ [24] (which takes into the account the effect of three-body force ($TBF$)) and the $A_{18}$ [25] interactions as the input phenomenological nucleon-nucleon potentials, and reformulated them in the radial and density-dependent parts as well as its direct and exchange components. Note that the radial parts are fixed and density dependent functions only depend on density which becomes a constant at fix density, i.e. similar to the $M3Y$ calculations. Here as we stated above, we only use the $LOCV AEI$ with the input $Reid68$ potential into the folding model and compare our results with those coming from the $DDM3Y1-Reid$ which uses a finite range potential as the direct and exchange components i.e.
M3Y interactions [4]. The LOCV effective two-body interactions were tested by calculating the properties of the light and the heavy closed shell nuclei [26,28], and recently it was used to calculate the in-medium \textit{nn} cross section, the transport properties of neutron matter [29,30] and the normal liquid Helium-3 [31]. In these works, it was shown that the \textit{LOCV AEI} gave the reasonable results in comparison to the corresponding available data.

So, this article is organized as follows: In the section 2, we briefly review the theoretical formalism of the double folding model. The density distributions and the different kinds of the effective interactions used into the folding model as well as the computational procedure are also discussed in this section. Finally, while the results of the calculations and discussions are given in the section 3, the section 4 is devoted to the summary and conclusions.

II. THE THEORETICAL FORMALISM

A. The double folding model

Satchler and Love [32] presented the basic idea of the folding model in detail and in the reference [4], an improved version of folding model was introduced to calculate the exchange part of the HI potential. We give here only a brief description of this model and refer the reader to the references [1,2,32-35] for details. In the first order of Feshbach’s theory for the optical potential, the microscopic nucleus-nucleus potential can be evaluated as an antisymmetrized \textit{HartreeFock} type potential for the \textit{dinuclear} system [1,2,4]:

\[
U = U_D + U_{EX} = \sum_{i \in A_1, j \in A_2} \left[ \langle ij | v_D | ij \rangle - \langle ij | v_{EX} | ji \rangle \right],
\]

where \(|i\rangle\) and \(|j\rangle\) refer to the single-particle wave functions of nucleons in the two colliding nuclei \(A_1\) and \(A_2\), respectively; \(v_D\) and \(v_{EX}\) are the direct and the exchange parts of the effective \textit{NN} interaction. After doing some algebra, one can explicitly write the energy-dependent direct and exchange potentials as,

\[
U_D(E, \mathbf{R}) = \int d\mathbf{r}_p d\mathbf{r}_t \rho_p(\mathbf{r}_p) \rho_t(\mathbf{r}_t) v_D(\rho, E, s) \; s = \mathbf{r}_p - \mathbf{r}_t + \mathbf{R},
\]

\[
U_{EX}(E, \mathbf{R}) = \int d\mathbf{r}_p d\mathbf{r}_t \rho_p(\mathbf{r}_p; \mathbf{r}_p + s) \rho_t(\mathbf{r}_t; \mathbf{r}_t - s) v_{EX}(\rho, E, s) e^{ik_{rel}s/A_{red}}.
\]

Note that, in general the one-body density is written as \(\rho(\mathbf{r}, \mathbf{r}')\). In the case of direct term, it becomes \(\rho(\mathbf{r}_p)\) or \(\rho(\mathbf{r}_t)\), i.e. the diagonal terms, where \(\mathbf{r}_p\) and \(\mathbf{r}_t\) are the positions of
the two nucleons in the nuclei p (projectile) and t (target), respectively, \( s = \mathbf{r}_p - \mathbf{r}_t + \mathbf{R} \) corresponds to the distance between the two specified interacting points of the projectile and the target, and \( \mathbf{R} \) is a vector from the center of the t nucleus to that of p nucleus. But in case of the exchange terms, we have \( \rho(\mathbf{r}, \mathbf{r}') \) for each nucleus, i.e. nondiagonal terms, with \((\mathbf{r} = \mathbf{r}_p, \mathbf{r}' = \mathbf{r}_p + s) \) or \((\mathbf{r} = \mathbf{r}_t, \mathbf{r}' = \mathbf{r}_t - s) \). So for the exchange term the densities are the functions of two different coordinates [4]. In the above equations, the wave number \( k_{\text{rel}} \) associated with the relative motion of colliding nuclei, which is given by:

\[
k_{\text{rel}}^2 (\mathbf{R}) = 2m_n A_{\text{red}} \left[ E_{\text{c.m.}} - U(E, \mathbf{R}) - V_C(\mathbf{R}) \right]/\hbar^2,
\]

where \( A_{\text{red}} = A_p A_t/(A_p + A_t) \), \( m_n \), \( E_{\text{c.m.}} \) and \( E \) are the reduced mass number, the bare nucleon mass, the center-of-mass (c.m.) energy and the incident laboratory energy per nucleon, respectively. Here \( U(E, \mathbf{R}) = U_D(E, \mathbf{R}) + U_{\text{EX}}(E, \mathbf{R}) \) and \( V_C(\mathbf{R}) \) are the total nuclear and the Coulomb potentials, respectively. It can be seen from the equation (3) that the energy-dependent HI potential is nonlocal through its exchange term. For simplicity of the numeric calculations, a realistic local expression for the density matrix is usually used [36]:

\[
\rho(\mathbf{R}, \mathbf{R} + s) \simeq \rho(\mathbf{R} + \frac{s}{2}) \hat{j}_1(k_F(\mathbf{R} + \frac{s}{2}) s),
\]

where \( \hat{j}_1(x) = 3(\sin x - x \cos x)/x^3 \). The explicit form of \( k_F(\mathbf{R}) \) is given in the reference [4]. In order to specify the overlap density during the HI collision, we have applied the procedure used in the reference [4] that is called frozen density approximation (FDA). In this approach, the overlap density, \( \rho \), is taken to be the sum of the densities of the target and the projectile densities at the midpoint of the inter-nucleon separation, i.e.,

\[
\rho = \rho_p(\mathbf{r}_p + \frac{s}{2}) + \rho_t(\mathbf{r}_t - \frac{s}{2}).
\]

This procedure simply corresponds to the local density approximation assumed in the different nuclear matter studies [4, 26, 28].

After performing some transformations one can obtain the exchange potential in the following local form:

\[
U_{\text{EX}}(E, \mathbf{R}) = 4\pi \int_0^\infty v_{\text{EX}}(s, E) s^2 ds \hat{j}_0(k(\mathbf{R}) s/M) \times \int f_1(\mathbf{r}, s) f_2(\mathbf{r} - \mathbf{R}, s) F[\rho_p(\mathbf{r}) + \rho_t(\mathbf{r} - \mathbf{R})] d\mathbf{r},
\]

5
where \( F(\rho) \) will be defined later on, i.e. see the equations (19) to (25) in the subsection II-B),
\[
f_{1(2)}(r, s) = \rho(\rho(r) \hat{j}_1(k_{F1(2)}(r)s), \quad \hat{j}_0(x) = \frac{\sin x}{x}.
\]
Applying the folding formulas in the momentum space \[36\], one can write the exchange potential as:
\[
U_{EX}(E, R) = 4\pi \int_{0}^{\infty} G(R, s) \hat{j}_0(k(R)s/M) \nu_{EX}(s, E)s^2 ds.
\]

The explicit form of \( G(R, s) \) function can be found in the reference \[4\].

As it can be seen from the equation (4), the wave number of relative motion, \( k_{rel}(R) \),
depends on the total HI potential, so, we encounter with a self-consistency problem in
obtaining the exchange part of HI potential at each radial point. In general, this problem
can be overcome by applying an iterative procedure, as it was performed for the first time
by Chaudhuri et al. \[37\]. However, in the references \[34, 35\] a closed expression was used
to obtain the exchange potential by using the multiplication theorem of the Bessel function
\( \hat{j}_0(k(R)s/M) \). In this paper, we use the iterative method to ensure the self-consistency at
all the radial point, in which, we chose \( U_D(E, R) \) as the starting potential to enter in the
\( \hat{j}_0(k(R)s/M) \) term in the exchange integral, the equation (9).

Since the effective NN interactions applied into the folding model are real, the calculated
HI potentials are also real, so, the imaginary part of HI potential, is usually treated
phenomenologically and its parameters are adjusted to optimize the fit to the observed scattering.
In the most cases, the Woods-Saxon (WS) shape (with volume or the surface type)
is used for the imaginary potential. Finally the HI potential can be written in the general
form as:
\[
U(E, R) = N_R[U_D(E, R) + U_{EX}(E, R)] - iW_V[1 + \exp(\frac{R - R_V}{a_V})]^{-1}
+ 4iW_Da_D\frac{d}{dR}[1 + \exp(\frac{R - R_D}{a_D})]^{-1},
\]

where the renormalization coefficient \( N_R \) together with the parameters of the imaginary potential
are adjusted to give the best fit to the scattering data. The renormalization coefficient \( N_R \) is needed to account roughly for the many-nucleon exchange effects and the
dynamical polarization potential \( (\Delta U) \) \[32\]. The volume or the surface WS (the second and
the third terms at above formula) are usually used as the imaginary potential in the elastic scattering analysis. However, we only use the volume term in our present calculations.
In the calculation of the exchange potential, we need also the Coulomb potential, \( V_C(R) \). According to the reference \[38\], the different models for the Coulomb potential do not have serious effect on the theoretical predictions. So, in our optical model (OM) calculations, we chose the Coulomb potential to be a simple interaction between a point charge and a uniform one with the radius \( R_C \) \[3\],

\[
V_C(R) = Z_p Z_t e^2 \begin{cases} 
\frac{1}{R} & R > R_C \\
\frac{1}{2R_C} \left[ 3 - \left( \frac{R}{R_C} \right)^2 \right] & R < R_C.
\end{cases}
\]

with \( e^2 = 1.44 \text{ MeV.fm} \) and \( R_C = R_p + R_t \), \( R_i = 1.76 Z_i^{1/3} - 0.96 \text{ fm} \), with \( i = p, t \).

B. The choice of the effective interaction and the density distribution

As it can be seen from the equations (2) and (3), the basic inputs into the folding model are the nuclear densities of the colliding nuclei in their ground state and the effective \( NN \) interaction. The density distributions should be normalized as:

\[
\int \rho_i(r_i) dr_i = A_i
\]

where \( A_i \) is the mass number of the projectile or the target nucleus. In this paper, the nuclear densities of two colliding nuclei are approximated by the two-parameter Fermi distribution: \( \rho(r) = \rho_0 [1 + \exp((r - c)/a)]^{-1} \) with parameters taken from the table 1 of the reference \[39\].

Given correct nuclear densities as inputs for the folding calculations, it is still necessary to have an appropriate \( NN \) interaction for a reasonable prediction of the nucleus-nucleus potential. The bare nucleon-nucleon interaction, obtained from analysis of \( NN \) scattering measurements, is too strong to be used directly in the folding model, so, it is common to use an effective in-medium interaction \[1,2\]. To evaluate an in-medium \( NN \) interaction starting from a realistic free \( NN \) interaction, still remains a challenge for the nuclear many-body theory. Therefore, most of the microscopic nuclear reaction calculations so far, still use different kinds of effective in-medium \( NN \) interaction \[4\]. One of the most popular choice for the \( NN \) interactions, were based on the \( M3Y \) interactions and its density dependent versions \[7,13\]. These interactions are designed to reproduce the \( G \)-matrix elements of the \textit{Reid} \[40\] and the \textit{Paris} \[41\] \( NN \) interactions in an oscillator basis \[1,26,28\]. We refer to
these as the $M3Y$-Reid and the $M3Y$-Paris interactions, respectively. The explicit forms for the direct part of interactions are [1, 2]:

$$M3Y - Reid : \, v_D(r) = \left[ 7999 \frac{e^{-4r}}{4r} - 2134 \frac{e^{-2.5r}}{2.5r} \right] \text{MeV}, \quad (13)$$

$$M3Y - Paris : \, v_D(r) = \left[ 11062 \frac{e^{-4r}}{4r} - 2538 \frac{e^{-2.5r}}{2.5r} \right] \text{MeV} \quad (14)$$

whereas the exchange parts of interactions in the finite-range-exchange (FRE) form ($M3Y/FRE$) are written as [1–4]:

$$M3Y - Reid : \, v_{EX}(r) = \left[ 4631 \frac{e^{-4r}}{4r} - 1787 \frac{e^{-2.5r}}{2.5r} - 7.847 \frac{e^{-0.7072r}}{0.7072r} \right] \text{MeV}, \quad (15)$$

$$M3Y - Paris : \, v_{EX}(r) = \left[ -1524 \frac{e^{-4r}}{4r} - 518.8 \frac{e^{-2.5r}}{2.5r} - 7.847 \frac{e^{-0.7072r}}{0.7072r} \right] \text{MeV} \quad (16)$$

However, in many other calculations, the zero-range pseudo-potential ($M3Y/PP$) is used to represent the knock-on exchange [1, 2]. But in this work we focus on the finite range interactions i.e. equations (13) and (15).

The older potentials based upon the density-independent $M3Y$ interactions could reasonably reproduce the data of HI scattering at the forward angle, or low energies [1, 2]. Also, the ground-state energy of nuclear matter (in a standard Hartree – Fock calculation) using the $M3Y$ interactions is calculated in the reference [7]. One can realize that, the density-independent M3Y interactions do not fulfill the saturation condition for cold nuclear matter, i.e. leading to collapse. To ensure the prediction of the nuclear matter saturation, an appropriate density-dependent factor is introduced into the original $M3Y$ interaction. It is usually taken as an independent factor that multiplied to the original radial $M3Y$ interaction, i.e. $v_{D(EX)}(r, \rho) = F(\rho)v_{D(EX)}(r)$. As it is stated in the references [1, 2], there is no theoretical justification for this factorization, but it leads to improve the description of nuclear matter properties and the HI scattering data. Various forms for $F(\rho)$ were proposed. In the $DDM3Y$ and $CDM3Yn$ ($n = 1 – 6$), the following form is assumed for the density dependent of the potential:

$$F(\rho) = C[1 + \alpha \exp(-\beta \rho) - \gamma \rho] \quad (17)$$

In $BDM3Yn$ ($n = 0 – 3$) interactions, a power-law dependent on $\rho$ is supposed:

$$F(\rho) = C(1 - \alpha \rho^\beta) \quad (18)$$
The parameters $C$, $\alpha$, $\beta$ and $\gamma$ are adjusted to reproduce the saturation of cold symmetric nuclear matter at $\rho_0 = 0.17 \text{ fm}^{-3}$ and a binding energy per nucleon of about 16 $MeV$. The values of these parameters for $CDM3Yn$ and $DDM3Y1$ and $BDM3Yn$ interactions are given in the references [1, 2, 7, 38, 42]. As we pointed out before for comparison we focus on the finite range $DDM3Y1$ interaction [4].

In the course of these application to the $NN$ scattering data, it is necessary to introduce an additional energy dependent factor over which provided by localization of the exchange potential:

$$
\nu_{D(EX)}^M(r, \rho, E) = \nu_{D(EX)}^M(r)F(\rho)g(E)
$$

(19)

where $g(E) = [1 - k(E/A)]$ with $k = 0.002 \text{ MeV}^{-1}$ or $k = 0.003 \text{ MeV}^{-1}$ for the Reid interaction or the Paris interaction [3], respectively. However none of the above potentials come from a Hamiltonian based many-body microscopic calculations.

In the present work, the $LOCV$ density dependent averaged effective two-body interaction ($AEI$) is generated though the $LOCV$ method with the bare nucleon-nucleon phenomenological Reid68 potential, and inserted as an input to the folding model calculations. In our previous work [20], we obtained the direct and the exchange parts of the density dependent nucleon-nucleon $AEI$ as follows (see the appendix for the definition of $a$ and $\nu$):

$$
\tilde{V}_{eff}^D(r, \rho, \rho) = \sum_{\alpha, i, j, k}(2T + 1)(2J + 1)\frac{1}{2}\nu_{a^i}^{j,k}(r, \rho)a_{\alpha}^{(i)}(r, \rho)
$$

(20)

$$
\tilde{V}_{eff}^{EX}(r, \rho) = \frac{\sum_{\alpha, i, j, k}(2T + 1)(2J + 1)\frac{1}{2}[(-1)^{L+S+T}]\nu_{a^i}^{j,k}(r, \rho)a_{\alpha}^{(i)}(r, \rho)}{\sum_{\alpha, i}(2T + 1)(2J + 1)\frac{1}{2}[(-1)^{L+S+T}]a_{\alpha}^{(i)}(r, \rho)},
$$

(21)

where $\alpha = JLST$, $J$ is the total orbital angular momentum of two nucleons i.e. $L$ plus $S$, and $T$, is the total iso-spin of two nucleons. Then we have reformulated these interactions as the product of a pure radial and a pure density-dependent parts:

$$
\tilde{V}_{eff}^{D(EX)}(r, \rho) = \tilde{V}^{D(EX)}(r)F^{D(EX)}(\rho).
$$

(22)

Here, we chose $\tilde{V}^{D(EX)}(r)$ and $F^{D(EX)}(\rho)$ to give the best fit to the $LOCV V_{eff}^{D(EX)}(r, \rho)$ and the corresponding equation of state ($LOCV$-EOS) of nuclear matter. The reader should note that, by this statement, we mean that the fitted potentials should again reproduce the SNM saturation properties given in the table 1.
There are many different functions which can fit $F^{D(\text{EX})}(\rho)$ well enough. A detailed role of description of density-dependent factor ($F$) can be found in our previous work, the reference [20], where we stated that the $\text{LOCVAEI}$ includes a radial part and a density-dependent part and we show that, the radial part form of the $\text{LOCVAEI}$ is fixed in any density (exactly like the M3Y type interactions) and the $\text{EOS}$ of $SNM$ without taking into account the density-dependent factor did not fulfill the saturation condition and the system was collapsed (see the figure 7 of the reference [20]). But one should notice that our density-dependent factor is not an external factor and it comes from the $\text{LOCV}$ calculations. So, we just parameterized it in a suitable form (i.e. see below, the equation (23)) (the exponential dependent form for density). In the reference [21], we compared the direct and exchange parts of the $\text{LOCVAEI}$ with the corresponding results of the M3Y interactions.(see the figures (1) and (4) of the reference [20])

So as we stated above, similar to our previous work [20], in order to reproduce the $\text{LOCV}$-$\text{EOS}$ of nuclear matter properly, we use the power-law-dependent on $\rho$: $F^{D(\text{EX})}(\rho) = C^{D(\text{EX})}(1 - \alpha^{D(\text{EX})} \rho^{\beta^{D(\text{EX})}})$. In this paper, we use the exponential dependent form for $\rho$ (similar to the $\text{DDM3Y1}$ interaction):

$$F^{D(\text{EX})}(\rho) = C^{D(\text{EX})}(1 + \alpha^{D(\text{EX})} \exp(-\beta^{D(\text{EX})} \rho)).$$

(23)

This choice allows us to easily calculate the integration of the double-folding equations in the momentum space [1, 2]. The parameters of equation (23) are given in the table 2.

Similar to the $M3Y$ interactions, in order to apply the $\text{LOCV AEI}$ to the $\text{NN}$ scattering data, we need to add an explicit energy-dependent factor to our $\text{LOCV AEI}$ to obtain the best description of HI scattering by taking into account the variation in the incident energy. We found that this factor can be assumed as the linear dependent to the incident energy per nucleon, which is similar to the $M3Y$ interactions i.e. $g(E) = [1 - k(E/A)]$. So, we can rewrite the $\text{LOCV AEI}$ as:

$$\tilde{V}^{D(\text{EX})}(r, \rho, E) = \tilde{V}^{D(\text{EX})}(r)F^{D(\text{EX})}(\rho)g(E).$$

(24)

Here, as in other $HI$ works, the $k$ is chosen to give the best fit to the $\text{NN}$ scattering data. It is shown that in the case of our $\text{LOCV AEI}$ by choosing $k = 0.003\text{MeV}^{-1}$, the optimized fit will be acquired. However, the calculation is not very sensitive to this parameter if it is chosen in its order.
C. The Computational procedure

At first, we calculate the real part of the folded potential for $^{12}\text{C} - ^{12}\text{C}$ and $^{16}\text{O} - ^{16}\text{O}$ elastic scattering by the double folding formula, i.e. the equations (2) and (3). Then we use the \textit{LOCV AEI} as the effective NN interactions and the two-parameter Fermi distribution for the nuclear densities of the projectile and the target nuclei. Now, in order to compute the scattering differential cross section, we also use the \textit{FRESCO} code developed by Ian Thompson \cite{43} which is developed for the calculation of different types of nucleon-nucleus and nucleus-nucleus scattering cross-sections. This code is capable to use our folded potential directly, to calculate the elastic scattering cross section.

We will discuss our resulting potentials and the elastic scattering cross section for $^{12}\text{C} - ^{12}\text{C}$ and $^{16}\text{O} - ^{16}\text{O}$ systems in the next section. Generally, the goodness of our resulting cross section is quantified via the $\chi^2$ expression \cite{11, 2},

$$\chi^2 = \frac{1}{N_{\sigma}} \sum_{i=1}^{N_{\sigma}} \frac{(\sigma_{th} - \sigma_{ex})^2}{(\Delta \sigma_{ex})^2}$$ \hspace{1cm} (25)

where $\sigma_{th}$ and $\sigma_{ex}$ are the theoretical and the experimental cross sections and $\Delta \sigma_{ex}$ are defined as the uncertainties in the experimental cross sections, respectively. $N_{\sigma}$ is the total number of angles at which measurements are made.

III. RESULTS AND DISCUSSIONS

As it was pointed out in the previous section, in order to calculate the direct and the exchange components of the real part of the HI optical potential, we use the direct and the exchange parts of the \textit{LOCV AEI} as the effective NN potential in the double folding formula (the equations (2) and (3)). Since the wave number of relative motion $k_{rel}(R)$, the equation (4), depends on the total HI potential, we are faced with a self-consistency problem in obtaining the exchange part of the HI potential at each radial point. So, we apply the iterative method at each point and use $U_D(E, R)$ as the starting potential to enter $\hat{j}_0(k(R)s/M)$ in the exchange integral, the equation (9), i.e. as it is performed when one considers the \textit{M3Y} interactions in the folding formula \cite{4}.

Unfortunately at small internuclear distances ($R \leq 1\text{fm}$), the iterative method for calculating the exchange potential based on the \textit{LOCV AEI}, does not converge reasonably. Of
course, with increasing the incident energy, this problem will be solved. Due to this low convergence speed of iterative method in case of the insertion of the \textit{LOCV AEI} in the folding formula, we need much more number of iterations with respect to the \textit{M3Y} interactions, in obtaining the exact self-consistent results for $U_{EX}(E,R)$, especially at small internuclear distances. According to the reference [4], in the case of the \textit{M3Y} interactions, the number of iterations required is around 20 at smallest radii and ranges from 3 to 5 at the surface region, while, in case of the \textit{LOCV AEI}, it is around 150 to 200 at smallest radii and around 2 or 3 at the surface region. For this reason, too much CPU computer time is needed to calculate the exchange part of the HI potential in case of the \textit{LOCV AEI}. For example for the $^{12}C-^{12}C$ elastic scattering at the $E_{lab}=300\,\text{MeV}$, it took about 50 hours computer CPU time by using the high performance computing (HPC) machine of the university of Tehran. Because of the different radial shapes of the \textit{LOCV AEI} with respect to the \textit{M3Y} interactions at the small distances, this problem is expected. Conversely to the \textit{M3Y} potentials, due to short range correlations coming from the channel-dependent correlation functions, at very small distances, the direct and the exchange components of the \textit{LOCV AEI} go to zero (see the figures 1 to 4 of the reference [20]) and this behavior makes the iterative method not to converge at these distances as quicker as for the \textit{M3Y} interactions. While, since the \textit{M3Y} interactions are constructed from the selected channels of, for example the \textit{Reid68} potential, i.e. the singlet and the triplet even and odd components, one does not faced with this problem.

So in the figures 1 and 2, we plot the calculated direct, exchange and also the total components of the folded potential by using the \textit{LOCV AEI} for $^{12}C-^{12}C$ and $^{16}O-^{16}O$ systems at several incident energies i.e. 112, 126.7, 240, 300 and 360 \text{MeV} for $^{12}C-^{12}C$ and 124, 145, 250, 350 and 480 \text{MeV} in the case of $^{16}O-^{16}O$ (note that we extrapolate the folded potential at the small distances ($R < 1\,\text{fm}$) for some points that the iterative method is not converge rapidly for calculation of the exchange potential based on \textit{LOCV – AEI}). Comparing the exchange parts with the direct parts at each incident energy, one can observe that the most of energy dependence of the HI potential is arising from the exchange part, as one should expects. We also notice that at small internuclear distances, which corresponds to large overlap densities ($\rho > \rho_0$), the exchange potential is more deep than the direct potential, especially at lower energies, and this shows that the density-dependent contribution of HI potential predominately comes from the exchange term. On
the other hand, in the surface region, which corresponds to the small overlap densities, all the calculated direct and exchange potentials are close in the strength and the slope. The figures 1 and 2 also show that with increasing the incident energy of projectile, the depth of the HI potential at the origin is decreased systematically. Similar results already reported in calculating folded potential using the M3Y interactions, for example see the references [4, 5].

We compare our calculated folded potential, using the LOCVAEI with the corresponding results of DDM3Y1 [4] for the cases of the $^{12}C - ^{12}C$ at $E_{lab} = 300$ MeV and the $^{12}O - ^{12}O$ at $E_{lab} = 350$ MeV in the figures 3 and 4, respectively. It can be observed that the folded potentials by using the LOCVAEI are more deep than the DDM3Y1 ones. For the other energies, the similar results are obtained.

The results of our folding analysis for the $^{12}C - ^{12}C$ elastic scattering, at incident energies ranging from 112 to 360 MeV with FRESCO code are presented in the figure 5 while the table 3 shows the WS parameters of the imaginary part of HI potential for the same system and at the same energies as well as $\sigma_R$ and $\chi^2$ (with respect to the experimental data, see the next paragraph). In this paper we take the imaginary part of HI potential as the conventional WS form and adjust its parameters to obtain the best description of the experimental scattering data in the whole angular range at each incident energy. The parameters in the table 3 are close to those found in earlier analysis for DDM3Y1 – Reid (see the table 2 of the reference [4]). The table 3 also shows that the best fit to the scattering data, can be found by using the values of $N_R$ which are slightly deviated from the unity. This result indicates that the high-order effects are negligible in our calculations.

The different panels of figure 5 (a to e) show the calculated cross section of $^{12}C - ^{12}C$ elastic scattering at several incident energies, i.e. 112, 126.7, 240, 300 and 360 MeV, by using the LOCVAEI folded potential in the FRESCO code. The scattering experimental data [44–52] and the resulting cross sections of the DDM3Y1 [4] are also presented. It is observed that a quite good description of data scattering can be obtained by using the LOCVAEI and adjusting the imaginary potential parameters and renormalization coefficient. However, in comparison to the DDM3Y1 (Reid) results [4], our results may not be too satisfactory, especially at forward angles, but one should notice that DDM3Y1 potential was constructed from the selected channels of the Reid68 potential and its density dependent factor was added to it later, to provide a reasonable description of HI scattering data and the equation
of state (EOS) of nuclear matter, while the \textit{LOCV AEI} are constructed based on the many-body calculations without any free parameters in the \textit{LOCV} calculations and its density dependent part comes directly from the \textit{LOCV} formalism (obviously \textit{LOCV} formalism has its owns EOS, i.e. \textit{LOCV-EOS}). It is worth to say that, by increasing the incident energy a better fit to the scattering data is achieved using the \textit{LOCV AEI} at forward angles.

The calculated cross sections using the \textit{LOCV AEI} for $^{16}O - ^{16}O$ elastic scattering at incident energies ranging from 124 to 480 MeV are plotted in the different panels (a to e) of the figure 6. The scattering experimental data [44–52] show a clear refractive pattern at large angles and a diffractive pattern produced by an interference between nearside and farsight components of the scattering amplitude at the small angles. The refractive pattern can be clearly distinguished from the diffractive structure, i.e. it is shifting substantially towards the small angles with increasing the incident energy [5].

One can realize that our calculated cross sections can predict reasonably the behavior of scattering data on large ranges of scattering angles [44–52]. Similar to the results obtained above for $^{12}C - ^{12}C$ system, there exist considerable differences between our results with respect to the experimental data and those coming from \textit{DDM3Y1}. Again, the similar discussion can be made for these results as the one we made above for $^{12}C$. In this case, it can also be observed that the agreement of our calculations to the scattering data are getting better as the energies of projectile are increased. To improve the agreement of the calculated cross sections using the \textit{DDM3Y1-Reid} and \textit{DDM3Y1-Paris} with data in the large-angle region, in the references [4, 5] a surface (\textit{WSD}) term was included into the imaginary part of potential. We hope, in our future works, we could investigate the inclusion of the \textit{WSD} term for improving our results.

The table 4 shows the parameters of our \textit{WS} imaginary potential and renormalization coefficient for $^{16}O - ^{16}O$ system at different incident energies as above. Again, we can see the values of $N_{R}$ are close to the unity and our \textit{WS} parameters are in agreement to the \textit{WS} parameters of \textit{DDM3Y1} analysis [4].

IV. SUMMARY

In conclusion, we analyzed the experimental data of $^{12}C - ^{12}C$ and $^{16}O - ^{16}O$ elastic scattering at different incident energies, within the standard optical model (\textit{OM}), using
the density-dependent \textit{LOCV AEI}. The direct and the exchange parts of \textit{LOCV AEI} were generated based on the \textit{LOCV} method for the symmetric nuclear matter, using the \textit{Reid} 68 interaction as the input phenomenological potential. In order to use our interaction into the folding model, we separated the radial and the density-dependent parts of the \textit{LOCV AEI}. Our calculated cross sections for $^{12}$\textit{C} and $^{16}$\textit{O} systems, indicate that a quite reasonable description of data scattering can be obtained by using the \textit{LOCV AEI} and adjusting the imaginary potential parameters and the renormalization coefficient. Our calculations favor a rather weak imaginary potential and a small deviation of the renormalization factor from the unity. Comparing our calculations with corresponding results of the \textit{DDM3Y1}, show some considerable differences. But one should notice that the \textit{M3Y} interactions are semi-phenomenological potentials and they are constructed from the selected channels of the \textit{Reid} potential, i.e. the singlet and the triplet even and odd components and the parameters of its density dependent part are adjusted to gain a reasonable description of HI scattering data and the EOS of nuclear matter. So, it is natural to fit the scattering data better than ours. While the \textit{LOCV AEI} are based on the many-body calculation with the phenomenological \textit{NN} potential without any free parameters, i.e. there are no free parameters in the \textit{LOCV} formalism besides the \textit{NN} potential and its density dependent part comes directly from the self consistent \textit{LOCV} calculations. So it is meaningful to apply the \textit{LOCV AEI} interaction to the heavy-ion scattering as the first attempt, but we hope the improvement of the present model could be committed in the near future.

The spite of the slow convergence speed of iterative procedure in using the \textit{LOCV AEI} in calculating of the exchange potential, especially at small internuclear distances which increases the computing time, since the \textit{LOCV AEI} are based on the many-body calculations, they are more trustable for the \textit{NN} collision calculations. So, with respect to the above arguments, because the \textit{LOCV AEI} provides a reasonable description of the normal nuclear matter \cite{20} as well as the HI elastic scattering data simultaneously, we can claim the \textit{LOCV AEI} is a good candidate to approximate the \textit{NN} interaction for the nuclear matter and finite nuclei.

Finally we should make this comment that the insertion of other phenomenological nucleon-nucleon potential such as the \textit{Av18} potential, should not have any dramatic change on our present results, but it is worth to be investigated.
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TABLE I: The nuclear matter saturation parameters (for Reid and $\triangle$-Reid potentials) extracted from reference \[22\] ($E_3$ denotes the inclusion of the three-body cluster energy, see the appendix A).

|                      | With Reid |                      | With $\triangle$-Reid |                      | Empirical |
|----------------------|-----------|----------------------|-----------------------|----------------------|-----------|
|                      | $E_1 + E_2$ | $E_1 + E_2 + E_3$ | $E_1 + E_2$ | $E_1 + E_2 + E_3$ |           |
| Saturation Fermi momentum ($fm^{-1}$) | 1.61 | 1.46 | 1.55 | 1.44 | 1.38 |
| Saturation binding energy ($MeV$) | 22.54 | 21.85 | 16.28 | 15.52 | 15.86 |
| Compressibility ($MeV$) | 340 | 298 | 300 | 277 | (200-300) |
| Convergence parameter | 0.127 | 0.085 | 0.093 | 0.062 | |

TABLE II: The parameters of the density-dependent part of the direct and the exchange components ($F^{D(EX)}(\rho)$) of the LOCV AEI using the Reid68 interaction as the input potentials.

|                      | $C$ | $\alpha$ | $\beta$ |
|----------------------|-----|----------|---------|
| Direct component     | 0.38 | 5.03     | 3.22    |
| Exchange component   | 13.57 | -0.9    | 0.12    |
TABLE III: The WS parameters of the imaginary part of HI potential used in our folding analysis of the $^{12}$C − $^{12}$C elastic scattering at $E_{lab} = 112, 126.7, 240, 300, 360$ MeV.

| $E_{lab}(MeV)$ | $N_R$ | $W_V(MeV)$ | $R_V(fm)$ | $a_V(fm)$ | $\sigma_R(mb)$ | $\chi^2$ |
|----------------|-------|------------|-----------|-----------|---------------|---------|
| 112            | 0.9383| 17.4       | 5.403     | 0.70      | 1526.79       | 36.52   |
| 126.7          | 0.9230| 19.10      | 5.128     | 0.79      | 1563.51       | 41.86   |
| 240            | 1.0207| 28.90      | 5.266     | 0.69      | 1551.95       | 39.34   |
| 300            | 0.9731| 33.82      | 4.991     | 0.72      | 1497.85       | 18.33   |
| 360            | 0.9684| 34.5       | 4.808     | 0.70      | 1374.73       | 9.81    |
TABLE IV: The same as the table 5 but for the $^{16}O - ^{16}O$ elastic scattering at $E_{lab} = 124, 145, 250, 350, 480$ MeV.

| $E_{lab}$(MeV) | $N_R$ | $W_V$(MeV) | $R_V$(fm) | $a_V$(fm) | $\sigma_R$(mb) | $\chi^2$ |
|----------------|-------|------------|----------|-----------|----------------|--------|
| 124            | 0.9455 | 15.3       | 6.30     | 0.93      | 2201.99        | 34.34  |
| 145            | 1.007  | 16.4       | 6.199    | 0.95      | 2226.17        | 37.07  |
| 250            | 1.011  | 31.6       | 5.695    | 0.86      | 2091.89        | 39.71  |
| 350            | 0.9890 | 36.76      | 5.544    | 0.77      | 1876.58        | 21.19  |
| 480            | 0.9703 | 42.65      | 5.241    | 0.79      | 1778.03        | 42.37  |
FIG. 1: The calculated direct and the exchange components and the total folded potential, by using LOCV AEI for the $^{12}C-^{12}C$ system at the several incident energies, i.e. $E_{lab} = 112$ (the full curve), 126.7 (the short-dash curve), 240 (the long-dash curve), 300 (the long-short-dash curve), 360 (long-double-short-dash curve) MeV.
FIG. 2: As the figure 1 but for the $^{16}O -^{16}O$ system and $E_{lab}=124$ (the full curve), 145 (the short-dash curve), 250 (the long-dash curve), 350 (the long-short-dash curve), 480 (the long-double-short-dash curve) $MeV$. 
The comparison of the calculated folded potentials using the \textit{LOCV AEI} (the full curve) and the DDM3Y1 \cite{4} (the short-dash curve) potential for the $^{12}\text{C} - ^{12}\text{C}$ scattering at $E_{\text{lab}} = 300$ $MeV$.

\textbf{Appendix: A brief introduction to the \textit{LOCV} formalism with the \textit{Reid68} interaction}

In the LOCV method, we use an ideal Fermi gas type wave function for the single particle states and the variational techniques, to find the wave function of interacting system \cite{22,53}.
FIG. 4: As the figure 3 but for the $^{16}\text{O}-^{16}\text{O}$ scattering at $E_{\text{lab}} = 350 \text{ MeV}$.

\[ \psi = \mathcal{F}\Phi \]  \hspace{1cm} (A.1)

where ($S$ is a symmetrizing operator)

\[ \mathcal{F} = S \prod_{i>j} F(ij). \]  \hspace{1cm} (A.2)
The correlation functions $F(ij)$ are operators and they are written as:

$$F(ij) = \sum_{\alpha,k} f^{(k)}_{\alpha} (ij) O^{(k)}_{\alpha}(ij).$$ \hspace{1cm} (A.3)

In above equation $\alpha = \{S, L, J, T\}$, $k = 1, 3$ and

$$O^{k=1,4}_{\alpha} = 1, \left(\frac{2}{3} + \frac{1}{6} S^{I}_{12}\right), \left(\frac{1}{3} - \frac{1}{6} S^{I}_{12}\right).$$ \hspace{1cm} (A.4)

In the case of the Reid68 potential, the spin-singlet channels with the orbital angular momentum $L \neq 0$ and the spin-triplet channels with $L \neq J \pm 1, k$ is superfluous and set only
to unity, while for $L = J \pm 1$ it takes the values of 2 and 3. All of the channel correlation functions $f^{(1)}_\alpha$, $f^{(2)}_\alpha$ and $f^{(3)}_\alpha$ heal to the modified Pauli function $f_P(r)$,

$$f_P(r) = [1 - l(k_F r)^2]^{-\frac{1}{2}} \tag{A.5}$$

with

$$l(x) = \frac{3}{2x} J_1(x) \tag{A.6}$$

where $J_1(x)$ are the familiar spherical Bessel functions and the Fermi momenta $k_F$ is fixed by the nuclear matter density i.e., $k_F = (\frac{3\pi^2}{2}\rho)^{\frac{1}{3}}$. 

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The nuclear matter energy per nucleon is \cite{22, 54–57},

\[ E_{\text{in}} = T_F + E_{MB}[F]. \]  \hspace{1cm} (A.7)

\( T_F \) is simply the Fermi gas kinetic energy and it is written as

\[ T_F = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m}. \]  \hspace{1cm} (A.8)

The many-body energy term \( E_{MB}[F] \) is calculated by constructing a cluster expansion for
the expectation value of our Hamiltonian,

\[ H = \sum_i \frac{p_i^2}{2m} + \sum_{i>j} V_{ij} \]  

(A.9)

where \( V_{ij} \) is the bare N-N interaction. Then, we keep only the first two terms in a cluster expansion of the energy functional:

\[ E[F] = \frac{1}{A} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = T_F + E_{MB} = T_F + E_2 + E_3 + \ldots \]  

(A.10)
FIG. 5: The calculated cross sections of the $^{12}$C – $^{12}$C elastic scattering at $E_{\text{lab}} = 112, 126.7, 240, 300, 360$ MeV by using the LOCV AEI (the full curve) using the FRESO code. The experimental scattering data (the full dotted points) and the resulting cross section of the finite range interaction DDM3Y1 [4] (the dash curve) are also presented. The experimental data are taken from the references [44–47].
The two-body energy term is defined as,

\[ E_2 = (2A)^{-1} \sum_{ij} <ij | \mathcal{V}(12) | ij > \]  \hspace{1cm} (A.11)

where

\[ \mathcal{V}(12) = -\frac{\hbar^2}{2m} [F(12), [\nabla^2_{12}, F(12)]] + F(12)V(12)F(12) \]  \hspace{1cm} (A.12)

and the two-body antisymmetrized matrix element \(<ij | \mathcal{V} | ij >\) are taken with respect to the single-particle functions composing \(\Phi\) i.e. the plane-waves. In the \(LOCV\) formalism \(E_{MB}\) is approximated by \(E_2\) and one hopes that the normalization constraint makes the
cluster expansion to converge very rapidly and bring the many-body effect into $E_2$ term. By inserting a complete set of two-particle state twice in the equation (A.11) and performing some algebra, we can rewrite the two-body term as following:

$$E_2 = E_{c}^{NN} + E_{T}^{NN}$$

where ($c$ and $T$ stand for the central and tensor parts, respectively)

$$E_i^j = \frac{2}{\pi^4 \rho} \sum_\alpha (2T + 1)(2J + 1) \frac{1}{2} \{1 - (-1)^{L+S+T}\} \int_0^\infty r^2 dr V^{ij}_\alpha(r, \rho)a^{(1)}_\alpha(r)$$

(A.14)
\( ^{16}\text{O} ^{16}\text{O}, \text{E}_{\text{lab}}=250 \text{ MeV} \)

\[
\frac{d\sigma}{d\Omega_{\text{Mott}}} (c)
\]

and \((i = c \text{ and } T)\)

\[
\mathcal{V}^{c,NN}(r, \rho) = \frac{\hbar^2}{m} \{ f_{a}^{(1)} \}^2 + \frac{m}{\hbar^2} V_{\alpha}^c f_{a}^{(1)}^2 \} \tag{A.15}
\]

\[
\mathcal{V}^{T,NN}(r, \rho) = \left\{ \frac{\hbar^2}{m} \{ f_{a}^{(2)} \}^2 + \frac{m}{\hbar^2} (V_{\alpha}^c + 2V_{\alpha}^T - V_{\alpha}^{LS}) f_{a}^{(2)} \} a_{a}^{(2)}(r)^2 + \frac{\hbar^2}{m} \{ f_{a}^{(3)} \}^2
+ \frac{m}{\hbar^2} (V_{\alpha}^c - 4V_{\alpha}^T - 2V_{\alpha}^{LS}) f_{a}^{(3)} \} a_{a}^{(3)}(r) + \{ r^{-2}(f_{a}^{(2)} - f_{a}^{(3)})^2 + \frac{m}{\hbar^2} V_{\alpha}^{LS} f_{a}^{(2)} f_{a}^{(3)} \} b_{a}^2 \} a_{a}^{(1)}(r)^2 \right\} \tag{A.16}
\]

\[
a_{a}^{(1)}(r, \rho) = I_J(r, \rho) \tag{A.17}
\]

\[
a_{a}^{(2)}(r, \rho) = (2J + 1)^{-1}[(J + 1)I_{J-1}(r, \rho) + J I_{J+1}(r, \rho)] \tag{A.18}
\]

\[
a_{a}^{(3)}(r, \rho) = (2J + 1)^{-1}[J I_{J-1}(r, \rho) + (J + 1) I_{J+1}(r, \rho)] \tag{A.19}
\]
The potential functions $V_{\alpha}, V_{T}^{\alpha}, \ldots$ etc., are given in the references [26, 27]. The calculation of $E_{3}$ is discussed in the reference [58] and the references therein.

The normalization constraint as well as the coupled and uncoupled differential equations for the NN-channels, coming from the Euler-Lagrange equations, are similar to those were described in the references [22, 54–57].

The following important points consider in the LOCV formalism: (i) Beside the inter-

\[ b_{\alpha}^{2}(r, \rho) = 2J(J + 1)(2J + 1)^{-1}[I_{J-1}(r, \rho) - I_{J+1}(r, \rho)] \quad (A.20) \]

\[ I_{J}(r, \rho) = (2\pi^{6} \rho^{2})^{-1} \int_{|k_{1}|,|k_{2}| \leq k_{F}} d\mathbf{k}_{1}d\mathbf{k}_{2}J_{J}^{2}(|\mathbf{k}_{1} - \mathbf{k}_{2}|r). \quad (A.21) \]
FIG. 6: As the figure 5 but for the $^{16}$O$-^{16}$O scattering at $E_{lab} = 124, 145, 250, 350, 480$ MeV. The experimental scattering data are taken from the references [48–52].

particle potentials, no free parameter is used in the LOCV method, i.e. it is fully self-consistent. (ii) To keep the higher cluster terms as small as possible, it considers the constraint in the form of a normalization condition [22, 54–57]. This was tested by calculating the three-body cluster terms with both the state-averaged and the state-dependent correlation functions [58]. (iii) In order to perform an exact functional minimization of the two-body
cluster energy with respect to the short-range behavior of correlation functions, it assumes
a particular form for the long-range part of correlation functions. (iv) Rather than simply
parameterizing the short-range behavior of the correlation functions, it performs an exact
functional minimization \[59\]. So, in this respect it also saves an enormous amount of the
computational time. For example, a nuclear matter LOCV calculation with the Nijmegen
group potentials at the given density takes a few minutes CPU time on a 1.8 GHz personal
computer.

Recently \[60\], it was shown that the neutron (nuclear) matter LOCV calculations with
the various two-body interactions, e.g. the Bethe homework potential and the Argonne Av'\[s
interaction \[59\], reasonably agree with those of FHNC and Auxiliary Field Diffusion Monte
Carlo (AFDMC) \[61–66\] methods. Moreover, it was realized that the different many-body
methods such as the LOCV and the fermions hyernetted chain FHNC approaches give
results close to each other when the normalization constraint is imposed in its correct form.
Therefore, the normalization constraint plays an important role in the minimizing of the
many-body terms.

So in the LOCV framework by using e.g. the Reid68 interaction, we solve the set of
Euler-Lagrange differential equations to find the correlation functions. Then we can find the
SNM-EOS by calculating the expectation value of the Hamiltonian. The minimization of
the LOCV – EOS gives some values for the binding and saturation density of the SNM,
which are demonstrated in the tables 1 and A.1. Obviously, as it is well known one should
not expect to get the exact SNM empirical values. But in the M3Y type interactions, the
situation is different, in order to ensure the empirical saturation density and the binding
energy as well as incompressibility of the symmetric nuclear matter, an external density
dependent factor is multiplied to the original radial M3Y interactions and the constants of
this density dependent function are obtained such that one could reproduce these empirical
saturation properties for the SNM. So the case of the LOCV method is different from
the M3Y type interactions. The separation of radial and density dependent parts of the
LOCV – AEI is done only to make it possible to use the LOCV – AEI in the double
folding procedure.

In the table A.1 we compare the LOCV results on the saturation properties of SNM by
using different interactions with other many body techniques (The BB, BHF, CBF and
BHF – ESC stand for the Brueckner, Bethe, Brueckner, Hartree, Fock, correlated-basis-
function and $BHF$ using extended-soft-core interactions, see the references [22] and [14], and the references therein, for detail, respectively). So the $EOS$ of SNM is directly calculated by the $LOCV$ formalism and there is no other constraint for obtaining the saturation properties of $SNM$.

Finally we should mention that the effect of $TBF$ have been fully discussed especially in the references [22, 54, 56].

TABLE A.1: The saturation energy and the density of nuclear matter as well as its incompressibility for different potentials and many-body methods. See reference [22] for detail.

| Potential | Method | Author | $\rho_0 (fm^{-3})$ | $E(\rho_0)(MeV)$ | $K(MeV)$ |
|-----------|--------|--------|---------------------|------------------|----------|
| $AV_{18}$ | LOCV   | BM     | 0.310               | -18.46           | 302      |
| $AV_{14}$ | LOCV   | BM     | 0.290               | -15.99           | 248      |
|          | FHNC   | WFF    | 0.319               | -15.60           | 205      |
|          | BB     | DW     | 0.280               | -17.80           | 247      |
|          | BHF    | BBB    | 0.256               | -18.26           | -        |
| $UV_{14}$ | LOCV   | BM     | 0.366               | -21.20           | 311      |
|          | FHNC   | CP     | 0.349               | -20.00           | -        |
|          | FHNC   | WFF    | 0.326               | -17.10           | 243      |
| $UV_{14} + TBF$ | LOCV | BM     | 0.170               | -17.33           | 276      |
|          | FHNC   | WFF    | 0.157               | -16.60           | 261      |
|          | CBF    | FFP    | 0.163               | -18.30           | 269      |
| $\triangle-\text{Reid}$ | LOCV | MI     | 0.258               | -16.28           | 300      |
| $\text{Reid}$ | LOCV | OBI    | 0.294               | -22.83           | 340      |
|          | LOCV   | MO     | 0.230               | -14.58           | 238      |
| $\text{ESC}$ | BHF   | FSY    | $\sim 0.14$         | $\sim -12.00$   | $\sim 84$ |
| $\text{ESC-TBA}$ | BHF  | FSY    | $\sim 0.16$         | $\sim -14.00$   | $\sim 173$ |
| $\text{ESC-TBA-Strong}$ | BHF  | FSY    | $\sim 0.19$         | $\sim -16.00$   | $\sim 260$ |
| Empirical | 0.170  | -15.86 | (200-300)           |                  |          |