Remarks on the Controllability of Some Quasilinear Equations*

Xu Zhang
Academy of Mathematics and Systems Sciences,
Chinese Academy of Sciences, Beijing 100190, China; and
Yangtze Center of Mathematics,
Sichuan University, Chengdu 610064, China.
E-mail: xuzhang@amss.ac.cn

Abstract
In this Note, we review the main existing results, methods, and some key open problems on the controllability of nonlinear hyperbolic and parabolic equations. Especially, we describe our recent universal approach to solve the local controllability problem of quasilinear time-reversible evolution equations, which is based on a new unbounded perturbation technique. It is also worthy to mention that the technique we develop can also be applied to other problems for quasilinear equations, say local existence, stabilization, etc.

1 Introduction
Consider the following controlled evolution equation:

\[
\begin{cases}
\frac{d}{dt}y(t) = A(y(t))y(t) + Bu(t), & t \in (0, T), \\
y(0) = y_0.
\end{cases}
\] (1.1)

Here, the time \( T > 0 \) is given, \( y(t) \in Y \) is the state variable, \( u(t) \in U \) is the control variable, \( y_0(\in Y) \) is the initial state; \( Y \) and \( U \) are respectively the state space and control space, both of which are some Hilbert space; \( A(\cdot) \) is a suitable (nonlinear and usually unbounded) operator on \( Y \), while the control operator \( B \) maps \( U \) into \( Y \). Many control problems for relevant nonlinear Partial Differential Equations (PDEs, for short) enter into this context. For instance, the quasilinear/semilinear parabolic equation, wave equation, plate equation, Schrödinger equation, Maxwell equations, and Lamé system, etc.

*This work was supported by the NSF of China under grants 1052105, 10831007 and 60821091, the Chunhui program (State Education Ministry of China) under grant Z007-1-61006, and the project MTM2008-0541 of the Spanish Ministry of Science and Innovation. Part of this work was done when the author visited Fudan University, with a financial support from the “French-Chinese Summer Institute on Applied Mathematics” (September 1-21, 2008).
In this Note, we shall describe some existing methods, results and main open problems on the controllability of these systems, especially these for nonlinear hyperbolic and parabolic equations.

System (1.1) is said to be exactly controllable in $Y$ at time $T$ if for any $y_0, y_1 \in Y$, there is a control $u \in L^2(0,T;U)$ such that the solution of system (1.1) with this control satisfies

$$y(T) = y_1. \quad (1.2)$$

When $\dim Y = \infty$ (We shall focus on this case later unless other stated), sometimes one has to relax the requirement (1.2), and this leads to various notions and degrees of controllability: approximate controllability, null controllability, etc. Note however that, for time reversible system, the notion of exact controllability is equivalent to that of null controllability.

Roughly speaking, the controllability problem for an evolution equation consists in driving the state of the system (the solution of the controlled equation under consideration) to a prescribed final target state (exact or in some approximate way) in finite time. Problems of this type are common in science and engineering and, in particular, they arise often in the context of flow control, in the control of flexible structures appearing in flexible robots and in large space structures, in quantum chemistry, etc.

The controllability theory for finite dimensional linear systems was introduced by R.E. Kalman [14] at the very beginning of the 1960s. Thereafter, many authors were devoted to develop it for more general systems including infinite dimensional ones, and its nonlinear and stochastic counterparts.

The controllability theory of PDEs depends strongly on its nature and, in particular, on its time-reversibility properties. To some extent, the study of controllability for linear PDEs is well-developed although many challenging problems are still unsolved. Classical references in this field are D.L. Russell [29] and J.L. Lions [21]. Updated progress can be found in a recent survey by E. Zuazua ([43]). Nevertheless, much less are know for nonlinear controllability problems for PDEs although several books on this topic are available, say J.M. Coron [6], A.V. Fursikov & O.Yu. Imanuvilov [11], T.T. Li [16], and X. Zhang [36]. Therefore, in this Note, we concentrate on controllability problems for systems governed by nonlinear PDEs.

The main result in this Note can be described as follows: Assume that $(A(0), B)$ is exact controllable in $Y$. Then, under some assumptions on the structure of $A(y)$ (for concrete problems, which needs more regularity on the state space, say $D(A(0)^k)$ for sufficiently large $k$), system (1.1) is locally exact controllable in $D(A(0)^{k})$.

The main approach that we employ to show the above controllability result is a new perturbation technique. The point is that, the perturbation is unbounded but small. Note however that this approach does NOT work for the null controllability problem of the time-irreversible systems, and therefore, one has to develop different method to solve the local null controllability of quasilinear parabolic equations.
For simplicity, in what follows, we consider mainly the case of internal control, i.e. $B \in \mathcal{L}(U,Y)$. Also, we will focus on the local controllability of the quasilinear wave equation. However, our approach is universal, and therefore, it can be extended to other quasilinear PDEs, say quasilinear plate equation, Schrödinger equation, Maxwell equations, and Lamé system, etc.

On the other hand, we mention that the technique developed in this Note can also be applied to other problems for quasilinear equations. For example, stabilization problem for system (1.1) (with small initial data) can be considered similarly. Indeed, although there does not exist the same equivalence between exact controllability and stabilization in the nonlinear setting, the approaches to treat them can be employed each other.

The rest of this Note is organized as follows. In Section 2, we review the robustness of the controllability in the setting of Ordinary Differential Equations (ODEs, for short). In Section 3, we recall some known perturbation result on the exact controllability of abstract evolution equations. Then, in Section 4, we show a new perturbation result on the exact controllability of general evolution equations. Sections 5 and 6 are addressed to present local controllability results for multidimensional quasilinear hyperbolic equations and parabolic equations, respectively. Finally, in Section 7, we collect some open problems, which seem to important in the field of controllability of PDEs.

2 Starting point: the case of ODEs

Consider the following controlled ODE:

\[
\begin{cases}
\frac{d}{dt} y = Ay + Bu, & t \in (0, T), \\
y(0) = y_0,
\end{cases}
\]

(2.1)

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. It is well-known ([14]) that system (2.1) is exact controllable in $(0, T)$ if and only if

$B^* e^{At} x_0 = 0, \quad \forall \ t \in (0, T) \Rightarrow x_0 = 0.$

Note that this condition is also equivalent to the following Kalman rank condition:

\[
\text{rank}(B, AB, A^2B, \ldots, A^{n-1}B) = n.
\]

(2.2)

From (2.2), it is clear that if $(A, B)$ is exact controllable, then there exists a small $\varepsilon = \varepsilon(A, B) > 0$ such that $(A, B)$ is still exact controllable provided that $||\tilde{A} - A|| + ||\tilde{B} - B|| < \varepsilon$. Therefore, the exact controllability of system (2.1) is robust under small perturbation.

Because of the above robustness, the local exact controllability of nonlinear ODEs is quite easy. Indeed, consider the following controlled system:

\[
\begin{cases}
\frac{d}{dt} y = Ay + f(y) + Bu, & t \in (0, T), \\
y(0) = y_0,
\end{cases}
\]

(2.3)
with \( f(\cdot) \in C^1(\mathbb{R}^n) \) and \( f(y) = O(|y|^{1+\delta}) \) when \( y \) is small, for some \( \delta > 0 \). The local exact controllability of system (2.3) follows from a standard perturbation argument.

However, the corresponding problem in PDE setting is much more complicated, as we shall see below.

### 3. Known perturbation result on exact controllability

In this section, we recall some known perturbation result on the exact controllability of abstract evolution equations. These results are based on the following two tools:

- **Duality argument** (e.g. [20, 21, 36]): In the linear setting (i.e., \( A(y) \equiv A \) is independent of \( y \) and linear, and further \( A \) generates an \( \mathcal{C}_0 \)-group \( \{e^{At}\}_{t \in \mathbb{R}} \) on \( Y \)), the null controllability of system (1.1) is equivalent to the following observability estimate:

\[
|e^{AT}z^*|_{Y^*}^2 \leq C \int_0^T |B^*e^{A^Ts}z^*|_{U^*}^2 ds, \quad \forall z^* \in Y^*,
\]

for some constant \( C > 0 \).

- **Variation of constants formula**: In the setting of semigroup, for a bounded perturbation \( P \in \mathcal{L}(Y) \):

\[
e^{(A+P)t}x = e^{At}x + \int_0^t e^{A(t-s)}P x ds, \quad \forall x \in Y.
\]

Combining (3.1) and (3.2), it is easy to establish the following well-known (bounded) perturbation result on the exact controllability:

**Theorem 3.1** Assume that \( A \) generates an \( \mathcal{C}_0 \)-group \( \{e^{At}\}_{t \in \mathbb{R}} \) on \( Y \) and \( B \in \mathcal{L}(U,Y) \). If \((A,B)\) is exact controllable, then so is \((A+P,B)\) provided that \( ||P||_{\mathcal{L}(Y)} \) is small enough.

The above perturbation \( P \) can also be time-dependent. In this case, one needs the language of evolution system. In the sequel, for a simple presentation, we consider only the time-independent case.

As a consequence of Theorem 3.1 and the standard fixed point technique, one can easily deduce a local exact controllability result for some semilinear equations, say the counterpart of system (2.3):

\[
\begin{cases}
\frac{d}{dt}z = Az + f(z) + Bv, & t \in (0,T), \\
z(0) = z_0.
\end{cases}
\]

More precisely, we have
Corollary 3.1 Assume that $A$ generates an $C_0$-group $\{e^{At}\}_{t \in \mathbb{R}}$ on $Y$, $B \in \mathcal{L}(U,Y)$, and $(A,B)$ is exact controllable. If the nonlinearity $f(\cdot) : Y \to Y$ satisfies $f(\cdot) \in C^1(Y)$ and, for some $\delta > 0$, $|f(z)|_Y = O(|z|_Y^{1+\delta})$ as $|z|_Y \to 0$, then system (3.3) is locally exact controllable in $Y$.

Clearly, both the time reservability of the underlying system and the variation of constants formula (3.2) plays a key role in the above perturbation-type results.

When the system is time-irreversible, the above perturbation technique does not work. The typical example is the controlled heat equation. In this case, one has to search for other robust method to derive the desired controllability, say, Carleman estimate. We shall consider this case in Section 6.

When the perturbation operator $P$ is unbounded, formula (3.2) may fail to work, and in this case things become much more delicate even for the semigroup theory itself. Nevertheless, there do exist some special case, for which the perturbation $P$ is unbounded but the above variation of constants formula still works (in the usual sense), say when the semigroup $\{e^{At}\}_{t \geq 0}$ has some smooth effect. In this case, one can find some perturbation result for exact controllability in S. Boulite, A. Idrissi and L. Maniar [3], S. Hadd [12], and H. Leiva [15]. However, it does not seem that these perturbation results can be adapted to solve the nonlinear controllability problems, especially for quasilinear equations.

4 A new perturbation result on exact controllability

In this section, we present a new perturbation result on the exact controllability of general evolution equations. The idea is simple, and the key point is that the generation of an $C_0$-semigroup $\{e^{At}\}_{t \geq 0}$ is robust with respect to a small perturbation of the same “order” with respect to the generator $A$.

Stimulated by quasilinear problem, we consider the following small perturbation of the same “order”:

$$P = P_0A,$$

where $P_0 \in \mathcal{L}(Y)$ and $||P_0|| < 1$. That is, the perturbed operator reads: $(I + P_0)A$. It is easy to show that, if $A$ generated a contractive $C_0$-semigroup, then so is $(I + P_0)A$. Indeed, it is obvious that $(I + P_0)A$ is dissipative in $Y$ with the new scalar product $((I + P_0)^{-1} \cdot, \cdot)$, which induces a norm, equivalent to the original one. Nevertheless, we remark that the variation of constants formula does not work for $e^{(I+P_0)At}$ for this general case.

Thanks to the above observation, a new perturbation result for exact controllability is shown in [38], which reads as follows:

Theorem 4.1 Assume that $A$ generates an unitary group $\{e^{At}\}_{t \in \mathbb{R}}$ on $Y$ and $B \in \mathcal{L}(U,Y)$. If $(A,B)$ is exact controllable, then so is $(A + P, B)$ provided that $||P_0||_{\mathcal{L}(Y)}$ is small enough.
Since the variation of constants formula does not work for \( e^{(I + P_0)A_t} \), the above result cannot be derived as Theorem 3.1. Instead, we need to use Laplace transform and some elementary tools from complex analysis to prove the desired result.

The above simple yet useful perturbation-type controllability result can be employed to treat the local controllability problems for quasilinear evolution-type PDEs with time-reversibility, as we shall see in the next section.

5 Local exact controllability for multidimensional quasilinear hyperbolic equations

This section is addressed to the local exact controllability of quasilinear hyperbolic equations in any space dimensions.

To begin with, let us recall the related known controllability results for controlled quasilinear hyperbolic equations. The problem is well-understood in one space dimension. To the author’s best knowledge, the first paper in this direction is M. Cirina [5]. Recent rich results are available in T.T. Li & B.P. Rao [17], T.T. Li & B.Y. Zhang [23], T.T. Li & L.X. Yu [19], Z.Q. Wang [32], and especially the above mentioned book by T.T. Li ([16]). As for the corresponding controllability results in multi-space dimensions, we refer to P. F. Yao ([35]) and Y. Zhou & Z. Lei [41].

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a sufficiently smooth boundary \( \Gamma \). Put \( Q = (0, T) \times \Omega \) and \( \Sigma = (0, T) \times \Gamma \). Let \( \omega \) be a nonempty open subset of \( \Omega \). We consider the following controlled quasilinear hyperbolic equations:

\[
\begin{align*}
  z_{tt} - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)z_{x_j}) &= G(t, x, z, \nabla_t x z, \nabla^2_t x z) + \phi_\omega(x)u, \quad \text{in} \ Q, \\
  z &= 0, \quad \text{in} \ \Sigma, \\
  z(0) = z_0, z_t(0) = z_1, \quad \text{in} \ \Omega,
\end{align*}
\]

(5.1)

where the coefficients \( a_{ij}(\cdot) \in C^2(\overline{\Omega}) \) \( (i, j = 1, \cdots, n) \) satisfy \( a_{ij} = a_{ji} \), and for some constant \( \rho > 0, \)

\[
\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \rho|\xi|^2, \quad \forall \ (x, \xi) = (x, \xi_1, \cdots, \xi_n) \in \overline{\Omega} \times \mathbb{R}^n;
\]

and following [41], the nonlinearity \( G(\cdot) \) is taken to be of the form

\[
G(t, x, \nabla_t x z, \nabla^2_t x z) = \sum_{i=1}^n \sum_{\alpha=0}^n g_{i\alpha}(t, x, \nabla_t x z)\partial^2_{x_i x_\alpha} z + O(|u|^2 + |\nabla_t x z|^2),
\]
\( g_{t\alpha}(t, x, 0, 0) = 0 \) and \( x_0 = t \); \( \phi_\omega \) is a nonnegative smooth function defined on \( \Omega \) and satisfying \( \min_{x \in \omega} \phi_\omega(x) > 0 \).

Denote by \( \chi_\omega \) the characteristic function of \( \omega \). We need to introduce the following Assumption (H):

Assume the linear hyperbolic equation

\[
\begin{align*}
\frac{\partial^2 y}{\partial t^2} - \sum_{i,j=1}^n \partial_x (a_{ij}(x)y_x) &= \chi_\omega(x)u, \quad \text{in } Q \\
y &= 0, \quad \text{in } \Sigma \\
y(0) &= y_0, \quad y_t(0) = y_1, \quad \text{in } \Omega
\end{align*}
\]  

(5.2)

is exact controllable in \( H^s(\Omega) \times L^2(\Omega) \).

The following controllability result for quasilinear hyperbolic equations is shown in [38]:

**Theorem 5.1** Let Assumption (H) hold. Then, for any \( s > \frac{n}{2} + 1 \), system (5.1) is local exact controllable in \( (H^{s+1}(\Omega) \cap H^0_0(\Omega)) \times H^s(\Omega) \) (provided that some compatible conditions are satisfied for the initial and final data).

Clearly, Theorem 5.1 covers the main results in [35, 41]. The above result follows by combining our new perturbation result for exact controllability, i.e. Theorem 4.1 and the fixed point technique developed in [41].

**Remark 5.1** The boundary control problem can be considered similarly although the technique is a little more complicated.

**Remark 5.2** The key point of our approach is to reduce the local exact controllability of quasilinear equations to the exact controllability of the linear equation. This method is general and simple. The disadvantage is that we can not construct the control explicitly. Therefore, this approach does not replace the value of [41], and the deep results for the corresponding 1-d problem, obtained by T. T. Li and his collaborators, as mentioned before. Especially, from the computational point of view, the later approach might be more useful.

We now return to Assumption (H), and review the known results and unsolved problems for exact controllability of the linear hyperbolic equation but we concentrate on the case of boundary control although similar things can be said for the case of internal control.

Denote by \( \mathcal{A} \) the elliptic operator appeared in the first equation of system (5.2). We consider the following controlled linear hyperbolic equation with a boundary controller:

\[
\begin{align*}
\frac{\partial^2 y}{\partial t^2} + \mathcal{A} y &= 0, \quad \text{in } Q \\
y &= \chi_\Sigma u, \quad \text{in } \Sigma \\
y(0) &= y_0, \quad y_t(0) = y_1, \quad \text{in } \Omega
\end{align*}
\]  

(5.3)
where \( \emptyset \neq \Sigma_0 \subset \Sigma \) is the controller. It is easy to show that, system (5.3) is exactly controllable in \( L^2(\Omega) \times H^{-1}(\Omega) \) at time \( T \) by means of control \( u \in L^2(\Sigma_0) \) if and only if there is a constant \( C > 0 \) such that solutions of its dual system

\[
\begin{aligned}
& w_{tt} + Aw = 0, & \text{in } Q \\
& w = 0, & \text{in } \Sigma \\
& w(0) = w_0, w_t(0) = w_1, & \text{in } \Omega
\end{aligned}
\]  

(5.4)

satisfies the following observability estimate:

\[
|w_0|^2_{H^1_0(\Omega)} + |w_1|^2_{L^2(\Omega)} \leq C \int_{\Sigma_0} \left| \frac{\partial_A w}{\partial \nu} \right|^2 d\Sigma_0, \quad \forall (w_0, w_1) \in H^1_0(\Omega) \times L^2(\Omega). 
\]  

(5.5)

When \( A = -\Delta, \Sigma_0 = (0, T) \times \Gamma_0 \) with \( \Gamma_0 \) to be a suitable subset of \( \partial \Omega \), L.F. Ho [13] establish (5.5) by means of the classical Rellich-type multiplier. Later, K. Liu [22] gave a nice improvement for the case of internal control. When \( A \) is a general elliptic operator of second order, and \( \Sigma_0 \) is a general (maybe non-cylinder) subset of \( \Sigma \), J.L. Lions [21] posed an open problem on “under which condition, inequality (5.5) holds?”. When \( \Sigma_0 = (0, T) \times \Gamma_0 \) is a cylinder subset of \( \Sigma \), Lions’s problem is almost solved. In this case, typical results are as follows:

1) Geometric Optics Condition (GOC for short) introduced by C. Bardos, G. Lebeau & J. Rauch [1], which is a sufficient and (almost) necessary condition for inequality (5.5) to hold. GOC is perfect except the three disadvantage: One is that it needs considerably high regularity on both the coefficients and \( \partial \Omega \) (N. Burq [4] gives some improvement in this respect); One is that this condition is not easy to verify; The other is that the observability constant derived from GOC is not explicit because it involves the contradiction argument to absorb the undesired lower order terms appeared in the observability estimate.

2) Rellich-type multiplier conditions introduced by L.F. Ho [13], K. Liu [22], A. Osses [27], etc., which require less smooth conditions than GOC but they are not necessary conditions for inequality (5.5) to hold.

3) There exist some other sufficient condition for inequality (5.5) to hold, say the vector field condition by A. Wyler [33], and the curvature condition by P.F. Yao ([34]. Later, it is shown by S.J. Feng & D.X. Feng [9] that these two conditions are equivalent although they are introduced through different tools.

4) Mixed tensor/vector field condition introduced by X. Zhang & E. Zuazua [40], which covers the conditions in 2) and 3).

Remark 5.3 It is shown by L. Miller [26] that when the data are sufficiently smooth, the conditions in 2) and 3) are special cases of GOC. Nevertheless, as
far as I know, it is an unsolved problem on the minimal assumption on data for GOC.

When $\Sigma_0 \neq (0, T) \times \Gamma_0$, especially when it is NOT a cylinder subset of $\Sigma$, there exist almost no nontrivial progress on Lions’s problem (which seems to be a challenging mathematical problem), even for the simplest $1-d$ wave equation! The only related results are as follows:

a) For $1-d$ wave equation and $\Sigma_0 = E \times \Gamma_0$ with $E \subset (0, T)$ to be a Lebesgue measurable set with positive measure, P. Martinez & J. Vancostenoble [24] show that (5.5) holds.

b) G. Wang [31] obtains an interesting internal observability estimate for the heat equation in multi-space dimensions, where the observer is $E \times \omega$ with $E$ being the same as in the above case and $\omega$ to be any nonempty open subset of $\Omega$.

6 Local null controllability for quasilinear parabolic equations

In this section, we consider the local exact controllability of quasilinear parabolic equations in any space dimensions.

As mentioned before, the perturbation technique does not apply to the time irreversible system, exactly the case of parabolic equations. Therefore, one has to search for other robust method to derive the desired null controllability, say, Carleman estimate even if the perturbation to the null-controllable system is very small (even in the linear setting!).

We consider the following controlled quasilinear parabolic system

$$\begin{cases}
y_t - \sum_{i,j=1}^n (a_{ij}(y)y_{x_i})_{x_j} = \chi_\omega u & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } \Omega,
\end{cases} \quad (6.1)$$

where $a_{ij}(\cdot) : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable functions satisfying similar conditions in the last section.

In the last decades, there are many papers devoted to the controllability of linear and semilinear parabolic equations (see e.g. [11, 43] and the rich references therein). However, as far as we know, nothing is known about the controllability of quasilinear parabolic equations except for the case of one space dimension. In [2], the author proves the local null controllability of a $1-d$ quasilinear diffusion equation by means of the Sobolev embedding relation $L^\infty(0, T; H^1_0(\Omega)) \subseteq L^\infty(Q)$, which is valid only for one space dimension.

The following local null controllability result for a class of considerably general multidimensional quasilinear parabolic equations, system (6.1), is shown in [23].
Theorem 6.1 There is a constant $\gamma > 0$ such that, for any initial value $y_0 \in C^{2+\frac{1}{2}}(\Omega)$ satisfying $|y_0|_{C^{2+\frac{1}{2}}(\Omega)} \leq \gamma$ and the first order compatibility condition, one can find a control $u \in C^{\frac{1}{2}}(\bar{Q})$ with $\text{supp } u \subseteq \omega \times [0, T]$ so that the solution $y$ of system (6.1) satisfies $y(T) = 0$ in $\Omega$. Moreover,

$$|u|_{C^{\frac{1}{2}}(\bar{Q})} \leq Ce^{C|y_0|_{L^2(\Omega)},}$$

where $A = \sum_{i,j=1}^{n} \left( 1 + \sup_{|s| \leq 1} |a_{ij}(s)|^2 + \sup_{|s| \leq 1} |a_{ij}'(s)|^2 \right)$, and $C$ depends only on $\rho$, $n$, $\Omega$ and $T$.

The key point in the proof of Theorem 6.1 is to improve the regularity of the control function for smooth data, which is a consequence of a new observability inequality for linear parabolic equations with an explicit estimate on the observability constant in terms of the $C^1$-norm of the coefficients in the principle operator. The later is based on a new global Carleman estimate for the parabolic operator.

7 Open problems

Although great progress have been made on the controllability theory of PDEs, the field is still full of open problems. In some sense, the linear theory is well-understood and there exist extensive works on the controllability of linear PDEs. But, still, even for the linear setting, some fundamental problems remain to be solved, as we shall explain later. The controllability theory of nonlinear system originated in the middle of 1960s but the progress is very slow. Similar to other nonlinear problems, controllability of infinite dimensional nonlinear system is usually very difficult. Due to the underlying properties of the equation, the progress of the exact controllability theory for nonlinear hyperbolic equations is even slower. Nevertheless, nonlinear problems are not always difficult than linear ones. Indeed, as we have shown in Theorem 5.1, local exact controllability of quasilinear hyperbolic equations is a consequence of the exact controllability of linear hyperbolic equations. One may then ask such a question: “How to judge a nonlinear result is good or not?” To the author’s opinion, except for some famous unsolved problem, the point is either “whether the result is optimal or not in some nontrivial sense?”, or “whether some new phenomenon is discovered or not?”.

From the above “criteria”, our result on the local exact controllability of quasilinear hyperbolic equations is not good at all. Indeed, there is no evidence to show that the result is optimal. Therefore,

How to establish the “optimal” local exact controllability result for quasilinear equations?

is one of the most challenging problems in the field of control of PDEs. As we shall see below, this problem is also highly nontrivial even in the semilinear setting!
We now review the exact controllability for the following semilinear hyper-
bolic equations:

\[
\begin{aligned}
z_{tt} + \mathcal{A}z &= f(z) + \chi_\omega(x)u(t, x), & \text{in } Q, \\
z &= 0, & \text{in } \Sigma, \\
z(0) &= z_0, z_t(0) = z_1, & \text{in } \Omega.
\end{aligned}
\]

(7.1)

For some very general nonlinearity \(f(\cdot)\) and a suitable controller \(\omega\), E. Zuazua [42] obtains the local exact controllability for system (7.1). Recently, B. Dehman & G. Lebeau [7] gave a significant improvement. However, as far as I know, no optimality on the controllability results are analyzed in these works, which seems also to be a challenging problem.

**Remark 7.1** The possible optimality on the local exact controllability for semi-linear equations should be strongly related to PDEs with lower regularity data. This is a very rapid developing field in recent years.

**Remark 7.2** There exists big difference between the controllability problems and pure PDEs problems. Indeed, the exact controllability problem for the system

\[
\begin{aligned}
z_{tt} + \mathcal{A}z &= f(z_t) + \chi_\omega(x)u(t, x), & \text{in } Q, \\
z &= 0, & \text{in } \Sigma, \\
z(0) &= z_0, z_t(0) = z_1, & \text{in } \Omega.
\end{aligned}
\]

(7.2)

in the natural energy space \(H^1_0(\Omega) \times L^2(\Omega)\) is not clear even if \(f(\cdot)\) is global Lipchtiz continuous. But, of course, the well-posedness of the corresponding pure PDE problem (i.e. the control \(u \equiv 0\)) is trivial.

Global exact controllability for semilinear equations is generally a very difficult problem. We refer to [36] for known global controllability results for the semilinear hyperbolic equation when the nonlinearity is global Lipschitz continuous. For system (7.1), if the nonlinearity \(f(\cdot)\) grows too fast, say

\[
\lim_{|s| \to \infty} |f(s)||s|^{-1}\log^{-r} |s| = 0, \quad r > 2,
\]

(7.3)

the solution may blowup, and therefore, global exact controllability is impossible in this case. Recently, based on X. Fu, J. Yong & X. Zhang [10] and V.Z. Meshkov [25], T. Duyckaerts, X. Zhang & E. Zuazua [8] showed that, if

\[
\lim_{|s| \to \infty} |f(s)||s|^{-1}\log^{-r} |s| = 0, \quad r < 3/2,
\]

(7.4)

then system (7.1) is globally exact controllable. Moreover, it is also shown that the above index “3/2” is optimal in some sense (i.e., wether the linearization argument works or not) when \(n \geq 2\). (But this number is not optimal in \(1 - d\)).

**Remark 7.3** The same “3/2”-phenomenon happens also for parabolic equations when \(n \geq 2\). Surprisingly, the \(1 - d\) problem is unsolved. That is, it is not clear whether the index “3/2” is optimal or not in \(1 - d\)! This means, sometimes, the \(1 - d\) problem is difficult than the multidimensional ones.
Remark 7.4 Note that for the pure PDE problems, the same phenomenon described above does not happen. This indicates that the study of the controllability problem for nonlinear PDEs has some independent interest, which is far from a sub-PDE-problem.

Remark 7.5 Another strongly related longstanding unsolved problem is the exact controllability of the linear time- and space-dependent hyperbolic equation under the GOC. It seems that, this needs to combine cleverly the tool from micro-local analysis and the technique of Carleman estimate. But nobody knows how to do it.

To end this Note, we list the following further open problems.

- **Controllability of the coupled and/or higher order systems by using minimal number of controls.** As shown in X. Zhang & E. Zuazua [39], the study of the related controllability problem is surprisingly complicated and highly nontrivial even for the systems in one space dimension!

- **Constrained controllability.** As shown in K.D. Phung, G. Wang & X. Zhang [28], the problem is unexpected difficult even for the simplest $1−d$ wave equation and heat equation.

- **Controllability of parabolic PDEs with memory, or retard argument and/or other nonlocal terms.** Consider the following controlled heat equations with a memory term:

\[
\begin{align*}
z_t - \Delta z &= \int_0^t a(s, x)z(s)ds + \chi_\omega(x)u, \quad &\text{in } Q, \\
z &= 0, \quad &\text{in } \Sigma, \\
z(0) &= z_0, \quad &\text{in } \Omega.
\end{align*}
\]

The PDE problem itself is not difficult. But, as far as I know, the controllability problem for the above equation is unsolved even if the memory kernel $a(\cdot, \cdot)$ is small!

- **Controllability/observability of stochastic PDEs.** There exists only very few nontrivial results, say [30, 37] and the reference cited therein. I believe this is a very hopeful direction for the control of PDEs in the near future.

- **Controllability of PDEs in non-reflexive space.** There exists almost no nontrivial results in this direction!

- **Other types of controllability.** Different notions of controllability, say, periodic controllability, may lead to new and interesting problems for PDEs.
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