Some examples of absolute continuity of measures in stochastic fluid dynamics

B. Ferrario
Dipartimento di Matematica - Università di Pavia

Abstract
A non linear Itô equation in a Hilbert space is studied by means of Girsanov theorem. We consider a non linearity of polynomial growth in suitable norms, including that of quadratic type which appears in the Kuramoto–Sivashinsky equation and in the Navier–Stokes equation. We prove that Girsanov theorem holds for the 1-dimensional stochastic Kuramoto–Sivashinsky equation and for a modification of the 2- and 3-dimensional stochastic Navier–Stokes equation; this modification consists in substituting the Laplacian $-\Delta$ with $(-\Delta)^\alpha$, where $\alpha > \frac{d}{2} + 1$ ($d = 2, 3$). In this way, we prove existence and uniqueness of solutions for these stochastic equations. Moreover, the asymptotic behaviour for $t \to \infty$ is characterized.

Key words: stochastic fluid dynamics, Girsanov theorem, existence and uniqueness of strong solutions, regularity results, uniqueness of invariant measures.

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1 Introduction
The study of non linear equations requires some skill to deal successfully with the non linearity. As far as stochastic differential equations are concerned, a possible technique to study a non linear equation is the Girsanov transform. Indeed, given a non linear stochastic Itô equation

$$du(t) + [Lu(t) + F(u(t))] \, dt = Gdw(t), \quad t \in [0, T]; \quad u(0) = x$$

it is possible to analyze it as a perturbation of the linear equation

$$dz(t) + Lz(t) \, dt = Gdw(t), \quad t \in [0, T]; \quad z(0) = x$$

by means of Girsanov theorem. It is well known that this theorem holds if for instance Novikov condition

$$E\left[\exp \left(\frac{1}{2} \int_0^T |G^{-1}F(z(t))|^2 dt\right)\right] < \infty$$

is satisfied. We are interested in the case in which Novikov condition is not fulfilled, but it is if the non linear term $F$ is suitably truncated. By an approximating procedure we can apply Girsanov transform so to get that equation (1) has a weak solution having the same regularity as $z$ and the law $\mathcal{L}_u$ of $u$ is absolutely continuous with respect to the law $\mathcal{L}_z$ of $z$ ($\mathcal{L}_u \prec \mathcal{L}_z$) and
possibly the converse too, so to get the equivalence of \( \mathcal{L}_u \) and \( \mathcal{L}_z \) \( (\mathcal{L}_u \sim \mathcal{L}_z) \). We recall that if \( \mathcal{L}_u \prec \mathcal{L}_z \), uniqueness for equation (2) implies uniqueness in law for equation (1). Moreover, if \( \mathcal{L}_u \sim \mathcal{L}_z \), each property holding P-a.s. for the process \( z \) must hold also for the process \( u \) and vice versa.

Our analysis to verify if Girsanov transform can be used is quite standard. We formalize it here in order to apply it in the next sections to some models in stochastic fluid dynamics, in which the equations are set in a infinite dimensional Hilbert space and the non linearity \( F \) is of quadratic type; however, the result holds true for \( F \) of polynomial growth.

As to the structure of the paper, in Section 2 two abstract results are presented; in the first it is proved that \( \mathcal{L}_u \prec \mathcal{L}_z \) and in the second that \( \mathcal{L}_u \sim \mathcal{L}_z \). Then, in the other two sections these results are applied to a stochastic Kuramoto–Sivashinsky equation and to a modified stochastic Navier–Stokes equation, respectively. For these non linear equations (which have a similar non linearity), we obtain results of existence and uniqueness of the solution; further, uniqueness of the invariant measure is proved, so to characterize the asymptotic behaviour.

### 2 Absolute continuity of laws

We are given a separable Hilbert space \( H \), equipped with a complete orthonormal system \( \{e_j\}_{j=1}^{\infty} \), and a complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\). We denote by \( \mathbb{E} \) the expectation with respect to the measure \( P \).

As far as equation (2) is concerned, we assume that \( L \) and \( G \) are linear operators in \( H \) and \( G \) is invertible. The process \( w \) is a cylindrical Wiener process in \( H \), defined on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\). This means that, given a sequence \( \{\beta_j\}_{j=1}^{\infty} \) of i.i.d. one dimensional Wiener processes defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), we represent \( w(t) = \sum_j \beta_j(t)e_j \).

Moreover, we assume that there exists a unique strong solution \( z \) (in the stochastic sense) which is a Markov process such that

\[
\mathbb{E} \|z\|_{C([0,T];E)}^{2p} < \infty
\]  

(3)

for some \( p > 1 \), where \( E \) is a separable dense subset of \( H \). Actually, it would be enough \( z \) to be a weak solution; but in our applications in Sections 3 and 4 \( z \) will be a strong solution and thus we assume it since now.

From now on, we denote by \( z(t; x) \), or simply by \( z(t) \), the solution of (1) evaluated at time \( t \) (thus \( z(0; x) = x \) and, for \( t > 0 \), \( z(t; x) \) is a random variable) and by \( z \) the solution process \( \{z(t; x)\}_{0 \leq t \leq T} \) on a time interval \([0,T]\).

The main assumption on the non linear term is that the operator \( G^{-1}F : E \rightarrow H \) is measurable and

\[
|G^{-1}F(v)|_H \leq c(1 + |v|_E^p) \quad \forall v \in E,
\]

(4)

where \( c \) is a suitable constant and \( p > 1 \) is the same as in (3). This implies that

\[
\int_0^T |G^{-1}F(z(t))|_H^2 \, dt \leq 2Tc^2 \left(1 + \mathbb{E} \|z\|_{C([0,T];E)}^{2p}\right)
\]

(5)

so

\[
\mathbb{E} \int_0^T |G^{-1}F(z(t))|_H^2 \, dt \leq 2Tc^2 \left(1 + \mathbb{E} \|z\|_{C([0,T];E)}^{2p}\right) < \infty.
\]

(6)
In particular
\[ P\left\{ \int_0^T \|G^{-1}F(z(t))\|_H^2 dt < \infty \right\} = 1. \] (7)

This condition is necessary for Novikov condition
\[ \mathbb{E}\left[ \exp\left( \frac{1}{2} \int_0^T \|G^{-1}F(z(t))\|_H^2 dt \right) \right] < \infty \] (8)

to hold. It is well known (see, e.g., [2] for stochastic PDE’s in Hilbert spaces) that condition (8) implies that
\[ \rho_{u/z}^T := \mathbb{E}\left[ \exp\left( \int_0^T H \langle G^{-1}F(z(s)), dw(s) \rangle_H - \frac{1}{2} \int_0^T \|G^{-1}F(z(s))\|_H^2 ds \right) \right] \] (9)

is a probability density. Here \( \sigma^T(z) \) denotes the \( \sigma \)-algebra generated by \( \{z(t)\}_{0 \leq t \leq T} \). The stochastic integral in the exponent has to be understood as \( \sum_j \int_0^T H \langle G^{-1}F(z(s)), e_j \rangle_H d\beta_j(s) \) and is well defined because of (8) (see [2], Chapter 4).

As soon as we know that \( \mathbb{E}[\rho_{u/z}^T] = 1 \), we apply Girsanov theorem to get that \( \mathcal{L}_u \prec \mathcal{L}_z \). We remind it here, for reader’s convenience (see, e.g., [2, 11, 12]). Defined the probability measure \( P^* \) on \( (\Omega, F) \) by \( dP^* = \rho_{u/z}^T dP \), Girsanov theorem states that
\[ w^*(t) = w(t) + \int_0^t G^{-1}F(z(s)) ds \]

is a cylindrical Wiener process on \( (\Omega, F, \{F_t\}_{0 \leq t \leq T}, P^*) \). So, if \( z \) solves equation (1) with Wiener process \( w \), then \( z \) solves equation (1) with Wiener process \( w^* \), since
\[ z(t) = x - \int_0^t Lz(s) ds + \int_0^t Gdw(s) \]
\[ = x - \int_0^t Lz(s) ds - \int_0^t F(z(s)) ds + \int_0^t Gdw^*(s). \]

Thus, \( P\{u \in \Lambda \} = P^*\{z \in \Lambda \} \) for every Borel set \( \Lambda \subset C([0, T]; E) \). Then \( P\{z \in \Lambda \} = 0 \) implies \( P^*\{z \in \Lambda \} = 0 \) and so \( P\{u \in \Lambda \} = 0 \), that is \( \mathcal{L}_u \prec \mathcal{L}_z \).

Summing up, assuming that the solution \( z \) to equation (1) is such that \( \mathbb{E}[\rho_{u/z}^T] = 1 \), then equation (1) has a weak solution having the same regularity as \( z \) and \( \mathcal{L}_u \prec \mathcal{L}_z \); moreover, uniqueness in law for \( z \) implies uniqueness in law for \( u \).

If \( \mathcal{L}_u \sim \mathcal{L}_z \), then each property holding \( \mathbb{P} \)-a.s. for the process \( z \) must hold also for the process \( u \) and vice versa. Also the laws of \( u(t; x) \) and \( z(t; x) \) are equivalent. In fact, \( P\{u(t; x) \in \Gamma \} = P^*\{z(t; x) \in \Gamma \} \) for every Borel set \( \Gamma \subset H \). In this way, if we can prove easily strong Feller property and irreducibility for the linear equation, these properties will be inherited by the non linear equation.

However, by (10) it does not follow that Novikov condition holds. Anyway, we can approximate the non linearity \( F \) in such a way that Novikov condition holds for the approximate equation and by this we obtain \( \mathbb{E}[\rho_{u/z}^T] = 1 \). The procedure is standard, but the results available in the literature do not apply here. For instance, there are similar techniques in [12] (but, even if they deal with a stochastic Navier–Stokes equation, the important issue there is the existence of weak solutions; Girsanov theorem is proved for other stochastic PDE’s)
even if they deal with a stochastic Kuramoto–Sivashinsky equation, the Novikov condition and Girsanov theorem are analyzed in a finite dimensional context. We point out that in this paper we prove Girsanov theorem for a 1D stochastic Kuramoto–Sivashinsky equation and for a modification of the 2D and 3D stochastic Navier–Stokes equation. Further, our results give regularity of strong solutions of equation (1) (we shall deal with a variety of spaces $E \subset H$) and the equivalence of all its transition functions so to characterize the asymptotic behaviour by means of Doob theorem.

We now state a first result on the absolutely continuity of the measures.

**Proposition 2.1** Assume (4) holds and that for every $x \in E$ there exists a unique strong solution $z$ of equation (2) on the time interval $[0,T]$, satisfying (3).

Then, given $u(0) = x$ there exists a unique weak solution $u$ to equation (1) on the time interval $[0,T]$ and the law of the process $u$ is absolutely continuous with respect to the law of the process $z$ solving (2), with density given by (9).

**Proof.** Let us define the approximating equation by

$$
\begin{cases}
    du^N(t) + Lu^N(t)dt + \chi_t^N(u^N(t))F(u^N(t))dt = Gdw(t)
\
u^N(0) = x
\end{cases}
$$

(10)

where for each $N = 1, 2, \ldots$, the truncation function $\chi^N$ is defined as follows:

$$
\chi_t^N(v) = \begin{cases}
1 & \text{if } \int_0^t |G^{-1}F(v(s))|^2_H ds \leq N \\
0 & \text{otherwise}
\end{cases}
$$

Notice that $\chi^N(z)$ is a progressively measurable process. Novikov condition

$$
\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^T |G^{-1}F(z(s))|^2_H ds\right)\right] < \infty
$$

now is trivially satisfied, since by the definition of $\chi_t^N$ we have

$$
\int_0^T |G^{-1}F(z(s))|^2_H ds \leq N \quad \text{a.s.}
$$

Hence, for any $N = 1, 2, \ldots$

$$
\mathbb{E}[e^{V_{T,N}}] = 1,
$$

where $V_{T,N} = \int_0^T \chi_N^N(z)\int H(G^{-1}F(z(s)), dw(s))_H - \frac{1}{2} \int_0^T \chi_N^N(z)|G^{-1}F(z(s))|^2_H ds$, and by Girsanov theorem we have that $\mathcal{L}_{u^N} \prec \mathcal{L}_z$ with the density

$$
\rho_{u^N/z} = \mathbb{E}[e^{V_{T,N}} | \sigma(T)].
$$

Now we want to prove that $\mathbb{E}[e^{V_T}] = 1$, where the exponent is $V_T = \int_0^T H(G^{-1}F(z(s)), dw(s))_H - \frac{1}{2} \int_0^T |G^{-1}F(z(s))|^2_H ds$.

We know that $\mathbb{E}[e^{V_{T,N}}] = 1$; moreover

$$
\mathbb{E}[e^{V_{T,N}}] = \mathbb{E}[\chi_N^N(z)e^{V_{T,N}}] + \mathbb{E}[(1 - \chi_N^N(z))e^{V_{T,N}}]
= \mathbb{E}[\chi_T^N(z)e^{V_T}] + P\{\chi_T^N(z) = 0\}.
$$
By monotone convergence, \( \lim_{N \to \infty} \mathbb{E}[\chi_N^T(z)e^{V_T}] = \mathbb{E}[e^{V_T}] \). On the other hand, \( \lim_{N \to \infty} P\{\chi_N^T(z) = 0\} = \lim_{N \to \infty} P\{\int_0^T |G^{-1}F(z(s))|^2 ds > N\} = 0. \)

Therefore \( \mathbb{E}[e^{V_T}] = 1 \) so that \( \mathbb{E}[e^{V_T}|\sigma^T(z)] \) is a probability density. Then, as explained before, the probability measure \( dP^* = \rho^T_{u/z} dP \) (with \( \rho^T_{u/z} \) given by (9)) defines a weak solution to equation (1).

Uniqueness (in law) of \( u \) is a consequence of uniqueness of \( z \) and \( \mathcal{L}u < \mathcal{L}z \). \( \square \)

Now, besides the previous conditions, let us assume that also equation (1) has a unique strong solution \( u \), enjoying the same property (3) as \( z \). We obtain a stronger result.

**Proposition 2.2** If (1) holds and for any \( x \in E \) both equations (1) and (2) have a unique strong solution on the time interval \( [0, T] \) satisfying (3), then the laws \( \mathcal{L}u \) and \( \mathcal{L}z \) are equivalent and the densities are given, respectively, \( \rho^T_{u/z} \) by (9) and

\[
\rho^T_{z/u} = \mathbb{E}\left[ \exp\left(-\int_0^T H(G^{-1}F(u(s)), dw(s)) + \frac{1}{2} \int_0^T |G^{-1}F(u(s))|^2 ds\right) | \sigma^T(u) \right].
\]

**Proof.** According to the previous proposition, we know that \( \mathcal{L}u < \mathcal{L}z \). On the other hand, interchanging the role of \( u \) and \( z \), again Proposition 2.1 provides that \( \mathcal{L}z < \mathcal{L}u \). Therefore, they are mutually absolutely continuous, i.e. equivalent. \( \square \)

As a consequence, also the laws of \( z(t; x) \) and \( u(t; x) \) are equivalent. Before stating the last result, we need to recall some definitions. A Markov process \( u \) is said to be strongly Feller in \( E \) at time \( t > 0 \) if \( P_t \) maps \( B_k(E) \) into \( C_k(E) \), where \( (P_t\phi)(x) := \mathbb{E}[\phi(u(t; x))] \); and irreducible in \( E \) at time \( t > 0 \) if \( P(t, x, \Gamma) > 0 \) for any \( x \in E \), \( 0 \neq \Gamma \subset E \) open, where \( P(t, x, \Gamma) := P\{u(t; x) \in \Gamma\} \).

**Corollary 2.3** Under the assumptions of Proposition 2.2 the process \( z \) is strongly Feller and irreducible if and only if so is the process \( u \).

In the next sections, we shall study first the linear equation so to check condition (3) and then estimate (1).

### 3 The 1D stochastic Kuramoto–Sivashinsky equation

We refer to [7] for the abstract setting, in which the stochastic Kuramoto–Sivashinsky equation in written as

\[
\begin{aligned}
\frac{du(t)}{dt} + [\nu A^2 u(t) - Au(t) + B(u(t), u(t))] dt &= A^\gamma dv(t) \\
u(0) &= x
\end{aligned}
\]

and the linear equation associated is

\[
\begin{aligned}
\frac{dz(t)}{dt} + [\nu A^2 z(t) - Az(t) + az(t)] dt &= A^\gamma dw(t) \\
z(0) &= x
\end{aligned}
\]

The unknown \( u \) can be interpreted as a one-dimensional velocity field in a compressible fluid (see [15]).

With respect to the setting of Section 2 we have that the linear operator is

\[
Lu = \nu A^2 u - Au + au
\]
with $a > 0$ large enough and $\nu > 0$, and the non linear operator is

$$ F(u) = B(u, u) - au. $$

The operator $G$ in front of the Wiener process is taken of the form $A^\gamma$ ($\gamma \in \mathbb{R}$). $w$ is a cylindrical Wiener process in $H$ on a probability space $(\Omega, F, P)$; $\{F_t\}_{t \in [0, T]}$ is the canonical filtration associated to the Wiener process.

The functional spaces are (given $L > 0$, so the spatial domain is $[-\frac{L}{2}, \frac{L}{2}]$)

$$ H = \{ u = u(\xi) \in L^2(-\frac{L}{2}, \frac{L}{2}) : \int_{-L/2}^{L/2} u \, d\xi = 0 \}, $$

$$ E = D(A^0) \text{ for some } \theta > 0, $$

where

$$ Au = -u'' $$

$$ D(A) = H \cap \{ u = u(\xi) \in H^2(-\frac{L}{2}, \frac{L}{2}) : u(-\frac{L}{2}) = u(\frac{L}{2}), u'(-\frac{L}{2}) = u'(\frac{L}{2}) \}. $$

The operator $A$ is a strictly positive unbounded self-adjoint operator in $H$, whose eigenvectors $\{e_j\}_{j=1}^\infty$ form a complete orthonormal basis of the space $H$. The powers $A^\theta$ are defined for any $\theta \in \mathbb{R}$: if $Ae_j = \lambda_j e_j$ then $A^\theta v = \sum_j \lambda_j^\theta \langle v, e_j \rangle e_j$, $D(A^\theta) = \{ v = \sum_j v_j e_j : \sum_j \lambda_j^{2\theta} v_j^2 < \infty \}$. Moreover, $\lambda_j \sim j^2$ as $j \to \infty$.

The operator $-(\nu A^2 - A + a)$ generates in $H$ (and in any $D(A^\theta)$) an analytic semigroup of negative type of class $C_0$.

The operator $B$ is the bilinear operator defined by

$$ B(u, v) = uv'. $$

For instance, $B$ maps $D(A^{1/2}) \times D(A^{1/2})$ into $H$; other domains of definition of $B$ are given in $\ref{7}$.

First, let us consider the linear equation. We are interested in the regularity of the solution $z$ and in the asymptotic behaviour for $t \to \infty$. For this, we denote by $R(t, x, \cdot)$ the transitions functions for $\ref{12}$, i.e. $R(t, x, \Gamma) = P\{ z(t; x) \in \Gamma \}$, and by $R_t$ the Markovian semigroup, i.e. $(R_t \phi)(x) = \mathbb{E}[\phi(z(t; x))]$. We say that a measure $m$ is invariant for equation $\ref{12}$ if $\int R_t \phi \, dm = \int \phi \, dm$ for every $t \geq 0, \phi \in C_b(D(A^\theta))$. We collect the results in the following proposition.

**Proposition 3.1** If $\theta + \gamma < \frac{3}{4}$, then for any $x \in D(A^\theta)$ equation $\ref{12}$ has a unique strong solution $z$ such that

$$ \mathbb{E}[\|z\|^{2p}_{C([0, T]; D(A^\theta))}] < \infty $$

for any $p \geq 1$ and $T < \infty$; this is a Markov process, strongly Feller and irreducible in $D(A^\theta)$ for any $t > 0$. The Gaussian measure $\mu_l = \mathcal{N}(0, \frac{1}{4} A^2 \gamma \nu A^2 - A + a)^{-1}$ is the unique invariant measure, all transition functions $R(t, x, \cdot)$ are equivalent to $\mu_l$ and

$$ \lim_{t \to +\infty} R_t \phi(x) = \int \phi \, d\mu_l, $$

$$ \lim_{t \to +\infty} R(t, x, \Gamma) = \mu_l(\Gamma) $$

for any $x \in D(A^\theta), \phi \in C_b(D(A^\theta))$ and Borel set $\Gamma \subset D(A^\theta)$. 

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Proof. From (3.10) in [7], we know that, given \( x \in D(A^0) \), if \( \theta + \gamma < \frac{3}{4} \) equation (12) has a unique strong solution \( z \)

\[
z(t) = e^{-(\nu A^2 - A + a)t} x + \int_0^t e^{-(\nu A^2 - A + a)(t-s)} A^\gamma dw(s)
\]

whose paths are, \( P \)-a.s., in \( C([0,T]; D(A^\beta)) \). This is a Markov process; many of its properties are easy to check, since the semigroup \( \{e^{-(\nu A^2 - A + a)t}\}_{t \geq 0} \) and the covariance of the noise are diagonal operators and commute.

We recall the basic steps for checking the regularity of \( z \) (the result follows rigorously, e.g., from [2], Chapter 5, and is proved in [7]):

\[
|A^\theta e^{-(\nu A^2 - A + a)t} x|_H \leq |A^\theta x|_H \quad \forall t \geq 0
\]

\[
E\left| \int_0^t A^\theta e^{-(\nu A^2 - A + a)(t-s)} A^\gamma dw(s) \right|^2_H = E\left[ \sum_{j=1}^\infty \lambda_j^{\theta+\gamma} \int_0^t e^{-(\nu \lambda_j^2 - \lambda_j + a)(t-s)} d\beta_j(s) e_j \right]^2_H
\]

\[
= \sum_{j=1}^\infty \lambda_j^{2(\theta+\gamma)} \int_0^t e^{-2(\nu \lambda_j^2 - \lambda_j + a)(t-s)} ds
\]

\[
\leq \sum_{j=1}^\infty \frac{\lambda_j^{2(\theta+\gamma)}}{2(\nu \lambda_j^2 - \lambda_j + a)} \quad \forall t > 0.
\]

The last series is convergent if \( \theta + \gamma < \frac{3}{4} \), since \( \lambda_j \sim j^2 \) as \( j \to \infty \).

According to Burkholder-Davis-Gundy inequality, the second estimate provides that inequality (13) holds for any \( p \).

The result on the invariant measure is obtained as in [2], Chapter 11. Actually, the result is trivial if we work first on each component \( z_j \) and then we recover the infinite dimensional result for \( z (z(t) = \sum_{j=1}^\infty z_j(t)e_j) \). Indeed, each component \( z_j \) satisfies

\[
dz_j(t) + [\nu \lambda_j^2 - \lambda_j + a]z_j(t) dt = \lambda_j \beta_j(t) dt, \quad z_j(0) = x_j;
\]

its law is \( \mathcal{N}\left(e^{-(\nu \lambda_j^2 - \lambda_j + a)t} x_j, \frac{1}{2} \frac{\lambda_j^{2\gamma}}{\nu \lambda_j^2 - \lambda_j + a} \right) \) and for \( t \to +\infty \) the density of this Gaussian measure converges to the density of the Gaussian measure \( \mathcal{N}(0, \frac{1}{2} \frac{\lambda_j^{2\gamma}}{\nu \lambda_j^2 - \lambda_j + a}) \), which is the unique stationary measure. Therefore, equation (12) has a unique invariant measure; this is the Gaussian measure with mean 0 and covariance operator \( Q_\infty = \frac{1}{2} A^{2\gamma}[\nu A^2 - A + a]^{-1} \).

It is easy to check that \( \int |A^\theta x|^2_H d\mu(x) < \infty \) and that \( \mu(\Gamma) > 0 \) for any open and non empty set \( \Gamma \subset D(A^0) \).

We expect that irreducibility and strong Feller property hold, because the noise acts on all directions \( e_j \) and the semigroup \( e^{-(\nu A^2 - A + a)t} \) makes \( z \) depending very regularly on the initial data \( x \).

As far as strong Feller property is concerned, by [2] (Chapter 9) we know that condition \( \text{Ran}(Q_1^{1/2}) \supset \text{Ran}(e^{-(\nu A^2 - A + a)t}) \) is equivalent to the strong Feller property, where \( Q_t \) is the covariance operator of the Gaussian random variable \( z(t;x) \). Since \( Q_t = \frac{1}{2} A^{2\gamma}[I - e^{-2(\nu A^2 - A + a)t}][\nu A^2 - A + a]^{-1} \) and for \( t > 0 \) the range of the operator \( e^{-(\nu A^2 - A + a)t} \) is contained in any space \( D(A^\beta) \) for \( \beta > 0 \), we see that this condition is trivially satisfied.
According to Theorem 11.13 in [2], (14) holds and all the transition measures $R(t, x, \cdot)$ are absolutely continuous with respect to $\mu$. Irreducibility comes straightforward. Let us point out that in the proof of this theorem, it is shown also that the law of $z(t; x)$ is equivalent to the law of $z(s; y)$ for any $t, s > 0$ and $x, y \in D(A^\theta)$; actually, this follows by Feldman-Hajek theorem, which is easy to verify in this case of diagonal operators. □

To set our problem as in Section 2, we have to fix some space $E = D(A^\theta)$. The interesting spaces are $D(A^\theta)$ for $\theta \geq 0$: $D(0^\theta) = H$ is the basic space of finite energy and, for $\theta > 0$, $D(\theta^\theta)$ is a subspace of $H$. In practise, given $\theta \geq 0$ we choose $\gamma$ as big as possible ($\gamma < \frac{4}{3} - \theta$) so to make to weakest assumption on the covariance of the noise. Or, given $\gamma < \frac{\theta}{3}$ (the limitation is due to $\theta \geq 0$), we choose $\theta$ as big as possible ($\theta < \frac{4}{3} - \gamma$). Decreasing $\gamma$, the operator $A^\gamma$ is "more regular" (in the sense that, for instance, $A^\gamma$ is a bounded operator for $\gamma \leq 0$) and this stronger assumption provides a more regular solution $z$ with paths in $C([0, T]; D(A^\theta))$.

Now, we deal with estimate (14). We have the following result.

**Lemma 3.2** Let parameters $\gamma$ and $\theta$ be chosen as follows:

for $\frac{1}{4} < \gamma < \frac{3}{4}$: $\frac{3}{4} \gamma - \frac{2}{3} \leq \theta < \frac{3}{4} - \gamma$

for $0 \leq \gamma \leq \frac{1}{4}$: $\frac{3}{4} \gamma - \theta \leq \frac{3}{4} - \gamma$

for $\gamma < 0$: $\frac{3}{4} \gamma - \theta < \frac{3}{4} - \gamma$.

Then there exists a constant $c$, depending on $\gamma, \theta$ and $a$, such that

$$|A^{-\gamma}[B(v, v) - av]|_H \leq c \left(1 + |A^\theta v|^2_H\right) \quad \forall v \in D(A^\theta).$$

**Proof.** Notice that (15) imply the bounds $\gamma < \frac{3}{4}$, $\theta > 0$ and $\theta + \gamma < \frac{3}{4}$. The non linear term is estimated as follows:

$$|A^{-\gamma}B(v, \tilde{v})|_H \leq C_1|A^{\frac{3}{4} - \gamma}v|_H|A^{\frac{3}{4} - \gamma}\tilde{v}|_H$$

if $\frac{1}{4} \leq \gamma < \frac{3}{4}$

(16)

$$|A^{-\gamma}B(v, \tilde{v})|_H \leq C_2|A^{\frac{3}{4} - \gamma}v|_H|A^{\frac{3}{4} - \gamma}\tilde{v}|_H$$

if $0 \leq \gamma \leq \frac{1}{4}$

(17)

$$|A^{-\gamma}B(v, \tilde{v})|_H \leq C_3|A^{\frac{3}{4} - \gamma}v|_H|A^{\frac{3}{4} - \gamma}\tilde{v}|_H$$

if $\gamma < 0$

(18)

The two first inequalities come from the proof of Lemma 2.2 in [9]. The latter is proved in Proposition 2.1 in [7]. By the way, recalling that $B(v_1, v_1) - B(v_2, v_2) = B(v_1 - v_2, v_1) + B(v_2, v_1 - v_2)$ by bilinearity, the above inequalities show that the operator $A^{-\gamma}B(v, v)$ is continuous (hence, measurable) in the spaces where it is defined.

Notice that if (15) are satisfied, then $\theta > -\gamma$. Therefore, choosing $\theta$ as in (15) we get

$$|A^{-\gamma}[B(v, v) - av]|_H \leq |A^{-\gamma}B(v, v)|_H + a|A^{-\gamma}v|_H$$

$$\leq C_4|A^\theta v|^2_H + aC_5|A^\theta v|_H$$

$$\leq C_6 \left(1 + |A^\theta v|^2_H\right).$$

□

**Remark 3.3** The case $\theta = 0$ is not included. Indeed, we have

$$|A^{-\gamma}B(v, v)|_H \leq c|v|^2_H$$
for $\gamma > \frac{2}{3}$, because

$$|(B(v, v), x)| = \int_{-L/2}^{L/2} \frac{1}{2} (v^2)'x dξ = \frac{1}{2} \int_{-L/2}^{L/2} v^2 x' dξ$$

$$\leq \frac{1}{2} |v^2|_{L^1} |x'|_{L^\infty}$$

$$\leq c|v|^2_{L^2} |x'|_{D(A^{m})} \text{ for } m > \frac{1}{4}$$

$$= c|v|^2_{L^2} |x|_{D(A^{\frac{1}{4}+m})} \text{ for } m > \frac{1}{4}.$$ 

But the condition $\gamma > \frac{2}{3}$ is incompatible with $\theta + \gamma < \frac{2}{3}, \theta = 0$.

Now, we consider equation (11). Let us denote by $P(t, x, \cdot)$ its transitions functions.

**Theorem 3.4** For every $\gamma < \frac{2}{3}$ and choosing $\theta$ as in (13), we have the following result. Given $x \in D(A^0)$ there exist unique strong solutions of equations (11) and (12) on any finite time interval $[0, T]$, with paths in $C([0, T]; D(A^0))$, $P$-a.s.. We have $\mathcal{L}_u \sim \mathcal{L}_z$, with the densities

$$\rho_{u/z}^T = \mathbb{E} \left[ e^{\gamma T} |\sigma^T(z) \right] \text{ with } V_{+}^T = \int_0^T \langle A^{-\gamma} [B(z(s), z(s)) - az(s)], dw(s) \rangle_H$$

$$- \frac{1}{2} \int_0^T |A^{-\gamma} [B(z(s), z(s)) - az(s)]|^2_H ds$$

$$\rho_{z/u}^T = \mathbb{E} \left[ e^{\gamma T} |\sigma^T(u) \right] \text{ with } V_{-}^T = - \int_0^T \langle A^{-\gamma} [B(u(s), u(s)) - au(s)], dw(s) \rangle_H$$

$$- \frac{1}{2} \int_0^T |A^{-\gamma} [B(u(s), u(s)) - au(s)]|^2_H ds$$

for any $T > 0$.

Further, $P(t, x, \cdot) \sim \mu_t$ for any $t > 0, x \in D(A^0)$, where $\mu_t = N(0, \frac{1}{2} A^2 \gamma [\nu A^2 - A + a]^{-1})$ is the unique invariant measure for (12). The process $u$ is strongly Feller and irreducible in $D(A^0)$ at any time $t > 0$.

Finally, there exists only one invariant measure $\mu_{KS}$ for (11) which is equivalent to $\mu_t$.

$\mu_{KS}$ is ergodic, i.e.

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \phi(u(t; x)) dt = \int \phi d\mu_{KS}$$

$P$-a.s. for every $x \in D(A^0)$, $\phi \in L^1(\mu_{KS})$, and strongly mixing, i.e.

$$\lim_{t \to -\infty} P(t, x, \Gamma) = \mu_{KS}(\Gamma)$$

for every $x \in D(A^0)$ and Borel set $\Gamma \subset D(A^0)$.

**Proof.** If $\theta$ and $\gamma$ are chosen as in (13), from Proposition 3.1 and Lemma 3.2 we know that the assumptions of Proposition 2.1 (with $p = 2$ and $E = D(A^0)$) are satisfied. This implies that for $x \in D(A^0)$ equation (11) has a weak solution $u$ living in $C([0, T]; D(A^0))$ and $\mathcal{L}_u \ll \mathcal{L}_z$; but Theorem 4.3 in [7] provides existence and uniqueness of a strong solution $u$ for
any \( u(0) \in H = D(A^0) \) and \( \gamma < \frac{3}{4} \). Thus, we have the regularity result: given \( x \in D(A^\theta) \) equation (11) has a unique strong solution \( u \) with paths in \( C([0,T]; D(A^0)) \). By Proposition 2.2 we obtain that \( \mathcal{L}_u \sim \mathcal{L}_z \); moreover, \( P(t,x,\cdot) \sim R(s,y,\cdot) \sim \mu_t \) and Corollary 2.3 holds. We conclude our proof, bearing in mind Doob theorem for uniqueness of invariant measures (see [3]). The existence of an invariant measure has been proved in [7].

Let us notice that, as far as the regularity of solutions is concerned, this result improves that of Proposition 6.5 in [7], since now we can consider any space \( D(A^\theta) \) with \( \theta > 0 \). However, we are not able to prove the absolute continuity result in \( H = D(A^0) \), as explained in Remark 3.3, even if we know from [7] that for any \( u(0) \in H \) there exists a unique solution \( u \) such that \( u \in C([0,T]; H) \) (P-a.s.).

Moreover, the results of this section hold true if the operator in front of the Wiener process in equation (11) is of the form \( \mathcal{L}^\alpha \), where \( \mathcal{L} \) is an isometry in \( H \) (e.g., in [7] we considered \( \mathcal{L}^\alpha w(t) = \sum_{j=1}^\infty \chi_j \beta_j(t)(-1)^j e_{j+(-1)j+1} \); this includes interesting cases from the physical point of view as explained in [7]).

### 4 A modified stochastic Navier–Stokes equation

Since the quadratic term in the Kuramoto–Sivashinsky equation is similar to that in the Navier–Stokes equation, the only difference being that the Navier–Stokes equation is set in spaces of divergence free vectors, it is appealing to investigate if Girsanov transform holds for the stochastic Navier–Stokes equation. Unfortunately, the answer is negative. Anyway, let us analyse this problem modifying the linear part. Our issue is to determine how to modify the Navier–Stokes equation to apply our procedure.

Therefore, instead of the stochastic Navier–Stokes equation

\[
\begin{align*}
\frac{du(t)}{dt} + [\nu A u(t) + B(u(t), u(t))] \, dt &= A^\gamma \, dw(t) \\
\end{align*}
\]

(studied, e.g., in [1], [16], [8]), we introduce a modification in the linear part; given any \( \alpha \geq 1 \) we consider

\[
\begin{align*}
\begin{cases}
\frac{du(t)}{dt} + [\nu A^\alpha u(t) + B(u(t), u(t))] \, dt &= A^\gamma \, dw(t) \\
u(0) &= x
\end{cases}
\end{align*}
\]

(19)

This corresponds to replace the Laplacian \( -\Delta \) with \( (-\Delta)^\alpha \) in the Navier–Stokes equations in order to seek which values of \( \alpha \) provide the absolute continuity of \( \mathcal{L}_u \) with respect to the law of the linear equation associated to (19), which is the modified stochastic Stokes equation:

\[
\begin{align*}
\begin{cases}
\frac{dz(t)}{dt} + \nu A^\alpha z(t) \, dt &= A^\gamma \, dw(t) \\
z(0) &= x
\end{cases}
\end{align*}
\]

(20)

In this sense, our analysis reminds that of [11] to investigate for which values of \( \alpha \) the modified deterministic Navier–Stokes equation

\[
\frac{du}{dt}(t) + \nu A^\alpha u(t) + B(u(t), u(t)) = f(t)
\]

is well posed for \( d = 3 \) (we recall that for \( d = 2 \) there is no need of modification to get existence and uniqueness of a global solution).
With respect to the setting of Section 2, we have that the linear operator is
\[ Lu = \nu A^\alpha u \]
with \( \nu > 0, \alpha \geq 1 \), and the non-linear operator is
\[ F(u) = B(u, u). \]
The operator \( G \) in front of the Wiener process is taken of the form
\[ A^\gamma (\gamma \in \mathbb{R}). \]
\( \mathbb{w} \) is a cylindrical Wiener process in \( H \) on a probability space \((\Omega, F, P); \{ \mathcal{F}_t \}_{t \in [0,T]} \) is the canonical filtration associated to the Wiener process.

The functional setting is defined as usual (see [14]). The symbols \( A \) and \( B \) will denote different operators from those of Section 3, but we use the same symbols because of the analogy between these quantities in equations (11) and (19).

For \( d = 2, 3 \), let \( \mathcal{D} \) be the \( d \)-dimensional torus \( \mathbb{R}^d / (2\pi \mathbb{Z})^d \), i.e. we consider our problem on the spatial domain \([0, 2\pi]^d\) with periodic boundary conditions.

Set
\[ H = \{ u = \mathbb{w}(\mathcal{Q}) \in [L^2(\mathcal{D})]^d : \ \text{div} \ u = 0, \gamma_nu \text{ periodic}, \int_\mathcal{D} u \ d\mathcal{Q} = 0 \} \]
\[ E = D(A^\theta) \quad \text{for some } \theta > 0 \]
where \( \gamma_nu \) is the trace of the normal component of \( u \) on \( \partial\mathcal{D} \).

Let \([H^m_p(\mathcal{D})]^d, m \in \mathbb{N}\setminus\{0\}, \) be the space of functions of \([H^m_{loc}(\mathbb{R}^d)]^d, \) periodic with period \([0, 2\pi]^d\) and with zero average. Then the Stokes operator is defined as
\[ Au = -\Delta u, \quad u \in D(A) = [H^2_p(\mathcal{D})]^d \cap H. \]
\( A \) is a strictly positive unbounded self-adjoint operator in \( H \), whose eigenvectors \( \{ e_j \}_{j=1}^\infty \) form a complete orthonormal basis of the space \( H \). The powers \( A^\alpha \) are defined for any \( \alpha \in \mathbb{R} \). The operator \(-A \) generates in \( H \) (and in any \( D(A^\beta) \)) an analytic semigroup of negative type \( e^{-tA} \) of class \( C_0 \). Moreover, \( Ae_j = \lambda_j e_j \) with \( \lambda_j \sim j^{2/d} \) as \( j \to \infty \).

Now, consider the bilinear operator \( B \) from \( D(A^{1/2}) \times D(A^{1/2}) \) into \( D(A^{-1/2}) \) defined as
\[ \langle B(u, v), z \rangle = \int_\mathcal{D} z \cdot [(u \cdot \nabla)v] \ d\mathcal{Q} \quad \forall u, v, z \in D(A^{1/2}). \]

By the incompressibility condition we have
\[ \langle B(u, v), v \rangle = 0, \quad \langle B(u, v), z \rangle = -\langle B(u, z), v \rangle. \]

Other domains of definition of \( B \) are given below in [24].

First, let us consider the linear equation. Similarly to the previous section, we have

**Proposition 4.1** If
\[ \alpha - 2(\theta + \gamma) > \frac{d}{2}, \quad (21) \]
then for any \( x \in D(A^\theta) \) equation \([20] \) has a unique strong solution \( z \) such that
\[ \mathbb{E}\|z\|^2_{L^p([0,T]; D(A^\theta))} < \infty \quad (22) \]
for any \( p \geq 1 \) and \( T < \infty \); this is a Markov process, strongly Feller and irreducible in \( D(A^\theta) \) for any \( t > 0 \). The transition functions \( \tilde{R}(t, x, \cdot) \) are equivalent to \( \tilde{\mu}_t \) for any \( t > 0, x \in D(A^\theta) \), where \( \tilde{\mu}_t = \mathcal{N}(0, \frac{1}{\nu^2} A^{2\gamma - \alpha}) \) is the unique invariant measure, and

\[
\lim_{t \to +\infty} \tilde{R}_t \phi(x) = \int \phi \, d\tilde{\mu}_t
\]

(23)

\[
\lim_{t \to +\infty} \tilde{R}(t, x, \Gamma) = \tilde{\mu}_t(\Gamma)
\]

for any \( x \in D(A^\theta), \phi \in C_b(D(A^\theta)) \) and Borel set \( \Gamma \subset D(A^\theta) \).

**Proof.** The solution of equation (20) is given by

\[
z(t) = e^{-\nu A^\alpha t} x + \int_0^t e^{-\nu A^\alpha (t-s)} A^\gamma dw(s).
\]

If (21) holds, then there exists a continuous version with values in \( D(A^\theta) \). Indeed, the basic estimates are

\[
|A^\theta e^{-\nu A^\alpha t} x|_H \leq |A^\theta x|_H \quad \forall t \geq 0
\]

\[
\mathbb{E} \left( \int_0^t A^\theta e^{-\nu A^\alpha (t-s)} A^\gamma dw(s) \right)^2_H = \mathbb{E} \left( \sum_{j=1}^{\infty} \lambda_j^{\theta + \gamma} \int_0^t e^{-\nu \lambda_j(t-s)} d\beta_j(s) e_j \right)^2_H
\]

\[
= \sum_{j=1}^{\infty} \lambda_j^{2(\theta + \gamma)} \int_0^t e^{-2\nu \lambda_j(t-s)} ds
\]

\[
\leq \sum_{j=1}^{\infty} \frac{\lambda_j^{2(\theta + \gamma)}}{2\nu \lambda_j^{\alpha}} \quad \forall t > 0.
\]

The last series is convergent if (21) is fulfilled, since \( \lambda_j \sim j^{2/d} \) as \( j \to \infty \). According to Burkholder-Davis-Gundy inequality, the second estimate provides that inequality (22) holds for any \( p \).

The unique invariant measure is the Gaussian measure with mean 0 and covariance operator \( \frac{1}{\nu^2} A^{2\gamma - \alpha} \); indeed, each component \( z_j \) satisfies

\[
dz_j(t) + \nu \lambda_j^{\alpha} z_j(t) dt = \lambda_j^{\gamma} d\beta_j(t); \quad z_j(0) = x_j
\]

and this equation has only one invariant measure which is the 1-dimensional Gaussian measure \( \mathcal{N}(0, \frac{1}{2\nu} \lambda_j^{2\gamma - \alpha}) \).

(23) and the equivalence \( \tilde{R}(t, x, \cdot) \sim \tilde{\mu}_t \) can be shown as in Proposition 3.1.

Now, we have to choose the space \( E \). Let us consider \( \theta \geq 1 \). Why? Because the easiest estimate for \( B(v, v) \) is in the spaces \( D(A^m) \) with \( m \geq \frac{1}{2} \); indeed, for these values the space \( D(A^m) \) is a multiplicative algebra and therefore

\[
|A^m B(v, v)|_H \leq c_m |A^m v|_H |A^{m+\frac{1}{2}} v|_H
\]

(24)
This estimate shows that in these spaces the operator $A_\theta B(v,v)$ is well defined and continuous. In particular

$$|A^{\theta-\frac{1}{2}} B(v,v)|_H \leq c_\theta |A_\theta v|^2_H \quad \text{for } \theta \geq 1.$$  \hspace{1cm} (25)

To check inequality (11) in our context, the latter result suggests to set

$$-\gamma = \theta - \frac{1}{2}.$$ 

In this case, from (21) we know that the process $z$ will have paths in $C([0,T]; D(A_\theta))$ if

$$\alpha > \frac{d}{2} + 1.$$ 

**Remark 4.2** This condition shows that $\alpha = 1$ is not allowed. That is, our procedure does not work for the Navier–Stokes equation; only taking $\alpha$ sufficiently large we can prove Girsanov theorem and the absolute continuity of the laws. In particular, for $d = 2$ we require $\alpha > 2$ and for $d = 3$ we require $\alpha > \frac{5}{2}$. In the same way we can prove this result of absolute continuity for the stochastic 1D Burgers equation if $\alpha > \frac{3}{2}$.

It is interesting to compare which values of $\alpha$ provide that the Navier–Stokes equation is well posed, that is it has a unique global solution. For the deterministic equation, when $d = 2$ there is well posedness for $\alpha = 1$ whereas when $d = 3$ there is well posedness for $\alpha > \frac{5}{3}$ (see [11]). For the stochastic problem, when $d = 2$ it is enough to take $\alpha = 1$ (see, e.g., [8], [6]). We guess that when $d = 3$ there is well posedness again for $\alpha > \frac{5}{4}$; this result will be proved in a future work.

At this point, we prefer to fix a value of $\theta$; indeed, there are three quantities involved in the study of equation (19): $\alpha, \gamma, \theta$. To get not too involved relations to determine the "good" values of these parameters, we reduce the number of parameters setting $\theta = 1$. We point out that all the following results can be obtained in the same way for any $\theta > 1$, because of (24).

However, the technicalities are more involved for $0 \leq \theta < 1$ (see also Remark 4.4 below).

Having set $-\gamma = \theta - \frac{1}{2}$, the choice $\theta = 1$ implies $\gamma = -\frac{1}{2}$. For these values of the parameters, we have a pathwise uniqueness result. This is stronger that uniqueness in law, which would not need to be proved, as soon as Girsanov transformation holds; indeed, if $\mathcal{L}_u \prec \mathcal{L}_z$ then uniqueness of $z$ implies uniqueness in law of $u$.

**Proposition 4.3 (Pathwise uniqueness)** For $\gamma = -\frac{1}{2}$ and $\alpha > \frac{d}{2} + 1$, given $x \in D(A)$ any two $C([0,T]; D(A))$-valued strong solutions of (19) coincide $P$-a.s.

**Proof.** Let $u_1, u_2$ be two strong solutions on the probability space $(\Omega,F,\{F_t\}_{t \geq 0}, P)$. Set $U = u_1 - u_2$. Then $U$ satisfies, $P$-a.s.,

$$\frac{dU}{dt}(t) + \nu A_\theta U(t) + B(u_1(t), u_1(t)) - B(u_2(t), u_2(t)) = 0$$ \hspace{1cm} (26)

with initial data $U(0) = 0$. We proceed pathwise.
By bilinearity, \( B(u_1, u_1) - B(u_2, u_2) = B(u_1, U) + B(U, u_2) \). We multiply both sides of (26) by \( A^2 U(t) \); then (all the norms are in \( H \))

\[
\frac{1}{2} \frac{d}{dt} |AU(t)|^2 + \nu A^{1+\frac{\alpha}{2}} U(t)|^2 = - \langle B(u_1(t), U(t)) \rangle + B(U(t), u_2(t)), A^2 U(t) \rangle 
= - \langle A^{\frac{\alpha}{2}} [B(u_1(t), U(t)) + B(U(t), u_2(t))], A^{\frac{\alpha}{2}} U(t) \rangle.
\]

Using (23), we have \( |A^{\frac{\alpha}{2}} [B(u_1, U) + B(U, u_2)]| \leq c[|Au_1| + |Au_2||AU|] \); thus

\[
|\langle A^{\frac{\alpha}{2}} [B(u_1, U) + B(U, u_2)], A^{\frac{\alpha}{2}} U \rangle| \leq c[|Au_1| + |Au_2||AU|] |A^{\frac{\alpha}{2}} U| \leq c \left[ |Au_1| + |Au_2||AU| |A^{1+\frac{\alpha}{2}} U| \right] \leq \frac{c}{2} |A^{1+\frac{\alpha}{2}} U| |AU| + c \left[ |Au_1|^2 + |Au_2|^2 \right]|AU|^2.
\]

Hence

\[
\frac{d}{dt} |AU(t)|^2 + \nu |A^{1+\frac{\alpha}{2}} U(t)|^2 \leq C_T \left[ |Au_1(t)|^2 + |Au_2(t)|^2 \right]|AU(t)|^2.
\]

In particular

\[
\frac{d}{dt} |AU(t)|^2 \leq C_T [ |Au_1(t)|^2 + |Au_2(t)|^2 ] |AU(t)|^2.
\]

Since the paths \( u_1, u_2 \in C([0, T]; D(A)) \) and \( U(0) = 0 \), Gronwall lemma implies that

\[
|AU(t)| = 0 \quad \forall t \in [0, T],
\]

that is \( u_1(t) = u_2(t) \) for all \( t \in [0, T] \).

**Remark 4.4** The estimates of the proof remain valid for any \( \alpha \geq 1 \); in fact, inequality (*) holds for \( \alpha \geq 1 \). Therefore, we could have stated the proposition assuming only \( \alpha \geq 1 \). This depends strongly on the choice of \( \theta \). We point out that for \( \theta < 1 \) uniqueness in \( C([0, T]; D(A^\theta)) \) can be proved along the same lines, but \( \alpha \) must be larger than 1.

For example, in the case \( \theta = 0 \) we estimate the non linearity by

\[
|A^{-\frac{1}{2}+\frac{\alpha}{4}+\epsilon} B(v, \tilde{v})|_H \leq c|v|_H |\tilde{v}|_H,
\]

which holds for any \( \epsilon > 0 \). This is proved by means of the embeddings \( D(A^{\frac{1}{2}+\frac{\alpha}{4}+\epsilon}) \subset [H^{1+\frac{\alpha}{4}+2\epsilon}]^d \)
and \( [H^{1+\frac{\alpha}{4}+2\epsilon}]^d \subset [L^\infty(D)]^d \), that generalize the estimate of Remark 5.3 (proved there for \( d = 1 \)). In the proof of pathwise uniqueness (for \( \theta = 0, \gamma = \frac{1}{2} + \frac{\alpha}{4} + \epsilon \) we would use

\[
|\langle B(u_1, U) + B(U, u_2), U \rangle| = |\langle A^{-\frac{1}{2}+\frac{\alpha}{4}+\epsilon} [B(u_1, U) + B(U, u_2)], A^{\frac{1}{2}+\frac{\alpha}{4}+\epsilon} U \rangle| 
\leq c[|u_1|_H + |u_2|_H] |U|_H |A^{\frac{1}{2}+\frac{\alpha}{4}+\epsilon} U|_H.
\]

If \( |A^{\frac{1}{2}+\frac{\alpha}{4}+\epsilon} U|_H \leq c |A^{\frac{\alpha}{2}} U|_H \), that is if \( \alpha > 1 + \frac{\alpha}{2} \), we would get that

\[
\frac{d}{dt} |U(t)|_H^2 + \nu |A^{\frac{\alpha}{2}} U(t)|_H^2 \leq C_s [ |u_1(t)|_H^2 + |u_2(t)|_H^2 ] |U(t)|_H^2,
\]

so to conclude by Gronwall lemma that \( |U(t)|_H = 0 \) for all \( t \in [0, T] \).
Hence, we can prove pathwise uniqueness in $C([0,T]; D(A^0))$ if $\alpha > 1 + \frac{d}{2}$. On the other hand, chosen $\theta = 0$ and $\gamma = \frac{1}{2} + \frac{d}{2} + \varepsilon$ so to estimate the quadratic term as in (27), it follows that inequality (21) holds for $\alpha > 1 + d$.

Summing up, we have checked that to apply our procedure for $\theta = 0$ we need a stronger assumption on $\alpha$: $\alpha > 1 + d$. This is the reason for choosing $\theta \geq 1$ so to make the minimal assumption on $\alpha$.

Here is our main result.

**Theorem 4.5** For $\gamma = -\frac{1}{2}$ and $\alpha > \frac{d}{2} + 1$, given $x \in D(A)$ there exist unique strong solutions of equations (19) and (20) on any finite time interval $[0,T]$; the laws $L_u$ and $L_z$ are equivalent as measures on the space $C([0,T]; D(A))$.

In particular, the densities are

$$
ρ_u^{T} = \mathbb{E} \left[ e^{\int_0^T \langle A^\frac{1}{2} B(z(s), z(s)), dw(s) \rangle - \frac{1}{2} \int_0^T |A^\frac{1}{2} B(z(s), z(s))|^2 \gamma ds | \sigma(s) \rangle \right]
$$

$$
ρ_z^{T} = \mathbb{E} \left[ e^{-\int_0^T \langle A^\frac{1}{2} B(u(s), u(s)), dw(s) \rangle - \frac{1}{2} \int_0^T |A^\frac{1}{2} B(u(s), u(s))|^2 \gamma ds | \sigma(s) \rangle \right]
$$

for any $T > 0$.

Moreover, $P(t,x,\cdot) \sim \tilde{\mu}_t$ for any $t > 0, x \in D(A)$, where $\tilde{\mu}_t = \mathcal{N}(0, \frac{1}{2\gamma} A^{-1-\alpha})$ is the unique invariant measure for (21). In particular, the Markov process $u$ is strongly Feller and irreducible in $D(A)$ at any time $t > 0$; hence there exists at most one invariant measure for (11).

**Proof.** For $\theta = 1$, $\gamma = -\frac{1}{2}$ and $\alpha > \frac{d}{2} + 1$, (21) shows that the linear equation has a unique strong solution $z$ with paths in $C([0,T]; D(A))$ and satisfying (3) for any $p$. Moreover, by (25) we see that (3) holds for $p = 2$. According to Proposition 2.1 we conclude that equation (19) has a weak solution $u$ living in $C([0,T]; D(A))$ and satisfying (3) for any $p$. This result of weak existence and the pathwise uniqueness result of Proposition 4.3 imply the existence of a strong solution $u$ to equation (19) (see, e.g., [13], Chapter IX, Th. 1.7). By Proposition 2.2 we obtain $L_u \sim L_z$ and also $P(t,x,\cdot) \sim \tilde{\mu}_t$. Then, Corollary 2.3 gives strong Feller property and irreducibility for every $t > 0$. By Doob theorem, we have uniqueness of invariant measures for equation (19).

\[\square\]

**Remark 4.6** In this section we have assumed periodic boundary conditions so to give a meaning to terms as $A^\frac{1}{2} B(z,z)$. The reader can consult [3] for instance, to see for which values of $\beta$ the expression $A^\beta B(z,z)$ is well defined when working in a bounded spatial domain $D \subset \mathbb{R}^d$, assuming the velocity vanishes on the boundary $\partial D$. However, no problem arises in the periodic case.

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