Regular orbit closures in module varieties

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August 1, 2018

Abstract

Let $A$ be a finitely generated associative algebra over an algebraically closed field. We characterize the finite dimensional modules over $A$ whose orbit closures are regular varieties.

1 Introduction and the main result

Throughout the paper $k$ denotes a fixed algebraically closed field. By an algebra we mean an associative finitely generated $k$-algebra with identity, and by a module a finite dimensional left module. Let $d$ be a positive integer and denote by $\mathbb{M}(d)$ the algebra of $d \times d$-matrices with coefficients in $k$. For an algebra $A$ the set $\text{mod}_A(d)$ of the $A$-module structures on the vector space $k^d$ has a natural structure of an affine variety. Indeed, if $A \cong k\langle X_1, \ldots, X_t \rangle/J$ for $t > 0$ and a two-sided ideal $J$, then $\text{mod}_A(d)$ can be identified with the closed subset of $(\mathbb{M}(d))^t$ given by vanishing of the entries of all matrices $\rho(X_1, \ldots, X_t)$ for $\rho \in J$. Moreover, the general linear group $\text{GL}(d)$ acts on $\text{mod}_A(d)$ by conjugation and the $\text{GL}(d)$-orbits in $\text{mod}_A(d)$ correspond bijectively to the isomorphism classes of $d$-dimensional $A$-modules. We shall denote by $\mathcal{O}_M$ the $\text{GL}(d)$-orbit in $\text{mod}_A(d)$ corresponding to (the isomorphism class of) a $d$-dimensional $A$-module $M$. It is an interesting task to study geometric properties of the Zariski closure $\overline{\mathcal{O}_M}$ of $\mathcal{O}_M$. We note that using a geometric equivalence described in [1], this is closely related to a similar problem for representations of quivers. We refer to [2], [3], [4], [5], [6], [9], [10], [11], [12], [13] and [14] for results concerning geometric properties of orbit closures in module varieties or varieties of representations.

Mathematics Subject Classification (2000): 14B05 (Primary); 14L30, 16G20 (Secondary).

Key Words and Phrases: Module varieties, orbit closures, regularity.
The main result of the paper concerns the global regularity of such varieties. Let \( \text{Ann}(M) \) denote the annihilator of a module \( M \). It is the kernel of the algebra homomorphism \( A \to \text{End}_k(M) \) induced by the module \( M \), and therefore the algebra \( B = A/\text{Ann}(M) \) is finite dimensional. Obviously \( M \) can be considered as a \( B \)-module.

**Theorem 1.1.** Let \( M \) be an \( A \)-module and let \( B = A/\text{Ann}(M) \). Then the orbit closure \( \overline{O}_M \) is a regular variety if and only if the algebra \( B \) is hereditary and \( \text{Ext}_B^1(M,M) = 0 \).

Let \( d = \dim_k M \). Observe that \( \text{mod}_B(d) \) is a closed GL\( (d) \)-subvariety of \( \text{mod}_A(d) \) containing \( \overline{O}_M \). Moreover, \( M \) is faithful as a \( B \)-module. Hence we may reformulate Theorem 1.1 as follows:

**Theorem 1.2.** Let \( M \) be a faithful module over a finite dimensional algebra \( B \). Then the orbit closure \( \overline{O}_M \) is a regular variety if and only if the algebra \( B \) is hereditary and \( \text{Ext}_B^1(M,M) = 0 \).

The next section contains a reduction of the proof of Theorem 1.2 to Theorem 2.1 presented in terms of properties of regular orbit closures for representations of quivers. Sections 3 and 4 are devoted to the proof of Theorem 2.1. For basic background on the representation theory of algebras and quivers we refer to [1].

## 2 Representations of quivers

Let \( Q = (Q_0, Q_1; s, t : Q_1 \to Q_0) \) be a finite quiver, i.e. \( Q_0 \) is a finite set of vertices, and \( Q_1 \) is a finite set of arrows \( \alpha : s(\alpha) \to t(\alpha) \). By a representation of \( Q \) we mean a collection \( V = (V_i, V_\alpha) \) of finite dimensional \( k \)-vector spaces \( V_i, i \in Q_0 \), together with linear maps \( V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)} \), \( \alpha \in Q_1 \). The dimension vector of the representation \( V \) is the vector

\[
\dim V = (\dim_k V_i) \in \mathbb{N}^{Q_0}.
\]

By a path of length \( m \geq 1 \) in \( Q \) we mean a sequence of arrows in \( Q_1 \):

\[
\omega = \alpha_m \alpha_{m-1} \ldots \alpha_1,
\]

such that \( s(\alpha_{l+1}) = t(\alpha_l) \) for \( l = 1, \ldots, m - 1 \). In the above situation we write \( s(\omega) = s(\alpha_1) \) and \( t(\omega) = t(\alpha_m) \). We agree to associate to each \( i \in Q_0 \) a path \( \varepsilon_i \) in \( Q \) of length zero with \( s(\varepsilon_i) = t(\varepsilon_i) = i \). The paths of \( Q \) form a \( k \)-linear basis of the path algebra \( kQ \). We define

\[
V_\omega = V_{\alpha_m} \circ V_{\alpha_{m-1}} \circ \ldots \circ V_{\alpha_2} \circ V_{\alpha_1} : V_{s(\omega)} \to V_{t(\omega)}
\]
for a path \( \omega = \alpha_0 \ldots \alpha_1 \) and extend easily this definition to \( V_\rho : V_i \to V_j \) for any \( \rho \) in \( \varepsilon_j \cdot kQ \cdot \varepsilon_i \), where \( i, j \in Q_0 \), as \( \rho \) is a \( k \)-linear combination of paths \( \omega \) with \( s(\omega) = i \) and \( t(\omega) = j \). Finally, we set

\[
\text{Ann}(V) = \{ \rho \in kQ \mid V_{\varepsilon_j \cdot \rho \cdot \varepsilon_i} = 0 \text{ for all } i, j \in Q_0 \},
\]

which is a two-sided ideal in \( kQ \). In fact, it is the annihilator of the \( kQ \)-module induced by \( V \) with underlying \( k \)-vector space \( \bigoplus_{i \in Q_0} V_i \).

Let \( d = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0} \) be a dimension vector. Then the representations \( V = (V_\alpha) \) of \( Q \) with \( V_i = k^{d_i}, i \in Q_0 \), form a vector space

\[
\text{rep}_Q(d) = \bigoplus_{\alpha \in Q_1} \text{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)}) = \bigoplus_{\alpha \in Q_1} \mathcal{M}(d_{t(\alpha)} \times d_{s(\alpha)}),
\]

where \( \mathcal{M}(d' \times d'') \) stands for the space of \( d' \times d'' \)-matrices with coefficients in \( k \). For abbreviation, we denote the representations in \( \text{rep}_Q(d) \) by \( V = (V_\alpha) \).

The group \( \text{GL}(d) = \bigoplus_{i \in Q_0} \text{GL}(d_i) \) acts regularly on \( \text{rep}_Q(d) \) via

\[
(g_i)_{i \in Q_0} \cdot (V_\alpha)_{\alpha \in Q_1} = (g_{t(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1})_{\alpha \in Q_1}.
\]

Given a representation \( W = (W_i, W_\alpha) \) of \( Q \) with \( \text{dim } W = d \), we denote by \( \mathcal{O}_W \) the \( \text{GL}(d) \)-orbit in \( \text{rep}_Q(d) \) of representations isomorphic to \( W \).

Let \( M \) be a faithful module over a finite dimensional algebra \( B \). It is well known that the algebra \( B \) is Morita-equivalent to the quotient algebra \( kQ/I \), where \( Q \) is a finite quiver and \( I \) an admissible ideal in \( kQ \), i.e. \( I \) is a two-sided ideal such that \( (\mathcal{R}_Q)^r \subseteq I \subseteq (\mathcal{R}_Q)^2 \) for some positive integer \( r \), where \( \mathcal{R}_Q \) denotes the two-sided ideal of \( kQ \) generated by the paths of length one (arrows) in \( Q \). Furthermore, the algebra \( B \) is hereditary if and only if \( I = \{0\} \) (in particular, the quiver \( Q \) has no oriented cycles, i.e. paths \( \omega \) of positive lengths with \( s(\omega) = t(\omega) \)). According to the above equivalence, the faithful \( B \)-module \( M \) corresponds to a representation \( N = (N_\alpha) \) in \( \text{rep}_Q(d) \) for some \( d \), such that \( \text{Ann}(N) = I \). Applying the geometric version of the Morita equivalence described by Bongartz in [4], \( \overline{\mathcal{O}}_M \) is isomorphic to an associated fibre bundle \( \text{GL}(d) \times_{\text{GL}(d)} \overline{\mathcal{O}}_N \). In particular, \( \overline{\mathcal{O}}_M \) is regular if and only if \( \overline{\mathcal{O}}_N \) is.

By the Artin-Voigt formula (see [8]):

\[
\text{codim}_{\text{rep}_Q(d)} \overline{\mathcal{O}}_N = \text{dim}_k \text{Ext}^1_Q(N, N),
\]

the vanishing of \( \text{Ext}^1_Q(N, N) \) means that \( \overline{\mathcal{O}}_N = \text{rep}_Q(d) \). Consequently, one implication in Theorem [12] is proved and it suffices to show the following fact:

**Theorem 2.1.** Let \( N \) be a representation in \( \text{rep}_Q(d) \) such that \( \text{Ann}(N) \) is an admissible ideal in \( kQ \) and \( \overline{\mathcal{O}}_N \) is a regular variety. Then \( \text{Ann}(N) = \{0\} \) and \( \overline{\mathcal{O}}_N = \text{rep}_Q(d) \).
3 Tangent spaces of orbit closures and nilpotent representations

From now on, \( N \) is a representation in \( \text{rep}_Q(d) \) such that \( \text{Ann}(N) \) is an admissible ideal in \( kQ \) and \( \mathcal{O}_N \) is a regular variety. The aim of the section is to prove that the quiver \( Q \) has no oriented cycles.

Let \( S[j] = (S[j]_i, S[j]_{\alpha}) \) stand for the simple representation of \( Q \) such that \( S[j]_j = k \) is the only non-zero vector space and all linear maps \( S[j]_{\alpha} \) are zero, for any vertex \( j \in Q_0 \). Observe that the point 0 in \( \text{rep}_Q(d) \) is the semisimple representation \( \bigoplus_{i \in Q_0} S[i]^d \). A representation \( W = (W_i, W_\alpha) \) of \( Q \) is said to be nilpotent if one of the following equivalent conditions is satisfied:

1. The endomorphism \( W_\omega \in \text{End}_k(W_{s(\omega)}) \) is nilpotent for any oriented cycle \( \omega \) in \( Q \).
2. The ideal \( \text{Ann}(W) \) contains \( (R_Q)_r \) for some positive integer \( r \).
3. Any composition factor of \( W \) is isomorphic to some \( S[i], i \in Q_0 \).
4. The orbit closure \( \mathcal{O}_W \) in \( \text{rep}_Q(d) \) contains 0.

Obviously the representation \( N \) is nilpotent. Thus the set \( \mathcal{N}_Q(d) \) of nilpotent representations in \( \text{rep}_Q(d) \) is a closed \( \text{GL}(d) \)-invariant subset which contains \( \mathcal{O}_N \). Furthermore, \( \mathcal{N}_Q(d) \) is a cone, i.e. it is invariant under multiplication by scalars in the vector space \( \text{rep}_Q(d) \).

We shall identify the tangent space \( T_{\text{rep}_Q(d),0} \) of \( \text{rep}_Q(d) \) at 0 with \( \text{rep}_Q(d) \) itself. Thus the tangent space \( T_{\mathcal{O}_N,0} \) is a subspace of \( \text{rep}_Q(d) \) and is invariant under the action of \( \text{GL}(d) \), i.e. it is a \( \text{GL}(d) \)-subrepresentation of \( \text{rep}_Q(d) \). Since \( \mathcal{O}_N \) is a regular variety, the tangent space \( T_{\mathcal{O}_N,0} \) is the tangent cone of \( \mathcal{O}_N \) at 0 (see \([7, \text{III.4}]\) ), and the latter is contained in the tangent cone of \( \mathcal{N}_Q(d) \) at 0. Therefore

\[
T_{\mathcal{O}_N,0} \subseteq \mathcal{N}_Q(d). \tag{3.1}
\]

**Lemma 3.1.** Let \( W = (W_\alpha) \) be a tangent vector in \( T_{\mathcal{O}_N,0} \). Then \( W_\gamma = 0 \) for any loop \( \gamma \in Q_1 \).

**Proof.** Suppose that the nilpotent matrix \( W_\gamma \in M(d_j) \) is non-zero for some loop \( \gamma : j \to j \) in \( Q_1 \). Then there are two linearly independent vectors \( v_1, v_2 \in k^{d_j} \) such that \( W_\gamma \cdot v_1 = v_2 \) and \( W_\gamma \cdot v_2 = 0 \). We choose \( g = (g_i) \) in \( \text{GL}(d) \) such that \( g_j \cdot v_1 = v_2 \) and \( g_j \cdot v_2 = v_1 \). Then \( U = W + g \cdot W \) belongs to \( T_{\mathcal{O}_N,0} \). Observe that \( U_\gamma \cdot v_1 = v_2 \) and \( U_\gamma \cdot v_2 = v_1 \). Hence the representation \( U \) is not nilpotent, contrary to (3.1). \( \square \)
Let $V_i = k^{d_i}$ and $R_{i,j}$ be the vector space of formal linear combinations of arrows $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$, for any $i,j \in Q_0$. We shall identify:

$$\text{rep}_{Q}(d) = \bigoplus_{i,j \in Q_0} \text{Hom}_k(R_{i,j}, \text{Hom}_k(V_i, V_j)) \quad \text{and} \quad \text{GL}(d) = \bigoplus_{i \in Q_0} \text{GL}(V_i).$$

Applying Lemma 3.1 we get

$$\mathcal{T}_{N,0} \subseteq \bigoplus_{i,j \in Q_0, i \neq j} \text{Hom}_k(R_{i,j}, \text{Hom}_k(V_i, V_j)).$$

Since the $\text{GL}(d)$-representations $\text{Hom}_k(V_i, V_j)$, $i \neq j$, are simple and pairwise non-isomorphic, we have

$$\mathcal{T}_{N,0} = \bigoplus_{i,j \in Q_0, i \neq j} \{ \varphi : R_{i,j} \to \text{Hom}_k(V_i, V_j) \mid \varphi(U_{i,j}) = 0 \}$$

for some subspaces $U_{i,j}$ of $R_{i,j}$, $i \neq j$.

The spaces $U_{i,j}$ are not necessarily spanned by arrows $\alpha : i \to j$ in $Q_1$, and we are going to replace $N$ by a “better” representation in $\text{rep}_Q(d)$. The group $\tilde{G} = \bigoplus_{i,j \in Q_0} \text{GL}(R_{i,j})$ can be identified naturally with a subgroup of automorphisms of the path algebra $kQ$ which change linearly the paths of length 1 but do not change the paths of length 0. Let $\tilde{g} = (\tilde{g}_{i,j})$ be an element of $\tilde{G}$. Then $\tilde{g} \ast (\mathcal{R}_Q)^p = (\mathcal{R}_Q)^p$ for any positive integer $p$, where $\ast$ denotes the action of $\tilde{G}$ on $kQ$. For a representation $W$ of $Q$ presented in the form

$$W = (W_i, W_{i,j} : R_{i,j} \to \text{Hom}_k(W_i, W_j))_{i,j \in Q_0},$$

we define the representation

$$\tilde{g} \ast W = (W_i, W_{i,j} \circ (\tilde{g}_{i,j})^{-1})_{i,j \in Q_0}.$$ 

Hence $\tilde{G}$ acts regularly on $\text{rep}_Q(d)$ and this action commutes with the $\text{GL}(d)$-action. Therefore the orbit closure $\mathcal{O}_{\tilde{g} \ast N} = \tilde{g} \ast \mathcal{O}_N$ is a regular variety, $\mathcal{T}_{\mathcal{O}_{\tilde{g} \ast N},0} = \tilde{g} \ast \mathcal{T}_{\mathcal{O}_N,0}$ and the ideal $\text{Ann}(\tilde{g} \ast N) = \tilde{g} \ast \text{Ann}(N)$ is admissible as

$$(\mathcal{R}_Q)^* = \tilde{g} \ast (\mathcal{R}_Q)^* \subseteq \tilde{g} \ast \text{Ann}(N) \subseteq \tilde{g} \ast (\mathcal{R}_Q)^2 = (\mathcal{R}_Q)^2.$$

Hence, replacing $N$ by $\tilde{g} \ast N$ for an appropriate $\tilde{g}$, we may assume that the spaces $U_{i,j}$, $i \neq j$, are spanned by arrows in $Q_1$. Consequently,

$$\mathcal{T}_{\mathcal{O}_N,0} = \text{rep}_{Q'}(d) \subseteq \text{rep}_Q(d)$$

for some subquiver $Q'$ of $Q$ such that $Q'_0 = Q_0$ and $Q'_1$ has no loops.
Lemma 3.2. The quiver $Q'$ has no oriented cycles.

Proof. Suppose there is an oriented cycle $\omega$ in $Q'$. Let $W = (W_\alpha)$ be a tangent vector in $T_{Q_0} = \text{rep}_Q(d)$ such that each $W_\alpha$, $\alpha \in (Q')_1$, is the matrix whose $(1,1)$-entry is 1, while the other entries are 0. Then the matrix $W_\omega$ has the same form, contrary to Lemma 3.3. \hfill \Box

Let $W = (W_t, W_a)$ be a representation of $Q$. We denote by $\text{rad}(W)$ the radical of $W$. In case $W$ is nilpotent, $\text{rad}(W) = \sum_{\alpha \in Q_1} \text{Im}(W_\alpha)$. We write $\langle w \rangle$ for the subrepresentation of $W$ generated by a vector $w \in \bigoplus_{i \in Q_0} W_i$.

Lemma 3.3. Let $\alpha : i \to j$ be an arrow in $Q_1$ such that $N_\alpha(v)$ does not belong to $\text{rad}^2 \langle v \rangle$ for some $v \in V_i$. Then $\alpha \in Q_1'$.

Proof. Let $d = \sum_{i \in Q_0} d_i$ and $c = \dim_k \langle v \rangle$. Then $\dim_k \text{rad} \langle v \rangle = c - 1$ and $d \geq c \geq 2$. Since $N_\alpha(v)$ does not belong to $\text{rad} \langle \text{rad} \langle v \rangle \rangle$, there is a codimension one subrepresentation $W$ of $\text{rad} \langle v \rangle$ which does not contain $N_\alpha(v)$. We choose a basis $\{\epsilon_1, \ldots, \epsilon_d\}$ of the vector space $\bigoplus_{i \in Q_0} V_i$ such that:

- the vector $\epsilon_b$ belongs to $V_{i_b}$ for some vertex $i_b \in Q_0$, for any $b \leq d$;
- the vectors $\epsilon_1, \ldots, \epsilon_d$ span a subrepresentation, say $N(b)$, of $N$ for any $b \leq d$; 
- $N(c-2) = W$, $\epsilon_{c-1} = N_\alpha(v)$, $N(c-1) = \text{rad} \langle v \rangle$, $\epsilon_c = v$ and $N(c) = \langle v \rangle$.

In fact, $0 = N(0) \subset N(1) \subset N(2) \subset \cdots \subset N(d) = N$ is a composition series of $N$. In particular, $N_\beta(\epsilon_b)$ belongs to $N(b-1)$, for any $b \leq d$ and any arrow $\beta : i_b \to j$ in $Q_1$. We take a decreasing sequence of integers

$$p_1 > p_2 > \ldots > p_d$$

and define a group homomorphism $\varphi : k^* \to \text{GL}(d) = \bigoplus_{i \in Q_0} \text{GL}(V_i)$ such that $\varphi(t)(\epsilon_b) = t^{p_b} \cdot \epsilon_b$ for any $b \leq d$. Observe that

$$N_\beta(\epsilon_b) = \sum_{i < b} \lambda_i \cdot \epsilon_i,$$

for any $b \leq d$ and any arrow $\beta : i_b \to j$ in $Q_1$. This leads to a regular map

$$\psi : k \to T_{QN}$$

such that $\psi(t) = \varphi(t) \ast N$ for $t \neq 0$ and $\psi(0) = 0$.

Assume now that $p_{c-1} - p_c = 1$. Applying the induced linear map $T_{\psi,0} : T_{Q_0} \to T_{QN}$, and using the fact that $N_\alpha(\epsilon_c) = \epsilon_{c-1}$, we obtain a tangent vector $W = (W_\alpha) \in T_{QN}$ such that $W_\alpha(\epsilon_c) = \epsilon_{c-1} \neq 0$. Thus $\alpha \in Q_1$'.

Lemma 3.4. For any arrow $\alpha : i \to j$ in $Q_1$, there exists a path $\omega$ in $Q'$ of positive length such that $s(\omega) = i$ and $t(\omega) = j$. 

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Proof. Since $\text{Ann}(N)$ is an admissible ideal in $kQ$, there is a vector $v \in V_i$ such that $N_\alpha(v) \neq 0$. Let $\omega = \alpha_m \ldots \alpha_2 \alpha_1$ be a longest path from $i$ to $j$ with $N_\omega(v) \neq 0$. Hence $N_\rho(v) = 0$ for any $\rho \in \epsilon_j \cdot (R_Q)^m \cdot \epsilon_i$. We show that the path $\omega$ satisfies the claim. Let $v_0 = v$ and $v_l = N_{\alpha_l}(v_{l-1})$ for $l = 1, \ldots, m$. According to Lemma 3.3 it is enough to show that $v_l \not\in \text{rad}^2 \langle v_{l-1} \rangle$ for any $1 \leq l \leq m$. Indeed, if $v_l \in \text{rad}^2 \langle v_{l-1} \rangle$ for some $l$, then $v_m \in \text{rad}^{m+1} \langle v_0 \rangle$, or equivalently, $N_{\omega}(v) = N_\rho(v)$ for some $\rho \in \epsilon_j \cdot (R_Q)^m \cdot \epsilon_i$, a contradiction. \qed

Combining Lemmas 3.2 and 3.4 we get

**Corollary 3.5.** The quiver $Q$ does not contain oriented cycles.

### 4 Gradings of polynomials on $\text{rep}_Q(d)$

Let $\pi : \text{rep}_Q(d) \to \text{rep}_{Q'}(d)$ denote the obvious $\text{GL}(d)$-equivariant linear projection and let $N' = \pi(N)$. Then $\pi(O_N) = O_{N'}$, and we get a dominant morphism

$$\eta = \pi|_{\mathcal{O}_N} : \mathcal{O}_N \to \mathcal{O}_{N'}.$$  

**Lemma 4.1.** $\mathcal{O}_{N'} = \text{rep}_{Q'}(d)$. 

Proof. Since $\text{Ker}(\pi) \cap \mathcal{T}_{\mathcal{O}_d,0} = \{0\}$, the morphism $\eta$ is étale at $0$. This implies that the variety $\mathcal{O}_{N'}$ is regular at $\eta(0) = 0$ (see [7] III.5 for basic information about étale morphisms). Since it is contained in $\text{rep}_{Q'}(d)$, it suffices to show that $\mathcal{T}_{\mathcal{O}_{N'},0} = \text{rep}_{Q'}(d)$. The latter can be concluded from the induced linear map $\mathcal{T}_{\eta,0} : \mathcal{T}_{\mathcal{O}_d,0} \to \mathcal{T}_{\mathcal{O}_{N'},0}$, which is the restriction of $\mathcal{T}_{\eta,0} = \pi$. \qed

Let $R = k[X_{\alpha,p,q}]_{\alpha \in Q_1, p \leq d_{\alpha}(d), q \leq d_{\alpha}(d)}$ denote the algebra of polynomial functions on the vector space $\text{rep}_Q(d)$ and $m = (X_{\alpha,p,q})$ be the maximal ideal in $R$ generated by variables. Here, $X_{\beta,p,q}$ maps a representation $W = (W_\alpha)$ to the $(p,q)$-entry of the matrix $W_\beta$. Using $\pi$, the polynomial functions on $\text{rep}_Q(d)$ form the subalgebra $R' = k[X_{\alpha,p,q}]_{\alpha \in Q_1, p \leq d_{\alpha}(d), q \leq d_{\alpha}(d)}$ of $R$. By Lemma 4.1

$$I(\mathcal{O}_N) \cap R' = \{0\},$$  

where $I(\mathcal{O}_N)$ stands for the ideal of the set $\mathcal{O}_N$ in $R$.

Let $X_\alpha$ denote the $d_{\alpha}(d) \times d_{\alpha}(d)$-matrix whose $(p,q)$-entry is the variable $X_{\alpha,p,q}$, for any arrow $\alpha$ in $Q_1$. We define the $d_j \times d_i$-matrix $X_\rho$ for $\rho \in \epsilon_j \cdot kQ \cdot \epsilon_i$, with coefficients in $R$, in a similar way as for representations of $Q$.

The action of $\text{GL}(d)$ on $\text{rep}_Q(d)$ induces an action on the algebra $R$ by $(g \ast f)(W) = f(g^{-1} \ast W)$ for $g \in \text{GL}(d), f \in R$ and $W \in \text{rep}_Q(d)$. We choose a standard maximal torus $T$ in $\text{GL}(d)$ consisting of $g = (g_i)$, where

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all $g_i \in \text{GL}(d_i)$ are diagonal matrices. Let $\widetilde{Q}_0$ denote the set of pairs $(i, p)$ with $i \in Q_0$ and $1 \leq p \leq d_i$. Then the action of $T$ on $R$ leads to a $\mathbb{Z}^{\widetilde{Q}_0}$-grading on $R$ with
\[
\deg(X_{\alpha,p,q}) = e_{s(\alpha),q} - e_{t(\alpha),p},
\]
where $\{e_{i,p}\}_{(i,p) \in \widetilde{Q}_0}$ is the standard basis of $\mathbb{Z}^{\widetilde{Q}_0}$.

**Proposition 4.2.** $Q' = Q$.

**Proof.** Suppose the contrary, which means there is an arrow $\beta$ in $Q_1 \setminus Q'_1$. Since the quiver $Q$ has no oriented cycles, we can choose $\beta$ minimal in the sense that any path $\omega$ in $Q$ of length greater than 1 with $s(\omega) = s(\beta)$ and $t(\omega) = t(\beta)$ is in fact a path in $Q'$. We conclude from (3.2) that $X_{\beta,u,v} \in m^2 + I(\overline{Q}^N)$ for $u \leq d_t(\beta)$ and $v \leq d_s(\beta)$. Since the polynomials $X_{\beta,u,v}$ as well as the ideals $m^2$ and $I(\overline{Q}^N)$ are homogeneous with respect to the above grading, there are homogeneous polynomials $f_{\beta,u,v}$ in the ideal $m^2$ such that
\[
X_{\beta,u,v} - f_{\beta,u,v} \in I(\overline{Q}^N) \quad \text{and} \quad \deg(f_{\beta,u,v}) = \deg(X_{\beta,u,v}) = e_{s(\beta),v} - e_{t(\beta),u}.
\]

Let $\prod_{i \leq n} X_{\alpha_i,p_i,q_i}$ be a monomial in $R$ of degree $e_{s(\beta),v} - e_{t(\beta),u}$. Then
\[
\#\{1 \leq l \leq n | s(\alpha_l) = i, q_l = r\} - \#\{1 \leq l \leq n | t(\alpha_l) = i, p_l = r\} = \begin{cases} 1 & (i, r) = (s(\beta), v), \\
-1 & (i, r) = (t(\beta), u), \\
0 & \text{otherwise}. \end{cases}
\]

Thus by (1.2), up to a permutation of the above variables, we get that $\omega = \alpha_m \ldots \alpha_1$ is a path in $Q$ for some $m \leq n$ such that $(s(\alpha_1), q_1) = (s(\beta), v)$, $(t(\alpha_m), p_m) = (t(\beta), u)$ and $q_l = p_{l-1}$ for $l = 2, \ldots, m$. Consequently, $\deg(X_{\alpha_{m+1},p_{m+1},q_{m+1}} \cdot \ldots \cdot X_{\alpha_{m},p_{m},q_{m}}) = 0$. Since $Q$ has no oriented cycles, the only monomial in $R$ with degree zero is the constant function 1. Hence $m = n$ and the homogeneous polynomial $f_{\beta,u,v}$ is the following linear combination:
\[
f_{\beta,u,v} = \sum \lambda(u, \alpha_m, p_{m-1}, \alpha_{m-1}, \ldots, p_1, \alpha_1, v) \cdot X_{\alpha_m,u,p_{m-1}} \cdot X_{\alpha_{m-1},p_{m-1},p_{m-2}} \cdot \ldots \cdot X_{\alpha_2,p_2,p_1} \cdot X_{\alpha_1,p_1,v},
\]
where the sum runs over all paths $\omega = \alpha_m \ldots \alpha_1$ in $Q$ with $s(\omega) = s(\beta)$, $t(\omega) = t(\beta)$ and positive integers $p_l \leq d_t(\alpha_l)$ for $l = 1, \ldots, m - 1$. Since $f_{\beta,u,v}$ belongs to the ideal $m^2$, we may assume that $m \geq 2$. Then the arrows $\alpha_1, \ldots, \alpha_m$ belong to $Q'_1$, by the minimality of $\beta$. In particular, $f_{\beta,u,v}$ belongs to $R'$. 

8
We claim that the scalars \(\lambda(u, \alpha_m; p_{m-1}, \alpha_{m-1}, \ldots, p_1, \alpha_1, v)\) do not depend on the integers \(u, p_{m-1}, \alpha_{m-1}, \ldots, p_1, \alpha_1, v\). Indeed, take \(u' \leq d_{t(\beta)}, v' \leq d_{s(\beta)}\) and \(p'_l \leq d_{t(\alpha_l)}\) for \(l = 1, \ldots, m-1\). We choose \(g = (g_l)\) in \(\text{GL}(d)\) with each \(g_l\) being the permutation matrix associated to a specific permutation \(\sigma_l \in S_d\). Then the multiplication by \(g\) in the algebra \(R\) permutes the monomials in \(R\).

We assume that
\[
\sigma_{s(\beta)}(v) = v', \quad \sigma_{s(\beta)}(v') = v, \quad \sigma_{t(\beta)}(u) = u', \quad \sigma_{t(\beta)}(u') = u,
\]
\[
\sigma_{t(\alpha_l)}(p_l) = p'_l \quad \text{and} \quad \sigma_{t(\alpha_l)}(p'_l) = p_l, \quad \text{for} \ l = 1, \ldots, m-1.
\]

Since \(g \ast X_{\beta, u', v'} = X_{\beta, u, v}\), the polynomial
\[
f_{\beta, u, v} - g \ast f_{\beta, u', v'} = g \ast (X_{\beta, u', v'} - f_{\beta, u', v'}) - (X_{\beta, u, v} - f_{\beta, u, v})
\]
belongs to the ideal \(I(\overline{\mathcal{O}}_N)\), as the latter is \(\text{GL}(d)\)-invariant. Thus \(f_{\beta, u, v} = g \ast f_{\beta, u', v'}\), by (4.3). Hence the claim follows from the fact that the monomial
\[
X_{\alpha_m, u, p_{m-1}} \cdot X_{\alpha_{m-1}, p_{m-1}, p_{m-2}} \cdot \cdots \cdot X_{\alpha_2, p_2, p_1} \cdot X_{\alpha_1, p_1, v}
\]
appears in \(g \ast f_{\beta, u', v'}\) with coefficient \(\lambda(u', \alpha_m, p'_{m-1}, \alpha_{m-1}, \ldots, p'_1, \alpha_1, v')\).

Let \(\Xi\) denote the set of all paths \(\xi \in Q'\) of length greater than 1 with \(s(\xi) = s(\beta)\) and \(t(\xi) = t(\beta)\). Then there are scalars \(\lambda(\xi), \xi \in \Xi\), such that
\[
f_{\beta, u, v} = \sum_{\xi = \alpha_m, \ldots, \alpha_1 \in \Xi} \lambda(\xi) \cdot \sum_{p_1 \leq d_{t(\alpha_1)}} \cdots \sum_{p_{m-1} \leq d_{t(\alpha_{m-1})}} X_{\alpha_m, u, p_{m-1}} \cdot \cdots \cdot X_{\alpha_1, p_1, v}
\]
for any \(u \leq d_{t(\beta)}\) and \(v \leq d_{s(\beta)}\). This equality means that \(f_{\beta, u, v}\) is the \((u, v)\)-entry of the matrix \(X_{\rho}\), where \(\rho = \sum_{\xi \in \Xi} \lambda(\xi) \cdot \xi \in kQ'\). Consequently, the entries of the matrix \(X_{\beta - \rho}\) belong to the ideal \(I(\overline{\mathcal{O}}_N)\). This implies that \(\beta - \rho\) does not belong to \((R_Q)^2\), the ideal \(\text{Ann}(N)\) is not admissible, a contradiction.

Combining Lemma 4.1 and Proposition 4.2 we get
\[
\overline{\mathcal{O}}_N = \text{rep}_Q(d).
\] (4.3)

Hence the following lemma finishes the proof of Theorem 2.1.

**Lemma 4.3.** \(\text{Ann}(N) = \{0\}\).

**Proof.** Suppose the contrary, that there is a non-zero element \(\rho\) in \(\varepsilon_j \cdot \text{Ann}(N)\). \(\varepsilon_i\) for some vertices \(i\) and \(j\). Observe that the set of representations \(W = (W_\rho)\) in \(\text{rep}_Q(d)\) such that \(W_\rho = 0\) is closed and \(\text{GL}(d)\)-invariant. Hence \(W_\rho = 0\) for any representation \(W = (W_\alpha)\) in \(\text{rep}_Q(d)\), by (4.3). Of course, \(\rho\)
is a linear combination of paths in $Q$ of length greater than 1 with $s(\omega) = i$ and $t(\omega) = j$. Let $\omega_0$ be a path appearing in $\rho$ with coefficient $\lambda \neq 0$. We choose a representation $W = (W_\alpha)$ in $\mathrm{rep}_Q(d)$ such that $W_\alpha$ is the matrix whose $(1, 1)$-entry is 1 and the other entries are 0 if the arrow $\alpha$ appears in the path $\omega_0$, and $W_\alpha = 0$ otherwise. Then the $(1, 1)$-entry of $W_\rho$ equals $\lambda$, a contradiction. 

\section*{Acknowledgments}

The second author gratefully acknowledges support from the Polish Scientific Grant KBN No. 1 P03A 018 27.

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