A compactness result for a Gelfand-Liouville system with Lipschitz condition

Samy Skander Bahoura*
Equipe d’Analyse Complexe et Géométrie.
Université Pierre et Marie Curie, 75005 Paris, France.

Abstract

We give a quantization analysis to an elliptic system (Gelfand-Liouville type system) with Dirichlet condition. An application, we have a compactness result for an elliptic system with Lipschitz condition.

Keywords: quantization, blow-up, boundary, Gelfand-Liouville system, Dirichlet condition, a priori estimate, Lipschitz condition.

MSC: 35J60, 35B44, 35B45

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on open set $\Omega$ of $\mathbb{R}^2$ with a smooth boundary.

We consider the following equation:

$$(P) \begin{cases} -\Delta u = Ve^v & \text{in } \Omega \subset \mathbb{R}^2, \\ -\Delta v = We^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial \Omega, \\ v = 0 & \text{in } \partial \Omega. \end{cases}$$

Here:

$0 \in \partial \Omega$

When $u = v$, the above system is reduced to an equation which was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1, 2, 9], we can find some existence and compactness results.

Among other results, we can see in [6] the following important Theorem,

**Theorem. (Brezis-Merle [6]).** If $(u_i)_i = (v_i)_i$ and $(V_i)_i = (W_i)_i$ are two sequences of functions relatively to the problem $(P)$ with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set $K$ of $\Omega$,

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

*e-mails: samybahoura@yahoo.fr, samybahoura@gmail.com
If we assume $V$ with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafrir see [10], that, if $(u_i), (V_i)$, are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_{K} u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose $(V_i)$, uniformly Lipschitzian with $A$ the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see [5].

Here, we give a caracterization of the behavior of the blow-up points on the boundary and also a proof of compactness of the solutions to Gelfand-Liouville type system with Lipschitz condition.

Here, we give an extenstion of Brezis-Merle Problem (see [6]) is:

**Problem (Brezis-Merle Type problem).** Suppose that $V_i \rightarrow V$ and $W_i \rightarrow W$ in $C^0(\Omega)$, with, $0 \leq V_i \leq b_1$ and $0 \leq W_i \leq b_2$ for some positive constants $b_1, b_2$. Also, we consider a sequence of solutions $(u_i), (v_i)$ of $(P)$ relatively to $(V_i), (W_i)$ such that,

$$\int_{\Omega} e^{u_i} \, dx \leq C_1, \quad \int_{\Omega} e^{v_i} \, dx \leq C_2,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C_3 = C_4(b_1, b_2, C_1, C_2, \Omega)?$$

and,

$$\|v_i\|_{L^\infty} \leq C_4 = C_4(b_1, b_2, C_1, C_2, \Omega)?$$

Here, we give a caracterization of the behavior of the blow-up points on the boundary. The conditions $V_i \rightarrow V$ and $W_i \rightarrow W$ in $C^0(\Omega)$ are not necessary.

But for the new proof for the Gelfand-Liouville type system (Brezis-Merle type problem) we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A_1, \quad \|\nabla W_i\|_{L^\infty} \leq A_2.$$

We have the following caracterization of the behavior of the blow-up points on the boundary.

**Theorem 1.1** Assume that $\max_{\Omega} u_i \rightarrow +\infty$ and $\max_{\Omega} v_i \rightarrow +\infty$ Where $(u_i)$ and $(v_i)$ are solutions of the probleme $(P)$ with:

$$0 \leq V_i \leq b_1, \quad \text{and} \quad \int_{\Omega} e^{u_i} \, dx \leq C_1, \; \forall \; i,$$
and,

\[ 0 \leq W_i \leq b_2, \quad \text{and} \quad \int_{\Omega} e^{w_i} dx \leq C_2, \quad \forall \, i, \]

then, after passing to a subsequence, there is a function \( u \), there is a number \( N \in \mathbb{N} \) and \( N \) points \( x_1, x_2, \ldots, x_N \in \partial \Omega \), such that,

\[ \partial_{\nu} u_i \rightarrow \partial_{\nu} u + \sum_{j=1}^{N} \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi \text{ weakly in the sense of measure } L^1(\partial \Omega). \]

and,

\[ u_i \rightarrow u \text{ in } C^1_{\text{loc}}(\bar{\Omega} - \{x_1, \ldots, x_N\}). \]

\[ \partial_{\nu} v_i \rightarrow \partial_{\nu} v + \sum_{j=1}^{N} \beta_j \delta_{x_j}, \quad \beta_j \geq 4\pi \text{ weakly in the sense of measure } L^1(\partial \Omega). \]

and,

\[ v_i \rightarrow v \text{ in } C^1_{\text{loc}}(\bar{\Omega} - \{x_1, \ldots, x_N\}). \]

In the following theorem, we have a proof for the global a priori estimate which concern the problem \((P)\).

**Theorem 1.2** Assume that \((u_i), (v_i)\) are solutions of \((P)\) relatively to \((V_i), (W_i)\) with the following conditions:

\[ x_1 = 0 \in \partial \Omega, \]

and,

\[ 0 \leq V_i \leq b_1, \quad ||\nabla V_i||_{L^{\infty}} \leq A_1, \quad \text{and} \quad \int_{\Omega} e^{w_i} \leq C_1, \]

\[ 0 \leq W_i \leq b_2, \quad ||\nabla W_i||_{L^{\infty}} \leq A_2, \quad \text{and} \quad \int_{\Omega} e^{v_i} \leq C_2, \]

We have,

\[ ||u_i||_{L^{\infty}} \leq C_3(b_1, b_2, A_1, A_2, C_1, C_2, \Omega), \]

We have,

\[ ||v_i||_{L^{\infty}} \leq C_4(b_1, b_2, A_1, A_2, C_1, C_2, \Omega), \]
2 Proof of the theorems

Proof of theorem 1.1:

Since $V_i e^{v_i}$ and $W_i e^{u_i}$ are bounded in $L^1(\Omega)$, we can extract from those two sequences two subsequences which converge to two nonegative measures $\mu_1$ and $\mu_2$.

If $\mu_1(x_0) < 4\pi$, by a Brezis-Merle estimate for the first equation, we have $e^{u_i} \in L^{1+\epsilon}$ around $x_0$, by the elliptic estimates, for the second equation, we have $v_i \in W^{2,1+\epsilon} \subset L^\infty$ around $x_0$, and, returning to the first equation, we have $u_i \in L^\infty$ around $x_0$.

If $\mu_2(x_0) < 4\pi$, then $u_i$ and $v_i$ are also locally bounded around $x_0$.

Thus, we take a look to the case when, $\mu_1(x_0) \geq 4\pi$ and $\mu_2(x_0) \geq 4\pi$. By our hypothesis, those points $x_0$ are finite.

We will see that inside $\Omega$ no such points exist. By contradiction, assume that, we have $\mu_1(x_0) \geq 4\pi$. Let us consider a ball $B_R(x_0)$ which contain only $x_0$ as nonregular point. Thus, on $\partial B_R(x_0)$, the two sequence $u_i$ is uniformly bounded. Let us consider:

$$
\begin{cases}
-\Delta z_i = V_i e^{v_i} & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\
z_i = 0 & \text{in } \partial B_R(x_0).
\end{cases}
$$

By the maximum principle we have:

$$z_i \leq u_i$$

and $z_i \rightarrow z$ almost everywhere on this ball, and thus,

$$\int e^{z_i} \leq \int e^{u_i} \leq C,$$

and,

$$\int e^z \leq C.$$

but, $z$ is a solution to the following equation:

$$
\begin{cases}
-\Delta z = \mu_1 & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\
z = 0 & \text{in } \partial B_R(x_0).
\end{cases}
$$

with, $\mu_1 \geq 4\pi$ and thus, $\mu_1 \geq 4\pi \delta x_0$ and then, by the maximum principle:

$$z \geq -2 \log |x - x_0| + C$$

thus,

$$\int e^z = +\infty,$$

which is a contradiction. Thus, there is no nonregular points inside $\Omega$.

Thus, we consider the case where we have nonregular points on the boundary, we use two estimates:
\[ \int_{\partial \Omega} \partial_{\nu} u_i d\sigma \leq C_1, \quad \int_{\partial \Omega} \partial_{\nu} v_i d\sigma \leq C_2, \]
and,
\[ ||\nabla u_i||_{L^q} \leq C_q, \quad ||\nabla v_i||_{L^q} \leq C'_q, \quad \forall \; i \text{ and } 1 < q < 2. \]

We have the same computations, as in the case of one equation.

We consider a points \( x_0 \in \partial \Omega \) such that:
\[ \mu_1(x_0) < 4\pi. \]

We consider a test function on the boundary \( \eta \) we extend \( \eta \) by a harmonic function on \( \Omega \), we write the equation:
\[-\Delta ((u_i - u) \eta) = (V_i e^{v_i} - V e^v) \eta + <\nabla (u_i - u) | \nabla \eta > = f_i \]
with,
\[ \int |f_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi, \]
\[-\Delta ((v_i - v) \eta) = (W_i e^{u_i} - W e^u) \eta + <\nabla (v_i - v) | \nabla \eta > = g_i, \]
with,
\[ \int |g_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi, \]

By the Brezis-Merle estimate, we have uniformly, \( e^{u_i} \in L^{1+\epsilon} \) around \( x_0 \), by the elliptic estimates, for the second equation, we have \( v_i \in W^{2,1+\epsilon} \subset L^\infty \) around \( x_0 \), and, returning to the first equation, we have \( u_i \in L^\infty \) around \( x_0 \).

We have the same thing if we assume:
\[ \mu_2(x_0) < 4\pi. \]

Thus, if \( \mu_1(x_0) < 4\pi \) or \( \mu_2(x_0) < 4\pi \), we have for \( R > 0 \) small enough:
\[ (u_i, v_i) \in L^\infty (B_R(x_0) \cap \Omega). \]

By our hypothesis the set of the points such that:
\[ \mu_1(x_0) \geq 4\pi, \quad \mu_2(x_0) \geq 4\pi, \]
is finite, and, outside this set \( u_i \) and \( v_i \) are locally uniformly bounded. By the elliptic estimates, we have the \( C^1 \) convergence to \( u \) and \( v \) on each compact set of \( \Omega - \{x_1, \ldots, x_N\} \).

**Proof of theorem 1.2:**

Without loss of generality, we can assume that 0 is a blow-up point (either, we use a translation). Also, by a conformal transformation, we can assume that \( \Omega = B^+_1 \), the half ball, and \( \partial^+ B^+_1 \) is the exterior part, a part which not contain 0 and on which \( u_i \) and \( v_i \) converge in the \( C^1 \) norm to \( u \) and \( v \). Let us consider \( B^+_\epsilon \), the half ball with radius \( \epsilon > 0 \).
The Pohozaev identity gives:

$$
\int_{B^+} \Delta u_i x|\nabla v_i > dx = - \int_{B^+} \Delta v_i x|\nabla u_i > dx + \int_{\partial B^+} g(\partial_u u_i, \partial_v v_i) d\sigma,
$$

(1)

Thus,

$$
\int_{B^+} V_i e^{v_i} x|\nabla v_i > dx = - \int_{B^+} W_i e^{u_i} x|\nabla u_i > dx + \int_{\partial B^+} g(\partial_u u_i, \partial_v v_i) d\sigma,
$$

(2)

After integration by parts, we obtain:

$$
\int_{B^+} V_i e^{v_i} dx + \int_{B^+} x|\nabla V_i > e^{v_i} dx + \int_{\partial B^+} \nu|\nabla V_i > d\sigma +
$$

$$
+ \int_{B^+} W_i e^{u_i} dx + \int_{B^+} x|\nabla W_i > e^{u_i} dx + \int_{\partial B^+} \nu|\nabla W_i > d\sigma =
$$

$$
g(\partial_u u_i, \partial_v v_i) d\sigma,
$$

Also, for $u$ and $v$, we have:

$$
\int_{B^+} V e^{v} dx + \int_{B^+} x|\nabla V > e^{v} dx + \int_{\partial B^+} \nu|\nabla V > d\sigma +
$$

$$
+ \int_{B^+} W e^{u} dx + \int_{B^+} x|\nabla W > e^{u} dx + \int_{\partial B^+} \nu|\nabla W > d\sigma =
$$

$$
g(\partial_u u, \partial_v v) d\sigma,
$$

If, we take the difference, we obtain:

$$
(1 + o(\epsilon))(\int_{B^+} V_i e^{v_i} dx - \int_{B^+} V e^{v} dx) +
$$

$$
+ (1 + o(\epsilon))(\int_{B^+} W_i e^{u_i} dx - \int_{B^+} W e^{u} dx) =
$$

$$
= \alpha_1 + \beta_1 + o(\epsilon) + o(1) = o(1),
$$

a contradiction.
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