ON THE QCD EVOLUTION OF THE TRANSVERSITY DISTRIBUTION

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Abstract

The QCD evolution of the transversity distributions is investigated and compared to that of the helicity distributions. It is shown that they differ largely in the small–$x$ region. It is also proved that the evolution preserves Soffer’s inequality among the three leading–twist distribution functions.
The transversity distribution, originally introduced by Ralston and Soper \cite{1} (see also \cite{2, 3, 4}), and now customarily called $h_1$, measures the polarization asymmetry of quarks (or antiquarks) in a transversely polarized hadron. More explicitly, $h_1(x)$ is the number density of quarks polarized in a transverse direction $+\hat{n}$ with a given longitudinal momentum fraction $x$ minus the number density of quarks polarized in the opposite direction $-\hat{n}$, when the hadron’s spin points in the direction $+\hat{n}$, \textit{i.e.}

$$h_1^q(x) = q_{+\hat{n}}(x) - q_{-\hat{n}}(x).$$

(1)

In the Operator–Product–Expansion language $h_1^q(x)$ is a leading twist quantity \cite{2} and therefore has the same status as the other better known leading twist distribution functions, the unpolarized density $q(x)$ and the helicity distribution $\Delta q(x)$. However, $h_1^q$ is chirally odd \cite{2} and decouples from inclusive deep inelastic scattering. This makes it a rather elusive observable. Its measurement is possible only in polarized hadron–hadron scattering or in semi-inclusive reactions \cite{5, 6, 7}, the best method being probably the Drell–Yan dimuon production with two transversely polarized proton beams, an experiment planned at RHIC \cite{8} (interest in this process has been also expressed in the HERA-$\vec{N}$ proposal \cite{9}).

Due to the lack of experimental data, our knowledge of the shape and magnitude of $h_1^q$ relies on model calculations \cite{10, 11}. An alternative adopted by some authors \cite{5, 12} to obtain predictions for measurable quantities related to $h_1$ is to use fits to the longitudinally polarized data with the assumption $h_1^q \simeq \Delta q$ at all momentum scales. This means that the difference in the QCD evolution of $h_1^q$ and $\Delta q$ in the $x$–space is considered to be irrelevant. We shall show that this is definitely not the case: even though the first moments $h_1^q(1,Q^2)$, $\Delta q(1,Q^2)$ do not evolve very differently at first order (actually $\Delta q(1,Q^2)$ is constant whereas $h_1^q(1,Q^2)$ decreases very slowly with $Q^2$, due to the smallness of its first anomalous dimension), the evolution in the shape of the
two distributions is dramatically different, especially at small $x$, and should certainly be taken into account. A calculation of an important class of observables, the Drell–Yan double–spin transverse asymmetries, which correctly treats the evolution of $h_1^{q\bar{q}}$ is contained in [13].

Another issue which deserves some attention is the inequality among the three leading twist distribution functions, $q$, $\Delta q$ and $h_1^q$, recently discovered by Soffer [14]. The theoretical status of Soffer’s inequality is matter of discussion. This inequality was proved in a parton model framework and it has been argued [15] that it is spoiled by radiative corrections, much like the Callan–Gross relation. A question of interest is whether it is preserved by the QCD evolution. We shall answer positively this question.

Let us start by looking in detail at the QCD evolution of the transverse polarization distribution. Being chirally odd, $h_1^q(x, Q^2)$ does not mix with gluon distributions, which are chirally even. Thus its $Q^2$ evolution at leading order is governed only by the process of gluon emission. The Altarelli–Parisi equation for the QCD evolution of $h_1^q(x, Q^2)$ at order $\alpha_s$ is ($t \equiv \log \frac{Q^2}{\mu^2}$)

$$\frac{dh_1^{q\bar{q}}(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dz}{z} P_h(z) h_1^{q\bar{q}}(\frac{x}{z},t),$$

(2)

where the leading order splitting function $P_h(z)$ has been computed by Artru and Mekhfi [3] and reads

$$P_h(z) = \frac{4}{3} \left[ \frac{2}{(1-z)_+} - 2 + \frac{3}{2} \delta(z-1) \right].$$

(3)

Inserting (3) in eq. (2) gives (for each quark and antiquark flavor)

$$\frac{dh_1^{q\bar{q}}(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi} \left\{ \left[ 2 + \frac{8}{3} \log (1-x) \right] h_1(x,t) \right\}$$

$$+ \frac{8}{3} \int_x^1 \frac{dz}{z} \left[ \frac{1}{1-z} [h_1(\frac{x}{z},t) - z h_1(x,t)] \right] - \frac{8}{3} \int_x^1 \frac{dz}{z} h_1(\frac{x}{z},t) \right\}. \quad (4)$$

Notice that $P_h$ can be conveniently decomposed as

$$P_h(z) = P_{qq}(z) - \frac{4}{3} (1-z),$$

(5)
where $P_{qq}$ is the usual quark splitting function for gluon emission, that is

$$P_{qq}(z) = \frac{4}{3} \left[ \frac{1 + z^2}{(1 - z)_+} + \frac{3}{2} \delta(z - 1) \right]. \tag{6}$$

and the second term in the r.h.s. of (6), which we shall call $\delta P_h(z)$, is responsible for the peculiar evolution of $h_1$. Note that $\delta P_h$ is always negative. The decomposition (5) will prove useful in the following.

The $Q^2$ dependence of the moments of $h_1$, $h_1(N, Q^2) \equiv \int_0^1 dx x^{N-1} h_1(x, Q^2)$, is governed by the anomalous dimensions $\gamma_N^h$ (i.e. the Mellin transforms of the splitting function $P_h(z)$), according to the multiplicative rule

$$h_1(N, Q^2) = h_1(N, Q_0^2) \left[ \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{\gamma_N^h/(33-2n_f)}, \tag{7}$$

where $n_f$ is the number of flavors. Explicitly, the leading order anomalous dimensions are given by

$$\gamma_N^h = \frac{4}{3} \left( \frac{3}{2} - 2 \sum_{j=1}^N \frac{1}{j} \right)$$

$$= \frac{4}{3} \left\{ \frac{3}{2} - 2 \left[ \psi(N + 1) + \gamma_E \right] \right\} \tag{8}$$

where $\psi(z) \equiv \frac{d \ln \Gamma(z)}{dz}$ is the digamma function and $\gamma_E$ is the Euler–Mascheroni constant.

In particular, since $\gamma_1^h = -\frac{2}{3}$, the first moment of $h_1$ and the tensor charge $\delta q \equiv \int dx \left( h_1^q - h_1^\bar{q} \right)$ decrease with $Q^2$ as

$$\delta q(Q^2) = \delta q(Q_0^2) \left[ \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{-4/27}. \tag{9}$$

The smallness of the exponent $-\frac{4}{27}$ might induce one to think that the evolution of $h_1^q(x, Q^2)$ is not much different from that of the helicity distributions $\Delta q(x, Q^2)$ (remember that the $q \to qG$ anomalous dimension $\gamma_1^{qq}$ vanishes at all orders and the $G \to q\bar{q}$ polarized anomalous dimension $\gamma_1^{Gq}$ is zero at leading order, so that $\Delta q(1, Q^2)$
is constant). As a matter of fact, the evolution in the $x$–space is sensibly different, especially at small $x$.

This can be seen analytically by an argument based on the double–log approximation. The leading behavior of the parton distributions at small $x$ is governed by the rightmost singularity of their anomalous dimensions in the $N$–space. For $h_1^q$ eq. (8) shows that this singularity is located at $N = -1$. Expanding $\gamma^h_N$ around this point gives

$$\gamma^h_N \sim -\frac{1}{N+1} + O(1) \quad (10)$$

In the $x$–space, expanding the splitting function $P_h$ in powers of $x$ yields

$$P_h(x) \sim \frac{8}{3} x + O(x^2) . \quad (11)$$

By contrast, in the longitudinally polarized case it is known [16] that for $\Delta q$ the rightmost singularity in the space of moments is located at $N = 0$ and the splitting functions $\Delta P_{qq}$ and $\Delta P_{qg}$ behave as constants as $x \to 0$. This means that, in the QCD evolution at small $x$, $h_1^q$ is suppressed by a power of $x$ with respect to $\Delta q$.

We can investigate numerically this problem by solving the Altarelli–Parisi equation (2) with a suitable input for $h_1$. We assume $h_1^q$ and $\Delta q$ to be equal at a small scale $Q_0^2$ and let the two distributions evolve differently, according to their own evolution equation. The assumption $h_1^q(x, Q_0^2) = \Delta q(x, Q_0^2)$ is suggested by quark model calculations [2, 10] of $h_1^q$ and $\Delta q$, which show that these two distributions are almost equal at a scale $Q_0^2 \lesssim 0.5$ GeV$^2$. For $\Delta q(x, Q_0^2)$ we use the leading order GRV parametrization [17] whose input scale is $Q_0^2 = 0.23$ GeV$^2$. The result for the $u$ distributions is shown in Fig. 1 (the situation is similar for the other flavors). The dashed line is the input, the solid line and the dotted line are the results of the evolution of $h_1^u$ and $\Delta u$, respectively, at $Q^2 = 25$ GeV$^2$. For completeness the evolution of $h_1^u$ driven only by $P_{qq}$, with the $\delta P_h$ term turned off – see eq. (3) – is also shown (dot-dashed line). The large difference
in the evolution of $h_1^q$ (solid curve) and $\Delta u$ (dotted curve) at small $x$ is evident. Notice also the discrepancy between the correct evolution of $h_1^q$ and the evolution driven by $P_{qq}$ (dot-dashed curve).

Let us come now to Soffer’s inequality among the three leading twist distribution functions. It reads \[ q(x) + \Delta q(x) \geq 2 |h_1^q(x)|, \tag{12} \]
or, equivalently, $q_+(x) \geq |h_1^q(x)|$, having introduced $q_\pm \equiv \frac{1}{2}(q \pm \Delta q)$. This relation has been rigorously proved by Soffer in the parton model, by relying on a positivity bound on the quark–nucleon forward amplitudes. A rederivation of the inequality was offered in [14], where it was pointed out that it is spoiled by radiative corrections and it was claimed that its status is similar to that of the Callan–Gross relation. The question arises whether Soffer’s inequality is preserved by the QCD evolution (at least at leading order, which is all we know at present). We shall show that the answer to this question is positive and can be obtained in a very simple manner.

Explicitly stated, the problem is: assuming (12) to be valid at some scale, will it hold at any larger scale? In order to prove that it is indeed so, it suffices to show that the rate of evolution of $|h_1^q|$ is always smaller than that of $q_+$, namely

\[ \frac{d|h_1^q|}{dt} \leq \frac{dq_+}{dt}. \tag{13} \]

Now, the Altarelli–Parisi equation for $q_+$ is ($\otimes$ denotes convolution)

\[ \frac{dq_+}{dt} = \frac{\alpha_s(t)}{2\pi} \left( P_{qq} \otimes q_+ + P_{qG}^{(+)} \otimes G_+ + P_{qG}^{(-)} \otimes G_- \right), \tag{14} \]

where $G_\pm(x,Q^2)$ are the gluon helicity distributions, and the two $G \to q\bar{q}$ splitting functions are $P_{qG}^{(+)}(z) = \frac{1}{2}z^2$ and $P_{qG}^{(-)}(z) = \frac{1}{2}(1-z)^2$.

Comparing the evolution equations (14) and (2) (the latter written for $|h_1^q|$), the inequality (13) follows immediately from two facts:
1. The splitting functions $P_{qG}^{(\pm)}(z)$ are positive definite.

2. $P_{h}$ always gives a contribution to the convolution integral smaller than that of $P_{qq}$, since $\delta P_{h}$ is negative, see eq. (5).

Therefore, Soffer’s inequality is not spoiled by the QCD evolution, in much the same way as the positivity bound $|\Delta q| \leq q$ is protected, because the evolution can never make the probabilities $q_\pm$ become negative. Hence, it is with this positivity relation that an analogy can be made, rather than with the Callan–Gross relation. The latter involves structure functions, i.e. physical quantities, whereas Soffer’s inequality is a relation among distribution functions. The reason why eq. (12) has eluded for such a longtime the attention of physicists is simply that it does not have a probabilistic interpretation, for it involves nondiagonal quark–nucleon amplitudes.

In conclusion, let us summarize our results. First of all, we have shown that the QCD evolution of $h_1(x, Q^2)$ possesses some relevant peculiarities which make it largely different from that of the helicity distributions at small $x$. This difference can by no means be neglected if one wants to make reliable predictions on experimentally accessible quantities (especially when these are sensitive to the small-$x$ region, as it is the case of some double–spin transverse asymmetries [13]) Second, we have shown that Soffer’s inequality is protected by the QCD evolution and thus it is precisely on the same ground as the positivity relation between polarized and unpolarized distribution functions.

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Figure 1: Evolution of the helicity and transversity distributions for the $u$ flavor. The dashed curve is the input $h_1^u \equiv \Delta u$ at $Q_0^2 = 0.23$ GeV$^2$ taken from the GRV [17] parametrization. The solid (dotted) curve is $h_1^u (\Delta u)$ at $Q^2 = 25$ GeV$^2$. The dot-dashed curve is the result of the evolution of $h_1^u$ at $Q^2 = 25$ GeV$^2$ driven by $P_{qq}$, i.e. with the term $\delta P_h$ turned off in $P_h$. 