S-transformations for CFT$_2$ as linear mappings from closed to open sector linear spaces

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Abstract

We make the first attempt to define S-transformations for CFT$_2$ as linear mappings from closed to open sector linear spaces. The definition is based on closed-open sector linear space isomorphisms and boundary condition completeness. Diagonal RCFTs can be applied to our definition straight-forwardly, while more classes of CFT$_2$ are expected to be applicable. An unconventional open sector sewing, not among open sector sewing introduced by Lewellen, rises naturally and generalizes the definition. Our geometrical approach, partially inspired by string field theory, reveals the relationship between algebraic information in CFT$_2$ and curvature on surfaces.

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1 Introduction

Two-dimensional conformal field theories (CFT$_2$) play an essential role in the studies of string theory and condensed matter physics. The foundations for CFT$_2$ defined on compact surfaces with no boundary were constructed mainly in the 1980s [1–8]. In the meantime, CFT$_2$ defined on surfaces with boundaries were developed in parallel [9–17].

Modular transformations, which are ‘large’ diffeomorphisms not smoothly connected to the identity, are a significant part of conformal transformations. One class of modular transformations is the S-transformation. In the early years, S-transformations for rational conformal field theories (RCFTs) are
defined at the level of cylinder or torus one-point functions [8,13]. Later, S-transformations are generalized to a continuum form, as a modular-S kernel, to transform general conformal blocks [18–23]. In conventional definition, the basis used in S-transformation are conformal blocks for closed sector interactions, assigned to punctured Riemann surfaces with no boundary.

We shall propose a different approach in this work: S-transformations for CFT$_2$ are defined as linear mappings from closed sector linear spaces, assigned to punctured Riemann surfaces, to open sector linear spaces, assigned to Riemann surfaces with boundaries. CFT$_2$ models that are applicable to our definition need to satisfy two conditions:

- The open sector representation coefficients $n^\omega(P_1)$ and $n^\omega(P_2)$ need to be identical to fusion coefficients $N^{P_3}_{P_1P_2}$.
- Boundary condition completeness.

Both conditions will be explained explicitly. S-transformation coefficients in our definition are constrained by partition function modular invariance on three boundaries sphere $C_{0,0,3}$. An unconventional open sector sewing, where the identity operator $I$ is expressed by complete orthonormal boundary states, will play an important role in our setup. Such unconventional open sector sewing, together with the conventional closed sector sewing [15,24–27], generalizes our S-transformation definition. General S-transformations are linear mappings from closed sector linear spaces, assigned to genus-$g$, $n$-punctured surfaces $C_{g,n,0}$, to open sector interaction linear spaces, assigned to genus-$g$, $n$-boundaries surfaces $C_{g,0,b=n}$.

Our definition can be applied to diagonal RCFTs straight-forwardly, due to the fact that open sector representation coefficients $n^\omega_{ij}$ and $n^\omega_{kj}$ are identical to fusion coefficients $N^k_{ij}$ in diagonal RCFTs [13]. Our unconventional open sector sewing is the sewing of Cardy boundary state for diagonal RCFTs, when boundary conditions are complete [13,16,17]

\begin{equation}
I = \sum_k |\tilde{k}\rangle \langle \tilde{k}|.
\end{equation}

For RCFTs, there are only a finite number of representations in the spectrum, and the total number of interaction linear spaces is also finite. This makes solving the S-transformation coefficients from the finite number of partition function modular invariance equations possible in principle.

Besides diagonal RCFTs, we expect that our S-transformation definition is applicable to more classes of CFT$_2$. Currently, we are investigating on whether Liouville field theory is applicable [28–38].

We will use two types of surfaces in our discussion: the light-cone type surfaces, which are constructed by light-cone string vertices [39–43]. And the covariant type surfaces, which are constructed from all kind of covariant
string vertices \{V_{g,n}\} \[44\] \[46\]. We denote both two types of genus-\(g\), \(n\)-punctured, \(\tilde{b}\)-boundaries surfaces as \(C_{g,n,\tilde{b}}\), adding the type name in the front when distinguishing is needed. The moduli space of \(C_{g,n,\tilde{b}}\) is denoted as \(\mathcal{M}_{g,n,\tilde{b}}\).

This article is organized as follows. In section 2, we introduce the time evolution on two types of \(C_{0,3,0}\) and \(C_{0,0,3}\). We will explain why changing the time evolution vector field by an imaginary \(i\), can be regarded as a global S-transformation. This logic can be applied to time evolution vector fields for general \(C_{g,n,\tilde{b}}\) and \(C_{g,0,b=n}\). In section 3, we will introduce the linear spaces assigned to \(C_{0,3,0}\) and \(C_{0,0,3}\). Then, we will define the boundary conditions completeness, and introduce S-transformation definition when boundary conditions are complete. Further, we will explain how S-transformation coefficients are constrained by partition function modular invariance on \(C_{0,0,3}\). Finally, we generalize our definition by sewing. In section 4, we show that our setup can be applied straight-forwardly for diagonal RCFTs. In section 5, we summarize the contents and state some open questions for future studies.

2 Expressing S-transformation by time evolution changing

Before discussing the definition of S-transformation, we need to introduce some geometry that will serve our propose. First, the time evolution generating vector fields for light-cone and covariant type \(C_{0,3,0}\) and \(C_{0,0,3}\) will be introduced. Then, we explain why changing from time evolution for \(C_{0,3,0}\) to time evolution for \(C_{0,0,3}\) can be regarded as a global S-transformation for both types of surfaces. Finally, we show that introducing a time evolution generating vector field on the surface will localize the curvature as curvature singularities at specific points on the surface.

2.1 Light-cone type surfaces

First, we consider the time evolution on the light-cone type surface, named after light-cone string theory, where the closed string interactions are achieved by point contacting. This indicates that the equal-time contours collide by point contacting on light-cone type surfaces.

2.1.1 Time evolution on light-cone type \(C_{0,3,0}\)

The time evolution on light-cone type \(C_{0,3,0}\) is defined as following

\[
\frac{\partial}{\partial t} := f(z) \frac{\partial}{\partial z} + \bar{f}(\bar{z}) \frac{\partial}{\partial \bar{z}},
\]  

(2)
Figure 1: Stream plotting of time evolution generated by \( f(z) = \frac{(z-2)z(z+1)}{(z-1)} \) (blue streams) and \( if(z) \) (yellow streams).

which gives every point on the surface a value of time

\[
t = \text{Re} \left[ \int \frac{dz}{f(z)} \right].
\] (3)

The form of generating \( f(z) \) can be obtained directly from the Mandelstam map \[39\]

\[
f(z) = N \frac{(z - z_1)(z - z_2)(z - z_3)}{(z - z_0)}.
\] (4)

Three first-order zeros of \( f(z) \) correspond to three punctures, and the one single pole of \( f(z) \) corresponds to the contact interacting point. These statements are concluded as following: the time for \( z_i, i = 1, 2, 3 \) diverge, which are expected for punctures; and \( z_0 \) is the only point which allows two different points to collide (proof shown in appendix [A]). Thus, the single pole \( z_0 \) is the point for contact interaction. One of the cases, when the time is generated by \( f(z) = \frac{(z-2)z(z+1)}{(z-1)} \), is shown by the blue streams in Figure 1.

For each puncture, we can define geometrical quantities \( \alpha_i \), whose absolute value is the radius of the external cylinders on the worldsheet \[39\]

\[
\alpha_i := \text{Res} \left[ \frac{1}{f(z)} \right] \big|_{z = z_i} \in \mathbb{R}, \quad i = 1, 2, 3.
\] (5)
Figure 2: Equal-time contours on light-cone $C_{0,3,0}$, whose time evolution is generated by $f(z) = \frac{(z-2)(z+1)}{(z-1)}$. From left to right, we have the equal-time contours at time $t$ being $t_c - 0.02$, $t_c$ and $t_c + 0.02$.

In light-cone string theory, $\alpha_i$ are called momentum since they are proportional to $P^+$ in light-cone string theory. The value of time can be written as

$$t(z) = \text{Re}\left[ \sum_{i=1}^{3} \alpha_i \ln(z - z_i) \right] + C,$$

(6)

where $C$ is a real finite integral constant. Now, we can conclude that, when $\alpha_i > 0$, the vertex operator inserted at puncture $i$ corresponds to an initial state, since $t(z_i) \to -\infty$. When $\alpha_j < 0$, the vertex operator inserted at puncture $j$ corresponds to a final state, since $t(z_j) \to \infty$. There exist a critical time $t_c = t(z_0)$ for $C_{0,3,0}$, which is the time of contact interacting pole $z_0$. The value of $t_c$ is finite and arbitrary, since it can be shifted by changing $C$. For our objective, we set $\alpha_{1,2} > 0$ and $\alpha_3 < 0$. This means when $t < t_c$, there exist two branches of equal-time contours $C_i$ and $C_j$, one surrounding puncture $i$ and another surrounding puncture $j$. Two equal-time contours are combined by making point contact at the pole $z_0$. When $t > t_c$, there is only one branch of equal-time contour $C_k$. The time evolution is shown in Figure 2. We can find a local disk $|w_i^{lc}| \leq 1$, corresponding to the external cylinder $i$, for each puncture $i$. The disk local coordinates are determined only up to a overall phase factor by

$$\alpha_i^{-1} w_i^{lc} (dw_i)^{-1} = f(z) (dz)^{-1}, \quad t = t_c \Leftrightarrow |w_i^{lc}| = 1.$$

(7)

At $t = t_c$ equal-time contour, the three local coordinate disks overlap, and the local coordinates are related by

$$(w_1^{lc})^{\alpha_1} = (w_3^{lc})^{\alpha_3}, \quad (w_1^{lc})^{\alpha_1} = (w_2^{lc})^{\alpha_2},$$

(8)

$$\alpha_1 \times 2\pi = \alpha_3 \times \theta_1, \quad \alpha_2 \times 2\pi = \alpha_3 \times \theta_2,$$

(9)

which can be seen from the middle picture of Figure 2.
\( \alpha_2 = 0 \) case: This case occurs when zero point \( z_2 \) collide with the pole \( z_0 \)

\[
\alpha_2 = \frac{1}{N} \prod_{i \neq 2} z_{2i} = 0 \quad \Leftrightarrow \quad z_{20} = 0, 
\]

(10)

where \( z_{ij} := z_i - z_j \). In this case \( f(z) \) reduces into a quadratic function

\[
f(z) = N(z - z_1)(z - z_3).
\]

(11)

Quadratic \( f(z) \) can be transformed to the conventional radial time evolution vector field \( f(w) = w \) by a Mobius transformation

\[
w = \frac{az + b}{cz + d}, \quad ad - bc = 1; \quad N = ac, \quad ad + bc = -N(z_1 + z_3), \quad N z_1 z_3 = bd.
\]

(12)

The zero-radius puncture 2, does not process an external cylinder corresponding to the external propagator. From the left picture of Figure (5), we see that the \( \alpha_2 = 0 \) case of light-cone \( C_{0,3,\tilde{0}} \) limit is a cylinder with three vertex operators at \( z_1, z_0 = z_2 \) and \( z_3 \). The non-existence of local disk for puncture 2 can also been seen from equation (7).

2.1.2 Time evolution on light-cone type \( C_{0,0,3} \)

Start from light-cone type \( C_{0,3,\tilde{0}} \) with time evolution generated by \( f(z) \) given by (4). We can cut three boundaries along local coordinate circles \(|w_{lc}^i| = a_i| \) \( a_i \in (0,1) \). We define the length of external cylinders as \( l_i := -\ln(a_i) \in (0,\infty) \).

Consider changing the time evolution to following form

\[
\frac{\partial}{\partial t} = if(z) \frac{\partial}{\partial z} - i\bar{f}(\bar{z}) \frac{\partial}{\partial \bar{z}}.
\]

(13)

From the yellow stream in Figure (4), we can see the time evolution generated by \( if(z) \) is periodic, with two different regions

- Region a: the equal-time curves have their endpoints on boundary 1 \(|w_{lc}^1| = a_1| \) and on boundary 3 \(|w_{lc}^3| = a_3| \).
- Region b: the equal-time curves have their endpoints on boundary 2 \(|w_{lc}^2| = a_2| \) and on boundary 3 \(|w_{lc}^3| = a_3| \).
- Region a and region b interchanges when the equal time curves hit the pole \( z_0 \).

The time evolution on \( C_{0,0,3} \) defined by (13) in the right picture of Figure (5).
$\alpha_2 = 0$ cases: If we set $\alpha_2 = 0$ in $f(z)$, and then trying to perform the cutting to construct $C_{0,0,3}$. We will be unable to cut puncture 2 out, since there is no external cylinder 2. Thus, we shall simply left it, and obtain a $C_{0,1,2}$, which is a finite cylinder with a puncture at $z = z_2 = z_0$.

### 2.1.3 S-transformation by time evolution changing

Now we explain that why changing the time evolution generating vector field from $f(z)$ to $if(z)$ can be regarded as a global S-transformation. An S-transformation is expected to change the modular parameter as following

$$S: \quad \tau \rightarrow -1/\tau. \quad (14)$$

Consider $C_{0,0,3}$, which can be decomposed into three finite external cylinders (ring domains). When time evolution is generated by $f(z)$, three moduli for three ring domains are

$$\tau_j = \frac{il_j}{2\pi \alpha_j}, \quad j = 1, 2, 3. \quad (15)$$

When time evolution is generated by $if(z)$, three moduli become

$$\tau'_j = \frac{i2\pi \alpha_j}{l_j} = -\frac{1}{\tau_j}, \quad j = 1, 2, 3. \quad (16)$$

Thus, changing time evolution generating vector field from $f(z)$ to $if(z)$ results in a global S-transformation on $C_{0,0,3}$.

**Changing time evolution as S-transformation on general light-cone type surfaces:** Adding an imaginary $i$ to the time generating vector field can be regarded as a global S-transformation for general light-cone type surfaces. General surfaces can be decomposed into cylinders (ring domains), and adding the $i$ in time evolution results in S-transformations in all ring domains. Thus, it can be regarded as a global S-transformation on general light-cone surfaces.

### 2.1.4 Index theorem and curvature localization

By assigning $f(z)$ as the time evolution generating vector field, we make a localization of the curvature on the surface. When the zeros and poles of $f(z)$ are isolated on the surface, we can define the index for $f(z)$ at any given point. For two-dimensional cases, the index at a given point is the order of $f(z)$ at the given point (please refer to mathematics literature for the formal definition). Thus, every zero in $f(z)$ processes index +1, the single-pole processes index −1, all regular points process index 0.
The index theorem for smooth vector field on Riemann surface, namely the Poincaré–Hopf index theorem is given by
\[ \sum \text{Ind } f(z) = \chi, \quad (17) \]
where the summation covers all poles and zeros, and \( \chi \) denotes the Euler character of the surface. It is easy to check our choice of \( f(z) \) in (4) satisfies the index theorem
\[ 3 \times (+1) + (-1) = 2 = \chi(S^2). \quad (18) \]

Two-dimensional surfaces satisfy the Gauss-Bonnet theorem:
\[ \int_M K \, dA = 2\pi\chi, \quad (19) \]
where \( M \) denotes the surface, \( K \) denotes Gaussian curvature, and \( dA \) denotes the area element. We expect the index theorem to be equivalent to the Gauss-Bonnet theorem, which means that the curvature on the surfaces are localized at isolated zeros and poles [40]
\[ K = \sum_{i=1}^{3} 2\pi \delta(z-z_i) - 2\pi \delta(z-z_0). \quad (20) \]

2.2 Covariant type surfaces

Another type of surface is the covariant type, named after covariant string field theory, where closed strings interact by half-string identification [44–47]. We have to define time evolution generating vector fields different from light-cone cases to generate the time evolution on covariant-type surfaces.

2.2.1 Time evolution on covariant type \( C_{0,3,0} \)

The half-string identification nature determines that a branch cut in the time evolution generating vector field \( v(z) \) is required. The reason is that \( \partial/\partial t \) process opposite direction on the two sides of the branch cut. Thus, the form of \( v(z) \) is expected to be
\[ v(z) = \mathcal{N} \frac{(z-z_1)(z-z_2)(z-z_3)}{(z-z_{b1})^{1/2}(z-z_{b2})^{1/2}}, \quad z_{b1} \neq z_{b2}. \quad (21) \]

Some restriction are needed, the time \( t \) defined by \( v(z) \)
\[ t = \text{Re} \left[ \int \frac{dz}{v(z)} \right], \quad (22) \]
at the two branch points need to be identical \( t(z_{b1}) = t(z_{b2}) \). The absolute value \( |\alpha_i| \) for all punctures are identical and non-zero. Disk local coordinates are given by
\[ \alpha_i^{-1} w_i^{co} (dw_i^{co})^{-1} = v(z)(dz)^{-1}, \quad t = t_c \Leftrightarrow |w_i^{co}| = 1. \quad (23) \]
Figure 3: Stream plotting of time evolution generated by $v(z)$ in equation (24) (blue streams) and $iv(z)$ (yellow streams).

One choice of $v(z)$ is given by the considering the doubling of cubic string vertex. Three punctures at $z = \pm \sqrt{3}, 0$, and the two branch points at $z = \pm i$. For this case, the explicit $v(z)$ is given by following expressions (the derivation of this expression, together with a quick review on Witten’s open string field theory, is given in appendix B)

$$v(z) = \frac{i}{6} \left( \frac{1 + iz}{1 - iz} \right)^3 \left( \frac{1 - iz}{1 + iz} \right)^{1/2} (1 - iz)^2$$

$$= -\left(1 + z^2\right)^3 \sin \left[ \frac{3 \arctan (z)}{3} \right].$$

The time evolution generated by $v(z)$ in equation (24) is shown in the blue streams in Figure (3). The zero and branch points behaviour can be seen from the asymptotic behaviour of $v(z)$

$$v(z) \sim C_i(z - z_i), \quad z \to z_i = 0, \pm \sqrt{3};$$

$$v(z) \sim C_{bi}(z - z_{bi})^{-1/2}, \quad z \to z_{bi} = \pm i.$$

The blue streams in Figure (3) shows how equal-time contours evaluate on covariant $C_{0,3,0}$, where we can see the half-string identification occurs on $[i, i\infty]$ and $[-i\infty, -i]$.
2.2.2 Time evolution on covariant type $C_{0,0,\tilde{3}}$

Changing the time evolution vector field to $iv(z)$, and cut three punctures out at $|w_{0i}| = a_i, a_i \in (0, 1)$. We shall only consider equal-time curves having endpoints on boundary 1 and boundary 3, or on boundary 2 and boundary 3 (those having endpoints on the branch cut are not considered). Again, we obtain two regions:

- Region a: the equal-time curves have their endpoints at boundary 1 $|w_{1}^0| = a_1$ and boundary 3 $|w_{3}^0| = a_3$.
- Region b: the equal-time curves have their endpoints at boundary 2 $|w_{2}^0| = a_2$ and boundary 3 $|w_{3}^0| = a_3$.
- Region a and region b interchanges when the equal time curves hit one of the branch points $z_{bi}$.

2.2.3 S-transformation by time evolution changing

By defining the length of the external cylinders $l_i := -\ln(a_i)$, and repeating the same procedure in analysing $\tau_i$ given by $v(z)$ and $iv(z)$ on $C_{0,0,\tilde{3}}$. Again, we conclude that changing the time evolution generating vector field from $v(z)$ to $iv(z)$ can be regarded as a global S-transformation for covariant type $C_{0,0,\tilde{3}}$. The conclusion can be extended to general covariant type of surfaces.

2.2.4 Index theorem and curvature localization

On covariant type $C_{0,3,\tilde{0}}$, with time evolution generated by $v(z)$, the index theorem is given by

$$3 \times (+1) + 2 \times (-\frac{1}{2}) = 2 = \chi(S^2),$$

(25)

where we have generalized the concept of index to rational numbers $\mathbb{Q}$ in order to define the index for branch points. Equating it with Gauss-Bonnet...
theorem, we have the Gaussian curvature localized at three zeros and two branch points
\[ K = \sum_{i=1}^{3} 2\pi \delta(z - z_i) - \sum_{j=1}^{2} \pi \delta(z - z_{bj}). \]  
(26)

3 Definition of S-transformations for CFT\(_2\)

We have introduced the time evolution for \(C_{0,3,0}\) and \(C_{0,0,3}\). We have also explained that if we cut three punctures on \(C_{0,3,0}\) out, then changing the time evolution can be regarded as a global S-transformation. If we assign two sets of linear spaces to \(C_{0,3,0}\) and \(C_{0,0,3}\), the S-transformation can be defined as linear mappings from closed sector spaces, assigned to \(C_{0,3,0}\), to open sector spaces, assigned to \(C_{0,0,3}\).

First, we need to define closed and open sector linear spaces. Then, we define the completeness condition of admissible boundary conditions. When admissible boundary conditions are complete, we can define the S-transformation as linear mappings from closed to open sector linear spaces. The coefficients of S-transformations are constrained by the partition function modular-S invariance on \(C_{0,0,3}\). This partition function modular-S invariance can be reduced into conventional partition function modular-S invariance on torus or cylinder, at the \(\alpha_2 = 0\) limit.

Then, we will introduce open sector sewing achieved by the sewing of complete boundary states. Such unconventional open sector sewing, together with conventional closed sector sewing generalize the S-transformation definition to linear mappings from spaces assigned to \(C_{g,n,0}\) to spaces assigned to \(C_{g,0,b=n}\).

Finally, we shall discuss what would be the difference if the setup is done on covariant type of the surface. A relation between CFT\(_2\) algebraic information changing, and the curvature on the surface is observed.

3.1 Definition of closed and open sector linear spaces

Consider CFT\(_2\), whose spectrum \(\mathcal{H}\) is parameterized by two sets of parameter \(\{P\}\) and \(\{\bar{P}\}\), with a coupling isomorphism \(\sigma\) between the two sets, describing how two sets of parameters are coupled in the spectrum. The admissible boundary conditions of the theory are parameterized by a set of parameters \(\{\mu\}\). We assume that exist an injective mapping \(x\) from \(\{\mu\}\) to \(\{P\}\), expressed as \(x(\mu_i) = P_i\), such that the Jacobian is trivial \(d\mu = dP\).
3.1.1 Closed sector linear space definition

The spectrum of the CFT\(_2\) of interest given by

\[ \mathcal{H} := \int_{\{P\}}^{\oplus} dP \int_{\{\bar{P}\}}^{\oplus} d\bar{P} \rho(\{P\}, \{\bar{P}\}) (\mathcal{H}_P \otimes \mathcal{H}_{\bar{P}}), \tag{27} \]

where \(\mathcal{H}_P \otimes \mathcal{H}_{\bar{P}}\) are tensor products of irreducible representations of two chiral algebras \(A \otimes A\) \cite{2,8,29,30,50}. \(\int_{\{P\}}^{\oplus} dP\) denotes the direct integral over representations, and the weight density \(\rho(\{P\}, \{\bar{P}\})\) takes the form of delta functions

\[ \rho(\{P\}, \{\bar{P}\}) = N_P \delta[\bar{P} - \sigma(P)], \tag{28} \]

where \(N_P\) is the weight for \(\mathcal{H}_P \otimes \mathcal{H}_{\sigma(P)}\). \(\rho(\{P\}, \{\bar{P}\})\) is strongly constrained by the partition function modular invariance on cylinder or torus, explicit examples of \(\rho\) are mainly found in RCFTs \cite{4,51–54}. For each chiral representation \(\mathcal{H}_P\), there exist a dual representation \(\mathcal{H}_{P^*}\) in the spectrum, representing its charge dual. If \(P = P^*\), then the representation \(\mathcal{H}_P\) is called self-dual.

The tensor product of two holomorphic irreducible representations can be written as a direct integral

\[ \mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2} = \int_{\{P_3\}}^{\oplus} (V_{P_3}^{P_{12}} \otimes \mathcal{H}_{P_3}), \tag{29} \]

where closed sector interaction linear spaces \(V_{P_3}^{P_{12}}\) are defined as the space of coupling. The dimension of \(V_{P_3}^{P_{12}}\) is defined to be the Clebsch–Gordan coefficient (fusion coefficient)

\[ \dim(V_{P_3}^{P_{12}}) := N_{P_3}^{P_{12}}. \tag{30} \]

\(N_{P_3}^{P_{12}}\) is not necessarily a non-negative integers in general, non-integer coefficients can be found in models like Liouville field theory \cite{35}. All \(N_{P_3}^{P_{12}}\) are assumed to be finite.

Linear spaces \(V_{P_3}^{P_{12}}\) process the so-called permutation symmetries

\[ V_{P_1}^{P_{12}} \cong V_{P_2}^{P_{12}} \cong V_{P_3}^{P_{12}} \tag{31} \]

The (light-cone type) surface assigned with the linear space \(V_{P_3}^{P_{12}}\) is shown in the left picture in Figure (5). When \(t < t_c\), the two branches of the equal-time contour (closed strings) carry algebraic information on bulk representations \(\mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2}\). When \(t > t_c\), the single branch equal-time contour carries algebraic information \(\mathcal{H}_{P_3}\). There must exist an operator to achieve the projection from the whole tensor product to \(\mathcal{H}_{P_3}\). We expect this is a local operator positioned at the pole \(z_0\)

\[ \Phi_{P_3}^{P_{12}}(z_0) : \mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2} \rightarrow \mathcal{H}_{P_3}. \tag{32} \]
The reason that we expect \( \Phi_{P_1 P_2}^3(z_0) \) to be a local operator is that the curvature on light-cone \( C_{0,3,0} \) is the localized at \( z_0 \). Note that, if we cut the three punctures out, but do not change the time evolution, the linear space assigned to the surface in still \( V_{P_1 P_2}^3(z_0) \).

Closed sector linear spaces \( V_{P_1 P_2}^3 \) can be used to construct a basis for a full closed sector interaction space \( V_{0,3,0} \), which describes all closed sector tree-level three-point interactions

\[
V_{0,3,0} := \prod_{i=1}^3 \int_{P_i}^\oplus \{ P_i \} dP_i V_{P_1 P_2}^3, \quad \dim_{\mathbb{R}}[V_{0,3,0}] = \prod_{i=1}^3 \int_{P_i}^\oplus dP_i N_{P_1 P_2}^3.
\]  (33)

The dimension of \( V_{0,3,0} \) can be either finite or infinite.

3.1.2 Open sector linear space definition

For CFT\(_2\) defined on surfaces with boundaries, we need the symmetry between two chiral algebras \( \mathcal{A} = \bar{\mathcal{A}} \) to be preserved at all boundaries. Boundary conditions preserving conformal symmetry and satisfying necessary non-linear constraints are admissible. The admissible boundary conditions are parameterized by a set \( \{ \mu \} \). For each admissible boundary condition \( \mu_i \), there exist a corresponding boundary state \( |\mu_i\rangle \), satisfying the gluing conditions

\[
(L_n - L_{-n})|\mu_i\rangle = 0, \quad (W^j_n - (-1)^{h_j} W_{-n}^j)|\mu_i\rangle = 0,
\]  (34)

and necessary extra non-linear constraints. \( W^j \) are quasi-primaries for extended symmetries.

Now, assigning boundary states \( |\mu_1\rangle, |\mu_2\rangle \) and \( \langle \mu_3| \) to the three boundaries of \( C_{0,0,3} \), as shown in the right picture of Figure 3. Denote the basis of boundary states by \( \mathcal{H}_{\mu_i} \). We propose that the bulk representations allowed in the region sandwiched by \( |\mu_1\rangle \) and \( \langle \mu_3| \) defines a tensor product structure similar to the closed sector fusion product

\[
\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_3} = \int_{\{ P \}}^\oplus dP \left( V_{\mu_3 \mu_1}^P \otimes \mathcal{H}_{P} \right).
\]  (35)

For the \( C_{0,0,3} \) we are considering, we only desire one specific value \( P_2 = x(\mu_2) \). \( \omega \) is some automorphism for the bulk representation parameter \( \{ P \} \). \( \rho_{\omega(P_2)} \mathcal{H}_{\omega(P_2)} \) is the only bulk presentation allowed in region a of right picture of Figure 3.

Similar tensor product structure can be defined in the regions sandwiched by \( |\mu_2\rangle \) and \( \langle \mu_3| \)

\[
\mathcal{H}_{\mu_2} \otimes \mathcal{H}_{\mu_3} = \int_{\{ P' \}}^\oplus dP' \left( V_{\mu_3 \mu_2}^{P'} \otimes \mathcal{H}_{P'} \right).
\]  (36)
Again, we pick out the specific value $P_1$ such that $P_1 = x(\mu_1)$, and allow only $n_{\kappa \mu_3}^{\omega(P_1)} \mathcal{H}_{\omega(P_1)}$ in region b. The dimension of the two open sector spaces are the open sector representation coefficients

$$\dim_\mathbb{R} [V_{\mu_3 \mu_1}^{\omega(P_3)}] = n_{\kappa \mu_3}^{\omega(P_1)}, \quad \dim_\mathbb{R} [V_{\mu_3 \mu_2}^{\omega(P_2)}] = n_{\kappa \mu_3}^{\omega(P_2)}.$$  \hspace{1cm} \text{(37)}

The CFT\textsubscript{2} models that are applicable to our definition need to satisfy the following condition

$$n_{\mu_3 \mu_1}^{\omega(P_1)} = n_{\mu_3 \mu_2}^{\omega(P_1)} = N_{P_1}.$$  \hspace{1cm} \text{(38)}

Thus, we can conclude that

$$V_{P_1 P_2} \cong V_{\mu_3 \mu_1}^{\omega(P_1)} \cong V_{\mu_3 \mu_2}^{\omega(P_2)}.$$  \hspace{1cm} \text{(39)}

There exist a local operator $\Phi_{\mu_3 \mu_2}(z_0)$ at the pole $z_0$ to realize the interchanging between linear spaces assigned to region a and b

$$\Phi_{\mu_3 \mu_2}(z_0) : \quad n_{\mu_3 \mu_2}^{\omega(P_2)} \mathcal{H}_{\omega(P_2)} \leftrightarrow n_{\mu_3 \mu_2}^{\omega(P_1)} \mathcal{H}_{\omega(P_1)}.$$  \hspace{1cm} \text{(40)}

$\Phi_{\mu_3 \mu_2}(z_0)$ can be regarded as the matrix and the inversion matrix that interchange $V_{\mu_3 \mu_2}^{\omega(P_1)}$ and $V_{\mu_3 \mu_2}^{\omega(P_2)}$, and its rank is the dimension of the two linear spaces.

The open sector linear space $V_{\mu_3 \mu_2}^{\omega(P_1)}$, assigned to the surface $C_{0,0,3}$ on the right picture of Figure (5), is schematically defined as a region-dependent or time-dependent space as

$$V_{\mu_3 \mu_2}^{\omega(P_1)} := \begin{cases} V_{\mu_3 \mu_1}^{\omega(P_2)} & \text{when equal-time curve in region a;} \\ V_{\mu_3 \mu_2}^{\omega(P_1)} & \text{when equal-time curve in region b.} \end{cases}$$  \hspace{1cm} \text{(41)}

The bulk representation allowed on surface $C_{0,0,3}$ on the right picture of Figure (5) is $n_{\mu_3 \mu_1}^{\omega(P_1)} \mathcal{H}_{\omega(P_1)}$ in region a, and $n_{\mu_3 \mu_2}^{\omega(P_2)} \mathcal{H}_{\omega(P_2)}$ in region b.

By definition, we know that the dimension for $V_{\mu_3 \mu_2}^{\omega(P_1)}$ is identical to $N_{P_1 P_2}$

$$\dim_\mathbb{R} [V_{\mu_3 \mu_2}^{\omega(P_1)}] = \begin{cases} n_{\kappa \mu_3}^{\omega(P_1)} & \text{when equal-time curve in region a;} \\ n_{\kappa \mu_3}^{\omega(P_2)} & \text{when equal-time curve in region b.} \end{cases} = \text{rank}[\Phi_{\mu_3 \mu_2}(z_0)] = N_{P_1 P_2}.$$  \hspace{1cm} \text{(42)}

Immediately, we conclude that the open sector interaction linear space $V_{\mu_3 \mu_2}^{\omega(P_1)}$ is isomorphic to the closed sector interaction linear space

$$V_{\mu_3 \mu_2}^{\omega(P_1)} \cong V_{P_1 P_2}^{P_3}.$$  \hspace{1cm} \text{(43)}

The injective mapping $x$ from $\{\mu\}$ to $\{P\}$ ensures that for all $V_{\mu_3 \mu_2}^{\omega(P_1)}$, the corresponding isomorphic $V_{P_1 P_2}^{P_3}$ does exist.
Figure 5: The left picture shows the $C_{0, 3, 0}$ assigned with closed sector linear space $V_{P_3 P_2}$. The right picture shows the $C_{0, 0, 3}$ assigned with open sector linear space $V_{\mu_1 \mu_2}$. The two spaces are isomorphic $V_{P_3 P_2} \cong V_{\mu_1 \mu_2}$ by definition.

Open sector linear spaces also process permutation symmetries

$$V_{\mu_1 \mu_2}^{\mu_3} \cong V_{\mu_2 \mu_1}^{\mu_3} \cong V_{\mu_3 \mu_2}^{\mu_1}, \quad (44)$$

which have lead to the concept of dual boundary condition $\mu^*$ in the set of admissible boundary conditions $\{\mu\}$, as an open sector analog of dual representation in closed sector. We can call a boundary condition $\mu$ self-dual if $\mu = \mu^*$.

Full open sector interaction linear space can be schematically defined in parallel to the full closed sector interaction linear space

$$V_{0, 0, 3} := \prod_{i=1}^{3} \int_{\{\mu_i\}} d\mu_i \ V_{\mu_1 \mu_2}^{\mu_3}, \quad \dim_{\mathbb{R}}[V_{0, 0, 3}] = \prod_{i=1}^{3} \int_{\{\mu_i\}} d\mu_i \ n_{\mu_1 \mu_2}^{\omega(P_1)}, \quad (45)$$

where $\mu_1$ dependence is inside $\omega(P_1)$. $\dim_{\mathbb{R}}[V_{0, 0, 3}]$ can also be either finite or infinite.

The surface $C_{0, 0, 3}$ can be regarded as an open string worldsheet with the open string making a jump between boundary 1 and boundary 2, when hitting the bulk singularity $z_0$. In target space, this boundary jumping corresponds to a D-brane jumping phenomenon since D-branes are formulated by BCFT boundary states $[55, 57]$.

3.1.3 $\alpha_2 = 0$ limit

Consider the $\alpha_2 = 0$ limit for the surfaces in Figure [3].

Closed sector: For closed sector case, the vertex operator inserted at puncture 2, $\mathcal{O}_{P_2}(z_2)$ collapses with the local operator $\Phi_{P_1 P_2}(z_0)$, defining a
product operator
\[ \phi_{P_1 P_3}^{P_3} (z_0) := \lim_{z_2 \to z_0} O_{P_3} (z_2) \Phi_{P_1 P_2}^{P_3} (z_0), \] (46)
whose function is the following
\[ \phi_{P_1 P_3}^{P_3} (z_0) [P_1 \to P_3] : H_{P_1} \to H_{P_3}. \] (47)
Performing a coordinate transformation on \( \phi_j (z_0) \) into the coordinate \( w \) defined in equation (12), we recover conventional chiral vertex operators (CVOs) defined in [8].

Open sector: For \( \alpha_2 = 0 \) limit on the right picture, the puncture 2 cannot be cut out. \( C_{0,0,3} \) reduced to a finite cylinder, and the vertex operator \( O_{\omega(P_2)} (z_2) \) collides with \( \Phi_{\mu_1 \mu_2}^{\mu_3} (z_0) \), resulting in the following product operator
\[ \mu_1 \psi_{\omega(P_2)}^{\mu_3} (z_0) := \lim_{z_2 \to z_0} O_{\omega(P_2)} (z_2) \Phi_{\mu_1 \mu_2}^{\mu_3} (z_0), \] (48)
whose function is to generate representation \( H_{\omega(P_2)} \) on the cylinder.

This product operator \( \mu_1 \psi_{\omega(P_2)}^{\mu_3} (z_0) \) can be transformed into conventional boundary condition changing operator (BCCO) \( \mu_1 \Psi_{\omega(P_2)}^{\mu_3} \) by a few steps. First, we cut the cylinder to a strip and extend its length to infinity. Then, we set the value of time at \( z = z_0 \) to be \( t \to -\infty \), and perform an exponential map to the upper half-plane (UHP), a conventional BCCO \( \mu_1 \Psi_{\omega(P_2)}^{\mu_3} (0) \) at the origin can be recovered.

The linear space isomorphism \( V_{P_1 P_2} \cong V_{\mu_1 \mu_2}^{\mu_3} \) reduce into a CVO-BCCO isomorphism at \( \alpha_2 = 0 \) limit
\[ \phi_{P_2} [P_1 \to P_3] \cong \mu_1 \Psi_{\omega(P_2)}^{\mu_3}. \] (49)
In diagonal RCFT, such CVO-BCCO isomorphisms are well-known [13,17].

3.2 Boundary condition completeness and S-transformations

Definition (completeness of boundary condition): The admissible boundary conditions are called complete, if the injective mapping \( x \) from \{\( \mu \}\) to \{\( P \}\) is surjective. In other words, admissible boundary conditions are complete if \{\( \mu \}\} \cong \{\( P \}\).

From (43), (44), (45), and \( d\mu = dP \), we conclude that
\[ \text{dim}_R (V_{0,3,\bar{0}}) = \text{dim}_R (V_{0,0,3}), \] (50)
where the dimension for the two spaces can be either finite or infinite. Equation (50) enables the definition of S-transformation between the basis of \( V_{0,3,\bar{0}} \) and \( V_{0,0,3} \).
Definition (S-transformations as linear mappings from closed sector to open sector linear spaces): S-transformations for CFT are defined as following

\[ S[V_{0,0,3}] := V_{0,0,3}, \quad S[V_{P_1 P_2}] := \prod_{i=1}^{3} \int_{\{\mu_i\}} d\mu_i C_{P_1 P_2 \mu_3} P_{\mu_3}, \quad (51) \]

where \( C_{P_1 P_2 \mu_3} \) are named S-transformation coefficients, describing how much a specific open sector space \( V_{\mu_3} \) contributes to \( S[V_{P_1 P_2}] \). All \( C_{P_1 P_2 \mu_3} \) are unique, from the linear independent condition for \( \{V_{\mu_3}\} \).

The inversion S-transformation is defined as

\[ S^{-1}[V_{0,0,3}] := V_{0,0,3}, \quad S^{-1}[V_{\mu_3}] := \prod_{i=1}^{3} \int_{\{P_i\}} dP_i \inv C_{\mu_3 P_1 P_2 P_3} V_{P_1 P_2}, \quad (52) \]

where \( \inv C_{\mu_3 P_1 P_2 P_3} \) are named inversion S-transformation coefficients, describing how much specific \( V_{P_1 P_2} \) contributes to \( S^{-1}[V_{\mu_3}] \). All \( \inv C_{\mu_3 P_1 P_2 P_3} \) are also unique, from the linear independent condition for \( \{V_{P_1 P_2}\} \).

The fact that

\[ S^{-1} \cdot S = I, \quad (53) \]

gives the following result

\[ \prod_{i=1}^{3} \int_{\{\mu_i\}} d\mu_i C_{P_1 P_2 \mu_3} \inv C_{\mu_3 P_1 P_2 P_3} = \prod_{j=1}^{3} \delta(P_j - P'_j). \quad (54) \]

Further, we expect that \( S^2 = C \), where \( C \) is the charge conjugate. \( C \)'s action on closed sector linear spaces is given by

\[ C[V_{P_1 P_2}] = V_{P_1 P_2}^*, \quad (55) \]

where we have ignored the phase from the inversion transformation. Inserting our S-transformation definition into equation (55), we obtain

\[ V_{P_1 P_2}^* = \prod_{i=1}^{3} \int_{\{\mu_i\}} d\mu_i C_{P_1 P_2 \mu_3} S[V_{\mu_3}], \quad (56) \]

If we define the S-transformation of open sector linear spaces as

\[ S[V_{\mu_3}] := \prod_{i=1}^{3} \int_{\{P_i\}} dP_i \conj C_{\mu_3 P_1 P_2 P_3} V_{P_1 P_2}, \quad (57) \]

and insert to equation (56), we obtain that

\[ \prod_{i=1}^{3} \int_{\{\mu_i\}} d\mu_i C_{P_1 P_2 \mu_3} \conj C_{\mu_3 P_1 P_2 P_3} = \prod_{j=1}^{3} \delta(P_j^* - P'_j). \quad (58) \]
The role of anti-holomorphic sector spaces: S-transformations are defined purely out of holomorphic closed sector linear spaces $V_{P_3}^{P_1 P_2}$ in equation (51). Actually, an equivalent definition can be made using purely anti-holomorphic linear spaces $V_{\bar{P}_3}^{\bar{P}_1 \bar{P}_2}$. We have assumed the existence of a coupling isomorphism between the two sets of bulk representation parameters $\{P\} \cong \{\bar{P}\}$. A direct conclusion from here is that 

$$V_{P_3}^{P_1 P_2} \cong V_{\sigma[P_3]}^{\sigma[P_1] \sigma[P_2]} \cong V_{\mu_1 \mu_2}^{\mu_3}.$$ (59)

This indicates that we can use anti-holomorphic spaces to form a basis of $V_{0,0,3}$, and define S-transformation as following

$$S[V_{P_1 P_2}] := \prod_{i=1}^{3} \int_{\{P_i\}} C_{P_1 P_2 \mu_3}^{P_3 \mu_1 \mu_2} V_{\mu_1 \mu_2}^{\mu_3}.$$ (60)

This work will only use the holomorphic closed sector linear spaces definition.

3.3 Partition function modular-S invariance on $C_{0,0,3}$

We discuss about the partition function modular-S invariance induced by the S-transformation definition (51), and how to reduce this new scheme to conventional partition function modular-S invariance at $\alpha_2 = 0$ limit, and relating the S-transformation coefficients $C_{P_1 P_2 \mu_3}^{P_3 \mu_1 \mu_2}$ with familiar unitary S-matrices $S_{P_1 P_2}$ in the process.

3.3.1 Setup

Assuming that the CFT$_2$ model of interest process the modular invariance property. We propose that the modular-S invariance on $C_{0,0,3}$ gives constraints to coefficients $C_{P_1 P_2 \mu_3}^{P_3 \mu_1 \mu_2}$. First, consider the $C_{0,3,0}$ assigned with $V_{P_1 P_2}^{P_3}$, we cut the three punctures out and create a three boundaries sphere $C_{0,3,0}$.

We will keep the time evolution and linear space $V_{P_1 P_2}^{P_3}$. The representations on the three finite external cylinders are still $H_{P_1}$, $H_{P_2}$ and $H_{P_3}$. The (holomorphic) partition function on such $C_{0,0,3}$ given by $V_{P_1 P_2}^{P_3}$ is

$$N_{P_1 P_2}^{P_3} \prod_{i=1}^{3} \chi_{P_i}(q_i),$$ (61)

where the character $\chi_{P_i}$ for representation $H_{P_i}$ is defined as

$$\chi_{P_i}(q_i) := \text{Tr}_{H_{P_i}} q_i^{L_0 - \frac{c}{24}}, \quad q_i := \exp \left( i \frac{l_i}{2 \pi \alpha_i} \right).$$ (62)

1There are CFT$_2$ models which do not satisfy modular invariance conditions [58], like the Monster CFT [59].
Note that, for degenerate representations, we need to eliminate the contributions from null representations to obtain the correct expression for characters $\chi_P$.

Consider the partition function on $C_{0,0,\tilde{3}}$ assigned with $V_{\mu_1\mu_2}^{\mu_3}$. There exist only two regions on such $C_{0,0,\tilde{3}}$, with the value of $q$ given by

$$q_a = \exp \left( i \frac{2\pi \alpha_1}{l_1 + l_3} \right), \quad q_b = \exp \left( i \frac{2\pi \alpha_2}{l_2 + l_3} \right),$$  \hspace{1cm} (63)

where $l_i$ are set to be same as the those in $q_i$ in (62). Thus, we have the partition function given by

$$V_{\mu_1\mu_2}^{\mu_3} = n_{\mu_3\mu_2}^{\omega(P_1)} \chi_{\omega(P_1)}(q_a) \times n_{\mu_3\mu_1}^{\omega(P_2)} \chi_{\omega(P_2)}(q_b).$$  \hspace{1cm} (64)

Thus, from the definition of the $S$-transformation (51), we have the partition function modular invariance under $S$-transformation given by

$$N_{P_1P_2}^{P_3} \prod_{i=1}^{3} \chi_{P_i}(q_i) = \prod_{i=1}^{3} \int d\mu_i \; C_{P_1P_2\mu_3}^{P_3} \chi_{\mu_1\mu_2}(q_i),$$  \hspace{1cm} (65)

which constraint the $S$-transformation coefficients $C_{P_1P_2\mu_3}^{P_3}$.

Equation (65) is the strong unintegrated version for partition function modular invariance, where modular invariance is required at all points in the moduli space $M_{0,0,\tilde{3}}$. There also exists a weak integrated version for partition function modular invariance, where modular invariance is required only after integral over the moduli space $M_{0,0,\tilde{3}}$. The weak integrated version is automatically satisfied when the strong unintegrated version is satisfied. In our work, partition function modular invariance will always refer to the strong unintegrated version.

### 3.3.2 Reducing to conventional partition function modular invariance

Consider the $S$-transformation of closed sector linear space $V_{P_1}^{P}$ in our definition

$$S[V_{P_1}^{P}] = \prod_{i=1}^{3} \int_{\{\mu_i\}} d\mu_i \; C_{P_1P_2\mu_3}^{\mu_1\mu_2} V_{\mu_1\mu_2}^{\mu_3}.$$  \hspace{1cm} (66)

We are willing to reduce our definition to conventional partition function modular-$S$ invariance

$$\chi_{P}(q) = \int dP' \; S_{PP'} \chi_{P'}(\tilde{q}),$$  \hspace{1cm} (67)

at the $\alpha_2 = 0$ limit. In this case, $q = \exp \left[ i \frac{\alpha_1}{2l_1 + l_3} \right]$ and $\tilde{q} = \exp \left[ i \frac{2\alpha_1 \alpha_2}{l_1 + l_3} \right]$.
Figure 6: The figure showing the direct sum \((V^\mu_1\mu_2 + V^\mu_5_1\mu_2)\). The two operators generating dual representations collide to create identity operator (and other operators allowed), with the representation generated on the cylinder being the dual representations \(n_{\mu_1}\mu_3\mu_1\chi_{\omega}[P_2] + n_{\mu_1}\mu_3\mu_3\chi_{\omega}[P_2]^*\). The boundary conditions allowed on two boundaries are those in tensor products of \(H_{\mu_1} \otimes H_{\mu_3}\).

Take the \(\alpha_2 = 0\) limit, the operator on the surface is the identity operator, which is \(\phi_I(z_0) : \mathcal{H}_P \rightarrow \mathcal{H}_P\) on the LHS surface, and \(\mu_\psi(z_0)\) on the RHS surface.

Now, we need to determine what \(V^\mu_1\mu_2\) contribute to the \(\mu_\psi(z_0)\). This is rather complicated since the contributions are divided into orders, where order-\(k\) means that the contribution from combining \(k\) spaces by direct sum. We shall show what is the effect on the surface by taking the direct sum, using the example \(V^\mu_1\mu_2 + V^\mu_5_1\mu_2\). The two surfaces assigned with the spaces are taken together, and result in a 'mixture' of algebraic information on the resulting surface, which is shown in Figure (6). Each part of the algebraic information are combined relatively by tensor product.

We should not confuse what happens in the direct sum \(V^\mu_1\mu_2 + V^\mu_5_1\mu_2\) with the operator product of BCCOs

\[
\mu_1 \Psi^\mu_3_{\omega[x(\mu_2)]}(x_1) \times \mu_3 \Psi^\mu_1_{\omega[x(\mu_2)]}(x_2) \sim \frac{\mu_1 \Psi^\mu_1_{\omega}[x(\mu_2)](x_2)}{(x_1 - x_2)^{n_{\omega}[x(\mu_2)] + n_{\omega}[x(\mu_2)]^*}} + \cdots \quad (68)
\]

The BCCOs operator product is related to the tensor product of linear spaces \(V^\mu_1\mu_2 \otimes V^\mu_5_1\mu_2\), which is shown in Figure (7).

Now, we come back to consider reducing to conventional modular-S invariance. At first order, we got the contribution from the spaces \(V^\mu_1\mu_2\), where \(\omega[x(\mu_2)] = I\). Each type of space \(V^\mu_1\mu_2\) is weighted by a factor \(A_{\mu_1}(1)\). At the second order, the contributions are form pairs taking the form of \(V^\mu_1\mu_2 + V^\mu_5_1\mu_2\), with each pair weighted by a factor \(A_{\mu_1\mu_2}(2)\). The logic to collect more higher order contributions are similar to the second order case, where the identity exist in the bulk indices product caused of direct sum. Thus, we expect that the partition function modular-S invariance for \(V^P_{11}\) is
\[ \mathcal{H}_{\mu_1} V_{\mu_1 \mu_2}^{\mu_3} \mathcal{H}_{\mu_2} \mathcal{H}_{\mu_3} V_{\mu_1 \mu_2}^{\mu_5} \mathcal{H}_{\mu_3} \mathcal{H}_{\mu_1} V_{\mu_1 \mu_2}^{\mu_3} \mathcal{H}_{\mu_2} \mathcal{H}_{\mu_3} = \mathcal{H}_{\omega(P_2)} \mathcal{H}_{\omega(P_2)^*} \mathcal{H}_{\omega(P_2)} \mathcal{H}_{\omega(P_2)^*} \]

Figure 7: The figure showing the tensor product \( V_{\mu_1 \mu_2}^{\mu_3} \otimes V_{\mu_1 \mu_2}^{\mu_5} \), which is related with conventional BCCOs OPE.

Given by

\[
\chi_{P}(q) = \prod_{i=1}^{3} \int d\mu_{i} \left[ A_{\mu_1}(1) \chi_{I}(\tilde{q}) + A_{\mu_1}^{\mu_3}(2) \left( n_{\omega_{\mu_3}}^{\omega(\mu_2)}(\tilde{q}) + n_{\omega_{\mu_3}}^{\omega(\mu_2)}(\tilde{q}) + \cdots \right) \right] \tag{69}
\]

Both \( S_{PP'} \) and \( C_{P_{1}P_{3}}^{P_{1}P_{3}} \) can be read off from equation (69) in principle

\[
S_{P_{I}} = \int d\mu_{1} \int d\mu_{3} \left[ A_{\mu_1}(1) + 2A_{\mu_1 \mu_2}^{\mu_3}(2) + \cdots \right], \quad \omega(x(\mu_2)) = I; \tag{70}
\]

\[
S_{PP'} = \int d\mu_{1} \int d\mu_{3} \left[ A_{\mu_1 \mu_2}^{\mu_3}(2)n_{\omega_{\mu_3}}^{\omega(\mu_2)}(\tilde{q}) + A_{\mu_1 \mu_2}^{\mu_3}(2)n_{\omega_{\mu_3}}^{\omega(\mu_2)}(\tilde{q}) + \cdots \right], \quad P' \neq I; \tag{71}
\]

\[
C_{P_{1}P_{3}}^{P_{1}P_{3}} = \left[ \delta_{\omega(x(\mu_2))}, I \delta_{\mu_1, \mu_3} A_{\mu_1} + A_{\mu_1 \mu_2}^{\mu_3}(2) + A_{\mu_1 \mu_2}^{\mu_3}(2) + \cdots \right] \tag{72}
\]

indicating that \( S_{PP'} \) and \( C_{P_{1}P_{3}}^{P_{1}P_{3}} \) are related in a complicated fashion. \( \delta_{a,b} \) denotes Kronecker delta.

### 3.4 Conventional closed sector sewing and unconventional open sector sewing

We shall introduce an unconventional open sector sewing which is different from conventional open sector sewing introduced by Lewellen [15]. In conventional sewing (closed and open sector), the identity operator is expressed by a complete set of states \( |P_{i}\rangle \) corresponding to bulk primary operators \( O_{P_{i}} \) by the state-operator corresponding

\[
I = \int dP_{i} |P_{i}\rangle \langle P_{i}|. \tag{73}
\]

The directions of time evolution are orthogonal to the sewing boundaries.
When the boundary conditions are complete, the identity operator can be expressed by orthonormal boundary states $|\mu_i\rangle$

$$I = \int d\mu_i |\mu_i\rangle\langle\mu_i|.\quad (74)$$

This fact can be used to introduce unconventional open sector sewing. The direction of the directions of time evolution are parallel to the sewing boundaries. Unconventional open sector sewing is Cardy boundary state sewing for diagonal RCFTs [16].

With unconventional open sector sewing and conventional closed sector sewing, the linear spaces assigned to $C_{g,n,\tilde{0}}$ and $C_{\tilde{g},\tilde{0},b=\tilde{n}}$ can be obtained straight-forwardly. General S-transformations can be defined between the two sets of spaces. Assuming a factorization property for S-transformations, the general S-transformation coefficients can be written as integrals over products of basic S-transformation coefficients $C_{P_3P_1P_2}$.

We can define open sector fusion matrices, different from the conventional definition from the associativity of BCCOs four-point functions, directly from the open sector linear spaces.

### 3.4.1 Separating sewing

Separating sewing in closed sector is carried out as following: we take two initial surfaces $C_{g_1,n_1,\tilde{0}}$ and $C_{g_2,n_2,\tilde{0}}$, cut out one puncture from each surface, and sew the two leftovers together. The resulting surface is a $C_{g_1+g_2,n_1+n_2-2,\tilde{0}}$.

More explicitly, we cut the two boundaries out by along local coordinate circles $|w_i| = \sqrt{|b|}$, and sew the two surfaces by boundary identification

$$w_i w_j = b, \quad b \in \mathbb{C}, \quad |b| \in (0,1).\quad (75)$$

The procedure is denoted as $C_{g_1,n_1,\tilde{0}} \# C_{g_2,n_2,\tilde{0}} \rightarrow C_{g_1+g_2,n_1+n_2-2,\tilde{0}}$.

The linear space assigned to the resulting surface, can be represented as direct integral over tensor products of linear spaces assigned to two initial surfaces. We take the simplest example to show this explicitly: $C_{0,3,\tilde{0}} \# C_{0,3,\tilde{0}} \rightarrow C_{0,4,\tilde{0}}$. The linear space assigned to the resulting $C_{0,4,\tilde{0}}$ is given by

$$V_{P_3P_5}^{P_1P_2} = \int_{\{P_3\}}^\oplus dP_3 \left( V_{P_3P_2}^{P_3} \otimes V_{P_3}^{P_4P_5} \right).\quad (76)$$

Note that $V_{P_3}^{P_5}$ type of space is not defined in our work yet, their are defined by the comultiplication of chiral algebra $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$

$$\int_{\{P_3\}}^\oplus V_{P_3}^{P_5} \otimes \mathcal{H}_{P_3} = \mathcal{H}_{P_4} \otimes \mathcal{H}_{P_5}.\quad (77)$$
The dimension of $V^P_{P_1 P_2}$ is denoted by $N^P_{P_1 P_2}$, and is identical to $N^P_{P'_1 P'_2}$.

From the fact that the bulk representation cannot change during the propagation of internal propagator, we conclude that

$$V^P_{P_1 P_2} \otimes V^P_{P'_3} = 0, \quad \text{if} \quad P_3 \neq P'_3.$$  \hspace{1cm} (78)

The full closed space $V_{0,4,0}$ is defined as the multiple direct integral over all bulk representation indices.

The open sector separating sewing is given by $C_{g_1,0,\mu_3} \# C_{g_2,0,\mu_3} \to C_{g_1+g_2,0,\mu_3}$. The complete set of boundary states are used to represent the identity operator (74). The simplest case is $C_{0,0,\mu} \# C_{0,0,\mu} \to C_{0,0,\mu}$. The linear space assigned to resulting $C_{0,0,\mu}$ is given by

$$V_{\mu_1 \mu_2}^{\mu_4 \mu_5} = \int_{\{\mu_3\}} d\mu_3 \left( V_{\mu_1 \mu_2}^{\mu_3} \otimes V_{\mu_3}^{\mu_4 \mu_5} \right). \hspace{1cm} (79)$$

Similarly, if the boundary conditions do not match, the tensor products vanish

$$V_{\mu_1 \mu_2}^{\mu_3} \otimes V_{\mu_3}^{\mu_4 \mu_5} = 0, \quad \text{if} \quad \mu_3 \neq \mu'_3. \hspace{1cm} (80)$$

Also, the open closed space $V_{0,0,4}$ is defined as the multiple direct integral over all boundary condition indices.

We can immediately conclude following relations from boundary condition completeness

$$\text{dim}_R [V^P_{P_1 P_2}] = \text{dim}_R [V_{\mu_1 \mu_2}^{\mu_4 \mu_5}], \quad \text{dim}_R [V_{0,4,0}] = \text{dim}_R [V_{0,0,4}]. \hspace{1cm} (81)$$

The definition of S-transformation from $V_{0,4,0}$ to $V_{0,0,4}$ is given by

$$S[V_{0,4,0}] := V_{0,0,4}, \quad S[V^P_{P_1 P_2}] := \prod_{i=1}^4 \int_{\{\mu_i\}} d\mu_i C^P_{P_1 P_2, \mu_{i+1} \mu_{i+2}} V_{\mu_{i+1} \mu_{i+2}}^{\mu_{i+3} \mu_{i+4}}. \hspace{1cm} (82)$$

Assuming the following factorizing property of S-transformation on tensor product

$$S[\otimes A_i] = \otimes S[A_i]. \hspace{1cm} (83)$$

Then we have

$$S[V^{P_3}_{P_1 P_2}] = \int_{\{P_3\}} S[V^P_{P_1 P_2}] \otimes S[V^P_{P_3}]$$

$$= \int_{\{P_3\}} \prod_{\mu_3} \int_{\{\mu\}} C^{P_3}_{P_1 P_2, \mu_3} V_{\mu_3}^{\mu_1 \mu_2} \otimes C^{P_3}_{P_3, \mu_3} V_{\mu_3}^{\mu_4 \mu_5}. \hspace{1cm} (84)$$

Tensor products with $\mu_3 \neq \mu'_3$ do not contribute to the sewing

$$V_{\mu_1 \mu_2}^{\mu_3} \otimes V_{\mu_3}^{\mu_4 \mu_5} = 0, \quad \mu_3 \neq \mu'_3. \hspace{1cm} (85)$$
Using the linear independent condition of open sector basis \( \{V_{\mu_3}^{\mu_1, \mu_2}\} \), we reach the final result

\[
C_{P_1 P_2 \mu_3 \mu_2}^{P_3 P_4 \mu_1 \mu_2} = \int_{P_3} \int_{\mu_3} dP_3 d\mu_3 \ C_{P_1 P_2 \mu_3}^{P_3 P_4 \mu_2} C_{P_3 \mu_4}^{P_5 \mu_3} . \tag{86}
\]

**Closed and open sector fusing matrices:** In conventional studies, fusing matrices \( F \) are introduced to interchange s-channel basis and t-channel basis \([8]\). In the language of linear spaces, \( F \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} \) are defined by (slightly different from conventional notation)

\[
F \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} : \int_{\{P_3\}} V_{P_1 P_2}^{P_3 P_5} \otimes V_{P_3 P_5}^{P_4 P_6} \rightarrow \int_{\{P'_3\}} V_{P_1 P_2}^{P'_3 P_5} \otimes V_{P_3 P_5}^{P'_4 P_6} . \tag{87}
\]

By fixing the specific value of \( P_3 \) and \( P'_3 \), we define

\[
F_{P_3 P'_3} \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} : V_{P_1 P_2}^{P_3 P_5} \otimes V_{P_3 P_5}^{P_4 P_6} \rightarrow V_{P_2 P_4}^{P'_3 P_5} \otimes V_{P'_3 P_5}^{P'_4 P_6} . \tag{88}
\]

\( F_{P_3 P'_3} \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} \) are the linear transformation coefficients for the basis

\[
V_{P_1 P_2}^{P_3 P_5} \otimes V_{P_3 P_5}^{P_4 P_6} = \int_{\{P'_3\}} dP'_3 F_{P_3 P'_3} \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} (V_{P_1 P_2}^{P'_3 P_5} \otimes V_{P'_3 P_5}^{P'_4 P_6}) . \tag{89}
\]

From the fact that the linear mappings preserve operations of addition and scalar multiplication. We expect S-transformations act trivially (commute with) on fusing matrices \( F \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} \), then by acting S-transformation on both sides of \([89]\), we can relate S-transformation coefficients with fusing matrices \( F_{P_3 P'_3} \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} \)

\[
\int d\mu_3 \ C_{P_1 P_2 \mu_3}^{P_3 P_4 \mu_3} C_{P_3 \mu_4}^{P_5 \mu_3} = \int dP'_3 \ C_{P_1 P_2 \mu_3}^{P'_3 P_4 \mu_3} C_{P_3 \mu_4}^{P_5 \mu_3} \tag{90}
\]

where the uniqueness of the S-transformation coefficients are used in concluding \([90]\).

As an analog to the conventional fusing matrices \( F \begin{bmatrix} P_4 & P_2 \\ P_3 & P_1 \end{bmatrix} \), we define open sector fusing matrices \( F^o \begin{bmatrix} \mu_4 & \mu_2 \\ \mu_5 & \mu_1 \end{bmatrix} \) as

\[
F^o \begin{bmatrix} \mu_4 & \mu_2 \\ \mu_5 & \mu_1 \end{bmatrix} : \int_{\{\mu_3\}} V_{\mu_1 \mu_2}^{\mu_3 \mu_5} \otimes V_{\mu_3}^{\mu_4 \mu_5} \rightarrow \int_{\{\mu'_3\}} V_{\mu_2 \mu_4}^{\mu'_3 \mu_5} \otimes V_{\mu'_3}^{\mu_1 \mu_5} . \tag{91}
\]
Figure 8: $C_{0,0,4}$ constructed by unconventional separating open sector sewing, with linear space given by equation (79). Analysing the bulk representations in any regions of $C_{0,0,4}$ will show that open sector coefficients $n$ forming a matrix representation of fusion algebra. One example is shown in equation (95).

By fixing the specific value of $\mu_3$ and $\mu'_3$, we define

$$F_{\mu_3\mu'_3}^{\mu_4\mu_2\mu_5\mu_1} : V_{\mu_3\mu_2}^{\mu_3} \otimes V_{\mu_4\mu_5}^{\mu_3} \rightarrow V_{\mu_3\mu_4}^{\mu'_3} \otimes V_{\mu_3}^{\mu_1\mu_5}.$$

(92)

$F_{\mu_3\mu'_3}^{\mu_4\mu_2\mu_5\mu_1}$ are the linear transformation coefficients for the basis

$$V_{\mu_3\mu_2}^{\mu_3} \otimes V_{\mu_3}^{\mu_4\mu_5} = \int d\mu'_3 F_{\mu_3\mu'_3}^{\mu_4\mu_2\mu_5\mu_1} \left( V_{\mu_3\mu_4}^{\mu'_3} \otimes V_{\mu_3}^{\mu_1\mu_5} \right).$$

(93)

Our definition for open sector fusing matrices $F_{\mu_3\mu'_3}^{\mu_4\mu_2\mu_5\mu_1}$ is different from conventional definition from the associativity of BCCOs four-points functions [14,15,60]. This definition (91) of $F_{\mu_3\mu'_3}^{\mu_4\mu_2\mu_5\mu_1}$ is closer to closed sector fusing matrices $F_{P_4 P_2 P_5 P_1}^{\mu_4 \mu_2 \mu_5 \mu_1}$ [8]. By acting an inversion S-transformation on both sides, we can relate inversion S-transformation coefficients in $C$ with open sector fusing matrices $F_{\mu_3\mu'_3}^{\mu_4\mu_2\mu_5\mu_1}$ similar to (90). The detailed expressions are not written down explicitly.

**Bulk representations on $C_{0,0,4}$:** We analyze the bulk representations on $C_{0,0,4}$ and give an explicit expression for $V_{\mu_3\mu_2}^{\mu_4\mu_5}$. As shown in Figure (8), we have the product $C_{0,0,4}$ defining the linear space $V_{\mu_3\mu_2}^{\mu_4\mu_5}$. There are four different regions, classified by which two boundaries the equal-time curves (open strings) are ending on. We take the region sandwiched by $|\mu_1\rangle$ and $\langle \mu_4|$ to analyze the bulk representation. Recall how
the sewing was accomplished, we have four initial boundaries, with two of
them being sewn
\[ \int_{\{\mu_3\}} H_{\mu_1} \otimes (H_{\mu_3} \otimes H_{\mu_3}) \otimes H_{\mu_3}^*. \]  
(94)
Assuming the associativity and the commutivity of \( V \) and \( H \), we conclude
\[ \int_{\{\mu_3\}} d\mu_3 (H_{\mu_1} \otimes H_{\mu_3}^*) \otimes (H_{\mu_3} \otimes H_{\mu_3}^*) = \int_{\{\mu_3\}} d\mu_3 (V_{\mu_3}^P \otimes V_{\mu_3}^P) \otimes H_{\mu_3} \]  
(95)
To conclude the last equating relation, we have used relations
\[ \int_{\{P\}} dP' V_{P'P} = \int_{\{\mu_3\}} d\mu_3 (V_{\mu_3}^P \otimes V_{\mu_3}^P), \]  
(96)
and
\[ V_{P'P} \otimes H_{P} = 0, \quad \text{if} \quad P' \neq P. \]  
(97)
Both of them are unproven, but it is not difficult to convenience ourselves that they should be correct.

The result indicates that the open sector representation coefficients \( n_{\mu_3\mu_2}^{(P_1)} \)
form a matrix representation (\( \mu \) are matrix indices) of fusion algebra, which is a generalization of fact that \( n_{k,l}^{(P)} \), forming a non-negative integer matrix representation of fusion algebra in diagonal RCFTs [16,17].

3.4.2 Non-separating sewing

Now we turn to the non-separating sewing. The closed sector sewing are expressed as \#\( C_{g,n,\tilde{0}} \rightarrow C_{g+1,n-2,\tilde{0}} \) with \( n \geq 2 \). The linear space assigned to the resulting \( C_{g+1,n-2,\tilde{0}} \) is a direct integral over the indices of two punctures used in the sewing. Take \#\( C_{0,4,\tilde{0}} \rightarrow C_{1,2,\tilde{0}} \) as an example
\[ V_{P_1}^{P_2}(1, 2, 0; A) := \int_{\{P_2\}} dP_2 V_{P_1P_2}^{P_1P_2}, \]  
(98)
where \( A \) denotes that the time evolution in the genus (handle) region is along \( A \)-cycles. The closed sector full linear space is defined as
\[ V_{1,2,0;A} := \int_{\{P_1\}} \int_{\{P_2\}} dP_1 dP_2 V_{P_1}^{P_1}(1, 2, 0; A). \]  
(99)
In open sector, the sewing are expressed as \#\( C_{g,0,\tilde{0}} \rightarrow C_{g+1,0,\tilde{0}-2} \). The linear space assigned to the resulting surface is a direct integral over the
The coefficients $C$ from spaces assigned to $S$-transformation. General $S$-transformations are defined as linear mappings

$$
\text{Closed sector sewing and open sector sewing generalize the definition for } S\text{-cycles. The open sector full linear space is defined as}
$$

$$
V_{1,0,\tilde{2};B} := \int_{\{\mu_1\}}^{\oplus} \int_{\{\mu_4\}}^{\oplus} d\mu_1 d\mu_4 V_{\mu_1}^{\mu_4}(1,0,\tilde{2};B).
$$

Straight-forwardly, we have the following relations

$$
\dim_{\mathbb{R}}[V_{P_1}(1,2,\tilde{0};A)] = \dim_{\mathbb{R}}[V_{\mu_1}^{\mu_4}(1,0,\tilde{2};B)],
$$

and the definition of $S$-transformation from $V_{1,2,\tilde{0};A}$ to $V_{1,0,\tilde{2};B}$ is given by

$$
S[V_{1,2,\tilde{0};A}] := V_{1,0,\tilde{2};B},
$$

$$
S[V_{P_1}(1,2,\tilde{0};A)] := \int_{\{\mu_1\}}^{\oplus} \int_{\{\mu_4\}}^{\oplus} d\mu_1 d\mu_4 C_{P_1\mu_4}^{P_4\mu_1}(1,2) V_{\mu_1}^{\mu_4}(1,0,\tilde{2};B).
$$

The coefficients $C_{P_1\mu_4}^{P_4\mu_1}(1,2)$ given by

$$
C_{P_1\mu_4}^{P_4\mu_1}(1,2) = \int_{P_2} dP_2 \int_{P_3} dP_3 \int_{\mu_2} d\mu_2 d\mu_3 C_{P_1\mu_3}^{P_2\mu_3} C_{P_2\mu_4}^{P_3\mu_2}. \quad (104)
$$

Note that, if we do not cut the boundaries, our global $S$-transformation shares some similarities in transforming from the necklace channel block basis into OPE channel block basis by crossing kernels [23,61]. However, our definition is different since the basis we used in $S$-transformations are not identical.

### 3.4.3 General S-transformations

Closed sector sewing and open sector sewing generalize the definition for $S$-transformation. General $S$-transformations are defined as linear mappings from spaces assigned to $C_{g,n,\tilde{0}}$ to spaces assigned $C_{g,0,\tilde{b}=n}$.

Denote the closed sector linear space assigned to $C_{g,n,\tilde{0}}$ as $V_{P_1}^{g,n,\tilde{0}}(g,n,\tilde{0};A^g)$, and the open sector linear space assigned to $C_{g,0,\tilde{b}=n}$ as $V_{\mu_1}^{\mu_4}(g,0,\tilde{b} = n; B^g)$. $A^g$ and $B^g$ denote that time evolution in all $g$-handles region are along $A$-cycles or $B$-cycles. Corresponding closed and open sector linear spaces have identical dimension, which is guaranteed by boundary condition completeness

$$
\dim_{\mathbb{R}}[V_{P_1}^{g,n,\tilde{0}}(g,n,\tilde{0};A^g)] = \dim_{\mathbb{R}}[V_{\mu_1}^{\mu_4}(g,0,\tilde{b} = n; B^g)], \quad (105)
$$

28
\[ V(g, n, \tilde{0}; A^g) := \prod_{\nu \in P} \int_{\{\nu\}} V^g_{\nu_1 \ldots \nu_n}(g, n, \tilde{0}; A^g), \]

\[ V(g, 0, \tilde{b} = n; B^g) := \prod_{\eta \in \mu} \int_{\{\mu\}} \bigoplus_{\mu} V^g_{\mu_1 \ldots \mu_n}(g, 0, \tilde{b} = n; B^g), \quad (106) \]

\[ \dim_\mathbb{R}[V(g, n, \tilde{0}; A^g)] = \dim_\mathbb{R}[V(g, 0, \tilde{b} = n; B^g)], \quad (107) \]

The definition of S-transformation is given by

\[ S[V(g, n, \tilde{0}; A^g)] := V(g, 0, \tilde{b} = n; B^g), \]

\[ S[V^g_{\mu_1 \ldots \mu_n}(g, n, \tilde{0}; A^g)] := \prod_{j=1}^n \int_{\{\mu_j\}} d\mu_j \ C^g_{\mu_1 \ldots \mu_n}(g, n) V^g_{\mu_1 \ldots \mu_n}(g, 0, \tilde{b} = n; B^g), \quad (108) \]

where the coefficients \( C^g_{\mu_1 \ldots \mu_n}(g, n) \) can be represented as integral over products of basic S-transformation coefficients \( C^g_{\mu_1 \mu_2 \mu_3} \).

### 3.5 Setup on covariant type surfaces

Up to now, our discussion on S-transformation and sewing are restricted to light-cone type of surfaces. We may also assign linear space to covariant type of surfaces, and define the S-transformation. We shall roughly describe the difference when using covariant type of surface \[44-46\].

**Bi-local operators:** For covariant \( C_{0,3,\tilde{0}} \), negative curvature are localized at two branch points \( z_{b_1} \) of \( v(z) \) with identical value of time \( t \), we expect the closed sector projection mappings are achieved by bi-local operators at the two branch points \[40\]

\[ \Phi^g_{P_1 P_2}(z_{b_1}; z_{b_2}) : \mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2} \rightarrow \mathcal{H}_{P_3}, \quad (109) \]

which are shown on the left picture of Figure \[9\]. The linear space assigned to covariant type \( C_{0,3,\tilde{0}} \) are identical to the space assigned to light-cone type of surfaces.

The open sector bulk representation interchanging are also achieved by bi-local operators

\[ \Phi^\mu_{P_1 P_2}(z_{b_1}; z_{b_2}) : n^\sigma_{\mu \pi \mu_1}(P_3) \mathcal{H}_\sigma(P_3) \leftrightarrow n^\sigma_{\mu \mu_2 \pi}(P_1) \mathcal{H}_\sigma(P_1), \quad (110) \]

which are shown on the right picture of Figure \[9\]. Again, the linear space assigned to covariant type \( C_{0,0,3} \) are identical to that assigned to light-cone type surface.
Operators achieving the mappings are both bi-local at two branch points $z_{b1}$, $z_{b2}$.

**Moduli space covering:** It is well-known that sewing of surfaces constructed from covariant string vertices, cannot make a full covering on the moduli space of the resulting surface [44–46]. The simplest example is that sewing of covariant vertices $V_{0,3,0}\#\emptyset$ cannot fully cover $\mathcal{M}_{0,4}$, there is a region $\mathcal{V}_{0,4}$ corresponding to fundamental tree-level, four-point interaction. When turning to the open sector, there is also vertex region $\mathcal{V}_{0,0,4}$ which cannot be reached by open sector sewing.

However, this is not a problem for our objective, since we can simply assign the surfaces in the string vertices region with the identical linear spaces with surfaces in Feynman regions (regions that can be reached by sewing). For example

$$V_{P_{1}P_{2}}^{P_{3}}(\mathcal{V}_{0,4}) := \int_{\{P_{3}\}} \circ \bigotimes_{\mu_{3}} V_{P_{1}P_{2}}^{P_{3}} \otimes V_{P_{1}P_{2}}^{P_{3}},$$

(111)

$$V_{\mu_{4}\mu_{5}}^{\mu_{1}\mu_{2}}(\mathcal{V}_{0,0,4}) := \int_{\{\mu_{3}\}} \circ \bigotimes_{\mu_{3}} V_{\mu_{1}\mu_{2}}^{\mu_{3}} \otimes V_{\mu_{1}\mu_{2}}^{\mu_{3}}.$$

(112)

The operators to achieve bulk representation mappings on $\mathcal{V}_{0,4}$ and $\mathcal{V}_{0,0,4}$ are operators having dependence on four positions. For example

$$\Phi_{P_{1}P_{2}}^{P_{3}}(\mathcal{V}_{0,4})(z_{b1}; z_{b2}; z_{b3}; z_{b5}) : \mathcal{H}_{P_{1}} \otimes \mathcal{H}_{P_{2}} \rightarrow \mathcal{H}_{P_{1}} \otimes \mathcal{H}_{P_{3}}.$$

(113)

We can see that $\Phi_{P_{1}P_{2}}^{P_{3}}(\mathcal{V}_{0,4})$ do not process a internal propagator, which is expected.

Linear spaces assigned to general surfaces in string vertices regions can be defined in similar fashion, which means we are free to define S-transformations for CFT$_{2}$ on both type of surfaces.
3.6 Algebraic information of CFT$_2$ and curvature singularities on surfaces

We discuss the relations between the algebraic information of CFT, that includes bulk representations and boundary conditions, with the curvature on the surface. At first look, they seem to be entirely distinguished topics. However, we have observed that the vertex operator insertion results in positive curvature localization on the surface, and interaction results in negative curvature localization on the surface, so curvature on the surface and algebraic information changing are related. In summary, we conclude:

- String vertex operator insertions occurs at the first order zeros of the time evolution generating vector fields, which are points where positive curvature are localized. This indicates that CFT$_2$ algebraic information in and output occurs at points with positive curvature singularities $K > 0$.

- String interaction occurs at branch points or poles of the time evolution generating vector fields, which are points where negative curvature are localized. This indicates that CFT$_2$ algebraic information occurs at points with negative curvature singularities $K < 0$.

- Regular points of time evolution generating vector field, corresponding to points with zero curvature $K = 0$, indicating that there is no algebraic information change when there is no non-zero curvature.

These conclusions are only true when all $\alpha \neq 0$. If some of the punctures have $\alpha = 0$, then they collide with interacting points, algebraic changing happens at points with zero curvature. However, if we order all punctures to process a local disk, corresponding to external string propagator, all $\alpha \neq 0$ is a necessary condition.

4 Application to diagonal RCFTs

Diagonal RCFTs can be applied to our setup straightforwardly. By diagonal, we mean that the anti-holomorphic sector representation is the charge conjugate dual representation of the holomorphic sector $\mathcal{H}_i \otimes \bar{\mathcal{H}}_{i\ast}$.

The description in this section will be parallel with section 3 but we will only describe the points we regard as crucial, since the general setup is already finished.

4.1 Facts about diagonal RCFTs

First, we shall quickly review some facts about diagonal RCFTs.
**Spectrum:** Only a finite number of representations are involved

\[ \mathcal{H} = \bigoplus_i d_i(\mathcal{H}_i \otimes \mathcal{H}_{i^*}). \]  

(114)

**Cardy boundary states:** For diagonal RCFTs, the solutions to the gluing conditions (34) are known as the Ishibashi states \[ |i\rangle\rangle, \] which are labeled by bulk chiral representation indices \[ \{i\} \] [11,12]. Physical boundary states for RCFTs are Cardy boundary states, which is the following linear combination of Ishibashi states [13]

\[ |\tilde{\ell}\rangle = \sum_j S_j^i \sqrt{S_j^0} |j\rangle\rangle, \]  

(115)

where \( S_j^i \) are unitary modular S-matrices, describing the S-transformation of chiral algebra characters on a torus.

Cardy boundary states correspond to admissible boundary conditions in diagonal RCFTs, and the labeling \( \{\tilde{\ell}\} \) indicate the existence of injective mapping from boundary conditions \( \{\tilde{i}\} \) to chiral bulk representations \( \{i\} \). The boundary conditions are complete if \( \#\{\tilde{\ell}\} = \#\{i\} \) [16], where the identity operator can be represented by Cardy boundary states

\[ I = \sum_k |\tilde{k}\rangle\langle\tilde{k}|. \]  

(116)

### 4.2 Linear space isomorphisms and S-transformation definition

The closed sector linear spaces \( V_{ij}^k \) are the spaces of representation coupling

\[ \mathcal{H}_i \otimes \mathcal{H}_j = \bigoplus_k (V_{ij}^k \otimes \mathcal{H}_k), \]  

(117)

whose dimensions are non-negative integers

\[ N_{ij}^k \in \mathbb{N}. \]  

(118)

Full closed sector interaction space is given by a triple direct sum

\[ V_{0,3,\bar{0}} = \bigoplus_{i,j,k} V_{ij}^k, \quad \dim_{\mathbb{R}}[V_{0,3,\bar{0}}] = \sum_{i,j,k} N_{ij}^k. \]  

(119)

Both \( N_{ij}^k \) and \( \dim_{\mathbb{R}}[V_{0,3,\bar{0}}] \) are finite.

The bulk representations in region a and b define two different linear spaces \( V_{ji}^{ij} \) and \( V_{kj}^{ij} \)

\[ \mathcal{H}_i \otimes \mathcal{H}_{i^*} = \bigoplus_{j^*} V_{ji}^{ij} \otimes \mathcal{H}_{j^*}, \]  

(120)
\[ \mathcal{H}_j \otimes \mathcal{H}_{k^r} = \bigoplus_i \mathcal{V}^j_{ki} \otimes \mathcal{H}_{k^r}, \tag{121} \]

whose dimensions are \( n^j_{ki} \) and \( n^i_{kj} \). The linear space \( \mathcal{V}^k_{ij} \) is then defined as

\[ \mathcal{V}^k_{ij} := \begin{cases} \mathcal{V}^j_{ki} & \text{when equal-time curve in region a;} \\ \mathcal{V}^i_{kj} & \text{when equal-time curve in region b.} \end{cases} \tag{122} \]

The automorphism \( \omega \) is the charge dual for diagonal RCFTs. From the well-known fact that for diagonal RCFTs \( n^i_{kj} = n^j_{ki} = N^k_{ij} \), we can conclude that

\[ \mathcal{V}^k_{ij} \cong \mathcal{V}^k_{ij}. \tag{123} \]

The full open sector interaction space is the triple direct sum

\[ \mathcal{V}_{0,3} = \bigoplus_{i,j,k} \mathcal{V}^k_{ij}, \quad \dim_{\mathbb{R}}[\mathcal{V}_{0,3}] = \sum_{i,j,k} n^i_{kj} = \dim_{\mathbb{R}}[\mathcal{V}_{0,3,0}]. \tag{124} \]

When the boundary conditions are complete, we can conclude that

\[ \mathcal{V}_{0,3,0} \cong \mathcal{V}_{0,0,3}. \tag{125} \]

The definition of S-transformation coefficients for diagonal RCFTs is given by

\[ S[\mathcal{V}_{0,3}] = \mathcal{V}_{0,3}, \quad S[\mathcal{V}^k_{ij}] = \bigoplus_{i,j,k} C^k_{ijk} \mathcal{V}^k_{ij}. \tag{126} \]

The partition functions modular invariance on \( C_{0,3} \) are given by

\[ N^k_{ij} \chi_i(q) \chi_j(q) \chi_k(q) = \sum_{i,j,k} C^k_{ijk} n^i_{kj} \chi_i(q_a) \chi_j(q_b). \tag{127} \]

Since there are in total a finite number of such equations for RCFTs, attempts to solve these equations are possible in principle.

Inversion S-transformations are given by

\[ S^{-1}[\mathcal{V}_{0,3}] = \mathcal{V}_{0,3}, \quad S^{-1}[\mathcal{V}^k_{ij}] = \bigoplus_{i,j,k} \text{inv} C^k_{ijk} \mathcal{V}^k_{ij}. \tag{128} \]

with the product of S-transformation coefficients and inversion S-transformation coefficients given by

\[ \sum_{i,j,k} \text{inv} C^k_{ijk} C^l_{ijk} = \delta_{ij} \delta_{jk} \delta_{kl}. \tag{129} \]
\[ I = \sum_k |\tilde{k}\rangle\langle\tilde{k}| \]

Figure 10: Example of separating Cardy boundary states sewing. We can take one of the four regions to analyze the bulk representation, as equation (95). The result indicates that coefficients \( n_{ji} \) form a non-negative integer matrix representation of the fusion algebra.

### 4.3 Cardy boundary state sewing

Cardy boundary state sewing is enabled by boundary condition completeness \( \#\{i\} = \#\{\tilde{i}\} \)

\[ I = \sum_k |\tilde{k}\rangle\langle\tilde{k}|. \tag{130} \]

Consider the simplest separating \( C_{0,0,\tilde{3}} \# C_{0,0,\tilde{3}} \rightarrow C_{0,0,\tilde{4}} \) sewing. The linear spaces assigned to \( C_{0,0,\tilde{4}} \) are given by a direct sum over tensor product of linear spaces defined by two initial \( C_{0,0,\tilde{3}} \)

\[ V_{ij}^{\tilde{l}\tilde{m}} = \bigoplus_k (V_k^l \otimes V_k^{\tilde{m}}). \tag{131} \]

The dimension of \( V_{ij}^{\tilde{l}\tilde{m}} \) is

\[ \dim \left( V_{ij}^{\tilde{l}\tilde{m}} \right) = \sum_k n_{kj}^l n_{ik}^m = \cdots = \sum_k N_{ij}^k N_{km}^l, \tag{132} \]

where \( \cdots \) denotes four ways to represent the result, corresponding to four different regions.

\( n_{kj}^l \) as a non-negative integer matrix representation for fusion algebra: Repeat the analysis in (95) will show the open sector representation coefficients \( n_{kj}^l \) forming a matrix representation for fusion algebra [17].

### 4.4 Example: Ising model \( \mathcal{M}(4,3) \)

Consider a detailed diagonal RCFT model, namely the Ising model \( \mathcal{M}(4,3) \) [2]. Ising model is one of the Virasoro minimal models \( \mathcal{M}(q,p) \), which consist of three primary operators: unit operator \( I = \phi(1,1) = \phi(2,3) \), \( h_I = (1,1) \); spin operator \( \sigma = \phi(2,2) = \phi(1,2) \), \( h_\sigma = (\frac{1}{16}, \frac{1}{16}) \); and energy density
operator $\epsilon = \phi_{(2,1)} = \phi_{(1,3)}$, $h_\epsilon = (\frac{1}{2}, \frac{1}{2})$. All the three presentations are self-dual. $\mathcal{H}_{\text{Ising}}$ is given by

$$\mathcal{H}_{\text{Ising}} = (\mathcal{H}_I \otimes \bar{\mathcal{H}}_I) \oplus (\mathcal{H}_\sigma \otimes \bar{\mathcal{H}}_\sigma) \oplus (\mathcal{H}_\epsilon \otimes \bar{\mathcal{H}}_\epsilon).$$  \hspace{1cm} (133)

Non-trivial fusion products (trivial refers to fusion products with $\mathcal{H}_I$ and those with zero-dimension) are listed

$$\mathcal{H}_\sigma \otimes \mathcal{H}_\sigma = \mathcal{H}_I \oplus \mathcal{H}_\epsilon, \quad \mathcal{H}_\epsilon \otimes \mathcal{H}_\epsilon = \mathcal{H}_I, \quad \mathcal{H}_\sigma \otimes \mathcal{H}_\epsilon = \mathcal{H}_\sigma.$$  \hspace{1cm} (134, 135, 136)

All non-zero dimensional closed sector interaction linear spaces are listed

$$V^I_{II}, V^\sigma_{I\sigma}, V^\epsilon_{I\epsilon}, V^\sigma_{\sigma\epsilon}, \cdots,$$  \hspace{1cm} (137)

where $\cdots$ denote the permutations of the spaces written down. All spaces process dimension 1, and the dimension for full closed sector space of Ising CFT is 10

$$\text{dim}_R[V_{0,3;0;\text{Ising}}] = 10.$$  \hspace{1cm} (138)

Now, we turn to the open sector. There are in total three admissible boundary conditions for boundary Ising CFT$_2$: $+$, $-$ and $f$ [9]. The Cardy boundary states correspond to the boundary conditions in the following sense [13]

$$|+\rangle = |\bar{0}\rangle, \quad |\!-\!\rangle = |\bar{\frac{1}{2}}\rangle, \quad |f\rangle = |\bar{\frac{1}{16}}\rangle.$$  \hspace{1cm} (139)

The non-zero open sector representation coefficients can be obtained from $n^I_{kj} = N^k_{ij}$ relation. We shall put the corresponding bulk representation on different regions of surfaces and assigned the open sector linear spaces to them. For example, for surface with all boundary conditions being $+$, we put $\mathcal{H}_I$ in both two regions. The two regions define two identical linear spaces $V^+_{++}$ and $V^+_{++}$. By the time dependent definition (41) of open sector linear spaces, the surface is assigned with $V^+_{++}$.

Non-zero dimensional open sector linear spaces are listed

$$V^+_{++}, V^f_{++}, V^-_{++}, V^{-f}_{++}, \cdots,$$  \hspace{1cm} (140)

with in total 10 non-zero dimensional spaces, giving the fact that the total dimension for interaction in

$$\text{dim}_R[V_{0,0,3;\text{Ising}}] = 10.$$  \hspace{1cm} (141)

The isomorphic relations between closed and open sector linear spaces are straight-forward

$$V^I_{II} \cong V^+_{++}, \quad V^\sigma_{I\sigma} \cong V^f_{++}, \quad \cdots.$$  \hspace{1cm} (142)
The S-transformation from closed to open sector linear spaces are given by

\[ S[V_{ij}^k] = \bigoplus_{i,j,k} C_{ijk}^{\tilde{i}\tilde{j}\tilde{k}} V_{ij}^k. \]  

(143)

where the RHS of equation (143) include totally 10 terms, consisting of all non-zero dimensional \( V_{ij}^k \). The total number of such S-transformation is also 10, since non-zero dimensional closed sector spaces \( V_{ij}^k \) for Ising model is 10.

Partition function modular invariance on are induced from S-transformation. Takes the equation from S-transformation of \( V_{IJ} \), we obtain

\[ \chi_0(q_1)\chi_0(q_2)\chi_0(q_3) = \sum_{i,j,k} C_{0ij}^{00k} \chi_i^*(q_a) \chi_j^*(q_b) \]

\[ = C_{00+}^{00+} \chi_0(q_a)\chi_0(q_b) + C_{00f}^{00f} \chi_0(q_a)\chi_0(q_b) + \cdots \]  

(144)

The closed and open sector sewing for Ising model is also straight-forward. \( V_{ij}^{lm} \) are constructed from 10 one-dimensional \( V_{ij}^k \) by

\[ V_{IJ}^{lm} = V_{IJ}^{l} \otimes V_{IJ}^{m}, \quad V_{\sigma\sigma}^{\epsilon\epsilon} = (V_{\sigma\sigma}^{l} \otimes V_{\sigma\sigma}^{m}) \oplus (V_{\epsilon\epsilon}^{l} \otimes V_{\epsilon\epsilon}^{m}), \cdots \]

Similarly, \( V_{ij}^{lm} \) are constructed from 10 one-dimensional \( V_{ij}^k \) by

\[ V_{++}^{++} = V_{++}^{l} \otimes V_{++}^{m}, \quad V_{ff}^{ff} = (V_{ff}^{l} \otimes V_{ff}^{mf}) \oplus (V_{ff}^{-l} \otimes V_{ff}^{ml}), \cdots \]

Generalize S-transformation definitions are also straight-forward.

5 Summary and open questions

S-transformations for CFT\(_2\) are defined as linear mappings from closed to open sector interaction linear spaces. The setup for the definition is based on closed-open linear space isomorphisms and boundary condition completeness. The S-transformation coefficients are constrained by partition function modular invariance on \( C_{00\tilde{3}} \). An unconventional open sector sewing rises naturally from the setup, which is the sewing of complete boundary states. Such unconventional open sector sewing, together with conventional closed sector sewing, generalize the definition of S-transformations. General S-transformations are defined as linear mappings from closed sector linear spaces, assigned to \( C_{g,n,\tilde{0}} \) to open sector linear spaces, assigned to \( C_{g,0,b=n} \). Diagonal RCFTs can be applied to our definition straight-forwardly. We also observed the relation between algebraic information changing and curvature on the surface.
Open question (application to Liouville field theory): We would like to investigate whether Liouville field theory is applicable to our definition [29,30,33,35–38].

Open question (more precise construction on linear spaces): The closed and open sector linear spaces introduced are constructed only to the level of representations and boundary conditions. We are willing to construct the closed and open linear spaces defined by precise vectors from different representation $H_P$, and from boundary state basis $H_\mu$. This will be necessary if we want to consider practical problems in modular-S bootstrap problems using our definition [58].

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A Proof: Poles in $f(z)$ are contact interacting points on light-cone type surfaces

Consider an order-$k$ pole $z_0$ in $f(z)$

$$f(z_0) \sim \frac{C}{(z - z_0)^k}, \quad k \in \mathbb{N}. \quad (145)$$

There exist regular points $z_a$ near $z_0$, which will eventually hit $z_0$ when we tune the value of time $t$.

Now, we want to determine how many $z_a$ are there in total. This can be solved by a simple integral

$$\int_{z_a}^{z_0} d(t + is) = \int_{z_a}^{z_0} \frac{dz}{f(z)} \sim \frac{1}{C} \int_{z_a}^{z_0} (z - z_0)^k. \quad (146)$$

By setting $(t + is)(z_0) = 0$, which does not affect anything, we obtain that

$$(t + is)(z_a) = \frac{1}{C}(z_a - z_0)^{k+1}, \quad (147)$$

indicating that there are in total $k + 1$ $z_a$ near $z_0$ which will hit $z_0$. This means that an order-$k$ pole of $f(z)$ is a point where $k + 1$ points will collide. The result also stands for $k = 0$ regular points of $f(z)$.

Now, we turn back to $f(z)$ for light-cone type $C_{0,3,\hat{0}}$. The first order pole $z_0$ is the only point where two points can collide, thus, $z_0$ is the contact interacting point on light-cone type $C_{0,3,\hat{0}}$.  

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B  Derivation for the time evolution generating vector field on covariant cubic string vertex

In this appendix, we show how equation (24) is derived. First, we need to review some basic knowledge in Witten’s open string field theory (OSFT) defined on the upper half-plane (UHP) \([47, 49, 63, 64]\). This appendix overlaps largely with the lecture notes by Taylor and Zwiebach \([49]\).

**Basic facts for Witten’s OSFT:** Open strings interaction by half-string identifications in Witten’s OSFT. An off-shell action for Witten’s OSFT was proposed \([47]\)

\[
S = \frac{-1}{2g^2} \left( \int \Psi \star Q \Psi + \frac{2}{3} \int \Psi \star \Psi \star \Psi \right),
\]

(148)

where \(\Psi\) is the string field, and \(g\) is the string coupling. The string field \(\Psi\) takes value in a graded algebra \(A\), associated with a star product

\[
\star : \ A \otimes A \rightarrow A,
\]

(149)

with the degree \(G\) being additive under star product \((G_{\Psi_1 \star \Psi_2} = G_{\Psi_1} + G_{\Psi_2})\).

There exist a BRST operator \(Q\) with degree 1

\[
Q : \ A \rightarrow A, \quad G_{Q \Psi} = 1 + G_{\Psi}.
\]

(150)

Integration over string fields are defined as mappings from \(A\) to complex numbers

\[
\int : \ A \rightarrow \mathbb{C}.
\]

(151)

The elements \(Q, \star, \int\) in Witten’s OSFT satisfy the following five axioms

- Nilpotency of \(Q\): \(Q^2 \Psi = 0\), \(\forall \Psi \in A\).
- \(\int Q \Psi = 0\), \(\forall \Psi \in A\).
- Derivation property of \(Q\):

\[
Q(\Psi \star \Phi) = (Q \Psi) \star \Phi + (-1)^{G_\Psi} \Psi \star (Q \Phi), \quad \forall \Psi, \Phi \in A.
\]

- Cyclicality: \(\int \Psi \star \Phi = (-1)^{G_\Psi G_\Phi} \int \Phi \star \Psi\), \(\forall \Psi, \Phi \in A\).
- Associativity: \((\Psi \star \Phi) \star \Omega = \Psi \star (\Phi \star \Omega)\), \(\Psi, \Phi, \Omega \in A\).

The OSFT action (148) is invariant under gauge transformation

\[
\delta \Psi = QA + \Psi \star A - A \star \Psi,
\]

(152)
where $\Lambda$ is the degree 0 gauge parameter (classical OSFT is considered here, not the quantum OSFT with full degree expansion). The equation of motion for the string field $\Psi$ can be obtained by verifying the SFT action

$$Q\Psi + \Psi \ast \Psi = 0.$$  \hspace{1cm} (153)

From now on we shall write the star product into the form of inner products defined by

$$\langle A, B \rangle := \int A \ast B,$$  \hspace{1cm} (154)

and the multi-linear objects mapping three string fields into a number

$$\langle A, B, C \rangle := \langle A, B \ast C \rangle.$$  \hspace{1cm} (155)

**Cubic vertex computation from CFT approach:** Following the work [63], the inner products (154) and three strings multi-linear objects (155) can be defined as CFT two-point and three-point functions

$$\langle \Psi_1, \Psi_2 \rangle := \langle I \circ \Psi_2(0) \Psi_1(0) \rangle,$$  \hspace{1cm} (156)

$$\langle \Psi_1, \Psi_2, \Psi_3 \rangle := \langle f_1 \circ \Psi_1(0) f_2 \circ \Psi_2(0) f_3 \circ \Psi_3(0) \rangle,$$  \hspace{1cm} (157)

where $I$ is the inversion mapping and $f_i$ are specific conformal mappings from local half disks to the UHP.

Now, we shall combine the local coordinate half-disks \{\(w_1^\text{co} | \leq 1, \text{Im}(w_1^\text{co}) \geq 0\}\}. Three half-string identifications are carried out by

\[
\begin{align*}
w_1^\text{co} w_2^\text{co} &= -1, & |w_1^\text{co}| &= 1, & \text{Re}(w_1^\text{co}) &\leq 0; \\
w_2^\text{co} w_3^\text{co} &= -1, & |w_2^\text{co}| &= 1, & \text{Re}(w_2^\text{co}) &\leq 0; \\
w_3^\text{co} w_1^\text{co} &= -1, & |w_3^\text{co}| &= 1, & \text{Re}(w_3^\text{co}) &\leq 0.
\end{align*}
\]

(158)

Then, three local half-disks will be combined into a unit disk by following steps. First, we map the upper half-disks to right half-disks by a $SL(2, \mathbb{C})$ transformation $h$

$$h(w_i^\text{co}) = \frac{1 + i w_i^co}{1 - i w_i^co}. \hspace{1cm} (159)$$

Then, the three right half-disks are squeezed into three 120 degrees wedges, and rotated to a unit disk. Now, the maps $f_i$ are clear:

$$f_1(w_1^\text{co}) = e^{\frac{2\pi i}{3}} \left( \frac{1 + i w_1^\text{co}}{1 - i w_1^\text{co}} \right)^{2/3},$$

$$f_2(w_2^\text{co}) = \left( \frac{1 + i w_2^\text{co}}{1 - i w_2^\text{co}} \right), \hspace{1cm} (160)$$

$$f_3(w_3^\text{co}) = e^{-\frac{2\pi i}{3}} \left( \frac{1 + i w_3^\text{co}}{1 - i w_3^\text{co}} \right)^{-2/3}.$$

An inversion transformation $h^{-1}$ of (159) maps the unit disk to UHP, with three string punctures at $z = 0, \pm \sqrt{3}$, and three-strings midpoint at $z = i$.
Derivation for $v(z)$: Now, we are ready to obtain the time evolution vector for cubic string vertex. We order puncture 1 and 3 being initial string punctures, and puncture 2 being the final product string puncture. The time evolution given by disk local coordinates are

$$\frac{\partial}{\partial t} = w_{1}^{\text{co}} \frac{\partial}{\partial w_{1}^{\text{co}}} + \text{c.c.} = w_{3}^{\text{co}} \frac{\partial}{\partial w_{3}^{\text{co}}} + \text{c.c.} = -w_{2}^{\text{co}} \frac{\partial}{\partial w_{2}^{\text{co}}} + \text{c.c.} \quad (161)$$

in the regions where $w_{i}^{\text{co}}$ cover, and c.c. denotes complex conjugate. The relation between the local disk coordinates and the UHP global coordinate is determined from the setup. Thus, we obtain the time evolution generating vector field $v(z)$ in equation (21). If we extend the evolution to $\mathbb{C} \cup \{\infty\}$, then $v(z)$ can be considered as the time evolution generating vector for covariant vertex $\mathcal{V}_{0,3}$.

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