Poisson Brackets on the Space of Histories

Donald Marolf

Physics Department, The Pennsylvania State University, University Park, PA 16802
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Abstract

We extend the Poisson bracket from a Lie bracket of phase space functions to a Lie bracket of functions on the space of canonical histories and investigate the resulting algebras. Typically, such extensions define corresponding Lie algebras on the space of Lagrangian histories via pull back to a space of partial solutions. These are the same spaces of histories studied with regard to path integration and decoherence. Such spaces of histories are familiar from path integration and some studies of decoherence. For gauge systems, we extend both the canonical and reduced Poisson brackets to the full space of histories. We then comment on the use of such algebras in time reparameterization invariant systems and systems with a Gribov ambiguity, though our main goal is to introduce concepts and techniques for use in a companion paper.
I. INTRODUCTION

Formulations of quantum theories can be roughly divided into two classes. The first follows the algebraic approach in which a commutator \(*\)-Algebra is defined and a Hilbert space representation is sought, while the second defines “transition amplitudes” by path integration over some space \(\mathcal{H}\) of histories and then interprets these amplitudes as matrix elements of operators in a Hilbert space. Both here and in the companion paper \([1]\), we will be concerned with the algebraic approach and, more specifically, with the classical (commuting) \(*\)-Lie algebra on which the quantum commutator is often based.

This classical algebra is the algebra of complex functions on some space with the usual operations of multiplication, addition, and complex conjugation (\(*\)) supplemented by a Lie bracket operation. For gauge-free systems, this may be the Poisson algebra \(\mathcal{A}_H(\Gamma)\) of complex functions on the phase space \(\Gamma\) or the Peierls algebra \(\mathcal{A}_L(S)\) of functions on the space \(S\) of solutions to the equations of motion. The subscripts \(H\) and \(L\) refer to the Hamiltonian and Lagrangian methods associated with the construction of these algebras. \(\mathcal{A}_H(\Gamma)\) and \(\mathcal{A}_L(S)\) are isomorphic \([2]\) under any map that takes \(S\) to \(\Gamma\) by evaluating the phase space coordinates at some time \(t\).

For systems with gauge symmetries we may pursue either option in the full theory and, in addition, we may choose to keep the gauge symmetry intact, or we may take the quotient by gauge transformations and consider reduced dynamics. This leads to two Poisson algebras, \(\mathcal{A}_H(\Gamma)\) of Dirac \([3]\) on the full phase space \(\Gamma\) and \(\mathcal{A}_H(\Gamma_r)\) on the reduced phase space \(\Gamma_r\), as well as two Peierls algebras, \(\mathcal{A}_L(S_r)\) on the space \(S_r\) of reduced solutions and \(\mathcal{A}_L^{GI}(S)\) defined on gauge invariant functions on \(S\). The Peierls bracket of gauge dependent functions is not defined.

Each algebra has advantages and disadvantages. For example, \(\mathcal{A}_H(\Gamma)\), \(\mathcal{A}_H(\Gamma_r)\), and \(\mathcal{A}_L(S_r)\) include all smooth functions on the relevant spaces and may often be expressed as the set of suitable combinations of a small number of functions, such as the canonical coordinates and momenta, that have simple algebraic properties. This greatly simplifies
factor ordering during quantization as well as many Lie bracket computations. However, the
reduced spaces, $\Gamma_r$ and $S_r$, and algebras, $A_H(\Gamma_r)$ and $A_H(\Gamma)$, may be difficult to construct,
especially when gauge fixing is impossible. Additionally, it may be desirable to display the
gauge symmetries explicitly.

For relativistic field theories, another important distinction is that $A_L(S_r)$ and $A_L(S)$ are
manifestly covariant while $A_H(\Gamma_r)$ and $A_H(\Gamma)$ are defined only after a 3+1 decomposition of
spacetime. Manifest covariance simplifies verification of the quantum theory and facilitates
such tasks as the derivation [4] of the measure for the covariant path integral. A related
remark is that, since $A_H(\Gamma)$ and $A_H(\Gamma_r)$ are defined on functions at some single time, they
lead most naturally to Schrödinger picture quantum mechanics while the Peierls bracket
leads most directly to Heisenberg picture quantization since $A_L(S_r)$ and $A_L(S)$ include
functions defined at all times and even functions with nonlocal spacetime support. We
note that a Heisenberg picture is particularly useful for a time reparameterization invariant
system since it allows reparameterization invariant operators to be defined by integration
over time.

Despite these differences, three of these algebras are isomorphic. Functions on $S_r$ are just
gauge invariant functions on $S$ and evaluating the reduced phase space coordinates at some
time $t$ maps $S_r$ to $\Gamma_r$. $A_H(\Gamma)$ is different, however, as some points in $\Gamma$ do not correspond to
solutions or points in $\Gamma_r$ and evolution on $\Gamma$ is not uniquely defined. We will find it useful
to partially bridge this gap by defining a space $E$ of all possible evolutions of points in $\Gamma$.

Algebraic structures on these spaces have been compared in [3], [4], and [5]. Of particular
interest is [3] which considers structures on the space $H$ of histories as well. $H$ is the domain
space of the system’s action functional so that it is typically the space of suitable fields on
some manifold $M$. This space is called $F$ in [3] and $\Phi$ in [4]; we use $H$ to avoid confusion
with other notation introduced here and in [4]. $H$ is an especially useful space to consider
since it contains both $S$ and $E$ as subspaces and from these $\Gamma$, $\Gamma_r$, and $S_r$ can be reached by
projection. In addition, studies of structures on $H$ may help to connect the algebraic and
path integral approaches to quantization.
References [5], [6], and [7] discuss a presymplectic structure on $S$ and $H$. While useful for other purposes, the presymplectic form is degenerate and cannot be inverted to define a Lie bracket of functions on $S$ and $H$. Quantization on $S$ and $H$ cannot proceed until this degeneracy is removed.

Our goal here and in [1] is the construction of Lie algebras $A_L(H)$ of complex functions on $H$ and thus on $E$ and $S$ by pull back and on $\Gamma$, $\Gamma_r$, and $S_r$ by projection. As with the presymplectic structure, these algebras reflect the gauge structure – often by becoming degenerate. As a result, the contravariant tensor field that defines the Lie bracket cannot be inverted to define a symplectic form. Thus, our study is complementary to that of [5], [6], and [7].

The construction of these algebras will be presented in [1] and is based on the Peierls bracket. In particular, it extends $\mathcal{A}^GI(S)$ to $\mathcal{A}^GI(H)$ and then generalizes this algebra to $\mathcal{A}_L(H)$ using the methods of [4]. However, much of this development can be described in terms of extensions $\mathcal{A}_H(H)$ of the Poisson algebras $\mathcal{A}_H(\Gamma)$ and $\mathcal{A}_H(\Gamma_r)$ to $H$. Such a description serves two purposes. By acting as an intermediate step, it greatly facilitates comparisons of $\mathcal{A}_L(H)$ with $\mathcal{A}_H(\Gamma)$ and $\mathcal{A}_H(\Gamma_r)$ and thus with conventional quantization methods and, because the Poisson bracket is more familiar than the Peierls bracket, it allows the introduction of useful concepts, such as the space $E$ of evolutions, and techniques, such as defining algebras locally, pulling back algebras from $H$, and “gauge breaking” without heavy machinery. This is the subject of the work below.

Our study begins in section II where $\mathcal{A}_H(H)$ is introduced for gauge-free systems. For gauge systems, $\mathcal{A}_H(H)$ is defined in IIIA as an extension of $\mathcal{A}_H(\Gamma)$ and in IIIB as an extension of $\mathcal{A}_H(\Gamma_r)$ using “gauge breaking” – a sort of generalization of gauge fixing that may be performed in the presence of a Gribov ambiguity. Because the Poisson bracket is grounded in the canonical formalism, sections II and III refer to the space $H$ of canonical histories of phase space coordinates, but section IV shows that such extensions also exist on spaces of so-called Lagrangian histories defined by more general fields. Section V explores the implications of using $\mathcal{A}_H(E)$ and $\mathcal{A}_H(S)$ for quantization, comparing this with quantization
of $\mathcal{A}_H(\Gamma)$ and $\mathcal{A}_H(\Gamma_r)$. We then close with a summary discussion and two appendices. Appendix A constructs a globally defined Lie algebra of functions on a manifold from a set of compatible algebras defined locally and appendix B describes the extension of the Dirac bracket to $\mathcal{H}$.

II. UNCONSTRAINED SYSTEMS

In this section we describe the extension of the Poisson bracket to a Lie bracket on the space $\mathcal{H}$ of histories for a system presented in unconstrained canonical form. Recall that a Lie bracket is a bilinear antisymmetric operation $(,)$ that satisfies the Jacobi identity and the derivation requirement:

$$(AB, C) = A(B, C) + (A, C)B \quad (2.1)$$

Consider a system with no gauge symmetries which is kinematically described by a phase space $\Gamma$ during the time interval $I = [t_i, t_f]$. Histories of this system lie in the space $\Gamma^I$ given by the direct product of one copy $\Gamma_t$ of the phase space $\Gamma$ for each time $t$ in the interval $I$. If $z^A$ for $A$ in some index set $\mathcal{I}$ are canonical coordinates on $\Gamma$ then $z^i$ for $i = (A, t) \in \mathcal{I} \times I$ are coordinates on $\Gamma^I$. We take the dynamics of this system to be governed by a Hamiltonian $H(t) = H(z^A(t), t)$ and a Poisson bracket $\{z^A, z^B\} = \Omega^{AB}$ or, equivalently, by a first order action principle of the form

$$S(z^i) = \int_{t_i}^{t_f} dt \left( \frac{1}{2} \Omega^{-1}_{AB} z^A(t) \dot{z}^B(t) - H(t) \right) \quad (2.2)$$

for some time and field independent invertible antisymmetric matrix $\Omega^{AB}$. We now define our space $\mathcal{H}$ of histories to be that subspace of $\Gamma^I$ that lies in the domain of $S$.

As described in the introduction, the phase space $\Gamma$ is isomorphic to the space $\mathcal{S}$ of solutions. This isomorphism carries the Poisson bracket $\{,\}$ to a Lie bracket $(,)_S$ on $\mathcal{S}$. We seek an extension of the Poisson bracket to $\mathcal{H}$ in the sense that any Lie bracket $(,)_\mathcal{H}$ that we define on $\mathcal{H}$ should have a well-defined pull back through the inclusion map $i : \mathcal{S} \rightarrow \mathcal{H}$ that coincides with $(,)_S$: 
\{ F \circ i, G \circ i \}_S = (F, G)_H \circ i \quad (2.3)

If this is to be well-defined, we must have \((F, G)_H \circ i = (J, K)_H \circ i\) whenever \(F \circ i = J \circ i\) and \(G \circ i = K \circ i\). Equivalently, we could have \((F, G)_H \circ i = 0\) whenever \(F \circ i = 0\) and in particular \((S, i), A)_H \circ i = 0\) for all \(A\). Here, the comma denotes the (functional) derivative with respect to the coordinate \(z^i\) and we use the condensed notation of [4].

Assuming that \(S\) is sufficiently smooth and \(S, i\) is appropriately behaved, the condition \((S, i), A)_H = 0\) would be enough to guarantee that the bracket of sufficiently smooth functions pulls back to \(S\). However, we choose to impose the stronger condition that \((S, i), A\) should vanish identically on \(H\). We will refer to this as the condition for \((,)_H\) to “respect” the equations of motion \(\{ S, i = 0 \}\). Note that this choice is coordinate dependent as it is not preserved under passing to new coordinates \(Z^j = Z^j(z^i)\) unless \(\frac{\partial z^i}{\partial Z^j}\) has identically vanishing bracket. If this change of variables ultralocal in time, 2.3 implies that the transformation is linear.

This coordinate dependence could be removed by choosing a suitable set of functions \(\{ f \}\) on \(H\) that vanish on \(S\) to satisfy \((f, A)_H = 0\) for all \(A\) in place of \(\{ S, i \}\). The coordinate dependence thus becomes a dependence on the set of functions chosen. However, because the definition of “suitable” functions is complicated and because the form of the equations of motion \(S, i\) defined by the coordinates \(z^i\) is particularly convenient, we will use the coordinate dependent formulation. This also eases comparison with [4] in which coordinate independent extensions of the Peierls bracket are defined by torsion-free connections instead of by sets of functions.

Appendix A shows how a Lie bracket of functions on a manifold \(M\) can be assembled from a set of Lie brackets of functions on patches \(U_i \subset M\) when these algebras agree in the overlap regions \(U_i \cap U_j\). To ensure this agreement, we choose coordinates defined globally on \(\Gamma\) or defined in patches such that the transitions functions are linear. We will refer to a manifold described by such coordinates as a linearized manifold or as a manifold with a linearized structure. While it is not entirely clear which differential manifolds are diffeomorphic to
linearized manifolds, we note that linearized structures for the sphere, torus, and other simple but nontrivial manifolds can be readily constructed and proceed with our discussion. The conditions $(S, i, A) = 0$ are then imposed on the local algebra of each patch.

Because of the derivation requirement 2.1, which implies that $(F, G)_H = F_{;i} (z^i, z^j)_H G_{;j}$, the extended bracket is entirely determined by its action $(z^i, z^j)_H$ on coordinate functions. Since $(z^A(t), z^B(t))_H \circ i = \{z^A(t), z^B(t)\}_S \equiv \Omega^{AB}$ and $\Omega^{AB}$ is a matrix of constant functions on $S$, we define $(z^A(t), z^B(t))_H$ to be the corresponding matrix of constant functions on $H$.

The use of this extension represents another choice that we have made, but this time a coordinate independent one.

From this choice, the bracket $(z^i, z^j)_H$ of coordinate functions at different times is specified uniquely by our condition that the algebra respect the equations of motion $\{S, i = 0\}$. To see this, we note that for $i = (A, t)$ we have:

$$0 = (S, i, z^B(t'))_H = (\Omega^{-1}_{CA} \dot{z}^A(t) - H_{[C}(t), z^B(t'))_H$$

where $|C$ denotes a derivative with respect to $z^C$ on $\Gamma$. It follows that

$$0 = \frac{\partial}{\partial t} (z^A(t), z^B(t'))_H - \Omega^{AC} H_{[CD}(z^D(t), z^B(t'))_H$$

in which the combination $(\Omega \circ H)^B_C = \Omega^{BD} H_{[DC}$ acts like a connection and propagates $(z^A(t), z^B(t'))_H$ from one time to another. The solution to 2.5 is thus

$$(z^A(t_1), z^B(t_2))_H = T_L(t_2, t_1)^A_C \Omega^{CB} = \Omega^{AC} T_R(t_2, t_1)_C^B$$

where

$$T_L(t_2, t_1)^A_C = \mathcal{P} \exp[\int_{t_2}^{t_1} dt \Omega_{|C}]$$

and $\mathcal{P}$ denotes path ordering. Our bracket $(\cdot, \cdot)_H$ is then defined by 2.6 and 2.1.

To see that $(\cdot, \cdot)_H$ is in fact a Lie bracket on $H$, note that the derivation property is manifest from the construction and that antisymmetry and the Jacobi identity will follow if we establish that $(z^i, z^j)_H = -(z^j, z^i)_H$ and

7
\[ \sum_{i,j,k \in T} \epsilon_{ijk} ((z^i, z^j)_{\mathcal{H}}, z^k)_{\mathcal{H}} = 0 \quad (2.8) \]

for any three element subset \( T \) of \( I \times I \) and any antisymmetric symbol \( \epsilon_{ijk} \) on \( T \). The symmetry of \( H_{|AB} \) implies that \( H_{|AC} \Omega_{CB} = -\Omega_{BC} H_{|CA} \) and we have the useful property:

\[ \mathcal{P} \exp\left[ \int_{t_1}^{t_2} dt \ H \circ \Omega_{|C}^B \right] = \mathcal{P} \exp\left[ \int_{t_2}^{t_1} dt \ \Omega \circ H_{|C}^B \right] \quad (2.9) \]

Antisymmetry then follows directly:

\[ (z^A(t_1), z^B(t_2))_{\mathcal{H}} = \Omega_{AC}^B T_R(t_1, t_2)_{B}^B = -T_L(t_2, t_1)_{C}^B \Omega_{CA}^C = -(z^B(t_2), z^A(t_1))_{\mathcal{H}} \quad (2.10) \]

Properties 2.6 and 2.9 can also be used to verify the Jacobi identity. A short calculation shows that

\[ ((z^{A_1}(t_1), z^{A_2}(t_2))_{\mathcal{H}}, z^{A_3}(t_3))_{\mathcal{H}} = -\int_{t_1}^{t_2} dt \ H_{|B_1B_2B_3}(t) \prod_{i=1}^{3} (z^{B_i}(t), z^{A_i}(t))_{\mathcal{H}} \quad (2.11) \]

so that the cyclic sum in 2.8 vanishes due to the symmetry of \( H_{|B_1B_2B_3} \). We have thus succeeded in extending the Poisson bracket for an unconstrained system through 2.6 and 2.1 to a Lie bracket \((,)_\mathcal{H}\) on \( \mathcal{H} \). We turn now to systems with gauge symmetries and constraints.

### III. GAUGE SYSTEMS

This section develops extended Poisson algebras for gauge systems and other constrained systems. Two types of Poisson algebra will be of interest, those defined through canonical procedures [3] and those defined through gauge fixing or other reduced phase space procedures. Each of these will be addressed in a separate subsection, though we will see that the two are quite similar and that both follow from a more general “gauge breaking” scheme. The last subsection compares the canonical case with canonical gauge fixing.

#### A. Phase Spaces with Constraints

Consider a system described not by an action of the form 2.2, but by an action of the form:

\[ \sum_{i,j,k \in T} \epsilon_{ijk} ((z^i, z^j)_{\mathcal{H}}, z^k)_{\mathcal{H}} = 0 \]
\[
S = \int_{t_i}^{t_f} dt [\frac{1}{2} \Omega_{AB}^1 z^A \dot{z}^B - H_0(t) - \lambda^a(t) \phi_a(t)]
\] (3.1)

for \( a \) in some index set \( \mathcal{G} \), \( A \in \mathcal{I} \), \( \Omega^{AB} \) an invertible antisymmetric time and field independent matrix, \( H_0(t) = H_0(z^A(t), t) \), and \( \phi^a = \phi^a(z^A(t), t) \). For convenience, we introduce an index \( \alpha = (a, t) \in \mathcal{G} \times \mathcal{I} \) such that \( \lambda^\alpha = \lambda^a(t) \).

Let the Lagrange multipliers \( \lambda^a(t) \) at time \( t \) take values in some space \( \Lambda \). \( S \) is then a function on a space \( \mathcal{H} \subset \Gamma^I \times \Lambda^I \) of paths through \( \Lambda \) and paths through \( \Gamma \). The variation of this action with respect to the Lagrange multipliers \( \lambda^\alpha \) enforces the constraints \( \phi_\alpha = 0 \).

It will be convenient to regard \( \mathcal{H} \) as a union of the subspaces \( \mathcal{H}_{c^\alpha} = \{(x, c^\alpha) | x \in \Gamma^I, (x, c^\alpha) \in \mathcal{H}\} \) on which \( \lambda^\alpha = c^\alpha \) and to consider the spaces of solutions \( \mathcal{S}_{c^\alpha} \subset \mathcal{H}_{c^\alpha} \) on which the restriction \( S_{c^\alpha} \) of \( S \) to \( \mathcal{H}_{c^\alpha} \) is stationary. Stationarity of \( S_{c^\alpha} \) imposes no constraints on the phase space so that, using some set of coordinates, section \ref{sec:extension} defines an extension \( \{,\}_{c^\alpha} \) of the Poisson bracket from \( \mathcal{S}_{c^\alpha} \) to \( \mathcal{H}_{c^\alpha} \). To build a Lie bracket on \( \mathcal{H} \) from the \( \{,\}_{c^\alpha} \), define

\[
(F, G)_\mathcal{H}(p) \equiv F_i^a(p) \{z^i, z^j\}_{c^\alpha}(x) G_{ij}(p)
\] (3.2)

for \( p = (x, c^\alpha) \in \mathcal{H}, x \in \Gamma^I, F, G \) any two smooth functions on \( \mathcal{H} \), and where the functions in the bracket \( \{,\}_{c^\alpha} \) are the restrictions of the indicated functions on \( \mathcal{H} \) to \( \mathcal{H}_{c^\alpha} \). The Lie bracket properties of \( (,)_\mathcal{H} \) follow from those of \( (,)_{c^\alpha} \). Note that \ref{eq:lie_bracket} depends on the Lagrange multipliers \( \lambda^\alpha \) through the parameterized Poisson bracket \( (,)_{c^\alpha} \) as well as through \( F \) and \( G \).

The resulting algebra once again depends on the choice of coordinates on \( \Gamma \) but is invariant under linear changes of coordinates. It follows that we may consistently define \( (,)_\mathcal{H} \) when \( \Gamma \) is a linearized manifold. Note, however, that \( (,)_\mathcal{H} \) is completely independent of the choice of coordinates on \( \Lambda^I \).

Because \( (A, \phi_\alpha)_\mathcal{H} \neq 0 \) for some \( A \), this algebra does not have a well-defined pull back to \( \mathcal{S} \). It does, however, have a well defined pull back to the space \( \mathcal{E} = \cup_{\lambda^\alpha} \mathcal{S}_{\lambda^\alpha} \) which we will call the space of canonical evolutions. This pull back is independent of smooth coordinate.
transformations on either $\Gamma$ or $\Lambda^I$. In this way, $\mathcal{E}$ can play a role for constrained systems similar to that of $\mathcal{S}$ for unconstrained systems and we will use it to build a quantum theory in $\mathcal{V}$. Note that $\mathcal{E}$ is isomorphic to $\Gamma \times \Lambda^I$ as a point in $\mathcal{E}$ describes the evolution of a point in phase space under the parameterized Hamiltonian $H + \lambda^a(t)\phi_a(t)$. An equivalent definition of $\mathcal{E}$ is therefore “that subspace of $\mathcal{H}$ on which the equations of motion $\{S, i = 0\}$ are satisfied but $\{S, \alpha = 0\}$ are not.”

We now have a Lie bracket $(\cdot, \cdot)_\mathcal{H}$ on $\mathcal{H}$ for constrained systems that maps to the Poisson bracket under pull back to $\mathcal{E}$ and projection to $\Gamma$. Interestingly, the above procedure defines such an extension even when some constraints are second class, though, the presence of second class constraints would allow us to extend the Dirac bracket as well. This option is discussed in appendix B.

### B. Gauge Breaking

Following the canonical approach of is not essential as Lie brackets associated with gauge systems may also be defined through gauge fixing or other reduced phase space techniques. The reduced Poisson algebra $\mathcal{A}_H(\Gamma_r)$ extends to $\mathcal{A}_H(\mathcal{H}_r)$ on $\mathcal{H}_r$ by section and we will see that it extends to $\mathcal{H}$ as well. We first consider the case where $\mathcal{H}_r$ has been embedded as a surface $\mathcal{H}_{gf}$ in $\mathcal{H}$ by some gauge fixing procedure.

Much as with our extension of the Poisson bracket from $\mathcal{S}$ to $\mathcal{H}$ in section, we will see that this extension depends not so much on the space $\mathcal{H}_{gf}$ as on a particular set of functions (the gauge fixing functions $P^\alpha$) that define $\mathcal{H}_{gf}$ through the condition $P^\alpha = c^\alpha$ for some $c^\alpha \in \mathcal{R}$. If these same functions are used to define a different gauge fixed slice $\mathcal{H}'_{gf}$ by $P^\alpha = c'^\alpha$, the resulting bracket $(\cdot, \cdot)_\mathcal{H}$ on $\mathcal{H}$ will not be altered. However, if a different set of functions $P'^\alpha$ are used to define the same surface $\mathcal{H}_{gf}$, our construction may define a different bracket $(\cdot, \cdot)'_\mathcal{H}$ on $\mathcal{H}$. It would thus be incorrect to describe $(\cdot, \cdot)_\mathcal{H}$ as a gauge fixed algebra and we will refer to it as “gauge broken.”

We will define $(\cdot, \cdot)_\mathcal{H}$ by introducing a local product structure on $\mathcal{H}$ that selects gauge
fixed local slices. Such a structure can be introduced more generally than a gauge fixed global slice and exists whenever \( \mathcal{H} \) is a fibre bundle of gauge orbits over \( \mathcal{H}_r \). When this product structure is global, the gauge broken algebra pulls back to the appropriate gauge fixed algebra on any cross section.

However, since our general construction will involve local gauge fixing, it will be simplest to disentangle locality from the procedure by first examining the case in which the product structure is global. We will then see that this analysis could have been performed locally, in patches, and that the resulting local algebras may be assembled into a globally defined Lie bracket as described in Appendix A.

We therefore begin with a system described by an action \( S \) on a space of histories \( \mathcal{H} \) with gauge invariances indexed at each time by \( a \in \mathcal{G} \). We assume that this system may be gauge fixed to a slice \( \mathcal{H}_{c^\alpha} \) by choosing values \( c^\alpha \) for a set of global gauge fixing functions \( P^\alpha \).

Specifically, we require that the variation of the restriction \( S_{c^\alpha} \) of \( S \) to \( \mathcal{H}_{c^\alpha} \) has no gauge invariances and that \( \mathcal{H}_{c^\alpha} \) is transverse to the orbits. If \( S_{c^\alpha} \) takes the canonical form 2.2, it defines a bracket on \( \mathcal{H}_{c^\alpha} \) in the manner discussed in section II after choosing linearized coordinates \( z^A \) on the gauge fixed phase space \( \Gamma_{c^\alpha} \).

However, if \( S \) is in the canonical form 2.2 and the gauge breaking is canonical (that is, if \( P_a(t) \) depends only on \( z^A(t) \)) then \( S_{c^\alpha} \) will take the form 2.2 only after pull back to a smaller space \( \mathcal{H}'_{c^\alpha} \) in which some of the equations of motion generated by \( S_{c^\alpha} \) have been solved. The relevant equations can be divided into the constraints \( \phi_{c^\alpha} \) and another set also indexed by \( \Lambda^I \). This other set arises by varying \( S_{c^\alpha} \) with respect to the quantities conjugate to \( P_a(t) \) in the sense of the canonical Poisson bracket and which take the form of a further constraint on \( \Gamma_{t,c^\alpha(t)} \times \Lambda_t \). Here, \( \Gamma_{t,c^\alpha(t)} \) is the subspace of \( \Gamma_t \) on which \( P_a(t) = c_a(t) \) and \( \Lambda_t \) is the appropriate copy of \( \Lambda \).

These equations can then be solved for the Lagrange multipliers \( \lambda^a(t) \) and some set of fields \( q^a(t) \) on \( \Gamma_t \) in the form

\[
\lambda^a(t) = \lambda^a(z^A(t), t) \quad (3.3a)
\]
so that we can use much the same technique as in III A to define \( q^a(t) = q^a(z^{A'}(t), t) \) \(^{(3.3b)}\)

where \( z^{A'} \) and \( q^a \) are independent functions on \( \Gamma_{tc^a(t)} \) so that the \( z^{A'}(t) \) pull back to coordinates on the phase space \( \Gamma'_{tc^a(t)} \subset \Gamma_{tc^a(t)} \) in which Eq. \(^{(3.3b)}\) hold. We assume that \( q^a \) and \( z^{i''} \) on \( \mathcal{H}_{c^a} \) are the pull backs of smooth functions \( q^a \) and \( z^{i''} \) on \( \mathcal{H} \) and require that \( \{q^a(t), z^{A'}(t)\}_t = 0 \) and \( \{q^a(t), P^{\beta}(t)\}_t = \gamma^{ab} \) where \( \{\cdot, \cdot\}_t \) is the canonical Poisson bracket on \( \Gamma_t \) and \( \gamma^{ab} \) is some matrix that depends on fields only through \( P^a \), so that the choice of \( z^{i''} \) determines \( q^a \) up to the choice of \( \gamma^{ab} \). For canonical gauge fixing, the restriction \( S'_{c^a} \) to \( \mathcal{H}'_{c^a} \) can be written in the form \(^{[72]}\) using the coordinates \( z^{A'}(t) \) so that the methods of section \(^{[72]}\) define a Lie bracket \( (\cdot, \cdot)_{c^a} \) on \( \mathcal{H}'_{c^a} \). We refer to this type of gauge breaking as “based on canonical gauge fixing.”

The form of Eq. \(^{(3.3)}\) allows \( (\cdot, \cdot)_{c^a}' \) to be extended to a bracket \( (\cdot, \cdot)_{c^a} \) on \( \mathcal{H}_{c^a} \). To do so, we first introduce a derivative operator: \( \iota' \) that takes derivatives with respect to \( z^{i''} \) along curves of constant \( \frac{\delta S}{\delta q} \bigg|_{z^{i''}, P^\gamma, \lambda^\beta} \) and constant \( \frac{\delta S}{\delta A} \bigg|_{z^{i''}, P^\gamma, q^\beta} \). This poses no difficulties since these two expressions are ultralocal in time on \( \mathcal{H}_{c^a} \). We then define:

\[
(F, G)_{c^a}(p) = F_{i''}(p) \ (z^{i''}, z^{j''})'_{c^a} (z'(x)) \ G_{j''}(p) \quad (3.4)
\]

for \( i' = (A', t), \ p = (x, c^a) \in \Gamma' \times \Lambda' = \mathcal{H}_{c^a}, \ z' \) the map from \( \Gamma' \) to \( \Gamma'' \) with components \( z^{i''}, F, G \) any two smooth functions on \( \mathcal{H}_{c^a} \), and where the functions in the bracket \( (\cdot, \cdot)'_{c^a} \) are the restrictions of the indicated functions on \( \mathcal{H}_{c^a} \) to \( \mathcal{H}'_{c^a} \).

Whether or not the gauge breaking was canonical, we now have a Lie bracket \( \{\cdot, \cdot\}_{c^a} \) on each slice \( \mathcal{H}_{c^a} \) of a foliation of \( \mathcal{H} \) that was defined using the pull back to \( \mathcal{H}_{c^a} \) of some set of functions \( \phi^\mu \) on \( \mathcal{H} \) as coordinates on the slices. Together, \( \phi^\mu \) and \( P^a \) form coordinates on \( \mathcal{H} \) so that we can use much the same technique as in \(^{[72]}\) to define \( (\cdot, \cdot)_{\mathcal{H}} \):

\[
(F, G)_{\mathcal{H}}(p) = F_{,\mu}(p) \ (\phi^\mu, \phi^\nu)_{c^a}(x) \ G_{,\nu}(p) \quad (3.5)
\]

for \( p = (x, c^a) \in \mathcal{H}, \ x \in \mathcal{H}_{c^a}, \ F, \ G \) any two functions on \( \mathcal{H} \), and where \( F_{,\mu} \) is a derivative with respect to \( \phi^\mu \) in \( (\phi^\nu, P^a) \) coordinates, and the functions in the bracket \( (\cdot, \cdot)_{c^a} \) are the
restrictions to $\mathcal{H}_{c\alpha}$ of the indicated functions on $\mathcal{H}$. The Lie bracket properties of $(\cdot, \cdot)_{\mathcal{H}}$ follow from those of $(\cdot, \cdot)_{c\alpha}$ and $(\cdot, \cdot)'_{c\alpha}$.

We now note that this bracket is independent of linear changes of the coordinates $\phi^\mu$ and is completely invariant under replacement of the $P^\alpha$ by arbitrary functions of themselves. We may therefore relax our assumption that $(\phi^\mu, P^\alpha)$ form a global coordinate chart on $\mathcal{H}$ and that $\mathcal{H}_{c\alpha}$ is a global section transverse to the gauge orbits. We need only assume that our $(P^\alpha, \phi^\mu)$ coordinates form a local product structure on $\mathcal{H}$—that is, they form coordinates in patches on $\mathcal{H}$ such that the transition functions map $P'$s to $P'$s and map $\phi'$s linearly to $\phi'$s and that in each patch $\mathcal{H}_{c\alpha}$ is a section transverse to the gauge orbits.

Finally, since every fibre bundle of gauge orbits over a linearized base space $\mathcal{H}_r$ of reduced histories has such a structure, we may use gauge breaking to extend any Lie algebra on $\Gamma_r$. We take $\phi^\mu$ to be linearized coordinates on $\mathcal{H}_r$ and $P^\alpha$ to be functions on the orbits, defined locally on $\mathcal{H}$, so that the local slice $\mathcal{H}_{c\alpha}$ is transverse to the orbits. Since $\mathcal{H}_r$ is just a piece of $\mathcal{H}_r$, $A_{\mathcal{H}}(\mathcal{H}_r)$ defines $(\cdot, \cdot)_{c\alpha}$ and 3.3 extends this algebra to $A_{\mathcal{H}}(\mathcal{H})$.

### C. Properties of Gauge Broken Algebras

Note that when the constraints are first class, the discussion of [III A] is identical to the construction of a gauge broken algebra in [III B] using the conditions $P^\alpha = \lambda^\alpha$ and the global product structure $\mathcal{H} = \mathcal{H}_\Gamma \times \Lambda^I$ for the appropriate space $\mathcal{H}_\Gamma \subset \Gamma^I$. In this way, both Dirac analysis and gauge fixing may be regarded as examples of “gauge breaking.”

The canonical case is in fact typical and we can generalize elements of [III A] to all gauge breaking procedures. For example, we now define a space $\mathcal{E} = \cup_{c\alpha} \mathcal{S}_{c\alpha}$ of gauge broken evolutions where $\mathcal{S}_{c\alpha}$ is the subspace of $\mathcal{H}_{c\alpha}$ on which $\mathcal{S}_{c\alpha}$ is stationary. The extended algebra has a well-defined pull back to $\mathcal{E}$ which is invariant under any change of coordinates that respect the local product structure on $\mathcal{H}$ (i.e., that does not mix the $P^\alpha$ with the $\phi^\mu$). As before, $\mathcal{E}$ can be defined as the subspace on which the equations of motion $S_{\cdot \mu} = 0$ hold and therefore on which 3.3 also holds if the gauge breaking is based on canonical gauge
fixing. As in III A, \( A_H(\mathcal{H}) \) does not in general have a further pull back to \( \mathcal{S} \).

It may, however, be pulled back to \( \mathcal{S} \) when the gauge breaking is based on canonical gauge fixing. To see this when the system is finite dimensional, recall that since the action is invariant under gauge transformations, the equations of motion are linearly dependent:

\[
0 = \delta_\alpha S = S_{\gamma \beta} \delta_\alpha \lambda^{\beta} + \int_{t_i}^{t_f} dt \ S_{\gamma (A, t)} \delta_\alpha z^A(z^B(t)) \tag{3.6}
\]

where \( \delta_\alpha \) generates the gauge transformation labelled by \( \alpha \), \( S_{\gamma \alpha} = \frac{\delta S}{\delta \lambda^\alpha}, \ S_{\gamma (A, t)} = \frac{\delta S}{\delta z^A(t)} \) and the notation \( \delta_\alpha z^A(z^B(t)) \) is to emphasize that the gauge transformation of \( z^A(t) \) depends only on the gauge parameters and the canonical coordinates \( z^B(t) \) at the same time \( t \). This equation can be solved to express \( \frac{\delta S}{\delta P^\alpha} \bigg|_{\phi^\mu} = \frac{\delta S}{\delta z^\mu} \bigg|_{q^\beta, z^\gamma} \frac{\delta S}{\delta z^\gamma} \) as a function of the equations of motion \( S_{\gamma \mu} = \frac{\delta S}{\delta \phi^\mu} \bigg|_{P^\alpha} \) that vanishes when \( S_{\gamma \mu} = 0 \). It follows that in fact all of the equations of motion in \( (\phi^\mu, P^\alpha) \) coordinates vanish on \( \mathcal{E} \). Thus, \( \mathcal{E} \) and \( \mathcal{S} \) are identical.

Unfortunately, this argument does not go through for infinite dimensional systems without considering the details of the boundary conditions (say, at spatial infinity). Such conditions enter when solving 3.6 for \( \frac{\delta S}{\delta P^\alpha} \bigg|_{\phi^\mu} \). It happens, however, that even in the infinite dimensional case an algebra defined by canonical gauge breaking has a well-defined pull back to \( \mathcal{S} \). A proof of this will be given in [1] using the methods of generalized Peierls brackets.

IV. ALGEBRAS ON SPACES OF LAGRANGIAN HISTORIES

In sections II and III B we considered Lie algebras on spaces of histories associated with the canonical formulation and a phase space \( \Gamma \). In typical cases, it is straightforward to define analogous Lie brackets on any space \( \mathcal{L} \) of Lagrangian histories, by which we mean the domain space of an action that may not be in the canonical form 2.2 or 3.1. This happens when \( \mathcal{L} \) embeds in \( \mathcal{H} \) in the same way as \( \mathcal{S} \) and \( \mathcal{E} \).

Typical covariant Lagrangians can be derived from the canonical Lagrangians by first splitting the coordinates \( z^A(t) \) into configuration variables \( q^I(t) \) and momenta \( p_I(t) \). The
covariant action is then the restriction of the canonical action to the subspace \( L \) on which it is stationary with respect to variations in the momenta – effectively solving the equations 

\[
\frac{\delta S}{\delta p_I(t)} = 0
\]

to express \( p_I(t) \) in terms of the velocities \( \dot{q}^I(t) \) and the Lagrange multipliers \( \lambda^a(t) \).

Since every algebra on \( \mathcal{H} \) that we have discussed respects equations of motion that arise by variation of phase space coordinates, every \((,)_{\mathcal{H}}\) has a well-defined pull back \((,)_{\mathcal{L}}\) to \( \mathcal{L} \).

Note that no constraints have been solved in passing to \( \mathcal{L} \) so that \( \mathcal{L} \) contains both the space \( \mathcal{S} \) of solutions and the spaces \( \mathcal{E} \) of evolutions defined in [IIA] and [IIC]. As a result, when the Hamiltonian \( H \) is quadratic in momenta, some of the equations of motion that follow from the restriction \( S_{\mathcal{L}} \) of \( S \) to \( \mathcal{L} \) will be of less than second order in time derivatives when expressed in terms of the coordinates \( q^I(t) \) and \( \lambda^a(t) \) on \( \mathcal{L} \). The bracket of these equations of motion with functions on \( \mathcal{L} \) will not in general vanish, just as was the case on \( \mathcal{H} \).

V. QUANTIZATION

We now investigate quantization of the extended Poisson algebras defined in [I] and [II]. Our interest will be focussed on gauge systems and we discuss gauge-free systems only as a part of the general introduction. Following the introductory comments, two subsections describe quantization of our extended canonical algebra and of gauge broken algebras, comparing them with the usual constraint quantization of Dirac [3] and with gauge fixing methods.

We first note that (at least when global coordinates exist on \( \mathcal{H} \)) any extended Poisson bracket algebra has a nontrivial center since there are equations of motion \( S_{,i} = 0 \) such that \( (S_{,i}, A) = 0 \). If this property persists after quantization, the relevant operators \( \mathcal{O} \) are proportional to the identity: \( \mathcal{O} = c_{\mathcal{O}} \mathbb{1} \) in any irreducible representation of the commutator algebra. The resulting representation may be thought of as a quantization of the pull back of \((,)_{\mathcal{H}}\) to the subspace on which such \( \mathcal{O} \) take the values \( c_{\mathcal{O}} \).

We are thus let to consider quantization of the algebra pulled back to spaces on which
\[ S_i = c_i \text{ for } i \text{ in some set } T \subset I \times I \text{ such that and some } c_i \in R. \] However, such spaces are incompatible with the dynamics unless \( c_i = 0 \) so we consider only such spaces of partial solutions. Observe that, for the algebras constructed in sections III and IV, the smallest space of this kind is either \( \mathcal{E} \) or \( \mathcal{S} \). For this reason, we now confine ourselves to a discussion of \( (,)_E \) for the extended canonical bracket and \( (,)_S \) for gauge broken algebras based on canonical gauge fixing.

Recall that such a pull back endows the algebra with additional coordinate invariance properties. Also recall that, in general, \( (,)_E \) and \( (,)_S \) have nontrivial centers as well, though this remaining center can be removed only by pulling back the algebra further onto a gauge fixed slice. It will be convenient not to do so even when such a slice is available. In what follows, \( I \) is the embedding of \( \mathcal{E} \) or \( \mathcal{S} \) into \( \mathcal{H} \).

A. Canonical Quantization

We first examine quantization of \( (,)_E \) defined in III A from the canonical Poisson bracket. Recall that \( \mathcal{E} = \Gamma \times A^I \) since we have \( \dot{z}^A(t) \circ I = \{ z^A, H(t) + \lambda^a(t) \phi_a(t) \}_t \circ I \). If the constraints are first class with respect to \( (,)_E \), we quantize the algebra through

\[
[z^A(t), z^B(t)] = i\Omega^{AB}, \quad [z^A(t), \lambda^\alpha] = 0, \quad [\lambda^\alpha, \lambda^\beta] = 0, \quad (5.1)
\]

and

\[
z^A(t_2) = \mathcal{P} \exp(i \int_{t_1}^{t_2} (H(t) + \lambda^a(t) \phi_a(t)) dt) \ z^A(t_1) \ \mathcal{P} \exp(i \int_{t_2}^{t_1} (H(t) + \lambda^a(t) \phi_a(t)) dt) \quad (5.2)
\]

where \( \mathcal{P} \) denotes path ordering. When some constraints are classically second class, the Dirac bracket on \( \Gamma \) may be used in place of \( \Omega^{AB} \) in (5.1) so that we quantize the extended Dirac bracket of appendix B.

The constraints are then to be factor ordered in such a way that they are first class with respect to (5.1) and imposed as conditions that select physical states as in [3]. Due to their first class nature and (5.2), imposing the entire set \( \{ \phi_a \} \) of constraints is equivalent to imposing the constraints \( \phi_a(t) \) at any single time \( t \).
Although this strongly resembles the usual Dirac quantization, the two are not identical. One difference is that the constraints in \[3\] generate arbitrary gauge transformations but, from \[5.1\], we have \[[\lambda^\alpha, \phi_\beta] = 0\] even though \(\lambda^\alpha\) is not gauge invariant. Also, as noted before, any combination \(c^\beta \phi_\beta\) of constraints can be expressed as a combination of the constraints at any single time \(t\): \(c^\beta \phi_\beta = c^a(t)\phi_a(t)\). Thus, transformations generated by the constraints may be parameterized by the values \(c^a\) for \(a \in G\), while the space of gauge transformations is parameterized by \(c^\alpha\) for \(\alpha \in G^I\). In addition, every transformation generated by the constraints \(\phi^\alpha\) extends arbitrarily far to the future, whereas gauge transformations should have compact support. More careful consideration shows that the constraints generate the transformations \(\delta z^i = e^\alpha \delta_\alpha z^i\) where \(\delta_\alpha\) is the gauge transformation labelled by \(\alpha\), for those parameters \(e^\alpha\) such that \(e^\alpha \lambda_\alpha = 0\). This is exactly that part of the gauge freedom not fixed by pulling back to a subspace of constant \(\lambda^\alpha\).

Perhaps the most apparent distinction between the above prescription and that of \[3\] is that \[5.1\] considers the Lagrange multipliers \(\lambda^\alpha\) to be operators whereas in \[3\] they are functions to be specified by hand. As a result, while any particular choice of these functions gives a formulation identical to some irreducible representation of \[5.1\] and \[5.2\], the original prescription of \[3\] gives no way to define the evolution of gauge dependent operators without choosing values for \(\lambda^\alpha\) and thereby performing a (partial) gauge fixing. However, since we have introduced the operators \(z^A(t)\) and \(\lambda^\alpha(t)\) for all times \(t\), the evolution of every operator is determined.

This feature is especially useful in the study of time reparameterization invariant systems where it allows us to construct gauge invariant operators by integrating over time. For example, in General Relativity, we might be interested in the total curvature \(\int_M d^4x R\) of some four-manifold \(M\) and in the study of the relativistic free particle, we might be interested in the proper time accumulated between \(x^0 = \alpha\) and \(x^0 = \beta\):

\[
\tau = - \int dt \theta(x^0(t) - \alpha)\theta(\beta - x^0(t))\sqrt{-\dot{x}^2(t)}
\]  
(5.3)
or in the value \([x^\alpha]_{a_\mu x^\nu = \tau}\) of some coordinate \(x^\alpha\) when the particle crosses the hypersurface
While such definitions are unlikely to simplify any computation, they may provide a conceptual advantage over building operators explicitly through phase space functions. For example, we would be free to consider the commutator algebra defined by 5.1 and 5.2 of all gauge invariants built from the operators $z^i$ and $\lambda^\alpha$. It is straightforward to show that this is a Lie algebra, though defined on a monstrously overcomplete set of operators and for which explicit commutation relations are difficult to compute. However, these difficulties may now be considered technical complications to be explored in each model. As a final comment we note that a similar construction can be performed in any quantization based on an algebra of functions of histories.

### B. Quantization in the presence of a Gribov Ambiguity

As mentioned in [III B], gauge fixed algebras are given by the pull back of a corresponding gauge broken algebra when $\mathcal{H}$ has the structure of a trivial fibre bundle of gauge orbits. Thus, gauge fixing and gauge breaking are nearly equivalent in the absence of a Gribov ambiguity. We now investigate the case where a Gribov ambiguity is present.

Recall that such a discussion is possible since a gauge broken algebra may be defined without reference to global gauge fixing conditions. All that is required is for the $P^\alpha$ to form local gauge fixing conditions and that $\mathcal{H}$ have a linearized structure. Note, however, that if the $P^\alpha$ are global gauge fixing conditions then $(A, P^\alpha) = 0$ for any function $A$ on $\mathcal{H}$. If the factor ordering preserves this feature after quantization then the $P^\alpha$ are proportional to the identity operator in any irreducible representation: $P^\alpha = c^\alpha \mathbb{1}$. Such representations are just the irreducible representations of the algebra pulled back to $\mathcal{H}_{c^\alpha}$ – i.e., the “gauge fixed representations.”

Now, suppose that $P^\alpha$ is defined only locally. In particular, consider a case in which the phase space is the cotangent bundle over some configuration space $Q$, the gauge conditions
\(P^a\) depend only on the configuration variables, \(P^a(t)\) is independent of time, and the gauge transformations generate translations on \(Q\). Similarly, we take the patches on \(\Gamma\) to be cotangent bundles over configuration space patches and consider corresponding patches on \(H\) and \(S\). We note that this gauge breaking is based on canonical gauge fixing so that we may pull back the algebra to \(S\).

In this case, the bracket \((A, P^a)_H\) is not defined since \(P^a\) has not been defined as a function on \(S\) but only on some patch \(U\). This makes quantization of \((,)_S\) more difficult.

We begin by defining:

\[
\phi^A(t_2) = \mathcal{P} \exp\left(i \int_{t_1}^{t_2} (H(t) + \lambda^a(t)\phi_a(t)) dt\right) \phi^A(t_1) \mathcal{P} \exp\left(i \int_{t_2}^{t_1} (H(t) + \lambda^a(t)\phi_a(t)) dt\right)
\]

and

\[
P^a(t) = P^a(t')
\]

where \(\mathcal{P}\) denotes path ordering. Thus, every operator may be built from \(\phi^A(t)\) and \(P^a(t)\). Except for \(\lambda^a(t)\), these are functions on \(T^*_Q\), which are in turn built from functions and vector fields on \(Q\). We then define \(\lambda^a(t)\) to be built from functions and vector fields on \(Q\) through some factor ordering of \([3,3]\).

To define the commutator algebra, let \(P^a_U\) be the gauge breaking functions on the patch \(U\). Then, for all vector fields \(v_1\) and \(v_2\) such that \(\mathcal{L}_{v_1}(P^a_U) = 0 = \mathcal{L}_{v_2}(P^a_U)\) on \(U\) for all patches \(U\), and all functions \(f\) on \(Q\) we define:

\[
[v_1, f] = i\mathcal{L}_{v_1}f \quad \text{and} \quad [v_1, v_2] = i\{v_1, v_2\}_L
\]

where \(\{v_1, v_2\}_L\) is the Lie bracket of vector fields. Additionally, if \(q_{Ua} = (\gamma^{-1})_{ab}q^b_U\) is the function on \(U\) conjugate to \(P^a_U\) in the sense of the canonical Poisson bracket, then for any function \(\rho_U\) on \(Q\) with support on \(U\) we define:

\[
[\rho_U q_{UA}, A] = (\gamma^{-1})_{ab}[\rho_U q^b(z^{At}), A]
\]

where \(A\) is some function on \(T^*_Q\) and \(q^b(z^{At})\) is some factor ordering of the solution in \([3,3]\). Note that the entire algebra is defined through the action of vector fields on functions by
infinitesimal translation. However, because the algebra is degenerate, not all infinitesimal translations are generated.

Thus, $Q$ may be partitioned into equivalence classes such that points in each class are joined by a series of infinitesimal translations generated by $\delta \xi$. The set of functions with support on any such equivalence class carries a representation of our algebra. If, in addition, there is an open set $V \neq Q$ in $Q$ such that every point $p \in V$ lies in some equivalence class $C_p$ that is entirely contained in $V$, then any representation carried by $L^2$ functions on $Q$ in which the vector fields act by translations generated by $\delta \xi$ and the functions act by multiplication is topologically reducible as well. Regardless of this, if $Q$ is a fibre bundle of gauge orbits over some base space $B$, gauge transformations act on functions with support a single equivalence class only through a representation of $\pi_1(B)$. Thus, the action of the gauge group may be much simpler on an irreducible representation than on the full configuration space $Q$.

**VI. DISCUSSION**

We have seen that the Poisson bracket can be extended to a Lie bracket of functions on the space $H$ of canonical histories and on spaces $L$ of Lagrangian histories. For gauge systems, we extended both the canonical Poisson bracket and reduced phase space Poisson bracket, observing that both are examples of “gauge breaking.” Gauge breaking is an interesting technique in itself, as it resembles gauge fixing yet may be performed in the presence of a Gribov ambiguity.

We then investigated quantization of such algebras. This reduces to a study of $(,)_{E}$ and $(,)_{S}$. We found that while gauge breaking may produce constraints that resemble those of Gribov, the interpretation of these constraints is different and they generate residual symmetry transformations instead of gauge transformations. We also saw that the Heisenberg picture nature of our quantized algebra allows the construction of invariant operators in time reparameterization invariant systems by integration over time and that a quantum theory based
on a gauge broken algebra in the presence of a Gribov ambiguity may still be reducible to a representation in which the gauge symmetry acts simply.

All of this was intended, however, to set the stage for [1]. We have presented an introductions to algebras on $\mathcal{H}$ and their pull backs while introducing the space $\mathcal{E}$ of evolutions and the concepts of locally defined algebras and gauge broken algebras. Since our construction was based on the Poisson bracket, we were also able to provide a straightforward comparison with more familiar techniques. In [1], we will place these ideas in the more general and unified framework of the generalized Peierls algebra.

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APPENDIX A: LOCALLY DEFINED ALGEBRAS

In this appendix we define a Lie bracket of functions on a manifold $M$ given a set of Lie brackets $\{,\}_i$ of functions $F_i$ whose support lies in patches $U_i$ when these algebras are compatible in the overlap regions and the patches cover $M$. Our specific compatibility assumption is that for two such brackets $\{,\}_i$ and $\{,\}_j$ on patches $U_i$ and $U_j$ and all smooth functions $F$ and $G$ with support in $U_i \cap U_j$, we have $\{F,G\}_i = \{F,G\}_j$. Note that $\text{supp}(\{F,G\}_i) \subset \text{supp}(F) \cap \text{supp}(G)$.

Now, given any functions $F$ and $G$ on $M$, we write

$$F = \sum_{\text{patches}} F_i, \quad G = \sum_{\text{patches}} G_i$$

(A1)
where the supports of $F_i$ and $G_i$ both lie in the patch $U_i$. We then define the bracket of $F$ and $G$ by

$$(F, G) \equiv \sum_{\text{patches}} \{F_i, G_i\}_i$$

(A2)

Note that our compatibility assumption guarantees that $A2$ is independent of the decomposition $A1$.

APPENDIX B: EXTENSION OF THE DIRAC BRACKET

In this appendix, we describe how the techniques of II may also be used to extend the Dirac bracket [3] to $\mathcal{H}$ when the constraints are entirely second class. We consider an action of the form

$$S = \int_{t_1}^{t_2} \left[ \frac{1}{2} \Omega^{-1}_{AB} z^A \dot{z}^B - H - \lambda^a \xi_a \right]$$

(B1)

for $\Omega^{AB}$, $z^A$, and $H$ as in II and some $\xi_a(t) = \xi_a(z^A(t), t)$. The $\lambda^a(t)$ are Lagrange multipliers that enforce the constraints $\xi_a(t) = 0$. If the constraints are second class, the matrix $\Delta_{ab} = \xi_a|A \Omega^{AB} \xi_b|B$ is invertible. The Dirac bracket $\{,\}^D$ is then defined [3] by

$$\{z^A, z^B\}^D = \Sigma^{AB} = \Omega^{AB} - \Omega^{AC} \xi_a|C (\Delta^{-1})^{ab} \xi_b|D \Omega^{DB}$$

(B2)

While the Dirac bracket is not defined on $\lambda^a$, the equations of motion $\xi_a(t) = 0$ place a constraint $\xi_a|A(t)(z^A(t), z^B(t'))^D_{\mathcal{H}} = 0$ on any extension $(,)^D_{\mathcal{H}}$ of $\{,\}^D$. These two features combine in such a way that the requirements $(S_{,i}, A)^D_{\mathcal{H}} = 0$ and $\Sigma^{AB}$ uniquely define the extension $(,)^D_{\mathcal{H}}$ to be given by:

$$(z^A(t_1), z^B(t_2))^D_{\mathcal{H}} = T_L^D(t_2, t_1)A^C \Sigma^{CB}$$

(B3a)

$$(\lambda^a(t_1), z^B(t_2))^D_{\mathcal{H}} = -\chi^a|A(t_1) [\Omega^{-1}_{AC} \partial_{t_1}(z^C(t_1), z^B(t_2))^D_{\mathcal{H}}$$

$$+ (H_{AC}(t_1) + \lambda^a(t_1) \xi_a|AC(t_1)) (z^C(t_1), z^B(t_2))^D_{\mathcal{H}}]$$

(B3b)

$$(\lambda^a(t_1), z^B(t_2))^D_{\mathcal{H}} = -\chi^a|A(t_1) [\Omega^{-1}_{AC} \partial_{t_1}(z^C(t_1), \lambda^b(t_2))^D_{\mathcal{H}}$$
\[ + (H_{AC}(t_1) + \lambda^a(t_1)\xi_{a|AC}(t_1)) (z^C(t_1), \lambda^b(t_2))_{H}^D \]  
(B3c)

where

\[ T_L(t_2, t_1)_{C}^A = \mathcal{P} \exp[\int_{t_2}^{t_1} dt \, Q(t)]^A_C, \]  
(B3d)

\[ Q_C^A = \Sigma^{AB} (H_{|BC} + \lambda^a \xi_{a|BC}) - \Omega_{AB} \xi_{a|B} \Delta^{-1} \frac{\partial}{\partial \xi_{b|C}} \]  
(B3e)

and

\[ \chi^{bA} \xi_{a|A} = \delta_a^b. \]  
(B3f)

We note that such a \( \chi^{aA} \) exists when the constraints \( \xi_a \) are independent.

As with the Poisson bracket, the extended Dirac bracket may be pulled back to spaces of partial solutions. In particular, such pull backs may be used to define the Dirac bracket on spaces \( \mathcal{L} \) of Lagrangian histories. Similarly, when both first and second class constraints are present, the techniques of section III may be used to define extensions either of the canonical or reduced Dirac bracket.
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* Electronic address: marolf@hbar.phys.psu.edu.

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