Quantum $N$-toroidal algebras and extended quantized GIM algebras of $N$-fold affinization

Dedicated to R.V. Moody on the occasion of his 80th birthday

Yun Gao, Naihuan Jing, Limeng Xia, and Honglian Zhang

ABSTRACT. We introduce the notion of quantum $N$-toroidal algebras as natural generalization of the quantum toroidal algebras as well as extended quantized GIM algebras of $N$-fold affinization. We show that the quantum $N$-toroidal algebras are quotients of the extended quantized GIM algebras of $N$-fold affinization, which generalizes a well-known result of Berman and Moody for Lie algebras.

1. Introduction

One of the most important generalizations of the finite dimensional complex Lie algebra $\mathfrak{g}$ is the (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$ (without derivation), the central extension of $\mathfrak{g} \otimes \mathbb{C}[t^\pm_0]$ by the one-dimensional center $\mathbb{C}c_0$. The $N$-toroidal Lie algebra $\mathfrak{g}_{N,tor}$ is a further generalization and the infinite dimensional universal central extension of $\mathfrak{g} \otimes \mathbb{C}[t^\pm_0, \cdots, t^\pm_{N-1}]$ (c.f. [RM] etc.). The algebra $\mathfrak{g}_{2,tor} = \mathfrak{g}_{tor}$ is usually called the toroidal Lie algebra or simply referred to as the double affine Lie algebra. The $N$-toroidal Lie algebra $\mathfrak{g}_{N,tor}$ has close connections with algebraic geometry, finite groups, conformal field theory, vertex algebras, Yangians, and differential equations and so on, and there are extensive works on the general toroidal Lie algebra (c.f. [ABFP]).

The quantum toroidal algebra $U_q(\mathfrak{g}_{tor})=U_q(\mathfrak{g}_{2,tor})$ in type A was introduced by Ginzburg, Kapranov, and Vasserot [GKV] in connection with geometric realization and Langlands reciprocity for algebraic surfaces. Besides the realization of Hecke operators for vector bundles on algebraic surfaces, Varagnolo and Vasserot [VV1] proved a Schur-Weyl duality between representations of the quantum toroidal algebras $U_q(\mathfrak{g}_{tor})$ and elliptic Cherednik algebras. Vertex representations of the quantum toroidal algebras in ADE types were also realized via the McKay correspondence [FJW]. In a series of papers [M1]-[M6], Miki studied the structures and representations of the quantum toroidal algebra $U_q(\mathfrak{g}_{tor})$ exclusively in type A. In [GJ], the authors constructed explicitly an irreducible vertex representation of the quantum toroidal algebra $U_q(\mathfrak{g}_{tor})$ of type A on the basic module for the affine Lie algebra $\hat{\mathfrak{g}}_1$. In the review [H2], the representation theory of general
quantum toroidal algebras $U_q(\mathfrak{g}_{\text{tor}})$ was understood as quantum affinizations (see also [J2]). Numerous important works on the quantum toroidal algebras and quantum affinizations were carried out in [STU], [Sy], [H1], [H2], [VV2], [Na1], [Na2], [FJM1], [FJM2], [GTL], [GM], [GNW] etc. Note that most of these works studied the structures and representations of the quantum toroidal algebra in type $A$, which further admits a two-parameter deformation $U_{q,\kappa}(\mathfrak{g}_{\text{tor}})$. Despite of all these, it is still far from complete understanding of the quantum toroidal algebras in type $A$, and even less is known for the representation theory of the quantum toroidal algebras in other types.

In [BM], the authors introduced the generalized intersection matrix (GIM) of $N$-fold affinization. The GIM algebra is defined by Chevalley generators subject to Serre-like relations defined by the GIM matrix, which is similar to the Cartan matrix but with (possible) positive off-diagonal entries (c.f. [Sl, Sk]). The $N$-toroidal Lie algebras were then proved to be quotient algebras of the GIM algebra of $N$-fold affinization (c.f. Proposition 4.15 in [BM]). The quantum GIM algebra was introduced [K] based on its relation with the 2-fold affinization found in [T1, T2, LT], however, it is still mysterious in general on its relation with a possible quantum $N$-toroidal algebra. Furthermore, we notice that the quantized GIM algebras for simply-laced cases are isomorphic to subalgebras of the quantum universal enveloping algebras [GHX].

This leads to an important question on how to generalize the quantum toroidal algebra $U_q(\mathfrak{g}_{\text{tor}})$ to the quantum $N$-toroidal algebra $U_q(\mathfrak{g}_{N,\text{tor}})$ for general $N$ and uncover their relations with other important algebraic structures such as quantum GIM algebras. In the present paper, we introduce the notion of quantum $N$-toroidal algebras for all types uniformly as natural generalization of the usual quantum toroidal algebra, just like the relation between 2-toroidal Lie algebras and $N$-toroidal Lie algebras. We find out that the novel quantum $N$-toroidal algebras are closely related to general extended quantized GIM algebras of $N$-fold affinization by using some simplified Drinfeld-type generators.

In [JZ1] and [JZ2], we formulated a simplified set of Drinfeld generators for the quantum affine algebras and quantum toroidal algebras in type $A$, respectively, to simplify practical computations. In the same way, the quantum $N$-toroidal algebra can be realized as a quotient algebra of certain quantum affine algebra generated by simplified generators. Interestingly, this formulation leads to an identification of the quantum $N$-toroidal algebra as a quotient algebra of the extended quantized GIM algebra of $N$-affinization, which is consistent with the case of Lie algebras [BM]. The corresponding GIMs of $N$-fold affinization, as well as the Dynkin diagrams for $N = 3$ of the subalgebras were given case by case. Furthermore we can realize our new algebras as certain subalgebras of the quantum toroidal algebras, thus showing that our new algebras have nontrivial (vertex) representations. We remark that in type $A$ the vertex representation can also be constructed using the two-parameter deformation $U_{q,\kappa}(\mathfrak{g}_{N,\text{tor}})$, and the latter algebra is a generalization of the quantum 2-toroidal algebra for type $A$ given in [Sy].

On the other hand, we note that the quantum GIM algebras carry nontrivial finite dimensional representations [X], while it is known that the quantum toroidal algebras don’t have such representations unless the centers are trivial. This shows that the newly defined quantum $N$-toroidal algebras may help with further investigation on their categorification.

In section 3, we define the quantum $N$-toroidal algebra for all types uniformly. At the same time, we find a subset of Drinfeld generators for the quantum $N$-toroidal algebra. It is shown that the algebra generated by this subset can be realized as the quotient of the extended quantized GIM algebra of $N$-fold affinization. It turns out that the quantum 2-toroidal algebra is isomorphic to a quotient algebra of the algebra for type $A$ and equals to the algebra for other types. In general the
quantum $N$-toroidal algebra for $N \geq 2$ is isomorphic to the quotient algebra of the algebra. This main result will be verified in the next two sections for $N = 2$ and $N \geq 3$, respectively, which implies that the quantum $N$-toroidal algebra is isomorphic to the quotient algebra of the extended quantized GIM algebra of $N$-affinization. In section 6, a vertex realization of the quantum $N$-toroidal algebra is given. In the Appendix, we list the Dynkin diagrams of the GIMs of $N$-fold affinization case by case for $N = 2$ and $N = 3$ for example.

2. Extended quantized GIM algebras of $N$-fold affinization

In this section, we first recall the definition of a generalized intersection matrix (GIM for short) (c.f. [Ne], [G]) and then give a general definition of extended quantized GIM algebras of $N$-fold affinization (c.f. [K]).

DEFINITION 2.1. Let $J$ be a finite index set, a square matrix $M = (m_{ij})_{i,j \in J}$ over $\mathbb{Z}$ is called a generalized intersection matrix if it satisfies:

(C1) $m_{ii} = 2$ for $i \in J$;
(C2) $m_{ij} \cdot m_{ji}$ are nonnegative integers for $i \neq j$;
(C3) $m_{ij} = 0$ implies $m_{ji} = 0$.

REMARK 2.2. As $m_{ij}$ can be positive, the notion of GIM generalizes that of a generalized Cartan matrix.

In this paper, we only consider the symmetrizable intersection matrix (IM) $M = (m_{ij})_{i,j \in J}$, i.e. there exists an integral diagonal matrix $D$ such that $DM$ is symmetric. We fix the notation $D = \text{diag}(d_i \in \mathbb{Z}_+ | i \in J)$.

The GIM algebras were introduced by P. Slodowy as generalization of the Kac-Moody Lie algebras [Si]. Similar to the latter, a GIM algebra $\mathcal{L}(M)$ associated to a GIM $M = (m_{ij})$ can be defined by generators and relations (c.f. [BM]).

DEFINITION 2.3. The GIM algebra $\mathcal{L}(M)$ associated to a GIM $M = (m_{ij})_{i,j \in J}$ is the Lie algebra over $\mathbb{C}$ generated by $e_i, f_i, h_i$ for $i \in J$ satisfying the following relations,

(R1) For $i,j \in J$,

$$[h_i, e_j] = m_{ij}e_j, \quad [h_i, f_j] = -m_{ij}f_j, \quad [e_i, f_i] = h_i.$$ 

(R2) For $m_{ij} \leq 0$,

$$[e_i, f_j] = 0 = [f_i, e_j], \quad (ade_i)^{-m_{ij}+1}e_j = 0 = (adf_i)^{-m_{ij}+1}f_j.$$

(R3) For $m_{ij} > 0$ and $i \neq j$,

$$[e_i, e_j] = 0 = [f_i, f_j], \quad (ade_i)^{m_{ij}+1}f_j = 0 = (adf_i)^{m_{ij}+1}e_j.$$ 

Let $I_0 = \{1, 2, \ldots, n\}$ and $\tilde{J} = \{-N + 1, \ldots, -1, 0, 1, \ldots, n\}$.

DEFINITION 2.4. Let $A = (a_{ij})_{i,j \in I_0}$ be a Cartan matrix of finite type. Define

$$M = (m_{ij})_{i,j \in J} = \begin{pmatrix} T & P \\ Q & A \end{pmatrix},$$

where $T$ is the $N \times N$ matrix $\sum_{i,j} 2E_{ij}$, and $P = (p_{ij})$ (resp. $Q = (q_{ij})$) is the $N \times n$ (resp. $n \times N$) matrix given by $p_{ij} = a_{0j}$ (resp. $q_{ij} = a_{i0}$).
Remark 2.5. Note that $M$ is an $N$-fold affinization of $A$, and is exactly the GIM introduced in [BM] after reordering the index.

In [TL1, TL2, LT, GHX], the authors studied the quantized GIM algebras for simply-laced cases associated to a GIM of 2-fold affinization. We will study a more general algebraic structure, namely (for any finite simple type, which will be called the extended quantized GIM algebra of $N$-fold affinization for simplicity. Here “extended” refers to adding a derivation to the algebra. Let $\tilde{J} = J_1 \cup J_2$ be a disjoint decomposition with $\text{card}(J_1) = N$ and $\mathbb{K} = \mathbb{C}(q)$.

Definition 2.6. The extended quantized GIM algebra $U_q(\mathcal{L}(M))$ of $N$-fold affinization is the unital associative algebra over $\mathbb{K}$ generated by the elements $E_i, F_i, K_i^{\pm 1}, q^{\pm d}(i \in \tilde{J})$, satisfying the following relations:

(M1) For $i, j \in \tilde{J}, K_i^{\pm 1} K_j^{\pm 1} = 1, q^{\pm d}$ and $K_i^{\pm 1}$ commute with each other.

(M2) For $i \in J_1$ and $j \in J_2$,
\begin{align*}
q^d E_i q^{-d} &= q E_i, & q^d F_i q^{-d} &= q^{-1} F_i, \\
q^d E_j q^{-d} &= E_j, & q^d F_j q^{-d} &= F_j.
\end{align*}

(M3) For $i \in \tilde{J}$ and $j \in \tilde{J}$,
\begin{align*}
K_i E_j K_i^{-1} &= q_i^{m_{ij}} E_j, & K_i F_j K_i^{-1} &= q_i^{-m_{ij}} F_j.
\end{align*}

(M4) For $i \in \tilde{J}$, we have that
\[ [E_i, F_i] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}. \]

(M5) For $m_{ij} < 0$, we have that
\begin{align*}
1^{1-m_{ij}} \sum_{s=0}^{1-m_{ij}} (-1)^s \left[ 1 \right]_{s} E_i^{1-m_{ij}-s} E_j E_i^s &= 0,
1^{1-m_{ij}} \sum_{s=0}^{1-m_{ij}} (-1)^s \left[ 1 \right]_{s} F_i^{1-m_{ij}-s} F_j F_i^s &= 0.
\end{align*}

(M6) For $m_{ij} > 0$ and $i \neq j$, we have that
\begin{align*}
1^{1+m_{ij}} \sum_{s=0}^{1+m_{ij}} (-1)^s \left[ 1 \right]_{s} E_i^{1+m_{ij}-s} F_j E_i^s &= 0,
1^{1+m_{ij}} \sum_{s=0}^{1+m_{ij}} (-1)^s \left[ 1 \right]_{s} F_i^{1+m_{ij}-s} E_j F_i^s &= 0.
\end{align*}

(M7) For $m_{ij} = 0$ and $i \neq j$, we have that
\[ [E_i, E_j] = 0 = [E_i, F_j] = [F_i, F_j], \]
where \( q_i = q^{d_i} \), \([m]_i = \frac{q^{m} - q^{-m}}{q_i - q_i^{-1}}\), \([m]_i! = [m]_i \cdots [2]_i[1]_i\), \([m]_i^n = \frac{[m]_i!}{[n]_i! [m-n]_i!}\).

3. Quantum \( N \)-toroidal algebras \( U_q(\mathfrak{g}_{N,\text{tor}}) (N \geq 2) \)

3.1. Quantum \( q \)-bracket. We recall the quantum \( q \)-brackets for later use (c.f. \cite{GKV}). For \( v_i \in \mathbb{K}\backslash\{0\} (i = 1, \cdots, s - 1)\), the quantum \( q \)-bracket \([a_1, a_2, \cdots, a_s]_{(v_1, \cdots, v_{s-1})}\) is defined inductively by

\[
[a_1, a_2]_{v_1} = a_1 a_2 - v_1 a_2 a_1,
\]
\[
[a_1, a_2, \cdots, a_s]_{(v_1, v_2, \cdots, v_{s-1})} = [a_1, \cdots, [a_{s-1}, a_s]_{v_1}, v_2, \cdots, v_{s-1}].
\]

It follows immediately that

\[
[a, [b, c]]_{v} = [[a, b]_q, c]_{\frac{v}{q}} + q[b, [a, c]]_{\frac{v}{q}} - [a, [b, c]]_{v} + q[a, [c, b]]_{v} = 0,
\]

\[
[[a, b]_q, c]_{v} = [a, [b, c]]_{\frac{v}{q}} + q[[a, c]]_{\frac{v}{q}} - [a, [b, c]]_{v} + q[a, [c, b]]_{v} = 0.
\]

3.2. Quantum \( N \)-toroidal algebra \( U_q(\mathfrak{g}_{N,\text{tor}}) \) via generating functions. The quantum toroidal algebra \( U_q(\mathfrak{g}_{L,\text{tor}}) \) for type \( A \) was introduced in [GKV] as a two-parameter deformation. It admits the Schur-Weyl duality \([\text{VV1}]\). In \([\text{H2}]\), the quantum toroidal algebras \( U_q(\mathfrak{g}_{\text{tor}}) \) for general types were introduced as Drinfeld quantum affinizations. We will define the quantum \( N \)-toroidal algebra uniformly through the process of Drinfeld-like quantum \( N \)-affinizations. In particular, the new algebra is a natural generalization of the quantum toroidal algebras \( U_q(\mathfrak{g}_{\text{tor}}) \) (see [H2] etc.).

Let \( I = \{0, 1, \cdots, n\} \) and \( I_0 = \{1, \cdots, n\} \). Set \( \mathfrak{g} \) a complex simple Lie algebra of rank \( n \), \( \alpha_i \in I_0 \) the simple roots of \( \mathfrak{g} \) and \( \hat{\alpha} \) the non-twisted affine Lie algebra associated to \( \mathfrak{g} \). Let \( A = (a_{ij})_{i,j \in I_0} \) be the Cartan matrix of \( \mathfrak{g} \) and \( \hat{\mathfrak{h}} \) the Cartan subalgebra. Let \( \delta \) denote the primitive imaginary root of the affine Lie algebra \( \hat{\mathfrak{g}} \) and \( \theta \) the highest root of \( \mathfrak{g} \), take \( \alpha_0 = \delta - \theta \), then \( \Pi = \{\alpha_i \mid i \in I\} \) is a basis of simple roots of the affine Lie algebra \( \hat{\mathfrak{g}} \).

Let \( \hat{A} = (a_{ij})_{i,j \in \ell} \) be the generalized Cartan matrix of the affine Lie algebra \( \hat{\mathfrak{g}} \) and \( \hat{\mathfrak{h}} \) the Cartan subalgebra of \( \hat{\mathfrak{g}} \). There exists a diagonal matrix \( D = \text{diag}(d_i \mid i \in I) \) such that \( D \hat{A} \) is symmetric. The non-degenerate symmetric bilinear form \( (\cdot | \cdot) \) on \( \hat{\mathfrak{h}}^\ast \) satisfies for all \( i, j \in I \),

\[
(\alpha_i | \alpha_j) = d_i a_{ij}, \quad (\delta | \alpha_i) = (\delta | \delta) = 0,
\]

where \( \hat{\mathfrak{h}}^\ast \) denotes the dual Cartan subalgebra of \( \hat{\mathfrak{g}} \).

Let \( q_i = q^{d_i} \), \( J = \{1, \cdots, N - 1\} \), \( \ell = (k_1, k_2, \cdots, k_{N-1}) \in \mathbb{Z}^{N-1} \), and \( e_s = (0, \cdots, 0, 1, 0, \cdots, 0) \) the \( s \)th standard unit vector of \((N - 1)\)-dimensional vector. We also denote by \( \underline{0} \) the \((N - 1)\)-dimensional zero vector.

**Definition 3.1.** The quantum \( N \)-toroidal algebra \( U_q(\mathfrak{g}_{N,\text{tor}}) \) is the associative algebra over \( \mathbb{K} \) generated by \( x_i^{\pm}(k), a_i^{(s)}(\ell), K_i^{\pm 1}, \gamma_s^{\pm \frac{1}{2}}, q^{\pm d} \), \( (i \in I, s \in J, k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}) \) satisfying the
following relations,

\[(3.3) \quad \gamma_{ij}^{\pm} \text{ are central such that } \gamma_{ij}^{\pm} = 1 \text{ and } K_i^{\pm 1} K_i^{\mp 1} = 1 = q^{\pm d} q^{\mp d},\]

\[(3.4) \quad [\alpha_i^{(s)}(\ell), K_j^{\pm 1}] = 0 = [K_j^{\pm 1}, q^{\pm 1}] = [\alpha_i^{(s)}(\ell), q^{\pm 1}];\]

\[(3.5) \quad [\alpha_i^{(s)}(\ell), \alpha_j^{(s)}(\ell')] = \delta_{\ell+\ell',0} \frac{[\ell \alpha_{ij}]_i}{\ell} \cdot \left( (\gamma\gamma')^{\frac{\ell}{2}} - (\gamma\gamma')^{\frac{-\ell}{2}} \right),\]

\[(3.6) \quad q^d x_i^\pm(k) q^{-d} = q^{\pm \delta_{i,0}} x_i^\pm(k),\]

\[(3.7) \quad K_i x_j^\pm(k) K_i^{-1} = q^{\pm a_{ij}} x_j^\pm(k),\]

\[(3.8) \quad [\alpha_i^{(s)}(\ell), x_j^\pm(k)] = \pm \frac{[\ell \alpha_{ij}]_i}{\ell} \gamma_{ij}^{\pm} x_j^\pm(k + e_s),\]

\[(3.9) \quad [x_i^\pm(k e_s), x_j^\pm(k e'_s)] = 0,\]

\[(3.10) \quad [x_i^\pm((t+1)e_s), x_j^\pm(t' e_s)]_{q_i^{a_{ij}}} + [x_j^\pm((t'+1)e_s), x_i^\pm(t e_s)]_{q_i^{a_{ij}}} = 0,\]

\[(3.11) \quad [x_i^\pm(t e_s), x_j^\pm(t' e_s)] = \frac{\delta_{ij}}{q_i - q_i'} \left( \gamma_{ij}^{\frac{t-t'}{2}} \phi_i^{(s)}(t+t') - \gamma_{ij}^{-\frac{t-t'}{2}} \phi_i^{(s)}(t+t') \right),\]

where \(\phi_i^{(s)}(t), \varphi_i^{(s)}(-t) \in \mathbb{Z}_{\geq 0}\) such that \(\phi_i^{(s)}(0) = K_i, \varphi_i^{(s)}(0) = K_i^{-1}\) are defined as below:

\[
\sum_{m=0}^{\infty} \phi_i^{(s)}(m) z^{-m} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{\ell=1}^{\infty} a_i^{(s)}(\ell) z^{-\ell} \right),
\]

\[
\sum_{m=0}^{\infty} \varphi_i^{(s)}(m) z^{-m} = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{\ell=1}^{\infty} a_i^{(s)}(\ell) z^{-\ell} \right),
\]

\[(3.12) \quad \text{Sym}_{m_1, \ldots, m_n} \sum_{k=0}^{n=1-a_{ij}} (-1)^k \binom{n}{k} x_i^\pm(m_1 e_s) \cdots x_i^\pm(m_k e_s) x_j^\pm(\ell e_s) \times x_i^\pm(m_{k+1} e_s) \cdots x_i^\pm(m_n e_s) = 0, \quad i \neq j \text{ and } s \in J,
\]

\[(3.13) \quad \sum_{k=0}^{3} (-1)^k \binom{3}{k} x_i^\pm(m_1 e_s) \cdots x_i^\pm(m_{3-k} e_s) x_j^\pm(\ell e_s') \times x_i^\pm(m_{4-k} e_s) \cdots x_i^\pm(m_k e_s) = 0, \quad \text{for } i \in I \text{ and } m_1 m_2 m_3 \ell \neq 0, \quad s \neq s' \in J,
\]

where \(\text{Sym}_{m_1, \ldots, m_n}\) denotes the symmetrization with respect to the indices \((m_1, \ldots, m_n)\).

**Remark 3.2.** When \(N = 2\), Definition 3.1 is just that of the quantum toroidal algebra \((H2)\) etc.). Note that for the case of \(\mathfrak{sl}_{n+1}\), an additional parameter can be added in the definition of the quantum toroidal algebra \((GKV)\).

**Remark 3.3.** For each fixed \(s \in J\), let \(U_q^{(s)}\) be the subalgebra of \(U_q(\mathfrak{g}_{N,tor})\) generated by the elements \(x_i^\pm(k e_s), a_i^{(s)}(\ell), K_i^{\pm 1}, q^{\pm d} \text{ for } i \in I, \text{ then } U_q^{(s)}\) is isomorphic to the quantum toroidal algebra \(U_q(\mathfrak{g}_{2,tor})\).
REMARK 3.4. There exists another central element $\gamma_0 = K_0 \theta$, where $\theta$ is the highest root of the simple Lie algebra $g$.

3.3. Simplified generators and the algebra $\mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$. In this subsection, we define an algebra $\mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$ generated by finite Drinfeld generators with finitely many Drinfeld relations and we prove that the quantum $N$-toroidal algebra $\mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$ is isomorphic to a quotient of $\mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$ or $\mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$ itself (c.f. Theorem 3.12 and Theorem 3.13). We will prove these theorems in the next two subsections respectively.

It is easy to see that the elements $x_i^\pm(0), x_i^0(\theta)$, $K_i^\pm$, $q^\pm d$ and $\gamma_s^\pm$ $(\epsilon = \pm 1$ or $\pm, i \in I, s \in J)$ form a reduced set of generators for the algebra $\mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$.

DEFINITION 3.5. Denote by $\mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$ the associative algebra over $\mathbb{K}$ generated by $x_i^\pm(0), x_i^0(\theta)$, $K_i^\pm$, $q^\pm d$ and $\gamma_s^\pm$ $(\epsilon = \pm 1$ or $\pm, i \in I, s \in J)$ satisfying the following relations:

\begin{align}
(3.14) & \quad \gamma_s^\pm \text{are central such that } \gamma_s^\frac{1}{2} \gamma_s^{-\frac{1}{2}} = 1, \\
(3.15) & \quad q^\pm d \text{ and } K_i^\pm \text{ commute with each other and } K_i^\pm K_i^\mp = 1 = q^\pm d q^\mp d, \\
(3.16) & \quad K_i x_i^\pm(0) K_i^{-1} = q_i^{\pm t} x_i^\pm(0), \quad K_i x_i^0(\theta) K_i^{-1} = q_i^{-\epsilon_0} x_i^0(\theta), \\
(3.17) & \quad [x_i^0(0), x_i^0(\theta)] = 0, \quad \text{for } i \neq 0, \\
(3.18) & \quad [x_i^0(0), x_j^0(0)] = \delta_{ij} K_i - K_i^{-1} q_i - q_i^{-1}, \quad [x_i^0(\theta), x_i^0(\theta)] = \gamma_s^{-1} K_0 - \gamma_s K_0^{-1}, \\
(3.19) & \quad q^d x_i^\pm(0) q^{-d} = q^\pm d x_i^\mp(0), \quad q^d x_i^0(\theta) q^{-d} = q^{-\epsilon} x_i^0(\theta), \\
(3.20) & \quad [x_i^0(\theta), x_j^0(0)] q_{ij} = 0, \quad [x_i^0(\theta), x_j^0(\theta)] q_{ij} = 0, \quad \text{for } s \neq s' \in J, \\
(3.21) & \quad \sum_{t=0}^{\ell=1-a_{ij}} (-1)^t \left[ \begin{array}{l} \ell \\ t \end{array} \right] x_i^0(0)^{\ell-t} x_j^0(0)x_i^0(0) = 0, \quad a_{ij} \leq 0, \\
(3.22) & \quad \sum_{t=0}^{\ell=1-a_{ij}} (-1)^t \left[ \begin{array}{l} \ell \\ t \end{array} \right] x_i^0(0)^{\ell-t} x_j^0(\theta) x_i^0(\theta) = 0, \quad a_{ij} \leq 0, \\
(3.23) & \quad \sum_{t=0}^{\ell=1-a_{ij}} (-1)^t \left[ \begin{array}{l} \ell \\ t \end{array} \right] x_i^0(\theta)^{\ell-t} x_j^0(0) x_i^0(\theta) = 0, \quad a_{ij} \leq 0, \\
(3.24) & \quad \text{Sym}_{0,-\epsilon_0} \sum_{t=0}^{\ell=1-a_{ij}} (-1)^t \left[ \begin{array}{l} \ell \\ t \end{array} \right] x_i^0(\theta)^{\ell-t} x_j^0(\theta) x_i^0(\theta) = 0, \quad a_{ij} \leq 0 \text{ and } k = 0, \\
(3.25) & \quad \sum_{t=0}^{3} (-1)^t q_{ij}^{2t} \left[ \begin{array}{l} 3 \\ t \end{array} \right] x_i^0(0)^{3-t} x_j^0(\theta) x_i^0(\theta) = 0, \quad \text{for } s \in J, \\
(3.26) & \quad \sum_{t=0}^{3} (-1)^t q_{ij}^{2t} \left[ \begin{array}{l} 3 \\ t \end{array} \right] x_i^0(\theta)^{3-t} x_j^0(\theta) x_i^0(\theta) = 0, \quad \text{for } s \in J,
\end{align}
where Sym_{m_1,m_2} denotes the symmetrization with respect to the indices \((m_1, m_2)\).

**Remark 3.6.** Notice that all the relations above are part of those in Definition 3.1 as they only involve with special modes or generators. Though relations (3.25) and (3.26) seem new, they can be deduced from relations (3.10) by (3.11) and (3.3) respectively. In fact, relations (3.25) and (3.26) are the following two relations in Definition 3.1 respectively,

\[
[x_0^s(0), x_0^s(\varepsilon s_1)]_{q_0^2} = 0,
\]

\[
[x_0^s(\varepsilon s_1), x_0^s(-2\varepsilon s_2)]_{q_0^2} = 0.
\]

From the definition of \(U_0(g_{N,tor})\), we have the following proposition.

**Proposition 3.7.** For \(s \in J\), the following map \(\tau_s\) defines an automorphism of \(U_0(g_{N,tor})\):

\[
\tau_s(x_0^s(-\varepsilon s_i)) = \begin{cases} (q^d)^{(1-\varepsilon)d_0} x_0^s(0)(q^d)^{(1-\varepsilon)d_0}, & \text{if } s = s'; \\ x_0^s(-\varepsilon s_i), & \text{if } s \neq s', \end{cases}
\]

\[
\tau_s(\gamma_{s'}) = \begin{cases} \gamma_{s'}^{-1}, & \text{if } s = s'; \\ \gamma_{s'} \gamma_{s}^{-1}, & \text{if } s \neq s', \end{cases}
\]

\[
\tau_s(x_0^s(0)) = (q^d)^{(\varepsilon+1)d_0} x_0^s(\varepsilon s_1)(q^d)^{(\varepsilon-1)d_0},
\]

\[
\tau_s(x_0^s(0)) = x_0^s(0),
\]

\[
\tau_s(K_0) = \gamma_{s}^{-1} K_0,
\]

\[
\tau_s(K_i) = K_i,
\]

\[
\tau_s(q^d) = q^d,
\]

where \(i = 1, 2, \ldots, n, s' \in J\) and \(\varepsilon = \pm 1\).

**Proof.** It suffices to check that \(\tau_s\) keeps the relations (3.14)-(3.27). First we verify the relations (3.18), and note that \([x_0^s(0), x_0^s(0)] = \delta_{ij} K_{i-j}^{-1} q_{i-j}^{-1}\) follows from the definition of \(\tau_s\). By definition

\[
\tau_s([x_0^s(-\varepsilon s), x_0^s(\varepsilon s)]) = [(q^d)^{-2d_0} x_0^s(0), x_0^s(0)(q^d)^{2d_0}] = \frac{K_0 - K_0^{-1}}{q_0 - q_0^{-1}},
\]

which matches with \(\tau_s(\gamma_{s}^{-1} K_0 - \gamma_{s} \gamma_{s}^{-1} K_0^{-1})\) due to \(\tau_s(K_0) = \gamma_{s}^{-1} K_0\). Similarly the equality is checked for \(s \neq s'\).

To check relations (3.20), by definition it follows that

\[
\tau_s([x_0^s(-\varepsilon s), x_0^s(0)])_{q_0^2} = [(q^d)^{-2d_0} x_0^s(0), (q^d)^{2d_0} x_0^s(-\varepsilon s)]_{q_0^2} = q_0^{-2}[x_0^s(0), x_0^s(-\varepsilon s)]_{q_0^2} = 0.
\]

Similarly,

\[
\tau_s([x_0^s(-\varepsilon s), x_0^s(0)]) = (q^d)^{-2d_0} [x_0^s(0), x_0^s(-\varepsilon s)]_{q_0^2} = 0.
\]

For relation (3.27), one has that

\[
\tau_s(\sum_{t=0}^{3} (-1)^t \begin{bmatrix} 3 \\ t \end{bmatrix} x_0^s(-\varepsilon s)^{3-t} x_0^s(\varepsilon s')(x_0^s(-\varepsilon s))^t)
\]

\[
= (q^d)^{-6d_0} \sum_{t=0}^{3} (-1)^t q_0^{-2t} \begin{bmatrix} 3 \\ t \end{bmatrix} x_0^s(0)^{3-t} x_0^s(\varepsilon s')(x_0^s(0))^t = 0.
\]

We can verify that \(\tau_s\) keeps the other relations in the same way.

\qed
Proposition 3.7 reveals a symmetry of the algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$, which will be shown by the Dynkin diagram in the next subsection.

3.4. $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ and the extended quantized GIM algebra of $N$-fold affinizations. In this subsection, we focus on showing that the algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ can be realized as a quotient of the extended quantized GIM algebra of $N$-fold affinization.

First let us denote the following elements of $\mathcal{U}_0(\mathfrak{g}_{N,tor})$: for $i \in I$ and $s \in J$,

\begin{align*}
F_{-s} &= x_0^{-}(e_s)(q^{-d})^{2d_0}, \quad E_{-s} = (q^d)^{2d_0}x_0^{+}(-e_s), \quad K_{-s} = \gamma_s^{-1}K_0, \\
E_i &= x_i^{+}(0), \quad F_i = x_i^{-}(0), \quad K_i = K_i,
\end{align*}

and set $d_s = d_0$ for $s \in \{-N + 1, \cdots, -1\}$ and $q_i = q^{d_i}$ for $i \in \tilde{J}$.

**Remark 3.8.** Note that $M$ given in Definition 2.4 is a symmetrizable GIM of $N$-fold affinization of $A$: $D_M M$ is symmetric for the diagonal matrix $D_M = \sum_{i \in J} d_i E_{ii} = \begin{pmatrix} d_0 I_N & 0 \\ 0 & D_0 \end{pmatrix}$ where $D_0 = \text{diag}(d_i | i \in I_0)$, $d_i$ was given in Section 3.2.

Based on the aforementioned elements we have the following result.

**Proposition 3.9.** The algebra $\mathcal{U}_0(\mathfrak{g}_{N,tor})$ is the associative algebra over $\mathbb{K}$ generated by $E_i, F_i, K_i, q^{\pm d}(i \in \tilde{J})$, satisfying the following relations:

\begin{align*}
K_i^{\pm 1}K_i^{\pm 1} &= q^{\pm d}q^{\mp d} = 1, \quad q^{\pm d} \text{ and } K_i^{\pm} \text{ commute with each other}, \\
K_iE_jK_i^{-1} &= q_i^{m_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{-m_{ij}}F_j, \\
[E_i, E_j] &= 0, \\
q^dE_iq^{-d} &= qE_i, \quad q^dF_iq^{-d} = q^{-1}F_i, \quad i \in \{-N + 1, \cdots, -1, 0\}, \\
q^dE_iq^{-d} &= E_j, \quad q^dF_iq^{-d} = F_j, \quad j \in \{1, 2, \cdots, n\}, \\
\sum_{s=0}^{-m_{ij}} (-1)^s \left[ \begin{array}{c} 1-m_{ij} \\ s \end{array} \right] E_i^{1-m_{ij}-s}E_jE_i^s &= 0, \quad i \neq j \in \tilde{J}, m_{ij} < 0; \\
\sum_{s=0}^{-m_{ij}} (-1)^s \left[ \begin{array}{c} 1-m_{ij} \\ s \end{array} \right] F_i^{1-m_{ij}-s}F_jF_i^s &= 0,
\end{align*}

\begin{align*}
[E_i, E_j] &= 0 = [E_i, F_j], \\
\sum_{s=0}^{m_{ij}} (-1)^s \left[ \begin{array}{c} 1+m_{ij} \\ s \end{array} \right] E_i^{1+m_{ij}-s}F_jE_i^s &= 0, \quad i \neq j \in \tilde{J}, m_{ij} > 0; \\
\sum_{s=0}^{m_{ij}} (-1)^s \left[ \begin{array}{c} 1+m_{ij} \\ s \end{array} \right] F_i^{1+m_{ij}-s}E_jF_i^s &= 0, \\
[E_i, E_j] &= 0 = [E_i, F_j] = [F_i, E_j], \quad i \neq j \in \tilde{J}, m_{ij} = 0; \\
[E_{-j}, [E_0, E_i]_{q_0}]_{q_0} + [E_0, [E_{-j}, E_i]_{q_0}]_{q_0^{-2}} &= 0, \\
[F_{-j}, [F_0, F_i]_{q_0^{-1}}]_{q_0^2} + [F_0, [F_{-j}, F_i]_{q_0^{-1}}]_{q_0} &= 0, \quad m_{i0} = 1, 0 \neq i \in I, j \in J;
\end{align*}
\[
\begin{align*}
[ E_{-j}, [ E_0, [ E_0, E_i ]_{q_0^2}^{-1} ]_{q_0^2} ]_{q_0^2} & + [ [ E_0, [ E_{-j}, E_i ]_{q_0^2}^{-1} ]_{q_0^2}^{-1} ]_{q_0^2} = 0, \\
[ F_{-j}, [ F_0, [ F_0, E_i ]_{q_0^2}^{-1} ]_{q_0^2} ]_{q_0^2} & + [ [ E_0, [ E_{-j}, E_i ]_{q_0^2}^{-1} ]_{q_0^2}^{-1} ]_{q_0^2} = 0,
\end{align*}
\]

(3.34)

\[
\begin{align*}
[ E_{-j}, [ E_0, [ E_i ]_{q_0^2}^{-1} ]_{q_0^2} ]_{q_0^2} & + [ [ E_0, [ E_{-j}, E_i ]_{q_0^2}^{-1} ]_{q_0^2}^{-1} ]_{q_0^2} = 0, \\
[ F_{-j}, [ F_0, [ F_0, E_i ]_{q_0^2}^{-1} ]_{q_0^2} ]_{q_0^2} & + [ [ E_0, [ E_{-j}, E_i ]_{q_0^2}^{-1} ]_{q_0^2}^{-1} ]_{q_0^2} = 0,
\end{align*}
\]

where \( m_{ij} \in M \) defined in Definition 2.4.

**Proof.** First of all, we remark that all generators \( E_i, F_i, K_i, q^d \) are simply rewriting of the generators of \( \mathcal{U}_0(\mathfrak{g}_{N,\text{tor}}) \), therefore the isomorphism follows by listing the corresponding relations. In fact, relation (3.28) holds by relations (3.14)-(3.18). It is easy to see that relation (3.29) follows from relation (3.27) for the first case. For the last two cases, it holds from (3.25) and (3.26). For relation (3.31), it suffices to check the relations involving with \( m_{ij} = 2 \) for \( i \neq j \), since the other relations can be verified directly. Specifically we need to show that for \( i \neq j \in \{-N + 1, \ldots, -2, -1, 0\} \),

\[
\sum_{s=0}^{1+m_{ij}} (-1)^s \binom{3}{s} E_i^{1+m_{ij}-s} F_j E_i^s = 0.
\]

We will check it for two cases: \( m_{-s-s'} = 2 \) for \( s \neq s' \in J \), \( m_{0-s} = 2 \) and \( m_{-s-0} = 2 \) for \( s \in J \). It follows from relation (3.27) for the first case. For the last two cases, it holds from (3.25) and (3.26).

For relation (3.32), we consider the case of \( m_{0-0} = 0 \) for example, that is,

\[
[ E_0, E_{-s} ] = [ x_0^+ (0), (q^d)^{2d_0} x_0^+ (-e_s) ] = q^{-2} (q^d)^{2d_0} [ x_0^+ (0), x_0^+ (-e_s) ]_{q^2} = 0,
\]

which follows from relation (3.27) by using (3.20).

Relations (3.33)-(3.34) hold by the Serre relations (3.21)-(3.26) directly. \( \square \)

**Remark 3.10.** From Proposition 3.7 there exists an automorphism \( \tau_\sigma \) of the algebra \( \mathcal{U}_0(\mathfrak{g}_{N,\text{tor}}) \) for \( \sigma \in S_X \) where \( X = \{0, -1, \ldots, -N + 1\} \), such that \( \tau_\sigma(q^d) = q^d, \tau_\sigma(g_s) = \gamma_{-\sigma(-i)} \) for \( s \in J \) and \( i \in K \),

\[
\begin{align*}
\tau_\sigma(E_j) &= \begin{cases} E_{\sigma(j)}, & \text{if } j \in X; \\
E_j, & \text{if } j \notin X, \end{cases} \\
\tau_\sigma(F_j) &= \begin{cases} F_{\sigma(j)}, & \text{if } j \in X; \\
F_j, & \text{if } j \notin X, \end{cases} \\
\tau_\sigma(K_j) &= \begin{cases} K_{\sigma(j)}, & \text{if } j \in X; \\
K_j, & \text{if } j \notin X, \end{cases}
\end{align*}
\]

where \( \gamma_{0} \) is defined in Remark 3.4.

Therefore we have the following Corollary immediately.
COROLLARY 3.11. The algebra $U_0(g_{N,\text{tor}})$ is isomorphic to the quotient algebra $U_q(\mathcal{L}(M))/K$ of the extended quantized GIM algebra of the $N$-fold affinization $U_q(\mathcal{L}(M))$. That is,

$$U_0(g_{N,\text{tor}}) \cong U_q(\mathcal{L}(M))/K,$$

where $K$ is the ideal of $U_q(\mathcal{L}(M))$ generated by Serre relations (3.33) and (3.34).

Moreover, we have the following two results, whose proofs will be given in the following two sections for the cases of $N = 2$ and $N \geq 3$, respectively.

THEOREM 3.12. As an associative algebra, the quantum 2-toroidal algebra $U_q(g_{2,\text{tor}})$ is isomorphic to a quotient algebra of $U_0(g_{2,\text{tor}})$ for type $A$ and $U_0(g_{2,\text{tor}})$ itself for other types. More specifically, one has

$$U_q(g_{2,\text{tor}}) \cong \begin{cases} U_0(g_{2,\text{tor}})/H_1, & \text{for type } A; \\ U_0(g_{2,\text{tor}}), & \text{otherwise}, \end{cases}$$

where $H_1$ is an ideal defined in Section 4.

THEOREM 3.13. As an associative algebra, the algebra $\overline{U}_q(g_{N,\text{tor}})$ ($N > 2$) is isomorphic to a quotient algebra of $U_0(g_{N,\text{tor}})$:

$$\overline{U}_q(g_{N,\text{tor}}) \cong U_0(g_{N,\text{tor}})/H_2,$$

where the ideal $H_2$ will be defined in Section 5.

Combining Theorem 3.12 and Theorem 3.13 with Corollary 3.11 we obtain the following main theorem, which generalizes a well-known result of Berman and Moody for Lie algebras [BM].

THEOREM 3.14. The algebra $\overline{U}_q(g_{N,\text{tor}})$ are isomorphic to quotient algebras of the extended quantized GIM algebras of $N$-fold affinization $U_q(\mathcal{L}(M))$.

4. Proof of Theorem 3.12

4.1. The algebra $U_0(g_{2,\text{tor}})$. Recall that the algebra $U_0(g_{2,\text{tor}})$ was defined in Definition 3.5 for $N = 2$ and $J = \{1\}$. To simplify notation, for $i \in I$ and $k \in \mathbb{Z}$, we denote that $x_i^+(0) = x_i^+(1)$, $x_i^-(k) = x_i^-(ke_1)$ and $\gamma^\pm \equiv \gamma_i^\pm \frac{1}{2}$.

The following result is immediate by definition.

PROPOSITION 4.1. There is a $\mathbb{C}$-algebra anti-involution $\iota$ of $U_0(g_{2,\text{tor}})$ such that $\iota : x_i^+ \rightarrow x_i^-(k), K_i \rightarrow K_i^{-1}, q^d \mapsto q^{-d}, \gamma^\pm \mapsto \gamma^{-\frac{1}{2}}$ and $q \mapsto q^{-1}$.

Before proving Theorem 3.12 we recall a useful result, which can be checked easily (cf. [JZT], Lemma 4.1).

LEMMA 4.2. Suppose the associative algebras $A = \langle x_i \rangle/(R_1)$ and $B = \langle \hat{x}_i, y_j \rangle/(\hat{R}_1, R_2, R_{12})$ with respective relations $R_1 = R_1(x_i), R_1 = R_1(\hat{x}_i), R_2 = R_2(y_j), R_{12} = R_{12}(\hat{x}_i, y_j)$. Define the map $\phi : A = \langle x_i \rangle/(R_1) \rightarrow B = \langle \hat{x}_i, y_j \rangle/(\hat{R}_1, R_2, R_{12})$ such that $x_i \mapsto \hat{x}_i$. If $y_j \in \text{Im} \phi$ inside $B$, and $R_2(y_j) \subset (\hat{R}_1), R_{12}(\hat{x}_i, y_j) \subset (\hat{R}_1)$, then $A \simeq B$ as associative algebras.

By this lemma, verification is made easy by only checking the relations (3.14) - (3.27) for the set of simplified generators of the quantum 2-toroidal algebra displayed in the algebra $U(g_{2,\text{tor}})$. 
4.2. Proof of Theorem 3.12

We now set out to prove Theorem 3.12. The idea is to show that a simplified set of relations are satisfied for the quantum algebra \( U_0(\mathfrak{g}_{2,\text{tor}}) \) in the toroidal case. Some of the computations are made in [121] for the affine type \( A \), so we will focus on the relations special for the toroidal case. The proof is divided into four steps:

**Step 1:** We will prove that \( U_0(\mathfrak{g}_{2,\text{tor}}) \) contains all other elements \( x_0^\pm(\epsilon k), a_0(\epsilon k) \) involving the index \( i = 0 \) in Definition 3.1, these elements satisfy relations consistent with Definition 3.1. Indeed, there exists a subalgebra of \( U_0(\mathfrak{g}_{2,\text{tor}}) \) generated by \( x_0^\pm(\epsilon k), a_0(\epsilon k), \gamma_0^{\pm} \) and \( K_0^{\pm} \), and we denote it by \( U_q(\hat{\mathfrak{sl}}_2)_0 \).

Here we use induction on degree \( k \in \mathbb{N} \), first we introduce the following useful elements for \( k = 1 \) in \( U_0(\mathfrak{g}_{2,\text{tor}}) \):

\[
\begin{align*}
    a_0(1) &= K_0^{-1} \gamma_0^{1/2} \left[ x_0^0(0), x_0^- (1) \right] \in U_0(\mathfrak{g}_{2,\text{tor}}), \\
    a_0(-1) &= K_0 \gamma_0^{-1/2} \left[ x_0^0(-1), x_0^- (0) \right] \in U_0(\mathfrak{g}_{2,\text{tor}}),
\end{align*}
\]

which are used to inductively generate higher degree elements using a spiral argument based on Lemma 4.2.

Furthermore, for \( \epsilon = \pm \) or \( \pm 1 \) we have that

\[
    x_0^\epsilon(\epsilon) = [2]_0^{-1/2} \gamma_0^{1/2} \left[ a_0(\epsilon), x_0^0(0) \right] \in U_0(\mathfrak{g}_{2,\text{tor}}).
\]

Now we check the relation (3.10) involving with the elements \( x_0^\epsilon(0) \) and \( x_0^\epsilon(\epsilon) \).

**Proposition 4.3.** Using the above notations, we have that

\[
[x_0^\epsilon(0), x_0^\epsilon(\epsilon)]_{q_0^{-2}} = 0.
\]

**Proof.** We only check in the case of \( \epsilon = + \) or \( +1 \), it is similar for the case of \( \epsilon = - \) or \( -1 \). It follows from (4.3) and (4.1) that

\[
[x_0^\epsilon(0), x_0^\epsilon(1)]_{q_0^{-2}} = [x_0^\epsilon(0), [2]_0^{-1} \gamma_0^{1/2} \left[ a_0(1), x_0^\epsilon(0) \right]]
= -q_0^4 [2]_0^{-1} \gamma_0^{1/2} K_0^{-1} \left[ x_0^\epsilon(0), [x_0^\epsilon(0), [x_0^\epsilon(0), x_0^\epsilon(1)]_{q_0^{-2}} \right]_{q_0^{-4}} = 0,
\]

where we have used the relation (3.25). \( \square \)

The following proposition gives key relations among the degree-1 elements \( K_0^{\pm}, x_0^\epsilon(0), a_0(\epsilon), x_0^\epsilon(\epsilon) \) and \( x_0^\epsilon(-\epsilon) \), which are consistent with Definition 3.1.
Proposition 4.4. Using the above notations, we have the following relations (as above and below $\epsilon = \pm$ or $\pm 1$):

\begin{align*}
(4.5) & \quad [a_0(\epsilon), x_0^-(\epsilon)] = \epsilon [2] \gamma^{-\frac{\ell}{2}} x_0^-(0), \quad [a_0(-\epsilon), x_0^+(0)] = \epsilon [2] \gamma^{-\frac{\ell}{2}} x_0^+(\epsilon), \\
(4.6) & \quad [a_0(1), a_0(-1)] = [2] \gamma - \gamma^{-1}_{q_0 - q_0^{-1}}, \\
(4.7) & \quad [x_0^+(\epsilon), x_0^-(0)] = \gamma [x_0^+(0), x_0^-(\epsilon)], \\
(4.8) & \quad a_0(\epsilon) = \epsilon K_0^{-\epsilon} \gamma^{\frac{\ell}{2}} [x_0^+(0), x_0^-(\epsilon)] = \epsilon K_0^{-\epsilon} \gamma^{-\frac{\ell}{2}} [x_0^+(\epsilon), x_0^-(0)], \\
(4.9) & \quad [x_0^+(1), x_0^+(\ell)]_{q_0^{\ell}} + [x_0^+(0), x_0^+(\ell)]_{q_0^{\ell}} = 0, \\
(4.10) & \quad [a_0(-\epsilon), x_0^+(\epsilon)] = \epsilon [2] \gamma^{-\frac{\ell}{2}} x_0^+(0), \\
(4.11) & \quad [x_0^+(1), x_0^-(\ell)] = \gamma K_0 - \gamma^{-1} K_0^{-1}_{q_0 - q_0^{-1}}.
\end{align*}

Proof. Most of the relations can be easily checked as in [JZ1] for type A. Here we check relation (4.9) for the case of $+$, others can be verified by the construction directly. Note that $[x_0^+(0), x_0^+(\ell)]_{q^2} = 0$ by (3.20), then

\begin{align*}
0 & = [a_0(1), [x_0^+(0), x_0^-(\ell)]_{q^2}] \\
& = \left( [a_0(1), x_0^+(0)]_{q^2} + [x_0^+(0), a_0(1)]_{q^2} \right) \left( [x_0^+(0), x_0^-(\ell)]_{q^2} + [x_0^+(0), x_0^+(\ell)]_{q^2} \right) \\
& = [2] \gamma^{-\frac{\ell}{2}} \left( [x_0^+(0), x_0^+(\ell)]_{q^2} + [x_0^+(0), x_0^+(\ell)]_{q^2} \right).
\end{align*}

It means that $[x_0^+(1), x_0^+(\ell)]_{q^2} + [x_0^+(0), x_0^+(\ell)]_{q^2} = 0$, which is consistent with the defining relation in Definition 3.1. □

Now we construct all degree-$k$ elements $x_0^+(k), x_0^+(\ell), a_0(\pm k)$ involving with index $i = 0$ by inductively as follows. For $\epsilon = \pm$ or $\pm 1$, we set that

\begin{align*}
(4.12) & \quad x_0^+(\ell k) = \pm [2] \gamma^{-\frac{\ell}{2}} \left[ a_0(\epsilon), x_0^+(\ell (k - 1)) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), \\
(4.13) & \quad \phi_0(k) = (q_0 - q_0^{-1}) \gamma^{\frac{k}{2}} \left[ x_0^+(k - 1), x_0^+(1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), \\
(4.14) & \quad \varphi_0(\ell k) = -(q_0 - q_0^{-1}) \gamma^{\frac{k-\ell}{2}} \left[ x_0^+(\ell - 1), x_0^+(\ell - k + 1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}),
\end{align*}

where $a_0(\pm k)$ are defined by $\phi_0(k)$ and $\varphi_0(\ell k)$ ($k \geq 0$) as follows:

\begin{align*}
\sum_{m=0}^{\infty} \phi_0(r) z^{-r} & = K_0 \exp \left( (q_0 - q_0^{-1}) \sum_{\ell=1}^{\infty} a_0(\ell) z^{-\ell} \right), \\
\sum_{r=0}^{\infty} \varphi_0(-r) z^{r} & = K_0^{-1} \exp \left( -(q_0 - q_0^{-1}) \sum_{\ell=1}^{\infty} a_0(-\ell) z^{\ell} \right).
\end{align*}

A partition of $k$, denoted $\lambda \vdash k$, is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = k$, where $l(\lambda) = l$ is called the number of parts.
A partition \( \lambda = (\lambda_1 \lambda_2 \cdots) \) can also be denoted as \((1^{m_1} 2^{m_2} \cdots)\) with multiplicity of \( i \) being \( m_i \). Then we obtain the following formulas between \( a_0(\pm k) \) and \( \phi_0(\pm k) \):

\[
\phi_0(k) = K_0 \sum_{\lambda \prec k} \frac{(q_0 - q_0^{-1})(\lambda)}{m_\lambda!} a_0(\lambda),
\]

\[
\phi_0(-k) = K_0^{-1} \sum_{\lambda \prec k} \frac{(q_0^{-1} - q_0)(\lambda)}{m_\lambda!} a_0(-\lambda),
\]

where \( m_\lambda! = \prod_{i \geq 1} m_i! \) and \( a_0(\pm \lambda) = a_0(\pm \lambda_1) a_0(\pm \lambda_2) \cdots \). It is easy to see \( \iota(\phi_0(k)) = \varphi_0(-k) \).

By the inductive hypothesis on degree, assume that all degree-\( n \) elements for \( n \in \mathbb{N} \) satisfy the relevant relations in Definition 3.1 now we are left to check that so do all degree-\((n + 1)\) elements for \( n \in \mathbb{N} \). The proof proceeds in the following propositions (cf. [JZ1 Sect. 4] for type \( A \)).

**Proposition 4.5.** From the above construction, we have the following relations for \( \epsilon = \pm 1 \), \( -n + 1 \leq l_1 \leq n - 1 \) and \( -n \leq l_2 < n \),

\[
K_1 x_0^2 \left( \epsilon(n + 1) \right) K_i^{-1} = q_i^{a_0} x_0^2 \left( \epsilon(n + 1) \right),
\]

\[
\phi_i^d x_0^2 \left( \epsilon(n + 1) \right) q_i^{-d} = q_i^{(n+1)} x_0^2 \left( \epsilon(n + 1) \right),
\]

\[
[x_0^2(n + 1), x_0^2(n - 1)]_{q_0^2}^+ + [x_0^2(n), x_0^2(n)]_{q_0^2}^+ = 0,
\]

\[
[x_0^2(n + 2), x_0^2(n + 1)]_{q_0^2}^+ + [x_0^2(n), x_0^2(n + 1)]_{q_0^2}^+ = 0,
\]

\[
[x_0^2(n + 1), x_0^2(l_1)]_{q_0^2}^+ + [x_0^2(l_1 + 1), x_0^2(n)]_{q_0^2}^+ = 0,
\]

\[
[x_0^2(n + 1), x_0^2(l_2)] = \frac{\gamma^{-1} \phi_0(n + l_2 + 1)}{q_0 - q_0^{-1}},
\]

\[
[x_0^2(n + 1), x_0^2(-n - 1)] = \frac{\gamma^{n+1} K_0 - \gamma^{-n-1} K_0^{-1}}{q_0 - q_0^{-1}}.
\]

**Proof.** We only check the “+” case, as the “−” case can be obtained similarly. (4.17) and (4.18) hold directly by the construction. The proof of relation (4.19) is similar to that of (4.20). Let’s consider (4.20), by the inductive hypothesis, take the bracket of \( a_0(2) \) and \([x_0^2(n), x_0^2(n - 1)]_{q_0^2} = 0\), then

\[
0 = \left[ a_0(2), [x_0^2(n), x_0^2(n - 1)]_{q_0^2} \right] = (\left[ [a_0(2), x_0^2(n)], x_0^2(n - 1) \right]_{q_0^2} + [x_0^2(n), [a_0(2), x_0^2(n - 1)]]_{q_0^2}) = [4]_2 \gamma^{-1} \left( [x_0^2(n + 2), x_0^2(n - 1)]_{q_0^2} + [x_0^2(n), x_0^2(n + 1)]_{q_0^2} \right),
\]

which implies

\[
[x_0^2(n + 2), x_0^2(n - 1)]_{q_0^2} + [x_0^2(n), x_0^2(n + 1)]_{q_0^2} = 0.
\]

In a similar manner, we can prove relations (4.21) and (4.22). Note that by (4.19)

\[
[x_0^2(n + 1), x_0^2(n - 1)]_{q_0^2} + [x_0^2(n), x_0^2(n)]_{q_0^2} = 0.
\]
Therefore, it is easy to see that
\[
0 = \left[ a_0(1), [x_0^+(n + 1), x_0^+(n - 1)]_{q_0^2} + [x_0^+(n), x_0^+(n)]_{q_0^2} \right]
\]
\[
= [2]_0 \gamma^{-\frac{1}{2}} \left( [x_0^+(n + 2), x_0^+(n - 1)]_{q_0^2} + [x_0^+(n + 1), x_0^+(n)]_{q_0^2} + [x_0^+(n), x_0^+(n + 1)]_{q_0^2} \right)
\]
\[
= 2[2]_0 \gamma^{-\frac{1}{2}} [x_0^+(n + 1), x_0^+(n)]_{q_0^2},
\]
where we have used (4.20). It yields that \([x_0^+(n + 1), x_0^+(n)]_{q_0^2} = 0\).

In order to check (4.23), we consider that for \(-n \leq l_2 \leq n\),
\[
[x_0^+(n + 1), x_0^-(l_2)] = [2]_0^{-1} \gamma^{\frac{1}{2}} \left( [a_0(1), x_0^-(l_2)], x_0^+(n) \right) + a_0(1), [a_0^+(n), x_0^-(l_2)] \right) \biggr) 
\]
\[
= -\gamma [x_0^-(l_2 + 1), x_0^+(n)] = \frac{2^{n+1/2} \phi_0(n + l_2 + 1)}{q_0 - q_0^{-1}}.
\]

For (4.24), one has that
\[
[x_0^+(n + 1), x_0^-(n + 1)]
\]
\[
= -[2]_0^{-2} \left( [a_0(1), x_0^+(n), a_0(-1)], x_0^-(n) \right) + [a_0(-1), [a_0(1), x_0^-(n)], x_0^+(n)] \right) 
\]
\[
= \frac{\gamma^{n+1} K_0 - \gamma^{-n-1} K_0^{-1}}{q_0 - q_0^{-1}}.
\]

**Proposition 4.6.** The following relations hold for \(d = \gamma^{-\frac{1}{2}} q_0^2\).

\begin{align*}
(4.25) \quad [\tilde{\phi}_0(r), x_0^+(m)] &= [2]_0 \gamma^{-\frac{1}{2}} \left( \sum_{t=1}^{r-1} (q_0 - q_0^{-1}) d^{t-1} x_0^+(m + t) \tilde{\phi}_0(r - t) + d^{r-1} x_0^+(r + m) \right), \\
(4.26) \quad [\tilde{\varphi}_0(-r), x_0^-(m)] &= -[2]_0 \gamma^{-\frac{1}{2}} \left( \sum_{t=1}^{r-1} (q_0 - q_0^{-1}) d^{t-1} \tilde{\varphi}_0(-r + t) x_0^-(m - t) + d^{r-1} x_0^-(r - m) \right).
\end{align*}
Proposition 4.6. Here we only check relation (4.25) for the case of \( m = -1 \), other cases are similar.

\[
\begin{align*}
&[\bar{\phi}_0(n + 1), x_0^+(-1)] \\
&= \gamma \frac{\ln n}{n} K_0^{-1} \left( [x_0^+(n), x_0^+(1), x_0^+(n + 1)] \right) + [x_0^+(n), x_0^+(1)]
\end{align*}
\]

Notice that by (4.15),

\[
\sum_{t=1}^{n-1} d^t(q_0 - q_0^{-1})q_0^{-2} [2]_0 x_0^+(t - 1)\bar{\phi}_0(n + 1 - t) = 0.
\]

\[
\begin{align*}
&= [2]_0 \gamma^{-\frac{1}{2}} \sum_{t=1}^{n} d^{t-1}(q_0 - q_0^{-1})x_0^+(t - 1)\bar{\phi}_0(n + 1 - t) + d^n x_0^+(n).
\end{align*}
\]

\[
\square
\]

Proposition 4.7. The following relations hold for \( \epsilon = \pm \) or \( \pm 1 \),

\[
\begin{align*}
&[a_0(\epsilon(n + 1)), x_0^\epsilon(-\epsilon)] = \frac{[2(n + 1)]_0}{n + 1} \gamma^{-\frac{n+1}{2}} x_0^\epsilon(\epsilon n), \\
&[a_0(\epsilon(n + 1)), x_0^{-\epsilon}(\epsilon n)] = \frac{[2(n + 1)]_0}{n + 1} \gamma^{-\frac{n+1}{2}} x_0^{-\epsilon}(\epsilon), \\
&[a_0(n + 1), a_0(-n - 1)] = \frac{[2(n + 1)]_0}{n + 1} \gamma^{n+1} - \gamma^{-(n+1)}.
\end{align*}
\]

Proof. Note that (4.28) is similar to (4.27), so we only deal with (4.27). It follows from Proposition 4.6 that

\[
\begin{align*}
&[a_0(n + 1), x_0^+(-1)] \\
&= [\bar{\phi}_0(n + 1), x_0^+(-1)] - \frac{(q - q_0^{-1})}{n + 1} \sum_{t=1}^{n} t [\bar{\phi}_0(n + 1 - t)a_0(t), x_0^+(-1)]
\end{align*}
\]

\[
= \frac{[2(n + 1)]_0}{(n + 1)!} \gamma^{-\frac{n+1}{2}} x_0^+(n).
\]

Notice that by (4.15),

\[
- \gamma^{-\frac{1}{2}} \frac{K_0^{-1}}{q_0 - q_0^{-1}} [x_0^+(0), \phi_0(n)]_{q_0^{-2}} = - \gamma^{-\frac{1}{2}} q_0^{-2} \sum_{\lambda^2 = n} \frac{(q_0 - q_0^{-1})^{\lambda - 1}}{m_\lambda!} [x_0^+(0), a_0(\lambda)]_{q_0^{-2}}.
\]
As a consequence, one has that
\[
[a_0(n + 1), x^+_0(-1)]
\]
\[= -\gamma^{-\frac{1}{2}}q_0^{-2} \sum_{t=1}^{n} \frac{(q_0 - q_0^{-1})^{t-1}}{t!} \sum_{1 \leq i_1, \ldots, i_t \leq n-t+1, \atop i_1 + i_2 + \cdots + i_t = n} [x^+_0(0), a_0(i_1) \cdots a_0(i_t)] q^t
\]
\[= \sum_{t=2}^{n+1} \frac{(q_0 - q_0^{-1})^{t-1}}{t!} \sum_{1 \leq i_1, \ldots, i_t \leq n-t+2, \atop i_1 + i_2 + \cdots + i_t = n+1} [a_0(i_1) \cdots a_0(i_t), x^+_0(-1)] = \frac{2(n+1)!}{(n+1)!} \gamma^{-\frac{n+1}{2}} x^+_0(n).
\]

For (4.29), it follows from (4.27)-(4.28) and (4.15) by the inductive hypothesis that
\[
[a_0(n + 1), a_0(-n - 1)]
\]
\[= \gamma^{-\frac{n+1}{2}} K_0[a_0(n+1), [x^+_0(-1), x^-_0(-n)]] - \sum_{\lambda \vdash n+1 \atop \lambda \neq (n+1)} \frac{(q_0 - q_0^{-1})^{(\lambda)-1}}{m_\lambda!} [a_0(n+1), a_0(\lambda)]
\]
\[= \gamma^{-\frac{n+1}{2}} K_0([[a_0(n+1), x^+_0(-1)], x^-_0(-n)] + [[x^+_0(-1), a_0(n+1), x^-_0(-n)])
\]
\[= \frac{2(n+1)!}{n+1} \gamma^{-\frac{n+1}{2}} - \gamma^{-(n+1)} \frac{1}{q_0 - q_0^{-1}}.
\]

So far, we have shown that the algebra \(\mathcal{U}_q(\mathfrak{g}_{2,tor})\) contains the subalgebra \(U_q(\hat{sl}_2)_0\), which is isomorphic to the quantum affine algebra for type \(\hat{sl}_2\).

**Step 2:** We will construct generators \(x^+_1(-1)\) and \(x^-_1(1)\) in \(\mathcal{U}_q(\mathfrak{g}_{2,tor})\) and prove that \(x^+_1(-1)\) and \(x^-_1(1)\) satisfy the same relations as those of \(x^+_0(-1), x^-_0(1)\) in Definition 3.5. Furthermore, we can construct another subalgebra \(U_q(\hat{sl}_2)_1\) generated by the node-1 elements such as \(x^+_1(\epsilon k), a_1(k), x^+_1(0), K^+_1\) and \(\gamma^H\) by repeating step 1. Moreover, we will check the relations between \(U_q(\hat{sl}_2)_0\) and \(U_q(\hat{sl}_2)_1\).

For \(\epsilon = \pm 1\) or \(\pm\), we define that
\[
(4.30)\quad x^\pm_1(\epsilon) = \pm \gamma^\pm \frac{1}{2} [a_0(\epsilon), x^\pm_1(0)] \in \mathcal{U}_q(\mathfrak{g}_{2,tor}),
\]

Using this construction, one has that
\[
(4.31)\quad K_i x^\pm_1(\epsilon) K_i^{-1} = q_i^{\pm a_{1i}} x^\pm_1(1), \quad q^d x^\pm_1(\epsilon) q^{-d} = q^{\pm 1} x^\pm_1(1),
\]
\[
(4.32)\quad [x^+_1(-\epsilon), x^-_1(0)]_{q_1^{-2}} = 0,
\]
\[
(4.33)\quad [x^+_i(-1), x^-_i(-\epsilon)] = 0, \quad \text{for} \ i \neq 1,
\]
\[
(4.34)\quad [x^+_1(-1), x^-_1(1)] = \frac{\gamma^{-1} K_1 - \gamma K_1^{-1}}{q_1 - q_1^{-1}}.
\]
To verify the Serre relations involving $x^\pm_1(-\epsilon)$ for $\epsilon = \pm 1$, we need to do some preparation. Similar to step 1, we construct

\begin{align}
(4.35) \quad a_1(1) &= \gamma^{1/2} K_1^{-1} \left[ x^+_1(0), x^-_1(1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), \\
(4.36) \quad a_1(-1) &= \gamma^{-1/2} K_1 \left[ x^+_1(-1), x^-_1(0) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}).
\end{align}

Similar to Proposition 4.4, we have the following relations.

**PROPOSITION 4.8.** It is easy to see that for $\epsilon = \pm 1$,

\begin{align}
(4.37) \quad \left[ x^+_1(\epsilon), x^-_1(0) \right] &= \gamma \left[ x^+_1(0), x^-_1(\epsilon) \right], \\
(4.38) \quad \left[ a_1(1), a_1(-1) \right] &= \left[ 2 \right]_1 \gamma - \gamma^{-1}, \\
(4.39) \quad a_1(1) &= K_1^{-1} \gamma^{1/2} \left[ x^+_1(0), x^-_1(1) \right] = K_1^{-1} \gamma^{1/2} \left[ x^+_1(1), x^-_1(0) \right], \\
(4.40) \quad a_1(-1) &= K_1 \gamma^{-1/2} \left[ x^+_1(-1), x^-_1(0) \right] = K_1 \gamma^{-1/2} \left[ x^+_1(0), x^-_1(-1) \right].
\end{align}

Now we proceed to check that $x^+_1(-\epsilon)$ keeps the Serre relations involving $x^0_1(-\epsilon)$ in Definition 3.5 (c.f. 3.24)-(3.26).

**PROPOSITION 4.9.** From the above construction, we have the following relations, which are consistent with the defining relations of $U_q(\mathfrak{g}_{2,tor})$ for $\epsilon = \pm 1$.

\begin{align}
(4.41) \quad \left[ x^0_1(0), \left[ x^+_1(0), x^-_1(-\epsilon) \right]_{q_1^{-1}} \right]_{q_0} &= 0, \\
(4.42) \quad \left[ x^+_1(-\epsilon), \left[ x^+_1(0), x^0_1(0) \right]_{q_1} \right]_{q_1^{-1}} + \left[ x^+_1(0), \left[ x^+_1(-\epsilon), x^0_1(0) \right]_{q_1^{-1}} \right]_{q_1} &= 0, \\
(4.43) \quad \left[ x^+_1(0), \left[ x^+_1(0), \left[ x^+_1(\epsilon), x^-_1(0) \right]_{q_1} \right]_{q_1^{-2}} \right]_{q_1^{-4}} &= 0.
\end{align}

**PROOF.** To check (4.41), by using (4.30) together with (3.1), we have that for $\epsilon = -1$,

\begin{align*}
\left[ x^-_1(0), \left[ x^+_1(0), x^-_1(1) \right]_{q_1^{-1}} \right]_{q_0} &= \gamma^{-1/2} \left[ x^-_1(0), \left[ x^-_1(0), x^-_1(0) \right]_{q_1^{-1}} \right]_{q_0} \\
&= \gamma^{-1/2} \left( \left[ x^-_1(0), \left[ x^-_1(0), x^-_1(0) \right]_{q_1^{-1}} \right]_{q_0} + \left[ x^-_1(0), \left[ x^-_1(0), x^-_1(0) \right]_{q_1^{-1}} \right]_{q_0} \right) \\
&= \left[ 2 \right]_0 \left[ x^-_1(0), \left[ x^-_1(0), x^-_1(0) \right]_{q_1^{-1}} \right]_{q_0} + \left[ x^-_1(0), \left[ x^-_1(0), x^-_1(0) \right]_{q_1^{-1}} \right]_{q_0} \\
&= 0,
\end{align*}

where the last equality uses the relations (3.24) and (3.21).

By using the $q$-bracket, the left hand side of (4.42) for the case of $\epsilon = -1$ can be seen as follows.

\begin{align*}
\text{LHS of (4.42)} &= \gamma^{-1/2} \left[ a_0(1), \left[ x^-_1, \left[ x^-_1(0), x^-_1(0) \right]_{q_1} \right]_{q_1^{-1}} \right].
\end{align*}

Thus (4.42) follows from the Serre relation $\left[ x^-_1(0), \left[ x^-_1(0), x^-_1(0) \right]_{q_1} \right]_{q_1^{-1}} = 0$.

In fact, relation (4.43) holds since it is equivalent to $\left[ x^+_1(0), x^+_1(\epsilon) \right]_{q_1^{-2}} = 0$. □

Next we turn to check the inter-relations between subalgebras $U_q(\mathfrak{sl}_2)_0$ and $U_q(\mathfrak{sl}_2)_1$. 
Proposition 4.10. From the above construction, we have that for $\epsilon = \pm$ or $\pm 1$

\begin{align}
(4.44) \quad \left[ a_0(1), a_1(-1) \right] &= [a_{01}]_0 \frac{\gamma - \gamma^{-1}}{q_1 - q_1^{-1}}, \\
(4.45) \quad \left[ x_1^\pm(1), x_0^\pm(0) \right] \sum_{s_{10}} + \left[ x_0^\pm(1), x_1^\pm(0) \right] \sum_{s_{10}} = 0, \\
(4.46) \quad \left[ a_0(\epsilon), x_1^\pm(-\epsilon) \right] = [a_{01}]_0 \gamma^{\mp \frac{1}{2}} x_1^\pm(0), \\
(4.47) \quad \left[ x_0^\pm(0), [x_{0}^{\pm}(0), x_{1}^{\pm}(1)]_{q_{0}^{-1}} \right]_{q_{0}} = 0.
\end{align}

To complete Step 2, we have to check the Serre relations for non-simply laced cases. Without loss of generality, it is sufficient to show the Serre relations for type $C_n$.

Proposition 4.11. In the case of type $C_n$ for $\epsilon = 0$, it holds that

\begin{equation}
(4.48) \quad \text{Sym}_{\epsilon_1, \epsilon_2, \epsilon_3} \sum_{s=0}^{3} (-1)^s \left[ x_1^-(\epsilon_1) \cdots x_1^-(\epsilon_s) x_0^-(0) x_1^-(\epsilon_{s+1}) \cdots x_1^-(\epsilon_3) = 0. \right.
\end{equation}

Proof. The proof is divided into four cases.

(1) The case of $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ is trivial.

(2) For the case of $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon_3 = 0$. Note that by the Serre relation in terms of the simple generators (3.21, 3.22) (using q-brackets), one has that

\begin{align*}
A_1 &= [x_1^-(0), [x_1^-(0), [x_1^-(0), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 = 0, \\
B_1 &= [x_1^-(0), [x_1^-(0), [x_1^-(0), x_0^-(1)]_{q_1^2}]]_{q_1^{-2}}]_1 = 0.
\end{align*}

Therefore, $[a_0(1), A_1] = 0$ implies that

\begin{align*}
&\left[ x_1^-(1), [x_1^-(0), [x_1^-(0), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 + [x_1^-(0), [x_1^-(1), [x_1^-(0), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 \\
&+ [x_1^-(0), [x_1^-(0), [x_1^-(1), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 - [2][x_1^-(0), [x_1^-(0), [x_1^-(1), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 = 0.
\end{align*}

Notice that the last summand is killed by $B_1 = 0$. So we obtain that

\begin{align*}
C_1 &= [x_1^-(1), [x_1^-(0), [x_1^-(0), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 + [x_1^-(0), [x_1^-(1), [x_1^-(0), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 \\
&+ [x_1^-(0), [x_1^-(0), [x_1^-(1), x_0^-(0)]_{q_1^2}]]_{q_1^{-2}}]_1 = 0.
\end{align*}
The case of $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = 0$. Similar to case (2) and using $[a_0(2), A_1] = [a_0(1), C_1] = 0$, we have that

\[
0 = [a_0(1), C_1] = \gamma_{12}^2 \left( [x_1^{-}(2), [x_1^{-}(0), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(1), [x_1^{-}(1), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(1), [x_1^{-}(0), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
- 2 [x_1^{-}(1), [x_1^{-}(0), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1 \right)
+ \gamma_{12}^2 \left( [x_1^{-}(1), [x_1^{-}(1), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(2), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(1), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
- 2 [x_1^{-}(0), [x_1^{-}(0), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1 \right)
+ \gamma_{12}^2 \left( [x_1^{-}(1), [x_1^{-}(0), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(1), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(0), [x_1^{-}(2), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
- 2 [x_1^{-}(0), [x_1^{-}(0), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1 \right)
= \gamma_{12}^2 \left( [x_1^{-}(1), [x_1^{-}(1), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(1), [x_1^{-}(0), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(1), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1 \right)
+ \gamma_{12}^2 \left( [x_1^{-}(2), [x_1^{-}(0), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(2), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(0), [x_1^{-}(2), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1 \right)
- \gamma_{12}^2 \left( 2 [x_1^{-}(1), [x_1^{-}(0), [x_1^{-}(0), x_0^{-}(1)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(1), [x_1^{-}(0), x_0^{-}(1)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(0), [x_1^{-}(1), x_0^{-}(1)]_{q^2_i}]_{q_i^{-2}}]_1 \right)
= \gamma_{12}^2 \left( [x_1^{-}(1), [x_1^{-}(1), [x_1^{-}(0), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(1), [x_1^{-}(0), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1
+ [x_1^{-}(0), [x_1^{-}(1), [x_1^{-}(1), x_0^{-}(0)]_{q^2_i}]_{q_i^{-2}}]_1 \right)"
which implies that (4.16) holds for \( \epsilon_1 = \epsilon_2 = 1 \) and \( \epsilon_3 = 0 \).

(4) The case of \( \epsilon_1 = \epsilon_2 = \epsilon_3 = 1 \) is checked similarly. Therefore, Proposition 4.11 has been proved.

Now we can repeat step 1 to construct the generators involving the index \( i = 1 \) as follows.

\[(4.49) \quad x_1^\pm(ek) = \pm [2]^{-1} \gamma^{\pm \frac{k}{2}} \left[ a_1(\epsilon), x_1^\pm(\epsilon(k - 1)) \right] \in U_0(g_{2,tor}),\]
\[(4.50) \quad \phi_1(k) = (q_1 - q_1^{-1}) \gamma^{\frac{k-1}{2}} \left[ x_1^+(k - 1), x_1^-(1) \right] \in U_0(g_{2,tor}),\]
\[(4.51) \quad \varphi_1(-k) = -(q_1 - q_1^{-1}) \gamma^{\frac{k-1}{2}} \left[ x_1^-(1), x_1^-(k + 1) \right] \in U_0(g_{2,tor}).\]

The elements \( x_1^\pm(ek) \) and \( a_1(k) \) also satisfy the relevant relations consistent with Definition 3.1 similar to Proposition 4.5 to Proposition 4.7.

In the remaining part of Step 2, we will check the Serre relations on higher degree elements \( x_1^\pm(ek) \) by induction on \( k \in \mathbb{N} \). By the inductive hypothesis, we assume that the Serre relations involving \( x_1^\pm(em) \) for \( m \leq n - 1 \) hold. Then we have the following proposition.

**Proposition 4.12.** For \( m, n, k \in \mathbb{N} \), the Serre relations holds

\[(4.52) \quad [x_0^\pm(ek), [x_0^\pm(ek), x_1^\pm(en)]_{q_0^{-1}}]_{q_0} = 0,\]
\[(4.53) \quad [x_1^\pm(ek), [x_1^\pm(ek), x_0^\pm(en)]_{q_1^{-1}}]_{q_1} = 0,\]
\[(4.54) \quad [x_1^\pm(en), [x_1^\pm(e(n + t)), x_0^\pm(em)]_{q_1^{-1}}]_{q_1} + [x_1^\pm(e(n + t)), [x_1^\pm(en), x_1^\pm(em)]_{q_1^{-1}}]_{q_1} = 0.\]

**Proof.** Here we only check (4.52) for the case \( - \) and \( \epsilon = + \), it is similar for the other cases. By (4.30) and (3.1), one has that

\[
\left[ x_0^-(k), [x_0^-(k), x_1^-(n)]_{q_0^{-1}} \right]_{q_0} = \gamma^{-\frac{k}{2}} \left[ x_0^-(k), [x_0^-(k), [a_0(1), x_1^-(n - 1)]_{q_0^{-1}}]_{q_0} \right.
\]
\[= \gamma^{-\frac{k}{2}} \left( [x_0^-(k), [x_0^-(k), a_0(1), x_1^-(n - 1)]_{q_0^{-1}}]_{q_0} + [x_0^-(k), [a_0(1), [x_0^-(k), x_1^-(n - 1)]_{q_0^{-1}}]_{q_0} \right)
\]
\[= [2]_1 \left[ x_0^-(k), [x_0^-(k + 1), x_1^-(n - 1)]_{q_0^{-1}} \right]_{q_0} + [2]_1 \left[ x_0^-(k + 1), [x_0^-(k), x_1^-(n - 1)]_{q_0^{-1}} \right]_{q_0}
\]
\[+ \gamma^{-\frac{k}{2}} \left[ a_0(1), [x_0^-(k), x_1^-(n - 1)]_{q_0^{-1}} \right]_{q_0} = 0,
\]

where we have used the inductive hypothesis

\[
[x_0^-(k), [x_0^-(k + 1), x_1^-(n - 1)]_{q_0^{-1}}]_{q_0} + [x_0^-(k + 1), [x_0^-(k), x_1^-(n - 1)]_{q_0^{-1}}]_{q_0} = 0,
\]
\[x_0^-(k), [x_0^-(k), x_1^-(n - 1)]_{q_0^{-1}} = 0.
\]

The proof of relation (4.53) is almost as same as that of relation (4.52).

Similarly we only check relation (4.54) for the case \( - \) and \( \epsilon = + \), first it follows from (4.53) for all \( m, n \in \mathbb{N} \)

\[
A_2 = [x_1^-(n), [x_1^-(n), x_0^-(m)]_{q_1^{-1}}]_{q_1} = 0.
\]
Therefore it is clear that
\[ 0 = [a_1(t), A_2] = [a_1(t), [x_1^-(n), [x_1^-(n), x_0^-(m)]_{q_1^{-1}}]_{q_1}] \]
\[ = -\frac{[2t]}{t} \gamma \frac{[\text{]}}{[\text{]}} \left( [x_1^-(n + t), [x_1^-(n), x_0^-(m)]_{q_1^{-1}}]_{q_1} + [x_1^-(n), [x_1^-(n + t), x_0^-(m)]_{q_1^{-1}}]_{q_1} \right) \]
\[ + \frac{[t]}{t} \gamma \frac{[\text{]}}{[\text{]}} [x_1^-(n), [x_1^-(n), x_0^-(m + t)]_{q_1^{-1}}]_{q_1}, \]
which implies that
\[ [x_1^-(n + t), [x_1^-(n), x_0^-(m)]_{q_1^{-1}}]_{q_1} + [x_1^-(n), [x_1^-(n + t), x_0^-(m)]_{q_1^{-1}}]_{q_1} = 0. \]

**Step 3:** Repeating the above two steps, we can obtain all elements \( x_i^\pm(ek) \) and \( a_i(ek) \) at the remaining nodes \( 1 < i \in I \) similarly, which satisfy all relevant relations in Definition 4.11.

\[
(x_{i\pm}(\epsilon) = \pm \gamma^{\frac{1}{2}} \left[ a_{i-1}(\epsilon), x_i^\pm(0) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), (4.55) \\
a_i(1) = \gamma \frac{1}{2} K_i^{-1} \left[ x_i^+(0), x_i^-(1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), (4.56) \\
a_i(-1) = \gamma^{-\frac{1}{2}} K_i \left[ x_i^-(0), x_i^+(1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), (4.57) \\
x_i^\pm(ek) = \pm [2]^{-1} \gamma^{\frac{1}{2}} \left[ a_i(\epsilon), x_i^\pm(\epsilon(ek) - 1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), (4.58) \\
\phi_i(k) = (q_i - q_i^{-1}) \gamma^{\frac{2-k}{2}} \left[ x_i^+(k-1), x_i^-(1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}), (4.59) \\
\varphi_i(-k) = -(q_i - q_i^{-1}) \gamma^{\frac{k-2}{2}} \left[ x_i^+(1), x_i^-(k+1) \right] \in \mathcal{U}_0(\mathfrak{g}_{2,tor}). (4.60)
\]

Actually we can check all defining relations involving these elements in Definition 4.11 similar to the above two steps, except for the Serre relations of non-simply-laced cases. We verify the Serre relations of type \( B_n \) and \( G_2 \) as follows.

**Proposition 4.13.** For type \( B_n \), the following Serre relation holds.

\[
\text{Sym}_{\epsilon_1, \epsilon_2, \epsilon_3} \sum_{s=0}^{3} (-1)^s \left( \begin{array}{c} 3 \\ s \end{array} \right) x_n^{-}(\epsilon_1) \cdots x_n^{-}(\epsilon_s)x_n^{-}(\epsilon_{s+1}) \cdots x_n^{-}(\epsilon_3) = 0. (4.61) 
\]

**Proof.** It holds by induction on the index that
\[ A_3 = [x_n^-(0), [x_n^-(0), [x_n^-(0), x_n^-(m)]_{q_n^{-1}}]_{q_n}]_{q_n} = 0, \]
\[ B_3 = [x_n^-(0), [x_n^-(0), x_n^-(m)]_{q_n^{-1}}]_{q_n} = 0. \]

the remaining proof is almost as the same as that of Proposition 4.11.

**Proposition 4.14.** For type \( G_2 \), we have that

\[
\text{Sym}_{\epsilon_1, \cdots, \epsilon_4} \sum_{s=0}^{4} (-1)^s \left( \begin{array}{c} 4 \\ s \end{array} \right) x_2^{-}(\epsilon_1) \cdots x_2^{-}(\epsilon_s)x_2^{-}(\epsilon_{s+1}) \cdots x_2^{-}(\epsilon_4) = 0. (4.62) 
\]

**Proof.** The proof is divided into five cases according to the value of \( \epsilon_i \).
Then \( U \) is injective except for \( a = 0 \) and \( B = 0 \), since all relations of the latter (in the same notation) do form a simplified set of generators for the latter. From \( \pi \), and they keep all defining relations in Definition 3.1, so it follows that \( \pi(x) = 0 \) for all \( x \) in the former. From \( \pi \), we have been using the same notation for the elements in the former.

For the case of type \( A \), note that the Dynkin diagram is cyclic, so for \( \epsilon = \pm 1 \) or \( \pm \), we can also define that

\[
y_n^{\pm}(\epsilon) = \pm \gamma^{\pm \epsilon} \left[ a_0(\epsilon), \ x_n^{\pm}(0) \right],
\]

\[
b_n(1) = \gamma^{\frac{\epsilon}{2}} K^{-1}_1 \left[ x_n^{+}(0), \ y_n(1) \right],
\]

\[
b_n(-1) = \gamma^{-\frac{\epsilon}{2}} K_1 \left[ y_n^{+}(-1), \ x_n^{-}(0) \right].
\]
We define the elements $y_i^\pm(\epsilon), b_i(1)$ and $b_i(-1)$ for $i = n - 1, n - 2, \ldots, 1, 0$ inductively by

$$y_i^\pm(\epsilon) = \pm\gamma^{\pm_1} [b_{i+1}(\epsilon), x_i^\pm(0)],$$
$$b_i(1) = \gamma^{\mp_1} K_1^{-1} [y_i^+(0), y_i^-(1)],$$
$$b_i(-1) = \gamma^{-\mp_1} K_1 [y_i^+(1), x_i^-(0)].$$

Arguing by the degree, one can define the higher degree elements $y_i^\pm(k)$ and $b_i(k)$ similar to $x_i^\pm(k)$ and $a_i(k)$ for $i \in I$ and $k \in \mathbb{Z}^*$ (c.f. Step 3). Therefore

$$\ker \pi = \{y_i^\pm(k) - x_i^\pm(k), b_i(k) - a_i(k) | i \in I, k \in \mathbb{Z}^*\}.$$

It is easy to see that $x_n^\pm(\epsilon) - y_n^\pm(\epsilon) \in \ker \pi$ for $\epsilon = \pm 1$ or $\pm$. Let $H_1$ be the ideal of $\mathcal{U}_0(\mathfrak{g}_{2,\text{tor}})$ generated by $(x_n^\pm(\epsilon) - y_n^\pm(\epsilon))$ for $\epsilon = \pm 1$ or $\pm$. Then $H_1 \subseteq \ker \pi$.

Denote the quotient algebra $\mathcal{U}_0(\mathfrak{g}_{2,\text{tor}})/H_1$ by $\mathcal{U}_0(\mathfrak{g}_{2,\text{tor}})$. Actually, in the quotient algebra $\mathcal{U}_0(\mathfrak{g}_{2,\text{tor}})$, we have that

$$b_n(1) = \gamma^{\pm_1} K_1^{-1} [x_n^+(0), y_n^-(1)] = \gamma^{\pm_1} K_1^{-1} [x_n^+(0), x_n^-(1)] = a_n(1),$$
$$b_n(-1) = \gamma^{\mp_1} K_1^{-1} [y_n^+(1), y_n^-(0)] = \gamma^{\mp_1} K_1^{-1} [x_n^+(1), x_n^-(0)] = a_n(-1).$$

Then in the quotient algebra $\mathcal{U}_0(\mathfrak{g}_{2,\text{tor}})$, by induction we have for $i = 0, 1, \ldots, n$ that

$$y_i^\pm(\epsilon) = \pm\gamma^{\pm_1} [b_{i+1}(\epsilon), x_i^\pm(0)] = \pm\gamma^{\pm_1} [a_{i+1}(\epsilon), x_i^\pm(0)] = x_i^\pm(\epsilon),$$
$$b_i(1) = \gamma^{\mp_1} K_1^{-1} [x_i^+(0), y_i^-(1)] = \gamma^{\mp_1} K_1^{-1} [x_i^+(0), x_i^-(1)] = a_i(1),$$
$$b_i(-1) = \gamma^{-\mp_1} K_1^{-1} [y_i^+(1), x_i^-(0)] = \gamma^{-\mp_1} K_1^{-1} [x_i^+(1), x_i^-(0)] = a_i(-1).$$

Arguing by the degree, one has that $y_i^\pm(k) = x_i^\pm(k)$ and $b_i(k) = a_i(k)$ for $k \in \mathbb{Z}^*$, which implies $\ker \pi \subseteq H_1$. Thus when $\mathfrak{g}$ is of type $A$, $\mathcal{U}_0(\mathfrak{g}_{2,\text{tor}})/H_1 \cong U_q(\mathfrak{g}_{2,\text{tor}})$.

When $\mathfrak{g}$ is not of type $A$, $\mathcal{U}_0(\mathfrak{g}_{2,\text{tor}}) \cong U_q(\mathfrak{g}_{2,\text{tor}})$.

\[\square\]

5. Proof of Theorem 3.13

Theorem 3.13 is equivalent to the following statement.

**Theorem 5.1.** There exists an epimorphism $\pi_1 : \mathcal{U}_0(\mathfrak{g}_{N,\text{tor}}) \rightarrow \mathcal{U}_q(\mathfrak{g}_{N,\text{tor}})$ such that $\pi_1(a) = a$ for $a \in \mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})$.

To show Theorem 5.1, we note that $\pi_1$ is an algebra homomorphism, as $\pi_1$ preserve all relations from (3.14) to (3.27) since they are part of the defining relations according to Definition 3.1.

Now we are left to show $\pi_1$ is surjective. Actually we can define all generators of $\mathcal{U}_q(\mathfrak{g}_{N,\text{tor}})$ inductively in a similar manner as to Section 4. First of all, for $s \in J$, $k \in \mathbb{N}$ and $\epsilon = \pm 1$, we define...
that 

\[ (3). \text{ In type } U \text{ which represent the same elements in the algebra of Theorem 3.13.} \]

\[ \text{Notice that we can also construct that for } \epsilon, \epsilon' = \pm 1 \text{ and } s \neq s' \in J, \]

\[ x_0^{\pm}(ee_1 + \epsilon' e_2) = \pm [2]_0^{-1} \gamma_1^{\pm 1/2} \left[ a_0^{(1)}(\epsilon), x_0^{\pm}(\epsilon' e_2) \right]. \]

Let \( k = k_1 e_1 + \cdots + k_{N-1} e_{N-1} \), we define that

\[ x_0^{\pm}(k) = \pm [2]_0^{-1} \gamma_1^{\pm 1/2} \left[ a_0^{(N-1)}(k_{N-1}), x_0^{\pm}(k_1 e_1 + \cdots + k_{N-2} e_{N-2}) \right]. \]

Moreover, we can also define the elements \( a_i^{(s)}(k) \) and \( x_i^{\pm}(ke_s) \) for \( i \in I \) and \( k \in \mathbb{Z}^s \) by the same method in section 4. Furthermore, we can construct that for \( \epsilon, \epsilon' = \pm 1 \text{ and } s \neq s' \in J, \)

\[ x_0^{\pm}(ee_1 + \epsilon' e_2) = \pm [2]_0^{-1} \gamma_1^{\pm 1/2} \left[ a_0^{(1)}(\epsilon), x_0^{\pm}(\epsilon' e_2) \right]. \]

Let \( k = k_1 e_1 + \cdots + k_{N-1} e_{N-1} \), we define that

\[ x_0^{\pm}(k) = \pm [2]_0^{-1} \gamma_1^{\pm 1/2} \left[ a_0^{(N-1)}(k_{N-1}), x_0^{\pm}(k_1 e_1 + \cdots + k_{N-2} e_{N-2}) \right]. \]

Then we can define \( x_i^{\pm}(k) \) using the same way. As a consequence, \( \pi_1 \) is surjective.

Let \( H_2 = \ker \pi_1 \), then \( \mathcal{U}_0(\mathfrak{g}_{N,\text{tor}})/H_2 \cong U_q(\mathfrak{g}_{N,\text{tor}}) \). Therefore, we have completed the proof of Theorem 3.13.

**Remark 5.2.** Unlike the case of \( N = 2 \), it is complicated to describe \( \ker \pi_1 \). We have the following observations.

1. Notice that we can also construct that for \( \epsilon, \epsilon' = \pm 1, i \in I \) and \( s \neq s' \in J, \)

\[ x_i^{\pm}(ee_s + \epsilon' e_{s'}) = \pm [2]_0^{-1} \gamma_1^{\pm 1/2} \left[ a_i^{(s)}(\epsilon), x_i^{\pm}(\epsilon' e_{s'}) \right], \]

\[ \tilde{x}_i^{\pm}(ee_s + \epsilon' e_{s'}) = \pm [2]_0^{-1} \gamma_1^{\pm 1/2} \left[ a_i^{(s')}(\epsilon'), x_i^{\pm}(ee_s) \right], \]

which represent the same elements in the algebra \( U_q(\mathfrak{g}_{N,\text{tor}}) \). So for \( i \in I \) and \( s \neq s' \in J, \)

\[ x_i^{-\epsilon}(ee_s + \epsilon' e_{s'}) - \tilde{x}_i^{-\epsilon}(ee_s + \epsilon' e_{s'}) \in \ker \pi_1. \]

2. Note that in the algebra \( U_q(\mathfrak{g}_{N,\text{tor}}) \), we have that \( [x_i^{\pm}(ke_s), x_i^{\pm}(le_{s'})] = 0 \), so

\[ [x_i^{\pm}(ke_s), x_i^{\pm}(le_{s'})] \in \ker \pi_1. \]

3. In type \( A \), since the affine Dynkin diagram is a cycle, for \( \epsilon = \pm 1 \) or \( \pm \text{ and } s \in J, \) we define that

\[ y_n^{\pm}(ee_s) = \pm \gamma_n^{\pm 1/2} \left[ a_0^{(s)}(\epsilon), x_n^{\pm}(0) \right], \]

\[ b_n^{(s)}(1) = \gamma_n^{1/2} K^{-1}_n \left[ x_n^{\pm}(0), y_n^{(s)}(e_s) \right], \]

\[ b_n^{(s)}(-1) = \gamma_n^{1/2} K_n \left[ y_n^{\pm}(-e_s), x_n^{-}(0) \right]. \]
Moreover, we inductively define for $i = n - 1, \cdots, 0$
\[
y_i^+(e_s) = \pm\gamma_s^{\frac{1}{2}} \left[b_i^{(s)}(e), x_i^+(0)\right],
\]
\[
b_i^{(s)}(1) = \gamma_s^{\frac{1}{2}} K_i^{-1} \left[x_i^+(0), y_i^-(e_s)\right],
\]
\[
b_i^{(s)}(-1) = \gamma_s^{\frac{1}{2}} K_i \left[y_i^+(e_s), x_i^-(0)\right].
\]
Furthermore, we also define that for $k \in \mathbb{Z}^{N-1}$ and $i \in I$
\[
y_i^\pm(k) = \pm[2]^{N-1} \gamma_1^{1/2} \left[b_i^{(N-1)}(k_{N-1}), x_i^\pm(k_1 e_1 + \cdots + k_{N-2} e_{N-2})\right].
\]
For $k \in \mathbb{Z}^{N-1}, l \in \mathbb{Z}^s, i \in I$ and $s \in J$
\[
x_i^\pm(k) - y_i^\pm(k) \in \ker \pi_1, \quad a_i^{(s)}(\ell) - b_i^{(s)}(\ell) \in \ker \pi_1.
\]

6. Vertex representations of quantum $N$-toroidal algebras $U_q(\mathfrak{g}_{N,\text{tor}})$ for simply-laced type

In this section, we will construct a level-one vertex representation of the quantum $N$-toroidal algebra for simply-laced type via generating functions (c.f. Def. 3.2).

Let $I = \{0, 1, \cdots, n\}$ and $I_0 = \{1, \cdots, n\}$. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of simply-laced type over $\mathbb{R}$ with the Cartan matrix $-(a_{ij})_{i,j \in I_0}$. Denote by $\hat{\mathfrak{g}}$ the affine Kac-Moody Lie algebra associated to $\mathfrak{g}$ and its Cartan matrix by $(\hat{a}_{ij})_{i,j \in I}$. Let $\mathfrak{h}$ and $\hat{\mathfrak{h}}$ be their Cartan subalgebras, $\Delta$ and $\hat{\Delta}$ their root systems, respectively. Also let $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ be a basis of $\Delta$, where $\alpha_0, \alpha_1, \cdots, \alpha_n$ are the simple roots of $\hat{\mathfrak{g}}$.

Let $\hat{Q} = \bigoplus_{i=1}^n \mathbb{Z} \hat{\alpha}_i$ and $Q = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i$ be the root lattices of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ respectively. The affine weight lattice is $P = \bigoplus_{i=1}^n \mathbb{Z} \hat{\lambda}_i \oplus \mathbb{Z} \delta$, where $\lambda_0, \cdots, \lambda_n$ are the fundamental weights of $\hat{\mathfrak{g}}$ and $\delta$ is the basic imaginary root.

Let $U_q(\hat{\mathfrak{h}})$ be the associative algebra generated by $\{a_i(l) \mid l \in \mathbb{Z}\{0\}, i \in I\}$, satisfying the following relations for $m, l \in \mathbb{Z}\{0\}$,
\[
[a_i(m), a_j(l)] = \delta_{m+l,0} \frac{[ma_{ij}]}{m}[m].
\]
The algebra $U_q(\hat{\mathfrak{h}})$ is a Weyl algebra that deforms the enveloping algebra of the Heisenberg algebra.

We denote by $U_q(\hat{\mathfrak{h}}^+)$ (resp. $U_q(\hat{\mathfrak{h}}^-)$) the commutative subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $a_i(l)$ (resp. $a_i(-l)$) with $l \in \mathbb{Z}_{>0}$, $i \in I$, $j \in J$. Let $S(\hat{\mathfrak{h}}^-)$ be the symmetric algebra generated by $a_i(-l)$ with $l \in \mathbb{Z}_{>0}$. Then $S(\hat{\mathfrak{h}}^-)$ is a $U_q(\hat{\mathfrak{h}})$, $N$-toroidal-module with the action defined by
\[
\gamma_{\alpha_i}^{1/2} \cdot v = q^{\frac{1}{2}} \cdot v,
\]
\[
a_i^{(s)}(-l) \cdot v = a_i(-l) \cdot v,
\]
\[
a_i^{(s)}(l) \cdot v = \frac{[l a_{ij}]}{l} q^l - q^{-l} \frac{d v}{d a_i(-l)}.\]
for any $v \in S(\hat{\mathfrak{h}}^-)$, $l \in \mathbb{Z}_{>0}$ and $i \in I$.

Let $\varepsilon(\ , \ )$ be the 2-cocycle of the root lattice $Q$ such that
\[
\varepsilon(\alpha, \beta) = (-1)^{(\alpha, \beta)} \varepsilon(\beta, \alpha).
\]
Let $\mathbb{K}\{Q\} = \sum_{\alpha \in Q} \mathbb{K}e^\alpha$ be the twisted group algebra spanned by $e^\alpha$ ($\alpha \in Q$) with multiplication: $e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}$. Define the Fock space $F = S(\hat{h}^-) \hat{\otimes} \mathbb{K}\{Q\}$, and extend the action of $U_q(\hat{h}^+)$ naturally. The operators $K_i$, $q^d$ and $z^{a_i(0)}$ act on $F$ as follows ($v \otimes e^\beta \in F$):

$$e^\alpha(v \otimes e^\beta) = v \otimes e^\delta,$$

$$a_i(0)(v \otimes e^\beta) = (\alpha_i, \beta)v \otimes e^\beta,$$

$$z^{a_i(0)}(v \otimes e^\beta) = z^{(\alpha_i, \beta)}v \otimes e^\beta,$$

$$q^d(v \otimes e^\beta) = q^{m_0}(v \otimes e^\beta),$$

where $\beta = \sum_{i=0}^n m_i \alpha_i \in Q$. Let $\varepsilon \::: \ottimes \mathbb{N}$ be the usual normal order defined by moving modes with lower degrees to the left and $e^{\varepsilon a_i}a_i(0) := a_i(0)e^{\varepsilon a_i} := e^{\varepsilon a_i}a_i(0)$.

Now define the vertex operators:

$$Y_i^\pm(z) = \exp \left( \pm \sum_{k=1}^{\infty} \frac{a_i(-k)}{[k]}q^{\mp k/2}z^k \right) \exp \left( \mp \sum_{k=1}^{\infty} \frac{a_i(k)}{[k]}q^{\mp k/2}z^{-k} \right) x^{\varepsilon a_i}z^{a_i(0)} = \sum_{n \in \mathbb{Z}} Y_i^\pm(n)z^{-n},$$

$$\Phi_i(z) = q^{a_i(0)} \exp \left( (q-q^{-1}) \sum_{\ell=1}^{\infty} a_i(\ell)z^{-\ell} \right),$$

$$\Psi_i(z) = q^{-a_i(0)} \exp \left( -(q-q^{-1}) \sum_{\ell=1}^{\infty} a_i(-\ell)z^{\ell} \right).$$

**Theorem 6.1.** For $i \in I$ and $s \in J$, the Fock space $F$ is a level one $U_q(\mathfrak{g}_{N,tor})$-module for simply-laced types under the action $\rho$ defined by:

$$\gamma_s^\pm \mapsto q^{\pm \frac{1}{2}},
q^\pm d \mapsto q^\pm d,
K_i \mapsto q^{a_i(0)},
x_i^\pm(k) \mapsto Y_i^\pm(ht(k)),
\phi_i^{(s)}(z) \mapsto \Phi_i(z),
\varphi_i^{(s)}(z) \mapsto \Psi_i(z),$$

where $ht(k) \doteq k_1 + \cdots + k_{N-1}$ for $k = (k_1, \cdots, k_{N-1})$.

This result can be checked directly by noting that the map specified in the theorem is in fact a homomorphism from the quantum $\hat{N}$-toroidal algebra to the quantum toroidal algebra and then using the Fock space representation constructed in [Sy].

**7. Appendix**

In the appendix, we will list the Dynkin diagrams case by case according to the type of $\mathfrak{g}$ and GIM $M$ given in Definition 2.4. Here if $m_{ij} \in \hat{M}$ such that $m_{ij} > 0$ for $i \neq j$, we use dotted lines to replace the edges of the Dynkin diagram for general Cartan matrix, and we keep other rules of the Dynkin diagram for Cartan matrix. We give the Dynkin diagrams for the case of $N = 2$ and $N = 3$, respectively.
7.1. Dynkin diagrams for the case of $N = 2$. (I). Type $A_n(n > 1)$:

(II). Type $B_n(n > 2)$:

(III). Type $C_n(n > 1)$:

(IV). Type $D_n(n > 3)$:
(V). Type $E_6$:

(VI). Type $E_7$:

(VII). Type $E_8$:
(VIII). Type $F_4$: 

(IX). Type $G_2$: 

7.2. Dynkin diagrams for the case of $N = 3$. (I). Type $A_n$, ($n > 1$):
For $N=3$

(II). Type $B_n$, ($n > 2$):

(III). Type $C_n$, ($n > 1$):

(IV). Type $D_n$, ($n > 3$):
For N=3

(V). Type $E_6$ and $\tilde{\mathcal{J}} = \{-N+1, \ldots, -1, 0, 1, \ldots, 6\}$:

(VI). Type $E_7$ and $\tilde{\mathcal{J}} = \{-N+1, \ldots, -1, 0, 1, \ldots, 7\}$:

(VII). Type $E_8$ and $\tilde{\mathcal{J}} = \{-N+1, \ldots, -1, 0, 1, \ldots, 8\}$:
(VIII). Type $F_4$ and $\tilde{J} = \{-N + 1, \cdots, -1, 0, 1, 3, 4\}$:

(IX). Type $G_2$ and $\tilde{J} = \{-N + 1, \cdots, -1, 0, 1, 2\}$:

ACKNOWLEDGMENT

Y. Gao would like to thank the support of NSERC of Canada and NSFC grant 11931009. N. Jing would like to thank the support of Simons Foundation grant 523868 and NSFC grants 11531004 and 12171303. L. Xia would like to thank the support of NSFC grants 11871249 and 12171155. H. Zhang would like to thank the support of NSFC grant 11871325.

Statement on Conflict of interest: The authors declare that they have no conflict of interest.
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DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, TORONTO, ON M3J 1P3, CANADA
*Email address*: ygao@yorku.ca

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27695, USA
*Email address*: jing@ncsu.edu

INSTITUTE OF APPLIED SYSTEM ANALYSIS, JIANGSU UNIVERSITY, ZHENJIANG, JIANGSU 212013, CHINA
*Email address*: xialimeng@ujs.edu.cn

DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, SHANGHAI 200444, CHINA
*Email address*: hlzhangmath@shu.edu.cn