Applications of Gaussian model of the vortex tangle in the superfluid turbulent HeII

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Abstract. In spite of an appearance of some impressive recent results in understanding of the superfluid turbulence in HeII they fail to evaluate many characteristics of vortex tangle needed for both applications and fundamental study. Early we reported the Gaussian model of the vortex tangle in superfluid turbulent HeII. That model is just trial distribution functional in space of vortex loop configurations constructed on the basis of well established properties of vortex tangle. It is designed to calculate various averages taken over stochastic vortex loop configurations. In this paper we use this model to calculate some important characteristics of the vortex tangle. In particular we evaluate the average superfluid mass current $J$ induced by vortices and the average energy $E$ associated with the chaotic vortex filament.

1 Introduction

The presence of vortex tangle appearing in the superfluid turbulent HeII essentially changes hydrodynamic properties of the latter (see e.g. [1], [2], [3]). According phenomena are studied in frame of so called Phenomenological Theory (PT) pioneered by Vinen [4] and greatly modified by Schwarz [5]. The PT describes superfluid turbulence (ST) in terms of the total length of vortex lines (per unit of volume) or the vortex line density (VLD) $L(t)$ and of the structure parameters of the VT. Knowledge of these quantities allows to calculate some of hydrodynamic characteristics of superfluid turbulent HeII such as a mutual friction, sound attenuation etc. Meanwhile there exist many other physical quantities connected to distribution of the filaments and their interaction related with other physical phenomena which can not be expressed in terms of the PT. The relevant phenomena should be covered by appropriate stochastic theory of chaotic vortex filaments. Of course, the most honest way to develop such theory is to study stochastic dynamics of vortex filaments on the base of equations of motion with some source of chaos. However due to extremely involved dynamics of vortex lines this way seems to be almost hopeless. Thus, a necessity of a developing an advanced phenomenological approach appeared. We offer one variant of such approach. The main idea and the main strategy are the following. Although the phenomenological theory of the superfluid turbulence deals with macroscopical characteristics of the vortex tangle, it conveys the rich information concerning the instantaneous structure of the vortex tangle. Namely we know that the VT consists of the closed loops labelled by $s_j(\xi)$, uniformly distributed in space and having the total length $L(t)$ per unit of volume. From acoustical experiments it follows that filaments are distributed in anisotropic manner and quantitative
characteristics of this anisotropy can be expressed by some structure parameters (see [1], [3], [5]). Beside this usual anisotropy there is more subtle anisotropy connected with averaged polarization of the vortex loops. Furthermore there are some proofs that the averaged curvature of the vortex lines is proportional to the inverse interline space and coefficient of this proportionality (which is of order of unit) was obtained in numerical simulations made by Schwarz [4].

The master idea of our proposal is to construct a trial distribution function (TDF) in the space of the vortex loops of the most general form which satisfies to all of the properties of the VT introduced above. We assume that this trial distribution function will enable us to calculate any physical quantities due to evaluating of the correlation functions. We also calculate the average hydrodynamic impulse (or Lamb impulse) $J_V$ in the counterflowing superfluid turbulent HeII and the average kinetic energy $E$ associated with the chaotic vortex loop.

2 Constructing of the trial distribution function

According to general prescriptions the average of any quantity $\langle \mathcal{B}([s_j(\xi_j)]) \rangle$ depending on vortex loop configurations is given by

$$\langle \mathcal{B}([s_j(\xi_j)]) \rangle = \sum_{\{s_j(\xi_j)\}} \mathcal{B}([s_j(\xi_j)]) \mathcal{P}([s_j(\xi_j)]).$$

(1)

Here $\mathcal{P}([s_j(\xi_j)])$ is a probability of the vortex tangle to have a particular configuration $\{s_j(\xi_j)\}$. Index $j$ distinguishes different loops. The meaning of summation over all vortex loop configurations $\sum_{\{s_j(\xi_j)\}}$ in formula (1) will be clear from further presentation. We put the usual in the statistical physics supposition that all configuration corresponding to the same macroscopic state have equal probabilities. Thus the probability $\mathcal{P}([s_j(\xi_j)])$ for vortex tangle to have a particular configuration $\{s_j(\xi_j)\}$ should be proportional to $1/N_{\text{allowed}}$, where $N_{\text{allowed}}$ is the number of allowed configurations, of course infinite

$$\mathcal{P}([s_j(\xi_j)]) \propto \frac{1}{N_{\text{allowed}}}.$$ (2)

Under term "allowed configurations" $N_{\text{allowed}}$ we mean only the configurations that will lead to the correct values for all average quantities known from experiment and numerical simulations. Formally it can be expressed as a path integral in space of three-dimensional (closed) curves supplemented with some constrains connected to properties of the VT.

$$N_{\text{allowed}} \propto \prod_j \int \mathcal{D}[s_j(\xi)] \times \text{constrains } \{s_j(\xi)\}.$$ (3)

The constrains entering this relation are expressed by delta functions expressing fixed properties of the VT. For instance constrain $\delta([s'_j(\xi)]^2 - 1)$ expresses that parameter $\xi$ is the arc length. However this condition will lead to not tractable
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We will use a trick known from the theory of polymer chains (see e.g. [3]), namely we will relax rigorous condition and change delta function by continuous (Gaussian) distribution of the link length with the same value of integral. This trick leads to the following expression for number of way:

\[ N_{\text{allowed}} \propto \prod_j \int \mathcal{D}\{s_j(\xi)\} \times e^{-\lambda_1 \int_0^L |\mathbf{s}'|^2 d\xi}. \]  

In the same manner we are able to introduce and treat other constraints connected to the known properties of the VT structure. The detailed calculations are exposed in paper of one of the author [7], now we write down final expression for probability of configurations

\[ N_{\text{allowed}} \propto \int \mathcal{D}\{s(\kappa)\} \exp \left( -\mathcal{L}\{s(\kappa)\} \right). \]  

Here \( s(\kappa) \) is one-dimensional Fourier transform of variable \( s(\xi) \) and Lagrangian \( \mathcal{L}\{s(\kappa)\} \) is a quadratic form of the components of the vector variable \( s(\kappa) \)

\[ \mathcal{L}\{s(\kappa)\} = \sum_{\kappa \neq 0} s_{\kappa}^\mu(\kappa) A^{\alpha\beta}(\kappa) s_{\kappa}^\nu(\kappa). \]  

In practice to calculate various averages it is convenient to work with the characteristic (generating) functional (CF) which is defined as a following average:

\[ W(\{P_j(\kappa)\}) = \langle \exp \left( -\sum_j \sum_{\kappa \neq 0} P_j^\mu(\kappa) s_j^{\mu}\{s(\kappa)\} \right) \rangle. \]

Due to that our Lagrangian is a quadratic form (in \( s(\kappa) \)) and, consequently, the trial distribution function is a Gaussian one, calculation of the CF can be made by accomplishing the full square procedure to give a result

\[ W(\{P_j(\kappa)\}) = \exp \left( -\sum_j \sum_{\kappa \neq 0} P_j^\mu(\kappa) N_j^{\mu\nu}(\kappa) P_j^\nu(\kappa) \right). \]

Elements of matrix \( N_j^{\mu\nu}(\kappa) \) are specified from calculation of total length, anisotropy coefficient, curvature and polarization. The explicit form of them is written down in [7].

Thus we reached the put goal and have written the expression for trial CF which, we repeat, enables us to calculate any averaged of the vortex filament configuration. For instance calculating some of the correlation functions we are able to describe a typical shape of the averaged curve. It is sketched out in Fig. 2.
Fig. 1. A snapshot of the averaged vortex loop obtained from analysis of the statistical properties. Position of the vortex line element is described as $s_j(\xi_j)$, where $\xi_j$ is arc length, $s'_j(\xi_j) = ds_j(\xi_j)/d\xi_j$ is a tangent vector, unit vector along the vortex line; $s''_j(\xi_j) = d^2s_j(\xi_j)/d\xi_j^2$ is the local curvature vector; vector production $s'_j(\xi_j) \times s''_j(\xi_j)$ is binormal which is responsible for mutual orientation of the tangent vector and vector of curvature. Close ($\Delta \xi \ll R$, where $R$ is the mean curvature ) parts of the line are separated in 3D space by distance $\Delta \xi$. The distant part ($R \ll \Delta \xi$) are separated in 3D space by distance $\sqrt{2\pi R} \Delta \xi$ (with correction due to the closeness). The latter property reflects a random walk structure of the vortex loops. As a whole the loop is not isotropic having a "pancake" form. In addition it has a total polarization $\langle \int s'_j(\xi_j) \times s''_j(\xi_j) d\xi_j \rangle$ forcing the loop to drift along vector $V_n$ and to produce nonzero superfluid mass current in z-direction

3 Hydrodynamic impulse of vortex tangle

As an one more illustration to the developed theory we discuss hydrodynamic impulse of the vortex tangle $J_V$ which is defined as

$$J_V = \frac{\rho_s}{2} \sum_j \int s_j(\xi_j) \times s'_j(\xi_j) \ d\xi_j$$

(8)

The quantity $J_V$ is closely related to momentum of fluid (see [8]). The averaged $\langle s_j(\xi_j) \times s'_j(\xi_j) \rangle$ is immediately evaluated by use of CF (7) to give the following result:

$$J^z_v = -\left[ \frac{\rho_s}{\rho_n c_s^2} \beta_v \right] \rho_s V_s$$

(9)

Note that the coefficient includes no fitting parameters but only characteristics known from the Phenomenological Theory (see [3]). Relation (8) shows that the
vortex tangle induces the superfluid current directed against the external superfluid current. It should be expected since there is some preferable polarization of the vortex loops. In the experiments this additional superfluid current should display itself as suppression of the superfluid density. This effect is 3D analog to the famous Kosterlitz-Thouless effect except of that distribution of the vortex lines is not calculated but is obtained appealing to the experimental data.

Since superfluid density enters an expression for second sound velocity, it seems attractive to detect it using transverse second sound testing. To do it we have firstly to evaluate transverse change of the $\rho_s$ and, secondly, to develop the theory to match it to nonstationary case. The general theory asserts that while applying a harmonic external second sound field suppression of superfluid density becomes the function of frequency $\omega$ of the following form:

$$\Delta\rho_s^x(\omega) = \left( \frac{\delta J^x_V}{\delta V^x_s} \right)_{\text{transv}} \frac{1}{1 + i\omega\tau_J}. \quad (10)$$

Here transverse $(\delta J^x_V/\delta V^x_s)_{\text{transv}}$ is half of the one given by rel. (9). The quantity $\tau_J$ is the time of relaxation of the superfluid current $J_V$, which is to be found from dynamical consideration. First, we have to derive $dJ^x_V/dt$ with help of the equation of motion of the vortex line elements and, second, to evaluate various averaged appearing in right-hand side. Function we obtain the following final result for change of the second sound velocity. Performing all of described procedures one obtains that the relative change $\Delta u_2/u_2$ of the second sound velocity is given by

$$\frac{\Delta u_2}{u_2} = -f(T) \frac{V_{ns}^4}{\omega^2}. \quad (11)$$

Here the function $f(T)$ is composed of the structure parameters of the vortex tangle

$$f(T) = \frac{4\rho\kappa I^2_2\alpha^2 (1 - I_{zz})^2}{\rho_n C_2^{\gamma^3}}. \quad (12)$$

Decreasing of the second sound velocity in the counterflowing HeII has been really observed about two decades ago by Vidal with coauthors [9]. Let us compare our result (11) with the Vidal’s experiment. Using the data on the structure parameters one obtains that e.g. for the temperatures $1.44K$ the value of function $f(T)$ is about $620 \, s^2/cm^4$. Taking the frequency $\omega = 4.3 \, rad/s$, used in [9], and $V_{ns} = 2 \, cm/s$ one obtains that $\Delta u_2/u_2 \approx 4 \times 10^{-4}$, which is very close to the observed value.

4 Energy of vortex tangle

In this section we calculate the averaged energy of the stochastic vortex loop distributed according trial distribution function [8]. The general expression for the energy associated with linear vortices can be written as (see e.g. [8])
\[ E = \left< \frac{1}{2} \int \rho_s \mathbf{v}_s^2 \, d^3r \right> = \left< \frac{\rho_s \kappa^2}{8\pi} \sum_{i,j} \int_{0}^{L_i} \int_{0}^{L_j} \frac{s_i'(\xi_i)s_j'(\xi_j)}{|s_i(\xi_i) - s_j(\xi_j)|} \, d\xi_i \, d\xi_j \right> \].  

In 3D Fourier space the average energy \( E \) [13] can be rewritten as

\[ E = \left< \frac{\rho_s \kappa^2}{2} \sum_{i,j} \int_{\mathbf{k}} \left( \frac{d^3k}{(2\pi)^3k^2} \int_{0}^{L_i} \int_{0}^{L_j} s_j'(\xi_j)d\xi_j \right) d\xi_i e^{ik(s_i(\xi_i) - s_j(\xi_j))} \right> \].  

Comparing [14] and [10] it is possible to express the energy \( E \) in terms of the characteristic Functional

\[ \langle E \rangle = \frac{\rho_s \kappa^2}{2} \sum_{i,j} \int_{\mathbf{k}} \left( \frac{d^3k}{(2\pi)^3k^2} \int_{0}^{L_i} \int_{0}^{L_j} s_j'(\xi_j) d\xi_j e^{ik(s_i(0) - s_j(0))} \right) \times \frac{\delta^2W}{\delta \mathbf{P}_i(\xi_i) \delta \mathbf{P}_j(\xi_j)} \]  

Here set of \( \mathbf{P}_n(\xi'_n) \) in CF \( W(|\{ \mathbf{P}_n(\xi'_n) \}|) \) is again determined with help of the \( \theta \)-functions

\[ \mathbf{P}_i(\xi'_i) = k\theta(\xi'_i)\theta(\xi'_i - \xi'_j), \quad \mathbf{P}_j(\xi'_j) = k\theta(\xi'_j)\theta(\xi'_j - \xi'_j), \quad \mathbf{P}_n(\xi'_n) = 0, \quad n \neq i, j \]  

The relation [14] implies that we have to choose in integrand in exponent of CF only points lying in interval from 0 to \( \xi_i \) on \( i \)-curve and from 0 to \( \xi_j \) on \( j \)-curve. While evaluation of self-energy of the same loop, \( i = j \), one has to distinguish points \( \xi_i \), and to put them to be e.g. \( \xi'_i \) and \( \xi''_i \). Further results concern the case of the only loop of length \( L \). Omitting tremendous calculations we write down the final answer in the following form:

\[ E = \frac{\rho \kappa^2 L}{4\pi} \ln \frac{R}{a_0} + \frac{\rho \kappa^2 L}{4\pi} \left( 1 - \frac{2}{\sqrt{\pi}} (f_2 - f_1) \right) \ln \frac{R}{a_0} \]

\[ + \frac{\rho \kappa^2 L}{4\pi} \left[ \frac{1}{(\sqrt{\pi} - 1)^{1/2}} \frac{2f_3}{\pi^{5/2}} \frac{I_2^2}{c_3^2} \right] + \frac{f_2}{\pi^{1/2} (\sqrt{\pi} - 1)^{1/2}} \]

where the quantities \( f \) (of order of unit) are expressed via the structure parameters of the VT as follows (below \( \beta = \sqrt{I_x - I_z/I_z} \))

\[ f_1(\beta) = \sqrt{2 \left( 3 - \beta^2 \right)} \left( \arcsin(\beta)/\beta \right), \]

\[ f_2(\beta) = \left( \sqrt{1 - \beta^2} + (2 - \beta^2) \arcsin(\beta)/\beta \right) / \sqrt{2 \left( 3 - \beta^2 \right)}, \]

\[ f_3(\beta) = 2 \left( 3 - \beta^2 \right)^{3/2} \left( \sqrt{1 - \beta^2} - (\arcsin(\beta)/\beta) \right) / \beta \]
Let us comment expression (17). The first term in the right-hand side of (17) is just the energy of unit of length of a straight vortex filament (see e.g. [1]) multiplied by its length. In this form it is frequently used in theory of superfluid turbulence (see e.g. [3]) and in other applications. But there are additional terms. The third and forth terms appeared from long-range interaction, they are smaller then logarithmic ones (about ten percents). The third term is of especial interest. It appeared due to polarization of the vortex loop and its presence implies that there is some elasticity of the vortex tangle in $V_{ns}$ direction. Results of the previous section showed that the VT induces some additional superfluid flow. Therefore one can expect that combination of longitudinal elasticity combining with inertia of additional will lead to appearing of elastic waves, 3D analog of the Tkachenko waves.

The second one is also logarithmically large. Logarithmic behaviour points out that this contribution came from denominator $|s(\xi) - s(\xi')|$. But it was the first (local) term which collected contributions from neighbor points along the line. Therefore the third term appeared from accidental self-crossing of remote (along the line) parts of the vortex filament. The fact that this term is proportional to $L$ and is of of the first (local) contribution is due to that the line is fractal object with Hausdorff $H_d$ dimension equal $H_d = 2$. According to general theory of fractal lines it has an infinite number of self-crossing with cardinal number $2H_d - 3$ i.e. it is equivalent to line.

5 Conclusion

We briefly exposed an essence of Gaussian model of the vortex tangle and give several examples how it can be used for evaluation of important physical characteristics such as induced momentum and energy of interaction. These characteristics has been discussed early (see e.g. [1]), however their evaluation has not been performed because of lack of a proper theory. We think that these illustrations convince that Gaussian model can serve as effective tool to study chaotic vortex filaments.

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