AN OVERDETERMINED PROBLEM FOR SIGN-CHANGING EIGENFUNCTIONS IN UNBOUNDED DOMAINS

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Abstract. We study the existence of non-trivial unbounded domains of \( \Omega \subset \mathbb{R}^2 \) where the equation
\[
-\lambda u_{xx} - u = u \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial \Omega,
\]
is solvable subject to the conditions
\[
\frac{\partial u}{\partial \eta} = -1 \quad \text{on } \partial \Omega^+ \quad \text{and} \quad \frac{\partial u}{\partial \eta} = +1 \quad \text{on } \partial \Omega^-.
\]
For every integer \( m \geq 0 \), we prove the existence of a family of unbounded domains \( \Omega \subset \mathbb{R}^2 \) indexed by \( 0 \leq \ell \leq 2m \), where the above problem admits periodic sign-changing solutions. The domains we construct are periodic in the first coordinate in \( \mathbb{R}^2 \), and they bifurcate from suitable strips.

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1. Introduction and main result

This paper is concerned with the existence of periodic sign-changing solutions to a particular prototype of overdetermined boundary value problem on strip domains of the plane. In 1971, Serrin [37] proved by Alexandrov [2] moving plane method that the only bounded and regular domains in the Euclidean space \( \mathbb{R}^N \) where the overdetermined problem
\[
-\Delta u = 1 \quad \text{in } \Omega \quad (1.1)
\]
and
\[
u = 0, \quad \partial_\nu u = \text{const} \quad \text{on } \partial \Omega \quad (1.2)
\]
is solvable are balls. Here \( \nu \) is the unit outer to the boundary.

Soon after this celebrate result was communicated to the PDE community, several authors have developed interest in the study of symmetry properties as well as rigidity results of the overdetermined problem \( (1.1) - (1.2) \) including the more general equation
\[
-\Delta u = f(u) \quad \text{in } \Omega, \quad u = 0, \quad \partial_\nu u = \text{const} \quad \text{on } \partial \Omega, \quad (1.3)
\]
where \( f : [0, \infty) \to \mathbb{R} \) is a locally Lipschitz function. We refer the reader to [1, 8, 21, 25, 27, 30, 33, 35, 34, 16, 20]. In the classical literature a smooth domain \( \Omega \) where the problem \( (1.3) \) is solvable is called \( f \)-extremal domain. In 1997, Berestycki, Caffarelli and Nirenberg [4] addressed the classification of unbounded...
$f$-extremal domains and conjectured that if $\Omega \subset \mathbb{R}^N$ is an $f$-extremal domain such that $\mathbb{R}^N \setminus \overline{\Omega}$ is connected and (1.3) admits a bounded solution, then $\Omega$ is either a half plane, or a generalized cylinder $\mathbb{R}^j \times B$ where $B$ is the unit Euclidean ball in $\mathbb{R}^{N-j}$, or the complement $B^c$ of a ball $B \subset \mathbb{R}^N$ or a cylinder. For $f(u) = \lambda_1 u$, where $\lambda_1$ is the first eigenvalue of the Laplacian with 0-Dirichlet boundary condition, this conjecture was disproved in dimension $N \geq 3$ by Sicbaldi [39], and later in dimension $N \geq 2$ by Sicbaldi and Schlenk in [38], where they proved existence of periodic and unbounded extremal domains bifurcating from straight cylinder $\mathbb{R} \times B$. Subsequently, we considered the case $f \equiv 1$ in [14] and proved the existence of periodic unbounded domains bifurcating from generalized-type cylinder domains in $\mathbb{R}^N$. We also refer to [13,28,29], where the conjecture in [4] is addressed in space forms.

We note that the results in the previous works all assume one-sign solutions, while the existence of sign-changing solutions in the context of overdetermined boundary value problems is a subject of intensive investigation in the literature.

In the present paper, we deal with the existence of sign-changing solutions to a specific prototype of overdetermined boundary value problem given by

$$
\begin{cases}
-\lambda u_{xx} - u_{tt} = u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \eta} = -1 & \text{on } \partial \Omega^+, \\
\frac{\partial u}{\partial \eta} = +1 & \text{on } \partial \Omega^-
\end{cases}
$$

for some $\lambda > 0$. Here, $\nu$ is the outer unit normal vector field to the boundary of $\Omega$,

$$
\partial \Omega^+ = \{(x,t) \in \partial \Omega, \ t > 0\} \quad \text{and} \quad \partial \Omega^- = \{(x,t) \in \partial \Omega, \ t < 0\}.
$$

Our aim is to prove the existence of sign-changing solutions to problem (1.4) on unbounded domains $\Omega \subset \mathbb{R}^2$.

Related to this paper are the following conjectures of Schiffer in spectral theory, see [8,30,41].

**Conjecture 1:** Let $\Omega$ be a simply connected domain in $\mathbb{R}^N$ and $\lambda \neq 0$. Then there exists a Dirichlet eigenfunction $u \neq 0$ satisfying

$$
(P_{\lambda}) : \begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \eta} = \text{const.} & \text{on } \partial \Omega.
\end{cases}
$$

if and only if $\Omega$ is a ball.
Conjecture 2: Let $\Omega$ be a simply connected domain in $\mathbb{R}^N$. If $\lambda > 0$, then there exists $u \not= 0$ such that

\[
(Q_{\lambda}) : \begin{cases} 
-\Delta u = \lambda u & \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial \Omega, \\
u = \text{const.} & \text{on } \partial \Omega.
\end{cases}
\] (1.6)

if and only if $\Omega$ is a ball.

Indeed if $u$ is a solution of (1.4), then the function

\[W^\lambda(y, \zeta) := u(\lambda y, \sqrt{\lambda} \zeta)\]

solves the overdetermined problem

\[
\begin{cases} 
-\Delta W^\lambda = \lambda W^\lambda & \text{in } \Omega^\lambda, \\
W^\lambda = 0 & \text{on } \partial \Omega^\lambda,
\end{cases}
\] (1.7)

\[\frac{\partial W^\lambda}{\partial \eta} = -1 \quad \text{on } \partial \Omega^\lambda^+ \quad \text{and} \quad \frac{\partial W^\lambda}{\partial \eta} = +1 \quad \text{on } \partial \Omega^\lambda_+,
\] (1.8)

where

\[\Omega^\lambda := \{(x, \frac{\tau}{\sqrt{\lambda}}), \quad (x, \tau) \in \Omega}\}.
\]

The solution $W^\lambda$ does not assume a constant Neumann data at boundary $\Omega^\lambda$ as (1.8) reveals. Nevertheless the problem (1.7)-(1.8) can be seen as a refined version of conjecture 1 above. In contrast, interchanging the boundary conditions in (1.4) and considering constant zero Neumann data and constant Dirichlet equals $-1$, we are led to the problem

\[
\begin{cases} 
-\Delta W^\lambda = \lambda W^\lambda & \text{in } \Omega^\lambda, \\
W^\lambda = -1 & \text{on } \partial \Omega^\lambda,
\end{cases}
\] (1.9)

\[\frac{\partial W^\lambda}{\partial \eta} = 0 \quad \text{on } \partial \Omega^\lambda,
\] (1.10)

which is a prototype in the conjecture 2. We announce here our forthcoming paper where we are addressing (1.9)-(1.10). We alert that this problem leads to a serious loss of regularity which we hope to overcome soon.

Up to now, only few works have been carried out regarding the existence on sign-changing solutions of overdetermined boundary value problems and the most recent in this context are [6, 7, 12, 34]. In [6] and [7], the authors addressed the following
question: for $\omega \in \mathbb{R}$, is it true that the only domain such that there exists a solution to the overdetermined problem

$$\begin{cases}
\Delta u + \omega^2 u = -1 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases} \tag{1.11}$$

and

$$\partial_{\nu} u = c \quad \text{on } \partial \Omega. \tag{1.12}$$

is a ball?

Moreover, the authors in [7] proved the under suitable assumptions on $\omega$ that the only bounded domain $\Omega$ such that there exists a solution to (1.11)-(1.12) is the ball $B_1$, independent on the sign of $u$, provided $\partial \Omega$ is a perturbation of the unit sphere $\partial B_1$ in $\mathbb{R}^N$. A similar result was derived in [6] by considering a different Neumann boundary condition.

Concerning the case of unbounded domains, our paper appears to be the first and in fact, we are not aware of any other existing result for sign-changing solutions in unbounded domains arising in the context of overdetermined boundary value problems.

To state our main result, we consider the reference domain

$$\Omega_* = \mathbb{R} \times (-2m+1,2m+1) \subset \mathbb{R}^2$$

for some fixed non-negative integer $m$. Then problem (1.4) is solved on $\Omega_*$ by the ($\lambda$-independent) solution $u_*(x,t) = \sin t$ for all $\lambda > 0$. Our aim is construct domains $\Omega$ close to $\Omega_*$ with the property that the overdetermined boundary value problem (1.4) is solvable on $\Omega$ by some sign-changing solution $u$.

We consider the open set

$$Y_2^+ := \{ h \in C^{2,\alpha}_{p,e}(\mathbb{R}) : h > 0 \quad h \text{ is even in } x \},$$

where $C^{2,\alpha}_{p,e}(\mathbb{R})$ stands for the space of 2$\pi$ periodic functions in $\mathbb{R}$. For a function $h \in Y_2^+$, we define the domain

$$\Omega_h := \{(x, \frac{\tau}{h(x)}) : (x, \tau) \in \Omega_* \}. \tag{1.13}$$

Our main result states the existence of sign-changing solutions $u$ to the overdetermined problem (1.4) on domains of the form $\Omega = \Omega_h$.

**Theorem 1.1.** Let $m \in \mathbb{N}$ and $\alpha \in (0,1)$. There exists a strictly decreasing and finite sequence $(\mu(\lambda^e_{\ell}))(0 \leq \ell \leq 2m)$ of real numbers in $(0,1)$ with the following properties: for every $0 \leq \ell \leq 2m$, there exists $\varepsilon_\ell > 0$ and a smooth curve

$$(-\varepsilon_\ell, \varepsilon_\ell) \to (0, +\infty) \times C^{2,\alpha}(\Omega_*^e) \times C^{2,\alpha}(\mathbb{R}), \quad s \mapsto (\lambda_\ell(s), \varphi_\ell(s), \psi_\ell(s))$$
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with \( \lambda_\ell(0) = \mu_\ell(m) \), \((\varphi_0^\ell, \psi_0^\ell) \equiv (0,0) \) and such that for every \( s \in (-\varepsilon_\ell, \varepsilon_\ell) \), there exists a solution \( u_s^\ell \in C^{2,\alpha}(\Omega_{1+\psi_s^\ell}) \) of the overdetermined problem

\[
\begin{align*}
-\lambda_s^\ell u_{xx} - u_{tt} &= u & \text{in } \Omega_{1+\psi_s^\ell}, \\
\varphi_0^\ell &= 0 & \text{on } \partial \Omega_{1+\psi_s^\ell}, \\
\varphi_{\nu}^\ell &= -1 & \text{on } \partial \Omega_{1+\psi_s^\ell}^+, \\
\varphi_{\nu}^\ell &= +1 & \text{on } \partial \Omega_{1+\psi_s^\ell}^-.
\end{align*}
\]  

(1.14)

Moreover, the function \( \psi_s^\ell \in C^{2,\alpha}(\mathbb{R}) \) is even and \( 2\pi \) periodic and

\[
\psi_s^\ell = s(\cos(x) + \kappa_s^\ell).
\]  

(1.15)

Furthermore, setting

\[
\tilde{u}_s^\ell := \sin(t) + s(\varphi_{\nu}^\ell + \mu_s^\ell),
\]

where

\[
\varphi_{\nu}^\ell(x,t) = \left((-1)^\ell(2m+1)\pi \sin\sqrt{1 - \mu_\ell(m)t} + t \cos(t)\right)\cos(x),
\]

the solution \( u_s^\ell \in C^{2,\alpha}(\Omega_{1+\psi_s^\ell}) \) to (1.14) is of the form

\[
u_s^\ell(x,t) = \tilde{u}_s^\ell\left(x,(1 + \psi_s^\ell(x))t\right),
\]

(1.16)

with a smooth curve

\[
(-\varepsilon_0, \varepsilon_0) \to C^{2,\alpha}(\Omega_s^+) \times C^{2,\alpha}(\mathbb{R}), \quad s \mapsto (\mu_s^\ell, \kappa_s^\ell)
\]

satisfying

\[
\int_{(-\pi, \pi) \times (-2m+1)\pi, (2m+1)\pi)} \mu_s^\ell \varphi_{\nu}^\ell(x,t) \, dx \, dt + \int_{(-\pi, \pi)} \kappa_s^\ell \cos(x) \, dx = 0.
\]

The fact that the solution \( u_s^\ell \in C^{2,\alpha}(\Omega_{1+\psi_s^\ell}) \) to (1.14) changes sign is precise in Section 3 below. Indeed \( \Omega_{1+\psi_s^\ell} \) is symmetric in the \( t \) coordinate and \( u_s^\ell \) is odd with respect to this variable. Furthermore, the sequence \( (\mu_\ell(m))_{0 \leq \ell \leq 2m} \) whose existence is claimed in Theorem 1.1 is explicitly given by

\[
\mu_\ell(m) = 1 - \frac{1}{4} \left(1 + 2\ell + 2m\right)^2.
\]

In particular \( \mu_0(0) = \mu_m(m) = \frac{3}{4} \).

We now describe the proof of Theorem 1.1 while presenting the contents of the paper.

The proof of Theorem 1.1 is achieved by the use of Crandall-Rabinowitz bifurcation theorem, [9]. Our aim is solve the problem (1.4) on the domain \( \Omega_h \) given by (1.13). In Section 2 we transform (1.4) to the fixed domain \( \Omega_s \), see (2.9). Next we write (2.9) into the solvability of a bifurcation equation \( F_\lambda(u,h) = 0 \) defined in (3.5), and we give the expression of linearized operator \( DF_\lambda(0,0) \) in (3.9). In Section 4 we compute
the kernel of $DF_\lambda(0,0)$ and derive the spectral properties of the operator $DF_\lambda(0,0)$ in Lemma 5.1 and Proposition 5.2 as well as all the preliminary assumptions of Crandall-Rabinowitz bifurcation theorem [9]. The proof of Theorem 1.1 is completed in Section 6. In section 7, we give the Crandall-Rabinowitz bifurcation theorem for the reader convenience.

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2. The transformed problem and its linearization

For some fixed non-negative integer $m$, we consider the domain

$$
\Omega_* = \mathbb{R} \times \left( -(2m+1)\pi, (2m+1)\pi \right) \subset \mathbb{R}^2
$$

and the open set

$$
Y_2^+ := \{ h \in C^{2,\alpha}_{p,e}(\mathbb{R}) : h > 0 \quad \text{h is even in x} \},
$$

where $C^{2,\alpha}_{p,e}(\mathbb{R})$ stands for the space of $2\pi$ periodic functions in $\mathbb{R}$.

For a function $h \in Y_2^+$, we define the domain

$$
\Omega_h := \{ (x, \frac{\tau}{h(x)}) : (x, \tau) \in \Omega_* \}. \tag{2.1}
$$

We are looking for a solution $u$ of the overdetermined problem

$$
\begin{align*}
-\lambda u_{xx} - u_{tt} &= u & \text{in } \Omega_h, \\
u &= 0 & \text{on } \partial \Omega_h, \\
\frac{\partial u}{\partial \eta} &= -1 & \text{on } \partial \Omega_h^+, \\
\frac{\partial u}{\partial \eta} &= +1 & \text{on } \partial \Omega_h^-.
\end{align*} \tag{2.2}
$$

for some $\lambda > 0$. Here, $\nu$ is the outer unit normal vector field to the boundary of $\Omega_h$,

$$
\partial \Omega_h^+ = \{(x, t) \in \partial \Omega_h, \quad t > 0\} \quad \text{and} \quad \partial \Omega_h^- = \{(x, t) \in \partial \Omega_h, \quad t < 0\}.
$$

Observe that $\Omega_h$ is parametrized by the mapping

$$
\Psi_h : \Omega_* \to \Omega_h, \quad (x, \tau) \mapsto (x, t) = (x, \frac{\tau}{h(x)}),
$$

with inverse given by $\Psi_h^{-1} : \Omega_h \to \Omega_*$, \quad (x, t) \mapsto (x, h(x)t).
The aim is to pull back problem (2.2) on the fixed unperturbed domain $\Omega_*$ via the ansatz
\[ v(x, t) = u(x, h(x)t) = u(x, \tau) \quad \text{for some function } u : \Omega_* \to \mathbb{R}. \] (2.3)
For this we define
\[ L_\lambda := \partial_{tt} + \lambda \partial_{xx} + \text{id}, \]
and we first to compute the differential operator $L^h_\lambda$ with the property that
\[ [L^h_\lambda u](x, h(x)t) = [L_\lambda v](x, t) \quad \text{for } (x, t) \in \Omega_h \] (2.4)
for the function $v : \Omega_h \to \mathbb{R}$, $v(x, t) = u(x, h(x)t)$.

Setting $\xi := (x, h(x)t)$ in the following, we compute that
\[ v_x(x, t) = u_x(\xi) + h'(x)tu_\tau(\xi), \]
\[ v_{xx}(x, t) = u_{xx}(\xi) + 2h'(x)tu_{\tau\tau}(\xi) + h''(x)tu_{\tau}(\xi) + t^2[h'(x)]^2u_{\tau\tau}(\xi) \]
and
\[ v_{tt}(x, t) = h(x)^2u_{\tau\tau}(\xi). \]
Consequently,
\[ L_\lambda v(r, t) = v(r, t) + \lambda v_{xx}v(r, t) + v_{tt}v(r, t) \]
\[ = u(\xi) + \lambda u_{xx}(\xi) + (h^2(x) + \lambda t^2[h'(x)]^2)u_{\tau\tau}(\xi) + \lambda h''(x)tu_{\tau}(\xi) + 2\lambda h'(x)tu_{\tau\tau}(\xi) \]
and
\[ L_\lambda v(x, \frac{\tau}{h(x)}) = u(x, \tau) + \lambda u_{xx}(x, \tau) + \left(h^2(x) + \lambda \tau^2\frac{h'(x)^2}{h(x)^2}\right)u_{\tau\tau}(x, \tau) \]
\[ + \lambda \frac{h''(x)}{h(x)}\tau u_{\tau}(x, \tau) + 2\lambda \frac{h'(x)}{h(x)}\tau u_{\tau\tau}(x, \tau). \]
Therefore the differential operator $L^h_\lambda$ is given in coordinates $(x, \tau)$ by
\[ L^h_\lambda = \text{id} + \lambda \partial_{xx} + \left(h^2(x) + \lambda \tau^2\frac{h'(x)^2}{h(x)^2}\right)\partial_{\tau\tau} + 2\lambda \frac{h'(x)}{h(x)}\tau \partial_{\tau\tau} + \lambda \frac{h''(x)}{h(x)}\tau \partial_{\tau}. \] (2.5)
Next we express the normal derivative of $u$ with respect to the outer normal vector field on $\partial\Omega_*$ induced by the parametrization $\Psi_h : \Omega_* \to \Omega_h$, $(x, \tau) \mapsto (x, \frac{\tau}{h(x)})$.

Let the metric $g_h$ be defined as the pull back of the euclidean metric $g_{\text{eucl}}$ under the map $\Psi_h$, so that $\Psi_h : (\Omega_*, g_h) \to (\Omega_h, g_{\text{eucl}})$ is an isometry. Denote by
\[ \eta_h : \partial\Omega_* \to \mathbb{R}^2 \]
the unit outer normal vector field on $\partial\Omega_*$ with respect to $g_h$. Since $\Psi_h : (\Omega_*, g_h) \to (\Omega_h, g_{\text{eucl}})$ is an isometry, we have
\[ \eta_h = [d\Psi_h]^{-1} \mu_h \circ \Psi_h \quad \text{on } \partial\Omega_*, \] (2.6)
where \( \mu_h : \partial \Omega_h \to \mathbb{R}^2 \) denotes the outer normal on \( \partial \Omega_h \) with respect to the Euclidean metric \( g_{eucl} \) given by

\[
\mu_h(x,t) = \frac{1}{\sqrt{1 + \frac{(2m+1)^2 \pi^2 h'(x)^2}{h(x)^2}}} \left( \frac{(2m+1) \pi h'(x)}{h(x)}, \frac{t}{|t|} \right) \in \mathbb{R}^2 \quad \text{for} \ (x,t) \in \partial \Omega_h. \tag{2.7}
\]

Moreover, by (2.6) we have \( \mu_h(\Psi_h(x,\tau)) = d\Psi_h(x,\tau) \eta_h(x,\tau) \) and therefore

\[
\partial_{\eta_h} u(x,\tau) = du(x,\tau) \eta_h(x,\tau) = dv(\Psi_h(x,\tau)) \eta_h(x,\tau) = dv(\Psi_h(x,\tau)) \mu_h(\Psi_h(x,\tau)) = \langle \mu_h(\Psi_h(x,\tau)), \nabla_{(x,t)} v(\Psi_h(x,\tau)) \rangle_{g_{eucl}}.
\]

From (2.3)

\[
\nabla_{(x,t)} v(x,t) = \left( u_x(x,h(x)t) + h'(x)t u_x(x,h(x)t), h(x) u_r(x,h(x)t) \right)
\]

and we have

\[
\nabla_{(x,t)} v(\Psi_h(x,\tau)) = \nabla_{(x,t)} v(x,\frac{\tau}{h(x)}) = \left( u_x(x,\tau) + \frac{h'(x)}{h(x)} \tau u_x(x,\tau), h(x) u_r(x,\tau) \right)
\]

and hence,

\[
\partial_{\eta_h} u(x,\tau) = \frac{1}{\sqrt{1 + \frac{(2m+1)^2 \pi^2 h'(x)^2}{h(x)^2}}} \left[ \frac{(2m+1) \pi h'(x)}{h(x)} u_x(x,\tau) + \frac{h'(x)}{h(x)} \tau u_x(x,\tau) \right] + \frac{\tau}{|\tau|} h(x) u_r(x,\tau) \tag{2.8}
\]

From (2.3) and (2.4) the original problem (2.2) is equivalent to

\[
\begin{cases}
[L_h^{\lambda} u](x,\tau) = 0 & \text{in } \Omega_0 \\
u = 0 & \text{on } \partial \Omega_0 \\
\partial_{\eta_h} u \equiv -1 & \text{on } \partial \Omega_+^* \\
\partial_{\eta_h} u \equiv +1 & \text{on } \partial \Omega_-^*,
\end{cases} \tag{2.9}
\]

where \( \partial_{\eta_h} u \) is expressed in (2.8).

Our aim is then to find \((u,h)\) such that (2.9) holds. Before proving the existence of such a couple \((u,h)\), we make the following which allow to reduce the Neumann conditions in (2.9) to a single equation.

**Remark 2.1.** Let \( u = u(x,t) \) satisfying

\[
u(x,\pm(2m+1)\pi) = \text{Const.}
\]
Then \( u_x(x, \pm (2m + 1)\pi) = 0 \) so that (2.8) yields

\[
\partial_{t^h} u(x, +(2m + 1)\pi) = \frac{1}{\sqrt{1 + \frac{(2m+1)\pi^2 h^2(x)}{h^4(x)}}} \left[ \frac{(2m+1)\pi^2 h^2(x)}{h^4(x)} + h(x) \right] u_t(x, (2m + 1)\pi)
\]

\[
\partial_{t^h} u(x, -(2m + 1)\pi) = -\frac{1}{\sqrt{1 + \frac{(2m+1)\pi^2 h^2(x)}{h^4(x)}}} \left[ \frac{(2m+1)\pi^2 h^2(x)}{h^4(x)} + h(x) \right] u_t(x, -(2m + 1)\pi).
\]

In addition to \( u(x, \pm (2m + 1)\pi) = \text{Const.} \), if we assume that \( u = u(x,t) \) is odd in \( t \), then \( u_t(x, -(2m + 1)\pi) = u_t(x, (2m + 1)\pi) \) and

\[
\partial_{t^h} u(x, +(2m + 1)\pi) = -\partial_{t^h} u(x, -(2m + 1)\pi).
\] (2.10)

In this case the equation

\[
\partial_{t^h} u \equiv c \neq 0 \quad \text{on} \quad \partial \Omega^*_x
\] (2.11)

has no solution. Nevertheless, one can consider the overdetermined problem

\[
\begin{cases}
-\lambda u_{xx} - u_{tt} = u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
u_r = -1 & \text{on } \partial \Omega^+, \\
u_r = +1 & \text{on } \partial \Omega^-,
\end{cases}
\] (2.12)

where \( \partial \Omega^+ = \{ (x,t) \in \partial \Omega, \quad t > 0 \} \) and \( \partial \Omega^- = \{ (x,t) \in \partial \Omega, \quad t < 0 \} \). It is obvious that the function \( \sin(t) \) solves (2.12) on \( \Omega^*_x \).

From (2.10), the Neumann boundary conditions in (2.12) therefore reduce to the single equation

\[
\frac{1}{\sqrt{1 + \frac{(2m+1)\pi^2 h^2(x)}{h^4(x)}}} \left[ \frac{(2m+1)\pi^2 h^2(x)}{h^4(x)} + h(x) \right] u_t(x, (2m + 1)\pi) = -1.
\] (2.13)

3. The setting of the pull back problem and computation of the linearized operator

To set up a framework for problem (2.9), we define the function spaces

\[ C_{p,e}^{2,\alpha}(\overline{\Omega}_x) := \{ u \in C^{2,\alpha}(\overline{\Omega}_x) : u = u(x,t) \text{ is } 2\pi \text{ periodic } x \}, \]

\[ X_2 := \{ u \in C_{p,e}^{2,\alpha}(\overline{\Omega}_x) : u \text{ is odd in } t \text{ and even in } x, \text{ and } u \equiv 0 \text{ on } \partial \Omega^*_x \}. \]

as well as

\[ X_0 := \{ u \in C^{0,\alpha}(\overline{\Omega}_x) : u \text{ is odd in } t \text{ and even in } x, \text{ and } 2\pi \text{ periodic in } x \} \]

and

\[ Y_2 := \{ h \in C_{p,e}^{2,\alpha}(\mathbb{R}) : h \text{ is even in } x \} \quad Z_2 := \{ z \in C_{p,e}^{1,\alpha}(\mathbb{R}) : z \text{ is even in } x \}. \]

We also recall \( Y_2^+ := \{ h \in Y_2 : h > 0 \} \) and \( u_*(x,t) = \sin t. \)
Our aim is to prove that for some parameter $\lambda$, we can find the functions $(u, h) \in X_2 \times Y_2$ such that

$$
\begin{cases}
[L_\lambda^h u](x, \tau) = 0 & \text{in } \Omega_* \\
u = 0 & \text{on } \partial\Omega_* \\
\sqrt{1 + \frac{(2m+1)^2 \pi^2 h'^2(x)}{h^3(x)}} \left[ \frac{(2m+1)^2 \pi^2 h'^2(x)}{h^3(x)} + h(x) \right] u_\tau(x, (2m+1)\pi) = -1 & \text{on } \mathbb{R}.
\end{cases}
$$

(3.1)

Define the mapping $Q : X_2 \times Y_2^+ \to Z_2$

by

$$
Q(u, h) := \frac{1}{\sqrt{1 + \frac{(2m+1)^2 \pi^2 h'^2(x)}{h^3(x)}}} \left[ \frac{(2m+1)^2 \pi^2 h'^2(x)}{h^3(x)} + h(x) \right] u_\tau(x, (2m+1)\pi) + 1.
$$

and

$$
\widetilde{Q}(u, h) := Q(u + u_*, 1 + h).
$$

(3.2)

Then $\widetilde{Q}(0, 0) = 0$ and by computation

$$
D\widetilde{Q}(0, 0)(v, g) = v_\tau(\cdot, (2m+1)\pi) - g(\cdot).
$$

(3.3)

Next, we define

$$
F_\lambda : X_2 \times Y_2^+ \to X_0 \times Z_2, \quad F_\lambda(u, h) := (L_\lambda^{1+h}(u + u_*), \widetilde{Q}(u, h)),
$$

(3.4)

where $u_*(x, t) = \sin(t)$ and consider the equation

$$
F_\lambda(u, h) = 0.
$$

(3.5)

By construction, if $F_\lambda(u, h) = 0$, then the function $\tilde{u} = u_* + u$ solves the problem (3.1).

Moreover, we have

$$
F_\lambda(0, 0) = 0 \quad \text{for all } \lambda > 0.
$$

To check if this trivial branch of solutions admits bifurcation, we need to consider the derivative $DF_\lambda(0, 0)$. By direct computation, the operator $DF_\lambda(0, 0)$ is given by

$$
DF_\lambda(0, 0)(v, g) = \left( L_\lambda^1 v + L_\lambda^g u_*, v_\tau(\cdot, (2m+1)\pi) - g(\cdot) \right),
$$

(3.6)

with

$$
L_\lambda^g := \left( \frac{\partial}{\partial h} \Big|_{h=1} L_\lambda^h \right) g = 2g(x)\partial_t + 2\lambda g'(x)t\partial_x + \lambda g''(x)t\partial_t.
$$

For a function $w$ depending only on $t$, we have

$$
L_\lambda^g w = 2g(x)\partial_t + \lambda g''(x)tw_t.
$$

Consequently, the first coordinate in $DF_\lambda(0, 0)(v, g)$ is given by

$$
L_\lambda^1 v + L_\lambda^g u_* = v + \lambda v_{xx} + v_{tt} - 2g(x)\sin t + \lambda g''(x)t \cos t,
$$
while the second is
\[ v_\tau(\cdot, (2m + 1)\pi) - g(\cdot). \]
We set
\[ U(x,t) := v(x,t) + g(x)t \cos(t). \]
(3.7)
Then we have by direct computation
\[
L_1^\lambda v + L_2^\lambda u = v + \lambda v_{xx} + v_{tt} - 2g(x) \sin t + \lambda g''(x)t \cos t
\]
\[ = U + \lambda U_{xx} + U_{tt} \]
\[ v_\tau(\cdot, (2m + 1)\pi) - g(\cdot) = U_\tau(\cdot, (2m + 1)\pi) \]
(3.8)
and
\[
DF_\lambda(0,0)(v,g) = \left( U + \lambda U_{xx} + U_{tt}, U_\tau(\cdot, (2m + 1)\pi) \right). \]
(3.9)

4. Computation of the kernel of \( DF_\lambda(0,0) \)

In this section, we determine the kernel of the operator \( DF_\lambda(0,0) \).

So suppose that \((v,g)\) is an element of the kernel. Then from (3.6) and (3.8), the function \( U \) is an odd function in \( t \) and \( 2\pi \) periodic and even in \( x \) satisfying
\[ U + \lambda U_{xx} + U_{tt} = 0 \]
(4.1)

In what follows, we consider the Fourier coefficients
\[ U_k(t) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} U(x,t) \cos(kx) \, dx, \quad g_k := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g(x) \cos(kx) \, dx. \]
(4.3)

Multiplying (4.1) and (4.2) with \( \cos(kx) \) and integrating in the \( x \)-variable from 0 to \( 2\pi \), \( U_k \) is an odd function of class \( C^{2,\alpha}([-2\pi, 2\pi]) \) solving the initial value problem
\[ U''_k + (1 - k^2 \lambda) U_k = 0 \]
(4.4)
and
\[ U'_k(\pm(2m + 1)\pi) = 0 \quad \text{and} \quad U_k(\pm(2m + 1)\pi) = \mp(2m + 1)\pi) g_k. \]
(4.5)

4.1. Exkursion on the initial value problem (4.4)-(4.5).

In the case \( g_k = 0 \). If \( g_k = 0 \), then \( U_k \) solves
\[ U''_k + (1 - k^2 \lambda) U_k = 0, \quad U_k((2m + 1)\pi) = 0 = U'_k(\pm(2m + 1)\pi) \]
and we see that \( U_k = 0 \). In this case, we see with (3.7) that the kernel of \( DF_\lambda(0,0) \) is trivial.
In the case \( g_k \neq 0 \). If \( g_k \neq 0 \), we may assume \( g_k = 1 \) by considering \( U_k/g_k \) in place of \( U_k \), so we check whether the linear initial value problem
\[
U''_k + (1 - k^2 \lambda)U_k = 0, \\
U'_k(\pm(2m+1)\pi) = 0 \quad \text{and} \quad U_k(\pm(2m+1)\pi) = \mp(2m+1)\pi).
\] (4.6)
We put \( \mu = k^2 \lambda \), and let \( u_\mu : \mathbb{R} \to \mathbb{R} \) denote the unique solution of the initial value problem
\[
u''_\mu + (1 - \mu)u_\mu = 0 \quad \nu_\mu(\pm(2m+1)\pi) = \mp(2m+1)\pi, \quad (4.7)
\]
together with
\[
u'_\mu(\pm(2m+1)\pi) = 0. \quad (4.8)
\]
\textbf{The case} \( \mu = 1 \). In this case the \((4.7)\) reads
\[
u''_1 = 0 \quad \nu_1(\pm(2m+1)\pi) = 1,
\]
and we have \( u_1(t) = -t \) and \( u'_1(\mp(2m+1)\pi)) = -1 < 0 \). Therefore, \( u_\mu \) is not solution to \((4.7)\).

The case \( 0 \leq \mu < 1 \). In this case, a fundamental system of the linear equation is given by
\[
\phi_1^\mu(t) = 1, \quad \phi_2^\mu(t) = t,
\]
and we have \( u_\mu(t) = -t \) and \( u'_\mu(\mp(2m+1)\pi)) = -1 < 0 \). Therefore, \( u_\mu \) is not solution to \((4.7)\).

In the special case \( \mu = 0 \), we get
\[
u_0(t) = B \sin t.
\]
But then \( u_\mu((\mp(2m+1)\pi))) = 0 \neq \mp(2m+1)\pi \), meaning we have no solution in this case.

In the general case \( \mu \in (0,1) \),

If \( \sqrt{1-\mu}(2m+1) \) is an integer, we see with \((4.9)\) that
We therefore assume that $\sqrt{1-\mu(2m+1)}$ is not an integer. Then the conditions $u_\mu(\pm(2m+1)\pi) = \mp(2m+1)\pi$ with (4.9) yield

$$u_\mu(t) = -\frac{(2m+1)\pi}{\sin(\sqrt{1-\mu(2m+1)}\pi)} \sin(\sqrt{1-\mu t})$$

and

$$u'_\mu((2m+1)\pi) = -\frac{(2m+1)\pi}{\sin(\sqrt{1-\mu(2m+1)}\pi)} \cos(\sqrt{1-\mu(2m+1)}\pi).$$

Consequently

$$u'_\mu((2m+1)\pi) = 0$$

iff and only if

$$\sqrt{1-\mu(2m+1)} = \frac{1}{2} + \ell, \quad \ell \in \mathbb{N}. \quad (4.10)$$

Since $\mu \in (0,1)$, $\sqrt{1-\mu} < 1$ and hence $0 \leq \ell \leq 2m$.

For $m = 0$ and $\mu \in (0,1)$, $\sqrt{1-\mu}$ is not an integer and $\sqrt{1-\mu} = \frac{1}{2} + \ell$ if and only if $\ell = 0$ and

$$\mu = \frac{3}{4}. \quad (4.11)$$

4.2. The case $\mu > 1$. A fundamental system of the linear equation is then given by

$$\phi_1^\mu(t) = e^{-\sqrt{\mu-1}t}, \quad \phi_2^\mu(t) = e^{\sqrt{\mu-1}t}$$

and

$$u_\mu(t) = -\frac{(2m+1)\pi}{\sinh(\sqrt{1-\mu(2m+1)}\pi)} \sinh(\sqrt{1-\mu t}) \quad (4.12)$$

and

$$u'_\mu(t) = -\frac{(2m+1)\pi \sqrt{1-\mu}}{\sinh(\sqrt{1-\mu(2m+1)}\pi)} \cosh(\sqrt{1-\mu(2m+1)}\pi) < 0. \quad (4.13)$$

4.3. Computation of the kernel. Summarizing what we have got so far, for every $m \in \mathbb{N}$, (4.10) yields strictly decreasing sequence of $(\mu_\ell(m))_{0 \leq \ell \leq 2m}$ made of zeroes of the function

$$V : \mu \mapsto u'_\mu((2m+1)\pi), \quad (4.14)$$

and such that

$$u_\ell(t) = -(-1)\ell(2m+1)\pi \sin(\sqrt{1-\mu_\ell(m)t}) \quad (4.15)$$

is solution to (4.7).
Moreover, for each $0 \leq \ell \leq 2m$, $\mu_\ell(m) \in (0, 1)$ and

$$
\mu_\ell(m) = 1 - \frac{1}{4} \left( \frac{1 + 2\ell}{1 + 2m} \right)^2 \frac{\left( 1 + 2(2m - \ell) \right) \left( 3 + 2(2m + \ell) \right)}{\left( 1 + 2(2m - \ell_0) \right) \left( 3 + 2(2m + \ell_0) \right)}.
$$

(4.16)

Recalling (4.8) it follows that for a suitable $\lambda > 0$, the kernel of the operator $DF_\lambda(0, 0)$ is determined by finding the functions $U_k$, where the integers $k$ satisfy the restriction

$$
0 < k^2 \lambda < 1.
$$

(4.17)

Together with

$$
k^2 \lambda = \mu_\ell(m), \quad \text{for some } 0 \leq \ell \leq 2m
$$

(4.18)

Now choose any $0 \leq \ell_0 \leq 2m$ and take $\lambda = \mu_{\ell_0}(m)$. Then (4.18) we are led to finding the $k$’s such that

$$
k^2 = \frac{\mu_\ell(m)}{\mu_{\ell_0}(m)} = \frac{\left( 1 + 2(2m - \ell) \right) \left( 3 + 2(2m + \ell) \right)}{\left( 1 + 2(2m - \ell_0) \right) \left( 3 + 2(2m + \ell_0) \right)}
$$

for some $0 \leq \ell \leq 2m$. (4.19)

Since the sequence $(\mu_\ell(m))_{0 \leq \ell \leq 2m}$ is strictly decreasing, $\frac{\mu_\ell(m)}{\mu_{\ell_0}(m)} < 1$ for every $\ell > \ell_0$. So we might assume in (4.19) that $\ell \leq \ell_0$.

Obviously $k = 1$ for $\ell = \ell_0$.

For $\ell < \ell_0$, we check that the ratio in the right hand side of (4.19) is not an integer. Indeed, if there exists some $\ell < \ell_0$ such that

$$
\frac{\left( 1 + 2(2m - \ell) \right) \left( 3 + 2(2m + \ell) \right)}{\left( 1 + 2(2m - \ell_0) \right) \left( 3 + 2(2m + \ell_0) \right)} \in \mathbb{N}.
$$

Then $(3 + 2(2m + \ell_0))$ must divide at least one of the factors $(1 + 2(2m - \ell))$ or $(1 + 2(2m - \ell_0))$.

If $(3 + 2(2m + \ell_0))$ divides $(1 + 2(2m - \ell))$, necessarily $(3 + 2(2m + \ell_0)) \leq (1 + 2(2m - \ell))$. That is $2 + 2\ell_0 \leq -2\ell$.

If $(3 + 2(2m + \ell_0))$ divides $(3 + 2(2m + \ell))$, the same argument implies the inequality $\ell_0 \leq \ell$, which is in contradiction with that assumption $\ell < \ell_0$.

We can therefore derive from (3.7) and (4.15) the that for all $0 \leq \ell \leq 2m$, the kernel of the operator $DF_{\mu_\ell}(0, 0)$ is one-dimensional and spanned by $(v_\ell, g_\ell)$ with

$$
v_\ell(x, t) = w_{\mu_\ell(m)}(t) \cos x, \quad g_\ell(x) = \cos x,
$$

with

$$
w_{\mu_\ell(m)}(t) = -(-1)^\ell (2m + 1) \pi \sin(t - \mu_\ell(m) t) - t \cos(t).
$$
and
\[ \sqrt{1 - \mu_\ell(m)(2m + 1)} = \frac{1}{2} + \ell. \]

We underline that the particular case \( m = 0 \) gives \( \mu_0 = \frac{3}{4} \) and the kernel of \( \text{DF}_{\mu_0}(0, 0) \) spanned by \( (v_0, g_0) \) with
\[ v_0(x, t) = w_{\mu_*}(t) \cos x, \quad g_0(x) = \cos x, \]
with
\[ w_{\mu_*}(t) = -\pi \sin(t/2) - t \cos(t). \]

We also observe from (4.16) that \( \mu_m(m) = \frac{3}{4} \) and therefore,
\[ \mu_\ell(m) \geq \mu_m(m) = \frac{3}{4} \quad \text{for all} \quad 0 \leq \ell \leq m. \quad (4.20) \]

In the next section, we gather the required assumptions to apply the bifurcation result from simple eigenvalues of Crandall-Rabinowitz to prove existence of branches of solution to the equation (3.5).

5. Properties of the linearized operator

In this section, we analyse the operator \( \text{DF}_{\mu_\ell}(0, 0) \) and gather the assumptions that enable us to apply the Crandall-Rabinowitz bifurcation theorem.

In the following, we choose \( \ell \) between 0 and \( 2m \) and let
\[ \ker(\text{DF}_{\mu_\ell(m)}(0, 0))^\perp \subseteq X_2 \times Y_2 \subseteq L_{2,\mu}^2((-\pi, \pi) \times (-2m + 1)\pi, (2m + 1)\pi)) \]
be the complement of \( \text{DF}_{\mu_\ell(m)}(0, 0) \) with respect to the scalar product
\[ \langle (v, g), (w, h) \rangle := \int_{(-\pi, \pi) \times (-2m+1)\pi, (2m+1)\pi)) v(x, t)w(x, t) \, dxdt + \int_{(-\pi, \pi)} g(x)h(x) \, dx. \quad (5.1) \]

Next we consider the restriction mapping
\[ \text{DF}_{\mu_\ell(m)}(0, 0) : \ker(\text{DF}_{\mu_\ell(m)}(0, 0))^\perp \ni (w, h) \mapsto \text{DF}_{\mu_\ell(m)}(0, 0) \cdot \begin{bmatrix} w \\ h \end{bmatrix} \in X_0 \times Z_2 \]
and set
\[ A(w)(x) := \int_{(-\pi, \pi)} t \cos(t)\partial_{xx}w(x, t) \, dt, \quad B(w)(x) := \int_{(-\pi, \pi)} \sin(t)w(x, t) \, dt \]
\[ \mathcal{L}(z) \cdot v := \int_{(-\pi, \pi)} v_t(x, \pi_m)z(x) \, dx, \quad \mathcal{C}(w) \cdot v := 2 \int_{(-\pi, \pi)} v_t(x, \pi_m)w(x, \pi_m) \, dx \]
\[ \mathcal{K}(z) \cdot g := -\int_{(-\pi, \pi)} g(x)z(x) \, dx. \quad (5.2) \]

With these notations, we have
Lemma 5.1. Let \((v, g) \in \ker(DF_{\mu_\ell(m)}(0, 0))\) and \((w, z) \in X_0 \times Z_2\). Then,
\[
\langle DF(0, 0) \begin{bmatrix} v \\ g \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \rangle = \langle \begin{bmatrix} \text{id} + \lambda \partial_{xx} + \partial_{tt} + \mathcal{C} & \mathcal{L} \\ \lambda A - 2B & \mathcal{K} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \begin{bmatrix} v \\ g \end{bmatrix} \rangle. \tag{5.3}
\]
Moreover for every \(0 \leq \ell \leq 2m\), we have
\[
X_0 \times Z_2 = DF_{\mu_\ell(m)}(0, 0) \left( \ker(DF_{\mu_\ell(m)}(0, 0)) \right) \oplus E_\ell, \tag{5.4}
\]
where
\[
E_\ell := \text{span} \left( \begin{bmatrix} \overline{w}' \\ \overline{z}' \end{bmatrix} \right) = \{ a(\overline{w}', \overline{z}') : a \in \mathbb{R} \}, \tag{5.5}
\]
and
\[
\overline{w}'(x, t) := \sin(\sqrt{1 - \mu_\ell(m)} t) \cos(x) \quad \text{and} \quad \overline{z}'(x) := -2(-1)^t \cos(x). \tag{5.6}
\]

Proof. Let \((v, g) \in \ker(DF_{\mu_\ell(m)}(0, 0))\) and \((w, z) \in X_0 \times Z_2\). To shorten we let \(\pi_m := (2m + 1)\pi\). Then
\[
\langle DF(0, 0) \begin{bmatrix} v \\ g \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \rangle
\]
\[
= \int_{(-\pi, \pi) \times (-\pi_m, \pi_m)} (v + \lambda v_{xx} + v_{tt} - 2g(x) \sin(t) + \lambda g''(x) t \cos(t)) w(x, t) \, dx \, dt
\]
\[
+ \int_{(-\pi, \pi)} (v_t(x, \pi_m) - g(x)) z(x) \, dx
\]
Performing an integration by parts, we find
\[
\langle DF(0, 0) \begin{bmatrix} v \\ g \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \rangle
\]
\[
= \int_{(-\pi, \pi) \times (-\pi_m, \pi_m)} w(v + \lambda v_{xx} + v_{tt}) \, dx \, dt + \lambda \int_{(-\pi, \pi) \times (-\pi_m, \pi_m)} t \cos(t) g(x) w_{xx}(x, t) \, dx \, dt
\]
\[
- 2 \int_{(-\pi, \pi) \times (-\pi_m, \pi_m)} \sin(t) g(x) w(x, t) \, dx \, dt + \lambda \int_{-\pi_m}^{\pi_m} (w v_x - v w_x) \bigg|_{-\pi}^{\pi} \, dt + \int_{-\pi}^{\pi} (v_t w - w_t v) \bigg|_{-\pi_m}^{\pi_m} \, dx
\]
\[
+ \lambda \int_{-\pi_m}^{\pi_m} t \cos(t) (g' w - g w_x) \bigg|_{-\pi}^{\pi} \, dt + \int_{(-\pi, \pi)} (v_t(x, \pi) - g(x)) z(x) \, dx
\]
\[
= \int_{(-\pi, \pi) \times (-\pi_m, \pi_m)} v(w + \lambda w_{xx} + w_{tt}) \, dx \, dt + \int_{-\pi}^{\pi} g(x) (\lambda A(w) - 2B(w)) \, dx
\]
\[
+ \int_{(-\pi, \pi)} (v_t(x, \pi_m) - g(x)) z(x) \, dx + 2 \int_{-\pi}^{\pi} v_t(x, \pi_m) w(x, \pi_m) \, dx + S(w, v, g)
\]
\[
= \langle \begin{bmatrix} \text{id} + \lambda \partial_{xx} + \partial_{tt} + \mathcal{C} & \mathcal{L} \\ \lambda A - 2B & \mathcal{K} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \begin{bmatrix} v \\ g \end{bmatrix} \rangle + S(w, v, g)
\]
\[
= \langle \mathcal{F}_\lambda \begin{bmatrix} w \\ z \end{bmatrix}, \begin{bmatrix} v \\ g \end{bmatrix} \rangle + S(w, v, g),
\]
where we have used (5.2) and $S(w,v,g)$ is given by
\[
S(w,v,g) := \lambda \int_{-\pi}^{\pi} (w\partial_x v - v\partial_x w) \mid_{-\pi}^{+\pi} dt + \lambda \int_{-\pi}^{\pi} t \cos(t)(g'w - gw_x) \mid_{-\pi}^{+\pi} dt \\
- 2\int_{(-\pi,\pi)} v(x,\pi_m)w_t(x,\pi_m) \, dx.
\]

Since $v = 0$ on $\partial\Omega$, $\partial_x w, \partial_x v$ and $g'$ are $2\pi$ periodic, and $w$ is even in $x$ and $t$, it follows that $S(w,v,g) = 0$. Therefore we have
\[
\langle DF_\lambda(0,0) \begin{bmatrix} v \\ g \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \rangle = \langle F_\lambda \begin{bmatrix} w \\ z \end{bmatrix}, \begin{bmatrix} v \\ g \end{bmatrix} \rangle. \tag{5.7}
\]
This proves (5.3).

To prove the splitting (5.4), we let $(w, z)$ such that $F_\lambda \begin{bmatrix} w \\ z \end{bmatrix} = 0$, that is
\[
\begin{cases}
0 = w + \lambda \partial_{xx} w + \partial_t w + C(w) + L(z) \\
\lambda A(w) - 2B(w) + K(z) = 0. \tag{5.8}
\end{cases}
\]

We claim that
The PDEs in (5.8) is uniquely solvable in $X_2 \times Y_2$ (up to a multiplicative constant). \tag{5.9}

Indeed writing $\overline{w}$ and $\overline{z}$ in the form
\[
\overline{w} = \sum_{k \geq 0} \overline{w}_k(t) \cos(kx) \quad \text{and} \quad \overline{z} = \sum_{k \geq 0} \overline{z}_k \cos(kx)
\]
the second equation in (5.8) reads
\[
\lambda \int_{-\pi}^{\pi} \cos(t)\partial_{xx} \overline{w}(x,t) \, dt - 2\int_{-\pi}^{\pi} \sin(t)\overline{w}(x,t) \, dt + K(\overline{z}) = 0, \quad x \in (-\pi, \pi). \tag{5.10}
\]
Next we apply $g(x) = \cos(kx)$ to (5.10) and integrate (by using Fubini Theorem) to get
\[
\int_{-\pi}^{\pi} \left( k^2 \lambda t \cos(t) + 2 \sin(t) \right) \overline{w}_k(t) \, dt + \overline{z}_k = 0. \tag{5.11}
\]
Similarly we apply the first equation in (5.8) to $v(x,t) = \cos(kx)$ and integrate to find
\[
\overline{w}_k''(t) + (1 - k^2 \lambda)\overline{w}_k(t) = 0. \tag{5.12}
\]
Next we apply it again to $v(x,t) = \sin(t)\cos(kx)$ and derive
\[
\overline{z}_k = -2\overline{w}_k(\pi_m). \tag{5.13}
\]
We set $\mu := k^2 \lambda = k^2 \mu_\ell(m)$. Then
If $\mu = 1$, then the solution $\overline{w}(t)$ to the equation
\[
\overline{w}''(t) + (1 - \mu)\overline{w}(t) = 0. \tag{5.14}
\]
is given by $\bar{w} = At$ and integrating by parts, we see with (5.13) that $A = 0$.

If $0 \leq \mu < 1$, the general solution of the equation (5.12) is given by

$$w(t) = B \sin(\sqrt{1-\mu}t),$$

(5.15)

for some real constant $B$ and we deduce

$$z_k = -2B \sin(\sqrt{1-\mu\pi_m}).$$

(5.16)

To check if $\bar{w}$ and $\bar{z}$ satisfy (5.11), we rewrite this condition in term of $\mu$.

We set $\xi := \sqrt{1-\mu}$. Then a straightforward computation allows to get

$$\int_0^{\pi_m} \sin(t) \sin(\xi t) \, dt = \frac{1}{2} \int_0^{\pi_m} \left( \cos((1-\xi)t) - \cos((1+\xi)t) \right) \, dt = \frac{1}{\mu} \sin(\xi \pi_m).$$

(5.17)

We also have

$$\int_0^{\pi_m} (\mu t \cos(t) \sin(\xi t)) \, dt = \frac{\mu}{2} \int_0^{\pi_m} t \left( \sin((1+\xi)t) - \sin((1-\xi)t) \right) \, dt$$

$$= \frac{\mu \pi_m}{2} U(\pi_m) - \frac{2-\mu}{\mu} \sin(\xi \pi_m),$$

(5.18)

where the last line follows after integrating by parts and

$$U(t) = -\frac{1}{1+\xi} \cos((1+\xi)t) + \frac{1}{1-\xi} \cos((1-\xi)t) \quad \text{and} \quad U(\pi_m) = -\frac{2\sqrt{1-\mu}}{\mu} \cos(\sqrt{1-\mu\pi_m}).$$

Gathering (5.17) and (5.18), it follows that

$$\int_{-\pi_m}^{\pi_m} (\mu t \cos(t) + 2 \sin(t)) \sin(\xi t) \, dt = 2 \sin(\sqrt{1-\mu\pi_m}) - 2\sqrt{1-\mu\pi_m} \cos(\sqrt{1-\mu\pi_m}).$$

(5.19)

This with (5.16) give

$$B \int_{-\pi_m}^{\pi_m} (\mu t \cos(t) + 2 \sin(t)) \sin(\xi t) \, dt + \bar{z}_k = -2B \sqrt{1-\mu\pi_m} \cos(\sqrt{1-\mu\pi_m}).$$

(5.20)

It is plain from (5.20) that $B = 0$ for $\mu = 0$. In this case the solution of (5.8) given (5.15) and (5.16) is trivial. In particular $\bar{z}_0 = 0$ and $\bar{w}_0 = 0$. Furthermore, we see with (5.20) that the condition (5.11) is fulfilled with $B \neq 0$ if and only if

$$k^2 \mu_\ell(m) = \mu_\ell(m) \quad \text{for some } 0 \leq \ell \leq 2m,$$

(5.21)

which from the analysis in Subsection 4.3 is equivalent to $k = 1$.

To complete the proof, we show that $\bar{z}_k = 0$ and $\bar{w}_k = 0$ for every $k \geq 2$. To see this, we distinguish two cases:
If $k \geq 2$ is such that $\mu = k^2 \mu_\ell(m) < 1$, then from the argument following (5.21), we must have $B = 0$.

Now if $k \geq 2$ is such that $\mu = k^2 \mu_\ell(m) > 1$, then the general solution of (5.12) in this case is given by

$$\overline{w}_k(t) = A(e^{\mu k}t - e^{-\mu k}t) = 2A \sinh(\mu k) t,$$

where

$$\mu(k) := \sqrt{k^2 \mu_\ell(m) - 1}.$$

Using [26, Page 231, 1], we find

$$\int_0^{\pi m} \sin(t) \sinh(\mu k) t \, dt = \frac{1}{1 + \mu(k)^2} \sinh(\mu(k) \pi m). \quad (5.22)$$

Also [26, Page 230, 6] yields,

$$\int_0^{\pi m} (t \cos(t) e^{\mu k} t) \, dt = \frac{1}{1 + \mu(k)^2} \left[ e^{\mu k} \pi m \left( \frac{\mu(k)^2}{1 + \mu(k)^2} - \mu k \pi m \right) + \frac{\mu(k)^2 - 1}{1 + \mu(k)^2} \right],$$

$$\int_0^{\pi m} (t \cos(t) e^{-\mu k} t) \, dt = \frac{1}{1 + \mu(k)^2} \left[ e^{-\mu k} \pi m \left( \frac{\mu(k)^2}{1 + \mu(k)^2} + \mu k \pi m \right) + \frac{\mu(k)^2 - 1}{1 + \mu(k)^2} \right],$$

so that

$$\int_0^{\pi m} \mu t \cos(t) \sinh(\mu k) t \, dt = \frac{\mu}{1 + \mu(k)^2} \left[ \left( \frac{\mu(k)^2 - 1}{1 + \mu(k)^2} \right) \sinh(\mu(k) \pi m) - \mu k \pi m \cosh(\mu(k) \pi m) \right], \quad (5.23)$$

and

$$\int_{-\pi m}^{\pi m} (\mu t \cos(t) + 2 \sin(t)) \sinh(\mu k) t \, dt$$

$$= \frac{2}{1 + \mu(k)^2} \left[ \mu \left( \frac{\mu(k)^2 - 1}{1 + \mu(k)^2} \right) + 2 \right] \sinh(\mu(k) \pi m) - \mu k \pi m \cosh(\mu(k) \pi m) \right]$$

$$= 2 \left( \sinh(\mu(k) \pi m) - \mu(k) \pi m \cosh(\mu(k) \pi m) \right). \quad (5.24)$$

From this

$$\int_{-\pi m}^{\pi m} (\mu t \cos(t) + 2 \sin(t)) \overline{w}_k(t) = 2A \int_{-\pi m}^{\pi m} (\mu t \cos(t) + 2 \sin(t)) \sinh(\mu k) t \, dt + \overline{z}_k$$

$$= 4A \left( \sinh(\mu(k) \pi m) - \mu(k) \pi m \cosh(\mu(k) \pi m) \right) - 4A \sinh(\mu(k) \pi m)$$

$$= -4A \mu(k) \pi m \cosh(\mu(k) \pi m) \quad (5.25)$$

and we deduce that $A = 0$.

Next, if $k \geq 2$ is such that $\mu = k^2 \mu_\ell(m) = 1$, we already know from the argument following (5.14) that the corresponding $\overline{w}_k$ vanishes. Gathering all the steps, we get $\overline{z}_k = 0$ and $\overline{w}_k = 0$ for every $k \geq 2$. 

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If $k \geq 2$ is such that $\mu = k^2 \mu_\ell(m) < 1$, then from the argument following (5.21), we must have $B = 0$.

Now if $k \geq 2$ is such that $\mu = k^2 \mu_\ell(m) > 1$, then the general solution of (5.12) in this case is given by

$$\overline{w}_k(t) = A(e^{\mu k}t - e^{-\mu k}t) = 2A \sinh(\mu k) t,$$

where

$$\mu(k) := \sqrt{k^2 \mu_\ell(m) - 1}.$$
Finally, the solution $w$ to (5.8) is given by
\[ w(x,t) := \sin(\sqrt{1 - \mu_\ell(m)} t) \cos(x) \quad \text{and} \quad z(x) := -2(-1)^\ell \cos(x) \] (5.26)
and setting
\[ E_\ell := \text{span} (\overline{w}, \overline{z}) = \{ a(\overline{w}, \overline{z}) : a \in \mathbb{R} \}, \] (5.27)
we obtain (5.4).

**Proposition 5.2.** Let $0 \leq \ell \leq 2m$. Then the linear operator
\[ \mathcal{H}_\ell := DF_{\mu_\ell(m)}(0,0) : X_2 \times Y_2 \to X_0 \times Z_2 \]
has the following properties:

(i) The kernel $N(\mathcal{H}_\ell)$ of $\mathcal{H}_\ell$ is spanned by $(v_\ell, g_\ell)$ with
\[ v_\ell(x,t) = w_{\mu_\ell(m)}(t) \cos x, \quad g_\ell(x) = \cos x, \] (5.28)
with
\[ w_{\mu_\ell(m)}(t) = -(-1)^\ell (2m + 1) \pi \sin(\sqrt{1 - \mu_\ell(m)} t) - t \cos(t) \]
and
\[ \sqrt{1 - \mu_\ell(m)(2m + 1)} = \frac{1}{2} + \ell. \]

(ii) The range of $\mathcal{H}_\ell$ is given by
\[ R(\mathcal{H}_\ell) = \left\{ (v, g) \in X_0 \times Z_2 : \int_{(-\pi,\pi) \times (-\pi_m,\pi_m)} \overline{w}(x,t)v(x,t) \, dx \, dt + \int_{(-\pi,\pi)} \overline{z}(x)g(x) \, dx = 0 \right\}, \]
with
\[ \overline{w}(x,t) := \sin(\sqrt{1 - \mu_\ell(m)} t) \cos(x) \quad \text{and} \quad \overline{z}(x) := -2(-1)^\ell \cos(x). \] (5.29)
Moreover,
\[ \partial_{\lambda} \bigg|_{\lambda = \mu_\ell(m)} DF_{\lambda}(0,0)(v_\ell, g_\ell) \notin R(\mathcal{H}_\ell). \] (5.30)

The remaining part of the paper is to prove Proposition 5.2.

**Proof.** The proof of (i) is already done at the end of Section 4. Moreover, the mapping $DF_{\mu_\ell(m)}(0,0)$ is injective.

To prove (ii) we first observe from (5.4) in Lemma 5.1 that
\[ \text{Im} \left( DF_{\mu_\ell(m)}(0,0) \right) \subseteq E_\ell^+, \] (5.31)
where
\[ E_\ell^+ := \left\{ (v, g) \in X_0 \times Y_2 : \int_{(-\pi,\pi) \times (-\pi_m,\pi_m)} \overline{w}(x,t)v(x,t) \, dx \, dt + \int_{(-\pi,\pi)} \overline{z}(x)g(x) \, dx = 0 \right\}. \] (5.32)
Now we will prove that the mapping
\[
DF_{\mu_\ell}(0, 0) : \ker(DF_{\mu_\ell}(0, 0))^\perp \rightarrow E_\ell^\perp
\]
is surjective.

By Fredholm alternative theorem, for any \((v, z) \in X_0 \times Z_2\), the PDEs
\[
DF_{\mu_\ell}(0, 0) \cdot \begin{bmatrix} w \\ h \end{bmatrix} = (v, z)
\]
is solvable in \(L^2_{p,e}(-\pi, \pi) \times L^2_{p,e}(-\pi_m, \pi_m)\) in the sense of distribution. Moreover by using Fourier method, it is not also difficult to show that if \(v\) and \(g\) are both periodic, then one can find a solution \((w, h)\) that is also periodic. By elliptic regularity we have \((w, h) \in C^{2,\alpha}(\Omega^\ell) \times C^{2,\alpha}(\mathbb{R})\). Indeed, letting
\[
U(x, t) := w(x, t) + h(x)t \cos(t),
\]
a direct calculation gives
\[
DF_{\mu_\ell}(0, 0) \cdot \begin{bmatrix} w \\ h \end{bmatrix} = (v, z) \iff \begin{cases} 
\mathcal{L}_{\mu_\ell}U + U = v & \text{in } \Omega^\ast \\
U = \mp \pi_m h & \text{on } \partial\Omega^\ast_+ \\
\partial U = z & \text{on } \partial\Omega^\ast,
\end{cases}
\]
with \(\mathcal{L}_{\mu_\ell} = \mu_\ell \partial_{xx} + \partial_{tt}\). We now have \(\partial_0 U = z \in C^{1,\alpha}(\mathbb{R})\) and by classical elliptic regularity, we deduce that \(U \in C^{2,\alpha}(\Omega^\ast)\). There result from \(U = \mp \pi_m h\) on \(\partial\Omega^\ast_+\) that \(h \in C^{2,\alpha}(\Omega^\ast)\) and \(w \in C^{2,\alpha}(\Omega^\ast)\). Finally, we observe that the functions \((x, t) \mapsto -U(x, -t)\) and \((x, t) \mapsto \frac{1}{2} \{U(x, t) + U(-x, t)\}\) also solve the equation \((5.35)\).

In particular \(U\) is odd in \(t\), which implies that \(w\) is odd in \(t\). In addition, \(U\) in \(x\) and we deduce from \(U = \mp \pi_m h\) on \(\partial\Omega^\ast_+\) that \(h\) and \(w\) are even in \(x\).

We complete the proof, we have to check \((5.30)\). That is
\[
\frac{d}{d\lambda}_{\lambda = \mu_\ell} DF_{\lambda}(0, 0) \begin{bmatrix} v_\ell \\ g_\ell \end{bmatrix} \notin E_\ell^\perp,
\]
with
\[
w_{\mu_\ell}(t) = -(-1)^\ell (2m + 1)\pi \sin(\sqrt{1 - \mu_\ell(m)}t) - t \cos(t)
\]
and
\[
\sqrt{1 - \mu_\ell(m)(2m + 1)} = \frac{1}{2} + \ell.
\]
Following \((3.9)\), we set
\[
U_\ell(x, t) := v_\ell(x, t) + g_\ell(x)t \cos(t) = -(-1)^\ell \pi_m \sin(\sqrt{1 - \mu_\ell(m)}t) \cos x.
\]

Then
\[
DF_{\lambda}(0, 0) \begin{bmatrix} v_\ell \\ g_\ell \end{bmatrix} = \left(\lambda - \mu_\ell(m)\right)\begin{bmatrix} -\pi_m \sin(\sqrt{1 - \mu_\ell(m)}t) \cos x \\ 0 \end{bmatrix}
\]
and
\[
\begin{align*}
\langle (\bar{\pi}', \bar{z}'), ((-1)^{\ell}\pi_m \sin(\sqrt{1 - \mu_\ell(m)}t) \cos x, 0) \rangle_{L^2((-\pi,\pi) \times (-\pi_m,\pi_m))} \\
= (-1)^{\ell}\pi_m \int_{(-\pi,\pi) \times (-\pi_m,\pi_m)} \cos^2(x) \sin^2(\sqrt{1 - \mu_\ell(m)}t) \, dx \, dt = (-1)^{\ell}\pi_m^2 \neq 0. \tag{5.37}
\end{align*}
\]

\[\]

6. Solving Problem (3.1)

In this section we complete the proof of Theorem 1.1.

**Proof of Theorem 1.1 (completed).** Recalling (3.4), the proof of Theorem 1.1 will be established by applying the Crandall-Rabinowitz Bifurcation Theorem to solve the equation
\[
F_\lambda(u, h) = (L_1 + h_\lambda(u + \mu^*_\ell), \tilde{Q}(u, h)) = (0, 0). \tag{6.1}
\]

where \((u, h) \in X_2 \times Y_2^+\) and \(u_*(x, t) = \sin(t)\). As already observed in Section 3, if (6.1) holds, then the function \(\tilde{u} = u_* + u\) solves the problem \((Q)\) in (3.1).

To solve equation (6.1), we fix \(\ell\) between 0 and 2, and consider the operator
\[
\mathcal{H}_\ell := DF_{\mu_\ell(m)}(0, 0).
\]

in Proposition (5.2), as well as the space
\[
X^+_\ell := \left\{(v, g) \in X_2 \times Y_2^+ : \int_{(-\pi,\pi) \times (-\pi_m,\pi_m)} v(x, t) v(x, t) \, dx \, dt + \int_{(-\pi,\pi)} g(x) g(x) \, dx = 0 \right\}. \tag{6.2}
\]

By Proposition (5.2) and the Crandall-Rabinowitz Theorem (see [9, Theorem 1.7]), we then find \(\varepsilon_\ell > 0\) and a smooth curve
\[
(-\varepsilon_\ell, \varepsilon_\ell) \to \mathbb{R}_+ \times X_2 \times Y_2^+, \quad s \mapsto (\lambda^\ell_s, \varphi^\ell_s, \psi^\ell_s)
\]
such that

(i) \(F_{\lambda_s}(\varphi^s_s, \psi^s_s) = 0\) for \(s \in (-\varepsilon_0, \varepsilon_0)\),

(ii) \(\lambda^\ell_0 = \mu_\ell(m)\), and

(iii) \((\varphi^\ell_s, \psi^\ell_s) = s((v_\ell, g_\ell) + (\mu^\ell_s, \kappa^\ell_s))\) for \(s \in (-\varepsilon_0, \varepsilon_0)\) with a smooth curve
\[
(-\varepsilon_0, \varepsilon_0) \to X^+_\ell, \quad s \mapsto (\mu^\ell_s, \kappa^\ell_s)
\]
satisfying \((\mu^\ell_0, \kappa^\ell_0) = (0, 0)\) and
\[
\int_{(-\pi,\pi) \times (-\pi_m,\pi_m)} \mu^\ell_s v_\ell(x, t) \, dx \, dt + \int_{(-\pi,\pi)} \kappa^\ell_s g_\ell(x) \, dx = 0.
\]

Since \((\lambda^\ell_s, \varphi^\ell_s, \psi^\ell_s)\) is a solution to (6.1) for every \(s \in (-\varepsilon_0, \varepsilon_0)\), the function
\[
\tilde{u}^\ell_s = u_* + s(v_\ell + \mu^\ell_s).
\]
solves the over-determined boundary value problem \((Q)\) in \((3.1)\). Recalling the ansatz \((2.3)\), the function
\[ v_s^\ell(x, t) = \tilde{u}_s^\ell(x, 1 + s(g_\ell(x) + \kappa_s^\ell(x))t) \] (6.3)
solves the problem \((P)\) in \((1.4)\) on the domain
\[ \Omega_s^\ell := \left\{ \left(x, \frac{\tau}{1 + s(g_\ell(x) + \kappa_s^\ell(x))} \right) : (x, \tau) \in \Omega_s \right\}. \]
The proof is complete.

7. Crandall-Rabinowitz bifurcation theorem

**Theorem 7.1** (Crandall-Rabinowitz bifurcation theorem, [9]). Let \(X\) and \(Y\) be two Banach spaces, \(U \subset X\) an open set of \(X\) and \(I\) an open interval of \(\mathbb{R}\). We assume that \(0 \in U\). Denote by \(\varphi\) the elements of \(U\) and \(\lambda\) the elements of \(I\). Let \(F : I \times U \to Y\) be a twice continuously differentiable function such that

(i) \(F(\lambda, 0) = 0\) for all \(\lambda \in I\),

(ii) \(\ker(D_{\varphi}F(\lambda_*, 0)) = \mathbb{R}\varphi_*\) for some \(\lambda_* \in I\) and \(\varphi_* \in X \setminus \{0\}\),

(iii) \(\text{Codim } \text{Im}(D_{\varphi}F(\lambda_*, 0)) = 1\),

(iv) \(D_{\lambda}D_{\varphi}F(\lambda_*, 0)(\varphi) \notin \text{Im}(D_{\varphi}F(\lambda_*, 0))\).

Then for any complement \(Z\) of the subspace \(\mathbb{R}\varphi_*\), spanned by \(\varphi_*\), there exists a continuous curve
\[ (-\varepsilon, \varepsilon) \to \mathbb{R} \times Z, \quad s \mapsto (\lambda(s), \varphi(s)) \]
such that

(i) \(\lambda(0) = \lambda_*, \quad \varphi(0) = 0\),

(ii) \(s(\varphi_* + \varphi(s)) \in U\),

(iii) \(F(\lambda(s), s(\varphi_* + \varphi(s))) = 0\).

Moreover, the set of solutions to the equation \(F(\lambda, u) = 0\) in a neighborhood of \((\lambda_*, 0)\) is given by the curve \(\{(\lambda, 0), \lambda \in \mathbb{R}\}\) and \(\{s(\varphi_* + \varphi(s)), s \in (-\varepsilon, \varepsilon)\}\).

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