Anomalous Discrete Flavor Symmetry and Domain Wall Problem

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Abstract

Discrete flavor symmetry is often introduced for explaining quark/lepton masses and mixings. However, its spontaneous breaking leads to the appearance of domain walls, which is problematic for cosmology. We consider a possibility that the discrete flavor symmetry is anomalous under the color SU(3) so that it splits the energy levels of degenerate discrete vacua as a solution to the domain wall problem. We find that in most known models of flavor symmetry, the QCD anomaly effect can only partially remove the degeneracy and there still remain degenerate vacua.
1 Introduction

Discrete flavor symmetry is often introduced in order to explain the observed patterns of neutrino masses and mixings. Depending on the choice of flavor symmetry and its way of spontaneous breaking, various predictions have been obtained (for reviews, see Refs. [1–4]). However, spontaneous breaking of a discrete symmetry leads to the formation of domain walls [5]. Once formed, domain walls are topologically stable and come to dominate the universe, which is a disaster for cosmology. There are several solutions. First, if inflation happens after the discrete symmetry breaking, domain walls are inflated away and no domain walls are left in our observable universe and hence there is no problem. Second, if the flavor symmetry is not exact but only an approximate one, domain walls are unstable and collapse at some instance [6].

Whether the first option works or not depends on the energy scale of inflation, that of flavor symmetry breaking, and the reheating temperature. If the flavor symmetry breaking scale is sufficiently larger than the inflation scale and maximum temperature of the universe, the flavor symmetry may not be restored after inflation. However, most concrete models introduce supersymmetry (SUSY) [7] in order to simplify the interaction among various flavon fields and there often appear flat directions in the scalar potential in SUSY [8]. These flat directions obtain masses of soft SUSY breaking, which is usually much lower than the flavor symmetry breaking scale. In this case, we need an inflation scale lower than the SUSY breaking scale (e.g. $\mathcal{O}(\text{TeV})$) for solving the hierarchy problem to solve the domain wall problem. Such a low-scale inflation is not excluded, but rather unlikely. Thus we do not pursue this option in this paper.

As for the second option, it is always possible to introduce small explicit symmetry breaking terms by hand unless the discrete symmetry is a remnant of some gauge symmetry. But there are so many ways to introduce such breaking terms, which are not under control and may (partly) destroy original motivation to consider flavor symmetry to explain observed data. Thus we restrict ourselves to a special case: the discrete flavor symmetry is exact at the classical level and only broken by the quantum anomaly. The concept is the same as the Peccei-Quinn (PQ) symmetry for solving the strong CP problem [9, 10].

The role of anomaly (in particular, anomaly under the color SU(3)) for Abelian discrete symmetry to solve the domain wall problem was explored in Ref. [11] and further applied to some concrete models in Refs. [12–17]. On the other hand, the anomaly of general (non-)Abelian discrete groups was extensively studied in Refs. [18–25]. In the context of model-building, non-anomalous discrete symmetry is often used in order to embed the discrete symmetry into some continuous gauge group. Our attitude is completely different: we want to make use of anomalous discrete flavor symmetry to find models without domain wall problem.

In Sec. 2 we summarize the structure of the discrete anomaly in general and examine all the non-Abelian discrete groups listed in Ref. [2] to see whether the anomalous breaking of the symmetry is allowed or not. We will find that none of them can be completely anomalous and hence the domain wall problem is not solved solely by the anomaly effect. In Sec. 3 we
will check this statement with some explicit examples. We conclude in Sec. 4.

2 Discrete anomaly and degenerate vacua

2.1 Discrete anomaly

The anomaly of general non-Abelian discrete groups has been analyzed in Refs. [21–25] by using the Fujikawa method [26,27]. It was pointed out that the measure of the path integral necessarily transforms as a one-dimensional representation of the discrete group and hence perfect groups, which do not have non-trivial one-dimensional representations, are anomaly-free [24, 25]. It means that we need to only care about the Abelian subgroups, $Z_n$, of the non-Abelian discrete groups.

Let us consider a non-Abelian discrete group $D$. What we seek is an anomaly of the $D$-SU(3)-SU(3) type, with SU(3) being the standard model QCD gauge group, since it is the only one that can significantly change the potential to solve the domain wall problem through the QCD instanton effect. If there is another strong hidden gauge interaction and a fermion charged under both this strong gauge group and the discrete group, it can also be important. Taking this case into account, we consider an anomaly of the $D$-SU($N$)-SU($N$) type.

Let us suppose that there is a set of left-handed chiral fermions $\psi$ that transforms as a representation $R$ of the gauge group SU($N$) and also transforms as

$$
\psi \to U_\psi(u)\psi,
$$

under an element $u$ of the discrete group $D$. Here, $U_\psi(u)$ is a matrix representation of $u$. Then we will have a following Lagrangian term

$$
\delta L = A \frac{2\pi}{n} \frac{1}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a},
$$

where $F^a_{\mu\nu}$ and $\tilde{F}^{\mu\nu a}$ denote the field strength of the SU($N$) gauge and its dual respectively, $n$ is a minimum integer such that $u^n = 1$ (hence $u$ generates a $Z_n$ subgroup of $D$). The coefficient $A$ is given by

$$
A = \sum_{\psi} q_\psi \frac{2\ell_\psi(R)}{},
$$

where $\ell_\psi(R)$ denotes the Dynkin index of the representation $R$, which is 1/2 for the fundamental representation of SU($N$). Here $q_\psi$ is a “charge” of $\psi$ under $Z_n$ rotation, defined by

$$
q_\psi = \frac{n}{2\pi i} \ln \det U_\psi(u).
$$
If $\psi$ transforms as a one dimensional representation of $D$, we have $U_\psi = e^{2\pi i q_\psi / n}$. The anomaly-free condition is given by

$$A \equiv 0 \mod n. \tag{5}$$

If this condition is violated, the discrete group $D$ is anomalous under the SU($N$). Then the anomaly (2) has a physical meaning that several vacua connected by the discrete transformation have different energy due to the instanton effect. Thus it can serve as a bias that renders domain walls unstable. It is completely analogous to the PQ solution to the strong CP problem: if we would consider global U(1) instead of discrete group $D$ and if it is anomalous under the QCD, the flat potential along the U(1) direction (axion) is lifted up by the instanton effect.

If there are several discrete groups $D$ under which the Lagrangian is classically invariant, as is often the case for models with flavor symmetry, we have

$$\delta \mathcal{L} = \sum_D \left( A_D \frac{2\pi}{n_D} \right) \frac{1}{32\pi^2} F^{\mu \nu} a \tilde{F}_{\mu \nu a}. \tag{6}$$

If the condition (5) is satisfied for each discrete group, the theory is obviously anomaly-free. If this condition is violated, the discrete symmetry is anomalous. Still, however, it is possible that there remain some subgroups of the original discrete symmetry free from anomalies. In order to ensure the non-existence of domain walls, there must be no remnant discrete symmetry.#1 For this purpose, we consider all the elements of $\mathbb{Z}_n$ rotations:

$$\psi \rightarrow (U_\psi(u))^k \psi, \tag{7}$$

with $k = 0, \ldots, n - 1$. Then the anomaly coefficient becomes

$$\mathcal{A} = \sum_D k_D \frac{A_D}{n_D}, \tag{8}$$

with $k_D = 0, \ldots, n_D - 1$. In order to remove all the degeneracy of the vacua, $\mathcal{A}$ should not be an integer for any choice of $k_D$, except for the trivial one (i.e., all $k_D$ are equal to zero). Otherwise, there remain discrete degenerate vacua.

From the above discussion, it is obvious that we need at least one representation for each $Z_n$ subgroup of $D$ which transforms non-trivially under $Z_n$. In Table 1, Table 2 and Table 3, we show a list of discrete groups frequently used in flavor models and their representations that take some non-trivial values of $\det U_\psi$ (see Eq. (4)) [2]. Representations with $\det U_\psi = 1$ or $q_\psi = 0$ for all $Z_n$ subgroups are omitted since they do not contribute to the anomaly. For example, the discrete group $S_3$, which is isomorphic to $Z_3 \times Z_2$, possesses a non-trivial singlet

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#1 The underlying assumption here is that vacuum expectation values (VEVs) of the flavon fields spontaneously break the discrete symmetry $D$ completely. If some subgroup is not spontaneously broken by VEVs of the flavons, what we need for the non-existence of domain walls is to explicitly break only the spontaneously broken part of the discrete symmetry.
Discrete group | det $U_\psi$ of representations
--- | ---
$S_3 \cong Z_3 \times Z_2$ | $1' \rightarrow (1, -1)$
 | $2 \rightarrow (1, -1)$

$S_4 \supset Z_3, Z_4$ | $1' \rightarrow (1, -1)$
 | $2 \rightarrow (1, -1)$
 | $3 \rightarrow (1, -1)$

$A_4 \cong (Z_2 \times Z_2) \times Z_3$ | $1' \rightarrow (1, 1, \omega)$
 | $1'' \rightarrow (1, 1, \omega^2)$

$A_5 \supset Z_2, Z_3$ | None

$T' \supset Z_3, Z_4$ | $1' \rightarrow (\omega, 1)$
 | $1'' \rightarrow (\omega^2, 1)$
 | $2' \rightarrow (\omega^2, 1)$
 | $2'' \rightarrow (\omega, 1)$

Table 1: List of frequently used discrete groups and their representations which can contribute to the anomaly. In the left column, discrete groups and their Abelian subgroups are shown. In the right column, det $U_\psi$ (see Eq. (4)) of relevant representations under the Abelian subgroups are shown. Here we defined $\omega = e^{2\pi i/3}$ and $\rho = e^{2\pi i/N}$.

expression $1'$, which has a charge $q_\psi = 1$ under $Z_2$, but is not charged ($q_\psi = 0$) under $Z_3$. From the tables, we can see that some of the groups (such as $A_4$) have an Abelian subgroup ($Z_2 \times Z_2$ in the case of $A_4$), which do not transform any fermion non-trivially. Then we can see that such subgroups cannot be anomalous. On the other hand, there are also examples such as $D_{2N}$, which do not have a $Z_N$ subgroup under which all the fermions are trivial. Note also that $A_5$, which is perfect, do not have non-trivial representations as we mentioned above.

Possible candidates of the groups that may be completely anomalous are $D_{2N}$, $Q_{4N}$, $Q_{4N+2}$, $\Sigma(2N^2)$, $\Delta(3N^2)$ ($N/3 = \text{integer}$), and $\Sigma(81)$. For $D_{2N}$, we see that all non-trivial representations have charge $q_\psi = N$ under $Z_{2N}$ and hence there must remain unbroken $Z_N$ symmetry. Similarly, for $Q_{4N}$ and $Q_{4N+2}$, $Z_{2N}$ and $Z_{2N+1}$ subgroup cannot be anomalous, respectively. For $\Sigma(2N^2)$, all relevant representations have the same charge under two $Z_N$ subgroups. Thus one linear combination of each $Z_N$ remains unbroken. Similar argument also applies to $\Delta(3N^2)$ ($N/3 = \text{integer}$) and $\Sigma(81)$. After all, all the groups listed here cannot be completely broken solely by the anomaly effect, and hence the domain wall problem is not solved.

In Sec. 3, we will explicitly show that vacuum degeneracy is partially removed by the QCD anomaly effect, but it is not enough to completely solve the domain wall problem. In particular, we focus on the examples with $A_4$ and $D_4$. 


Table 2: Continuation from Table 1.

2.2 The case with global U(1)

Discrete flavor symmetry may naturally explain observed neutrino masses and mixings, but often it cannot explain the hierarchical mass structure of the quarks and charged leptons. In many concrete models of flavor symmetry, a global U(1) symmetry is introduced in order to explain quark/lepton masses [1]. Hierarchical Yukawa couplings may be explained through Froggatt-Nielsen (FN) mechanism [28]. In the FN mechanism, quarks and/or leptons are assumed to have charges under global U(1), called U(1)\textsubscript{FN}, and the VEV of a flavon field with U(1)\textsubscript{FN} charge naturally explains the hierarchical mass structure. Even if the U(1)\textsubscript{FN} is exact at the classical level, it may have a QCD anomaly. In such a case, we can identify the U(1)\textsubscript{FN} as the PQ symmetry to solve the strong CP problem [29–34].

Let us consider the case where a global U(1) is introduced in addition to the discrete flavor group \(D\), and suppose that both the U(1) and \(D\) is anomalous under the QCD. Then the anomaly coefficient (8) becomes

\[
\mathcal{A} = \frac{\theta}{2\pi n_{U(1)}} + \sum_D k_D \frac{A_D}{n_D},
\]

where \(\theta = 0 \sim 2\pi\) is the U(1) rotation angle, corresponding to the Nambu-Goldstone (NG) boson, and \(n_{U(1)}\) is a non-zero integer determined by the U(1) charge of quarks. This \(\theta\) is a dynamical field and it is dynamically relaxed to the potential minimum so that \(\mathcal{A} = 0\).
Table 3: Continuation from Table 2.

due to the QCD instanton effect and solves the strong CP problem. In this case, the QCD anomaly effect works as a solution to the strong CP problem, but it has nothing to do with the domain wall problem associated with the spontaneous breaking of the discrete group $D$ [11]. Even if it is possible to assign the quark charges so that the discrete group $D$ is completely anomalous under the QCD, such an anomaly is canceled by the shift of the NG boson and the discrete minima reappear, independently of the value of $n_{U(1)}$. Thus we do not consider a global $U(1)$ hereafter.

### 3 Examples

#### 3.1 $A_4$ model

The so-called tri-bimaximal neutrino mixing pattern [35] can be achieved by models with $A_4$ flavor symmetry [8, 36]. However, the discovery of $\theta_{13}$ [37] requires modification to the tri-bimaximal mixing. Introducing additional scalar $\xi'$ which transforms as $1'$ under $A_4$ can explain small but non-zero $\theta_{13}$ [38] (see also Ref. [39]). Table 4 summarizes charge
assignments of quarks, leptons, Higgs and flavon fields under $A_4$ and additional $Z'_N$. We sometimes use the notation e.g. $d^c_i$ with $i = 1, 2, 3$ corresponding to $d^c, s^c, b^c$ respectively, and so on.

The Yukawa interactions responsible for lepton mass matrices are given as

$$\mathcal{L} = y_{le} \frac{e^c(L\phi_l)H_d}{M} + y_{\mu} \frac{\mu^c(L\phi_\mu)^cH_d}{M} + y_{\tau} \frac{\tau^c(L\phi_\tau)^cH_d}{M} + y_\nu \frac{(\phi_\nu LL)H_uH_u}{M^2} + y_\xi \frac{\xi(LL)H_uH_u}{M^2} + y_\xi' \frac{\xi'(LL)H_uH_u}{M^2} + h.c.,$$

where $(\cdots)$, $(\cdots)'$ and $(\cdots)''$ denote $1$, $1'$ and $1''$ product of $A_4$ triplets, respectively. After having VEVs of

$$\langle \phi_l \rangle = (v_l, 0, 0), \quad \langle \phi_\mu \rangle = (v_\nu, v_\nu, v_\nu), \quad \langle \xi \rangle = v_\xi, \quad \langle \xi' \rangle = v_\xi',$n

it can reproduce the charged lepton masses, neutrino masses and mixings as discussed in detail in Ref. [38], so we do not repeat here.

As for the quark sector, we can write the Lagrangian as follows:

$$\mathcal{L} = y_{ij}^d d^c_i Q_j H_d + y_{ij}^{u} u^c_i Q_j H_u + y_{ij}^{1} u^c_i Q_j H_u + y_{ij}^{1'} u^c_i Q_j H_u + h.c.,$$

where $i, j = 1, 2, 3$ and $I = 2, 3$. After having VEV of $\xi'$, this reduces to completely general Yukawa matrices for up and down quarks, so clearly there are enough degrees of freedom to reproduce observed quark masses and mixings. The reason for this particular choice of charge in the quark sector is that $A_4$ and $Z'_N$ become anomalous under the QCD since only one of the right-handed up-quarks is charged under these discrete groups. This is just one simple example and there are many other possible choices.

Now let us explore the vacuum structure of the model at the classical level. Details of the shape of the entire scalar potential are not needed for our purpose. The only information we need is that (11) is one of the potential minima. Any transformation of (11) by the $A_4$ and $Z'_N$ group element is also a vacuum. In the following, we see that there should be $12N$

| $Q_i$ | $u^c_i$ | $c^c_i$ | $t^c_i$ | $d^c_i$ | $L_i$ | $e^c_i$ | $\mu^c_i$ | $\tau^c_i$ | $H_u$ | $H_d$ | $\phi_l$ | $\phi_\mu$ | $\xi$ | $\xi'$ |
|-------|--------|--------|--------|--------|-------|--------|--------|--------|-------|-------|--------|--------|------|------|
| $A_4$ | 1      | $1''$  | 1      | 1      | 3     | 1      | $1''$  | $1'$   | 1     | 1     | 3      | 3      | 1    | 1    |
| $Z'_N$| 1      | $\omega^2_N$ | 1      | 1      | $\omega_N$ | $\omega^{-1}_N$ | $\omega^{-1}_N$ | $\omega^{-1}_N$ | 1     | 1     | $\omega^{-2}_N$ | $\omega^{-2}_N$ | $\omega^{-2}_N$ |
(N = odd) or 6N (N = even) discrete vacua with degenerate energy (at the classical level). First, for fixed \(Z_N\) angle, we find 12 vacua consisting of some combinations of the following \(\langle \phi_l \rangle\) and \(\langle \phi_\nu \rangle\) where

\[
\langle \phi_l \rangle = v_l(1, 0, 0), \quad \frac{v_l}{3}(-1, 2, 2), \quad \frac{v_l}{3}(-1, 2\omega, 2\omega^2), \quad \frac{v_l}{3}(-1, 2\omega^2, 2\omega),
\]

(13)

where \(\omega \equiv e^{2\pi i/3}\) and

\[
\langle \phi_\nu \rangle = \pm v_\nu(1, 1, 1), \quad \pm v_\nu(1, \omega, \omega^2), \quad \pm v_\nu(1, \omega^2, \omega).
\]

(14)

For each vacuum, \(\langle \phi_\nu \rangle\) transformed by the \(Z_N'\) symmetry is also a vacuum and hence we have \(12N\) (N = odd) or \(6N\) (N = even) vacua in total.\(^4\) The existence of 12 discrete vacua (for fixed \(Z_N'\) phase) reflects the fact that all the \(A_4\) symmetry is spontaneously broken by the VEVs of flavon fields. (If some subgroup of the \(A_4\) symmetry remains unbroken, some part of the 12 vacua may coincide with each other.)

The degeneracy of these vacua at the classical level is partly lifted by the effect of the QCD anomaly. Since the up-quark transforms non-trivially under \(A_4\), in particular under its \(Z_3\) subgroup, vacua which are connected by the \(T\) transformation will actually have a different energy of order \(\sim (m_\pi f_\pi)^2\) with \(m_\pi\) and \(f_\pi\) being the pion mass and decay constant, respectively.\(^5\) Similarly, the \(Z_N'\) symmetry is also anomalous and vacua connected by the \(Z_N'\) transformation are also lifted. The anomaly coefficient (8) is

\[
\mathcal{A} = k_1 \frac{2}{3} + k_2 \frac{2}{N},
\]

(15)

where \(k_1 = 0, 1, 2\) and \(k_2 = 0, \ldots, N - 1\). For \(N = 5\), for example, there are no solutions of \((k_1, k_2)\) that make \(\mathcal{A}\) integer except for the trivial one \(k_1 = k_2 = 0\). Thus \(Z_3 \times Z_N'\) may be completely broken by the QCD anomaly and vacuum degeneracy associated with these groups is solved. However, the QCD anomaly has no effects on the \(Z_2 \times Z_2\) subgroup of \(A_4\), since both \(1'\) and \(1''\) do not transform under this subgroup. Explicitly, we can find vacua connected by \(S, TST^2, T^2ST\) transformation from one vacuum and they have degenerate energy.

Therefore there exist stable domain walls associated with them. This argument does not depend on the structure of the (one of the) vacuum (11). The discussion is completely parallel for more general VEVs of flavons. After all, the QCD anomaly of the discrete group is not enough to solve the domain wall problem in this class of models.\(^6\)

\(^4\) Since the “charge” of the flavon \(\phi_\nu\) under the \(Z_N'\) symmetry is \(-2\), there are only \(N/2\) different transformations when \(N\) is even: rotations by an overall factor \(\omega^n_{\nu}\) with \(n = 0, 2, \ldots, N - 2\). On the other hand, there are \(N\) different transformations with \(n = 0, 1, \ldots, N - 1\) when \(N\) is odd.

\(^5\) We follow the notation of Ref. [1] for the \(A_4\) group element: \(S^2 = 1\) and \(T^3 = 1\) in the basis where \(T\) is diagonal. See App. A.1.

\(^6\) In a concrete setup of the flavon sector [8], there is a flat direction in the scalar potential whose VEV spontaneously breaks \(Z_3 \subset A_4\). In such a case, it is possible that inflation happens after the flavor symmetry breaking but before the \(Z_3\) breaking and hence the QCD anomaly effect may be enough to solve the domain wall problem [14].
operators, are given as
\[ \text{where} \]
leptons and various Higgs fields are listed in Table 5. We basically follow those of Ref. [42]. An example of charge assignments under \( D_4 \) model based on the \( 3.2 \)
\[ \text{the degree of freedom of the hypercharge U(1) rotation.} \]
\[ \text{where} \]
\[ \text{matrices for up and down type quarks, so it can clearly reproduce the observed quark masses} \]
\[ \text{As for the quark sector, we can write the Lagrangian as follows:} \]
\[ \text{where} \]
\[ \text{As for the} \] #7

\[ \text{Here} \]
\[ \text{we will obtain desired charged lepton masses, neutrino masses, and mixings with maximal atmospheric mixing angle and vanishing} \]
\[ \text{Higher order corrections modify this prediction and non-zero} \]
\[ \text{discrete degenerate vacua connected by} \]
\[ \text{As for the} \]
\[ \text{for quarks, leptons and various Higgs fields.} \]
\[ \text{We basically follow those of Ref. [42] where} \]
\[ \text{The Yukawa interactions responsible for lepton mass matrices, up to the dimension six operators, are given as} \]
\[ \text{where} \]
\[ \text{product of} \]
\[ \text{where} \]
\[ \text{charge assignments under} \]
\[ \text{where} \]
\[ \text{for quarks, leptons and various Higgs fields.} \]

\[ \text{Table 5: Charge assignments under} \]

\[ \text{3.2} \]

\[ \text{A model based on the} \]

\[ \text{for quarks, leptons and various Higgs fields are listed in Table 5. We basically follow those of Ref. [42] where} \]

\[ \text{The Yukawa interactions responsible for lepton mass matrices, up to the dimension six operators, are given as} \]

\[ \text{where} \]

\[ \text{As for the quark sector, we can write the Lagrangian as follows:} \]

\[ \text{where} \]

\[ \text{As for the} \]

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and mixings by choosing parameters appropriately. Since $1_{-+}$ changes its sign under both the $Z_4$ and $Z_2$ rotation, which are subgroups of the $D_4$, they are explicitly broken by the QCD anomaly and similarly $Z'_N$ is also broken. The anomaly coefficient (8) becomes

$$\mathcal{A} = k_1 \frac{1}{2} + k_2 \frac{1}{2} + k_3 \frac{1}{N},$$

where $k_1 = 0, \ldots, 3$, $k_2 = 0, 1$, and $k_3 = 0, \ldots, N - 1$ correspond to $Z_4 \subset D_4$, $Z_2 \subset D_4$, and $Z'_N$ discrete rotations, respectively. Unfortunately, there are non-trivial cases that make $\mathcal{A}$ integer for any choice of $N$: $(k_1, k_2, k_3) = (2, 0, 0), (1, 1, 0)$. The former means that the $Z_2$ rotation generated by the group element $A^2$ (see App. A.2) remains unbroken and the latter means that another $Z_2$ rotation $AB(= BA^3)$ remains unbroken. The latter $Z_2$ may be made anomalous if there exists another strong gauge interaction, but the former $Z_2$ always remain unbroken independently of the details of the model, since there is no non-trivial one-dimensional representation that is rotated by the angle $e^{\pm i\pi/2}$ under $A$. It means that the QCD anomaly does not completely break the flavor symmetry but there remains at least one unbroken $Z_2$ symmetry. Thus domain walls exist in this class of models.

## 4 Conclusions

We have considered the effect of the QCD anomaly on the structure of discrete vacua in models with discrete flavor symmetry assuming that some of the quarks are charged under the discrete group. It is found that the anomaly only partially removes the vacuum degeneracy and cannot remove all the degeneracy as far as the discrete groups listed in Ref. [2] are considered. It means that the domain wall problem is not solved solely by the QCD anomaly effect.

Several remarks are in order. The above conclusion was derived under the assumption that the flavon VEVs spontaneously break the discrete symmetry $D$ completely. It might be possible that flavon VEVs leave some part of the original discrete symmetry $D$ unbroken. Then, if the QCD anomaly explicitly breaks a part of the symmetry that will be spontaneously “broken”, domain walls become unstable. For example, let us suppose that $D = A_4$ is spontaneously broken down to $Z_2 \times Z_2$ by the flavon VEVs. The domain walls associated with the spontaneous breaking of $Z_3$ can be unstable since $Z_3$ can be anomalous due to fermions in $1'$ or $1''$ representation. It is non-trivial whether such a partial breaking of some discrete symmetry could lead to phenomenologically interesting predictions for quark/lepton masses and mixings. Furthermore, even if such a model-building will be successful, we may need a relatively low scale of the flavor symmetry breaking, in order for the domain walls to collapse due to the QCD anomaly effect before the domain wall domination of the universe. It may be worth pursuing this possibility further.

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In this case, it is possible that $Z_4 \subset D_4$ is broken by the QCD anomaly, while $Z_2 \subset D_4$ is broken by the hidden SU($N$) anomaly or vice versa. Then there only remains an unbroken $Z_2$ symmetry generated by the group element $A^2$.  

10
We also remark that the existence of degenerate vacua does not necessarily cause the cosmological domain wall problem. As noted in Introduction, if both the inflation scale and the reheating temperature are sufficiently low, the flavor symmetry may never be restored after inflation and there do not appear domain walls in the observable universe.\(^9\) The realistic situation can be more involved. In most known models of the discrete flavor symmetry, there are several flavons that make the whole scalar potential complicated and the VEV or mass of each flavon can take hierarchically different value. It means that the flavor symmetry breaking scale may not necessarily be parametrized by just one scale, and it can happen that only some part of the flavor symmetry breaking occurs after inflation, and so on [14]. It should also be noticed that we may put explicit symmetry breaking terms by hand at the classical level unless the discrete symmetry is a remnant of some gauge symmetry. Although it modifies the original prediction for the flavor structure, the correction might be small enough to be neglected phenomenologically while it can serve as a bias to make domain walls unstable.

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**A Notes on discrete groups**

**A.1 A\(_4\) group**

\(A_4\) is isomorphic to \((Z_2 \times Z_2) \rtimes Z_3\) and it has 12 group elements. There are four representations (and hence four conjugacy classes), three of which are one-dimensional \(1, 1', 1''\) and the other is three-dimensional \(3\). Here we summarize explicit matrix form of the three-dimensional

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\(^9\) Inflation models in which the flavon takes a role of inflaton have been proposed [33, 43, 44].
representation of the $A_4$ group elements [1]:

\[
1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad ST = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega \end{pmatrix}, \quad ST^2 = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2 & -\omega^2 & 2\omega \\ 2 & 2\omega^2 & -\omega \end{pmatrix},
\]

\[
T^2S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2\omega^2 & -\omega^2 & 2\omega^2 \\ 2\omega & 2\omega & -\omega \end{pmatrix}, \quad T^2ST = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega^2 \\ 2\omega & -\omega^2 & 2 \omega^2 \\ 2\omega^2 & 2\omega & -\omega \end{pmatrix}, \quad T^2ST^2 = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & -\omega & 2 \omega \\ 2\omega & 2\omega & -\omega \end{pmatrix},
\]

where $\omega = e^{2\pi i/3}$. Note that $T^2ST^2 = STS$. The product of two $A_4$ triplets is decomposed as $3 \times 3 = 1 + 1' + 1'' + 3_S + 3_A$. Explicitly, for two triplets $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, we find

\[
1 \sim a_1b_1 + a_2b_3 + a_3b_2 \equiv (ab),
\]

\[
1' \sim a_3b_3 + a_1b_2 + a_2b_1 \equiv (ab)',
\]

\[
1'' \sim a_2b_2 + a_3b_1 + a_1b_3 \equiv (ab)'',
\]

\[
3_S \sim \begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_3b_1 - a_1b_3 \end{pmatrix}, \quad 3_A \sim \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}.
\]

### A.2 $D_4$ group

$D_4$ is isomorphic to $Z_4 \times Z_2$ and it has 8 group elements. There are five representations (and hence five conjugacy classes), four of which are one-dimensional $1_{++}, 1_{--}, 1_{+-}, 1_{-+}$ and the other is two-dimensional $2$. Two-dimensional representation of the $D_4$ group elements are given by

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},
\]

\[
B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad BA^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad BA^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
\]
Note that $ABA = B$. The product of two $D_4$ doublets is decomposed as $2 \times 2 = 1_{++} + 1_{--} + 1_{+-} + 1_{-+}$. Explicitly, for two doublets $a = (a_1, a_2)$ and $b = (b_1, b_2)$, we find

$$(ab)_{++} = a_1 b_2 + a_2 b_1, \quad (ab)_{--} = a_1 b_2 - a_2 b_1, \quad (ab)_{+-} = a_1 b_1 + a_2 b_2, \quad (ab)_{-+} = a_1 b_1 - a_2 b_2.$$  \hfill (30)

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