On an asymptotic estimate of the $n$-loop correction in perturbative QCD

J.Chýla, J.Fischer and P.Kolář

Abstract

A recently proposed method of estimating the asymptotic behaviour of QCD perturbation theory coefficients is critically reviewed and shown to contain numerous invalid mathematical operations and unsubstantiated assumptions. We discuss in detail why this procedure, based solely on renormalization group (RG) considerations and analyticity constraints, cannot lead to such estimates. We stress the importance of correct renormalization scheme (RS) dependence of any meaningful asymptotic estimate and argue that the unambiguous summation of QCD perturbation expansions for physical quantities requires information from outside of perturbation theory itself.
1. Introduction

Because of computational complications only the lowest two or three terms in QCD perturbation expansions are available for phenomenologically interesting quantities. Taking into account that the relevant expansion parameter (strong coupling constant $\alpha_s/\pi$) is of the order of 0.1, and for certain quantities even bigger, we face the following two important questions:

1. how to reliably estimate higher order terms in perturbation expansions of physical quantities and, having obtained them,

2. how to formulate a physically well-motivated summation method in the situation when these asymptotic estimates prevent the use of the familiar Borel summation technique.

A number of theorists have struggled over the last two decades to evaluate multiloop Feynman diagrams in QCD and other theories and we seem currently to be at the border of what can be done with the available analytical as well as numerical tools. The state of the art in this field has recently been reached by the first three loop QCD calculation of a physical quantity, the familiar $R$-ratio in $e^+e^-$ annihilation into hadrons $[1]$

$$R(q^2) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \left(3 \sum_{i=1}^{n_f} Q_i^2\right) \sum_{n=0}^{\infty} r_n(q^2/\mu^2) \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n,$$

(1)

where $q^2$ denotes the square of the center of mass energy in $e^+e^-$ collisions and $Q_i$ is the electric charge of the produced quark with flavour $i$. These, as well as the analogous results on certain structure functions sum rules $[2]$ have been used in phenomenological analyses of experimental data and the inclusion of NNLO correction has been shown to reduce theoretical uncertainties $[3, 4]$ and, in the case of the Gross-Llewelyn-Smith sum rule, also to improve agreement with experiment $[4]$. There seems, however, to be little chance of going in a foreseeable future to still higher orders for any of the physically measurable quantities. The quest for at least a reliable estimate of these higher order contributions is therefore highly commendable, in particular in view of the fact that the relevant perturbation series are expected to be factorially divergent with asymptotically sign-definite terms.

In fact the question of the possible divergence of QCD (as well as other field theories) perturbation expansions, which goes back to the original argument of Dyson $[5]$, still defies definite and clear answer. The original conjecture that the divergence of perturbation expansions is directly related to the discontinuities of Green’s functions in unphysical domains of the coupling constant $[6]$ has been seriously questioned by several authors $[7, 8, 9]$. The knowledge of the discontinuity itself is simply insufficient for the determination of large order behaviour of
perturbation theory coefficients and one needs much more detailed information on the behaviour of Green’s functions in the vicinity of the corresponding cut in order to establish this relation.

We share the viewpoint of Stevenson [8] that there is little evidence that QCD perturbation expansions for physical quantities do indeed diverge. The examples discussed in [8] indicate that in fact most of the information on the analytic properties of relevant Green’s functions near \( g^2 = 0 \) [10, 11] may actually be invisible in perturbation expansions themselves. Moreover, the fact, contained already in [7] and later rediscovered and advocated by Stevenson [8], that the summability of perturbation expansions is not the same as the possibility of recovering from such a series the corresponding “full” physical quantity, has now gained wider acceptance. As the summability of QCD perturbation expansions is determined by the large order behaviour of their coefficients we primarily need to know which kind of information is necessary and sufficient to derive it.

Quite recently two new attempts have been undertaken in order to derive the asymptotic behaviour of the perturbation-theory coefficients for certain physical quantities [9, 12]. They use completely different ideas but both arrive at the same conclusion: QCD perturbation expansions are indeed factorially divergent. The novel argument of [9] is tailored for the special case of particle fractions in \( e^+e^- \) annihilation and can not be used for such simple quantity as (1). Contrary to the original approach of [6] it, however, doesn’t employ any information on analytic properties of corresponding Green’s functions. Although it is based on several strong and questionable assumptions, which render its conclusions open to doubt, we can at least imagine that more sophisticated procedure can be formulated along similar lines.

This is not the case with the procedure suggested in [12], which is claimed to answer both of the above questions. In a subsequent paper [13], West uses his estimates to formulate a summation procedure with truly remarkable properties. In particular they make the use of a few lowest terms in (1) a reliable approximation even in the low \( q^2 \) region where the expansion parameter \( \alpha_s(q^2)/\pi \) is so large that this approximation is normally considered meaningless.

Soon after it appeared, [12] was criticized by Brown and Yaffe [14], who have argued that the assumptions employed by West, i.e. the analyticity constraints coupled with RG considerations, allow one to find the relation between the Borel transform of (1) and that of the dispersive part of \( \Pi(q^2) \) but are not sufficient to unambiguously determine the asymptotic behaviour of the coefficients \( r_k \) themselves. We fully agree with these authors but consider it useful and illuminating to go through the procedure of [12] carefully step by step in order to see exactly where and why it breaks down if carried out correctly. This will be done in Section 3. Our analysis will also show why there is no hope to remedy the derivation in [12]. Finally in Section 4 we shall comment on the summation procedure suggested in [13], which uses the factorially divergent nature of the coefficients \( r_k \) but is in principle independent of the particular way of deriving it. We shall argue
that this algorithm is just one of an infinite number of equally plausible “sums” of $\lbrack 1 \rbrack$ compatible with the factorial growth of $r_k$.

In renormalized field theories like QCD the question of the large order behaviour of the coefficients like $r_k$ in $\lbrack 1 \rbrack$ is complicated by their inherent RS dependence. Consequently even the convergence property of the expansion $\lbrack 1 \rbrack$ does in fact crucially depend on the RS chosen. We therefore start our analysis with general remarks concerning the constraints following from the formal RG invariance of expansions like $\lbrack 1 \rbrack$ which any asymptotic estimate of $r_k$ must obey in order to make sense.

2. RG constraints and RS dependence of asymptotic estimates

The expansion parameter $\alpha_s$ as well as the coefficients $r_k$ in $\lbrack 1 \rbrack$ depend on the RS chosen. A part of this dependence is labelled by the scale $\mu$ but the full RS ambiguity is much wider. As this point is often a source of misunderstanding let us specify very carefully what is understood under this term and where it differs from the so called renormalization convention (RC), the notion introduced in $\lbrack 13 \rbrack$. We have decided to include this detailed exposition after getting acquainted with West’s response $\lbrack 16 \rbrack$ to $\lbrack 14 \rbrack$ and other critics $\lbrack 17 \rbrack$.

We start with the usual RG equation

$$\frac{dg(\mu^2)}{d \ln \mu} \equiv \beta(g) = -g^3 \left( b_1 + b_2 g^2 + b_3 g^4 + \cdots \right),$$

which represents basically a definition of the expansion parameter

$$a(\mu^2) \equiv \alpha_s(\mu^2)/\pi = g^2(\mu^2)/4\pi^2$$

used in $\lbrack 1 \rbrack$. While $b_1$ as well as $b_2$ are unique numbers, fixed by the number $n_f$ of quark flavours (we consider massless quarks only), all the higher order coefficients $b_n$, $n > 2$, are arbitrary numbers defining the renormalization convention. For physical quantities the notion of the renormalization scheme involves, besides the arbitrariness in the $\beta$-function coefficients, another degree of freedom, not related to them. To exemplify this point let us just recall the difference between the MS and $\overline{\text{MS}}$ RS. Both RS have exactly the same $\beta$-function coefficients but they are nevertheless associated with different values of the couplant $a(\mu^2)$ as well as the coefficients $r_k, k \geq 2$. In fact there is an infinite set of RS which have the same $\beta$-function as $\overline{\text{MS}}$, but which correspond to different values of $r_k, k \geq 2$. This dependence of the coefficients $r_k$ on the RS chosen is formally to the k-th order compensated by that of the couplant $a(\mu^2, \text{RS})$ in a way which guarantees the internal consistency of perturbation theory $\lbrack 13 \rbrack$. There are various ways of parametrizing this degree of freedom, the most conventional one using just the value of renormalization scale $\mu$. 

4
Mathematically, this additional degree of freedom in the definition of the RS is related to the simple fact that the RG equation (4) defining the renormalized coupling constant \( g(\mu^2) \) (and thus \( \alpha_s(\mu^2) \)) has, even for fixed values of all the coefficients \( b_i \), an infinite number of solutions. For any well-defined r.h.s. of (4), like for instance in the so-called ’t Hooft’s RC, (or in any of the “analytical” RC discussed in [17]) where all free higher order coefficients \( b_i, i \geq 3 \) are set identical to zero [10], the behaviour of \( g(\mu^2) \) for \( \mu \to \infty \) is essentially fixed by the first term in (4). At finite \( \mu \), however, the differences between various solutions of (4) may be arbitrarily large.

Specifying the RS then means besides fixing the \( \beta \)-function coefficients also choosing the renormalization scale \( \mu \). Variation of \( \alpha_s(\mu^2) \) with this scale is then compensated to any finite order by that of the coefficients \( r_k \). For a fixed \( \mu \) there is, however, an infinite set of the values \( g(\mu^2) \) corresponding to different solutions of (4). So even setting \( \mu \) equal to some external kinematical variable (like the total CMS energy \( \sqrt{q^2} \) in (1)), we still face the necessity to specify which of the solutions to (4) we have in mind. This fact is frequently overlooked and one often finds statements that the fixing of the scale \( \mu \) and the \( \beta \)-function coefficients \( b_k \) defines uniquely the renormalization scheme. This is, however, inaccurate and the selection of the “referential” RS, i.e. the selection of one of the solutions to (4) as referential is as important as the choice of the scale \( \mu \) itself. Indeed, the variation of the scale \( \mu \) for fixed referential RS (RRS) is equivalent to the variation of the RRS for fixed \( \mu \). In the approach suggested and advocated in [15], we select once for all one of these solutions and label different RS by different values of \( \mu \) and \( b_j \). The selection of this referential RS is of course merely a matter of bookkeeping and has nothing to do with the actual choice of a particular RS. The RS is thus uniquely defined by either of the sets \{\( \mu, b_i \), RRS\}, \{\( a, b_i \)\} or \{\( r_2, b_i \)\}. In the last two cases there is no need to select the RRS.

There are various ways how to label the solutions of (4), the most convenient one employing the renormalization group invariant, dimensional parameter denoted usually \( \Lambda \). Each of the solutions of (4) is then associated with one particular value of \( \Lambda \) (we denote it \( \Lambda_{\text{RRS}} \)) and the couplant \( a(\mu^2) \) is then in fact a function of the ratio \( \mu^2/\Lambda^2_{\text{RRS}} \). The precise definition of \( \Lambda_{\text{RRS}} \) being again a matter of convention, we can define it, for example, by the relation \( a(\mu^2 = \Lambda^2_{\text{RRS}}) = \infty \). Knowing the dependence of the couplant \( a(\mu^2/\Lambda^2; \text{RRS}) \) on both the scale and the RRS we come now to that of the coefficients \( r_k \). For them the internal consistency of perturbation theory implies the following recurrence relations [13]:

\[
\frac{dr_k(q^2/\mu^2; \text{RRS})}{d \ln \mu} = 8\pi^2 b_1 (k-1) r_{k-1} + 32\pi^4 b_2 (k-2) r_{k-2} + \mathcal{O}(b_3). \tag{3}
\]

For \( r_2 \) this yields (supplemented with the initial condition \( r_0 = r_1 = 1 \)) the following explicit dependence on \( \mu \)

\[
r_2(q^2/\mu^2; \text{RRS}) = 4\pi^2 b_1 \ln(\mu^2/q^2) + r_2(1; \text{RRS}) \tag{4}
\]
\[ = 8\pi^2 b_1 \ln(\mu/\Lambda_{\text{RRS}}) - \rho_1(q^2), \]

while for \( r_3 \) we get
\[ r_3 = \rho_2 - c_2 + (r_2 + c/2)^2, \tag{5} \]

where \( c = b_2/b_1 \) and \( c_2 = b_3/b_1 \). As a consequence of \((3)\) analogous algebraic relations hold also for higher order coefficients \( r_k \). Similar consistency conditions can also be obtained for the variation of \( r_k \) with respect to other RS parameters, i.e., \( b_i, i \geq 3 \).

The quantities \( \rho_1 \) and \( \rho_2 \) (and analogous quantities at higher orders) entering the formulae \((3)\) and \((4)\) are RS invariants, \( \rho_1 \) being a function of \( q^2 \) while \( \rho_2 \) (and all higher order invariants \( \rho_i, i > 2 \) as well) is a pure number. As \( \mu \) and \( r_2 \) are (in a given RRS) in one-to-one correspondence we can, instead of \( \mu \), alternatively use the value of \( r_2 \) itself to label the same degree of freedom as \( \mu \). In other words, the fixing of \( r_2 \) is an alternative way of fixing (together with the coefficients \( b_n, n > 2 \)) the RS. We stress that it is already the value of \( r_2 \) which is different in the \( \overline{\text{MS}} \) and the MS RS, although the corresponding \( \beta \)-functions have exactly the same coefficients \( b_i \).

We mention this alternative labelling of the RS by means of \( \{r_2, b_i\} \) for the following reason. It is in fact difficult to translate the results on the coefficients \( r_k \) obtained by means of any method not explicitly using Feynman diagram renormalization (like the one in \([12]\)) into the language of the conventional perturbation theory and vice versa. For instance, if, in West’s approach, one were to carry out the evaluation of \( r_k \) in, say, \( \overline{\text{MS}} \) RS there seems to be no obvious meaning of the term “minimum subtraction”, which is inherent to the usual Feynman rules calculations. The same holds of course for any other conventional RS. Using \((3)\) we can, in any selected RRS, define \( \mu \) associated with the \( \overline{\text{MS}} \) RS by demanding that the corresponding \( r_2(\mu^2/q^2; \text{RRS}) = 1.41 \) but this is possible only if the dependence of \( r_2(\mu^2/q^2; \text{RRS}) \) on \( \mu \) is consistent with \((3)\). In the notation where \( \mu \) is fixed, being set equal to, say, \( \sqrt{q^2} \), similar statements hold for the dependence on the RRS. In \([12]\), however, the asymptotic behaviour of the coefficients \( r_k \)
\[ r_n(1) \approx -\frac{e^{1+b'}}{\pi} (4\pi^2 e b_1)^{n-1} \frac{\Gamma(n+b')}{(n+b')^2}, \tag{6} \]
is manifestly RRS independent. Indeed it was the striking lack of any RRS dependence of \((3)\) which has originally attracted our attention and cast doubts on their validity. The absence of appropriate RRS dependence of \( r_n(1) \), prevents, however, the association of \((3)\) with any well-defined RS. This crucial fact has not been properly appreciated in \([12]\).

To see clearly the implications of the consistency conditions \((3)\) for the RRS dependence of the large order behaviour of the coefficients \( r_k \) let us consider the (for technical reasons somewhat simplified) case when \( b_k = 0 \) for all \( k > 1 \), i.e.,
also \( b_2 = 0 \). Then (3) can be explicitly integrated to yield

\[
r_k(q^2/\mu^2; \text{RRS}) = \sum_{j=0}^{k-1} \tau^j r_{k-j}(q^2/\mu_0^2) \binom{k-1}{j},
\]

where \( \tau = 8\pi^2 b_1 \ln(\mu/\mu_0) \), \( \mu_0 \) is some “initial” value of \( \mu \) (corresponding to some RRS\( _0 \)) and \( r_k(q^2/\mu_0^2) \) are the associated expansion coefficients of (1) in RRS\( _0 \). The above formula can simply be rewritten to express the RRS dependence for fixed \( \mu^2 = q^2 \):

\[
r_k(1; \text{RRS}) = \sum_{j=0}^{k-1} \tau^j r_{k-j}(1; \text{RRS}_0) \binom{k-1}{j},
\]

where now

\[
\tau = 8\pi^2 b_1 \ln(\Lambda_{\text{RRS}_0}/\Lambda_{\text{RRS}}).
\]

Assuming, as an example, the factorial behaviour of the coefficients, \( r_k(1; \text{RRS}_0) = k! \), we get from (8) for general RRS

\[
r_k(1; \text{RRS}) = k! \sum_{i=0}^{k-1} \frac{\tau^i}{i!} \Rightarrow \lim_{k \to \infty} \frac{r_k}{k!} = \exp \tau.
\]

This means that although the character of the large order behaviour of the coefficients \( r_k(1; \text{RRS}) \) is not changed (in the above example it is merely multiplied by the \( k \)-independent factor \( \exp \tau \)) when we vary the RRS, the magnitude of the coefficients \( r_k(1; \text{RRS}) \) obviously is. As \( \tau \) can in principle be any real number, the magnitude of the coefficients \( r_k(1; \text{RRS}) \) can, in our example, be arbitrarily large.

Although the RRS dependence of the large order behaviour of \( r_k(1) \) is irrelevant for considerations concerning the summability of our series, it is crucial when we are interested in their numerical values. For the realistic case with \( b_2 \neq 0 \) the dependence of \( r_k(1; \text{RRS}) \) on the RRS is more complicated than (10) but its main feature i.e. the arbitrariness of the magnitude of \( r_k(1; \text{RRS}) \) on the RRS persists. We want to emphasize this point very strongly as in his reply [16] West explicitly claims that the large order behaviour of the coefficients \( r_k(1) \) should be “scheme invariant” with “scheme dependence” entering only via the dependence of nonleading corrections on the free coefficients \( b_k, k \geq 3 \). The relation (14) demonstrates that this claim is definitely wrong as the dependence of \( r_k(1; \text{RRS}) \) on the RRS may change the magnitude of them essentially at will. For example, the value of \( \tau \) corresponding to the change from the \( \overline{\text{MS}} \) to the MS RRS equals, for \( n_f = 5 \)

\[
\tau = 8\pi^2 b_1 \ln(\Lambda_{\overline{\text{MS}}}/\Lambda_{\text{MS}}) = 3.83 \ln 2.65 = 3.73.
\]

There is thus no obvious reason to compare \( r_3 = -13.4 \) from (6) with its exact value in the \( \overline{\text{MS}} \) RS (where \( r_3(1; \overline{\text{MS}}) = -12.8 \)) and not, for instance, in MS which has exactly the same \( \beta \)-function coefficients but where, however, \( r_3(1; \text{MS}) = 16.3 \). The agreement of (8) with \( r_3(1; \overline{\text{MS}}) \) should therefore be considered as a
pure coincidence. This has also been demonstrated in [17] on the basis of the comparison of the corresponding $n_f$ dependences of (3) and the exact results [3]. There is in fact no agreement of (6) even in the MS RS and even following uncritically the whole derivation of [12]. The “correct” (i.e. disregarding all the objections we shall discuss in the next section) result thereof reads

$$r_n(1) \approx -\frac{e^{1+b'}}{2\pi} (4\pi^2 e b_1)^{n-1} \frac{\Gamma(n+b')}{(n+b')^{2+b'}}; \quad (12)$$

which differs from (6) asymptotically by a factor $(n^{-b'})/2$ and by a factor of about 1/4 for $n = 3$. So even within the framework of West’s approach the starting point for his considerations, i.e., the coincidence with exact results of [3] is actually absent. On the other hand it is clear that for any value $\tilde{r}_3$ there is always such a RRS that $r_3(1; \text{RRS}) = \tilde{r}_3$. We also stress that the ambiguity in association of (6) with a particular RS has nothing to do with the neglected higher order $\beta$-function coefficients $b_k$ but reflects the fact that even for a given RC, i.e. for fixed $\mu$ and $b_k, k \geq 3$ there is still a degree of freedom connected with RRS.

In perturbation theory the quantity (1) is described by a pair of expansions (1) and (2), neither of which has a physical meaning of its own. Except for the first two coefficients in both (1) and (2) all other ones are arbitrary, subject only to the internal consistency constraints like (5) which follow from (3). We can choose the RS=$\{\mu, b_k, \text{RRS}\}$ in such a way that either (1) or (2) is a well-defined, convergent series but we can not do this for both of them simultaneously. For instance, the “effective charges” approach of Grunberg [18] sets $r_k = 0$ for all $k > 1$ so that one has to worry about the convergence property of (2) only. On the contrary, in ’t Hooft’s RC $b_k = 0$, $k > 2$ so that the definition equation (2) of the couplant $a(\mu^2)$ is well-defined and the nontrivial question concerns the expansion coefficients $r_k$. In a computationally simple RS, like MS, both of the expansions (1) and (2) are presumably divergent. This makes their use for the purpose of summation considerations rather unsuitable as the couplant $\alpha_s/\pi$ itself is ill-defined in the limit of infinite order. Consequently, any statement concerning the large order behaviour of the coefficients $r_k$ is inextricably connected with that of the coefficients $b_k$ of (2) and makes sense only when it is clearly and unambiguously related to some definite RS.

3. Discussion of West’s approach

We now turn to a discussion of the West approach advocated in [12]. (For reader’s convenience we basically use the original notation of [12].) We start from the definition of the vacuum polarization function $\Pi(q^2, g^2)$ in terms of the time-ordered product of currents:

$$\langle e^{i q x} \langle 0| T j_\mu(x) j_\nu(0)|0 \rangle \rangle,$$
where $j_\mu(x)$ is the electromagnetic current of the quarks in massless QCD. The function $\Pi$ is assumed to be analytic in the whole $q^2$ complex plane cut along the positive real axis. The imaginary part of $\Pi$ is related to the ratio $R$ (see eq. (1))

$$\text{Im } \Pi(q^2/\mu^2, \alpha_s(\mu^2)) = \frac{1}{12\pi} R(q^2/\mu^2, \alpha_s(\mu^2)).$$

(14)

We further introduce the function

$$D(q^2/\mu^2, g^2) = \frac{\partial}{\partial t} \Pi(q^2/\mu^2, \alpha_s),$$

where $t = \ln(q^2/\mu^2)$. $R$ and $D$ satisfy the homogeneous renormalization group equation

$$[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}] R(q^2/\mu^2, \alpha_s) = 0$$

(15)

and analogously for $D$, where the Callan-Symanzik function $\beta$ is defined in (2).

As a consequence the functions $R$ and $D$ depend (in massless QCD) on the single variable $z = (q^2/\mu^2) e^{2K(g)} [10, 12]$. Here the function $K(g)$ is defined by

$$K(g) = \int_{g_0}^g \frac{dg'}{\beta(g')} \approx \frac{1}{2b_1g^2} - \frac{b'}{2} \ln(b_1/g^2 + b_2) + \cdots,$$

(16)

where $b' = b_2/b_1^2$. (In obtaining eq. (16) a convenient choice of $g_0$ has been made.) It is useful to define the running coupling constant $\bar{g}(t)$ by $2K(\bar{g}(t)) = 2K(g) = t$. Then the eq. (1) can be rewritten as

$$R(q^2/\mu^2, \alpha_s(\mu^2)) = R(1, \bar{g}_s(q^2)) \approx (3 \sum Q_i^2) \sum_{n=0}^\infty r_n(1) (\bar{g}_s(q^2)/\pi)^n.$$

(17)

The dispersion relation for $D$ can be given the form

$$D(q^2/\mu^2, g^2) = A(q^2/\mu^2, g^2) \int_0^\infty dz \frac{f(z)}{[z - A(q^2/\mu^2, g^2)]^2},$$

(18)

where

$$f(z) = \frac{1}{12\pi^2} R(q^2/\mu^2, \alpha_s(\mu^2)),$$

(19)

and

$$A(q^2/\mu^2, g^2) = \frac{q^2}{\mu^2} e^{2K(g)}.$$  

(20)

(In the following we shall use the reduced notation $A(k) = A(q^2/\mu^2, 1/k)$, where $k = 1/g^2$.)

The aim of West’s considerations is to use eq. (18) in the analysis of large $n$ behaviour of the expansion coefficients $d_n$ defined by formal (asymptotic) expansion

$$D(q^2/\mu^2, g^2) \approx \sum_{n=0}^\infty (-1)^n d_n(q^2/\mu^2) g^{2n}.$$

(21)
The expansion coefficients \( r_n \), defined in (1), are then connected with the \( d_n \) by the relation

\[
\left( \sum_i Q_i^2 \right) r_n = -(-4\pi^2)^{n+1} \frac{1}{\pi n b_1} \left[ \text{Im} \ d_{n+1} + \frac{b_2}{b_1} \ \text{Im} \ d_n + \cdots \right], \quad (22)
\]
as a consequence of the equation

\[
\text{Im} \ D = \frac{g\beta(g)}{12\pi} \frac{\partial R}{\partial g^2} \approx -\left( \sum_i Q_i^2 \right) b_1 g^4 \frac{1}{16\pi^3} \left[ 1 + \left( \frac{b_2}{b_1} + \frac{r_2}{2\pi^2} \right) g^2 + \cdots \right]. \quad (23)
\]

This suggests that we can get the large order behaviour of \( r_n \) once we know that of \( d_n \) (see however [14]).

We now focus on three crucial points in the method of calculation of the coefficients \( d_n \).

(A) If eq. (21) can be understood as an asymptotic expansion of \( D \) then the \( d_n \) are given by (see Wightman [10])

\[
d_n = -\lim_{g_0 \to 0^+} \oint_{C(g_0)} \frac{dg^2}{2\pi i} \frac{1}{k_0^n} (g^2 + g_0^2)^{-1-n} D(q^2/\mu^2, g^2). \quad (24)
\]

\( C(g_0) \) is a closed contour around the point \( g_0^2 \). The size of the contour has to be chosen sufficiently small in order not to hit the complicated singularity structure of the function \( D \) around the origin. In particular, \( C(g_0) \) must not leave, with \( g_0 \) approaching zero, the wedge bisected by the real axis and bounded above and below by circles that are tangent to the real axis at the origin [10]. (There are arguments that this horn-shaped region is the analyticity region of the function \( D \) [10].) Making the transformation \( k = 1/g^2 \) and choosing the corresponding contour \( C'(k_0) \) around the point \( k_0 = 1/g_0^2 \) (see Fig.1), we can rewrite eq. (24) into the form

\[
d_n = \lim_{k_0 \to \infty} \oint_{C'(k_0)} \frac{dk}{2\pi i} \left( k - k_0 \right)^{-1-n} D(q^2/\mu^2, 1/k). \quad (25)
\]

One particularly convenient choice of \( C'(k_0) \) is the contour \( C'_0 \) (see Fig.1) which is independent of the position of \( k_0 \) provided that \( k_0 \) is sufficiently large. (We denote as \( C_0 \) the corresponding contour in the \( g^2 \)-plane.) We do not extend the contour \( C'_0 \) up to \( \text{Re} \ k \to -\infty \), in order to avoid the problem with a singularity of \( D \) at \( |k| = \infty \). Thus, in the contour \( C'_1 \) depicted in Fig.1 one cannot neglect the contribution along the direction perpendicular to the real axis at \( \text{Re} \ k \to -\infty \). The contour mentioned in [10] should therefore be understood as the \( \text{Re} \ k \to -\infty \) limit of \( C'_1 \).

Obviously in eqs. (24) and (25) one can not interchange the order of the limit \( k_0 \to \infty \) and the integration. Otherwise we would obtain the following result

\[
d_n = \oint_{C'_0} \frac{dk}{2\pi i} (-k)^{n-1} D(q^2/\mu^2, 1/k). \quad (26)
\]
Using the expansion (21) in the neighbourhood of Re \( k \to -\infty \) we conclude that the integral (26) does not exist for any \( n \geq 0 \). Rewriting (26) back to the \( g^2 \)-plane we obtain the expression

\[
d_n = -\oint_{C_0} \frac{dg^2}{2\pi i} (-g^2)^{-n-1} D(q^2/\mu^2, g^2),
\]

which coincides with West’s formula (with the integration contour properly defined). Thus, the \( d_n \) presented in [12] cannot be understood as well-defined quantities.

(B) The next step in West’s approach is the substitution of (18) into (25)

\[
d_n = \lim_{k_0 \to \infty} \oint_{C'_0} \frac{dk}{2\pi i} k^{n-1} k_0^{n+1} (k - k_0)^{-1-n} A(k) \int_0^\infty dz \frac{f(z)}{[z - A(k)]^2},
\]

and the interchange of the order of integrations. We shall however show that the \( d_n \) computed in this way are incorrect (this suspicion was also voiced in ref. [17]). For this purpose we introduce the quantity

\[
\bar{d}_n = \lim_{k_0 \to \infty} \int_0^\infty dz \ f(z) \frac{\partial}{\partial t} \Phi(n, t, z, k_0),
\]

where

\[
\Phi(n, t, z, k_0) = \frac{1}{2\pi i} \oint_{C_0} \frac{dk}{k^{n-1} k_0^{n+1} (k - k_0)^{n+1}} \frac{1}{z - A(k)}.
\]

Note that the formal limit \( k_0 \to \infty \) would recast this relation to the form

\[
\Phi(n, t, z) = \frac{1}{2\pi i} \oint_{C'_0} \frac{dk}{z - A(k)},
\]

used in [12].

Applying the Cauchy theorem to (30) we get

\[
\Phi(n, t, z, k_0) = k_0^{n+1} \left\{ \frac{1}{n!} \left. \frac{\partial^n}{\partial k^n} \frac{k^{n-1}}{z - A(k)} \right|_{k=k_0} \right. - \\
- \frac{k_p^{n-2} (b_1 k_p + b_2)}{(k_p - k_0)^{n+1} A(k_p)} \delta_{C'} \right\} \equiv \Phi_1(n, t, z, k_0) + \Phi_2(n, t, z, k_0).
\]

The contour is chosen such that at most one of the poles given by

\[
A(k) = z - e^{t+2K(k)} = 0,
\]

can lie inside it. We will denote its position by \( k_p \) (see Appendix). \( \delta_{C'} \) is a contour dependent factor, \( \delta_{C'} = 1 \) if the pole is inside \( C' \), otherwise \( \delta_{C'} = 0 \).
This contour-dependence is apparently undesirable and it is the first signal that \( \tilde{d}_n \) given by (29) differs from \( d_n \) (\( d_n \) is contour-independent provided that \( k_0 \) lies inside \( C'_0 \)).

Using (32) it is easy to show by induction over \( n \) that

\[
\lim_{k_0 \to \infty} \int_0^\infty dz \ f(z) \frac{\partial}{\partial t} \Phi_1(n, t, z, k_0) = d_n.
\] (34)

Thus, we get

\[
\tilde{d}_n - d_n = \lim_{k_0 \to \infty} \int_0^\infty dz \ f(z) \frac{\partial}{\partial t} \Phi_2(n, t, z, k_0).
\] (35)

We conclude that this contour-dependent difference is given by the pole contribution to \( \tilde{d}_n \). The position of the pole depends on the value of \( z \). (When \( b_2 \approx 0 \) we have \( k_p \approx b_1 (\ln z - t) \)). In general, the pole moves to the right in the \( k \)-plane when \( z \) increases. The integration in (35) can therefore be restricted to an interval \( (z_1, z_2) \) defined by the occurrence of pole inside \( C' \). In the case of the contour \( C'_0 \) depicted in Fig.1 we have \( z_2 = \infty \). For our aims it is sufficient to prove that the difference (35) does not vanish. In fact we prove in the appendix that the difference can be divergent for some contours. Consequently, one cannot interchange the order of integrations in (28). The same objection has been raised in [11] (in this paper, however, the contribution of the pole to \( \tilde{d}_n \) is ignored).

(C) In [12] the integral (31) has been estimated by the saddle point technique which might be expected to provide a large-\( n \) approximation to (32). An essential feature within the saddle point method is a convenient choice of the integration contour. Representing the integrand in the form of \( \exp h(k) \), where \( h(k) \) is a complex function analytic in some vicinity of the contour, we should choose the contour so that the following conditions are satisfied:

(i) The integration contour can be deformed (without changing the value of the integral) so that it passes through the saddle point \( k_s \).

(ii) \( \text{Re} \ h(k) \) has its maximum on the contour just at the saddle point \( k_s \).

(iii) The contour can not be deformed so that a new maximum of \( \text{Re} \ h(k) \) is below \( \text{Re} \ h(k_s) \).

A detailed inspection of the integrand in (31) reveals the fact that the condition (ii) is not satisfied if we integrate in the direction perpendicular to the real axis in the complex \( k \)-plane. The curve of the steepest descent passes along the real axis. This fact invalidates the approach of [12] where the integration contour is not specified and seems to be chosen perpendicular to the real axis at the saddle point. Moreover, a detailed analysis of the saddle points shows their strong dependence on the value of \( z \). This crucial fact was also overlooked in [12], where \( k_s \approx b_1 (n - 1 + b') \) was taken. We illustrate the situation in Figs.2-5, where the positions of poles and saddle points of the integrand in (31) are displayed for
different values of \( z \) and for \( n = 4 \). One can notice an interesting interplay of poles and saddle points resulting in the situation when two complex conjugate saddle points become leading (i.e. with maximal \( \text{Re} \, k \)) for sufficiently large \( z \). The \( z \) dependence of the saddle point \( k_s = k_s(z) \) does not allow one to choose any universal (\( z \)-independent) integration contour in the \( k \)-plane. Hence, the use of the saddle point method is plagued with the occurrence of the moving leading saddle point \( k_s(z) \). Consequently, one cannot obtain the relation (\( \Phi(k) \) is defined in [12])

\[
d_n(q^2/\mu^2) \approx \frac{1}{2} \left[ \frac{2}{\pi \Phi(k_s)} \right]^{1/2} k_s^{n-1} D(q^2/\mu^2, 1/k_s) \cos n\pi, \tag{36}
\]

which plays a crucial role in the derivation of West’s result. (We have added the factor 1/2 which is missing in [12].) It was just this relation which made it possible, employing (22) and (23), to derive the simple relation (4) for the coefficients of the expansion (17). We would like to stress that even within West’s approach, based on the saddle point method, the \( d_n \) will also get contributions that cannot simply be related to the value of \( D(q^2/\mu^2, 1/k) \) for some large \( k \) (in (36) \( k_s \to \infty \) when \( n \to \infty \)). This follows from the \( z \)-dependence of the leading saddle point \( k_s \) mentioned above. The same argument can also be used for the case of finite \( k_0 \) in (30).

Moreover, the pole of the integrand in (34) generates, at finite \( k = k_0 \), an additional saddle point which disappears when \( k_0 \to \infty \), and for some \( z \), becomes the leading one. For these values of \( z \) saddle point method can be used to estimate the integral (34) because the conditions (i)–(ii) are satisfied. This is another signal that the saddle point method of approximating the \( d_n \) is not applicable in the form advocated by West.

4. Summing perturbation series in QCD

Using the asymptotic estimate (3) West has in his subsequent paper suggested [13] a way of handling the divergence of QCD perturbation series which has two remarkable properties:

1. it is extraordinarily close to the “correct” sum using only a few lowest order terms and, moreover,

2. the number of terms to be taken into account in his procedure is inversely proportional to the value of the couplant, which implies that it decreases as we approach the infrared region, i.e. as \( q^2 \to 0 \) in (2).

These properties are remarkable and, were they true, would have serious repercussions as the conventional wisdom tells us that at low \( q^2 \) the contribution of nonperturbative effects are expected to become dominant [19, 20]. As the message of [13] is, from the viewpoint of phenomenological applications, very appealing we
wish to point out several simple and obvious facts which show the arbitrariness of West’s construction. Basically what he does in [13] is that he inadmissibly generalizes the conclusions valid for one particular definition of the sum, which really has the above-mentioned properties, to the case of QCD perturbation expansion ([1]). Let us recall the basic fact that for the divergent asymptotic series

\[ \sum_{n=0}^{\infty} a_n z^n \]  

(37)

there is no “natural”, unique, sum. There is an infinite number of functions \( F(z) \) which have the series (37) as its asymptotic one and these functions while sharing the same behaviour in the vicinity of \( z = 0 \) may for any fixed \( z \neq 0 \) be arbitrarily far apart. Perturbation theory itself does not tell us which one of them to choose and we have to look somewhere else for ideas what to do with it [20, 21]. Without first defining what we mean under the “sum” of the asymptotic series (37), there is no sense in talking about the “error” of any finite order approximation. To claim, as is done in [13], that this error, defined as

\[ R_N = \left| F(z) - \sum_{n=0}^{N-1} a_n z^n \right|, \]  

(38)

\[ R_N \approx a_N z^N \]  

(39)

is therefore unsubstantiated.

The asymptoticity of the series (37) merely means that

\[ \lim_{z \to 0} \frac{R_N(z)}{a_{N-1} z^{N-1}} = 0 \]  

(40)

but, for a fixed \( z \neq 0 \), \( R_N \) may be quite arbitrary. In other words any statement concerning the error estimate (38) is meaningless unless we unambiguously specify the definition of the sum \( F(z) \) appearing in (38) as well. This holds equally well for Borel summable as well as non Borel summable series. In [13] the author bases his claims essentially on the property of a particular Borel summable series with the coefficients \( r_k = (-1)^k k! \). He demonstrates that by choosing the number \( N(a) \) of terms in (38), where this value is roughly inversely proportional to the expansion parameter \( a \) and given by the local minimum of the contribution of \( a_n z^n \) to the partial sum, he gets a very good approximation of the corresponding Borel sum. This is by no means the only possible approach, even for Borel summable series: one needs further information on the function \( F(z) \) itself (i.e. information not contained in the perturbation expansion) to guarantee the uniqueness of the Borel sum. To generalize the properties of this toy example series (coupled with one ad hoc definition of the sum) to the non Borel summable series in realistic QCD is an entirely arbitrary procedure. Moreover, the procedure suggested in
has numerous concrete shortcomings. We mention just the most important
ones:

- It is obvious that by adding to this toy example series any convergent one
we get the expansion with the coefficients which have the same large-$n$
behaviour while to any finite order the partial sums may be arbitrarily far
apart. Any meaningful statement concerning the accuracy of a given sum-
mation technique must take into account the true values of the expansion
coefficients, not merely their asymptotic behaviour. To determine the value
of $N(a)$ from this asymptotic behaviour is unsubstantiated.

- The simple picture in which the contributions of the terms $a_n z^n$ first mono-
tonously decrease and then starting at $N(a)$ increase forever is a property
of the toy example mentioned but in realistic QCD the behaviour of this
contribution may be much more complicated. In fact, for a function $F(z)$
to have (37) as its asymptotic series we can take $N(a)$ entirely arbitrary,
subject only to the condition that as $a \to 0$, $N(a) \to \infty$. For $a \to 0$
all these procedures do, of course, converge to the same result but for any
nonvanishing $a$ they may yield vastly different numbers. Consequently, also
the estimate (17) in [13] of the error of the procedure suggested is valid for
the example discussed but has no general validity. To give his claims some
justification West would have to argue why his definition of the estimate is
to be preferred to the infinite number of other possibilities.

- For the asymptotic series of the type $r_k \approx k!$, expected to occur in QCD,
there is no “natural” sum like the Borel one for the sign alternating series
$r_k \approx (-1)^k k!$. To define in these circumstances the estimate by means of the
same, above mentioned, procedure is unsubstantiated and has no physical
motivation behind it.

- Even if all the coefficients $r_k$ in (2) were known exactly (as in the toy exa-
ample of the series with coefficients $r_k = (-1)^k k!$), the summation procedure
suggested in [13] would still be itself badly ambiguous as the value of $N(a)$
and consequently also of the estimate defined as

$$\sum_{k=0}^{N(a)} r_k (\alpha_s/\pi)^k$$

would depend on the RS in that the truncation procedure is carried out.

5. Summary and conclusion

We have examined in detail the method recently proposed by West for the de-
termination of the large order behaviour of perturbation expansion coefficients in
QCD. Going carefully through his derivation, which is based exclusively on the renormalization group considerations and analyticity properties of the vacuum polarization function, we have identified several points where strong and largely unsubstantiated assumptions have to be made in order to arrive at the result (6); in particular

(A) the interchange of the limit $k_0 \to \infty$ and the integration over $k$ in (25);

(B) the interchange of integrations in the double integral in (28), which results in the inadmissible contour dependence of the result;

(C) the use of the saddle-point technique for the estimate of the contour integrals, in spite of the ignorance of the $z$-dependence of the positions of the leading saddle points.

A first signal that there is something wrong with the result (6) comes already from the fact that this formula for $r_n$ is independent of the RRS chosen. However, as is obvious from his reply [16] to the early critical voices [14, 17], West considers this property of his formula (6) on the contrary a merit. To clarify this crucial point we have discussed at length in particular the correct RS dependence of any consistent large order behaviour of the coefficients $r_k$. In general, we thus agree with the claim of Brown and Yaffe [14] that the information exploited by West is in principle insufficient for the derivation of the asymptotic estimates of these coefficients. Compared with their work, which shows what can be derived from these assumptions, we have attempted to demonstrate why the construction suggested by West doesn’t in fact work if carried out with proper care, and why there is no conceivable way to improve it and still arrive at an asymptotic estimate of the type (6). The large order behaviour of QCD perturbation expansions thus remains to large extent an open question which certainly deserves further attention. In our view, to give these expansions good mathematical as well as physical meaning we have to go beyond the purely perturbative framework and look, as in [20, 21, 22], to their interplay with various nonperturbative effects.

Acknowledgement

We wish to thank A.Kataev for careful reading of the manuscript and many useful comments and suggestions.

Appendix

First, let us derive the equation for the boundary of the principal strip in the $k$-plane. The relation

$$ q^2 / \mu^2 e^{2\kappa(g)} = z $$

(42)
transforms, at $q^2$ fixed, the complex $z$-plane into a strip in the $k$-plane. $K(g)$ is given by (46). Let us introduce the notation

$$k = u + iv, \quad q^2/\mu^2 = \sigma e^{i\varphi}, \quad \ln \sigma = t, \quad z = re^{i\psi}$$

(43)

and take the logarithm of (12) (ln stands for the principal branch):

$$\ln \sigma + u/b_1 - \frac{1}{2} b' \ln[(b_1u + b_2)^2 + b_1^2v^2] = \ln r$$

(44)

and

$$2\pi N + \varphi - \psi + v/b_1 - b'\arg(b_1k + b_2) = 0$$

(45)

The borders of the strip are given by the inequality

$$-\pi \leq \psi < \pi,$$

(46)

which represents the whole complex cut $z$-plane. Choosing $N = 0$ and $\varphi = 0$, we obtain the implicit equation for the upper bound of the strip in the form

$$v - b_2 \left( \tan(\tilde{v}_p + \gamma) - \frac{1}{2} \ln \tilde{v}_p^2 + \frac{1}{2} \ln[\sin^2(\tilde{v}_p + \gamma)] - B \right) = 0, \quad v > 0,$$

(47)

the lower border being symmetric with respect to the real axis.

We now turn to the contour dependence of the difference defined in (35):

$$\Delta d_n \equiv \tilde{d}_n - d_n = \lim_{k_0 \to \infty} \int_{z_1}^{z_2} dz f(z) \frac{\partial}{\partial t} \Phi_2(n, t, z, k_0),$$

(48)

where

$$\Phi_2(n, t, z, k_0) = -k_0^{n+1} \frac{k_p^{n-2}(b_1k_p + b_2)}{(k_p - k_0)^{n+1}A(k_p)},$$

(49)

and for $z \in (z_1, z_2)$ the pole $k_p$ occurs inside the integration contour. We note that a usual regularization is understood in the denominator factor $(k_p - k_0)$ of (49): we can add a small imaginary part to $k_0$. The function $k_p(z)$ solves the equation (33). Denoting $k_p = u_p + iv_p$ one can show that the positions of poles satisfy the equation

$$\frac{\tilde{v}_p}{\tan(\tilde{v}_p + \gamma)} - \frac{1}{2} \ln \tilde{v}_p^2 + \frac{1}{2} \ln[\sin^2(\tilde{v}_p + \gamma)] - B = 0,$$

(50)

where

$$\tilde{v}_p = \frac{b_1}{b_2} v_p,$$

$$\gamma = 2\pi N b_1^2/b_2, \quad N = 0, \pm 1, \pm 2, \ldots,$$

$$B = 1 + \ln b_2 + \frac{b_1^2}{b_2} (\ln z - t),$$

17
provided that \( \bar{v} + \gamma \neq n\pi, \ n = 0, \pm 1, \pm 2 \ldots \). Knowing \( v_p \) the real part \( u_p \) can then be calculated from the equation

\[
\tilde{u}_p \equiv u_p \frac{b_1}{b_2} = \frac{\tilde{v}_p}{\tan(\tilde{v}_p + \gamma)} - 1
\]

A different regime can be observed if \( \bar{v} + \gamma = n\pi \). These are the relevant poles because the integration contour is situated in the right half plane. (We consider only such contours which can be hit only by one pole, i.e., which are placed in the strip.) The poles on the real axis satisfy the equation

\[
\tilde{u}_p - \frac{1}{2} \ln[(1 + \tilde{u}_p)^2] - B = 0
\]  

Here the corresponding \( v_p \) equals, of course, zero. In the 5-flavour case, i.e., \( b_1 = 0.0485 \) and \( b_2 = 0.00155 \), we get that for \( z > 71.024 \) the equation (52) has two solutions, one with \( u_p \) positive, the other one with \( u_p \) negative. They move to opposite sides if \( z \) increases starting from the origin. If \( z < 71.024 \) we get no real poles in the strip. However, one can find two complex conjugate poles in the left half of the complex \( k \)-plane. The \( z \)-evolution of the poles is depicted in Figs.2-5. As is seen there are two poles which collide at \( z = 71.024 \) from either side along the imaginary axis and scatter along the real axis. In the following we will consider the pole moving along the positive real axis; let us denote its position by \( k_p(z) \), for \( z > 71.024 \).

Taking the derivative of (49), substituting the variable \( y = k_p(z) \) into the integral over \( z \) and defining

\[
F_{n,m}(k_0, y_1, y_2) = \int_{y_1}^{y_2} dy \ \bar{f}(y) \ \frac{y^n}{(y - k_0)^m}
\]

we get

\[
\Delta d_n = \lim_{k_0 \to \infty} k_0^{n+1} \{ b_1^2(n + 1)F_{n-1,n+2} + b_1b_2(n + 1)F_{n-2,n+2} - b_1^2(n - 1)F_{n-2,n+1} - b_1b_2(n - 2)F_{n-3,n+1} \}.
\]

Here \( y_i = k_p(z_i), \ i = 1, 2 \) are the intersections of the path of the pole with the contour, \( \bar{f}(y) = f(k_p^{-1}(y)) \); for simplicity, we have omitted the arguments of the functions \( F_{n,m} \).

Obviously, the behaviour of \( \Delta d_n \) depends on the contour chosen. The general proof of the behaviour of \( \Delta d_n \) is, however, complicated because the function \( f(z) \) is unknown. In realistic situations \( f(z) \) is positive and bounded from above by a constant. In the asymptotic region we consider the usual asymptotic expansion (17); which suggests to expand \( f(y) \) in powers of \( 1/y \). It is therefore instructive
to consider the case \( f(z) = 1/y^l \). For \( l = 0 \), we have

\[
F_{n,m|\tilde{f}=1} = -\sum_{j=0}^{n} \binom{n}{j} \frac{k_0^{n-j}}{m-j-1} \times \\
\left( \frac{1}{(y_2 - k_0)^{m-j-1}} - \frac{1}{(y_1 - k_0)^{m-j-1}} \right).
\]

(55)

We can now check the different contours; simple examples are (a) \( y_1 - k_0 = \text{const} \) and \( y_2 - k_0 = \text{const} \) and (b) the contour \( C_0' \), i.e., \( y_1 = \text{const} \) and \( y_2 = \infty \). It can be seen from eq. (54) that in these examples \( \Delta d_n \) diverges when \( k_0 \to \infty \).

A similar analysis can also be carried out for \( f(z) = 1/y^l, \ l > 0 \). The divergent parts cannot compensate each other, so we are led to the conclusion that the \( \Delta d_n \) are typically divergent. Concluding this appendix we would like to stress that the contour dependence of \( \tilde{d}_n \) is quite unacceptable.

References

[1] S.G. Gorishny, A.L. Kataev and S.A. Larin, Phys. Lett. 259B, 144 (1991).
[2] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. 259B, 345 (1991).
[3] J. Chýla, A.L. Kataev and S.A. Larin, Phys. Lett. 267B, 269 (1991).
[4] J. Chýla and A.L. Kataev, preprint CERN-TH.6604/92.
[5] F.J. Dyson, Phys.Rev. 85, 861 (1952).
[6] G. Parisi, Phys. Lett. 66B, 167 (1977); Phys. Lett. 76B, 65 (1978).
[7] N.N. Khuri, Phys. Rev. D 16, 1754 (1977).
[8] P.M. Stevenson, Nucl. Phys. B 231, 65 (1984).
[9] D.T. Barclay and C.J. Maxwell, Phys. Rev. D 45, 1760 (1992).
[10] G.’t Hooft in The Whys of Subnuclear Physics, Proceedings of the 1977 International School of Subnuclear Physics, Erice, Sicily 1977, Plenum Press, New York 1979, Ed. A. Zichichi, p.943; A.S. Wightman, ibid., p.983.
[11] A.H. Mueller, Nucl. Phys. B 250, 327 (1985).
[12] Geoffrey B. West, Phys. Rev. Lett. 67 1388 (1991), Erratum Phys. Rev. Lett. 3732 67 (1991).
[13] Geoffrey B. West, Los Alamos National Laboratory preprint LA-UR-91-3460 (1991).

[14] L.S. Brown and L.G. Yaffe, Phys. Rev. D 45, R398 (1992).
    L.S. Brown, L.G. Yaffe and C. Zhai, University of Washington preprint UW/PT-92-07 (1992).

[15] P.M. Stevenson, Phys. Rev. D 23, 2916 (1981).

[16] Geoffrey B. West, Los Alamos National Laboratory preprint LA-UR-92-1849 (1992).

[17] D.T. Barclay and C.J. Maxwell, University of Durham preprint DTP-91/72, October 1991.

[18] G. Grunberg, Phys. Lett. 95B, 70 (1980); Phys. Rev. D29, 2315 (1984).

[19] E. Bogomolny and V.A. Fateev, Phys. Lett. 71B, 93 (1977).

[20] V.I. Zakharov, Max Planck Institute preprint MPI-Ph/92-33, Munich, April 1992.

[21] A.H. Mueller, talk given at the Conference QCD-20 Years Later, Aachen 1992.

[22] J. Chýla, Czech. J. Phys. 42, 263 (1992).
Figure Captions

Fig.1 The integration contours in the $k$-plane (see Sec. 3).

Fig.2 The $z$-evolution of the positions of poles inside the principal strip in the $k$-plane. Two complex conjugate poles “collide” at the origin (for $z = 71.024$) and “scatter” along the real axis.

Fig.3 The $z$-evolution of the positions of the leading saddle points. Two real saddle points “collide” and set out to the complex plane. The saddle point near $k_0$ completely disappears for $k_0 \to \infty$ (see eq. (31)). The stars correspond to the positions of the saddle points at $z = 71.024$.

Fig.4 The plot of the real part of the logarithm of the integrand in eq. (30). The values chosen are: $n = 4$, ln $z = 10$, $t = 0$ and $k_0 = 1$.

Fig.5 The contour plot corresponding to the situation of Fig.4. The resolution is not sufficient to detect the pole located to the left of the origin.