POSITIVITY OF THE CM LINE BUNDLE FOR FAMILIES OF K-STABLE KLT FANOS

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Abstract. The Chow-Mumford (CM) line bundle is a functorial line bundle on the base of any family of polarized varieties, in particular on the base of families of klt Fano varieties (also called sometimes Q-Fano varieties). It is conjectured that it yields a polarization on the conjectured moduli space of K-semi-stable klt Fano varieties. This boils down to showing semi-positivity/positivity statements about the CM-line bundle for families with K-semi-stable/K-polystable fibers. We prove the necessary semi-positivity statements in the K-semi-stable situation, and the necessary positivity statements in the uniform K-stable situation, including in both cases variants assuming K-stability only for very general fibers. Our statements work in the most general singular situation (klt singularities), and the proofs are algebraic, except the computation of the limit of a sequence of real numbers via the central limit theorem of probability theory. We also present an application to the classification of Fano varieties. Furthermore, in the semi-positivity case we may allow log-Fano pairs.

1. INTRODUCTION

Throughout the article, the base field is an algebraically closed field k of characteristic zero.

1.1. Boundary free statements

For a flat family of n-dimensional Fano family \( f : X \to T \) (that is, \( X \) and \( T \) are normal and projective, and \( -K_{X/T} \) is an \( f \)-ample \( \mathbb{Q} \)-Cartier divisor) the Chow-Mumford (CM) line bundle is the pushforward cycle

\[
\lambda_f := -f_* \left( c_1(-K_{X/T})^{n+1} \right).
\]

This cycle, up to multiplying with a positive rational number, is the first Chern class of the a functorial line bundle on \( T \) defined in [PT06, PT09] (see also [FS90, FR06, PRS08]).

Our main motivation to consider the CM line bundle originates from the classification theory of algebraic varieties: the birational part of classification theory, also called the Minimal Model Program, predicts that up to specific birational equivalences, each projective variety decomposes into iterated fibrations with general fibers of 3 basic types: Fano, weak Calabi-Yau, and general type (to be precise one here needs to allow pairs, see Section 1.2, but the boundary free case is a good first approximation). These 3 types are defined by having a specific class of mild singularities and negative/numerically trivial/positive canonical bundles. Then the moduli part of the classification theory is supposed to construct compactified moduli spaces for the above 3 basic types of varieties. According to our current understanding the moduli part seems to be doable only in the presence of a (singular) Kähler-Einstein metric, which is predicted to be equivalent to the algebraic notion of \( K \)-polystability. We note that the above predictions are proven in large cases, e.g., MMP: [BCHM10, HM10, HX13, Bir12, Fuj09, Fli92, KMM94, AHK07, Bir10, Bir07]; connections between \( K \)-stability and Kähler-Einstein metrics: [BG14, CDS15a, CDS15b, CDS15c, Tia15a, Oda13, Oda12, OX12, LW17].

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In particular, on the Fano side for the moduli part one should construct algebraically a moduli space conjectured below. Furthermore, the polarization on this moduli space should be given by the CM line bundle, as the connection between the Kähler-Einstein theory and $K$-stability originates from Tian’s idea to view the metrics as polystable objects in an infinite dimensional GIT type setting with respect to the CM line bundle. (See Definition 4.7 and Corollary 4.8 for a precise definition and characterization used in the present article for $K$-semistability and see Section 1.6 for an explanation on $K$-polystability). The precise conjecture is as follows:

**Conjecture 1.1.** (e.g., a combination of [Oda14b, Conj. 3.1] and [Oda14a, Qtn 3, page 15], but also implicit to different degrees for example in [Tia97, PT09, CDS15c, LWX15, LWX16, SSY16])

(a) The stack $\mathcal{M}_{n,v}^{K,ss}$ of $K$-semistable klt Fano varieties of fixed dimension $n > 0$ and anti-canonical volume $v > 0$ is an Artin stack of finite type over $k$.

(b) The stack $\mathcal{M}_{n,v}^{K,ss}$ admits a good moduli space $M_{n,v}^{K,ps}$ (in the sense of [Alp13]), which is proper over $k$, and which parametrizes $K$-polystable klt Fano varieties of dimension $n$ and volume $v$. Furthermore, the CM line bundle descends onto $M_{n,v}^{K,ps}$.

(c) $M_{n,v}^{K,ps}$ is a projective scheme via the polarization given by the CM line bundle. In particular, the CM line bundle is

1. nef in families of $K$-semi-stable klt Fanos, and
2. nef and big in maximally varying families of $K$-polystable klt Fanos.

Our main result concerns point (c) of Conjecture 1.1. We completely solve point (c(ii)) and we solve (c(iii)) on the uniformly $K$-stable locus (Notes: $K$-polystability usually occurs at the boundary of the uniformly $K$-stable locus, see Section 1.6 for the definitions; see also Remark 1.11 for the reasons of the specific generality in point (c(ii)), and see Section 1.2 for the results for pairs).

**Theorem 1.2.** Let $f : X \to T$ be a flat morphism with connected fibers between normal projective varieties such that $-K_{X/T}$ is $\mathbb{Q}$-Cartier and $f$-ample, and let $\lambda_f$ be the CM line bundle defined in equation (1.0.a).

(a) **Pseudo-effectivity:** If $T$ is smooth and the very general geometric fibers of $f$ are $K$-semistable, then $\lambda_f$ is pseudo-effective.

(b) **Nefness:** If all the geometric fibers of $f$ are $K$-semi-stable then $\lambda_f$ is nef.

(c) **Bigness:** If $T$ is smooth, the very general geometric fibers of $f$ are uniformly $K$-stable, the variation of $f$ is maximal (i.e., there is a non-empty open set of $T$ over which the isomorphism equivalence classes of the fibers are finite), and either $\dim T = 1$ or the fibers of $f$ are reduced, then $\lambda_f$ is big.

(d) **Ampleness:** If all the geometric fibers of $f$ are uniformly $K$-stable and the isomorphism equivalence classes of the fibers are finite, then $\lambda_f$ is ample.

(e) **Quasi-projectivity:** If $T$ is only assumed to be a proper normal algebraic space, all the geometric fibers are $K$-semi-stable and there is an open set $U \subseteq T$ over which the geometric fibers are uniformly $K$-stable and the isomorphism classes of the fibers are finite, then $U$ is a quasi-projective variety.

**Remark 1.3.** Notably, Theorem 1.2 deals with non-smoothable singular Fanos too, about which we remark that:

(a) This is the first result about (semi-)positivity of the CM line bundle dealing with non-smoothable singular Fanos, as explained in Remark 1.10.

(b) Non-smoothable singular Fanos is the general case, for example in the sense that smoothable $K$-semi-stable Fanos are bounded, even without fixing the anti-canonical volume (one can put this together from [KMM92, Jia17]). On the other hand, non-smoothable $K$-semi-stable Fanos are unbounded if one does not fix the volume, as can be seen by considering quasi-étale quotients by bigger and bigger finite groups of $\mathbb{P}^2$, which are $K$-semi-stable according to [Fuj17c, Cor. 1.7].
Remark 1.4. The proof of Theorem 1.2 uses the Central Limit Theorem of probability theory. See Section 1.7.1 for an outline of our argument or Theorem 5.11 for the precise place where the Central Limit Theorem is used.

Remark 1.5. Either all or no very general geometric fibers is $K$-semi-stable (resp. uniformly $K$-stable), as shown by the constancy of the $\delta$-invariant on very general generic fibers (Proposition 4.14). In particular, if $k$ is uncountable, say $\mathbb{C}$, the assumptions of Theorem 1.2 can be checked on closed fibers. Furthermore, in the $K$-semistable cases, so for points (a) and (b) of Theorem 1.2, it is enough to find a single $K$-semistable closed fiber, according to [BL18, Thm 3]. The uniformly $K$-stable version of [BL18, Thm 3] is not known, but it is expected too.

We also remark that in Theorem 1.2 we carefully said “geometric fiber” instead of just “fiber”. The reason is that we use the $\delta$-invariant description of $K$-stability, and the $\delta$-invariant of a variety is not invariant under base extension to the algebraic closure (see Remark 4.15). So, for scheme theoretic fibers over non algebraically closed fields the $\delta$-invariant can have non semi-continuous behavior.

Remark 1.6. We chose the actual generality for Theorem 1.2, as it is the generality in which the relative canonical divisor exists and admits reasonable base-change properties (see Section 2.3 for details) on very general curves in moving families of curves on the base. Nevertheless, in situations where this base-change is automatic, Theorem 1.2 directly implies statements over non-normal, non-projective, and even non-scheme bases. This is made precise in the following statement:

Corollary 1.7. Let $f : X \to T$ be a flat, projective morphism with connected fibers to a proper algebraic space, such that there is an integer $m > 0$ for which $\omega^{[m]}_{X/T}$ is a line bundle and all the geometric fibers are $K$-semi-stable klt Fano varieties. Let $N$ be the CM-line bundle associated to the polarization $\omega^{[m]}_{X/T}$ as defined over general bases in [PT09] (see Notation 3.6). Then, $N$ is nef, and if the variation of $f$ is maximal and the very general geometric fiber is uniformly $K$-stable, then $N$ is big.

Remark 1.8. Note that over $\mathbb{C}$ the positivity properties of Theorem 1.2 (nefness, pseudo-effectivity, bigness, ampleness) can be characterized equivalently analytically, e.g., [Dem92, Prop 4.2]

Remark 1.9. Negativity of $-K_{X/T}$ point of view. Unwinding definition (1.0.a), we obtain that Theorem 1.2 in the case of one dimensional base states that $(-K_{X/T})^{n+1}$ is at most zero/smaller than 0. Using this in conjunction with the base-change property with the CM line bundle (Proposition 3.8) we obtain that Theorem 1.2, especially the last 3 points, prove strong negativity property of $-K_{X/T}$ for klt Fano families.

There does exist birational geometry statements claiming that $-K_{X/T}$ is not nef, e.g., [Zha96, Prop 1]. Our negativity statements point in this direction but go further. However, it is not a coincidence that strong negativity statements on $-K_{X/T}$ did not show up earlier, as in fact Theorem 1.2 is not true for every family of klt Fano varieties. Indeed, Example 11.1 shows that in Theorem 1.2 one cannot relax the $K$-semi-stable Fano assumption to just assuming klt Fano. The development of the notions of $K$-stability in the past decade was essential for creating the chance of proving negativity statements for $-K_{X/T}$ of the above type.

We also note that as $-K_{X/T}$ is not nef usually in the above situation (c.f., Theorem 1.22 and Example 11.3), the negativity of $(-K_{X/T})^{n+1}$ is independent of the negativity of $\kappa(-K_{X/T})$. In fact, assuming the former, $\kappa(-K_{X/T})$ can be $-\infty$ (Example 11.4), 0 (Example 11.3), dim $X$ (Example 11.2), and also something in between the latter two values (Example 11.4).

Remark 1.10. The following are the already known partial results on Conjecture 1.1.

(a) On the algebraic side, aiming for all klt singularities, there were no results on point (c) of Conjecture 1.1 earlier (although the second author with Xu proved the canonically polarized
Speaking about points (a) and (b) of Conjecture 1.1, they decompose into statements about different properties of the moduli functor: boundedness, separatedness, properness, openness of K-semi-stability, and contrary to the canonically polarized case, an analysis of the action of automorphisms on the CM-line bundle is also necessary to have it descend to the coarse moduli space. Out of these, only boundedness is known according to [Jia17, Cor 1.7], which uses also crucially the seminal papers of Birkar [Bir16a, Bir16b].

(b) On the other hand, using analytic methods (sometimes in conjunction with algebraic ones), there are plenty of results about Conjecture 1.1 on the closure of the locus of smooth Fanos: [LWX15, LWX16, SSY16, Oda14a].

The only significant piece missing from the above results is that the positivity of the CM line bundle is not known on closed subspaces \( V \) lying in the boundary of the closure of the locus of smooth Fanos. Our theorem in particular remedies this if the very general Fano parametrized by \( V \) is uniformly K-stable (and necessarily singular), see Corollary 1.18.

Remark 1.11. There are two main reasons why our positivity statements (points (c), (d) and (e) of Theorem 1.2) work in the uniformly K-stable case, but not in the K-polystable case:

(a) We rely on the characterization of K-semistability and uniform K-stability via the \( \delta \) invariant given by [FO16, BJ17]. Such characterization is not available for the K-polystable case.

(b) Our theorem on the nef threshold (Theorem 1.22 below, on which the above 3 points of Theorem 1.2 depend) fails in the K-polystable case according to Example 11.3. Hence, one would need a significantly different approach to extend points (c), (d) and (e) of Theorem 1.2 to the K-polystable case.

Remark 1.12. One could make definition (1.0.a) also without requiring flatness. We do not know if Theorem 1.2 holds in this situation. Nevertheless, we note that it would be interesting to pursue this direction for example for applications to Mori-fiber spaces with higher dimensional bases (see Corollary 1.19).

Also we expect that the reduced fiber assumption of point (c) of Theorem 1.2 can be removed, as we needed it for technical reasons (certain base changes over movable curves are nice), and also because the conjectured K-semi-stable reduction should eliminate it.

1.2. Logarithmic statements

We prove points (a) and (b) of Theorem 1.2 in the generality of pairs. We state this separately, in the present subsection, as the statements are more cumbersome (e.g., one needs to guarantee that the boundary can restrict to fibers, etc.).

If \( f : (X, \Delta) \to T \) is a flat morphism of relative dimension \( n \) from a projective normal pair to a normal projective variety such that \( -(K_X/T + \Delta) \) is \( \mathbb{Q} \)-Cartier and \( f \)-ample. Then we define the CM line bundle by

\[
\lambda_{f,\Delta} := -f_*(-(K_X/T + \Delta)^{n+1}).
\]

Theorem 1.13. Let \( f : X \to T \) be a flat morphism of relative dimension \( n \) with connected fibers between normal projective varieties and let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( -(K_X/T + \Delta) \) is \( \mathbb{Q} \)-Cartier and \( f \)-ample. Let \( \lambda_{f,\Delta} \) be the CM line bundle on \( T \) as defined in (1.12.a).

(a) **Pseudo-effectivity:** If \( T \) is smooth and \( (X_t, \Delta_t) \) is K-semi-stable for very general geometric fibers \( X_t \), then \( \lambda_{f,\Delta} \) is pseudo-effective.

(b) **Nefness:** If all fibers \( X_t \) are normal, \( \Delta \) does not contain any fibers (so that we may restrict \( \Delta \) on the fibers), and \( (X_t, \Delta_t) \) is K-semi-stable for all geometric fibers \( X_t \), then \( \lambda_{f,\Delta} \) is nef.

Remark 1.14. We note that positivity statements, that is, points (d), (e) and (f) of Theorem 1.2, are much trickier to prove in the logarithmic case than in the boundary free case (c.f. [KP17]). The main issue is that a family of pairs can have maximal variations in a way that the underlying
family of varieties has no variation, but at the same time the underlying family of varieties is not trivializable on a finite cover (e.g., take the family of Example 11.1 and put a pair structure on it using a small multiple of varying anti-pluricanonical divisors). The same issue in the canonically polarized case involved a decent amount of work [KP17, PX17].

1.3. Applications

Our applications of the theorems above are of two type:
(a) moduli theoretic, as we have already suggested in Section 1.1, and
(b) birational geometric, aiming to understand Mori-fiber spaces with $K$-semistable fibers.

We start with the precise statements of the moduli theoretic ones (Corollary 1.15 and Corollary 1.18):

Corollary 1.15. If $M$ is a proper algebraic space which is the moduli space for some class $X$ of $K$-semi-stable Fanos with the uniform $K$-stable locus $M^u \subseteq M$ being open (see Definition 10.4 for the precise definitions), then the normalization of $M^u$ is a quasi-projective scheme over $k$.

Remark 1.16. The space $M^u$ of Corollary 1.15 is many times smooth already, in which case the normalization can be certainly dropped from the statement. In fact, we know that it is smooth at the points corresponding to smooth Fanos [Ran92, Kaw92], and to terminal Fano 3-folds [San16, Thm 1.7]. Unfortunately, these unobstructedness statements do not hold for all Fanos, as [San16, Rem 2.13] gives a counterexample. However, the counterexample is a cone over a Del-Pezzo surface of degree 6. Hence, it has infinite automorphism group, and in particular it is not uniformly $K$-stable. This leads to the following question.

Question 1.17. Is the deformation space of uniformly $K$-stable Fanos (including general klt ones) unobstructed?

In the next corollary, we extend the locus of the moduli space of $K$-semistable Fanos that is known to be quasi-projective from the smooth locus to the union of the smooth locus and the largest open set in the uniformly $K$-stable locus.

Corollary 1.18. If $M$ is the moduli space of smoothable $K$-semistable Fanos (which is known to exist as an algebraic space according to [LWX15, LWX16, SSY16, Oda14a]), and $M^0$ is an open set parametrizing Fanos that are either smooth or uniformly $K$-stable, then the normalization of $M^0$ is quasi-projective.

Fujita showed in [Fuj15, Thm 1.1] that $\text{vol}(-K_X) \leq (n+1)^n$ for every $K$-semistable Fano variety $X$ of dimension $n$ (see [Lin16, Thm 3] for better bounds in the presence of quotient singularities). Using Theorem 1.13 we can show similar bounds for (non-necessarily klt) Fano $X$ admitting a Fano fibration structure with $K$-semi-stable general fiber.

Corollary 1.19. If $(X, \Delta)$ is a normal Fano pair, and $f : X \to \mathbb{P}^1$ is a fibration with $K$-semi-stable very general geometric fibers $F$, then

$$\text{vol}(- (K_X + \Delta)) \leq 2 \dim(X) \text{vol}(- (K_F + \Delta_F)).$$

If furthermore $\Delta = 0$, then

$$\text{vol}(- K_X) \leq 2 \dim(X)^{\dim(X)}.$$

Remark 1.20. Corollary 1.19 is sharp for surfaces and threefolds. Indeed, a del Pezzo surface of degree 8 and the blow-up of $\mathbb{P}^3$ at a line (whose anti-canonical volume is 54) can be fibred over $\mathbb{P}^1$ with $K$-semistable fibres.

Remark 1.21. Classification of (uniform) $K$-(semi/poly)-stable Fanos: to understand the power of Corollary 1.19, it is useful to know which Fanos are $K$-semi-stable and which are not. In fact, then one want to figure this out for all the four $K$-stability properties (see Section 1.6),
which has been an active area of research recently. To start with, let us recall that $K$-semi-stable Fano varieties are always klt.

A Del-Pezzo surface is $K$-polystable if and only if it is not of degree 8 or 7 [TY87, Tia90]. Smooth Fano surfaces with discrete automorphism groups are even uniformly $K$-stable, and their delta invariant (see Section 4) is bounded away from 1 in an effective way [SS17]. Smoothable singular K-stable Del-Pezzo surfaces are classified in [OSS16].

K-stable proper intersection of two quadrics in an odd dimensional projective space are classified in [SS17] (also [AGP06]); in particular, smooth varieties of these types are always K-stable. Cubic 3-folds are studied in [LX17], where again smooth ones are $K$-stable, and so are the ones containing only $A_k$ singularities for $k \leq 4$. Under adequate hypotheses, in [Der16a], it is shown that Galois covers of K-semistable Fano varieties are K-stable. This can be applied for instance to double solids. Furthermore, birational superigid Fano varieties are K-stable under some addition mild hypothesis [OO13, ZS18, Zhu18]. However, according to the best knowledge of the authors, there is not a complete classification of K-stable smooth Fano threefolds.

If one wants to study klt Fano varieties from the point of view of the MMP, it is particularly relevant to see if one can apply Corollary 1.19 to the case of Mori Fibre Spaces. In [CFST16, Corollary 1.11], it is shown that if a smooth Fano surface or a smooth toric variety can appear as a fibre of MFS, then it is K-semistable. We do not know if the analogous result holds in dimension 3. However, there are examples of smooth Fano fourfolds with Picard number one (which then can be general fibers of MFS’s) that are not K-semistable [Fuj17a], see also [CFST18].

1.4. **Byproduct statements**

As a byproduct of our method we obtain the following bound on the nef threshold of $-(K_{X/T}+\Delta)$ with respect to $\lambda$ in the uniformly $K$-stable case.

**Theorem 1.22.** Let $f : X \to T$ be a flat morphism with connected fibers from a normal projective variety of dimension $n + 1$ to a smooth curve and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that

- $-(K_{X/T} + \Delta)$ is $\mathbb{Q}$-Cartier and $f$-ample, and
- $(X_\tau, \Delta_\tau)$ is uniformly $K$-stable for very general geometric fibers $X_\tau$.

Set

- set $\delta := \delta(X_\tau, \Delta_\tau)$ for $\tau$ very general geometric point, and
- let $v := ((-K_{X/T} - \Delta))_t^n$ for any $t \in T$.

Then, $-K_{X/T} - \Delta + \frac{\delta}{(\delta - 1)(n + 1)} f^*\lambda_f, \Delta$ is nef.

**Remark 1.23.** One cannot have a nef threshold statement as in Theorem 1.22 for uniformly $K$-stable replaced with $K$-polystable. Indeed, take the family $f : X \to T$ given by Example 11.3. It has $K$-polystable fibers, deg $\lambda_f = 0$, but $K_{X/T}$ is not nef. In particular, for any $a \in \mathbb{Q}$, $-K_{X/T} + a f^*\lambda_f \equiv K_{X/T}$, and hence for any $a \in \mathbb{Q}$, $-K_{X/T} + a f^*\lambda_f$ is not nef.

And we have a structure theorem when $\lambda$ is not positive:

**Theorem 1.24.** Let $f : X \to T$ be a flat morphism of relative dimension $n$ with connected fibers between normal projective varieties and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $-(K_{X/T} + \Delta)$ is $\mathbb{Q}$-Cartier and $f$-ample. Assume that $(X_\tau, \Delta_\tau)$ is uniformly $K$-stable for very general geometric fibers $X_\tau$. If $H$ is an ample divisor on $T$, such that $\lambda_{f, \Delta} \cdot H^{\dim T - 1} = 0$, then for every integer $q > 0$ divisible enough, $f_*\mathcal{O}_X(q(-K_{X/T} - \Delta))$ is an $H$-semi-stable vector bundle of slope 0.

1.5. **Similar results in other contexts**

Roughly, there are three types of statements above: (semi-)positivity results, moduli applications, inequality of volumes of fibrations. Although in the realm of $K$-stability ours are the first general algebraic results, statements of these types were abundant in other, somewhat related,
contexts: KSBA stability, GIT stability, and just general algebraic geometry. Our setup and our methods are different from these results, still we briefly list some of them for completeness of background. We note that KSBA stability is related to our framework as it is shown to be exactly the canonically polarized \( K \)-stable situation [Oda13, Oda12, OX12]. Also, GIT stability is related, as \( K \)-stability originates from an infinite dimensional GIT, although it is shown that it cannot be reproduced using GIT (e.g., [WX14]).

| (semi-)positivity | general algebraic geometry | KSBA stability | GIT stability |
|-------------------|---------------------------|----------------|--------------|
|                    | [Gri70, Fuj78, Kaw81, Vie83, Kol87] | [Kol90, Fuj12, KP17, PX17] | [CH88] |
| moduli applications | [Vie95] | [Kol90, KP17, AB17] | [CH88] |
| volume (slope) inequalities | [Xia87] | [Par05, BS14] |

1.6. **Overview of \( K \)-stability for Fano varieties**

In the present article we define \( K \)-semi-stability and uniform \( K \)-stability using valuations (Definition 4.7), which is equivalent then to the \( \delta \)-invariant definition (Corollary 4.8). These definitions were shown to be equivalent in [BJ17, Theorem B] to the more traditional ones that use test configurations. Also, they have a serious disadvantage: there is no known delta invariant type definition of \( K \)-polystability, which although we do not use in any of the statements or in the proofs, they are important notions for the big picture. Hence, for completeness we mention below their definitions using test configurations. We refer the reader to [DT92, Don05] or more recent papers such as [Der16b, BHJ17] for more details:

**K-semi-stability:** For every normal test configuration, the Donaldson-Futaki invariant is non-negative.

**K-stability:** For every normal test configuration, the Donaldson-Futaki invariant is non-negative, and it is equal to zero if and only if the test configuration is a trivial test configuration. In particular, there is no 1-parameter subgroup of \( \text{Aut}(X) \).

**K-poly-stability:** For every normal test configuration the Donaldson-Futaki invariant is non-negative, and it is equal to zero if and only if the test configuration is a product test configuration, i.e. it comes from a one parameter subgroup of the automorphism group of \( X \).

**Uniform K-stability:** There exists a positive real constant \( \delta \) such that for every normal test configuration the Donaldson-Futaki invariant is at least \( \delta \) times the \( L^1 \) norm (or, equivalently, the minimum norm) of the test configuration. This notion implies K-stability, and in the case of smooth complex Fanos that the automorphism group of \( X \) is finite [BHJ16, Cor E].

We also note that the Yau-Tian-Donaldson conjecture asserts that a klt Fano variety admits a singular Kähler-Einstein metric if and only if it is K-polystable. This is known for smooth [CDS15a, CDS15b, CDS15c, Tia15b] and smoothable Fano varieties [LWX16] (and independently [SSY16] in the finite automorphism case), and for singular ones admitting a crepant resolution [LW17]. In the literature, there are also many proposed strengthening of the notion of K-stability; they should be crucial to extend the YTD conjecture to the case of constant scalar curvature Kähler metrics. In this paper we are interested in uniform K-stability [Der16b, BHJ17, BBJ15], which at least for smooth Fano manifold is known to be equivalent to K-stability (we should stress that the proof is via the equivalence with the existence of a Kähler-Einstein metric). One can also strengthen the notion of K-stability by looking at possibly non-finitely generated filtration of the coordinate ring, see [WN12, Szé15, Cod18].

1.7. **Outline of the proof**

Our proof for the semi-postivity (that is, nefness and pseudo-effectivity) and the positivity (that is, ampleness and bigness) statements are different. Hence, we discuss the corresponding
outlines separately in Section 1.7.1 and in Section 1.7.3, respectively. Additionally, as it is an indispensable link between semi-positivity and positivity, we present the ideas behind the nefness threshold statement of Theorem 1.22 in Section 1.7.2. For simplicity, we restrict in all cases to the non-logarithmic situation, that is, to statements about $-K_{X/T}$ instead of $-(K_{X/T} + \Delta)$. As all the assumptions and consequences are invariant under base-extension to another algebraically closed field, we may also assume that $k$ is uncountable. In particular, the very general geometric fibers assumed then exist also as closed fibers.

### 1.7.1. Semi-positivity statements

As nefness and pseudo-effectivity can be checked via non-negative intersection with effective or moving 1-cycles, respectively, points (a) and (b) of Theorem 1.13 can be reduced to the case of a curve base. Hence, we assume that the base of our fibration $f : X \to T$ is a curve, in which case pseudo-effectivity and nefness are both equal to the degree being at least zero. So, we are supposed to prove that $\deg \lambda_f \geq 0$ or equivalently that $(-K_{X/T})^{n+1} \leq 0$ (see (1.12.a)).

We argue by contradiction, so we assume that $(-K_{X/T})^{n+1} > 0$. If we fix a $\mathbb{Q}$-divisor $H$ on $T$ of small enough positive degree, then by the continuity of the intersection product $(-K_{X/T} - f^*H)^{n+1} > 0$ also holds. As $X$ is normal and fibered over the curve $T$ over which $-K_{X/T}$ is ample, this implies via a Riemann-Roch computation that the $\mathbb{Q}$-linear system $|-K_{X/T} - f^*H|$ is non-empty (see Remark A.3). Our initial idea is to obtain a contradiction from this, as Proposition 7.3 tells us that there can exist no $\Gamma \in |-K_{X/T} - f^*H|_T$ such that $(X_t, \Gamma_t)$ is klt for general $t \in T$. The only problem is that there are examples where $|-K_{X/T} - f^*H|$ is non-empty but the above klt condition fails. Indeed, every family with negative CM line bundle has to be an example like that according to Proposition 7.3. An explicit example is given in Example 11.1.

Our second idea is that maybe the $K$-stable assumption leads us to a $\Gamma$ as above that also satisfies the klt condition. According to the delta invariant description of $K$-semi-stability (Corollary 4.8), if $X_t$ is $K$-semi-stable, then up to a little perturbation one can obtain klt divisors the following way: for $q \gg 0$, let $D_1, \ldots, D_t$ be divisors corresponding to any basis of $H^0(X_t, -qK_{X_t})$; then the divisor $D := \sum_{i=1}^t \frac{D_i}{ql} \in |-K_{X_t}|_T$ is such that $(X_t, D)$ is klt.

Now, we would like to lift such a divisor to $|-K_{X/T} - f^*H|_T$. To this end, it is enough to lift for $q \gg 0$, every element of a basis of $H^0(X_t, -qK_{X_t})$ to elements of $H^0(X, q(-K_{X/T} - f^*H))$. Using some perturbation argument, one can get by by finding linearly independent sections $s_1, \ldots, s_l \in H^0(X_t, -qK_{X_t})$ such that $s_i$ lifts, and $\frac{1}{\sum_{i=1}^l q^{n(q(-K_{X_t}))}}$ is close enough to 1.

This in turn would be implied by the following: let $E_q$ be the subsheaf of $f_*\mathcal{O}_X(-qK_{X/T})$ spanned by the global sections. Then we would need to show that

$$(1.24.a) \quad \lim_{q \to \infty} \frac{\text{rk } E_q}{\text{rk } f_*\mathcal{O}_X(q(-K_{X/T} - f^*H))} = 1.$$  

(For the readers more familiar with the language of volumes and restricted volumes, we note that (1.24.a) is equivalent to showing that the restricted volume of $-K_{X/T}$ over a general fiber is equal to the anti-canonical volume of the fibers.)

Unfortunately, (1.24.a) is still not doable. For example, if one takes the isotrivial family

$$X := \mathbb{P}_T(\mathcal{O}_T(-n) \oplus \mathcal{O}_T(1) \oplus \cdots \oplus \mathcal{O}_T(1))$$

of $\mathbb{P}^n$'s over $T := \mathbb{P}^1$ (as in Example 11.1 for $n = 2$), then

$$f_*\mathcal{O}_X(-qK_{X/T}) \cong S^{(n+1)q}(\mathcal{O}_T(-n) \oplus \mathcal{O}_T(1) \oplus \cdots \oplus \mathcal{O}_T(1)).$$
In this situation $\mathcal{E}_q$ is the direct sum of the factors with degree greater than $q \deg H \sim q \varepsilon$ (here $1 \gg \varepsilon > 0$). Then one can compute that (1.24.a) does not hold. For example, in the case of $n = 1$,

$$S^{2q}(\mathcal{O}_T(-1) \oplus \mathcal{O}_T(1)) = \mathcal{O}_T(-q) \oplus \mathcal{O}_T(-q + 1) \oplus \cdots \oplus \mathcal{O}_T(q).$$

So, we see that the limit of (1.24.a) is $\frac{1}{2} - \varepsilon$.

The idea that saves the day at this point is the **product trick**, which was pioneered in the case of semi-positivity questions by Viehweg [Vie83]. The precise idea is to replace $X$ by an $m$-times self fiber product $X^{(m)}$ over $T$. Let $f^{(m)} : X^{(m)} \to T$ be the induced morphism (Section 2.1). Then, one can replace the initial goal with showing that there exists $\Gamma \in \bigl[ -K_{X^{(m)}/T} - (f^{(m)})^* mH \bigr] \mathbb{Q}$ such that $\bigl(X^{(m)}_t, \Gamma\bigr)$ is klt for $t \in T$ general. Running through the previous arguments for $X^{(m)}$ instead of $X$, this would boil down to showing that

$$\lim_{m \to \infty} \frac{\text{rk} \mathcal{E}_{q,m}}{\text{rk} f^{(m)}_* \mathcal{O}_{X^{(m)}}\left( q \left( -K_{X^{(m)}/T} - (f^{(m)})^* mH \right) \right)} = 1,$$

where $\mathcal{E}_{q,m}$ is a subsheaf given by certain condition specified below of the subsheaf generated by global sections of

$$f^{(m)}_* \mathcal{O}_{X^{(m)}}\left( q \left( -K_{X^{(m)}/T} - (f^{(m)})^* mH \right) \right) \cong \bigotimes_{m \text{ times}} f_* \mathcal{O}_X(q(-K_{X/T} - f^*H)).$$

The extra condition in the definition of $\mathcal{E}_{q,m}$ is due to the need that $\Gamma$ has to be klt on a general fiber. This would be automatic if the conjecture that products of $K$-semi-stable klt Fanos are $K$-semi-stable was known. Unfortunately this is a surprisingly hard unsolved conjecture in the theory of $K$-stability. Hence, we evade it by considering only bases of $H^0\left(X^{(m)}_t, -qK_{X^{(m)}_t}\right)$ that are induced from bases of $H^0\left(X_t, -qK_{X_t}\right)$. As log canonical thresholds are known to behave well under taking products (Proposition 4.13), if the restriction $\Gamma|_{X^{(m)}_t}$ to a general fiber is a divisor corresponding to such basis, the $K$-stability of $X_t$ implies that $\left(X^{(m)}_t, \Gamma|_{X^{(m)}_t}\right)$ is klt. Hence, the additional condition in the definition of $\mathcal{E}_{q,m}$ is that it is the biggest subsheaf as above such that $\mathcal{E}_{q,m}|_t$ is spanned by simple tensors for a basis $t_1, \ldots, t_l$ of $f_* \mathcal{O}_X(q(-K_{X/T} - f^*H))$ to be specified soon.

So, we are left to specify a basis of $\left(f_* \mathcal{O}_X(q(-K_{X/T} - f^*H))\right)_t \cong H^0(X_t, -qK_{X_t})$ for which (1.24.b) holds. For that we use the Harder-Narasimhan filtration $0 = \mathcal{F}^0 \subseteq \cdots \subseteq \mathcal{F}^r$ of $f_* \mathcal{O}_X(q(-K_{X/T} - f^*H))$. Let the basis $v_1, \ldots, v_l$ be any basis adapted to the restriction of this filtration over $t$, that is, to $0 = \mathcal{F}^0_t \subseteq \cdots \subseteq \mathcal{F}^r_t$. The lower part of the filtration, until the graded pieces reach slope $2g$ (where $g$ is the genus of $T$), is globally generated. Furthermore, there is an induced Harder-Narasimhan filtration on (1.24.c). The slope at least $2g$ part of the latter filtration is globally generated such that its restriction over $t \in T$ is generated by simple tensors in $v_i$ (Proposition 5.9). Hence, if $\mathcal{E}'_{q,m}$ is this part of the Harder-Narasimhan filtraton, then it is enough to prove that

$$\lim_{m \to \infty} \frac{\text{rk} \mathcal{E}'_{q,m}}{\text{rk} f^{(m)}_* \mathcal{O}_{X^{(m)}}\left( q \left( -K_{X^{(m)}/T} - (f^{(m)})^* mH \right) \right)} = 1,$$

The final trick of the semi-positivity part is then that (1.24.d) can be translated to a probability limit, which then is implied by the central limit theorem of probability theory (Theorem 5.11).
We explain here the probability theory argument via the example of
\[ \mathcal{F}_m := \bigotimes_{m \text{ times}} (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)). \]
The claim then is that as \( m \) goes to infinity the rank of the non-negative degree part of \( \mathcal{F}_m \) over the rank of \( \mathcal{F}_m \) converges to 1. It is easy to see that this is the following limit:
\[
\sum_{0 \leq i \leq m, 2i - (m - i) \geq 0} \binom{m}{i} \left( \frac{1}{2} \right)^n = \sum_{0 \leq i \leq m, i \geq \frac{m}{2}} \binom{m}{i} \left( \frac{1}{2} \right)^n \geq \sum_{0 \leq i \leq m, i \geq \frac{m}{2} - A \frac{m}{2}} \binom{m}{i} \left( \frac{1}{2} \right)^n
\]
for \( m \) big enough, where \( A > 0 \) is an arbitrary fixed real number.

The latter is the probability when flipping a coin \( m \) times one gets at least \( \frac{m}{2} - A \frac{\sqrt{m}}{4} \) heads. Note that for this \( m \)-times flipping the expected value is \( \frac{m}{2} \) and \( \sqrt{m} \)-times the square deviation is \( \frac{m}{4} \). Hence, the above probability converges to \( \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \) by the classical De Moivre-Laplace theorem, a special case of the central limit theorem. We obtain (1.24.d) by taking \( A \to \infty \) limit, and using that the above integral integrates the density function of the standard Gaussian normal distribution.

1.7.2. Nefness threshold. This part uses the same ideas as the above semi-positivity part, but in a different logical framework. That is, the argument is not a proof by contradiction. Instead, the starting point is that \( \left( -K_{X/T} + (f^{(m)})^* \left( \frac{\lambda_f}{v(n+1)} + H \right) \right)^{n+1} \geq 0 \). Hence, again up to a little perturbation and by using the ideas of the previous point, there is an integer \( m > 0 \) such that there exists a \( \Gamma \in \left[ -\delta K_{X^{(m)} / T} + (f^{(m)})^* m \left( \frac{\delta \lambda_f}{v(n+1)} + H \right) \right] \), for which \( \left( X^{(m)}_t, \Gamma_t \right) \) is klt for \( t \in T \) general. Then standard semi-positivity argument (Proposition 6.3) shows that
\[
K_{X^{(m)} / T} - \delta K_{X^{(m)} / T} + (f^{(m)})^* m \left( \frac{\delta \lambda_f}{v(n+1)} + H \right) = (1-\delta)K_{X^{(m)} / T} + (f^{(m)})^* m \left( \frac{\delta \lambda_f}{v(n+1)} + H \right)
\]
is nef. Lastly, one divides by \( \delta - 1 \), converges to 0 with \( H \), and lastly by a standard lemma (Lemma 8.1) removes the \( \left( \omega \right)^{(m)} \).

1.7.3. Positivity. The rough idea here is to use a twisted version of the ampleness lemma ([Kol90, 3.9 Ampleness Lemma], with little modifications in [KP17, Thm 5.1]). We need a twisted version of the ampleness lemma as the techniques developed until this point in the article do not work directly over higher dimensional bases. The main idea here is that to get bigness of \( \lambda_f \) it is enough to show positivity of \( \lambda_f \) over a very general element \( C \) of each moving family of curves of \( T \), in a bounded way. Below we explain how we do this.

The main benefit of proving the nefness threshold result above is the following: one can prove, again using standard semi-positivity arguments (Proposition 6.3), that \( \mathcal{Q} := f_* \mathcal{O}_X(-rK_{X/T} + \alpha f^* \lambda_f) \) is nef, for some constants \( r \) and \( \alpha \). Furthermore, these constants \( r \) and \( \alpha \) can be chosen to be uniform, as \( f \) runs through all families obtained by base-changing on a very general element \( C \) of a moving family of curves on \( T \). Then, the ampleness lemma (Theorem 9.8) gives an ample line bundle \( B \) on \( T \) such that for all curves \( C \) as above, \( C \cdot B \leq C \cdot \det \mathcal{Q} \). Then one can use another trick from (semi-)positivity theory, which we also learned from Viehweg’s works. That is, for \( q := \text{rk} \mathcal{Q} \), there is an embedding
\[
\det \mathcal{Q} \to \bigotimes_{q \text{ times}} f_* \mathcal{O}_X(-rK_{X/T} + \alpha f^* \lambda_f) \cong f_*^{(q)} \mathcal{O}_X \left( -rK_{X^{(q)} / T} + q \alpha \left( f^{(q)} \right)^* \lambda_f \right),
\]
Using the adjunction of $f^*_{(q)}$ and $(f^{(q)})^*$ this implies the inequality of divisors

$$(f^{(q)})^* B \leq (f^{(q)})^* \det Q \leq -rK_{X^{(q)}/T} + qa(f^{(q)})^* \lambda_f,$$

which survives the restriction over $C$ by the genericity assumption in the definition of the latter. From here, a simple intersection computation shows that $C \cdot B$ bounds $\deg \lambda_f|_C$ from below up to some uniform constants, not depending on the choice of $C$ (see the end of the proof of point (c) of Theorem 1.2).

1.8. **Organization of the paper**

See Section 1.7 for a thorough explanation on which part of the argument can be found where. Here we only note that the actual argument, so what is explained in Section 1.7, starts in Section 5, and lasts until Section 11, where we present some examples important for our statements (mostly showing sharpness of the latter). After Section 11, we only have Appendix A, with some computations related to the definition of the CM line bundle.

Before the argument starts, in Section 2, Section 3 and Section 4 we present notation and background, as well as, simpler statements. The division of this part between the above 3 sections is based on topics. Section 2 contains general topics, Section 3 contains the definition of the CM line bundle and the related statements, and Section 4 contains the definition and the basics about the $\delta$-invariant and $K$-stability. In particular, the latter contains the definition of $K$-stability (Definition 4.7 and Corollary 4.8).

We also include a table on the location of the proofs of the theorems stated in the introduction.

| Statements of the introduction | their proofs |
|--------------------------------|-------------|
| Theorem 1.2 (a) & (b)          | Section 7.2 |
| (c) & (d) & (e)               | Section 9.4 |
| Theorem 1.13                   | Section 7.2 |
| Corollary 1.17 & Theorem 1.24  | Section 9.4 |
| Corollary 1.15 & Corollary 1.18 & Corollary 1.19 | Section 10 |
| Theorem 1.22                   | Section 8   |

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2. **Notation**

2.1. **Fiber product notation**

The most important particular notation used in the article is that of fiber products. That is, for a family $f : X \to T$ of varieties we denote the $m$-times fiber product of $X$ with itself over $T$ by $X^{(m)}$. As in our situation the base is always clear, we omit it from the notation. So, for $X^{(m)}$ means $m$-times fiber product over $T$, and $X^{(m)}_t$ means $m$-times fiber product over $t$. In this situation $p_i : X^{(m)} \to X$ denotes the projection onto the $i$-th factor, and we set for any divisor $D$
or line bundle $\mathcal{L}$:

$$D^{(m)} := \sum_{i=1}^{m} p_i^* D, \text{ and } \mathcal{L}^{(m)} := \bigotimes_{i=1}^{m} p_i^* \mathcal{L}.$$ 

### 2.2. General further notation

A variety is an integral, separated scheme of finite type over $k$. A $(X, \Delta)$ is a pair if $X$ is a normal variety, and $\Delta$ is an effective $\mathbb{Q}$-divisor, called the boundary. A projective pair $(X, \Delta)$ over $k$ is Fano if $(X, \Delta)$ has klt singularities, and $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-Cartier divisor. To avoid confusion, many times we say klt Fano instead of Fano, nevertheless we mean the same by the two. If there is not boundary, we mean taking the empty boundary. A projective pair $(X, \Delta)$ is a normal Fano pair, if the klt condition is not assumed, that is, we only assume that $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-Cartier divisor.

A big open set $U$ of a variety $X$ is an open set for which $\text{codim}_X (X \setminus U) \geq 2$.

A vector bundle is a locally free sheaf of finite rank.

The $\mathbb{Q}$-linear system of a $\mathbb{Q}$-divisor $D$ on a normal variety is $|D|_\mathbb{Q} := \{ L \text{ is a } \mathbb{Q}\text{-divisor} | \exists m \in \mathbb{Z}, m > 0 : mL \sim mD \}$.

A geometric fiber of a morphism $f : X \to T$ is a fiber over a geometric point, that is over a morphism $\text{Spec } K \to T$, where $K$ is an algebraically closed field extension of $k$. We say that a condition holds for a very general geometric point/fiber, if there are countably many proper closed sets, outside of which it holds for all geometric points/fibers. General point/fiber is defined the same way but with excluding only finitely many proper closed subsets. The (geometric) generic point/fiber on the other hand denotes the scheme theoretic (geometric) generic point/generic fiber.

### 2.3. Relative canonical divisor

For a flat family $f : X \to T$ the relative dualizing complex is defined by $\omega^\bullet_{X/T} := f^! O_T$, where $f^!$ is Grothendieck upper shriek functor as defined in [Har66]. If $f$ is also a family of pure dimension $n$, then the relative canonical sheaf is the lowest non-zero cohomology sheaf $\omega_{X/T} := h^{-n}(\omega^\bullet_{X/T})$ of the relative dualizing complex. To obtain the absolute versions of these notions one uses the above definition for $T = \text{Spec } k$. The important facts for this article are:

(a) The sheaf $\omega_{X/T}$ is reflexive if the fibers are normal [PSZ13, Prop A.10].

(b) If $T$ is Gorenstein, then $\omega_{X/T} \cong \omega_X \otimes^f \omega^{-1}$ [Pat15, Lemma 2.4], and then as $\omega_X$ is $S_2$ [KM98, Cor 5.69], it is also reflexive [Har80].

(c) By the previous two points, if $f$ is flat and either $T$ is smooth or the fibers are normal, then $\omega_{X/T}$ is reflexive, and then if $X$ is normal, it corresponds to a divisor linear equivalence class which we denote by $K_{X/T}$.

(d) On the relative Cohen-Macaulay locus $U \subseteq X$ (that is, on the open set where the fibers are Cohen-Macaulay), $\omega_{U/T} \cong \omega_{X/T}|_U$ is compatible with base-change [Con00, Thm 3.6.1].

In particular, by the above we always have the following assumptions on our families: $f : X \to T$ is flat with fibers being of pure dimension $n$, and either $T$ is smooth, or the fibers of $f$ are normal. In both cases we discuss base-change properties of the relative canonical divisor below.

#### 2.3.1. Base-change of the relative log-canonical divisor when the fibers are normal

Let us assume that $f : X \to T$ is a projective, flat morphism to normal projective variety with normal, connected fibers (in particular $X$ is also normal), and $\Delta$ is an effective $\mathbb{Q}$-divisor on $X$, such that $\Delta$ does not contain any fiber, and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Let $U \subseteq X$ be the smooth locus of $f$, which is an open set, and by the normality assumption on the fibers, $U \cap X_t$ is a big open set on each fiber $X_t$ (see Section 2.2 for the definition of a big open set).

Let $S \to T$ be a morphism from another normal projective variety. Then, we may define a pullback $\Delta_S$ as the unique extension of the pullback of $\Delta|_U$ to $U_S$ (the key here is that $\Delta|_U$ is
\(\mathbb{Q}\)-Cartier. In particular, \(f_S : X_S \to S\) and \(\Delta_S\) satisfies all the assumptions we had for \(f : X \to T\) and \(\Delta\). Moreover, if \(\sigma : X_S \to X\) is the induced morphism, then as \(\mathbb{Q}\)-Cartier divisors
\[(2.0.a) \quad K_{X_S/S} + \Delta_S \sim_{\mathbb{Q}} \sigma^*(K_{X/T} + \Delta).\]
Indeed, it is enough to verify this isomorphism on \(U\) (as \(U\) is big in \(X\), and \(U_S\) is big in \(X_S\)). However, over \(U\) the linear equivalence \((2.0.a)\) holds by the definition of \(\Delta_S\) and by the base-change property of point \((d)\) above.

2.3.2. Base-change of the relative log-canonical divisor when the base is smooth. Let \(f : X \to T\) be a flat morphism from a normal projective variety to a smooth, projective variety with connected fibers. Let \(\Delta\) be an effective \(\mathbb{Q}\)-divisor on \(X\) such \(K_{X/T} + \Delta\) is \(\mathbb{Q}\)-Cartier. Let \(T_{\text{norm}} \subseteq T\) be the open set over which the fibers of \(X\) are normal.

Note that by the smoothness assumption on \(T\), at a point \(x \in X\), the fiber \(X_f(x)\) is Gorenstein if and only if \(x\) is relatively Gorenstein if and only if \(x\) is Gorenstein. Let \(U \subseteq X\) be the open set of relatively Gorenstein points over \(T\). Let \(\iota : C \to T\) be a finite morphism from a smooth, projective curve such that \(\iota(C) \cap T_{\text{norm}} \neq \emptyset\), and denote by \(\sigma : X_C \to X\) the natural morphism.

We claim that \(\sigma^{-1}U\) is big in \(X_C\). This is equivalent to show that for each \(c \in C\), \(X_c\) is Gorenstein at some point, and that for general \(c \in C\), there is a big open set of \(X_c\) where \(X_c\) is Gorenstein. The former is true for all schemes of finite type over \(k\) (hence also for \(X_c\), and the latter is true by the \(\iota(C) \cap T_{\text{norm}} \neq \emptyset\) assumption. This concludes our claim.

Now, let \(\pi : Z \to X_C\) be the normalization of \(X_C\), \(\rho : Z \to X\) and \(g : Z \to C\) the induced morphisms and set \(W := \rho^{-1}U\). The notations are summarized in the following diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{\sigma^{-1}U} & U \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\pi} X_C & \xrightarrow{\sigma} X \\
\downarrow & & \downarrow \sigma \\
C & \xrightarrow{g} T & \xrightarrow{f} T_{\text{norm}}
\end{array}
\]

Then, [KP17, Lem. 9.13] tells us that there is a natural injection \(\omega_{W/C} \to (\pi|_W)^*\omega_{\sigma^{-1}U/C}\). To be precise, [KP17, Lem. 9.13] assumes \(\sigma^{-1}U\) to be normal, but as the proof does not use it, this is an unnecessary assumption. Combining this injection with the isomorphism \((\sigma|_{\sigma^{-1}U})^*\omega_{U/T} \cong \omega_{\sigma^{-1}U/C}\) given by point \((d)\) above we obtain
\[(2.0.b) \quad \omega_{W/C} \hookrightarrow (\pi|_W)^*\omega_{\sigma^{-1}U/C} \cong (\pi|_W)^*(\sigma|_{\sigma^{-1}U})^*\omega_{U/T} \cong (\rho|_W)^*\omega_{U/T},\]
which is an isomorphism over the locus \(T_{\text{red}}\) over which the fibers of \(f\) are reduced. Indeed, over \(T_{\text{red}}\) fibers of \(X_C \to C\) are all reduced, and by the \(\iota(C) \cap T_{\text{norm}} \neq \emptyset\) assumption the general fiber of \(X_C \to C\) is normal. In particular, over \(T_{\text{red}}\), \(X_C\) is \(R_1\) and \(S_2\), and hence normal. So, \(\pi\) is the identity over \(T_{\text{red}}\).

Let \(m > 0\) be then an integer such that \(m(K_{X/T} + \Delta)\) is Cartier. That is, \(\mathcal{L} := \mathcal{O}_X(m(K_{X/T} + \Delta))\) is a line bundle, and furthermore, \(m\Delta\) yields an embedding \(\omega_{U/T}^{\otimes m} \hookrightarrow \mathcal{L}|_U\). Composing this with the \(m\)-th power of the homomorphism of \((2.0.b)\) we obtain:
\[(2.0.c) \quad \omega_{W/C}^{\otimes m} \to (\rho|_W)^*\mathcal{L} \cong \mathcal{O}_W(mp^*(K_{X/T} + \Delta)|_W),\]
which map over \(T_{\text{red}}\) is given by “multiplying with \((\rho|_{g^{-1}T_{\text{red}}})^{-1}\) \(m\Delta\). Indeed, for the latter remark, the main thing to note is that the regular locus of \(X\), over which \(m\Delta\) is necessarily Cartier, pulls back to a big open set of \(g^{-1}T_{\text{red}}\) (as general fiber of \(f_C\) is normal and special fiber of \(f_C\) over \(T_{\text{red}}\) are reduced). Hence \(\pi\) is an isomorphism over \(g^{-1}T_{\text{red}}\) and also the
pullback \((\rho|_{g^{-1}L_{\text{red}}})*m\Delta\) is sensible the usual way: restricting to the regular locus, performing the pullback there, and then taking divisorial extension using bigness of the open set.

Lastly, the map (2.0.c) is given by an effective divisor \(D\). If we set \(\Delta_Z := \frac{D}{n}\), using that \(W\) is big in \(Z\), we obtain:

**Proposition 2.1.** In the above situation, there is an effective \(\mathbb{Q}\)-divisor \(\Delta_Z\) on \(Z\) such that:

(a) \(K_{\mathcal{Z}/\mathcal{C}} + \Delta_Z \sim_{\mathbb{Q}} \rho^*(K_{\mathcal{X}/\mathcal{T}} + \Delta)\),

(b) \(X_C\) is normal over \(T_{\text{red}}\) and \(\Delta_Z|_{g^{-1}L_{\text{red}}} = (\rho|_{g^{-1}L_{\text{red}}})*\Delta\), and

(c) \(\Delta_Z|_{g^{-1}L_{\text{norm}}}\) agrees with the pullback of \(\Delta|_{f^{-1}L_{\text{norm}}}\) in the sense of Section 2.3.1.

### 3. The definition of the CM line bundle

Here we present the definition of the CM line bundle in two cases:

(a) in the non-logarithmic case for arbitrary polarizations, and

(b) in the logarithmic case for the anticanonical polarization.

In the first case, we also connect it to the other existing definitions in the literature. In the second case, we are not able to present such connections, because in this case, the lack of literature would force us to work out many details about the Paul-Tian type definition [PT06, PT09], and then prove the equivalence with that. This would be beyond the scope of the present article.

In any case, it is important to stress that the definitions are different in the two cases: One does not obtain the logarithmic version by simply plugging in the logarithmic relative anti-canonical divisor into the polarization of the non-logarithmic case. The reason for the difference is that in the logarithmic case the CM line bundle has to take into account also the variation of the boundary, see Remark 1.14.

**Definition 3.1.** CM line bundle in the non-logarithmic setting. Let \(f : X \to T\) be a flat morphism of normal projective varieties of relative dimension \(n\), and \(L\) an \(f\)-ample \(\mathbb{Q}\)-Cartier divisor on \(X\). For every integer \(q\) divisible enough, the Hilbert polynomial of a (equivalently any) fiber \(X_t\) is

\[
\chi(X_t, qL) = a_0q^n + a_1q^{n-1} + O(q^{n-2}).
\]

Set \(\mu_L := \frac{2a_1}{a_0}\). We define the Chow-Mumford line bundle as the pushforward cycle

\[
\lambda_{f, L} := f_*(\mu_LL^{n+1} + (n + 1)L^n \cdot K_{X/T}),
\]

which is an abuse of language as it is not line bundle but a \(\mathbb{Q}\)-Cartier divisor class, according to Proposition 3.7. We would also like to stress that \(\lambda_{f, L}\) is a divisor class (in the Weil group, or equivalently the first Chow group), as opposed to a fixed divisor.

If \(L\) is not indicated, then we take \(L = -K_{X/T}\), which we assume to be an \(f\)-ample \(\mathbb{Q}\)-Cartier divisor, and we use the notation \(\lambda_f := \lambda_{f, L}\).

**Remark 3.2.** Note that in the \(L = -K_{X/T}\) case:

\[
\lambda_f = f_*(\mu_L(-K_{X/T})^{n+1} + (n + 1)(-K_{X/T})^n \cdot K_{X/T}) = f_*(\mu_L - (n + 1))(-K_{X/T})^{n+1}
\]

As \(X\) in Definition 3.1 is assumed to be normal, so is \(X_t\) for \(t\) a general closed point. In particular, Lemma A.2 implies that

\[
\mu_L = \frac{2a_1}{a_0} = \frac{2\left(\frac{K_X \cdot L^{n-1}_{\text{red}}}{2(n-1)!}\right)}{\frac{\ell_T^n}{n!}} = n\frac{-K_{X_t} \cdot L^{n-1}_t}{L_t^n}.
\]

In particular if \(L = -K_{X/T}\) we obtain that \(\mu_L = n\). Hence, we obtain the definition we used in (1.0.a):

\[
\lambda_f = f_*(\mu_L - (n + 1))(-K_{X/T})^{n+1} = -f_*(-K_{X/T})^{n+1}.
\]
We only define the logarithmic version of the CM line bundle in the anti-log-canonically polarized case. If $\Delta = 0$, this definition agrees with the case of $L = -K_{X/T}$ of the non-logarithmic definition, according to the final formula of Remark 3.2.

**Definition 3.3.** CM line bundle in the logarithmic setting. If $f : (X, \Delta) \to T$ is a flat morphism of relative dimension $n$ from a projective normal pair to a normal projective variety such that $-(K_{X/T} + \Delta)$ is $\mathbb{Q}$-Cartier and $f$-ample. Then we define the CM line bundle by

$$\lambda_{f, \Delta} := -f_*(-(K_{X/T} + \Delta))^{n+1}.$$

**Notation 3.4.** In the set-up of Definition 3.1 (resp. of Definition 3.3 with setting $L := -(K_{X/T} + \Delta)$), fix an integer $s$ such that $sL$ is an $f$-very ample Cartier divisor. Following [MFK94, Appendix to Chapter 5, Section D] and [KM76, Theorem 4], consider the Mumford-Knudsen expansion of $O_X(sL)$:

$$(3.4.b) \quad \det f_*O_X(qsL) \cong \bigotimes_{i=0}^{n+1} M_i^{(q)}_i,$$

where $M_i$ are uniquely determined line bundles on $T$.

For future reference, we note that as the left side of (3.4.b) is invariant under base-change for $q \gg 0$, the above unicity of $M_i$ implies that:

**Lemma 3.5.** In the situation of Notation 3.4, the formation of $M_i$ is compatible with base-change. That is, if $S \to T$ is a base-change, and $M_i^S$ are the coefficients of the Knudsen-Mumford expansion of $sL_S$, then $M_i^S \cong (M_i)_S$.

**Notation 3.6.** In the case of Definition 3.1, according to [PT09, Definition 1] (see also [PT06, Section 2.4, page 11] and [KM76, Theorem 4] for the role of $M_{n+1}$), the CM line bundle is defined as

$$L_{CM,f,sL} := M_{n+1}^{(n+1)+\mu_sL} \otimes M_n^{-2(n+1)}.$$

For simplicity we regard $L_{CM,f,sL}$ as a Cartier divisor. (As we explained earlier in the case of Definition 3.3 a definition as above is not worked out in the literature to such an extent, and hence we do not consider it here.)

The proof of the following proposition will be given in Appendix A.

**Proposition 3.7.** (a) Connection with the Paul-Tian definition. In the situation of Notation 3.6, if $T$ is smooth or the fibers of $f$ are normal, then $s^n \lambda_{f,L} = c_1(L_{CM,f,sL})$.

(b) Connection with the leading term of the Knudsen-Mumford expansion. In the situation of Notation 3.4, if

- we consider the case of Definition 3.3 (which includes the case of Definition 3.1 with $L = -sK_{X/T}$)
- either $T$ is smooth or the fibers of $f$ are normal, and $\Delta$ does not contain any fiber,

then $-s^{n+1} \lambda_{f,\Delta} = c_1(M_{n+1})$.

In particular, $\lambda_{f,L}$ and $\lambda_{f,\Delta}$ are $\mathbb{Q}$-Cartier.

**Proposition 3.8.** Base-change for the CM-line bundle. Let $f : X \to T$ be a flat morphism between projective normal varieties, let $\Delta$ be an effective $\mathbb{Q}$-divisor such that $-(K_{X/T} + \Delta)$ is an $f$-ample $\mathbb{Q}$-Cartier divisor, and let $\tau : S \to T$ be a morphism from a normal projective variety. Assume either:

(a) the fibers of $f$ are normal and $\Delta$ does not contain any fiber, in which case set $g := f_S, Z := X_S$, and let $\Delta_Z$ be the pullback of $\Delta$ as explained in Section 2.3.1.
(b) \( T \) is smooth and \( \tau \) is a finite morphism from a curve, such that some of the fibers of \( f \) over \( \tau(S) \) are normal and not contained in \( \Delta \). In this case, set \( Z \) to be the normalization of \( X_S \), \( \rho : Z \to X \) and \( g : Z \to S \) the induced morphisms and \( \Delta_Z \) the effective \( \mathbb{Q} \)-divisor on \( Z \) given by Proposition 2.1.

Then, the CM line bundle satisfies the base-changes \( \tau^* \lambda_{f, \Delta} = \lambda_{g, \Delta_Z} \).

**Proof.** Set \( V := X_S \), \( L := -(K_{X/T} + \Delta) \) and let \( h : V \to S \) and \( \sigma : V \to X \) be the induced morphisms. Fix an integer \( s > 0 \) be such that \( sL \) and \( s\rho^*L \) are relatively very ample over \( T \) and \( S \), respectively. Note that according, to point (a) of Proposition 2.1, \( s\rho^*L \cong -s(K_Z/T + \Delta_Z) \).

Furthermore, set \( \mathcal{M}_{n+1}^f \) and \( \mathcal{M}_{n+1}^h \) be the leading terms of the Knudsen-Mumford expansions of \( sL \) and \( s\rho^*L \), respectively. Then,

\[
\tau^* \lambda_{f, \Delta} = \frac{\tau^* c_1 \left( \mathcal{M}_{n+1}^f \right)}{-s^{n+1} \Delta} = \frac{c_1 \left( \mathcal{M}_{n+1}^h \right)}{-s^{n+1} \Delta} = \frac{c_1 \left( \mathcal{M}_{n+1}^h \right)}{-s^{n+1} \Delta} = \lambda_{g, \Delta_Z}.
\]

4. **THE DELTA INVARIANT AND K-STABILITY**

Here we give the definitions and the properties used in the present article of \( \delta \)-invariants, as well as we present the definition of \( K \)-semi-stability and uniform \( K \)-stability in Definition 4.7. For all the later parts of the article we use the characterizations of \( K \)-semi-stability and \( K \)-stability via \( \delta \)-invariants given in Corollary 4.8. We also prove in the present section that the \( \delta \)-invariant is constant at very general points (Proposition 4.14).

4.1. **Definitions**

Basis-type divisors and the delta invariant have been introduced by K. Fujita and Y. Odaka in [FO16], see also [BJ17]; in this section we recall their definitions.

**Definition 4.1.** Assume we are in the following situation:

- \( Z \) is a variety over \( k \),
- \( L \) is a \( \mathbb{Q} \)-Cartier divisor on \( Z \), and
- \( q > 0 \) is an integer for which \( qL \) is Cartier.

A divisor \( D \in |L|_\mathbb{Q} \) is of \( q \)-basis type if there are \( D_i \in |qL| \ (1 \leq i \leq h^0(X, qL)) \), for which the corresponding \( s_i \in H^0(Z, qL) \) form a \( k \)-basis of \( H^0(Z, qL) \), and \( D \) can be expressed as

\[
D = \frac{1}{qh^0(Z, qL)} \sum_{i=1}^{h^0(Z, qL)} D_i.
\]

\( D \) is of basis type if it is of \( q \)-basis type for some integer \( q > 0 \).

Let \( \Delta \) be a fixed effective \( \mathbb{Q} \)-divisor on \( Z \) such that \((Z, \Delta)\) is a klt pair. Given a \( \mathbb{Q} \)-Cartier effective divisor \( D \) on \( Z \), we define its log canonical threshold as

\[
lct(Z, \Delta; D) := \sup \{ t \mid (Z, \Delta + tD) \text{ is klt} \}.
\]

Remark that since \((Z, \Delta)\) is klt, the above threshold is a positive number. Let us recall the definition of the \( \alpha \) invariant.
Definition 4.2. Let \((Z, \Delta)\) be a klt pair and let \(L\) be an effective \(\mathbb{Q}\)-Cartier divisor on \(Z\). The alpha invariant of \((Z, \Delta; L)\) is
\[
\alpha(Z, \Delta; L) := \inf_{D \in |L|_{\mathbb{Q}}} \text{lct}(Z, \Delta; D).
\]
We write \(\alpha(Z, \Delta)\) for \(\alpha(Z, \Delta; -K_Z - \Delta)\).

The \(\alpha\) invariant has been introduced by Tian in relation with the existence problem for Kähler-Einstein metrics. The delta invariant is a variation on the alpha invariant. The main difference is that in the case of \(\alpha\) invariant one considers the log canonical threshold of all divisors in the \(\mathbb{Q}\)-linear system, while in the \(\delta\) invariant is defined using only basis type divisors. In particular, while \(\alpha(X) \geq \dim X \dim X + 1\) only implies \(K\)-semi-stability [Tia87, OS12], \(\delta(X) \geq 1\) happens to be equivalent to it [BJ17, Theorem B], see also Corollary 4.8. The delta invariant was introduced in [FO16, Definition 0.2]. In [BJ17], although it was also denoted by \(\delta\), it is called the stability threshold.

Definition 4.3. Let \((Z, \Delta)\) be a klt pair and let \(L\) be a \(\mathbb{Q}\)-Cartier divisor on \(Z\).

(a) For every positive integer \(q\) for which that \(qL\) is Cartier and \(h^0(Z, qL) > 0\), the \(q\)-th delta invariant of \(L\) with respect to the pair \((Z, \Delta)\) is
\[
\delta_q(Z, \Delta; L) := \inf_{D \in |L|_{\mathbb{Q}}} \text{lct}(Z, \Delta; D) \text{ of } q\text{-basis type}
\]
According to [KP17, Lem 8.8], this infimum is in fact a minimum.

(b) Assume that \(L\) is big, and fix an integer \(s > 0\) such that \(sL\) is Cartier and \(h^0(Z, sL) > 0\), which conditions then also hold for every positive multiple of \(s\). The delta invariant of \(L\) with respect to \((Z, \Delta)\) is
\[
\delta(Z, \Delta; L) := \limsup_{q \to \infty} \delta_{sq}(Z, \Delta; L).
\]

According to Corollary 4.6, the above definition does not depend on the choice of \(s\), and the limsup is in fact a limit.

(c) If \((Z, \Delta)\) is a klt Fano pair, we let \(\delta_q(Z, \Delta) := \delta_q(Z, \Delta; -K_Z - \Delta)\) and \(\delta(Z, \Delta) := \delta(Z, \Delta; -K_Z - \Delta)\).

4.2. Relation to K-stability

In this section we follow closely [BJ17], as we want to adapt some of their result from Fano varieties over \(\mathbb{C}\) to Fano pairs over \(k\). Similar adaptation was done also in [Blu18]. Consider the situation:

Notation 4.4. \((Z, \Delta)\) is a klt pair, \(L\) is a \(\mathbb{Q}\)-Cartier divisor on \(Z\), and \(s > 0\) is an integer such that \(sL\) is Cartier and \(h^0(Z, sL) \neq 0\).

Let \(v\) be a non-trivial divisorial valuation on \(Z\) associated to a prime divisor \(E\) over \(Z\), we consider the filtration
\[
F_i H^0(Z, qsL) := \{ t \in H^0(Z, qsL) | \text{ such that } v(t) \geq i \} = H^0(V, qsn^*L - iE),
\]
and the invariant
\[
S_q(v) := \frac{1}{qsh^0(Z, qsL)} \sum_i i \dim_k \left( F_i H^0(Z, qsL)/F_{i+1} H^0(Z, qsL) \right)
= \frac{1}{qsh^0(Z, qsL)} \sum_{i \geq 1} \dim_k F_i H^0(Z, qsL).
\]
Denote by $B_q$ the set of $qS$-basis type divisors with respect to $qS$. As observed for instance in [FO16, proof of Lemma 2.2],

\[(4.4.a) \quad S_q(v) = \max_{D \in B_q} v(D),\]

and the maximum is attained exactly for bases adapted to the filtration $F_i$. When $L$ is big, the asymptotic of $S_q$ is well-understood, see for instance [FO16, proof of Theorem 1.3], [BJ17, Corollary 2.12] and [BHJ17, Corollary 3.2];

\[(4.4.b) \quad S(v) := \lim_{q \to \infty} S_q(v) = \frac{1}{\Vol(L)} \int_0^{\infty} \Vol(\pi^*L - xE)dx\]

The next statement is a logarithmic version of [BJ17, Theorem 4.4], following very closely the arguments given there.

**Theorem 4.5.** (a) If $L$ is a big $\Q$-Cartier divisor, such that $sL$ is a Cartier divisor and $h^0(Z, sL) \neq 0$, then the sequence $\delta_{qs}(Z, \Delta; L)$ converges to $\delta(Z, \Delta; L)$, i.e. the delta invariant is a limit and not only a limsup; moreover

\[\delta(Z, \Delta; L) = \inf_v \frac{A(v)}{S(v)},\]

where $A(v)$ is the log-discrepancy of $v$ with respect to the klt pair $(Z, \Delta)$, and the inf is taken over all non-trivial divisorial valuations. In particular, $\delta(Z, \Delta; L)$ is independent of the choice of $s$.

(b) Assuming furthermore that $L$ is ample, the following bounds hold

\[\frac{\dim Z + 1}{\dim Z} \alpha(Z, \Delta; L) \leq \delta(Z, \Delta; L) \leq (\dim Z + 1) \alpha(Z, \Delta; L).\]

**Proof.** **POINT (A).** Set $\delta_q := \delta_{qs}(Z, \Delta; L)$ and $\delta := \delta(Z, \Delta; L)$. We first prove the inequality

\[(4.5.c) \quad \limsup_{q \to \infty} \delta_q \leq \inf_v \frac{A(v)}{S(v)}\]

Thanks to Equations (4.4.a) and (4.4.b), we can write

\[
\begin{align*}
\inf_v \frac{A(v)}{S(v)} &= \inf_v \lim_{q \to \infty} \inf_{D \in B_q} \frac{A(v)}{v(D)} \\
&\geq \limsup_{q \to \infty} \left( \inf_{D \in B_q} \inf_v \frac{A(v)}{v(D)} \right) = \limsup_{q \to \infty} \delta_q.
\end{align*}
\]

We now prove the inequality

\[(4.5.d) \quad \liminf_{q \to \infty} \delta_q \geq \inf_v \frac{A(v)}{S(v)}\]

This follows from the key uniform convergence result [BJ17, Corollary 3.6]: for every $\varepsilon > 0$ there exists a $q_0 = q_0(\varepsilon)$ such that for all $q > q_0$ and all divisorial valuations $v$ we have

\[(1 + \varepsilon)S(v) \geq S_q(v)\]

(The quoted result is above the complex numbers, however its proof works also on $k$..) We thus have for $q$ big enough

\[
\frac{1}{1 + \varepsilon} \inf_v \frac{A(v)}{S(v)} \leq \inf_v \frac{A(v)}{S_q(v)} = \inf_{D \in B_q} \inf_v \frac{A(v)}{v(D)} = \delta_q
\]

\[\inf_v \frac{A(v)}{v(D)} = \lct(Z, \Delta; D)\]
taking the liminf on $q$ on the right hand side, and then letting $\varepsilon$ go to zero, we get the requested inequality. We obtain point (a) combining Equations 4.5.c and 4.5.d.

Point (b). Given a divisorial valuation $v$, we define its $q$-th pseudo-effective threshold as

$$ T_q(v) := \max \left\{ \frac{v(D)}{qs} \mid D \in |qsL| \right\} $$

and we have

$$ \alpha(Z, \Delta; L) = \inf_v \frac{A(v)}{T_q(v)}. $$

When $L$ is ample, [BJ17, Prop. 3.11] gives the following bounds

$$ \frac{\dim(Z)}{\dim(Z) + 1} \inf_q T_q(v) \geq S(v) \geq \left( \frac{1}{\dim(Z) + 1} \right) \inf_q T_q(v), $$

which imply point (b) (again, the proof in [BJ17] is over the complex numbers, but it works also over $k$). \hfill \square

**Corollary 4.6** (Invariance of the delta invariant by scaling). In the situation of Definition 4.3.(b), for every positive integer $r > 0$, $\delta(Z, \Delta; L) = r\delta(Z, \Delta; rL)$. Equivalently,

$$ \limsup_{q \to \infty} \delta_{sq}(Z, \Delta; L) = \limsup_{q \to \infty} \delta_{sq}(Z, \Delta; L). $$

**Proof.** By Theorem 4.5, the limsup appearing in Equation (4.6.e) is a limit, so the claim. \hfill \square

We give the following definition of $K$-stability, which is equivalent to the more classical one by [OS15, Theorem 6.1 (ii)] and [Fuj17c, Theorem 1.5].

**Definition 4.7.** A normal Fano pair $(Z, \Delta)$ is

(a) $K$-semi-stable if it is klt and for every divisorial valuation $v$, one has $A(v) \geq S(v)$;

(b) uniformly $K$-stable if it is klt and there exists a positive constant $\varepsilon$ such that for every divisorial valuation $v$, one has $A(v) \geq (1 + \varepsilon)S(v)$.

Here $A(v)$ denotes the log-discrepancy with respect to the pair $(Z, \Delta)$. The following corollary is now an immediate consequence of the above definition and Theorem 4.5.

**Corollary 4.8** (Characterization of $K$-stability). Let $(Z, \Delta)$ be a normal Fano pair. Then, $(Z, \Delta)$ is

(a) $K$-semi-stable if and only if $(Z, \Delta)$ is klt and $\delta(Z, \Delta) \geq 1$,

(b) uniformly $K$-stable if and only if $(Z, \Delta)$ is klt and $\delta(Z, \Delta) > 1$.

Moreover, if $(Z, \Delta)$ is klt and $\alpha(Z, \Delta) \geq \frac{\dim(Z)}{\dim(Z) + 1}$ (resp. $> \frac{\dim(Z)}{\dim(Z) + 1}$), then $(Z, \Delta)$ is $K$-semi-stable (resp. uniformly $K$-stable); if $(Z, \Delta)$ is klt and $\alpha(Z, \Delta) \leq \frac{1}{\dim(Z) + 1}$ (resp. $< \frac{1}{\dim(Z) + 1}$), then $(Z, \Delta)$ is not uniformly $K$-stable (resp. not $K$-semi-stable).

**4.3. Products**

The following conjecture is motivated by the equivalence between $K$-stability and K"ahler-Einstein metrics in the Fano setting, it has been already proposed in [PW17, Conjecture 1.11].

**Conjecture 4.9.** Given two klt Fano pairs $(W, \Delta_W)$ and $(Z, \Delta_Z)$, one has

$$ \delta(W \times Z, \Delta_W \boxtimes \Delta_Z) = \min\{\delta(W, \Delta_W), \delta(Z, \Delta_Z)\} $$

The analogue result for the alpha invariant and any polarization was written down for example in [KP17, Proposition 8.11], but used to be present much earlier in a smaller generality for example in Viehweg’s works. See also [PW17, Thm. 1.10] and [CS08, Lemma 2.29] for the Fano case. We can prove a weaker result for the delta invariant in Proposition 4.13, before which we need a definition and a lemma.
**Definition 4.10** (Product basis type divisor). Let \((W, \Delta_W)\) and \((Z, \Delta_Z)\) be two klt pairs, let \(L_W\) and \(L_Z\) \(\mathbb{Q}\)-Cartier divisors on \(W\) and \(Z\), respectively, and let \(q > 0\) be an integer such that both \(qL_W\) and \(qL_Z\) are Cartier. A divisor \(D\) on \(W \times Z\) is of *q-product basis type* if there exist \(q\)-basis type divisors \(D_W\) on \(W\) and \(D_Z\) on \(Z\) such that
\[
D = p_W^*D_W + p_Z^*D_Z
\]
where \(p_W\) and \(p_Z\) are the projections.

**Remark 4.11.** In Definition 4.10, if \(D_W\) is associated to a basis \(s_i\) and \(D_Z\) to a basis \(t_i\), then \(D\) is associated to the basis \(s_i \boxtimes t_j\).

**Lemma 4.12.** Let \((W, \Delta_W)\) and \((Z, \Delta_Z)\) be two klt (resp. lc) pairs, then also \((W \times Z, \Delta_W \boxtimes \Delta_Z)\) is klt (resp. lc).

**Proof.** As we work in characteristic zero, we may take the product of a log resolution of \((W, \Delta_W)\) and of \((Z, \Delta_Z)\). This will be a log-resolution for \((W \times Z, \Delta_W \boxtimes \Delta_Z)\), with the union of the discrepancies of the original two log-resolutions, so the claim.

**Proposition 4.13.** With the notations of Definition 4.10, let \(D\) be a q-product basis type divisor. Then,
\[
\text{let}(W \times Z, \Delta_W \boxtimes \Delta_Z, D) \geq \min\{\delta_q(W, \Delta_W; L_W), \delta_q(Z, \Delta_Z; L_Z)\}
\]

**Proof.** Take \(t < \min\{\delta_q(W, \Delta_W; L_W), \delta_q(Z, \Delta_Z; L_Z)\}\). We have to show that \((W \times Z, \Delta_W \boxtimes \Delta_Z + tD)\) is log canonical. Recall that
\[
(W \times Z, \Delta_W \boxtimes \Delta_Z + tD) = (W \times Z, (\Delta_W + tD_W) \boxtimes (\Delta_Z + tD_Z))
\]
and both \((W, \Delta_W + tD_W)\) and \((Z, \Delta_Z + tD_Z)\) are log canonical because of the hypothesis on \(t\), so the claim follows from Lemma 4.12.

**4.4. Behavior in families**

Here we prove that the \(\delta\)-invariant is constant on very general geometric points. Recall that a *geometric point* of \(T\) is a map from the spectrum of an algebraically closed field to \(T\). Key examples are the closed points and the geometric generic point (i.e. the algebraic closure of the function fields) of \(T\).

**Proposition 4.14.** If \(f : (X, \Delta) \to T\) is a flat, projective family of normal pairs over a normal variety (that is \(\text{Supp} \Delta\) does not contain any fiber and \(K_{X/T}\) is \(\mathbb{Q}\)-Cartier), and \(L\) is an \(f\)-ample \(\mathbb{Q}\)-Cartier divisor on \(X\), then there is a very general value of \(\delta(X_\overline{T}, \Delta_\overline{T}; L_\overline{T})\). More precisely, there is a real number \(d \geq 0\) and there are countably many Zariski closed subsets \(T_i \subseteq T\) such that for any geometric point \(\overline{t} \in T \setminus (\bigcup_i T_i), \delta(X_\overline{T}, \Delta_\overline{T}; L_\overline{T}) = d\).

**Proof.** We may fix an integer \(s > 0\) such that \(sL\) is Cartier and \(f_*O_X(qsL)\) is non-empty and commutes with base-change for any integer \(q > 0\). In particular, then for all \(t \in T\), \(sL_t\) is Cartier and \(h^0(X_t, qsL_t)\) is positive and independent of \(t\) for any integer \(q > 0\).

We prove the statement that for each integer \(q > 0\) there is a real number \(d > 0\) and a non-empty Zariski open set \(U \subseteq T\) such that for each geometric point \(\overline{t} \in U, \delta_{qs}(X_\overline{T}, \Delta_\overline{T}; L_\overline{T}) = d\). Setting \(T_q := T \setminus U\) implies then the statement of the proposition. So, we fix an integer \(q > 0\), and in the rest of the proof we show the above statement in italics. We also set \(r := h^0(X_t, qsL_t)\) and \(l := qsr\), where the former is independent of \(t \in T\) by the above choice of \(s\).

Set \(W := \mathbb{P}((f_*O_X(qsL))^*)\). Then, for any geometric point \(\overline{t} \in T\) we have natural bijections:

\[
\text{(4.14.a)} \quad \begin{cases} k(\overline{t})\text{-rational points of } W_{\overline{t}} & \leftrightarrow \text{lines through the origin in } H^0(X_{\overline{T}}, O_X(qsL)|_{X_{\overline{T}}}) \leftrightarrow D \in |qsL|_{X_{\overline{T}}}
\end{cases}
\]
We consider the open subset
\[ Y \subseteq W \times_T W \times_T \cdots \times_T W \]
corresponding to linearly independent lines. That is, for any geometric point \( \overline{t} \in T \), using (4.14.a), we have a natural bijection
\[(4.14.b) \quad k(\overline{t})\text{-rational points of } Y_{\overline{t}} \leftrightarrow (D_i) = (D_1, \ldots, D_r) \text{ is a basis of } \vert qsL\vert_{X_{\overline{t}}} \]
Denote by \( \overline{y}_{(D_i)} \) the geometric point of \( Y \) corresponding to \((D_i)\) via the correspondence (4.14.b), where \( D_i \in \vert qsL\vert_{X_{\overline{t}}} \).

Consider the universal family of \( q \)-basis type divisors
\[ g : (Z := X \times_T Y, \Delta' := \Delta_Y; \Gamma) \rightarrow Y \]
such that for any geometric point \( \overline{y} := \overline{y}_{(D_i)} \in Y \), \( \Gamma_{\overline{y}} = \sum_{i=1}^r D_i \). Denote by \( \pi : Y \rightarrow T \) the natural projection.

According to [KP17, Lem 8.8], the log canonical threshold function \( \overline{t} \mapsto \text{lct} \left( \Gamma_{\overline{y}}, Z_{\overline{y}}, \Delta_{\overline{y}} \right) \) is lower semi-continuous. Furthermore, the second paragraph of [KP17, Lem 8.8] shows that there is a dense open set \( Y_0 \subseteq Y \) such that \( \text{lct} \left( \Gamma_{\overline{y}}, Z_{\overline{y}}, \Delta_{\overline{y}} \right) \) is the same for every \( \overline{y} \in Y_0 \). Applying this iteredly to the complement of \( Y_0 \), we obtain that \( \overline{y} \mapsto \text{lct} \left( \Gamma_{\overline{y}}, Z_{\overline{y}}, \Delta_{\overline{y}} \right) \) takes only finitely many values on \( Y \), say \( r_1 > r_2 > \cdots > r_t \), and the level sets are constructible subsets of \( Y \). Hence,
\[ L_i := \{ \overline{y} \in Y \mid \text{lct} \left( \Gamma_{\overline{y}}, Z_{\overline{y}}, \Delta_{\overline{y}} \right) \geq r_i \} \]
are open sets, and for any geometric point \( \overline{y} := \overline{y}_{(D_j)} \) of \( Y \),
\[ \text{lct} \left( \Gamma_{\overline{y}}, \Delta_{\overline{y}}; \Gamma_{\overline{y}} \right) = \text{lct} \left( \sum_{i=1}^r D_i \right) = \max\{ r_i | (D_j) \in L_i \}. \]
It follows that for any geometric point \( \overline{t} \in T \),
\[(4.14.c) \quad \delta_{qs} \left( X_{\overline{t}}, \Delta_{\overline{t}} \right) = \max\{ r_i | Y_{\overline{t}} \subseteq (L_i)_{\overline{t}} \}. \]

After the above discussion, our claim follows immediately. Indeed, we just need to choose \( a \) to be the smallest integer such that \( L_a \) contains the generic fiber of \( \pi \). Then there is a non-empty open set \( U \subseteq T \) contained in
\[ (T \setminus \pi(Y \setminus L_a)) \cap \pi(L_a \setminus L_{a-1}). \]
In particular, for any geometric point \( \overline{t} \in U \):
(a) \( Y_{\overline{t}} \subseteq (L_a)_{\overline{t}} \), and
(b) \( (L_a \setminus L_{a-1})_{\overline{t}} \neq \emptyset \) and hence \( Y_{\overline{t}} \nsubseteq (L_{a-1})_{\overline{t}} \).
Therefore, by setting \( d := r_a \), (4.14.c) implies that \( \delta_{qs} \left( X_{\overline{t}}, \Delta_{\overline{t}} \right) = d \) for all geometric points \( \overline{t} \in U \).

\[ \square \]

Remark 4.15. We note that one could define the \( \delta \)-invariant also over over non algebraically closed base fields, with verbatim the same definition as Definition 4.3. If \( (Y_K, \Delta_K) \) is a projective klt pair and \( N_K \) is a \( \mathbb{Q} \)-Cartier divisor defined over a non-closed field \( K \), and furthermore we choose a basis type divisor \( D = \sum_{i=1}^{h^0(Y_K, qN_K)} D_i \) (that is, \( D_i \) form a \( K \)-basis of \( H^0(Y_K, qN_K) \)), then \( \text{lct}(Y_K, \Delta_K, D) \) = \( \text{lct}(Y_K, \Delta_K, D_K) \), where \( D_K \) is a basis type divisor for \( N_K \). Hence, \( \delta_q(Y_K, \Delta_K, N_K) \geq \delta_q \left( Y_K, \Delta_K, N_K \right) \). However, \( \delta_q(Y_K, \Delta_K, N_K) > \delta_q \left( Y_K, \Delta_K, N_K \right) \) could happen as not all basis type divisors of \( N_K \) come from basis type divisors of \( N_K \). A simple example
is if $Y_K$ is a conic not isomorphic to $\mathbb{P}^1_K$, $\Delta_K = 0$, and $N_K = K_{Y_K}^{-1}$. Then, $\delta_q(Y_K, \Delta_K; N_K) = 3$, but $\delta_q(Y_K, \Delta_K; N_K) = 1$.

In particular, if one takes a conic bundle $f : X \to T$ without a section, and $\eta$ is the generic point of $T$, then for the generic fiber we have $\delta(X_\eta) = 2$, but for all geometric fiber (including the geometric generic fiber) outside of the discriminant locus we have $\delta(X_\eta) = 1$. So, the $\delta$-invariant is not the same for a general and for the generic point (in general). In particular, one cannot replace "any geometric point $\overline{t} \in T$" in Proposition 4.14 with just "any point $t \in T$".

Remark 4.16. The special case of Proposition 4.14 when $d = 1$ and $\Delta = 0$ (so for $K$-semi-stability via [BJ17]) was shown in [Liu17, Thm 3] with other methods.

Remark 4.17. Proposition 4.14 is very weak version of what is expected. It is conjectured (c.f., [BL18]) that $\delta$ is lower semi-continuous, and furthermore the $\delta \geq 1$ set is also open. Some of this has been proven in [LWX16, Thm 1.1(i)] and [BL18].

5. Growth of sections of vector bundles over curves

In this section, we present results about the growth of the number of sections of vector bundles over curves. We apply these in Section 7 and Section 9 to vector bundles of the form $f_* \mathcal{O}_X(q(-K_X/T - \Delta - f^* H))$ to obtain many sections of divisors of type $q(-K_X/T - \Delta - f^* H)^{(m)}$, where $f : (X, \Delta) \to T$ is a log-Fano family, $H$ is an auxiliary divisor and $(\cdot)^{(m)}$ is the fiber product notation of Section 2.1. The precise statement is given in Theorem 5.11.

Notation 5.1. Let $T$ be a smooth projective curve of genus $g$ over $k$, let $\mathcal{E}$ be a vector bundle on $T$. Let $\mu(\mathcal{E})$ be the slope of $\mathcal{E}$, namely $\mu(\mathcal{E}) := \text{deg} \mathcal{E} / \text{rk} \mathcal{E}$.

First we recall well known statements in Proposition 5.2, Proposition 5.3 and Proposition 5.4 concerning semi-stable bundles.

Proposition 5.2. In the situation of Notation 5.1, given two vector bundles $\mathcal{E}$ and $\mathcal{E}'$, we have $\mu(\mathcal{E} \otimes \mathcal{E}') = \mu(\mathcal{E}) + \mu(\mathcal{E}')$: moreover, if both $\mathcal{E}$ and $\mathcal{E}'$ are semi-stable, then so is $\mathcal{E} \otimes \mathcal{E}'$.

Proof. For the first statement, just remark that $\text{det}(\mathcal{E} \otimes \mathcal{E}') = \text{det}(\mathcal{E})^{\otimes \text{rk}(\mathcal{E}')} \otimes \text{det}(\mathcal{E}')^{\otimes \text{rk}(\mathcal{E})}$. The second statement is [Laz04b, Corollary 6.4.14].

Proposition 5.3. In the situation of Notation 5.1, if $\mathcal{E}$ is semi-stable with $\mu(\mathcal{E}) > 2g - 2$, then $h^1(\mathcal{E}) = h^0(\omega_T \otimes \mathcal{E}^*) = 0$.

Proof. We prove the $h^0$ vanishing, and then the $h^1$ vanishing follows by Serre-duality. The bundle $\omega_T \otimes \mathcal{E}^*$ is also semi-stable and $\mu(\omega_T \otimes \mathcal{E}^*) = \mu(\omega_T) - \mu(\mathcal{E}) < 0$. Hence, $h^0(\omega_T \otimes \mathcal{E}^*) = 0$, as a section would give a subbundle of $\mathcal{E}$ of slope 0.

Proposition 5.4. In the situation of Notation 5.1, if $\mathcal{E}$ is semi-stable with $\mu(\mathcal{E}) \geq 2g$, then $\mathcal{E}$ is globally generated.

Proof. Fix a closed point $t \in T$, and let $\mathcal{G}$ be either $\mathcal{E}$ or $\mathcal{E}(-t)$. Riemann-Roch tells us that

$$h^0(\mathcal{G}) = h^0(\mathcal{G}) - h^1(\mathcal{G}) = \text{deg} \mathcal{G} + \text{rk}(\mathcal{G})(1 - g) = \text{rk}(\mathcal{G})(\mu(\mathcal{G}) + 1 - g).$$

In particular, $h^0(\mathcal{E}) = h^0(\mathcal{E}(-t)) + \text{rk}(\mathcal{G})$. So, by looking at the usual exact sequence:

$$0 \to H^0(T, \mathcal{E}(-t)) \to H^0(T, \mathcal{E}) \to H^0(k(t), \mathcal{E} \otimes k(t))$$

we see that $\mathcal{E}$ is in fact generated at $t$. As $t$ was chosen arbitrarily, $\mathcal{E}$ is globally generated.
**Notation 5.5.** In the situation of Notation 5.1, let $0 = F^0 \subsetneq F^1 \subsetneq \cdots \subsetneq F^{\ell} \subsetneq F^\ell = E$ be the Harder-Narasimhan filtration [HN75, Lem 1.3.7 & 1.3.8] of $E$. Set $\mu_i := \mu(G^i)$ and $r_i := \text{rk}(G^i)$, where $G^i := F^i/F^{i-1}$. (In particular, we have $\mu_1 > \mu_2 > \cdots > \mu_\ell$ [HN75, Lem 1.3.8].)

**Remark 5.6.** When $T = \mathbb{P}^1$, we have a canonical decomposition
\[ E = \bigoplus_{1 \leq j \leq \ell} O_T(a_j) \otimes O_T^{\oplus n_j}, \]
with $a_j < a_{j+1}$. In this case, the Harder-Narasimhan filtration turns out to be
\[ F^i = \bigoplus_{1 \leq j \leq i} O_T(a_j) \otimes O_T^{\oplus n_j}, \]
and the slope $\mu_i$ is just $a_i$. In the study of K-stability, a key situation is when $T = \mathbb{P}^1$ is the base of a test configuration $f : (X, L) \to T$ (trivially compactified at infinity), and $E = f_*(mL)$ for some $m > 0$. In this case, the HN filtration is equal to the classical weight filtration; this follows from the classical localization formula, see for instance [Wan12, Example 1], which relates the degree of a line bundle on $\mathbb{P}^1$ with a $\mathbb{G}_m$ action to the weight of the action. We can thus think at the Harder-Narasimhan filtration as a generalization of the weight filtration.

On the other hand, we also note that the Harder-Narasimhan filtration is much more general than the weight filtration as it exists for any family not only for test configurations, in particular for non-isotrivial families over arbitrary curve bases. This is a crucial point for our argument.

**Proposition 5.7.** In the situation of Notation 5.5, if $\mu_i \geq 2g$ for every $i$, then:
(a) $H^1(T, E) = 0$
(b) $E$ is globally generated.

**Proof.** We prove both statements at once, by induction on $\ell$. If $\ell = 1$, both statements were shown in Proposition 5.3. So, we may assume that $\ell > 1$. However, then we may include $E$ in an exact sequence
\[ (5.7.a) \quad 0 \longrightarrow G \longrightarrow E \longrightarrow E' \longrightarrow 0, \]
where $G$ is semi-stable of rank at least $2g$ and $E'$ also satisfies the assumption of the proposition, but with $\ell$ replaced by $\ell - 1$. Hence we know both statements for $E$ replaced by $E'$. Applying now long exact sequence of cohomology to (5.7.a) yields:
\[ (5.7.b) \quad 0 \longrightarrow H^0(T, G) \longrightarrow H^0(T, E) \longrightarrow H^0(T, E') \longrightarrow 0 = H^1(T, G) \longrightarrow H^1(T, E) \longrightarrow H^1(T, E') = 0, \]
where the two vanishings are due to Proposition 5.3 and induction, respectively. This concludes our cohomology vanishing statement. For the global generation statement, we just use that where both $G$ and $E'$ are globally generated again by Proposition 5.3 and induction, respectively. Hence, according to (5.7.b), the sections generating at a given $t$ of these two lift to sections generating $E$ at $t$.

□

After the above basic statements, we work towards Theorem 5.11. This is a statement about tensor powers of vector bundles of positive degree. In particular, we need to understand the Harder-Narasimhan filtration of a tensor power, in terms of the Harder-Narasimham filtration of the original vector bundle. The necessary notation is introduced in Notation 5.8.

**Notation 5.8.** In the situation of Notation 5.5, fix also a closed point $t \in T$ (which will be the point at which the global sections we are interested in would need to become simple tensors). Then the Harder Narasimhan filtration induces a filtration $0 = F^0_t \subsetneq F^1_t \subsetneq \cdots \subsetneq F^{\ell-1}_t \subsetneq F^\ell_t = E_t$. Let...
\{e_i\} be a basis of \(E_i\) adapted to this filtration. By this, we mean that the intersection of \(\mathcal{F}_i^{j}\) and \(\{e_i\}\) gives a basis of \(\mathcal{F}_i^j\) for every \(j\).

Fix an integer \(m > 0\) (this integer will be the power of the tensor-power of \(E\) that we are examining). We will parametrize subsheaves of \(E^\otimes m\) that are tensor products of the \(\mathcal{F}^i\)'s by elements of \(\{1, \ldots, \ell\}^m\). Because of Proposition 5.7, we will be particularly interested in subsheaves with slope at least \(2g\). Consider the subset of \(\{1, \ldots, \ell\}^m\) defined by:

\[
S_m := \left\{ (s_1, \ldots, s_m) \in \{1, \ldots, \ell\}^m \mid \sum_{j=1}^m \mu_{s_j} \geq 2g \right\}
\]

As we are interested in a filtration of \(E^\otimes m\), we will need an ordering on \(\{1, \ldots, \ell\}^m\). First we introduce a partial ordering: for any two \(s, s' \in \{1, \ldots, \ell\}^m\), we say that

- \(s \geq s'\) if \(s_j \geq s'_j\) for all \(1 \leq j \leq m\), and
- \(s > s'\) if \(s \geq s'\) and there is a \(1 \leq j \leq m\) such that \(s_j > s'_j\).

Note that \(S_m\) is closed in the downwards direction, that is, whenever \(s \in S\), and \(s' < s\), then \(s' \in S\).

We also assign a minimal slope \(\nu(s)\) to \(s \in \{1, \ldots, \ell\}^m\) given by \(\sum_{j=1}^m \mu_{s_j}\), which will be the actual minimal slope of the corresponding sheaf in the Harder-Narasimhan filtration of \(E^\otimes m\). Note that \(\nu(s) \geq 2g\) if and only if \(s \in S_m\).

After the above, we arrange the elements of \(\{1, \ldots, \ell\}^m\) in a decreasing order with respect to \(\nu\), such that \(S_m\) consists of the first \(d\) elements.

\[
S_m = \{ s^1, \ldots, s^d \}, \quad \text{and} \quad \{1, \ldots, \ell\}^m = S_m \cup \left\{ s^{d+1}, \ldots, s^m \right\},
\]

where \(\nu(s^c)\) is a (not necessarily strictly) decreasing function of \(c\). In particular, whenever \(s^{c'} < s^c\), then \(c' < c\). As expected, \(s^c_j\) denotes the coordinates of \(s^c\), that is, \(s^c = (s^c_1, \ldots, s^c_m)\).

For any integer \(1 \leq c \leq e\), we define then the following subbundles of \(E^\otimes m\):

\[
\mathcal{F}^c := \bigotimes_{j=1}^m \mathcal{F}^{s^c_j}, \quad \text{and} \quad \mathcal{H}^c := \sum_{i=1}^c \mathcal{F}^i.
\]

In fact, it is not clear from the definition that \(\mathcal{H}^c\) is a subbundle as opposed to just a coherent subsheaf. We prove in Proposition 5.9 that it is indeed a subbundle. For simplicity we also define

\[
\tilde{\mathcal{G}}^c := \bigotimes_{j=1}^m \mathcal{G}^{s^c_j}.
\]

Recall that

\[
(5.8.c) \quad \operatorname{rk} \left( \tilde{\mathcal{G}}^c \right) = \prod_{j=1}^m r_{s^c_j}, \quad \text{and} \quad \mu \left( \tilde{\mathcal{G}}^c \right) = \sum_{j=1}^m \mu \left( \mathcal{G}^{s^c_j} \right) = \sum_{j=1}^m \mu_{s^c_j}.
\]

After the above notational preparation, it is quite straightforward to state and prove the description of the Harder-Narasimhan filtration of \(E^\otimes m\) that we need:

**Proposition 5.9.** In the situation of Notation 5.8:

(a) For each integer \(1 \leq c \leq e\), \(\mathcal{H}^c\) is a subbundle of \(E^\otimes m\).

(b) The filtration \(0 \subsetneq \mathcal{H}^1 \subsetneq \mathcal{H}^2 \subsetneq \cdots \subsetneq \mathcal{H}^d\) is a refinement of the Harder-Narasimhan filtration of \(\mathcal{H}^d\). More precisely, the quotients are semi-stable with (not necessarily strictly) decreasing slopes. Furthermore, all these slopes are at least \(2g\). Even more precisely, for each integer \(1 \leq c \leq e\),

\[
\mathcal{H}^c / \mathcal{H}^{c-1} \cong \tilde{\mathcal{G}}^c.
\]
(c) $\mathcal{H}_t^d \subseteq \mathcal{E}_t^{\otimes m}$ is spanned by simple tensors of $e_i$.

Proof. For each integer $1 \leq c \leq e$ we have a surjective homomorphism:

$$
(5.9.d) \quad \mathcal{H}^c/\mathcal{H}^{c-1} = \left( \mathcal{H}^{c-1} + \bar{\mathcal{F}}^c \right) / \mathcal{H}^{c-1} \cong \bar{\mathcal{F}}^c / \left( \bar{\mathcal{F}}^c \cap \mathcal{H}^{c-1} \right) \cong \bar{\mathcal{G}}^c
$$

by definition

$$
\text{isomorphism theorem}
$$

$$
\sum_{c', c' < c} \bar{\mathcal{F}}^{c'} \subseteq \bar{\mathcal{F}}^c \cap \mathcal{H}^{c-1}
$$

So,

$$
(5.9.e) \quad \operatorname{rk} (\mathcal{E}^{\otimes m}) = \sum_{c=1}^e \operatorname{rk} \left( \mathcal{H}^c/\mathcal{H}^{c-1} \right) \leq \sum_{c=1}^e \operatorname{rk} \left( \bar{\mathcal{G}}^c \right) = \sum_{(i_1, \ldots, i_m) \in \{1, \ldots, \ell\}^m} \left( \prod_{j=1}^m r_{i_j} \right) = \left( \sum_{i=1}^{\ell} r_i \right)^m = \operatorname{rk} (\mathcal{E}^{\otimes m}).
$$

Hence, we have equality in the middle of (5.9.e), and hence the homomorphism of (5.9.d) is an isomorphism for all $c$.

In particular, for all $1 \leq c \leq e$, there is an exact sequence:

$$
(5.9.f) \quad 0 \longrightarrow \mathcal{H}^{c-1} \longrightarrow \mathcal{H}^c \longrightarrow \bar{\mathcal{G}}^c \longrightarrow 0
$$

This concludes (a), as both $\mathcal{H}^c$ and $\mathcal{E}^{\otimes m}/\mathcal{H}^c$ are iterated extensions of vector bundles, hence they are both vector bundles.

Point (c) also follows immediately from the definition of $\mathcal{H}^c$ as it is a sum of product type subbundles. Lastly, (b) also follows directly from (5.9.f).

□

Notation 5.10. In the situation of Notation 5.8 (in fact, for introducing the following notation we only need the first two paragraphs of Notation 5.8), let $G_{m,t}$ be the $k$-linear subspace of

$$
\operatorname{im} \left( H^0(T, \mathcal{E}^{\otimes m}) \rightarrow (\mathcal{E}^{\otimes m})_t \right)
$$

spanned by pure tensor in the $e_i$.

Theorem 5.11. In the situation of Notation 5.10, if $\deg \mathcal{E} > 0$, then

$$
\lim_{m \to \infty} \frac{\dim_k G_{m,t}}{\dim_k \mathcal{E}_t^{\otimes m}} = 1.
$$

Proof. Combining Proposition 5.7 and Proposition 5.9 yields $\mathcal{H}_t^d \subseteq G_{m,t}$. So, it is enough to prove that

$$
\lim_{m \to \infty} \frac{\operatorname{rk} \mathcal{H}_t^d}{\operatorname{rk} \mathcal{E}_t^{\otimes m}} = 1.
$$

By Proposition Proposition 5.9, item (b), and by equation (5.8.c) if we set $r := \operatorname{rk} \mathcal{E}$ and $p_i := \frac{r}{r_i}$, then

$$
(5.11.g) \quad \frac{\operatorname{rk} \mathcal{H}_t^d}{\operatorname{rk} \mathcal{E}_t^{\otimes m}} = \sum_{c=1}^d \frac{\operatorname{rk} \bar{\mathcal{G}}^c}{\operatorname{rk} \mathcal{E}_t^{\otimes m}} = \sum_{c=1}^d \prod_{j=1}^m s_j \frac{r_j}{r} = \sum_{s \in S_m} \prod_{j=1}^m s_j \frac{r_j}{r} = \sum_{s \in S_m} \prod_{j=1}^m p_{s_j},
$$
As $\sum_{i=1}^{\ell} p_i = 1$, we may define a discrete probability space $X$ on $\ell$ elements $\{1, \ldots, \ell\}$ with measures $p_1, \ldots, p_\ell$ respectively. Let $X_i$ be a sequence of independent random variables of $X$ that take value $\mu_i$ on $i$, and let $Z_m := \sum_{i=1}^{m} X_i$. On this language (5.11.g) tells us that
\[
\frac{\text{rk } H^d}{\text{rk } E \otimes m} = P\left( \sum_{i=1}^{m} X_i \geq 2g \right) = P\left( Z_m \geq 2g \right),
\]
where $P(\ldots)$ denotes the probability of the given condition. Hence we are left to show that
\[
\lim_{m \to \infty} P\left( Z_m \geq 2g \right) = 1.
\]

Consider now, the Central Limit Theorem of probability theory as for example in [Dur10, Thm 3.4.1]. Note that as $X$ is a finite metric space both the expected value $\mu$ and the variance $\sigma^2$ of $X_i$ are finite. Then the central limit theorem states that the random variable $Z_m - m\mu \sqrt{m}$ weakly converges to a normal distribution $\Phi$ with expected value 0 and covariance $\sigma^2$. In particular, this induces a convergence on the level of distribution functions, or more precisely we would like to use the following convergence, which holds for each real number $A$ and it is shown for example in [Dur10, Thm 3.2.5.iv]:
\[
\lim_{m \to \infty} P\left( Z_m - m\mu \sqrt{m} \geq A \right) = P\left( \Phi \geq A \right).
\]

We claim that for each fixed real number $A$ there is an integer $m_A > 0$ such that for all integers $m \geq m_A$:
\[
\frac{Z_m - m\mu}{\sqrt{m}} \geq A \quad \Rightarrow \quad Z_m \geq 2g
\]

For this, note first that
\[
\mu = \sum_{i=1}^{\ell} \mu_i p_i = \sum_{i=1}^{\ell} \mu_i \frac{r_i}{\text{deg } E} = \frac{\text{deg } E}{\text{rk } E} > 0.
\]

Hence, if we assume that $\frac{Z_m - m\mu}{\sqrt{m}} \geq A$ then there is an integer $m_A$ such that
\[
2g \leq A\sqrt{m + m\mu} \leq Z_m \quad \text{for } m \geq m_A.
\]

This concludes our claim.

Combining our claim and (5.11.i) we obtain that
\[
\liminf_{m \to \infty} P\left( Z_m \geq 2g \right) \geq P\left( \Phi \geq A \right)
\]

As this is true for all real numbers $A$, and $\lim_{A \to -\infty} P(\Phi \geq A) = 1$, we obtain that
\[
\liminf_{m \to \infty} P\left( Z_m \geq 2g \right) = 1 \Rightarrow \lim_{m \to \infty} P\left( Z_m \geq 2g \right) = 1
\]

This is exactly the statement of (5.11.h), which was our goal to prove.

\[ \square \]

Remark 5.12. We note that in the proof of Theorem 5.11, one can replace the Central Limit Theorem by the weaker statement of Chebyshev’s inequality. Indeed, using the notation of the proof, as the variance of $Z_m$ is $m\sigma^2$:
\[
1 - P(Z_m - m\mu \geq \sqrt{m}A) \leq P\left( |Z_m - m\mu| \geq \frac{A}{\sigma} \sqrt{m}\sigma \right) \leq \frac{\sigma^2}{A^2} \to 0 \quad \text{(as } A \to -\infty).\]

Chebyshev’s inequality
6. Ancillary statements

Here we gather smaller statements that are used multiple times in Section 7, Section 8 and Section 9.

6.1. Normality of total spaces

Lemma 6.1. In the situation of Notation 7.1, there exists a finite morphism from a smooth projective curve \( \tau : S \to T \) such that if \( g : Y \to S \) is the normalized pullback of \( f \) (so the normalization of the pullback) and \( \pi : Y \to X \) the induced morphism, then \( g \) has reduced fibers, and there is an effective \( \mathbb{Q} \)-divisor \( \Gamma \) on \( Y \) such that

\[
(a) \quad \pi^*(K_X/T + \Delta) = K_{Y/S} + \Gamma,
(b) \quad \lambda_g = \sigma^* \lambda,
\]

where \( \lambda_g \) is the CM line bundle for \( g \).

Proof. Let \( \tau \) be any finite cover such that at the closed points \( t \in T \) over which the fiber \( X_t \) is non-reduced, the ramification order of \( \tau \) is divisible by all the multiplicities of all the components of \( \tau \). Then, \( g \) will have reduced fibers, and Section 2.3.2 implies the existence of \( \Gamma \) (denoted by \( \Delta_Z \) there). Finally, Proposition 2.1.(a) yields point (a), and Proposition 3.8.(b) yields point (b). □

Lemma 6.2. If \( f : X \to T \) is a surjective morphism from a normal variety to a smooth projective curves with reduced fibers, \( m > 0 \) is an integer and \( \tau : S \to T \) is a finite morphism from another smooth curve, then

\[
(a) \quad X \times_T S \text{ is normal, and}
(b) \quad X^{(m)} \text{ is normal (see Section 2.1 for the product notation)}.
\]

Proof. First we note that \( f \) is flat and hence so is \( f^{(m)} : X^{(m)} \to T \) by induction on \( m \) and the stability of flatness under base-change.

We know that a variety \( Z \) is normal if and only if it is \( S_2 \) and \( R_1 \). In the particular case, when \( Z \) maps to a smooth curve \( U \) via a flat morphism \( g \), then \( Z \) is \( S_2 \) if and only if the general fibers of \( g \) are \( S_2 \) and the special ones are \( S_1 \) (so without embedded points) [Gro65, 6.3.1] [Gro66, 12.2.4.i], and it is \( R_1 \) if the general fibers are \( R_1 \) and the special ones are \( R_0 \) (so reduced) [Gro66, 12.2.4.ii]. It is immediate then that this characterization of \( S_2 \) and \( R_1 \) propagates both to fiber powers and to base-changes. □

6.2. Semi-positivity engine

Proposition 6.3. Let \( f : (X, \Delta) \to T \) be a surjective morphism from a normal, projective pair to a smooth curve such that \((X_t, \Delta_t)\) is klt for general \( t \in T \) (as \( X_t \) is normal for \( t \in T \) general, \( \Delta \) is \( \mathbb{Q} \)-Cartier at the codimension 1 points of \( X_t \), and hence \( \Delta_t \) makes sense), and let \( L \) be a Cartier divisor on \( X \) such that \( L - K_{X/T} - \Delta \) is an \( f \)-ample and nef \( \mathbb{Q} \)-Cartier divisor. Then, \( f_* \mathcal{O}_X(L) \) is a nef vector bundle.

Proof. According to Lemma 6.1 we may assume that the fibers of \( f \) are reduced. According to [Pat14, Lem 3.4], it is enough to prove that for all integers \( m > 0 \), the following vector bundle is generated at a general \( t \in T \) by global sections:

\[
\omega_T(2t) \otimes \bigotimes_{i=1}^m f_* \mathcal{O}_X(L) \cong f_*^{(m)} \mathcal{O}_X^{(m)} \left( L^{(m)} + \left( f^{(m)} \right)^* K_T + 2X^{(m)}_t \right).
\]

[KP17, Lem 3.6], and see Section 2.1 for the fiber product notation.
For that it is enough to prove that the natural restriction homomorphism $H^0(X^{(m)}, N) \to H^0\left(X_t^{(m)}, N_t\right)$ is surjective, where

$$N := L^{(m)} + f^{(m)}/K_T + 2X_t^{(m)} = K_{X^{(m)}} + \Delta^{(m)} + (L - K_{X/T} - \Delta)^{(m)} + 2X_t^{(m)}.$$  

We note here that according to Lemma 6.2, $X^{(m)}$ is normal. Furthermore, $K_{X^{(m)}} + \Delta^{(m)} = (K_{X/T} + \Delta)^{(m)} + (f^{(m)})^*K_T$ is $\mathbb{Q}$-Cartier. We also note that the only generality property of $t$ that we use below is that $X_t$ is normal, $X_t \subseteq \text{Supp}\Delta_t$ and $(X_t, \Delta_t)$ is klt. Hence, at this point, we fix a $t$ with such properties.

Set $\mathcal{I} := J(X^{(m)}, \Delta^{(m)})$, where $J$ denotes the multiplier ideal of the corresponding pair. Then for the above surjectivity the next diagram, the top row of which is exact, shows that it is enough to prove the vanishing of $H^1\left(X^{(m)}, \mathcal{I} \otimes O_{X^{(m)}}(N - X_t^{(m)})\right)$.

$$
\begin{array}{ccc}
H^0\left(X^{(m)}, \mathcal{I} \otimes O_{X^{(m)}}(N)\right) & \rightarrow & H^0\left(X_t^{(m)}, N|_{X_t^{(m)}}\right) \rightarrow H^1\left(X^{(m)}, \mathcal{I} \otimes O_{X^{(m)}}(N - X_t^{(m)})\right) \\
\downarrow & & \downarrow \\
H^0\left(X^{(m)}, N\right) & \rightarrow & H^1\left(X^{(m)}, \mathcal{I} \otimes O_{X^{(m)}}(N - X_t^{(m)})\right)
\end{array}
$$

We note that here we used that $(X_t^{(m)}, \Delta_t^{(m)})$ is klt by Lemma 4.12, and hence by inversion of adjunction [KM98, Thm 5.50] so does $(X^{(m)}, \Delta^{(m)})$ in a neighborhood of $X_t^{(m)}$. This then implies that $\mathcal{I}$ is trivial in a neighborhood of $X_t^{(m)}$.

We conclude by noting that the above cohomology vanishing is given by Nadel-vanishing as

$$N - X_t^{(m)} = K_{X^{(m)}} + \Delta^{(m)} + (L - K_{X/T} - \Delta)^{(m)} + X_t^{(m)}.$$  

\[\Box\]

**Corollary 6.4.** Let $f : (X, \Delta) \to T$ be a surjective morphism from a normal, projective pair to a smooth curve such that $(X_t, \Delta_t)$ is klt for some (or equivalently general) $t \in T$, and let $L$ be a $\mathbb{Q}$-Cartier divisor on $X$ such that

(a) $L_t$ is globally generated for $t \in T$ general,

(b) $L - K_{X/T} - \Delta$ is an $f$-ample and nef $\mathbb{Q}$-Cartier divisor, and

(c) there is a $\mathbb{Q}$-Cartier divisor $N$ on $T$ such that $L + f^*N$ is Cartier.

Then $L$ is nef.

**Proof.** According to Lemma 6.1 we may assume that the fibers of $f$ are reduced, and by further pullback (using Lemma 6.2.(b)) we may also assume that $N$ is Cartier, whence $L$ is also Cartier. Then, we may apply Proposition 6.3 yielding that $f_*O_X(L)$ is nef. Then, our proof is concluded by the natural homomorphism $f^*f_*O_X(L) \to O_X(L)$, which is surjective over a non-empty open set of $T$ according to assumption (a) and the fact that cohomology and base change always holds over an opens set. \[\Box\]

7. **Semi-positivity**

In this section we prove our semi-positivity results. Here, and also in Section 9 we use extensively the fiber product notation explained in Section 2.1.
7.1. **Framework and results**

In this and the next sections we work in the following setup:

**Notation 7.1.** Let \( f : (X, \Delta) \to T \) satisfy the following assumptions:

(a) \( T \) is a smooth, projective curve,
(b) \( X \) is a normal, projective variety of dimension \( n + 1 \),
(c) \( f \) is a projective and surjective morphism with connected fibers,
(d) \( \Delta \) is an effective \( \mathbb{Q} \)-divisor on \( X \),
(e) \( -(K_X + \Delta) \) is an \( f \)-ample \( \mathbb{Q} \)-Cartier divisor.
(f) \((X_t, \Delta_t)\) is klt for general \( t \in T \).

The main result of the section is the following, from which the statements of the introduction will follow in a quite straightforward manner.

**Theorem 7.2.** In the situation of Notation 7.1, if \( \delta(X_\tau, \Delta_\tau) \geq 1 \) for very general geometric points \( t \in T \), then \( \deg \lambda_{f, \Delta} \geq 0 \).

7.2. **Proofs**

The proof of Theorem 7.2 will be by contradiction with the next proposition.

**Proposition 7.3.** In the situation of Notation 7.1, let \( H \) be an ample \( \mathbb{Q} \)-divisor on \( T \). Then, there do not exist \( \mathbb{Q} \)-Cartier divisors \( \Gamma \) and \( \tilde{\Gamma} \) on \( X \) such that:

(a) \( \Gamma + \tilde{\Gamma} \sim_{\mathbb{Q}} -K_{X/T} - \Delta - f^*H \),
(b) \( \tilde{\Gamma} \) is nef, and
(c) \((X_t, \Delta_t + \Gamma_t)\) is klt for \( t \in T \) general.

**Proof.** Assume that there exist \( \Gamma \) and \( \tilde{\Gamma} \) as above. Let \( a > 0 \) be a rational number such that \( -K_{X/T} - \Delta + af^*H \) is ample. Fix a rational number \( \varepsilon > 0 \) such that \( \varepsilon a - (1 - \varepsilon) < 0 \). The following computation the yields a contradiction, as Corollary 6.4 yields that the right side is nef.

\[
\frac{(\varepsilon a - (1 - \varepsilon))f^*H}{\varepsilon a - (1 - \varepsilon) < 0 \Rightarrow \text{this is not nef}} \sim_{\mathbb{Q}} K_{X/T} + \Delta + (1 - \varepsilon)\Gamma + (1 - \varepsilon)\tilde{\Gamma} + \varepsilon \left(-K_{X/T} - \Delta + af^*H \right)_{\text{ample}}
\]

\[= (X_t, \Delta_t + (1 - \varepsilon)\Gamma_t) \text{ is klt} \]

\[\square\]

**Proof of Theorem 7.2.** As both the consequences and the conditions of the theorem are invariant under base-extension to another algebraically closed field, we may assume that \( k \) is uncountable. In particular, the very general geometric fibers show up also amongst closed fibers.

First, according to Lemma 6.1 we may assume that all fibers of \( f \) are reduced. This is to guarantee that the \( m \)-times iterated fiber product \( X^{(m)} \) is normal for any integer \( m > 0 \), according to Lemma 6.2.

We argue by contradiction, so assume that \( \deg \lambda_{f, \Delta} < 0 \). For \( m \) big enough, we are going to produce divisors \( \Gamma \) and \( \tilde{\Gamma} \) on \( (X^{(m)}, \Delta^{(m)}) \) whose existence contradicts Proposition 7.3.

Fix a closed point \( t \) in \( T \) such that \( X_t \) is normal, \( X_t \not\subseteq \text{Supp} \Delta_t \), \((X_t, \Delta_t)\) is klt and \( \delta(X_t, \Delta_t) \geq 1 \) (using Proposition 4.14). Let \( H \) be an ample line bundle on \( T \). Fix rational numbers \( a, \varepsilon > 0 \) and \( 0 < c < 1 \) and an integer \( q > 0 \), such that:

(a) the intersection product inequality \( -(K_{X/T} - \Delta - \varepsilon f^*H)^{n+1} > 0 \) holds. This is possible because Definition 3.3 and the assumption \( \deg \lambda_{f, \Delta} < 0 \) implies that \( -(K_{X/T} - \Delta)^{n+1} > 0 \).

Set \( M := -K_{X/T} - \Delta - \varepsilon f^*H \).

(b) \( D := -K_{X/T} - \Delta + af^*H \) is ample.

(c) \( c < \frac{\varepsilon}{a + \varepsilon} \).
(d) \(qM\) is Cartier, which is possible, as \(M\) is \(\mathbb{Q}\)-Cartier.

(e) \(R^i f_* \mathcal{O}_X(qM) = 0\) for all \(i > 0\), which is possible, as \(M\) is \(f\)-ample.

(f) \(\deg (f_* \mathcal{O}_X(qM)) > 0\), using Lemma A.2.

(g) \(\delta_q(X_t, \Delta_t) > 1 - c\), using Theorem 4.5.

From now on, let \(\mathcal{E} := f_* \mathcal{O}_X(qM)\). Remark that according to [KP17, Lemma 3.6] for every integer \(m > 0\),

\[
\mathcal{E}^\otimes m = f_*^{(m)} \mathcal{O}_{X^{(m)}}(qM^{(m)}) \cong f_*^{(m)} \mathcal{O}_{X^{(m)}} \left( q \left( -K_{X^{(m)}/T} - \Delta^{(m)} - m \varepsilon f^{(m)} H \right) \right),
\]

and by item (e), the following base change holds

\[
\mathcal{E}^\otimes m = H^0 \left( X_t^{(m)}, q \left( -K_{X_t^{(m)}} - \Delta_t^{(m)} \right) \right).
\]

In general, it is not possible to lift a basis of \(\mathcal{E}_t\) to sections of \(\mathcal{E}\). However, thanks to Theorem 5.11, we can choose a basis \(e_i\) of \(\mathcal{E}_t\), an integer \(m > 0\), and \(\ell\) global sections \(s_i\) of \(\mathcal{E}^\otimes m\) so that the sections \(s_i\), when restricted over \(t\), are linearly independent pure tensor in the \(e_i\), and furthermore

\[
\frac{\ell}{h^0 \left( X_t^{(m)}, -q \left( K_{X_t^{(m)}} + \Delta_t^{(m)} \right) \right)} > \frac{1 - c}{\delta_q(X_t, \Delta_t)} < 1 \text{ according to assumption (g)}.
\]

We are now ready to construct \(\Gamma\) and \(\tilde{\Gamma}\) on \(X^{(m)}\) as in Proposition 7.3. We let

\[
\Gamma := (1 - c) \frac{1}{q \ell} \sum_{i=1}^{\ell} \{ s_i = 0 \},
\]

and

\[
\tilde{\Gamma} := c D^{(m)}.
\]

To complete the proof of Theorem 7.2, we have to prove that \(\Gamma\) and \(\tilde{\Gamma}\) are as in Proposition 7.3, with \(f\) replaced by \(f^{(m)}\). To check item (a), remark that

\[
\Gamma + \tilde{\Gamma} \sim q - K_{X^{(m)}/T} - \Delta^{(m)} + m (ca - (1 - c) \varepsilon) f^{(m)} H.
\]

Furthermore, because of assumption (c), \(ca - (1 - c) \varepsilon < 0\) holds; so, we may apply Proposition 7.3 replacing \(H\) by \(-m (ca - (1 - c) \varepsilon) H\). Item (b) of Proposition 7.3 follows from the ampleness of \(D\).

To prove of item (c) of Proposition 7.3, we compute the log canonical threshold. We first remark that, since the sections \(s_i\) restricted to \(X_t^{(m)}\) are linearly independent pure tensors in the \(e_i\), we have that

\[
\frac{\ell}{h^0 \left( X_t^{(m)}, -q \left( K_{X_t^{(m)}} + \Delta_t^{(m)} \right) \right)} \Gamma_t \leq (1 - c) P
\]

for the \(q\)-product basis type divisor \(P\) on \(X_t^{(m)}\) associated to \(\{ e_i \}\), as in Definition 4.10 and Remark 4.11. Using Proposition 4.13, we obtain that \(\lct \left( X_t^{(m)}, \Delta_t^{(m)}; P_t \right) \geq \delta_q(X_t, \Delta_t)\). This yields

\[
\lct \left( X_t^{(m)}, \Delta_t^{(m)}; \Gamma_t \right) \geq \frac{\delta_q(X_t, \Delta_t) \ell}{(1 - c) h^0 \left( X_t^{(m)}, -q \left( K_{X_t^{(m)}} + \Delta_t^{(m)} \right) \right)} > 1.
\]

Hence, all assumptions of Proposition 7.3 are verified, implying that \(\Gamma\) and \(\tilde{\Gamma}\) cannot exist. Therefore, we obtained a contradiction with our initial assumption that \(\deg \lambda_f, \Delta < 0\). \(\square\)
**Proof of Theorem 1.13.** The proof of point (a): As at the beginning of the proof of Theorem 7.2, we may assume that \( k \) is uncountable. According to [BDPP13, Thm 0.2], it is enough to show that \( \lambda_{f,\Delta} \cdot C \geq 0 \) for every morphism \( \iota : C \to X \) from a smooth projective curve such that \( C \to \iota(C) \) is the normalization and \( \iota(C) \) is a very general curve in a family covering \( T \). In particular, for a very general closed point \( t \in \iota(C) \), \( X_t \) is normal, \( (X_t, \Delta_t) \) is klt and \( \delta(X_t, \Delta_t) \geq 1 \). Let \( Z \) be the normalization, \( g : Z \to C \) the induced morphism and \( \Delta_Z \) the boundary induced by \( \Delta \) on \( Z \) as explained in Section 2.3.2. According to Proposition 2.1.(a), \( g : (Z, \Delta_Z) \to C \) satisfies the assumptions of Theorem 7.2. Hence the following computation concludes the proof of point (a):

\[
0 \leq \deg \lambda_{g,\Delta_Z} = C \cdot \lambda_{f,\Delta}.
\]

**The proof of point (b):** In this case for each finite morphism \( C \to T \) from a smooth projective curve, according to Section 2.3.1, \( f_C : (X_C, \Delta_C) \to C \) satisfy the assumptions of Theorem 7.2. So:

\[
0 \leq \deg \lambda_{f_C,\Delta_C} = C \cdot \lambda_{f,\Delta}.
\]

\[\square\]

**Proof of points (a) and (b) of Theorem 1.2.** These are special cases of Theorem 1.13. \[\square\]

### 8. Bounding the nef threshold

**Lemma 8.1.** If \( f : X \to T \) is a morphism between projective varieties, \( m > 0 \) is an integer and \( M \) is a \( \mathbb{Q} \)-Cartier divisor on \( X \), then \( M \) is nef if and only if \( M^{(m)} \) is nef.

**Proof.** If \( M \) is nef, \( M^{(m)} \) is nef by definition. For the other direction, assume that \( M^{(m)} \) is nef. Let \( \iota : C \to X \) be a morphism from a smooth, projective curve. Take then the diagonal morphism \( \Delta : C \to X^{(m)} \), which is defined by the equality \( p_i \circ \Delta = \iota \) for each \( i \). Then:

\[
0 \leq M^{(m)} \cdot \Delta(C) = \left( \sum_{i=1}^{m} p_i^* M \right) \cdot \Delta(C) = \sum_{i=1}^{m} (p_i^* M \cdot \Delta(C)) = \sum_{i=1}^{m} (M \cdot C) = m(M \cdot C).
\]

Hence, \( M \cdot C \geq 0 \). As this works for any curve \( C \) in \( X \) we see that \( M \) is nef. \[\square\]

**Proof of Theorem 1.22.** As both the consequences and the conditions of the theorem are invariant under base-extension to another algebraically closed field, we may assume that \( k \) is uncountable. In particular, the very general geometric fibers show up also amongst closed fibers.

According to Lemma 6.1 we may assume that all fibers of \( f \) are reduced. In particular then for all integers \( m > 0 \), \( X^{(m)} \) is normal according to Lemma 6.2. Set \( \lambda := \lambda_{f,\Delta} \).

Fix the following:

(a) let \( H \) be an ample divisor on \( T \) of degree 1,
(b) let \( t \in T \) be a closed point such that \( X_t \) is normal, \( X_t \subseteq \text{Supp} \Delta, (X_t, \Delta_t) \) is klt and \( \delta(X_t, \Delta_t) = \delta \),
(c) let \( 0 < \varepsilon < \delta - 1 \) be an arbitrary rational number, and
(d) let \( 0 < \varepsilon' \ll \varepsilon \) be another rational number.

It is enough to prove that

\[
(8.1.a) \quad N := \varepsilon(-K_{X/T} - \Delta) + \left( \frac{(1 + \varepsilon) \deg \lambda}{v(n + 1)} + \varepsilon' \right) f^* H
\]

is nef, as we may converge with \( \varepsilon \) and \( \varepsilon' \) to \( \delta - 1 \) and to 0, respectively. Set

\[
(8.1.b) \quad M := (1 + \varepsilon)(-K_{X/T} - \Delta) + \left( \frac{(1 + \varepsilon) \deg \lambda}{v(n + 1)} + \varepsilon' \right) f^* H = N - K_{X/T} - \Delta.
\]
Note that

\[(8.1.c) \quad M^{n+1} = \left( (1 + \varepsilon)(-K_{X/T} - \Delta) + \left( \frac{(1 + \varepsilon) \deg \lambda}{v(n + 1)} + \varepsilon' \right) f^*H \right)^{n+1} \]

\[= (1 + \varepsilon)^n(-K_{X/T} - \Delta)^n \left( (1 + \varepsilon)(-K_{X/T} - \Delta) + (n + 1) \left( \frac{(1 + \varepsilon) \deg \lambda}{v(n + 1)} + \varepsilon' \right) f^*H \right) \]

\[= (1 + \varepsilon)^n(\deg \lambda(-(1 + \varepsilon) + (1 + \varepsilon)) + (n + 1)v\varepsilon') = (1 + \varepsilon)^n(n + 1)\varepsilon'v > 0. \]

We now fix a positive integer \(q\) so that the following hold:

(e) \(qM\) is Cartier,
(f) \(q \in \mathbb{N}\),
(g) \(R^i f_* O_X(qM) = 0\) for all \(i > 0\), which is doable as \(M\) is \(f\)-ample,
(h) \(\deg (f_* O_X(qM)) > 0\), which is doable according to Lemma A.2 and (8.1.c), and
(i) \(\delta_{q'}(X_t, \Delta_t) > 1 + \varepsilon\), where \(q' := q(1 + \varepsilon)\). This is doable according to Corollary 4.6 and assumption (c).

From now on, let \(E := f_* O_X(qM)\). Remark that according to [KP17, Lemma 3.6] for every integer \(m > 0\),

\[E^{\otimes m} \cong f_*^{(m)} O_{X^{(m)}}(qM^{(m)}) \]

\[\cong f_*^{(m)} O_{X^{(m)}}(q'(-K_{X^{(m)}/T} - \Delta_{X^{(m)}}^{(m)}) + qm \left( \frac{(1 + \varepsilon) \deg \lambda}{v(n + 1)} + \varepsilon' \right) (f^{(m)})^* H) \]

and, by item (g), the following base change holds

\[E_{t}^{\otimes m} = H^0 \left( X_t^{(m)}, q'(-K_{X_t^{(m)}} - \Delta_t^{(m)}) \right). \]

According to Theorem 5.11, we may find a basis \(\{ e_i \}\) of \(E_t\), an integer \(m > 0\), and \(\ell\) global sections \(s_1, \ldots, s_\ell\) of \(E^{\otimes m}\) so that the sections \(s_j\), when restricted over \(t\), are linearly independent pure tensors in the \(e_i\), and furthermore

\[(8.1.d) \quad \frac{\ell}{h^0 \left( X_t^{(m)}, -q' \left( K_{X_t^{(m)}} + \Delta_t^{(m)} \right) \right)} > \frac{1 + \varepsilon}{\delta_{q'}(X_t, \Delta_t)} < 1 \text{ by assumption (i)} \cdot \]

Define \(\Gamma\) as

\[\Gamma := \frac{1}{q\ell} \sum_{i=1}^\ell \{s_i = 0\} \sim_Q M^{(m)}. \]

Note that according to (8.1.b),

\[K_{X^{(m)}/T} + \Delta_{X^{(m)}}^{(m)} + \Gamma \sim_Q N^{(m)}. \]

So, to show (8.1.a), according to Lemma 8.1 it is enough to prove that \(K_{X^{(m)}/T} + \Delta_{X^{(m)}}^{(m)} + \Gamma\) is nef, and for that according to [Fuj12, Thm 1.13] it is enough to show that \(\left( X_t^{(m)}, \Delta_t^{(m)} + \Gamma_t \right)\) is klt. For this we compute the log canonical threshold. We first remark that, since the sections \(s_i\) restricted to \(X_t^{(m)}\) are linearly independent pure tensors in the \(e_i\), we have that

\[\frac{q\ell}{q' h^0 \left( X_t^{(m)}, -q' \left( K_{X_t^{(m)}} + \Delta_t^{(m)} \right) \right)} \Gamma_t \leq P \]
for the \( q' \)-product basis type divisor \( P \) on \( X_t^{(m)} \) associated to \( \{e_i\} \), as in Definition 4.10 and Remark 4.11. Using Proposition 4.13, we obtain that \( \text{let } \left( X_t^{(m)} , \Delta_t^{(m)} ; P \right) \geq \delta_q(X_t, \Delta_t) \); this yields

\[
\text{lct} \left( X_t^{(m)} , \Delta_t^{(m)} ; \Gamma_t \right) \geq \frac{\delta_{q'}(X_t, \Delta_t) \ell_q}{h^0 \left( X_t^{(m)} , -q' \left( K_{X_t^{(m)}} + \Delta_t^{(m)} \right) \right) q'} = \frac{\ell_q \delta_{q'}(X_t, \Delta_t)}{h^0 \left( X_t^{(m)} , -q' \left( K_{X_t^{(m)}} + \Delta_t^{(m)} \right) \right) (1 + \varepsilon)} \geq \frac{1}{(8.1.d)} \tag{by the definition of \( q' \) in (i)}
\]

\[ \square \]

9. Positivity

9.1. Variation

**Definition 9.1.** Let \( f : X \to T \) be a flat morphism between normal projective varieties, with \(-K_{X/T} \) \( \mathbb{Q} \)-Cartier and \( f \)-ample. Let \( q_0 \) be an integer such that \( q_0K_{X/T} \) is Cartier, and for all positive integers \( q_0 | q \), set \( \mathcal{L}_q := \mathcal{O}_X(-qK_{X/T}) \). As \( \mathcal{L}_q \) provides a relatively ample polarization, the Isom scheme \( I := \text{Isom}_{T \times T}(p_1^*f, p_2^*f) \) exists together with the two natural projections \( q_i : I \to T \). Let \( I' \) be the image of \((q_1, q_2) : I \to T \times T \). Then, there is a non-empty open set \( U \subseteq T \) where the fibers of \( p_1 | _U : I' \to T \) have the same dimension, say \( d \). This dimension is the dimension of a general isomorphism equivalence class of the fibers of \( f \). As these isomorphism equivalence classes (at least general ones) would be exactly the fibers of any reasonable moduli map, one defines the **variation** of \( f \) as

\[
\text{Var}(f) := \dim T - d.
\]

\( f \) has **maximal variation**, if \( \text{Var}(f) = \dim T \).

9.2. Curve base

**Notation 9.2.** In the situation of Notation 7.1, assume that

(a) \( \delta > 1 \), where \( \delta = \delta (X_T, \Delta_T) \) for every general geometric points \( T \in T \), and

(b) \( \deg \lambda_{f, \Delta} = 0 \).

**Theorem 9.3.** In the situation of Notation 9.2, for each ample \( \mathbb{Q} \)-divisor \( L \) on \( T \), \( | -K_{X/T} - \Delta - f^*L| \).

Using [Fuj17b, Thm 1.2], Corollary 4.8 and Proposition 4.14, choose a small rational number \( \varepsilon > 0 \) such that for every general geometric points \( T \in T \) we have \( \delta (X_T, \Delta_T + \varepsilon \Gamma_T) \geq 1 \). Then,

\[
0 \geq \left( -K_{X/T} - \Delta - \varepsilon \Gamma \right)^{n+1} = (1 - \varepsilon)^n \left( -K_{X/T} - \Delta \right)^{n+1} + (n + 1)\varepsilon ( -K_{X/T} - \Delta ) f^*L = (n + 1)\varepsilon ( -K_{X/T} - \Delta ) f^*L = (n + 1)\varepsilon ( -K_{X/T} - \Delta ) f^*L \geq 0.
\]

This is a contradiction.

**Notation 9.4.** In the situation of Notation 9.2,

(a) let \( q_0 > 0 \) be an integer such that \( q_0(-K_{X/T} - \Delta) \) is Cartier,
(b) for each integer \( q_0 | q \), define \( E_q := f_* O_X (q (-K_{X/T} - \Delta)) \), and set \( 0 = F^0_q \subseteq F^1_q \subseteq \cdots \subseteq F^{n-1}_q \subseteq F^n_q \) be the Harder-Narasimhan filtration of \( E_q \). Set \( G^i_q := F^i_q / F^{i-1}_q \),

(c) let \( g \) be the genus of \( T \).

**Lemma 9.5.** In the situation of **Notation 9.4**, for every positive integer \( q_0 | q, \mu(F^1_q) \leq 2g \).

**Proof.** Assume the contrary, that is, \( \mu(F^1_q) > 2g \), and let \( t \in T \) be an arbitrary closed point. According to **Proposition 5.4**, \( F^1_q (-t) \) is globally generated. So, there is a \( \Gamma' \in |q (-K_{X/T} - \Delta) - f^* L'| \), where \( L' \) is the divisor determined by \( t \) on \( T \). Hence, for \( \Gamma := \frac{
abla}{q} \) and \( L := \frac{L}{q} \) we have

\[ \Gamma \in | - K_{X/T} - \Delta - f^* L|_Q \]

This contradicts **Theorem 9.3**.

**Proposition 9.6.** In the situation of **Notation 9.4**, for every positive integer \( q_0 | q, \mu(F^1_q) \leq 0 \).

**Proof.** Assume that \( \mu (F^1_q) > 0 \), and let \( \mathcal{H} \) be the image of \( \xi : (F^1_q)^{\otimes m} \to E_{qm} \)

for some \( m \gg 0 \). We claim that \( \mathcal{H} \) is not zero because of the following: Let \( \eta \) be the generic point of \( T \). Then any \( x \in (F^1_q)_\eta \) can be identified with some \( \tilde{x} \in H^0 (X_\eta, q (-K_{X/T} - \Delta_\eta)) \), in which case \( \xi (x^{\otimes m}) \) gets identified with \( \tilde{x}^m \in H^0 (X_\eta, m q (-K_{X/T} - \Delta_\eta)) \). In particular, the following implications conclude our claim: \( x \neq 0 \Rightarrow \tilde{x} \neq 0 \Rightarrow \tilde{x}^m \neq 0 \Rightarrow \xi (x^{\otimes m}) \neq 0 \).

Let then \( j \) be the smallest integer such that \( F^j_{qm} \) contains \( \mathcal{H} \), and let \( \mathcal{H}' \) be the image of \( \mathcal{H} \) in \( G^j_{qm} \). By the choice of \( j \), \( \mathcal{H}' \neq 0 \), and as \( \mathcal{H}' \) is a surjective image of \( (F^1_q)^{\otimes m} \):

\[
\mu (F^1_{mq}) \geq \mu (G^m_{qm}) \geq \mu (\mathcal{H}') > \mu (F^1_q)^{\otimes m} > m \mu (F^1_q) > 2g
\]

This contradicts **Lemma 9.5**.

**Theorem 9.7.** In the situation of **Notation 9.2**, if \( q > 0 \) is an integer such that \( -q(K_{X/T} + \Delta) \) is Cartier, then \( f_* O_X (-q(K_{X/T} + \Delta)) \) is a semi-stable vector bundle of slope 0.

**Proof.** First, **Theorem 1.22** yields that \( -K_{X/T} - \Delta \) is nef. Then, \( f_* O_X (q (-K_{X/T} - \Delta)) \) is also nef, by **Proposition 6.3** taking into account the \( \mathbb{Q} \)-linear equivalence

\[ q (-K_{X/T} - \Delta) \sim_{\mathbb{Q}} K_{X/T} + \Delta + (q + 1) (-K_{X/T} - \Delta) \]

Finally, **Proposition 9.6**, concludes our proof.

### 9.3. Ampleness lemma

**Theorem 9.8** is an extract of the argument of the ampleness lemma of [Kol90] (one assumption removed in [KP17]). It will be one of the main technical ingredients for the proof of items (c) and (d) of **Theorem 1.2** given in **Section 9.4**.

**Theorem 9.8.** Let \( V \) be a vector bundle of rank \( v \) on a normal projective variety \( T \) over \( k \), and let \( \phi : W := \text{Sym}^d (V) \to Q \) be a surjective homomorphism onto another vector bundle, where the ranks are \( w \) and \( q \), respectively. Assume that there is an open set, where the map of sets \( T(k) \to \text{Gr}(w, k) / \text{GL}(v, k) \) induced by \( \phi \) is finite to one. Then, for each ample Cartier divisor \( B \) on \( T \) there is an integer \( m > 0 \) and a non-zero homomorphism

\[ \text{Sym}^m \left( \bigoplus_{i=1}^w W_i \right) \to O_T (-B) \otimes (\text{det} Q)^m. \]
**Proof.** The proof is contained in [KP17], but not presented there ideally for our purposes, so we give a recipe of how to turn [KP17, Thm 5.5] into the above statement. First, specialize [KP17, Thm 5.5] to the case of projective base and the special choice of $W = \text{Sym}^d(V)$ and $G = \text{GL}(v, k)$, where the latter two choices are fine according to [KP17, Rem 5.3]. At this point the assumptions of [KP17, Thm 5.5] become identical to ours, except that in [KP17, Thm 5.5] was assumed to be weakly positive. However, this assumption is not used until the last three lines of the proof. In particular, the existence of the non-zero homomorphism of the displayed equation [KP17, (5.5.5)] exists even without the weakly positive assumption. This is exactly our statement, taking into account the isomorphism of the displayed equation of the proof of [KP17, Thm 5.5] after [KP17, (5.5.5)].

\[\square\]

### 9.4. Arbitrary base

**Proof of point (c) of Theorem 1.2.** As in the proofs of Theorem 7.2 and Theorem 1.22, we may assume that $k$ is uncountable. Let $\eta$ be the generic point of $T$.

(a) Set $n := \dim X - \dim T$, $v := K^\eta_{X_n}$, $\delta := \delta(X_\eta)$.

(b) Fix a rational number $\alpha$ such that $\alpha > \max\left\{1, \frac{1}{|\delta-1|\eta (n+1)}\right\}$.

Throughout the proof $\iota : C \to T$ denotes the normalization of a very general member of an arbitrary covering curve family of $T$. Very general here means that it is not contained in countably many divisors $S_i$, which we will specify during the proof explicitly. Set:

- $\eta_C$ to be the generic point of $C$,
- $Z := X_C$ (note that as the fibers of $f$ are reduced, and the general ones are normal, $Z$ is normal),
- $\sigma : Z \to X$ and $g : Z \to C$ be the induced morphisms,
- $\lambda := \lambda_g$.

Then the following holds:

- $\sigma^* K_{X/T} \cong K_{Z/C}$ by Proposition 2.1(a), and $\lambda = \lambda_f|_C$ by Proposition 3.8(b).
- a $\mathbb{Q}$-Cartier divisor $L$ is pseudo-effective if and only if $L \cdot C \geq 0$ (for any such $C$),
- according to Proposition 4.14, $\delta(X_\overline{C}) = \delta$ (assuming we add the countably many divisors to $S_i$ over which $\delta(X_t) < \delta$, which are given by Proposition 4.14). In particular, as $\delta > 1$ the very general fibers of $g$ are uniformly $K$-stable, and hence klt, see [Oda13, Theorem 1.3].
- in particular, by Theorem 7.2, $\deg \lambda \geq 0$,
- by Theorem 1.22, $-K_{Z/C} + \alpha \eta \lambda$ is nef and $g$-ample.

It is important that throughout the proof all constants (so all rational numbers) will be fixed independently of the particular choice of $C$ (for which there are two choices, first one choses a covering family, and then a very general member of that). For this reason, whenever such a constant is fixed, we do it in a numbered list item (see points above and below).

Choose integers $r \geq 2$ and $d > 0$ such that

(c) $rK_{X/T}$ and $r\alpha \lambda_f$ are Cartier,

(d) $h^0(X_t, -rK_{X_t}) = 0$ for all $i > 0$ and all $t \in T$,

(e) $-rK_{X_t}$ is very ample for all $t \in T$,

(f) the multiplication maps $W := \text{Sym}^d f_* O_X(-rK_{X/T}) \to f_* O_X(-drK_{X/T}) =: Q$ are surjective, and

(g) for all $t \in T$, $K_t := \text{Ker} \left(\text{Sym}^d H^0(X_t, -rK_{X_t}) \to H^0(X_t, -drK_{X_t})\right)$ generates $\mathcal{I}(d)$, where $\mathcal{I}$ is the ideal of $X_t$ via the embedding $\varphi_{-rK_{X_t}} : X_t \to \mathbb{P}^{v-1}$, where $v := \text{rk} f_* O_X(-rK_{X/T})$ and $\varphi_{-rK_{X_t}}$ is defined only up to the action of $\text{GL}(v, k)$ on the target. Note that this is achievable because $\mathcal{I}$ form a flat family as $t$ varies.
In particular, if we set \( w := \text{rk} W \) and \( q := \text{rk} Q \), then for every \( t \in T(k) \), \( K_t \subseteq W_t \) determines \( X_t \hookrightarrow \mathbb{P}^{w-1} \) up to the action of \( \text{GL}(v, k) \), which then means that the orbit of \( K_t \) in \( \text{Gr}(w, q)/G(v, k) \) determines \( X_t \) up to isomorphism. Therefore if we apply Theorem 9.8 for \( W \to Q \), then the fibers of the classifying map \( T(k) \to \text{Gr}(w, q)/G(v, k) \) are contained in the isomorphism classes of the fibers of \( f \) and hence, by the maximal variation assumption, there is an open set where these fibers are finite.

As,

\[
r(-K_{Z/C} + 2\alpha g^* \lambda) = K_{Z/C} + (r + 1)(-K_{Z/C} + \alpha g^* \lambda) + (r - 1)\alpha g^* \lambda,
\]

by Proposition 6.3, \( g_* \mathcal{O}_Z(r(-K_{Z/C} + 2\alpha g^* \lambda)) \) is a nef vector bundle. Set

\[
M := r(-K_{X/T} + 2\alpha f^* \lambda_f).
\]

Note that the conclusions of point (g) about the finiteness of the classifying map hold also for \(-rK_X\) replaced by \( M \), as \( f_* \mathcal{O}_X(M) \) and \( f_* \mathcal{O}_X(dM) \) differs from \( f_* \mathcal{O}_X(-rK_{X/T}) \) and \( f_* \mathcal{O}_X(-rdK_{X/T}) \) only by a twist with \( r2\alpha \lambda_f \) and \( dr2\alpha \lambda_f \), respectively. So, Theorem 9.8 yields an ample divisor \( B \) on \( T \), an integer \( m > 0 \) and a non-zero homomorphism as follows (see point (g) above for the definition of \( w \) and \( q \)):

\[
\xi : \text{Sym}^m \left( \bigoplus_{i=1}^w \text{Sym}^d(f_* \mathcal{O}_X(M)) \right) \to \mathcal{O}_X(-B) \otimes (\text{det} g_* \mathcal{O}_X(dM))^m.
\]

As the target of \( \xi \) is a line bundle, there exists a divisor, on the complement of which \( \xi \) is surjective. Let us add this divisor to \( S_t \). Then \( \xi|_C \) is a non-zero homomorphism as follows:

\[
\xi_C : \text{Sym}^m \left( \bigoplus_{i=1}^w \text{Sym}^d(g_* \mathcal{O}_Z(MC)) \right) \to \mathcal{O}_C(-BC) \otimes (\text{det} g_* \mathcal{O}_Z(dMC))^m.
\]

Define

\[
A := \text{det} f_* \mathcal{O}_X(dr(-K_{X/T} + 2\alpha f^* \lambda_f)) = \text{det} f_* \mathcal{O}_X(dM),
\]

and let \( A \) be a divisor corresponding to \( A \). As \( g_* \mathcal{O}_Z(M) \) is nef and hence so is every bundle that admits a generically surjective map from the left side of \( \xi_C \), we obtain that

\[
\text{(9.8.a)} \quad \deg A|_C = \deg \text{det} g_* \mathcal{O}_Z(dMC) \geq \frac{B \cdot C}{m}.
\]

Consider now, the natural embedding:

\[
\alpha : \text{det} f_* \mathcal{O}_X(dM) \hookrightarrow \bigotimes_{i=1}^q f_* \mathcal{O}_X(dM) \cong f_*^{(q)} \mathcal{O}_{X^{(q)}} \left( dM^{(q)} \right),
\]

given by the embedding of representations \( \text{det} \to \bigotimes_{i=1}^q \) of \( \text{GL}(q, k) \). Hence, by adjunction of \( f_*^{(q)}(\cdot) \) and \( (f^{(q)})^* (\cdot) \) one can write \( (f^{(q)})^* A + D = dM^{(q)} \), where \( D \) is an effective divisor on \( X^{(q)} \). Furthermore, as \( \alpha \) is a morphism of vector bundles, and the formation of \( f_* \mathcal{O}_X(dM) \) is compatible with base-change, \( D \) does not contain any fiber. By the continuity of log canonical threshold, there is a \( 0 < \varepsilon < \frac{1}{m} \) such that \( \left( X_t^{(q)}, \varepsilon D_t \right) \) is klt for general closed points \( t \in T \). In particular by the genericity of \( C \) the same holds also for general \( t \in C \). Then, if we define \( N := dr(-K_{X/T} + 3\alpha f^* \lambda_f) \), according to Corollary 6.4, the following divisor is nef (\( Z^{(q)} \) is normal...
by Lemma 6.2.(b)).

\[
K_{Z(q)/C} + \varepsilon DC_t + (dr + 1 - \varepsilon rd) \left( -K_{Z(q)/C} + 2aq \left( g^{(q)} \right)^* \lambda \right) + (dr - 2)\alpha q \left( g^{(q)} \right)^* \lambda
\]

is nef and $f$-ample $(r \geq 2, d > 0)$

\[
\sim \left( N_C^q - \varepsilon \left( g^{(q)} \right)^* AC \right) = \left( N_C - g^{q} AC \right)^{(q)}
\]

Set $\varepsilon' := \frac{\varepsilon}{qd}$. Then we have that $N_C - \varepsilon' g^* AC$ is nef according to Lemma 8.1. So,

\[
0 \leq (-K_{Z/C} + 3\alpha g^* \lambda - \varepsilon' g^* AC)^{n+1} = \left( -K_{Z/C} + \frac{g^* \lambda}{v(n+1)} + \left( 3\alpha - \frac{1}{v(n+1)} \right) g^* \lambda + \varepsilon' g^* AC \right) \leq (n+1)v \deg \left( 3\alpha - \frac{1}{v(n+1)} \right) \lambda - \varepsilon' \frac{B_C}{m}
\]

Hence, $3\alpha - \frac{1}{v(n+1)} \lambda_f - \varepsilon' \frac{B_C}{m}$ is pseudo-effective (as it dots to at least zero with each movable class). Therefore, $\lambda_f$ is the sum of an ample and a pseudo-effective $Q$-divisor, so $\lambda_f$ is big.

Proof of point (d) of Theorem 1.2. By Nakai-Moishezon it is enough to prove that for all normal varieties $V$ mapping finitely to $X$, $(\lambda_f|_V)^{\dim V} > 0$. However, using Proposition 3.8, this we may obtain by replacing $f : X \to T$ with $f_V : X \times_T V \to V$, and applying point (c) to $f_V$.

Proof of point (e) of Theorem 1.2. Let $m > 0$ be the integer such that $L := -mK_{X/T}$ is an $f$-very ample divisor, and let $N$ be the line bundle, which is the inverse of the leading term of the Knudsen-Mumford expansion for $L$ (Notation 3.6). According to [LWX15, Thm 6.1] it is enough to prove that $N$ is nef and that for all closed subvarieties $V$ of $T$ intersecting $U$, $(N|_V)^{\dim V} > 0$. In fact, also nefness is a similar intersection question, that is, $\deg N|_C \geq 0$ for all curves $C$ of $T$. Note that the normalization of $C$ is automatically a scheme, hence we may assume that $C$ is a smooth scheme. Furthermore, $V$ has a finite cover by a scheme [sta, Tag 04V1] and by Nagata’s theorem and resolution of singularities we may also assume that $V$ is projective and smooth. Therefore, by replacing $V$ by this generically finite cover we may assume that $V$ is also a smooth scheme. As $N$ is compatible with base-change (Lemma 3.5), by relaxing the isomorphism class assumption, we can replace $T$ by $B$ or $V$. Then, the base is smooth and projective, and we have to prove that without any variation assumption $N$ is nef, and if furthermore the variation is maximal, then $N$ is even big. By Proposition 3.7.(b), we can prove this for the CM line bundle, instead of $N$, which is then shown in points (b) and (c) of Theorem 1.2.

Proof of Theorem 1.2d. Choose $q$ big enough such that $-q(K_{X/T} + \Delta)$ is Cartier and without higher cohomology on the fibers. Let $H_i \in |H|$ be general for $i = 1, \ldots, \dim T - 1$, and set $C := \bigcap_{i=0}^{\dim T-1} H_i$. By the above generic choices, $Z := X_C$ is normal. Furthermore, $C$ lies in the smooth locus of $T$, hence for base-change properties along $C \to T$ we may assume that $T$ is smooth. In particular, there is an induced boundary $\Delta_Z$ on $Z$ (Section 2.3.2), for which $K_{X/T} + \Delta|_Z = K_{Z/C} + \Delta_Z$ (Proposition 2.1), and consequently

\[
f_* O_X(-q(K_{X/T} + \Delta))|_C \cong (f_C)_* O_Z(-q(K_{Z/C} + \Delta_Z)).
\]
Furthermore,
\[ 0 = \lambda_{f*,\Delta} \cdot H^{\dim T - 1} \quad \text{assumption} = \deg \lambda_{f*,\Delta} |_C = \deg \lambda_{f,\Delta,\Delta_Z}. \]

Therefore, according to Theorem 9.7, \((f_C)_* \mathcal{O}_Z (-q (K_{Z/C} + \Delta_Z))\) is a semi-stable vector bundle of slope 0. However, then the isomorphism (9.8.b) implies that \(f_* \mathcal{O}_X (-q(K_{X/T} + \Delta))\) is H-semi-stable of slope 0: if it had a subsheaf \(\mathcal{F}\) of \(H\)-slope bigger than 0, then for the saturation \(\mathcal{F}'\) of \(\mathcal{F}\), \(\mathcal{F}'|_C\) would be a subbundle of positive degree of \((f_C)_* \mathcal{O}_Z (-q (K_{Z/C} + \Delta_Z))\), which is a contradiction.

**Proof of Corollary 1.7.** The proof is very similar to that of point (e) of Theorem 1.2 above. As in the above proof, \(T\) has a generically finite cover by a smooth, projective scheme. By base-changing over this cover one may assume that the base is smooth and projective. By Proposition 3.7, we may replace \(N\) by the CM-line bundle notion used in the present article (Definition 3.1), and then points (b) and (c) of Theorem 1.2 concludes the proof.

\[ \square \]

## 10. Applications

### 10.1. Moduli applications

**Definition 10.1.** Let \(\mathcal{X}\) be a class of \(K\)-semi-stable Fano varieties over \(k\). A pseudo-functor (or equivalently category fibered in groupoid) \(\mathcal{M}_\mathcal{X}\) is called a moduli functor for \(\mathcal{X}\), if for normal test scheme \(T\) of finite type over \(k\):

\[ (10.1.a) \quad \mathcal{M}_\mathcal{X}(T) = \left\{ \begin{array}{c} X \quad \text{(a) } f \text{ is a flat morphism,} \\ f \quad \text{(b) } K_{X/T} \text{ is } \mathbb{Q}\text{-Cartier,} \\ T \quad \text{(c) for each geometric point } t \in T, \; X_t \in \mathcal{X}. \end{array} \right\}, \]

with arrows being given by Cartesian diagrams.

**Remark 10.2.** We did not require anything in Definition 10.1 for the value of \(\mathcal{M}_\mathcal{X}\) on non-normal test schemes to keep the definition open to future developments of the theory.

**Remark 10.3.** In higher dimensional moduli theory generally it is required also Kollár’s condition, that is, for all integer \(m\), \(\omega_{X/T}^{|m|}\) is supposed to be flat and compatible with base-change. Here, the latter compatibility precisely means that \(\left( \omega_{X/T}^{|m|} \right)_S \cong \omega_{X_S/S}^{|m|}\) for every base change \(S \to T\).

We did not require this condition in Definition 10.1 as it is known that over reduced bases in characteristic zero, this condition follows from \(K_{X/T}\) being \(\mathbb{Q}\)-Cartier [Kol14, Thm 3.68]

**Definition 10.4.** Let \(\mathcal{X}\) be a class of \(K\)-semistable Fano varieties over \(k\) and \(\mathcal{M}_\mathcal{X}\) a moduli functor for \(\mathcal{X}\) (as in Definition 10.1). We say that a proper algebraic space \(M\) is a moduli space for \(\mathcal{X}\) with the uniform \(K\)-stable locus \(M^u \subseteq M\) being open, if

(a) \(M^u \subseteq M\) is an open sub-algebraic space,
(b) \(\mu : \mathcal{M}_\mathcal{X} \to M\) is universal among maps to algebraic spaces (that is, for any algebraic space \(N\), composition with \(\mu\) induces a bijection \(\text{Hom}(\mathcal{M}_\mathcal{X}, N) \cong \text{Hom}(M, N)\)),
(c) \(\mu(k)\) takes exactly the uniformly \(K\)-stable varieties from \(\mathcal{X}(k)\) to \(M^u(k)\),
(d) \(\mu(k)\) is surjective and it is a bijection when restricted to the uniform \(K\)-stable varieties,
(e) there is a generically finite, proper cover \(\pi : Z \to M\) by a scheme, given by a family \(f : X \to Z\) in \(\mathcal{M}_\mathcal{X}(Z)\), such that
   i. \(\pi\) is finite over \(M^u\), and that
   ii. some positive multiple of the CM line bundle \(\lambda_f\) on \(Z\) is numerically equivalent to a line bundle that descends to \(M\).
Proof of Corollary 1.15. Consider the generically finite cover \( \pi : Z \rightarrow M \) given by Definition 10.4. Also by the assumptions of Definition 10.4, there is an integer \( r > 0 \) and a line bundle \( \mathcal{L} \) on \( M \) such that \( \pi^* \mathcal{L} \equiv r \lambda_f \). Let \( \rho : M' \rightarrow M \) be the normalization of \( M \) and let \( \tau : Z \rightarrow M' \) be the induced morphism. Set \( \mathcal{N} := \rho^* \mathcal{L} \). We note that \( \rho^{-1}(M^u) \) is the normalization of \( M^u \).

According to \cite[Thm 6.1]{LWX15} to prove that \( \rho^{-1}(M^u) \) is quasi-projective, we have to show that \( \mathcal{N} \) is nef and for all closed, irreducible subspaces \( V \subseteq \rho^{-1}(M^u) \), \( c_1(\mathcal{N}) \dim V > 0 \). However, these are immediate, as:

(a) \( \tau^* \mathcal{N} \equiv \tau \lambda_f \) is nef according to Theorem 1.2, hence so is \( \mathcal{N} \).
(b) \( \tau^{-1}V \) is a closed subspace of \( Z \) intersecting non-trivially \( \pi^{-1}(M^u) \). Let \( \iota : W \rightarrow Z \) be the resolution of a component of \( \tau^{-1}V \) dominating \( V \). By Nagata’s theorem we may assume that \( W \) is projective (as opposed to just proper). Then \( \iota(W) \) has to intersect \( \pi^{-1}(M^u) \), and as \( \pi^{-1}(M^u) \) is finite, \( \dim W = \dim V \). By the assumption \( \tau^* \mathcal{N} \equiv \tau \lambda_f \), it is enough to prove that \( \lambda_f |_{\tau^{-1}V} > 0 \). As \( \lambda_f \) is nef, for this in turn it is enough to show that \( \lambda_f |_W = \lambda_f |_W \) is big, where the latter equality is given by Proposition 3.8. However, \( f_W \) is of maximal variation by the finiteness of \( \tau|_{\pi^{-1}(M^u)} \). So, we are done by Theorem 1.2(c) applied to \( f_W \).

\( \square \)

Proof of Corollary 1.18. The proof is a verbatim copy of the above proof of Corollary 1.15 with two differences:

- One does not have to assume the existence of the moduli space \( M_X \), as its semi-normalization is known to exist by \cite[Thm 1.3]{LWX16} and \cite[Thm 1.1]{Oda14a}. As we work on the normalization this is enough for our purposes because the normalization maps to the semi-normalization.
- The argument eventually boils down to showing that if \( g : Y \rightarrow V \) is a family of \( K \)-semi-stable klt Fanos of maximal variation such that the very general fiber is either smooth or uniformly \( K \)-stable, then \( \lambda_g \) is big. In the uniformly \( K \)-stable case this is Corollary 1.7, and in the smooth case this is \cite[Thm 1.1]{LWX15}.

\( \square \)

10.2. Applications to anticanonical volumes

Proof of Corollary 1.19. We have

\[
\begin{align*}
\text{vol}(-K_X - \Delta) &= \underbrace{(-K_X - \Delta)^{\dim X}}_{-K_X - \Delta \text{ is ample}} \equiv \left((-K_{X/P^1} - \Delta) - f^* K_{P^1}\right)^{\dim X} \\
&= (-K_{X/P^1} - \Delta)^{\dim X} + (\dim X)2 \text{vol}(-K_F - \Delta_F) = -\deg \lambda_f,\Delta + (\dim X)2 \text{vol}(-K_F - \Delta_F) \\
&\leq (\dim X)2 \text{vol}(-K_F - \Delta_F)
\end{align*}
\]

For the second claim, if \( F \) is smooth, we apply the bound on the volume of \( K \)-semi-stable Fano varieties obtained in \cite[Thm 1.1]{Fuj15} to \( F \); if \( F \) is singular, we apply \cite[Thm 3]{Liu16}.

\( \square \)

11. Examples

In this section, we give examples showing the sharpness of our theorems.

Example 11.1. Here, we give an example of a family of Fano varieties which are not \( K \)-semistable and such that the degree of the Chow-Mumford line bundle is strictly negative. The members of this family are smooth del Pezzo surfaces of degree 8, and the family is isotrivial but not trivial. The relevance of this example for the study of the Chow-Mumford line bundle was already pointed
out by J. Fine and J. Ross in [FR06, Example 5.2] (let us warn the reader that, in contrast with [FR06], our projective bundles parametrizes rank one quotients rather than sub-bundles). Let

(a) \( T := \mathbb{P}^1 \),
(b) \( V := \mathcal{O}_T(-2) \oplus \mathcal{O}_T(1) \oplus \mathcal{O}_T(1) \) (note that \( \deg V = 0 \)),
(c) \( p \) : \( Y := \mathbb{P}V \rightarrow T \) the natural projection,
(d) \( C \) the curve on \( Y \) defined by the quotient \( V \rightarrow \mathcal{O}_T(-2) \),
(e) \( X := \text{Bl}_C Y \), \( \pi : X \rightarrow Y \) the natural morphism, and \( E \) the exceptional divisor of \( \pi \), and
(f) \( f : X \rightarrow T \) the natural morphism.

Then, \( f \) is a family of smooth degree 8 del Pezzo surfaces. We want to compute

\[
\deg \lambda_f = -(-K_{X/T})^3 = -(\pi^*\mathcal{O}_Y(3) - E)^3
\]

We compute the four monomials appearing in the above expression separately.

- \( (\pi^*\mathcal{O}_Y(3))^3 = \mathcal{O}_Y(3)^3 \) (projection formula \( \dim T = 1 \), and [Ful08, Rem 3.2.4] \( \deg V = 0 \)).
- \( (\pi^*\mathcal{O}_Y(3))^2 \cdot E = \mathcal{O}_Y(3)^2 \cdot \pi_*E = 0 \) (projection formula \( \pi_*E = 0 \)).

Before describing the other two terms we need to have a better understanding of \( E \). The ideal \( J_C \) of \( C \) corresponds to the graded ideal \( I \) of \( \text{Sym} V \) generated by the degree 1 "monomials" \( \mathcal{O}_T(1) \oplus \mathcal{O}_T(1) \). Hence, the sheaf \( J_C/J_C^2 \) corresponds to the rank 2 locally free graded module over \( \text{Sym} (\mathcal{O}_T(-2)) \) generated again by \( \mathcal{O}_T(1) \oplus \mathcal{O}_T(1) \) in degree 1, or equivalently to the rank 2 locally free graded module generated by \( \mathcal{O}_T(3) \oplus \mathcal{O}_T(3) \) in degree 0. Hence, \( E \cong \mathbb{P}W \), for \( W := \mathcal{O}_C(3) \oplus \mathcal{O}_C(3) \), and \( \mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbb{P}W}(1) \). In particular, the natural map \( \mathbb{P}W \rightarrow C \) can be identified with \( \pi|_E : \mathbb{P}^1 \times C \rightarrow C \cong \mathbb{P}^1 \), and \( \mathcal{O}_{\mathbb{P}W}(1) \equiv D + 3F \) (see [Har77, Lemma II.7.9]), where \( D \) and \( F \) are the horizontal and the vertical rulings of \( W \) over \( C \). We have:

- \( \pi^*\mathcal{O}_Y(3) \cdot E^2 = \mathcal{O}_Y(3) \cdot \pi_*E^2 = \mathcal{O}_Y(3) \cdot \pi_*(-D - 3F) = \mathcal{O}_Y(3) \cdot (-C) = 6 \).
- \( E^3 = (-D - 3F)^2 = 6 \).

Wrapping up, we obtain

\[
\deg \lambda_f = -(\pi^*\mathcal{O}_Y(3))^3 + 3(\pi^*\mathcal{O}_Y(3))^2 \cdot E - 3\pi^*\mathcal{O}_Y(3) \cdot E^2 + E^3 = -0 + 3 \cdot 0 - 3 \cdot 6 + 6 = -12 < 0.
\]

**Example 11.2.** In this example we exhibit a family \( f : X \rightarrow T \) of smooth degree 8 del Pezzo surfaces over a curve such that \( \deg \lambda_f > 0 \), or equivalently \( (-K_{X/T})^3 < 0 \), but \(-K_{X/T}\) is big. So, the statement \(-K_{X/T})^3 < 0 \) is a negativity condition independent of \(-K_{X/T}\) being big or not. However, we also note that there is one missing piece of our example: it is a family of non-\(K\)-semi-stable Fanos, although we suspect that a \(K\)-semi-stable one exists also.

Modify Example 11.1 replacing \( V \) with its dual; so we take \( V = \mathcal{O}_T(2) \oplus \mathcal{O}_T(-1) \oplus \mathcal{O}_T(-1) \) and we blow-up the curve defined by the quotient \( V \rightarrow \mathcal{O}_T \). In this case, \( \deg \lambda_f = 12 > 0 \). However, \(-K_{X/T}\) is still big. Indeed, write \( V = A \oplus B \oplus C \), where \( A = \mathcal{O}_T(2) \) and \( B = C = \mathcal{O}_T(-1) \). Then for every integer \( m > 0 \):

\[
H^0(X, -mK_{X/T}) = H^0(X, \pi^*\mathcal{O}_Y(3) - E) = H^0(Y, \mathcal{O}_Y(3) \otimes I_C^m) \subseteq H^0(Y, \mathcal{O}_Y(3))
\]

\[
= \bigoplus_{i, j, l \geq 0 \atop i + j + l = 3m, \atop 2i - j - l \geq 0} H^0(T, \mathcal{O}_T(3i - 3m)) \oplus 3m - i + 1
\]
The above computation shows that $H^0(X, -K_{X/T})$ is the part of the latter group that vanishes on $C$ to order $m$. As $I_C$ is generated in Sym $V$ by $B \oplus C$, we obtain that (11.2.a)

$$H^0(X, -mK_{X/T}) = \bigoplus_{i,j,l \geq 0} H^0(T, A^i B^j C^l) = \bigoplus_{i=m}^{2m} H^0(T, \mathcal{O}_T(3i - 3m)) \oplus 3m - i + 1.$$

To show that $-K_{X/T}$ is big, it is enough to prove that $\lim_{m \to \infty} \frac{h^0(m - K_{X/T})}{m^3} > 0$. Equation (11.2.a) yields:

$$\frac{h^0(X, -mK_{X/T})}{m^3} = \sum_{i=m}^{2m} (3i - 3m + 1)(3m - i + 1) = \frac{1}{m} \sum_{i=m}^{2m} \left( \frac{3i}{m} - 3 + \frac{1}{m} \right) \left( \frac{3i}{m} + \frac{1}{m} \right)$$

Hence,

$$\lim_{m \to \infty} \frac{h^0(X, -mK_{X/T})}{m^3} = \int_1^2 (3x - 3)(3 - x) dx = 2$$

So, we showed indeed that $-K_{X/T}$ is big, and we even computed that $\text{vol}(-K_{X/T}) = 12$ (a coincidence with the previous number 12 above).

**Example 11.3.** Here we give an example of a family $f : X \to T$ of smooth Del-Pezzo surfaces of degree 6 such that $\delta_X = 1$ for all closed point $t \in T$, $\deg \lambda_f = 0$ but $-K_{X/T}$ not nef. This shows that the hypothesis $\delta > 1$ in Theorem 1.22 is necessary.

For this, we modify Example 11.1 in two respects:

(a) We take $V$ to be the dual vector bundle, that is, $V := \mathcal{O}_T(2) \oplus \mathcal{O}_T(-1) \oplus \mathcal{O}_T(-1)$.

(b) Instead of one curve, we blow up 3 curves. That is, we set $X := \text{Bl}_{C_1, C_2, C_3} Y$, where $C_i$ is the curve defined by the quotient $V \to \mathcal{L}_i$, where $\mathcal{L}_i$ is the $i$-th direct summand of $V$.

Let $E_i, F_i$ and $W_i$ (and for $i = 1$ also $D_1$) to be defined for each $C_i$ as $E$, $F$ and $W$ (and for $i = 1$ also $D$) was defined for $C$ in Example 11.1. We do not define $D_i$ also as in Example 11.1 because for $i = 2, 3$, $W_i = \mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(3)$, so $E_2, E_3 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$. Instead, for $i = 2, 3$, set $D_i$ to be the divisor class of $\mathcal{O}_{F_i}(1)$.

Note that the $E_i$ are disjoint, and hence any intersection product involving different $E_i$ is automatically 0. We write out below the computations where the result is different than in Example 11.1, where $i = 2$ or 3:

- $\pi^* \mathcal{O}_Y(3) \cdot E_i^2 = \mathcal{O}_Y(3) \cdot \pi_*(E_i^2) = \mathcal{O}_Y(3) \cdot \pi_*(-D_1 + 3F_1) = \mathcal{O}_Y(3) \cdot (-C_1) = -6$.

- $E_1^3 = (-E_1 + 3F_1)^2 = -6$.

- $\pi^* \mathcal{O}_Y(3) \cdot E_i^2 = \mathcal{O}_Y(3) \cdot \pi_*(E_i^2) = \mathcal{O}_Y(3) \cdot \pi_*(-D_i) = \mathcal{O}_Y(3) \cdot (-C_i) = 3$.

- $E_i^3 = (-D_i)^2 = D_i^2 = c_1(\pi^* W_i) D_i = \deg(W_i) F_i \cdot D_i = \deg W_i = 3$.

Set $E := E_1 + E_2 + E_3$. Then, we conclude that

$$\deg \lambda_f = -3(\pi^* \mathcal{O}_Y(3) - E)^3 = -\pi^* \mathcal{O}_Y(3) \cdot E^2 + E^3$$

$$= -3\pi^* \mathcal{O}_Y(3) \cdot (E_1^3 + E_2^3 + E_3^3) + (E_1^3 + E_2^3 + E_3^3) = -3 \cdot (-6) - 2 \cdot 3 \cdot 3 + (-6) + 2 \cdot 3 = 0$$

The fibres of $f$ are smooth del Pezzo surfaces of degree 6, they are well-known to be K-poly-stable (so, in particular, K-semi-stable), but they are not uniformly K-stable because they have a positive
Corollary 4.8, so \( \deg X \).

These are verifications of \( K \).

Theorem 1.2 tells us that \( \deg X \).

Example 11.4. This example is interesting also, because for each integer \( m > 0 \), \( | - mK_X/T | \) has a single element. Indeed, via the containment

\[
H^0(\mathscr{O}_X (m)) \subseteq \bigoplus_{i,j,l \in I} H^0 \left( T, \mathcal{O}_T (2) \otimes \mathcal{O}_T (-1) \otimes \mathcal{O}_T (-1)^l \right),
\]

where \( I \) runs through all the triples such that the corresponding "monomial" vanish to the order \( m \) along all \( C_1, C_2 \) and \( C_3 \). However, there is only one triple satisfying this requirement \( (i, j, l) = (m, m, m) \). So,

\[
H^0(-mK_X/T) = H^0 (T, \mathcal{O}_T (2)^m \otimes \mathcal{O}_T (-1)^m \otimes \mathcal{O}_T (-1)^l) = H^0 (T, \mathcal{O}_T) = k.
\]

Example 11.4. In this example, for each choice of an integer \( d > 0 \) we exhibit families \( f: X \rightarrow T \) of uniformly \( K \)-stable del Pezzo surfaces of degree 4 over a smooth projective curve. In this situation, Theorem 1.2.(c) tells us that \( \deg \lambda_f > 0 \), or equivalently \( (K_X/T)^3 < 0 \). So, one would expect \( -K_X/T \) to have only a few sections. Here, we show that both the expected and the unexpected behavior can happen. More precisely, \( | - K_X/T | = 0 \) for \( d > 3 \), and \( \kappa(-K_X/T) \geq 1 \) for \( d = 1 \).

Let \( p_1, \ldots, p_4 \) be four points in \( \mathbb{P}^2 \) in general position, and denote by \( L_{ij} \) the line through \( p_i \) and \( p_j \). Let \( \iota: T \rightarrow \mathbb{P}^2 \) be a degree \( d \) smooth curve in \( \mathbb{P}^2 \) which avoids the four points. Let \( \Gamma \cong T \) be the graph of \( \iota \) in \( \mathbb{P}^2 \times T \), and let \( T_i \) be the curve \( \{p_i\} \times T \) in \( \mathbb{P}^2 \times T \). We want to look at the blow-up \( \pi: Y \rightarrow \mathbb{P}^2 \times T \) of \( \Gamma \) and \( T_i \), for \( i = 1, \ldots, 4 \). Denote by \( g: Y \rightarrow T \) the natural projection.

The family \( g: Y \rightarrow T \) is generically a family of degree 4 smooth del Pezzo surfaces of maximal variation. The only exception is at the points \( t \in T \) where \( \iota(T) \) intersects one of the lines \( L_{ij} \). In these cases, \( Y_t = \text{Bl}_{p_1, p_2, p_3, p_4, p} \mathbb{P}^2 \), where \( p \) lies on \( L_{ij} \). In particular, \( -K_{Y_t} \) is big and semi-ample, and there is a unique curve \( C \) for which \( C \cdot -K_{Y_t} = 0 \): the proper transform of \( L_{ij} \). The anti-canonical model is the contraction of \( L_{ij} \) to an \( A_1 \) singularity, so in particular it has canonical singularities.

Let \( f: X \rightarrow T \) be the relative anti-canonical model of \( g \) (remark that \( R^ig_{*}(-mK_Y/T) = 0 \) for \( i > 0 \) and \( m \) big enough, by Kawamata-Viehweg vanishing theorem, so we do have base change). The family \( f \) satisfies the hypotheses of Theorem 1.2(c), so \( \deg \lambda_f > 0 \).

We show that, if \( d > 3 \), then \( | - K_X/T \mathbb{P}^2 \) and if \( d = 1 \), then \( \kappa(-K_X/T) \geq 1 \). In either case the crucial remark is that \( H^0(X, -mK_X/T) \) can be identified with the subspace of \( H^0(\mathbb{P}^2 \times T, \mathcal{O}_{\mathbb{P}^2 \times T} (3m)) \) which vanish along \( \iota(T) \) and \( p_i \) with multiplicity at least \( m \). Hence:

- If \( d > 3 \), then there are no such sections, as \( d \) is exactly the degree of \( \iota(T) \).
- If \( d = 1 \), then \( \iota(T) \) is a line \( L \). So, \( | - K_X/T \mathbb{P}^2 \) is the set of cubics \( C \) on \( \mathbb{P}^2 \) such that \( C \) goes through \( p_i \) and \( \text{Supp} C \) contains \( L \). Hence, \( C = L + C' \), where \( C' \) is a conic through \( p_i \). There is a one parameter family of such conics.

**Appendix A. Computations concerning the definition of the CM line bundle**

The following work is needed to prove the statements of Section 3. These are verifications of technical issues concerning the singular situation.
We need the following lemmas as we work with singular varieties, and hence Riemann-Roch computations do not work directly. It turns out that if the spaces are normal then singularities do not mess up any of the terms involving any of the definitions of the CM line bundle. However, in the non-normal situation (which we do not deal with in the present article), Lemma A.2 seems to suggest that one has to be careful.

**Lemma A.1.** Consider the following situation:

- $f : X \to T$ is a projective morphism to a normal quasi-projective variety (allowing $T = \text{Spec} k$),
- $M$ is an $f$-ample $\mathbb{Q}$-divisor on $X$,
- $E$ is a coherent sheaf on $X$, and
- $r \geq 0$ is an integer such that $(\dim \text{Supp} \mathcal{E})_y \leq r$ for $y \in Y$ the generic point and $(\dim \text{Supp} \mathcal{E})_y \leq r + 1$ for $y \in Y$ a codimension 1 point.

Then there are $\mathbb{Q}$-divisors $D_i$ (resp. $d_i \in \mathbb{Q}$), determined uniquely up to $\mathbb{Q}$-linear equivalence (resp. determined uniquely), such that for all $q$ divisible enough, if $\dim T > 0$, then

$$
c_1(f_*(\mathcal{O}_X(qM) \otimes E)) = \sum_{i=0}^{1+r} q^i D_i,
$$

(resp. if $T = \text{Spec} k$, then

$$
h^0(X, \mathcal{O}_X(qM) \otimes E) = \sum_{i=0}^{\dim \text{Supp} \mathcal{E}} q^i d_i).
$$

**Proof.** In the case of $T = \text{Spec} k$, $h^0(X, \mathcal{O}_X(qM) \otimes E)$ equals the Hilbert polynomial for $q$ divisible enough, and hence the statement follows. So, from now we assume that $\dim T > 0$.

Let $s > 0$ be an integer such that $sM$ is relatively very ample. As the statement is for all $q$ divisible enough, by replacing $M$ with $sM$ we may assume that $M$ is relatively very ample and $f_* \mathcal{O}_X(M)$ is locally free, in which case we will exhibit $\mathbb{Z}$-divisors $D_i$. Furthermore, as the statement is about codimension 1 behavior over $T$, and $T_{\text{reg}}$ is a big open set of $T$, by replacing $T$ with $T_{\text{reg}}$ we may assume that $T$ is regular.

As $M$ is relatively very ample, it induces an embedding $\iota : X \hookrightarrow P := \text{Proj} \ f_* \mathcal{O}_X(M)$. Let $\pi : P \to X$ be the natural morphism. As $P$ is regular, $\iota_* \mathcal{E}$ has a locally free resolution $\mathcal{P}^*$, which in particular is a perfect complex on $P$. Hence, for $q$ divisible enough the following holds (where $\det$ is taken in the sense of [KM76], that is, as the alternating tensor product of the determinants of the elements of a locally free resolution, where the latter exists as $T$ is regular):

$$
c_1(f_*(\mathcal{O}_X(qM) \otimes \mathcal{E})) = c_1(\det f_*(\mathcal{O}_X(qM) \otimes \mathcal{E})) = c_1(\det Rf_*(\mathcal{O}_X(qM) \otimes \mathcal{E}))
$$

relative Serre vanishing

$$
= c_1(\det R\pi_*(\mathcal{O}_P(q) \otimes \iota_* \mathcal{E})) = c_1(\det R\pi_*(\mathcal{O}_P(q) \otimes \mathcal{P}^*)) = c_1\left(\bigotimes_{i=0}^{1+r} \mathcal{M}_i^q \right) = \sum_{i=0}^{1+r} q^i D_i.
$$

for some line bundles $\mathcal{M}_i$ according to [KM76], Thm 4, p 55] (see p 50 for the definition of $Q_{(c)}$)

$D_i := c_1(\mathcal{M}_i)$.

Note that in the following lemma we do not assume any $\mathbb{Q}$-Cartier hypothesis on $K_{X/T}$. Still, our intersection in (1.2.a) is well defined as $M$ is $\mathbb{Q}$-Cartier.
**Lemma A.2.** Let $f : X \to T$ be a surjective morphism from a normal projective variety of dimension $n + d$ to a smooth variety of dimension $d \geq 0$ with $n \geq 1$, and let $M$ be a $\mathbb{Q}$-Cartier $f$-ample divisor on $X$.

(a) If $\dim T > 0$, then for all divisible enough integers $q > 0$,

$$
c_1(f_*\mathcal{O}_X(qM)) = \frac{q^{n+1}}{(n+1)!} f_* (M^{n+1}) + \frac{q^n}{2 \cdot n!} f_* (K_{X/T} \cdot M^n) + p^{n-1}(q),$$

where $p^{n-1}(x)$ is polynomial of degree at most $n - 1$ with $x$ as a variable and $\mathbb{Q}$-divisors as coefficients.

(b) If $T = \text{Spec } k$, then $\chi(X, qM) = \frac{M^n}{n!} q^n - \frac{K_X \cdot M^{n-1}}{2(n-1)!} q^{n-1} + O(q^{n-2}).$

In particular, if $T$ is a curve and $M^{n+1} > 0$, then $\deg f_*\mathcal{O}_X(qM) > 0$ for all positive integers $q$ divisible enough.

**Proof.** As Grothendieck-Riemann-Roch works directly only for smooth $X$ (or also on locally complete intersection singularities, which does not include klt singularities with Cartier index greater than 1), we need to compare $X$ with a resolution. Let $\sigma : Z \to X$ be a resolution of singularities and set $g := f \circ \sigma$.

First, we claim that for all integers $i > 0$ and $1||q$, in the respective cases:

(a) $\deg R^i g_* \mathcal{O}_Z(q\sigma^* M) = p_i^{n-1}(q)$ for some polynomial $p_i^{n-1}(x)$ of degree at most $n - 1$ and $\mathbb{Q}$-divisor coefficients, and

(b) $h^i(Z, q\sigma^* M) = O(q^{n-2}).$

Indeed, fix an integer $i > 0$. There is a spectral sequence with $E^2$-terms $R^p f_* (\mathcal{O}_X(qM) \otimes R^r \sigma_* \mathcal{O}_Z)$ abutting to $R^i g_* \mathcal{O}_Z(q\sigma^* M)$ for $i = p + r$. As $M$ is $f$-ample and $q$ is divisible enough, this spectral sequence degenerates. Therefore,

$$R^i g_* \mathcal{O}_Z(q\sigma^* M) \cong f_* (\mathcal{O}_X(qM) \otimes R^i \sigma_* \mathcal{O}_Z).$$

Then Lemma A.1 applied to $E := R^i \sigma_* \mathcal{O}_Z$ concludes our claim, using that $\text{Supp} R^i \sigma_* \mathcal{O}_Z$ is contained in the non-normal locus, which is at most $n - 2$ dimensional in the generic fiber and at most $n - 1$ dimensional over the fibers over codimension 1 points.

Having shown our claim, in the $\dim T > 0$ case the statement of the proposition is shown by the following computation, which holds for every $q$ divisible enough (so $qM$ is $f$-very ample, Cartier and without higher cohomologies on the fibers):

$$c_1(f_*\mathcal{O}_X(qM)) = c_1(f_*\mathcal{O}_X(qM)) = \underbrace{\chi(\mathcal{O}_Z(q\sigma^* M) \cdot \text{td}(T_y))_{n+1}}_{\text{Grothendieck-Riemann-Roch, as } Z \text{ and } T \text{ are smooth, and}} + p^{n-1}(q) = \underbrace{g_* \left( \sum_{i=0}^{n+1} q^i (\sigma^* M)^i \cdot \text{td}(T_y) \right)}_{\text{the above claim}} + p^{n-1}(q)$$

$$= \underbrace{\frac{q^{n+1}}{(n+1)!} f_* \sigma_* (\sigma^* M)^{n+1}}_{\sigma_* (\sigma^* M)^{n+1} = M^{n+1}, \text{ and } \sigma_* ((\sigma^* M)^n \cdot (-K_{Z/T})) = M^n \sigma_* (-K_{Z/T}) = -M^n \cdot K_{X/T} \text{ by the projection formula}} + \underbrace{\frac{q^n}{2n!} f_* \sigma_* ((\sigma^* M)^n \cdot (-K_{Z/T}))}_{\sigma_* ((\sigma^* M)^n \cdot (-K_{Z/T})) = M^n \sigma_* (-K_{Z/T}) = -M^n \cdot K_{X/T} \text{ by the projection formula}} + p^{n-1}(q)$$

$$= \frac{q^{n+1}}{(n+1)!} f_* (M^{n+1}) - \frac{q^n}{2n!} f_* (M^n \cdot K_{X/T}) + p^{n-1}(q).$$
In the case of \( T = \text{Spec} \, k \), a similar computation concludes the proof:

\[
\chi(X, qM) = h^0(X, qM) = h^0(Z, q\sigma^*M) = \chi(Z, q\sigma^*M) + O(q^{n-2})
\]

where \( \chi \) is normal over \( X \).

\[
= \int_Z \left( \sum_{i=1}^{n} q^i \frac{(\sigma^*M)^i}{i!} \right) \text{td}(T_Z) + O(q^{n-2})
\]

\[
= \frac{q^n}{n!}(\sigma^*M)^n + \frac{q^{n-1}}{2(n-1)!}(\sigma^*M)^{n-1} \cdot (\Delta - K_Z) + O(q^{n-2}) = \frac{q^n}{n!}M^n - \frac{q^{n-1}}{2(n-1)!}M^{n-1} \cdot K_X + O(q^{n-2})
\]

\[
(\sigma^*M)^n = M^n, \quad \text{using our assumption and that } \sigma \text{ is birational, and } (\sigma^*M)^{n-1} \cdot (\Delta - K_Z) = -M^{n-1} \cdot K_X \text{ by the projection formula.}
\]

**Remark A.3.** In the situation of Lemma A.2, we also have that if \( M^{n+1} > 0 \) then \( M \) is big on \( X \). Let us stress that \( M \) is not assumed to be nef on \( X \), hence this does not follow directly from standard criteria such as [Laz04a, Theorem 2.2.14]. Indeed:

\[
h^0(X, qM) = h^0(T, f_*O_X(qM)) \geq \chi(T, f_*O_X(qM)) = \text{deg } f_*O_X(qM) + \text{rk } f_*O_X(qM)(1 - g) = \frac{q^{n+1}}{(n+1)!}M^{n+1} + O(q^n).
\]

**Proof of Proposition 3.7.** Step 1: We may assume that \( T \) is smooth. If \( T \) is already smooth, there is nothing to prove, so assume that it is not smooth. Hence, by our assumptions, the fibers are normal and \( \text{Supp } \Delta \) does not contain any of the fibers. Take a resolution \( \tau : T' \to T \). Then, according to Section 2.3.1, in the respective cases,

(a) \( f_{T'} : X_{T'} \to T' \) and \( L_{T'} \), and

(b) \( f_{T'} : (X_{T'}, \Delta_{T'}) \to T' \) and \( L_{T'} \),

satisfy all our original assumptions, including that \( sL \sim -(K_{X/T} + \Delta) \) in the case of point (b) by Proposition 2.1.(a).

We claim that \( \tau_* \lambda_{f_{T'}, L_{T'}} = \lambda_{f, L} \) (resp. \( \tau_* \lambda_{f_{T'}, \Delta_{T'}} = \lambda_{f, \Delta} \)). This is verified in the following computations, where \( \sigma : \tilde{X}_{T'} \to X \) is the induced morphism:

(a) \( \tau_* \lambda_{f_{T'}, L_{T'}} = \tau_* (f_{T'})(\mu L_{T'}^{n+1} + (n + 1)L_{T'}^n \cdot K_{X_{T'}/T'}) = f_* \sigma_* (\mu \sigma^*L^{n+1} + (n + 1)\sigma^*L^n \cdot \sigma^*K_{X/T}) \)

\[= f_* \left( (L_{T'}^{n+1} + (n + 1)L_{T'}^n \cdot K_{X/T}) \right) = \lambda_{f, L}, \quad \text{and} \]

(b) \( \tau_* \lambda_{f_{T'}, \Delta_{T'}} = -\tau_* (f_{T'})(- (K_{X_{T'}/T'} + \Delta_{T'})^{n+1}) = -f_* \sigma_* \left( - (K_{X_{T'}/T'} + \Delta_{T'})^{n+1} \right) \)

\[= -f_* \left( (K_{X/T} + \Delta)^{n+1} \right) = \lambda_{f, \Delta} \].

Having shown our claim, Step 1 follows. Indeed, if we prove, in the case of point (a), that \( s^n \lambda_{f_{T'}, L_{T'}} = c_1(L_{CM,f_{T'}, sL_{T'}}) \), then

\[s^n \lambda_{f, L} = s^n \tau_* \lambda_{f_{T'}, L_{T'}} = \tau_* c_1(L_{CM,f_{T'}, sL_{T'}}) = \tau_* \tau^* c_1(L_{CM,f,sL}) = c_1(L_{CM,f,sL}) \]

The case of (b) is verbatim the same with \( s^n \), \( \lambda_{f, L} \) and \( L_{CM,f,sL} \) replaced by \(-s^{n+1}, \lambda_{f, \Delta} \) and \( M_{n+1} \), respectively.
Step 2: The proof assuming that $T$ is smooth. Set $M_i := c_1(M_i)$. Taking into account that
\[
\frac{q^{n+1}}{(n+1)!} - \frac{q(q-1) \cdots (q-n)}{(n+1)!} = \frac{nq^n}{2n!} + O(q^{n-1}),
\]
according to Lemma A.2,
\[(1.3.b) \quad M_{n+1} = f_*(sL)^{n+1}, \text{ and } M_n = f_*(\frac{-(sL)^n \cdot K_{X/T}}{2} + \frac{n(sL)^{n-1}}{2}).\]

(Where $L = -(K_{X/T} + \Delta)$ in the case of point (b).) Hence, the next computation concludes the proof in the respective cases:

(a)
\[
c_1(L_{CM,f,sL}) = (n(n+1) + \mu sL) M_{n+1} - 2(n+1) M_n
\]
\[
= (n(n+1) + \mu sL) f_*((sL)^{n+1}) - 2(n+1) f_* \left( \frac{-K_{X/T} \cdot (sL)^n}{2} + \frac{n(sL)^{n-1}}{2} \right)
\]
\[
= \frac{\mu L}{s} s^{n+1} f_*(\lambda L)^{n+1} - s^n(n+1) f_*(K_{X/T} \cdot L^n) = s^n \lambda_{f,L}
\]

(b)
\[
M_{n+1} = f_* ((-(K_{X/T} + \Delta))^{n+1}) = -s^{n+1} \lambda_{f,\Delta}
\]

The next lemma is a technical statement used in Proposition 3.8.

Lemma A.4. Let $h : V \to S$ be a flat $n$-relative dimensional morphism from a reduced projective scheme to a smooth projective curve, and let $L$ be an $h$-very ample line bundle on $V$. Let $\pi : Z \to V$ be the normalization of $V$ with $g : Z \to S$ being the induced morphism, and assume also that $\pi^*L$ is $g$-very ample. Then the $n+1$-th (so highest) Knudsen-Mumford coefficients of $L$ with respect to $g$ (as in Notation 3.6) agrees with that of $\pi^*L$.

Proof. Consider the exact sequence on $V$ given by the normalization:
\[
0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow 0
\]
This yields a natural inclusion $h_*(\mathcal{L}^q) \subset g_*(\pi^*\mathcal{L}^q)$. Hence, it is enough to prove that for $q$ divisible enough, $\deg h_*(\mathcal{L}^q \otimes \mathcal{E}) = O(q^n)$, which is given by Lemma A.1 as dim Supp $\mathcal{E} \leq n$. 

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