PT symmetric Hamiltonian model and Dirac equation in 1+1 dimensions

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Abstract
In this paper, we have introduced a PT symmetric non-Hermitian Hamiltonian model which is given as
\[ \hat{H} = \omega (\hat{b}\hat{b}^\dagger + \frac{1}{2}) + \alpha (\hat{b}^2 - (\hat{b}^\dagger)^2) \]
where \( \omega \) and \( \alpha \) are real constants, \( \hat{b} \) and \( \hat{b}^\dagger \) are first-order differential operators. The Hermitian form of the Hamiltonian \( \hat{H} \) is obtained by suitable mappings and it is interrelated to the time-independent one-dimensional Dirac equation in the presence of position-dependent mass. Then, Dirac equation is reduced to a Schrödinger-like equation and two new complex non-PT symmetric vector potentials are generated. We have obtained a real spectrum for these new complex vector potentials using the shape invariance method. We have searched the real energy values using numerical methods for the specific values of the parameters.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The nature of quantization arises due to the symmetry of the equations governing the physics. So, symmetry has long been known as a powerful and computational topic in quantum mechanics. Recently, there have been many studies of PT symmetric non-Hermitian systems with real energy since the original work of Bender and Boettcher [1] and the literature on such systems has expanded rapidly [2–7]. One of the key points of the investigation was the PT symmetry generated by the product of the parity, \( P \), and time, \( T \), linear and anti-linear inversion operators

\[ P x P = -x, \quad T x T = x, \quad T i T = -i. \]

The operator \( T \) is anti-linear because it changes the sign of \( i \). If PT symmetry of the Hamiltonian is unbroken; the eigenfunction of the operator \( \hat{P} \) is simultaneously an eigenstate of Hamiltonian \( \hat{H} \), i.e. \( [\hat{H}, \hat{P}] = 0 \). Later, it was realized that the existence of real eigenvalues can be associated with a non-Hermitian Hamiltonian provided it is \( \eta \)-pseudo-Hermitian [8]:

\[ \eta \hat{H} = \hat{H}^\dagger \eta \]

where \( \eta \) is a Hermitian linear automorphism which can be given as \( \eta = (O O^\dagger)^{-1} \), \( O \) is a linear invertible operator. Here, the Hilbert space equipped with the inner product \( \langle ., \eta . \rangle \) is identified as the physical Hilbert space.
space. And the observable $\Theta$ which is the element of physical Hilbert space is related to the Hermitian operator $\theta$ by means of a similarity transformation $\Theta = \rho^{-1}\theta\rho$ where $\rho = \eta^2$. At the same time Bagchi and Quesne have established the twin concepts of pseudo-Hermiticity and weak-pseudo-Hermiticity \cite{9}. Thus, the concept of pseudo-Hermiticity has attracted much interest on behalf of physicists \cite{10–13}.

In \cite{14}, the Kepler problem solutions are investigated in Dirac theory for the particle whose mass is position dependent and the effective mass is given in the form of a multipole expansion, existence of the bound states are discussed in detail. Earlier, using a standard expansion of radial functions as a different approach, the Dirac equation spectrum was obtained for the mixed potentials \cite{15}. The results of \cite{14}, which are about large quantum numbers not leading to inverse mass and momentum independent energy, are also consistent with those found in \cite{16}. In \cite{17}, Dirac matrices $\hat{\alpha}$ and $\hat{\beta}$ have space factors as $f$ and $f_1$ functions where $f$ is responsible for the deformation of the Heisenberg algebra for the coordinates and momentum operators, $f_1$ is responsible for the dependence of the particle mass on its position. Exact solutions are found for the fermion in a Coulomb field with the function $f$ which depends on $r$ linearly while the function $f_1$ depends on $r$ inversely. It is pointed out that the spectrum results of \cite{17} can be useful for nanoheterosystems. Similar arguments about the Dirac oscillator with deformed commutation relations leading to the existence of the minimal length of space can be found in \cite{18}.

Lately, non-Hermitian potentials for the fermions have been studied in the literature \cite{19–23} and fermion models interacting with $PT$ symmetric potentials in the presence of effective mass have attracted interest \cite{24–30}. In \cite{31}, the one-dimensional effective mass Dirac equation bound states are studied within the interactions of non-$PT$-symmetric, and non-Hermitian, exponential type potentials. Moreover, $(1 + 1)$ Dirac equation with position-dependent mass (PDM) and complexified Lorentz scalar interactions is discussed through supersymmetric quantum mechanics (SUSY QM) \cite{32}. More references about the complex potentials and Dirac theory can be found in \cite{33}.

Also, pseudo-Hermitian interaction in relativistic quantum mechanics is studied with the positive definite metric operator $\eta$ calculations for the state vectors \cite{34}. Using the spin and pseudo-spin concept, the spectrum of $PT$ symmetric Rosen–Morse potential is studied and analytical methods are used in \cite{35}. The Dirac equation with position-dependent effective mass transformed into a Schrödinger-like equation is studied in a general context and Lévi’s method is used \cite{36}. SUSY QM provides elegant procedures to solve some classes of potentials with unbroken SUSY and shape invariance (SI), which is one of the standard ways, and it is known that the potential algebras of these systems have been investigated to find exact solutions \cite{37–46}. SUSY QM methods and relativistic extensions have been used by many authors \cite{47–50}.

The purpose of the present paper is to explore new relativistic complex vector potentials of the non-Hermitian bosonic Hamiltonians which may be unsolvable and map them into solvable but real effective potentials. In the literature, bosonic/fermionic Hamiltonians with two modes have physical importance, such as the Jaynes–Cummings model in solid-state physics \cite{51}, Bose–Einstein condensate \cite{52}, squeezed states in a condensate of ultracold bosonic atoms confined by a double-well potential \cite{53}.

Using the methods of SUSY QM, we have obtained solutions of complex vector potentials and showed that in the Dirac equation, decomposing the vector potential into the real and imaginary parts leads to derivation of both exactly and conditionally exactly solvable potentials. The paper is organized as follows. In section 2, a non-Hermitian Hamiltonian model is introduced by us and mapped into its Hermitian form. Shape invariance, which is one of the effective tools in SUSY QM, is given briefly in section 3. Section 4 includes the
mapping of the Dirac equation into a Schrödinger-like equation and obtaining new complex and effective potentials with their exact solutions.

2. The non-Hermitian model and Hermitian equivalents

Previous works by the author have included many aspects of a non-Hermitian su(2) Hamiltonian known as the Swanson Hamiltonian [12, 54–58]. The Swanson Hamiltonian is given by

$$H = \omega(a^\dagger a + 1/2) + \alpha a^2 + \beta a^2,$$

where \(a^\dagger\), \(a\) are annihilation and creation operators, \(\omega, \alpha\) and \(\beta\) are real constants. In this paper, we let consider a \(\mathcal{PT}\) symmetric non-Hermitian model with two parameters given by

$$\mathcal{H} = \omega \left( \frac{\beta}{\rho} + \frac{i}{2} \right) + \alpha (\beta^2 - (\beta^2))$$

where \(\dagger\) is Hermitian adjoint, \(\beta\) is the annihilation operator given in a general form

$$\beta = A(x) \frac{d}{dx} + B(x)$$

and \(A(x), B(x)\) are real functions. The \(\mathcal{PT}\) operator has an effect as \(x \to -x, p \to p\) and \(i \to -i\) in the Hamiltonian, if the operators are taken as \(\beta = \frac{\beta}{\rho} + \frac{i}{2} \rho\), it can be seen that the Hamiltonian is \(\mathcal{PT}\) symmetric. Now, in terms of differential operators, (1) becomes

$$\mathcal{H} = -\omega A(x)^3 \frac{d^2}{dx^2} + (4\omega A(x)B(x) - 2\omega A(x)A(x)) \frac{d}{dx}$$

$$- (\omega - 2\alpha)A(x)B(x) - (\omega - 2\alpha)A(x)B(x)$$

$$+ \omega B(x)^2 \alpha (A(x)A(x)^2 + (A(x)^2)^2 + \frac{\omega}{2}}.$$ (3)

We may write the eigenvalue equation for (1) as given below

$$\mathcal{H}\psi = \epsilon\psi.$$ (4)

Here, the pseudo-Hermitian Hamiltonian (3) can be mapped into a Hermitian operator form by using a mapping function \(\rho\)

$$h = \rho \mathcal{H}\rho^{-1}$$

where

$$\rho = e^{-\frac{2i}{\omega} \int \frac{dx}{A(x)}}.$$ (6)

Here we note that \(h\psi = \epsilon\psi, \psi = \rho^{-1}\xi\). So we can introduce operator \(h\) which is Hermitian equivalent of \(\mathcal{H}\) as

$$h = -\omega A(x)^3 \frac{d}{dx} + U_{\text{eff}}(x)$$

(7)

here \(U_{\text{eff}}(x)\) takes the form

$$U_{\text{eff}}(x) = \frac{\omega}{2} - \omega (A(x)B(x)) \alpha (A(x)B(x)) + A(x)^2 (x) A(x)^2 + \left( \omega + \frac{4\alpha^2}{\omega} \right) B^2(x),$$

(8)

where the primes denote the derivatives. Then (7) can be mapped into a Schrödinger-like form by using

$$\xi(x) = \frac{1}{A(x)} \Phi(x).$$ (9)

Hence, Schrödinger-like equation becomes

$$-\Phi''(x) + \left( \frac{\omega\alpha}{\omega^2} - \frac{A(x)B(x)}{A^2(x)} \right) \Phi(x) + \omega A(x) A(x) \Phi(x) = \frac{\omega}{\omega^2} \left( \Phi(x) \right)$$

(10)
3. Shape invariance

It is very well known that a quantum system having a square-integrable ground state with finite/infinite discrete energy levels \( E_0 < E_1 < E_2 < \cdots \) where the ground-state energy is chosen to be zero \( E_0 = 0 \) is a fundamental idea in SUSY QM. Generally, we can denote the positive semi-definite Hamiltonian by \( \mathbb{H} \) which can be given in a factorized form [46]:

\[
\mathbb{H} = A^\dagger A = -\frac{d^2}{dx^2} + v(x) \tag{11}
\]

\[
A = \frac{d}{dx} - W(x), \quad A^\dagger = -\frac{d}{dx} + W(x) \tag{12}
\]

\[
v^\pm(x) = W^2(x) \pm W'(x). \tag{13}
\]

We used the unit system \( \hbar = 2m = 1 \). Here \( W(x) \) is the function which is real and smooth known as the superpotential and the ground-state wavefunction \( \zeta_0(x) = e^{-\int^x W(y) \, dy} \) is nodeless. It is noted that \( A\zeta_0(x) = 0 \). In this approach, the potential depends on a set of parameters \( a = (a_0, a_1, a_2, \ldots) \) to be expressed by \( W(x, a), A(a), E(a), \ldots \). The shape invariance condition is

\[
A(a)A^\dagger (a) = A^\dagger (a + \Delta)A(a + \Delta) + E_1(a), \tag{14}
\]

in which \( \Delta \) is the shift of the parameters. The entire set of discrete eigenvalues and corresponding eigenfunctions are \( E_n(a) \) and \( \zeta_n(x, a) \) and can be written as

\[
E_n(a) = \sum_{k=0}^{n-1} E_1(a + k\Delta) \tag{15}
\]

\[
\zeta_n(x, a) \sim A^\dagger (a)A^\dagger (a + \Delta) \cdots A^\dagger (a + (n - 1)\Delta) e^{-\int^x W(y, a+n\Delta)}. \tag{16}
\]

4. Dirac equation

The Dirac equation which plays an important role in relativistic quantum mechanics describes relativistic effects due to the speed and spin of particles. The one-dimensional time-independent Dirac equation with effective mass \( M(x) \) and vector potential \( V(x) \) is

\[
(\bar{\alpha} \cdot \vec{p} + \bar{\beta} M(x) + \bar{V} \hat{I})\Psi(x) = E\hat{I}\Psi(x), \tag{17}
\]

where \( \Psi \) is the two component spinor wavefunction, \( E \) is the energy, \( \vec{p} \) is the momentum operator, \( M(x) \) denotes the position-dependent mass and \( \bar{\alpha} \) and \( \bar{\beta} \) are \( 2 \times 2 \) Dirac matrices in standard representation and \( \hbar = c = 1 \) atomic units are chosen. Let us show the upper and lower components by \( \phi(x) \) and \( \theta(x) \). Using \( a = \sigma_3, \beta = \sigma_1, \) where \( \sigma_1 \) and \( \sigma_3 \) are Pauli matrices, and multiplying (17) by \( \sigma_1 \) we obtain [19]

\[
-i\frac{d\theta}{dx} + (E - V(x))\theta - M(x)\phi = 0
\]

\[
i\frac{d\phi}{dx} + (E - V(x))\phi - M(x)\theta = 0.
\]

If we terminate \( \theta \) in above coupled differential equations, we obtain

\[
\begin{align*}
-\frac{d^2\phi}{dx^2} + \frac{1}{M(x)} \frac{dM(x)}{dx} \frac{d\phi}{dx} + \left( 2EV(x) - V(x)^2 - i \frac{dV(x)}{dx} - i \frac{1}{M(x)} \frac{dM(x)}{dx} (E - V(x)) \right) \phi \\
= (E^2 - M(x)^2)\phi. \tag{18}
\end{align*}
\]
We use a transformation of the upper component wavefunction which is \( \phi(x) = \sqrt{M(x)} \psi(x) \) in (18), we find that
\[
-\frac{d^2\phi}{dx^2} + V_{\text{eff}}(x)\phi = E^2\phi.
\] (19)
Here, effective potential \( V_{\text{eff}}(x) \) reads
\[
V_{\text{eff}}(x) = -V^2(x) - i\frac{dV(x)}{dx} + M^2(x) + i\left(\frac{V(x)}{M(x)} \frac{dM(x)}{dx}\right) + E\left(2V(x) - \frac{i}{M(x)} \frac{dM(x)}{dx}\right) - \frac{1}{2M(x)} \frac{d^2M(x)}{dx^2} + \frac{3}{4} \left(\frac{1}{M(x)} \frac{dM(x)}{dx}\right)^2.
\] (20)
Now we decompose the vector potential \( V(x) \) into the real and imaginary parts in (20) as
\[
V(x) = V_r(x) + iV_I(x)
\] (21)
which leads to
\[
V_{\text{eff}}(x) = -V_r^2(x) + V_I^2(x) + M^2(x) + 2EV_r(x) - \frac{M'(x)}{2M(x)} V_I(x) + \frac{3}{4} \left(\frac{M'(x)}{M(x)}\right)^2 + V'_I(x)
\]
\[
- \frac{M'(x)}{M(x)} V_r(x) + i \left(-2V_I(x)V_r(x) + 2EV_I(x) - V_r(x) + \frac{M'(x)}{M(x)} V_I(x) - E\frac{M'(x)}{M(x)}\right).
\] (22)
We may terminate the imaginary part of \( V_{\text{eff}}(x) \) by using
\[
V_I = \frac{M'(x)}{2M(x)} V_r(x) + \frac{V'_I(x)}{2(E - V_r(x))}.
\] (23)
Because we have obtained a real effective potential expression for the non-Hermitian Hamiltonian in the last section. Now, we can give \( V_{\text{eff}}(x) \) in the form of
\[
V_{\text{eff}}(x) = -V_r(x)^2 + M(x)^2 + 2EV_r(x) + \frac{3(V_r(x))'}{4(E - V_r(x))} + \frac{V''_r(x)}{2(E - V_r(x))}.
\] (24)
In order to compare \( U_{\text{eff}}(x) \) and \( V_{\text{eff}}(x) \), we may choose \( M(x) \) and \( V_r(x) \) as
\[
M(x) = m_1 \frac{A'(x)}{A(x)} + m_2 \frac{B(x)}{A(x)}
\] (25)
\[
V_r(x) = E - \frac{E}{A(x)}
\] (26)
and put in (24) where \( m_1 \) and \( m_2 \) are real constants. Thus, we give another ansatz for \( B(x) \) as
\[
B(x) = \gamma A(x) + \beta A'(x),
\] (27)
where \( \gamma \) and \( \beta \) are real constants. Afterward, \( V_{\text{eff}}(x) \) takes the form given below:
\[
V_{\text{eff}}(x) = -\frac{E^2}{A(x)^2} + m_2^2 \gamma^2 + 2\gamma m_2 (m_1 + \beta m_2) \frac{A(x)'}{A(x)}
\]
\[
+ \left( (m_1 + \beta m_2)^2 - \frac{1}{4} \right) \frac{(A(x)')^2}{A(x)^2} + \frac{A(x)''}{2A(x)}.
\] (28)
This time, we shall use (27) in (10) so that we would compare (28) and (10); then we obtain
\[
-\Phi''(x) + \left[ \frac{\omega^2 + 4\alpha^2}{\omega^2} \gamma^2 + \varepsilon + \frac{\omega/2 - \varepsilon\omega}{\omega^2} A(x)'' + \left( \beta^2 \frac{\omega^2 + 4\alpha^2}{\omega^2} - \beta - \frac{\alpha}{\omega} \right) \frac{(A(x)')^2}{A(x)^2}
\]
\[
+ \left( \frac{\omega - \alpha}{\omega} - \beta \right) \frac{A(x)''}{A(x)} + 2\gamma \left( \frac{\omega^2 + 4\alpha^2}{\omega^2} \beta - 1 \right) \frac{A(x)'}{A(x)} \right] \Phi(x) = \varepsilon \Phi(x)
\] (29)
and we can also give $U_{\text{eff}}(x)$ as

$$
U_{\text{eff}}(x) = \frac{\omega^2 + 4\alpha^2}{\omega^2} \gamma^2 + \varepsilon + \frac{\omega/2 - \varepsilon}{A(x)^2} + \left( \frac{\alpha^2}{\omega} - \beta - \frac{\alpha}{\omega} \right) \frac{(A(x)')^2}{A(x)^2}
$$

$$
+ \left( \frac{\omega - \alpha}{\omega} - \beta \right) \frac{A(x)''}{A(x)} + 2\gamma \left( \frac{\omega^2 + 4\alpha^2}{\omega^2} \beta - 1 \right) \frac{A(x)'}{A(x)}. \tag{30}
$$

Hence, we can compare (30) and (28); then we find this set of equations

$$
\varepsilon = \gamma^2 \frac{m^2}{2} - \frac{\omega^2}{2} + \frac{4\alpha^2}{\omega^2} \gamma^2 \tag{31}
$$

$$
\beta = \frac{\omega - 2\alpha}{2\omega} \tag{32}
$$

$$
E^2 = \frac{\omega}{2} - \varepsilon \tag{33}
$$

$$
m_2(m_1 + \beta m_2) = \frac{\omega^2 + 4\alpha^2}{\omega^2} \beta - 1. \tag{34}
$$

From the last relation, we can find

$$
m_1 = \frac{1}{2\omega} (-\beta \omega m_2 \pm \sqrt{\omega^2(1 + \beta^2 m_2^2) - 4\alpha \omega}) \tag{35}
$$

and then, we can give $E$ in terms of parameters $\omega$, $\alpha$ as

$$
E^2 = \frac{\omega}{2} - \gamma^2 \left( m_1^2 - \frac{\omega^2}{2} + \frac{4\alpha^2}{\omega^2} \right). \tag{36}
$$

Now we will give two potential models.

4.1. Example 1: non-$\mathcal{PT}$ symmetric vector potential

Using some special values of $A(x)$ may give rise to solvable effective potential models. For instance, if $A(x) = \delta \cosh x$ is chosen, one obtains

$$
V(x) = E - E \text{sech} x + \frac{1}{2} \frac{\text{sech} x}{\mu \cosh x + \sinh x}, \tag{37}
$$

that is not a solvable non-$\mathcal{PT}$ symmetric potential, at the same time, the mass expression is given by

$$
M(x) = m_2 \gamma + m_2^{-1} \left( \frac{\omega}{\omega^2} + \frac{4\alpha^2}{\omega^2} \frac{\beta}{\beta} - 1 \right) \tanh x. \tag{38}
$$

In this case, $V_{\text{eff}}(x)$ is obtained as

$$
V_{\text{eff}}(x) = E^2 - \left( E^2 - \frac{1}{4} + (m_1 + \beta m_2) \right) \text{sech} x^2
$$

$$
+ 2\gamma m_2(m_1 + \beta m_2) \tanh x + \gamma^2 m_2^2 + \frac{1}{4} + (m_1 + \beta m_2)^2. \tag{39}
$$

We can give (39) in terms of $\omega$ and $\alpha$ constants by the aid of (31)–(34):

$$
V_{\text{eff}}(x) = V_0 - V_1 \text{sech}^2 x + V_2 \tanh x, \quad -\infty < x < \infty, \tag{40}
$$

where

$$
V_0 = \frac{\omega}{2} + \gamma^2 \sigma + \frac{1}{4} + \left( \frac{\sigma \beta - 1}{m_2} \right)^2, \quad \sigma = \frac{\omega^2 + 4\alpha^2}{\omega^2} \tag{41}
$$
\[ V_1 = \frac{\omega}{2} - \gamma^2 (m_2^2 - \sigma) - \frac{1}{4} + \frac{(\sigma \beta - 1)^2}{m_2^2} \]
\[ V_2 = 2\gamma (\sigma \beta - 1). \]

If we recall the form of the Schrödinger-like equation, which is
\[ -\phi'' + V_{\text{eff}} \phi = \tilde{E} \phi, \quad \tilde{E} = E^2 - V_0, \]
then, we would write the ground-state wavefunction in terms of super-potential \( W(x) \) as
\[ \phi_0(x) = \exp \left( - \int x^2 W(y) \, dy \right). \]

We shall put the super-potential in the form of
\[ W(x) = C_1 + C_2 \tanh x, \]
where \( C_1 \) and \( C_2 \) are constants; using this relation we obtain the ground-state wavefunction \( \phi_0(x) \) as
\[ \phi_0(x) = e^{-C_1 x} (\cosh x)^{-C_2}. \]

There are boundary conditions as \( C_2 > 0 \) and \( |C_1| < C_2 \) such that \( \phi_0(x) \to 0 \) when \( x \to \pm \infty \). The partner potentials can be given in the following manner:
\[ V_{\text{eff}}^+(x) = W^2(x) + W'(x) = C_1^2 + C_2^2 - (C_2^2 - C_2) \text{sech}^2 x + V_2 \tanh x \]
and
\[ V_{\text{eff}}^-(x) = W^2(x) - W'(x) = C_1^2 + C_2^2 - (C_2^2 + C_2) \text{sech}^2 x + V_2 \tanh x. \]

If we show the ground-state energy with \( \tilde{E}_0 \), then we may give the expression as below
\[ W^2(x) - W'(x) = -V_1 \text{sech}^2 x + V_2 \tanh x - \tilde{E}_0. \]

Now, we can match (49) with (40), one obtains
\[ C_1^2 + C_2^2 = -\tilde{E}_0 \]
\[ C_2^2 = V_1 \]
\[ 2C_1 C_2 = V_2. \]

Solving these equations, we obtain \( C_1, C_2, \tilde{E}_0 \) as follows:
\[ C_2 = \frac{1}{2} (-1 \pm \sqrt{1 + 4V_1}) \]
and we must choose the positive sign in (54) because of the boundary conditions; this also leads to \( V_1 > 0 \). The other constant \( C_1 \) is given by
\[ C_1 = \frac{2V_2}{-1 + \sqrt{1 + 4V_1}} \]
and
\[ -\tilde{E}_0 = \frac{1}{4} (-1 + \sqrt{1 + 4V_1})^2 + \frac{V_2^2}{(-1 + \sqrt{1 + 4V_1})^2}. \]

It is seen that two partner potentials satisfy the well-known shape invariant relationship
\[ V_{\text{eff}}^+(x; a_0) = V_{\text{eff}}^-(x; a_1) + R(a_1), \]
where \( a_0 = C_2 \) and \( a_1 = C_2 - 1 \). The reminder \( R(a_1) \) is not dependent on \( x \) and it contributes to the energy spectrum as
\[ \tilde{E}_0 = 0 \]
\[ E_n^- = \sum_{k=1}^{n} R(a_k) \]
\[ = \frac{V_2^2}{4C_2^2} + C_2^2 - \frac{V_2^2}{4(C_2 - n)^2} + (C_2 - n)^2, \quad n = 0, 1, 2, \ldots \]

Eventually, using (56) we obtain the relativistic energy spectrum for (37) as

\[ E_n = \pm \sqrt{V_0 - \frac{V_2^2}{4\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4V_1 - n}\right)^2} + \left(\frac{1}{2}(-1 + \sqrt{1 + 4V_1}) - n\right)^2}. \]

For real energies, \(1 + 4V_1\) must be positive, i.e.

\[ 2\omega + 4\gamma^2(\sigma - m_2^2) + \frac{4(\sigma \beta - 1)^2}{m_2^2} > 0 \]

and

\[ V_0 > \frac{V_2^2}{4\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4V_1 - n}\right)^2} - \left(\frac{1}{2}(-1 + \sqrt{1 + 4V_1}) - n\right)^2. \]

Hereafter, we shall find the wavefunction \(\varphi(x)\). In that case, using (59) in (44) we obtain

\[ -\varphi'' + (V_2 \tanh x - (C_2 + C_2^2) \sech^2 x)\varphi = \left((C_2 - n)^2 - \frac{V_2^2}{4(C_2 - n)^2}\right)\varphi, \]

and if we use a new variable \(z = -\tanh x\) in the above equation and write the function as

\[ \varphi = \left(\frac{1 - z}{2}\right)^{-r} \left(\frac{1 + z}{2}\right)^{-s} P(z), \]

then we obtain

\[ (1 - z^2)P'(z) + (-2s + 2r - (2 - 2r - 2s)z)P'(z) + n(n - 2r - 2s + 1)P(z) = 0 \]

where

\[ r = \frac{1}{2} \left(n + \frac{1}{2}(1 - \sqrt{1 + 4V_1}) - \frac{V_2}{2n + \frac{1}{2}(1 - \sqrt{1 + 4V_1})}\right) \]

\[ s = \frac{1}{2} \left(n + \frac{1}{2}(1 - \sqrt{1 + 4V_1}) + \frac{V_2}{2n + \frac{1}{2}(1 - \sqrt{1 + 4V_1})}\right). \]

Thus, the unnormalized wavefunction and upper spinor component \(\phi_n(x)\) are given by

\[ \varphi_n(x) = \left(\frac{1 + \tanh x}{2}\right)^{-r} \left(1 - \tanh x\right)^{-s} P_n^{(2r-2s)}\left(-\tanh x\right) \]

\[ \phi_n = \sqrt{m_2 \gamma + m_2^{-1} \frac{\omega^2 + 4\alpha^2}{\omega^2} - \beta - 1} \tanh x \]

\[ \times \left(\frac{1 + \tanh x}{2}\right)^{-r} \left(1 - \tanh x\right)^{-s} P_n^{(2r-2s)}\left(-\tanh x\right), \]

where \(P_n^{(2r-2s)}\left(-\tanh x\right)\) are the Jacobi polynomials. In addition to the results here, in [28, 19] the authors obtained the spectrum of the Dirac equation with scalar, vector and pseudoscalar potentials. Our results are consistent with [28, 19] in the case of \(V_2 \to \imath V_2\).
4.2. Example 2: non-$PT$ symmetric vector potential

The choice of $A(x) = \delta \coth cx$ gives a non-$PT$ symmetric potential which is given by

$$V(x) = E - \frac{E}{\delta} \tanh cx + i \left( -\frac{c}{2} \csch cx \sech cx + \frac{2c^2(m_1 + m_2\beta) \coth 2cx}{-2c(m_1 + m_2\beta) + m_2\gamma \sinh 2cx} \right),$$  \hspace{1cm} (70)

where $E$ was given in (36). And the mass expression reads

$$M(x) = m_2\gamma - 2m_2^{-1} \csch 2x.$$  \hspace{1cm} (71)

Thus, $A(x)$, $V(x)$ and $M(x)$ yields the effective potential given below

$$V_{\text{eff}}(x) = E^2 + c^2 \sech^2 cx \left( 1 + \frac{E^2}{\delta^2 c^2} + \frac{3}{4} c^2 \sech^2 cx \csch^2 cx - \frac{E^2}{\delta^2} + \gamma^2 m_2^2 \right).$$  \hspace{1cm} (72)

Let us take $\beta$ as

$$\beta = -\frac{m_1}{m_2}$$  \hspace{1cm} (73)

to terminate the term $\csch cx \sech cx$ in (72), then (72) turns into

$$V_{\text{eff}}(x) = E^2 + c^2 \sech^2 cx \left( 1 + \frac{E^2}{\delta^2 c^2} + \frac{3}{4} c^2 \sech^2 cx \csch^2 cx - \frac{E^2}{\delta^2} + \gamma^2 m_2^2 \right).$$  \hspace{1cm} (74)

To obtain a solvable effective potential, we shall add and subtract $\frac{3}{2} c^2 \sech^2 cx$ to (74), we obtain

$$V_{\text{eff}}(x) = E^2 \left( 1 - \frac{1}{\delta^2} \right) + c^2 \left( \frac{1}{4} + \frac{E^2}{\delta^2 c^2} \right) \sech^2 cx + \frac{3}{4} c^2 \sech^2 cx + \gamma^2 m_2^2, \hspace{1cm} 0 < x < \infty.$$  \hspace{1cm} (75)

It is recalled that $V(x)$ turns into

$$V(x) = E - \frac{E}{\delta} \tanh cx - i \frac{c}{2} \csch cx \sech cx.$$  \hspace{1cm} (76)

Next, we shall give the super-potential in this form

$$W(x) = A \tanh cx - B \coth cx$$  \hspace{1cm} (77)

then we obtain the partner potentials and ground-state wavefunction as

$$W^2(x) - W'(x) = V_{\text{eff}}(x) = (A - B)^2 + B(B - c) \csch^2 cx - A(A + c) \sech^2 cx$$  \hspace{1cm} (78)

and

$$W^2(x) + W'(x) = V_{\text{eff}}(x) = (A - B)^2 + B(B + c) \csch^2 cx - A(A - c) \sech^2 cx$$  \hspace{1cm} (79)

and

$$\phi_0(x) = (\cosh cx)^{-\frac{\delta}{c}} (\sinh cx)^{\frac{\beta}{c}}$$  \hspace{1cm} (80)

here $\frac{\delta}{c} > 0$ and $\frac{\beta}{c} > 0$ is taken owing to the boundary conditions. Now, let us compare (78) and (75)

$$(A - B)^2 = E^2 \left( 1 - \frac{1}{\delta^2} \right) + \gamma^2 m_2^2$$  \hspace{1cm} (81)

$$B(B - c) = \frac{3}{4} c^2$$  \hspace{1cm} (82)

$$A(A + c) = -c^2 \left( \frac{1}{4} + \frac{E^2}{\delta^2 c^2} \right)$$  \hspace{1cm} (83)
hence we obtain \( B = \frac{3c}{2} \), \( A = \frac{3}{2} - \frac{\omega/2 - \gamma^{2}(m_{0}^{2} - \sigma) + \delta^{2}m_{0}^{2}}{4c} \). The shape invariance relation is written as

\[
V^{+}_{\text{eff}}(x, a_{0}) = V^{-}_{\text{eff}}(x, a_{1}) + R(a_{1}), \quad (84)
\]

where \( a_{0} \) and \( a_{1} \) are given as \( a_{0} = \{A, B\} \) and \( a_{1} = \{A - c, B + c\} \). If we use the expressions \( \tilde{E} = E^{2} - V_{0} \), we find

\[
\tilde{E}_{n}^{-} = \sum_{k=1}^{n} R(a_{k}) = (A-B)^{2} - (A-B - 2cn)^{2}. \quad (85)
\]

Finally, the following relativistic energy spectrum of (70) equals

\[
E_{n} = \pm \delta \sqrt{(\gamma m_{2})^{2} + (A-B)^{2} - (A-B - 2cn)^{2}} \quad (86)
\]

where the term inside the square root must be positive owing to obtaining real energies. Substituting (85) in (44) we obtain

\[
-\varphi''(x) + ((A-B)^{2} + B(B - c) \text{csch}^{2}cx - A(A + c) \text{sech}^{2}cx)\varphi(x) = ((A-B)^{2} - (A-B - 2cn)^{2})\varphi(x) \quad (87)
\]

and we use a new variable \( y = \cosh 2cx \) and express the function \( \varphi(x) = (1-y)^{B/c}(1+y)^{-A/c}P(y) \), then the above equation becomes

\[
(1-y^{2})P''(y) + (-A - B - (B - A + 1)y)P'(y) + n(n + B - A)P(y) = 0, \quad (88)
\]

thus, the wavefunction is given in terms of Jacobi polynomials \( P_{n}^{B/c-1/2, -A/c-1/2}(y) \)

\[
\varphi_{n}(x) = (1-y)^{B/c}(1+y)^{-A/c}P_{n}^{B/c-1/2, -A/c-1/2}(y). \quad (89)
\]

Hence, the upper component reads

\[
\phi_{n}(x) = \sqrt{m_{2} \gamma - 2c(m_{1} + m_{2} \beta)} \text{csch} 2cx(1 - \cosh 2cx)^{B/c}\times (1 + \cosh 2cx)^{-A/c}P_{n}^{B/c-1/2, -A/c-1/2}(\cosh 2cx). \quad (90)
\]

Results agree with those obtained earlier in [42].

Figure 1. Graph of (60) with respect to \( m_{2} \), for the red curve: \( n = 0, \alpha = 2, \omega = 3, \gamma = 0.1, \beta = 6 \); for the blue curve: \( n = 3, \alpha = 2, \omega = 3, \gamma = 0.1, \beta = 6 \).
5. Conclusion

In this work, we have introduced a Hamiltonian model $\mathcal{H}$ which is in non-Hermitian form and mapped $\mathcal{H}$ into a physical Hamiltonian $h$. The time-independent Dirac equation with effective mass in one dimension is related to $h$ and transformed into the Schrödinger-like equation with the new complex vector potentials $V(x)$ which are (37) and (76) derived using the algebraic methods. In [19], the authors used real or pure imaginary vector potentials $V(x)$. It is seen that composing $V(x)$ into its real and imaginary components leads to more general effective potentials which are the elements of the Schrödinger-like equation. In examples 1 and 2, terminating the imaginary part of the effective potential we have derived hyperbolic Rosen–Morse II-type solvable effective potential and hyperbolic generalized Pöschl–Teller potential II potential. We note that the mass relations for each case are more general. We have obtained the solutions of these effective potential models using the shape invariance method. We have seen that the real spectrum of the Hamiltonian given for solvable potentials cannot be obtained by using $\beta = -\alpha$ in the Swanson Hamiltonian. Thus, the metric operator which is positive definite for the so-called Hamiltonian can be searched in the next studies.

We have introduced some graphs for the energy eigenvalues with respect to $m_2$. Equation (60) is used in figure 1 and we note that the different values of the parameters can lead to real or pure imaginary energy. For the red curve, the energy is real for the chosen parameters but it can be seen that between $m_2 = 4.2145$ and $m_2 = 5.6142$, we have imaginary energy values as $i0.0565786$ and $i0.0310165$ for the blue curve. If we compare these results, we see that when $n$ takes the larger values, energy may take imaginary values for some specific values of $m_2$. When it comes to figure 2, we have real energies for the chosen parameters but when $n$ becomes larger again, the energy is imaginary for some values of $m_2$ which is $0 \leq m_2 \leq 1.404$.

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