Main result
The following well-known result is my starting point.

Theorem 1 (Bogomolov–Sommese vanishing, see [1]). Let $X$ be a complex projective manifold and $D \subset X$ a divisor with simple normal crossings. For any invertible subsheaf $\mathcal{L} \subset \Omega^p_X(\log D)$, we have $\kappa(\mathcal{L}) \leq p$, where $\kappa(\mathcal{L})$ denotes the Kodaira–Litaka dimension of $\mathcal{L}$.

Building on the Extension Theorem of Greb–Kebekus–Kovács–Peternell [2], I generalized this to the setting of reflexive differential forms on log canonical pairs as follows.

Theorem 2 (Bogomolov–Sommese vanishing on lc $C$-pairs). Let $(X, D')$ be a complex projective log canonical pair, and let $D \subset D'$ be a divisor such that $(X, D)$ is a $C$-pair. If $\mathcal{A} \subset \text{Sym}^k \Omega^p_X(\log D)$ is a Weil divisorial subsheaf, then $\kappa(\mathcal{A}) \leq p$.

A $C$-pair is a pair $(X, D)$ where all the coefficients of $D$ are of the form $1 - 1/n$ for $n \in \mathbb{N} \cup \{\infty\}$. This notion was introduced by Campana under the name orbifolds géométriques. The $C$-Kodaira dimension $\kappa_C$ of a Weil divisorial sheaf of differential forms on $(X, D)$ is a natural generalization of the Kodaira dimension of a line bundle, which takes into account the fractional part of $D$.

Adjunction on dlt $C$-pairs
In the course of the proof, I showed that on dlt $C$-pairs, there is a version of the adjunction formula as well as a residue map for symmetric differential forms, and that these two are compatible with each other in the following sense.

Theorem 3 (Residues of symmetric differentials). Let $(X, D)$ be a dlt $C$-pair and $D_0 \subset [D]$ a component of the reduced boundary. Set $D_0 := \text{Diff}_{D_0}(D - D_0)$, such that $(K_X + D)|_{D_0} = K_{D_0} + D_0$. Then the pair $(D_0, D_0)$ is also a dlt $C$-pair, and for any integer $p \geq 1$, there is a map
\[ \text{res}^p_{D_0} : \text{Sym}^p \Omega^p_X(\log D) \to \text{Sym}^p \Omega^p_{D_0}(\log D_0) \]
which on the snc locus of $(X, [D])$ coincides with the $k$-th symmetric power of the usual residue map for snc pairs.

Corollary: A Kodaira–Akizuki–Nakano-type vanishing result

Corollary 4 (KAN-type vanishing). Let $(X, D)$ be a complex projective log canonical pair of dimension $n$, $\mathcal{A}$ a Weil divisorial sheaf on $X$. Then
\[ H^n(X, \Omega^p_X(\log D) \otimes \mathcal{A}^*) = 0 \quad \text{and} \quad H^n(X, \Omega^p_X(\log D) \otimes \mathcal{A}) = 0 \]
for $p \geq n - \kappa(\mathcal{A}) + 1$.

Idea of proof of Theorem 2
The basic idea is to pull back the sheaf $\mathcal{A}$ to a log resolution $(\tilde{X}, \tilde{D})$ of $(X, D)$ and apply Theorem 1. By the Extension Theorem of [2], this should be possible. However, since pulling back is not functorial for Weil divisorial sheaves, the Kodaira dimension of $\mathcal{A}$ might drop in this process. Therefore we enlarge the pulled back sheaf by taking its saturation $\mathcal{B}$ in $\Omega^p_{\tilde{X}}(\log \tilde{D})$. We prove that sections of $\mathcal{A}^{[k]}$ extend to sections of $\mathcal{B}^{[k]}$.

A major issue is that we cannot really work on a log resolution, because it extracts too many divisors. Therefore we pass to a minimal dlt model $(Z, D_Z)$ of $(X, D)$. This is possible by the minimal model program as proved by BCHM. However, $(Z, D_Z)$ is not an snc pair, which makes the proof rather involved. In particular, we have to use Theorem 3.

Sharpness of Theorem 2
Theorem 2 fails if one replaces log canonical by Du Bois singularities. A counterexample can be obtained as follows. Catanese constructed smooth projective surfaces $S$ such that $K_S$ is ample, but the Hodge numbers $h^{0,1}(S)$ and $h^{0,2}(S)$ are zero. Let $X$ be the cone over such an $S$ with respect to a sufficiently high pluricanonical embedding. Then $X$ even has rational singularities, but the pullback of $\omega_S$ to $X$ is a $\mathbb{Q}$-ample subsheaf of $\Omega^2_X$.

References
[1] Fedor A. Bogomolov, Holomorphic tensors and vector bundles on projective varieties, Math. USSR Izvestija 13 (1979), 499–555.
[2] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, Differential forms on log canonical spaces, Publications Mathématiques de L’IHÉS 114 (2011), 1–83.