Abstract

It is known that symmetric orbits in $g^*$ for any simple Lie algebra $g$ are equipped with a Poisson pencil generated by the Kirillov-Kostant-Souriau bracket and the reduced Sklyanin bracket associated to the "canonical" R-matrix. We realize quantization of this Poisson pencil on $\mathbb{CP}^n$ type orbits (i.e. orbits in $sl(n+1)^*$ whose real compact form is $\mathbb{CP}^n$) by means of q-deformed Verma modules.

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1 Introduction

The problem of quantization of Poisson brackets is one of the most important in mathematical physics. In the framework of the deformation quantization scheme going back to the works by A. Lichneriwicz and his school (cf. [BFLLS]) it can be formulated as follows. Given a variety $M$ equipped with a Poisson bracket, it is necessary to construct a flat deformation $\mathcal{A}_\hbar$ of an algebra $\mathcal{A} = \text{Fun}(M)$ of functions over $M$ such that the corresponding Poisson bracket (which exists for any flat deformation of a commutative algebra) coincides with the initial one.

The existence of such a quantization for any nowhere degenerated (i.e. defined by a symplectic structure) Poisson bracket had been shown in [DL]. Recently, M. Kontsevich [K] has proved that any Poisson bracket is quantizable in the above sense.

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1 We say that an algebra $\mathcal{A}_\hbar$ depending on a formal parameter $\hbar$ is a flat deformation (or simply, deformation) of $\mathcal{A}$ if $\mathcal{A} = \mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ and $\mathcal{A}_\hbar$ is isomorphic to $\mathcal{A}[[\hbar]]$ as $\mathbb{C}[[\hbar]]$-modules. Hereafter $V[[\hbar]]$ where $V$ is a linear space stands for the completion in $\hbar$-adic topology of $V \otimes_\mathbb{C} \mathbb{C}[[\hbar]]$ (in what follows the basic field is $k = \mathbb{C}$). Abusing notation we will let $\mathcal{A} \to \mathcal{A}_\hbar$ denote the deformation in question. Two parameter flat deformation can be defined in a similar way.
Nevertheless, physicists are interested in an operator quantization, i.e. they want to realize the quantum algebra $\mathcal{A}_h$ as an operator algebra in a linear (ideally, Hilbert) space. This enables them to carry out a spectral analysis of Hamiltonians and to compute partition functions and other numerical characteristics of quantum models. Such a quantization of nondegenerated Poisson bracket (on any compact smooth variety) has been realized by B. Fedosov [F]. In fact, the famous Kirillov-Duflo orbit method which consists in assigning a representation $\rho : g \rightarrow \text{End } V$ of a Lie algebra $g$ to an orbit $O \subset g^*$ can be considered as a particular case of the Fedosov approach. (We do not discuss here the limits of the orbit method, in the sequel we will restrict ourselves to semisimple orbits in $g^*$ for simple Lie algebras $g$).

The quantization procedure suggested by Fedosov leads to an operator algebra equipped with a commutative trace. In fact, such a trace is delivered for appropriated quantum algebras by the Liouville mesure of the initial Poisson bracket. However, a generic Poisson bracket does not possess any invariant measure and consequently it is not clear what is a trace in the corresponding quantum algebra.

In the earlier 90’s one of the authors (D.G.) suggested certain Poisson brackets associated to classical R-matrices whose quantization leads to operator algebras in twisted categories. Traces in such algebras are also twisted (cf. [G3], [GRZ]). These algebras arise from quantization of Poisson pencils generated by the linear Poisson-Lie bracket on $g^*$ or its restriction to an orbit, called the Kirillov-Kostant-Souriau (KKS) one, and by a bracket naturally associated to a solution $R \in \wedge^2(g)$ of the classical nonmodified Yang-Baxter equation

$$[[R, R]] = [R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}] = 0. \quad (1.1)$$

Here, as usual $R^{12} = R \otimes \text{id}$ etc.

Let us describe the latter bracket. Let $M$ be a variety equipped with a representation $\rho : g \rightarrow \text{Vect } (M)$ where $\text{Vect } (M)$ stands for the space of vector fields. Then the following bracket

$$\{f, g\}_R = \mu < \rho^{\otimes 2}(R), df \otimes dg >, \quad f, g \in \text{Fun } (M) \quad (1.2)$$

is Poisson. Here $< , >$ is the pairing between the differential forms and vector fields on $M$ extended to their tensor powers and $\mu$ is the usual commutative product in the space $\text{Fun } (M)$. The bracket $\{ , \}_R$ is called the $R$-matrix bracket. If $M = g^*$ or $M = O \subset g^*$ is an orbit we take as $\rho$ the coadjoint representation or its restriction to the orbit.

It is not difficult to see that in the latter case any bracket of the family

$$\{ , \}_a = a \{ , \}_\text{KKS} + b \{ , \}_R \quad (1.3)$$

is Poisson. Here by $\{ , \}_\text{KKS}$ we mean either the KKS bracket or the linear Poisson-Lie one on $g^*$. Thus, we have a Poisson pencil well defined on $g^*$ or on an orbit in $g^*$.

A procedure of quantizing this Poisson pencil can be realized in two steps (we call such procedures "double quantization"). In the first step one quantizes only the KKS bracket by means of the orbit method or by means of generalized Verma modules as it is discribed in Section 3. Then one twists the quantum operator algebra as it is described in Section 2. The resulting object is a two parameter operator algebras in a twisted category. It comes with a deformed trace which is no longer commutative but S-commutative in the spirit of a super-trace. Here $S$ is an involutive ($S^2 = \text{id}$) twist, i.e. an operator acting in tensor square of this algebra and satisfying quantum Yang-Baxter equation (QYBE)

$$S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}. \quad (2)$$
Let us remark that by means of a similar twisting one can introduce natural "S-analogues" of basic objects of geometry and analysis. Thus, S-analogues of commutative algebras, vector fields, Lie algebras, (formal) Lie groups were defined, in a spirit of super-theory, in [G1], [G4] (cf. also [GRR], [GRZ]). However the straightforward generalization of these notions to noninvolutive twists (connected, say, to the quantum group (QG) $U_q(\mathfrak{g})$) leads as a rule to a nonflat deformation. (From our viewpoint the principle "raison d’être" for objects belonging to the category of $U_q(\mathfrak{g})$-modules is that they should represent a flat deformation of their classical counterparts.)

The main purpose of the paper is to realize an operator quantization of Poisson pencils (1.3) associated to the "canonical" classical R-matrix

\[
R = \sum_{\alpha \in \Omega^+} \frac{X_\alpha \wedge X_{-\alpha}}{\langle X_\alpha, X_{-\alpha} \rangle} \in \bigwedge^2(\mathfrak{g}),
\]

where $\mathfrak{g}$ is a complex simple Lie algebra, $\Omega^+$ stands for the set of its positive roots with respect to a fixed triangular decomposition of $\mathfrak{g}$ and $\langle , \rangle$ stands for the Killing form.

This R-matrix satisfies the so-called classical modified YBE which means that the element $[[R, R]]$ is nontrivial and $g$-invariant. Since this element is not identically zero the associated R-matrix bracket is Poisson only on varieties where the three-vector field $\rho^{\otimes 3}([[R, R]])$ vanishes. Such varieties were called in [GP] R-matrix type ones. All R-matrix type orbits in $\mathfrak{g}^*$ were classified in [GP]. In particular, all symmetric orbits in $\mathfrak{g}^2$ are R-matrix type varieties. (Let us recall that an orbit $O_\sigma$ of a point $x$ is called symmetric if there exists a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where $\mathfrak{k}$ is the stabilizer of $x$ such that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$).

Moreover, the R-matrix bracket over a symmetric orbit coincides with one of the two left- or right- invariant components of the Sklyanin bracket reduced to the orbit (recall that the Sklyanin bracket equals to a difference between left- and right- invariant brackets defined by (1.2), where $\rho$ is the natural homomorphism of $\mathfrak{g}$ into the space of left- or right- invariant vector fields on the corresponding group $G$). Meanwhile, the other component being reduced becomes proportional to the KKS bracket. This implies that on any symmetric orbit the R-matrix bracket and the KKS one are compatible and therefore they generate the Poisson pencil (1.3).

Note that the one-sided invariant components of the Sklyanin bracket can be reduced to any semisimple (ss) orbit in $\mathfrak{g}^*$ (i.e., to that of an ss element), but each of them becomes Poisson brackets only on symmetric orbits (cf. [KRR] and [DG2]).

The Poisson pencil (1.3) with R-matrix (1.4) on symmetric orbits has been quantized in the spirit of deformation quantization in [DS]. The resulting object of the quantization procedure suggested in [DS] is a two parameter family of associative $U_q(\mathfrak{g})$-invariant algebras. Let us make precise that an algebra $A$ is called $U_q(\mathfrak{g})$-invariant (or $U_q(\mathfrak{g})$-covariant) if

\[
u \cdot (x_1 x_2) = \left(u(1) \cdot x_1\right) \left(u(2) \cdot x_2\right), \quad \forall u \in U_q(\mathfrak{g}), \; x_1, x_2 \in A.
\]

Hereafter $u(1) \otimes u(2)$ stands for $\Delta(u)$ (the Sweedler’s notations). Algebras $A$ possessing this property will be called quantum or braided ones, while by twisted algebras we mean the algebras belonging to a twisted category.

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2All these objects are also well defined for some nonquasiclassical twists (i.e. twists which cannot be obtained by a deformation of the ordinary flip $S = \sigma$). One can naturally associate to an involutive twist $S: V^{\otimes 2} \to V^{\otimes 2}$ where $V$ is a linear space $S$-symmetric and $S$-skewsymmetric algebras of $V$. The Poincaré series of these algebras related to quasiclassical twists, i.e. those being deformation of the flip, coincide with classical ones while those series corresponding to nonquasiclassical twists can differ drastically from the classical (and super-) ones. The first examples of such twists were given in [G1] (similar twists of Hecke type has been introduced in [G2]).
However, the product in the quantum algebra is realized in $\mathbb{CP}^n$ by a series in two formal parameters, meanwhile the QG $U_q(\mathfrak{g})$ appears as $U(\mathfrak{g})[[\hbar]]$ but equipped with a deformed coproduct (the so-called, Drinfeld’s realization, cf. Section 2).

In the present paper we perform a double quantization of $\mathbb{CP}^n$ type orbits by an operator method. By $\mathbb{CP}^n$ type orbits we mean the orbits in $sl(n)^*$ of elements $\mu \omega_1$ or $\mu \omega_{n-1}$ where $\omega_1 (\omega_{n-1})$ is the first (the last) fundamental weight of $sl(n)$ and $\mu \in \mathbb{C}$ is an arbitrary nontrivial factor. Compact forms of these complex orbits are just $\mathbb{CP}^{n-1}$ embedded as closed algebraic varieties in $su(n)^*$.

More precisely, we represent our two parameter quantum object $A_{h,q}$ as operator algebra in braided (or q-deformed) generalized Verma modules. Similiarly to the previous case arising from the classical nonmodified YBE, our quantization procedure consists of two steps.

The first, "classical", step is realized as follows. There exists a natural way to quantize ss orbits in $\mathfrak{g}^*$ for any simple Lie algebra $\mathfrak{g}$ by generalized Verma modules. Let $M_\omega$ be such a module of highest weight $\omega$ (its construction is given in Section 3) and $\rho_\omega : T(\mathfrak{g}) \rightarrow \text{End} M_\omega$ be the corresponding representation of the free tensor algebra $T(\mathfrak{g})$. Then, the operator algebras $A_\hbar = \text{Im} \ h \rho_{\omega/h} \subset \text{End} M_\omega[[h]]$ can be treated as a quantum object with respect to the KKS bracket on the orbit $O_\omega \subset \mathfrak{g}^*$ of the element $\omega$ (we regard $\omega$ as an element of $\mathfrak{g}^*$ as explained below). The passage from the representation $\rho_\omega$ to that $h \rho_{\omega/h}$ will be referred below as a renormalization procedure.

Let us remark that operator algebra $A_\hbar$ is an object of the category of $\mathfrak{g}$-invariant algebras similarly to the initial function algebra $A = \text{Fun}(O_\omega)$.

The second step consists in a braiding of the algebras $A_\hbar$. As a result we get the mentioned above two parameter $U_q(\mathfrak{g})$-invariant operator algebra $A_{h,q}$. Let us emphasize that our approach to represent quantum algebras by means of braided generalized Verma modules has the following advantage. The parameters $\hbar$ and $q$ can be specialized: the operator realization of the algebra $A_{h,q}$ is well defined for any value of $\hbar$ and a generic $q$.

Moreover, the flatness of deformation $A \rightarrow A_{h,q}$ is assured automatically. Let us remark that the quantum algebra $A_{h,q}$ can be also represented by a system of some algebraic equations. For the $\mathbb{CP}^n$ type orbits these equations are quadratic-linear-constant. It is not so difficult to guess a general form of these equations. The problem is to find the exact meaning of factors occurring in them which ensure flatness of deformation of the corresponding quotient algebras. Different ways to look for these factors were discussed in [DG2], [GR], [DGR2], [G5]. The "operator method" presented here is the most adequate way to solve this problem.

Thus, compared with $\mathbb{CP}^n$ our approach enables us to realize quantum counterparts of the Poisson pencil in question explicitly in the spirit of noncommutative algebraic geometry.

The paper is organized as follows. In the next section we describe different algebraic structures connected to involutive twists arising from quantization of R-matrices satisfying the classical nonmodified YBE. We show that certain quotients of twisted Hopf algebras are the appropriate objects to describe explicitly quantized orbits in $\mathfrak{g}^*$. We also analyze the difference between this case referred in what follows as triangular or involutive and that connected to the quasitriangular QG $U_q(\mathfrak{g})$.

Section 3 is devoted to the "classical step" of quantization. The final object of this step is the mentioned above algebras $A_\hbar$. Then we realize a q-deformation of these algebras as follows. We equip $\mathfrak{g} = sl(n)$ with a structure of a $U_q(\mathfrak{g})$-module, extend the action of the QG $U_q(\mathfrak{g})$ to its enveloping algebra and represent the latter algebra in the q-deformed generalized Verma modules considered on the first step. These constructions are discribed in Sections 4 and 5. They result in a two parameter family $A_{h,q}$ presented in the last section.

Completing the introduction we want to put the following question: how is it possible to
define a proper trace in a quantum algebra arising in virtue of \( K \) from a given Poisson bracket? As our examples show, such traces are not necessary commutative. (Although we are dealing with the complexification of \( \mathbb{C}P^n \) the trace defined by a projection of the algebra \( A_{h,q} \) to its trivial component is well defined on this algebra since it corresponds to the compact form of the orbits in question, cf. \( \cite{GV} \) where such a trace in \( sl(2) \) case is studied).

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2 Triangular and quasitriangular cases: comparative description

2.1. Let us first consider certain algebraic structures arising from R-matrices satisfying the classical YBE \((\ref{1.1})\). Let us fix such an R-matrix \( R \).

By Drinfeld’s result \( \cite{D} \) there exists a series \( F = F_\nu \in U(g)^{\otimes 2}[[\nu]] \) quantizing the R-matrix \( R \) in the following sense \( F_\nu = 1 + \nu P + ... \) where \( P \in g^{\otimes 2} \), \( P - P^{21} = 2R \), and

\[
\Delta^{12} F^{12} = \Delta^{23} F^{23}, \quad \varepsilon^1 F = \varepsilon^2 F = 1. \tag{2.1}
\]

Here \( \Delta : U(g) \to U(g)^{\otimes 2} \) is the usual coproduct and \( \varepsilon : U(g) \to \mathbb{C} \) is the counit in \( U(g) \) (all operators are assumed to be extended to \( U(g)^{[[\nu]]} \) in a natural way). Using \( F \) one can deform the usual Hopf structure of the algebra \( U(g) \) in (at least) two different ways.

The first way consists of the following procedure. Let us introduce a new coproduct setting

\[
\Delta_F(u) = F^{-1} \Delta(u) F = F_{(1)}^{-1} u_{(1)} F_{(2)}(1) \otimes F_{(2)}^{-1} u_{(2)} F_{(2)}. \]

Here \( F_{(1)} \otimes F_{(2)} \) (reps., \( F_{(1)}^{-1} \otimes F_{(2)}^{-1} \)) stands for \( F \) (resp., \( F^{-1} \)).

Then the algebra \( U(g)^{[[\nu]]} \) equipped with the initial product and unit, the coproduct \( \Delta_F \) and the uniquely defined counit and antipode (cf. \( \cite{D2} \) and \( \cite{GM} \) where the antipode is expressed via \( F \)) becomes an Hopf algebra looking like the famous QG \( U_q(g) \). Let us denote it \( H \).

Another way consists in simultaneous deformation of the product and coproduct as follows

\[
\overline{\Delta}(u) = \text{ad} F^{-1}(\Delta(u)) = \text{ad} F_{(1)}^{-1}(u_{(1)}) \otimes \text{ad} F_{(2)}^{-1}(u_{(2)})
\]

and

\[
\overline{\mu}(u_1 \otimes u_2) = \mu (\text{ad} F(u_1 \otimes u_2)) = \mu (\text{ad} F_{(1)}(u_1) \otimes \text{ad} F_{(2)}(u_2)). \tag{2.2}
\]

Here \( \mu \) is the initial product in \( U(g) \) and \( \text{ad} F^{\pm 1} \) is defined by

\[
\text{ad} X(Y) = [X, Y] \quad \text{and} \quad \text{ad} (X_1 X_2 ... X_p)(Y) = \text{ad} X_1 (\text{ad} X_2 (... \text{ad} X_p(Y) ...)).
\]

The space \( U(g)^{[[\nu]]} \) being equipped with these product, coproduct, the classical unit, counit and antipode becomes a twisted Hopf algebra. Essentially, this means that

\[
\overline{\Delta} \overline{\mu}(u_1 \otimes u_2) = (\overline{\mu} \otimes \overline{\mu}) (\text{id} \otimes S \otimes \text{id}) (\overline{\Delta}(u_1) \otimes \overline{\Delta}(u_2)) \tag{2.3}
\]

where \( S = S_\nu = F^{-1} \sigma F \) and \( \sigma \) is the flip (\( F \) and \( F^{-1} \) act in the above sense by \( \text{ad} \otimes \text{ad} \)). Let us denote \( \overline{\mu} \) this twisted Hopf algebra. We leave to the reader to verify that the operator \( S \) satisfies the QYB.
Let us observe that the $\Delta$-primitive elements $X \in U(\mathfrak{g})$, i.e., such that $\Delta(X) = X \otimes 1 + 1 \otimes X$ (they are just the elements of the algebra $\mathfrak{g}$) are still $\Delta$-primitive. This follows from the second relation (2.1).

The algebra $\overline{H}$ can be treated as the enveloping algebra of a generalized (or $S$-)Lie algebra defined by the deformed Lie bracket $[\ , \ ]_\nu = [\ , \ ]_{F_\nu}$ or in more detailed form

$$[X, Y]_\nu = [\text{ad } F_{(1)}(X), \text{ad } F_{(2)}(Y)].$$

An axiomatic description of such a type bracket is given, for example, in [34]. We will design the space $\mathfrak{g}[[\nu]]$ equipped with the bracket $[\ , \ ]_\nu$ by $\mathfrak{g}_\nu$. Its enveloping algebra defined naturally by

$$U(\mathfrak{g}_\nu) = T(\mathfrak{g}_\nu)[[\nu]]/\{x \otimes y - S(x \otimes y) - [x, y]_\nu\} \tag{2.4}$$

is filtered quadratic (more precisely, the ideal is generated by quadratic-linear elements).

Hereafter $T(V)$ stands for the free tensor algebra of a linear space $V$ and $\{I\}$ stands for its ideal generated by a subset $I \subset T(V)$.

We need also the algebra $A_{h, \nu} = U(\mathfrak{g}_\nu)_h$ defined by the formula (2.4) but with the bracket $[\ , \ ]_\nu$ replaced by $h[\ , \ ]_\nu$. The algebra $A_{h, \nu}$ is also filtered quadratic and moreover, possesses a twisted Hopf structure. Moreover, we have by construction the following

**Theorem 1** The two parameter family $A_{h, \nu}$ is a flat deformation of the algebra $\text{Sym}(\mathfrak{g}) = \text{Fun} (\mathfrak{g}^*)$. The corresponding Poisson pencil is just $\{I, \}^{KKS}$ where $\{I, \}^{KKS}$ is the linear extension of KKS bracket (Poisson-Lie one) and the bracket $\{I, \}^R$ is associated to the initial $R$-matrix.

By passing to the quotient $A_\nu = A_{h, \nu}/hA_{h, \nu}$ we get an $S$-commutative algebra which also is a flat deformation of the algebra $\text{Fun}(\mathfrak{g}^*)$. Let us precise that by this we mean an algebra $A = A_\nu$ equipped with an associative product $\mu : A^\otimes 2 \to A$ and an involutive twist $S : A^\otimes 2 \to A^\otimes 2$ such that $\mu S = \mu$ and $S \mu^{12} = \mu^{23} S^{12} S^{23}$. The latter relation signifies that the product $\mu$ is $S$-invariant.

Now, let $\mathfrak{g}$ be a simple Lie algebra. Then the enveloping algebra $\overline{H} = U(\mathfrak{g}_\nu)$ is isomorphic to $U(\mathfrak{g})[[\nu]]$. So, we can treat it as the algebra $U(\mathfrak{g})[[\nu]]$ but equipped with a new coproduct (still denoted by $\Delta$). Thus, we have equipped the algebra $U(\mathfrak{g})[[\nu]]$ with two deformed coassociative coalgebraic structures converting it respectively into an Hopf algebra $H$ and a twisted Hopf algebra $\overline{H}$.

However, in some sense the properties of the latter algebra are closer to those of the usual enveloping algebra $U(\mathfrak{g})$. In the first place it is due to the fact that the algebra $\overline{H}$ possesses a generating set formed by $\Delta$-primitive elements. Moreover, for this algebra its $S$-commutative analogue, i.e., the algebra $A_\nu$, is well defined and being equipped with the coproduct $\overline{\Delta}$ is still a twisted Hopf algebra, as in the classical case. The passage from the latter algebra to $A_{h, \nu}$ can be regarded as a twisted version of the quantization procedure of the linear Poisson-Lie bracket on $\mathfrak{g}^*$ consisting in a passage from the symmetric algebra of $\mathfrak{g}$ to the enveloping algebra $U(\mathfrak{g})$.

By means of $\Delta$-primitive elements it is not difficult to introduce the notion of twisted (or $S$-)vector fields: the twisted version of the Leibnitz rule for an involutive $S$ is well known. It is worth noticing that the twisted vector fields are just classical ones but their action on functions is deformed as follows:

$$\rho_\nu(X) \cdot a = \rho(\text{ad } F_{(1)}(X)) \cdot \rho(F_{(2)})a, \ X \in \mathfrak{g}, \ a \in \text{Fun} (M) \tag{2.5}$$

where $\rho : \mathfrak{g} \to \text{Vect} (M)$ is a representation of $\mathfrak{g}$ into the space of vector fields on a variety $M$ extended to $U(\mathfrak{g})$. 

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Unfortunately, the Hopf algebra \( H \) does not possess, in general, any generating set formed by \( \Delta F \)-primitive elements (the \( \Delta \)-primitive elements are no longer \( \Delta F \)-primitive). This is a reason why it is not so clear what is the natural analogue of the Leibnitz rule related to the quantum group \( H \) (although some palliative forms of "quantum Leibnitz rule" can be sometimes suggested).

Let us consider now the category of \( U(\mathfrak{g}) \)-modules. It can be equipped with the twist

\[
S^{U,V}_\nu = (\rho_V \otimes \rho_U)^{-1}\sigma(\rho_U \otimes \rho_V)F : U \otimes V[[\nu]] \to V \otimes V[[\nu]]
\]

where \( \rho_U \) is the representation of \( U(\mathfrak{g}) \) in \( U \). Thus, we have a twisted (symmetric monoidal in MacLane’s terminology) category consisting of the same objects as the initial one but equipped with a new transposition.

This twisted category can be regarded as that of \( H \)-modules and that of \( \overline{\Pi} \)-modules. However, the action of an element \( X \in \overline{\Pi} \) to a tensor product of two modules \( U \) and \( V \) must be defined in spirit of the formula (2.3) by means of the twist \( S_\nu \) transposing \( X(2) \) and \( U \) (here \( X(1) \otimes X(2) = \overline{\Delta}(X) \)). In particular, in this way we can deform all (generalized) Verma modules into twisted ones.

Let us remark that the renormalization procedure mentioned in the introduction (cf. also Section 3) has its twisted analogue. If in a classical case the map \( h_{\rho,\omega/h} \) sends \( U(\mathfrak{g}) \) into \( \text{End} \, M_\omega[[h]] \) (i.e., the image does not contain negative powers of \( h \)), in a deformed case such a property is satisfied only for an appropriated base in the deformed algebra. In the algebra \( \overline{\Pi} \) (which is isomorphic to \( H \) as an algebra) such a base is delivered by \( \overline{\Delta} \)-primitive elements.

Let us also mention the algebras dual to those \( H \) and \( \overline{\Pi} \). Both of them can be treated as deformations of the function algebra \( \text{Fun}(G) \) on the group \( G \). However, if the former one looks like the famous "RTT=TTR" algebra and possesses a Hopf algebra structure, the latter one looks like the reflection equation (RE) algebra. For involutive twists it has been introduced in \([G1], [G4]\) under the name of monoidal group. In more general setting RE algebras appear as dual objects of Majid’s braided groups, cf. \([M]\). Majid has also suggested a transmutation procedure converting one algebra to the other one. RE algebras associated to twists depending on a spectral parameter were considered in \([KS]\).

2.2 Let us pass to a quasitriangular case, i.e., that related to the QG \( U_q(\mathfrak{g}) \) where \( \mathfrak{g} \) is a complex simple Lie algebra. In this case there also exits a series \( F_\nu \) quantizing the R-matrix (1.4) in the above sense. However, the first equation (2.1) takes another form containing Drinfeld’s associator \( \Phi \) (cf. \([CP]\)). Moreover, the corresponding twist takes the form

\[
S = S_\nu = F^{-1}\sigma e^{it}F
\]

where \( t \) is the split Casimir.

In this case the Hopf algebra \( H \) can be constructed in the same way as above. It is just the famous QG \( U_q(\mathfrak{g}) \) but realized in an equivalent way as the algebra \( U(\mathfrak{g})[[\nu]] \) equipped with the deformed coproduct \( \Delta_F \) (we call this form of the QG \( U_q(\mathfrak{g}) \) its Drinfeld’s realization). However, the above construction of the twisted algebra \( \overline{\Pi} \) is no longer valid because the product \( \overline{\Delta} \) defined as above is not associative (the associativity default is due to the Drinfeld’s associator).

Nevertheless, a twisted Hopf algebra arising from the QG \( H \) exists: it can be obtained from \( H \) by means of a transmutation procedure which is dual to that mentioned above. In fact, this procedure does not deform the algebraic structure and transforms the coproduct \( \Delta_F \) into a new one \( \overline{\Delta} \) converting the QG into a "braided group".
However, this braided Hopf algebra is rather useless for us since it does not apparently possess any base of \( \Delta \)-primitive elements. In fact, instead of looking for an appropriated base in \( U_q(\mathfrak{g}) \) we construct another, complementary, algebra which possesses such a base. More precisely, we will introduce a space \( \mathfrak{g}_q \) being nothing but \( \mathfrak{g} \) itself equipped with an action \( U_q(\mathfrak{g}) \to \text{End} \mathfrak{g}_q \) of the GQ and represent the tensor algebra \( T(\mathfrak{g}_q) \) into a \( q \)-deformed generalized Verma module \( M_\omega^q \) with \( \omega = \mu \omega_1 \). Namely, the image of the algebra \( T(\mathfrak{g}_q) \) with \( \mu \) expressed via \( \hbar \) in a proper way provides us with the quantum counterpart of the Poisson pencil (1.3) on the \( \mathbb{C}P^n \) type orbits (in a classic case \( \hbar \) is proportional to \( \mu^{-1} \) but in quantum case their relation is a little bit more complicated). Hopefully, this method is valid for any symmetric orbit in \( \mathfrak{g}^* \) for any simple Lie algebra \( \mathfrak{g} \).

Note that although we do not embed the space \( \mathfrak{g}_q \) into the GQ \( U_q(\mathfrak{g}) \), such an embedding exists in \( sl(n) \) case in virtue of [LS]. Using this embedding the authors of [LS] have introduced a version of quantum Lie \( sl(n) \) bracket.

Completing this section we want to stress that in our approach the QG \( U_q(\mathfrak{g}) \) play an auxiliary role. We use it only to describe the category which our quantum algebras \( A_{\hbar,q} \) belong to. Let us note that in a case when such a category is related to nonquasiclassical twists mentioned in footnote 2 the algebras looking like \( A_{\hbar,q} \) can be constructed without any QG like objects. (In this case algebras of "RTT=TTR" type can be introduced in the usual way, cf. [G4], but their dual algebras differ drastically from the QG \( U_q(\mathfrak{g}) \). We refer the reader to the paper [KC] where an attempt to describe these algebras is undertaken.)

3 \( \mathbb{C}P^n \) type orbits and their quantization by generalized Verma modules

Let us realize now the first, classical, step of double quantization procedure for orbits in question.

Let \( \mathfrak{g} \) be a simple complex Lie algebra and \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- \) be a fixed triangular decomposition where \( \mathfrak{h} \) is a Cartan subalgebra and \( \mathfrak{n}_\pm \) are Borel subalgebras. Consider a nontrivial element \( \omega \in \mathfrak{h}^* \) and extend it by 0 to the subalgebra \( \mathfrak{n}_\pm \). Thus, we can treat \( \omega \) as an element of \( \mathfrak{g}^* \). Let \( \mathcal{O}_\omega \) be \( G \)-orbit of \( \omega \) in \( \mathfrak{g}^* \) where \( G \) is the Lie group corresponding to \( \mathfrak{g} \) acting on \( \mathfrak{g}^* \) by coadjoint operators and

\[
\{f,g\}_{KKS}(x) = \langle df, dg \rangle, \quad x \in \mathcal{O}_\omega
\]

be the KKS bracket on \( \mathcal{O}_\omega \).

It is well known that the orbit \( \mathcal{O}_\omega \) is a closed algebraic variety in \( \mathfrak{g}^* \). Moreover, the space of (polynomial) functions \( \mathcal{A} = \text{Fun} (\mathcal{O}_\omega) \) can be identified with a quotient \( T(\mathfrak{g})/\{I\} \) where \( I \) is some finit subset in \( T(\mathfrak{g}) \).

Thus, if \( \mathcal{O}_\omega \) is a generic semisimple orbit (this means that in the decomposition \( \omega = \sum \mu_i \omega_i \) where \( \omega_i \) are fundamental weights \( \mu_i \neq 0 \) for any \( i \) the family \( I \) consists of elements \( x_ix_j - x_jx_i, \ 1 \leq i, j \leq \dim \mathfrak{g} \) and \( C_i - c_i(\omega), \ 1 \leq i \leq \text{rank} \mathfrak{g} \) where \( C_i \) are invariant (Casimir) functions and \( c_i(\omega) \) are certain constants depending on \( \omega \).

Let us consider another example of such type orbits, namely, those in \( \mathfrak{g}^* = sl(n)^* \) of elements \( \omega = \mu \omega_1 \) or \( \omega = \mu \omega_{n-1} \) for some \( \mu \in \mathbb{C} \). These orbits (or more precisely, their real compact forms in \( su(n)^* \)) can be identified with \( \mathbb{C}P^{n-1} \). They are called \( \mathbb{C}P^n \) type orbits. It is well known that these orbits can be described by means of a system of quadratic equations. An explicit form of this system follows from the structure of \( \mathfrak{g}^{\otimes 2} \) as a \( \mathfrak{g} \)-module. Let us exhibit such an analysis.
Proposition 1 Let $\mathfrak{g} = \mathfrak{sl}(n)$, $n \geq 4$. Then highest weights of irreducible components of $\mathfrak{g}$-module $\mathfrak{g}^{\otimes 2}$ are:

$$2\omega_1 + 2\omega_{n-1}, \quad \omega_1 + \omega_{n-1}, \quad \omega_2 + 2\omega_{n-1}, \quad 2\omega_1 + \omega_{n-2}, \quad \omega_2 + \omega_{n-2}, \quad \text{and} \quad 0 \quad (3.1)$$

All the irreducibles from (3.1) occur in $\mathfrak{g}^{\otimes 2}$ with multiplicity one except the irreducible with highest weight $\omega_1 + \omega_{n-1}$ (being highest weight of $\mathfrak{g}$ itself) which occurs twice, once in the symmetric part $I_+$ of $\mathfrak{g}^{\otimes 2}$ and once in the skewsymmetric part $I_-.$

Note that in the $\mathfrak{sl}(2)$ case the decomposition (3.1) contains only the components with highest weights

$$0, \quad 2\omega_1, \quad 4\omega_1$$

all with multiplicity one and in the $\mathfrak{sl}(3)$ case the component of highest weight $\omega_2 + \omega_{n-2}$ does not appear.

Let us denote the finite dimensional irreducible $\mathfrak{g}$-module with highest weight $\lambda$ by $V_\lambda$. A corresponding highest weight vector (assuming $V_\lambda$ to be imbedded in $\mathfrak{g}^{\otimes 2}$) will be denoted by $s_\lambda$. For highest weight $\omega_1 + \omega_{n-1}$ which occurs twice in (3.1) we denote $V_{\omega_1 + \omega_{n-1}}$ (resp. $V^+_{\omega_1 + \omega_{n-1}}$) the component of highest weight $\omega_1 + \omega_{n-1}$ belonging to $I_+$ (resp. $I_-$). Their highest weight vectors will be designed by $s_{\omega_1 + \omega_{n-1}}^+$ (resp. $s_{\omega_1 + \omega_{n-1}}^-$). The precise expressions for the corresponding highest weight vectors are presented in Proposition 3 with a specialization $q = 1$.

Then the orbit under consideration can be defined by the following system (here $n \geq 4$, the cases $n = 2, 3$ are left to the reader)

$$V_{\omega_1 + 2\omega_{n-1}} = 0, \quad V_{2\omega_1 + \omega_{n-2}} = 0, \quad V_{\omega_1 + \omega_{n-1}}^+ = 0 \quad \text{(or, equivalently,} \quad x_ix_j - x_jx_i = 0 \quad \forall \ i, j) \quad (3.2)$$

$$V_{\omega_2 + \omega_{n-2}} = 0, \quad s_0 - c_0(\omega) = 0, \quad V_{\omega_1 + \omega_{n-1}}^+ - c_1(\omega)\mathfrak{g} = 0 \quad (3.3)$$

where $s_0 = C_1$ is a generator of the trivial module (Casimir element) and the constants $c_i(\omega)$, $i = 0, 1$ are:

$$c_0(\mu\omega_1) = \frac{n - 1}{n}\mu^2, \quad c_1(\mu\omega_1) = \frac{2n - 2}{n}\mu \quad (3.4)$$

(if we normalize $s_0$, $s_{\omega_1 + \omega_{n-1}}^1$ and $s_{\omega_1 + \omega_{n-1}}^2$ by (4.13)-(4.17) with $q = 1$ and put $s_{\omega_1 + \omega_{n-1}}^{+} = s_{\omega_1 + \omega_{n-1}}^1 + s_{\omega_1 + \omega_{n-1}}^2$). The last equation (3.3) is a symbolic form of the relation $s_{\omega_1 + \omega_{n-1}}^{+} = c_1(\omega)g_{1,n}$ and all the descendents of this relation.

Thus, we have $\mathcal{A} = \text{Fun}(\mathcal{O}_\omega) = T(\mathfrak{g})/\{I\}$ with the family $I \subset \mathbb{C} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}$ generated by the l.h.s. of the formulae (3.2), (3.3). So, the algebra $\mathcal{A}$ is filtred quadratic. (Note that this system was given in DG2 in a nonconsistent form.)

Since the orbit $\mathcal{O}_\omega$ is a symmetric space it is a spherical or multiplicity free variety, i.e. in the decomposition of the space $\text{Fun}(\mathcal{O}_\omega)$ into a direct sum of irreducibles their multiplicities are at most one.

It is well known that for the orbits of $\mathbb{C}P^n$ type

$$\text{Fun}(\mathcal{O}_\omega) \approx \bigoplus_{k=0}^{\infty} V_{k(\omega_1 + \omega_{n-1})}.$$ 

---

Let us remark that the only symmetric orbits corresponding to Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$ are

$$\mathcal{O}_x = SL(n)/S(L(k) \times L(n - k)), \quad 1 \leq k \leq n - 1$$

whose real compact forms are Grassmanians (the cases $k = 1$ and $k = n - 1$ correspond to $\mathbb{C}P^{n-1}$). Symmetric orbits in $\mathfrak{g}^*$ for other simple Lie algebras $\mathfrak{g}$ have been classified by E.Cartan (cf. [4], [KRR]).
Let us discuss now a way to quantize the KKS bracket well defined in the algebras \( A = \text{Fun}(\mathcal{O}_\omega) \) by means of generalized Verma modules.

Let \( K \) be the stabilizer of the point \( \omega \in g^* \). So, \( \mathcal{O}_\omega = G/K \). Let \( k = \text{Lie}(K) \) be the Lie algebra of the group \( K \) and \( p = k + n_+ \) be a parabolic subalgebra of \( g \). Let us consider the induced \( g \)-module

\[
M_\omega = \text{Ind}_{p}^{g} \mathbf{1}_\omega = U(g) \otimes_{U(p)} \mathbf{1}_\omega
\]

where \( \mathbf{1}_\omega \) is the one dimensional \( p \)-module equipped with a representation \( \rho_\omega(x)e = \langle \omega, x \rangle e, \; x \in p \) (\( e \) is a generator of the module). The \( g \)-module \( M_\omega \) is usually called generalized Verma module. Let us denote \( \rho_\omega : g \to \text{End} M_\omega \) the induced representation.

The operator algebra \( \text{End} M_\omega \) is quantum object with respect to the algebra of functions \( \text{Fun}(\mathcal{O}_\omega) \). To give an exact meaning to this statement let us introduce an associative algebra \( \mathbf{A}_h \) depending on a parameter \( \bar{\hbar} \). The elements \( x, \bar{\hbar} \rho_\omega/h : g \to \text{End} M_\omega \). We demonstrate it (in the sequel we omit this precision) Verma module. Let us denote \( \mathbf{A}_h \) as subalgebra of \( \text{End} M_\omega[[h]] \) being, by definition, the image \( \mathbf{A}_h(T(g)) \).

**Proposition 2** The algebra \( \mathbf{A}_h \) is a flat deformation of that \( A = \text{Fun}(\mathcal{O}_\omega) \) and the corresponding Poisson bracket is just the KKS one.

This statement is valid for any simple Lie algebra and for any ss orbit. We demonstrate it for the orbits of \( CP^n \) type where all the calculations can be easily done. In fact we will see that in this case the algebras \( A \) and \( \mathbf{A}_h[[h]] \) are isomorphic as \( g \)-modules (they are consisting of the same irreducibles with multiplicity one).

Let us first consider the finite dimensional \( sl(n) \)-modules \( V_\omega, \; \omega = \mu \omega_1, \; \mu \in \mathbb{Z}_+ \) where \( \mathbb{Z}_+ \) stands for the set of nonnegative integers. Such a module can be naturally identified with symmetric power of the vector fundamental space \( V_\omega \) (in the \( sl(2) \) case the factor \( \mu \) is just spin of the module). Its dimension is equal to \( \binom{\mu + n - 1}{n - 1} \).

Let us fix in the space \( V_\omega \) the base

\[
| m_1, \ldots, m_n > = x_1^{m_1} \cdots x_n^{m_n}, \; \sum m_i = \mu.
\]

(3.5)

Let \( h_i \in h, \; e_i \in n_+, \; f_i \in n_- \), \( 1 \leq i \leq n - 1 \) be a standard Chevelley base in the Lie algebra \( sl(n) \). The elements \( h_i, e_i, f_i \) act in the module \( V_\omega \) as first order differential operators

\[
e_i = x_i \frac{\partial}{\partial x_{i+1}}, \quad f_i = x_{i+1} \frac{\partial}{\partial x_i}, \quad h_i = x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}}.
\]

(3.6)

In the base (3.5) the operators (3.6) look like

\[
e_i | m_1, \ldots, m_n > = m_i | m_1, \ldots, m_i+1, m_{i+1} - 1, \ldots, m_n >,
\]

\[
f_i | m_1, \ldots, m_n > = m_i | m_1, \ldots, m_i-1, m_{i+1} + 1, \ldots, m_n >,
\]

\[
h_i | m_1, \ldots, m_n > = ( m_i - m_{i+1} ) | m_1, \ldots, m_i, m_{i+1}, \ldots, m_n >.
\]

(3.7)

It is well known that \( sl(n) \)-module \( \text{End} V_{\mu \omega_1} \), \( \mu \in \mathbb{Z}_+ \) is isomorphic to the following multiplicity free direct sum:

\[
\text{End} V_{\mu \omega_1} \cong \bigoplus_{k=0}^{\mu} V_{k(\omega_1 + \omega_n - 1)}.
\]

(3.8)
Let us pass now to the Verma module $M_\omega, \omega = \mu\omega_1, \mu \in \mathbb{C}$. Similarly to the above finite dimensional modules $V_\omega$ it possesses the following base

$$|m_1, \ldots, m_n>, \quad \sum m_k = \mu, \quad m_k \in \mathbb{Z}_+, \ k = 2, \ldots, n.$$  \hfill (3.9)

Thus, the elements of the base (3.9) are labeled by the vectors $(m_2, \ldots, m_n), \ m_k \in \mathbb{Z}_+. \ \text{The action of } sl(n) \ \text{on } M_{\mu\omega_1} \ \text{is given by the formulae} (3.6)-(3.7) \ \text{as well.}

The formula (3.8) must be modified as follows

$$\text{Im } \rho_\omega(T(\mathfrak{g})) \approx \bigoplus_{k=0}^{\infty} V_{k(\omega_1+\omega_{n-1})}.$$  \hfill (3.10)

We have replaced $\text{End} V_{\mu\omega_1}$ by $\text{Im } \rho_\omega(T(\mathfrak{g}))$ since for infinite dimensional modules the map $\rho_\omega$ is no longer surjective.

Let us now go back to the algebra $A_{\bar{h}} = \text{Im } \overline{\rho}_h(T(\mathfrak{g})) \subset \text{End} M_\omega[[h]]$. By using the decomposition (3.10) it is easy to show that $A_{\bar{h}}$ is a flat deformation of the algebra $A$. The map $\overline{\rho}_h$ sends the Chevalley generators into operators acting with respect to the formulae (3.7) but with $m_2, \ldots, m_n$ replaced by $hm_2, \ldots, hm_n$. The commutators between the images of the Chevalley generators are those in $sl(n)$ multiplied by $\bar{h}$. This implies that the corresponding Poisson bracket is equal to the KKS one.

Let us represent now the algebra $A_{\bar{h}}$ as a quotient

$$A_{\bar{h}} = \text{Im } \overline{\rho}_h(T(\mathfrak{g})) = T(\mathfrak{g})[[h]]/\text{Ker } \overline{\rho}_h.$$  

In the case under consideration ($\mathfrak{g} = sl(n), \ \omega = \mu\omega_1$) this quotient is also a quadratic algebra. More precisely, the ideal $\text{Ker } \overline{\rho}_h$ is generated by a finite family $I_h \in \mathbb{C} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}$, looking like that defined by the l.h.s. of (3.2)-(3.3) but with some evident modifications: the elements $x_ix_j - x_jx_i$ must be replaced by those $x_ix_j - x_jx_i - \bar{h}[x_i, x_j]$ and the factors $c_i(\omega), \ i = 0, 1$ must be deformed to those $c_i(\omega, h)$ depending on $h$ (with $c(\omega, h) = c(\omega) \mod \bar{h}$). Namely, with $s_0$ and $s_{\omega_1}^{\omega_1+\omega_{n-1}}$ normalized as in (3.4) we have

$$c_0(\mu\omega_1, h) = \frac{n-1}{n} \mu(\mu + n\bar{h}), \quad c_1(\mu\omega_1, h) = \frac{n-2}{n} (2\mu + n\bar{h}).$$

4 Braided algebras

Our next aim is to braid the above quantization procedure. Let us begin with a description of the space $\mathfrak{g}_q$ mentioned in Section 2.

Let $U_q(sl(n))$ be the quantum enveloping algebra corresponding to $sl(n)$. In Chevalley generators $e_i, f_i, h_i, \ i = 1, \ldots, n$ it could be described by the relations

$$[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i,$$  \hfill (4.1)

$$[h_i, e_{i\pm 1}] = -e_{i\pm 1}, \quad [h_i, f_{i\pm 1}] = f_{i\pm 1}$$  \hfill (4.2)

$$[h_i, e_j] = [h_i, f_j] = 0, \quad |i - j| > 1,$$  \hfill (4.3)

$$[e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$  \hfill (4.4)

$$e_i^2 e_{i\pm 1} - [2]_q e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0,$$  \hfill (4.5)

$$f_i^2 f_{i\pm 1} - [2]_q f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0,$$  \hfill (4.6)
with
\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad q^{\alpha h_i} = \exp(\nu \alpha h_i). \]

We choose a comultiplication map as follows:
\[ \Delta h_i = h_i \otimes 1 + 1 \otimes h_i, \quad \Delta e_i = e_i \otimes 1 + q^{-h_i} \otimes e_i, \quad \Delta f_i = 1 \otimes f_i + f_i \otimes q^{h_i}. \]

Then the antipode has a form:
\[ s(h_i) = -h_i, \quad s(e_i) = -q^{h_i} e_i, \quad s(f_i) = -f_i q^{-h_i}. \]

Let \( g_\mathbb{Q} \) be a \( q \)-analogue of an adjoint representation of Lie algebra \( sl(n) \) on itself, i.e., \( g_\mathbb{Q} \) is \( (n^2 - 1) \)-dimensional \( U_q(sl(n)) \)-module with highest weight \( \omega_1 + \omega_{n-1} \). We want to describe the action of the QG \( U_q(sl(n)) \) to \( g_\mathbb{Q} \) explicitly in a fixed base of \( g_\mathbb{Q} \). We denote further this action by \( \text{ad} = \text{ad}_q \). Namely, the vector space \( g_\mathbb{Q} \) is generated by the elements \( g_{i,j}, i, j = 1, \ldots, n, i \neq j \) and \( t_i, i = 1, \ldots, n - 1 \). An action of Cartan elements coincides with the classical one:
\[ \text{ad} h_i (t_k) = 0, \]
\[ \text{ad} h_i (g_{k,l}) = (\delta_{i,k} - \delta_{i,l} - \delta_{i+1,k} + \delta_{i+1,l}) g_{k,l}. \]

Nontrivial matrix coefficients of the action of Chevalley generators \( e_i \) and \( f_i \) of \( U_q(sl(n)) \) look as follows:
\[ \text{ad} e_i (g_{a,i+1}) = -g_{a,i+1}, \quad \text{ad} e_i (g_{i+1,a}) = g_{i,a}, \quad a \neq i, i + 1, \]
\[ \text{ad} e_i (g_{i+1,i}) = t_i, \quad \text{ad} e_i (t_i) = -[2]_q g_{i,i+1}, \quad \text{ad} e_i (t_{i+1}) = g_{i,i+1}, \]
\[ \text{ad} f_i (g_{a,i}) = -g_{a,i}, \quad \text{ad} f_i (g_{i,a}) = g_{i+1,a}, \quad a \neq i, i + 1, \]
\[ \text{ad} f_i (g_{i,i+1}) = -t_i, \quad \text{ad} f_i (t_i) = [2]_q g_{i,i+1}, \]
\[ \text{ad} f_i (t_{i+1}) = -g_{i+1,i}. \]

So, we get the matrix coefficients of this action from the classical ones replacing the coefficient 2 by its \( q \)-analogue \([2]_q = q + q^{-1}\).

Since \( g_\mathbb{Q} \) is a \( U_q(sl(n)) \)-module, the tensor algebra \( T(g_\mathbb{Q}) \) can be equipped with a \( U_q(sl(n)) \)-invariant product in sense of [1](7). In the what follows the algebra \( T(g_\mathbb{Q}) \) and all its \( U_q(g) \)-invariant quotients will be called braided ones.

In fact, the braided algebra \( T(g_\mathbb{Q}) \) is "too big" for us. We are rather interested in its quotient over the kernel of map sending this algebra into \( \text{End} \, M_3^\mathbb{Q} \) where \( M_3^\mathbb{Q} \) is a \( q \)-analogue of the above Verma modules with \( \omega = \mu \omega_1 \). Namely this quotient with \( \mu \) properly expressed via the parameter \( h \) plays the role of our "double quantum" object \( A_{h,q} \).

Let us describe the mentioned kernel. To do this we need a decomposition of the \( U_q(sl(n)) \)-module \( g_\mathbb{Q}^{\otimes 2} \) into a direct sum of irreducibles.

**Proposition 3** The formulas below describe all highest weight vectors of reducibles in \( U_q(sl(n)) \)-module \( g_\mathbb{Q}^{\otimes 2} \) \( (n \geq 4) \):

\[ s_{2\omega_1 + 2\omega_{n-1}} = g_{1,n} \otimes g_{1,n}, \]
\[ s_{2\omega_1 + \omega_{n-2}} = g_{1,n} \otimes g_{1,n-1} - q^{-1} g_{1,n-1} \otimes g_{1,n}, \]
\[ s_{2\omega_2 + 2\omega_{n-1}} = g_{1,n} \otimes g_{2,n} - q^{-1} g_{2,n} \otimes g_{1,n}, \]
\[ s_{\omega_2 + \omega_{n-2}} = gg_{1,n} \otimes g_{2,n-1} + q^{-1} g_{2,n-1} \otimes g_{1,n} - g_{1,n-1} \otimes g_{2,n} - g_{2,n} \otimes g_{1,n-1}, \]
\[ s_{\omega_1+\omega_{n-1}}^1 = g_{1,2} \otimes g_{2,n} + q g_{1,3} \otimes g_{3,n} + \ldots + q^{n-3} g_{1,n-1} \otimes g_{n-1,n} + \]

\[+ q^{-2} \sum_{k=1}^{n-1} \frac{[n-k]}{[n]} q t_k \otimes g_{1,n} - q^{-2} \sum_{k=1}^{n-1} g_{1,n} \otimes \frac{[k]}{[n]} q t_k , \quad (4.15) \]

\[ s_{\omega_1+\omega_{n-1}}^2 = g_{2,n} \otimes g_{1,2} + q^{-1} g_{3,n} \otimes g_{1,3} + \ldots + q^{-n+3} g_{n-1,n} \otimes g_{1,n-1} - \]

\[ - q^{-1-n} \sum_{k=1}^{n-1} \frac{[k]}{[n]} q t_k \otimes g_{1,n} + q \sum_{k=1}^{n-1} g_{1,n} \otimes \frac{[n-k]}{[n]} q t_k , \quad (4.16) \]

\[ s_0 = \sum_{i,j=1,i \leq j}^{n-1} \frac{i q[n-j]}{[n]} q t_i \otimes t_j + \sum_{i,j=1,i > j}^{n-1} \frac{j q[n-j]}{[n]} q t_i \otimes t_j + q \sum_{i < j} q^{i-j} g_{i,j} \otimes g_{j,i} + q^{-1} \sum_{i > j} q^{i-j} g_{i,j} \otimes g_{j,i} . \quad (4.17) \]

Because of multiplicity in this decomposition of the highest weight \( \omega_1 + \omega_{n-1} \) component it is not so clear what are natural \( q \)-analogues \( I^q_\pm \) of symmetric \( I_+ \) and skew-symmetric \( I_- \) components in \( g^{\otimes 2}_q \) (except the \( sl(2) \) case)\(^4\).

Let us consider a way to introduce a decomposition \( g^{\otimes 2}_q = I^q_+ \oplus I^q_- \) arising from an operator \( \tilde{S} \) discussed in [DS] and [DG2]. In Drinfeld’s realization of the QC \( U_q(\mathfrak{g}) \) the operator \( \tilde{S} \) is defined by the formula (2.8) but without the factor \( e^{it} \). So, it is evident that this operator is involutive. Moreover, being restricted on \( g^{\otimes 2}_q \) it has the same eigenspaces as the YB operator \( S \) has but with eigenvalues \( \pm 1 \). Namely, to pass from \( S \) to \( \tilde{S} \) we must replace the eigenvalues of \( S \) close to 1 (resp. -1) by 1 (resp -1) assuming that \( |q-1| \ll 1 \).

We complete this Section with describing the action of \( \tilde{S} \) on the isotypical component of highest weight \( \omega_1 + \omega_{n-1} \) (its action on other components of \( g^{\otimes 2}_q \) contains no new information for us). We use this computation in the last Section.

To do this we need a partial information on the quantum universal \( R \)-matrix in an \( sl(n) \) case.

It is well known that the universal \( R \)-matrix \( R \) for the algebra \( U_q(\mathfrak{g}) \) can be presented by

\[ R = R_0 \cdot q \sum c_{i,j} h_i \otimes h_j , \quad (4.18) \]

where \( (c_{i,j}) \) is the matrix inverse to the Cartan matrix of \( \mathfrak{g} \) and \( R_0 \) belongs to a tensor product of quantized enveloping algebras of nilpotent subalgebras \( \mathfrak{n}_\pm \) of \( \mathfrak{g} \): \( R_0 \in U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{n}_-) \).

Moreover, \( R_0 = 1 \mod \mathfrak{n}_+ U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{n}_-) \).

In \( sl(n) \) case formula (4.18) has especially simple form after embedding of \( R \) into \( U_q(sl_n) \otimes U_q(gl_n) \):

\[ R = R_0 \cdot q \sum_{i=1}^{n-1} \varepsilon_i \otimes \varepsilon_i - \frac{1}{2} (\sum_{i=1}^{n} \varepsilon_i) \otimes (\sum_{i=1}^{n} \varepsilon_i) , \quad (4.19) \]

where \( h_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1 \).

Let us compute the expressions \( S(s_{\omega_1+\omega_{n-1}}^i), i = 1, 2 \), where \( S = \sigma(ad \otimes ad)R \) is the image of the universal \( R \)-matrix in tensor square of adjoint representation multiplied by the flip. The commutativity of \( S \) and \( \Delta(x) \) implies that \( S(s_{\omega_1+\omega_{n-1}}^i) = \sum_j a_{i,j} s_{\omega_1+\omega_{n-1}}^j, i, j = 1, 2 \) for some constants \( a_{i,j} \).

\(^4\)For other simple Lie algebras \( \mathfrak{g} \) such natural \( q \)-analogues \( I^q_\pm \) exist since \( g^{\otimes 2}_q \) is multiplicity free as a \( \mathfrak{g} \)-module but deformations \( T(\mathfrak{g})/\{I_\pm\} \rightarrow T(\mathfrak{g}_q)/\{I^q_\pm\} \) are not flat. In \( sl(n) \) case one can split \( g^{\otimes 2}_q \) into a direct sum of components \( I^q_\pm \) in such a way that these deformations are flat. It is shown in [DS] by means of an embedding \( \mathfrak{g}_q \rightarrow U_q(\mathfrak{g}) \) which is slightly different from that considered in [LS].
The space $g_q$ can be decomposed into three parts:

$$g_q = g_+ + t + g_-$$

where $g_+$ is generated by the vectors $g_{i,j}$, $i < j$; $g_-$ is generated by the vectors $g_{i,j}$, $i > j$; and $t$ is generated by elements $t_i$. Their crucial properties are:

$$U_q(n_\pm)t \subset g_\pm \text{ and } U_q(n_\pm)g_\pm \subset g_\pm.$$ 

One can observe from the explicit expressions for $s^i_{\omega_1+\omega_{n-1}}$ that

$$s^1_{\omega_1+\omega_{n-1}} = q^{-2} \sum_{k=1}^{n-1} \frac{[n-k]_q}{[n]_q} t_k \otimes g_{1,n} + \sum x_i \otimes y_i \ , \quad x_i \in g_+ , \quad (4.20)$$

$$s^2_{\omega_1+\omega_{n-1}} = q \sum_{k=1}^{n-1} \frac{[n-k]_q}{[n]_q} g_{1,n} \otimes t_k + \sum u_i \otimes v_i \ , \quad v_i \in g_+ . \quad (4.21)$$

Due to (4.19) we have

$$S(s^1_{\omega_1+\omega_{n-1}}) = q^{-3} s^2_{\omega_1+\omega_{n-1}} . \quad (4.22)$$

Analogously,

$$S(s^2_{\omega_1+\omega_{n-1}}) = q^{3-2n} s^1_{\omega_1+\omega_{n-1}} . \quad (4.23)$$

Formulas (4.22) and (4.23) show that the operator $S$ is diagonal on isotypical component $V^q_{\omega_1+\omega_{n-1}} \oplus V^q_{\omega_1+\omega_{n-1}}$ (as well as in the whole space $g_q^{\otimes 2}$), has on it eigenvalues $\pm q^{-n}$, and the corresponding eigenvectors are

$$s_\pm = q^{2-n} s^1_{\omega_1+\omega_{n-1}} \pm q^{-1} s^2_{\omega_1+\omega_{n-1}} . \quad (4.24)$$

This result is true for $n \geq 3$. The case $n = 2$ is left for the reader (here $s_\pm = 0$).

5 Braided modules

**Definition 1** We say that $M$ is a braided $T(g_q)$–module (or, simply braided module), if $M$ is equipped with a structure of $U_q(sl(n))$–module and of $T(g_q)$–module, and these structures are related as

$$u \cdot (gm) = \left(u^{(1)} \cdot g\right) \left(u^{(2)} \cdot m\right) \quad (5.1)$$

for any $u \in U_q(sl(n))$, $g \in T(g_q)$ and $m \in M$.

The braided algebra $T(g_q)$ together with the category of its braided representations can be described also in a language of intertwining operators. Let $M$ be a $U_q(sl(n))$–module. Then, by definition, the (second type) intertwining operator $\Psi^g_q$ is a $U_q(sl(n))$–morphism.

$$\Psi^g_q : g_q \otimes M \rightarrow M \quad (5.2)$$
The components \( \Psi_{a}^{g_{q}} : M \to M \) are defined via fixing a base \( g_{a} \) in \( \mathfrak{g}_{q} \):

\[
\Psi_{a}^{g_{q}}(m) = \Psi^{g_{q}}(g_{a} \otimes m).
\] (5.3)

Thus, \( M \) is a braided \( T(\mathfrak{g}_{q}) \)-module if and only if there exists an action of intertwining operator \( \Psi^{g_{q}} \) on \( M \) in the above sense.

Our next aim is to perform an explicit construction of certain braided modules. More precisely, we will define a \( U_{q}(\mathfrak{sl}(n)) \)-morphism

\[
T(\mathfrak{g}_{q}) \to \text{End} V_{\omega}^{q}, \quad \omega = \mu \omega_{1}, \quad \mu \in \mathbb{Z}_{+}
\] (5.4)

where \( V_{\omega}^{q} \) is a \( q \)-deformed finite dimensional module. After that we will extend this construction to the \( q \)-deformed Verma modules.

Since the module \( \text{End} V_{\omega} \) is multiplicity free the component isomorphic to \( \mathfrak{g} \) is represented in it only once. The same is true for the \( U_{q}(\mathfrak{sl}(n)) \)-module \( V_{\omega}^{q} \). This enables us to define the \( U_{q}(\mathfrak{g}) \)-modules possessing these two properties were called in [G 5] braided, here we use this term in more general sense.

Let us describe now the map (5.4) explicitly. Since the algebra \( T(\mathfrak{g}_{q}) \) is generated by the space \( \mathfrak{g}_{q} \) and \( U_{q}(\mathfrak{sl}(n)) \) is generated by Chevalley base, it suffices to ensure the relation (5.1) for \( u = h_{i}, e_{i}, f_{i} \) and for \( g \in \mathfrak{g}_{q} \). Below we write down these relations using the following traditional notation. Let \( M \) be a \( U_{q}(\mathfrak{sl}(n)) \)-module and for \( x \in M \) be an eigenvector of the action of Cartan subalgebra \( \mathfrak{h} \) of \( U_{q}(\mathfrak{sl}(n)) \). Then we denote its eigenvalue by \( \lambda(x) \in \mathfrak{h}^{\ast} \) such that

\[
h_{i}(x) = (\varepsilon_{i} - \varepsilon_{i+1}, \lambda(x)) \cdot x, \quad \varepsilon_{i} = \text{diag} (0, \ldots, 0, 1, \ldots, 0), \quad 1 \text{ at } i\text{-th place}.
\]

For instance, the weights \( \lambda(g) \) of the representation \( \text{ad} \) in a space \( \mathfrak{g}_{q} \) coincide with the classical ones:

\[
\lambda(g_{i,j}) = \varepsilon_{i} - \varepsilon_{j}, \quad \lambda(t_{i}) = 0.
\]

and the action of the Cartan elements in \( \mathfrak{g}_{q} \) is given by the relation \( \text{ad} h_{i}(g) = (\varepsilon_{i} - \varepsilon_{i+1}, \lambda(g)) \cdot g \).

**Proposition 4** Let \( M \) be a finite dimensional \( U_{q}(\mathfrak{sl}(n)) \)-module. Then \( M \) is braided \( T(\mathfrak{g}_{q}) \)-module if and only if for any \( g \in \mathfrak{g}_{q} \) the following relations hold:

\[
[h_{i}, g] = \text{ad} h_{i}(g),
\] (5.5)

\[
[e_{i}, g]_{q}^{-}(\varepsilon_{i} - \varepsilon_{i+1}, \lambda(g)) = \text{ad} e_{i}(g),
\] (5.6)

\[
[f_{i}, g] = \text{ad} f_{i}(g) \cdot q^{h_{i}}.
\] (5.7)

Here

\[
[a, b]_{q} = ab - qba
\]

and all brackets are understood in operator sense.

**Proof** It suffices to say that for any finite dimensional \( U_{q}(\mathfrak{sl}(n)) \)-module \( M \) we identify

\[
\text{End} M = M \otimes M^{\ast}
\]
as left $U_q(sl(n))$-modules where an action of $U_q(sl(n))$ to $M^*$ is defined by means of an antipode $s$:

$$ (v, u \cdot \xi) = (s(u) \cdot v, \xi), \quad v \in M, \xi \in M^* \; u \in U_q(sl(n)). $$

The rest is a substitution of (4.7) in (5.1).

Let $V_{\omega_1}^q$ be the first (vector) fundamental representation of the algebra $U_q(sl(n))$. Similarly to classical case let us consider the irreducible finite dimensional representations $V_{\mu \omega_1}^q, \mu \in \mathbb{Z}_+$ of $U_q(sl(n))$ with highest weights $\mu \omega_1$ (in what follows we will omit $q$).

One can easily check that the operators $e_i, f_i, h_i \in \text{End} \ V_{\mu \omega_1}$, whose nontrivial matrix elements are described in (5.8) satisfy the relations (4.1)-(4.6) and thus define an action of the algebra $U_q(sl(n))$ in vector space $V_{\mu \omega_1}$:

$$ e_i \mid m_1, \ldots, m_n > = [m_{i+1}]_q \mid m_1, \ldots, m_i + 1, m_{i+1} - 1, \ldots, m_n > , 
$$

$$ f_i \mid m_1, \ldots, m_n > = [m_i]_q \mid m_1, \ldots, m_i - 1, m_{i+1} + 1, \ldots, m_n > , 
$$

$$ h_i \mid m_1, \ldots, m_n > = (m_i - m_{i+1}) \mid m_1, \ldots, m_i, m_{i+1}, \ldots, m_n > . 
$$

**Proposition 5** There is a unique structure (up to a multiplicative constant $\alpha \in \mathbb{C}$) of braided $T(g_q)$-module on $U_q(sl(n))$-module $V_{\mu \omega_1}$.

The action of generators of $T(g_q)$ in braided module $V_{\mu \omega_1}$ is:

$$ g_{i,j} \mid m_1, \ldots, m_n > = \alpha(\mu) \ q^{i+(m_1+\ldots+m_i)-(m_{j+1}+\ldots+m_n)} \cdot [m_j]_q \mid m_1, \ldots, m_i + 1, \ldots, m_{j-1}, \ldots, m_n > 
$$

for $i < j$

$$ g_{j,i} \mid m_1, \ldots, m_n > = \alpha(\mu) \ q^{-1+(m_1+\ldots+m_{i-1})-(m_{j+1}+\ldots+m_n)} \cdot [m_i]_q \mid m_1, \ldots, m_i - 1, \ldots, m_{j+1}, \ldots, m_n > 
$$

for $i < j$

$$ t_i \mid m_1, \ldots, m_n > = \alpha(\mu) \ q^{i+(m_1+\ldots+m_{i-1})-(m_{i+2}+\ldots+m_n)} \cdot \frac{[2]_q q^{m_i-m_{i+1}} - q^{m_i+m_{i+1}} - q^{-m_i-m_{i+1}-1}}{q - q^{-1}} \mid m_1, \ldots, m_n > . 
$$

**Proof** The proof goes by induction on the rank $n$. Let us start from the $U_q(sl(2))$ case. In this case the relation (5.7) implies that

$$ [g_{2,1}, f_1] = 0 . 
$$

From the commutation relation (5.5) of $g_{2,1}$ with Cartan element we see that in addition $g_{2,1}$ has the same matrix structure as $f_1$ and thus these two operators are proportional to each other. Applying twice the relation (5.6) we get the description of the operators $t_1$ and $g_{1,2}$. Finally we check that all the relations (5.5)-(5.7) are satisfied.

The passage to $U_q(sl(3))$ looks as follows. We know from $sl(2)$ case that the operator $g_{2,1}$ has a form

$$ g_{2,1} = \alpha(m_3) f_1 . 
$$

We wish to find out the normalization constant $\alpha(m_3)$. Applying the following particular cases of (5.4) and of (5.7):

$$ g_{3,1} = [f_2, g_{2,1}] q^{-h_2}, \quad g_{3,2} = -[e_1, g_{3,1}] $$

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to the ansatz (5.13), we get the description of the operator \( g_{3,2} \), depending on a choice of \( \alpha(m_{3}) \).

But we know again that

\[
g_{3,2} = \alpha(m_{1}) f_{2}.
\]

This gives a recurrence equation on \( \alpha(m_{3}) \) which unique (up to a constant factor) solution is \( \alpha(m_{3}) = q^{-m_{3}} \). Then we get from (5.7) the description of others generators of \( g_{q} \) in \( sl(3) \) case.

The general induction step is similar.

Let us pass now to \( q \)-deformed Verma modules. Let \( M_{\omega} = M_{\omega}^{q} \) be such a module. It also possesses a base labeled by \( (m_{2}, \ldots, m_{n}) \), \( m_{i} \in \mathbb{Z}_{+} \). For any \( \mu \in \mathbb{C} \) there exists a \( U_{q}(sl(n)) \)-invariant map

\[
\overline{\rho}_{\omega} : T(g_{q}) \rightarrow \text{End} M_{\omega}, \omega = \mu \omega_{1}.
\]

This map is also defined (uniquely up to a factor) by the formulae (5.9)–(5.11).

We are interested now in the ideal \( \text{Ker} \overline{\rho}_{\omega} \). It is also generated by its quadratic part \( I_{q}(\mu, \alpha) = \text{Ker} \overline{\rho}_{\omega} \cap (\mathbb{C} \oplus g_{q} \oplus g_{q}^{\otimes 2}) \). To describe this quadratic part we consider the images of the highest weight vectors \( s_{0}, s_{\omega_{1}+\omega_{n-1}} \) and \( s_{\omega_{1}+\omega_{n-1}}^{2} \) in \( g_{q}^{\otimes 2} \) with respect to the map \( \overline{\rho}_{\omega} \).

The operator \( s_{0} \) is a scalar (we omit the symbol \( \overline{\rho}_{\omega} \)):

\[
s_{0} = \alpha(\mu)^{2} q^{n} \frac{[n-1]_{q} [\mu]_{q} [\mu + n]_{q}}{[n]_{q}} \text{Id},
\]

and the operators \( s_{\omega_{1}+\omega_{n-1}}^{1} \) and \( s_{\omega_{1}+\omega_{n-1}}^{2} \) are proportional to the operator \( g_{1,n} \):

\[
s_{\omega_{1}+\omega_{n-1}}^{1} = \alpha(\mu) q^{n-2} \frac{[n-1]_{q} [\mu]_{q} [\mu + n]_{q} - [\mu]_{q}}{[n]_{q}} g_{1,n}, \tag{5.16}
\]

\[
s_{\omega_{1}+\omega_{n-1}}^{2} = \alpha(\mu) q^{n} \frac{[n-1]_{q} [\mu]_{q} [\mu + n]_{q}}{[n]_{q}} g_{1,n}. \tag{5.17}
\]

Finally, we have the following description of \( I_{q}(\mu, \alpha) \).

**Proposition 6** The subspace \( I_{q}(\mu, \alpha) \subset (\mathbb{C} \oplus g_{q} \oplus g_{q}^{\otimes 2}) \) is a \( U_{q}(sl(n)) \)-module generated by

(i) the highest weight vectors \( s_{2\omega_{1}+\omega_{n-2}}, s_{\omega_{2}+2\omega_{n-1}}, s_{\omega_{2}+\omega_{n-2}} \)

(ii) the following combinations of highest weight vectors:

\[
s_{\omega_{1}+\omega_{n-1}}^{1} = \alpha(\mu) q^{n-2} \frac{[n-1]_{q} [\mu]_{q} [\mu + n]_{q} - [\mu]_{q}}{[n]_{q}} g_{1,n}, \tag{5.18}
\]

\[
s_{\omega_{1}+\omega_{n-1}}^{2} = \alpha(\mu) q^{n} \frac{[n-1]_{q} [\mu]_{q} [\mu + n]_{q}}{[n]_{q}} g_{1,n}. \tag{5.19}
\]

\[
s_{0} = \alpha(\mu)^{2} q^{n} \frac{[n-1]_{q} [\mu]_{q} [\mu + n]_{q}}{[n]_{q}} g_{1,n} + 1, \tag{5.20}
\]

Therefore for the elements \( s_{\pm} \) defined by (4.24) we have the following formula

\[
s_{\pm} = \alpha(\mu) q^{n} \frac{[n-1]_{q} [\mu]_{q} [\mu + n]_{q} + 1}{[n]_{q}} g_{1,n}. \tag{5.21}
\]
6 The algebra $A_{h,q}$ and quantum CP$^n$ type orbits

Let us define now the two parameter algebra $A_{h,q}$ using the results of the previous Sections. To do this we must express $\mu$ via $\hbar$ and choose the factor $\alpha(\mu)$ in a proper way. In the classical case ($q = 1$) by setting $\hbar = \mu^{-1}$, $\alpha(\mu) = \alpha_0 h$ we get an algebra which differs from the above algebra $A_h$ by a renormalization of the parameter. Thus, the algebra $A_h/hA_h$ is just function algebra on the corresponding orbit (labeled by $\alpha_0$).

In the quantum case ($q \neq 1$) we suppose that $|q| \neq 1$. This condition is motivated by our desire to have $|\mu| \to \infty$ as $\mu \to \infty$.

Let us set

$$\alpha(\mu) = \frac{\alpha_0}{|\mu|} \text{ and } \frac{|\mu + n|_q}{|\mu|_q} = \gamma(q) + \hbar, \quad (6.1)$$

where $\gamma(q) = q^n$ if $|q| > 1$ and $\gamma(q) = q^{-n}$ if $|q| < 1$. We use (6.1) as the definition of the parameter $\hbar$.

Then the elements (5.18)-(5.20) become

$$s^1_{\omega_1+\omega_{n-1}} = \alpha_0 q^{n-2} \left( \frac{n-1}{n} q^{\gamma(q) + \hbar} - 1 \right) g_{1,n}, \quad (6.2)$$

$$s^2_{\omega_1+\omega_{n-1}} = \alpha_0 q \left( \frac{n-1}{n} q - (\gamma(q) + \hbar) \right) g_{1,n}, \quad (6.3)$$

$$s_0 = \alpha_0^2 q^n \left( \frac{n-1}{n} q (\gamma(q) + \hbar) + 1 \right). \quad (6.4)$$

Meanwhile, the element $s_-$ defined by the formula (5.21) takes the form

$$s_- = \alpha_0 \left( \frac{n-1}{n} q + 1 \right) \left( (\gamma(q) + \hbar) - 1 \right) g_{1,n}. \quad (6.5)$$

Let us introduce now the algebra $A_{h,q}$ as the quotient of $T(g_q)$ by the ideal generated by the elements listed in Proposition 6 (i), the elements (5.2)- (5.4) and all their descendants. Expressing $\mu$ via $\hbar$ and substituting it in the formulae (5.9)-(5.11) we can realize the algebra $A_{h,q}$ as some subalgebra in End $V_q[[h]]$ with a fixed $\omega$ ($\alpha(\mu)$ is assumed to be expressed via the formula (6.1)).

This operator realization of the algebra $A_{h,q}$ implies that deformation $A \to A_{h,q}$ is flat. In fact, it suffices to note that the algebra $A_{h,q}$ contains all components $V_k(\omega_1+\omega_{n-1})$, $k = 0, 1, 2, \ldots$. This follows from the fact that the images of the elements $g_{1,n}^k$ are not trivial operators for any $k = 0, 1, 2, \ldots$ (recall that $|q| \neq 1$ and therefore $q$ is not any root of the unity).

The arguments analogous to Nakayama lemma (cf. [AM]) show that the algebra $A_q = A_{h,q}/hA_{h,q}$ also contains all components $V_k(\omega_1+\omega_{n-1})$ and therefore the deformation $A \to A_q$ is flat.

Our next aim is to verify that the Poisson pencil corresponding to the algebra $A_{h,q}$ is just that (1.3) with $R$-matrix (1.4). To do this it suffices to compute the brackets corresponding to one parameter deformations $A \to A_q$ and $A \to A_{h,q}/(q - 1)A_{h,q}$. It is easy to see that the algebra $A_{h,q}/(q - 1)A_{h,q}$ is just that discussed in the beginning of this Section. Therefore the corresponding Poisson bracket is proportional to the KKS one.

Consider now the algebra $A_q$. Let us remark that this algebra differs from analogous one parameter algebras from [DS] and [DG2]. The latter algebras were $\tilde{S}$-commutative where the operator $\tilde{S}$ is defined in Section 4 and the algebra $A_q$ is no longer $\tilde{S}$-commutative. Instead of the relation $s_- = 0$ taking place in an $\tilde{S}$-commutative algebra we have now (6.3) with $h = 0$. 

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In [DG2] it has been shown that Poisson bracket corresponding to ˜S-commutative algebra on a symmetric orbit is proportional to the R-matrix one.

The ˜S-commutativity default of the algebra A_q is measured by the r.h.s. of the formula (6.5). In the quasiclassical limit this term gives rise to a contribution proportional to the KKS bracket. This completes the proof.

In this connection the following question arises: what algebra of the family A_{h,q} can be considered as a q-analogue of commutative algebra of functions on the CP^n type orbits and therefore it can be called quantum or braided CP^n type orbit? In [DG2] (following [DS]) ˜S-commutative algebras were considered in such a role.

However, from representation theory point of view it is more reasonable to consider as "quantum (braided) orbit of CP^n type" the algebra A_q since it is the only algebra from the family A_{h,q} which cannot be represented in a q-deformed Verma module V_{µω1} with any µ. From this point of view it is a singular point like in a classical case.

One way more to define a version of a q-commutative algebra is discussed in [Do] (cf. footnote 4).

So, there is no universal way to single out from the family A_{h,q} a braided analogue of a commutative algebra. All the above candidates for this role have their own motivations.

Let us remark that in sl(2) case our approach leads to Podles’ quantum sphere (cf. [3] where this algebra is also equipped with an involution *). We do not consider here the problem of a proper definition of involution operators (cf. [DGR1] for a discussion on this problem). We could only emphasize that in our approach all representations of algebras in question are U_q(g)-morphisms. So, if we want to consider a *-representation theory of this algebra we must first introduce *-operator in the space End V in the spirit of super-theory: the classical property (AB)^* = B^*A^* will be failed.

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