Wegner estimate and Anderson localization for random magnetic fields

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Abstract

We consider a two dimensional magnetic Schrödinger operator with a spatially stationary random magnetic field. We assume that the magnetic field has a positive lower bound and that it has Fourier modes on arbitrarily short scales. We prove the Wegner estimate at arbitrary energy, i.e. we show that the averaged density of states is finite throughout the whole spectrum. We also prove Anderson localization at the bottom of the spectrum.

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1 Introduction

We consider a spinless quantum particle in the two dimensional Euclidean space $\mathbb{R}^2$ subject to a random magnetic field $B: \mathbb{R}^2 \to \mathbb{R}$. The energy is given by the magnetic Schrödinger operator, $H = (p - A)^2 + V$, where $p = -i\nabla$, $A: \mathbb{R}^2 \to \mathbb{R}^2$ is a random magnetic vector potential satisfying $\nabla \times A = B$ and $V$ is a deterministic external potential. In contrast to the standard Anderson model for localization with a magnetic field (see, e.g., [2, 3, 5, 8, 19]), we consider a model where the external potential is deterministic, and only the magnetic field carries randomness in the system.

The existence of the integrated density of states and its independence of the boundary conditions in the thermodynamic limit have been proven for both the discrete and the continuous model and Lifschitz tail asymptotics have also been obtained [10, 14, 15].

However, Anderson localization for the random field model has only been shown under an additional condition that the random part of the magnetic flux is locally zero [13]. Since a deterministic constant magnetic field localizes, one could expect that its random perturbation even enhances localization, so the zero flux condition in [13] should physically be unnecessary. Technically, however, random magnetic fields are harder to fit into the standard proofs of localization mainly because the vector potential is nonlocal while a spatially stationary magnetic field typically does not lead to a stationary Hamiltonian.

To circumvent this difficulty, Hislop and Klopp [9] and later Ueki [18] have considered spatially stationary random vector potential of the form

$$A_\omega(x) = \sum_{z \in \mathbb{Z}^2} \omega_z u(x - z), \quad (1.1)$$

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where $\omega = \{\omega_z : z \in \mathbb{Z}^2\}$ is a collection of i.i.d. real random variables with some moment condition and $u : \mathbb{R}^2 \to \mathbb{R}$ is a fixed vectorfield with a fast decay at infinity. For such random field, Anderson localization was shown in [18, 6], motivated by a method in [9], that gave the first Wegner estimate for this model. The method works only for energies away from the spectrum of the deterministic part of the Hamiltonian, mainly because the Wegner estimate is shown only in that regime.

Note that for the magnetic field $B_\omega = \nabla \times A_\omega$ generated by (1.1), the fluctuation of the total flux $\int_{\Lambda_L} B_\omega(x)dx$ within a large box $\Lambda_L = [-L/2, L/2]^2$ is of order of the square root of the boundary, $|\partial \Lambda_L|^{1/2} \sim L^{1/2}$ by the central limit theorem. In contrast, if the magnetic field $B_\omega(x)$ itself were given by a spatially stationary random process with a sufficient correlation decay, e.g.,

$$B_\omega(x) = \sum_{z \in \mathbb{Z}^2} \omega_z u(x - z)$$  \hspace{1cm} (1.2)

with some decaying scalar function $u : \mathbb{R}^2 \to \mathbb{R}$, then $\int_{\Lambda_L} B_\omega(x)dx$ would fluctuate on a scale of order square root of the area, $|\Lambda_L|^{1/2} \sim L$. Assuming stationarity on the vector potential thus imposes an unnatural constraint on the physically relevant gauge-invariant quantity, i.e., on the magnetic field.

The analogous problem for the lattice magnetic Schrödinger operator has been studied with different methods. For the discrete magnetic Schrödinger operator on $\mathbb{Z}^2$, the magnetic field is given by its flux on each plaquet of the lattice. Extending the method of Nakamura [14], Anderson localization was proven for this model [13] near the spectral edge, however, the zero flux condition was enforced in a strong sense. Instead of considering the more natural i.i.d. (or weakly correlated) random fluxes on each plaquet, the neighboring plaquets were domino-like paired and the magnetic fluxes were opposite within each domino. Such magnetic field again has much less fluctuation than the i.i.d. case, moreover the flux is deterministically zero on each domino.

The main technical reason for the zero flux condition in both the continuous and the discrete model was that the proof of the Wegner estimate required it. The Wegner estimate is a key element in any known mathematical proof of the Anderson localization since it provides an a-priori bound for the resolvent with a high probability. Typically, the statement is formulated for the finite volume truncation of the Anderson localization since it provides an a-priori bound for the resolvent with a finite volume truncation.

We work in $\mathbb{R}^2$ and set $|x|_\infty := \max\{|x_1|, |x_2|\}$ for any $x \in \mathbb{R}^2$. We are given two positive numbers, $b_0$ and $K_0 > 3$, and a deterministic (possibly nonconstant) magnetic field $B_{\text{det}}(x)$ with

$$0 < 2b_0 \leq B_{\text{det}}(x) \leq (K_0 - 1)b_0.$$  \hspace{1cm} (2.1)

# Definition of the random magnetic field

We work in $\mathbb{R}^2$ and set $|x|_\infty := \max\{|x_1|, |x_2|\}$ for any $x \in \mathbb{R}^2$. We are given two positive numbers, $b_0$ and $K_0 > 3$, and a deterministic (possibly nonconstant) magnetic field $B_{\text{det}}(x)$ with

$$0 < 2b_0 \leq B_{\text{det}}(x) \leq (K_0 - 1)b_0.$$  \hspace{1cm} (2.1)
We perturb this magnetic field by a random one, i.e., we consider
\[ B = B^\omega = B^\det + \mu B^\ran^\omega, \]  
where the random field is assumed to be \(|B^\ran^\omega| \leq b_0\) and \(0 < \mu \leq 1\) is a coupling constant. In particular,
\[ 0 < b_0 \leq B^\omega \leq K_0b_0. \]  

Now we define the random field \(B^\ran^\omega\) more precisely. We will need the assumption that \(B^\ran^\omega\) has components on arbitrary small scales, but these components decay in size. For simplicity we present a class of magnetic fields for which our method works, but our approach can be extended to more general fields with a similar structure. We remark that the analogous result in the discrete setup [4] will not require such assumption on the structure of the random field.

We choose a smooth profile function \(u \in C^1_0(\R^2), 0 \leq u \leq 1\), that satisfies one of the following two conditions for some sufficiently small \(\delta\): either
\[ u(x) \equiv 0 \quad \text{for} \quad |x|_{\infty} \geq \frac{1}{2} + \delta \quad \text{and} \quad u(x) \equiv 1, \quad \text{for} \quad |x|_{\infty} \leq \frac{1}{2} - \delta \]  
or
\[ u(x) = \delta^2 u_0(x\delta) \quad \text{with some} \ u_0 \in C^1_0(\R^2), \quad \int_{\R^2} u_0 = 1, \quad u_0(x) \equiv 0 \quad \text{for} \quad |x|_{\infty} \geq 1. \]  

In both cases \(\delta\) will be chosen as a sufficiently small positive number \(\delta \leq \delta_0 \leq 1\). The threshold \(\delta_0\) can be chosen as
\[ \delta_0 = \frac{1}{3200} \quad \text{under condition} \ (2.4) \]  
\[ \delta_0 = \frac{1}{640 + 32\|\nabla u_0\|_{\infty}^2} \quad \text{under condition} \ (2.5). \]  

Fix \(k \in \N\) and define the lattice \(\Lambda^{(k)} = (2^{-k}\Z)^2\). For \(z \in \Lambda^{(k)}\) define
\[ \beta^{(k)}_z(x) := u(2^k(x - z)). \]  
The randomness is represented by a collection of independent random variables
\[ \omega = \{\omega^{(k)}_z : k \in \N, z \in \Lambda^{(k)}\}. \]  

We assume that all \(\omega^{(k)}_z\) have zero expectation, and they satisfy a bound that is uniform in \(z\)
\[ |\omega^{(k)}_z| \leq \sigma^{(k)} := e^{-\rho k} \]  
with some \(\rho > 0\). We assume that the distribution of \(\omega^{(k)}_z\) is absolutely continuous, its density function \(v^{(k)}_z\) is in \(C^2_0(\R)\) and satisfies
\[ \int_{\R} \left| \frac{d^2 v^{(k)}_z}{ds^2}(s) \right| ds \leq C|\sigma^{(k)}|^{-2} = C e^{2\rho k}. \]  

Note that we do not require identical distribution. Thus for each \((k, z) \in \mathcal{L} := \bigcup_{k \in \N} \{k\} \times \Lambda^{(k)}\) we have a probability measure with density \(v^{(k)}_z\). The associated product measure, \(P\), is probability measure on \(\Omega = \R^\mathcal{L}\), and we denote expectation with respect to this measure by \(\mathbb{E}\).

For example, one can assume that for each fixed \(k\), the random variables \(\{\omega^{(k)}_z : z \in \Lambda^{(k)}\}\) are i.i.d. and they all live on a scale \(\sigma^{(k)}\), e.g. \(v^{(k)}_z(s) = [\sigma^{(k)}]^{-1} v(s/\sigma^{(k)})\) for all \(z\) with some smooth, compactly supported density function \(v\).
We define the random magnetic field as
\[ B^\omega_{\text{ran}}(x) = B_{\text{ran}}(x) := \sum_{k=0}^{\infty} B^{(k)}(x), \quad B^{(k)}(x) := \sum_{z \in \Lambda^{(k)}} B_z^{(k)}(x), \quad B_z^{(k)}(x) := \omega_z^{(k)} \beta_z^{(k)}(x), \] (2.10)
i.e. \( B^\omega_{\text{ran}} \) is the sum of independent local magnetic fields on each scale \( k \) and at every \( z \in \Lambda^{(k)} \). We assume that
\[ \sum_{k=0}^{\infty} \sigma^{(k)} \leq b_0, \] (2.11)
i.e. \( (1 - e^{-\rho})b_0 \geq 1 \), then clearly \( |B_{\text{ran}}(x)| \leq b_0 \). Note that \( B^\omega_{\text{ran}} \) is differentiable if \( \rho > \ln 2 \). We will make the following assumption

\( (R) \) \( B_{\omega} \) is a random magnetic field constructed in (2.2), (2.7), and (2.10), and it satisfies (2.1), (2.3), (2.8), (2.9), (2.11), and one of the conditions (2.4) or (2.5).

Let \( \Lambda \subset \mathbb{R}^2 \) be square and we will consider the magnetic Schrödinger operator with Dirichlet boundary conditions on \( \Lambda \). We will work in the Hilbert space \( L^2(\Lambda) \) and denote the scalar product by \( \langle \cdot, \cdot \rangle \) and the norm by \( \| \cdot \| \). Let \( A \) be a magnetic vector potential such that \( \nabla \times A = B \). By \( H_A(\Lambda) \) we denote the magnetic Schrödinger operator on \( L^2(\Lambda) \) with Dirichlet boundary conditions, i.e., \( H_A(\Lambda) = (p - A)^2 + V \).

Here \( V \) is a bounded external potential. In the special case where \( \Lambda = [-L/2, L/2]^2 \subset \mathbb{R}^2 \) with \( L \in \mathbb{N} \) we will write
\[ H_L(A) = H_{\Lambda_L}(A). \] (2.12)

By \( H(A) = (p - A)^2 + V \), we denote the magnetic Schrödinger operator on \( L^2(\mathbb{R}^2) \). The magnetic Hamilton operators can be realized by the Friedrichs extension. If we refer to statements which are independent of the particular choice of gauge, with a slight abuse of notation, we shall occasionally write \( H_A(\Lambda) \) and \( H(B) \). If \( \nabla \times A_{\omega} = B_{\omega} \) and \( B_{\omega} \) satisfies \( (R) \), then \( \omega \to H_L(A_{\omega}) \) is measurable. This follows for example from an application of Proposition 1.2.6 [17].

### 3 Main results

The first result is a Wegner estimate. Fix an energy \( E \) and a window of width \( \eta \leq 1 \) about \( E \). Let \( \chi_{E, \eta} \) be the characteristic function of the interval \([E - \eta/2, E + \eta/2]\).

**Theorem 3.1** Let \( K_0 > 3 \). We assume that \( B_{\omega} \) is a random magnetic field satisfying \( (R) \). Let \( A_{\omega} \) be a vector potential with \( \nabla \times A_{\omega} = B_{\omega} = B_{\text{det}} + \mu B_{\text{ran}} \) with \( \mu \in (0, 1) \). We assume \( \rho \geq \ln 2, \ (1 - e^{-\rho})b_0 \geq 1, \) and \( \|V\|_{\infty} \leq b_0/4 \). Let \( \delta \leq \delta_0 \) and \( K_1 \geq 1 \). Then there exist positive constants \( C_0 = C_0(K_0, K_1), \ C_1 = C_1(K_0, K_1), \) and \( L_0^* = L_0^*(K_0, K_1, \delta) \) such that for any \( 0 < \kappa \leq 1 \)

\[ \mathbf{E} \text{Tr} \chi_{E, \eta}(H_L(A)) \leq C_0 \eta^{2} L^{C_1(\kappa^{-1} + \rho)}, \]

for all \( E \in [\frac{1}{2}, K_1 b_0], \ 0 < \eta \leq 1, \) and \( L \geq L_0^* b_0^{\kappa} \).

The next theorem is a standard result stating that the spectrum is deterministic. For this we need that the random magnetic field is stationary on each scale:

\( \text{(i.i.d.)} \) For any fixed \( k \in \mathbb{N} \), \( \{\omega_z^{(k)} : z \in \Lambda^{(k)}\} \) are i.i.d., i.e., \( v^{(k)} = v_z^{(k)} \).

**Theorem 3.2** Suppose \( B_{\omega} \) is a random magnetic field such that \( (R) \) and \( \text{(i.i.d.)} \) hold. For \( \nabla \times A_{\omega} = B_{\omega} \), the function \( \omega \to H(A_{\omega}) \) is measurable. There exists a set \( \Sigma \subset \mathbb{R} \) and a set \( \Omega_1 \subset \Omega \) with \( \mathbf{P}(\Omega_1) = 1 \) such that for all \( \omega \in \Omega_1 \)

\[ \sigma(H(B_{\omega})) = \Sigma. \]
For completeness, we give a proof of Theorem 3.2 in Appendix A.

The second main result is localization at the bottom of the spectrum. We make additional assumptions on the profile function, namely that

\[ U(x) := \sum_{z \in \mathbb{Z}^2} u(x - z) \geq c_u \quad \text{and} \quad \sup_{x \in \mathbb{R}^2} U(x) = 1, \]

for some positive constant \( c_u > 0 \).

The result about localization will hold under the following Hypotheses.

(A) \( V \) and \( B_{\text{det}} \) are \( \mathbb{Z}^2 \)-periodic, and \( B_\omega \) is a random magnetic field satisfying (R) with profile function satisfying (3.1). Hypothesis (i.i.d.) holds and \( \text{supp} v(k) \) is a compact interval \([m_-^{(k)}, m_+^{(k)}] \).

Second we assume a polynomial bound on the lower tail of \( v(0) \). To this end we introduce the probability distribution function

\[ \nu(h) := \int_{m_-^{(0)}}^{m_-^{(0)} + h} v(0)(x)dx. \]

(A+) Hypotheses (A) holds and there exists a constant \( c_v \) such that for all \( h \geq 0 \) we have \( \nu(h) \leq c_v h^\tau \).

The next theorem states that we have Anderson localization at the bottom of the spectrum. Recall that \( \Sigma \) denotes the almost sure deterministic spectrum of \( H(B_\omega) \), see Theorem 3.2, and let \( \Sigma_{\text{inf}} \) be its infimum. We will assume that the following quantity is finite

\[ K_2 := \max_{|\alpha|=1} \left( (1 - 2e^{-\rho})^{-1} \| D^\alpha U \|_\infty + \| D^\alpha B_{\text{det}} \|_\infty + \| D^\alpha V \|_\infty \right). \]

where we used the multi-indices notation \( D^\alpha = \partial_x^\alpha \partial_y^\beta \) with \( \alpha, \beta \in \mathbb{N}_0^2 \) and \( |\alpha| = \alpha_1 + \alpha_2 \). To show localization we will need that \( K_2b_0^{-1/2} \) is small. A more explicit relation between \( b_0 \) and derivatives of \( U, B_{\text{det}}, \) and \( V \) can be obtained from the first inequality of (10.7) given in the proof.

**Theorem 3.3** Let \( K_0 > 3 \), \( K_1 \geq 1 \), \( \rho > \ln 2 \), \( (1 - e^{-\rho})b_0 \geq 1 \), \( \| V \|_\infty \leq b_0/4 \). Suppose (A+) holds for some \( \tau > 2 \), and let \( B_\omega = B_{\text{det}} + \mu B_\text{ran} \) with \( \mu \in (0, 1] \) be the random magnetic field with a vector potential \( A = A_\omega \). If \( K_2b_0^{-1/2} \) is sufficiently small, then there exists an \( \varepsilon_0 > 0 \) such that for almost every \( \omega \) the operator \( H(A_\omega) \) has in \( [\Sigma_{\text{inf}}, \Sigma_{\text{inf}} + \varepsilon_0] \) dense pure point spectrum with exponentially decaying eigenfunctions. For \( p < 2(\tau - 2) \), there exists an \( \varepsilon_0 > 0 \) such that for any subinterval \( I \subset [\Sigma_{\text{inf}}, \Sigma_{\text{inf}} + \varepsilon_0] \) and any compact subset \( K \subset \mathbb{R}^2 \), we have

\[ \mathbb{E} \left\{ \sup_{t} \| X|e^{-iH(A)t}1_I(H(A))\chi_K \| \right\} < \infty. \]

We will use the notation that \( 1_S \) as well as \( \chi_S \) denotes the characteristic function of a set \( S \).

**Remark.** We note that if \( K_2 = 0 \), then no large \( b_0 \) assumption is necessary, that is, the assertion of the theorem holds for any \( b_0 \geq 2 \). Now \( K_2 = 0 \) holds provided \( B_{\text{det}} \) and \( V \) are constant and \( U = 1 \). The condition \( U = 1 \) can be realized for example as follows. We choose \( \varphi \in C_0^\infty(\mathbb{R}^2; [0, 1]) \) with \( \varphi(x) = 0 \), if \( |x| \geq 1 \), \( \varphi = 1 \), and set, for \( s > 0 \), \( u = 1_{|x| \leq 1/2} \ast \varphi_\sigma \) and \( \varphi_s(x) = s^{-2}\varphi(x/s) \). Conditions (2.4) or (2.5) can be satisfied by taking \( s \) sufficiently small or sufficiently large, respectively.

The next theorem provides estimates on the location of the deterministic spectrum \( \Sigma \) of \( H(B_\omega) \), under the influence of the random potential. It will be used in the proof of Theorem 3.3. To formulate it, we define two specific configurations of the collection of random variables, \( \omega^+ \) and \( \omega^- \), by \( \omega^{(k)}_{\pm} := m^{(k)}_{\pm} \), and we set

\[ E_{\text{inf}} := \inf_{x \in \mathbb{R}^2} \left[ B_{\omega^-}(x) + V(x) \right], \quad E_{\text{sup}} := \inf_{x \in \mathbb{R}^2} \left[ B_{\omega^+}(x) + V(x) \right]. \]

Moreover, we will write \( M_{\pm} = \sum_{k=0}^{\infty} m^{(k)}_{\pm} \). Note that \( |m^{(k)}_{\pm}| \leq \sigma^{(k)} \) and thus \( |M_{\pm}| \leq (1 - e^{-\rho})^{-1} \).
Theorem 3.4  Suppose (A) holds, and let $\rho > \ln 2$. Then the following statements hold:

(a) We have
\[ E_{\inf} \leq \Sigma_{\inf} \leq E_{\inf} + 4K_2^2b_0^{-2} + \min(K_2b_0^{-1/2}, K_3b_0^{-1}), \] (3.5)
where we defined
\[ K_3 := 2 \max_{|\alpha|=2} \{ \|D^\alpha B_{\det}\|_{\infty} + (1 - 4e^{-\rho})^{-1}\|D^\alpha U\|_{\infty} + \|D^\alpha V\|_{\infty} \}, \] (3.6)
if $4e^{-\rho} < 1$, and $K_3 := \infty$ otherwise.

(b) We have $E_{\inf} + \mu c_u(M_+ - M_-) \leq E_{\sup}$.

(c) If $\Sigma_{\inf} < E_{\sup}$, then
\[ \Sigma \supset [\Sigma_{\inf}, E_{\sup}]. \] (3.7)

(d) In the special case when $U = 1$, $B_{\det}$ is constant and $V = 0$, then $\Sigma_{\inf} = B_{\det} + \mu M_-$ and
\[ \Sigma \supset \bigcup_{n \in \mathbb{N}_0} \{(1 + 2n)(B_{\det} + \mu[M_-, M_+])\}. \]

Remark. The finiteness of $K_3$ improves the upper bound on $\Sigma_{\inf}$ in the large $b_0$ regime, see (3.5), but it requires higher regularity on the data. We also remark that in view of (a) and (b) the condition $\Sigma_{\inf} < E_{\sup}$ in (c) can be guaranteed if $c_u > 0$ and $b_0$ is sufficiently large.

The paper is organized as follows. In Section 4 some previous methods to obtain a Wegner estimate are presented. Sections 5–7 are devoted to the proof of the Wegner estimate as stated in Theorem 3.1. Its proof is given in Section 5 modulo the key Proposition 5.1, whose proof is given in Section 6. Section 7 contains some elliptic regularity estimates needed in Section 6. The ergodicity property needed to show Theorem 3.2 will be given in Appendix A. In Sections 8–10 we explain how the Wegner estimate leads to Anderson localization. In Section 8 an inner bound on the deterministic spectrum is shown, i.e., a proof of Theorem 3.4 will be given. In Section 9, an initial length scale estimate will be proven. This estimate will then be used in Section 10, where the localization result, Theorem 3.3, will be shown. We will use the multiscale analysis following the approach presented in Stollmann’s book [17]. We remark that we could alternatively have followed the setup presented by Combes and Hislop in [2] to prove the initial length scale estimate by verifying their Hypothesis $[H1](\gamma_0, l_0)$.

We will use the convention that unspecified positive constants only depending on $K_0$ and $K_1$ are denoted by $C, C_0, C_1, \ldots$ or $c, c_0, c_1, \ldots$ whose precise values are irrelevant and may change from line to line.

4  Main ideas of the proof of the Wegner estimate

The standard approach to prove Wegner estimate for random external potential is to use monotonicity of the eigenvalues as a function of the random coupling parameters (see, e.g. [17] for an exposition). Consider the simplest Anderson model of the form $H_L = -\Delta + V_\omega(x)$ with Dirichlet boundary conditions on $\Lambda_L$. The random potential is given by
\[ V_\omega(x) = \sum_{z \in \mathbb{Z}^d} \omega_z u(x - z) \] (4.1)
with i.i.d. random variables $\omega = \{ \omega_z, z \in \mathbb{Z}^d \}$ and with a local potential profile function $u(x) : \mathbb{R}^d \to \mathbb{R}$. By the first order perturbation formula for any eigenvalue $\lambda$ with normalized eigenfunction $\psi$ we have
\[ \frac{\partial \lambda}{\partial \omega_z} = \langle \psi, u(\cdot - z)\psi \rangle = \int |\psi(x)|^2 u(x - z)dx. \] (4.2)
We define the vector field \( Y = \sum_{z \in \Lambda} (\partial/\partial \omega_z) \) on the space of the random couplings \( \omega \), where the summation is over all \( z \in \Lambda := \Lambda_L \cap \mathbb{Z}^d \). If, additionally, \( \sum_z u(x - z) \geq c \) with some positive constant \( c \), then \( Y \lambda \geq c \).

This estimate guarantees that each eigenvalue moves with a positive speed as the random couplings vary in the direction of \( Y \). In particular if \( \omega_z \) are continuous random variables with some mild regularity condition on their density function \( v_z(\omega_z) \) then no eigenvalue can stick to any fixed energy \( E \) when taking the expectation.

More precisely, if \( \chi = \chi_{E, \eta} \) is the characteristic function of the spectral interval \( I = [E - \eta/2, E + \eta/2] \) and \( F \) is its antiderivative, \( F' = \chi \), with \( F(-\infty) = 0 \), then the expected number of eigenvalues in \( I \) is estimated by

\[
E \text{Tr} \chi(H_L) \leq c^{-1} E \text{Tr} Y F(H_L) = c^{-1} \int_{\mathbb{R}^\Lambda} \left( \prod_{z \in \Lambda} v_z(\omega_z) d\omega_z \right) \sum_{z \in \Lambda} \frac{\partial}{\partial \omega_z} \text{Tr} F(H_L). \tag{4.3}
\]

If \( v_z \) is sufficiently regular, then, after performing an integration by parts and using that \( 0 \leq F \leq \eta \) together with some robust Weyl-type bound for the number of eigenvalues, one obtains the Wegner estimate. Note that the proof essentially used that \( \sum_z u(x - z) \geq c > 0 \), in particular it does not apply to sign indefinite potential profile \( u \). We remark that for a certain class of random displacement models a different mechanism of monotonicity has been established in [12] to prove the a Wegner estimate and Anderson localization.

For random vector potential of the form (1.1), the first order perturbation formula gives

\[
\frac{\partial \lambda}{\partial \omega_z} = \langle u(\cdot - z), j_\psi \rangle, \tag{4.4}
\]

where \( j_\psi = 2 \text{Re} \bar{\psi}(p - A) \psi \) is the current of the eigenfunction. Unlike the non-negative density \( |\psi(x)|^2 \), the current is a vector and no apparent condition on \( u(\cdot - z) \) can guarantee that \( Y \lambda \geq c > 0 \) for some \( \psi \)-independent vectorfield of the form \( Y = \sum_c c_z(\omega)(\partial/\partial \omega_z) \).

The method of [9] addresses the issue of the lack of positivity of \( Y \lambda \) for both the sign non-definite random potential (4.1) case and the random vector potential (1.1) case but it does not seem to apply for random magnetic fields (1.2) due to the long-range dependence of \( A_\omega \) generating \( B_\omega \). Moreover, it uses the Birman-Schwinger kernel, i.e. it is restricted for energies below the spectrum of the deterministic part \( H_{\text{det}} \) of the total Hamiltonian.

To outline our approach, we go back to (4.4), and will exploit that

\[
\sum_z \left( \frac{\partial \lambda}{\partial \omega_z} \right)^2 = \sum_z |\langle u(\cdot - z), j_\psi \rangle|^2 \tag{4.5}
\]

is non-negative, and, in fact, it has an effective positive lower bound (Proposition 5.1). The proof relies on three observations. First, \( \|j_\psi\|_2 \) has an effective lower bound because we assume that there is a strictly positive background magnetic field (Lemma 6.2). Second, \( \|\nabla j_\psi\|_2 \) has an upper bound following from elliptic regularity (Lemma 6.1). This will ensure that most of the \( L^2 \)-norm of \( j_\psi \) comes from low momentum modes. Finally, assuming that the random magnetic field has modes on arbitrarily short scales, i.e. the summation over \( z \) in (4.5) is performed on a fine lattice, we see that a substantial part of the low modes of \( j_\psi \) is captured by the right hand side of (4.5), giving a positive lower bound \( c \) on (4.5).

Using this lower bound we can estimate, similarly to (4.3),

\[
E \text{Tr} \chi(H_L) = \sum_\ell \chi(\lambda_\ell) \leq c^{-1} E \sum_z (Y_z \lambda_\ell)^2 \chi(\lambda_\ell),
\]

where \( Y_z = (\partial/\partial \omega_z) \) and \( \lambda_\ell \) are the eigenvalues of \( H_L \). The square of the derivative, \( (Y_z \lambda_\ell)^2 \), can be estimated in terms of the second derivatives of the eigenvalues (see (5.9)). By usual perturbation theory, to compute second derivatives of the eigenvalues requires first derivatives of eigenfunctions which seems to be a hopeless task in case of possible multiple or near-multiple eigenvalues. However, a key inequality in Lemma 5.2 ensures that the sum of second derivatives can be estimated by the trace of the second derivative of the Hamiltonian itself. Since the Hamiltonian is quadratic in the random parameters, this latter quantity can be computed.
5 Proof of the Wegner estimate

In the following proof we consider $L$ fixed. Set $k = k(L)$ such that

$$\frac{1}{2} L^{-K} \leq 2^{-k} \leq L^{-K}$$

(5.1)

with some fixed exponent $K$ to be determined later. For brevity, we denote $\varepsilon = \varepsilon(L) := 2^{-k(L)}$. Set $\Lambda_\varepsilon = (\varepsilon\mathbb{Z})^2 \cap \Lambda_{L+1} = \Lambda^{(k)} \cap \Lambda_{L+1}$. Note that $|\Lambda_\varepsilon| \leq CL^2\varepsilon^{-2}$. For this given $L$, we decompose the magnetic field (2.2) as follows

$$B = \widetilde{B} + \mu B^{(k)}, \quad \widetilde{B} := B_{\text{det}} + \mu \sum_{m=0}^{\infty} B^{(m)}.$$  

We will use only the random variables in $B^{(k)}$ and we fix all random variables in $B^{(m)}$, $m \neq k$, i.e. we consider $\widetilde{B}$ deterministic. We will choose a divergence free gauge for $\widetilde{B}$, i.e. $\nabla \times \widetilde{A} = \widetilde{B}, \nabla \cdot \widetilde{A} = 0$. Since $k$ is fixed, we can drop the $k$ superscript in the definitions of $B_z^{(k)}, \omega_z^{(k)}, \beta_z^{(k)}, \sigma_z^{(k)}$ and $\sigma^{(k)}$, i.e.

$B_z^{(k)}(x) = \sum_{z \in \Lambda_\varepsilon} B_z(x), \quad B_z(x) = \omega_z \beta_z(x), \quad \beta_z(x) = u((x - z)/\varepsilon).$

We define two different vector potentials for $B_z$ by setting

$$a_z^{(1)} := \omega_z \alpha_z^{(1)}, \quad a_z^{(2)} := \omega_z \alpha_z^{(2)}$$

with

$$\alpha_z^{(1)}(x_1, x_2) := \left( \int_{-\infty}^{x_2} \beta_z(x_1, s) ds \right) e_1, \quad \alpha_z^{(2)}(x_1, x_2) := -\left( \int_{-\infty}^{x_1} \beta_z(s, x_2) ds \right) e_2,$$

(5.2)

where $e_1 = (1, 0), e_2 = (0, 1)$ are the standard unit vectors. Then $\nabla \times \alpha_z, 1 = \nabla \times \alpha_z, 2 = \beta_z$ and $\nabla \times a_z^{(1)} = \nabla \times a_z^{(2)} = B_z$ and notice that

$$\|a_\tau^{(\tau)}\|_\infty \leq \varepsilon, \quad \tau = 1, 2,$$

(5.3)

and, actually, under condition (2.5) we even have $\|a_\tau^{(\tau)}\|_\infty \leq 2\varepsilon$. Let

$$A^{(1)} := \widetilde{A} + \mu \sum_{z \in \Lambda_\varepsilon} \omega_z \alpha_z^{(1)}, \quad A^{(2)} := \widetilde{A} + \mu \sum_{z \in \Lambda_\varepsilon} \omega_z \alpha_z^{(2)}$$

(5.4)

then $\nabla \times A^{(\tau)} = B$, $\tau = 1, 2$. We consider the two unitarily equivalent random Hamiltonians

$$H_L(A^{(\tau)}) := (p - A^{(\tau)})^2 + V, \quad \tau = 1, 2$$

with Dirichlet boundary conditions on $\Lambda_L$. For a while we will neglect the $\tau = 1, 2$ indices; all arguments below hold for both cases.

Let $\lambda$ be an eigenvalue of $H_L(A)$ with eigenfunction $\psi$. We consider $\lambda$ as a function of the collection of random variables $\{\omega_z\}$. For each fixed $z$,

$$Y_z \lambda := \frac{\partial \lambda}{\partial \omega_z} = 2\mu \text{Re} \int \psi \overline{\alpha_z} \cdot (p - A) \psi = \mu \int \alpha_z \cdot j_\psi,$$

(5.5)

where $j_\psi = j = (j_1, j_2) = 2\text{Re} \overline{\psi} (p - A) \psi$ is the current of the eigenfunction. Short calculation shows that $j$ is gauge invariant and divergence free.
Thus Lemma 5.2. We have for any
Proof of Lemma 5.2. We use spectral decomposition, $T := t(H_L(A))$. Clearly
since the derivative $t'$ is bounded by 1. In the sequel we set $\chi = \chi_{t(E), \eta}$. Let $F(u)$ such that $F' = \chi$ with $F(u) = 0$ for $u \leq t(E) - \eta/2$ and let $G(u)$ such that $G' = F$ and $G(u) = 0$ for $u \leq t(E) - \eta/2$.

Let $\lambda_1, \lambda_2, \ldots$ denote the eigenvalues of $H_L(A)$ and let $\tau_\ell = t(\lambda_\ell)$ be the eigenvalues of $T$. In Section 6 we will prove the following key technical estimate:

**Proposition 5.1** With the notations above, and assuming $\rho \geq \ln 2$ (i.e. $\sigma \leq \varepsilon$) there exist positive constants $C_0$ and $C_1$, depending only on $K_0$ and $K_1$, and a constant $L_0^*$, depending on $K_0$, $K_1$, and $\delta$, such that for any $0 < \kappa \leq 1$ and $a = C_1 \kappa^{-1}$

$$\sum_z \left( \frac{\partial \lambda_\ell}{\partial \omega_z} \right)^2 = \sum_z (Y_\ell \lambda_\ell)^2 \geq C_0^{-1} L^{-a} \varepsilon^2 \mu^2$$

(5.7)

for any eigenvalue $\lambda_\ell$ of $H_L(A)$ and all $L \geq L_0^* b_0^*$.

From (5.7) it easily follows that

$$\text{Tr} \chi(T) = \sum_\ell \chi(\tau_\ell) \leq C L \varepsilon^{-2} \mu^{-2} \sum_\ell \sum_z (Y_\ell \tau_\ell)^2 \chi(\tau_\ell),$$

(5.8)

since $Y_\ell \tau_\ell = g'(\lambda_\ell) Y_\ell \lambda_\ell$ and for $\tau_\ell$ in the support of $\chi$ the number $|g'(\lambda_\ell)|$ is bounded from below by a universal constant.

Notice that for any $Y = Y_z$ and any $\ell$ we have

$$Y^2 G(\tau_\ell) = Y((Y \tau_\ell) F(\tau_\ell)) = (Y^2 \tau_\ell) F(\tau_\ell) + (Y \tau_\ell)^2 \chi(\tau_\ell).$$

Thus

$$\sum_\ell \sum_z (Y_\ell \tau_\ell)^2 \chi(\tau_\ell) = \sum_\ell \sum_z Y^2 \tau_\ell G(\tau_\ell) - \sum_\ell \sum_z (Y^2 \tau_\ell) F(\tau_\ell) - \sum_\ell \sum_z (Y^2 \tau_\ell) F(\tau_\ell)$$

$$= \sum_\ell \text{Tr} Y^2 \tau_\ell G(T) - \sum_\ell \sum_z (Y^2 \tau_\ell) F(\tau_\ell).$$

(5.9)

**Lemma 5.2** We have for any $Y = Y_z$

$$\text{Tr} (Y^2 T) F(T) \leq \sum_\ell (Y^2 \tau_\ell) F(\tau_\ell).$$

(5.10)

**Proof of Lemma 5.2.** We use spectral decomposition, $T = \sum_\alpha \tau_\alpha |u_\alpha \rangle \langle u_\alpha|$, 

$$\text{Tr} (Y^2 T) F(T) = \sum_\alpha F(\tau_\alpha) \langle u_\alpha| Y^2 (Y \sum_\beta \tau_\beta |u_\beta \rangle \langle u_\beta|) |u_\alpha \rangle$$

$$= \sum_\alpha F(\tau_\alpha) \langle u_\alpha| \left( \sum_\beta (Y^2 \tau_\beta) |u_\beta \rangle \langle u_\beta| + 2 \sum_\beta (Y \tau_\beta) Y(|u_\beta \rangle \langle u_\beta|) + \sum_\beta \tau_\beta Y^2(|u_\beta \rangle \langle u_\beta|) \right) |u_\alpha \rangle$$

$$= \sum_\alpha F(\tau_\alpha) (Y^2 \tau_\alpha) + 2 \sum_{\alpha, \beta} F(\tau_\alpha) (Y \tau_\beta) \langle u_\alpha| Y(|u_\beta \rangle \langle u_\beta|) |u_\alpha \rangle + \sum_{\alpha, \beta} F(\tau_\alpha) \tau_\beta (u_\alpha | Y^2(|u_\beta \rangle \langle u_\beta|) |u_\alpha \rangle.$$
The second term is zero, since
\[ \langle u_\alpha | Y (| u_\beta \rangle \langle u_\beta |) | u_\alpha \rangle = \langle u_\alpha | Y u_\beta \rangle \langle u_\beta | u_\alpha \rangle + \langle u_\alpha | u_\beta \rangle \langle Y u_\beta | u_\alpha \rangle = \delta_{\alpha\beta} (\langle u_\alpha | Y u_\alpha \rangle + \langle Y u_\alpha | u_\alpha \rangle) = \delta_{\alpha\beta} Y \langle u_\alpha | u_\alpha \rangle = 0 \] (5.12)
since \( \langle u_\alpha | u_\alpha \rangle = 1 \). In the last term in (5.11), we use that
\[ 0 = Y \langle u_\alpha | u_\beta \rangle = \langle Y u_\alpha | u_\beta \rangle + \langle u_\alpha | Y u_\beta \rangle \]
and differentiating it once more:
\[ 0 = Y \left( \langle Y u_\alpha | u_\beta \rangle + \langle u_\alpha | Y u_\beta \rangle \right) = (Y^2 u_\alpha | u_\beta \rangle + 2 \langle Y u_\alpha | Y u_\beta \rangle + \langle u_\alpha | Y^2 u_\beta \rangle. \]
Thus
\[ \langle u_\alpha | Y^2 (| u_\beta \rangle \langle u_\beta |) | u_\alpha \rangle = \langle u_\alpha | \left( | Y^2 u_\beta \rangle \langle u_\beta | + 2 | Y u_\beta \rangle \langle Y u_\beta | + | u_\beta \rangle \langle Y^2 u_\beta \rangle \right) | u_\alpha \rangle = \delta_{\alpha\beta} \left( \langle u_\alpha | Y^2 u_\alpha \rangle + \langle Y^2 u_\alpha | u_\alpha \rangle \right) + 2 \| Y u_\alpha \| u_\alpha \|^2 \]
\[ = 2 \| Y u_\alpha \| u_\alpha \|^2 - 2 \delta_{\alpha\beta} \langle Y u_\alpha | Y u_\alpha \rangle. \]
So for the last term in (5.11),
\[ \sum_{\alpha, \beta} F(\tau_\alpha) \tau_\beta \langle u_\alpha | Y^2 (| u_\beta \rangle \langle u_\beta |) | u_\alpha \rangle \]
\[ = 2 \sum_{\alpha, \beta} F(\tau_\alpha) \tau_\beta \left( \langle u_\beta | Y u_\alpha \rangle \|^2 - \delta_{\alpha\beta} \langle Y u_\alpha | Y u_\alpha \rangle \right) \]
\[ = 2 \sum_{\alpha} F(\tau_\alpha) \tau_\alpha \left( \sum_{\beta} \langle u_\beta | Y u_\alpha \rangle \|^2 - \langle Y u_\alpha | Y u_\alpha \rangle \right) + 2 \sum_{\alpha, \beta} F(\tau_\alpha) (\tau_\beta - \tau_\alpha) \langle u_\beta | Y u_\alpha \rangle \|^2. \]
The first term is zero since \( u_\beta \) is an orthonormal basis. In the second term we use that \( \| u_\beta | Y u_\alpha \rangle \|^2 \) is symmetric in the \( \alpha, \beta \) indices and write
\[ \sum_{\alpha, \beta} F(\tau_\alpha) (\tau_\beta - \tau_\alpha) \langle u_\beta | Y u_\alpha \rangle \|^2 = \sum_{\alpha < \beta} \left[ F(\tau_\alpha) (\tau_\beta - \tau_\alpha) + F(\tau_\beta) (\tau_\alpha - \tau_\beta) \right] \langle u_\beta | Y u_\alpha \rangle \|^2 \]
\[ = - \sum_{\alpha < \beta} [ F(\tau_\alpha) - F(\tau_\beta) ] (\tau_\alpha - \tau_\beta) \langle u_\beta | Y u_\alpha \rangle \|^2 \leq 0 \] (5.14)
since \( F \) is monotone increasing. This proves Lemma 5.2. □

Thus combining (5.8), (5.9) and (5.10), we have
\[ \text{Tr} \chi(T) \leq C L^a \varepsilon^{-2} \mu^{-2} \sum_z \left( \text{Tr} Y_z^2 G(T) - \text{Tr} (Y_z^2 T) F(T) \right). \] (5.15)

We compute \( Y_z^2 T \). First, to present the idea, imagine that we did not have the high energy cutoff, i.e. \( T \) were simply \( (p - A)^2 + V \). Then
\[ Y_z [(p - \tilde{A} - \mu \sum_\zeta \omega_\zeta \alpha_\zeta)^2 + V] = -\mu \alpha_z \cdot (p - \tilde{A} - \mu \sum_\zeta \omega_\zeta \alpha_\zeta) - \mu (p - \tilde{A} - \mu \sum_\zeta \omega_\zeta \alpha_\zeta) \cdot \alpha_z \]
and
\[ Y_z^2 (p - \tilde{A} - \mu \sum_\zeta \omega_\zeta \alpha_\zeta)^2 = 2 \mu^2 \alpha_z^2, \]
thus, using $|F| \leq \eta$ and (5.3), we would have to compute $|\text{Tr} (Y^2 T F(T))| \leq C\varepsilon^2 \eta \text{Tr} 1 (T \geq E - \eta/2)$. Unfortunately, this trace is unbounded. The smooth high energy cutoff ensures the finiteness of the trace and gives a bound $C\varepsilon^2 \eta L^2$, but it makes the second derivative calculation more complicated.

To make the argument more precise, we go back to $T = t(H_L(A))$ with eigenvalues $\tau_\ell = t(\lambda_\ell)$ where $\lambda_\ell$ are the eigenvalues of $H_L(A)$. Then

$$\text{Tr} F(T) = \sum_\ell F(\tau_\ell) \leq \eta \sum_\ell 1 (t(\lambda_\ell) \geq t(E) - \eta/2) \leq \eta \sum_\ell 1 (E - C\eta \leq \lambda_\ell \leq E^* + C\eta)$$

where $E^* > E$ is the other root of the equation $t(E^*) = t(E)$. Using $E \leq K_1 b_0 = \frac{\pi}{10}$ it is easy to see that $E^* \leq C(E + s) \leq CK_1 b_0$ and $|t'(E)|, |t'(E^*)|$ are bounded from below by a universal constant which was used in the last inequality. Thus, by applying Weyl's bound (5.17) on the number of eigenvalues below a fixed threshold, we obtain

$$\text{Tr} F(T) \leq C b_0 \eta L^2,$$  \hspace{1cm} (5.16)

where $C$ depends $K_1$. We note that the Weyl bound holds for magnetic Schrödinger operators as well, namely, for any $K > 0$ we have

$$\# \{\lambda_\ell \leq K\} \leq \# \{\text{eigenvalues of } (p - A)^2 + V - 2K 1_{\Lambda_L} \text{ between } [-K, -2K]\} \leq CK^{-1} \text{Tr} \left[ (p - A)^2 + V - 2K 1_{\Lambda_L} \right]_- \leq CK^{-1} \int_{\Lambda_L} \left[ 2K 1_{\Lambda_L} \right]^2 = CKL^2,$$  \hspace{1cm} (5.17)

with some universal constant $C$. Here $\text{Tr} [h]_-$ denotes the sum of absolute values of the negative eigenvalues of the operator $h$ and we applied it to $h = (p - A)^2 + V - 2K 1_{\Lambda_L}$ with Dirichlet boundary conditions on $\Lambda_L$. The last inequality is the Lieb-Thirring inequality that holds for magnetic Schrödinger operators as well.

To compute $Y^2 T$, we define the resolvent

$$R = \frac{1}{s + (p - A)^2 + V},$$

where $(p - A)^2 + V$ is understood with Dirichlet boundary conditions. We have

$$Y \frac{(p - A)^2 + V}{s + (p - A)^2 + V} = [Y(p - A)^2] R^3 - \sum_{k=1}^3 [(p - A)^2 + V] R^k [Y(p - A)^2] R^{4-k},$$  \hspace{1cm} (5.18)

and thus

$$Y^2 \frac{(p - A)^2 + V}{s + (p - A)^2 + V} = [Y^2(p - A)^2] \frac{1}{s + (p - A)^2 + V} - 2 \sum_{k=1}^3 [Y(p - A)^2] R^k [Y(p - A)^2] R^{4-k} - \sum_{k=1}^3 [(p - A)^2 + V] R^k [Y^2(p - A)^2] R^{4-k} + 2 \sum_{\ell=1}^4 \sum_{k, k+i \leq 4} \sum_{k \leq i \leq 4} [(p - A)^2 + V] R^k [Y(p - A)^2] R^\ell [Y(p - A)^2] R^{5-k-\ell}. $$  \hspace{1cm} (5.19)

Let $P = 1(H_L(A) \leq E^* + C\eta)$ be the spectral projection. Since $F(T) = 0$ on the complement of $P$, we can insert $P$ as

$$\text{Tr} (Y^2 T) F(T) = \text{Tr} P (Y^2 T) PF(T).$$

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Using that
\[ Y_z(p - A)^2 = -\mu \alpha_z \cdot (p - A) - \mu (p - A) \cdot \alpha_z, \quad Y_z^2(p - A)^2 = 2\mu^2 \alpha_z^2, \]
the estimates
\[ \left\| (p - A) \frac{1}{(s + (p - A)^2 + V)^{1/2}} \right\| \leq C, \quad \left\| P[(p - A)^2 + V] \right\| \leq E^* + C\eta \leq CK_1b_0 \leq Cs, \]
the fact that \( \|V\|_\infty \leq s \) and the bound \( |\alpha_z| \leq \varepsilon \) we can estimate the right hand side of (5.19) and we obtain
\[ \| P(Y_z^2 T) \| \leq C\varepsilon^2 \]
with \( C \) depending on \( K_1 \). Thus
\[ |\text{Tr} (Y_z^2 T) F(T)| \leq C\varepsilon^2 \text{Tr} F(T) \leq Cb_0 L^2 \varepsilon^2 \eta \]
using the positivity of \( F(T) \) and the bound (5.16).

Recalling (5.15) and \( |\Lambda_\varepsilon| \leq CL^2 \varepsilon^{-2} \), we proved that
\[ \text{Tr} \chi(T) \leq C\mu^{-2}L^a \varepsilon^{-2} \sum_z \text{Tr} (Y_z^2 G(T)) + Cb_0 \mu^{-2} L^{a+4} \varepsilon^{-2} \eta \]
under (5.7). After taking expectation with respect to the collection
\[ \{\omega_z : z \in \Lambda_\varepsilon \} = \{\omega_z^{(k)} : z \in \Lambda^{(k)} \cap \Lambda_L \}, \]
we integrate by parts
\[ \mathbb{E} \text{Tr} Y_z^2 G(T) = \int \prod_{\zeta \in \Lambda_\varepsilon} v_\zeta(\omega_\zeta) d\omega_\zeta \frac{\partial^2}{\partial \omega_\zeta} \text{Tr} G(T) = \int \prod_{\zeta \neq \zeta} v_\zeta(\omega_\zeta) d\omega_\zeta \int v_\zeta(\omega_\zeta) d\omega_\zeta \text{Tr} G(T). \]

To compute \( \text{Tr} G(T) \) we use that \( G(u) \leq C\eta u \), and that \( t(u) \leq s^3(s + u)^{-2} \), we have
\[ \text{Tr} G(T) \leq C\eta s^3 \text{Tr} \left( \frac{1}{s^3 + (p - A)^2} \right) \leq C\eta L^2 s^3 \int \frac{dp}{s^3 + p^2} \leq C\eta s^2 L^2 \]
using the integral representation \( \alpha^{-2} = \int_0^\infty te^{-at} dt \), with \( \alpha > 0 \), Feynman-Kac-Itô formula, and the diamagnetic inequality. Using (2.9) and (5.7), we get from (5.21) that
\[ \mathbb{E} \text{Tr} \chi(T) \leq Cb_0 L^2 \varepsilon^{-4} \mu^{-2} \varepsilon^{-2} + Cb_0 \mu^{-2} L^{a+4} \varepsilon^{-2} \eta. \]
Considering the choice of \( \varepsilon = 2^{-k} \sim L^{-K}, (5.1) \) and (2.8), we get
\[ \mathbb{E} \text{Tr} \chi(T) \leq Cb_0 \eta \mu^{-2} L^{a+4+2K+2(\ln 2)^{-1}} \rho \]
and together with (5.6) this completes the proof of Theorem 3.1. \( \square \)

### 6 Proof of Proposition 5.1

In this section we prove that the lower bound (5.7) on \( (Y_z \lambda)^2 \) holds for any eigenvalue. Fix \( \ell \) and denote \( \lambda = \lambda_\varepsilon \). Using (5.5), we have
\[ \sum_z (Y_z \lambda)^2 = 4\mu^2 \sum_z (|\alpha_z^{(1)}|, j_\psi)^2 = 4\mu^2 \sum_z (|\alpha_z^{(2)}|, j_\psi)^2 \]
which we write as
\[ \mu^{-2} \sum_z (Y_z \lambda)^2 = 2 \sum_z (|\alpha_z^{(1)}|, j_\psi)^2 + 2 \sum_z (|\alpha_z^{(2)}|, j_\psi)^2. \]
In the sequel \((\cdot, \cdot)\) denotes the scalar product on \( L^2(\Lambda_L) \). We will prove the following two lemmas:
Lemma 6.1 There are positive \( d' \) and \( g' \) and a constant \( C = C(K_0, K_1) \) such that

\[
\int_{\Lambda_{\epsilon}} |\nabla j_{\psi}|^2 \leq CL^{d'}b_0^{g'}
\]

for all normalized eigenfunction \( \psi \) with energy \( E \leq K_1b_0 \). From the proof, \( d' = 126 \) and \( g' = 60 \). Here we adopted the notation \( |\nabla j_{\psi}|^2 = \sum_{i,k=1}^2 |\partial_k(j_{\psi})|^2 \).

Lemma 6.2 There are positive \( d'' \) and \( g'' \) and a positive constant \( c = c(K_0, K_1) \) such that

\[
\int_{\Lambda_{\epsilon}} |j_{\psi}|^2 \geq cb_0^{-g''}L^{-d''}
\]

uniformly for all normalized eigenfunction \( \psi \), with energy \( E \leq K_1b_0 \). From the proof, \( d'' = 100 \) and \( g'' = 46 \).

First we show that from these two Lemmas, (5.7) follows if \( \delta_0 \) in (2.4) or (2.5) is sufficiently small. Let \( N := [\delta^{-2}] + 1 \), where \( [\cdot] \) denotes the integer part, and define the square

\[
Q_z := \{ x \in \mathbb{R}^2 : |x - z|_{\infty} \leq N\epsilon \}
\]

and the \( L^2 \)-normalized vector-valued functions

\[
M_z^{(1)}(x) = \frac{1}{2N\epsilon}1_{Q_z}(x)e_1 \quad M_z^{(2)}(x) = \frac{1}{2N\epsilon}1_{Q_z}(x)e_2.
\]

We set \( \Lambda_\epsilon' = (2N\epsilon\mathbb{Z})^2 \cap \Lambda_{\epsilon+1} \subset \Lambda_{\epsilon} \) to be a sublattice of \( \Lambda_{\epsilon} \). The \( \tau \)-indices of \( M_z^{(j)}, j = 1, 2 \), will always run over this sublattice \( z \in \Lambda_\epsilon' \). We write

\[
M_z^{(1)} = \frac{1}{2N\epsilon^2} \sum_{k=-N}^N (\alpha_{z+k\epsilon e_1-N\epsilon e_2}^{(1)} - \alpha_{z+k\epsilon e_1+N\epsilon e_2}^{(1)}) + \mathcal{E}_{\epsilon}^{(1)}
\]

\[
M_z^{(2)} = \frac{1}{2N\epsilon^2} \sum_{k=-N}^N (\alpha_{z+k\epsilon e_2-N\epsilon e_1}^{(2)} - \alpha_{z+k\epsilon e_2+N\epsilon e_1}^{(2)}) + \mathcal{E}_{\epsilon}^{(2)}
\]

with errors \( \mathcal{E}_{\epsilon}^{(\tau)} \) that are defined by these equations. Let \( Q_z' = \{ x \in \mathbb{R}^2 : |x - z|_{\infty} \leq (N+\delta^{-1})\epsilon \} \). We will prove the following estimates at the end of this section.

Proposition 6.3 With the notations above, we have

\[
supp \mathcal{E}_{\epsilon}^{(\tau)} \subset Q_z', \quad \tau = 1, 2,
\]

\[
\int |\mathcal{E}_{\epsilon}^{(\tau)}|^2 \leq \Theta \delta, \quad \tau = 1, 2.
\]

where \( \Theta = 100 \) under the condition (2.4) and \( \Theta = 20 + \|\nabla u_0\|^2 \) under the condition (2.5).

Suppose that (5.7) is wrong, then, by (6.1), we have

\[
\sum_{\tau=1,2} \sum_{z \in \Lambda_{\epsilon}} |(\alpha_z^{(\tau)}, j)|^2 \leq CL^{-a} \epsilon^2
\]

after dropping the subscript \( \psi \). Then, in particular

\[
\sum_{\tau=1,2} \sum_{z \in \Lambda_{\epsilon}'} |(M_z^{(\tau)}, j)|^2 \leq CL^{-a} \epsilon^{-2} + 2 \sum_{\tau=1,2} \sum_{z \in \Lambda_{\epsilon}'} |(\mathcal{E}_{\epsilon}^{(\tau)}, j)|^2,
\]
and using (6.5), we get
\[
2 \sum_{\tau=1,2} \sum_{z \in \Lambda'_z} |(\mathcal{E}'_{z}(\tau), j)|^2 \leq 2 \sum_{\tau=1,2} \sum_{z \in \Lambda'_z} \|\mathcal{E}'_{z}(\tau)\|^2 \|j_1 Q_z\|^2 \leq 16\Theta \delta \|j\|^2. \quad (6.8)
\]

For \( z \in \Lambda'_z \) and \( \tau = 1, 2 \), we let
\[
(j_{\tau})_z := (2N\varepsilon)^{-2} \int_{Q_z} j_{\tau} = (2N\varepsilon)^{-1}(M^{(\tau)}_{z}, j).
\]

We write
\[
j_{\tau} = \sum_{z \in \Lambda'_z} (j_{\tau})_z 1_{Q_z} + J_{\tau} = \sum_{z \in \Lambda'_z} (M^{(\tau)}_{z}, j)(2N\varepsilon)^{-1} 1_{Q_z} + J_{\tau},
\]
where \( J_{\tau} \) is orthogonal to all \( 1_{Q_z}, z \in \Lambda'_z \). Since the \((2N\varepsilon)^{-1} 1_{Q_z}\) functions are orthonormal, we have from (6.7) and (6.8) that
\[
\|j\|^2 = \sum_{\tau=1,2} \int_{\Lambda_L} |j_{\tau}|^2 = \sum_{\tau=1,2} \sum_{z \in \Lambda'_z} |(M^{(\tau)}_{z}, j)|^2 + \sum_{\tau=1,2} \|J_{\tau}\|^2 \leq CL^{-a}\varepsilon^{-2} + 16\Theta \delta \|j\|^2 + \sum_{\tau=1,2} \|J_{\tau}\|^2. \quad (6.9)
\]

Choosing \( \delta_0 = (32\Theta)^{-1} \), for any \( \delta \leq \delta_0 \) we get
\[
\|j\|^2 \leq CL^{-a}\varepsilon^{-2} + 2 \sum_{\tau=1,2} \|J_{\tau}\|^2.
\]

However, by Poincaré inequality
\[
\int_{\Lambda_L} |j_{\tau}|^2 = \sum_{z \in \Lambda'_z} \int_{Q_z} |j_{\tau} - (j_{\tau})_z|^2 \leq \sum_{z \in \Lambda'_z} (CN\varepsilon)^2 \int_{Q_z} |\nabla j_{\tau}|^2 \leq C\delta^{-4}\varepsilon^2 \int_{\Lambda_L} |\nabla j|^2 \leq C\delta^{-4}\varepsilon^2 L^d b_0^{d'}
\]
by Lemma 6.1, where the last constant depends on \( K_0, K_1 \). Thus, from (6.9) and Lemma 6.2, we have
\[
cL^{-d'}b_0^{d''} \leq CL^{-a}\varepsilon^{-2} + C\delta^{-4}\varepsilon^2 L^d b_0^{d'} \leq CL^{-a+2K} + CL^{d+4-2K} b_0^{d'},
\]
where we used that \( \varepsilon \sim L^{-K} \) from (5.1) and we assumed \( L \geq \delta^{-4} \). Using \( \sigma \leq \varepsilon \), we get
\[
cL^{-d'}b_0^{d''} \leq CL^{-a+2K} + CL^{d+4-2K} b_0^{d'}.
\]
Choosing first \( K \) such that \( g' + g'' < \kappa(2K - d' - d'' - 4) \) and then \( a \) such that \( g'' < \kappa(a - 2K - d'') \), we see that (6.10) is a contradiction if \( L \geq (2Cc^{-1}b_{0})^{\kappa} \) and \( L \geq \delta^{-4} \). \( \Box \)

**Proof of Proposition 6.3.** To see the support property (6.5), we note that \( \text{supp } u \subset [-\delta^{-1}, \delta^{-1}]^2 \) under either conditions (2.4) or (2.5) on the profile function \( u \). Therefore \( \text{supp } \beta_{\zeta} \subset Q_{\zeta}^1 \) for any \( \zeta \in \Lambda_z \) with \( |\zeta - z|_{\infty} \leq N\varepsilon \) and (6.5) follows immediately from the definitions of \( \alpha_{\zeta}^{(\tau)} \), \( \tau = 1, 2 \), see (5.2).

For the proof of (6.6) we distinguish between the two alternative conditions (2.4) and (2.5). If \( u(x) \) satisfies (2.4) then \( \mathcal{E}^{(1)}_{z}(x) = 0 \) unless \( x \) satisfies either \( (N - 1)\varepsilon \leq |x - z|_{\infty} \leq (N + 1)\varepsilon \) or \( |x_1 - z_1 - k \varepsilon - \frac{\zeta}{2}| \leq \delta \varepsilon \) for some \( k \in \mathbb{Z} \) with \( |k| \leq N + 1 \). Therefore \( \mathcal{E}^{(1)}_{z}(x) \) is nonzero on a set with measure at most \( 20N\varepsilon^2 + 20(N\varepsilon)^2 \delta \) and \( \|\mathcal{E}^{(1)}_{z}\|_{\infty} \leq 2(N\varepsilon)^{-1} \) by the support properties of \( \alpha_{\zeta}^{(1)} \), thus (6.6) follows from \( N \geq \delta^{-1} \) for \( \tau = 1 \). The case \( \tau = 2 \) is analogous.

If \( u(x) \) satisfies (2.5), then we still have \( \|\mathcal{E}^{(2)}_{z}\|_{\infty} \leq 2(N\varepsilon)^{-1} \) since in the \( k \)-summation in (6.4) at most \( 2\delta^{-1} \) terms overlap and \( \|\alpha_{\zeta}^{(2)}\|_{\infty} \leq \delta \varepsilon \). We can use this \( L^\infty \) bound in the regime \( (N - \delta^{-1})\varepsilon \leq |x - z|_{\infty} \leq (N + \delta^{-1})\varepsilon \),
which gives a contribution of at most $20\delta^{-1}N^{-1} \leq 20\delta$ to the integral in (6.6) as before. In the complementary regime we claim that

$$|\mathcal{E}_x^{(p)}(x)| \leq \delta(N\varepsilon)^{-1}\|\nabla u_0\|_{\infty} \quad \text{for} \quad |x-z|_{\infty} \leq (N-\delta^{-1})\varepsilon. \quad (6.11)$$

which would give a contribution of at most $\delta^2\|\nabla u_0\|_{\infty}^2$ to the integral in (6.6).

To see (6.11), we introduce a new variable $v = \delta^{-1}(x-z)$, with components $v = (v_1, v_2)$ and note that $|x-z|_{\infty} \leq (N-\delta^{-1})\varepsilon$ implies $|v|_{\infty} \leq N\delta - 1$. Using (2.5), (2.7), (5.2) and (6.4) and after changing variables we have

$$E^{(1)}_x(x) = \frac{1}{2N\varepsilon} \left[ 1_{Q_2}(x) - \sum_{k=-N}^{N} \int_{-\infty}^{\infty} \delta \left( u_0(v_1 - k\delta, s + N\delta) - u_0(v_1 - k\delta, s - N\delta) \right) ds \right] e_1$$

$$= \frac{1}{2N\varepsilon} \left[ 1 - \delta \sum_{k=-N}^{N} \int_{-1-N\delta}^{1-N\delta} u_0(v_1 - k\delta, s + N\delta) ds \right] e_1, \quad (6.12)$$

where we used that $u_0$ is supported on $[-1,1]$ and that $|v|_{\infty} \leq N\delta - 1$ implies $1 - N\delta \leq v_2 \leq N\delta - 1$ to restrict the regime of integration for the first term and to conclude that the second integrand is zero. Since $|v|_{\infty} \leq N\delta - 1$, we see that $|v_1 - k\delta| \geq 1$ if $|k| > N$, thus the summation can be extended to all $k \in \mathbb{Z}$ without changing the value of the right hand side, since $u_0$ is supported on $[-1,1]$. We use the fact that for any $f \in C^1(\mathbb{R})$ function with compact support

$$\left| \sum_{k \in \mathbb{Z}} \delta f(\delta k) - \int_\mathbb{R} f(t)dt \right| \leq \delta \|f'\|_{\infty}|\text{supp } f|$$

that can be easily obtained by Taylor expansion. Thus

$$|E_x^{(1)}(x)| \leq \frac{\delta}{N\varepsilon} \|\partial_1 u_0\|_{\infty}. \quad \text{The proof for } E^{(2)}_x \text{ is analogous and this completes the proof of Proposition 6.3.} \quad \Box.$$

7 Proof of the regularity lemmas

Since $j_{\psi}$ is gauge invariant, to prove Lemmas 6.1 and 6.2, we can work in an appropriate gauge $\tilde{A}$ for the deterministic part $\tilde{B}$ of the magnetic field. Since $\psi$ is supported in $\Lambda_L$, it is sufficient to construct $\tilde{A}$ on $\Lambda_L$.

**Proposition 7.1** Given a bounded magnetic field $\tilde{B}$ on $\Lambda_L$, there exists a vector potential $\tilde{A}$, $\nabla \times \tilde{A} = \tilde{B}$, that is divergence free, $\nabla \cdot \tilde{A} = 0$, and for any $1 < p < \infty$

$$\|\tilde{A}\|_p \leq C_pL\|\tilde{B}\|_{\infty} \quad (7.1)$$

with some constant $C_p$ depending only on $p$. Here $\| \cdot \|_p$ denotes the $L^p(\Lambda_L)$-norm.

**Proof.** Let $A^*$ be the Poincaré gauge for $\tilde{B}$, i.e.

$$A_{\ast}^1(x) = -\int_{0}^{1} t\tilde{B}(tx)x_2dt, \quad A_{\ast}^2(x) = \int_{0}^{1} t\tilde{B}(tx)x_1dt,$$

then clearly $\|A^*\|_p \leq L\|\tilde{B}\|_{\infty}$ and $\nabla \times A^* = \tilde{B}$. Define

$$w(x) := \frac{1}{2\pi} \int_{\Lambda_L} \log |x-y|A^*(y)dy,$$
i.e. \( \Delta w = A^* \), and \( \phi := \nabla \cdot w \). Note that \( \nabla \cdot \nabla \phi = \Delta \nabla \cdot w = \nabla \cdot \Delta w = \nabla A^* \). By the Calderon-Zygmund inequality (see Theorem 9.9 of [7]) we have
\[
\|\nabla \phi\|_p = \|\nabla (\nabla \cdot w)\|_p \leq C_p \|\Delta w\|_p = C_p \|A^*\|_p \leq C_p L \|\tilde{B}\|_\infty
\]
for any \( 1 < p < \infty \). We define \( \tilde{A} = A^* - \nabla \phi \), then \( \nabla \times \tilde{A} = \tilde{B} \), \( \nabla \cdot \tilde{A} = 0 \) and (7.1) holds. \( \square \)

**Proof of Lemma 6.1.** Since \( \psi \) is a Dirichlet eigenfunction, we have
\[
-\Delta \psi + 2iA \cdot \nabla \psi + i(\nabla \cdot A)\psi + (A^2 + V - E)\psi = 0 \tag{7.2}
\]
and
\[
\int |(\nabla - iA)\psi|^2 \leq \tilde{E} \|\psi\|^2,
\]
with \( \tilde{E} := E + \|V\|_\infty \). Since \( \psi \) is supported on \( \Lambda_L \), all integrals and norms in this proof will be in \( \Lambda_L \).

By the Gagliardo-Nirenberg inequality
\[
\|\psi\|_6 \leq C \|\nabla \psi\|_3/2 \leq C \|\nabla \nabla \psi\|_3/2 + C \|A\psi\|_3/2 \\
\leq C L^{1/3} \|\nabla - iA\psi\|_2 + \|A\|_6 \|\psi\|_2
\]
\( \tag{7.3} \)

and similarly
\[
\|\psi\|_4 \leq C \|\nabla \nabla \psi\|_3/4 \leq C (L^{1/2} E^{1/2} + \|A\|_4) \|\psi\|_2.
\]
\( \tag{7.4} \)

Then
\[
\|\nabla \psi\|_2 \leq \|(\nabla - iA)\psi\|_2 + \|A\psi\|_2 \leq \tilde{E}^{1/2} \|\psi\|_2 + \|A\|_4 \|\psi\|_4 \leq C (E^{1/2} + L^{1/2} E^{1/2} \|A\|_4 + \|A\|_4^2) \|\psi\|_2.
\]
\( \tag{7.5} \)

We use (7.2) and Calderon-Zygmund inequality in the form given in Corollary 9.10 of [7] for \( \psi \in W_0^{2, p}(\Lambda_L) \)
\[
\|D^2 \psi\|_4 \leq C \|\Delta \psi\|_4 \leq C (\|2A\nabla \psi\|_4 + \|\nabla \cdot A\|_\infty \|\psi\|_4 + \|(A^2 + V - E)\psi\|_4) \\
\leq C \left( 2 \|A\|_{20} \|\nabla \psi\|_5 + \|\nabla \cdot A\|_\infty + \tilde{E} \right) \|\psi\|_4 + \|A\|_{24} \|\psi\|_6.
\]
\( \tag{7.6} \)

We can estimate
\[
\|\nabla \psi\|_5 \leq C \|\nabla \psi\|^{9/10}_6 ||\nabla \psi||^{1/10}_2 \leq C \kappa \|\nabla \psi\|_6 + C \kappa^{-9} \|\nabla \psi\|_2
\]
\( \leq C \kappa \|D^2 \psi\|_3/2 + C \kappa^{-9} ||\nabla \psi||_2 \)
\( \tag{7.7} \)
\]

for any \( \kappa > 0 \), where we used the Gagliardo-Nirenberg inequality once more. Choosing \( \kappa = (4 CL^{5/6} \|A\|_{20})^{-1} \)
we can absorb the first term in the right hand side of (7.6) into the left term and we obtain
\[
\|D^2 \psi\|_4 \leq C \left( L^{15/2} \|A\|_{20}^9 \|\nabla \psi\|_2 + \|\nabla \cdot A\|_\infty + \tilde{E} \right) \|\psi\|_4 + \|A\|_{24} \|\psi\|_6.
\]

The vector potential is given by (5.4), \( A = A + \sum_{z \in \Lambda} \omega_z \alpha_z \), with \( |\omega_z| \leq \sigma \), \( \|\alpha_z\|_\infty \leq \varepsilon \), so \( \|A\| \leq |A| + C \sigma \)

since among all \( \alpha_z \) at most \( C \varepsilon^{-1} \) of them overlap. Thus, from (7.1) we have
\[
\|A\|_p \leq C_p L K_0 b_0 \quad 1 < p < \infty.
\]
\( \tag{7.8} \)

Moreover, \( |\nabla \cdot A| \leq C \sigma / \varepsilon \) since \( \nabla \cdot \tilde{A} = 0 \) and \( |\nabla \alpha_z| \leq C \). Using these estimates together with (7.3), (7.4) and (7.5), we have proved
\[
\|D^2 \psi\|_4 \leq C L^{39/2} b_0^9 \|\psi\|_2
\]
\( \tag{7.9} \)
with $C = C(K_0, K_1)$. By Hölder inequality we also get

$$
\|D^2 \psi\|_2 \leq C L^{20} b_0^{10} \|\psi\|_2.
$$  \tag{7.10}

Going back to the estimate on $\|\nabla \psi\|_6$ used in (7.7), we also have

$$
\|\nabla \psi\|_6 \leq C L^{5/6} \|D^2 \psi\|_4 \leq C L^{21} b_0^{10} \|\psi\|_2
$$  \tag{7.11}

using $L \geq 1$, and we have

$$
\|\nabla \psi\|_4 \leq L^{1/6} \|\nabla \psi\|_6.
$$  \tag{7.12}

Moreover, from (7.3) and (7.4)

$$
\|\psi\|_6, \|\psi\|_4 \leq C L \|\psi\|_2.
$$  \tag{7.13}

From (7.9) and Sobolev inequality applied to $\nabla \psi$, we have

$$
\|\nabla \psi\|_\infty \leq \|D^2 \psi\|_4 + \|\nabla \psi\|_4 < \infty.
$$

Then$j = 2 \text{Re} \left[ -i \bar{\psi} \nabla \psi - A |\psi|^2 \right]$ vanishes at the boundary, since $\nabla \psi$ is bounded and $\psi$ vanishes at the boundary. To prove (6.2), we use that

$$
\int |\nabla j|^2 = \int |\nabla \times j|^2
$$

since $\nabla \cdot j = 0$ and $j$ vanishes on the boundary. Now we compute

$$
|\nabla \times j| \leq 2|\nabla \times [\bar{\psi} (p-A) \psi]| \leq C \left( |\nabla \psi|^2 + \|D^2 \psi\|_4 |\psi| + |A| \|\psi\| \|\nabla \psi\| + |B| \|\psi\|^2 \right).
$$

Thus, using the estimates (7.8), (7.11), (7.12) and (7.13)

$$
\int |\nabla j|^2 \leq C \left( \|\nabla \psi\|_4^4 + \|D^2 \psi\|_4^4 + \|\psi\|_4^4 + \|\nabla \psi\|_6^6 + \|\nabla \psi\|_8^8 + \|A\|_6^6 + \|B\|_\infty^2 \|\psi\|_4^4 \right) \leq C L^{126} b_0^{60} \|\psi\|_2^4.
$$

This bound proves Lemma 6.1. \qed

**Proof of Lemma 6.2.** We first we need a lower bound on the eigenfunction. Let $\psi$ be a normalized Dirichlet eigenfunction of $H_L(A)$, i.e.

$$
\left[ (-i \nabla - A)^2 + V - E \right] \psi = 0.
$$

Let $x_0$ be the point where $|\psi(x)|$ reaches its maximum. Since

$$
1 = \int_{\Lambda_L} |\psi|^2 \leq L^2 |\psi(x_0)|^2
$$

we have

$$
|\psi(x_0)| \geq \frac{1}{L}.
$$  \tag{7.14}

In particular, $x_0 \in \text{int}(\Lambda_L)$. Now we consider a disk $\tilde{D}$ of radius $\ell$ about $x_0$, where $\ell > 0$ is sufficiently small so that $\tilde{D} \subset \Lambda_L$. Let

$$
\langle \psi \rangle := \frac{1}{|\tilde{D}|} \int_{\tilde{D}} \psi, \quad \langle \nabla \psi \rangle := \frac{1}{|\tilde{D}|} \int_{\tilde{D}} \nabla \psi.
$$

Notice that from (7.5) and (7.8)

$$
\langle \nabla \psi \rangle^2 \leq \frac{1}{|\tilde{D}|} \int_{\tilde{D}} |\nabla \psi|^2 \leq \ell^{-2} \|\nabla \psi\|_{L^2(\tilde{D})}^2 \leq C L^4 b_0^4 \ell^{-2}.
$$  \tag{7.15}
For $x \in \tilde{D}$ define
\[ f(x) := \psi(x) - \langle \psi \rangle - \langle \nabla \psi \rangle \cdot (x - x_0). \]
Then
\[ \int_{\tilde{D}} |\nabla f|^2 = \int_{\tilde{D}} |\nabla \psi - \langle \nabla \psi \rangle|^2 \leq C\ell^2 \int_{\tilde{D}} |D^2 \psi|^2, \]
by Poincare inequality in $\tilde{D}$. Thus, applying Sobolev inequality for $f$, we have
\[ \|f\|_{\infty} \leq C \left( \|D^2 f\|_{L^2(\tilde{D})} + \epsilon^{-1} \|f\|_{L^2(\tilde{D})} \right) \leq C \ell \|D^2 f\|_{L^2(\tilde{D})}, \]
where in the last step we used (7.18) and Lemma 7.2
Thus
\[ \|f\|_{L^2(\tilde{D})} \leq C\ell \|\nabla f\|_{L^2(\tilde{D})} \leq C \ell^2 \|D^2 f\|_{L^2(\tilde{D})}, \]
by (7.10) and so, by (7.15),
\[ |\psi(x) - \langle \psi \rangle| \leq C\ell L^{20} b_0^{10} + |x - x_0| \|\nabla \psi\| \leq C\ell L^{20} b_0^{10} + C \ell^2 b_0^{12} |x - x_0| \leq C L^{11} b_0^{10} |x - x_0|^{1/2} \]
(7.16)
after choosing $\ell = L^{-9} b_0^{-4} |x - x_0|^{1/2}$. From (7.14) this guarantees that there is a disk $D = D_R$ about $x_0$ of radius $R = cL^{-24} b_0^{-12}$ so that $|\psi(x)| \geq \frac{1}{2L}|\psi(x_0)|$ for $x \in D$, i.e.
\[ |\psi(x)| \geq \frac{1}{2L}, \quad x \in D. \]
(7.17)

Now we give a lower bound on the current. On $D$ the wave function $\psi$ does not vanish, so we can write it as $\psi = |\psi|e^{i\theta}$ with some real phase function $\theta$. Then
\[ \text{Re} \ \bar{\psi}(p - A) \psi = \text{Re} \ |\psi| e^{-i\theta} (-i \nabla - (A - \nabla \theta)) |\psi| = \text{Re} \ |\psi| (-i \nabla - (A - \nabla \theta)) |\psi| = -(A - \nabla \theta) |\psi|^2. \]
Thus
\[ \int_D |j|^2 = 4 \int_D (A - \nabla \theta)^2 |\psi|^4 \geq cL^{-4} \int_D (A - \nabla \theta)^2, \]
(7.18)
where we used (7.17). Finally, we will need the following elementary lemma:

**Lemma 7.2** Let $D = D_R$ be a disk of radius $R$ and let $A$ be a vector potential generating $B$ with a lower bound $B(x) \geq b_0$. Then
\[ \int_{D_R} A^2 \geq \frac{\pi}{8} b_0^2 R^4. \]

**Proof.** Let $S_r$ be the circle of radius $r$ with the same center as $D$.
\[ \int_D A^2 = \int_0^R \left( \int_{S_r} A^2 \right) dr \geq \int_0^R \frac{1}{2\pi r} \left( \int_{S_r} A d\theta \right)^2 dr \]
\[ = \int_0^R \frac{1}{2\pi r} \left( \int_{D_r} B \right)^2 dr \geq \int_0^R \frac{1}{2\pi r} \left( \pi r^2 b_0 \right)^2 dr = \frac{\pi}{8} b_0^2 R^4. \]
\[ \square \]

Lemma 6.2 now follows from (7.18) and Lemma 7.2
\[ \int_D |j|^2 \geq cL^{-4} \int_D (A - \nabla \theta)^2 \geq cL^{-4} b_0^2 |D|^2 = c b_0^{-46} L^{-100}. \]
\[ \square \]
8 Deterministic Spectrum

The goal of this section is to prove Theorem 3.4. For \( l > 0 \) and \( x \in \mathbb{R}^2 \) we denote by

\[
\Lambda_l(x) := \{ y \in \mathbb{R}^2 : |y - x|_\infty < l/2 \}
\]

the open square of sidelength \( l \) centered at \( x \). We introduce the constant

\[
c_\delta = \begin{cases} 
\frac{3}{2}, & \text{in case (2.4)} \\
\delta^{-1}, & \text{in case (2.5)} 
\end{cases}
\]  

which gives the distance beyond which the random magnetic field is independent. From Theorem 3.2 recall that \( \Sigma \) denotes the almost surely deterministic spectrum.

**Theorem 8.1** Let (R) and (i.i.d.) hold with \( \rho > \ln 2 \). Assume that \( B_{\text{det}} \) and \( V \) are \( \mathbb{Z}^2 \) periodic. Then

\[
\Sigma \supset \bigcup_{L \in \mathbb{N}} \bigcup_{\omega \in \mathcal{V}_L} \sigma(H(B_\omega)),
\]

where \( \mathcal{V}_L = \{ \omega \in \Omega : \forall k \in \mathbb{N}, \omega_z^{(k)} \in [m_-^{(k)}, m_+^{(k)}], \omega_z^{(k)} = \omega_z^{(k)}, \forall n \in \mathbb{Z}^2 \} \) is the set of \( L \)-periodic configurations.

**Proof.** By unitary equivalence, we can fix a gauge. Given a magnetic \( B \)-field, for any \( y \in \mathbb{R}^2 \) we define the vector potential

\[
A_y[B](x) := \left( 0, \int_0^{x_1} B(x'_1, x_2)dx'_1 \right).
\]

Fix \( \omega_0 \in \mathcal{V}_L \) for some \( L \in \mathbb{N} \). Let \( E \in \sigma(H(B_{\omega_0})) \). Since the magnetic Hamiltonian is essentially self-adjoint on \( C_0^\infty \) functions, it follows that there exists a normalized sequence \( \varphi_n \in C_0^\infty \) such that

\[
\|(H(A_0[B_{\omega_0}]) - E)\varphi_n\| \to 0, \quad (n \to \infty).
\]  

(8.2)

Let \( l_n \in 2\mathbb{N} + 1 \) be such that \( \text{supp}(\varphi_n) \subset \Lambda_{l_n}(0) \) and \( l_n \geq n \). For \( x \in \mathbb{Z}^2 \), we introduce the following random variables

\[
B_{x,l_n,\omega_0}(\omega) := \|(B_{\text{ran}}^x - B_{\text{ran}}^{\omega_0}) \downarrow \Lambda_l(x)\|_\infty,
\]

\[
B_{x,l_n,\omega_0}'(\omega) := \sum_{i=1,2} \| \partial_{x_i} (B_{\text{ran}}^x - B_{\text{ran}}^{\omega_0}) \downarrow \Lambda_l(x)\|_\infty,
\]

and we define the set

\[
\Omega_n(x) = \Omega_n,\omega_0(x) := \{ \omega \in \Omega | B_{x,l_n,\omega_0}(\omega) \leq l_n^{-3}, B_{x,l_n,\omega_0}'(\omega) \leq l_n^{-3} \}.
\]  

(8.3)

Using the properties of the random potential, it is straightforward to verify that \( P(\Omega_n(x)) \) is independent of \( x \) and strictly positive. Moreover, if \( \text{dist}(\Lambda_{l_n}(x), \Lambda_{l_n}(y)) \geq 2c_\delta \) then \( \Omega_n(x) \) and \( \Omega_n(y) \) are independent. It now follows that for a.e. \( \omega \in \Omega \) there exists an \( x_n = x_n(\omega,\omega_0) \in L\mathbb{Z}^2 \) such that \( \omega \in \Omega_n,\omega_0(x_n) \). We set

\[
\overline{A} = \overline{A}_n,\omega_0 = A_{x_n}[B_{\omega}], A_{x_n}[B_{\omega}] = A_{x_n}[\mu B_{\text{ran}} - \mu B_{\text{ran}}^{\omega_0}],
\]

Then setting \( \varphi_n^x(\cdot) = \varphi_n(\cdot - x_n) \), we have

\[
\|(H(A_{x_n}[B_{\omega}]) - E)\varphi_n^x\| \leq \|(H(A_{x_n}[B_{\omega}]) - E)\varphi_n\| + 2R_1 + R_2 + R_3,
\]  

(8.4)
with

\[ R_1 := \|\bar{A} \cdot (p - A_{x_n} [B_{\omega_0}]) \varphi_n^x\|, \]
\[ R_2 := \|\bar{A}^2 \varphi_n^x\|, \]
\[ R_3 := \|((\nabla \cdot \bar{A}) \varphi_n^x)\|. \]

Let \( \chi_n \) denote the characteristic function of \( \Lambda_n(x_n) \). We estimate

\[ R_1 \leq \|\bar{A} \chi_n\|_\infty \|p - A_0[B_{\omega_0}]\| \varphi_n \leq C l^{-1}_n (E + 1)^{1/2}, \]

where in the first inequality we used the \( L \)-periodicity of \( B_{\omega_0} \) and in the second inequality we used (8.2) and the definition of \( \bar{A} \). Using again the definition of \( \bar{A} \), we similarly find \( R_2 \leq C l^{-1}_n \) and \( R_3 \leq C l^{-2}_n \). Using a gauge transformation such that

\[ H(\Lambda_{x_n} [B_{\omega_0}]) = e^{i\lambda_n \omega \omega_0} H(\Lambda_0[B_{\omega_0}]) e^{-i\lambda_n \omega \omega_0} \]

it now follows from (8.4) and (8.2) that

\[ \lim_{n \to \infty} \| (H(\Lambda_0[B_{\omega}]) - E) e^{-i\lambda_n \omega \omega_0} \varphi_n \| = 0. \]

This yields the theorem. \( \square \)

**Lemma 8.2** Let \( \Lambda \) be a square or \( \mathbb{R}^2 \) and \( B = \nabla \times A \). As an inequality in the sense of forms in \( L^2(\Lambda) \)

\[ H_\Lambda(A) \geq \pm B + V. \]

**Proof.** Let \( \sigma_i \) denote the \( i \)-th Pauli matrix. Then for \( \varphi \in C_0^\infty (\Lambda; \mathbb{C}^2) \) we have

\[ \langle \varphi, (p - A)^2 \varphi \rangle = \langle \varphi, \left( \sum_{i=1}^2 (p_i - A_i) \sigma_i \right)^2 - \sigma_3 B \varphi \rangle \geq \langle \varphi, -\sigma_3 B \varphi \rangle. \]

The lemma now follows by density argument. \( \square \)

**Proof of Theorem 3.4.** (d). First observe that \( \Sigma_{\text{inf}} \geq B_{\text{det}} + \mu M_- \), by Lemma 8.2. From Theorem 8.1 and the fact that a magnetic Hamiltonian with a constant magnetic field is explicitly solvable, we find that

\[ \sigma(H(\Lambda)) \supset \bigcup_{n \in \mathbb{N}_0} \left\{ (1 + 2n)(B_{\text{det}} + \mu I_v) \right\}, \]

where \( I_v = [M_-, M_+] \). (d) now follows.

(b). This follows directly from the definition of \( E_{\text{inf}} \) and \( E_{\text{sup}} \).

(a). First observe that \( \Sigma_{\text{inf}} \geq E_{\text{inf}} \) follows from Lemma 8.2 and the definition of \( E_{\text{inf}} \). Next we show that

\[ E_{\text{inf}} + 4 K_2 b_0^{-2} + \min(K_2 b_0^{-1/2}, K_3 b_0^{-1}) \geq \Sigma_{\text{inf}} \] (8.5)

using a trial state. By continuity and periodicity of \( V + B_{\omega_-} \) we have

\[ E_{\text{inf}} = (V + B_{\omega_-})(\bar{x}), \]

for some \( \bar{x} \in \mathbb{R}^2 \). We choose the gauge

\[ A_{\text{inf}}(x) = \frac{1}{2} \left( - \int_{\bar{x}_2}^{x_2} B_{\omega_-}(x_1, y_2) dy_2, \int_{\bar{x}_1}^{x_1} B_{\omega_-}(y_1, x_2) dy_1 \right) \]

and we set \( A_0(x) := \frac{1}{2} B_{\text{inf}}^{(0)}(-(x_2 - \bar{x}_2), x_1 - \bar{x}_1) \) with \( B_{\text{inf}}^{(0)} := B_{\omega_-}(\bar{x}) \). Let us consider the trial state

\[ \varphi_0(x) = \exp(-\frac{1}{4} B_{\text{inf}}^{(0)} |x - \bar{x}|^2), \] (8.6)
which satisfies \((p - A_0) + i(p - A_0)\varphi_0 = 0\). Using a straightforward calculation we find
\[
\langle \varphi_0, H(A_{\inf})\varphi_0 \rangle = \| (p - A_{\inf}) + i(p - A_{\inf})\varphi_0 \|^2 + \langle \varphi_0, (B_{\omega} + V)\varphi_0 \rangle
\]
\[
= E_{\inf}\|\varphi_0\|^2 + \| (A_0 - A_{\inf}) + i(A_0 - A_{\inf})\varphi_0 \|^2 + \langle \varphi_0, (B_{\omega} + V - E_{\inf})\varphi_0 \rangle.
\]
By Taylor expansion with remainder it is straightforward to see that
\[
(A_0(x) - A_{\inf}(x))^2 \leq \frac{1}{4} \max_{|\alpha| = 1} \| D^\alpha B_{\omega} \|_\infty \left\{ 4|x_1 - \bar{x}_1||x_2 - \bar{x}_2| + \frac{1}{2}|x_1 - \bar{x}_1|^2 + \frac{1}{2}|x_2 - \bar{x}_2|^2 \right\}.
\]
Using this estimate and evaluating a Gaussian integral we find,
\[
\| (A_0 - A_{\inf}) + i(A_0 - A_{\inf})\varphi_0 \|^2 \leq \frac{1}{b_0^2} \|\varphi_0\|^2,
\]
where we used that \(0 < b_0 \leq B_{\inf}^{(0)}\), which follows from (2.3). Using a Taylor expansion up to the first respectively second order and that \(V + B_{\omega} \) attains in \(\bar{x}\) its minimum \(E_{\inf}\) we find, similarly,
\[
\langle \varphi_0, (B_{\omega} + V - E_{\inf})\varphi_0 \rangle \leq \min \left\{ \max_{|\alpha| = 2} \| D^\alpha (B_{\omega} + V) \|_\infty \frac{2}{b_0}, \max_{|\alpha| = 1} \| D^\alpha (B_{\omega} + V) \|_\infty \frac{2}{\pi b_0} \right\} \|\varphi_0\|^2.
\]
Now inserting the above estimates into the right hand side of (8.7) and using Theorem 8.1 and the estimate
\[
\| D^\alpha B_{\omega}\|_\infty \leq \sum_{k=0}^{\infty} 2^{\frac{1}{2}k} |m^{(k)}| \| D^\alpha U \|_\infty \leq \sum_{k=0}^{\infty} 2^{\frac{1}{2}k} e^{-\rho k} \| D^\alpha U \|_\infty,
\]
we obtain (8.5). Thus we have shown (a).
(c) Now we estimate the interior of the spectrum. Let \(\varepsilon > 0\). Then by Theorem 8.1 there exists an \(\omega^\varepsilon\) in the support of the probability measure and a normalized \(\varphi \in C_0^\infty\) such that
\[
\langle \varphi, H(A_{\omega^\varepsilon})\varphi \rangle \leq \Sigma_{\inf} + \varepsilon.
\]
Choose \(L_0'\) such that \(\text{supp}\varphi \subset A_{L_0'}\). Now choose \(L_0 \geq L_0' + c_\varepsilon \in \mathbb{N}\). To show (3.7) we consider the path
\[
(\omega_{s'})_{z'} = (\omega^\varepsilon')_{z'} + s(m^{'(k)} - (\omega^\varepsilon')_{z'}), \quad 0 \leq s \leq 1,
\]
where \(z = nL_0 + z' \in A_{L_0}\) and \(n \in \mathbb{Z}\). Note that the configuration \(\omega_s\) is \(L_0\)-periodic. We have
\[
\inf \sigma(H(B_{\omega_0})) \leq \Sigma_{\inf} + \varepsilon, \quad E_{\sup} \leq \inf \sigma(H(B_{\omega_1})),
\]
where the first inequality follows from (8.8) and (8.9), and the second inequality follows from Lemma 8.2. By perturbation theory it is known that for any \(L > 0\), \(\inf \sigma(H_L(B_{\omega_s}))\) is a continuous function of \(s\). In Lemma 8.3 below we will show the limit of \(\inf \sigma(H_{nL_0}(B_{\omega_s}))\) as \(n \to \infty\) converges to \(\inf \sigma(H(B_{\omega_s}))\) uniformly in \(s\). Thus \(s \to \inf \sigma(H(B_{\omega_s}))\) is a continuous function of \(s \in [0, 1]\). In view of Theorem 8.1 this continuity property and (8.10) imply the inclusion (3.7), since \(\varepsilon > 0\) is arbitrary.

\(\square\)

**Lemma 8.3** Suppose the assumptions of Theorem 3.4 hold and suppose \(\omega_s \in \Omega\) is as defined in (8.9). Then there exists a universal constant \(C\) such that
\[
|\inf \sigma(H_L(B_{\omega_s})) - \inf \sigma(H(B_{\omega_s}))| \leq \frac{C}{L^2},
\]
for all \(s \in [0, 1]\) and \(L = nL_0\) with \(n \in \mathbb{N}\).
Proof. Set \( B = B_{\omega} \) and \( E_L(B) := \inf \sigma(H_L(B)) \). For notational simplicity we drop the \( \omega \) dependence, the estimate will be uniform in \( \omega \). By \( L_0\mathbb{Z}^2 \)-periodicity of the \( B \) field, we have for any \( n \in \mathbb{N} \),

\[
\inf \sigma(H(B)) \leq E_L(B). \tag{8.11}
\]

To find a lower bound we use the I.M.S. localization formula,

\[
H(A) = \sum_{z \in L\mathbb{Z}^2} J_z H(A) J_z - \sum_{z \in L\mathbb{Z}^2} |\nabla J_z|^2,
\]

where we introduced a partition of unity \( J_z = \varphi((z - z)/L) \), with \( \varphi \in C_0^\infty(\mathbb{R}^2; [0, 1]) \), supp\( \varphi \in [-1, 1]^2 \), \( \sum_{z \in \mathbb{Z}} \varphi^2(x - z) = 1 \), and \( C_\varphi := \| \sum_{z \in \mathbb{Z}} (\nabla \varphi)^2(x - z) \|_\infty < \infty \). By the \( L_0\mathbb{Z}^2 \)-periodicity of the \( B \) field, we find for any vector potential \( A \) with \( \nabla \times A = B \) and any normalized \( \psi \in C_0^\infty \),

\[
\langle \psi, H(A) \psi \rangle = \sum_{z \in L\mathbb{Z}^2} \langle \psi, J_z H(A) J_z \psi \rangle - \sum_{z \in L\mathbb{Z}^2} \langle \psi, |\nabla J_z|^2 \psi \rangle \geq E_{2L}(B) \sum_{z \in L\mathbb{Z}^2} \|J_z \psi\|^2 - \frac{C_\varphi}{L^2}.
\]

This implies

\[
\inf \sigma(H(B)) \geq E_{2L}(B) - \frac{C_\varphi}{L^2},
\]

which, together with (8.11), yields the lemma.

\[\square\]

9 Initial length scale estimates

In this section we show an initial length scale estimate. We define \( \tilde{\Lambda} := \Lambda + [-c_\delta, c_\delta]^2 \), with \( c_\delta \) as defined in (8.1).

Theorem 9.1 Assume that (A) holds and recall the definition of \( \nu(\cdot) \) from (3.2). Then for \( h > 0 \)

\[
P\{ \text{dist}(\inf \sigma(HA)), E_{inf} \geq \mu h \} \geq 1 - |\tilde{\Lambda}| \nu(c_u^{-1} h).
\]

Proof. By Lemma 8.2, \( E_{inf} \) is a lower bound of the infimum of the spectrum, thus

\[
\text{l.h.s. of (9.1)} \geq P\{ B_{det}(x) + \mu B_{ran}(x) + V(x) \geq \mu h + E_{inf}, \ \forall x \in \Lambda \} \\
\geq \left[ P\{ \omega_0^{(0)} \geq m_\omega^{(0)} + c_u^{-1} h \} \right]^{|\Lambda|} \\
\geq 1 - |\tilde{\Lambda}| \nu(c_u^{-1} h). \tag{9.2}
\]

The second line follows, since \( \omega_z^{(0)} \geq m_\omega^{(0)} + c_u^{-1} h \) for all \( z \in \tilde{\Lambda} \) implies that for all \( x \in \Lambda \)

\[
B_{det}(x) + \mu B_{ran}(x) + V(x) \geq B_{det}(x) + V(x) + \mu \sum_{k=0}^{\infty} \sum_{z \in \Lambda^{(k)}} \omega_z^{(k)} u(x - z) \\
\geq B_{det}(x) + V(x) + \mu \sum_{z \in \Lambda^{(0)}} c_u^{-1} h u(x - z) + \mu \sum_{k=0}^{\infty} \sum_{z \in \Lambda^{(k)}} m_\omega^{(k)} u(x - z) \\
\geq B_{det}(x) + V(x) + \mu h + \mu \sum_{k=0}^{\infty} \sum_{z \in \Lambda^{(k)}} m_\omega^{(k)} u(x - z) \\
\geq E_{inf} + h,
\]

where we used the notation \( \Lambda^{(k)} = \Lambda^{(k)} \cap \tilde{\Lambda} \). Now (9.2) follows from the binomial formula.

\[\square\]
Corollary 9.2 Assume that \((A_\tau)\) holds for some fixed \(\tau > 2\) and \(c_\omega\). For any \(\xi \in (0, \tau - 2)\) set \(\beta := \frac{1}{2}(1 - \frac{\xi + 2}{\tau}) \in (0, 1)\), then there is an \(l_{\text{initial}} = l_{\text{initial}}(\tau, \xi, c_\omega, c_\delta)\) such that

\[
P\left\{ \text{dist}(\inf \sigma(H_\Lambda), E_{\text{inf}}) \geq \mu l^{\beta - 1} \right\} \geq 1 - l^{-\xi},
\]

for any \(\Lambda = \Lambda_t(x)\), with \(x \in \mathbb{Z}^2\) and \(l \geq l_{\text{initial}}\).

**Proof.** Set \(h = l^{\beta - 1}\) in Theorem 9.1. Then

\[
|\bar{A}|v(c_u^{-1} h) \leq |\bar{A}|v(c_u^{-1} h)^T = c_u^{-1} c_v(l + c_\delta)^2 l^{(\beta - 1)\tau} \leq l^{-\xi},
\]

where the first inequality follows from assumption \((A_\tau)\), and the second inequality holds for large \(l\). \(\square\)

## 10 Multiscale analysis

The goal of this section is to prove Theorem 3.3. We will essentially follow the setup presented in [17] and indicate the necessary modifications for magnetic fields. Alternatively, one could follow the setup of [2] and verify their key hypothesis \([H1](\gamma_0, l_0)\).

We assume \((A_\tau)\) throughout this section for some fixed \(\tau > 2\) and \(c_\omega\). The constants \(b_0, \rho, \delta\) are as in the assumptions of Theorem 3.1. We write

\[
R_\Lambda(z) = R_\Lambda(A, z) = (H_\Lambda(A) - z)^{-1} = (H_\Lambda(A) - z^{-1}).
\]

For notational simplicity we will occasionally drop the \(A\) and \(z\), and mostly the \(\omega\) dependence. Boxes with sidelength \(l \in 2\mathbb{N} + 1\) and center \(x \in \mathbb{Z}^2\) are called *suitable*. For a suitable square \(\Lambda = \Lambda_t(x)\), we set

\[
\Lambda_{\text{int}} := \Lambda_{l/3}(x), \quad \Lambda_{\text{out}} := \Lambda_t(x) \setminus \Lambda_{l-2}(x),
\]

and we set \(\chi_{\text{int}} = \chi_{\Lambda_{\text{int}}}\) and \(\chi_{\text{out}} = \chi_{\Lambda_{\text{out}}}\). For \(A\) an operator in a Hilbert space we will denote by \(\rho(A)\) the resolvent set of \(A\).

**Definition 10.1** A square \(\Lambda\) is called \((\gamma, \Lambda)\)-good for \(\omega \in \Omega\) if

\[
\|\chi_{\text{out}}R_\Lambda(B_\omega, E)\chi_{\text{int}}\| \leq \exp(-\gamma l),
\]

where \(E \in \rho(H_\Lambda(B_\omega))\).

Let us introduce the multiscale induction hypotheses. Below we denote by \(I \subset \mathbb{R}\) an interval and assume \(l \in 2\mathbb{N} + 1\). First, for \(\gamma > 0\), and \(\xi > 0\) we introduce the following hypothesis.

\(G(I, l, \gamma, \xi)\): \(\forall x, y \in \mathbb{Z}^2\), \(|x - y|_\infty \geq l + c_\delta\), the following estimate holds:

\[
P\{\forall E \in I \mid \Lambda_t(x) \text{ or } \Lambda_t(y) \text{ is } (\gamma, E)\text{-good for } \omega \} \geq 1 - l^{-2\xi}.
\]

Note that this definition includes a security distance \(c_\delta\), to ensure the independence of squares.

**Lemma 10.1** For any \(\xi \in (0, \tau - 2)\) there is an \(l_G = l_G(\tau, \xi, c_\omega, c_\delta)\) such that for all \(l \geq l_G\), \(G(I, l, \gamma, \xi)\) holds with \(\gamma = l^{\beta - 1}\), \(I = E_{\text{inf}} + [0, \frac{1}{2}\mu l^{\beta - 1}]\), and \(\beta = \frac{1}{2}(1 - \frac{\xi + 2}{\tau}) \in (0, 1)\).

**Proof.** Consider \(\omega\) such that

\[
\text{dist}(\inf \sigma(H_\Lambda(\omega)), E_{\text{inf}}) \geq \mu l^{\beta - 1}. \tag{10.1}
\]

If \(E \in I\), then \(\text{dist}(H_\Lambda(\omega), E) \geq \frac{\mu}{2} l^{\beta - 1}\). Thus by the resolvent decay estimate, see Theorem C.2, we find

\[
\|\chi_{\text{int}}(H_\Lambda(\omega) - E)^{-1}\chi_{\text{out}}\| \leq \frac{4}{\mu} l^{1-\beta} \exp(-\mu l^{\beta - 1}/4) \|1/4\),
\]

\[
\text{dist}(\inf \sigma(H_\Lambda(\omega)), E_{\text{inf}}) \geq \mu l^{\beta - 1}. \tag{10.1}
\]

If \(E \in I\), then \(\text{dist}(H_\Lambda(\omega), E) \geq \frac{\mu}{2} l^{\beta - 1}\). Thus by the resolvent decay estimate, see Theorem C.2, we find

\[
\|\chi_{\text{int}}(H_\Lambda(\omega) - E)^{-1}\chi_{\text{out}}\| \leq \frac{4}{\mu} l^{1-\beta} \exp(-\mu l^{\beta - 1}/4) \|1/4\),
\]

\[
\text{dist}(\inf \sigma(H_\Lambda(\omega)), E_{\text{inf}}) \geq \mu l^{\beta - 1}. \tag{10.1}
\]

If \(E \in I\), then \(\text{dist}(H_\Lambda(\omega), E) \geq \frac{\mu}{2} l^{\beta - 1}\). Thus by the resolvent decay estimate, see Theorem C.2, we find

\[
\|\chi_{\text{int}}(H_\Lambda(\omega) - E)^{-1}\chi_{\text{out}}\| \leq \frac{4}{\mu} l^{1-\beta} \exp(-\mu l^{\beta - 1}/4) \|1/4\),
\]
for $l \geq 4$. Since by Corollary 9.2 the bound (10.1) holds with probability greater than $1 - l^{-\xi}$ for any large $l \geq l_{\text{initial}}$, it follows that for sufficiently large $l$, $G(I, l, \gamma, \xi)$ is valid for $\gamma = l^{\beta - 1}$.

For $\Theta > 0$, and $q > 0$ we introduce the following hypothesis.

$W(I, l, \Theta, q)$: For all $E \in I$ and $\Lambda = \Lambda(t)$, $x \in \mathbb{Z}^2$, the following estimate holds:

$$P\{\sigma(H_\Lambda(\omega)), E \leq \exp(-l^{\Theta})\} \leq l^{-q}.$$  

**Lemma 10.2** Suppose the assumptions of Theorem 3.1 hold. Let $\Theta > 0$, $q > 0$, and $0 < \kappa \leq 1$. Let $I \subset \mathbb{R}$ be a finite interval with $\inf I \geq b_0/2$. Then there exists a constant $L_0 = L_0^\kappa(I, \Theta, q, K_0, K_1, \delta, \mu, \kappa, \rho)$ such that $W(I, l, \Theta, q)$ holds for all $l \geq L_0^\kappa b_0^\rho$.

**Proof.** Let $0 \leq \eta \leq 1$, and $\Lambda = \Lambda(t)$. Then using Markov inequality and Theorem 3.1 we have

$$P\{\sigma(H_\Lambda(A)), E \leq \eta/2\} = P(\text{Tr} \chi_{\eta}(H_\Lambda(A)) \geq 1) \leq E(\text{Tr} \chi_{\eta}(H_\Lambda(A))) \leq C_0 \eta \mu^{-2} c_1(\kappa^{-1} + \rho),$$

for some constants $C_0$ and $C_1$, and $l$ sufficiently large. In fact, by Theorem 3.1 there exists an $L_0^\kappa$ such that (10.2) holds for all $l \geq L_0^\kappa b_0^\rho$. Now we choose $\eta = 2 \exp(-l^\Theta)$. Then by possibly choosing $L_0^\kappa$ larger the right hand side of (10.2) is bounded by $l^{-q}$ for all $l \geq L_0^\kappa b_0^\rho$.

Thus we have shown that under certain conditions the induction hypothesis of the multiscale analysis can be verified. The following three technical lemmas will be needed for the multiscale analysis. The have been verified for nonmagnetic random Schrödinger operators, see [17]. Here we prove that they also hold for magnetic Schrödinger operators.

**Lemma 10.3** (INDY) $H_\Lambda(A, \omega)$ is measurable with respect to $\omega \in \Omega$, the Hamiltonian $H_{\Lambda, \omega}(x), A, \omega)$ is stationary in $x \in \mathbb{Z}^2$ in the sense of (A.5), and $|R_{\Lambda}(A, \omega, z)(x, y)|$ for $x, y \in \Lambda$ and $|R_{\Lambda}^\prime(A, \omega, z)(x', y')|$ for $x', y' \in \Lambda'$ are independent for disjoint suitable squares $\Lambda$ and $\Lambda'$ with $\text{dist}(\Lambda, \Lambda') \geq c_\delta$.

**Proof.** The measurability follows from standard arguments see for example [17] Proposition 1.2.6 or see also [1]. The stationarity is shown in Theorem A.1 (b). The independence follows from the independence of the magnetic fields when restricted to squares which are separated by a distance which is larger than $c_\delta$.

**Lemma 10.4** Let $J \subset \mathbb{R}$ be a bounded interval.

(a) (WEYL) There is a constant $C = C(J, \|V\|_{\infty})$ such that

$$\text{Tr} \chi_\Lambda 1_J(H(A)) \leq C|\Lambda| \quad \text{for all} \quad \omega \in \Omega$$

and every square $\Lambda$.

(b) $\chi_\Lambda 1_J(H(A)) \chi_\Lambda$ is trace class and there exists a constant $C$ such that for every square $\Lambda$

$$\text{Tr} \chi_\Lambda 1_J(H(A)) \leq C|\Lambda|.$$  

(10.3)

**Proof.** Part (a) follows from an application of the Lieb-Thirring inequality, see (5.17). For part (b), by cyclicity of the trace $\text{Tr} \chi_\Lambda 1_J(H(A)) = \text{Tr} \chi_\Lambda 1_J(H(A)) \chi_\Lambda$. By the spectral theorem

$$0 \leq \chi_\Lambda 1_J(H(A)) \chi_\Lambda \leq C_t J \chi_\Lambda e^{-2tH(A)} \chi_\Lambda,$$

and by the diamagnetic inequality

$$\text{Tr} \chi_\Lambda e^{-2tH(A)} \chi_\Lambda \leq e^{2||V||_{\infty}} \int_{\Lambda} e^{2t\Delta(x, x)} dx \leq C t^{-1} e^{2||V||_{\infty}} |\Lambda|,$$

where $V_- := \min(0, V)$. Choosing $t = 1$, we obtain (b).
Lemma 10.5 (GRI) There is a $C_{\text{geom}} = C_{\text{geom}}(\|V\|_{\infty})$ such that for $\Lambda, \Lambda'$ suitable squares with $\Lambda \subset \Lambda'$, and $\Gamma_1 \subset \Lambda^{\text{int}}, \Gamma_2 \subset \Lambda' \setminus \Lambda$, the following inequality holds for all $z \in \rho(H_{\Lambda'}) \cap \rho(H_{\Lambda'})$,

$$\|\chi_{\Gamma_2}R_{\Lambda'}(z)\chi_{\Gamma_1}\| \leq C_{\text{geom}}(1 + |z|)\|\chi_{\Gamma_2}R_{\Lambda'}(z)\chi^\text{out}_\Lambda\|\|\chi^\text{out}_\Lambda R_{\Lambda}(z)\chi_{\Gamma_1}\|,$$

where the norms are operator norms.

Proof. Let $\Lambda = \Lambda_t(x)$. Choose $\phi \in C_c^\infty(\Lambda_{t-1/2}(x))$ which is 1 on $\Lambda_{t-1}(x)$. Let $\Omega$ be the interior of $\Lambda^{\text{out}}$. Then $\text{dist}(\partial\Omega, \text{supp } \nabla\phi) \geq 1/4 =: d$. Moreover, $\phi$ can be chosen such that $\|\nabla\phi\|_{\infty}$ is bounded, independent of $\Lambda$. Then we have

$$\|\chi_{\Gamma_2}(H_{\Lambda'} - z)^{-1}\chi_{\Gamma_1}\| = \|\chi_{\Gamma_2}[\phi(H_{\Lambda'} - z)^{-1} - (H_{\Lambda'} - z)^{-1}\phi]\chi_{\Gamma_1}\|

= \|\chi_{\Gamma_2}(H_{\Lambda'} - z)^{-1} W(\phi)(H_{\Lambda'} - z)^{-1}\chi_{\Gamma_1}\|

\leq \|\chi_{\Gamma_2}R_{\Lambda}(z)(p - A)\cdot (\nabla\phi) R_{\Lambda'}(z)\chi_{\Gamma_1}\| + \|\chi_{\Gamma_2}R_{\Lambda}(z)(\nabla\phi) \cdot (p - A) R_{\Lambda'}(z)\chi_{\Gamma_1}\|,$$

where in the second line we used the geometric resolvent identity,

$$(H_{\Lambda'} - z)^{-1}\phi = \phi(H_{\Lambda'} - z)^{-1} + (H_{\Lambda'} - z)^{-1} W(\phi)(H_{\Lambda'} - z)^{-1},$$

(10.4)

with $W(\phi) := [\phi, H_{\Lambda'}] = i\nabla\phi \cdot (p - A) + (p - A) \cdot i\nabla\phi$. Now we estimate the first term on the right hand side. Choose $\Omega$ with $\text{supp } \nabla\phi \subset \Omega \subset \Omega$, and $\text{dist}(\partial\Omega, \partial\Omega) \geq d/2$. We estimate

$$I = \|\chi_{\Gamma_2}R_{\Lambda}(z)(p - A) \cdot (\nabla\phi) R_{\Lambda'}(z)\chi_{\Gamma_1}\|

\leq \|\chi_{\Gamma}(p - A) R_{\Lambda}(z)\chi_{\Gamma_2}\| \|\chi_{\Omega} R_{\Lambda}(z)\chi_{\Gamma_1}\| \|\nabla\phi\|_{\infty}.$$

We now claim that the first term can be estimated by

$$\|\chi_{\Gamma}(p - a) R_{\Lambda}(z)\chi_{\Gamma_2}\| \leq C(1 + |z| + \|V\|_{\infty})\|\chi_{\Omega} R_{\Lambda}(z)\chi_{\Gamma_2}\|.$$

To see this we use Lemma B.1 from Appendix B, with $u = (H_{\Lambda} - z)^{-1}\chi_{\Gamma_2} f$ and $g = \chi_{\Gamma_2} f$, for some $f \in L^2(\Omega)$, and note that $\chi_{\Gamma_2} f = 0$ in $\Omega$. This yields the desired bound on Term $I$. The second term, Term $II$, can be estimated similarly. □

Lemma 10.6 Let $H(A)$ be a magnetic Schrödinger operator with $A \in C^1$ and $\nabla \cdot A = 0$ such that for $|\alpha| = 1$ we have $\sup_{x \in \mathbb{R}^2} |D^\alpha A(x)|(1 + |x|)^{-1} < \infty$ and $\|D^\alpha V\|_{\infty} < \infty$.

(a) For spectrally almost every $E \in \sigma(H(A))$ there exists a polynomially bounded eigenfunction corresponding to $E$, i.e., $1_\Delta(H(A)) = 0$ where $\Delta$ is the set of all energies in $\mathbb{R}$ for which there does not exist a polynomially bounded eigenfunction.

(b) For every bounded set $J \subset \mathbb{R}$ there exists a constant $C_J$ such that every generalized eigenfunction $u$ of $H(A)$ corresponding to $E \in J \setminus \sigma(H(A))$ satisfies

$$(\text{EDI}) \quad \|\chi_{\Lambda}^\text{int} u\| \leq C_J \|\chi_{\Lambda}^\text{out}(H_{\Lambda}(A) - E)^{-1}\chi_{\Lambda}^\text{int}\|\|\chi_{\Lambda}^\text{out} u\|,$$

where $H_{\Lambda}(A)$ denotes the restriction of $H(A)$ to $L^2(\Lambda)$ with Dirichlet, Neumann or periodic boundary conditions.

Proof. (a) Follows from a generalization of Theorem C.5.4 in [16] to magnetic Schrödinger operators. The proof given there generalizes to magnetic Schrödinger operators by means of the diamagnetic inequality and the following modification. The $L^2$ growth estimate stated in (ii) of Theorem C.5.2 [16] can be shown as in that paper by means of the diamagnetic inequality. To show that (ii) of Theorem C.5.2 [16] implies (iii)
of that same theorem one has to use elliptic regularity instead of the Harnack inequality which was used in [16]. (b) Follows with minor modifications as in the proof of Lemma 3.3.2 in [17] and Lemma B.1.

**Proof of Theorem 3.3.** Fix \( \xi \in (0, \tau - 2) \) and let \( \beta = \frac{1}{2}(1 - \frac{\xi + 2}{\tau}) \). By Lemma 10.1 there exists an \( l_G = l_G(\tau, \xi, c_u, c_o, c_0) \) such that \( G(I_l, l, \gamma_l, \xi) \) holds with \( I_l := E_{inf} + [0, \frac{1}{4}\mu l^{\beta - 1}] \) and \( \gamma_l := l^{\beta - 1} \) for all \( l \geq l_G \). Then choose \( 0 < \Theta < \beta / 2 \) and \( q > 2 \) and \( 0 < \kappa < \min(2 - 2\beta, 1) \). By Lemma 10.2 there exists an \( l^* \) (depending on \( \Theta, q, K_0, K_1, \delta, \mu, \kappa, \rho \)) such that \( W(I_l, l, \Theta, q) \) is satisfied for \( l \geq l^* \) and thus also for \( l \geq l_0 := \max(l^*, b_0^\beta, l_G) \). Moreover, by Lemmas 10.4, 10.3, and 10.5 we can now apply the multiscale analysis as outlined in [17] for the interval \( J_0 := I_{l_0} \) (Specifically the assumptions of Theorem 3.2.2 and Corollary 3.2.6 in [17] are now verified). Note that the properties stated in Lemma 10.3 are weaker than the corresponding properties stated in [17], but one readily verifies that they are sufficient for the multiscale analysis. Namely, there is a minor modification necessary due to the security distance, which we introduced in the definition of \( G(I, l, \gamma, \xi) \). For a detailed discussion of the necessary changes, see for example [11].

Fix \( \omega \in \Omega \). Having established the application of the multiscale analysis we can now show that \( H(A_{\omega}) \) has pure point spectrum in \( J_0 \) using the following standard argument. By Lemma 10.6 (a) there is a set \( \tilde{J}_0 \subseteq J_0 \) with the following properties: (i) for every \( E \in \tilde{J}_0 \) there is a polynomially bounded eigenfunction \( u \) of \( H(A_{\omega}) \) corresponding to \( E \), (ii) \( J_0 \setminus \tilde{J}_0 \) is a set of measure zero for the spectral resolution of \( E_{H(A_{\omega})} \).

Take a generalized eigenfunction \( u \) with energy \( E \in \tilde{J}_0 \). By Lemma 10.6 (b) it satisfies (EDI). Thus by Proposition 3.3.1 in [17] \( u \) must be exponentially decaying. Thus \( E \) is an eigenvalue. Since the Hilbert space is separable, it follows that \( \tilde{J}_0 \) must be countable. Thus the restriction of the spectral measure to \( \tilde{J}_0 \) is supported on the countable set \( \tilde{J}_0 \), and therefore it must be purely discontinuous. Thus the spectrum of \( H(A_{\omega}) \) in \( J_0 \) is pure point. Moreover, the eigenfunctions are exponentially decaying. Dynamical localization, i.e. (3.3), follows from an application of Theorem 3.4.1. in [17]. A necessary condition for the application of Theorem 3.4.1. in [17] is that

\[
p < \min(2\xi, 1 - q/2).
\]

If \( p < 2(\tau - 2) \), we can choose \( \xi \) and \( q \), such that the multiscale analysis can be applied, i.e., \( \xi < \tau - 2 \) and \( q > 2 \), and that (10.5) holds. (Notice that different choices for \( \xi \) and \( q \), will affect the right endpoint of \( J_0 \). Hence the interval for which we are able prove dynamical localization might be smaller than the interval for which we can prove pure point spectrum.) We thus proved that that the spectrum in \( J_0 = [E_{inf}, E_{inf} + e_0] \), with \( e_0 := \frac{1}{2}\mu l_0^{\beta - 1} \), is pure point.

It remains to show that \( J_0 \) contains indeed spectrum. For simplicity, we first consider the case \( K_2 = 0 \) and \( V = 0 \). We know from Theorem 3.4 (d) that in that case \( E_{inf} = \Sigma_{inf} \) and hence \( J_0 = [\Sigma_{inf}, \Sigma_{inf} + c_0] \subseteq \Sigma \).

Now let us assume the general case. By possibly choosing \( l_0 \) larger we can assume by Theorem 3.4 (b) that \( E_{sup} \geq E_{inf} + c_0 \). From Theorem 3.4 (a) we know that

\[
E_{inf} \leq \Sigma_{inf} \leq E_{inf} + K(b_0),
\]

with

\[
K(b_0) := 4K_2b_0^{-2} + \min(K_2b_0^{-1/2}, K_3b_0^{-1}).
\]

For \( b_0 \) sufficiently large, we have one the one hand \( l_0 = l^*_Wb_0^\beta \) and on the other

\[
K(b_0) \leq \frac{1}{4}\mu(l^*_Wb_0^\beta)^{\beta - 1} = \frac{1}{2}c_0.
\]

In particular \( \Sigma_{inf} < E_{sup} \). Applying Theorem 3.4 (c) we get

\[
[\Sigma_{inf}, E_{sup}] \subseteq \Sigma.
\]

Now (10.8) and (10.7) imply that

\[
J_0 \cap \Sigma = [\Sigma_{inf}, E_{inf} + c_0] = [\Sigma_{inf}, \Sigma_{inf} + e_1].
\]
for some $\epsilon_1 > 0$, see the figure below.

\[
\begin{array}{c}
E_{\inf} & \leq K(b_0) \leq \frac{1}{2} \epsilon_0 & e_1 & \Sigma \\
\Sigma_{\inf} & E_{\inf} + \epsilon_0 & & E_{\sup}
\end{array}
\]

\[\square\]

A  
Ergodicity

**Proof of Theorem 3.2, Part 1.** The measurability of $H_A(A_\omega)$ for a finite box follows from an easy application of Proposition 1.2.6. in [17]. Let $f, g \in C_0^\infty$. For any $z$ with nonzero imaginary part we have

\[\lim_{t \to \infty} \langle f, (H_A(A_\omega) - z)^{-1} g \rangle = \langle f, (H(A_\omega) - z)^{-1} g \rangle. \quad (A.1)\]

To this end we can use the geometric resolvent equation (10.4), and the resolvent decay estimate of Theorem C.2. Since the limit of measurable functions is measurable (A.1) implies the measurability of the magnetic Hamiltonian on $\mathbb{R}^2$. 

For $a \in \mathbb{R}^2$ we define the shift operator $T_a$ acting on functions $f$ on $\mathbb{R}^2$ by $(T_a f)(x) = f(x - a)$. The operator $T_a$ acts unitarily on the Hilbert space $L^2(\mathbb{R}^2)$ and in that case we denote it by $U_a$. Given a magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$ we fix a gauge for the vector potential $A[B] : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

\[A[B](x_1, x_2) := \left(0, \int_0^{x_1} B(x_1', x_2)dx_1'\right).\]

Note that $T_a B = \nabla \times (T_a A[B])$. We define the function

\[\lambda_a[B](x) := \int_{\gamma_x} \{T_a(A[B]) - A[T_a B]\} ds,
\]

where $\gamma_x$ is a path in $\mathbb{R}^2$ connecting the origin with $x$ and $ds$ is the line integration measure. Since $\mathbb{R}^2$ is simply connected and the rotation of the integrand is zero, the explicitly choice of $\gamma_x$ is not important.

From the identity $e^{i\lambda_a[B]}(p - T_a A[B])e^{-i\lambda_a[B]} = p - A[T_a B]$ it follows that

\[e^{i\lambda_a[B]}U_a H(A[B])U_a^* e^{-i\lambda_a[B]} = H(A[T_a B]). \quad (A.2)\]

We define a family $(T_a)_{a \in \mathbb{Z}^2}$ of shift operators acting on $\Omega$ as $(T_a \omega)(m) := \omega_{z-a}$. As a trivial consequence of the definitions we have

\[B_{T_a \omega} = \omega_{T_a \omega}. \quad (A.3)\]

**Proposition A.1** Let $a \in \mathbb{Z}^2$ and $V_a = e^{i\lambda_a[B]}U_a$. Then the following holds.

(a) We have

\[V_a H(A[B_\omega])V_a^* = H(A[B_{T_a \omega}]), \quad (A.4)\]

i.e., $\omega \mapsto H(A[B_\omega])$ is ergodic with respect to the family $(T_a)_{a \in \mathbb{Z}^2}$.

(b) For all $\psi, \varphi \in C_0^\infty$,

\[\langle V_a \psi, H(A[B])V_a \varphi \rangle \text{ and } \langle \psi, H(A[B]) \varphi \rangle \quad (A.5)\]

have the same probability distribution.
Proof. (a) is a direct consequence of (A.2) and (A.3). (b) From (A.2) it also follows that
\[ \langle \psi, H(A[B]) \phi \rangle = \langle V_\alpha \psi, H(A[T_\alpha B]) V_\alpha \phi \rangle. \]
Now using (A.3) and the measure preserving property of \( T_\alpha \) part (b) follows.

Proof of Theorem 3.2, Part 2. By the ergodicity property as stated in Proposition A.1 (a), Theorem 3.2 can be obtained the same way as Theorem 1.2.5 in [17] using the invariance of the trace under conjugation by a unitary operator.

B Bound on the Magnetic Gradient

We set \( \nabla_A = \nabla - i A \). Let \( z \in \mathbb{C} \). We say that \( u \) is a weak solution of \( ((p - A)^2 + V) u = g \) in \( \Omega \), if \( u \in W^{1,2}(\Omega) \) and, for every \( \varphi \in C^\infty_c(\Omega) \),
\[
\langle \nabla_A \varphi, \nabla_A u \rangle + \langle \varphi, Vu \rangle = \langle \varphi, g \rangle. \tag{B.1}
\]
The following lemma is a minor modification of Lemma 2.5.3 in [17].

Lemma B.1 Let \( \tilde{\Omega} \subset \Omega \subset \mathbb{R}^2 \) with \( \text{dist}(\partial \Omega, \partial \tilde{\Omega}) =: d > 0 \). Then there exists a constant \( C = C(d) \) such that every weak solution of \( Hu = g \) in \( \Omega \) satisfies
\[
\| \nabla_A u \|_{L^2(\tilde{\Omega})} \leq C(1 + \| V \|_\infty) \left( \| u \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \right).
\]

Proof. Since \( C^\infty_c(\Omega) = W^{1,2}_0(\Omega) \), Equation (B.1) holds for all \( \varphi \in W^{1,2}_0(\Omega) \). We can choose a function \( \Psi \in C^\infty_c(\Omega) \), \( 0 \leq \Psi \leq 1 \) with \( \Psi \equiv 1 \) on \( \Omega \) and \( \| \nabla \Psi \|_\infty \leq Cd^{-1} \), where \( C \) depends on the dimension \( d \). Set \( w := \Psi^2 u \). Then \( w \in W^{1,2}_0(\Omega) \) and \( \nabla w = \Psi^2 \nabla u + 2u \Psi \nabla \Psi \). It follows from (B.1) that
\[
\langle \nabla_A w, \nabla_A u \rangle + \langle w, Vu \rangle = \langle w, g \rangle.
\]
We obtain
\[
\| \Psi \nabla_A u \|^2 = \langle \nabla_A w, \nabla_A u \rangle - 2 \langle u \nabla \Psi, \Psi \nabla_A u \rangle = \langle w, g \rangle + \langle w, Vu \rangle - 2 \langle u \nabla \Psi, \Psi \nabla_A u \rangle \leq \| g \| \| u \| + \| V \|_\infty \| \Psi u \|^2 + \frac{1}{2} \| \Psi \nabla_A u \|^2 + 4 \| u \|^2 \| \nabla \Psi \|^2_\infty.
\]
By the choice of \( \Psi \) this now yields the claim.

C Resolvent Decay Estimates

Define the function \( \rho(x) = (1 + |x|^2)^{1/2} \). Let \( \bar{H} \) be an operator of the form \( H_\Lambda(A) \). Define
\[
\bar{H}(\alpha) := e^{i \alpha \rho} \bar{H} e^{-i \alpha \rho} = \bar{H} - \alpha \nabla \rho \cdot (-i \nabla - a) - (-i \nabla - a) \cdot \alpha \nabla \rho + \alpha^2 | \nabla \rho |^2.
\tag{C.1}
\]
Since \( | \nabla \rho | \) and \( | \Delta \rho | \) are bounded and \( (-i \nabla - a) \) is infinitesimally small with respect to \( \bar{H} \), we obtain that \( \bar{H}(\alpha) \) is an analytic family of type \( \Lambda \) on \( \mathbb{C} \).

Lemma C.1 Let \( \beta \in \mathbb{R} \). Then \( (-\infty, \inf \sigma(\bar{H}) - \beta^2) \subset \rho(\bar{H}(i \beta)) \). Let \( z \in \mathbb{C} \) and \( \text{Rez} < \inf \sigma(\bar{H}) - \beta^2 \), then
\[
\| (\bar{H}(i \beta) - z)^{-1} \| \leq \frac{1}{\inf \sigma(\bar{H}) - \beta^2 - \text{Rez}}.
\]
Proof. Using $|\nabla \rho| \leq 1$, we find
\[
\|\psi\|\| (\bar{H}(i\beta) - z) \psi \| \geq |\langle \psi, (\bar{H}(i\beta) - z) \psi \rangle| \geq |\text{Re}(\psi, (\bar{H}(i\beta) - z) \psi)| \\
\geq \langle \psi, (\bar{H} - \beta^2 - \text{Re}z) \psi \rangle \geq (\inf \sigma(\bar{H}) - \beta^2 - \text{Re}z) \|\psi\|^2.
\]
The lemma follows from this estimate.

Theorem C.2 Let $\Lambda = \Lambda_4 \subset \mathbb{R}^2$. Let $E < \inf \sigma(H_\Lambda)$ and $\eta = \text{dist}(E, \inf \sigma(H_\Lambda))$. Then, for $l \geq 4$,
\[
\| \chi^\text{int}(H_\Lambda - E)^{-1} \chi^\text{out} \| \leq \frac{2}{\eta} \exp \left( - \sqrt{\frac{\eta}{2}} \frac{l}{4} \right).
\]

Proof. Let $\varphi_1, \varphi_2 \in C^\infty_c(\Lambda)$, and $\alpha \in \mathbb{R}$. Then by unitarity
\[
I = \langle \chi^\text{int} \varphi_1, (H_\Lambda - E)^{-1} \chi^\text{out} \varphi_2 \rangle = \langle e^{i\alpha \rho} \chi^\text{int} \varphi_1, (H_\Lambda(\alpha) - E)^{-1} \chi^\text{out} e^{i\alpha \rho} \varphi_2 \rangle.
\]
By Lemma C.1, we can analytically continue the resolvent occurring of the right hand side to a strip around the real axis of width $\eta^{1/2}$. Thus we find for $\alpha = i\beta$ with $\beta = \sqrt{\eta/2}$,
\[
I = \langle e^{\beta \rho} \chi^\text{int} \varphi_1, (H_\Lambda(i\beta) - E)^{-1} \chi^\text{out} e^{-\beta \rho} \varphi_2 \rangle.
\]
Using the resolvent estimate of Lemma C.1 and inserting the definition of $\rho$, we find
\[
|I| \leq \|\varphi_1\| \|\varphi_2\| \frac{2}{\eta} \exp \left( - \sqrt{\frac{\eta}{2}} \frac{l}{4} \right).
\]
The theorem now follows.

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