Barcodes for Hamiltonian homeomorphisms of surfaces

Joly Benoît

Ruhr University Bochum

March 25 2022
Barcodes

Definition (Barcode)

A barcode is a countable set of intervals \((a, b]\) or \((c, +\infty)\) of \(\mathbb{R}\) called bars.
A barcode is a countable set of intervals \((a, b]\) or \((c, +\infty)\) of \(\mathbb{R}\) called bars.

Let us consider a Morse function \(\mathcal{H} : S^2 \to \mathbb{R}\) as follows:

\[
\forall \lambda' > \lambda \in \mathbb{R}, \quad H_\ast(\{\mathcal{H} < \lambda\}) \xrightarrow{\ i_\ast \ } H_\ast(\{\mathcal{H} < \lambda'\}).
\]
**Definition (Barcode)**

A barcode is a countable set of intervals \((a, b]\) or \((c, +\infty)\) of \(\mathbb{R}\) called bars.

Let us consider a Morse function \(\mathcal{H} : S^2 \to \mathbb{R}\) as follows:

\[
\forall \lambda' > \lambda \in \mathbb{R}, \quad H_*\left(\{\mathcal{H} < \lambda\}\right) \overset{i_*}{\rightarrow} H_*\left(\{\mathcal{H} < \lambda'\}\right).
\]
Barcodes

Definition (Barcode)
A barcode is a countable set of intervals \((a, b]\) or \((c, +\infty)\) of \(\mathbb{R}\) called bars.

Let us consider a Morse function \(\mathcal{H} : S^2 \rightarrow \mathbb{R}\) as follows:

\[
\begin{align*}
\forall \lambda' > \lambda \in \mathbb{R}, \quad H_\ast(\{\mathcal{H} < \lambda\}) & \overset{i_\ast}{\rightarrow} H_\ast(\{\mathcal{H} < \lambda'\}).
\end{align*}
\]
A barcode is a countable set of intervals \((a, b]\) or \((c, +\infty)\) of \(\mathbb{R}\) called bars.

Let us consider a Morse function \(\mathcal{H} : S^2 \to \mathbb{R}\) as follows:

\[
\forall \lambda' > \lambda \in \mathbb{R}, \quad H_*(\{\mathcal{H} < \lambda\}) \xrightarrow{i_*} H_*(\{\mathcal{H} < \lambda'\}).
\]
Why do we study barcodes?

*In symplectic geometry:*

- The notion was introduced by Polterovich-Shelukhin and Usher-Zang,
Why do we study barcodes?

In symplectic geometry:

- The notion was introduced by Polterovich-Shelukhin and Usher-Zang,
- Barcodes are the filtered Floer Homology of time-dependent Hamiltonian functions $\mathcal{H} : [0, 1] \times \Sigma \to \mathbb{R}$ where the filtration is given by the action function,
In symplectic geometry:

- The notion was introduced by Polterovich-Shelukhin and Usher-Zang,
- Barcodes are the filtered Floer Homology of time-dependent Hamiltonian functions $\mathcal{H} : [0, 1] \times \Sigma \to \mathbb{R}$ where the filtration is given by the action function,
- Endpoints of bars are the values of the spectrum of $\mathcal{H}$,
Why do we study barcodes?

*In symplectic geometry*:

- The notion was introduced by Polterovich-Shelukhin and Usher-Zang,
- Barcodes are the filtered Floer Homology of time-dependent Hamiltonian functions $\mathcal{H} : [0, 1] \times \Sigma \to \mathbb{R}$ where the filtration is given by the action function,
- Endpoints of bars are the values of the spectrum of $\mathcal{H}$,
- The barcode is a conjugacy invariant,
Why do we study barcodes?

In symplectic geometry:

- The notion was introduced by Polterovich-Shelukhin and Usher-Zang,
- Barcodes are the filtered Floer Homology of time-dependent Hamiltonian functions $\mathcal{H} : [0, 1] \times \Sigma \to \mathbb{R}$ where the filtration is given by the action function,
- Endpoints of bars are the values of the spectrum of $\mathcal{H}$,
- The barcode is a conjugacy invariant,
- The barcodes are $C^0$-continuous and extend to homeomorphisms. (Kislev-Shelukhin, Le Roux-Seyfaddini-Viterbo, Jannaud, Buhovski-Humilière-Seyfaddini)
Main goal

We want to construct barcodes for Hamiltonian homeomorphisms of surfaces without using Floer homology.
Main goal

We want to construct barcodes for Hamiltonian homeomorphisms of surfaces without using Floer homology.

Definition (Hamiltonian homeomorphisms)

An isotopy $I = (f_t)_{t \in [0,1]}$ induces a Hamiltonian homeomorphism if its flux through every closed loop $\gamma \subset \Sigma$ is zero:

$$\int_{\Sigma} \gamma \wedge I(z) \omega = 0,$$

where $I(z) : t \mapsto f_t(z)$.
Context and main difficulties

From now on we consider:

- a compact oriented surface $\Sigma$,
- a Hamiltonian homeomorphism $f$ on $\Sigma$ s.t. $\text{Fix}(f)$ is finite.

Action function (Jian Wang)

A Hamiltonian homeomorphism $f$ is equipped with an action function $A_f : \text{Fix}(f) \rightarrow \mathbb{R}$, extending the classical notion for diffeomorphisms.

Difficulties:

- There is no function defined everywhere,
- We cannot compute directly a filtered homology on $\Sigma$. 

Joly Benoît

March 25 2022 5 / 15
From now on we consider:

- a compact oriented surface $\Sigma$, 
- a Hamiltonian homeomorphism $f$ on $\Sigma$ s.t. $\text{Fix}(f)$ is finite.

**Action function (Jian Wang)**

A Hamiltonian homeomorphism $f$ is equipped with an action function $A_f : \text{Fix}(f) \to \mathbb{R}$, extending the classical notion for diffeomorphisms.
From now on we consider:

- a compact oriented surface $\Sigma$,
- a Hamiltonian homeomorphism $f$ on $\Sigma$ s.t. $\text{Fix}(f)$ is finite.

**Action function (Jian Wang)**

A Hamiltonian homeomorphism $f$ is equipped with an action function $A_f : \text{Fix}(f) \to \mathbb{R}$, extending the classical notion for diffeomorphisms.

**Difficulties**:

- There is no function defined everywhere,
- We can not compute directly a filtered homology on $\Sigma$. 
There exists an application

$$\beta : \mathcal{G} \mapsto \text{Barcodes},$$

where $\mathcal{G}$ is the set of couples $(G, A)$ s.t.

- $G$ is a finite oriented and connected graph,
- $A : V \rightarrow \mathbb{R}$ decreasing along the edges,

where $V$ is the set of vertices of $G$. 
The graph is defined by the gradient lines, $\nabla \mathcal{H}$. 

Joly Benoît  
March 25 2022 7 / 15
The graph is defined by the gradient lines, $A = \mathcal{H}$.
The graph is defined by the gradient lines, $A = \mathcal{H}$. 

\[ G(\nabla \mathcal{H}) \quad \nabla \mathcal{H} \quad B(\mathcal{H}) \]
Positively transverse foliations

Theorem (Le Calvez)

For every maximal isotopy $I$ of a homeomorphism $f$ of a surface $\Sigma$ there exists a foliation positively transverse to $I$. 
Positively transverse foliations

**Theorem (Le Calvez)**

For every maximal isotopy $I$ of a homeomorphism $f$ of a surface $\Sigma$ there exists a foliation positively transverse to $I$.

- We consider an isotopy $I = (f_t)_{t \in [0,1]}$ from id to $f$ of $\Sigma$, 
Theorem (Le Calvez)

For every maximal isotopy $I$ of a homeomorphism $f$ of a surface $\Sigma$ there exists a foliation positively transverse to $I$.

- We consider an isotopy $I = (f_t)_{t \in [0,1]}$ from id to $f$ of $\Sigma$,
- We set $\text{Sing}(I) = \{ z \in \Sigma | \forall t \in [0,1], f_t(z) = z \} \subset \text{Fix}(f)$,
For every maximal isotopy $I$ of a homeomorphism $f$ of a surface $\Sigma$ there exists a foliation positively transverse to $I$.

- We consider an isotopy $I = (f_t)_{t \in [0,1]}$ from $\text{id}$ to $f$ of $\Sigma$,
- We set $\text{Sing}(I) = \{z \in \Sigma | \forall t \in [0,1], f_t(z) = z\} \subset \text{Fix}(f)$,
- $I$ is maximal if $\text{Sing}(I)$ is maximal for the inclusion,
**Theorem (Le Calvez)**

For every maximal isotopy $I$ of a homeomorphism $f$ of a surface $\Sigma$ there exists a foliation positively transverse to $I$.

- We consider an isotopy $I = (f_t)_{t \in [0,1]}$ from $\text{id}$ to $f$ of $\Sigma$,
- We set $\text{Sing}(I) = \{ z \in \Sigma | \forall t \in [0,1], f_t(z) = z \} \subset \text{Fix}(f)$,
- $I$ is maximal if $\text{Sing}(I)$ is maximal for the inclusion,
- An oriented topological foliation $\mathcal{F}$ on $\Sigma \setminus \text{Sing}(I)$ is positively transverse if $\forall x \in \Sigma \setminus \text{Sing}(I)$ the path $I(x) : t \mapsto f_t(x)$ is as follows:
Fact

Every foliation $\mathcal{F}$ positively transverse to a maximal isotopy of a Hamiltonian homeomorphism $f$ is gradient-like.
Fact

Every foliation $\mathcal{F}$ positively transverse to a maximal isotopy of a Hamiltonian homeomorphism $f$ is *gradient-like*.

\[ \forall \phi \in \mathcal{F}, \ A_f(\alpha(\phi)) > A_f(\omega(\phi)). \]
Construction of a barcode

Definition

Let $G(F)$ be a graph where the set of vertices is $\text{Sing}(I)$ and there exists an oriented edge from $x$ to $y$ if there exists a leaf $\phi$ of $F$ from $x$ to $y$.

The graph $G(F)$ is naturally equipped with a filtration by $A_f$. 


Construction of a barcode

Definition

Let $G(\mathcal{F})$ be a graph where the set of vertices is $\text{Sing}(I)$ and there exists an oriented edge from $x$ to $y$ if there exists a leaf $\phi$ of $\mathcal{F}$ from $x$ to $y$.

The graph $G(\mathcal{F})$ is naturally equipped with a filtration by $A_f$.

Construction

$$(f, I) \leadsto \mathcal{F} \leadsto (G(\mathcal{F}), A_f) \xrightarrow{\beta} B(\mathcal{F}) \subset \text{Barcodes},$$

where $I$ is a maximal isotopy of $f$. 
Construction of a barcode

Definition

Let $G(\mathcal{F})$ be a graph where the set of vertices is $\text{Sing}(I)$ and there exists an oriented edge from $x$ to $y$ if there exists a leaf $\phi$ of $\mathcal{F}$ from $x$ to $y$.

The graph $G(\mathcal{F})$ is naturally equipped with a filtration by $A_f$.

Construction

$$(f, I) \leadsto \mathcal{F} \leadsto (G(\mathcal{F}), A_f) \xrightarrow{\beta} B(\mathcal{F}) \subset \text{Barcodes},$$

where $I$ is a maximal isotopy of $f$.

Theorem

$B(\mathcal{F})$ is independent of $\mathcal{F}$, it depends only on $I$. 
A graph without a foliation: An order on $\text{Sing}(I)$

We consider a maximal isotopy $I = (f_t)_{t \in [0,1]}$ from $\text{id}$ to $f$, 

$\Sigma'$

$\Sigma'$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$
We consider a maximal isotopy \( I = (f_t)_{t \in [0,1]} \) from id to \( f \),

**Definition**

\( \forall x, y \in \text{Sing}(I) \) we set \( x > y \) if there exists an oriented path \( \gamma \) from \( x \) to \( y \) s.t. every lift \( \tilde{\gamma} \) of \( \gamma \) is an oriented *Brouwer line* of \( \tilde{f} \).

It induces an order on \( \text{Sing}(I) \).
We consider a maximal isotopy \( l = (f_t)_{t \in [0,1]} \) from \( \text{id} \) to \( f \),

**Definition**

\( \forall x, y \in \text{Sing}(l) \) we set \( x > y \) if there exists an oriented path \( \gamma \) from \( x \) to \( y \) s.t. every lift \( \tilde{\gamma} \) of \( \gamma \) is an oriented *Brouwer line* of \( \tilde{f} \).

It induces an order on \( \text{Sing}(l) \).
We consider a maximal isotopy $I = (f_t)_{t\in[0,1]}$ from $\text{id}$ to $f$.

**Definition**

$\forall x, y \in \text{Sing}(I)$ we set $x > y$ if there exists an oriented path $\gamma$ from $x$ to $y$ s.t. every lift $\tilde{\gamma}$ of $\gamma$ is an oriented *Brouwer line* of $\tilde{f}$. It induces an order on $\text{Sing}(I)$. 
A graph without a foliation: An order on $\text{Sing}(I)$

We consider a maximal isotopy $I = (f_t)_{t \in [0,1]}$ from $\text{id}$ to $f$.

**Definition**

$\forall x, y \in \text{Sing}(I)$ we set $x > y$ if there exists an oriented path $\gamma$ from $x$ to $y$ s.t. every lift $\tilde{\gamma}$ of $\gamma$ is an oriented *Brouwer line* of $\tilde{f}$.

It induces an order on $\text{Sing}(I)$. 

\[ \Sigma \setminus \text{Sing}(I) \]
A graph without a foliation: An order on Sing(I)

We consider a maximal isotopy \( \lambda = (f_t)_{t \in [0,1]} \) from id to \( f \),

**Definition**

\( \forall x, y \in \text{Sing}(\lambda) \) we set \( x > y \) if there exists an oriented path \( \gamma \) from \( x \) to \( y \) s.t. every lift \( \tilde{\gamma} \) of \( \gamma \) is an oriented Brouwer line of \( \tilde{f} \). It induces an order on \( \text{Sing}(\lambda) \).
A graph without a foliation: An order on $\text{Sing}(I)$

We consider a maximal isotopy $I = (f_t)_{t \in [0, 1]}$ from $\text{id}$ to $f$.

**Definition**

$\forall x, y \in \text{Sing}(I)$ we set $x > y$ if there exists an oriented path $\gamma$ from $x$ to $y$ s.t. every lift $\tilde{\gamma}$ of $\gamma$ is an oriented Brouwer line of $\tilde{f}$. It induces an order on $\text{Sing}(I)$. 

![Diagram](image)
A graph without a foliation : An order on $\text{Sing}(I)$

**Definition**

We set $G(\rangle)$ the graph s.t. $\text{Sing}(I)$ is the set of vertices and there exists an oriented edge from $x$ to $y$ if $x \rangle y$.

$G(\rangle)$ is naturally equipped with a filtration by $A_f$. 
A graph without a foliation: An order on $\text{Sing}(I)$

**Definition**

We set $G(>)$ the graph s.t. $\text{Sing}(I)$ is the set of vertices and there exists an oriented edge from $x$ to $y$ if $x > y$.

$G(>)$ is naturally equipped with a filtration by $A_f$.

**Construction**

$$(f, I) \leadsto \text{order } > \leadsto (G(>), A_f) \overset{\beta}{\mapsto} B(>) \subseteq \text{Barcodes},$$

where $I$ is a maximal isotopy of $f$. 
Equality of the barcodes

Theorem 1 (J.)

∀\mathcal{F} foliation positively transverse to I, we have \( B(>) = B(\mathcal{F}) \).
Equality of the barcodes

\textbf{Theorem 1 (J.)}

\forall \mathcal{F} \text{ foliation positively transverse to } I, \text{ we have } B(>) = B(\mathcal{F}).

\textbf{Summary}

\[ f \rightarrow I \rightarrow \mathcal{F} \rightarrow G(\mathcal{F}) \rightarrow B(\mathcal{F}) \]

\[ \rightarrow > \rightarrow G(>) \rightarrow B(>) \]
Equality of the barcodes

Theorem 1 (J.)

∀\mathcal{F} foliation positively transverse to \( I \), we have \( B(>) = B(\mathcal{F}) \).

Summary

\[ f \rightarrow I \quad \rightarrow \mathcal{F} \rightarrow G(\mathcal{F}) \rightarrow B(\mathcal{F}) \]

\[ > \rightarrow G(>) \rightarrow B(>) \]

Question 1

Can we construct barcodes which depend only on \( f \)?
**Theorem 2 (J.)**

Let $f \in \text{Ham}(\Sigma)$ be $C^2$-close to the identity s.t the fixed points are nondegenerate.

For every foliation $\mathcal{F}$ positively transverse to a maximal isotopy, the barcode $B(\mathcal{F})$ is equal to the Floer homology barcode of $f$. 

**Question 2**

Is Theorem 2 more general?
Do we obtain Floer Homology barcodes?

**Theorem 2 (J.)**

Let $f \in \text{Ham}(\Sigma)$ be $C^2$-close to the identity s.t the fixed points are nondegenerate. For every foliation $\mathcal{F}$ positively transverse to a maximal isotopy, the barcode $B(\mathcal{F})$ is equal to the Floer homology barcode of $f$.

**Question 2**

Is Theorem 2 more general?
Thank you!