Recent Advances in Asymptotic Analysis

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Abstract

This is a survey article on an old topic in classical analysis. We present some new developments in asymptotics in the last fifty years. We start with the classical method of Darboux and its generalizations, including an uniformity treatment which has a direct application to the Heisenberg polynomials. We then present the development of an asymptotic theory for difference equations, which is a major advancement since the work of Birkhoff and Trjitzinsky in 1933. A new method was introduced into this field in the nineteen nineties, which is now known as the nonlinear steepest descent method or the Riemann-Hilbert approach. The advantage of this method is that it can be applied to orthogonal polynomials which do not satisfy any differential or difference equations neither do they have any integral representations. As an example, we mention the case of orthogonal polynomials with respect to the Freud weight. Finally, we show how the Wiener-Hopf technique can be used to derive asymptotic expansions for the solutions of an integral equation on a half line.

2020 mathematics subject classification: 41-02; 41A60; 39A06; 45A05; 45E10

Keywords and phrases: Asymptotics, uniform asymptotic expansions, Darboux’s method, second order difference equations, Riemann-Hilbert approach, parametrix, Wiener-Hopf equations

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Funding: The second author was supported in part by the National Natural Science Foundation of China under grant numbers 11571375 and 11971489.

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1 Introduction

Asymptotics is one of the oldest topics in classical analysis; see [5]. Examples from this topic include Stirling’s formula [33, 79], prime number theorem [81], and Hardy-Ramanujan’s formula [43]. Despite of the fact that it is an old topic, new methods continue to be discovered. For instance, in the construction of error bounds for asymptotic expansions of the Stieltjes transform and the fractional integral transform, a method was introduced which makes use of distribution theory; see [61] and [62]. This method was later extended to derive asymptotic expansions of the generalized Mellin transform [101], which is a divergent integral but exists in the sense of generalized functions. In a study of the behavior of polynomials orthogonal with respect to the Freud weight \( \exp(-|x|^{\beta}) \), \( \beta > 0 \), another new method was brought into the field of asymptotics, which is the Riemann-Hilbert approach introduced by Deift-Zhou [30] in their investigation of the behavior of solutions to the mKdV equation.

The development of an asymptotic theory for ordinary differential equations was quite successful in the last century. After Poincaré [75] introduced a proper definition for asymptotic expansion and established rigorously the results for irregular singularity at infinity, several people obtained asymptotic expansions for solutions of differential equations with a large parameter, including Schlesinger [78], Birkhoff [6], and Turrittin [85]. More important results were given by Langer [54] [55] [56] on uniform asymptotic solutions to turning-point problems, and his results were substantially extended by Olver [67, 68, 69] including extensions to unbounded domains of the independent variable and explicit bounds for the error terms [70]. The corresponding development for difference equations, however, was not as successful as for differential equations. Formal theory of irregular linear difference equations was first given by Birkhoff [7], and rigorous analysis was later given by Birkhoff and Trjitzinsky [8]. The first paper appeared in 1930, and the second paper was published in 1933. For the remaining part of the century, there were essentially no significant results in this area of research. The first paper on turning-point problem for difference equation [86] did not appear until 2002; see also [88]. Subsequently, progress seems to have picked up, and results in this area are now nearly as complete as they are for differential equations; see [87], [88], [13] and [57].

While there are now plenty of results on finding asymptotic solutions to differential equations and to difference equations, the same is not true for integral equations. The problem is not that difficult if the equation is either of Volterra type or Fredholm type with a difference kernel, since the equation can be reduced to an algebraic equation by taking either Laplace transforms or Fourier transforms, and asymptotic behaviour of the solution can be obtained by taking their inverse transformations; see, e.g., [95], [4, p. 286]. The situation is quite different if the equation is of the form

\[
    u(t) = f(t) + \int_0^\infty k(t-\tau)u(\tau)d\tau.
\]

As far as we are aware, there is only one paper on asymptotic solutions to such an equation [63]. Equation in (1.1) is known as of Wiener-Hopf type or renewal equation on half
The famous Wiener-Hopf technique is to construct a contour-integral representation for the solution of (1.1), but to find the behaviour of the solution is another story since the functions involved in the integrand are very complicated and contain Stieltjes and Hilbert transforms. A rather satisfactory solution to this problem was given only very recently; see [58].

The purpose of this survey is to bring people’s attention to some of the developments in asymptotic analysis in the past 50 years. In Section 2, we begin with Darboux’s method. Even this classical result has some very attractive extensions. We will first present a generalization that involves logarithmic singularities [102], and mention couple examples to show that this generalization is indeed useful. Next, we give a result on the behavior of the coefficients in a Maclaurin series, whose generating function has two coalescing algebraic singularities on its circle of convergence, and provide as an application the asymptotic expansion of the Heisenberg polynomials as the degree of the polynomial goes to infinity; see [60] and [103].

In Section 3, we briefly review the results that have been obtained for difference equations during the past twenty years. As examples, we present the asymptotic expansions of the Racah and the Wilson polynomials; these polynomials are listed at the top level of the Askey scheme [74, p. 464].

In Section 4, we introduce the Riemann-Hilbert approach, and show how to use it to derive the asymptotic behavior of the polynomials orthogonal with respect to the Freud weight \(\exp(-|x|^\beta), \beta > 0\). If \(\beta \geq 1\), then the problem was completely solved by Kriecherbauer and McLaughlin [50] in 1999. The difficulty of this problem occurs when \(0 < \beta < 1\), in which case it is not known what is the dominant approximant in the asymptotic expansion of these polynomials; see [29, p. 62]. This problem was finally settled in 2016, and the answer is that the dominant approximant does not involve special functions such as Airy or Bessel; it involves two solutions to a pair of decoupled scalar linear integral equations.

In the final section, Section 5, we return to the integral equation (1.1). We first recall what is the Wiener-Hopf technique [92], the solution to equation (1.1) is then given by a contour integral. Thus, we reduce the problem to an asymptotic evaluation of a Fourier integral whose integrand involves Stieltjes and Hilbert transforms.

## 2 Darboux’s method

The following combinatorial problem has appeared in Whitworth [90].

*If there be \(n\) straight lines in one plane, no three of which meet in a point, no two lines are parallel, there will be \(n(n-1)/2\) points of intersection. Put the intersection points into groups of \(n\), in each of which no three points lie in one of the straight lines. Find the number \(g_n\) of such groups.*

The answer given in [90] is \(g_n = \frac{1}{2}(n-1)!\) for \(n = 3, 4, \cdots\). However, in 1951, Robinson [77] considered the problem and showed that the answer is wrong. He found the correspondence between \(n\)-point groups and \(n\)-polygons. By counting \(g_n = \frac{1}{2}(n-1)!\), one misses the composite \(n\)-gons: those consist of several separate simple polygons with
totally $n$ sides. Instead, $u_n = g_n/\left(\frac{1}{2}(n-1)\right)$ satisfies the recurrence relation
\[ u_{n+3} = u_{n+2} + \frac{1}{2n}u_n, \quad n = 1, 2, 3, \cdots, \] (2.1)
with $u_1 = u_2 = 0$ and $u_3 = 1$. After confirming the existence of the number
\[ b = \lim_{n \to \infty} \frac{u_n}{n} = 0.284098 \cdots, \]
as noted by Robinson [77], $b$ assumes the role of an absolute geometric constant. Robinson further asked whether it is an algebraic number, which seems unlikely, whether it is expressible in terms of $\pi$ and $e$, or whether it is entirely new.

The answers came fast. It is stated that the constant is given by
\[ b = 4e^{-3/2}/\pi; \]
cf. an editorial note [34]. Thirteen readers, including Harry Pollard, John Riordan, George Szekeres, Fritz Ursell and Morgan Ward, wrote to the editor about Robinson’s constant. Several of them used the generating function
\[ f(x) = -2 + 2(1-x)^{-1/2} \exp\left(-\frac{x^2 + 2x}{4}\right) = \sum_{n=3}^{\infty} \frac{u_n}{n} x^n, \] (2.2)
which can readily be derived from (2.1) via the differential equation
\[ 2(1-x)f'(x) = x^2 \left[f(x) + 2\right], \quad f(0) = 0. \]

Mention had also been made of a theorem from Titchmarsh [82, p. 224]:

**Proposition 2.1.** Let
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n, \quad |x| < 1, \]
where $a_n \geq 0$, $b_n \geq 0$, and $\sum_{n=0}^{\infty} b_n$ diverges. If $\lim_{n \to \infty} a_n/b_n = C$, then $\lim_{x \to 1} f(x)/g(x) = C$.

To determine the behavior of $a_n$, it seems that the converse of Proposition 2.1 matters much. Yet Titchmarsh [82, p. 226] claimed that there is no general converse of the above proposition: *from the asymptotic behavior of* $f(x)$ *we can not deduce that of* $a_n$.

Nevertheless, Darboux’s method (G. Darboux, 1878, [24]) demonstrates how to deduce the asymptotic behavior of $a_n$ from that of $f(x)$. Such problems arise in various mathematical fields, including number theory, combinatorics, and orthogonal polynomials. As a start, one may use the Cauchy integral formula
\[ a_n = \frac{1}{2\pi i} \int_C f(z)z^{-n-1}dz, \]
where $C$ is a circle encircling the origin, but not any singularities of $f(z)$. Expanding the contour $C$ if possible, and applying the residue calculus if a pole is encountered, we obtain the leading contribution to the asymptotic behavior of $a_n$. This procedure fails, however, if an algebraic singularity is the closest to the origin, and the classic Darboux’s method ([24]; see also [80] and [97]) becomes useful.

To describe briefly how this method works, we assume that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

(2.3)

is analytic in $|z| < 1$, with only one singularity on the circle $|z| = 1$, say, at $z = 1$, and that in a neighborhood of $z = 1$, $f(z)$ is of the form

$$f(z) = (1 - z)^\alpha g(z),$$

(2.4)

where $g(z)$ is analytic at $z = 1$, and $\alpha \not\in \mathbb{Z}$. To find the large-$n$ behavior of $a_n$, we expand

$$g(z) = \sum_{r=0}^{\infty} c_r (1 - z)^r$$

(2.5)

at $z = 1$. The $m$-th Darboux approximant of $f(x)$ is then defined by

$$f_m(z) = \sum_{r=0}^{m} c_r (1 - z)^{\alpha + r},$$

which is also analytic in $|z| < 1$; cf. (2.3). Expanding $f_m(z)$ into a Maclaurin series

$$f_m(z) = \sum_{n=0}^{\infty} b_{mn} z^n,$$

it turns out $a_n \sim b_{mn}$ for finite $m$, as $n \to \infty$. Indeed, we can write $a_n = b_{m,n} + \varepsilon_m(n)$, and show that the error term

$$\varepsilon_m(n) = \frac{1}{2\pi i} \int_C [f(z) - f_m(z)] z^{-n-1} dz = o(n^{-N}),$$

where $\text{Re} \alpha + m + 1 \leq N < \text{Re} \alpha + m + 2$. The above error estimate has been proved rigorously; see, e.g., Wong [97, Ch. 2.6]. Thus we have the following result.

**Theorem 2.1.** With $f(z)$ given in (2.3), (2.4) and (2.5), assume that $\alpha \not\in \mathbb{Z}$ is a complex constant, and $z = 1$ is the only singularity on the circle of convergence $|z| = 1$. For any $m \geq 0$, the Maclaurin coefficient of $f(z)$ has the asymptotic expansion

$$a_n = \sum_{r=0}^{m} c_r \binom{\alpha + r}{n} (-1)^n + o\left(n^{-\text{Re} \alpha - m - 1}\right)$$

(2.6)

as $n \to \infty$. 

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Formula (2.6) is referred to as a generalized asymptotic expansion in [97], which is more general than the usual Poincaré expansion; see Erdély and Wyman [36]. The asymptotic nature of the expansion is rooted in
\[
\left( \frac{\alpha + r}{n} \right) (-1)^n = \frac{\Gamma(n - \alpha - r)}{n! \Gamma(-\alpha - r)} \sim \frac{n^{-\alpha-1-r}}{\Gamma(-\alpha - r)}, \quad r = 0, 1, \ldots \quad \text{as } n \to \infty.
\]

When there are finite number of algebraic singularities on the circle of convergence, the final result can be extended by adding up the contributions from all singular points; see Szegő [80, p. 207, Thm. 8.4] and Wong [97, p. 119, Thm. 5].

Applying Theorem 2.1 to the generating function (2.2) (cf. [97, p. 120]), one finds that
\[
u_n \sim b \sqrt{n} \quad \text{as } n \to \infty,
\]
with
\[b = \frac{4e^{-3/2}}{\pi};
\]
this is Robinson’s constant. The theorem can also be used to solve a problem of Erlebach and Ruehr in counting Hamiltonian cycles for bipartite graphs; see Knuth [47]. Darboux’s method plays an important role as well in determining orthogonal measures, especially those with explicit three-term recurrence relations; see Ismail [45].

We complete our discussion of the classical Darboux’s method by mentioning the following interesting example investigated by Olver [71]. It is well known that the Legendre polynomials possess the generating function
\[
f(z) = \left( e^{i\theta} - z \right) \left( e^{-i\theta} - z \right)^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos \theta) z^n, \quad (2.7)
\]
where the branches are chosen such that \((e^{\pm i\theta} - z)^{-1/2} \to e^{\mp \frac{1}{2} i\theta}\) as \(z \to 0\). Now applying Darboux’s method yields
\[
P_n(\cos \theta) \sim \left( \frac{2 \sin \theta}{\sin \theta} \right)^{1/2} \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k \left( \frac{k - 1}{n} \right) \cos \theta_{n,k} (2 \sin \theta)^k, \quad n \to \infty,
\]
where
\[
\theta_{n,k} = \left( n - k + \frac{1}{2} \right) \theta + \left( n - \frac{1}{2} k - \frac{1}{4} \right) \pi.
\]
On the other hand,
\[
P_n(\cos \theta) = \left( \frac{1}{2 \sin \theta} \right)^{1/2} \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k \left( \frac{k - 1}{n} \right) \cos \theta_{n,k} (2 \sin \theta)^k
\]
for \(\frac{1}{6} \pi < \theta < \frac{5}{6} \pi\). A paradox now arises, since one has
\[
P_n(\cos \theta) \sim 2 P_n(\cos \theta), \quad \frac{1}{6} \pi < \theta < \frac{5}{6} \pi, \quad n \to \infty; \quad (2.8)
\]
cf. Olver [71].
Darboux’s method dates back to 1878. Although frequently used, it does not appear to have been extended until around 1970. In 1974, Wong and Wyman [102] have given a generalization of Darboux’s method, which allows the generating function $f(z)$ in (2.3) to have logarithmic-type singularities on its circle of convergence. This paper was considered an important early (and unduly neglected) reference in analytic combinatorics; see Flajolet and Sedgewick [38, p. 438]. First we quote a couple of examples of this type.

**Problem of Pólya** (1954, [76, Ex. 8, p. 8 and p. 213]): Define $A_n$ by

$$\frac{z}{\ln(1 + z)} = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!}, \quad |z| < 1.$$  

(2.9)

Show that $A_n \sim (-1)^{n-1} \frac{1}{n \ln^2 n}$ as $n \to \infty$. (2.10)

**Problem of Knuth** (2004, [48]): What is the asymptotic behavior of the constant $l_n$ defined by the generating function

$$\frac{1}{\ln(1 - z)} + \frac{1}{z} = \sum_{n=0}^{\infty} l_n z^n, \quad |z| < 1.$$  

(2.11)

In [102], Wong and Wyman dealt with the general case

$$f(z) = (1 - z)^{\alpha} [\ln(1 - z)]^{\mu} g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1,$$  

(2.12)

where $\alpha$ and $\mu$ are complex constants, and $g(z)$ is analytic in $|z| \leq 1 + \delta$ for some $\delta > 0$. Now we briefly outline their treatment. Again the Cauchy integral formula applies, and one has, by deforming the contour,

$$2\pi i a_n = \oint_{|z| = 1 + \delta_n} f(z) z^{-n-1} dz - \oint_{|z-1| = \delta_n} f(z) z^{-n-1} dz,$$

where the integration paths are counterclockwisely oriented, and $\delta_n = n^{-1/2}$. The first integral is of the order $O(e^{-c\sqrt{n}})$ for some positive constant $c$; the second integral alone contributes to the full asymptotic expansion. The approximants are “special” functions with integral representations

$$M(\alpha, \mu, n) = -\frac{1}{2\pi i} \int_{-\infty}^{(0+)} (-t)^{\alpha} [\ln(-t)]^{\mu} e^{-(n+1)t} dt, \quad n = 0, 1, 2, \cdots,$$  

(2.13)

and have full asymptotic expansions of the form

$$M(\alpha, \mu, n) \sim \frac{(-\ln(n+1))^{\mu}}{(n+1)^{\alpha+1}} \sum_{k=0}^{\infty} \left(\frac{\mu}{k}\right) \frac{d^k}{d\alpha^k} \left[ \frac{1}{\Gamma(-\alpha)} \right] \left(\frac{1}{(-\ln(n+1))^{k}} \right).$$
Wong and Wyman further introduced the function
\[ J_m = -\frac{1}{2\pi i} \oint_{|t|=\delta_n} (-t)^{\alpha+m} \ln(-t)^\mu P_m((n+1)t)e^{-(n+1)t} dt, \quad (2.14) \]
where \( P_m(w) \) is a polynomial given by
\[ P_m(w) = \frac{1}{m!} \frac{d^m}{dt^m} \left[ g(t+1) \exp \left\{ -\frac{1}{2} wt \left[ \frac{2(\ln(1+t) - t)}{t^2} \right] \right\} \right]_{t=0} = \sum_{s=0}^m p_s w^s, \]
so that
\[ J_m \sim \sum_{s=0}^m p_s (n+1)^s M(\alpha+m+s,\mu,n) \quad \text{as } n \to \infty. \]

**Theorem 2.2.** (Wong-Wyman, [102]) If \( f(z) \) is the function given in (2.12), then its Maclaurin coefficient \( a_n \) has the asymptotic expansion
\[ a_n \sim \sum_{m=0}^\infty (-1)^m J_m, \]
where \( J_m \) is defined in (2.14) and has the asymptotic expansion
\[ J_m \sim \frac{(-\ln(n+1))^{\mu}}{(n+1)^{\alpha+m+1}} \sum_{k=0}^\infty \left\{ \frac{\mu}{k} \sum_{s=0}^m p_s \frac{d^k}{d\alpha^k} \left[ \frac{1}{\Gamma(-\alpha-m-s)} \right] \right\} \frac{1}{(-\ln(n+1))^k}, \quad n \to \infty. \]

The above result includes Theorem 2.1 as a special case if \( \mu = 0 \), and can be extended to cases with several fixed singularities on the circle of convergence.

Now it is readily seen that (2.10) follows accordingly, by applying Theorem 2.2 to (2.9). For the generating function (2.11), we take \( \alpha = 0, \mu = -1 \) and \( g(z) \equiv 1 \) in (2.12). Then, Theorem 2.2 applies and we have
\[ l_n \sim \frac{1}{(n+1)\ln^2(n+1)} \quad \text{as } n \to \infty. \]

As a consistence check, it is worth noting that (2.10) and (2.11) are related in the straightforward manner: \( A_n/n! = (-1)^{n-1} l_{n-1} \) for positive integers \( n \).

When the singularities are free to move on the circle of convergence, Darboux’s method will continue to work only if their essential configuration remains the same while the relative positions vary. However, this method breaks down when two or more singularities coalesce with each other. For example, the generating function (2.7) for the Legendre polynomials has a pair of algebraic singularities \( e^{\pm i\theta} \). Darboux’s method fails to apply when \( e^{\pm i\theta} \to 1 \) as \( \theta \to 0^+ \), and \( e^{\pm i\theta} \to -1 \) as \( \theta \to \pi^- \); a difficulty arises. In the cases of coalescing singular points, the asymptotic expansion may involve transcendental functions instead of elementary ones.
In 1967, Fields \[37\] presented a uniform treatment of Darboux’s method when two or three singularities coalesce. More precisely, he considered the case in which

\[ f(z, \theta) = (1 - z)^{-\lambda} \left[ (e^{i\theta} - z) (e^{-i\theta} - z) \right]^{-\Delta} g(z, \theta) = \sum_{n=0}^{\infty} a_n(\theta) z^n, \quad (2.15) \]

where the Maclaurin expansion converges for \(|z| < 1\), \(\lambda\) and \(\Delta\) are bounded quantities, the branches of \((1 - z)^{-\lambda}\) and \((e^{i\theta} - z) (e^{-i\theta} - z)^{-\Delta}\) are chosen such that each of them reduces to 1 at \(z = 0\), and \(g(z, \theta)\) is analytic in \(|z| \leq e^\eta\) \((\eta > 0)\), uniformly for \(\theta \in [0, \pi]\).

In \[37\], Fields first expressed \(a_n(\theta)\) as a Cauchy integral, then made a change of variable and a rescaling, and finally obtained a generalized asymptotic expansion in the sense of Erdélyi and Wyman \[36\]. His results are uniform in certain \(\theta\)-intervals depending on \(n\).

Despite the fact that Fields’ results have achieved the so-called uniform reduction in the sense of Olver \[73, p. 102\], they are found to be too complicated for any practical application; see, e.g., Erdélyi \[35, p. 167\], Olver \[73, pp. 112-113\], and Wong \[97, p. 145\]. For example, Olver commented that It may be desirable to investigate whether any simplifications are feasible since the results in \[37\] are rather complicated to apply in their present form.

A motivation for the uniform treatment of Darboux’s method came from the uniform asymptotic expansions of integrals. As a point of information, we mention a few relevant references in this respect. For example, Chester, Friedman, and Ursell (1957, \[19\]) first presented a uniform treatment of the steepest descent method, Bleistein (1967, \[10\]) considered the problem of many nearby stationary points and algebraic singularities, Wong (1989, \[97, Ch. VII\]) provided various types of coalescence of singular points, in particular uniform asymptotics of orthogonal polynomials, and Olde Daalhuis and Temme (1994, \[66\]) introduced a class of rational functions based on which error terms can be estimated.

The work of Wong and Zhao \[103\] in 2005 seems to have responded to Olver’s request mentioned above, i.e., to derive simpler forms of uniform asymptotic expansions when two or more algebraic singularities, on the circle of convergence, coalesce with each other as certain parameter approaches a critical value. To begin with, Wong and Zhao first considered the simplest case with two singularities, namely

\[ f(z, \theta) = \left[ (e^{i\theta} - z) (e^{-i\theta} - z) \right]^{-\alpha} g(z, \theta) = \sum_{n=0}^{\infty} a_n(\theta) z^n, \quad (2.16) \]

where \(g(z, \theta)\) is analytic in \(|z| \leq e^\eta\), \(\eta > 0\), which is a special case of Fields; cf. \[2.15\] with \(\lambda = 0\). Contribution to the large-\(n\) behavior still comes from the singular points \(e^{\pm i\theta}\), which are now subject to vary as \(\theta \to 0^+\). The approximants in this case are

\[ T_1(x) = \frac{1}{2\pi i} \int_{\Gamma_0} (s^2 + 1)^{-\alpha} e^{xs} ds, \quad T_2(x) = \frac{1}{2\pi i} \int_{\Gamma_0} s(s^2 + 1)^{-\alpha} e^{xs} ds, \]

where \(\Gamma_0\) is a Hankel-type loop which starts and ends at \(-\infty\), and encircles \(s = \pm i\) in
the positive sense. An asymptotic expansion of the form
\[
a_n(\theta) = \theta^{1-2\alpha} T_1(n\theta) \sum_{k=0}^{m-1} \frac{\alpha_k(\theta)}{n^k} + \theta^{1-2\alpha} T_2(n\theta) \sum_{k=0}^{m-1} \frac{\beta_k(\theta)}{n^k} + \varepsilon(\theta, m)
\]
was obtained for \( m = 1, 2, 3, \cdots \), with the error term \( \varepsilon(\theta, m) \) given explicitly and estimated, and the coefficients determined recursively as follows: Begin with an analytic function
\[
h_0(s, \theta) := g(e^{-s\theta}, \theta) \left[ \left( \frac{e^{-s\theta} - e^{i\theta}}{(-s - i)\theta} \right) \frac{e^{-s\theta} - e^{-i\theta}}{(-s + i)\theta} \right]^{-\alpha}
\]
for \( \text{Re } s \geq -\eta/\theta \) and \( |s \pm i| < 2\pi/\theta \), and make use of the iteration
\[
\begin{align*}
\begin{cases}
  h_k(s, \theta) := \alpha_k(\theta) + s\beta_k(\theta) + (s^2 + 1) g_k(s, \theta), \\
  h_{k+1}(s, \theta) = -\frac{1}{\theta} \left[ (s^2 + 1) \frac{d}{ds} + 2(1 - \alpha)s \right] g_k(s, \theta),
\end{cases}
k = 0, 1, 2, \cdots.
\end{align*}
\]
Assuming analyticity of each \( h_k(s, \theta) \) and \( g_k(s, \theta) \) at \( s = \pm i \), one determines the coefficients \( \alpha_k(\theta) \) and \( \beta_k(\theta) \). Furthermore, straightforward verification shows that
\[
T_1(x) = \sqrt{\frac{\pi}{\Gamma(\alpha)}} \left( \frac{x}{2} \right)^{\alpha - \frac{1}{2}} J_{\alpha - \frac{1}{2}}(x),
\]
and
\[
T_2(x) = \sqrt{\frac{\pi}{\Gamma(\alpha)}} \left( \frac{x}{2} \right)^{\alpha - \frac{1}{2}} \left[ \frac{2\alpha - 1}{x} J_{\alpha - \frac{1}{2}}(x) - J_{\alpha + \frac{1}{2}}(x) \right] = \sqrt{\frac{\pi}{\Gamma(\alpha)}} \left( \frac{x}{2} \right)^{\alpha - \frac{1}{2}} J_{\alpha - \frac{1}{2}}(x).
\]
Hence we have

\textbf{Theorem 2.3.} (Wong-Zhao \[\text{[103]}\]; see also \[\text{[60, Cor. 1]}\]) If \( f(z, \theta) \) is the function given in \textbf{(2.10)}. Then its Maclaurin coefficient has the asymptotic expansion
\[
a_n(\theta) \sim \frac{\sqrt{\pi}}{\Gamma(\alpha)} \left( \frac{n}{2\theta} \right)^{\alpha - \frac{1}{2}} \left[ J_{\alpha - \frac{1}{2}}(n\theta) \sum_{k=0}^{\infty} \frac{\alpha_k(\theta)}{n^k} + J_{\alpha - \frac{1}{2}}(n\theta) \sum_{k=0}^{\infty} \frac{\beta_k(\theta)}{n^k} \right]
\]
as \( n \to \infty \), holding uniformly for \( \theta \in [0, \pi - \delta] \), \( \delta > 0 \), with coefficients \( \alpha_k(\theta) \) and \( \beta_k(\theta) \) given by \textbf{(2.17)} and \textbf{(2.18)}. The leading coefficients are \( \alpha_0(\theta) = \cos(\alpha\theta) \left( \frac{\sin\theta}{\theta} \right)^{-\alpha} \) and \( \beta_0(\theta) = \sin(\alpha\theta) \left( \frac{\sin\theta}{\theta} \right)^{-\alpha} \).

The above result can be extended to handle cases with many coalescing algebraic singularities. For instance, we have
\[
f(z, \theta) = \left\{ \prod_{k=1}^{q} (z_k(\theta) - z)^{-\alpha_k} \right\} g(z, \theta) = \sum_{n=0}^{\infty} a_n(\theta) z^n,
\]
where \( g(z, \theta) \) is analytic in \( |z| \leq e^\eta \) with \( \eta > 0 \), \( z_k(\theta) \equiv e^{i\theta s_k(\theta)} \) so that \( z_k(\theta) \to 1 \) as \( \theta \to 0 \). In this case, the approximants are the special functions

\[
T_l(x) := \frac{1}{2\pi i} \int_{\Gamma_0} s^{l-1} e^{sx} \prod_{k=1}^q (s + i s_k(\theta))^{-\alpha_k} ds
\]

for \( l = 1, 2, \cdots, q \), where \( \Gamma_0 \) is also a Hankel-type loop which starts and ends at \( -\infty \), encircling all points \( s = -i s_k(\theta) \) in the positive direction.

**Theorem 2.4. (Wong-Zhao [103])** Let \( f(z, \theta) \) be the function given in (2.20). If \( z_k(\theta) = e^{i s_k \theta} \), \( s_k \) being real constants, then

\[
a_n(\theta) \sim \theta^{1-\alpha} \sum_{l=1}^q T_l(n\theta) \sum_{k=0}^{\infty} \frac{\beta_{k,l}(\theta)}{n^k}
\]

as \( n \to \infty \), uniformly in \( \theta \in [0, \nu] \) with \( \nu < \min_{1 \leq k \leq q} \{\pi/|s_k|\} \), where \( \alpha = \sum_{k=1}^q \alpha_k \), and the coefficients \( \beta_{k,l}(\theta) \) can be determined successively.

Properties of the special functions \( T_l(x) \) in (2.21) with \( s_k(\theta) \equiv s_k \) are considered in [103]. For instance, from (2.21) we have

\[
T_l(x) = \frac{d^{l-1}}{dx^{l-1}} T_1(x) \quad \text{for} \quad l = 1, 2, \cdots, q,
\]

and that \( w = T_1(x) \) solves the differential equation

\[
x \frac{d^q w}{dx^q} + \sum_{l=0}^{q-1} (d_l x + c_l) \frac{d^l w}{dx^l} = 0,
\]

where the coefficients \( c_l \) and \( d_l \) are determined by

\[
\prod_{k=1}^q (s + i s_k) := s^q + \sum_{l=0}^{q-1} d_l s^l, \quad \sum_{k=1}^q (1 - \alpha_k) \prod_{k \neq k}^q (s + i s_k) := \sum_{l=0}^{q-1} c_l s^l.
\]

Now we check the special case (2.15) investigated by Fields [37]. Theorem 2.4 applies, and we may set

\[
q = 3; \quad s_1 = 0, \alpha_1 = \lambda; \quad s_2 = 1, \alpha_2 = \Delta; \quad s_3 = -1, \alpha_3 = \Delta.
\]

By expanding a part of the integrand in (2.21) into Laurent series, we find that in terms of the generalized hypergeometric functions, we have

\[
T_l(x) = \frac{x^{\lambda + 2\Delta - l}}{\Gamma(\lambda + 2\Delta - l + 1)} \left[ \frac{\lambda + 2\Delta - l + 1}{2}, \frac{\lambda + 2\Delta - l + 2}{2}; -\frac{x^2}{4} \right]
\]
for \( l = 1, 2, \) and 3; cf. [74] (16.2.1).

We complete our discussion on uniform treatment of Darboux’s method by listing several follow-up progresses. In 2007, Bai and Zhao [2] derived uniform asymptotics for the Jacobi polynomials via uniform treatment of Darboux’s method, using not the results but the ideas discussed above. Later in 2013, Liu, Wong and Zhao [60] apply the uniform Darboux’s method to analyze the Heisenberg polynomials. The asymptotic expansion obtained involves the Kummer function and its derivative.

We mention yet another situation, which may be termed a singular case. We take as an example the Pollaczek polynomials

\[
(1 - ze^{i\theta})^{-\frac{1}{2} + ih(\theta)}(1 - ze^{-i\theta})^{-\frac{1}{2} - ih(\theta)} = \sum_{n=0}^{\infty} P_n(x; a, b)z^n, \tag{2.23}
\]

each of the factors in the generating function reduces to 1 for \( z = 0, \) where \( h(\theta) = \frac{a\cos\theta + b}{2\sin\theta}, a > |b|. \) There are two singularities \( z = e^{\pm i\theta} \) on the circle of convergence, coalescing with each other when \( \theta \to 0. \) Meanwhile the exponent \( h(\theta) \sim \frac{a+b}{\theta} \) as \( \theta \to 0, \) demonstrating a singular behavior. The reader is referred to [11] for an asymptotic analysis of these polynomials using integral methods. Such types of generating function with varying exponents, regular or singular, seem to be of interest.

### 3 Difference equations

As remarked earlier in Section 1, the development of the asymptotic theory for difference equations took a halt in the thirties of the last century, and did not make any progress until the turn of this century. It is interesting to note that in his lecture, given during the conference in honor of his 80th birthday, Frank Olver expressed that In my view Birkhoff and Trjitzinsky [8] set back all research into the asymptotic solutions of difference equations for most of the 20th century. Also, in 1985, Wimp and Zeilberger [94] made the interesting remark that Once on the forefront of mathematical research in America, the asymptotics of the solutions of linear recurrence equations is now almost forgotten, especially by the people who need it most, namely combinatorics and computer scientists. Later, in 1991, Wimp [93] also made the statement that There are still vital matters to be resolved in asymptotic analysis. At least one widely quoted theory, the asymptotic theory of irregular difference equations expounded by G. D. Birkhoff and W. R. Trjitzinsky [7] [8] in the early 1930’s, is vast in scope; but there is now substantial doubt that the theory is correct in all its particulars. The computations involved in the algebraic theory alone are truly mind-boggling.

The above comments were certainly the driving force behind the work carried out by Wong and Li [99] [100] in the 1990’s. In order to understand the problem better, they first studied in [99] the simple second-order difference equation

\[
y(n + 2) + a(n)y(n + 1) + b(n)y(n) = 0,
\]
where \( a(n) \) and \( b(n) \) have infinite expansions of the form

\[
a(n) \sim \sum_{s=0}^{\infty} \frac{a_s}{n^s} \quad \text{and} \quad b(n) \sim \sum_{s=0}^{\infty} \frac{b_s}{n^s}
\]  

(3.1)

for large values of \( n \), and \( b_0 \neq 0 \). Then in [100], they extended their investigation to include the more general equation

\[
y(n + 2) + n^p a(n)y(n + 1) + n^q b(n)y(n) = 0,
\]

(3.2)

where \( p \) and \( q \) are integers, and \( a(n) \) and \( b(n) \) are as given in (3.1) with the leading coefficients \( a_0 \) and \( b_0 \) being nonzero.

The basic technique adopted in [99, 100] is essentially the method of successive approximations, which is customarily used in the asymptotic theory of differential equations. The ultimate goal of Wong's investigation was to develop a turning point theory for the three-term recurrence relation

\[
y_{n+1} = (a_n x + b_n)y_n - c_n y_{n-1}, \quad n = 1, 2, \ldots
\]

(3.3)

where \( a_n, b_n \) and \( c_n \) are constants. If \( x \) is a fixed number, then the recurrence relation is equivalent to the second-order linear difference equation (3.2). The importance of developing a turning point theory for the recurrence relation (3.3) lies in the fact that many special functions of mathematical physics (Bessel functions, parabolic cylinder functions, Legendre functions, etc.) satisfy such an equation. In fact, any sequence of orthogonal polynomials satisfies an equation of the form (3.3); see [80, p. 42]. This statement, of course, applies to classical orthogonal polynomials such as Hermite \( H_n(x) \), Laguerre \( L_n^{(a)}(x) \) and Jacobi \( P_n^{(\alpha,\beta)}(x) \); more importantly, it includes all those that do not satisfy any second-order linear differential equations, e.g., Charlier \( C_n^{(a)}(x) \), Meixner \( m_n(x; \beta, c) \), Pollaczek \( P_n(x; a, b) \), Meixner-Pollaczek \( M_n(x; \delta, \eta) \) and Krawtchouk polynomials \( K_n^N(x; p, q) \).

While a turning-point theory for second-order differential equations has been satisfactorily developed from 1930 to 1960 by Langer [54], Cherry [18], Olver [69] and others, not much progress has been made in the development of a corresponding theory for the recurrence equation (3.3). A possible explanation is that difference equations are considerably more complicated to analyze than differential equations; see a remark made by Iserles [44, p. 743].

To illustrate the difficulty in hand, we consider the Bessel function \( J_\nu(\nu x) \), which has been used as a typical example in some of the most profound research in asymptotic analysis; see, e.g., [19], [67] and [68]. The differential equation approach proceeds with the equation

\[
\frac{d^2 w}{dx^2} = \left\{ \nu^2 \frac{1 - x^2}{x^2} - \frac{1}{4x^2} \right\} w,
\]

(3.4)

which is satisfied by \( x^\frac{1}{2} J_\nu(\nu x) \). From (3.4), it is clear that \( x = \pm 1 \) are two turning points, i.e., zeros of the coefficients function multiplied by the large parameter \( \nu^2 \). An
application of the transformations

\[
\frac{2}{3} \zeta^\frac{3}{2} = \ln \frac{1 + (1 - x^2)^\frac{1}{2}}{x} - (1 - x^2)^\frac{1}{2}
\]  (3.5)

and

\[
W = f^\frac{1}{2} w, \quad f = \frac{1 - x^2}{x^2 \zeta}
\]  (3.6)

gives

\[
\frac{d^2 W}{d \zeta^2} = \{ \nu^2 \zeta + \psi(\zeta) \} W,
\]

where

\[
\psi(\zeta) = \frac{5}{16 \zeta^2} + \frac{\zeta x^2(x^2 + 4)}{4(x^2 - 1)}.
\]  (3.7)

The basic approximation equation is

\[
\frac{d^2 W}{d \zeta^2} = \nu^2 \zeta W;
\]  (3.8)

two linearly independent solutions are the Airy functions \( \text{Ai}(\nu^\frac{2}{3} \zeta) \) and \( \text{Bi}(\nu^\frac{2}{3} \zeta) \). After identifying \( J_\nu(\nu x) \) and \( \text{Ai}(\nu^\frac{2}{3} \zeta) \) and matching their behavior as \( x \to \infty \), one obtains the expansion

\[
J_\nu(\nu x) \sim \left( \frac{4 \zeta}{1 - x^2} \right)^\frac{1}{4} \left\{ \text{Ai}(\nu^\frac{2}{3} \zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\nu^{2s}} + \frac{\text{Ai}'(\nu^\frac{2}{3} \zeta)}{\nu^\frac{2}{3}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\nu^{2s}} \right\}
\]  (3.9)

as \( \nu \to \infty \), uniformly with respect to \( x \geq 0 \). The coefficients \( A_s(\zeta) \) and \( B_s(\zeta) \) are defined recursively by \( A_0(\zeta) = 1 \),

\[
B_s(\zeta) = \frac{1}{2 \zeta^2} \int_0^\zeta \{ \psi(v) A_s(v) - A''_s(v) \} \frac{dv}{v^2}
\]  (3.10)

and

\[
A_{s+1}(\zeta) = -\frac{1}{2} B'_s(\zeta) + \frac{1}{2} \int \psi(\zeta) B_s(\zeta) d\zeta + \text{constants}. \tag{3.11}
\]

From a difference-equation point of view, it is natural to suggest that the same result can be obtained directly from the three-term recurrence equation

\[
J_{\nu+1}(x) - \frac{2\nu}{x} J_\nu(x) + J_{\nu-1}(x) = 0.
\]  (3.12)

This problem turns out to be considerably more difficult to tackle than what we would have anticipated. To start with the very form of expansion (3.9) immediately raises the following questions:
(i) Since the Airy function $\text{Ai}(x)$ does not satisfy any second-order difference equation, how is it going to arise from the three-term recurrence relation (3.12)?

(ii) Since we cannot change the discrete variable $n$ (or $\nu$) to a new independent variable, like what we have done with the Langer transformation (3.5), how are we going to produce the function $\zeta$ in (3.9) (cf. (3.5))?

(iii) Even if we can derive an asymptotic expansion similar to that in (3.9), can the coefficients be determined recursively like (3.10)-(3.11)?

(iv) Can the region of validity of our expansion be as large as the one for (3.9), i.e., the whole positive real axis $x \geq 0$; or is it just a finite interval $0 \leq x \leq M$ like in the case of integral approach (see [19, p. 610])?

Answers to these questions are given in [86].

In 1967, Dingle and Morgan [31, 32] made an attempt to study second-order difference equations of the form

$$f_{n+\omega}(x) + f_{n-\omega}(x) = 2\sigma(n, x) f_n(x),$$

(3.13)

where $\sigma(n, x)$ is a slowly varying function of the independent variable $n$. For convenience, they expressed $\sigma(n, x)$ as

$$\sigma(n, x) = a(n, x) + \omega^2 b(n, x) + \omega^4 c(n, x) + \cdots$$

(3.14)

with the understanding that $b(n, x)$ is two “order” smaller than $a(n, x)$, and that $c(n, x)$ is four “order” smaller than $a(n, x)$ and so on; see [32, eq. (35)]. In (3.14), $\omega$ is used as an “ordering” parameter which picks out terms of comparable magnitude. These authors called the zeros $n_0(x)$ of the equation

$$a(n, x) = 1$$

(3.15)

turning points, and suggested that near $a = 1$ (i.e., in a neighborhood of $n_0(x)$), two linearly independent asymptotic expansions are given by

$$f_n^{(1)}(x) = J_{n+F(x)-n_0(x)}(F(x))$$

(3.16)

and

$$f_n^{(2)}(x) = Y_{n+F(x)-n_0(x)}(F(x)),$$

(3.17)

where

$$F(x) = \left( \frac{da(n, x)}{dn} \right)^{-1} \Big|_{n=n_0(x)}.$$  

(3.18)

In the case of (3.12), $\omega = 1$ and $\sigma(n, x) = n/x$. Thus, in (3.14) we just take $a(n, x) = \sigma(n, x) = n/x$ and $b(n, x) = c(n, x) = \cdots = 0$. The turning point (in the sense of Dingle and Morgan) is at $n = n_0(x) = x$, and the function in (3.18) is given by $F(x) = x$. As a result, the asymptotic solution $f_n^{(1)}(x)$ in (3.16) is just the function $J_n(x)$ itself, which is
what we wish to approximate. In any case, the arguments in [31, 32] are too sketchy and non-rigorous. Nevertheless, in view of (3.9), the result of Dingle and Morgan is most likely correct.

Another paper that is relevant to our discussion here is by Costin and Costin [21]. These authors have studied the asymptotic behavior of the solutions of recurrence relations of the form

\[ a_2(k\varepsilon, \varepsilon) y_{k+2} + a_1(k\varepsilon, \varepsilon) y_{k+1} + a_0(k\varepsilon, \varepsilon) y_k = 0 \]  

(3.19)
as \( \varepsilon \to 0^+ \), where the coefficients \( a_i(x, \varepsilon) \), \( i = 0, 1, 2 \), are \( C^\infty \)-functions in \( x \) and \( \varepsilon \) in some domain \( I \times [0, \varepsilon_0] \), \( I \) being a compact interval. Furthermore, they assume that for any \( n \geq 0 \),

\[ a_i(x, \varepsilon) = \sum_{s=0}^{n} a_{i,s}(x) \varepsilon^s + O(\varepsilon^{n+1}) \], \( i = 0, 1, 2 \),

(3.20)where \( a_{i,s}(x) \) are \( C^\infty \)-functions in \( x \) and the \( O \)-symbol is uniform with respect to \( x \). Let \( \lambda_1(x) \) and \( \lambda_2(x) \) be the roots of the characteristic polynomial

\[ a_2(x, 0)\lambda^2 + a_1(x, 0)\lambda + a_0(x, 0) = 0, \]  

(3.21)and assume that \( \lambda_1(0) = \lambda_2(0) \) but \( \lambda_1(x) \neq \lambda_2(x) \) when \( x \neq 0 \). Fix \( \frac{1}{4} < \alpha < \frac{1}{2} \). One of the major results in [21] is that for \( k < \varepsilon^{-\alpha} \), equation (3.19) has two solutions of the form

\[ \exp \left\{ F_\pm \left( k\varepsilon^{\frac{1}{4}}, \varepsilon^{\frac{1}{4}} \right) \right\} \],

where \( \exp \{ F_\pm (x, 0) \} = \text{Ai}(\Theta x) \pm \text{Bi}(\Theta x) \) and \( \Theta \) is an explicitly given constant in terms of \( a_i(0, 0) \) and \( D_x a_i(0, 0) \). A particular solution has the behavior

\[ y_{k, \varepsilon} \sim \text{Ai} \left( \Theta k\varepsilon^{\frac{1}{4}} \right) \left[ 1 + \varepsilon^{\frac{1}{4}} A_1 \left( \Theta k\varepsilon^{\frac{1}{4}} \right) + \cdots \right] \]  

(3.22)for large \( k < \varepsilon^{-\alpha} \). A comparison of equations (3.12) and (3.19) readily shows that the results of Costin and Costin can not be used to derive a uniform asymptotic expansion for \( J_{\nu}(\nu x) \) such as the one given in (3.9).

We are now ready to present a summary of the investigation carried out by Wong and his collaborators in the last twenty years. Returning to equation (3.3), we first define a sequence \( \{K_n\} \) recursively by

\[ K_{n+1}/K_n - 1 = c_n, \]  

and assume that \( K_n(0) = \lambda_n(0) \) but \( K_n(x) \neq \lambda_n(x) \) when \( x \neq 0 \). Fix \( \frac{1}{4} < \alpha < \frac{1}{2} \). One of the major results in [21] is that for \( k < \varepsilon^{-\alpha} \), equation (3.19) has two solutions of the form

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where \( \exp \{ F_\pm (x, 0) \} = \text{Ai}(\Theta x) \pm \text{Bi}(\Theta x) \) and \( \Theta \) is an explicitly given constant in terms of \( a_i(0, 0) \) and \( D_x a_i(0, 0) \). A particular solution has the behavior

\[ y_{k, \varepsilon} \sim \text{Ai} \left( \Theta k\varepsilon^{\frac{1}{4}} \right) \left[ 1 + \varepsilon^{\frac{1}{4}} A_1 \left( \Theta k\varepsilon^{\frac{1}{4}} \right) + \cdots \right] \]  

(3.22)for large \( k < \varepsilon^{-\alpha} \). A comparison of equations (3.12) and (3.19) readily shows that the results of Costin and Costin can not be used to derive a uniform asymptotic expansion for \( J_{\nu}(\nu x) \) such as the one given in (3.9).

We are now ready to present a summary of the investigation carried out by Wong and his collaborators in the last twenty years. Returning to equation (3.3), we first define a sequence \( \{K_n\} \) recursively by

\[ K_{n+1}/K_n - 1 = c_n, \]  

with \( K_0 \) and \( K_1 \) depending on the particular sequence \( \{y_n\} \) satisfying (3.3). Then we put \( A_n = a_n K_n/K_{n+1} \), \( B_n = b_n K_n/K_{n+1} \) and \( x_n = y_n/K_n \), so that (3.3) becomes

\[ x_{n+1} - (A_n x + B_n) x_n + x_{n-1} = 0. \]  

(3.23)
The coefficients \( A_n \) and \( B_n \) are assumed to have asymptotic expansions of the form

\[ A_n \sim n^{-\delta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s} \quad \text{and} \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}, \]  

(3.24)
where $\theta$ is a real number and $\alpha_0 \neq 0$.

Let $\tau_0$ be a constant and put $\nu := n + \tau_0$. Clearly, the expansions in (3.24) can be recast in the form

$$A_n \sim \nu^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha'_s}{\nu^s} \quad \text{and} \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{\nu^s}.$$  (3.25)

In (3.23), we now let $x = \nu^\theta t$ and $x_n = \lambda_n$. Substituting (3.25) into (3.23) and letting $n \to \infty$ (and hence $\nu \to \infty$), we obtain the characteristic equation

$$\lambda^2 - (\alpha'_0 t + \beta'_0) \lambda + 1 = 0.$$  (3.26)

The roots of this equation are given by

$$\lambda = \frac{1}{2} \left[ (\alpha'_0 t + \beta'_0) \pm \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4} \right],$$  (3.27)

and they coincide when $t = t_\pm$, where

$$\alpha'_0 t_\pm + \beta'_0 = \pm 2.$$  (3.28)

The values $t_\pm$ play an important role in the asymptotic theory of the three-term recurrence relation (3.23), and they correspond to the transition points (i.e., turning points and poles) occurring in differential equations; [72, p. 362]. For this reason, we shall also call them transition points. Since $t_+ > t_-$, we may restrict ourselves to consider the case $t_+ > 0$. Furthermore, to make the presentation simpler, we also assume that $t_- < 0$. These two assumptions together are equivalent to the condition $|\beta_0| < 2$. Before stating our first theorem, we need to impose one more condition; that is,

$$\alpha'_1 = \beta'_1 = 0.$$  (3.29)

It is interesting to note that this condition holds in most of the classical cases. In fact, in [81] Dingle and Morgan have assumed that $\alpha_{2s+1} = \beta_{2s+1} = 0$ for $s = 0, 1, 2, \ldots$. For a result without this assumption, we refer the reader to [87]. Now we define the function

$$\begin{cases}
\frac{2}{3} \left[ \zeta(t) \right]^3 = \alpha'_0 t_+^\frac{1}{\theta} \int_{t_+}^{t} \frac{s^{-1/\theta} ds}{\sqrt{(\alpha'_0 s + \beta'_0)^2 - 4}} - \ln \frac{\alpha'_0 t + \beta'_0 + \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4}}{2}, & t \geq t_+,
\frac{2}{3} \left[ -\zeta(t) \right]^3 = -\alpha'_0 t_+ \int_{t}^{t_+} \frac{s^{-1/\theta} ds}{\sqrt{4 - (\alpha'_0 s + \beta'_0)^2}} + \arccos \left( \frac{\alpha'_0 t + \beta'_0}{2} \right), & t < t_+,
\end{cases}$$  (3.30)

which plays the role of the Langer transform for differential equations; cf. [35].
Theorem 3.1. Assume that the coefficients $A_n$ and $B_n$ in the recurrence relation (3.23) have asymptotic expansions of the form given in (3.24) with $\theta \neq 0$ and $|\beta_0| < 2$. (See the statement just before (3.29).) Let $\zeta(t)$ be defined as in (3.30), and recall the number $\nu := n + \tau_0$ in the asymptotic expansions in (3.26). Then, with $x = \nu^\theta t$, equation (3.28) has a pair of solutions $P_n(x)$ and $Q_n(x)$ given by

$$
P_n\left(\nu^\theta t\right) \sim \left(\frac{4\zeta}{(\alpha_0^2 + \beta_0^2)^2 - 4}\right)^{1/2} \left[A(\nu^2 \zeta) \sum_{s=0}^{\infty} \frac{\tilde{A}_s(\zeta)}{\nu^{s+\frac{1}{2}}} + A'(\nu^2 \zeta) \sum_{s=0}^{\infty} \frac{\tilde{B}_s(\zeta)}{\nu^{s+\frac{1}{2}}} \right],
$$

and

$$
Q_n\left(\nu^\theta t\right) \sim \left(\frac{4\zeta}{(\alpha_0^2 + \beta_0^2)^2 - 4}\right)^{1/2} \left[B(\nu^2 \zeta) \sum_{s=0}^{\infty} \frac{\tilde{A}_s(\zeta)}{\nu^{s+\frac{1}{2}}} + B'(\nu^2 \zeta) \sum_{s=0}^{\infty} \frac{\tilde{B}_s(\zeta)}{\nu^{s+\frac{1}{2}}} \right],
$$

where the coefficients $\tilde{A}_s(\zeta)$ and $\tilde{B}_s(\zeta)$ are determined successively from some recursive formulas, beginning with $A_0(\zeta) = 1$ and $B_0(\zeta) = 0$.

Case (ii) Historically, this case was dealt with nearly 10 years after Case (iii) was settled. One of the reasons is that it was not clear what exactly the dominant approximant should be, although the example of Laguerre polynomials suggested that it most likely is a Bessel function. Since $\theta \neq 0$ in this case, we may let $\tau_0 := -\alpha_1/(\alpha_0 \theta)$ and $N := n + \tau_0$. Then, from (3.23) and (3.24), we have

$$A_n x + B_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s} x + \sum_{s=0}^{\infty} \frac{\beta_s}{n^s} x := \sum_{s=0}^{\infty} \frac{\alpha_s' t + \beta_s'}{N^s},
$$

(3.31)

where $x := N^\theta t$. A simple calculation gives

$$\alpha_0' = \alpha_0, \quad \alpha_1' = 0, \quad \beta_0' = \beta_0, \quad \beta_1' = \beta_1, \quad \beta_2' = \beta_2 + \beta_1 \tau_0.
$$

(3.32)

If $\beta_0 = 2$ (or $-2$), from (3.28) it follows that one of the transition points is zero. Without loss of generality, we assume that $t_+ = 0 < t_-$ and $\beta_0 = 2, \alpha_0 < 0$. (For other cases, see Remark 3.1 below.) As in Case (i), we assume that $\beta_1' = \beta_1' = 0$ so that

$$\alpha_1' t_+ + \beta_1' = 0.
$$

(3.33)

(In most of the classical cases, $\beta_1' = \beta_1 = 0$; see the statement following (3.29)).

With $\alpha_0' < 0$ and $\beta_0 = 2$, we now define the function

$$\pm \zeta(t) = \arccos \left(\frac{\alpha_0' t + \beta_0'}{2}\right) + \alpha_0' t \int_{\alpha}^{t} \frac{\phi^{-1/\theta}}{\sqrt{4 - (\alpha_0' \phi + \beta_0')^2}} d\phi
$$

(3.34)

for $t < t_-$, where “+” sign is taken for $\theta < 2$ and “−” sign is taken for $\theta > 2$. The sign is chosen so that $\zeta(t)$ is positive for $t \in (0, t_-)$. The choice of the lower limit of integration
is just for the purpose of convergence of the integral. For instance, we can choose \( a = 0 \) if \( \theta < 0 \) or \( \theta > 2 \) and

\[
a = \begin{cases} 
-\infty, & t < 0, \\
0, & 0 \leq t \leq t_- \\
t_-, & 0 < \theta < 2
\end{cases}
\]

(3.35)

With this choice, it can be shown that \( \zeta(0) = 0 \).

**Theorem 3.2.** Assume that the coefficients \( A_n \) and \( B_n \) in the recurrence relation (3.23) have asymptotic expansions given in (3.24) with \( \theta \neq 0, 2 \) and \( \beta_0 = 2 \). Let \( t_+ = 0 \) be a transition point defined in (3.28), and the function \( \zeta(t) \) be given as in (3.34) and (3.35). Then, (3.23) has a pair of linearly independent solutions

\[
P_n \left( N^\theta t \right) \sim N^{\frac{1}{2}} \left( \frac{4c^2}{4 - (\alpha_0^2 + \beta_0^2)^2} \right)^{1/2} \left[ J_{\nu}(N\zeta) \sum_{s=0}^{\infty} \frac{\tilde{A}_s(\zeta)}{N^s} + J_{\nu+1}(N\zeta) \sum_{s=0}^{\infty} \frac{\tilde{B}_s(\zeta)}{N^s} \right]
\]

and

\[
Q_n \left( N^\theta t \right) \sim N^{\frac{1}{2}} \left( \frac{4c^2}{4 - (\alpha_0^2 + \beta_0^2)^2} \right)^{1/2} \left[ W_{\nu}(N\zeta) \sum_{s=0}^{\infty} \frac{\tilde{A}_s(\zeta)}{N^s} + W_{\nu+1}(N\zeta) \sum_{s=0}^{\infty} \frac{\tilde{B}_s(\zeta)}{N^s} \right]
\]

for \(-\infty < t \leq t_- \), \( \delta \) being an arbitrary positive constant. Here, \( W_{\nu}(x) := Y_{\nu}(x) - iJ_{\nu}(x) \), \( x = N^\theta t, N = n + \tau_0, \tau_0 = -\alpha_1/(\alpha_0\theta) \) and \( \nu \) is given by

\[
\nu = \sqrt{\frac{1 + 4\beta_0^2}{(\theta - 2)^2}}.
\]

The coefficients \( \tilde{A}_s(\zeta) \) and \( \tilde{B}_s(\zeta) \) are determined successively from some recursive formulas, beginning with \( \tilde{A}_0(\zeta) = 1 \) and \( \tilde{B}_0(\zeta) = 0 \).

In the above theorem, we have used the functions \( J_{\nu}(x) \) and \( W_{\nu}(x) := Y_{\nu}(x) - iJ_{\nu}(x) \) as two linearly independent solutions of the Bessel equation. This is because \( J_{\nu}(x) \) and \( H_{\nu}^{(1)}(x) \) form a numerically satisfactory pair when \( 0 < \arg x < \pi \); see [7.4 §10.2(iii)]. Note that \( i\zeta(t) \) is negative for \( t < 0 \); thus \( J_{\nu}(N\zeta) \) is exponentially large and \( H_{\nu}^{(1)}(N\zeta) \) is exponentially small as \( N \to \infty \).

**Remark 3.1.** For the case \( \alpha_0 > 0 \) and \( \beta_0 = -2 \), the two transition points \( t_\pm \) satisfy \( t_- = 0 < t_+ \). Note that we again have one transition point at the origin and the other being positive. Put \( \mathcal{P}_n(x) := (-1)^n P_n(-x) \). Theorem 3.2 then applies to \( \mathcal{P}_n(x) \). If the two transition points satisfy \( t_- < 0 = t_+ \), we set \( x := -N^\theta t \), instead of \( x := N^\theta t \). Theorem 3.2 again holds with \( P_n \) replaced by \( \mathcal{P}_n \) (in the case \( \alpha_0 > 0, \beta_0 = 2 \)) or \( (-1)^n P_n \) (in the case \( \alpha_0 < 0, \beta_0 = -2 \)).

**Remark 3.2.** In Theorem 3.2, we assume \( \theta \neq 0, 2 \). The situation with \( \theta = 0 \) was studied in [86], and will be discussed in Case (iii) below. The result in that case holds regardless
whether the transition point is at 0 or not. \( \theta = 2 \) is exceptional in our present case (i.e., Case (ii)), since \( \theta - 2 \) appears in the denominator of (3.37). However, \( \theta = 2 \) and one of the transition point being zero is exactly the case of Wilson polynomials. For the results in that case, we refer to [57].

The presentation of Case (ii) here is based on the work of Cao and Li [13].

Case (iii) We now consider the case \( \theta = 0 \). Here, there is no need to make the scale change \( x = \nu^\theta t \) in Case (i) or \( x = N^\theta t \) in Case (ii); cf. (3.25) and (3.31). Also, we will not be concerned with whether the transition point \( t_+ (= x_+ ) \) in (3.28) is at the origin or not. However, we still need to make the assumptions \( \alpha_1 = \beta_1 = 0 \); see (3.29) and (3.33). Let

\[
\tau_0 := -\frac{\alpha_3 x_+ + \beta_3}{2(\alpha_2 x_+ + \beta_2)}, \quad N := n + \tau_0,
\]

and define

\[
\nu = \left( \alpha'_{2x_+} + \beta'_{2} + \frac{1}{4} \right)^{1/2},
\]

\[
\zeta_1^\frac{1}{2}(x) = \text{arccosh} \left( \frac{\alpha_0 x + \beta_0}{2} \right).
\]

Under the assumption in (3.29), it is easily verified that \( \alpha'_{2} = \alpha_2 \) and \( \beta'_{2} = \beta_2 \); see (3.32).

**Theorem 3.3.** Assume that the coefficients \( A_n \) and \( B_n \) in the recurrence relation (3.23) have asymptotic expansions given in (3.24) with \( \theta = 0 \). Let \( x_+ (= t_+ ) \) be the transition points defined by (3.28). Then, equation (3.23) has a pair of linearly independent solutions

\[
P_n(x) \sim \left( \frac{4 \zeta}{(\alpha_0 x + \beta_0)^2 - 4} \right)^{\frac{1}{2}} \left[ N^\frac{1}{2} I_\nu \left( N \zeta^\frac{1}{2} \right) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \right. \\
\left. + N^\frac{1}{2} \zeta^\frac{1}{2} I_{\nu - 1} \left( N \zeta^\frac{1}{2} \right) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \right],
\]

and

\[
Q_n(x) \sim \left( \frac{4 \zeta}{(\alpha_0 x + \beta_0)^2 - 4} \right)^{\frac{1}{2}} \left[ N^\frac{1}{2} K_\nu \left( N \zeta^\frac{1}{2} \right) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \\
- N^\frac{1}{2} \zeta^\frac{1}{2} K_{\nu - 1} \left( N \zeta^\frac{1}{2} \right) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \right]
\]

as \( n \to \infty \), for \( x_+ = \delta \leq x < \infty \), \( \delta \) being an arbitrary positive constant, where \( N = n + \tau_0 \), \( \tau_0 \) is given in (3.38), \( \nu \) is given in (3.39) and \( \zeta(x) \) is defined by (3.40). The coefficients \( A_s(\zeta) \) and \( B_s(\zeta) \) can be determined successively for any given \( A_0(\zeta) \) and \( B_0(\zeta) \).
4 Riemann-Hilbert approach

A significant connection between orthogonal polynomials and matrix-valued Riemann-Hilbert problems was established in the work of Fokas, Its and Kitaev [40]. Assuming that there is a sequence of polynomials \( \{p_n\} \), orthonormal with respect to the weight function \( d\alpha(\zeta) = w(\zeta)d\zeta \) supported on a curve \( \Gamma \), the formulation of Fokas, Its and Kitaev is the following Riemann-Hilbert problem for a \( 2 \times 2 \) matrix-valued function \( Y(z) \):

\[
\begin{align*}
(Y_1) \quad & Y(z) \text{ is analytic in } \mathbb{C} \setminus \Gamma. \\
(Y_2) \quad & \text{The jump condition on } \Gamma \text{ is} \\
& \quad Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix}
1 & w(\zeta) \\
0 & 1
\end{pmatrix}, \quad \zeta \in \Gamma, \\
& \quad \text{where } Y_\pm(\zeta) \text{ denote the limits of } Y(z) \text{ as } z \to \zeta \in \Gamma \text{ from the left and right of the oriented curve } \Gamma, \text{ respectively.}
\end{align*}
\]

\[
(Y_3) \quad \text{The behavior at infinity is}
\]

\[
Y(z) = \left( I + O \left( \frac{1}{z} \right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty. 
\]  

(4.2)

The significance lies in that the unique solution to the above Riemann-Hilbert problem can be determined explicitly in terms of the orthogonal polynomials, that is,

\[
Y(z) = \begin{pmatrix} \pi_n(z) & -2\pi i \gamma_n \int_{\Gamma} \frac{\pi_n(\zeta)w(\zeta)d\zeta}{\zeta - z} \\ -2\pi i \gamma_n \int_{\Gamma} \frac{\pi_n(\zeta)w(\zeta)d\zeta}{\zeta - z} & \pi_n(z) - \gamma_n \int_{\Gamma} \frac{\pi_{n-1}(\zeta)w(\zeta)d\zeta}{\zeta - z} \end{pmatrix}, \quad z \in \mathbb{C} \setminus \Gamma,
\]

(4.3)

where \( \pi_n(z) \) is the monic polynomial corresponding to \( p_n(z) \), and \( \gamma_n \) is the leading coefficient of \( p_n(z) \) such that \( p_n(z) = \gamma_n \pi_n(z) \). The uniqueness of the solution \( Y(z) \) can be justified by using Liouville’s theorem, with asymptotic behavior appropriately preassigned to each critical point such as an endpoint or a possible point of intersection of \( \Gamma \), etc.

To extract asymptotics from (4.3), we may use a steepest descent type method for oscillatory Riemann-Hilbert problems introduced by Deift and Zhou [30]; see Deift et al. [29] for a brief historical account of how this steepest descent method, also termed the Deift-Zhou method or Riemann-Hilbert approach, was further developed. This new approach to study asymptotic questions for orthogonal polynomials is rigorous, essentially global, and has proved to be very powerful. The pioneering works in which the nonlinear steepest descent method was actually applied to orthogonal polynomials are by Deift et al. [27] on orthogonal polynomials with varying exponential weight, with an application to a rigorous proof of the universality in random matrix theory, and by Bleher and Its [9] on semi-classical asymptotics with applications in random matrix theory; see also [28].
There are many more publications afterwards. Here we mention only a very few. In [52], Kuijlaars et al. studied the polynomials orthogonal with respect to the modified Jacobi weight. This provides a situation where one seems to have no choice but to use the Riemann-Hilbert approach, since the orthogonal measure is the only property known. Discrete orthogonal polynomials are investigated in Johansson [46] and Baik et al. [3]. The orthogonal polynomials with respect to Freud weights are considered in Kriecherbauer and McLaughlin [50]. Wong and his coauthors have made attempts to bring in a global treatment to the Riemann-Hilbert approach; see [89] and a brief account in [104], in which all zeros of the polynomials belong to one single domain of uniformity.

There have been excellent introductions to the Riemann-Hilbert approach. The reader is referred to the books of Deift [26] and Fokas et al. [39], and the paper of Kuijlaars [51], for a better understanding of this method; see also [104] and [110] for some more recent progress.

The asymptotic analysis of the Riemann-Hilbert problem consists of the following successive transformations, starting from the Riemann-Hilbert problem for $Y$.

(i) The first transformation, if applicable, is a re-scaling of the variable to make the equilibrium measure compactly supported.

(ii) The second transformation is a normalization of the behavior at infinity, involving a function $g$ which is the log transformation of the equilibrium measure, and turning the problem into an oscillatory Riemann-Hilbert problem.

(iii) The next transformation is the central piece of the analysis, which involves a factorization of the jump matrix and a deformation of the contour; oscillatory terms are transformed into exponentially decaying terms and may be neglected along the deformed contour, which reminds us of Debye’s classical method of steepest descent for integrals.

(iv) The last transformation combines together the parametrices at all critical points such as the endpoints of the contour, and edges of the support of the equilibrium measure. In many circumstances technical difficulties may arise in the construction of such a parametrix, that is, an explicitly given local solution to the modified Riemann-Hilbert problem in a neighborhood of the critical point.

Now we distinguish two types of parametrices. For the first type, an example is the frequently used parametrix involving the Airy functions ([26, 41, 83]), usually constructed in a neighborhood of the soft edge of the spectrum. An application of it to the random matrix theory is to describe the soft edge behavior by using the Airy kernel

$$A(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}.$$ 

While the Bessel kernel,

$$J_\alpha(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J_\alpha'(\sqrt{x})}{2(x - y)},$$ 

22
resulted from a Bessel parametrix, is often used to characterize the behavior at the hard edge of the spectrum; cf. [41, 52, 84]. The list of these parametrices is much longer, including special functions such as the confluent hypergeometric functions, the parabolic cylinder functions, the Painlevé functions, and more general Painlevé-type functions. There is a quite vast literature on these more transcendental parametrices; see, e.g., [39], [106] and [107] and the references therein.

A common feature of the above parametrices, is that the special function involved satisfies a certain differential equation, linear or nonlinear. The reason for this phenomenon is that each of these parametrices can be transformed, by making use of a certain local conformal mapping, into a model Riemann-Hilbert problem, say $P(\zeta)$, with constant jump.

In the constant jump case, $P(\zeta)$ and $P'(\zeta)$ share the same jump condition. Hence all jumps vanish for $A(\zeta) := P'(\zeta)/P(\zeta)^{-1}$. Then $A(\zeta)$ is analytic in $\mathbb{C}$, with only isolated singularities of known types. This would give a differential equation $P'(\zeta) = A(\zeta)P(\zeta)$.

However, there are still parametrices of another type, in which the model problems could not be turned into the ones with constant jumps. In what follows, we briefly describe three examples of this type. These examples deal with polynomials orthogonal with respect to a logarithmic weight, a Bessel-function weight, and the Freud weight, separately.

Example 4.1. Logarithmic weight. In 2018, Conway and Deift [20] derived the asymptotics of the recurrence coefficients for the polynomials orthogonal with respect to the logarithmic weight $w(x)dx$, where

$$w(x) = \ln \frac{2k}{1-x}, \quad x \in [-1, 1], \quad (4.4)$$

and $k > 1$ so that the measure is strictly positive on its support.

The formulation in (4.1) and (4.2) fits well with $\Gamma = [-1, 1]$, the weight $w$ specified in (4.4), and $Y(z)$ having at most weak singularities at the endpoints $z = \pm 1$. Following a now standard series of transformations, we change the original Riemann-Hilbert problem for $Y$ into a modified problem for $Q$, of which the jumps on curves adjacent to $z = 1$ are

$$J_Q(s) = \begin{cases} \left( \begin{array}{cc} 1 & 0 \\ w^2 \phi^{-2n} & 1 \end{array} \right), & s \in \Sigma_1 \cup \Sigma_2, \\
\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), & s \in (-1, 1), \\
\left( \begin{array}{cc} 1 & 0 \\ w^2 + w^2 \phi^{-2n} & 1 \end{array} \right), & s \in (1, +\infty); \end{cases} \quad (4.5)$$

cf. [20]. Here, $(\ )_\pm$ denote the boundary values (cf. (4.1)), all curves are in a rightward direction, $\Sigma_1$ and $\Sigma_2$ are respectively the upper and lower boundaries of the lens opening.
from \((-1, 1), w(z) = \ln \frac{2i}{z-1}\) is analytic in \(\mathbb{C} \setminus [1, +\infty)\), the mapping \(\phi(z)\) and the Szegő function \(F(z)\) (cf. [52]) defined by

\[
\phi(z) = z + \sqrt{z^2 - 1}, \quad F(z) = \exp \left( \frac{(z^2 - 1)^{1/2}}{2\pi i} \int_{-1}^{1} \frac{\ln w(s)}{(s^2 - 1)^{1/2}} \frac{ds}{s - z} \right)
\]

are analytic in \(\mathbb{C} \setminus [-1, 1]\). The new difficulty is that no explicit local solution is known near the logarithmic singularity \(x = 1\) of the weight \((4.4)\), since the jump matrices in \((4.5)\) are unlikely to be simultaneously turned into constant ones.

Fortunately, for the above problem with logarithmic weight, no local solution is needed. The strategy in [20] is to compare the above modified Riemann-Hilbert problem with the corresponding modified Riemann-Hilbert problem for the Legendre case. The comparison is not to examine the quotient, but to estimate the difference of solutions to these two Riemann-Hilbert problems. The estimates given in [20] seem to allow an effective comparison of two Riemann-Hilbert problems on the same contour in the general case.

**Example 4.2. Bessel function weight.** In 2016, Deaño, Kuijlaars and Román [25] investigated the asymptotics of the polynomials \(P_n(z)\) that are orthogonal with respect to the weight function \(J_{\nu}(x)\) on \([0, \infty)\), where \(J_{\nu}\) is the Bessel function of order \(\nu \geq 0\). The Bessel function is oscillatory with an amplitude that decays like \(O(x^{-1/2})\) as \(x \to +\infty\), and therefore the moments \(\int_0^\infty x^j J_{\nu}(x)dx\) do not exist. However, the polynomials \(P_n\) can be defined via a regularization of the weight with an exponential factor; see Asheim and Huybrechs [1].

\[
\int_0^\infty P_n(x; s)x^j J_{\nu}(x)e^{-sx}dx = 0, \quad j = 0, 1, \cdots, n - 1, \quad (4.6)
\]

and then

\[
P_n(x) = \lim_{s \to 0^+} P_n(x; s). \quad (4.7)
\]

From numerical experiments, Asheim and Huybrechs [1] observe that the zeros of \(P_n(z)\) seem to cluster along the vertical line \(\text{Re } z = \nu \pi / 2\). When \(0 \leq \nu \leq 1/2\), Deaño, Kuijlaars and Román [25] derive large-\(n\) asymptotic behavior of \(P_n(in\pi z)\) by using the Riemann-Hilbert approach, and give a rigorous proof of the observation in this case. More precisely, they have shown that for the scaled zeros, that is, dividing the imaginary parts of the zeros by \(n\) while keeping the real parts fixed, the limiting curve is a vertical line segment of \(\text{Re } z = \nu \pi / 2\). They further conjecture, based also on numerical evidence, that there is a limiting curve for the scaled zeros that differs from the vertical line segment when \(\nu > 1/2\), partially because in this case the method in [25] to construct a local parametrix at the origin fails: this difficulty may very well be related to the different behavior of the zeros. Yet a very recent research [109] shows that for \(\nu > 1/2\), the limiting curve for the scaled zeros is still a vertical segment of \(\text{Re } z = \nu \pi / 2\).

The most technical part of [25] (3.50)-(3.51), Prop. 3.16] is the analysis of the local parametrix at the origin, with jump matrices involving the Bessel functions in a complicated way. Since by no means one can transform the jumps into constant matrices,
an integral operator argument was brought in to show that the modified local Riemann-Hilbert problem has a solution for \( n \) large enough. The leading order behavior of the parametrix has not yet been explicitly given, though it does not affect the determination of zero distribution.

In 1982, Wong \[96\] considered quadrature formulas for oscillatory integral transforms with Fourier kernel and Bessel kernels. Also, in a paper \[22\], Dai, Hu and Wang have proposed a study of the asymptotics of orthogonal polynomials whose weight function is closely related to the Bessel weight stated above.

**Example 4.3. Freud weight.** In 1999, Kriecherbauer and McLaughlin \[50\] used the Riemann-Hilbert approach to study the strong asymptotics of the polynomials orthogonal with respect to the Freud weight

\[
 w_\beta(x)dx = e^{-\kappa_\beta|x|^\beta}dx, \quad x \in \mathbb{R}, \quad \kappa_\beta = \frac{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}\right)}, \quad \beta > 0. \quad (4.8)
\]

Although the Riemann-Hilbert approach works, an obstacle arises, that is, to find the explicit local solution in a neighborhood of the origin.

When \( 0 < \beta < 1 \), the problem is reduced to solving a model Riemann-Hilbert problem for a certain \( 2 \times 2 \) matrix-valued function \( L : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}^{2 \times 2} \) is analytic,

\[
 L_+(s) = L_-(s)\nu_L(s) \quad \text{for} \quad s \in \mathbb{R}, \quad (4.9)
\]

\[
 L(s) = O(\ln|s|) \quad \text{as} \quad s \to 0, \quad (4.10)
\]

\[
 L(s) \to I \quad \text{for} \quad s \to \infty, \quad (4.11)
\]

where the jump matrix is given by

\[
 \nu_L(s) = \begin{pmatrix} 1 & -\eta_L(s) \\ \eta_L(-s) & 1 \end{pmatrix}, \quad s \in \mathbb{R},
\]

with

\[
 \eta_L(s) = 2i(-1)^{n+1}e^{-|s|^\beta\sin(\pi\beta/2)}\sin\left(|s|^{\beta}\cos\frac{\pi\beta}{2}\right)1_{[0,\infty)}(s), \quad s \in \mathbb{R},
\]

\( 1_{[0,\infty)}(s) \) denoting the indicator function of the set \([0,\infty)\), and \( n \) being the polynomial degree; see \[50\] (6.41)-(6.43).

In \[50\], the existence and uniqueness of \( L(s) \) were established in the Wiener class. However, as commented in Deift et al. \[29\], p. 62, Rmk. 3.6,

For \( 0 < \beta < 1 \) the leading order behavior of the solution to the model problem at the origin has not been determined explicitly. It is defined through a Riemann-Hilbert problem which only depends on \( \beta \) and on the parity of \( n \).

The reader is referred to \[50\] and \[105\] and the references there for polynomials orthogonal with respect to weights similar to the Freud weight. For example, Chen and Ismail \[17\] considered the Freud-like orthogonal polynomials arising from a recurrence
relation related to the indeterminate moment problems; see also Dai, Ismail and Wang \cite{23}. In fact, the polynomials orthogonal with respect to the Bessel weight in \cite{25}; cf. (4.6) and (4.7), provides another example of Freud-type weights. Expressing $J_\nu(x)$ as a linear combination of $K_\nu(\pm ix)$, one encounters a varying exponential weight of Freud-type $e^{-n|x|}$ with a potential function $\pi|x|$. The jump (4.9) with constraint (4.11), cannot be transformed into a constant jump. Further, to solve the model problem, one may either represent them in terms of known special functions, or alternatively use new special functions to serve the same purpose.

It is well known that the Riemann-Hilbert problems are closely related to singular integral equations; see, e.g., Muskhelishvili \cite{64}. Assume
\[ Y(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi(\zeta)d\zeta}{\zeta - z}, \quad z \notin \Gamma, \tag{4.12} \]
where $\Phi(\zeta)$ for $\zeta \in \Gamma$ is the new matrix-valued unknown function. From the Plemelj formula we have
\[ Y_\pm(\zeta) = \pm \frac{1}{2} \Phi(\zeta) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi(\tau)d\tau}{\tau - \zeta}, \quad \zeta \in \Gamma, \]
where the bar indicates that the integral is a Cauchy principal value; cf. (5.41). Substituting the above formulas into the jump condition (4.1), we obtain a homogeneous singular integral equation
\[ \Phi(\zeta) - \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi(\tau)d\tau}{\tau - \zeta} \begin{pmatrix} 0 & w(\zeta) \\ 0 & 0 \end{pmatrix} = 0 \tag{4.13} \]
with index 0. Finding asymptotic results from (4.13) for specific $\Gamma$ or more general integral equations is an important topic worth exploring.

The rest of the present section will be devoted to the discussion of special functions determined by integral equations. Returning to Example 4.3, we examine the model problem for $L$. To facilitate our discussion, instead of the behavior (4.10) of $L(s)$ at the origin, we assume
\[ L(s) = O(\epsilon_\beta(s)) \quad \text{as} \quad s \to 0, \quad \epsilon_\beta(s) := \begin{cases} 1, & 1/2 < \beta < 1, \\ \ln|s|, & \beta = 1/2, \\ |s|^{1/2}, & 0 < \beta < 1/2. \end{cases} \tag{4.14} \]

The objective is to introduce a pair of special functions, $u_\beta$ and $v_\beta$, as solutions of two scalar linear integral equations. Based on these functions, the solution to the model problem $L$ of Kriecherbauer and McLaughlin \cite{50} can readily be constructed. More precisely, we assume that $u_\beta$ and $v_\beta$ solve, respectively, the following integral equations
\[ u(x) = 1 + \int_0^\infty \frac{K(t)u(t)dt}{t + x}, \quad x \in (0, \infty), \tag{4.15} \]
\[ v(x) = 1 - \int_0^\infty \frac{K(t)v(t)dt}{t + x}, \quad x \in (0, \infty), \tag{4.16} \]
where
\[
K(t) = \frac{1}{2\pi i} \left[ \exp \left( e^{\left( \frac{\pi}{2} + \frac{i\pi}{2} \right)t^2} \right) - \exp \left( e^{-\left( \frac{\pi}{2} + \frac{i\pi}{2} \right)t^2} \right) \right] \quad (4.17)
\]
\[
= \frac{1}{\pi} \exp \left( -t^2 \sin \frac{\pi \beta}{2} \right) \sin \left( t^2 \cos \frac{\pi \beta}{2} \right). \quad (4.18)
\]

Note that \( \eta_L(t) = 2\pi i(-1)^{n+1}K(t) \) for \( t \in [0, +\infty) \).

First, we formally derive the integral equations from the model problem for \( L \) satisfying (4.9), (4.11) and (4.14). From the jump condition (4.9) and the behavior in (4.11) and (4.14), it is readily seen that the \((1, 1)\)-entry \( L_{11}(s) \) is analytic in \( \mathbb{C} \setminus (-\infty, 0] \), such that
\[
(L_{11})_+ (s) - (L_{11})_- (s) = \eta_L(\tau)L_{12}(s), \ s \in (-\infty, 0),
\]
\[
L_{11}(s) = 1 + o(1), \ s \to \infty,
\]
\[
L_{11}(s) = O(\epsilon_\beta(s)), \ s \to 0.
\]

Hence, in view of the behavior of \( L_{12} \) obtained from (4.11) and (4.14), we have from the Plemelj formula
\[
L_{11}(s) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{\eta_L(\tau)L_{12}(\tau)d\tau}{\tau - s} = 1 + \frac{1}{2\pi i} \int_{0}^{\infty} \frac{\eta_L(\tau)L_{12}(\tau)d\tau}{-\tau - s} \quad (4.19)
\]
for \( s \in \mathbb{C} \setminus (-\infty, 0] \), especially for \( s \in (0, \infty) \). Similarly, it is also seen from (4.9) that \( L_{12}(s) \) is analytic in \( \mathbb{C} \setminus [0, \infty) \), and solves the scalar Riemann-Hilbert problem
\[
(L_{12})_+ (s) - (L_{12})_- (s) = -\eta_L(s)L_{11}(s), \ s \in (0, \infty),
\]
\[
L_{12}(s) = o(1), \ s \to \infty,
\]
\[
L_{12}(s) = O(\epsilon_\beta(s)), \ s \to 0.
\]

Hence, we have
\[
L_{12}(s) = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{-\eta_L(\tau)L_{11}(\tau)d\tau}{\tau - s} \quad \text{for} \ s \in \mathbb{C} \setminus [0, \infty), \quad (4.20)
\]
especially for \( s \in (-\infty, 0) \). For \( x \in (0, +\infty) \), now we define
\[
u_\beta(x) = L_{11}(x) + (-1)^nL_{12}(-x), \quad v_\beta(x) = L_{11}(x) - (-1)^nL_{12}(-x), \quad (4.21)
\]
n being the integer appearing in the definition of \( \eta_L(s) \). From (4.19)-(4.20) and (4.21), it is readily verified that \( u_\beta \) and \( v_\beta \) solve the integral equations (4.15) and (4.16), respectively.

It is worth noting that similar coupled scalar integral equations have been derived in [25], although the equations in that paper were used only to construct a contraction mapping. Also, it is mentioned in Fokas et al. [39, p. 161] that a function \( u(x) \) can be
parametrized via the solution of a linear integral equation. This is exactly what we are doing: We are rigorously defining a pair of new special functions, using the above integral equations, to construct a parametrix.

We now outline the investigation carried out in Wong and Zhao [105]. To begin with, we have

**Lemma 4.1.** The operator $T$ is a compact operator on $L^2(0, +\infty)$, where

$$(Tu)(x) = \int_0^\infty \frac{K(t)u(t)dt}{t + x}.$$  \hfill (4.22)

The reason is that $\frac{K(t)}{t+x} \in L^2([0, +\infty) \times [0, +\infty))$, with $K(t)$ given by (4.18). Thus, $T$ is a Hilbert-Schmidt operator, and hence is compact; cf. e.g. [108, p. 277].

The next step is to show that the operators $I \pm T$ has trivial null space by using a vanishing lemma technique. To this aim, we need to know more about the analytic structure of the solutions to (4.15) and (4.16). It is readily seen that $(Tu)(z)$ in (4.22) is a Stieltjes transform, and is hence analytic in the cut plane $|\arg z| < \pi$. For later use, we need the following behavior of $(Tu)(z)$ at the origin:

**Lemma 4.2.** For $u \in L^2(0, +\infty)$, we have

$$(Tu)(z) = O(\epsilon_\beta(z)) \quad \text{for} \quad |\arg z| \leq \pi \quad \text{as} \quad z \to 0,$$

where $\epsilon_\beta(z)$ is given by (4.14).

A description of the null space is as follows.

**Lemma 4.3.** Assume that $u \in L^2(0, +\infty)$ and $u - Tu = 0$ (or $u + Tu = 0$) in $L^2(0, +\infty)$. Then, $u \equiv 0$ for $x \in [0, +\infty)$ in $L^2(0, +\infty)$.

To conclude, we have the unique existence of $u_\beta(x)$ and $v_\beta(x)$ in $L^2$ sense.

**Theorem 4.1.** (Wong-Zhao [105]) There exist unique solutions $u(x)$ and $v(x)$ to the integral equations (4.15) and (4.16), respectively, such that $u(x) - 1 \in L^2(0, +\infty)$ and $v(x) - 1 \in L^2(0, +\infty)$.

Further analysis of the integral equations (4.15) and (4.16) show that $u_\beta(x)$ and $v_\beta(x)$ are in fact bounded for all $\beta \in (0, 1)$.

**Theorem 4.2.** (Wong-Zhao [105]) Let $u_\beta(x)$ and $v_\beta(x)$ be solutions of the integral equations in (4.15) and (4.16), respectively. Then

$$u_\beta(x), v_\beta(x) \in L^\infty[0, +\infty) \quad \text{and} \quad u_\beta(x), v_\beta(x) \in C[0, +\infty).$$

Since $u_\beta(x)$ and $v_\beta(x)$ solve the integral equations (4.15) and (4.16), with $u_\beta - 1, v_\beta - 1 \in L^2(0, +\infty)$, we can deduce that $u_\beta(z), v_\beta(z) = 1 + O(1/z)$ as $z \to \infty$ for $|\arg z| < 3\pi/2$ and are of the order $O(\epsilon_\beta(z))$ as $z \to 0$ for $|\arg z| \leq \pi$. Thus we have
Theorem 4.3. (Wong-Zhao [105]) The piece-wise analytic function

\[
L(s) = \begin{cases} 
(L_\beta(s) \quad (-1)^n U_\beta(se^{-\pi i}) 
\left( -1 \right)^n U_\beta(se^{-\pi i}) 
\right), & 0 < \arg s < \pi; \\
(L_\beta(s) \quad (-1)^n U_\beta(se^{\pi i}) 
\left( -1 \right)^n U_\beta(se^{\pi i}) 
\right), & -\pi < \arg s < 0
\end{cases}
\]

(4.23)

solves the Riemann-Hilbert problem (4.9), (4.11) and (4.14) for \( L \), where

\[
L_\beta(z) = \frac{1}{2} (u_\beta(z) + v_\beta(z)), \quad U_\beta(z) = \frac{1}{2} (u_\beta(z) - v_\beta(z))
\]

(4.24)

for \( \arg z \in (-\infty, \infty) \), \( z \neq 0 \).

Regarding \( u_\beta(z) \) and \( v_\beta(z) \) as a pair of special functions, a very preliminary analysis has been carried out in [105]. For example, for analytic continuation, applying the Plemelj formula to the integral equations (4.15)-(4.16), we have

\[
u(z e^{-\pi i}) - u(z e^{\pi i}) = 2\pi i K(z) u(z), \quad v(z e^{-\pi i}) - v(z e^{\pi i}) = -2\pi i K(z) v(z),
\]

(4.25)

initially for real and positive \( z \), and then for complex \( z \) with \( \arg z \in (-\infty, \infty) \); cf. (4.17).

Also, treating the integral equations as Stieltjes transform and applying Theorem 4.1, we readily obtain the asymptotic approximations

\[
u_\beta(z) \sim 1 + \sum_{k=1}^{\infty} c_k z^k \quad \text{and} \quad v_\beta(z) \sim 1 + \sum_{k=1}^{\infty} d_k z^k \quad \text{as} \quad z \to +\infty, \quad |\arg z| < \frac{3\pi}{2},
\]

(4.26)

where

\[
c_k = (-1)^{k-1} \int_{0}^{\infty} K(t) t^{k-1} u_\beta(t) dt \quad \text{and} \quad d_k = (-1)^{k} \int_{0}^{\infty} K(t) t^{k-1} v_\beta(t) dt
\]

(4.27)

for \( k = 1, 2, \ldots \).

The Stokes phenomenon is also addressed in [105]. Assume that the functions \( u_\beta(z) \) and \( v_\beta(z) \) solve respectively (4.15) and (4.16), such that \( u_\beta - 1, v_\beta - 1 \in L^2[0, \infty) \). Then the Stokes lines for both functions are

\[
\arg z = \pm \left\{ \frac{\pi}{2}(3 - \alpha) + 2(l - 1)\pi \right\},
\]

where \( \alpha = \frac{1}{\beta} \in (4l - 3, 4l + 1], \ l = 1, 2, \ldots \).

Much remains to be done. We propose a thorough investigation of the analytic and asymptotic properties of the functions \( u_\beta \) and \( v_\beta \), such as their zeros, modulus function [74, §10.18], kernel [53], and differential or difference equations, etc. Here, we list a few.

(i) Determine the coefficients \( c_k \) and \( d_k \) in the expansions at infinity given by (4.26)-(4.27), which are now expressed in terms of \( u_\beta \) and \( v_\beta \). It would be interesting and challenging to decode the coefficients from (4.15) and (4.16) in an explicit manner.
As a refinement of Theorem 4.2, one might reasonably expect that the behaviors of \( u_\beta \) and \( v_\beta \) at the origin are of the form \( \sum_{k=0}^{\infty} \tilde{c}_k z^k + \sum_{k=1}^{\infty} \tilde{d}_k z^{\beta k} \). A natural problem is to determine the small-\( z \) asymptotic formula and to evaluate explicitly the coefficients \( \tilde{c}_k \) and \( \tilde{d}_k \), at least the leading coefficients such as \( k = 0 \) and \( k = 1 \).

The eigenvalues of the compact operator \( T \) given by (4.22), that is, find \( \lambda \) such that the null space of \( \lambda I - T \) is nontrivial in \( L^2[0, +\infty) \).

It has been conjectured in [105] that the function \( u_\beta(x) \) is monotonically decreasing in \( x \in [0, \infty) \), and also that the initial value \( u_\beta(0) \) is decreasing in \( \beta \in (0, 1) \). Corresponding conjectures can be made for the function \( v_\beta(x) \), such as it is increasing in \( x \in [0, \infty) \), and also that the initial value \( v_\beta(0) \) is increasing in \( \beta \in (0, 1) \). It is also of interest to consider the complete monotonicity of these special functions.

5 Wiener-Hopf equation

As far as we are aware, the first paper that addresses asymptotic solutions to an integral equation of the form in (1.1) is by Muki and Sternberg [63] in 1970. But, their approach assumes that the solution behaves like \( t^{-\delta}/M \) as \( t \to +\infty \), where \( \delta \) is a positive number and \( M \) is a constant. An attempt was made by Li and Wong [59] in 1994 to remove such an assumption and to derive the asymptotic solution directly from the integral equation, but their attempt was not successful. Later, it was realized that equation (1.1) is known as of Wiener-Hopf type. This naturally led Li and Wong [58] to use the famous Wiener-Hopf technique [92]. Here we first give an outline of this technique, leading to an integral representation of the solution. For convenience we reproduce (1.1) below:

\[
 u(t) = f(t) + \int_0^{\infty} k(t - \tau)u(\tau)d\tau, \quad t > 0. \tag{5.1}
\]

The following notations will be used throughout this section:

\[
 u_+(t) = \begin{cases} 
 u(t), & t > 0, \\
 0, & t < 0
\end{cases} \tag{5.2}
\]

and

\[
 u_-(t) = \begin{cases} 
 0, & t > 0, \\
 f(t) + \int_0^{\infty} k(t - \tau)u(\tau)d\tau, & t < 0
\end{cases} \tag{5.3}
\]

Since we are only concerned with \( t > 0 \) in equation (5.1), in most cases the function \( f(t) \) is identically zero for \( t \) negative. Equation (5.1) can now be written as

\[
 u_+(t) + u_-(t) = f(t) + \int_0^{\infty} k(t - \tau)u_+(\tau)d\tau, \quad -\infty < t < \infty. \tag{5.4}
\]

Taking Fourier transforms on both sides of this equation gives

\[
 \left[ 1 - \sqrt{2\pi}K(\lambda) \right] U_+(\lambda) = F(\lambda) - U_-(\lambda), \tag{5.5}
\]
where
\[ K(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(t)e^{i\lambda t} dt, \quad F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\lambda t} dt, \]
\[ U_{\pm}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{\pm}(t)e^{i\lambda t} dt, \quad (5.6) \]
and \( \lambda = \sigma + i\tau \). In (5.5), we note that \( U_{+}(\lambda) \) and \( U_{-}(\lambda) \) are unknown functions, and that \( K(\lambda) \) and \( F(\lambda) \) are known functions. To solve this functional equation involves a factorization of the function \( 1 - \sqrt{2\pi}K(\lambda) \), which is the crux of the idea in the Wiener-Hopf technique.

Let \( L(\lambda) \) be an analytic function of \( \lambda = \sigma + i\tau \) in the strip \( \tau_{-} < \tau < \tau_{+} \) such that \( |L(\sigma + i\tau)| \leq C|\sigma|^{-p} \), \( p > 0 \) and \( C > 0 \), as \( |\sigma| \to \infty \). This inequality holds uniformly for \( \tau \) in the strip \( \tau_{-} + \varepsilon \leq \tau \leq \tau_{+} - \varepsilon \), \( \varepsilon > 0 \). For \( \tau_{-} < c < \tau < d < \tau_{+} \), we have by Cauchy’s theorem
\[ L(\lambda) = \frac{1}{2\pi i} \int_{C} \frac{L(\zeta)}{\zeta - \lambda} d\zeta, \]
where \( C \), a positively oriented contour, is the boundary of the rectangle whose four corners are at \( -R + ic \), \( R + ic \), \( R + id \) and \( -R + id \). Letting \( R \to +\infty \) yields the decomposition
\[ L(\lambda) = L_{+}(\lambda) - L_{-}(\lambda), \quad (5.7) \]
where
\[ L_{+}(\lambda) = \frac{1}{2\pi i} \int_{-\infty+ic}^{+\infty+ic} \frac{L(\zeta)}{\zeta - \lambda} d\zeta, \quad L_{-}(\lambda) = \frac{1}{2\pi i} \int_{-\infty+id}^{+\infty+id} \frac{L(\zeta)}{\zeta - \lambda} d\zeta, \quad (5.8) \]
and \( \tau_{-} < c < d < \tau_{+} \). It is easily seen that \( L_{+}(\lambda) \) is analytic for all \( \tau > c \) and \( L_{-}(\lambda) \) is analytic for all \( \tau < d \).

If \( \ln K(\lambda) \) satisfies the conditions imposed on the function \( L(\lambda) \) above, then we immediately have
\[ K(\lambda) = \frac{K_{+}(\lambda)}{K_{-}(\lambda)}, \quad (5.9) \]
where
\[ K_{+}(\lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty+ic}^{+\infty+ic} \frac{\ln K(\zeta)}{\zeta - \lambda} d\zeta \right\}, \quad (5.10) \]
and
\[ K_{-}(\lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty+id}^{+\infty+id} \frac{\ln K(\zeta)}{\zeta - \lambda} d\zeta \right\}. \quad (5.11) \]
It is not difficult to see that \( K_{+}(\lambda) \) and \( K_{-}(\lambda) \) are analytic, bounded and nonzero in the half-planes \( \tau > \tau_{-} \) and \( \tau < \tau_{+} \), respectively.

Returning to (5.5), we assume that \( 1 - \sqrt{2\pi}K(\lambda) \) has zeros \( a_{1}, \ldots, a_{m} \). As in (5.9), we can find \( K_{+}(\lambda) \) and \( K_{-}(\lambda) \), which are analytic and zero-free in the respective half-plane \( \tau > \tau_{-} \) and \( \tau < \tau_{+} \), such that
\[ 1 - \sqrt{2\pi}K(\lambda) = \frac{K_{+}(\lambda)}{K_{-}(\lambda)} (\lambda - a_{1})(\lambda - a_{2}) \cdots (\lambda - a_{m}). \quad (5.12) \]
Equation (5.55) may now be written as

\[ U_+(\lambda)K_+(\lambda)(\lambda - a_1)(\lambda - a_2) \cdots (\lambda - a_m) = K_-(-\lambda)F(\lambda) - K_-(-\lambda)U_-(\lambda). \]  \hspace{1cm} (5.13)

As in (5.7), the term \( K_-(-\lambda)F(\lambda) \) can be decomposed into the form

\[ K_-(-\lambda)F(\lambda) = C_+(\lambda) + C_-(-\lambda), \]  \hspace{1cm} (5.14)

where \( C_+(\lambda) \) and \( C_-(-\lambda) \) are analytic in \( \tau > \tau_- \) and \( \tau < \tau_+ \), respectively. With (5.14), we rearrange (5.13) so that we can define a function \( E(\lambda) \) by

\[ E(\lambda) = U_+(\lambda)K_+(\lambda)(\lambda - a_1)(\lambda - a_2) \cdots (\lambda - a_m) - C_+(\lambda) = C_-(\lambda) - K_-(-\lambda)U_-(\lambda). \]  \hspace{1cm} (5.15)

This equation defines \( E(\lambda) \) only in the strip \( \tau_- < \tau < \tau_+ \). But, by the first equality \( E(\lambda) \) is actually defined and analytic in \( \tau > \tau_- \), and by the second equality \( E(\lambda) \) is defined and analytic in \( \tau < \tau_+ \). Hence, using analytic continuation, we can define \( E(\lambda) \) in the whole \( \lambda \)-plane. Suppose that it can be shown

\[ |U_+(\lambda)K_+(\lambda)(\lambda - a_1)(\lambda - a_2) \cdots (\lambda - a_m) - C_+(\lambda)| \leq |\lambda|^p, \quad \text{as} \ \lambda \to \infty \ \text{in} \ \tau > \tau_-; \]

\[ |C_-(\lambda) - K_-(-\lambda)U_-(\lambda)| \leq |\lambda|^q, \quad \text{as} \ \lambda \to \infty \ \text{in} \ \tau < \tau_+. \]

Then, by the extended form of the Liouville theorem, \( E(\lambda) \) is a polynomial \( P(\lambda) \) of degree less than or equal to the integer part of \( \min\{p, q\} \), i.e.,

\[ U_+(\lambda)K_+(\lambda)(\lambda - a_1)(\lambda - a_2) \cdots (\lambda - a_m) - C_+(\lambda) = P(\lambda), \]

\[ C_-(\lambda) - K_-(-\lambda)U_-(\lambda) = P(\lambda). \]  \hspace{1cm} (5.16)

To determine the degrees \( p \) and \( q \), we need to know the behaviors of \( U_+(\lambda) \) and \( U_-(\lambda) \). In special cases, these can be found from the behavior of \( u(t) \). For example, if \( u(t) \sim t^{-1/2} \) as \( t \to 0^+ \), then by the Abelian theorem \( U_+(\lambda) \sim \lambda^{1/2} \) as \( \lambda \to \infty \) in the upper half-plane \( \tau > 0 \); see [65] p. 36 and p. 69]. In equations in (5.16), \( U_+(\lambda) \) and \( U_-(\lambda) \) are determined only within an arbitrary polynomial \( P(\lambda) \); that is, within a finite number of arbitrary constants, which must be determined by some other methods.

By using the analytic properties of the function \( U_+(\lambda) \) and Cauchy’s theorem, we first derive the integral representation

\[ U_+(\lambda) = \frac{1}{2\pi i} \int_{-\infty + i\alpha}^{\infty + i\alpha} \frac{U_+(\zeta)}{\zeta - \lambda} d\zeta, \]  \hspace{1cm} (5.17)

where \( \alpha \) is a real number bigger than \( \tau_- \) and \( \lambda \) is any point in the upper half-plane \( \tau > \alpha \). The denominator of the above Cauchy integral can be written as

\[ \frac{1}{i(\zeta - \lambda)} = \int_0^\infty e^{-i(\zeta - \lambda)t} dt, \]  \hspace{1cm} (5.18)
where $\lambda$ is any point in the half-plane $\tau > \alpha$. Substituting (5.18) into (5.17) and interchanging the order of integration give

$$U_+(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{it\lambda} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty+ia}^{\infty+ia} U_+(\zeta)e^{-it\zeta}d\zeta \right] dt.$$  

Comparing the integral on the right side of this equation with the Fourier transforms given in (5.6), we have

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+ia}^{\infty+ia} U_+(\zeta)e^{-it\zeta}d\zeta, \quad t > 0. \quad (5.19)$$

On account of (5.16), the solution of (5.1) has the integral representation

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+ia}^{\infty+ia} \frac{P(\lambda) + C_+(\lambda)}{K_+(\lambda)(\lambda - a_1)(\lambda - a_2)\cdots(\lambda - a_m)} e^{-it\lambda}d\lambda, \quad t > 0, \quad (5.20)$$

for any $\alpha$, $\tau_- < \alpha < \tau_+$. A major assumption in the preceding analysis is that the kernel function $k(t)$ in equation (5.1) is exponentially decaying at infinity. This enables one to factor

$$\frac{1 - \sqrt{2\pi} K(\lambda)}{(\lambda - a_1)(\lambda - a_2)\cdots(\lambda - a_m)}$$

into two components $K_+(\lambda)$ and $K_-(\lambda)$, one analytic in an upper half-plane, the other analytic in a lower half-plane, and the two half-planes overlapping in an infinite strip parallel to the real axis.

If the kernel is not exponentially decaying, e.g., the Cauchy density function

$$k(t) = \frac{1}{\pi(1 + t^2)},$$

then one would not be able to decompose $\frac{1 - \sqrt{2\pi} K(\lambda)}{(\lambda - a_1)(\lambda - a_2)\cdots(\lambda - a_m)}$ into two factors analytic in two overlapping half-planes. A natural way to extend the method presented above is to introduce an exponential function, such as $e^{-\varepsilon|t|}$, in the kernel. This idea has been used by Carlson and Heins [15] and by Carrier [16], but the arguments in both papers are only formal. Rigorous justifications were given by Widom [91] for the case $k \in L^2(\mathbb{R})$, and by Krein for $k \in L^1(\mathbb{R})$. (For a brief introduction to the work of Krein [49] and Gohberg and Krein [42], see [4, Ch. 20].) Here, we give a quick review of the argument given in Widom [91].

Let $\{\varepsilon_n\}$ be a monotonically decreasing sequence tending to zero as $n \to \infty$. Define

$$\tilde{\varepsilon}_n(t) = \begin{cases} 
 e^{-\varepsilon_n t}, & t > 0 \\
 0, & t < 0.
\end{cases}$$

We call $u(t)$ a solution of (5.1), if for any $\eta > 0$ it satisfies

$$\lim_{n \to \infty} \left\| e^{-\eta|t|} \int_{-\infty}^{\infty} k(t - \tau)\tilde{\varepsilon}_n(\tau)u(\tau)d\tau - e^{-\eta|t|} [u(t) - f(t)] \right\|_2 = 0. \quad (5.21)$$
In (5.21), \(e^{-\eta \| t \|}\) is inserted for some technical reasons. If such a solution does exist, then it is well known that there will exist a subsequence \(\{\varepsilon_{n_k}\}\) such that \(\varepsilon_{n_k} \to 0\) as \(k \to \infty\) and

\[
\lim_{k \to \infty} \int_0^\infty k(t - \tau) e_{n_k}(\tau) u(\tau) d\tau = u(t) - f(t)
\]

holds almost everywhere; see [12, Prop. 0.1.10]. To make this argument work, Widom first proved that the functional equation

\[
U_-(\zeta) = F(\zeta) + \left[\sqrt{2\pi} K(\zeta) - 1\right] U_+(\zeta), \quad \zeta \in \mathbb{R},
\]

holds almost everywhere; cf. (5.23). Next, he introduced the function

\[
\Psi(\zeta) := \frac{\left[1 - \sqrt{2\pi} K(\zeta)\right]}{(\zeta - \alpha_1)^{m_1}(\zeta - \alpha_2)^{m_2} \cdots (\zeta - \alpha_l)^{m_l}} \left(\frac{\zeta - i}{\zeta + i}\right)^k,
\]

which has no zero on the real line. Here, \(\alpha_j\) for \(1 \leq j \leq l\) is a zero of \(1 - \sqrt{2\pi} K(\zeta)\) with multiplicity \(m_j\), and \(n = m_1 + \cdots + m_l\). To obtain a factorization formula like the one in (5.12), he set

\[
\chi(\lambda) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \Psi(\xi)}{\xi - \lambda} d\xi,
\]

cf. (5.10) and (5.11). Note that this integral defines an analytic function in \(\mathbb{C} \setminus \mathbb{R}\). Put

\[
\phi_+(\lambda) := (\lambda + i)^{\frac{\theta}{2}} e^{-\chi_+(\lambda)}, \quad \text{Im} \lambda > 0,
\]

\[
\phi_-(\lambda) := (\lambda - i)^{\frac{\theta}{2}} e^{-\chi-(\lambda)}, \quad \text{Im} \lambda < 0,
\]

where

\[
\chi_+(\lambda) = \chi(\lambda) \quad \text{in} \quad \text{Im} \lambda > 0, \quad \chi-(\lambda) = \chi(\lambda) \quad \text{in} \quad \text{Im} \lambda < 0
\]

with boundary values

\[
\chi_+(\zeta) = \lim_{\varepsilon \to 0^+} \chi(\zeta + i\varepsilon) = \frac{1}{2} \ln \Psi(\zeta) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \Psi(\xi)}{\xi - \zeta} d\xi,
\]

\[
\chi-(\zeta) = \lim_{\varepsilon \to 0^+} \chi(\zeta - i\varepsilon) = -\frac{1}{2} \ln \Psi(\zeta) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \Psi(\xi)}{\xi - \zeta} d\xi
\]

for \(\zeta \in \mathbb{R}\), where the bar indicates that the integral is a Cauchy principal value. It is now readily seen that

\[
1 - \sqrt{2\pi} K(\zeta) = \frac{\phi_-(\zeta)}{\phi_+(\zeta)} (\zeta - \alpha_1)^{m_1}(\zeta - \alpha_2)^{m_2} \cdots (\zeta - \alpha_l)^{m_l};
\]

cf. (5.12). Applying (5.28) to the functional equation (5.23) yields

\[
\frac{U_+(\zeta)}{\phi_+(\zeta)} (\zeta - \alpha_1)^{m_1}(\zeta - \alpha_2)^{m_2} \cdots (\zeta - \alpha_l)^{m_l} = \frac{U_-(\zeta)}{\phi_-(\zeta)} + \frac{F(\zeta)}{\phi_-(\zeta)}.
\]

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To obtain an analogue of (5.14), Widom set
\[ H(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\zeta) \, d\zeta}{\phi_{-}(\zeta) (\zeta - \lambda)} \] (5.30)
with
\[ H_{+}(\lambda) = H(\lambda) \quad \text{in} \quad \text{Im} \lambda > 0, \quad H_{-}(\lambda) = H(\lambda) \quad \text{in} \quad \text{Im} \lambda < 0 \]
and
\[
H_{+}(\zeta) = \frac{1}{2} \frac{F(\zeta)}{\phi_{-}(\zeta)} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\xi) \, d\xi}{\phi_{-}(\xi) (\xi - \zeta)}, \\
H_{-}(\zeta) = -\frac{1}{2} \frac{F(\zeta)}{\phi_{-}(\zeta)} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(\xi) \, d\xi}{\phi_{-}(\xi) (\xi - \zeta)}
\] (5.31)
for \( \zeta \in \mathbb{R} \); cf. (5.27). From (5.31), it follows
\[ \frac{F(\zeta)}{\phi_{-}(\zeta)} = H_{+}(\zeta) - H_{-}(\zeta). \] (5.32)

Coupling (5.29) and (5.32) yields
\[ \frac{U_{+}(\zeta)}{\phi_{+}(\zeta)}(\zeta - \alpha_{1})^{m_{1}}(\zeta - \alpha_{2})^{m_{2}} \cdots (\zeta - \alpha_{l})^{m_{l}} - H_{+}(\zeta) = -\frac{U_{-}(\zeta)}{\phi_{-}(\zeta)} - H_{-}(\zeta). \] (5.33)

The left-hand side of (5.33) defines an analytic function on the upper half-plane \( \mathbb{C}_{+} \), and the right-hand side defines an analytic function on the lower half-plane \( \mathbb{C}_{-} \); these two functions agree on the real line \( \mathbb{R} \). By using a theorem of Carleman [14, p. 40], Widom showed that these two functions are analytic continuations of each other and together represent an entire function \( P(\lambda) \). Furthermore, by using an estimate of the function \( \phi_{-}(\lambda) \) in (5.29), Widom concluded that \( P(\lambda) \) is in fact a polynomial of degree less than \( \lfloor \frac{n}{2} + k \rfloor \). From (5.33), we have
\[ U_{+}(\lambda) = \frac{[P(\lambda) + H_{+}(\lambda)] \phi_{+}(\lambda)}{(\lambda - \alpha_{1})^{m_{1}}(\lambda - \alpha_{2})^{m_{2}} \cdots (\lambda - \alpha_{l})^{m_{l}}}. \] (5.34)

By inversion, we obtain
\[
u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{[P(\lambda) + H_{+}(\lambda)] \phi_{+}(\lambda)}{(\lambda - \alpha_{1})^{m_{1}}(\lambda - \alpha_{2})^{m_{2}} \cdots (\lambda - \alpha_{l})^{m_{l}}} e^{-it\lambda} d\lambda, \quad t > 0,
\] (5.35)
for any \( \gamma > 0 \); cf. (5.19). In his final step, Widom established the limit in (5.21), thus proving that \( u(t) \) is a solution of (5.1).

For illustration purpose, it suffices to consider just the special case in which all zeros of \( 1 - \sqrt{2\pi}K(\lambda) \) are simple; i.e., \( m_{1} = \cdots = m_{l} = 1 \), and \( K(\lambda) \) and \( F(\lambda) \) are both smooth, except possibly for a finite number of points in \( \mathbb{R} \setminus \{\alpha_{1}, \cdots, \alpha_{l}\} \). By Cauchy’s residue theorem, the integral in (5.35) can be expressed as
\[
u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{[P(\lambda) + H_{+}(\lambda)] \phi_{+}(\lambda)}{(\lambda - \alpha_{1})(\lambda - \alpha_{2}) \cdots (\lambda - \alpha_{l})} e^{-it\lambda} d\lambda \\
- i \sqrt{\frac{\pi}{2}} \sum_{j=1}^{l} G_{j}(\alpha_{j}) e^{-it\alpha_{j}}, \] (5.36)
where
\[ G_j(\lambda) = \frac{[P(\lambda) + H_+(\lambda)] \phi_+(\lambda)(\lambda - \alpha_j)}{(\lambda - \alpha_1)(\lambda - \alpha_2) \cdots (\lambda - \alpha_l)}. \] (5.37)

Note that here we have made use of the formula
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi} d\xi = -\sqrt{\frac{\pi}{2}} i. \] (5.38)

Finally, choose \( l \) cut-off functions \( \eta_j(\lambda), j = 1, \cdots, l, \) such that \( \eta_j(\lambda) \) is equal to 1 for \( \lambda \) near \( \alpha_j, j = 1, \cdots, l, \) and \( \text{supp} \eta_j \cap \text{supp} \eta_i = \emptyset \) for each \( j \neq i. \) Let \( \eta_0 = 1 - \sum_{j=1}^{l} \eta_j, \) the integral in (5.36) can now be written as
\[
\begin{align*}
    u(t) &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{l} e^{-i\alpha_j} \int_{-\infty}^{\infty} G_j(\xi) \eta_j(\xi) - G_j(\alpha_j) e^{-i\alpha_j} d\xi \\
    &\quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{[P(\xi) + H_+(\xi)] \phi_+(\xi) \eta_0(\xi)}{(\xi - \alpha_1)(\xi - \alpha_2) \cdots (\xi - \alpha_l)} e^{-i\xi} d\xi \\
    &\quad - \sqrt{\frac{\pi}{2}} i \sum_{j=1}^{l} G_j(\alpha_j) e^{-i\alpha_j}.
\end{align*}
\] (5.39)

Asymptotic behavior of the solution \( u(t) \) to equation (5.1) can be obtained from the Fourier integrals in (5.39); see [97, Thm. 1, p. 199 and Ex. 11, p. 234]. But, the amount of work to derive just the leading term in the asymptotic expansion of \( u(t) \) is still tremendous. This is because the integrands in (5.39) involve functions \( \phi_+(\lambda) \) and \( H_+(\lambda), \) which are given in terms of Hilbert transforms; see (5.26), (5.27) and (5.31). Fortunately, asymptotic expansions of Hilbert transforms can be obtained via corresponding results for the Stieltjes transform
\[ S_\phi(z) = \int_{0}^{\infty} \frac{\phi(t)}{t + z} dt, \quad |\arg z| < \pi, \] (5.40)

and the one-sided Hilbert transform
\[ H_+^\phi(x) = \int_{0}^{\infty} \frac{\phi(t)}{t - x} dt, \quad x \in \mathbb{R}^+, \] (5.41)

where again the bar indicates that the integral is a Cauchy principal value. In deriving the asymptotic expansions of these transforms, we also run into the Mellin transform of \( \phi, \) which is defined by
\[ M[\phi; z] = \int_{0}^{\infty} t^{z-1} \phi(t) dt, \quad \text{Re } z > 0. \] (5.42)

We assume that \( \phi \in L^1(\mathbb{R}^+) \) has an asymptotic expansion of the form
\[ \phi(t) \sim \sum_{s=0}^{\infty} \sum_{i=0}^{s} \alpha_{s,i} t^{-s-\theta} \ln^{i} t, \quad t \to \infty, \] (5.43)
for fixed \( \theta \in (0, 1) \). Let
\[
\varphi_n(t) = \varphi(t) - \sum_{s=0}^{n-1} \sum_{i=0}^{s} \alpha_{s,i} t^{-s-\theta} \ln^i t. \tag{5.44}
\]

For convenience, we also introduce a new notation. Recall the Pochhammer symbol
\[\left(\theta\right)_n = \theta(\theta + 1) \cdots (\theta + n - 1)\]. Since
\[(-1)^n \frac{d^n}{dt^n} \left(\frac{t^{-\theta}}{(\theta)_n}\right) = t^{-n-\theta}\]
for any \( \theta \in \mathbb{R} \) and \( n \in \mathbb{N} \) with \((\theta)_n \neq 0\), we have
\[
(-1)^{n+i} \frac{d^n}{dt^n} \left(\frac{t^{-\theta}}{(\theta)_n}\right) = t^{-n-\theta} \ln^i t \tag{5.45}
\]
for any \( i \in \mathbb{N} \). Let
\[
H_{\theta,n,i}(t) = (-1)^{n+i} \frac{d^n}{dt^n} \left(\frac{t^{-\theta}}{(\theta)_n}\right) = \sum_{j=0}^{i} C_{\theta,n,i}^{j} t^{-\theta} \ln^j t, \tag{5.46}
\]
where
\[
C_{\theta,n,i}^{j} = (-1)^{n+i+j} \frac{i}{j} \frac{d^{i-j}}{d\theta^{i-j}} \left(\frac{1}{(\theta)_n}\right). \tag{5.47}
\]
Note that from (5.46), we have
\[
H_{\theta,n,i}^{(n)}(t) = t^{-n-\theta} \ln^i t.
\]

**Theorem 5.1.** Suppose that \( \varphi \) satisfies (5.43) with \( \theta \in (0, 1) \). Then for any \( z \in \mathbb{C} \) with \( |\arg z| < \pi \) and \( n \in \mathbb{N}^+ \), we have
\[
S_{\varphi}(z) = \sum_{s=0}^{n-1} \sum_{i=0}^{s} \sum_{j=0}^{i} (-1)^j \pi \alpha_{s,i} C_{\theta,s,i}^{j} \frac{d^j}{d\theta^j} \left[ \frac{(\theta)_s}{\sin \theta \pi z^{-\theta}} \right] z^{-s} - \sum_{s=1}^{n} (s-1)! c_s z^{-s} + \frac{(-1)^n}{z^n} \int_0^\infty \frac{x^n \varphi_n(\tau)}{\tau + z} d\tau, \tag{5.48}
\]
where \( c_s = \frac{(-1)^s}{(s-1)!} M[\varphi; s] \) and \( M[\varphi; s] \) is the Mellin transform of \( \varphi \).

To state the next result, we need another notation, namely
\[
\gamma_{k,s} = \int_0^\infty \frac{\ln^k t}{(t + 1)^{s+2}} dt, \tag{5.49}
\]
where \( s, k \in \mathbb{N} \). Direct calculation gives
\[
\gamma_{0,s} = \frac{1}{s + 1}, \quad \gamma_{1,s} = -\frac{1}{s + 1} \sum_{i=1}^{s} \frac{1}{i}. \tag{5.50}
\]
\[
\gamma_{2,s} = \frac{4}{s+1} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^2} - \frac{2}{s+1} \sum_{i=1}^{s} \int_{0}^{\infty} \frac{\ln t}{(t+1)^{i+1}} dt
\]

and
\[
\gamma_{k,s} = \frac{[1+(-1)^k]k!}{s+1} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^k} - \frac{k}{s+1} \sum_{i=1}^{s} \int_{0}^{\infty} \frac{\ln^{k-1} t}{(t+1)^{i+1}} dt
\]

\[
= \frac{[1+(-1)^k]k!}{s+1} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^k} - \frac{k}{s+1} \sum_{i=1}^{s} \gamma_{k-1,i-1},
\]

empty sum being understood to be zero.

**Theorem 5.2.** Suppose that \( \varphi \) satisfies (5.43) with \( \theta = 1 \). Then for any \( z \in \mathbb{C} \) with \(|\arg z| < \pi\) and \( n \in \mathbb{N}^+ \), we have
\[
S_\varphi(z) = \sum_{s=0}^{n-1} \sum_{i=0}^{s} \sum_{j=0}^{j+1} (s+1)! \alpha_{s,i} C_{1,s,i} \binom{j+1}{l} \gamma_{j+1-l,s} \frac{z^{-s-1} \ln^l z}{j+1} - \sum_{s=1}^{n} (s-1)! c_s^* z^{-s} + \frac{(-1)^n}{z^n} \int_{0}^{\infty} \frac{\tau^n \varphi_n(\tau)}{\tau + z} d\tau,
\]

where \( c_s^* = \varphi_{s,s}(1) - \int_{0}^{1} \varphi_{s-1,s-1}(t) dt \), \( \varphi_{s,s}(t) = \frac{(-1)^s}{(s-1)!} \int_{t}^{\infty} (\tau-t)^{s-1} \varphi_s(\tau) d\tau \) for \( s \in \mathbb{N}^+ \), and \( \varphi_{0,0}(t) = \varphi(t) \).

In view of the well-known formula of Plemelj
\[
H^+_{\varphi}(x) = \frac{1}{2} \lim_{\varepsilon \to 0^+} \int_{0}^{\infty} \left[ \frac{1}{t-x+i\varepsilon} - \frac{1}{t-x-i\varepsilon} \right] \varphi(t) dt,
\]

the results of the last two theorems also give the asymptotic expansions of the one-side Hilbert transform of \( \varphi \).

**Theorem 5.3.** Suppose that \( \varphi \) satisfies (5.43) with \( \theta \in (0,1) \). Then for any \( n \in \mathbb{N}^+ \), we have
\[
H^+_{\varphi}(x) = \sum_{s=0}^{n-1} \sum_{i=0}^{s} \sum_{j=0}^{j+1} (-1)^{s+j} \pi \alpha_{s,i} C_{1,s,i} \frac{d^j}{d\theta^j} \left[ (\theta)_s \cot(\theta \pi) x^{-\theta} \right] x^{-s} - \sum_{s=1}^{n} (-1)^s (s-1)! c_s x^{-s} + \frac{1}{\pi^n} \int_{0}^{\infty} \frac{\tau^n \varphi_n(\tau)}{\tau-x} d\tau,
\]

where \( c_s = \frac{(-1)^s}{(s-1)!} M[\varphi;s] \).
Theorem 5.4. Suppose that $\varphi$ satisfies (5.43) with $\theta = 1$. Then for any $n \in \mathbb{N}^+$, we have

$$H_\varphi^+(x) = \sum_{s=0}^{n} \sum_{i=0}^{s} \sum_{j=0}^{i+1} \sum_{l=0}^{j} \frac{(-1)^{s+1}(s+1)!\alpha_{s,i}C_{l,s,i} \gamma_{j+1-l,s}}{2(j+1)} \times \left[(\ln x + i\pi)^l + (\ln x - i\pi)^l\right] x^{-s-1}$$

$$- \sum_{s=1}^{n} (-1)^s(s-1)!c_s^1x^{-s} + \frac{1}{x^n} \int_0^\infty \frac{\tau^n\varphi_n(\tau)}{\tau - x} d\tau,$$

where $c_s^1 = \varphi_{s,s}(1) - \int_0^1 \varphi_{s-1,s-1}(t)dt$, $\varphi_{s,s}(t) = \frac{(-1)^s}{(s-1)!} \int_t^\infty (\tau - t)^{s-1}\varphi(\tau) d\tau$ for $s \in \mathbb{N}^+$, and $\varphi_{0,0}(t) = \varphi(t)$.

To obtain the asymptotic behavior of the solution to equation (5.1), one would have to make use of the results in the above four theorems. For examples, see [58].

References

[1] A. Asheim and D. Huybrechs, Complex Gaussian quadrature for oscillatory integral transforms, *IMA J. Numer. Anal.*, 33 (2013), 1322-1341.

[2] X.-X. Bai and Y.-Q. Zhao, A uniform asymptotic expansion for Jacobi polynomials via uniform treatment of Darboux’s method, *J. Approx. Theory*, 148 (2007), 1-11.

[3] J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin and P. D. Miller, *Discrete orthogonal polynomials. Asymptotics and applications*, Annals of Mathematics Studies, 164, Princeton University Press, Princeton, NJ, 2007.

[4] R. Beals and R. Wong, *Explorations in complex functions*, Springer, Cham, 2020.

[5] G. Birkhoff (Ed.), *A source book in classical analysis*, Harvard University Press, Cambridge, MA, 1973, assisted by U. Merzbach.

[6] G. D. Birkhoff, On the asymptotic character of the solutions of certain linear differential equations containing a parameter, *Trans. Amer. Math. Soc.*, 9 (1908), 219-231.

[7] G. D. Birkhoff, Formal theory of irregular linear difference equations, *Acta Math.*, 54 (1930), 205-246.

[8] G. D. Birkhoff and W. J. Trjitzinsky, Analytic theory of singular difference equations, *Acta Math.*, 60 (1933), 1-89.
[9] P. Bleher and A. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model, *Ann. Math.*, **150** (1999), 185-266.

[10] N. Bleistein, Uniform asymptotic expansions of integrals with many nearby stationary points and algebraic singularities, *J. Math. Mech.*, **17** (1967), 533-560.

[11] R. Bo and R. Wong, Asymptotic behavior of the Pollaczek polynomials and their zeros, *Stud. Appl. Math.*, **96** (1996), 307-338.

[12] P. L. Butzer and R. J. Nessel, *Fourier analysis and approximation I*, Academic Press, New York, 1971.

[13] L.-H. Cao and Y.-T. Li, Linear difference equations with a transition point at the origin, *Anal. Appl.*, **12** (2014), 75-106.

[14] T. Carleman, *L’intégrale de Fourier et questions qui s’y rattachent*, Uppsala, 1944.

[15] J. F. Carlson and A. E. Heins, The reflection of an electromagnetic plane wave by an infinite set of plates I, *Quart. Appl. Math.*, **4** (1947), 313-329.

[16] G. F. Carrier, Sound transmission from a tube with flow, *Quart. Appl. Math.*, **13** (1956), 457-461.

[17] Y. Chen and M. Ismail, Some indeterminate moment problems and Freud-like weights, *Constr. Approx.*, **14**(1998), 439-458.

[18] T. M. Cherry, Uniform asymptotic expansions, *J. London Math. Soc.*, **24** (1949), 121-130.

[19] C. Chester, B. Friedman, F. Ursell, An extension of the method of steepest descents, *Proc. Cambridge Philos. Soc.*, **53** (1957), 599-611.

[20] T. O. Conway and P. Deift, Asymptotics of polynomials orthogonal with respect to a logarithmic weight, *SIGMA Symmetry Integrability Geom. Methods Appl.*, **14** (2018), Paper No. 056, 66 pp.

[21] O. Costin and R. Costin, Rigorous WKB for finite-order linear recurrence relations with smooth coefficients, *SIAM J. Math. Anal.*, **27** (1996), 110-134.

[22] D. Dai, W.-Y. Hu and X.-S. Wang, Uniform asymptotics of orthogonal polynomials arising from coherent states, *SIGMA Symmetry Integrability Geom. Methods Appl.*, **11** (2015), Paper 070, 17 pp.

[23] D. Dai, M. E. H. Ismail and X.-S. Wang, Plancherel-Rotach asymptotic expansion for some polynomials from indeterminate moment problems, *Constr. Approx.* **40** (2014), 61-104.
[24] G. Darboux, Mémoire sur l’approximation des fonctions de très grands nombres, et sur une classe étendue de développements en série, *J. Math. Pures. Appl.*, 4 (1878), 5-56, 377-416.

[25] A. Deaño, A. B. J. Kuijlaars and P. Román, Asymptotic behavior and zero distribution of polynomials orthogonal with respect to Bessel functions, *Constr. Approx.*, 43 (2016), 153-196.

[26] P. Deift, *Orthogonal polynomials and random matrices: A Riemann-Hilbert approach*, Courant Lecture Notes, Vol.3, New York University, 1999.

[27] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.*, 52 (1999), 1335-1425.

[28] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.*, 52 (1999), 1491-1552.

[29] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, A Riemann-Hilbert approach to asymptotic questions for orthogonal polynomials, *J. Comput. Appl. Math.*, 133 (2001), 47-63.

[30] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.*, 137 (1993), 295-368.

[31] R. B. Dingle and G. J. Morgan, WKB methods for difference equations. I, *Appl. Sci. Res.*, 18 (1967), 221-237.

[32] R. B. Dingle and G. J. Morgan, WKB methods for difference equations. II, *Appl. Sci. Res.*, 18 (1967), 238-245.

[33] J. Dutka, The early history of the factorial function, *Arch. Hist. Exact Sci.*, 43 (1991), 225-249.

[34] Editorial note, Robinson’s Constant, *Amer. Math. Monthly*, 59 (1952), 296-297.

[35] A. Erdély, Uniform asymptotic expansion of integrals, In: *Analytic methods in mathematical physics* (R. P. Gilbert, R. G. Newton, eds.), New York, Gordon and Breach, 149-168, 1970.

[36] A. Erdély and M. Wyman, The asymptotic evaluation of certain integrals, *Arch. Rational Mech. Anal.*, 14 (1963), 217-260.

[37] J. L. Fields, A uniform treatment of Darboux’s method, *Arch. Rational Mech. Anal.*, 27 (1967), 289-305.
[38] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.

[39] A. S. Fokas, A. R. Its, A. A. Kapaev and V. Y. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach*, Mathematical Surveys and Monographs, **128**, American Mathematical Society, Providence, RI, 2006.

[40] A. S. Fokas, A. R. Its and A. V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.*, **147** (1992), 395-430.

[41] P. J. Forrester, The spectrum edge of random matrix ensembles, *Nucl. Phys. B*, **402** (1993), 709-728.

[42] I. C. Gohberg and M. G. Krein, Systems of integral equations on a half line with kernels depending on the difference of arguments, *Uspehi Mat. Nauk*, **13**(1958), 3-72; *Amer. Math. Soc. Transl. Ser. (2)*, **14** (1960), 217-287.

[43] G. H. Hardy and S. Ramanujan, Asymptotic formulæ in combinatory analysis, *Proc. London Math. Soc. (2)*, **17** (1918), 75-115.

[44] A. Iserles, Stephen Smale: the mathematician who broke the dimension barrier by Steve Batterson, *SIAM Review*, **42** (2000), 739-745.

[45] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of Mathematics and its Applications, 98. Cambridge University Press, Cambridge, 2005.

[46] K. Johansson, Discrete orthogonal polynomial ensembles and the Plancherel measure, *Ann. Math.*, **153** (2001), 259-296.

[47] D. E. Knuth, Asymptotic Behavior of a Sequence, *SIAM Rev.*, (Problem Sect., Prob. 79-5 by L. Erlebach and O. Ruehr), **22** (1980), 101-102.

[48] D. E. Knuth, Communication between D. E. Knuth and F. W. J. Olver, 2004.

[49] M. G. Krein, Integral equations on a half-line with a kernel depending on the difference of the arguments, *Uspehi Mat. Nauk*, **13**(1958), 3-120; *Amer. Math. Soc. Transl. Ser. (2)*, **22** (1962), 163-288.

[50] T. Kriecherbauer and K. T.-R. McLaughlin, Strong asymptotics of polynomials orthogonal with respect to Freud weights, *Int. Math. Res. Not.*, **6** (1999), 299-333.

[51] A. B. J. Kuijlaars, Riemann-Hilbert analysis for orthogonal polynomials, *Orthogonal polynomials and special functions (Leuven, 2002)*, 167-210, Lecture Notes in Math., 1817, Springer, Berlin, 2003.

[52] A. B. J. Kuijlaars, K. T.-R. McLaughlin, W. Van Assche and M. Vanlessen, The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on \([-1,1]\), *Adv. Math.*, **188** (2004), 337-398.
[53] A. B. J. Kuijlaars and M. Vanlessen, Universality for eigenvalue correlations from the modified Jacobi unitary ensemble, *Int. Math. Res. Not.*, 2002 (2002), 1575-1600.

[54] R. E. Langer, On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order, *Trans. Amer. Math. Soc.*, 33 (1931), 23-64.

[55] R. E. Langer, On the asymptotic solutions of differential equations, with an application to the Bessel functions of large complex order, *Trans. Amer. Math. Soc.*, 34 (1932), 447-480.

[56] R. E. Langer, The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point, *Trans. Amer. Math. Soc.*, 67 (1949), 461-490.

[57] Y.-T. Li, X.-S. Wang and R. Wong, Asymptotics of the Wilson polynomials, *Anal. Appl.*, 18 (2020), 237-270.

[58] K. Li and R. Wong, Asymptotic expansions for Wiener-Hopf equations, *Anal. Appl.*, 19 (2021), 1059-1092.

[59] X. Li and R. Wong, Error bounds for asymptotic expansions of Laplace convolutions, *SIAM J. Math. Anal.*, 25 (1994), 1537-1553.

[60] S.-Y. Liu, R. Wong and Y.-Q. Zhao, Uniform treatment of Darboux’s method and the Heisenberg polynomials, *Proc. Amer. Math. Soc.*, 141 (2013), 2683-2691.

[61] J. P. McClure and R. Wong, Explicit error terms for asymptotic expansions of Stieltjes transforms, *J. Inst. Math. Appl.*, 22 (1978), 129-145.

[62] J. P. McClure and R. Wong, Exact remainders for asymptotic expansions of fractional integrals, *J. Inst. Math. Appl.*, 24 (1979), 139-147.

[63] R. Muki and E. Sternberg, Note on an asymptotic property of solutions of a class of Fredholm integral equations, *Quart. Appl. Math.*, 28 (1970), 277-281.

[64] N. I. Muskhelishvili, *Singular integral equations*, Groningen, Noordhoff, 1953; reprinted by Dover Publications, New York, 1992.

[65] B. Noble, *Methods based on the Wiener-Hopf technique for the solution of partial differential equations*, Pergamon Press Inc., New York, 1958.

[66] A. B. Olde Daalhuis and N. M. Temme, Uniform Airy-type expansions of integrals, *SIAM J. Math. Anal.*, 25 (1994), 304-321.

[67] F. W. J. Olver, The asymptotic solution of linear differential equations of the second order for large values of a parameter, *Phil. Trans. R. Soc. Lond. A*, 247 (1954), 307-327.
[68] F. W. J. Olver, The asymptotic expansion of Bessel functions of large order, *Phil. Trans. R. Soc. Lond. A*, **247** (1954), 328-368.

[69] F. W. J. Olver, The asymptotic solution of linear differential equations of the second order in a domain containing one transition point, *Phil. Trans. R. Soc. Lond. A*, **249** (1956), 65-97.

[70] F. W. J. Olver, Error bounds for the Liouville-Green (or WKB) approximation, *Math. Proc. Cambridge Philos. Soc.*, **57** (1961), 790-810.

[71] F. W. J. Olver, A paradox in asymptotics, *SIAM J. Math. Anal.*, **1** (1970), 533-534.

[72] F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974; reprinted by A. K. Peters, Ltd., Wellesley, 1997.

[73] F. W. J. Olver, Unsolved problems in the asymptotic estimation of special functions, In: *Theory and application of special functions* (R. Askey, ed.), New York, Academic Press, 99-142, 1975.

[74] F. Olver, D. Lozier, R. Boisvert and C. Clark, *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge, 2010.

[75] H. Poincaré, Sur les intégrales irrégulières des équations linéaires, *Acta Math.*, **8** (1886), 295-344.

[76] G. Pólya, *Mathematics and plausible reasoning, Volume 1: Induction and analogy in mathematics*, Princeton University Press, Princeton, N. J., 1954.

[77] R. Robinson, A new absolute geometric constant? *Amer. Math. Monthly*, **58** (1951), 462-469.

[78] L. Schlesinger, Über asymptotische darstellungen der lösungen linearer differentialsysteme als funktionen eines parameters, *Math. Ann.*, **63** (1907), 277-300.

[79] J. Stirling, *Methodus differentialis*, London, 1730.

[80] G. Szegő, *Orthogonal polynomials*, fourth edition, American Mathematical Society, Providence, Rhode Island, 1975.

[81] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, Oxford, 1951.

[82] E. C. Titchmarsh, *The theory of functions*, reprint of the second edition, Oxford University Press, Oxford, 1958.

[83] C. A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, *Comm. Math. Phys.*, **159** (1994), 151-174.

[84] C. A. Tracy and H. Widom, Level spacing distributions and the Bessel kernel, *Comm. Math. Phys.*, **161** (1994), 289-309.
[85] H. L. Turrittin, Asymptotic solutions of certain ordinary differential equations associated with multiple roots of the characteristic equation, *Amer. J. Math.*, 66 (1936), 364-376.

[86] Z. Wang and R. Wong, Uniform asymptotic expansion of $J_\nu(\nu a)$ via a difference equation, *Numer. Math.*, 91 (2002), 147-193.

[87] Z. Wang and R. Wong, Asymptotic expansions for second-order linear difference equations with a turning point, *Numer. Math.*, 94 (2003), 147-194.

[88] Z. Wang and R. Wong, Linear difference equations with transition points, *Math. Comp.*, 74 (2005), 629-653.

[89] Z. Wang and R. Wong, Bessel-type asymptotic expansions via the Riemann-Hilbert approach, *Proc. R. Soc. Lond. A*, 461 (2005), 2839-2856.

[90] W. A. Whitworth, Choice and chance, 5th ed., George Bell, London, 1901.

[91] H. Widom, Equations of Wiener-Hopf type, *Illinois J. Math.*, 2 (1958), 261-270.

[92] N. Wiener and E. Hopf, Über eine Klasse singulärer integral gleichungen, *Sitz. Ber. Preuss. Akad. Wiss.*, 30 (1931), 696-706.

[93] J. Wimp, Review of R. Wong’s book ‘Asymptotic approximations of integrals’, *Math. Comp.*, 56 (1991), 388-394.

[94] J. Wimp and D. Zeilberger, Recurrenting the asymptotics of linear recurrences, *J. Math. Anal. Appl.*, 111 (1985), 162-176.

[95] J. S. W. Wong and R. Wong, Asymptotic solutions of linear Volterra integral equations with singular kernels, *Trans. Amer. Math. Soc.*, 189 (1974), 185-200.

[96] R. Wong, Quadrature formulas for oscillatory integral transforms, *Numer. Math.*, 39 (1982), 351-360.

[97] R. Wong, *Asymptotic approximations of integrals*, Academic Press, Boston, 1989; reprinted by SIAM, Philadelphia, PA, 2001.

[98] R. Wong, Asymptotics of linear recurrences, *Anal. Appl.*, 12 (2014), 463-484.

[99] R. Wong and H. Li, Asymptotic expansions for second-order linear difference equations, *J. Comput. Appl. Math.*, 41 (1992), 65-94.

[100] R. Wong and H. Li, Asymptotic expansions for second-order linear difference equations II, *Stud. Appl. Math.*, 87 (1992), 289-324.

[101] R. Wong and J. P. McClure, Generalized Mellin convolutions and their asymptotic expansions, *Canad. J. Math.*, 36 (1984), 924-960.
[102] R. Wong and M. Wyman, The method of Darboux, *J. Approximation Theory*, **10** (1974), 159-171.

[103] R. Wong and Y.-Q. Zhao, On a uniform treatment of Darboux’s method, *Constr. Approx.*, **21** (2005), 225-255.

[104] R. Wong and Y.-Q. Zhao, Asymptotics of orthogonal polynomials via the Riemann-Hilbert approach, *Acta Math. Sci. Ser. B*, **29** (2009), 1005-1034.

[105] R. Wong and Y.-Q. Zhao, Special functions, integral equations and a Riemann-Hilbert problem, *Proc. Amer. Math. Soc.*, **144** (2016), 4367-4380.

[106] S.-X. Xu, D. Dai and Y.-Q. Zhao, Critical edge behavior and the Bessel to Airy transition in the singularly perturbed Laguerre unitary ensemble, *Comm. Math. Phys.*, **332** (2014), 1257-1296.

[107] S.-X. Xu and Y.-Q. Zhao, Gap probability of the circular unitary ensemble with a Fisher-Hartwig singularity and the coupled Painlevé V system, *Comm. Math. Phys.*, **377** (2020), 1545-1596.

[108] K. Yosida, *Functional analysis*, reprint of the sixth edition, Springer-Verlag, Berlin, 1995.

[109] C.-R. Zhao, X.-B. Wu and Y.-Q. Zhao, Asymptotic zero distribution of orthogonal polynomials with respect to Bessel functions via Riemann-Hilbert approach, *preprint*, 2021.

[110] Y.-Q. Zhao, Uniform asymptotics for orthogonal polynomials via the Riemann-Hilbert approach, *Appl. Anal.*, **85** (2006), 1165-1176.