ON COMPACT FINSLER SPACES OF
POSITIVE CONSTANT CURVATURE

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Abstract

An $n$-dimensional ($n \geq 2$) simply connected, compact without boundary Finsler space of positive constant sectional curvature is conformally homeomorphic to an $n$-sphere in the Euclidean space $\mathbb{R}^{n+1}$.

Résumé

Sur les espaces finslériennes compactes à courbure constante positive.
Un espace de Finsler de dimension $n$ ($n \geq 2$), simplement connexe compacte, non-bornée, à courbure sectionnelle constante positive est conformément homéomorphe à une $n$-sphère dans l’espace euclidien $\mathbb{R}^{n+1}$.

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Introduction.

The normal coordinates has been proved to be an extremely useful tool in the theory of global Riemannian geometry, while it is not so useful in Finsler geometry. In fact, in the latter case the exponential map is only $C^1$ at the zero section of $TM$ while in the former case it is $C^\infty$. More intuitively H. Akbar-Zadeh proved that exponential map is $C^2$ at the zero section if and only if the Finsler structure is of Berwald type, cf. [1]. In 1950s Y. Tashiro have defined adapted coordinates which was used by many authors to obtain global results in Riemannian geometry, which will not be mentioned here. Effectively, definition of this coordinate system on a manifold $M$, is equivalent to the existence of a non-trivial solution for a certain second order differential equation which describes circles preserving transformations on $M$. Recently, the circle-preserving transformations are studied in Finsler geometry by present author and Z. Shen, cf. [5]. Previously, inspiring the work of Tashiro, cf. [8], the present author have specialized adapted coordinates to Finsler setting in a joint paper, and proved the following theorem, cf. [2].

**Theorem A:** Let $(M, g)$ be a connected complete Finsler manifold of dimension $n \geq 2$. If $M$ admits a non-trivial solution of

$$\nabla_i \nabla_j \rho = \phi g_{ij},$$

(1)

where, $\nabla$ is the Cartan h-covariant derivative, then depending on the number of critical points of $\rho$, i.e. zero, one or two respectively, it is conformal to

(a) A direct product $J \times \overline{M}$ of an open interval $J$ of the real line and an $(n - 1)$-dimensional complete Finsler manifold $\overline{M}$.

(b) An $n$-dimensional Euclidean space.

(c) An $n$-dimensional unit sphere in an Euclidean space.

Here, we show that if $(M, g)$ is compact then only the third case may occur. More
Theorem 1: Let \((M, g)\) be an \(n\)-dimensional \((n \geq 2)\) without boundary compact simply connected Finsler manifold. In order that \((M, g)\) admits a non-trivial solution \(\rho\) of Eq.(1), it is necessary and sufficient that \((M, g)\) be conformally homeomorphic to an standard \(n\)-sphere in the Euclidean space \(\mathbb{R}^{n+1}\).

Theorem 2: Let \((M, g)\) be an \(n\)-dimensional \((n \geq 2)\) without boundary compact simply connected Finsler manifold of positive constant flag curvature. Then it is conformally homeomorphic to an standard \(n\)-sphere in the Euclidean space \(\mathbb{R}^{n+1}\).

1 Preliminaries.

Let \(M\) be a real \(n\)-dimensional manifold of class \(C^\infty\). We denote by \(TM \rightarrow M\) the bundle of tangent vectors and by \(\pi : TM_0 \rightarrow M\) the fiber bundle of non-zero tangent vectors. A Finsler structure on \(M\) is a function \(F : TM \rightarrow [0, \infty)\), with the following properties: \(F\) is differentiable \((C^\infty)\) on \(TM_0\); \(F\) is positively homogeneous of degree one in \(y\), i.e. \(F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0\), where we denote an element of \(TM\) by \((x, y)\). The Hessian matrix of \(F^2\) is positive definite on \(TM_0\), i.e. \((g_{ij}) := \left(\frac{1}{2} \left[ \frac{\partial^2}{\partial y^i \partial y^j} F^2 \right] \right)\). A Finsler manifold \((M, g)\) is a pair of a differentiable manifold \(M\) and a tensor field \(g = (g_{ij})\). Let \((x, y)\) be the line element of \(TM\) and \(P(y, X) \subset T_x(M)\) a 2-plane generated by the vectors \(y\) and \(X\) in \(T_x M\). Then the sectional or flag curvature \(K(x, y, X)\) with respect to the plane \(P(y, X)\) at a point \(x \in M\) is defined by \(K(x, y, X) := \frac{g(R(X, y)y, X)}{g(X, X)g(y, y) - g(X, y)^2}\), where \(R(X, y)y\) is the \(h\)-curvature tensor of Cartan connection. If \(K\) is independent of \(X\), then \((M, g)\) is called space of scalar curvature. If \(K\) has no dependence on \(x\) or \(y\), then the Finsler manifold is said to be of constant curvature. We say that a curve \(\gamma\) on \(M\)
is a \textit{geodesic} of a Finsler connection $\nabla$, if its natural lift $\tilde{\gamma}$ to $TM$, is a geodesic of $\nabla$, or equivalently $\nabla \frac{d\tilde{\gamma}}{dt} = 0$. Two points $p$ and $q$ are said to be \textit{conjugate points} along a geodesic $\gamma$ if there exists a non-zero Jacobi field along $\gamma$ that vanishes at $p$ and $q$, cf. [3]. Throughout this paper, all manifolds are supposed to be connected.

Let $\rho : M \to [0, \infty)$ be a scalar function on $M$ and $\nabla_i \nabla_j \rho = \phi g_{ij}$, a second order differential equation, where $\nabla_i$ is the Cartan horizontal covariant derivative and $\phi$ is a function of $x$ alone, then we say that Eq. (1) has a solution $\rho$. The solution $\rho$ is said to be \textit{trivial} if it is constant. Existence of solution of Eq. (1) is equivalent to the existence of some special conformal change of metric on $M$. We denote by $\text{grad} \rho = \rho^i \partial/\partial x^i$ the gradient vector field of $\rho$, where $\rho^i = g^{ij} \rho_j$, $\rho_j = \partial \rho/\partial x^j$ and $i, j, ...$ run over the range $1, ..., n$. We say the point $o$ of $(M, g)$ is a \textit{critical point} of $\rho$ if the vector field $\text{grad} \rho$ vanishes at $o$, or equivalently if $\rho'(o) = 0$, where $\rho' = d\rho/dt$. All other points are called \textit{ordinary points} of $\rho$ on $M$. It’s noteworthy to recall that the partial derivatives $\rho_j$ are defined on the manifold $M$, while $\rho^i$ the components of $\text{grad} \rho$ are defined on the slit tangent bundle $TM_0$. Hence, $\text{grad} \rho$ can be considered as a section of $\pi^*TM \to TM_0$, the pulled-back tangent bundle over $TM_0$, and its trajectories lie on $TM_0$. Let the Finsler manifold $(M, g)$ admits a non-trivial solution $\rho$ of (1), then for any ordinary point $p \in M$ there exists a coordinate neighborhood $U$ of $p$ which contains no critical point, and where we can choose a system of coordinates $(u^1 = t, u^2, ..., u^n)$ having the following properties, cf. [2];

- The function $\rho$ depends only on the first variable $u^1 = t$ on $U$.

- The integral curve of $\text{grad} \rho$ is a geodesic and geodesic containing such a curve is called a $\rho$-\textit{curve} or a $t$-\textit{geodesic} of $\rho$.

- The connected component of a regular hyper-surface defined by $\rho = \text{constant}$, is called a \textit{level set} of $\rho$ or simply a $t$-level. Given a solution $\rho$ and a point $q \in U$, there exists one and only one $t$-level set of $\rho$ passing through $q$. The $t$-geodesics form the
normal congruence to the family of $t$-level sets of $\rho$.

- The curves defined by $u^\alpha = \text{const}$ are $t$-geodesics of $\rho$, and the parameter $u^1 = t$ may be regarded as the arc-length parameter of $t$-geodesics.

- The components $g_{ij}$ of the Finsler metric tensor $g$ satisfy $g_{\alpha 1} = g_{1 \alpha} = 0$, where the Greek indices $\alpha, \beta$ run over the range $2, 3, ..., n$ and the Latin indices $i, j$ run over the range $1, 2, ..., n$.

- In adapted coordinates the first fundamental form of $(M, g)$ is given by

$$ds^2 = (dt)^2 + \rho'^2 f_{\gamma \beta} du^\gamma du^\beta,$$

(2)

where $f_{\gamma \beta}$ given by $g_{\gamma \beta} = \rho'^2 f_{\gamma \beta}$ are components of a metric tensor on a $t$-level of $\rho$ and $g_{\gamma \beta}$ is the induced metric tensor of this $t$-level. For more details about our purpose on adapted coordinates, one can refer to [2418].

2 Compact Finsler spaces of constant curvature

**Proof of Theorem 1.** Let $(M, g)$ be a an $n$-dimensional $n \geq 2$ Finsler manifold which admits a non trivial $C^\infty$ solution $\rho$ of Eq.(1). Consider the so called $t$-geodesic which is integral curve of the gradient vector field $\text{grad} \rho$ on $M$. It is well known that every $t$-geodesic is a geodesic on $M$. Since $M$ is compact by extension of Extreme Value Theorem to differentiable manifolds every solution $\rho$ of Eq.(1) is bounded and attains its extremum values on $M$. Once the assumption is made as $M$ is without boundary, differentiability of $\rho$ requires that these extremum values are critical points. Let $O$ be a critical point for a $t$-geodesic on $M$. By compactness, $M$ must have finite diameter $D$ and no $t$-geodesic longer than $D$ can remain minimizing. Thus every $t$-geodesic longer than $D$ issuing from $O$ contains at least two critical points.

Before proceeding further, we shall recall that on a Finsler manifold there is no more
than two critical points of \( \rho \) on every \( t \)-geodesic emanating from \( O \), cf. [2]. Therefore, every \( t \)-geodesic on \((M,g)\) contains exactly two critical points. Thus by means of Theorem A, \((M,g)\) is conformal to an \( n \)-dimensional sphere in the Euclidean space \( \mathbb{R}^{n+1} \), with the first fundamental form [2]. Moreover, \( M \) is assumed to be simply connected and an extension of the Milnor theorem to Finslerian category, cf. [6], implies that \( M \) is topologically homeomorphic to an \( n \)-sphere.

Conversely, let \((M,g)\) be compact and conformally homeomorphic to the \( n \)-sphere \( S^n \subset \mathbb{R}^{n+1} \). The first fundamental form of \( S^n \) is given by

\[
g_{S^n} = dt^2 + \sin^2 t g_{S^{n-1}},
\]

where \( g_{S^{n-1}} \) is the first fundamental form of the hypersphere \( S^{n-1} \), cf. [7]. Let \( \gamma := x^i(t) \) be a geodesic on \((M,g)\), by definition its differential equation is given by

\[
\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = \varphi \frac{dx^i}{dt},
\]

where \( t \) is an arbitrary parameter and \( \varphi \) is a function of \( t \). If we denote \( \frac{dx^j}{dt} = \gamma^j \), by virtue of Eq.(4) we have

\[
\gamma^k \frac{d\gamma^j}{dx^k} + \Gamma^l_{jk} \gamma^j \gamma^k = \varphi \gamma^l.
\]

This is equivalent to \( \gamma^k (\nabla_k \gamma^l) = \varphi \gamma^l \), where \( \nabla_k \) is the Cartan h-covariant derivative. Denoting \( \gamma_i := g_{il} \gamma^l \) and contracting with \( g_{il} \), we obtain \( \gamma^k (\nabla_k \gamma_i) = \varphi \gamma_i \), which leads to

\[
\gamma^k (\nabla_k \gamma_i - \varphi g_{ik}) = 0.
\]

Conformal assumption of \((M,g)\) to the standard sphere \((S^n, g_{S^n})\) implies that the Finsler metric \( g \) is positively proportional to \( g_{S^n} \), that is \( g = e^{2\psi} g_{S^n} \) where, by the Knebelman theorem \( \psi \) is a function on \( M \). Therefore \( g \) is also a function on \( M \) and
hence a Riemannian metric. By compactness of $M$ the vector field $\gamma^k$ is complete and Eq. (6) leads to $\nabla_k \gamma_i = \varphi g_{ik}$ which is equivalent to Eq. (1). This completes the proof of Theorem. \hfill \Box

**Proposition 2.1** Let $(M, g)$ be an $n$-dimensional compact Finsler manifold of constant flag curvature $K$. Then the following SODE

$$\frac{d^2 \rho}{dt^2} + K \rho = 0,$$

(7)

admits a non-trivial solution on $(M, g)$ if and only if $K > 0$.

**Proof.** Let $(M, g)$ be a Finsler manifold of constant flag curvature $K$, then the following equation holds well, cf. [1], see also [3].

$$\ddot{A}_{ijk} + KA_{ijk} = 0,$$

(8)

where $A_{ijk}$ is the Cartan torsion tensor, $\dot{A}_{ijk} := (\nabla_s A_{ijk}) \ell^s$ and $\ddot{A}_{ijk} := (\nabla_s \nabla_t A_{ijk}) \ell^s \ell^t$ and $\ell^i := y^i / F$. Let $\gamma : \mathbb{R} \to M$ be any geodesic parameterized by arc length $t$ on $(M, g)$ passing through $\gamma(0) = p$, having tangent vector $\frac{d\gamma}{dt} = \ell^i$ and the canonical lift $\dot{\gamma} := \frac{d\gamma}{dt}$ to $TM_0$. To differentiate the Cartan torsion tensor along $\gamma$, we consider the linearly independent parallel sections $X(t)$, $Y(t)$ and $Z(t)$ of $\pi^*TM$ along $\dot{\gamma}$ with $X(0) = X$, $Y(0) = Y$ and $Z(0) = Z$ at the point $p$. By a direct computation, for two linearly independent parallel vector fields $X(t)$ and $Y(t)$ along $\gamma$, we have $\frac{d}{dt} g_{\alpha(t)}(X(t), Y(t)) = 0$, see for instance [1] or [3]. In this sense, a $g$-orthonormal basis for $\pi^*TM$ remains $g$-orthonormal at every point $(x(t), y(t))$ along $\dot{\gamma}$. Therefore, by assuming $A(t) = A(X(t), Y(t), Z(t))$, $\dot{A}(t) = \dot{A}(X(t), Y(t), Z(t))$ and $\ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t))$ along $\gamma$, we have $\frac{dA}{dt} = \dot{A}$, $\frac{d\dot{A}}{dt} = \ddot{A}$ and Eq. (8) becomes

$$\frac{d^2 A(t)}{dt^2} + KA(t) = 0.$$
By assuming $\rho(t) = A(t)$ we obtain Eq. (7). To complete the proof we consider three cases.

*Case $K > 0$.* In this case $\rho(t) = a \cos \sqrt{K}t + b \sin \sqrt{K}t$ is a non-trivial general solution for Eq. (7).

*Case $K < 0$.* In this case the general solution is given by

$$A(t) = \alpha e^{\sqrt{-K}t} + \beta e^{-\sqrt{-K}t}.$$  
(10)

For $v \in TM_0$, assume that the norm of Cartan torsion tensor is $\|A\|_v := \sup A(X,Y,Z)$, where the supremum is taken over all unit vectors of $\pi^* v TM$. Suppose that $S_x M = \{ w \in T_x M, \quad F(w) = 1 \}$ is the indicatrix and $\|A\| = \sup_{v \in SM} \|A\|_v$, where $SM = \bigcup_{x \in M} S_x M$. Since $M$ is compact the norm $\|A\|$ is bounded. On the other hand $M$ is compact and therefore geodesically complete and the parameter $t$ takes all the values in $(-\infty, +\infty)$. Letting $t \to +\infty$ or $t \to -\infty$, then Eq. (10) implies that $A(0) = 0$. In fact as $t$ approaches to $t \to \pm \infty$ the left hand side of the equation is bounded and the right hand side is infinity, so Eq. (10) can be hold only if the coefficients $\alpha$ and $\beta$ vanish. Replacing it on the equation (10), we obtain $A(t) = 0$. That is to say the solution $\rho(t) = A(t)$ of Eq. (7) is trivial.

*Case $K = 0$.* In this case the general solution of Eq. (7) is given by $A(t) = \alpha + \beta t$, where $\alpha$ and $\beta$ are constant. Following the procedure described above we obtain $\beta = 0$ which implies that $A(t) = \alpha$, is constant and hence a trivial solution of Eq. (7). This completes proof of the proposition.

Proof of Theorem 2. Let $(M, g)$ be a compact Finsler manifold of positive constant flag curvature $K$. If we assume $\phi = -K \rho$ then Eq. (1) reduces to

$$\nabla_i \nabla_j \rho + K \rho g_{ij} = 0.$$  
(11)

Following an argument similar to the one in the proof of above proposition Eq. (11)
reduces to Eq. (7) along geodesics. By virtue of Proposition 2.1 there is a non-trivial solution say $\rho$ for Eq. (7) and hence for Eq. (11) on $M$. Therefore $(M, g)$ admits a non-trivial solution $\rho$ for Eq. (1). A simple application of Theorem 1, completes proof of Theorem 2. \qed

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