SURGERY ON Aut(F₂)

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Abstract. We study a geometric construction of certain finite index subgroups of Aut(F₂).

We recall that Aut(F₂) admits an isometric properly discontinuous action with compact quotient on a CAT(0) complex X₀, called the Brady complex, which was introduced in [4].

In §1, we show that Aut(F₂) can be presented (virtually) in a very simple manner from a labelling of a flat torus. Starting from a torus of size 6 × n, for some fixed integer n ≥ 1 (we shall discuss the case n = 5 in details), we associate to it, via a “pinching and (systolic) filling” construction, a 2-complex Bₙ, with fundamental group a group Gₙ which is of finite index in Aut(F₂). The universal cover Xₙ of Bₙ is a CAT(0) space. We show in §1 that the Xₙ’s are pairwise isometric for every n ≥ 1, and that X₀ and Xₙ are locally isometric, in the sense that their vertex links are pairwise isometric (Lemma 1.3) for every n ≥ 0. This implies, by the result below, that Xₙ is isometric to X₀ for every n.

In §2, we prove a geometric rigidity theorem for the Brady complex. Roughly speaking, the result states that X₀ is the “free complex” on one (any) of its face, among the complexes locally isomorphic to X₀ (see Th. 2.3 for a precise statement). This seems to be a rather special property of X₀, which is not very often satisfied among the 2-complexes we have studied.

Theorem 2.3 implies that every CAT(0) 2-complex locally isometric to X₀ is isometric to X₀. The notion of local isomorphism in this statement is slightly more restrictive than requiring the existence of an abstract isometry between the links shown in Lemma 1.3; the two complexes must be of the same (local) type (see §2). The additional conditions are however immediate to verify for the Xₙ’s for n ≥ 1.

In §3, we show that every torsion free finite index orientable subgroup of Aut(F₂) can be constructed abstractly by a pinching—and—filling construction, similar to the one given in §1 applied to finitely many tori. It is not clear however how to extend the explicit procedure given in §1 to describe, e.g., the family of torsion free finite index subgroups which are associated with a fixed number of tori.

In §4, we explain the origin of the toric presentation given in §1. The present paper can be seen as a continuation of an earlier work [3], in which we introduce a cobordism category Bord₄ which can be used to construct groups acting on complexes of a given (local) type A. We show below that the techniques of [3] can be applied to the case of Aut(F₂). It gives rise groups acting on complexes of type Aut(F₂) as defined in [2].

In the case of Aut(F₂), however, the spaces constructed by surgery in this way must, by the results in §2, be quotients of the Brady complex X₀, and the resulting fundamental groups, subgroups of Aut(F₂). This is not true of many cobordism categories, and contrasts for example with the categories studied in [3], in which...
the groups accessible by surgery in a given category (of a fixed local type, e.g., Moebius–Kantor) are typically not pairwise commensurable. Again, the category \( \text{Bord}_3 \) is rather special in this respect when \( A \) is the type \( \text{Aut}(F_2) \).

Finally, we give in §5 an example of a CAT(0) 2-complex \( X' \) which is locally isomorphic to the Brady complex to \( X_0 \), but not isometrically isomorphic to it. Here "locally isomorphic" refers to the fact that the links in \( X' \) are isometric to the links in the complex \( X_0 \).

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1. The toric presentation

Consider the flat torus \( T_5 \) of size \( 6 \times 5 \) defined as follows:

Every edge in \( T_5 \) is oriented and labelled. The boundary is identified in the standard way respecting both the orientation and the labelling of the boundary edges.

Note that there is a non trivial Dehn twist, that we will denote \( \tau_{-6} \), in the vertical direction.

We endow the torus \( T_5 \) with the standard Euclidean metric, in which the cells are (as shown in the figure) lozenges with sides of length 1.

Here is the basic construction.

The figure contains a total of 20 letters. They are denoted \( A_r, B_r, C_r, D_r, \) \( 0 \leq r \leq 4 \). Let \( L \) be a letter. For every triple \( K \) of the form

\[
K = (L, L', L'')
\]

consider an oriented triangle with edges labelled by \( K \) in the given order. We attach this triangle to the torus \( T_5 \) along its boundary, respecting the orientation and labelling for the boundary edges. This operation, repeated for the twenty triples \( K \), defines a 2-complex \( B_5 \).

Let \( X_5 := \tilde{B}_5 \) denote the universal cover of \( B_5 \), and \( G_5 := \pi_1(B_5) \) denote its fundamental group of \( B_5 \).
Note that the canonical map $T_5 \to B_5 = X_5/G_5$ is not injective on vertices. One may view $B_5$ as a “wrinkled presentation” of the group $G_5$ and the map $T_5 \to B_5$ as the “sewing map”. Observe furthermore that every triple $K$ “jumps” on the torus $T_5$. (We call $K$ a “knight”.)

By definition, a jump on $T_5$ is an oriented edge between two vertices of $T_5$. Every triple $K$ defines three jumps, from the extremity of an edge in $K$ to the origin of the consecutive edge, modulo 3.

**Lemma 1.1.** Jumps are either disjoint or they share a common support.

**Proof.** A jump associated with a triple $K$ corresponds either to the affine transformation

$$
\begin{align*}
x &\mapsto x + 1 \mod 6 \\
y &\mapsto y - 2 \mod 5
\end{align*}
$$

where $x$ is even modulo 6, or to its inverse

$$
\begin{align*}
x &\mapsto x - 1 \mod 6 \\
y &\mapsto y + 2 \mod 5
\end{align*}
$$

where $x$ is odd modulo 6. It is not difficult to show that these two transformations do not depend on $K$. Since they are inverse of each other, jumps with a common vertex must have the same support. \hfill \square

In particular, the jumps define an involution $\sigma$ of the vertex set of $T_5$, whose orbit partition $T_5/\langle \sigma \rangle$ coincides with the vertex set of $X_5/G_5$.

Let us orient the torus $T_5$ counterclockwise, and consider the positive labelling $\in \{1, 2, 3, 4\}$ of the edges issued from a vertex, where 1 refers to the positive real axis. The basic construction induces a permutation of the labels associated with every jump. We shall now describe this permutation.

**Lemma 1.2.** The permutation of $\{1, 2, 3, 4\}$ associated with the jump

$$
\begin{align*}
x &\mapsto x + 1 \mod 6 \\
y &\mapsto y - 2 \mod 5
\end{align*}
$$

is the 4-cycle $(1, 2, 4, 3)$.

This shows that the resulting permutation does not depend on $K$; the permutation associated with the opposite jump is the inverse permutation. \hfill \square

The following shows that $X_5$ is locally isomorphic to the Brady complex in the (usual) sense that their links are pairwise isomorphic.

**Lemma 1.3.** Every link in $X_5$ is isomorphic to the link of the Brady complex.
**Proof.** We shall compute the links in $X_5$. That it is isomorphic to that of the Brady complex follows from [4, 7] (see also §2 below). Since the expression for $\sigma$ is independent of the base point in $T_5$, it is enough to check the link of the origin. We represent the links at the origin and its image in $T_5$ as follows (the drawings respects the scale provided by the angle metric):

![Diagram of links]

The prime labels correspond to the image $(1, -2)$. According to the previous lemma, edges in the link of $X_5$ corresponds to the permutation $s = (1, 2, 4, 3)$. This defines four additional edges in the above figure: $(x, s(x)')$ for every $x \in \{1, 2, 3, 4\}$. It is straightforward to check that this graph is the link of $X_0$ (compare §2). □

The basic construction can be generalized to an arbitrary integer $n \geq 1$ in the following way.

Suppose first that $n$ is a sufficiently large integer (e.g., $n \geq 4$). Consider a torus $T_n$ of size $6 \times n$, where the vertical identification involves a Dehn twist $\tau_n$. For every letter $L$ on $((x, y), (x, y - 1))$, where $x$ is even, write labels $L'$ and $L''$ on, respectively, $((x, y - 2), (x + 1, y - 2))$ and $((x + 1, y - 3), (x + 1, y - 4))$; for every letter $L$ on $((1, y), (2, y))$, write labels $L'$ and $L''$ on, respectively, $((3, y - 2), (4, y - 2))$ and $((5, y - 4), (6, y - 4))$. Then the same construction for every triple $K = (L, L', L'')$ on $4n$ letters defines a 2-complex $B_n$ which is locally isomorphic to the Brady complex.

The notation $B_n$ is consistent with the previous notation $B_5$.

One can further extend this construction of $B_n$ to every integer $n \geq 1$ as follows. Let $T_\infty := \lim_{n \to \infty} T_n$ (with respect to partial embeddings from a base point) is a cylinder with an obvious action of $\mathbb{Z}$. Since the set of triples (knights) is $\mathbb{Z}$-invariant, this action descends to the basic construction $B_\infty$; we let, by definition, $X_n$ is the universal cover of the quotient $B_n$ of this space by $n\mathbb{Z}$. The notation $B_n$ is again consistent. Note however than the description using knights is only clearly visible for $n$ sufficiently large ($n \geq 4$ is large enough).

This shows the following:

**Proposition 1.4.** The space $X_n$ are pairwise isomorphic for $n \geq 1$.

**Proof.** They have a common cover $B_\infty$. □

In the next section we give a different proof of this fact, which includes isomorphism with the Brady complex $X_0$.

2. **Geometric rigidity**

We shall describe the local data by a type (or “local type”), following [3 §4]. In the latter paper we were interested in two sorts of types, simplicial and metric. In the present paper, we shall use *labelled types*, which add connecting maps to mark the link edges using angle labels as follows (cf. [3 Rem. 4.5]).
Definition 2.1. A labelled type (in dimension 2) is

1. a set of graphs (the links);
2. a set of marked shapes, i.e., polygons with filled interior and labelled angles;
3. a set of connecting maps marking every link edge with an angle label.

We define the type $\text{Aut}(F_2)$ as follows:

1. the link of the Brady complex; it is isomorphic to the graph (see [7, Fig. 6])

The letters are associated (see [7, §3] for details) with the presentation

\[
\langle a, b, c, d, e, f \mid ba = ae = eb, \ de = cc = cd, \ bc = cf = f b, \ df = fa = ad, \ ca = ac, \ ef = fe \rangle
\]

of the braid group $B_4$.

2. two shapes, a lozenge and an equilateral triangle, labelled in the following way:

3. a connecting map defined by

Let $T$ be a labelled type. We say that a 2-complex with labelled face angles is of type $T$ if it has the correct links and shapes, and the induced marking of the link edges corresponds to a connecting map. A homomorphism between two complexes of type $T$ is a 2-complex homomorphism which preserves the angle labels.

The following is straightforward to verify from, e.g., the original description of $X_0$ in [4].

Proposition 2.2. The Brady complex $X_0$ is of type $\text{Aut}(F_2)$. 

Our main theorem in this section is a converse of this statement. More precisely, we prove that the complex $X_0$ satisfies a universal property: it is freely generated by any of its faces.
Theorem 2.3. Let $X$ be a 2-complex of type $\text{Aut}(F_2)$. Let $S$ be a face in $X_0$ and let $f:S \to X$ be a label and shape preserving map from $S$ to a face in $X$. There exists a unique homomorphism $f:X_0 \to X$ whose restriction to $S$ coincides with $f$. Furthermore, $f$ is a covering map onto its image.

Every 2-complex of type $\text{Aut}(F_2)$ can be naturally endowed with a metric structure, in which the triangle face is equilateral and the lozenge a union of two equilateral triangles. By the link condition, every such a complex is locally CAT(0). Every homomorphism between complexes of type $\text{Aut}(F_2)$ is isometric, and conversely, every isometry preserves the angle labels. The universal property in the metric situation states that if $f:S \to X$ is an isometry between a face $S$ of $X_0$ and a face of $X$, then there exists a unique isometry $\tilde{f}:X_0 \to X$ whose restriction to $S$ coincides with $f$.

Lemma 2.4. Let $X$ be a 2-complex of type $\text{Aut}(F_2)$. Let $S$ be a face in $X_0$ and let $f:S \to X$ be a map identifying $S$ with a face in $X$. Let $p$ be a vertex of $S$. There exists a unique label preserving extension $f: \text{St}_p(X_0) \to X$ of $f$ to the star of $p$ in $X_0$.

Proof. Let $L_0$ denote the link of $p$ in $X_0$ and $L$ the link of $f(p)$ in $X$. The map $f$. The map $f$ induces a label preserving map from an edge $e_0$ in $L_0$ to an edge $e$ in $L$. Since the labels incident to an arbitrary vertex in $L_0$ and $L$ are identical, and the labels around a vertex are pairwise distinct, there exists a unique label preserving extension of $f$ to the faces adjacent to $S$ containing $p$. More generally, it is easy to check that the map $e_0 \to e$ admits a unique label preserving extension to a graph isomorphism $L_0 \to L$. This shows that $f$ admits a unique label preserving extension $\tilde{f}:\text{St}_p(X_0) \to X$. □

Proof of Theorem 2.3. We refer to the standard CAT(0) structure on $X_0$ defined before the lemma. Let $C$ be a maximal ball in $X_0$ centred in $S$ to which $f$ admits an unique extension. We let $f$ denote this extension. Suppose for towards a contradiction that $C$ has a finite radius.

Let $p \in \partial C$. If $p$ belongs to the interior of a face, it is obvious how to extends $f$ to an $\varepsilon$-neighbourhood of $p$ in $X_0$. Suppose that $p$ belongs to the interior of an edge $e$, and let $f$ be the unique face containing $e$ and intersecting the interior of $C$. Since both $X_0$ and $X$ are of type $\text{Aut}(F_2)$, there exists a unique extension of $f$ to an $\varepsilon$-neighbourhood of $p$ in $X_0$. Assume now that $p$ is a vertex of $\partial C$. In this case, $C$ contains a face, and the previous lemma shows that $f$ can be extended in a unique way to an $\varepsilon$-neighbourhood of $p$.

Furthermore, that if $p, p'$ are two points in $\partial C$ at distance $\leq 1$, then the two extensions of $f$ from $p$ and $p'$ coincide on their intersection. Since $\partial C$ is compact, this shows that $f$ can be extended to an $\varepsilon$-neighbourhood of $C$, contradicting the maximality of $C$.

Finally, $\tilde{f}$ is a covering map by construction. □

Corollary 2.5. The spaces $X_n$ are pairwise isomorphic for every $n \geq 0$.

Proof. Since $X_n$ is of type $\text{Aut}(F_2)$, we have a covering map $X_0 \to X_n$. Since $X_n$ is simply connected, this map is an isomorphism. □

Corollary 2.6. The groups $G_n$ are of finite index in $\text{Aut}(F_2)$.
Proof. The special automorphism group $\text{SAut}(F_2)$, which is of index 2 in $\text{Aut}(F_2)$, acts transitively on the set of triangles in $X_0$ by the description in [7]. If $f$ is a triangle in $X_0$, and $s$ an element in $G_n$, then there exists a unique element $t_s \in \text{SAut}(F_2)$ whose restriction to $f$ coincide with $s$. By the theorem, $s$ and $T - s$ coincide on $X_0$, and the map $s \mapsto t_s$ provides an embedding of $G_n$ into $\text{SAut}(F_2)$. □

Another corollary, Theorem 2.8 below, shows that the Brady complex admits a “frame” in the following sense.

We recall that a flat plane in $X_0$ is an isometric embedding $\mathbb{R}^2 \to X_0$ of the standard Euclidean plane in $X_0$.

**Definition 2.7.** A frame on $X_0$ is an orientation, and a labelling by two letters $e$ and $f$, of the edge set of $X_0$, such that for every flat plane $\Pi$ in $X_0$ which is a union of lozenges, the following holds

1. the ordered set $B_x := (e_x, f_x)$ of outgoing edges at a vertex $x$ in $\Pi$, with respective labels $e$ and $f$, forms a basis of $\Pi$,
2. the unique translation of $\Pi$ which takes a vertex $x$ to a vertex $y$ takes the ordered set $B_x$ to the ordered set $B_y$.

Thus a frame is a way to move a basis consistently along the various embeddings $\Pi$ into $X_0$.

**Theorem 2.8.** There exists a frame on $X_0$.

*Proof.* The map $X_0 \to B_5$ is a covering map. We consider the obvious frame on the torus $T_5$, and the induced orientation and labelling of the edge set of $B_5$ by the sewing map $T_5 \to B_5$ (which is a bijection on the edge set), and lift the orientation and labelling to $X_0$ using the map $X_0 \to B_5$. Since every flat plane in $X_5$ maps onto the image of $T_5$ in $B_5$, this defines a frame on $X_0$. □

We shall say that a group of automorphisms of $X_0$ is orientable if it preserves the frame constructed in Theorem 2.8.

Note that every finite index subgroup of $\text{Aut}(F_2)$ contains a finite index subgroup which is orientable.

*Proof.* Let $G$ be a finite index subgroup of $\text{Aut}(F_2)$. Then the group $G \cap G_5$ is of finite index in $\text{Aut}(F_2)$:

$$[\text{Aut}(F_2) : G \cap G_5] \leq [\text{Aut}(F_2) : G][\text{Aut}(F_2) : G_5].$$

Furthermore, $G \cap G_5$ is orientable since $G_5$ is. □

### 3. Pinching and filling tori

The spaces in 1 are obtained in two steps, by a procedure which can be described as “pinching and systolic filling” starting from a flat torus.

Theorem 2.3 shows that every such a construction, using a family of flat tori, will have $X_0$ as a universal cover, provided it satisfies a few basic conditions, described in the following proposition.

**Proposition 3.1.** Let $t \geq 1$ be an integer. Suppose that:

1. $T_1, \ldots, T_t$ is a finite family of flat tori, endowed with a simplicial metric structure in which every cell is a lozenge with sides of length 1
(2) $\sigma$ is a fixed point free involution on the vertex set of $T := \bigcup_{k=1}^{t} T_k$
(3) the systolic length in $T' := T/\langle \sigma \rangle$ is 3
(4) every edge in $T'$ belongs to a unique systole of length 3
(5) the systolic filling $B'$ of $T'$, obtained by attaching isometrically an equilateral triangle to every systole in $T'$, is locally CAT(0) (i.e., the link girth in $B'$ is $\geq 2\pi$)

then $B \cong X_0$.

**Proof.** By Theorem 2.3 it is enough to prove that $B$ is of type $\text{Aut}(F_2)$. Since $\sigma$ is fixed point free, the link at a vertex in $B$ contains a union of two disjoint circles of length $2\pi$.

We shall use the notation in Lemma 1.3. Since every edge belongs to a unique systole of length 3, the systolic filling provides an involution $\tau$ of the set $\{1, 2, 3, 4\} \cup \{1', 2', 3', 4'\}$ of vertices in the link. Since $B$ is locally CAT(0), and the edge length from the systoles are $\pi/3$, the involution $\tau$ induces a bijection from $\{1, 2, 3, 4\}$ to $\{1', 2', 3', 4'\}$. We may assume without loss of generality that $\tau(1) = 2'$. By the CAT(0) condition, it follows that $\tau(4) \neq 1'$.

Suppose that $\tau(4) = 4'$. Then the distance between 3 and 3' is $\leq \pi$, which implies $\tau(2) = 3'$ and $\tau(3) = 1'$. In this case, however, the cycle $212'3'$ is of length $< 2\pi$, which is a contradiction. Thus, $\tau(4) \neq 4'$. This implies that $\tau(4) = 3'$. Applying again the CAT(0) condition, we must have $\tau(2) = 4'$ and $\tau(3) = 1'$.

Labelling the angles of the faces as in §2 the above shows that the link of vertex in $B$ is label isomorphic to the link of type $\text{Aut}(F_2)$, where the labels $t$ are associated with the systolic filling. This implies that $B$ is of type $\text{Aut}(F_2)$ and therefore that $X_0 \cong B_0$.

Furthermore, every (sufficiently deep) orientable torsion free subgroup of finite index is constructed in this way:

**Proposition 3.2.** Let $G$ be an orientable torsion free subgroup of finite index in $\text{Aut}(F_2)$. Suppose that the injectivity radius of $X_0/G$ is $> 1$. Then there exists a finite family of tori $T_1, \ldots, T_i$ and a fixed point free involution $\sigma$ on the vertex set of $T := \bigcup_{k=1}^{t} T_k$, such that $T' := T/\langle \sigma \rangle$ satisfies the condition in the previous proposition, and the systolic filling $B$ of $T'$ is isometric to $X_0/G$.

**Proof.** We say that two edges $e$ and $f$ in $X_0$ (or $X_0/G$) are equivalent if there exists a gallery $(f_1, \ldots, f_n)$ containing them, such that $f_i$ is a losenge for every $i$.

Let $e$ be an edge in $X_0$ and $\tilde{e}$ be a lift of $e$ in $X_0$. It is clear that the equivalence class $[\tilde{e}]$ of $\tilde{e}$ maps surjectively onto the equivalence class of $[e]$ under the covering map $\pi: X_0 \to X_0/G$. The convex hull $H$ of $[\tilde{e}]$ is isometric to a flat plane tessellated by lozenges. We let $T_\sigma$ denote the image of $H$ under $\pi$.

Say that a vertex $x \in T_\sigma$ is a double point if the link of $T_\sigma$ at $x$ is a disjoint union of circles. The map $H \to T_\sigma$ factorize through a map $H \to T_\sigma \to T_\sigma$, where $T_\sigma$ is obtained from $T_\sigma$ by blowing up every double point. Since $X_0/G$ is compact, so is $T_\sigma$. Therefore, $T_\sigma$ is compact. Since $H \to T_\sigma$ is a covering map and $G$ is orientable, it follows that $T_\sigma$ is a torus. We let $\sigma_k$ be the partially defined involution on $T_\sigma$ inducing the quotient map $T_\sigma \to T_\sigma'$. Let $e_1, \ldots, e_t$ be a representative set of equivalence classes of edges in $X_0/G$. Associated with the $e_k$’s are tori $T_k$ and partially defined involution $\sigma_k$ on $T_k$ such that the edge set of $T_k/\langle \sigma \rangle_k \subset X_0/G$ coincides with the equivalence class of $e_k$.
Furthermore, for every vertex \( x \in T_k \) not in the domain of \( \sigma_k \), there exists a unique \( k' \neq k \) such that \( x \) is a vertex of \( T'_{k'} \). This defines an involution \( \sigma_0 \) on the complement of \( \cup \sigma \) to itself on \( T = \cup_k T_k \). This involution is a fixed point free involution on \( T \). Since the injectivity radius of \( X_0/G \) is \( > 1 \), the systolic length of \( T/(\sigma) \) is \( \geq 3 \), and therefore \( X_0/G \) is the systolic completion of \( T/(\sigma) \) in the sense of the previous proposition.

The map \( T \rightarrow B \) can be viewed as a structure of “space with jumps” on the torus (or union of tori) \( T \). The geodesics with respect to such a structure in \( T \) are allowed to jump between certain transverse codimension 1 subspaces they cross (in the present situation, it is the 1-skeleton, which are the sides of the triangles). The length of the jump, and the incidence angles are described by the geometry of the added triangles. A pinching occurs along a codimension 2 subspaces (intersection of codimension 1 subspaces), which are singular sets, corresponding to instantaneous jumps of a geodesic between two points in \( T \). We will not attempt to formalize this notion further in the present paper.

**Remark 3.3.** The number \( t \) of tori, and the geometric parameters of the individual tori, provide conjugacy invariants for the given subgroup \( G \). As mentioned in the introduction, the description of the family of subgroups with a prescribed invariant, e.g., the torsion free finite index subgroups \( G \) of \( \text{Aut}(F_2) \) with a given torus number \( t(G) \), seems rather involved however.

4. A GROUP COBORDISM FOR \( \text{Aut}(F_2) \)

In this section we show that the surgery techniques from \([3]\) which were used to construct (in many cases, infinitely many) groups of a given type, can be applied to the group \( \text{Aut}(F_2) \). (Indeed, this is how the toric presentation in \([4]\) and the groups \( G_n \) were found.)

Let \( A \) be a (e.g., labelled) type. A category \( \text{Bord}_A \) of group cobordisms of type \( A \) can be defined as follows. The objects in this category are called collars, and the arrows, group cobordisms; in the present paper we only discuss the case where \( A \) is the type \( \text{Aut}(F_2) \) defined in \([2]\).

Let us first review the notion of collar. An (abstract) open collar is a topological space of the form \( H \times (0, 1) \) where \( H \) is a graph (not necessarily connected). If \( X \) is a 2-complex, an open collar in \( X \) is, by definition, an embedding \( C : H \times (0, 1) \rightarrow X \). We shall refer to the domain \( H \times (0, 1) \) as the abstract collar defining \( C \). The dual of an open collar of \( X \) is the open collar \( C' : H \times (0, 1) \rightarrow X \) defined by \( C'(x, t) := C(x, 1-t) \). The collar closure of \( C \) the topological closure \( \overline{C} \) of the image of \( C \) in \( X \); the span of \( C \) in \( X \) is the set span(\( C \)) of vertices of \( X \) contained in collar closure of \( C \); the simplicial closure of \( C \) is is the union of all the open edges and open faces it intersects. As in \([2]\) we only consider collars which are simplicially closed and vertex free.

We shall denote by \( \text{Bord}_{\text{Aut}(F_2)} \) the category of group cobordisms of type \( \text{Aut}(F_2) \). We construct an object \( C \) in \( \text{Bord}_{\text{Aut}(F_2)} \) as follows.

Fix an integer \( y \in \mathbb{N} \). We use the notation introduced at the end of \([4]\). We will view \( C \) as a “slice” of the cylinder \( T_{\infty} \). We fix four letters \( A_y, B_y, C_y, D_y \) respectively on \( ((x, y), (x, y-1)) \) where \( x = 0, 2, 4 \) and on \( ((1, y), (2, y)) \). Recall that for every letter \( L \) on \( ((x, y), (x, y-1)) \), where \( x \) is even, we write labels \( L' \) and \( L'' \) on, respectively, \( ((x, y-2), (x+1, y-2)) \) and \( ((x+1, y-3), (x+1, y-4)) \).
while for a letter \( L \) on \(((1, y), (2, y))\), we write labels \( L' \) and \( L'' \) on, respectively, \(((3, y - 2), (4, y - 2))\) and \(((5, y - 4), (6, y - 4))\).

By definition, the cylinder \( T_\infty \) is a quotient of a strip \([0, 6] \times \mathbb{R}\) using the twist \( \tau_{-6} \) in the vertical direction. Recall that a gallery is a sequence of faces \((f_1, \ldots, f_n)\) such that \( f_i \cap f_{i+1} \) is an edge.

We say that a gallery in \( T_\infty \) is generating if it is closed (i.e., cyclic permutations remain galleries) and homotopic to an element generating \( \pi_1(T_\infty) \).

**Lemma 4.1.** The minimal generating gallery has length \( n = 12 \).

**Proof.** Indeed, writing \( T_\infty \) as a quotient of a strip \([0, 6] \times \mathbb{R}\) of size \( 6 \times \infty \) by \( \tau_{-6} \), the gallery distance between a boundary edge on \([0] \times \mathbb{R}\) and its image by \( \tau_{-6} \) in \([6] \times \mathbb{R}\) is \( 12 = 6 + 6 \).

The collar \( C \) will be built from a minimal generating gallery on \( T_\infty \). Starting from the edge labelled \( A_y \), the gallery is defined by the succession of edges \( f_i \cap f_{i+1} \).

The edges have the following labels:

\[
A_y, A_y', A_y'', A_y'+1, A_y''+1, B_y-2, B_y'-2, B_y''-2, B_y'+1, B_y'''+1, C_y-4, C_y'-4, C_y'''-4, C_y'''+3.
\]

Note the corresponding gallery \((f_1, \ldots, f_{12})\) is closed: every change of letter occurs with a drop of \(-2\) for a total drop of \(-6\), which is consistent with \( \tau_{-6} \). This defines a “zig-zag” gallery generating \( \pi_1(T_\infty) \).

As a topological space the gallery \((f_1, \ldots, f_{12})\) is homeomorphic to \([0,1] \times S^1\). We shall refer to the gallery minus its boundary as open.

**Definition 4.2.** Let \( C \) be the union of

1. the image of the open generating gallery \((f_1, \ldots, f_{12})\) in the basic construction \( B_\infty \).
2. the triangles in \( B_\infty \) associated with the following six triples (knights) \( K = (L, L', L'') \) on the letters

\[
L = A_y+1, A_y, B_y-1, B_y-2, C_y-3, C_y-4
\]

where every triangle associated with a triple \( K \) is semi-open, in the sense that it does not contain the (unique) edge not belonging to the image of the gallery.

**Lemma 4.3.** \( C \) is a product space.

**Proof.** It is clear that the open gallery is a product space homeomorphic to \((0,1) \times S^1\). Under this identification, the added triples \( K \) define a space of the form \((0,1) \times H\) where \( H \) is a finite graph (the nerve) obtained by adding 6 edges to \( S^1 \).

One can of course give an explicit description of \( H \):

**Lemma 4.4.** The graph \( H \) is isomorphic to the Cayley graph of \( \mathbb{Z}/12\mathbb{Z} \), with respect to 1, together with an additional edge \((n, n+2)\) for every \( n \equiv 0, 1 \mod 4 \).

Therefore, we may view \( C \) as a open collar in \( B_\infty \) under the identity mapping \( C \to B_\infty \).

**Lemma 4.5.** \( C \) is a full collar in \( B_\infty \)

**Proof.** Every open edge \( e = f_i \cap f_{i+1} \) in \( C \) belongs to a (unique) triple \( K \), and therefore every point in \( e \) has an open neighbourhood included in \( C \).
Since $B_\infty$ is a complex of type $\text{Aut}(F_2)$, the above shows that the isomorphism class of $C$ is an object in the category $\text{Bord}_{\text{Aut}(F_2)}$.

The arrows in $\text{Bord}_{\text{Aut}(F_2)}$ are group cobordisms:

**Definition 4.6.** A group cobordism is a 2-complex $B$ together with a pair $(C, D)$ of collars of $B$ whose boundaries $\partial^- C$ and $\partial^+ D$ form a partition of the topological boundary of $B$:

$$\partial B = \partial^- C \cup \partial^+ D.$$ 

Let us construct the group cobordism $B$ of type $\text{Aut}(F_2)$. The collar $C$ depends on $y \in \mathbb{N}$, however, it is clear that $C_y \cong C_{y+1}$. The cobordism $B$ has $C$ as domain and codomain.

**Definition 4.7.** Let $B$ be the union of

1. $C_y \cup C_{y+1}$
2. the closed triangle in $B_\infty$ associated with the triple $K = (L, L', L'')$ on the letters $L = D_{y-2}$.

Again, $B$ depends on $y$, where $B_y$ is isomorphic to $B_{y+1}$ and defines a unique arrow, again denoted $B$, in $\text{Bord}_{\text{Aut}(F_2)}$. The inclusion map $L_B, R_B: C \to B$ (left and right collar boundary) and the obvious inclusion of $C$ as $C_y$ and $C_{y+1}$.

In particular:

**Theorem 4.8.** The map taking 1 to $B$ induces a unital inclusion $\mathbb{N} \to \text{Bord}_{\text{Aut}(F_2)}$.

**Proof.** Indeed, $B^n \neq B^m$ if $n \neq m$, where $B^n$ refers to the $n$-fold composition $B \circ \cdots \circ B$ in $\text{Bord}_{\text{Aut}(F_2)}$. □

In the language of [3], the above shows the following:

**Theorem 4.9.** $\text{Aut}(F_2)$ is virtually accessible by surgery.

This means that $\text{Aut}(F_2)$ admits a finite index subgroup which is the fundamental group of a complex obtained by a surgery construction in a cobordism category (see [3] §10). Here the groups $G_n$ are of finite index in $\text{Aut}(F_2)$ and the fundamental groups of the complexes $B_n$, which are of type $\text{Aut}(F_2)$ defined by a surgery construction in $\text{Bord}_{\text{Aut}(F_2)}$.

We take this opportunity to make a correction to [2, Lemma 17]. At the bottom of the page it is stated that “there are two extensions of this section”: it should be “three extensions”. Namely, in the first case (when the lozenges on the south-east triangles are oriented pointing south) one extension is the 3-strip, as indicated, which amounts to extending the lozenges with two triangles. A third sort of extension uses lozenges instead. In this case, the lozenges belong to a (using the terminology in [2]) semi-infinite $\circ$-strip of type $2 \times \infty$. This can be visualized using the surgery construction above: starting from the closed triangle defined in $B$ above. Def. 4.7 (2), one may use three lozenges belonging to a single collar (either all belonging to $C_y$, or all in $C_{y+1}$) which can be extended into three semi-infinite $\circ$-strip of type $2 \times \infty$ in the universal cover (so the resulting puzzle has an order 3 symmetry).

5. **Complements to Theorem 2.3**

We conclude some remarks on Theorem 2.3 regarding spaces locally isometric to $X_0$. It is an interesting exercise to construct groups acting freely uniformly on
a CAT(0) 2-complex locally isometric to the 2-complex $X_0$ of $\text{Aut}(F_2)$ (but not isometric to it), in the sense that their link are isometric to the link of $X_0$. In the present section, we provide one example.

By Theorem 2.3 such a complex $X'$ is not of type $\text{Aut}(F_2)$. The example will be of the following type.

Let $A$ denote the metric type (i.e., a set of metric graphs, and a sets of shapes) defined by:

1. Graph: the link of the Brady complex with the angular metric (see §2).
2. Shapes: an equilateral triangle, and an hexagon with sides of length 1. Both are viewed as standard polygons in the Euclidean plane with the induced metric.

By definition, every CAT(0) 2-complex of type $A$ is locally isometric to $X_0$ but not isometric to it.

**Proposition 5.1.** There exists a group $G'$ acting freely uniformly isometrically on a CAT(0) 2-complex $X'$ of type $A$.

The construction is as follows. We begin with a single hexagon on a set of 6 vertices, which we denote

$$\{1^+, 2^-, 3^+, 1^-, 2^+, 3^-\},$$

and edges labelled from 1 to 6 in a cyclic order as follows.

We shall realize these 6 vertices as the vertex set of a locally CAT(0) space of type $A$, containing the hexagon as a face.

Consider additional edges between these vertices:

- two edges between $i^-$ and $i^+$
- two edges between $i^+$ and $(i + 1)^+$
- two edges between $i^-$ and $(i + 1)^-$

(where $i$ is an index modulo 3) organized and named as follows:
Together with the edges of the hexagon, this defines a regular graph of order 8. Note that this graph has a natural symmetry $\sigma$ of order 3 taking every letter $l_i$ to the letter $l_{i+1}$ (modulo 3).

Consider the following hexagon and four triangles

$$(a_1, c_2, f_1, e_1, d_2, b_1)$$
$$(d_1, a_1, 4) (f_1, d_1, 1')$$
$$(b_1, c_1, 4) (c_1, e_1, 1)$$

Together with their images under $\sigma$, this defines 3 triangles and 12 triangles. In addition to these triangle add the four triangles:

$$(a_1, a_2, a_3) (b_1, b_2, b_3)$$
$$(e_1, e_2, e_3) (f_1, f_2, f_3)$$

This defines a 2-complex, whose fundamental group is $G'$ and universal cover $X'$. It is immediate to check that:

**Lemma 5.2.** The link of $X'$ is isometric to the link of $X_0$.

**Proof.** Note that it is enough to check a single vertex, since $\sigma$ and the reflection with respect to the horizontal axis extend to the 2-complex.

We may index the vertex set of the link by $a_1, b_1, a_3, b_3, c_1, d_1, 1, 2$, where the latter two numbers are associated with the initial hexagon. There are four hexagon edges: $(a_1, b_1), (a_3, c_1), (b_3, d_1)$, and $(1, 2)$ (for the first hexagon). One can then draw the edge associated with triangles: these are $(a_1, d_1), (d_1, 1), (b_1, c_1), (c_1, 1)$, from the images under $\sigma$: $(b_3, 2), (a_3, 2)$, and finally, $(a_1, a_3)$ and $(b_1, b_3)$.

It is not difficult to show that this graph is isometric to the link of $X_0$. □

We also note that:

**Proposition 5.3.** Aut($X'$) is vertex transitive.

This is part of the argument in the previous lemma.

**References**

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