FIXED POINTS OF A FINITE SUBGROUP OF THE
PLANE CREMONA GROUP

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Abstract. We classify all finite subgroups of the plane Cremona group
which have a fixed point. In other words, we determine all rational
surfaces $X$ with an action of a finite group $G$ such that $X$ is equivariantly
birational to a surface which has a $G$-fixed point.

1. Introduction

Let $G$ be a finite subgroup of the plane Cremona group, $\text{Cr}(2)$, the group
of birational transformations of the complex projective plane. We say that
$G$ has a fixed point if there exists a smooth rational projective surface $X$
with a faithful $G$-action $\rho : G \hookrightarrow \text{Aut}(X)$, and a birational map $\phi : X \dasharrow \mathbb{P}^2$
such that $\phi \circ \rho(G) \circ \phi^{-1} = G$ and $X$ has a $G$-fixed point. This definition
depends only on the conjugacy class of $G$ in $\text{Cr}(2)$. In this paper we present
a classification of conjugacy classes of subgroups of $\text{Cr}(2)$ with fixed point
and, for each class, we find a representative $G$-surface.

For abelian finite groups acting on smooth proper varieties, the presence
of a fixed point is a birational invariant (see Proposition A.2 of [RY00]).
In general, however, this is not true; for example, the exceptional divisor
of a blow up of a fixed point may not have any fixed points. However, if
$f : X \to X'$ is a morphism of $G$-surfaces, then a fixed point on $X$ maps to
a fixed point on $X'$. Thus, the theory of minimal models of $G$-surfaces tells us
that it suffices to find minimal $G$-surfaces of one of the following two types:

- **Conic bundles**: there exists a regular map $f : X \to \mathbb{P}^1$ such that
  the general fiber is isomorphic to $\mathbb{P}^1$ and the subgroup $\text{Pic}(X)^G$
of $G$-invariant invertible sheaves on $X$ is generated over $\mathbb{Q}$ by the
  canonical class $K_X$ and the class of a fiber of $f$.
- **del Pezzo $G$-surfaces**: the anticanonical class $-K_X$ is ample and
  $\text{Pic}(X)^G$ is generated over $\mathbb{Q}$ by $K_X$.

An important tool for solving our problem is the classification of con-
jugacy classes of finite subgroups of $\text{Cr}(2)$ from [DI09]. Although we use
some results from [DI09], many of our proofs do not directly rely on this
work. In fact, our work led to a discovery of some gaps in the classification

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and we use this opportunity to fill these gaps in that paper. Note however, that this classification is incomplete in the case of conic bundles (see also [Tsy11]). One may also find an independent classification of abelian subgroups of Cr(2) in [Bla07].

By considering the action on the tangent space of a fixed point, we see that any finite group $G$ acting on a smooth surface with a fixed point must be isomorphic to a subgroup of GL(2). Also, it is well-known that a cyclic group always has a fixed point on a rational variety (for example, as was noticed by J.-P. Serre, this follows from the Lefschetz fixed-point formula applied to the structure sheaf). Consequently, we restrict our attention to $G$ not cyclic.

Recall that a del Pezzo surface has degree $d = K_X^2$. A del Pezzo surface $X$ of degree 4 can be written by two equations in $\mathbb{P}^4$ defined by diagonal quadrics. The coordinate hyperplanes cut out 5 elliptic curves $E_1, \ldots, E_5$ on $X$. A del Pezzo surface $X$ of degree 3 is a cubic surface in $\mathbb{P}^3$. An Eckardt point on $X$ is a point where three lines on the surface meet. A del Pezzo surface $X$ of degree 2 is a double cover of $\mathbb{P}^2$ branched over a smooth plane quartic curve $B$.

**Theorem 1.1.** Suppose $G$ is a finite non-cyclic subgroup of the Cremona group admitting a fixed point. Then there exists a minimal $G$-surface $X$ realizing a fixed point $p$ of $G$ of one of the following forms:

| Case | Description |
|------|-------------|
| 1    | $X$ is $\mathbb{P}^2$, |
| 4    | $X$ is a del Pezzo surface of degree 4 and $p$ lies on exactly two curves $E_i, E_j$, both of which are equianharmonic, |
| 3    | $X$ is a cubic surface and the tangent space to $p$ contains three Eckardt points, |
| 2A   | $X$ is a del Pezzo surface of degree 2 and $p$ lies on the ramification divisor $R$, |
| 2B   | $X$ is a del Pezzo surface of degree 2 and $p$ is the intersection point of four exceptional curves, |
| 1    | $X$ is a del Pezzo surface of degree 1 and $p$ is the base point of the anti-canonical linear system, |
| C    | $X$ is a minimal conic bundle. |

Note that there may be some overlap between these cases as there may be more than one minimal $G$-surface in an equivalence class. Occurrences of this phenomenon, along with the specific groups that occur in each case, will be discussed in the sections below. For the readers convenience, we consolidate those groups acting on del Pezzo surfaces of degree 2–4 in Table [Table 1]. We use the notations for finite groups employed in [DI09] borrowed from [Atlas].

We thank Vladimir Popov who asked the first author about the classification of finite groups of automorphisms of rational surfaces admitting a fixed point.
Table 1. Non-cyclic subgroups $G$ of $\text{Cr}(2)$ with a fixed point realized by a minimal del Pezzo $G$-surface of degree 2–4, but not a minimal conic bundle.

2. Preliminaries

Many of our notations are the same as those in [DI09]. Let $G$ be a finite group. A $G$-surface is a pair $(X, \rho)$ where $X$ is a smooth projective surface and $\rho : G \hookrightarrow \text{Aut}(X)$ is a faithful $G$-action. We will often refer to the pair $(X, G)$ or simply $X$ when the context is clear. A morphism of $G$-surfaces $(X, \rho) \rightarrow (X', \rho')$ is a morphism of the underlying surfaces $f : X \rightarrow X'$ such that $\rho'(G) \circ f = f \circ \rho(G)$. Similarly, one defines rational maps, birational maps and birational morphisms of $G$-surfaces.

A $G$-surface $X$ is minimal if any birational morphism $X \rightarrow X'$ of $G$-surfaces is an isomorphism. We say that an action of $G$ on a surface $X$ is a minimal group of automorphisms if the corresponding $G$-surface is minimal. As in the introduction, minimal $G$-surface are either minimal conic bundles or minimal del Pezzo $G$-surfaces.

A minimal conic bundle $f : X \rightarrow \mathbb{P}^1$ is either a minimal ruled surface with $f$ being one of its rulings, or it has $k \geq 3$ degenerate fibers isomorphic to the
union of two $\mathbb{P}^1$‘s intersecting transversally at one point. Recall that a del Pezzo surface $X$ is a smooth projective surface such that the anticanonical divisor $-K_X$ is ample. The degree of a del Pezzo surface is $d = K_X^2$, which takes values $1 \leq d \leq 9$.

We caution the reader that a minimal $G$-surface may be a del Pezzo surface but not be a minimal del Pezzo $G$-surface! We shall see an example of such a surface in Section 7.

With the notable exception of $\mathbb{P}^1 \times \mathbb{P}^1$, every del Pezzo surface is a blow up of $\mathbb{P}^2$ at $9 - d$ points $x_1, \ldots, x_{9-d}$ in general position, with corresponding exceptional divisors $R_1, \ldots, R_{9-d}$. Conversely, any set of $9 - d$ disjoint $(-1)$-curves can be blown down to $\mathbb{P}^2$; giving rise to a plane model of $X$. Each such choice is called a geometric marking and gives rise to a choice of basis for the orthogonal complement $\mathcal{R}_X$ of $K_X$ in Pic($X$).

For $d \leq 6$, the action of Aut($X$) on $\mathcal{R}_X$ defines a homomorphism

$$
\rho : \text{Aut}(X) \to W_{9-d},
$$

where $W_n$ denotes the Weyl group of a simple root system of type $E_n$ (by definition, $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 + A_1$). If $d \leq 5$, then $\rho$ is injective. It follows that in this case any subgroup $G$ of Aut($X$) defines a conjugacy class of $W_n$ which is independent of a choice of a basis in Pic($X$).

A $G$-surface $X$ is a minimal del Pezzo $G$-surface if Pic($X$)$^G$ is generated over $\mathbb{Q}$ by $K_X$. A minimal del Pezzo $G$-surface of degree 8 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; the other surface of degree 8 is never minimal. Similarly, the surface of degree 7 is never minimal.

In order to determine whether a del Pezzo surface is minimal, we will use the following consequence of the Lefschetz fixed-point formula (as was used in Section 6 of [DI09]):

**Proposition 2.1.** Let $X$ be a del Pezzo surface. If $\sigma$ is an automorphism of $X$, then the trace of $\sigma^*$ on $\mathcal{R}_X$ is given by

$$
\text{Tr}(\sigma^*|\mathcal{R}_X) = s - 3 + \sum_{i=1}^{n} (2 - 2g_i),
$$

where $s$ is the number of isolated fixed points and $g_1, \ldots, g_n$ are the genera of the fixed curves. Moreover, for a finite group $G$, the surface $X$ is $G$-minimal if and only if

$$
\sum_{\sigma \in G} \text{Tr}(\sigma^*|\mathcal{R}_X) = 0.
$$

We are classifying $G$-surfaces up to birational equivalence. It may happen that two minimal $G$-surfaces are birationally equivalent. Indeed, we will see in Sections 3, 4, 5, and 6 that all del Pezzo $G$-surfaces of degree $\geq 5$ with a fixed point are isomorphic to $\mathbb{P}^2$ with a fixed point. On the other hand, from Section 8 of [DI09] we have that any minimal del Pezzo $G$-surface of degree $\leq 3$ is rigid; thus we have the following.
Lemma 2.2. Every minimal del Pezzo $G$-surface $X$ of degree $\leq 3$ is the unique minimal $G$-surface in its birational $G$-equivalence class.

The remaining case of degree 4 is more subtle and will be discussed in Section 7.

In Theorems 8.1, 9.1 and 9.5, we will describe families of del Pezzo surfaces via normal forms involving parameters. For a given family $A$, there is some collection of groups $G$ which fix a point and for which the $G$-surface is minimal. For certain special values of these parameters, the set of possible groups $G$ may be larger and we have a new family $B$. We say that $A$ specializes to $B$, say that $B$ is a specialization of $A$, or write $A \rightarrow B$. Conversely, we say that $A$ is a generalization of $B$.

This language is justified in view of the following:

Proposition 2.3. Let $X \rightarrow T$ be a flat family of del Pezzo surfaces of degree $\leq 5$ over a base scheme $T$. For each conjugacy class $C$ of subgroups in $W_{9-d}$, the set

$$\{ t \in T : \text{Aut}(X_t) \text{ contains a subgroup } G \text{ representing } C \}$$

is closed in $T$.

Proof. Since the monodromy group of a smooth flat family of del Pezzo surfaces is a finite subgroup of the Weyl group $W_{9-d}$, after passing to a certain finite cover of $T$, we may trivialize the local coefficient system on $T$ defined by the second cohomology group of fibers. Choosing simultaneously a geometric marking in each fiber, we may define a map from $T$ to the GIT-quotient $P^g_{9-d}$ of $(\mathbb{P}^2)^{9-d}$ by the group PGL(3). Since the preimage of a closed set is closed, it suffices to assume that $T$ is an open subset $U$ of $P^g_{9-d}$ parameterizing point sets whose blow-up is a del Pezzo surface. From [DO], the group $W_{9-d}$ acts biregularly on $U$ via Cremona transformations and the stabilizer of a point $t \in T$ is equal to the image of Aut($X_t$) under the homomorphism (2.1).

Let $a : \Gamma \times V \rightarrow V$ be an action of a finite group $\Gamma$ on an algebraic variety $V$. The pre-image $Z$ of the diagonal $\Delta$ of $V$ under the map $(a, \text{id}) : \Gamma \times V \rightarrow V \times V$ consists of points $(g, v)$ such that $g \in \Gamma_v$. For any subgroup $H$ of $\Gamma$, the pre-image of $H$ under the first projection $Z \rightarrow \Gamma$ is a closed subset $W$ of $Z$. Since $\Gamma$ is finite, the image of $W$ via the second projection $Z \rightarrow V$ is a closed subset of $V$ consisting of points whose stabilizer contains $H$. Applying this to our situation, where $\Gamma = W_{9-d}$ and $V = U$, we obtain the assertion of the proposition. \hfill $\square$

3. del Pezzo surfaces of degree 9

In this case, $X \cong \mathbb{P}^2$ and we will classify finite subgroups of Aut($\mathbb{P}^2) \cong \text{PGL(3)}$ that have a fixed point.
Theorem 3.1. Conjugacy classes of finite subgroups of GL(2) are in bijection with conjugacy classes of finite subgroups of Aut(P²) with a fixed point.

Proof. Let $G$ be a subgroup of PGL(3) and let $\tilde{G}$ be a preimage in GL(3). We have a three dimensional representation $\rho$ of $\tilde{G}$. The existence of a $G$-fixed point on $X$ is equivalent to the existence of a 1-dimensional subrepresentation $\chi$ of $\rho$. Note that multiplying $\rho$ by a character does not change the image $G$ in PGL(3), thus we may assume that $\chi$ is trivial. An invariant complement to $\chi$ is thus a faithful representation of $G$ of dimension 2. □

4. del Pezzo surfaces of degree 8

In this case $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Assume $G$ has a fixed point $p$. Let $\ell_1, \ell_2$ be the two fibers passing through $p$. Their union is $G$-invariant. The group $G$ contains a subgroup $G'$ of index 1 or 2 such that each ruling $\pi_i : X \to \mathbb{P}^1$ is invariant.

Since each ruling $\pi_i$ is $G'$-equivariant, there must be a $G'$-fixed point on each image $\pi_i(X) \cong \mathbb{P}^1$. Note that any group of automorphisms which fixes one point on $\mathbb{P}^1$ must fix another. Thus there exists another pair of lines $\ell'_1, \ell'_2$ whose intersection is another $G'$-fixed point $p'$ on $X$.

Blowing up $p$, the strict transforms of $\ell_1$ and $\ell_2$ both become exceptional curves. Since they do not intersect and their union is $G$-invariant, we may blow them down $G$-equivariantly to $X'$. The variety $X'$ is isomorphic to $\mathbb{P}^2$ and has a $G'$-fixed point since the birational map $X \to X'$ is defined at $p'$. Thus, we have the following:

Theorem 4.1. If $G$ has a fixed point on $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ then $X$ is $G$-birationally equivalent to $\mathbb{P}^2$.

5. del Pezzo surfaces of degree 6

The surface $X$ is isomorphic the blow-up of three non-collinear points $x_1, x_2, x_3$ in the plane. The strict transforms of the lines $\ell_{12}, \ell_{13}, \ell_{23}$ through each pair of points are $(-1)$-curves on $X$. Along with the exceptional divisors sitting above each point $x_i$, these form a hexagon of $(-1)$-curves.

Let $p$ be a fixed point of $G$. If $p$ is on the hexagon, then it must be one of its vertices since otherwise the side of the hexagon containing $p$ is $G$-invariant and hence can be equivariantly blown down. But, if $p$ is a vertex, then the opposite vertex is also fixed, and there will be two skew lines that are left invariant and can be equivariantly blown down. This contradicts the minimality assumption. Thus $p$ is not on the hexagon.

Since $p$ is not on the hexagon, it must be the preimage of a point $x_0$ in the plane. The three lines joining $x_0$ with the points $x_1, x_2, x_3$ intersect the lines $\ell_{12}, \ell_{13}, \ell_{23}$ at three points $x', x'_2, x'_3$. The preimage of these points on $X$ are three points on three skew sides of the hexagon. Since the set of such points is $G$-invariant, we obtain that the three sides are invariant and can
be equivariantly blown down, contradicting the minimality condition. In summary:

**Theorem 5.1.** No minimal del Pezzo $G$-surface of degree 6 has a fixed point.

### 6. del Pezzo surfaces of degree 5

The surface is isomorphic to the blow up of four points $x_1, \ldots, x_4$ in $\mathbb{P}^2$, no three of which are collinear. In this case we know from Theorem 6.4 of [DI09] that $\text{Aut}(X) \cong S_5$. The 10 exceptional curves along with their intersections are in bijective correspondence with vertices and lines of the Peterson graph. Alternatively, the 10 exceptional curves are in bijection with pairs of elements of $\{1, 2, 3, 4, 5\}$; two curves intersect if and only if the pairs have no common elements.

The maximal subgroups of $S_5$ are $S_3 \times 2$, $S_4$, $A_5$ and $5 : 4$. Note that $S_3 \times 2 \cong \langle (123), (45) \rangle$ is not minimal since it fixes the exceptional curve corresponding to $\{4, 5\}$. The group $S_4$ is not minimal since it leaves invariant the 4 skew lines $\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}$. Any group containing an element of order 5 must be minimal since 5 lines cannot be skew. Thus, the groups $S_5, A_5$ and $5 : 4$ are minimal; however they do not have 2-dimensional representations and thus cannot have fixed points. Among their subgroups, the only non-cyclic group not yet considered is $G \cong D_{10}$.

The group $D_{10}$ is minimal and has two fixed points. To see this, we use the well-known $S_5$-equivariant isomorphism between a del Pezzo surface $X$ of degree 5 and the GIT-quotient $P^5_1$ of $(\mathbb{P}^1)^5$ by $\text{PGL}(2)$. Represented as point sets, the points

$$p_0 = (1, \epsilon_5, \epsilon_5^2, \epsilon_5^3, \epsilon_5^4) \text{ and } p_1 = (1, \epsilon_5^3, \epsilon_5, \epsilon_5^4, \epsilon_5^2)$$

on $X$ are fixed by the group $G = \langle \sigma, \tau \rangle \cong D_{10}$ where

$$\sigma = (12345) \text{ and } \tau = (25)(34).$$

Indeed, $\sigma(p_1) \equiv p_1$ since it amounts to multiplication by a constant; while $\tau$ corresponds to $z \mapsto z^{-1}$ on each $\mathbb{P}^1$.

While this $G$-surface is minimal, it is birationally equivalent to $\mathbb{P}^2$. Note that neither fixed point lies on an exceptional divisor since every $G$-orbit of exceptional divisors contains skew divisors. Considering $X$ as the blow-up of four points in $\mathbb{P}^2$, the linear system of cubic curves through the four points and a double point at the image of $p_0$ in the plane is of dimension 2. Thus we have an equivariant birational map from $X$ to $\mathbb{P}^2$ which maps $p_1$ to a fixed point. We conclude:

**Theorem 6.1.** Suppose $(X, G)$ is a minimal del Pezzo surface of degree 5 with a fixed point and $G$ non-cyclic. Then $G \cong D_{10}$ and $X$ is $G$-birational to $\mathbb{P}^2$ with a fixed point.
7. del Pezzo surfaces of degree 4

We recall several facts from Section 6.4 of [DI09]. Any quartic del Pezzo surface \( X \) is isomorphic to a smooth surface in \( \mathbb{P}^4 \) given by the equations

\[
\sum_{i=1}^{5} t_i^2 = \sum_{i=1}^{5} a_i t_i^2 = 0,
\]

where \( a_i \neq a_j \) whenever \( i \neq j \).

The natural representation of \( \text{Aut}(X) \) on the Picard group of \( X \) defines an isomorphism \( \rho \) of \( \text{Aut}(X) \) onto a subgroup of the Weyl group \( W(D_5) \cong 2^4 \cdot \mathfrak{S}_5 \). The normal subgroup \( 2^4 \) is always in the image of \( \rho \) and acts on \( X \) by multiplying an even number of coordinates by \(-1\). The image of \( \text{Aut}(X) \) in \( \mathfrak{S}_5 \) could be one of the following groups: \( 1, 2, 3, 4, \) and \( D_{10} \).

Each element of \( 2^4 \) is represented by a subset \( A \) of \( \{1, 2, 3, 4, 5\} \) corresponding to the indices of the coordinates \( t_i \) that are multiplied by \(-1\).

We denote by \( E_k \) the elliptic curve cut out by the hyperplane section \( t_k = 0 \). The group \( \text{Aut}(X) \) acts on the set of such curves with kernel of the action equal to \( 2^4 \). The fixed point set on \( X \) of each \( \iota_k \) is precisely the corresponding elliptic curve \( E_k \).

We now discuss how to see the action of \( W(D_5) \) on the exceptional divisors of \( X \) and its connection to the plane model. Recall that \( X \) is isomorphic to the blow-up of 5 points \( x_1, \ldots, x_5 \) in the projective plane. We label the 16 exceptional divisors of \( X \): let \( R_1, \ldots, R_5 \) be the exceptional curves corresponding to the points \( x_i \), let \( R_{ij} \) be the strict transforms of the lines \( x_i, x_j \), and let \( R_0 \) be the strict transform of the conic through the points \( x_1, \ldots, x_5 \). Each geometric marking corresponds to a choice of the 5 disjoint lines \( R_i \). There are \( 2^4 \) such subsets and the Weyl group \( W(D_5) \) has \( 2^4 \) conjugate subgroups isomorphic to \( \mathfrak{S}_5 \); each of them leaves invariant the set of the divisor classes of 5 disjoint lines.

Each of the involutions \( \iota_k \) is given by a de Jonquières involution of the plane model with center at the point \( x_k \) (see Section 2.3 of [DI09]). The involution is given by the linear system of cubics through the points \( x_i, i \neq k \), and a singular point at \( x_k \). The image of \( E_k \) is the unique plane cubic curve that passes through the points \( x_1, \ldots, x_5 \) with tangent direction at each point \( x_j, j \neq k \), equal to the line \( x_j, x_k \). The de Jonquières involution preserves the pencil of lines through the point \( x_k \).

It follows from the construction of de Jonquières involutions that \( \iota_k \) interchanges \( R_i \) with \( R_{ik} \), and \( R_k \) with \( R_0 \). The remaining set of 6 lines \( R_{ij} \), where \( i, j \neq k \), consist of three orbits of pairs of intersecting lines. Note that, even though no orbits of \((−1)\)-curves can be blown down, the subgroup generated by \( \iota_k \) does not give \( X \) the structure of a \( G \)-minimal del Pezzo surface.
However, $X$ is $G$-minimal considered as a conic bundle defined by the pencil of conics given by the proper inverse transforms of the lines through $x_k$.

It follows that the involution $i_{kl} = i_k \circ i_l$ interchanges the disjoint lines $R_0$ and $R_{kl}$; thus, it does not act minimally. The fixed points of $i_{kl}$ are precisely the four intersection points of the two elliptic curves $E_k$ and $E_l$. The only minimal subgroups of $2^4$ with fixed points are those that contain exactly two involutions of the first kind.

In Section 8 of [D109], it is shown that any minimal del Pezzo $G$-surface with a fixed point is birationally equivalent to a $G$-minimal conic bundle. However, the conic bundle may not have a fixed point. We clarify the situation as follows:

**Lemma 7.1.** Suppose $X$ is a minimal del Pezzo $G$-surface.

1. If $G$ has more than one fixed point or $G$ is abelian, then $X$ is birationally equivalent to a minimal conic bundle with a fixed point.
2. If $G$ has exactly one fixed point and $G$ is non-abelian, then $X$ is the unique minimal $G$-surface with a fixed point up to birational equivalence.

**Proof.** Let $p$ be a $G$-fixed point on $X$. Blowing up the point $p$ we obtain a weak del Pezzo surface $X'$ of degree 3 with $\text{Pic}(X')^G \cong \mathbb{Z}^2$. The linear system $|-K_{X'} - R|$, where $R$ is the exceptional curve of the blow-up, defines on $X'$ a structure of a $G$-minimal conic bundle.

If $X$ has more than one fixed point, then $X'$ also has a fixed point. Also, if $G$ is abelian then the induced action on $R \cong \mathbb{P}^1$ is cyclic and thus $X'$ again has a fixed point.

However, if $X$ has a unique fixed point and $G$ is non-abelian, then the new surface $X'$ does not have a fixed point. Indeed, the exceptional curve $R$ has an action of $G$; since $G$ is not abelian the image of its action is not cyclic and $R \cong \mathbb{P}^1$ cannot have a fixed point.

Conceivably, there might be a third minimal $G$-surface $X''$ birational to $X$ which does have a fixed point. We consult the classification of elementary links in Section 7.4 of [D109]. From $X$, the link of Type I to $X'$ as described above is the only link which changes the isomorphism class of $X$. The conic bundle $X'$ satisfies $K_{X'}^2 = 3$ and the only links which change the isomorphism class are links of type II. These are simply compositions of elementary transformations and cannot introduce new fixed points, nor change the value of $K_{X'}^2$. Thus, there are no other minimal $G$-surfaces in the equivalence class of $X$.

We now prove the main result of this section.

**Theorem 7.2 (Case 4).** Let $X$ be a minimal del Pezzo $G$-surface of degree 4. Suppose $G$ has a fixed point and is not birationally equivalent to a minimal conic bundle with a fixed point. Then $X$ is isomorphic to the $G$-surface

$$t_1^2 + e_3 t_2^2 + e_3^2 t_3^2 + t_4^2 = t_1^2 + e_3^2 t_2^2 + e_3 t_3^2 + t_4^2 = t_5^2 = 0, \quad e_3 = e^{2 \pi i / 3},$$
whose automorphism group (as an ordinary surface) is generated by $2^4$ along with the transformations

\[
g_1 : (t_1 : t_2 : t_3 : t_4 : t_5) \mapsto (t_2 : t_3 : t_1 : \epsilon_3 t_4 : \epsilon_2^3 t_5) \\
g_2 : (t_1 : t_2 : t_3 : t_4 : t_5) \mapsto (t_1 : t_3 : t_2 : t_5 : t_4)
\]

The group $G$ is isomorphic to one of the following groups:

\[
2^2 : \mathfrak{S}_3, \quad 3 : 4
\]

with the unique fixed point $p = (1 : 1 : 1 : 0 : 0)$.

**Proof.** From the Lemma, it suffices to find $G$-minimal del Pezzo surfaces with a unique fixed point and $G$ non-abelian.

It is known that any minimal subgroup of $\text{Aut}(X)$ contains a non-trivial subgroup of $2^4$. Hence, a fixed point $p$ of $G$ must lie on one of the curves $E_i$. Since no three elliptic curves $E_1, \ldots, E_5$ have a common point, the group $G$ contains a subgroup $G'$ of index $\leq 2$ that leaves $E_i$ invariant. We may consider $E_i$ as an abelian curve with the zero element $p$. Let $A$ be the image of $G'$ in the automorphism group of the abelian curve $E_i$. It is known that $A$ is of order 2, 3, 4, or 6. It has 4, 3, 2, or 1 fixed points, respectively. Thus $A$ must be of order 6, hence the order of $G$ is divisible by 3.

Let $G$ be a group of automorphisms of $X$ of order divisible by 3. It is known that $X$ is isomorphic to the surface from the assertion of the theorem. Also, the automorphism group of $X$ is generated by involutions $\iota_A$ and the subgroup $H = \langle g_1, g_2 \rangle \cong \mathfrak{S}_3$. We fix a plane model of $X$ as above to assume that $g_1$ acts on $\text{Pic}(X)$ by permuting cyclically the classes of the exceptional curves $R_1, R_2, R_3$ and fixing the curves $R_4, R_5$. The element $g_2$ acts by switching $R_2, R_3$ and $R_4, R_5$.

There are four subgroups of order 3 in $\text{Aut}(X)$:

\[
\langle \iota_1 \iota_2 \rangle, \quad \langle \iota_1 \iota_3 \rangle, \quad \langle \iota_1 \iota_5 \rangle, \quad \langle \iota_1 \rangle
\]

but they are all conjugate. We may assume without loss of generality that $g_1$ is in $G$.

Let $K$ be the kernel of the homomorphism $G \to \mathfrak{S}_3$ and $\bar{G}$ be the image of this homomorphism. We enumerate all the possible subgroups $K$ of rank $\leq 2$ which are invariant under $g_1$:

\[
\langle \iota_4 \rangle, \quad \langle \iota_5 \rangle, \quad \langle \iota_{45} \rangle, \quad \langle \iota_4, \iota_5 \rangle, \quad \langle \iota_{12}, \iota_{23} \rangle.
\]

Note that $\langle \iota_{12}, \iota_{23} \rangle$ does not fix a point and can be eliminated. The remaining groups are fixed pointwise by $g_1$. Thus, if $\bar{G}$ is cyclic of order 3 then $G$ is abelian and can be eliminated. It remains to consider $G \cong S_3$. In this case, only the subgroups $\langle \iota_{45} \rangle$ and $\langle \iota_4, \iota_5 \rangle$ are invariant under $g_2$; so these are the only possibilities for $K$.

Consider the set $\Gamma \subset 2^4$ of all elements $\iota_A$ such that $g_2 \iota_A$ is in $G$. Note that $g_3 g_2 \iota_A g_3 = g_2 \iota_{123} A$ and $(g_2 \iota_A)^{-1} = g_2 \iota_{12} \iota_{45} A$ are also in $G$. Also, we note that $(g_2 \iota_A)^{-1}(g_2 \iota_B) = \iota_{AB}$. Thus $\Gamma$ is an $\mathfrak{S}_3$-invariant set such that
the product of any two elements in $\Gamma$ is in $K$. We conclude that $\Gamma$ contains only id, $\iota_4$, $\iota_5$ and $\iota_{45}$.

Thus the only possibilities for $G$ are

$$\langle g_2, g_3, \iota_{45} \rangle \cong 2 \times \mathfrak{S}_3$$

$$\langle g_2, g_3, \iota_4 \rangle \cong 2^2 : \mathfrak{S}_3$$

$$\langle g_2\iota_4, g_3 \rangle \cong 3 : 4 .$$

All of these leave fixed the point $(1 : 1 : 1 : 0 : 0)$. Appealing to Proposition 2.1 we see that $2 \times \mathfrak{S}_3$ is not minimal while the other two groups are minimal. □

**Remark 7.3.** As was first noticed by Yuri Prokhorov (see [Pro13]), the groups $3 : 4$ and $2^2 : \mathfrak{S}_3$ above were missing from the classification in [DI09]. We found additional missing groups isomorphic to $2 \times D_8$, $M_{16}$, $2^3 : \mathfrak{S}_3$, and $L_{16} : 3$; as well as a second copy of $L_{16}$ which is not conjugate to existing group in the list. In addition, the group of order 32 identified as $2^2 : 8$ should instead be $2^3 : 4$. Here $L_{16}$ and $M_{16}$ are certain groups of order 16 whose structure is described in Table 3 from [DI09]. One finds the corrected statements and the corrected proofs in a version of the paper at [http://www.math.lsa.umich.edu/~idolga/papers.html](http://www.math.lsa.umich.edu/~idolga/papers.html).

8. **del Pezzo surfaces of degree 3**

Recall that a del Pezzo surface of degree 3 is a smooth cubic surface in $\mathbb{P}^3$. Here we prove the following:

**Theorem 8.1** (Case 3). Suppose $G$ is a non-cyclic group and $X$ is a minimal cubic $G$-surface with a fixed point $p$. Then $X$ is equivariantly projectively equivalent to the surface in $\mathbb{P}^3$ cut out by

$$F = t_0^3 + t_1^3 + t_2^3 + t_4^3 + t_0t_1(at_2 + bt_3)$$

with fixed point $p = (0 : 0 : 1 : -1)$, where $a$ and $b$ are parameters. The tangent plane at $p$ contains three Eckardt points. The different possibilities are given in the following table

| Name | Possible $G$ | Parameters | Surface type from [DI09] |
|------|--------------|------------|-------------------------|
| 3.1  | $\mathfrak{S}_3$ |             | I–VI, VIII, V            |
| 3.2  | $\mathfrak{S}_3 \times 2$ | $a = b$ | $I$, $II$, $VI$          |
| 3.3  | $\mathfrak{S}_3 \times 6$, $\mathfrak{S}_3 \times 3$ (twice), $6 \times 3, 3 \times 3$ | $a = b = 0$ | $I$                        |

which have specializations

```
3.1   3.2   3.3
```

Note that we do not list those $G$ which already occur in generizations.
Proof. We begin by considering the case 3.3. Here $X$ is the Fermat cubic surface

$$X : t_0^3 + t_1^3 + t_2^3 + t_3^3 = 0.$$ 

The automorphism group of $X$ is $3^3 : S_4$ of order 648 (see [Dol12]). The surface $X$ has 18 Eckardt points; one of which is $p = (0 : 0 : 1 : -1)$ and all the others are obtained from $p$ by automorphisms. The stabilizer $\text{Aut}(X, p)$ is isomorphic to $S_3 \times 6$ of order 36. We will show that every case specializes to this case.

Now, let $X$ be general as in the theorem. Since $G$ is minimal, the cardinality of any orbit on the 27 lines must be divisible by 3 (if the sum of $k$ lines is linearly equivalent to $mK_X$, then $k = 3m$). Thus, $G$ has order divisible by 3.

Let $g$ be an element of order 3 in $G$. From Table 9.5 of [Dol12], up to projective equivalence, we have three different options for the action of $g$ on $\mathbb{P}^3$:

- (3A): $g(t_0 : t_1 : t_2 : t_3) = (\epsilon_3 t_0 : t_1 : t_2 : t_3)$
- (3C): $g(t_0 : t_1 : t_2 : t_3) = (\epsilon_3 t_0 : \epsilon_3 t_1 : t_2 : t_3)$
- (3D): $g(t_0 : t_1 : t_2 : t_3) = (\epsilon_3 t_0 : \epsilon_3^2 t_1 : t_2 : t_3)$

where $\epsilon_3$ is a primitive 3rd root of unity.

If we assume that $g$ is of the class (3C) then $X$ must be the Fermat cubic (see Section 9.5.1 of [Dol12]). The fixed points are all Eckardt points and so we are in case 3.3.

Now, we assume that $g$ is of class (3D). As in Section 9.5.1 of [Dol12], up to a projective change of coordinates, the surface $X$ is one of the surfaces stated in the theorem. Set $\ell_1 : t_2 = t_3 = 0$ and $\ell_2 : t_0 = t_1 = 0$. Note that $\ell_1$ and $\ell_2$ are canonically defined given $g$. The three points $\ell_2 \cap X$ are the only fixed points of $g$ on $X$; they are of the form $(0 : 0 : 1 : -a)$ where $a^3 = 1$. Without loss of generality we may take $p = (0 : 0 : 1 : -1)$.

There is always an involution $\sigma$ which interchanges $t_0$ and $t_1$. Thus $G$ always acts on $X$ fixing $p$. The line $\ell_1$ is stable under $G$ and the three points $X \cap \ell_1$ are all Eckardt points by Proposition 9.1.27 of [Dol12].

Consider the tangent space $T_pX \subset \mathbb{P}^3$. One checks that $T_pX$ contains $\ell_1$. The intersection $C = T_pX \cap X$ is either a nodal cubic or three concurrent lines. There is a faithful action of $G$ on $T_pX$ which must leave $C$ invariant.

In the case of $C$ a nodal cubic, we see that $G \subset \mathbb{G}_m : 2$. Since, $G$ contains an element of order 3 and the three points in $\ell_1 \cap C$ must be $G$-invariant. We see that $G$ is isomorphic to $S_3$ and we are in case 3.1.

When $C$ is three concurrent lines, the point $p$ is an Eckardt point and we have $S_3 \times 2 \subset \text{Aut}(X, p)$ by Proposition 9.1.26 of [Dol12]. The automorphism group of 3 concurrent lines in $\mathbb{P}^2$ is $\mathbb{G}_m \times S_3$. The polar $P$ of $p$ in $X$ is a union of two planes, the tangent plane $t_2 + t_3 = 0$ and the plane $t_2 - t_3 = 0$. Since both these planes and the line $\ell_1$ must be $G$-invariant, the only other automorphisms fixing $p$ must be of the form
\[(t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : t_1 : \lambda t_2 : \lambda t_3)\] where \(\lambda\) is in \(\mathbb{C}^\times\). This \(\lambda\) can be non-trivial only in the case where \(X\) is the Fermat cubic (and we are then in case \([3.3]\)). If \(\lambda\) is forced to be trivial, then we are in case \([3.2]\).

Finally, if we assume that \(g\) is of the class \((3A)\), then \(X\) is cyclic cubic surface

\[X : t_0^3 + F(t_1, t_2, t_3) = 0.\]

It is a triple cover of \(\mathbb{P}^2\) ramified at the smooth elliptic curve cut out by the plane \(t_0 = 0\). The Hessian quartic surface is a union of the plane \(t_0 = 0\) and a cone over the Hessian cubic curve \(H\) associated to \(\mathbb{P}^2\). If \(X\) has an additional cyclic structure, then the curve \(H\) is a union of 3 concurrent lines and \(X\) must be isomorphic to the Fermat cubic (see Lemma 3.2.4 of [Dol12]). Since the Fermat cubic was already considered above, we may assume the cyclic structure is unique and, thus, \(G\) leaves invariant an elliptic curve containing the fixed point \(p\). This means that \(G\) is a central extension of a cyclic group \(H\) by 3. The group \(G\) is thus cyclic unless \(H\) has order divisible by 3. In this latter case, \(G\) contains a subgroup isomorphic to \(C_3^2\). This means that \(G\) must contain an element of order 3 whose class is not of the form \((3A)\) and thus was already discussed above.

It remains to determine which subgroups \(G\) of \(\text{Aut}(X, p)\) are minimal. It suffices to consider only \([3.3]\) since the others are generalizations of this case. First, we list all non-cyclic subgroups of \(S_3 \times 6\) up to conjugacy:

\[2^2, \, S_3 \text{ (twice)}, \, 3^2 \times 2, \, S_3 \times 2, \, 6 \times 3, \, S_3 \times 3 \text{ (twice)}, \, S_3 \times 6.\]

We compute the traces on the space \(\mathcal{R}_X\) using Proposition 2.1 as in Table 9.4 of [Dol12]:

| Eigenvalues on \(\mathbb{P}^1\) | \(\text{Tr}(-|\mathcal{R}_X|)\) |
|-----------------------------|-----------------|
| 1 1 1 1                     | 6               |
| 1 1 1 -1                    | -2              |
| 1 1 -1 -1                   | 2               |
| 1 1 1 \(\epsilon\)         | -3              |
| 1 1 \(\epsilon\) \(\epsilon\)| 3               |
| 1 1 \(\epsilon\) \(\epsilon^2\)| 0              |
| 1 1 \(\epsilon\) \(-\epsilon\)| 1             |
| 1 -1 \(\epsilon\) \(-\epsilon\)| -1             |
| 1 -1 \(\epsilon\) \(\epsilon^2\)| -2             |

where \(\epsilon\) is a primitive third root of unity.

Note that the subgroup generated by an element with eigenvalues 1,1,1,\(\epsilon\) give traces which sum to 0, thus any group containing this element is minimal. Thus, \(3 \times 3, \, 6 \times 3, \, S_3 \times 3\) and \(S_3 \times 6\) are minimal groups. We remark that the eigenvalues of the involutions in the two different classes of \(S_3 \times 3\) are different; thus the two conjugacy classes are distinct in \(\text{Cr}(2)\).

Additionally, the group \(S_3\) generated by elements with eigenvalues 1,1,\(\epsilon\),\(\epsilon^2\) and 1,1,1,-1 gives traces which sum to 0. Thus, \(S_3\) and \(S_3 \times 2\) are minimal. It remains to establish that \(6 \times 2, \, 2 \times 2\) and the other conjugacy class of \(S_3\) are not minimal. The group \(6 \times 2\) and the group \(S_3\) both have traces
which sum to 12, so neither is minimal; the group $2^2$ is a subgroup of $6 \times 2$ and thus, is not minimal.

9. del Pezzo surfaces of degree 2

Throughout this section, $G$ is a non-cyclic finite group, $X$ is a minimal del Pezzo $G$-surface of degree 2, and $p$ is a $G$-fixed point on $X$.

We recall some features of such surfaces from Section 6.6 [DI09]. Any such surface has an involution $\gamma$ called the *Geiser involution*. Its set of fixed points is a smooth curve $R$ of genus 3. The quotient by $\gamma$ induces a degree 2 map

$$\pi : X \to \mathbb{P}^2$$

with branch locus $B \cong R$ a smooth quartic curve.

We may write $X$ as:

$$F(t_0, t_1, t_2) + t_3^2 = 0$$

in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, where $F$ is the degree 4 form which defines $B$ in $\mathbb{P}^2$. The Geiser involution is simply the map which takes $t_3$ to $-t_3$. We have a decomposition $\text{Aut}(X) \cong \text{Aut}(B) \times \langle \gamma \rangle$. Note that $\text{Aut}(B)$ is a finite subgroup of $\text{PGL}(3)$ since $F = 0$ is the canonical embedding of $B$. The possible $\text{Aut}(B)$ can be found in Theorem 6.5.2 of [Dol12].

**Theorem 9.1 (Case 2A).** If $p$ lies on the ramification curve $R$ then the group $\text{Aut}(X, p)$ is abelian of the form $H \times \langle \gamma \rangle$ where $H$ is a cyclic subgroup of $\text{Aut}(B)$. We have the following possibilities

| Name  | Possible $G$ | Surface type from [DI09] |
|-------|--------------|--------------------------|
| 2A.1  | $2^2$        | I–V, VII–X, XII          |
| 2A.2  | $6 \times 2$ | III, VIII                |
| 2A.3  | $4 \times 2$ | II–III, V                |
| 2A.4  | $12 \times 2$| III                      |
| 2A.5  | $8 \times 2$ | II                       |

satisfying the specializations

$$2A.1 \rightarrow 2A.2 \rightarrow 2A.4$$

$$2A.3 \rightarrow 2A.5$$

Note that we do not list those $G$ which already occur in generizations.

**Proof.** Since $p$ lies on $R$, $\text{Aut}(X, p)$ contains $\gamma$. It remains only to classify the possible $H$. Since $H$ acts faithfully on the tangent space to $R$ at $p$, we see that $H$ is cyclic. Since $G$ is not cyclic, $\text{Aut}(X, p)$ is not cyclic. Thus, the possible $H$ are precisely the maximal cyclic subgroups of $\text{Aut}(B)$ of even order which fix a point on $B$. From Lemma 6.5.1 of [Dol12], we obtain the
Now, suppose $G$ does not fix any points on the ramification curve $R$. Then $p$ is not fixed by $\gamma$ and we may assume that $G$ is an isomorphic lift of a subgroup $\tilde{G}$ of $\text{Aut}(B)$ fixing a point $q = \pi(p)$ in $\mathbb{P}^2$ not lying on $B$.

A del Pezzo surface has 56 exceptional curves (lines) $E_i$ on which $G$ acts. Any orbit of $G$ on the lines consists of $k$ lines whose sum is linearly equivalent to a multiple of $K_X$. Since $K_X^2 = 2$, this implies that $k$ is even. Thus the order of $G$ is even and $G$ contains an involution $\tilde{\tau}$, a lift of an involution $\tau$ of $\mathbb{P}^2$ that leaves $B$ invariant. The set $(\mathbb{P}^2)^\tau$ of fixed points of $\tau$ is equal to $\{q\}$ plus a line $L$ that intersects $B$ at four fixed points (counted with multiplicities). The set of fixed points of $\tilde{\tau}$ is the set containing the two points $p$ and $\gamma(p)$ along with an elliptic curve $\pi^{-1}(L)$.

We claim that $q$ is the intersection point of four bitangents. Choose the projective coordinates $(t_0, t_1, t_2)$ in $\mathbb{P}^2$ such that $q = (0 : 0 : 1)$ and $L : t_2 = 0$. Then the equation of $B$ has the form

\begin{equation}
(9.1) \quad t_1^2 + 2f_2(t_0, t_1)t_2^2 + f_4(t_0, t_1) = 0
\end{equation}

where the involution $\tau$ acts by $(t_0 : t_1 : t_2) \mapsto (t_0 : t_1 : -t_2)$ and $f_2$ and $f_4$ are homogeneous polynomials of degree 2 and 4, respectively. Note that we can rewrite the equation in the form

\begin{equation}
(9.2) \quad (t_2^2 + f_2(t_0, t_1))^2 + (f_4(t_0, t_1) - f_2(t_0, t_1))^2 = 0.
\end{equation}

This shows that each line $bt_0 - at_1 = 0$, where $(f_4(a, b) - f_2(a, b)^2) = 0$, is a bitangent of $B$ passing through the point $q$. Thus $q$ is the intersection point of four bitangents of $B$ as claimed.

The converse was first proven by Sonya Kowalevski [Kow]. Although we do not use this result we give a proof.

**Proposition 9.2.** Suppose a smooth plane quartic curve $B$ has four bitangents meeting at a point $q$. Then there exists a projective involution $\tau$ of $\mathbb{P}^2$ that leaves $B$ invariant and has the point $q \in B$ as an isolated fixed point.

**Proof.** By Proposition 6.1.4 from [Dol12], any three of the bitangent lines form a syzygetic triad of bitangents, i.e. the corresponding six tangency points lie on a conic. This implies that all eight tangency points lie on a conic. Choose coordinates so that $q = (0 : 0 : 1)$. Let $\ell_i : l_i = a_it_0 + b_it_1 = 0$ and let $C_2(t_0, t_1, t_2) = 0$ be the equation of the conic $K$ passing through the eight tangency points. Then the polynomials $C_2^2$ and $l_1 \cdots l_4$ define the same divisor on $B$, hence the equation of $B$ can be written in the form $F = C_2^2 + l_1l_2l_3l_4 = 0$. Let $C_2 = t_2^2 + 2t_2l(t_0, t_1) + q(t_0, t_1) = (t_2 + l(t_0, t_1))^2 + q(t_0, t_1) - l(t_0, t_1)^2 = 0$. After we change again the coordinates $t_2 \mapsto t_2 + l(t_0, t_1)$, the equation of $B$ is reduced to the form (9.1). The involution $(t_0 : t_1 : t_2) \mapsto (t_0 : t_1 : -t_2)$ is the projective involution of $B$. \(\square\)
The involution $\tau$ has four fixed points $(a : b : 1)$ on $B$, where $f_4(a, b) = 0$, and the quotient $E = B/\langle \tau \rangle$ is an elliptic curve with equation

$$z^2 + 2zf_2(x, y) + f_4(x, y) = 0$$

in the weighted projective space $\mathbb{P}(1, 1, 2)$.

Lemma 9.3. The involution $\tau$ of $B$ belongs to the center of the group $G$.

Proof. For any $\sigma \in G$, the element $\tau' = \sigma \tau \sigma^{-1}$ fixes $q$ and leaves invariant the set of the bitangents of $B$ that contain $q$. Thus it leaves invariant the pencil of lines through $q$. This shows that $\tau'$ is an involution of $\mathbb{P}^2$ with the same isolated fixed point as $\tau$. Thus $\tau$ and $\tau'$ must coincide. \qed

Since the polynomial $f_4$ has four distinct roots, we can choose projective coordinates $t_0, t_1, t_2$ in the plane to assume that

$$f_2(t_0, t_1) = at_0^2 + bt_0 t_1 + ct_1^2, \quad f_4(t_0, t_1) = t_0^4 + dt_0^2 t_1^2 + t_1^4.$$  

The only condition on the coefficients here is $d^2 \neq 4$ expressing the fact that $f_4$ has no multiple roots, or, equivalently, the curve $B$ is nonsingular.

We may assume that $G$ acts via its lift to $\mathrm{GL}(3)$ as the group of matrices of the form $\begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus the group $\tilde{G}$ is naturally identified with a subgroup of $\mathrm{GL}(2)$. The transformation $\tau$ is defined by the matrix $-I_2$.

We want to find a list of maximal non-cyclic subgroups of $\mathrm{GL}(2)$, up to conjugacy, that leave $f_2$ and $f_4$ invariant. The automorphism $\tau$ is always present. Let $H$ be the automorphism group of $f_4 = 0$ viewed as a set of 4 points in $\mathbb{P}^1$. Let $K$ be the image of $G$ in $\mathrm{PGL}(2)$; note that $K \subset H$.

Consulting Section 5.5 of [DI09], we see that for $f_4$ in the coordinates above, $H$ is either $2^2$ for general $d$, $\mathfrak{A}_4$ for $d^2 = -12$, or $D_8$ for $d = 0$. If $f_2 = 0$ then the kernel of $G \to K$ is cyclic of order 4 and the possible maximal $G$ are, respectively, $4.2^2$, $4.\mathfrak{A}_4$ and $4.D_8$.

Suppose $f_2 \neq 0$. The kernel of $G \to K$ is precisely $\langle \tau \rangle$. Since $K$ must leave a pair of points invariant, it is isomorphic to one of $1$, $2$, $3$, $4$ or $2^2$. We may discount 1 and 3 since we consider non-cyclic $G$. Since $G$ must be a subgroup of $4.2^2$ or $4.D_8$, all of its elements act by scaling $t_0$ and $t_1$ while possibly interchanging them. Thus, all the remaining possibilities arise when $a$, $b$, or $c$ is zero, or when $a = c$.

Note that if $a = c$ then we may instead assume $b = 0$ via the the linear change of variables

$$(t_0, t_1) \mapsto (\delta (t_0 - t_1), \delta (t_0 + t_1))$$

for some $\delta$ satisfying $\delta^4 = (2 + d)^{-1}$. Accounting also for the symmetry between $a$ and $c$, we enumerate the possibilities in Table 2.

Our group $G$ is a minimal isomorphic lift of a subgroup $\tilde{G}$ of $\mathrm{Aut}(B)$ as above. Following Section 6.6 of [DI09], we say a lift is even if the group $G$ in its representation in $W(E_2)$ is contained in the normal subgroup $W(E_7)^+$ of index 2, and a lift is odd otherwise.
Remark 9.4. The classification of minimal groups of automorphisms of degree 2 del Pezzo surfaces from [DI09] has the following errors.

1. $\langle \gamma \rangle$ is missing from all types (except XII).
2. Type XIII is missing completely.
3. $2^2 \times \langle \gamma \rangle$ was omitted in Type I surfaces.
4. An even lift of $Q_8$ was omitted in Types II, III and V.
5. $2 \cdot \mathfrak{A}_4 \cong Q_8 : 3$ in Type III (not $D_8 : 3$).
6. $\mathfrak{A}_4 \times \langle \gamma \rangle$ was omitted in Type IV.
7. $C_3 \times \langle \gamma \rangle$ was omitted in Type III.

One finds the corrected statements and the corrected proofs in a version of the paper at [http://www.math.lsa.umich.edu/~idolga/papers.html](http://www.math.lsa.umich.edu/~idolga/papers.html).

Theorem 9.5 (Case 2B). Let $G$ be a minimal group with a fixed point $p$ is not in the ramification curve $R$, then $B$ is isomorphic to the plane quartic curve

$$F = t_2^4 + t_2^2(a t_0^2 + ct_1^2) + t_0^4 + dt_0^2 t_1 + t_1^4 = 0$$

and the fixed point is a lift of $(0:0:1)$.

We have the following cases:

| Name | Possible $G$ | Parameters | Surface type from [DI09] |
|------|--------------|------------|--------------------------|
| 2B.1 | $D_8$        | $a = c \neq 0$ | I-V, VII                 |
| 2B.2 | $2 \times 4$ | $a = d = 0$ | II, III, V               |
| 2B.3 | $4 \cdot 2^2 \cong 2 \cdot D_8, Q_8$ | $a = c = 0$ | II, III, V               |
| 2B.4 | $4 \cdot \mathfrak{A}_4, 2 \cdot \mathfrak{A}_4$ | $a = c = 0, d^2 = -12$ | III                      |
| 2B.5 | $4 \cdot D_8, 4 \times 4$ | $a = c = d = 0$ | II                       |

satisfying the specializations

$$\begin{array}{ccc}
2B.1 & \rightarrow & 2B.3 \\
 & \rightarrow & 2B.4 \\
2B.2 & \rightarrow & 2B.5
\end{array}$$

Note that we do not list those $G$ which already occur in generizations.
Proof. We may assume that the fixed point is $(0 : 0 : 1)$ and that $G$ acts as a subgroup of $GL(2)$ on the coordinates $(t_0, t_1)$. The maximal $G$ and the appropriate parameters can be obtained from Table 2. It remains only to determine which subgroups make $X$ minimal. Our main tool is Proposition 2.1 and the classification in Table 7 of [DI09] (which was derived using the same method).

Observe that any involution in $PGL(3)$ fixes an isolated point and a line. One lift to $Aut(X)$ fixes points only on $R$ and thus is excluded. The other fixes a pair of points and an elliptic curve, thus every involution has trace $-1$ on $R_X$. In particular, $G \cong 2^2$ has sum of traces equal to 4 and cannot be minimal.

An element of order 4 in $G$ with eigenvalues $i, -i$ in $GL(2)$ fixes 2 points on $X$ and thus has trace $-1$ on $R_X$. From this we conclude that both $D_8$ and $Q_8$ are minimal groups. In particular, the group $D_8$ in [2B.1] is minimal.

The element of order 4 with a generator having eigenvalues 1, $i$ in $GL(2)$ fixes an elliptic curve on $X$ and thus has trace $-3$. The group it generates is minimal. Thus the group $2 \times 4$ from [2B.2] is minimal.

Now we refer to Table 7 of [DI09]. The cases [2B.1] and [2B.2] are finished. For [2B.3], we note that $Q_8$ appears and that $4 \cdot 2$ is minimal since it contains $D_8$. For [2B.4], $2 \cdot A_4$ and $4 \cdot A_4$ contain $Q_8$ and are therefore minimal.

In the case of [2B.5], we only need to consider subgroups of a Sylow 2-subgroup of $Aut(X)$ isomorphic to $4 \cdot D_8$. The group $4^2$ contains $2 \times 4$ and is thus minimal. The group $4 \cdot D_8$ is similarly minimal. It remains to show that the even lift of $M_{16}$ does not fix $p$. We will do this by showing that the cyclic subgroup of order 8 within is an odd lift.

An automorphism of order 8 of $B$ acts by $(t_0 : t_1 : t_2) \mapsto (\epsilon^2 t_0 \epsilon^{-1} t_1 : t_2)$ in coordinates where $B$ is given by the equation $t_0^2 + t_0 t_1 (t_0 + t_1) = 0$. It has 3 fixed points in $\mathbb{P}^2$, two of which are on $B$. Thus, its lift must have 4 fixed points. The trace of an even lift of $g$ is equal to $-1$ and has 2 fixed points. We conclude that an element of order 8 in $G$ is an odd lift. Thus the subgroup isomorphic to $M_{16}$ in [2B.5] is not minimal. \hfill $\square$

10. del Pezzo surfaces of degree 1

Theorem 10.1 (Case 1). Let $X$ be a minimal del Pezzo $G$-surface of degree 1. Then $G$ has a fixed point.

This is immediate since the unique base point of $| - K_X|$ is canonical, and thus must be fixed by any automorphism of $X$. A list of all the minimal groups in this case can be found in Section 6.7 of [DI09].

11. Conic Bundles

Throughout this section, $\pi : X \to B$ is a minimal conic $G$-bundle with $B \cong \mathbb{P}^1$. Let $G_K$ be the kernel of the action of $G$ on $B$ and let $G_B$ be the
image. We have an exact sequence

\[ 1 \to G_K \to G \to G_B \to 1. \]

Let \( \Sigma \subset B \) be the set of points whose preimages under \( \pi \) are singular.

**Lemma 11.1.** The group \( G_K \) acts faithfully on each fiber.

**Proof.** This is a variant of Lemma 5.3 of [Dun13]. Let \( F \) be a fiber on which \( G_K \) has an element \( \sigma \neq 1 \) that acts identically on \( F \) and let \( q = \pi(F) \). Since the fixed locus of any automorphism of finite order is smooth, we may assume that \( F \) is nonsingular. Let \( A \) be the formal completion of the local ring \( \mathcal{O}_{B,q} \). Since \( X \) is locally trivial in a Zariski neighborhood of \( q \), the element \( \sigma \) acts on the base change \( X \times_B \text{Spec}(A) \cong \mathbb{P}^1_A \) via an element of \( \text{PGL}(2, A) \cong \text{PGL}(2, \mathbb{C}[[t]]) \). By assumption, \( \sigma \) belongs to the kernel of the reduction homomorphism \( r : \text{PGL}(2, \mathbb{C}[[t]]) \to \text{PGL}(2, \mathbb{C}) \). By multiplying \( \sigma \) by a constant scalar matrix, we may assume that \( \sigma \) is represented by a matrix of the form \( 1 + tM \) for some matrix \( M \). Since \( (1 + tM)^n = 1 + nMt + t^2M' \) for some matrix \( M' \), we obtain that \( \sigma \) is of infinite order. This contradiction proves the lemma. \( \square \)

Let \( G_0 \) be the kernel of the action of \( G \) on \( \text{Pic}(X) \). We use the trichotomy of conic bundles as in [DI09]:

1. \( X \to B \) is a ruled surface,
2. \( X \to B \) is non-exceptional: \( G_0 = 1 \) and \( X \) is not ruled,
3. \( X \to B \) is exceptional: \( G_0 \neq 1 \) and \( X \) is not ruled.

We will consider each case in turn.

We begin by considering the case of a minimal ruled surface. Recall that the case of \( F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) was shown to be birationally equivalent to \( \mathbb{P}^2 \) with a \( G \)-fixed point in Section 4. In fact, this is true for all ruled \( G \)-surfaces.

**Theorem 11.2.** Suppose \( X \cong F_n \) is a ruled \( G \)-surface with a fixed point where \( n \geq 2 \). Then \( G \) is abelian and \( X \) is birationally equivalent to \( \mathbb{P}^2 \) with a \( G \)-fixed point.

**Proof.** We recall some facts from the proof of Theorem 4.10 of [DI09]. Let \( S \) be the exceptional section. Here \( G \) acts on the cone \( \tilde{X} \) obtained from \( X \) by blowing down the exceptional section \( S \). The cone \( \tilde{X} \) can be identified with the weighted projective plane \( \mathbb{P}(1,1,n) \) whose group of automorphisms consists of transformations

\[ (t_0 : t_1 ; t_2) \mapsto (at_0 + bt_1 : ct_0 + dt_1 ; et_2 + f_n(t_0, t_1)) \]

where \( f_n \) is a homogeneous polynomial of degree \( n \). Since \( G \) is finite, we have \( f_n = 0 \).

The action of \( G \) on the coordinates \( (t_0, t_1) \) is isomorphic to the action of \( G \) on \( S \). Since the restriction \( \pi : S \to B \) is equivariant and \( B \) has a fixed point, there must be a fixed point on \( S \) (in fact, at least two). Thus, any element of \( g \) has two fixed points on the base equal to, say, \( (t_0 : t_1) = (1 : 0), (0 : 1) \). Thus \( g \) has the form from above with \( b = c = 0 \). Thus \( G \) is abelian.
The group $G$ has at least 4 fixed points on $X$: the two on $S$ and the preimages of the points $(1 : 0 : 0)$ and $(0 : 1 : 0)$.

We now show that $X$ is birationally equivalent to $\mathbb{P}^2$ with a fixed point. Since $G$ fixes a point $p$ not on $S$, we may perform an elementary transformation at that point to obtain a ruled $G$-surface $X'$ isomorphic to $F_{n-1}$ which must also have a fixed point since $G$ is abelian. By applying this procedure inductively, we eventually find a birational equivalence to a $G$-surface isomorphic to $F_1$. Blowing down the exceptional divisor we have the desired result.

□

Theorem 11.3 (Case C.ne). Let $G$ be a non-cyclic finite group and let $X$ be a non-exceptional $G$-minimal conic bundle. Assume that $G$ has a fixed point. Then, we have one of the following cases:

(i) $G_K \cong 2, G_B \cong 2n, G \cong 2 \times 2n$,

(ii) $G_K \cong 2^2, G_B \cong n, G \cong 2 \times 2n$,

(iii) $G_K \cong 2^2, G_B \cong n, G \cong (2m : 2) \times q$.

where $n = mq$ is a positive integer, $m$ is a power of 2 and $q$ is odd.

Proof. Recall that in this case $G_0$ is trivial and $X$ is not a ruled surface. Here $G_K \cong 2^a$ with $a = 1$ or 2 ([DI09], Theorem 5.7). Since $G$ has a fixed point, the group $G_B$ is cyclic.

First, consider $G_K \cong 2$. Since we assume that $G$ is not cyclic, $G \cong 2 \times 2n$ for some positive integer $n$.

Assume now that $G_K \cong 2^2$. The order of $G_B$ is $n = mq$ where $m$ is a power of 2 and $q$ is a positive odd integer. Thus there is a homomorphism from $G$ to a cyclic group of order $m$ whose kernel is a 2-group. Since the quotient and kernel have coprime orders, the extension splits. The 2-group will also be minimal since it contains $G_K$. Thus, it suffices to assume the order of $G_B$ is of the form $n = m$.

Since $G$ must embed into $\text{GL}(2)$, we see that some element $z$ in $G_K$ must map to the matrix $-\text{id}$. Let $x$ be a non-trivial element of $G_K$ not equal to $z$. Let $g$ be a lift to $G$ of a generator of $G_B$.

Since $g$ must normalize $G_K$ and $z$ must be central, we see that either

(1) $gxg^{-1} = x$, or
(2) $gxg^{-1} = xz$.

In case (1), the group $G$ is abelian. Since $G$ must have rank $\leq 2$, we see that $G \cong 2 \times 2m$.

In case (2), rearranging we obtain $xgx^{-1} = gz$. Note that $xg^m x^{-1} = g^m$ since $m$ is even. Thus, the group generated by $\langle x, z, g^m \rangle$ must be abelian of rank 2. Since $G_K = \langle x, z \rangle$, and $x, g$ do not commute, we conclude that either

(a) $g^m = 1$, or
(b) $g^m = z$.

In case (a), we conclude $g$ has order 2 and $G \cong D_8$; otherwise, we would have a contradiction as the abelian group $\langle x, z, g^{m/2} \rangle$ would have rank 3. In
case (b), we conclude that our group $G$ is a semidirect product $2m : 2$ where the involution $x$ acts by $g \mapsto g^{m+1}$. 

**Example 11.4.** Let $X$ be a del Pezzo surface of degree 4 and $G$ be a subgroup of automorphisms generated by two involutions of the first kind, say $\iota_1, \iota_2$. The group has 4 fixed points $E_1 \cap E_2 = \{p_1, p_2, p_3, p_4\}$. Let $\sigma : X' \to X$ be the blowing up of $p_1$. From the description of the involutions in Section 4, we see that $p_1$ does not lie on any exceptional divisors. Thus, the surface $X'$ is a del Pezzo surface of degree 3. In its anti-canonical model, it is a cubic surface.

The image of the exceptional curve $E$ of $\sigma$ is a line on $X'$ invariant with respect to $G$. The pencil of planes through $R$ has $R$ as its fixed component and the residual pencil is a pencil of conics invariant under $G$. It equips $X'$ with a structure of a minimal conic bundle $G$-surface with a 2-section $R$.

Since $G \cong 2^2$ acts faithfully on the tangent space of $X$ at $p_1$, and has two invariant tangent directions of $E_1$ and $E_2$ at $p_1$, we see that the involution of the second kind $\iota_{12}$ acts identically on $R$ while the other two involutions act non-trivially. We have a 2 to 1 morphism $R \cong \mathbb{P}^1 \to B \cong \mathbb{P}^1$ which is equivariant with respect to a cyclic group of order 2; this forces the action on $B$ to be trivial. Thus the group $G_B$ is trivial and $G = G_K \cong 2^2$.

As $G$ acts faithfully on each fiber by Lemma 11.1, we conclude that all of the fixed points must be the singular points of the singular fibers of $\sigma$. A conic bundle on a cubic surface has 5 singular fibers, so we have 5 fixed points. Alternatively, we note that there are 4 fixed points on $X$, but there are 2 fixed points on $R$; thus $X'$ has 5 fixed points.

Note in the plane model of $X'$ as the blow-up of 6 points $x_1, \ldots, x_5, p_1$, the pencil of conics arises from the pencil of cubics through $x_1, \ldots, x_5$ and a double point at $p_1$. The singular fibers are the unions of the line $\ell_i = \overline{p_1, x_i}$ and the conic $C_i$ through the points $x_k, k \neq i$ and $p_1$. Three of such pairs $(\ell_i, C_i)$ intersect at $p_1$ and $p_i, i = 2, 3, 4$ and the remaining two are tangent at $p_1$ with the tangent directions corresponding to the cubics defined by $E_1$ and $E_2$.

**Theorem 11.5** (Case C.ex). Let $G$ be a non-cyclic finite group and let $X$ be an exceptional $G$-minimal conic bundle with a fixed point $p$. Then $G_K$ is a dihedral group or a cyclic group of even order, $G_B$ is cyclic or trivial, and $G$ is a subgroup of $D_{2m} \times n$ for some integers $m$ and $n$. Furthermore, $p$ is a singular point of a singular fibre of $\pi$.

**Proof.** Here $G_0$ is non-trivial, but $X$ is not a ruled surface. Recall from Section 5 of [DI09] that $X$ has 2 disjoint sections $S_0$ and $S_\infty$ that can be blown down to obtain a hypersurface

$$X' : H_{2g+2}(t_0, t_1) + t_2 t_3 = 0$$

in the weighted projective space $\mathbb{P}(1 : 1 : g+1 : g+1)$ for $g$ a positive integer. The map $\pi : X \to B$ is given by the morphism $(t_0 : t_1) \mapsto (t_0 : t_1)$. 

Since $G_B$ is cyclic or trivial, by the proof of Proposition 5.3 of [DI09] we see that $G$ is a subgroup of $G_B \times N$ where $G_B$ acts on $(t_0 : t_1)$ linearly and $N$ is the subgroup $\mathbb{C}^\times : 2$ of $\text{SL}_2(\mathbb{C})$ which preserves $t_2t_3$. Note that if $G$ is minimal then there must exist an element in $G_K$ which swaps $t_2$ and $t_3$ and thus has even order. Since $G_K$ is a subgroup of even order of a dihedral group it must be of the form in the statement of the theorem.

Finally, we establish that $G$ fixes a singular point of a singular fiber. The subgroup $G_0$ leaves invariant each singular fiber and each section $S_0$ and $S_\infty$. Since there are $\geq 3$ singular fibers, the sections $S_0$ and $S_\infty$ have $\geq 3$ points fixed by $G_0$. Thus the action of $G_0$ on $S_0$ and $S_\infty$ is trivial. Since $G_0$ is a subgroup of $G_K$, by Lemma 11.1, it acts faithfully on each fiber $F$. In particular, it can only fix the points $F \cap S_0$ and $F \cap S_\infty$ on a non-singular fiber. Since an element of $G_K$ swaps the two sections and $X^G \subset X^{G_0}$ we see that $G$ can only fix the singular points of singular fibers. □

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