AN APPLICATION OF COHOMOLOGICAL INVARIANTS

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ABSTRACT. Let $G$ be a finite group, $k$ be a field and $G \to GL(V_{\text{reg}})$ be the regular representation of $G$ over $k$. Then $G$ acts naturally on the rational function field $k(V_{\text{reg}})$ by $k$-automorphisms. Define $k(G)$ to be the field $k(V_{\text{reg}})^G$. Noether’s problem asks whether $k(G)$ is rational (resp. stably rational) over $k$.

When $k = \mathbb{Q}$ and $G$ contains a normal subgroup $N$ with $G/N \simeq C_8$ (the cyclic group of order 8), Jack Sonn proves that $\mathbb{Q}(G)$ is not stably rational over $\mathbb{Q}$, which is a non-abelian extension of a theorem of Endo-Miyata, Voskresenskii, Lenstra and Saltman for the abelian Noether’s problem $\mathbb{Q}(C_8)$. Using the method of cohomological invariants, we are able to generalize Sonn’s theorem as follows. Theorem. Let $G$ be a finite group and $N \trianglelefteq G$ such that $G/N \simeq C_{2^n}$ with $n \geq 3$. If $k$ is a field satisfying that $\text{char } k = 0$ and $k(\zeta_{2^n})/k$ is not a cyclic extension where $\zeta_{2^n}$ is a primitive $2^n$-th root of unity, then $k(G)$ is not stably rational (resp. not retract rational) over $k$.

1. INTRODUCTION

Let $G$ be a finite group, $k$ be a field and $\rho : G \to GL(V_{\text{reg}})$ be the regular representation of $G$ over $k$ where $V_{\text{reg}} = \bigoplus_{g \in G} k \cdot e_g$ is the regular representation space and $h \cdot e_g = e_{hg}$ for any $g, h \in G$. Let $k(V_{\text{reg}})$ be the function field of $V_{\text{reg}}$. Define $k(G) := k(V_{\text{reg}})^G = \{ f \in k(V_{\text{reg}}) : \sigma \cdot f = f \text{ for any } \sigma \in G \}$, the fixed field of $k(V_{\text{reg}})$ under the action of $G$. Noether’s problem asks whether $k(G)$ is stably rational over $k$.

The answer to Noether’s problem depends on the group $G$ and on the field $k$. For examples, Swan shows that $\mathbb{Q}(C_{47})$ is not stably rational over $\mathbb{Q}$ where $C_n$ denotes the cyclic group of order $n$, and it is well-known that, if $G$ is a finite abelian group of exponent $e$ and $k$ is a field containing a primitive $e$-th root of unity, then $k(G)$ is rational over $k$ (Fischer’s Theorem). On the other hand, Saltman shows that, for any prime number $p$, there is a non-abelian group $G$ of order $p^9$ such that $\mathbb{C}(G)$ is not stably rational [Sa3]. We recommend Swan’s paper [Sw] for a survey of Noether’s problem.

Suppose that $\rho : G \to GL(V)$ is a faithful representation of $G$ over $k$. Then $k(G)$ is stably rational over $k$ if and only if so is $k(V)^G$ over $k$, by the No-Name Lemma (see, for examples, [Ka3 Theorem 5.1]). Thus some authors would formulate Noether’s problem in the following equivalent form : whether $BG$ is stable rational over $k$ [Mc2 page 283] where $BG$ denotes the classifying space of $G$ over $k$; for the definition of the classifying space, see [To].

Now we turn to the stable rationality of $k(C_{2^n})$.

It is Endo and Miyata, Voskresenskii who prove that $\mathbb{Q}(C_{2^n})$ is not stably rational if $n \geq 3$ [LM Corollary 3.1]. If $k$ is a field satisfying that $k(C_{2^n})/k$ is not a cyclic extension, it is also known that $k(C_{2^n})$ is not stable rational (see Lenstra’s paper [Lc page 310, Proposition 3.2] and the proof of the main theorem on page 319 therein). We remark that $k(C_{2^n})$ is retract rational if and only if either $\text{char } k = 2$ or $k(\zeta_{2^n})/k$ is a cyclic extension (Saltman); see, for examples, [Ka3 Theorem 3.7].

A new perspective of the above theorem arose from an alternative proof found by Saltman [Sa1 Theorem 5.11], who related the negation of the stable rationality of $\mathbb{Q}(C_{2^n})$ ($n \geq 3$) to the non-existence of a degree 8 cyclic unramified extension of $\mathbb{Q}_2$ (the 2-adic local field), i.e. Shianghaw Wang’s counter-example to Grunwald’s theorem [Wa].

Following the line of Saltman’s idea, Jack Sonn found an unexpected extension to the non-abelian situation.

Theorem 1.1 (Sonn [So]). Let $G$ be a finite group containing a normal subgroup $N$ such that $G/N \simeq C_8$. Then $\mathbb{Q}(G)$ is not stably rational over $\mathbb{Q}$.

The proof of Theorem 1.1 relies heavily on many delicate arithmetic properties of number fields pertaining to the field $\mathbb{Q}$. It is not completely obvious whether Theorem 1.1 is valid for a field $k$ other than $\mathbb{Q}$.

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The theory of cohomological invariants is developed by Serre, Rost, Merkurjev etc. (see [GMS] and [Mc1]). By using the results of Merkurjev [Mc2] and V. J. Bailey [Ba], we are able to generalize Theorem 1.1 as follows.

**Theorem 1.2.** Let $G$ be a finite group containing a normal subgroup $N$ such that $G/N \cong C_{2^n}$ (with $n \geq 3$). If $k$ is a field of char $k = 0$ satisfying that $k(\zeta_{2^n})/k$ is not a cyclic extension where $\zeta_{2^n}$ is a primitive $2^n$-th root of unity, then $k(G)$ is not stable rational (resp. not retract rational) over $k$.

The proof of Theorem 1.2 will be given in Section 3. For the convenience of the reader, we include a brief introduction to some basic notions of cohomological invariants in Section 2.

In the remaining part of this section, we digress to explain the notion of retract rationality which appears in Theorem 1.2.

**Definition 1.3.** According to Saltman [Sa2] Definition 3.1, a field extension $L$ of an infinite field $k$ is called retract rational over $k$ if there are an affine domain $A$ over $k$ and $k$-algebra morphisms $\varphi : A \to k[X_1, \ldots, X_n][1/f]$, $\psi : k[X_1, \ldots, X_n][1/f] \to A$ where $k[X_1, \ldots, X_n]$ is the polynomial ring of $k$, $f \in k[X_1, \ldots, X_n] \setminus \{0\}$ satisfying that

(i) $L$ is the quotient field of $A$, and
(ii) $\psi \circ \varphi = 1_A$, the identity map of $A$.

Saltman’s notion of retract rationality is generalized by Merkurjev by waiving the assumption that the base field $k$ is an infinite field and is formulated as follows (see [Mc2] page 280, Proposition 3.1): Let $L/k$ be a field extension. $L$ is called retract rational over $k$ if there are irreducible quasi-projective varieties $U$, $Y$, $X$ defined over $k$ and $k$-morphisms $\varphi : U \to Y$, $\psi : Y \to X$ of algebraic varieties such that

(i) $L$ is the function field of $X$,
(ii) $U$ is an open subset of $X$, and $\psi \circ \varphi : U \to X$ is the inclusion map of $U$ into $X$, and
(iii) $Y$ is an open subset of $A^n_k$, the affine space of dimension $n$ over $k$ for some positive integer $n$.

To distinguish the above two definitions, we will called them (S) retract rational and (M) retract rational where (S) and (M) stand for Saltman and Merkurjev respectively. If the base field $k$ is an infinite field, then it is obvious that “(S) retract rational” $\Rightarrow$ “(M) retract rational”. In this note, we will focus on (M) retract rational unless otherwise specified; thus the prefix (M) is omitted henceforth.

It is easy to verify that “rational” $\Rightarrow$ “stably rational” $\Rightarrow$ “retract rational” $\Rightarrow$ “unirational”.

In the lectures of Serre [GMS] page 86], two notions $	ext{Noe}(G/k)$ and $	ext{Rat}(G/k)$ are introduced. Note that “$	ext{Noe}(G/k)$ is true” $\Rightarrow$ “$k(G)$ is stably rational” $\Rightarrow$ “$	ext{Rat}(G/k)$ is true” $\Rightarrow$ “$k(G)$ is retract rational” (see [Ka3] page 25, the third paragraph).

Standing notation. Throughout this note, we consider finite groups $G$. If $k$ is a field, we regard $G$ as the constant group scheme over $k$ associated to $G$ [KMRT] page 332]; thus all the results of cohomological invariants developed in [GMS], [Mc2], [Ba] may be applied to the situation where $G$ is a finite group. We denote by $C_n$ the cyclic group of order $n$. If $k$ is a field with gcd$(n, \text{char} k) = 1$, then $\zeta_n$ denotes a primitive $n$-th root of unity in some extension field of $k$. For emphasis, recall that $k(G)$ is the fixed field $k(V_{\text{reg}})^G = k(x_g : g \in G)^G$ where $h \cdot x_g = x_{hg}$ for all $g, h \in G$. Remember that the classifying variety $BG$ over $k$ is stably rational (resp. retract rational) if and only if $k(G)$ is stably rational (resp. retract rational); see [Mc2] page 283, [Ba] Section 2.2.

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2. Cohomological invariants

For a general definition of invariants (the invariant of a smooth algebraic group over a field $k$ in a cycle module), the reader may look into [Mc1] Section 2 or [GMS] page 7]. In this note, we are concerned only with the cohomological invariants, and the smooth algebraic groups we consider are restricted to finite groups.
Definition 2.1. Let $k$ be the base field and $G$ be a finite group over $k$, i.e. the finite constant group schemes over the field $k$ \cite{KMRT} page 332.

Denote by $\text{Fields}_{/k}$ the category of field extensions $K$ of $k$, by Sets the category of (pointed) sets, by Abelian Groups the category of abelian groups. Let $A$ and $H$ be two functors defined by

$$A : \text{Fields}_{/k} \to \text{Sets}$$

$$K \mapsto H^1(K, G) := H^1(\Gamma_K, G)$$

where $\Gamma_K = \text{Gal}(K_{\text{sep}}/K)$ the absolute Galois group of $K$,

$$H : \text{Fields}_{/k} \to \text{Abelian Groups}$$

$$K \mapsto H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) := H^d(\Gamma_K, \mathbb{Q}/\mathbb{Z}(d-1))$$

where $d$ is a positive integer and the Galois cohomology $H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$ is defined as in \cite{Me2} page 278, Section 2.1], \cite{Ba} pages 7–8, Section 2.3].

An $H$-invariant of $A$ (or a cohomological invariant of $A$, in short) is a natural transformation $a : A \to H$ of the functor $A$ to the functor $H$.

Explicitly, for any $K \in \text{Fields}_{/k}$, there is a map

$$a_K : H^1(K, G) \to H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$$

satisfying the conditions: If $K \subseteq K'$ are two objects in $\text{Fields}_{/k}$, then the following diagram commutes:

$$
\begin{array}{ccc}
H^1(K, G) & \xrightarrow{\alpha_K} & H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) \\
\downarrow & & \downarrow \\
H^1(K', G) & \xrightarrow{\alpha_{K'}} & H^d(K', \mathbb{Q}/\mathbb{Z}(d-1))
\end{array}
$$

Since $G$ is the constant group scheme, it follows that $\Gamma_K$ acts trivially on $G(K_{\text{sep}}) \simeq G$. Thus we may regard the set $H^1(K, G)$ as the set of all group homomorphisms from $\Gamma_K$ to the finite group $G$. On the other hand, the set $H^1(K, G)$ is naturally bijective to the set of $G$-torsors over $\text{Spec}(K)$, equivalently, the set of $G$-Galois algebras over $K$ \cite{KMRT} pages 388–389. Note that a $G$-Galois algebra $A$ over a field $K$ \cite{KMRT} page 288 is a $G$-Galois extension $A/K$ of commutative rings in the sense of Galois extensions of commutative rings in \cite{DI} page 80-84.

In $H^1(k, G)$, there is a distinguished element $\varphi_0 : \Gamma_k \to G$ which maps every element of $\Gamma_k$ to the identity element of $G$. A cohomological invariant $a$ is called normalized if $a_k(\varphi_0) = 0$ in $H^d(k, \mathbb{Q}/\mathbb{Z}(d-1))$.

A cohomological invariant $a$ is called constant if there is a $d$-cocycle $\gamma$ for the group $\Gamma_k$ such that for any field extension $f : k \to K$, $a_K$ is the constant map whose image is $[f^*(\gamma)]$ where $[f^*(\gamma)]$ is the cohomological class associated to the cocycle $f^*(\gamma)$ and $f^* : \Gamma_K \to \Gamma_k$ is the map induced by $f : k \to K$ \cite{GMS} page 10.

Every invariant $a$ can be written as (constant)+(normalized) (see \cite{GMS} page 11).

The set of all the $H$-invariants of $A$ is denoted by

$$\text{Inv}_{K}^d(G, \mathbb{Q}/\mathbb{Z}(d-1)) ;$$

it is an abelian group under the pointwise addition.

The set of all the normalized $H$-invariants of $A$ is denoted by

$$\text{Inv}_{K}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\text{norm}} ;$$

it is a subgroup of $\text{Inv}_{K}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$.

Definition 2.2. An invariant $a \in \text{Inv}_{K}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))$ is called unramified if for any field extension $K/k \in \text{Fields}_{/k}$, any $\varphi \in H^1(K, G)$, the image $a_K(\varphi)$ belongs to the kernel of $r_v : H^d(K, \mathbb{Q}/\mathbb{Z}(d-1)) \to H^{d-1}(k_v, \mathbb{Q}/\mathbb{Z}(d-2))$ for all $v$ where $v$ is a discrete $k$-valuation on $K$ with residue field $k_v$ (see \cite{GMS} page 19).

The map $r_v$ is called the residue map.

The subgroup of all the unramified normalized $H$-invariants of $A$ is denoted as

$$\text{Inv}_{\text{nr},K}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\text{norm}}.$$
Theorem 2.3 ([GMS] page 87, Proposition 33.10, [Mc2] page 289, Corollary 6.6). If \( k \) is retract rational over \( k \), then \( \text{Inv}_{nr,k}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\text{norm}} = 0 \). Consequently, if \( \text{Inv}_{nr,k}^d(G, \mathbb{Q}/\mathbb{Z}(d-1))_{\text{norm}} \neq 0 \), then \( k \) is not stably rational (resp. not retract rational) over \( k \).

Using Theorem 2.3, we may prove that \( \mathbb{Q}(C_8) \) is not retract rational over \( \mathbb{Q} \) by showing the non-triviality of \( \text{Inv}_{nr,k}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \) where \( G = C_8 \); this is carried out in [Mc2] page 292, Example 8.3, [Ba] Section 4.1 (and also in [GMS] pages 87–88, Proposition 33.15). Such proofs are different from those of the original proofs of Endo-Miyata [EM], Lenstra [Le], Saltman [Sa1].

Theorem 2.4. (1) ([Me2] page 290, Proposition 7.1, [Ba] Chapter 3)

\[
\text{Inv}_k^1(G, \mathbb{Q}/\mathbb{Z})_{\text{norm}} \simeq H^1(G, \mathbb{Q}/\mathbb{Z}),
\]

\[
\text{Inv}_k^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \simeq H^2(G, k^\times).
\]

(2) (V. J. Bailey [Ba] page 31) \( \text{Inv}_{nr,k}^1(G, \mathbb{Q}/\mathbb{Z})_{\text{norm}} = 0 \).

Definition 2.5 (V. J. Bailey [Ba] page 34). By Theorem 2.3, we have \( \text{Inv}_k^2(G, \mathbb{Q}/\mathbb{Z})_{\text{norm}} \simeq H^1(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, \mathbb{Z}) \) where the last isomorphism is given by the map \( \delta : H^1(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, \mathbb{Z}) \), the connecting homomorphism associated to the short exact sequence \( 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \). Since we have the cup product

\[
\cup : H^2(G, \mathbb{Z}) \otimes \mathbb{Z} H^0(G, k^\times) \to H^2(G, k^\times) \simeq \text{Inv}_k^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}
\]

which is injective by [Ba] page 34, Proposition, we define a degree two decomposable invariant as an element in the image of the following map:

\[
H^1(G, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} k^\times \to H^2(G, k^\times) \simeq \text{Inv}_k^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}
\]

\[
\chi \otimes b \mapsto \delta(\chi) \cup (b)
\]

where \( (b) \in H^0(G, k^\times) \) is identified with \( b \in k^\times \) through the isomorphism \( H^0(G, k^\times) \simeq (k^\times)^G = k^\times \).

A decomposable invariant is normalized [Ba] page 34. The subgroup of decomposable invariants in \( \text{Inv}_k^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \) is denoted as

\[
\text{Inv}_k^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}}.
\]

The subgroup of unramified invariants of \( \text{Inv}_k^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}} \) is denoted as

\[
\text{Inv}_{nr,k}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}}.
\]

Theorem 2.6 (V. J. Bailey). Let \( k \) be a field and \( G \) be a finite abelian group. Let \( G_0 \) be the 2-Sylow subgroup of \( G \) and write \( G_0 \simeq \mathbb{Z}/2^d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2^d_t \mathbb{Z} \) where \( d_1 \geq \cdots \geq d_t \geq 0 \).

(1) If \( \text{char} k = 2 \), then \( \text{Inv}_{nr,k}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}} = 0 \);

(2) If \( \text{char} k \neq 2 \) and \( k(\zeta_{2^d_i})/k \) is a cyclic extension, then \( \text{Inv}_{nr,k}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}} = 0 \);

(3) If \( \text{char} k \neq 2 \) and \( k(\zeta_{2^{d_i+1}})/k \) is not a cyclic extension for \( 1 \leq i \leq e (\leq t) \) while \( k(\zeta_{2^{d_i}})/k \) is cyclic, then

\[
\text{Inv}_{nr,k}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}} \simeq (\mathbb{Z}/2^d_i \mathbb{Z})^{e_i}.
\]

Proof. Let \( p_1, \ldots, p_m \) be the odd prime divisors of \( |G| \) and \( G_i \) be the \( p_i \)-Sylow subgroup of \( G \) for \( 1 \leq i \leq m \). Then \( G \simeq \bigoplus_{0 \leq i \leq m} G_i \). Thus we have

\[
\text{Inv}_{nr,k}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}} \simeq \bigoplus_{0 \leq i \leq m} \text{Inv}_{nr,k}(G_i, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}^{\text{dec}}
\]

by [Ba] page 42, Lemma 5. In other words, to prove Theorem 2.6 we may assume that \( G \) is an abelian \( p \)-group.

If \( \text{char} k \neq p \), we may apply Bailey’s theorem in [Ba] page 48. If \( \text{char} k = p \), since \( k(G) \) is rational over \( k \) by Gaschütz-Kuniyoshi’s Theorem [Ga], [Ku], we may apply Theorem 2.3 and we find that \( \text{Inv}_{nr,k}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} = 0 \).
3. Proof of Theorem 1.2

Lemma 3.1. Let $G$ be a finite group and $G^{ab} = G/[G, G]$ where $[G, G]$ is the commutator subgroup of $G$. Then the canonical projection $f : G \to G^{ab}$ induces an injective morphism from $\text{Inv}^2_{nr,k}(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}$ to $\text{Inv}^2_{nr,k}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}}$.

Proof. Step 1. Note that $\text{Hom}(G^{ab}, \mathbb{Q}/\mathbb{Z}) \simeq \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$. In fact, if $\chi : G \to \mathbb{Q}/\mathbb{Z}$ is any group homomorphism, then $\text{Kernel}(\chi) \supseteq [G, G]$ because the image of $\chi$ is abelian. Thus $\chi$ factors through $G^{ab}$, i.e. there is a group homomorphism $\chi : G^{ab} \to \mathbb{Q}/\mathbb{Z}$ such that $\chi \cdot f = \chi$. Conclusion: $\chi \mapsto \chi \cdot f$ provides an isomorphism for $H^1(G^{ab}, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z})$.

Thus we have a commutative diagram

$$
\begin{array}{ccc}
H^1(G^{ab}, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} k^x & \overset{\simeq}{\longrightarrow} & \text{Inv}^2_k(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}} \\
\downarrow & & \downarrow \phi \\
H^1(G, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} k^x & \overset{\simeq}{\longrightarrow} & \text{Inv}^2_k(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}}
\end{array}
$$

where the horizontal morphisms and the left vertical morphism are isomorphisms. It follows that the right vertical morphism $\phi$ is also an isomorphism. Note that $\phi$ is defined by $\phi(\delta(\chi) \cup (b)) = \delta(\chi \cdot f) \cup (b)$ for any $\chi \in H^1(G^{ab}, \mathbb{Q}/\mathbb{Z})$, any $b \in k^x$.

Step 2. Recall the definition of a degree two decomposable invariant.

Let $\chi : G \to \mathbb{Q}/\mathbb{Z}$ be a group homomorphism and $b \in k^x$. Consider the invariant $\delta(\chi) \cup (b)$. Write $H$ for the kernel of $\chi$, and let the image of $\chi$ be $(1/n) \subset \mathbb{Q}/\mathbb{Z}$. Choose $\sigma \in G/H$ such that $\chi$ induces an isomorphism of $G/H \cong (1/n)$ onto $(1/n)$ with $\chi(\sigma) = 1/n$.

By [La] pages 37-40 $\delta(\chi) \cup (b)$ is defined as follows: for any field extension $K/k$, for any $G$-Galois algebra $L/K$, $(\delta(\chi) \cup (b))_K(L)$ is the similarity class, in $Br(K) \simeq H^2(K, \mathbb{Q}/\mathbb{Z}(1))$, of the cyclic algebra $[E/K, \sigma, b]$ where $E = L^H$ (the fixed ring of $L$ under $H$). It is known that $E$ is a $G/H$-Galois algebra over $K$ by the fundamental theorem of Galois theory (see [Di] page 80, page 143]). For the definition of the cyclic algebra $[E/K, \sigma, b]$, see [Di] page 49, page 71] and [DI] page 121].

Step 3. There is a morphism $f^* : \text{Inv}^2_k(G^{ab}, \mathbb{Q}/\mathbb{Z}(1)) \to \text{Inv}^2_k(G, \mathbb{Q}/\mathbb{Z}(1))$ associated to the group homomorphism $f : G \to G^{ab}$ (see [GMS] page 33)).

If $a \in \text{Inv}^2_k(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))$, then $f^*(a) \in \text{Inv}^2_k(G, \mathbb{Q}/\mathbb{Z}(1))$ is defined as follows: for any field extension $K/k$, for any $G$-Galois algebra $L/K$, $f^*(a)_K(L) := a_K(L_0)$ where $L_0 = L^{[G,G]}$ (the fixed ring of $L$ under the action of $[G, G]$) by the construction of the $G^{ab}$-torsor $T$ in the proof of [GMS] page 33, Proposition 13.1]. Note that $L_0$ is a $G^{ab}$-Galois algebra over $K$; thus $a_K(L_0)$ is well-defined. Also note that $f^*$ sends the normalized invariants (resp. the unramped invariants) to the normalized invariants (resp. the unramped invariants).

Step 4. Consider the following diagram:

$$
\begin{array}{ccc}
\text{Inv}^2_{nr,k}(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}} & \overset{\phi}{\longrightarrow} & \text{Inv}^2_{nr,k}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}} \\
\downarrow f^* & & \downarrow f^* \\
\text{Inv}^2_{nr,k}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}} & \longrightarrow & \text{Inv}^2_{nr,k}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}}
\end{array}
$$

We will show that the above diagram is a commutative diagram.

For any field extension $K/k$, for any $G$-Galois algebra $L/K$, we will show that $(\delta(\chi \cdot f) \cup (b))_K(L)$ and $f^*(\delta(\chi \cdot f) \cup (b))_K(L)$ represent similar central simple $K$-algebras.

Write $H_0$ for the kernel of $\chi$, and let $H$ be the subgroup of $G$ such that $[G, G] \subset H \subset G$ and $H/[G, G] \simeq H_0$. Then $H$ is the kernel of $\chi \cdot f$. Define $E := L^H$ as in Step 2. Then $(\delta(\chi \cdot f) \cup (b))_K(L)$ is the similarity class of the cyclic algebra $[E/K, \sigma, b]$. For the notation of the cyclic algebra, see [Dr], page 49, page 70]; note that the definition $[E/K, \sigma, b]$ is still valid when the field extension $E/K$ is replaced by a Galois algebra (see [DI]).

On the other hand, by Step 3, $(\delta(\chi \cdot f) \cup (b))_K(L_0) = (\delta(\chi) \cup (b))_K(L_0)$ where $L_0 = L^{[G,G]}$. Since $L_0^{H_0} = (L^{[G,G]})^{H_0} = L^H = E$, we find that $(\delta(\chi) \cup (b))_K(L_0)$ is the similarity class of $[E/K, \sigma, b]$. Hence the result.

Step 5. From the above commutative diagram, we find that $\text{Inv}^2_{nr,k}(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}}$ is mapped injectively by $f^*$ into $\text{Inv}^2_{nr,k}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}}$. Thus the image of $\text{Inv}^2_{nr,k}(G^{ab}, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}}$ lies in $\text{Inv}^2_{nr,k}(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{dec}}$, because $f^*$ preserves the unramped invariants.

□
Proof of Theorem 1.2

Let $N \triangleleft G$ and $G/N \cong C_{2^n}$ where $n \geq 3$. It follows that $N \supset [G,G]$ and there is a surjection $G^{ab} \to G/N \cong C_{2^n}$.

Let $G_0$ be the 2-Sylow subgroup of $G^{ab}$ and write $G_0 \cong \mathbb{Z}/2^{d_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2^{d_i}\mathbb{Z}$ where $d_1 \geq \cdots \geq d_i \geq 1$. Since $G^{ab}$ has a quotient isomorphic to $C_{2^n}$ with $n \geq 3$, it follows that $d_i \geq n$.

By the assumption that $k(\zeta_{2^n})/k$ is not cyclic, it follows that $k(\zeta_{2^n})/k$ is not cyclic. By Theorem 2.6, $\text{Inv}_{nr,k}(G^{ab},\mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \neq 0$. Hence $\text{Inv}_{nr,k}(G,\mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \neq 0$ by Lemma 5.1. By Theorem 2.3, we find that $k(G)$ is not retract rational over $k$.

Example 3.2. In [GMS] page 88, Theorem 33.16, Serre proves the following result: Let $G$ be a finite group whose 2-Sylow subgroup $P$ is a cyclic group of order $\geq 8$. Then $\text{Rat}(G/\mathbb{Q})$ (resp. $\text{Noe}(G/\mathbb{Q})$) is false.

By Burnside’s normal $p$-complement theorem (see [BS] page 160, Corollary 5.14)], $G = N \rtimes P$ where $N \triangleleft G$ and $|N|$ is odd. It follows that $G/N \cong C_{2^n}$ where $n \geq 3$. Thus we may apply Theorem 1.2 to obtain an alternative proof of Serre’s theorem when the field $\mathbb{Q}$ is replaced by a field $k$ such that $\text{char } k = 0$ and $k(\zeta_{2^n})/k$ is not cyclic.

The conclusion, for the split case $G = N \rtimes H$ where $N$ and $H$ are any subgroups, is anticipated by Saltman. In fact, Saltman proves that, for a finite group $G = N \rtimes H$ (where $N \triangleleft G$), if $k(G)$ is (S) retract rational over an infinite field $k$, so is $k(H)$ over $k$ (see [Sa1] page 265, Theorem 3.1 (b)). Note that, in [Sa1], the result is formulated in terms of generic Galois extensions. However, it is known that a group $G$ has a generic Galois extension over $k$ if and only if $k(G)$ is (S) retract rational over $k$ by [Sa2] Theorem 3.12 (also see [Ka3] Theorem 1.2 and pages 28–29).

Example 3.3. We will exhibit a non-split extension $1 \to N \to G \to G/N \to 1$. Define $G = \langle \sigma, \tau : \sigma^{64} = \tau^{32} = 1, \sigma^{32} = \tau^{16}, \tau^{-1} \sigma \tau = \sigma^7 \rangle$, $N = \langle \sigma \rangle$. Then $G/N \cong \langle \tau \rangle \cong C_{32}$. This example is taken with the parameters $(r, s, u, t, w) = (3, 1, 0, 1, 0, 1)$ in [Ka2] Definition 5.5] and [Ka2] Lemma 5.6 (ii)].

By Theorem 1.3, if $k$ is a field with $\text{char } k = 0$ such that $k(\zeta_{32})/k$ is not cyclic, then $k(G)$ is not retract rational over $k$. However, if $k' := k(\zeta_{64})$, then $k'(G)$ is rational over $k'$ by [Ka2] Theorem 1.5].

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