On the second boundary value problem for Lagrangian mean curvature equation

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Abstract
Considering the second boundary value problem of the Lagrangian mean curvature equation, we obtain the existence and uniqueness of the smooth uniformly convex solution, which generalizes the Brendle–Warren’s theorem about minimal Lagrangian diffeomorphism in Euclidean metric space.

Mathematics Subject Classification 53D12 · 35J66

1 Introduction
The main aim of this article is to study the existence and uniqueness of the smooth uniformly convex solution for the second boundary value problem of the Lagrangian mean curvature equation

\[
\begin{align*}
F_T(\lambda(D^2 u)) &= \kappa \cdot x + c, \quad x \in \Omega, \\
Du(\Omega) &= \tilde{\Omega},
\end{align*}
\]

(1.1)

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where \( \Omega \) and \( \hat{\Omega} \) are two uniformly convex bounded domains with smooth boundary in \( \mathbb{R}^n \), \( \kappa \in \mathbb{R}^n \) is a constant vector, \( \lambda(D^2u) = (\lambda_1, \ldots, \lambda_n) \) are the eigenvalues of Hessian matrix \( D^2u \), \( c \) is a constant to be determined and

\[
F_\tau(\lambda) := \begin{cases} 
\frac{1}{2} \sum_{i=1}^{n} \ln \lambda_i, & \tau = 0, \\
\frac{\sqrt{a^2 + 1}}{2b} \sum_{i=1}^{n} \ln \frac{\lambda_i + a - b}{\lambda_i + a + b}, & 0 < \tau < \frac{\pi}{4}, \\
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{1 + \lambda_i}, & \tau = \frac{\pi}{4}, \\
\frac{\sqrt{a^2 + 1}}{b} \sum_{i=1}^{n} \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
\sum_{i=1}^{n} \arctan \lambda_i, & \tau = \frac{\pi}{2},
\end{cases}
\]  

(1.2)

where \( a = \cot \tau, b = \sqrt{|\cot^2 \tau - 1|} \).

Let

\[
g_\tau = \sin \tau \delta_0 + \cos \tau g_0, \quad \tau \in \left[0, \frac{\pi}{2}\right]
\]

be the linear combined metric of the standard Euclidean metric

\[
\delta_0 = \sum_{i=1}^{n} dx_i \otimes dx_i + \sum_{j=1}^{n} dy_j \otimes dy_j
\]

and the pseudo-Euclidean metric

\[
g_0 = \frac{1}{2} \sum_{i=1}^{n} dx_i \otimes dy_i + \frac{1}{2} \sum_{j=1}^{n} dy_j \otimes dx_j
\]

in \( \mathbb{R}^n \times \mathbb{R}^n \).

In 2010, Warren \[1\] firstly obtained that if \((x, Du(x))\) is a minimal Lagrangian graph in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\), then \(u\) satisfies

\[
F_\tau(\lambda(D^2u)) = c,
\]

(1.3)

which is a special case of (1.1) when \( \kappa \equiv 0 \).

If \( \tau = 0 \), (1.3) is the famous Monge-Ampère equation

\[
det D^2u = e^{2c},
\]

which general form is

\[
det D^2u = f(x, u, Du).
\]

(1.4)

If \( \tau = \frac{\pi}{2} \), (1.3) becomes the special Lagrangian equation

\[
\sum_{i=1}^{n} \arctan \lambda_i(D^2u) = c.
\]

(1.5)
The special Lagrangian equation was first introduced by Harvey and Lawson in [2]. They proved that a Lagrangian graph \((x, Du(x))\) in \((\mathbb{R}^n \times \mathbb{R}^n, \delta_0)\) is minimal if and only if the Lagrangian angle is a constant, that is, \((1.5)\) holds. According to \((1.5)\), several authors obtained the same Bernstein type theorems simultaneously using different techniques. Jost and Xin [3] used the properties of harmonic maps into convex subsets of Grassmannians. Yuan [4] used the geometric measure theory.

We will be considering the Lagrangian graphs of prescribed constant mean curvature vector \((0, 0, \ldots, 0, \kappa)\perp\) in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\) and \(Du\) is a diffeomorphism from \(\Omega\) to \(\tilde{\Omega}\). For \(\kappa \equiv 0\), finding a minimal Lagrangian diffeomorphism between two uniformly convex bounded domains in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\) and \(Du\) is a diffeomorphism from \(\Omega\) to \(\tilde{\Omega}\).

For \(\kappa \equiv 0\), finding a minimal Lagrangian diffeomorphism between two uniformly convex bounded domains in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\) is equivalent to solving \((1.3)\) with second boundary condition

\[
Du(\Omega) = \tilde{\Omega},
\]

that is,

\[
\begin{align*}
F_\tau(\lambda(D^2 u)) &= c, \quad x \in \Omega, \\
Du(\Omega) &= \tilde{\Omega}.
\end{align*}
\]

Here \(Du\) is a minimal Lagrangian diffeomorphism from \(\Omega\) to \(\tilde{\Omega}\) in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\).

In dimension 2, Delanoë [5] obtained a unique smooth solution for the second boundary value problem of the Monge-Ampère equation for \(\tau = 0\) in \((1.7)\) if both domains are uniformly convex. Later the generalization of Delanoë’s theorem to higher dimensions was given by Caffarelli [6] and Urbas [7]. Using the parabolic method, Schnürer and Smoczyk [8] also obtained the existence of solutions to \((1.7)\) for \(\tau = 0\).

As far as \(\tau = \frac{\pi}{2}\) is concerned, Brendle and Warren [9] proved the existence and uniqueness of the solution by the elliptic method, and the second author [10] obtained the existence of solution by the parabolic method. Then by the elliptic and parabolic method, the second author with Ou [11], Ye [12] [13] and Chen [13] proved the existence and uniqueness of the solution to \((1.7)\) for \(0 < \tau < \frac{\pi}{2}\).

Motivated by the works of [1], for \(\tau \in \left[0, \frac{\pi}{2}\right]\) we obtain a generalization of Warren’s result.

**Proposition 1.1** For \(f \in C^1(\Omega)\), if \(u\) satisfies

\[
F_\tau(\lambda(D^2 u)) = f(x), \quad x \in \Omega.
\]

Then \((0, 0, \ldots, 0, Df(x))\perp\) is the mean curvature vector of the gradient graph \(\{(x, Du(x)) | x \in \Omega\}\) in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\), where \(\perp\) means projecting the vector to the normal space of the gradient graph in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\).

For \(f(x) = \kappa \cdot x + c\), Proposition 1.1 becomes

**Corollary 1.2** If \(u\) satisfies \((1.1)\), then \((0, 0, \ldots, 0, \kappa)\perp\) is the mean curvature vector of the gradient graph \(\{(x, Du(x)) | x \in \Omega\}\) in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\).

By the continuity method, for the second boundary value problem \((1.1)\), we have

**Theorem 1.3** For \(\tau \in \left(0, \frac{\pi}{2}\right]\), there exists some positive constant \(\varepsilon_0\) depending only on \(\Omega\) and \(\tilde{\Omega}\), such that if \(|\kappa| \leq \varepsilon_0\), then there exists a uniformly convex solution \(u \in C^\infty(\tilde{\Omega})\) and a unique constant \(c\) solving \((1.1)\), and \(u\) is unique up to a constant.

**Remark 1.4** For \(\tau = 0\), the problem \((1.1)\) was solved by J. Urbas [7], O.C. Schnürer and K. Smoczyk [8].
The geometric meaning of this theorem is that if $\Omega$ and $\Omega'$ are two uniformly convex bounded domains with smooth boundary in $\mathbb{R}^n$, then there exists a diffeomorphism $\psi = Du : \Omega \to \Omega'$ such that

$$\Sigma := \{(x, \psi(x)) : x \in \Omega\}$$

is a Lagrangian submanifold, of which the mean curvature vector is $(0, 0, 0, \ldots, 0, \kappa)^\perp$ in $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$.

In another paper, we shall point out by the parabolic method that $c$ means the coefficient of time variable for the translating solution of the parabolic problem corresponding to (1.1), and therefore $c$ cannot be given in advance.

Theorem 1.3 presents an extension of the previous work on $\kappa = 0$ done by Brendle–Warren [9], Huang [10], Huang-Ou [11], Huang-Ye [12] and Chen-Huang-Ye [13].

The rest of this article is organized as follows. In Section 2, we give the proof of Proposition 1.1 and introduce a class of fully nonlinear elliptic equation containing (1.1). Then we present a theorem on the corresponding second boundary value problem, which is a generalization of Theorem 1.3. To prove this new theorem, we verify the strictly oblique estimate in Section 3, present the $C^2$ estimate in Section 4 and give the proof of this theorem by the continuity method in Section 5.

Throughout the following, Einstein’s convention of summation over repeated indices will be adopted. We denote, for a smooth function $u$,

$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \ldots$$

### 2 A generalization of Theorem 1.3

We begin with the proof of Proposition 1.1. One can see the details for the submanifold geometry of high codimension in [14], [15].

**Proof** Assume that $e_i = (0, \ldots, 1, \ldots, 0)$ be the $i$-th axis vector in $\mathbb{R}^n \times \mathbb{R}^n, i = 1, 2, \ldots, 2n$ and $\Xi = \{(x, Du(x)) | x \in \Omega\}$. Let $\overrightarrow{E} : \Xi \to (\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$ be a spacelike immersion of an $n$-dimensional manifold $\Xi$ into a semi-Riemannian manifold $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$ of dimension $2n$. Without loss of generality, suppose that $u$ is smooth function in $\Omega$. Then $\overrightarrow{E}$ is smooth and the pullback $\overrightarrow{E}^* g_\tau$ is a Riemannian metric on $\Xi$. We see that the tangential vector fields of $\overrightarrow{E} := (x, Du(x))$ are

$$E_i = e_i + u_{ij} e_{n+j}, \quad i = 1, \ldots, n.$$ 

Therefore,

$$\nabla_{E_i}^{E_j} = u_{ijk} e_{n+k},$$

where $\nabla$ is the Levi-Civita connection of $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$. Then the pullback metric $\overrightarrow{E}^* g_\tau$ on $\Xi$ is given by

$$g_{ij} = \langle E_i, E_j \rangle = \langle e_i + u_{ik} e_{n+k}, e_j + u_{jl} e_{n+l} \rangle = \sin \tau (\delta_{ij} + u_{ik} u_{kj}) + 2 \cos \tau u_{ij}.$$ 

Denote $(g^{ij}) = (g_{ij})^{-1}$ and write down the normal vector fields of $\Xi$ as follows

$$M_\alpha = M^i_\alpha e_i + \tilde{M}^j_\alpha e_{n+j}, \quad \alpha = 1, \ldots, n.$$ 

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We use \( \perp \) to denote projecting the vector to the normal space of \( \Xi \). Consequently, the part of the mean curvature vector

\[
\vec{H} = g^{ij} (\nabla^E_i) \perp
\]

on \( M_\alpha \) is

\[
H_\alpha := \langle \vec{H}, M_\alpha \rangle = g^{ij} (\nabla^E_i) \perp \langle u_{ijk} e_{n+k}, M^l_\alpha e_l + \tilde{M}^l_\alpha e_{n+p} \rangle = g^{ij} \left( \cos \tau u_{ijk} M^k_\alpha + \sin \tau u_{ijk} \tilde{M}^k_\alpha \right).
\]

(2.1)

Let \( \vec{F} = (0, 0, \ldots, 0, Df(x))\perp \), then

\[
F_\alpha := \langle \vec{F}, M_\alpha \rangle = \langle f_ie_{n+l}, M^j_\alpha e_i + \tilde{M}^j_\alpha e_{n+j} \rangle = \cos \tau f_i M^i_\alpha + \sin \tau f_i \tilde{M}^i_\alpha.
\]

(2.2)

By (1.8), we know that if for any \( k = 1, \ldots, n \) we have

\[
g^{ij} u_{ijk} = f_k,
\]

(2.3)

Comparing (2.1) with (2.2), we see that \( H_\alpha = F_\alpha, \alpha = 1, \ldots, n \). Then

\[
\vec{H} = (0, 0, \ldots, 0, Df(x))\perp.
\]

In conclusion, we complete the proof of Proposition 1.1.

In order to prove Theorem 1.3 in four cases of all together, we reduce it to a more general form. For the convenience, we introduce some notations.

It is obvious that \( F_\tau (\lambda_1, \ldots, \lambda_n), \tau \in \left(0, \frac{\pi}{2}\right] \) is a smooth symmetric function defined on \( \Gamma^+_n \), where

\[
\Gamma^+_n := \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0, i = 1, \ldots, n \}.
\]

By direct calculation, we get

\[
F_\tau (0, \ldots, 0) = \begin{cases}
\frac{n\sqrt{a^2 + b^2} + 1}{2b} \ln \frac{a-b}{a+b}, & 0 < \tau < \frac{\pi}{4}, \\
-\sqrt{2}n, & \tau = \frac{\pi}{4}, \\
\frac{n\sqrt{a^2 + b^2} + 1}{b} \arctan \frac{a-b}{a+b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
0, & \tau = \frac{\pi}{2}.
\end{cases}
\]

\[
F_\tau (+\infty, \ldots, +\infty) = \begin{cases}
0, & 0 < \tau < \frac{\pi}{4}, \\
0, & \tau = \frac{\pi}{4}, \\
\frac{n\pi \sqrt{a^2 + b^2}}{4b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
\frac{n\pi}{2}, & \tau = \frac{\pi}{2}.
\end{cases}
\]
\[ \frac{\partial F_\tau}{\partial \lambda_i} = \begin{cases} \sqrt{a^2 + 1}, & 0 < \tau < \frac{\pi}{4}, \\ \frac{\sqrt{2}}{1 + \lambda_i}, & \tau = \frac{\pi}{4}, \\ \frac{\sqrt{a^2 + 1}}{(1 + \lambda_i)^2}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \frac{1}{1 + \lambda_i^2}, & \tau = \frac{\pi}{2}, \end{cases} \]

and

\[ \frac{\partial^2 F_\tau}{\partial \lambda_i \partial \lambda_j} = \begin{cases} -2 \frac{\sqrt{a^2 + 1} \delta_{ij}}{(1 + \lambda_i)^2}, & 0 < \tau < \frac{\pi}{4}, \\ -2 \frac{\delta_{ij}}{(1 + \lambda_i)^2}, & \tau = \frac{\pi}{4}, \\ -2 \frac{\sqrt{a^2 + 1} \delta_{ij}}{(1 + \lambda_i)^2}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ -2 \frac{\lambda_j \delta_{ij}}{(1 + \lambda_i^2)}, & \tau = \frac{\pi}{2}, \end{cases} \]

for \( i, j = 1, \ldots, n \). Then

\[ -\infty < F_\tau(0, \ldots, 0) < F_\tau(+\infty, \ldots, +\infty) < +\infty, \quad \tau \in \left(0, \frac{\pi}{2}\right], \quad (2.4) \]

\[ \frac{\partial F_\tau}{\partial \lambda_i} > 0, \quad 1 \leq i \leq n \text{ on } \Gamma_n^+, \quad (2.5) \]

and

\[ \left( \frac{\partial^2 F_\tau}{\partial \lambda_i \partial \lambda_j} \right) \leq 0 \text{ on } \Gamma_n^+. \quad (2.6) \]

For any \( s_1 > 0, s_2 > 0 \), define

\[ \Gamma_{s_1, s_2}^+ = \{ (\lambda_1, \ldots, \lambda_n) \in \Gamma_n^+ : 0 \leq \min_{1 \leq i \leq n} \lambda_i \leq s_1, \max_{1 \leq i \leq n} \lambda_i \geq s_2 \}. \]

Then for any \( (\lambda_1, \ldots, \lambda_n) \in \Gamma_{s_1, s_2}^+ \), we have

\[ \sum_{i=1}^n \frac{\partial F_\tau}{\partial \lambda_i} \in \begin{cases} \left[ \frac{\sqrt{a^2 + 1}}{(s_1 + a)^2}, \frac{n \sqrt{a^2 + 1}}{a^2 - b^2} \right], & 0 < \tau < \frac{\pi}{4}, \\ \left[ \frac{\sqrt{\pi}}{(s_1 + a)^2}, n \sqrt{2} \right], & \tau = \frac{\pi}{4}, \\ \left[ \frac{\sqrt{a^2 + 1}}{(s_1 + a)^2}, \frac{n \sqrt{a^2 + 1}}{a^2 + b^2} \right], & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \left[ \frac{1}{1 + s_1^2}, n \right], & \tau = \frac{\pi}{2}, \end{cases} \quad (2.7) \]
and

\[
\sum_{i=1}^{n} \frac{\partial F_{\tau}}{\partial \lambda_i} \lambda_i^2 \in \begin{cases} 
  \left[ \frac{s_i^2 \sqrt{a^2+1}}{(s_i+a)^2+b^2}, n\sqrt{a^2+1} \right], & 0 < \tau < \frac{\pi}{4}, \\
  \left[ \frac{s_i^2}{1+s_i^2}, n\sqrt{2} \right], & \tau = \frac{\pi}{4}, \\
  \left[ \frac{s_i^2 \sqrt{a^2+1}}{(s_i+a)^2+b^2}, n\sqrt{a^2+1} \right], & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
  \left[ \frac{s_i^2}{1+s_i^2}, n \right], & \tau = \frac{\pi}{2}.
\end{cases}
\] (2.8)

For any \((\mu_1, \ldots, \mu_n) \in \Gamma_n^+\), denote

\[
\lambda_i = \frac{1}{\mu_i}, \quad 1 \leq i \leq n,
\]

and

\[
\tilde{F}_{\tau}(\mu_1, \ldots, \mu_n) := -F_{\tau}(\lambda_1, \ldots, \lambda_n).
\]

Then

\[
\frac{\partial \tilde{F}_{\tau}}{\partial \mu_i} = \lambda_i^2 \frac{\partial F_{\tau}}{\partial \lambda_i}, \quad \mu^2 \frac{\partial \tilde{F}_{\tau}}{\partial \mu_i} = \frac{\partial F_{\tau}}{\partial \lambda_i},
\]

and

\[
\frac{\partial^2 \tilde{F}_{\tau}}{\partial \mu_i \partial \mu_j} = -\lambda_i^3 \left( \lambda_i \frac{\partial^2 F_{\tau}}{\partial \lambda_i^2} + 2 \frac{\partial F_{\tau}}{\partial \lambda_i} \right) \delta_{ij}
\]

\[
= \begin{cases} 
  -\frac{2 \sqrt{a^2+1}(\mu_i+a)}{(1+a\mu_i)^2-(b\mu_i)^2} \delta_{ij}, & 0 < \tau < \frac{\pi}{4}, \\
  \frac{2\sqrt{2}\delta_{ij}}{(1+\mu_i)^{3/2}}, & \tau = \frac{\pi}{4}, \\
  -\frac{2 \sqrt{a^2+1}(\mu_i+a)}{(1+a\mu_i)^2+(b\mu_i)^2} \delta_{ij}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
  -\frac{2\mu_i \delta_{ij}}{(1+\mu_i^2)^{3/2}}, & \tau = \frac{\pi}{2}.
\end{cases}
\]

Therefore, we have

\[
\frac{\partial \tilde{F}_{\tau}}{\partial \mu_i} > 0, \quad 1 \leq i \leq n \text{ on } \Gamma_n^+,
\]

and

\[
\left( \frac{\partial^2 \tilde{F}_{\tau}}{\partial \mu_i \partial \mu_j} \right) \leq 0 \text{ on } \Gamma_n^+.
\] (2.9)

Motivated by (2.4)-(2.9), in order to prove Theorem 1.3, we introduce a class of nonlinear functions containing \(F_{\tau}(\lambda), \tau \in (0, \frac{\pi}{2})\).

Let \(F(\lambda_1, \ldots, \lambda_n)\) be a smooth symmetric function defined on \(\Gamma_n^+\), and satisfy

\[
-\infty < F(0, \ldots, 0) < F(+\infty, \ldots, +\infty) < +\infty,
\] (2.10)
\[
\frac{\partial F}{\partial \lambda_i} > 0, \quad 1 \leq i \leq n \quad \text{on} \quad \Gamma_n^+, \quad (2.11)
\]

and

\[
\left( \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} \right) \leq 0 \quad \text{on} \quad \Gamma_n^+. \quad (2.12)
\]

For any \((\mu_1, \ldots, \mu_n) \in \Gamma_n^+\), denote

\[
\hat{\lambda}_i = \frac{1}{\mu_i}, \quad 1 \leq i \leq n,
\]

and

\[
\hat{F}(\mu_1, \ldots, \mu_n) := -F(\lambda_1, \ldots, \lambda_n).
\]

Assume that

\[
\left( \frac{\partial^2 \hat{F}}{\partial \mu_i \partial \mu_j} \right) \leq 0 \quad \text{on} \quad \Gamma_n^+. \quad (2.13)
\]

In addition, for \(s_1, s_2 > 0\), we assume that there exist positive constants \(\Lambda_1\) and \(\Lambda_2\), depending on \(s_1\) and \(s_2\), such that for any \((\lambda_1, \ldots, \lambda_n) \in \Gamma_{[s_1,s_2]}^+\),

\[
\Lambda_1 \leq \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \leq \Lambda_2, \quad (2.14)
\]

and

\[
\Lambda_1 \leq \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \leq \Lambda_2. \quad (2.15)
\]

**Remark 2.1** Since

\[
\frac{\partial^2 \hat{F}}{\partial \mu_i \partial \mu_j} = -\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} \lambda_i^2 \lambda_j^2 - 2\lambda_i^3 \delta_{ij} \frac{\partial F}{\partial \lambda_i},
\]

we cannot deduce (2.13) from (2.11) and (2.12).

By the discussion above, we have

**Proposition 2.2** The operator \(F_\tau(\lambda), \tau \in (0, \frac{\pi}{2})\) satisfies the structure conditions (2.10)-(2.15).

In fact, there are more operators satisfying the structure conditions (2.10)-(2.15). For any constant \(\alpha > 1\), define the operator as follow

\[
S^\alpha(\lambda_1, \ldots, \lambda_n) = -\sum_{i=1}^n \frac{1}{(1 + \lambda_i)^\alpha}.
\]

Therefore, if

\[
F\left[D^2u\right] = S^\alpha(\lambda(D^2u)) \quad (2.16)
\]

then \(F\left[D^2u\right]\) satisfies the structure conditions (2.10)-(2.15).
For \( f(x) \in C^\infty(\bar{\Omega}) \), we define
\[
\text{osc}(f) := \max_{\Omega} |f(x) - f(y)|,
\]
and
\[
\mathcal{A}_\delta := \left\{ f(x) \in C^\infty(\bar{\Omega}) : \text{osc}(f) \leq \delta \right\}.
\]
The constant \( \delta \) is any positive constant satisfying
\[
\delta < \min \{ F(+\infty, \ldots, +\infty) - F(\Theta_0, +\infty, \ldots, +\infty), F(0, \ldots, 0, \Theta_0) - F(0, \ldots, 0) \},
\]
where \( \Theta_0 := \left( \frac{1}{12} \right)^{1/n} \).

Considering the more general second boundary value problem
\[
\begin{cases}
F(\lambda(D^2 u)) = f(x) + c, & x \in \Omega, \\
Du(\Omega) = \tilde{\Omega},
\end{cases}
\tag{2.17}
\]
we claim that

**Theorem 2.3** Let \( F \) satisfy the structure conditions (2.10)-(2.15) and \( f \) is concave which belongs to \( \mathcal{A}_\delta \). If
\[
|Df| \leq \frac{\theta \Lambda_1}{2} \cdot \frac{1}{\max_{\Omega} |Dh|}
\tag{2.18}
\]
holds, where \( \theta \) and \( h \) depending only on \( \tilde{\Omega} \) appear in Definition 3.3. Then there exists a uniformly convex solution \( u \in C^\infty(\bar{\Omega}) \) and a unique constant \( c \) solving (2.17), and \( u \) is unique up to a constant.

**Remark 2.4** Combining with Legendre transformation of \( u \), the concave condition of \( f \) is used to reduce the positive lower bound of \( D^2 u \) to the boundness on the boundary and estimate the pure tangential second derivatives of the dual solution on the boundary, seeing lemma 4.7 and lemma 4.9.

**Remark 2.5** It’s not hard to deduce that if (2.18) holds, \( |Df| < \frac{\delta}{\text{diam}(\Omega)} \), then \( f \in \mathcal{A}_\delta \). One can merge (2.18) and \( \text{osc}_{\bar{\Omega}}(f) \leq \delta \) into another condition. But we keep two conditions for ease of exposition in this paper.

By Proposition 2.2 we know that, once Theorem 2.3 is proved, Theorem 1.3 holds.

In the next three sections, we are going to prove Theorem 2.3 through the continuity method, which is based on the strictly oblique estimate and the \( C^2 \) estimate.

### 3 The strict obliqueness estimate

To prove the strict obliqueness estimate, first we need

**Lemma 3.1** Let \( \lambda_1(x), \ldots, \lambda_n(x) \) be the eigenvalues of \( D^2 u \) at \( x \). Suppose that (2.10) and (2.11) hold, if \( \text{osc}_{\tilde{\Omega}}(f) \leq \delta \) and \( u \in C^\infty(\bar{\Omega}) \) is a uniformly convex solution of (2.17), then there exists \( \mu > 0 \) and \( \omega > 0 \) depending only on \( F, \Theta_0 \) and \( \delta \) such that
\[
\min_{1 \leq i \leq n} \lambda_i(x) \leq \mu, \quad \max_{1 \leq i \leq n} \lambda_i(x) \geq \omega.
\tag{3.1}
\]
Proof By $D_{u}(\Omega) = \tilde{\Omega}$, we have
\[
\int_{\Omega} \det D^{2}u(x)dx = |\tilde{\Omega}|.
\]
Then we can find $\bar{x} \in \tilde{\Omega}$ such that
\[
\prod_{i=1}^{n} \lambda_{i}(\bar{x}) = \det D^{2}u(\bar{x}) = \frac{|\tilde{\Omega}|}{|\Omega|} = \Theta_{0}^{n}.
\]
Therefore,
\[
\min_{1 \leq i \leq n} \lambda_{i}(\bar{x}) \leq \Theta_{0} \leq \max_{1 \leq i \leq n} \lambda_{i}(\bar{x}).
\]
By (2.17) and condition (2.11), we obtain
\[
F \left( \min_{1 \leq i \leq n} \lambda_{i}(x), \ldots, \min_{1 \leq i \leq n} \lambda_{i}(x) \right) \leq F \left( \lambda_{1}(x), \ldots, \lambda_{n}(x) \right)
= F \left( \lambda_{1}(\bar{x}), \ldots, \lambda_{n}(\bar{x}) \right) + f(x) - f(\bar{x})
\leq F \left( \Theta_{0}, +\infty, \ldots, +\infty \right) + \frac{\text{osc}(f)}{\Omega}
\leq F \left( \Theta_{0}, +\infty, \ldots, +\infty \right) + \delta
< F \left( +\infty, \ldots, +\infty \right),
\]
and
\[
F \left( \max_{1 \leq i \leq n} \lambda_{i}(x), \ldots, \max_{1 \leq i \leq n} \lambda_{i}(x) \right) \geq F \left( \lambda_{1}(x), \ldots, \lambda_{n}(x) \right)
= F \left( \lambda_{1}(\bar{x}), \ldots, \lambda_{n}(\bar{x}) \right) + f(x) - f(\bar{x})
\geq F \left( 0, \ldots, 0, \Theta_{0} \right) - \frac{\text{osc}(f)}{\Omega}
\geq F \left( 0, \ldots, 0, \Theta_{0} \right) - \delta
> F \left( 0, \ldots, 0 \right).
\]
By the monotonicity of $F$ and condition (2.10), we get the desired result. \qed

By Lemma 3.1, the points $(\lambda_{1}, \ldots, \lambda_{n})$ are always in $\Gamma_{\mu, \omega}^{+}$ under the problem (2.17). Then there exists $\Lambda_{1} > 0$ and $\Lambda_{2} > 0$ depending only on $F, \Theta_{0}$ and $\delta$, such that $F$ satisfies the structure conditions (2.14) and (2.15). In the following, we always assume that $\Lambda_{1}$ and $\Lambda_{2}$ are universal constants depending only on the known data.

For technical needs below, we introduce the Legendre transformation of $u$. For any $x \in \mathbb{R}^{n}$, define
\[
\tilde{x}_{i} := \frac{\partial u}{\partial x_{i}}(x), \quad i = 1, 2, \ldots, n,
\]
and
\[
\tilde{u}(\tilde{x}_{1}, \ldots, \tilde{x}_{n}) := \sum_{i=1}^{n} x_{i} \frac{\partial u}{\partial x_{i}}(x) - u(x).
\]
In terms of $\tilde{x}_{1}, \ldots, \tilde{x}_{n}$ and $\tilde{u}(\tilde{x}_{1}, \ldots, \tilde{x}_{n})$, we can easily check that
\[
\left( \frac{\partial^{2} \tilde{u}}{\partial \tilde{x}_{i} \partial \tilde{x}_{j}} \right) = \left( \frac{\partial^{2} u}{\partial x_{i} \partial x_{j}} \right)^{-1}.
\]
Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $D^2\bar{u}$ at $\bar{x} = Du(x)$. We denote
\[
\mu_i = \lambda_i^{-1}, \quad i = 1, 2, \ldots, n.
\]
Then
\[
\frac{\partial \bar{F}}{\partial \mu_i} = \lambda_i^2 \frac{\partial F}{\partial \lambda_i}, \quad \mu_i \frac{\partial \bar{F}}{\partial \mu_i} = \frac{\partial F}{\partial \lambda_i}.
\]
Moreover, it follows from (2.17) that
\[
\left\{ \begin{array}{l}
\bar{F} (\lambda (D^2\bar{u})) = -f (D\bar{u}) - c, \quad \bar{x} \in \bar{\Omega}, \\
D\bar{u}(\bar{\Omega}) = \Omega.
\end{array} \right. \tag{3.2}
\]
**Remark 3.2** By Lemma 3.1, if $u$ is a smooth uniformly convex solution of (2.17), then the eigenvalues of $D^2u$ and $D^2\bar{u}$ must be in $\Gamma_1$ and $\Gamma_1^+$, respectively. Therefore, $\bar{F}$ also satisfies the structure conditions (2.14) and (2.15).

Next, we will carry out the strictly oblique estimate. Let $\mathcal{P}_n$ be the set of positive definite symmetric $n \times n$ matrices, and $\lambda_1 (A), \ldots, \lambda_n (A)$ be the eigenvalues of $A$. For $A = (a_{ij}) \in \mathcal{P}_n$, denote
\[
F[A] := F (\lambda_1 (A), \ldots, \lambda_n (A))
\]
and
\[
(a^{ij}) = (a_{ij})^{-1}, \quad F^{ij} = \frac{\partial F}{\partial a_{ij}}, \quad F^{ij, rs} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{rs}}.
\]

**Definition 3.3** A smooth function $h : \mathbb{R}^n \to \mathbb{R}$ is called the defining function of $\bar{\Omega}$ if
\[
\bar{\Omega} = \{ p \in \mathbb{R}^n : h(p) > 0 \}, \quad |Dh|_{\partial \bar{\Omega}} = 1,
\]
and there exists $\theta > 0$ such that for any $p = (p_1, \ldots, p_n) \in \bar{\Omega}$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,
\[
\frac{\partial^2 h}{\partial p_i \partial p_j} \xi_i \xi_j \leq -\theta |\xi|^2.
\]
Therefore, the diffeomorphism condition $Du(\Omega) = \bar{\Omega}$ in (2.17) is equivalent to
\[
h(Du) = 0, \quad x \in \partial \Omega. \tag{3.3}
\]
Then (2.17) can be rewritten as
\[
\left\{ \begin{array}{l}
F[D^2u] = f(x) + c, \quad x \in \Omega, \\
h(Du) = 0, \quad x \in \partial \Omega.
\end{array} \right. \tag{3.4}
\]
This is an oblique boundary value problem of second order fully nonlinear elliptic equation. We also denote $\beta = (\beta^1, \ldots, \beta^n)$ with $\beta^i := h_{pi} (Du)$, and $\nu = (\nu_1, \ldots, \nu_n)$ as the unit inward normal vector at $x \in \partial \Omega$. The expression of the inner product is
\[
\langle \beta, \nu \rangle = \beta^i \nu_i.
\]

**Lemma 3.4** (See J. Urbas [7].) Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be the unit inward normal vector of $\partial \Omega$. If $u \in C^2(\bar{\Omega})$ with $D^2u \geq 0$, then there holds $h_{pi} (Du) \nu_k \geq 0$. 
Now, we can present

**Lemma 3.5** Let $F$ satisfy the structure conditions (2.10)-(2.15) and $f \in \mathcal{A}_\delta$ satisfying (2.18). If $u$ is a uniformly smooth convex solution of (2.17), then the strict obliqueness estimate

$$\langle \beta, v \rangle \geq \frac{1}{C_1} > 0$$

(3.5)

holds on $\partial \Omega$ for some universal constant $C_1$, which depends only on $F$, $\Omega$, $\tilde{\Omega}$ and $\delta$.

**Remark 3.6** Without loss of generality, in the following we set $C_1, C_2, \ldots$, to be constants depending only on the known data.

**Proof** Define

$$v = \langle \beta, v \rangle + h(Du).$$

Let $x_0 \in \partial \Omega$ such that

$$\langle \beta, v \rangle(x_0) = h_{p_k}(Du(x_0))v_k(x_0) = \min_{\partial \Omega} \langle \beta, v \rangle.$$

By rotation, we may assume that $v(x_0) = (0, \ldots, 0, 1) =: e_n$. Using the above assumptions and the boundary condition, we obtain

$$v(x_0) = \min_{\partial \Omega} v = h_{p_n}(Du(x_0)).$$

By the convexity of $\Omega$ and its smoothness, we extend $v$ smoothly to a tubular neighborhood of $\partial \Omega$ such that in the matrix sense

$$(v_{kl}) := (D_k v_l) \leq -\frac{1}{C_2} \text{diag}(1, \ldots, 1, 0),$$

where $C_2$ is a positive constant depending only on $\partial \Omega$. By Lemma 3.4, we see that $h_{p_n}(Du(x_0)) \geq 0$.

At $x_0$ we have

$$0 = v_r = h_{p_n p_k} u_{kr} + h_{p_k} v_{kr} + h_{p_k} u_{kr}, \quad 1 \leq r \leq n - 1.$$  

(3.7)

We assume that the following key estimate

$$v_n(x_0) > -C_3$$

holds which will be proved later, where $C_3$ is a constant depending only on $\Omega, h, \tilde{h}$ and $\delta$.

It’s not hard to check that (3.8) can be rewritten as

$$h_{p_n p_k} u_{kn} + h_{p_k} v_{kn} + h_{p_k} u_{kn} > -C_3.$$  

(3.9)

Multiplying (3.9) with $h_{p_n}$ and (3.7) with $h_{p_r}$ respectively, and summing up together, we obtain

$$h_{p_k} h_{p_l} u_{kl} \geq -C_3 h_{p_n} - h_{p_k} h_{p_l} v_{kl} - h_{p_k} h_{p_n p_l} u_{kl}.$$  

(3.10)

Using (3.6) and

$$1 \leq r \leq n - 1, \quad h_{p_k} u_{kr} = \frac{\partial h(Du)}{\partial x_r} = 0, \quad h_{p_k} u_{kn} = \frac{\partial h(Du)}{\partial x_n} \geq 0, \quad -h_{p_n p_n} \geq 0,$$

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we have
\[ h_{pk} h_{pl} u_{kl} \geq -C_3 h_{pn} + \frac{1}{C_2} |Dh|^2 - \frac{1}{C_2} h_{pn}^2 \]
\[ \geq -C_4 h_{pn} + \frac{1}{C_4} h_{pn}^2, \]
where we use $|Dh|^2 - h_{pn}^2 = \sum_{k=1}^{n-1} h_{pk}^2$ and let $C_4 = \max\{C_2, C_3\}$. For the last term of the above inequality, we distinguish two cases at $x_0$.

Case (i). If
\[ -C_4 h_{pn} + \frac{1}{C_4} h_{pn}^2 \leq \frac{1}{2C_4}, \]
then
\[ h_{pk}(Du)\nu_k = h_{pn} \geq \sqrt{\frac{1}{2} + \frac{C_4^4}{4} - \frac{C_2^2}{2}}. \]
It shows that there is a uniform positive lower bound for the quantity $\min_{\partial\Omega} \langle \beta, \nu \rangle$.

Case (ii). If
\[ -C_4 h_{pn} + \frac{1}{C_4} h_{pn}^2 > \frac{1}{2C_4}, \]
then we obtain a positive lower bound of $h_{pk} h_{pl} u_{kl}$.

Let $\tilde{u}$ be the Legendre transformation of $u$, then $\tilde{u}$ satisfies
\[
\begin{cases}
\tilde{F}[D^2 \tilde{u}] = -f(D\tilde{u}) - c, & \tilde{x} \in \tilde{\Omega}, \\
\tilde{h}(D\tilde{u}) = 0, & \tilde{x} \in \partial\tilde{\Omega},
\end{cases}
\]
(3.11)
where $\tilde{h}$ is the defining function of $\tilde{\Omega}$. That is,
\[ \Omega = \{ \tilde{p} \in \mathbb{R}^n : \tilde{h}(\tilde{p}) > 0 \}, \quad |D\tilde{h}|_{\tilde{\Omega}} = 1, \quad D^2 \tilde{h} \leq -\tilde{\theta} I, \]
where $\tilde{\theta}$ is some positive constant. The unit inward normal vector of $\partial\Omega$ can be expressed by $\nu = D\tilde{h}$. For the same reason, $\tilde{\nu} = Dh$, where $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_n)$ is the unit inward normal vector of $\partial\tilde{\Omega}$.

Let $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_n)$ with $\tilde{\beta}_k := \tilde{h}_{pk}(D\tilde{u})$. We also define
\[ \tilde{\nu} = \langle \tilde{\beta}, \tilde{\nu} \rangle + \tilde{h}(D\tilde{u}), \]
in which
\[ \langle \tilde{\beta}, \tilde{\nu} \rangle = \langle \beta, \nu \rangle. \]

Denote $\tilde{x}_0 = Du(x_0)$. Then $\tilde{\nu}(\tilde{x}_0) = \nu(x_0) = \min_{\partial\tilde{\Omega}} \tilde{\nu}$. Using the same methods, under the assumption of
\[ \tilde{\nu}_n(\tilde{x}_0) \geq -C_5, \]
(3.12)
we obtain the positive lower bounds of $h_{pk} h_{pl} \tilde{u}_{kl}$, or
\[ h_{pk}(Du)\nu_k = \tilde{h}_{pk}(D\tilde{u})\tilde{\nu}_k \tilde{h}_{pn} \geq \sqrt{\frac{1}{2} + \frac{C_5^4}{4} - \frac{C_5^2}{2}}. \]
We notice that
\[ \tilde{h}_{pkl} = v_i v_j u^{ij}. \]

Then by the positive lower bounds of \( \tilde{h}_{pkl} \), the lemma follows from
\[
\langle \beta, \nu \rangle = \sqrt{\tilde{h}_{pkl} u^{ij} v_i v_j},
\]
which is proved in [7].

It remains to prove the key estimate (3.8) and (3.12). We prove (3.8) first. By \( D^2 \tilde{h} \leq -\tilde{\theta}I \) and (2.14) we have
\[
L \tilde{h} \leq -\tilde{\theta} \sum_{i=1}^{n} F^{ii},
\]
where \( L := F^{ij} \partial_{ij} \). On the other hand,
\[
L v = h_{pkpi} F^{ijk} u^{ij} u_{mj} + 2 h_{pkpi} F^{ij} v_{kj} u^{ij} u_{li} + h_{pkpi} F^{ij} u_{lj} u_{ki} + h_{pkpi} v_{k} L u_{l} + h_{pi} L v_{k} + h_{pi} L u_{k}.
\]

At first we estimate the first term on the right hand side of (3.15). By the diagonal basis and (2.15), we have
\[
|h_{pkpi} v_{k} F^{ij} u^{ij} u_{mj}| \leq C \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \leq C_6,
\]
where \( C_6 \) is a constant depending only on \( h, \Omega, \Lambda_2 \). Similarly, we also get
\[
|h_{pkpi} F^{ij} u_{lj} u_{ki}| \leq C \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \leq C_7.
\]

For the second term, by Cauchy inequality, we obtain
\[
|2 h_{pkpi} F^{ij} v_{kj} u^{ij}| \leq C \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i = C \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \sqrt{\sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2}
\leq C \left( \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \right)^{1/2}
\leq C_8.
\]

By (2.17) we have \( L u_l = f_l \). Then we get
\[
|h_{pkpi} v_{k} L u_{l}| \leq C_9, \quad |h_{pi} L u_{k}| \leq C_{10}.
\]

It follows from (2.14) that
\[
|h_{pi} L v_{k}| \leq C_{11} \sum_{i=1}^{n} F^{ii}.
\]

Inserting these into (3.15) and using (2.14), it is immediate to check that there exists a positive constant \( C_{12} \) depending only on \( h, \Omega, \Lambda_1, \Lambda_2, \Theta_0 \) and \( \delta \), such that
\[
L v \leq C_{12} \sum_{i=1}^{n} F^{ii}.
\]
Denote a neighborhood of $x_0$ in $\Omega$ by
\[ \Omega_\rho := \Omega \cap B_\rho(x_0), \]
where $\rho$ is a positive constant such that $v$ is well defined in $\Omega_\rho$. To obtain the desired results, it suffices to consider the function
\[ \Phi(x) := v(x) - v(x_0) + C_0 \tilde{h}(x) + A|x - x_0|^2, \]
where $C_0$ and $A$ are positive constants to be determined. On $\partial \Omega$, it is clear that $\Phi \geq 0$. Since $v$ is bounded, we can choose $A$ large enough such that on $\Omega_\rho \cap \partial B_\rho(x_0)$
\[ \Phi(x) = v(x) - v(x_0) + C_0 \tilde{h}(x) + A \rho^2 > 0. \]
It follows from (3.14) that
\[ L(C_0 \tilde{h}(x) + A|x - x_0|^2) \leq (-C_0 \bar{\theta} + 2A) \sum_{i=1}^{n} F^{ii}. \]
Then using (3.17) and choosing $C_0 \gg A$ we have
\[ L \Phi(x) \leq 0. \]
Therefore,
\[ \begin{cases} 
L \Phi \leq 0, & x \in \Omega_\rho, \\
\Phi \geq 0, & x \in \partial \Omega_\rho. 
\end{cases} \tag{3.18} \]
We apply the maximum principle to get
\[ \Phi|_{\Omega_\rho} \geq \min_{\partial \Omega_\rho} \Phi \geq 0. \]
Combining it with $\Phi(x_0) = 0$, we obtain $\partial_n \Phi(x_0) \geq 0$, which gives the desired estimate (3.8).

Finally, we prove (3.12). The proof of (3.12) is similar to the one of (3.8). Define
\[ \tilde{L} := \tilde{F}^{ij} \tilde{\partial}_{ij} + f_{\tilde{p}} \tilde{\partial}_{\tilde{p}}. \]
By (3.11), we see that $\tilde{L} \tilde{u}_l = 0$, and thus
\[ \tilde{L} \tilde{v} = \tilde{F}^{ij} \tilde{u}_{mj} \tilde{u}_{li} \tilde{h}_{\tilde{p}k} \tilde{u}_{\tilde{p}m} \tilde{v}_k + 2 \tilde{h}_{\tilde{p}k} \tilde{F}^{ij} \tilde{u}_{li} \tilde{v}_{kj} + \tilde{F}^{ij} \tilde{h}_{\tilde{p}k} \tilde{v}_{ki} + \tilde{h}_{\tilde{p}k} \tilde{F}^{ij} \tilde{u}_{lj} \tilde{u}_{ki} + \tilde{h}_{\tilde{p}k} f_{\tilde{p}} \tilde{v}_k. \]
By making use of the following identities
\[ \frac{\partial \tilde{F}}{\partial \mu_i} = \lambda^2_i \frac{\partial F}{\partial \lambda_i}, \quad \mu_i^2 \frac{\partial \tilde{F}}{\partial \mu_i} = \frac{\partial F}{\partial \lambda_i}. \]
we deduce that $\tilde{F}$ satisfies the structure conditions (2.10)-(2.15). Repeating the proof of (3.17), we have
\[ \tilde{L} \tilde{v} \leq C_{13} \sum_{i=1}^{n} \tilde{F}^{ii}, \tag{3.19} \]
where $C_{13}$ depends only on $\Omega$, $\tilde{\Omega}$, $\Theta_0$ and $\delta$.

Denote a neighborhood of $\tilde{x}_0$ in $\tilde{\Omega}$ by
\[ \tilde{\Omega}_r := \tilde{\Omega} \cap B_r(\tilde{x}_0), \]
where \( r \) is a positive constant such that \( \tilde{\nu} \) is well defined in \( \tilde{\Omega}_r \). Consider

\[
\tilde{\Phi}(y) \equiv \tilde{\nu}(y) - \tilde{\nu}(\tilde{x}_0) + \tilde{C}_0 h(y) + \tilde{A}|y - \tilde{x}_0|^2,
\]

where \( \tilde{C}_0 \) and \( \tilde{A} \) are positive constants to be determined. It is clear that \( \tilde{\Phi} \geq 0 \) on \( \partial \tilde{\Omega} \). Since \( \tilde{\nu} \) is bounded, we can choose \( \tilde{A} \) large enough such that on \( \tilde{\Omega} \cap \partial B_r(\tilde{x}_0) \)

\[
\tilde{\Phi}(y) = \tilde{\nu}(y) - \tilde{\nu}(\tilde{x}_0) + \tilde{C}_0 h(y) + \tilde{A}|y - \tilde{x}_0|^2 > 0.
\]

By (2.15) and (3.19), it is not difficult to show that

\[
\tilde{L} \tilde{\Phi}(y) \leq \left( C_{13} - \frac{\tilde{C}_0 \theta}{2} + 2\tilde{A} \right) \sum_{i=1}^{n} \tilde{F}^{ii} + 2\tilde{A} \tilde{f}_{\tilde{P}_i}(y_i - \tilde{x}_0) - \tilde{C}_0 \left( \frac{\theta}{2} \sum_{i=1}^{n} \tilde{F}^{ii} - \tilde{f}_{\tilde{P}_i} \tilde{h}_i \right).
\]

In order to make

\[
\tilde{L} \tilde{\Phi}(y) \leq 0,
\]

we only need to choose \( \tilde{C}_0 \gg \tilde{A} \) and

\[
|Df| \leq \frac{\theta \Lambda_1}{2} \cdot \frac{1}{\max_{\tilde{\Omega}} |Dh|}.
\]

Consequently,

\[
\begin{cases}
\tilde{L} \tilde{\Phi} \leq 0, & y \in \tilde{\Omega}_r, \\
\tilde{\Phi} \geq 0, & y \in \partial \tilde{\Omega}_r.
\end{cases}
\]

Therefore, we get (3.12) as same as the argument in (3.8). Thus the proof of (3.5) is completed.

\[\square\]

### 4 The \( C^2 \) estimate

The following definition provides a basic connection between (3.2) and (2.17) and will be used frequently in the sequel.

**Definition 4.1** We say that \( \tilde{\nu} \) in (3.2) is a dual solution to (2.17).

We now proceed to carry out the \( C^2 \) estimate. The strategy is to reduce the \( C^2 \) global estimate of \( u \) and \( \tilde{\nu} \) to the boundary.

**Lemma 4.2** If \( u \) is a smooth uniformly convex solution of (2.17) and there hold (2.11), (2.12) and (2.14), then there exists a positive constant \( C_{14} \) depending only on \( n, \Omega, \tilde{\Omega}, \Lambda_1 \) and \( \text{diam}(\Omega) \), such that

\[
\sup_{\Omega} |D^2 u| \leq \max_{\partial \Omega} |D^2 u| + C_{14} \sup_{\Omega} |D^2 f|.
\]

**Proof** Without loss of generality, we may assume that \( \Omega \) lies in cube \([0, d]^n\). Let

\[
L := F_{ij} \tilde{\partial}_j.
\]

For any unit vector \( \xi \), differentiating the equation in (2.17) twice in direction \( \xi \) gives

\[
L u_{\xi \xi} + F_{ij,rs} u_{ij \xi} u_{rs \xi} = f_{\xi \xi}.
\]
Then by the concavity of $F$ on $\Gamma_1^+$, we have

$$Lu_{\xi\xi} = -F^{ij,rs}u_{ij\xi}u_{rs\xi} + f_{\xi\xi} \geq f_{\xi\xi}. \quad (4.2)$$

Let

$$v = \sup_{\partial\Omega} u_{\xi\xi} + \frac{1}{\Lambda_1} \left( ne^d - \sum_{i=1}^n e^{xi} \right) \sup_\Omega |f_{\xi\xi}|.$$

By direct calculation and (2.14), we obtain

$$Lv = -\frac{1}{\Lambda_1} \sup_\Omega |f_{\xi\xi}| \left( \sum_{i=1}^n e^{xi} F^{ii} \right) \leq -\frac{1}{\Lambda_1} \sup_\Omega |f_{\xi\xi}| \left( \sum_{i=1}^n F^{ii} \right) \leq -\sup_\Omega |f_{\xi\xi}|. \quad (4.3)$$

Combining (4.2) with (4.3), we have

$$L(v - u_{\xi\xi}) \leq -\left( \sup_\Omega |f_{\xi\xi}| + f_{\xi\xi} \right) \leq 0.$$

It is obvious that $v - u_{\xi\xi} \geq 0$ on $\partial\Omega$. Then by the maximum principle we obtain

$$\sup_\Omega u_{\xi\xi} \leq \sup_\Omega v \leq \sup_{\partial\Omega} u_{\xi\xi} + \frac{ne^d}{\Lambda_1} \sup_\Omega |f_{\xi\xi}|.$$

This completes the proof of (4.1). \qed

Next, we estimate the second order derivative on the boundary. By differentiating the boundary condition $h(Du) = 0$ in any tangential direction $\varsigma$, we have

$$u_{\beta\varsigma} = h_{pk} (Du) u_{k\varsigma} = 0. \quad (4.4)$$

The second order derivative of $u$ on the boundary is controlled by $u_{\beta\varsigma}$, $u_{\beta\beta}$ and $u_{\varsigma\varsigma}$. In the following we give the arguments as in [7], one can see there for more details.

At $x \in \partial\Omega$, any unit vector $\xi$ can be written in terms of a tangential component $\varsigma(\xi)$ and a component in the direction $\beta$ by

$$\xi = \varsigma(\xi) + \frac{\langle v, \xi \rangle}{\langle \beta, v \rangle} \beta,$$

where

$$\varsigma(\xi) := \xi - \langle v, \xi \rangle v - \frac{\langle v, \xi \rangle}{\langle \beta, v \rangle} \beta^T,$$

and

$$\beta^T := \beta - \langle \beta, v \rangle v.$$
By the strict obliqueness estimate (3.5), we have
\[
|\varsigma(\xi)|^2 = 1 - \left( 1 - \frac{|\beta^T|^2}{\langle \beta, v \rangle^2} \right) \langle v, \xi \rangle^2 - 2 \langle v, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, v \rangle} \leq 1 + C_{15} \langle v, \xi \rangle^2 - 2 \langle v, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, v \rangle} \langle \beta, v \rangle^2 - 2 \langle v, \xi \rangle \langle \beta^T, \xi \rangle \langle \beta, v \rangle \leq 1 + C_{16}. \tag{4.5}
\]

Denote \( \varsigma := \varsigma(\xi)/|\varsigma(\xi)| \), then by (4.4), (4.5) and (3.5) we obtain
\[
u_{xx} = |\varsigma(\xi)|^2 \nu_{ss} + 2 |\varsigma(\xi)| \frac{\langle v, \xi \rangle}{\langle \beta, v \rangle} \nu_{s \beta} + \frac{\langle v, \xi \rangle^2}{\langle \beta, v \rangle^2} \nu_{\beta \beta}
\]
\[
= |\varsigma(\xi)|^2 \nu_{ss} + \frac{\langle v, \xi \rangle^2}{\langle \beta, v \rangle^2} \nu_{\beta \beta} \leq C_{17} (\nu_{ss} + \nu_{\beta \beta}). \tag{4.6}
\]

where \( C_{17} \) depends only on \( \Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta \) and the constant \( C_1 \) in (3.5). Therefore, we only need to estimate \( \nu_{\beta \beta} \) and \( \nu_{ss} \) respectively.

First we have

**Lemma 4.3** Let \( F \) satisfy the structure conditions (2.10)-(2.15) and \( f \in \mathcal{A}_\delta \). If \( u \) is a smooth uniformly convex solution of (2.17), then there exists a positive constant \( C_{18} \) depending only on \( \Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta \), such that
\[
\max_{\partial \Omega} \nu_{\beta \beta} \leq C_{18}. \tag{4.7}
\]

**Proof** Let \( x_0 \in \partial \Omega \) satisfy \( \nu_{\beta \beta}(x_0) = \max_{\partial \Omega} \nu_{\beta \beta} \). Consider the barrier function
\[
\Psi := -h(Du) + C_0 \tilde{h} + A|x - x_0|^2.
\]

For any \( x \in \partial \Omega \), \( Du(x) \in \partial \tilde{\Omega} \), then \( h(Du) = 0 \). It is clear that \( \tilde{h} = 0 \) on \( \partial \tilde{\Omega} \). As the proof of (3.18), we can find the constants \( C_0 \) and \( A \) such that
\[
\begin{cases}
L \Psi \leq 0, & x \in \Omega_\rho, \\
\Psi \geq 0, & x \in \partial \Omega_\rho.
\end{cases} \tag{4.8}
\]

By the maximum principle, we get
\[
\Psi(x) \geq 0, \quad x \in \Omega_\rho.
\]
Combining it with \( \Psi(x_0) = 0 \) we obtain \( \Psi_{\beta}(x_0) \geq 0 \), which implies
\[
\frac{\partial h}{\partial \beta} (Du(x_0)) \leq C_0.
\]

On the other hand, we see that at \( x_0 \),
\[
\frac{\partial h}{\partial \beta} = (Dh(Du), \beta) = \frac{\partial h}{\partial p_k} u_{kl} \beta^l = \beta^k u_{kl} \beta^l = \nu_{\beta \beta}.
\]

Let \( C_{18} = C_0 \). Therefore,
\[
\nu_{\beta \beta} \leq C_{18}.
\]

\( \square \)
Next, we estimate the double tangential derivative.

**Lemma 4.4** Let $F$ satisfy the structure conditions (2.10)-(2.15) and $f \in \mathcal{A}_\delta$ satisfying (2.18). If $u$ is a smooth uniformly convex solution of (2.17), then there exists a positive constant $C_{19}$ depending only on $F$, $\Omega$, $\tilde{\Omega}$, $\Lambda_1$, $\Lambda_2$, $\delta$, and $\sup_{\Omega} |D^2 f|$, such that

$$\max_{\partial \Omega} u_{\xi\xi} \leq C_{19}. \quad (4.9)$$

**Proof** Assume that $u_{\xi\xi}|_{\partial \Omega}$ attains its maximum at $x_0 \in \partial \Omega$. Let

$$M := u_{11}(x_0) = \max_{\partial \Omega} u_{\xi\xi},$$

and $e_n$ be the unit inward normal vector of $\partial \Omega$ at $x_0$.

For any $x \in \partial \Omega$, we have by (4.5),

$$u_{\xi\xi} = |\xi(\xi)|^2 u_{\xi\xi} + \frac{\langle v, \xi \rangle^2}{\langle \beta, v \rangle^2} u_{\beta\beta} \leq \left( 1 + C_{20} |\xi|^2 - 2 \frac{\langle v, \xi \rangle}{\langle \beta, v \rangle} \right) M + \frac{\langle v, \xi \rangle^2}{\langle \beta, v \rangle^2} u_{\beta\beta}. \quad (4.10)$$

Without loss of generality, we assume that $M \geq 1$. Then by (3.5) and (4.7) we have

$$\frac{u_{\xi\xi}}{M} + 2 \frac{\langle v, \xi \rangle}{\langle \beta, v \rangle} \frac{\langle \beta^T, \xi \rangle}{\langle \beta, v \rangle} \leq 1 + C_{21} |\xi|^2. \quad (4.11)$$

If $\xi = e_1$, then

$$\frac{u_{11}}{M} + 2 \frac{\langle v, e_1 \rangle}{\langle \beta, v \rangle} \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, v \rangle} \leq 1 + C_{21} |e_1|^2. \quad (4.12)$$

As in the proof of Proposition 2.14 in [9], let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function satisfying $\eta(s) = s$ for $s \geq \frac{1}{C_1}$ and $\eta(s) \geq \frac{1}{2C_1}$ for all $s \in \mathbb{R}$. We see that the function

$$w := A|x - x_0|^2 - \frac{u_{11}}{M} - 2 \frac{\langle v, e_1 \rangle}{\eta(\langle \beta, v \rangle)} \frac{\langle \beta^T, e_1 \rangle}{\eta(\langle \beta, v \rangle)} + C_{21} |v, e_1|^2 + 1 \quad (4.13)$$

satisfies

$$w|_{\partial \Omega} \geq 0, \quad w(x_0) = 0.$$ 

Then, it follows by (4.1) that we can choose the constant $A$ large enough such that

$$w|_{\Omega \cap \partial B_{\rho}(x_0)} \geq 0.$$ 

Remembering the operator $L := F^{ij} \partial_{ij}$. We see that

$$L(A|x - x_0|^2) = 2A \sum_{i=1}^n F^{ii},$$

$$L(-2 \frac{\langle v, e_1 \rangle}{\eta(\langle \beta, v \rangle)} \frac{\langle \beta^T, e_1 \rangle}{\eta(\langle \beta, v \rangle)} + C_{21} |v, e_1|^2 + 1) \leq \hat{C} \sum_{i=1}^n F^{ii}. \quad (4.14)$$

By (2.17), we deduce that

$$Lu_{11} = -F^{ij,rs} u_{ij1} u_{rs1} + f_{11} \geq f_{11},$$
where we use the concavity of $F$ on $\Gamma^+_n$. It yields
\[
L \left( -\frac{u_{11}}{M} \right) \leq -\frac{f_{11}}{M}.
\]
Thus we obtain
\[
Lw \leq \hat{C} \sum_{i=1}^n F^{ii}
\]
for some constant $\hat{C}$ on the known data. As in the proof of Lemma 4.3, we consider the function
\[
\Upsilon := w + C_0\tilde{h}.
\]
A standard barrier argument shows that
\[
\Upsilon_\beta(x_0) \geq 0.
\]
Therefore,
\[
u_{11\beta}(x_0) \leq \hat{C}_0 M.
\]
for some constant $\hat{C}_0$ on the known data.

On the other hand, differentiating $h(Du)$ twice in the direction $e_1$ at $x_0$, we have
\[
h_{p_k}u_{k11} + h_{p_k p_l}u_{k1}u_{l1} = 0.
\]
The concavity of $h$ yields that
\[
h_{p_k}u_{k11} = -h_{p_k p_l}u_{k1}u_{l1} \geq \theta M^2.
\]
Combining it with $h_{p_k}u_{k11} = u_{11\beta}$, and using (4.14) we obtain
\[
\theta M^2 \leq \hat{C}_0 M.
\]
Then we get the upper bound of $M = u_{11}(x_0)$ and thus the desired result follows. \qed

By Lemma 4.3, Lemma 4.4 and (4.6), we obtain the $C^2$ a-priori estimate on the boundary.

**Lemma 4.5** Let $F$ satisfy the structure conditions (2.10)-(2.15) and $f \in \mathcal{A}_\delta$ satisfying (2.18). If $u$ is a smooth uniformly convex solution of (2.17), then there exists a positive constant $C_{22}$ depending only on $F$, $\Omega$, $\bar{\Omega}$, $\Lambda_1$, $\Lambda_2$, $\delta$ and $\sup_{\Omega_1} |D^2 f|$, such that
\[
\max_{\partial \Omega} |D^2 u| \leq C_{22}.
\]

In terms of Lemma 4.2 and Lemma 4.5, we see that

**Lemma 4.6** Let $F$ satisfy the structure conditions (2.10)-(2.15) and $f \in \mathcal{A}_\delta$ satisfying (2.18). If $u$ is a smooth uniformly convex solution of (2.17), then there exists a positive constant $C_{23}$ depending only on $F$, $\Omega$, $\bar{\Omega}$, $\Lambda_1$, $\Lambda_2$, $\delta$ and $\sup_{\Omega_1} |D^2 f|$, such that
\[
\max_{\Omega} |D^2 u| \leq C_{23}.
\]
In the following, we describe the positive lower bound of $D^2 u$. For (3.11), in consider of the Legendre transformation of $u$, one can take the linearized operator

$$
\tilde{L} := \tilde{F}^{ij} \partial_{ij} + f_{\tilde{p}_i} \partial_{i}.
$$

Then our goal is to show the upper bound of $D^2 \tilde{u}$ and the argument is very similar to the one used in the proof of Lemma 4.6 by the concavity of $f$ and the condition that $|Df|$ is sufficiently small. For the convenience of readers, we give the details.

At the beginning of the repeating procedure, we have

**Lemma 4.7** Suppose that $f$ is concave on $\Omega$. If $\tilde{u}$ is a smooth uniformly convex solution of (3.11), then there holds

$$
\sup_{\tilde{\Omega}} |D^2 \tilde{u}| \leq \max_{\partial \tilde{\Omega}} |D^2 \tilde{u}|.
$$

**Proof** For any unit vector $\tilde{\xi}$, differentiating the equation in (3.11) twice in direction $\tilde{\xi}$ gives

$$
\tilde{L} \tilde{u}_{\tilde{\xi} \tilde{\xi}} + \tilde{F}^{ij,rs} \tilde{u}_{ij} \tilde{u}_{rs} \tilde{\xi} + \frac{\partial^2 f}{\partial \tilde{p}_i \partial \tilde{p}_j} \tilde{u}_{i \tilde{\xi}} \tilde{u}_{j \tilde{\xi}} = 0.
$$

Then by the concavity of $\tilde{F}$ on $\Gamma_n^+$ and $f$ on $\Omega$, we have

$$
\tilde{L} \tilde{u}_{\tilde{\xi} \tilde{\xi}} = -\tilde{F}^{ij,rs} \tilde{u}_{ij} \tilde{u}_{rs} \tilde{\xi} - \frac{\partial^2 f}{\partial \tilde{p}_i \partial \tilde{p}_j} \tilde{u}_{i \tilde{\xi}} \tilde{u}_{j \tilde{\xi}} \geq 0.
$$

Then by the maximum principle we obtain

$$
\sup_{\tilde{\Omega}} \tilde{u}_{\tilde{\xi} \tilde{\xi}} \leq \sup_{\partial \tilde{\Omega}} \tilde{u}_{\tilde{\xi} \tilde{\xi}}.
$$

This completes the proof of (4.17).

Recall that $\tilde{\beta} = (\tilde{\beta}^1, \ldots, \tilde{\beta}^n)$ with $\tilde{\beta}^k := \tilde{h}_{\tilde{p}_k}(D\tilde{u})$ and $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_n)$ is the unit inward normal vector of $\partial \tilde{\Omega}$. Similar to the discussion of (4.4), (4.5) and (4.6), for any tangential direction $\tilde{\xi}$, we have

$$
\tilde{u}_{\tilde{\beta} \tilde{\xi}} = \tilde{h}_{\tilde{p}_k}(D\tilde{u}) \tilde{u}_{k \tilde{\xi}} = 0.
$$

Then the second order derivative of $\tilde{u}$ on the boundary is also controlled by $\tilde{u}_{\tilde{\beta} \tilde{\xi}}$, $\tilde{u}_{\tilde{\beta} \tilde{\beta}}$ and $\tilde{u}_{\tilde{\xi} \tilde{\xi}}$.

At $\tilde{x} \in \partial \tilde{\Omega}$, any unit vector $\tilde{\xi}$ can be written in terms of a tangential component $\tilde{\xi}(\tilde{\xi})$ and a component in the direction $\tilde{\beta}$ by

$$
\tilde{\xi} = \tilde{\xi}(\tilde{\xi}) + \frac{\langle \tilde{\nu}, \tilde{\xi} \rangle}{\langle \tilde{\beta}, \tilde{\nu} \rangle} \tilde{\beta},
$$

where

$$
\tilde{\xi}(\tilde{\xi}) := \tilde{\xi} - \langle \tilde{\nu}, \tilde{\xi} \rangle \tilde{\nu} - \frac{\langle \tilde{\nu}, \tilde{\xi} \rangle}{\langle \tilde{\beta}, \tilde{\nu} \rangle} \tilde{\beta},
$$

and

$$
\tilde{\beta}^T := \tilde{\beta} - \langle \tilde{\beta}, \tilde{\nu} \rangle \tilde{\nu}.
$$
Therefore,

$$|\tilde{\xi}(\tilde{\xi})| \leq C_{24},$$

(4.19)

and

$$\tilde{u}_{\tilde{\xi}\tilde{\xi}} \leq C(\tilde{u}_{\tilde{\xi}\tilde{\xi}} + \tilde{u}_{\tilde{\beta}\tilde{\beta}}),$$

(4.20)

where $\tilde{\xi} := \tilde{\xi}(\tilde{\xi})$ and $C_{24}$ depends only on $\Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta$ and the constant $C_1$ in (3.5).

Then we also only need to estimate $\tilde{u}_{\tilde{\beta}\tilde{\beta}}$ and $\tilde{u}_{\tilde{\xi}\tilde{\xi}}$ respectively.

Indeed, as shown by Lemma 4.3, we state

**Lemma 4.8** Let $F$ satisfy the structure conditions (2.10)-(2.15) and $f \in \mathcal{A}_\delta$ satisfying (2.18). If $\tilde{u}$ is a smooth uniformly convex solution of (3.11), then there exists a positive constant $C_{25}$ depending only on $F, \Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta$, such that

$$\max_{\partial \tilde{\Omega}} \tilde{u}_{\tilde{\beta}\tilde{\beta}} \leq C_{25}. \quad (4.21)$$

**Proof** Let $\tilde{x}_0 \in \partial \tilde{\Omega}$ satisfy $\tilde{u}_{\tilde{\beta}\tilde{\beta}}(\tilde{x}_0) = \max_{\partial \tilde{\Omega}} \tilde{u}_{\tilde{\beta}\tilde{\beta}}$. To estimate the upper bound of $\tilde{u}_{\tilde{\beta}\tilde{\beta}}$, we consider the barrier function

$$\tilde{\Psi} := -\tilde{h}(\partial \tilde{\Omega}) + C_0 h + A|y - \tilde{x}_0|^2.$$

For any $y \in \partial \tilde{\Omega}$, $\partial \tilde{\Omega}(y) \in \partial \Omega$, then $\tilde{h}(\partial \tilde{\Omega}) = 0$. It is clear that $h = 0$ on $\partial \tilde{\Omega}$.

Similar to the proof of (3.20), first we have

$$\tilde{L}(C_0 h) = C_0 \left( \tilde{F}_{ij} h_{ij} + f \tilde{p}_i h_i \right) \leq C_0 \left( -\theta \sum_{i=1}^n \tilde{F}^{ii} + |Df| \cdot |Dh| \right),$$

and

$$\tilde{L} \left( A|y - \tilde{x}_0|^2 \right) = 2A \sum_{i=1}^n \tilde{F}^{ii} + 2Af \tilde{p}_i (y_i - \tilde{x}_0 i).$$

Similar to the proof of (3.16), we get

$$\tilde{L} \left( -\tilde{h}(\partial \tilde{\Omega}) \right) = \tilde{F}^{ij} \left( -\tilde{h} \tilde{p}_k \partial_{ki} \tilde{u} \right) \cdot \left( -\tilde{h} \tilde{p}_l \partial_{lj} \tilde{u} \right) \leq C_{26} \sum_{i=1}^n \tilde{F}^{ii}. $$

Therefore, we obtain

$$\tilde{L} \tilde{\Psi}(y) \leq \left( C_{26} - \frac{C_0 \theta}{2} + 2A \right) \sum_{i=1}^n \tilde{F}^{ii} + 2Af \tilde{p}_i (y_i - \tilde{x}_0 i) - C_0 \left( \theta \sum_{i=1}^n \tilde{F}^{ii} - f \tilde{p}_i \partial_i h \right).$$

Since (2.18) holds, we can find the constants $C_0, A$ and $r$ such that

$$\begin{cases} 
\tilde{L} \tilde{\Psi} \leq 0, & y \in \tilde{\Omega}_r, \\
\tilde{\Psi} \geq 0, & y \in \partial \tilde{\Omega}_r.
\end{cases} \quad (4.22)$$

By the maximum principle, we get

$$\tilde{\Psi}(y) \geq 0, \quad y \in \tilde{\Omega}_r.$$
Combining it with $\tilde{\Psi}(\tilde{x}_0) = 0$ we obtain $\tilde{\Psi}_\beta(\tilde{x}_0) \geq 0$, which implies

$$\frac{\partial \tilde{h}}{\partial \beta}(D\tilde{u}(\tilde{x}_0)) \leq C_{27}.$$  

On the other hand, we see that at $\tilde{x}_0$,

$$\frac{\partial \tilde{h}}{\partial \beta} = (D\tilde{h}(D\tilde{u}), \tilde{\beta}) = \frac{\partial \tilde{h}}{\partial \tilde{\beta}_k} \tilde{u}_{kl} \tilde{\beta}^l = \tilde{\beta}^k \tilde{u}_{kl} \tilde{\beta}^l = \tilde{u}_{\beta \beta}.$$  

We choose $C_{25} = C_{27}$. Therefore,

$$\tilde{u}_{\beta \beta} = \frac{\partial \tilde{h}}{\partial \beta} \leq C_{25}.$$  

☐

Next, we estimate the double tangential derivative of $\tilde{u}$.

Lemma 4.9 Let $F$ satisfy the structure conditions (2.10)-(2.15) and $f \in \mathcal{A}_\delta$ which is concave and satisfies (2.18). If $\tilde{u}$ is a smooth uniformly convex solution of (3.11), then there exists a positive constant $C_{26}$ depending only on $F, \Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta$ and $\sup_{\Omega} D^2 f$, such that

$$\max_{\partial \tilde{\Omega}} \tilde{u}_{\xi \xi} \leq C_{26}. \quad (4.23)$$

Proof Assume that $\tilde{u}_{\xi \xi} |_{\partial \tilde{\Omega}}$ attains its maximum at $\tilde{x}_0 \in \partial \tilde{\Omega}$. Let

$$\tilde{M} := \tilde{u}_{11}(\tilde{x}_0) = \max_{\partial \tilde{\Omega}} \tilde{u}_{\xi \xi},$$

and $e_n$ be the unit inward normal vector of $\partial \tilde{\Omega}$ at $\tilde{x}_0$.

For any $y \in \partial \tilde{\Omega}$, we have by (4.19),

$$\tilde{u}_{\xi \xi} = |\tilde{\xi}|^2 \tilde{u}_{\xi \xi} + \frac{\langle \tilde{v}, \tilde{\xi} \rangle^2}{\langle \tilde{\beta}, \tilde{v} \rangle^2} \tilde{u}_{\beta \beta} \leq \left(1 + C_{27} \langle \tilde{v}, \tilde{\xi} \rangle^2 - 2 \langle \tilde{v}, \tilde{\xi} \rangle \frac{\langle \tilde{\beta}^T, \tilde{\xi} \rangle}{\langle \tilde{\beta}, \tilde{v} \rangle}\right) \tilde{M} + \frac{\langle \tilde{v}, \tilde{\xi} \rangle^2}{\langle \tilde{\beta}, \tilde{v} \rangle^2} \tilde{u}_{\beta \beta}. \quad (4.24)$$

Without loss of generality, we assume that $\tilde{M} \geq 1$. Then by (3.5) and (4.21) we have

$$\frac{\tilde{u}_{\xi \xi}}{\tilde{M}} + 2 \langle \tilde{v}, \tilde{\xi} \rangle \frac{\langle \tilde{\beta}^T, \tilde{\xi} \rangle}{\langle \tilde{\beta}, \tilde{v} \rangle} \leq 1 + C_{28} \langle \tilde{v}, \tilde{\xi} \rangle^2. \quad (4.25)$$

Let $\tilde{\xi} = e_1$, then

$$\frac{\tilde{u}_{11}}{\tilde{M}} + 2 \langle \tilde{v}, e_1 \rangle \frac{\langle \tilde{\beta}^T, e_1 \rangle}{\langle \tilde{\beta}, \tilde{v} \rangle} \leq 1 + C_{28} \langle \tilde{v}, e_1 \rangle^2. \quad (4.26)$$

As same as (4.13), let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function satisfying $\eta(s) = s$ for $s \geq \frac{1}{C_1}$ and $\eta(s) \geq \frac{1}{2C_1}$ for all $s \in \mathbb{R}$. We see that the function

$$\bar{w} := A|y - \tilde{x}_0|^2 - \frac{\tilde{u}_{11}}{\tilde{M}} - 2 \langle \tilde{v}, e_1 \rangle \frac{\langle \tilde{\beta}^T, e_1 \rangle}{\eta(\langle \tilde{\beta}, \tilde{v} \rangle)} + C_{28} \langle \tilde{v}, e_1 \rangle^2 + 1 \quad (4.27)$$
satisfies
\[ \tilde{w}|_{\partial \tilde{\Omega}} \geq 0, \quad \tilde{w}(\tilde{x}_0) = 0. \]

Then, by (4.17) we can choose the constant \( A \) large enough such that
\[ \tilde{w}|_{\tilde{\Omega} \cap \partial B_{r}(\tilde{x}_0)} \geq 0. \]

By making use of the concavity of \( \tilde{F} \) and \( f \), it yields
\[ \tilde{L}\tilde{u}_{11} = -\tilde{F}^{ijs} \tilde{u}_{ij1} \tilde{u}_{rs1} - \frac{\partial^2 f}{\partial \tilde{p}_i \partial \tilde{p}_j} \tilde{u}_{i1} \tilde{u}_{j1} \geq 0. \]

By (3.11), \( f \in \mathcal{A}_\delta \) satisfying (2.18), as same as the proof Lemma 4.4, we can show that
\[ \tilde{L}\tilde{w} \leq \tilde{C} \sum_{i=1}^{n} \tilde{F}^{ii}. \]

Here \( \tilde{C} \) depends only on the known data.

Since the proof of Lemma 4.8, let us define
\[ \tilde{\Upsilon} := \tilde{w} + C_0 \tilde{h}. \]

By (2.18), we have
\[ \tilde{L}\tilde{\Upsilon} \leq 0. \]

Then a standard barrier argument makes conclusion of
\[ \tilde{\Upsilon}_{\tilde{\beta}}(\tilde{x}_0) \geq 0. \]

Therefore,
\[ \tilde{u}_{11\tilde{\beta}}(\tilde{x}_0) \leq \tilde{C} \tilde{M}. \] (4.28)

On the other hand, differentiating \( \tilde{h}(D\tilde{u}) \) twice in the direction \( e_1 \) at \( \tilde{x}_0 \), we have
\[ \tilde{h}_{p_k} \tilde{u}_{k11} + \tilde{h}_{p_k p_l} \tilde{u}_{k1} \tilde{u}_{l1} = 0. \]

The concavity of \( \tilde{h} \) yields that
\[ \tilde{h}_{p_k} \tilde{u}_{k11} = -\tilde{h}_{p_k p_l} \tilde{u}_{k1} \tilde{u}_{l1} \geq \tilde{\theta} \tilde{M}^2. \]

Combining it with \( \tilde{h}_{p_k} \tilde{u}_{k11} = \tilde{u}_{11\tilde{\beta}} \), and using (4.28) we obtain
\[ \tilde{\theta} \tilde{M}^2 \leq \tilde{C} \tilde{M}. \]

Then we get the upper bound of \( \tilde{M} = \tilde{u}_{11}(\tilde{x}_0) \) and thus the desired result follows. \( \square \)

By Lemma 4.8, Lemma 4.9 and (4.20), we obtain the \( C^2 \) a-priori estimate of \( \tilde{u} \) on the boundary.

**Lemma 4.10** Let \( F \) satisfy the structure conditions (2.10)-(2.15) and \( f \in \mathcal{A}_\delta \) which is concave and satisfies (2.18). If \( \tilde{u} \) is a smooth uniformly convex solution of (3.11), then there exists a positive constant \( C_{29} \) depending only on \( F, \Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta \), and \( \sup_{\tilde{\Omega}} |D^2 f| \), such that
\[ \max_{\partial \tilde{\Omega}} |D^2 \tilde{u}| \leq C_{29}. \] (4.29)
By Lemma 4.7 and Lemma 4.10, one can see that

\textbf{Lemma 4.11} Let \( F \) satisfy the structure conditions (2.10)-(2.15) and \( f \in \mathcal{A}_\delta \) which is concave and satisfies (2.18). If \( \tilde{u} \) is a smooth uniformly convex solution of (3.11), then there exists a positive constant \( C_{30} \) depending only on \( F, \Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta \) and \( \sup_{\Omega} |D^2 f| \), such that

\[
\max_{\tilde{\Omega}} |D^2 \tilde{u}| \leq C_{30}. \quad (4.30)
\]

By Lemma 4.6 and Lemma 4.11, we conclude that

\textbf{Lemma 4.12} Let \( F \) satisfy the structure conditions (2.10)-(2.15) and \( f \in \mathcal{A}_\delta \) which is concave and satisfies (2.18). If \( u \) is a smooth uniformly convex solution of (2.17), then there exists a positive constant \( C_{31} \) depending only on \( F, \Omega, \tilde{\Omega}, \Lambda_1, \Lambda_2, \delta \) and \( \sup_{\Omega} |D^2 f| \), such that

\[
\frac{1}{C_{31}} I_n \leq D^2 u(x) \leq C_{31} I_n, \quad x \in \tilde{\Omega}, \quad (4.31)
\]

where \( I_n \) is the \( n \times n \) identity matrix.

\section{5 Proof of Theorem 2.3}

In this section, we will prove Theorem 2.3. Two lemmas will be needed when we start the proof. Thanks to the discussion on the uniqueness in the fifth section of \cite{9}, we deduce that

\textbf{Lemma 5.1} If (2.11) holds and \( u \in C^\infty(\tilde{\Omega}) \) is a uniformly convex solution of (2.17), then \( u \) is unique up to a constant.

Next, similar to Corollary 1.2 in \cite{12}, we obtain

\textbf{Lemma 5.2} If (2.11) holds and \( u \in C^2(\tilde{\Omega}) \) is a uniformly convex solution of (2.17), then \( u \in C^\infty(\tilde{\Omega}) \).

\textbf{Proof} By Evans-Krylov theorem, it’s obvious that

\[ u \in C^\infty(\Omega). \]

In the following, we describe the regularity of \( u \) on the boundary.

Suppose that \( x_0 \in \partial \Omega \) and in the neighbourhood of \( x_0 \),

\[ \partial \Omega = \{ x_n = \varphi(x') : x' \in \Omega' \} \]

for some domain \( \Omega' \subset \mathbb{R}^{n-1} \). Without loss of generality, we set

\[
\begin{align*}
x_0 &= (0, \ldots, 0, 0), \quad \varphi(0, \ldots, 0) = 0, \quad D\varphi(0, \ldots, 0) = (0, \ldots, 0), \quad D^2 \varphi(0, \ldots, 0) > 0.
\end{align*}
\]

Let \( m_0 \) be small enough such that

\[ \Omega_{m_0} := \Omega \cap \{ x_n < m_0 \} = \{ (x', x_n) : x' \in \Omega'', \varphi(x') < x_n < m_0 \} \]

for some domain \( \Omega'' \subset \mathbb{R}^{n-1} \). And we denote \( \Omega'' \) also by \( \Omega' \).

We take the coordinate transformation in the following

\[ T : \Omega_{m_0} \to T(\Omega_{m_0}), \quad x = (x', x_n) \mapsto z = (z', z_n) = (x', x_n - \varphi(x')), \]

\( \square \) Springer
where
\[
T(\Omega_{m_0}) := \{(z', z_n) : z' \in \Omega', \ 0 < z_n < m_0 - \varphi(z')\}.
\]

We set \( w(z) := u(T^{-1}(z)) \). Then \( x(z) = w(x', x_n - \varphi(x')) \). Such that for \( 1 \leq i \leq n - 1 \),
we see that \( u_i = w_i - w_n \varphi_i \) and \( u_n = w_n \). Furthermore, for \( 1 \leq k, l \leq n - 1 \), a direct computation shows that
\[
\begin{align*}
    u_{kl} &= w_{kl} - w_{kn} \varphi_l - w_{nl} \varphi_k + w_{nn} \varphi_k \varphi_l, \\
    u_{kn} &= w_{kn} - w_{nn} \varphi_k, \\
    u_{ln} &= w_{ln} - w_{nn} \varphi_l, \\
    u_{nn} &= w_{nn}.
\end{align*}
\]

It is obvious that by
\[
\|D^2 u\|_{C(\tilde{\Omega})} + \|Du\|_{C(\tilde{\Omega})} \leq C
\]
we get
\[
\|D^2 w\|_{C(T(\Omega_{m_0}))} + \|D w\|_{C(T(\Omega_{m_0}))} \leq C, \tag{5.1}
\]
where \( C \) is a constant depending only on the known data.

Since \( w(z) = u(T^{-1}(z)) \) and \( x = T^{-1}(Z) \), we obtain
\[
\begin{align*}
    D^2 u &= D(DT \cdot Dw), \\
    D_x u &= D_z w - w_n (D_{z'} \varphi, 0)^t,
\end{align*}
\]
where \( D_x, D_z \) and \( D_{z'} \) denote the gradient according to the variables \( x, z \) and \( z' \).

Then by (2.17), \( w \) satisfies
\[
\begin{cases}
    F[\lambda(D(DT \cdot Dw))] = f(T^{-1}(z)) + c, & z \in T(\Omega_{m_0}), \\
    h(D_z w - w_n (D_{z'} \varphi, 0)^t) = 0, & z \in \partial T(\Omega_{m_0}) \cap \{z_n = 0\}.
\end{cases} \tag{5.2}
\]

When \( 1 \leq s \leq n - 1 \), we set \( v = \frac{\partial w}{\partial z_s} \). Let \( a^{ij} (1 \leq i, j \leq n) \), \( \beta^i (1 \leq i \leq n) \) be the definitions in Section 3.

In the following, the repeated indices \( k, l \) denote the summation from 1 to \( n - 1 \) and the repeated indices \( i, j \) present the summation from 1 to \( n \). Define
\[
[A^{ij}]_{1 \leq i, j \leq n} = \begin{cases}
    a^{kl}, & 1 \leq k, l \leq n - 1, \ (i, j) = (k, l), \\
    a^{kn} - a^{kl} \varphi_l, & 1 \leq k \leq n - 1, \ (i, j) = (k, n), \\
    a^{nn} + a^{kl} \varphi_k \varphi_l - 2a^{kn} \varphi_k, & (i, j) = (n, n).
\end{cases}
\]

Then differentiating the equation (5.2) with respect to \( z_s \), we see that \( v \) solves the following linear equation with oblique boundary condition
\[
\begin{cases}
    L_s v = \frac{\partial}{\partial z_s} f(T^{-1}(z)), & z \in T(\Omega_{m_0}), \\
    \gamma^k v_k + \gamma^n v_n = \phi, & z \in \partial T(\Omega_{m_0}) \cap \{z_n = 0\},
\end{cases} \tag{5.3}
\]
where
\[
L_s = A^{ij} \frac{\partial^2}{\partial z_i \partial z_j}.
\]
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\[ \gamma^k = \beta^k (1 \leq k \leq n - 1), \quad \gamma^n = \beta^n - \beta^k \varphi_k, \]
\[ \phi = w_n \beta^k \varphi_{ks}. \]

By Theorem 1.1 in [16], we get \( u \in C^{2,\alpha}(\Omega) \) for some \( 0 < \alpha < 1 \), from which one can deduce that
\[ A^{ij} \in C^{\alpha}(T(\Omega_{m_0})), \]
\[ \gamma^i \in C^{1,\alpha}(T(\Omega_{m_0})), \]
\[ \phi \in C^{1,\alpha}(T(\Omega_{m_0})). \]

It's obvious that the unit inward normal vector of \( \partial \Omega \) at \( x_0 = (0, \ldots, 0) \) is \( \nu = (0, 0, \ldots, 0, 1) \).

Using lemma 3.5, we can show that
\[ \beta^n |_{x_0} = \langle \beta, \nu \rangle |_{x_0} \geq \frac{1}{C_1}. \]

By \( D\varphi(0, \ldots, 0) = (0, \ldots, 0) \), taking \( r > 0 \) small enough such that for any \( z' \in B'_r(\theta) \subset \Omega' \), we obtain
\[ 3n \Lambda_2 \geq [A^{ij}] \geq \frac{1}{3n} \Lambda_1. \]

and
\[ \langle \gamma, \nu \rangle = \beta^n - \beta^k \varphi_k \geq \frac{1}{3C_1}, \]

where \( \gamma = (\gamma^1, \gamma^2, \ldots, \gamma^n) \) and \( \nu = (0, 0, \ldots, 0, 1) \) is the unit inward normal vector of \( \partial T(\Omega_{m_0}) \cap \{z_n = 0\} \). This implies that the equation (5.3) is uniformly elliptic with strict oblique boundary condition.

Set
\[ \mathcal{G}_r := \{(z', z_n) : z' \in B'_r(0), 0 < z_n < m_0 - \varphi(z')\}. \]

Then it follows by Theorem 6.30 and Theorem 6.3.1 in [17], for \( 1 \leq s \leq n - 1 \), we deduce that
\[ \frac{\partial w}{\partial z_s} =: v \in C^{2,\alpha}(\mathcal{G}_r). \]

Let \( Y = \frac{\partial w}{\partial z_s} \). Then the differentiation of the equation (5.2) to \( z_n \) gives
\[ a^{nn}Y_{nn} + a^{kn}(Y_{kn} - Y_{nnp}_k) + a^{ln}(Y_{ln} - Y_{nnp}_l) + a^{kl}(Y_{kl} - Y_{knp}_l - Y_{nlp}_k + Y_{nlp}_k) = \frac{\partial f}{\partial z_n}. \]

It reads as
\[ (a^{nn} + a^{kl} \varphi_k \varphi_l - a^{kn} \varphi_k - a^{ln} \varphi_l)Y_{nn} = R.H.S, \quad (5.4) \]

where
\[ R.H.S := -a^{kn}Y_{kn} - a^{ln}Y_{ln} - a^{kl}Y_{kl} + a^{kl}Y_{knp}_l + a^{kl}Y_{nlp}_k + \frac{\partial f}{\partial z_n}. \]

By \( v = \frac{\partial w}{\partial z_s} \in C^{2,\alpha}(\mathcal{G}_r) \), we have \( R.H.S \in C^{\alpha}(\mathcal{G}_r) \).

Recalling the proof of \( 3n \Lambda_2 \geq [A^{ij}] \geq \frac{1}{3n} \Lambda_1 \) and taking \( r > 0 \) small enough, we can also obtain
\[ a^{nn} + a^{kl} \varphi_k \varphi_l - a^{kn} \varphi_k - a^{ln} \varphi_l \geq \frac{1}{3n} \Lambda_1. \]
Then by (5.4) and \((a^{nn} + a^{kl}\varphi_k\varphi_l - a^{kn}\varphi_k - a^{ln}\varphi_l) \in C^\alpha(\overline{G_r})\), we see that 
\[ Y_{nn} \in C^\alpha(\overline{G_r}) \]
and therefore 
\[ w \in C^{3,\alpha}(\overline{G_r}) \]
Consequently, 
\[ u \in C^{3,\alpha}(\overline{\Omega_1}) \]
By Finite Covering Theorem, we get 
\[ u \in C^{3,\alpha}(\overline{\Omega_1}) \]
Using bootstrap argument, we obtain 
\[ u \in C^\infty(\overline{\Omega_1}) \]
\[ \square \]
Now, by the continuity method, we can show that:

**Proof of Theorem 2.3** For each \( t \in [0, 1] \), consider
\[
\begin{align*}
  F[D^2u] &= tf(x) + c(t), \quad x \in \Omega, \\
  Du(\Omega) &= \tilde{\Omega},
\end{align*}
\]
which is equivalent to
\[
\begin{align*}
  F[D^2u] &= tf(x) + c(t), \quad x \in \Omega, \\
  h(Du) &= 0, \quad x \in \partial\Omega.
\end{align*}
\]
By the main results in [11] and [12], (5.5) is solvable if 
\( t = 0 \). By Lemma 5.2, denote the closed subset
\[ X := \{ u \in C^{2,\alpha}(\overline{\Omega}) : \int_{\Omega} u = 0 \} \]
in \( C^{2,\alpha}(\overline{\Omega}) \) and 
\[ Y := C^\alpha(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega) \]
Define a map from \( X \times \mathbb{R} \) to \( Y \) as 
\[ \mathfrak{F}_t := (F(D^2u) - tf(x) - c(t), h(Du)). \]
Then the linearized operator 
\[ D\mathfrak{F}_t(u, c) : X \times \mathbb{R} \rightarrow Y \]
is given by
\[ D\mathfrak{F}_t(u, c)(w, a) = \left(F^{ij}(D^2u)\partial_{ij}w - a, h_{pi}(Du)\partial_iw\right). \]
Repeating the proof of Proposition 3.1 in [9], we know that \( D\mathfrak{F}_t(u, c) \) is invertible for any 
\( t \in [0, 1] \) and \( (u, c) \) being the solution to (5.5).
Define the set
\[ I := \{ t \in [0, 1] : (5.5) \text{ has at least one convex solution} \} \]
Since \( 0 \in I \), by Huang-Ou’s theorem [11] or Huang-Ye’s theorem [12], \( I \) is not empty. We claim that \( I = [0, 1] \), which is equivalent to the fact that \( I \) is not only open, but also closed. It follows from Proposition 3.1 in [9] again and Theorem 17.6 in [17] that \( I \) is open. So we only need to prove that \( I \) is a closed subset of \([0, 1]\).
That \( I \) is closed is equivalent to the fact that for any sequence \( \{ t_k \} \subset I \), if \( \lim_{k \to \infty} t_k = t_0 \), then \( t_0 \in I \). For \( t_k \), denote \((u_{t_k}, c(t_k))\) solving
\[
\begin{align*}
  F[D^2u_{t_k}] &= t_k f(x) + c(t_k), \quad x \in \Omega, \\
  Du_{t_k}(\Omega) &= \tilde{\Omega}.
\end{align*}
\]
It follows from Lemma 4.12 that \( \|u_k\|_{C^{2,\alpha}(\tilde{\Omega})} \leq C \), where \( C \) is independent of \( t_k \). Since
\[
|c(t)| = \left| F \left[ D^2 u \right] - tf(x) \right| \leq F (+\infty, \ldots, +\infty) + \max_{\tilde{\Omega}} |f(x)|,
\]
by Arzela-Ascoli Theorem we know that there exists \( \hat{u} \in C^{2,\alpha}(\tilde{\Omega}) \), \( \hat{c} \in \mathbb{R} \) and a subsequence of \( \{t_k\} \), which is still denoted as \( \{t_k\} \), such that letting \( k \to \infty \),
\[
\begin{cases}
\|u_k - \hat{u}\|_{C^2(\tilde{\Omega})} \to 0, \\
c(t_k) \to \hat{c}.
\end{cases}
\]
Since \( (u_k, c(t_k)) \) satisfies
\[
\begin{align*}
F \left[ D^2 u_k \right] &= t_k f(x) + c(t_k), & x \in \Omega, \\
h(Du_k) &= 0, & x \in \partial \Omega,
\end{align*}
\]
letting \( k \to \infty \), we have
\[
\begin{align*}
F \left[ D^2 \hat{u} \right] &= t_0 f(x) + \hat{c}, & x \in \Omega, \\
h(D\hat{u}) &= 0, & x \in \partial \Omega.
\end{align*}
\]
Therefore, \( t_0 \in I \), and thus \( I \) is closed. Consequently, \( I = [0, 1] \). By Lemma 5.1 we know that the solution of (5.5) \( u \) is unique up to a constant.

Then we complete the proof of Theorem 2.3. \( \square \)

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