I. INTRODUCTION

Quantum Chromodynamics (QCD) [1, 2] is widely accepted as a realistic quantum field gauge theory of strong interactions not only at the fundamental (microscopic) quark-gluon level but at the hadronic (macroscopic) level as well. This means that in principle it should describe the properties of experimentally observed hadrons in terms of experimentally never seen quarks and gluons, i.e., to describe the hadronic world from first principles – an ultimate goal of any fundamental theory. But this is a formidable task because of the color confinement phenomenon, the dynamical mechanism of which is not yet understood, and therefore the confinement problem remains unsolved up to the present days. It prevents colored quarks and gluons to be experimentally detected as physical ("in" and "out" asymptotic) states which are colorless (i.e., color-singlets), by definition, so color confinement is permanent and absolute [1].

Today there is no doubt left that color confinement and other dynamical effects, such as spontaneous breakdown of chiral symmetry, bound-state problems, etc., being essentially nonperturbative (NP) effects, are closely related to the large-scale (low-energy/momentum) structure of the true QCD ground state and vice-versa (2 and references therein). The perturbation theory (PT) methods in general fail to investigate them. If QCD itself is a confining theory then a characteristic scale has to exist. It should be directly responsible for the above-mentioned structure of the true QCD vacuum in the same way as \( \Lambda_{QCD} \) is responsible for the nontrivial perturbative dynamics there (scale violation, asymptotic freedom (AF) [1]).

However, the Lagrangian of QCD [1, 2] does not contain explicitly any of the mass scale parameters which could have a physical meaning even after the corresponding renormalization program is performed. The main goal of this paper is to show how the characteristic scale (the mass gap, for simplicity) responsible for the NP dynamics may explicitly appear in QCD. This becomes an imperative especially after Jaffe and Witten have formulated their theorem "Yang-Mills Existence And Mass Gap" [3]. In order to make the existence of a mass gap perfectly clear it is defined as the difference between the regularized full gluon self-energy and its subtracted (also regularized) counterpart. The mass gap is mainly generated by the nonlinear interaction of massless gluon modes. A self-consistent violation of SU(3) color gauge invariance/symmetry is discussed in order to realize a mass gap in QCD. For this purpose, we propose not to impose the transversality condition on the full gluon self-energy, while restoring the transversality of the full gluon propagator relevant for the non-perturbative QCD at the final stage. At the same time, the Slavnov-Taylor identity for the full gluon propagator is always preserved. All this allows one to establish the general structure of the full gluon propagator in the presence of a mass gap. In this case, two independent types of formal solutions for the full gluon propagator have been established. The nonlinear iteration solution at which the gluons remain massless is explicitly present. The existence of the solution with an effective gluon mass is also demonstrated.

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As mentioned above, there is no place for the mass gap in the QCD Lagrangian, so the only place when the mass gap may appear is the corresponding system of dynamical equations of motion, the so-called Schwinger-Dyson (SD) equations, which should be complemented by the ST identities (1 and references therein). The propagation of gluons is one of the main dynamical effects in the true QCD vacuum. It is described by the above-mentioned corresponding SD quantum equation of motion for the full gluon propagator. The importance of this equation is due to the fact that its solutions reflect the quantum-dynamical structure of the true QCD ground state. The color gauge structure of this equation is also one of the main subjects of our investigation in order to find a way how to realize a mass gap in QCD.

In the presence of a mass gap two different and independent types of formal solutions for the regularized full gluon propagator have been established. The general nonlinear iteration solution is always severely singular at small gluon momentum, i.e., the gluons remain massless, and this does not depend on the gauge choice. The massive solution leads to an effective gluon mass, which depends on the gauge choice. No truncations/approximations/assumptions and no special gauge choice are made for the skeleton loop contribution to the photon self-energy (the so-called vacuum polarization tensor).

II. QED

It is instructive to begin with a brief explanation why a mass gap does not occur in quantum electrodynamics (QED). The photon SD equation can be symbolically written down as follows:

\[ D(q) = D^0(q) + D^0(q)\Pi(q)D(q), \] (2.1)

where we omit, for convenience, the dependence on the Dirac indices, and \( D^0(q) \) is the free photon propagator. \( \Pi(q) \) describes the electron skeleton loop contribution to the photon self-energy (the so-called vacuum polarization tensor). Analytically it looks

\[ \Pi(q) \equiv \Pi_{\mu\nu}(q) = -g^2 \int \frac{id^4p}{(2\pi)^4} Tr[\gamma_\mu S(p-q)\Gamma_\nu(p-q, q)S(p)], \] (2.2)

where \( S(p) \) and \( \Gamma_\mu(p-q, q) \) represent the full electron propagator and the full electron-photon vertex, respectively. Here and everywhere below the signature is Euclidean, since it implies \( q_i \to 0 \) when \( q^2 \to 0 \) and vice-versa. This tensor has the dimensions of a mass squared, and it is quadratically divergent at least in the PT. To make the formal existence of a mass gap perfectly clear, let us now, for simplicity, subtract its value at zero. One obtains

\[ \Pi^*(q) \equiv \Pi^*_{\mu\nu}(q) = \Pi_{\mu\nu}(q) - \Pi_{\mu\nu}(0) = \Pi_{\mu\nu}(q) - \delta_{\mu\nu}\Delta^2(\lambda). \] (2.3)

The explicit dependence on the dimensionless ultraviolet (UV) regulating parameter \( \lambda \) has been introduced into the mass gap \( \Delta^2(\lambda) \), given by the integral (2.2) at \( q^2 = 0 \), in order to assign a mathematical meaning to it. In this connection a few remarks are in order in advance. The dependence on \( \lambda \) (when it is not shown explicitly) is assumed in all divergent integrals here and below in the case of the gluon self-energy as well (see next section). This means that all the expressions are regularized (including photon/gluon propagator), and we can operate with them as with finite quantities. \( \lambda \) should be removed on the final stage only after performing the corresponding renormalization program. So through this paper the mass gap is a ”bare” one, i.e., it is only regularized. Whether the regulating parameter \( \lambda \) has been introduced in a gauge-invariant way (though this always can be achieved) or not, and how it should be removed is not important for the problem if a mass gap can be ”released/liberated” from the corresponding vacuum. We will show in the most general way (not using the PT and not choosing any special gauge) that this is impossible in QED and might be possible in QCD.

The decomposition of the subtracted vacuum polarization tensor (2.3) into the independent tensor structures is

\[ \Pi^*_{\mu\nu}(q) = T_{\mu\nu}(q)q^2\Pi^*(q^2) + q_\mu q_\nu(q)\tilde{\Pi}^*(q^2), \] (2.4)

where both invariant functions \( \Pi^*(q^2) \) and \( \tilde{\Pi}^*(q^2) \) are, by definition, dimensionless and regular at small \( q^2 \), since \( \Pi^*_{\mu\nu}(0) = 0 \) identically due to the subtraction (2.3); otherwise they remain arbitrary. From this relation it follows
that $\Pi_{\mu\nu}(q) = O(q^2)$, i.e., it is always of the order $q^2$. In this connection a few remarks are in order. The subtraction (2.3) at zero point in QED is justified, since it is abelian gauge theory, and therefore there is no the self-interaction of massless photons, which can be source of the singularities in the $q^2 \to 0$ limit. The vacuum polarization tensor (2.2) has no infrared (IR) singularities in this limit, at least in the PT. So in what follows we will consider the above-mentioned invariant functions as regular at small $q^2$, indeed. Also, here and everywhere below

$$T_{\mu\nu}(q) = \delta_{\mu\nu} - q_\mu q_\nu/q^2 = \delta_{\mu\nu} - L_{\mu\nu}(q).$$

In the same way, the photon self-energy (2.2) in terms of the independent tensor structures is

$$\Pi_{\mu\nu}(q) = T_{\mu\nu}(q)q^2\Pi(q^2) + q_\mu q_\nu\tilde{\Pi}(q^2),$$

where both invariant functions $\Pi(q^2)$ and $\tilde{\Pi}(q^2)$ are dimensionless functions; otherwise they remain arbitrary. Due to the transversality of the photon self-energy

$$q_\mu \Pi_{\mu\nu}(q) = q_\nu \Pi_{\mu\nu}(q) = 0,$$

which comes from the current conservation condition in QED, one has $\tilde{\Pi}(q^2) = 0$, i.e., it should be the purely transversal

$$\Pi_{\mu\nu}(q) = T_{\mu\nu}(q)q^2\Pi(q^2).$$

On the other hand, from the subtraction (2.3), on account of the relation (2.4), and the transversality condition (2.7) it follows that

$$\tilde{\Pi}^*(q^2) = -\frac{\Delta^2(\lambda)}{q^2},$$

which, however, is impossible since $\tilde{\Pi}^*(q^2)$ is a regular function of $q^2$, by definition. So the mass gap should be discarded, i.e., put formally to zero and, consequently, $\tilde{\Pi}^*(q^2)$ as well, i.e.,

$$\Delta^2(\lambda) = 0, \quad \tilde{\Pi}^*(q^2) = 0.$$ 

Thus the subtracted photon self-energy is also transversal, i.e., it satisfies the transversality condition

$$q_\mu \Pi_{\mu\nu}(q) = q_\nu \Pi_{\mu\nu}(q) = 0,$$

and coincides with the photon self-energy (see Eq. (2.3) at zero mass gap). Moreover, this means that the photon self-energy does not have a pole in the $q^2 \to 0$ limit in its invariant function $\Pi(q^2) = \Pi^*(q^2)$. As mentioned above, in obtaining these results neither the PT has been used nor a special gauge has been chosen. So there is no place for quadratically divergent constants in QED, while logarithmic divergence still can be present in the invariant function $\Pi(q^2) = \Pi^*(q^2)$. It is to be included into the electric charge through the corresponding renormalization program (for these detailed gauge-invariant derivations explicitly done in lower order of the PT see Refs. [2, 6, 7, 8, 9]).

Taking into account the subtraction (2.3), on account of the relations (2.10), the photon SD equation (2.1) becomes equivalent to

$$D(q) = D^0(q) + D^0(q)\Pi^*(q)D(q).$$

It can be summed up into geometric series, so one obtains

$$D(q) = \frac{D^0(q)}{1 - \Pi^*(q)D^0(q)} = D^0(q) + D^0(q)\Pi^*(q)D^0(q) - D^0(q)\Pi^*(q)D^0(q)\Pi^*(q)D^0(q) + \ldots.$$
Since \( \Pi^s(q) = O(q^2) \) and \( D^0(q) \sim (q^2)^{-1} \), the IR singularity of the full photon propagator is determined by the IR singularity of the free photon propagator, i.e., \( D(q) = O(D^0(q)) \) with respect to the behavior at small photon momentum.

In fact, the current conservation condition (2.7), i.e., the transversality of the photon self-energy lowers the quadratic divergence of the corresponding integral (2.2) to a logarithmic one. That is the reason why in QED only logarithmic divergences survive. Thus in QED there is no mass gap and the relevant photon SD equation is shown in Eq. (2.13). From it follows that the behavior of the full gluon propagator at small gluon momentum is determined by the behavior of its free PT counterpart. In other words, in QED we can replace \( \Pi(q) \) from the very beginning (\( \Pi(q) \rightarrow \Pi^s(q) \)), totally discarding the quadratically divergent constant \( \Delta^2(\lambda) \) from all the equations and relations. The current conservation condition for the photon self-energy (2.7), i.e., its transversality, and the condition \( q_\mu q_\nu D_{\mu\nu}(q) = i \xi \) (here and everywhere below \( \xi \) is the gauge-fixing parameter) imposed on the full gluon propagator are consequences of gauge invariance. They should be maintained at every stage of the calculations, since the photon is a physical state. In other words, at all stages the current conservation plays a crucial role in extracting physical information from the S-matrix elements in QED, which are usually proportional to the combination \( j^a_\mu(q)D_{\mu\nu}(q)j^a_\nu(q) \). The current conservation condition \( j^a_\mu(q)q_\mu = j^a_\nu(q)q_\nu = 0 \) implies that the unphysical (longitudinal) component of the full photon propagator does not change the physics of QED, i.e., only its physical (transversal) component is important. In its turn this means that the transversality condition imposed on the photon self-energy is also important, because \( \Pi_{\mu\nu}(q) \) itself is a correction to the amplitude of the physical process, for example such as electron-electron scattering.

Concluding, let us emphasize that the photon is always a massless state, since in QED (unlike QCD, see below) the mass gap cannot be realized.

III. QCD

For our purposes just like in QED it is convenient to begin with the general description of the SD equation for the full gluon propagator, and not for its inverse. Symbolically it can be written down as follows:

\[
D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^1_{\mu\nu}(q)\Pi_{\rho\sigma}(q; D)D_{\rho\sigma}(q),
\]  

(3.1)

where \( D^0_{\mu\nu}(q) \) is the free gluon propagator. \( \Pi_{\rho\sigma}(q; D) \) is the gluon self-energy, and in general it depends on the full gluon propagator due to non-abelian character of QCD (see below as well). Thus the gluon SD equation is highly NL, while the photon SD equation (2.1) is a linear one. In what follows we omit the color group indices, since for the gluon propagator (and hence its self-energy) they are reduced to the trivial \( \delta \)-function, for example \( D^{ab}_{\mu\nu}(q) = D_{\mu\nu}(q)\delta^{ab} \).

Also, for convenience, we introduce \( i \) into the gluon SD equation (3.1).

In comparison with the photon self-energy (2.2), the gluon self-energy \( \Pi_{\rho\sigma}(q; D) \) is the sum of a few terms,

\[
\Pi_{\rho\sigma}(q; D) = \Pi^s_{\rho\sigma}(q) + \Pi^g_{\rho\sigma}(q) + \Pi^1_{\rho\sigma}(q; D^2) + \Pi^{(2)}_{\rho\sigma}(q; D^4) + \Pi^{(2)}_{\rho\sigma}(q; D^3),
\]  

(3.2)

where \( \Pi^s_{\rho\sigma}(q) \) describes the skeleton loop contribution due to quark degrees of freedom (it is an analog of the vacuum polarization tensor (2.2) in QED), while \( \Pi^g_{\rho\sigma}(q) \) describes the skeleton loop contribution due to ghost degrees of freedom. Both skeleton loop integrals do not depend on the full gluon propagator \( D \), so they represent the linear contribution to the gluon SD equation. \( \Pi^s_{\rho\sigma}(D) \) represents the so-called constant skeleton tadpole term. \( \Pi^{(1)}_{\rho\sigma}(q; D^2) \) represents the skeleton loop contribution, which contains the triple gluon vertices only. \( \Pi^{(2)}_{\rho\sigma}(q; D^4) \) and \( \Pi^{(2)}_{\rho\sigma}(q; D^3) \) describe topologically independent skeleton two-loop contributions, which combine the triple and quartic gluon vertices. The last four terms explicitly contain the full gluon propagators in the corresponding powers symbolically shown above, that is why they form the NL part of the gluon SD equation. The analytical expressions for the corresponding skeleton loop integrals \( \Pi^{(1)}_{\rho\sigma}(q; D^2) \) (in which the symmetry coefficients can be included) are of no importance here, since we are not going to introduce into them any truncations/approximations as well as to choose some special gauge. Let us note that like in QED these skeleton loop integrals are quadratically divergent at least in the PT, and therefore they are assumed to be regularized (see remarks above and below).

A. Subtractions

Quite similar to the subtraction (2.3), let us formally subtract from the full gluon self-energy (3.2) its value at zero point (see, however, remarks below). Thus, one obtains
\[ \Pi^s_{\rho\sigma}(q; D) = \Pi_{\rho\sigma}(q; D) - \Pi_{\rho\sigma}(0; D) = \Pi_{\rho\sigma}(q; D) - \delta_{\rho\sigma} \Delta^2(\lambda; D). \]  

(3.3)

In this connection let us make in advance a few general remarks. Contrary to QED, QCD being non-abelian gauge theory can suffer from the IR singularities in the \( q^2 \to 0 \) limit due to the self-interaction of massless gluon modes. Thus the initial subtraction at zero point in the definition (3.3) may be dangerous [1], indeed. That is why in all quantities below the dependence on the finite (slightly different from zero) dimensionless subtraction point \( \alpha \) is to be understood. In other words, all the subtractions at zero and the Taylor expansions around zero should be understood as the subtractions at \( \alpha \) and the Taylor expansions near \( \alpha \), where they are justified to use. From the technical point of view it is convenient to put formally \( \alpha = 0 \) in all derivations below, and to restore the explicit dependence on non-zero \( \alpha \) in all quantities at the final stage only. However, we will restore the explicit dependence on \( \alpha \), when it will be necessary for better understanding of the corresponding derivations (see section IV below).

Let us remind once more that by mass gap we understand the difference between the regulated gluon self-energy and its subtracted (also regularized) counterpart. To demonstrate a possible existence of a mass gap \( \Delta^2(\lambda; D) \) in QCD, it is not important how \( \lambda \) has been introduced and how it should be removed at the final stage. The mass gap itself is mainly generated by the nonlinear interaction of massless gluon modes, slightly corrected by the linear contributions coming from the quark and ghost degrees of freedom, namely

\[ \Delta^2(\lambda; D) = \Pi^s(D) + \sum_a \Pi^a(0; D) = \Delta^2_D + \sum_a \Delta^2_a(0; D), \]  

(3.4)

where index "a" runs as follows: \( a = q, gh, 1, 2, 2' \), and here, obviously, the tensor indices are omitted. In these relation all the quadratically divergent constants \( \Pi^a(D) \) and \( \Pi^a(0; D) \), having the dimensions of a mass squared, are given by the corresponding skeleton loop integrals at \( q^2 = 0 \), which appear in Eq. (3.2). In this connection, let us remind that by the quadratic divergences we conventionally understand the divergent constants having the dimensions of a mass squared and summed up into the mass gap (3.4). Then not losing generality, we can put \( \Delta^2(\lambda) = m^2 f(\lambda) \), where \( m^2 \) is some fixed mass squared, and \( f(\lambda) \) is some dimensionless function. Its dependence on \( \lambda \) is determined by the divergences of the above-mentioned skeleton loop integrals. However, due to AF the dependence is linear one (up to AF logarithm), so the divergence becomes the quadratic one \( \Delta^2(\lambda) \sim m^2 \lambda \sim \Lambda^2 \), indeed, like in the PT.

The subtracted gluon self-energy (3.3)

\[ \Pi^s_{\rho\sigma}(q; D) \equiv \Pi^s(q; D) = \sum_a \Pi^s_a(q; D) \]  

(3.5)

is free from the tadpole contribution, because \( \Pi^s(D) = \Pi(D) - \Pi(D) = 0 \), by definition, at any \( D \), while in the gluon self-energy (3.2) it is explicitly present

\[ \Pi_{\rho\sigma}(q; D) \equiv \Pi(q; D) = \Pi(D) + \sum_a \Pi^s_a(q; D). \]  

(3.6)

The general decomposition of the subtracted gluon self-energy into the independent tensor structures can be written down as follows:

\[ \Pi^s_{\rho\sigma}(q; D) = T^s_{\rho\sigma}(q) q^2 \Pi^s(q^2; D) + q_\rho q_\sigma \tilde{\Pi}^s(q^2; D), \]  

(3.7)

where both invariant functions \( \Pi^s(q^2; D) \) and \( \tilde{\Pi}^s(q^2; D) \) are dimensionless functions of their argument \( q^2 \). The subtracted gluon self-energy does not contain the tadpole contribution, see Eq. (3.5). Let us note in advance (see subsection B below) that in this case we can impose the color current conservation condition on it, i.e., to put

\[ q_\rho \Pi^s_{\rho\sigma}(q; D) = q_\sigma \Pi^s_{\sigma\rho}(q; D) = 0, \]  

(3.8)

which implies \( \tilde{\Pi}^s(q^2; D) = 0 \). So the subtracted gluon self-energy finally becomes the purely transversal

\[ \Pi^s_{\rho\sigma}(q; D) = T^s_{\rho\sigma}(q) q^2 \Pi^s(q^2; D). \]  

(3.9)
Let us remind once more that we can expand $\Pi^i(q^2; D)$ in a Taylor series near the subtraction point $\alpha$ at any $D$. Thus the subtracted quantities are free from the quadratic divergences, but the logarithmic ones at large $q^2$ can be still present in $\Pi^0(q^2; D)$, like in QED.

Concluding, let us note that we are not going to impose the transversality condition on the gluon self-energy (3.6) itself (see below). That is why we need no its decomposition into the independent tensor structures.

**B. A self-consistent violation of color gauge invariance/symmetry (SCVCGI/S)**

QCD is $SU(3)$ color gauge invariant theory, but however:

(i). Due to color confinement, the gluon (unlike the photon) is not a physical state. Moreover, there is no such physical amplitude to which the gluon self-energy (like the photon self-energy) may directly contribute. For example, quark/quark and quark/antiquark scattering are not a physical processes.

(ii). Contrary to the conserved currents in QED, the color conserved currents do not play any role in the extraction of physical information from the $S$-matrix elements for the corresponding physical processes and quantities in QCD. In other words, not the conserved color currents, but only their color-singlet counterparts, which can even be partially conserved, contribute directly to the $S$-matrix elements describing this or that physical process/quantity. For example, such an important physical QCD parameter as the pion decay constant is given by the following $S$-matrix element: $<0|J_{\mu}(0)|\pi^i(q) >= iq_\mu F_\nu^{\delta i j}$, where $J_{\mu}(0)$ is just the axial-vector current, while $|\pi^i(q) >$ describes the pion bound-state amplitude, and $i,j$ are flavor indices.

(iii). Moreover, in QCD (contrary to QED) exists a direct evidence/indication that the transversality of the full gluon self-energy may be violated beyond the PT, indeed.

The color gauge invariance condition for the full gluon self-energy (3.2) can be reduced to the three independent transversality conditions imposed on it. It is well known that the quark contribution can be made transversal independently of the pure gluon contributions within any regularization scheme which preserves gauge invariance, for example such as the dimensional regularization method (DRM) [11] (see Refs. [1, 2, 8, 9] as well). So, we can put

$$q_\rho \Pi_{\rho\sigma}^g(q) = q_\sigma \Pi_{\rho\sigma}^g(q) = 0. \quad (3.10)$$

Explicitly it can be shown in lower order of the PT (see, for example Refs. [2, 6, 7, 8, 9]). It is assumed, however, that in principle it should be valid in every order of the PT, thus going beyond the PT.

In the same way the sum of the gluon contributions can be made transversal by taking into account the ghost contribution, so again one can put

$$ q_\rho \left[ \Pi_{\rho\sigma}^{(1)}(q; D^2) + \Pi_{\rho\sigma}^{(2)}(q; D^4) + \Pi_{\rho\sigma}^{(2)}(q; D^3) + \Pi_{\rho\sigma}^{g h}(q) \right] = 0. \quad (3.11)$$

The role of ghost degrees of freedom is to cancel the unphysical (longitudinal) component of gauge boson (gluon) propagator in every order of the PT, i.e., going beyond the PT and thus being the general one. The previous general condition of cancellation (3.11) just demonstrates this, since it contains the corresponding skeleton loop integrals. As in a quark case, the explicit cancellation can be shown, nevertheless, only in lower order of the PT. For this we should put $D = D^0 \equiv D_0$ and approximate all other quantities, entering the corresponding skeleton loop integrals in the relation (3.11), by their free PT counterparts (see, for example Refs. [2, 8, 9]).

However, there is no such regularization scheme (preserving or not gauge invariance) in which the transversality condition for the full gluon self-energy could be satisfied unless the constant skeleton tadpole term

$$\Pi_t(D) = g^2 \int \frac{id^q q_1}{(2\pi)^d} T^0_T(q_1), \quad (3.12)$$

is disregarded from the very beginning (here $T^0_T$ is the four-gluon point-like vertex and we omit the tensor and color indices, as unimportant for further purpose). It is nothing else but the quadratically divergent in the PT constant. It explicitly violets the transversality condition for the full gluon self-energy, since formally $q_\rho \Pi_{\rho\sigma}^i(D) = q_\sigma \delta_{\rho\sigma} \Delta^i_T(D) = q_\sigma \Delta^i_T(D) \neq 0$. In the PT, when the full gluon propagator is always approximated by the free one, the constant tadpole term is set to be zero within the DRM [2, 8, 9], i.e., $\Pi_{\rho\sigma}^i(D_0) = 0$. So in the PT the transversality condition for the full gluon self-energy is always satisfied. However, even in the DRM this is not an exact result, but rather an embarrassing prescription, as pointed out in Ref. [8]. To show explicitly that even in the PT there are still problems, it is instructive.
to substitute the first iteration of the gluon SD equation (3.1) into the previous expression. Symbolically it looks like $D = D_0 + D_0 \Pi(D_0) D_0 + ..., $ where we omit all the indices. Doing so, one obtains

$$
\Pi_t(D) = \Pi_t(D_0) + g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_0^0[D_0(q_1)]^2 i \Pi(q_1; D_0) + ...
$$

$$
\Pi_t(D_0) + \Pi(0; D_0) g^2 \int \frac{i^2 d^4 q_1}{(2\pi)^4} T_0^0[D_0(q_1)]^2 + g^2 \int \frac{i^2 d^4 q_1}{(2\pi)^4} T_0^0[D_0(q_1)]^2 q_1^2 \Pi(q_1^2; D_0) + ... .
$$

(3.13)

Here we introduce the subtraction as follows: $\Pi^*(q_1; D_0) = \Pi(q_1; D_0) - \Pi(0; D_0), $ and $\Pi(0; D_0) = \Pi_t(D_0) + ...,$ where all other quadratically divergent constant terms are omitted, for simplicity. In the second line of Eq. (3.13) the first integral is not only UV divergent but it is IR singular as well. If now we omit the first term in accordance with the above-mentioned prescription, the product of this integral and the tadpole term $\Pi_t(D_0)$ remains, nevertheless, undetermined. Moreover, the structure of the second integral in this line is completely different from the divergent constant integral $\Pi_t(D_0)$. This constant term is also not determined, since in general we do not know the behavior of $\Pi^*(q_1^2; D_0)$ at small and large $q_1^2$. All this reflects the general problem that all such kind of massless integrals $\int (d^4 q(2\pi))^4 (q_{\mu_1}, ..., q_{\mu_n})/(q^2)^n$ are ill defined, since there is no dimension where they are meaningful; they are either IR singular or UV divergent. This prescription clearly shows that the DRM, preserving gauge invariance, nevertheless, does not alone provide us insights into the correct treatment of the power-like IR singularities (we will address this problem in the second (II) part of our investigation). However, in the PT we can adhere to the prescription that such massless tadpole integrals can be discarded in the DRM.[8] This is the only way for ghosts to validate the transversality condition on the full gluon self-energy in PT QCD. It makes the full gluon propagator the purely transversal. Then the $S$-matrix elements for physical quantities and processes in PT QCD become free from unphysical degrees of freedom of gauge bosons, maintaining thus the unitarity of $S$-matrix in this theory.

So, we conclude that beyond the PT, i.e., in the general case the transversality of the full gluon self-energy may be violated. In other words, in the general case, i.e., beyond the PT, we cannot discard the tadpole term (3.12) from the very beginning. If we do not know how to treat it, this does not mean that we should neglect it at all. Its regularized version should be explicitly taken into account. Moreover, all other quadratically divergent constants, which have been summed up into the mass gap (3.4), cannot be discarded like in QED, since the transversality condition for the gluon self-energy is not going to be imposed (see below). In other words, in QCD the quadratic divergences of the corresponding skeleton loop integrals cannot be lowered to the logarithmic ones, and therefore the mass gap (3.4) should be explicitly taken into account in this theory.

Thus in order to realize the mass gap (3.4) our proposal is not to impose the transversality condition on the gluon self-energy (3.2), i.e., to admit that in general case

$$
q_\rho \Pi_{\rho\sigma}(q; D) = q_\rho \Pi_{\rho\sigma}(q; D) \neq 0,
$$

(3.14)

indeed. At the same time, we would like to preserve the color gauge invariance condition for the full gluon propagator, i.e., within our approach the relation

$$
q_\rho q_\nu D_{\rho\nu}(q) = i \xi,
$$

(3.15)

which is nothing else but the ST identity, always holds. This is important for the renormalization.

The lesson which comes from QED is that if one preserves the transversality of the photon self-energy at every stage, then there is no mass gap. Thus in order to realize a mass gap in QCD, our proposal is not to impose the transversality condition on the gluon self-energy, Eq. (3.14), but preserving the ST identity for the full gluon propagator, Eq. (3.15).

Concluding, a few general remarks are in order:

1. We would like to emphasize the special role of the constant skeleton tadpole term (3.12) in the NP QCD dynamics. Its existence is a direct evidence that the transversality of the gluon self-energy may be violated beyond the PT theory.

2. The second important observation is that the ghosts themselves cannot now automatically provide the transversality of the gluon propagator in NP QCD. Thus, we sacrifice the general role of ghosts in order to realize a mass gap. To realize a mass gap is much more necessary than to maintain the general role of ghosts. At long last, the role of ghosts is mainly kinematical, while the mass gap dominates the dynamics of QCD at large distance (see below). This is important for understanding of the confinement mechanism.
3. However, let us note in advance that how to restore the transversality of the full gluon propagator relevant for NP QCD will be explained in part II of our investigation. In other words, at the initial stage we violate the transversality of the gluon self-energy in order to realize a mass gap, while restoring the transversality of the full gluon propagator relevant for NP QCD at the final stage. At the same time, the ST identity for the gluon propagator is always valid within our approach. All this will make it possible to maintain the unitarity of the $S$-matrix in NP QCD.

4. The discussion above does not mean that we need no ghosts at all. We need them in other sectors of QCD, for example in the quark-gluon ST identity, which contains the so-called ghost-quark scattering kernel explicitly [3]. It provides an important piece of information on quark degrees of freedom themselves. If one omits the ghosts, then it will be totally lost (for details see Ref. [12, 13, 14], a recent publication [15] and references therein).

5. The transversality condition for the gluon self-energy can be satisfied partially, i.e., if one imposes it on quark and gluon (along with ghost) degrees of freedom, as it follows from the relations (3.10) and (3.11). Then the mass gap is to be reduced to $\Pi^i(D)$, since all other constants $\Pi^a(0; D)$ can be discarded in this case, see Eq. (3.4). However, we will stick to our proposal not to impose the transversality condition on the gluon self-energy at all, and thus to deal with the mass gap on account of all the possible contributions.

C. General structure of the gluon SD equation

Our strategy is not to impose the transversality condition on the gluon self-energy in order to realize a mass gap despite whether or not the tadpole term is explicitly present. At the same time, we would like to preserve the ST identity (3.15), as underlined above. It implies that the general tensor decomposition of the full gluon propagator becomes the standard one, namely

$$D_{\mu\nu}(q) = i \{ T_{\mu\nu}(q)d(q^2) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2},$$

(3.16)

where $d(q^2) \equiv d(q^2; \xi)$ is the full gluon invariant function (the full gluon form factor or equivalently the full effective charge (“running’’)). To show that our strategy works, let us substitute the subtraction (3.3), on account of the relation (3.9), into the initial gluon SD equation (3.1). Then one obtains

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) iT_{\rho\sigma}(q)q^2\Pi^\sigma(q^2; D)\Pi^\rho(0; D) + D^0_{\mu\sigma}(q)i\Delta^2(\lambda; D)\Pi^\rho(0; D)\Pi^\nu(0; D)\Pi^\rho(0; D)\Pi^\nu(0; D).$$

(3.17)

In the presence of a mass gap, it is instructive to introduce the general tensor decomposition of the auxiliary free gluon propagator as follows:

$$D^0_{\mu\nu}(q) = iT_{\mu\nu}(q) + L_{\mu\nu}(q)d_0(q^2)(1/q^2).$$

(3.18)

Evidently, any temporary deviation in the auxiliary free gluon propagator from the standard free gluon propagator in the presence of a mass gap may appear only in its unphysical (longitudinal) component. There is no dynamics in any free gluon propagator provided by the nontrivial (not equal to one) form factor, affiliated with its transversal component.

Substituting all these decompositions (3.16) and (3.18) into the gluon SD equation (3.17), one obtains

$$d(q^2) = \frac{1}{1 + \Pi^\sigma(q^2; D) + (\Delta^2(\lambda; D)/q^2)},$$

(3.19)

and

$$d_0(q^2) = \frac{\xi}{1 - \xi(\Delta^2(\lambda; D)/q^2)}.$$  

(3.20)

However, we need the standard free gluon propagator rather than its auxiliary counterpart, despite the latter one being reduced to the former one in the formal PT $\Delta^2(\lambda; D) = 0$ limit. To achieve this goal, the auxiliary free gluon propagator defined in Eqs. (3.18) and (3.20) is to be equivalently replaced as follows:

$$D^0_{\mu\nu}(q) \Rightarrow D^0_{\mu\nu}(q) + i\xi L_{\mu\nu}(q)d_0(q^2)\Delta^2(\lambda; D)/q^2,$$

(3.21)
where $D^0_{\mu\nu}(q)$ in the right-hand-side is the standard free gluon propagator now, i.e.,

$$ D^0_{\mu\nu}(q) = i \{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2}. \tag{3.22} $$

Then the gluon SD equation in the presence of a mass gap (3.17), after the replacement (3.21) and doing some algebra, on account of the explicit expression for the auxiliary free gluon form factor (3.20), becomes

$$ D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\nu}(q) i T_{\mu\nu}(q) q^2 \Pi^*(q^2; D) D_{\sigma\nu}(q) + D^0_{\mu\nu}(q) i \Delta^2(\lambda; D) D_{\sigma\nu}(q) + i \xi^2 L_{\mu\nu}(q) \frac{\Delta^2(\lambda; D)}{q^2}. \tag{3.23} $$

Here and from now on $D^0_{\mu\nu}(q)$ is the standard free gluon propagator (3.22). Using it explicitly, and on account of the decomposition (3.16), this equation can be further simplified to

$$ D_{\mu\nu}(q) = D^0_{\mu\nu}(q) - T_{\mu\nu}(q) \left[ \Pi^*(q^2; D) + \frac{\Delta^2(\lambda; D)}{q^2} \right] D_{\sigma\nu}(q). \tag{3.24} $$

It is easy to check that the full gluon propagator given by this equation (and hence by Eq. (3.23)) satisfies the ST identity (3.15), indeed, even in the presence of a mass gap. So the full gluon propagator is the expression (3.16) with the full gluon form factor given in Eq. (3.19), which obviously satisfies Eqs. (3.17), (3.23) and (3.24) simultaneously. The only price we have paid so far is the gluon self-energy, while its subtracted counterpart is always transversal. At the same time, the full gluon propagator (3.16) and the free PT gluon propagator (3.22) automatically satisfy the ST identity (3.15). So, one can conclude that our mechanism for the realization of the mass gap is rather self-consistent.

Let us emphasize that the expression for the full gluon form factor shown in the relation (3.19) cannot be considered as the formal solution for the full gluon propagator $D$ (see Eq. (3.16)), since both the mass gap $\Delta^2(\lambda; D)$ and the invariant function $\Pi^*(q^2; D)$ depend on $D$ themselves. It clearly follows from the relation (3.19) that the effect of the mass gap dominates the IR region when the gluon momentum goes to zero, and this effect vanishes when the gluon momentum goes to infinity. This once more underlines a close intrinsic link between the NP dynamics governed by the mass gap and the structure of the true QCD vacuum at large distances ($q^2 \to 0$). It is worth recalling once more that in the opposite limit, i.e., at large $q^2$, the subtracted gluon self-energy $\Pi^*(q^2; D)$ may still suffer from the logarithmic divergences, like in QED.

In the formal PT $\Delta^2(\lambda; D) = 0$ limit, from the gluon SD equation (3.23) one recovers the standard gluon SD equation (3.1), and the gluon self-energy coincides with its subtracted counterpart like in QED, see Eq. (3.3). Then the formal ”solution” (3.19) will not depend on the mass gap. The above-mentioned general role of ghosts is to be automatically restored. They will again provide the cancellation of the longitudinal component of the full gluon propagator in this limit.

Concluding, we have established the general structure of the full gluon propagator in the presence of a mass gap. Moreover, we have explicitly shown that the initial gluon SD equation (3.17) has the same ”solution” as the final gluon SD equation (3.23), that is Eq. (3.16) along with the relation (3.19).

### IV. NONLINEAR ITERATION SOLUTION

In order to find a formal solution for the regularized full gluon propagator (3.16), on account of its effective charge (3.19) (or equivalently the full gluon form factor), let us rewrite the latter one in the form of the corresponding transcendental (i.e., not algebraic) equation, namely

$$ d(q^2) = 1 - \left[ \Pi(q^2; d) + \frac{\Delta^2(\lambda; d)}{q^2} \right] d(q^2) = 1 - P(q^2; d) d(q^2), \tag{4.1} $$

suitable for the formal nonlinear iteration procedure. Here we replace the dependence on $D$ by the equivalent dependence on $d$. Also, for simplicity, we replaced $\Pi^*(q^2; d) \to \Pi(q^2; d)$. For future purposes, it is convenient to introduce short-hand notations as follows:
\[ \Delta^2(\lambda; d = d^{(0)} + d^{(1)} + d^{(2)} + \ldots + d^{(m)} + \ldots) = \Delta^2_m = \Delta^2 c_m(\lambda, \alpha, \xi, g^2) \equiv c_m \Delta^2, \]
\[ \Pi(q^2; d = d^{(0)} + d^{(1)} + d^{(2)} + \ldots + d^{(m)} + \ldots) = \Pi_m(q^2), \] (4.2)
and
\[ P_m(q^2) = \left[ \Pi_m(q^2) + \frac{\Delta_m^2}{q^2} \right], \quad m = 0, 1, 2, 3, \ldots. \] (4.3)

In these relations \( \Delta^2_m \) are the auxiliary mass squared parameters, while \( \Delta^2 \equiv \Delta^2(\lambda; d) \) is the mass gap itself. Via the corresponding subscripts the dimensionless constants \( c_m \) depend on which iteration for the gluon form factor \( d \) is actually done. They may depend on the dimensionless coupling constant squared \( q^2 \), as well as on the gauge-fixing parameter \( \xi \). We also introduce in advance the explicit dependence on the finite (slightly different from zero) dimensionless subtraction point \( \beta \), as pointed out above. The dependence of \( \Delta^2 \) on all these parameters, as well as on the number of different flavors \( N_f \) and colors \( N_c \), is not shown explicitly, and if necessary it can be restored any time. Let us also recall that all the invariant functions \( \Pi_m(q^2) \) can be expand in a formal Taylor series near the finite subtraction point \( \alpha \). If it were possible to express the full gluon form factor \( d(q^2) \) in terms of these quantities then it would be the formal solution for the full gluon propagator. In fact, this is nothing but the skeleton loops expansion, since the regularized skeleton loop integrals, contributing to the gluon self-energy, have to be iterated. This is the so-called general nonlinear iteration solution. As mentioned above, no truncations/approximations/assumptions and no special gauge choice have been made. This formal expansion is not a PT series. The magnitude of the coupling constant squared and the dependence of the regularized skeleton loop integrals on it is completely arbitrary. Let us emphasize once more that through this paper the mass gap \( \Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2) \) is a “bare” one, i.e., it is only regularized (with the help of \( \lambda \) and \( \alpha \)) in order to assign a mathematical meaning to all derivations involving it.

It is instructive to describe the general iteration procedure in some details. Evidently, \( d^{(0)} = 1 \), and this corresponds to the approximation of the full gluon propagator by its free counterpart. Doing the first iteration in Eq. (4.1), one thus obtains

\[ d(q^2) = 1 - P_0(q^2) + \ldots = 1 + d^{(1)}(q^2) + \ldots, \] (4.4)
where obviously
\[ d^{(1)}(q^2) = -P_0(q^2). \] (4.5)
Carrying out the second iteration, one gets
\[ d(q^2) = 1 - P_1(q^2)[1 + d^{(1)}(q^2)] + \ldots = 1 + d^{(1)}(q^2) + d^{(2)}(q^2) + \ldots, \] (4.6)
where
\[ d^{(2)}(q^2) = -d^{(1)}(q^2) - P_1(q^2)[1 - P_0(q^2)]. \] (4.7)
Doing the third iteration, one further obtains
\[ d(q^2) = 1 - P_2(q^2)[1 + d^{(1)}(q^2) + d^{(2)}(q^2)] + \ldots = 1 + d^{(1)}(q^2) + d^{(2)}(q^2) + d^{(3)}(q^2) + \ldots, \] (4.8)
where
\[ d^{(3)}(q^2) = -d^{(1)}(q^2) - d^{(2)}(q^2) - P_2(q^2)[1 - P_1(q^2)(1 - P_0(q^2))], \] (4.9)
and so on for the next iterations.
Thus up to the third iteration, one finally arrives at
\[ d(q^2) = \sum_{m=0}^{\infty} d^{(m)}(q^2) = 1 - \left[ \Pi_2(q^2) + \frac{\Delta_2^2}{q^2} \right] - \left[ \Pi_1(q^2) + \frac{\Delta_1^2}{q^2} \right] - \left[ \Pi_0(q^2) - \frac{\Delta_0^2}{q^2} \right] + \ldots. \] (4.10)

We restrict ourselves by the iterated gluon form factor up to the third term, since this already allows to show explicitly some general features of the nonlinear iteration solution.
A. Splitting/shifting procedure

Doing some tedious algebra, the previous expression (4.10) can be rewritten as follows:

\[ d(q^2) = \left[ 1 - \Pi_2(q^2) + \Pi_1(q^2)\Pi_2(q^2) - \Pi_0(q^2)\Pi_1(q^2)\Pi_2(q^2) + \ldots \right] \]
\[ + \frac{1}{q^2} \left[ \Pi_2(q^2)\Delta^2 + \Pi_1(q^2)\Delta_2^2 + \Pi_0(q^2)\Pi_1(q^2)\Delta^2_2 - \Pi_0(q^2)\Pi_2(q^2)\Delta^2_1 - \Pi_1(q^2)\Pi_2(q^2)\Delta^2_2 + \ldots \right] \]
\[ - \frac{1}{q^2} \left[ \Pi_0(q^2)\Delta^2_2\Delta^2_1 + \Pi_1(q^2)\Delta^2_2\Delta^2_0 + \Pi_2(q^2)\Delta^2_0\Delta^2_1 + \ldots \right] \]
\[ - \frac{1}{q^2} \left[ \Delta^2_2 - \frac{\Delta^2_1\Delta^2_2}{q^2} + \frac{\Delta^2_1\Delta^2_1\Delta^2_0}{q^4} + \ldots \right]. \] (4.11)

This formal expansion contains three different types of terms. The first type are the terms which contain only different combinations of \( \Pi_m(q^2) \) (they are not multiplied by inverse powers of \( q^2 \)); the third type of terms contains only different combinations of \( (\Delta^2_2/q^2) \). The second type of terms contains the so-called mixed terms, containing the first and third types of terms in different combinations. The two last types of terms are multiplied by the corresponding powers of \( 1/q^2 \). Such structure of terms will be present in each iteration term for the full gluon form factor. However, any of the mixed terms can be split exactly into the first and third types of terms. For this purpose the formal Taylor expansions for \( \Pi_m(q^2) \) around the finite subtraction point \( \alpha \) should be used. Thus an exact IR structure of the full gluon form factor (which just is our primary goal to establish) is determined not only by the third type of terms. It gains contributions from the mixed terms as well, but without changing its functional dependence (see remarks below). To demonstrate this in some detail, it is convenient to express the previous expansion (4.11) in terms of dimensionless variable and parameters, namely

\[ x = \frac{q^2}{M^2}, \quad c = \frac{\Delta^2}{M^2}, \quad \alpha = \frac{\mu^2}{M^2}. \] (4.12)

where \( M^2 \) is some auxiliary fixed mass squared, and \( \mu^2 \) is the point close to \( q^2 = 0 \) (to be not mixed up with the tensor index). Also, in the formal PT \( \Delta^2 = 0 \) limit \( c = c(\lambda, \alpha, \xi, g^2) = 0 \) (unlike \( c_m \) in the relations (4.2)), since \( M^2 \) is fixed. Using further relations (4.2), and on account of the relations (4.12), the expansion (4.11) becomes

\[ d(x) = \left[ 1 - \Pi_2(x) + \Pi_1(x)\Pi_2(x) - \Pi_0(x)\Pi_1(x)\Pi_2(x) + \ldots \right] \]
\[ + \frac{c}{x} \left[ \Pi_2(x)c_1 + \Pi_1(x)c_2 - \Pi_0(x)\Pi_1(x)c_2 - \Pi_0(x)\Pi_2(x)c_1 - \Pi_1(x)\Pi_2(x)c_2 + \ldots \right] \]
\[ - \frac{c^2}{x^2} \left[ \Pi_0(x)c_1c_2 + \Pi_1(x)c_0c_2 + \Pi_2(x)c_0c_1 + \ldots \right] \]
\[ - \frac{c}{x} \left[ c_2 - \frac{c_1c_2c}{x} + \frac{c_0c_1c_2c^2}{x^2} + \ldots \right]. \] (4.13)

Taking into account the above-mentioned formal Taylor expansions

\[ \Pi_m(x) = \sum_{n=0}^{\infty} (x - \alpha)^n \Pi_m^{(n)}(\alpha) = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} p_{nk}x^k\alpha^{n-k} \right] \Pi_m^{(n)}(\alpha), \] (4.14)

for example, the mixed term \( (c_1c/x)\Pi_2(x) \) can be then exactly split/decomposed as follows:

\[ \frac{c_1c}{x} \Pi_2(x) = \frac{c_1c}{x} \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} p_{nk}x^k\alpha^{n-k} \right] \Pi_2^{(n)}(\alpha) = \left( \frac{c_1c}{x} \right) P_1(\alpha) + P_0(\alpha) + O_2(x). \] (4.15)

Here and below the dependence on all other possible parameters is not shown, for simplicity. The dimensionless function \( O_2(x) \) is of the order \( x \) at small \( x \); otherwise it remains arbitrary. The first term now is to be shifted to the third type of terms, while the remaining terms are to be shifted to the first type of terms. All other mixed terms of similar structure should be treated absolutely in the same way.
The mixed term \( \frac{c_1c_2c^2}{x^2} \Pi_0(x) \) can be split as

\[
\frac{c_1c_2c^2}{x^2} \Pi_0(x) = \frac{c_1c_2c^2}{x^2} \sum_{n=0}^\infty \left[ \sum_{k=0}^n \frac{p_{nk} x^k \alpha^{n-k}}{\alpha^{n-k}} \right] \Pi_0^{(n)}(\alpha) = \left( \frac{c}{x} \right)^2 \Pi_0^{(1)}(\alpha) + \left( \frac{c}{x} \right) \Pi_0^{(2)}(\alpha) + O_0(x),
\]

where the dimensionless function \( O_0(x) \) is of the order \( x \) at small \( x \); otherwise it remains arbitrary. Again the first two terms should be shifted to the third type of terms, while the last two terms should be shifted to the first type of terms.

Similarly to the formal Taylor expansion (4.14), we can write

\[
\Pi_m(x) \Pi_{m'}(x) = \Pi_{mm'}(x) = \sum_{n=0}^\infty (x - \alpha)^n \Pi_{mm'}^{(n)}(\alpha) = \sum_{n=0}^\infty \left[ \sum_{k=0}^n \frac{p_{nk} x^k \alpha^{n-k}}{\alpha^{n-k}} \right] \Pi_{mm'}^{(n)}(\alpha).
\]

Then, for example the mixed term \( \frac{c_2c}{x} \Pi_0(x) \Pi_1(qx) \) can be split as

\[
\frac{c_2c}{x} \Pi_0(x) \Pi_1(x) = \frac{c_2c}{x} \Pi_0(x) = \frac{c_2c}{x} \sum_{n=0}^\infty \left[ \sum_{k=0}^n \frac{p_{nk} x^k \alpha^{n-k}}{\alpha^{n-k}} \right] \Pi_{01}^{(n)}(\alpha) = \left( \frac{c}{x} \right) M_1(\alpha) + M_0(\alpha) + O_{01}(x),
\]

where the dimensionless function \( O_{01}(x) \) is of the order \( x \) at small \( x \); otherwise it remains arbitrary. Again the first term should be shifted to the third type of terms, while other two terms are to be shifted to the first type of terms.

Completing this exact splitting/shifting procedure in the expansion (4.13), and restoring the explicit dependence on the dimensional variable and parameters (4.12), one can equivalently present the initial expansion (4.11) as follows:

\[
d(q^2) = \left( \frac{\Delta^2}{q^2} \right) B_1(\lambda, \alpha, \xi, g^2) + \left( \frac{\Delta^2}{q^2} \right)^2 B_2(\lambda, \alpha, \xi, g^2) + \left( \frac{\Delta^2}{q^2} \right)^3 B_3(\lambda, \alpha, \xi, g^2) + \ldots + d_3(q^2; \Delta^2) + \ldots,
\]

where we use notations (4.2) explicitly now, since the coefficients of the above-used expansions depend in general on the same set of parameters: \( \lambda, \alpha, \xi, g^2 \). The invariant function \( d_3(q^2; \Delta^2) \) is dimensionless and it is free from the power-type IR singularities; otherwise it remains arbitrary. We have restored the dependence on the mass gap \( \Delta^2 \) instead the dependence on the parameter \( c \). In the formal \( \text{PT} \Delta^2 = 0 \) limit it survives, and is to be reduced to the sum of the first type of terms in the expansion (4.11). The generalization on the next iterations is almost obvious.

Concluding, let us underline that the splitting/shifting procedure does not change the structure of the nonlinear iteration solution at small \( q^2 \). It only changes the coefficients at inverse powers of \( q^2 \) in the corresponding expansion. In other words, it makes it possible to rearrange the terms in the initial expansion (4.11) in order to get it in the final form (4.19). Also, in the \( q^2 \rightarrow 0 \) limit, it is legitimate to suppress the subtracted gluon self-energy in comparison with the mass gap term in the initial Eq. (4.1). Nevertheless, as a result of the splitting/shifting procedure, which becomes almost trivial in this case, one will obtain the same expansion (4.19) with only different residues, as just mentioned above, and with \( d_3(q^2; \Delta^2) = 1 \) in this case.

**B. The exact structure of the general nonlinear iteration solution**

Substituting the generalization of the expansion (4.19) on all iterations and doing some algebra, the general nonlinear iteration solution for the regularized full gluon propagator (3.16) can be exactly decomposed as the sum of the two principally different terms as follows:

\[
D_{\mu\nu}(q) = i \left\{ T_{\mu\nu}(q)d(q^2; \xi) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2} = D^{\text{INP}}_{\mu\nu}(q; \Delta^2) + D^{\text{PT}}_{\mu\nu}(q; \Delta^2),
\]

so that \( D_{\mu\nu}(q) = D_{\mu\nu}(q; \Delta^2) \). Here

\[
D^{\text{INP}}_{\mu\nu}(q; \Delta^2) = i T_{\mu\nu}(q)d^{\text{INP}}(q^2; \Delta^2) \frac{1}{q^2} = i T_{\mu\nu}(q) \frac{\Delta^2}{(q^2)^2} L(q^2; \Delta^2),
\]

(4.21)
\[ L(q^2; \Delta^2) = \sum_{k=0}^{\infty} \left( \frac{\Delta^2}{q^2} \right)^k \Phi_k(\lambda, \alpha, \xi, g^2) = \sum_{k=0}^{\infty} \left( \frac{\Delta^2}{q^2} \right)^k \sum_{m=0}^{\infty} \Phi_{km}(\lambda, \alpha, \xi, g^2), \quad (4.22) \]

while

\[ D_{\mu\nu}^{PT}(q; \Delta^2) = i \left[ T_{\mu\nu}(q)d^{PT}(q^2; \Delta^2) + \xi L_{\mu\nu}(q) \right] \frac{1}{q^2}. \quad (4.23) \]

In Eqs. (4.20) and (4.21) the superscript "INP" stands for the intrinsically NP part of the full gluon propagator. Let us emphasize that the general problem of convergence of formal (but regularized) series in Eq. (4.20) is irrelevant here. In other words, it does not make any sense to discuss the convergence of such kind of series before the renormalization program is performed (which will allow to see whether the mass gap survives it at all or not). Anyway, the problem how to remove the UV overlapping divergences (Ref. [16] and for a more detail remarks see appendix C) and usual overall ones [1, 2, 7, 8, 9] is a standard one. Our problem will be how to deal with severe IR singularities due to their novelty and genuine NP character. Fortunately, there already exists a well-elaborated mathematical formalism for this purpose, namely the distribution theory (DT) [17] into which the DRM [11] should be correctly implemented (see also Refs. [18, 19]). It will be explicitly used in part II of our investigation.

We distinguish between the two terms in Eq. (4.20) first of all by the character of the corresponding IR singularities, when the gluon momentum goes to zero. Secondly, by the explicit presence of the mass gap, when it formally goes to zero then the INP term vanishes, while the PT term survives. The third point is that the INP part depends only on the transversal degrees of freedom of gauge bosons, while the PT part has the longitudinal component as well.

The INP part of the full gluon propagator is characterized by the presence of severe power-type (or equivalently NP) IR singularities \((q^2)^{-2-k}, k = 0, 1, 2, 3, \ldots\). So these IR singularities are defined as more singular than the power-type IR singularity of the free gluon propagator \((q^2)^{-1}\), which thus can be defined as the PT IR singularity. The INP part of the full gluon propagator (4.21), apart from the structure \((\Delta^2/q^2)\), is nothing but the corresponding Laurent expansion (explicitly shown in Eq. (4.22)) in integer powers of \(q^2\) accompanied by the corresponding powers of the mass gap squared and multiplied by the \(q^2\)-independent factors, the so-called residues \(\Phi_k(\lambda, \alpha, \xi, g^2) = \sum_{m=0}^{\infty} \Phi_{km}(\lambda, \alpha, \xi, g^2)\). The sum over \(m\) indicates that an infinite number of iterations (all iterations) of the corresponding regularized skeleton loop integrals invokes each severe IR singularity labelled by \(k\). It is worth emphasizing that now this Laurent expansion cannot be summed up into anything similar to the initial Eq. (3.19), since its residues at poles gain additional contributions due to the splitting/shifting procedure, i.e., they become arbitrary. However, this arbitrariness is not a problem. The functional dependence, which has been established exactly, is all that matters (a correct treatment of severe IR singularities will be given in part II of our investigation, as mentioned above). It is worth emphasizing once more that just the mass gap term in Eq. (4.1) determines the functional structure of the INP part (4.21)-(4.22) of the full gluon propagator.

Thus the true QCD vacuum is really beset with severe IR singularities. Within the general nonlinear iteration solution they should be summarized (accumulated) into the full gluon propagator and effectively correctly described by its structure in the deep IR domain, exactly represented by its INP part (4.21). The second step is to assign a mathematical meaning to the integrals, where such kind of severe IR singularities will explicitly appear, i.e., to define them correctly in the IR region [10, 17, 18]. Just this violent IR behavior makes QCD as a whole an IR unstable theory, and therefore it may have no IR stable fixed point [1]. This means that QCD itself might be a confining theory without involving some extra degrees of freedom [12, 20, 21, 22, 23, 24, 25, 26]. In part II we will show that this is so, indeed.

It is instructive to discuss the principal difference between QED and QCD from the point of view of the structures of the full photon and gluon propagators, respectively. In section II it has been explained that the mass gap cannot occur in QED because the photon is a pure PT-like IR singularity, i.e., \(1/q^2\), see Eq. (2.13). This is in agreement with the cluster property of the Wightman functions, that is, correlation functions of observables in QED [27]. It forbids a more singular behavior of the full photon propagator in the IR than the behavior of its free photon counterpart. In QCD, due to color confinement the gluon is not a physical state. From our investigation then it follows that the mass gap can be realized, and the general nonlinear iteration solution for the full gluon propagator (4.20) becomes inevitably severely IR singular. This is in agreement with the so-called Strocchi theorem [28], which validates such singular structure of the full gluon propagator in the IR. So the existence of a mass gap is a primary reason for a possible violation of the cluster property of the Wightman functions in QCD (to our best knowledge there is no rigorous proof of this property in this theory, see discussion in Ref. [3] as well). At the fundamental quark-gluon level only these remarks about the structure of the Wightman functions in QCD make sense before the solution of the color confinement problem. Concluding, let
us note in advance that in the presence of a mass gap there also exists the so-called massive solution. It becomes explicitly smooth at small $q^2$ in the Landau gauge only (see appendix A).

The PT part of the full gluon propagator (4.23), which has the power-type PT IR singularity only, remains undetermined. Its functional structure is due to the unknown in general $II(q^2; D)$ term in Eq. (4.1). This is the price we have paid to fix exactly the functional dependence of the INP part of the full gluon propagator. What we know about the PT effective charge $d^{PT}(q^2; \Delta^2)$ is that it cannot have the power-type IR singularities; otherwise it remain arbitrary. Also, in the formal PT $\Delta^2 = 0$ limit it survives, i.e., $d^{PT}(q^2; \Delta^2 = 0) = d^{PT}(q^2)$. This can be shown directly by restoring the explicit dependence on the mass gap instead the dependence on the coefficient $c = (\Delta^2/M^2)$, which appears in the splitting/shifting procedure. According to this procedure, the PT effective charge can be explicitly written down as follows:

$$d^{PT}(q^2; \Delta^2) = \sum_{k=0}^{\infty} \left( \frac{\Delta^2}{M^2} \right)^k \sum_{m=0}^{\infty} A_{km}(q^2),$$

(4.24)

where $A_{km}(q^2)$ are dimensionless functions, which cannot have the power-type IR singularities; otherwise they remain arbitrary (their dependence on the parameters are not shown, for simplicity). The explicit presence of the mass gap (or equivalently the above-mentioned coefficient $c$) in this expansion just prevents the ghosts to cancel the longitudinal component in the full gluon propagator (4.20). However, in the formal PT $\Delta^2 = 0$ limit the role of ghosts will be automatically restored, as pointed out above in section III. At the same time, it is worth emphasizing that due to the character of the IR singularity the longitudinal component of the full gluon propagator should be included into its PT part. That is the reason why its INP part becomes automatically transversal, as emphasized above. This also means that the PT part of the full gluon propagator (4.23) contains the free PT gluon propagator, so that we can put $d^{PT}(q^2) = 1 + d^{AF}(q^2)$, where $d^{AF}(q^2)$ should satisfy AF in the $q^2 \to \infty$ limit (see appendix B).

Both terms in Eq. (4.20) are valid in the whole energy/momentum range, i.e., they are not asymptotics. At the same time, we have achieved the exact separation between the two terms responsible for the NP (dominating in the IR ($q^2 \to 0$)) and the nontrivial PT (dominating in the UV ($q^2 \to \infty$)) dynamics in the true QCD vacuum. The structure of this solution clearly confirms our conclusion driven above that the deep IR region interesting for confinement and other NP effects is dominated by the presence of a mass gap. In the formal PT $\Delta^2 = 0$ limit, the nontrivial PT dynamics is all that matters. Let us note in advance that the separation is not only exact but it is unique as well. There exists a special regularization expansion for severe (i.e., NP) IR singularities, while for the PT IR singularity, which is only one present in the PT part of the full gluon propagator (4.23), such kind of expansion does not exist.

We came to the same structure (4.20)-(4.23) but in a rather different way in Refs. [10, 18, 19].

V. GENERAL DISCUSSION

The mass gap $\Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2)$ has not been introduced by hand. It is hidden in the skeleton loop integrals, contributing to the full gluon self-energy. No truncations/approximations/assumptions and no special gauge choice are made for these integrals. An appropriate subtraction scheme has been applied to make the existence of a mass gap perfectly clear. It is dynamically generated mainly by the NL interaction of massless gluon modes. The Lagrangian of QCD does not contain a mass gap, while it explicitly appears in the gluon SD equation of motion. This once more underlines the importance of the investigation of the SD system of equations and the corresponding ST identities [11, 17, 19] and references therein) for understanding of the true structure of the QCD ground state.

From the relation (3.19) it follows that the mass gap shows up explicitly when the gluon momentum goes to zero. Especially this is clearly seen in the general nonlinear iteration solution (4.20), where it indeed explicitly determines its IR structure. There are no doubts that the mass gap plays a dominant role in the dynamics of QCD at large distances. So the problem how to "liberate/release" it from the QCD vacuum becomes vitally important to understand correctly the mechanism of color confinement. It turned out that for this aim we need to sacrifice the role of ghosts at the initial stage, while restoring the transversality of the full gluon propagator relevant for NP QCD at the final stage.

In order to realize a mass gap (more precisely its regularized version), we propose not to impose the transversality condition on the full gluon self-energy, see Eq. (3.14), while always preserving the ST identity for the full gluon propagator, see Eq. (3.15). Such a self-consistent violation of color gauge invariance/symmetry (SCVCGI/S) is completely NP effect, since in the formal PT $\Delta^2 = 0$ limit this effect vanishes. The first necessary condition for the
SCVCGI/S is color confinement, due to which the gluon is not a physical state. The second sufficient condition for the SCVCGI/S is the explicit presence of the skeleton tadpole term in the full gluon self-energy. In other words, not to request the transversality of the gluon self-energy is the first necessary condition to realize a mass gap. In this case whether the tadpole term is explicitly present or not in the gluon self-energy becomes not important. It plays only a temporary role, indicating that the above-mentioned transversality may be violated at the initial stage, indeed. All this makes it possible to establish the structure of the regularized full gluon propagator and the corresponding gluon SD equation in the presence of a mass gap. After the restoration of the transversality of the full gluon propagator relevant for NP QCD in part II of our investigation, the SCVCGI/S in QCD will have no direct physical consequences. None of physical quantities/processes in low-energy QCD will be affected by this proposal, i.e., the unitarity of $S$-matrix in NP QCD will not suffer, as emphasized above.

In QED a mass gap is always in the "gauge prison". It cannot be realized even temporarily, since the photon is a physical state. However, in QCD a door of the "color gauge prison" can be opened for a moment in order to realize a mass gap, because the gluon is not a physical state. A key to this "door" is the constant skeleton tadpole term, which explicitly violates the transversality of the full gluon self-energy. On the other hand, this "door" can be opened without a key (as any door) by not imposing the transversality condition on the full gluon self-energy. So in QED a mass gap cannot be "liberated/released" from the vacuum, while photons and electrons can be liberated/released from the vacuum in order to be physical states. In QCD a mass gap can be "liberated/released" from the vacuum, while gluons and quarks cannot be liberated/released from the vacuum in order to be physical states. In other words, there is no breakdown of $U(1)$ gauge symmetry in QED because the photon is a physical state. At the same time, a temporary breakdown of $SU(3)$ color gauge symmetry in QCD is possible because the gluon is not a physical state.

In summary, QCD as a theory of quark-gluon interactions may have a mass gap $\Delta_2$, possibly realized in accordance with our proposal. The dynamically generated mass is usually related to breakdown of some symmetry (for example, the dynamically generated quark mass is an evidence of chiral symmetry breakdown). Here a mass gap is an evidence of the SCVCGI/S. In the presence of a mass gap the coupling constant plays no role. Thus the SCVCGI/S is also a direct evidence of the "dimensional transmutation", $g^2 \to \Delta_2^2(\lambda, \alpha, \xi, g^2)$ [30, 31], which occurs whenever a massless theory acquires masses dynamically. It is a general feature of spontaneous symmetry breaking in field theories.

### VI. CONCLUSIONS

A self-consistent violation of $SU(3)$ color gauge invariance/symmetry for the realization of a mass gap as it has been described in this investigation (section III) and briefly discussed in the previous section is our first main result.

The structures of the regularized full gluon propagator and the corresponding gluon SD equation in the presence of a mass gap have been firmly established. This is our second main result (subsection C in section III).

The general nonlinear iteration solution (4.20) for the full gluon propagator explicitly depends on the mass gap. However, it is always severely singular in the $q^2 \to 0$ limit. So the gluons remain massless, and this does not depend on the gauge choice. This solution is our third main result (section IV).

In the presence of a mass gap gluon may acquire an effective gluon mass, depending on the gauge choice (the so-called massive solution). This is our fourth main result. Nevertheless, we put it into appendix A, since its relation to the solution of the color confinement problem is unclear, even after the renormalization program is performed.

Thus, we have shown explicitly that in the presence of a mass gap at least two independent and different types of formal solutions for the regularized full gluon propagator exist (let us remind that the gluon SD equation is highly NL one, so the number of independent solutions is not fixed). No truncations/approximations/assumptions and no special gauge choice are made in order to show the existence of these general types of solutions.

The nonlinear iteration solution (4.20) is interesting for confinement. However, three important problems remain to solve:

- **A.** How to make this solution to be the purely transversal.
- **B.** To perform the renormalization program for the mass gap, and to see whether the mass gap survives it or not.
- **C.** How to treat correctly severe IR singularities inevitably present in this solution.

Let us now explain each of these points in more detail.

**A.** As we already know, in the presence of a mass gap the ghosts alone cannot provide the cancellation of the longitudinal component of the full gluon propagator. So a universal method of the restoration of the transversality of the full gluon propagator relevant for NP QCD should be formulated. This will make it possible to eliminate the explicit dependence on the gauge-fixing parameter from all the corresponding expressions in NP QCD. It is worth emphasizing in advance that due to the above-mentioned universal method, the PT part of the general nonlinear iteration solution (4.23), which remains undetermined, will be of no importance for us. Only its INP counterpart (4.21)-(4.22), which functional dependence is exactly determined up to the above-mentioned residues, will matter.
B. As repeatedly mentioned above, in this paper the mass gap has been only regularized, i.e., \( \Delta^2 = \Delta^2(\lambda, \alpha, \xi, g^2) \). To perform the renormalization program means to remove the dependence on the above shown parameters in a self-consistent way. We should prove that the product \( \Delta_{JW}^2 = Z(\lambda, \alpha, \xi, g^2)\Delta^2(\lambda, \alpha, \xi, g^2) \) exists in the \( \lambda \to \infty \) and \( \alpha \to 0 \) limits. The mass gap’s renormalization constant \( Z(\lambda, \alpha, \xi, g^2) \) should appear naturally, i.e., it should not be introduced by hand in order not to compromise the general renormalizability of QCD. In other words, the renormalized mass gap should not depend on the gauge-fixing parameter, should be finite, positive definite, etc. Only after performing this program, we can assign to the Jaffe-Witten (JW) mass gap \( \Delta_{JW}^2 \) a physical meaning to be responsible for the NP dynamics in QCD.

C. This solution is characterized by the explicit presence of severe IR singularities \( (q^2)^{-2-k} \), \( k = 0, 1, 2, 3..., \) which, in principle, are independent from each other. The only method to treat them in a self-consistent way is the DRM [11], correctly implemented into the DT [16], as emphasized above.

All these problems will be addressed and solved in our subsequent paper (part II of our investigation).

Concluding, a few preliminary remarks are in order. We have already emphasized that the most appropriate place where the mass gap may appear is the system of the SD equations (complemented by the corresponding ST identities) for the QCD Green’s functions in momentum space. The gauge dependence of the gluon Green’s function (4.20) in this space is in general twofold: the explicit dependence on the gauge-fixing parameter via the longitudinal component and the implicit gauge dependence in the Lorentz structure, affiliated with its transversal component. Within our approach the former will be solved if we will be able to formulate a general method how to restore the transversality of the full gluon propagator relevant for NP QCD (the problem A). The latter one will be solved if we will be able to find how to renormalize the mass gap in a gauge-invariant way, i.e., the mass gap should survive the renormalization program (the problem B). Solving the above-mentioned problems, we will achieve an explicit gauge independence at the fundamental quark-gluon level, while it is too earlier to discuss the gauge independence of S-matrix elements in NP QCD within our approach. Working always in momentum space, we avoid thus the problem of gauge ambiguity (uncertainty) [32], which has been discovered in much more complicated functional space (see Ref. [33] as well).

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APPENDIX A: MASSIVE SOLUTION

One of the direct consequences of the explicit presence of a mass gap in the full gluon propagator is that the gluon may indeed acquire an effective mass. From Eq. (3.19) it follows that

\[
\frac{1}{q^2}d(q^2) = \frac{1}{q^2 + q^2\Pi(q^2; \xi) + \Delta^2(\lambda, \xi)}.
\]

(A1)

where instead of the dependence on \( D \) the dependence on \( \xi \) is explicitly shown (let us remind that as in the case of the iteration solution, we replace \( \Pi^*(q^2; \xi) \to \Pi(q^2; \xi) \)). The full gluon propagator (3.16) may have a pole-type solution at the finite point if and only if the denominator in Eq. (A1) has a zero at this point \( q^2 = -m_g^2 \) (Euclidean signature), i.e.,

\[
-m_g^2 - m_g^2\Pi(-m_g^2; \xi) + \Delta^2(\lambda, \xi) = 0,
\]

(A2)

where \( m_g^2 \equiv m_g^2(\lambda, \xi) \) is an effective gluon mass, and the previous equation is a transcendental equation for its determination. Evidently, the number of the solutions of this equation is not fixed, \textit{a priori}. Excluding the mass gap, one obtains that the denominator in the full gluon propagator becomes

\[
q^2 + q^2\Pi(q^2; \xi) + \Delta^2(\lambda, \xi) = q^2 + m_g^2 + q^2\Pi(q^2; \xi) + m_g^2\Pi(-m_g^2; \xi).
\]

(A3)

Let us now expand \( \Pi(q^2; \xi) \) in a Taylor series near \( m_g^2 \),

\[
\Pi(q^2; \xi) = \Pi(-m_g^2; \xi) + (q^2 + m_g^2)\Pi'(-m_g^2; \xi) + O\left((q^2 + m_g^2)^2\right).
\]

(A4)
Substituting this expansion into the previous relation and after doing some tedious algebra, one obtains

$$q^2 + m_g^2 + q^2 \Pi(q^2; \xi) + m_g^2 \Pi(-m_g^2; \xi) = (q^2 + m_g^2)[1 + \Pi(-m_g^2; \xi) - m_g^2 \Pi'(-m_g^2; \xi)][1 + \Pi^R(q^2; \xi)],$$  \hspace{1cm} (A5)

where $\Pi^R(q^2; \xi) = 0$ at $q^2 = -m_g^2$, otherwise it remains arbitrary. Thus the full gluon propagator (3.16) now looks

$$D_{\mu\nu}(q; m_g^2) = iT_{\mu\nu}(q) \frac{Z_3}{(q^2 + m_g^2)[1 + \Pi(q^2; m_g^2)]} + i\xi L_{\mu\nu}(q) \frac{1}{q^2},$$  \hspace{1cm} (A6)

where, for future purpose, in the invariant function $\Pi^R(q^2; m_g^2)$ instead of $\xi$ we introduced the dependence on the gluon effective mass squared $m_g^2$ which depends on $\xi$ itself. The gluon renormalization constant is

$$Z_3 = \frac{1}{1 + \Pi(-m_g^2; \xi) - m_g^2 \Pi'(-m_g^2; \xi)}.$$  \hspace{1cm} (A7)

In the formal PT limit $\Delta^2(\lambda, \xi) = 0$, an effective gluon mass is also zero, $m_g^2(\lambda, \xi) = 0$, as it follows from Eq. (A2). So an effective gluon mass is the NP effect. At the same time, it cannot be interpreted as the "physical" gluon mass, since it remains explicitly gauge-dependent quantity (at least at this stage). In other words, we were unable to renormalize an effective gluon mass squared, but its interpretation from the physical point of view is clear. Evidently, the massive solution (A6) is completely independent from the general nonlinear iteration solution (4.20), and it is difficult to relate it to confinement. However, its existence shows the general possibility for a vector particles to acquire masses dynamically, i.e., without so-called Higgs mechanism (which in its turn requires the existence of not yet discovered Higgs particle). The above-mentioned possibility is due only to the internal dynamics and symmetries of the corresponding gauge theory.

**APPENDIX B: REMARKS ON ASYMPTOTIC FREEDOM**

As mentioned above, our general solution for the "running" effective charge (3.19) should satisfy AF at large $q^2$. We already know that in the $q^2 \to \infty$ limit the subtracted gluon self-energy $\Pi(q^2; D)$ can be only logarithmically divergent at any $D$. So neglecting the mass gap term in the relation (3.19), to leading order one obtains

$$g^2(q^2; \Lambda^2) = \frac{g^2(\lambda)}{1 + bg^2(\lambda) \ln(q^2/\Lambda^2)},$$  \hspace{1cm} (B1)

where $\Lambda^2$ is the UV cutoff squared, and $b > 0$ is the standard color group factor [12]. We introduce into the numerator $g^2(\lambda)$, so that when it formally zero (no interaction at all) then the full gluon propagator should be reduced to its free PT counterpart. For convenience, we also denote $d(q^2)$ as $g^2(q^2; \Lambda^2)$, i.e., put $d(q^2) \equiv g^2(q^2; \Lambda^2)$. This expression represents the summation of the so-called main PT logarithms. However, nothing should depend on $\Lambda$ (and hence on $\lambda$) when they go to infinity in order to recover the finite effective charge in this limit. To show explicitly that this finite limit exists, let us rewrite the previous expression in the symmetric form [34]

$$\frac{g^2(\lambda_1)}{1 + bg^2(\lambda_1) \ln(q^2/\Lambda^2)} = \frac{g^2(\lambda_2)}{1 + bg^2(\lambda_2) \ln(q^2/\Lambda^2)}.$$  \hspace{1cm} (B2)
since \( g^2(q^2; \Lambda^2_1) = g^2(q^2; \Lambda^2_2) \), i.e., nothing should depend on how we denote the UV cutoff, indeed. In the \( \Lambda_{1,2} \to \infty \) (and hence \( \lambda_{1,2} \to \infty \)) limits neglecting the dependence on \( \ln q^2 \), from the relation (B2) one obtains

\[
\ln \Lambda_2 - \frac{1}{2b q^2(\lambda_2)} = \ln \Lambda_1 - \frac{1}{2b q^2(\lambda_1)},
\]

and this relation becomes more and more exact with all the UV cutoffs becoming bigger and bigger (and thus the suppression of \( \ln q^2 \) becoming more and more justified). Evidently, this relation is equivalent to

\[
\Lambda_2 \exp\left(-\frac{1}{2b q^2(\lambda_2)}\right) = \Lambda_1 \exp\left(-\frac{1}{2b q^2(\lambda_1)}\right).
\]

Thus there exists indeed the limit

\[
\lim_{(\Lambda, \lambda) \to \infty} \Lambda \exp\left(-\frac{1}{2b q^2(\lambda)}\right) = \Lambda_{QCD}
\]

at which it is finite and does not depend on the UV cutoff or the renormalization point (evidently, not losing generality, we can estimate that \( 2b q^2(\lambda) \sim 1/\ln \lambda \) in the \( \lambda \to \infty \) limit). This finite limit is nothing but \( \Lambda_{QCD} = \Lambda_{PT} \), which governs the nontrivial dynamics of PT QCD in asymptotic regime at large \( q^2 \). Thus, using the limit (B5), we can rewrite the initial expression (B1) in terms of the finite quantities. It reproduces the well known AF behavior of the effective charge in QCD at large \( q^2 \), namely

\[
\alpha_s(q^2) = \frac{1}{b \ln(q^2/\Lambda^2_{QCD})},
\]

where the standard notation \( \alpha_s(q^2) = g^2(q^2)/4\pi \) has been used. It can be generalized like Eq. (B1) as follows:

\[
\alpha_s(q^2) = \frac{\alpha_s}{1 + b \alpha_s \ln(q^2/\Lambda^2_{QCD})},
\]

where \( \alpha_s = g^2/4\pi \) is the fine-structure constant of strong interactions, calculated at some fixed scale, for example at \( Z \) boson mass. At a very large \( q^2 \) one recovers the previous expression.

Concluding, within our approach we have shown explicitly the AF behavior of QCD at short distances (\( q^2 \to \infty \)) not using the renormalization group equations [1, 2, 8, 9, 34]. There is no relation between the mass gap (even renormalized) and the asymptotic scale parameter, at least to leading order.

**APPENDIX C: REMARKS ON OVERLAPPING DIVERGENCES**

In order to unravel overlapping UV divergences problem in QCD, in each of the standard SD equations the necessary number of the differentiation with respect to the external momentum should be done first (in order to lower divergences). Then the point-like vertices, which are present in the corresponding skeleton loop integrals should be replaced by their full counterparts via the corresponding integral equations. Finally, one obtains the corresponding SD equations which are much more complicated than the previous (standards) ones, containing different scattering amplitudes. These skeleton expansions are, however, free of the above-mentioned overlapping divergences. Of course, the real procedure ([16] and references therein) is much more tedious than briefly described above.

However, even at this level it is clear that by taking derivatives with respect to the external momentum \( q \) in the gluon SD equation (3.17), which is convenient to rewrite as follows:

\[
D^{-1}(q) = D_0^{-1}(q) - q^2 \Pi(q^2; D) - \Delta^2(\lambda; D),
\]

the main initial information on the mass gap will be totally lost (we omit the tensor indices, for simplicity). Just the mass gap which appears first in this equation is the main object we were worried about to demonstrate explicitly its crucial role within our approach. Whether the above-mentioned information will be somehow restored or not at the
later stages of the renormalization program is not clear at all. Thus in order to remove overlapping UV divergences ("the water") from the SD equations and skeleton expansions, we are in danger to completely lose the information on the dynamical source of the mass gap ("the baby") within our approach. In order to avoid this danger and to be guaranteed that no dynamical information are lost, we are using the standard gluon SD equation (3.17) in the presence of a mass gap. The existence of any kind of UV divergences (overlapping and usual (overall)) in the skeleton expansions will not cause any problems in order to detect the mass gap responsible for the IR structure of the true QCD vacuum. As emphasized above, the problem of convergence of the regularized skeleton loop series which appear in Eq. (4.20) is completely irrelevant in the context of the present investigation. Anyway, we keep any kind of UV divergences under control within our method, since we are working with the regularized quantities. At the same time, the existence of a mass gap responsible for the IR structure of the full gluon propagator does not depend on whether overlapping divergences are present or not in the SD equations and corresponding skeleton expansions. As argued above, the existence of a mass gap is only due to the SCVCGI/S. All this is the main reason why our starting point is the standard gluon SD equation (3.1) for the unrenormalized (but necessarily regularized) Green’s functions (this also simplifies notations).

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