Modified wave operators for discrete Schrödinger operators with long-range perturbations

Shu Nakamura

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Abstract

We consider the scattering theory for discrete Schrödinger operators on \( \mathbb{Z}^d \) with long-range potentials. We prove the existence of modified wave operators constructed in terms of solutions of a Hamilton-Jacobi equation on the torus \( \mathbb{T}^d \).

1 Introduction

We consider the discrete Schrödinger operator

\[
Hu[n] = -\frac{1}{2} \Delta u[n] + V[n]u[n], \quad n \in \mathbb{Z}^d,
\]

for \( u \in \mathcal{H} = \ell^2(\mathbb{Z}^d) \), where

\[
\Delta u[n] = \sum_{|m-n|=1} u[m],
\]

and \( V \) is a real-valued function on \( \mathbb{Z}^d \). We denote discrete variables using the square braces \([\cdot]\), and continuous variables using the round braces \((\cdot)\). If \( V \) is bounded, \( H \) is a bounded self-adjoint operator on \( \ell^2(\mathbb{Z}^d) \).

The discrete Schrödinger operator \( H \) has many common properties as the continuous Schrödinger operator on \( \mathbb{R}^d \). For example, if \( V \) is short-range type, i.e.,

\[
|V[n]| \leq C(1 + |n|)^{-\mu}, \quad n \in \mathbb{Z}^d,
\]

with some \( \mu > 1 \) and \( C > 0 \), then the scattering theory is constructed in the standard way. Namely, by setting \( H_0 = -\frac{1}{2} \Delta \), we can show the wave operators

\[
W_\pm = \text{s-lim}_{t \to \pm \infty} e^{itH}e^{-itH_0}
\]

*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro, Tokyo, 153-8914 Japan. E-mail: shu@ms.u-tokyo.ac.jp. The research was partially supported by JSPS Grant Kiban (A) 21244008.
exist; it is an isometry into $\mathcal{H}_{ac}(H)$, the absolutely continuous subspace of $H$; the intertwining property: $HW_{\pm} = W_{\pm}H_0$ holds; moreover they are asymptotically complete: $\text{Ran } W_{\pm} = \mathcal{H}_{ac}(H)$ (see, e.g, Boutet de Monvel, Sahbani \cite{1}, Isozaki, Korotyaev \cite{7} and references therein).

We will consider the long-range case, i.e., when $0 < \mu \leq 1$. If $V$ is long-range type, the wave operators do not exist in general, and we need to introduce \textit{modified} wave operators. In order to state our main result, we introduce several notations.

Following the standard notation, we denote $W \in S^m(\mathbb{R}^d)$, $m \in \mathbb{R}$, if $W \in C^\infty(\mathbb{R}^d)$ and for any multi-index $\alpha \in \mathbb{Z}^d$ there is $C_\alpha > 0$ such that

$$|\partial^\alpha_x W(x)| \leq C_\alpha \langle x \rangle^{m-|\alpha|}, \quad x \in \mathbb{R}^d,$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$. We note that when we consider scattering theory for (continuous) Schrödinger operators, $V \in S^{-\mu}(\mathbb{R}^d)$ with $\mu > 0$ is a standard assumption. A natural analogue for the discrete case is the following. We denote

$$\tilde{\partial}_j u[n] = u[n] - u[n - e_j], \quad n \in \mathbb{Z}^d, \quad j = 1, \ldots, d,$$

where $\{e_j\}$ is the standard orthonormal basis of $\mathbb{R}^d$, and $u[\cdot]$ is a function on $\mathbb{Z}^d$. We denote $\tilde{\partial}^\alpha = \prod_{j=1}^d \tilde{\partial}^{\alpha_j}$ for $\alpha \in \mathbb{Z}^d_+$ as usual.

\textbf{Definition 1.1.} Let $V$ be a function on $\mathbb{Z}^d$, and let $m \in \mathbb{R}$. We denote $V \in S^m(\mathbb{Z}^d)$ if for any $\alpha \in \mathbb{Z}^d_+$ there is $C_\alpha > 0$ such that

$$|\tilde{\partial}^\alpha V[n]| \leq C_\alpha \langle n \rangle^{m-|\alpha|}, \quad n \in \mathbb{Z}^d.$$

We suppose $V \in S^{-\mu}(\mathbb{Z}^d)$ with $\mu > 0$. Then, the essential spectrum on $H$ is $[-d, d]$, and we are interested in the structure of the spectrum of $H$ in $[-d, d]$.

In the next section, we show that we can extend $V \in S^m(\mathbb{Z}^d)$ to an element in $S^m(\mathbb{R}^d)$. We denote $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$. Then we may consider the classical mechanics generated by $p(x, \xi)$, the symbol of $H$ on $T^*\mathbb{T}^d \cong \mathbb{R}^d \times \mathbb{T}^d$, that is

$$p(x, \xi) = \sum_{j=1}^d \cos(\xi_j) + V(x), \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{T}^d.$$

We denote the set of threshold energies by $\mathcal{T} = \{-d, -d+2, \ldots, d-2, d\}$. For a given interval $I \subset [-d, d] \setminus \mathcal{T}$, we can find solutions to the Hamilton-Jacobi equation on $\mathbb{T}^d$:

$$\frac{\partial}{\partial t} \Phi_{\pm}(t, \xi) = p(\partial_\xi \Phi_{\pm}(t, \xi), \xi), \quad \xi \in \mathbb{T}^d, \quad \pm t \geq 0, \quad p_0(\xi) \in I,$$

such that

$$|\Phi_{\pm}(t, \xi) - tp_0(\xi)| = O(|t|^{1-\mu})$$
as \( t \to \pm \infty \), where \( p_0(\xi) = \sum_{j=1}^{d} \cos(\xi_j) \).

We denote the discrete Fourier transform by \( F \), i.e.,

\[
F u(\xi) := (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} e^{-in \cdot \xi} u[n], \quad \xi \in \mathbb{T}^d = (\mathbb{R}/2\pi \mathbb{Z})^d.
\]

We note \( F \) is a unitary map from \( \ell^2(\mathbb{Z}^d) \) to \( L^2(\mathbb{T}^d) \). For a function \( f(\xi) \) on \( \mathbb{T}^d \), we denote

\[
f(D_x)u = F^* (f(\xi)(Fu)(\xi)), \quad u \in \ell^2(\mathbb{Z}^d).
\]

Using these, we can state our main result:

**Theorem 1.1.** Suppose \( V \in S^{-\mu}(\mathbb{Z}^d) \) with \( \mu > 0 \). Let \( I \subseteq [-d, d] \setminus \mathcal{T} \), and and let \( \Phi_{\pm}(t, \xi) \) be as above (or as constructed in Section 4). Then the modified wave operators

\[
W_{\Phi}^\pm(I) = \lim_{t \to \pm \infty} e^{itH} e^{-i\Phi(t, D_x)} E_I(H_0)
\]

exist and they are isometry from \( \ell^2(\mathbb{Z}^d) \) to \( \mathcal{H}_{ac}(H) \). Moreover, the intertwining property:

\[
HW_{\Phi}^\pm(I) = W_{\Phi}^\pm(I) H_0
\]

holds.

We expect the asymptotic completeness: \( \text{Ran} \ W_{\Phi}^\pm(I) = \mathcal{H}_{ac}(H) \) holds, though we do not discuss it in this paper.

The long-range scattering theory for Schrödinger equation has long history, starting from the pioneering work by Dollard [4]. We refer Reed-Simon [9] §XI.9, Yafaev [10] Chapter 10, Derezinski-Gérard [3] §4.7, and references therein. We follow the argument of Hörmander [5] in this paper to construct modified wave operators.

We prepare a simple extension lemma in Section 2. We discuss the classical mechanics on \( \mathbb{T}^d \) in Section 3, and solutions to the Hamilton-Jacobi equation are constructed in Section 4. We prove Theorem 1.1 in Section 5.

## 2 Preliminaries

Here we construct an extension of \( V \in S^m(\mathbb{Z}^d) \) to a smooth function \( \tilde{V}(x) \) on \( \mathbb{R}^d \).

**Lemma 2.1.** Suppose \( V[\cdot] \in S^m(\mathbb{Z}^d) \). Then there is \( \tilde{V} \in S^m(\mathbb{R}^d) \) such that it is real-valued and \( \tilde{V}(n) = V[n] \) for any \( n \in \mathbb{Z}^d \).

**Proof.** We interpolate \( V[n] \) using a window function (see, e.g., Oppenheim-Schafer-Buck [8] §7.2). Let \( \chi_0 \in C_0^\infty(\mathbb{R}^d) \) such that
Since $\chi$ is supported in $[-\frac{3}{2}\pi, \frac{3}{2}\pi]^d$, we may write
\[
F(\partial_j \tilde{V})(\xi) = \left[ \frac{e^{i\xi_j/2}\xi_j}{2\sin(\xi_j/2)} \right] \hat{\chi}_0(\xi) F(\partial_j \tilde{V})(\xi) = \hat{\chi}_j(\xi) F(\partial_j \tilde{V})(\xi),
\]
with $\hat{\chi}_j \in C_c^\infty(\mathbb{R}^d)$. Thus we learn
\[
\partial_j \tilde{V}(x) = (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} \chi_j(x - n) \partial_j \tilde{V}[n],
\]
where $\chi_j = \mathcal{F} \hat{\chi}_j$. Since $\partial_j V[n] = O((n/m)^{-1})$ as $|n| \to \infty$, we have $\partial_j \tilde{V}(x) = O((x/m)^{-1})$ as $|x| \to \infty$. Repeating this procedure, we conclude $\tilde{V} \in S^m(\mathbb{R}^d)$.

3 Classical mechanics

In the following we suppose $V \in S^{-\mu}(\mathbb{Z}^d)$ with $\mu > 0$. Let $\tilde{V}(x) \in S^m(\mathbb{R}^d)$ be an extension of $V[n]$, and we write $\tilde{V}(x) = \bar{V}(x)$ for simplicity. The existence of such $\tilde{V}$ is shown in Lemma 2.1, but it is not unique, and we may choose different extension. For example, if $V[n] = c(n)^{-\mu}$, then it is natural to choose $\tilde{V}(x) = c(n)^{-\mu}$, which is different from the extension in Lemma 2.1.

We now construct a classical mechanics on $T^*T^d$ corresponding to the discrete Schrödinger operator. We write the symbols of $H$ and $H_0$ by

\[ p(x, \xi) = p_0(\xi) + V(x), \quad p_0(\xi) = \sum_{j=1}^{d} \cos(\xi_j) \]

for $(x, \xi) \in \mathbb{R}^d \times \mathbb{T}^d \cong T^*T^d$, respectively. It is easy to see

\[ H_0 u = F^*(p_0(\xi)(F u)(\xi)) \quad \text{for} \quad u \in \ell^2(\mathbb{Z}^d). \]

We consider the solutions to the Hamilton equation

\[
\frac{d}{dt} x_j(t) = \frac{\partial p}{\partial \xi_j}(x, \xi) = \sin(\xi_j), \quad \frac{d}{dt} \xi_j(t) = -\frac{\partial p}{\partial x_j}(x, \xi) = -\frac{\partial V}{\partial x_j}(x),
\]

with an initial condition $(x(0), \xi(0)) = (x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{T}^d$. In this section, we study long-time behavior of the solution.

Let $\mathcal{I} = \{-d, -d+2, \ldots, d-2, d\}$ be the set of threshold energies for $p_0(\xi)$. Note

\[ \sin(\xi_j) = 0, \quad j = 1, \ldots, d \iff \xi_j \in \pi \mathbb{Z}, \quad j = 1, \ldots, d \]

\[ \implies p_0(\xi) \in \mathcal{I}. \]

We denote the velocity by

\[ v(\xi) = (\sin(\xi_1), \ldots, \sin(\xi_d)) \in \mathbb{R}^d, \quad \xi \in \mathbb{T}^d. \]

We compute

\[
\frac{d}{dt} |x(t)|^2 = 2 \sum_{j=1}^{d} v_j(\xi(t)) x_j(t),
\]

\[
\frac{d^2}{dt^2} |x(t)|^2 = 2 \sum_{j=1}^{d} v_j(\xi(t))^2 - 2 \sum_{j=1}^{d} x_j(t) \frac{\partial V}{\partial x_j}(x(t)). \quad (3.1)
\]

We write

\[ k(\xi) = |v(\xi)|^2 = \sum_{j=1}^{d} (\sin(\xi_j))^2. \]

We note $k(\xi) > 0$ if $p_0(\xi) \notin \mathcal{I}$. For $I \subset (-d, d)$, we set

\[ \Omega_\pm(I, R) = \{(x, \xi) \in \mathbb{R}^d \times \mathbb{T}^d \mid p(x, \xi) \in I, |x| \geq R, \pm x \cdot v(\xi) \geq 0\}. \]
Proposition 3.1. Let $I \in [-d, d] \setminus \mathcal{T}$ and $\Omega_{\pm}(I, R)$ be as above. Then there are $R_0 > 0$ and $\delta > 0$ such that

$$|x(t)| \geq \sqrt{R^2 + \delta t^2} \quad \text{for } \pm t \geq 0,$$

if the initial condition $(x_0, \xi_0) \in \Omega_{\pm}(I, R)$ with $R \geq R_0$.

Proof. We choose $\delta > 0$ so that $(I + [-\delta, \delta]) \cap \mathcal{T} = \emptyset$ and

$$\inf \{ k(\xi) \mid p_0(\xi) \in I + [-\delta, \delta] \} \geq 2\delta.$$

We also choose $R_0 > 0$ so that $|V(x)| \leq \delta$, $|x \cdot \nabla V(x)| \leq \delta$ if $|x| \geq R_0$.

If $|x| \geq R_0$ and $p(x, \xi) \in I$, then $p_0(x, \xi) \in I + [-\delta, \delta]$, and hence $k(\xi) \geq 2\delta$. By virtue of (3.1), we then learn

$$\frac{d^2}{dt^2}|x(t)|^2 = 2k(\xi) - 2x \cdot \nabla V(x) \geq 2\delta$$

for such $(x, \xi)$. If $(x_0, \xi_0) \in \Omega_{\pm}(I, R)$ then $\pm \frac{d}{dt}|x(t)|^2 \geq 0$ at $t = 0$, and hence

$$|x(t)|^2 \geq |x_0|^2 + \delta t^2 \quad \text{for } \pm t \geq 0.$$

This implies, in particular,

$$|x(t)| \geq \frac{1}{\sqrt{2}}(R + \sqrt{\delta}|t|) \quad \text{for } \pm t \geq 0.$$

Once this estimate is established, following estimates are proved exactly same way as in the Euclidean space case (see, e.g., [5], [2]). We denote the solution to the Hamilton equation with the initial condition $(y, \eta)$ by

$$x(t) = x(t, y, \eta), \quad \xi(t) = \xi(t, y, \eta).$$

Proposition 3.2. Let $(x_0, \xi_0) \in \Omega_{\pm}(I, R)$ and $x(t) = x(t, x_0, \xi_0)$, $\xi(t) = \xi(t, x_0, \xi_0)$. Then

$$\xi_\pm = \lim_{t \to \pm \infty} \xi(t)$$

exists. Moreover,

$$|\xi(t) - \xi_\pm| \leq C \langle t \rangle^{-\mu} \quad \pm t \geq 0, \quad (3.2)$$

$$|x(t) - tv(\xi_\pm)| \leq C \langle t \rangle^{1-\mu} \quad \pm t \geq 0. \quad (3.3)$$
We note,
\[ |x(t) - tv(\xi(t))| \leq C(t)^{1-\mu} \]
is proved at first by the Hamilton equation, and combining this with (3.2) we obtain (3.3).

**Proposition 3.3.** There is \( C > 0 \) such that
\[
\left| \frac{\partial}{\partial y} \xi(t, y, \eta) \right| \leq CR^{1-\mu}, \quad \left| \frac{\partial}{\partial \eta} \xi(t, y, \eta) \right| \leq CR^{-\mu}
\]
for \( R \geq R_0, \ (y, \eta) \in \Omega_\pm(I, R), \pm t \geq 0 \). Moreover, for any \( \alpha, \beta \in \mathbb{Z}_+^d \), there is \( C_{\alpha\beta} > 0 \) such that
\[
\left| \partial_\alpha^\nu \partial_\eta^\beta (x(t, y, \eta) - y) \right| \leq C_{\alpha\beta}|t|,
\left| \partial_\alpha^\nu \partial_\eta^\beta \xi(t, y, \eta) \right| \leq C_{\alpha\beta}
\]
for any \( (y, \eta) \in \Omega_\pm(I, R), \pm t \geq 0 \).

## 4 Construction of solutions to the Hamilton-Jacobi equation

As before, we set \( p(x, \xi) = \sum \cos(\xi_j) + V(x) \), and construct solutions to the Hamilton-Jacobi equation
\[
\frac{\partial}{\partial t} \Phi_\pm(t, \xi) = p(\partial_\xi \Phi_\pm(t, \xi), \xi), \quad \pm t \geq 0
\]
for \( \xi \in \{ \xi \mid p_0(\xi) \in I \} \), where \( I \in (-d, d) \setminus \mathcal{T} \). We suppose \( I + [-\delta, \delta] \subset (-d, d) \setminus \mathcal{T} \) with some \( \delta > 0 \).

The characteristic equations for (4.1) is
\[
\xi' = -\nabla V(x), \quad x' = v(\xi), \quad u' = p(x, \xi) - x \cdot \nabla V(x),
\]
where \( x = \frac{\partial}{\partial \xi} \Phi_\pm \) and \( u = \Phi_\pm \) on the characteristic curves. The first two equations are the Hamilton equation, and we solve the equation with the initial condition:
\[
u(0) = \pm R_1 p_0(\eta), \quad \xi(0) = \eta \in \mathbb{T}^d,
\]
with sufficiently large \( R_1 > 0 \). Then
\[
x(0) = \frac{\partial u}{\partial \xi}(0) = \pm R_1 v(\eta),
\]
and we choose \( R_1 \) so large that
\[
R := R_1 \cdot \inf \{ |v(\eta)| \mid p_0(\eta) \in I + [-\delta, \delta] \}
\]
satisfies the condition of Proposition 3.1, and \( CR^{-\mu} \ll 1 \) in Proposition 3.3. We denote

\[
\Lambda_t : \eta \mapsto \xi(t, \pm R_1 v(\eta), \eta), \quad \pm t \geq 0.
\]

Then \( \Lambda_t \) is locally diffeomorphic, and the derivatives are uniformly bounded in \( t \). If \( R_1 \) is sufficiently large, we can easily show that \( \Lambda_t^{-1} \) is well-defined on \( \{ \xi \mid p_0(\xi) \in I \} \), and the image is contained in \( \{ \xi \mid p_0(\xi) \in I + [-\delta, \delta] \} \). Thus the solution to (4.1) is given by

\[
\Phi_\pm(t, \xi) = u \circ \Lambda_t^{-1}, \quad u(t, \eta) = R_1 p_0(\eta) + \int_0^t (p(x(s), \xi(s)) - x(s) \cdot \nabla V(x(s))) ds.
\]

Moreover, by the construction, we have

\[
\partial_\xi \Phi_\pm(t, \xi) = x(t, Rv(\eta), \eta) = \Lambda_t^{-1}(\xi).
\]

Thus properties of \( \partial_\xi^\alpha \Phi_\pm, \alpha \neq 0 \), follow from properties of \( x(t, y, \eta) \). In particular, we have

\[
|\partial_\xi^\alpha \Phi_\pm(t, \xi)| \leq C(t), \quad \pm t \geq 0, \quad \xi \in \{ \xi \mid p_0(\xi) \in I \},
\]

if \( \alpha \neq 0 \). Also, by the definition of \( u \), we learn

\[
|\Phi_\pm(t, \xi) - tp_0(\xi)| \leq C(t)^{1-\mu}, \quad \pm t \geq 0,
\]

\[
|\partial_\xi (\partial_\xi \Phi_\pm(t, \xi) - tv(\xi))| \leq C(t)^{1-\mu}, \quad \pm t > 0,
\]

for any \( \alpha \in \mathbb{Z}^d_+ \).

## 5 Existence of modified wave operators

Let \( \Phi_\pm(t, \xi) \), etc., be as in Section 3. We fix \( I \subseteq (-d, d) \setminus \mathcal{T} \). We show

**Theorem 5.1.** Suppose \( V \in S^{\mu}(\mathbb{Z}^d) \) with \( \mu > 0 \). Then the modified wave operators

\[
W^\Phi_\pm(I) = \text{s-lim}_{t \to \pm \infty} e^{itH} e^{-i\Phi_\pm(t, D_\xi)} E_I(H_0)
\]

exists.

Generally, we follow the argument of Hörmander [5] to prove Theorem 5.1. We denote

\[
D(I) = \{ \xi \in \mathbb{T}^d \mid p_0(\xi) \in I, \cos(\xi_j) \neq 0, j = 1, \ldots, d \}.
\]

Then \( C^\infty_0(D(I)) \) is dense in \( \text{Ran} \ E_I(H_0) \). Thus it suffices to show the existence of \( W_\pm(I) \varphi \) for \( \varphi \in C^\infty_0(D(I)) \). Moreover, by the partition of unity, we
may suppose \( \varphi \) is supported in an arbitrarily small neighborhood of a point \( \xi_0 \in D(I) \). We compute
\[
\varphi(t, x) := e^{-i\Phi_{\pm}(t, D_x) \varphi(x)} = (2\pi)^{-d/2} \int_{\mathbb{T}^d} e^{i(x \cdot \xi - \Phi_\pm(t, \xi))} \hat{\varphi}(\xi) d\xi
\]
for \( x \in \mathbb{Z}^d \), where \( \hat{\varphi} = F\varphi \). We write
\[
x \cdot \xi - \Phi_{\pm}(t, \xi) = t \left( \frac{x}{t} \cdot \xi - \frac{1}{t} \Phi_\pm(t, \xi) \right)
\]
and consider \( t \) as the large parameter when we apply the stationary phase method. The stationary phase point is then given by
\[
x \frac{1}{t} = \frac{1}{t} \partial_\xi \Phi_\pm(t, \xi) = \nu(\xi) + O((t)^{-\mu}), \quad \pm t > 0.
\]
From this, we also learn that the determinant of the Hessian of \( \frac{1}{t} \Phi_{\pm}(t, \xi) \) is \( \prod \cos(\xi_j) + O((t)^{-\mu}) \), and hence its absolute value is uniformly bounded from below by a positive constant on the support of \( \hat{\varphi} \) with large \( t \). Also the derivatives of \( \frac{1}{t} \Phi_{\pm}(t, \xi) \) in \( \xi \) are uniformly bounded in \( t \) on the support of \( \hat{\varphi} \).

Let \( D' \subset D(I) \) be a small neighborhood of \( \text{supp } \hat{\varphi} \). We denote
\[
G_t : \xi \mapsto \partial_\xi \Phi_{\pm}(t, \xi).
\]
We may suppose \( G_t \) is diffeomorphis on \( D' \), and we note \( \text{vol}(G_t(D')) = O((t)^d) \) as \( t \to \pm \infty \).

By the non stationary phase method, we first learn
\[
|\varphi(t, x)| \leq C_N(|x| + |t|)^{-N} \quad \text{if } x \notin G_t(D'), \quad (5.1)
\]
with any \( N \). On the other hand, by the stationary phase method, we have
\[
\varphi(t, x) = t^{-d/2} J(t, \eta) \hat{\varphi}(\eta) + O((t)^{-d/2-1}) \quad \text{for } x \in G_t(D'), \quad (5.2)
\]
where \( \eta = G_t^{-1}(x) \), \( J(t, \eta) \) is a uniformly bounded function of \( t, \eta \), depending only on \( \Phi_{\pm}(t, \eta), \partial_\xi \Phi_{\pm}(t, \eta) \) and \( \partial_\xi \partial_\xi \Phi_{\pm}(t, \eta) \). In particular, \( J(t, \eta) \) is independent of \( \varphi \). (see, e.g., [6], §7.7).

Now we estimate
\[
\frac{d}{dt} \left( e^{itH} e^{-i\Phi_{\pm}(t, D_x) \varphi} \right) = i e^{itH} (H - \partial_\xi \Phi_{\pm}(t, D_x)) e^{-i\Phi_{\pm}(t, D_x)} \varphi
\]
and apply the Cook-Kuroda method. We note, by (4.1), it suffices to show
\[
\int_{\mathbb{R}^d} \left\| (H - p(\partial_\xi \Phi_{\pm}(t, D_x), D_x)) e^{-i\Phi_{\pm}(t, D_x)} \varphi \right\| dt < \infty. \quad (5.3)
\]
By (5.1), it is easy to see
\[
\int_{0}^{\pm}\int_{\chi_{G_{t}(D')}}\chi(x)(V(x) - V(\partial_{\xi}\Phi_{\pm}(t, D_{x})))(\varphi(t, x))\, dt < \infty.
\]
On the other hand, by (5.2), on $G_{t}(D')$ we have
\[
V(x)\varphi(t, x) = t^{-d/2}V(x)J(t, \eta)(\hat{\varphi})(\eta) + O(|t|^{-d/2-1-\mu}),
\]
\[
V(\partial_{\xi}\Phi_{\pm}(t, D_{x}))(\varphi(t, \xi)) = t^{-d/2}J(t, \xi)V(\partial_{\xi}\Phi_{\pm}(t, \eta))\hat{\varphi}(\eta) + O(|t|^{-d/2-1-\mu}),
\]
where $x = \partial_{\xi}\Phi_{\pm}(t, \eta)$, and the leading terms of these coincide. Thus we learn
\[
\int_{0}^{\pm}\int_{\chi_{G_{t}(D')}}\chi(x)(V(x) - V(\partial_{\xi}\Phi_{\pm}(t, D_{x})))(\varphi(t, x))\, dt
\]
\[
\leq \int_{0}^{\pm}\int_{\chi_{G_{t}(D')}}\chi(x)(V(x) - V(\partial_{\xi}\Phi_{\pm}(t, D_{x})))(\varphi(t, x))\, dt
\]
\[
\leq C\int_{0}^{\pm}\int_{\chi_{G_{t}(D')}}\chi(x)(V(x) - V(\partial_{\xi}\Phi_{\pm}(t, D_{x})))(\varphi(t, x))\, dt
\]
\[
\leq C\int_{0}^{\pm}\int_{\chi_{G_{t}(D')}}\chi(x)(V(x) - V(\partial_{\xi}\Phi_{\pm}(t, D_{x})))(\varphi(t, x))\, dt
\]
\[
\leq C\int_{0}^{\pm}\int_{\chi_{G_{t}(D')}}\chi(x)(V(x) - V(\partial_{\xi}\Phi_{\pm}(t, D_{x})))(\varphi(t, x))\, dt
\]
\[
\leq C\int_{0}^{\pm}\int_{\chi_{G_{t}(D')}}\chi(x)(V(x) - V(\partial_{\xi}\Phi_{\pm}(t, D_{x})))(\varphi(t, x))\, dt
\]
Combining these, we conclude (5.3), and completes the proof.

Remark 1. We note that the choice of the extension $V(x)$ on $\mathbb{R}^{d}$ is not unique, and hence $\Phi_{\pm}(t, \xi)$ are not unique either. The modified wave operators $W_{\Phi}^{\pm}$ depend on such construction, but in general, they are equivalent up to multiplication by unitary Fourier multipliers from the right. Actually, if
\[
W_{\Phi}^{\pm} = \text{s-lim}_{t \to \pm\infty} e^{iHt}e^{-i\Phi_{\pm}(t, D_{x})} \quad \text{and} \quad W_{\Psi}^{\pm} = \text{s-lim}_{t \to \pm\infty} e^{iHt}e^{-i\Psi_{\pm}(t, D_{x})}
\]
each exist, then the limit
\[
G_{\pm} = \text{s-lim}_{t \to \pm\infty} e^{i\Phi_{\pm}(t, D_{x})}e^{-i\Phi_{\pm}(t, D_{x})}
\]
exist. $G_{\pm}$ are obviously unitary Fourier multipliers, and $W_{\Phi}^{\pm} = W_{\Psi}^{\pm}G_{\pm}$.

Remark 2. If $V \in S^{-\mu}(\mathbb{Z}^{d})$ with $\mu > 1/2$, then we may employ a simpler approximate solution to the Hamilton-Jacobi equation.
\[
U_{D}(t) = e^{-i\Phi_{D}(t, D_{x})}, \quad \Phi_{D}(t, \xi) = tp_{0}(\xi) + \int_{0}^{t} V(sv(\xi))ds,
\]
is called the Dollard-type modifier, and we can show the existence of the modified wave operators similarly (see, e.g., [9] §XI.9, [3] §4.9).
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