FUKUSHIMA’S DECOMPOSITION
FOR DIFFUSIONS ASSOCIATED
WITH SEMI-DIRICHLET FORMS

LI MA
Department of Mathematics
Hainan Normal University
Haikou, 571158, China
mary_henan@yahoo.com.cn

ZHI-MING MA
Institute of Applied Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, China
mazm@amt.ac.cn

WEI SUN
Department of Mathematics and Statistics
Concordia University
Montreal, H3G 1M8, Canada
wsun@mathstat.concordia.ca

Abstract
Diffusion processes associated with semi-Dirichlet forms are studied in the paper. The main results are Fukushima’s decomposition for the diffusions and a transformation formula for the corresponding martingale part of the decomposition. The results are applied to some concrete examples.

Keywords: Fukushima’s decomposition, semi-Dirichlet form, diffusion, transformation formula.

AMS Subject Classification: 31C25, 60J60
1 Introduction

It is well known that Doob-Meyer decomposition and Itô’s formula are essential in the study of stochastic dynamics. In the framework of Dirichlet forms, the celebrated Fukushima’s decomposition and the corresponding transformation formula play the roles of Doob-Meyer decomposition and Itô’s formula, which are available for a large class of processes that are not semi-martingales. The classical decomposition of Fukushima was originally established for regular symmetric Dirichlet forms (cf. [4] and [5, Theorem 5.2.2]). Later it was extended to the non-symmetric and quasi-regular cases, respectively (cf. [16, Theorem 5.1.3] and [12, Theorem VI.2.5]). Suppose that \((E, D(E))\) is a quasi-regular Dirichlet form on \(L^2(E; m)\) with associated Markov process \((X_t)_{t \geq 0}, (P_x)_{x \in E}\) (we refer the reader to [12], [5] and [13] for notations and terminologies of this paper). If \(u \in D(E)\), then there exist unique martingale additive functional (MAF in short) \(M[u]_t\) of finite energy and continuous additive functional (CAF in short) \(N[u]_t\) of zero energy such that

\[
\tilde{u}(X_t) - \tilde{u}(X_0) = M[u]_t + N[u]_t,
\]

where \(\tilde{u}\) is an \(E\)-quasi-continuous \(m\)-version of \(u\) and the energy of an AF \(A := (A_t)_{t \geq 0}\) is defined to be

\[
e(A) := \lim_{t \to 0} \frac{1}{2t} E_m[A_t^2] (1.1)
\]

whenever the limit exists in \([0, \infty)\).

The aim of this paper is to establish Fukushima’s decomposition for some Markov processes associated with semi-Dirichlet forms. Note that the assumption of the existence of dual Markov process plays a crucial role in all the Fukushima-type decompositions known up to now. In fact, without that assumption, the usual definition (1.1) of energy of AFs is questionable. To tackle this difficulty, we employ the notion of local AFs (cf. Definition 2.2 below) introduced in [5] and introduce a localization method to obtain Fukushima’s decomposition for a class of diffusions associated with semi-Dirichlet forms. Roughly speaking, we prove that for any \(u \in D(E)_{loc}\), there exists a unique decomposition

\[
\tilde{u}(X_t) - \tilde{u}(X_0) = M[u]_t^{loc} + N[u]_t^{loc},
\]

where \(M[u]^{loc} \in M^{[0, \infty]}_{loc}\) and \(N[u]^{loc} \in N_{c, loc}\). See Theorem 2.4 below for the involved notations and a rigorous statement of the above decomposition.

Next, we develop a transformation formula of local MAFs. Here we encounter the difficulty that there is no LeJan’s transformation rule available for semi-Dirichlet forms. Also we cannot replace a \(\gamma\)-co-excessive function \(g\) in the Revuz correspondence (cf. (2.2) below) by an arbitrary \(g \in B^+(E) \cap D(E)\), provided the corresponding smooth measure is not of finite energy integral. Borrowing some ideas of [9, Theorem 5.4] and [17] Theorem 5.3.2], but putting more extra efforts, we are able to build up an analog of LeJan’s formula (cf. Theorem 3.4 below). By
virtue of LeJan’s formula developed in Theorem 3.4 employing again the localization method developed in this paper, finally we obtain a transformation formula of local MAFs for semi-Dirichlet forms in Theorem 3.10.

The main results derived in this paper rely heavily on the potential theory of semi-Dirichlet forms. Although they are more or less parallel to those of symmetric Dirichlet forms, we cannot find explicit statements in literature. For the solidity of our results, also for the interests by their own, we checked and derived in detail some results on potential theory and positive continuous AFs (PCAFs in short) for semi-Dirichlet forms. These results are presented in Section 5 at the end of this paper as an Appendix. In particular, we would like to draw the attention of the readers to two new results, Theorem 5.3 and Lemma 5.9.

The rest of the paper is organized as follows. In Section 2, we derive Fukushima’s decomposition. Section 3 is devoted to the transformation formula. In Section 4, we apply our main results to some concrete examples. In these examples the usual Doob-Meyer decomposition and Itô’s formula for semi-martingales are not available. Nevertheless we can use our results to perform Fukushima’s decomposition and apply the transformation formula in the semi-Dirichlet forms setting. Section 5 is the Appendix consisting of some results on potential theory and PCAFs in the semi-Dirichlet forms setting.

2 Fukushima’s decomposition

We consider a quasi-regular semi-Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) on \(L^2(E; m)\), where \(E\) is a metrizable Lusin space (i.e., topologically isomorphic to a Borel subset of a complete separable metric space) and \(m\) is a \(\sigma\)-finite positive measure on its Borel \(\sigma\)-algebra \(\mathcal{B}(E)\). Denote by \((T_t)_{t \geq 0}\) and \((G_\alpha)_{\alpha \geq 0}\) (resp. \((\hat{T}_t)_{t \geq 0}\) and \((\hat{G}_\alpha)_{\alpha \geq 0}\)) the semigroup and resolvent (resp. co-semigroup and co-resolvent) associated with \((\mathcal{E}, D(\mathcal{E}))\). Let \(M = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta})\) be an \(m\)-tight special standard process which is properly associated with \((\mathcal{E}, D(\mathcal{E}))\) in the sense that \(P_tf\) is an \(E\)-quasi-continuous \(m\)-version of \(T_tf\) for all \(f \in \mathcal{B}_b(E) \cap L^2(E; m)\) and all \(t > 0\), where \((P_t)_{t \geq 0}\) denotes the semigroup associated with \(M\) (cf. [13, Theorem 3.8]).

Below for notations and terminologies related to quasi-regular semi-Dirichlet forms we refer to [13] and Section 5 of this paper.

Recall that a positive measure \(\mu\) on \((E, \mathcal{B}(E))\) is called smooth (w.r.t. \((\mathcal{E}, D(\mathcal{E}))\)), denoted by \(\mu \in S\), if \(\mu(N) = 0\) for each \(\mathcal{E}\)-exceptional set \(N \in \mathcal{B}(E)\) and there exists an \(\mathcal{E}\)-nest \(\{F_k\}\) of compact subsets of \(E\) such that

\[\mu(F_k) < \infty \text{ for all } k \in \mathbb{N}.\]

A family \((A_t)_{t \geq 0}\) of functions on \(\Omega\) is called an additive functional (AF in short) of \(M\) if:

(i) \(A_t\) is \(\mathcal{F}_t\)-measurable for all \(t \geq 0\).
(ii) There exists a defining set \( \Lambda \in \mathcal{F} \) and an exceptional set \( N \subset E \) which is \( \mathcal{E} \)-exceptional such that \( P_x[\Lambda] = 1 \) for all \( x \in E \setminus N \), \( \theta_t(\Lambda) \subset \Lambda \) for all \( t > 0 \) and for each \( \omega \in \Lambda \), \( t \to A_t(\omega) \) is right continuous on \((0, \infty)\) and has left limits on \((0, \zeta(\omega))\), \( A_0(\omega) = 0 \), \( |A_t(\omega)| < \infty \) for \( t < \zeta(\omega) \), \( A_t(\omega) = A_\zeta(\omega) \) for \( t \geq \zeta(\omega) \), and
\[
A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega), \quad \forall \, s, t \geq 0.
\] (2.1)

Two AFs \( A = (A_t)_{t \geq 0} \) and \( B = (B_t)_{t \geq 0} \) are said to be equivalent, denoted by \( A = B \), if they have a common defining set \( \Lambda \) and a common exceptional set \( N \) such that \( A_t(\omega) = B_t(\omega) \) for all \( \omega \in \Lambda \) and \( t \geq 0 \). An AF \( A = (A_t)_{t \geq 0} \) is called a continuous AF (CAF in short) if \( t \to A_t(\omega) \) is continuous on \((0, \infty)\). It is called a positive continuous AF (PCAF in short) if \( A_t(\omega) \geq 0 \) for all \( t \geq 0 \), \( \omega \in \Lambda \).

In the theory of Dirichlet forms, it is well known that there is a one to one correspondence between the family of all equivalent classes of PCAFs and the family \( S \) (cf. \([3]\)). In \([3]\), Fitzsimmons extended the smooth measure characterization of PCAFs from the Dirichlet forms setting to the semi-Dirichlet forms setting. Applying \([3]\) Proposition 4.12, following the arguments of \([5]\) Theorems 5.1.3 and 5.1.4 (with slight modifications by virtue of \([13, 14, 10]\) and \([1, \text{Theorem 3.4}]\)), we can also obtain a one to one correspondence between the family of all equivalent classes of PCAFs and the family \( S \). The correspondence, which is referred to as Revuz correspondence, is described in the following lemma.

**Lemma 2.1.** Let \( A \) be a PCAF. Then there exists a unique \( \mu \in S \), which is referred to as the Revuz measure of \( A \) and is denoted by \( \mu_A \), such that:

For any \( \gamma \)-co-excessive function \( g (\gamma \geq 0) \) in \( D(\mathcal{E}) \) and \( f \in \mathcal{B}^+(E) \),
\[
\lim_{t \downarrow 0} \frac{1}{t} E_{g \cdot m}((fA)_t) = \langle f, \mu, g \rangle. \tag{2.2}
\]

Conversely, let \( \mu \in S \), then there exists a unique (up to the equivalence) PCAF \( A \) such that \( \mu = \mu_A \).

See Theorem 5.8 in the Appendix at the end of this paper for more descriptions of the Revuz correspondence (2.2).

From now on we suppose that \( (\mathcal{E}, D(\mathcal{E})) \) is a quasi-regular local semi-Dirichlet form on \( L^2(E; m) \). Here “local” means that \( \mathcal{E}(u, v) = 0 \) for all \( u, v \in D(\mathcal{E}) \) with \( \text{supp}[u] \cap \text{supp}[v] = \emptyset \). Then, \( (\mathcal{E}, D(\mathcal{E})) \) is properly associated with a diffusion process \( \mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta}) \) (cf. \([10, \text{Theorem 4.5}]\)). Here “diffusion” means that \( \mathbf{M} \) is a right process satisfying
\[
P_x[t \to X_t \text{ is continuous on } [0, \zeta]] = 1 \text{ for all } x \in E.
\]

Throughout this paper, we fix a function \( \phi \in L^2(E; m) \) with \( 0 < \phi \leq 1 \) \( m \)-a.e. and set \( h = G_1\phi, \hat{h} = \hat{G}_1\phi \). Denote \( \tau_B := \inf\{t > 0 \mid X_t \notin B\} \) for \( B \subset E \).

Let \( V \) be a quasi-open subset of \( E \). We denote by \( X^V = (X^V_t)_{t \geq 0} \) the part process of \( X \) on \( V \) and denote by \( (\mathcal{E}^V, D(\mathcal{E})_V) \) the part form of \((\mathcal{E}, D(\mathcal{E}))\) on \( L^2(V; m) \). It is
known that $X^V$ is a diffusion process and $(\mathcal{E}^V, D(\mathcal{E})_V)$ is a quasi-regular local semi-
Dirichlet form (cf. [10]). Denote by $(T^V_t)_{t \geq 0}$, $(\dot{T}^V_t)_{t \geq 0}$, $(G^V_\alpha)_{\alpha \geq 0}$ and $(\dot{G}^V_\alpha)_{\alpha \geq 0}$ the
semigroup, co-semigroup, resolvent and co-resolvent associated with $(\mathcal{E}^V, D(\mathcal{E})_V)$,
respectively. One can easily check that $\hat{h}_{|V}$ is 1-co-excessive w.r.t. $(\mathcal{E}^V, D(\mathcal{E})_V)$.
Define $\bar{h}^V := \hat{h}_{|V} \wedge \dot{G}^V_1 \phi$. Then $\bar{h}^V \in D(\mathcal{E})_V$ and $\bar{h}^V$ is 1-co-excessive.

For an AF $A = (A_t)_{t \geq 0}$ of $X^V$, we define
\[ e^V(A) := \lim_{t \downarrow 0} \frac{1}{2t} E_{\hat{h}^V,m}(A_t^2) \] (2.3)
whenever the limit exists in $[0, \infty]$. Define
\[ \dot{\mathcal{M}}^V := \{ M \mid M \text{ is an AF of } X^V, E_x(M_t^2) < \infty, E_x(M_t) = 0 \]
for all $t \geq 0$ and $\mathcal{E}$-q.e. $x \in V, e^V(M) < \infty \}, \]
\[ \mathcal{N}^V_c := \{ N \mid N \text{ is a CAF of } X^V, E_x(|N|) < \infty \text{ for all } t \geq 0 \]
and $\mathcal{E}$-q.e. $x \in V, E^V(N) = 0 \}, \]
\[ \Theta := \{ \{ V_n \} \mid V_n \text{ is } \mathcal{E} \text{-quasi-open, } V_n \subset V_{n+1} \text{ }\mathcal{E} \text{-q.e.,} \]
$\forall n \in \mathbb{N}$, and $E = \cup_{n=1}^\infty V_n \text{ }\mathcal{E} \text{-q.e.} \} \]
and
\[ D(\mathcal{E})_{\text{loc}} := \{ u \mid \exists \{ V_n \} \in \Theta \text{ and } \{ u_n \} \subset D(\mathcal{E}) \]
such that $u = u_n$ m-a.e. on $V_n, \forall n \in \mathbb{N} \}. \]

For our purpose we shall employ the notion of local AFs introduced in [5] as follows.

**Definition 2.2.** (cf. [4, page 226]) A family $A = (A_t)_{t \geq 0}$ of functions on $\Omega$ is
called an local additive functional (local AF in short) of $\mathcal{M}$, if $A$ satisfies all the
requirements for an AF as stated in above (i) and (ii), except that the additivity
property (2.1) is required only for $s, t \geq 0$ with $t + s < \zeta(\omega)$.

Two local AFs $A^{(1)}$, $A^{(2)}$ are said to be equivalent if for $\mathcal{E}$-q.e. $x \in E$, it holds that
\[ P_x(A_t^{(1)} = A_t^{(2)}; t < \zeta) = P_x(t < \zeta), \forall t \geq 0. \]

Define
\[ \dot{\mathcal{M}}_{\text{loc}} := \{ M \mid M \text{ is a local AF of } \mathcal{M}, \exists \{ V_n \}, \{ E_n \} \in \Theta \text{ and } \{ M^n \mid M^n \in \dot{\mathcal{M}}^V \}
\text{ such that } E_n \subset V_n, M_{t \wedge E_n} = M^n_{t \wedge E_n}, t \geq 0, n \in \mathbb{N} \}
\]
and
\[ \mathcal{N}_{\text{c,loc}} := \{ N \mid N \text{ is a local AF of } \mathcal{M}, \exists \{ V_n \}, \{ E_n \} \in \Theta \text{ and } \{ N^n \mid N^n \in \mathcal{N}^V_c \}
\text{ such that } E_n \subset V_n, N_{t \wedge E_n} = N^n_{t \wedge E_n}, t \geq 0, n \in \mathbb{N} \}. \]

We use $\mathcal{M}_{\text{loc}}^{[0, \zeta]}$ to denote the family of all local martingales on $[0, \zeta]$ (cf. [6 §8.3]).

We put the following assumption:
Assumption 2.3. There exists \( \{V_n\} \in \Theta \) such that, for each \( n \in \mathbb{N} \), there exists a Dirichlet form \((\eta^{(n)}, D(\eta^{(n)}))\) on \( L^2(V_n; m) \) and a constant \( C_n > 1 \) such that \( D(\eta^{(n)}) = D(\mathcal{E})_{V_n} \) and for any \( u \in D(\mathcal{E})_{V_n} \),
\[
\frac{1}{C_n} \eta_1^{(n)}(u, u) \leq \mathcal{E}_1(u, u) \leq C_n \eta_1^{(n)}(u, u).
\]

Now we can state the main result of this section.

Theorem 2.4. Suppose that \((\mathcal{E}, D(\mathcal{E}))\) is a quasi-regular local semi-Dirichlet form on \( L^2(E; m) \) satisfying Assumption 2.3. Then, for any \( u \in D(\mathcal{E})_{\text{loc}} \), there exist \( M^{[u]} \in \mathcal{M}_{\text{loc}} \) and \( N^{[u]} \in \mathcal{N}_{e, \text{loc}} \) such that
\[
\tilde{u}(X_t) - \tilde{u}(X_0) = M^{[u]}_t + N^{[u]}_t, \quad t \geq 0, \quad \mathcal{P}_\pi \text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E. \tag{2.4}
\]
Moreover, \( M^{[u]} \in \mathcal{M}^{[0, \infty]}_{\text{loc}} \).

Decomposition (2.4) is unique up to the equivalence of local AFs.

Before proving Theorem 2.4 we present some lemmas.

We fix a \( \{V_n\} \in \Theta \) satisfying Assumption 2.3. Without loss of generality, we assume that \( \tilde{h} \) is bounded on each \( V_n \), otherwise we may replace \( V_n \) by \( V_n \cap \{\tilde{h} < n\} \). To simplify notations, we write
\[
\tilde{h}_n := \tilde{h}^{V_n}, \quad \text{and} \quad D(\mathcal{E})_{V_n, b} := B_b(E) \cap D(\mathcal{E})_{V_n}.
\]

By the definition (2.3), employing some potential theory developed in the Appendix of this paper (cf. Lemma 5.9 Theorem 5.8 and Theorem 5.3), following the argument of [5, Theorem 5.2.1], we can prove the following lemma.

Lemma 2.5. \( \tilde{M}^{V_n} \) is a real Hilbert space with inner product \( e^{V_n} \). Moreover, if \( \{M_t\} \subset \tilde{M}^{V_n} \) is \( e^{V_n} \)-Cauchy, then there exist a unique \( M \in \tilde{M}^{V_n} \) and a subsequence \( \{l_k\} \) such that \( \lim_{k \to \infty} e^{V_n}(M_{l_k} - M) = 0 \) and for \( \mathcal{E}\text{-q.e. } x \in V_n \),
\[
\mathcal{P}_\pi(\lim_{k \to \infty} M_{l_k}(t) = M(t) \text{ uniformly on each compact interval of } [0, \infty)) = 1.
\]

Next we give Fukushima’s decomposition for the part process \( X^{V_n} \).

Lemma 2.6. Let \( u \in D(\mathcal{E})_{V_n, b} \). Then there exist unique \( M^{n,[u]}_t \in \tilde{M}^{V_n} \) and \( N^{n,[u]}_t \in \mathcal{N}^{V_n}_{e} \) such that for \( \mathcal{E}\text{-q.e. } x \in V_n \),
\[
\tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n}) = M^{n,[u]}_t + N^{n,[u]}_t, \quad t \geq 0, \quad \mathcal{P}_\pi \text{-a.s.} \tag{2.5}
\]

Proof. Note that if an AF \( A \in \tilde{M}^{V_n} \) with \( e^{V_n}(A) = 0 \) then \( \mu^{(n)}_{<A>} (\tilde{h}_n) = 2e^{V_n}(A) = 0 \) by Theorem 5.8 in the Appendix and (2.3). Here \( \mu^{(n)}_{<A>} \) denotes the Revuz measure of \( A \) w.r.t. \( X^{V_n} \). Hence \( < A >= 0 \) since \( \tilde{h}_n > 0 \) \( \mathcal{E}\text{-q.e. on } V_n \). Therefore
\[ \mathcal{M}^n \cap \mathcal{N}^n = \{0\} \text{ and the proof of the uniqueness of decomposition (2.3) is complete.} \]

To obtain the existence of decomposition (2.5), we start with the special case that \( u = R^n f \) for some bounded Borel function \( f \in L^2(V_n; m) \), where \( (R^n t)_{t \geq 0} \) is the resolvent of \( X^n \). Set

\[
\begin{align*}
N^n_{t,u} &= \int_0^t (u(X^n_t) - f(X^n_s)) ds, \\
M^n_{t,u} &= u(X^n_t) - u(X^n_t) - N^n_{t,u}, \quad t \geq 0. 
\end{align*}
\]

(2.6)

Then \( N^n_{t,u} \in \mathcal{N}^n \) and \( M^n_{t,u} \in \mathcal{M}^n \). In fact,

\[
e^{V_n}(N^n_{t,u}) = \lim_{t \downarrow 0} \frac{1}{2t} E_{\bar{h},m}[(\int_0^t (u - f)(X^n_t) ds)^2] \\
\leq \lim_{t \downarrow 0} \frac{1}{2} E_{\bar{h},m}[(\int_0^t (u - f)^2(X^n_t) ds)] \\
= \lim_{t \downarrow 0} \frac{1}{2} \left[ \int_0^t \int_{V_n} \bar{h}_n T^n u (u - f)^2 dm ds \right] \\
\leq \|u - f\|_\infty \lim_{t \downarrow 0} \frac{1}{2} \left[ \int_0^t \int_{V_n} |u - f| T^n \bar{h}_n dm ds \right] \\
\leq \|u - f\|_\infty \lim_{t \downarrow 0} \frac{1}{2} \left[ \int_0^t \left( \int_{V_n} (u - f)^2 dm \right)^{1/2} \left( \int_{V_n} (\bar{h}_n)^2 dm \right)^{1/2} ds \right] \\
\leq \|u - f\|_\infty \left( \int_{V_n} (u - f)^2 dm \right)^{1/2} \left( \int_{V_n} \bar{h}_n^2 dm \right)^{1/2} \lim_{t \downarrow 0} \frac{t}{2} \\
= 0. \quad (2.7)
\]

By Assumption 2.3, \( u^2 \in D(\mathcal{E})_{V_n,b} \) and \( u\bar{h}_n \in D(\mathcal{E})_{V_n,b} \). Then, by (2.6), (2.7), [1] Theorem 3.4 and Assumption 2.3 we get

\[
e^{V_n}(M^n_{t,u}) = \lim_{t \downarrow 0} \frac{1}{2} E_{\bar{h},m}[(u(X^n_t) - u(X^n_t))^2] \\
= \lim_{t \downarrow 0} \left\{ \frac{1}{t} (u\bar{h}_n, u - T^n u) - \frac{1}{2t} (\bar{h}_n, u^2 - T^n u^2) \right\} \\
= \mathcal{E}^{V_n}(u, u\bar{h}_n) - \frac{1}{2} \mathcal{E}^{V_n}(u^2, \bar{h}_n) \\
\leq \mathcal{E}^{V_n}_1(u, u\bar{h}_n) \\
\leq K \mathcal{E}^{V_n}_1(u, u^{1/2})^{1/2} \mathcal{E}^{V_n}_1(u\bar{h}_n, u\bar{h}_n)^{1/2} \\
\leq K \mathcal{C}^{1/2}_n \mathcal{E}^{V_n}_1(u, u^{1/2})^{1/2} \bar{h}_n^{1/2} (u\bar{h}_n, u\bar{h}_n)^{1/2} \\
\leq K \mathcal{C}^{1/2}_n \mathcal{E}^{V_n}_1(u, u^{1/2})^{1/2} (\|u\|_\infty \bar{h}^{1/2}_n (\bar{h}_n, \bar{h}_n)^{1/2} + \|\bar{h}_n\|_\infty \bar{h}^{1/2}_n (u, u)^{1/2}) \\
\leq K C \mathcal{E}^{V_n}_1(u, u^{1/2})^{1/2} (\|u\|_\infty \mathcal{E}^{V_n}_1(\bar{h}_n, \bar{h}_n)^{1/2} + \|\bar{h}_n\|_\infty \mathcal{E}^{V_n}_1(u, u)^{1/2}), \quad (2.8)
\]

7
where $K$ is the continuity constant of $(\mathcal{E}, D(\mathcal{E}))$ (cf. [5.1] in the Appendix).

Next, take any bounded Borel function $u \in D(\mathcal{E})_{V_n}$. Define
\[
    u_t = lR_{t+1}^{V_n}u = R_1^{V_n}g_t, \quad g_t = l(u - lR_{t+1}^{V_n}u).
\]
By the uniqueness of decomposition (2.5) for $u$'s, we have $M^{n,[u]} - M^{n,[u_k]} = M^{n,[u-u_k]}$. Then, by (2.8), we get
\[
e^{V_n}(M^{n,[u]} - M^{n,[u_k]}) = e^{V_n}(M^{n,[u-u_k]}) \leq KC_n\mathcal{E}^{V_n}(u_t - u_k, u_t - u_k)^{1/2}(\|u_t - u_k\|_{\infty}\mathcal{E}_1^{V_n}(\bar{h}_n, \bar{h}_n)^{1/2} + \|\bar{h}_n\|_{\infty}\mathcal{E}_1^{V_n}(u_t - u_k, u_t - u_k)^{1/2}).
\]
Since $u_t \in D(\mathcal{E})_{V_n}$, bounded by $\|u\|_{\infty}$, and $\mathcal{E}^{V_n}$-convergent to $u$, we conclude that $\{M^{n,[u]}\}$ is an $e^{V_n}$-Cauchy sequence in the space $\hat{M}^{V_n}$. Define
\[
M^{n,[u]} = \lim_{l \to \infty} M^{n,[u]}(\hat{M}^{V_n}, e^{V_n}), \quad N^{n,[u]} = \tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n}) - M^{n,[u]}.
\]
Then $M^{n,[u]} \in \hat{M}^{V_n}$ by Lemma 2.5.

It only remains to show that $N^{n,[u]} \in N^{V_n,c}$. By Lemma 5.6 in the Appendix and Lemma 2.5, there exists a subsequence $\{l_k\}$ such that for $\mathcal{E}$-q.e. $x \in V_n$,
\[
P_x(N^{n,[u_k]}) \text{ converges to } N^{n,[u]} \text{ uniformly on each compact interval of } [0, \infty)) = 1.
\]
From this and (2.6), we know that $N^{n,[u]}$ is a CAF. On the other hand, by
\[
N_t^{n,[u]} = A_t^{n,[u-u]} - (M_t^{n,[u]} - M_t^{n,[u]}) + N_t^{n,[u]},
\]
we get
\[
e^{V_n}(N^{n,[u]}) \leq 3e^{V_n}(A^{n,[u-u]}) + 3e^{V_n}(M^{n,[u]} - M^{n,[u]}),
\]
which can be made arbitrarily small with large $l$ by (2.8). Therefore $e^{V_n}(N^{n,[u]}) = 0$ and $N^{n,[u]} \in N^{V_n,c}$.

We now fix a $u \in D(\mathcal{E})_{loc}$. Then there exist $\{V_n^1\} \in \Theta$ and $\{u_n\} \subset D(\mathcal{E})$ such that $u = u_n$ m.a.e. on $V_n^1$. By [13 Proposition 3.6], we may assume without loss of generality that each $u_n$ is $\mathcal{E}$-quasi-continuous. By [13 Proposition 2.16], there exists an $\mathcal{E}$-nest $\{F_n^2\}$ of compact subsets of $E$ such that $\{u_n\} \subset C\{F_n^2\}$. Denote by $V_n^2$ the finely interior of $F_n^2$ for $n \in \mathbb{N}$. Then $\{V_n^2\} \in \Theta$. Define $V_n' = V_n^1 \cap V_n^2$. Then $\{V_n'\} \in \Theta$ and each $u_n$ is bounded on $V_n'$. To simplify notation, we still use $V_n$ to denote $V_n \cap V_n'$ for $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we define $E_n = \{x \in E : \bar{h}_n(x) > \frac{1}{n}\}$, where $h_n := G_{V_n}^{1,\phi}$. Then $\{E_n\} \in \Theta$ satisfying $E_n^{\mathcal{E}} \subset E_{n+1}$ $\mathcal{E}$-q.e. and $E_n \subset V_n$ $\mathcal{E}$-q.e. for each $n \in \mathbb{N}$ (cf. [10 Lemma 3.8]). Here $E_n^{\mathcal{E}}$ denotes the $\mathcal{E}$-quasi-closure of $E_n$. Define $f_n = nh_n \vee 1$. Then $f_n = 1$ on $E_n$ and $f_n = 0$ on $V_n^c$. Since $f_n$ is a 1-excessive function of
(\(E^0_n, D(\mathcal{E})_V\)) and \(f_n \leq n\tilde{h}_n \in D(\mathcal{E})_V\), hence \(f_n \in D(\mathcal{E})_V\) by [13] Remark 3.4(ii). Denote by \(Q_n\) the bound of \(|u_n|\) on \(V_n\). Then \(u_n f_n = ((-Q_n) \vee u_n \wedge Q_n) f_n \in D(\mathcal{E})_V\).}

For \(n \in \mathbb{N}\), we denote by \(\{\mathcal{F}_t^n\}\) the minimal completed admissible filtration of \(X^{V_n}\). For \(n < l\), \(\mathcal{F}_t^n \subset \mathcal{F}_t^l \subset \mathcal{F}_t\) since \(E_n \subset V_n\); \(\tau_{E_n}\) is an \(\{\mathcal{F}_t^n\}\)-stopping time.

**Lemma 2.7.** For \(n < l\), we have \(M^{n,[u_n,f_n]}_{t \wedge \tau_{E_n}} = M^{l,[u_l,f_l]}_{t \wedge \tau_{E_n}}\) and \(N^{n,[u_n,f_n]}_{t \wedge \tau_{E_n}} = N^{l,[u_l,f_l]}_{t \wedge \tau_{E_n}}\), \(t \geq 0\), \(P_x\)-a.s. for \(\mathcal{E}\)-q.e. \(x \in V_n\).

**Proof.** Let \(n < l\). Since \(M^{n,[u_n,f_n]} \in \mathcal{M}^{V_n}\), \(M^{n,[u_n,f_n]}\) is an \(\{\mathcal{F}_t^n\}\)-martingale by the Markov property. Since \(\tau_{E_n}\) is an \(\{\mathcal{F}_t^n\}\)-stopping time, \(\{M^{n,[u_n,f_n]}_{t \wedge \tau_{E_n}}\}\) is an \(\{\mathcal{F}_t^n\}_{t \wedge \tau_{E_n}}\)-martingale. Denote \(\Upsilon^n_t = \sigma\{X^{V_n}_{s \wedge \tau_{E_n}} | 0 \leq s \leq t\}\). Then \(\{M^{n,[u_n,f_n]}_{t \wedge \tau_{E_n}}\}\) is a \(\{\Upsilon^n_t\}\)-martingale. Denote \(\Upsilon^n_{l,t} = \sigma\{X^{V_l}_{s \wedge \tau_{E_n}} | 0 \leq s \leq t\}\). Similarly, we can show that \(\{M^{l,[u_l,f_l]}_{t \wedge \tau_{E_n}}\}\) is a \(\{\Upsilon^n_t\}\)-martingale. By the assumption that \(\tilde{M}\) is a diffusion, the fact that \(f_n\) is quasi-continuous and \(f_n = 1\) on \(E_n\), we get \(f_n(X^{V_n}_{s \wedge \tau_{E_n}}) = 1\) if \(0 < s \wedge \tau_{E_n} < \zeta\). Hence \(X^{V_n}_{s \wedge \tau_{E_n}} \in V_n\), if \(0 < s \wedge \tau_{E_n} < \zeta\), since \(f_n = 0\) on \(V^c_n\). Therefore

\[
X^{V_l}_{s \wedge \tau_{E_n}} = X^{V_n}_{s \wedge \tau_{E_n}} = X^{V_n}_{s \wedge \tau_{E_n}}; \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in V_n,
\]

which implies that \(\{M^{l,[u_l,f_l]}_{t \wedge \tau_{E_n}}\}\) is a \(\{\Upsilon^n_t\}\)-martingale.

Let \(N \in \mathcal{N}^{V_l}_{c}\) for some \(j \in \mathbb{N}\). Then, for any \(T > 0\),

\[
\sum_{k=1}^{[rT]} E_h^{j,m}[((N_{k+1} - N_k)^2)] \leq \sum_{k=1}^{[rT]} e^T(E(N^2_{k+1}), e^{-h_j T} \tilde{h}_j)
\]

\[
\leq \sum_{k=1}^{[rT]} e^T(E(N^2_{k+1}), \bar{h}_j)
\]

\[
\leq rT e^T E_h^{j,m}(N^2_{k+1}) \to 0 \quad \text{as } r \to \infty.
\]

Hence

\[
\sum_{k=1}^{[rT]} (N_{k+1} - N_k)^2 \to 0, \quad r \to \infty, \quad \text{in } P_m,
\]

which implies that the quadratic variation process of \(N\) w.r.t. \(P_m\) is 0.

By [10] Proposition 3.3, \((\tilde{G}_1^V)^1_{V_c} = \tilde{G}^V_1 \phi - \tilde{G}^V_1 \phi\). Since \(V^c_n \supset V^c_l\), \((\tilde{G}_1^V)^1_{V_c} \geq (\tilde{G}_1^V)^1_{V_l}\). Then \(\tilde{G}^V_1 \phi \leq \tilde{G}^V_1 \phi\) and thus

\[
\bar{h}_n \leq \bar{h}_l.
\]

Therefore

\[
e^{V_n}(A) \leq e^{V_l}(A)
\]
for any AF $A = (A_t)_{t \geq 0}$ of $X^n$.

Note that $M^{t,[u_n,f_n]} = \widetilde{u_n f_n}(X^n) - \tilde{u_n f_n}(X^n) - M^{t,[u_n,f_n]} \in \Upsilon^n_\tau = \Upsilon^n_\tau \subset \mathcal{F}_t^{\tau_\tau_\tau}$. By the analog of [3], Lemma 5.5.2 in the semi-Dirichlet forms setting, \{\lambda^{t,[u_n,f_n]}\} is a CAF of $X^n$. By (2.11), $e^{V_n}(N^{t,[u_n,f_n]}_n) \leq e^{V_n}(N^{t,[u_n,f_n]}_n) = 0$. Hence $N^{t,[u_n,f_n]}_n \geq 0 \in \mathcal{N}_c^{V_n}$, which implies that the quadratic variation process of \{\lambda^{t,[u_n,f_n]}\} w.r.t. $P$ may be 0. Since for $\mathcal{E}$-q.e. $x \in V_n$, by (2.9),

$$M^{n,[u_n,f_n]} + N^{n,[u_n,f_n]} = \widetilde{u_n f_n}(X^n) - \widetilde{u_n f_n}(X^n) = M^{t,[u_n,f_n]} + N^{t,[u_n,f_n]}, \quad P_x-a.s.,$$

and both \{\lambda^{n,[u_n,f_n]}\} and \{\lambda^{t,[u_n,f_n]}\} are \{(\Upsilon^n)\}-martingale, hence $M^{n,[u_n,f_n]} = M^{t,[u_n,f_n]}$ and $N^{n,[u_n,f_n]} = N^{t,[u_n,f_n]}, \quad P_x-a.s.$ for m-a.e. $x \in V_n$. This implies that $E_m(\tau_{\tau_\tau_\tau} < M^{n,[u_n,f_n]} - M^{t,[u_n,f_n]} \geq 1) = 0$, \forall $t \geq 0$. Then, by Theorem (3.8)(i) in the Appendix, $M^{t,[u_n,f_n]} = M^{t,[u_n,f_n]}, \quad \forall t \geq 0, P_x-a.s.$ for $\mathcal{E}$-q.e. $x \in V_n$. Hence $N^{n,[u_n,f_n]} = N^{t,[u_n,f_n]}$, \forall $t \geq 0, P_x-a.s.$ for $\mathcal{E}$-q.e. $x \in V_n$. 

Since $u_n f_n = u t f_t = u_n f_t$ on $E_n$, similar to [11], Lemma 2.4], we can show that $M^{t,[u_n,f_n]} = M^{t,[u_t f_t]}$ when $t < \tau_{\tau_\tau_\tau}$. In $\mathcal{E}$-q.e. $x \in V_t$. If $\tau_{\tau_\tau_\tau} = \zeta$, then by the fact $u_n f_n(X^n) = u t f_t(X^n)$ and the continuity of $N^{t,[u_n,f_n]}$ and $N^{t,[u_t f_t]}$, one finds that $M^{t,[u_n,f_n]} = M^{t,[u_t f_t]}$. By the quasi-continuity of $u_n f_n$, $u_t f_t$ and the assumption that $\mathcal{M}$ is a diffusion, one finds that $M^{t,[u_n,f_n]}$ and $M^{t,[u_t f_t]}$ are continuous on $[0, \zeta)$, $P_x-a.s.$ for $\mathcal{E}$-q.e. $x \in V_t$. Hence, if $\tau_{\tau_\tau_\tau} = \zeta$ we have $M^{t,[u_n,f_n]} = M^{t,[u_t f_t]}$. Therefore $M^{n,[u_n,f_n]} = M^{t,[u_t f_t]}$ and $N^{n,[u_n,f_n]} = N^{t,[u_t f_t]}, \forall t \geq 0, P_x-a.s.$ for $\mathcal{E}$-q.e. $x \in V_n$. 

**Proof of Theorem 2.2.** We define $M^{t,[u]} := \lim_{t \to \infty} M^{t,[u_t f_t]}$ and $M^{t,[u]} := 0$ for $t > \zeta$ if there exists some $n$ such that $\tau_{\tau_\tau_\tau} = \zeta$ and $\zeta < \infty$; or $M^{t,[u]} := 0$ for $t \geq \zeta$, otherwise. By Lemma 2.7 $M^{t,[u]}$ is well defined. Define $M^{t,[u]} := M^{t,[u_t f_t]}$ for $t \geq 0$ and $n \in \mathbb{N}$. Then $M^{t,[u]} = M^{t,[u_t f_t]}$ for $\mathcal{E}$-q.e. $x \in V_{n+1}$ by Lemma 2.7. Since $t \in E^{\tau_{\tau_\tau_\tau}} \subset E_{n+1} \subset V_{n+1} \mathcal{E}$-q.e. implies that $P_x(\tau_{\tau_\tau_\tau} = 0) = 1$ for $x \not\in V_{n+1}$, $M^{t,[u]} = M^{t,[u_t f_t]}$ for $\mathcal{E}$-q.e. $x \in E$. Similar to (2.10) and (2.11), we can show that $e^{V_n}(M^n) \leq e^{V_n}(M^n)$ for each $n \in \mathbb{N}$. Then $M^n \in \mathcal{M}^{V_n}$ and hence $M^n \in \mathcal{M}^{V_n}$. Define $N^{t,[u]} = \tilde{u}(X_t) - \tilde{u}(X_0) - M^{t,[u]}$. Then, we have $N^{t,[u]} \in \mathcal{N}^{V_n}_{\text{loc}}$. Moreover $N^{t,[u]} \in \mathcal{N}^{V_n}_{\text{loc}}$. 

Next we show that $M^n$ is also an $\{F_t\}$-martingale, which implies that $M^{t,[u]} \in \mathcal{M}^{V_{n+1}}_{\text{loc}}$. In fact, by the fact that $\tau_{\tau_\tau_\tau}$ is an $\{F^{t+1}_t\}$-stopping time, we find that $I_{\tau_{\tau_\tau_\tau} \leq s}$ is $\mathcal{F}_{s+\tau_{\tau_\tau_\tau}}$ measurable for any $s \geq 0$. Let $0 \leq s_1 < \ldots < s_k \leq s < t$ and $g \in \mathcal{B}_d(\mathbb{R}^k)$. Then, we obtain by (2.9) and the fact $M^{n+1,[u_t f_t]} \in \mathcal{M}^{V_{n+1}}$ that
In this section, we adopt the setting of Section 2. Suppose that $(\mathcal{E},D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption 2.3. We fix a $\{V_n\} \in \Theta$ satisfying Assumption 2.3 and satisfying that $\tilde{h}$ is bounded on each $V_n$. Let $X^{V_n}$, $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$, $\tilde{h}_n$, etc. be the same as in Section 2. For $u \in D(\mathcal{E})_{V_n,b}$, we denote by $\mu_{<u,v>}$ the Revuz measure of $< M_n[u] >$ (cf. Lemma 2.16 and Theorem 5.8 in the Appendix). For $u, v \in D(\mathcal{E})_{V_n,b}$, we define
\[
\mu_{<u,v>}^{(n)} := \frac{1}{2}(\mu_{<u+v>}^{(n)} - \mu_{<u>}^{(n)} - \mu_{<v>}^{(n)}).
\]
Lemma 3.1. Let $u, v, f \in D(\mathcal{E})_{V_n,b}$. Then
\[
\int_{V_n} \tilde{f} d\mu^{(n)}_{<u,v>} = \mathcal{E}(u, vf) + \mathcal{E}(v, uf) - \mathcal{E}(uv, f). \tag{3.2}
\]

Proof. By the polarization identity, (3.2) holds for $\mathcal{E}(u, v, f) \in D(\mathcal{E})_{V_n,b}$ is equivalent to
\[
\int_{V_n} \tilde{f} d\mu^{(n)}_{<u,v>} = 2\mathcal{E}(u, uf) - \mathcal{E}(u^2, f), \quad \forall u, f \in D(\mathcal{E})_{V_n,b}. \tag{3.3}
\]

Below, we will prove (3.3). Without loss of generality, we assume that $f \geq 0$.

For $k, l \in \mathbb{N}$, we define $f_k := f \wedge (k\tilde{h}_n)$ and $f_{k,l} := l\tilde{G}^V_{k+1}f_k$. By [12, (3.9)], $f_k \in D(\mathcal{E})_{V_n,b}$ and
\[
\mathcal{E}_1(f_k, f_k) \leq \mathcal{E}_1(f, f_k). \tag{3.4}
\]

By [12, Proposition III.1.2], $f_{k,l}$ is $(l + 1)$-co-excessive. Since $\tilde{h}_n$ is 1-co-excessive,
\[
0 \leq f_{k,l} \leq k\tilde{h}_n. \tag{3.5}
\]

Hence $f_{k,l} \in D(\mathcal{E})_{V_n,b}$ by noting that $\tilde{h}_n$ is bounded.

Note that by (3.5)
\[
\lim_{t \downarrow 0} \frac{1}{t} E_{f_{k,l},m}[(N^m_t)^2] \leq \lim_{t \downarrow 0} \frac{1}{t} E_{\tilde{h}_n,m}[(N^m_t)^2] = 2k e^V_n (N^m_t) = 0. \tag{3.6}
\]

Then, by Theorem 5.8(i) in the Appendix and (3.6), we get
\[
\int_{V_n} \tilde{f}_{k,l} d\mu^{(n)}_{<u,v>} = \lim_{t \downarrow 0} \frac{1}{t} E_{f_{k,l},m}[(N^m_t)^2] \geq \lim_{t \downarrow 0} \frac{1}{t} E_{\tilde{h}_n,m}[(N^m_t)^2] = 0.
\]

By [12, Theorem I.2.13], for each $k \in \mathbb{N}$, $f_{k,l} \to f_k$ in $D(\mathcal{E})_{V_n}$ as $l \to \infty$. Furthermore, by Assumption 2.3 [12, Corollary I.4.15] and (3.5), we can show that $\sup_{l \geq 1} E(u f_{k,l}, u f_{k,l}) < \infty$ Thus, we obtain by [12, Lemma I.2.12] that $u f_{k,l} \to u f_k$ weakly in $D(\mathcal{E})_{V_n}$ as $l \to \infty$. Note that $\int_{V_n} \tilde{h}_n d\mu^{(n)}_{<u,v>} = 2e^V_n (M^m_t) < \infty$ for any $u \in D(\mathcal{E})_{V_n,b}$. Therefore, we obtain by (3.7), (3.5) and the dominated convergence theorem that
\[
\int_{V_n} \tilde{f}_{k} d\mu^{(n)}_{<u,v>} = 2\mathcal{E}(u, uf_k) - \mathcal{E}(u^2, f_k), \quad \forall u \in D(\mathcal{E})_{V_n,b}. \tag{3.8}
\]

By (3.4) and the weak sector condition, we get $\sup_{k \geq 1} \mathcal{E}_1(f_k, f_k) < \infty$. Furthermore, by Assumption 2.3 and [12, Corollary I.4.15], we can show that
\[ \sup_{k \geq 1} E(u f_k, u f_k) < \infty. \] Thus, we obtain by [12, Lemma I.2.12] that \( f_k \to f \) and \( u f_k \to u f \) weakly in \( D(\mathcal{E})_{\nu_n} \) as \( k \to \infty \). Therefore [3,3] holds by [3,8] and the monotone convergence theorem.

For \( u \in D(\mathcal{E})_{\nu_n,b} \), we denote by \( M^{n,[u],c} \) and \( M^{n,[u],k} \) the continuous and killing parts of \( M^{n,[u]} \), respectively; denote by \( \mu^{n,c}_{<u>} \) and \( \mu^{n,k}_{<u>} \) the Revuz measures of \( < M^{n,[u],c} > \) and \( < M^{n,[u],k} > \), respectively. Then \( M^{n,[u]} = M^{n,[u],c} + M^{n,[u],k} \) with

\[
M^{n,[u],k} = -\tilde{u}(X_{\zeta(n)-}^{V_n}) I(\zeta(n) \leq t) - (-\tilde{u}(X_{\zeta(n)-}^{V_n}) I(\zeta(n) \leq t))^p,
\]

where \( \zeta(n) \) denotes the life time of \( X^{V_n} \) and \( p \) denotes the dual projection, and

\[
\mu^{n,(n)}_{<u>} = \mu^{n,c}_{<u>} + \mu^{n,k}_{<u>}. \tag{3.9}
\]

Let \( (N^{(n)}(x, dy), H^{(n)}) \) be a Lévy system of \( X^{V_n} \) and \( \nu^{(n)} \) be the Revuz measure of \( H^{(n)} \). Define \( K^{(n)}(dx) := N^{(n)}(x, \Delta) \nu^{(n)}(dx) \). Similar to [5] (5.3.8) and (5.3.10), we can show that

\[
<M^{n,[u],k} >_t = (\tilde{u}^2(X_{\zeta(n)-}^{V_n}) I(\zeta(n) \leq t))^p = \int_0^t \tilde{u}^2(X_{s}^{V_n}) N^{(n)}(X_{s}^{V_n}, \Delta) dH^{(n)}_s \tag{3.10}
\]

and

\[
\mu^{n,k}_{<u>}(dx) = \tilde{u}^2(x) K^{(n)}(dx). \tag{3.11}
\]

For \( u, v \in D(\mathcal{E})_{\nu_n,b} \), we define

\[
\mu^{n,c}_{<u,v>} := \frac{1}{2}(\mu^{n,c}_{<u+v>} - \mu^{n,c}_{<u>} - \mu^{n,c}_{<v>}), \quad \mu^{n,k}_{<u,v>} := \frac{1}{2}(\mu^{n,k}_{<u+v>} - \mu^{n,k}_{<u>} - \mu^{n,k}_{<v>}). \tag{3.12}
\]

**Theorem 3.2.** Let \( u, v, w \in D(\mathcal{E})_{\nu_n,b} \). Then

\[
d\mu^{n,c}_{<u,v,w>} = -\tilde{u}d\mu^{n,c}_{<v,u,w>} + \tilde{v}d\mu^{n,c}_{<u,u,w>}. \tag{3.13}
\]

**Proof.** By quasi-homeomorphism and the polarization identity, [3,13] holds for \( u, v, w \in D(\mathcal{E})_{\nu_n,b} \) is equivalent to

\[
\int_{V_n} \tilde{f} d\mu^{n,c}_{<u^2+w>} = 2 \int_{V_n} \tilde{f} \tilde{u} d\mu^{n,c}_{<u,u,w>}, \quad \forall f, u, w \in D(\mathcal{E})_{\nu_n,b}. \tag{3.14}
\]

By (3.1) and (3.9)–(3.12), we find that (3.14) is equivalent to

\[
\int_{V_n} \tilde{f} d\mu^{(n)}_{<u^2,w>} + \int_{V_n} \tilde{f} \tilde{u}^2 w dK^{(n)} = 2 \int_{V_n} \tilde{f} \tilde{u} d\mu^{(n)}_{<u,u,w>}, \quad \forall f, u, w \in D(\mathcal{E})_{\nu_n,b}. \tag{3.15}
\]

For \( k \in \mathbb{N} \), we define \( v_k := kR_{V_n}^{V_n} u \). Then \( v_k \to u \) in \( D(\mathcal{E})_{\nu_n} \) as \( k \to \infty \). By Assumption 2.3 and [12, Corollary I.4.15], we can show that \( \sup_{k \geq 1} E(v_k w, v_k w) < \infty \). Then, by [12, Lemma I.2.12], there exists a subsequence \( \{(v_{k_l})\}_{l \in \mathbb{N}} \) of \( \{v_k\}_{k \in \mathbb{N}} \).

13
such that $u_k w \to uw$ in $D(\mathcal{E})_{V_n}$ as $k \to \infty$, where $u_k := \frac{1}{k} \sum_{i=1}^{k} v_{k_i}$. Note that $u_k \to u$ in $D(\mathcal{E})_{V_n}$ as $k \to \infty$ and $\|u_k\|_{\infty} \leq \|u\|_{\infty}$ for $k \in \mathbb{N}$. Moreover, $\|L^nu_k\|_{\infty} < \infty$ for $k \in \mathbb{N}$, where $L^nu$ is the generator of $X^V_n$.

By Assumption 2.3 and [12, Corollary I.4.15], we can show that $\sup_{k \geq 1} [\mathcal{E}(u_k f w, u_k f w) + \mathcal{E}(u_k^2 f, u_k^2 f) + \mathcal{E}(u_k f, u_k f)] < \infty$. Then, we obtain by [12, Lemma I.2.12] that $u_k f w \to uf w$, $u_k^2 f \to u^2 f$ and $u_k f \to uf$ weakly in $D(\mathcal{E})_{V_n}$ as $k \to \infty$. Hence by (3.16) and the fact $\sup_{k \geq 1} [\mathcal{E}(u_k f w, u_k f w) + \mathcal{E}(u_k f, u_k f)] < \infty$ we get

$$
\int_{V_n} \tilde{f} d\mu^{(n)}_{<u,w>} = \mathcal{E}(u, uf w) + \mathcal{E}(w, u^2 f) - \mathcal{E}(uw, uf) \\
\quad = \lim_{k \to \infty} \mathcal{E}(u, u_k f w) + \mathcal{E}(w, u_k^2 f) - \mathcal{E}(uw, u_k f) \\
\quad = \lim_{k \to \infty} \mathcal{E}(u_k f w) + \mathcal{E}(w, u_k^2 f) - \mathcal{E}(u_k w, u_k f) \\
\quad = \lim_{k \to \infty} \int_{V_n} \tilde{f} d\mu^{(n)}_{<u_k,w>}.
$$

By Assumption 2.3 and [12, Corollary I.4.15], we can show that $\sup_{k \geq 1} [\mathcal{E}(u_k^2 f, u_k^2 f) + \mathcal{E}(u_k^2 f, u_k^2 w)] < \infty$. Then, we obtain by [12, Lemma I.2.12] that $u_k^2 \to u^2$, $u_k^2 f \to u^2 f$ and $u_k^2 w \to u^2 w$ weakly in $D(\mathcal{E})_{V_n}$ as $k \to \infty$. Hence by (3.16) we get

$$
\int_{V_n} \tilde{f} d\mu^{(n)}_{<u^2,w>} = \mathcal{E}(u^2, f w) + \mathcal{E}(w, u^2 f) - \mathcal{E}(u^2 w, f) \\
\quad = \lim_{k \to \infty} \mathcal{E}(u_k^2, f w) + \mathcal{E}(w, u_k^2 f) - \mathcal{E}(u_k^2 w, f) \\
\quad = \lim_{k \to \infty} \int_{V_n} \tilde{f} d\mu^{(n)}_{<u_k^2,w>}.
$$

By (3.16), (3.17) and the dominated convergence theorem, to prove (3.15), we may assume without loss of generality that $u$ is equal to some $u_k$. Moreover, we assume without loss of generality that $f \geq 0$.

For $k, l \in \mathbb{N}$, we define $f_k := f \wedge (k\varrho_n)$ and $f_{k,l} := l\tilde{G}_{l+1}f_k$. By [14, (3.9)], $f_k \in D(\mathcal{E})_{V_n,b}$; by [12, Proposition III.1.2], $f_{k,l}$ is $(l + 1)$-co-excessive. Since $\varrho_n$ is 1-co-excessive,

$$
0 \leq f_{k,l} \leq k\varrho_n.
$$

Hence $f_{k,l} \in D(\mathcal{E})_{V_n,b}$ by noting that $\varrho_n$ is bounded. By the dominated convergence theorem, to prove that (3.15) holds for any $f \in D(\mathcal{E})_{V_n,b}$, it suffices to prove that (3.15) holds for any $f_{k,l}$.

Below, we will prove (3.15) for $u = u_k$ and $f = f_{k,l}$.

Note that for any $g \in D(\mathcal{E})_{V_n,b}$,

$$
\lim_{t \to 0} \frac{1}{t} E_{f_{k,l},m}[(N^{|g|}_t)^2] \leq k \lim_{t \to 0} \frac{1}{t} E_{\varrho_n,m}[(N^{|g|}_t)^2] = 2k\varrho_n(N^{|g|}) = 0.
$$

(3.18)
By Theorem 5.8(i) in the Appendix and (3.18), we get
\[
\int_{V_n} \overline{f_{k,t}} d\mu^{(n)}_{<u_{k,w}>} = \lim_{t \uparrow 0} \frac{1}{t} E_{f_{k,t}m} [< M^{n,[u_k]}, M^{n,w} > t]
\]
\[
= \lim_{t \uparrow 0} \frac{1}{t} E_{f_{k,t}m} [\overline{u_k}^2(X_{V_n}^t) - \overline{u_k}^2(X_{V_0}^t)](\overline{w}(X_{V_n}^t) - \overline{w}(X_{V_0}^t))
\]
\[
= \lim_{t \uparrow 0} \frac{2}{t} E_{(f_{k,t}u_k)m} [\overline{u_k}(X_{V_n}^t) - \overline{u_k}(X_{V_0}^t)](\overline{w}(X_{V_n}^t) - \overline{w}(X_{V_0}^t))
\]
\[
+ \lim_{t \uparrow 0} \frac{1}{t} E_{f_{k,t}m} [\overline{u_k}(X_{V_n}^t) - \overline{u_k}(X_{V_0}^t)]^2(\overline{w}(X_{V_n}^t) - \overline{w}(X_{V_0}^t))
\]
\[
:= \lim_{t \uparrow 0} [I(t) + II(t)].
\] (3.19)

By (3.18), Theorem 5.8(iii) in the Appendix and (3.2), we get
\[
\lim_{t \uparrow 0} I(t) = \lim_{t \uparrow 0} \frac{2}{t} E_{(f_{k,t}u_k)m} [< M^{n,[u_k]}, M^{n,w} > t]
\]
\[
= \lim_{t \uparrow 0} \frac{2}{t} \int_{0}^{t} \overline{d\mu^{(n)}_{<u_{k,w}>}} > ds
\]
\[
= \lim_{t \uparrow 0} \frac{2}{t} \int_{0}^{t} [\mathcal{E}(u_k, w\overline{T}_{V_n}^{s}(f_{k,t}u_k)) + \mathcal{E}(w, u_k\overline{T}_{V_n}^{s}(f_{k,t}u_k))]
\]
\[
- \mathcal{E}(u_k w, w\overline{T}_{V_n}^{s}(f_{k,t}u_k))] ds.
\] (3.20)

By [1] Theorem 3.4, \(\overline{T}_{V_n}^{s}(f_{k,t}u_k) \rightarrow f_{k,t}u_k\) in \(D(\mathcal{E})_{V_n}\) as \(s \rightarrow 0\). Furthermore, by Assumption 2.3 [12] Corollary I.4.15] and the fact that \(|e^{-s\overline{T}_{V_n}^{s}(f_{k,t}u_k)}| \leq k\|u_k\|_{\infty} V_n, s > 0\), we can show that \(\sup_{s > 0} \mathcal{E}(w\overline{T}_{V_n}^{s}(f_{k,t}u_k), w\overline{T}_{V_n}^{s}(f_{k,t}u_k)) < \infty\). Thus, we obtain by [12] Lemma I.2.12] that \(w\overline{T}_{V_n}^{s}(f_{k,t}u_k) \rightarrow w f_{k,t}u_k\) weakly in \(D(\mathcal{E})_{V_n}\) as \(s \rightarrow 0\). Similarly, we get \(u_k\overline{T}_{V_n}^{s}(f_{k,t}u_k) \rightarrow u_k f_{k,t}u\) weakly in \(D(\mathcal{E})_{V_n}\) as \(s \rightarrow 0\). Therefore, by (3.20) and (3.2), we get
\[
\lim_{t \uparrow 0} I(t) = 2 \int_{V_n} \overline{f_{k,t}u_k} d\mu^{(n)}_{<u_{k,w}>}.
\] (3.21)

Note that
\[
II(t) = \frac{1}{t} E_{f_{k,t}m} [(M^{n,[u_k],c} t^{M^{n,w},c})^2 M^{n,[w],k} t^{M^{n,w},k}]
\]
\[
:= III(t) + IV(t).
\] (3.22)

By Burkholder-Davis-Gundy inequality, we get
\[
\lim_{t \uparrow 0} III(t) \leq (\lim_{t \uparrow 0} \frac{1}{t} E_{f_{k,t}m} [(M^{n,[u_k],c} t^{M^{n,w},c})^4])^{1/2} (\lim_{t \uparrow 0} \frac{1}{t} E_{f_{k,t}m} [< M^{n,[u_k],c} > t])^{1/2}
\]
\[
\leq C(2k E^{V_n}(M^{n,[w]}))^{1/2} (\lim_{t \uparrow 0} \frac{1}{t} E_{f_{k,t}m} [< M^{n,[u_k],c} > t])^{1/2}
\] (3.23)

15
for some constant $C > 0$, which is independent of $t$.

By Theorem 5.8(i) in the Appendix, for any $\delta > 0$, we get

$$\lim_{t \downarrow 0} \frac{1}{t} E_{f_k,m}[<M^{n,[u_k],c}_t >]$$

$$= \lim_{t \downarrow 0} \frac{2}{t} E_{f_k,m}\left[ \int_0^t <M^{n,[u_k],c}_{t-s}>_s \circ \theta_s d <M^{n,[u_k],c}_s> \right]$$

$$= \lim_{t \downarrow 0} \frac{2}{t} E_{f_k,m}\left[ \int_0^t E_{V_n}[<M^{n,[u_k],c}_{t-s}>_s] d <M^{n,[u_k],c}_s> \right]$$

$$\leq 2 < E[<M^{n,[u_k]}>_\delta] \cdot \mu_{\leq u_k}^{(n)}, \tilde{f}_{k,t} >. \quad (3.24)$$

Note that by our choice of $u_k$, there exists a constant $C_k > 0$ such that $E_{x}[<M^{n,[u_k]}>_\delta] = E_{x}[(M^{n,[u_k]}_t)^2] = E_{x}[(\tilde{u}_k(X^V_t)) - \tilde{u}_k(X^V_0) - \int_0^\delta L_{V_n} u_k(X^V_s)ds]^2 \leq C_k$ for any $\delta \leq 1$ and $\mathcal{E}$-q.e. $x \in V_n$. Letting $\delta \to 0$, by (3.24), the dominated convergence theorem and (3.23), we get

$$\lim_{t \downarrow 0} III(t) = 0. \quad (3.25)$$

By [17], Theorem II.33, integration by parts (page 68) and Theorem II.28, we get

$$IV(t) = \frac{1}{t} E_{f_k,m}\left[ I_{\zeta(n) \leq t} \{-((\tilde{u}_k^2 \tilde{w})(X^V_{\zeta(n)})) \right.$$

$$+ 2((\tilde{u}_k \tilde{w})(X^V_{\zeta(n)})) I_{\zeta(n) \leq t} \right]$$

$$\left. + 2 ((\tilde{u}_k^2)(X^V_{\zeta(n)})) I_{\zeta(n) \leq t}) \right]$$

$$\leq \frac{1}{t} E_{f_k,m}\left[ -((\tilde{u}_k^2 \tilde{w})(X^V_{\zeta(n)})) I_{\zeta(n) \leq t}) \right]$$

$$+ \frac{2}{t} E_{f_k,m}^{1/2} \left[ (((\tilde{u}_k \tilde{w})(X^V_{\zeta(n)})) I_{\zeta(n) \leq t}) \right]^{1/2} E_{f_k,m}^{1/2} [<M^{n,[u_k],c}_t >]$$

$$+ E_{f_k,m}^{1/2} \left[ ((\tilde{u}_k^2)(X^V_{\zeta(n)})) I_{\zeta(n) \leq t}) \right]^{1/2} E_{f_k,m}^{1/2} [<M^{n,[u_k],c}_t >]. \quad (3.26)$$

By Theorem 5.8(i) in the Appendix, (3.10)-(3.12), we obtain that for $\psi_1, \psi_2 \in D(\mathcal{E})_{V_n,b}$,

$$\lim_{t \downarrow 0} \frac{1}{t} E_{f_k,m}\left[ (((\tilde{u}_k \tilde{w})(X^V_{\zeta(n)})) I_{\zeta(n) \leq t}) \right] = \int_{V_n} \tilde{f}_{k,t} \mu_{\leq \psi_1,\psi_2}^{(n)}$$

$$= \int_{V_n} \tilde{f}_{k,t} \tilde{\psi}_1 \tilde{\psi}_2 dK^{(n)} \quad (3.27)$$
and
\[ \lim_{t \downarrow 0} \frac{1}{t} E_{f_{k,l},m}[<(M^{n,[\psi]}_1)_k>^2] = \int_{V_n} \tilde{f}_{k,l} d\mu^{n,k}_{<\psi>_1}. \] (3.28)

Furthermore, for any \( \delta > 0, \)
\[ \lim_{t \downarrow 0} \frac{1}{t} E_{f_{k,l},m}[\{(\tilde{\psi}_1 \tilde{\psi}_2)(X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq \delta)^p\}] \]
\[ = \lim_{t \downarrow 0} \frac{2}{t} E_{f_{k,l},m} \left[ \int_{0}^{t} (\tilde{\psi}_1 \tilde{\psi}_2)(X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq (t-s))^p \circ \theta_s d((\tilde{\psi}_1 \tilde{\psi}_2)(X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq s)^p] \right. \]
\[ = \lim_{t \downarrow 0} \frac{2}{t} E_{f_{k,l},m} \left[ \int_{0}^{t} E_{X^{V_n}}[(\tilde{\psi}_1 \tilde{\psi}_2)(X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq (t-s))^p] d((\tilde{\psi}_1 \tilde{\psi}_2)(X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq s)^p] \right] \]
\[ \leq \Delta E[I\tilde{\psi}_1 \tilde{\psi}_2](X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq \delta)^p] \cdot \mu^{n,k}_{<\psi>_1}, \tilde{f}_{k,l} > \]
\[ = \Delta E[I\tilde{\psi}_1 \tilde{\psi}_2](X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq \delta)^p] \cdot \mu^{n,k}_{<\psi>_1}, \tilde{f}_{k,l} > . \] (3.29)

Letting \( \delta \to 0, \) by (3.29) and the dominated convergence theorem, we get
\[ \lim_{t \downarrow 0} \frac{1}{t} E_{f_{k,l},m}[\{(\tilde{\psi}_1 \tilde{\psi}_2)(X^{V_n}_{\zeta(n)} I_{\zeta(n)} \leq \delta)^p\}] = 0. \] (3.30)

By (3.26)–(3.28) and (3.30), we get
\[ \lim_{t \downarrow 0} IV(t) = - \int_{V_n} \tilde{f}_{k,l} \tilde{u}_k^2 \tilde{w} dK^{(n)}. \] (3.31)

Therefore, the proof is completed by (3.19), (3.21), (3.22), (3.25) and (3.31).

**Remark 3.3.** When deriving formula (3.13) for non-symmetric Markov processes, we cannot apply Theorem 3.8(vi) or (vii) in the Appendix of this paper to smooth measures which are not of finite energy integral. To overcome that difficulty and obtain (3.13) in the semi-Dirichlet forms setting, we have to make some extra efforts as shown in the above proof. The proof uses some ideas of [7, Theorem 5.4] and [10, Theorem 5.2].

**Theorem 3.4.** Let \( m \in \mathbb{N}, \Phi \in C^1(\mathbb{R}^m) \) with \( \Phi(0) = 0, \) and \( u = (u_1, u_2, \ldots, u_m) \) with \( u_i \in D(\mathcal{E})_{V_n,b}, 1 \leq i \leq m. \) Then \( \Phi(u) \in D(\mathcal{E})_{V_n,b} \) and for any \( v \in D(\mathcal{E})_{V_n,b}, \)
\[ d\mu^{n,c}_{\Phi(u),v} = \sum_{i=1}^{m} \Phi_{x_i}(u) d\mu^{n,c}_{<u_i,v>}. \] (3.32)

**Proof.** \( \Phi(u) \in D(\mathcal{E})_{V_n,b} \) is a direct consequence of Assumption 2.3 and the corresponding property of Dirichlet form. Below we only prove (3.32). Let \( v \in D(\mathcal{E})_{V_n,b}. \) Then (3.32) is equivalent to
\[ \int_{V_n} \tilde{f} - h_n d\mu^{n,c}_{\Phi(u),v} = \sum_{i=1}^{m} \int_{V_n} \tilde{f} h_n \Phi_{x_i}(u) d\mu^{n,c}_{<u_i,v>}, \forall f \in D(\mathcal{E})_{V_n,b}. \] (3.33)
Let \( \mathcal{A} \) be the family of all \( \Phi \in C^1(\mathbb{R}^m) \) satisfying (3.32). If \( \Phi, \Psi \in \mathcal{A} \), then \( \Phi \Psi \in \mathcal{A} \) by Theorem 3.2. Hence \( \mathcal{A} \) contains all polynomials vanishing at the origin. Let \( O \) be a finite cube containing the range of \( u(x) = (u_1(x), \ldots, u_m(x)) \). We take a sequence \( \{\Phi^k\} \) of polynomials vanishing at the origin such that \( \Phi^k \to \Phi \), \( \Phi^k \to \Phi_{x_i}, 1 \leq i \leq m \), uniformly on \( O \). By Assumption 2.3 and [5, (3.2.27)], \( \Phi^k(u) \) converges to \( \Phi(u) \) w.r.t. \( \mathcal{E}_1^{V_n} \) as \( k \to \infty \). Then, by (2.8), we get
\[
\begin{align*}
\left| \int_{V_n} \tilde{f} \bar{h}_n d\mu_{<\Phi(u),v>} - \int_{V_n} \tilde{f} \bar{h}_n d\mu_{<\Phi^k(u),v>} \right| & \leq \|f\|_\infty \int_{V_n} \tilde{h}_n d\mu_{<\Phi^k(u),v>} |1/2| \int_{V_n} \tilde{h}_n d\mu_{<\Phi(u),v>} |1/2 \\
& \leq \|f\|_\infty \int_{V_n} \tilde{h}_n d\mu_{<\Phi^k(u),v>} |1/2| \int_{V_n} \tilde{h}_n d\mu_{<\Phi(u),v>} |1/2 \\
& = 2 \|f\|_\infty \|e_{V_n}(M_n, [\Phi^k(u) - \Phi(u)])^{1/2} \| e_{V_n}(M_n, [\Phi^k(u)])^{1/2} \\
& \leq 2 \|f\|_\infty \|e_{V_n}(M_n, [\Phi^k(u) - \Phi(u)])^{1/2} \| [K C_n e_{V_n}(\Phi(u) - \Phi^k(u), \Phi(u) - \Phi^k(u))]^{1/2} \\
& = [\|\Phi(u) - \Phi^k(u)\|_{\mathcal{E}_1^{V_n}(\tilde{h}_n, \tilde{h}_n)]^{1/2} \\
& \leq \|\bar{h}_n\|_{\mathcal{E}_1^{V_n}(\Phi(u) - \Phi^k(u), \Phi(u) - \Phi^k(u))]^{1/2}]^{1/2}.
\end{align*}
\]

Hence
\[
\int_{V_n} \tilde{f} \bar{h}_n d\mu_{<\Phi(u),v>} = \lim_{k \to \infty} \int_{V_n} \tilde{f} \bar{h}_n d\mu_{<\Phi^k(u),v>}.
\]

It is easy to see that
\[
\int_{V_n} \tilde{f} \bar{h}_n \Phi_{x_i}(\tilde{u}) d\mu_{<\Phi_{x_i},v>} = \lim_{k \to \infty} \int_{V_n} \tilde{f} \bar{h}_n \Phi^k_{x_i}(\tilde{u}) d\mu_{<\Phi_{x_i},v>}, \quad 1 \leq i \leq m.
\]

Therefore (3.33) holds. \( \square \)

For \( M, L \in \mathcal{M}^{V_n} \), there exists a unique CAF \( < M, L > \) of bounded variation such that
\[
E_x(M_t L_t) = E_x(< M, L >), \quad t \geq 0, \quad \mathcal{E}-q.e. \ x \in V_n.
\]

Denote by \( \mu_{<M,L>}^{(n)} \) the Revuz measure of \( < M, L > \). Then, similar to [4, Lemma 5.6.1], we can prove the following lemma.

**Lemma 3.5.** If \( f \in L^2(V_n; \mu_{<M>}^{(n)}) \) and \( g \in L^2(V_n; \mu_{<L>}^{(n)}) \), then \( fg \) is integrable w.r.t. \( |\mu_{<M,L>}^{(n)}| \) and
\[
\left( \int_{V_n} |fg| d|\mu_{<M,L>}^{(n)}| \right)^2 \leq \int_{V_n} f^2 d\mu_{<M>}^{(n)} \int_{V_n} g^2 d\mu_{<L>}^{(n)}.
\]

**Lemma 3.6.** Let \( M \in \mathcal{M}^{V_n} \) and \( f \in L^2(V_n; \mu_{<M>}^{(n)}) \). Then there exists a unique element \( f \cdot M \in \mathcal{M}^{V_n} \) such that
\[
e_{V_n}(f \cdot M, L) = \frac{1}{2} \int_{V_n} \tilde{f} \bar{h}_n d\mu_{<M,L>}^{(n)}, \quad \forall L \in \mathcal{M}^{V_n}.
\]
The mapping $f \to f \cdot M$ is continuous and linear from $L^2(V_n; \mu^{(n)}_{<M>})$ into the Hilbert space $(\mathcal{M}^{V_n}; e^{V_n})$.

**Proof.** Let $L \in \mathcal{M}^{V_n}$. Then, by Lemma 3.5, we get
\[
\left| \frac{1}{2} \int_{V_n} \tilde{h}_n d\mu^{(n)}_{<f,M,L>} \right| \leq \frac{1}{\sqrt{2}} \sqrt{\left\langle f^2 \tilde{h}_n d\mu^{(n)}_{<M,L>} \right\rangle^{1/2} \left\langle \tilde{h}_n d\mu^{(n)}_{<L,M>} \right\rangle^{1/2}} \\
\leq \frac{\|\tilde{h}_n\|_\infty}{\sqrt{2}} \| f \|_{L^2(V_n; \mu^{(n)}_{<M>})} \sqrt{e^{V_n}(L)}.
\]
Therefore, the proof is completed by Lemma 2.5. \qed

Similar to [5, Lemma 5.6.2, Corollary 5.6.1 and Lemma 5.6.3], we can prove the following two lemmas.

**Lemma 3.7.** Let $M, L \in \mathcal{M}^{V_n}$. Then

(i) $d\mu^{(n)}_{<f,M,L>} = f d\mu^{(n)}_{<M,L>}$ for $f \in L^2(V_n; \mu^{(n)}_{<M>})$.

(ii) $g \cdot (f \cdot M) = (gf) \cdot M$ for $f \in L^2(V_n; \mu^{(n)}_{<M>})$ and $g \in L^2(V_n; f^2 d\mu^{(n)}_{<M>})$.

(iii) $e^{V_n}(f \cdot M, g \cdot L) = \frac{1}{2} \int f g \tilde{h}_n d\mu^{(n)}_{<M,L>}$ for $f \in L^2(V_n; \mu^{(n)}_{<M>})$ and $g \in L^2(V_n; \mu^{(n)}_{<L>})$.

**Lemma 3.8.** The family $\{\tilde{f} \cdot M^n \mid f \in D(\mathcal{E})_{V_n,b}\}$ is dense in $(\mathcal{M}^{V_n}, e^{V_n})$.

**Theorem 3.9.** Let $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$ with $\Phi(0) = 0$, and $u = (u_1, u_2, \ldots, u_m)$ with $u_i \in D(\mathcal{E})_{V_n,b}$, $1 \leq i \leq m$. Then
\[
M_{[\Phi(u)],c} = \sum_{i=1}^m \Phi_i(u) \cdot M_{[u_i],c}, \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in V_n. \tag{3.34}
\]

**Proof.** Let $v \in D(\mathcal{E})_{V_n,b}$ and $f, g \in D(\mathcal{E})_{V_n,b}$. Then, by Lemma 3.7(iii) and Theorem 3.4, we get
\[
e^{V_n}(\tilde{f} \cdot M^{n,[\Phi(u)],c}, \tilde{g} \cdot M^{n,[v]}) = \frac{1}{2} \int_{V_n} \tilde{f} \tilde{g} \tilde{h}_n d\mu^{(n)}_{<M^{n,[\Phi(u)],c},M^{n,[v]>}} = \frac{1}{2} \int_{V_n} \tilde{f} \tilde{g} \tilde{h}_n d\mu^{n,c}_{<\Phi(u),v>} = \frac{1}{2} \sum_{i=1}^m \int_{V_n} \tilde{f} \tilde{g} \tilde{h}_n \Phi_i(u) d\mu^{n,c}_{<u_i,v>} = \frac{1}{2} \sum_{i=1}^m \int_{V_n} \tilde{f} \tilde{g} \tilde{h}_n \Phi_i(u) d\mu^{(n)}_{<M^{n,[u_i],c},M^{n,[v]>}} = e^{V_n}(\sum_{i=1}^m (\tilde{f} \Phi_i(u)) \cdot M^{n,[u_i],c}, \tilde{g} \cdot M^{n,[v]}).
By Lemma 3.8, we get
\[ \tilde{f} \cdot M^{n,[\Phi(u)],c} = \sum_{i=1}^{m} (\tilde{f} \Phi_{x_i}(u)) \cdot M^{n,[u_i],c}, \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in V_n. \]

Therefore, (3.34) is satisfied by Lemma 3.7(ii), since \( f \in D(\mathcal{E}) \) is arbitrary.

Let \( M \in \mathcal{M}_{\text{loc}} \). Then, there exist \( \{V_n\}, \{E_n\} \in \Theta \) and \( \{M^n \mid M^n \in \mathcal{M}^{V_n}\} \) such that \( E_n \subset V_n, M_{t \wedge \tau_{E_n}} = M^n_{t \wedge \tau_{E_n}}, t \geq 0, n \in \mathbb{N} \). We define
\[ <M>_t := \lim_{s \uparrow \zeta} <M>_s \text{ for } t \geq \zeta. \]

Then, we can see that \( <M>_t \) is well-defined and \( <M>_t \) is a PCAF. Denote by \( \mu_{<M>_t} \) the Revuz measure of \( <M>_t \). We define
\[ L^2_{\text{loc}}(E; \mu_{<M>_t}) := \{ f \mid \exists \{V_n\}, \{E_n\} \in \Theta \text{ and } \{M^n \mid M^n \in \mathcal{M}^{V_n}\} \text{ such that } E_n \subset V_n, M_{t \wedge \tau_{E_n}} = M^n_{t \wedge \tau_{E_n}}, f \cdot I_{E_n} \in L^2(E; \mu^{(n)}_{<M>_t}), t \geq 0, n \in \mathbb{N} \}. \]

For \( f \in L^2_{\text{loc}}(E; \mu_{<M>_t}) \), we define \( f \cdot M \) on \([0, \zeta]\) by
\[ (f \cdot M)_{t \wedge \tau_{E_n}} := ((f \cdot I_{E_n}) \cdot M^n)_{t \wedge \tau_{E_n}}, \quad t \geq 0, n \in \mathbb{N}. \]

Then, we can see that \( f \cdot M \) is well-defined and \( f \cdot M \in \mathcal{M}_{\text{loc}}^{[0,\zeta]} \). Denote by \( M^c \) the continuous part of \( M \).

Finally, we obtain the main result of this section.

**Theorem 3.10.** Suppose that \((\mathcal{E}, D(\mathcal{E}))\) is a quasi-regular local semi-Dirichlet form on \( L^2(\mathcal{E}; m) \) satisfying Assumption 2.3. Let \( m \in \mathbb{N}, \Phi \in C^1(\mathbb{R}^m), \) and \( u = (u_1, u_2, \ldots, u_m) \) with \( u_i \in D(\mathcal{E})_{\text{loc}}, 1 \leq i \leq m \). Then \( \Phi(u) \in D(\mathcal{E})_{\text{loc}} \) and
\[ M^{[\Phi(u)],c} = \sum_{i=1}^{m} \Phi_{x_i}(u) \cdot M^{[u_i],c} \text{ on } [0, \zeta], \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E. \quad (3.35) \]

**Proof.** Since \( 1 \in D(\mathcal{E})_{\text{loc}}, \Phi(u) \in D(\mathcal{E})_{\text{loc}} \) by Theorem 3.4. Hence (3.35) is a direct consequence of (3.34).

**4 Examples**

In this section we investigate some concrete examples.

**Example 4.1.** We consider the following bilinear form
\[ \mathcal{E}(u, v) = \int_0^1 u'v' dx + \int_0^1 bu' dx, \quad u, v \in D(\mathcal{E}) := H^1_0(0, 1). \]
(i) Suppose that \( b(x) = x^2 \). Then one can show that \((\mathcal{E}, D(\mathcal{E}))\) is a regular local semi-Dirichlet form (but not a Dirichlet form) on \( L^2((0,1); dx) \) (cf. [13, Remark 2.2(ii)]). Note that any \( u \in D(\mathcal{E}) \) is bounded and \( \frac{1}{2} \)-Hölder continuous by the Sobolev embedding theorem. Then we obtain Fukushima’s decomposition, \( u(X_t) - u(X_0) = M^u_t + N^u_t \), by Lemma [2.6], where \( X \) is the diffusion process associated with \((\mathcal{E}, D(\mathcal{E}))\), \( M^u_t \) is an MAF of finite energy and \( N^u_t \) is a CAF of zero energy.

(ii) Suppose that \( b(x) = \sqrt{x} \). By [13, Remark 2.2(ii)], \((\mathcal{E}, D(\mathcal{E}))\) is a regular local semi-Dirichlet form but not a Dirichlet form. Let \( u \in D(\mathcal{E})_{\text{loc}} \). Then we obtain Fukushima’s decomposition (2.4) by Theorem 2.4.

If \( u \in D(\mathcal{E}) \) satisfying \( \text{supp}[u] \subset (0,1) \), then we may choose an open subset \( V \) of \((0,1)\) such that \( \text{supp}[u] \subset V \subset (0,1) \). Let \( X^V \) be the part process of \( X \) w.r.t. \( V \). Then we obtain Fukushima’s decomposition, \( u(X^V_t) - u(X^V_0) = M^V_t[u] + N^V_t[u] \), by Lemma [2.6], where \( M^V_t[u] \) is an MAF of finite energy and \( N^V_t[u] \) is a CAF of zero energy w.r.t. \( X^V \).

Example 4.2. Let \( d \geq 3, U \) be an open subset of \( \mathbb{R}^d \), \( \sigma, \rho \in L^1_{\text{loc}}(U; dx), \sigma, \rho > 0 \) \( dx \)-a.e. For \( u, v \in C^\infty_0(U) \), we define

\[
\mathcal{E}_\rho(u, v) = \sum_{i,j=1}^{d} \int_U \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \rho dx.
\]

Assume that

\[
(\mathcal{E}_\rho, C^\infty_0(U)) \text{ is closable on } L^2(U; \sigma dx).
\]

Let \( a_{ij}, b_i, d_i \in L^1_{\text{loc}}(U; dx), 1 \leq i, j \leq d \). For \( u, v \in C^\infty_0(U) \), we define

\[
\mathcal{E}(u, v) = \sum_{i,j=1}^{d} \int_U \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} a_{ij} dx + \sum_{i=1}^{d} \int_U \frac{\partial u}{\partial x_i} b_i dx + \sum_{i=1}^{d} \int_U u \frac{\partial v}{\partial x_i} d_i dx + \int_U uv dx.
\]

Set \( \tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji}), \tilde{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji}), b := (b_1, \ldots, b_d), \) and \( d := (d_1, \ldots, d_d). \) Define \( F \) to be the set of all functions \( g \in L^1_{\text{loc}}(U; dx) \) such that the distributional derivatives \( \frac{\partial g}{\partial x_i}, 1 \leq i \leq d, \) are in \( L^1_{\text{loc}}(U; dx) \) such that \( \| \nabla g \| (g \sigma)^{-\frac{1}{2}} \in L^\infty(U; dx) \) or \( \| \nabla g \| (g^{p+1} \sigma^{p/q})^{-\frac{3}{2}} \in L^d(U; dx) \) for some \( p, q \in (1, \infty) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p < \infty \), where \( \| \cdot \| \) denotes Euclidean distance in \( \mathbb{R}^d \). We say that a \( \mathcal{B}(U) \)-measurable function \( f \) has property \( (A_{p,\sigma}) \) if one of the following conditions holds:

(i) \( f(\rho \sigma)^{-\frac{1}{2}} \in L^\infty(U; dx). \)

(ii) \( f^p(\rho^{p+1} \sigma^{p/q})^{-\frac{3}{2}} \in L^d(U, dx) \) for some \( p, q \in (1, \infty) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p < \infty \), and \( \rho \in F \).

Suppose that
(C.I) There exists $\eta > 0$ such that $\sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \geq \eta \xi_i^2$, $\forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

(C.II) $a_{ij} \rho^{-1} \in L^\infty(U; dx)$ for $1 \leq i, j \leq d$.

(C.III) For all $K \subseteq U$, $K$ compact, $1_K \|b + d\|_1$ and $1_K c^{1/2}$ have property $(A_{\rho,\sigma})$, and $(c + \alpha_0 \sigma) dx - \sum_{i=1}^{d} \frac{\partial \alpha_i}{\partial x_i}$ is a positive measure on $\mathcal{B}(U)$ for some $\alpha_0 \in (0, \infty)$.

(C.IV) $\|b - d\|$ has property $(A_{\rho,\sigma})$.

(C.V) $b = \beta + \gamma$ such that $\|\beta\|, \|\gamma\| \in L^1_{\text{loc}}(U, dx)$, $(\alpha_0 \sigma + c) dx - \sum_{i=1}^{d} \frac{\partial \alpha_i}{\partial x_i}$ is a positive measure on $\mathcal{B}(U)$ and $\|\beta\|$ has property $(A_{\rho,\sigma})$.

Then, by [18, Theorem 1.2], there exists $\alpha > 0$ such that $(\mathcal{E}_\alpha, C^\infty_0(U))$ is closable on $L^2(U; dx)$ and its closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a regular local semi-Dirichlet form on $L^2(U; dx)$. Define $\eta_\alpha(u, u) := \mathcal{E}_\alpha(u, u) - \int \langle \nabla u, \beta \rangle dx$ for $u \in D(\mathcal{E}_\alpha)$. By [18, Theorem 1.2 (ii) and (1.28)], we know $(\eta_\alpha, D(\mathcal{E}_\alpha))$ is a Dirichlet form and there exists $C > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C} \eta_\alpha(u, u) \leq \mathcal{E}_\alpha(u, u) \leq C \eta_\alpha(u, u).$$

Let $X$ be the diffusion process associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. Then, by Theorem 2.4, Fukushima’s decomposition holds for any $u \in D(\mathcal{E})_{\text{loc}}$. Moreover, the transformation formula [3.33] holds for local MAFs.

**Example 4.3.** Let $S$ be a Polish space. Denote by $\mathcal{B}(S)$ the Borel $\sigma$-algebra of $S$. Let $E := \mathcal{M}_1(S)$ be the space of probability measures on $(S, \mathcal{B}(S))$. For bounded $\mathcal{B}(S)$-measurable functions $f, g$ on $S$ and $\mu \in E$, we define

$$\mu(f) := \int_S f \, d\mu, \quad \langle f, g \rangle_\mu := \mu(fg) - \mu(f) \cdot \mu(g), \quad \|f\|_\mu := \langle f, f \rangle_\mu^{1/2}.$$  

Denote by $\mathcal{FC}_0^\infty$ the family of all functions on $E$ with the following expression:

$$u(\mu) = \varphi(\mu(f_1), \ldots, \mu(f_k)), \quad f_i \in C_c(S), 1 \leq i \leq k, \varphi \in C_0^\infty(\mathbb{R}^k), k \in \mathbb{N}.$$  

Let $m$ be a finite positive measure on $(E, \mathcal{B}(E))$, where $\mathcal{B}(E)$ denotes the Borel $\sigma$-algebra of $E$. We suppose that $\text{supp}[m] = E$. Let $b : S \times E \to \mathbb{R}$ be a measurable function such that

$$\sup_{\mu \in E} \|b(\mu)\|_\mu < \infty,$$

where $b(\mu)(x) := b(x, \mu)$.

For $u, v \in \mathcal{FC}_0^\infty$, we define

$$\mathcal{E}^b(u, v) := \int_E \langle \nabla u(\mu), \nabla v(\mu) \rangle_\mu + \langle b(\mu), \nabla u(\mu) \rangle_\mu v(\mu) \, d\mu,$$

where

$$\nabla u(\mu) := (\nabla_x u(\mu))_{x \in S} := \left( \frac{d}{ds} u(\mu + s \xi_x) \right|_{s=0} \right)_{x \in S}.$$
We suppose that \((\mathcal{E}^0, \mathcal{FC}_c^\infty)\) is closable on \(L^2(E; m)\). Then, by [13, Theorem 3.5], there exists \(\alpha > 0\) such that \((\mathcal{E}^b, \mathcal{FC}_c^\infty)\) is closable on \(L^2(E; m)\) and its closure \((\mathcal{E}^b, D(\mathcal{E}^b))\) is a quasi-regular local semi-Dirichlet form on \(L^2(E; m)\). Moreover, by [13, Lemma 2.5], there exists \(C > 1\) such that for any \(u \in D(\mathcal{E}^a)\),

\[
\frac{1}{C} \mathcal{E}^0(u, u) \leq \mathcal{E}^b(u, u) \leq C \mathcal{E}^0(u, u).
\]

Let \(X\) be the diffusion process associated with \((\mathcal{E}^b, D(\mathcal{E}^b))\), which is a Fleming-Viot type process with interactive selection. Then, by Theorem 2.4, Fukushima’s decomposition holds for any \(u \in D(\mathcal{E}^b)\_\text{loc}\). Moreover, the transformation formula \((\mathcal{E}, \mathcal{E}_1)\) holds for local MAFs.

## 5 Appendix: some results on potential theory and PCAFs for semi-Dirichlet forms

Let \(E\) be a metrizable Lusin space and \(m\) be a \(\sigma\)-finite positive measure on its Borel \(\sigma\)-algebra \(\mathcal{B}(E)\). Suppose that \((\mathcal{E}, D(\mathcal{E}))\) is a quasi-regular semi-Dirichlet form on \(L^2(E; m)\). Let \(K > 0\) be a continuity constant of \((\mathcal{E}, D(\mathcal{E}))\), i.e.,

\[
|\mathcal{E}_1(u, v)| \leq K \mathcal{E}_1(u, u)^{1/2} \mathcal{E}_1(v, v)^{1/2}, \quad \forall u, v \in D(\mathcal{E}). \quad (5.1)
\]

Denote by \((T_t)_{t \geq 0}\) and \((G_\alpha)_{\alpha \geq 0}\) (resp. \((\hat{T}_t)_{t \geq 0}\) and \((\hat{G}_\alpha)_{\alpha \geq 0}\)) the semigroup and resolvent (resp. co-semigroup and co-resolvent) associated with \((\mathcal{E}, D(\mathcal{E}))\). Then there exists an \(m\)-tight special standard process \(\text{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta})\) which is properly associated with \((\mathcal{E}, D(\mathcal{E}))\) (cf. [13, Theorem 3.8]). It is known that any quasi-regular semi-Dirichlet form is quasi-homeomorphic to a regular semi-Dirichlet form (cf. [7, Theorem 3.8]). By quasi-homeomorphism and the transfer method (cf. [2] and [12] VI, especially, Theorem VI.1.6), without loss of generality we can restrict to Hunt processes when we discuss the AFs of \(\text{M}\).

Let \(A \subset E\) and \(f \in D(\mathcal{E})\). Denote by \(f_A\) (resp. \(\hat{f}_A\)) the 1-balayaged (resp. 1-cobalayaged) function of \(f\) on \(A\). We fix \(\phi \in L^2(E; m)\) with \(0 < \phi \leq 1\) \(m\)-a.e. and set \(h = G_1\phi, \hat{h} = \hat{G}_1\phi\). Define for \(U \subset E, U\) open,

\[
\text{cap}_\phi(U) := (h_U, \phi)
\]

and for any \(A \subset E\),

\[
\text{cap}_\phi(A) := \inf\{\text{cap}_\phi(U) \mid A \subset U, U\text{ open}\}.
\]

Hereafter, \((\cdot, \cdot)\) denotes the usual inner product of \(L^2(E; m)\). By [13, Theorem 2.20], we have

\[
\text{cap}_\phi(A) = (h_A, \phi) = \mathcal{E}_1(h_A, \hat{G}_1\phi).
\]
Definition 5.1. A positive measure \(\mu\) on \((E, \mathcal{B}(E))\) is said to be of finite energy integral, denoted by \(S_0\), if \(\mu(N) = 0\) for each \(\mathcal{E}\)-exceptional set \(N \in \mathcal{B}(E)\) and there exists a positive constant \(C\) such that

\[
\int_E |\tilde{v}(x)|\mu(dx) \leq C\mathcal{E}_1(v, v)^{1/2}, \quad \forall v \in \mathcal{D}(\mathcal{E}).
\]

Remark 5.2. (i) Assume that \((\mathcal{E}, D(\mathcal{E}))\) is a regular semi-Dirichlet form. Let \(\mu\) be a positive Radon measure on \(E\) satisfying

\[
\int_E |v(x)|\mu(dx) \leq C\mathcal{E}_1(v, v)^{1/2}, \quad \forall v \in C_0(E) \cap D(\mathcal{E})
\]

for some positive constant \(C\), where \(C_0(E)\) denotes the set of all continuous functions on \(E\) with compact supports. Then one can show that \(\mu\) charges no \(\mathcal{E}\)-exceptional set (cf. [8, Lemma 3.5]) and thus \(\mu \in S_0\).

(ii) Let \(\mu \in S_0\) and \(\alpha > 0\). Then there exist unique \(U_\alpha \mu \in D(\mathcal{E})\) and \(\hat{U}_\alpha \mu \in D(\mathcal{E})\) such that

\[
\mathcal{E}_\alpha(U_\alpha \mu, v) = \int_E \tilde{v}(x)\mu(dx) = \mathcal{E}_\alpha(v, \hat{U}_\alpha \mu).
\]

We call \(U_\alpha \mu\) and \(\hat{U}_\alpha \mu\) \(\alpha\)-potential and \(\alpha\)-co-potential, respectively.

Let \(u \in D(\mathcal{E})\). By quasi-homeomorphism and similar to [5, Theorem 2.2.1] (cf. [8, Lemma 1.2]), one can show that the following conditions are equivalent to each other:

(i) \(u\) is \(\alpha\)-excessive (resp. \(\alpha\)-co-excessive).

(ii) \(u\) is an \(\alpha\)-potential (resp. \(\alpha\)-co-potential).

(iii) \(\mathcal{E}_\alpha(u, v) \geq 0\) (resp. \(\mathcal{E}_\alpha(v, u) \geq 0\)), \(\forall v \in D(\mathcal{E}), v \geq 0\).

Theorem 5.3. Define

\[
\hat{S}_0^* := \{\mu \in S_0 | \hat{U}_1 \mu \leq c\hat{G}_1 \phi \text{ for some constant } c > 0\}.
\]

Let \(A \in \mathcal{B}(E)\). If \(\mu(A) = 0\) for all \(\mu \in \hat{S}_0^*\), then \(\text{cap}_\phi(A) = 0\).

Proof. By quasi-homeomorphism, without loss of generality, we suppose that \((\mathcal{E}, D(\mathcal{E}))\) is a regular semi-Dirichlet form. Assume that \(A \in \mathcal{B}(E)\) satisfying \(\mu(A) = 0\) for all \(\mu \in \hat{S}_0^*\). We will prove that \(\text{cap}_\phi(A) = 0\).

Step 1. We first show that \(\mu(A) = 0\) for all \(\mu \in S_0\). Suppose that \(\mu \in S_0\). By [14, Proposition 4.13], there exists an \(\mathcal{E}\)-nest \(\{F_k\}\) of compact subsets of \(E\) such that \(\hat{G}_1 \phi, \hat{U}_1 \mu \in C(\{F_k\})\) and \(\hat{G}_1 \phi > 0\) on \(F_k\) for each \(k \in \mathbb{N}\). Then, there exists a sequence of positive constants \(\{a_k\}\) such that

\[
\hat{U}_1 \mu \leq a_k \hat{G}_1 \phi \text{ on } F_k \text{ for each } k \in \mathbb{N}.
\]
Define $u_k = \tilde{U}_1(I_{F_k} \cdot \mu)$ and set $v_k = u_k \wedge a_k \tilde{G}_1 \phi$ for $k \in \mathbb{N}$. Then $\tilde{u}_k \leq \tilde{U}_1 \mu \leq a_k \tilde{G}_1 \phi \ EF$-q.e. on $F_k$. By (5.2), we get

$$\mathcal{E}_1(v_k, u_k) = \int_{F_k} \tilde{v}_k(x) \mu(dx) = \int_{F_k} \tilde{u}_k(x) \mu(dx) = \mathcal{E}_1(u_k, u_k).$$

Since $v_k$ is a 1-co-potential and $v_k \leq u_k$ m-a.e., $\mathcal{E}_1(v_k - u_k, v_k - u_k) = \mathcal{E}_1(v_k - u_k, v_k) - \mathcal{E}_1(v_k - u_k, u_k) \leq 0$, proving that $u_k = v_k \leq a_k \tilde{G}_1 \phi$ m-a.e. Hence $I_{F_k} \cdot \mu \in \hat{S}_{0^*}$.

Therefore $\mu(A) = 0$ by the assumption that $A$ is not charged by any measure in $\hat{S}_{0^*}$.

**Step 2.** Suppose that $\text{cap}_\phi(A) > 0$. By [13, Corollary 2.22], there exists a compact set $K \subset B$ such that $\text{cap}_\phi(K) > 0$. Note that $(\tilde{G}_1 \phi)_K \in D(\mathcal{E})$ is 1-co-excessive.

By Remark 5.2(ii), there exists $\mu_{(\tilde{G}_1 \phi)_K} \in S_0$ such that

$$\text{cap}_\phi(K) = \mathcal{E}_1((G_1 \phi)_K, \tilde{G}_1 \phi) = \mathcal{E}_1(G_1 \phi, (\tilde{G}_1 \phi)_K) = \int_E \tilde{G}_1 \phi d\mu_{(\tilde{G}_1 \phi)_K} \leq \mu_{(\tilde{G}_1 \phi)_K}(E). \quad (5.3)$$

For any $v \in C_0(K^c) \cap D(\mathcal{E})$, we have $\int \tilde{v} \mu_{(\tilde{G}_1 \phi)_K} = \mathcal{E}_1(v, (\tilde{G}_1 \phi)_K) = 0$. Since $C_0(K^c) \cap D(\mathcal{E})$ is dense in $C_0(K^c)$, the support of $\mu_{\tilde{G}_1 \phi}$ is contained in $K$. Thus, by (5.3), we get $\mu_{\tilde{G}_1 \phi}(K) > 0$. Therefore $\text{cap}_\phi(A) = 0$ by Step 1. \hfill \square

**Theorem 5.4.** The following conditions are equivalent for a positive measure $\mu$ on $(E, \mathcal{B}(E))$.

(i) $\mu \in S$.

(ii) There exists an $\mathcal{E}$-nest $\{F_k\}$ satisfying $I_{F_k} \cdot \mu \in S_0$ for each $k \in \mathbb{N}$.

**Proof.** (ii) $\Rightarrow$ (i) is clear. We only prove (i) $\Rightarrow$ (ii). Let $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ be the symmetric part of $(\mathcal{E}, D(\mathcal{E}))$. Then $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is a symmetric positivity preserving form. Denote by $(\tilde{G}_n)_{n \geq 0}$ the resolvent associated with $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ and set $\tilde{h} := \tilde{G}_1 \varphi$. Then $(\tilde{\mathcal{E}}_1^h, D(\mathcal{E}_1^h))$ is a quasi-regular symmetric Dirichlet form on $L^2(E; \tilde{h}^2 \cdot m)$ (the $\tilde{h}$-transform of $(\tilde{\mathcal{E}}_1, D(\mathcal{E}))$).

By [10] pages 838-839, for an increasing sequence $\{F_k\}$ of closed sets, $\{F_k\}$ is an $\mathcal{E}$-nest if and only if it is an $\tilde{\mathcal{E}}_1^h$-nest. We select a compact $\tilde{\mathcal{E}}_1^h$-nest $\{F_k\}$ such that $\tilde{h}$ is bounded on each $F_k$. Let $\mu \in S(\mathcal{E})$, the family of smooth measures w.r.t. $(E, \mathcal{B}(E))$. Then $\mu \in S(\tilde{\mathcal{E}}_1^h)$, the family of smooth measures w.r.t. $(\tilde{\mathcal{E}}_1^h, D(\mathcal{E}_1^h))$. By [3, Theorem 2.2.4] and quasi-homeomorphism, we know that there exists a compact $\tilde{\mathcal{E}}_1^h$-nest (hence $\mathcal{E}$-nest) $\{J_k\}$ such that $I_{J_k} \cdot \mu \in S_0(\tilde{\mathcal{E}}_1^h)$. Then, there exists
a sequence of positive constants \( \{C_k\} \) such that
\[
\int_E |\tilde{g}| I_{J_k} d\mu \leq C_k \tilde{\mathcal{E}}_1^k(g, g)^{1/2}, \quad \forall g \in D(\tilde{\mathcal{E}}^k).
\]

We now show that each \( I_{F_k \cap J_k} \cdot \mu \in S_0(\mathcal{E}) \), which will complete the proof. In fact, let \( f \in D(\mathcal{E}) \). Then \( \frac{f}{\tilde{h}} \in D(\mathcal{E}) \) and
\[
\int_E |\tilde{f}| I_{F_k \cap J_k} d\mu \leq \|\tilde{h}\|_{F_k} \int_E |\tilde{f}| I_{I_{F_k \cap J_k}} d\mu \\
\leq \|\tilde{h}\|_{F_k} \int_E |\tilde{f}| I_{I_{J_k}} d\mu \\
\leq \|\tilde{h}\|_{F_k} \int_E \tilde{C}_k \tilde{\mathcal{E}}_1^k(f/\tilde{h}, f/\tilde{h})^{1/2} \\
= \|\tilde{h}\|_{F_k} \|\tilde{C}_k \tilde{\mathcal{E}}_1^k(f, f)^{1/2}.
\]

Since \( f \in D(\mathcal{E}) \) is arbitrary, this implies that \( I_{F_k \cap J_k} \cdot \mu \in S_0(\mathcal{E}) \).

**Lemma 5.5.** For any \( u \in D(\mathcal{E}) \), \( \nu \in S_0 \), \( 0 < T < \infty \) and \( \varepsilon > 0 \),
\[
P_\nu(\sup_{0 \leq t \leq T} |\tilde{u}(X_t)| > \varepsilon) \leq \frac{2K^{5/2}e^T}{\varepsilon} \mathcal{E}_1(u, u)^{1/2} \tilde{\mathcal{E}}_1(\tilde{U}_1 \nu, \tilde{U}_1 \nu)^{1/2}.
\]

**Proof.** We take an \( \mathcal{E} \)-quasi-continuous Borel version \( \tilde{u} \) of \( u \). Let \( A = \{x \in E \mid |\tilde{u}(x)| > \varepsilon\} \) and \( \sigma_A := \inf\{t > 0 \mid X_t \in A\} \). By [10, Theorem 4.4], \( H_A^1|u| := E[e^{-\sigma_A}|u|(X_{\sigma_A})] \) is an \( \mathcal{E} \)-quasi-continuous version of \( |u|_A \). Then, by [13, Proposition 2.8(i) and (2.1)], we get
\[
P_\nu(\sup_{0 \leq t \leq T} |\tilde{u}(X_t)| > \varepsilon) \leq \frac{e^T E_\nu[e^{-\sigma_A}|u|(X_{\sigma_A})]}{\varepsilon} \\
= \frac{e^T}{\varepsilon} \int_E |u|_A d\nu \\
= \frac{e^T}{\varepsilon} \mathcal{E}_1(|u|_A, \tilde{U}_1 \nu) \\
\leq \frac{Ke^T}{\varepsilon} \mathcal{E}_1(|u|_A, |u|_A)^{1/2} \mathcal{E}_1(\tilde{U}_1 \nu, \tilde{U}_1 \nu)^{1/2} \\
\leq \frac{e^T}{\varepsilon} \mathcal{E}_1(|u|, |u|)^{1/2} \mathcal{E}_1(\tilde{U}_1 \nu, \tilde{U}_1 \nu)^{1/2} \\
\leq \frac{Ke^T}{\varepsilon} \mathcal{E}_1(|u|, |u|)^{1/2} \mathcal{E}_1(\tilde{U}_1 \nu, \tilde{U}_1 \nu)^{1/2}.
\]

By Lemma 5.5 and Theorem 5.3 similar to [5, Lemma 5.1.2], we can prove the following lemma.
Lemma 5.6. Let \( \{u_n\} \) be a sequence of \( \mathcal{E} \)-quasi continuous functions in \( D(\mathcal{E}) \). If \( \{u_n\} \) is an \( \mathcal{E}_1 \)-Cauchy sequence, then there exists a subsequence \( \{u_{n_k}\} \) satisfying the condition that for \( \mathcal{E} \)-q.e. \( x \in E \),
\[
P_x(u_{n_k}(X_t)) \text{ converges uniformly in } t \text{ on each compact interval of } [0, \infty) = 1.
\]

In [3], Fitzsimmons extended the smooth measure characterization of PCAFs from the Dirichlet forms setting to the semi-Dirichlet forms setting (see [3, Theorem 4.22]). In particular, the following proposition holds.

Proposition 5.7. (cf. [3, Proposition 4.12]) For any \( \mu \in S_0 \), there is a unique finite PCAF \( \mathcal{A} \) such that \( E_x(\int_0^\infty e^{-t}dA_t) \) is an \( \mathcal{E} \)-quasi-continuous version of \( U_1\mu \).

By Proposition 5.7 and Theorem 5.4, following the arguments of [5, Theorems 5.1.3 and 5.1.4] (with slight modifications by virtue of [13, 14, 10] and [11, Theorem 3.4]), we can obtain the following theorem.

Theorem 5.8. Let \( \mu \in S \) and \( \mathcal{A} \) be a PCAF. Then the following conditions are equivalent to each other:

(i) For any \( \gamma \)-co-excessive function \( g \ (\gamma \geq 0) \) in \( D(\mathcal{E}) \) and \( f \in \mathcal{B}^+(E) \),
\[
\lim_{t \downarrow 0} \frac{1}{t} E_{g,m}((fA)_t) = < f \cdot \mu, \tilde{g} >.
\] (5.4)

(ii) For any \( \gamma \)-co-excessive function \( g \ (\gamma \geq 0) \) in \( D(\mathcal{E}) \) and \( f \in \mathcal{B}^+(E) \),
\[
\alpha(g, U_{\mathcal{A}}^{\alpha+\gamma}f) \uparrow < f \cdot \mu, \tilde{g} >, \quad \alpha \uparrow \infty,
\]
where \( U_{\mathcal{A}}^{\alpha}f(x) := E_x(\int_0^\infty e^{-\alpha t}f(X_t)dA_t) \).

(iii) For any \( t > 0 \), \( g \in \mathcal{B}^+(E) \cap L^2(E;m) \) and \( f \in \mathcal{B}^+(E) \),
\[
E_{g,m}((fA)_t) = \int_0^t < f \cdot \mu, \tilde{T}_g > ds.
\]

(iv) For any \( \alpha > 0 \), \( g \in \mathcal{B}^+(E) \cap L^2(E;m) \) and \( f \in \mathcal{B}^+(E) \),
\[
(g, U_{\mathcal{A}}^{\alpha}f) = < f \cdot \mu, \tilde{G}_\alpha g >.
\]

When \( \mu \in S_0 \), each of the above four conditions is also equivalent to each of the following three conditions:

(v) \( U_{\mathcal{A}}^{1}1 \) is an \( \mathcal{E} \)-quasi-continuous version of \( U_1\mu \).

(vi) For any \( g \in \mathcal{B}^+(E) \cap D(\mathcal{E}) \) and \( f \in \mathcal{B}^+_b(E) \),
\[
\lim_{t \downarrow 0} \frac{1}{t} E_{g,m}((fA)_t) = < f \cdot \mu, \tilde{g} >.
\]
(vii) For any \( g \in \mathcal{B}^+(E) \cap D(\mathcal{E}) \) and \( f \in \mathcal{B}_b^+(E) \),
\[
\lim_{\alpha \to \infty} \alpha (g, U^\alpha f) = \langle f \cdot \mu, \tilde{g} \rangle.
\]

The family of all equivalent classes of PCAFs and the family \( S \) are in one to one correspondence under the Revuz correspondence \((5.4)\).

Given a PCAF \( A \), we denote by \( \mu_A \) the Revuz measure of \( A \).

**Lemma 5.9.** Let \( A \) be a PCAF and \( \nu \in \hat{S}^*_0 \). Then there exists a positive constant \( C_\nu \) such that for any \( t > 0 \),
\[
E_\nu(A_t) \leq C_\nu (1 + t) \int_E \tilde{h} d\mu_A.
\]

**Proof.** By Theorem \( 5.4 \), we may assume without loss of generality that \( \mu_A \in S_0 \).

Set \( c_t(x) = E_x(A_t) \). Similar to [16, page 137], we can show that \( c_t \in D(\mathcal{E}) \) and for any \( v \in D(\mathcal{E}) \)
\[
\mathcal{E}(c_t, v) = \langle \mu_A, v - \dot{T}_t v \rangle.
\]

Let \( \nu \in \hat{S}^*_0 \). Then
\[
E_\nu(A_t) = \langle \nu, c_t \rangle = \mathcal{E}_1(c_t, \dot{U}_t \nu) \leq \langle \mu_A, \dot{U}_t \nu \rangle + \langle c_t, \dot{U}_t \nu \rangle \leq C_\nu \langle \mu_A, \hat{h} \rangle + E_{\hat{h},m}(A_t)
\]
for some constant \( C_\nu > 0 \). Therefore the proof is completed by \((5.4)\). \( \square \)

**Acknowledgments**

We are grateful to the support of NSFC (Grant No. 10961012), 973 Project, Key Lab of CAS (Grant No. 2008DP173182), and NSERC (Grant No. 311945-2008).

**References**

[1] S. Albeverio, R.Z. Fan, M. Röckner and W. Stannat, A remark on coercive forms and associated semigroups, *Partial Differential Operators and Mathematical Physics, Operator Theory Advances and Applications* 78 (1995) 1-8.

[2] Z.Q. Chen, Z.M. Ma and M. Röckner, Quasi-homeomorphisms of Dirichlet forms, *Nagoya Math. J.* 136 (1994) 1-15.

[3] P.J. Fitzsimmons, On the quasi-regularity of semi-Dirichlet forms, *Potential Anal.* 15 (2001) 158-185.
[4] M. Fukushima, A decomposition of additive functionals of finite energy, *Nagoya Math. J.* **74** (1979) 137-168.

[5] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, first edition, 1994; second revised and extended edition, 2011. (In the present paper, the sections and pages are quoted from the first edition.)

[6] S.W. He, J.G. Wang and J.A. Yan, *Semimartingale Theory and Stochastic Calculus*, Science Press, Beijing, 1992.

[7] Z.C. Hu, Z.M. Ma and W. Sun, Extensions of Lévy-Khintchine formula and Beurling-Deny formula in semi-Dirichlet forms setting, *J. Funct. Anal.* **239** (2006) 179-213.

[8] Z.C. Hu and W. Sun, Balayage of semi-Dirichlet forms, To appear in *Can. J. Math.*

[9] J.H. Kim, Stochastic calculus related to non-symmetric Dirichlet forms, *Osaka J. Math.* **24** (1987) 331-371.

[10] K. Kuwae, Maximum principles for subharmonic functions via local semi-Dirichlet forms, *Can. J. Math.* **60** (2008) 822-874.

[11] K. Kuwae, Stochastic calculus over symmetric Markov processes without time reversal, *Ann. Probab.* **38** (2010) 1532-1569.

[12] Z.M. Ma and M. Röckner, *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Springer-Verlag, 1992.

[13] Z.M. Ma, L. Overbeck and M. Röckner, Markov processes associated with semi-Dirichlet forms, *Osaka J. Math.* **32** (1995) 97-119.

[14] Z.M. Ma and M. Röckner, Markov processes associated with positivity preserving coercive forms, *Can. J. Math.* **47** (1995) 817-840.

[15] L. Overbeck, M. Röckner and B. Schmuland, An analytic approach to Fleming-Viot processes with interactive selection, *Ann. Probab.* **23** (1995) 1-36.

[16] Y. Oshima, *Lecture on Dirichlet Spaces*, Univ. Erlangen-Nürnberg, 1988.

[17] P.E. Protter, *Stochastic Integration and Differential Equations*, Springer, Berlin-Heidelberg-New York, 2005.

[18] M. Röckner and B. Schmuland, Quasi-regular Dirichlet forms: examples and counterexamples, *Can. J. Math.* **47** (1995) 165-200.