Adler-Bell-Jackiw anomaly, the Nieh-Yan form and vacuum polarization

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We show from first principles, using explicitly invariant Pauli-Villars regularization of chiral fermions, that the Nieh-Yan form does contribute to the Adler-Bell-Jackiw (ABJ) anomaly for spacetimes with generic torsion, and comment on some of the implications. There are a number of interesting and important differences with the usual ABJ contribution in the absence of torsion. For dimensional reasons, the Nieh-Yan term is proportional to the square of the regulator mass. In spacetimes with flat vierbein but nontrivial torsion, the associated diagrams are actually vacuum polarization rather than triangle diagrams and the Nieh-Yan contribution to the ABJ anomaly arises from the fact that the axial torsion “photon” is not transverse.

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I. PRELIMINARIES

The Adler-Bell-Jackiw (ABJ) anomaly [1] paved the way for the elucidation of anomalies in quantum field theories, and continues to be a fertile link to many diverse topics in elementary particle physics and gravitation.

Recently, there have been some discussions, and also controversy, on the question of further contributions to the ABJ anomaly [2–6]. We shall show from fundamental principles using Pauli-Villars regularization that there are indeed further contributions to the ABJ anomaly. These come in two forms. In addition to the regulator scale independent $\text{Tr}(\gamma^5 \partial_2)$ contributions, there is also the interesting Nieh-Yan term [7] which diverges as the square of the regulator mass. With flat vierbein but non-vanishing axial torsion, this further ABJ anomaly term is associated with vacuum polarization diagrams with two external axial torsion vertices, rather than with the usual triangle diagrams.

Let us begin by first recalling some basic relations to establish the notations. The basic independent ingredients of Riemann-Cartan spacetimes are the spin connection $A_{AB}$ and the vierbein $e_A$ one-forms. Lorentz indices are denoted by uppercase Latin indices while Greek indices are spacetime indices. From the definition of the torsion

$$T_A = de_A + A_{AB} \wedge e^B, \quad (1.1)$$

a generic spin connection can be written (provided the vierbein is invertible) as the sum of the torsionless spin connection $\omega_{AB}$ and terms involving the torsion and vierbein. Specifically,

$$A_{AB} = \omega_{AB} - \frac{1}{2} [T_{\mu AB}E^\mu_B - T_{\mu B}E^\mu_A - T_{\sigma C}e^\sigma_{\mu C}E^\mu_A E^\mu_B] \quad (1.2)$$

with $E^\mu_A$ being the inverse of the vierbein $e_{\mu A}$ while $\omega_{AB}$ satisfies $de_A + \omega_{AB} \wedge e^B = 0$, and can be solved as

$$\omega_{\mu AB} = \frac{1}{2} [E^\nu_A (\partial_\nu e_{\mu B} - \partial_\mu e_{\nu B}) - E^\nu_B (\partial_\nu e_{\mu A} - \partial_\mu e_{\nu A})]$$

$$- E^\alpha_A E^\beta_B (\partial_\alpha e_{\beta C} - \partial_\beta e_{\alpha C}) e^C_{\mu} \quad (1.3)$$

Spin 1/2 fermions couple to torsion through the spin connection $\frac{i}{2} A_{\mu AB} \sigma^{AB}$, $(\sigma^{AB} = \frac{i}{4}[\gamma^A, \gamma^B])$, in the covariant derivative $i\mathcal{D} = \gamma^\mu (i\partial_\mu + \frac{1}{2} A_{\mu AB} \sigma^{AB} + W_{\mu a} T^a)$. Here $W_{\mu a}$ denotes the generic internal gauge connection in the $T^a$ representation. A number of interesting identities are worth mentioning. Note that

$$\epsilon^{\frac{i}{2}} \bar{\Psi} A_{AB} \sigma^{AB} \Psi = \epsilon^{\frac{i}{4}} A_{\mu AB} (\bar{\Psi} \{\gamma^\mu, \sigma^{AB}\} \Psi + \bar{\Psi} [\gamma^\mu, \sigma^{AB}] \Psi)$$

$$= \frac{1}{2} (-iA_{\mu AB} J^\mu - \frac{1}{2} \epsilon_{AB}^{CD} A_{\mu CD} J^{5\nu}) E^{\mu B} e^A_{\nu} \quad (1.4)$$

where $J^\mu = \bar{\Psi} \gamma^\mu \Psi$ and $J^{5\mu} = \bar{\Psi} \gamma^\mu \gamma^5 \Psi$ are the densitized vector and axial-vector currents. The anti-commutator term within brackets is anti-Hermitian while the commutator term is Hermitian. Moreover in Eq. (1.4) the spin
connection coupling (with three $\gamma$-matrices) has been reduced to vector and axial-vector couplings. In particular, for chiral fermions,

$$i \overline{\Psi} L,R A_{AB} \sigma^{AB} \Psi_{L,R} = \frac{1}{2} (-i A_{\mu AB} \pm \frac{1}{2} \epsilon_{AB} C D A_{\mu CD}) E^{\mu B} e^A_{\nu} J^\nu_{L,R}$$

(1.5)

with $J^\mu_{L,R} = \overline{\Psi} L,R e^\gamma_{\mu} \Psi_{L,R} = \mp J^\mu_{L,R}$. This shows that left(right)-handed chiral fermions couple to the left(right)-handed (or anti-self-dual(self-dual)) projection of the spin connection in $i \overline{\Psi}$. By substituting for $A_{\mu AB}$ we may further isolate the torsion couplings as

$$e^{i/2} \overline{\Psi} L,R A_{AB} \sigma^{AB} \Psi_{L,R} = \frac{1}{2} (-i \omega_{\mu AB} \pm \frac{1}{2} \epsilon_{AB} C D \omega_{\mu CD}) E^{\mu B} e^A_{\nu} J^\nu_{L,R}$$

$$- \left( i B_{\mu} \pm \frac{1}{4e} A_{\mu} \right) J^\mu_{L,R}$$

(1.6)

where $B_{\mu} \equiv \frac{1}{2} T_{A\mu} E^{\nu A}$ and $A_{\mu} \equiv \frac{1}{2} e^\gamma_{\mu} e_{\nu A} T^A_{\alpha \beta}$ are the trace and axial parts of the torsion respectively. Similarly, the $B_{\mu}$ piece is anti-Hermitian while the term associated with $A_{\mu}$ is Hermitian. There are again a few noteworthy remarks. Both $B_{\mu}$ and $A_{\mu}$ are explicitly invariant under local Lorentz transformations while the vierbein and torsion transform covariantly as rank one Lorentz tensors. Note that $A_{\mu} dx^\mu$ is parity-odd (this property is required for the consistency of the ABJ anomaly equation if the Nieh-Yan four-form, $d(e_A \wedge T^A)$, contributes to the anomaly). Indeed $A_{\mu} dx^\mu$ is the Hodge dual, $\ast$, of the 3-form $e_A \wedge T^A$. Thus its divergence is related to the Nieh-Yan form through

$$\partial_\mu A^\mu = \ast d(e_A \wedge T^A).$$

(1.7)

All the currents in this article are densitized tensors of weight one. Thus, for instance, $\partial_\mu J^\mu_L = \partial_\mu (e \overline{\Psi} L e^\gamma_{\mu} \Psi_L)$ is a total divergence while

$$e \nabla_\mu (e \overline{\Psi} L e^\gamma_{\mu} \Psi_L) = \partial_\mu (e \overline{\Psi} L e^\gamma_{\mu} \Psi_L) - 2e B_{\mu} \overline{\Psi} L e^\gamma_{\mu} \Psi_L$$

(1.8)

is not when the torsion trace $B_{\mu}$ is non-vanishing. So for the ABJ anomaly, the correct divergence to consider for chiral fermions is $\partial_\mu (e \overline{\Psi} L e^\gamma_{\mu} \Psi_L) = - \partial_\mu (e \overline{\Psi} L e^\gamma_{\mu} \Psi_L)$.

Does $B_{\mu}$ interact at all with spin 1/2 chiral fermions? This depends on whether we couple chiral fermions to gravity through the conventional Majorana or Hermitian Weyl prescription, or adopt the view that chirality supersedes Hermiticity and left-handed chiral fermions interact only with the left-handed part of the spin connection (for further details, please see [8]). The latter point of view is required by the (anti)self-dual description of gravity [9] which extends the Weyl nature of the interaction between matter and the forces to the gravitational sector [8]. In terms of possible ABJ contributions, the latter is more general since $B_{\mu}$ couplings and effects will also be included. Thus we shall assume the latter point of view here and point out the differences with the conventional picture so the reader may also deduce what happens then. To wit, the bare chiral (Weyl) fermion action is

$$S = - \int d^4 x e \overline{\Psi} L i \overline{\Psi} P_L \Psi_L,$$

(1.9)

where $P_L = \frac{1}{2}(1 - \gamma^5)$ is the left-handed projection operator. We adopt the convention

$$\{ \gamma^A, \gamma^B \} = 2 \eta^{AB},$$

(1.10)

with $\eta^{AB} = \text{diag}(-1, +1, +1, +1)$. It is clear from the previous comment after Eq. (1.6) that hermitizing the Weyl action kills the $B_{\mu}$ coupling completely, and it is for this reason that one often sees statements to the effect that spin 1/2 fermions interact only with the axial part of the torsion $A_{\mu}$.

In general, the fermion multiplet $\Psi_L$ is in a complex representation. This is true of the standard model where there are no gauge and Lorentz invariant bare masses. Consequently, this poses a challenge for the usual invariant Pauli-Villars regularization, even though the chiral fermions may belong to an anomaly-free representation. An explicitly gauge and Lorentz invariant regularization actually exists for the standard model; and it can be achieved through an infinite tower of Pauli-Villars regulators which are doubled in the internal space (see Ref. [10] for further details). Specifically, the internal space is doubled from $T^a$ to

$$T^a = \begin{pmatrix} (-T^a)^* & 0 \\ 0 & T^a \end{pmatrix},$$

(1.11)
and the original fermion multiplet is projected as $\Psi_L = \frac{1}{2}(1 - \sigma^3)\Psi_L$, where

$$\sigma^3 = \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix},$$

and $d$ is the number of Weyl fermions in the $\Psi_L$ multiplet. With the regularization, we are ready to compute the ABJ anomaly.

II. THE ABJ OR $\gamma^5$ ANOMALY

The ABJ anomaly arises because regularization of the axial vector current violates the symmetry of the bare action under $\gamma^5$-rotations. This happens for instance when Pauli-Villars regularization which maintains gauge invariance breaks the symmetry through the presence of regulator masses. We shall calculate the ABJ anomaly first through Pauli-Villars regularization of the full standard model and show how it is related to heat kernel operator regularization.

Under a singlet chiral $\gamma^5$ rotation,

$$\bar{\Psi}_L \rightarrow e^{i\alpha} \bar{\Psi}_L = e^{-i\alpha} \bar{\Psi}_L,$$

$$\bar{\Psi}_L \rightarrow \bar{\Psi}_L e^{i\alpha} = \bar{\Psi}_L e^{i\alpha}.$$  \hspace{1cm} (2.1)

For curved spacetimes, we use densitized variables $\bar{\Psi}_L = e^{*\Psi}_L, \bar{\Psi}_L = e^{*\Psi}_L$. The bare massless action is invariant under such a global axial transformation, and the associated ABJ or $\gamma^5$ current

$$J^\mu_5 = \bar{\Psi}_L \gamma^\mu \gamma^5 \bar{\Psi}_L = -\bar{\Psi}_L \gamma^\mu \bar{\Psi}_L$$

is conserved classically, i.e. $\partial_\mu J^\mu_5 = 0$. However, the bare quantum composite current

$$\langle J^\mu_5 \rangle_{bare} = -\lim_{x \rightarrow y} Tr \left\{ \gamma^\mu(x) P_L \left[ \frac{1}{i\partial^\mu} \frac{1}{2} (1 - \sigma^3) \right] \delta(x - y) \right\}$$

is divergent. The regularized current is however not conserved, as we shall show. In Eq. (2.3), (and henceforth in $i\partial$), because of the doubling in internal space, we write the representation of the internal gauge field as $W_{\mu a} T^a$ in $\mathcal{D} = e^{*\Psi}_L e^{-*\Psi}_L$ and insert the $\frac{1}{2}(1 - \sigma^3)$ projection.

As demonstrated in [11], the expectation value of the Pauli-Villars regularized ABJ current is

$$\langle J^\mu_5(x) \rangle_{reg} = -\lim_{x \rightarrow y} Tr \left\{ \gamma^\mu(x) P_L \left[ \frac{1}{2} (1 - \sigma^3) \right] \frac{1}{(i\partial^\mu)(i\partial^\mu)} \right\}$$

$$+ \sum_{r=2,3,\ldots} \frac{1}{r^2 \Lambda^2 + (i\partial^\mu)(i\partial^\mu)} - \sum_{s=1,3,\ldots} \frac{1}{s^2 \Lambda^2 + (i\partial^\mu)(i\partial^\mu)} \delta(x - y) \right\}$$

$$= -\lim_{x \rightarrow y} Tr \left\{ \gamma^\mu(x) \frac{1}{2} (1 - \gamma^5) \frac{1}{i\partial^\mu} \frac{1}{2} \left[ (f(\mathcal{D}^\dagger/\Lambda^2) - \sigma^3) \right] \delta(x - y) \right\}. \hspace{1cm} (2.4)$$

where $f(y) = \frac{\pi y}{\sinh(\pi \sqrt{y})}$ is the regulator function. So in effect the regulators serve to replace the $\frac{1}{2}(1 - \sigma^3)$ projection in the bare current by $\frac{1}{2}[f(\mathcal{D}^\dagger/\Lambda^2) - \sigma^3]$ in the regularized ABJ current. The current is regularized for finite values of the regulator mass scale $\Lambda$ if $\mathcal{D}$ is a perturbative anomaly-free representation.

The ABJ anomaly can be explicitly computed by taking the divergence of the expectation value of the regularized expression of Eq. (2.4) as

$$\langle \partial_\mu J^\mu_5 \rangle_{reg} = \partial_\mu \lim_{x \rightarrow y} Tr \left\{ -\gamma^\mu(x) \frac{1}{2} (1 - \gamma^5) \frac{1}{i\partial^\mu} \frac{1}{2} \left[ f(\mathcal{D}^\dagger/\Lambda^2) - \sigma^3 \right] \delta(x - y) \right\}. \hspace{1cm} (2.5)$$

To evaluate the trace, we make use of the complete sets of eigenvectors, $\{X_n\}$ and $\{\tilde{Y}_n\}$, of the positive-semidefinite Hermitian operators in Euclidean signature with

$$\mathcal{D}^\dagger \tilde{X}_n = \lambda_n^2 \tilde{X}_n,$$
\[ \mathcal{D}^\dagger \mathcal{D} \gamma_n = \lambda_n^2 \gamma_n. \] (2.6)

For the modes with nonzero eigenvalues, \( \gamma_n \) and \( \gamma_n \) are paired by

\[ \tilde{X}_n = \mathcal{D} \tilde{Y}_n/\lambda_n, \quad \tilde{Y}_n = \mathcal{D} \tilde{X}_n/\lambda_n. \] (2.7)

Consequently,

\[ \langle \partial_{\mu} J^\mu \rangle_{\text{reg}} = -i \partial_{\mu} \left[ \sum_n \tilde{X}_n^\dagger \gamma^\mu P_L (i\mathcal{D})^\dagger \frac{1}{i \mathcal{D}^\dagger} \left( f(\mathcal{D}^\dagger / \Lambda^2) - \sigma^3 \right) \tilde{X}_n \right] \]

\[ = i \partial_{\mu} \left[ \sum_n \tilde{X}_n^\dagger \gamma^\mu P_L \frac{1}{2 \lambda_n} (f(\lambda_n^2 / \Lambda^2) - \sigma^3) \tilde{Y}_n \right] \]

\[ = i \sum_n \partial_{\mu} (\tilde{X}_n^\dagger \gamma^\mu) P_L \frac{1}{2 \lambda_n} (f(\lambda_n^2 / \Lambda^2) - \sigma^3) \tilde{Y}_n \]

\[ + \sum_n \tilde{X}_n^\dagger P_L \frac{1}{2 \lambda_n} (f(\lambda_n^2 / \Lambda^2) - \sigma^3) \gamma^\mu \partial_{\mu} \tilde{Y}_n \]

\[ = -i \sum_n \tilde{Y}_n^\dagger \frac{1}{2} (1 - \gamma^5) (f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) - \sigma^3) \tilde{X}_n \]

\[ - \tilde{X}_n^\dagger \frac{1}{2} (1 + \gamma^5) (f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) - \sigma^3) \tilde{Y}_n. \] (2.8)

The traces over \( \sigma^3 \) as well as the parity-even part drop out, and the result for Euclidean signature is

\[ \langle \partial_{\mu} J^\mu \rangle_{\text{reg}} = \lim_{\Lambda \to \infty} \frac{i}{4} \sum_n [\tilde{Y}_n^\dagger \gamma^5 f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) \tilde{Y}_n + \tilde{X}_n^\dagger \gamma^5 f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) \tilde{X}_n] \]

\[ = \lim_{\Lambda \to \infty} \frac{i}{4} \sum_n [e Y_n^\dagger \gamma^5 f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) Y_n + e X_n^\dagger \gamma^5 f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) X_n] \]

\[ = \lim_{\Lambda \to \infty} \lim_{x \to x'} \frac{i}{4} \text{Tr} \left\{ e \gamma^5 f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) \frac{\delta(x - x')}{e} I(x, x') \right\} \]

\[ + \text{Tr} \left\{ e \gamma^5 f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) \frac{\delta(x - x')}{e} I(x, x') \right\}. \] (2.9)

We have used

\[ \sum_n e X_n(x) X_n^\dagger (x') = \delta(x - x') I(x, x'); \quad \sum_n e Y_n(x) Y_n^\dagger (x') = \delta(x - x') I(x, x') \] (2.10)

with \( \tilde{X}_n = e^{\hat{x}} X_n \) and \( \tilde{Y}_n = e^{\hat{x}} Y_n \) and \( I(x, x') \) is the displacement bispinor.

A few remarks are in order. The operator \( i\mathcal{D} \) is actually not self-adjoint in the presence of generic torsion terms. With respect to the Euclidean inner product \( \langle W | Z \rangle = \int_M e W^\dagger Z \),

\[ (i\mathcal{D})^\dagger = e^{-1} i D_\mu \gamma^\mu e = i\mathcal{D} + 2i B_\mu \gamma^\mu \] (2.11)

Thus \( (i\mathcal{D})^2 \) is not positive-definite, so it is questionable whether regularization of the axial anomaly by \( \exp[-(i\mathcal{D})^2 / \Lambda^2] \) insertion as in Ref. \[ \text{[1]} \] is completely justified in the presence of generic torsion which includes nonvanishing \( B_\mu \). It is alright assuming zero \( B_\mu \). On the other hand, it is clear that regularization by the \( f(\mathcal{D}^\dagger \mathcal{D} / \Lambda^2) \) and

\[ \text{[1]} \]

It is assumed that zero modes have been subtracted from the expectation value of the current. They do not occur in the action in the path integral formulation \[ [1] \].
\( f(\mathcal{D}^\dagger \mathcal{D}/\Lambda^2) \) pair presented here does not suffer from this defect. From Eq. (2.11), the self-adjoint Dirac operator is
\[ i\Delta = i \mathcal{D} + i \mathcal{B}. \]
Using this and Eq. (2.11), we have
\[ \mathcal{D}^\dagger \mathcal{D} + \mathcal{D} \mathcal{D}^\dagger = -2\Delta^2 + 2B^\mu B_\mu. \tag{2.12} \]
This relates the operators on the L.H.S. with the square of the self-adjoint Dirac operator. Moreover, for Euclidean signature, every term in the equation is a positive-definite operator. If the Hermitized version of Eq. (1.9) is assumed, the relevant operator to consider is the square of the self-adjoint Dirac operator rather than the \( \mathcal{D}^\dagger \mathcal{D} \) and \( \mathcal{D} \mathcal{D}^\dagger \) pair.

There is an intimate relation between the Pauli-Villars regularization presented here and the heat kernel method. This can be seen as follows. In the form of a power series with Bernoulli numbers \( B_k \),
\[ \ln\left( \frac{\sinh y}{y} \right) = \sum_{k=1}^{\infty} b_k y^{2k} \tag{2.13} \]
with \( b_k = \frac{2^{2k-1}B_k}{k(2k)!} \). Thus the regulator function takes the form
\[ f(y) = \exp\left( -\sum_{k=1}^{\infty} b_k \pi^k y^k \right) \tag{2.14} \]
with \( y = \mathcal{D}\mathcal{D}^\dagger/\Lambda^2 \). Therefore for \( \Lambda \to \infty \), we may omit terms \( k > 1 \) in the series, and the regularization in effect gives the same result as regularization by \( f(\mathcal{D}\mathcal{D}^\dagger/\Lambda^2) = \exp(-t\mathcal{D}\mathcal{D}^\dagger)/\Lambda^2; \lim t \to 0 \), with \( t = b_1 \pi/\Lambda^2 \). This allows a direct comparison with heat kernel methods since the operator \( \exp(-t\hat{O}) \) satisfies the heat equation
\[ -\frac{\partial K(x, x'; t)}{\partial t} = \hat{O}K(x, x'; t) \tag{2.15} \]
with \( K(x, x'; t) = \langle x | \exp(-t\hat{O}) | x' \rangle \).

In order to evaluate the ABJ anomaly, we have only to compute terms such as \( \lim_{t \to 0} \lim_{x \to x'} Tr[e^{\gamma^5} \exp(-t\hat{O})e^{-1}\delta(x - x')I(x, x')] \) in Eq. (2.9) for which the operator \( \hat{O} \) assumes the form
\[ \hat{O} = -g^{\mu\nu}D_\mu D_\nu - 2Q^\mu D_\mu + Z \]
\[ = -g^{\mu\nu}D'_\mu D'_\nu + X \tag{2.16} \]
where \( D'_\mu \equiv D_\mu + Q_\mu \), and \( X \equiv D_\mu Q^\mu + Q^\mu Q_\mu + Z \).

The evaluation of anomalies using heat kernel techniques for operators of the above form has been pursued in a series of careful papers by Yajima [12]. We summarize the essential steps and quote the relevant results. First we expand
\[ \delta(x - x') = \int \frac{dk}{(2\pi)^4} \exp[i\gamma^5 \sigma^{\mu\nu}(x, x')] \]
with the biscalar \( \sigma(x, x') \) being the geodetic interval. It is the generalization of the flat spacetime quantity \( \frac{1}{2}(x - x')^2 \) for curved spacetimes, and obeys
\[ \sigma(x, x') = \frac{1}{2}g_{\mu\nu}(x)\sigma^{\mu\nu}(x, x')\sigma(x, x') \]
\[ = \frac{1}{2}g_{\mu'\nu'}(x')\sigma^{\mu'\nu'}(x, x')\sigma^{\nu'}(x, x'). \tag{2.17} \]
The heat kernel of Eq. (2.15) may then be expressed as
\[ K(x, x'; t) = \int \frac{e^{ikx}}{(2\pi)^4} e^{-t\hat{O}} e^{ik_\mu \sigma^{\mu\nu}} I(x, x'). \tag{2.18} \]
Thus

\[ \text{In the heat kernel regularization of Ref. [12], it is assumed that spinors couple only to the axial part of the torsion.} \]
\[
\lim_{t \to 0} \lim_{x' \to x} Tr[e^{\gamma^5 t} \exp(-it\hat{O}) e^{-1/2 \delta(x-x')} I(x, x')] = \lim_{t \to 0} \lim_{x' \to x} Tr[\gamma^5 K(x, x'; t)] = \lim_{t \to 0} \frac{1}{(4\pi t)^2} Tr[\sum_{n=0}^{\infty} \epsilon \gamma^5 a_n(x) t^n] \tag{2.19}
\]

if we employ the DeWitt ansatz

\[
K(x, x'; t) = \frac{1}{(4\pi t)^2} [\det(\sigma^{\mu\nu}(x, x'))]^{1/2} \exp\left(\frac{\sigma(x, x')}{2t}\right) \sum_{n=0}^{\infty} a_n(x, x') t^n \tag{2.20}
\]

which gives the \((x \to x')\) coincidence limit as \(K(x, x; t) = \frac{e}{(4\pi t)^2} \sum_{n=0}^{\infty} a_n(x) t^n\). By substituting the DeWitt ansatz into the heat equation and matching the coefficients of powers of \(t\), the recursive relation for \(a_n\) can be obtained; from which \(a_0 = 1, a_1 = (\frac{1}{2!} R - X)\) and so on \([12]\).

The form of the ABJ anomaly is obtained by identifying \(Z\) and \(Q^\mu\) for the specific operators. To wit,

\[
\hat{O} = \partial^\dagger \psi
\]

\[
= -g^{\mu\nu} D_\mu D_\nu + [\Gamma^\mu_{\alpha\nu} \gamma^\nu \gamma^\alpha - 2B_\nu \gamma^\nu \gamma^\mu] D_\mu - \sigma^{\mu\nu}[D_\mu, D_\nu], \tag{2.21}
\]

yields

\[
-2Q^\mu = [\Gamma^\mu_{\alpha\nu} \gamma^\nu \gamma^\alpha - 2B_\nu \gamma^\nu \gamma^\mu],
\]

\[
Z = -\sigma^{\mu\nu}[D_\mu, D_\nu]
\]

\[
= -\frac{1}{2}\sigma^{\mu\nu} \sigma^{AB} F_{\mu\nu AB} - \sigma^{\mu\nu} G_{\mu\nu a} T^a, \tag{2.22}
\]

with \(G_{\mu\nu a}\) and \(F_{\mu\nu AB}\) being respectively the curvatures of \(W_{\mu a}\) and \(A_{\mu AB}\). Similarly,

\[
\hat{O} = \partial^\dagger \psi
\]

\[
= -g^{\mu\nu} D_\mu D_\nu + [\Gamma^\mu_{\alpha\nu} \gamma^\nu \gamma^\alpha - 2B_\nu \gamma^\nu \gamma^\mu] D_\mu
\]

\[
- \sigma^{\mu\nu}[D_\mu, D_\nu] - 2\gamma^\mu \gamma^\nu \nabla_\mu B_\nu, \tag{2.23}
\]

leads to the identification for this latter case of

\[
-2Q^\mu = [\Gamma^\mu_{\alpha\nu} \gamma^\nu \gamma^\alpha - 2B_\nu \gamma^\nu \gamma^\mu],
\]

\[
Z = -\frac{1}{2}\sigma^{\mu\nu} \sigma^{AB} F_{\mu\nu AB} - \sigma^{\mu\nu} G_{\mu\nu a} T^a - 2\gamma^\mu \gamma^\nu \nabla_\mu B_\nu \tag{2.24}
\]

Since \(Tr(\gamma^5 a_0) = 0\), the first contribution to the ABJ anomaly comes from the term proportional to \(1/t\) or \(\Lambda^2\). As dictated by Eq. (2.9), we need to compute the sum of the traces of \(\gamma^5\) with the \(a_1\)'s of the two operators \(\partial^\dagger \psi\) and \(\psi \partial\). To order \(1/t\) or \(\Lambda^2\), the result is

\[
\langle \partial^\dagger J_5^\mu \rangle_{\text{reg.}} = -\frac{i}{4} \frac{(2d)}{(4\pi)^2 t} \left( - \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} e^A_{\alpha} e^B_{\beta} F_{\mu\nu AB} + 2(g_{\mu\nu} \epsilon^{\alpha\beta\mu\nu} \Gamma^\mu_{[\alpha\eta]} \Gamma^\nu_{[\beta]}) \right)
\]

\[
= -\frac{i \times d}{(4\pi)^2 t} \left( - \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} e^A_{\alpha} e^B_{\beta} F_{\mu\nu AB} + \frac{1}{4} \epsilon^{\alpha\beta\mu\nu} T_{\alpha\beta A} T_{\mu\nu} \right)
\]

\[
= -\frac{i \times d}{(4\pi)^2 t} \epsilon [e^A \wedge e^B + T^A \wedge T_A]
\]

\[
= -\frac{i \times d}{(4\pi)^2 t} \epsilon [d(e^A \wedge T_A)]. \tag{2.25}
\]
In the above, we have used $\Gamma_{\alpha\beta\gamma}^\mu = -\frac{1}{2} e^{\mu A} T_{\alpha\beta\gamma}$ while $d(e^A \wedge T_A)$ is precisely the Nieh-Yan four-form \[7\]. We have therefore confirmed from first principles using Pauli-Villars regularization of chiral fermions that the Nieh-Yan four-form indeed contributes to the ABJ anomaly \[8\]. In the absence of torsion, the $A$-independent terms from $Tr(\gamma^5 a_2)$ are the familiar curvature-squared ABJ contributions. Using the heat kernel method, Obukhov et al \[9\] have also found the Nieh-Yan contribution to the ABJ anomaly using the operator $\hat{O} = -\Delta^2$. It is implicitly present in $Tr(\gamma^5 a_1)$ in the series of papers by Yajima \[12\], and also in Ref. \[13\].

**III. FURTHER REMARKS**

The Nieh-Yan term is proportional $1/t$ and hence to the square of the Pauli-Villars regulator mass scale $\Lambda^2$ for dimensional reasons. Moreover, it is also clear from the discussion here that the Nieh-Yan contribution is indeed due to regularization and is compatible with the general understanding of the origin of anomalies in quantum field theories. It must be stressed that we do not have to make statements to the effect that the integration over $k$ in Eq. (2.18) is truncated at some scale $M_{\text{cut-off}}$ so that $\int d^4k \approx M_{\text{cut-off}}^4$ and so on. These statements have the essence of introducing an extra cut-off scale $M_{\text{cut-off}}$ which may or may not be $\Lambda$; and can lead to confusion on the dependence of divergent ABJ contributions on the regulator scale $\Lambda$. The cut-off is unnecessary because the integrals are really well-defined due to the presence of the regulating function $f$ and its derivatives (for instance, $\int d^4k \exp(-k^2/\Lambda^2) = \pi^2\Lambda^4$). The upper limit of the integrals is really $\infty$; and no extra cut-off mass scale is needed. There is only one regulator mass scale $\Lambda$.

In covariant operator regularization, a regulating function $f(\hat{O})$ is inserted in the trace with $\gamma^5$ in Eq. (2.9). This does not necessarily imply that the anomaly is independent of $\hat{O}$ although it is independent of the specific form of $f$ with the same boundary conditions, as has been emphasized by Fujikawa \[11\]. There may still be some leeway and ambiguity in selecting the operator $\hat{O}$. In our case, the operator in Eq. (2.9) is dictated by the requirement that to regularize gauge and spin currents and also the energy-momentum of the full theory using the generalized Pauli-Villars method \[10\], it is essential that the regulators couple to chiral fermions in the same manner as specified by the bare Lagrangian.

In dimensional regularization of fermion loops, there is the subtlety with $\gamma^5$ leading to inconsistencies (see for instance Ref. \[14\] and references therein) if $\{\gamma^\mu, \gamma^5\} = 0$ is maintained for arbitrary values of the spacetime dimension. A consistent set of relations is $\{\gamma^\mu, \gamma^5\} = 0$ if $\mu = 0, 1, 2, 3$; $\{\gamma^5, \gamma^\mu\} = 0$ otherwise. This implies that global $\gamma^5$ rotations no longer generate symmetries if the dimension of spacetime is not 4. Consequently, in the axial vector current we have $\partial_\mu J^5_\mu = \nabla e^{\gamma^5 i L}$. On taking the expectation value of this equation, we see the ABJ anomaly as the expectation value of the R.H.S.

Mielke and Kriemer \[15\] have argued that the Nieh-Yan form cannot contribute to the ABJ anomaly because perturbatively it cannot come from triangle diagrams. The first part of the argument is incorrect and it is therefore instructive to see what Feynman diagram processes are associated with the Nieh-Yan contribution. Things are much clearer if we specialize to flat vierbein $e^A_{\mu} = \delta^A_{\mu}$, but with non-trivial axial torsion $\hat{A}_{\mu}$. This allows us to retain the essential information regarding the Nieh-Yan contribution without having to worry about background graviton fluctuations from $e_{A\mu}$. Let us also set $W_{\mu\alpha} = 0$ and $J_{\mu} = 0$ for convenience. Then the action of Eq. (1.9) reduces to

$$S = -\int d^4x \nabla_L \gamma^\mu (ie \partial_\mu - \frac{1}{4} \hat{A}_\mu) \Psi_L,$$

and the torsion coupling is “QED-like”. It is also clear that the ABJ current $J^5_\mu$ is coupled to $\hat{A}_\mu$. Thus for chiral fermions the ABJ current is the source for axial torsion. In Pauli-Villars regularization, fermion loops with background $\hat{A}_\mu$ vertices are obtained by functionally differentiating the regularized current with respect to $\hat{A}_\mu$. As computed, the Nieh-Yan contribution (after continuation to Lorentzian signature) is

$$\partial_\mu \langle J^5_\mu \rangle = \frac{d}{(4\pi)^2 t} \partial_\mu \hat{A}_\mu.$$

This implies

$$\partial_\mu \frac{\delta \langle J^5_\mu (x) \rangle}{\delta \hat{A}_\mu(x')} = \frac{d}{(4\pi)^2 t} \partial^\nu \delta(x-x').$$

But the vacuum polarization amplitude $P^\mu_{\nu}$ with two external background $\hat{A}$ vertices is proportional to the Fourier transform of the functional derivative of the current with respect to $\hat{A}$, i.e.
\[
\frac{\delta(J_{\mu}^{\nu}(x))}{\delta A_{\nu}(x')} \bigg|_{A=0} \propto \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \Pi^{\mu\nu}.
\]

In momentum space, Eq. (3.3) means that the Ward identity which corresponds to the Nieh-Yan contribution of the ABJ anomaly reads

\[
k_{\mu} \Pi^{\mu\nu} \propto k^{\nu}.
\]

This is consistent with

\[
\Pi^{\mu\nu} = (g^{\mu\nu} - (k^{\mu}k^{\nu}/k^2))\Pi(k^2) + \Pi' g^{\mu\nu}
\]

If \( \Pi' = 0 \), we recover then the usual “transverse photon” condition of “gauge” invariance i.e. \( k_{\mu} \Pi^{\mu\nu} = 0 \) and \( \langle \partial_{\mu} J_{\nu}^{\mu} \rangle = 0 \) for the vacuum polarization diagram. However, we must remember that \( \hat{A}_{\mu} \) is not a gauge potential, but a composite; and is completely invariant (as emphasized in Section I) under local Lorentz transformations which are actually gauged with the spin connection \( A_{\mu AB} \). Thus local Lorentz invariance is not anomalous as evidenced by the explicitly Lorentz (and also gauge) invariant regularization scheme \[10\]. Even if we include the full \( W_{\mu}T^{a} \) and \( B_{\mu} \) couplings, there are no perturbative chiral \textit{gauge} anomalies provided \( Tr(T^{a}) = Tr(T^{a}\{T^{b},T^{c}\}) = 0 \) \[13\]. However, the current that is coupled to the parity-odd \( \hat{A}_{\mu} \) composite is none other than the ABJ current which is anomalous (see Eq. (3.2)) because the “photon” \( \hat{A}_{\mu} \) is not transverse i.e. \( \partial_{\mu}A^{\mu} \neq 0 \) precisely when the Nieh-Yan form is non-vanishing.

Since \( \hat{A}^{\mu} \) is actually local Lorentz invariant and transforms covariantly as a general coordinate tensor density, it may be possible to redefine the current (and the corresponding charge) generating axial rotations by

\[
\langle \delta_{\mu}J_{\nu}^{\mu} \rangle \propto \frac{1}{(4\pi)^2} \hat{A}_{\mu}.
\]

This is in contradistinction with the usual \( \epsilon^{\alpha\beta\gamma\delta} G_{\alpha\beta a} G_{\mu\nu}^{a} \) contribution for which we cannot construct a gauge invariant physical current by absorbing the gauge-dependent Chern-Simons current (similarly, in the gravitational case, the associated Chern-Simons current also does not transform covariantly under general coordinate transformations). However, absorbing \( \hat{A}^{\mu} \) into \( J^{5\mu} \) can lead to interesting changes in the scaling behaviour of the redefined current and the renormalization properties. These are currently under investigation. It is also clear that \( \epsilon_{A} \wedge T^{A} \) is local Lorentz invariant and globally defined (even if the vierbein and spin connections are defined only locally), in the sense that in the overlap of patches 1 and 2, \( \langle \epsilon_{A} \rangle_{1} \wedge T_{1}^{A} = \langle \epsilon_{A} \rangle_{2} \wedge T_{2}^{A} \). So the Nieh-Yan term of the ABJ anomaly gives zero contribution when integrated over compact manifolds without boundaries \[13\], and therefore does not affect the Atiyah-Singer index theorem in such cases. However for manifolds with boundaries, \( \int_{\partial M} \epsilon_{A} \wedge T^{A} \) can be non-trivial \[3\]. So axial rotations of fermions in the presence of torsion will then lead to extra \( P \), CP and T violations from Nieh-Yan ABJ contributions over and beyond the usual instanton terms.

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