On finite generation of self-similar groups of finite type

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Abstract

A self-similar group of finite type is the profinite group of all automorphisms of a regular rooted tree that locally around every vertex act as elements of a given finite group of allowed actions. We provide criteria for determining when a self-similar group of finite type is finite, level-transitive, or topologically finitely generated. Using these criteria and GAP computations we show that for the binary alphabet there is no infinite topologically finitely generated self-similar group given by patterns of depth 3, and there are 32 such groups for depth 4.

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1 Introduction

There are two important classes of groups acting on regular rooted trees that have arisen as a generalization of the Grigorchuk group: self-similar groups and branch groups. An automorphism group of a regular rooted tree is self-similar if the restriction of the action of every its element onto every subtree can be given again by an element of the group. There are many examples of self-similar groups with numerous extreme properties (like the Grigorchuk group) and this class of groups is very promising for looking different counterexamples. At the same time, self-similar groups appear naturally in many areas of mathematics and have strong connections with fractal geometry, dynamical systems, automata theory (see [8] and the references therein). Branch groups are automorphism groups of a tree whose subgroup lattice is similar to the tree [1]. This class plays an important role in classification of just-infinite groups [5].

Self-similar groups of finite type have arisen as the closure of certain self-similar branch groups in the topology of the tree. It was noticed in [1] Section 7] that the closure of the Grigorchuk group is a profinite self-similar group that can be described by a finite group of allowed local actions on a finite tree (obtained from the binary tree by truncating at some depth). R.I. Grigorchuk used this observation to define a self-similar group of finite type
as the group of all tree automorphisms that locally around every vertex act as elements of a given finite group (see precise definition in the next section). The term “group of finite type” comes from the analogy with shifts of finite type in symbolic dynamics \cite{7} (note that a different term, namely finitely constrained group, is used in \cite{9} \cite{10}). Every self-similar group of finite type with transitive action on levels of the tree is a profinite branch group by \cite[Proposition 7.5]{4}, and conversely, the closure (and profinite completion) of a self-similar regular branch group with congruence subgroup property is a self-similar group of finite type by \cite[Theorem 3]{9}. The last observation was the main ingredient to compute the Hausdorff dimension of such branch groups in \cite{9}.

Although a self-similar group of finite type is easy to define by a finite group of patterns, it is not clear what are the properties of the group. In particular, R.I. Grigorchuk asked in \cite[Problem 7.3(i)]{4} under what conditions a self-similar group of finite type is topologically finitely generated. In this note we address this question and establish certain criterion in Theorem 3 as well as some necessary and sufficient conditions. We also answer such basic questions like how to check whether a self-similar group of finite type is trivial, finite, or acts transitively on levels of the tree.

The closure of the Grigorchuk group is a self-similar group of finite type defined by patterns of depth 4 over the binary tree. The closure of groups defined in \cite{9} give examples of infinite finitely generated self-similar groups of finite type defined by patterns of depth \( d \) for any \( d \geq 4 \). For depth 2 and binary tree every self-similar group of finite type is either finite or not finitely generated as shown in \cite{10}. The only unknown case was for depth 3. Using developed criteria and GAP computations we prove that there is no infinite finitely generated self-similar group of finite type defined by patterns of depth 3 over the binary tree. For depth 4 there are 32 such groups (including the closures of the Grigorchuk group and the iterated monodromy group of \( z^2 + i \) \cite{6}).

\section{Self-similar groups of finite type}

In this section we first recall all needed definitions and introduce self-similar groups of finite type (see \cite{8} \cite{4} for more information). After that we study conditions when a self-similar group of finite type is trivial, finite, or level-transitive.

**Tree** \( X^* \). Let \( X \) be a finite alphabet with at least two letters. Let \( X^* \) be the free monoid freely generated by \( X \). The elements of \( X^* \) are all finite words \( x_1 x_2 \ldots x_n \) over \( X \) (including the empty word). We also use notation \( X^* \) for the tree with the vertex set \( X^* \) and edges \((v, vx)\) for all \( v \in X^* \) and \( x \in X \). The set \( X^n \) is the \( n \)-th level of the tree \( X^* \). The subtree of \( X^* \) induced by the set of vertices \( \cup_{i=0}^n X^i \) is denoted by \( X^{[n]} \).

Self-similar groups of finite type are defined as special subgroups of the group \( \text{Aut} X^* \) of all automorphisms of the tree \( X^* \). The group \( \text{Aut} X^* \) is profinite; it is the inverse limit of the sequence
\[ \ldots \rightarrow \text{Aut} X^{[3]} \rightarrow \text{Aut} X^{[2]} \rightarrow \text{Aut} X, \]
where the homomorphisms are given by restriction of the action.
Sections of automorphisms. For every automorphism \( g \in \text{Aut} \, X^* \) and every word \( v \in X^* \) define the section \( g(v) \in \text{Aut} \, X^* \) of \( g \) at \( v \) by the rule: \( g(v)(x) = y \) for \( x, y \in X^* \) if and only if \( g(vx) = g(v)y \). In other words, the section of \( g \) at \( v \) is the unique automorphism \( g(v) \) of \( X^* \) such that \( g(vx) = g(v)g(u)(x) \) for all \( x \in X^* \). Sections have the following properties:

\[
g(vu) = g(v)(u), \quad (g^{-1})(v) = (g^{-1}(v))^{-1}, \quad (gh)(v) = g(h(v))h(v)
\]

for all \( v, u \in X^* \) (we are using left actions, i.e., \((gh)(v) = g(h(v))\)).

A subgroup \( G < \text{Aut} \, X^* \) is called self-similar if \( g(v) \in G \) for every \( g \in G \) and \( v \in X^* \).

The restriction of the action of an automorphism \( g \in \text{Aut} \, X^* \) to the subtree \( X^{[d]} \)

\]
denoted by \( g|_{X^{[d]}} \in \text{Aut} \, X^{[d]} \). To every \( g \in \text{Aut} \, X^* \) there corresponds a collection \((g(v)|_{X^{[d]}})_{v \in X^*}\) of automorphisms from \( \text{Aut} \, X^{[d]} \) which completely describe the action of \( g \) on the tree \( X^* \).

Self-similar groups of finite type. A subgroup \( \mathcal{P} \) of \( \text{Aut} \, X^{[d]} \) will be called a group of patterns of depth \( d \) (or a pattern group of depth \( d \)), \( d \geq 1 \). We say that an automorphism \( g \in \text{Aut} \, X^* \) agrees with \( \mathcal{P} \) if every section \( g(v), \ v \in X^*, \) acts on \( X^{[d]} \) in the same way as some element in \( \mathcal{P} \), i.e., \( g(v)|_{X^{[d]}} \in \mathcal{P} \) for all \( v \in X^* \). Since \( \mathcal{P} \) is a group, the inverse \( g^{-1} \) and all sections \( g(v) \) of such an element \( g \) agree with \( \mathcal{P} \), the product of two elements that agree with \( \mathcal{P} \) also agrees with \( \mathcal{P} \). We obtain the self-similar group \( G_\mathcal{P} \) of all automorphisms \( g \in \text{Aut} \, X^* \) that agree with \( \mathcal{P} \), i.e., we define the group

\[
G_\mathcal{P} = \{ g \in \text{Aut} \, X^*: g(v)|_{X^{[d]}} \in \mathcal{P} \text{ for every } v \in X^* \},
\]

called the self-similar group of finite type given by the pattern group \( \mathcal{P} \). Note that Grigorchuk in [1] introduced these groups using finite sets of forbidden patterns, while we are using “allowed” patterns.

Every group \( G_\mathcal{P} \) is closed in the topology of \( \text{Aut} \, X^* \). Indeed, if for an element \( g \in \text{Aut} \, X^* \) the restriction \( g|_{X^{[n]}} \) belongs to \( G_\mathcal{P}|_{X^{[n]}} \) for every \( n \in \mathbb{N} \), then \( g(v)|_{X^{[d]}} \in \mathcal{P} \) for every \( v \in X^* \) and thus \( g \in G_\mathcal{P} \). Hence \( G_\mathcal{P} \) is a profinite group.

Let us consider a few simple examples. If \( \mathcal{P} \) is trivial then \( G_\mathcal{P} \) is trivial. If \( \mathcal{P} = \text{Aut} \, X^{[d]} \) then \( G_\mathcal{P} = \text{Aut} \, X^* \) (for any \( d \in \mathbb{N} \)). For every subgroup \( \mathcal{P} < \text{Sym}(X) \) the infinitely iterated permutational wreath product \( \ldots \mathcal{P} \ldots \mathcal{P} \) is a self-similar group of finite type, where \( \mathcal{P} \) is the corresponding group of patterns of depth 1, and every self-similar group of finite type given by patterns of depth 1 is of this form.

Minimal pattern groups. The same self-similar group of finite type may be given by different groups of patterns of depth \( d \) and we want to choose a unique pattern group in each class. Let \( G \) be a self-similar group of finite type given by a group of patterns of depth \( d \) and consider the pattern group \( \mathcal{P} = G|_{X^{[d]}} \). Since the group \( G \) is self-similar, \( g(v)|_{X^{[d]}} \in \mathcal{P} \) for every \( g \in G \) and \( v \in X^* \), and thus \( G < G_\mathcal{P} \). On the other hand, it is clear from the definition that every pattern group of depth \( d \) that produces \( G \) contains \( \mathcal{P} \) as a subgroup. Hence \( G = G_\mathcal{P} \) and \( \mathcal{P} \) is the smallest group of patterns of depth \( d \) with this property. A pattern group \( \mathcal{P} \) of depth \( d \) will be called minimal if the equality \( G_\mathcal{P} = G_Q \) for \( Q < \text{Aut} \, X^{[d]} \) implies \( \mathcal{P} < Q \). It follows from the above arguments that a pattern group \( \mathcal{P} \)
of depth $d$ is minimal if and only if $\mathcal{P} = G\mathcal{P}|_{X^{|d|}}$, in other words, if every pattern in $\mathcal{P}$ is realized as a restriction of an element of $G\mathcal{P}$. Every self-similar group of finite type given by patterns of depth $d$ is represented by a unique minimal pattern group of depth $d$.

**Pattern graph.** Let $\mathcal{P}$ be a group of patterns of depth $d$. In order to minimize $\mathcal{P}$ one can use a directed labeled graph $\Gamma\mathcal{P}$ which we call the pattern graph associated to $\mathcal{P}$. The vertices of $\Gamma\mathcal{P}$ are the elements of $\mathcal{P}$ and for $a, b \in \mathcal{P}$ and $x \in X$ we put a labeled arrow $a \xrightarrow{x} b$ whenever $a(x)|_{X^{|d-1|}} = b|_{X^{|d-1|}}$. Informally, the arrow $a \xrightarrow{x} b$ shows that we can use the pattern $b$ to extend the action of $a$ on the subtree $xX^{[d]}$ (see Fig. 1). If a vertex $a \in \mathcal{P}$ does not have an outgoing edge labeled by some letter $x \in X$, then the action of $a$ cannot be extended to the next level using patterns from $\mathcal{P}$; in other words $a$ is not a restriction of an element of $G\mathcal{P}$. Now it is clear how to minimize $\mathcal{P}$: we just remove every vertex that does not have an outgoing edge labeled by some $x \in X$ and repeat this reduction as long as possible. The remaining patterns will form a minimal pattern group for $G\mathcal{P}$. In particular, $\mathcal{P}$ is minimal if and only if every vertex of $\Gamma\mathcal{P}$ has an outgoing edge labeled by every $x \in X$.

![Figure 1: Coordination between patterns.](image)

The graph $\Gamma\mathcal{P}$ can be used to represent elements of the group $G\mathcal{P}$ by graph homomorphisms as follows. Let us take the tree $X^*$ and add direction and label to every edge by $v \xrightarrow{x} vx$ for every $v \in X^*$ and $x \in X$. Then every element $g \in G\mathcal{P}$ defines a homomorphism $\phi : X^* \to \Gamma\mathcal{P}$ of labeled directed graphs by the rule $\phi(v) = g(v)|_{X^{|d|}}$. Indeed, for every arrow $v \xrightarrow{vx}$ in the tree $X^*$ the elements $g(v)(x)$ and $g(vx)$ are the same and we have the arrow $\phi(v) \xrightarrow{x} \phi(vx)$ in the graph $\Gamma\mathcal{P}$. And vise versa, every homomorphism $\phi : X^* \to \Gamma\mathcal{P}$ defines an element $g \in G\mathcal{P}$ by its restrictions $g(v)|_{X^{|d|}} = \phi(v)$. This description is an analog of a standard statement in symbolic dynamics that every shift of finite type is sofic (see [7, Theorem 3.1.5]), and pattern graphs play a role of recognition graphs. One can use this observation to introduce the notion of a self-similar group of sofic type which we will discuss elsewhere.

**Branching properties.** Let us explain the connection between self-similar groups of finite type and branch groups mentioned in Introduction.

Let $G$ be a subgroup of $\text{Aut} X^*$. The vertex stabilizer $\text{St}_G(v)$ of a vertex $v \in X^*$ is the subgroup of all $g \in G$ such that $g(v) = v$. The $n$-th level stabilizer $\text{St}_G(n)$ is the subgroup of all $g \in G$ such that $g(v) = v$ for every $v \in X^n$. Notice that $\text{St}_G(v)$ and $\text{St}_G(n)$ have finite index in $G$. The rigid vertex stabilizer $\text{RiSt}_G(v)$ of a vertex $v \in X^*$ is the subgroup
of all \( g \in G \) such that \( g(u) = u \) for every vertex \( u \in X^* \setminus vX^* \). The set of all sections \( g(v) \) for \( g \in \text{RiSt}_G(v) \) forms a group which we call the section group of \( \text{RiSt}_G(v) \) at the vertex \( v \). The group \( G \) is called level-transitive if it acts transitively on all levels \( X^n \) of the tree. The group \( G \) is called regular branch branching over its subgroup \( K \) if \( G \) is level-transitive, \( K \) is a normal subgroup of finite index, and the group of all automorphism \( g \in \text{St}_{\text{Aut}X^*}(1) \) such that the tuple \((g(x))_{x \in X}\) belongs to \( \prod_X K \) is a subgroup of finite index in \( K \). Note that the last condition implies that the section group of \( \text{RiSt}_K(v) \) at \( v \) contains \( K \) for every vertex \( v \in X^* \).

Every level-transitive self-similar group \( G_P \) of finite type given by patterns of depth \( d \) is regular branch over its level stabilizer \( \text{St}_{G_P}(d-1) \) (see [1, Proposition 7.15]). Indeed, notice that for every element \( h \in \text{St}_{G_P}(d-1) \) and any vertex \( v \in X^* \) the unique automorphism \( g \in \text{RiSt}_{\text{Aut}X^*}(v) \) such that \( g(v) = h \) agrees with the pattern group \( P \) and hence belongs to \( G_P \). It follows that \( \text{St}_{G_P}(n) \) for \( n \geq d \) decomposes into the direct product

\[
\text{St}_{G_P}(n) \cong \text{St}_{G_P}(d-1) \times \ldots \times \text{St}_{G_P}(d-1)
\]

of \( |X|^{n-d+1} \) copies of \( \text{St}_{G_P}(d-1) \), where each factor acts on the corresponding subtree \( vX^* \) for \( v \in X^{n-d+1} \). The last condition in the definition of a regular branch group follows. Conversely, if \( G \) is a self-similar regular branch group branching over its level stabilizer \( \text{St}_G(d-1) \) then the closure of \( G \) in \( \text{Aut} X^* \) is a self-similar group of finite type given by patterns of depth \( d \) (see [9, Theorem 3]).

**Triviality, finiteness, and level-transitivity of \( G_P \).** Given a pattern group \( P \) we want to understand whether the group \( G_P \) is trivial, finite, or acts transitively on the levels of the tree \( X^* \). The answer to the question about triviality of \( G_P \) directly follows from the definition of a minimal pattern group. Namely, the group \( G_P \) is trivial if and only if minimizing \( P \) we obtain the trivial group.

The finiteness of \( G_P \) can be effectively checked using the next statement.

**Proposition 1.** Let \( P \) be a minimal pattern group of depth \( d \). The group \( G_P \) is finite if and only if the stabilizer \( \text{St}_P(d-1) \) is trivial, and in this case \( G_P \) is isomorphic to \( P \).

**Proof.** Let \( \Gamma_P \) be the pattern graph of \( \mathcal{P} \) and put \( m = |\text{St}_P(d-1)| \). Notice that \((bc)|_{X^{d-1}} = b|_{X^{d-1}} \) for every \( b \in \mathcal{P} \) and \( c \in \text{St}_P(d-1) \). Hence if \( a \xrightarrow{x} b \) is an arrow in \( \Gamma_P \) then \( a \xrightarrow{x} bc \) is also an arrow in \( \Gamma_P \) for every \( c \in \text{St}_P(d-1) \), and every outgoing arrow at \( a \) with label \( x \) is of this form. Therefore, since \( \mathcal{P} \) is minimal, every vertex of \( \Gamma_P \) has precisely \( m \) outgoing edges labeled by \( x \) for every \( x \in X \). It follows that for every \( a \in \mathcal{P} \) there are precisely \( m_{|X^d} \) elements \( g \in G_P|_{X^{d+1}} \) such that \( g|_{X^d} = a \); in other words, every pattern in \( \mathcal{P} \) has \( m_{|X^d} \) extensions to the next level. Then for each level \( n > d \) and for every \( f \in G_P|_{X^n} \) there are precisely \( m_{|X|^{n-d+1}} \) elements \( g \in G_P|_{X^{n+1}} \) such that \( g|_{X^n} = f \). Now we can compute the total number of elements in the restriction \( G_P|_{X^n} \):

\[
|G_P|_{X^n} = |\mathcal{P}| \cdot m_{|X^d}|X|^{2+\ldots+n}|X|^{n-d}, \text{ for } n > d.
\]

Therefore the group \( G_P \) is finite if and only if \( m = 1 \), i.e., when the group \( \text{St}_P(d-1) \) is trivial. In this case, \( |G_P| = |\mathcal{P}| \) and the restriction \( g \mapsto g|_{X^d} \) is an isomorphism between \( G_P \) and \( \mathcal{P} \). \( \square \)
It follows from the proof that we can also use the pattern graph $\Gamma_{\mathcal{P}}$ to check the finiteness of $G_{\mathcal{P}}$. If $\mathcal{P}$ is minimal, then the group $G_{\mathcal{P}}$ is finite if and only if some (equivalently, every) vertex of $\Gamma_{\mathcal{P}}$ has only one outgoing edge labeled by $x$ for each $x \in X$.

Let us treat transitivity on levels. We will use the standard observation that a subgroup $G < \text{Aut} X$ acts transitively on $X^{n+1}$ if and only if it acts transitively on $X^n$ and the stabilizer $\text{St}_G(v)$ of some (every) vertex $v \in X^n$ acts transitively on $vX$.

Let $\mathcal{P}$ be a minimal pattern group of depth $d$ and consider the self-similar group of finite type $G_{\mathcal{P}}$. We fix a letter $x \in X$ and use notation $x^n$ for the word $x \ldots x$ ($n$ times). Let $\mathcal{P}_n$ be the group of all elements $a \in \mathcal{P}$ for which there exists $g \in \text{St}_{G_{\mathcal{P}}}(x^n)$ such that $g(x^n)|_{X^{[d]}} = a$. Then $\text{St}_{G_{\mathcal{P}}}(x^n)$ is transitive on $x^nX$ if and only if $\mathcal{P}_n$ is transitive on $X$. It follows that $G_{\mathcal{P}}$ is level-transitive if and only if each group $\mathcal{P}_n$ for $n \geq 0$ is transitive on $X$. Notice that the groups $\mathcal{P}_n$ can be computed recursively by the rule: $\mathcal{P}_0 = \mathcal{P}$ and

$$\mathcal{P}_{n+1} = \{a \in \mathcal{P}_n : \text{there exists } b \in \text{St}_{\mathcal{P}_n}(x) \text{ such that } b(x)|_{X^{[d-1]}} = a|_{X^{[d-1]}}\}.$$ 

We obtain a decreasing sequence $\mathcal{P} > \mathcal{P}_1 > \ldots$ of finite groups which should stabilize on some subgroup $\mathcal{Q} < \mathcal{P}$, $\mathcal{Q} = \cap_{n \geq 0} \mathcal{P}_n$. Moreover, if we take the smallest $n$ such that $\mathcal{P}_n = \mathcal{P}_{n+1}$ then $\mathcal{P}_n = \mathcal{P}_{n+k}$ for every $k \in \mathbb{N}$, and thus $\mathcal{Q} = \mathcal{P}_n$. Hence the group $\mathcal{Q}$ can be algorithmically computed. We have proved the following effective criterium.

**Proposition 2.** The group $G_{\mathcal{P}}$ is level-transitive if and only if the group $\mathcal{Q}$ is transitive on $X$.

### 3 Finite generation of groups $G_{\mathcal{P}}$

In this section we study when the group $G_{\mathcal{P}}$ is topologically finitely generated. Further we omit the word “topologically”.

**Theorem 3.** Let $G$ be a level-transitive self-similar group of finite type given by patterns of depth $d$. The group $G$ is finitely generated if and only if there exists $n \geq d$ such that the commutator of $\text{St}_G(d-1)|_{X^{[n]}}$ contains $\text{St}_G(n-1)|_{X^{[n]}}$.

**Proof.** Let $G = G_{\mathcal{P}}$ for a minimal pattern group $\mathcal{P}$ of depth $d$.

First we prove the necessity. The proof will not use transitivity on levels. Assume that the commutator of $\text{St}_G(d-1)|_{X^{[n]}}$ does not contain $\text{St}_G(n-1)|_{X^{[n]}}$ for every $n \geq d$. Let us prove that $\text{St}_G(d-1)$ and thus $G$ are not finitely generated. In the proof we will use notations $S = \text{St}_G(d-1)$ and $S_n = S|_{X^{[n]}}$. For each $m \geq d$ consider the homomorphism

$$\varphi : S \to \prod_{n=d}^m S_n /[S_n, S_n], \quad \varphi(g) = (g|_{X^{[n]}}[S_n, S_n])_{n=d}^m.$$ 

Recall that the stabilizer $\text{St}_G(n-1)|_{X^{[n]}}$ decomposes into the direct product $\text{St}_{\mathcal{P}}(d-1) \times \ldots \times \text{St}_{\mathcal{P}}(d-1)$ of $|X|^{n-d}$ copies of $\text{St}_{\mathcal{P}}(d-1)$. By our assumption there exists an element $g_n = (1, \ldots, a_n, \ldots, 1)$, $a_n \in \text{St}_{\mathcal{P}}(d-1)$, of this product that does not belong to the
commutator \([S_n, S_n]\). Let \(A_n\) be the group generated by the image of \(g_n\) in the quotient of \(\text{St}_G(n - 1)|_{X[n]}\) by \([S_n, S_n]\). The group \(A_n\) is a nontrivial subgroup of the finite abelian group \(S_n/[S_n, S_n]\). Hence \(A_n\) is also a quotient of \(S_n/[S_n, S_n]\). Composing with \(\varphi\) we obtain a homomorphism from \(S\) to \(\prod_{n=d}^m A_n\). Moreover, for \(i < n\) the \(i\)-th component of the image of \(g_n\) in this direct product is trivial. It follows that \(\prod_{n=d}^m A_n\) is a homomorphic image of \(S\). Since \(|A_n| \leq |P|\) for all \(n\), the number of generators of \(\prod_{n=d}^m A_n\) goes to infinity as \(m\) goes to infinity. Hence \(S\) is not finitely generated.

Let us prove the converse. Fix \(k \geq d\) such that the commutator of \(\text{St}_G(d - 1)|_{X[k]}\) contains \(\text{St}_G(k - 1)|_{X[k]}\). We construct a finitely generated dense subgroup of \(G\) using the techniques from branch groups (see [1, 2]). Let \(f_1, \ldots, f_l\) and \(h_1, \ldots, h_m\) be the elements of \(G\) such that

\[
\langle f_1, \ldots, f_l \rangle|_{X[1+d+k]} = G|_{X[1+d+k]} \quad \text{and} \quad \langle h_1, \ldots, h_m \rangle|_{X[k]} = \text{St}_G(d - 1)|_{X[k]}.
\]

The group \(\text{St}_P(d - 1)\) is nontrivial by Proposition [1] and we can find \(v \in X^d\) and \(a \in \text{St}_G(d - 1)\) such that \(a(v) \neq v\) (the element \(a\) will be used to shift the section of certain automorphisms at the vertex \(v\)). Fix two letter \(x, y \in X\), \(x \neq y\). Define the automorphisms \(g_1, \ldots, g_m\) recursively by their sections:

\[
g_i(yv) = h_i \quad \text{and} \quad g_i(x) = g_i, \quad i = 1, \ldots, m,
\]

and the other sections are trivial (see Fig. [2]). Notice that \(g_1, \ldots, g_m\) belong to \(G\).

Consider the group \(H = \langle f_1, \ldots, f_l, h_1, \ldots, h_m, g_1, \ldots, g_m \rangle\) and let us show that \(H\) is dense in \(G\). We need to prove that \(H|_{X[n]} = G|_{X[n]}\) for all \(n \in \mathbb{N}\). The statement holds for \(n \leq 1 + d + k\) by construction. By induction on \(n\) assume that we have proved it for all levels \(\leq n + d + k\). There exists an element \(g \in G\) such that

\[
g(x^iy) = a \quad \text{for} \quad i = 0, \ldots, n - 1
\]

and the other sections are trivial (see Fig. [2]). By inductive hypothesis there exists \(h \in H\) such that \(h|_{X[n+d+k]} = g|_{X[n+d+k]}\). Then the commutator \([h^{-1}g_i h, g_j]\) acts trivially on the vertices in \(X^{n+d+k+1}\setminus x^n y v X[k]\) and at the vertex \(x^n y v\) has section

\[
[h^{-1}g_i h, g_j](x^n y v) = [h^{-1}g_i h, h(x^n y v), h_j].
\]
Conjugating by generators $g_1, \ldots, g_m$ we obtain that the section group of $\text{RiSt}_H(x^n yv)\big|_{X^{[n+d+k+1]}}$ at $x^n yv$ contains the commutator of $\text{St}_G(d-1)\big|_{X^k}$ and hence $\text{St}_G(k-1)\big|_{X^k}$. Since the action is transitive this holds for every vertex of the level $X^{n+d+1}$. Hence $\text{St}_H(n + d + k)\big|_{X^{[n+d+k+1]}} = \text{St}_G(n + d + k)\big|_{X^{[n+d+k+1]}}$ and the statement follows.

Remark 1. An automorphism of the tree $X^*$ is called finite-state if it has finitely many sections (the term comes from automata theory); a subgroup is finite-state if it consists of finite-state automorphisms. We can always choose elements $f_1, \ldots, f_i$ and $h_1, \ldots, h_m$ so that they are finite-state. Then the elements $g_1, \ldots, g_m$ and the group $H$ constructed in the proof will be also finite-state. Adding sections of elements we obtain a finitely generated finite-state self-similar dense subgroup in $G$.

Remark 2. The condition of level-transitivity cannot be dropped in Theorem 3. For example, consider the alternating group $A_5$ with the natural action on $\{1, 2, 3, 4, 5\}$, extend the action to the alphabet $X = \{0, 1, 2, 3, 4, 5\}$ by putting $\pi(0) = 0$ for every $\pi \in A_5$, and consider the infinitely iterated permutational wreath product $G_{A_5} = \ldots \wr_X A_5 \wr_X A_5$. The group $A_5$ is perfect, i.e., $[A_5, A_5] = A_5$, hence the condition in Theorem 3 holds for $n = d = 1$. However the group $G$ is not finitely generated, because the map $g \mapsto (g(0^n)|_X)_{n \in \mathbb{N}}$ is a surjective homomorphism from $G$ to the product $\prod_{n \in \mathbb{N}} A_5$ which is not finitely generated.

Remark 3. It is not difficult to see that for a group $G_P$ given by a transitive pattern group $P$ of depth 1 the condition in the theorem holds for some $n$ if and only if the group $P$ is perfect. Hence Theorem 3 generalizes Corollary 3.6 in [2] about finite generation of iterated permutational wreath products $\ldots \wr_X P \wr_X P$.

Proposition 4. Let $G$ be a self-similar group of finite type given by patterns of depth $d$. If there exists $n \geq d$ such that the commutator of $G\big|_{X^{[n]}}$ does not contain $\text{St}_G(n-1)\big|_{X^{[n]}}$ then the group $G$ is not finitely generated.

Proof. The proof uses the same arguments as in the first part of the proof above. Fix $n \geq d$ such that the commutator of $G_n := G\big|_{X^{[n]}}$ does not contain $\text{St}_G(n-1)\big|_{X^{[n]}}$. For every $k \in \mathbb{N}$ consider the map

$$\varphi_k : G \to G_n/[G_n, G_n], \quad \varphi_k(g) = \prod_{v \in X^k} g(v)\big|_{X^{[n]}}[G_n, G_n].$$

Since $G_n/[G_n, G_n]$ is abelian every map $\varphi_k$ is a homomorphism. Now for every $m \in \mathbb{N}$ consider the homomorphism $\varphi : G \to \prod_{k=1}^m G_n/[G_n, G_n], \quad \varphi(g) = (\varphi_k(g))_{k=1}^m$. For every $k$ and every pattern $a \in \text{St}_G(n-1)\big|_{X^{[n]}}$ there exists $g$ in the rigid stabilizer $\text{RiSt}_G(v)$ of a vertex $v \in X^k$ such that $g(v)|_{X^{[n]}} = a$, and thus $\varphi_k(g) = a$ and $\varphi_i(g) = e$ for $i < k$. Since $\text{St}_G(n-1)\big|_{X^{[n]}}/[G_n, G_n]$ is a homomorphic image of $G_n$, it follows that the abelian group $\prod_{k=1}^m \text{St}_G(n-1)\big|_{X^{[n]}}/[G_n, G_n]$ is a homomorphic image of $G$ for every $m$. Hence $G$ is not finitely generated.

The next statement generalizes Proposition 2 in [10].
Corollary 5. Let $\mathcal{P}$ be an abelian pattern group. The group $G_{\mathcal{P}}$ is finitely generated if and only if it is finite.

Proof. The statement follows from Proposition 4 and Proposition 5 with $n = d$. □

Corollary 6. Take a cyclic subgroup $C < \text{Sym}(X)$ and consider the group $C \wr X C$ as a natural subgroup of $\text{Aut} X^{[2]} \cong \text{Sym}(X) \wr X \text{Sym}(X)$. Then for any nilpotent pattern group $\mathcal{P} < C \wr X C$ the group $G_{\mathcal{P}}$ is finitely generated if and only if it is finite.

Proof. Since $\mathcal{P}/\text{St}_{\mathcal{P}}(1)$ is cyclic, the commutator $[\mathcal{P}, \mathcal{P}]$ is a subgroup of $\text{St}_{\mathcal{P}}(1)$. If it is a proper subgroup then the group $G_{\mathcal{P}}$ is not finitely generated by Proposition 4. Suppose $[\mathcal{P}, \mathcal{P}] = \text{St}_{\mathcal{P}}(1)$. For any $a, b \in \mathcal{P}$ there exists $k \in \mathbb{N}$ such that $a^k b$ or $b^k a$ belongs to $\text{St}_{\mathcal{P}}(1)$. Using the equality $[a, b] = [a, a^k b] = [b^k a, b]$ we obtain that $[\mathcal{P}, \mathcal{P}] = [\mathcal{P}, \text{St}_{\mathcal{P}}(1)]$. Since $\mathcal{P}$ is nilpotent, the last equality implies that $[\mathcal{P}, \mathcal{P}] = \text{St}_{\mathcal{P}}(1) = \{1\}$ and hence the group $G_{\mathcal{P}}$ is finite by Proposition 1. □

4 A few classification results

In this section we classify self-similar groups of finite type for the binary alphabet $X = \{0, 1\}$ and depth $\leq 4$. All computations were made in GAP. Our strategy for classifying self-similar groups of finite type of a given depth $d$ is the following. First we find all subgroups in $\text{Aut} X^{[d]}$, then minimize all subgroups and obtain the number of all minimal pattern groups, which is equal to the number of self-similar groups of finite type of a given depth as subgroups in $\text{Aut} X^*$. Further we distinguish all finite groups using Proposition 1. Then we apply Proposition 4 for small values of $n$ to distinguish groups that are not finitely generated. An infinite self-similar group over the binary alphabet is level-transitive (see [3, Lemma 3]), hence the rest of the groups are level-transitive and we can apply Theorem 3. In this way it was possible to obtain the following results.

Depth $d = 2$. This case was treated in [10]. There are ten subgroups in $\text{Aut} X^{[2]}$, six minimal pattern subgroups, and hence six self-similar groups of finite type. Among them there are three finite groups, namely the trivial group and two groups isomorphic to $C_2$, and the other three groups are not finitely generated (Proposition 4 works with $n = 2$).

Depth $d = 3$. There are 576 subgroups in $\text{Aut} X^{[3]}$, 60 minimal pattern subgroups, and hence 60 self-similar groups of finite type. Among them there are 23 finite groups, namely the trivial group, two groups isomorphic to $C_2$, four groups isomorphic to $C_2 \times C_2$, 16 groups isomorphic to the dihedral group $D_8$. The other 37 groups are not finitely generated (27 groups satisfy Proposition 4 with $n = 3$ and 10 groups with $n = 4$).

Corollary 7. A self-similar group of finite type given by patterns of depth $d \leq 3$ over the binary alphabet is either finite or not finitely generated.

Depth $d = 4$. There are 4544 self-similar groups of finite type. Among them there are 1535 finite groups, namely the trivial group, two groups isomorphic to $C_2$, four groups isomorphic to $C_2 \times C_2$, 16 groups isomorphic to $D_8$, eight groups isomorphic to $C_2 \times C_2 \times C_2$, 80 groups isomorphic to $C_2 \times C_2 \times C_2 \times C_2$, 320 groups isomorphic to $C_2 \times C_2 \times C_2 \times C_2 \times C_2$, and the rest are not finitely generated.
96 groups isomorphic to $C_2 \times D_8$, 128 groups isomorphic to $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$, 256 groups isomorphic to $(((C_4 \times C_2) \times C_2) \times C_2) \times C_2$, and 1024 groups isomorphic to $\text{Aut} X[3] \cong C_2 \wr (C_2 \wr C_2)$. Among the rest of the groups there are 2977 not finitely generated (1235 groups satisfy Proposition 4 with $n = 4$, 778 groups with $n = 5$, 508 groups with $n = 6$, 200 groups with $n = 7$, and 256 groups with $n = 8$) and 32 finitely generated groups that satisfy Theorem 3 with $n = 6$. The pattern groups of these 32 self-similar groups of finite type all have order 4096, their restriction on $X[3]$ is equal to $\text{Aut} X[3]$, and among them there are 20 pairwise non-isomorphic groups. These pattern groups can be described as follows. Let us consider the group $\text{Aut} X[4]$ as a natural subgroup of the symmetric group $\text{Sym}(16)$ on the set \{1, 2, \ldots , 16\} ↔ $X^4$ and fix the permutations:

\[
\begin{align*}
a_1 &= (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) \\
a_2 &= (1,10,2,9)(3,11)(4,12)(5,14,6,13)(7,15)(8,16) \\
a_3 &= (1,10)(2,9)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) \\
a_4 &= (1,9,2,10)(3,11)(4,12)(5,14,6,13)(7,15)(8,16)
\end{align*}
\]

\[
\begin{align*}
b_1 &= (1,5)(2,6)(3,7)(4,8)(9,10) & c_1 &= (1,3)(2,4) & c_3 &= (1,3)(2,4)(5,6) \\
b_2 &= (1,6)(2,5)(3,7)(4,8)(9,10) & c_2 &= (1,4,2,3) & c_4 &= (1,4,2,3)(5,6)
\end{align*}
\]

Then the 32 pattern groups mentioned above is the family of groups $P_{ijk} = \langle a_i, b_j, c_k \rangle$. In this family: the self-similar group of finite type $G_{P_{123}}$ is the closure of the Grigorchuk group and $G_{P_{111}}$ is the closure of the iterated monodromy group of $z^2 + i$ [9].

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