Baryon Charges in 4d Superconformal Field Theories and Their AdS Duals

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We consider general aspects of the realization of R and non-R flavor symmetries in the AdS$_5 \times H_5$ dual of 4d $\mathcal{N} = 1$ superconformal field theories. We find a general prescription for computing the charges under these symmetries for baryonic operators, which uses only topological information (intersection numbers) on $H_5$. We find and discuss a new correspondence between the nodes of the SCFT quiver diagrams and certain divisors in the associated geometry. We also discuss connections between the non-R flavor symmetries and the enhanced gauge symmetries in non-conformal theories obtained by adding wrapped branes.
1. Introduction

Interesting gauge theories arise in string theory from D-branes at geometric singularities. This was first studied for orbifold singularities [1,2] and, more recently, for more general singularities. In this paper, we will be especially interested in 4d $\mathcal{N} = 1$ superconformal field theories (SCFTs) which can be string engineered by placing $N$ D3 branes at the conical singularity of a general (local) Calabi-Yau 3-fold $X_6$. For large $N$, it is useful to consider the AdS dual [3] description of these SCFTs, which is IIB string theory on $AdS_5 \times H_5$ with $H_5$ the horizon manifold [4] of $X_6$. A mirror IIA construction is to wrap $N$ D6 branes on the SYZ [5] $T^3$ of the mirror geometry $\tilde{X}_6$.

String theory and AdS/CFT provide useful insights into the SCFTs which can be so constructed. For example, AdS/CFT implies that the D3 brane world-volume gauge theory actually does flow to an interacting RG fixed point in the IR. It is thus interesting to study generally which quiver gauge theories can be string engineered, and what sorts of general predictions string theory makes about these SCFTs.

Many techniques have been developed over the past several years to help determine, given a general singularity $X_6$, precisely what is the associated world-volume quiver gauge theory (or rather theories, since there can be many Seiberg dual descriptions of the same superconformal field theory). As of yet, however, there is no completely general method to systematically answer this question. A partial answer can be found via partial resolution of orbifold singularities [6], but there is no systematic method for following the RG flow from the orbifold SCFT to that of the partially resolved singularity. Another method, which is useful for toric singularities, is to go to the mirror IIA description, where the gauge group and matter content can be determined in terms of intersections of 3-cycles [7,8,9]. Still another method, which seems to give correct results [10] even outside of its expected regime of validity, is to analyze the IIB D3-brane gauge theories (“bundles on cycles”) in the large volume limit, and then just extrapolate to the opposite limit of singular $X_6$. For a selection of additional relevant references, see [11,12,13,14,15,16,17].

We will here study some general aspects of the world-volume SCFTs which can be constructed via IIB D3-branes at singularities and aspects of their AdS duals, focusing particularly on the geometric realization of the flavor symmetries of SCFTs. We will here only consider cases where the world-volume gauge theory has already been determined by the above mentioned methods, but we hope that some of the methods we discuss could also be helpful in determining the world-volume gauge theories for more general singularities.
The world-volume gauge theories thus obtained are of the general quiver form

$$
\prod_\alpha U(Nd_\alpha),
$$

(1.1)

where $\alpha$ run over the nodes of the quiver. The quiver and coefficients $d_\alpha$ depend on the particular singularity. The theories are generally chiral, with $n_{\alpha\beta} > 0$ chiral superfield bifundamentals $Q_{\alpha\beta}^i$, $i = 1 \ldots n_{\alpha\beta}$, in the $(Nd_\alpha, Nd_\beta)$ of $U(Nd_\alpha) \times U(Nd_\beta)$. We take $n_{\beta\alpha} = -n_{\alpha\beta}$, and draw the arrows on the quiver so that $n_{\alpha\beta} > 0$ means that the arrow(s) point from node $\alpha$ to node $\beta$. Absence of gauge anomalies requires each node to have the same number of incoming and outgoing arrows, so $\sum_\beta n_{\alpha\beta}d_\beta = 0$ for every $\alpha$.

The SCFT is specified by the quiver diagram and the superpotential, which is also determined from the geometry and is a sum of terms of the form

$$
W = a_{i_1 \ldots i_k} \operatorname{Tr} Q_{\alpha_1}^{i_1} Q_{\alpha_2}^{i_2} \ldots Q_{\alpha_{i+1}}^{i_{i+1}} \ldots Q_{\alpha_k}^{i_k}
$$

(1.2)

with the bifundamental gauge indices contracted in a closed loop to form a gauge invariant meson of the quiver theory. There are also gauge invariant baryons, but these generally do not enter into the superpotential.

We will be interested in studying the bifundamentals $Q_{\alpha\beta}$ and their charges under the flavor symmetries. In the AdS dual, we only see the gauge invariant operators; in particular, we see the baryons formed from the bifundamentals rather than the bifundamentals themselves. To simplify our discussion, we will here only consider cases where all $d_\alpha = 1$ in (1.1), so the baryons are simply $B_{\alpha\beta} = \det_{N \times N}(Q_{\alpha\beta})$. Thus, the charges of the $B_{\alpha\beta}$ are just a factor of $N$ times those of the $Q_{\alpha\beta}$. In AdS/CFT the baryonic operators map to particles in the $AdS_5$ bulk, which arise as D3 branes wrapping 3-cycles of $H_5$.

The 4d $\mathcal{N} = 1$ SCFT has a global symmetry group $SU(2,2|1) \otimes \mathcal{F}$, where $SU(2,2|1)$ contains the superconformal $U(1)_R$ symmetry whose existence is necessary for a SCFT and where $\mathcal{F}$ are non-R flavor symmetries. We will be especially interested in a $U(1)^n$ subgroup of $\mathcal{F}$; these are the flavor symmetries under which all $n_{\alpha\beta}$ bifundamentals $Q_{\alpha\beta}$ carry the same charge, so the baryons are charged under these. In the AdS dual, the continuous global symmetries are all gauge symmetries in the $AdS_5$ bulk. In particular, $U(1)_R$ arises as a Kaluza-Klein gauge field coming from the metric; it is associated with a geometric isometry of the horizon manifold $H_5$. The $U(1)^n \subset \mathcal{F}$ gauge fields in $AdS_5$ arise via reduction of the IIB RR gauge field $C_4$ on $n$ independent 3-cycles of $H_5$. Since the baryons are wrapped D3s, they are charged under these gauge fields.
Supposing that $H_5$ is a regular Einstein-Sasaki manifold (this assumption might not actually be necessary for our discussion to apply), it can be written as a $U(1)$ fibration over a four dimensional surface $V_4$ \([13]\). The $U(1)$ fiber is the isometry associated with the $U(1)_R$ symmetry, and the baryons $B_{\alpha\beta}$ must wrap this fiber since they are charged under the superconformal $U(1)_R$ \([12,14]\). In addition, the baryons wrap certain holomorphic 2-cycles $L_{\alpha\beta} \subset V_4$. The holomorphic condition on the 2-cycles is necessary for the 3-cycle obtained via including the $U(1)_R$ fiber to be supersymmetric.

Therefore the baryons, and thus also the bifundamentals $Q_{\alpha\beta}$ in our quiver gauge theory, are associated with divisors $L_{\alpha\beta}$ on $V_4$. All $n_{\alpha\beta}$ bifundamentals connecting nodes $\alpha$ and $\beta$ are associated with the same divisor $L_{\alpha\beta}$, and we take $L_{\alpha\beta} = L_{\beta\alpha}$. As far as we know, a general method for determining the correct $L_{\alpha\beta}$ has not appeared in the literature, though they were discussed in detail for a particular example, $V_4 = dP_3$, in \([12]\). As we discuss, the $U(1)_R$ and flavor charges of the $Q_{\alpha\beta}$ are determined via topological intersections with the corresponding $L_{\alpha\beta}$. For example, the $U(1)_R$ charge of the baryons is related to their dimension via $R[B_{\alpha\beta}] = \frac{2}{3} \Delta[B_{\alpha\beta}]$, which is proportional to the volume of the 3-cycles which the baryon wraps \([12,14]\). This yields (when all $d_\alpha = 1$ in (1.1))

$$ R[Q_{\alpha\beta}] = \frac{2c_1 \cdot L_{\alpha\beta}}{c_1 \cdot c_1}, \quad (1.3) $$

measured by the intersection of the divisor with the first Chern class of $V_4$.

The $U(1)^n$ non-R flavor charges in $\mathcal{F}$ are given by all possible independent divisors $J_i$ of $V_4$ which are orthogonal to the first Chern class of $V_4$:

$$ J_i \cdot c_1 = 0, \quad i = 1 \ldots n. \quad (1.4) $$

This condition, via (1.3), is required for the flavor current to be $U(1)_R$ neutral. We can pick an arbitrary basis of such $J_i$, satisfying $J_i \cdot c_1 = 0$. The charges of the bifundamentals under these flavor symmetries is

$$ F_i[Q_{\alpha\beta}] = J_i \cdot L_{\alpha\beta}. \quad (1.5) $$

While the overall normalization of the R-symmetry is fixed, that of the other flavor symmetries is irrelevant.

It is interesting that string theory “knows” which is the correct superconformal $U(1)_R$ symmetry, i.e. precisely which $U(1)_R$ is the one which is in the same supermultiplet as the stress tensor. In the geometry, this preferred $U(1)_R$ is precisely that which is measured
by $c_1$, rather than some linear combination of $c_1$ and the $J_i$. Finding the correct superconformal $U(1)_R$ directly via field theory methods was, until recently, an open problem. Inspired by our geometric results discussed here, we have very recently found [19] a field theory method to determine the superconformal $U(1)_R$. We will verify in examples that our field theory condition [19] agrees with the result (1.3).

We find some interesting properties which the divisors $L_{\alpha\beta}$, which describe the bifundamentals in the quiver, must satisfy. We now summarize these results for the simplifying case where all $d_\alpha = 1$ in (1.1). First, our $U(1)_R$ and flavor symmetries (1.3) and (1.5) must not have any ABJ anomalies. This is equivalent to the requirement that, for every node $\alpha$, we must have

$$\sum_\beta |n_{\alpha\beta}|L_{\alpha\beta} = (N_f(\alpha) - 1)c_1. \tag{1.6}$$

$N_f(\alpha)$ is the total number of flavors at node $\alpha$: $N_f(\alpha) = \frac{1}{2} \sum_\beta |n_{\alpha\beta}|$. In addition, the superpotential must respect these charges. This implies that every term in the superpotential must have net divisor equal to $c_1$, since then (1.3) and (1.5) properly assign the superpotential R-charge 2 and flavor charge 0. Hence, for every non-zero superpotential term, $\Pi_{\alpha\beta}Q_{\alpha\beta}^{m_{\alpha\beta}}$, we must have

$$\sum_{\alpha\beta} m_{\alpha\beta}L_{\alpha\beta} = c_1. \tag{1.7}$$

Furthermore, we find that the $L_{\alpha\beta}$ can be written as differences of divisors $L_\alpha$, which are associated with the nodes of the quiver:

$$L_{\alpha\beta} = \frac{n_{\alpha\beta}}{|n_{\alpha\beta}|}(L_\beta - L_\alpha) + c_1\theta_{\alpha\beta} \quad \text{where} \quad \theta_{\alpha\beta} \equiv \begin{cases} 0 & \text{if } \frac{n_{\alpha\beta}}{|n_{\alpha\beta}|}(L_\beta - L_\alpha) \geq 0 \\ 1 & \text{if } \frac{n_{\alpha\beta}}{|n_{\alpha\beta}|}(L_\beta - L_\alpha) < 0. \end{cases} \tag{1.8}$$

By taking $\beta$ to be the endpoint of the $|n_{\alpha\beta}|$ arrows, and $\alpha$ the start, the factors $n_{\alpha\beta}/|n_{\alpha\beta}|$ become +1. The sign of $L$ in (1.8) refers to the sign of $c_1 \cdot L$, and we’ll always choose the $L_\alpha > 0$ in this sense. We require that all $L_{\alpha\beta} \geq 0$ because the expression (1.3) must assign non-negative R-charge to all chiral superfields. For most $L_{\alpha\beta}$, the $\theta_{\alpha\beta}$ term in (1.8) vanishes. In fact, every term in the superpotential (1.2) has precisely one $Q_{\alpha\beta}$ for which the associated $\theta_{\alpha\beta} = 1$, with the others having $\theta_{\alpha\beta} = 0$, and this ensures that (1.7) is satisfied: every superpotential term has net divisor $c_1$.

The anomaly free condition (1.6) implies that, for every node $\alpha$ of the quiver,

$$\sum_\beta n_{\alpha\beta}L_\beta = \sum_{\text{outgoing } \beta} |n_{\alpha\beta}|L_\beta - \sum_{\text{incoming } \beta} |n_{\alpha\beta}|L_\beta = 0 \mod c_1; \tag{1.9}$$
we could write the specific coefficient of \( c_1 \) on the RHS in terms of \( N_f(\alpha) \) and the \( \theta_{\alpha\beta} \), but (1.3) suffices for a later application. Outgoing \( \beta \) means those nodes where the arrow goes out from \( \alpha \), toward \( \beta \). We used the fact that, mod \( c_1 \), \( 0 = \sum_{\beta} |n_{\alpha\beta}|L_{\alpha\beta} = \sum_{\beta} n_{\alpha\beta}(L_{\beta} - L_{\alpha}) \), and \( \sum_{\beta} n_{\alpha\beta} = 0 \).

Using (1.8), the superconformal \( U(1)_R \) charges (1.3) and other flavor charges (1.5) can be expressed as differences of charges associated with the nodes of the quiver:

\[
R[Q_{\alpha\beta}] = R(\beta) - R(\alpha) + 2\theta_{\alpha\beta}, \quad R(\alpha) \equiv \frac{2c_1 \cdot L_{\alpha}}{c_1 \cdot c_1},
\]

\[
F_i[Q_{\alpha\beta}] = F_i(\beta) - F_i(\alpha) \quad F_i(\alpha) \equiv J_i \cdot L_{\alpha}.
\] (1.10)

As we discuss in section 2, the \( L_{\alpha} \) in (1.8) are expected to have some natural mathematical meaning, in some way related to a dual version of the collection of bundles on divisors \( \mathcal{O}(D_{\alpha}) \). However, we were not able to make this precise here, and did not find a fully general method to independently obtain the \( L_{\alpha} \) from first principles.

As we also discuss, Seiberg duality [20] has a simple action on the \( L_{\alpha} \). To simplify the discussion, we consider the case where the dualized gauge group at node \( \alpha \) has \( N_f = 2N_c \), so that the rank of the dualized gauge group is the same as it was originally; then all \( d_{\alpha} = 1 \), for both the original and also the dualized quivers. We write the bifundamentals associated with node \( \alpha \) as \( Q_{\alpha\beta} \), \( Q_{\alpha\gamma} \), \( Q_{\rho\alpha} \), and \( Q_{\sigma\alpha} \) with the arrows going out from node \( \alpha \) out to \( \beta \) and \( \gamma \) (which could be the same node) and into node \( \alpha \) from \( \rho \) and \( \sigma \) (which could also be the same). The dualized quiver has dual quark bifundamentals, with reversed arrows, and also bifundamentals corresponding to the mesons of the original \( U(N)_{\alpha} \) theory. We show that the duality correspondences and R-charge and flavor charge assignments imply that Seiberg dualizing node \( \alpha \) only changes the \( L_{\alpha} \) of that node, as

\[
L'_{\alpha} = L_{\beta} + L_{\gamma} - L_{\alpha}, \quad (1.11)
\]

with the \( L \)'s associated with the other nodes remaining unchanged after Seiberg duality.

One can also construct non-conformal theories, e.g. by wrapping D5 branes on cycles \( \Sigma_i \subset X_6 \), with the other directions filling the uncompactified 4d space transverse to \( X_6 \). As discussed in [13], there is a flux condition which requires that the two-cycles \( \Sigma_i \) of \( X_6 \) not intersect any compact 4-cycles (this condition also rules out wrapping D7s on 4-cycles). The cycles \( \Sigma_i \) which the D5’s wrap correspond to divisors in \( V_4 \), and the flux condition implies that they must have zero intersection with \( c_1(V_4) \). Thus the total 5-brane charge must be that of \( N_i \) D5s wrapped on divisors \( J_i \) of \( V_4 \), i.e. \( \sum_i N_i J_i \) where every \( J_i \) satisfies
c_1 \cdot J_i = 0. These are the same J_i in (1.4), corresponding to the non-R flavor symmetries of the SCFT theory without wrapped D5s. Indeed, the flavor symmetries of the SCFT without wrapped D5s become part of the gauge symmetry in the theory with wrapped D5’s:

\[ \prod_{\alpha} U(N + M_{\alpha}) \quad \text{with} \quad M_{\alpha} = \sum_i N_i J_i \cdot L_\alpha. \quad (1.12) \]

Because the flavor charges (1.5) of the bifundamentals have become part of their gauge charge in the theory (1.12) with added wrapped D5s, consistency of the theory (1.12) requires that the flavor symmetries \( F_i \) have vanishing ’t Hooft anomalies

\[ \text{Tr} F_i = 0 \quad \text{and} \quad \text{Tr} F_i F_j F_k = 0 \quad \text{for all} \quad i, j, k. \quad (1.13) \]

This can be seen to be the case from the origin of these symmetries in the \( AdS_5 \times H_5 \) dual, as the reduction of \( C_4 \) on 3-cycles of \( H_5 \): the \( C_4 \) gauge field does have the particular Chern-Simons type terms which would be needed to yield non-zero ’t Hooft anomalies upon reduction on \( H_5 \).

The outline of this paper is as follows: In section 2, we discuss how our main results, reviewed above, are obtained. In section 3, we discuss aspects of ’t Hooft anomalies and our field theory result \[19\] for determining the superconformal \( U(1)_R \). In section 4, we illustrate our ideas for the examples of certain toric and non-toric del Pezzo surfaces. We expect that the methods apply more generally.

While this paper was in preparation, Chris Herzog and James McKernan alerted us to their related work \[21\].

**Note added in revised version, July ’03:** Several of the loose ends raised in this paper were subsequently analyzed and clarified in a nice paper by Herzog and Walcher \[22\]. Among other things, they presented a precise notion of the “dual” to the exceptional collection, which is related to the \( L_\alpha \) that we introduced in (1.8).

## 2. Some string predictions

One way to find the quiver gauge theory associated with a singularity is in terms of a collection of sheaves. These often can be written as \( \mathcal{O}(D_\alpha) \), where \( D_\alpha \) is some set of divisors of \( V_4 \). Given such an collection, the number of bifundamentals can then be computed by the formula

\[ n_{\alpha \beta} = \chi(\mathcal{O}(D_\alpha), \mathcal{O}(D_\beta)) - \chi(\mathcal{O}(D_\beta), \mathcal{O}(D_\alpha)), \quad (2.1) \]
where \( \chi(\mathcal{O}(D_\alpha), \mathcal{O}(D_\beta)) = \sum_{i=0}^{3} (-1)^i \dim \text{Ext}^i(\mathcal{O}(D_\alpha), \mathcal{O}(D_\beta)) \) is the relative Euler characteristic of the two sheaves. (For an exceptional collection one of the two terms in (2.1) vanishes.) A small modification of (2.1), though, is generally needed: at certain nodes, the directions of the bifundamentals need to be flipped. This is seen in the gauge theory because otherwise some gauge groups would be anomalous. The flip is a continuation of \( N \rightarrow -N \) for the corresponding node. Precisely which nodes require such a flip can be determined by the methods of \([7,8]\).

We consider the situation where all gauge groups at nodes \( \alpha \) are \( U(N) \), to simplify the analysis of the baryons \( B_{\alpha\beta} = \det_{N \times N}(Q_{\alpha\beta}) \). If there is a multiplicity \( n_{\alpha\beta} > 1 \) of bifundamentals, there will be a corresponding multiplicity of baryons; we do not consider such baryon multiplicities further. In AdS/CFT, the baryons arise as D3 branes wrapped on 3-cycles of \( H_5 \), with the dimension of the corresponding operator directly proportional to the volume of the corresponding 3-cycle, see \([12,14]\). This yields \( R(B_{\alpha\beta}) = \frac{2}{3} \Delta(B_{\alpha\beta}) = \frac{2}{3} \mu_3 L^4 \text{Vol}(\Sigma_{\alpha\beta}^3) \), where \( \mu_3 \) is the tension of the brane. The \( \Sigma_{\alpha\beta}^3 \) corresponds to a holomorphic divisor \( L_{\alpha\beta} \) of \( V \), combined with the \( S^1 \) fiber. As discussed in \([12,14]\), we have

\[
\text{Vol}(H_5) = \left( \frac{2\pi q}{3} \right) \text{Vol}(V) = \left( \frac{\pi^3 q}{27} \right) c_1 \cdot c_1, \tag{2.2}
\]

where \( 2\pi q/3 \) is the length of the \( U(1) \) fiber and \( c_1 \) is the first Chern class of \( V \), with \( c_1 \cdot c_1 \equiv \int_V c_1 \wedge c_1 \). Here \( q \) is defined by \( c_1(V) = qc_1(U(1)) \), with \( c_1(V) \) the first Chern class of the 2 complex dimensional Kähler-Einstein manifold \( V \), which satisfies \( \omega = \frac{\pi}{3} c_1(V) \) with \( \omega \) the Kähler form of \( V \), and \( c_1(U(1)) \) is the first Chern class of the \( U(1) \) line bundle. Then \( R(B_{\alpha\beta}) = \frac{2}{3} \mu_3 L^4 \text{Vol}(\Sigma_{\alpha\beta}^3) \) yields

\[
R(B_{\alpha\beta}) = \left( \frac{2}{3} \right) \left( \frac{N\pi}{2 \text{Vol}(H_5)} \right) \left( \frac{2\pi q}{3} \right) \frac{\pi}{3} c_1 \cdot L_{\alpha\beta} = 2N \frac{c_1 \cdot L_{\alpha\beta}}{c_1 \cdot c_1}. \tag{2.3}
\]

Since \( R[B_{\alpha\beta}] = NR[Q_{\alpha\beta}] \), this yields (1.3).

The non-R flavor symmetries \( \mathcal{F} \) under which the baryons are charged come from reducing the IIB gauge field \( C_4 \) on 3-cycles of \( H_5 \). These 3-cycles must include the \( U(1) \) fiber direction, along with some divisors \( J_i \) of \( V_4 \). Since these flavor symmetries must be R-neutral, (1.3) implies that the \( J_i \) must satisfy \( c_i \cdot J_i = 0 \). A similar consideration as in (2.3) then leads to the flavor charge assignments of the baryons and hence the bifundamentals, as in (1.3). Again, the overall normalization of these non-R \( U(1) \) flavor charges is irrelevant, so we drop the normalization factor of \( 2/(c_1 \cdot c_1) \) for these.
The condition that the $U(1)_R$ symmetry be anomaly free is that, at every node $\alpha$,

$$2d_\alpha + \sum_\beta |n_{\alpha\beta}|(R[Q_{\alpha\beta}] - 1)d_\beta = 0. \quad (2.4)$$

In our situation, where all $d_\alpha = 1$, this together with (1.3) requires

$$c_1 \cdot \sum_\beta |n_{\alpha\beta}| L_{\alpha\beta} = (N_f(\alpha) - 1)c_1 \cdot c_1, \quad (2.5)$$

where $N_f(\alpha) = \frac{1}{2} \sum_\beta |n_{\alpha\beta}|$ is the number of flavors at node $\alpha$. Likewise, the condition that the $U(1)_i$ flavor symmetries associated with $J_i$ have vanishing ABJ anomaly at every node $\alpha$ is

$$\sum_\beta |n_{\alpha\beta}| F_i[Q_{\alpha\beta}] = 0. \quad (2.6)$$

This, together with (1.5), implies that

$$J_i \cdot \sum_\beta |n_{\alpha\beta}| L_{\alpha\beta} = 0 \quad \text{for all } J_i \cdot c_1 = 0. \quad (2.7)$$

Taken together, (2.5) and (2.7) imply (1.6), which is a very restrictive condition on the $L_{\alpha\beta}$.

In addition, the $L_{\alpha\beta}$ must satisfy another condition in order that the superpotential respect the $U(1)_R$ and flavor symmetries. Since every term in the superpotential must have R-charge 2 and non-R flavor charge zero, the total divisor associated with any superpotential term must be precisely $c_1$. Thus a necessary (but generally not sufficient) condition for a non-zero superpotential term $\Pi_{\alpha\beta} Q_{\alpha\beta}^{m_{\alpha\beta}}$ is

$$\sum_{\alpha\beta} m_{\alpha\beta} L_{\alpha\beta} = c_1. \quad (2.8)$$

We have found that, furthermore, the $L_{\alpha\beta}$ can be written as a difference of divisors $L_\alpha$ which are associated with the nodes of the quiver, as in (1.8). The condition (1.8) is sufficiently restrictive so that, given the $L_{\alpha\beta}$, the $L_\alpha$ can be determined up to the addition of an overall constant divisor to all $L_\alpha$, which would cancel on the RHS of (1.8).

We expect that the $L_\alpha$ must have a natural mathematical interpretation, which could be used to independently determine them. To get some insight into what this direct interpretation of the $L_\alpha$ might be, consider the process of partially resolving the geometric singularity. This corresponds to turning on a FI term at some node, which forces some
bifundamentals to then have a non-zero expectation value, Higgsing the world-volume gauge theory down to that of the resolved singularity. Thus the bifundamental divisors $L_{\alpha\beta}$ are naturally associated with differences of FI terms at the nodes $\alpha$ and $\beta$: roughly, $L_{\alpha\beta} \sim \zeta_\beta - \zeta_\alpha$. We interpret this as in (1.8). Note that the FI terms are dual to the bundles at the nodes, since we have a corresponding coupling $\int d^4x d^4\theta \zeta_\alpha V_\alpha$, so this suggests that the $L_\alpha$ are dual to the $O(D_\alpha)$. We do not presently, however, have a prescription for how to make this precise.

So, for the present work, we found the $L_\alpha$ on a case-by-case basis by first obtaining the $L_{\alpha\beta}$, and then using (1.8). We have found that, at least in all those cases which we have considered, it is possible to take $L_\alpha = D_\alpha$ for most, but not all, of the nodes. In particular, in the examples that we considered, we can take $L_\alpha = D_\alpha$ for all of those nodes for which no $N \to -N$ flip is required to get the bifundamentals via (2.1). For those nodes which do require such a flip, the $L_\alpha \neq D_\alpha$ and, since we do not yet know the general procedure for determining these $L_\alpha$, we determined them on a case-by-case basis in the examples by imposing the very restrictive consistency conditions, discussed above, which the $L_{\alpha\beta}$ must satisfy.

As mentioned in the introduction, we can also consider non-conformal theories, obtained by wrapping $N_i$ D5 branes on 2-cycles of $X_6$. Doing so leads to a quiver of the same form as in the conformal case, but with the gauge group modified, as in (1.12); as indicated, the $L_\alpha$ determine the gauge group modification:

$$\prod_\alpha U(N + M_\alpha) \quad \text{with} \quad M_\alpha = \sum_i N_i J_i \cdot L_\alpha. \quad (2.9)$$

Absence of gauge anomalies at every node $\alpha$ requires

$$\sum_\beta n_{\alpha\beta}(N + M_\beta) = 0, \quad (2.10)$$

which is indeed satisfied thanks to (1.9), since $J_i \cdot c_1 = 0$.

These theories with the wrapped D5s effectively gauge our previously global $U(1)^n$ flavor symmetries. Consider, as an example, the case where $N_1 = 1$ and all other $N_i = 0$. The gauge group is then $\prod_\alpha U(N + J_1 \cdot L_\alpha)$, which has as a subgroup $U(1) \times \prod_\alpha U(N)$, with bifundamentals $Q_{\alpha\beta}$ having charge $J_1 \cdot (L_\beta - L_\alpha)$ under the $U(1)$. The additional gauged $U(1)$ here is just the flavor $U(1)$ associated with $J_1$. Likewise, the general gauge

\footnote{We thank M. Douglas for suggesting this to us.}
theory (2.9) has as a subgroup $U(1)^n \times \prod_\alpha U(N)_\alpha$, where the $U(1)^n$ correspond to what were flavor symmetries before we added the wrapped D5s.

There is another condition on the $M_\alpha$ appearing above, which is required by the flux condition discussed e.g. in [13]:

$$c_1 \cdot \sum_\alpha (N + M_\alpha) \epsilon_\alpha D_\alpha = 0,$$

(2.11)

where $\epsilon_\alpha = +1$ for all nodes except for those which require a $N \to -N$ sign flip in (2.1); for these flipped nodes, $\epsilon_\alpha = -1$. Considering the case of no wrapped D5s, the $D_\alpha$ must satisfy

$$c_1 \cdot \sum_\alpha \epsilon_\alpha D_\alpha = 0 \quad \text{thus} \quad \sum_\alpha \epsilon_\alpha D_\alpha = \sum_i \tilde{N}_i J_i.$$  

(2.12)

If we took for the $\epsilon_\alpha D_\alpha$ the “fractional brane” charges of the sort discussed e.g. in [9], we would have $\tilde{N}_i = 0$ in (2.12), but generally we will not make that choice, instead taking the $D_\alpha$ to satisfy the weaker condition (2.12). Including wrapped D5s, we must have $\sum_\alpha M_\alpha \epsilon_\alpha c_1 \cdot D_\alpha = 0$, implying that $\sum_\alpha \epsilon_\alpha (J_i \cdot L_\alpha) D_\alpha$ is in the span of the $J_i$.

Finally, consider the action of Seiberg duality [20] on a node $\alpha$, which we suppose has $N_f = 2N_c$ in order for the gauge group to be self-dual. We also suppose that all $d_\alpha = 1$ in (1.1). We write the bifundamentals associated with node $\alpha$ as $Q_{\alpha \beta}$, $Q_{\alpha \gamma}$, $Q_{\rho \alpha}$, and $Q_{\sigma \alpha}$ with the arrows going out from node $\alpha$ out to $\beta$ and $\gamma$ (which could be the same node) and into node $\alpha$ from $\rho$ and $\sigma$ (which could also be the same). The dualized quiver has dual quark bifundamentals, with reversed arrows, and also bifundamentals corresponding to the mesons of the original $U(N)_\alpha$ theory. We write the dual quarks as $Q'_{\beta \alpha}$, $Q'_{\gamma \alpha}$, $Q'_{\rho \alpha}$, and $Q'_{\sigma \alpha}$. The bifundamentals coming from the mesons are $Q'_{\rho \beta} = Q_{\rho \alpha} Q_{\alpha \beta}$, $Q'_{\sigma \beta} = Q_{\sigma \alpha} Q_{\alpha \beta}$, $Q'_{\rho \gamma} = Q_{\rho \alpha} Q_{\alpha \gamma}$, and $Q'_{\sigma \gamma} = Q_{\sigma \alpha} Q_{\alpha \gamma}$. Since the R-charge and flavor charges, given by (1.3) and (1.5), must respect this map, the divisors associated with the meson legs of the dual quiver must satisfy

$$L'_{\rho \beta} = L_{\rho \alpha} + L_{\alpha \beta}, \quad L'_{\sigma \beta} = L_{\sigma \alpha} + L_{\alpha \beta},$$

$$L'_{\rho \gamma} = L_{\rho \alpha} + L_{\alpha \gamma}, \quad L'_{\sigma \gamma} = L_{\sigma \alpha} + L_{\alpha \gamma}.$$  

(2.13)

Duality maps the baryons of the original theory to those of the dual, as $Q_{f_1} \ldots Q_{f_{N_c}} = \epsilon_{f_1} \ldots \epsilon_{f_{N_f}} q^{1 N_c + 1} \ldots q^{f_{N_f}}$ [20]. For our theory, this implies a mapping det $Q_{\alpha \beta} = \det Q'_{\gamma \alpha}$, and hence the map $Q_{\alpha \beta} \leftrightarrow Q'_{\gamma \alpha}$. We reversed the direction of the arrows, because the dual quarks transform in the conjugate flavor representation (and we then need to apply charge
conjugation on node $\alpha$ to get back bifundamentals). We exchanged the $\beta$ and $\gamma$ because of the $\epsilon f_1...f_{N_f}$ in the baryon map, which maps e.g. $\det Q_{\alpha\beta}$ to $\det q_{\gamma\alpha}$. Likewise, the other bifundamentals map as $Q_{\alpha\gamma} \leftrightarrow Q'_{\beta\alpha}$, $Q_{\rho\alpha} \leftrightarrow Q'_{\sigma\alpha}$, and $Q_{\sigma\alpha} \leftrightarrow Q'_{\alpha\rho}$. Since the R-charge and flavor charge assignments, given by (1.3) and (1.5), must respect this map, the divisors associated with the dual quark legs of the dual quiver must satisfy

$$L'_{\gamma\alpha} = L_{\alpha\beta}, \quad L'_{\beta\alpha} = L_{\alpha\gamma}, \quad L'_{\alpha\rho} = L_{\sigma\alpha}, \quad L'_{\alpha\sigma} = L_{\rho\alpha}. \tag{2.14}$$

The dual theory has superpotential terms such as $W = \frac{1}{\mu} Q'_{\rho\beta} Q'_{\beta\alpha} Q'_{\alpha\rho} + \ldots$, which must respect the $U(1)_R$ and flavor symmetries, and hence must have total divisor $c_1$:

$$L'_{\rho\beta} + L'_{\beta\alpha} + L'_{\alpha\rho} = c_1. \tag{2.15}$$

Finally, all of the other nodes and legs of the original quiver are otherwise untouched by the Seiberg duality on node $\alpha$, so their charges, and hence leg divisor assignments, are the same in the dual as in the original theory.

All of these conditions can be satisfied very simply in terms of our relation (1.8) for writing the divisors of the quiver’s legs in terms of divisors associated with the nodes. Seiberg duality only acts on the $L_{\alpha}$ of the dualized node $\alpha$, with the $L$’s of all other nodes unchanged. The conditions (2.13) are then almost immediately satisfied, though there is apparently non-trivial conditions coming from the terms proportional to $c_1$: $\theta_{\rho\beta} = \theta_{\rho\alpha} + \theta_{\alpha\beta}$ etc.; we verified that these conditions are indeed satisfied in all of our examples. The conditions in (2.14) are also satisfied by $L'_{\alpha}$ as in (1.11), with the other node $L$’s untouched. For example, (1.8) gives for the first relation in (2.14): $L'_{\alpha} - L_{\gamma} + c_1 \theta(L'_{\alpha} - L_{\gamma}) = L_{\beta} - L_{\alpha} + c_1 \theta(L_{\beta} - L_{\alpha})$, which is indeed satisfied when $L'_{\alpha} = L_{\beta} + L_{\gamma} - L_{\alpha}$. Note also that using (2.13), (2.14), and $N_f(\alpha) = 2$, (2.15) is equivalent to (1.6).

3. ’t Hooft anomalies

It is useful to consider the ’t Hooft anomalies of the global flavor symmetries $U(1)_R$ and $U(1)_{F_i}$ of the SCFTs. The conditions (2.4) and (2.6), that $U(1)_R$ and $U(1)_{F_i}$ have vanishing ABJ anomalies at each node, ends up implying that they also have vanishing linear ’t Hooft anomalies (relevant for coupling to gravity):

$$\text{Tr} R = \text{Tr} F_i = 0. \tag{3.1}$$
This is a consequence of the quiver gauge group form (1.1), with only purely chiral, bifundamental matter.

The various cubic ’t Hooft anomalies (again, taking all \( d_\alpha = 1 \) in (1.1) to simplify) are

\[
\begin{align*}
\text{Tr} R^3 &= N^2 \sum_\alpha \left( 1 + \frac{1}{2} \sum_\beta |n_{\alpha\beta}|(R[Q_{\alpha\beta}] - 1)^3 \right), \\
\text{Tr} RF_i F_j &= \frac{1}{2} N^2 \sum_{\alpha\beta} |n_{\alpha\beta}|(R[Q_{\alpha\beta}] - 1) F_i[Q_{\alpha\beta}] F_j[Q_{\alpha\beta}], \\
\text{Tr} R^2 F_i &= \frac{1}{2} N^2 \sum_{\alpha\beta} |n_{\alpha\beta}| (R[Q_{\alpha\beta}] - 1)^2 F_i[Q_{\alpha\beta}], \\
\text{Tr} F_i F_j F_k &= \frac{1}{2} N^2 \sum_{\alpha\beta} |n_{\alpha\beta}| F_i[Q_{\alpha\beta}] F_j[Q_{\alpha\beta}] F_k[Q_{\alpha\beta}].
\end{align*}
\]

(3.2)

We can evaluate these in terms of the geometry via (1.3) and (1.5).

Interestingly, for each of the ’t Hooft anomalies (3.2), we can also make an independent prediction. For example, using the AdS/CFT prediction for the central charges \( a \) and \( c \) \cite{23} and their relation with the \( \text{Tr} R^3 \) ’t Hooft anomaly \cite{24,25} leads to the prediction

\[
\text{Tr} R^3 = \frac{8}{9} N^2 \frac{\text{Vol}(S^5)}{\text{Vol}(H_5)},
\]

(3.3)

which we can write in terms of \( q \) and \( c_1 \cdot c_1 \) using \cite{22}. (And also \( \text{Tr} R = 0 \), which we’ve already seen to indeed be the case.) So both (3.2) and (3.3) compute \( \text{Tr} R^3 \) via geometric data; hence, some mathematical identity must ensure that the two, apparently different, geometric computations always agree. We do not yet have a general understanding of this expected identity, but check that the computations indeed agree in all of our examples.

As discussed in \cite{19}, the superconformal \( U(1)_R \) charge has the property that, among all possibilities, it maximizes \( 3 \text{Tr} R^3 - \text{Tr} R \). If we write the most general \( U(1)_R \) symmetry as \( R = R_0 + \sum_i s_i F_i \), where \( R_0 \) is an arbitrary initial R-symmetry and \( s_i \) are real parameters, maximizing \( 3 \text{Tr} R^3 - \text{Tr} R \) with respect to the \( s_i \) yields: \( 9 \text{Tr} R^2 F_i = \text{Tr} F_i \) and \( \text{Tr} RF_i F_j < 0 \) \cite{19}. In the present context, where all \( U(1)_{F_i} \) have \( \text{Tr} F_i = 0 \), we thus must have

\[
\text{Tr} R^2 F_i = 0 \quad \text{and} \quad \text{Tr} RF_i F_j < 0,
\]

(3.4)

specifically the latter matrix in \( i \) and \( j \) must have all negative eigenvalues. We check in all cases that (3.2), using (1.3) and (1.5), indeed satisfies (3.4). Again, we suspect that some
mathematical identity ensures that this is indeed always the case, e.g. the first identity in \((3.4)\) requires
\[
\sum_{\alpha\beta}|n_{\alpha\beta}|(2\frac{c_1 \cdot L_{\alpha\beta}}{c_1 \cdot c_1} - 1)^2(J_i \cdot L_{\alpha\beta}) = 0,
\]
for all \(J_i\) satisfying \(c_1 \cdot J_i = 0\).

Finally, the flavor symmetries \(F_i\) are expected to have all vanishing cubic 't Hooft anomalies
\[
\text{Tr } F_i F_j F_k = 0 \quad \text{for all } i, j, k.
\]

Any non-zero such cubic 't Hooft anomalies would require the presence of Chern-Simons 5-forms \(\sim A \wedge F \wedge F\) in the \(AdS_5\) bulk \([26,27]\), but such a term does not have a candidate 10d origin, in terms of the 10d gauge fields \(C_4 \sim A \wedge \eta\) and \(F_5 = F \wedge \eta\), with \(\eta\) a 3-form on \(H_5\). Further, as we mentioned above, the \(F_i\) flavor symmetries become part of the gauge symmetry upon including wrapped D5s. Hence absence of gauge anomalies of those theories requires \((3.6)\). Again we can check in all examples that, indeed,
\[
\sum_{\alpha\beta}|n_{\alpha\beta}|(J_i \cdot L_{\alpha\beta})(J_i \cdot L_{\alpha\beta})(J_k \cdot L_{\alpha\beta}) = 0,
\]
for all \(i, j, k\), with \(J_{i,j,k} \cdot c_1 = 0\).

4. del Pezzo examples

Consider the case where \(X_6\) is a local Calabi-Yau which is a complex cone over the del Pezzo surface \(dP_n\). Recall that \(dP_n\) is a copy of \(\mathbb{P}^2\) blown up at \(n\) points, where \(0 \leq n \leq 8\). Each blown-up point corresponds to an exceptional divisor \(E_i\), and there is also a divisor \(D\) which is the pullback of a hyperplane on \(\mathbb{P}^2\). The intersection numbers of these divisors are
\[
D \cdot D = 1, \quad E_i \cdot E_j = -\delta_{ij}, \quad D \cdot E_i = 0,
\]
and the first Chern class (anti-canonical class) is
\[
c_1 = 3D - \sum_{i=1}^{n} E_i.
\]

There are \(n\) linearly independent divisors \(J_i\) satisfying \(J_i \cdot c_1 = 0\), so there will be a non-R \(U(1)^n\) flavor symmetry under which the baryons are charged. These \(J_i\) correspond to the
root lattice of the exceptional group $E_n$, with $E_1 = A_1$, $E_2 = A_1 + A_1$, $E_3 = A_3$, $E_4 = A_4$, $E_5 = D_5$, and $E_{6,7,8}$ as expected. In particular, if we take for our basis

$$J_i = E_i - E_{i+1} \quad \text{for } i = 1 \ldots n - 1, \quad \text{and} \quad J_n = D - E_1 - E_2 - E_3 \quad (4.3)$$

their intersections $J_i \cdot J_j$ are given by the $E_n$ Cartan matrix. The $dP_n$ automorphisms correspond to the $E_n$ Weyl reflections on the $J_i$.

Rewriting (2.3) in this language, the R-charge of a baryon $B_{\alpha\beta}$ corresponding to a holomorphic 2-cycle $L_{\alpha\beta}$ is

$$R(B_{\alpha\beta}) = 2N \frac{c_1 \cdot L_{\alpha\beta}}{c_1 \cdot c_1} = 2N \frac{c_1 \cdot L_{\alpha\beta}}{9 - n}. \quad (4.4)$$

Since the numerator is an integer, this implies that the R-charge of any baryon in the $dP_n$ theory is an integer multiple of $\frac{2N}{9 - n}$. Also, using (3.3) and (2.2), we get that the cubic 't Hooft anomaly must be

$$\text{Tr} R^3 = \frac{24N^2}{9 - n}. \quad (4.5)$$

In the following sections, we will work out our prescription in detail for the case of the toric del Pezzos $dP_{n \leq 3}$ and the non-toric del Pezzo $dP_4$. The $dP_3$ case was studied extensively in [12], so we start with that case first.

4.1. Cone over $dP_3$.

The are four known field theories that arise from the cone over $dP_3$ which are related to each other via Seiberg duality. One of these (usually called Model III) is described by the $U(N)^6$ theory given by the quiver in Figure 1.

![Figure 1: The Model III $dP_3$ quiver.](image)
The correspondence between divisors and bifundamentals has already been worked out in [12] and is given in the following table:

\[
\begin{array}{cccc}
Q_{\alpha\beta} & L_{\alpha\beta} \\
X_{51} & E_1 \\
X_{24} & E_2 \\
X_{53} & E_3 \\
X_{43} & D - E_1 - E_2 \\
X_{25} & D - E_1 - E_3 \\
X_{41} & D - E_2 - E_3 \\
X_{16} & D - E_1 \\
X_{62} & D - E_2 \\
X_{36} & D - E_3 \\
X_{64} & D \\
X_{65} & 2D - E_1 - E_2 - E_3
\end{array}
\]

Note that, because the assignment of divisors and charges is the same for any of the \(|n_{\alpha\beta}|\) bifundamentals connecting the same nodes, we have not explicitly written the fields \(Y_{16}, Y_{36}, Y_{62}\) in this table. It is of course important to include these multiplicities in all computations, e.g. when computing traces.

It is easily verified that the \(L_{\alpha\beta}\) (4.6) indeed satisfy our vanishing ABJ anomaly condition (1.6). Furthermore, the superpotential (found in [12,15]) respects the symmetries, because every term in the superpotential has exactly one field for which \(\theta_{\alpha\beta} = 1\).

We now write these \(L_{\alpha\beta}\) as in (1.8). As seen in the table below, we can take the \(L_{\alpha}\) to equal the \(D_{\alpha}\) which define the collection of bundles, except at nodes 2,4,5. These are precisely the nodes where a flip is required [13] to obtain the quiver diagram; this seems to be a general connection. We also include in the table below the \(M_{\alpha} = \sum_i N_i J_i \cdot L_{\alpha}\), which give the modification of the gauge groups in the quiver diagram with added wrapped D5’s, as in (1.12). The \(J_i\) are as in (4.3): \(J_1 = E_1 - E_2\), \(J_2 = E_2 - E_3\), \(J_3 = D - E_1 - E_2 - E_3\).

\[
\begin{array}{cccc}
\text{Node} & L_{\alpha} & D_{\alpha} & M_{\alpha} \\
1 & E_1 & E_1 & N_3 - N_1 \\
2 & 2D - E_2 & E_2 & N_3 - N_1 + N_2 \\
3 & E_3 & E_3 & N_2 + N_3 \\
4 & 2D & D - E_2 & 2N_3 \\
5 & 0 & D - E_3 & 0 \\
6 & D & D & N_3
\end{array}
\]

(4.7)

One can readily check that these \(L_{\alpha}\) and (1.8) reproduce the required \(L_{\alpha\beta}\) in (1.6).
Since $dP_3$ has $c_1 = 3D - E_1 - E_2 - E_3$, we write the three flavor currents as $J_1 = E_2 - E_1$, $J_2 = E_2 - E_3$, and $J_3 = D - E_1 - E_2 - E_3$. Then, we can use (1.3) and (1.5) to read off the charges:

$$
\begin{array}{cccc}
Q_{\alpha\beta} & J_1 & J_2 & J_3 & R \\
X_{51} & -1 & 0 & 1 & 1/3 \\
X_{24} & 1 & -1 & 1 & 1/3 \\
X_{53} & 0 & 1 & 1 & 1/3 \\
X_{43} & 0 & 1 & -1 & 1/3 \\
X_{25} & 1 & -1 & -1 & 1/3 \\
X_{41} & -1 & 0 & -1 & 1/3 \\
X_{16} & 1 & 0 & 0 & 2/3 \\
X_{62} & -1 & 1 & 0 & 2/3 \\
X_{36} & 0 & -1 & 0 & 2/3 \\
X_{64} & 0 & 0 & 1 & 1 \\
X_{65} & 0 & 0 & -1 & 1 \\
\end{array} \tag{4.8}
$$

These are exactly the $-U(1)_C$, $U(1)_D$, $-U(1)_E$, and R charges found in [12].

Let’s now examine a Seiberg dual theory, known as Model IV. The quiver for this theory is obtained by Seiberg dualizing node 2; see Figure 2.

![Figure 2: The Model IV $dP_3$ quiver.](image)

The bifundamental/divisor correspondence has been worked out already in [12], so we only need to check that our prescription for Seiberg dualizing the $L_\alpha$ agrees. The only difference is in $L_2$, which becomes $L'_2 = L_4 + L_6 - L_2 = E_2$, which checks with the results of [12].

4.2. Cone over $dP_2$

Since blowing down a divisor is equivalent to Higgsing an appropriate bifundamental, one can easily obtain the $dP_2$ quiver by Higgsing any bifundamental field in the $dP_3$ quiver.
that corresponds to an exceptional divisor. Depending on which divisor gets Higgsed, the resulting quiver will be one of two possible Seiberg dual theories. We choose to blow down $E_2$ or, equivalently, Higgs $X_{24}$. The resulting quiver appears in Figure 3 (for simplicity, we have relabeled nodes $6 \to 4$ and $4/2 \to 2$).

![Figure 3: The $dP_2$ quiver resulting from Higgsing $X_{24}$.](image)

It is easy to figure out the appropriate assignment of divisors here: We simply take the divisors from our $dP_3$ model and remove any $E_2$’s. In the case where bifundamental fields combine and point in the same direction, the divisors never differ by more than an $E_2$, and thus there is no ambiguity. In the case where bifundamentals combine that point in opposite directions, the resulting divisor is the one corresponding to the bifundamental that did not change direction, i.e. the one that had more flavors.

The $L_\alpha$ can be easily derived from our $dP_3$ example by simply blowing down the divisor $E_2$. Since the $L_\alpha$ for the nodes from the $dP_3$ quiver that get combined are the same up to an $E_2$, there is no ambiguity in how to assign the $L_\alpha$ for the $dP_2$ theory. We note that this also is true for any of the $E_i$ we could have chosen to blow down. This yields (relabeling $E_3 \to E_2$)

| Node | $L_\alpha$ | $M_\alpha$ |
|------|------------|------------|
| 1    | $E_1$      | $2N_2 - N_1$ |
| 2    | $2D$       | $2N_2$     |
| 3    | $E_2$      | $N_1 + N_2$ |
| 4    | $D$        | $N_2$      |
| 5    | 0          | 0          |

The $M_\alpha$ in (4.9) give the gauge groups in the theory with added wrapped D5’s. We take $J_1 = E_1 - E_2$ and $J_2 = D - 2E_1 - E_2$, which satisfy $J_i \cdot c_1 = 0$ with $c_1 = 3D - E_1 - E_2$ for $dP_2$. For the theory without wrapped branes, we get the following assignment of divisors and flavor charges:
\[ Q_{\alpha\beta} \quad L_{\alpha\beta} \quad J_1 \quad J_2 \quad R \]

\[
\begin{array}{|l|c|c|c|c|}
\hline
X_{51} & E_1 & -1 & 2 & 2/7 \\
X_{53} & E_2 & 1 & 1 & 2/7 \\
X_{25} & D - E_1 - E_2 & 0 & -2 & 2/7 \\
X_{23} & D - E_1 & 1 & -1 & 4/7 \\
X_{14} & D - E_1 & 1 & -1 & 4/7 \\
X_{21} & D - E_2 & -1 & 0 & 4/7 \\
X_{34} & D - E_2 & -1 & 0 & 4/7 \\
X_{42} & D & 0 & 1 & 6/7 \\
X_{45} & 2D - E_1 - E_2 & 0 & -1 & 8/7 \\
\hline
\end{array}
\]

Notice that the non-R flavor charges here are given by linear combinations of the \( U(1)'s \) from \( dP_3 \) under which \( X_{42} \) is neutral, \( J_1^{dP_2} = J_1^{dP_3} + J_3^{dP_3} \) and \( J_2^{dP_2} = J_2^{dP_3} - J_1^{dP_3} \). This is also consistent with the divisors assigned to these flavor charges, as one sees by taking the appropriate linear combinations and removing any instances of the blown-down divisor.

The superpotential for this theory is \[ \text{(4.10)} \]

\[
W_{dP_2} = X_{51}X_{14}X_{45} + X_{53}X_{34}X_{45} + X_{51}Y_{14}X_{42}X_{25} + X_{53}Y_{34}Y_{42}X_{25} + X_{21}X_{14}Y_{42} + X_{23}X_{34}X_{42} + X_{21}Y_{14}Z_{42} + X_{23}Y_{34}Z_{42},
\]

where we don’t bother recording the exact coefficients. This indeed obeys the condition that every term has precisely one field with \( \theta_{\alpha\beta} = 1 \).

The reader can easily verify that our ‘t Hooft anomaly conditions are also satisfied: \( \text{Tr}R^3 \) is indeed given by \( \text{(4.5)} \) for \( n = 2 \), as required by \( \text{(3.3)} \). The condition \( \text{(3.4)} \) of \[ \text{(19)} \] is indeed satisfied, showing that the geometry knows how to pick out the correct superconformal \( U(1)_R \), via \( c_1 \). Finally, the flavor ‘t Hooft anomalies \( \text{(3.6)} \) vanish, as generally happens for these string-constructed theories.

We also check that our prescription for Seiberg duality works. Dualizing on node 3 yields the other phase of \( dP_2 \), given in Figure 4:
The only $L$ that changes is $L_3$, which becomes $L'_3 = L_2 + L_5 - L_3 = 2D - E_2$. It is easy to check that this is consistent with the divisors one gets by appropriately Higgsing $dP_3$.

4.3. Cone over $dP_1$

It is useful here to Higgs the $dP_2$ theory to the $dP_1$ theory, since this is an especially simple example. To obtain this theory, Higgs the field $X_{51}$ from $dP_2$, which corresponds to blowing down the exceptional curve $E_1$. This yields the quiver in Figure 5.

![Figure 5: The $dP_1$ quiver](image)

The $L_\alpha$, $D_\alpha$, and $M_\alpha$ for wrapped branes are given by

| Node | $L_\alpha$ | $D_\alpha$ | $M_\alpha$ |
|------|------------|------------|------------|
| 1    | $2D - E$   | $E$        | $-N_1$     |
| 2    | $D$        | $D$        | $N_1$      |
| 3    | $2D$       | $D - E$    | $2N_1$     |
| 4    | 0          | 0          | 0          |

Note that here the flipped nodes are 1 and 3, where the $L_\alpha$ and $D_\alpha$ differ. Here, $c_1 = 3D - E$, so we take $J = D - 3E$. This yields the following fields and charges.

| $Q_{\alpha\beta}$ | $L_{\alpha\beta}$ | $J$  | $R$  |
|-------------------|-------------------|------|------|
| $X_{13}$          | $E$               | 3    | 1/4  |
| $X_{21}$          | $D - E$           | -2   | 1/2  |
| $X_{34}$          | $D - E$           | -2   | 1/2  |
| $X_{14}$          | $D$               | 1    | 3/4  |
| $X_{23}$          | $D$               | 1    | 3/4  |
| $X_{42}$          | $D$               | 1    | 3/4  |

(4.12)
The superpotential here is
\[
W = X_{42}X_{21}X_{14} + X_{42}X_{23}X_{34} + X_{42}X_{21}X_{13}X_{34} \tag{4.14}
\]
which obeys the required conditions.

Seiberg dualizing on either node 1 or node 3 yields the same theory; one can check that the new \( L_\alpha \) are identical to the original after relabeling nodes.

4.4. Cone over \( dP_4 \)

As with the other del Pezzo surfaces, there are many different Seiberg dual quiver theories possible for \( dP_4 \). Here, we will use the one given in Figure 6 \[7,\,9\]. It is straightforward to check that by Higgsing \( X_{67} \), one returns to the Model III \( dP_3 \) quiver.

![Figure 6: One possible quiver for \( dP_4 \)](image)

There is a unique assignment of \( L_\alpha \) which reproduces the divisors on the above \( dP_3 \) Model III theory. These were found by enforcing that the field \( X_{67} \) corresponds to the exceptional curve we’re blowing down, \( L_{67} = E_4 \), and that the remaining divisors can only differ from their \( dP_3 \) counterparts by this same exceptional curve. We also list the \( M_\alpha = \sum_i N_i J_i \cdot L_\alpha \), relevant for the theory with added wrapped D5s.

| Node | \( L_\alpha \) | \( M_\alpha \) |
|------|----------------|---------------|
| 1    | \( E_1 \)      | \( N_4 - N_1 \) |
| 2    | \( 2D - E_2 - E_4 \) | \( N_4 - N_1 + N_2 - N_3 \) |
| 3    | \( E_3 \)      | \( N_2 - N_3 + N_4 \) |
| 4    | \( 2D - E_4 \) | \( 2N_4 - N_3 \) |
| 5    | \( 0 \)        | \( 0 \)       |
| 6    | \( D - E_4 \)  | \( N_4 - N_3 \) |
| 7    | \( D \)        | \( N_4 \)     |

(4.15)
On $dP_4$, the first Chern class is $c_1 = 3D - E_1 - E_2 - E_3 - E_4$. Thus, we can take as our $J_i$ to be $J_1 = E_1 - E_2$, $J_2 = E_2 - E_3$, $J_3 = E_3 - E_4$, and $J_4 = D - E_1 - E_2 - E_3$. We thus find the divisors and charges to be

\[
\begin{array}{ccccccc}
Q_{\alpha\beta} & L_{\alpha\beta} & J_1 & J_2 & J_3 & J_4 & R \\
X_{51} & E_1 & -1 & 0 & 0 & 1 & 2/5 \\
X_{24} & E_2 & 1 & -1 & 0 & 1 & 2/5 \\
X_{53} & E_3 & 0 & 1 & -1 & 1 & 2/5 \\
X_{67} & E_4 & 0 & 0 & 1 & 0 & 2/5 \\
X_{43} & D - E_1 - E_2 & 0 & 1 & 0 & -1 & 2/5 \\
X_{25} & D - E_1 - E_3 & 1 & -1 & 1 & -1 & 2/5 \\
X_{16} & D - E_1 - E_4 & 1 & 0 & -1 & 0 & 2/5 \\
X_{41} & D - E_2 - E_3 & -1 & 0 & 1 & -1 & 2/5 \\
X_{72} & D - E_2 - E_4 & -1 & 1 & -1 & 0 & 2/5 \\
X_{36} & D - E_3 - E_4 & 0 & -1 & 0 & 0 & 2/5 \\
X_{17} & D - E_1 & 1 & 0 & 0 & 0 & 4/5 \\
X_{62} & D - E_2 & -1 & 1 & 0 & 0 & 4/5 \\
X_{37} & D - E_3 & 0 & -1 & 1 & 0 & 4/5 \\
X_{74} & D - E_4 & 0 & 0 & -1 & 1 & 4/5 \\
X_{75} & 2D - \sum_i E_i & 0 & 0 & 0 & -1 & 4/5 \\
\end{array}
\]

The superpotential for this theory \cite{10} indeed obeys the condition that each term has precisely one field with nonzero $\theta_{\alpha\beta}$. (These charge and divisor assignments also apply for the $PdP_4$ case considered in \cite{17}, which has a slightly different superpotential.) We can also check that our ’t Hooft anomaly conditions \eqref{4.5}, \eqref{3.4} and \eqref{3.6} are also satisfied.

It is also worth checking that one can Higgs this theory to $dP_3$ and watch the divisor $E_4$ collapse in the same manner we observed in the Higgsing of $dP_3$ down to $dP_2$. This indeed works; we note that Higgsing $X_{67}$ and relabeling the node 6/7 $\rightarrow$ 7 produces exactly the results found above.

Finally, we can immediately construct the quivers and $L_\alpha$ for Seiberg dual theories. For example, dualizing on node 2 yields a quiver with $L'_2 = L_7 + L_6 - L_2 = L_4 + L_5 - L_2 = E_2$ and all other $L_\alpha$ unchanged.

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