Abstract

In four space-time dimensions, there are good theoretical reasons for believing that General Relativity is the correct geometrical theory of gravity, at least at the classical level. If one admits the possibility of extra space-time dimensions, what would we expect classical gravity to be like? It is often stated that the most natural generalisation is Lovelock’s theory, which shares many physical properties with GR. But there are also key differences and problems. A potentially serious problem is the breakdown of determinism, which can occur when the matrix of coefficients of second time derivatives of the metric degenerates. This can be avoided by imposing inequalities on the curvature. Here it is argued that such inequalities occur naturally if the Lovelock action is obtained from Weyl’s formulae for the volume and surface area of a tube. Part of the purpose of this article is to give a treatment of the Weyl tube formula in terminology familiar to relativists and to give an appropriate (straightforward) generalisation to a tube embedded in Minkowski space.

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I. INTRODUCTION

A. Generalising GR to higher dimensions

The Einstein equation can be cast as a hyperbolic system of equations. Therefore deterministic evolution of the metric from an initial geometric data (satisfying the initial value constraints) is guaranteed provided that spacetime is globally hyperbolic. This is an important result which establishes GR as a legitimate classical theory— one might say that determinism is the defining characteristic of a classical theory of physics.

In dimensions greater than four, there are other symmetric tensors $H_{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma})$, known as Lovelock tensors\[1\], that one can add which satisfy an identity $\nabla_\mu H^\mu_\nu = 0$ derived from the Bianchi identity. Therefore, if we add these tensors to the Einstein equation, we still expect to have just the right number of independent equations to determine the metric up to diffeomorphisms. Furthermore the Lovelock tensors are second order in derivatives, so one expects to have the same initial data (which we may assume to be the spatial metric and its first time derivative). The Lovelock tensors are polynomials in the curvature of the form:

$$\left(H^{(n)}\right)_{\mu\nu} := -\frac{1}{2n+1} \delta^{\mu\rho_1 \cdots \rho_{2n}}_{\sigma_1 \cdots \sigma_{2n}} R^\sigma_1 \rho_1 \cdots R^\sigma_{2n-1} \rho_{2n-1} \rho_{2n}.$$

(1)

The equation of Lovelock gravity in $D = 2m + 1$ or $2m + 2$ dimensions will be

$$\Lambda g_{\mu\nu} + G_{\mu\nu} + \alpha_2 (H^{(2)})_{\mu\nu} + \cdots + \alpha_m (H^{(m)})_{\mu\nu} = \kappa T_{\mu\nu}$$

(2)

with $(H^{(m)})_{\mu\nu}$ being the highest order term which does not vanish identically.

Lovelock gravity has been studied in various contexts: compactified\[2\] and brane-world\[3\] cosmological models; Kaluza-Klein theory\[4\][5] (a more recent work is \[6\]); black holes\[7\][8]; Chern-Simons theories of (super)-gravity\[9\][10][11], to name but some. Mathematical properties of the Lovelock terms have been studied in Refs. \[12\] and \[13\].

B. Determinism and curvature inequalities

However, there is a problem which afflicts Lovelock’s theory. This problem was encountered some time ago by Teitelboim and Zanelli\[14\] working in the Hamiltonian formalism and by Choquet-Bruhat who considered the Cauchy problem\[15, 16\]. Here we shall briefly
review the problem, following Ref. [16]. Let us introduce a time-slicing, writing the metric in ADM form and setting the shift to zero. Let $h_{ab}$ be the intrinsic metric on the constant-time hypersurface. The curvature component containing the second time derivatives is $R^0_{a0b} \simeq \frac{1}{2\alpha^2} g_{ab,00}$ where $\alpha$ is the lapse function and $\simeq$ means equality modulo terms of lower order in time derivatives. Looking at the field equations (in empty space), one finds that $H^0_0 = 0$ and $H^0_a = 0$ contain only first time derivatives and therefore will be initial-value constraints; $H^a_b = 0$ contains terms $g_{ab,00}$ and therefore describes the evolution of the system.

The relevant part of the Lovelock tensors is

$$(H^{(n)})_{ab} \simeq \frac{1}{\alpha^2} \left( \Xi^{(n)} \right)_{ab}^{cd} g_{cd,00}, \quad \left( \Xi^{(n)} \right)_{ab}^{cd} := -n \frac{1}{2^n} g^{ef} g^{ac} \delta_{abcd}^{b_1 \ldots b_{2d-2n} \ldots b_{2n-1} b_{2n}}.$$

It is helpful to use the trace of the equations to cast them in the form $R_{ab} +$ Lovelock corrections. Then we get

$$\frac{1}{2\alpha^2} \left( \delta^c_a \delta^d_b - Y_{ab}^{cd} \right) g_{cd,00} = f_{ab}(g, \dot{g}, g', g'', \ldots) \simeq 0,$$ (3)

$$Y_{ab}^{cd} := \sum_{n=2}^{[d/2]} 2\alpha_n \left( \left( \Xi^{(n)} \right)_{ab}^{cd} - \frac{1}{D-2} g_{ab} g^{ef} \left( \Xi^{(n)} \right)_{ef}^{cd} \right).$$ (4)

(Note that $\alpha$ is not a dynamical variable, which corresponds to the fact that locally one can always choose Gaussian normal coordinates). Above, the term $f_{ab}$ denotes a matrix which depends only on the initial data $g_{ab}$, $g_{ab,0}$ and spatial derivatives, but not on $g_{ab,00}$.

It is useful to combine the symmetrised pairs of indices into a single index $I := ab$, $J := cd$, so that $\delta^I_J + Y^I_J$ is a $d(d + 1)/2$-by-$d(d + 1)/2$ matrix. The system is solvable for $g_{cd,00}$ iff

$$\det(\delta^I_J + Y^I_J) \neq 0.$$

Unlike for Einstein’s theory, in Lovelock gravity those coefficients are functions, and so it may be that the determinant is non-zero in some regions but vanishing in other regions. At such points where the determinant vanishes, there is an ambiguity of the continuation of space-time into the future.\(^1\)

\(^1\) In considering the Cauchy problem in this way, one treats the intrinsic metric and its time derivative (i.e. extrinsic curvature) as the initial data. In the Hamiltonian approach one has $g_{ab}$ and the canonical momenta $\Pi_{ab}$. Hamilton’s equation is of the form $\Pi_{ab} = \ldots$. However, non-determinism enters when one faces the fact that one cannot always invert $g_{ab}$, appearing on the r.h.s., to express it as a function of $\Pi_{ab}$. Of course $\det(\delta^I_J + Y^I_J) = 0 \iff \det \left( \frac{\partial \Pi_I}{\partial K_J} \right) = 0$, so the ill-posed Cauchy problem and the breakdown
If the matrix $\mathcal{Y}_I^J$ is small, then the determinant is positive definite and deterministic evolution is guaranteed. Roughly speaking, this will be true if the curvature components are small compared with lengthscales$^{-2}$ constructed from the coupling constants. Is there some interpretation of the theory in which such inequalities on the curvature arise naturally?

II. WELLY’S TUBE FORMULA

In what follows, we develop some new ideas concerning the relation between Weyl’s classic formulae for the volume and area of a tube on the one hand, and Lovelock gravity and the problem of determinism on the other.

A. Euclidean tube formula

Let $M$ be a $D$-dimensional submanifold of $\mathbb{R}^N$. It’s $l$-tube is defined to be the set of all points in $\mathbb{R}^n$ with distance $\leq l$ from $M$ along a geodesic which intersects $M$ normally (if $M$ has no boundary, this is the same as the set of all points of shortest distance $\leq l$ from $M$). If $l$ is small enough compared to the curvature radii of $M$ at every point, then the tube is diffeomorphic to $M \times B_{N-D}$, where $B_{N-D}$ is the unit ball of dimension $N-D$. For small enough $l$, a formula due to Weyl says that the volume of the $l$-tube is:

$$V = \text{Vol}(B_{N-D}) \sum_{n=0}^{[D/2]} \frac{(N-D)!!}{(N-D+2n)!!(2n)!!} l^{N-D+2n} \int_M L^{(n)},$$

(5)

where

$$L^{(n)} := \frac{1}{2^n} \delta^{\rho_1 \cdots \rho_{2n}}_{\sigma_1 \cdots \sigma_{2n}} R^{\sigma_1 \sigma_2}_{\rho_1 \rho_2} \cdots R^{\sigma_{2n-1} \sigma_{2n}}_{\rho_{2n-1} \rho_{2n}} \sqrt{g} d^D x.$$ (6)

(See [20] for an interesting review.) If any of the curvature radii are small compared to $l$, we expect the formula to break down because different sections of the tube associated with different regions of $M$ can intersect. This would cause the formula to overcount the volume.

of the Hamiltonian method are closely related. However they are not quite equivalent. For example non-invertibility can even occur on a hypersurface in Minkowski space where $\delta_{ab}$ can jump dramatically without discontinuity in $\Pi_{ab}$ (this solution was found explicitly in Ref. [17]). In that case $\det(\delta_I^J + \mathcal{Y}_I^J)$ is certainly not zero. Similar issues are discussed in Ref. [18].

This appears to be related to the results of Ref. [19] where it was shown that the Hamiltonian evolution normal to a boundary (in that case at infinity) is equivalent to the Lagrangian treatment only if additional Dirichlet boundary terms are added to the Lagrangian.
It was recently pointed out by Labbi[13] that the curvature invariants appearing in Weyl’s formula are the same as those appearing in the Lagrangian of Lovelock’s theory. So the volume of a tube coincides with the action of Euclidean Lovelock theory with a special choice of coupling constants. It would be interesting to generalise the tube formula to a Minkowski space background. Also, it may be of interest to find a tube formula in (A)dS space. The generalisation to hyperbolic space is well known[20].

B. Minkowski space tube formula

The first question which arises in generalising to Minkowski space is how to define the tube. In the Euclidean case the definition is motivated by the intuitive fact that the shortest route from a point to a surface is the line that hits the surface normally. In Minkowski space this is no longer true. Indeed it would be futile to define the tube as the locus of points of less than \( l \) spacelike proper distance from \( M \) for a simple reason. Let \( p \) be a point on \( M \). Then any points which are infinitesimally close to the lightcone of \( p \) and which have spacelike separation from \( p \) must be included in the tube. So a tube thus defined would stretch all the way out to future and past null infinity. However, even though the meaning is not quite the same as as in the Euclidean case, we can still define the tube in the same way:

**Definition II.1.** The tube of \( M^d \) in Minkowski space \( \mathbb{M}^n \) is the set of all points of proper distance less than \( l \) along a geodesic which intersects \( M^d \) normally.

According to the above definition, the tube will not extend out towards null infinity unless the normal vector of \( M^d \) becomes null at some point. So for an embedded submanifold of strictly Minkowski signature, the tube is bounded.

It is curious that, although the geometry of Minkowskian tubes is quite different compared to their Euclidean counterparts, the formula for the volume turns out to be the same. Before considering the general proof of this, let us check it explicitly with a pair of examples.

First we consider the embedding of an \((N - 1)\)-sphere \( S_{r}^{N-1} \) into \( \mathbb{R}^N \) and then the Lorentzian equivalent, de Sitter space embedded as a hyperboloid in Minkowski space. In the first case, the volume of the tube is the volume contained between two concentric spheres.
FIG. 1: The tube of a section of de Sitter space $dS^{N-1}$ embedded in Minkowski space $\mathbb{M}^N$. A slice through the $(X,T)$ plane is shown. The tube is the shaded region between concentric $dS$ spaces of curvature radii $r-l$ and $r+l$. The normal vectors are aligned with rays through the origin.

of radius $r-l$ and $r+l$, i.e.

$$V = N\text{Vol}(S^{N-1}) \{(r+l)^N - (r-l)^N\}$$

Using $\int L^n = r^{-N+2n+1} \frac{(N-1)!}{(N-2n-1)!} \text{Vol}(S^{N-1})$ for the Lovelock scalars of the sphere of radius $r$ we can expand the tube volume as:

$$V = 2 \sum_{n=0}^{[(N-1)/2]} \frac{l^{2n+1}}{(2n+1)!} \int_M L^{(n)}$$

with $M = S^{N-1}_r$.

In the second case, we have $dS^{N-1}_\rho$ realised by the embedding $-dT^2 + d\vec{X} \cdot d\vec{X} = \rho^2$ in $\mathbb{M}^N$. It is useful to parametrise this by $T = \rho \sinh \chi$ etc. As with the sphere, the normal vectors lie along rays through the origin and one finds that the tube is delimited by two concentric embedded $dS$ spaces of curvature radii $\rho = r-l$ and $\rho = r+l$ respectively. The volume element is $\rho^{N-1}d\rho d\Omega_{dS}$ where $d\Omega_{dS}$ is the volume element on the hyperboloid of unit curvature $^2$. So we obtain for the volume:

$$V = N\text{Vol}(dS^{N-1}) \{(r+l)^N - (r-l)^N\}$$

Since the Lovelock curvature scalars are the same for $dS$ as for the sphere, we obtain the same formula (7).

Let us now consider the tube of a general Lorentzian manifold $M^D \subset \mathbb{M}^N$. Since the definition of a tube in terms of the normal vectors is the same as for the Euclidean case, one

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^2 The volume diverges, but we can restrict to the region $\chi_i \leq \chi \leq \chi_f$. This correctly accounts for the edge effects of the tube, because the lines $\chi = \text{const.}$ coincide with the normal vectors (see fig 1).
would expect Weyl’s formula in terms of extrinsic curvatures\[20\] to be the same. Also, since the Gauss-Coddazzi equations are the same, we expect the formula \((5)\) in terms of intrinsic curvatures to apply also to Minkowski space. In order to confirm this, let us briefly revisit the proof of the tube formula, formulating things in a terminology familiar to relativists using Minkowskian signature.

Let us consider an infinitesimal region on \(M^D\) and let \((e_1, \ldots, e_D)\) be a set of orthonormal vectors forming a basis of the tangent space. In a local neighbourhood, this can be extended to an orthonormal basis of \(T^\ast M^N\) which can also be interpreted as a set of direction vectors in \(M^N\), denoted \((\vec{e}_1, \ldots, \vec{e}_D, \vec{n}_{(D+1)}, \ldots, \vec{n}_{(N)})\), where the \(n_{(i)}\) are normal vectors to \(M^D\).

Consider an infinitesimal \(D\)-cube defined by the vectors \(\vec{v}_1 = \theta^{(1)} \vec{e}_1, \ldots, \vec{v}_D = \theta^{(D)} \vec{e}_D\) where \(\theta^{(i)} = \theta^{(i)} \delta x^\nu\) are the infinitesimal line elements in the direction of the vector. Now we displace the vertices of the cube by a vector \(\sum_i z_i \vec{n}_i\) in a normal direction. If there is extrinsic curvature then \(\vec{n}_i\) will vary from one vertex to another. Therefore the displaced infinitesimal vectors will be

\[
\vec{v}_a(z) = \vec{v}_a + z^i \nabla_{v_a} \vec{n}_i = \vec{v}_a - \vec{v}_b z^i K^b_{\ a i}
\]

where \(K^b_{\ a i}\) is the extrinsic curvature tensor w.r.t. the normal \(\vec{n}_i\). The displaced \(D\)-volume element is therefore

\[
\det \left( \delta^a_b - z^i K^a_{\ bi} \right) \theta^{(1)} \wedge \cdots \wedge \theta^{(d)}.
\]

Integrating these elements over \(z^i\) and over \(M^D\), assuming that they do not intersect each other, gives:

\[
\int_{M^D} \int_{z^i \leq D^2} \det \left( \delta^a_b - z^i K^a_{\ bi} \right) dV(M^D).
\]

The calculation of combinatorial factors amounts to calculating the moments of the \((N-D)\)-ball, \(\langle z^i \rangle, \langle z^i z^j \rangle\) etc. Since odd moments vanish, the extrinsic curvatures will always appear in pairs, \((K^a_{\ c i}, K^b_{\ d i}) - K^b_{\ c i} K^a_{\ d i})\) \cdots when we expand out the determinant. In this way, the extrinsic curvatures can always be substituted for intrinsic curvatures using the Gauss formula. Since the normal space is Euclidean, the Gauss formula is the same as in the Euclidean case \(R^{ab}_{\ cd} = \sum_i (K^a_{\ c d i}, K^b_{\ d i} - K^b_{\ c i} K^a_{\ d i})\). Therefore, the same combinatorial factors and signs must arise\(^3\). So we conclude:

\(^3\) For details of the calculation of the combinatorial coefficients, see \[20\]. Alternatively, following Weyl, one
Proposition II.2. Let $M^D$ be a manifold with metric of Minkowski signature embedded in Minkowski space $\mathbb{M}^N$. Let $T(M)$ be the l-tube of $M^D$. Assuming that every point in the tube has a unique geodesic which connects it with $M^D$ and intersects $M^D$ normally, the world-volume of the tube is correctly described by formula $\text{(5)}$.

C. Example in D=5 illustrating the failings of the volume formula

In this case we have:

$$V = \frac{\text{Vol}(B_{N-5}) l^{N-3}}{2(N-3)} \int_{M^5} \left( -2\Lambda + R + \alpha(R^2 - 4R_{\mu\nu}R^\mu\nu + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right) \sqrt{-g} d^5x \quad (11)$$

with

$$\Lambda = -\frac{N-3}{l^2}, \quad \alpha = \frac{l^2}{4(N-1)}. \quad (12)$$

We note that the “magic” combination of coupling constants has the value $x := \frac{4\alpha\Lambda}{3} = -\frac{N-3}{3(N-1)}$. For all dimensions $N > 5$ this is in the range $-1/3 < x \leq -1/5$. The Chern-Simons gravity theories correspond to $x = -1$ and so it is not possible to obtain their action as a tube volume. Previously, the value $x = -1/3$ has been shown to be an exceptional case in the context of product spacetime solutions $[21]$. Also, in the context of the first order theory the value was found to be special, since this fine-tuning permitted compactified solutions with constant torsion on a three-sphere $[22]$. Here in the context of tube volumes (torsion-free by construction) we find $\lim_{N \to \infty} x(N) = -1/3$, providing further evidence that this value is special in some sense. In fact, since a general 5-manifold may need up to 26 dimensions in order for an embedding to exist, we should take $N >> 3$ and so $x = -1/3$ to good approximation.

The Lovelock theory defined by the above action admits two constant curvature solutions $R_{\mu\nu\rho\sigma} = \lambda \delta_{\mu\nu}$ with Gaussian curvature given by the roots of a quadratic equation $\lambda = \frac{2(N-1)}{l^2} \left( -1 \pm \sqrt{1 - \frac{N-3}{3(N-1)}} \right)$. So for large $N$ the characteristic curvature radius of the spacetime is given by $lN^{-1/2}$. Therefore the size of the (5-dimensional) universe is much smaller then the thickness of the tube. However, in this regime the tube formula is not valid and so we can not regard the solutions as meaningful. It can be checked that this appearance of an

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**can take an example where $M^D$ is of constant curvature, and read off the coefficients.**
enormous effective cosmological constant is a generic feature of for all $D$. We therefore look for an appropriate term to add to the action, which may cancel the cosmological constant.

**D. The addition of a term proportional to the tube surface area**

If we think of the tube volume as an action functional of the intrinsic metric, it is interesting to ask what are the extrema of the action. If one looks for a maximally symmetric solution, with constant curvature $\lambda$, the Lovelock field equations will give a polynomial of order $[D/2]$ for $\lambda$. So for $D$ odd one always has at least one solution. A preliminary investigation suggests that, for real roots $\lambda$ tends to be large compared to $1/l^2$ i.e. it describes a geometry where the volume formula is expected to break down. Also $\lambda$ is always negative.

In order to ensure a maximally symmetric solution with small curvature, it seems to be necessary to use the tube area formula. The surface area of the tube is:

$$A = \text{Vol}(B_{N-D}) \sum_{n=0}^{[D/2]} \frac{(N-D)!!}{(N-D+2n-2)!!(2n)!!} l^{N-D+2n-1} \int_{M} \mathcal{L}^{(n)}, \quad (13)$$

A more general action would then be $S = \rho V + \sigma A$, with $\rho, \sigma$ constants. The analogy would be with a drop of fluid, whose internal energy has an extensive part and also a contribution from the surface energy.

We shall consider the simple choice

$$S \propto -V + \frac{l}{N-D} A. \quad (14)$$

This choice allows us to cancel completely the term proportional to the area of $M$. The resulting action depends only on the curvature terms,

$$S \propto \text{Vol}(B_{N-D}) \sum_{n=1}^{[D/2]} \frac{(N-D-2)!!}{(N-D+2n)!!(2n-2)!!} l^{N-D+2n} \int_{M} \mathcal{L}^{(n)}, \quad (15)$$

and therefore Minkowski space will be a solution. More generally, the absence of the bare cosmological constant ($n = 0$) term means that there will be a branch of the solutions where the curvature is small compared to $1/l^2$. These solutions will be like solutions of Einstein’s equation with higher order corrections in $l^2$. Solutions for $M$ belonging to this branch can have tubes that do not self intersect.
Normalising so that the coefficient of the Einstein-Hilbert term is unity, the coefficients of the Lovelock series are:

\[
\begin{align*}
\Lambda &= 0, \\
\alpha_2 &= \frac{l^2}{(N - D + 4)2!!}, \\
\alpha_3 &= \frac{l^4}{(N - D + 4)(N - D + 6)4!!}.
\end{align*}
\]

etc. Generally

\[
\alpha_n = \frac{(N - D + 2)!! l^{2n-2}}{(N - D + 2n)!!(2n - 2)!!}.
\]

(16)

III. VALIDITY OF THE TUBE FORMULA AND DETERMINISM

A. Domain of validity of determinism

To see when determinism breaks down in this theory\(^4\), we need to examine the determinant \(\det(1 + \mathcal{Y})\). First, in order to simplify the expression for \(\mathcal{Y}\), let us introduce Lovelock tensors, with two and four free indices, of the spatial components of the curvature:

\[
\begin{align*}
(\mathcal{H}^{(p)})^{ab}_{a b} := & -\frac{1}{2p+1} \delta^{a_{b1} \cdots a_{2p}} b_{1b_2} \cdots R^{a_{2p-1} a_{2p}} b_{2p-1b_{2p}}, \\
(\mathcal{I}^{(p)})^{ac}_{b c d} := & -\frac{1}{2p+1} \delta^{a_{b1} \cdots a_{2p}} b_{1b_2} \cdots R^{a_{2p-1} a_{2p}} b_{2p-1b_{2p}}.
\end{align*}
\]

Then we obtain the general formula:

\[
\mathcal{Y}^{cd}_{ab} = -2 \sum_{n=2}^{[D/2]} n \alpha_n ((\mathcal{I}^{(n-1)})^{cd}_{ab} - g_{ab} (\mathcal{H}^{(n-1)})^{cd}_{ab}).
\]

(17)

Now we shall evaluate this for the choice of coupling coefficients (16) obtained in the preceding section. Let us assume that the embedding space is high dimensional: \(N >> D\).

\(^4\) In Ref. \[15\][16], determinism is defined in terms of solving for \(\ddot{\tilde{g}}_{ab}\) given initial data \(g_{ab}\) and \(\dot{g}_{ab}\) on a space-like hypersurface. As discussed in footnote 1, this is not always equivalent to the Hamiltonian evolution. The former approach will arise naturally when integrating by finite element approximation. As such it is relevant to numerical evolution of solutions. The latter approach is more correct from the point of view of taking limits, for example when we consider classical solutions as arising from the method of stationary phase\[23\]. Here we follow the definition of Ref. \[15\][16], because it allows us to restore determinism by imposing a simple inequality on the Riemann tensor. For the Hamiltonian evolution no such simple condition exists.
Therefore

\[ \alpha_n \sim \left( \frac{l^2}{2N} \right)^{n-1} \frac{1}{(n-1)!} \cdot \]

\[ \mathcal{Y}_{ab}^{\,cd} \sim -2 \sum_{p=1}^{[(D-2)/2]} \left( \frac{l^2}{2N} \right)^p \frac{1}{p!} \left( (\mathcal{T}^{(p)})_{ab}^{\,cd} - g_{ab}(\mathcal{H}^{(p)})^{cd} \right). \quad (18) \]

The determinant will never vanish if all eigenvalues of \( \mathcal{Y}_I^J \) are much smaller than unity (in an appropriate frame, e.g. an orthonormal frame, we may say that all the components are much less than unity). This will always be the case provided all \( R_{ab}^{\,cd} \ll 2N/l^2 \). Determinism will only be in danger of breaking down once Riemann tensor components become of order \( 2N/l^2 \).

B. Domain of validity of the tube formula

As mentioned previously, the tube formula breaks down if the tube intersects itself in some way. The tube formula is valid provided every point in the tube has a unique geodesic which connects it with \( M^D \) and intersects \( M^D \) normally\(^5\). It is easy to check that:

The tube formula in Minkowski space breaks down locally around a point \( x \in M \) if any of the eigenvalues of the extrinsic curvature matrices \( K_i(x) \) is greater than or equal to \( 1/l \) in magnitude. Furthermore, at least locally, requiring the absolute value of all the eigenvalues of the \( K_i \) to be less than \( 1/l \) is a sufficient condition for the validity of the tube formula. In view of the Gauss relation, this means that the magnitude of components of the Riemann tensor in an appropriate basis are certainly less than \( 2N/l^2 \). In fact, since the sectional curvatures will be less than \( 1/l^2 \), the tube formula is expected to break down when the Riemann tensor components (in an orthonormal frame say) are of order \( 1/l^2 \).

C. Physical implications

Although we have only given an order of magnitude estimate, the result is quite compelling. It gives us strong evidence that the domain of validity of the tube formula is

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\(^5\) i.e. provided that the exponential map from the bundle of normal vectors into \( M^N \) is bijective for normal vectors of length \( \leq l \)
contained within the domain of validity of determinism. *If this is so, it means that in regions where the curvature blows up, the tube formula breaks down before determinism breaks down.*

When the tube formula breaks down, it is because elements of the extrinsic geometry interfere with the simple expression of the volume and area in terms of intrinsic geometry of $M$. If we regard (14) as the fundamental definition of the action, then formula (15) is an effective description only when extrinsic curvatures are small. Once they become large, there is a phase transition to a regime where the geometrical degrees of freedom are different.

Therefore, instead of a phase transition to a nondeterministic (and therefore classically ill-defined) theory, we have a phase transition to a different sector of the theory where the tube volume and area are not described entirely in terms of the intrinsic geometry of $M$, but where extrinsic geometry of the embedding becomes relevant as a physical degree of freedom.

In all of this we are assuming that in the sector described by (15) it is legitimate to vary the action with respect to the intrinsic metric of $M$, rather than w.r.t. the embedding itself. This is potentially a rather big weakness, which we will pick up on again in the concluding section.

**IV. EMBEDDING SPACE-TIMES INTO MINKOWSKI SPACE**

We have treated space-time and its tube as embedded in some Minkowski space of higher dimension in such a way that the intrinsic geometry of spacetime coincides with the induced geometry of the embedding. So far we have just assumed that such isometric embeddings (of the appropriate level of smoothness) exist. Now it is necessary to take this question seriously. For Riemannian manifolds, it is a classic result of J. Nash that any manifold may be smoothly isometrically embedded into Euclidean space of large enough dimension. For manifolds of Lorentzian signature, we need to know what kind of manifolds have such an embedding in $M^N$. Fortunately, in recent years a very satisfactory answer to this question has emerged.

So how does Nash’s embedding theorem generalise to Minkowski space? Clearly, not every space-time admits such an embedding. For example, if space-time is not time-orientable it
can not be embedded\(^6\). Using straightforward arguments, Penrose showed that the manifold must admit a spacelike surface separating space-time into two disconnected regions (past and future), such that every causal path cuts the surface no more than once and every timelike curve ending on the surface has bounded proper time\(^{20}\) (note that this is weaker than global hyperbolicity- for example take a globally hyperbolic space-time and remove some points or timelike surfaces. The resulting spacetime will not be globally hyperbolic but it will still obey the above condition). A highly non-trivial result - almost the converse of Penrose’s - obtained recently, is the following remarkable theorem\(^{27}\):

**Theorem IV.1** (Müller, Sánchez). *Any globally hyperbolic space-time manifold \(M^D\) admits a global smooth isometric embedding into Minkowski space \(M^N\) for large enough \(N\).*

The current upper bound for what is a sufficiently large value of \(N\) is \(\max\{(D^2 + 5D + 2)/2, (D^2 + 3D + 12)/2\}\) i.e. one higher than the corresponding upper bound for Euclidean manifolds. So if we want to study four-dimensional space-times, we should embed them in at least 19 dimensions to be sure that an embedding exists. For five dimensions, we should embed them in 26 dimensions etc.

As mentioned above, global hyperbolicity is not a necessary condition for the embedding. However, the slightly weaker condition of causal simplicity is a necessary condition\(^{27}\).

It is quite wonderful that the existence of the embedding is guaranteed by only one requirement- and a very welcome requirement it is too. A globally hyperbolic space-time is the arena for deterministic physics. This complements rather well the (heuristic) results of the previous section.

V. CONCLUSIONS AND FURTHER DISCUSSION

Weyl’s formulae for the volume \(V\) and surface area \(A\) of a tube in Euclidean space have been shown to generalise straightforwardly to a tube surrounding a pseudo-Riemannian manifold embedded in Minkowski space. The resulting formulae correspond to the action of

\(^6\) It may be possible to embed such a space-time into a pseudo-Euclidean space \(\mathbb{E}^{p,q}\) with \(q > 1\) time dimensions. In fact Greene\(^{24}\) and Clarke\(^{25}\) independently showed that any pseudo-Riemannian manifold can be isometrically embedded into \(\mathbb{E}^{p,q}\) for large enough \(p\) and \(q\). However, for the purposes of the tube formula, such embeddings are not acceptable, due to the problem of defining a tube when there are null geodesics in the normal space.
Lovelock gravity, with a special combination of the coefficients. We have focussed on just one spacial case, taking a combination \(-V + \frac{1}{N-D} A\) so that the bare cosmological constant term vanishes from the Lovelock series. In this case evidence was found that the Lovelock description of the tube volume breaks down before determinism breaks down. Therefore, instead of the theory itself breaking down, one would have a phase transformation to a different sector, governed by different geometrical degrees of freedom. For the future, a more careful study of the curvature inequalities is needed to verify this, also taking into account other linear combinations of \(V\) and \(A\).

In order to obtain the equations of Lovelock gravity, we have not varied w.r.t. the embedding, but rather the intrinsic geometry of \(M\). This means, for instance that we regard translations of the tube in Minkowski space as pure gauge. Also, in higher dimensions, it is possible to have changes in the extrinsic curvature which preserve the intrinsic metric and curvature. This is known as isometric bending. These are also treated as pure gauge. However, one is at liberty to question this approach. If the embedding is regarded as real rather than just metaphorical, the rigid motions would then be more correctly regarded as zero modes of the theory. Also the isometric bending would become something like zero modes. If we follow the string/brane approach in constructing the variational principle, we should regard the embedding coordinates \(X^A(x^\mu)\) as the degrees of freedom. To derive the field equations, one follows exactly the same argument as with standard Regge-Teitelboim geodetic brane gravity\[28\]\[29\]. Noting that \(\delta g_{\mu\nu} = \delta (X^A X^B) \eta_{AB}\) and using the Bach-Lanczos identity \(H^\mu_{\mu\rho} = 0\) we get:

\[
\left( \sum_n c_n (H^{(n)})^{\mu\nu} \right) X^A_{;\mu\nu} = 0 . 
\]

So we see that the solutions of Lovelock gravity would be a subset of the resulting solutions. However, there are also other solutions such as the rather trivial \(X^A_{;\mu\nu} = 0\). The possible degeneration of this term multiplying the field equations will affect any conclusions regarding determinism. Therefore it may be desirable to avoid varying w.r.t. \(X^A(x^\mu)\). More study is needed.

That there is a formal connection\[13\] between Weyl’s tube formula and Lovelock gravity is, in the authors opinion, of undoubted interest. It remains to be checked more carefully if this truly provides a resolution to the problem of indeterminism (or any other physical problems). In our method there is perhaps some mixing of philosophies between the realist
and the metaphorical interpretation of the embedding space, which needs to be untangled in a satisfactory manner. This work is offered as an introduction and perhaps an invitation to further study of the subject.

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