A COMPUTER-ASSISTED UNIQUENESS PROOF FOR A SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

PATRICK J. MCKENNA, FILOMENA PACELLA, MICHAEL PLUM, AND DAGMAR ROTH

Abstract. A wide variety of articles, starting with the famous paper [11], is devoted to the uniqueness question for the semilinear elliptic boundary value problem $-\Delta u = \lambda u + u^p$ in $\Omega$, $u > 0$ in $\Omega$, $u = 0$ on $\partial \Omega$, where $\lambda$ ranges between 0 and the first Dirichlet Laplacian eigenvalue. So far, this question was settled in the case of $\Omega$ being a ball and, for more general domains, in the case $\lambda = 0$. In [16], we proposed a computer-assisted approach to this uniqueness question, which indeed provided a proof in the case $\Omega = (0,1)^2$, and $p = 2$. Due to the high numerical complexity, we were not able in [16] to treat higher values of $p$. Here, by a significant reduction of the complexity, we will prove uniqueness for the case $p = 3$.

Dedicated to the memory of Wolfgang Walter

1. Introduction

The semilinear elliptic boundary value problem

$$-\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega \quad (1.1)$$

has attracted a lot of attention since the 19th century. Questions of existence and multiplicity have been (are still being) extensively studied by means of variational methods, fixed-point methods, sub- and supersolutions, index and degree theory, and more.

In this article, we will address the question of uniqueness of solutions for the more special problem

$$\begin{cases} 
-\Delta u = \lambda u + u^p & \text{in } \Omega \\
 u > 0 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega 
\end{cases} \quad (1.2)$$

where $\lambda$ ranges between 0 and $\lambda_1(\Omega)$, the first eigenvalue of the Dirichlet Laplacian. It has been shown in a series of papers [19], [29], [28], [1], [2] that the solution of (1.2) is indeed unique when $\Omega$ is a ball, or when $\Omega$ is more general but $\lambda = 0$ ([30], [12], [9], [10]).

We will concentrate on the case where $\Omega = (0,1)^2$ and $p = 3$, and prove that uniqueness holds for the full range $[0, \lambda_1(\Omega))$ of $\lambda$. Thus, our paper constitutes the first uniqueness result for this situation. More precisely we prove

**Theorem 1.1.** Let $\Omega$ be the unit square in $\mathbb{R}^2$, $\Omega = (0,1)^2$. Then the problem

$$\begin{cases} 
-\Delta u = \lambda u + u^3 & \text{in } \Omega \\
 u > 0 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega 
\end{cases} \quad (1.3)$$

admits only one solution for any $\lambda \in [0, \lambda_1(\Omega))$. 

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Remark 1.1. a) A simple scaling argument shows that our uniqueness result carries over to all squares $\Omega_l := (0, l)^2$ (and thus, to all squares in $\mathbb{R}^2$): If $u$ is a positive solution of $-\Delta u = \lambda u + u^3$ in $\Omega_l$, $u = 0$ on $\partial \Omega_l$, for some $\lambda \in [0, \lambda_1(\Omega_l))$, then $v(x, y) := lu(lx, ly)$ is a solution of (1.3) for $\lambda = \lambda l^2 \in [0, \lambda_1(\Omega))$.

b) Since we also show that the unique solution in the square is nondegenerate, by a result of [10] we deduce that the solution is unique also in domains “close to” a square.

c) Finally we observe that having shown in [16] (case $p = 2$) and in this paper (case $p = 3$) that the unique solution is nondegenerate then uniqueness follows also for other nonlinearities of the type $\lambda u + u^p$ for $p$ close to 2 and 3. Indeed, by standard arguments (see for example [9]) nonuniqueness of positive solutions in correspondence to sequences of exponents converging to 3 (resp. to 2) would imply degeneracy of the solution for $p = 3$ (resp. $p = 2$).

Our proof heavily relies on computer-assistance. Such computer-assisted proofs are receiving an increasing attention in the recent years since such methods provided results which apparently could not be obtained by purely analytical means (see [6], [5], [24], [17], [18]).

We compute a branch of approximate solutions and prove existence of a true solution branch close to it, using fixed point techniques. By eigenvalue enclosure methods, and an additional analytical argument for $\lambda$ close to $\lambda_1(\Omega)$ we deduce the non-degeneracy of all solutions along this branch, whence uniqueness follows from the known bifurcation structure of the problem.

In [16] we give a general description of these computer-assisted means and use them to obtain the desired uniqueness result for the case $\Omega = (0, 1)^2$, $p = 2$. To make the present paper dealing with the case $p = 3$ more self-contained, we recall parts of the content of [16] here. We remark that the numerical tools used in [16] turned out not to be sufficient to treat the case $p = 3$. Now, by some new trick to reduce the numerical complexity, we are able to handle this case.

2. Preliminaries

In the following, let $\Omega = (0, 1)^2$. We remark that the results of this section can be carried over to the more general case of a “doubly symmetric” domain; see [16] for details.

First, note that problem (1.2) can equivalently be reformulated as finding a non-trivial solution of

$$
\begin{cases}
-\Delta u = \lambda u + |u|^p & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(2.1)
since, for $\lambda < \lambda_1(\Omega)$, by the strong maximum principle (for $-\Delta - \lambda$) every non-trivial solution of (2.1) is positive in $\Omega$. In fact, this formulation is better suited for our computer-assisted approach than (1.2).

As a consequence of the classical bifurcation theorem of [25] and of the results of [9] the following result was obtained in [20]:

**Theorem 2.1.** All solutions $u_\lambda$ of (1.2) lie on a simple continuous curve $\Gamma$ in $[0, \lambda_1(\Omega)) \times C^{1,\alpha}(\bar{\Omega})$ joining $(\lambda_1(\Omega), 0)$ with $(0, u_0)$, where $u_0$ is the unique solution of (1.2) for $\lambda = 0$.

We recall that the uniqueness of the solution of (1.2) for $\lambda = 0$ was proved in [10] and [9]. As a consequence of the previous theorem we have
Corollary 2.1. If all solutions on the curve \( \Gamma \) are nondegenerate then problem (1.2) admits only one solution for every \( \lambda \in [0, \lambda_1(\Omega)) \).

Proof. The nondegeneracy of the solutions implies, by the Implicit Function Theorem, that neither turning points nor secondary bifurcations can exist along \( \Gamma \). Then, for every \( \lambda \in [0, \lambda_1(\Omega)) \) there exists only one solution of (1.2) on \( \Gamma \). By Theorem 2.1 all solutions are on \( \Gamma \), hence uniqueness follows.

Theorem 2.1 and Corollary 2.1 indicate that to prove the uniqueness of the solution of problem (1.2) for every \( \lambda \in [0, \lambda_1(\Omega)) \) it is enough to construct a branch of nondegenerate solutions which connects \((0, u_0)\) to \((\lambda_1(\Omega), 0)\). This is what we will do numerically in the next sections with a rigorous computer-assisted proof.

However, establishing the nondegeneracy of solutions \( u_\lambda \) for \( \lambda \) close to \( \lambda_1(\Omega) \) numerically can be difficult, due to the fact that the only solution at \( \lambda = \lambda_1(\Omega) \), which is the identically zero solution, is obviously degenerate because its linearized operator is \( L_0 = -\Delta - \lambda_1 \) which has the first eigenvalue equal to zero. The next proposition shows that there exists a computable number \( \bar{\lambda}(\Omega) \in (0, \lambda_1(\Omega)) \) such that for any \( \lambda \in [\bar{\lambda}(\Omega), \lambda_1(\Omega)) \) problem (1.2) has only one solution which is also nondegenerate. Of course, from the well-known results of Crandall and Rabinowitz, \([7, 8]\), one can establish that for \( \lambda \) “close to” \( \lambda_1 \), all solutions \( u_\lambda \) are nondegenerate. However, in order to complete our program, we need to calculate a precise and explicit estimate of how close they need to be. This allows us to carry out the numerical computation only in the interval \([0, \bar{\lambda}(\Omega))\) as we will do later.

Let us denote by \( \lambda_1 = \lambda_1(\Omega) \) and \( \lambda_2 = \lambda_2(\Omega) \) the first and second eigenvalue of the operator \(-\Delta\) in \( \Omega \) with homogeneous Dirichlet boundary conditions. We have

Proposition 2.1. If there exists \( \bar{\lambda} \in (0, \lambda_1) \) and a solution \( u_{\bar{\lambda}} \) of (1.2) with \( \lambda = \bar{\lambda} \) such that

\[
\|u_{\bar{\lambda}}\|_\infty < \left( \frac{\lambda_2 - \lambda_1}{p} \right)^{\frac{1}{p-1}} \cdot \left( \frac{\bar{\lambda}}{\lambda_1} \right)^{\frac{1}{p-1}} \tag{2.2}
\]

then

\[
\|u_\lambda\|_\infty < \left( \frac{\lambda_2 - \lambda_1}{p} \right)^{\frac{1}{p-1}} \tag{2.3}
\]

and \( u_\lambda \) is non-degenerate, for all solutions \( u_\lambda \) of (1.2) belonging to the branch \( \Gamma_2 \subset \Gamma \) which connects \((\bar{\lambda}, u_{\bar{\lambda}})\) to \((\lambda_1, 0)\).

(Recall that \( \Gamma \) is the unique continuous branch of solutions given by Theorem 2.1)

Proof. see \([16]\)

Corollary 2.2. If on the branch \( \Gamma \) there exists a solution \( u_{\bar{\lambda}}, \bar{\lambda} \in (0, \lambda_1) \) such that:

i) on the sub-branch \( \Gamma_1 \) connecting \((0, u_0)\) with \((\bar{\lambda}, u_{\bar{\lambda}})\) all solutions are nondegenerate and

\[
\|u_{\bar{\lambda}}\|_\infty < \left( \frac{\lambda_2 - \lambda_1}{p} \right)^{\frac{1}{p-1}} \cdot \left( \frac{\bar{\lambda}}{\lambda_1} \right)^{\frac{1}{p-1}, \frac{1}{p-1}} \tag{2.4}
\]

then all solutions of (1.2) are nondegenerate, for all \( \lambda \in (0, \lambda_1) \), and therefore problem (1.2) admits only one solution for every \( \lambda \in [0, \lambda_1(\Omega)) \).
Proof. We set \( \Gamma = \Gamma_1 \cup \Gamma_2 \) with \( \Gamma_1 \) connecting \((0, u_0)\) to \((\lambda, u_\lambda)\). On \( \Gamma_1 \) we have that all solutions are nondegenerate by i). On the other hand the hypothesis ii) allows us to apply Proposition 2.1 which shows nondegeneracy of all solutions on \( \Gamma_2 \). Hence there is nondegeneracy all along \( \Gamma \) so the assertion follows from Corollary 2.1.

The last corollary suggests the method of proving the uniqueness through computer assistance: first we construct a branch of nondegenerate “true” solutions near approximate ones in a certain interval \([0, \overline{\lambda}]\) and then verify ii) for the solution \( u_\lambda \). Note that the estimate (2.4) depends only on \( p \) and on the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of the operator \(-\Delta\) in the domain \( \Omega \). So the constant on the right-hand side is easily computable. When \( \Omega \) is the unit square which is the case analyzed in the next sections, the estimate (2.4) becomes:

\[
\|u_\lambda\|_\infty < \left( \frac{3\pi^2}{p} \right)^{\frac{1}{p-1}} \cdot \left( \frac{\lambda_1}{2\pi^2} \right)^{\frac{1}{p-1}} = \left( \frac{3\lambda_2}{2p} \right)^{\frac{1}{p-1}}
\]

because \( \lambda_1 = 2\pi^2 \) and \( \lambda_2 = 5\pi^2 \).

Fixing \( p = 3 \) we finally get the condition

\[
\|u_\lambda\|_\infty < \sqrt{\frac{\lambda}{2}}. \tag{2.5}
\]

3. The basic existence and enclosure theorem

We start the computer-assisted part of our proof with a basic theorem on existence, local uniqueness, and non-degeneracy of solutions to problem (2.1), assuming \( p = 3 \) now for simplicity of presentation. In this section, the parameter \( \lambda \in [0, \lambda_1(\Omega)) \) is fixed.

Let \( H^1_0(\Omega) \) be endowed with the inner product \( \langle u, v \rangle_{H^1_0} := \langle \nabla u, \nabla v \rangle_{L^2} + \sigma \langle u, v \rangle_{L^2} \); actually we choose \( \sigma = 1 \) in this paper, but different (usually positive) choices of \( \sigma \) are advantageous or even mandatory in other applications, whence we keep \( \sigma \) as a parameter in the following.

Let \( H^{-1}(\Omega) \) denote the (topological) dual of \( H^1_0(\Omega) \), endowed with the usual operator sup \(-\) norm.

Suppose that an approximate solution \( \omega_\lambda \in H^1_0(\Omega) \) of problem (2.1) has been computed by numerical means, and that a bound \( \delta_\lambda > 0 \) for its defect is known, i.e.

\[
\| -\Delta \omega_\lambda - \lambda \omega_\lambda - |\omega_\lambda|^3 \|_{H^{-1}} \leq \delta_\lambda, \tag{3.1}
\]

as well as a constant \( K_\lambda \) such that

\[
\|v\|_{H^1_0} \leq K_\lambda \|L(\lambda, \omega_\lambda)[v]\|_{H^{-1}} \text{ for all } v \in H^1_0(\Omega). \tag{3.2}
\]

Here, \( L(\lambda, \omega_\lambda) \) denotes the operator linearizing problem (2.1) at \( \omega_\lambda \); more generally, for \((\lambda, u) \in \mathbb{R} \times H^1_0(\Omega)\), let the linear operator \( L(\lambda, u) : H^1_0(\Omega) \to H^{-1}(\Omega) \) be defined by

\[
L(\lambda, u)[v] := -\Delta v - \lambda v - 3|u|^2uv \quad (v \in H^1_0(\Omega)). \tag{3.3}
\]

The practical computation of bounds \( \delta_\lambda \) and \( K_\lambda \) will be addressed in Sections 6, 7 and 8. Let \( C_4 \) denote a norm bound (embedding constant) for the embedding \( H^1_0(\Omega) \hookrightarrow L^4(\Omega) \), which is bounded since \( \Omega \subset \mathbb{R}^2 \). \( C_4 \) can be calculated e.g. according to the explicit formula given in [23, Lemma 2]. Finally, let

\[
\gamma := 3C_4^3.
\]

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In our example case where $\Omega = (0, 1)^2$, the above-mentioned explicit formula gives (with the choice $\sigma := 1$)
\[
\gamma = \frac{3\sqrt{2}}{4(\pi^2 + 1)^{3/4}} \left( < \frac{1}{5} \right).
\]

**Theorem 3.1.** Suppose that some $\alpha_\lambda > 0$ exists such that
\[
\delta_\lambda \leq \frac{\alpha_\lambda}{K_\lambda} - \gamma\alpha_\lambda^2 (\|\omega_\lambda\|_{L^4} + C_4\alpha_\lambda)
\]
and
\[
2K_\lambda\gamma\alpha_\lambda (\|\omega_\lambda\|_{L^4} + C_4\alpha_\lambda) < 1.
\]
Then, the following statements hold true:

a) (existence) There exists a solution $u_\lambda \in H^1_0(\Omega)$ of problem (2.1) such that
\[
\|u_\lambda - \omega_\lambda\|_{H^1_0} \leq \alpha_\lambda.
\]

b) (local uniqueness) Let $\eta > 0$ be chosen such that (3.5) holds with $\alpha_\lambda + \eta$ instead of $\alpha_\lambda$. Then,
\[

\begin{align*}
\|u - \omega_\lambda\|_{H^1_0} \leq \alpha_\lambda + \eta
\end{align*}
\implies u = u_\lambda.
\]

c) (nondegeneracy)
\[

\begin{align*}
\|u - \omega_\lambda\|_{H^1_0} \leq \alpha_\lambda \implies L(\lambda, u) : H^1_0(\Omega) \to H^{-1}(\Omega) \text{ is bijective,}
\end{align*}
\]
whence in particular $L(\lambda, u_\lambda)$ is bijective (by (3.6)).

For a proof, see [16].

**Corollary 3.1.** Suppose that (3.4) and (3.5) hold, and in addition that $\|\omega_\lambda\|_{H^1_0} > \alpha_\lambda$. Then, the solution $u_\lambda$ given by Theorem 3.1 is non-trivial (and hence positive).

Remark 3.1. a) The function $\psi(\alpha) := \frac{\alpha}{K_\lambda} - \gamma\alpha^2 (\|\omega_\lambda\|_{L^4} + C_4\alpha)$ has obviously a positive maximum at $\bar{\alpha} = \frac{1}{3C_4} \left( \sqrt{\|\omega_\lambda\|_{L^4}^2 + \frac{2C_4}{\gamma K_\lambda^2}} - \|\omega_\lambda\|_{L^4} \right)$, and the crucial condition (3.4) requires that
\[
\delta_\lambda \leq \psi(\bar{\alpha}) = \frac{4C_4 + \gamma K_\lambda \|\omega_\lambda\|_{L^4}^2}{K_\lambda \left( \sqrt{\gamma K_\lambda^2(\gamma K_\lambda \|\omega_\lambda\|_{L^4}^2 + 3C_4) + \gamma K_\lambda \|\omega_\lambda\|_{L^4}} \right) - \gamma K_\lambda \|\omega_\lambda\|_{L^4} + 6C_4} \cdot \frac{1}{\left( \sqrt{\gamma K_\lambda^2(\gamma K_\lambda \|\omega_\lambda\|_{L^4}^2 + 3C_4) + \gamma K_\lambda \|\omega_\lambda\|_{L^4} + 6C_4} \right)}
\]
i.e. $\delta_\lambda$ has to be sufficiently small. According to (3.1), this means that $\omega_\lambda$ must be computed with sufficient accuracy, which leaves the “hard work” to the computer!
Furthermore, a “small” defect bound $\delta_\lambda$ allows (via (3.4)) a “small” error bound $\alpha_\lambda$, if $K_\lambda$ is not too large.

b) If moreover we choose the minimal $\alpha_\lambda$ satisfying (3.4), then the additional condition (3.5) follows automatically, which can be seen as follows: the minimal choice of $\alpha_\lambda$ shows
that $\alpha_\lambda \leq \bar{\alpha}$. We have

$$2K\gamma\bar{\alpha}(\|\omega\|_{L^4} + C_4\bar{\alpha}) =
1 - \frac{C_4}{3C_4 + 2\gamma K\|\omega\|_{L^4}^2 + 2\sqrt{\gamma K\bar{\lambda}}(\gamma K\bar{\lambda}\|\omega\|_{L^4}^2 + 3C_4)\|\omega\|_{L^4}} < 1 \quad (3.10)$$

and thus condition (3.5) is satisfied.

Since we will anyway try to find $\alpha_\lambda$ (satisfying (3.4)) close to the minimal one, condition (3.5) is “practically” always satisfied if (3.4) holds. (Nevertheless, it must of course be checked.)

4. The branch $(u_\lambda)$

Fixing some $\bar{\lambda} \in (0, \lambda_1(\Omega))$ (the actual choice of which is made on the basis of Proposition 2.1; see also Section 5), we assume now that for every $\lambda \in [0, \bar{\lambda}]$ an approximate solution $\omega_\lambda \in H^1_0(\Omega)$ is at hand, as well as a defect bound $\delta_\lambda$ satisfying (3.1), and a bound $K_\lambda$ satisfying (3.2). Furthermore, we assume now that, for every $\lambda \in [0, \bar{\lambda}]$, some $\alpha_\lambda > 0$ satisfies (3.4) and (3.5), and the additional non-triviality condition $\|\omega_\lambda\|_{H^1_0} > \alpha_\lambda$ (see Corollary 3.1). We suppose that some uniform ($\lambda$-independent) $\eta > 0$ can be chosen such that (3.5) holds with $\alpha_\lambda + \eta$ instead of $\alpha_\lambda$ (compare Theorem 3.1 b)). Hence Theorem 3.1 gives a positive solution $u_\lambda \in H^1_0(\Omega)$ of problem (2.1) with the properties (3.6), (3.7), (3.8), for every $\lambda \in [0, \bar{\lambda}]$.

Finally, we assume that the approximate solution branch $([0, \bar{\lambda}] \to H^1_0(\Omega), \lambda \mapsto \omega_\lambda)$ is continuous, and that $([0, \bar{\lambda}] \to \mathbb{R}, \lambda \mapsto \omega_\lambda)$ is lower semi-continuous.

In Sections 6, 7 and 8, we will address the actual computation of such branches $(\omega_\lambda), (\delta_\lambda), (K_\lambda), (\alpha_\lambda)$.

So far we know nothing about continuity or smoothness of $([0, \bar{\lambda}] \to H^1_0(\Omega), \lambda \mapsto u_\lambda)$, which however we will need to conclude that $(u_\lambda)_{\lambda \in [0, \bar{\lambda}]}$ coincides with the sub-branch $\Gamma_4$ introduced in Corollary 2.2.

Theorem 4.1. The solution branch

$$\begin{align*}
\left\{ \begin{array}{l}
[0, \bar{\lambda}] \to H^1_0(\Omega) \\
\lambda \mapsto u_\lambda
\end{array} \right. 
\end{align*}$$

is continuously differentiable.

Proof: The mapping

$$\mathcal{F} : \begin{cases}
\mathbb{R} \times H^1_0(\Omega) & \to H^{-1}(\Omega) \\
(\lambda, u) & \mapsto -\Delta u - \lambda u - |u|^3
\end{cases}$$

is continuously differentiable, with $(\partial \mathcal{F}/\partial u)(\lambda, u) = L(\lambda, u)$ (see (3.3)), and $\mathcal{F}(\lambda, u_\lambda) = 0$ for all $\lambda \in [0, \bar{\lambda}]$. Using the Mean Value Theorem one can show that $L(\lambda, u)$ depends indeed continuously on $(\lambda, u)$; see [16] Lemma 3.1 for details.

It suffices to prove the asserted smoothness locally. Thus, fix $\lambda_0 \in [0, \bar{\lambda}]$. Since $L(\lambda_0, u_{\lambda_0})$ is bijective by Theorem 3.1 e), the Implicit Function Theorem gives a $C^1$-smooth solution branch

$$\begin{align*}
\left\{ \begin{array}{l}
(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to H^1_0(\Omega) \\
\lambda \mapsto \hat{u}_\lambda
\end{array} \right. 
\end{align*}$$

to problem (2.1), with $\hat{u}_{\lambda_0} = u_{\lambda_0}$. By (3.6),

$$\|\hat{u}_{\lambda_0} - \omega_{\lambda_0}\|_{H^1_0} \leq \alpha_{\lambda_0} \ . \quad (4.1)$$
Since $\hat{u}_\lambda$ and $\omega_\lambda$ depend continuously on $\lambda$, and $\alpha_\lambda$ lower semi-continuously, (4.1) implies 
\[ \|\hat{u}_\lambda - \omega_\lambda\|_{H^1_0} \leq \alpha_\lambda + \eta \quad (\lambda \in [0, \bar{\lambda}] \cap (\lambda_0 - \bar{\varepsilon}, \lambda_0 + \bar{\varepsilon})) \]
for some $\bar{\varepsilon} \in (0, \varepsilon)$. Hence Theorem 3.1 b) provides 
\[ \hat{u}_\lambda = u_\lambda \quad (\lambda \in [0, \bar{\lambda}] \cap (\lambda_0 - \bar{\varepsilon}, \lambda_0 + \bar{\varepsilon})) , \]
implying the desired smoothness in some neighborhood of $\lambda_0$ (which of course is one-sided if $\lambda_0 = 0$ or $\lambda_0 = \bar{\lambda}$).

As a consequence of Theorem 4.1, $u_\lambda$ is a continuous solution curve connecting the point $(0, u_0)$ with $(\bar{\lambda}, u_\lambda)$, and thus must coincide with the sub-branch $\Gamma_1$, connecting these two points, of the unique simple continuous curve $\Gamma$ given by Theorem 2.1. Using Theorem 5.1 c), we obtain

**Corollary 4.1.** On the sub-branch $\Gamma_1$ of $\Gamma$ which connects $(0, u_0)$ with $(\bar{\lambda}, u_\lambda)$, all solutions are nondegenerate.

Thus, if we can choose $\bar{\lambda}$ such that condition (2.4) holds true, Corollary 2.2 will give the desired uniqueness result.

### 5. Choice of $\bar{\lambda}$

We have to choose $\bar{\lambda}$ such that condition (2.4) is satisfied. For this purpose, we use computer-assistance again. With $x_M$ denoting the intersection of the symmetry axes of the (doubly symmetric) domain $\Omega$, i.e. $x_M = (\frac{1}{2}, \frac{1}{2})$ for $\Omega = (0, 1)^2$, we choose $\bar{\lambda} \in (0, \lambda_1(\Omega))$, not too close to $\lambda_1(\Omega)$, such that our approximate solution $\omega_\lambda$ satisfies

\[ \omega_\lambda(x_M) < \left( \frac{\lambda_2(\Omega) - \lambda_1(\Omega)}{3} \right)^{\frac{1}{2}} \left( \frac{\bar{\lambda}}{\lambda_1(\Omega)} \right)^{\frac{1}{2}} \]

with “not too small” difference between right- and left-hand side. Such a $\bar{\lambda}$ can be found within a few numerical trials.

Here, we impose the additional requirement

\[ \omega_\lambda \in H^2(\Omega) \cap H^1_0(\Omega) , \]

which is in fact a condition on the numerical method used to compute $\omega_\lambda$. (Actually, condition (5.2) could be avoided if we were willing to accept additional technical effort.) Moreover, exceeding (3.1), we will now need an $L^2$-bound $\hat{\delta}_\lambda$ for the defect:

\[ \| - \Delta \omega_\lambda - \bar{\lambda} \omega_\lambda - |\omega_\lambda|^3 \|_{L^2} \leq \hat{\delta}_\lambda \]

(5.3)

Finally, we note that $\Omega$ is convex, and hence in particular $H^2$-regular, whence every solution $u \in H^1_0(\Omega)$ of problem (2.1) is in $H^2(\Omega)$.

Using the method described in Section 3, we obtain, by Theorem 3.1 a), a positive solution $u_\lambda \in H^2(\Omega) \cap H^1_0(\Omega)$ of problem (2.1) satisfying

\[ \|u_\lambda - \omega_\lambda\|_{H^1_0} \leq \alpha_\lambda \]

(5.4)

provided that (3.4) and (3.5) hold, and that $\|\omega_\lambda\|_{H^1_0} > \alpha_\lambda$.

Now we make use of the explicit version of the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ given in [21]. There, explicit constants $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$ are computed such that

\[ \|u\|_{\infty} \leq \hat{C}_0 \|u\|_{L^2} + \hat{C}_1 \|\nabla u\|_{L^2} + \hat{C}_2 \|u_{xx}\|_{L^2} \quad \text{for all } u \in H^2(\Omega) , \]
with \(\|u_{xx}\|_{L^2}\) denoting the \(L^2\)-Frobenius norm of the Hessian matrix \(u_{xx}\). E.g. for \(\Omega = (0, 1)^2\), \cite{21} gives

\[
\hat{C}_0 = 1, \quad \hat{C}_1 = 1.1548 \cdot \sqrt{\frac{2}{3}} \leq 0.9429, \quad \hat{C}_2 = 0.22361 \cdot \sqrt{\frac{28}{45}} \leq 0.1764.
\]

Moreover, \(\|u_{xx}\|_{L^2} \leq \|\Delta u\|_{L^2}\) for \(u \in H^2(\Omega) \cap H^1_0(\Omega)\) since \(\Omega\) is convex (see e.g. \cite{14}). Consequently,

\[
\|u_\lambda - \omega_\lambda\|_\infty \leq \hat{C}_0\|u_\lambda - \omega_\lambda\|_{L^2} + \hat{C}_1\|u_\lambda - \omega_\lambda\|_{H^1_0} + \hat{C}_2\|\Delta u_\lambda - \Delta \omega_\lambda\|_{L^2}. \tag{5.5}
\]

To bound the last term on the right-hand side, we first note that

\[
\|[u_\lambda] - [\omega_\lambda]\|_{L^2} = \left\|3 \int_0^1 [\omega_\lambda + t(u_\lambda - \omega_\lambda)](\omega_\lambda + t(u_\lambda - \omega_\lambda))dt \cdot (u_\lambda - \omega_\lambda) \right\|_{L^2}
\]

\[
\leq 3 \int_0^1 \|[\omega_\lambda + t(u_\lambda - \omega_\lambda)]^2 \cdot |u_\lambda - \omega_\lambda|\|_{L^2}dt
\]

\[
\leq 3 \int_0^1 \|[\omega_\lambda + t(u_\lambda - \omega_\lambda)]^2\|_{L^2} \|u_\lambda - \omega_\lambda\|_{L^2}dt
\]

\[
\leq 3 \int_0^1 (\|[\omega_\lambda]\|_{L^6} + tC_6\alpha_\lambda)^2 dt \cdot C_6\alpha_\lambda \tag{5.6}
\]

\[
= 3C_6 \left(\|[\omega_\lambda]\|_{L^6}^2 + C_6\|\omega_\lambda\|_{L^6}\alpha_\lambda + \frac{1}{3}C_6^2\alpha_\lambda^2\right)\alpha_\lambda, \tag{5.7}
\]

using \(5.4\) and an embedding constant \(C_6\) for the embedding \(H^1_0(\Omega) \rightarrow L^6(\Omega)\) in the last but one line; see e.g. \cite{23, Lemma 2} for its computation. Moreover, by \(5.4\) and \(5.3\),

\[
\|\Delta u_\lambda - \Delta \omega_\lambda\|_{L^2} \leq \hat{\delta}_\lambda + \hat{\lambda}\|u_\lambda - \omega_\lambda\|_{L^2} + \|[u_\lambda] - [\omega_\lambda]\|_{L^2}. \tag{5.8}
\]

Using \(5.4\) - \(5.8\), and the Poincaré inequality

\[
\|u\|_{L^2} \leq \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}}\|u\|_{H^1_0}(u \in H^1_0(\Omega)), \tag{5.9}
\]

we finally obtain

\[
\|u_\lambda - \omega_\lambda\|_\infty \leq \left[\frac{\hat{C}_0 + \hat{\lambda}\hat{C}_2}{\sqrt{\lambda_1(\Omega) + \sigma}} + \hat{C}_1 + 3C_6\hat{C}_2 \left(\|[\omega_\lambda]\|_{L^6}^2 + C_6\|\omega_\lambda\|_{L^6}\alpha_\lambda + \frac{1}{3}C_6^2\alpha_\lambda^2\right)\right] \cdot \alpha_\lambda + \hat{C}_2\hat{\delta}_\lambda, \tag{5.10}
\]

and the right-hand side is “small” if \(\alpha_\lambda\) and \(\hat{\delta}_\lambda\) are “small”, which can (again) be achieved by sufficiently accurate numerical computations.

Finally, since

\[
u_\lambda(x_M) \leq \omega_\lambda(x_M) + \|u_\lambda - \omega_\lambda\|_\infty,
\]

\(5.10\) yields an upper bound for \(u_\lambda(x_M)\) which is “not too much” larger than \(\omega_\lambda(x_M)\). Hence, since \(u_\lambda(x_M) = \|[u_\lambda]\|_\infty\) by \(11\), condition \(2.4\) can easily be checked, and \(5.1\) (with “not too small” difference between right- and left-hand side) implies a good chance that this check will be successful; otherwise, \(\hat{\lambda}\) has to be chosen a bit larger.
6. Computation of $\omega_{\lambda}$ and $\delta_{\lambda}$ for fixed $\lambda$

In this section we report on the computation of an approximate solution $\omega_{\lambda} \in H^2(\Omega) \cap H_0^1(\Omega)$ to problem (2.1), and of bounds $\delta_{\lambda}$ and $K_{\lambda}$ satisfying (3.1) and (3.2), where $\lambda \in [0, \lambda_1(\Omega))$ is fixed (or one of finitely many values). We will again restrict ourselves to the unit square $\Omega = (0,1)^2$.

An approximation $\omega_{\lambda}$ is computed by a Newton iteration applied to problem (2.1), where the linear boundary value problems

$$L_{(\lambda,\omega_{\lambda}^{(n)})}[v_n] = \Delta \omega_{\lambda}^{(n)} + \lambda \omega_{\lambda}^{(n)} + |\omega_{\lambda}^{(n)}|^3$$

(6.1)

occurring in the single iteration steps are solved approximately by an ansatz

$$v_n(x_1,x_2) = \sum_{i,j=1}^{N} \alpha_{ij}^{(n)} \sin(i\pi x_1) \sin(j\pi x_2)$$

(6.2)

and a Ritz-Galerkin method (with the basis functions in (6.2)) applied to problem (6.1). The update $\omega_{\lambda}^{(n+1)} := \omega_{\lambda}^{(n)} + v_n$ concludes the iteration step.

The Newton iteration is terminated when the coefficients $\alpha_{ij}^{(n)}$ in (6.2) are “small enough”, i.e. their modulus is below some pre-assigned tolerance. To start the Newton iteration, i.e. to find an appropriate $\omega_{\lambda}^{(0)}$ of the form (6.2), we first consider some $\lambda$ close to $\lambda_1(\Omega)$, and choose $\omega_{\lambda}^{(0)}(x_1,x_2) = \alpha \sin(\pi x_1) \sin(\pi x_2)$; with an appropriate choice of $\alpha > 0$ (to be determined in a few numerical trials), the Newton iteration will “converge” to a non-trivial approximation $\omega_{\lambda}^{(\lambda)}$. Then, starting at this value, we diminish $\lambda$ in small steps until we arrive at $\lambda = 0$, while in each of these steps the approximation $\omega_{\lambda}^{(\lambda)}$ computed in the previous step is taken as a start of the Newton iteration. In this way, we find approximations $\omega_{\lambda}$ to problem (2.1) for “many” values of $\lambda$. Note that all approximations $\omega_{\lambda}$ obtained in this way are of the form (6.2).

The computation of an $L^2$-defect bound $\delta_{\lambda}$ satisfying

$$\| - \Delta \omega_{\lambda} - \lambda \omega_{\lambda} - |\omega_{\lambda}|^3 \|_{L^2} \leq \delta_{\lambda}$$

(6.3)

amounts to the computation of an integral over $\Omega$.

Due to [11] every solution of (2.1) is symmetric with respect to reflection at the axes $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$. Therefore it is useful to look for approximate solutions of the form

$$\omega_{\lambda}(x_1,x_2) = \sum_{i,j=1}^{N} \alpha_{ij} \sin(i\pi x_1) \sin(j\pi x_2).$$

(6.4)

Using sum formulas for sin and cos one obtains for all $n \in \mathbb{N}_0, x \in \mathbb{R}$

$$\sin((2n+1)\pi x) = 2 \sum_{k=1}^{n} \cos(2k\pi x) + 1 \sin(\pi x)$$

and thus $\omega_{\lambda}$ can be written as follows:

$$\omega_{\lambda}(x_1,x_2) = \alpha_{11} \sin(\pi x_1) \sin(\pi x_2) +$$

$$\sum_{k,l=1}^{\lfloor N/2 \rfloor} \alpha_{2k+1,2l+1} \left( 2 \sum_{i=1}^{k} \cos(2i\pi x_1) + 1 \right) \left( 2 \sum_{j=1}^{l} \cos(2j\pi x_2) + 1 \right) \sin(\pi x_1) \sin(\pi x_2).$$

(6.5)
Since \( \cos(x) \) ranges in \([-1, 1]\) and \(\sin(\pi x_1) \sin(\pi x_2)\) is positive for \((x_1, x_2) \in \Omega = (0, 1)^2\), \(\omega_\lambda\) will be positive if
\[
\alpha_{11} + \sum_{k,l=1}^{N-1} \alpha_{2k+1,2l+1} \left([-2k + 1, 2k + 1]\right) \left([-2l + 1, 2l + 1]\right) \subset (0, \infty). \tag{6.6}
\]

Condition (6.6) can easily be checked using interval arithmetic and is indeed always satisfied for our approximate solutions, since \(\alpha_{11}\) turns out to be “dominant” and the higher coefficients decay quickly. Hence \(\omega_\lambda\) is positive and one can omit the modulus in the computations. Therefore the integral in (6.3) can be computed in closed form, since only products of trigonometric functions occur in the integrand. After calculating them, various sums \(\sum_{i=1}^{N}\) remain to be evaluated. In order to obtain a rigorous bound \(\delta_\lambda\), these computations (in contrast to those for obtaining \(\omega_\lambda\) as described above) need to be carried out in interval arithmetic [13, 27], to take rounding errors into account.

Note that the complexity in the evaluation of the defect integral in (6.3), without any further modifications, is \(O(N^{12})\) due to the term \(\omega_\lambda^3\). Using some trick, it is however possible to reduce the complexity to \(O(N^6)\):

Applying the sum formulas \(\sin(a) \sin(b) = \frac{1}{2} \left[ \cos(a - b) - \cos(a + b) \right] \) and \(\cos(a) \cos(b) = \frac{1}{2} \left[ \cos(a - b) + \cos(a + b) \right] \) one obtains:

\[
\begin{align*}
\sin(i_1 \pi x) \sin(i_2 \pi x) \sin(i_3 \pi x) \sin(i_4 \pi x) \sin(i_5 \pi x) \sin(i_6 \pi x) &= -\frac{1}{32} \sum_{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \in \{-1, 1\}} \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \cos \left( (i_1 + \sigma_2 i_2 + \sigma_3 i_3 + \sigma_4 i_4 + \sigma_5 i_5 + \sigma_6 i_6) \pi x \right).
\end{align*}
\]

Since \(\int_0^1 \cos(n \pi x) \, dx = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \in \mathbb{Z} \setminus \{0\} \end{cases} =: \delta_n\), we get

\[
\int_\Omega \omega_\lambda(x_1, x_2)^6 \, d(x_1, x_2) = \frac{1}{1024} \sum_{\sigma_2, ..., \sigma_N \in \{-1, 1\}} \sum_{\rho_2, ..., \rho_N \in \{-1, 1\}} \sigma_2 \cdot \ldots \cdot \sigma_N \cdot \rho_2 \cdot \ldots \cdot \rho_N \cdot 
\sum_{i_1, ..., i_N = 1}^{N} \delta_{i_1 + \sigma_2 i_2 + \ldots + \sigma_N i_N} \delta_{j_1 + \rho_2 j_2 + \ldots + \rho_N j_N} \alpha_{i_1 j_1} \cdot \ldots \cdot \alpha_{i_N j_N}.
\]

Setting \(\alpha_{ij} := 0\) for \((i, j) \in \mathbb{Z}^2 \setminus \{1, \ldots, N\}^2\) the previous sum can be rewritten as

\[
\frac{1}{1024} \sum_{\sigma_2, ..., \sigma_N, \rho_2, ..., \rho_N \in \{-1, 1\}} \sigma_2 \cdot \ldots \cdot \sigma_N \cdot \rho_2 \cdot \ldots \cdot \rho_N \cdot 
\sum_{k=-2N+1}^{3N} \sum_{l=-2N+1}^{3N} \left( \sum_{i_1, i_2, j_1, j_2}^{N} \alpha_{i_1 j_1} \alpha_{i_2 j_2} \alpha_{i_3 j_3} \right) \left( \sum_{i_4, i_5, j_4, j_5}^{N} \alpha_{i_4 j_4} \alpha_{i_5 j_5} \alpha_{i_6 j_6} \right).
\]

For fixed \(\sigma_i, \rho_i, k\) and \(l\) each of the two double-sums in parentheses is \(O(N^4)\). Since they are independent, the product is still \(O(N^4)\). The sums over \(k\) and \(l\) then give \(O(N^6)\), whereas the sums over \(\sigma_i\) and \(\rho_i\) do not change the complexity.
Moreover the sum $\sum_{k=-2N+1}^{3N}$ is only
\[
\begin{cases}
\sum_{k=3}^{3N} & \text{if } \sigma_2 = 1, \sigma_3 = 1 \\
\sum_{k=2-N}^{2N-1} & \text{if } \sigma_2 \cdot \sigma_3 = -1 \\
\sum_{k=-2N+1}^{N-2} & \text{if } \sigma_2 = -1, \sigma_3 = -1.
\end{cases}
\]
Similarly, also certain constellations of $\sigma_4, \sigma_5, \sigma_6$ reduce the $k$-sum, and of course analogous reductions are possible for the $l$-sum. Since $\alpha_{ij} = 0$ if $i$ or $j$ is even, the result does not change if the sum is only taken over odd values of $i, j, n, k$ and $l$.

**Remark 6.1.** a) Computing trigonometric sums in an efficient way is an object of investigation since a very long time, but up to our knowledge the above complexity reduction has not been published before.
b) As an alternative to the closed form integration described above, we also tried quadrature for computing the defect integral, but due to the necessity of computing a safe remainder term bound in this case, we ended up in a very high numerical effort, since a large number of quadrature points had to be chosen. So practically closed-form integration turned out to be more efficient, although its complexity (as $N \to \infty$) is higher than the quadrature complexity.

Once an $L^2$-defect bound $\hat{\delta}_\lambda$ (satisfying (6.3)) has been computed, an $H^{-1}$-defect bound $\delta_\lambda$ (satisfying (3.1)) is easily obtained via the embedding
\[
\|u\|_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \|u\|_{L^2} \quad (u \in L^2(\Omega))
\]
which is a result of the corresponding dual embedding (5.9). Indeed, (6.3) and (6.7) imply that
\[
\delta_\lambda := \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \hat{\delta}_\lambda
\]
satisfies (3.1).

The estimate (6.7) is suboptimal but, under practical aspects, seems to be the most suitable way for obtaining an $H^{-1}$-bound for the defect. At this point we also wish to remark that, as an alternative to the weak solutions approach used in this paper, we could also have aimed at a computer-assisted proof for strong solutions (see [23]), leading to $H^2$-and $C^0$-error bounds; in this case an $L^2$-bound is needed directly (rather than an $H^{-1}$-bound).

### 7. Computation of $K_\lambda$ for fixed $\lambda$

For computing a constant $K_\lambda$ satisfying (3.2), we use the isometric isomorphism
\[
\Phi : \begin{cases}
H_0^1(\Omega) & \to \ H^{-1}(\Omega) \\
u & \mapsto -\Delta u + \sigma u
\end{cases},
\]
and note that $\Phi^{-1}L(\lambda, \omega_\lambda) : H_0^1(\Omega) \to H_0^1(\Omega)$ is $(\cdot, \cdot)_{H_0^1}$-symmetric since
\[
(\Phi^{-1}L(\lambda, \omega_\lambda)[u], v)_{H_0^1} = \int_{\Omega} [\nabla u \cdot \nabla v - \lambda uv - 3|\omega_\lambda|\omega_\lambda uv] \, dx,
\]
and hence selfadjoint. Since $\|L(\lambda, \omega_\lambda)[u]\|_{H^{-1}} = \|\Phi^{-1}L(\lambda, \omega_\lambda)[u]\|_{H_0^1}$, (3.2) thus holds for any
\[
K_\lambda \geq \left[ \min \{ |\mu| : \mu \text{ is in the spectrum of } \Phi^{-1}L(\lambda, \omega_\lambda) \} \right]^{-1},
\]
provided the min is positive.
A particular consequence of (7.2) is that
\[ \langle (I - \Phi^{-1}L_{\lambda,\omega}) [u], u \rangle_{H^1_0} = \int_{\Omega} W_\lambda u^2 dx \quad (u \in H^1_0(\Omega)) \tag{7.4} \]
where
\[ W_\lambda(x) := \sigma + \lambda + 3|\omega_\lambda(x)|\omega_\lambda(x) \quad (x \in \Omega). \tag{7.5} \]
Note that, due to the positivity of our approximate solutions \( \omega_\lambda \) established in Section 6, the modulus can be omitted here, which again facilitates numerical computations. Choosing a positive parameter \( \sigma \) in the \( H^1_0 \)-product (recall that we actually chose \( \sigma := 1 \)), we obtain
\[ W_\lambda > 0 \text{ on } \overline{\Omega}. \]
Thus, (7.4) shows that all eigenvalues \( \mu \) of \( \Phi^{-1}L_{\lambda,\omega_\lambda} \) are less than 1, and that its essential spectrum consists of the single point 1. Therefore, (7.3) requires the computation of eigenvalue bounds for the eigenvalue(s) \( \mu \) neighboring 0.

Using the transformation \( \kappa = 1/(1 - \mu) \), the eigenvalue problem \( \Phi^{-1}L_{\lambda,\omega_\lambda}[u] = \mu u \) is easily seen to be equivalent to
\[ -\Delta u + \sigma u = \kappa W_\lambda u, \]
or, in weak formulation,
\[ \langle u, v \rangle_{H^1_0} = \kappa \int_{\Omega} W_\lambda uv dx \quad (v \in H^1_0(\Omega)), \tag{7.6} \]
and we are interested in bounds to the eigenvalue(s) \( \kappa \) neighboring 1. It is therefore sufficient to compute two-sided bounds to the first \( m \) eigenvalues \( \kappa_1 \leq \cdots \leq \kappa_m \) of problem (7.6), where \( m \) is (at least) such that \( \kappa_m > 1 \). In all our practical examples, the computed enclosures \( \kappa_i \in [\underline{\kappa}_i, \bar{\kappa}_i] \) are such that \( \bar{\kappa}_1 < 1 < \underline{\kappa}_2 \), whence by (7.3) and \( \kappa = 1/(1 - \mu) \) we can choose
\[ K_\lambda := \max \left\{ \frac{1}{\bar{\kappa}_1}, \frac{\underline{\kappa}_2}{\kappa_1 - 1} \right\}. \tag{7.7} \]

**Remark 7.1.** By [11] and the fact that \( \omega_\lambda \) is symmetric with respect to reflection at the axes \( x_1 = \frac{1}{2} \) and \( x_2 = \frac{1}{2} \), all occurring function spaces can be replaced by their intersection with the class of reflection symmetric functions. This has the advantage that some eigenvalues \( \kappa_i \) drop out, which possibly reduces the constant \( K_\lambda \).

The desired eigenvalue bounds for problem (7.6) can be obtained by computer-assisted means of their own. For example, upper bounds to \( \kappa_1, \ldots, \kappa_m \) (with \( m \in \mathbb{N} \) given) are easily and efficiently computed by the Rayleigh-Ritz method [26]:

Let \( \tilde{\varphi}_1, \ldots, \tilde{\varphi}_m \in H^1_0(\Omega) \) denote linearly independent trial functions, for example approximate eigenfunctions obtained by numerical means, and form the matrices
\[ A_1 := \left( \langle \tilde{\varphi}_i, \tilde{\varphi}_j \rangle_{H^1_0} \right)_{i,j=1,\ldots,m}, \quad A_0 := \left( \int_{\Omega} W_\lambda \tilde{\varphi}_i \tilde{\varphi}_j dx \right)_{i,j=1,\ldots,m}. \]

Then, with \( \Lambda_1 \leq \cdots \leq \Lambda_m \) denoting the eigenvalues of the matrix eigenvalue problem
\[ A_1 x = \Lambda A_0 x \]
(which can be enclosed by means of verifying numerical linear algebra; see [3]), the Rayleigh-Ritz method gives
\[ \kappa_i \leq \Lambda_i \text{ for } i = 1, \ldots, m. \]
However, also lower eigenvalue bounds are needed, which constitute a more complicated task than upper bounds. The most accurate method for this purpose has been proposed by Lehmann [15], and improved by Goerisch concerning its range of applicability [4]. Its numerical core is again (as in the Rayleigh-Ritz method) a matrix eigenvalue problem, but the accompanying analysis is more involved. In particular, in order to compute lower bounds to the first \( m \) eigenvalues, a rough lower bound to the \((m+1)\)-st eigenvalue must be known already. This a priori information can usually be obtained via a homotopy method connecting a simple “base problem” with known eigenvalues to the given eigenvalue problem, such that all eigenvalues increase (index-wise) along the homotopy; see [22] or [5] for details on this method, a detailed description of which would be beyond the scope of this article. In fact, [5] contains the newest version of the homotopy method, where only very small \((2 \times 2)\) or even \((1 \times 1)\) matrix eigenvalue problems need to be treated rigorously in the course of the homotopy.

Finding a base problem for problem (7.6), and a suitable homotopy connecting them, is rather simple here since \( \Omega \) is a bounded rectangle, whence the eigenvalues of \( -\Delta \) on \( H_0^1(\Omega) \) are known: We choose a constant upper bound \( c_0 \) for \( |\omega_\lambda| \omega_\lambda = \omega_\lambda^2 \) on \( \Omega \), and the coefficient homotopy

\[
W_\lambda^{(s)}(x) := \sigma + \lambda + 3[(1-s)c_0 + s\omega_\lambda(x)^2] \quad (x \in \Omega, 0 \leq s \leq 1).
\]

Then, the family of eigenvalue problems

\[
-\Delta u + \sigma u = \kappa^{(s)} W_\lambda^{(s)} u
\]

connects the explicitly solvable constant-coefficient base problem \((s = 0)\) to problem (7.6) \((s = 1)\), and the eigenvalues increase in \( s \), since the Rayleigh quotient does, by Poincaré’s min-max principle.

8. Computation of branches \((\omega_\lambda), (\delta_\lambda), (K_\lambda), (\alpha_\lambda)\)

In the previous section we described how to compute approximations \( \omega_\lambda \) for a grid of finitely many values of \( \lambda \) within \([0, \lambda_1(\Omega))\). After selecting \( \bar{\lambda} \) (among these) according to Section 5, we are left with a grid

\[
0 = \lambda^0 < \lambda^1 < \cdots < \lambda^M = \bar{\lambda}
\]

and approximate solutions \( \omega^i = \omega_{\lambda^i} \in H_0^1(\Omega) \cap L^\infty(\Omega) \) \((i = 0, \ldots, M)\). Furthermore, according to the methods described in the previous sections, we can compute bounds \( \delta^i = \delta_{\lambda^i} \) and \( K^i = K_{\lambda^i} \), such that (3.1) and (3.2) hold at \( \lambda = \lambda^i \).

Now we define a piecewise linear (and hence continuous) approximate solution branch \([(0, \bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto \omega_\lambda)\) by

\[
\omega_\lambda := \frac{\lambda^i - \lambda}{\lambda^i - \lambda^{i-1}} \omega^{i-1} + \frac{\lambda - \lambda^{i-1}}{\lambda^i - \lambda^{i-1}} \omega^i \quad (\lambda^{i-1} \leq \lambda < \lambda^i, i = 1, \ldots, M).
\]

(8.1)

To compute corresponding defect bounds \( \delta_\lambda \), we fix \( i \in \{1, \ldots, M\} \) and \( \lambda \in [\lambda^{i-1}, \lambda^i] \), and let \( t := (\lambda - \lambda^{i-1})/(\lambda^i - \lambda^{i-1}) \in [0, 1] \), whence

\[
\lambda = (1 - t)\lambda^{i-1} + t\lambda^i, \quad \omega_\lambda = (1 - t)\omega^{i-1} + t\omega^i.
\]

(8.2)
Using the classical linear interpolation error bound we obtain, for fixed $x \in \Omega$,
\[
|\lambda \omega(x) - [(1-t)\lambda^i \omega^i(x) + t \lambda \omega^i(x)]| 
\leq \frac{1}{2} \max_{s \in [0,1]} \left| \frac{d^2}{ds^2} \left[ (1-s)\lambda^i \omega^i(x) + s \omega^i(x) \right] \right| \cdot t(1-t)
\leq \frac{3}{4} \max_{s \in [0,1]} \left[ (1-s)\lambda^i \omega^i(x) + s \omega^i(x) \right] \cdot (\lambda^i - \omega^i(x))^2
\leq \frac{3}{4} \max \{ \|\omega^i\|_\infty, \|\omega^i\|_\infty \} \|\omega^i - \omega^{i-1}\|_\infty^2,
\]
(8.3)
\[
|\lambda \omega(x) - [(1-t)\lambda^i \omega^i(x) + t \lambda \omega^i(x)]| 
\leq \frac{1}{2} \max_{s \in [0,1]} \left| \frac{d^2}{ds^2} \left[ (1-s)\lambda^i + s\lambda^i \right] \right| \cdot t(1-t)
\leq \frac{1}{4} (\lambda^i - \lambda^{i-1}) \|\omega^i - \omega^{i-1}\|_\infty.
\]
(8.4)

Since $\|u\|_{H^{-1}} \leq C_1 \|u\|_\infty$ for all $u \in L^\infty(\Omega)$, with $C_1$ denoting an embedding constant for the embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ (e.g.
\[ C_1 = \sqrt{|\Omega|} C_2 \]
\[ (8.3) \]
\[ (8.4) \]

and
\[ \omega_\lambda^3 - [(1-t)(\omega^{i-1})^3 + t(\omega^i)^3]\|_{H^{-1}} \]
\[ \leq \frac{3}{4} C_1 \max \{ \|\omega^{i-1}\|_\infty, \|\omega^i\|_\infty \} \|\omega^i - \omega^{i-1}\|_\infty^2 =: \rho_i, \]
\[ (8.5) \]

\[ \|\lambda \omega - [(1-t)(\lambda^{i-1})^i + t \lambda \omega^i]\|_{H^{-1}} \leq \frac{1}{4} C_1 (\lambda^i - \lambda^{i-1}) \|\omega^i - \omega^{i-1}\|_\infty =: \tau_i. \]
\[ (8.6) \]

Now (8.2), (8.5), (8.6) give
\[ \| - \Delta \lambda \omega - \lambda \omega - \omega_\lambda^2 \|_{H^{-1}} \]
\[ \leq (1-t)\| - \Delta \omega^{i-1} - \lambda^{i-1} \omega^{i-1} - (\omega^{i-1})^3\|_{H^{-1}} + t\| - \Delta \omega^i - \lambda^i \omega^i - (\omega^i)^3\|_{H^{-1}} + \tau_i + \rho_i \]
\[ \leq \max \{ \delta^{-1}, \delta^i \} + \tau_i + \rho_i =: \delta_\lambda. \]
\[ (8.7) \]

Thus, we obtain a branch $\delta_\lambda$ of defect bounds which is constant on each subinterval $[\lambda^{i-1}, \lambda^i]$. In the points $\lambda^1, \ldots, \lambda^{M-1}, \delta_\lambda$ is possibly doubly defined by (8.7), in which case we choose the smaller of the two values. Hence, $([0, \lambda] \rightarrow \mathbb{R}, \lambda \mapsto \delta_\lambda)$ is lower semicontinuous.

Note that $\delta_\lambda$ given by (8.7) is “small” if $\delta^{-1}$ and $\delta^i$ are small (i.e. if the approximations $\omega^{i-1}$ and $\omega^i$ have been computed with sufficient accuracy; see Remark 3.1a) and if $\rho_i, \tau_i$ are small (i.e. if the grid is chosen sufficiently fine; see (8.5), (8.6)).

In order to compute bounds $K_\lambda$ satisfying (3.2) for $\lambda \in [0, \lambda]$, with $\omega_\lambda$ given by (8.1), we fix $i \in \{1, \ldots, M-1\}$ and $\lambda \in \left[ \frac{1}{2}(\lambda^{i-1} + \lambda^i), \frac{1}{2}(\lambda^i + \lambda^{i+1}) \right]$. Then,
\[ |\lambda - \lambda^i| \leq \frac{1}{2} \max \{ \lambda^i - \lambda^{i-1}, \lambda^{i+1} - \lambda^i \} =: \mu_i, \]
\[ (8.8) \]
\[ \|\omega_\lambda - \omega^i\|_{H_0^1} \leq \frac{1}{2} \max \{ \|\omega^i - \omega^{i-1}\|_{H_0^1}, \|\omega^{i+1} - \omega^i\|_{H_0^1} \} =: \nu_i, \]
whence a coefficient perturbation result given in [16, Lemma 3.2] implies: If
\[ \zeta_i := K^i \left[ \frac{1}{\lambda_i(\Omega)} + \mu_i + 2\gamma(\|\omega^i\|_{L^4} + C_4\nu_i)\nu_i \right] < 1, \]  

(8.9)

then (3.2) holds for

\[ K_\lambda := \frac{K^i}{1 - \zeta_i}. \]  

(8.10)

Note that (8.9) is indeed satisfied if the grid is chosen sufficiently fine, since then \( \mu_i \) and \( \nu_i \) are “small” by (8.8).

Analogous estimates give \( K_\lambda \) also on the two remaining half-intervals \([0, \frac{1}{2}\lambda^1]\) and \([\frac{1}{2}(\lambda^{M-1} + \lambda^M), \lambda^M]\).

Choosing again the smaller of the two values at the points \( \frac{1}{2}(\lambda^{-i-1} + \lambda^i) \) for \( i = 1, \ldots, M \)

where \( K_\lambda \) is possibly doubly defined by (8.10), we obtain a lower semi-continuous, piecewise constant branch \(([0, \bar{\lambda}] \to \mathbb{R}, \lambda \mapsto K_\lambda)\).

According to the above construction, both \( \lambda \mapsto \delta_\lambda \) and \( \lambda \mapsto K_\lambda \) are constant on the \( 2M \) half-intervals. Moreover, (8.1) implies that, for \( i = 1, \ldots, M \),

\[ \|\omega_\lambda\|_{L^4} \leq \left\{ \begin{array}{l} \max\{\|\omega_i^{-1}\|_{L^4}, \frac{1}{2}(\|\omega_i^{-1}\|_{L^4} + \|\omega_i^1\|_{L^4})\} \text{ for } \lambda \in [\lambda_i^{-1}, \frac{1}{2}(\lambda_i^{i-1} + \lambda^i)] \text{ and } \lambda \in [\frac{1}{2}(\lambda_i^{i-1} + \lambda^i), \lambda^i] \end{array} \right\} \]

and again we choose the smaller of the two values at the points of double definition.

Using these bounds, the crucial inequalities (3.4) and (3.5) (which have to be satisfied for all \( \lambda \in [0, \bar{\lambda}] \)) result in finitely many inequalities which can be fulfilled with “small” and piecewise constant \( \alpha_\lambda \) if \( \delta_\lambda \) is sufficiently small, i.e. if \( \omega^0, \ldots, \omega^M \) have been computed with sufficient accuracy (see Remark 3.1a)) and if the grid has been chosen sufficiently fine (see (8.5) - (8.7)). Moreover, since \( \lambda \mapsto \delta_\lambda \), \( \lambda \mapsto K_\lambda \) and the above piecewise constant upper bound for \( \|\omega_\lambda\|_{L^4} \) are lower semi-continuous, the structure of the inequalities (3.4) and (3.5) clearly shows that also \( \lambda \mapsto \alpha_\lambda \) can be chosen to be lower semi-continuous, as required in Section 4. Finally, since (3.5) now consists in fact of finitely many strict inequalities, a uniform (\( \lambda \)-independent) \( \eta > 0 \) can be chosen in Theorem 3.1b), as needed for Theorem 4.1.

9. Numerical results

All computations have been performed on an AMD Athlon Dual Core 4800+ (2.4GHz) processor, using MATLAB (version R2010a) and the interval toolbox INTLAB [27]. For some of the time consuming nested sums occurring in the computations, we used moreover mexfunctions to outsource these calculations to C++. For these parts of the program we used C-XSC [13] to verify the results. Our source code can be found on our webpage.
Using $\bar{\lambda} = 18.5$ (which is not the minimally possible choice; e.g. $\bar{\lambda} = 15.7$ could have been chosen) and $M + 1 = 94$ values $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{93} = 18.5$ (with $\lambda_1 = 0.1$, $\lambda_2 = 0.3$ and the remaining gridpoints equally spaced with distance 0.2) we computed approximations $\omega^0, \ldots, \omega^{93}$ with $N = 16$ in (6.2), as well as defect bounds $\delta^0, \ldots, \delta^{93}$ and constants $K^0, \ldots, K^{93}$, by the methods described in Section 6 and 7. Figure 1 shows an approximate branch $[0, 2\pi^2) \to \mathbb{R}, \lambda \mapsto \|\omega_\lambda\|_\infty$. The continuous plot has been created by interpolation of the above grid points $\lambda_j$, plus some more grid points between 18.5 and 2$\pi^2$, where we computed additional approximations.

For some selected values of $\lambda$, Table 1 shows, with an obvious sub- and superscript notation for enclosing intervals, the computed eigenvalue bounds for problem (7.6) (giving $K_\lambda$ by (7.7)). These were obtained using the Rayleigh-Ritz and the Lehmann-Goerisch method, and the homotopy method briefly mentioned at the end of Section 7 (exploiting also the symmetry considerations addressed in Remark 7.1). The integer $m$, needed for these procedures, has been chosen different (between 3 and 10) for different values of $\lambda$, according to the outcome of the homotopy. This resulted in a slightly different quality of the eigenvalue enclosures.
Table 1. Eigenvalue enclosures for the first two eigenvalues

| λ-interval | δ_λ | K_λ | α_λ |
|------------|------|------|------|
| (0,0.05)   | 0.0005943 | 1.7443526 | 0.0010378 |
| (2.2,2.1)  | 0.0023344 | 1.7707941 | 0.0041521 |
| (6.6,6.1)  | 0.0022937 | 1.6669879 | 0.0038369 |
| (10,10.1)  | 0.0023644 | 1.5677657 | 0.0037168 |
| (14,14.1)  | 0.0026980 | 1.9582604 | 0.0053028 |
| (16,16.1)  | 0.0031531 | 3.2267762 | 0.0102701 |
| (18.4,18.5)| 0.0050056 | 13.8930543 | 0.0882899 |

Table 2 contains, for some selected of the 186 λ-half-intervals,

a) the defect bounds δ_λ obtained by (8.7) from the grid-point defect bounds δ^{i-1}, δ^i, and from the grid-width characteristics ρ_i, τ_i defined in (8.5), (8.6),
b) the constants K_λ obtained by (8.10) from the grid-point constants K^i and the grid-width parameters ν_i defined in (8.8) (note that μ_i = 0.1 for all i),
c) the error bounds α_λ computed according to (3.4), (3.5).

Thus, Corollary 2.1, together with all the considerations in the previous sections, proves Theorem 1.1

| λ-interval | δ_λ | K_λ | α_λ |
|------------|------|------|------|
| (0,0.05)   | 0.0005943 | 1.7443526 | 0.0010378 |
| (2,2.1)    | 0.0023344 | 1.7707941 | 0.0041521 |
| (6,6.1)    | 0.0022937 | 1.6669879 | 0.0038369 |
| (10,10.1)  | 0.0023644 | 1.5677657 | 0.0037168 |
| (14,14.1)  | 0.0026980 | 1.9582604 | 0.0053028 |
| (16,16.1)  | 0.0031531 | 3.2267762 | 0.0102701 |
| (18.4,18.5)| 0.0050056 | 13.8930543 | 0.0882899 |

Table 2.

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Dep. of Mathematics, University of Connecticut, Storrs, CT 06269-3009, MSB 328, USA
E-mail address: mckenna@math.uconn.edu

Dipartimento di Matematica, Università di Roma “La Sapienza”, P.le A. Moro 2, 00185 Roma, Italy
E-mail address: pacella@mat.uniroma1.it

Institut für Analysis, Karlsruher Institut für Technologie (KIT), 76128 Karlsruhe, Germany
E-mail address: michael.plum@kit.edu

Institut für Analysis, Karlsruher Institut für Technologie (KIT), 76128 Karlsruhe, Germany
E-mail address: dagmar.roth@kit.edu