Cosmographic analysis of the equation of state of the universe through Padé approximations

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(Dated: May 11, 2014)

Cosmography is used in cosmological data processing in order to constrain the kinematics of the universe in a model-independent way, providing an objective means to evaluate the agreement of a model with observations. In this paper, we extend the conventional methodology of cosmography employing Padé expansions of observables by an alternative approach using Padé approximations. Due to the superior convergence properties of Padé expansions, it is possible to improve the fitting analysis to obtain numerical values for the parameters of the cosmographic series. From the results, we can derive the equation of state parameter of the universe and its first derivative and thus acquire information about the thermodynamic state of the universe. We carry out statistical analyses using observations of the distance modulus of type Ia supernovae, provided by the union 2.1 compilation of the supernova cosmology project, employing a Markov chain Monte Carlo approach with an implemented Metropolis algorithm. We compare the results of the original Taylor approach to the newly introduced Padé formalism. The analyses show that experimental data constrain the observable universe well, finding an accelerating universe and a positive jerk parameter. We demonstrate that the Padé convergence radii are greater than standard Taylor convergence radii, and infer a lower limit on the acceleration of the universe solely by requiring the positivity of the Padé expansion. We obtain fairly good agreement with the Planck results, confirming the ΛCDM model at small redshifts, although we cannot exclude a dark energy density varying in time with negligible speed of sound.

PACS numbers: 98.80.-k, 98.80.Jk, 98.80.Es

\section{I. INTRODUCTION}

Over the last decades, several experiments have gathered significant evidence suggesting that our universe is currently undergoing an accelerated phase of expansion \cite{1,2}. Indications are coming from the detection of type Ia supernovae (SNeIa) \cite{3,4}, from measurements of the Hubble space telescope, from galaxy redshift surveys, cosmic microwave background detection, baryonic acoustic oscillations and so forth (see \cite{5} and references therein). A wide number of theoretical models has been investigated in order to clarify the physical origin of such a cosmic speed up \cite{6}, although so far no self-consistent solution has been found. Among the multitude of approaches, ideas range from postulating a new ingredient dubbed dark energy (DE) driving the acceleration, to modifications of the spacetime geometry of the universe itself \cite{8}. A widely accepted and overall quite successful framework is the so-called ΛCDM model \cite{9}, until now the standard paradigm of cosmology. Here, CDM stands for cold dark matter, and Λ represents a cosmological constant \cite{10}. Unfortunately the model comprises two precarious issues, namely the problems of fine tuning and coincidence, which do not allow us to consider ΛCDM as the final paradigm for describing the universe’s dynamics \cite{11}. The fine tuning problem refers to the large difference between the energy density driving the cosmic expansion and the value for the vacuum energy density predicted by quantum field theory \cite{12}. The coincidence problem addresses the fact that the densities of the fluid driving the cosmic acceleration and pressure-less matter are comparatively similar at the present time. In other words, although these two quantities evolved differently during the history of the universe, their contributions to the overall energy density of the universe are of the same order of magnitude today \cite{13,14}. In the absence of a self-consistent theoretical scheme to explain DE, the search for new cosmological models able to overcome both the fine tuning and coincidence problems is an open task of modern cosmology. Any viable model attempting to describe the dynamics of the universe must also provide agreement with the most recent observations \cite{15}. As ΛCDM is already quite successful in doing so, there is the common consensus that any feasible new model should reproduce the effects of ΛCDM in the low redshift regime. Thus arises the conclusion that ΛCDM may be viewed as a first approximation of a more complicated paradigm \cite{16,17}. Literature presents us with a huge amount of models \cite{15,20} with diverse approaches, aiming at improving the shortcomings of ΛCDM and explaining the aforementioned problems. Many of them are quite successful in describing the observed phenomena, but with the growing number

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of models it becomes increasingly difficult to fairly discriminate between them and favor a particular one over others. Indeed, the biggest problem of confronting models with data is the fact that in such analyses the model in question is usually \textit{a priori} postulated to be the correct one \cite{21}. This leads to a degeneracy among models, since the comparison of every model to data favors the one being tested. Thus, many models seem to work better than others, leading to a desperate need of independent methods to test cosmological paradigms \cite{22}. To this end, increasing efforts have been devoted to the development of the so called cosmography of the universe \cite{23, 24}. Cosmography represents the branch of cosmology attempting to obtain insights into the cosmological picture by exploring only the universe kinematics, relying on as few assumptions as possible in order to keep a viewpoint as neutral as possible \cite{25}. We assume the validity of the cosmological principle only, i.e. the universe is supposed to be homogeneous and isotropic. The dynamics of the universe can then be formulated in terms of a Friedmann-Robertson-Walker (FRW) metric, $ds^2 = dt^2 - a(t)^2 \left( \frac{d\rho}{1 - \kappa r^2} + r^2 \sin^2 \theta d\phi^2 + r^2 d\theta^2 \right)$ \cite{26, 27}. The methodology of cosmography is essentially based on expanding measurable cosmological quantities into Taylor series around the present time, providing model-independent constraints on the universe’s kinematics or energy density \cite{25}. The present work’s main purpose is to introduce an alternative technique to Taylor expansions to the framework of cosmography. In particular, in order to carry out the cosmographic analyses, the formalism of Padé approximations (PAs) is proposed \cite{29}. The use of PAs eliminates the convergence problem, e.g. the systematic errors due to truncated Taylor series, which does not permit cosmography to provide reliable constraints at higher redshifts. Efforts to alleviate the convergence problem have been employing the reparametrization of the redshift $z$ with \textit{ad hoc} auxiliary variables, compressing data to shorter intervals of redshift where convergence of Taylor series is given \cite{26}. Instead of using artificial redshift constructions, we demonstrate that the PAs may be a viable alternative to improve the cosmographic fitting procedure. Indeed, the convergence radii of PAs exceed those of the Taylor approach, justifying the use of data from a larger redshift regime. The paper is organized as follows: in Section II we give a general introduction of the fundamental principles of cosmography and describe the technicalities of the analysis, as well as define the conventional fitting functions in the Taylor formalism used in the literature. We shortly sketch how to obtain certain important cosmological quantities that can be derived from the fitting parameters, like the equation of state parameter of the universe. We will address some concerns about problems occurring in the cosmographic procedure, and address the question of how to alleviate them in Section III. In Section IV introducing the concept of Padé approximations as a way to circumvent the convergence problem, we will describe in detail how to calculate the functions for the fits in the Padé formalism, and state their explicit form. Furthermore the convergence radius of a Padé expansion will be introduced as a quantitative measure of the range of validity of the approximation with respect to a Taylor expansion. In Section VI after devoting some attention to describing the statistical methods used, we will present the numerical results of the analyses, i.e. the obtained values for the parameters of the CS, as well as the results for the convergence radius of the Padé expansion and the equation of state of the universe obtained from those values. In Section VII we will conclude our work.

II. COSMOGRAPHY OF THE UNIVERSE

The present section is devoted to describing in detail the role played by cosmography in the analysis of the observable universe. As already mentioned before, the main feature of cosmography is its aim to rely on as few underlying assumptions as possible. In particular, it is based on the validity of the cosmological principle. In this work we will further use the assumption of a spatially flat universe, i.e. $k = 0$.

The standard procedure is to expand the scale factor $a(t)$ parameterizing the expansion of the universe in the FRW-metric into a Taylor series with respect to time $t$ around the present time $t_0$, as

$$a(t) \equiv 1 + \sum_{\kappa=1}^{\infty} \frac{1}{\kappa!} \frac{d^\kappa a}{dt^\kappa} \bigg|_{t=t_0} (t - t_0)^\kappa, \quad (1)$$

with the coefficients in terms of scale factor derivatives evaluated at current time. From these coefficients, we can define a set of parameters, given for a generic time $t$ as

$$\mathcal{H} = \frac{1}{a} \frac{da}{dt}, \quad (2a)$$

$$q = -\frac{1}{aH^2} \frac{d^2a}{dt^2}, \quad (2b)$$

$$j = \frac{1}{aH^3} \frac{d^3a}{dt^3}. \quad (2c)$$

They are known in the literature as the Hubble parameter, the acceleration parameter and the jerk parameter, respectively \cite{30}, and evaluated at the present time $t_0$ customarily termed the cosmographic series (CS) \cite{23–25, 27}. The CS can be extended to higher orders defining further parameters, but in this work we restrict ourselves to the first three. Each parameter of the CS has its distinct physical meaning. The acceleration parameter $q_0$ describes the behavior of the universe’s expansion, quantifying its acceleration. In our currently accelerating universe, we expect $q_0$ to be negative. The jerk parameter in turn gives information about inflection points in the expansion history of the universe. A positive $j_0$ implies
that the universe has gone through a change of sign of the acceleration parameter in the past, meaning that there has been a transition from deceleration to acceleration.

For our purposes, it is useful to combine the CS among themselves and express Eqs. (2) in terms of each other, yielding

\[ q = -\frac{\dot{H}}{H^2} - 1, \quad j = \frac{\dot{H}}{H^3} - 3q - 2. \]  \hspace{1cm} (3)

Moreover, it is worth noting that to convert the time variable \( t \) to the redshift \( z \), the following identity

\[ \frac{dz}{dt} = -\mathcal{H}(z)(1 + z) \]  \hspace{1cm} (4)

can be used. One of the most relevant consequences derived from the CS bases on the fact that universe’s energy density can be related to cosmographic parameters without invoking a model \textit{a priori}. This property has been extensively demonstrated for simple barotropic fluids, in which the pressure is a function of the total density, i.e. \( P = \omega \rho \).

We limit our attention to pressureless matter, denoted by the subscript \( m \), comprising both baryonic and cold dark matter, and dark energy. This choice is justified by the fact that present contributions due to neutrinos, dark matter, and dark energy. This choice is justified by the fact that present contributions due to neutrinos, photons, scalar curvature, and so forth are negligible. Hence, we end up with an overall EoS parameter of the form

\[ \omega = \frac{\mathcal{P}_{DE}}{\rho_{DE} + \rho_m}. \]  \hspace{1cm} (5)

Using the Friedmann equations with \( 8\pi G = c = 1 \),

\[ \mathcal{H}^2 = \frac{\rho}{3}, \]  \hspace{1cm} (6a)
\[ \mathcal{H} + \mathcal{H}^2 = -\frac{1}{6}(3\mathcal{P} + \rho), \]  \hspace{1cm} (6b)

and the continuity equation

\[ \frac{dp}{dt} + 3\mathcal{H}(\mathcal{P} + \rho) = 0, \]  \hspace{1cm} (7)

we obtain the pressure of the universe in terms of \( q \) as

\[ \mathcal{P} = \mathcal{H}^2 (2q - 1), \]  \hspace{1cm} (8)

and thus the corresponding expressions for \( \omega \) and its first derivative \( \omega' \equiv \frac{d\omega}{dz} \) in terms of the CS read

\[ \omega = \frac{2q - 1}{3}, \]  \hspace{1cm} (9a)
\[ \omega' = \frac{2(j - q - 2q^2)}{3(1 + z)}. \]  \hspace{1cm} (9b)

These two quantities give important information about current universe’s expansion and recent changes in expansion behavior. Equations (9a) and (9b) show that constraining the current values of \( q \) and \( j \) corresponds to fixing limits on the current thermodynamic state of the universe. In other words, possible changes of cosmographic parameters correspond to changes of the EoS of the universe and its derivatives. Since we aspire to determine the current values of the CS without the need of assuming an EoS of any given model, the results of our fits will provide direct constraints on the current EoS parameter \( \omega \) and its derivative \( \omega' \) by employing Eqs. (5a) and (5b).

Finding numerical values for the CS can be achieved by fitting appropriate data, e.g. the apparent luminosity of type Ia supernovae as a function of the redshift. It is possible to express the distance modulus \( \mu_D \), given by the difference of the apparent (\( \mu_{app} \)) and absolute luminosity (\( \mu_{abs} \)) of an object, in terms of the luminosity distance \( d_L \) as

\[ \mu_D = \mu_{app} - \mu_{abs} = 25 + \frac{5}{\ln 10} \ln \left( \frac{d_L}{1 \text{ Mpc}} \right). \]  \hspace{1cm} (10)

and in turn the luminosity distance as a function of the scale factor as

\[ d_L = R_0(1 + z) = R_0 \frac{1}{a(t)}, \]  \hspace{1cm} (11)

where we used the identity

\[ a = \frac{1}{1 + z}. \]  \hspace{1cm} (12)

Here \( R_0 \) is the distance that a photon travels from a light source at \( r = R_0 \) to our position at \( r = 0 \), defined as

\[ R_0 = \int_0^{t_0} \frac{d\xi}{a(\xi)}. \]  \hspace{1cm} (13)

We can calculate this quantity by using the power series expansion for the inverse of the scale factor and integrating each term in the sum separately. By inserting the expansion of \( a(t) \) and \( R_0 \) into the luminosity distance, we then obtain a Taylor series expansion of \( d_L \). Substituting the time variable \( t \) by the redshift \( z \) according to Eq. (11), it is possible to obtain the luminosity distance \( d_L \), and the distance modulus \( \mu_D \) in terms of a Taylor expansion with respect to the redshift \( z \).

Thus, we obtain for the luminosity distance \( d_L \)

\[ d_L = d_H z \left[ 1 + z \left( \frac{1}{2} - \frac{q_0}{2} \right) + z^2 \left( -\frac{1}{6} + \frac{j_0}{6} + \frac{q_0}{6} + \frac{j_0^2}{2} \right) + ... \right], \]  \hspace{1cm} (14)

where \( d_H = 1/\mathcal{H}_0 \) is the Hubble distance. After straightforward calculations, we can obtain the distance modulus in the form

\[ \mu_D = 25 + \frac{5}{\ln 10} \left[ \ln \left( \frac{d_H}{1 \text{ Mpc}} \right) + \ln z + \zeta_1 z + \zeta_2 z^2 + ... \right], \]  \hspace{1cm} (15)
where \( \zeta_1 \) and \( \zeta_2 \) are as yet undetermined coefficients. We have to expand the logarithm of the luminosity distance for small \( z \), which leads to the following results for the coefficients:

\[
\begin{align*}
\zeta_1 &= \frac{1}{2} - \frac{q_0}{2}, \\
\zeta_2 &= \frac{5q_0}{12} + \frac{3q_0^2}{8} - \frac{j_0}{6} - \frac{7}{24}.
\end{align*}
\]

Equations (14) and (15) are commonly used in cosmography to fit supernovae data in order to obtain numerical values for the CS. In the next sections, we will address some problematic issues connected to this formalism, and in this context introduce the concept of Padé approximations.

### III. THE PROBLEMS WITH COSMOGRAPHY

As previously described, cosmography considers Taylor expansions of relevant observables, which are then constrained by directly fitting cosmological data. Its methodology permits us to assume that the cosmographic procedure is model-independent from any particular cosmological model, because in any of the expansions used we do not rely on model-dependent assumptions a priori. However, the introduced formalism of determining cosmological bounds on the CS entails some other difficulties, which have to be addressed. In the following subsections, we describe in detail each of these problems, which must be alleviated through the use of either theoretical or statistical techniques.

#### A. Truncated approximations of the Taylor series

Taylor expansions are approximations to an exact expression, and coincide with the original function for the limit of infinite terms in the expansion. As it is impossible to consider an infinite number of terms in numerical analyses, the series has to be truncated at some finite order. This introduces errors into the analysis, since the formulae used for fitting only represent approximations to the true expressions. This problem can be moderated by including higher orders of the series, but this comes at the expense of introducing more fitting parameters and considerably complicating the corresponding statistical analysis. Every extension of parameter space implies a broadening of the posterior distributions for each parameter, and thus in principle it is desirable to keep the number of fitting parameters as low as possible.

#### B. Convergence at higher redshifts

A second issue is related to the range of convergence of Taylor series used for cosmographic expansions. By definition, the Taylor series we constructed converges only for small \( z \). Hence, it may happen that for higher redshifts the series diverges, being unable to correctly represent the distance modulus or the luminosity distance for the whole set of cosmic data. SNeIa data reach up to a redshift of about \( z \approx 1.414 \), which lies outside the expected convergence range of the series; and the inclusion of data from other astrophysical sources can extend the fitting regime even to redshifts higher than that. Thus, the attempt to improve the statistics of the analysis by including higher redshift data actually jeopardizes the original aim and leads to a less accurate result due to the lack of convergence of the fitting functions used. This problem is known in the literature as convergence problem [27]. To avoid divergences, some modifications of the formalism have been proposed, like e.g. rephrasing the series in terms of a new variable, which compresses the data to a region in which the convergence of the series is still guaranteed. One possibility for such a new variable, i.e. an alternative definition of redshift, has been suggested in earlier work as \( y = z/(1 + z) \), which exhibits improved convergence properties as compared to the conventional redshift \( z \). This idea has been extended to further notions of redshift by [28]. The new redshift variable \( y \) shows better convergence behavior in particular in the past redshift regime; for \( z \in [0, \infty) \), the new redshift is bound in \( y \in [0, 1] \). Less fortunately, in the future regime, it fails to converge, as with \( z \in [-1, 0] \) we obtain \( y \in (-\infty, 0] \). Independently of the exact choice of a new redshift variable, this seems to be a common problem, i.e. by requiring that possible re-parametrizations must result in a smooth function, the new variable \( z_{\text{new}} \), given by \( z_{\text{new}} = Z(z) \), with \( Z(z) \) a generic function of the redshift \( z \) converging to \( Z(z) \to 1 \) as \( z \to \infty \), generally shows divergences at \( z \to -1 \). Thus, although the past behavior is of crucial importance, usually the future regime is unfortunately not well constrained and there the convergence problem remains. We aim at building up an alternative method to alleviate the convergence problem in both past and future regimes by bringing Padé expansions into play. This idea will be further elaborated in the following sections.

### IV. PRINCIPLES OF PADÉ EXPANSION

For a function \( f(x) \), the Padé approximant of order \((m, n)\) is given by

\[
P_{mn}(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n}.
\]

In principle, it is possible to express the luminosity distance \( d_L \) not in form of a conventional Taylor series \( d_L = d_L(z) (1 + \alpha z + \beta z^2 + \ldots) \), where \( \alpha, \beta, \ldots \) are dependent on the parameters of the CS, but in the shape of a Padé approximant expression. As well as in the case of a Taylor expansion, the coefficients \( a_i, b_i \) depend on the parameters of the CS. The correspondence between the
two functional forms and their sets of parameters, i.e. expressing the coefficients \( a_i, b_i \) by their equivalents \( \xi_1, \xi_2 \), can be obtained from the requirement that for \( z = 0 \) the two different parametrizations and their derivatives be equal. In order to achieve a balanced correspondence between Taylor and Padé form, we should match the number of coefficients, and so we choose \( a_0 = 0 \) and look for a Padé approximant of the form \((1, 2)\),

\[
d_{L, \text{Padé}} = \frac{a_1 z}{1 + b_1 z + b_2 z^2}.
\]

The Padé form of the luminosity distance then reads

\[
d_{L, \text{Padé}} = \frac{12d_H z}{12 + 6(q_0 - 1)z + (5 + 2j_0 - 8q_0 - 3q_0^2)z^2},
\]

where

\[
a_1 = d_H, \\
b_1 = \frac{1}{2}(q_0 - 1), \\
b_2 = \frac{1}{12}(5 + 2j_0 - 8q_0 - 3q_0^2).
\]

The thus formulated \( d_{L, \text{Padé}} \) cannot diverge, since no real poles occur in the observational range for \( z \) and for any \( j_0, j_0 \). The Padé-expanded parametrization of the magnitude \( \mu_D \) can then be easily calculated by using Eqs. (15) and (19) and results in a rather involved expression due to the logarithmic form, reported in Appendix A.

While on first sight this may not reveal any improvements with respect to the Taylor expression, some considerations show that the Padé form, besides some possible drawbacks, does have its advantages. Within the conventional Taylor treatment, the quantity to be expanded originally is the scale factor \( a(t) \). This expansion defines the cosmographic series to begin with. Since the luminosity distance is a function of \( a(t) \), it can be directly given as a function of the CS. In the Padé treatment, we have to continue using the Taylor expansion of \( a(t) \), since we want to keep the CS as our parameter set. Expanding \( d_L \) into a Padé approximant, and linking the coefficients \( a_i, b_i \) to the CS, thus means to approximate an already approximated expression anew.

To avoid this, it would be necessary to go back to the expansion of the scale factor itself and start with Padé approximants already at that stage. However, since the CS is defined from a Taylor expansion, and as such offers a very direct and intuitive interpretation of the dynamics of the universe’s expansion, retaining the set of cosmographic parameters as defined from the conventional Taylor approach is preferable. Following the outlined procedure, the statistical analyses will show that the reexpression of the Taylor series in terms of Padé approximants does not significantly propagate systematic errors, and that the use of Padé expansions can be justified retrospectively.

Moreover, in another concern the Padé approximant method has its undeniable advantages, and that is the question of convergence. Padé approximants are known to have a much larger radius of convergence, while a Taylor series fails to converge for \( z > 1 \) or even earlier. Since in usual cosmographic analyses supernova data from a range of redshifts \( z \in [0, 1.414] \) is used, and the extension of data sets to information from higher redshift sources is desirable, significant problems with the Taylor formalism are expected for the high redshift regime of data. With a Padé approximant, it would be no problem to not only use the full range of supernova data, but to even expand the data sets to include new high redshift sources. In this context, we regard it as highly useful to consider a cosmographic analysis in the framework of a Padé expansion.

### A. The convergence radius

In this subsection, we will explicitly demonstrate why the PA represents a better alternative to standard Taylor expansions. To quantify the advantages of Padé over Taylor formalism, a viable tool is to evaluate their convergence radii \( R \). Let us consider the case of a Padé expansion of the luminosity distance in analogy to Eqs. (18) and (19), but to order \((1, 1)\), as a simple example for evaluating the convergence radius. The choice of a Padé approximant of order \((1, 1)\) corresponds to a second order Taylor series, for which we will calculate the convergence radius as well. Forms of the luminosity distance with higher orders of expansions would refine the result for the convergence radius, but not significantly change it.

Restating the luminosity distance in the form

\[
d_{L, \text{Padé}} = d_H \frac{A(z)}{B + Cz},
\]

with the identifications

\[
A = 1, \\
B = 1, \\
C = \frac{1}{2}(q_0 - 1),
\]

we can, by demanding the positivity of the Padé expansion of \( d_L \), derive the following condition on the acceleration parameter:

\[
q_0 > -1,
\]

for the choice \( z = 1 \). This result naturally predicts an accelerated universe in a rather stringent form, but is only a consequence of restricting the luminosity distance Eq. (21) in its Padé form to positive values, regardless of the correct cosmological model. For the sake of clearness, the value presented in Eq. (23) only gives a lower limit for the acceleration parameter. The Padé expansions thus theoretically exclude \( q_0 = -1 \), which corresponds to a pure de Sitter universe.
By recalling the definition of the geometrical series for a generic variable \( x < 1 \),
\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},
\]
and using a reformulation of the luminosity distance obtained after cumbersome algebra,
\[
d_{L,\text{Padé}} = \frac{2}{q_0-1} \left[ 1 - \frac{2}{2 + (q_0 - 1)z} \right],
\]
we can rewrite Eq. (20) in terms of a geometric series as
\[
d_{L,\text{Padé}} = \frac{2}{q_0-1} \left[ 1 - \sum_{n=0}^{\infty} \left( \frac{2}{2 + (q_0 - 1)z} \right)^n \right],
\]
which converges if the argument of the series is smaller than unity, i.e. if
\[
z < \frac{2}{1 - q_0},
\]
implying that the convergence radius of the Padé expansion is
\[
R_{\text{Padé}} = \frac{2}{1 - q_0},
\]
which for typical values of \( q_0 \sim [0.7, -0.4] \) results in values of \( R_{\text{Padé}} \sim [1.18, 1.42] \). Since the regime of convergence of usual cosmographic Taylor expansions has an upper limit of unity, i.e. the highest convergence radius is \( R_{\text{Taylor}} = 1 \), we hope to obtain a higher value for \( R_{\text{Padé}} \), which would show that the use of Padé approximants extends the convergence region of the analysis and thus significantly reduces the problem of convergence plaguing standard cosmography. However, the typical convergence radius of Taylor series can also be less than the unity. In fact, using the Taylor expansion of \( d_L \) up to second order as given by Eq. (14), it can be approximately given as
\[
R_{\text{Taylor}} \simeq \frac{1 - q_0}{2} = R_{\text{Padé}},
\]
which typically lies in the range of \( R_{\text{Taylor}} \sim [0.70, 0.85] \). In order to obtain numerical values for the expressions of the convergence radii of Padé and Taylor expansions, we should consult the fitting results for \( q_0 \). We show the values for the convergence radii in Tab. I, calculated from the later obtained results for the CS. We find that, in all cases, the convergence radius of Padé is greater than the convergence radius of Taylor series, suggesting that the Padé convergence always extends to a larger regime than the one predicted by standard Taylor formalism.

V. FITTING RESULTS

In this section, we describe the statistical procedure for fitting the cosmological data through the use of the PAs. All fits have been carried out for the luminosity \( \mu_D \), adopting the union 2.1 compilation [3] of distance moduli of supernovae as a function of the redshift \( z \), combined with diverse priors on one or more of the fitting parameters.

A. Supernova data and statistics

Type Ia supernovae represent well-consolidated rulers for fitting cosmological data, since they are considered standard candles and have become primary distance indicators. The most recent supernova survey is based on the union 2.1 compilation of the supernova cosmology project [3] and spans a wide range of supernovae in \( z \in [0.015, 1.414] \). The list of supernovae includes 580 measurements of the distance modulus, extending previous compilations, e.g. the union 2 [31] and union 1 [32] sets.

In order to fix cosmographic constraints on the fitting parameters, we use a Monte Carlo analysis employing Markov chains. The adopted Metropolis algorithm [33] enables us to reduce the dependence of the analysis on the initial statistical distribution by modifying the statistics of the proposal distribution during the run of the Monte Carlo simulation. In particular, the statistical distribution in one step then depends on the previously used one, leading to normally distributed numerical results [34].

In the analyses, one maximizes the following likelihood function
\[
\mathcal{L} \propto \exp(-\chi^2/2),
\]
where \( \chi^2 \) is the chi-square function, defined as
\[
\chi^2 = \sum_{k=0}^{580} \frac{(\mu_{D,k}^{\text{th}} - \mu_{D,k}^{\text{obs}})^2}{\sigma_k^2},
\]
with \( \mu_{D,k}^{\text{th}} \) the theoretical value of \( \mu_D \), predicted by the fitting function used, and \( \mu_{D,k}^{\text{obs}} \) the observed luminosities given by data catalogs. The corresponding 1σ error to each supernova is denoted by \( \sigma_k \) and is reported by the supernova survey as well. The investigated ranges of parameters adopted for the analyses are
\[
\begin{align*}
    h & \in [0.4, 0.9], \\
    q_0 & \in [-1.5, 0], \\
    j_0 & \in [-2, 2],
\end{align*}
\]
where \( h \) is defined via \( H_0 = 100 h \text{ km/s/Mpc} \). These ranges agree with theoretical predictions [23–27] and are in agreement with the Planck results [15]. Different analyses were performed without and then with the assumptions of priors imposed on the fitting parameters. For nearly all of the fits we used the complete range of data with redshifts \( z \in [0, 1.414] \), while one fit was carried out for a restricted sample of supernova data with \( z \in [0, 0.36] \). Further, one fit was performed using Taylor parametrization to show the differences between the two approaches.
B. Cosmological priors

Cosmological priors are used in order to simplify or concretize the numerical analysis. In fact, it is possible that using a single fitting function depending on many parameters is not enough to separately or sufficiently constrain all parameters due to occurring degeneracies among them. By fixing viable priors on parameters, the total phase space is reduced, and complementing different priors with each other may lead to new insights and compelling results for the fitting parameters. For those reasons, we carried out our numerical analyses with and without adopting numerical priors. For $H_0$, two different priors were imposed; one being obtained from the best fit for $H_0$ extracted from the most recent observations by the Planck collaboration [15], i.e. $H_0 = 67.11\text{ km/(s Mpc)}$. The second was obtained by fitting the union 2.1 supernova data in the range $z < 0.36$, expanding the luminosity distance into a power series to first order, where the Taylor and the Padé approach coincide, i.e.,

$$d_L \simeq \frac{1}{H_0} z .$$

This procedure results in a value of $H_0 = 69.96^{+1.12}_{-1.16}\text{ km/(s Mpc)}$, which was used as a prior in one of the fits.

Furthermore, an additional prior on $q_0$ may be imposed assuming that the ΛCDM model is the limiting case of a more general paradigm, using a constant DE term combined with the Planck results. In a universe containing baryonic and dark matter with the density $\Omega_m$ and a cosmological constant with the density $\Omega_{\Lambda} = 1 - \Omega_m$, the Hubble parameter evolves as

$$H = H_0 \sqrt{\Omega_m (1 + z)^3 + 1 - \Omega_m} ,$$

leading to an acceleration parameter of

$$q_0 = -1 + \frac{3}{2} \Omega_m .$$

Hence, by using the result from the Planck mission for the matter density, $\Omega_m = 0.3175$, the prior on the acceleration parameter turns out to be $q_0 = -0.5132$. This numerical value for $q_0$ is used for one of the fits performed, which we can then compare to the results obtained without fixing any parameter a priori, with the ones where a prior on $H_0$ alone is used, and the one where $H_0$ and $q_0$ are both fixed.

As mentioned before, we performed several fits, distinguished by numbers in Tab. I. Without priors, we carried out analyses using the Taylor approach (1), the Padé parametrization (2), and the Padé parametrization using the short redshift range $z \in [0, 0.36]$ (3). Further, we presumed priors from Planck’s results on $H_0$ only (4), on $q_0$ only (5) and on both $H_0$ and $q_0$ (6), as well as a prior on $H_0$ from the first-order fit of the luminosity distance (7), for a total of seven different fits. For each fit, the respective $p$-values have been reported in Tab. I representing the probabilities that the result obtained by a single fit is observed, supposing the null hypothesis to hold. Since it is a qualitative measure for the likelihood of a certain outcome of a fit, it is expected to be as close as possible to unity [35, 36].

All the numerical results for the CS have been reported in Tab. I with the presumption of priors indicated in each column, and the corresponding contour and distribution plots in Figures 1-6.
FIG. 3: (color online) Contour plots and posterior distributions for $H_0$, $q_0$ and $j_0$, for a fit with Padé parametrization and without any priors imposed, for the short redshift range.

FIG. 4: (color online) Contour plots and posterior distributions for $H_0$, $q_0$ and $j_0$, for a fit with Padé parametrization and with a prior on $H_0$ from the Planck results.

The results of the fits are as different as the priors that have been involved in the analysis. The implications from the obtained $p$-values also permit us to conclude that some hypotheses are disfavored strongly in comparison to others. In particular, from the fits (1) and (2), where no priors have been used, we can draw a comparison of the Taylor and the Padé treatments. Using Taylor expansions leads to a slightly lower Hubble parameter than derived from the Padé treatment, as well as a less negative acceleration parameter and a much smaller jerk parameter. Overall, the results from the Taylor treatment resemble much more the ΛCDM predictions, but predict a significantly higher value of the Hubble parameter than ΛCDM. The $p$-values of (1) and (2) are nearly equal, albeit $p$ for fit (2) is slightly higher. The outcomes of the results of fit (3), which was carried out for the restricted sample of redshifts, support the results from the Padé approach in fit (2), but predict a slightly smaller Hubble parameter than (2), whereas the acceleration and jerk parameter are rather close to the values of (2). The $p$-value of (3) is however significantly larger, which can be accounted for by the smaller sample of redshifts used, leading to higher accuracy of the result. The similarity of (2) and (3) indicates however the validity of the Padé approach regardless of the regime of redshifts used in the analysis.

From the results of fits (4) and (5), it seems that the imposition of priors leads to rather disadvantageous results. The predicted values of $q_0$ and $j_0$ in fit (4) do not make much sense, since $q_0$ is barely negative, which indicates an only slowly expanding universe, and $j_0$ is neg-
ative, which is in contradiction with the assumption of changes in the direction of expansion in the cosmological history. Correspondingly, the $p$-value of this fit seems to indicate a very low likelihood for these results. From fit (5), only imposing the Planck prior on $q_0$, the predicted Hubble parameter is again larger than expected from the Planck data analysis, and $j_0$ again negative, with a surprisingly large $p$. Fixing both $H_0$ and $q_0$ finally, the obtained jerk parameter is very large and positive, but at the disadvantage of a very low $p$. However, generally $p$-values below a certain significance level are only taken as an indication for the rejection of the null hypothesis, i.e., the assumption of no connection between the input and the outcome for the fitting parameter, therefore the particularly low $p$ seems to indicate a deeper connection of the three parameters $H_0$, $q_0$ and $j_0$, and not a particularly bad fitting likelihood.

Ultimately, the last fit using the prior on the Hubble parameter obtained from the first-order fit of the luminosity distance seems to be the most successful of the analysis. The Hubble parameter is higher than predicted by ΛCDM, but in close accordance with the results from fits (2) and (3). Furthermore, the results for acceleration and jerk parameter are both reasonable simultaneously, $q_0$ being slightly less negative than the values predicted from (2) and (3), but more negative than the Planck prior; whereas the jerk parameter is positive and nearly identical to $j_0 = 1$ as predicted by the ΛCDM model. The $p$-value of fit (6) is the highest of all, and identical to $p$ from fit (2).

In summary, the results seem to indicate that the current Hubble parameter is higher than the value claimed in the cosmological analyses by the Planck collaboration [15], which can be inferred from the results of the (statistically favorable) fits (2), (3) and (7), as well as from the poor results in fits (4) and (5). The acceleration parameter is comparable to the one predicted by the ΛCDM model, while the jerk parameter is positive, although tendentially larger than the ΛCDM value $j_0 = 1$ [37].

C. Implications for the EoS parameter

The EoS parameter $\omega$ and its first derivative with respect to the redshift $\omega'$ can be directly inferred from the fitting results for the CS via expressions (36) and (37), which were introduced in Section IV.5. At the present time, we can evaluate the two quantities using the different values of $q_0$ and $j_0$ obtained in the numerical analyses by

$$\omega_0 = \frac{2q_0 - 1}{3}, \quad \omega'_0 = \frac{2}{3}(j_0 - q_0 - 2q_0^2).$$

The results can be found in Tab. IV. Disregarding fit (4), which has proven to lead to dubious results, also regarding the EoS, the values of the EoS parameter vary in the expected range of $\omega_0 \in [-0.789, -0.675]$. The results from fits (5) and (6) with the Planck priors on $q_0$ are the least negative, whereas those from fits (2) and (3) have larger absolute values. It seems that in general cosmography predicts a slightly more negative EoS than the one predicted by the ΛCDM model,

$$\omega_0 = -1 + \Omega_m,$$

which, with an overall matter density of $\Omega_m = 0.314 \pm 0.02$ is constrained in the interval $\omega_0 \in [-0.688, -0.684]$. The variation of the EoS parameter given by its first derivative is clearly positive in all cases except fits (4) and (5), which confirms that the universe is in a state of transition between different equations of state, evolving towards a more negative EoS parameter in the future [38]. In this regard, the ΛCDM model predicts

$$\omega'_0 = 3(1 - \Omega_m)\Omega_m,$$

which lies in the range $\omega'_0 \in [0.644, 0.648]$. The results inferred for $\omega'_0$ show that models with negative constant pressure seem to be favored in describing the acceleration of the universe. However, degeneracies between models occur, since besides the concordance ΛCDM there exist others, as e.g. the one proposed in [39], where the dark energy density evolves in time, but has constant negative pressure and a vanishing speed of sound. From the above fitting results, it is not possible to distinguish between those two models. Summarizing, the statistical fitting results are all compatible with the ΛCDM model, however, they do not exclude other models as e.g. ones with a varying DE term and vanishing speed of sound [26, 39, 42].

VI. CONCLUSIONS AND OUTLOOK

In this paper we used a model-independent procedure based on cosmography to fix constraints on cosmological parameters in order to investigate observational data. In particular, in order to fit the CS, i.e. a set of observables characterizing the kinematics of the universe, we introduced a new technique to parameterize the fitting functions in form of Padé expansions, extending conventional treatments with Taylor series used in cosmography.

The Padé approach turns out to be advantageous in the reduction of systematic weaknesses of the Taylor treatment, alleviating the convergence problem associated to the truncation and the range of validity of a Taylor series being limited to the regime $z < 1$. We demonstrated the derivation of Padé expansions from the well-known Taylor expressions, and provided the parametrization of the distance modulus $\mu_D$, as well as the luminosity distance $d_L$ in terms of the CS. Solely from the expression for the luminosity distance in Padé form, it is possible to infer the condition $q_0 > -1$ on the acceleration parameter, which is not a consequence of fitting cosmological data, but a mere theoretical prediction originating from
the definition of the luminosity distance. Further, we defined the convergence radius of the Padé expansion in terms of the CS, to be calculated from the fitting results and compared to the convergence regime of Taylor series.

Fits were carried out using the union 2.1 compilation of the supernova cosmology project, and with fitting functions in Taylor and Padé parametrizations. We adopted a Monte Carlo analysis with Markov chains implementing a Metropolis algorithm. We used different or no priors on the fitting parameters \( H_0 \), \( q_0 \) and \( j_0 \), and obtained the best fit values including the 1σ errors. In general, the results seem to indicate a larger Hubble parameter and a slightly more negative acceleration parameter than the ones found by the Planck mission [15]. Further, a clearly positive jerk parameter has been found, implying a transition in the expansion dynamics of the universe at a finite past redshift.

The imposition of priors from the Planck mission leads to rather conflicting results and statistically unfavorable fitting behaviors, whereas the prior on \( H_0 \) from a first-order expansion of the luminosity distance produced one of the two statistically best fits, the other one being a fit without any priors assumed. Comparing the outcome of fits with Taylor and Padé approaches, the Taylor treatment yields results closer to the Planck predictions, but with a slightly lower \( p \)-value than the corresponding Padé approach. This can be directly interpreted as an improvement of the convergence problem.

From the results for \( H_0 \), \( q_0 \) and \( j_0 \) we further calculated the current values of the EoS parameter \( \omega_0 \) and its first derivative \( \omega_0' \) at the present time. The results allow for the possibility of the EoS of the universe being in a transition between a matter-dominated state and a de Sitter expansion, and indicate a decreasing EoS parameter in the future. Fairly good agreement with the ACtDM predictions has been found, although none of the results permit us to conclude with certainty that the DE density is constant in time. In fact, our results do not exclude a priori a varying DE term with vanishing speed of sound, since our numerical bounds comply with the theoretical ranges predicted by such a model [39–41].

Summarizing, we introduced Padé expansions as a new technique to perform cosmographic fits, yielding improved results with respect to the standard Taylor treatment, in terms of convergence radii and data analyses. Our approach represents a viable alternative to standard methods and shows good statistical fitting behaviors, yielding bounds compatible with Planck’s first results. These investigations will be object of future efforts and more precise analyses, by expanding series to higher orders and refining the Padé expansions by working with even more accurate observational datasets, which would provide further insights on the form of both the EoS of DE and its evolution in time, reconstructing its functional behavior as the universe expands.

| Fit | \( q_0 \) | \( j_0 \) | \( R_{\text{Padé}} \) | \( R_{\text{Taylor}} \) | \( \omega_0 \) | \( \omega_0' \) |
|-----|--------|--------|----------------|----------------|--------|--------|
| fit (1) | \(-0.528^{+0.092}_{-0.088}\) | \(0.506^{+0.489}_{-0.428}\) | \(1.309^{+0.075}_{-0.075}\) | \(0.764^{+0.046}_{-0.044}\) | \(-0.685^{+0.061}_{-0.059}\) | \(0.317^{+0.333}_{-0.293}\) |
| fit (2) | \(-0.683^{+0.084}_{-0.105}\) | \(2.044^{+1.002}_{-0.705}\) | \(1.188^{+0.059}_{-0.074}\) | \(0.842^{+0.042}_{-0.052}\) | \(-0.789^{+0.056}_{-0.072}\) | \(1.196^{+0.675}_{-0.485}\) |
| fit (3) | \(-0.658^{+0.098}_{-0.098}\) | \(2.412^{+1.956}_{-0.978}\) | \(1.206^{+0.071}_{-0.071}\) | \(0.829^{+0.049}_{-0.049}\) | \(-0.772^{+0.065}_{-0.061}\) | \(1.469^{+0.718}_{-0.661}\) |
| fit (4) | \(-0.069^{+0.051}_{-0.055}\) | \(-0.955^{+0.228}_{-0.175}\) | \(1.870^{+0.099}_{-0.096}\) | \(0.535^{+0.025}_{-0.027}\) | \(-0.380^{+0.034}_{-0.036}\) | \(-0.597^{+0.154}_{-0.12}\) |
| fit (5) | \(-0.513\) | \(-0.785^{+0.220}_{-0.208}\) | \(1.322\) | \(0.757\) | \(-0.675\) | \(-0.532^{+0.147}_{-0.138}\) |
| fit (6) | \(-0.513\) | \(2.227^{+0.245}_{-0.237}\) | \(1.322\) | \(0.757\) | \(-0.675\) | \(1.476^{+0.164}_{-0.158}\) |
| fit (7) | \(-0.561^{+0.055}_{-0.042}\) | \(0.999^{+0.346}_{-0.468}\) | \(1.281^{+0.045}_{-0.034}\) | \(0.780^{+0.027}_{-0.021}\) | \(-0.707^{+0.037}_{-0.028}\) | \(0.620^{+0.235}_{-0.314}\) |

Note. \( H_0 \) is given in \( \text{km/s/Mpc} \).
O.L. is grateful to M. Scinta for his support during the course of this research, and wishes to express his grat-
Appendix A: Padé expansion of the distance modulus

The distance modulus formulated in terms of a Padé approximant as a function of the redshift \( z \) reads

\[
\mu_{D, \text{Padé}} = \frac{5}{\ln 10} \left[ \ln z + \frac{D}{\mathcal{E}} \right],
\]

(A1)

where

\[
D = a_0 + a_1 z,
\]

(A2a)

\[
\mathcal{E} = b_0 + b_1 z + b_2 z^2,
\]

(A2b)

and

\[
a_0 = -24 \left[ -6 - 35 \ln 10 - 20 j_0 \ln 10 + q_0^2 (-6 + 45 \ln 10) + 2 q_0 (6 + 25 \ln 10) \right.
\]

\[
+ (9 q_0^2 + 10 q_0 - 4 j_0 - 7) \ln \left( \frac{dH}{1\text{Mpc}} \right) \left[ 5 \ln 10 + \ln \left( \frac{dH}{1\text{Mpc}} \right) \right],
\]

(A3a)

\[
a_1 = 24 (q_0 - 1) \left[ -3 - 35 \ln 10 - 20 j_0 \ln 10 + q_0^2 (-3 + 45 \ln 10) + q_0 (6 + 50 \ln 10) \right.
\]

\[
+ (9 q_0^2 + 10 q_0 - 4 j_0 - 7) \ln \left( \frac{dH}{1\text{Mpc}} \right) \left[ 5 \ln 10 + \ln \left( \frac{dH}{1\text{Mpc}} \right) \right],
\]

(A3b)

\[
b_0 = 24 (4 j_0 - 9 q_0^2 - 10 q_0 + 7) \ln \left( \frac{dH}{1\text{Mpc}} \right) + 480 j_0 \ln 10 + 144 q_0^2 - 1080 q_0^2 \ln 10
\]

\[
- 288 q_0 - 1200 q_0 \ln 10 + 144 + 840 \ln 10,
\]

(A3c)

\[
b_1 = -12 (4 j_0 + 17) q_0 + 48 j_0 + 108 q_0^3 + 12 q_0^2 + 84,
\]

(A3d)

\[
b_2 = 16 j_0^2 - 2 (36 j_0 + 13) q_0^2 - 20 (4 j_0 + 7) q_0 + 56 j_0 + 81 q_0^4 + 180 q_0^3 + 49.
\]

(A3e)