Shock formation for forced Burgers equation and application

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Abstract. We study the inviscid Burgers equation in the presence of spatially periodic potential force. We prove that for foliated initial value problem there are always solutions developing shocks in a finite time. We give an application of this result to a quasi-linear system of conservation laws which appeared in the study of integrable Hamiltonian systems with 1.5 degrees of freedom.

1 Introduction

Usually it is not a simple task to prove the shock formation for quasi-linear equations. We refer to [6] for an exposition of techniques. For the forced Burgers equation there are usually some initial data which do not lead to formation of shocks. In this paper we use integral geometry approach to prove the shock formation for at least some leaves of foliation which is composed by the graphs of solutions of the forced Burgers equation. This approach is inspired by Hopf’s famous theorem [5] in Riemannian geometry, which can be naturally interpreted as a result on formation of shocks.

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Consider the inviscid Burgers equation

$$f_t + ff_q + F = 0 \quad (2.1)$$

We shall assume that the force $F(q, t) = u_q(q, t)$, where the potential function $u(q, t)$ satisfies the following requirements.

(2.2a) $u$ is of class $C^2$, 1-periodic in $q$, $u(q + 1, t) = u(q, t)$.

(2.2b) for a positive constant $K$, $\int_0^1 u_q^2(q, t) dq < K$ holds for all $t$.

We consider the initial values for $f(q, t)$ depending on parameter $\alpha$

$$f_\alpha(q, t)|_{t=0} = \varphi(\alpha, q)$$

where $\varphi$ is a $C^2$-function satisfying:

(2.3a) $\varphi(\alpha, q)$ is a 1-periodic in $q$, for any $\alpha$

(2.3b) for any fixed $q$, the mapping $\alpha \mapsto \varphi(\alpha, q)$ is a $C^2$-diffeomorphism of $\mathbb{R}$.

Geometrical meaning of (2.3) is that the graphs of $\varphi(\alpha, q)$ for a $C^2$-foliation of the cylinder $S^1 \times \mathbb{R}$. That is why we refer to such initial data as foliated initial data.

**Theorem 1** Let $u(q, t)$ be the potential of a non-zero force $F$ satisfying (2.2a,b). Then for any foliated initial data $\varphi(\alpha, q)$ satisfying (2.3a,b) there always exist the values of $\alpha$ such that the corresponding solutions of Burgers equation (2.1) develop shock singularities in a finite (positive or negative) time.

**Remark 1** The only case when shocks are not created is the case of zero force, with the initial data $\varphi(\alpha, q) = \varphi(\alpha)$. It should be mentioned that there are many potentials satisfying (2.2a,b) such that some solutions of (2.1) do not form shocks. For instance, this is always the case if $u$ is smooth enough and periodic in both $q$ and $t$. In this case KAM theory applies and yields that there are many solutions for (2.1) periodic in $q$ and $t$.

The next result shows that it is not necessarily true that the shocks described by Theorem 1 always appear in a forward time.

**Theorem 2** Let $u$ be any $C^2$-function periodic in $q$ satisfying the following

(2.4a) $u(q, t) \equiv 0$ for $0 < T \leq t$

(2.4b) for all $0 \leq t \leq T$, $|u_{qq}(q, t)| < \left(\frac{\pi}{T}\right)^2$

Then there exists a foliated initial data at $t = 0$ for (2.1) such that all the shocks are created in a backward time.
3 Proofs

For the proof of Theorem 1 we will need the following

**Lemma 1** Let \( V \) be a \( C^1 \)-vector field on \( \mathbb{R}^2 \) with the following property

\[
\text{div} V \geq C ||V||^2
\]

(3.1)

for a positive constant \( C \). Then \( V \equiv 0 \).

**Remark 2** This lemma does not generalise to higher dimensions. There are smooth vector fields on \( \mathbb{R}^n \) for \( n \geq 3 \) satisfying (3.1) everywhere.

**Proof of Lemma 1** Integrate (3.1) over the circle \( S_r \). We obtain

\[
\int_{S_r} \text{div} V d\omega \geq C \int_{S_r} ||V||^2 d\omega
\]

(3.2)

where \( d\omega \) is the standard measure on \( S_r \). The left hand side of (3.2) can be easily written in the form

\[
\int_{S_r} \text{div} V d\omega = \frac{d}{dr} \int_{B_r} \text{div} V d(\text{vol}) = \frac{d}{dr} \left( \int_{S_r} \langle V, n \rangle d\omega \right)
\]

(3.3)

where \( \partial B_r = S_r \) and \( n \) is a unite normal to \( S_r \). The right hand side of (3.2) can be estimated by Cauchy–Shwarz inequality

\[
\int_{S_r} ||V||^2 d\omega \geq \left( \int_{S_r} \langle V, n \rangle d\omega \right)^2 / \int_{S_r} ||n||^2 d\omega = \frac{1}{2\pi r} \left( \int_{S_r} \langle V, n \rangle d\omega \right)^2
\]

(3.4)

Combining (3.2) with (3.3) and (3.4) we obtain the following

\[
\varphi'(r) \geq \frac{C}{2\pi r} \quad \text{where} \quad \varphi(r) = \int_{S_r} \langle V, n \rangle d\omega
\]

(3.5)

It is easy to see that the only solution of (3.5) which is finite for all \( r > 0 \) is \( \varphi \equiv 0 \). But then \( V \equiv 0 \), by (3.3) and (3.2). \( \square \)

**Proof of Theorem 1** Proof goes by contradiction. Let \( \varphi(\alpha, q) \) be a foliated initial data for (2.1) which does not lead to formation of shocks. Note that the characteristics of (2.1) are given by the Newton Equations

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -u_q
\end{align*}
\]

(3.6)

The periodicity assumption (2.2a) implies that the flow \( g^t \) of (3.6) is complete. Then we have that the graphs \( \{ p = f_{\alpha}(q, t) \} \) form a \( C^2 \)-foliation of the space \( \mathbb{R}(p) \times S^1(q) \times \mathbb{R}(t) \). Define the function \( \omega(p, q, t) \) by the rule

\[
\omega(f_{\alpha}(q, t), q, t) = \frac{\partial f_{\alpha}}{\partial q}(q, t).
\]
Then \( \omega \) is \( C^1 \), and it is easy to see that the following equation holds true:
\[
\omega_t + p\omega_q - u_q\omega_p + \omega^2 + u_{qq} = 0 \tag{3.7}
\]
Integrate (3.7) with respect to \( q \) over \( S^1 \) and obtain
\[
-\frac{\partial}{\partial t} \int \omega dq + \frac{\partial}{\partial p} \int \omega u_q dq = \int \omega^2 dq \tag{3.8}
\]
Denote by \( V_1(p, t) = -\int \omega dq, V_2(p, t) = \int \omega u_q dq \) and set \( V = (V_1, V_2) \). Then the equation (3.8) implies for the field \( V \) on the plane \( \mathbb{R}^2(p, t) \) satisfies
\[
\text{div} V = \int \omega^2 dq \tag{3.9}
\]
Cauchy-Shwarz inequality applied to the right handside of (3.9) together with the assumption (2.2b) imply that the field \( V \) meets the assumption (3.1) of the lemma. But then \( V \) vanishes identically and so does \( \omega \) (by (3.9)) and also \( u_q \) (by (3.7)). This completes the proof. \( \blacksquare \)

**Proof of Theorem 2** Fix a number \( \alpha \) and consider the family \( M_\alpha \) consisting of those characteristics which for \( t \geq T \) can be written
\[
q(t, \beta) = \alpha(t - T) + \beta \tag{3.10}
\]
It follows that \( M_\alpha \) is ordered and defines a smooth foliation of the semi-cylinder \( S^1 \times \{ t \geq 0 \} \).
Indeed, the Jacobi field \( \xi_\beta(t) = \frac{\partial q}{\partial \beta}(t, \beta) \) satisfies the linearised equation
\[
\dot{\xi}_\beta + u'''_{qq}(q(t, \beta), t) \xi_\beta = 0 \tag{3.11}
\]
with \( \dot{\xi}_\beta(T) = 0 \) by (3.10). Comparing the equation (3.11) with \( \ddot{\xi} + \left( \frac{k}{T} \right)^2 \xi = 0 \) on the segment \( [0, T] \) one immediately concludes by (2.4b) that \( \xi_\beta(t) \neq 0 \), for all \( t \) in \( [0, T] \). This implies that \( \frac{\partial q}{\partial \beta}(t, \beta) > \text{const} > 0 \). And thus \( M_\alpha \) is a smooth foliation.

To each \( M_\alpha \) naturally corresponds the solution \( f_\alpha(q, t) \) of (2.1) defined by the rule
\[
f_\alpha(q(t, \beta), t) = \frac{\partial q}{\partial t}(t, \beta).
\]
The family of solutions \( f_\alpha(q, t) \) define the foliated initial data \( \varphi_\alpha(q) = f_\alpha(q, 0) \) for which, obviously, the solutions exist infinite positive time. The proof is completed. \( \blacksquare \)

**4 An Application**

Consider the quasi-linear system \( U_t = A(U)U_q \) for \( U = (u_1(q, t) \ldots u_n(q, t)) \) of the following form:
\[
(u_k)_t = (n - k + 1)u_{k-1}(u_1)_q - (u_{k+1})_q \quad \text{for} \quad k = 1, \ldots, n \tag{4.1}
\]
where \( u_{n+1} \equiv 0 \) and \( u_0 \) is a constant parameter. This system naturally appears in the study of polynomial integrals for the Hamiltonian system (3.6). It turns out \([1]\) that (4.1) has a remarkable Hamiltonian form and infinitely many conservation laws. It is an open question if all smooth solutions of (4.1) defined on the whole cylinder \( S^1(q) \times \mathbb{R}(t) \) are simple waves solutions (see \([2]\) for partial results).

We apply Theorem 1 for the elliptic regimes of (4.1), i.e. for those solutions for which \( A(U) \) has no real eigenvalues \((n\) is automatically even in this case).\(^2\) Let me start with the

**Example** In the case \( n = 2 \), the system has the form

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_t = \begin{pmatrix}
  2u_0 & -1 \\
  u_1 & 0
\end{pmatrix} \begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_q
\]

(4.2)

For a solution \( U = (u_1, u_2) \) lying in the elliptic domain, i.e. satisfying \( u_1(q,t) > u_0^2 \) introduce the function

\[
E(t) = \int_{S^1} (u_1)^\gamma dq \quad \text{for} \quad \gamma \in (0,1).
\]

(4.3)

Then the direct computation using (4.2) yields

\[
\ddot{E}(t) = \gamma(\gamma - 1) \int_{S^1} (u_1)^{\gamma-2} \left( (u_1)_t^2 - 2u_0(u_1)_t(u_1)_q + (u_1)_q^2 \right) dq
\]

Since \( u_1 > u_0^2 \) the integrand is non-negative and hence by the choice of \( \gamma \) in \((0,1)\), one obtains that the function \( E(t) \) is a concave positive function. Thus \( E \equiv 0 \) and then \( u_1, u_2 \) are constants.

For higher \( n \) we prove the following:

**Theorem 3** Let \( n \) be even and greater than two and let \( U = (u_1(q,t) \ldots u_n(q,t)) \) be a smooth solution for (4.1) defined on \( S^1(q) \times \mathbb{R}(t) \) satisfying

\[
\begin{align*}
(4.3a) & \quad U \text{ is such that the matrix } A(U) \text{ has no real eigenvalues.} \\
(4.3b) & \quad \int_{S^1} u_1^2(q,t) dq < K, \text{ for some positive constant } K.
\end{align*}
\]

Then \( U \equiv \text{const.} \)

**Proof of theorem 3** For a solution \( U \) introduce a polynomial function

\[
F = \frac{1}{n+1} p^{n+1} + u_0 p^n + u_1 p^{n-1} + \cdots + u_n
\]

It can be easily checked that the system (4.1) expresses the fact that the function \( F(p,q,t) \) satisfies the equation

\[
F_t + pF_q - (u_1)_q F_p = 0
\]

\(^2\)We shall report on strictly hyperbolic case elsewhere.
ie. $F$ has constant values along the Hamiltonian flow of (3.6) (with $u = u_1$). Moreover, it turns out that characteristic polynomial of $A(U)$ satisfies

$$\det(A(U) - \lambda I) = F_p(-\lambda, u_0, u_1, \ldots u_n)$$

But then by the assumption (4.3a) $F_p$ does not vanish and hence the levels of $F$ determine the foliation consisting of graphs of solutions of (2.1) with the potential $u_1(q, t)$. Then it follows from Theorem 1 that $U$ must be a constant. $\square$

5 Open questions

We formulate here some natural open questions

1. It is not clear if the growth condition (2.2b) is really essential for theorem 1 to be true. Moreover, since theorem 1 is used for the proof of theorem 3 we had to assume (4.3b). But the argument in the Example indicates that probably this assumption may be omitted.

2. The lemma used for the proof of theorem 1 does not generalise to highest dimensions. It would be interesting to find some other integral geometric tools applicable in higher dimensions. One such tool was suggested in [3] for potentials periodic both in space and in time.

3. It is important to understand if there exist non-zero potential forces satisfying (2.2 a, b) such that all orbits have no conjugate points. Our method does not imply that the force $F$ must vanish in this case, though it is close to that. Such a dichotomy is well known in this type of question (see for example [4]).

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