Low temperature relations in QCD

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Abstract

In this talk I discuss the low temperature relations for the trace of the energy-momentum tensor in QCD with two and three quarks. It is shown that the temperature derivatives of the anomalous and normal (quark massive term) contributions to the trace of the energy-momentum tensor in QCD are equal to each other in the low temperature region. Leading corrections connected with $\pi\pi$-interactions and thermal excitations of $K$ and $\eta$ mesons are calculated.

The investigation of the vacuum state behavior under the influence of various external factors is known to be one of the central problems of quantum field theory. In the realm of strong interactions (QCD) the main factors are the temperature and the baryon density. At low temperatures, $T < T_c$ (temperature of the "hadron–quark-gluon" phase transition), the dynamics of QCD is essentially nonperturbative and is characterized by confinement and spontaneous breaking of chiral symmetry (SBCS). In the hadronic phase the partition function of the system is dominated by the contribution of the lightest particles in the physical spectrum. It is well known that due to the smallness of pion mass as compared to the typical scale of strong interactions, the pion plays a special role among other strongly-interacting particles. Therefore for many problems of QCD at zero temperature the chiral limit, $M_\pi \to 0$, is an appropriate one. On the other hand a new mass

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scale emerges in the physics of QCD phase transitions, namely the critical transition temperature $T_c$. Numerically the critical transition temperature turns out to be close to the pion mass, $T_c \approx M_\pi$. However hadron states heavier than pion have masses several times larger than $T_c$ and therefore their contribution to the thermodynamic quantities is damped by Boltzmann factor $\exp\{-M_{\text{hadr}}/T\}$. Thus the thermodynamics of the low temperature hadron phase, $T \lesssim M_\pi$, is described basically in terms of the thermal excitations of relativistic massive pions.

The low-energy theorems, playing an important role in the understanding of the vacuum state properties in quantum field theory, were discovered almost at the same time as quantum field methods have been applied in particle physics (see, for example, Low theorems [2]). In QCD, they were obtained in the beginning of eighties [3]. The QCD low-energy theorems, being derived from the very general symmetry considerations and not depending on the details of confinement mechanism, sometimes give information which is not easy to obtain in another way. Also, they can be used as "physically sensible" restrictions in the constructing of effective theories. Recently, they were generalized to finite temperature [4] and a magnetic field case [5]. These theorems were used for investigation of QCD vacuum phase structure in a magnetic field at finite temperature [6].

In this talk I will discuss the low temperature relations for the trace of the energy-momentum tensor in QCD with two and three light quarks. These relations is based on the general dimensional and renormalization-group properties of the QCD partition function and dominating role of the pion thermal excitations in the hadronic phase. The physical consequences of these relations are discussed as well as the possibilities to use it in the lattice studies of the QCD at finite temperature.

For non-zero quark mass ($m_q \neq 0$) the scale invariance is broken already at the classical level. Therefore the pion thermal excitations would change, even in the ideal gas approximation, the value of the gluon condensate with increasing temperature [6].

A relation between the trace anomaly and thermodynamic pressure in the chiral limit of QCD was first written in [7]. Rigorous derivation of this...
relation in the framework of the renormalization group (RG) method in pure-glue QCD was performed in \[8\] and in QCD with non-zero quark masses in \[9, 10\].

Trace anomaly is \[10\]

\[
\langle \theta_{\mu\nu} \rangle = \frac{\beta(\alpha_s)}{16\pi\alpha_s^2} \langle G^2 \rangle = -(4 - T \frac{\partial}{\partial T} - \sum_q (1 + \gamma_{m_q}) m_q \frac{\partial}{\partial m_q}) P_R, \tag{1}
\]

where \(P_R\) is the renormalized pressure, \(\beta(\alpha_s) = d\alpha_s(M)/d\ln M\) is the Gell-Mann-Low function and \(\gamma_{m_q}\) is the anomalous dimension of the quark mass. It is convenient to choose such a large scale that one can take the lowest order expressions, \(\beta(\alpha_s) \rightarrow -b\alpha_s^2/2\pi\), where \(b = (11N_c - 2N_f)/3\) and \(1 + \gamma_m \rightarrow 1\). Thus, we have the following equations for condensates

\[
\langle G^2 \rangle(T) = \frac{32\pi^2}{b}(4 - T \frac{\partial}{\partial T} - \sum_q m_q \frac{\partial}{\partial m_q}) P_R \equiv \hat{D} P_R, \tag{2}
\]

\[
\langle \bar{q}q \rangle(T) = -\frac{\partial P_R}{\partial m_q}. \tag{3}
\]

In the hadronic phase the effective pressure from which one can extract the condensates \(\langle \bar{q}q \rangle(T)\) and \(\langle G^2 \rangle(T)\) using the general relations (2) and (3) has the form

\[
P_{\text{eff}}(T) = -\varepsilon_{\text{vac}} + P_h(T), \tag{4}
\]

where \(\varepsilon_{\text{vac}} = \frac{1}{4} \langle \theta_{\mu\nu} \rangle\) is the nonperturbative vacuum energy density at \(T = 0\) and

\[
\langle \theta_{\mu\nu} \rangle = \frac{b}{32\pi^2} \langle G^2 \rangle + \sum_{q=u,d} m_q \langle \bar{q}q \rangle \tag{5}
\]

is the trace of the energy-momentum tensor. In Eq.(4) \(P_h(T)\) is the thermal hadrons pressure. The quark and gluon condensates are given by the equations

\[
\langle \bar{q}q \rangle(T) = -\frac{\partial P_{\text{eff}}}{\partial m_q}, \tag{6}
\]

\[
\langle G^2 \rangle(T) = \hat{D} P_{\text{eff}}, \tag{7}
\]
where the operator \( \hat{D} \) is defined by the relation (2)

\[
\hat{D} = \frac{32\pi^2}{b} (4 - T \frac{\partial}{\partial T} - \sum_q m_q \frac{\partial}{\partial m_q}).
\] (8)

Consider the \( T = 0 \) case. One can use the low energy theorem for the derivative of the gluon condensate with respect to the quark mass [3]

\[
\frac{\partial}{\partial m_q} \langle G^2 \rangle = \int d^4x \langle G^2(0)\bar{q}q(x) \rangle = -\frac{96\pi^2}{b} \langle \bar{q}q \rangle + O(m_q),
\] (9)

where \( O(m_q) \) stands for the terms linear in light quark masses. Then one arrives at the following relation

\[
\frac{\partial \varepsilon_{\text{vac}}}{\partial m_q} = -\frac{b}{128\pi^2} \frac{\partial}{\partial m_q} \langle G^2 \rangle + \frac{1}{4} \langle \bar{q}q \rangle = \frac{3}{4} \langle \bar{q}q \rangle + \frac{1}{4} \langle \bar{q}q \rangle = \langle \bar{q}q \rangle.
\] (10)

Note that three fourths of the quark condensate stem from the gluon part of the nonperturbative vacuum energy density. Along the same lines one arrives at the expression for the gluon condensate

\[
-\hat{D} \varepsilon_{\text{vac}} = \langle G^2 \rangle.
\] (11)

Let us consider \( N_f = 2 \) case. In order to get the dependence of the quark and gluon condensates upon \( T \) use is made of the Gell-Mann- Oakes-Renner (GMOR) relation (\( \Sigma = |\langle \bar{u}u \rangle| = |\langle \bar{d}d \rangle| \))

\[
F_\pi^2 M_\pi^2 = -\frac{1}{2} (m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle = (m_u + m_d) \Sigma.
\] (12)

Then we can find the following relations

\[
\frac{\partial}{\partial m_q} = \frac{\Sigma}{F_\pi^2} \frac{\partial}{\partial M_\pi^2},
\] (13)

\[
\sum_{q=u,d} m_q \frac{\partial}{\partial m_q} = (m_u + m_d) \frac{\Sigma}{F_\pi^2} \frac{\partial}{\partial M_\pi^2} = M_\pi^2 \frac{\partial}{\partial M_\pi^2},
\] (14)

\[
\hat{D} = \frac{32\pi^2}{b} (4 - T \frac{\partial}{\partial T} - M_\pi^2 \frac{\partial}{\partial M_\pi^2}).
\] (15)
At low temperature the main contribution to the pressure comes from thermal excitations of massive pions. The general expression for the pressure reads

\[ P_\pi = T^4 \varphi(M_\pi/T) , \]  

(16)

where \( \varphi \) is a function of the ratio \( M_\pi/T \). Then the following relation is valid

\[ (4 - T \frac{\partial}{\partial T} - M_\pi^2 \frac{\partial}{\partial M_\pi^2} ) P_\pi = M_\pi^2 \frac{\partial P_\pi}{\partial M_\pi^2} . \]  

(17)

With the account of (6,7), (10,22) and (17) one gets

\[ \Delta \langle \bar{q}q \rangle = - \frac{\partial P_\pi}{\partial m_q}, \quad \Delta \langle G^2 \rangle = \frac{32\pi^2}{b} M_\pi^2 \frac{\partial P_\pi}{\partial M_\pi^2} , \]  

(18)

where \( \Delta \langle \bar{q}q \rangle = \langle \bar{q}q \rangle_T - \langle \bar{q}q \rangle \) and \( \Delta \langle G^2 \rangle = \langle G^2 \rangle_T - \langle G^2 \rangle \). In view of (14) one can recast (18) in the form

\[ \Delta \langle G^2 \rangle = - \frac{32\pi^2}{b} \sum_{q=u,d} m_q \Delta \langle \bar{q}q \rangle . \]  

(19)

Differentiating (19) with respect to \( T \) one obtains

\[ \frac{\partial \langle G^2 \rangle}{\partial T} = \frac{32\pi^2}{b} \sum_{q=u,d} m_q \frac{\partial \langle \bar{q}q \rangle}{\partial T} . \]  

(20)

This can be rewritten as

\[ \frac{\partial \langle \theta_{\mu\nu}^q \rangle}{\partial T} = \frac{\partial \langle \theta_{\mu\nu}^g \rangle}{\partial T} , \]  

(21)

where \( \langle \theta_{\mu\nu}^q \rangle = \sum m_q \langle \bar{q}q \rangle \) and \( \langle \theta_{\mu\nu}^g \rangle = (\beta(\alpha_s)/16\pi\alpha_s^2) \langle G^2 \rangle \) are correspondingly the quark and gluon contributions to the trace of the energy-momentum tensor. Note that in deriving this result use was made of the low energy GMOR relation, and therefore the thermodynamic relation (20,21) is valid in the light quark theory. Thus in the low temperature region when the excitations of massive hadrons and interactions of pions can be neglected, equation (21) becomes a rigorous QCD theorem.

As it was mentioned above the pion plays an exceptional role in thermodynamics of QCD due to the fact that its mass is numerically close to the phase transition temperature while the masses of heavier hadrons are several
times larger than $T_c$. It was shown in [12] that at low temperatures, the contribution to $\langle \bar{q}q \rangle$ generated by the massive states is very small, less than 5% if $T$ is below 100 MeV. At $T = 150$ MeV, this contribution is of the order of 10%. The influence of thermal excitations of massive hadrons on the properties of the gluon and quark condensates in the framework of the conformal-nonlinear $\sigma$-model was studied in detail in [13].

Let us consider leading corrections to relation (19-21) within the described above framework. Clearly, leading corrections are connected with the $\pi\pi$-interaction, since it’s contribution to the pressure is $\propto e^{-2M_\pi/T}$. Also, account for $s$-quark leads to the contributions to the pressure $\propto e^{-M_K/T}$, $e^{-M_\eta/T}$, which are related to the thermal excitations of $K$ and $\eta$-mesons. Then pressure in hadronic phase can be recast in the following form

$$P_h(T) = P_g(T) + P_{\pi\pi}(T),$$

(22)

$$P_g(T) = \sum_{i=\pi,K,\eta} P_i(T),$$

(23)

where $P_i(T) = g_i T^4 \varphi(M_i/T)$ is gas pressure of $i = \pi, K, \eta$-meson and $g_i$ is the number of degrees of freedom of $i$-state, $g_\pi = 3$, $g_K = 4$, $g_\eta = 1$. Pressure related to $\pi\pi$ interaction in two-loop ChPT in general form is

$$P_{\pi\pi} = T^4 \frac{M_\pi^2}{F_\pi^2} f \left( \frac{M_\pi}{T} \right),$$

(24)

here $f$ is function of ratio $M_\pi/T$, and factor $M_\pi^2/F_\pi^2$ is connected with the $\pi\pi$ interaction vertex.

Making use of Gell-Mann-Okubo relations one gets (analogous to (17))

$$\dot{P}_g(T) = \frac{32\pi^2}{b} \left( 4 - T \frac{\partial}{\partial T} - \sum_{q=u,d,s} m_q \frac{\partial}{\partial m_q} \right) P_g(T)$$

(25)

$$= \frac{32\pi^2}{b} \sum_{i=\pi,K,\eta} M_i^2 \frac{\partial P_i}{\partial M_i^2}.$$

For the temperature shift of quark condensates on has

$$\frac{\Delta \Sigma(T)}{\Sigma} = \frac{\partial P_g}{\partial m_u} = \frac{1}{F_\pi^2} \left( \frac{\partial P_\pi}{\partial M_\pi^2} + \frac{\partial P_K}{\partial M_K^2} + \frac{1}{3} \frac{\partial P_\eta}{\partial M_\eta^2} \right)$$

(26)
\[
\frac{\Delta \Sigma_s(T)}{\Sigma_s} = \frac{\partial P_g}{\partial m_s} = \frac{1}{F_\pi^2} \left( \frac{\partial P_K}{\partial M_K^2} + \frac{4}{3} \frac{\partial P_\eta}{\partial M_\eta^2} \right)
\]  

(27)

Note, that light \( \pi \)-meson does not carry strangeness and thus does not participate in \( \langle \bar{s}s \rangle \) condensate "evaporation". Leading contribution to \( \Delta \langle \bar{s}s \rangle(T) \) comes from thermal excitations of lightest strange \( K \)-meson with the mass several times larger than \( M_\pi \). Therefore it is obvious that \( \langle \bar{s}s \rangle(T) \) decreases more slowly than \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \) with the increase of \( T \). In the gas approximation one finds

\[
\frac{\Delta \Sigma_s(T)/\Sigma_s}{\Delta \Sigma(T)/\Sigma} = \frac{4}{3} \left( \frac{M_K}{M_\pi} \right)^{1/2} \left( \frac{F_\pi}{F_K} \right)^2 e^{(M_\pi-M_K)/T}
\]  

(28)

and this ratio is of order of \( \sim 0.13 \) at \( T \sim 140 \) MeV. Analogous to the derivation of equation (19) (with the account of the (24-27)) one gets

\[
-\frac{b}{32\pi^2} \Delta \langle G^2 \rangle = \sum_{q=u,d,s} m_q \Delta \langle \bar{q}q \rangle + 2P_{\pi\pi} + \frac{1}{2} M_\pi^2 \frac{\partial P_K}{\partial M_K^2}
\]  

(29)

Let us introduce functions

\[
\theta^{\pm}(T) = \langle \theta^{\mu\mu}_g \pm \theta^{\mu\mu}_q \rangle(T) - \langle \theta^{\mu\mu}_g \pm \theta^{\mu\mu}_q \rangle(0).
\]  

(30)

\( \theta^+(T) \) is the thermal part of the trace of the energy-momentum tensor and \( \Delta \langle \theta^{\mu\mu}_{tot} \rangle(T) = \varepsilon - 3P \), where \( \varepsilon = T dP/dT - P \) is energy density.

Then the function

\[
\delta_\theta(T) = \frac{\theta^-(T)}{\theta^+(T)} = \frac{2P_{\pi\pi} + \frac{1}{2} M_\pi^2 \frac{\partial P_K}{\partial M_K^2}}{\varepsilon - 3P}
\]  

(31)

can be considered as a measure of the deviation from low temperature relation (19). Let us estimate this correction numerically. One has for \( P_{\pi\pi} \) [14].

\[
P_{\pi\pi} = -\frac{1}{6} \left( \frac{M_\pi^2}{F_\pi^2} \right) [g_1(M_\pi/T)]^2
\]  

(32)

\[
g_1 = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\omega_\pi (e^{\omega_\pi/T} - 1)}, \quad \omega_\pi = \sqrt{p^2 + M_\pi^2}
\]  

(33)

\footnote{Contribution of \( K \)-meson to the \( \Delta \Sigma_s(T) \) can be obtained from low temperature expression for the condensate \( \Delta \Sigma(T) \) (see [11]), with the obvious substitution of \( M_\pi \to M_K \), \( F_\pi \to F_K \) and multiplication by the factor 4/3.}

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For $i$-meson gas

$$P_i = -g_i T \int \frac{d^3 p}{(2\pi)^3} \ln(1 - e^{-\sqrt{p^2 + M_i^2}/T}). \quad (34)$$

Choosing $M_\pi = 140 \text{ MeV}$, $M_K = 493 \text{ MeV}$, $F_\pi = 93 \text{ MeV}$, one can see from numerical calculations that

$$\delta \theta(T < 150 \text{ MeV}) < 0.04 \quad (35)$$

Consequently, leading corrections to the low temperature relation (19) amounts to several percent up to the critical temperature.

Thus the function $\delta \theta(T)$ at low temperatures is, with good accuracy, close to zero. In the vicinity and at the phase transition point, i.e. in the region of nonperturbative vacuum reconstruction this function changes drastically. To see it, we first consider pure gluodynamics. It was shown in [15] using the effective dilaton Lagrangian, that gluon condensate decreases very weakly with the increase of temperature, up to phase transition point. This result is physically transparent and is the consequence of Boltzmann suppression of thermal glueball excitations in the confining phase.

Further, the dynamical picture of deconfinement based on the reconstruction of the nonperturbative gluonic vacuum was suggested in [16]. Namely, confining and deconfining phases differ first of all in the vacuum fields, i.e., in the value of the gluon condensate and in the gluonic field correlators. The color-magnetic (CM) correlators and their contribution to the condensate are kept intact across the temperature phase transition, while the confining color-electric (CE) part abruptly disappears above $T_c$. Furthermore, there exist numerical lattice measurements of field correlators near the critical transition temperature $T_c$, performed by the Pisa group [17], where both CE and CM correlators are found with good accuracy. These data clearly demonstrate the strong suppression of CE component above $T_c$ and persistence of CM component. Thus, the function $\delta \theta(T)$ can be presented as a $\delta$-function smeared around the critical point $T_c$ with the width $\sim \Delta T$ which defines the fluctuation region of phase transition.

Similar, but more complicated and interesting situation takes place in the theory with quarks. The function $\delta \theta(T)$ contains the quark term, proportional to the chiral phase transition order parameter $\langle \bar{q}q \rangle(T)$. So it is interesting to check the relation (21) and to study the behavior of the function $\delta \theta(T)$ in the lattice QCD at finite temperature. It would allow both
to test the nonperturbative QCD vacuum at the low temperatures in the
confining phase and to extract additional information on the thermal phase
transitions in QCD.

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