A Categorification of the Vandermonde Determinant

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Categorification (a philosophy)

**Categorification** is the process of finding category theoretic analogues of set theoretic ideas by "adding extra structure":

| Categorification | Decategorification |
|------------------|--------------------|
| sets             | categories         |
| elements         | objects            |
| equations between elements | isomorphisms between objects |
| functions        | functors           |

Decategorification is the reverse process (forgetting the extra structure)
Example: a categorification of $\mathbb{N}$

Objects: f.d. $k$-vector spaces

Morphisms: $k$-linear maps

$k$ a field. The category $k$-Vect categorifies $\mathbb{N}$. Decategorify by taking dimension.

- $V$ decategorifies to $\dim V$
- $V \oplus W$ decategorifies to $\dim V + \dim W$
- $V \otimes W$ decategorifies to $\dim V \dim W$
Categorifying $\mathbb{Z}$

Objects: bounded chain complexes
Morphisms: chain maps

The category $C^b(k\text{-Vect})$ categorifies $\mathbb{Z}$.

- $C_*$ decategorifies to $\chi(C_*) = \sum (-1)^i \dim C_i$
- $C_* \oplus D_*$ decategorifies to $\chi(C_*) + \chi(D_*)$
- $C_* \otimes D_*$ decategorifies to $\chi(C_*)\chi(D_*)$
A classic example from topology

- \( \Delta \): a simplicial complex, \( c_i = \# \) faces of dim \( i \)
- The Euler characteristic of \( \Delta \) is
  \[
  \chi(\Delta) = \sum_{i \geq 0} (-1)^i c_i
  \]
- \( C_i(\Delta) \): free abelian group generated by faces of dimension \( i \)
- \( d : C_k \to C_{k-1} \) sends
  \[
  [v_{i_1}, \ldots, v_{i_k}] \mapsto \sum_j (-1)^j [v_{i_1}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_k}]
  \]
- \( \chi(C_*(\Delta)) = \chi(\Delta) \)
The plan for this talk

The Vandermonde determinant is defined as

\[
V_n = \begin{vmatrix}
    x_1 & x_1^2 & \cdots & x_1^n \\
    x_2 & x_2^2 & \cdots & x_2^n \\
    \vdots & \vdots & \ddots & \vdots \\
    x_n & x_n^2 & \cdots & x_n^n \\
\end{vmatrix} = \sum_{\pi \in S_n} (-1)^{\text{inv}(\pi)} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}
\]

- Categorify (evaluations of) \( V_n \) for \( x_1, \ldots, x_n \in \mathbb{N} \)
- Accomplish this in a way analogous to Khovanov’s categorification of the Kauffman bracket
Khovanov homology

Categorifies the Kauffman bracket

\[ \langle K \rangle = \sum_{\alpha \in \{0,1\}^n} (-1)^{h(\alpha)} q^{h(\alpha)} (q + q^{-1})^{s(\alpha)} \]

where \( K \) is a link diagram with \( n \) crossings,

\[ h(\alpha) = \#1's \ in \ \alpha, \]

\[ s(\alpha) = \#circles \ in \ \alpha\text{-smoothing} \]
The $n$-cube $C_n = \{0, 1\}^n$

- Partially ordered set with vertices $\{0, 1\}^n$
- Cover relation (edge) when you change a 0 to a 1:
$n$-tuples of 1’s and 0’s encode smoothings of $K$
Edges encode **cobordisms** between smoothings

- Start with \((100\text{-smoothing}) \times [0, 1]\)
- Remove cylindrical neighborhood of changing crossing
- Replace with a saddle
Category of cobordisms

Objects: closed 1-dim manifolds

Morphisms: 2-dim cobordisms

The category $\text{Cob}_2$ contains smoothings and cobordisms as its objects and morphisms
Replace vertices in $C_n$ by corresponding smoothings.
Replace edges in $C_n$ by cobordisms

A commutative diagram in $\text{Cob}_2$
2D TQFTs and Frobenius algebras

A 2D TQFT is a monoidal functor from $\text{Cob}_2$ to $k$-$\text{Vect}$.

- Assigns a $k$-vector space $A$ to each circle

$$
\begin{align*}
\text{Circle} & \quad \rightarrow \quad A \\
\text{Circle with 4 disks} & \quad \rightarrow \quad A \otimes A \otimes A \otimes A
\end{align*}
$$

- Assigns linear maps to cobordisms

$$
\begin{align*}
\text{Diagram 1} & \quad \rightarrow \quad \left( m : A \otimes A \rightarrow A \right) \\
\text{Diagram 2} & \quad \rightarrow \quad \left( \Delta : A \rightarrow A \otimes A \right)
\end{align*}
$$

- Multiplication

$$
\begin{align*}
\text{Diagram 3} & \quad \rightarrow \quad \left( \eta : R \rightarrow A \right) \\
\text{Diagram 4} & \quad \rightarrow \quad \left( \epsilon : A \rightarrow R \right)
\end{align*}
$$

- Unit

- Comultiplication

- Counit
Apply a 2D TQFT (with $q \dim A = q + q^{-1}$)

An anti-commutative diagram in $R$-gmod
Direct sum down ranks and get a chain complex

\[ A \otimes A \rightarrow A \oplus A \rightarrow (A \otimes A) \oplus^3 \rightarrow A \otimes A \otimes A \]
The (shifted) homology groups of this chain complex are link invariants and the graded Euler characteristic of this complex is equal to the Kauffman bracket

$$\sum_{i \in \mathbb{Z}} (-1)^i q \dim H^i = \langle K \rangle$$

- The (shifted) Khovanov homology groups give a strictly stronger link invariant than the Jones polynomial
- Khovanov homology is a functor. That is, cobordisms between links induce maps between Khovanov homology groups
Why did this construction work?

- The Kauffman bracket is a rank alternating sum over a ranked poset \( P = C_n \)
  \[
  \sum_{x \in P} (-1)^{r(x)} f(x)
  \]
- Every interval of length 2 in \( C_n \) is a diamond (i.e. \( C_n \) is thin)
- There is a \( \{+1, -1\} \) edge coloring of \( C_n \) for which each diamond has an odd number of -1’s (a balanced coloring)
Categorifying Vandermonde

\[ V_n = \sum_{\pi \in S_n} (-1)^{\text{inv}(\pi)} x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(n)} \]

- \( S_n \) has a thin poset structure: the **Bruhat order**
- \( \text{inv}(\pi) \) is the rank function for this ordering
- The Bruhat order has a balanced coloring

**We're in business!**
Bruhat Order on $S_n$

- An **inversion** in a permutation $\pi$ is a pair $(i, j)$ with $i < j$ and $\pi(i) > \pi(j)$
- $\text{inv}(\pi)$ denotes the number of inversions of $\pi$
- Bruhat order on $S_n$ has a vertex for each $\pi \in S_n$
- Has an edge (cover relation) $\pi \lessdot \sigma$ whenever $\sigma$ is gotten from $\pi$ (in one line notation) by transposing a non-inversion pair for which $\text{inv}(\sigma) = \text{inv}(\pi) + 1$
E.g. Bruhat Order on $S_3$
Colored Cobordisms: \( \text{Cob}_2^n \)

Objects: \([n]\)-colored closed 1-manifolds

Morphisms: color preserving cobordisms

Let \([n] = \{1, 2, \ldots, n\}\). The category \(\text{Cob}_2^n\) has

- Objects: closed 1-manifolds with each connected component given a color from \([n]\)
- Morphisms: 2-dimensional manifolds for which each connected component has monochromatic boundary
For example, let $M = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$ and $N = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$

is a colored cobordism from $M$ to $N$, but not
Permutations (vertices) encode colored smoothings

$K$ a link diagram with crossings $c_1, \ldots, c_n$ and $\pi \in S_n$

- The $\pi$-smoothing of $K$ is

$$K^\pi = K_1^\pi \amalg K_2^\pi \amalg \ldots \amalg K_n^\pi \in \text{Ob Cob}_2^n$$

- $K_i^\pi$ is gotten from $K$ by giving $c_1, c_2, \ldots, c_{\pi(i)}$ 1-smoothings and all other crossings 0-smoothings

- All components of $K_i^\pi$ are colored $i$
\( \pi \)-Smoothing Example

\[ K = \begin{array}{c}
\varepsilon_3 \\
\varepsilon_2 \\
\varepsilon_1
\end{array} \quad \pi = 213 \text{ smoothing} \]

\[ K_1^\pi \quad K_2^\pi \quad K_3^\pi \]
Edges encode colored cobordisms

If $\pi \preceq \sigma$ then $K^{\pi} = \bigoplus_{i=1}^{n} K_{i}^{\pi}$ and $K^{\sigma} = \bigoplus_{i=1}^{n} K_{i}^{\sigma}$ differ at exactly two colors. Use connected genus 0 cobordisms on the colored pieces which differ, and identity (cylinders) on pieces which do not change.
Replace vertex $\pi$ with $\pi$-smoothing
Replace edges with colored cobordisms

A (non) commutative diagram in $\text{Cob}_2^n$
2D colored TQFTs

Definition

- A colored TQFT is a monoidal functor $F : \text{Cob}_2^n \to k\text{-Vect}$ which restricts to a TQFT on each color.

\[ \sim A_{x_1} \quad \text{dim} = x_1 \]
\[ \sim A_{x_2} \quad \text{dim} = x_2 \]
\[ \ldots \]
\[ \sim A_{x_n} \quad \text{dim} = x_n \]
Special TQFTs and special Frobenius algebras

A 2D TQFT $F$ is \textbf{special} if the following condition holds:

$$F\left(\begin{array}{c}
\end{array}\right) = F\left(\begin{array}{c}
\end{array}\right) \iff \mu \circ \Delta = 1$$

A 2D colored TQFT is special if its restriction to each color is a special TQFT.
Apply a Special Colored TQFT

\[
\begin{align*}
\Delta \otimes m \otimes \text{Id} & \quad \rightarrow \quad \Delta^{2} \otimes \Delta \otimes \text{Id} \\
\text{Id} \otimes \Delta^{2} \otimes m^{2} & \quad \rightarrow \quad \Delta^{2} \otimes m \otimes \text{Id} \\
\Delta \otimes \text{Id} \otimes m & \quad \rightarrow \quad \Delta \otimes \text{Id} \otimes m^{2} \Delta \\
\text{Id} \otimes \Delta \otimes m & \quad \rightarrow \quad \Delta^{2} \otimes m \otimes \Delta m^{2} \\
A_{x_1} \otimes A_{x_2} \otimes A_{x_3} & \quad \rightarrow \quad A_{x_1}^{3} \otimes A_{x_2}^{2} \otimes A_{x_3}^{3} \\
A_{x_1} \otimes A_{x_2}^{2} \otimes A_{x_3}^{3} & \quad \rightarrow \quad A_{x_1}^{2} \otimes A_{x_3}^{3} \otimes A_{x_3} \\
A_{x_1} \otimes A_{x_2} \otimes A_{x_3} & \quad \rightarrow \quad A_{x_1} \otimes A_{x_2} \otimes A_{x_3}^{2} \\
\Delta^{2} \otimes m^{2} \otimes \text{Id} & \quad \rightarrow \quad \Delta^{2} \otimes m \otimes \Delta m^{2}
\end{align*}
\]

An anti-commutative diagram in \( k \)-Vect

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A Categorification of the Vandermonde Determinant
Direct sum down ranks to get a chain complex

\[ A_{x_1} \otimes A_{x_2} \otimes A_{x_3} \rightarrow A_{x_1} \otimes A_{x_2} \otimes A_{x_3} \]

\[ \Delta \otimes m \otimes \text{Id} \]

\[ \text{Id} \otimes \Delta^2 \otimes m^2 \]

\[ \Delta \otimes \text{Id} \otimes m \]

\[ \Delta^2 m \otimes \Delta m^2 \otimes \text{Id} \]

\[ \Delta \otimes \text{Id} \otimes m \]

\[ \text{Id} \otimes \Delta \otimes m \]

\[ \Delta^2 \otimes m \otimes \text{Id} \]

\[ \Delta^2 \otimes m \otimes \Delta^2 \]

\[ \Delta \otimes \text{Id} \otimes \Delta \]

\[ \text{Id} \otimes \Delta \otimes \Delta \]

\[ \Delta^2 \otimes m^2 \otimes \text{Id} \]
Theorem (C.)

Let $K$ be the alternating two strand braid diagram of the $(2, n)$-torus knot. Then the Euler characteristic of this chain complex is equal to the Vandermonde determinant

$$V_n = \sum_{i \geq 0} (-1)^i \dim H^i$$
What’s Next?

Questions:

- Is this categorification functorial?
- What kinds of polynomials do we recover for arbitrary knots?
- Do specific classes of knots correspond to known classes of polynomials?
- Relation to $V_n = x_1 \ldots x_n \prod_{i<j}(x_j - x_i)$
Thank you!