A Mechanical Implementation and Diagrammatic Calculation of Entangled Basis States

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Abstract

We give for the first time a diagrammatic calculational tool of quantum entanglement. We present a pedagogical and simple mechanical implementation of quantum entanglement or "spooky action at a distance" to give a tangible realization of this weird quantum mechanical concept alien to classical physics. When two or more particles are correlated in a certain way, no matter how far apart they are in space, their states remain correlated. Their correlation, which is instantaneous, does not seem to involve any communication which is limited by the speed of light. The same mechanical implementation demonstrates the fundamental physical limits of any computational processes. The analytical derivations of calculational entangled basis states are given and their corresponding diagrammatic representations give an efficient aid in determining the calculational entangled basis states. A quantum Fourier transform for the two-state diagrams representing entangled basis states ('renormalized qubits') can also be formulated. Our results seem to advocate the idea that quantum entanglement generates the extra dimensions of the gravitational theory, indeed quantum entanglement is related to deep issues in the unification of general relativity and quantum mechanics. This extra dimensions of spacetime entanglement are currently being speculated in the literature.

1 INTRODUCTION

The principal characteristics of quantum mechanics, which are alien to classical physics, are as follows. First is the notion of uncertainty, i.e., not all the classical physical observable properties of a system can be simultaneously determined with exact precision. For example, position and momentum cannot both be known at the same time. There may other sets of observable properties that share the uncertainty properties. Second, the notion of superposition, for
example, a cat is a superposition of alive and dead cat. In other words an arbitrary quantum state is a superposition of several quantum states that describe measurable events. Third, the notion of entanglement, for example, if two photons or two particle spins become entangled, i.e., they are allowed to interact initially so that they will subsequently be defined by a single quantum state, then once they are separated, they will still share the same quantum state. So measuring one will determine the state of the other: for example, with a spin-zero entangled state, if one particle is measured to be in a spin-up state, the other is instantly forced to be in a spin-down state. In this paper, we would like to address the weirdest aspect of quantum mechanics, i.e., the notion of quantum entanglement.

Today, quantum entanglement forms the basis of several quantum technologies. In quantum teleportation and quantum cryptography, entangled particles are used to transmit signals that cannot be intercepted by an eavesdropper. The preliminary viable quantum cryptography systems are now being used by banks. And the exploding field of quantum computation uses superposition and entangled quantum states to perform computational calculations in parallel and at ultra-speed, so that some types of classically impractical calculations can be done by quantum computers in reasonable length of time.

Historically, in a 1935 paper, Einstein, Boris Podolsky and Nathan Rosen argue that quantum mechanics was not a complete physical theory. Known today as the “EPR paradox,” the thought experiment was meant to demonstrate the inherent conceptual, in terms of classical analogies, of quantum theory. Entanglement claims that the result of a measurement on one particle of an entangled quantum system can have an instantaneous effect on another particle, regardless of their distance. But the EPR paradox did help deepen our understanding of quantum mechanics by exposing the fundamentally non-classical characteristics of the measurement process. Before that paper, most physicists viewed a measurement as a physical disturbance inflicted directly on the measured system: one shines light onto an electron to determine its position, but this disturbs the electron and produces uncertainties. The EPR paradox shows that a “measurement” can be performed on a particle without disturbing it directly, by performing a measurement on a distant entangled particle.

Einstein postulated the existence of hidden variables, yet unknown local properties of the system which should account for the discrepancy, so that no instantaneous spooky action would be necessary. Bohr disagreed vehemently with this view and defended the far stricter Copenhagen interpretation of quantum mechanics. The two men were passionate about the subject, especially at the Solvay Conferences of 1927 and 1930, neither Einstein nor Bohr conceded defeat.

In this paper, we try to propose a simple mechanical model of a classical analogy to the ‘spooky at a distance’ property of quantum entanglement. This is a simple mechanical model, an ideal classical analogy for Einstein arguments. For what it is worth, it gives a simple classical/mechanical implementation and pedagogical value to the concept of quantum entanglement, which may revive interest in some intermediary medium (not hidden variables, but part of the
whole system, i.e., nonlocality and hence does not violate Bell’s theorem) in quantum entanglement.

2 ANALYTICAL DERIVATION OF ENTANGLED BASIS STATES

First, we give formal derivation of calculational entangled basis states as our foundation to the diagrammatic techniques. This was treated before in the author’s book [1]. Since the treatment there is not widely announced, we give it here for completeness. Consider the identity

\[ |p⟩ = \sum_q \langle q|p⟩ |q⟩, \]

where the \( \langle q|p⟩ \) is the transformation function. For discrete quantum mechanics, this is given by the discrete Fourier transform function,

\[ \langle q|p⟩ = \frac{1}{\sqrt{N}} \exp\left(-\frac{i}{\hbar} p \cdot q \right). \]

Therefore,

\[ |p⟩ = \frac{1}{\sqrt{N}} \sum_q \exp\left(-\frac{i}{\hbar} p \cdot q \right) |q⟩. \]

The product state in the 'momentum' basis is expanded in terms of the 'position' basis

\[ |p'⟩|p''⟩ = \frac{1}{\sqrt{N}} \sum_{q'} \exp\left(-\frac{i}{\hbar} p' \cdot q' \right) |q'⟩ \frac{1}{\sqrt{N}} \sum_{q''} \exp\left(-\frac{i}{\hbar} p'' \cdot q'' \right) |q''⟩ \]

\[ = \frac{1}{N} \sum_{q',q''} \exp\left(-\frac{i}{\hbar} p' \cdot q' \right) \exp\left(-\frac{i}{\hbar} p'' \cdot q'' \right) |q'⟩ |q''⟩. \]

We can expressed the right hand side in terms of 'correlated' (entangled) product states by writing, \( q'' = q' + m \), where \( m \) is the quantum-correlation 'distance',

\[ |p'⟩|p''⟩ = \frac{1}{N} \sum_{q',m} \exp\left(-\frac{i}{\hbar} p' \cdot q' \right) \exp\left(-\frac{i}{\hbar} p'' \cdot (q' + m) \right) |q'⟩ |q' + m⟩ \]

\[ = \frac{1}{N} \sum_{q',m} \exp\left(-\frac{i}{\hbar} (p' + p'') \cdot q' \right) \exp\left(-\frac{i}{\hbar} p'' \cdot m \right) |q'⟩ |q' + m⟩. \]
Now writing $p' + p'' = p$, we have

$$|p' \rangle |p - p'\rangle = \frac{1}{\sqrt{N}} \sum_{q',m} \exp \left( -\frac{i}{\hbar} (p' \cdot q') \right) \exp \left( -\frac{i}{\hbar} (p - p') \cdot m \right) |q' \rangle |q' + m\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{q'} \exp \left( -\frac{i}{\hbar} (p - p') \cdot q' \right) \left\{ \frac{1}{\sqrt{N}} \sum_{q'} \exp \left( -\frac{i}{\hbar} p' \cdot q' \right) |q' \rangle |q' + m\rangle \right\}$$

$$= \frac{1}{\sqrt{N}} \sum_{m} \exp \left( -\frac{i}{\hbar} (p - p') \cdot m \right) |\psi_{p,m}\rangle.$$ 

The inverse transformation gives $|\psi_{p,m}\rangle$ in terms of the ‘momentum’ basis products

$$|\psi_{p,m}\rangle = \frac{1}{\sqrt{N}} \sum_{p'} \exp \left( \frac{i}{\hbar} (p - p') \cdot m \right) |p' \rangle |p - p'\rangle.$$ 

(1)

Clearly, the correlated basis defined by $|\psi_{p,m}\rangle$ forms orthonormal and complete set.

### 2.1 Bell Basis

Thus, for $N = 2$, we have the standard momentum product states in terms of the so-called Bell basis, $|\psi_{p,m}\rangle$, for example,

$$|k' \rangle |k - k'\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} \exp (-\pi i (k - k') \cdot m) |\psi_{k,m}\rangle,$$

which yields

$$|0 \rangle |k\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} \exp (-\pi i k \cdot m) |\psi_{k,m}\rangle,$$

and using the identities

$$|\psi_{00}\rangle = |\Phi^+\rangle,$$

$$|\psi_{01}\rangle = |\Psi^+\rangle,$$

$$|\psi_{10}\rangle = |\Phi^-\rangle,$$

$$|\psi_{11}\rangle = |\Psi^-\rangle,$$

we have,

$$|0 \rangle |0\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} \exp (-\pi i k \cdot m) |\psi_{0,m}\rangle$$

$$= \frac{1}{\sqrt{2}} (|\psi_{00}\rangle + |\psi_{01}\rangle)$$

$$= \frac{1}{\sqrt{2}} (|\Phi^+\rangle + |\Psi^+\rangle),$$
\[ |0\rangle |1\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} \exp(-\pi i \cdot m) |\psi_{1,m}\rangle \]
\[ = \frac{1}{\sqrt{2}} (|\psi_{1,0}\rangle - |\psi_{1,1}\rangle) \]
\[ = \frac{1}{\sqrt{2}} (|\Phi^-\rangle - |\Psi^-\rangle), \]

We also have
\[ |1\rangle |k + 1\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} \exp(-\pi i (k + 1) \cdot m) |\psi_{k,m}\rangle \pmod{2}, \]
which yields
\[ |1\rangle |1\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} \exp(-\pi i (1) \cdot m) |\psi_{0,m}\rangle \]
\[ = \frac{1}{\sqrt{2}} (|\psi_{0,0}\rangle - |\psi_{0,1}\rangle) \]
\[ = \frac{1}{\sqrt{2}} (|\Phi^+\rangle - |\Psi^+\rangle), \]

\[ |1\rangle |1 + 1\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} \exp(-\pi i (1 + 1) \cdot m) |\psi_{1,m}\rangle \pmod{2}, \]
\[ |1\rangle |0\rangle = \frac{1}{\sqrt{2}} (|\psi_{1,0}\rangle + |\psi_{1,1}\rangle) \]
\[ = \frac{1}{\sqrt{2}} (|\Phi^-\rangle + |\Psi^-\rangle). \]

Therefore, for \( N = 2 \), we have the following transformation from the maximally entangled Bell basis to the standard ‘momentum’-state product basis given by
\[
\begin{pmatrix}
|0\rangle |0\rangle \\
|0\rangle |1\rangle \\
|1\rangle |0\rangle \\
|1\rangle |1\rangle
\end{pmatrix}
\]
\[ = \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
|\Phi^+\rangle \\
|\Psi^+\rangle \\
|\Phi^-\rangle \\
|\Psi^-\rangle
\end{pmatrix}_{Bell}.
\]

Hence,
\[
\begin{pmatrix}
|\Phi^+\rangle \\
|\Psi^+\rangle \\
|\Phi^-\rangle \\
|\Psi^-\rangle
\end{pmatrix}_{Bell}
\]
\[ = \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
|0\rangle |0\rangle \\
|0\rangle |1\rangle \\
|1\rangle |0\rangle \\
|1\rangle |1\rangle
\end{pmatrix}_p
\]
\[ = \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
|0\rangle |0\rangle \\
|0\rangle |1\rangle \\
|1\rangle |0\rangle \\
|1\rangle |1\rangle
\end{pmatrix}_p. \quad (2)
The above result also follows from the general inverse expression, Eq. (1), in terms of 'momentum' product basis,

\[ |\psi_{p,m}\rangle = \frac{1}{\sqrt{N}} \sum_{p'} \exp \left( \frac{i}{\hbar} (p - p') \cdot m \right) |p'\rangle |p - p'\rangle, \]

so that for \( N = 2 \),

\[ |\psi_{k,m}\rangle = \frac{1}{\sqrt{2}} \sum_{k'} \exp \left( i\pi (k - k') \cdot m \right) |k'\rangle |k - k'\rangle, \]

which yields

\[ |\psi_{00}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle + |1\rangle |1\rangle \right), \]

\[ |\psi_{01}\rangle = |\Psi^+\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle - |1\rangle |1\rangle \right), \]

\[ |\psi_{10}\rangle = |\Phi^-\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle - |1\rangle |0\rangle \right), \]

\[ |\psi_{11}\rangle = |\Psi^-\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |1\rangle + |1\rangle |0\rangle \right). \]

We also have

\[ \begin{pmatrix} |\Phi^+\rangle \\ |\Psi^+\rangle \\ |\Phi^-\rangle \\ |\Psi^-\rangle \end{pmatrix} \text{_{Bell}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} |0\rangle |0\rangle \\ |0\rangle |1\rangle \\ |1\rangle |0\rangle \\ |1\rangle |1\rangle \end{pmatrix} q. \]

Therefore, we have the identity

\[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} |0\rangle |0\rangle \\ |0\rangle |1\rangle \\ |1\rangle |0\rangle \\ |1\rangle |1\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} |0\rangle |0\rangle \\ |0\rangle |1\rangle \\ |1\rangle |0\rangle \\ |1\rangle |1\rangle \end{pmatrix} q, \]

6
which gives

\[
\begin{pmatrix}
|0\rangle |0\rangle \\
|0\rangle |1\rangle \\
|1\rangle |0\rangle \\
|1\rangle |1\rangle \\
\end{pmatrix}_p = \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & -1 \\
\end{pmatrix}^{-1}
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
|0\rangle |0\rangle \\
|0\rangle |1\rangle \\
|1\rangle |0\rangle \\
|1\rangle |1\rangle \\
\end{pmatrix}_q
\]

\[
= \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
\end{pmatrix}_q
\]

\[
= \frac{1}{2}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
\end{pmatrix}_q
\]

Therefore, 'momentum' basis is given by the transformation of the 'position' basis as

\[
\begin{pmatrix}
|0\rangle |0\rangle \\
|0\rangle |1\rangle \\
|1\rangle |0\rangle \\
|1\rangle |1\rangle \\
\end{pmatrix}_p
= \frac{1}{2}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
|0\rangle |0\rangle \\
|0\rangle |1\rangle \\
|1\rangle |0\rangle \\
|1\rangle |1\rangle \\
\end{pmatrix}_q
\]

2.2 Three-Qubit Entangled Basis

As in the two-qubit, the product state in the 'momentum' basis is expanded in terms of the 'position' basis

\[
|p\rangle' |p''\rangle |p'''\rangle
= \frac{1}{N^2}
\sum_{q'} \exp\left(-\frac{i}{\hbar} p' \cdot q'\right) |q'\rangle \sum_{q''} \exp\left(-\frac{i}{\hbar} p'' \cdot q''\right) |q''\rangle \sum_{q'''} \exp\left(-\frac{i}{\hbar} p''' \cdot q'''\right) |q'''\rangle
\]

\[
= \frac{1}{N^2}
\sum_{q',q'',q'''} \exp\left(-\frac{i}{\hbar} p' \cdot q'\right) \exp\left(-\frac{i}{\hbar} p'' \cdot q''\right) \exp\left(-\frac{i}{\hbar} p''' \cdot q'''\right) |q'\rangle |q''\rangle |q'''\rangle.
\]

We can expressed the right hand side in terms of 'correlated' (entangled) product states by writing, \(q'' = q' + m\), \(q''' = q' + l\) where \(m\) and \(l\) are the quantum-
correlation ‘distances’,

\[ |p'\rangle |p''\rangle |p'''\rangle = \frac{1}{N^2} \sum_{q',m,l} \exp \left( -\frac{i}{\hbar} p' \cdot q' \right) \exp \left( -\frac{i}{\hbar} p'' \cdot (q' + m) \right) \times \exp \left( -\frac{i}{\hbar} p''' \cdot (q' + l) \right) \]

\[ |q'\rangle |q' + m\rangle |q' + l\rangle = \frac{1}{N^2} \sum_{q',m,l} \exp \left( -\frac{i}{\hbar} (p' + p'' + p''') \cdot q' \right) \times \exp \left( -\frac{i}{\hbar} p'' (q' + l) \right) \exp \left( -\frac{i}{\hbar} p''' (l - m) \right) \times |\psi_{p,m,l}\rangle , \]

Now writing \( p' + p'' + p''' = p \), we have

\[ |p'\rangle |p - p' - p'''\rangle |p'''\rangle = \frac{1}{(N^2)} \sum_{m,l} \exp \left( -\frac{i}{\hbar} (p - p') \cdot m \right) \exp \left( -\frac{i}{\hbar} p'' \cdot (l - m) \right) \times \sum_{q'} \exp \left( -\frac{i}{\hbar} p \cdot q' \right) |q'\rangle |q' + m\rangle |q' + l\rangle \]

\[ = \frac{1}{N} \sum_{m,l} \exp \left( -\frac{i}{\hbar} (p - p') \cdot m \right) \exp \left( -\frac{i}{\hbar} p'' \cdot (l - m) \right) \times |\psi_{p,m,l}\rangle , \]

where

\[ |\psi_{p,m,l}\rangle = \frac{1}{\sqrt{N}} \sum_{q'} \exp \left( -\frac{i}{\hbar} p \cdot q' \right) |q'\rangle |q' + m\rangle |q' + l\rangle , \]

is the three-qubit correlated (entangled) state. Substituting \( N = 2 \), and \( p = \pi \hbar k \), we have

\[ |\psi_{k,m,l}\rangle = \frac{1}{\sqrt{2}} \sum_{q'=0}^1 \exp \left( -ik \cdot q' \right) |q'\rangle |q' + m\rangle |q' + l\rangle , \]

and obtain

\[ |\psi_{0,0,0\rangle} = \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle + |1\rangle |1\rangle \right) = \Theta^+ _3 , \]

which is the well-known Greenberger-Horne-Zeilinger state, \(|GHZ\rangle\), which is the maximally entangled state of three qubits since \( m = 0 \), and \( l = 0 \) (a complete correlation). We give the expressions for the rest of \(|\psi_{k,m,l}\rangle\),

\[ |\psi_{0,0,1}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |1\rangle + |1\rangle |0\rangle \right) = \Gamma^+ , \]

\[ |\psi_{0,1,0}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |1\rangle |0\rangle + |1\rangle |0\rangle |1\rangle \right) = \Omega^+ , \]

8
\[ |\psi_{0,1,1}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \langle 1| |1\rangle + |1\rangle \langle 0| |0\rangle \right) = \Xi^+, \]
\[ |\psi_{1,0,0}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \langle 0| |0\rangle - |1\rangle \langle 1| |1\rangle \right) = \Theta_3^-, \]
\[ |\psi_{1,0,0}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \langle 0| |0\rangle + |1\rangle \langle 1| |1\rangle \right) = \Theta_3^+, \]
\[ |\psi_{1,0,1}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \langle 0| |1\rangle - |1\rangle \langle 1| |0\rangle \right) = \Gamma^-, \]
\[ |\psi_{1,1,0}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \langle 1| |0\rangle - |1\rangle \langle 0| |1\rangle \right) = \Omega^-, \]
\[ |\psi_{1,1,1}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \langle 1| |1\rangle - |1\rangle \langle 0| |0\rangle \right) = \Xi^- . \]

Just like the Bell basis, the set of three-qubit correlated states, \{ |\psi_{k,m,l}\rangle \}, is a complete and orthonormal basis set, since each element is connected to the complete orthonormal product states by unitary transformations. In what follows, we will to \{ |\psi_{k,m,l}\rangle \} as the three-particle correlated basis (TPCB).

**Remark 1** There is another three-qubit entangled state, the so-called W state. A W state is a collective spin state with one ‘excitation’ as compared to a coherent spin state which is fully aligned. With all but one spin aligned in the x-direction, the state can be written:

\[ |W\rangle = \frac{1}{\sqrt{N}} \left( |\downarrow\uparrow\uparrow \ldots \uparrow\rangle_1 + |\uparrow\downarrow\uparrow \ldots \uparrow\rangle_2 + \ldots |\uparrow\uparrow\uparrow \ldots \downarrow\rangle_N \right) \]

The \( |W\rangle \) may be viewed as a 'zero momentum' superposition of 'position coordinates' indicated by the position of an excitation in an array of fully-aligned spin

\[ |p\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{i \bar{p} \cdot R_i} |R_i\rangle \]
\[ |W\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{i \bar{p} \cdot R_i} |R_i\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |R_i\rangle \]

where the 'position' coordinate, \( |R_i\rangle \), correspond to the different position of one excitation, \( \downarrow \), in a spin array \( |\downarrow \rangle \) as indicated above (more appropriately in a spin ring). However, the 'position' coordinates used is only a small subspace of the complete and orthonormal product Hilbert space of an array of spin, moreover \( |W\rangle \) corresponds only to \( |p\rangle = |p = 0\rangle \), i.e., zero-phase superposition. In general there are also \( N \) values of \( p \) producing other \( |W\rangle \)-type entangled states of array of spins with one excitation. Furthermore, corresponding to the superposition of position eigenstates to produce a Schroedinger wavefunction,
one can also form an arbitrary superposition of spin array $|R_i\rangle$ to form other correlated (entangled) state, for example,

$$|\Psi\rangle = \sum_{i=1}^{N} \psi(R_i) |R_i\rangle$$

where $\psi(R_i)$ may no longer be obtained from Schroedinger equation, but from some optimization criteria, say for optimal condition of a universal cloning machine, which for three-particle entangled state, in the form (Phys. Rev. A 57, 2368 (1998))

$$|\Psi\rangle = \sqrt{\frac{2}{3}} |100\rangle - \sqrt{\frac{1}{6}} |010\rangle - \sqrt{\frac{1}{6}} |001\rangle$$

It is worth pointing out that the method of generating entangled states using the product of ‘momentum’ basis states, $|p\rangle$, in the manner given above automatically yield complete orthonormal entangled basis states.

### 2.3 A qubit Teleportation Using Three-Particle Entanglement

Here we will used as the quantum channel shared by Alice and Bob to be the GHZ state

$$|GHZ\rangle = \Theta_3^+ = \frac{1}{\sqrt{2}} (|0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_1 |1\rangle_2 |1\rangle_3) .$$

As before Alice wants to send to Bob the unknown state, $|\psi\rangle = \alpha |0\rangle_U + \beta |1\rangle_U$. To set up the teleportation, let us assume that particles 1 and 2 are kept by Alice and particle 3 is sent to Bob. So, Alice has three particles ($U$, the one she wants to teleport, and particles 1 and 2, two of the entangled three qubits), and Bob has one particle 3. In the total system, the state of these four particles is given by

$$|\psi\rangle \otimes |\Phi_3^+\rangle = (\alpha |0\rangle_U + \beta |1\rangle_U) \otimes \frac{1}{\sqrt{2}} (|0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_1 |1\rangle_2 |1\rangle_3)$$

$$= \frac{\alpha}{\sqrt{2}} (|0\rangle_U |0\rangle_1 |0\rangle_2 |0\rangle_3 + |0\rangle_U |1\rangle_1 |1\rangle_2 |1\rangle_3)$$

$$+ \frac{\beta}{\sqrt{2}} (|1\rangle_U |0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_U |1\rangle_1 |1\rangle_2 |1\rangle_3) ,$$

where we have enclosed in square bracket the particles belonging to Alice. We will now express Alice particle states in terms of the TPCB basis. We have

$$|0\rangle_U |0\rangle_1 |0\rangle_2 = \frac{1}{\sqrt{2}} (\Theta_3^+ + \Theta_3^-)_{U12} ,$$

$$|1\rangle_U |1\rangle_1 |1\rangle_2 = \frac{1}{\sqrt{2}} (\Theta_3^+ - \Theta_3^-)_{U12} ,$$

10
\[ |0\rangle_U |1\rangle_1 |1\rangle_2 = \frac{1}{\sqrt{2}} (\Xi^+ + \Xi^-)_{U12}, \]
\[ |1\rangle_U |0\rangle_1 |0\rangle_2 = \frac{1}{\sqrt{2}} (\Xi^+ - \Xi^-)_{U12}. \]

Therefore we can write
\[
|\psi\rangle \otimes |\Phi^+\rangle = \frac{\alpha}{\sqrt{2}} \left( \left[ \frac{1}{\sqrt{2}} (\Theta^+ + \Theta^-)_{U12} \right] |0\rangle_3 + \left[ \frac{1}{\sqrt{2}} (\Xi^+ + \Xi^-)_{U12} \right] |1\rangle_3 \right)
+ \frac{\beta}{\sqrt{2}} \left( \left[ \frac{1}{\sqrt{2}} (\Xi^+ - \Xi^-)_{U12} \right] |0\rangle_3 + \left[ \frac{1}{\sqrt{2}} (\Theta^+ - \Theta^-)_{U12} \right] |1\rangle_3 \right),
\]

\[
|\psi\rangle \otimes |\Phi^+\rangle = \frac{1}{2} (\Theta^+)_{U12} (\alpha |0\rangle_3 + \beta |1\rangle_3) + \frac{1}{2} (\Xi^+ + \Xi^-)_{U12} (\frac{1}{2}) |0\rangle_3 + \frac{1}{2} (\Xi^+ - \Xi^-)_{U12} (\frac{1}{2}) |1\rangle_3
+ \frac{1}{2} (\Theta^+ + \Theta^-)_{U12} I \left( \left( \frac{\alpha}{\delta} \right) \right) + \frac{1}{2} (\Xi^+ - \Xi^-)_{U12} I \left( \left( \frac{\alpha}{\delta} \right) \right).
\]

\[
|\psi\rangle \otimes |\Phi^+\rangle = \frac{1}{2} (\Theta^+)_{U12} I \left( \left( \frac{\alpha}{\delta} \right) \right) + \frac{1}{2} (\Xi^-)_{U12} (\frac{1}{2} \alpha) \sigma_x \left( \left( \frac{\alpha}{\delta} \right) \right)
+ \frac{1}{2} (\Theta^+ - \Theta^-)_{U12} I \left( \left( \frac{\alpha}{\delta} \right) \right).
\]

Hence, it follows that regardless of the unknown state \(|\psi\rangle\), by using the maximally-entangled quantum channel, \(|GHZ\rangle\), Alice can still perform only four measurements with outcomes equally likely with probability equal to \(\frac{1}{4}\). After Alice measurement, Bob’s particle 3 will have been projected to one of the four pure states, after which Bob has to perform the necessary transformation to regain the original state that Alice has, as indicated in the following table,

| Alice measurement | Bob’s particle state | Bob’s transformation |
|--------------------|----------------------|---------------------|
| \((\Theta^+)_{U12}\) | \((\alpha |0\rangle_3 + \beta |1\rangle_3)\) | I |
| \((\Theta^-)_{U12}\) | \((\alpha |0\rangle_3 - \beta |1\rangle_3)\) | \(\sigma_z\) |
| \((\Xi^+)_{U12}\) | \((\alpha |1\rangle_3 + \beta |0\rangle_3)\) | \(\sigma_x\) |
| \((\Xi^-)_{U12}\) | \((\alpha |1\rangle_3 - \beta |0\rangle_3)\) | \(i\sigma_y\) |

To transmit Alice’s classical (measurement) result to Bob, Alice needs a two-bit classical channel to transmit which one of the four equally likely results. This is in contrast with other proposed more complex teleportation scheme using the same maximally entangled \(|GHZ\rangle\) quantum channel which employs Bob as an ancilla to transmit the unknown state to Cliff, and use two classical bit between Alice and Bob and one classical bit between Bob and Cliff. Here, whether we use the Bell basis or TPCB basis to transmit an unknown quantum state, only two-bit classical channel is simply needed directly between the sending and receiving parties, and without the use of intermediary ancilla and only one local measurement is involved.
2.4 Teleportation Using Three-Particle Entanglement and an Ancilla

Instead of keeping the particles 1 and 2 as done above, Alice only keep particle 1 and send particle 2 to Bob and particle 3 to Cliff. The idea here is that Alice will locally only measure one of the orthonormal two-particle entangled Bell states, $|\Phi^\pm\rangle$, $|\Psi^\pm\rangle$, instead of one of the complete orthonormal three-particle correlated (entangled) states given above. The price to pay in this scheme is that there is a need to have two bits of classical communication channel between Alice and Bob and one more bit of classical communication channel between Bob and Cliff. Moreover, two local measurements has to be done, one local measurement by Alice on her Bell states and one local measurement by Bob on his qubit, to implement the teleportation of one qubit from Alice to Cliff. Thus, Bob becomes an ancilla in this teleportation scheme.

Again, using the Alice wants to send to Bob the unknown state, $|\psi\rangle = \alpha |0\rangle_U + \beta |1\rangle_U$, using the $|GHZ\rangle$ shared quantum channel. To set up the teleportation, Alice keep particle 1 and send particle 2 to Bob, and particle 3 to Cliff. So, Alice has two particles ($U$, the unknown qubit she wants to teleport and particle 1, one of the entangled three $|GHZ\rangle$ qubits). In the total system, the state of these four particles is given by

$$
|\psi\rangle \otimes |GHZ\rangle = (\alpha |0\rangle_U + \beta |1\rangle_U) \otimes \frac{1}{\sqrt{2}} ((|0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_1 |1\rangle_2 |1\rangle_3)
$$

$$
= \frac{\alpha}{\sqrt{2}} ((|0\rangle_U |0\rangle_1 |0\rangle_2 |0\rangle_3 + |0\rangle_U |1\rangle_1 |1\rangle_2 |1\rangle_3)
$$

$$
+ \frac{\beta}{\sqrt{2}} ((|1\rangle_U |0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_U |1\rangle_1 |1\rangle_2 |1\rangle_3),
$$

where the particles enclosed in square brackets belongs to Alice. Upon changing the Alice product basis states to Bell basis, we have

$$
|\psi\rangle \otimes |GHZ\rangle = (\alpha |0\rangle_U + \beta |1\rangle_U) \otimes \frac{1}{\sqrt{2}} ((|0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_1 |1\rangle_2 |1\rangle_3)
$$

$$
= \frac{1}{2} \left( |\Phi^+_U1 (\alpha |0\rangle_2 |0\rangle_3 + \beta |1\rangle_2 |1\rangle_3) + |\Psi^+_U1 (\alpha |1\rangle_2 |1\rangle_3 + \beta |0\rangle_2 |0\rangle_3) \right)
$$

$$
+ \frac{1}{2} \left( |\Phi^-_U1 (\alpha |0\rangle_2 |0\rangle_3 - \beta |1\rangle_2 |1\rangle_3) + |\Psi^-_U1 (\alpha |1\rangle_2 |1\rangle_3 - \beta |0\rangle_2 |0\rangle_3) \right).
$$
A local measurement of one of the four Bell states, which have equal probability equal to $\frac{1}{4}$, will project the joint state of the particles 2 and 3 possessed by Bob and Cliff, respectively, into one of the entangled states shown above. Assume that Alice local measurement yields $|\Phi^+\rangle_{U1}$, which is communicated to Bob through a classical two-bit channel. Then the state of particles 2 and 3 is

$$|\psi_{23}\rangle = \alpha |0\rangle_2 |0\rangle_3 + \beta |1\rangle_2 |1\rangle_3.$$ 

In order to effect a transfer of the unknown qubit to Cliff, Bob must now unentangle his qubit from that of Cliff. To do this Bob has to make a local measurement on his qubit 2.

Suppose Bob’s measurement apparatus collapses his qubit state to two possible outcomes, namely $k_1$ and $k_2$. Then Bob can decompose his incoming state to the new basis states, namely, $|k_1\rangle$ and $|k_2\rangle$, and write the unitary/orthogonal transformation from the old basis to the new basis as

$$|0\rangle_2 = \sin \theta \, |k_1\rangle + \cos \theta \, |k_2\rangle,$$

$$|1\rangle_2 = \cos \theta \, |k_1\rangle - \sin \theta \, |k_2\rangle.$$

Then $|\psi_{23}\rangle$ in terms of Bob’s new basis is

$$|\psi_{23}\rangle = (\sin \theta \, |k_1\rangle + \cos \theta \, |k_2\rangle) \alpha |0\rangle_3 + (\cos \theta \, |k_1\rangle - \sin \theta \, |k_2\rangle) \beta |1\rangle_3$$

$$= \sin \theta \, \alpha |0\rangle_3 |k_1\rangle + \cos \theta \, \alpha |0\rangle_3 |k_2\rangle + \cos \theta \, \beta |1\rangle_3 |k_1\rangle - \sin \theta \beta |1\rangle_3 |k_2\rangle$$

$$= (\sin \theta \, \alpha |0\rangle_3 + \cos \theta \, \beta |1\rangle_3) |k_1\rangle + (\cos \theta \, \alpha |0\rangle_3 - \sin \theta \beta |1\rangle_3) |k_2\rangle.$$ 

In general, the new basis and the old basis are mutually-unbiased basis which means $\sin \theta = \cos \theta$ or $\theta = \frac{\pi}{4}$. Then $|\psi_{23}\rangle$ becomes

$$|\psi_{23}\rangle = (\alpha |0\rangle_3 + \beta |1\rangle_3) \frac{1}{\sqrt{2}} |k_1\rangle + (\alpha |0\rangle_3 - \beta |1\rangle_3) \frac{1}{\sqrt{2}} |k_2\rangle.$$ 

Whatever Bob’s measurement outcome is, Bob has to communicate to Cliff his result through a one-bit classical channel. If the outcome of Bob’s local measurement is $k_1$ then Cliff has the unknown qubit from Alice and do nothing. On the other hand if Bob’s measurement yields $k_2$, then Cliff has to perform a $\sigma_z$ transformation on his qubit to obtain the unknown qubit from Alice.

### 2.5 Two-Qubit Teleportation Using Three-Particle Entanglement

We will now show that a full use of the capability of the GHZ quantum channel is achieved when teleporting two qubit of information. As before, the quantum channel shared by Alice and Bob to be the GHZ state

$$|GHZ\rangle = \Theta_3^+ = \frac{1}{\sqrt{2}} (|0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_1 |1\rangle_2 |1\rangle_3).$$

---

Example of mutually-unbiased bases are the position basis states and momentum basis states.
But now Alice wants to send an unknown two-particle bits (triplet) to Bob, namely,

$$|\psi\rangle = \alpha |0\rangle_{U1} |0\rangle_{U2} + \delta |1\rangle_{U1} |1\rangle_{U2},$$

which we may write in vector form as

$$|\psi\rangle = \begin{pmatrix} \alpha |0\rangle_{U1} |0\rangle_{U2} \\ \delta |1\rangle_{U1} |1\rangle_{U2} \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \delta \end{pmatrix},$$

where $|\alpha|^2 + |\delta|^2 = 1$. To set up the two-bit teleportation, particle 1 is kept by Alice and particle 2 are3 are sent to Bob. In the total system, the state of these three particles is given by

$$|\psi\rangle \otimes |GHZ\rangle = (\alpha |0\rangle_{U1} |0\rangle_{U2} + \delta |1\rangle_{U1} |1\rangle_{U2})$$

$$\otimes \frac{1}{\sqrt{2}} (|0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_1 |1\rangle_2 |1\rangle_3)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha (|0\rangle_{U1} |0\rangle_{U2} |0\rangle_1 |0\rangle_2 |0\rangle_3 + |0\rangle_{U1} |0\rangle_{U2} |1\rangle_1 |1\rangle_2 |1\rangle_3) \\ + \delta (|1\rangle_{U1} |1\rangle_{U2} |0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_{U1} |1\rangle_{U2} |1\rangle_1 |1\rangle_2 |1\rangle_3) \end{pmatrix}$$.

Now we change basis using the complete orthonormal set of three-particle correlated (entangled) states

$$|\psi_{0,0,0}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle |0\rangle + |1\rangle |1\rangle |1\rangle) = \Theta_3^+,\quad$$

$$|\psi_{0,0,1}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle |1\rangle + |1\rangle |1\rangle |0\rangle) = \Gamma^+,\quad$$

$$|\psi_{0,1,0}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |1\rangle |0\rangle + |1\rangle |0\rangle |1\rangle) = \Omega^+,\quad$$

$$|\psi_{0,1,1}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |1\rangle |1\rangle + |1\rangle |0\rangle |0\rangle) = \Xi^+,\quad$$

$$|\psi_{1,0,0}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle |0\rangle - |1\rangle |1\rangle |1\rangle) = \Theta_3^-,\quad$$

$$|\psi_{1,0,1}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle |1\rangle - |1\rangle |1\rangle |0\rangle) = \Gamma^-,\quad$$

$$|\psi_{1,1,0}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |1\rangle |0\rangle - |1\rangle |0\rangle |1\rangle) = \Omega^-,\quad$$

$$|\psi_{1,1,1}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |1\rangle |1\rangle - |1\rangle |0\rangle |0\rangle) = \Xi^-,$$

to obtain

$$|0\rangle_{U1} |0\rangle_{U2} |0\rangle_1 = \frac{1}{\sqrt{2}} (\Theta_3^+ + \Theta_3^-),$$

$$|0\rangle_{U1} |0\rangle_{U2} |1\rangle_1 = \frac{1}{\sqrt{2}} (\Gamma^+ + \Gamma^-),$$
\[ |0\rangle_{U_1} |1\rangle_{U_2} |0\rangle_1 = \frac{1}{\sqrt{2}} (\Omega^+ + \Omega^-), \]
\[ |1\rangle_{U_1} |1\rangle_{U_2} |0\rangle_1 = \frac{1}{\sqrt{2}} (\Xi^+ + \Xi^-), \]
\[ |1\rangle_{U_1} |0\rangle_{U_2} |1\rangle_1 = \frac{1}{\sqrt{2}} (\Xi^- - \Xi^-), \]
\[ |1\rangle_{U_1} |1\rangle_{U_2} |0\rangle_1 = \frac{1}{\sqrt{2}} (\Omega^- - \Omega^-), \]
\[ |1\rangle_{U_1} |1\rangle_{U_2} |1\rangle_1 = \frac{1}{\sqrt{2}} (\Theta_1^+ - \Theta_3^-). \]

Upon changing the basis of Alice particles, we have

\[ |\psi\rangle \otimes |GHZ\rangle = \frac{1}{\sqrt{2}} \left( \alpha \left( \frac{1}{\sqrt{2}} (\Theta_3^+ + \Theta_3^-) |0\rangle_2 |0\rangle_3 + \frac{1}{\sqrt{2}} (\Gamma^+ + \Gamma^-) |1\rangle_2 |1\rangle_3 \right) \right. + \delta \left( \frac{1}{\sqrt{2}} (\Gamma^+ - \Gamma^-) |0\rangle_2 |0\rangle_3 \right.
\[ + \frac{1}{\sqrt{2}} (\Theta_3^+ - \Theta_3^-) |1\rangle_2 |1\rangle_3 \left. \right) \right), \]

\[ |\psi\rangle \otimes |GHZ\rangle = \frac{1}{2} \left( \begin{array}{c}
\Theta_3^+ (\alpha |0\rangle_2 |0\rangle_3 + \delta |1\rangle_2 |1\rangle_3) \\
+ \Theta_3^- (\alpha |0\rangle_2 |0\rangle_3 - \delta |1\rangle_2 |1\rangle_3) \\
+ \Gamma^+ (\delta |0\rangle_2 |0\rangle_3 + \alpha |1\rangle_2 |1\rangle_3) \\
+ \Gamma^- (-\delta |0\rangle_2 |0\rangle_3 + \alpha |1\rangle_2 |1\rangle_3)
\end{array} \right)
\]

Thus, depending on the result of Alice measurement using the equi-probable, with probability = \( \frac{1}{4} \), three-particle entanglement basis, \( \Theta_3^+ \), \( \Theta_3^- \), \( \Gamma^+ \), and \( \Gamma^- \), which is communicated to Bob via classical two-bit channel, Bob will then use the inverse transformation to recover the original triplet sent by Alice.

Alice can also send an unknown singlet to Bob, namely,

\[ |\psi\rangle = \alpha |0\rangle_{U_1} |1\rangle_{U_2} + \delta |1\rangle_{U_1} |0\rangle_{U_2}. \]

However, the quantum channel shared by Alice and Bob must now be chosen to be given by

\[ |Q_{ch}\rangle = \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 |0\rangle_3 + |1\rangle_1 |0\rangle_2 |1\rangle_3). \]
instead of $|GHZ\rangle$ used above. To set up the two-bit teleportation, particle 1 is kept by Alice and particle 2 are sent to Bob. In the total system, the state of these three particles is given by

$$|\psi\rangle \otimes |Q_{ch}\rangle = \{\alpha |0\rangle_U |1\rangle_U + \delta |1\rangle_U |0\rangle_U \} \otimes \left\{ \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 |0\rangle_3 + |1\rangle_1 |0\rangle_2 |1\rangle_3) \right\}$$

Upon changing the basis to three-particle entanglement basis, we have

$$|\psi\rangle \otimes |Q_{ch}\rangle = \frac{1}{\sqrt{2}} \left( \alpha \left( \frac{1}{\sqrt{2}} (\Omega^+ + \Omega^-) |1\rangle |0\rangle |1\rangle_3 + \frac{1}{\sqrt{2}} (\Xi^+ + \Xi^-) |0\rangle |1\rangle |1\rangle_3 \right) + \delta \left( \frac{1}{\sqrt{2}} (\Xi^- - \Xi^+) |1\rangle |0\rangle |1\rangle_3 + \frac{1}{\sqrt{2}} (\Omega^- - \Omega^+) |0\rangle |1\rangle |1\rangle_3 \right) \right)$$

Thus, depending on the result of Alice measurement using the equi-probable, with probability $= \frac{1}{4}$, three-particle entanglement basis, $\Omega^+$, $\Omega^-$, $\Xi^+$, and $\Xi^-$, which is communicated to Bob via classical two-bit channel, Bob will then use the inverse transformation to recover the original singlet sent by Alice.

### 3 MECHANICAL MODELING OF QUANTUM ENTANGLEMENTS

Here, we now give a simple and pedagogical implementation of quantum entanglement using a chain of mechanical inverters, see-saws or swings. This mechanical model has been previously used by the author in investigating the fundamental physical limits of computational processes [2].

#### 3.1 Mechanical model for entangled two qubits

A schematic picture of an entangled two qubits is shown in Figs. [1] and [2]. The frictionless guides and fulcrum would make the inverter chain reversible.
What is more important though, from the point of view of entanglement, is the presence of rigid coupling. Rigid coupling allows for simultaneous events to take place irrespective of the length of the intermediary chain, thus capturing the essence of quantum entanglement.

No communication is involved since the whole process is a reconfiguration of the whole chain system. No hidden variables and the deterministic reconfiguration process does not violate Bell’s inequality. In other words, this mechanical model is conceptually an ideal model of quantum entanglement. One is tempted to speculate whether spacetime provides rigid coupling in realistic situations, indeed due to spacetime entanglement [3].

Figure 1 shows a configuration for a triplet entanglement.

![Figure 1: An entangled spin triplet.](image)

Another independent configuration of the chain embodies the spin singlet configuration shown in Fig. 2.

![Figure 2: An entangled spin singlet.](image)
4 Diagrammatic Techniques

In the following diagrammatic analyses, the inverter chains of Figs. 1 and 2 are abstracted into line segment representations.

4.1 Two-level diagram representations of entangled two qubits

The line diagrams below is an abstraction of the inverter chain of Fig. 1 and Fig. 2, respectively. The first line of Fig. 3 represents a two-state diagram, with entangled basis $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$. The second line represents a two-state diagram, with entangled basis $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$.

![Figure 3: Two two-state diagrams for an entangled two qubits. The calculational entangled basis is obtained using Hadamard transformation or discrete Fourier transform for two-state systems.](image)

4.1.1 Hadamard transformation for calculational entangled basis states

The Hadamard transform of the entangled basis states of two qubits given above is what we referred to as the calculational entangled basis states. These are the Bell entangled basis states derived formally before. For the first diagram of Fig. 3 we have,

$$
\left( \begin{array}{c} |\Phi^+\rangle \\ |\Phi^-\rangle \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right)
$$

and for the second diagram,

$$
\left( \begin{array}{c} |\Psi^+\rangle \\ |\Psi^-\rangle \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right)
$$

4.2 Mechanical modeling of entangled three qubits

For entangled $n$ qubits, there are $2^n$ integer number of two-state entangled basis diagrams. Thus, for three qubits the four diagrams shown in Fig. 4.

From the diagrams of Fig. 4 and using the Hadamard transformation, we can immediately write down the calculational entangled basis states for three qubits.

$$
\left( \begin{array}{c} \Xi^+ \\ \Xi^- \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right) \left( \begin{array}{c} |1\rangle \\ |0\rangle \end{array} \right)
$$

$$
\left( \begin{array}{c} \Theta_3^+ \\ \Theta_3^- \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} |0\rangle \\ |0\rangle \end{array} \right) \left( \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right) \left( \begin{array}{c} |1\rangle \\ |1\rangle \end{array} \right)
$$
4.3 Mechanical modeling of entangled four qubits

We can consider eight two-entangled-state diagrams for entangled four qubits. These are shown in Fig. 5.

Following the same procedure using Hadamard Fourier transformation, we can also immediately write down the calculational entangled basis states. Here, we use our own notations, $\Phi_j^\pm$, for the transformed calculational basis states. Following the order of diagrams in Fig. 5, we have

\[
\begin{align*}
\begin{pmatrix} \Phi_1^+ \\ \Phi_1^- \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |0\rangle |1\rangle |0\rangle |0\rangle \\ |1\rangle |0\rangle |1\rangle |0\rangle \end{pmatrix} \\
\begin{pmatrix} \Phi_2^+ \\ \Phi_2^- \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |0\rangle |0\rangle |1\rangle |0\rangle \\ |1\rangle |1\rangle |1\rangle |0\rangle \end{pmatrix} \\
\begin{pmatrix} \Phi_3^+ \\ \Phi_3^- \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |0\rangle |0\rangle |1\rangle |0\rangle \\ |1\rangle |1\rangle |0\rangle |1\rangle \end{pmatrix}
\end{align*}
\]
In general, one can systematically write down the calculational entangled basis states for any number, \( n \), of qubits by using the physically meaningful diagrammatic techniques prescribed here. One does not even need to draw the diagrams themselves, but simply list the \( 2^n \) entangled basis states to deduce the calculational entangled basis states by Hadamard transformation of the two configurations of imagined entangled diagrams.

5 Concluding Remarks

The apparently perfect scheme of the mechanical model in describing quantum entanglement seems to advocate the existence of entangled spacetime-medium to take the place of the chain of inverters in some unknown forms. An interesting analogy is the bound/entangled pair of vortices and antivortex on the surface of fluids which are really the ends of a vortex chain (tube) forming a U-shape underneath the surface. For example, in 3-D the superfluid quantized vortices form a metastable closed ring or open chain ending at the surface. A vortex chain with both ends ending at the same surface appears as a bound
vortex/anti-vortex pair at the surface. Thus, it seems extra dimensions are needed in spacetime to have a theory of entanglement. The work of Ooguri [5] and collaborators shows that this quantum entanglement generates the extra dimensions of the gravitational theory. "It appears that it seems possible to generate a geometric connection between entangled qubits, even though there is no direct interaction between the two systems [4]. Furthermore, the solid and reliable structure of spacetime is due to the ghostly features of entanglement.

Could it be that besides the geometrical spacetime aspects of gravity, there is a purely quantum mechanical aspect of spacetime geometry with extra dimensions that give rise to entanglement? A hint along this idea is also given by Malcedona [4] when he stated that "One can consider, therefore, a pair of black holes where all the microstates are “entangled.” Namely, if we observe one of the black holes in one particular microstate, then the other has to be in exactly the same microstate. A pair of black holes in this particular EPR entangled state would develop a wormhole, or Einstein-Rosen bridge, connecting them through the inside. The geometry of this wormhole is given by the fully extended Schwarzschild geometry. It is interesting that both wormholes and entanglement naively appear to lead to a propagation of signals faster than light.” "It was known that quantum entanglement is related to deep issues in the unification of general relativity and quantum mechanics, such as the black hole information paradox and the firewall paradox,” says Hirosi Ooguri” [5].

In any case, from the computational point of view, deriving the computational entangled basis states have now been given for the first-time a systematic and physically meaningful procedure using a diagrammatic foundation. One can also define a closed group of entanglement diagrams. For example, a direct product of entangled diagrams in Fig. 3 generates the ordered fusion algebra:

\[
\text{triplet} \otimes \text{triplet} = \text{triplet} \\
\text{triplet} \otimes \text{singlet} = \text{singlet} \\
\text{singlet} \otimes \text{singlet} = \text{triplet}
\]

References

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21
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