THE GENERALIZED HÖLDER’S INEQUALITIES AND THEIR APPLICATIONS IN MARTINGALE SPACES

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ABSTRACT. We give the generalized Hölder’s inequalities for integral and conditional expectation. Moreover, a generalized Doob maximal operator is introduced and weighted inequalities for the operator are established by the applications of the generalized Hölder’s inequalities.

1. Introduction

1.1. Weighted Inequalities for the Hardy-Littlewood Maximal Operator and the Multisublinear One in \(R^n\). Let \(R^n\) be the \(n\)-dimensional real Euclidean space and \(f\) a real valued measurable function. The classical Hardy-Littlewood maximal operator \(M\) is defined by

\[
Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,
\]

where \(Q\) is a non-degenerate cube with its sides parallel to the coordinate axes and \(|Q|\) is the Lebesgue measure of \(Q\).

Let \(u, v\) be two weights, i.e., positive measurable functions. As is well known, for \(p \geq 1\), Muckenhoupt [15] showed that the inequality

\[
\lambda^p \int_{\{Mf > \lambda\}} u(x)dx \leq C \int_{R^n} |f(x)|^p v(x)dx, \ \lambda > 0, \ f \in L^p(v)
\]

holds if and only if \((u, v) \in A_p\), i.e., for any cube \(Q\) in \(R^n\) with sides parallel to the coordinates

\[
\left( \frac{1}{|Q|} \int_Q u(x)dx \right) \left( \frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p-1}}dx \right)^{p-1} < C, \ p > 1;
\]

\[
\frac{1}{|Q|} \int_Q u(x)dx \leq C \text{ess inf}_{Q} v(x), \ p = 1.
\]

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Suppose that $u = v$ and $p > 1$, Muckenhoupt [15] also proved that
\[ \int_{\mathbb{R}^n} (Mf(x))^p v(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \ \forall f \in L^p(v) \]
holds if and only if $v$ satisfies
\[ (\int_Q v(x) dx) \left( \frac{1}{|Q|} \int_Q v(x) 1^{-p'} dx \right)^{p-1} < C, \ \forall Q. \tag{1.1} \]
The crucial step is to show that if $v$ satisfies $A_p$, then there is an $\varepsilon > 0$ such that $v$ also satisfies $A_{p-\varepsilon}$. But, the problem of finding all $u$ and $v$ such that
\[ \int_{\mathbb{R}^n} (Mf(x))^p u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \ \forall f \in L^p(v) \]
is much hard and complicated. In order to solve the problem, Sawyer [18] established the testing condition $S_{p,q}$, i.e., for any cube $Q$ in $\mathbb{R}^n$ with sides parallel to the coordinates
\[ \left( \int_Q (Mf(x))^q u(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_Q v(x)^1 dx \right)^{\frac{1}{p}}, \ \forall Q \]
where $1 < p \leq q < \infty$. The condition $S_{p,q}$ is a sufficient and necessary condition such that the weighted inequality
\[ \left( \int_{\mathbb{R}^n} (Mf(x))^q u(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{\frac{1}{p}}, \ \forall f \in L^p(v) \]
holds. In this case, the method of proof is very interesting. Motivated by these results, the theory of weighted inequalities developed rapidly in the last years, not only for the Hardy-Littlewood maximal operator but also for some of the main operators in Harmonic Analysis like Caldersor-Zygmund operators (see [5] and [4] for more information).

Recently, the multisublinear maximal function
\[ \mathcal{M}(f_1, \ldots, f_m)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i \tag{1.2} \]
associated with cubes with sides parallel to the coordinate axes was studied in [12]. The importance of this operator is that it generalizes the Hardy–Littlewood maximal function (case $m = 1$) and in several ways it controls the class of multilinear Calderon–Zygmund operators as it is shown in [12]. The relevant class of multiple weights for $\mathcal{M}$ is given by the condition $A_{\vec{p}}$ : for $\vec{p} = (p_1, p_2, \ldots, p_m)$, $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)$ and a weight $v$, the weight vector $(v, \vec{\omega}) \in A_{\vec{p}}$ if
\[ \sup_Q \frac{v(Q)}{|Q|} \prod_{i=1}^m (\sigma_i(|Q|) \omega_i)^{\frac{1}{p_i}} < \infty, \]
where $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ and $1 \leq p_1, p_2, \ldots, p_m < \infty$. 
It is easy to see that in the linear case (that is, if \( m = 1 \)), condition \( A_{\frac{1}{2}} \) is the usual \( A_{p} \). In [12] the following multilinear extension of the Muckenhoupt \( A_p \) theorem for the maximal function was obtained: the inequality

\[
\|\mathcal{M}(\mathcal{f})\|_{L^p,\infty(v)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}, \ \forall f_i \in L^{p_i}(\omega_i)
\]

holds if and only if \((v, \mathcal{F}) \in A_{\frac{1}{2}}\). Moreover, if \( 1 < p_1, p_2, \ldots, p_m \) and \( v = \prod_{i=1}^{m} w_i^{p_i/p_i} \), then the inequality

\[
\|\mathcal{M}(\mathcal{f})\|_{L^p(v)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}, \ \forall f_i \in L^{p_i}(\omega_i)
\]

holds if and only if \((v, \mathcal{F}) \in A_{\frac{1}{2}}\). The more general case was extensively discussed in [7, 6].

In order to establish the generalization of Sawyer’s theorem to the multilinear setting, a kind of monotone property and a reverse Hölder’s inequality on the weights were introduced in [11] and [2], respectively. They both obtained the multilinear version of Sawyer’s result.

In this paper, we define a new generalized maximal function

\[(1.3) \quad \mathcal{M}(\mathcal{f})(x) \triangleq \sup_{x \in Q} \prod_{i=1}^{\infty} \frac{1}{|Q|} \int_{Q} |f_i(y_i)|dy_i \]

for suitable \( \mathcal{f} = (f_1, f_2, \ldots) \). If \( \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty \) and \( \sum_{i=1}^{\infty} \frac{1}{p_i} = \frac{1}{p} \), then

\[
\|\mathcal{M}(\mathcal{f})\|_{L^p} \leq \prod_{i=1}^{\infty} \|Mf_i\|_{L^p} \leq \left( \prod_{i=1}^{\infty} p_i' \right) \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty,
\]

where we have used Theorem 2.11 and Theorem 2.12 in Section 2. Then it is natural to establish weighted inequalities for it. But, the method of [12] is not suitable. One reason is that Calderon–Zygmund decomposition deeply depends on the constant \( m \), which appears in (1.2). However, this is not the end of the story. We can establish related theory in martingale setting.

1.2. Weighted Inequalities for Doob Maximal Operator and Multisublinear One in Martingale Setting. Let \((\Omega, \mathcal{F}, \mu)\) be a complete probability space and let \((\mathcal{F}_n)_{n \geq 0}\) be an increasing sequence of sub-\(\sigma\)-fields of \( \mathcal{F} \) with \( \mathcal{F} = \bigvee_{n \geq 0} \mathcal{F}_n \). A weight \( \omega \) is a random variable with \( \omega > 0 \) and \( E(\omega) < \infty \). For any \( n \geq 0 \) and integral function \( f \), we denote the conditional expectation with respect to \( \mathcal{F}_n \) by \( E_n(f) \) or \( E(f|\mathcal{F}_n) \), then \((E_n(f))_{n \geq 0}\) is an uniformly integral martingale. For \((\Omega, \mathcal{F}, \mu)\) and \((\mathcal{F}_n)_{n \geq 0}\), the family of all stopping times is denoted by \( \mathcal{T} \). Given \( \tau \in \mathcal{T} \), let

\[
\mathcal{F}_\tau = \{ F \in \mathcal{F} : F \cap \{ \tau \leq n \} \in \mathcal{F}, \ \forall n \geq 0 \},
\]

then \( \mathcal{F}_\tau \) is a sub-\(\sigma\)-fields of \( \mathcal{F} \). For an integral function \( f \), we denote the conditional expectation with respect to \( \mathcal{F}_\tau \) by \( E_\tau(f) \). Moreover, if we define \( f_\tau(x) \triangleq f_{\tau(x)}(x)\chi_{\{\tau < \infty\}} + f(x)\chi_{\{\tau = \infty\}} \),
then $E_{\tau}(f) = f_{\tau}$ (see [16, 13] for more information). Let $B \in \mathcal{F}$, we always denote $\int_{\Omega} \chi_{B} d\mu$ and $\int_{\Omega} \chi_{B \omega} d\mu$ by $|B|$ and $|B|_{\omega}$, respectively.

Suppose that functions $f$, $g$ are integrable on the probability space $(\Omega, \mathcal{F}, \mu)$, then the Doob maximal operator and the bilinear one are defined by

$$Mf = \sup_{n \geq 0} |E_{n}(f)| \quad \text{and} \quad \mathcal{M}(f, g) = \sup_{n \geq 0} |E_{n}(f)||E_{n}(g)|,$$

respectively.

In regular martingale spaces, Izumisawa and Kazamaki [8] characterized the inequality

$$\left( \int_{\Omega} (Mf)^p v d\mu \right)^{\frac{1}{p}} \leq C \left( \int_{\Omega} |f|^p v d\mu \right)^{\frac{1}{p}},$$

where $p > 1$ and $v$ is a weight. In addition, Long and Peng [14] obtained probabilistic $A_p$ condition and $S_p$ condition, which were also discussed in [10] and [1], respectively.

Let $v$, $\omega_1$, $\omega_2$ be weights and $1 < p_1$, $p_2 < \infty$. Suppose that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(\omega_1, \omega_2) \in RH(p_1, p_2)$, for the bilinear Doob maximal operator $\mathcal{M}$, Chen and Liu [3] characterized the weights for which $\mathcal{M}$ is bounded from $L^{p_1}(\omega_1) \times L^{p_2}(\omega_2)$ to $L^{p, \infty}(v)$ or $L^p(v)$. If $v = \omega_2^{p_1} \omega_2^{p_2}$, they also have a bilinear version for the convergence of martingale.

In this paper, we define the generalized Doob maximal operator $\mathfrak{M}$ in the following way:

$$\mathfrak{M}(\vec{f}) \triangleq \sup_{n \geq 0} \prod_{i=1}^{\infty} |E_{n}(f_i)|,$$

where $\vec{f} = (f_1, f_2, \ldots)$ and $\vec{f}$ is subjected to suitable restrictions. The suitable restrictions can be found in Proposition 2.15 and Remark 3.1.

In order to discuss weighted inequalities for the operator, we should introduce generalizations of probabilistic $A_p$ condition, $S_p$ condition and Hölder’s inequality which can be found in Sections 2 and 3. Now, we state our main results. Some notations and assumptions can be found in Section 3.

**Theorem 1.1.** Let $v$ be a weight and $\vec{f} \in RH_{\vec{p}}$, then the following statements are equivalent:

1. There exists a positive constant $C$ such that

$$\left( \int_{\tau<\infty} \prod_{i=1}^{\infty} E_{\tau}(f_i)^p v d\mu \right)^{\frac{1}{p}} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}, \quad \forall \tau \in \mathcal{T}, \ f_i \in L^{p_i}(\omega_i), \ i \in N,$$

where $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)} < \infty$;

2. There exists a positive constant $C$ such that

$$\|\mathfrak{M}(\vec{f})\|_{L^{p, \infty}(v)} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}, \quad \forall f_i \in L^{p_i}(\omega_i), \ i \in N,$$

where $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)} < \infty$;
(3) The weight vector \((v, \overrightarrow{ω})\) satisfies the condition \(A\overrightarrow{p}\), i.e.,

\[(v, \overrightarrow{ω}) \in A\overrightarrow{p}.
\]

**Theorem 1.2.** Let \(v\) be a weight and \(\overrightarrow{ω} \in RH\overrightarrow{p}\), then the following statements are equivalent:

1. There exists a positive constant \(C\) such that

\[
\|M(\overrightarrow{f})\|_{L^p(v)} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^p(ω_i)}, \forall f_i \in L^p(ω_i), \ i \in N,
\]

where \(\prod_{i=1}^{\infty} \|f_i\|_{L^p(ω_i)} < \infty;\)

2. There exists a positive constant \(C\) such that

\[
\|M(\overrightarrow{g})\|_{L^p(v)} \leq C \prod_{i=1}^{\infty} \|g_i\|_{L^p(σ_i)}, \forall g_i \in L^p(σ_i), \ i \in N,
\]

where \(\prod_{i=1}^{\infty} \|g_i\|_{L^p(σ_i)} < \infty;\)

3. The weight vector \((v, \overrightarrow{ω})\) satisfies the condition \(S\overrightarrow{p}\), i.e.,

\[(v, \overrightarrow{ω}) \in S\overrightarrow{p}.
\]

The remainder of this paper is organized as follows. In Section 2, we prove the generalized Hölder’s inequalities for integral and conditional expectation in details, which will be used in Section 3. The proofs of Theorem 1.1 and Theorem 1.2 are contained in Section 3. In this paper, for simplicity, we omit the annotation ‘almost everywhere’ in the following statements.

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2. **Generalized Hölder’s Inequalities for Integral and Conditional Expectation**

The section consists of a series of Lemmas. If the readers are familiar with them, they could omit them and read Theorems 2.11, 2.12 and 2.14 directly.

2.1. **Some Properties of Series, Lebesgue’s Integral and Infinity Product.** Let \(\{a_i\}\) be a sequence of real numbers. Let \(\{s_n\}\) be the sequence obtained from \(\{a_i\}\), where for each \(n \in N, \ s_n = \sum_{i=1}^{n} a_i\). If \(s_n\) converges in \(R\) or diverges to \(+\infty\) (or \(-\infty\)), we say that the sum of the series is well defined and we denote the sum as \(\sum_{i=1}^{\infty} a_i\). Let \(λ_i \in (0, 1), b_i \in R, \ i \in N,\) and let \(\sum_{i=1}^{\infty} λ_i = 1\). It is known that \((N, 2^N)\) is a measurable space. By the sequences \(\{λ_i\}\) and
\{b_i\}, we can define a measure \(\lambda\) and a measurable function \(b\) on the space in the following way

\[
\lambda(i) = \lambda_i \quad \text{and} \quad b(i) = b_i, \quad \forall i \in \mathbb{N}.
\]

Then \((\mathbb{N}, 2^\mathbb{N}, \lambda)\) is a probability space. Applying Levi’s Lemma, we have

\[
\sum_{i=1}^{\infty} \lambda_i b_i^+ = \lim_{k \to \infty} \sum_{i=1}^{k} \lambda_i b_i^+ = \lim_{k \to \infty} \int_{\Omega} b^+ \chi_{\{1,2,\ldots,k\}} \, d\lambda = \int_{\Omega} b^+ \, d\lambda
\]

and

\[
\sum_{i=1}^{\infty} \lambda_i b_i^- = \lim_{k \to \infty} \sum_{i=1}^{k} \lambda_i b_i^- = \lim_{k \to \infty} \int_{\Omega} b^- \chi_{\{1,2,\ldots,k\}} \, d\lambda = \int_{\Omega} b^- \, d\lambda,
\]

For simplicity, we denote \(\sum_{i=1}^{\infty} \lambda_i b_i^+\) and \(\sum_{i=1}^{\infty} \lambda_i b_i^-\) by \(A\) and \(B\), respectively. It follows that \(A, B \in [0, +\infty]\). If \(A\) or \(B\) is finite, then \(\sum_{i=1}^{\infty} \lambda_i b_i\) is well defined, integral of \(b\) exists and

\[
\sum_{i=1}^{\infty} \lambda_i b_i = \int_{\Omega} b \, d\lambda.
\]

This paper also involve the concept of an infinite product. Let us recall the definition (see, e.g., [17, p. 298]).

**Definition 2.1.** Suppose \(\{c_n\}\) is a sequence of complex number,

\[
p_n = \prod_{i=1}^{n} c_i,
\]

and \(p = \lim_{n \to \infty} p_n\) exists. Then we write

\[
(2.1) \quad p = \prod_{i=1}^{\infty} c_i.
\]

The \(p_n\) are the partial products of the infinite product \((2.1)\). We should say that the infinite product \((2.1)\) converges if the sequence \(\{p_n\}\) converges.

**Remark 2.2.** Suppose \(\{c_n\}\) and \(\{c'_n\}\) are nonnegative sequences, and the infinite product \(\prod_{i=1}^{\infty} c_i\) converges. If \(c'_n \leq c_n, \quad n \in \mathbb{N}\), then the infinite product \(\prod_{i=1}^{\infty} c'_i\) also converges.

**Remark 2.3.** Suppose \(\{f_i\}\) is a sequence of measurable functions on a measurable space \((\Omega, \mathcal{F})\), and suppose that the sequence of numbers \(\{f_i(x)\}\) converges for every \(x \in \Omega\). We can then define a function \(\prod_{i=1}^{\infty} f_i\) by

\[
\prod_{i=1}^{\infty} f_i(x) = \lim_{n \to \infty} \prod_{i=1}^{n} f_i(x).
\]
We should say that the function $\prod_{i=1}^{\infty} f_i(x)$ is well defined.

**Lemma 2.4.** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. If the measurable function $f : \Omega \to R$ such that $\exp(f)$ is integrable, then integral of the function $f$ exists and

$$\exp \left( \int_{\Omega} f d\mu \right) \leq \int_{\Omega} \exp(f) d\mu.$$ 

**Proof of Lemma 2.4** It is clear that $f^+ \leq \exp(f^+)$ and $\max\{\exp(f), 1\} \leq \exp(f) + 1$, then

$$\int_{\Omega} f^+ d\mu \leq \int_{\Omega} \exp(f) d\mu + 1 < \infty.$$ 

Thus integral of the measurable function $f$ exists. If $\int_{\Omega} f^- d\mu < \infty$, it follows from Jensen’s inequality that

$$\exp \left( \int_{\Omega} f d\mu \right) \leq \int_{\Omega} \exp(f) d\mu.$$ 

If $\int_{\Omega} f^- d\mu = +\infty$, we have $\int_{\Omega} f d\mu = -\infty$ and $\exp \left( \int_{\Omega} f d\mu \right) = 0$. We are done. ■

**Corollary 2.5.** Let $\lambda_i \in (0, 1), i \in N, \sum_{i=1}^{\infty} \lambda_i = 1$. If $b_i \in R, i \in N$ and $\sum_{i=1}^{\infty} \lambda_i \exp(b_i) < \infty$, then $\sum_{i=1}^{\infty} \lambda_i b_i$ is well defined and

$$\exp\left( \sum_{i=1}^{\infty} \lambda_i b_i \right) \leq \sum_{i=1}^{\infty} \lambda_i \exp(b_i).$$

**Proof of Corollary 2.5** The corollary is another version of Lemma 2.4. We can prove the corollary in the way of Lemma 2.4 with obvious changes and we omit it. ■

**Lemma 2.6.** Let $\lambda_i \in (0, 1), i \in N$ and $\sum_{i=1}^{\infty} \lambda_i = 1$. If $a_i \geq 0, i \in N$ and $\sum_{i=1}^{\infty} \lambda_i a_i < \infty$, then

$$\prod_{i=1}^{\infty} a_i^{\lambda_i} \leq \sum_{i=1}^{\infty} \lambda_i a_i.$$ 

**Proof of Lemma 2.6** Without loss of generalization, we assume $a_i > 0, i \in N$. Substituting $b_i = \ln a_i, i \in N$ into Corollary 2.5, we have

$$\exp\left( \sum_{i=1}^{\infty} \lambda_i \ln a_i \right) \leq \sum_{i=1}^{\infty} \lambda_i \exp(\ln a_i).$$

It follows that

$$\prod_{i=1}^{\infty} a_i^{\lambda_i} \leq \sum_{i=1}^{\infty} \lambda_i a_i.$$ ■
Lemma 2.7. Let \( 1 < p_i < \infty, i \in N \) and \( \sum_{i=1}^{\infty} \frac{1}{p_i} = 1 \). If \( a_i \geq 0, i \in N \) and \( \sum_{i=1}^{\infty} \frac{a_i}{p_i} < \infty \), then
\[
\prod_{i=1}^{\infty} a_i^{\frac{1}{p_i}} \leq \sum_{i=1}^{\infty} \frac{a_i}{p_i}.
\]

Proof of Lemma 2.7 Substituting \( \lambda_i = \frac{1}{p_i}, i \in N \) into Lemma 2.6, we have Lemma 2.7.\[\square\]

Lemma 2.8. Let \( 1 < p_i < \infty, i \in N \) and \( \sum_{i=1}^{\infty} \frac{1}{p_i} = 1 \). If \( c_i \geq 0, i \in N \) and \( \sum_{i=1}^{\infty} \frac{c_i^{p_i}}{p_i} < \infty \), then
\[
\prod_{i=1}^{\infty} c_i \leq \sum_{i=1}^{\infty} \frac{c_i^{p_i}}{p_i}.
\]

Proof of Lemma 2.8 Substituting \( a_i = c_i^{p_i}, i \in N \) into Lemma 2.7, we have Lemma 2.8.\[\square\]

2.2. Generalized Hölder’s Inequality for Integral. In the subsection, we suppose that \((\Omega, \mathcal{F}, \mu)\) is a measure space and \(\{f_i\}\) is a sequence of nonnegative measurable functions on \((\Omega, \mathcal{F}, \mu)\). This kind of inequality also discussed on the \(\sigma\)-finite measure space in [9].

Lemma 2.9. Let \( 1 < p_i < \infty \) and \( \|f_i\|_{L^{p_i}} = 1, i \in N \). If \( \sum_{i=1}^{\infty} \frac{1}{p_i} = 1 \), then the function \( \prod_{i=1}^{\infty} f_i \) is well defined and
\[
\|\prod_{i=1}^{\infty} f_i\|_{L^1} \leq 1.
\]

Proof of Lemma 2.9 Since \( \|f_i\|_{L^{p_i}} = 1, i \in N \) and \( \sum_{i=1}^{\infty} \frac{1}{p_i} = 1 \), we have
\[
\int_{\Omega} \sum_{i=1}^{\infty} \frac{f_i^{p_i}}{p_i} d\mu = \sum_{i=1}^{\infty} \int_{\Omega} \frac{f_i^{p_i}}{p_i} d\mu = \sum_{i=1}^{\infty} \int_{\Omega} f_i^{p_i} d\mu = \sum_{i=1}^{\infty} \frac{1}{p_i} = 1 < \infty,
\]
where we have used the monotone convergence theorem. It follows that
\[
\sum_{i=1}^{\infty} \frac{f_i^{p_i}}{p_i} < \infty.
\]
Combining this with Lemma 2.8, we get that \( \prod_{i=1}^{\infty} f_i \) is well defined and
\[
\prod_{i=1}^{\infty} f_i \leq \sum_{i=1}^{\infty} \frac{f_i^{p_i}}{p_i} < \infty.
\]
Hence,
\[
\int_{\Omega} \prod_{i=1}^{\infty} f_i d\mu \leq \sum_{i=1}^{\infty} \frac{\int_{\Omega} f_i^{p_i} d\mu}{p_i} = \sum_{i=1}^{\infty} \frac{1}{p_i} = 1.\[\square\]
Lemma 2.10. Let $1 < p_i < \infty$, $i \in N$ and $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$. If $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty$, then the function $\prod_{i=1}^{\infty} f_i$ is well defined and $\|\prod_{i=1}^{\infty} f_i\|_{L^1} \leq \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}}$.

Proof of Lemma 2.10 We split the proof into three cases.

Firstly, we assume that $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} = 0$ and there exists an $i_0 \in N$ such that $\|f_{i_0}\|_{L^{p_i}} = 0$.

It is clear that the function $\prod_{i=1}^{\infty} f_i$ is well defined and $\|\prod_{i=1}^{\infty} f_i\|_{L^1} \leq \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}}$.

Secondly, we assume that $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} = 0$ and $\|f_i\|_{L^{p_i}} > 0$, $\forall i \in N$. Let $\hat{f}_i = \frac{f_i}{\|f_i\|_{L^{p_i}}}$, $i \in N$. Then $\|\hat{f}_i\|_{L^{p_i}} = 1$, $i \in N$. It follows from Lemma 2.9 that $\prod_{i=1}^{\infty} \hat{f}_i$ is well defined. Combining this with $f_i = \|f_i\|_{L^{p_i}} \cdot \hat{f}_i$, $i \in N$, we obtain that $\prod_{i=1}^{\infty} f_i$ is well defined and

$$\prod_{i=1}^{\infty} f_i = \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} \prod_{i=1}^{\infty} \hat{f}_i = 0.$$ 

Thus, $\|\prod_{i=1}^{\infty} f_i\|_{L^1} \leq \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}}$.

Finally, we suppose that $0 < \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty$. Let $\hat{f}_i = \frac{f_i}{\|f_i\|_{L^{p_i}}}$, $i \in N$. Then $\|\hat{f}_i\|_{L^{p_i}} = 1$, $i \in N$. It follows from Lemma 2.9 that $\prod_{i=1}^{\infty} \hat{f}_i$ is well defined and

$$\|\prod_{i=1}^{\infty} \hat{f}_i\|_{L^1} \leq 1.$$ 

Thus the function $\prod_{i=1}^{\infty} f_i$ is also well defined and $\|\prod_{i=1}^{\infty} f_i\|_{L^1} \leq \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}}$.

Theorem 2.11. Let $0 < p_i < \infty$, $i \in N$ and $\sum_{i=1}^{\infty} \frac{1}{p_i} = \frac{1}{p}$. If $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty$, then the function $\prod_{i=1}^{\infty} f_i$ is well defined and $\|\prod_{i=1}^{\infty} f_i\|_{L^p} \leq \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}}$.

Proof of Theorem 2.11 It is clear that Theorem 2.11 follows from Lemma 2.10.

Theorem 2.12. Let $1 < p_i < \infty$, $i \in N$ and $\sum_{i=1}^{\infty} \frac{1}{p_i} = \frac{1}{p}$. Then

$$\prod_{i=1}^{\infty} p_i' < \infty,$$

where $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $i \in N$. 

Proof of Theorem 2.12 It suffices to prove $\sum_{i=1}^{\infty} \ln p_i' < \infty$. Because of $p_i' = (1 - \frac{1}{p_i})^{-1}$, we should prove $\sum_{i=1}^{\infty} \ln(1 - \frac{1}{p_i})^{-1} < \infty$. Since $\lim_{i \to \infty} \frac{\ln(1 - \frac{1}{p_i})^{-1}}{p_i'} = 1$ and $\sum_{i=1}^{\infty} \frac{1}{p_i} = \frac{1}{p}$, we have $\sum_{i=1}^{\infty} \ln(1 - \frac{1}{p_i})^{-1} < \infty$ by the Limit Comparison Test. ■

2.3. Generalized Hölder’s Inequality for Conditional Expectation. In the subsection, we suppose that $(\Omega, \mathcal{F}, \mu)$ is a complete probability space and $\{f_i\}$ is a sequence of nonnegative measurable functions on $(\Omega, \mathcal{F}, \mu)$.

Proposition 2.13. Let $1 < p_i < \infty, i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$. Suppose that $\mathcal{F}'$ be a sub-$\sigma$-fields of $\mathcal{F}$. If $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty$, then

$$E_{\mathcal{F}'}(\prod_{i=1}^{\infty} f_i) \leq \prod_{i=1}^{\infty} E_{\mathcal{F}'}(f_i^{p_i})^{\frac{1}{p_i}} < \infty.$$  

Proof of Proposition 2.13 Because of $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty$, it follows from Lemma 2.10 that the function $\prod_{i=1}^{\infty} f_i$ is well defined and $\prod_{i=1}^{\infty} \|f_i\|_{L^1} \leq \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}}$. Since $\|f_i\|_{L^{p_i}} = \|f_i^{p_i}\|_{L^1}^{\frac{1}{p_i}} = \|E_{\mathcal{F}'}(f_i^{p_i})\|_{L^1}^{\frac{1}{p_i}} = \|E_{\mathcal{F}'}(f_i^{p_i})^{\frac{1}{p_i}}\|_{L^{p_i}}, \forall i \in \mathbb{N}$, we have that $\prod_{i=1}^{\infty} E_{\mathcal{F}'}(f_i^{p_i})^{\frac{1}{p_i}}$ is well defined and

$$\|\prod_{i=1}^{\infty} E_{\mathcal{F}'}(f_i^{p_i})^{\frac{1}{p_i}}\|_{L^1} \leq \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty.$$  

Moreover, $\prod_{i=1}^{\infty} E_{\mathcal{F}'}(f_i^{p_i})^{\frac{1}{p_i}} < \infty$. So we will focus on proving $E_{\mathcal{F}'}(\prod_{i=1}^{\infty} f_i) \leq \prod_{i=1}^{\infty} E_{\mathcal{F}'}(f_i^{p_i})^{\frac{1}{p_i}}$.

For $k \in \mathbb{N}$, we define $q_k = \frac{1}{\sum_{i=1}^{k} \frac{1}{p_i}}$, then $\sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{q_k}$. Applying Fatou’s Lemma and Hölder’s
inequality for conditional expectation, we have
\[ E_{\mathcal{F}'}\left(\prod_{i=1}^{\infty} f_i\right) \leq \liminf_{k \to \infty} E_{\mathcal{F}'}\left(\prod_{i=1}^{k} f_i^q\right)^{\frac{1}{q_k}} \leq \liminf_{k \to \infty} \prod_{i=1}^{k} E_{\mathcal{F}'}(f_i^p)^{\frac{1}{p_i}} = \prod_{i=1}^{\infty} E_{\mathcal{F}'}(f_i^p)^{\frac{1}{p_i}}. \]

**Theorem 2.14.** Let \(0 < p_i < \infty, i \in N\) and \(\sum_{i=1}^{\infty} \frac{1}{p_i} = \frac{1}{p}\). Suppose that \(\mathcal{F}'\) be a sub-\(\sigma\)-fields of \(\mathcal{F}\). If \(\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty\), then
\[ E_{\mathcal{F}'}\left(\prod_{i=1}^{\infty} f_i^p\right)^{\frac{1}{p}} \leq \prod_{i=1}^{\infty} E_{\mathcal{F}'}(f_i^p)^{\frac{1}{p_i}}. \]

**Proof of Theorem 2.14** It is clear that Theorem 2.14 follows from Proposition 2.13.■

**Proposition 2.15.** Let \(1 < p_i < \infty, i \in N\) and \(\sum_{i=1}^{\infty} \frac{1}{p_i} = \frac{1}{p}\). If \(\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty\), then \(\mathcal{M}(\mathbf{f})\) is well defined.

**Proof of Proposition 2.15** Let \(q > 1\). It is well known that conditional expectation \(E_n(\cdot)\) on \(L^q(\Omega, \mathcal{F}, \mu)\) is a contraction, and maps \(L^q(\Omega, \mathcal{F}, \mu)\) onto \(L^q(\Omega, \mathcal{F}_n, \mu)\). Combining this with Theorem 2.11 and Remark 2.2, we have \(\prod_{i=1}^{\infty} E_n(f_i)\) is well defined. Then \(\mathcal{M}(\mathbf{f})\) is well defined.■

### 3. Weighted Inequalities in Martingale Spaces

There are a lot of assumptions and notations which will be used in the section. For convenience, we state them at the beginning of this part. In addition, \(C\) will denote a constant not necessarily the same at each occurrence.

**ASSUMPTIONS** Let \(\omega_i \in L^1\) and \(1 < p_i < \infty, i \in N\), and let \(\{f_i\}\) be a sequence of nonnegative measurable function on the probability space \((\Omega, \mathcal{F}, \mu)\). Suppose that \(\frac{1}{p} = \sum_{i=1}^{\infty} \frac{1}{p_i}\) and \(\sigma_i = \omega_i^{-\frac{1}{p_i-1}} \in L^1, i \in N\). We always suppose that \(\prod_{i=1}^{\infty} \|\sigma_i\|_{L^{p_i}(\omega_i)} < \infty\), \(\prod_{i=1}^{\infty} E_n(\omega_i^{-1/p_i})^{\frac{1}{p_i}} < \infty\), and \(\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)} < \infty\). Moreover, we assume that \(\prod_{i=1}^{\infty} \sigma_i\).
NOTATIONS We denote that $\vec{p} = (p_1, p_2, \cdots)$, $\vec{ω} = (ω_1, ω_2, \cdots)$, $\vec{f} = (f_1, f_2, \cdots)$. Moreover, we also denote $\vec{f}_Q = (f_1 χ_Q, f_2 χ_Q, \cdots)$ and $\vec{σ}_Q = (σ_1 χ_Q, σ_2 χ_Q, \cdots)$, where $Q$ is a measurable set.

Remark 3.1. It follows from generalized Holder’s inequality for integral that

$$
\int_{Ω} \prod_{i=1}^{∞} E_n(f_i^p ω_i)^{\frac{1}{p_i}} dμ \leq \prod_{i=1}^{∞} \left( \int_{Ω} E_n(f_i^p ω_i) dμ \right)^{\frac{1}{p_i}} = \prod_{i=1}^{∞} \left( \int_{Ω} f_i^p ω_i dμ \right)^{\frac{1}{p_i}} < ∞.
$$

Hence, $\prod_{i=1}^{∞} E_n(f_i^p ω_i)^{\frac{1}{p_i}} < ∞$. By Holder’s inequality for conditional expectation and Remark 2.2, we have

$$
\prod_{i=1}^{∞} E_n(f_i) \leq \prod_{i=1}^{∞} E_n(f_i^p ω_i)^{\frac{1}{p_i}} E_n(ω_i^{-\frac{1}{p_i-1}})^{\frac{1}{p_i'}} = \prod_{i=1}^{∞} E_n(f_i^p ω_i)^{\frac{1}{p_i}} \prod_{i=1}^{∞} E_n(ω_i^{-\frac{1}{p_i-1}})^{\frac{1}{p_i'}} < ∞.
$$

Then $\mathcal{M}(\vec{f})$ is well defined. Let $f_i = σ_i$, we also have $\prod_{i=1}^{∞} E_n(σ_i) < ∞$ and $\mathcal{M}(\vec{σ})$ is well defined.

Definition 3.2. We say that the weight vector $\vec{ω}$ satisfies the reverse Hölder’s condition $RH_{\vec{p}}$, if there exists a positive constant $C$ such that

$$
\prod_{i=1}^{∞} \left( \int_{\{τ < ∞\}} σ_i dμ \right)^{\frac{1}{p_i'}} \leq C \int_{\{τ < ∞\}} \prod_{i=1}^{∞} σ_i^{\frac{1}{p_i'}} dμ, \ \forall τ ∈ T.
$$

Definition 3.3. Let $v$ be a weight. We say that the weight vector $(v, \vec{ω})$ satisfies the condition $A_{\vec{p}}$, if there exists a positive constant $C$ such that

$$
E_n(v)^{\frac{1}{p}} \prod_{i=1}^{∞} E_n(ω_i^{1-\nu_i})^{\frac{1}{ν_i'}} \leq C, \ \forall n ≥ 0,
$$

where $\frac{1}{p_i} + \frac{1}{ν_i'} = 1, \ i ∈ N$.

Definition 3.4. Let $v$ be a weight. We say that the weight vector $(v, \vec{ω})$ satisfies the condition $S_{\vec{p}}$, if there exists a positive constant $C$ such that

$$
\left( \int_{\{τ < ∞\}} \mathcal{M}(σχ_{\{τ < ∞\}})^p v dμ \right)^{\frac{1}{p}} \leq C \prod_{i=1}^{∞} |\{τ < ∞\}|^{\frac{1}{p_i}}, \ \forall τ ∈ T.
$$

Proof of Theorem 1.1 We shall follow the scheme: $(2) ⇔ (1) ⇔ (3)$. 

(1) $\Rightarrow$ (2). Let $f_i \in L^{p_i}(\omega_i), \ i \in N$ and let $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)} < \infty$. For $\lambda > 0$, define

$$\tau = \inf \{ n : \prod_{i=1}^{\infty} E_n(f_i) > \lambda \}.$$

It follows from (1.4) that

$$\lambda \left\| \{ \mathfrak{M}(\vec{f}) > \lambda \} \right\|_{\nu}^{\frac{1}{p}} = \left( \int_{\{ \tau < \infty \}} \lambda^p v d \mu \right)^{\frac{1}{p}} \leq \left( \int_{\{ \tau < \infty \}} \prod_{i=1}^{\infty} E_\tau(f_i)^p v d \mu \right)^{\frac{1}{p}} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}.$$

Thus (1.5) is valid.

(2) $\Rightarrow$ (1). Let $f_i \in L^{p_i}(\omega_i), \ i \in N$ and let $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)} < \infty$. Fix $n \in N$ and $B \in \mathcal{F}_n$. Let

$$F_i = f_i \chi_B, \ i \in N.$$

Then $E_n(F_i) = E_n(f_i) \chi_B$. Moreover

$$\prod_{i=1}^{\infty} E_n(f_i) \chi_B \leq \mathfrak{M}(\vec{F}).$$

Combining with (1.5), we have

$$\lambda^p \int_{B \cap \{ \prod_{i=1}^{\infty} E_n(f_i) > \lambda \}} v d \mu \leq \lambda^p \int_{\{ \mathfrak{M}(\vec{F}) > \lambda \}} v d \mu \leq C \prod_{i=1}^{\infty} \|F_i||_{L^{p_i}(\omega_i)^p} = C \prod_{i=1}^{\infty} \left( \int_B f_i^{p_i}(\omega_id\mu) \right)^{\frac{p_i}{p}}.$$

For $k \in \mathbb{Z}$, let

$$B_k = \{ 2^k < \prod_{i=1}^{\infty} E_n(f_i) \leq 2^{k+1} \}.$$

Note that

$$\{ 2^k < \prod_{i=1}^{\infty} E_n(f_i) \leq 2^{k+1} \} \subseteq \{ 2^k < \prod_{i=1}^{\infty} E_n(f_i) \}.$$
\[ \int_{\Omega} \left( \prod_{i=1}^{\infty} E_n(f_i) \right)^p v d\mu = \sum_{k \in \mathbb{Z}} \int_{B_k} \prod_{i=1}^{\infty} E_n(f_i)^p v d\mu \]

\[ \leq C \sum_{k \in \mathbb{Z}} \int_{B_k \cap \{ \prod_{i=1}^{\infty} E_n(f_i) > 2^k \}} 2^{kp} v d\mu \]

\[ \leq C \sum_{k \in \mathbb{Z}} \prod_{i=1}^{\infty} \left( \int_{B_k} f_i^p \omega_i d\mu \right)^{\frac{p_i}{p}} \]

\[ \leq C \prod_{i=1}^{\infty} \left( \sum_{k \in \mathbb{Z}} \int_{B_k} f_i^p \omega_i d\mu \right)^{\frac{p_i}{p}} \]

\[ \leq C \prod_{i=1}^{\infty} \left( \int_{\Omega} f_i^p \omega_i d\mu \right)^{\frac{p_i}{p}}. \]

where we have used the generalized Hölder’s inequality. As for \( \tau \in \mathcal{T} \), it is easy to see that

\[ \int_{\{\tau < \infty\}} \prod_{i=1}^{\infty} E_\tau(f_i)^p v d\mu = \sum_{n \geq 0} \int_{\{\tau = n\}} \prod_{i=1}^{\infty} E_n(f_i)^p v d\mu \]

\[ \leq C \sum_{n \geq 0} \prod_{i=1}^{\infty} \left( \int_{\Omega} (f_i \chi_{\{\tau = n\}})^{p_i} \omega_i d\mu \right)^{\frac{p_i}{p}} \]

\[ \leq C \prod_{i=1}^{\infty} \left( \sum_{n \geq 0} \int_{\Omega} (f_i \chi_{\{\tau = n\}})^{p_i} \omega_i d\mu \right)^{\frac{p_i}{p}} \]

\[ \leq C \prod_{i=1}^{\infty} \left( \int_{\Omega} f_i^p \omega_i d\mu \right)^{\frac{p_i}{p}}. \]

Therefore,

\[ \left( \int_{\{\tau < \infty\}} \prod_{i=1}^{\infty} E_\tau(f_i)^p v d\mu \right)^{\frac{1}{p}} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}. \]

(3) \( \Rightarrow \) (1). Let \( f_i \in L^{p_i}(\omega_i), \ i \in \mathbb{N} \) and let \( \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)} < \infty \). Applying Hölder’s inequality for conditional expectation, we get

\[ E_n(f_i) \leq E_n(f_i^p \omega_i)^{\frac{1}{p}} E_n(\omega_i^{-\frac{1}{p_i-1}})^{\frac{1}{p_i}}. \]
Furthermore,

\[
\prod_{i=1}^{\infty} E_n(f_i)^p \leq \prod_{i=1}^{\infty} E_n(f_i^p \omega_i)^{\frac{p}{p_i}} E_n(\omega_i^{-\frac{1}{p_i-1}})^{\frac{p_i-1}{p_i}},
\]

where \( E_n^v(\cdot) \) is the conditional expectation relative to the probability measure \( \frac{v}{|\Omega_v|} d\mu \). Because of (1.6), we get

\[
\prod_{i=1}^{\infty} E_n(f_i)^p \leq C \prod_{i=1}^{\infty} E_n^v(f_i^p \omega_i v^{-1})^{\frac{p}{p_i}}.
\]

From this, using the generalized Hölder’s inequality, we have

\[
\| \prod_{i=1}^{\infty} E_n(f_i) \|_{L^p(v)} \leq C \prod_{i=1}^{\infty} \| E_n^v(f_i^p \omega_i v^{-1})^{\frac{1}{p_i}} \|_{L^p(v)}
\]

\[
\leq C \prod_{i=1}^{\infty} \| E_n^v(f_i^p \omega_i v^{-1})^{\frac{1}{p_i}} \|_{L^1(v)}
\]

\[
= C \prod_{i=1}^{\infty} \| f_i^p \omega_i \|_{L^1(v)}^{\frac{1}{p_i}}
\]

\[
= C \prod_{i=1}^{\infty} \| f_i^p \|_{L^p(\omega_i)}.
\]

(1) \(\Rightarrow\) (3). For any \( n \geq 0, i \in N \) and \( B \in F_n \), set \( f_i = \omega_i^{-\frac{1}{p_i-1}} \chi_B \). Then

\[
\left( \int_{B} \prod_{i=1}^{\infty} E_n(\omega_i^{-\frac{1}{p_i-1}})^p v d\mu \right)^{\frac{1}{p}} \leq C \prod_{i=1}^{\infty} \left( \int_{\Omega} \omega_i^{-\frac{1}{p_i-1}} \chi_B d\mu \right)^{\frac{1}{p_i}}.
\]

Furthermore,

\[
\int_{B} \prod_{i=1}^{\infty} E_n(\omega_i^{-\frac{1}{p_i-1}})^p E_n(v) d\mu \leq C \prod_{i=1}^{\infty} \left( \int_{B} \sigma_i d\mu \right)^{\frac{p}{p_i}}.
\]

Note that \( \omega \in RH_{\frac{\omega}{\chi}} \), we have

\[
\int_{B} \prod_{i=1}^{\infty} E_n(\omega_i^{-\frac{1}{p_i-1}})^p E_n(v) d\mu \leq C \int_{B} \prod_{i=1}^{\infty} \sigma_i^p d\mu.
\]
It follows from the generalized Hölder’s inequality for conditional expectation that
\[
\int_B \prod_{i=1}^{\infty} E_n(\omega_i^{-\frac{1}{p_i-1}})^p E_n(v) d\mu \leq C \int_B \prod_{i=1}^{\infty} E_n(\sigma_i)^{\frac{1}{p_i}} d\mu.
\]

Thus, there exists a constant \(C\) such that
\[
\left( \prod_{i=1}^{\infty} E_n(\omega_i^{-\frac{1}{p_i-1}})^p E_n(v) \right)^{\frac{1}{p}} \leq C \prod_{i=1}^{\infty} E_n^p(\sigma_i)^{\frac{1}{p_i}}.
\]

Then
\[
E_n^p(v) \prod_{i=1}^{\infty} E_n(\omega_i^{1-p_i})^{\frac{1}{p_i}} \leq C.
\]

**Proof of Theorem 1.2** It is clear that \((1) \iff (2) \implies (3)\), so we omit them. To prove \((3) \implies (2)\), we proceed in the following way. Let \(g_i \in L^p(\sigma_i), \ i \in \mathbb{N}\) and let \(\prod_{i=1}^{\infty} \|g_i\|_{L^p(\sigma_i)} < \infty\). For all \(k \in \mathbb{Z}\), define stopping times
\[
\tau_k = \inf\{n : \prod_{i=1}^{\infty} E_n(g_i\sigma_i) > 2^k\}.
\]

Set
\[
A_{k,j} = \{\tau_k < \infty\} \cap \{2^j < \prod_{i=1}^{\infty} E_{F_{\tau_k}}(\sigma_i) \leq 2^{j+1}\};
\]
\[
B_{k,j} = \{\tau_k < \infty, \tau_{k+1} = \infty\} \cap \{2^j < \prod_{i=1}^{\infty} E_{F_{\tau_k}}(\sigma_i) \leq 2^{j+1}\}, \ j \in \mathbb{Z}.
\]

Then \(A_{k,j} \in \mathcal{F}_{\tau_k}, B_{k,j} \subseteq A_{k,j}\) and
\[
E_{F_{\tau_k}}(g_i\sigma_i) = E_{F_{\tau_k}}^\sigma(g_i) E_{F_{\tau_k}}(\sigma_i).
\]

Moreover, \(\{B_{k,j}\}_{k,j}\) is a family of disjoint sets and
\[
\{2^k < M(\bar{g}) \leq 2^{k+1}\} = \{\tau_k < \infty, \tau_{k+1} = \infty\} = \bigcup_{j \in \mathbb{Z}} B_{k,j}, \ k \in \mathbb{Z}.
\]

On each \(A_{k,j}\), we have
\[
2^{kp} \leq \text{ess inf}_{A_{k,j}} \prod_{i=1}^{\infty} E_{F_{\tau_k}}(g_i\sigma_i)^p \leq \text{ess inf}_{A_{k,j}} \prod_{i=1}^{\infty} E_{F_{\tau_k}}^\sigma(g_i)^p \text{ ess sup}_{A_{k,j}} \prod_{i=1}^{\infty} E_{F_{\tau_k}}(\sigma_i)^p \leq 2^p \text{ess inf}_{A_{k,j}} \prod_{i=1}^{\infty} E_{F_{\tau_k}}^\sigma(g_i)^p |B_{k,j}|_v^{-1} \int_{B_{k,j}} \prod_{i=1}^{\infty} E_{F_{\tau_k}}(\sigma_i)^p v d\mu.
\]
To estimate \( \int_{\Omega} \mathcal{M}(\overline{g_\sigma})^p v d\mu \), firstly we have

\[
\int_{\Omega} \mathcal{M}(\overline{g_\sigma})^p v d\mu = \sum_{k \in \mathbb{Z}} \int_{\{2^k < \mathcal{M}(\overline{g_\sigma}) \leq 2^{k+1}\}} \mathcal{M}(\overline{g_\sigma})^p v d\mu \\
\leq 2^p \sum_{k \in \mathbb{Z}} \int_{\{2^k < \mathcal{M}(\overline{g_\sigma}) \leq 2^{k+1}\}} 2^{kp} v d\mu \\
= 2^p \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} 2^{kp} \int_{B_{k,j}} v d\mu \\
\leq 4^p \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \operatorname{ess inf}_{A_{k,j}} \prod_{i=1}^{\infty} E_{\mathcal{F}_{r_k}}(g_i)^p \int_{B_{k,j}} \prod_{i=1}^{\infty} E_{\mathcal{F}_{r_k}}(\sigma_i)^p v d\mu.
\]

It is clear that \( \vartheta \) is a measure on \( X = \mathbb{Z}^2 \) with

\[
\vartheta(k, j) = \int_{B_{k,j}} \prod_{i=1}^{\infty} E_{\mathcal{F}_{r_k}}(\sigma_i)^p v d\mu.
\]

For the above \( \{g_i\} \), define

\[
T_{\overline{g}}(k, j) = \operatorname{ess inf}_{A_{k,j}} \prod_{i=1}^{\infty} E_{\mathcal{F}_{r_k}}(g_i)^p
\]

and denote

\[
E_\lambda = \left\{ (k, j) : \operatorname{ess inf}_{A_{k,j}} \prod_{i=1}^{\infty} E_{\mathcal{F}_{r_k}}(g_i)^p > \lambda \right\} \quad \text{and} \quad G_\lambda = \bigcup_{(k,j) \in E_\lambda} A_{k,j}
\]

for each \( \lambda > 0 \). Then we have

\[
|\{T_{\overline{g}}(k, j) > \lambda\}|_\vartheta = \sum_{(k,j) \in E_\lambda} \int_{B_{k,j}} \prod_{i=1}^{\infty} E_{\mathcal{F}_{r_k}}(\sigma_i)^p v d\mu \\
= \sum_{(k,j) \in E_\lambda} \int_{B_{k,j}} \prod_{i=1}^{\infty} E_{\mathcal{F}_{r_k}}(\sigma_i \chi_{G_\lambda})^p v d\mu \\
\leq \int_{G_\lambda} \mathcal{M}(\overline{\sigma \chi_{G_\lambda}})^p v d\mu.
\]
Let $\tau = \inf \{ n : \prod_{i=1}^{\infty} E_{n_i}(g_i)^p > \lambda \}$, we have $G_{\lambda} \subseteq \left\{ \mathcal{M}(\overrightarrow{g})^p > \lambda \right\} = \{ \tau < \infty \}$. It follows from $S_{\overrightarrow{p}}$ and $RH_{\overrightarrow{p}}$ that

$$
\left| \{ T_{\overrightarrow{g}}(k, j) > \lambda \} \right|_{\sigma} \leq \int_{\{ \tau < \infty \}} \mathcal{M}(\overrightarrow{\sigma}(\tau < \infty))^p v d\mu.
$$

$$
\leq C \prod_{i=1}^{\infty} \left| \{ \tau < \infty \} \right|_{\sigma_i}^{\frac{p}{\sigma_i}}
$$

$$
\leq C \int_{\{ \tau < \infty \}} \prod_{i=1}^{\infty} \sigma_i^{\frac{p}{\sigma_i}} d\mu.
$$

Therefore,

$$
\int_{\Omega} \mathcal{M}(\overrightarrow{g})^p v d\mu \leq 4^p \int_{X} T_{\overrightarrow{g}} d\theta = 4^p \int_{0}^{\infty} \left| \{ T_{\overrightarrow{g}} > \lambda \} \right|_{\sigma} d\lambda
$$

$$
\leq C \int_{0}^{\infty} \int_{\{ \tau < \infty \}} \prod_{i=1}^{\infty} \sigma_i^{\frac{p}{\sigma_i}} d\mu d\lambda
$$

$$
= C \int_{0}^{\infty} \int_{\{ \mathcal{M}(\overrightarrow{g})^p > \lambda \}} \prod_{i=1}^{\infty} \sigma_i^{\frac{p}{\sigma_i}} d\mu d\lambda
$$

$$
= C \int_{0}^{\infty} \mathcal{M}(\overrightarrow{g})^p \prod_{i=1}^{\infty} \sigma_i^{\frac{p}{\sigma_i}} d\mu
$$

$$
\leq C \int_{\Omega} \prod_{i=1}^{\infty} M^{\sigma_i}(g_i)^p \sigma_i^{\frac{p}{\sigma_i}} d\mu
$$

$$
\leq C \prod_{i=1}^{\infty} \left( \int_{\Omega} M^{\sigma_i}(g)^p \sigma_i d\mu \right)^{\frac{p}{\sigma_i}}
$$

$$
\leq C \left( \prod_{i=1}^{\infty} p_i^i \right)^p \prod_{i=1}^{\infty} \| g_i \|_{L^p(\sigma_i)}^{p}
$$

where we have used Hölder’s inequality and Doob’s inequality. Whence (1.7) is valid, because of $\prod_{i=1}^{\infty} p_i^i < \infty$. 

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