Relativistic magnetic flux quantum - a unique radiation carrier

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Abstract. The work found the shape of spectral lines of electromagnetic radiation produced by a relativistic quantum of magnetic flux moving in a Josephson transmission line under the action of a purely fluctuating external current and dissipation. It is shown that even in the case of a Gaussian noise current, the shape of the spectral lines of electromagnetic radiation has an essentially non-Gaussian form. The limited dispersion of the frequency of electromagnetic radiation (which is determined by the pinching of the spectral lines of radiation) is proved, due to the relativistic properties of the quantum magnetic flux. This is another unique aspect of the relativistic quantum magnetic flux - as a carrier of electromagnetic radiation in the Josephson transmission line.

1. Introduction
Currently, one of the important problems of applied high – temperature superconductivity is the theoretical justification and practical development of so-called Josephson generators or Terahertz detectors operating at liquid nitrogen temperatures or higher temperatures [1-3].

But the solution to this problem faces a number of significant difficulties. For example, the statistical nature of current fluctuations in high-temperature superconductors is not yet fully understood. As is known, these noise superconducting currents play a critical role in such systems, namely, they determine the magnitude of the frequency broadening of electromagnetic radiation generated by Josephson vortices [4,5]. The latter includes, in particular, fluxons, i.e., magnetic flux quanta.

Further, we limit ourselves only to the study of the characteristics of electromagnetic radiation produced by relativistic quanta of the magnetic flux-the velocities of which can be of the order of the so-called Swichart velocity, i.e., the limiting velocity of magnetic waves in the Josephson transmission line. Our choice is due to the following reasons:

1. Fluxone is an exceptionally stable formation and is easy to control with the help of external influences. Therefore, it can be used as a basic unit of information, since the fluxon, when propagating along the Josephson transmission line, carries one quantum of magnetic flux;

2. In the Josephson transmission line, relativistic effects are easy to investigate experimentally-due to the fact that the speed of the magnetic flux quantum can be changed by means of an external distributed current source. In other words we can supply energy to fluxons by changing the amount of external current;

3. In experiments, the effects of fluctuations can be detected by the radiation spectrum taken from the end of the Josephson transmission line.
In the existing literature [2,4-6] devoted to the study of the radiative characteristics of the Josephson transmission line, the effects of relatively weak current fluctuations on the broadening of the frequencies of resonant lines in the non-relativistic approximation, i.e., at fluxon velocities significantly lower than the Swichart velocity, were mainly studied.

In this paper, for the first time, we consider the features of electromagnetic radiation produced by a relativistic quantum of magnetic flux under the action of sufficiently strong current fluctuations. In the first section, a function describing the shape of the spectral radiation lines is defined. In the second section, the characteristic average radiation frequencies are calculated and based on them, their (mean-square) dispersion is proved to be limited at any noise level in the Josephson transmission line. In conclusion, a comparison of the results obtained, as well as suggestions for the future prospects of research on this topic, is presented.

2. The shape of the spectral lines of the electromagnetic radiation produced by the fluxon

It is well known, when considering vibrational, wave and radiative processes occurring under the influence of random influences of various physical nature, the statistical approach is fundamental. Simply for the reason that in such situations, the system under study shows a probabilistic character.

Naturally, the existence of numerous statistical methods (both physical and mathematical) solutions to such problems sometimes will not help, and as bitter experience shows, it puts the researcher in a very difficult position. A well-known example is the Fokker–Planck equation method for describing the distribution function (or probability density) [7]. As rightly noted in [8], applying different "rules" of averaging-Ito, Stratonovich and Klimontovich-Hanggi from the same Fokker–Planck equation, we obtain three different distribution functions [8-11]. Yes, the authors [8] are right, who called this situation an "unsolvable tri-lemma", i.e. it is not possible to give a definite answer to the question – which of these results is reliable. This is, indeed, an absurd circumstance and the way out of the resulting maze with the help of the above-mentioned "path"she cannot find, in principle.

Below we give an original solution to this "unsolvable" problem. Josephson vortices moving in Josephson transmission lines associated with an external distributed random current source are affected by the Lorentz force [12], which is directly proportional to the external current. This force in this system is directed along the Josephson transmission line. In addition, these magnetic vortices are affected by forces of a dissipative nature – which, as always, are directed against their velocity vector. To simplify the problem under study, we will take into account only the dissipation, which is caused by the tunneling of normal (i.e., non-superconducting) electrons across the Josephson transmission line. As was shown in [14-16], in this case, the magnetic flux quanta perform relativistic Brownian motion, which is described by the following stochastic differential equation:

\[
\frac{du}{dt} = -\alpha u(1 - u^2) + \frac{\pi}{4} f(t) (1 - u^2)^{3/2}
\]  

(1)

Here: \(u\) is the dimensionless fluxon velocity in the Josephson transmission line, which varies in the interval (-1;1) and is determined by the following formula:

\[ u = \frac{\vartheta_x}{c} \]  

(2)

where: \(\vartheta_x\) is the velocity of relativistic quanta of the magnetic flux, which, according to equation (1), varies randomly, both in magnitude and direction; \(c\) is the limiting velocity of magnetic waves in the Josephson transmission line or the Swichart velocity.

In (1), \(\alpha\) is the dissipation coefficient determined by the resistance of the Josephson contact in the non-superconducting state, i.e., in the normal state. The function \(f(t)\) describes fluctuations of an external current random in time.

In the latter case, there is no universal, general statistical description-stochastic (or random) processes, suitable for any systems. The reason for this circumstance is well known and it is due to the variety of physical mechanisms of the occurrence of fluctuations [7-17].
Therefore, for the sake of concreteness, we restrict ourselves to studying the effect of a purely fluctuating external current on the dynamics of relativistic magnetic flux quanta, assuming that the current is a time-correlated Gaussian noise with the following statistical characteristics:

\[ \langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 2\sigma^2 \sigma(t - t'). \]  \hfill (3)

Here: angle brackets mean averaging over all possible realizations of the random function \( f(t) \). \( \sigma^2 \) expresses the normalized intensity of current fluctuations.

As can be seen from (1), it is a nonlinear differential stochastic equation and the random force \( f(t) \) enters it multiplicatively, i.e., it is multiplied by \( (1 - u^2)^{3/2} \). Here: \( u(t) \) is the desired solution (1), which is expressed by a nonlinear functional of \( f(t) \). This makes it much more difficult to find both the solution to equation (1) itself and to calculate the average values of \( \langle u^n \rangle \) (here: \( n = 1, 2, \ldots \))

Therefore, we will try to transform the nonlinear stochastic differential equation (1) into a linear equation of this type. To do this, we take into account that the relativistic quanta of the magnetic flux are a relativistic quasiparticle, i.e., its momentum and velocity are related by the expression [15].

\[ p = \frac{u}{\sqrt{1 - u^2}}, \]  \hfill (4)

which is similar to an ordinary relativistic particle.

Now let's perform the reverse transformation

\[ u = \frac{p}{\sqrt{1 + p^2}}. \]  \hfill (5)

Then, as you can easily see, the nonlinear stochastic equation (1) for the velocity \( u(t) \) is transformed into a linear stochastic equation for the momentum \( p(t) \):

\[ \frac{dp}{dt} = -\alpha p + \frac{\pi}{4} f(t). \]  \hfill (6)

In addition, as can be seen from (6), now the random force \( f(t) \) enters it is additive.

Therefore, figuratively speaking, in this case it was possible with one bullet to kill two birds with one stone: 1) linearize the nonlinear equation; 2) transform the multiplicative noise into additive noise. This circumstance greatly simplifies the further solution of the problem, since several powerful research methods have been developed for linear stochastic differential equations with additive noise [7,17-21].

Using the formulas for the transformation of the distribution function (see, for example, [7,18]), we easily find the velocity distribution function of the relativistic quantum of the magnetic flux:

\[ \Phi(u) = \frac{1}{\sqrt{\pi}} \left( 1 - u^2 \right)^{-3/2} \exp \left[ -\frac{au^2}{1 - u^2} \right], \]  \hfill (7)

\[ a = \frac{\alpha}{2D}. \]

Here: \( \alpha \) is the dissipation factor, \( D = \frac{\pi}{4} \sigma^2 \) is the renormalized intensity of current fluctuations.

\[ |u| \leq 1. \]

As can be seen from the last expression, the velocity distribution function \( f(u) \) of the relativistic quantum of the magnetic flux is essentially non-Gaussian, although the current fluctuations are Gaussian. This is a consequence of the nonlinear transformation from the velocities \( u \) to the pulses \( p \), which is determined by the formula (5).

Now it is time to discuss the unsolvable tri-lemma discussed in [8-11]. A thorough analysis of these works shows that their authors did not provide for non-trivial transformations (4) ↔ (5). Therefore, they failed to ”get rid” of the multiplicative noise in both the equation of motion and the Fokker – Planck equation – which describes the change in the distribution function (of impulses or velocities). This did them a disservice and the authors were forced to resort to various” rules ” of averaging – which, as emphasized earlier in the literature (see, for example, [7]), are not always correct from a physical point
of view and lead as a consequence to ambiguous results, which, unfortunately, happened in the above cases.

We do not hide to our happiness, we paid attention to the transformation (4) ↔ (5) unexpected for mathematics and quite understandable – from the point of view of relativistic physics.

The main reason for these failures, in our opinion, is that the "Lorentz transformation of the non-relativistic equation of motion" [10] was performed, which, as is known, is physically incorrect. This predetermined-initially, if I may say so, the sad "fate" of these "tricks".

We find the shape of the spectral lines of electromagnetic radiation produced by relativistic quanta of the magnetic flux. To do this, we use the following well-known expression for the generated frequencies of electromagnetic radiation by fluxons:

\[ f = \frac{\vartheta}{2L} \]  

(8)

Here: \( \vartheta \) is the velocity modulus of relativistic magnetic flux quanta, \( L \) - is the length of the Josephson transmission line.

Applying the formulas for the transformation of the distribution function and taking into account the expressions (7) – (8), we obtain the following expression for the shape of the spectral lines of electromagnetic radiation produced by relativistic quanta of the magnetic flux:

\[ \psi(f) = 2 \sqrt{\frac{\pi}{a}} \left[ 1 - \left( \frac{f}{f_m} \right)^2 \right]^{-\frac{3}{2}} \exp \left[ -a \left( \frac{f}{f_m} \right)^2 \right] \]  

(9)

Here: \( f_m = \frac{c}{2L} \) - is the maximum frequency of generated electromagnetic radiation in Josephson transmission lines.

Comparing the expressions for the distribution function (7) and the shape of the spectral lines of electromagnetic radiation (9), we see that they are quite similar. This is to be expected, since the distribution function retains its form in a linear transformation (see, for example, [18]).

Note that the factor of two in the expression the shape of the spectral lines of electromagnetic radiation appeared due to the fact that

\[ \int_0^{f_m} \psi(f) df = 1. \]  

(10)

The latter condition holds for any distribution function – since the total probability is always one for all possible events and zero for impossible events.

A remarkable and non-trivial feature of the found form of the spectral lines of electromagnetic radiation is that it is at the values of the parameter \( a \) satisfying the condition

\[ 0 < a < \frac{3}{2} \]  

(11)

the function \( \psi(f) \) has two extreme points

\[ f_1 = 0, \quad f_2 = f_m \sqrt{1 - \frac{2a}{3}}. \]  

(12)

Here, the 1st and 2nd extreme points correspond to the minimum and maximum of this function (Fig. 1). Figuratively speaking, the shape of this function takes on a wonderful appearance: from a semi – bell shape, it will turn into a single-humped one.
Figure 1. The shape of the spectral lines of radiation produced by fluxons.

Note that in the non-relativistic approximation, i.e. at $\vartheta \ll c$ and in the case of small current fluctuations, i.e. at $a >> 1$ (or $2D << c$), the shape of the spectral lines of electromagnetic radiation becomes Gaussian and is described by the expression

$$\psi_{r}(f) = 2\frac{a}{\sqrt{\pi}} \exp \left[ -a \left( \frac{f}{f_{m}} \right)^{2} \right].$$

(13)

This function, for any value of the parameter $a$, has a single maximum point at $f = 0$ and always keeps its appearance unchanged.

We emphasize that such an extraordinary "behavior" of the shape of the spectral lines of electromagnetic radiation $\psi(f)$ (9) could not have been foreseen in advance! Therefore, this hyperfine shape of the spectral lines of electromagnetic radiation, as well as the velocity distribution function of relativistic magnetic flux quanta, has not been discovered by anyone in the more than a century-old history of the theory of Brownian motion.

3. Limitation of the root-mean-square dispersion of the frequency of electromagnetic radiation produced by relativistic quanta of the magnetic flux

It is well known that when the generated radiation is of a statistical nature – and in this case it is precisely so, since the relativistic quantum of the magnetic flux performs a Brownian motion-it has a random character. In such situations, it is customary to describe the radiation process using the following so-called statistical characteristics, namely: $\langle f \rangle$ - the average frequency, the average square of the frequency $\langle f^{2} \rangle$ and the mean-square dispersion of the radiation frequency determined by them

$$\bar{D}f = \langle f^{2} \rangle - \langle f \rangle^{2}. \quad \text{(14)}$$

Next, we take into account that according to (8) and (7):

$$\langle f^{2} \rangle = f_{m}^{2} \int_{-1}^{1} u^{2} \Phi(u) du \equiv f_{m}^{2} \bar{D}[u]. \quad \text{(15)}$$

Now, using (14) – (15), we find the following beautiful expression for the root-mean-square dispersion of the frequency of electromagnetic radiation produced by the relativistic quantum of magnetic flux:

$$\bar{D}f = f_{m}^{2} (\langle u^{2} \rangle - \langle |u| \rangle^{2}) \equiv f_{m}^{2} \bar{D}|u|. \quad \text{(16)}$$

Here $\bar{D}u$ -is the mean-square fluctuation of the fluxon velocity. $\langle |u| \rangle$ and $\langle u^{2} \rangle$ -are the mean of the modulus and the square of the modulus of the normalized velocity of relativistic magnetic flux quanta.

Then using the known expressions for the statistical characteristics of the velocity of the relativistic quantum magnetic flux
\[ \langle |u| \rangle = \left[ 1 - \text{erf}(\sqrt{a}) \right] \exp a , \]  
\[ \langle |u|^2 \rangle = 1 - \sqrt{a\pi} \cdot \langle |u| \rangle , \]  

obtained in [14-16], we obtain
\[ \frac{\bar{D}f}{f} = 1 - \sqrt{a\pi} \langle |u| \rangle - \langle |u|^2 \rangle . \]  

As follows from the last expression, the normalized root-mean-square dispersion of the radiation frequency depends on the parameter \( a \) in a non-monotonic way and has its maximum value at a certain critical value (Fig. 2).

![Figure 2. Normalized root-mean-square dispersion of the radiation frequency.](image)

To prove the non-triviality of the unexpected result obtained, we give an expression for the root-mean-square dispersion of the fluxon velocity at very small fluctuations of the random external current in the non-relativistic approximation [16]:
\[ \langle |u|^2 \rangle = \frac{1}{\sqrt{a\pi}} , \quad \langle u^2 \rangle = \frac{1}{2a} , \quad \bar{D}u = \left( 1 - \frac{1}{2} \right) \cdot \frac{1}{2a} . \]  

Further, given that \( \frac{1}{2a} = \frac{D}{\alpha} \), we conclude that the mean – square dispersion of the radiation frequency is unlimited for relatively large current fluctuations.

4. Conclusion
Let us summarize the results of the research carried out in this work:

1. We will express the parameter \( \alpha \) in terms of the physical quantities that characterize the relativistic quantum of the magnetic flux and the Josephson transmission line. To do this, we use the equality of the statistical and thermodynamic mean, i.e.
\[ \langle u^2 \rangle = \frac{\langle \theta^2 \rangle}{c^2} . \]  

But in a state of thermal equilibrium
\[ \langle \theta^2 \rangle = \frac{kT}{m} . \]  

Therefore, according to (22) – (23) we have
\[ \langle u^2 \rangle = \frac{kT}{mc^2} \equiv \frac{kT}{E_0} . \]  

Here: \( E_0 = mc^2 \) -is the rest energy of fluxons, which is defined by the following expression
\[ E_0 = \frac{I_c \Phi_0}{2\pi} \]  

(24)

Where: \( I_c \) - critical (or maximum) Josephson current, \( \Phi_0 = \frac{h}{2e} \) - the magnitude of the magnetic flux quantum, \( h \) - Planck's constant, \( e \) - the electron charge modulus.

Now we take into account that in the non-relativistic approximation under consideration

\[ \langle u^2 \rangle = \frac{1}{2a} \equiv \frac{D}{\alpha} \]  

(25)

Thus, in accordance with the above, we get:

\[ \frac{D}{\alpha} = \frac{kT}{E_0}, \quad D = \frac{2\pi akT}{I_c \Phi_0}. \]  

(26)

The latter formula relates the intensity of random effects on the fluxon \( D \), the dissipation factor \( \alpha \), and the effective temperature \( T \) of the system under consideration. As you know, the last expression is the Einstein relation (see, for example, [9]) and it is valid only for linear Brownian motion. The latter result is consistent with the results of [4,5], which are valid only for very small fluctuations in the noise current and fluxon velocities much lower than the Swichart velocity. The latter formula relates the intensity of random effects on the fluxon \( D \), the dissipation coefficient \( \alpha \), and the effective temperature \( T \) of the system under consideration. As you know, the last expression is the Einstein relation (see, for example, [9]) and it is valid only for linear Brownian motion. The latter result is consistent with the results of [4,5], which are valid only for very small fluctuations in the noise current and fluxon velocities much lower than the Swichart velocity.

But the expressions for the characteristic fluxon velocities obtained by us are valid for arbitrary values of the parameter, i.e., for any ratio of the thermal energy \( kT \) and the rest energy of the fluxon \( E_0 \).

2. The most intriguing conclusion follows when \( \alpha = 0 \), i.e. for a dissipative Josephson transmission line. According to the results of the theory of nonrelativistic Brownian motion (25), we get \( \langle u^2 \rangle \) equal to infinity - which has no physical meaning.

But according to our formulas (17) – (18), both the arithmetic mean and the quadratic mean velocity of the magnetic flux quanta are equal to the limiting velocity \( c \) in the system under consideration. Even more striking is that at \( \alpha \to 0 \), the mean-square dispersion of the frequency of electromagnetic radiation produced by fluxons: is equal to infinity-according to the non-relativistic theory of Brownian motion, and is equal to zero – within the framework of the exact relativistic theory of Brownian motion.

The latter comparison proves the necessity and correctness of the relativistic consideration of Brownian motion in such critical cases.

3. Now we determine the value of the critical Josephson current under the conditions when the shape of the spectral lines of electromagnetic radiation is transformed – which is observed when the condition is met

\[ 2a = \frac{\alpha}{D} \equiv \frac{E_0}{kT} < 3. \]  

(27)

Then, taking into account (25), we obtain the following inequality

\[ I_c < I_f, \quad I_f = \frac{6\pi akT}{\Phi_0} \equiv AT, \]  

(28)

where \( I_f \) is the fluctuating current at temperature \( T \). \( A \) - is the proportionality coefficient.

Given that the Boltzmann constant \( k = 1.38 \times 10^{-23} \text{ J/K} \), and the magnitude of the magnetic flux quantum \( \Phi_0 \approx 2.06875 \times 10^{-15} \text{ Vb} \), we obtain

\[ A \approx 0.126 \text{ mA/K}. \]  

(29)

We find the value of the fluctuating current at the nitrogen temperature \( T=77 \text{ K} \). Then, given (28) – (29) we get \( I_f = 10 \text{ mA} \). Therefore, for low-current superconductors with an electric current of the order of one mA, condition (29) is realized. And for high-current superconductors with a critical current of
the order of 100 mkA, \( I_i > I_f \) i.e., the inverse inequality is satisfied and the shape of the spectral lines of electromagnetic radiation is not transformed.

Thus, in the presented work, the shape of spectral lines of electromagnetic radiation produced by relativistic quanta of magnetic flux moving in Josephson transmission lines under the action of a purely fluctuating external current and dissipation was found for the first time. For the first time, it was also proved that the rms dispersion of the frequency of generated electromagnetic radiation by fluxons is limited at any level of noise current.

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