Dynamical behaviour of multiplanet systems close to their stability limit

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ABSTRACT

The dynamics of systems of two and three planets, initially placed on circular and nearly coplanar orbits, is explored in the proximity of their stability limit. The evolution of a large number of systems is numerically computed and their dynamical behaviour is investigated with the frequency map analysis as chaos indicator. Following the guidance of this analysis, it is found that for two–planet systems the dependence of the Hill limit on the planet mass, usually made explicit through the Hill’s radius parametrization, does not appear to be fully adequate. In addition, frequent cases of stable chaos are found in the proximity of the Hill limit. For three–planet systems, the usual approach adopted in numerical explorations of their stability, where the planets are initially separated by multiples of the mutual Hill radius, appears too reducing. A detailed sampling of the parameter space reveals that systems with more packed inner planets are stable well within previous estimates of the stability limit. This suggests that a two–dimensional approach is needed to outline when three–planet systems are dynamically stable.

Key words: planetary systems; planets and satellites: dynamical evolution and stability

1 INTRODUCTION

Most observed extrasolar planets detected up to date move on highly elliptical orbits, much larger than any solar system planet. The largest known eccentricity is 0.93, for HD 80606 b (Naef et al. 2001). This behaviour is at odds with formation theories predicting planets on circular orbits. Even migration by tidal interaction with the harboring disks is characterized by dissipative interactions mostly leading to a reduction of also initially small eccentricities. Different mechanisms may be invoked to excite planet eccentricities (Namouni 2007), but planet–planet scattering appears to be the most valid candidate for exciting large orbital eccentricities (Rasio & Ford 1996; Weidenschilling & Marzari 1996; Marzari & Weidenschilling 2004; Juric & Tremaine 2008; Nagasawa et al. 2008; Chatterjee et al. 2008; Marzari 2011; Nagasawa & Ida 2011; Beaugé & Nesvorný 2012). In a multi–planet system, mutual gravitational perturbations between the planets may cause an eccentricity growth leading to orbital crossing. Eventually, the resulting dynamically violent chaotic phase is dominated by close encounters between the planets ending when one or more planets are ejected from the system on hyperbolic trajectories. The initial orbital structure of the multi–planet system is dramatically altered and a new stable configuration is reached with the surviving planets left on highly eccentric non–interacting orbits possibly inclined respect to the initial orbital plane. If tidal interaction with the host star is invoked to circularize the orbit of the planet to the periastron distance, this mechanism may explain a fraction of the observed “Hot Jupiters”. In addition, it could also account for the production of pairs of planets in mean motion resonances (Raymond et al. 2008).

A classical simple example of this scenario is that of two jovian–size planets embedded in a circumstellar disk either trapped in resonance or evolving independently. Once the disk is dispersed either by strong stellar winds associated to the T–Tauri phase of the star or by the progressive UV photo–evaporation induced by the central, or nearby stars, the gas damping on the planet orbits ceases and the system may become dynamically unstable undergoing a prolonged phase of chaotic evolution characterized by repeated close encounters. Potentially, leftover planetesimals may still affect the planet evolution after the gas dissipation (Raymond et al. 2008) but here I focus on the phase where the only relevant forces are the mutual gravitational attractions between the planets. In this context, it is important to understand when a system of 2 or more planets, even far from resonances, may become chaotic due to their mutual perturbations. This may tell us how frequently planet–planet scattering occurs on the basis of planet formation models.
A significant effort has been devoted to outline a threshold value $\Delta$ for the initial separation between the planet orbits within which planet–planet scattering always occurs while beyond it the system is macroscopically stable and close encounters do not occur at least over an extended timespan possibly comparable with the stellar age. The value of $\Delta$ has been extensively explored for systems of 2, 3 and more planets and lead to the introduction of the notion of "Hill stability". For a system of 2 planets, Hill-stable orbits are those for which the planets will conserve their ordering in terms of distance from the star. A more stringent formulation described in Barnes & Greenberg (2006) and Veras & Mustill (2013) called "Lagrangian stability", requires that the outer planet cannot escape from the system on a hyperbolic trajectory. As shown in Barnes & Greenberg (2006) these two last criteria give a very similar value of the initial separation between the two planets. In addition, the simulations performed by Veras & Mustill (2013) show also that for initially circular orbits nearly all Hill stable systems are also Lagrange stable. Since in this paper only orbits with initial small proper eccentricity are considered, we will adopt the Hill stability value $\Delta_h$ as estimate of the threshold value marking the transition from stable to unstable orbits. However, while analytical estimated of $\Delta_h$ can be found in the literature, the dynamics close to the Hill stability limit has not been explored in details. Ad example, the orbits close to the Hill stability limit are stable in the sense of "quasi periodic" or are there cases of "stable chaos"? Stable chaos is an oxymoron indicating orbits that exhibit a chaotic evolution but that, at the same time, are long-term macroscopically stable possibly because the chaotic domain is thin. An example of this behaviour is provided by asteroid (522) Helga as described in Milani & Nobili (1992).

In this paper, by using the frequency map analysis (hereinafter FMA), the stability properties of systems of two and three planets will be explored close to the Hill stability limit defined by an initial separation $\Delta_h$. If this limit marks a macroscopic transition between stability and instability, one would expect a sudden drop of the frequency diffusion speed. This would mark the end of the resonance overlap region so that beyond $\Delta_h$ strong instability will potentially develop only at individual low order resonances located farther out. We will see that this is not the case and that examples of "stable chaos" beyond $\Delta_h$ are frequently encountered. In addition, the FMA analysis will also show that the analytical estimate of $\Delta_h$ is not very precise. In particular, the dependence of $\Delta_h$ on the planet mass is not fully accounted by expressing it as a function of the Hill’s radius.

In Sect. 2 the dynamics of two planets will be investigated close to the Hill’s stability limit by integrating a large number of systems and analysing their stability properties with the frequency map analysis. In Sect. 3 the dynamics of systems with three planets will be analysed. It will be shown that the parametrization that uses multiples of the mutual Hill’s radius to sample the initial parameter space and adopted in all previous stability studies misses many stable systems with an inner more packed planet pair.

2 ANALYTICAL AND NUMERICAL ESTIMATES OF $\Delta_h$

For the case of two planets around a massive star, the elliptic three-body problem formulated in barycentric coordinates leads to the following analytical estimate for $\Delta_h$ given in implicit form Gladman (1993):

$$\alpha^{-3} \left( \mu_1 + \frac{\mu_2}{(1 + \Delta_h/\alpha_1)^3} \right) (\mu_1 \gamma_1 + \mu_2 \gamma_2) \left( 1 + \left( \frac{\Delta_h}{\alpha_1} \right)^{1/3} \right)^2 > 1 + 3^{4/3} \mu_1 \mu_2 \alpha^{4/3}$$

(1)

where $\mu_i = m_i/M$, $\alpha = \mu_1 + \mu_2$, $\gamma_i = (1 - e_i^2)^{1/2}$, $m_i$ is the mass of planet $i$, $M$ is the mass of the star, $e_i$ is the eccentricity of planet $i$. In the case of initially circular (and coplanar) orbits the above equation can be solved and, to the lowest order in mass, it gives:

$$\Delta_h \sim 2.40 (\mu_1 + \mu_2)^{1/3} a_1$$

(2)

When the eccentricities and inclinations of the planets are not small, more complex expressions can be derived Donnison (2006) but we will concentrate in this paper on the more simple case of nearly coplanar and initially circular orbits. The above formula has been numerically tested by Chambers et al. (1996) showing that a more precise value is given by:

$$\Delta_h \sim 2 \sqrt{3} R_H$$

(3)

where $R_H$ is the planets’ mutual Hill radius given by

$$R_H = \left( \frac{m_1 + m_2}{3M_\odot} \right)^{1/3} \left( \frac{a_1 + a_2}{2} \right)$$

(4)

For systems of three planets, numerical integrations of orbits have been performed showing that the timescale for the onset of instability grows logarithmically by increasing the mutual separation between the planets (Marzari & Weidenschilling 2002, Chambers et al. 1996, Chatterjee et al. 2008). This has been interpreted as a decrease of the chaotic diffusion which should finally lead to stable quasi periodic orbits beyond a threshold value of separation. These studies are based on a parametrization of the initial separation between the planets based on multiples of the Hill radius. Once fixed the inner planet semimajor axis, the next one is placed at:

$$a_{i+1} = a_i + K R_H$$

(5)

In all these numerical simulations it was found that, whenever a third planet is added, the Hill-stable region is placed beyond the $\Delta_h$ value computed independently for each pair of planets. In addition, the value of $K$ defining the $\Delta_h$ for which systems are stable depends on the mass of the planets.
3 USE OF THE FMA TO DETECT CHAOTIC BEHAVIOURO

There is a variety of numerical tools useful for the detection of chaos. The computation of the Maximum Lyapunov Exponent (MLE) which measures the exponential divergence of close orbits (Benettin et al. 1976) has the drawback of being difficult to be implemented in an automatic way for exploring a large sample of orbits. More suited for this task is the computation of the Mean Exponential Growth Factor of Nearby Orbits (MEGNO) introduced by (Cincotta & Simó 2000) and used ad example by Go´zdziewski (2002, 2003), or the Frequency Map Analysis that is adopted in this paper. The FMA (Frequency Map Analysis) (Laskar 1993), implemented as described in Marzari et al. (2003), is used to measure the diffusion in the phase space of the fundamental frequencies of a dynamical system. The algorithm of the FMA method computes the strongest peak in the Fourier transform of a time series with the MFT (Modified Fourier Transform) high precision method (Laskar 1993; Sidíchkovsky & Nesvorny 1996). For a system of 2 planets we consider the non-singular variables $h, k$ in their complex-$number$-form, and with the MFT we compute one of the main frequencies $g_j$ predicted by the traditional Laplace-Lagrange theory (Murray & Dermott 1999). They are the eigenvalues of the 2x2 matrix $A$ function only of the masses and semimajor axes of the two planets whose elements are given as:

$$A_{jj} = \frac{n_j}{4 M_0 + m_j} \alpha_{12}^2 b_{j/2}(\alpha_{12})$$

$$A_{jk} = \frac{n_j}{4 M_0 + m_j} \alpha_{12}^2 b_{j/2}(\alpha_{12}),$$

where $j = 1, 2, k = 2, 1, j \neq k$ and $\alpha_{12} = a_1/a_2$ since in our case $a_1 \leq a_2$. The $b_{j/2}(\alpha)$ are the Laplace coefficients whose expression can be found in Murray & Dermott (1999). The precise value of one of the two frequencies $g_j$ (the stronger one in the power spectrum of the inner planet) is computed over running windows covering the whole timespan of the numerical integration of the planet orbits. Its standard deviation $\sigma_j$ is hence used to compute a chaos indicator $c_v = \log(\sigma_j)/g$. The system is quasi-periodic only if the proper frequency is not changing with time, otherwise it is chaotic (Laskar 1992) and $c_v$ measures the variation in time of the proper frequency and then the diffusion speed of the orbit in the phase space.

### 3.1 The FMA set up

To investigate the dynamics of a two-planet system close to the Hill stability limit, the initial semimajor axis of the inner one $a_1$ is fixed at 3 AU while $a_2$ ranges from 3 AU to 6 AU. The initial eccentricity is set to 0 for both planets while the mutual inclination is randomly selected in the range $[0^\circ, 5^\circ]$, a reasonable initial value for planets formed within a protostellar disk. The other orbital angles are all randomly selected between $[0^\circ, 360^\circ]$. The masses of the planets are all equal and varied from $5.1 \times 10^{-5} M_\odot$ (Neptune-size planets) to $2.8 \times 10^{-4} M_\odot$ (Saturn-size planets) and to $9.5 \times 10^{-3} M_\odot$ (Jupiter-size planets). The trajectories of the bodies are integrated over 10 Myr and the FMA is applied to the $h, k$ variables of the inner planet over running windows of 2 Myr. The symplectic integrator SyMBA (Levison & Duncan 1994; Duncan et al. 1998) is used with a timestep of 10 days and the output is recorded every 5 yrs. In this version of SyMBA, whenever two bodies suffer a mutual encounter the time step for the involved bodies is recursively subdivided to whatever level is required. However, it must be noted that the numerical integration is halted after the first close encounter between planets since it is a clear indication of instability and the FMA analysis is not needed in this case. The timestep of 10 days adopted in the simulations is approximately $1/190$ of the orbital period of the inner planet and, as a consequence, the numerical algorithm can comfortably handle the eccentricity built up by the dynamical instability that leads to the first close approach.

4 STABILITY OF TWO–PLANET SYSTEMS

In Fig.1 the three panels show the outcome of the FMA analysis for the different values of the planet masses. The inner planet is on a fixed orbit at 3 AU so we plot the value of $c_v$, the chaos indicator, as a function of the outer planet semimajor axis $a_2$. One would expect a sudden drop of the stability index $c_v$ beyond the Hill stability limit but this occurs only partly and at a slightly different value respect to the analytical prediction $\Delta_h$ (Eq. (4)). The mismatch depends on the mass of the planets.

For Neptune–size planets, $\Delta_h$ is underestimated and the decrease in the stability properties shown by the FMA analysis begins farther out. This is confirmed by the long term integration of two cases, marked by green squares in Fig.1, top panel, which have an initial separation larger than $\Delta_h$. Both systems become macroscopically chaotic before 1 Gyr and undergo close encounters. This indicates that the Hill’s stability limit is farther out respect to the analytical prediction. The minimum value of $c_v$ is reached at about 3.7 AU, just after the 4:3 mean motion resonance (hereinafter MMR). Beyond the 4:3, unstable orbits are found only at individual and well separated mean motion resonances like, for example, the 2:1. From the analysis of $c_v$, a good choice for an initial estimate of $\Delta$, marking the transition between stable and unstable orbits, could be the separation between the location of the inner planet and the 4:3 resonance. In this case $\Delta$ would be about 0.63 AU against the value of $\Delta_h$ that is 0.36 AU.

For Saturn mass planets (Fig.1 middle panel), the value of $\Delta_h$ appears to better outline the transition from stability to instability. It is very close to the 4:3 MMR and the drop in the value of $c_v$ begins very close to it. However, beyond $\Delta_h$ a consistent fraction of planet pairs have still a large value of $c_v$ until the 3:2 MMR is encountered. Once the resonance is passed, a minimum of $c_v$ is met and it remains small up to the 2:1 resonance and beyond, apart from unstable orbits at lower order mean motion resonances. The region between the Hill stability boundary and the 3:2 MMR is populated by cases of “stable chaos”. The irregular motion is limited in scale and it does not lead to large eccentricities, close encounters and then macroscopic chaotic evolution. Their behaviour is clearly illustrated in Fig.2 where the long term evolution of the eccentricity of the inner planet of a few selected systems located between the Hill stability boundary and the 3:2 MMR is shown. For these systems it looks like
the chaotic domain, which is due to overlap of multiple resonances, does not contain a region where the planets may increase their eccentricities by mutual perturbations up to the point of triggering close encounters.

In the case of Jupiter-size planets (Fig.1 bottom panel), the dynamics appears to be complex. Only a few cases are stable within the 3:2 MMR beyond which the $c_v$ quickly drops. The value of $\Delta_h$ overestimates the transition from stable to unstable systems and at smaller separations there are cases that are not chaotic and stable over 5 Gyr. However, this is not an effect due to the presence of the 3:2 MMR. Barnes & Greenberg (2007) have shown that within a MMR the stability properties of planetary systems may be increased. A similar result was obtained by Marzari et al. (2006) for the 2:1 resonance where long term stable and chaotic zones can be found. In the case of Jupiter-pair systems, the decrease of the stability properties begin outside the 3:2 MMR and the system marked by the green square in the bottom panel of Fig.1 is not in a 3:2 MMR (no libration or slow circulation of the critical arguments is observed) and it is stable over 5 Gyr. As a consequence, the analytical estimate of $\Delta_h$ indeed overestimates the location of the transition from stable to unstable orbits. In addition, the stability marker $c_v$ in the bottom panel of Fig.1 decreases away from the location of the 3:2 MMR while, usually, getting closer to the separatrix leads to systems with lower values of $c_v$.

The green square in Fig.1 bottom panel, marks a case which is stable over 5 Gyr in spite of being located within $\Delta_h$. The evolution of the eccentricity of both planets is periodic and it does not ever show signs of stable chaos. For Jupiter-size planets, the location of the 3:2 MMR is a better indicator of stability respect to $\Delta_h$. The region between the 3:2 and 2:1 MMR are characterized by the presence of both unstable and "stable chaos" orbits in particularly close to the 7:4 MMR.

In conclusion, the transition between stable and unstable orbits in two-planet systems appears to be more complicated than thought. The accuracy of the semi-analytical estimate of the initial separation $\Delta_h$ depends on the mass of the planets in the system. In addition, the transition does not occur as a sudden change from chaotic to non-chaotic orbits since cases of stable chaos are found close to the threshold value. It must also be noted that the offset of $\Delta_h$ respect to the real stability limit observed in Fig.1 cannot be ascribed to the difference between the definition of Hill stability and Lagrange stability. The main reason is that the mismatch between the semi-analytical estimate of $\Delta_h$ and the real beginning of instability has a systematic trend ranging from $\Delta_h$ being too small for Neptune-size planets to being too large for Jupiter-size planets while it is a very good approximation for Saturn-size planets. This trend does not reflect the fact that the Lagrange stability limit is more restrictive being then either coincident or systematically larger than $\Delta_h$. Moreover, as stated by Barnes & Greenberg (2003) and Veras & Mustill (2015), the two limits for systems with initially circular orbits practically coincide.

It is noteworthy that, for any value of the planet masses considered here, there are cases of stable Trojan planet configurations. Ad example, in the case of Jupiter-size planet pairs long term numerical simulations of a few sample cases have shown that a low value of $c_v$ implies dynamical stability over 5 Gyr. The possibility that extrasolar planets may be trapped in a 1:1 resonance has been explored in details in previous publications (Schwarz et al. 2007; Hadjidemetriou & Vassiliev 2009). In particular, Laughlin & Chambers (2002) have shown that Trojan-type orbits are stable for a mass ratio $\mu = \frac{(m_1 + m_2 + M_0)}{(m_1 + m_2)} \leq 0.03812$ while the condition for horseshoe orbit stability is $\mu \leq 0.0004$. Hence, we expect that only among Neptune and Saturn-size planets (top and middle panels of Fig.1) stable pairs of planets in horseshoe orbits can be found.

In all 3 panels of Fig.1 after a marked decrease of $c_v$ in the proximity of $\Delta_h$, its value appears to slowly increase again for large values of $a_2$. This growing trend is due to a degradation of the precision in measuring the variation of the frequencies $g_1$ and $g_2$. This is related to the decrease of both $g_1$ and $g_2$ for a larger separation between the planets. Since the time span of the integration and of the running windows where the frequencies are computed is constant, less periods of the $h,k$ variables are covered for larger values of $a_2$ and, as a consequence, the precision decreases. At the same time, the forced eccentricity of each planet is smaller for larger values of $a_2$ and the $h,k$ variables become progressively smaller, increasing the numerical error in the computation of the associated frequencies. This does not affect the overall outcome of the FMA analysis, since this study is focused on what happens close to the value of $\Delta_h$ and in its surroundings.

5 THE CASE OF THREE EQUAL-MASS PLANETS

Observed systems Ups And, 55 Cnc, and HD 37124 have proven that it is indeed possible to form more than two giant planets around solar type stars. A system with three giant planets has a more rich and complex dynamical behaviour compared to two planet systems. The onset of instability depends on a wider parameter space and its outcome is dynamically more various. It has been a custom in previous studies of dynamical scattering of systems of 3 or more planets (Chambers et al. 1996; Marzari & Weidenschilling 2002; Chatterjee et al. 2008) to explore the stability limit using the parametrization described by Eq. (3) with $K$ depending on the planet masses (assuming they are all equal). The initial semimajor axis of each of the three planets are usually selected according to the relations $a_2 = a_1 + K R H_{1,2}$ and $a_3 = a_2 + K R H_{2,3}$. However, the problem has an intrinsic higher dimensionality and this parametrization, as it will be shown here, appears inadequate to fully describe the stability properties of three planets and to derive where the transition between stable and unstable systems occurs.

As for two planets, to explore the dynamics of three-planet systems close to their stability threshold, we integrated the orbits of a large number (about $10^5$) of putative systems and computed, via the FMA, the $c_v$ coefficient. In this case, we have randomly sampled both $a_2$ and $a_3$, the initial semimajor axis of the second and third planet. The initial eccentricity is set to 0 for all three planets and their inclination is sampled in the same range of the 2-planet cases. All other angles are randomly chosen. The FMA is performed on the $h,k$ variables of the inner planet. To illustrate the behaviour of the chaos indicator $c_v$, it is necessary
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Figure 1. Values of the FMA chaos indicator $c_v$ for different values of the second planet semimajor axis $a_2$. The vertical lines mark the location of the Hill stability limit, estimated by $\Delta h$, and of the main first order mean motion resonances. The green squared points indicate those cases that have been integrated over 5 Gyr to explore their long term dynamical behaviour. In the top panel the mass of the two planet is $5.1 \times 10^{-5} M_\odot$ (Neptune–size planets), in the middle panel the planet mass is set to $2.8 \times 10^{-4} M_\odot$ (Saturn–size planets) and in the bottom panel the mass is $9.5 \times 10^{-3} M_\odot$ (Jupiter–size planets).

Figure 2. Long term evolution of 3 cases ($a_2 = 3.77, 3.79$, and $3.92$) of stable chaos highlighted in Fig.1 (middle panel) as green squares. The eccentricity evolution of the inner planet confirms the chaotic character of the systems in spite of their macroscopic stability over 5 Gyr.

In this case to resort to color mapping plots where the semimajor axis of the two outer planets are adopted as variables.

In Fig.3 $c_v$ is drawn as a function of $a_2$ and $a_3$ for the same three values of planet masses adopted in the previous section. We also plot the curve given by the function:

$$a_3 = a_2 + KR_H$$

with ticks marking increasing integer values of $K$. Hereinafter we will call this parametrization curve the $K$–curve. By inspecting these plots we can test the robustness of the strategy of following the $K$–curve to explore the stability of three–planet systems. The first thing that can be noted in each stability map of Fig.3 is the dense web of resonances in the proximity of the instability limit, more marked in the case of more massive planets since the resonance width is larger. Vertical stripes with low values of $c_v$ correspond to MMR between the first and second planet, while inclined straight lines mark MMR between the second and third planet.

The weakness of the choice of the $K$–curve as a guiding
parametrization to explore the stability properties of these systems is clearly shown in all panels of Fig. 3. For Neptune-size planets, the value of $K$ for which the first stable systems are met is about 8, marked by a square in the figure. However, there are two critical aspects in this finding. First of all, beyond $K = 8$ resonances destabilize again the system until $K$ is about 9 and, finally, a large region of stability is met. Second and most important aspect, stable systems can be found outside the $K$–curve for smaller $a_2$ and values of $a_3$ which lay above the $K$–curve. Ad example, all three systems marked by red circles are stable over 5 Gyr (no sign of chaos in the orbital elements) and they approximately corresponds to $a_2 = a_1 + 5R_{H1.2}$ and $a_3 = a_2 + KR_{H2.3}$ with $K$ ranging from about 11, for the case with smaller $a_3$, and beyond. The circled cases are representative of a large region of the phase space populated by stable systems that are not met and identified by following the $K$–curve parametrization. The same effect can be seen in the other two panels of Fig. 3. In all these systems the two inner planets are more packed respect to what it would be predicted by the $K$–curve parametrization. For Saturn–size planets (middle panel), the first stable systems along the $K$–curve appear at $K \approx 7$. However, moving farther along the $K$–curve, the 2:1 MMR between the two inner planets is crossed and only beyond $K \approx 8$ the systems become stable again. Even in this case the red circles indicate systems far from the $K$–curve which are stable and non–chaotic over 5 Gyr and with a more packed inner planet pair. The value of $K$ for the two inner planets is about 3.5 while for the outer planet it is about 10 for the system closer to the $K$–curve and larger for the other two marked systems. For Jupiter–size planets (bottom panel), the first stable systems along the $K$–curve are found for $K$ close to 6 (squared value in panel 3, Fig. 3), in good agreement with Marzari & Weidenschilling (2002); Chatterjee et al. (2008). Also in this case there is a large portion of the parameter space, above the $K$–curve and for small values of $K$, where stable systems can be found. Ad example, the red circles label cases that do not comply with the standard parametrization given by Eq. (3) but are stable for 5 Gyr without any trace of chaotic behaviour. In conclusion, the $K$–curve used to probe the stability of 3–planet systems is misleading since it neglects a significant amount of internally more packed cases which are stable at least for 5 Gyr and beyond.

It is also interesting to observe that Trojan planets, either the two inner ones or the two outer ones, can coexist with a close external perturber. In all three panels, the inner vertical stripe represent systems where the two inner planets are in a 1:1 resonance. The inclined straight line on the bottom shows instead systems with the two outer planets in a Trojan configuration. In both groups there are cases with very low values of $c_v$ and then stable on the long term.

As in Fig. 2 a slow degradation is observed in the FMA analysis and higher values of $c_v$ are found for larger values of both $a_2$ and $a_3$. As for the case of two planets, this is due to the decrease of the frequencies and eccentricities of the more detached planetary systems.

![Figure 3](image)  
Figure 3. Stability maps for systems of 3 planets. The top panel is for Neptune–size planets, the middle for Saturn–size planets and the bottom for Jupiter–size planets. The color coding gives the value of the chaos indicator $c_v$. Brighter colors indicate unstable orbits. The black dashed line represents the curve given by $a_3 = a_2 + KR_H$ for different values of $K$. The crosses on the curve mark the location of integer values of $K$. The red circles label cases that have been integrated over 5 Myr and proven to be
6 SUMMARY AND CONCLUSIONS

In this paper, the dynamics of two and three equal-mass planet systems has been explored in detail close to the stability limit under the guidance of the frequency map analysis. For two planet systems, the main findings concern the presence of stable chaos beyond the separation $\Delta_h = 2\sqrt{3R_H}$ which makes it difficult to talk about a ‘stability limit’. This stable chaos is limited in phase space and it does not lead to planet–planet scattering. Another aspect coming out from the FMA analysis is that $\Delta_h$ depends on the planet mass in a way that is not fully accounted for by the the definition of Hill’s radius. For Neptune–size planets the value of $\Delta_h$ appears to be underestimated, while it is a bit overestimated for Jupiter class planets. A better and more complete estimate of $\Delta_h$, derived from long term numerical simulations, is then needed. The FMA can be used to outline the most promising regions where this exploration must be undertaken.

For three–planet systems, the parametrization based on the relations $a_2 = a_1 + KR_{H1,2}$ and $a_3 = a_2 + KR_{H2,3}$, widely studied to study the stability of these systems, does not appear a reliable choice. As shown by the stability maps, there is a large portion of systems which are stable but violate the above mentioned parametrization and do not lay on the $K$–curve. A two dimensional approach is needed to find the transition between stable and unstable systems that may lead to planet–planet scattering. This has important consequences in modeling the initial stages of evolution of a planetary system that has reached completion. The initial separation, marking the transition from stable to unstable orbits, must be estimated via numerical simulations and cannot be easily determined on the basis of semi-empirical formulas. Care must also be used when analysing a newly discovered system and predictions based only on the above mentioned formulas should be tested against a more accurate dynamical analysis based on chaos indicators and direct numerical integration.

The models presented in this paper are limited since they consider equal mass planets in initially circular and almost coplanar orbits. More detailed explorations have to be performed for more dynamically complex systems where the eccentricity or mutual inclinations are not negligible. In this case, the reference formulas will be those described in Donnison [2006].

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