On global regularity and singularities of Navier-Stokes- and Euler equation solutions

Jörg Kampen

February 6, 2015

Abstract

Euler-Leray data functions of first and second order are defined by first and second order derivatives of the nonlinear spatial part of the incompressible Euler equation operator in Leray projection form applied to Cauchy data. The Lipschitz continuity of these functions for strong Cauchy data in $H^m \cap C^m$, $m \geq 2$ is sufficient for the existence of global regular upper bounds of incompressible Navier Stokes equation solutions (in case of Cauchy data in $H^m \cap C^m$, $m \geq 2$). Global regular upper bounds of global solution branches of the incompressible Euler equation can be obtained (in case of Cauchy data in $H^m \cap C^m$, $m \geq 3$), if the Cauchy data satisfy an additional condition of strong polynomial decay at spatial infinity. Furthermore, if a Lipschitz condition for the Euler-Leray data function of second order is satisfied, then there are long time vorticity blow ups of the incompressible Euler equation, and, correspondingly, short time and long time vorticity blow ups or singular solutions of incompressible Navier Stokes equations with time-dependent forces in $H^{m-2} \cap C^{m-2}$. A further consequence is that multiple global solutions of the incompressible Euler equations exist. The so-called degeneracy issue of convolutions with first order spatial derivatives of the Gaussian is addressed. For the Navier Stokes equation Lipschitz continuity of the Leray projection term in local time has the effect that a rough upper bound estimate of the Leray projection term in terms of the second moment of the Gaussian implies the existence of global regular upper bounds due to stronger damping of spatially scaled solutions. These simple estimates can be refined in order to obtain global regular upper bounds of solution branches in the viscosity limit.

2010 Mathematics Subject Classification. 35Q31, 76N10

1 Euler-Leray data functions of first order and global regular upper bounds of the Navier Stokes equation

Consider the Cauchy problem for the incompressible Navier Stokes equation for regular velocity data $v_i^0(t_0,.) \in H^m \cap C^m$, $m \geq 2$, $1 \leq i \leq D$ on a domain $[t_0, t_0 + \Delta_0] \times \mathbb{R}^D$ at time $t_0 \geq 0$ and for a small time horizon $\Delta_0 > 0$. The short time solution of the incompressible Navier Stokes equation Cauchy problem with constant positive viscosity $\nu > 0$ and without external forces has the
representation
\[ v_i^\nu = v_i^\nu(t_0, \cdot) *_{\text{sp}} G_\nu - \sum_{j=1}^D \left( v_j^\nu \frac{\partial v_i^\nu}{\partial x_j} \right) * G_\nu \]
\[ + \left( \sum_{j,m=1}^D \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_j} K_D(\cdot - y) \right) \sum_{j,m=1}^D \left( \frac{\partial v_i^\nu}{\partial x_j} \frac{\partial v_j^\nu}{\partial x_m} \right) (\cdot, y)dy \right) * G_{\nu, i} \]  
(1)

Here, the symbol ‘*’ denotes convolution with respect to space and time and the symbol ‘*_{sp}’ denotes convolution with respect to the spatial variables. Furthermore, the symbol \( K_D \) refers to the Laplacian kernel of dimension \( D \), and the symbol \( G_\nu \) denotes the Gaussian fundamental solution of the heat equation
\[ p_t - \nu \Delta p = 0. \]  
(2)

The validity of the local-time representation in (1) is due to a local time contraction result with respect to the norm \( \max_{1 \leq t \leq r} \sup_{\|v\| \leq \Delta_0} |v(t, \cdot)|_{H^{m+\gamma} \cap C^m} \) (for a small time interval size \( \Delta_0 > 0 \) and regularity order \( m \geq 2 \)). The solution has then an upper bound with respect to this norm on the time interval \([t_0, t_0 + \Delta_0]\). We may use the incompressibility condition
\[ \sum_{j=1}^D \frac{\partial v_j}{\partial x_j} = 0, \]
(3)
in order to rewrite the Burgers term. Indeed, the incompressibility condition in (3) implies that
\[ \sum_{j=1}^D \frac{\partial(v_i v_j)}{\partial x_j} = \sum_{j=1}^D v_j \frac{\partial v_i}{\partial x_j} + v_i \sum_{j=1}^D \frac{\partial v_j}{\partial x_j} = \sum_{j=1}^D v_j \frac{\partial v_i}{\partial x_j}. \]
(4)

Hence we may rewrite (1) such that the nonlinear terms are convolutions with the first order spatial derivative of the Gaussian. We have
\[ v_i^\nu = v_i^\nu(t_0, \cdot) *_{\text{sp}} G_\nu - \sum_{j=1}^D \left( v_j^\nu \frac{\partial v_i^\nu}{\partial x_j} \right) * G_{\nu, j} \]
\[ + \left( \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_j} K_D(\cdot - y) \right) \sum_{j,m=1}^D \left( \frac{\partial v_i^\nu}{\partial x_j} \frac{\partial v_j^\nu}{\partial x_m} \right) (\cdot, y)dy \right) * G_{\nu, i}, \]  
(5)

where we use the convolution rule for derivatives. We shall observe that the viscosity damping of the term \( v_i^\nu(t_0, \cdot) *_{\text{sp}} G_\nu \) in (5) is stronger than possible growth of the nonlinear terms due to the spatial effects related to \( G_{\nu, j} \) if the data \( |v_i^\nu(t_0, \cdot)|_{H^{m+\gamma} \cap C^m}, m \geq 2 \) exceed a certain level, and if the Leray projection term satisfies a Lipschitz condition. For the scaled Gaussian \( G^\rho_{\nu, r} \) (obtained by replacement of \( \nu \) by \( \rho r^2 \nu \), cf. below), we compute

\[ |G^\rho_{\nu, r}(\tau, \cdot)| \leq \frac{1}{(4\pi \rho r^2 \nu \tau)^{D/2}} \exp \left( -\frac{|y|^2}{4\pi \rho r^2 \nu \tau} \right) \]  
(6)

\[ \leq \frac{1}{(4\pi \rho r^2 \nu \tau)^{D/2}} \left( \frac{|y|^2}{4\pi \rho r^2 \nu \tau} \right)^{D/2 + 1 - \delta} \exp \left( -\frac{|y|^2}{4\pi \rho r^2 \nu \tau} \right). \]

Hence, we have for \( \delta \in (0, 1) \) and all \( \rho, r > 0 \)
\[ |G^\rho_{\nu, r}(\tau, \cdot)| \leq \frac{C}{(4\pi \rho r^2 \nu \tau)^{D/2}} |y|^{D+1-2\delta}. \]
(7)
where the constant
\[ C = \sup_{|z| > 0} (z)^{D/2 + 1 - \delta} \exp \left( -z^2 \right) > 0 \]
is independent of \( \nu > 0 \). The estimate in (7) may be used locally, i.e., on compact subsets of \( \mathbb{R}^D \). For a Lipschitz continuous function \( L(x) = L(x - y) \) with \( |L(x)| \leq l|y| \) (for a Lipschitz constant \( l \) which is independent of \( x \)) we and on a small time interval of length \( \Delta \) get the direct rough estimate (via the transformation \( y_i = \sqrt{4\pi\rho\tau^2} y_i \))
\[
L \ast G^{\rho,\tau}_\nu(x, t) = \int_{\mathbb{R}^D} L(x) G^{\rho,\tau}_\nu(\sigma, y) d\sigma dx \leq lG_2 \Delta, \tag{8}
\]
where the finite constant \( G_2 \) is the second moment of the Gaussian (or an upper bound of the second moment). Here we avoid the factor
\[
\frac{C}{(4\pi\rho\tau^2\nu)^\delta} \tag{9}
\]
Even for this rough estimate we have a scaling \( \rho\tau \) of the nonlinear terms and an upper bound proportional to \( \rho\tau\Delta^{1-\delta} \), while we shall observe a damping of order \( \rho\tau\Delta^3 \) on a small time scale and for data which exceed a certain threshold. This implies that upper bounds are preserved for some strong spatial scaling \( r > 1 \) which depends reciprocally on \( \Delta \) and \( \nu \). Note that a strong spatial scaling is related to a deviation from a strong semigroup contraction. However, refined estimates compensate the factor (9) in (7) times a Lipschitz factor \( l |y| \) by upper bound estimates
\[
\int_{|y| \leq 4\pi \rho \tau^2 \nu} \frac{lC}{(4\pi\rho\tau^2\nu)^\delta} |y|^D dy \sim (4\pi\rho\tau^2\nu)^\delta \tag{10}
\]
for strong damping parameter \( \tau \) with \( 4\pi\rho\tau^2\nu \geq 1 \). Convolutions with Lipschitz functions then satisfy
\[
|L \ast G^{\rho,\tau}_\nu(\Delta, x)| = \int_{\mathbb{R}^D} L(x) G^{\rho,\tau}_\nu(\sigma, y) d\sigma dx \leq lC(4\pi\rho\tau^2\nu)^\delta \Delta^{1-\delta} \tag{11}
\]
for any \( \delta \in (0, 1) \). Note the difference to upper bounds of first order derivatives of the Gaussian, which are integrable for \( \delta > 0.5 \), due to Lipschitz continuity of the Leray projection term. This upper bound can become smaller than the damping as we shall observe next. Indeed the scaled (transformed) equation for \( v^{\rho,\nu}_i(\tau, y) = v_i(t, x) \) we get
\[
v^{\rho,\nu}_i = v^{\rho,\nu}_i(t_0, \cdot) \ast_{sp} G^{\rho,\tau}_\nu - \rho\tau \sum_{j=1}^D \left( v^{\rho,\tau,\nu}_j v^{\rho,\tau,\nu}_i \right) G^{\rho,\tau}_\nu + \rho\tau \left( \int_{\mathbb{R}^D} (K_D(\cdot, y)) \sum_{j,m=1}^D \left( \frac{\partial v^{\rho,\tau,\nu}_j}{\partial x_j} \frac{\partial v^{\rho,\tau,\nu}_m}{\partial x_m} \right)(\cdot, y) dy \right) G^{\rho,\tau}_\nu, \tag{12}
\]
such that the scaling of an upper bound is even
\[
\rho\tau(4\pi\rho\tau^2\nu)^\delta \text{ for } \delta \in (0, 1). \tag{13}
\]
Let us be a bit more specific. We search for conditions such that the viscosity damping encoded in the first term on the right side of (5) offsets possible growth
caused by the nonlinear terms. For this task it is convenient to consider the 
transformation 
\[ v_\nu^i(t, x) = v_{\rho, r, \nu}^i(t, z), \quad z_i = rx_i, \quad 1 \leq i \leq D, \quad t - t_0 = \rho r. \] (14)

For all \( 1 \leq i, j \leq D \) we have 
\[ v_\nu^{i,j}(t, x) = v_{\rho, r, \nu}^{i,j}(t, z), \quad v_\nu^{i,j}(t, x) = v_{\rho, r, \nu}^{i,j}(t, z)r^2. \] (15)

Hence, under the transformation \( z_i = rx_i, \quad 1 \leq i \leq D \) the original Cauchy problem for the incompressible Navier Stokes equation (cf. \[1\] for the modeling) 
\[ \partial v_\nu^i / \partial t - \nu \Delta v_\nu^i \sum_{j=1}^{D} (v_\nu^j \partial v_\nu^i / \partial x_j) = -\nabla p, \] \[ \sum_{i=1}^{D} \partial v_\nu^i / \partial x_i = 0, \quad v_\nu^i(0, .) = f_i \] (16)
becomes 
\[ \partial v_{\rho, r, \nu}^i / \partial \tau - \rho r^2 \nu \Delta v_{\rho, r, \nu}^i + \rho r \sum_{j=1}^{D} (v_{\rho, r, \nu}^j \partial v_{\rho, r, \nu}^i / \partial z_j) = -\rho r \nabla p^r, \] \[ r \sum_{i=1}^{D} \partial v_{\rho, r, \nu}^i / \partial z_i = 0, \quad v_{\rho, r, \nu}^i(0, .) = f_i, \] (17)
where for all \( t \geq 0 \) and \( z = rx \in \mathbb{R}^D \) we have 
\[ p_{\rho, r}^{\nu}(t, z) = p(t, x), \quad p_{\rho, i}(t, x) = p_{\rho, i}^{\nu}(t, z)r. \] (18)

As usual the elimination of the pressure \( p^r \) is by application of the divergence operator. From the first equation in (17) we obtain 
\[ r^2 \sum_{j=1}^{D} v_{\rho, r, \nu}^{i,j} v_{\rho, r, \nu}^{i,j} = -r^2 \Delta p_{\rho, r}, \] (19)
which is of the same form as the usual Poisson equation of the non-parametrized velocity, i.e., we have 
\[ \sum_{j=1}^{D} v_{\rho, r, \nu}^{i,j} v_{\rho, r, \nu}^{i,j} = -\Delta p_{\rho, r}. \] (20)

Hence 
\[ \int_{\mathbb{R}^D} K_{D, i}(., - y) \sum_{j=1}^{D} \left( \partial v_{\rho, r, \nu}^{i,j} / \partial x_j \right) (., y) dy \] (21)
and the transformed equation becomes 
\[ \partial v_{\rho, r, \nu}^i / \partial \tau - \rho r^2 \nu \Delta v_{\rho, r, \nu}^i + \rho r \sum_{j=1}^{D} (v_{\rho, r, \nu}^j \partial v_{\rho, r, \nu}^i / \partial z_j) \]
\[ -\rho r \int_{\mathbb{R}^D} K_{D, i}(., - y) \sum_{j=1}^{D} \left( \partial v_{\rho, r, \nu}^{i,j} / \partial x_j \right) (., y) dy = 0. \] (22)

Note that the fundamental solution of 
\[ p_{\rho, \nu} = -\rho r^2 \nu \Delta p = 0. \] (23)
is explicitly given by
\[ G_{\nu}^{p,r} = \frac{1}{\sqrt{4\pi \rho r^2 \nu (\tau - \sigma)}} \exp \left( -\frac{(x - y)^2}{4\rho r^2 \nu (\tau - \sigma)} \right). \] (24)

We get
\[ v_i^{p,r,\nu} = v_i^{p,r,\nu}(t_0, \cdot) *_{sp} G_{\nu}^{p,r} - \rho r \sum_{j=1}^{\mathcal{D}} \left( v_j^{p,r,\nu} v_i^{p,r,\nu} \right) * G_{\nu}^{p,r} \]
\[ + \rho r \left( \int_{\mathbb{R}^D} (K_D(\cdot - y)) \sum_{j,m=1}^{\mathcal{D}} \left( \frac{\partial v_j^{p,r,\nu}}{\partial x_j} \frac{\partial v_m^{p,r,\nu}}{\partial x_m} \right) (\cdot, y)dy \right) * G_{\nu}^{p,r}. \] (25)

Now the first derivative of the scaled Gaussian \( G_{\nu}^{p,r} \) is given by
\[ G_{\nu,i}^{p,r} = \frac{-2(x - y)}{4\rho r^2 \nu (\tau - \sigma)} \frac{1}{\sqrt{4\pi \rho r^2 \nu (\tau - \sigma)}} \exp \left( -\frac{(x - y)^2}{4\rho r^2 \nu (\tau - \sigma)} \right). \] (26)

Hence,
\[ v_i^{p,r,\nu}(\tau, \cdot) = v_i^{p,r,\nu}(t_0, \cdot) *_{sp} G_{\nu}^{p,r}(\tau, \cdot) \]
\[-\rho r \int_0^\tau \int_{\mathbb{R}^D} \sum_{j=1}^{\mathcal{D}} \left( v_j^{p,r,\nu} v_i^{p,r,\nu} \right) (\sigma, y) \left( \frac{-2(\cdot - y)}{4\rho r^2 \nu (\tau - \sigma)} \right) G_{\nu}^{p,r} \]
\[ + \rho r \int_0^\tau \int_{\mathbb{R}^D} \left( \int_{\mathbb{R}^D} (K_D(y - z)) \sum_{j,m=1}^{\mathcal{D}} \left( \frac{\partial v_j^{p,r,\nu}}{\partial x_j} \frac{\partial v_m^{p,r,\nu}}{\partial x_m} \right) (\tau, z)dz \right) \left( \frac{-2(\cdot - y)}{4\pi \rho r^2 \nu (\tau - \sigma)} \right) \times \]
\[ G_{\nu}^{p,r}(\tau - \sigma, \cdot - y)dyds. \] (27)

First we consider the damping estimates.

i) First we consider \( L^2 \)-estimates. At time \( t_0 \) we estimate the growth of the functions \( D_x^2 v_i(t, \cdot) \), \( 0 \leq |\beta| \leq m \), \( [t \in [t_0, t_0 + \Delta]] \), where
\[ \max_{1 \leq i \leq n} |D_x^3 v_i(t_0, \cdot)|_{L^2_{t,x} \mathcal{C}} \geq 1. \] (28)

If the latter condition is not satisfied for some \( \beta \), then there is a \( \Delta > 0 \) such that the respected norm is less equal 1 for some time \( t \in [t_0, t_0 + \Delta] \) (because the solutions are locally continuous time curves with values in \( H^m \cap C^m \) according to local contraction result), and we need no damping estimate for this part of the \( H^m \cap C^m \)-norm in the interval \([t_0, t_0 + \Delta] \).

Proceeding this way we contract and upper bound which is close to the constant \( \max_{1 \leq i \leq D} |v_i(t_0, \cdot)|_{H^m \cap C^m} \) times the number of terms in the standard definition of \( H^m \)-norms. We apply a Fourier transform with respect to the spatial variables, i.e., the operation
\[ \mathcal{F}(u)(\tau, \xi) = \int_{\mathbb{R}^D} \exp(-2\pi ix\xi) u(\tau, x)dx, \] (29)
in order to analyze the viscosity damping encoded in the first term on the right side of (1) on a time interval \([t_0, t_0 + \Delta]\), where \( \Delta_0 = \rho \Delta \). For \( \tau \in [t_0, t_0 + \Delta] \) and parameters \( r, \rho > 0 \) we have
\[ \mathcal{F} \left( v_i^{p,r,\nu}(t_0, \cdot) *_{sp} G_{\nu}^{p,r}(\tau - t_0, \cdot) \right) = \mathcal{F} \left( v_i^{p,r,\nu}(t_0, \cdot) \right) \mathcal{F} \left( G_{\nu}^{p,r}(\tau - t_0, \cdot) \right) \]
\[ = \mathcal{F} \left( v_i^{p,r,\nu}(t_0, \cdot) \right) \exp \left( -4\pi^2 \rho r^2 \nu (\tau - t_0)(\cdot)^2 \right), \] (30)
The latter constant is finite since the latter value is an upper bound of \( \sqrt{\frac{1}{4\pi^2\rho r^2\nu \tau}} \exp \left( -\frac{(\cdot)^2}{4\pi^2\rho r^2\tau} \right) \) for \( \tau > 0 \) we have

\[
\mathcal{F} (G_{\nu}^{\rho, r}(\tau, \cdot))(\tau, \xi) = \mathcal{F} \left( \frac{1}{\sqrt{4\pi^2\rho r^2\nu \tau}} \exp \left( -\frac{(\cdot)^2}{4\pi^2\rho r^2\tau} \right) \right)(\tau, \xi)
\]

\[
= \exp \left( -4\pi^2\rho r^2\nu \tau|\xi|^2 \right).
\]

Let us set \( t_0 = 0 \) for simplicity (and without loss of generality of the argument). For \( \Delta > 0 \) small enough (such that, say, \( 8\pi^2\rho r^2\nu\tau\Delta^2 \leq 1 \)), and for \( \tau \in [0, \Delta] \) we get

\[
|v_i^{\rho, r, \nu}(t_0, \cdot)|_{L^2}^2 = \int_{\mathbb{R}^3} \left( \mathcal{F} \left( v_i^{\rho, r, \nu}(t_0, \cdot) \right)(\xi) \exp \left( -4\pi^2\rho r^2\nu\tau|\xi|^2 \right) \right) d\xi
\]

\[
= \int_{\mathbb{R}^3} \left( \mathcal{F} \left( v_i^{\rho, r, \nu}(t_0, \cdot) \right)(\xi) \exp \left( -8\pi^2\rho r^2\nu\tau|\xi|^2 \right) \right) d\xi
\]

\[
= \int_{\mathbb{R}^3 \setminus \{ |\xi| \leq \Delta, 1 \leq j \leq D \}} \left( \mathcal{F} \left( v_i^{\rho, r, \nu}(t_0, \cdot) \right)(\xi) \exp \left( -8\pi^2\rho r^2\nu\tau|\xi|^2 \right) \right) d\xi
\]

\[
+ \int_{\{ |\xi| \leq \Delta, 1 \leq j \leq D \}} \left( \mathcal{F} \left( v_i^{\rho, r, \nu}(t_0, \cdot) \right)(\xi) \exp \left( -8\pi^2\rho r^2\nu\tau|\xi|^2 \right) \right) d\xi
\]

\[
\leq \int_{\mathbb{R}^3} \left( \mathcal{F} \left( v_i^{\rho, r, \nu}(t_0, \cdot) \right)(\xi) \exp \left( -8\pi^2\rho r^2\nu\tau|\xi|^2 \right) \right) d\xi
\]

\[
+ \left| \int_{\{ |\xi| \leq \Delta, 1 \leq j \leq D \}} \left( \mathcal{F} \left( v_i^{\rho, r, \nu}(t_0, \cdot) \right)(\xi) \times \exp \left( -8\pi^2\rho r^2\nu\tau|\xi|^2 \right) \right) d\xi \right|
\]

\[
\leq |\mathcal{F}(v_i^{\rho, r, \nu})(t_0, \cdot)|_{L^2}^2 \exp \left( -8\pi^2\rho r^2\nu\tau \Delta^2 \right) + c_m^\Delta \left( 8D\pi^2\rho r^2\nu\tau\Delta^{1+D} \right).
\]

Here, we use the assumption that \( \Delta > 0 \) is small enough (especially \( 8\pi^2\rho r^2\nu\tau\Delta \leq 1 \)) and use the abbreviation

\[
c_m^\Delta := \sup_{|\xi| \leq \Delta} |\mathcal{F}(v_i^{\rho, r, \nu}(t_0, \cdot))|^2 (\xi).
\]

The latter constant is finite (since \( |v_i^i(t_0, \cdot)|_{H^2} \) is finite, and the square of the latter value is an upper bound of \( c_m^\Delta \) for sure).

If we take the square root we may use the asymptotics \( \sqrt{1 + a} = 1 + \frac{1}{2}a + O(a^2) \).

For \( \tau \in [0, \Delta] \) and

\[
0 < \Delta \leq \max \left\{ \frac{1}{8\pi^2\rho r^2\nu \max \{ c_m^\Delta, 1 \}}, \frac{1}{2} \right\}
\]
we get (the generous) estimate
\[
|v_{i}^{\rho,r,\nu}(t_0,\cdot)|_{L^2} \leq |F(v_{i}^{\rho,r,\nu})(t_0,\cdot)|_{L^2} \exp \left( -4\pi^2 \nu \rho r^2 \tau \Delta^2 \right) + c_n^\Delta \left( 8D\pi^2 \nu r^2 \tau \Delta^{1+D} \right)
\]
\[
\leq |v_{i}^{\rho,r,\nu}(t_0,\cdot)|_{L^2} \exp \left( -4\pi^2 \nu \rho r^2 \tau \Delta^2 \right) + c_n^\Delta \left( 8D\pi^2 \nu r^2 \tau \Delta^{1+D} \right).
\]
If \( |v_{i}^{\rho,r,\nu}(t_0,\cdot)|_{L^2} \) becomes large or \( \Delta > 0 \) is small enough, then the second summand on right side of (35) is small compared to the first summand. A similar observation holds for derivatives \( |D^2 v_{i}^{\rho,r,\nu}(t_0,\cdot)|_{L^2} \) for \( 0 \leq |\beta| \leq m \).

It is straightforward to obtain analogous estimates for spatial derivatives. If for \( 0 \leq |\beta| \leq m \) the initial data value \( |F(D^2 v_{i}^{\rho,r,\nu})(t_0,\cdot)|_{L^2} = |D^2 v_{i}^{\rho,r,\nu}(t_0,\cdot)|_{L^2} \) exceeds a certain level, then for small \( \Delta \) the viscosity damping is stronger than possible growth caused by the additional term of order \( \Delta^{D+1} \). As we are interested in the case \( D \geq 3 \), this is evident, and we may remark in addition that the effect of the additional term becomes smaller as dimension \( D \) increases. In items a) and b) below we observe that the damping effect is strong enough in order to offset possible growth caused by the nonlinear terms. Finally note that for \( t_0 > 0 \) we get the analogous estimate
\[
|v_{i}^{\rho,r,\nu}(t_0,\cdot)|_{L^2} \leq |v_{i}^{\rho,r,\nu}(t_0,\cdot)|_{L^2} \exp \left( -4\pi^2 \nu \rho r^2 (\tau - t_0) \Delta^2 \right) + c_n^\Delta \left( 8D\pi^2 \nu r^2 (\tau - t_0) \Delta^{1+D} \right).
\]

ii) Similar considerations hold in other \( L^p \)-spaces.

Next we show that the viscosity damping can offset possible growth caused by the nonlinear terms. We consider again two different estimates in items a) and b) below which can be combined with the estimates in item i) and item ii) above in order to obtain time-uniform global regular upper bounds which are independent of \( \nu \) (cf. estimates in item b) below) or which are independent of \( \nu \) (cf. estimates in item a) below). Next we introduce Euler Leray data functions of order \( l \geq 1 \).

**Definition 1.1.** For \( l \geq 1 \) and data \( g = (g_1, \cdots, g_D)^T \) with \( g_i \in H^{l+1} \cap C^{l+1} \) for \( 1 \leq i \leq D \) the Euler-Leray data function of order \( l \) and of type 1 is defined by
\[
EL^1_l(g) : \mathbb{R}^D \to \mathbb{R},
\]
\[
EL^1_l(g)(y) = \sum_{|\gamma| \leq l} \sum_{j=1}^D D^\gamma_j (g_j g_{i,j}) (y) + \sum_{|\gamma| \leq l} \left( \int_{\mathbb{R}^D} (K_D,\gamma, (y - z)) \sum_{j,m=1}^D D^\gamma_j (g_{m,j} g_{j,m}) (z) dz \right).
\]
Furthermore, we define the Euler-Leray data function of type 0 by
\[ EL^l_0(g) : \mathbb{R}^D \to \mathbb{R}, \]
\[ EL^l_0(g)(y) = \sum_{|\gamma| \leq l} \sum_{j=1}^D D^\gamma_x (g_j g_{i,j}) (y) \]
\[ + \sum_{|\gamma| \leq l} \left( \int_{\mathbb{R}^D} (K_D(y-z)) \sum_{j,m=1}^D D^\gamma_z (g_{m,j} g_{j,m}) (z) dz \right), \]
(38)

The Euler-Leray functions of order \( l \geq 1 \) are Lipschitz continuous for regular data \( g_i \in H^{l+1} \cap C^{l+1}, 1 \leq i \leq D, \) i.e., for \( g_i \in H^{l+1} \cap C^{l+1} \)
\[ |EL^l_i(g)(y) - EL^l_i(g)(y')| \leq L_i |y - y'| \]
(39)
for some finite Lipschitz constant \( L_i \). for \( g_i \in H^{l+1} \cap C^{l+1} \)
\[ |EL^l_0(g)(y) - EL^l_0(g)(y')| \leq L^0_1 |y - y'| \]
(40)
for some finite Lipschitz constant \( L^0_1 \). These Lipschitz constants depend on the data norms \( |g_k|_{H^{l+1} \cap C^{l+1}} \). Nevertheless Lipschitz continuity of the Euler-Leray functions can be computed explicitly for local representations of solutions as in (39) for strong data due to local contraction results.

This Lipschitz continuity can be applied for data at any time \( \sigma_0 \), i.e., (for fixed argument \( x \in \mathbb{R}^D \) and \( 0 \leq |\beta| \leq l \))
\[ g_i(\cdot) = D^\beta_{x_i} v^{\rho,\tau,\nu}_i (\sigma_0, x - \cdot) \]
(41)
a) We consider data in \( H^m \cap C^m \) at time \( t_0 \geq 0 \). Using the convolution rule from (27), we get for all \( \tau \in [t_0, t_0 + \Delta] \) and \( x \in \mathbb{R}^D \)
\[ v^{\rho,\tau,\nu}_i (\tau, x) = v^{\rho,\tau,\nu}_i (t_0, .) *_{sp} G^{\rho,\tau}_\nu (\tau - t_0, x) \]
\[ - \rho \int_{t_0}^\tau \int_{\mathbb{R}^D} \sum_{j=1}^D (1, v^{\rho,\tau,\nu}_j v^{\rho,\tau,\nu}_i) (\sigma, x - y) \left( \frac{-2(y)_i}{4 \rho \nu (\tau - \sigma)} \right) G^{\rho,\tau}_\nu (\tau - \sigma, y) dyd\sigma \]
\[ + \rho \int_{t_0}^\tau \int_{\mathbb{R}^D} \left( \int_{\mathbb{R}^D} (K_D(y - z)) \sum_{j,m=1}^D \left( \frac{\partial v^{\rho,\tau,\nu}_j}{\partial x_j} \frac{\partial v^{\rho,\tau,\nu}_m}{\partial x_m} \right) (\sigma, x - z) dz \right) \times \]
\[ \times \left( \frac{-2(y)_i}{4 \rho \nu (\tau - \sigma)} \right) G^{\rho,\tau}_\nu (\tau - \sigma, y) dyd\sigma. \]
(42)

For multivariate derivatives of order \( m \geq |\beta| = |\gamma| + 1, \gamma_i + 1 = \beta_i \), we have
\[ D^\beta_{x_i} v^{\rho,\tau,\nu}_i (\tau, x) = D^\beta_{x_i} v^{\rho,\tau,\nu}_i (t_0, .) *_{sp} G^{\rho,\tau}_\nu \]
\[ - \rho \int_{t_0}^\tau \int_{\mathbb{R}^D} \sum_{j=1}^D D^\gamma_{x_j} \left( v^{\rho,\tau,\nu}_j v^{\rho,\tau,\nu}_i \right) (\sigma, x - y) dyd\sigma \]
\[ \left( \frac{-2(y)_i}{4 \rho \nu (\tau - \sigma)} \right) G^{\rho,\tau}_\nu (\tau - \sigma, y) dyd\sigma \]
\[ + \rho \int_{t_0}^\tau \int_{\mathbb{R}^D} \left( \int_{\mathbb{R}^D} (K_D(y - z)) \sum_{j,m=1}^D D^\gamma_{x_j} \left( \frac{\partial v^{\rho,\tau,\nu}_j}{\partial x_j} \frac{\partial v^{\rho,\tau,\nu}_m}{\partial x_m} \right) (\tau - \sigma, x - z) dz \right) \times \]
\[ \times \left( \frac{-2(y)_i}{4 \rho \nu (\tau - \sigma)} \right) G^{\rho,\tau}_\nu (\tau - \sigma, y) dyd\sigma, \]
(43)
where in the last step we indicate that we can have the difference \( \tau - \sigma \) in the Leray-projection term alternatively. Recall from (7) that for \( \sigma > 0 \) we have the Gaussian upper bound

\[
\forall \delta \in (0, 1), r, \rho > 0 \quad \left| G^{\rho, r}_{\nu, \sigma}(\sigma, y) \right| \leq \frac{C}{(4\pi \rho r^2 \nu)^{\frac{d}{2}} |y|^{d+1-2\delta}}.
\]

where (due to Lipschitz continuity of the Leray projection term) \( \delta > 0 \) can be chosen small if this function is convoluted with a Lipschitz continuous function (or a H"older continuous function) and \( C > 0 \) is independent of \( \rho, r \). There are different estimates for different \( \delta \in (0, 1) \) as we shall see. In order to apply Lipschitz continuity of the Euler-Leray function we use local time contraction. We have

**Lemma 1.2.** Let \( t_0 \geq 0 \) and assume that for some \( m \geq 2 \) we have

\[
\left| v^{\rho, r, \nu}_i(t_0, \cdot) \right|_{H^m \cap C^m} \leq C_0
\]

For \( \delta v^{\rho, r, \nu, k+1} = v^{\rho, r, \nu, k+1} - v^{\rho, r, \nu, k} \), \( 1 \leq j \leq D \) and \( v^{\rho, r, \nu, 0} = v^{\rho, r, \nu}(t_0, \cdot) \), \( 1 \leq j \leq D \) and \( \Delta \leq \delta \) we have

\[
\sup_{\tau \in [t_0, t_0 + \Delta]} \left| \delta v^{\rho, r, \nu, k+1}_j(\tau, \cdot) \right|_{H^m \cap C^m} \leq \frac{1}{2} \sup_{\tau \in [t_0, t_0 + \Delta]} \left| \delta v^{\rho, r, \nu, k+1}_j(\tau, \cdot) \right|_{H^m \cap C^m}
\]

and

\[
\sup_{\tau \in [t_0, t_0 + \Delta]} \left| \delta v^{\rho, r, \nu, 1}_i(\tau, \cdot) \right|_{H^m \cap C^m} \leq \frac{1}{2}.
\]

For \( v^{\rho, r, \nu}_i(\tau, \cdot) \in H^m \cap C^m \), \( 1 \leq i \leq D \), \( \tau \in [t_0, t_0 + \Delta] \) and all \( x \in \mathbb{R}^D \) the function

\[
y \rightarrow \int_{\mathbb{R}^D} (K_{D,i}(y - z)) \sum_{j, m=1}^D D^2_{x'} \left( \frac{\partial v^{\rho, r, \nu}_j}{\partial x_m} \frac{\partial v^{\rho, r, \nu}_m}{\partial x_j} \right)(\tau, x - z) dz,
\]

\[0 \leq |\gamma| \leq m - 1\]

is Lipschitz continuous with a constant \( L_{m-1} > 0 \) which is independent of \( x \in \mathbb{R}^D \). In the following we shall consider a refined estimate, where we use a strong damping parameter \( r \) such that \( 4\pi^2 r^2 \nu \leq 1 \). The argument below will show that this choice is consistent with the choices of other parameters. The advantage of the refined estimate is that it can be used in order to construct regular upper bounds in the viscosity limit.

The function in (13) is well-defined for all \( y \in \mathbb{R}^D \). First we consider the case \( |\beta| \geq 1 \). From (13) (using the convolution rule with respect to time), Lipschitz continuity of the Leray data function with Lipschitz constant \( L_m \), and (14) we have

\[
\left| D^\beta_{x'} v^{\rho, r, \nu}_i(\tau, x) \right| \leq \left| D^\beta_{x'} v^{\rho, r, \nu}_i(t_0, \cdot) *_{sp} G^{\rho, r}_{\nu}(\tau - t_0, x) \right|
\]

\[
+ r \rho L_m \int_{t_0}^{\tau} \int_{B^D} |y| \frac{C}{(4\pi r^2 \nu)^{\frac{d}{2}} |y|^{d+1-2\delta}} dyd\sigma + \epsilon,
\]

where \( B^D \) is the ball of radius \( 4\pi r^2 \nu \geq 1 \) (around the origin) and

\[
\epsilon = r \rho L_m \int_{t_0}^{t_0 + \Delta} \int_{\mathbb{R}^D \setminus B^D} |y| \left| G^{\rho, r}_{\nu, \sigma}(\sigma, y) \right| dyd\sigma
\]

(50)
becomes small for a small time interval length $\Delta$. For $\tau \in [t_0, t_0 + \Delta]$ we get

$$\left| D_x^3 v^{\rho,\nu}_i (\tau, \cdot) \right| \leq \left| D_x^3 v^{\rho,\nu}_i (t_0, \cdot) *_{sp} G^\nu_\nu (\tau, \cdot) \right| + pr L_m \int_{t_0}^{\tau} (4\pi \rho r^2 \nu)^{-\delta} C \rho r^2 \nu \sigma^{-\delta} d\sigma + \epsilon$$

$$= \left| D_x^3 v^{\rho,\nu}_i (t_0, \cdot) *_{sp} G^\nu_\nu (\tau, \cdot) \right| + pr L_m (4\pi \rho r^2 \nu)^{\delta} (\tau^{1-\delta} - t_0^{1-\delta}) C + \epsilon. \quad (51)$$

**Remark 1.3.** If we do not apply the convolution rule with respect to time to (43) first, then we get estimates $\sim (\tau - t_0)^{1-\delta}$. However this does not matter really.

Recall from (35) for $t_0 = 0$ and in general from (36) for $t_0 \geq 0$ and analogous estimates for spatial derivatives that we have a damping estimate

$$\left| D_x^3 v^{\rho,\nu}_i (t_0, \cdot) *_{sp} G_\nu (\tau, \cdot) \right|_{L^2} \leq \left| v_i^{\rho,\nu} (t_0, \cdot) \right|_{L^2} \exp \left( -4\pi \rho r^2 \tau \Delta^2 \right) + c_\nu^3 (8\pi^2 \rho r^2 \nu \tau \Delta^{1+D}), \quad (52)$$

which becomes effective for small $\Delta > 0$. The last upper bound term in (49) is a constant with respect to space. Define

$$c_\nu^3 := c_\nu (8\pi^2 \rho r^2 \nu \tau \Delta^{1+D}) \quad (53)$$

For $\tau \in [t_0, t_0 + \Delta]$ we have

$$\left| D_x^3 v^{\rho,\nu}_i (t_0 + \Delta, \cdot) \right|_{L^2} \leq \left| v_i^{\rho,\nu} (t_0, \cdot) \right|_{L^2} \exp \left( -4\pi \rho r^2 \tau \Delta^2 \right) + c_\nu^3$$

$$+ pr L_m (4\pi \rho r^2 \nu)^{\delta} (\Delta^{1-\delta} - t_0^{1-\delta}) C + \epsilon. \quad (54)$$

The hardest estimate is for $t_0 = 0$. In this case the relation on (51) shows us that

$$\left| D_x^3 v^{\rho,\nu}_i (\Delta, \cdot) \right|_{L^2} \leq \left| D_x^3 v^{\rho,\nu}_i (0, \cdot) \right|_{L^2} \exp \left( -4\pi \rho r^2 \tau \Delta^2 \right) + c_\nu^3 \quad (55)$$

if

$$\left| v_i^{\rho,\nu} (0, \cdot) \right|_{L^2} \exp \left( -4\pi \rho r^2 \tau \Delta^2 \right) - 1 + c_\nu^3 \quad (56)$$

$$+ pr L_m (4\pi \rho r^2 \nu)^{\delta} (\Delta^{1-\delta}) C + \epsilon \leq 0$$

Note that we realize a large damping parameter ($r \geq 1$, w.l.o.g.) in the refined estimate such that

$$4\pi \rho r^2 \nu \geq 1, \quad (57)$$

which imposes the condition

$$r^{1+2\delta} \leq r^2, \quad \text{or } \delta \in \left( 0, \frac{1}{2} \right). \quad (58)$$

For a small time interval $\Delta > 0$ the positive real number $\epsilon$ in (51) becomes arbitrarily small. Furthermore, as $c_\nu^3 \downarrow 0$ as $\Delta \downarrow 0$ with $\Delta^{D+1}$, and the damping factor $\left| v_i^{\rho,\nu} (0, \cdot) \right|_{L^2} (1 - \exp (-4\pi^2 \rho r^2 \tau \Delta^2))$ is dominant for $\tau = \Delta$ if $\left| v_i^{\rho,\nu} (0, \cdot) \right|_{L^2}$ is greater than a certain threshold (say
\( |v_i^{\rho,r,\nu}(0,.)|_{L^2} = 1 \) such that \( c_1^2 \) is relatively small compared to the modulus of the main part of this damping term, i.e., small compared to the modulus of

\[ |v_i^{\rho,r,\nu}(0,.)|_{L^2} (-4\pi^2 \nu \rho r \Delta \Delta^2) . \]  

(59)

Next we consider the conditions such that the modulus of the main damping part is larger than the last term (54) (the last term for \( \epsilon \) which is comparatively small and can be neglected). Here we observe the exponents of the parameters \( \rho, r, \nu, \Delta \) in (59) compared to the exponents of the parameters \( \rho, r, \Delta \) of the last term in (54). For \( \tau = \Delta \) in (59) we have the dependence

\[ \nu \rho r \Delta^3. \]  

(60)

and for the last term in (54) we have the dependence

\[ (\rho)^{1+\delta} (r)^{1+2\delta} (\nu)^{\delta} (\Delta^{1-\delta}) . \]  

(61)

First observe that viscosity limits can be obtained only if some parameter depends on \( \nu \), because the dependence of the nonlinear terms of form \( \nu^\delta \) is stronger than the damping of order \( \nu \) as \( \nu \downarrow 0 \). The natural parameter for such a dependence is the time parameter \( \rho \), as we have chosen \( r \) to be large. The viscosity limit is considered in the next two sections below.

Choosing a small step size parameter \( \rho \) and a small spatial parameter, say \( \rho = \Delta \mu \), we have

\[ (\rho)^{1+\delta} (r)^{1+2\delta} \Delta^{1-\delta} = \Delta^{\mu(1+\delta)+1-\delta} (r)^{1+2\delta} , \]  

(62)

and for this choice of parameters the damping term has the dependence

\[ \rho^2 \Delta^3 = \Delta^{\mu+3},2. \]  

(63)

Hence the damping is stronger than the possible growth of the nonlinear term for \( |v_i(t_0,.)|_{L^2} \geq 1, D \geq 3, \) and a small time interval \( \Delta \), if

\[ \mu(1+\delta)+1-\delta > \mu + 3 \]  

and \( \delta \in \left(0, \frac{1}{2}\right) \),

(64)

or

\[ \mu > \frac{2+\delta}{\delta} \]  

and \( \delta \in \left(0, \frac{1}{2}\right) \).

(65)

The estimates in the case \( |\beta| = 0 \) are analogous where the Lipschitz constant \( L_m \) has to be replaced by \( L_m^0 \).

Remark 1.4. For \( t_0 > 0 \) analogous estimates hold where \( \tau \) is replaced by \( \tau - t_0 \). It follows that we have constructed global regular upper bounds for multiples of a certain small time interval \( \Delta \), where we get a global upper bound for all time from this by the local time contraction result. This is a global regular upper bound for \( |v_i^{\rho,r,\nu}(t,.)|_{H^m \cap C^m} \) for \( m \geq 2 \) which transfers to a global regular upper bound for \( |v_i^\nu(t,.)|_{H^m \cap C^m} \) for \( m \geq 2 \) if multiplied by \( r^m \) (due to the terms of the highest regularity in the definition of the \( H^m \cap C^m \)-norm).

b) Similar estimates hold for \( L^p \) spaces related to item ii) above, but there is a loss of regularity in general as we pass from \( L^p \) spaces to \( L^2 \)-spaces.
2 Short-and Long time singularities of the Navier Stokes equations with time dependent force term

In the following we do not stick to the notational difference of an original time intervals length $\Delta$ and a scaled interval time length $\Delta_0$. It suffices to consider the scaled functions $v^{\rho, r, \nu}_i$, $1 \leq i \leq D$, and we just write $\Delta$ for the time length of intervals, where these functions are considered. The preceding $L^2$-argument together with the estimate in (66) shows that for $\Delta > 0$ small enough and data in $H^m \cap C^m$, $m \geq 2$ possible growth caused by the nonlinear terms is offset by viscosity damping in the sense that (e.g.) for $\delta \in (0, \frac{1}{2})$, a small time interval $\Delta$, small $\rho \sim \Delta^\mu$ with $(\mu, \delta)$ as (65) we get for all $0 \leq |\beta| \leq m$:

$$|D^{\beta}_x v^{\rho, r, \nu}_i(t_0 + \Delta, .)|_{L^\infty C} \leq |v^{\rho, \nu}_i(t_0, .)|_{L^2} \exp \left(-4\pi \rho r^2 (\tau - t_0) \Delta^2 \right) + c_n \Delta$$

$$+ \rho r L_m (4\pi \rho r^2 \nu)^\delta (\Delta^{1 - \delta}) C + \epsilon \leq |D^{\beta}_x v^{\rho, r, \nu}_i(t_0, .)|_{L^\infty C},$$

if the latter term is larger than a certain threshold, say, it satisfies

$$|D^{\beta}_x v^{\rho, r, \nu}_i(t_0, .)|_{L^\infty C} \geq 1.$$  \hspace{1cm} (67)

We have not observed explicitly that these upper bounds are independent of $\nu$, essentially, in the sense that for $\nu$ small, e.g., $\nu \sim \sqrt{\Delta}$ as the time interval becomes small, we can set up a scheme of step size $\frac{1}{\sqrt{\Delta}}$ solving for $v^{\rho, r, \nu}_i$, $1 \leq i \leq D$ which has a global regular upper bound which is independent of the size of $\Delta$, and, therefore, independent of $\nu$. Let us choose $\nu = \sqrt{\Delta}$ for small $\Delta > 0$, and

$$1 \geq r = \frac{1}{\sqrt{\Delta^{\mu + 1}}}, \quad \rho = \Delta^\mu, \quad \mu > \frac{2 + \delta}{\delta} \quad \text{and} \quad \delta \in \left(0, \frac{1}{2}\right).$$  \hspace{1cm} (68)

Then we have

$$4\pi \rho r^2 \nu^2 > 4\pi > 1,$$  \hspace{1cm} (69)

where the damping estimate has a stronger scaling with respect to $r$ than the upper bound of the nonlinear term in the sense that

$$r^2 > r^{1 + 2\delta} \quad \text{for} \quad r \geq 1,$$  \hspace{1cm} (70)

The growth caused by the nonlinear terms in an interval $[t_0, t_0 + \Delta]$ is offset by the damping for small $\Delta$ under the condition (68) if

$$\rho r (4\pi \rho r^2 \nu)^\delta (\Delta^{1 - \delta}) \leq \rho r^2 \Delta^{\delta},$$  \hspace{1cm} (71)

or

$$\Delta^{\mu - \frac{1}{\mu + 1}} \left(4\pi \Delta^{\mu - \frac{1}{\mu + 1}} \sqrt{\Delta}\right)^{\delta} (\Delta^{1 - \delta}) \leq \Delta^{\mu - \frac{1}{\mu + 1}} \Delta^{\delta},$$  \hspace{1cm} (72)

which means (for small $\Delta$)

$$\Delta^{\mu - \frac{\mu + 1}{2\mu + 1}} (4\pi \sqrt{\Delta})^{-\delta} (\Delta^{1 - \delta}) \leq \Delta^2, \quad \text{or} \quad \mu \geq 5 + 3\delta.$$  \hspace{1cm} (73)

Hence we have indeed upper bound estimates which are essentially independent of $\nu$. 

12
Remark 2.1. (so-called degeneracy issue). It is seems that there is still a believe that (even local-) time contraction results based on convolution with first order derivatives of the Gaussian cannot be used in order to prove local time contraction in the viscosity limit. More precisely it seems to be believed that, e.g., the iterative solution of an equation

$$v_t - \nu \Delta v + F(v, \nabla v) = 0, \quad v(t_0, \cdot) = v_{t_0}$$  \hspace{1cm} (74)

by an iterative convolution scheme of the form

$$v^{k+1}_t = v^k_t \ast s_p \ast G_\nu + F(v^k) \ast G_{\nu,j}$$  \hspace{1cm} (75)

causes principle problems in the limit $\nu \downarrow 0$ (even if considered on a compact domain $\Omega$) due to a degeneracy (with respect to $\nu$) of a typical upper bounds (for some $\delta \in (0, 1)$) obtained, e.g., from intergals of the upper bound

$$l |y| G^{\rho,r}_{\nu,i}(\sigma, y) \leq l C \left(\frac{4\pi \rho r^2}{\nu}\right)^{\frac{D-2\delta}{\delta}} |y|^{D-2\delta}$$  \hspace{1cm} (76)

on a ball of radius $4\pi \rho r^2 \nu \geq 1$. However if the analysis of (75) leads to a functional series $(v^{\nu_k})_{k \geq 0}$ for a sequence $(\nu_k) \downarrow 0$ such that $v^{\nu_k}$ solves (74) for $\nu_k$ on a time interval $[t_0, t_0 + \Delta]$ and we have an upper bound

$$\sup_{t \in [t_0, t_0 + \Delta]} |v^{\nu_k}(t, \cdot)|_{H^m(\Omega)} \leq C$$  \hspace{1cm} (77)

for a strong $H^m_0$-norm (where $H^m_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m$, and for a finite constant $C > 0$ which is independent of $\nu_k$ for all $k \geq 1$), then Rellich's compactness result gives a subsequence $(v^{\nu_k'})_{k \geq 0}$ with $\nu_k' \downarrow 0$ such that the limit is in the slightly weaker space $v^0(t, \cdot) \in H^{m-\epsilon}$ for any small $\epsilon > 0$. For large $m$ (dependent on dimension, of course) the limit function $v^0$ solves the equation in (74) pointwise for strong data in $H^m_0(\Omega)$. Such a reasoning can be transferred to infinite domains if we have strong polynomial decay of the data and the operator function $F$ preserves this strong polynomial decay. The latter property holds for the Burgers and Leray projection term for strong polynomial decay as we observe in the next sections.

In the next section we shall observe that global global viscosity limit schemes can be constructed. If strong polynomial decay of a solution family for each $\nu > 0$ allows for a compactness argument, then (in the next section) we shall show that this implies the existence of global regular solution branches for the Euler equation. In this section we consider the consequences for the Navier Stokes equation with force terms. The existence of singular solutions of the incompressible Euler equation implies the existence of singular solutions of the incompressible Navier Stokes equation with time dependent force terms. A classical solution

$$v_i, \quad 1 \leq i \leq D, \quad v_i(t, \cdot) \in H^m \cap C^m, \quad t \in [0, T), \quad m \geq 2$$  \hspace{1cm} (78)

of the incompressible Euler equation (on the interval $[0, T)$) with a blow-up of vorticity at time $T$ satisfies the incompressible Navier Stokes equation

$$\frac{\partial v_i}{\partial t} - \nu \Delta v_i + \sum_{j=1}^m v_j v_{i,j} - \int_{\mathbb{R}^D} K_{D,i}(\cdot, -z) \sum_{j,m=1}^D \left( \frac{\partial v^\nu_m}{\partial x_j} \frac{\partial v_{\nu,j}}{\partial x_m} \right)(\cdot, z)dz = F_i$$  \hspace{1cm} (79)
with force term $F = (F_1, F_2, \ldots, F_D)^T$ (on the same time interval) if
\[ F_i = -\nu \Delta u_i \in H^{m-2} \cap C^{m-2}, \quad 1 \leq i \leq D. \] (80)

The analysis of global regular solution branches and time reversed Euler equations below shows that $F_i$ is also $L^2$ with respect to time on the time interval $[0, T]$, where a possible large time $T > 0$ is the time where the vorticity of the Euler equation blows up. As a consequence of this analysis we note that there are long time singularities.

**Theorem 2.2.** The Navier Stokes equation with initial data $h_i \in H^m \cap C^m$, $m \geq 2$, $1 \leq i \leq D$ of strong polynomial decay, i.e., $h_i \in C^m_{\text{pol},m}$, $1 \leq i \leq D$ (cf. the definition of the latter function space in the next section below), and external time-dependent forces $F_i \in H^{m-2} \cap C^{m-2}$, $1 \leq i \leq D$ is not well-posed in general. More precisely, for the Navier Stokes equation Cauchy problem in (79) and for any time $T > 0$ there exist time dependent force terms
\[ F_i(t, \cdot) \in H^m \cap C^m, \quad k \geq 0, \quad \tau \in [0, T] \] (81)
and data $v_i(0, \cdot) \in H^m \cap C^m \cap C^m_{\text{pol},m}$, $1 \leq i \leq 3$, $m \geq 2$ such that a regular classical solution of the Navier Stokes equation on the time interval $[0, T]$ has a blow-up or a kink of order $k \geq 1$ at time $T$.

### 3 Global solution branches of the Euler equation

We have found upper bounds which are independent of the viscosity constant $\nu > 0$. This implies that there exists a finite constant $\tilde{C} > 0$ for the such that
\[ \max_{1 \leq i \leq D} \sup_{t > t_0} \| v_i(t, \cdot) \|_{H^m \cap C^m} \leq \tilde{C}. \] (82)
As we have an unbounded domain we have to be a little careful concerning compactness arguments. However, the schemes used here preserve strong polynomial decay. If for fixed $\rho, r > 0$ the solution $v_i^\rho, r, \nu, 1 \leq i \leq D$ has polynomial decay of order $2m(D + 1)$ (we are concerned with $D \geq 3$), then standard compactness arguments can be transferred. First we observe the inheritance of polynomial decay of order $2m(D + 1)$. We define a related function space.

**Definition 3.1.** For $l \geq 1$ and $m \geq 2$ we define a space of functions which satisfy polynomial decay of order $l \geq 1$ at spatial infinity for multivariate spatial derivatives up to order $m$. More precisely, we define
\[ C_l^{m, \text{pol},m} = \left\{ f : \mathbb{R}^D \to \mathbb{R} : \exists c > 0 \forall |x| \geq 1 \forall 0 \leq |\gamma| \leq m \left| D^{\gamma} f(x) \right| \leq \frac{c}{1 + |x|^l} \right\}. \] (83)

Assume at time $t_0 \geq 0$ we have Cauchy data
\[ v_i^{\nu, \beta}(t_0, \cdot) := v_i(t_0, \cdot) \in C^{m(D + 1)}_{\text{pol},m}. \] (84)

For $0 \leq |\beta| \leq m$ and $|\gamma| + 1 = |\beta|$, $\gamma_j + 1 = \beta_j$ if $|\beta| > 0$ define the local time iteration scheme
\[ D_{\infty}^\beta v_i^{\nu, k} = D_{\infty}^\beta v_i^{\nu, k}(t_0, \cdot) - D_{\infty}^\beta \left( \sum_{j=1}^D v_j^{\nu, k-1} \frac{\partial v_j^{\nu, k-1}}{\partial x_j} \right) * G_{\nu,j} \]
\[ + \left( \sum_{j,m=1}^D \int_{\mathbb{R}^D} D_{\infty}^\gamma \left( \frac{\partial}{\partial x_j} K_D(-y) \right) \sum_{i,m=1}^D \left( \frac{\partial v_i^{\nu, k-1}}{\partial x_m} \frac{\partial v_m^{\nu, k-1}}{\partial x_m} \right) (t, y) dy \right) * G_{\nu,j}. \] (85)
Here, recall $G_t$ is the fundamental solution of the heat equation $p_t - \nu \Delta p = 0$, $\ast$ denotes the convolution, $\ast_{sp}$ denotes the spatial convolution, and $K_D$ denotes the fundamental solution of the Laplacian equation for dimension $D \geq 3$. In the following, the constant $c > 0$ is generic. For $1 \leq i \leq D$ the initial data $v_i^\nu(t,.)$ are in $C_{pol,m}^{m(D+1)}$. Hence, for $k = 0$, for $0 \leq |\gamma| \leq m$, and for $|x| \geq 1$

$$|D_\gamma^\nu v_i^\nu,0(t_0, x)| \leq \frac{c}{1 + |x|^{m(D+1)}} \tag{86}$$

for some finite constant $c > 0$ and $t_0 \geq 0$. Assuming inductively that for $t \in [t_0, t_0 + \Delta]$ all $l\leq k-1 \forall 0 \leq |\gamma| \leq m |D_\gamma^\nu v_i^\nu (t,.)| \leq \frac{c}{1 + |x|^{m(D+1)}} \tag{87}$$

we have or some finite constant $c > 0$, for $0 \leq |\delta| \leq m - 1$ and for $|x| \geq 1$

$$|D_\delta^\nu B^{k-1}| := \sum_{j=1}^D D_\delta^j \left( v_j^{\nu, k-1} \frac{\partial v_j^{\nu, k-1}}{\partial x_j} \right) (t,.) \leq \frac{c}{1 + |x|^{2m(D+1)}}, \tag{88}$$

and

$$|D_\delta^\nu L^{k-1}| \leq \frac{c}{1 + |x|^{2m(D+1)-1}} \tag{89}$$

where

$$D_\delta^\nu L^{k-1} \equiv \sum_{j=1}^D \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_j} K_D(. - y) \right) \sum_{j=1}^D D_\delta^j \left( \frac{\partial v_j^{\nu, k-1}}{\partial x_j} \frac{\partial v_j^{\nu, k-1}}{\partial y_m} \right) (t, y) dy. \tag{90}$$

Convolutions with $G_t$ or $G_{\nu,i}$ weaken this polynomial decay by order $D$ at most such that we (generously) get for some finite constant $c > 0$, for $0 \leq |\delta| \leq m - 1$ and for $|x| \geq 1$

$$|D_\delta^\nu B^{k-1} \ast G_{\nu,i}| \leq \frac{c}{1 + |x|^{2m-1}(D+1)} \tag{91}$$

and

$$|D_\delta^\nu L^{k-1} \ast G_{\nu,i}| \leq \frac{c}{1 + |x|^{2m-1}(D+1)-1}. \tag{92}$$

Hence using the representation (11), (44), (45) we can complete the induction step and get

$$\forall l \leq k \forall 0 \leq |\gamma| \leq m |D_\gamma^{\nu, l} (.)| \leq \frac{c}{1 + |x|^{m(D+1)}} \tag{93}$$

and by (87) the same holds for the increments

$$D_\gamma^{\nu, k} v_i^\nu (.) = D_\gamma^{\nu, k} v_i^{\nu, k}(.) - D_\gamma^{\nu, k} \delta v_i^{\nu, k-1}(.).$$

For some $\Delta > 0$ we have local time contraction with respect to a $H^m \cap C^m$-norm, such that the limit

$$D_\gamma^{\nu, k} v_i^\nu = \lim_{k \geq 1} D_\gamma^{\nu, k} v_i^{\nu, k} \tag{94}$$

inherits this order of polynomial decay at spatial infinity for all $0 \leq |\gamma| \leq m$. More precisely, for $t \in [t_0, t_0 + \Delta]$, $|x| \geq 1$, and for all $0 \leq |\gamma| \leq m$ we have a finite constant $c > 0$ such that for all $x \in \mathbb{R}^D$

$$\forall 0 \leq |\gamma| \leq m |D_\gamma^{\nu, l} (t,.)| \leq \frac{c}{1 + |x|^{m(D+1)}}. \tag{95}$$
We choose a sequence \((\nu_p)_{p\geq 1}\) converging to zero and consider the spatial transformation
\[
v_i^{c,\nu_p}(t, y) = v_i^{\nu_p}(t, x)
\]
for \(y_j = \arctan(x_j), 1 \leq j \leq D\) and for all \(t \in [t_0, t_0 + \Delta]\). For multiindices \(\gamma\) with \(0 \leq |\gamma| \leq m\) for all \(t \in [0, T]\), all \(y \in (-2\pi, 2\pi)^D\) and all \(x \in \mathbb{R}^D\)
\[
|D_y^\gamma \delta v_i^{c,\nu_p}(t, y)| \leq c_0(1 + |x|^{2m})|D_y^\gamma \delta v_i^{\nu_p}(t, x)| \leq C
\]
for some finite constants \(c_0, C > 0\). Then for and \(\epsilon > 0\) there is a subsequence which we may denote again by \((\nu_p)_{p\geq 1}\) such that we have a limit
\[
\lim_{p \to \infty} v_i^{c,\nu_p}(t, \cdot) := \lim_{\nu \downarrow 0} v_i^{c,\nu_p}(t, \cdot) \in H^{m-\epsilon} \quad \text{for all} \quad t_0 \leq t \leq t_0 + \Delta
\]
by Rellich’s theorem. Moreover \(v_i^{c,\nu_p}(t, \cdot) \in C_0^{m-1}\left((-\frac{\pi}{2}, \frac{\pi}{2})^D\right)\). Here, \(C_0^{m-1}\left((-\frac{\pi}{2}, \frac{\pi}{2})^D\right)\) is the function space of \(m-1\)-times continuously differentiable functions which vanish if a component \(y_i\) becomes equal to \(-\frac{\pi}{2}\) or \(\frac{\pi}{2}\). This space is a Banach space if equipped with the usual \(C^{m-1}\)-supremum-norm on the bounded domain. The limit \(c_i(t, \cdot) = \lim_{p \to \infty} v_i^{c,\nu_p}(t, \cdot) \in H^{m-\epsilon} \cap C^{m-1}, 1 \leq i \leq D, t \in [t_0, t_0 + \Delta_0]\) satisfies the transformed Euler equation with respect to spatial coordinates \(y_i, 1 \leq i \leq D\), and the corresponding function limit \(\epsilon_i, 1 \leq i \leq D\) with \(\epsilon_i(t, x) = c_i(t, y)\) satisfies the original Euler equation with respect to spatial coordinates \(x_i, 1 \leq i \leq D\). The construction becomes global straightforwardly by application of the semigroup property.

4 Short and Long time singularities of the Euler equation

A characteristic difference of the Euler equation (compared to the Navier Stokes equation) is that it can be solved backwards in time for regular data. In the previous section we have observed that compactness arguments can be applied if we have strong polynomial decay at spatial infinity of the data. This strong polynomial decay is inherited by the natural local time iteration scheme considered above. We have observed that local-time contraction results hold in the viscosity limit for regular data. In the convolution representation of the velocity component functions the contribution of the Gaussian or its first order spatial derivative is concentrated on a ball of radius \(\sqrt{\nu}\) around the spatial argument \(x\) for positive viscosity \(\nu > 0\). We have observed subsequences in strong spaces with uniform upper bounds (independent of \(\nu\)) have natural pointwise limits. Note that spatial Fourier transforms of the convolution of regular data with (first order spatial derivatives) Gaussian have the effect of a multiplication with a linear term in the viscosity limit (at most) which is absorbed by the polynomial spatial decay of the Fourier transform of the regular data. If short time solutions \(\epsilon_i^\nu, 1 \leq i \leq D\) of the time-reversed Euler equation with weakly singular data gain 'enough' regularity then this implies the existence of weak short - and even long-time singularities of the original Euler equation. Here 'enough regularity' for long-time singularities means that the evaluation of a local time solution with weakly singular data at some time has sufficient regularity such that the semi-group property of the (time-reversed) Euler equation
can be combined with the argument for a global solution branch of the previous section. Next we consider this in detail. We consider positive viscosity \( \nu > 0 \) first, and consider the viscosity limit in a second step. We consider the time transformation \( t \rightarrow -t := s \) and the time reversed equation for

\[
v_{i}^{\nu,-}(s,.) = v_{i}^{\nu}(t,.) \quad 1 \leq i \leq D, \quad v_{i}^{\nu,-}(s0,.) = v_{i}^{\nu}(t1,.) \quad 1 \leq i \leq D.
\]

(99)

for some \( t_1 > 0 \). Since we have global solution branches for the class of strong data \( C^{m(D+1)} \) (cf. previous section) the initial time \( s_0 = -t_1 \) for the time-reversed equation can be chosen arbitrarily if a short time solution \( v_{i}^{\nu,-} \), \( 1 \leq i \leq D \) with weakly singular data \( v_{i}^{\nu,-}(s0,.) \), \( 1 \leq i \leq D \) gains enough regularity after short time such that \( v_{i}^{\nu,-}(s,.) \in C^{m(D+1)} \) for some \( s > s_0 \). We reconsider here a variation of a local construction which we have considered elsewhere and sharpen some results. We construct local time solutions for carefully chosen data via the iteration scheme

\[
v_{i}^{\nu,-k} = v_{i}^{\nu,-}(s0,.) *_{sp} G_{\nu} + \sum_{j=1}^{D} \left( v_{j}^{\nu,-k-1} \frac{\partial v_{j}^{\nu,-k-1}}{\partial x_{j}} \right) * G_{\nu}
\]

\[
- \left( \sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_{j}} K_{D}(.,-y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v_{m}^{\nu,-k-1}}{\partial x_{j}} \frac{\partial v_{m}^{\nu,-k-1}}{\partial x_{m}} \right) (.,y)dy \right) * G_{\nu}, \quad k \geq 2
\]

\[
v_{i}^{\nu,-0}(s0,.) := v_{i}^{\nu,-}(s0,.) := v_{i}^{\nu}(t1,.), \quad 1 \leq i \leq D,
\]

\[
v_{i}^{\nu,-1}(s0,.) := v_{i}^{\nu,-}(s0,.) *_{sp} G_{\nu}, \quad 1 \leq i \leq D.
\]

(100)

Local time contraction results are obtained as for the original Navier–Stokes equation, and we may use representations of solutions of the form

\[
v_{i}^{\nu,-} = v_{i}^{\nu,-2}(s0,.) + \sum_{k \geq 3} \delta v_{i}^{\nu,-k},
\]

(101)

where we denote \( \delta v_{i}^{\nu,-k} = v_{i}^{\nu,-k} - v_{i}^{\nu,-k-1} \) for \( k \geq 1 \). We choose data which are weakly singular in the sense that the vorticity (original Euler equation)

\[
\omega = \text{curl}(v) = \left( \frac{\partial v_{3}}{\partial x_{2}} - \frac{\partial v_{2}}{\partial x_{3}}, \frac{\partial v_{1}}{\partial x_{3}} - \frac{\partial v_{3}}{\partial x_{1}}, \frac{\partial v_{2}}{\partial x_{1}} - \frac{\partial v_{1}}{\partial x_{2}} \right)
\]

(102)

has no finite upper bound at time \( t_1 > 0 \). This means that there are data \( \text{curl}(h) \in C^{m(D+1)} \) such that a solution of the Cauchy problem for the incompressible Euler equation in vorticity form

\[
\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \frac{1}{2} (\nabla v + \nabla v^{T}) \omega,
\]

(103)

blows up after finite time \( t_1 \), where \( t_1 > 0 \) can be large. Note that a vorticity blow up means that the corresponding velocity solution has a kink as it is well-known (cf. [25]) that

\[
v(t,x) = \int_{\mathbb{R}^3} K_3(x - y)\omega(t,y)dy, \quad \text{where} \quad K_3(x)h = \frac{1}{4\pi} \frac{x \times h}{|x|^3}.
\]

(104)

We prove
Theorem 4.1. Let $D = 3$. For any (arbitrarily large) finite time $T > 0$ and all $m \geq 2(D + 1)$ there exist data $h_i \in C_{pol,m}$, $1 \leq i \leq D$ and a vorticity solution $\omega_i$, $1 \leq i \leq D$ of the $D$-dimensional incompressible Euler equation Cauchy problem such that there is a blow-up of the classical solution at time $T > 0$, i.e.,

i) there is a solution function $\omega_i : [0, T) \times \mathbb{R}^D \to \mathbb{R}$, $1 \leq i \leq D$ in $C^1([0, T) \times \mathbb{R}^D)$ which satisfies the incompressible Euler equation point-wise on the domain $[0, T) \times \mathbb{R}^D$ in a classical sense;

ii) for the solution in item i) we have

$$\sup_{t \in [0, T)} |\omega_i(t, x)| = \infty,$$

i.e., there is no finite upper bound for the left side of (105).

Note that the preceding theorem and the construction of global solution branches for data $h_i \in C_{pol,m}$, $1 \leq i \leq D$, $m \geq 2$ in the preceding section imply that incompressible Euler equation Cauchy problems do not have unique solutions in general. More precisely we have

Corollary 4.2. For data $D \geq 3$ $h_i \in C_{pol,m}$, $1 \leq i \leq D$, $m \geq 2$ the Cauchy problem of the incompressible Euler equation as infinitely many solutions. Next to a global solution branch there exist solution with blow up s of first derivatives at any time $T > 0$ and solution in $C^{k+1} \setminus C^k$, $k \geq 2$ or solutions with kinks of any order $k$ at any time $T > 0$.

Some arguments of the preceding section such as local time contraction can be transferred to the time-reversed equation straightforwardly. Additionally we have to show that for some data with weakly singular data in $H^2$ there is a local solution branch which gains enough regularity after short time in order to apply the arguments for global regular solution branches of the Euler equation for regular data obtained in the last section. Finally, we add for additional steps which we need for a proof of Theorem 4.1.

i) First we choose appropriate weakly singular data. For some time $s_0$, a positive real number $\beta_0 \in (1, 1+\alpha_0)$, and $\alpha_0 \in (0, \frac{1}{2})$ we consider velocity component data $v_i^{\alpha_0}(s_0, \cdot) \in H^2$, $1 \leq i \leq 3$. For one index $i_0 \in \{1, 2, 3\}$ we choose weak data, where we define $v_i^{\alpha_0}(s_0, x) = g_{i_0}(r)$ for some univariate function $g$. The function $g_{i_0}$ is dependent on $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq 0$. We define $g_{i_0} : \mathbb{R}_0^+ \to \mathbb{R}$ by

$$g_{i_0}(r) := \begin{cases} \phi_1(r)r^{\beta_0}\sin\left(\frac{1}{r^{\alpha}}\right) & \phi_1 \in C_c^\infty, \quad \text{and} \quad \phi_1(r) = \begin{cases} 1 & \text{if } r \leq 1, \\ \alpha_\ast(r) & \text{if } 1 \leq r \leq 2, \\ 0 & \text{if } r \geq 2. \end{cases} \end{cases}$$

where $\phi \in C_c^\infty$, and
Here, $\alpha_0$ is a smooth function with bounded derivatives for $1 \leq r \leq 2$, and $C^\infty_0$ denotes the function space of smooth functions with compact support. For $j \in \{1, 2, 3\} \setminus \{i_0\}$ we may choose regular velocity component data, i.e. we choose data
\[ v_j^{\nu,-}(s_0,.) \in C^\infty_0. \quad (108) \]
Note that for all $1 \leq i \leq D$ and multiindices $\alpha$ with $0 \leq |\alpha| = k$ we have
\[ |D^\alpha v_i^{\nu,-}(s_0,.)| \leq C r^{\beta_0-k(1+\alpha_0)}. \quad (109) \]
For the derivative of the data $v_{i_0}^{\nu,-}(s_0,.)$ we compute for $r \neq 0$ and $r \leq 1$
\[ g'(r) = \frac{d}{dr} r^{\beta_0} \sin \left( \frac{1}{r^{\beta_0}} \right) = \beta_0 r^{\beta_0-1} \sin \left( \frac{1}{r^{\beta_0}} \right) - \alpha_0 r^{\beta_0-1-\alpha_0} \cos \left( \frac{1}{r^{\beta_0}} \right). \quad (110) \]
The derivative $g'$ of the function $g$ at $r = 0$ is strongly singular for $\beta \in (1, 1 + \alpha_0)$ and $\alpha_0 \in (0, \frac{1}{2})$. Note that it is 'oscillatory' singular bounded for $\beta_0 - 1 = \alpha_0 \in (0, \frac{1}{2})$. Note that for data $v_{i_0}^{\nu,-}(s_0, x_1, x_2, x_3) = g(r)$ we have (for $r \neq 0$)
\[ v_{i_0}^{\nu,-}(s_0, x) = g'(r) \frac{\partial r}{\partial x_j} = g'(r) \frac{x_j}{r}. \quad (111) \]
In polar coordinates $(r, \theta, \phi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi]$ with
\[ x_1 = r \cos(\theta) \sin(\phi), \quad x_2 = r \sin(\theta) \sin(\phi), \quad x_3 = r \cos(\theta), \quad (112) \]
(where for $r \neq 0$ and $x_1 \neq 0$ we have $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\theta = \arccos \left( \frac{x_3}{r} \right)$, $\phi = \arctan \left( \frac{x_2}{x_1} \right)$) we get
\[ v_{i_0}^{\nu,-}(s_0, x) = g'(r) \frac{x_1}{r} = g'(r) \sin(\theta) \cos(\phi), \]
\[ v_{i_0}^{\nu,-}(s_0, x) = g'(r) \frac{x_2}{r} = g'(r) \sin(\theta) \sin(\phi), \]
\[ v_{i_0}^{\nu,-}(s_0, x) = g'(r) \frac{x_3}{r} = g'(r) \cos(\theta), \quad (113) \]
such that we have
\[ v_{i_0}^{\nu,-}(s_0,. \in H^1 \text{ obviously.} \quad (114) \]
The second derivative of $g$ is
\[ g''(r) = \frac{d^2}{dr^2} r^{\beta_0} \sin \left( \frac{1}{r^{\beta_0}} \right) \]
\[ = \frac{d}{dr} \left( \beta_0 r^{\beta_0-1} \sin \left( \frac{1}{r^{\beta_0}} \right) - \alpha_0 r^{\beta_0-1-\alpha_0} \cos \left( \frac{1}{r^{\beta_0}} \right) \right) \]
\[ = \beta_0 (\beta_0 - 1) r^{\beta_0-2} \sin \left( \frac{1}{r^{\beta_0}} \right) - \alpha_0 \beta_0 r^{\beta_0-3-\alpha_0} \cos \left( \frac{1}{r^{\beta_0}} \right) \]
\[ +(\alpha_0)(1 + \alpha_0 - \beta_0) r^{\beta_0-1-\alpha_0} \cos \left( \frac{1}{r^{\beta_0}} \right) \]
\[ - (\alpha_0)^2 r^{\beta_0-2-2\alpha_0} \sin \left( \frac{1}{r^{\beta_0}} \right). \quad (115) \]
We have $v_{i_0}^{\nu,-}(s_0,.) \in H^2$, since
\[ \beta_0 - 2 - 2\alpha_0 > -\frac{3}{2}. \quad (116) \]
Note that
\[ v_{\nu_i}^{\nu_0} (s_0, \cdot) \in C^\delta (\mathbb{R}^3), \quad (117) \]
for Hölder constants of order \( \delta \in (0, \beta_0 - \alpha_0) \). Hence Lipschitz continuity of the data does not hold, and we have to refine estimates of convolutions with (first spatial derivatives of) the Gaussian in order to extend the argument of the preceding section.

ii) For the scheme defined in (100) above and data \( v_{\nu_i}^{\nu_0} (s_0, \cdot) \) \( 1 \leq i \leq D \) as defined in item i) we first observe (based on similar reasons as in the previous section) that for first order multivariate spatial derivatives \( 0 \leq |\gamma| \leq 1 \) and \( |x| \geq 1 \), for some \( \Delta > 0 \) for all \( s \in [s_0, s_0 + \Delta] \) there exists a finite constant \( c \) which is independent of \( \nu > 0 \) such that
\[ \left| D_\gamma x v_{\nu_i}^{\nu_0} (s, x) \right| \leq c \frac{1}{1 + |x|^{2(D+1)}}, \quad (118) \]
and
\[ \forall \ k \geq 1 \left| D_\gamma x v_{\nu_i}^{\nu_0-k} (s, x) \right| \leq c \frac{1}{1 + |x|^{2(D+1)}}. \quad (119) \]
Note that this implies that for first order multivariate spatial derivatives \( 0 \leq |\gamma| \leq 1 \), some \( \Delta > 0 \) and \( s \in [s_0, s_0 + \Delta] \) there exists a finite constant \( c \) which is independent of \( \nu > 0 \) such that for \( \delta v_{\nu_i}^{\text{init},\nu_0,2} := v_{\nu_i}^{\nu_0,2} - v_{\nu_i}^{\nu_0,2} (s_0, \cdot) \ast G_\nu \) we have
\[ \left| D_\gamma x \delta v_{\nu_i}^{\text{init},\nu_0,2} (s, x) \right| \leq c \frac{1}{1 + |x|^{2(D+1)}}, \quad (120) \]
and
\[ \forall \ k \geq 1 \left| D_\gamma x \delta v_{\nu_i}^{\nu_0-k} (s, x) \right| \leq c \frac{1}{1 + |x|^{2(D+1)}}. \quad (121) \]
Next concerning the behavior at \( r = \sqrt{\sum_{i=1}^{D} x_i^2} = 0 \) we have to refine some observations of the preceding section since the data are only Hölder continuous with exponent \( \delta \in (0, \beta_0 - \alpha_0) \) and not Lipschitz in general as \( \beta_0 - \alpha_0 \) is close but smaller than 1 according to our choice. A choice \( \beta_0 = 1 + \alpha_0 \) implies a bounded oscillatory singularity for the vorticity. Here we want to prove a stronger result, i.e., a blow up for vorticity. We consider convolutions of (first order spatial derivatives of) the Gaussian \( G_\nu \) with Hölder continuous functions \( g \), where we have the initial data in mind first. It seems impossible to get \( \nu \)-independent estimates for the increments of the first iteration of the local scheme, but the local functional increments have \( \nu \)-independent estimates from the second iteration step on, and this insufficient for our purposes. Indeed, for some \( \Delta > 0 \) and for all \( s \in [s_0, s_0 + \Delta] \) we have \( \nu \)-independent upper bounds
\[ \left| D_\gamma x \delta v_{\nu_i}^{\text{init},\nu_0,2} (s, x) \right| \leq r^{\beta_0-|\gamma|}, \quad (122) \]
and
\[ \forall \ k \geq 3 \left| D_\gamma x \delta v_{\nu_i}^{\nu_0-k} (s, x) \right| \leq r^{\beta_0-|\gamma|}. \quad (123) \]
From these estimates in (122), (123), (120), and (121) local time contraction can be obtained straightforwardly. Then it follows that

\[ v_i^{\nu_1, \nu_3}(s_0, \cdot) = \delta v_i^{\nu_1, \nu_3}(s_0, \cdot) + \sum_{k=1}^{\infty} \delta v_i^{\nu_1, \nu_3}(s_0, \cdot) \in C^{1,2} \] (124)

has a uniform upper bound such that

\[ e_i^\nu = \lim_{k \to \infty} v_i^{\nu_1, \nu_3} \in C^{1,1} \cap \phi_{pol,1}^{3(D+1)} . \] (125)

Let us consider this in some more detail. Since we consider local time solutions we may consider representations of local solutions in terms of scaled Gaussian \( G_\nu \equiv G_\nu^1 \) as above with \( \nu' = 4\pi pr^2 \nu \). Recall that we start the iteration with the initial data

\[ v_i^{\nu_1, \nu_3}(s_0, \cdot) = v_i^{\nu_1, \nu_3}(s_0, \cdot), \quad 1 \leq i \leq D \] (126)

assuming that the data \( v_i^{\nu_1, \nu_3}(s_0, \cdot), \quad 1 \leq i \leq D \) are known at time \( s_0 \geq 0 \), and that for \( k = 1 \) we define

\[ v_i^{\nu_1, \nu_3}(s_0, \cdot) = v_i^{\nu_1, \nu_3}(s_0, \cdot) * G_\nu, \quad 1 \leq i \leq D \] (127)

In order to estimate \( \delta v_i^{\nu_1, \nu_3}(s_0, \cdot) = v_i^{\nu_1, \nu_3}(s_0, \cdot) - v_i^{\nu_1, \nu_3}(s_0, \cdot) * G_\nu \) and \( \delta v_i^{\nu_1, \nu_3}(s_0, \cdot) \geq 3 \) for \( 1 \leq i \leq D \) we have to plug in

\[ v_i^{\nu_1, \nu_3}(s_0, \cdot) * G_\nu, \quad 1 \leq i \leq D \] into the Burgers term and the Leray projection term. Using the convolution rule we observe that for \( s_1 > s_0 \) the first order spatial derivatives of

\[ (v_i^{\nu_1, \nu_3} * G_\nu)(s_1, x) = \int_{s_0}^{s_1} \int_{\mathbb{R}^D} v_i^{\nu_1, \nu_3}(s_0, x-y) \frac{1}{\sqrt{4\pi \nu' \sigma}} \exp \left(-\frac{|y|^2}{4\nu' \sigma} \right) dyd\sigma \] (128)

have the representation

\[ (v_i^{\nu_1, \nu_3}(s_0, \cdot) * G_\nu,i)(s_1, x) = \int_{s_0}^{s_1} \int_{\mathbb{R}^D} v_i^{\nu_1, \nu_3}(s_0, x-y)G_{\nu,i}(\sigma, y)dyd\sigma . \] (129)

Now the first order derivatives of the scaled Gaussian \( G_{\nu,i} \) are given by

\[ G_{\nu,i}(s, x) = \left( \frac{-2y}{4\pi \nu'^2 \sigma} \right) \frac{1}{\sqrt{4\pi \nu' \sigma}} \exp \left(-\frac{|y|^2}{4\nu' \sigma} \right) , \] (130)

and have the upper bound

\[ \left| G_{\nu,i}(\sigma, y) \right| \leq \left| \frac{-2y}{4\pi \nu'^2 \sigma} \right| \frac{1}{\sqrt{4\pi \nu' \sigma}} \exp \left(-\frac{|y|^2}{4\nu' \sigma} \right) \left| \right| \] (131)
where for $|z| = \frac{|y|}{\sqrt{4\pi \nu^2 \sigma}}$ and $\delta \in (0, 1)$

$$c_s := 2 \sup_{|z| \geq 0} \left( \frac{|y|^2}{4\pi \nu^2 \nu^2 \sigma} \right)^{D/2 + 1 - \delta} \exp \left( \frac{-|y|^2}{4\nu^2 \sigma} \right).$$

(132)

Since

$$|v_{\nu',-}(s_0, x - y)| \leq c_0|x - y|^{\beta_0}$$

for some finite constant $c_0 > 0$ we have

$$|v_{\nu',-}(s_0, \ast G_{\nu, i})| \leq c_1|r|^{\beta_0 - 1}$$

(133)

for some finite constant $c_1$. Here, note that the elliptic integral gives the upper bound $\int_{s_0}^{s_1} \nu^{\nu + 2 - 1} \nu^{(\sigma - s_0)} d\sigma$, and as we plug in this elliptic integral into the iteration formula for $v_{\nu',-}$ the contribution of the integral terms in the formula for $v_{\nu',-}$ is concentrated in the area $r^2 \leq \nu'$ as $\nu'$ becomes small. Concerning the behavior of the first order spatial derivatives of the Gaussian for $\sqrt{\sum_{j=1}^{D} y_j^2} = r > \sqrt{\nu'}$ note that

$$|G_{\nu', i}(\sigma, y)| \leq \left( \frac{2\pi}{4\sigma^2} \right)^{1/2} \exp \left( \frac{-|y|^2}{8\sigma^2} \right) \exp \left( \frac{-|y|^2}{8\sigma^2} \right) \left( -1 \right)$$

$$\leq \left( \frac{1}{4\pi \nu^2} \right)^{1/2} \exp \left( \frac{-|y|^2}{8\nu^2} \right),$$

where

$$\bar{c}_s := 2 \sup_{|z| \geq 0} \left( \frac{|y|^2}{4\pi \nu^2 \nu^2 \sigma} \right)^{D/2 + 1 - \delta} \exp \left( \frac{-|y|^2}{8\nu^2} \right).$$

(136)

Convolutions of this upper bound with data of order $|x - y|^{\beta_0}$ are integrable at $|y| = 0$ even for $\delta > 0$ close to zero. Furthermore for small $\nu'$ and at a point $|y| > \sqrt{\nu'}$ close to $\sqrt{\nu'}$ we have $\sqrt{\nu'}^{1-\epsilon}$ for small $\epsilon > 0$ such that the last factor in (135) becomes exp $\left( \frac{-|\nu'|^{1-\epsilon/2}}{8\nu^2} \right) = \exp \left( \frac{-1}{8\nu^2} \right)^{1/2} \downarrow 0$ as $\nu' \downarrow 0$. For $\epsilon = 2\delta$ we observe that the factor $|\nu'|^{-\delta}$ is damped by this exponential factor such that the upper bound in (134) is independent of $\nu'$ for these terms.

More explicitly, for $k = 2$ we have

$$v_{\nu',-2} = v_{\nu',-0}(s_0, \ast sp G_{\nu'} + \sum_{j=1}^{D} v_{\nu',-1} \frac{\partial v_{\nu',-1}}{\partial x_j} \ast G_{\nu'}$$

$$- \sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_j} K_B(\cdot, -y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v_{\nu',-1}}{\partial x_j} \frac{\partial v_{\nu',-1}}{\partial x_m} \right) (t, y) dy \ast G_{\nu'}$$

$$=: v_{\nu',-0}(s_0, \ast sp G_{\nu'} + B_1 \ast G_{\nu'} - L_1 \ast G_{\nu'},$$

(137)

where $B_1$ and $L_1$ denote abbreviations of the next order of approximation of the Burgers term and the Leray projection term. The estimate

$$|v_{\nu',-0}(s_0, \ast sp G_{\nu',-j})| \leq c\nu^{\beta_0 - 1},$$

(138)
Hence
\[
\left| \frac{v_\nu' \partial v_\nu', -1}{\partial x} \right| (s, \cdot) \leq cr^{2\beta_0 - 1} \leq (139)
\]
and as \( \frac{\partial}{\partial x_i} K_{D,i}(-y) \sim \frac{1}{r} \) for \( D = 3 \) we have
\[
\left| \int_{\mathbb{R}^D} K_{D,i}(-y) \sum_{j,m=1}^D \left( \frac{\partial v_{\nu'} \partial v_{\nu'}, -1}{\partial x_j} \right) (\sigma, y) dy \right| \leq cr^{2(\beta_0 - 1) + 1}. \leq (140)
\]
Using the \( \nu \)-independent Gaussian estimates above with \( \Delta r := |x - y| \) for the Burgers term \( B^1 \) we get
\[
\left| B^1 \star D^n G_\nu' (\sigma, \cdot) \right| \leq cr^{2\beta_0 - 1 - |\gamma|}, \leq (141)
\]
and for the Leray projection term \( L^0 \) we have
\[
\left| L^1 \star D^n G_\nu' (\sigma, \cdot) \right| \leq cr^{2(\beta_0 - 1) + 1 - |\gamma|}. \leq (142)
\]
Hence, for first order multivariate spatial derivatives \( 0 \leq |\gamma| \leq 1 \) and \( |x| \geq 1 \), for some \( \Delta > 0 \) for all \( s \in [s_0, s_0 + \Delta] \) there exists a finite constant \( c \) which is independent of \( \nu > 0 \) such that for all \( \epsilon > 0 \) small enough we have
\[
\left| D^n \delta v_i^{\text{init}, \nu', -2} (\sigma, x) \right| \leq cr^{2\beta_0 - 1 - \epsilon - |\gamma|}, \leq (143)
\]
and
\[
\forall k \geq 3 \left| D^n \delta v_i^{\nu', -k} (\sigma, x) \right| \leq cr^{2\beta_0 - 1 - \epsilon - |\gamma|}, \leq (144)
\]
where the constant \( c \) is independent of \( \nu' \). Local time contraction results are obtained as for the original Navier Stokes equation, and on some time interval \( [s_0, s_0 + \Delta] \) we may use representations of solutions of the form
\[
v_i^{\nu', -} = v_i^{\nu', -0} (s_0, \cdot) \star_\rho G_\nu' + \delta v_i^{\text{init}, \nu', -2} + \sum_{k \geq 3} \delta v_i^{\nu', -k}. \leq (145)
\]
Hence, for \( s \in [s_0, s_0 + \Delta] \)
\[
v_i^{\nu', -} (s, \cdot) \in C^{m(D + 1)} \leq (146)
\]
Using this strong polynomial decay and the compactness argument of the preceding section we get a subsequence \( \nu_k \downarrow 0 \) such that
\[
e^\nu \in C^{1,1} \text{ where } \forall s \in [s_0, s_0 + \Delta] \; e^\nu_i(s, \cdot) = \lim_{k \uparrow \infty} v_i^{\nu_k, -} (s, \cdot) \in C^{m(D + 1)} \leq (147)
\]

iii) In order to strengthen the regularity result we use the semi-group property of the Euler-and Navier stokes equation operator and the estimates of
the preceding item, where we consider the local representation for $\sigma \in [s_0, s_0 + \Delta]$

$$v_i^{\nu',-}(\sigma, .) = v_i^{\nu',-0}(s_0, .) * \delta \mu + \delta v_i^{\nu',-2}(\sigma, .) \\ + \sum_{k \geq 3} \delta v_i^{\nu',-k}(\sigma, .).$$

(148)

For $\sigma > s_0$ the first term in \[148\] $v_i^{\nu',-0}(s_0, .) * \delta \mu G_{\nu'}(\sigma, .)$ is a smooth function which implies that for some finite constant $c'$ and

$$\forall k \geq 2 \left| D^k G_{\nu'}(\sigma, x) \right| \leq c' \sigma^{2\beta_0 - 1 - \epsilon - |\gamma|},$$

(149)

Iterating the argument of the preceding item with initial data $v_i^{\nu',-0}(s_1, .)$ at $s_1 > s_0$ once we get

$$\left| L \ast D^k G_{\nu'}(\sigma, .) \right| \leq c_{4\beta_0-1+1-|\gamma|},$$

(150)

where $L$ denotes the Leray projection operator applied to the local time solution in $[s_1, s_1 + \Delta]$ for $\Delta > 0$ as above. Similar for the burgers term (where slightly stronger regularity can be proved after one iteration of the regularity argument of item ii) with data $v_i^{\nu',-0}(s_1, .)$). We get

$$\forall s \in [s_0, s_0 + \Delta] \epsilon_{i0}(\sigma, .) \ni \lim_{\nu \to 0} v_i^{\nu',-0}(s, .) \in C^{m(D+1)}_{\text{pol}, m}$$

(151)

for $s_0 + \Delta \geq s_1 > s_0$.

iv) Choose a time horizon $T > 0$. Let $s_0 = T$ and consider the time-reversed incompressible Euler equation. From the previous step we have a local solution $e_i^{\nu,-}$, $1 \leq i \leq D = 3$ with data in $H^2$ which correspond to a vorticity blow up at $s_0 = T$. We have $e_i^{\nu,-} \in C^{1,1}((s_0, s_0 + \Delta])$ for $1 \leq i \leq D$ and for some $\Delta > 0$. Moreover as in the previous step such that contraction holds for the higher order increments $\delta v_i^{\nu'-k}$ with $k \geq 3$ as in \[103\]. Moreover, $e_i^{-}(s, .) \in C^m \cap C^{m(D+1)}_{\text{pol}, m}$ for $s_0 < s \leq s_0 + \Delta$. Hence the global solution branch technique of the preceding section can be applied for the time-reversed Euler equation with data $e_i^{-}(s, .) \in C^m \cap C^{m(D+1)}_{\text{pol}, m}$ for $s > s_0$. It follows that there is a global solution branch $e_i^{-}$, $1 \leq i \leq D$ defined on the time interval $(s_0, s_0 + T]$. Then the time transformation $t = -s_0 + T$ implies that $t \to e_i(t, .) = e_i^{-}$, $1 \leq i \leq D$ is a global regular solution branch of the incompressible Euler equation on the time interval $[0, T]$ with a vorticity blow up at time $T$, where $T > 0$ is arbitrarily large.

References

[1] Landau, L., Lifschitz, E. Lehrbuch der Theoretischen Physik VI, Hydrodynamik, Akademie Verlag, Berlin. J., (1978).

[2] Majda, A., Bertozzi, L. Vorticity and Incompressible Flow (Cambridge Texts in Applied Mathematics) Cambridge University Press, 2001.

[3] Nirenberg, L. On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa, p. 115-162, vol. 13, (1959).