Loop and Path Spaces and Four-Dimensional BF Theories:
Connections, Holonomies and Observables

Alberto S. Cattaneo,\(^{(a)}\)
Paolo Cotta-Ramusino,\(^{(a)}\) and Maurizio Rinaldi\(^{(b)}\)

\(^{(a)}\)Dipartimento di Matematica
Università di Milano
Via Saldini 50
20133 Milano, Italy
and
I.N.F.N., Sezione di Milano

\(^{(b)}\)Dipartimento di Matematica
Università di Trieste
Piazzale Europa 1
34127 Trieste, Italy

Abstract
We study the differential geometry of principal \(G\)-bundles whose base space is the space of free paths (loops) on a manifold \(M\). In particular we consider connections defined in terms of pairs \((A, B)\), where \(A\) is a connection for a fixed principal bundle \(P(M, G)\) and \(B\) is a 2-form on \(M\). The relevant curvatures, parallel transports and holonomies are computed and their expressions in local coordinates are exhibited. When the 2-form \(B\) is given by the curvature of \(A\), then the so-called non-abelian Stokes formula follows.

For a generic 2-form \(B\), we distinguish the cases when the parallel transport depends on the whole path of paths and when it depends only on the spanned surface. In particular we discuss generalizations of the non-abelian Stokes formula. We study also the invariance properties of the (trace of the) holonomy under suitable transformation groups acting on the pairs \((A, B)\).

In this way we are able to define observables for both topological and non-topological quantum field theories of the \(BF\) type. In the non-topological case, the surface terms may be relevant for the understanding of the quark-confinement problem. In the topological case the (perturbative) four-dimensional quantum \(BF\)-theory is expected to yield invariants of imbedded (or immersed) surfaces in a 4-manifold \(M\).

This work has been partly supported by research grants of the Ministero dell’Università e della Ricerca Scientifica e Tecnologica (MURST). Part of this work was completed while A.S.C. was at Harvard University supported by I.N.F.N. Grant No. 5565/95 and DOE Grant No. DE-FG02-84ER25228. Amendment No. A003. P.C.-R. developed some of the work related to this paper while participating to the APCTP/Plms Summer Workshop at the University of Vancouver, B.C.

e-mail addresses: cattaneo@elanor.mat.unimi.it, cotta@mi.infn.it, rinaldi@univ.trieste.it.
1. Introduction

In this paper we consider the spaces $\mathcal{L}M$ and $\mathcal{P}M$ of free loops and paths of a compact manifold $M$ and the principal $G$-bundles on $\mathcal{L}M$ and $\mathcal{P}M$ obtained by pulling back, via the evaluation map, a fixed principal bundle $P(M, G)$. We are interested in the connections on such bundles that are determined by pairs $(A, B)$ where $A$ is a connection on $P(M, G)$ and $B$ is a 2-form of the adjoint type on $P(M, G)$. We study the properties of the curvature and of the holonomy of such connections.

The motivations for this study are rooted in the four-dimensional quantum field theories of the $BF$-type. One of our goals is to understand the relation between those QFT’s and the (smooth) invariants of four-manifolds and of surfaces imbedded (or immersed) in four-manifolds.

Before discussing the differential geometrical results, we comment briefly on quantum $BF$-theories.

1.1. Quantum Field Theory

Four-dimensional $BF$-theories may become increasingly relevant both to the quantum-field theoretical description of smooth four-manifold invariants and to the understanding of quark-confinement problems.

What characterizes $BF$-theories, and distinguishes them from ordinary gauge theories, is the fact that there are two fundamental fields: a connection $A$ for some principal $G$-bundle over a four-dimensional manifold $M$ and a 2-form field $B$ that transforms under gauge transformations as the curvature of $A$.

Various actions (and observables) can be constructed with the two fields $A$ and $B$ and, as a result, $BF$-theories can be both topological and non-topological (see section 8).

One of the relevant non-topological $BF$ theories is the first-order formalism of the Yang-Mills theory introduced in [1], [2], [3] and then modified in [4] by replacing $B$ with $B - dA\eta$, where the extra field $\eta$ is a 1-form. The resulting theory, which turns out to be a deformation of a topological theory, has been shown in [4] to be equivalent to the Yang-Mills theory.

For general topological field theories the reader is referred to [5].

Topological $BF$-theories have been introduced in [6] (see also [7]) and reviewed in [8]. The inclusion of observables in four-dimensional topological $BF$-theories is due to [9].

Here we begin a study of the geometry of four-dimensional $BF$-theories. The starting point is the observation that the spatial components of $B$ are the conjugate momenta to the spatial components of $A$. If we formally identify the tangent and cotangent spaces, we may see $B$ as an infinitesimal connection.

The natural way to interpret the 2-form $B$ as a tangent vector to a space of connections is to consider a principal $G$-bundle which has both as the total space and as the base manifold a space of loops or paths. By integrating the 2-form $B$ over a path, one obtains a 1-form. We show in this paper that pairs $(A, B)$ represent connections on a $G$-bundle over the path or loop space of our manifold $M$.

An ordinary connection $A$ yields the parallel transport along a path of geometrical
objects associated to points. Similarly a connection on the path or loop space, represented by a pair \((A, B)\), yields the parallel transport along a path of paths or a path of loops of geometrical objects associated to paths or loops.

Surfaces are spanned by paths of paths (though not in a unique way), so the first question one has to ask is when a Stokes-like formula holds for parallel transport. It is easy to see that when the field \(B\) is the (opposite of the) curvature \(F_A\) of \(A\), then the parallel transport along a path of paths spanning a surface \(S\) is uniquely determined by the \(A\)-parallel transport along the boundary of \(S\). This is a new version of the well-known non-abelian Stokes formula (see [10], [11]).

If \(B\) is a small deformation of a (small) curvature \(F_A\), then a surface term appears in the parallel transport with respect to the connection \((A, B)\). For large deformations the parallel transport with respect to \((A, B)\) depends on the whole path of paths structure and not only on the spanned surface.

We now recall that the ordinary holonomy along a loop in space–time has a physical interpretation: namely, it represents the contribution of a pair quark–antiquark forced to move along the loop. With our construction we have at our disposal more general objects.

For example we may consider the parallel transport with respect to the pair \((A, B)\) along a path of paths. This provides us with a simple generalization of the previous situation: we might think of this case as of a pair quark–antiquark with an interaction that is not given just by the \(A\)-parallel transport.

As a second example we may take the holonomy corresponding to the pair \((A, B)\) of a loop in the path (or loop) space. This should represent the contribution of a pair of open (or closed) strings in interaction.

Thus, the equivalence between the Yang–Mills theory and a deformation of the \(BF\)-theory plus the presence of surface terms associated to the parallel transport involving a non-trivial \(B\)-field may be relevant for the problem of quark confinement as formulated by Wilson [12].

A rôle of the \(BF\)-theories in the understanding of quark confinement has been discussed in [13] (see also [14]).

As far as topological \(BF\) theories are concerned, we notice that one can define, at least in principle, a four-dimensional analogue of the Witten–Chern–Simons theory, whose vacuum expectation values (v.e.v.) of (products of) Wilson loops represent link-invariants.

In four-dimensional topological \(BF\)-theories, one can compute v.e.v.’s of traces of holonomies of imbedded (or immersed) loops of paths (or of loops). Again one of the delicate points is to check when these invariants can be considered invariants of the surfaces spanned by loops of paths (or of loops). This requirement is close to the parameterization invariance for the surface since (for imbeddings) the loop-of-paths structure yields coordinates on the surface.

In this paper we show that when the fields \(A\) and \(B\), restricted to the surface spanned by a loop of paths, take values in an abelian subgroup \(T\) of \(G\), then the trace of the \((A, B)\)-holonomy depends only on the surface and not on the loop-of-paths structure.

This reducibility condition is related to the abelian projection considered in [15], [16].

The actual computation of v.e.v.’s for the topological \(BF\) theories will be carried out elsewhere. There is some indication that these v.e.v.’s may be related to the invariants of
imbedded (or immersed) surfaces considered by Kronheimer and Mrowka [17].

Even though the original motivations for this paper lies in the development of BF quantum field theories, our main goal here is to study connections, curvatures and holonomies of principal fiber bundles over path and loop spaces. Our language will be therefore the language of differential geometry.

1.2. Geometry

We begin Sect. 2 by considering a fixed principal fiber bundle $P(M,G)$ together with a connection $A$. The space of $A$-horizontal paths, denoted by $\mathcal{P}_A P$, is a principal $G$-bundle itself which has the space of paths on $M$ as base manifold.

We describe explicitly the tangent bundle of $\mathcal{P}_A P$ as a submanifold of the path space of $TP$. We consider connections on $\mathcal{P}_A P$ and particularly those connections that are defined in terms of a 2-form $B$. We call these connections on $\mathcal{P}_A P$ special connections.

The curvatures, horizontal distributions and parallel transports corresponding to special connections are computed. The explicit expression for the curvature involves Chen integrals.

In Sect. 3 we discuss the parallel transport of paths of paths with respect to special connections. We single out the case when $B$ is given by the (opposite) of the curvature $F_A$ of $A$: it is the only case where we have an abelian Stokes formula, namely a relation between the parallel transport of a path of paths and the $A$-holonomy of the loop given by the boundary of such path of paths.

More generally we study the possible conditions that force the (trace of the) parallel transport along a path of paths (or of loops) and the (trace of the) holonomy of a loop of paths (or of loops) to depend only on the spanned surface. The first of these conditions is a “perturbative” one: namely we assume that $P$ admits a flat connection and we expand both $B$ and $F_A$ around zero. Then, for imbedded paths of paths, the (trace of the) parallel transport with respect to a special connection $(A,B)$ depends only on the spanned surface, up to second-order terms.

In Sect. 4 we compute the expressions of the special connections and of the parallel transport of paths of paths in local coordinates.

After recalling the general transformation properties of the holonomy considered as a function of the space of connections for a generic principal bundle (Sect. 5), we discuss in Sect. 6 the “non-perturbative” conditions that guarantee that the holonomy of a loop of paths is independent of those automorphisms of $P$ that maps the spanned surface into itself.

Here we require both $B$ and $A$ to be reducible to an abelian subgroup of the structure group $G$ once they are restricted to the image of the spanned surface.

In Sect. 7 we study the action on the space of pairs $(A,B)$ of those transformations groups that happen to be symmetries for the BF-theories.

The invariance of the (trace of the) holonomy under those transformations is guaranteed provided that we ask again the reducibility conditions for both $A$ and $B$.

It is worthwhile noticing that the group of gauge transformations on $\mathcal{P}_A P$, which preserves the trace of the holonomy, is not a symmetry group for the BF-theories. More
precisely the symmetries of the $BF$ theories are “close” to being gauge transformations on $\mathcal{P}_A P$, the missing terms being boundary terms and higher-order Chen integrals.

The full group of gauge transformations on $\mathcal{P}_A P$, the space of all connections on $\mathcal{P}_A P$ and the relation between $\mathcal{P}_A P$ and the free loop bundle $\mathcal{L} P$ (whose structure group is the loop group of $G$) will be discussed in a forthcoming paper [18].

In section 8 we describe the observables for $BF$ quantum field theories, both in the topological and in the non-topological case.

2. Differential geometry of horizontal paths

We describe here the general setting of this paper. We consider a smooth manifold $M$ that is assumed to be closed, compact, oriented and Riemannian, a compact Lie Group $G$ with an $\text{Ad}$-invariant inner product on its Lie algebra $\mathfrak{g}$ and a fixed principal $G$-bundle $P = P(M, G)$ over $M$. The group of gauge transformations of $P$ will be denoted by $\mathcal{G}$, while the space of connections on $P$ will be denoted by $\mathcal{A}$.

Also we denote by $\Omega^* (M, \text{ad} P)$ the graded Lie algebra of forms on $M$ with values in the adjoint bundle $\text{ad} P = P \times \text{Ad} \mathfrak{g}$. We will consistently consider the elements of $\Omega^* (M, \text{ad} P)$ also as forms on $P$ that are both of the adjoint type and tensorial [19].

The group $\mathcal{G}$ acts on $\mathcal{A}$, and this action is free provided that we restrict $\mathcal{A}$ to be the space of irreducible connections and divide $\mathcal{G}$ by its center. We denote this action as follows:

$$\mathcal{A} \times \mathcal{G} \ni (A, g) \mapsto A^g \in \mathcal{A}.$$  

In the course of this paper we will have to consider other principal $G$-bundles, say $P_X (X, G)$, over some manifold $X$, possibly infinite-dimensional. We will then denote then $\mathcal{G}(P_X)$ and by $\mathcal{A}(P_X)$ the relevant group of gauge transformations and the space of connections. If no confusion arises, we use the symbol $\pi$ to denote the projection of any fiber bundle.

For any manifold $X$ we denote by $\mathcal{P} X$ the space of smooth paths on $X$. The space of smooth free loops on $X$ will be denoted by the symbol $\mathcal{L} X$ and the space of $x$-based loops ($x \in X$) by the symbol $\mathcal{L}_X X$. With some extra work we could consider also piecewise smooth paths and loops, but we do not wish to discuss this problem here.

We will also be interested in the space of smooth maps assigning to each point $x \in X$ a path or a loop with initial point $x$. We call such maps path-fields and, respectively, loop-fields. If we denote by $\text{Map}(X)$ the space of smooth maps of $X$ to itself, then a path field and a loop field on $X$ are represented respectively by a path or a loop on $\text{Map}(X)$ with initial point the identity map.

Most of this paper deals with horizontal lifts of paths on $M$ with respect to a given connection $A \in \mathcal{A}$. We use the following notation for horizontal lifts; for any $\gamma : [0, 1] \to M$ and for any $p \in P$ with $\pi(p) = \gamma(0)$, the $A$-horizontal lift of $\gamma$ with initial point $p$ is denoted by the symbol

$$\mathcal{L}(A, \gamma, p).$$  

Our first task is to study the differential geometry of the space of $A$-horizontal paths.
2.1. The principal bundle of horizontal paths and its tangent bundle

Let \( \mathcal{P}_A P \) denote the space of \( A \)-horizontal paths in \( P \). This is a principal \( G \)-bundle

\[
\mathcal{P}_A P \longrightarrow \mathcal{P} M
\]

where the right \( G \)-action is given by the right \( G \)-action on the initial points of the horizontal paths.

If we consider two distinct connections \( A, \bar{A} \in \mathcal{A} \), then we have two distinct and isomorphic principal \( G \)-bundles \( \mathcal{P}_A P \) and \( \mathcal{P}_{\bar{A}} P \). They are isomorphic since, for any connection \( A \), the bundle \( \mathcal{P}_A P \) is isomorphic to the pulled-back bundle \( ev_0^* P \). By the symbol \( ev \) we denote in general the evaluation map and, in this particular case, the map

\[
ev: \mathcal{P} M \times I \rightarrow M, \quad I = [0, 1], \quad ev_t \triangleq ev(\cdot, t).
\]

Let us call \( J_A \) the isomorphism between \( ev_0^* P \) and \( \mathcal{P}_A P \) given by

\[
J_A(\gamma, p) \triangleq \mathcal{L}(A, \gamma, p), \quad \gamma \in \mathcal{P} M, \quad p \in \pi^{-1} \gamma(0).
\]

We denote by \( j_A \) the evaluation map \( ev_0^* P \times I \rightarrow P \) given by

\[
j_A((\gamma, p), t) \equiv \mathcal{L}(A, \gamma, p)(t).
\]

We have the following bundle morphisms:

\[
\mathcal{P}_A P \times I \xrightarrow{ev} P; \quad \mathcal{P}_A P \xrightarrow{ev_t} P.
\]

As a particular case we can consider the loop space \( \mathcal{L} M \) on \( M \), instead of \( \mathcal{P} M \) and the corresponding principal bundle \( \mathcal{L}_A P \) whose elements are the \( A \)-horizontal paths on \( P \) whose projections are loops.

We now study the properties of the tangent bundle of the bundle \( \mathcal{P}_A P \).

Firstly we identify \( T \mathcal{P} M \) (the tangent bundle of the path space) with \( \mathcal{P}(TM) \) (the path space of the tangent bundle).

In other words, given any path \( \gamma \in \mathcal{P} M \), a vector \( X \in T_\gamma(\mathcal{P} M) \) is given by the assignment for each \( t \in I \) of a vector \( X(t) \in T_{\gamma(t)} M \). Equivalently the same vector \( X \) can be represented by a smooth map \( \Gamma: (-\epsilon, \epsilon) \times I \rightarrow M \), so that

\[
\Gamma(0, t) = \gamma(t), \quad \Gamma'(0, t) \equiv \frac{\partial \Gamma(s, t)}{\partial s}\bigg|_{s=0} = X(t).
\]

For any horizontal path \( q \) on \( P \), let us consider a tangent vector \( q \in T_q(\mathcal{P} P) \), defined by a smooth map \( Q: (-\epsilon, \epsilon) \times I \rightarrow P \) satisfying the following conditions

\[
Q(0, t) = q(t), \quad \left( Q \ast \frac{\partial}{\partial t} \right)_{Q(0,t)} \equiv \dot{q}(t), \quad \left( Q \ast \frac{\partial}{\partial s} \right)_{Q(0,t)} = q(t).
\]
The tangent vector \( q \) belongs to \( T_q(\mathcal{P}_A P) \) if and only if the following extra requirement is satisfied:

\[
\left( Q_s \frac{\partial}{\partial s} \right) A(\bar{Q}(s,t)) = 0, \quad s = 0, \forall t \in I. \tag{2.7}
\]

Here we used the dot to denote the derivative with respect to the variable \( t \in I \). We will use this notation also in the future. Moreover when dealing with two variables \((s,t) \in I \times I\) we will use the prime to denote the derivative with respect to the variable \( s \in I \).

Condition (2.7) is independent of the choice of the map \( Q(s,t) \) representing \( q \). In fact any two such choices \( Q(s,t) \) and \( \tilde{Q}(s,t) \) would satisfy (in local coordinates) the condition \( Q(s,t) - \tilde{Q}(s,t) = sg(s,t) \) for some map \( g \) with \( g(0,t) = 0 \).

By considering the Lie derivative \( L \) and inner product \( i \) operators, condition (2.7) is written as

\[
L_{\frac{\partial}{\partial s}} i_{\frac{\partial}{\partial t}} Q^* A = 0, \quad s = 0, \forall t \in I. \tag{2.8}
\]

An important consequence of (2.8) which will be used several times in the rest of this paper is given by the following

**Theorem 2.1.** For any element \( q \in T_q(\mathcal{P}_A P) \), the following equations hold:

\[
\frac{dA(q(t))}{dt} + F_A(q(t), \dot{q}(t)) = 0, \quad \forall t \in I,
\]

\[
A(q(t)) - A(q(0)) = - \int_0^t dt_1 F_A(q(t_1), \dot{q}(t_1)), \quad \forall t \in I,
\]

where \( F_A \) denotes the curvature of \( A \).

**Proof of Theorem 2.1**

Condition (2.8) and the commutation property \( \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0 \) imply

\[
(Q^*dA) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) = L_{\frac{\partial}{\partial t}} i_{\frac{\partial}{\partial s}} Q^* A, \quad s = 0, t \in I. \tag{2.10}
\]

Since \( q \) is horizontal we have also

\[
(Q^*[A,A]) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) = 0, \quad s = 0, t \in I. \tag{2.11}
\]

Equations (2.10) and (2.11) and the structure equation for the curvature imply

\[
(Q^*F_A) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) = L_{\frac{\partial}{\partial t}} i_{\frac{\partial}{\partial s}} Q^* A, \quad s = 0, t \in I. \tag{2.12}
\]
If we recall (2.6) then (2.12) becomes immediately the first equation of (2.9), namely

\[ F_A (\dot{q}(t), q(t)) = \frac{dA(q(t))}{dt}, \]

while the second equation of (2.9) is obtained by integrating the first.

To any path \((q, \dot{q})\) in \(TP\) and \(t \in I\) we associate the tangent vectors \((\dot{q}(t), \dot{q}(t)) \in T_{(q(t),q(t))}TP\). Altogether the quadruple \((q, q, \dot{q}, \dot{q})\) represents a path in \(TTP\).

To any connection \(A \in \mathcal{A}\) we can canonically associate a connection \(\hat{A}\) on the TG-bundle \(TP\) [20], [21], called the tangential connection.

We recall that the tangential connection \(\hat{A}\) applied to an element \(TTP\) represented by a smooth map \(Q: (-\epsilon, \epsilon) \times (t_0 - \epsilon, t_0 + \epsilon) \rightarrow P\) yields

\[ \left( A(\dot{Q})(0, t_0), \frac{\partial A(\dot{Q})(s, t_0)}{\partial s} \bigg|_{s=0} \right) \in \mathfrak{g} \times \mathfrak{g}. \]

(2.13)

We have then the following

**Remark 2.2.** A path \((q, \dot{q})\) in \(TP\) represents an element of \(TP_A P\) if and only if it is a \(\hat{A}\)-horizontal path in \(TP\), namely if we have \(\hat{A}(q, \dot{q}, \dot{q}, \ddot{q}) = 0\).

In other words \((q, \dot{q})\) is the \(\hat{A}\)-horizontal lift of a path \((\gamma, \rho)\) in \(TM\) with initial point \((q(0), \dot{q}(0)) \in \pi^{-1}(\gamma(0), \rho(0))\).

A *vertical* vector \(q \in T_q (\mathcal{P}_A P)\) is required to satisfy the extra condition

\[ q(t) \in V_{q(t)} P, \quad \forall t \in I, \]

(2.14)

where \(V_p P\) denotes the vertical subspace of \(T_p P\).

Finally we have the following

**Corollary 2.4.** As a consequence of Theorem 2.1 and condition (2.14), vertical vectors in \(T_q (\mathcal{P}_A P)\) satisfy the equation

\[ \frac{dA(q(t))}{dt} = 0, \quad \forall t \in I. \]
2.2. Connections and curvatures on the bundles of horizontal paths

We now consider connections on $\mathcal{P}_A P$. In particular we are interested here in those connections on $\mathcal{P}_A P$ that are determined by 2-forms in $\Omega^2(M, \text{ad} P)$, as shown in the following

**Theorem 2.4.** Let $B \in \Omega^2(M, \text{ad} P)$ and $A, \tilde{A}$ be any pair of connections on $P$. The form

$$ ev_0^* \tilde{A} + \int_I ev^* B $$

(2.15)

defines a connection on $\mathcal{P}_A P$.

**Proof of Theorem 2.4**

The $\mathfrak{g}$-valued 1-form $ev_0^* \tilde{A}$ is a connection on $\mathcal{P}_A P$. Moreover the 1-form $\int_I ev^* B$ is of the adjoint type and is tensorial, as can be seen by inspecting the explicit expression:

$$ \left( \int_I ev^* B \right)(q) = \int_0^1 dt \ B(q(t), \dot{q}(t)), \quad q \in T_q(\mathcal{P}_A P). $$

(2.16)

We will call any connection of the above form a *special connection* on $\mathcal{P}_A P$ and will denote it by the triple $(A, \tilde{A}, B)$. The space of special connections on $\mathcal{P}_A P$ is an affine space modeled on $\Omega^1(M, \text{ad} P) \oplus \Omega^2(M, \text{ad} P)$.

The reason why we are particularly interested in special connections is that elements of $\Omega^2(M, \text{ad} P)$ and connections are the essential ingredients of four-dimensional $BF$-theories [8], [9], [22], [4].

The space of special connections is a proper subspace of the space of all smooth connections on $\mathcal{P}_A P$. A simple example of a connection on $\mathcal{P}_A P$ that is not special is very easy to construct. Let $t \mapsto \tilde{B}_t$ be a path in $\Omega^2(M, \text{ad} P)$ and $A, \tilde{A}$ be any pair of connections on $P$. Then we have a connection on $\mathcal{P}_A P$, defined on $q \in T_q(\mathcal{P}_A P)$ as

$$ \tilde{A}(q(0)) + \int_0^1 dt \ \tilde{B}_t(q(t), \dot{q}(t)). $$

(2.17)

Other examples of connections on $\mathcal{P}_A P$ that are not special will be discussed extensively in a subsequent paper [18].

In (2.15) we often choose $\tilde{A} = A$ and denote the triple $(A, A, B)$ simply by a pair $(A, B)$. Here $A$ is kept fixed, so the space of special connections on $\mathcal{P}_A P$ that are represented by pairs $(A, B)$ is an affine space modeled on $\Omega^2(M, \text{ad} P)$. 

8
A vector \( q \in T_q(P_A P) \) is, by definition, horizontal with respect to the connection \((A, \bar{A}, B)\) if the following condition is satisfied:

\[
\bar{A}(q(0)) + \int_0^1 dt \left[ B(q(t), \dot{q}(t)) \right] = 0. \tag{2.18}
\]

We consider two particular connections on \( P_A P \):

1. The trivial connection \((A, 0)\) with curvature \( ev_0^* F_A \). Here condition (2.18) is equivalent to requiring that \( q(0) \) is \( A \)-horizontal.

2. The tautological connection \((A, -F_A)\).

As a consequence of Theorem 2.1, we have the following Corollary 2.6.

**Corollary 2.6.** The tautological connection is given by \( ev_1^* A \) and its curvature is given by \( ev_1^* F_A \). Condition (2.18) for the tautological connection is the requirement that \( q(1) \) is \( A \)-horizontal.

Let us add that on \( \mathcal{L}_A P \) the tautological and the trivial connections are gauge equivalent, but we refer to [18] for the study of the gauge group of \( \mathcal{L}_A P \).

The computation of the curvature for a generic 2-form \( B \) involves Chen integrals. These integrals are defined in [23] for scalar forms, but their extensions to forms in \( \Omega^*(M, \text{ad} P) \) is relatively easy and will be discussed extensively in [18].

Here it is enough to say that the Chen integral of a form \( w \in \Omega^{\deg(w)}(M, \text{ad} P) \) is just the form given by the ordinary integral

\[
\int_{\text{Chen}} w \triangleq \int_I ev^* w \in \Omega^{\deg(w)-1}(PM, \text{ad}(P_A P)),
\]

while the Chen bracket of two forms \( w_1, w_2 \in \Omega^*(M, \text{ad} P) \) is the form in \( \Omega^*(PM, \text{ad}(P_A P)) \) of degree \( \deg(w_1) + \deg(w_2) - 1 \) defined as

\[
\int_{\text{Chen}} \{ w_1; w_2 \} \triangleq \int_{0 < t_1 < t_2 < 1} \left[ w_1(\cdots, \dot{\gamma}(t_1)) dt_1, w_2(\cdots, \dot{\gamma}(t_2)) \right] dt_2 =
\]

\[
= (-1)^{\deg(w_2) - 1} \int_{0 < t_1 < t_2 < 1} \left[ w_1(\cdots, \dot{\gamma}(t_1)), w_2(\cdots, \dot{\gamma}(t_2)) \right] dt_1 dt_2,
\]

where for each value of \( t_i \) \( w_i(\cdots, \dot{\gamma}(t_i)) \) are \((\deg(w_i) - 1)\)-forms on \( P \) to be evaluated at tangent vectors in \( T_{\gamma(t_i)} P \). Notice that the Chen bracket is bilinear, but neither skew-symmetric nor graded-skew-symmetric.

We have the following
Theorem 2.6. The curvature $F_{(A,B)}$ of $(A, B)$ is given by the following 2-form on $\mathcal{P}_A P$:

$$
\begin{align*}
ev_0^* F_A - ev_1^* B + ev_0^* B + \int_I ev^* dAB &+ (1/2) \left[ \int_I ev^* B, \int_I ev^* B \right] - \int_I [ev^* A - ev_0^* A, ev^* B] \\
&= ev_0^* F_A - ev_1^* B + ev_0^* B + \int_I ev^* dAB + \int_{Chen} \{ B + F_A; B \}.
\end{align*}
$$

Proof of Theorem 2.6

The curvature of the connection $(A, B)$ is given by:

$$
F_{(A,B)} = ev_0^* F_A + d_{ev_0^* A} \int_I ev^* B + (1/2) \left[ \int_I ev^* B, \int_I ev^* B \right].
$$

We then recall that we have the following relation between exterior derivatives

$$
d|_{\mathcal{P}_A P \times I} = d|_{\mathcal{P}_A P} \pm d|_I,
$$

where the sign is given by the parity of the order of the form on $\mathcal{P}_A P$.

Hence we have the following chain of identities:

$$
\int_I ev^* dAB = \int_I d_{ev^* A} ev^* B = \int_I d ev^* B + [ev^* A, ev^* B] \\
= d \int_I ev^* B + \int_I [ev^* A, ev^* B] + ev_1^* B - ev_0^* B \\
= d_{ev_0^* A} \int_I ev^* B + \int_I [ev^* A - ev_0^* A, ev^* B] + ev_1^* B - ev_0^* B.
$$

We also have

$$
(1/2) \left[ \int_I ev^* B, \int_I ev^* B \right] = \int_{Chen} \{ B; B \}
$$

and by taking into account Theorem 2.1, we conclude the proof.

It is now natural to look for flat connections on $\mathcal{P}_A P$. If we restrict to special connections $(A, B)$, then $(A, 0)$ is a flat connection if $A$ is flat. In order to find other flat connections, we have to require some reducibility conditions.

Let $T$ be an abelian subgroup of $G$. We use the following
Definition 2.7. We say that a form $\omega \in \Omega^*(M, \text{ad} P)$ is reducible to $T$ if there exists a $T$-subbundle of $P$, such that $\omega$ restricted to it takes values in $\text{Lie}(T)$.

When we require the reducibility of the connection $A$ and of some forms $\omega_i \in \Omega^*(M, \text{ad} P)$, it will be understood that there will exist a $T$-subbundle of $P$ where the above forms are reducible simultaneously. When we restrict ourselves to considering the bundle $L_A P$ of horizontal paths whose projections are loops, then a sufficient condition for the flatness of $(A, B)$ is given by the following

Theorem 2.8. The curvature of the connection $(A, B)$ on $L_A P$ is zero if the following conditions are satisfied:

1. $F_A = 0$,
2. $d_A B = 0$,
3. $A$ and $B$ are reducible to $T$.

To conclude this section, we recall that the bundles $ev_0^* P$ and $\mathcal{P}_A P$ are isomorphic via (2.4). We denote by $\mathcal{A}(A, \bar{A}, B)$ and $\mathcal{A}(A, B)$ the connections on $ev_0^* P$ induced respectively by the connection $(A, \bar{A}, B)$ and $(A, B)$ on $\mathcal{P}_A P$.

Namely we set

$$
\mathcal{A}(A, \bar{A}, B) \triangleq ev_0^* \bar{A} + \int_I j_A^* B, \quad (2.19)
$$

$$
\mathcal{A}(A, B) \triangleq ev_0^* A + \int_I j_A^* B, \quad (2.20)
$$

where $j_A$ has been defined in (2.5). The curvature of (2.19) and (2.20) will be denoted respectively by the symbols $\mathcal{F}(A, \bar{A}, B)$ and $\mathcal{F}(A, B)$.

3. Horizontal lift of paths of paths and the non-abelian Stokes formula

In this section we consider the parallel transport of paths of paths and in particular of imbedded paths of paths. We discuss when the relevant parallel transport is invariant under isotopy.

Any connection on $\mathcal{P}_A P$ defines horizontal lifts of a path of paths $\Gamma$ in $M$, namely, of a map

$$
\Gamma: [0, 1] \times [0, 1] \to M.
$$

Each of these horizontal lifts depends on the choice of the initial path $q \in \mathcal{P}_A P$, with $\pi(q(t)) = \Gamma(0, t)$. In turn this initial path $q$, being $A$-horizontal, depends only on the choice of a initial point $q(0) \in \pi^{-1} \Gamma(0, 0)$. So we will speak of horizontal lift of paths of paths with respect to an initial point $p_0 \in \pi^{-1} \Gamma(0, 0)$.

The horizontal lift of a path of paths $\Gamma$ with respect to a given connection on $\mathcal{P}_A P$ is, by definition, a path of $A$-horizontal paths. So we have the following:
Theorem 3.1. The horizontal lift of $\Gamma: I \times I \to M$, with respect to a given connection on $P_A P$ and an initial point $p_0 \in \pi^{-1}(0,0)$, is uniquely determined by the lift of the path of the initial points of the given paths $s \sim \Gamma(s,0) \in M$.

Notice that the lift of the path of initial points considered in Theorem 3.1 coincides with the horizontal lift with respect to a connection $\bar{\alpha} \in A$ only if we choose the special connection $(\alpha, \bar{\alpha}, 0)$ on $P_A P$. For a general connection on $P_A P$, the lift of the path of initial points is more general than the horizontal lift: it is still $G$-equivariant but depends on the whole path of paths $\Gamma$.

It is therefore convenient to consider the following general definition of path-lifting for a principal bundle $P(M,G)$.

Definition 3.2. A lift is a smooth map: $\mathfrak{h}: ev_0^* P \to \mathcal{P}P$ satisfying the conditions

$$\mathfrak{h}(p,\gamma)(0) = p; \quad \pi(\mathfrak{h}(p,\gamma)) = \gamma,$$

and the $G$-equivariance

$$\mathfrak{h}(ph,\gamma) = [\mathfrak{h}(p,\gamma)]h, \quad \forall h \in G.$$

Our definition of lift is a smooth $G$-equivariant version of the definition of a “connection” for the fibration $\pi: P \to M$, as given, e.g., in [24]. But we will use the term “lift” instead of “connection” in order to avoid confusion with the ordinary connections on $P(M,G)$.

We recall that a path-field is a smooth map $M \to \mathcal{P}M$ that assigns to each $x \in M$ a path beginning at $x$. Each path-field $Z$ composed with $\gamma$ yields an element of $\mathcal{P}(\mathcal{P}M)$. When we lift $Z \circ \gamma$ via a connection on $P_A P$, we obtain a path of paths in $P$ whose initial points are a lift of $\gamma$ in the sense of Definition 3.2. Hence Theorem 3.1 can be rephrased as follows:

Theorem 3.3. If we denote by $\mathfrak{P}(M)$ the space of path-fields on $M$, by $\mathfrak{H}(P)$ the space of lifts as in Definition 3.2 and by $\mathcal{A}(ev_0^* P)$ the space of connections on $ev_0^* P$, then we have a map:

$$\mathfrak{P}(M) \times \mathcal{A}(ev_0^* P) \to \mathfrak{H}(P). \quad (3.2)$$

In particular standard horizontal lifts correspond either to the choice of a special connection of the type $(\alpha, \bar{\alpha}, 0)$ on $P_A P$ together with an arbitrary choice of a path-field or to an arbitrary choice of a connection on $P_A P$ together with the choice of the trivial path-field (i.e. the path field assigning the constant path to every point of $M$). This shows in particular that (3.2) is far from being injective.

Following definition (2.2) we denote the horizontal lift with respect to the connection $\omega$ on $P_A P$ by the symbol $\mathfrak{L}(\omega, \Gamma, p_0)$. The comparison of the lift of the initial points of $\Gamma \in \mathcal{P}(\mathcal{P}M)$ with respect to the two connections $\omega$ and $(\alpha, 0)$ defines a path $k_{\Gamma, \omega} \in \mathcal{P}G$ such that

$$\mathfrak{L}(\omega, \Gamma, p_0)(s,0) = \mathfrak{L}((\alpha,0), \Gamma, p_0)(s,0) \cdot k_{\Gamma, \omega}(s). \quad (3.3)$$
Due to the $A$-horizontality of the the lifted paths $t \sim L(\omega, \Gamma, p_0)(s, t)$, equation (3.3) holds also for a generic $t$, namely we have

$$L(\omega, \Gamma, p_0)(s, t) = L((A, 0), \Gamma, p_0)(s, t) \cdot k_{\Gamma,\omega}(s).$$  \hspace{1cm} (3.4)

When we work with a fixed path of paths $\Gamma$ and a fixed base point $p_0 \in \pi^{-1}\Gamma(0, 0)$ we use a simplified notation, i.e., we set

$$L_A(s, t) \equiv L((A, 0), \Gamma, p_0)(s, t).$$  \hspace{1cm} (3.5)

From (3.3) we conclude that $k_{\Gamma,\omega}$ satisfies the following differential equation:

$$\frac{dk_{\Gamma,\omega}(s)}{ds}k_{\Gamma,\omega}(s)^{-1} = -\omega(L'_A(s, \cdot)),$$  \hspace{1cm} (3.6)

where we have to keep in mind that for any $s \in I$, the map

$$L'_A(s, \cdot)(t) \equiv L'_A(s, t)$$

represents a tangent vector in $T\mathcal{P}_A P$.

We will use consistently in this paper the notation of (3.6). Namely for any function $f$ of several variables $a, b, c, d, \ldots$ we denote by $f(\cdot, b, c, d, \ldots)$ the function of one variable $(a)$ obtained by evaluating $f$ at $b, c, d, \ldots$.

When we choose $\omega$ to be the special connection $(A, A + \eta, B)$, then equations (3.4) and (2.18) imply the following differential equation:

$$\frac{dk_{\Gamma,A,\eta,B}(s)}{ds}k_{\Gamma,A,\eta,B}(s)^{-1} = -\int_0^1 B(L'_A(s, t), \dot{L}_A(s, t))dt - \eta(L'_A(s, 0)).$$  \hspace{1cm} (3.7)

The solution is given by a path-ordered exponential (in the variable $s$)

$$k_{\Gamma,A,\eta,B}(s) = \mathcal{P} \exp \left\{ -\int_{[0,s]} ds_1 \left[ \int_0^1 B(L'_A(s_1, t), \dot{L}_A(s_1, t))dt + \eta(L'_A(s_1, 0)) \right] \right\}. \hspace{1cm} (3.8)$$

If $G$ is an abelian group (e.g., $U(1)^n$), then path-ordering is not needed.

Consider now the evaluation map $ev: \mathcal{P}(\mathcal{P}M) \times I \times I \to M$ and the pulled-back bundle $ev^*_{0,0} P$ whose elements are represented precisely by pairs $(\Gamma, p_0)$ where $\Gamma$ is a path of paths on $M$ and $p_0 \in P$ is an element in the fiber over $\Gamma(0, 0)$.

We have the following

**Theorem 3.4.** Any connection $\omega$ on $\mathcal{P}_A P$ determines a map

$$\mathcal{H}_\omega: ev^*_{0,0} P \to G$$
of the adjoint type, i.e. satisfying the equation
\[ \mathcal{H}_\omega(\Gamma, pg) = \text{Ad}_{g^{-1}}(\mathcal{H}_\omega(\Gamma, p)), \quad \forall g \in G. \] (3.9)

In particular when \( \omega \) is a special connection \((A, A + \eta, B)\), then the map \( \mathcal{H}_{(A, A + \eta, B)} \) has the following properties
\[ \mathcal{H}_{(A^\psi, A^\psi + \text{Ad}_{\psi^{-1}} \eta, \text{Ad}_{\psi^{-1}} B)}(\Gamma, p) = \text{Ad}_{\psi^{-1}}(p) \left( \mathcal{H}_{(A, A + \eta, B)}(\Gamma, p) \right), \quad \forall \psi \in G. \] (3.10)

Proof of Theorem 3.4

We define \( \mathcal{H}_\omega(\Gamma, p) \equiv k_{\Gamma, \omega}(1) \), where the r.h.s. is in turn defined by (3.3). (3.10) is a consequence of (3.8).

Theorem 3.4 summarizes the properties of the horizontal lift of paths of paths. Let us now consider the “square” associated to (i.e. the image of) a path of paths \( I \times I \) to \( M \). If we compute the map \( \mathcal{H} \) applied to two different paths of paths with the same image in \( M \), is the result the same? The answer is in general no, but some special situations are worth of consideration.

First we consider the case of the trivial connection on \( \mathcal{P}_A P \). It follows from the definition, that for any \( \Gamma \) and for any \( A \in \mathcal{A} \), we have \( \mathcal{H}_{A,0}(\Gamma, p_0) = 1 \).

Next we consider the tautological connection \((A, -F_A)\). For this connection the non-abelian Stokes formula holds, namely we have the following

**Theorem 3.5.** For any path of paths \( \Gamma \in \mathcal{P}(\mathcal{P}M) \) and for any \( p_0 \in \pi^{-1}(0,0) \) we have
\[ \mathcal{H}_{(A, -F_A)}(\Gamma, p_0) = \text{Hol}_A(\partial \Gamma, p_0). \]

Here \( \partial \Gamma: [0,1] \to M \) denotes the (smooth)[1] loop defined by
\[
\begin{align*}
(\partial \Gamma)(\tau) &= \Gamma(0, 4\tau), & 0 \leq \tau \leq \frac{1}{4} \\
(\partial \Gamma)(\tau) &= \Gamma(4\tau - 1, 1), & \frac{1}{4} \leq \tau \leq \frac{3}{4} \\
(\partial \Gamma)(\tau) &= \Gamma(1, 3 - 4\tau), & \frac{3}{4} \leq \tau \leq \frac{1}{2} \\
(\partial \Gamma)(\tau) &= \Gamma(4 - 4\tau, 0). & \frac{1}{2} \leq \tau \leq 1.
\end{align*}
\]

Proof of Theorem 3.5

[1] We assume that, when needed, all corners are properly smoothed.
Here the connection on the bundle of horizontal paths, is given by $ev^*_1 A$. Hence the $(A, -F_A)$-horizontal lift at $p_0 \in P$ of the path of initial points $\Gamma(\bullet, 0)$ is obtained as follows.

We first consider the $A$-horizontal lift of $\Gamma(0, \bullet)$ and its end point $p_1 = \Gamma(0, 1)$. Then we consider the $A$-horizontal lift of the path of end-points $\Gamma(\bullet, 1)$ beginning at $p_1$ and the $A$-horizontal lift of all the paths $\Gamma(s, \bullet)$ for $s \in (0, 1]$ with assigned end-point. The resulting path of initial points is the $(A, -F_A)$-horizontal lift of $\Gamma(\bullet, 0)$. The theorem follows immediately from Theorem 3.4.

The non-abelian Stokes Formula has a long history, starting from [10], [11]. For some more recent papers see [25], [26], [27]. The treatment of the problem as a problem of parallel transport in a space of paths is new.

We now consider different paths of paths with the same image in $M$ and see if their image with respect to the map given in Theorem 3.4 is the same.

In this section, from now on, we limit ourselves to considering imbedded paths (or loops) of paths. We assume, in particular, that we have an isotopy $\Gamma_r : I \times I \to M$, $r \in [0, 1]$ satisfying the following assumptions:

$\textbf{G.1}$ $\Gamma_r(0, 0) = \Gamma_0(0, 0), \quad \forall r \in [0, 1]$
$\textbf{G.2}$ $\text{Im}(\Gamma_r) = \text{Im}(\Gamma_0), \quad \forall r \in [0, 1]$
$\textbf{G.3}$ $\text{Im}(\Gamma_r(\bullet, 0)) = \text{Im}(\Gamma_0(\bullet, 0)), \quad \forall r \in [0, 1].$

By taking derivatives of $\Gamma_r$ with respect to the parameter $r$, we define for each $r$ a smooth map $Z_r : I \times I \to TM$, with $Z_r(s, t) \in T_{\Gamma_r(s, t)}M$. The following conditions for $Z_r$ are a consequence of the corresponding conditions for $\Gamma_r$:

$\textbf{Z.1}$ $Z_r(0, 0) = 0$
$\textbf{Z.2}$ $Z_r(s, t)$ is tangent to $\text{Im}(\Gamma_0)$ and the restriction of $Z_r$ to $\partial(I \times I)$ is tangent to $\text{Im}(\partial \Gamma_0)$.
$\textbf{Z.3}$ $Z_r(s, 0)$ is tangent to $\text{Im}(\Gamma_0(\bullet, 0))$ and $Z_r(1, 0) = 0$

When do we have $\mathcal{H}(\Gamma_r, p_0) = \mathcal{H}(\Gamma_0, p_0)$? We have the following partial answer

**Theorem 3.6.** If $\Gamma_r : I \times I \to M$ is an isotopy satisfying the conditions G.1 and G.2 above and if, moreover, $F_A = 0$, then for any $B \in \Omega^2(M, \text{ad}P)$ we have

$$\mathcal{H}_{A, \lambda B}(\Gamma_r, p_0) = \mathcal{H}_{A, \lambda B}(\Gamma_0, p_0) + o(\lambda), \quad \forall r \in [0, 1]. \quad (3.11)$$

If, in addition, condition G.3 is satisfied, then we have also

$$\mathcal{H}_{(A, A + \lambda \eta, \lambda B)}(\Gamma_r, p_0) = \mathcal{H}_{(A, A + \lambda \eta, \lambda B)}(\Gamma_0, p_0) + o(\lambda), \quad \forall r \in [0, 1]. \quad (3.12)$$
Here we have assumed that some representation of the group $G$ has been chosen so that the sum in (3.11) and (3.12) makes sense. Even though we do not have in general a true horizontal lift of squares or of surfaces, the implications of Theorem 3.6 are that in some particular cases such horizontal lifts do exist. This is true if we consider imbeddings or immersions as paths of paths, and small deviations from a flat connection and from $B = 0$.

**Proof of Theorem 3.6**

Consider $\mathcal{L}((A,0), \Gamma_r, p_0)$, i.e., the $(A,0)$-horizontal lift of $\Gamma_r$. Since $A$ is flat, the image under $\mathcal{L}((A,0), \Gamma_r, p_0)$ of any curve in $I \times I$ is $A$-horizontal in $P$. We take derivatives with respect to $r$ in $\mathcal{L}((A,0), \Gamma_r, p_0)$ and obtain a map $\bar{Z}_r: I \times I \to TP$. For each $(s, t)$, $\bar{Z}_r(s, t)$ is now a horizontal lift of $Z_r(s, t)$. The map $r \rightsquigarrow \mathcal{H}_{A, \lambda B}(\Gamma_r, p_0)$ defines a curve in $G$. By taking the logarithmic derivative of the above map, we obtain an element of $\mathfrak{g}$. If this element is zero, up to terms of order $\lambda^2$ and for any $r$, then the theorem is proved.

We use again a simplified notation by setting $L_{A,r}(s, t) \equiv L_{((A,0), \Gamma_r, p_0)}(s, t)$. (3.13)

We first consider equation (3.11). By taking into account (3.8), we see that the element in $\mathfrak{g}$ we are looking for is

$$- \int_{I \times I} \frac{d(\mathcal{L}_{A,r}^* B)}{dr} + o(\lambda).$$

(3.14)

The integrand above coincides with

$$\mathcal{L}_{A,r}^* (L_{\bar{Z}_r} B).$$

But the Stokes theorem and property $Z_2$ imply that the integral of $\mathcal{L}_{A,r}^* di_{\bar{Z}_r} B$ vanishes. Moreover we have

$$\int_{I \times I} \mathcal{L}_{A,r}^* i_{\bar{Z}_r} dB = \int_{I \times I} \mathcal{L}_{A,r}^* i_{\bar{Z}_r} dAB = 0$$

since, for any $X, Y \in T_{s,t}(I \times I)$, the three vectors $(\mathcal{L}_{A,r})_* X$, $(\mathcal{L}_{A,r})_* Y$, $\bar{Z}_r(s, t)$ are $A$-horizontal and linearly dependent.

As for (3.12), we set

$$\mathcal{L}_{A,r,0}(s) \equiv \mathcal{L}((A,0), \Gamma_r, p_0)(s, 0).$$

(3.15)

In order to prove (3.12) we have to show the vanishing of the term

$$\int_I \frac{d(\mathcal{L}_{A,r,0}^* \eta)}{dr} = \int_I \mathcal{L}_{A,r,0}^* (L_{\bar{Z}_r} \eta).$$

(3.16)

The integral (3.16) vanishes since the Stokes theorem and conditions $Z.1$-$Z.2$ imply that $\int_I \mathcal{L}_{A,r}^* (di_{\bar{Z}_r} \eta)$ vanishes, while condition $Z.3$ implies $\int_I \mathcal{L}_{A,r}^* (i_{\bar{Z}_r} dA \eta) = 0$.  

16
Remark 3.7. If the image of $\Gamma_r$ is contained in a submanifold $i:N \hookrightarrow M$, then in order for the conclusions of Theorem 3.6 to remain true, it is enough to require $i^*F_A = 0$. Moreover in (3.11) and (3.12) we may replace $\lambda B$ with any $B(\lambda)$ such that $i^*B = o(\lambda)$.

Let us come back to the non-abelian Stokes formula. This formula implies that $\text{Tr} \, \mathcal{H}_{(A,-F_A)}(\Gamma,p_0)$ coincides with the Wilson loop of the boundary $\partial \Gamma$.

We recall that the Wilson loop is defined precisely as $\text{Tr} \, \text{Hol}_A(\gamma,p_0)$ for $\gamma \in \mathcal{L}M$, $A \in \mathcal{A}$ and $\pi(p_0) = \gamma(0)$.

When we consider instead of the tautological connection a generic special connection $(A,B)$, the corresponding generalized Wilson loop $\text{Tr} \, \mathcal{H}_{(A,B)}(\Gamma,p_0)$ depends on the path of paths $\Gamma$ and not only on $\partial \Gamma$.

This may be relevant for the understanding of the quark-confinement problem in the framework of $BF$-theories.

In particular we are interested in considering generalized Wilson loops represented by deformations of the ordinary Wilson loop, where, up to the second order in the perturbative expansion, $\text{Tr} \, \mathcal{H}_{(A,B)}(\Gamma,p_0)$ depends only on the surface $\text{Im}(\Gamma)$ and not on the particular path of paths $\Gamma$.

Accordingly we consider a special connection given by a perturbation series in a neighborhood of a flat connection $(A,0)$, where $F_A = 0$.

We may use two different variables $\kappa$ and $\lambda$ to describe respectively the deformation of the connection $A$ and of the 2-form field $B$, i.e. we set:

$$A(\kappa) \equiv A + \kappa \eta + o(\kappa), \quad B(\lambda) \equiv \lambda B + o(\lambda). \tag{3.17}$$

We now choose a smooth isotopy of imbeddings $\Gamma_r$ (or smooth homotopy of immersions) as before and set

$$\mathcal{H}(\kappa, \lambda, r, p_0) \equiv \mathcal{H}_{(A(\kappa),-F_{A(\kappa)}+B(\lambda))}(\Gamma_r,p_0). \tag{3.18}$$

If we have only one parameter $\kappa$, we set

$$\mathcal{H}(\kappa, r, p_0) \equiv \mathcal{H}_{(A(\kappa),-F_{A(\kappa)}+B(\kappa))}(\Gamma_r,p_0). \tag{3.19}$$

When $\lambda = 0$, then the non-abelian Stokes formula implies that $\mathcal{H}(\kappa, \lambda = 0, r, p_0)$ is independent of $r$.

In the general case the power series expansions of $\mathcal{H}(\kappa, \lambda, r, p_0)$ and $\mathcal{H}(\kappa, r, p_0)$ depend on $r$ but satisfy the following:

Theorem 3.8. Let $P(M,G)$ be a principal $G$-bundle admitting a flat connection $A$. Let $\Gamma_r$ satisfy G.1 and G.2 and let $A(\kappa)$ and $B(\lambda)$ be defined as above. For $\mathcal{H}(\kappa, \lambda, r, p_0)$ given by (3.18) we have the following equation:

$$\left. \frac{\partial^2 \mathcal{H}(\kappa, \lambda, r, p_0)}{\partial \lambda \partial r} \right|_{\kappa=\lambda=0} = 0. \tag{3.20}$$
If we have only one parameter $\kappa = \lambda$, and we assume also G.3, then we have
\[
\frac{\partial^2 \mathcal{H}(\kappa, r, p_0)}{\partial \kappa \partial r} \bigg|_{\kappa=0} = 0,
\] (3.21)
where definition (3.19) has been assumed.

Theorem 3.8 provides a surface law for the generalized Wilson loop in BF-theories. The main difference between Theorem 3.6 and Theorem 3.8 lies in the fact that in the latter the field $B$ deforms a (non-trivial) tautological connection (for which the non-abelian Stokes formula holds) at any order in $\kappa$.

Proof of Theorem 3.8

As in (3.3) and (3.5) we set
\[
\mathcal{L}((A(\kappa), -F_A(\kappa) + B(\lambda)), \Gamma_r, p_0)(s, t) \equiv \mathcal{L}((A, 0), \Gamma_r, p_0)(s, t)k_{(r, \kappa, \lambda)}(s),
\]
with $k_{(r, \kappa, \lambda)} \in \mathcal{P}G$ and
\[
\mathcal{L}_{(r, \kappa)}(s, t) \equiv \mathcal{L}((A, 0), \Gamma_r, p_0)(s, t).
\]
Analogously to (3.7) we have
\[
\frac{dk_{(r, \kappa, \lambda)}(s)}{ds} k_{(r, \kappa, \lambda)}(s)^{-1} = -\kappa \eta(\mathcal{L}'_{(r, \kappa)}(s, 0)) + \int_0^1 [F_A(\kappa) - B(\lambda)](\mathcal{L}'_{(r, \kappa)}(s, t), \dot{\mathcal{L}}_{(r, \kappa)}(s, t)) dt.
\] (3.22)

By taking the derivative of (3.22) with respect to $\lambda$ at $\kappa = \lambda = 0$ the r.h.s. of (3.22) becomes
\[
- \int_0^1 B \left( \mathcal{L}'_{(A, r)}(s, t), \dot{\mathcal{L}}_{(A, r)}(s, t) \right) dt,
\]
(see (3.13) for the notation) and the proof of Theorem 3.6 applies verbatim to our case.

As for (3.21) we notice that we have to replace $\lambda$ with $\kappa$ in (3.22). The derivative with respect to $\kappa$ at $\kappa = 0$ of the r.h.s of (3.22) becomes
\[
-\eta \left( \mathcal{L}'_{(A, r)}(s, 0) \right) + \int_0^1 \left[ -B + d_A \eta \right] \left( \mathcal{L}'_{(A, r)}(s, t), \dot{\mathcal{L}}_{(A, r)}(s, t) \right) dt.
\]
We differentiate again with respect to $r$. Using G.3 and the same arguments as in Theorem 3.6, we obtain (3.21).

We end this section by considering the special case of $\Gamma$ being an imbedded loop of paths. The holonomy with respect to the connection $(A, B)$ is then given by
\[
\text{Hol}_{(A, B)}(\Gamma, p_0) = \text{Hol}_A(\Gamma(\bullet, 0), p_0)\mathcal{P} \exp \left( - \int_{\mathcal{L}_A([0,1] \times [0,1])} B \right),
\] (3.23)
It is clear that $\text{Hol}_{A,B}(\Gamma, p_0)$ is given by the group element $g = g(\Gamma, (A, B), p_0)$ such that $p_0 g$ is the end-point of the $(A, B)$-horizontal lift of the loop of initial points $\Gamma(\bullet, 0)$. If $B = 0$, then the above holonomy is nothing else than the $A$-holonomy of the loop of initial points.

**Remark 3.9.** If $\Gamma \in \mathcal{L}(\mathcal{P}M)$ then we have:

$$\mathcal{H}_{(A,B)}(\Gamma, p_0) = \text{Hol}_A^{-1}(\Gamma, p_0)\text{Hol}_{(A,B)}(\Gamma, p_0)$$

and if $\Gamma$ is a loop of loops:

$$\mathcal{H}_{(A,-F_A)}(\Gamma, p_0) = \text{Hol}_A^{-1}(\Gamma(\bullet, 0), p_0)\text{Hol}_A(\Gamma(0, \bullet), p_0)\text{Hol}_A(\Gamma(\bullet, 0), p_0)\text{Hol}_A^{-1}(\Gamma(0, \bullet), p_0).$$

4. Local coordinates

Here we discuss the expressions of the special connections on $\mathcal{P}_A P$ and of the relevant parallel transport in *local coordinates*.

Let $U$ be the domain of a local chart in $M$. We denote by $\mathcal{P}U$ the space of paths in $U$ and by $\mathcal{P}_U M$ the space of paths in $M$ with initial point in $U$. Any section $\sigma: U \subset M \rightarrow P$ determines a section

$$\tilde{\sigma}: \mathcal{P}_U M \rightarrow \mathcal{P}_A P,$$

$$\gamma \rightsquigarrow \tilde{\sigma}(\gamma) \triangleq \mathcal{L}(A, \gamma, \sigma(\gamma(0))),$$

where, as before, $\mathcal{L}$ denotes the horizontal lift. So the bundle of horizontal paths is trivial if and only if the bundle $P$ is trivial.

**Definition 4.1.** For any section $\sigma: U \rightarrow P$ we define $h: \mathcal{P}U \times I \rightarrow G$ by the equation

$$\sigma(\gamma(t))h(\gamma, t) \equiv [\tilde{\sigma}(\gamma)](t).$$

The map $h$ allows us to compare the $A$-parallel transport with the image of a section $\sigma$ and is given by the standard path-ordered exponential of the integral

$$h(\gamma, t) = \mathcal{P} \exp \int_{[0, t]} \gamma^*(-\sigma^*A).$$
4.1. Connections

Given any \( X \in T_\gamma(\mathcal{P}M) \), we set \( h_\gamma(t) \equiv h(\gamma,t) \), \( q = \tilde{\sigma}(\gamma) \) and \( q \equiv \tilde{\sigma}_* X \). We have
\[
q = \sigma_* X h_\gamma + i((h^{-1}dh)X),
\]
where the map \( i: g \to \mathfrak{X}(P) \) is, by definition, the map yielding fundamental vector fields. Moreover we have
\[
\dot{q} = (\sigma_* \dot{\gamma}) h_\gamma + (\sigma \circ \gamma) h_\gamma.
\]
The second terms of both (4.3) and (4.4) are vertical vectors fields along \( q \), so we finally obtain
\[
\tilde{\sigma}^* \left( \int q^* B \right)(X) = \int_0^1 dt \ Ad_{h_\gamma^{-1}(t)} (\sigma^* B(X(t),\dot{\gamma}(t))).
\]
This is the expression in local coordinates of the difference between the connection \( (A, B) \) and the connection \( (A, 0) \).[2]

4.2. Horizontal lift of paths of paths

Now we compute the \( (A, B) \)-horizontal lift of \( \Gamma \in \mathcal{P}(\mathcal{P}M) \) in local coordinates.

Consider a map \( \Gamma: I \times I \to M \), where the first variable \( (s) \) describes the path of paths, while the second variable \( (t) \) describes each individual path. We assume that the image of \( \Gamma \) is all contained in the domain \( U \) of a local section \( \sigma: U \subset M \to P \).

We consider the section \( \tilde{\sigma} \) (4.2)) on \( \mathcal{P}A \). We have explicitly, for each fixed \( s \in I \),
\[
(\tilde{\sigma})_\Gamma(s, t) = \sigma(\Gamma(s, t)) h_{\Gamma(s, \bullet)}(t),
\]
where \( h \) is as in Definition 4.1 and \( \Gamma(s, \bullet) \) denotes the path in \( \mathcal{P}M \) given by \( t \sim \Gamma(s, t) \).

The \( (A, 0) \)-horizontal lift of \( \Gamma \) is given, in local coordinates, by
\[
(s, t) \sim (\tilde{\sigma})_\Gamma(s, t) h_{\Gamma(\bullet, 0)}(s) = \sigma(\Gamma(s, t)) h_{\Gamma(s, \bullet)}(t) h_{\Gamma(\bullet, 0)}(s).
\]

When we consider as in (3.3) the path \( k_{\Gamma, A, B}: I \to G \), then the \( (A, B) \)-horizontal lift of \( \Gamma \) is given, in local coordinates, by
\[
(s, t) \sim (\tilde{\sigma})_\Gamma(s, t) h_{\Gamma(\bullet, 0)}(s) k_{\Gamma, A, B}(s).
\]

We now consider the following vectors in \( T_{(\tilde{\sigma})_\Gamma(s, t)} P \):
\[
\Gamma_1(s, t) \triangleq \frac{\partial (\tilde{\sigma})_\Gamma}{\partial s}(s, t), \quad \Gamma_2(s, t) \triangleq \frac{\partial (\tilde{\sigma})_\Gamma}{\partial t}(s, t).
\]

We also set
\[
K_{\Gamma, A, B}(s) \triangleq h_{\Gamma(\bullet, 0)}(s) k_{\Gamma, A, B}(s).
\]

[2] The expression (4.5) was firstly considered in [9] where the notation \( \text{Hol}_A(\gamma)_0 \) for \( h_\gamma(t) \) was employed.
The \((A, B)\)-horizontality of (4.6) translates into the following equations

\[
\frac{dK_{\Gamma, A, B}(s)}{ds} K_{\Gamma, A, B}^{-1}(s) + A(\Gamma_1(s, 0)) + \int_0^1 dt \ B(\Gamma_1(s, t), \Gamma_2(s, t)) = 0, \tag{4.8}
\]

\[
\frac{dh_{\Gamma(\bullet, 0)}(s)}{ds} h_{\Gamma(\bullet, 0)}^{-1}(s) + A(\Gamma_1(s, 0)) = 0. \tag{4.9}
\]

Hence we have

\[
\frac{dk_{\Gamma, A, B}(s)}{ds} k_{\Gamma, B}^{-1}(s) + \text{Ad}_{h_{\Gamma(\bullet, 0)}^{-1}(s)} \int_0^1 dt \ B(\Gamma_1(s, t), \Gamma_2(s, t)) = 0, \tag{4.10}
\]

and

\[
\int_0^1 dt \ B(\Gamma_1(s, t), \Gamma_2(s, t)) = \int_0^1 dt \ \text{Ad}_{h_{\Gamma(s, \bullet)}^{-1}(t)}(\sigma^* B) \left( \Gamma'(s, t), \dot{\Gamma}(s, t) \right).
\]

The solution of (4.10) is finally given by

\[
k_{\Gamma, A, B}(s') = \mathcal{P}_{s'} \exp \left\{ - \int_{s'}^{s} ds \ \text{Ad}_{h_{\Gamma(s, \bullet)}^{-1}(s)} \int_0^1 dt \ \text{Ad}_{h_{\Gamma(s, \bullet)}^{-1}(t)}(\sigma^* B) \left( \Gamma'(s, t), \dot{\Gamma}(s, t) \right) \right\}, \tag{4.11}
\]

where \(\mathcal{P}_{s'}\) denotes path-ordering in the variable \(s'\).

5. Transformation properties of the holonomy as a function of the connection

In this section we consider a generic manifold \(X\) and a principal \(G\)-bundle \(\pi: P_X \to X\) (typically we have in mind either \(X = M\) or \(X = PM\)) and we recall the main properties of the parallel transport of paths and of the holonomy of loops, both seen as functions on the space of connections \(\mathcal{A}(P_X)\).

Let \(\omega \in \mathcal{A}(P_X), \eta \in T\mathcal{A}(P_X), \gamma \in \mathcal{P}X,\) and \(u \in \pi^{-1}\gamma(0) \subset P_X\). We consider the horizontal lift \(\mathfrak{L}(\omega, \gamma, u)\) (2.2). To the path of connections given by \(\kappa \leadsto \omega + \kappa \eta, \kappa \in (-\epsilon, \epsilon)\) we associate the path in \(\mathcal{P}G, \kappa \leadsto g_\kappa = g_\kappa(\gamma, \omega, \eta, u) \in \mathcal{P}G\) given by the solution of the following equation:

\[
\mathfrak{L}(\omega + \kappa \eta, \gamma, u)(t) = \mathfrak{L}(\omega, \gamma, u)(t)g_\kappa(t), \quad g_\kappa(0) = 1. \tag{5.2}
\]

By definition we have

\[
\left( \omega + \kappa \eta \right) \left( \frac{d\mathfrak{L}(\omega + \kappa \eta, \gamma, u)(t)}{dt} \right) = 0, \forall \kappa \in (-\epsilon, \epsilon), \forall t. \tag{5.3}
\]
In this section we use again a simplified notation for the horizontal lift by setting
\[ \mathcal{L}(t) \equiv \mathcal{L}(\omega, \gamma, u)(t), \]
and the relevant evaluation map
\[ ev : I \times \mathcal{L} \to P. \]
The paths \( g_\kappa(t) \) satisfy the following equation in the variable \( t \)
\[ g_\kappa^{-1} \dot{g}_\kappa + \kappa \text{Ad}_{g_\kappa}^{-1} \eta(\dot{\mathcal{L}}) = 0 \]  
\[(5.4). \]
The solution is the path-ordered exponential
\[ g_\kappa(t) = \mathcal{P} \exp \left( -\kappa \int_0^t d\tau \eta(\dot{\mathcal{L}}(\tau)) \right) = \mathcal{P} \exp \left( -\kappa \int_{[0,t]} ev^* \eta \right), \]  
\[(5.5) \]
We are interested in \( H(t) \equiv \frac{d g_\kappa(t)}{d\kappa} \bigg|_{\kappa=0}. \)
By differentiating at \( \kappa = 0 \) (5.4), we get
\[ H(t) = -\int_{[0,t]} dt \eta(\dot{\mathcal{L}}(t)) = -\int_{[0,t]} ev^* \eta. \]  
\[(5.6) \]
Thus we have proved the following

**Theorem 5.1.** For any loop \( \gamma \in \mathcal{L}X \), the logarithmic exterior derivative of the holonomy, seen as a function of the connection \( \omega \in \mathcal{A}X \), is given by
\[ \text{Hol}^{-1}_\omega(\gamma, u) \delta(\text{Hol}_\omega(\gamma, u)) (\eta) = -\int ev^* \eta. \]  
\[(5.7) \]

We denote by \( \text{Aut} \ P_X \) the group of automorphisms of \( P_X \) and by \( \text{aut} \ P_X \) the Lie algebra of infinitesimal automorphisms of \( P_X \). There is an action of \( \text{Aut} \ P_X \) on \( \mathcal{A}X \) and a projection (group homomorphism)
\[ \rho: \text{Aut} \ P_X \to \text{Diff} X \]  
\[(5.8) \]
whose kernel is the group of gauge transformations \( \mathcal{G}(P_X) \).
This projection allows us to define an action of \( \text{Aut} \ P_X \) on \( \mathcal{P}X \). Hence any \( \psi \in \text{Aut} \ P_X \) defines an isomorphism of bundles of horizontal paths
\[ \psi: \mathcal{P}_\omega P_X \to \mathcal{P}_{\psi^* \omega} P_X \]  
\[(5.9) \]
We now want to discuss the effect of this isomorphism on the parallel transport and the holonomy.
The isomorphism (5.9) satisfies the following equation
\[ \mathcal{L}(\psi^*\omega, \rho(\psi^{-1}) \circ \gamma, \psi^{-1}(u)) = \psi^{-1}(\mathcal{L}(\omega, \gamma, u)), \quad \gamma \in \mathcal{P}X. \] (5.10)

This implies that the infinitesimal action of \( \text{aut} \, P_X \) on \( \mathcal{P}_\omega P_X \) is just the opposite of the corresponding action on \( \mathcal{A}_X \). For any \( Z \in \text{aut} \, P_X \) we compute the corresponding Lie derivative
\[ L_Z \omega = d_\omega i_Z \omega + i_Z F_\omega. \]

By setting \( \eta = L_Z \omega \) in (5.7), we get
\[ \frac{d}{ds} \left[ \text{Hol}_{\omega+sL_Z \omega}(\gamma, u) \right]_{s=0} = -\text{Hol}_\omega(\gamma, u) \int_I ev^* (i_Z F_\omega + d_\omega i_Z \omega) = \]
\[ -\text{Hol}_\omega(\gamma, u) \left( \int_I ev^* i_Z F_\omega \right) - \text{Hol}_\omega(\gamma, u) (i_Z \omega(\mathcal{L}(1)) - i_Z \omega(\mathcal{L}(0))). \] (5.11)

If we consider the variation of the trace of the holonomy (in any representation of \( G \)), we have

**Theorem 5.2.** Let \( \omega \in \mathcal{A}_X, \gamma \in \mathcal{L}X, Z \in \text{aut} P_X, \) and \( u \in \pi^{-1}\gamma(0). \) Then we have
\[ (\delta \text{Tr} \, \text{Hol}_\omega(\gamma, u))(L_Z \omega) = -\text{Tr} \left( \text{Hol}_\omega(\gamma, u) \int_I ev^* i_Z F_\omega \right) \] (5.12)

**Proof of Theorem 5.2**

We have
\[ i_Z \omega(\mathcal{L}(1)) = i_Z \omega(\mathcal{L}(0)) \text{Hol}_\omega(\gamma, u) = \text{Hol}_\omega(\gamma, u)^{-1}(i_Z \omega(\mathcal{L}(0))) \text{Hol}_\omega(\gamma, u) \]
and therefore
\[ \text{Tr} \left( \text{Hol}_\omega(\gamma, u)i_Z \omega(\mathcal{L}(1)) - \text{Hol}_\omega(\gamma, u)i_Z \omega(\mathcal{L}(0)) \right) = 0 \]

The result now follows from (5.11).

**Corollary 5.4.** The variation (5.12) vanishes if the restriction of \( Z \) to the image of \( \gamma \) is proportional to the tangent vector \( \dot{\gamma} \).

In particular if the loop is an imbedding, then the corresponding trace of the holonomy is invariant under the action of any \( \psi \in \text{Aut} \, P_X \) connected to the identity for which \( \rho(\psi) \in \text{Diff} \, X \) maps the image of the loop into itself.
6. Holonomy of cylinders and the group of automorphisms of $P$.

In this and the following section we consider loops of paths and loops of loops and study the corresponding holonomies as functions on the space of (special) connections.

We will use the name cylinders to denote the image of loops of paths, even though we are not assuming that such loops of paths are necessarily imbeddings or immersions.

In this section we look for the conditions which guarantee that the (trace of the) holonomy of a loop of paths is invariant under those automorphisms of $P$ which project onto diffeomorphisms connected to the identity, that map the corresponding image (cylinder) into itself.

Since we are considering the action of $\text{Aut } P$ on the space of connections $\mathcal{A}$, it is convenient to work primarily with the bundle $ev_0^*P$ instead of $\mathcal{P}A_P$, for which the choice of a fixed connection $A$ is required. We will, though, make constantly use of the isomorphism $J_A: ev_0^*P \rightarrow \mathcal{P}A_P$ (2.4).

Equation (5.10) says that the group $\text{Aut } P$ of automorphisms of $P$ acts in a natural way on the bundle $ev_0^*P$. In fact we have

$$P \times \mathcal{PM} \ni (p, \gamma) \mapsto (\psi(p), \rho(\psi)(\gamma)), \quad \psi \in \text{Aut } P$$

with $p \in \pi^{-1}\gamma(0) \implies \psi(p) \in \pi^{-1}\rho(\psi)(\gamma(0))$.

The group $\text{Aut } P$ can be identified with a subgroup of $\text{Aut}(ev_0^*P)$. The Lie algebra $\text{aut } P$ can be accordingly identified with a subalgebra of $\text{aut}(ev_0^*P)$.

Given now $Z \in \text{aut } P$ and the corresponding element in $\text{aut}(ev_0^*P)$ which we denote by the same symbol, we want to describe $((J_A)_*Z) \in \text{aut}(\mathcal{P}A_P)$ explicitly. Consider $q \in \mathcal{P}A_P$.

The path

$$t \mapsto (\pi q(t), \rho_* (\pi q(t)))$$

is an element of $\mathcal{T}\mathcal{PM}$. We now lift $\hat{A}$-horizontally (6.2) (see Remark 2.2) with initial point $(p, Z(p)) \in TP$. This lifted path is $((J_A)_*Z)(q)$. For any $t$, $((J_A)_*Z)(q)(t)$ is a vector in $T_{q(t)}P$. Notice that in general $((J_A)_*Z)(q)(t)$ is different from $Z(q(t))$ unless $t = 0$.

The isomorphism (2.4) $J_A: ev_0^*P \rightarrow \mathcal{P}A_P$ and the corresponding evaluation map (2.5) $j_A: ev_0^*P \times I \rightarrow P$ allow us to transform forms on $\mathcal{P}A_P$ defined by Chen integrals into forms defined on $ev_0^*P$. The result of performing first Chen integrals and then pulling back the forms to $ev_0^*P$ via $J_A$ will be represented by the symbol $\int_{\text{Chen}(A)}$. In the special case of line-integrals, we have for a generic $k$-form $\phi$ on $P$

$$\int_{\text{Chen}(A)} \phi = \int_I j_A^* \phi.$$

Then we have the following
Theorem 6.1. The pullback of the connection \( \mathcal{A}(A, B) \) (2.20) via \( \psi \in \text{Aut} \mathcal{P} \) is given by

\[
\psi^*(\mathcal{A}(A, B)) = ev_0^*\psi^*A + \int_I j^*_\psi A \psi^* B. \tag{6.3}
\]

At the infinitesimal level, for any \( Z \in \text{aut} \mathcal{P} \), we have

\[
L_Z \mathcal{A}(A, B) = ev_0^*L_Z A + \int_{\text{Chen}(A)} L_Z B + \int_{\text{Chen}(A)} \{L_Z A; B\}. \tag{6.4}
\]

Proof of Theorem 6.1

We have

\[
L_Z \mathcal{A}(A, B) = ev_0^*L_Z A + \int_{\text{Chen}(A)} L_Z B + \frac{d}{dk}\bigg|_{k=0}^{} \int_{\text{Chen}(A+\kappa L_Z A)} \{L_Z A; B\}. \tag{6.5}
\]

If we are given \( \eta \in \Omega^1(M, \text{ad} \mathcal{P}) \) and \( \zeta \in \Omega^*(M, \text{ad} \mathcal{P}) \) we have

\[
\frac{d}{dk}\bigg|_{k=0}^{} \int_{\text{Chen}(A+\kappa \eta)} \zeta = \frac{d}{dk}\bigg|_{k=0}^{} \int_I j^*_A \eta \zeta = \frac{d}{dk}\bigg|_{k=0}^{} \int_I \text{Ad}_{g_k^{-1}} j^*_A \zeta
\]

where \( g_k \) is defined as in in (5.2). Now the proof follows from (5.6).

The curvature \( \mathcal{F}(A, B) \) of \( \mathcal{A}(A, B) \) at \( (q, p) \), is given by

\[
ev_0^*F_A - j_A(1)^*B + ev_0^*B + \int_{\text{Chen}(A)} d_A B + \int_{\text{Chen}(A)} \{B + F_A; B\}. \tag{6.6}
\]

A direct consequence of Theorem 5.2 is

Theorem 6.2. Let \( \Gamma \in \mathcal{L}(\mathcal{P}M) \) and \( Z \in \text{aut} \mathcal{P} \). The trace of the holonomy \( \text{Hol}_{\mathcal{A}(A, B)}(\Gamma, p) \) in \( ev_0^*P \) with respect to \( \mathcal{A}(A, B) \) transforms as follows

\[
\delta \text{Tr} \text{Hol}_{\mathcal{A}(A, B)}(\Gamma, p)(Z) = -\text{Tr} \left( \text{Hol}_{\mathcal{A}(A, B)}(\Gamma, p) \int_I ev^* i_Z \mathcal{F}(A, B) \right), \tag{6.7}
\]

with \( ev: I \times \mathcal{L}(\mathcal{A}(A, B), \Gamma, p) \to P \).

We now compute explicitly (6.7). First we set

\[
\mathfrak{L}_{A,B}(s, t) \equiv \mathfrak{L}(\mathcal{A}(A, B), \Gamma, p)(s)(t), \quad Z(s, t) \equiv ((J_A)_* Z)(\mathfrak{L}_{A,B}(s, \bullet))(t) \in T_{\mathfrak{L}_{A,B}(s, t)}\mathcal{P}. \tag{6.8}
\]
We can write down (6.7) as follows:

\[ \delta \text{Tr Hol}_{(A,B)}(\Gamma, p)(Z) = -\text{Tr Hol}_{(A,B)}(\Gamma, p) \left\{ \int_0^1 ds F_A \left( Z(s,0), \mathcal{L}'_{A,B}(s,0) \right) + \int_0^1 ds \int_0^1 dt d_A B \left( Z(s,t), \mathcal{L}'_{A,B}(s,t), \dot{\mathcal{L}}_{A,B}(s,t) \right) \right. \]

\[ \left. + \int_0^1 ds \int_0^1 dt \int_0^t d\tau \left[ (B + F_A) \left( Z(s,\tau), \mathcal{L}'_{A,B}(s,\tau) \right), B \left( \mathcal{L}'_{A,B}(s,\tau), \dot{\mathcal{L}}_{A,B}(s,\tau) \right) \right] \right. \]

\[ \left. - \int_0^1 ds \int_0^1 dt \int_0^t d\tau \left[ (B + F_A) \left( \mathcal{L}'_{A,B}(s,\tau), \dot{\mathcal{L}}_{A,B}(s,\tau) \right), B \left( Z(s,\tau), \dot{\mathcal{L}}_{A,B}(s,\tau) \right) \right] \right\}. \]

In order to obtain the vanishing of the previous expression, we make some extra assumptions on the vector field \( Z \), namely:

A) \( \pi_* Z(s,0) \) is proportional to the tangent vector to the path of initial points \( \Gamma(\bullet,0) \) i.e., \( Z(s,0) \) is proportional to \( \mathcal{L}'_{A,B}(s,0) \) up to vertical vectors,

B) \( \pi_* Z(s,t) \) is a linear combination of \( \Gamma'(s,t) \) and \( \dot{\Gamma}(s,t) \), with coefficients that, in general, are functions of \( s \) and \( t \).

Moreover let \( \Sigma \) be a submanifold of \( M \) containing \( \text{Im}(\Gamma) \). For the restriction of \( P \) to \( \Sigma \), we make the following assumptions:

C) the connection \( A \) restricted to the bundle \( P_\Sigma \) is reducible to an abelian subgroup \( T \) of \( G \),

D) the form \( B \in \Omega^2(M, \text{ad}P) \) restricted to \( P_\Sigma \) is (simultaneously) reducible to \( T \)

We have finally the following

**Theorem 6.3.** We have

\[ \delta \text{Tr Hol}_{(A,B)}(\Gamma, p)(Z) = 0 \] (6.9)

provided either that conditions A), B), C), D) are satisfied or that conditions A), B) are satisfied together with the extra requirement that on \( \Sigma \) we have either \( B = -F_A \) or \( B = 0 \).
7. **Invariance properties of the (trace of the) \((A, B)\)-holonomy**

The space of connections on \(ev_0^*P\) of the type \(\mathcal{A}(A, B)\) is isomorphic to the affine space \(\mathcal{A} \times \Omega^2(M, \text{ad}P)\) which is acted upon by some transformation groups, that arise in the framework of quantum field theories of the \(BF\) type (see below).

In this section we want to check under what conditions the trace of the \((A, B)\)-holonomy is invariant under those transformation group.

In quantum field theories one considers first of all the gauge group \(G\). If we divide \(G\) by its center and consider only irreducible connections, then \(G\) acts freely on \(\mathcal{A} \times \Omega^2(M, \text{ad}P)\)

\[(A, B)g = (A^g, \text{Ad}_{g^{-1}}B), \quad (7.2)\]

We have moreover the group \(G_T\) given by the semidirect product \(G \ltimes \Omega^1(M, \text{ad}P)\), where \(G\) acts on the abelian group \(\Omega^1(M, \text{ad}P)\) via the adjoint action. The group \(G_T\) acts non-freely in two ways on \(\mathcal{A} \times \Omega^2(M, \text{ad}P)\). The first action is given by the transformation

\[(A, B) \sim (A^g + \eta, \text{Ad}_{g^{-1}}B - d_{A^g} \eta - \frac{1}{2} [\eta, \eta]), \quad (g, \eta) \in G_T, \quad (7.3)\]

while the second action is given by

\[(A, B) \sim (A^g, \text{Ad}_{g^{-1}}B - d_{A^g} \eta), \quad (g, \eta) \in G_T. \quad (7.4)\]

Before seeing how the above transformation groups act on the holonomy, we compute the derivative of the \((A, B)\)-holonomy as a function on \(\mathcal{A} \times \Omega^2(M, \text{ad}P)\) at \((\eta, \beta) \in T\mathcal{A} \times T\Omega^2(M, \text{ad}P)\)

Under the transformation \(A \sim A + \eta, B \sim B + \beta\) the connection \(\mathcal{A}(A, B) = ev_0^*A + \int j_A^*B\) on the bundle \(ev_0^*P \to \mathcal{P} M\) transforms into

\[ev_0^*(A + \kappa \eta) + \int_{\text{Chen}(A + \kappa \eta)} B + \kappa \beta.\]

The corresponding derivative of the holonomy is given by:

\[\delta \text{Tr} \, \text{Hol}_{\mathcal{A}(A, B)}(\Gamma, p)(\eta, \beta) =\]

\[-\text{Tr} \left( \text{Hol}_{\mathcal{A}(A, B)}(\Gamma, p) \int_0^1 \left( ev_0^*\eta + \int_{\text{Chen}(A)} \beta + \int_{\text{Chen}(A)} \{\eta; B\} \right) \right), \quad (7.5)\]

The integral in the r.h.s. of (7.5) can be written explicitly as

\[\int_0^1 ds \left( \eta(L_{A, B}'(s, 0)) + \int_0^1 dt \beta(L_{A, B}'(s, t), \hat{L}_{A, B}(s, t)) \right) +\]

\[\int_0^1 ds \int_0^1 dt \left[ \int_0^t d\tau \eta(\hat{L}_{A, B}(s, \tau)), B(L_{A, B}'(s, t), \hat{L}_{A, B}(s, t)) \right],\]

where the prime denotes, as usual, the derivative with respect to the variable \(s\) and the dot the derivative with respect to the variable \(t\), and \(L_{A, B}'(s, t)\) has been defined in (6.8).

A direct consequence of Theorem 6.1 is the following
Theorem 7.1. Let $\Gamma: S^1 \times I \to M$ be any loop in $\mathcal{P}M$, let $p$ be such that $\pi(p) = \Gamma(0,0)$ and let $g \in \mathcal{G}$ be any gauge transformation. The trace of the $(A,B)$-holonomy of $\Gamma$ with initial point $p$ is invariant under the transformation

$$(A,B) \leadsto (A^g, \text{Ad}_g^{-1}B).$$

Now we can study the transformation properties of the $(A,B)$-holonomy under (7.3) and (7.4) in the special case when we restrict the elements of $\mathcal{G}$ to be the identity. In this case (7.3) becomes the transformation

$$A \leadsto A + \eta, \quad B \leadsto B - d_A\eta - \frac{1}{2}[\eta,\eta],$$

and we have the following:

Theorem 7.2. When $\beta = -d_A\eta - \frac{1}{2}[\eta,\eta]$ then (7.5) becomes

$$\delta \text{Tr Hol}_{A(A,B)}(\Gamma,p)(\eta,-d_A\eta) = -\text{Tr} \left( \text{Hol}_{A(A,B)}(\Gamma,p) \int_I \left( ev^*_1\eta - \frac{1}{2} \int_{\text{Chen}(A)} [\eta,\eta] - \int_{\text{Chen}(A)} \{B + F_A; \eta\} \right) \right).$$

Proof of Theorem 7.2

We have

$$\int_0^1 \int_0^1 dsdt \left[ \int_0^t d\tau B(\hat{\mathcal{L}}_{A,B}(s,\tau), \mathcal{L}_{A,B}'(s,\tau)) + \int_0^t d\tau F_A(\hat{\mathcal{L}}_{A,B}(s,\tau), \mathcal{L}_{A,B}'(s,\tau), \eta(\hat{\mathcal{L}}_{A,B}(s,\tau))) \right],$$

where we have used (2.18). Therefore

$$-\int_I \left( \int_{\text{Chen}(A)} [A,\eta] - \int_{\text{Chen}(A)} \{\eta; B\} \right) = -\int_0^1 ds \int_0^1 dt \left[ \int_0^t d\tau (F_A + B)(\hat{\mathcal{L}}_{A,B}(s,\tau), \mathcal{L}_{A,B}'(s,\tau), \eta(\hat{\mathcal{L}}_{A,B}(s,\tau))) \right].$$

28
Notice also that

\[
- \int_I \int_{\text{Chen}(A)} d\eta = \int_0^1 dt \left( -\eta(\mathcal{L}_{A,B}(1,t)) + \eta(\mathcal{L}_{A,B}(0,t)) \right) - \\
\int_0^1 ds \left( \eta(\mathcal{L}'_{A,B}(s,0)) - \eta(\mathcal{L}'_{A,B}(s,1)) \right) = \\
\int_0^1 dt \left( \eta(\mathcal{L}_{A,B}(0,t)) - \text{Hol}_{A,B}(\Gamma,p)^{-1}\eta(\mathcal{L}_{A,B}(0,t))\text{Hol}_{A,B}(\Gamma,p) \right) \\
- \int_0^1 ds \left( \eta(\mathcal{L}'_{A,B}(s,0)) - \eta(\mathcal{L}'_{A,B}(s,1)) \right). 
\]

Therefore

\[
- \text{Tr}\text{ Hol}_{A,B}(\Gamma,p) \left( \int_0^1 ds \left( \eta(\mathcal{L}'_{A,B}(s,0)) - \int_0^1 dt d_A\eta(\mathcal{L}'_{A,B}(s,t),\mathcal{L}_{A,B}(s,t)) \right) + \\
\int_0^1 ds \int_0^1 dt \left[ \int_0^t d\tau \eta(\mathcal{L}_{A,B}(s,\tau)), B(\mathcal{L}'_{A,B}(s,t),\mathcal{L}_{A,B}(s,t)) \right] \right) = \\
\int_I ev_1^*\eta - \int_{\text{Chen}(A)} \{F_A + B;\eta\}
\]

Finally we take into account Theorem 7.2 and consider the invariance properties under (7.4) of the trace of the holonomy of a loop of paths \(\Gamma: S^1 \times I \rightarrow M\). The previous discussion yields the following

**Theorem 7.3.** Corresponding to the action (7.4) we have

\[
\delta \text{Tr}\text{ Hol}_{A,B}(\Gamma,p)(0, d_A\eta) = \\
\text{Tr} \left[ \text{Hol}_{A,B}(\Gamma,p) \left( \int_I \left( \int_{\text{Chen}(A)} \{F_A;\eta\} + \left[ \int B, \int \eta \right] \right) + \\
\int_I \mathcal{L}_{A,B}(s,1)^*\eta - \int_I \mathcal{L}_{A,B}(s,0)^*\eta \right) \right].
\]

We now consider loops of loops. In this case we have:

**Corollary 7.5.** Let \(T\) be an abelian subgroup of \(G\). If conditions \(C)\) and \(D)\) of the previous section are satisfied and if the restriction of \(\eta \in \Omega^1(M, adP)\) to \(\Gamma: S^1 \times S^1 \rightarrow M\)
is also reducible to \( T \), then the trace of the \((A,B)\)-holonomy of the loop of loops \( \Gamma \) is invariant under (7.4). If, besides the above conditions, we have also

\[
\int_I ev^* \eta = 0, \quad ev: I \times \Gamma(\bullet, 0)(p) \to P,
\]

then the trace of the \((A,B)\)-holonomy for loops of loops is also invariant under (7.3).

The conclusion of this section is that the symmetry (7.4), which arises from the BF (quantum) field theory, does not leave the trace of the holonomy invariant, unless some reducibility constraints are imposed on the connection \( A \) and on the field \( B \).

In this sense the transformations (7.4) represent almost a good symmetry for the observable given by the trace of the \((A,B)\)-holonomy.

A good symmetry for the same observable would certainly be represented by the group of gauge transformations for \( \mathcal{P}_A P \). Unfortunately gauge transformations for \( \mathcal{P}_A P \) do not map special connections into special connections and hence are not good symmetries for the BF theories.

In general gauge transformations for \( \mathcal{P}_A P \) map special connections into special connections plus some extra terms given by Chen integrals and boundary terms. By neglecting these extra terms one obtains exactly (7.4). In this sense the transformations (7.4) are almost gauge transformations for \( \mathcal{P}_A P \).

### 8. Observables, actions and quantum field theories

An application of the ideas developed in this paper is the construction of new observables for quantum field theories (QFT).

A QFT is described by an action functional, and by observable one means another functional that is invariant under the same symmetries that leave the action functional unchanged. A weaker requirement for the observables is the invariance only on shell (i.e., upon using the Euler–Lagrange equations); in this case the quantization of the theory requires the use of the Batalin–Vilkovisky formalism [28], [29], but we will discuss this elsewhere. Throughout this section we will restrict ourselves to considering a four-dimensional manifold \( M \).

#### 8.1. Non-topological QFT’s

The first QFT we consider is the Yang–Mills theory described by the action functional

\[
S_{YM}[A] = \| F_A \|^2 = - \int_M \text{Tr} (F_A \wedge *F_A),
\]

where \( * \) is the Hodge dual with respect to the Riemannian metric on \( M \). The invariance group of the Yang–Mills action functional is the group of gauge transformations \( A \to A^g \).
In this framework we have two natural elements of $\Omega^2(M, \text{ad} P)$ at our disposal, viz., $F_A$ and $*F_A$. Therefore, we may consider the following family of observables:

$$O_{\alpha\beta}(\Gamma) = \text{Tr} \mathcal{H}_{(A,\alpha F_A + \beta * F_A)}(\Gamma, p),$$  \hspace{1cm} (8.2)

where $\Gamma$ is a path of paths or loops. Theorem 3.4 guarantees that this is indeed an observable.

Notice that for $\alpha = \beta = 0$ the observable reduces to the trace of the identity, while for $\alpha = -1$ and $\beta = 0$ it yields the trace of the $A$-holonomy along the boundary $\partial \Gamma$. Taking $\alpha = 0$ and $\beta = 1$ ($\beta = -1$) is an interesting choice if the background connection—i.e., the solution of the Euler–Lagrange equations $d^*_A F_A = 0$ around which we are working—is anti-self-dual (self-dual); in this case, on shell the observable is the $A$-holonomy along the boundary of $\Gamma$ but off shell it depends on $\Gamma$ (see Theorem 3.8).

Another family of observables can be obtained by replacing $\mathcal{H}$ by $\text{Hol}$ in the above formula.

As discussed in the Introduction, a physical interpretation of these observables may be the following: as the Wilson loop—i.e., the trace of the $A$-holonomy—describes the displacement of a point-like charge, so the observable $O$ describes the displacement of a path-like (or loop-like) charge, namely of an open or closed string.

Notice that, even if the image of $\Gamma$ represents a smooth surface, the observable $O$ depends in general on its underlying path-of-paths structure. If, however, we impose assumption $C$ of section 6 as a boundary condition for $A$, then $O$ will depend only on the surface represented by $\Gamma$ and on the loop of initial points.

There are other theories that are equivalent to the Yang–Mills theory, like the first order Yang–Mills theory \cite{1}\cite{2},

$$S_{YM'} = \frac{1}{4} ||B||^2 + i \int_M \text{Tr} (B \wedge F_A).$$  \hspace{1cm} (8.3)

In this case, however, we have at our disposal a bigger family of observables than those given by (8.2). In fact as our form in $\Omega^2(M, \text{ad} P)$, we can take a generic linear combination

$$\alpha F_A + \beta * F_A + \gamma B + \delta * B.$$

Another version of (8.3) is the so-called $BF$-Yang–Mills theory \cite{22}, \cite{4}, where $B$ is replaced by $B - d_A \eta$, $\eta \in \Omega^1(M, \text{ad} P)$, in the above action and, consequently, in the observable.

The $BF$-Yang–Mills theory has been extensively studied in \cite{4} where it has been shown to be equivalent to the Yang–Mills theory. This equivalence makes more interesting the appearance of a surface term for Wilson loops.

**8.2. Topological QFT’s**

Topological Quantum Field Theories are QFT’s whose action functional does not depend on the Riemannian structure of $M$ and so it is expected to yield topological or smooth invariants as its vacuum expectation values.
We consider the following TQFT's:

1) the topological Yang–Mills theory

\[ S_{tY.M} = \int_M \text{Tr} (F_A \wedge F_A), \]

2) the BF theory with a cosmological term

\[ S_{BF-BB} = \int_M \text{Tr} (B \wedge F_A) + \frac{1}{2} \int_M \text{Tr} (B \wedge B), \]

and

3) the pure BF theory

\[ S_{BF} = \int_M \text{Tr} (B \wedge F_A). \]

We do not have a non trivial loop-of-loops observable for the topological Yang–Mills theory.

As for the BF theory with a cosmological term, we notice that the symmetries read, at the infinitesimal level,

\[ \delta A = d_A \xi + \eta, \quad \delta B = [B, \xi] - d_A \eta, \]

with \( \xi \in \Omega^0(M, \text{ad } P) \) and \( \eta \in \Omega^1(M, \text{ad } P) \). These transformations correspond to (7.3).

Since the Euler–Lagrange equations are \( B + F_A = 0 \), then the trace of \( \text{Hol}_{(A,B)}(\Gamma, p) \) is almost invariant on shell. The problem is the presence of boundary terms in \( \eta \), see (7.6). To get a good on-shell observable for loops of paths \( \Gamma \), we have to eliminate these boundary terms; so we may consider

\[ \mathcal{O}(\Gamma) = \text{Tr} \left[ \text{Hol}_{(A,-F_A)}(\Gamma, p)^{-1} \text{Hol}_{(A,B)}(\Gamma, p) \right]. \]

Notice that on shell this observable is trivial. Off shell one must add Batalin–Vilkovisky corrections. Alternatively one can assume conditions C and D of section 6 as boundary conditions. In this case the above observable is automatically invariant both on shell and off shell.

In the case of the pure BF theory, the infinitesimal symmetries are (7.4), i.e.

\[ \delta A = d_A \xi, \quad \delta B = [B, \xi] - d_A \eta. \]

The Euler–Lagrange equations read \( F_A = 0, d_A B = 0 \). These conditions correspond almost to the flatness of the connection for loops of paths, the missing requirement being the reducibility of \( B \), see Theorem 2.8.

We have then a first observable for pure BF theory, namely,

\[ \mathcal{O}(\Gamma) = \text{Tr} \text{Hol}_{(A,B)}(\Gamma, p). \]
In fact, by Theorem 7.3 we get
\[ \delta \mathcal{O}(\Gamma) = 0, \]
provided that we assume conditions \( C \) and \( D \) of section 6 as boundary conditions. In this case the above observable is invariant both on shell and off shell.

Another possible choice for pure \( BF \) theory is given by the observable
\[ \tilde{\mathcal{O}}(\Gamma) = \text{Tr} \exp \left[ \frac{d}{dt} \bigg|_{t=0} \text{Hol}_{(A,tB)}(\Gamma,p) \right]. \quad (8.4) \]

On shell (i.e. when \( F_A = 0 \)) Theorem 3.6 guarantees that (8.4) is an observable that can be rightly associated to the surface spanned by a loop of paths. To compute the transformation properties of this observable, we must consider the transformation of the holonomy and not of its trace but only up to the first order in \( t \). So \( \tilde{\mathcal{O}} \) turns out to be invariant on shell if one requires \( \eta \) to vanish on the restriction of \( P \) over a submanifold \( \Sigma \) containing the image of \( \partial \Gamma \). To get a good observable also off shell, i.e., also in the case when \( A \) is not flat, one must add Batalin–Vilkovisky corrections. Notice that (8.4) is the exact counterpart of the observable for 3-dimensional \( BF \)-theory considered in [30], [31], [32].

Since the \( BF \) theories are topological—i.e., do not depend on the choice of the Riemannian metric on \( M \)—one expects that the vacuum expectation values of the above metric-independent observables will yield smooth invariants of the image of an imbedded (immersed) loop of paths (of loops).

When \( M \) is a four-dimensional simply connected manifold, we conjecture that these invariants are related to the Kronheimer–Mrowka invariants [17] of imbedded (immersed) surfaces. Both in their theory and in our framework, a special rôle is played by connections that are reducible when restricted to the given surface. Moreover both in [17] and in the preliminary perturbative calculations of the four-dimensional quantum \( BF \) theory (see [9], [22]), the reducible connections ("monopoles on the surface") yield loops and surfaces that are non-trivially linked.

9. Conclusions

The natural geometrical setting for field theories of the \( BF \) type is a principal bundle on the space of paths ("open strings") or loops ("closed strings") of a (four-dimensional) manifold \( M \). The fields \( A \) and \( B \) of the \( BF \) theory describe collectively a connection on such principal bundles.

Out of the trace of the corresponding holonomy one can define observables associated to paths (loops) of paths (of loops). These can be seen as associated to imbedded (or immersed) surfaces only if some extra conditions are met and if those extra conditions are taken into account in the calculations of Feynman integrals.

The geometrical analysis of \( BF \) theories suggests two physically relevant considerations:
1. In those $BF$ theories that are related (equivalent) to the Yang–Mills theory, one can consider $B$-dependent observables associated to paths of paths which, when $B$ is a deformation of the curvature, are a deformation of the Wilson loop along the boundary of the surface spanned by the path of paths. In other words a deviation from the non-abelian Stokes formula appears and this may be relevant for a correct understanding of the problem of quark-confinement.

2. Four-dimensional topological $BF$ theories yield invariants of the four-manifold. When no $B$-dependent observable is included, the invariants to be considered should be related to the Donaldson invariants. When $B$-dependent observables are considered, one expects the corresponding quantum field theory to yield invariants of imbedded (or immersed) surfaces (like the Kronheimer–Mrowka invariants).

Four-dimensional $BF$ theories can then be considered as a sort of gauge theories for loops and paths. The main difference is the fact that the action functional is not integrated over the whole space of paths (loops) but over the original four-manifold $M$. As a consequence, the action functional is not invariant under the full gauge group of the principal bundle over the path space but is only approximatively invariant (i.e. when one neglects boundary terms and higher-order Chen integrals).

The full structure of the gauge group, of the space of connections and of the space of gauge orbits for paths and loops as well as the relation with Hochschild (and cyclic) (co)homology, will be discussed elsewhere.

**Acknowledgments**

We thank A.Belli, L.Bonora, J.D.S Jones, M.Martellini for useful discussions. P.C.-R. thanks G. Semenoff for inviting him to Vancouver B.C. (July 1997, APCTP/Plms Summer Workshop).
10. Appendix: Iterated loop spaces

Most of the construction in this paper can be easily iterated, namely we can consider principal $G$-bundles on iterated free path and loop spaces. Let us denote those by the symbols $\mathcal{P}^n M$ and $\mathcal{L}^n M$.

We describe here the special connections and the relevant curvatures for iterated path spaces (in the case $n=2$).

If we are given a connection $(A, B)$ on $\mathcal{P}_A P$, then we can consider connections on the $G$-principal bundle of $(A, B)$-horizontal paths of paths

$$\mathcal{P}^2_{(A,B)} P \longrightarrow \mathcal{P}^2 M.$$  

We have the following diagram

$$
\begin{array}{ccc}
I \times I \times \mathcal{P}^2_{(A,B)} P & \xrightarrow{\text{ev}^{13}} & I \times \mathcal{P}_A P \\
\downarrow \text{id} \times \text{id} \times \pi & & \downarrow \text{id} \times \pi \\
I \times I \times \mathcal{P}^2 M & \xrightarrow{\text{ev}^{13}} & I \times \mathcal{P} M \longrightarrow M.
\end{array}
$$

Here $\text{ev}^{13}$ acts on the first and the third element of the product.

Elements of $\mathcal{P}^2_{(A,B)} P$ are maps

$$Q: I \times I \rightarrow P$$

$$(s, t) \mapsto Q(s, t)$$

satisfying

$$A\left(\dot{Q}(s, t)\right) = 0, \quad \forall s, t \in I$$

$$A\left(Q'(s, 0)\right) = \int_0^1 dt B\left(\dot{Q}(s, t), Q'(s, t)\right), \quad \forall s \in I.$$  

Vectors tangent to $\mathcal{P}^2_{(A,B)} P$ are maps from $I \times I$ to $TP$, which are in turn defined by maps

$$\tilde{R}: (-\epsilon, \epsilon) \times I \times I \rightarrow P$$

so that

$$\left.\frac{\partial}{\partial r}\right|_{r=0} A\left(R'(r, s, 0)\right) - \left.\frac{\partial}{\partial r}\right|_{r=0} \int_0^1 dt B\left(\dot{R}(r, s, t), R'(r, s, t)\right) = 0, \quad \forall s \in I.$$  

Following the definitions of sect. 2, in order to define a special connection on $\mathcal{P}^2_{(A,B)} P$ we need another connection $(\bar{A}, \bar{B})$ and a form $C \in \Omega^3(M, \text{ad} P)$. Here we choose

35
\( \bar{A} = A, \bar{B} = B. \) By considering the double evaluation map \( Ev: I \times I \times P^2_{(A,B)} \rightarrow P \) the special connection \((A, B, C)\) is explicitly given by:

\[
Ev^*_{(0,0)}A + \int_I Ev^*_{(0,\cdot)}B + \int_{I \times I} Ev^*C.
\]

The space of special connections considered above is an affine space modeled on \( \Omega^3(M, \text{ad } P) \).

We have:

\[
\int_{I \times I} Ev^*C = \int_I \left( \int_I \text{ev}^* C \right).
\]

Any tangent vector \( X \in T_Q P^2_{A,B} \) is a map \( I \times I \ni (s,t) \mapsto T_Q P(s,t) \). So we get

\[
\int_{[0,1] \times [0,1]} ds \, dt \, C \left( X(s,t), \dot{Q}(s,t), Q'(s,t) \right) = \int_{[0,1]} ds \left( \int_I \text{ev}^* C \right) (X(s,t), Q'(s,t)).
\]

The curvature of a special connection \((A, B, C)\) is obtained directly from Theorem 2.6 via the following replacements

\[
\begin{align*}
A &\rightarrow \text{ev}^*_0 A + \int_I \text{ev}^*_\cdot B \\
B &\rightarrow \int_I \text{ev}^* C.
\end{align*}
\]

References

[1] M.B. Halpern, Field Strength Formulation of Quantum Chromodynamics, Phys. Rev. D 16, 1798–1801 (1977);
[2] M.B. Halpern Gauge Invariant Formulation of the Selfdual Sector, Phys. Rev. D 16, 3515–3519 (1977);
[3] H. Reinhardt, Dual Description of QCD, hep-th/9608191;
[4] A.S. Cattaneo, P. Cotta-Ramusino, F. Fucito, M. Martellini, M. Rinaldi, A. Tanzini, M. Zeni, Four-Dimensional Yang–Mills Theory as a Deformation of Topological BF Theory, to be published in Commun. Math. Phys. (1998);
[5] E. Witten, Topological Quantum Field Theory, Commun. Math. Phys. 117, 353–386 (1988);
[6] A.S. Schwartz, The Partition Function of a Degenerate Quadratic Functional and Ray-Singer Invariants, Lett. Math. Phys. 2, 247–252 (1978);
[7] G.T. Horowitz, Exactly Soluble Diffeomorphism Invariant Theories, Commun. Math. Phys. 125, 417–436 (1989);
[8] D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Topological Field Theories, Phys. Rep. 209, 129-340 (1991);
[9] P. Cotta-Ramusino, M. Martellini, BF theories and 2-knots, in “Knots and Quantum Gravity”, edited by J. Baez (Oxford University, Oxford), 169–189 (1994);
[10] M.B. Halpern, Field Strength and dual variable formulation of Gauge theory, Phys. Rev. D 19, 517–530 (1979);
[11] I. Ya. Aref’eva, *Non-Abelian Stokes Formula*, Teor. Math. Fiz. 43, 111–116 (1980);
[12] Wilson, *Confinement of Quarks*, Phys. Rev. D, 10, 2445–2459 (1974);
[13] F. Fucito, M. Martellini, M. Zeni, *The BF Formalism for QCD and Quark Confinement*, Nucl. Phys. B 496, 259–284 (1997);
[14] K. Kondo, *Yang-Mills Theory as a Deformation of Topological Field Theory, Dimensional Reduction and Quark Confinement*, hep-th/9801024;
[15] G. ’t Hooft, *On the Phase Transition towards Permanent Quark Confinement*, Nucl. Phys. B 138, 1–25 (1978);
[16] G. ’t Hooft, *A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories*, Nucl. Phys. B 153, 141-160 (1979);
[17] P.B. Kronheimer, T.S. Mrowka, *Gauge theory for embedded surfaces, I,II*, Topology, 32, 4, 773–826, (1993) and 34, 1, 37–97 (1995);
[18] P. Cotta-Ramusino, M. Rinaldi, *in preparation*;
[19] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Interscience Publishers, New York (1963);
[20] S. Kobayashi, *Theory of Connections*, Ann. Mat. Pura Appl. 43, 119–194 (1967);
[21] A.S. Cattaneo, P. Cotta-Ramusino, M. Rinaldi, *BRST symmetries for the tangent gauge group*, J. Math. Phys. 39 1316–1339 (1998);
[22] A.S. Cattaneo, P. Cotta-Ramusino, A. Gamba, M. Martellini, *The Donaldson-Witten Invariants and Pure QCD with Order and Disorder ’t Hooft-like Operators*, Phys. Lett. B 355, 245–254 (1995);
[23] K. Chen, *Iterated integrals of differential forms and loop space homology*, Ann. of Math. 97, 2, 217–246 (1973);
[24] G.W. Whitehead, *Elements of Homotopy theory*, Springer Verlag, Berlin Heidelberg, New York (1979);
[25] B. Broda, *Non-Abelian Stokes Theorem*, in “Advanced Electromagnetism: Foundations, Theory and Application” (T. Barrett, D. Grimes eds.) World Scientific, Singapore, 496–505 (1995);
[26] D. Diakonov, V. Petrov, *Non-Abelian Stokes Theorem and Quark-Monopole Interaction*, hep-th 9606104;
[27] F.A. Lunev, *Pure Bosonic Worldline Path Integral Representation for Fermionic Determinants, Non-Abelian Stokes Theorem, and Quasiclassical Approximation in QCD*, Nucl. Phys. B 494, 433–470 (1997);
[28] I.A. Batalin and G.A. Vilkovisky, *Relativistic S-Matrix of Dynamical Systems with Boson and Fermion Constraints*, Phys. Lett. 69 B, 309–312 (1977);
[29] E.S. Fradkin and T.E. Fradkina, *Quantization of Relativistic Systems with Boson and Fermion First- and Second-Class Constraints*, Phys. Lett. 72 B, 343–348 (1978);
[30] A.S. Cattaneo, P. Cotta-Ramusino and M. Martellini, *Three-Dimensional BF Theories and the Alexander–Conway Invariant of Knots*, Nucl. Phys. B 346, 355–382 (1995);
[31] A.S. Cattaneo, P. Cotta-Ramusino, J. Fröhlich, M. Martellini, *Topological BF theories in 3 and 4 dimensions*, J. Math. Phys. 36, 6137–6160 (1995);
[32] A.S. Cattaneo, *Cabled Wilson loops in BF theories*, J. Math. Phys. 37, 3684–3703 (1996).