Lie algebras of curves and loop-bundles on surfaces

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Abstract
W. Goldman and V. Turaev defined a Lie bialgebra structure on the $\mathbb{Z}$-module generated by free homotopy classes of loops of an oriented surface (i.e. the conjugacy classes of its fundamental group). We generalize this construction to a much larger space of equivalence classes of curves by replacing homotopies by thin homotopies, following the combinatorial approach of M. Chas. As an application we use properties of the generalized bracket to give a geometric proof of a conjecture by Chas in the original setting of full homotopy classes, namely a characterization of homotopy classes of simple curves in terms of the Goldman–Turaev bracket.

Keywords Goldman–Turaev bracket · Loop spaces · Characterization of simple curves · Thin homotopies

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1 Introduction
Goldman [14] and Turaev [20] defined a Lie bialgebra structure on the $\mathbb{Z}$-module generated by the free homotopy classes of loops of an oriented surface $M$ (i.e. the conjugacy classes of $\pi_1(M)$). The bracket is defined by
\[
[X, Y]_{\pi_1(M)} = \sum_{p \in \alpha \cap \beta} \epsilon(p; \alpha, \beta) [\alpha_p \beta_p]
\]  
(1)

where \(\alpha\) and \(\beta\) are representatives of \(X\) and \(Y\) respectively, intersecting at most at transversal double points, the number \(\epsilon(p; \alpha, \beta) = \pm 1\) denotes the oriented intersection number of \(\alpha\) and \(\beta\) at \(p\), and \([\alpha_p \beta_p]\) is the conjugacy class of the element \(\alpha_p \cdot \beta_p \in \pi_1(M, p)\) where \(\alpha_p\) and \(\beta_p\) are the elements of \(\pi_1(M, p)\) that correspond to reparametrizing \(\alpha\) and \(\beta\) to start (and end) at \(p\).

Turaev showed in [20] that there is a colagebra structure that gives rise to a Lie bialgebra. Chas in [3] proposed a combinatorial model for this bialgebra structure. The aim of this paper is to generalize the Goldman–Turaev construction from homotopy classes to the much larger space of thin homotopy classes. Thin homotopies relate curves that differ by spikes, that is, by segments of the form \(aa^{-1}\) which trace a path and then immediately backtrack. They will be defined precisely in section 2.1.

Let \(M\) be an oriented surface endowed with any Riemannian metric. We denote by \(\mathcal{E}(M)\) the set of classes of piecewise geodesic curves in \(M\) modulo thin homotopies. For each \(x \in M\) we define \(\mathcal{E}_x(M)\) as the set of elements of \(\mathcal{E}(M)\) that start at \(x\), and \(\mathcal{L}_x(M)\) as the subset that both starts and ends at \(x\). It is easy to see that this is a group under concatenation, and \(\pi_1(M, x)\) is a quotient of it.

The free conjugacy class \(g \in \mathcal{L}_x(M)\) is the set of \(p^{-1}gp \in \mathcal{E}(M)\) such that \(p \in \mathcal{E}_x(M)\). Let \(S(M)\), the space of strings of \(M\), be the set of free conjugacy classes of \(\mathcal{L}_x(M)\), which does not depend on \(x\), as shown in Sect. 2.4. Let \(S(M)\) be the free abelian group generated by \(S(M)\). In this paper we shall define a bracket \([\cdot, \cdot]\) on \(S(M)\) following the approach of Chas in [3]. Then we shall show that

**Theorem 1.1** \((S(M), [\cdot, \cdot])\) is a Lie algebra.

It is also possible to give a coalgebra structure and show that \(S(M)\) is a Lie bi-algebra using the same techniques, though we will not present this construction explicitly. We will see that the Goldman–Turaev structure on \(M\) is the quotient of \(S(M)\) obtained by taking regular (non-thin) homotopies. This fact will emerge naturally from our construction of the bracket, and the case for the co-bracket is analogous. On surfaces with boundary the Goldman algebra is a subalgebra of \(S(M)\). (See the proof of Corollary 1.3).

Perhaps it would be natural to compare the bracket on \(S(M)\) with the Andersen-Mattes-Reshetikhin (AMR) bracket [2]. The Goldman bracket differs from the AMR bracket: The main result in [2] shows that if \(\alpha_1\) and \(\alpha_2\) are two free homotopy classes of loops on an oriented surface, then the number of terms in the AMR bracket of \(\alpha_1\) and \(\alpha_2\) equals the minimum number of intersection points of loops in the classes \(\alpha_1\) and \(\alpha_2\). This is not the case, in general, for the Goldman bracket of \(\alpha_1\) and \(\alpha_2\), as example 5.6 in [3] shows. The bracket defined here is therefore distinct from the AMR bracket both because it is defined on \(S(M)\) which is much larger than the space of homotopy classes, containing uncountably many non-trivial elements in arbitrarily small neighborhoods, and because it projects to the Goldman bracket, not the AMR bracket, on homotopy classes. After Definition 3.9 we give an explicit example of a string \(X\) having a representative contained in a contractible disk such that \([X, X^{-1}] \neq 0\). Since the AMR bracket vanishes on the sphere, we have again that the algebras are different.

The Goldman algebra is relevant in the study of spaces of representations of the fundamental groups of 2-manifolds, which in turn, can be regarded as moduli spaces of flat connections on orientable 2-manifolds [16]. Our construction, defined on a larger set of equivalence classes of curves which separate also non-flat connections, may similarly play a

[End of text]
role in the study of the space of all connections. This aspect will not be treated here. Although $S(M)$ is quite a large set (uncountable), the fact that its elements are equivalence classes of piecewise geodesic curves ensures that intersections are clean, which allows us to adapt the approach in [3].

Chas and Krongold obtained an algebraic characterization of homotopy classes of simple curves in terms of the bracket [9] and the cobracket [10]. (See also [7, 8]). In the present paper we give a geometric proof of a different characterization of simple curves in terms of the bracket, which was conjectured by Chas in [3]. For other related results see [1, 2, 4–6, 11–13].

A string $X$ is primitive if every representative $g \in L_x(M)$ for any $x$ in $M$ of $X$ is primitive in the sense of group theory: if there is no $h \in L_x(M)$ such that $g = h^n$ with $n > 1$. A piecewise geodesic closed curve will be called simple if it has no stable self-intersections, i.e. if there is a small perturbation that has no self-intersections (and thus is simple in the usual sense).

We shall show the following theorem:

**Theorem 1.2** Let $M$ be an oriented surface. A primitive $X \in S(M)$ has a simple representative if and only if $[X, X^{-1}] = 0$.

The geometric group theory approach of our proof of Theorem 1.2 may have some independent interest beyond the result itself. The aforementioned proof includes some tools, namely, the notion of $\alpha$-oriented subgroups (see Sect. 4.3), which could be useful in other contexts.

Theorem 1.2 will allow us to prove a conjecture posed by Chas [3]:

**Corollary 1.3** Let $M$ be a compact oriented surface with boundary. A primitive free homotopy class $X$ of $M$ has a simple representative if and only if $[X, X^{-1}]_{\pi_1(M)} = 0$.

**Proof** Let $p : \tilde{M} \to M$ be the universal covering of $M$. Let $D \subset \tilde{M}$ be a fundamental polygon as in [3]. Since $M$ has boundary the fundamental group is freely generated by the set $T$ of those $g$ in $\pi_1(M)$ such that $gD \cap D \neq \emptyset$. Choose on $\tilde{M}$ a metric of constant non-positive curvature and let $L_{x,0}(M) \subset L_x(M)$ be the subgroup of $L_x(M)$ generated by those elements of the form $h_g = [p \circ \alpha_g]$ where $\alpha_g$ is a geodesic (corresponding to the chosen metric) joining $x$ to $gx$ for each $g \in T$. Denote by $S_0(M)$ the set of strings corresponding to the loops in $L_{x,0}(M)$. Note that $\pi_1(M)$ is isomorphic to $L_{x,0}(M)$ via the standard quotient, i.e. taking non-thin homotopies. As we mentioned below Theorem 1.1, this quotient is a Lie algebra homomorphism, thus it gives an isomorphism between the subalgebra of $S(M)$ generated by $S_0(M)$ and the Goldman–Turaev algebra on the free homotopy classes of $M$. Then we apply Theorem 1.2 to strings in $S_0(M)$ to conclude this proof. \(\square\)

## 2 The loop bundle

### 2.1 Definitions of thin homotopies and the spaces of loops

Let $I$ be the unit interval and $M$ a Riemannian manifold. We begin by recalling some standard notations. A *path* in $M$ is a continuous function from $I$ to $M$, and we say that two paths $a, b : I \to M$ are equivalent modulo reparametrization if there is an orientation preserving homeomorphism $\sigma : I \to I$ such that $a \circ \sigma = b$. Denote by $\Omega_0$ the quotient set under this equivalence relation. If $a(1) = b(0)$ we define $ab$ and $a^{-1}$ as follows: $ab(t) = a(2t)$ if...
that if \( \delta \) is trivial. Namely that: \( \Omega_1 \)

In order to define what we call thin homotopy between piecewise geodesic paths we need to consider another preliminary equivalence, which amounts to collapsing constant sub-paths. Let \( a \) be a non-constant path in \( M \). We shall define a minimal form \( a_r \) for \( a \) as follows: let \( I_i \subseteq I \) be the maximal subintervals in which \( a \) is constant, and let \( \sigma : I \to I \) be a surjective non-decreasing continuous function, constant in each \( I_i \) and strictly increasing in \( I - \bigcup I_i \). Then there is \( a_r : I \to M \) such that \( a = a_r \circ \sigma \), which is non-constant on any subinterval of \( I \) (this map is obtained by a universal property of quotients). Different choices of the function \( \sigma \) give rise to minimal forms that are equivalent modulo reparametrization, and moreover, if two paths \( a \) and \( b \) are equivalent, so are any of their minimal forms \( a_r \) and \( b_r \). This allows us to define the minimal class of an element of \( \Omega_0 \) (as the class of any minimal form of any representative), and take a quotient \( \Omega_1 \) where we identify two elements of \( \Omega_0 \) if they have the same minimal class (extending the definition to constant paths in the trivial way). The product and inverse are well defined on \( \Omega_1 \), and the classes of constant paths are units for the product.

Let \( \Omega \subseteq \Omega_1 \) be the set of classes of either constant paths or paths that are piecewise geodesic, i.e. a finite concatenation of geodesic segments. Notice that for \( \alpha \in \Omega \) there are well defined notions of endpoints \( \alpha(0) \) and \( \alpha(1) \), of image \( \alpha(I) \), and of length \( l(\alpha) \). Throughout the paper we will refer to the elements \( \alpha \in \Omega \) as curves, and say that \( \alpha \) is a closed curve if \( \alpha(0) = \alpha(1) \).

In the set \( \Omega \) we consider the equivalence relation generated by the identifications \( aaaa^{-1}b \sim ab \). This is what we call equivalence under thin homotopies. With the formal definition in hand, we recall the concepts from the introduction: Let \( \mathcal{E}(M) \) denote the quotient set of \( \Omega \) under thin homotopies, and let \( \mathcal{L}_x(M) \) be the projection onto \( \mathcal{E}(M) \) of the set of closed curves starting and ending at \( x \). Note that \( \mathcal{L}_x(M) \) is a group under concatenation whose identity element, \( id_x \), is the equivalence class of \( e_x \), the constant path at \( x \).

### 2.2 Reductions and basic properties

A reduction for \( \alpha \in \Omega \) is a factorization of the form \( \alpha = aacc^{-1}d \) with non-trivial \( c \). We say that \( \alpha \) is reduced if it admits no such reduction. Since the curves in \( \Omega \) are classes of piecewise geodesic (or constant) paths, it is easy to show that every element of \( \mathcal{E}(M) \) has a unique reduced representative in \( \Omega \) (though the proof of uniqueness may be a bit cumbersome). The reduced form of \( \alpha \in \Omega \) is the unique reduced curve that is equivalent to \( \alpha \) under thin homotopy.

Next we point out some basic facts that will be used without explicit reference throughout the article. First we see that the concept of length in \( \Omega \) satisfies the expected properties, namely that:

- \( l(ab) = l(a) + l(b) \), and
- if \( ab = cd \) with \( l(a) = l(c) \), then \( a = c \) and \( b = d \).

Next notice that a curve \( \gamma \in \Omega \) satisfies \( \gamma = \gamma^{-1} \) only when it is of the form \( \gamma = cc^{-1} \) because if we write \( \gamma = cd \) with \( l(c) = l(d) \) we obtain \( d = c^{-1} \). Thus for a reduced curve \( \gamma \) we have \( \gamma = \gamma^{-1} \) only when \( \gamma \) is constant.

For \( \gamma, \delta \in \Omega \) we shall write \( \gamma \subseteq \delta \) if we have \( \delta = a\gamma b \) for \( a, b \in \Omega \). In case \( a \) is trivial we say that \( \gamma \) is an initial segment of \( \delta \), and if \( b \) is trivial that \( \gamma \) is a final segment of \( \delta \). Note that if \( \delta \) is reduced, so must be \( \gamma \).
We say that two curves $\gamma$ and $\delta$ overlap if an initial segment of one of them agrees with a final segment of the other, i.e., if we can write either $\gamma = ab$, $\delta = ca$ with $b$ non-constant, or $\gamma = ab$, $\delta = ca$ with $a$ non-constant. Note that if $\gamma$ is reduced and non-constant, then $\gamma$ and $\gamma^{-1}$ cannot overlap; for instance, if $\gamma = ab$ and $\gamma^{-1} = bc$, we get $b = b^{-1}$ where $b$ is reduced, so $b$ must be constant.

### 2.3 Definitions of loop bundle and horizontal lift

Consider the space $E(M)$ defined in the previous section, and let $[\alpha] \in E(M)$ stand for the equivalence class of $\alpha \in \Omega$. Let $E_x(M)$ be the set of the $[\alpha] \in E(M)$ such that $\alpha(0) = x$; define $\pi : E_x(M) \to M$ by $\pi([\alpha]) = \alpha(1)$, and observe that $L_x(M) = \pi^{-1}(x)$. The group $L_x(M)$ acts on $E_x(M)$ by left multiplication and for all $[\alpha] \in L_x(M)$ and $[\gamma] \in E_x(M)$ we have $\pi([\alpha][\gamma]) = \pi([\gamma])$; hence the quadruple $(E_x(M), L_x(M), M, \pi)$ is a principal fiber bundle over $M$, with structure group $L_x(M)$.

Let $\gamma$ be a path in $M$, and take $p \in E_x(M)$ with $\pi(p) = \gamma(0)$. We define the horizontal lift of $\gamma$ at $p$ to be the path $\tilde{\gamma}$ in $E_x(M)$ which is obtained in the following way. Take $\beta$ any representative of $p$ (i.e., $p = [\beta]$), and for each $s \in I$ set $\gamma_s$ to be the path in $M$ defined by $\gamma_s(t) = \gamma(st)$. Then $\tilde{\gamma}(s) = [\beta \gamma_s]$. This horizontal lift can be thought of as a topological connection in the bundle $(E_x(M), L_x(M), M, \pi)$ since we do not need a connection 1-form for its definition and moreover, this horizontal lift can be defined for continuous curves in any topological space. (Although we are not going to use it here, let us mention, for the sake of completeness, that in [18] it was shown that the bundle we have defined has the structure of a differentiable space and the horizontal lift we defined can be obtained from a connection 1-form defined in $E_x(M)$.)

We will say that a path in $E_x(M)$ is horizontal if it can be obtained by horizontal lift (see [17, 19]). Note that the concept of horizontal lift is well defined at the level of curves (i.e. in $\Omega$), thus we may speak of horizontal curves.

We define the length of an horizontal curve as the length of the projection. Observe that the action of $L_x(M)$ preserves the set of horizontal curves, as well as their lengths (by definition). We should clarify that we are not giving a metric on $E_x(M)$.

### 2.4 Conjugacy classes in $L_x$ and the space of strings.

Recall that the free conjugacy class of $g \in L_x(M)$ is the set of $p^{-1}gp$ with $p \in E_x(M)$ and that the space of strings $S(M)$ is the set of free conjugacy classes of $L_x(M)$. This does not depend on $x$ because of the following remark.

**Remark 2.1** (Change of basepoint) If $x$, $y \in M$ and $\gamma_0 \in \Omega$ has $\gamma_0(0) = x$ and $\gamma_0(1) = y$, let $p_0 = [\gamma_0]$ and define the maps $\psi : E_x(M) \to E_y(M)$ by $\psi(p) = p_0^{-1}p$ and $\phi : L_x(M) \to L_y(M)$ by $\phi(g) = p_0^{-1}gp_0$. Then $\phi$ is an isomorphism of groups, and $(\psi, \phi)$ is an isomorphism of fiber bundles over $M$, commuting with the horizontal lift.

We say that a closed curve $\alpha$ is cyclically reduced if $\alpha$ is reduced and it cannot be factorized as $cac^{-1}$ with non-trivial $c$. For a cyclically reduced curve $\gamma$, we say that $\beta \in \Omega$ is a permutation (or cyclical permutation) of $\gamma$ if there are $r, s \in \Omega$ such that $\gamma = rs$ and $\beta = sr$. If $s$ and $r$ are non-constant we say that $\beta$ is a non-trivial permutation of $\gamma$. Note that permutation is an equivalence relation among the cyclically reduced curves in $\Omega$.

Note that there is a bijection between the conjugacy classes in $L_x(M)$ and the free conjugacy classes of elements in $L_x(M)$. By Remark 2.1 every $h \in L_y(M)$, for any $y \in M$,
belongs to the free conjugacy class of some \( g \in \mathcal{L}_x(M) \). In particular, the class \([\alpha]\) of a cyclically reduced \( \alpha \in \Omega \) belongs to the free conjugacy class of some \( g \in \mathcal{L}_x(M) \). Reciprocally, for every \( g \in \mathcal{L}_x(M) \) there is a cyclically reduced \( \alpha \) such that \([\alpha]\) is freely conjugate to \( g \).

On the other hand if \( \hat{\alpha} \) is a permutation of a cyclically reduced \( \alpha \), then \([\alpha]\) and \([\hat{\alpha}]\) belong to the same free conjugacy class of some \( g \in \mathcal{L}_x(M) \). Therefore we have

**Remark 2.2** There is a bijection between the set \( S(M) \) and the permutation classes of cyclically reduced curves.

Throughout the paper, when we take representatives of strings we will always assume them to be cyclically reduced. We shall denote by \([\alpha]\), the free conjugacy class of \( \alpha \). If \( X \in S(M) \) and \( \alpha \) is a representative of it, we define \( X^{-1} \) as the permutation class of \( \alpha^{-1} \). It is straightforward to check that if \( \alpha \) is non-constant then \( \alpha^{-1} \) is not a permutation of \( \alpha \). Thus \( X \neq X^{-1} \) unless \( X \) is trivial.

A cyclically reduced curve \( \alpha \) is *primitive* if there is no \( \gamma \in \Omega \) such that \( \alpha = \gamma^n \) with \( n > 1 \). The following easy result is well known.

**Lemma 2.3** Let \( \alpha \) be a cyclically reduced curve. Then \( \alpha \) is not primitive if and only if \( \alpha \) has a non-trivial permutation \( \hat{\alpha} \) such that \( \alpha = \hat{\alpha} \).

Note that a string \( X \in S(M) \) is primitive, as defined in the introduction, if it has a cyclically reduced representative that is primitive.

### 3 Lie bialgebra structure

#### 3.1 Linked pairs

In order to define the bracket in \( S(M) \) we need a way of encoding the intersections of curves in \( \Omega \) that are stable under local homotopy. We do this by adapting the notion of *linked pairs* from Chas [3] to our context.

Let \( \alpha_1, \alpha_2 \) and \( \gamma \) in \( \Omega \) be classes of geodesic segments contained in a normal ball such that \( \alpha_1(1) = \alpha_2(0) = y \) and either \( \gamma(0) = y \) or \( \gamma(1) = y \). Assume that \( \alpha = \alpha_1 \alpha_2 \) is reduced and \( \gamma \) only meets \( \alpha \) at \( y \). Take \( \rho > 0 \) small enough so that \( B(y, \rho) \) is a normal ball and \( \alpha_1, \alpha_2 \) and \( \gamma \) are not contained in it, and let \( z_1, z_2, z \) be the intersection points of \( \alpha_1, \alpha_2 \) and \( \gamma \) with \( \partial B(y, \rho) \) respectively. Since \( M \) is oriented, the orientation of \( B(y, \rho) \) induces an orientation of \( \partial B(y, \rho) \cong S^1 \), which is equivalent to giving a circular order on \( \partial B(y, \rho) \).

We write \( \text{sign}(\alpha, \gamma) = 1 \) if either \( \gamma(0) = y \) and the order of the sequence \( z_2, z, z_1 \) coincides with the circular order of \( \partial B(y, \rho) \), or \( \gamma(1) = y \) and the order of the sequence \( z_2, z_1, z \) coincides with the circular order of \( \partial B(y, \rho) \). Otherwise we write \( \text{sign}(\alpha, \gamma) = -1 \). Notice that this sign does not depend on the choice of \( \rho \).

Informally one could say that \( \text{sign}(\alpha, \gamma) = 1 \) if \( \gamma \) is either outgoing at the “left” side of \( \alpha \) or incoming at the “right” side of \( \alpha \), while \( \text{sign}(\alpha, \gamma) = -1 \) if one of the reverse situations happens.

Since elements of \( \Omega \) are piecewise geodesic curves, the intersections between two elements are either transversal or along an interval. Taking this into account, we discuss the general forms of these intersections and indicate which ones will constitute linked pairs. A factorization of a curve \( \alpha \in \Omega \) is a sequence \((\alpha_1, \ldots, \alpha_n)\) such that \( \alpha = \alpha_1 \cdots \alpha_n \) where \( \alpha_i \in \Omega \).
Definition 3.1 Consider the following factorizations (of some curves)

\[
A = (a, \eta, b) \\
B = (c, \xi, d)
\]

where \(a, b, c, d\) are geodesics contained in normal balls. We say that \((A, B)\) is a linked pair if any of the following conditions hold

1. \(\eta = \xi = \text{point}, \ d \ \text{meets} \ ab \ \text{only at} \ d(0) \ \text{and} \ c \ \text{meets} \ ab \ \text{only at} \ c(1), \ \text{and}
   \[
   \text{sign}(ab, d) = \text{sign}(ab, c)
   \]
2. \(\eta = \xi \ \text{(non-constant)}, \ \text{if we factorize} \ \eta = \gamma_1 \eta_1 \gamma_2 \ \text{such that} \ \gamma_1 \ \text{and} \ \gamma_2 \ \text{are contained in normal balls, we have that} \ d \ \text{meets} \ \gamma_2 b \ \text{only at} \ d(0) \ \text{and} \ c \ \text{meets} \ a \gamma_1 \ \text{only at} \ c(1), \ \text{and}
   \[
   \text{sign}(\gamma_2 b, d) = \text{sign}(a \gamma_1, c)
   \]
3. \(\eta = \xi^{-1} \ \text{(non-constant)}, \ \text{if we factorize} \ \eta = \gamma_1 \eta_1 \gamma_2 \ \text{such that} \ \gamma_1 \ \text{and} \ \gamma_2 \ \text{are contained in normal balls, we have that} \ c \ \text{meets} \ \gamma_2 b \ \text{only at} \ c(1) \ \text{and} \ d \ \text{meets} \ a \gamma_1 \ \text{only at} \ d(0), \ \text{and}
   \[
   \text{sign}(a \gamma_1, d) = \text{sign}(\gamma_2 b, c)
   \]

We define the sign of the linked pair as follows: In case (1) we set \(\text{sign}(A, B) = \text{sign}(ab, d)\), in case (2) we set \(\text{sign}(A, B) = \text{sign}(\gamma_2 b, d)\) and in case (3) set \(\text{sign}(A, B) = \text{sign}(a \gamma_1, d)\).

If all but the orientation (sign) conditions hold we say that \((A, B)\) is an intersection pair. Notice that the intersections between two cyclically reduced curves in \(\Omega\) can locally be written in the form of intersection pairs. The orientation conditions say that an intersection pair \((A, B)\) is a linked pair exactly when the intersection between the underlying curves is stable under small perturbations. We shall refer to linked pairs of type (1), (2) or (3) according to which one of the conditions they satisfy in Definition 3.1, and we do the same for intersection pairs.

Next we turn to the intersections of cyclically reduced curves in \(\Omega\) in a global sense, i.e. in a way that takes account of multiplicities. Our goal is to define intersections for strings, so it will be useful to explicit the natural bijection between the factorizations of a closed curve and those of a permutation of it. Let \(\hat{\alpha}\) be a permutation of \(\alpha\). For each factorization \(R\) of \(\alpha\) we are going to associate \(\hat{R}\), a factorization of \(\hat{\alpha}\). Let \(R = (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_n), \ \alpha = \omega_1 \omega_2 \ \text{and} \ \hat{\alpha} = \omega_2 \omega_1\). Then \(\omega_1 = \gamma_1, \ldots, \gamma_i, 1, \omega_2 = \gamma_i, 2, \ldots, \gamma_n, \ \text{such that} \ \gamma_i = \gamma_i, 1, \gamma_i, 2, 1, \ldots, \). Set \(\hat{R} = (\gamma_{i,2}, \ldots, \gamma_n \gamma_1, \ldots, \gamma_{i,1})\). In the set of factorizations of \(\alpha\) we consider the equivalence relation generated by identifying \(R\) and \(\hat{R}\). We say that the class of a factorization \(P = (\gamma_1, \gamma_2, \gamma_3)\) is a cyclic factorization of \(\alpha\). We say that \(P^a\) is an abbreviated cyclic factorization associated to \(P\) if either \(P^a = (\gamma_2, \gamma_3 \gamma_1)\) or \(P^a = (\gamma_3 \gamma_1, \gamma_2)\). Note that \(P^a\) depends only on the class of \(P\).

The concept of cyclic factorization we defined here, and its difference with that of abbreviated cyclic factorizations, is most relevant when \(\alpha\) is not primitive, since in this case we have different cyclic factorizations yielding the same pair of abbreviated cyclic factorizations.

Definition 3.2 Let \(\alpha, \beta \in \Omega\) be cyclically reduced closed curves. A linked pair between \(\alpha\) and \(\beta\) is a pair \((P, Q)\) of cyclic factorizations of \(\alpha\) and \(\beta\) respectively, such that there are \(P^a = (\alpha_1, \eta)\), an abbreviated cyclic factorization associated to \(P\) and \(Q^a = (\beta_1, \xi)\), an abbreviated cyclic factorization is associated to \(Q\) so that if we write

- \(\alpha_1 = b \alpha_1 a\) and \(\beta_1 = d \beta_1 c\), where \(a, b, c, d\) are geodesics contained in normal balls, and
- \(A = (a, \eta, b)\) and \(B = (c, \xi, d)\),
then \((A, B)\) is a linked pair.

Notice that the concatenations \(a\eta b\) and \(c\xi d\) are well defined, so the above definition makes sense. Moreover, they are sub-curves of some permutations of \(\alpha\) and \(\beta\) respectively, thus saying that \((A, B)\) is a linked pair means that there is a stable intersection between \(\alpha\) and \(\beta\), or the strings they represent. Defining \(P\) and \(Q\) as cyclic factorizations keeps track of the position of the intersection segments relative to the parameter-basepoints of \(\alpha\) and \(\beta\), so intersections that repeat count as different linked pairs. This amounts to counting multiplicity, just as is usual in differential topology for the intersection between transversal smooth paths. Notice also that taking a permutation of \(\alpha\) or \(\beta\) induces a natural bijection between the sets of linked pairs.

We define the length of a linked pair as \(l(P, Q) = l(\eta) = l(\xi)\), i.e. as the length of the intersection segment. The type of \((P, Q)\) shall be the type of \((A, B)\) in Definition 3.2.

### 3.2 Definition of the string bracket

In this section we define the bracket following closely the presentation in [3]. Since the definition of the co-bracket involves no new ideas we omit it. Recall that \(S(M)\), the space of strings, is the set of conjugacy classes of \(L_1(M)\). Also recall that \(S(M)\) is the free abelian group generated by \(S(M)\), in which we shall define the bracket.

For \(X \in S(M)\) and an integer \(n > 0\), let \(X^n\) be the conjugacy class of \([\alpha]^n\), where \([\alpha]\) represents \(X\). Define also \(l(X) = l(\alpha)\) where \(\alpha\) is a cyclically reduced representative of \(X\), noting that different choices of such representative have the same length. Since \(\alpha\) is cyclically reduced, we have that \(l(X^n) = nl(X)\).

Although we will not present the definition of the co-bracket, we give the main definition in which it is based, for the sake of completeness.

**Definition 3.3** Let \(X\) be a string. We define \(LP_1(X)\), the set of linked pairs of \(X\), as the set of linked pairs of any two representatives of \(X\).

By the discussion at the end of the previous section, the choice of representatives of \(X\) in Definition 3.3 does not affect \(LP_1(X)\). It is possible to show, similarly as in [3], that this definition reflects the stable self-intersections of \(X\), at least when \(X\) is primitive, in a 2 to 1 correspondence: each stable self-intersection corresponds to two linked pairs of the form \((P, Q)\) and \((Q, P)\). Non-primitive closed curves have stable self-intersections, in the sense of the self-intersections of a transversal perturbation, that do not arise from linked pairs. Since we will not focus on the co-bracket, we shall not prove these assertions. Next we turn to the case of linked pairs between two strings, that will be the key for the construction of the bracket.

**Definition 3.4** Let \(X\) and \(Y\) be strings. Define \(LP_2(X, Y)\), the set of linked pairs of \(X\) and \(Y\), as the set of linked pairs \((P, Q)\) between representatives of \(X^n\) and \(Y^m\) for \(n, m \geq 1\), where \(l(X^{n-1}) \leq l(P, Q) < l(X^n)\) and \(l(Y^{m-1}) \leq l(P, Q) < l(Y^m)\). (Defining \(l(X^0) = l(Y^0) = 0\).

Again, different choices of representatives for the strings in Definition 3.4 yield sets \(LP_2(X, Y)\) that are in natural bijection.

**Remark 3.5** The powers are necessary:

Consider \(\alpha\) and \(\beta\), closed geodesics starting and ending at the same point \(x\) and meeting transversally at \(x\). Let \(X\) be the conjugacy class of \([\alpha]\) and \(Y\) the conjugacy class of \([\beta][\alpha^2]\).
There is no linked pair between $X$ and $Y$ but there is a linked pair between $X^3$ and $Y$. Note that the core segment of the linked pair is $\alpha^2$.

On the other hand, it can be shown that $LP_2(X, X) = LP_1(X)$, i.e. the powers are not needed in the case $X = Y$. We shall see later, in Lemma 3.13, that $LP_2(X, Y)$ captures the notion of stable intersections between $X$ and $Y$. It is not immediate from Definition 3.4 that $LP_2(X, Y)$ is finite, the proof of this fact will be based in the following result.

**Proposition 3.6** Let $U = \{\alpha_1, \ldots, \alpha_n\}$ be a finite set of piecewise geodesic curves. There are factorizations $\alpha_i = a_{i,1} \cdots a_{i,n_i}$ such that whenever $a_{i,j} \cap a_{k,l} \neq \emptyset$, either

1. $a_{i,j}$ and $a_{k,l}$ meet only at one endpoint.
2. $a_{i,j} = a_{k,l}$
3. $a_{i,j} = a_{k,-1}$

**Proof** Subdivide any factorization of the curves $\alpha_i$ until the desired properties are obtained. This will happen because of the transversality properties of the geodesics. \(\square\)

Given strings $X$ and $Y$, Proposition 3.6 allows us to find a finite set of curves that works as an alphabet for writing some representatives of $X$ and $Y$, as well as all the core curves of the intersection pairs between (powers of) these representatives. Thus we can write the cyclic factorizations that make up the elements of $LP_2(X, Y)$ as words in this alphabet.

**Lemma 3.7** For any strings $X$ and $Y$, $LP_2(X, Y)$ is finite.

**Proof** Using Proposition 3.6 as indicated above, this becomes a straightforward adaptation of Lemma 2.9 of [3]. \(\square\)

**Definition 3.8** Let $X$ and $Y$ be strings and $(P, Q) \in LP_2(X, Y)$. Let $P^\alpha = (\alpha_1, \eta)$ and $Q^\beta = (\beta_1, \xi)$ be abbreviated cyclic decomposition as in the definition of linked pairs, and let $\alpha$ and $\beta$ be the representatives of $X$ and $Y$ that satisfy the following:

- If $(P, Q)$ is of type (1) or (2), then
  
  $$\alpha^n = \alpha_1 \eta \text{ and } \beta^m = \beta_1 \xi$$

  (where $n, m \geq 1$ are the powers of $X$ and $Y$ that correspond to $(P, Q)$ in Definition 3.4).

- If $(P, Q)$ is of type (3), then
  
  $$\alpha^n = \alpha_1 \eta \text{ and } \beta^m = \xi \beta_1$$

  (for the same $n, m \geq 1$).

In any of the above cases, define $(X \cdot (P, Q) Y)$ to be the conjugacy class of $[\alpha][\beta]$.

We say that $(X \cdot (P, Q) Y)$ is the dot product of $X$ and $Y$ at $(P, Q)$. Notice that $\alpha$ and $\beta$ are the representatives of $X$ and $Y$ that we get by choosing parameter-basepoints at the “ending” of the linked pair’s core curve. They are indeed loops based at the same point, so the concatenation $[\alpha][\beta]$ is well defined.

**Definition 3.9** Let $X$ and $Y$ be strings, we define their bracket as

$$[X, Y] = \sum_{(P, Q) \in LP_2(X, Y)} \text{sign}(P, Q)(X \cdot (P, Q) Y)$$

Then we extend the definition to $S(M)$ so that the bracket is bilinear.
As an example consider the string $X$ represented by the curve $\gamma$ depicted in Fig. 1, which is assumed to be contained in a contractible disk. Namely, consider $X = \{\gamma\}$ where $\gamma = \beta\xi^{-1}\alpha\xi$, where $\alpha$, $\beta$ and $\xi$ are simple and $\xi$ meets $\alpha$ and $\beta$ at its endpoints.

We consider the following linked pairs of $LP_2(X, X^{-1})$: Firstly the pair $(P_1, Q_1)$ given by $P_1 = (\alpha, \xi, \beta\xi^{-1})$ and $Q_1 = (\alpha^{-1}, \xi, \beta^{-1}\xi^{-1})$ and secondly $(P_2, Q_2)$ where $P_2 = (\beta, \xi^{-1}, \alpha\xi)$ and $Q_2 = (\beta^{-1}, \xi^{-1}, \alpha^{-1}\xi)$. Note that sign $(P_1, Q_1) = 1$ and sign $(P_2, Q_2) = -1$, and also that $LP_2(X, X^{-1})$ consists exactly on these two linked pairs.

We compute

\[(X \cdot_{(P_1, Q_1)} X^{-1}) = \{\beta\xi^{-1}\alpha\xi\beta^{-1}\xi^{-1}\alpha^{-1}\xi\},\]

and

\[(X \cdot_{(P_2, Q_2)} X^{-1}) = \{\alpha\xi\beta\xi^{-1}\alpha^{-1}\xi\beta^{-1}\xi^{-1}\}.\]

Therefore

\[[X, X^{-1}] = \{\beta\xi^{-1}\alpha\xi\beta^{-1}\xi^{-1}\alpha^{-1}\xi\} - \{\alpha\xi\beta\xi^{-1}\alpha^{-1}\xi\beta^{-1}\xi^{-1}\},\]

which does not vanish since the curve $\alpha\xi\beta^{-1}$ is a factor of the first term but not of the second.

### 3.3 Linked pairs and differentiable curves

Our next goal is to show the correspondence between linked pairs and stable intersections, which will lead to the relationship between the bracket in Definition 3.9 and the Goldman–Turaev bracket given by Eq. (1). This will in turn allow us to prove Theorem 1.1, i.e. that Definition 3.9 gives a Lie algebra.

Let $C$ be a compact one dimensional complex on an oriented Riemannian surface $M$ whose edges are geodesic arcs, and take a basepoint $x \in C$. Let $\tilde{C}$ be the universal covering of $C$. Then the following lemma is straightforward.

**Lemma 3.10** $L_x(C) \cong \pi_1(C, x)$ and $E_x(C) \cong \tilde{C}$.

Moreover, these correspondences give an isomorphism of fiber bundles

\[(E_x(C), L_x(C), C, \pi) \cong (\tilde{C}, \pi_1(C, x), C, \pi)\]

where $\pi_1(C, x)$ acts on $\tilde{C}$ by deck transformations.

Let $S(C)$ be the set of strings contained in $C$, and note that the string bracket of Definition 3.9 can be restricted to $S(C)$, the free abelian group on $S(C)$. We will denote this bracket by $[\cdot, \cdot]_C$. 

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For any set $V \subset M$ let $V_{\varepsilon}$ denote an $\varepsilon$-neighborhood of $V$. The following lemma is well known, see [15] for a very general construction.

**Lemma 3.11** If $\varepsilon$ is small enough, there is a retraction $\chi : C_{\varepsilon} \to C$. It induces an isomorphism $\chi_\ast : \pi_1(C_{\varepsilon}, x) \to \pi_1(C, x)$ for any $x \in C$.

Let $S_1(M)$ be the free abelian group generated by the set of conjugacy classes of $\pi_1(M)$, where the Goldman–Turaev bracket is defined. Applying Lemma 3.10 we can see that $\chi$ induces an isomorphism of abelian groups $\chi_\ast : S_1(C_{\varepsilon}) \to S(C)$. We shall prove that $\chi_\ast$ sends the Goldman–Turaev bracket of the surface $C_{\varepsilon}$ to $[\cdot ]_C$.

Proposition 3.11 in [3] can be rephrased as

**Lemma 3.12** Let $[\alpha]$ and $[\beta]$ be representatives of strings in $S(C)$. Then there are differentiable curves $\gamma$ and $\delta$ in $C_{\varepsilon}$ such that

- $\gamma$ and $\delta$ are $\varepsilon$-perturbations of $\alpha$ and $\beta$ respectively,
- $\alpha$ and $\beta$ are the reduced forms of $\chi \circ \gamma$ and $\chi \circ \delta$ respectively, and
- $\gamma$ and $\delta$ intersect transversally, in at most double points, and determine no bigons.

We remark that in Lemma 3.12 the curves $\chi \circ \gamma$ and $\chi \circ \delta$ need not be reduced, but the curves removed in their reduction have length less than $\varepsilon$, and each one is contained in some geodesic edge of the complex $C$, assuming $\varepsilon$ is small enough. Now we can relate the linked pairs of two strings in $S(C)$ with the intersections that are stable under perturbation in $C_{\varepsilon}$.

**Lemma 3.13** Let $[\alpha]$ and $[\beta]$ be representatives of strings $X$ and $Y$ in $S(C)$, and let $\gamma$ and $\delta$ be curves given by Lemma 3.12.

Then for each intersection point $p$ of $\gamma$ and $\delta$ there are $n, m \geq 1$, curves $a \subset \gamma^n$ and $b \subset \delta^m$ meeting at $p$, and a linked pair $(P, Q)$ between $a^n$ and $b^m$ that satisfy the following:

If $P^a = (\alpha_1, \xi)$ and $Q^a = (\beta_1, \eta)$ are abbreviated cyclic factorizations, then the reduced forms of $\chi \circ a$ and $\chi \circ b$ can be written as $a_1\xi a_2$ and $b_1\eta b_2$ respectively, where $a_1, a_2, b_1, b_2$ are geodesic segments.

Moreover, this correspondence is a bijection between the intersection points of $\gamma$ and $\delta$, and $LP_2(X, Y)$.

**Proof** This is done with the techniques of [3]. Let $\pi : \tilde{C}_{\varepsilon} \to C_{\varepsilon}$ be the universal cover of $C_{\varepsilon}$. Then $\tilde{C}$ is embedded in $\tilde{C}_{\varepsilon}$ as a tree, made of piecewise geodesic curves, since $C$ is piecewise geodesic. Consider also $\tilde{\chi} : \tilde{C}_{\varepsilon} \to \tilde{C}$ the lift of $\chi$, and for each geodesic arc $c$ in the decomposition of $\tilde{C}$ let $V(c) = \tilde{\chi}^{-1}(c)$. Then the sets $V(c)$ are homeomorphic to closed disks, they cover $\tilde{C}_{\varepsilon}$, and their interiors are disjoint.

Now, for each $p$ in the intersection between $\gamma$ and $\delta$, we pick $\hat{p} \in \tilde{C}_{\varepsilon}$ projecting to $p$, and consider $\hat{\gamma}$ and $\hat{\delta}$ the infinite lifts of $\gamma$ and $\delta$ that meet only at $\hat{p}$ (recalling that $\gamma$ and $\delta$ have no bigons). Let $\hat{\alpha}$ and $\hat{\beta}$ be the respective reductions of $\hat{\chi} \circ \hat{\gamma}$ and $\hat{\chi} \circ \hat{\delta}$, which are infinite lifts of $\alpha$ and $\beta$. The set $\hat{\alpha}(I) \cap \hat{\beta}(I)$ is compact, since $\hat{\gamma}$ and $\hat{\delta}$ meet only once and are lifts of closed curves, and it is an arc, since $\hat{\alpha}$ and $\hat{\beta}$ are reduced. We shall write $\hat{\alpha}(I) \cap \hat{\beta}(I)$ as a curve in two ways, with possibly different orientations: we call $\hat{\xi}$ and $\hat{\eta}$ to the curves spanning $\hat{\alpha}(I) \cap \hat{\beta}(I)$ with the orientations given by $\hat{\alpha}$ and $\hat{\beta}$ respectively. It may be the case that $\hat{\alpha}(I) \cap \hat{\beta}(I)$ is just a point, then $\hat{\xi}$ and $\hat{\eta}$ will be constant (this will result in a linked pair of type (1)).

Let

$$V = \bigcup \{ V(c) : c \cap \hat{\alpha}(I) \cap \hat{\beta}(I) \neq \emptyset \}$$
Fig. 2 Proof of lemma 3.13. We depict the simple case when \( \hat{a} \cap \hat{b} \) does not contain branching vertices of \( \hat{C} \) in its interior. For this figure, \( a_1, \ldots, a_k \) denote the geodesic segments in \( \hat{a} \cap \hat{b} \)

Then \( V \) is, topologically, a closed disk, and we have \( \hat{a} \cap V = c_1 \hat{\xi} c_2 \) and \( \hat{b} \cap V = d_1 \hat{\eta} d_2 \), where \( c_1, c_2, d_1, d_2 \) are pairwise different geodesic segments.

For each geodesic segment \( c \) that meets \( \hat{a} \cap \hat{b} \) but is not contained in it, we define the set \( B(c) = \partial V(c) \cap \partial (\hat{C}_e - V) \). Note that \( B(c) \) is a segment in \( \partial V \cong S^1 \), and that \( c \) has an endpoint in \( B(c) \) and the other in \( \hat{a} \cap \hat{b} \). The segments \( B(c) \) just defined are pairwise disjoint, and their union is the relative boundary of \( V \) in \( \hat{C}_e \). Taking \( \epsilon \) small enough, we may assume that the \( \epsilon \)-neighborhood of \( \hat{a} \) only meets the relative boundary of \( V \) at the arcs \( B(c_1) \) and \( B(c_2) \). Thus \( \hat{\gamma} \cap V \) is an arc that enters \( V \) through \( B(c_1) \) and exits through \( B(c_2) \), meeting no other segment of the relative boundary of \( V \). Similarly we get that \( \hat{\delta} \cap V \) is an arc that traverses \( V \) from \( B(d_1) \) to \( B(d_2) \).

Let \( \hat{a} = \hat{\gamma} \cap V \) and \( \hat{b} = \hat{\delta} \cap V \). They must intersect at \( \hat{p} \), in particular \( \hat{p} \in V \), since the complementary arcs of \( \hat{\gamma} \) and \( \hat{\delta} \) are in different components of \( \hat{C}_e - V \). Note that \( \hat{\chi} \circ \hat{a} \) and \( \hat{\chi} \circ \hat{b} \) can be reduced, respectively, to \( \hat{a} \cap V = c_1 \hat{\xi} c_2 \) and \( \hat{b} \cap V = d_1 \hat{\eta} d_2 \). We define the curves \( a, b, \xi, \eta \) in the statement as the respective projections under \( \pi \) of \( \hat{a}, \hat{b}, \hat{\xi}, \hat{\eta} \).

By compactness, there are \( n, m \geq 1 \) such that \( \hat{a} \) and \( \hat{b} \) are contained in lifts of \( \gamma^n \) and \( \delta^m \) inside \( \hat{\gamma} \) and \( \hat{\delta} \) respectively. We choose \( n, m \) minimal for these inclusions to be strict. Thus \( \xi \) and \( \eta \) induce cyclic factorizations of \( a^n \) and \( b^m \) respectively, namely \( P \) and \( Q \). It only remains to show that

\[
(a_1, \xi, a_2) \quad \text{and} \quad (b_1, \eta, b_2)
\]
is a linked pair, where \( a_1, a_2, b_1, b_2 \) are the respective projections of the curves \( c_1, c_2, d_1, d_2 \). This is because \( \hat{\gamma} \) and \( \hat{\delta} \) meet transversally, and only once in \( V \), thus \( B(c_1) \cup B(c_2) \) separates \( B(d_1) \) from \( B(d_2) \) in \( \partial V \cong S^1 \). Using the orientation of \( V \) induced by lifting the one of \( C_e \subset M \), the last fact allows us to verify the sign conditions in the definition of linked pair. It also yields that \( sign(P, Q) = \epsilon(p, \gamma, \delta) \), which will be useful later.

Note that, by the minimality of \( n \) and \( m \), we have \( (P, Q) \in LP_2(X, Y) \). The reciprocal construction is now straightforward, and so is checking bijectivity.

\[\square\]

**Lemma 3.14** \((S(C), [\cdot, \cdot]_C)\) is a Lie algebra.

**Proof** Let \( X, Y \in S(C) \) and let \( \alpha, \beta, \gamma, \delta \) be as in Lemma 3.12. We need to show that \( \chi_\pi(L_\gamma, \delta) = [X, Y]_C \), where \( [\gamma, \delta]_{I_1(C_e)} \) stands for the Goldman–Turaev bracket between the free homotopy classes of \( \gamma \) and \( \delta \). This is a consequence of Lemma 3.13: If \( p \) corresponds to the linked pair \( (P, Q) \), then we have seen that \( sign(P, Q) = \epsilon(p, \gamma, \delta) \) in the proof of Lemma 3.13. On the other hand, \((X \cdot(P, Q))\) is the conjugacy class of the image under \( \chi \) of \( \gamma \cdot p \), which follows from Definition 3.8 and the properties of the correspondence between \( p \) and \( (P, Q) \) given by Lemma 3.13.

\[\square\]
We remark that we have obtained the isomorphism of Lie algebras
\[ \chi_* : (\mathcal{S}_1(C_\varepsilon), [, ]_{\pi_1(C_\varepsilon)}) \to (\mathcal{S}(C), [, ]_C) \]
that we desired.

### 3.4 Proof of Theorem 1.1

We shall check that the bracket on \( S(M) \) given by Definition 3.9 satisfies the axioms of a Lie algebra. It is bilinear by definition, and we would like to remark that anti-symmetry can be checked directly, showing that the bijection between \( LP^2(X, Y) \) and \( LP^2(Y, X) \) that sends \((P, Q)\) to \((Q, P)\) verifies that
\[ (X \cdot (P, Q) Y) = (Y \cdot (Q, P) X) \text{ and } \text{sign}(P, Q) = -\text{sign}(Q, P) \]
It can also be proved by the same method we use for the Jacobi identity, which we shall check next.

Consider strings \( X, Y, Z \in S(M) \), and cyclically reduced representatives \( \alpha, \beta, \gamma \) of them. Applying Proposition 3.6 to \( \mathcal{U} = \{\alpha, \beta, \gamma\} \) we see that the set \( C = \alpha(I) \cup \beta(I) \cup \gamma(I) \) is a one dimensional complex with geodesic edges. Recall that the string bracket \([, ]\) restricts to \( S(C) \subset S(M) \), where it defines a Lie algebra by Lemma 3.14. By construction we have that \( X, Y, Z \) are in \( S(C) \), thus the Jacobi identity between \( X, Y, Z \) is obtained.

That shows Theorem 1.1. We would also like to point out that there is a natural quotient \( S(M) \to \mathcal{S}_1(M) \), since free homotopy of closed curves is a coarser equivalence than the one defining \( S(X) \), and we can show that this map is a homomorphism
\[ (S(M), [, ]) \to (\mathcal{S}_1(M), [, ]_{\pi_1(M)}) \]
To check this we can consider \( X, Y \in S(M) \), take representatives \( \alpha, \beta \) and let \( \gamma, \delta \) be the curves given by Lemma 3.12 for \( C = \alpha(I) \cup \beta(I) \). Then \( \gamma \) and \( \delta \) are freely homotopic to \( \alpha \) and \( \beta \), and the same argument for Lemma 3.14 shows that \([X, Y]\) maps to \([\gamma, \delta]_{\pi_1(M)} \) under the natural quotient.

As we commented in the introduction, it is also possible to define a co-bracket in a similar fashion as we did for the bracket in Definition 3.9, this time involving \( LP^1 \). That gives a Lie bi-algebra structure on \( S(M) \), and the axioms can also be verified using one dimensional complexes and results of [3].

### 4 Infinite lifts and intersections

With the goal of proving Theorem 1.2 in mind, we will study the intersections of a cyclically reduced curve with its inverse by looking at the horizontal lifts in the loop bundle.

Throughout this section we fix a cyclically reduced, non-trivial closed curve \( \alpha \), and write \( x = \alpha(0) \). Let \( \tilde{\alpha} \) be the horizontal lift of \( \alpha \) to \( E_x(M) \) such that \( \tilde{\alpha}(0) = id_x \). We consider the set
\[ \Lambda_\alpha = \bigcup_{n \in \mathbb{Z}} [\alpha]^n \tilde{\alpha}(I) \]
which is nothing but the infinite lift of \( \alpha \) through \( id_x \). Since \( \alpha \) is cyclically reduced, \( \Lambda_\alpha \) is a line in \( E_x(M) \), i.e. is an embedding of \( \mathbb{R} \) (it has no “spikes”). We give it a standard orientation
induced by the orientation of \( \tilde{\alpha} \). Note that \( \Lambda_{\alpha}^{-1} \) agrees with \( \Lambda_{\alpha} \) as a set, but has the opposite orientation.

### 4.1 Intersections as elements of \( \mathcal{L}_x \).

For every \( A \subset \mathcal{E}_x(M) \) define

\[
T(A) = \{ g \in \mathcal{L}_x(M) \text{ such that } gA \cap A \neq \emptyset \}
\]

Consider \( g \in T(\Lambda_{\alpha}) \) so that \( g\Lambda_{\alpha} \neq \Lambda_{\alpha} \). We show that \( g\Lambda_{\alpha} \cap \Lambda_{\alpha} \) must be a compact arc (or a point): Note that \( \mathcal{E}_x(M) \) contains no non-trivial horizontal loops, and if \( g\Lambda_{\alpha} \cap \Lambda_{\alpha} \) contains a ray, then we note that \( g[\alpha]^n \tilde{\alpha}(I) \subset \Lambda_{\alpha} \) for some \( n \). Therefore \( g[\alpha]^n \tilde{\alpha}(0) = g[\alpha]^n \) belongs to some translated of \( \tilde{\alpha}(I) \), i.e. there is \( m \in \mathbb{Z} \) such that \( g[\alpha]^n = [\alpha]^m \tilde{\alpha}(I) \). Let \( \phi \) be a curve such that \( g[\alpha]^n = [\alpha]^m \) where \( \alpha = \phi_\eta \). Let \( \tilde{\eta} \) be the lift of \( \eta \) such that \( \tilde{\eta}(0) = g[\alpha]^n = [\alpha]^m [\phi] \) and let \( \tilde{\phi} \) be the lift of \( \phi \) such that \( \tilde{\phi}(0) = \tilde{\eta}(1) \). We consider \( \tilde{\alpha} \) the lift of \( \alpha \) starting at \( g[\alpha]^n = [\alpha]^m [\phi] \). We have two cases: either \( \tilde{\alpha} = \tilde{\eta} \phi \) or \( \tilde{\alpha} = \phi_1^{-1} \tilde{\eta}_1^{-1} \), where \( \phi_1 \) and \( \tilde{\eta}_1 \) are lifts of \( \phi \) and \( \eta \) such that \( \tilde{\eta}_1(1) = \tilde{\phi}_1(0) \). In the second case we have \( \alpha = \phi^{-1} \eta^{-1} \) which is absurd. In the first case we have \( \alpha = \eta \phi \) and hence there is a curve \( \omega \) such that \( \phi = \omega \phi \) and \( \eta = \omega \eta \). Then there is \( r \) such that \( g = [\omega]^r \). On the other hand note that \( \Lambda_{\alpha} \) is the union of the sets \( [\omega]^n \tilde{\omega}(I) \) where \( \tilde{\omega}(0) = id_x \). Consequently we obtain \( g[\omega]^n \tilde{\omega}(I) = [\omega]^n \tilde{\omega}(I) \) and this implies \( g\Lambda_{\alpha} = \Lambda_{\alpha} \).

Then to each \( g \in T(\Lambda_{\alpha}) \) with \( g\Lambda_{\alpha} \neq \Lambda_{\alpha} \) we can associate a horizontal curve \( b_g \subset \Lambda_{\alpha} \), with the same orientation as \( \Lambda_{\alpha} \), such that \( b_g(I) = g\Lambda_{\alpha} \cap \Lambda_{\alpha} \). Let \( a_g \subset \Lambda_{\alpha} \) be the horizontal curve such that \( ga_g(I) = b_g(I) \) and \( a_g \) has the orientation carried from \( b_g \) by the action of \( g \) (thus we may write \( ga_g = b_g \)). Note that the pair \( (a_g, b_g) \) determines \( g \), since \( \mathcal{L}_x(M) \) acts freely on \( \mathcal{E}_x(M) \).

**Definition 4.1** Let \( g \in T(\Lambda_{\alpha}) \).

- We say that \( g \) preserves orientation if either \( g\Lambda_{\alpha} = \Lambda_{\alpha} \) or the orientation of \( a_g \) agrees with that of \( \Lambda_{\alpha} \). Let \( T^+(\Lambda_{\alpha}) \) be the set of orientation preserving elements of \( T(\Lambda_{\alpha}) \).
- We say that \( g \) reverses orientation if the orientation of \( a_g \) is opposite to that of \( \Lambda_{\alpha} \). We denote by \( T^-(\Lambda_{\alpha}) \) the set of orientation reversing elements of \( T(\Lambda_{\alpha}) \).

The case when \( a_g \) is constant shall be regarded as both orientation preserving and reversing. When we want to exclude this case we say that \( g \) strictly preserves or reverses orientation.

**Remark 4.2** The sets \( T(\Lambda_{\alpha}) \), \( T^+(\Lambda_{\alpha}) \) and \( T^-(\Lambda_{\alpha}) \) are closed under taking inverses.

**Remark 4.3** If \( g \in T(\Lambda_{\alpha}) \) and \( g\Lambda_{\alpha} \neq \Lambda_{\alpha} \), then \( l(a_g) = l(b_g) < l(\alpha) \).

**Proof** Assume the contrary, i.e. that \( l(b_g) \geq l(\alpha) \). With a similar argument we applied to show that \( g\Lambda_{\alpha} \cap \Lambda_{\alpha} \) is a compact interval we conclude that if \( g \in T^+(\Lambda_{\alpha}) \) we get a contradiction by showing that \( g\Lambda_{\alpha} = \Lambda_{\alpha} \). On the other hand, if \( g \in T^-(\Lambda_{\alpha}) \) we can deduce that \( \alpha^{-1} \) is a permutation of \( \alpha \), which would mean that \( \alpha \) is trivial. \( \Box \)
Next we shall see that each \( g \in T(\Lambda_\alpha) \) with \( g \Lambda_\alpha \neq \Lambda_\alpha \) defines naturally an intersection pair between \( \alpha \) and \( \alpha^{-1} \). Figure 4 gives an idea of this situation, representing the intersection of the lifts in one of the possible cases. Let \( t_g = \pi \circ a_g = \pi \circ b_g \) and \( \epsilon = \pm 1 \) according to whether \( g \) preserves or reverses orientation (in case \( a_g \) is constant the pick makes no difference). Recalling that \( b_g(I) = g \Lambda_\alpha \cap \Lambda_\alpha \), we can find geodesic curves \( r, s, u, v \) such that \( (r, t_g, s), (u, t_g^{-\epsilon}, v) \) is an intersection pair with \( rt_g s \subset \alpha \) and \( ut_g v \subset \alpha^{-1} \), where the inclusions are modulo permutation (we do not have to consider powers of \( \alpha \) or \( \alpha^{-1} \) because of Remark 4.3).

Then we can give abbreviated cyclic factorizations \( P_g^\alpha = (\alpha_1, t_g) \) of \( \alpha \) and \( Q_g^\alpha = (\beta_1, t_g^{-\epsilon}) \) of \( \alpha^{-1} \) so that:

- the horizontal lift of \( t_g \alpha_1 \) at \( b_g(0) \) is contained in \( \Lambda_\alpha \), and
- the horizontal lift of \( t_g^{-\epsilon} \beta_1 \) at \( a_g^{-\epsilon}(0) \) is contained in \( \Lambda_\alpha \).

In other words, \( \alpha_1 \) can be obtained by projecting a curve spanning a component of

\[
\Lambda_\alpha - \bigcup_{n \in \mathbb{Z}} [\alpha]^n b_g(I)
\]

with the orientation given by \( \Lambda_\alpha \), while \( \beta_1 \) is the analog for the translates of \( a_g \) and the reverse orientation to that of \( \Lambda_\alpha \).

Notice that if \( g \) is orientation preserving, i.e. when \( \epsilon = 1 \), we get an intersection pair of type either (3) or (1) between \( \alpha \) and \( \alpha^{-1} \). In the orientation reversing case, when \( \epsilon = -1 \), we get an intersection pair of type either (2) or (1). In Fig. 4 we depict a situation where \( g \) is orientation preserving.

We are going to show that the map \( g \to (P_g^\alpha, Q_g^\alpha) \) factorizes through a map \( g \to (P_g, Q_g) \) such that \( (P_g, Q_g) \) is a linked pair and \( P_g^\alpha \) and \( Q_g^\alpha \) are abbreviated cyclic factorizations associated to \( P_g \) and \( Q_g \) respectively. First assume that there is \( n \in \mathbb{Z} \) such that \( [\alpha]^n \in b_g(I) \). Let \( \xi_1 \) be the horizontal lift of \( \xi_1 \) such that \( \xi_1(0) = [\alpha]^n \) and \( \xi_1(1) = b_g(1) \). Let \( \tilde{\xi}_2 \) be the horizontal lift of \( \xi_2 \) such that \( \tilde{\xi}_2(1) = [\alpha]^n \) and \( \tilde{\xi}_2(0) = b_g(0) \). In this case \( t_g = \xi_2 \xi_1 \). Define \( P_g \) as the class of \( (\xi_1, \alpha_1, \xi_2) \). Note that it is a cyclic decomposition such that \( P_g^\alpha \) is associated to \( P_g \). Now assume that \( b_g(I) \subset [\alpha]^n \tilde{\alpha}(I) \). In this case let \( \tilde{\xi}_1 \) be the horizontal
lift of $\xi_1$ such that $\hat{\xi}_1(0) = [\alpha]^n$ and $\hat{\xi}_1(1) = b_g(0)$. Let $\hat{\xi}_2$ be the horizontal lift of $\xi_2$ such that $\hat{\xi}_2(1) = [\alpha]^{n+1}$ and $\hat{\xi}_2(0) = b_g(1)$. Define $P_g$ as the class of $(\xi_1, t_g, \xi_2)$ and note that $\alpha_1 = \xi_2\xi_1$. Thus in this case it is also true that $P_g$ is a cyclic decomposition such that $P_a$ is associated to $P_g$. The cyclic decomposition $Q_g$ is defined in an analogous way.

Observe that an element of the form $h = [\alpha]^m g[\alpha]^{-m}$ induces the same intersection pair as $g$, since $b_h = [\alpha]^n b_g$ and $a_h = [\alpha]^{-m} a_g$ induce the same cyclic factorizations of $\alpha$ and $\alpha^{-1}$.

With this in mind we define

$$ T_1(\alpha) = \{ g \in T(\tilde{\alpha}(I)) : g \Lambda_{\alpha} \neq \Lambda_{\alpha} \text{ and } \tilde{\alpha}(1) \notin a_g(I) \cup b_g(I) \} $$

noting that the conditions amount to ask that $a_g$ and $b_g$ meet $\tilde{\alpha}(I)$ but not $\tilde{\alpha}(1)$. Since $\tilde{\alpha}([0, 1])$ is a fundamental domain for $\Lambda_{\alpha}$ under translations by powers of $[\alpha]$, we get the following:

**Remark 4.4** Let $g \in T(\Lambda_{\alpha})$ such that $g \Lambda_{\alpha} \neq \Lambda_{\alpha}$. Then there is a unique $g_1 \in T_1(\alpha)$ and integers $m, n$ such that

$$ g = [\alpha]^n g_1 [\alpha]^m. $$

The next result describes the intersections of a string with its inverse in terms of elements of the loop group. Let $T(\alpha, \alpha^{-1})$ denote the set of intersection pairs between $\alpha$ and $\alpha^{-1}$.

**Lemma 4.5** The map

$$ T_1(\alpha) \to T(\alpha, \alpha^{-1}) $$

that takes $g \to (P_g, Q_g)$ is a bijection.

**Proof** For injectivity, consider the way in which an horizontal curve $\nu$ contained in $\Lambda_{\alpha}$ with $l(\nu) < l(\alpha)$ defines a cyclic factorization of $\alpha$ or $\alpha^{-1}$, as was used in the construction of the map $g \to (P_g, Q_g)$. Then observe that two such curves $v_1$ and $v_2$ yield the same cyclic factorization iff $v_1 = [\alpha]^n v_2$ for some $n \in \mathbb{Z}$. Combining this fact with Remark 4.4 gives injectivity.

To show surjectivity consider $(P, Q) \in T(\alpha, \alpha^{-1})$ and let $P^a = (\alpha_1, \xi), Q^a = (\beta_1, \eta)$ be abbreviated cyclic factorizations associated to $P$ and $Q$ respectively as in Definition 3.2.

Then $P$ is the class of $(\omega_1, \nu, \omega_2)$ so that either $\nu = \xi$ or $\nu = \alpha_1$. Let $\tilde{\nu}$ be the lift of $\nu$ starting at $[\omega_1]$ and let $\tilde{\omega}_1$ be the lift of $\omega_2$ starting at $[\omega_1]$. These lifts determine lifts $\hat{\xi}$ and $\hat{\alpha}_1$, of $\xi$ and $\alpha_1$ respectively, that are contained in $\Lambda_{\alpha}$. In the same way $Q$ gives rise to lifts $\hat{\eta}$ and $\hat{\beta}_1$ also contained in $\Lambda_{\alpha}$. Let $\epsilon = 1$ such that $\eta = \xi^{-\epsilon}, \nu = \alpha_1$, and either one for type (2), and either one for type (1). Let $g$ be such that $g \eta^{-\epsilon} = \tilde{\xi}(I)$. It is straightforward to check that $(P_g, Q_g) = (P, Q)$.

Through the proof of Lemma 4.5 we see that $T_1(\alpha)$ is a choice of a restriction of domain, in order to obtain a bijection from the construction that associates $g \to (P_g, Q_g)$. This choice satisfies the following nice property:

**Remark 4.6** $g \in T_1(\alpha)$ iff $g^{-1} \in T_1(\alpha)$. Moreover,

$$ b_g^{-1} = a_g^\epsilon \text{ and } a_g^{-1} = b_g^\epsilon $$

where $\epsilon = \pm 1$ according to whether $g$ is orientation preserving or reversing.

In later sections we shall focus on the linked pairs, i.e. the intersection pairs that are relevant for the bracket.
**Definition 4.7** We define $T_0(\alpha) \subseteq T_1(\alpha)$ as the set of elements that correspond to linked pairs under the bijection of Lemma 4.5.

Note that by Remark 4.3 and Lemma 4.5, the set $T_0(\alpha)$ is in bijection with $LP_2(X, X^{-1})$, where $X$ is the conjugacy class of $[\alpha]$. From these same results we also get that $LP_1(X) = LP_2(X, X)$, which is in natural bijection with $LP_2(X, X^{-1})$. Observe also that by Lemma 3.13 a string $X$ is simple, as defined in the introduction, iff $LP_1(X) = \emptyset$, or equivalently, iff $LP_2(X, X^{-1}) = \emptyset$.

### 4.2 Orientation reversing elements and unique intersections.

The orientation properties of the elements of $T_0(\alpha)$, which correspond to the type of their associated linked pairs, will play a major role in proving Theorem 1.2. Next we study the key properties of the orientation reversing case.

**Lemma 4.8** Let $\xi$ and $\eta$ be non-constant positively oriented segments contained in $\Lambda_\alpha$. Then if $\pi \circ \xi = (\pi \circ \eta)^{-1}$ we have $\xi(I) \cap \eta(I) = \emptyset$ and $l(\alpha) > 2l(\xi)$.

**Proof** Write $\gamma = \pi \circ \xi$. Then an overlap between $\xi$ and $\eta$ would project to an overlap between $\gamma$ and $\gamma^{-1}$, and if $\xi$ and $\eta$ meet at an endpoint, that would project to a reduction of $\alpha$, of the form $\gamma \gamma^{-1}$ or $\gamma^{-1} \gamma$. Thus we get the first claim. The second one comes from considering a permutation $\alpha_0$ of $\alpha$ so that its horizontal lift starting at $\xi(0)$ is contained in $\Lambda_\alpha$. Note that such lift ends at $[\alpha] \xi(0)$, which is not in $\eta(I)$ by the first claim applied to $\eta$ and $[\alpha] \xi$. Thus we obtain $\alpha_0 = \gamma \alpha \gamma^{-1} b$ with $a$ and $b$ non-constant, so

$$l(\alpha) = l(\alpha_0) > 2l(\gamma) = 2l(\xi).$$

\[ \square \]

**Lemma 4.9** Let $g \in T^-(\Lambda_\alpha)$. Then $[\alpha^n]a_g(I) \cap [\alpha^m]b_g(I) = \emptyset$ for all $n, m \in \mathbb{Z}$.

**Proof** If $g$ reverses orientation strictly, we apply lemma 4.8 to the curves $[\alpha^n]a_g^{-1}$ and $[\alpha^m]b_g$. In case $a_g$ is constant, say $a_g = e_p$ for $p \in \Lambda_\alpha$, we get that $[\alpha^n]g[\alpha^{-m}]p = p$ which is absurd because the action of $L_x(M)$ is free and $g$, being orientation reversing, is not a power of $\alpha$.

\[ \square \]

**Remark 4.10** Note that for $g \in T^-(\Lambda_\alpha)$ we have $l(t_g) < l(\alpha)/2$.

We say that $\alpha$ has **unique intersection** if $T_1(\alpha)$ consists of only two elements, $g$ and $g^{-1}$ (by Remark 4.6). By Lemma 4.5 this is equivalent to say that $\mathcal{I}(\alpha, \alpha^{-1})$ has two elements. Observe that the definition of intersection pair makes sense for a curve in a general one dimensional complex, i.e. not necessarily embedded in a surface. Thus we may speak of unique intersection for curves in this more general setting.

Given a subgroup $G \subseteq L_x(M)$ we can consider $\Lambda_\alpha/G$, the image of $\Lambda_\alpha$ in the quotient $E_x(M)/G$, which is a one dimensional complex since $\Lambda_\alpha/L_x(M) = \alpha(I)$ and $\alpha$ is piecewise geodesic. For $g \in L_x(M)$, let $G_g$ be the subgroup generated by $g$ and $[\alpha]$.

**Lemma 4.11** Let $g \in T^-(\Lambda_\alpha)$ and $\bar{\alpha}$ be the projection of $\tilde{\alpha}$ onto $\Lambda_\alpha/G_g$. Then $\bar{\alpha}$ has a **unique intersection**.
Lemma 4.14 Suppose \( g, h_1, \ldots, h_n \in L_\alpha(M) \) satisfy that:

- \( g^{-1}h_1, g^{-1}h_n \in T(\Lambda_\alpha) \).
- \( g^{-1}h_i \notin T(\Lambda_\alpha) \) for \( i = 2, \ldots, n - 1 \).
- \( h_{i-1}^{-1}h_i \in T(\Lambda_\alpha) \) for \( i = 2, \ldots, n \).

Then \( h_1^{-1}h_n \in T(\Lambda_\alpha) \).

Proof Interpreting the hypotheses in terms of intersections of translates of \( \Lambda_\alpha \), we can find a curve \( \beta \subset h_1\Lambda_\alpha \cup \cdots \cup h_n\Lambda_\alpha \) that only meets \( g\Lambda_\alpha \) at its endpoints, with \( \beta(0) \in h_1\Lambda_\alpha \cap g\Lambda_\alpha \) and \( \beta(1) \in h_n\Lambda_\alpha \cap g\Lambda_\alpha \). Since \( g\Lambda_\alpha \cup h_1\Lambda_\alpha \cup \cdots \cup h_n\Lambda_\alpha \) has no non-trivial loops, we must have \( \beta(0) = \beta(1) \) (and \( \beta \) must be a trivial loop), which provides a point in \( h_1\Lambda_\alpha \cap h_n\Lambda_\alpha \) as desired. \( \square \)

The following is a straightforward observation.
Lemma 4.15 If \( g, h \in T^+(\Lambda_\alpha) \) and \( g^{-1}h \in T(\Lambda_\alpha) \), then \( g^{-1}h \in T^+(\Lambda_\alpha) \).

Next we present the main result of this subsection, concerning the subgroups generated by orientation preserving elements.

Lemma 4.16 Let \( G \subset L_1(M) \) be a finitely generated subgroup whose generators belong to \( T^+(\Lambda_\alpha) \). Then \( G \) is \( \alpha \)-oriented.

Proof By hypothesis we can write

\[ G = \bigcup_i G_i, \quad \text{where } G_0 = \{id\}, \text{ and } G_{i+1} = G_i \cup \{h_{i+1}\}, \]

such that there is \( h'_i \in G_i \) with \( h_{i+1}^{-1}h'_i \in T^+(\Lambda_\alpha) \) for every \( i \geq 0 \). We are going to show the lemma by induction on \( i \): assuming that \( g^{-1}h \in T(\Lambda_\alpha) \) implies \( g^{-1}h \in T^+(\Lambda_\alpha) \) for \( g, h \in G_i \), we shall show that this same property holds for \( g, h \in G_{i+1} \). The base case of this induction is trivial.

By Remark 4.2, we only need to consider the case when \( g = h_{i+1} \) and \( h \in G_i \). So let \( h \in G_i \) be such that \( h_{i+1}^{-1}h \in T(\Lambda_\alpha) \), and we are going to show that \( h_{i+1}^{-1}h \in T^+(\Lambda_\alpha) \).

By construction of the set \( G_i \) there is a sequence \( h'_i = k_1, \ldots, k_n = h \) in \( G_i \) such that \( k_j^{-1}k_{j+1} \in T^+(\Lambda_\alpha) \) for \( j = 1, \ldots, n-1 \). Since \( k_j \in G_i \) for every \( j \), the induction hypothesis gives us that \( k_j^{-1}k_l \in T^+(\Lambda_\alpha) \) whenever \( k_j^{-1}k_l \in T(\Lambda_\alpha) \).

By Lemma 4.14 we can assume, maybe after taking a subsequence, that \( h_{i+1}^{-1}k_j \in T(\Lambda_\alpha) \) for every \( j \). Now write

\[ h_{i+1}^{-1}k_2 = (h_{i+1}^{-1}k_1)(k_1^{-1}k_2) \]

and note that \( h_{i+1}^{-1}k_1 \) and \( k_1^{-1}k_2 \) are orientation preserving by construction. Therefore \( h_{i+1}^{-1}k_2 \) is orientation preserving by Lemma 4.15. Proceeding inductively we conclude that \( h_{i+1}^{-1}k_n \) is also orientation preserving, as desired.

\[ \square \]

5 Formulas for the terms of the bracket

In this section we study the dot products between a string and its inverse applying what we developed in section 4. Again we fix a cyclically reduced, non-trivial closed curve \( \alpha \), and let \( x = \alpha(0) \) and \( X \in S(M) \) be the conjugacy class of \( \alpha \).

5.1 Expressions for the dot product

Let \( g \in T_1(\alpha) \) and recall the horizontal curves \((a_g, b_g)\) defined in subsection 4.1. We introduce the following curves:

- \( \bar{\alpha}_g \) is the segment of \( \Lambda_\alpha \) starting at \( b_g(1) \) and ending at \([\alpha]\)\( b_g(1) \). Let \( \bar{\alpha}_g = \pi \circ \bar{\alpha}_g \).
- \( \bar{\beta}_g \) is the segment of \( g\Lambda_\alpha \) starting at \( b_g(1) \) and ending at \( g[\alpha^{-1}]a_g(1) \). Let \( \beta_g = \pi \circ \bar{\beta}_g \).
- \( \bar{\gamma}_g \) is the segment of \( \Lambda_\alpha \) starting at \( id_\alpha \) and ending at \( b_g(1) \). Let \( \gamma_g = \pi \circ \bar{\gamma}_g \).

We observe \( \alpha_g \) and \( \beta_g \) are permutations of \( \alpha \) and \( \alpha^{-1} \) respectively, and that \( \bar{\alpha}_g \) and \( \bar{\beta}_g \) are their respective horizontal lifts starting at \( b_g(1) \). This is easy to see for \( \alpha_g \), and in the
case of $β_g$ note that $\tilde{β}_g = g\tilde{β}'$ where $\tilde{β}'$ is the segment of $Λ_α$ starting at $a_β(1)$ and ending at $[α^{-1}]a_β(1)$. According to Definition 3.8 (which also makes sense for intersection pairs), the conjugacy class of $α_ββ_g$ is the dot product $(X^−(P, Q) X^{-1})$ where $(P, Q)$ is the intersection pair corresponding to $g$.

On the other hand, $γ_g$ is the curve that gives the change of basepoint conjugation so that $[γ_gα_βγ_g^{-1}]$ and $[γ_gβ_γγ_g^{-1}]$ belong to $L_α(M)$. We also have that $\tilde{γ}_g$ is the horizontal lift of $γ$ at $iδx$. For an example of these curves, in figure 4 we have $α_g = bca, β_g = a^{-1}c^{-1}e^{-1}f^{-1}$ and $γ_g = a$.

**Remark 5.1** By Remark 4.6 we have

- If $g$ preserves orientation, then $α_{g^{-1}} = β_{g^{-1}}$ and $β_{g^{-1}} = α_{g^{-1}}$.

- If $g$ reverses orientation, $α_{g^{-1}}$ and $β_{g^{-1}}$ are the respective reductions of $tgβ_g^{-1}t_1g$ and $t_1gα_1g^{-1}t_1g$.

(Note that in the first case $t_{g^{-1}} = t_g$, while in the second case $t_{g^{-1}} = t_g^{-1}$).

This gives a relationship between the dot products between $X$ and $X^{-1}$ associated to $g$ and $g^{-1}$.

**Remark 5.2** Let $(P, Q) ∈ I(α, α^{-1})$ correspond to $g ∈ T_1(α)$, and denote by $(Q′, P′)$ the intersection pair corresponding to $g^{-1}$. Then we have

$$(X^−(Q′, P′) X^{-1}) = (X^−(P, Q) X^{-1})^{-1}$$

In particular, these dot products cannot be equal, as a non-trivial loop is not conjugate to its inverse. Next we define the curves that will help us write reduced forms for the dot products.

**Definition 5.3** Let $g ∈ T_1(α)$. We define the curves $c_1(α, g)$ and $c_2(α, g)$ according to whether $g$ preserves or reverses orientation:

- If $g$ preserves orientation,

  $c_1(α, g)$ is the reduced form of $α_1gt_1g^{-1}$ and

  $c_2(α, g)$ is the reduced form of $t_1gβ_g$

- If $g$ reverses orientation, let

  $c_1(α, g) = α_g$ and $c_2(α, g) = β_g$

As an example, in Fig. 4 we have $c_1(α, g) = b$ and $c_2(α, g) = e^{-1}f^{-1}$.

Let us interpret this definition in terms of the construction of the intersection pair $(P_g, Q_g)$ given in subsection 4.1. In the orientation preserving case we had $P_α = (α_1, t_g)$ and $Q_α = (β_1, t_1g^{-1})$, and recalling the construction we get

$α_g = α_1t_g$ and $β_g = t_1g^{-1}β_1$, thus $c_1(α, g) = α_1$ and $c_2(α, g) = β_1$

In the orientation reversing case we had $P_α = (α_1, t_g)$ and $Q_α = (β_1, t_g)$ and we get

$α_g = α_1t_g$ and $β_g = β_1t_g$, and so $c_1(α, g) = α_1t_g$ and $c_2(α, g) = β_1t_g$

In both cases we have that the concatenation $c_1(α, g)c_2(α, g)$ is the cyclically reduced form of $α_gβ_g$. Also note that:
Remark 5.4 If \( g \in T_1(\alpha) \) is orientation reversing, then \( c_1(\alpha, g) \) is a permutation of \( \alpha \) and \( c_2(\alpha, g) \) is a permutation of \( \alpha^{-1} \). In particular we have \( l(c_1(\alpha, g)) = l(c_2(\alpha, g)) = l(\alpha) \).

Remark 5.5 On the other hand, if \( g \in T_1(\alpha) \) is orientation preserving we have \( l(c_1(\alpha, g)) = l(c_2(\alpha, g)) = l(\alpha) - l(t_g) \).

So in any case the lengths of \( c_1(\alpha, g) \) and \( c_2(\alpha, g) \) agree. We can also deduce the length of the dot product (i.e. the length of a cyclically reduced form), as follows:

Remark 5.6 For \( g \in T_1(\alpha) \) we have:

- \( l(c_1(\alpha, g)c_2(\alpha, g)) = 2l(\alpha) - 2l(t_g) \) if \( g \) preserves orientation, and
- \( l(c_1(\alpha, g)c_2(\alpha, g)) = 2l(\alpha) \) if \( g \) reverses orientation.

Recalling the relationship between orientation and the type of intersection pairs from subsection 4.1, Remark 5.6 implies that if \( c_1(\alpha, g)c_2(\alpha, g) \) is a permutation of \( c_1(\alpha, h)c_2(\alpha, h) \) for \( g, h \in T_1(\alpha) \) then either:

- \( (P_g, Q_g) \) and \( (P_h, Q_h) \) are both of types (2) or (1) (i.e. \( g \) and \( h \) are orientation reversing),
- \( (P_g, Q_g) \) and \( (P_h, Q_h) \) are both of type (3) (i.e. \( g \) and \( h \) are strictly orientation preserving),

and \( l(t_g) = l(t_h) \).

This observation is an example of recovering information about the intersection pair from the corresponding dot product. In the following sections we will be proving stronger results within this same idea, which will ultimately lead us to Theorem 1.2 by showing there can be no cancellations in the formula for \([X, X^{-1}]\).

Next we record the behaviour of the curves from Definition 5.3 under taking inverses in \( T_1(\alpha) \), which we can compute from Remark 5.1.

Remark 5.7 Let \( g \in T_1(\alpha) \), then

- if \( g \) preserves orientation,
  \[
  c_1(\alpha, g^{-1}) = c_2(\alpha, g)^{-1} \quad \text{and} \quad c_2(\alpha, g^{-1}) = c_1(\alpha, g)^{-1}
  \]
- if \( g \) reverses orientation,
  \[
  c_1(\alpha, g^{-1}) \text{ is the reduced form of } t_gc_2(\alpha, g)^{-1}t_g^{-1} \quad \text{and} \quad c_2(\alpha, g^{-1}) \text{ is the reduced form of } t_gc_1(\alpha, g)^{-1}t_g^{-1}
  \]

The next result lets us write the dot products as commutators in \( L_x(M) \).

Lemma 5.8 Let \( g \in T_1(\alpha) \), and put \( c_i = c_i(\alpha, g) \) for \( i = 1, 2 \). Then we have

1. \( [\alpha]g[\alpha]^{-1}g^{-1} = [\gamma_g c_1 c_2 \gamma_g^{-1}] \), and
2. if \( (P, Q) \in I(\alpha, \alpha^{-1}) \) corresponds to \( g \) by the bijection of Lemma 4.5, then \( (X \cdot (P, Q)X^{-1}) \) is the conjugacy class of \( [\alpha]g[\alpha]^{-1}g^{-1} \).

Proof The second point follows from the first and the fact that the cyclically reduced form of \( [\gamma_g c_1 c_2 \gamma_g^{-1}] \), which is \( c_1 c_2 \), represents the dot product \( (X \cdot (P, Q)X^{-1}) \) as discussed previously in this subsection.

We show the first point in the statement for \( g \) corresponding to an intersection pair of type (3), as the other cases result from a straightforward adaptation of the same computations.
In fact, type (1) can be considered within types either (2) or (3) by allowing constant core curves.

Within the case of \((P, Q)\) being of type (3), we distinguish 3 subcases according to whether \(id_x\) belongs to both \(a_g\) and \(b_g\), to only one of them, or to neither of them. First we assume that \(id_x\) is in \(b_g\) but not in \(a_g\). This is the situation shown in Fig. 4. Then we can write \(\alpha = abc = ecacf\) with \(g = [c^{-1}e^{-1}]\), noting that it corresponds to an intersection pair of type (3). We have \(\gamma_g = a, c_1 = b\) and \(c_2 = e^{-1}f^{-1}\). Thus we get \([\gamma_g c_1 c_2 \gamma_g^{-1}] = abe^{-1}f^{-1}a^{-1}\).

On the other hand we compute

\[
g[\alpha]^{-1}g^{-1} = [(c^{-1}e^{-1})(f^{-1}a^{-1}c^{-1}e^{-1})(ec)]
\]

Thus

\[
[\alpha]g[\alpha]^{-1}g^{-1} = [(abc)(c^{-1}e^{-1})(f^{-1}a^{-1}c^{-1}e^{-1})(ec)] = [\gamma_g c_1 c_2 \gamma_g^{-1}]
\]

as desired. The situation is symmetrical for \(id_x\) in \(a_g\) but not in \(b_g\).

In the second subcase, when \(id_x\) is in both \(a_g\) and \(b_g\), we have that \(g \in b_g(I)\). If \(g\) lies before \(id_x\) with respect to the orientation of \(\Lambda_a\) we can write \(\alpha = abc' = c'afc\), where \(t_g = cc'a\) and \(g = [c'^{-1}]\). Now \(\gamma_g = a, c_1 = b\) and \(c_2 = f^{-1}\), and we compute

\[
g[\alpha]^{-1}g^{-1} = [c'^{-1}(c^{-1}f^{-1}a^{-1}c'^{-1})c']
\]

and

\[
[\alpha]g[\alpha]^{-1}g^{-1} = [(abc)c'^{-1}(c^{-1}f^{-1}a^{-1}c'^{-1})c'] = [abf^{-1}a^{-1}],
\]

that is \([\gamma_g c_1 c_2 \gamma_g^{-1}]\). When \(g\) lies after \(id_x\) in \(\Lambda_a\) we write \(\alpha = aa'bc = a'fca\), where \(t_g = caa'\) and \(g = [a']\), and the computation is similar.

Finally, if \(id_x\) is neither in \(a_g\) nor \(b_g\), we have \(\alpha = abc = ebf\) with \(t_g = b\) and \(g = [ae^{-1}]\).

We see that \(\gamma_g = ab, c_1 = ca\) and \(c_2 = e^{-1}f^{-1}\). On the other hand

\[
g[\alpha]^{-1}g^{-1} = [(ae^{-1})(f^{-1}b^{-1}e^{-1})(ea^{-1})]
\]

thus we get

\[
[\alpha]g[\alpha]^{-1}g^{-1} = [ab(cae^{-1}f^{-1})b^{-1}a^{-1}] = [\gamma_g c_1 c_2 \gamma_g^{-1}]
\]

\(\square\)

### 5.2 Conjugate dot products

Our strategy for Theorem 1.2 is to show that for a primitive string \(X\) there can be no cancellations in the formula for \([X, X^{-1}]\) given in Definition 3.9. This would imply that \([X, X^{-1}] = 0\) only when \(LP_2(X, X^{-1}) = \emptyset\), provided that \(X\) is primitive, thus proving Theorem 1.2. So we need to study what happens if two linked pairs between \(X\) and \(X^{-1}\) yield the same dot product. Here we shall focus on what we can achieve for general intersection pairs, leaving the discussion of linked pairs and their signs for the next section.

Assume that \(g, h \in T_1(\alpha)\) are such that \(c_1(\alpha, g)c_2(\alpha, g)\) is a permutation (maybe trivial) of the curve \(c_1(\alpha, h)c_2(\alpha, h)\), which is to say that their corresponding intersection pairs yield the same dot product. So we have

\[
[c_1(\alpha, h)c_2(\alpha, h)] = [rc_1(\alpha, g)c_2(\alpha, g)r^{-1}]
\]

where \(r \in \Omega\) is an initial segment of \(c_1(\alpha, h)c_2(\alpha, h)\).
By Remark 5.6 and the discussion preceding it, the curves $c_1(\alpha, h)$, $c_2(\alpha, h)$, $c_1(\alpha, g)$ and $c_2(\alpha, g)$ have all the same length. Thus we may assume that $r$ is an initial segment of $c_1(\alpha, h)$ (otherwise we exchange the roles of $g$ and $h$), and find $s$, $t$, $u \in \Omega$ such that
\begin{align*}
c_1(\alpha, h) &= rs \quad c_2(\alpha, h) = tu \quad (5) \\
c_1(\alpha, g) &= st \quad c_2(\alpha, g) = ur \quad (6)
\end{align*}
where $l(r) = l(t)$ and $l(s) = l(u)$. In particular, $r$ is constant iff $t$ is constant (trivial permutation case), and $s$ is constant iff $u$ is constant. (Note: $t$ is not to be confused with $t_g$ nor $t_h$). Let $g = [\gamma_h \gamma_g^{-1}] \in L_x(M)$
\begin{equation}
\phi = [\gamma_h \gamma_g^{-1}] \in L_x(M) \quad (7)
\end{equation}
Then by Lemma 5.8 we have
\begin{equation}
\phi \alpha \phi^{-1} = \alpha \alpha^{-1}g^{-1}g^{-1} = [\alpha]h[\alpha]^{-1}h^{-1} \quad (8)
\end{equation}
Lemma 5.8 also gives the converse: if $g, h \in T_1(\alpha)$ are so that $[\alpha]g[\alpha]^{-1}g^{-1}$ and $[\alpha]h[\alpha]^{-1}h^{-1}$ are conjugate in $L_x(M)$, then $c_1(\alpha, g) c_2(\alpha, g)$ is a permutation of $c_1(\alpha, h) c_2(\alpha, h)$.

**Lemma 5.9** Assume that $\alpha$ is primitive, and that $g, h \in T_1(\alpha)$ are orientation reversing and so that $[\alpha]h[\alpha]^{-1}h^{-1}$ and $[\alpha]g[\alpha]^{-1}g^{-1}$ are conjugate. Then $g = h$

*in particular*
\begin{equation}
c_i(\alpha, g) = c_i(\alpha, h) \quad \text{for } i = 1, 2.
\end{equation}

**Proof** Recall that since $g$ and $h$ are orientation reversing we have that $c_1(\alpha, g) = \alpha g$, $c_2(\alpha, g) = \beta g$, $c_1(\alpha, h) = \alpha h$ and $c_2(\alpha, h) = \beta h$, which will be useful through the proof.

Let $\tilde{\alpha}$ be the projection of $\alpha$ onto $\Lambda_\alpha / G_h$ as in Lemma 4.11. Note that $E_{\alpha}(M) / G_h$ is an intermediate bundle over $M$, and has a notion of horizontal lift, by projecting the one in $E_{\alpha}(M)$ (which is equivariant). Throughout this proof we shall consider horizontal lifts to $E_{\alpha}(M) / G_h$ repeatedly, and refer to them simply as “lifts”.

Let $\tilde{g}$ be the lift of $g$ starting at $\tilde{g}(0) = \tilde{\alpha}(0)$, which is the same as the projection of $\tilde{g}_h$, thus it is contained in $\Lambda_\alpha / G_h$ and $\tilde{g}_h(1)$ is the projection of $b_h(1)$. Let $\tilde{c}_1(\alpha, h) \tilde{c}_2(\alpha, h)$ be the lift of $c_1(\alpha, h) c_2(\alpha, h)$ beginning at $\tilde{g}_h(1)$. We see that this curve is closed and contained in $\Lambda_\alpha / G_h$, by observing that $\tilde{c}_1(\alpha, h)$ and $\tilde{c}_2(\alpha, h)$ are the respective projections of $\tilde{\alpha}_h \subset \Lambda_\alpha$ and $h^{-1} \tilde{\beta}_h \subset \Lambda_\alpha$, thus each one is a closed curve at $\tilde{g}_h(1)$.

Next we take $\tilde{r}$ the lift of $r$ starting at $\tilde{g}_h(1)$, noting that it is an initial segment of $\tilde{c}_1(\alpha, h)$. Consider $\tilde{c}_1(\alpha, g) \tilde{c}_2(\alpha, g)$ the lift of $c_1(\alpha, g) c_2(\alpha, g)$ that starts at $\tilde{r}(1)$ (recalling that $c_1(\alpha, g)$ begins at $r(1)$).

*Claim 1*: $\tilde{c}_1(\alpha, g) \tilde{c}_2(\alpha, g)$ is closed and contained in $\Lambda_\alpha / G_h$.

To show this claim, write $c_1(\alpha, g) c_2(\alpha, g) = stur$, and start by taking the lift $\tilde{s}$ of $s$ starting at $\tilde{r}(1)$. Recalling that $rs = c_1(\alpha, h)$ we conclude that $\tilde{r} \tilde{s} = \tilde{c}_1(\alpha, h)$, and thus $\tilde{s} \subset \Lambda_\alpha / G_h$ and $\tilde{s}(1) = \tilde{g}_h(1)$.

We continue lifting $c_1(\alpha, g) c_2(\alpha, g) = stur$ by taking the lift of $tu$ beginning at $\tilde{s}(1) = \tilde{g}_h(1)$, and we notice that this lift agrees with $\tilde{c}_2(\alpha, h)$ since $tu = c_2(\alpha, h)$. In particular it ends at $\tilde{g}_h(1)$, so $\tilde{r}$ is the lift of $r$ that we need to complete the lifting of $c_1(\alpha, g) c_2(\alpha, g) = stur$. Therefore we obtain that $\tilde{c}_1(\alpha, g) \tilde{c}_2(\alpha, g) = \tilde{s} \tilde{c}_2(\alpha, h) \tilde{r}$, which is closed at $\tilde{r}(1)$ and clearly contained in $\Lambda_\alpha / G_h$.

*Claim 2*: For $i = 1, 2$ we have
\begin{equation}
\tilde{c}_i(\alpha, g) = \tilde{c}_i(\alpha, h).
\end{equation}
We can consider \( \mathcal{L}_{\alpha}(\Lambda_\alpha/G_h) \) as a subgroup of \( \mathcal{L}_x(M) \), since projection induces an injective homomorphism, and then Lemma 4.11 implies that \( T_1(\tilde{\alpha}) = \{h, h^{-1}\} \). By construction we have

\[
\tilde{c}_i(\alpha, h) = c_i(\tilde{\alpha}, h) \quad \text{for } i = 1, 2.
\]

On the other hand, by Claim 1 we have that \( \tilde{c}_1(\alpha, g) \) and \( \tilde{c}_2(\alpha, g) \) lie inside \( \Lambda_\alpha/G_h \), and we recall that they are lifts of \( \alpha_g \) and \( \beta_g \) respectively, which are permutations of \( \alpha \) and \( \alpha^{-1} \).
Therefore \( \tilde{c}_1(\alpha, g) \) and \( \tilde{c}_2(\alpha, g) \) are permutations of \( \tilde{\alpha} \) and \( \tilde{\alpha}^{-1} \) respectively (in particular they are closed). Then we get that \( \tilde{c}_1(\alpha, g)\tilde{c}_2(\alpha, g) \) represents the dot product for an intersection pair of \( \tilde{\alpha} \), and since it is cyclically reduced we must have

\[
\tilde{c}_i(\alpha, g) = c_i(\tilde{\alpha}, h^\epsilon) \quad \text{for } i = 1, 2 \text{ and for some } \epsilon = \pm 1.
\]

Since we have \( \tilde{c}_1(\alpha, h)\tilde{c}_2(\alpha, h) = \bar{r}\tilde{c}_1(\alpha, g)\tilde{c}_2(\alpha, g) \bar{r}^{-1} \), we can apply Remark 5.2 to get that \( \epsilon = 1 \), proving this claim.

To finish the proof of the lemma recall that, since we are in the orientation reversing case, we have \( c_1(\alpha, g) = \alpha_g \) and \( c_1(\alpha, h) = \alpha_h \), which are permutations of \( \alpha \). Equation (9) implies, by projecting, that \( \alpha_g = \alpha_h \), and then Lemma 2.3 gives \( g = h \), since \( \alpha \) is primitive.

Putting Lemma 5.9 together with Remark 5.6, we see that a dot product of the form \( (X \cdot (P, Q), X^{-1}) \) with length \( 2l(\alpha) \) comes from a unique intersection pair \( (P, Q) \), which is of type either (1) or (3).

One would like to remove the orientation reversal condition from the hypothesis of Lemma 5.9, for that would give a stronger result than Theorem 1.2, namely that the strings in the terms of the formula for \( [X, X^{-1}] \) cannot repeat. Unfortunately this remains open. Next is the result we can get when dropping said orientation condition, which will suffice for our purpose.

**Lemma 5.10** Assume that \( \alpha \) is primitive, and that \( g, h \in T_1(\alpha) \) are such that \( [\alpha]h[\alpha]^{-1}h^{-1} \) and \( [\alpha]g[\alpha]^{-1}g^{-1} \) are conjugate. Then

\[
c_i(\alpha, g) = c_i(\alpha, h) \quad \text{for } i = 1, 2.
\]

**Proof** We shall assume \( g \) and \( h \) preserve orientation strictly, since Remark 5.6 implies that the only other possible case is the one covered by Lemma 5.9.

Recall the notation from Eqs. (5) and (6), i.e. the curves \( r, s, t, u \) and their properties. Notice that we can prove the lemma by showing that \( t \), and hence \( r \), are constant.

Firstly we shall define some horizontal lifts in \( \mathcal{L}_x(M) \) that will be useful through the proof: By Eq. (6) we can take \( \tilde{s} \) the lift of \( s \) starting at \( \tilde{\gamma}_g(1) = b_g(1) \), and \( \tilde{t} \) the lift of \( t \) starting at \( \tilde{s}(1) \). Note that \( \tilde{s}t \) is the lift of \( c_1(\alpha, g) \) that is an initial segment of \( \tilde{\alpha}_g \), thus it is contained in \( \Lambda_\alpha \) and ends at \( \tilde{r}(1) = [\alpha]b_g(0) \), recalling Definition 5.3 in the orientation preserving case.
We also see that \( \tilde{s} \) and \( \tilde{t} \) are positively oriented in \( \Lambda_\alpha \).

We also define \( \tilde{r} \) as the lift of \( r \) starting at \( \tilde{\gamma}_h(1) = b_h(1) \), and \( \tilde{s} \) as the lift of \( s \) starting at \( \tilde{r}(1) \), which are well defined by Equation (5). Again we see that \( \tilde{r}s \) is the lift of \( c_1(\alpha, g) \) which is an initial segment of \( \tilde{\alpha}_h \) and so it is contained in \( \Lambda_\alpha \) and ends at \( \tilde{s}(1) = [\alpha]b_h(0) \).
Also, \( \tilde{r} \) and \( \tilde{s} \) are positively oriented in \( \Lambda_\alpha \).

Finally, let \( \tilde{r} \) be the lift of \( t \) starting at \( \tilde{s}(1) \), which is well defined since \( t(0) = s(1) \) from Eq. (6). We see from Eq. (5) that \( \tilde{r} \) is an initial segment of a lift of \( c_2(\alpha, h) \), namely the one contained in \( [\alpha] \tilde{\beta}_h \), since it starts at \( [\alpha]b_h(0) \). Thus \( \tilde{r} \) is contained in \( [\alpha]h\Lambda_\alpha \), meeting \( \Lambda_\alpha \).
only at \( \tilde{t}(0) \), and it goes in the negative direction with respect to the orientation of \([\alpha]h\Lambda_\alpha\) induced by \([\alpha]h\).

Next we recall Eq. (7), defining \( \phi = [\gamma_hr\gamma_g^{-1}] \). Notice that, by the above definitions, we have \( \phi\gamma_g(1) = \tilde{r}(1) \), in particular \( \phi \in T(\Lambda_\alpha) \). Since the action of \( L_x(M) \) preserves horizontal lifting, we also get that \( \phi\tilde{s} = \tilde{s} \) and \( \phi\tilde{t} = \tilde{t} \).

Claim 1: If \( \phi \in T^+(\Lambda_\alpha) \) then \( \tilde{t} \) is constant.

Let \( G \) be the subgroup generated by \( g, h, \phi \) and \([\alpha] \). Then \( G \) is \( \alpha \)-oriented by Lemma 4.16, so there is a \( G \)-invariant orientation on \( G\Lambda_\alpha \). On the other hand, \( \phi\tilde{t} = \tilde{t} \) where \( \tilde{t} \) has positive orientation in \( \Lambda_\alpha \) but \( \tilde{t} \) has negative orientation in \([\alpha]h\Lambda_\alpha \). This is a contradiction unless \( t \) is constant.

Claim 2: If \( s \) is non-constant, then \( \phi \in T^+(\Lambda_\alpha) \).

We have \( \phi\tilde{s} = \tilde{s} \) where both \( \tilde{s} \) and \( \tilde{s} \) are contained in \( \Lambda_\alpha \). Since \( \tilde{s} \subset b_\phi(I) \) and \( \tilde{s} \subset a_\phi(I) \). Recall that both \( \tilde{s} \) and \( \tilde{s} \) are positively oriented in \( \Lambda_\alpha \), thus showing that \( \phi \) preserves orientation, if \( s \) is non-constant.

Recall that we prove the lemma by showing that \( t \) is constant. Thus, in light of these claims, the only case that remains to be considered is when \( s \) is constant and \( \phi \) reverses orientation strictly. We shall show that this case is void, which makes sense as \( s \) and \( t \) cannot be both constant in Eq. (6). Thus we assume that \( s \) is constant and \( \phi \) strictly reverses orientation, aiming to reach a contradiction.

This case is the most complex part of this proof, the key will be to consider the dot product defined by \( \phi \). Note that this can be defined even if \( \phi \) is not in \( T_1(\alpha) \), through Lemma 4.4, and the same is true for the curves \( c_1(\alpha, \phi) \) and \( c_2(\alpha, \phi) \) from Definition 5.3. In order to simplify notation we put \( c_i(\phi) = c_i(\alpha, \phi) \) for \( i = 1,2 \), and define
\[
\omega = [\alpha]\phi[\alpha^{-1}]\phi^{-1} = [\gamma_\phi c_1(\phi)c_2(\phi)\gamma_\phi^{-1}]
\]

Claim 3: \( \omega \) is primitive.

By change of basepoint, i.e. Lemma 2.1, this is equivalent to show that \( c_1(\phi)c_2(\phi) \) is primitive. Assume the contrary, i.e. that there is a closed curve \( \tau \) and \( k > 1 \) so that \( \tau^k = c_1(\phi)c_2(\phi) \). Note that \( \tau \) must be cyclically reduced, since \( c_1(\phi)c_2(\phi) \) is, and that \( l(c_1(\phi)) = l(c_2(\phi)) \) by Remark 5.4.

If \( k \) is even, we write \( k = 2j \) and get that \( c_1(\phi) = \tau^j = c_2(\phi) \), which is absurd by Remark 5.4 and the assumption that \( \phi \) reverses orientation, since a permutation of \( \alpha \) cannot agree with a permutation of \( \alpha^{-1} \). If \( k \) is odd, we write \( k = 2j + 1 \) and we get that
\[
\tau = vw \quad \text{with} \quad c_1(\phi) = (vw)^j v \quad \text{and} \quad c_2(\phi) = w(vw)^j \]
where we also have \( l(v) = l(w) \). Again by Remark 5.4, we get that \( (w^{-1}v^{-1})^j w^{-1} \) is a permutation of \( (vw)^j v \). Since \( l(v) = l(w) \) and \( v \) cannot overlap \( w^{-1} \) (nor \( w \) overlap \( w^{-1} \)), we must have \( v = w^{-1} \), which is absurd since \( \tau \) cannot be trivial. Thus we have shown Claim 3.

Claim 4: There is a non-trivial \( \theta \in T(\Lambda_\alpha) \) such that \( \theta\omega\theta^{-1} = \omega \).

To show this claim it will be useful to write Eqs. (5) and (6) for the case when \( s \) is constant:
\[
c_1(\alpha, h) = c_2(\alpha, g) = r \quad \text{and} \quad c_1(\alpha, g) = c_2(\alpha, h) = t, \quad (10)
\]
and we also get that \( \tilde{t}(0) = \tilde{\gamma}_g(1) = b_\tilde{g}(1) \) and \( \tilde{r}(1) = \tilde{t}(0) = [\alpha]b_\tilde{h}(0) \). By Remark 5.7 we deduce that
\[
c_1(\alpha, h^{-1}) = c_2(\alpha, g^{-1}) = t^{-1} \quad \text{and} \quad c_1(\alpha, g^{-1}) = c_2(\alpha, h^{-1}) = r^{-1} \quad (11)
\]
Define
\[ \psi = h^{-1}\phi g. \]

Let us check that \( \psi \in T^{-}(\Lambda_{\alpha}) \). Note first that, since \( g \) preserves orientation, we have \( \tilde{\gamma}_{g^{-1}}(1) = b_{g^{-1}}(1) = a_{g}(1) \) by Remark 4.6. By Eq. (11) we may take \( \tilde{r} \) the lift of \( r \) that ends at \( \tilde{r}(1) = a_{g}(1) \), i.e. so that \( \tilde{r}^{-1} \) lifts \( c_{1}(\alpha, g^{-1}) \) starting at \( a_{g}(1) = \tilde{\gamma}_{g^{-1}}(1) \). Thus we see that \( \tilde{r}^{-1} \) is contained in \( \Lambda_{\alpha} \) and is positively oriented, since it is a segment of \( \tilde{\alpha}_{g^{-1}} \). We also see that \( \tilde{r}(0) = \tilde{r}^{-1}(1) = [\alpha]a_{g}(0) \).

We shall describe the action of \( \psi = h^{-1}\phi g \) on \( \tilde{r} \). First we notice that \( g\tilde{r} \) ends at \( ga_{g}(1) = b_{g}(1) = \tilde{\gamma}_{g}(1) \). Next we recall that \( \phi \tilde{\gamma}_{g}(1) = \tilde{r}(1) \), so by equivariance of the horizontal lifting we get that \( \phi g\tilde{r} = \tilde{r} \). Finally we get that \( \psi \tilde{r} = h^{-1}\tilde{r} \). Recalling Eq. (11) we see that this curve lifts \( c_{2}(\alpha, h^{-1}) \) starting at \( h^{-1}\tilde{r}(0) = h^{-1}b_{h}(1) = a_{h}(1) \). Thus \( \psi \tilde{r} \) is contained in \( h^{-1}\Lambda_{\alpha} \) and meets \( \Lambda_{\alpha} \) exactly at \( \psi \tilde{r}(0) = a_{h}(1) \). This shows that \( \psi \in T(\Lambda_{\alpha}) \), and it must reverse orientation strictly by Lemma 4.16 since \( \phi = h\psi g^{-1} \).

Now we set \( h^{*} = [\alpha]h[\alpha]^{-1} \), and a simple computation from Eq. 8 and the definition of \( \psi = h^{-1}\phi g \) gives us
\[ h^{n}[\alpha]\psi[\alpha]^{-1}\psi^{-1} = [\alpha]\phi[\alpha]^{-1}\phi^{-1} \]
(12)

By Remark 4.4 there are integers \( i, j, k, l \) such that \([\alpha]^{i}\phi[\alpha]^{j}\) and \([\alpha]^{k}\psi[\alpha]^{l}\) belong to \( T_{1}(\alpha) \).

Now we apply Lemma 5.9 to these elements and Eq. (12), hence for \( n = i - k, m = k - l \) we obtain
\[ \psi = [\alpha]^{n}\phi[\alpha]^{m} \]
and Eq. (12) becomes
\[ (h^{n}[\alpha]^{n})[\alpha]\phi[\alpha]^{-1}\phi^{-1}(h^{n}[\alpha]^{n})^{-1} = [\alpha]\phi[\alpha]^{-1}\phi^{-1} \]

So we set \( \theta = h^{n}[\alpha]^{n} \) and get that \( \theta_{i}\theta^{-1} = \omega \). Note that \( \theta = [\alpha]h[\alpha]^{-1} \) is not trivial, since \( h \) is not a power of \([\alpha] \), and belongs to \( T(\Lambda_{\alpha}) \), since \( \theta \Lambda_{\alpha} = [\alpha]h\Lambda_{\alpha} \) intersects \( \Lambda_{\alpha} \) in the segment \([\alpha]b_{h}(I) \). Thus we have shown Claim 4.

Putting \( C = \alpha(I) \) and recalling Lemma 3.10, we see that \( \omega, \theta \in \mathcal{L}_{x}(C) \) which is a free group, so Claims 3 and 4 imply that
\[ \theta = \omega^{i} \] for some \( i \in \mathbb{Z}, i \neq 0 \)
(13)

We shall reach a contradiction by showing that \( \theta \Lambda_{\alpha} \) is disjoint from \( \Lambda_{\alpha} \), i.e.that \( \theta \notin T(\Lambda_{\alpha}) \), against Claim 4. Since \( T(\Lambda_{\alpha}) \) is closed under taking inverses (Remark 4.2), we may assume that \( i > 0 \).

Changing the basepoint if necessary, we may assume that \( b_{\phi}(1) = id_{c} \). Note that such a change of basepoint ammounts to replace \( \alpha \) by a permutation of it, and does not change the curves \( c_{1}(\phi) \) and \( c_{2}(\phi) \). By Lemma 2.1 this change of basepoint also preserves Claims 3 and 4, and Eq. (13). From \( b_{\phi}(1) = id_{c} \) we get that \( \gamma_{\phi} \) is constant, \( c_{1}(\phi) = \alpha \) and \( c_{2}(\phi) \) is a non-trivial permutation of \( \alpha^{-1} \).

Let \( \tilde{c}_{2}(\phi) \) be the horizontal lift of \( c_{2}(\phi) \) that ends at \( b_{\phi}(1) = id_{c} \). Since \( \phi \) reverses orientation, the discussion after Definition 5.3 implies that \( \tilde{c}_{2}(\phi) \) intersects \( \Lambda_{\alpha} \) exactly in the segment \( b_{\phi}(I) \), and in particular \( \tilde{c}_{2}(\phi)(0) \notin \Lambda_{\alpha} \). On the other hand we take \( \eta \) the horizontal lift of \( (c_{1}(\phi)c_{2}(\phi))^{l} \) that begins at \( id_{c} \), so we have \( \eta(1) = \omega^{i} = \theta \). Note that since \( c_{1}(\phi) = \alpha \), we have a reduced factorization
\[ \eta = \tilde{\alpha} \psi \]
where \( v \) meets \( \Lambda_\alpha \) only at \( v(0) \), since the lift of \( c_2(\phi) \) starting at the point \([\alpha] = [\alpha]b_\phi(1)\) is an initial segment of \( v \). Observe also that \( \theta\tilde{c}_2(\phi) \), being the lift of \( c_2(\phi) \) that ends at \( \theta = v(1) \), is a final segment of \( v \) (if \( i = 1 \) it agrees with the initial segment just discussed). Since \( \mathcal{E}_i(C) \) is a tree, the line \( \theta\Lambda_\alpha \) cannot intersect \( \Lambda_\alpha \) without containing \( v(I) \), but acting by \( \theta \) we see that \( \theta\tilde{c}_2(\phi) \) intersects \( \theta\Lambda_\alpha \) only at \( \theta b_\phi(I) \), which does not contain \( \theta\tilde{c}_2(\phi)(0) \in v(I) \). Therefore \( \theta \notin T(\Lambda_\alpha) \) and we have a contradiction, concluding the proof.

\[ \square \]

6 Signs of the terms of the bracket

Here we shall study the signs of the linked pairs in \( LP_2(X, X^{-1}) \) for \( X \in S(M) \), showing that linked pairs yielding the same dot product have also the same sign, and finally arriving at the proof of Theorem 1.2. In order to do so, we may need to consider a small deformation of a curve \( \alpha \) representing \( X \).

6.1 Deformations of 1-complexes

Let \( C \) be a one dimensional complex, and consider \( p \in C \). Then a small enough neighborhood \( B \) of \( p \) in \( C \) is homeomorphic to a wedge of intervals, each one with an endpoint at \( p \) and the other in \( \partial B \). We call the number of such segments the \textit{valence} of \( p \) in \( C \). A point of valence 2 will be called a \textit{regular point} of \( C \). Observe that the set of non-regular points is discrete.

We will be interested in complexes of the form \( C = \alpha(I) \) where \( \alpha \in \Omega \) is a cyclically reduced, non-constant closed curve. Then \( C \) has no points of valence 1, and by compactness, the set of non-regular points (i.e. \textit{branching points}) is finite. Note that, replacing \( \alpha \) by a permutation if necessary, we can assume that \( x = \alpha(0) \) is a regular point.

Before introducing the perturbation of \( \alpha \) that we need, it is worth recalling that \( C_\varepsilon \) is the \( \varepsilon \)-neighborhood of \( C \) in \( M \), and for \( \varepsilon \) small enough, Lemma 3.11 states that \( C_\varepsilon \) retracts by deformation onto \( C \). Then the map induced by the inclusion \( i_* : \pi_1(C, x) \to \pi_1(C_\varepsilon, x) \) is an isomorphism, and its inverse is \( \chi_* : \pi_1(C_\varepsilon, x) \to \pi_1(C, x) \), which is induced by a retraction \( \chi : C_\varepsilon \to C \).

**Lemma 6.1** Let \( \alpha \) be a cyclically reduced closed curve and \( C = \alpha(I) \). Assume that \( x = \alpha(0) \) is a regular point of \( C \). Then there are \( \varepsilon > 0 \) and a closed, cyclically reduced curve \( \gamma \subset C_\varepsilon \) such that

1. \( \gamma \) is an \( \varepsilon \)-perturbation of \( \alpha \), with \( \gamma(0) = x \),
2. \( \Gamma = \gamma(I) \) has no points of valence greater than 3, and
3. The map \( i_* : \pi_1(\Gamma, x) \to \pi_1(C_\varepsilon, x) \) induced by the inclusion is an isomorphism.

Moreover, \( \gamma \) only differs from \( \alpha \) in an \( \varepsilon \)-neighborhood of the non-regular points of \( C \).

**Proof** Set \( \varepsilon \) as in Lemma 3.11, though we may need to reduce it further. For each non-regular point \( p \) of \( C \) we consider \( B_p \) a closed ball in \( M \) centered at \( p \) with radius \( \varepsilon/2 \). Note that if \( \varepsilon \) is small enough, then \( B_p \) is a normal ball and \( B_p \cap C \) is a union of geodesic segments joining \( p \) to \( \partial B_p \). Reducing \( \varepsilon \) if necessary, we can also make the sets \( B_p \) pairwise disjoint, and disjoint from \( x \) by our assumption.

First we will construct the complex \( \Gamma \). For each non-regular point \( p \) of \( C \), we take a segment \( \eta \subset \alpha \) with endpoints in \( \partial B_p \) and so that \( \eta(I) \subset B_p \), i.e. \( \eta \) is a segment of \( \alpha \) that traverses \( B_p \). Then we consider a piecewise geodesic complex \( Y_p \) contained in \( B_p \), with
$Y_p \cap \partial B_p = C \cap \partial B_p$, and so that $Y_p$ is a finite tree containing $\eta(I)$, whose branching points lie in $\eta(I)$ and have valence 3. Figure 5 shows an example of this construction. It can be interpreted as a deformation of $C \cap B_p$ that spreads out the segments that are not in $\eta$, so that their endpoints are spread along $\eta$ instead of converging at $p$. Then we define $\Gamma$ so that it agrees with $C$ in the complement of $B = \bigcup_p B_p$, and that $\Gamma \cap B_p = Y_p$ for every non-regular point $p$ in $C$.

By construction, $\Gamma$ is a connected piecewise geodesic complex and all its non-regular points are of valence 3. Note also that $B_p$ retracts by deformation to $Y_p$ for each $p$. These maps can be extended to a retraction by deformation $\chi_1 : C_\varepsilon \to \Gamma$, thus obtaining point (3) in the statement.

Now we describe the curve $\gamma$. Write

$$\alpha = \alpha_0 \beta_1 \alpha_1 \cdots \beta_n \alpha_n$$

where $\alpha_i$ is contained in the closure of the complement of the set $B$ for all $i = 0, \ldots, n$, and $\beta_j$ is contained in $B$, for $j = 1, \ldots, n$. None of these curves is constant since $x$ is outside $B$. Then for each $j$ there is some $p$ with $\beta_j \subset B_p$, and we have $\beta_j(0), \beta_j(1) \in \partial B_p$. Note that, by construction, there is a unique reduced curve $\bar{\beta}_j$ joining $\beta_j(0)$ to $\beta_j(1)$ in $Y_p$. Thus $\bar{\beta}_j \subset \Gamma \cap B_p$ with $\bar{\beta}_j(0) = \beta_j(0)$ and $\bar{\beta}_j(1) = \beta_j(1)$. We define

$$\gamma = \alpha_0 \bar{\beta}_1 \alpha_1 \cdots \bar{\beta}_n \alpha_n$$

i.e. we replace each $\beta_j$ with $\bar{\beta}_j$. This is an $\varepsilon$-perturbation of $\alpha$, since for each $j$, $\beta_j$ and $\bar{\beta}_j$ are in the same ball of radius $\varepsilon/2$. It is also clear that $\gamma$ admits no reductions, and that $\gamma'(0) = \gamma'(1) = x$. It only remains to show that $\gamma(I) = \Gamma$. By construction we have $\gamma(I) \subseteq \Gamma$, and it is also clear that the closure of $\Gamma - B$ is contained in $\gamma(I)$, writing this set as $\bigcup I_a(I)$. So we must show that $Y_p \subset \gamma(I)$ for each non-regular point $p$ of $C$. For one such $p$ we note that the curve $\eta$ used in the construction of $Y_p$ is in $\gamma(I)$: by its definition $\eta = \beta_j$ for some $j$ (maybe more than one), and in that case we also have $\bar{\beta}_j = \eta$. On the other hand, since we have $\partial B_p \cap \Gamma \subset \gamma(I)$, the rest of the segments making up $Y_p$ must also be contained in $\gamma(I)$.

$\square$
6.2 Signs and lifts

Now we shall interpret the signs of linked pairs in terms of horizontal lifting, in the same fashion as we did for intersection pairs and dot products in the previous sections. Let $\alpha \in \Omega$ be a non-trivial cyclically reduced closed curve and $x = \alpha(0)$.

**Definition 6.2** For $g \in T_0(\alpha)$ we write

$$\epsilon_g(\alpha) = \text{sign}(P_g, Q_g),$$

i.e. the sign of the linked pair of $\alpha$ corresponding to $g$. When the curve $\alpha$ is clear from the context, we just write $\epsilon_g$.

Let $C = \alpha(I)$ and recall from Lemma 3.10 that the universal cover $\tilde{C}$ is isomorphic, as a principal fiber bundle, to $E_x(C)$. We choose such an isomorphism by picking $\tilde{x} \in \tilde{C}$ a lift of $x$, and setting that $id_x$ corresponds to $\tilde{x}$. Then we can identify $\Lambda_\alpha$ with the infinite lift of $\alpha$ to $\tilde{C}$ that starts at $\tilde{x}$.

Let $C_\varepsilon$ be a neighborhood of $C$ satisfying Lemma 3.11. Then its universal cover $\tilde{C}_\varepsilon$ contains $\tilde{C}$, and retracts onto it by lifting the retraction map $\chi : C_\varepsilon \to C$. Note that the complement of $\Lambda_\alpha$ in $\tilde{C}_\varepsilon$ has two connected components. We give $\tilde{C}_\varepsilon$ the orientation lifted from that of $C_\varepsilon \subset M$, and let $C^+(\alpha)$ be the left side of $\Lambda_\alpha$. More precisely, $C^+(\alpha)$ is the component of $\tilde{C}_\varepsilon - \Lambda_\alpha$ so that the standard orientation of $\Lambda_\alpha$, given by $\hat{a}$, agrees with the orientation induced on $\Lambda_\alpha$ as part of the boundary $\partial C^+(\alpha)$. The other component, i.e. the right side of $\Lambda_\alpha$, will be denoted by $C^-(\alpha)$.

Let $a_1, a_2, v$ be small geodesic segments in $\tilde{C}_\varepsilon$, so that $a_1(1) = a_2(0)$ and $v$ meets $a_1a_2$ only at this point, which is an endpoint of $v$. Note that we have $\text{sign}(a_1a_2, v) = \text{sign}(\pi(a_1a_2), \pi(v))$, by definition of the orientation of $\tilde{C}_\varepsilon$. In case that $a_1a_2$ is contained in $\Lambda_\alpha$ with positive orientation, then we have $\text{sign}(a_1a_2, v) = 1$ if either $v(0) = a_1(1)$ and $v(1) \in C^+(\alpha)$, or $v(1) = a_2(1)$ and $v(0) \in C^-(\alpha)$. In the reverse cases we have $\text{sign}(a_1a_2, v) = -1$.

Now consider $g \in T_1(\alpha)$. We identify $L_x(C)$ with $\pi_1(C_\varepsilon, x)$ acting on $\tilde{C}_\varepsilon$ by deck transformations, using Lemmas 3.10 and 3.11. Then, by definition of $T_1(\alpha)$, we have that $g[\alpha] \tilde{x}$ and $g[\alpha^{-1}] \tilde{x}$ are not contained in $\Lambda_\alpha$. Due to the previous observations, we see that $g$ corresponds to a linked pair, i.e. $g \in T_0(\alpha)$, iff $g[\alpha] \tilde{x}$ and $g[\alpha^{-1}] \tilde{x}$ are in different components of $\tilde{C}_\varepsilon - \Lambda_\alpha$. In that case, we have $\epsilon_g = -1$ if $g[\alpha^{-1}] \tilde{x} \in C^-(\alpha)$, and thus $g[\alpha] \tilde{x} \in C^+(\alpha)$, which is to say that $g\Lambda_\alpha$ crosses $\Lambda_\alpha$ going from its right side and towards its left side. We have $\epsilon_g = 1$ if the reverse holds. We compile these results for future reference.

**Remark 6.3** Let $g \in T_1(\alpha)$, then

- $g \in T_0(\alpha)$ iff $g[\alpha] \tilde{x}$ and $g[\alpha^{-1}] \tilde{x}$ are on different sides of $\Lambda_\alpha$.
- In that case, $\epsilon_g = -1$ iff $g[\alpha^{-1}] \tilde{x} \in C^-(\alpha)$, i.e. is at the right side of $\Lambda_\alpha$.

In a similar manner, we see that for $g \in T_0(\alpha)$ we have $\epsilon_g = -1$ iff $\tilde{b}_g(1) \in C^-(\alpha)$. Equivalently, iff $g[\alpha]g^{-1}\tilde{b}_g(0) \in C^+(\alpha)$. Noting that $c_2(\alpha, g)$ ends at $t_g(1)$ by Definition 5.3, we have the following.

**Remark 6.4** For $g \in T_0(\alpha)$ let $\tilde{c}_2(\alpha, g)$ be the lift of $c_2(\alpha, g)$ that ends at $b_g(1)$ (namely, the one contained in $g[\alpha]g^{-1}\tilde{b}_g$). Then $\epsilon_g = -1$ iff $\tilde{c}_2(\alpha, g)$ is at the left of $\Lambda_\alpha$ (more precisely, is contained in the closure of $C^+(\alpha)$).
By the comments after Definition 5.3 we see that the intersection of \( \tilde{c}_2(\alpha, g) \) with \( \Lambda_\alpha \) is either the endpoint \( b_g(1) \), in case \( g \) preserves orientation, or the segment \( b_g(I) \), if \( g \) reverses orientation. We also point out that the terminology of orientation preserving or reversing elements of Definition 4.1 does not relate to the orientation of \( \tilde{C}_e \), which is preserved by every deck transformation.

6.3 Signs and deformations

Let \( \alpha \in \Omega \) be a cyclically reduced closed curve, and assume that \( x = \alpha(0) \) is a regular point of \( C = \alpha(I) \). Take \( \varepsilon \) and \( \gamma \) as given by Lemma 6.1, and let \( \Gamma = \gamma(I) \). Then we can identify \( \mathcal{E}_\varepsilon(\Gamma) \) with \( \tilde{\Gamma} \), the universal cover of \( \Gamma \), which is embedded in \( \tilde{C}_e \). This identifies \( \mathcal{L}_\varepsilon(\Gamma) \) with \( \pi_1(C, x) \), and thus with \( \mathcal{L}(C) \). We will be assuming these identifications in the sequel, though we should make clear that \( \mathcal{L}_\varepsilon(C) \) and \( \mathcal{L}_\varepsilon(\Gamma) \) are different as subgroups of \( \mathcal{L}(M) \), and that \( \mathcal{E}_\varepsilon(C) \cup \mathcal{E}_\varepsilon(\Gamma) \), as a subspace of \( \mathcal{E}_\varepsilon(M) \), is not homeomorphic to \( C \cup \tilde{\Gamma} \). In fact, this correspondence identifies \( \{\gamma\} \in \mathcal{L}(\Gamma) \) with \( [\alpha] \in \mathcal{L}(C) \).

We also identify \( \Lambda_\varepsilon \) with the infinite lift of \( \varepsilon \) at \( \tilde{\varepsilon} \), the lift of \( x \) used to define both correspondences \( \mathcal{E}_\varepsilon(C) \cong \tilde{C} \) and \( \mathcal{E}_\varepsilon(\Gamma) \cong \tilde{\Gamma} \). Thus the sets \( T(\Lambda_\alpha) \) and \( T(\Lambda_\gamma) \) can be considered as subsets of the same group \( \pi_1(\tilde{C}_e, x) \), which we shall see as the group of deck transformations of \( \tilde{C}_e \).

Lemma 6.5 Let \( \alpha \) and \( \gamma \) be as in Lemma 6.1, and take \( g \in \pi_1(\tilde{C}_e, x) \). Then \( g \in T_0(\alpha) \) iff \( g \in T_0(\gamma) \), in which case \( \epsilon_g(\alpha) = \epsilon_g(\gamma) \).

Proof Let \( B \) be the \( \varepsilon \)-neighborhood of \( \Lambda_\alpha \) in \( \tilde{C}_e \), which also contains \( \Lambda_\gamma \). Moreover, we have \( C^\pm(\alpha) \cap (\tilde{C}_e - B) = C^\pm(\gamma) \cap (\tilde{C}_e - B) \), i.e. points outside \( B \) are in the same side with respect to both \( \Lambda_\alpha \) and \( \Lambda_\gamma \). Note that the construction of Lemma 6.1 allows for reducing \( \varepsilon \) as necessary, so we can assume that the translates of \( \tilde{x} \) that do not belong to \( \Lambda_\alpha \) are outside \( B \).

If \( g \in T_0(\alpha) \) we observed in Remark 6.3 that \( g[\alpha^{-1} \tilde{x}] \) and \( g[\alpha] \tilde{x} \) are on different sides of \( \Lambda_\alpha \), and since they are not in \( B \), they are also on different sides of \( \Lambda_\gamma \). Since \( [\alpha] \) and \( [\gamma] \) agree when seen as elements of \( \pi_1(\tilde{C}_e, x) \), we see that \( g \Lambda_\gamma \) contains both \( g[\alpha^{-1}] \tilde{x} \) and \( g[\alpha] \tilde{x} \), implying that \( g \Lambda_\gamma \) and \( \Lambda_\gamma \) intersect. This intersection corresponds to a linked pair by Remark 6.3. We get that \( g \in T_1(\gamma) \) by recalling the definition of this set, together with the fact that \( \alpha \) and \( \gamma \) agree on a neighborhood of their basepoint, so every lift of \( \gamma \) is an \( \varepsilon \)-perturbation of the corresponding lift of \( \alpha \) that agrees with it in a neighborhood of its endpoints. With this we conclude that \( g \in T_0(\gamma) \).

The reciprocal argument is analogous. We have that \( \epsilon_g(\alpha) = \epsilon_g(\gamma) \) by Remark 6.3, together with the facts that \( g[\alpha^{-1}] \tilde{x} \) and \( g[\alpha] \tilde{x} \) lie outside \( B \) and the identity \( C^\pm(\alpha) \cap (\tilde{C}_e - B) = C^\pm(\gamma) \cap (\tilde{C}_e - B) \).

Let \( X \) and \( Y \) be the strings represented by \( \alpha \) and \( \gamma \) as in Lemma 6.1. Then Lemma 6.5 gives a bijection between \( LP_2(X, X^{-1}) \) and \( LP_2(Y, Y^{-1}) \) that preserves the sign. It also preserves the dot product, by the isomorphism between \( \mathcal{L}(C) \) and \( \mathcal{L}(\Gamma) \) and Lemma 5.8. Thus proving Theorem 1.2 for \( Y \) also implies it for \( X \), i.e. we may replace \( \alpha \) with \( \gamma \) whenever necessary in our proof. We will only be doing this replacement at the points of the argument that require it, namely in the next Lemma 6.6.

The proof of Lemma 6.5 clearly does not generalize for intersection pairs that are not linked. It is also possible to show Lemma 6.5 by following what happens to an intersection
pair during the construction of $\gamma$ in Lemma 6.1, though some cases may get cumbersome, as well as the assertion about the signs. That approach would give that intersection pairs of types (2) and (3) of $\alpha$ are also present in $\gamma$, maintaining their types. Since $\Gamma$ has valence 3 at every branching point, $\gamma$ has no intersection pairs of type (1), so the type (1) linked pairs of $\alpha$ will become linked pairs of type either (2) or (3) in $\gamma$. Some of the unlinked intersection pairs of type (1) of $\alpha$ may indeed be removed when passing to $\gamma$ (e.g. a suitable parametrization of a circle with three points identified). We do not need these assertions to show Theorem 1.2, so we will not prove them.

6.4 Signs and conjugation

Next we show the main lemma that implies there are no cancellations in the formula for $\left[ X, X^{-1} \right]$ when $X$ is primitive. After that, we shall finish the details of the proof of Theorem 1.2. Let $\alpha \in \Omega$ be a cyclically reduced non-trivial closed curve, and $x = \alpha(0)$. We consider $C = \alpha(I)$ and $C_x$ as in the rest of this section, identifying $\mathcal{E}_x(C)$ with $\mathcal{C}$ embedded in $\mathcal{C}_x$.

Lemma 6.6 Assume that $\alpha$ is primitive, and that $g, h \in T_0(\alpha)$ are such that $[\alpha]h[\alpha]^{-1}h^{-1}$ and $[\alpha]g[\alpha]^{-1}g^{-1}$ are conjugate. Then

$$\epsilon_g = \epsilon_h.$$

Proof If $g$ and $h$ reverse orientation this is a consequence of Lemma 5.9. So we assume that $g$ and $h$ preserve orientation, since by Remark 5.6 this is the other possible case. Recall the notation of Eqs. (5) and (6), and let $\phi$ be the element defined in Eq. (7). By Lemma 5.10 we have

$$c_i(\alpha, g) = c_i(\alpha, h) \quad \text{for} \quad i = 1, 2,$$  

(16)

so $r$ and $t$ are constant in Equations (5) and (6), and we have $\phi = [\gamma_h \gamma_g^{-1}]$. Therefore we get that $\phi b_g(1) = b_h(1)$.

Let $\tilde{c}_1(\alpha, g)$ be the lift of $c_1(\alpha, g)$ starting at $b_g(1)$, and $\tilde{c}_2(\alpha, g)$ the lift of $\tilde{c}_2(\alpha, g)$ ending at $b_g(1)$. Then $\tilde{c}_1(\alpha, g)$ is contained in $\Lambda_\alpha$ with positive orientation, and $\tilde{c}_2(\alpha, g)$ only meets $\Lambda_\alpha$ at its endpoint $b_g(1)$, since $g$ preserves orientation. By Remark 6.4, the sign $\epsilon_g$ is decided by the side of $\Lambda_\alpha$ that $\tilde{c}_2(\alpha, g)$ is on. We may write

$$\epsilon_g = \text{sign}(b_g \tilde{c}_1(\alpha, g), \tilde{c}_2(\alpha, g))$$

(17)

where we understand we are taking the intersections of these curves with a small enough ball centered at $b_g(1)$, as to follow the definition of sign in Sect. 3.1.

Similarly, we take $\tilde{c}_1(\alpha, h)$ as the lift of $c_1(\alpha, h)$ starting at $b_h(1)$, and $\tilde{c}_2(\alpha, h)$ as the lift of $\tilde{c}_2(\alpha, h)$ ending at $b_h(1)$. The same observations hold, in particular

$$\epsilon_h = \text{sign}(b_h \tilde{c}_1(\alpha, h), \tilde{c}_2(\alpha, h))$$

(18)

Since $\phi b_g(1) = b_h(1)$ and deck transformations preserve horizontal lifting, Eq. (16) implies that

$$\phi \tilde{c}_1(\alpha, g) = \tilde{c}_1(\alpha, h) \quad \text{and} \quad \phi \tilde{c}_2(\alpha, g) = \tilde{c}_2(\alpha, h).$$

(19)

The situation is depicted in Fig. 6.

First we shall complete the proof assuming the following extra condition:

Assumption 1: There exists $\eta$ a final segment of $b_g$ so that $\phi \eta$ is a final segment of $b_h$.
Under this assumption, we note that Eq. (19) gives that
\[ \text{sign}(\eta \tilde{c}_1(\alpha, g), \tilde{c}_2(\alpha, g)) = \text{sign}(\phi \eta \cdot \tilde{c}_1(\alpha, h), \tilde{c}_2(\alpha, h)) \]
since \( \phi \), as a deck transformation, preserves the orientation of \( \tilde{C}_e \). When we combine this with Eqs. (17) and (18), Assumption 1 gives us
\[ \epsilon_g = \epsilon_h \]
as desired.

Now observe that Assumption 1 holds if \( t_g(1) = t_h(1) \) has valence 3 in \( C \): for then \( b_g(1) \) and \( b_h(1) \) would have valence 3 in \( \tilde{C} \), and since \( \phi \) is bijective and satisfies Eq. 19, we get that a small enough final segment of \( b_g \) must be mapped by \( \phi \) to a final segment of \( b_h \). Figure 6 depicts this case.

Changing the basepoint of \( \alpha \) if necessary, we consider the curve \( \gamma \) in Lemma 6.1. By Lemma 6.5 we may replace \( \alpha \) with \( \gamma \) if needed to ensure Assumption 1.

\[ \square \]

We finally complete the proof of Theorem 1.2. Let \( X \in S(M) \) be non-trivial and primitive, and \( \alpha \) be a cyclically reduced representative of \( X \).

Proof of Theorem 1.2: As we pointed out after Definition 4.7, we have that \( LP_1(X) = LP_2(X, X) \cong LP_2(X, X^{-1}) \) which is in correspondence with \( T_0(\alpha) \), recalling Lemma 4.5 and Definition 4.7. We rewrite the formula for \( [X, X^{-1}] \) given in Definition 3.9 using this correspondence, Lemma 5.8 and Definition 6.2, to get
\[ [X, X^{-1}] = \sum_{g \in T_0(\alpha)} \epsilon_g \{[\alpha]g[\alpha]^{-1}g^{-1}\} \quad (20) \]
where \([\alpha]g[\alpha]^{-1}g^{-1}\) stands for the conjugacy class of \([\alpha]g[\alpha]^{-1}g^{-1}\).

We must show that if \( [X, X^{-1}] = 0 \) then \( LP_1(X) = \emptyset \), or equivalently, \( T_0(\alpha) = \emptyset \). We show it by contradiction: we assume that \( [X, X^{-1}] = 0 \) and \( T_0(\alpha) \) is non-empty. Since \( S(M) \) is a free abelian group of basis \( S(M) \), there must be cancellations in the second term of Eq. (20), so there must be \( g, h \in T_0(\alpha) \) with
\[ \{[\alpha]g[\alpha]^{-1}g^{-1}\} = \{[\alpha]h[\alpha]^{-1}h^{-1}\} \quad \text{and} \quad \epsilon_g = -\epsilon_h \]
contradicting Lemma 6.6. \[ \square \]
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