Finite temperature amplitudes and reaction rates in Thermofield dynamics.

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Abstract

We propose a method for calculating the reaction rates and transition amplitudes of generic processes taking place in a many body system in equilibrium at temperature $T$. The relationship of the scattering and decay amplitudes as calculated in Thermo Field Dynamics to the conventional techniques is established. It is shown that in many cases the calculations are relatively easy in TFD.

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I. INTRODUCTION

Recent experimental developments in heavy ion collision experiments have lead to the formation of hadronic matter at finite temperature with high densities. At collisions with relativistic heavy ions at facilities like RHIC at Brookhaven and beams at SPS, and eventually at LHC, lead to the deconfinement of quarks from the confined state in hadrons forming a quark gluon plasma. The phase transition is anticipated at temperature of \( T_c \approx 150 \text{MeV} - 250 \text{MeV} \). The quark gluon plasma above the critical temperature \( T_c \) and the hot and dense hadronic matter below \( T_c \) possess special problems to investigate their properties at finite temperature and high densities. We have to consider interaction of hadrons, the decay of mesons and neutrons and finally the behavior of quark-quark interactions and the screening effect in the quark gluon plasma (QGP). Such a study requires the development of a formalism of quantum field theory to deal with problems at finite temperature and high densities.

It should be stressed that, developments to deal with hot and dense nuclear matter and hot QGP would provide valuable tools to study early universe as well as the interior of stars. Of course there is a range of problems that would require time dependent treatment since the system may be in a non-equilibrium state that is changing rapidly with time. In general, therefore, it appears that both early Universe and relativistic heavy ion reactions would require a formalism that depends on time and temperature simultaneously.

We wish to explore methods to calculate the reaction rates and decay rates in a medium at temperature \( T \) using the real time finite temperature field theory - Thermo Field Dynamics (TFD) \([1]\). This theory was developed so that the time degree of freedom is not lost \([2]\). It was realized that, an extension of the usual field theory at zero temperature to finite temperatures would require doubling of the Hilbert space. At the same time the operators have to be doubled such that the new set of operators, designated tilde operators, act on the second Hilbert space, tilde space. Later on it was interpreted that, the second Hilbert space acts like a heat bath that ensures the dynamical system to stay at constant temperature \( T \). It is established that with TFD the Wick’s theorem can be applied and other features
like Ward - Takahshsi relations and Nambu - Goldstone theorem on spontaneously breaking symmetry can be established. Furthermore as in the case of the usual field theory at zero temperature, Feynman rules for scattering amplitude and decay rates can be established. Therefore the combination of all these feature in TFD allows us to carry out calculations as in the case of $T = 0$ field theory. Some of the essential aspects such as a definition of the vacuum and a definition of the creation and annihilation operators at finite temperature are given in the Appendix. It should be mentioned that for systems in equilibrium there are two other methods available. The first is the imaginary time method due to Matsubara [3] that has been extended to quantum field theory by Jackiw and Dolan [4]. The second method is the closed time path method due to Schwinger and Keldysh [5]. Both these methods are essentially based on Green’s function approach. Recently Niegawa [6] has developed a technique to use the closed time method to calculate the scattering amplitude and decay rates. We’ll compare this method with that of TFD later on.

It should be stressed that what has been done is to write down the decay amplitudes by using the method of Cutkosky to find the imaginary part of an amplitude. These rules are very helpful in our investigation.

Recently Jeon and Ellis [6] have developed a multiple scattering expansion of the self energy at finite temperature in the imaginary time method. They have calculated the response function in this approach. Brandt et al [6] have calculated the retarded thermal Green’s function and forward scattering amplitude at two loops. Using time-dependent two- and three-point functions in scalar theory they have calculated i) the forward scattering amplitude of two on-shell thermal particles and ii) forward scattering amplitude of a single on-shell thermal particle. The calculations include one-loop self energy and vertex corrections that are retarded but are at zero temperature. The present work attempts to calculate scattering and decay amplitudes at finite temperatures.

The paper is organised as follows. In sect. II we briefly present basic TFD formulas, which will be necessary in further calculations. In sect. III we describe our basic formalism. In sections IV, V and VI we consider decay and scattering processes respectively. In sect
II. THERMAL FIELDS AND S - MATRIX IN TFD.

The Thermo Field Dynamics is a real time operator formalism of quantum field theory at finite temperature. Any physical state can be constructed from a temperature dependant vacuum \( |0(\beta)\rangle \) which is a pure state. The main feature of TFD is that, the thermal average of any operator \( A \) is equal to its temperature dependent vacuum expectation value with the: vacuum, \( |0(\beta)\rangle \), being obtained from the usual vacuum by a Bogoliubov transformation. Therefore, we have

\[
<A> = \langle 0(\beta)|A|0(\beta)\rangle
\]

(2.1)

where \( \beta \equiv 1/k_B T \) with \( k_B \) being the Boltzmann constant. Doubling of degrees of freedom of all fields is required. This is achieved through a ”tilde” operation: to each zero temperature field \( \phi(x) \) a doublet of fields \( (\phi(x), \tilde{\phi}(x)) \) is attached, the dynamics of which is controlled by a thermal Lagrangian

\[
\hat{L} = L(\phi) - \tilde{L}(\phi) = L(\phi) - L^*(\tilde{\phi})
\]

(2.2)

The temperature dependent vacuum state \( |0(\beta)\rangle \) is annihilated by the temperature dependent physical annihilation operators

\[
a_k(\beta)|0(\beta)\rangle = c_k(\beta)|0(\beta)\rangle = d_k(\beta)|0(\beta)\rangle = 0
\]

\[
\tilde{a}_k(\beta)|0(\beta)\rangle = \tilde{c}_k(\beta)|0(\beta)\rangle = \tilde{d}_k(\beta)|0(\beta)\rangle = 0
\]

(2.3)

which are obtained through Bogoliubov transformation from usual annihilation, \( a_k \), and creation operators, \( a_k^\dagger \). For a Bose system

\[
a_k = \cosh \theta_k a_k(\beta) + \sinh \theta_k \tilde{a}_k(\beta), \quad a_k^\dagger = \cosh \theta_k a_k^\dagger(\beta) + \sinh \theta_k \tilde{a}_k^\dagger(\beta),
\]

\[
\tilde{a}_k^\dagger = \sinh \theta_k a_k(\beta) + \cosh \theta_k \tilde{a}_k(\beta), \quad \tilde{a}_k = \sinh \theta_k a_k^\dagger(\beta) + \cosh \theta_k \tilde{a}_k^\dagger(\beta),
\]

(2.4)

where

\[
sinh^2 \theta_k = n_B(k) = 1/(e^{\beta |k_0|} - 1), \quad \cosh \theta_k = e^{\beta k_0}/2 \sinh \theta_k.
\]

(2.5)
with \( k_0 \) being the energy associated with the four-vector, \( k \). The creation and annihilation operators satisfy ordinary commutation relations

\[
[a_k, a^\dagger_p] = [\bar{a}_k, \bar{a}^\dagger_p] = [a_k(\beta), a^\dagger_p(\beta)] = [\bar{a}_k(\beta), \bar{a}^\dagger_p(\beta)] = \delta(\vec{k} - \vec{p})
\]  

(2.6)

and all other commutators vanish.

For fermions, with the creation, \( c^\dagger_{p,s} \) and \( d^\dagger_{p,s} \), and annihilation, \( c_{p,s} \) and \( d_{p,s} \) operators the Bogoliubov transformations lead to the relations

\[
c_{p,s} = \cos \theta_p c_{p,s}(\beta) + i \sin \theta_p^\dagger d_{p,s}(\beta), \quad d_{p,s} = \cos \theta_p^\dagger d_{p,s}(\beta) + i \sin \theta_p c_{p,s}(\beta), \\
c^\dagger_{p,s} = i \sin \theta_p c^\dagger_{p,s}(\beta) + \cos \theta_p^\dagger d^\dagger_{p,s}(\beta), \quad d^\dagger_{p,s} = i \sin \theta_p^\dagger d^\dagger_{p,s}(\beta) + \cos \theta_p c^\dagger_{p,s}(\beta), \\
c^\dagger_{p,s} = -i \sin \theta_p^\dagger c^\dagger_{p,s}(\beta) + \cos \theta_p d^\dagger_{p,s}(\beta), \quad d^\dagger_{p,s} = -i \sin \theta_p d^\dagger_{p,s}(\beta) + \cos \theta_p^\dagger c_{p,s}(\beta),
\]  

(2.7)

where

\[
\sin^2 \theta_p = 1/(1 + e^{\beta(p_0 + \mu)}), \quad \cos \theta_p = e^{\beta p_0} (p_0 \pm \mu)/2 \sin \theta_p.
\]  

(2.8)

Here \( p_0 \) is the energy associated with the four-vector, \( p \) and \( \mu \) is the chemical potential. The anticommutation relations for creation and annihilation operators (c- for particles, and d - for antiparticles) are similar to those at zero temperature:

\[
[c_{p,s}' c^\dagger_{p,s}]_+ = [d_{p,s}' d^\dagger_{p,s}]_+ = [c_{p,s}'(\beta) c^\dagger_{p,s}(\beta)]_+ = [d_{p,s}'(\beta) d^\dagger_{p,s}(\beta)]_+ = \delta(\vec{p} - \vec{p}') \delta_{ss'} \\
[c^\dagger_{p,s}' c_{p,s}]_+ = [d^\dagger_{p,s}' d_{p,s}]_+ = [c^\dagger_{p,s}'(\beta) c_{p,s}(\beta)]_+ = [d^\dagger_{p,s}'(\beta) d_{p,s}(\beta)]_+ = \delta(\vec{p} - \vec{p}') \delta_{ss'}
\]  

(2.9)

All other anti-commutators are zero. In order to define the reaction rates and decay widths, we follow the prescription due to Feynman [1], which is best stated by Dyson [3], and this involves writing down an operator:

\[
H_F(x_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_n T[H^c(x_0) H_I(x_1) H_I(x_2) \ldots H_I(x_n)],
\]  

(2.10)

Then choosing \( H^c(x_0) = 1 \) this reduces to an expression that is evaluated between an initial and a final state. Therefore we get an expression that looks like and is equal to an \( S \) matrix
\[
S = \sum_{n=0}^{\infty} S^n = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 dx_2 \ldots dx_n T[H_I(x_1)H_I(x_2) \ldots H_I(x_n)], \tag{2.11}
\]

For the total transition in TFD we have
\[
\hat{S} = \sum_{n=0}^{\infty} \hat{S}^n = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 dx_2 \ldots dx_n T[\tilde{H}_I(x_1)\tilde{H}_I(x_2) \ldots \tilde{H}_I(x_n)], \tag{2.12}
\]

where \(\tilde{H}_I(x) = H_I(x) - \tilde{H}_I(x)\). Particularly, at the tree level
\[
\hat{S} \approx 1 - i \int d^4x (H_I(x) - \tilde{H}_I(x)) \tag{2.13}
\]

Now in these expressions \(H_I(x)\) and \(\tilde{H}_I(x)\) may be introduced explicitly and then Wick’s theorem is used as in the case of \(T = 0\) field theory to obtain all the contributions in perturbation theory. The evaluation proceeds by using a set of Feynman rules for evaluation of a reaction rate and decay widths. It is clear that for operators appearing in \(H_I(x)\) and \(\tilde{H}_I(x)\), the transformation of boson and Fermion operators given by Eqs. (2.4) and (2.7) respectively has to be used to bring in the temperature dependent factors. The one particle Green’s functions are given explicitly in the Appendix.

### III. CALCULATION OF REACTION RATES

Let us consider the process
\[
p_1 + p_2 + \ldots + p_n \rightarrow p'_1 + p'_2 + \ldots + p'_m \tag{3.1}
\]

At zero temperature, \(T = 0\), the amplitude of this process can be calculated by usual Feynman rules:
\[
\langle f | \hat{S} | i \rangle = \sum_{n=0}^{\infty} \langle f | \hat{S}^{(n)} | i \rangle \tag{3.2}
\]

where \(|i\rangle = a_{p_1}^\dagger a_{p_2}^\dagger \ldots a_{p_n}^\dagger |0\rangle\), \(|f\rangle = a_{p_1}^\dagger a_{p_2}^\dagger \ldots a_{p_m}^\dagger |0\rangle\) and \(|0\rangle\) is an ordinary vacuum state: \(a_p |0\rangle = 0\). In the present article we propose that the amplitude and, hence the cross section, of process (3.1) can be directly calculated perturbatively using Eqs. (2.12) at \(T \neq 0\) also. Namely, we propose that
1) The transition amplitudes are determined from Eq. (2.12) where for the process (3.1) we have
\[ |i⟩ = a_p^\dagger (β)a_p^\dagger (β)\ldots a_p^\dagger (β)0(β) \]
and
\[ |f⟩ = a_p'^\dagger (β)a_p'^\dagger (β)\ldots a_p'^\dagger (β)0(β) \].

2) The phase space factors, relating an amplitude to the rate are the same as in zero temperature case. For example, the differential cross section for the process
\[ p_1 + p_2 → p'_1 + p'_2 + \ldots + p'_m \]
is
\[ dσ = \frac{1}{4E_1E_2v_{rel}} \prod_i (2m_i) \prod_i^{m} \frac{dp_i'}{(2\pi)^32E_i}|M_{fi}|^2 \]  (3.3)

where \( E_p = \sqrt{m_i^2 + (\vec{p})^2} \) and \( v_{rel} \) is the relative velocity of the two initial particles with momenta \( p_1 \) and \( p_2 \). The amplitude is related to \( \hat{S} \) matrix in manner similar to the procedure carried out at zero temperature
\[ \langle f|\hat{S}|i⟩ = i(2\pi)^4δ^4(P_f - P_i)M_{fi} \prod_{ext}[(m_i\sqrt{E_i})^{1/2}] \prod_{ext}[(m_i\sqrt{\omega})^{1/2}] \].  (3.4)

Here \( P_i \) and \( P_f \) are the total four momenta in the initial and final states, and the products extend over all external fermions and bosons, \( E \) and \( ω \) being the energies of the individual external fermions and bosons respectively, and \( V \) is the volume. We propose that, in TFD this definition still holds.

### IV. DECAY OF PARTICLES

We know that at \( T = 0 \) the decay rates \( Γ \) can be calculated either by studying discontinuities of the self energy of a decaying particle (Cutkosky rules [11]) or by direct calculation of the amplitude perturbatively (Feynman rules). Obviously these two methods must lead to the same result.

For finite temperature the Cutkosky rules were generalized by Kobes and Semenoff [12]. In the next section we shall calculate the decay of a particle at finite temperature
a) by using generalized Cutkosky rules (GCR); and also
b) directly as a square of module of elements of \( \hat{S} \) matrix, i.e. by using the above method.
It will be established that, these two methods give the same result for \( Γ \). We shall present these to establish their equivalence.
A. \( \sigma(k) \rightarrow \pi^0(k_1) + \pi^0(k_2) \)

Firstly, we calculate the decay rate of sigma meson into two pions (we’ll omit isospin indices for simplicity) in real time formalism, using GCR. The interaction Lagrangian is 
\[ L_{\text{int}} = \frac{\lambda}{2} \sigma(x) \pi^2(x). \]

The decay rate of a boson with mass \( M \) and four momentum \( k = (\omega, \vec{k}) \) with \( \omega = \sqrt{M^2 + (\vec{k})^2} \), is related to the self energy \[\Sigma(\omega)\] by
\[
\Gamma_{GCR}(w) = -\frac{1}{\omega} Im \Sigma(\omega) = -\frac{(e^{\beta \omega} - 1)}{\omega(e^{\beta \omega} + 1)} Im \Sigma_{11}(k) \quad (4.1)
\]
where
\[
i \Sigma_{11}(k) = \lambda^2 \int \frac{dp^4}{(2\pi)^4} i \Delta_1^0(p) i \Delta_1^0(p - k) \quad (4.2)
\]
The GCR \[\Sigma_{11}(k)\], as illustrated in Fig. 1 give:
\[
2 Im \Sigma_{11}(k) = \lambda^2 \int \frac{dp^4}{(2\pi)^4} [i \Delta^+(p) i \Delta^-(p - k) + i \Delta^-(p) i \Delta^+(p - k)] \quad (4.3)
\]
where \( i \Delta^\pm(p) = 2\pi[\Theta(\pm p_0) + n_B(p)]\delta(p^2 - m^2). \) The two terms in the integrand are related by
\[
i \Delta^\pm(p) = ie^{\pm \beta p_0} \Delta^\mp(p) \quad (4.4)
\]
from \[(4.1) - (4.4)\) we get:
\[
\Gamma_{GCR}(w) = \frac{(e^{\beta \omega} - 1)(e^{-\beta \omega} + 1)\lambda^2}{2\omega(e^{\beta \omega} + 1)} \int \frac{dp^4}{(2\pi)^4} (2\pi)^2 \delta(p^2 - m^2) \delta((p - k)^2 - m^2) \times \]
\[
\times[\Theta(p_0) + n_B(p_0)][\Theta(-p_0 + \omega) + n_B(p_0 - \omega)] \quad (4.5)
\]
We shall proceed and evaluate \( \Gamma_{GCR} \) explicitly. Note, however that, at finite temperature Lorentz invariance is lost, and hence, the decay rate will no longer be Lorentz invariant, but dependent on the reference frame. In the following examples we will choose the rest frame of the decaying particle, as the reference frame, that is we set \( \vec{k} = 0 \). First, we consider the product of the mass shell \( \delta \) functions. With \( k = (M, 0, 0, 0) \) the compatible zeros are easily found to be \( p_0 = \omega_p = M/2 \). Hence the delta functions reduce to:

8
\[
\delta(p^2 - m^2)\delta((p - k)^2 - m^2) = \frac{1}{4M\omega_p}\delta(p_0 - \omega_p)\delta(\omega_p - M/2)
\]

(4.6)

and fix the momentum dependence of the integrand completely.

The thermal factors in (4.5) can be reduced on mass shell: \( p_0 = M/2, \omega = M \)

\[
\frac{(e^{\beta \omega} - 1)(e^{-\beta \omega} + 1)}{(e^{\beta \omega} + 1)}[\Theta(p_0) + n_B(p_0)][\Theta(-p_0 + \omega) + n_B(p_0 - \omega)] =
\]

\[
= (1 - e^{-\beta \omega})(1 + n_B(p_0))[1 + n_B(p_0 - \omega)] = (1 - e^{-\beta M})(1 + n_B(M/2))^2
\]

(4.7)

now inserting (4.6), (4.7) into (4.5) we get:

\[
\frac{\Gamma_{GCR}(T \neq 0)}{\Gamma(T = 0)} = \frac{(1 + n_B(M/2))^2}{1 + n_B(M)} = (1 + n_B(M/2))^2 - n_B^2(M/2)
\]

(4.8)

Here we used the simple relations:

\[
e^{-\beta M} = e^{-\beta M/2}e^{-\beta M/2}
\]

\[
e^{-\beta \omega} = n_B(\omega)/(1 + n_B(\omega))
\]

(4.9)

which will be useful for further calculations.

Now we calculate the rate \( \Gamma(\sigma \rightarrow \pi\pi) \) directly. Because of the doubling of degrees of freedom the whole interaction Lagrangian will be \( \tilde{\mathcal{L}}_{\text{int}} = \lambda\sigma(x)\pi^2(x) - \lambda\tilde{\sigma}(x)\tilde{\pi}^2(x) \). The initial and final states are \( |i\rangle = a_k^\dagger(\beta)|0(\beta)\rangle \) and \( |f\rangle = b_{k_1}(\beta)b_{k_2}(\beta)|0(\beta)\rangle \) where operators \( a(\beta) \) and \( b(\beta) \) stand for \( \sigma \) and \( \pi \) fields respectively. At the tree level (??) the matrix element of \( \hat{S} \) matrix is given by

\[
\langle f|\hat{S}|i\rangle = i\lambda \int dx(0(\beta)|b_{k_1}(\beta)b_{k_2}(\beta)[\sigma(x)\pi^2(x) - \tilde{\sigma}(x)\tilde{\pi}^2(x)]a_k^\dagger(\beta)|0(\beta)\rangle
\]

(4.10)

where the boson fields \( \sigma(x) \) and \( \pi(x) \) are defined as in eq. (A.22). Using Bogoluibov transformations (2.4) and commutation relations (2.6) it is easy to show that:

\[
\langle 0(\beta)|\sigma(x)a_k^\dagger(\beta)|0(\beta)\rangle = e^{-ikx}\cosh \theta_k,
\]

\[
\langle 0(\beta)|\tilde{\sigma}(x)a_k^\dagger(\beta)|0(\beta)\rangle = e^{-ikx}\sinh \theta_k,
\]

(4.11)

\[
\langle 0(\beta)|b_{k_1}(\beta)b_{k_2}(\beta)\pi^2(x)|0(\beta)\rangle = e^{i(k_1 + k_2)x}\cosh \theta_{k_1}\cosh \theta_{k_2}
\]

\[
\langle 0(\beta)|b_{k_1}(\beta)b_{k_2}(\beta)\tilde{\pi}^2(x)|0(\beta)\rangle = e^{i(k_1 + k_2)x}\sinh \theta_{k_1}\sinh \theta_{k_2}.
\]

From Eqs (4.10), (4.11) and (2.12) we obtain the amplitude:
\[ \mathcal{M}_{fi}(T) = \lambda [\cosh \theta_k \cosh \theta_{k_1} \cosh \theta_{k_2} - \sinh \theta_k \sinh \theta_{k_1} \sinh \theta_{k_2}] . \quad (4.12) \]

The decay rate, can be calculated directly from (4.12) without introducing any additional thermal factor:

\[
\Gamma(\omega) = \frac{1}{2\omega} \int \frac{d\vec{k}_1 d\vec{k}_2 (2\pi)^4 \delta(k - k_1 - k_2)|\mathcal{M}_{fi}(T)|^2}{(2\omega_1)(2\omega_2)(2\pi)^3(2\pi)^3} = \]

\[
= \frac{\lambda^2}{32\omega\pi^2} \int \frac{d\vec{k}_1 d\vec{k}_2 \delta^4(k - k_1 - k_2)W_B(\omega, \omega_1, \omega_2)}{\omega_1 \omega_2} = \frac{\lambda^2 I_B(T)}{32\omega\pi^2} \quad (4.13) \]

where

\[
W_B(\omega, \omega_1, \omega_2) = [\cosh \theta_k \cosh \theta_{k_1} \cosh \theta_{k_2} - \sinh \theta_k \sinh \theta_{k_1} \sinh \theta_{k_2}]^2 , \quad (4.14) \]

\[
\omega_i = \sqrt{(\vec{k}_i)^2 + m^2} \quad \text{and} \quad \omega = \sqrt{(\vec{k})^2 + M^2} . \]

Now using the properties of the hyperbolic functions and Eqs (2.5), and (1.9) this may be further simplified as follows:

\[
\delta(\omega - \omega_1 - \omega_2)W_B(\omega, \omega_1, \omega_2) = n_1 n_2 n_\omega [e^{\beta(\omega + \omega_1 + \omega_2)/2} - 1]^2 \delta(\omega - \omega_1 - \omega_2) = \]

\[
= n_1 n_2 n_\omega [e^{\beta \omega} - 1] [e^{\beta(\omega_1 + \omega_2 - 1)} - 1] \delta(\omega - \omega_1 - \omega_2) = \]

\[
= n_1 n_2 n_\omega \left\{ \frac{1 + n_\omega}{n_\omega} - 1 \right\} \left\{ \frac{(1 + n_1)(1 + n_2)}{n_1 n_2} - 1 \right\} \delta(\omega - \omega_1 - \omega_2) = \]

\[
= (1 + n_1 + n_2) \delta(\omega - \omega_1 - \omega_2) \quad (4.15) \]

where for simplicity we denote \( n_i \equiv n_B(k_i) \), \( i = 1, 2 \), and \( n_\omega \equiv n_B(k) \).

The integral \( I_B(T) \) in Eq. (4.13) can be explicitly calculated in the rest frame of the decaying particle: \( \omega = M, \vec{k} = 0, \omega_i = \sqrt{(\vec{k}_i)^2 + m^2} = \sqrt{(q)^2 + m^2} = \omega_q \):

\[
I_B(T) \equiv \int \frac{d\vec{k}_1 d\vec{k}_2 \delta^4(k - k_1 - k_2)W_B(\omega, \omega_1, \omega_2)}{\omega_1 \omega_2} = \]

\[
= 4\pi \int \frac{dqq^2 \delta(2\omega_q - M)W_B(M, \omega_q, \omega_q)}{\omega_q^2} = 8\pi \sqrt{1/4 - (m/M)^2} W_B(M, M/2, M/2) \quad (4.16) \]

using (4.15) and (4.16) in (4.13) we finally obtain:

\[
\frac{\Gamma(T \neq 0)}{\Gamma(T = 0)} = W_B(M, M/2, M/2) = (1 + 2n_B(M/2)) \quad (4.17) \]

this formulae is the same as obtained by the GCR method eq. (1.8)
B. \( H(k) \rightarrow e^-(k_1) + e^+(k_2) \)

As a second example we consider the decay of a boson into two fermions, namely, the decay of Higgs boson into electron-positron pair. The decay rate of this process has been calculated in detail using GCR by Keil [14]. The final tree level result is:

\[
\frac{\Gamma_{GCR}(T \neq 0)}{\Gamma(T = 0)} = \frac{(1 - n_F(M/2))(1 - \bar{n}_F(M/2))}{1 + n_B(M)}
\]

(4.18)

which is obtained in the rest frame of \( H \) boson with a mass \( M \).

Now we evaluate the same rate by direct calculation of the amplitude. The interaction Lagrangian is

\[
\hat{L}_{int} = L_{int} - \tilde{L}_{int}, \quad L_{int} = -igH(x)\bar{\psi}(x)\psi(x), \quad \tilde{L}_{int} = +ig\tilde{H}(x)\bar{\psi}(x)\psi(x).
\]

(4.19)

The matrix element of \( \hat{S} \) matrix from an initial state \( |i⟩ = a^\dagger_k(\beta)|0(\beta)⟩ \) to the final state \( |f⟩ = c^\dagger_{k_1}(\beta)d^\dagger_{k_2}(\beta)|0(\beta)⟩ \) at the tree level is

\[
\langle f|\hat{S}|i⟩ = i(-ig)\int dx^4 \langle i|[H(x)\bar{\psi}(x)\psi(x) + \tilde{H}(x)\bar{\psi}(x)\psi(x)]|f⟩
\]

(4.20)

where we omit the spin indices for simplicity. As in previous example, Eq. (4.11), we have

\[
\langle f|H(x)|i⟩ = e^{-ikx}\cosh \theta_k, \quad \langle f|\tilde{H}(x)|i⟩ = e^{-ikx}\sinh \theta_k.
\]

(4.21)

For the fermions, using (A23) in (4.20) we have

\[
\langle f|\bar{\psi}(x)\psi(x)|i⟩ = \int d\vec{p}d\vec{p}'N_pN'_p\langle 0(\beta)|c^\dagger_{k_1}(\beta)d_{k_2}(\beta)[c^\dagger_p\bar{u}(p)e^{ipx} + d_p\bar{v}(p)e^{-ipx}]×
\]

\[
\times[c_{p'}u(p')e^{-ip'x} + d^\dagger_{p'}v(p')e^{ip'x}]|0(\beta)⟩
\]

(4.22)

and

\[
\langle f|\bar{\psi}(x)\tilde{\psi}(x)|i⟩ = \int d\vec{p}d\vec{p}'N_pN'_p\langle 0(\beta)|c^\dagger_{k_1}(\beta)d_{k_2}(\beta)[\tilde{c}^\dagger_p\bar{u}(p)e^{-ipx} + \tilde{d}_p\bar{v}(p)e^{ipx}]×
\]

\[
\times[\tilde{c}_{p'}u(p')e^{ip'x} + \tilde{d}^\dagger_{p'}v(p')e^{-ip'x}]|0(\beta)⟩
\]

(4.23)

The only "surviving" term in (4.22) is that one which includes
\[
\langle 0(\beta) \vert c_k(\beta) d_{k_2}(\beta) c_{p'}^\dagger d_{p'}^\dagger \vert 0(\beta) \rangle = \langle 0(\beta) \vert c_k(\beta) d_{k_2}(\beta) [\cos \theta_{+p'}c_{p'}^\dagger(\beta) - i \sin \theta_{+p'}c_{p'}(\beta)] = \\
[\cos \theta_{-p'}d_{p'}^\dagger(\beta) - i \sin \theta_{-p'}d_{p'}(\beta)] \vert 0(\beta) \rangle = \langle 0(\beta) \vert c_k(\beta) d_{k_2}(\beta) c_{p'}^\dagger(\beta) d_{p'}^\dagger(\beta) \vert 0(\beta) \rangle \times (4.24)
\]
\[
\times \cos \theta_{+p'} \cos \theta_{-p'} \cos \theta_{+k_1} \cos \theta_{-k_2} \delta(\vec{k}_1 - \vec{p}) \delta(\vec{k}_2 - \vec{p'}).
\]
Similarly, the nonzero contribution to (4.23) involves
\[
\langle 0(\beta) \vert c_k(\beta) d_{k_2}(\beta) \bar{c}_{p'} \vert 0(\beta) \rangle = \langle 0(\beta) \vert c_k(\beta) d_{k_2}(\beta) [\cos \theta_{-p'}d_{p'}(\beta) - i \sin \theta_{-p'}d_{p'}^\dagger(\beta)] \times \\
[\cos \theta_{+p'}c_{p'}(\beta) - i \sin \theta_{+p'}c_{p'}^\dagger(\beta)] \vert 0(\beta) \rangle = (4.25)
\]
\[
= -\langle 0(\beta) \vert c_k(\beta) d_{k_2}(\beta) c_{p'}^\dagger(\beta) d_{p'}(\beta) \vert 0(\beta) \rangle \sin \theta_{-p} \sin \theta_{+p'}
\]
\[
= -\sin \theta_{+k_1} \sin \theta_{-k_2} \delta(\vec{k}_1 - \vec{p}) \delta(\vec{k}_2 - \vec{p'}).
\]

The decay of the process is given by a general formula of the zero temperature field theory:
\[
\Gamma(\omega) = \frac{1}{2\omega} \int \frac{d\vec{k}_1 d\vec{k}_2 (2\pi)^4 \delta^4(k - k_1 - k_2)(2m)^2 |M_{fi}(T)|^2}{(2\omega_1)(2\omega_2)(2\pi)^3(2\pi)^3}
\]
(4.26)
where the transition amplitude is given by Eqs (3.4), and for the present case it has the form
\[
M_{fi}(T) = (-ig) [\cos \theta_{k_1} \cos \theta_{-k_2} \cosh \theta_\omega \bar{u}(k_1)v(k_2) - \sin \theta_{k_1} \sin \theta_{-k_2} \sinh \theta_\omega \bar{v}(k_2)\bar{u}(k_1)].
\]
(4.27)

Then Eq. (4.26) can be further simplified by using
\[
\sum_{\text{spins}} |M_{fi}(T)|^2 = g^2 [\cos \theta_{k_1} \cos \theta_{-k_2} \cosh \theta_\omega - \sin \theta_{k_1} \sin \theta_{-k_2} \sinh \theta_\omega]^2 Tr \frac{(k_1 + m)(k_2 + m)}{2m}
\]
(4.28)
and
\[
\delta(\omega - \omega_1 - \omega_2)[\cos \theta_{k_1} \cos \theta_{-k_2} \cosh \theta_\omega - \sin \theta_{k_1} \sin \theta_{-k_2} \sinh \theta_\omega]^2 = \\
= \delta(\omega - \omega_1 - \omega_2) \cos^2 \theta_{k_1} \cos^2 \theta_{-k_2} \cosh^2 \theta_\omega (1 - e^{-\beta(\omega + \omega_1 + \omega_2)})^2 = \\
= \delta(\omega - \omega_1 - \omega_2)(1 - n_F(k_1)(1 - \bar{n}_F(k_2))(1 - e^{-\beta \omega}) \equiv \delta(\omega - \omega_1 - \omega_2)W_F(\omega, \omega_1, \omega_2).
\]
(4.29)

Now, inserting (4.27) - (4.29) into (4.26) and performing integration in the rest frame of the decaying boson we obtain
\[
\frac{\Gamma(T \neq 0)}{\Gamma(T = 0)} = W_F(M, M/2, M/2) = (1 - n_F(M/2))(1 - \bar{n}_F(M/2))(1 - e^{-\beta M}) \quad (4.30)
\]
which is exactly the same as Eq. (4.18) given by GCR, since \((1 - e^{-\beta \omega}) = (\cosh^2 \theta_\omega)^{-1} = (1 + n_B(\omega))^{-1} .

C. \(\Phi(k) \rightarrow \phi(k_1) + \phi(k_2) + \phi(k_3)\)

The next example is the decay rate of a boson \(-\Phi(x)\) into three bosons \(-\phi(x)\) with the interaction Lagrangian \(L_{\text{int}} = -\frac{\lambda}{3!} \Phi(x)\phi^3(x)\), and \(\hat{L}_{\text{int}} = L_{\text{int}} - \hat{L}_{\text{int}} \ldots\). The decay rate of the process (both in imaginary time formalism and in TFD) using GCR has been obtained by Fujimoto et al. \[13\] to be

\[
\Gamma_{GCR}(\omega) = \frac{1}{8\omega} \int \frac{d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 (2\pi)^3 \delta(k - k_1 - k_2 - k_3) |\mathcal{M}_{\tilde{\phi}}(T)|^2}{(2\omega_1)(2\omega_2)(2\omega_3)(2\pi)^3(2\pi)^3} \quad (4.31)
\]

where \(n_i = n_B(k_i), \omega_i = \sqrt{\langle k_i^2 \rangle + m^2}, (i = 1, 2, 3)\) and \(\omega = \sqrt{\langle k \rangle^2 + M^2}\). On the other hand the decay rate is related to the transition amplitude as:

\[
\Gamma(\omega) = \frac{1}{2\omega} \int \frac{d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 (2\pi)^3 \delta(k - k_1 - k_2 - k_3) |\mathcal{M}_{\tilde{\phi}}(T)|^2}{(2\omega_1)(2\omega_2)(2\omega_3)(2\pi)^3(2\pi)^3} \quad (4.32)
\]

where the amplitude \(\mathcal{M}_{\tilde{\phi}}(T)\) is defined by Eq. (3.4) with the elements of \(\hat{S} - \text{matrix:}\)

\[
\langle f|\hat{S}|i \rangle = i\lambda \int dx \langle 0(\beta)|b_{k_1}(\beta)b_{k_2}(\beta)b_{k_3}(\beta)(\Phi(x)\phi^3(x) - \tilde{\Phi}(x)\tilde{\phi}^3(x))a^\dagger_k(\beta)|0(\beta)\rangle \quad (4.33)
\]

As in the previous example, we get

\[
\delta(\omega - \omega_1 - \omega_2 - \omega_3)|\mathcal{M}_{\tilde{\phi}}(T)|^2 =
\]

\[
= \left[ \cosh \theta_k \cosh \theta_{k_1} \cosh \theta_{k_2} \cosh \theta_{k_3} - \sinh \theta_k \sinh \theta_{k_1} \sinh \theta_{k_2} \sinh \theta_{k_3} \right]^2 \delta(\omega - \omega_1 - \omega_2 - \omega_3) =
\]

\[
= n_1 n_2 n_3 n_\omega \left[ 1 - e^{\beta(\omega + \omega_1 + \omega_2 + \omega_3)/2} \right]^2 \delta(\omega - \omega_1 - \omega_2 - \omega_3) =
\]

\[
= n_1 n_2 n_3 n_\omega \left[ 1 - e^{\beta(\omega_1 + \omega_2 + \omega_3)} \right] \delta(\omega - \omega_1 - \omega_2 - \omega_3) =
\]

\[
= n_1 n_2 n_3 n_\omega \left[ 1 - \frac{1 + n_\omega}{n_\omega} \right] \left[ 1 - \frac{(1 + n_1)(1 + n_2)(1 + n_3)}{n_1 n_2 n_3} \right] \delta(\omega - \omega_1 - \omega_2 - \omega_3) =
\]

\[
= [(1 + n_1)(1 + n_2)(1 + n_3) - n_1 n_2 n_3] \delta(\omega - \omega_1 - \omega_2 - \omega_3)
\]

\[ (4.34) \]
where \( n_\omega = n_B(k) \), \( n_i = n_B(k_i) \). Then, by using (4.34) in (4.32) the result given in Eq. (4.31) is reproduced. Again the direct calculations and those from GCR agree.

V. THE DETAILED BALANCE.

The last example of the previous section can be easily generalized for the decay of a boson into \( m \) particles: \( \Phi(k) \to \phi(k_1) + \phi(k_2) + \ldots + \phi(k_m) \) with the decay rate

\[
\Gamma(\omega) = \frac{1}{2\omega} \int \frac{d\vec{k}_1 d\vec{k}_2 \ldots d\vec{k}_m (2\pi)^4 \delta^4(k - k_1 - k_2 - \ldots - k_m) |M_{fi}(T)|^2}{(2\omega_1)(2\omega_2)\ldots(2\omega_m)(2\pi)^{3m}}
\]  

(5.1)

where

\[
\delta(\omega - \omega_1 - \ldots - \omega_m)|M_{fi}(T)|^2 = \delta(\omega - \omega_1 - \ldots - \omega_m)
\]

\[
[cosh \theta_k cosh \theta_{k_1} \ldots cosh \theta_{k_m} - sinh \theta_k sinh \theta_{k_1} \ldots sinh \theta_{k_m}]^2 =
\]

\[
= n_1 \ldots n_m n_\omega [1 - e^{\beta \omega}] [1 - e^{\beta (\omega_1 + \ldots + \omega_m)}] =
\]

\[
= [(1 + n_1)(1 + n_2)\ldots(1 + n_m) - n_1 n_2 \ldots n_m] \delta(\omega - \omega_1 - \ldots - \omega_m).
\]  

(5.2)

It is clear from Eq. (5.1) and (5.2) that, the total decay rate \( \Gamma(\omega) \) is the difference of the forward rate of the boson decay - \( \Gamma_d(\omega) \) and the rate of the inverse process \( \Gamma_i(\omega) \):

\[
\Gamma(\omega) = \Gamma_d(\omega) - \Gamma_i(\omega),
\]

which are related by the principle of detailed balance:

\[
\frac{\Gamma_d(\omega)}{\Gamma_i(\omega)} = \frac{(1 + n_1)(1 + n_2)\ldots(1 + n_m)}{n_1 n_2 \ldots n_m} = e^{\beta (\omega_1 + \omega_2 + \ldots + \omega_m)} = e^{\beta \omega}
\]  

(5.3)

This relation was first shown by Weldon, [15] by analysing the discontinuities of the self energy function in the Matsubara formalism. Thus, we conclude that, our approach of direct calculation of rates through elements of the \( \hat{S} \) matrix leads to the correct relation for the detailed balance principle at finite temperature. Note that, the Eqs. (5.1) - (5.3) reveal the importance of doubling of degrees of freedom at nonzero temperature: If we had neglected the term \( \tilde{L}_{int} \) then the term proportional to "sinh \( \theta_k \)" in Eq. (5.2) would have vanished giving a wrong relation. Although the "tilde" particles are introduced as an essential part of the finite temperature formalism as fictitious particles, they play an important role in direct and inverse processes taking place at finite temperature. It is important to remark
that there are no transitions, and hence no matrix element, between the tilde and non-tilde particles.

However, following Keil [14], we underline that, $\Gamma(\omega)$ and not the partial rates $\Gamma_d(\omega)$ and $\Gamma_i(\omega)$ represent the physically measurable decay rate! The reason is that, it is the full decay width $\Gamma(\omega) = \Gamma_d(\omega) - \Gamma_i(\omega)$ that is connected to the pole of the Green’s function [13, 14].

\[
\Delta_{11}(k) = \frac{1 + n_B(k_0)}{k^2 - m_0^2 - \Sigma + i\varepsilon} - \frac{n_B(k_0)}{k^2 - m_0^2 - \Sigma^* - i\varepsilon} = \frac{1 + n_B(k_0)}{(k_0 + i\Gamma/2)^2 - \omega^2 + i\varepsilon} - \frac{n_B(k_0)}{(k_0 - i\Gamma/2)^2 - \omega^2 - i\varepsilon}
\]

(5.4)

where $\omega^2 = k_0^2 + Re\Sigma - \Gamma^2/4$, $\Gamma = -Im\Sigma/k_0$, $k_0 = \sqrt{(\vec{k})^2 + m^2}$.

VI. SCATTERING CROSS SECTION OF $1 + 2 \rightarrow 1' + 2'$.

As it was stated in Sect. III in our approach, in accordance with the concept of TFD, the relation between the rate (cross section) and the transition amplitude is the same as it is in the zero temperature quantum field theory. For example the cross section of the process $a(k_1) + b(k_2) \rightarrow a'(k_1') + b'(k_2')$ is [11]

\[
\frac{d\sigma}{d\Omega'} = \frac{|M_{fi}(T)|^2}{64\pi^2} |\prod_{\ell} (2m_{\ell})| |\vec{k}_1'|^2 \left\{ \frac{\partial (\omega'_1 + \omega'_2)}{\partial |\vec{k}_1'|} \right\}^{-1}
\]

(6.1)

in the usual notation. In particular, the cross section of the elastic scattering of two bosons in their center of mass (CoM) frame ($\vec{k}_1' = -\vec{k}_2'$) is

\[
\frac{d\sigma}{d\Omega} = \frac{|M_{fi}(T)|^2}{64\pi^2 (\omega_1 + \omega_2)^2}
\]

(6.2)

These equations imply that

\[
\frac{(d\sigma/d\Omega)|_{T \neq 0}}{(d\sigma/d\Omega)|_{T = 0}} = \frac{|M_{fi}(T \neq 0)|^2}{|M_{fi}(T = 0)|^2} = W(T)
\]

(6.3)

where the amplitudes may be determined through Eq.s (3.2)-(3.4). To see the consequences of this relation we will consider some examples.

A. Boson - Boson scattering.
Let us assume that both a and b particles are bosons with the interaction Lagrangian
\[ L_{\text{int}} = \lambda \phi_a^2(x)\phi_b^2(x) \] and \( \hat{L}_{\text{int}} = L_{\text{int}} - \tilde{L}_{\text{int}} \). At the tree level, using the results given in Eq.s (3.2) - (3.4) and Eq. (4.11) in (6.3) we get
\[ W_{BB}(T) = \left| M_{fi(T \neq 0)} \right|^2 \left| M_{fi(T = 0)} \right|^2 = n_1 n_2 n_1' n_2' \left( e^\beta (\omega_1 + \omega_2 + \omega_1' + \omega_2')/2 - 1 \right)^2, \]
where
\[ C(T) = \cosh \theta_{k_1} \cosh \theta_{k_2} \cosh \theta_{k_1'} \cosh \theta_{k_2'}, \]
\[ S(T) = \sinh \theta_{k_1} \sinh \theta_{k_2} \sinh \theta_{k_1'} \sinh \theta_{k_2'}, \]
and \( n_i = n_B(k_i), n_i' = n_B(k_i') \). Due to energy conservation we have
\[ \left( e^\beta (\omega_1 + \omega_2 + \omega_1' + \omega_2')/2 - 1 \right)^2 = (e^\beta (\omega_1 + \omega_2) - 1)(e^\beta (\omega_1' + \omega_2') - 1). \]

Now we may express the exponents via \( n_i, n_i' \)
\[ e^\beta \omega_i = \frac{1 + n_i}{n_i}, \quad e^\beta \omega'_i = \frac{1 + n_i'}{n_i'}, \]

\[ W_{BB}(T) = (1 + n_1 + n_2)(1 + n_1' + n_2') = [(1 + n_1)(1 + n_2) - n_1 n_2][(1 + n_1')(1 + n_2') - n_1' n_2']. \]

\[ W_{FF}(T) = \frac{|M_{fi(T \neq 0)}|^2}{|M_{fi(T = 0)}|^2} = \left[ \cos \theta_{k_1} \cos \theta_{k_2} \cos \theta_{k_1'} \cos \theta_{k_2'} - \sin \theta_{k_1} \sin \theta_{k_2} \sin \theta_{k_1'} \sin \theta_{k_2'} \right]^2 = n_F(k_1) n_F(k_2) n_F(k_1') n_F(k_2') \left( e^\beta (\omega_1 + \omega_2 + \omega_1' + \omega_2') - \mu_1 - \mu_2 - \mu_1' - \mu_2' \right)/2 - 1)^2. \]

At equilibrium \( \mu_1 + \mu_2 = \mu_1' + \mu_2' \), and hence, by using
Then using similar manipulations, as in previous examples we get

\[
\left[e^\beta(\omega_1 - \mu_1 + \omega_2 - \mu_2) - 1\right]^2 = \left[e^\beta(\omega_1 - \mu_1 + \omega_2 - \mu_2) - 1\right][e^\beta(\omega'_1 - \mu'_1 + \omega'_2 - \mu'_2) - 1] = \\
\left[\frac{1}{n_F(k_1)n_F(k_2)} - \frac{1}{n_F(k_1')} - \frac{1}{n_F(k_2')}\right][\left(\frac{1}{n_F(k_1)n_F(k_2')} - \frac{1}{n_F(k_1')} - \frac{1}{n_F(k_2')}\right],
\]

we get

\[
W_{FF}(T) = \frac{|M_{fi}(T \neq 0)|^2}{|M_{fi}(T = 0)|^2} = [(1 - n_1)(1 - n_2) - n_1n_2][(1 - n'_1)(1 - n'_2) - n'_1n'_2]. \quad (6.11)
\]

where \(n_i \equiv n_F(k_i)\), and \(n'_i \equiv n_F(k'_i)\).

C. Fermion - Boson scattering.

Let us assume that particle \(a\) is a fermion and particle \(b\) is a boson with the interaction lagrangian given as \(L_{\text{int}} = -ig\bar{\psi}(x)\Gamma\psi(x)\phi(x)\) (for example \(\gamma = \gamma_5\) for pion - nucleon scattering.) In this case we get at tree level

\[
W_{FB}(T) = \frac{|M_{fi}(T \neq 0)|^2}{|M_{fi}(T = 0)|^2} = [\cos \theta_{k_1}\cosh \theta_{k_2}\cos \theta_{k'_1}\cosh \theta_{k'_2} + \sin \theta_{k_1}\sinh \theta_{k_2}\sin \theta_{k'_1}\sinh \theta_{k'_2}]^2.
\]

(6.12)

Then using similar manipulations, as in previous examples we get

\[
W_{FB}(T) = \frac{|M_{fi}(T \neq 0)|^2}{|M_{fi}(T = 0)|^2} = [1 - n_F(k_1) + n_B(k_2)][1 - n_F(k'_1) + n_B(k'_2)]. \quad (6.13)
\]

The expressions (6.8), (6.11) and (6.13) for the relation of in medium cross sections of \(1 + 2 \rightarrow 1' + 2'\) scattering can be interpreted physically in the spirit of Weldon [15]. For example \(W_{FB}(T)\) can be rewritten as

\[
W_{FB}(T) = (1 - n_F(k_1))(1 + n_B(k_2))(1 - n_F(k'_1))(1 + n_B(k'_2)) + \\
+ n_F(k_1)n_B(k_2)n_F(k'_1)n_B(k'_2) + n_F(k_1)n_B(k_2)(1 - n_F(k'_1))(1 + n_B(k'_2)) + \\
+ n_F(k'_1)n_B(k'_2)(1 - n_F(k_1))(1 + n_B(k_2)). \quad (6.14)
\]

It is to be emphasized that the in-medium cross sections at finite temperature are not directly measureable in any many-body process. However such cross-sections are an important input to treat a gas at low density through a kinetic theory approach such as a Boltzmann equation. The finite temperature cross-sections will be a part of the collision term that would lead
to a distribution function by a solution of the Boltzmann equation. As indicated in the introduction these types of results are a necessary input to a better understanding of both the early Universe and the hadronic matter at finite temperature and density.

VII. LOOP CORRECTIONS

All the above results have been obtained at the tree level. Now we wish to show how the method can be handled beyond the tree level. For simplicity we choose the $\phi^4(x)$ interaction:

$$L_{\text{int}} = -\frac{\lambda}{4!} \phi^4(x).$$

The $\hat{S}$ matrix in Eq. (2.12) up to order $\lambda^2$ is given by:

$$\hat{S} \approx 1 - i \int dx \hat{H}_I(x) + \frac{(-i)^2}{2!} \int dx dy [\hat{H}_I(x) \hat{H}_I(y)] \equiv \hat{S}^{(1)} + \hat{S}^{(2)}.$$  (7.1)

For the scattering process from the initial state

$$|i\rangle = |\phi(k_1)\phi(k_2)\rangle = a^{\dagger}_{k_1}(\beta)a^{\dagger}_{k_2}(\beta)|0(\beta)\rangle$$  (7.2)

to a final state

$$|f\rangle = |\phi(k_1')\phi(k_2')\rangle = a^{\dagger}_{k_1'}(\beta)a^{\dagger}_{k_2'}(\beta)|0(\beta)\rangle$$  (7.3)

the matrix element $\langle f|\hat{S}^{(1)}|i\rangle$ and hence the amplitude $\mathcal{M}_{fi}(T) = \mathcal{M}^{(1)}_{fi}(T) + \mathcal{M}^{(2)}_{fi}(T)$, is defined as,

$$\langle f|\hat{S}^{(1)} + \hat{S}^{(2)}|i\rangle = \delta_{ij} + \delta^4(P - P') \frac{\mathcal{M}^{(1)}_{fi}(T) + \mathcal{M}^{(2)}_{fi}(T)}{(4\pi)^2 \sqrt{\omega_1\omega_2\omega_1'\omega_2'}}.$$  (7.4)

with $P = k_1 + k_2$ and $P' = k_1' + k_2'$. $\mathcal{M}^{(1)}_{fi}(T)$ and $\mathcal{M}^{(2)}_{fi}(T)$ can be simply calculated by using the procedure used in sec. VI. For instance,

$$\mathcal{M}^{(1)}_{fi}(T) = -i\lambda[C(T) - S(T)].$$  (7.5)

where $C(T)$ and $S(T)$ are defined by Eqs (6.5).

To evaluate $\mathcal{M}^{(2)}_{fi}(T)$, the generalized Wick theorem [16] is used and $2 \times 2$ Green’s function $\Delta_{ab}(q)$ (see Appendix) is introduced. In particular:
\[ i\Delta_{11}^0(x - y) = \langle 0(\beta)|T[\phi(x)\phi(y)]|0(\beta)\rangle = i \int \frac{d^4q}{(2\pi)^4} e^{-iq(x - y)\Delta_{11}^0(q)} \]

\[ i\Delta_{22}^0(x - y) = \langle 0(\beta)|T[\tilde{\phi}(x)\tilde{\phi}(y)]|0(\beta)\rangle = i \int \frac{d^4q}{(2\pi)^4} e^{-iq(x - y)\Delta_{22}^0(q)} \]

\[ i\Delta_{12}^0(x - y) = \langle 0(\beta)|T[\phi(x)\tilde{\phi}(y)]|0(\beta)\rangle = i \int \frac{d^4q}{(2\pi)^4} e^{-iq(x - y)\Delta_{12}^0(q)}. \tag{7.6} \]

In terms of these Green’s functions, the amplitude is proportional to \(\lambda^2\) and is

\[
\mathcal{M}_\mathrm{fi}^{(2)}(T) = \lambda^2 \{ C(T) \int \frac{dq^4}{(2\pi)^4} \Delta_{11}^0(q) \Delta_{11}^0(P - q) + S(T) \int \frac{dq^4}{(2\pi)^4} \Delta_{22}^0(q) \Delta_{22}^0(P - q) - \\
- \int \frac{dq^4}{(2\pi)^4} [C_s(k_1, k_2; k_1', k_2') \Delta_{12}^0(q) \Delta_{12}^0(P - q) + C_s(k_1', k_2'; k_1, k_2) \Delta_{21}^0(q) \Delta_{21}^0(P - q)]\}. \tag{7.7} \]

where \(C_s(k_1, k_2; k_1', k_2') = \cosh \theta_{k_1} \cosh \theta_{k_2} \sinh \theta_{k_1'} \sinh \theta_{k_2'}\). The terms in the integrand are related by

\[
\Delta_{11}^0(q) = -[\Delta_{22}^0(q)]^* = \frac{1}{q^2 - m^2 + i\varepsilon} - 2i\pi n_B(q)\delta(q^2 - m^2), \tag{7.8} \]

\[
\Delta_{12}^0(q) = \Delta_{21}^0(q) \]

and hence Eq. (7.7) simplifies to

\[
\mathcal{M}_\mathrm{fi}^{(2)}(T) = \lambda^2 \{ C(T)F(P, T) + S(T)F^*(P, T) + \\
+ 8\pi^2 e^{\beta F_0/2} I_3(P) [C_s(k_1, k_2; k_1', k_2') + C_s(k_1', k_2'; k_1, k_2)]\}, \tag{7.9} \]

\[
F(P, T) = \int \frac{dq^4}{(2\pi)^4} \Delta_{11}^0(q) \Delta_{11}^0(P - q) = I_1(P) - 4\pi i I_2(P) - 4\pi^2 I_3(P) \]

with

\[
I_1(P) = \int \frac{dq^4}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\varepsilon)((P - q)^2 - m^2 + i\varepsilon)} \]

\[
I_2(P) = \int \frac{dq^4}{(2\pi)^4} \frac{n_B(q)\delta(q^2 - m^2)}{(P - q)^2 - m^2} \tag{7.10} \]

\[
I_3(P) = \int \frac{dq^4}{(2\pi)^4} n_B(q)n_B(P - q)\delta(q^2 - m^2)\delta((P - q)^2 - m^2). \]

The integrals \(I_1(P), I_2(P), I_3(P)\) can be evaluated in the CoM frame of the two particles, where \(P^2 = (k_1^0 + k_2^0)^2 = E^2\) - total energy of the system.
The integral $I_1(P)$ is logarithmically divergent and can be treated using a renormalization procedure and this finite contribution is called the vacuum fluctuation correction. This term exists even at zero temperature, and can be calculated by the usual method of dimensional regularization \[17\]. As a result $I_1(P)$ may be written as:

$$I_1(P) = \frac{iN\varepsilon}{16\pi^2} + \frac{i}{16\pi^2}\left[b_0(E) - \ln \frac{m^2}{\mu^2}\right] - \frac{\sqrt{D_0}}{16\pi}$$

$$b_0(E) = 2 + \sqrt{D_0} \ln \left|\frac{1 - \sqrt{D_0}}{1 + \sqrt{D_0}}\right|$$

$$D_0 = 1 - 4m^2/E,$$

where the first term involving $N\varepsilon = 1/\varepsilon - \gamma_0 + \ln 4\pi$ is infinite. To eliminate the latter we introduce the following counterterm in the Lagrangian \[17\]

$$L_{\text{contr}} = -\frac{\lambda^2 N\varepsilon}{16\pi^2 4!} : \phi^4(x) :$$

which clearly does not depend on temperature. On the other hand, in accordance with the tilde operation rule, this automatically leads us to introduce its tilde partner also:

$$\tilde{L}_{\text{contr}} = -\frac{\lambda^2 N\varepsilon}{16\pi^2 4!} : \tilde{\phi}^4(x) :$$

which serves to compensate the divergence, coming from $S(T) \ast F^*(P,T)$ term in Eq. (7.9).

The $I_2(P)$ is convergent and can be evaluated numerically:

$$I_2(P) = \int \frac{dq}{(2\pi)^4} \int_{-\infty}^{+\infty} dq_0 n_B(q_0)\left[\delta(q_0 + |\omega_q|) - \delta(q_0 - |\omega_q|)\right]$$

$$= \frac{1}{4\pi^3} \int \frac{dqq^2n_B(q)}{E^2(1 - 4\omega_q^2/E^2)\omega_q} \equiv \frac{1}{64\pi^3} \tilde{I}_2(E,T)$$

The integral $I_3$ including two $\delta-$ functions may be evaluated in a similar way as in sect. IV (Eq.(7.4)) and is given as

$$I_3(P) = \int \frac{dq_0 dq B(P-q)n_B(q)}{(2\pi)^44E\omega_q} \delta(q_0 - \omega_q)\delta(\omega_q - E/2) = \frac{n_B^2(E/2)\sqrt{D_0}}{32\pi^3}. \quad (7.15)$$

Now, using Eqs. (7.3), (7.9)-(7.13) and introducing $\overline{M}_\text{fi} = iM_\text{fi}$ gives

$$\overline{M}_\text{fi}(T) = \text{Re}\overline{M}_\text{fi}(T) + i\text{Im}\overline{M}_\text{fi}(T)$$

$$\text{Re}\overline{M}_\text{fi}(T) = \lambda (C(T) - S(T))(1 - \tilde{\lambda}A(E,T)) \quad (7.16)$$

$$\text{Im}\overline{M}_\text{fi}(T) = -\lambda^2 \sqrt{D_0}[(C(T) + S(T))(1 + 2n_B^2) - 4C_s(T)e^{\beta E/2n_B^2}]/(16\pi),$$
where \( A(E, T) = b_0(E) - \ln \frac{m^2}{\mu^2} - \bar{I}_2(E, T) \), \( n_B = n_B(E/2) \), \( C(T) = (1+n_B)^2 \) and \( S(T) = n_B^2 \), \( C_s(T) = 2e^{\beta E/2}n_B \) and \( \bar{\lambda} = \lambda/16\pi^2 \).

From (7.16) one may get the relation for the in-medium elastic cross section in \( \phi^4(x) \) scalar theory up to one loop approximation:

\[
\left. \frac{(d\sigma/d\Omega)}{d\sigma/d\Omega} \right|_{T \neq 0} = \left. \frac{(C(T) - S(T))^2(1 - \bar{\lambda}A(E, T))^2 + \pi^2\bar{\lambda}^2D_0[(1 + 2n_B^2)(C(T) + S(T)) - 8e^{\beta E}n_B^3]^2}{[1 - \lambda(b_0(E) - \ln \frac{m^2}{\mu^2})]^2 + \pi^2D_0\bar{\lambda}^2} \right|_{T = 0}
\]

(7.17)

The appropriate Feynman rules for \( \phi^4 \) scalar theory at finite temperature can be explicitly obtained from Fig. 2. They are similar to those for zero temperature. The main differences are the following.

The amplitude of any order \( n \) consists of \( n \) vertices. There are two kinds of vertices: ”dotted” (ordinary) and ”circled” ones. To calculate \( \mathcal{M}_R(T) \)

1) Attach factor \(-i\lambda\) to each ”dotted” vertex.

2) Attach the factor \( \cosh \theta_p \) to each external leg (incoming or outgoing particle) with momentum \( p \) connected to the ”dotted” vertex.

3) Attach the factor \( \sinh \theta_p \) to each external leg (incoming or outgoing particle) with momentum \( p \) connected to the ”circled” vertex.

4) Attach \( i\Delta_{11}(q) \) to the propagator carrying momentum \( q \), connecting two ”dotted” vertices.

5) Attach \( i\Delta_{22}(q) \) to the propagator carrying momentum \( q \), connecting two ”circled” vertices.

6) Attach \( i\Delta_{12}(q) \) to the propagator carrying momentum \( q \), between a ”dotted” and a ”circled” vertices.

7) Attach \( i\Delta_{21}(q) \) to the propagator carrying momentum \( q \), between a ”circled” and a ”dotted” vertices.

Using these rules one may calculate any process in \( \phi^4 \) theory at finite temperature to any order of perturbative theory.

**Gluon Self-energy and Screening length.**

The gluon self-energy in one loop order is calculated in order to obtain the screening
length in a quark-gluon plasma. The calculation is straight forward in Thermo Field Dynamics. Working in Feynman gauge the ghost states do not contribute. Then the photon self-energy $\pi(\vec{k}, k_0; T)$ in the limit $\vec{k} \to 0$ and $k_0 \to 0$ but at finite temperature reduces to a simple form. The screening length is related to real part of the self energy which has the expression in one loop order as

$$Re\pi^{ab}_{00}(\vec{k} \to 0, k_0 \to 0; T \pm 0) = \frac{2g^2T^2\delta^{ab}}{\pi^2}(N + \frac{1}{2}N_f)\Gamma(2)\xi(2) = \delta^{ab}\frac{1}{3}(N + \frac{N_f}{2})g^2T^2$$

where $N_f = $ number of flavors and $a, b$ refer to the color indices of the group SU(N).

Here the term proportional to $N$ arises from the triplet and quartet couplings of gluons and the term proportional to $N_f$ arises from quark-anti-quark loops. Then the color electric mass of the gluon is

$$m^2_{el} = \frac{1}{3}(N + \frac{1}{2}N_f)g^2T^2$$

and the screening length at finite temperature is

$$L(T) = \frac{\sqrt{3}}{\sqrt{N + \frac{1}{2}N_fgT}}$$

The color magnetic mass of a gluon at finite temperature is zero. Details of this calculation are given by Zhao and Khanna [23].

VIII. DISCUSSIONS AND SUMMARY

In the present paper the decay amplitudes and scattering processes are studied in TFD. We have proved that, in the TFD formalism, the rate and transition amplitude of a process can be evaluated in a similar way as in the zero temperature case. Note that, both imaginary and real parts the amplitudes at finite temperature can be calculated directly. The amplitudes are written by establishing a set of Feynman rules using the Lagrangian $\hat{L} = L - \tilde{L}$. The Feynman diagrams are considered. Use of Bogoluibov transformations to define a pure vacuum state leads to Feynman rules for including temperature dependant factors for a process occurring in a system at finite temperature. It will be interesting to evaluate for example $NN$ scattering in nuclear matter at finite temperature. Temperature
dependence of coupling constants and masses have been explored in perturbative theory of TFD [18, ?]. At present we are attempting a self consistent calculation of these quantities. The role of Ward Takahashi relations at finite T will be crucial to obtaining a self consistent solution.

The present discussion for nuclear matter in equilibrium at finite T can be easily extended to QGP at temperature above the deconfinement phase transition. Some of the interesting properties like screening length and quark - quark interaction at finite temperature have been investigated at finite T [20]. The results are obtained in a simple and straightforward manner. There are numerous other properties of nuclear matter and QGP that may be calculated and would be helpful in studies of many particle hadronic systems.

The method developed in this paper using Thermo Field Dynamics are applicable to calculations at tree level as well as loop calculations. This will allow us to calculate processes in nuclear matter as well as in quark gluon plasma at finite temperature.

The present approach simplifies calculation of concrete processes taking place in hot and dense matter. Examples are neutrino electron scattering in thermal plasma [21, ?] , β - decay, which plays an important role in cosmology, in -medium nucleon - nucleon cross sections [23] taking place in ion - ion collisions etc. These processes are being investigated with the present approach.

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APPENDIX A:

A. The necessity of doubling.

In the statistical mechanics the statistical average of a quantity $A$ is given by

\[ <A> = Z^{-1}(\beta) \text{Tr}[Ae^{-\beta \mathcal{H}}] \]  

(A1)

where $\mathcal{H} = H - \mu N$, $Z(\beta) = \text{Tr}[e^{-\beta \mathcal{H}}]$. In the zero temperature field theory the average of an observable $A$ is determined by it’s vacuum expectation value:

\[ <A> = \langle 0|A|0 \rangle. \]  

(A2)

To make the temperature formalism completely parallel to the zero temperature case we assume that there exists a ”thermal vacuum state”, such that

\[ <A> = \langle 0(\beta)|A|0(\beta) \rangle = Z^{-1} \sum_n e^{-\beta E_n} \langle n|A|n \rangle \]  

(A3)

for any operator $A$. Since the Hilbert space is complete we can expand $|0(\beta)\rangle$ in terms of $|n\rangle$ as

\[ |0(\beta)\rangle = \sum_n |n\rangle \langle n|0(\beta)\rangle \equiv \sum_n f_n(\beta) |n\rangle. \]  

(A4)

From the last equation we obtain

\[ \langle 0(\beta)|A|0(\beta) \rangle = \sum_{n,m} f_n^*(\beta) f_m(\beta) \langle n|A|m \rangle \]  

(A5)

which will agree with eq. (A3) provided

\[ f_n^*(\beta) f_m(\beta) = Z^{-1} e^{-\beta E_n} \delta_{mn}. \]  

(A6)

However as we see from eq. (A4), $f_n(\beta)$’s are ordinary numbers and therefore it is not possible to satisfy (A6). This shows that as long as we restrict ourselves to the original Hilbert space, we cannot define a finite temperature vacuum which would satisfy eq. (A3). Therefore we have to double the Hilbert space, introducing a fictitious system identical to the original system. Let us denote this auxiliary system as a tilde system. Mathematically this means that
\[ |n, \tilde{m} \rangle = |n \rangle \otimes |\tilde{m} \rangle. \] (A7)

Now we can expand \(|0(\beta)\rangle\) in terms of these \(|n, \tilde{m} \rangle\) states

\[ |0(\beta)\rangle = \sum_n f_n(\beta) |n, \tilde{n} \rangle = \sum_n f_n(\beta) |n \rangle \otimes |\tilde{n} \rangle. \] (A8)

In this case we note that

\[ \langle 0(\beta)|A|0(\beta)\rangle = \sum_{n,m} f_m^*(\beta) f_n(\beta) \langle n, \tilde{n}|A|m, \tilde{m} \rangle = \sum_{n,m} f_m^*(\beta) f_n(\beta) \langle n|A|m \rangle \delta_{nm} = \sum_n f_n^*(\beta) f_n(\beta) \langle n|A|n \rangle. \] (A9)

Here we have used the fact that an operator of the physical system does not act on the Hilbert space of the tilde system and vice versa. So, using the orthonormality of the states we can write

\[ \langle n, \tilde{m}|A|n', \tilde{m}' \rangle = \langle n|A|n'|\langle \tilde{m}|\tilde{m}' \rangle = \delta_{mm'} \langle n|A|n' \rangle \] (A10)

\[ \langle n, \tilde{m}|\tilde{A}|n', \tilde{m}' \rangle = \langle n|\tilde{A}|n'|\langle \tilde{m}|\tilde{m}' \rangle = \delta_{mm'} \langle n|\tilde{A}|n' \rangle. \]

From (A3) and (A6) it is seen that

\[ f_m^*(\beta) = f_n(\beta) = Z^{-1/2}(\beta) \exp(-\beta E_n/2) \] (A11)

Now we can write the Fock space based on the vacuum \(|0(\beta)\rangle\) which is defined by a Bogoliubov transformation as

\[ |0(\beta)\rangle = U(\beta)|0, 0\rangle \] (A12)

where

\[ U(\beta) = \exp[-\theta(a\tilde{a} - a^\dag\tilde{a}^\dag)] \] (A13)

where \(\cosh(\theta) = 1/\sqrt{(1 - \exp(-\beta \omega))}\) and \(\omega\) is the energy. Then the annihilation operators at finite temperature are defined as

\[ a(\beta) = U^{-1}(\beta)aU(\beta), \quad \tilde{a}(\beta) = U^{-1}(\beta)\tilde{a}U(\beta) \] (A14)

such that
\[ a(\beta)|0(\beta)\rangle = 0 \; , \; \bar{a}(\beta)|0(\beta)\rangle = 0. \tag{A15} \]

Then Fock space is given as
\[
|0(\beta)\rangle, \quad a^\dagger(\beta)|0(\beta)\rangle, \quad \bar{a}^\dagger(\beta)|0(\beta)\rangle, \ldots
\]
\[\ldots \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{m!}} [a^\dagger(\beta)]^n [\bar{a}^\dagger(\beta)]^m |0(\beta)\rangle \tag{A16} \]
where \(n\) and \(m\) are the number of physical particles and tilde particles respectively. This therefore shows that we can introduce a temperature dependent vacuum state so that the statistical ensemble average of any operator can be identified with the expectation value of the operator in this state. This, however entails a doubling of the Hilbert space. The advantage, on the other hand, lies in the fact that the techniques of zero temperature field theory can now be carried over to the finite temperature case.

Here are the tilde conjugation rules \[24\]
\[
(\widetilde{AB}) = \tilde{A}\tilde{B},
\]
\[
(c_1A + c_2B) = c_1^*\tilde{A} + c_2^*\tilde{B},
\]
\[
(\tilde{A}^\dagger) = (\tilde{A})^\dagger,
\]
\[
|0(\beta)\rangle = |0(\beta)\rangle.
\]

For example the tilde partner of the weak interaction Lagrangian
\[
L_W = \frac{G_F}{\sqrt{2}} \bar{\nu}_\mu \Gamma_\alpha \mu \bar{e} \Gamma^\alpha e
\]
(in usual notations) is
\[
\tilde{L}_W = \frac{G_F}{\sqrt{2}} \bar{\nu}_e \Gamma_\alpha \bar{\mu} \tilde{\Gamma}^\alpha e \tilde{\nu}_\mu,
\]
but not \(\frac{G_F}{\sqrt{2}} \bar{\nu}_\mu \Gamma_\alpha \mu \bar{e} \tilde{\Gamma}^\alpha e \tilde{\nu}_e\) which would give zero contribution to the amplitude of the process \(\mu \to e\bar{\nu}_e \nu_\mu\) at tree level.

**B. Green’s functions.**

In TFD the thermal doublet for each field is defined as
\[
\phi^{(a)}(x) \equiv \begin{cases} 
\phi(x) \\
\tilde{\phi}(x) \end{cases} \equiv \begin{cases} 
\phi_1(x) \\
\phi_2(x) \end{cases} \tag{A20}
\]

\[
\psi^{(a)}(x) \equiv \begin{cases} 
\psi(x) \\
\tilde{\psi}^\dagger(x) \end{cases} \equiv \begin{cases} 
\psi_1(x) \\
\psi_2(x) \end{cases} \tag{A21}
\]

where \( t \) means transposition with respect the spinor index, and \( a(=1,2) \) specifies a component of the thermal doublet. The first component is physical, and the second is fictitious.

The scalar free field may be written as [3]:

\[
\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{2\omega_p}} [a_p e^{-ipx} + a_p^\dagger e^{ipx}],
\]

\[
\tilde{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{2\omega_p}} [\tilde{a}_p e^{ipx} + \tilde{a}_p^\dagger e^{-ipx}]. \tag{A22}
\]

A Fermion field may be written as:

\[
\psi(x) = \sum_s \int d\vec{p} N_p [c_{p,s} u(p,s) e^{-ipx} + d_{p,s}^\dagger v(p,s) e^{ipx}],
\]

\[
\tilde{\psi}(x) = \sum_s \int d\vec{p} N_p [\tilde{c}_{p,s} \tilde{u}(p,s) e^{ipx} + \tilde{d}_{p,s}^\dagger \tilde{v}(p,s) e^{-ipx}],
\]

\[
\bar{\psi}(x) = \sum_s \int d\vec{p} N_p [c_{p,s}^\dagger \tilde{u}(p,s) e^{ipx} + d_{p,s} \tilde{v}(p,s) e^{-ipx}],
\]

\[
\tilde{\bar{\psi}}(x) = \sum_s \int d\vec{p} N_p [\tilde{c}_{p,s}^\dagger \tilde{u}(p,s) e^{-ipx} + \tilde{d}_{p,s} \tilde{v}(p,s) e^{ipx}], \tag{A23}
\]

where \( N_p = \sqrt{m/(2\pi)^3} E_p \), \( E_p = \sqrt{m^2 + (\vec{p})^2} \), \( \tilde{u}(p,s) = u^*(p,s) \) etc.

The boson and fermion propagators, \( \Delta_{ab}(k) \) and \( S_{ab}(k) \) respectively, are 2 \( \otimes \) 2 matrices. They are defined as:

\[
i \Delta_{ab}(x - y) = \langle 0(\beta) | T[\phi_a(x) \phi_b(y)] | 0(\beta) \rangle = i \int \frac{dp^4}{(2\pi)^4} e^{-iq(x - y)} \Delta_{ab}^0(q)
\]

\[
i S_{ab}(x - y) = \langle 0(\beta) | T[\psi_a(x) \tilde{\psi}_b(y)] | 0(\beta) \rangle = i \int \frac{dp^4}{(2\pi)^4} e^{-iq(x - y)} S_{ab}^0(q). \tag{A24}
\]

The free propagators may be presented as a sum of Feynman propagator, \( \Delta_F^0(k) \) (\( S_F^0(k) \)) and temperature dependent part, \( \Delta_T^0(k) \), \( (S_T^0(k)) \) as:
where $\Theta$ is the step function: $\Theta(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k > 0 \end{cases}$

Here the following notations are used

$$
\Delta_{11}^0(k) = \Delta_{11}^F(k) + \Delta_{11}^B(k), \quad \Delta_{22}^0(k) = -(\Delta_{11}^0(k))^*
$$

$$
\Delta_{12}^0(k) = -2\pi i\delta(k^2 - m^2)m_B(k), \quad \Delta_{21}^0(k) = \Delta_{12}^0(k)
$$

$$
\Delta_F^0(k) = \frac{1}{k^2 - m^2 + i\epsilon}, \quad \Delta_B^0(k) = -2\pi i n_B(k)\delta(k^2 - m^2),
$$

$$
S_{11}^0(k) = (k + m)(S_F^0(k) + S_T^0(k)), \quad S_{22}^0(k) = (k + m)(S_F^0(k) + S_T^0(k))^* \quad (A25)
$$

$$
S_{12}^0(k) = 2i\pi(k + m)\delta(k^2 - m^2)m_F(k), \quad S_{21}^0(k) = S_{12}^0(k)
$$

$$
S_F^0(k) = \frac{1}{k^2 - m^2 + i\epsilon}, \quad S_B^0(k) = 2i\pi\delta(k^2 - m^2)\sin^2\theta_k.
$$

Here the following notations are used

$$
\sin^2\theta_k = \Theta(k_0)n_F(k) + \Theta(-k_0)\bar{n}_F(k),
$$

$$
m_F(k) = \frac{e^{x/2}[\Theta(k_0) - \Theta(-k_0)]}{e^x + 1} = \frac{\text{sign}(k_0)}{2}\sin 2\theta_x
$$

$$
m_B(k) = e^{\beta k_0/2}n_B(k) = \frac{1}{2}\sinh 2\theta_k, \quad x = \beta(k_0 - \mu)
$$

where $\Theta$ is the step function: $\Theta(k_0) = 1$ if $k_0 > 0$ and $\Theta(k_0) = 0$ otherwise and $\text{sign}(k_0)$ is the sign of $k_0$.

The full propagator satisfies the Dyson equation:

$$
\Delta_{ab}(k) = \Delta_{ab}^0(k) + \sum_{cd}^\Sigma_{ac}^B(k)\Sigma_{cd}^F(k)\Delta_{ab}(k)
$$

$$
S_{ab}(k) = S_{ab}^0(k) + \sum_{cd}^\Sigma_{ac}^F(k)\Sigma_{cd}^B(k)S_{ab}(k)
$$

where $\Sigma_{ab}^F(k)$ and $\Sigma_{ab}^B(k)$ are the proper self energies of a fermion and boson respectively.

By introducing a complex function $\Sigma_B = Re\Sigma_B + iIm\Sigma_B$ we can represent the general form of the self energy as follows:

$$
\Sigma_{11}^B(k) = Re\Sigma_B(k) + iIm\Sigma_B(k)\cosh 2\theta_k, \quad \Sigma_{22}^B(k) = -(\Sigma_{11}^B(k))^*
$$

$$
\Sigma_{12}^B(k) = -i\sinh 2\theta_kIm\Sigma_B(k), \quad \Sigma_{21}^B(k) = \Sigma_{12}^B(k). \quad (A28)
$$

Similarly for the fermion self energy we have:

$$
\Sigma_{11}^F(k) = Re\Sigma_F(k) + i\cos 2\theta_kIm\Sigma_F(k) \quad \Sigma_{22}^F(k) = (\Sigma_{11}^F(k))^*
$$

$$
\Sigma_{12}^F(k) = -2Im\Sigma_F(k)m_{F}(k), \quad \Sigma_{21}^F(k) = \Sigma_{12}^F(k). \quad (A29)
$$
The full propagator has the following compact form in terms of $\Sigma$:

$$
\Delta_{11}(k) = \frac{\cosh^2 \theta_k}{k^2 - m^2 - \text{Re} \Sigma_B(k) - i \text{Im} \Sigma_B(k) + i \varepsilon} - \frac{\sinh^2 \theta_k}{k^2 - m^2 - \text{Re} \Sigma_B(k) + i \text{Im} \Sigma_B(k) - i \varepsilon}
$$

$$
\Delta_{12}(k) = m_B(k) \{ \frac{1}{k^2 - m^2 - \text{Re} \Sigma_B(k) - i \text{Im} \Sigma_B(k) + i \varepsilon} - \frac{1}{k^2 - m^2 - \text{Re} \Sigma_B(k) + i \text{Im} \Sigma_B(k) - i \varepsilon} \}
$$

$$
\Delta_{22}(k) = - (\Delta_{11}(k))^*, \quad \Delta_{21}(k) = \Delta_{12}(k)
$$

(A30)

$$
S_{11}(k) = \frac{\cos^2 \theta_k_0}{\not{\mathcal{p}} - m - \text{Re} \Sigma_F(k) - i \text{Im} \Sigma_F(k) + i \varepsilon} + \frac{\sin^2 \theta_k_0}{\not{\mathcal{p}} - m - \text{Re} \Sigma_F(k) + i \text{Im} \Sigma_F(k) - i \varepsilon}
$$

$$
S_{12}(k) = - m_F(k) \{ \frac{1}{\not{\mathcal{p}} - m - \text{Re} \Sigma_F(k) - i \text{Im} \Sigma_F(k) + i \varepsilon} + \frac{1}{\not{\mathcal{p}} - m - \text{Re} \Sigma_F(k) + i \text{Im} \Sigma_F(k) - i \varepsilon} \}
$$

$$
S_{21}(k) = S_{12}(k) \quad S_{22}(k) = (S_{11}(k))^*,
$$

(A31)

where $\cosh^2 \theta_k = 1 + n_B$, $\sinh^2 \theta_k = n_B(k)$, $\cos^2 \theta_k_0 = 1 - \sin^2 \theta_k_0$ and $\sin^2 \theta_k_0$ was defined in Eq. (A26). The spectral representation for propagators and their matrix form may be found in ref. [9].
REFERENCES

[1] Ashok Das, Lectures on finite temperature field theory University of Rochester, 1996; F.C. Khanna and A.E. Santana, Topics in Thermo Field Dynamics, Lectures given at the second International school on field theory and gravitation. Vitoria, Brazil, 1999.

[2] Y. Takahashi and H. Umezawa, Int. Journ. Mod. Phys. 10, 1755 (1996), (original paper in Coll. Phenomenon 2, 55 (1975)).

[3] T. Matsubara, Progr. Theor. Phys., 14, 351 (1955).

[4] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).

[5] J. Schwinger, J. Math. Phys. 2, 407 (1961); L.V. Keldish, Sov. Phys. JETP, 20, 1018 (1964).

[6] A. Niegawa, Phys.Rev.D 57, 1379 (1998); S. Jeon and P.J. Ellis, Phys. Rev. D58, 04013 (1998); F.T. Brandt, A. Das and J. Frenkel, Phys. Rev. D60, 105008 (1999).

[7] R. Feynman, Rev. Mod. Phys. 20, 367 (1948).

[8] F. Dyson, Phys. Rev, 75, 1736 (1949).

[9] K. Saito, T. Maruyama and K. Soutome, Phys. Rev.C 40, 407 (1989).

[10] F. Mandl and G. Shaw, Quantum field theory. (A Wiley - Interscience 1986).

[11] F. Cutkosky, Jour. Math. Phys. 1, 429 (1960).

[12] R. Kobes and G. Semenoff, Nucl. Phys. B260, 714 (1985); R. Kobes and G. Semenoff, Nucl. Phys. B272, 329 (1986).

[13] Y. Fujimoto, M. Morikawa and M. Sasaki, Phys. Rev. D 33, 590 (1986).

[14] W. Keil, Phys. Rev. D 40, 1176 (1989).

[15] H. A. Weldon, Phys. Rev. D 28, 2007 (1983).

[16] M. Matsumoto, I. Ojima and H. Umezawa, Ann. Phys. (N.Y.) 152, 348 (1984).
[17] P. Ramond, Field theory of modern primer. (Benjamin/cumming publishing company) (1981).

[18] A.M. Rakhimov, U.T. Yakhshev and F.C. Khanna. Phys.Rev.C 61, 024907(2000)

[19] Yi - Jun Zhang, Song Gao and Ru - Keng Su. Phys. Rev. C 56, 3336 (1997) and references there in.

[20] W. Zhao and F.C. Khanna, to be published.

[21] S.J. Hardy and M.H. Thomas, Neutrino - electron processes in a strongly magnetized thermal field, astro-ph/0008473 and references therein.

[22] V.G. Bezchastnov and P. Haensel. Phys. Rev. D 54, 3706 (1996)

[23] C. Fuchs, I. Senh and H.H. Wolter, Nucl. Phys. A601, 505, (1996);
    R. Machleidt, Microscopic calculation of in - medium proton - proton cross sections. nucl-th/9308016

[24] H. Umezawa, H. Matsumoto and H. Tachiki, Thermo Field Dynamics and condensed states (North- Holland, Amsterdam, 1982);
    H. Umezawa, Advanced field theory, Micro, Macro and Thermal Physics, (AIP 1994).
Fig. 1. Generalized Cutkosky rules for the imaginary part of boson self energy. See ref. [11] for the details.
Fig. 2 Feynman diagrams for boson - boson scattering amplitude up $\lambda^2$ in $\lambda\phi^4(x)/4!$ model at finite temperature. Here the following notations are used: $C(k) \equiv \cosh \theta_k$, $S(k) \equiv \sinh \theta_k$, $K = k_1 + k_2 = k_1' + k_2'$ and $\lambda_e = \lambda^2 N_e/16\pi^2$. 

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