CONTINUOUS APPROXIMATION OF QUASIPLURISUBHARMONIC FUNCTIONS

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Abstract. Let $X$ be a compact Kähler manifold and $\theta$ a smooth closed $(1,1)$-real form representing a big cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$. The purpose of this note is to show, using pluripotential and viscosity techniques, that any $\theta$-plurisubharmonic function $\varphi$ can be approximated from above by a decreasing sequence of continuous $\theta$-plurisubharmonic functions with minimal singularities, assuming that there exists a single such function.

Dedicated to D.H.Phong on the occasion of his 60th birthday

1. Introduction

Let $X$ be a compact Kähler manifold and $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class. Recall that a cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ is big if it contains a Kähler current, i.e. a positive closed current which dominates a Kähler form.

Fix $\theta$ a smooth closed real $(1,1)$-form representing $\alpha$. We denote by $\text{PSH}(X, \theta)$ the set of all $\theta$-plurisubharmonic functions, i.e. those functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ which can be written locally as the sum of a smooth and a plurisubharmonic function and such that the current $\theta + dd^c \varphi$ is a closed positive current, i.e.: $\theta + dd^c \varphi \geq 0$ in the sense of currents. It follows from the $\partial \bar{\partial}$-lemma that any closed positive current $T$ in $\alpha$ can be written as $T = \theta + dd^c \varphi$ for some $\varphi \in \text{PSH}(X, \theta)$. We use the standard normalization $d = \partial + \bar{\partial}$, $d^c := \frac{1}{2i\pi} (\partial - \bar{\partial})$ so that $dd^c = \frac{i}{\pi} \partial \bar{\partial}$.

In general Kähler currents are too singular, so one usually prefers to work with positive currents in $\alpha$ having minimal singularities. A positive current $T = \theta + dd^c \varphi \in \alpha$ (resp. a $\theta$-plurisubharmonic function $\varphi$) has minimal singularities if for every other positive current $S = \theta + dd^c \psi \in \alpha$, there exists $C \in \mathbb{R}$ such that $\psi \leq \varphi + C$ on $X$. The function $V_\theta := \sup \{v | v \in \text{PSH}(X, \theta) \text{ and } \sup_X v \leq 0\}$

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is an example of $\theta$-psh function with minimal singularities. It satisfies $\sup_X V_\theta = 0$. We let
\[ P(\alpha) := \{ x \in X \mid V_\theta(x) = -\infty \} \quad \text{and} \quad NB(\alpha) := \{ x \in X \mid V_\theta \notin L^\infty_{\text{loc}}(\{x\}) \} \]
denote respectively the polar locus and the non bounded locus of $\alpha$. The definitions clearly do not depend on the choice of $\theta$ and coincide with the polar (resp. non bounded locus) of any $\theta$-psh function with minimal singularities.

The purpose of this note is to show that if a big cohomology class contains one current having minimal singularities and exponentially continuous potentials, then there is actually plenty of such currents:

**Theorem.** Let $X$ be a compact Kähler manifold and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class such that the polar locus $P(\alpha)$ coincides with the non-bounded locus $NB(\alpha)$.

Fix $\theta \in \alpha$ a smooth representative and $T = \theta + dd^c \varphi$ a positive current in $\alpha$, where $\varphi \in \operatorname{PSH}(X, \theta)$. Then there exists $\varphi_j \in \operatorname{PSH}(X, \theta)$ a sequence of exponentially continuous $\theta$-plurisubharmonic functions which have minimal singularities and decrease towards $\varphi$.

We say here that a $\theta$-psh function is exponentially continuous iff $e^\varphi : X \to \mathbb{R}$ is continuous. Observe that if there exists one exponentially continuous $\theta$-psh function with minimal singularities, then $P(\alpha) = NB(\alpha)$.

The technical condition $P(\alpha) = NB(\alpha)$ is thus necessary. It is obviously satisfied when $\alpha$ is semi-positive, or even bounded (i.e. there exists a positive closed current in $\alpha$ with bounded potentials, a condition that has become important in complex dynamics recently, see [DG09]), since $P(\alpha) = NB(\alpha) = \emptyset$ in this case. A more subtle example of a big and nef class $\alpha$ with $P(\alpha) = NB(\alpha) \neq \emptyset$ has been given in [BEGZ10, Example 5.4].

It is easy to construct $\theta$-plurisubharmonic functions $\psi$ with $P(\psi) \subseteq NB(\psi)$, however we do not know of a single example of a big class $\alpha$ for which $P(\alpha)$ is strictly smaller than $NB(\alpha)$.

Despite the relative modesty of its conclusion, this result relies on three important tools:
- the regularization techniques of Demailly as used in [BD12],
- the resolution of degenerate complex Monge-Ampère equations in big cohomology classes, as developed in [BEGZ10],
- and the viscosity approach to complex Monge-Ampère equations [EGZ11].

The latter was developed in the case where $\alpha$ is both big and semi-positive, hence we need here to extend this technique so as to cover the general setting of big classes. We stress that the global viscosity comparison principle holds for a big cohomology class $\alpha$ if and only if $P(\alpha) = NB(\alpha)$ (Theorem 3.3).

Our approximation result is new even when $\alpha$ is both big and semi-positive. Let us stress that continuous $\theta$-plurisubharmonic functions are
easy to regularize by using Richberg’s technique \cite{R68}. As a consequence we obtain the following:

**Corollary.** Let \((V, \omega_V)\) be a compact normal Kähler space and let \(\varphi\) be a \(\omega_V\)-plurisubharmonic function on \(V\). Then there exists a sequence \((\varphi_j)\) of smooth \(\omega_V\)-plurisubharmonic functions which decrease towards \(\varphi\).

When \(\omega_V\) is a Hodge form, this regularization can be seen as a consequence of the extension result of \cite{CGZ13}.

Approximation from above by regular objects is of central use in the theory of complex Monge-Ampère operators, as the latter are continuous along (and even defined through) such monotone sequences \cite{BT82}, while they are not continuous with respect to the weaker \(L^1\)-topology \cite{Ceg83}.

**Plan of the note.** We first establish our main result when the underlying cohomology class is also semi-positive (section 2), as the viscosity technology is already available \cite{EGZ11}; the corollary follows then easily by using Richberg’s regularization result. We then (section 3) adapt the techniques of \cite{EGZ11} to the general context of big cohomology classes. The technical condition \(P(\alpha) = NB(\alpha)\) naturally shows up as it is necessary for the global viscosity comparison principle to hold. We finally (section 4) use recent stability estimates for big cohomology classes \cite{GZ12} to obtain continuous solutions of slightly more general Monge-Ampère equations, which allow us to prove our main result.

**Dédicace.** C’est un plaisir de contribuer à ce volume en l’honneur de Duong Hong Phong, dont nous apprécions la générosité, la vision et le bon goût, tant mathématique que gastronomique !

2. **The Case of Semi-Positive Classes**

We fix once and for all \((X, \omega_X)\) a compact Kähler manifold of complex dimension \(n\), \(\alpha \in H^{1,1}(X, \mathbb{R})\) a big cohomology class and \(\theta\) a smooth closed \((1,1)\) form representing \(\alpha\).

2.1. **Minimal vs Analytic Singularities.** Recall that \(\alpha\) is semi-positive if \(\theta\) can be chosen to be a semi-positive form. In this case a \(\theta\)-psh function has minimal singularities if and only if its is bounded. The easiest example of \(\theta\)-psh functions with minimal singularities are constant functions which are indeed \(\theta\)-psh iff \(\theta\) is semipositive.

For more general \(\alpha\), \(\theta\)-psh function with minimal singularities can be constructed as enveloppes, e.g.:

\[
V_\theta := \sup \{ v \mid v \in PSH(X, \theta) \text{ and } \sup_X v \leq 0 \}.
\]

Note that if \(V \in PSH(X, \theta)\) is another function with minimal singularities, then \(V - V_\theta\) is globally bounded on \(X\). Also if \(\theta' = \theta + dd^c \rho\) is another smooth form representing \(\alpha\), then \(PSH(X, \theta') = PSH(X, \theta) - \rho\) where \(\rho \in C^\infty(X, \mathbb{R})\) hence \(V_\theta - V_{\theta'}\) is also globally bounded on \(X\).
Definition 2.1. The polar locus of $\alpha$ is 

$$P(\alpha) := \{ x \in X \mid V_{\theta}(x) = -\infty \}. $$

The non-bounded locus of $\alpha$ is 

$$NB(\alpha) := \{ x \mid V_{\theta} \notin L^\infty_{loc}(\{x\}) \}. $$

The observations above show that these definitions only depend on $\alpha$. Clearly $P(\alpha) \subset NB(\alpha)$ and $NB(\alpha)$ is closed. We shall assume in the sequel that $P(\alpha) = NB(\alpha)$ which, as will turn out, is equivalent to saying that there exists one exponentially continuous $\theta$-psh function with minimal singularities.

By definition $\alpha$ is big if it contains a Kähler current, i.e. there is a (singular) positive current $T \in \alpha$ and $\varepsilon > 0$ such that $T \geq \varepsilon \omega_X$. It follows from the regularization techniques of Demailly (see [Dem92]) that one can further assume that $T$ has analytic singularities:

Definition 2.2. A positive closed current $T$ has analytic singularities if it can be locally written $T = dd^c u$, with 

$$u = \frac{c}{2} \log \left[ \sum_{j=1}^{s} |f_j|^2 \right] + v,$$

where $c > 0$, $v$ is smooth and the $f_j$’s are holomorphic functions.

We let Amp($\alpha$) denote the ample locus of $\alpha$, i.e. the Zariski open subset of all points $x \in X$ for which there exists a Kähler current in $\alpha$ with analytic singularities which is smooth in a neighborhood of $x$.

It follows from the work of Boucksom [Bou04] that one can find a single Kähler current $T_0 = \theta + dd^c \psi_0$ with analytic singularities in $\alpha$ such that 

$$\text{Amp}(\alpha) = X \setminus \text{Sing} T_0.$$ 

In the sequel we fix such a Kähler current $T_0$ and assume for simplicity that 

$$T_0 \geq \omega_X.$$ 

Observe that $\psi_0$ is exponentially continuous, however $\psi_0$ does not have minimal singularities unless $\alpha$ is Kähler (see [Bou04]).

Bounded vs continuous approximations. Fix $\varphi \in PSH(X, \theta)$ a $\theta$-psh function. It is easy to approximate $\varphi$ from above by a decreasing sequence of $\theta$-psh functions with minimal singularities. Indeed we can set 

$$\varphi_j := \max(\varphi, V_0 - j) \in PSH(X, \theta).$$

The latter have minimal singularities and decrease to $\varphi$ as $j \nearrow +\infty$. This construction needs however to be refined to get exponentially continuous $\theta$-psh approximations with minimal singularities. The mere existence of exponentially continuous $\theta$-psh functions $\psi$ with minimal singularities is actually not obvious.
2.2. **Continuous approximations in the semi-positive case.** We show here our main result in the simpler case when $\alpha$ is both big and semi-positive.

**Theorem 2.3.** Assume $\alpha \in H^{1,1}(X, \mathbb{R})$ is big and semi-positive and let $\varphi \in PSH(X, \theta)$ be a $\theta$-plurisubharmonic function.

Then there exists a sequence of continuous $\theta$-plurisubharmonic functions which decrease towards $\varphi$.

**Proof.** Fix $h_j$ a sequence of smooth functions decreasing to $\varphi$ (recall that $\varphi$ is upper semi-continuous) and set

$$\varphi_j = P(h_j) := \sup \{ u \mid u \in PSH(X, \theta) \text{ and } u \leq h_j \}.$$ 

Observe that $\varphi_j \in PSH(X, \theta)$ and $\varphi_j \leq h_j$ hence

$$\varphi \leq \varphi_j \leq \varphi_{j+1}.$$

We claim that $\varphi = \lim \varphi_j$. Indeed set $\psi := \lim \varphi_j \geq \varphi$. Then $\psi \leq h_j$ for all $j$ and $\psi \in PSH(X, \theta)$, hence $\psi \leq \varphi$, so that $\psi = \varphi$ as claimed.

It thus suffices to check that $\varphi_j$ is continuous. It follows from the work of Berman and Demailly \[BD12\] that $\varphi_j$ has locally bounded Laplacian on the ample locus $\text{Amp}(\alpha)$ of $\alpha$, with

$$(\theta + dd^c \varphi_j)^n = 1_{\{ P(h_j) = h_j \}} (\theta + dd^c h_j)^n \text{ in } \text{Amp}(\alpha).$$

The measure on the right hand side is absolutely continuous with respect to Lebesgue measure, with bounded density. It therefore follows from \[EGZ11\] Theorem C that $\varphi_j = P(h_j)$ is continuous. \hfill $\Box$

**Remark 2.4.** Observe that a key point in the proof above is that if $h$ is a smooth function on $X$, then its $\theta$-plurisubharmonic projection $P(h)$ is a continuous $\theta$-plurisubharmonic function with minimal singularities.

Although the proof is quite short in appearance, it uses several important tools: Demailly’s regularization technique (which is heavily used in \[BD12\]), and the viscosity approach for degenerate complex Monge-Ampère equations developed in \[EGZ11\].

The proof of our main theorem follows exactly the same lines: the result of Berman-Demailly applies for general big classes, while \[BEGZ10\] produces solutions of complex Monge-Ampère equations with minimal singularities in big cohomology classes. It thus remains to extend the viscosity approach of \[EGZ11\] to the setting of big cohomology classes, which is the contents of the next section.

Since Richberg’s regularization technique \[R68\] applies in a singular setting, we obtain the following interesting consequence:

**Corollary 2.5.** Let $(V, \omega_V)$ be a compact normal Kähler space and let $\varphi$ be a $\omega_V$-plurisubharmonic function on $V$. Then there exists a sequence $(\varphi_j)$ of smooth $\omega_V$-plurisubharmonic functions which decrease towards $\varphi$. 
Proof. Fix $\varphi \in PSH(X, \omega_V)$. We can assume without loss of generality that $\varphi < 0$ on $V$. Let $\pi : X \to V$ be a desingularization of $V$ and set $\theta := \pi^*\omega_V$. Then $\psi := \varphi \circ \pi \in PSH(X, \theta)$.

Since $\pi^*\{\omega_V\} \in H^{1,1}(X, \mathbb{R})$ is big, it follows from our previous result that we can find continuous $\theta$-psh functions $\psi_j < 0$ which decrease towards $\psi$ on $X$. Since $\pi$ has connected fibers, one easily checks that $PSH(X, \theta) = \pi^*PSH(V, \omega_V)$, in particular there exists $\varphi_j \in PSH(V, \omega_V) \cap C^0(V)$ such that $\psi_j := \varphi_j \circ \pi$, with $\varphi_j$ decreasing to $\varphi$.

We can now invoke Richberg’s regularization result [R68] (see also [Dem92]): using local convolutions and patching, one can find smooth functions $(\varphi_j)_{k}$ on $V$ which decrease to $\varphi_j$ as $k \to +\infty$ and such that $\varphi_j \in PSH(V, (1 + \varepsilon_k)\omega_V)$ with $\varepsilon_k \searrow 0$. We can also assume that $\varphi_j < 0$ on $V$. Set finally

$$u_j := \frac{1}{1 + \varepsilon_j} \varphi_j \in PSH(V, \omega_V) \cap C^\infty(V).$$

We let the reader check that $(u_j)$ still decreases to $\varphi$. □

Remark 2.6. When $\omega_V$ has integer class, i.e. when it represents the first Chern class of an ample line bundle on $V$, the above result was obtained in [CGZ13] as a consequence of an extension result of $\omega_V$-psh functions.

3. Viscosity approach in a big setting

We set here the basic frame for the viscosity approach to the equation

$$\left( DMA_{\varepsilon} \right) (\theta + dd^c \varphi)v^n = e^{\varepsilon \varphi}v$$

where $v$ is a volume form with nonnegative continuous density and $\varepsilon > 0$ is a real parameter.

3.1. Viscosity sub/super-solutions for big cohomology classes. To fit in with the viscosity point of view, we rewrite the Monge-Ampère equation as

$$\left( DMA_{\varepsilon} \right) e^{\varepsilon \varphi}v - (\theta + dd^c \varphi)v^n = 0$$

Let $x \in X$. If $\kappa \in \Lambda^{1,1}T_xX$ we define $\kappa_+^n$ to be $\kappa^n$ if $\kappa \geq 0$ and 0 otherwise. For a technical reason, we will also consider a slight variant of $(DMA_{\varepsilon})$,

$$\left( DMA_{\varepsilon} \right)_+ e^{\varepsilon \varphi}v - (\theta + dd^c \varphi)_+^n = 0$$

If $\varphi^{(2)}_x$ is the 2-jet at $x \in X$ of a $C^2$ real valued function $\varphi$ we set

$$F(\varphi^{(2)}_x) = F_{\varepsilon}(\varphi_x) = \begin{cases} e^{\varepsilon \varphi(x)}v_x - (\theta_x + dd^c \varphi_x)v^n & \text{if } \theta + dd^c \varphi_x \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Recall the following definition from [CIL92], [EGZ11, Definition 2.3]:
Definition 3.1. A subsolution of $(DMA^ε)_{\nu}$ is an upper semi-continuous function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that $\varphi \not\equiv -\infty$ and the following property is satisfied: if $x_0 \in X$ and $q \in C^2$, defined in a neighborhood of $x_0$, is such that $\varphi(x_0) = q(x_0)$ and $\varphi - q$ has a local maximum at $x_0$, then $F(q^{(2)}_{x_0}) \leq 0$.

We say that $\varphi$ has minimal singularities if there exists $C > 0$ such that $V_\theta - C \leq \varphi \leq V_\theta + C$ on $X$.

It has been shown in [EGZ11, Corollary 2.6] that a viscosity subsolution $\varphi$ of $(DMA^ε)_{\nu}$ is a $\theta$-psh function which satisfies $(\theta + dd^c \varphi)^n \geq e^{\varepsilon \varphi \nu}$ in the pluripotential sense of [BT82] [BEGZ10].

We now slightly extend the concept of supersolution, so as to allow a supersolution to take $-\infty$ values:

Definition 3.2. A supersolution of $(DMA^ε)_{\nu}$ is a supersolution of $(DMA^ε)_{\nu^+}$, that is, a function $\varphi : X \to \mathbb{R} \cup \{\pm \infty\}$ such that $e^\varphi$ is lower semicontinuous, $\varphi \not\equiv +\infty$, $\varphi \not\equiv -\infty$ and the following property is satisfied: if $x_0 \in X$ and $q \in C^2$, defined in a neighborhood of $x_0$, is such that $\varphi(x_0) = q(x_0)$ and $\varphi - q$ has a local minimum at $x_0$, then $F_+(q^{(2)}_{x_0}) \geq 0$.

We say that $\varphi$ has minimal singularities if there exists $C > 0$ such that $(V_\theta)_* - C \leq \varphi \leq (V_\theta)_* + C$ on $X$.

Here $(V_\theta)_*$ denotes the lower semi-continuous regularization of $V_\theta$. It is important to allow $-\infty$ values since we are trying to build a $\theta$-psh viscosity solution of $(DMA^ε)_{\nu}$: in general such a function will be infinite at polar points $x_0 \in P(\alpha)$. Note that we don’t impose any condition at such points.

Definition 3.3. A viscosity solution of $(DMA^ε)_{\nu}$ is a function that is both a sub- and a supersolution. In particular a viscosity solution $\varphi$ is automatically an exponentially continuous $\theta$-plurisubharmonic function.

By comparison, a pluripotential solution of $(DMA^ε)_{\nu}$ is an usc function $\varphi \in L^\infty_{loc}(\text{Amp}(\alpha)) \cap \text{PSH}(X, \omega)$ such that $(\theta + dd^c \varphi)^n_{BT} = e^{\varepsilon \varphi \nu}$ in $\text{Amp}(\alpha)$.

in the sense of Bedford-Taylor [BT82] (see [BEGZ10] for the slightly more general notion of non-pluripolar products): it follows from [BEGZ10] that such a pluripotential solution automatically has minimal singularities, however there is no continuity information, especially at points in $X \setminus \text{Amp}(\alpha)$, as this set is pluripolar hence invisible from the pluripotential point of view.
3.2. The big viscosity comparison principle.

**Theorem 3.4.** Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class and assume $\varepsilon > 0$ and $v > 0$. Let $\varphi$ (resp. $\psi$) be a subsolution (resp. supersolution) of $(DMA_\varepsilon^\alpha)$ with minimal singularities, then

$$\varphi \leq \psi \text{ in } \text{Amp}(\alpha).$$

Moreover $\varphi \leq \psi$ on $X$ if and only if $P(\alpha) = NB(\alpha)$.

**Proof.** We can assume $\varepsilon = 1$ without loss of generality.

We let $x_0 \in X$ denote a point that realizes the maximum of $e^{\varphi} - e^\psi$ on $X$. If $x_0 \in P(\alpha)$, then we conclude trivially: $\varphi(x_0) = -\infty$, hence $\max_X(e^{\varphi} - e^\psi) \leq 0$.

Assume now $x_0 \not\in NB(\alpha)$. Then $\varphi$ and $\psi$ are locally bounded near $x_0$. Since $NB(\alpha)$ is closed, we can choose complex coordinates $(z^1, \ldots, z^n)$ near $x_0$ defining a biholomorphism identifying an open neighborhood of $x_0$ in $X - NB(\alpha)$ to the complex ball $B(0, 5) \subset \mathbb{C}^n$ of radius 5 sending $x_0$ to the origin.

We define $h \in C^2(B(0, 5), \mathbb{R})$ to be a local potential smooth up to the boundary for $\theta$ and extend it smoothly to $X$. In particular $dd^c h = \theta$ and $w_- := \varphi + h$ is a bounded viscosity subsolution of the equation

$$(dd^c w)^n = e^w W \text{ in } B(0, 5)$$

with $W$ a positive and continuous volume form. On the other hand $w^+ = \psi + h$ is a bounded viscosity supersolution of the same equation.

Now choose $C > 0$ such that $\sup_{x \in B(0, 4)} \max(|\varphi(x)|, |\psi(x)|) \leq C/1000$ and $\sup_{x \in B(0, 4)} |h(x)| \leq C/10$. With this constant $C > 0$, construct as in [EGZ11] p. 1076 a smooth auxiliary function $\varphi_3$ on $B(0, 4)^2$. Using the same notations as in [EGZ11] p. 1077, fix $\beta > 0$ and consider $(x_\beta, y_\beta) \in B(0, 4)^2$ such that:

$$M_\beta = \sup_{(x, y) \in B(0, 4)^2} w_-(x) - w^+(y) - \varphi_3(x, y) - \frac{\beta}{2} d^2(x, y)$$

By construction, $\varphi_3$ is big enough outside $B(0, 2)^2$ to ensure that the sup is achieved at some point $(x_\beta, y_\beta) \in B(0, 2)^2$. Limit points $(x, y)$ of $(x_\beta, y_\beta)$ satisfy $x = y$ and the construction forces $\varphi_3$ to vanish to high order at such a limit point. Then, the argument of [EGZ11] p. 1077-1078 based on Ishii’s version of the maximum principle (see [CIL92]) applies verbatim to prove that

$$\limsup_{\beta \to 0} w^+_\beta(x_\beta) - w^-_\beta(y_\beta) \geq 0$$

and enables us to conclude that $\varphi \leq \psi$.

However, if $x_0 \not\in NB(\alpha) \setminus P(\alpha)$, $\varphi(x_0) > -\infty$ since $\phi$ has minimal singularities, while $\psi(x_0) = -\infty$ since $\psi$ has minimal singularities in the sense
of definition 3.2 and $x_0 \in NB(\alpha)$ implies that $(V_0)_*(x_0) = -\infty$. The global comparison principle thus fails if $P(\alpha) \neq NB(\alpha)$.

Let us now justify that in general we do have $\varphi \leq \psi$ on $\text{Amp}(\alpha)$. Let $T_0 = \theta + dd^c \psi_0$ be a Kähler current such that $\text{Amp}(\alpha) = X \setminus \text{Sing}(T_0)$, $\psi_0 \leq \psi$ and $\psi_0 \leq 0$. Fix $\delta > 0$ and consider $\varphi_\delta := (1 - \delta)\varphi + \delta \psi_0 + n \log(1 - \delta)$. We claim that $\varphi_\delta$ is again a subsolution of $(DMA^c_\alpha)$. Indeed there is nothing to test on $\text{Sing}(T_0)$, while in $\text{Amp}(\alpha)$

$$(\theta + dd^c \varphi_\delta)^n \geq (1 - \delta)^n (\theta + dd^c \varphi)^n \geq (1 - \delta)^n e^\varphi v \geq e^{\varphi_\delta} v,$$

as follows easily by interpreting these inequalities in the pluripotential sense (see [EGZ11] Proposition 1.11).

Let $x_\delta$ be a point where the upper semi-continuous function $e^{\varphi_\delta} - e^\psi$ attains its maximum. If $x_\delta \in \text{Sing}(T_0)$, then $e^{\varphi_\delta(x_\delta)} = 0$ hence

$$e^{\varphi_\delta} \leq e^\psi \Rightarrow \varphi_\delta \leq \psi \text{ on } X.$$ 

If $x_\delta \in \text{Amp}(\alpha)$, then both $\varphi_\delta$ and $\psi$ are locally bounded near $x_\delta$ and the argument above leads to the conclusion that $e^{\varphi_\delta(x_\delta)} \leq e^{\psi(x_\delta)}$ hence $\varphi_\delta \leq \psi$ on $X$. Letting $\delta$ decrease to zero, we infer that $\varphi \leq \psi$ in $\text{Amp}(\alpha)$. \hfill \qed

### 3.3. Continuous solutions of big Monge-Ampère equations.

**Theorem 3.5.** Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class and assume $\varepsilon > 0$ and $v > 0$ is a continuous positive density. Then there exists a unique pluripotential solution $\varphi$ of $(DMA^c_\alpha)$ on $X$, such that

1. $\varphi$ is a $\theta$-plurisubharmonic function with minimal singularities,
2. $\varphi$ is a viscosity solution in $\text{Amp}(\alpha)$ hence continuous there,
3. Its lower semicontinuous regularisation $\varphi_*$ is a viscosity supersolution.

If $P(\alpha) = NB(\alpha)$ then $\varphi$ is a viscosity solution of $(DMA^c_\alpha)$ on $X$, hence $e^\varphi$ is continuous on $X$.

**Proof.** We can always assume that $\varepsilon = 1$. Since the comparison principle holds on $\text{Amp}(\alpha)$, we can use Perron’s method by considering the upper envelope of subsolutions.

It follows from [BEGZ10] that the equation $(DMA^c_\alpha)$ has a pluripotential solution $\phi_0$ which is a $\theta$–plurisubharmonic function on $X$ satisfying the equation $(\theta + dd^c \phi_0)^n = e^{\phi_0} v$ weakly on $X$. Since the right hand side is a bounded volume form, it follows from the big version of Kolodziej’s uniform estimates that $\varphi_0$ has minimal singularities.

Moreover from the definition of a subsolution in the big case and [EGZ11] Corollary 2.6), it follows that $\varphi_0$ is a viscosity subsolution to the equation $(DMA^c_\alpha)$.

On the other hand, since by [BDI2], $V_\theta$ satisfies the equation $(\theta + dd^c V_\theta)^n = 1_{\{V_\theta \leq 0\}} \theta^n$ in the pluripotential sense and the right hand side is a bounded volume form, it follows that for some constant $C >> 1$ the function $\phi_1 := V_\theta + C$ satisfies the inequality $(\theta + dd^c \phi_1)^n \leq e^{\phi_1} v$ in pluripotential sense. It
follows therefore from the proof of [EGZ11, Lemma 4.7(1)] that \( \psi_1 := (\phi_1)_* \) is a (viscosity) supersolution to the equation \((DMA^1_\nu)\).

We can now consider the upper envelope of (viscosity) subsolutions,

\[ \varphi := \sup\{\psi : \psi \text{ viscosity subsolution}, \phi_0 \leq \psi \leq \psi_1\}, \]

which is a subsolution with minimal singularities to the equation \((DMA^1_\nu)\).

Using the bump construction ([CIL92], [EGZ11]) we can show that the lower semi-continuous regularization \( \varphi^* \) of \( \varphi \) is a (viscosity) supersolution with minimal singularities to the equation \((DMA^1_\nu)\).

Therefore since the comparison principle holds on \( \text{Amp}(\alpha) \), it follows that \( \varphi \leq \varphi^* \) on \( \text{Amp}(\alpha) \), hence \( \varphi = \varphi^* \) on \( \text{Amp}(\alpha) \) is a viscosity solution to the equation \((DMA^1_\nu)\) on \( \text{Amp}(\alpha) \).

If moreover \( P(\alpha) = NB(\alpha) \) then we conclude by Theorem 3.4 that \( \varphi = \varphi^* \) on \( X \) is a viscosity solution to the equation \((DMA^1_\nu)\) on \( X \), hence \( e^\varphi \) is continuous on \( X \).

\[ \square \]

4. Pluripotential tools

4.1. Stability inequalities for big classes. The following result is the main stability inequality established in [GZ12]:

**Theorem 4.1.** Assume \( (\theta + dd^c \varphi_\mu)^n = f_\mu \omega^n_X, (\theta + dd^c \varphi_\nu)^n = f_\nu \omega^n_X \), where the densities \( 0 \leq f_\mu, f_\nu \) are in \( L^p(\omega^n_X) \) for some \( p > 1 \) and \( \varphi_\mu, \varphi_\nu \in PSH(X, \theta) \) are normalized by \( \sup_X \varphi_\mu = \sup_X \varphi_\nu = 0 \). Then

\[ \|\varphi_\mu - \varphi_\nu\|_{L^\infty(X)} \leq M_\tau \|f_\mu - f_\nu\|_{L^1(X)}, \]

where \( M_\tau > 0 \) only depends on upper bounds for the \( L^p \) norms of \( f_\mu, f_\nu \) and

\[ 0 < \tau < \frac{1}{2^n(nq+1)-1}, \quad \frac{1}{p} + \frac{1}{q} = 1. \]

**Corollary 4.2.** Let \( \alpha \in H^{1,1}(X, \mathbb{R}) \) be a big cohomology class and assume \( \varepsilon > 0 \) and \( v \geq 0 \) is a probability measure with \( L^p \)-density with respect to Lebesgue measure, where \( p > 1 \). Then there exists a unique \( \theta \)-plurisubharmonic function \( \varphi \) with minimal singularities which is a pluripotential solution of \((DMA^\varepsilon_\alpha)\) on \( X \).

Moreover \( \varphi \) is continuous on \( \text{Amp}(\alpha) \) and if \( P(\alpha) = NB(\alpha) \) then \( e^\varphi \) is also continuous on \( X \).

**Proof.** The first part follows from [BEGZ10] and we get a unique pluripotential solution \( \varphi \) with minimal singularities. Let \( f \) denote the density of \( v = f \omega^n_X \). We can approximate \( f \) by continuous and positive densities by using convolutions, locally \( f_\delta := f * \chi_\delta + \delta, \delta > 0 \). Theorem 3.5 insures that there exists a unique \( \varphi_\delta \in PSH(X, \theta) \) solution to

\[ (\theta + dd^c \varphi_\delta)^n = e^{\varepsilon \varphi_\delta} f_\delta \omega^n_X \]
which has minimal singularities and is continuous in Amp(\(\alpha\)). It follows moreover from [BEGZ10] Theorem 4.1 that the functions \(\varphi_\delta - V_\theta\) are uniformly bounded as \(\delta \to 0^+\).

The family \(\{\varphi_\delta\}_{\delta > 0}\) is compact in the \(L^1\) topology hence we can extract a sequence \((\delta_k)_k\) such that \(\varphi_{\delta_k}\) converges almost everywhere and \(\sup_X \varphi_{\delta_k}\) converges. We can now apply the stability inequality (Theorem 4.1) to \(\tilde{\varphi}_\delta := \varphi_\delta - \sup_X \varphi_\delta\) to check that the functions \(\tilde{\varphi}_\delta_k\) form a Cauchy sequence, so that the functions \(\varphi_{\delta_k}\) form a Cauchy sequence too. The uniform limit \(\phi = \lim \varphi_{\delta_k}\) has minimal singularities and satisfies \((\theta + dd^c \phi)^n = e^\phi f_\omega^X\) in the pluripotential sense, hence coincides with \(\varphi\). □

4.2. The flat setting.

**Theorem 4.3.** Let \(\alpha \in H^{1,1}(X, \mathbb{R})\) be a big cohomology class and assume \(v \geq 0\) is a volume form with non-negative \(L^p\)-density with respect to Lebesgue measure such that \(\int_X v = \text{Vol}(\alpha)\) and \(p > 1\). Then there exists a unique \(\theta\)-plurisubharmonic function \(\varphi\) with minimal singularities which is a pluripotential solution of \((DMA^\varepsilon_v)\) on \(X\) and such that \(\int_X \varphi dv = 0\).

Moreover \(\varphi\) is continuous on Amp(\(\alpha\)) and if \(P(\alpha) = NB(\alpha)\) then \(e^\varphi\) is also continuous on \(X\).

**Proof.** Fix \(\varepsilon > 0\) and let \(\varphi_\varepsilon\) be the unique solution of \((DMA^\varepsilon_v)\) given by Corollary 4.2. It follows from [BEGZ10] that the functions \(\varphi_\varepsilon - V_\theta\) are uniformly bounded, hence the argument of Corollary 4.2 enables to extract a Cauchy sequence whose limit \(\psi\) is a solution of \((DMA^0_v)\) which satisfies

\[
\int \psi dv = \lim_{\varepsilon \to 0} \int \left[ \frac{e^{\varepsilon \varphi_\varepsilon} - 1}{\varepsilon} \right] dv = \int \varphi dv = 0,
\]

hence \(\varphi = \psi\) (by the uniqueness result proved in [BEGZ10]) has all required properties. □

**Corollary 4.4.** The function \(V_\theta\) is continuous if and only if \(P(\alpha) = NB(\alpha)\).

**Proof.** Observe that if \(e^{V_\theta}\) is continuous then \(P(\alpha) = NB(\alpha)\), as points in \(NB(\alpha) \setminus P(\alpha)\) correspond to points where \(V_\theta\) is finite but not locally finite.

Conversely it follows from the work of Berman-Demailly [BD12] that \(V_\theta\) has locally bounded Laplacian in Amp(\(\alpha\)) and satisfies

\[
(\theta + dd^c V_\theta)^n = 1_{\{V_\theta = 0\}} \theta^n \text{ in Amp}(\alpha).
\]

Since \(\theta^n\) is smooth and the density \(1_{\{V_\theta = 0\}}\) is bounded, it follows from previous theorem that \(V_\theta\) is continuous on \(X\) if \(P(\alpha) = NB(\alpha)\). □

4.3. Conclusion. The proof of the main theorem proceeds now exactly as in that of Theorem 2.3 if \(\varphi\) is a given \(\theta\)-psh function, we approximate it from above by a decreasing sequence of smooth functions \(h_j\) and set \(\varphi_j := P(h_j) \in PSH(X, \theta)\).
These functions have minimal singularities, decrease to $\varphi$ and solve the complex Monge-Ampère equation

$$(\theta + dd^c \varphi_j)^n = 1_{\{\varphi_j = h_j\}} (\theta + dd^c h_j)^n = f_j \omega^n_X,$$

where $f_j$ is bounded. It follows therefore from Theorem 4.3 that $\varphi_j$ is continuous if $P(\alpha) = NB(\alpha)$.

Let us conclude by mentioning that we don’t know any example of a compact Kähler manifold $X$ and a big cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ such that $P(\alpha)$ does not coincide with $NB(\alpha)$.

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