A PASSIVATION ALGORITHM FOR LINEAR TIME-INVARIANT SYSTEMS
ANTONIO FAZZI∗, NICOLA GUGLIELMI†, AND CHRISTIAN LUBICH‡

Abstract. We propose and study an algorithm for computing a nearest passive system to a given non-passive linear time-invariant system (with much freedom in the choice of the metric defining ‘nearest’), and also a closely related algorithm for computing the structured distance of a given passive system to non-passivity. Both problems are addressed by solving eigenvalue optimization problems for Hamiltonian matrices that are constructed from perturbed system matrices. The proposed algorithms are two-level methods that optimize the Hamiltonian eigenvalue of smallest positive real part over perturbations of a fixed size in the inner iteration, using a constrained gradient flow, and optimize over the perturbation size in the outer iteration. For large systems, we propose a variant of the algorithm that takes advantage of the inherent low-rank structure of the problem. Numerical experiments illustrate the behavior of the proposed algorithms.

Key words. passive control system, structured passivity enforcement, distance to non-passivity, matrix nearness problem, structured eigenvalue optimization

AMS subject classifications. 65K05, 93B40, 93B60, 93C05, 15A18

1. Introduction.

1.1. Preliminaries. We consider linear time-invariant dynamical systems expressed by their state space representation:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) =Cx(t) + Du(t), \]

(1.1)

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m} \) and we take zero initial values. As for terminology, \( x(t) \in \mathbb{R}^n \) is the state vector, while \( u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^p \) are the input and the output vector, respectively. A system of the form (1.1) is identified by the block matrix

\[ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(n+p) \times (n+m)} \]

(1.2)

that collects all the parameters chosen for its representation.

A system (1.1) is called passive if every input \( u \in L^2(\mathbb{R}_+) \) bounds the corresponding output \( y \) by

\[ \int_0^\infty \|y(t)\|^2 \, dt \leq \int_0^\infty \|u(t)\|^2 \, dt, \]

where the norm \( \| \cdot \| \) is the Euclidean norm. By the Plancherel formula, this is equivalent to stating that the matrix transfer function \( T(s) = C(sI - A)^{-1}B + D \), which describes the relation between the Laplace transform of the input and the Laplace transform of the output through the expression \( \mathcal{L}y(s) = T(s)\mathcal{L}u(s) \), has spectral norm bounded by 1 for Re\( s \geq 0 \): \( \|T(s)\|_2 \leq 1 \).

∗Department ELEC, Vrije Universiteit Brussel (VUB), Pleinlaan 2, 1050 Brussels, Belgium. Email: Antonio.Fazzi@vub.be
†Gran Sasso Science Institute, via Crispi 7, L’ Aquila, Italy. Email: nicola.guglielmi@gssi.it
‡Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D–72076 Tübingen, Germany. Email: lubich@ma.uni-tuebingen.de

arXiv:2010.15954v1 [math.NA] 29 Oct 2020
Under the natural assumptions that all eigenvalues of \( A \) have negative real part and that \( \|D\|_2 < 1 \), passivity of the linear control system (1.1) is characterized by the condition that the associated Hamiltonian matrix \( M(X) \) has no eigenvalues on the imaginary axis [3, Theorem 2]; here,

\[
M(X) = \begin{pmatrix} F & G \\ H & -F^T \end{pmatrix} \in \mathbb{R}^{2n \times 2n} \quad \text{with}
\]

\[
F = A - BR^{-1}D^T C, \quad G = -BR^{-1}B^T, \quad H = C^T S^{-1} C,
\]

\[
R = D^T D - I, \quad S = D D^T - I.
\]

We recall that a real Hamiltonian matrix is a matrix \( M \in \mathbb{R}^{2n \times 2n} \) that satisfies

\[
(JM)^T = JM, \quad \text{with} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

The spectrum of a real Hamiltonian matrix is known to be symmetric with respect to both the real and the imaginary axis.

1.2. Content of the paper and contributions. In this paper we consider algorithms for the following problems, in which the notion ‘nearest’ is yet to be specified:

1. Passivity enforcement. Given a non-passive system (1.2) (i.e., the Hamiltonian matrix \( M(X) \) has some purely imaginary eigenvalues), compute the ‘nearest’ perturbed system \( X + \Delta X \) that is passive; more precisely, given \( \delta > 0 \), all eigenvalues of \( M(X + \Delta X) \) are to have a real part of absolute value at least \( \delta \).

2. Distance to the nearest non-passive system. Given a passive system (1.2) (i.e., the Hamiltonian matrix \( M(X) \) has no purely imaginary eigenvalues), compute the ‘nearest’ perturbed system \( X + \Delta X \) that is no more passive (i.e., the Hamiltonian matrix \( M(X + \Delta X) \) has at least one pair of purely imaginary eigenvalues).

We remark that Problems 1 and 2 can be combined in asking for the nearest passive system whose distance to non-passivity is at least \( \delta \). ‘Nearest’ could refer to the distance to \( X \) in the Frobenius norm or a weighted Frobenius norm, but different concepts of nearness are often imposed in the engineering literature. One approach, described in [5, 6], is to keep the matrices \( A, B \) and \( D \) fixed and to perturb only the state-output matrix \( C \) in such a way that the Frobenius norm of \( \Delta C Q^T \) is kept minimal, where \( Q^T \in \mathbb{R}^{n \times n} \) is a Choleski factor of the controllability Gramian \( G_c = Q^T Q \). This Gramian depends only on the matrices \( A \) and \( B \) and is positive definite for a controllable system, as will be assumed henceforth when we consider this guiding example. In other cases, nearness may also be understood under constraints of preserving a sparsity pattern or symmetry.

A framework that includes all nearness options in the literature as far as known to us, and which allows for structured passivation, is to specify a linear map \( L : \mathbb{R}^{k \times l} \to \mathbb{R}^{(n+p) \times (n+m)} \) and to consider only perturbations

\[
\Delta X = L[\Delta Z] \quad \text{and minimize} \quad \|\Delta Z\|_F.
\]

\[\hfill (1.5)\]

\[\hfill ^1\text{In this paper we put arguments of linear maps in square brackets.}\]
In the example considered above, where only $C$ is perturbed and $\|\Delta C Q^T\|_F$ is to be minimized, we have $k = p$ and $l = n$ and

$$L[\Delta Z] = \begin{pmatrix} 0_{n \times n} & 0_{n \times m} \\ \Delta Z Q^{-T} & 0_{p \times m} \end{pmatrix}.$$  \hspace{1cm} (1.6)

Another case of interest is sparsity-preserving passivation, where $L[\Delta Z]$ represents a sparse matrix whose nonzero entries are collected in a vector $\Delta Z$.

Our proposed algorithms are two-level iterative methods similar to \cite{7, 8}. In the inner iteration, for a given $\varepsilon > 0$, we use a gradient ascent / descent algorithm to solve the eigenvalue optimization problem, over $E \in \mathbb{R}^{k \times l}$ with $\|E\|_F = 1$,

$$E_\varepsilon = \arg \max_{\arg \min} \phi_\varepsilon(E) \quad \text{with} \quad \phi_\varepsilon(E) = \Re \lambda(M(X + L(E\varepsilon))),$$  \hspace{1cm} (1.7)

where the maximum is taken for Problem 1 and the minimum for Problem 2. Here, $\lambda(M)$ is an eigenvalue of minimal nonnegative real part (chosen with nonnegative imaginary part) of a real Hamiltonian matrix $M$. The outer iteration determines the smallest $\varepsilon$ such that $\phi_\varepsilon(E_\varepsilon) = \delta$ for a given small threshold $\delta > 0$.

For the passivation problem (Problem 1) we assume that a matrix $\Delta Z_0$ is known such that $X + L[\Delta Z_0]$ is passive. In the above example, this is trivially satisfied for $\Delta Z_0 = -CQ^T$, which yields $C + \Delta Z_0 Q^{-T} = 0$. More generally, $\Delta Z_0$ might be the result of some computationally inexpensive passivity enforcement algorithm such as given in \cite{6}. The matrix $\Delta Z_0$ provides an initial iterate for an iteration over passive matrices that tend to $X + L[\Delta Z]$, with $\Delta Z = \varepsilon E(\varepsilon)$ of (locally) minimal Frobenius norm.

Both Problems 1 and 2 are nonconvex and nonsmooth optimization problems. Our algorithm is not guaranteed to find the global optimum but it computes a passive perturbation of the original system for Problem 1 and a non-passive perturbation for Problem 2 that are at least locally optimal. The algorithm can also be used to improve the results of other passivation algorithms by taking their results as starting iterates.

The related problem of finding a nearest Hamiltonian matrix with / without eigenvalues on the imaginary axis has previously been addressed in \cite{11, 12}. However, there the resulting perturbed Hamiltonian matrix is not, in general, a matrix $M(X + \Delta X)$ associated with some linear control system $X + \Delta X$. In contrast, in this paper we put a structured perturbation $\Delta X = L[\Delta Z]$ directly on $X$ and not on the Hamiltonian matrix, and we minimize the Frobenius norm of $\Delta Z$ and not the Frobenius distance between the perturbed and unperturbed associated Hamiltonian matrices.

1.3. Outline. In Section 2 we study the optimization of the eigenvalue of smallest positive real part over structured perturbations $L[\Delta Z]$ with $\Delta Z$ of a fixed norm $\varepsilon$, using a norm-constrained gradient system.

In Section 3 we study, for Problem 1 (passivation), the equation for determining the optimal perturbation size $\varepsilon$ near a point of coalescence of a pair of eigenvalues of Hamiltonian matrices.

In Section 4 we describe the two-level algorithm for Problem 1.

There is a completely analogous algorithm for Problem 2 (distance to passivity), which is briefly addressed in Section 5.

In Section 6 we show how the inherent low-rank structure of the problem can be exploited for large sparse systems.

In Section 7 we present the results of numerical experiments.
2. Optimizing the eigenvalue of smallest positive real part over perturbations of a fixed norm. A main tool used in the paper is related to the perturbation of the eigenvalues of Hamiltonian matrices. We repeatedly use the following basic result.

**Lemma 2.1.** [10] Consider a differentiable matrix valued function \( C(t) \) for \( t \) in a neighborhood of \( t_0 \in \mathbb{R} \). Let \( \lambda(t) \) be a continuous path of simple eigenvalues of \( C(t) \) for \( t \) near \( t_0 \). Let \( x_0,y_0 \) be the left and right eigenvectors, respectively, of \( C(t_0) \) corresponding to \( \lambda(t_0) \). Then \( x_0^*y_0 \neq 0 \) and \( \lambda(t) \) is differentiable at \( t_0 \) with

\[
\dot{\lambda}(t_0) = \frac{x_0^* \dot{C}(t_0) y_0}{x_0^* y_0}
\]  

(2.1)

Since we have \( x_0^*y_0 \neq 0 \), we will always apply the normalization

\[
\|x_0\| = 1, \quad \|y_0\| = 1, \quad x_0^* y_0 \text{ is real and positive.}
\]

(2.2)

Given \( \varepsilon > 0 \) and a smooth path of matrices \( E(t) \in \mathbb{R}^{k \times l} \) of unit Frobenius norm, at \( X(t) = X + L[\varepsilon E(t)] \) we compute the change of \( \phi_{\varepsilon}(E(t)) \) of (1.7) as \( t \) varies. We use Lemma 2.1 for the eigenvalue \( \lambda(t) = \lambda(M(X + L[\varepsilon E(t)]) \) of minimal positive real part (with nonnegative imaginary part), which we assume to be simple, and denote by \( x(t), y(t) \) the corresponding left and right eigenvectors in the above normalization and we set

\[
\kappa(t) = \kappa_{\varepsilon}(E(t)) := \frac{1}{x(t)^* y(t)} > 0.
\]

(2.3)

We denote by \( \langle U, V \rangle = \text{tr}(U^T V) \) the Frobenius inner product on \( \mathbb{R}^{(n+p) \times (n+m)} \) or \( \mathbb{R}^{2n \times 2n} \) or \( \mathbb{R}^{k \times l} \), as will be clear from the context. We then obtain

\[
\frac{d}{dt} \phi_{\varepsilon}(E(t)) = \frac{d}{dt} \text{Re} \lambda(M(X + L[\varepsilon E(t)])
\]

\[
= \kappa(t) \text{Re} x(t)^* \frac{d}{dt} M(X + L[\varepsilon E(t)]) y(t)
\]

\[
= \kappa(t) \langle \frac{d}{dt} M(X + L[\varepsilon E(t)]), \text{Re} x(t)^* y(t) \rangle
\]

\[
= \kappa(t) \langle M'(X + L[\varepsilon E(t)])[L[\varepsilon \dot{E}(t)]], \text{Re} x(t)^* y(t) \rangle
\]

\[
= \kappa(t) \varepsilon \langle \dot{E}(t), L^*[M'(X + L[\varepsilon E(t)])^* \text{Re} x(t)^* y(t)] \rangle,
\]

where \( L^* : \mathbb{R}^{(n+p) \times (n+m)} \to \mathbb{R}^{k \times l} \) is the adjoint of the linear map \( L : \mathbb{R}^{k \times l} \to \mathbb{R}^{(n+p) \times (n+m)} \), i.e.,

\[
\langle L[U], V \rangle = \langle U, L^*[V] \rangle \quad \text{for all} \quad U \in \mathbb{R}^{k \times l}, V \in \mathbb{R}^{(n+p) \times (n+m)},
\]

and \( M'(X)^* : \mathbb{R}^{2n \times 2n} \to \mathbb{R}^{(n+p) \times (n+m)} \) is the adjoint of the linear map \( M'(X) : \mathbb{R}^{(n+p) \times (n+m)} \to \mathbb{R}^{2n \times 2n} \). Summarizing this calculation, we have the following result.

**Lemma 2.2.** With the above notation, if \( \lambda(t) \) is a simple eigenvalue, then

\[
\frac{1}{\varepsilon \kappa(t)} \frac{d}{dt} \phi_{\varepsilon}(E(t)) = \langle G_{\varepsilon}(E(t)), \dot{E}(t) \rangle
\]
here, sym(Z) = \frac{1}{2}(Z + Z^T) denotes the symmetric part of a matrix Z.

Lemma 2.3. For \( W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \in \mathbb{R}^{2n \times 2n} \) partitioned according to the \( n \times n \) blocks of \( M(X) \), we have \( M'(X)^* [W] = \begin{pmatrix} V_A & V_B \\ V_C & V_D \end{pmatrix} \in \mathbb{R}^{(n+p) \times (n+m)} \) partitioned according to the blocks of the system matrix \( X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), with

\[
\begin{align*}
V_A &= W_{11} - W_{22}^T \\
V_B &= -(W_{11} - W_{22}^T)C^T DR^{-1} - 2 \text{sym}(W_{12})BR^{-1} \\
V_C &= -DR^{-1}B^T(W_{11} - W_{22}^T) + 2S^{-1}C \text{sym}(W_{21}) \\
V_D &= -C(W_{11} - W_{22}^T)^TBK^{-1} + 2D \text{sym}(R^{-1}B^T(W_{11} - W_{22}^T)C^T DR^{-1}) \\
&\quad + 2DR^{-1}B^T \text{sym}(W_{12})BR^{-1} - 2S^{-1}C \text{sym}(W_{21})C^T S^{-1}D.
\end{align*}
\]

Proof. For any path \( X(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} \) we have \( \dot{M} = \frac{d}{dt} M(X) = M'(X)[\dot{X}] \) and hence

\[
\langle \dot{X}, M'(X)^*[W] \rangle = \langle M'(X)[\dot{X}], W \rangle = \langle \dot{M}, W \rangle.
\]

By the block structure of the two matrices, we have

\[
\langle \dot{M}, W \rangle = \langle \dot{F}, W_{11} \rangle + \langle \dot{G}, W_{12} \rangle + \langle \dot{H}, W_{21} \rangle + \langle -\dot{F}^T, W_{22} \rangle.
\]

Using the definitions of the matrices \( F, G, H \) given in \([1,3]\) we obtain

\[
\begin{align*}
\langle \dot{F}, W_{11} \rangle &= \langle \dot{A}, W_{11} \rangle - \langle \dot{B}, W_{11}(R^{-1}D^T C)^T \rangle + \langle \dot{R}, (BR^{-1})^T W_{11}(R^{-1}D^T C)^T \rangle \\
&\quad - \langle \dot{D}^T, (BR^{-1})^T W_{11} C^T \rangle - \langle \dot{C}, (BR^{-1}D^T)^T W_{11} \rangle,
\end{align*}
\]

where we used the formula \( \frac{d}{dt}(R^{-1}) = -R^{-1} \dot{R} R^{-1} \). Moreover we note that \( \dot{R} = \dot{D}^T D + DT \dot{D} \), hence for any matrix \( K \) of suitable dimension

\[
\langle \dot{R}, K \rangle = \langle \dot{D}, D(K^T + K) \rangle = \langle \dot{D}, 2D \text{sym}(K) \rangle.
\]

Similar calculations for the other inner products give

\[
\begin{align*}
\langle \dot{G}, W_{12} \rangle &= -\langle \dot{B}, \text{sym}(W_{12})BR^{-1} \rangle + \langle \dot{D}, 2D(BR^{-1})^T \text{sym}(W_{12})BR^{-1} \rangle; \\
\langle \dot{H}, W_{21} \rangle &= \langle \dot{C}, 2S^{-1}C \text{sym}(W_{21}) \rangle - \langle \dot{D}, 2S^{-1}C \text{sym}(W_{21})(S^{-1}C)^T D \rangle; \\
\langle -\dot{F}^T, W_{22} \rangle &= -\langle \dot{F}, W_{22}^T \rangle.
\end{align*}
\]
We sum the left-hand inner products with \( \bar{F}, \bar{G}, \bar{H}, \bar{F}^T \) to obtain \( \langle \bar{X}, M'(X)'[W] \rangle \). Since \( \bar{X} \) is an arbitrary matrix in \( \mathbb{R}^{(n+p)\times(n+m)} \), we find that \( V_A, V_B, V_C, V_D \) are as listed in (2.4). \( \square \)

We further note that for the particular choice (1.6) of \( L \), we have

\[
L^*[V] = V_C Q^{-1} \quad \text{for} \quad V = \begin{pmatrix} V_A & V_B \\ V_C & V_D \end{pmatrix} \in \mathbb{R}^{(n+p)\times(n+m)}
\] (2.5)

partitioned according to the blocks of \( X \).

A differential equation for \( E(t) \) of unit Frobenius norm along which \( \phi_\varepsilon(E(t)) \)

increases (as desired for Problem 1: passivation) or decreases (as desired for Problem 2: computing the distance to the nearest non-passive system), is given by a gradient flow constrained to the unit sphere \( \|E\|_F = 1 \) in the direction of steepest ascent (for Problem 1) or steepest descent (for Problem 2) with the + or – sign, respectively:

\[
\dot{E} = \pm (G_\varepsilon(E) - \mu E),
\] (2.6)

where the Lagrange multiplier \( \mu = \langle G_\varepsilon(E), E \rangle \) is chosen to ensure that \( \frac{d}{dt}\|E\|_F^2 = 2\langle E, \dot{E} \rangle \) is identically zero when \( \|E\|_F = 1 \).

This differential equation for \( E \) is solved numerically into a stationary point. We note the following properties (cf. [3] Theorem 3.1).

**Theorem 2.4.** Along solutions \( E(t) \) of (2.6) of unit Frobenius norm for which the eigenvalue \( \lambda(t) \) of \( M(X + L[\varepsilon E(t)]) \) with smallest positive real part is simple, we have

\[
\pm \frac{d}{dt} \phi_\varepsilon(E(t)) \geq 0,
\]

and the following statements are equivalent if \( G_\varepsilon(E) \neq 0 \):

1. \( \frac{d}{dt} \phi_\varepsilon(E(t)) = 0 \).
2. \( \dot{E} = 0 \).
3. \( E \) is a real multiple of \( G_\varepsilon(E) \).

**Proof.** The short proof is essentially the same as that of Theorem 3.1 in [3]. We include it for the convenience of the reader. Let \( G = G_\varepsilon(E) \) for short. Since \( \mu = \langle G, E \rangle \) and \( \|E\|_F = 1 \), 3. implies 2., and clearly 2. implies 1. So it remains to show that 1. implies 3. We note that

\[
\pm \frac{1}{\varepsilon \kappa(t)} \frac{d}{dt} \phi_\varepsilon(E(t)) = \pm \langle G, \dot{E} \rangle = \langle G, G - \mu E \rangle = \|G\|_F^2 - \langle G, E \rangle^2 \geq 0.
\]

The last inequality holds by the Cauchy-Schwarz inequality and \( \|E\|_F = 1 \). This inequality is strict unless \( G \) is a real multiple of \( E \). Finally, \( G \) is nonzero by assumption. Hence, 1. implies 3. \( \square \)

**Remark 2.1.** Along a trajectory \( E(t) \), the eigenvalue with smallest positive real part \( \lambda(t) = \lambda(M(X + L[\varepsilon E(t)]) \) may become discontinuous (because a different branch of eigenvalues gets to have smallest positive real part) or become a multiple eigenvalue at some instance \( t \). In the case of a discontinuity, the differential equation is further solved, with an ascent/descent of the eigenvalue with smallest positive real part until finally a stationary point is approximately reached. On the other hand, the simplicity of the eigenvalues is generic in an unstructured problem, and the steepest ascent/descent direction is insensitive to some eigenvalues being very close. Even if some continuous trajectory runs into an exceptional point with multiple eigenvalues, this is highly unlikely to happen after discretization of the differential equation, and so the computation will not be affected.
3. The equation for the perturbation size $\varepsilon$ for Problem 1 (passivation).
We consider Problem 1, which corresponds to the $+$ sign above.

In order to present the outer iteration for optimizing the perturbation size $\varepsilon$, we need to study the behavior of a pair of non-imaginary eigenvalues close to coalescence on the imaginary axis.

Let $E(\varepsilon)$ of unit Frobenius norm be a local maximizer of the optimization problem \(1.7\). We let $\lambda(\varepsilon)$ denote the eigenvalue of smallest positive real part (and nonnegative imaginary part) of $M(X + L[\varepsilon E(\varepsilon)])$ and $x(\varepsilon)$ and $y(\varepsilon)$ are the corresponding left and right eigenvectors, normalized according to \(2.2\). We let $\hat{\varepsilon}$ denote the smallest value of $\varepsilon$ such that

$$f(\varepsilon) := \phi_\varepsilon(\varepsilon E(\varepsilon)) = \text{Re} \lambda(\varepsilon)$$

becomes zero and is nonzero to the right of $\varepsilon$:

$$f(\varepsilon) = 0 \quad \text{for } 0 < \varepsilon \leq \hat{\varepsilon} \quad \text{and} \quad f(\varepsilon) > 0 \quad \text{for } \varepsilon > \hat{\varepsilon} \text{ near } \hat{\varepsilon}.$$ 

For a given small threshold $\delta > 0$, we denote by $\varepsilon_\delta > \hat{\varepsilon}$ the smallest value of $\varepsilon$ such that $f(\varepsilon)$ equals $\delta$:

$$f(\varepsilon_\delta) = \delta.$$

To determine $\varepsilon_\delta$, we are thus left with a one-dimensional root-finding problem. In the following we show that, under additional assumptions that appear generically satisfied, the function $f$ is differentiable in a right neighborhood of $\hat{\varepsilon}$ with the exception of $\hat{\varepsilon}$ and has a square-root behavior $f(\varepsilon) \sim \sqrt{\varepsilon - \delta}$ as $\varepsilon \searrow \hat{\varepsilon}$.

**Assumption 3.1.** For $\varepsilon$ in a right neighborhood of $\hat{\varepsilon}$, i.e., $\varepsilon \in (\hat{\varepsilon}, \varepsilon_\delta)$, we assume that the eigenvalue $\lambda(\varepsilon)$ of smallest positive real part with nonnegative imaginary part of the Hamiltonian matrix $M(\varepsilon) := M(X + L[\varepsilon E(\varepsilon)])$ is unique and a simple eigenvalue. Moreover, $E(\varepsilon)$ and $\lambda(\varepsilon)$ are assumed to be smooth functions of $\varepsilon$.

Under Assumption \(3.1\) also the associated normalized eigenvectors $x(\varepsilon), y(\varepsilon)$ with $x(\varepsilon)^* y(\varepsilon) > 0$ are smooth functions of $\varepsilon$. We refer to Remark 3.2 in [8] for useful comments, which apply also here, on an analogous assumption in a different context.

**Assumption 3.2.** We assume that the gradient $G(\varepsilon) := G_\varepsilon(E(\varepsilon))$ is different from zero for $\varepsilon \in (\hat{\varepsilon}, \varepsilon_\delta)$.

The following result provides an inexpensive formula for the computation of the derivative of $\varepsilon \mapsto \phi_\varepsilon(E(\varepsilon))$, which will be used in a Newton-type outer iteration of the method. We denote

$$f(\varepsilon) = \phi_\varepsilon(E(\varepsilon)), \quad G(\varepsilon) = G_\varepsilon(E(\varepsilon)), \quad \kappa(\varepsilon) = \kappa_\varepsilon(E(\varepsilon)) = \frac{1}{x(\varepsilon)^* y(\varepsilon)}.$$ 

**Lemma 3.1.** Under Assumption \(3.1\), the function $f$ is differentiable and monotone increasing on the interval $(\hat{\varepsilon}, \varepsilon_\delta)$, and its derivative satisfies (with $' = d/d\varepsilon$)

$$f'(\varepsilon) = \kappa(\varepsilon) \|G(\varepsilon)\|_F.$$  \hspace{1cm} (3.1)

**Proof.** Differentiating $f(\varepsilon) = \phi_\varepsilon(E(\varepsilon))$ with respect to $\varepsilon$ we obtain, by the same calculation that led to Lemma \(2.2\)

$$f'(\varepsilon) = \kappa(\varepsilon) \langle G(\varepsilon), E(\varepsilon) + \varepsilon E'(\varepsilon) \rangle.$$ \hspace{1cm} (3.2)
From here on the proof is completed by observing that $|\langle G(\varepsilon), E(\varepsilon) \rangle| = \|G(\varepsilon)\|_F$ and $\langle G(\varepsilon), E'(\varepsilon) \rangle = 0$, as in Lemma 3.2 in [8]. This yields $|f'(\varepsilon)| = \kappa(\varepsilon) \|G(\varepsilon)\|_F$. Since $f(\varepsilon) = 0$ and $f(\varepsilon) > 0$ for $\varepsilon > \varepsilon_0$ near $\varepsilon$, the derivative $f'$ must somewhere be positive. Since $\kappa(\varepsilon) > 0$ and $G(\varepsilon) \neq 0$ by assumption, $f'$ cannot change sign and is therefore positive for all $\varepsilon \in (\hat{\varepsilon}, \bar{\varepsilon})$.

**Assumption 3.3.** We assume that the limit $E(\varepsilon) := \lim_{\varepsilon \searrow \varepsilon_0} E(\varepsilon)$ exists and that the eigenvalue $\lambda(\varepsilon) = \lim_{\varepsilon \searrow \varepsilon_0} \lambda(\varepsilon)$ of $M(\varepsilon) = M(X + L[E(\varepsilon)])$ has algebraic multiplicity two and is defective (that is, the zero singular value of $M(\varepsilon) - \lambda(\varepsilon)I$ is simple).

An illuminating example. In order to make Assumption 3.3 plausible we consider the $2 \times 2$ system

\[
X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]  
(3.3)

with $a, b, c, d \in \mathbb{R}$. This gives

\[
M(X) = \frac{1}{d^2 - 1} \begin{pmatrix} a(d^2 - 1) - cb^2 & -b^2 \\ c^2 & -a(d^2 - 1) + cb^2 \end{pmatrix}
\]  
(3.4)

The condition to have a coalescent pair of eigenvalues at $z = 0$ is

\[
a = a_0 = \frac{(b^2 \pm b)c}{d^2 - 1}.
\]  
(3.5)

On the other hand the singular values of $A$ under condition (3.5) are $\sigma_1 = 0$ and

\[
\sigma_2^2 = \frac{(b^2 + c^2)^2}{(d^2 - 1)^2}.
\]  
(3.6)

We easily obtain that $\sigma_1 = \sigma_2 = 0$ implies $b = c = 0$, which implies that the only case where the eigenvalue is non defective is associated to

\[
X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

which is a codimension 3 phenomenon. So we can assert that the annihilation of both singular values is strongly non-generic.

We wish to investigate the behavior of a pair of eigenvalues close to coalescence on the imaginary axis. To give an indication, we consider again Example 3.3. Consider now a small perturbation of $a$ around $a_0$ (see (3.5)),

\[
a = \frac{(b^2 \pm b)c}{d^2 - 1} - \mu, \quad |\mu| \ll 1.
\]

Then we obtain the perturbed eigenvalues of $M(X)$,

\[
\lambda_{1,2} = \frac{\sqrt{\mu} \sqrt{2bc + d^2 \mu - \mu}}{d^2 - 1} = \sqrt{\mu} \sqrt{\frac{2bc}{d^2 - 1} + \frac{d^2 - 1 - \frac{3}{2} \mu^2 \sqrt{bc}}{2\sqrt{2bc}}} + O\left(|\mu|^{5/2}\right),
\]  
(3.7)

which shows that close to $\mu = 0$ the eigenvalues have a square root behavior which determines a loss of smoothness. Moreover, assuming $bc > 0$, for $\mu > 0$ the eigenvalues
are real and coalesce at 0. Afterwards when \( \mu < 0 \) the eigenvalues split along the imaginary axis, where they remain due to the Hamiltonian symmetry.

Under Assumption 3.3 it follows from [4] that then also left and right eigenvectors \( x(\hat{\varepsilon}) \) and \( y(\hat{\varepsilon}) \) of \( M(\hat{\varepsilon}) \) are obtained as the limits of corresponding eigenvectors as \( \varepsilon \searrow \hat{\varepsilon} \). Since \( \lambda(\hat{\varepsilon}) \) is a defective eigenvalue, we have \( x(\hat{\varepsilon})^* y(\hat{\varepsilon}) = 0 \). Moreover, since \( \text{Re} \lambda(\hat{\varepsilon}) = 0 \), we then have

\[
Jx(\hat{\varepsilon}) \propto y(\hat{\varepsilon}),
\]

where the proportionality is with a complex number of unit modulus. The next result shows that the factor of proportionality is real.

**Theorem 3.2.** Under Assumptions 3.1–3.3 and the normalization \( x(\varepsilon)^* y(\varepsilon) > 0 \) for \( \varepsilon > \hat{\varepsilon} \) we have in the limit

\[
y(\hat{\varepsilon}) = \pm Jx(\hat{\varepsilon}).
\]

**Proof.** By [2, Theorem 2.6], there exists a real symplectic matrix \( S_\varepsilon \), for \( \varepsilon > \hat{\varepsilon} \), such that

\[
M_0(\varepsilon) = S_\varepsilon^{-1} M(\varepsilon) S_\varepsilon, \quad M_0(\varepsilon) = \begin{pmatrix} F_\varepsilon & G_\varepsilon \\ 0 & -F_\varepsilon^T \end{pmatrix}
\]

(3.8)

with \( G_\varepsilon \) symmetric, i.e., \( M_0(\varepsilon) \) is real Hamiltonian quasi-triangular.

For the eigenvalue \( \lambda(\varepsilon) \), the left and right eigenvectors of \( M_0(\varepsilon) \) are related to those of \( M(\varepsilon) \) by

\[
x_0(\varepsilon) = S_\varepsilon^T x(\varepsilon), \quad y_0(\varepsilon) = S_\varepsilon^{-1} y(\varepsilon).
\]

(3.9)

We assumed that \( x(\varepsilon) \) and \( y(\varepsilon) \) are normalized such that \( x(\varepsilon)^* y(\varepsilon) > 0 \) for \( \varepsilon > \hat{\varepsilon} \), and hence we have also

\[
x_0(\varepsilon)^* y_0(\varepsilon) > 0.
\]

(3.10)

Noting that the lower half of the right eigenvector \( y_0(\varepsilon) \) to the block triangular matrix \( M_0(\varepsilon) \) consists only of zeros, we split the eigenvectors into the upper and the lower \( n/2 \)-dimensional subvectors as

\[
y_0(\varepsilon) = \begin{pmatrix} -p(\varepsilon) \\ 0 \end{pmatrix}, \quad x_0(\varepsilon) = \begin{pmatrix} -s(\varepsilon) \\ r(\varepsilon) \end{pmatrix}.
\]

(3.11)

By the Hamiltonian symmetry, we can represent the eigenvectors associated to the eigenvalue \(-\overline{\lambda}(\varepsilon)\) as \( \overline{y}_0(\varepsilon) = Jx_0(\varepsilon), \overline{x}_0(\varepsilon) = Jy_0(\varepsilon) \). This gives

\[
\overline{y}_0(\varepsilon) = \begin{pmatrix} r(\varepsilon) \\ s(\varepsilon) \end{pmatrix}, \quad \overline{x}_0(\varepsilon) = \begin{pmatrix} 0 \\ p(\varepsilon) \end{pmatrix}.
\]

(3.12)

By the convergence of the eigenvectors, we have that for some complex \( \xi, \eta \) of unit modulus,

\[
\lim_{\varepsilon \to \hat{\varepsilon}} \overline{y}_0(\varepsilon) = -\eta \lim_{\varepsilon \to \hat{\varepsilon}} y_0(\varepsilon), \quad \lim_{\varepsilon \to \hat{\varepsilon}} \overline{x}_0(\varepsilon) = \xi \lim_{\varepsilon \to \hat{\varepsilon}} x_0(\varepsilon).
\]

(3.13)
We thus obtain
\[
\lim_{\varepsilon \to \hat{\varepsilon}} r(\varepsilon) = \eta \lim_{\varepsilon \to \hat{\varepsilon}} p(\varepsilon), \quad \lim_{\varepsilon \to \hat{\varepsilon}} p(\varepsilon) = \xi \lim_{\varepsilon \to \hat{\varepsilon}} r(\varepsilon)
\] (3.14)
so that
\[
\xi = \bar{\eta}. \tag{3.15}
\]
By (3.10),
\[
s(\varepsilon)^*p(\varepsilon) \text{ is real and positive for } \varepsilon > \hat{\varepsilon}. \tag{3.16}
\]
As is noted in [15], we have
\[
s(\varepsilon)^*r(\varepsilon) \text{ is real for } \varepsilon > \hat{\varepsilon}. \tag{3.17}
\]
Premultiplying the terms in the first limit in (3.14) by \(s(\varepsilon)^*\) and scaling by \(|s(\varepsilon)^*p(\varepsilon)|\) we get
\[
\lim_{\varepsilon \to \hat{\varepsilon}} s(\varepsilon)^*r(\varepsilon) \left/ \left| s(\varepsilon)^*p(\varepsilon) \right| \right. = \eta \lim_{\varepsilon \to \hat{\varepsilon}} s(\varepsilon)^*p(\varepsilon),
\] (3.18)
which yields that \(\eta\) is real and hence \(\eta = \pm 1\). As a consequence, we obtain
\[
y_0(\hat{\varepsilon}) = \pm Jx_0(\hat{\varepsilon}). \tag{3.19}
\]
Then it is direct to conclude that
\[
x(\varepsilon) = \pm \left( S_{\varepsilon}^{-1} \right)^T x_0(\varepsilon) = \mp JS_{\varepsilon}Jx_0(\varepsilon), \quad y(\varepsilon) = \pm S_{\varepsilon}Jx_0(\varepsilon) \tag{3.20}
\]
and therefore
\[
y(\hat{\varepsilon}) = \pm Jx(\hat{\varepsilon}), \tag{3.21}
\]
which proves the result. □

**Theorem 3.3.** Under Assumptions 3.1–3.3 we have
\[
\Re \lambda(\varepsilon) = \gamma \sqrt{\varepsilon - \hat{\varepsilon}} (1 + o(1)) \quad \text{as } \varepsilon \searrow \hat{\varepsilon}
\]
for some positive constant \(\gamma\).

**Proof.** The proof adapts the proof of Theorem 5.2 in [7] to the current situation. We consider the nonnegative function
\[
\vartheta(\varepsilon) := \frac{1}{\kappa(\varepsilon)} = x(\varepsilon)^*y(\varepsilon) > 0 \text{ for } \varepsilon \in (\hat{\varepsilon}, \bar{\varepsilon}), \quad \vartheta(\bar{\varepsilon}) = 0.
\]
Based on results in [14, 9], the derivative \(\vartheta'(\varepsilon)\) is obtained in part (a) of the proof of Theorem 5.2 in [7] as
\[
\vartheta'(\varepsilon) = x(\varepsilon)^*M'(\varepsilon)Z(\varepsilon)x(\varepsilon)\vartheta(\varepsilon) + y(\varepsilon)^*Z(\varepsilon)M'(\varepsilon)y(\varepsilon)\vartheta(\varepsilon),
\] (3.22)
where the group inverse \(Z(\varepsilon)\) of \(N(\varepsilon) := M(\varepsilon) - \lambda(\varepsilon)I\) is related to the pseudoinverse \(N(\varepsilon)^\dagger\) by the formulas
\[
Z(\varepsilon) = \frac{1}{\vartheta(\varepsilon)^2} \hat{Z}(\varepsilon)
\]
\[
\hat{Z}(\varepsilon) = (\vartheta(\varepsilon)I - y(\varepsilon)x(\varepsilon)^*)N(\varepsilon)^\dagger(\vartheta(\varepsilon)I - y(\varepsilon)x(\varepsilon)^*). \tag{3.23}
\]
By Assumption 3.3, $N(\varepsilon)^\dagger$ remains bounded as $\varepsilon \searrow \hat{\varepsilon}$, and thus
\begin{equation}
\hat{Z}(\varepsilon) = y(\varepsilon)x(\varepsilon)^*N(\varepsilon)^\dagger y(\varepsilon)x(\varepsilon)^* + O(\vartheta(\varepsilon)) = \nu(\varepsilon)y(\varepsilon)x(\varepsilon)^* + O(\vartheta(\varepsilon)) \tag{3.24}
\end{equation}

with the factor
\[ \nu(\varepsilon) := x(\varepsilon)^*N(\varepsilon)^\dagger y(\varepsilon), \]
which has $\nu(\hat{\varepsilon}) \neq 0$ by part (b) of the proof of Theorem 5.2 in [7]. Furthermore, we set
\[ \mu(\varepsilon) := x(\varepsilon)^*M'(\varepsilon)y(\varepsilon), \]
for which we note that $\mu(\varepsilon) = \lambda'(\varepsilon)\vartheta(\varepsilon)$ and hence, by Lemma 3.1, $\Re \mu(\varepsilon) = \|G(\varepsilon)\|_F$, which has a well-defined nonzero limit as $\varepsilon \searrow \hat{\varepsilon}$.

We insert the expression for the group inverse $Z(\varepsilon)$ into (3.22) and note that $x(\varepsilon)^*M'(\varepsilon)N(\varepsilon)x(\varepsilon) = 0$ and $y(\varepsilon)^*N(\varepsilon)M'(\varepsilon)y(\varepsilon) = 0$ because of $N(\varepsilon)x(\varepsilon) = 0$ and $y(\varepsilon)^*N(\varepsilon) = 0$, which follows from $x(\varepsilon)^*N(\varepsilon) = 0$ and $N(\varepsilon)y(\varepsilon) = 0$, respectively. We then obtain
\begin{equation}
\dot{\vartheta}(\varepsilon) = x(\varepsilon)^*M'(\varepsilon)\hat{Z}(\varepsilon)x(\varepsilon) + y(\varepsilon)^*\hat{Z}(\varepsilon)M'(\varepsilon)y(\varepsilon) = 2\nu(\varepsilon)\mu(\varepsilon) + O(\vartheta(\varepsilon)\mu(\varepsilon)). \tag{3.25}
\end{equation}

In the limit $\varepsilon \searrow \hat{\varepsilon}$ we have, recalling that $Jx(\hat{\varepsilon}) = \pm y(\hat{\varepsilon})$ and noting that $JN(\hat{\varepsilon})$ is a hermitian matrix,
\[ \nu(\hat{\varepsilon}) = x(\hat{\varepsilon})^*N(\hat{\varepsilon})^\dagger y(\hat{\varepsilon}) = x(\hat{\varepsilon})^*(JN(\hat{\varepsilon}))^\dagger y(\hat{\varepsilon}) = \mp x(\hat{\varepsilon})^*(JN(\hat{\varepsilon}))^\dagger x(\hat{\varepsilon}) \in \mathbb{R}, \]
which implies that
\[ \mu(\varepsilon)(1 + O(\vartheta(\varepsilon))) = \frac{\vartheta'(\varepsilon)\vartheta(\varepsilon)}{2\nu(\varepsilon)} \in \mathbb{R}. \]

Using the fact that $\lim_{\varepsilon \searrow \hat{\varepsilon}} \Re \mu(\varepsilon) = \|G(\hat{\varepsilon})\|_F$ exists and is nonzero by assumption, this implies that
\[ \mu(\hat{\varepsilon}) := \lim_{\varepsilon \searrow \hat{\varepsilon}} \mu(\varepsilon) \in \mathbb{R} \text{ exists and is nonzero.} \]

Hence, the right-hand side of (3.25) has a nonzero finite limit as $\varepsilon \searrow \hat{\varepsilon}$. The proof is then concluded as in part (d) of the proof of Theorem 5.1 in [7]. This first shows that $\vartheta(\varepsilon) \sim \sqrt{\varepsilon - \hat{\varepsilon}}$ and then that $\Re \lambda(\varepsilon)$ is asymptotically proportional to $\vartheta(\varepsilon)$ in a right neighborhood of $\hat{\varepsilon}$. □

4. An algorithm to solve Problem 1 (passivity enforcement). To recap, Problem 1 is as follows: given a matrix $X$ such that $M(X)$ has a few eigenvalues on the imaginary axis and given a threshold $\delta > 0$, find a nearest (in the Frobenius-norm) structured matrix $X + L[\varepsilon E]$ such that the closest eigenvalues of $M(X + L[\varepsilon E])$ to the imaginary axis are $\delta$-close to it.

Preliminarily we have to find an initial perturbation $\varepsilon_0 E_0$ such that
\[ M(X + L[\varepsilon_0 E_0]) \text{ has no eigenvalues on the imaginary axis} \]
which implies
\[ \phi_{\varepsilon_0}(E_0) > 0, \]
where
\[ \phi_{\varepsilon}(E) = \text{Re} \lambda_1(M(X + L[\varepsilon E])) \]
with \( \lambda_1(M) \) denoting the leftmost eigenvalue of \( M \) in the right complex half-plane \( \mathbb{C}^+ = \{ z \in \mathbb{C} \mid \text{Re } z \geq 0 \} \).

This can be obtained by applying a few methods available in the literature (see e.g. [6]). Afterwards our aim is that of optimizing \( \varepsilon \).

In particular the algorithm has the goal to compute
\[ \hat{\varepsilon}_\delta = \inf \{ \varepsilon \geq 0 : \phi_{\varepsilon}(E(\varepsilon)) = \delta \}, \quad (4.1) \]
where
\[ E(\varepsilon) = \arg \max_{E : \|E\|_F = 1} \phi_{\varepsilon}(E). \]

Repeated use of Algorithm 1, which we are going to present in Section 4.1, is used to determine the extremizer \( E \) such that the leftmost eigenvalue of \( M(X + L[\hat{\varepsilon}_\delta E]) \) in the right half-plane has real part \( \delta \).

Since Algorithm 1 finds locally leftmost points, the algorithms we present here are guaranteed only to find upper bounds on \( \hat{\varepsilon}_\delta \); nevertheless they typically find good approximations for the cases we have tested.

4.1. Integration of the constrained gradient system. We assume that \( M(X + L[\varepsilon_0 E_0]) \) has no imaginary eigenvalues. For problems of moderate size we integrate the ODE
\[ \dot{E} = +G_{\varepsilon}(E) - \mu E, \quad \mu = \langle G_{\varepsilon}(E), E \rangle. \quad (4.2) \]

which - as we have seen - is the constrained gradient systems for the functional \( \phi_{\varepsilon}(E) \).

In the sequel we use the following notation: all quantities written as \( g(\varepsilon) \), like \( \lambda(\varepsilon), E(\varepsilon) \) and so on, are intended to be associated to stationary points (i.e., local extremizers) of (4.2).

We discretize the differential equation (4.2) by the explicit Euler method with an adaptively chosen stepsize. At each discretization step, we require – according to the monotonicity property of Theorem 2.4 – that the real part of the leftmost positive eigenvalue of \( A + \varepsilon E \) is decreased, for a given \( \varepsilon > 0 \).

In this way the method determines a sequence \( (\lambda_n, E_n) \) such that \( \text{Re } \lambda_n < \text{Re } \lambda_{n-1} \), until \( E_n \) approaches a stationary point.

4.2. An Euler step of integration. Given \( E_n \approx E(t_n) \) of unit Frobenius norm, and given \( x_n \) and \( y_n \) left and right eigenvectors of \( M(X + L[\varepsilon E]) \) associated with its leftmost positive eigenvalue \( \lambda_n \), with \( x_n^* y_n > 0 \), we determine the following approximations at time \( t_{n+1} = t_n + h_n \), by applying an Euler step to (4.2).

The stepsize control is only based on the preservation of the monotonicity property, since a very accurate approximation of the exact trajectory for \( E(t) \) is not necessary. The cost of one step of the algorithm is dominated by the computation of rightmost eigenvalues and associated eigenvectors of an Hamiltonian matrix.
Algorithm 1: Euler step, full rank integration

Data: \(E_n, x_n, y_n, \lambda_n, \gamma\) and \(\rho_n\) (step size predicted the previous step)

Result: \(E_{n+1}, x_{n+1}, y_{n+1}, \lambda_{n+1}\) and \(\rho_{n+1}\)

begin
1. Set \(h = \rho_n\), \(G_n = G(\varepsilon E_n) = L^* \left( M'(X + L[\varepsilon E_n])^* |\text{Re} (x_n y_n^*) \right) \), \(\mu_n = \langle G_n, E_n \rangle\)
2. Compute \(\hat{E}_{n+1} = E_n + h((G_n - \mu E_n)), \ E_{n+1} = \hat{E}_{n+1}/||\hat{E}_{n+1}||_F\)
3. Compute the leftmost positive eigentriple \(\hat{\lambda}, \hat{x}\) and \(\hat{y}\) of \(M(X + L[\varepsilon E_{n+1}])\)
4. if \(\text{Re}(\hat{\lambda}) \leq \text{Re}(\lambda_n)\) then
   - reject the step, reduce the step size as \(h := h/\gamma\) and repeat from 3
   else
   - accept the step: set \(h_{n+1} = h, \ \lambda_{n+1} = \hat{\lambda}, \ x_{n+1} = \hat{x}\) and \(y_{n+1} = \hat{y}\)
5. if \(h_{n+1} = \rho_n\) then
   - increase the stepsize as \(\rho_{n+1} := \gamma \rho_n\)
   else
   - set \(\rho_{n+1} = \rho_n\)
6. Proceed to next step

Initial conditions.
In the beginning we assume to be provided by some existing code a a matrix \(\varepsilon_0 E_0\) such that \(M(X + L[\varepsilon_0 E_0])\) has no imaginary eigenvalues.

In subsequent steps, for a given \(\varepsilon_{\ell+1}\) we consider as \(E(0) = E_{\ell}\), the final matrix computed by Algorithm 1 for \(\varepsilon = \varepsilon_{\ell}\).

Terminating at a globally positive leftmost point. In order to find a global (and not just local) extremizer, we run the algorithm with a set of different initial values to reduce the possibility of getting trapped in a point that is only locally leftmost, which cannot be excluded from following a single trajectory and which we have indeed occasionally observed in our numerical experiments. In view of the monotonicity of the real part of the eigenvalue along a trajectory (Theorem 2.4), it appears that only exceptional trajectories would run into a stationary point that does not correspond to a locally leftmost point of the pseudospectrum, and in fact we never observed such a situation in our extensive numerical experiments.

4.3. The outer iteration to approximate \(\hat{\varepsilon}\). In order to compute the value of \(\hat{\varepsilon}\) defined in (4.1), starting from \(\varepsilon > 0\) such that \(\phi_{\varepsilon}(E(\varepsilon)) > 0\), we want to compute a root \(\hat{\varepsilon}\) of the equation

\[
    f(\varepsilon) := \phi_{\varepsilon}(E(\varepsilon)) = \delta. \tag{4.3}
\]

Remark 4.1. Since the function \(\varepsilon \mapsto \phi_{\varepsilon}(E(\varepsilon))\) is monotonically increasing in a right neighbourhood of the minimal solution \(\hat{\varepsilon}\) of (4.3), and we have a formula for the derivative of \(\phi_{\varepsilon}(E(\varepsilon))\) with respect to \(\varepsilon\), we can apply a Newton-bisection method in order to compute a zero of Equation (4.3). \(\diamond\)
4.4. The case when $\delta$ is not too small; a Newton bisection method.

Recalling that we have introduced the notation $\phi_\varepsilon(E(\varepsilon)) = \text{Re}\lambda(\varepsilon)$ according to Theorem 3.3 we have that the function $\varepsilon \mapsto \phi_\varepsilon(E(\varepsilon))$ has a graph like that shown in Figure 4.4.

![Figure 4.1](image)

**Fig. 4.1.** The function $f(\varepsilon) = \phi_\varepsilon(E(\varepsilon))$ in a neighborhood of $\hat{\varepsilon}$.

For conciseness we omit the dependence on $E$ in the coded algorithms and denote it simply by $\phi_\varepsilon$. We consider a given $\delta > 0$.

In view of Lemma 3.1, it seems very natural to make use of a Newton iteration,

$$
\varepsilon_{n+1} = \varepsilon_n - \left(f(\varepsilon_n) + \delta\right) \frac{1}{\kappa(\varepsilon_n)\|G(\varepsilon_n)\|_F}, \quad n = 0, 1, \ldots
$$

possibly coupled with bisection (see Algorithm 2 where we denote by $f'$ the derivative of $\phi_\varepsilon$ with respect to $\varepsilon$ stated in Lemma 3.1).

The method yields quadratic convergence to $\hat{\varepsilon}_\delta$ when this is a simple root.

**Remark 4.2.** Although the Newton method proposed in this section relies on the requirement of a smooth dependence of $E(\varepsilon)$ on $\varepsilon$, we combine it with a bisection method to guarantee convergence also in cases where the solutions are not smooth. On the other hand, we note that in all numerical experiments we have performed, we have always observed - at least when $\delta$ is not extremely small - a fast convergence resulting from such a smoothness assumption. This leads us to conjecture that Assumption 3.1 holds true generically. ○

4.5. The case of very small $\delta$. Since we expect a defective coalescence on the imaginary axis, according to Theorem 3.3 we expect that the dependence $\varepsilon \mapsto \lambda(\varepsilon)$ is not smooth when $\text{Re}\lambda(\varepsilon)$ vanishes.

A modified method. The Newton method is expected to perform very well when $\delta$ is not too small since when $\delta$ is very small, that is when $\varepsilon$ is very close to $\hat{\varepsilon}$, the loss of differentiability of the function $f(\varepsilon)$ at $\hat{\varepsilon}$ leads to difficulties in the Newton iteration and results in a bisection-like behaviour of the method.

We emphasize that Theorem 3.3 is useful to justify a fast algorithm for the computation of the distances we are interested in, but it is not essential for the two-level approach taken here. In fact, we have to compute the zero of a monotonically in-
Algorithm 2: Newton-bisection method for Problem 1

**Data:** Matrix $X$, $\varepsilon_0$, $E_0$, $k_{\text{max}}$ (max number of iterations), tol (tolerance)

$\varepsilon_{\text{lb}}$ and $\varepsilon_{\text{ub}}$ (starting values for the lower and upper bounds for $\hat{\varepsilon}_{\delta}$)

**Result:** $\hat{\varepsilon}_{\delta}$ (upper bound for the measure)

begin
1. Solve the ode (4.2) with $\varepsilon = \varepsilon_0$ and initial datum $E(0) = E_0$
2. Set $\lambda(\varepsilon_0)$ positive leftmost eigenvalue of $M(X + L[\varepsilon_0 E(\varepsilon_0)])$ in the right complex plane, $x(\varepsilon_0)$ and $y(\varepsilon_0)$ the corresponding left and right eigenvectors
3. Compute $f(\varepsilon_0)$, $f'(\varepsilon_0)$
4. Set $\varepsilon_1 = -f(\varepsilon_0)/f'(\varepsilon_0)$
5. Set $k = 0$
6. Initialize lower and upper bounds: $\varepsilon_{\text{lb}} = 0$, $\varepsilon_{\text{ub}} = +\infty$

while $|f(\varepsilon_k) - f(\varepsilon_{k-1})| < \text{tol}$ do
7. Solve the ode (4.2) with $\varepsilon = \varepsilon_k$ and initial datum $E(0) = E(\varepsilon_{k-1})$
8. Set $\lambda(\varepsilon_k)$ positive leftmost eigenvalue of $M(X + L[\varepsilon_k E(\varepsilon_k)])$ in the right complex plane, $x(\varepsilon_k)$ and $y(\varepsilon_k)$ the corresponding left and right eigenvectors
9. Compute $f(\varepsilon_k)$, $f'(\varepsilon_k)$
10. Update upper and lower bounds $\varepsilon_{\text{lb}}$, $\varepsilon_{\text{ub}}$

if $f(\varepsilon_k) = 0$ then
11. Set $\varepsilon_{\text{ub}} = \min(\varepsilon_{\text{ub}}, \varepsilon_k)$
12. Compute $\varepsilon_{k+1} = (\varepsilon_{\text{lb}} + \varepsilon_{\text{ub}})/2$ (bisection step)
else
13. Set $\varepsilon_{\text{lb}} = \max(\varepsilon_{\text{lb}}, \varepsilon_k)$
14. Compute $f'(\varepsilon_k)$
15. Compute $\varepsilon_{k+1} = \varepsilon_k - f(\varepsilon_k)/f'(\varepsilon_k)$ (Newton step)

if $k = k_{\text{max}}$ then
16. Halt
else
17. Set $k = k + 1$

Return $\hat{\varepsilon}_{\delta} = \varepsilon_k$

creasing function, which can be done - although less efficiently - by a bisection-like method.

4.6. The algorithm. For $\varepsilon$ close to $\hat{\varepsilon}$, $\varepsilon > \hat{\varepsilon}$ we expect generically

$$
\begin{align*}
  f(\varepsilon) &= \gamma \sqrt{\varepsilon - \hat{\varepsilon}} + \mathcal{O}(\varepsilon - \hat{\varepsilon})^{3/2} \\
  f'(\varepsilon) &= \frac{\gamma}{2 \sqrt{\varepsilon - \hat{\varepsilon}}} + \mathcal{O}((\varepsilon - \hat{\varepsilon})^{1/2}),
\end{align*}
$$

(4.4)

which corresponds to the coalescence of two eigenvalues.

For an iterative process, given $\varepsilon_k$, we use Lemma [3.1] to compute $f'(\varepsilon)$ and estimate $\gamma$ and $\hat{\varepsilon}$ by solving (4.4) with respect to $\gamma$ and $\hat{\varepsilon}$. We denote the solution as $\gamma_k$
Algorithm 3: Basic algorithm for computing the optimal perturbation for small $\delta$

\begin{verbatim}
Data: $\delta$, tol, $\theta$ (default 0.8), and $\varepsilon_0$ (such $f(\varepsilon_0) > tol$) 
Result: $\hat{\varepsilon}_\delta$

begin 
1. Set Reject = False and $k = 0$
2. while $|f(\varepsilon_k) - \delta| \geq tol$ do
3.   if Reject = False then
4.     Set $\bar{\varepsilon} = \varepsilon_k$, $\bar{\theta} = \theta$
5.     Compute $\gamma_k$ and $\hat{\varepsilon}_k$ by (4.5)
6.     Set $\varepsilon_{k+1} = \hat{\varepsilon}_k - \frac{\delta^2}{\gamma_k^2}$
else
7.     Set $\varepsilon_{k+1} = \bar{\theta} \varepsilon_k + (1 - \bar{\theta}) \bar{\varepsilon}$
8.     Set $\bar{\theta} = \theta \bar{\theta}$
9. end
10. Set $k = k + 1$
11. Compute $f(\varepsilon_k)$ by integrating (4.2) (complex case) with initial datum $E(\varepsilon_{k-1})$ ($k \geq 1$)
12. Compute $f'(\varepsilon_k)$ by (3.1)
13. if $f(\varepsilon_k) < tol$ then
14.   Set Reject = True
else
15.   Set Reject = False
end
16. Print $\hat{\varepsilon}_\delta \approx \varepsilon_k$
17. Halt
\end{verbatim}

and $\hat{\varepsilon}_k$, i.e.

$$
\gamma_k = \sqrt{2f(\varepsilon_k)f'(\varepsilon_k)}, \quad \hat{\varepsilon}_k = \varepsilon_k - \frac{f(\varepsilon_k)}{2f'(\varepsilon_k)} \quad (4.5)
$$

and then compute

$$
\varepsilon_{k+1} = \hat{\varepsilon}_k + \delta^2/\gamma_k^2.
$$

An algorithm based on previous formulæ is Algorithm 3.

Usually it is recommended to use $\theta$ close to 1 since by second order convergence one has

$$
|\varepsilon_{k+1} - \hat{\varepsilon}_\delta| = \mathcal{O}(|\varepsilon_{k+1} - \hat{\varepsilon}_\delta|^2).
$$

The test at line 9 - when positive - means that for $\varepsilon = \varepsilon_k$ there are coalescing eigenvalues (up to a tolerance tol). The algorithm shows quadratic convergence as we will show in the forthcoming section of numerical illustrations.

5. Problem 2: distance to the nearest non-passive system. We consider next Problem 2, for which the ODE to solve in the inner iteration is given by (2.6) with minus sign, i.e.

$$
\dot{E} = -(G_\varepsilon(E) - \mu E)
$$
with $\mu = \langle G_\varepsilon(E), E \rangle$.

The outer iteration still pivots on the behavior of a pair of non-imaginary eigenvalues close to coalescence on the imaginary axis.

**5.1. Equation for the perturbation size $\varepsilon$.** Similar to Theorem 3.3, we have now the following analogous result.

**Theorem 5.1.** Under Assumptions 3.1–3.3 (now for $0 < \varepsilon < \hat{\varepsilon}$) we have

$$\text{Re } \lambda(\varepsilon) = \gamma \sqrt{\varepsilon - \sqrt{\varepsilon} (1 + o(1))}$$

as $\varepsilon \to \hat{\varepsilon}$

for some positive constant $\gamma$.

**Fig. 5.1.** The function $f(\varepsilon) = \phi_\varepsilon(E(\varepsilon))$ in a left neighborhood of $\hat{\varepsilon}$.

In this case, still considering

$$f(\varepsilon) := \phi_\varepsilon(\varepsilon E(\varepsilon)) = \text{Re } \lambda(\varepsilon)$$

we have a graph illustrated in Figure 5.1 with the minimal zero identified by $\hat{\varepsilon}$.

**5.2. Algorithms.** The algorithm described in Section 4 can be adapted in a straightforward way. The only significant difference is that although we generically expect that the function $f(\varepsilon)$ vanishes identically for $\varepsilon > \hat{\varepsilon}$, we cannot exclude cases where the function $f(\varepsilon)$ increases for $\varepsilon > \hat{\varepsilon}$.

**6. Low-rank dynamics for large systems.** We recall that the rescaled gradient is $G_\varepsilon(E) = L^* M'(X + L[E])^* \text{Re } (x y^*) \in \mathbb{R}^{k \times l}$, where we note that $\text{Re } (x y^*)$ has rank at most 2. Lemma 2.3 then shows that the matrix $M'(X + L[E])^* \text{Re } (x y^*) \in \mathbb{R}^{2n \times 2n}$ is of moderate rank independent of $n$. If $L^*$ maps matrices of this rank to matrices of moderate rank, say at most $r$, then also $G_\varepsilon(E)$ is of rank at most $r$ for any $E$. In particular, for $L$ chosen as in (1.6), Lemma 2.3 and (2.5) show that (with $k = p$ and $l = n$)

$$\text{rank } (G_\varepsilon(E)) \leq r = 8 \text{ for all } E \in \mathbb{R}^{k \times l},$$

independently of the dimension $n$. In this section we describe an algorithm that makes use of this low-rank structure. This low-rank algorithm appears particularly suited for large-scale passivation problems. Since Theorem 2.4 shows that in a stationary point,
E is proportional to \( G_\varepsilon(E) \) and hence of rank at most \( r \), we restrict the dynamics for \( E(t) \) to matrices of rank \( r \), which turns out to have exactly the same stationary points as the original gradient system. This leads to an algorithm that works with time-dependent factor matrices of dimensions \( k \times r \), \( l \times r \), and \( r \times r \) instead of solving matrix differential equations of dimension \( k \times l \) as in the previous section, and moreover, the low-rank structure can be beneficial in the computations of eigenvalues and left and right eigenvectors of the Hamiltonian matrices \( M(X + L[\varepsilon E(t)]) \). This makes the low-rank approach favourable for the passivation of high-dimensional systems.

The proposed algorithm and its properties are very similar to those of the low-rank algorithm in [8, Section 4].

### 6.1. Rank-\( r \) matrices and their tangent matrices.

Following [12], we first collect some basic properties. Matrices of rank \( r \) form a manifold, here denoted

\[
\mathcal{M}_r = \{ E \in \mathbb{R}^{k \times l} : \text{rank}(E) = r \}.
\]

For the computation with rank-\( r \) matrices, we represent \( E \in \mathcal{M}_r \) in a non-unique way as

\[
E = USV^T, \tag{6.1}
\]

where \( U, V \in \mathbb{R}^{k \times l} \) have orthonormal columns and \( S \in \mathbb{R}^{r \times r} \) is invertible. Unlike the singular value decomposition, we do not require \( S \) to be diagonal.

The tangent space \( T_E \mathcal{M}_r \) then consists of all matrices

\[
\delta E = \delta USV^T + U\delta SV^T + US\delta V^T, \tag{6.2}
\]

where \( U^T\delta U \) and \( V^T\delta V \) are skew-hermitian \( r \times r \) matrices, and \( \delta S \) is an arbitrary \( r \times r \) matrix. Moreover, \( (\delta U, \delta S, \delta V) \) are determined uniquely by \( \delta E \) if one imposes the gauge constraints \( U^T\delta U = 0 \) and \( V^T\delta V = 0 \). The orthogonal projection onto the tangent space \( T_E \mathcal{M}_r \) is given by [12, Lemma 4.1] as

\[
P_E[G] = GVV^T - UU^TGVV^T + UU^TG. \tag{6.3}
\]

### 6.2. The gradient system restricted to rank-\( r \) matrices.

We replace the matrix differential equation (2.6) on \( \mathbb{R}^{k \times l} \) with the projected differential equation on \( \mathcal{M}_r \):

\[
\dot{E} = \pm P_E \left[ G_\varepsilon(E) - \mu E \right], \quad \text{with} \quad \mu = \langle G_\varepsilon(E), E \rangle. \tag{6.4}
\]

The following result combines Theorems 5.1 and 5.2 of [8], which apply also to the gradient flow considered here with the same proofs. The result shows, in particular, that the functional \( \phi_\varepsilon \) ascends / descends along solutions of (6.4) and that the differential equations (2.6) and (6.4) have the same stationary points.

**THEOREM 6.1.** Along solutions \( E(t) \) of the matrix differential equation (6.4) on the rank-\( r \) manifold \( \mathcal{M}_r \), with an initial value \( E(0) \in \mathcal{M}_r \) of unit Frobenius norm we have \( \|E(t)\|_F = 1 \) for all \( t \), and

\[
\pm \frac{d}{dt} \phi_\varepsilon(E(t)) \geq 0.
\]

If \( \text{rank}(G_\varepsilon(E(t))) = r \), then the following statements are equivalent:

1. \( \frac{d}{dt} \phi_\varepsilon(E(t)) = 0 \).
2. \( \dot{E} = 0 \).
3. \( E \) is a real multiple of \( G_\varepsilon(E) \).
6.3. A numerical integrator for the rank-\(r\) differential equation \((6.4)\).
The \(k \times l\) matrix differential equation \((6.4)\) can be written equivalently as a system of differential equations for the factors \(U, S, V\); see \[12\]. The right-hand sides of the differential equations for \(U\) and \(V\) contain, however, the inverse of \(S\), which leads to difficulties with standard numerical integrators when \(S\) is nearly singular, that is, when \(E\) is close to a matrix of rank smaller than \(r\). We therefore follow the alternative approach of \[13\]. This uses an integration method that is based on splitting the tangent space projection \(P_E\), which by \((6.3)\) is an alternating sum of three subprojections. A time step of the numerical integrator based on the Lie–Trotter splitting corresponding to these three terms can then be implemented in the following way. Like in \[8\], we here consider a variant of the projector-splitting integrator of \[13\] such that the unit Frobenius norm is preserved.

The algorithm starts from the factorized rank-\(r\) matrix \(E_0 = U_0 S_0 V_0^T\) of unit norm at time \(t_0\) and computes the factors of the approximation \(E_1 = U_1 S_1 V_1^T\), again of unit Frobenius norm, at the next time \(t_1 = t_0 + h\):

1. With \(G_0 = G_\varepsilon(E_0)\), set
   \[
   K_1 = U_0 S_0 - hG_0 V_0
   \]
   and, via a QR decomposition, compute the factorization
   \[
   U_1 \tilde{S}_1 \tilde{\sigma}_1 = K_1
   \]
   with \(U_1 \in \mathbb{R}^{k \times r}\) having orthonormal columns and \(\tilde{S}_1 \in \mathbb{R}^{r \times r}\) of unit Frobenius norm, and a positive scalar \(\tilde{\sigma}_1\).
2. Set
   \[
   \tilde{\sigma}_0 \tilde{S}_0 = \tilde{S}_1 + U_1^T hG_0 V_0,
   \]
   where \(\tilde{S}_0 \in \mathbb{R}^{r \times r}\) is of unit Frobenius norm and \(\tilde{\sigma}_0 > 0\).
3. Set
   \[
   L_1 = V_0 S_0^T - hG_0^T U_1
   \]
   and, via a QR decomposition, compute the factorization
   \[
   V_1 S_1^T \sigma_1 = L_1,
   \]
   with \(V_1 \in \mathbb{R}^{l \times r}\) having orthonormal columns, with \(S_1 \in \mathbb{R}^{r \times r}\) of unit Frobenius norm, and a positive scalar \(\sigma_1\).

The algorithm computes a factorization of the rank-\(r\) matrix of unit Frobenius norm
\[
E_1 = U_1 S_1 V_1^T,
\]
which is taken as an approximation to \(E(t_1)\). As is shown in \[11\], this is a first-order method that is robust to possibly small singular values of \(E_0\) or \(E_1\).

6.4. Use of the low-rank structure in the eigenvalue computation. For the computation of the gradient matrix \(G_\varepsilon(E)\), one needs to compute the eigenvalue of smallest positive real part and the associated left and right eigenvectors of the Hamiltonian matrix \(M(X + L[\varepsilon E])\). Except in the very first step of the algorithm, one can make use of the eigenvalue of smallest real part of the previous step in an
inverse iteration (and possibly of the eigenvalues of second and third smallest real part etc. to account for a possible exchange of the leading eigenvalue).

Moreover, for the choice \((1.6)\) of \(L\) we get from a perturbation \(\Delta Z = \varepsilon E\) with 
\[E = U\Sigma V^T\] of rank 8 that \(C\) is perturbed by \(\Delta C = \Delta Z Q^{-T} = \varepsilon(U\Sigma)(Q^{-1}V)^T\) of the same rank 8, which yields the perturbed Hamiltonian matrix 
\[M(X + L[\Delta Z]) = M(X) + \Delta M,\]
where the perturbation 
\[
\Delta M = \begin{pmatrix}
-BR^{-1}D^T\Delta C & 0 \\
2\text{sym}(C^T S^{-1}\Delta C) + \Delta C^T S^{-1}\Delta C & \Delta C^T D R^{-1} B^T
\end{pmatrix}
\]
is still of moderate rank (at most 40, corresponding to 5 \(n \times n\) matrices of rank 8). This fact can be put to good use in the computation of the required eigenvalues in the case of a high-dimensional system, using the Sherman–Morrison–Woodbury formula in an inverse iteration.

We further note that if \(m, p \ll n\), then \(M(X)\) can be viewed as a low-rank perturbation to the matrix 
\[
\begin{pmatrix}
A & 0 \\
0 & -A^T
\end{pmatrix}
\]
With the Sherman–Morrison–Woodbury formula, this can yield an efficient inverse iteration when \(A\) is large and sparse.

7. Numerical illustrations. The goal of this section is to provide a few illustrative numerical examples. Both Problems 1 and 2 are considered. All the numerical values are approximations of the computed values to few decimal digits.

7.1. Example for Problem 1: passivity enforcement. We consider an example from [5]. In this case the data is a linear control system whose Hamiltonian has some purely imaginary eigenvalues. Our solution strategy is to consider as initial point a perturbation of the system which makes it passive (i.e. which moves all the eigenvalues of the Hamiltonian matrix out of the imaginary axis) and then to optimize such a solution. The starting solution is computed by means of the algorithm in [5]. This algorithm perturbs only the matrix \(C\) (so that \(L\) has the form given by \((1.6)\)).

In the following we perform two different experiments.

- We optimize in terms of the matrix \(C\) only;
- We run our optimization algorithm by perturbing the whole system;

The original system consists of the following matrices:
\[
A = \begin{pmatrix}
-1/2 & 1 \\
-1 & -1/2
\end{pmatrix}, \quad B = C^T = \begin{pmatrix}
1/2 \\
1/2
\end{pmatrix}, \quad D = (1/2)
\] (7.1)

and the associated Hamiltonian matrix is
\[
\begin{pmatrix}
1/3 & 7/6 & 1/3 & 1/3 \\
-7/6 & -1/3 & -1/3 & 1/3 \\
-1/3 & -1/3 & 1/3 & 5/6 \\
-1/3 & -1/3 & -7/6 & 1/3
\end{pmatrix}
\]
whose eigenvalues are two pairs of purely imaginary eigenvalues
\[
\pm 1.1902 \ldots i, \quad \pm 0.8660 \ldots i.
\] (7.2)
Perturbing the matrix $C$ only. The first step is to get an initial approximation $X_0$ which moves all the eigenvalues out of the imaginary axis.

The algorithm might be stopped earlier.

By running the algorithm proposed in [5] we get (recalling that only $C$ is perturbed)

$$C_0 = (0.2018 \quad 0.4615)$$

and the eigenvalues of the Hamiltonian matrix $M(X_0) = M(A, B, C_0, D)$ are

$$\pm 0.3199 \pm 1.0596i.$$  

With our algorithms we are able to optimize such a starting solution by moving the eigenvalues closer to the imaginary axis, at a distance $\delta = 10^{-2}$.

First, with $L$ given by (1.13), starting from the system $X_0 = (A, B, C_0, D)$ we run our algorithm by perturbing only the matrix $C_0$ leaving $A, B, D$ unperturbed. Here we compute $Q$ such that $Q^T \in \mathbb{R}^{n \times n}$ is a Choleski factor of the controllability Gramian $G_c = Q^T Q$, as follows:

$$Q = \begin{pmatrix} 0.5916 & 0.0845 \\ 0 & 0.3780 \end{pmatrix}$$

We get the solution

$$\hat{C} = (0.3703 \quad 0.6115),$$

which is closer than $C_0$ to the data matrix $C$; in fact

$$\| (\hat{C} - C)Q^T \|_F \approx 0.1014 < 0.4719 = \| (C_0 - C)Q^T \|_F.$$  

**Table 7.1**

| it. num. | $f(\varepsilon)$ | $\varepsilon$ |
|----------|-------------------|---------------|
| 0        | 0.435329          | 0.471928      |
| 1        | 0.424050          | 0.461975      |
| 2        | 0.340189          | 0.295810      |
| 3        | 0.219163          | 0.161991      |
| 4        | 0                 | 0.081834      |
| 5        | 0.206323          | 0.152732      |
| 6        | 0.168366          | 0.131636      |
| 7        | 0.110590          | 0.112236      |
| 8        | 0.005487          | 0.101320      |
| 9        | 0.067009          | 0.105804      |
| 10       | 0.036737          | 0.102600      |
| 11       | 0.018486          | 0.101618      |
| 12       | 0.009431          | 0.101376      |
| 13       | 0.011275          | 0.101412      |
| 14       | 0.009672          | 0.101380      |
| 15       | 0.009839          | 0.101383      |
| 16       | 0.009957          | 0.101386      |
| 17       | 0.010044          | 0.101387      |
| 18       | 0.009991          | 0.101386      |

Convergence for passivity enforcement optimization - perturbation on $C$ only: dynamic which moves an eigenvalue $\delta$-close to the imaginary axis.
The eigenvalues of the Hamiltonian matrix
\[ M \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
are
\[ \pm 0.9991 \cdot 10^{-2} \pm 1.0004i \]
so that the $\delta$-closeness to the imaginary axis is achieved. Similar convergence properties to the unconstrained case can be observed in Table 7.1.

**Perturbing the whole system.** In the second case examined, we perturb the whole system $X_0 = (A, B, C_0, D)$. The computed perturbed solution, corresponding to a passive system, is given by
\[
\hat{A} = \begin{pmatrix} -0.4592 & 1.0012 \\ -0.9968 & -0.4575 \end{pmatrix}, \hat{B} = \begin{pmatrix} 0.5169 \\ 0.5445 \end{pmatrix} \\
\hat{C} = (0.2477, 0.5029), \hat{D} = (0.5587)
\]
and the associated Hamiltonian matrix
\[ M \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \]
has eigenvalues
\[ \pm 0.9997 \cdot 10^{-2} \pm 1.0750i. \]
The computed distance is
\[ \| \hat{X} - X \|_F \approx 0.1141. \]

**7.2. Example with Problem 2: distance to the nearest non-passive system.** This example is taken from [2], where the Hamiltonian matrix nearness problem was solved without taking into account the original linear time-invariant system $X$. The data of the problem are the following matrices $A, B, C, D$:
\[
A = \begin{pmatrix} -8 & -4 & -1.5 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, C^T = \begin{pmatrix} 1 \\ 1 \\ 0.75 \end{pmatrix}, D = (-0.75) \]
(7.3)
hence the eigenvalues of the Hamiltonian matrix $M(X)$ are
\[ \pm 6.5856, \pm 2.5784, \pm 0.5173. \]
The positive eigenvalue which is closer to the imaginary axis is $\lambda = 0.5173 \ldots$, which we aim to move $\delta$-close to the imaginary axis (with $\delta = 10^{-2}$). By running the proposed algorithm we get the following perturbed system $\hat{X} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix}$:
\[
\hat{A} = \begin{pmatrix} -8.0008 & -4.0060 & -1.4577 \\ 3.9986 & -0.0102 & 0.0717 \\ -0.0023 & 0.9829 & 0.1196 \end{pmatrix}, \hat{B} = \begin{pmatrix} 2.0142 \\ 0.0240 \\ 0.0399 \end{pmatrix}, \\
\hat{C} = (0.9991, 0.9936, 0.7978), \hat{D} = (-0.7335). \]
The corresponding eigenvalues of the Hamiltonian matrix $M(\hat{X})$ are
\[ \pm 6.4472, \pm 2.7294, \pm 0.010. \]

As expected the Hamiltonian matrix is different with respect to the one computed in [7], which does not correspond to any perturbed system. Note that the perturbed system is relatively close to the original one indicating a small distance to non-passivity.

In Table 7.2, we report the computed values in the outer iteration (i.e. $f(\varepsilon) := \phi(\varepsilon E(\varepsilon)) = \lambda(\varepsilon)$), which alternates Newton steps to bisection steps. We observe that the solution computed at iteration 13 is good enough, though the algorithm needs few more iterations in order to achieve more accurate values in the computed solution of the optimization problem.

Acknowledgment. The authors thank M. Karow for useful suggestions and S. Grivet Talocia for providing interesting examples for Problem 2.

Nicola Guglielmi acknowledges that his research was supported by funds from the Italian MUR (Ministero dell’Università e della Ricerca) within the PRIN 2017 Project “Discontinuous dynamical systems: theory, numerics and applications” and by the INdAM Research group GNCS (Gruppo Nazionale di Calcolo Scientifico).

REFERENCES

[1] R. Alam, S. Bora, M. Karow, V. Mehrmann, and J. Moro. Perturbation theory for Hamiltonian matrices and the distance to bounded-realness. *SIAM J. Matrix Anal. Appt.*, 32(2):484–514, 2011.

[2] P. Benner. Symplectic balancing of Hamiltonian matrices. *SIAM J. Sci. Comput.*, 22(5):1885–1904, 2000.
[3] S Boyd, V. Balakrishnan, and P. Kabamba. A bisection method for computing the $H_\infty$ norm of a transfer matrix and related problems. *Mathematics of Control, Signals and Systems*, 2:207–219, 1989.

[4] J.B. Conway and P.R. Halmos. Finite-dimensional points of continuity of lat. *Linear Algebra Appl.*, 31:95–102, 1980.

[5] S. Grivet-Talocia. Passivity enforcement via perturbation of Hamiltonian matrices. *IEEE Trans. Circuits and Systems I: Regular Papers*, 51(9):1755–1769, 2004.

[6] S. Grivet-Talocia and B. Gustavsen. *Passive macromodeling: theory and applications*. Wiley Series in Microwave and Optical Engineering, Wiley, Hoboken, New Jersey, 2015.

[7] N. Guglielmi, D. Kressner, and C. Lubich. Low rank differential equations for Hamiltonian matrix nearness problems. *Numer. Math.*, 129(2):279–319, 2015.

[8] N. Guglielmi and C. Lubich. Matrix stabilization using differential equations. *SIAM J. Numer. Anal.*, 55(6):3097–3119, 2017.

[9] N. Guglielmi and M. Overton. Fast algorithms for the approximation of the pseudospectral abscissa and pseudospectral radius of a matrix. *SIAM J. Matrix Anal. Appl.*, 32:1166–1192, 2011.

[10] T. Kato. *Perturbation theory for linear operators; 2nd ed.* Grundlehren Math. Wiss. Springer, Berlin, 1976.

[11] E. Kieri, C. Lubich, and H. Walach. Discretized dynamical low-rank approximation in the presence of small singular values. *SIAM J. Numer. Anal.*, 54:1020–1038, 2016.

[12] O. Koch and C. Lubich. Dynamical low-rank approximation. *SIAM J. Matrix Anal. Appl.*, 29(2):434–454, 2007.

[13] C. Lubich and I. V. Oseledets. A projector-splitting integrator for dynamical low-rank approximation. *BIT*, 54:171–188, 2014.

[14] C.D. Meyer and G.W. Stewart. Derivatives and perturbations of eigenvectors. *SIAM J. Numer. Anal.*, 25:679–691, 1988.

[15] C. Paige and C. Van Loan. A Schur decomposition for Hamiltonian matrices. *Linear Algebra Appl.*, 41:11–32, 1981.