The Semiring Properties of Boolean Propositional Algebras

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Abstract

This paper illustrates the relationship between boolean propositional algebra and semirings, presenting some results of partial ordering on boolean propositional algebras, and the necessary conditions to represent a boolean propositional subalgebra as equivalent to a corresponding boolean propositional algebra. It is also shown that the images of a homomorphic function on a boolean propositional algebra have the relationship of boolean propositional algebra and its subalgebra. The necessary and sufficient conditions for that homomorphic function to be onto-order preserving, and also an extension of boolean propositional algebra, are explored.

Keywords: propositions, algebra, boolean algebra, semirings

1 Introduction

The English mathematician George Boole (1815–1864) sought to give symbolic form to Aristotle’s system of logic. Boole wrote a treatise on the subject in 1854, titled An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities, which codified several rules of relationship between mathematical quantities limited to one of two possible values: true or false, 1 or 0. His mathematical system became known as Boolean algebra.

Bourne [2] has discussed the homomorphism theorems for semirings. Allen [1] has discussed the extension of a theorem of Hilbert to semirings. Zeleznikow [12] has discussed the natural partial order on semirings. This paper illustrates the general idea of interpreting properties of boolean propositional algebras, including partial ordering, homomorphism, isomorphism and differences, when they are taken as semirings.

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Let \( p, q, r \) be propositions, and \( \mathcal{P} \) be the set of all propositions in the universe of discourse. A *Boolean Propositional Algebra (BPA)* \( \mathcal{B} \) is a six-tuple consisting of a set \( \mathcal{P} \), equipped with two binary operations \( \land \) (called ‘meet’ or ‘and’) and \( \lor \) (called ‘join’ or ‘or’), a unary operation \( \neg \) (called ‘complement’ or ‘not’) and two elements 0 and 1 and it is denoted by \( (\mathcal{P}, \land, \lor, \neg, 0, 1) \). If we use infix operators like \( \lt \) or \( \ltimes \) to compare propositions, we may write \( <_{\mathcal{B}} \) and \( \ltimes_{\mathcal{B}} \) to clarify which BPA is taken as the scope.

If there is some subset \( \hat{\mathcal{B}} \) of a boolean propositional algebra \( \mathcal{B} \) that is a BPA in its own right, then it may be called a *Boolean Propositional Subalgebra (BPSA)* \( \hat{\mathcal{B}} \) If we use infix operators like \( \lt \) or \( \ltimes \), we may write \( <_{\hat{\mathcal{B}}} \) and \( \ltimes_{\hat{\mathcal{B}}} \) to indicate the corresponding scope.

## 2 Boolean Propositional Algebra of Monoids and Semirings

It is possible to define an arithmetic on propositional logic in the obvious way: take + to mean \( \land \), and \( \times \) to mean \( \lor \).

**Proposition 2.1.** Given propositions \( p, q, \) and \( r \), we have the following.

(i) \( + \) and \( \times \) are commutative and associative: \( p \times q = q \times p; p + r = r + p; \)
\( p + (q + r) = (p + q) + r; \) and \( p \times (q \times r) = (p \times q) \times r. \)

(ii) \( \times \) distributes over \( +: \) \( p \times (q + r) = (p \times q) + (p \times r). \)

Given this propositional arithmetic, we can posit the existence of two identity operators, one each for + and \( \times \).

**Definition 2.2.** The multiplicative and additive identities are defined as follows.

(i) The additive identity \( \top \) is the proposition such that for any proposition \( p, p + \top = \top + p = p. \)

(ii) The multiplicative identity \( \bot \) is the proposition such that for any proposition \( p, p \times \bot = \bot \times p = p. \)

By the commutativity of the \( + \) and \( \times \) operators, we observe that the identity elements are two-sided.

Informally, we may describe these elements as follows:

(i) The additive identity \( \top \) is a proposition “that is always true.” The direct sum of such a proposition and \( p \) is obviously \( p \) itself.

(ii) The multiplicative identity \( \bot \) is a proposition “that is always false.” The direct product of such a proposition and \( p \) is likewise \( p \) itself.
Then \( \mathcal{P} \), combined with the + operator, is a monoid (a set with an associative operator and a two-sided identity element) \([7]\). Similarly, \( \mathcal{P} \) is also a monoid when considering the \( \times \) operator. For notational convenience, we denote these monoids as \((\mathcal{P}, +)\) and \((\mathcal{P}, \times)\).

It is further clear that the set \((\mathcal{P}, +, \times)\) is a semiring when taken with the operations + and \( \times \) because the following conditions \([4]\) for being a semiring are satisfied:

(i) \((\mathcal{P}, +)\) is a commutative monoid with identity element \( \top \);
(ii) \((\mathcal{P}, \times)\) is a monoid with identity element \( \bot \);
(iii) \( \times \) distributes over + from either side;
(iv) \( \top \times p = \top = p \times \top \) for all \( p \in \mathcal{P} \).

This proposition semiring will be denoted by \((\mathcal{P}, +, \times)\), and its properties are as indicated in the following.

**Remark 2.3.** The semiring \((\mathcal{P}, +, \times)\) is zerosumfree, because \( p + q = \top \) implies, for all \( p, q \in \mathcal{P} \), that \( p = q = \top \).

This property shows \([4]\) that the monoid \((\mathcal{P}, +)\) is completely removed from being a group, because no non-trivial element in it has an inverse.

A zerosumfree semiring is also called an antiring \([9]\), which is thus another term that can be used to describe \((\mathcal{P}, +, \times)\).

**Remark 2.4.** \((\mathcal{P}, +, \times)\) is entire, because there are no non-zero elements \( p, q \in \mathcal{P} \) such that \( p \times q = \top \).

This likewise shows that the monoid \((\mathcal{P}, \times)\) is completely removed from being a group, as there is no non-trivial multiplicative inverse.

**Remark 2.5.** \((\mathcal{P}, +, \times)\) is simple, because \( \bot \) is infinite, i.e., \( p + \bot = \bot, \forall p \in \mathcal{P} \).

We may state another important definition \([4]\) about semirings, and observe a property of \((\mathcal{P}, +, \times)\).

**Definition 2.6.** The center \( C(\mathcal{P}) \) of \( \mathcal{P} \) is the set \( \{ p \in \mathcal{P} \mid p \times q = q \times p, \forall q \in \mathcal{P} \} \).

**Remark 2.7.** The semiring \((\mathcal{P}, +, \times)\) is commutative because \( C(\mathcal{P}) = \mathcal{P} \).
3 Partial Ordering on a Boolean Propositional Algebra

Consider a partial ordering relation $\preceq$ on $\mathcal{P}$. Informally, $p \preceq q$ means that $p$ has a lower measure of some metric than $q$ (e.g., $p$ is less likely to be true than $q$, or is a weaker proposition than $q$).

Formally, $\preceq$ is a partial ordering on the semiring $(\mathcal{P}, +, \times)$ where the following conditions are satisfied [5].

**Definition 3.1.** If $(\mathcal{P}, +, \times)$ is a semiring and $(\mathcal{P}, \preceq)$ is a poset, then $(\mathcal{P}, +, \times, \preceq)$ is a *partially ordered semiring* if the following conditions are satisfied for all $p, q,$ and $r$ in $\mathcal{P}$.

(i) The *monotony law of addition*:

$$p \preceq q \rightarrow p + r \preceq q + r$$

(ii) The *monotony law of multiplication*:

$$p \preceq q \rightarrow p \times r \preceq q \times r.$$

It is assumed that $\top \preceq p, \forall p \in \mathcal{P}$, and that $p \preceq \bot$.

A semiring with a partial order defined on it is denoted as $(\mathcal{P}, +, \times, \preceq)$. Given Definition 3.1, it is instructive to consider the behavior of the partial order under composition. We begin with a couple of simple results.

**Lemma 3.2.** Given a partially-ordered semiring $(\mathcal{P}, +, \times, \preceq), \forall p, q \in \mathcal{P}$:

(i) $p \preceq p + q$, and

(ii) $p \times q \preceq q$.

**Proof.** For (i), consider that $\top \preceq q$. Using the monotony law of addition, we get $\top + p \preceq q + p$. Considering that $\top$ is the additive identity element and that addition is commutative, we get $p \preceq p + q$.

For (ii), consider that $q \preceq \bot$. Using the monotony law of multiplication, we get $q \times p \preceq \bot \times p$. Considering that $\bot$ is the multiplicative identity element and that multiplication is commutative, we get $p \times q \preceq p$. \qed

Given these, we can state the following result on $(\mathcal{P}, +, \times, \preceq)$.

**Theorem 3.3.** Given $p, q, r \in \mathcal{P}$,

(i) if $p + q \preceq r$, then $p \preceq r$ and $q \preceq r$; and

(ii) if $p \preceq q \times r$, then $p \preceq q$ and $p \preceq r$. 

4
Proof. For part (i): The proof is by contradiction. Assume the contrary. Then \( p + q \preceq r \), and at least one of \( p \preceq r \) or \( q \preceq r \) is false.

Without loss of generality, assume that \( r \preceq p \). Using the monotony law of addition and the commutativity of the + operator, \( q + r \preceq p + q \).

Now, by Lemma 3.2 (i), \( r \preceq q + r \). Given the transitivity of \( \preceq \), we get \( r \preceq p + q \), which is a contradiction.

For part (ii): The proof is again by contradiction. Assume the contrary. Then \( p \preceq q \times r \) and at least one of \( p \preceq q \) and \( p \preceq r \) is false.

Without loss of generality, assume that \( q \preceq p \). Using the monotony law of multiplication and the commutativity of the \( \times \) operator, we get \( q \times r \preceq p \times r \).

Now, by Lemma 3.2 (ii), \( p \times r \preceq p \). Given the transitivity of \( \preceq \), we get \( q \times r \preceq p \), which is a contradiction.

The following result is similar.

Theorem 3.4. Given \( p, q, r, s \in \mathcal{P} \), if \( p \preceq q \) and \( r \preceq s \), then,

(i) \( p + r \preceq q + s \), and

(ii) \( p \times r \preceq q \times s \).

Proof. These results can be proven directly. Only (i) is proved, the proof of (ii) being very similar.

We know the following:

\[ p \preceq q \quad (1) \]

and:

\[ r \preceq s \quad (2) \]

From (1) and the monotony law of addition (considering the direct sum of \( s \) and both sides), we have:

\[ p + s \preceq q + s \quad (3) \]

Similarly, from (2) and the monotony law (considering the direct sum of \( p \) and both sides), we have:

\[ p + r \preceq p + s \quad (4) \]

By considering transitivity in respect of (1) and (3), we get \( p + r \preceq q + s \). \[ \square \]

Remark 3.5. The positive cone \( \hat{\mathcal{P}} \) of \((\mathcal{P}, +, \preceq)\), which is the set of elements \( p \in \mathcal{P} \) for which \( p \preceq p + q, \forall q \in \mathcal{P} \), is the set \( \mathcal{P} \) itself. The negative cone is empty.
This is a direct consequence of Lemma \[3.2\] (i), and it also follows that the set of elements \( \{ p \mid p + q \preceq p \} = \emptyset \), showing that the negative cone is empty.

The analogous property of \( \mathcal{P} \) in consideration of the \( \times \) operator can also be noted.

**Definition 3.6.** (i) A semiring \((\mathcal{P}, +, \times)\) is called **additively cancellative** if \((\mathcal{P}, +)\) is cancellative, i.e., if \( a + x = a + y \) implies \( x = y \) for all \( a, x, y \in \mathcal{P} \).

(ii) Let \((\mathcal{P}, +, \times)\) be a semiring with a zero \( \top \). Then \( \top \) is called **multiplicatively absorbing** if \( \top \) is absorbing in \((\mathcal{P}, \times)\), i.e., if \( \top a = a \top = \top \) holds for all \( a \in \mathcal{P} \).

We have the following result.

**Theorem 3.7.** Let \( \mathcal{B} = (\mathcal{P}, +, \times, \preceq_B) \) be a partially ordered BPA, and \( \hat{\mathcal{B}} = (\hat{\mathcal{P}}, +, \times) \) where \( \hat{\mathcal{P}} \subseteq \mathcal{P} \), is a BPSA of \((\mathcal{P}, +, \times)\). Then,

(i) if \((\mathcal{P}, +, \times)\) contains an additively cancellable element, but not \( \top \), and

(ii) \( \hat{\mathcal{B}} \cup \{ \top \} = \mathcal{B} \) if \((\mathcal{P}, +, \times)\) has a \( \top \),

then \( \hat{\mathcal{B}} = \mathcal{B} \).

**Proof.** First we show that \( \hat{\mathcal{B}} \) is partially ordered.

(a) \( p \preceq p \) (reflexivity)

\( p \preceq p \) always holds good, for some \( p \in \hat{\mathcal{B}} \).

(b) if \( p \preceq q \) and \( q \preceq p \) then \( p = q \) (antisymmetry).

Since \( p \preceq q \), therefore by Lemma \[3.2\] (ii), \( p+q = p \). Similarly \( q+p = q \).

Hence \( p = q \) for some \( p, q \in \hat{\mathcal{B}} \).

(c) if \( p \preceq q \) and \( q \preceq r \) then \( p \preceq r \) (transitivity) for some \( p, q, r \in \hat{\mathcal{B}} \).

By associative property of proposition and by Theorem \[3.4\], it is transitive.

Hence, \((\hat{\mathcal{P}}, +, \times, \preceq_{\hat{\mathcal{B}}})\) is partially ordered BPSA. So, by Lemma \[3.2\] we can say that \( p \preceq q \) implies \( p + x = y \) for some \( x \in \hat{\mathcal{B}} \), which satisfies the monotony law of addition. Also, if \( p < q \) implies \( p \times r \preceq q \times r \) and \( r \times p \preceq r \times q \) for all \( p, q \in \mathcal{B} \) and \( r \in \hat{\mathcal{B}} \) which in turn satisfies monotony law of multiplication. Therefore, we can say that \( \hat{\mathcal{B}} \subseteq \mathcal{B} \) which means \( \preceq_{\mathcal{B}} \) and \( \preceq_{\hat{\mathcal{B}}} \) similar.

If \((\mathcal{P}, +, \times)\) has a cancellable element, say \( \top \), then \( \mathcal{B} \cap \hat{\mathcal{B}} \) is either empty or contains a single element, which has to be the \( \top \) of \((\mathcal{P}, +, \times)\). Also \( \mathcal{B} \cap \hat{\mathcal{B}} = \phi \) and therefore \( \hat{\mathcal{B}} = \mathcal{B} \).

We have \( \top \) as a cancellable element of \((\mathcal{P}, +, \times)\) and obtain \( \mathcal{B} \cap \hat{\mathcal{B}} \subseteq \{ \top \} \), i.e., \( \hat{\mathcal{B}} \cup \{ \top \} \). \( \square \)
4 Homomorphism and Isomorphism on Boolean Propositional Algebra

Let $Q = (P, +, \times)$ and $R = (P, \oplus, \otimes)$ be two BPA s, then a mapping $\psi : Q \rightarrow R$ of $Q$ into $R$ is called a homomorphism of $(P, +, \times)$ into $(P, \oplus, \otimes)$ if,

(i) $\psi(a + b) = \psi(a) + \psi(b)$ and

(ii) $\psi(a \times b) = \psi(a) \times \psi(b)$ are satisfied for all $a, b \in P$.

In other words we can say that $\psi$ is:

(i) Order preserving: For each $x, y \in P$, if $x \preceq y$, then $\psi(x) \preceq \psi(y)$

(ii) Operator preserving: For some operator $o$ and each $x_1, \ldots, x_n \in P$, $\psi(o(x_1, \ldots, x_n)) = o(\psi(x_1), \ldots, f(x_n))$

(iii) Each mapping $\psi$ from $(P, +, \times)$ into $(T, \oplus, \otimes)$ determines an equivalence relation $\tau$ on $P$ by $\tau = \psi^{-1} \circ \psi$, which may also be expressed by $a \tau a \equiv \psi(a) = \psi(\hat{a})$ for all $a, \hat{a} \in P$.

**Definition 4.1.** (i) An isomorphism $\psi : Q \rightarrow R$ is a homomorphism such that the inverse map $\psi^{-1} : R \rightarrow Q$ - given by setting $\psi^{-1}(y) = x$ where $\psi(x) = y$ is a homomorphism. Two BPA s are isomorphic if and only if there is an isomorphism from one to the other.

(ii) Let $(P, +, \times)$ be BPA and $\psi : (P, +, \times) \rightarrow (P, \oplus, \otimes)$ is a homomorphism. Then $(\psi(P), +, \times)$ is again a BPA.

Based on this, we have the following.

**Theorem 4.2.** Let $(P, +, \times), (P_1, +, \times), (P_2, +, \times)$ be BPA s and there exists two homomorphic functions such that, $\psi_1 : P \rightarrow P_1$ and $\psi_2 : P \rightarrow P_2$, then the homomorphic function $\psi : P_1 \rightarrow P_2$ establish the relation of BPA and BPSA between $P_1, P_2$.

**Proof.** The proof is by contradiction. Assume to the contrary that homomorphism $\psi$ exists. Since $\psi_1$ is surjective, for $a_1 \in P_1$, there is some $a \in P$ which satisfies $\psi_1(a) = a_1$. Also we know that $\psi \circ \psi_1 = \psi_2$ by Definition 4.1(ii), we can show that $\psi(a_1) = \psi(\psi_1(a)) = \psi_2(a)$. Hence,

$$\psi(a_1) = \psi_2(a) \forall a \in P \text{ such that } \psi_1(a) = a_1.$$  \hspace{1cm} (5)

This shows that $\psi_1(a) = \psi_1(\hat{a}) \implies \psi_2(a) = \psi_2(\hat{a})$ where $a, \hat{a} \in P$.

Therefore, $\tau_1 \subseteq \tau_2$

We have to show $\psi$ is surjective iff $\psi_2$ is surjective.

To prove by contradiction, assume that on the contrary we have $\tau_1 \subseteq \tau_2$.

For $\psi$ to be surjective, necessary conditions are:
(i) \(\psi(a_1 + b_1) = \psi(a_1) + \psi(b_1)\) and

(ii) \(\psi(\top) = \top\)

(iii) \(\psi(\bot) = \bot\) are satisfied for all \(a_1, b_1 \in \mathcal{P}\)

By (5), we have \(\psi_1(a) = a_1 = \psi_1(\hat{a}) \implies a\tau_1\hat{a} \implies a\tau_2\hat{a}\) This in return shows that \(\psi_2(a) = \psi_2(\hat{a})\)

Hence, \(\psi\) defines the mapping of \(\mathcal{S}_1\) into \(\mathcal{S}_2\) and \(\psi_2(a) = \psi(\psi_1(a)) \implies \psi \circ \psi_1 = \psi_2\)

Also, \(\psi(\top + a) = \psi(\top) + \psi(a)\) for some \(a \in \mathcal{P}\) Since, \(\psi(\top) \preceq \psi(a)\). □

**Theorem 4.3.** Let \((\mathcal{P}, +, \times)\) and \((\mathcal{P}_1, +, \times)\) are two BPA\(s\) and there exist a homomorphic function \(\psi : \mathcal{P} \to \mathcal{P}_1\). The function \(\psi\) is an onto order preserving iff it is an isomorphism.

**Proof.** If \(\psi\) is an onto order preserving, we first need to show that \(\psi^{-1}\) is well defined. Since \(\psi\) is onto, for every \(y\) there is at least one \(x \in \mathcal{P}\) where \(\psi(x) = y\). Since \(\psi\) is an order preserving, it is one to one, so there is not more than one \(x\) where \(\psi(x) = y\). Hence by definition of \(\psi^{-1}\) we can show that \(\psi(\psi^{-1}(y)) = y\) and \(\psi^{-1}(\psi(x)) = x\) for each \(y \in \mathcal{P}_1\) and \(x \in \mathcal{P}\).

Now \(\psi^{-1}\) is order preserving since if \(\psi^{-1}(y) \preceq \psi^{-1}(\hat{y})\), we have \(\psi(\psi^{-1}(y)) \preceq \psi(\psi^{-1}(\hat{y}))\) implies \(y \preceq \hat{y}\). Contraposing this, we have \(y \preceq \hat{y}\) only if \(\psi^{-1}(y) \preceq \psi^{-1}(\hat{y})\).

Similarly, \(\psi^{-1}\) preserve operators. For any \(n\) place operator \(\tau\), \(\psi^{-1}(\tau(y_1, \ldots, y_n)) \preceq \tau(\psi^{-1}(y_1), \ldots, \psi^{-1}(y_n))\), then since \(\psi\) is one to one, we have \(\psi(\psi^{-1}(\tau(y_1, \ldots, y_n))) \preceq \tau(\psi(\psi^{-1}(y_1)), \ldots, \psi(\psi^{-1}(y_n)))\).

But, we have \(\psi(\psi^{-1}(y)) = y\) for all \(y\), we get \(\tau(y_1, \ldots, y_n) \preceq \tau(y_1, \ldots, y_n)\), which is contradiction Therefore, \(\psi^{-1}\) must preserve operators, so \(\psi\) is an isomorphism.

Conversely, if \(\psi\) is an isomorphism, then we have to show that it is an onto order preserving. Primarily, If \(y \in \mathcal{P}_1\), we have \(\psi(\psi^{-1}(y)) = y\), and hence \(\psi\) is onto. Secondly, if \(\psi(x) \preceq \psi(y)\), then \(\psi^{-1}(\psi(x)) \preceq \psi^{-1}(\psi(y))\), will give \(x \preceq y\), and hence, \(\psi\) is an order preserving. □

**5 Differences in Boolean Propositional Algebra**

Given any BPA \(\mathcal{B} = (\mathcal{P}, +, \times, \preceq)\), its *ideal BPA* \([5]\) \([8]\) is \(\text{Ideal}(\mathcal{B})\), ordered under \(\subseteq\) and with operators \((+, \times)\).

Let \(\mathcal{B} = (\mathcal{P}, +, \times, \preceq)\) be BPA and (subtrahends \(\ominus\)) \([10]\) be BPSA of \(\mathcal{B}\) whose elements are cancellable in \((\mathcal{P}, +, \times, \preceq)\). We further assume the existence of a zero \(\top\) and contains an opposite element \(\neg\alpha\) for each \(\alpha \in \ominus\). Then without restriction of generality, \(\ominus\) can be chosen as \(\text{Ideal}(\mathcal{B})\) of \((\mathcal{P}, +, \times)\). For convinience, we can write BPA of differences w.r.t \(\mathcal{P}\) as \(\text{D}(\mathcal{P}, \ominus)\)
Definition 5.1. Let \((\mathcal{P}, +, \times, \preceq)\) be a BPA and \((\mathcal{P}, \oplus, \otimes) = D(\mathcal{P}, \ominus)\) a BPA of differences of \((\mathcal{P}, +, \times)\), then \(p \preceq q\) implies there exists some \(\Delta \in \ominus\) and \(p, q \in \mathcal{P}\) such that

\[
p + \Delta \preceq q + \Delta
\]

defines the smallest extension of \(\preceq\) within \(\mathcal{P}\). For convinience we can denote this smallest extension as \(\preceq'\). Further \((\mathcal{P}, +, \times, \preceq')\) is a partial order BPA with the property

\[
p \preceq' q \iff p + \xi \preceq' q + \xi \text{ for all } p, q \in \mathcal{P} \text{ and } \xi \in \ominus
\]

Moreover, \(\preceq'\) and \(\preceq\) are similar iff \((\mathcal{P}, +, \times, \preceq)\) itself satisfies

\[
p \preceq q \iff p + \xi \preceq q + \xi \text{ for all } p, q \in \mathcal{P} \text{ and } \xi \in \ominus
\]

We have the following result.

Theorem 5.2. Let \((\mathcal{P}, \oplus, \otimes) = D(\mathcal{P}, \ominus)\) be BPA of differences of BPA \((\mathcal{P}, +, \times)\) with respect to the ideal of subtrahends \(\ominus\), then if \((\mathcal{P}, +, \times)\) is multiplicatively left cancellative, the necessary and sufficient condition for \(D(\mathcal{P}, \ominus)\) to be multiplicatively left-cancellative is

\[
\Delta \neq c \text{ and } a \neq b \implies c \times a + \Delta b \neq c \times b + \Delta a
\]

for all \(a, b, c \in \mathcal{P}\) and \(\Delta \in \ominus\)

Proof. Let \((\mathcal{P}, \oplus, \otimes)\) be multiplicatively left cancellative. Then \(\Delta \neq c\) and \(a \neq b\) imply \((c - \Delta)a \neq (c - \Delta)b\) and thus \(c \times a + \Delta b \neq c \times b + \Delta a\), which proves (9). For the converse we assume \((c - \Delta)(a - \alpha) = (c - \Delta)(b - \alpha)\) for arbitrary elements \(a - \alpha, b - \alpha\) and \(c - \Delta \neq \top\) of \(\mathcal{P}\). This yields \(c \times a + \Delta a + \alpha b = c \times a + \Delta a + c \times b + \Delta a\). Since \(\Delta a \in \ominus\) and \(\alpha b \in \ominus\) are cancellable in \((\mathcal{P}, +)\), we obtain \(c \times a + \Delta b = c \times b + \Delta a\). This and \(c \neq \Delta\) imply \(a = b\) by (9) and hence \(a - \alpha = b - \alpha\). Therefore (9) implies that \((\mathcal{P}, \oplus, \otimes) = D(\mathcal{P}, \ominus)\) is multiplicatively left cancellative, which yields the same for \((\mathcal{P}, +, \times)\) and completes the proof.

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