Second order contributions to elastic large–angle Bhabha scattering

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Abstract

The cross section of (quasi–)elastic large–angle electron–positron scattering at high energies is calculated. Radiative corrections of the orders $O(\alpha^2 L^2)$ and $O(\alpha^2 L)$, except pure two–loop box contributions, are explicitly calculated. In the second order we considered the following sources of corrections: 1) Virtual photonic corrections coming from squares of 1–loop level amplitudes and their relevant interferences (vertex–type and box–type Feynman diagrams). 2) Double soft photon emission and one–loop corrections to single soft photon emission. The results are presented in an analytical form.

PACS: 12.20.–m Quantum electrodynamics, 12.20.Ds Specific calculations

1 Introduction

The importance to know the cross section of elastic and inelastic Bhabha scattering is emphasized mostly by experimental physics requirements, since these processes are used for calibration. Better than per–mille accuracy in theoretical calculations of the cross section is required in order to obtain an adequate level of accuracy to the present one of Standard Model parameters. Also, gaining more insight into the properties of hadrons extracted from experiments at colliders and meson factories, demands the above accuracy.

Much of attention was paid to a calculation of different contributions to the cross section to the second order of perturbation theory (PT). The detailed analysis of contribution coming from two hard photon emission [1], real and virtual pairs emission [4] was recently performed to the leading ($\alpha^2 L^2$) and next–to–leading ($\alpha^2 L$) accuracy. In the meantime, an investigation of the 2–loop box–type gauge–invariant set of Feynman diagrams is not completed to the moment (their total number is 44) [3]. Another problem is the calculation of radiative corrections (RC) to the single hard photon emission process. These tasks now are under close scrutiny. As for the present paper We summarize here all obtained in the past results concerning 2–loop contributions to the vertex functions (Sect. 1), then calculate the contributions coming from the squares of 1–loop Feynman amplitudes which correspond to the both vertex– and box–type Feynman diagrams (Sect. 2). In Sect. 3 we give the expressions of contributions of two soft photons emission processes. In this paper we do not consider the effects of vacuum polarization inserted into the virtual photon Green function since it was examined earlier in paper [2]. Our final result is given in Eq.(19).
2 2–loop vertex contribution

\[ \Gamma(q^2) = \gamma \left[ 1 + \frac{\alpha}{\pi} \Gamma^{(2)}(q^2) + \left( \frac{\alpha}{\pi} \right)^2 \Gamma^{(4)}(q^2) \right], \quad \text{where } q^2 = s > 0 \quad \text{or} \quad q^2 = t < 0 \]

\[ \Gamma^{(2)}(s) = \frac{L - \frac{1}{2} L_{\lambda}}{2} - \frac{1}{4} L^2 + \frac{3}{4} L + \frac{\pi^2}{3} - 1 + i\pi \left( \frac{1}{2} L - \frac{1}{2} L_{\lambda} - \frac{3}{4} \right), \]

\[ \Gamma^{(2)}(t) = (L_t - 1) \left( \frac{1}{2} L_{\lambda} + 1 \right) - \frac{1}{4} L_t^2 - \frac{1}{4} L_t + \frac{\pi^2}{12}, \]

\[ \text{Re } \Gamma^{(4)}(s) = \frac{1}{8} (L^2 - 2L + 1 - \pi^2) L_{\lambda}^2 + \frac{1}{2} L_{\lambda} \left[ (L - 1) \left( -\frac{1}{4} L^2 + \frac{3}{4} L - 1 + \frac{\pi^2}{3} \right) + \pi^2 \left( \frac{1}{2} L \right. \right. \]

\[ \left. - \frac{3}{4} \right) \right] + \frac{1}{32} L^4 - \frac{3}{16} L^3 + \left( \frac{17}{32} - \frac{5\pi^2}{24} \right) L^2 + \left( -\frac{21}{32} + \frac{3}{2} \zeta(3) + \frac{17\pi^2}{36} \right) L + \mathcal{O}(1), \]

\[ \Gamma^{(4)}(t) = \frac{1}{32} L_t^4 - \frac{3}{16} L_t^3 + \left( \frac{17}{32} - \frac{\pi^2}{48} \right) L_t^2 + \left( -\frac{21}{32} - \frac{\pi^2}{16} + \frac{3}{2} \zeta(3) \right) L_t + \frac{1}{8} L_{\lambda}^2 (L_t - 1)^2 \]

\[ + \frac{1}{2} L_{\lambda} (L_t - 1) \left( -\frac{1}{4} L_t^2 + \frac{3}{4} L_t - 1 + \frac{\pi^2}{12} \right) + \mathcal{O}(1), \quad (1) \]

Figure 1: Vertex diagrams up to 2–loop level.

The corresponding Feynman diagrams up to 2–loop level are depicted in Fig. 1 (there are four more diagrams coming from crossing channels to Fig. 1(g,h,i,j)). We use the following asymptotes of the fermion vertex function in the case of space–like and time–like 4–vectors of virtual photons [1]:

\[ L = \frac{1}{2} L_{\lambda} - \frac{1}{4} L^2 + \frac{3}{4} L + \frac{\pi^2}{3} - 1 + i\pi \left( \frac{1}{2} L - \frac{1}{2} L_{\lambda} - \frac{3}{4} \right), \]

\[ L_t = \frac{1}{4} L_t^2 + \frac{1}{4} L_t + \frac{\pi^2}{12}, \]

\[ \text{Re } \Gamma^{(4)}(s) = \frac{1}{8} (L^2 - 2L + 1 - \pi^2) L_{\lambda}^2 + \frac{1}{2} L_{\lambda} \left[ (L - 1) \left( -\frac{1}{4} L^2 + \frac{3}{4} L - 1 + \frac{\pi^2}{3} \right) + \pi^2 \left( \frac{1}{2} L \right. \right. \]

\[ \left. - \frac{3}{4} \right) \right] + \frac{1}{32} L^4 - \frac{3}{16} L^3 + \left( \frac{17}{32} - \frac{5\pi^2}{24} \right) L^2 + \left( -\frac{21}{32} + \frac{3}{2} \zeta(3) + \frac{17\pi^2}{36} \right) L + \mathcal{O}(1), \]

\[ \Gamma^{(4)}(t) = \frac{1}{32} L_t^4 - \frac{3}{16} L_t^3 + \left( \frac{17}{32} - \frac{\pi^2}{48} \right) L_t^2 + \left( -\frac{21}{32} - \frac{\pi^2}{16} + \frac{3}{2} \zeta(3) \right) L_t + \frac{1}{8} L_{\lambda}^2 (L_t - 1)^2 \]

\[ + \frac{1}{2} L_{\lambda} (L_t - 1) \left( -\frac{1}{4} L_t^2 + \frac{3}{4} L_t - 1 + \frac{\pi^2}{12} \right) + \mathcal{O}(1), \quad (1) \]
\[ L = \ln \frac{s}{m^2}, \quad L_t = \frac{-t}{m^2}, \quad L_\lambda = \ln \frac{\lambda^2}{m^2}, \quad \zeta(3) \approx 1.2020569. \]

In these formulæ we have retained only the Dirac form factor of electron and dropped the Pauli one, since its contribution is suppressed by the factor of \(m^2/s\).

The second order PT contribution to the matrix element squared reads

\[
\Delta |M|^2 = 2(M_a + M_b)^*(M_{2\text{-vertex}}) + |M_c + M_d + M_f + M_m + M_n + M_p + M_q|^2, \tag{2}
\]

where \(M_{2\text{-vertex}}\) is the matrix element of the ten 2-loop vertex Feynman diagrams. The matrix element of elastic Bhabha scattering, including relevant contributions up to the 2–loop level, may be written in the following form:

\[
M = M_{0t}(1 + \delta_t^{(1)} + \delta_t^{(2)}) - M_{0s}(1 + \delta_s^{(1)} + \delta_s^{(2)}) + B_1^{(1)} + B_2^{(1)} - B_3^{(1)} - B_4^{(1)} + B^{(2)}, \tag{3}
\]

where

\[
M_{0t} = \frac{4\pi\alpha}{t} \bar{u}(p'_1)\gamma_\mu u(p_1)\bar{v}(p_2)\gamma_\mu v(p'_2), \quad M_{0s} = \frac{4\pi\alpha}{t} \bar{v}(p'_2)\gamma_\mu u(p_1)\bar{u}(p'_1)\gamma_\mu v(p_2).
\]

The quantities \(B_i^{(1)}\) correspond to the 1–loop box–type diagrams (see Fig. 2(m–q)), whereas \(B^{(2)}\) comes from the 2–loop ones (some representatives are drawn in Fig. 2(r–t)). At the Born level we have

\[
\sum |M_{0t}^2| = (4\pi\alpha)^2 \frac{8}{t^2}(s^2 + u^2), \quad \sum |M_{0s}^2| = (4\pi\alpha)^2 \frac{8}{s^2}(t^2 + u^2),
\]

\[
\sum M_{0s}M_{0t}^* = -(4\pi\alpha)^2 \frac{u^2}{st}, \quad \sum |M_{0t} - M_{0s}|^2 = 16(4\pi\alpha)^2 \left(\frac{s}{t} + \frac{t}{s} + 1\right)^2, \tag{4}
\]

\[
s = (p_1 + p_2)^2, \quad t = (p_1 - p'_2)^2, \quad u = (p_1 - p_2)^2,
\]

\[
p_1 + p_2 = p'_1 + p'_2, \quad p_{1,2}^2 = p_{1,2}^2 = m^2.
\]
The quantities $\delta_t^{(1)}, \delta_t^{(2)}$ are real. They read
\[
\delta_t^{(1)} = \frac{\alpha}{\pi} \left[ 2\Gamma^{(2)}(t) + \Pi^{(2)}(t) \right], \\
\delta_t^{(2)} = \left( \frac{\alpha}{\pi} \right)^2 \left[ (\Gamma^{(2)}(t))^2 + 2\Pi^{(2)}(t)\Gamma^{(2)}(t) + (\Pi^{(2)}(t))^2 + \Pi^{(4)}(t) + 2\Gamma^{(4)}(t) \right]. \tag{5}
\]

Here $\Pi^{(i)}(t)$ are the vacuum polarization insertions. Similar expressions are held for $\delta_s^{(1)}, \delta_s^{(2)}$ and can be derived from (5) by using crossing relations (relevant quantities are, in general, complex valued).

The relevant second order PT contribution to the matrix element, squared and summed over spin states, looks as follows:
\[
\Delta \sum |M|^2 = \alpha^2 \left( \frac{\alpha}{\pi} \right)^2 (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4), \tag{6}
\]
where
\[
\frac{\alpha^4}{\pi^2} \Delta_1 = \sum |M_{st}|^2 (|\delta_t^{(1)}|^2 + 2\delta_t^{(2)}) + \sum |M_{os}|^2 (|\delta_s^{(1)}|^2 + 2\text{Re} \, \delta_s^{(2)}) \\
- 2\text{Re} \sum M^*_{os} M_{st} (\delta_t^{(1)} \delta_s^{(1)} + \delta_t^{(2)} + \delta_s^{(2)}),
\]
\[
\frac{\alpha^4}{\pi^2} \Delta_2 = 2\text{Re} \sum (M_{st} \delta_t^{(1)} - M_{os} \delta_t^{(1)}) \cdot (B_1^{(1)} + B_2^{(1)} - B_3^{(1)} - B_4^{(1)}),
\]
\[
\frac{\alpha^4}{\pi^2} \Delta_3 = \sum |B_1^{(1)} + B_2^{(1)} - B_3^{(1)} - B_4^{(1)}|^2,
\]
\[
\frac{\alpha^4}{\pi^2} \Delta_4 = 2\text{Re} \sum (M_{st} - M_{os})^* B^{(2)}. \tag{7}
\]

Quantity $B^{(2)}$, which enters into the definition of $\Delta_4$, is nowadays unknown. The aim of this work is to give explicit expressions for $\Delta_1, \Delta_2, \text{and } \Delta_3$. The first term $\Delta_1$ was given above. As for $\Delta_2$, using the usual notation $\hat{a} = \gamma_\mu a^\mu$, it can be cast down in the form
\[
\Delta_2 = (1 + \mathcal{P}_{st}) \text{Re} \left( \Gamma^{(2)}(t) \right) \frac{1}{t} \int \frac{d^4k}{i\pi^2} \left\{ \frac{\text{Tr} (\gamma_\mu (\hat{p}_1 + \hat{k}) \gamma_\nu \hat{p}_1 \gamma_\rho \hat{p}_1') \cdot \text{Tr} (\gamma_\mu (\hat{p}_2 + \hat{k}) \gamma_\nu \hat{p}_2 \gamma_\rho \hat{p}_2')}{A(p_1 + k)A(k - q_1)A(-p_2 + k)} \right\} + \delta \Delta_2,
\]
\[
+ \frac{2\text{Tr} (\hat{p}_2 \hat{p}_1' (\hat{p}_1 + \hat{k}) \hat{p}_2 \hat{p}_1') \cdot \text{Tr} (\gamma_\mu (\hat{p}_2' - \hat{k}) \gamma_\nu \hat{p}_2' \gamma_\rho \hat{p}_2')}{A(p_1 + k)A(k - q_1)A(p_1' + k)} + 2u^2(p_1 + k)(-p_2' - k) + A(p_1 + k)A(k - q_1)A(p_1' + k),
\]
\[
A(p_{1,2} \pm k) = (p_{1,2} \pm k)^2 - m^2, \quad A(p_{1,2} \pm k) = (p_{1,2} \pm k)^2 - m^2, \quad q = p_1' - p_1, \quad A(q - k) = \left( q - k \right)^2 - \lambda^2, \quad A(q_1 - k) = \left( q_1 - k \right)^2 - \lambda^2, \quad q_1 = -p_1 - p_2.
\]

where the permutation operator $\mathcal{P}_{st}$ acts as follows:
\[
\mathcal{P}_{st} A(s,t,u,L,L_t) = A(t,s,u,L_t,L).
\]

Calculating the first term in $\Delta_2$, we have to put (see Appendix A)
\[
L_s = L, \quad \psi_{1s} = \frac{1}{s} \left( \frac{1}{2} L^2 + \frac{\pi^2}{6} \right). \tag{8}
\]
The second term on the right hand side (rhs) of Eq. (8), $\delta \Delta_2$ arises from the product of imaginary parts of $\Gamma^{(2)}(s)$ and box structures (see Eq. (7) and Appendix A). It can be obtained by applying the following rules:

$$\delta \operatorname{Re} \Gamma^{(2)}(s)^* L_s = -\pi^2 \left( \frac{1}{2} L - \frac{1}{2} L_\lambda - \frac{3}{4} \right), \quad \delta \operatorname{Re} \Gamma^{(2)}(s)^* \psi_{1s} = -\frac{\pi^2}{s} L \left( \frac{1}{2} L - \frac{1}{2} L_\lambda - \frac{3}{4} \right),$$

$$\delta \operatorname{Re} L_s^* L_s = \pi^2, \quad \delta \operatorname{Re} L_s^* \psi_{1s} = \frac{\pi^2 L}{s}, \quad \delta \operatorname{Re} \psi_{1s}^* \psi_{1s} = \frac{\pi^2 L^2}{s^2}. \quad (9)$$

By performing the loop–momentum integration (relevant integrals are given in Appendix A) one arrives to the result (see Eq. (12)).

Consider now $\Delta_3$. The symmetry properties permit us to express it in the form

$$\Delta_3 = |B_1 + B_2 - B_3 - B_4|^2 = |1 + P_{su} + (1 + P_{tu})P_{st}| |B_1|^2 + 2(1 + P_{st})[B_1B_2^* - B_2B_3^* - \frac{1}{2} B_1B_3^* - \frac{1}{2} B_2B_4^*] + \delta \Delta_3. \quad (10)$$

The quantity $\delta \Delta_3$ is to be written according to the rules mentioned earlier [F]. In calculations of the first two terms in $\Delta_3$ we have to take $L_s$ and $\psi_{1s}$ as in (9). The remaining contributions to $\Delta_3$ are

$$|B_1|^2 = \int \frac{d^4k_1}{i\pi^2} \int \frac{d^4k}{i\pi^2} \operatorname{Tr} \left( \hat{p}_1^\gamma \gamma_\mu (\hat{p}_1 + \hat{k}) \gamma_\nu \hat{p}_1 \gamma_\xi (\hat{p}_1 + \hat{k}) \gamma_\eta \right) \operatorname{A}(p_1 + k_1)A(k)A(k - q)A(-p_2 + k_1) \right) \right) \right),$$

$$B_1B_2^* = \int \frac{d^4k_1}{i\pi^2} \int \frac{d^4k}{i\pi^2} \operatorname{Tr} \left( \hat{p}_1^\gamma \gamma_\mu (\hat{p}_1 + \hat{k}) \gamma_\nu \hat{p}_1 \gamma_\xi (\hat{p}_1 + \hat{k}) \gamma_\eta \right) \operatorname{A}(p_1 + k_1)A(k)A(k - q)A(p_2 - k_1) \right) \right),$$

$$B_2B_3^* = \int \frac{d^4k_1}{i\pi^2} \int \frac{d^4k}{i\pi^2} \operatorname{Tr} \left( \hat{p}_1^\gamma \gamma_\mu (\hat{p}_1 + \hat{k}) \gamma_\nu \hat{p}_1 \gamma_\eta (\hat{p}_1 + \hat{k}) \gamma_\xi \hat{p}_2 \gamma_\mu (\hat{p}_2 + \hat{k}) \gamma_\eta \right) \operatorname{A}(p_1 + k_1)A(k)A(k - q)A(p_2 + k_1) \right) \right),$$

$$B_3B_4^* = \int \frac{d^4k_1}{i\pi^2} \int \frac{d^4k}{i\pi^2} \operatorname{Tr} \left( \hat{p}_1^\gamma \gamma_\mu (\hat{p}_1 + \hat{k}) \gamma_\nu \hat{p}_1 \gamma_\eta (\hat{p}_1 + \hat{k}) \gamma_\xi \hat{p}_2 \gamma_\mu (\hat{p}_2 + \hat{k}) \gamma_\eta \right) \operatorname{A}(p_1 + k_1)A(k)A(k - q)A(p_2 + k_1) \right) \right),$$

Standard but rather tedious computation gives the following result:

$$\Delta_i = L_i^2(a_{1i} L^2 + a_{2i} L) + L_\lambda (a_{1i3} L^3 + a_{1i4} L^2 + a_{1i5} L) + a_{1i6} L^4 + a_{i7} L^3 + a_{i8} L^2 + a_{i9} L, \quad i = 1, 2, 3. \quad (12)$$
Coefficients $a_{ij}$ as the functions of $\theta = \mathbf{P}_1 \cdot \mathbf{P}'_1$ are given below:

$$a_{11} = 2F^2, \quad a_{12} = -4F^2 - 2 \left[ F + 2 \left( \frac{s^2}{t^2} + \frac{s}{t} + 1 \right) \right] L_{st},$$

$$a_{13} = -2F^2, \quad a_{14} = \frac{28}{3} F^2 + \left[ 3F + 6 \left( \frac{s^2}{t^2} + \frac{s}{t} + 1 \right) \right] L_{st},$$

$$a_{15} = \left( -6 \frac{s^2}{t^2} - 8 \frac{s}{t} - 2 \frac{t}{s} - 7 \right) L_{st}^2 + \left( -\frac{56 s^2}{3 t^2} - \frac{28 t}{3 s} - 28 \frac{s}{t} - 28 \right) L_{st},$$

$$+ \left( \frac{2 s^2}{3 t^2} + \frac{10 s}{3 t} + \frac{28 t}{3 s} + \frac{20 t^2}{3 s^2} + 9 \right) \pi^2 - \frac{158}{9} F^2,$$

$$a_{16} = \frac{1}{2} F^2, \quad a_{17} = -\frac{11}{3} F^2 - \left( \frac{2 s^2}{t^2} + \frac{3 s}{t} + \frac{t}{s} + 3 \right) L_{st},$$

$$a_{18} = \left( 3 \frac{s^2}{t^2} + 4 \frac{s}{t} + \frac{t}{s} + \frac{7}{2} \right) L_{st}^2 + \left[ \frac{11}{2} F + 11 \left( \frac{s^2}{t^2} + \frac{s}{t} + 1 \right) \right] L_{st},$$

$$- \left( \frac{1 s^2}{3 t^2} + \frac{5 s}{3 t} + \frac{14 t}{3 s} + \frac{10 t^2}{3 s^2} + \frac{9}{2} \right) \pi^2 + \frac{215}{18} F^2,$$

$$a_{19} = \left( -2 \frac{s^2}{t^2} - \frac{5 s}{2 t} - \frac{1 t}{2 s} - 2 \right) L_{st}^3 - \left( \frac{11 s^2}{3 t^2} + \frac{14 t}{t^2} + \frac{1 t}{3 s} + \frac{77}{6} \right) L_{st}^2$$

$$+ \left( \frac{2 s^2}{3 t^2} + \frac{3 s}{2 t} + \frac{5 t}{6 s} + 2 \right) \pi^2 L_{st} - \frac{215}{3} \left( \frac{1 s^2}{3 t^2} + \frac{1 s}{2 t} + \frac{1 t}{6 s} + 1 \right) L_{st}$$

$$+ \left( \frac{13 s^2}{18 t^2} + \frac{85 s}{18 t} + \frac{257 t}{18 s} + \frac{185 t^2}{18 s^2} + \frac{27}{2} \right) \pi^2 - \left( \frac{1313}{72} - 6 \zeta(3) \right) F^2,$$

$$a_{21} = 0, \quad a_{22} = 4 F^2 L_{su} - 2 \left[ F + 2 \left( \frac{t^2}{s^2} + \frac{t}{s} + 1 \right) \right] L_{st},$$

$$a_{23} = 0, \quad a_{24} = -6 F^2 L_{su} + 3 \left[ F + 2 \left( \frac{t^2}{s^2} + \frac{t}{s} + 1 \right) \right] L_{st},$$

$$a_{25} = \left( 12 \frac{s^2}{t^2} + 20 \frac{s}{t} + 6 \frac{t}{s} + 19 \right) L_{st} L_{su} + \left( \frac{34}{3} F^2 - \frac{s}{t} - \frac{t}{s} \right) L_{su}$$

$$- \frac{1}{3} \left( 34 F^2 - 17 F - 34 \frac{s^2}{t^2} - 37 \frac{s}{t} - 34 \right) L_{st} - \left( \frac{11}{2} F - \frac{s}{t} - 6 \right) L_{st}^2 - F L_{su}^2$$

$$+ \pi^2 \left( 4 F - \frac{s}{t} + \frac{7}{2} \right), \quad a_{26} = 0, \quad a_{27} = 2 F^2 L_{su} - \left[ F + 2 \left( \frac{t^2}{s^2} + \frac{t}{s} + 1 \right) \right] L_{st},$$

$$a_{28} = \left[ -2 F \left( 3 \frac{s}{t} + 2 \right) + 4 \frac{t}{s} + \frac{1}{2} \right] L_{st} L_{su} + \left[ \frac{11}{4} F + \frac{1 s}{2 t} + 3 \right] L_{st}^2 + \frac{1}{2} F L_{su}^2 - \left[ \frac{22}{3} F^2 \right.$$

$$- \frac{1}{2} \left( \frac{s}{t} + \frac{t}{s} \right) \right] L_{su} + \left[ \frac{19}{6} F + \frac{22}{3} \left( \frac{t^2}{s^2} + \frac{t}{s} + 1 \right) + \frac{1 t}{2 s} + \frac{1}{2} \right] L_{st} - \pi^2 \left[ 2 F + \frac{1 s}{2 t} - \frac{7}{4} \right],$$

$$a_{29} = - \left( F + \frac{5 s}{2 t} + 3 \right) L_{st}^3 + \left( \frac{6 s^2}{t^2} + 9 \frac{s}{t} + 1 \frac{s}{t} + 6 \right) L_{st} L_{su} - \left( \frac{s}{t} + \frac{1}{2} \right) L_{su} L_{su}$$

$$+ \frac{1}{3} \left( 44 \frac{s^2}{t^2} + 74 \frac{s}{t} + 22 \frac{t}{s} + \frac{143}{2} \right) L_{su} L_{su} - \left( \frac{77}{12} F - \frac{5 s}{6 t} - \frac{22}{3} \right) L_{st}^2.$$
\begin{align*}
- \frac{11}{6} F L_{su}^2 & - \left( \frac{2}{3} F + \frac{18 t^2}{3 s^2} + \frac{s}{t} + 9 \frac{t}{s} + 6 \right) \pi^2 L_{su} \\
+ \left[ \frac{92}{9} F^2 - \frac{11}{6} \left( \frac{s}{t} + \frac{t}{s} \right) \right] L_{su} + \left( \frac{20 t^2}{3 s^2} + 6 \frac{t}{s} + 4 \frac{s}{3 t} + \frac{9}{2} \right) \pi^2 L_{st} - \left( \frac{46}{3} F + \frac{92 t^2}{9 s^2} \right) \\
- \frac{217 s}{18 t} L_{st} + \left( \frac{11 s}{6 t} + \frac{8 t}{3 s} + \frac{77}{12} \right) \pi^2,
\end{align*}

\begin{align*}
a_{31} &= 0, \quad a_{32} = 0, \quad a_{33} = 0, \quad a_{34} = 0, \quad a_{35} = -2 F^2 L_{su}^2 + 2 F \left( \frac{2 t}{s} + 1 \right) L_{st} L_{su} \\
&- \left( \frac{2 t^2}{s^2} + \frac{t}{s} + 1 \right) L_{st}^2 - \left( \frac{2 s^2}{t^2} + \frac{2 s}{t} + 1 \right) \pi^2, \quad a_{36} = 0, \quad a_{37} = 0, \\
a_{38} &= \frac{1}{2} F^2 L_{su}^2 - F \left( \frac{2 t}{s} + 1 \right) L_{st} L_{su} + \left( \frac{2 t^2}{s^2} + \frac{t}{s} + \frac{1}{2} \right) L_{st}^2 + \pi^2 \left( \frac{s^2}{t^2} + \frac{s}{t} + \frac{1}{2} \right), \\
a_{39} &= \frac{1}{2} F^3 L_{su} + \frac{1}{2} \left( \frac{s}{t} + \frac{t}{s} \right) (L_{su}^2 - L_{st} L_{su}) - \left[ F^2 \left( \frac{2 s}{t} + 3 \frac{t}{s} \right) + \frac{1}{2} \left( \frac{s}{t} + \frac{t}{s} \right) \right] L_{su} L_{st} \\
&+ \left( \frac{7}{2} F + \frac{1}{2} \left( \frac{s}{t} + 3 \frac{t}{s} \right) \right) L_{st}^2 L_{su} + \left( F + \frac{1}{2} s - \frac{5}{4} \right) \pi^2 L_{su}^2 - \frac{3}{4} \left( \frac{t}{s} + 1 \right) L_{st}^3 \\
&- \left( F^2 + \left( \frac{s}{t} + \frac{t}{s} \right) \left( \frac{s}{t} - \frac{t}{s} - 3 \right) + \frac{1}{2} \left( \frac{s}{t} + \frac{t}{s} \right) \right) \pi^2 L_{st},
\end{align*}

where \( F = \frac{s}{t} + \frac{t}{s} + 1 \), \( L_{st} = \ln \frac{s}{-t} \), \( L_{su} = \ln \frac{s}{-u} \).

### 3 Emission of soft photons

Second order corrections to the 1-loop virtual photon emission corrected cross section, which arise from emission of a single real soft photon having energy less than \( \Delta \varepsilon \), can be written down in the factorized form:

\[
\frac{d\sigma_{SV}}{d\sigma_0} = \frac{\alpha_s}{\pi} \frac{\alpha_s}{\pi} \frac{\alpha_s}{\pi} = \left( \frac{\alpha}{\pi} \right)^2 \Delta_{SV}, \quad d\sigma_0 = \frac{\alpha^2}{s} \left( \frac{1 - \chi + \chi^2}{\chi} \right)^2,
\]

\[
\chi = \frac{1}{2} (1 - \cos \theta) = \sin^2 \frac{\theta}{2}, \quad \theta = \left( \mathbf{p}_1, \mathbf{p}'_1 \right), \quad \chi = \frac{-t}{s}, \quad 1 - \chi = \frac{-u}{s},
\]

\[
\delta_s = 4 \ln \frac{m \Delta \varepsilon}{\lambda \varepsilon} \left( L - 1 + \ln \frac{\chi}{1 - \chi} \right) + L^2 + 2L \ln \frac{\chi}{1 - \chi} + \ln^2 \chi - \ln^2 (1 - \chi) \quad (13)
\]

\[
- \frac{2\pi^2}{3} + 2 \text{Li}_2(1 - \chi) - 2 \text{Li}_2(\chi), \quad \text{Li}_2(x) = -\int_0^x \frac{dy}{y} \ln(1 - y),
\]

\[
\delta_V = 4 \ln \frac{m}{\lambda} \left( 1 - L + \ln \frac{1 - \chi}{\chi} \right) - L^2 + 2L \ln \frac{1 - \chi}{\chi} - \ln^2 \chi + \ln^2 (1 - \chi)
\]

\[
+ 3L - 4 + f(\chi),
\]

\[
f(\chi) = (1 - \chi + \chi^2)^{-2} \left[ \frac{\pi^2}{12} (-4 + 8\chi + 3\chi^2 - 10\chi^3 + 8\chi^4) + \frac{1}{2} (-2 + 5\chi - 7\chi^2 + 5\chi^3) \\
- 2\chi^4 \ln^2(1 - \chi) + \frac{1}{4} \chi (3 - \chi - 3\chi^2 + 4\chi^3) \ln^2 \chi + \frac{1}{6} (22 - 30\chi + 33\chi^2
\right),
\]

\[
\frac{d\sigma_{SV}}{d\sigma_0} = \frac{\alpha_s}{\pi} \frac{\alpha_s}{\pi} \frac{\alpha_s}{\pi} = \left( \frac{\alpha}{\pi} \right)^2 \Delta_{SV}, \quad d\sigma_0 = \frac{\alpha^2}{s} \left( \frac{1 - \chi + \chi^2}{\chi} \right)^2,
\]
\[- 11\chi^3 \ln \chi - \frac{1}{2} \chi (1 + \chi^2) \ln (1 - \chi) + \frac{1}{2} (4 - 8\chi + 7\chi^2 - 2\chi^3) \ln \chi \ln (1 - \chi) \].

The virtual corrections due to vacuum polarization were not taken into account in the expression for $\delta_V$. They give an additional contribution to the latter that looks:

$$\delta_{V_{II}} = \frac{2}{3} L - \frac{10}{9} - \frac{1}{3} (1 - \chi + \chi^2)^{-2} (2 - 3\chi + 3\chi^2 - \chi^3) \ln \chi.$$  

(14)

Consideration of emission of two soft photons having total energy $\omega_1 + \omega_2 \leq \Delta \varepsilon$ requires some caution. The final result (details of computations see in Appendix B) has the following form:

$$\frac{d\sigma_{SS}}{d\sigma_0} = \frac{1}{2!} \left( \frac{\alpha}{\pi} \right)^2 \left[ \delta_S^2 - \frac{8}{3} \pi^2 \left( L - 1 + \ln \frac{\chi}{1 - \chi} \right)^2 \right] \equiv \left( \frac{\alpha}{\pi} \right)^2 \Delta_{SS}.$$  

(15)

Note that in the case photons are emitted independently, i.e. $\omega_1 \leq \Delta \varepsilon$ and $\omega_2 \leq \Delta \varepsilon$, the second term in square brackets will be absent. The multiplier $1/2!$ is due to the identity of photons.

### 4 Conclusions

At first we would like to mention that using a set of integrals given in this paper one can perform a similar calculation for the case of Möller scattering.

What one would expect from the real 2–box amplitudes contribution is that the total correction must be free of infrared divergences supplying by cancellation of fourth and third power of large logarithms. We quote for completeness our final result derived by presenting it in the form

$$\Delta_{SV} + \Delta_{SS} = F^2 \left\{ L_\lambda^2 \cdot \left( L^2 - 2 \right) + L_\lambda^2 \cdot 4 \left( 1 - \ln \frac{\chi}{1 - \chi} \right) + L_\lambda L \cdot 2 + L^2 \cdot a_1 \right\},$$  

(16)

$$a_1 = 14 + 4 \ln^2 \left( \frac{\chi}{1 - \chi} \right) - 10 \ln \frac{\chi}{1 - \chi} + 2 \ln^2 \chi + 2 \ln^2 (1 - \chi) - 2f(\chi),$$

$$a_2 = -4 - \frac{8}{3} \pi^2 - 2 \ln^2 \left( \frac{\chi}{1 - \chi} \right) + 6 \ln \frac{\chi}{1 - \chi} - \ln^2 \chi + \ln^2 (1 - \chi) + f(\chi),$$

$$b = \frac{16}{3} \pi^2 \ln \frac{\chi}{1 - \chi} + \frac{10}{3} \pi^2 - 2 \ln \frac{\chi}{1 - \chi} \left( \ln^2 \chi - \ln^2 (1 - \chi) - f(\chi) + 4 \right) + 3 \ln^2 \chi - 3 \ln^2 (1 - \chi) + 6 \text{Li}_2(1 - \chi) - 6 \text{Li}_2(\chi),$$

$$z_1 = 12 \ln \frac{\chi}{1 - \chi} - \frac{8}{3} \pi^2 + 8 \text{Li}_2(1 - \chi) - 8 \text{Li}_2(\chi) + 4f(\chi),$$

8
\[ z_2 = -16 \left( 1 - \ln \frac{\chi}{1 - \chi} \right), \]

\[
\sum_{i=1}^{3} \Delta_i = F^2 \left[ L_\lambda^2 L^2 \cdot 2 + L_\lambda^2 L \cdot 4(L_{su} - L_{st} - 1) - L_\lambda L^3 \cdot 2 + L_\lambda L^2 \cdot \left( 6L_{st} - 6L_{su} \right) 
+ \frac{28}{3} + L^4 \cdot \frac{1}{2} + L^3 \cdot \left( 2L_{su} - 2L_{st} - \frac{11}{3} \right) \right] + L_\lambda L \cdot c_1 + L^2 \cdot c_2 + L \cdot d, \quad (17)
\]

\[
c_1 = L_{st}^2 \left( -6 \frac{s^2}{t^2} - 2 \frac{t^2}{s^2} - 25 \frac{s}{2 t} + \frac{19 t}{2} - \frac{15}{2} \right) + L_{st} L_{su} \left( 12 \frac{s^2}{t^2} + 4 \frac{t^2}{s^2} + 22 \frac{s}{t} + 12 \frac{t}{s} \right) 
+ 25 - L_{st} \left( \frac{56 s^2}{3 t^2} + \frac{34 t^2}{3 s^2} + \frac{79 t}{3} + \frac{98 s}{3 t} + 45 \right) + L_{su} \left( \frac{34}{3} F^2 - F + 1 \right) 
+ \pi^2 \left( -\frac{4 s^2}{3 t^2} + \frac{20 t^2}{3 s^2} + \frac{13 s}{3 t} + \frac{40 t}{3 s} + \frac{31}{2} \right) - \frac{158}{9} F^2,
\]

\[
c_2 = L_{st}^2 \left( \frac{3 s^2}{t^2} + \frac{t^2}{s^2} + \frac{29 s}{4 t} + \frac{19 t}{4 s} + \frac{39}{4} \right) + \frac{3}{2} F^2 L_{su}^2 + L_{st} L_{su} \left( -2F^2 
- F \left( \frac{4 s}{t} - 3 \right) + \frac{t}{s} + \frac{1}{2} \right) + L_{st} \left( 11F^2 - \frac{7}{3} F - \frac{11}{3} \left( \frac{t^2}{s^2} + \frac{t}{s} + 1 \right) + \frac{t}{2} + \frac{1}{2} \right) 
- L_{su} \left( \frac{22}{3} F^2 - \frac{1}{2} (F + 1) \right) - \pi^2 \left( \frac{1 s^2}{3 t^2} + \frac{4 t^2}{3 s^2} + \frac{25 s}{6 t} + \frac{14 s}{3} + \frac{7}{4} \right) - \frac{215}{18} F^2,
\]

\[
d = L_{st}^3 \left( -\frac{2 s^2}{t^2} - \frac{6 s}{t} - \frac{9 t}{4 s} - \frac{27}{4} \right) + \frac{1}{2} F^2 L_{su}^2 + L_{st} L_{su} \left( \frac{6 s^2}{t^2} + \frac{45 s}{4 t} + \frac{11 t}{4 s} + \frac{39}{4} \right) 
- L_{st} L_{su} \left( F \left( \frac{2 s}{t} + \frac{3}{2} \right) + \frac{3 s}{2 t} + \frac{3}{4} \right) - L_{st} \left( \frac{11 s^2}{4 t^2} + \frac{81 s}{4 t} + \frac{121 t}{12 s} + \frac{143}{12} \right) 
- L_{su} \left( \frac{4}{3} F + \frac{1}{2} \right) + L_{st} L_{su} \left( \frac{144 s^2}{3 t^2} + \frac{145 s}{6 t} + \frac{41 t}{6 s} + \frac{143}{6} \right) + \pi^2 L_{st} \left( \frac{4 s^2}{3 t^2} 
+ \frac{20 t^2}{3 s^2} + \frac{25 s}{3 t} + \frac{71 t}{6 s} + 8 \right) + \pi^2 L_{su} \left( -\frac{18 s^2}{3 t^2} - \frac{1 s}{6 t} - \frac{26 t}{3 s} - \frac{53}{12} \right) 
+ L_{st} \left( \frac{46 s^2}{3 t^2} + \frac{215 t^2}{9 s^2} + \frac{92 t^2}{9 s^2} + \frac{215 t}{18 s} + \frac{214 s}{9 s} - \frac{215}{6} \right) 
+ L_{su} \left( \frac{92}{9} F^2 - \frac{11 t}{6 s} - \frac{11 t}{6 s} \right) + \pi^2 \left( \frac{13 s^2}{18 t^2} + \frac{185 t^2}{18 s^2} + \frac{59 s}{9 t} + \frac{305 t}{18 s} + \frac{239}{12} \right) 
- \left( \frac{1313}{72} - 6 \zeta(3) \right) F^2.
\]

Here \( \Delta_{SS} \) and \( \Delta_{SV} \) denote quasi-elastic contributions, coming from double soft and soft–virtual photons emission. It immediately follows that all the terms proportional to \( L^4, L_\lambda^2 L^2, L_\lambda L^3 \) and \( L_\lambda^2 L \) disappear in the total sum. One must expect the cancellation of the third power of large logarithms as well as the rest of infrared singularities when contribution of 2–box diagrams will be taken into account. As for the terms containing \( L^2 \ln^2(\Delta \varepsilon/\varepsilon) \) and \( L^2 \ln(\Delta \varepsilon/\varepsilon) \), they are explicitly seen to agree with those which could be derived in the renormalization group approach. To show this, let us write down the expression for cross section according to the
renormalization group:

\[
\frac{d\sigma}{d\sigma_0} = \left(1 + \frac{\alpha}{2\pi}L\mathcal{P}_\Delta^{(1)} + \frac{1}{2!}\left(\frac{\alpha}{2\pi}L\right)^2 \mathcal{P}_\Delta^{(2)}\right)^4,
\]

\[
\mathcal{P}_\Delta^{(1)} = 2\ln\Delta + \frac{3}{2}, \quad \mathcal{P}_\Delta^{(2)} = \left(2\ln\Delta + \frac{3}{2}\right)^2 - 4\frac{\pi^2}{6}, \quad \Delta = \frac{\Delta\varepsilon}{\varepsilon}, \quad (18)
\]

and somewhat rewrite the main result of this paper:

\[
\Delta_{SS} + \Delta_{SV} + \sum_{i=1}^{3} \Delta_i = F^2 L^2 \left[\frac{1}{2}\mathcal{P}_\Delta^{(2)} + \frac{3}{2}\mathcal{P}_\Delta^{(1)}\right] + F^2 \cdot \frac{4}{3} \left[L_\lambda L^2 - L^3\right] + L_\lambda L \cdot \left[F^2 a_1 + c_1\right] + L \cdot \left[F^2 (b + z_1 + z_2) + d\right], \quad (19)
\]

Then, one can immediately be convinced that indeed an agreement takes place. Besides, the non–leading terms of the types \(L \ln^2(\Delta\varepsilon/\varepsilon), L \ln(\Delta\varepsilon/\varepsilon)\) are one of the new incomings obtained in this paper. We expect the 2–boxes contribution to compensate second, third and fourth terms on rhs of Eq. (19) and does modify the fifth one.

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References

[1] A.B. Arbuzov et al., Nucl. Phys. B 483 (1997) 83.
[2] A.B. Arbuzov et al., Nucl. Phys. B 474 (1996) 271; Phys. Atom. Nucl. 60 (1997) 591.
[3] G. Fäldt and P. Osland, Nucl. Phys. B 413 (1994) 16; B 413 (1994) 64.
[4] R. Barbieri, J. Miquaco and E. Remiddi, Il Nuovo Cimento A 11 (1972) 824.
[5] E.A. Kuraev, Preprint INP 80–155, Novosibirsk, 1980.

Appendix A. Set of integrals corresponding to the box–type Feynman diagrams

We consider the large–angle high–energy Bhabha elastic scattering process. An accuracy of formulae given below is predetermined by the omitted terms of order of \(m^2/s\) as compared to those of order unity.

The integrals for the scattering–type box diagram Fig. 2(m) with uncrossed photon lines are (see also [3])

\[
\int \frac{d^4k}{i\pi^2} \frac{\{1, k^\mu, k^\nu k^\nu\}}{A(p_1 + k)A(k)A(k - q)A(p_2 - k)} = \left\{G, \quad G_1 p_\mu^\mu + G_2 q^\mu, \right\}
\]
The integrals for the annihilation–type box diagram Fig. 2(p) with uncrossed photon lines are

\[
G_{00}g^{\mu\nu} + G_{11}p_{-}^\mu p_{-}^\nu + G_{22}q^\mu q^\nu + G_{33}p_{+}^\mu p_{+}^\nu + G_{12}(p_{-}^\mu q^\nu + p_{-}^\nu q^\mu) \}
\]

(A.1)

\[q = p_1' - p_1 = p_2 - p_2', \quad p_\pm = \frac{1}{2}(p_2 \pm p_1).
\]

Coefficients in the above expression look as follows:

\[
G = \frac{2}{st}L_u(L_t - L_\lambda), \quad G_1 = -\frac{2}{us}L_u(L_t - L_u) - \frac{2}{u}(\psi_s + \psi_t), \quad G_2 = \frac{1}{2}(G - G_1),
\]

\[
G_{00} = -\frac{s}{4}G_1 + \frac{1}{2}\psi_t, \quad G_{11} = \frac{s - t}{u}G_1 - \frac{4}{u}\psi_t + \frac{4}{tu}L_t + \frac{4}{us}L_s,
\]

\[
G_{22} = \frac{s - u}{2u}G_1 + \frac{1}{2}G + \frac{t - u}{st}L_s - \frac{1}{u}\psi_t + \frac{1}{tu}L_t, \quad G_{33} = G_1 - \frac{4}{st}L_t,
\]

\[
G_{12} = -\frac{s}{u}G_1 + \frac{2}{u}\psi_t - \frac{2}{tu}L_t - \frac{2}{us}L_s, \quad \psi_t = \frac{1}{t}\left(\frac{1}{2}L_t^2 + \frac{2}{3}\pi^2\right),
\]

\[
\psi_s = \frac{1}{s}\left(\frac{1}{2}L_s^2 + \frac{2}{3}\pi^2\right), \quad L_s = L - i\pi. \quad \text{(A.2)}
\]

The integrals for the scattering–type box diagram Fig. 2(n) with crossed photon lines are

\[
\int \frac{d^4k}{i\pi^2} A(p_1 + k)A(k)A(k - q)A(p_2' + k) = \left\{ \tilde{G}, \quad \tilde{G}_{11}p_{-}^\mu + \tilde{G}_{22}q^\mu, \quad \tilde{G}_{00}g^{\mu\nu} + \tilde{G}_{11}p_{-}^\mu p_{-}^\nu + \tilde{G}_{22}q^\mu q^\nu + \tilde{G}_{33}p_{+}^\mu p_{+}^\nu + \tilde{G}_{12}(\tilde{p}_{-}^\mu q^\nu + \tilde{p}_{-}^\nu q^\mu) \right\}, \quad \text{(A.3)}
\]

\[q = p_1' - p_1 = p_2 - p_2', \quad \tilde{p}_\pm = \frac{1}{2}(-p_2' \pm p_1).
\]

where

\[
\tilde{G} = \frac{2}{tu}L_u(L_t - L_\lambda), \quad \tilde{G}_1 = -\frac{2}{us}L_u(L_t - L_u) - \frac{2}{s}(\psi_s + \psi_t), \quad \tilde{G}_2 = \frac{1}{2}(\tilde{G} - \tilde{G}_1),
\]

\[
\tilde{G}_{00} = -\frac{u}{4}\tilde{G}_1 + \frac{1}{2}\psi_t, \quad \tilde{G}_{11} = \frac{u - t}{s}\tilde{G}_1 - \frac{4}{s}\psi_t + \frac{4}{st}L_t + \frac{4}{us}L_u,
\]

\[
\tilde{G}_{22} = \frac{u - s}{2s}\tilde{G}_1 + \frac{1}{2}\tilde{G} + \frac{t - s}{st}L_u - \frac{1}{s}\psi_t + \frac{1}{st}L_t, \quad \tilde{G}_{33} = \tilde{G}_1 - \frac{4}{st}L_t,
\]

\[
\tilde{G}_{12} = -\frac{u}{s}\tilde{G}_1 + \frac{2}{st}\psi_t - \frac{2}{st}L_t - \frac{2}{us}L_u, \quad \psi_u = \frac{1}{u}\left(\frac{1}{2}L_u^2 + \frac{\pi^2}{6}\right), \quad L_u = \ln\frac{-u}{m^2}. \quad \text{(A.4)}
\]

The integrals for the annihilation–type box diagram Fig. 2(p) with uncrossed photon lines are

\[
\int \frac{d^4k}{i\pi^2} A(p_1 + k)A(k)A(k - q_1)A(p_1' + k) = \left\{ H, \quad H_1p_{-}^\mu + H_2q^\mu, \quad H_{00}g^{\mu\nu} + H_{11}p_{-}^\mu p_{-}^\nu + H_{22}q^\mu q^\nu + H_{33}p_{+}^\mu p_{+}^\nu + H_{12}(p_{-}^\mu q^\nu + p_{-}^\nu q^\mu) \right\}, \quad \text{(A.5)}
\]

\[q_1 = -p_1 - p_2, \quad p_\pm' = \frac{1}{2}(-p_2' \pm p_1).
\]

Coefficients are:

\[
H = \frac{2}{st}L_u(L_s - L_\lambda), \quad H_1 = -\frac{2}{ut}L_t(L_s - L_t) - \frac{2}{u}(\psi_{1s} + \psi_t), \quad H_2 = \frac{1}{2}(H - H_1),
\]
\[ H_{00} = -\frac{t}{4}H_1 + \frac{1}{2}\psi_{1s}, \quad H_{11} = \frac{t-s}{u}H_1 - \frac{4}{u}\psi_{1s} + \frac{4}{us}L_s + \frac{4}{ut}L_t, \]
\[ H_{22} = \frac{t-u}{2u}H_1 + \frac{1}{2}H + \frac{s-u}{stu}L_t - \frac{1}{u}\psi_{1s} + \frac{1}{su}L_s, \quad H_{33} = H_1 - \frac{4}{stu}L_s, \]
\[ H_{12} = -\frac{t}{u}H_1 + \frac{2}{u}\psi_{1s} - \frac{2}{su}L_s - \frac{2}{stu}L_t, \quad \psi_{1s} = \frac{1}{s}\left(\frac{1}{2}L^2 + \frac{\pi^2}{6} - i\pi L\right), \quad (A.6) \]

The integrals for the annihilation–type box diagram Fig. 2(q) with crossed photon lines are
\[
\int \frac{d^4k}{i\pi^2} A(p_1 + k)A(k - q_1)A(p_2 + k) = \left\{ \widetilde{H}, \quad \widetilde{H}_{100}g^{\mu\nu} + \widetilde{H}_{111}p_1^\pm \psi + \widetilde{H}_{222}q_1^\mu + \widetilde{H}_{333}p_1^\nu + \widetilde{H}_{122}(p_1^\nu q_1^\mu + p_1^\mu q_1^\nu) \right\}, \quad (A.7)
\]
\[ q_1 = -p_1 - p_2, \quad \tilde{p}_\pm = \frac{1}{2}(-p_1^\pm \pm p_1). \]

And the corresponding coefficients have the form
\[
\widetilde{H} = \frac{2}{su}L_u(L_s - L_\lambda), \quad \widetilde{H}_{100} = \frac{2}{ut}L_u(L_s - L_\lambda) - \frac{2}{t}(\psi_{1s} + \psi_u), \quad \widetilde{H}_{22} = \frac{1}{2}(\widetilde{H} - \widetilde{H}_1),
\]
\[
\widetilde{H}_{11} = \frac{u}{t}H_1 + \frac{1}{2}\psi_{1s}, \quad \widetilde{H}_{12} = \frac{u}{t}\psi_{1s} - \frac{2}{stu}L_s - \frac{2}{tst}L_u.
\]
\[ (A.8) \]

**Appendix B. Two soft photons emission**

In this appendix we give the explicit expressions of integrals which can be encountered in considering a contribution coming from two soft photons emission. We have to calculate the following expression:
\[
\frac{d\sigma^{SS}}{d\sigma_0} = \frac{1}{2!} \left(\frac{-4\pi\alpha}{16\pi^3}\right)^2 \int \frac{d^3k_1}{\omega_1} \int \frac{d^3k_2}{\omega_2} \left( \frac{p_1'}{p_1k_1} - \frac{p_1}{p_1k_1} + \frac{p_2}{p_2k_2} - \frac{p_2'}{p_2k_2} \right)^2 \times \left( \frac{p_1'}{p_1k_2} - \frac{p_1}{p_1k_2} + \frac{p_2}{p_2k_2} - \frac{p_2'}{p_2k_2} \right)^2. \quad (B.1)
\]

The region of integration over the energies of photons obeys the strict inequality \( D : \omega_1 + \omega_2 < \Delta E \). The above formula when elaborated has three different structures. The first one looks as follows:
\[
m^2 \int \frac{d^3k_1}{\omega_1} \left( \frac{1}{(p_1'k_1)^2} + \frac{1}{(p_1k_1)^2} + \frac{1}{(p_2k_1)^2} + \frac{1}{(p_2'k_1)^2} \right)
= 16\pi \int \frac{\sigma^2}{\sqrt{1 + x_1^2}} \left(1 + \sigma^2 x_1^2\right)^{-1} \equiv 16\pi \int d\sigma f_1(x_1). \quad (B.2)
\]
Here \( x_1 = k_1 / \lambda, \sigma^2 = m^2 / \varepsilon^2 \) and the region of integration transforms into

\[ \mathcal{D}' : \quad 0 < x_1 + x_2 < N = \Delta E / \lambda. \]

The second reads:

\[
\int \frac{d^3 k_1}{\omega_1} \left( \frac{2p_1p_2}{p_1k_1 \cdot p_2k_1} + \frac{2p_1'p_2'}{p_1'k_1 \cdot p_2'k_1} \right) = 16\pi \int_{\mathcal{D}'} \frac{x_1 dx_1}{1 + x_1^2} \ln \left( \frac{x_1^2 + 1 + x_1}{\sigma^2 x_1^2 + 1} \right) \equiv 16\pi \int_{\mathcal{D}'} dx_1 f_2(x_1),
\]

and the third one is

\[
\int \frac{d^3 k_1}{\omega_1} \left( \frac{2p_1p_2}{p_1k_1 \cdot p_2k_1} + \frac{2p_1p_2'}{p_1k_1 \cdot p_2'k_1} - \frac{2p_1p_2'}{p_2k_1 \cdot p_2'k_1} - \frac{2p_1p_2'}{p_2k_1 \cdot p_2'k_1} \right) = 16\pi \int_{\mathcal{D}'} \frac{x_1 dx_1}{\sqrt{1 + x_1^2}} \ln \left( \frac{x_1^2 + 1 + x_1 \sqrt{1 + \frac{1}{2}}}{\sqrt{1 + x_1^2 + x_1 \sqrt{1 + \frac{1}{2}}} \sigma^2 x_1^2 + 1} \right) \equiv 16\pi \int_{\mathcal{D}'} dx_1 [f_3^u(x_1) - f_3^l(x_1)].
\]

Further evaluation gives for the terms of (1) * (1) type:

\[
\int_{\mathcal{D}'} d^2x f_1(x_1) f_1(x_2) = \left( \ln(2N) - \frac{1}{2}L \right)^2 - \frac{\pi^2}{6}.
\]

For those of (1) * (2) type we have:

\[
\int_{\mathcal{D}'} d^2x f_1(x_1) f_2(x_2) = -L \frac{\pi^2}{6} + \left( \ln(2N) - \frac{1}{2}L \right) \left[ L \left( \ln(2N) - \frac{1}{2}L \right) + \frac{1}{4}L^2 - \frac{\pi^2}{6} \right].
\]

And the remaining looks as follows:

(1) * (3) : \[
\int_{\mathcal{D}'} d^2x f_1(x_1) f_3^l(x_2) = \left( \ln(2N) - \frac{1}{2}L \right) \left[ L_u \left( \ln(2N) - \frac{1}{2}L \right) + \frac{1}{4}L^2 \right] - \frac{\pi^2}{6} + \Phi_u - \frac{\pi^2}{6} L_u,
\]

(2) * (2) : \[
\int_{\mathcal{D}'} d^2x f_2(x_1) f_2(x_2) = \left[ L \left( \ln(2N) - \frac{1}{2}L \right) + \frac{1}{4}L^2 - \frac{\pi^2}{6} \right]^2 - L^2 \frac{\pi^2}{6},
\]

(2) * (3) : \[
\int_{\mathcal{D}'} d^2x f_2(x_1) f_3^u(x_2) = \left[ L \left( \ln(2N) - \frac{1}{2}L \right) + \frac{1}{4}L^2 - \frac{\pi^2}{6} \right] L_u \left( \ln(2N) - \frac{1}{2}L \right) + \frac{1}{4}L^2 - \frac{\pi^2}{6},
\]

(3) * (3) : \[
\int_{\mathcal{D}'} d^2x f_3^l(x_1) f_3^l(x_2) = \left[ L_i \left( \ln(2N) - \frac{1}{2}L \right) + \frac{1}{4}L^2 - \frac{\pi^2}{6} + \Phi_i \right] L_u \left( \ln(2N) - \frac{1}{2}L \right) + \frac{1}{4}L^2 - \frac{\pi^2}{6},
\]

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\[ i = u, t, \quad L_u = \ln \frac{-u}{m^2}, \quad L_t = \ln \frac{-t}{m^2}, \quad \Phi_u = \text{Li}_2 \left( \frac{s + u}{s} \right), \quad \Phi_t = \text{Li}_2 \left( \frac{s + t}{s} \right). \]

Summing up all the expressions derived and making definite rearrangements we arrive to the final result given above in (15).