Generalized asymptotic Sidon basis

Sándor Z. Kiss ‡, Csaba Sándor †

Abstract

Let \( h, k \geq 2 \) be integers. We say a set \( A \) of positive integers is an asymptotic basis of order \( k \) if every large enough positive integer can be represented as the sum of \( k \) terms from \( A \). A set of positive integers \( A \) is called \( B_{h}[g] \) set if all positive integers can be represented as the sum of \( h \) terms from \( A \) at most \( g \) times. In this paper we prove the existence of \( B_{h}[1] \) sets which are asymptotic bases of order \( 2h + 1 \) by using probabilistic methods.

2010 Mathematics Subject Classification: 11B34, 11B75.

Keywords and phrases: additive number theory, general sequences, additive representation function.

1 Introduction

Let \( \mathbb{N} \) denote the set of positive integers. Let \( h, k \geq 2 \) be integers. Let \( A \subset \mathbb{N} \) be an infinite set of positive integers and let \( R_{h,A}(n) \) denote the number of solutions of the equation

\[
a_1 + a_2 + \cdots + a_h = n, \quad a_1 \in A, \ldots, a_h \in A, \quad a_1 \leq a_2 \leq \ldots \leq a_h,
\]

where \( n \in \mathbb{N} \). A set of positive integers \( A \) is called \( B_{h}[g] \) set if for every \( n \in \mathbb{N} \), the number of representations of \( n \) as the sum of \( h \) terms in the form (1) is at most \( g \), that is \( R_{h,A}(n) \leq g \). We denote the fact that \( A \) is a \( B_{h}[g] \) set by \( A \in B_{h}[g] \). We say a set \( A \subset \mathbb{N} \) is an asymptotic basis of order \( k \), if \( R_{k,A}(n) > 0 \) for all large enough positive integer \( n \), i.e., if there exists a positive integer \( n_0 \) such that \( R_{k,A}(n) > 0 \) for \( n > n_0 \). In [4] and [5] P. Erdős, A. Sárközy and V. T. Sós asked if there exists a Sidon set (or \( B_{2}[1] \) set) which is an asymptotic basis of order 3. It is easy to see that a Sidon set cannot be an asymptotic basis of order 2. J. M. Deshouillers and A. Plagne in [3] constructed a Sidon

‡ Institute of Mathematics, Budapest University of Technology and Economics, H-1529 B.O. Box, Hungary; kisspest@cs.elte.hu; This author was supported by the National Research, Development and Innovation Office NKFIH Grant No. K115288 and K129335. This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Supported by the ÚNKP-18-4 New National Excellence Program of the Ministry of Human Capacities. Supported by the ÚNKP-19-4 New National Excellence Program of the Ministry for Innovation and Technology

† Institute of Mathematics, Budapest University of Technology and Economics, MTA-BME Lendület Arithmetic Combinatorics Research Group H-1529 B.O. Box, Hungary, csandor@math.bme.hu. This author was supported by the NKFIH Grants No. K129335. Research supported by the Lendület program of the Hungarian Academy of Sciences (MTA), under grant number LP2019-15/2019.
set which is an asymptotic basis of order at most 7. In [7] it was proved the existence of Sidon sets which are asymptotic bases of order 5 by using probabilistic methods. In [1] and [9] this result was improved on by proving the existence of a Sidon set which is an asymptotic basis of order 4. It was also proved [1] that there exists a $B_2[2]$ set which is an asymptotic basis of order 3. In this paper we will prove a similar but more general theorem. Namely, we prove the existence of an asymptotic basis of order $2h + 1$ which is a $B_h[1]$ set.

Theorem 1. For every $h \geq 2$ integer there exists a $B_h[1]$ set which is an asymptotic basis of order $2h + 1$.

Before we prove the above theorem, we give a short survey of the probabilistic method we are working with.

2 Probabilistic tools

To prove Theorem 1 we use the probabilistic method due to Erdős and Rényi. There is an excellent summary of this method in the book of Halberstam and Roth [6]. In this paper we denote the probability of an event by $\mathbb{P}$, and the expectation of a random variable $Y$ by $\mathbb{E}(Y)$. Let $\Omega$ denote the set of the strictly increasing sequences of positive integers.

Lemma 1. Let

$$\alpha_1, \alpha_2, \alpha_3 \ldots$$

be real numbers satisfying

$$0 \leq \alpha_n \leq 1 \quad (n = 1, 2, \ldots).$$

Then there exists a probability space $(\Omega, X, \mathbb{P})$ with the following two properties:

(i) For every natural number $n$, the event $E^{(n)} = \{ A : A \in \Omega, n \in A \}$ is measurable, and $\mathbb{P}(E^{(n)}) = \alpha_n$.

(ii) The events $E^{(1)}, E^{(2)}, \ldots$ are independent.

See Theorem 13. in [6], p. 142. We denote the characteristic function of the event $E^{(n)}$ by $\varrho(A, n)$:

$$\varrho(A, n) = \begin{cases} 1, & \text{if } n \in A \\ 0, & \text{if } n \notin A. \end{cases}$$

Furthermore, for some $A = \{a_1, a_2, \ldots \} \in \Omega$ we denote the number of solutions of $a_{i_1} + a_{i_2} + \ldots + a_{i_h} = n$ with $a_{i_1} \in A, a_{i_2} \in A, \ldots, a_{i_h} \in A$, $1 \leq a_{i_1} < a_{i_2} \ldots < a_{i_h} < n$ by $r_h(n)$. Then

$$r_{h,A}(n) = r_h(n) = \sum_{\substack{(a_1, a_2, \ldots, a_h) \in \mathbb{N}^h \\ 1 \leq a_1 < \ldots < a_h < n \\ a_1 + a_2 + \ldots + a_h = n}} \varrho(A, a_1)\varrho(A, a_2)\ldots\varrho(A, a_h). \quad (2)$$

Let $R_h^*(n)$ denote the number of those representations of $n$ in the form (1) in which there are at least two equal terms. Thus we have

$$R_{h,A}(n) = r_h(n) + R_h^*(n). \quad (3)$$

In the proof of Theorem 1 we use the following lemma:
Lemma 2. (Borel-Cantelli) Let $X_1, X_2, \ldots$ be a sequence of events in a probability space. If
\[ \sum_{j=1}^{+\infty} P(X_j) < \infty, \]
then with probability 1, at most a finite number of the events $X_j$ can occur.

See [6], p. 135.

3 Proof of Theorem 1

Let $h$ be fixed and let $\alpha = \frac{2}{2h+1}$. Define the sequence $\alpha_n$ in Lemma 1 by
\[ \alpha_n = \frac{1}{n^{1-\alpha}}, \]
so that $P(\{A: A \in \Omega, n \in A\}) = \frac{1}{n^{1-\alpha}}$. The proof of Theorem 1 consists of three parts. In the first part we prove similarly as in [8] that with probability 1, $A$ is an asymptotic basis of order $2h+1$. In particular, we show that $R_{2h+1,A}(n)$ tends to infinity as $n$ goes to infinity. In the second part we show that deleting finitely many elements from $A$ we obtain a $B_h[1]$ set. Finally, we show that the above deletion does not destroy the asymptotic basis property.

By (3), to prove that $A$ is an asymptotic basis of order $2h+1$ it is enough to show $r_{2h+1,A}(n) > 0$ for every $n$ large enough. To do this, we apply the following lemma with $k = 2h+1$.

Lemma 3. Let $k \geq 2$ be a fixed integer and let $P(\{A: A \in \Omega, n \in A\}) = \frac{1}{n^{1-\alpha}}$ where $\alpha > \frac{1}{k}$. Then with probability 1, $r_{k,A}(n) > cn^{k\alpha-1}$ for every sufficiently large $n$, where $c = c(\alpha,k)$ is a positive constant.

The proof of Lemma 3 can be found in [8]. It is clear from (3) that
\[ P(\mathcal{E}) = 1, \quad (4) \]
where $\mathcal{E}$ denotes the event
\[ \mathcal{E} = \{A: A \in \Omega, \exists n_0(A) = n_0 \text{ such that } R_{2h+1,A}(n) \geq cn^{1/k} \text{ for } n > n_0\}, \]
where $c$ is a suitable positive constant. In the next step we prove that removing finitely elements from $A$ we get a $B_h[1]$ set with probability 1. To do this, it is enough to show that with probability 1, $R_{h,A}(n) \leq 1$ for every $n$ large enough. Note that in a representation of $n$ as the sum of $h$ terms there can be equal summands. To handle this situation we consider the terms of a representation $a_1 + \ldots + a_h = n$ as a vector $(a_1, \ldots, a_h) \in \mathbb{N}^h$. We denote the set which elements are the coordinates of the vector $\bar{x}$ as $\text{Set}(\bar{x})$. Of course, if two or more coordinates of $\bar{x}$ are equal, this value appears only once in $\text{Set}(\bar{x})$. We say that two vectors $\bar{x}$ and $\bar{y}$ are disjoint if $\text{Set}(\bar{x})$ and $\text{Set}(\bar{y})$ are disjoint sets. We define $r_{l,A}^*(n)$ as the maximum number of pairwise disjoint representations of $n$ as sum of $l$ elements of $A$, i.e., the maximum number of pairwise disjoint vectors of $R_l(n)$ with their coordinates in $A$. We say that $A$ is a $B_l^*[g]$ sequence if $r_{l,A}^*(n) \leq g$ for every $n$. 

3
Lemma 4. Let \( \mathbb{P}(\{A: A \in \Omega, n \in A\}) = \frac{1}{n^{\frac{1}{\alpha}}} \), where \( \alpha = \frac{2}{4h+1} \).

(i) For every \( 2 \leq k \leq h \) almost always there exists a finite set \( A_k \) such that \( r^*_{k,A\setminus A_k}(n) \leq 1 \).

(ii) For every \( h+1 \leq k \leq 2h \) almost always there exists a finite set \( A_k \) such that \( r^*_{k,A\setminus A_k}(n) \leq 4h+1 \).

Proof. We need the following proposition (see Lemma 3.7 in [2]).

Proposition 1. For a sequence \( A \in \Omega \), for every \( k \) and \( n \)
\[
\mathbb{P}(r^*_{k,A}(n) \geq s) \leq C_{k,\alpha,s} n^{(k\alpha-1)s}\]
where \( C_{k,\alpha,s} \) depends only on \( k, \alpha \) and \( s \).

We apply Proposition 1 by \( s = 2 \). Then we have
\[
\mathbb{P}(r^*_{k,A}(n) \geq 2) \leq C_{k,\alpha} n^{2(k\alpha-1)} = C_{k,\alpha} n^{-\frac{8h+4k+2}{4h+1}}.
\]
Since \( 2 \leq k \leq h \), we have
\[
\mathbb{P}(r^*_{k,A}(n) \geq 2) \leq n^{-\frac{4h+2}{4h+1}}
\]
then by the Borel-Cantelli lemma we get that almost always there exists an \( n_k \) such that \( r^*_{k,A}(n) \leq 1 \) for \( n \geq n_k \). It follows that
\[
r^*_{k,A\setminus A_k}(n) \leq 1,
\]
where \( A_k = A \cap [0,n_k] \).

Assume that \( h < k \leq 2h \). We apply Proposition 1 by \( s = 4h+2 \). Then we have
\[
\mathbb{P}(r^*_{k,A}(n) \geq 4h+2) \leq C_{k,h,\alpha} n^{(4h+2)(k\alpha-1)} = C_{k,\alpha} n^{-(2h+1)\frac{8h+4k+2}{4h+1}}.
\]
Since \( h < k \leq 2h \), we have
\[
\mathbb{P}(r^*_{k,A}(n) \geq 4h+2) \leq n^{-\frac{4h+2}{4h+1}}
\]
then by the Borel-Cantelli lemma we get that almost always there exists an \( n_k \) such that \( r^*_{k,A}(n) \leq 4h+1 \) for \( n \geq n_k \). It follows that
\[
r^*_{k,A\setminus A_k}(n) \leq 4h+1,
\]
where \( A_k = A \cap [0,n_k] \).

It follows from (4) and Lemma 4 that there exists a set \( A \) and for every \( 2 \leq k \leq h \) finite sets \( A_k \subset A \) such that
\[
R_{2h+1,A}(n) \geq cn^{\frac{1}{4h+1}}
\]
for \( n \geq n_0 \) and for every \( 2 \leq k \leq h \),
\[
r^*_{k,A\setminus A_k}(n) \leq 1,
\]
for every \( h < k \leq 2h \),
\[
r^*_{k,A}(n) \leq 4h+1.
\]
Set \( B = A \setminus \bigcup_{k=1}^{2h} A_k \). In the next step we show that \( B \) is both a \( B_h[1] \) set and a \( B_{2h}[g] \) set for some \( g \). We apply the following proposition (see Remark 3.10 in [2]).
Proposition 2.

\[ B^*_h[g] \cap B_{h-1}[l] \subseteq B_h[g(h(l - 1) + 1)]. \]

By using the definition of \( B \), the fact that \( B^*_2[1] = B_2[1] \) and (6), (7) it follows that

\[ B \in B_2[1] \cap B^*_3[1] \cap \ldots \cap B^*_h[1] \cap B^*_{h+1}[4h + 1] \cap \ldots \cap B^*_{2h}[4h + 1]. \]

Applying Proposition 2 with \( g = l = 1 \) we get by induction that for every \( 2 \leq s \leq h \) if \( B \in B^*_s[1] \cap B_{s-1}[1] \) then \( B \in B_s[1] \), thus \( B \) is a \( B_h[1] \) set. Applying Proposition 2 with \( g = 4h + 1, l = 1 \) we get that \( B \in B_{h+1}[4h+1] \). Using Proposition 2 again with \( g = 4h + 1, l = 4h + 1 \) we get that if \( B \in B^*_{h+2}[4h+1] \cap B_{h+1}[4h+1] \) then \( B \in B_{h+2}[(4h+1)(h-4h+1)] \).

Continuing this process we obtain that for every \( 1 < k \leq 2h \) we have \( B \in B_k[g_k] \) for some positive integer \( g_k \). Let \( \cup_{k=1}^{2h} A_k = \{d_1, \ldots, d_w\} \) \((d_1 < \ldots < d_w)\). Now we show that \( A \in B_{2h}[G] \) where

\[ G = 2^w \cdot \max_{1<k\leq2h} g_k. \]

We prove by contradiction. Assume that there exists a positive integer \( n \) with \( R_{2h,A}(n) > 2^w \cdot \max_{1<k\leq2h} g_k \). Then there exist indices \( 1 \leq i_1 < i_2 < \ldots < i_j \leq w \) such that the number of representations of \( n \) in the form \( n = d_{i_1} + \ldots + d_{i_j} + c_{j+1} + \ldots + c_{2h} \), where \( c_{j+1}, \ldots, c_{2h} \in B \) is more than \( \max_{1<k\leq2h} g_k \). It follows that

\[ R_{2h-j,B}(n - (d_{i_1} + \ldots + d_{i_j})) > \max_{1<k\leq2h} g_k \geq g_{2h-j} \]

which is a contradiction.

Finally, we prove similarly as in [7] that \( B \) is an asymptotic basis of order \( 2h + 1 \), i.e., the deletion of the “small” elements of \( A \) does not destroy its asymptotic basis property. We prove by contradiction. Assume that there exist infinitely many positive integers \( M \) which cannot be represented as the sum of \( 2h + 1 \) numbers from \( B \). Choose such an \( M \) large enough. In view of (5), we have \( R_{2h+1,A}(M) > cM^{\frac{1}{2h+1}} \). It follows from our assumption that every representations of \( M \) as the sum of \( 2h + 1 \) numbers from \( A \) contains at least one element from \( A \setminus B = \cup_{k=1}^{2h} A_k \). Then by the pigeon hole principle there exists an \( y \in \cup_{k=1}^{2h} A_k \) which is in at least \( \frac{R_{2h+1,A}(M)}{w} \) representations of \( M \). As \( A \in B_{2h}[G] \), it follows that with probability \( 1 \),

\[ \frac{c_3 M^{\frac{1}{2h+1}}}{w} < \frac{R_{2h+1,A}(M)}{w} \leq R_{2h,A}(M - y) \leq G, \]

which is a contradiction if \( M \) is large enough.

4 Acknowledgement

Supported by the ÚNKP-18-4 New National Excellence Program of the Ministry of Human Capacities.
References

[1] J. Cilleruelo. *On Sidon sets and asymptotic bases*, Proc. Lond. Math. Soc., **111** (2015), 1206-1230.

[2] J. Cilleruelo, S. Z. Kiss, I. Z. Ruzsa, C. Vinuesa. *Generalization of a theorem of Erdős and Rényi on Sidon sequences*, Random Structures and Algorithms, **37** (2010), 455-464.

[3] J. M. Deshouillers, A. Plagne. *A Sidon basis*, Acta Mathematica Hungarica, **123** (2009), 233-238.

[4] P. Erdős, A. Sárközy, V. T. Sós. *On additive properties of general sequences*, Discrete Mathematics, **136** (1994), 75-99.

[5] P. Erdős, A. Sárközy, V. T. Sós. *On sum sets of Sidon sets I.*, Journal of Number Theory, **47** (1994), 329-347.

[6] H. Halberstam, K. F. Roth. *Sequences*, Springer - Verlag, New York, 1983.

[7] S. Z. Kiss. *On Sidon sets which are asymptotic bases*, Acta Mathematica Hungarica, **128** (2010), 46-58.

[8] S. Z. Kiss. *On generalized Sidon sets which are asymptotic bases*, Annales Univ. Sci. Budapest. Eötvös, **57** (2014), 149-160.

[9] S. Z. Kiss, E. Rozgonyi, Cs. Sándor. *On Sidon sets which are asymptotic bases of order 4*, Functiones et Approximatio Comm. Math., **51** (2014), 393-413.