COMPLEX STRUCTURES ON TANGENT AND COTANGENT LIE ALGEBRAS OF DIMENSION SIX

RUTWIG CAMPOAMOR-STURSBERG AND GABRIELA P. OVANDO

Abstract. This paper deals with complex structures on Lie algebras $T_\pi \mathfrak{h} = \mathfrak{h} \ltimes_\pi V$, where $\pi$ is either the adjoint or the coadjoint representation. The main topic is the existence question of complex structures on $T_\pi \mathfrak{h}$ for $\mathfrak{h}$ a three dimensional real Lie algebra. First it was proposed the study of complex structures $J$ satisfying the constrain $J\mathfrak{h} = V$. Whenever $\pi$ is the adjoint representation this kind of complex structures are associated to non singular derivations of $\mathfrak{h}$. This fact derives different kind of applications. Finally an approach to the pseudo Kähler geometry was done.

1. Introduction

While the existence of complex structures on reductive Lie algebras of even dimension was solved in different steps (starting with [Sm] and [Wa]), the solvable case remains an open problem. For dimensions up to four, complex structures were studied in [Sa1, SJ, O]; in dimension six the classification and induced complex geometry has been considered in the nilpotent case in [CFGU1, CFGU2, CFU, FPS, GR, Mg, Sl, Ug]. Since those works are mainly done on the basis of a case by case study, one of the principal obstructions in classifying complex (and more general) structures on solvable Lie groups of dimensions equal or greater than six relies in the high number of isomorphism classes. This implies that different criteria have to be developed in order to describe any kind of geometry on Lie groups. One of these is the notion of generalized complex structure, introduced by Hitchin in [Hi] and treated by various authors (see for instance [ABDF, CG, G] and references therein). On the other hand, in order to study the complex geometry, special types of complex structures were considered, the so called abelian [BDM] and nilpotent [CFGU1, CFGU2], specific for nilpotent Lie algebras, and which have been shown to be of considerable interest, in particular in combination with other compatible geometric structures.

The aim of this work is the study of complex structures on tangent and cotangent Lie algebras, that is Lie algebras which are semidirect products $T_\pi \mathfrak{h} = \mathfrak{h} \ltimes_\pi V$, for $\dim V = \mathfrak{h}$, originally for $\pi$ the adjoint or the coadjoint representation. We focus in the following existence questions:

i) complex structures satisfying the condition $J\mathfrak{h} = V$,

ii) complex structures on $T\mathfrak{h}$ and $T^*\mathfrak{h}$ where $\mathfrak{h}$ is a three dimensional real Lie algebra,
iii) symplectic structures which are compatible for a complex structure in ii), therefore
inducing pseudo Kähler geometries.

Complex structures appearing in i) are called totally real. They have become objects of
importance in the construction of weak mirror pairs (see for instance [CLP] and references
therein).

Complex and symplectic geometry are extremal special cases of generalized complex ge-
ometry. Once a Lie algebra \( h \) was fixed, the corresponding underlying geometric structure
arises either as a complex structure on \( h \) or as a totally real complex structure on \( T^* h \),
which is Hermitian for the canonical metric on \( T^* h \).

For the adjoint representation we prove that a totally real complex structure corresponds
to a non singular derivation of \( h \). Therefore the existence of such a totally real complex
structure on \( T^* h \) imposes the condition on \( h \) to be nilpotent (Theorem (3.5)). In dimension
three only \( T^* h_1 \), where \( h_1 \) denotes the Heisenberg Lie algebra of dimension three, possesses
a totally real complex structure. As application one proves the existence of a generalized
complex structure of symplectic type on some kind of nilpotent Lie algebras \( h \) and the
existence of lagrangian symplectic structures on \( T^* h \).

For the coadjoint representation, we give the general form of totally real complex struc-
tures \( J \) on \( T^* h \), proving the existence when \( h \) is one of the following three dimensional Lie
algebras: the Heisenberg Lie algebra, the Lie algebra of the group of rigid motions of the
Minkowski 2-space \( \mathfrak{t}_3, -1 \), the Lie algebra of the group of rigid motions of the Euclidean
2-space \( \mathfrak{t}_3, 0 \) and the one dimensional trivial central extension of the Lie algebra of the
group of affine motions, usually denoted as \( \text{aff}(\mathbb{R}) \).

In addition to the Lie algebras obtained in i) the six dimensional tangent Lie algebras
admitting complex structures correspond to a Lie algebra \( h \) which is either simple: \( \mathfrak{sl}(2) \),
\( \mathfrak{so}(3) \) or solvable and isomorphic to \( \mathbb{R} \times \text{aff}(\mathbb{R}) \). In the cotangent case we add \( \mathfrak{sl}(2) \), \( \mathfrak{so}(3) \)
\( \mathfrak{t}_3, 1 \) and \( \mathfrak{t}_3, \eta \) for \( \eta > 0 \).

Concerning iii) we study the geometry that derives from the Kähler pairs. The only Lie
algebras carrying such a structure are: the tangent and the cotangent of the Heisenberg
Lie algebra and the tangent of \( \mathbb{R} \times \text{aff}(\mathbb{R}) \). The study in the nilpotent case says that there
are flat and non flat pseudo Kähler metrics, result which extends those in [CFU]. Again
in this case totally real complex structures provide examples for Kähler pairs.

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2. Generalities on Complex structures

An almost complex structure on a Lie algebra \( g \) is an endomorphism \( J : g \to g \) satisfying
\( J^2 = -I \), where \( I \) is the identity map.

Let \( g^\mathbb{C} = g \otimes \mathbb{C} \) denote the complexification of \( g \) whose elements have the form \( v \otimes c \),
with \( v \in g \), \( c \in \mathbb{C} \). An almost complex structure \( J \) on \( g \) can be extended to a complex
linear endomorphism of \( g^\mathbb{C} \) that we also denote by \( J \), by setting \( J(v \otimes c) = Jv \otimes c \).

As usual, we identify \( v \in g \), with \( v \otimes 1 \in g^\mathbb{C} \), and hence any element in \( g^\mathbb{C} \) can be written
as \( x + iy \) where \( x, y \in g \). With this identification, the eigenspace corresponding to the
imaginary eigenvalue \( i \) of \( J \) is the subspace \( m \) of \( g^\mathbb{C} \) given by

\[ m = \{ x - iJx : x \in g \} \].
If we denote by $\sigma$ the conjugation map on $g^C$, that is, $\sigma(x + iy) = x - iy$, the eigenspace corresponding to $-i$ is $\sigma m$, and we obtain the direct sum of vector spaces

$$g = m \oplus \sigma m.$$ 

Conversely any decomposition of type (1) induces an almost complex structure on $g$. In fact if $x \in g \subseteq g^C$ and $x$ can be written uniquely as $x = u + v \in m \oplus \sigma m$, define an endomorphism $J$ by $Jx = iu - iv$. Since $\sigma \circ J = J \circ \sigma$, the map $J$ gives rise to an almost complex structure on $g$.

The integrability condition of an almost complex structure $J$ is expressed in terms of the Nijenhuis tensor $N_J$

$$N_J(x, y) = [Jx, Jy] - [x, y] - J[Jx, y] - J[x, y], \quad \text{for all } x, y \in g.$$

It is straightforward to verify that $N_J(Jx, Jy) = -N_J(x, y)$ for any $x, y \in g$. Hence, if $g = u \oplus Ju$ is a direct sum as vector subspaces, then $N_J \equiv 0$ if and only if $N_J(u, v) = 0 = N_J(J(u, v))$ for all $u, v \in u$.

An almost complex structure $J$ on $g$ is called integrable if $N_J \equiv 0$. In this case $J$ is called a complex structure on $g$. Equivalently, $J$ is integrable if and only if $m$ (and hence $\sigma m$) satisfying (1) is a complex subalgebra of $g^C$.

Special types of almost complex structures are determined by those endomorphisms $J : g \rightarrow g$ satisfying $J^2 = -1$ and one of the following conditions for any $x, y \in g$:

$$c1) \quad J[x, y] = [x, Jy] \quad \quad \quad \quad \quad \quad c2) \quad [Jx, Jy] = [x, y].$$

In any case they are integrable. Complex structures of type $c1)$ determine a structure of complex Lie algebra on $g$, they are sometimes called bi-invariant. The subalgebra corresponding to the eigenvalue $\pm i$ is actually an ideal of $g^C$. Structures of type $c2)$ are called abelian, and the corresponding eigenspaces for the eigenvalues $\pm i$ are complex abelian subalgebras of $g^C$. It should be remarked that not any Lie algebra admitting a complex structure can be endowed with an abelian complex structure, as shown by the following example.

**Example 2.1.** Let $h_1$ be the Heisenberg Lie algebra of dimension three and let $T^* h_1$ denote the cotangent Lie algebra of $h_1$. This is spanned by the vectors $e_1, e_2, e_3, e_4, e_5, e_6$ with the non trivial Lie bracket relations

$$[e_1, e_2] = e_3, \quad [e_1, e_6] = -e_5, \quad [e_2, e_6] = e_4.$$ 

Assume $J : T^* h_1 \rightarrow T^* h_1$ is an endomorphism satisfying $[Jx, Jy] = [x, y]$ or, in equivalent form, $[Jx, y] = -[x, Jy]$ for all $x, y \in g$. If $y$ belongs to the center of $T^* h_1$, by $c2)$ one has that $Jy \in z(T^* h_1)$, thus $J$ restricts to the center $z(T^* h_1)$, and therefore $J$ cannot be almost complex since the dimension of $z(T^* h_1)$ is odd. To see that $T^* h_1$ admits a complex structure see (4.12).

As proved in the previous example, if $g$ carries an abelian complex structure, then the center of $g$ must be $J$-invariant and therefore even dimensional. Another necessary condition to have abelian complex structures is that $g$ is 2-step solvable, which means that the commutator subalgebra $C(g)$ is abelian (see [12] for instance). For the sake of completeness we include here a proof.

Let $g$ be a Lie algebra with a decomposition as direct sum of vector subspaces $g = a \oplus b$, where $a$ and $b$ are abelian subalgebras. Denote $[a, b'] = a^* + b''$ and $[a', b] = a'' + b'$,
Let \( g \) be a Lie algebra and let \( J \) be a fixed almost complex structure on \( g \). For any \( l \geq 0 \) we define the set \( a_l(J) \) inductively as:

\[
a_0(J) = \{0\}, \quad a_l(J) = \{X \in g, [X, g] \subset a_{l-1} \text{ and } [JX, g] \subset a_{l-1}\} \quad l \geq 1.
\]

It is easy to verify that

\[
a_0(J) \subseteq a_1(J) \subseteq a_2(J) \subseteq \ldots
\]

For a fixed \( X \in a_{i+1}(J) \) we have that \([X, Y] \in a_i(J) \subseteq a_{i+1}(J)\) for all \( Y \in g \), and clearly \([J[X, Y], Z] \in a_i(J) \subset a_{i+1}(J)\) for all \( Y, Z \in g \). Therefore \( a_i(J) \) is a \( J \)-invariant ideal of \( g \) for any \( i \geq 0 \).

The almost complex structure \( J \) is called \emph{nilpotent} if there exists a \( t \) such that \( a_t(J) = g \). This implies that \( g \) must be nilpotent. For a nilpotent almost complex structure \( J \) on a \( s \)-step nilpotent Lie algebra of even dimension we shall say that it is \( r \)-step nilpotent if \( r \) is the first nonnegative integer such that \( a_r(J) = g \); this satisfies the inequality \( s \leq r \leq n \) \cite{CFGU2}. In the following sections examples are given that show that these bounds are actually reached (see \((4.12)\) and \((4.5)\)). Notice that if \( J \) is a nilpotent almost complex structure on a nilpotent Lie algebra \( g \), then any term of the ascending series of \( g \) admits a two dimensional \( J \)-invariant subspace. Clearly, if \( J \) is integrable, the condition of being nilpotent is stronger than asking the corresponding \( m \) for \( J \) to be nilpotent.

Notice \( a_1(J) \subseteq \mathfrak{z}(g) \). If \( J \) is abelian then the equality holds \( a_1(J) = \mathfrak{z}(g) \).

\textbf{Example 2.2.} The canonical complex structure of a nilpotent complex Lie algebra is nilpotent (see \((4.11)\)).

An equivalence relation is defined among Lie algebras with complex structures. Lie algebras with complex structures \((g_1, J_1)\) and \((g_2, J_2)\) are called \emph{holomorphically equivalent} if there exists an isomorphism of Lie algebras \( \alpha : g_1 \to g_2 \) such that \( J_2 \circ \alpha = \alpha \circ J_1 \). In particular when \( g_1 = g_2 \) we simply say that \( J_1 \) and \( J_2 \) are \emph{equivalent} and a classification of complex structures can be done.

\textbf{Lemma 2.3.} Let \( g \) be an even dimensional real Lie algebra.

1. The class of an abelian complex structure, if non-empty, consists only of abelian complex structures.
2. Let \( J, J' \) be complex structures on \( g \) such that \( J' = \sigma J \sigma^{-1} \). Then \( \sigma a_l(J) = a_l(J') \) for any \( l \geq 0 \).

In particular the class of a nilpotent complex structure on a given nilpotent Lie algebra consists only of nilpotent complex structures, all of them being nilpotent of the same type.

3. The class of a bi-invariant complex structure has only bi-invariant complex structures.

\textbf{Proof.} 1) Let \( J \) an abelian complex structure on \( g \). If \( J' = \sigma J \sigma^{-1} \), the result follows using that \( \sigma \) is an automorphism and \( J \) is abelian.
ii) For $i = 0$ it is clear that $\sigma a_0(J) = a_0(J')$. Assume that $\sigma a_l(J) = a_l(J')$ for all $l \leq i$. Let $X \in a_{i+1}(J)$. It follows at once that the following identity is satisfied
\[ [\sigma X, g] = [\sigma X, \sigma g] = \sigma [X, g] \in \sigma a_i(J) = a_i(J'). \]
On the other hand,
\[ [J' \sigma X, g] = [\sigma JX, g] = \sigma [JX, g] \in \sigma a_i(J) = a_i(J'). \]
Therefore $\sigma a_{i+1}(J) \subset a_{i+1}(J')$. A similar argument, interchanging the roles of $J$ and $J'$, proves that $\sigma a_i(J) = a_i(J')$.

iii) follows by direct application of the definitions. \hfill \Box

Remark. From the definitions above it is not immediately clear which is the relationship between nilpotent complex structures and complex structures whose corresponding $i$-eigenspace $m$ is nilpotent. In (4.5) we show that the tangent Lie algebra of the Heisenberg Lie algebra $h_1$ carries only 2-step nilpotent complex structures, some of them being abelian, and others having $\pm i$-eigenspaces which are 2-step nilpotent subalgebras.

3. Totally real complex structures on tangent and cotangent Lie algebras

The aim of this section is the study of totally real complex structures on tangent and cotangent Lie algebras, that is complex structures $J$ on $T\pi h$ such that $Jh = V$.

We briefly recall the construction. Let $h$ denote a real Lie algebra and let $(\pi, V)$ be a finite dimensional representation of $h$. By endowing $V$ with the trivial Lie bracket, consider the semidirect product of $h$ and $V$ relative to $\pi$, $T\pi h := h \ltimes \pi V$, where the Lie bracket is:
\[ [(x, v), (x', v')] = ([x, x'], \pi(x)v' - \pi(x')v) \quad x, x' \in h, v, v' \in V. \]

In this work we focus on the the adjoint and the coadjoint representations. In both cases $V$ is a real vector space with the same dimension as that of $h$. The adjoint representation $\text{ad} : h \to \mathfrak{gl}(h)$ is given by $\text{ad}(x)y = [x, y]$, and it defines the tangent Lie algebra that we denote with $T\pi h$.

For the coadjoint representation $\text{ad}^* : h \to \mathfrak{gl}(h^*)$, that is $V = h^*$, and
\begin{equation}
\text{ad}(x)^*\varphi(y) = -\varphi \circ \text{ad}(x)y \quad \text{for } x, y \in h, \varphi \in h^*;
\end{equation}
we call the resulting Lie algebra cotangent Lie algebra and we denote it as $T^* h$.

A question concerning complex structures when we look at the algebraic structure of the Lie algebra $T\pi h = h \ltimes \pi V$ is whether there exists an almost complex structure $J$ such that $Jh = V$. Such a $J$ induces a linear isomorphism $j : h \to V$, and conversely any such $j : h \to V$ determines an almost complex structure on $T\pi h$ such that $Jh = V$, by means of
\begin{equation}
J(x, v) = (-j^{-1}v, jx) \quad x \in h, v \in V, j : h \to V.
\end{equation}

It follows that both $h$ and $V$ are totally real with respect to $J$. We adopt the next terminology, following [CLP].

Definition 3.1. Let $T\pi h := h \ltimes \pi V$ be the semidirect product of a Lie algebra $h$ with the real vector space $V$ such that $\dim V = \dim h$ and let $J$ denote an (almost) complex structure on $T\pi h$. If $Jh = V$ we say that $J$ is a totally real (almost) complex structure on $g$. 
Assume \((\pi, V)\) denotes a finite dimensional representation of \(\mathfrak{h}\) and let \(J\) be a totally real almost complex structure on \(T_\pi \mathfrak{h} = \mathfrak{h} \ltimes_\pi V\) like in (5). In this case, the integrability condition for \(J\) reduces to

\[
0 = [x, y] - j^{-1}\pi(x)jy + j^{-1}\pi(y)jx \quad \text{for all } x, y \in \mathfrak{h}.
\]

Fix a Lie algebra \(\mathfrak{h}\), recall that the representations \((V, \pi)\) and \((V', \pi')\) are called equivalent if there is a linear isomorphism \(T : V \rightarrow V'\) such that \(T^{-1}\pi'(x)T = \pi(x)\) for all \(x \in \mathfrak{h} \).

Actually for any \(\psi \in \text{Aut}(\mathfrak{h})\), the map \(\varphi : T_\pi \mathfrak{h} \rightarrow T_{\pi'} \mathfrak{h}\) given by \(\varphi = \psi + T\) is a Lie algebra isomorphism. In fact for all \(x, y \in \mathfrak{h}, u, v \in V\)

\[
\varphi[x + u, y + v] = \varphi([x, y] + \pi(x)v - \pi(y)u) = \psi[x, y] + T\pi(x)v - T\pi(y)u = [\psi x, \psi y] + \pi'(x)Tv - \pi'(y)Tu = [\varphi(x + u), \varphi(y + v)].
\]

And therefore if \(J\) denote a complex structure on \(T_\pi \mathfrak{h}\) then \(J' := \psi \circ J \circ \varphi^{-1}\) denotes a complex structure on \(T_{\pi'} \mathfrak{h}\) making of \((T_\pi \mathfrak{h}, J)\) and \((T_{\pi'} \mathfrak{h}, J')\) a pair of holomorphically equivalent Lie algebras.

In particular if \(J\) is a totally real complex structure on \(T_\pi \mathfrak{h}\), then \((T_\pi \mathfrak{h}, J)\) is holomorphically equivalent to \((T_{\pi'} \mathfrak{h}, J')\) where \(\tilde{J}_\mathfrak{h} : \mathfrak{h} \rightarrow V'\) is \(\tilde{J} = T \circ J_{\mathfrak{h}}\) and extended as in (5). The proof of the following result follows by using these relations and the integrability condition (6).

**Proposition 3.2.** Let \((V, \pi)\) and \((V', \pi')\) be equivalent representations of a Lie algebra \(\mathfrak{h}\) such that \(\dim V = \dim V' = \dim \mathfrak{h}\). Complex structures on \(T_\pi \mathfrak{h}\) are in one to one correspondence with complex structures on \(T_{\pi'} \mathfrak{h}\).

In particular, totally real complex structures on \(T_\pi \mathfrak{h}\) are homomorphically equivalent to totally real complex structures on \(T_{\pi'} \mathfrak{h}\).

A first consequence of (6) concerns abelian complex structures.

**Corollary 3.3.** Let \(\mathfrak{h}\) be a Lie algebra and let \(V\) denote the underlying vector space of \(\mathfrak{h}\). Let \(T_\pi \mathfrak{h} := \mathfrak{h} \ltimes_\pi V\) denote the semidirect product and let \(J\) be an abelian totally real complex structure on \(T_\pi \mathfrak{h}\). Then \(\mathfrak{h}\) is abelian and \(\pi\) and \(J\) are related by \(\pi(x)Jy = \pi(y)Jx\) for all \(x, y \in \mathfrak{h}\).

**Remark.** The converse of the previous corollary is true. Let \(\mathfrak{h}\) denote an abelian Lie algebra and let \(\pi\) be a representation of \(\mathfrak{h}\) into \(\mathfrak{h}\). Then if \(J : \mathfrak{h} \rightarrow \mathfrak{h}\) is a non singular map such that \(\pi(x)Jy = \pi(y)Jx\) for all \(x, y \in \mathfrak{h}\), then the almost complex structure on \(\mathfrak{h} \ltimes_\pi \mathfrak{h}\) given as in (5) is integrable and totally real with respect to \(\mathfrak{h}\). See the final section for an explicit example.

**Example 3.4.** Consider \(\mathbb{R}^n\) with the canonical basis \(\{e_1, e_2, \ldots, e_n\}\) and let \(A\) be a non singular \(n \times n\) real matrix. Let \(C_A\) denote the centralizer of \(A\) in \(gl(n, \mathbb{R})\) that is, the set of \(n \times n\) matrices \(B\) such that \(BA = AB\). Let \(B_1, B_2, \ldots, B_n\) be matrices in \(C_A\) such that they are pairwise in involution, \(B_iB_j = B_jB_i\) for all \(i, j\). Take \(\pi\) the representation of \(\mathbb{R}^n\) which extends linearly the mapping \(e_i \rightarrow B_i\) (notice that this could be trivial depending on \(A\).) The map \(j\) represented by \(A\) amounts to a totally real abelian complex structure on \(T_\pi \mathbb{R}^n\).
When \( \pi \) is the adjoint representation, the solutions of (6) have an algebraic interpretation as it will be seen next. Recall first that a derivation of a Lie algebra \( \mathfrak{h} \) is a linear map \( d : \mathfrak{h} \to \mathfrak{h} \) such that
\[
d[x, y] = [dx, y] + [x, dy]
\]
for all \( x, y \in \mathfrak{h} \).

Jacobson proved that if a Lie algebra \( \mathfrak{h} \) admits a non singular derivation then it must be nilpotent [Ja].

**Theorem 3.5.** Let \( T\mathfrak{h} \) denote the tangent Lie algebra of \( \mathfrak{h} \). The set of totally real complex structures on \( T\mathfrak{h} \) is in one to one correspondence with the set of non singular derivations of \( \mathfrak{h} \). If one (and therefore both) of these sets is non empty, then \( \mathfrak{h} \) is nilpotent.

**Proof.** Let \( ad \) denote the adjoint representation of \( \mathfrak{h} \). The integrability condition (5) becomes
\[
0 = j[x, y] - ad(x) jy + ad(y) jx \quad \text{for all } x, y \in \mathfrak{h}.
\]
This shows, via identifications, that the complex structure \( J \) determined by \( j \) (5) corresponds to a non singular derivation of \( \mathfrak{h} \). The proof is completed after the application of the result of Jacobson.

**Example 3.6.** Let \( \mathfrak{h}_1 \) denote the Heisenberg Lie algebra of dimension three (see 3.7). By Lemma (3.5) if \( J \) is a totally real complex structure on \( T\mathfrak{h}_1 \), then it corresponds to a non singular derivation of \( \mathfrak{h}_1 \). Any non singular derivation of \( \mathfrak{h}_1 \) has a matrix representation in the basis of (3.7) given by:
\[
\begin{pmatrix}
A & 0 \\
* & tr(A)
\end{pmatrix}, \quad \text{with } A \in GL(2, \mathbb{R}) \quad \text{and } tr(A) \neq 0
\]
where \( tr \) denotes the trace of the matrix \( A \).

More generally any non singular derivation \( d \) of the Heisenberg Lie algebra \( \mathfrak{h}_n \) of dimension \( 2n + 1 \), has a matrix representation as above with \( A \in GL(2n, \mathbb{R}) \) and \( tr(A) \neq 0 \). This induces a totally real complex structure \( J \) on the tangent Lie algebra \( T\mathfrak{h}_n \).

### 3.1. Totally real complex structures on six dimensional cotangent Lie algebras.

We now proceed to analyze the existence of totally real complex structures on six dimensional cotangent Lie algebras \( T^* \mathfrak{h} \). To this extend, recall the classification of three dimensional Lie algebras as given e.g. in [GOV] or [Mi].

**Theorem 3.7.** Let \( \mathfrak{h} \) be a real Lie algebra of dimension three spanned by \( e_1, e_2, e_3 \). Then it is isomorphic to one and only one in the following list:

\[
\begin{align*}
\mathfrak{h}_1 & \quad [e_1, e_2] = e_3 \\
\mathfrak{r}_3 & \quad [e_1, e_2] = e_2, \quad [e_1, e_3] = e_2 + e_3 \\
\mathfrak{r}_{3,\lambda} & \quad [e_1, e_2] = e_2, \quad [e_1, e_3] = \lambda e_3, \quad |\lambda| \leq 1 \\
\mathfrak{r}_{3,\eta} & \quad [e_1, e_2] = \eta e_2 - e_3, \quad [e_1, e_3] = e_2 + \eta e_3, \quad \eta \geq 0 \\
\mathfrak{so}(2) & \quad [e_1, e_2] = e_3, \quad [e_3, e_1] = 2e_1, \quad [e_3, e_2] = -2e_2 \\
\mathfrak{so}(3) & \quad [e_1, e_2] = e_3, \quad [e_3, e_1] = e_2, \quad [e_3, e_2] = -e_1
\end{align*}
\]

A Lie algebra \( \mathfrak{g} \) which satisfies \( tr(ad(x)) = 0 \) for all \( x \in \mathfrak{g} \) is called unimodular. Among the Lie algebras above, the unimodular ones are: \( \mathfrak{h}_1, \mathfrak{r}_{3,-1} \) and \( \mathfrak{r}_{3,0} \). The Lie algebra \( \mathfrak{h}_1 \) is known as the Heisenberg Lie algebra, while \( \mathfrak{r}_{3,-1} \) is the Lie algebra of the group of rigid motions of the Minkowski 2-space and \( \mathfrak{r}_{3,0} \) corresponds to the Lie algebra of the group of rigid motions of the Euclidean 2-space. The Lie algebra \( \mathfrak{r}_{3,0} \) denotes the central extension of the Lie algebra of the group of affine motions in \( \mathbb{R} \), usually denoted as \( \text{aff}(\mathbb{R}) \).
Proposition 3.8. Let $T^* \mathfrak{h} = \mathfrak{h} \times \mathfrak{h}^*$ be a cotangent Lie algebra of a three dimensional Lie algebra $\mathfrak{h}$. Then totally real complex structures on $T^* \mathfrak{h}$ exist whenever $\mathfrak{h}$ is either unimodular or isomorphic to $\mathbb{R} \times \mathfrak{aff}(\mathbb{R})$. In those cases the map $j : \mathfrak{h} \to \mathfrak{h}^*$ admits a matrix representation as follows

$$
\begin{align*}
T^* \mathfrak{h}_1 & \left( \begin{array}{ccc} a_{41} & a_{42} & a_{43} \\
 & a_{52} & a_{53} \\
( & a_{43} & -a_{53} \\
\end{array} \right); & T^* r_{3,-1} & \left( \begin{array}{ccc} a_{41} & a_{42} & a_{43} \\
 & -a_{42} & a_{53} \\
( & -a_{43} & a_{53} \\
\end{array} \right); \\
T^* r_{3,0} & \left( \begin{array}{ccc} a_{41} & a_{42} & a_{43} \\
 & -a_{42} & 0 \\
( & a_{61} & a_{63} \\
\end{array} \right); & T^* r'_{3,0} & \left( \begin{array}{ccc} a_{41} & a_{42} & a_{43} \\
 & -a_{42} & 0 \\
( & -a_{43} & a_{53} \\
\end{array} \right); \\
\end{align*}
$$

where the matrix should be non singular.

Proof. The proof follows by direct computation of (6) taking $\pi$ as the coadjoint representation. In the cases not listed above, the maps $j$ solving (7) are singular, hence they cannot induce a complex structure on $T^* \mathfrak{h}$. \qed

Example 3.9. Let $\mathfrak{s}$ denote a semisimple Lie algebra. Since the Killing form is non degenerate this induces an ad-invariant metric on $\mathfrak{s}$. Therefore the adjoint and coadjoint representation are equivalent. By (3.2) the existence of totally real complex structures on $T^* \mathfrak{s}$ reduces to the existence of totally real complex structures on $T \mathfrak{s}$ and this cannot admit a totally real complex structure by (3.5). Observe that in dimension three the simple Lie algebras are $\mathfrak{sl}(2)$ and $\mathfrak{so}(3)$ (see (3.7)).

4. Complex structures on tangent and cotangent Lie algebras of dimension six

Examples of six dimensional real Lie algebras with complex structures arise from three dimensional complex Lie algebras. In fact let $\tilde{\mathfrak{g}}$ denote a three dimensional complex Lie algebra, then the underlying real Lie algebra $\mathfrak{g} := \tilde{\mathfrak{g}}_{\mathbb{R}}$, is naturally equipped with a bi-invariant complex structure induced by the multiplication by $i$ on $\tilde{\mathfrak{g}}$. In this way this complex structure on $\mathfrak{g}$ is bi-invariant.

Example 4.1. Let $\mathfrak{g}$ denote a six dimensional two-step nilpotent Lie algebra equipped with bi-invariant complex structure $J$. We shall see that $\mathfrak{g}$ is isomorphic to the real Lie algebra underlying $\mathfrak{h}_1 \otimes \mathbb{C}$, the complexification of the Heisenberg Lie algebra of dimension three.

Since $J \circ \text{ad}(x) = \text{ad}(x) \circ J$ for any $x \in \mathfrak{g}$, we obtain the inclusions $J^3(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$ and $J^3C(\mathfrak{g}) \subseteq C(\mathfrak{g})$. Furthermore, there is a central ideal $\mathfrak{v} \subseteq \mathfrak{z}(\mathfrak{g})$ such that

$$
\mathfrak{z}(\mathfrak{g}) = C(\mathfrak{g}) \times \mathfrak{v} \quad \text{direct sum of } J\text{-invariant Lie algebras}.
$$

A way to produce such a $\mathfrak{v}$ is the following. Take $\langle , \rangle$ an inner product on $\mathfrak{z}(\mathfrak{g})$ which is Hermitian for $J$ and let $\mathfrak{v} = C(\mathfrak{g})^\perp$.

Thus $\mathfrak{g} = \mathfrak{v} \times \mathfrak{n}$ where $\mathfrak{n}$ is a two-step nilpotent Lie algebra such that $C(\mathfrak{g}) = C(\mathfrak{n}) = \mathfrak{z}(\mathfrak{n})$ and $\mathfrak{n}$ is equipped with a bi-invariant complex structure $J$, induced from that one on $\mathfrak{g}$. Now if $\mathfrak{v}$ is non empty then it may be two or four dimensional. If it is two dimensional, then $\mathfrak{n}$ is four dimensional and is equipped with a bi-invariant complex structure, therefore it must be abelian (in dimension four a solvable Lie algebra endowed with a bi-invariant complex structure is either abelian or isomorphic to $\mathfrak{aff}(\mathbb{C})$, see [O2] for instance). A similar reasoning applies when $\mathfrak{v}$ has dimension four, and therefore $\mathfrak{z}(\mathfrak{g}) = C(\mathfrak{g})$. 


Now let \( z \in \mathfrak{z}(\mathfrak{g}) = C(\mathfrak{g}) \). Then there exist \( x, y \in \mathfrak{g} \) such that \([x, y] = z\). The set \( \{ x, y, z \} \) is linearly independent and the set \( \{ x, y, z, Jx, Jy, Jz \} \) is a basis of \( \mathfrak{g} \). Due to the bi-invariance condition on \( J \) one has the following Lie bracket relations

\[
[x, y] = z \quad [Jx, y] = Jz \quad [x, Jy] = Jz \quad [Jx, Jy] = -z
\]

moreover \([x, Jx] = 0 = [y, Jy]\), and therefore the Lie algebra \( \mathfrak{g} \) is isomorphic to the real Lie algebra of dimension six underlying \( \mathfrak{h}_1 \otimes \mathbb{C} \).

In the previous section we gave examples of complex structures in tangent and cotangent Lie algebras of dimension six. Now we shall study the existence problem of complex structures on any tangent or cotangent Lie algebra corresponding to a three dimensional real Lie algebra as in (3.7). Some considerations about abelian and nilpotent complex structures are also be given.

**The simple case.** Among the Lie algebras listed in (3.7) the simple ones are \( \mathfrak{sl}(2) \) and \( \mathfrak{so}(3) \). Since the Killing form is non degenerate in both cases, the adjoint and the coadjoint representations are equivalent. After (3.2) for a semisimple Lie algebra \( \mathfrak{s} \), the existence of complex structures on \( \mathfrak{T} \mathfrak{s} \) determines one on \( \mathfrak{T}^* \mathfrak{s} \) and viceversa. Recall that complex structures on compact semisimple and more generally on reductive Lie algebras were extended studied (see for instance \[ \text{[Sm]} \text{[Wa]} \text{[Sa1]} \text{[Sa2]} \text{[SD]} \]). We perform below a construction of a complex structure on \( \mathfrak{T} \mathfrak{s}(2) \) and \( \mathfrak{T} \mathfrak{so}(3) \). Explicitly the Lie brackets are given by:

\[
\begin{align*}
\mathfrak{T} \mathfrak{s}(2) & : [e_1, e_2] = e_3 \quad [e_3, e_1] = 2e_1 \quad [e_3, e_2] = -2e_2 \\
& \quad [e_1, e_5] = e_6 \quad [e_1, e_6] = -2e_4 \quad [e_2, e_4] = e_5 \\
& \quad [e_2, e_6] = 2e_5 \quad [e_3, e_1] = 2e_4 \quad [e_3, e_2] = -e_6
\end{align*}
\]

\[
\begin{align*}
\mathfrak{T} \mathfrak{so}(3) & : [e_1, e_2] = e_3 \quad [e_3, e_1] = e_2 \quad [e_3, e_2] = -e_1 \\
& \quad [e_1, e_5] = -e_6 \quad [e_1, e_6] = e_5 \quad [e_2, e_4] = -e_6 \\
& \quad [e_2, e_6] = e_4 \quad [e_3, e_4] = -e_5 \quad [e_3, e_5] = e_4
\end{align*}
\]

In any case an almost complex structure \( J \) can be defined by

\[
Je_3 = e_6 \quad Je_2 = e_1 \quad Je_4 = e_5.
\]

By calculating \( N_J \) one verifies that \( J \) is integrable. Hence

**The tangent Lie algebras \( \mathfrak{T} \mathfrak{so}(3) \) and \( \mathfrak{T} \mathfrak{s}(2) \) (and therefore \( \mathfrak{T}^* \mathfrak{so}(3) \) and \( \mathfrak{T}^* \mathfrak{s}(2) \)) carry complex structures**

**The solvable case.** Suppose that \( \mathfrak{g} \) is a six dimensional tangent \( \mathfrak{T} \mathfrak{h} \) or cotangent Lie algebra \( \mathfrak{T}^* \mathfrak{h} \) being \( \mathfrak{h} \) a solvable real Lie algebra of dimension three. It admits a complex structure if and only if \( \mathfrak{g}^\mathbb{C} \) decomposes as a direct sum of vector subspaces \( \mathfrak{g}^\mathbb{C} = \mathfrak{m} \oplus \mathfrak{m} \mathfrak{o} \), where \( \mathfrak{m} \) (resp. \( \mathfrak{m} \mathfrak{o} \)) is a complex subalgebra. Without lost of generality assume that \( \mathfrak{m} \) is spanned by vectors \( U, V, W \) as follows:

\[
(8) \quad U = e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 \quad V = b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6,
\]

\[
W = c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6, \quad a_i, b_j, c_k \in \mathbb{C}, \forall i,j,k = 2, \ldots, 6.
\]

Let \( \mathfrak{a} := \text{span}\{V, W\} \). We claim that \( \mathfrak{a} \) is a ideal in \( \mathfrak{m} \). In fact, according to the Lie brackets in \( \mathfrak{g} \) (see (4.2) and (4.3) below), one verifies that \( U \notin C(\mathfrak{g}) \), hence for any \( x, y \in \mathfrak{m} \), \([x, y] \in C(\mathfrak{m}) \subseteq \mathfrak{a} \). Thus \( \mathfrak{m} = \mathfrak{C} U \ltimes \mathfrak{a} \), being \( \mathfrak{a} \) a ideal of \( \mathfrak{m} \) of dimension two and therefore isomorphic either to i) \( \mathbb{C}^2 \) or to ii) \( \mathfrak{a}\mathfrak{f}(\mathbb{R}) \), the two dimensional complex Lie algebra spanned by vector \( X, Y \) with \([X, Y] = Y\). We may assume in the last situation that \( V, W \) satisfy the Lie bracket relation \([V, W] = W\).
In case \( m = \mathbb{C} U \ltimes \mathbb{C}^2 \), the action of \( U \) on \( a \) admits a basis whose matrix is one of the following ones

\[
(9) \quad (1) \quad \begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}, \quad \nu, \mu \in \mathbb{C}; \quad (2) \quad \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}, \quad \nu \in \mathbb{C}.
\]

In case \( m = \mathbb{C} U \ltimes \text{aff}(\mathbb{R}) \) the action of \( U \) on \( a \) is a derivation of \( \text{aff}(\mathbb{R}) \) thus over the basis \( \{V, W\} \) we have a matrix

\[
(10) \quad \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad a, b \in \mathbb{C}
\]

By making use of this we shall derive the existence or non existence of complex structures on any tangent or cotangent Lie algebra corresponding to a three dimensional solvable real Lie algebra.

4.1. **Complex structures on six dimensional tangent Lie algebras.** If \( H \) denotes a Lie group, its tangent bundle \( TH \) is identified with \( H \ltimes \mathfrak{h} \), which inherits a natural Lie group structure as the semidirect product under the adjoint representation. Its Lie algebra, the tangent Lie algebra \( T \mathfrak{h} \), is the semidirect product via the adjoint representation \( \mathfrak{h} \ltimes_{\text{ad}} V \), where \( V \) is the underlying vector space to \( \mathfrak{h} \) equipped with the trivial Lie bracket.

**Proposition 4.2.** Let \( \mathfrak{h} \) be a solvable real Lie algebra of dimension three and let \( T \mathfrak{h} \) denote the tangent Lie algebra spanned by \( e_1, e_2, e_3, e_4, e_5, e_6 \). Then the non zero Lie brackets are presented in the following list:

\[
\begin{align*}
T \mathfrak{h}_1 : & \quad [e_1, e_2] = e_3, \quad [e_1, e_5] = e_6, \quad [e_2, e_4] = -e_6 \\
T \mathfrak{h}_3 : & \quad [e_1, e_2] = e_2, \quad [e_1, e_3] = e_2 + e_3, \\
& \quad [e_1, e_5] = e_5, \quad [e_1, e_6] = e_5 + e_6, \quad [e_2, e_4] = -e_5, \quad [e_3, e_4] = -e_5 - e_6 \\
T \mathfrak{h}_{3,\lambda} : & \quad [e_1, e_2] = e_2, \quad [e_1, e_3] = \lambda e_3 \\
& \quad [e_1, e_5] = e_5, \quad [e_1, e_6] = \lambda e_6, \quad [e_2, e_4] = -e_5, \quad [e_3, e_4] = -\lambda e_6 \\
T \mathfrak{h}_{3,\eta} : & \quad [e_1, e_2] = \eta e_2 - e_3, \quad [e_1, e_3] = e_2 + \eta e_3, \quad [e_1, e_5] = \eta e_5 - e_6, \\
& \quad [e_1, e_6] = e_5 + \eta e_6, \quad [e_2, e_4] = -\eta e_5 + e_6, \quad [e_3, e_4] = -e_5 - \eta e_6 \\
\end{align*}
\]

**Theorem 4.3.** Let \( \mathfrak{h} \) denote a three dimensional Lie algebra, then \( T \mathfrak{h} \) admits a complex structure if and only if \( \mathfrak{h} \) is either isomorphic to \( \mathfrak{h}_1 \) or \( \mathbb{R} \ltimes \text{aff}(\mathbb{R}) \).

The proof can be derived from the next paragraphs.

**Lemma 4.4.** If \( m \) is a complex subalgebra of \( T \mathfrak{h} \) being \( \mathfrak{h} \) a three dimensional solvable real Lie algebra such that \( T \mathfrak{h}^C = m \oplus \sigma m \) then \( m \simeq \mathbb{C} \ltimes \mathbb{C}^2 \).

**Proof.** According to the previous paragraphs it should hold \( m \simeq \mathbb{C} \ltimes \mathbb{C}^2 \) or \( m \simeq \mathbb{C} \ltimes \text{aff}(\mathbb{R}) \). We shall prove that the last situation is not possible. In fact, from the Lie brackets above \( (4.2) \) we see that \( [V, W] \in \text{span}\{e_5, e_6\} \) so that \( c_2 = 0 = c_3 = c_4 \). But by computing one has \( [V, W] = 0 \) implying \( W = 0 \) and therefore no complex structure can be derived from this situation.

With the previous Lemma it follows to analyze next the existence of complex structures attached to complex Lie subalgebras \( m \) such that \( m \simeq \mathbb{C} \ltimes \mathbb{C}^2 \).

We already know that the tangent Lie algebra of the three dimensional Heisenberg Lie algebra \( T \mathfrak{h}_1 \), admits complex structures, moreover totally real ones. Recall that any totally real complex structure on \( T \mathfrak{h}_1 \) corresponds to a non singular derivation of \( \mathfrak{h}_1 \) \( (3.6) \). No one of these complex structures is abelian. However \( T \mathfrak{h}_1 \) can be equipped with abelian complex structures as we show below.
Let $\mathfrak{m}$ be a complex subalgebra of $\mathfrak{T}\mathfrak{h}_1$ spanned by vectors $U, V, W$ as in $[8]$. The subspace $\mathfrak{a} = \text{span}\{V, W\}$ is an ideal of $\mathfrak{m}$ and $\mathfrak{m} = \mathbb{C}U \ltimes \mathfrak{a}$. Since $\mathfrak{T}\mathfrak{h}_1$ is nilpotent, $\mathfrak{a}$ is abelian and the action of $U$ on $\mathfrak{a}$ is of type (9) and moreover case 1) holds for $\mu = \nu = 0$ while case 2) holds for $\nu = 0$. Case 1) gives rise to abelian complex structures, while case 2) corresponds to non abelian ones.

Computing the Lie brackets $[V, W]$, $[U, V]$ and $[U, W]$, and imposing these brackets to be zero, we get

$$U = e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6,$$

$$V = b_3 e_3 + b_4 e_4 - a_2 b_4 e_5 + b_6 e_6, \quad W = c_3 e_3 + c_4 e_4 - a_2 c_4 e_5 + c_6 e_6.$$

If the set $\{U, V, W, \sigma U, \sigma V, \sigma W\}$ spans a basis of $(\mathfrak{T}\mathfrak{h}_1)^C$, the tangent algebra $\mathfrak{T}\mathfrak{h}_1$ carries an abelian complex structure $J$. For instance the linear homomorphism $J$ given by

$$Je_1 = e_2 \quad Je_6 = e_3 \quad Je_4 = e_5,$$

and such that $J^2 = -I$ defines an abelian complex structure on $\mathfrak{T}\mathfrak{h}_1$. After $[Mg]$ there is only one class among abelian complex structures.

Any abelian complex structure is 2-step nilpotent. In fact, since $J$ is abelian $a_1(J) = \mathfrak{z}(\mathfrak{T}\mathfrak{h}_1)$ and clearly the condition $C(\mathfrak{T}\mathfrak{h}_1) = \mathfrak{z}(\mathfrak{T}\mathfrak{h}_1)$ shows that $a_2(J) = \mathfrak{T}\mathfrak{h}_1$.

On the other hand the following set of vectors on $\mathfrak{T}\mathfrak{h}_1^C$ is a basis of the complex subalgebra $\mathfrak{m}$ corresponding to a totally real complex structure on $\mathfrak{T}\mathfrak{h}_1$:

$$e_1 - i(ae_4 + be_5 + ce_6); \quad e_2 - i(ce_4 + de_5 + fe_6), \quad e_3 - i(a + d)e_6$$

with $a, b, c, d, e, f \in \mathbb{R}$, $a + d \neq 0$ and $ad - bc \neq 0$. They induce non abelian complex structures, and furthermore there are more non abelian complex structures than the totally real ones. Let $\mathfrak{m}$ be a complex subalgebra of $(\mathfrak{T}\mathfrak{h}_1)^C$ spanned by $U, V, W$ as in $[S]$. Requiring that $[U, V] = 0 = [V, W]$ and $[U, W] = V$ we deduce that any complex subalgebra $\mathfrak{m}$ of $(\mathfrak{T}\mathfrak{h}_1)^C$ spanned by

$$U = e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6,$$

$$V = c_2 e_3 + (c_5 - a_2 c_4 + a_4 c_2)e_6 \quad W = c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6,$$

and such that $U, V, W, \sigma U, \sigma V, \sigma W$ is a basis of $(\mathfrak{T}\mathfrak{h}_1)^C$, induces a non abelian complex structure on $\mathfrak{T}\mathfrak{h}_1$.

The class of non abelian complex structures $J$ is 2-step nilpotent. Actually the vector $X := W + \sigma W$ belongs to the center of $\mathfrak{T}\mathfrak{h}_1$ and also $JX \in \mathfrak{z}(\mathfrak{T}\mathfrak{h}_1)$. Since $a_1(J) = \text{span}\{X, JX\} = \mathfrak{z}(\mathfrak{T}\mathfrak{h}_1)$ and $C(\mathfrak{T}\mathfrak{h}_1) = \mathfrak{z}(\mathfrak{T}\mathfrak{h}_1)$, we conclude that $a_2(J) = \mathfrak{T}\mathfrak{h}_1$.

After $[Mc]$ in the set of non abelian complex structures, one has the following non equivalent complex structures (the extension is such that $J^2 = -I$):

$$J_s e_1 = e_4 \quad J_s e_2 = -s e_4 + e_5 \quad J_s e_3 = 2e_6 \quad s = 0, 1,$$

which are totally real, and next

$$J_s e_1 = e_2 + (1 - \nu)e_4 + \frac{1 - \nu}{\nu}e_5 \quad J_s e_2 = -\nu e_1 + (1 - \nu)e_4 \quad J_s e_3 = e_6 \quad \nu \in \mathbb{R} - \{0\}$$

which are neither abelian nor totally real.

**Proposition 4.5.** The tangent Lie algebra $\mathfrak{T}\mathfrak{h}_1$ admits abelian and nonabelian complex structures, which are in every case 2-step nilpotent.

**Remark.** The Lie algebra $\mathfrak{T}\mathfrak{h}_1$ is isomorphic to $\mathfrak{G}_{6,4}$ in $[Mg]$ and to $\mathfrak{h}_4$ in $[CFU]$.

Next we show a family of tangent Lie algebras which cannot be equipped with a complex structure.
Proposition 4.6. The Lie algebras $T_{3}$ and $T_{3,\eta}$ $\eta \geq 0$ do not admit complex structures.

Proof. We shall give the complete proof for the tangent Lie algebra $T_{3}$. The proof for $T_{3,\eta}$ can be achieved with a similar reasoning.

Assume that $m$ is a complex subalgebra of $T_{3}^{C}$ corresponding to a complex structure of $T_{3}$. Let $U, V, W \in m$ be linearly independent vectors as in (3) such that $m = \mathbb{C} U \times a$ with $a = \text{span}\{V, W\}$.

Let $m$ be a complex subalgebra of $T_{3}^{C}$ spanned by vectors $U, V, W$ as in (3) and assume that $a = \text{span}\{V, W\}$ is abelian. The Lie bracket relations on $m$ are:

\[
\begin{align*}
[V, W] &= (b_{4}c_{2} - b_{2}c_{4} + b_{4}c_{3} - b_{3}c_{4})c_{5} + (b_{4}c_{3} - b_{3}c_{4})c_{6}, \\
[U, V] &= (b_{4} + b_{3})e_{2} + b_{3}e_{3} + (b_{5} + b_{6} + a_{4}b_{3} - a_{3}b_{4})c_{5} + \\
&\quad + (b_{6} + a_{4}b_{3} - a_{3}b_{4})c_{6}, \\
[U, W] &= (c_{2} + c_{3})e_{2} + c_{3}e_{3} + (c_{5} + c_{6} + a_{4}c_{2} - a_{2}c_{4} + a_{4}c_{3} - a_{3}c_{4})c_{5} + \\
&\quad + (c_{6} + a_{4}c_{3} - a_{3}c_{4})c_{6}.
\end{align*}
\]

We see that $\nu b_{4} = 0$ implies either $\nu = 0$ or $b_{4} = 0$.

If $\nu = 0$, from $[U, V] = 0$ one has $b_{2} = 0 = b_{3}$ and $V$ has the form $V = b_{4}(e_{4} + a_{2}e_{5} + a_{3}e_{6})$.

We also have $\mu c_{4} = 0$. If $\mu = 0$ then $c_{2} = 0 = c_{3}$ and so $U, V, \sigma V \in \text{span}\{e_{4}, e_{5}, e_{6}\}$; therefore this set is not linearly independent. Hence $\mu \neq 0$ and $c_{4} = 0$. But from $[V, W] = 0$ it must hold $b_{4}c_{3} = 0 = b_{4}c_{2}$ and since $b_{4} \neq 0$ one has $c_{2} = 0 = c_{3}$ but then $V, W, \sigma V \in \text{span}\{e_{4}, e_{5}, e_{6}\}$ which gives no basis of $T_{3}^{C}$.

From this contradiction we get that $\nu \neq 0$ and $b_{4} = 0$.

From $[V, W] = 0$ it may hold $b_{2}c_{4} = 0 = c_{4}b_{3}$. If $c_{4} \neq 0$ then $b_{2} = 0 = b_{3}$. Moreover if $\nu \neq 1$ then $b_{5} = 0 = b_{6}$ which derives in the contradiction $V = 0$. Therefore $\nu = 1$ implies $b_{6} = 0$ and so $V \in \text{span}\{e_{5}\}$ which does not allow a basis of $T_{3}^{C}$. Hence $c_{4} = 0$. If $\nu \neq 1$ then $b_{2} = 0 = b_{5} = b_{6}$ thus $V = 0$ is a contradiction. Now $\nu = 1$ implies $b_{3} = 0$ and $b_{6} = a_{4}b_{2}$ so that $V \in \text{span}\{e_{2}, e_{5}, e_{6}\}$.

From $[U, W] = \mu W$ we have that if $\mu \neq 1$ then $c_{2} = 0 = c_{3}$ and in this way $V, W, \sigma V, \sigma W \in \text{span}\{e_{2}, e_{5}, e_{6}\}$ which cannot build a basis of $T_{3}^{C}$. Thus $\mu = 1$ and so $c_{3} = 0$, and therefore the set $V, W, \sigma V, \sigma W \in \text{span}\{e_{2}, e_{5}, e_{6}\}$ does not induce a basis of $T_{3}^{C}$ implying the non existence of a complex subalgebra $m$ in this case.

Assume now $m = \text{span}\{U, V, W\}$ is a complex subalgebra such that $T_{3}^{C} = m \oplus \sigma m$ and $[V, W] = 0 \quad [U, V] = \nu V \quad [U, W] = V + \nu W$ for some $\nu \in \mathbb{C}$.

If $\nu \neq 1$ it follows that $b_{2} = 0 = b_{3}$ which also implies $c_{2} = 0 = c_{3}$. Therefore $V, W, \sigma V, \sigma W \in \text{span}\{e_{4}, e_{5}, e_{6}\}$ and we have no complex structure.

Hence $\nu = 1$. Thus $b_{3} = 0 = b_{4} = c_{4}$ and $c_{3} = b_{2}$. From $[U, V] = V$ one gets $b_{6} + a_{4}b_{2} = 0$ and from $[U, V] = V + W$ one has $b_{6} - a_{4}b_{2} = 0$; therefore $b_{6} = 0 = a_{4}b_{2}$. Since $a_{4} \neq 0$ then $b_{2} = 0$ but this implies $V = b_{5}e_{5}$ and so $V, \sigma V$ are not linearly independent. Hence no complex structure arise in this case.

Proposition 4.7. The following statements are equivalent:

(i) $T_{3,\lambda}$ can be endowed with a complex structure;

(ii) $T_{3,\lambda}$ carries an abelian complex structure;

(iii) $\lambda = 0$.

Proof. One proves i) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i). It is easy to see ii) $\Rightarrow$ iii).

We will give a general line for the proof.

Let $m$ denote a complex subalgebra of $T_{3,\lambda}^{C}$ spanned by vectors $U, V, W$ as in (3), with $[V, W] = 0$. 

The Lie brackets follow

\[ [V, W] = (b_1c_2 - b_2c_1)e_5 + \lambda(b_4c_3 - b_3c_4)e_6 \]
\[ [U, V] = b_2e_2 + \lambda b_3e_3 + b_5e_5 + \lambda b_6e_6 + (a_4b_2 - a_2b_4)e_5 + \lambda(a_4b_3 - a_3b_4)e_6 \]
\[ [U, W] = c_2e_2 + \lambda c_3e_3 + c_5e_5 + \lambda c_6e_6 + (a_4c_2 - a_2c_4)e_5 + \lambda(a_4c_3 - 3a - 3c_4)e_6 \]

If the action of \( U \) on \( \text{span}\{V, W\} \) is of type (1) in (9), by solving the corresponding system, one gets a basis of \((T_e r_3, \lambda)^c\) only if \( \lambda = 0 \), with the additional constraints \( \nu = 0 = \mu \). Explicitly, the vectors adopt the form

\[ U = e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \]

whenever \( U, V, W, \sigma U, \sigma V, \sigma W \) is a basis of \((T_0 r_3, \lambda)^c\). It follows at once that the induced complex structure on \( T_0 r_3, \lambda \) is abelian.

If the action of \( U \) on \( \text{span}\{V, W\} \cong \mathbb{C}^2 \) is of type (2) in (9), then we cannot find a complex structure for any value of \( \lambda \). This argument shows \( i) \iff iii) \).

For \( ii) \iff iii) \) one works out the equations deriving from \( [V, W] = 0 = [U, V] = [U, W] \) to obtain that a solution exists only for \( \lambda = 0 \). In this case, one gets the vectors \( U, V, W \) above (14). For instance the following \( J \) gives rise to a abelian complex structure on \( T_0 r_3, \lambda \):

\[ J e_1 = e_2, \quad J e_3 = -e_6, \quad J e_4 = e_5. \]

To prove \( iii) \implies ii) \) one must solve the equation \( [V, W] = 0, [U, V] = \nu V \) and \( [U, W] = \mu W \) for \( \lambda = 0 \). It is possible to see that the only way to get solutions is for \( \nu = \mu = 0 \), finishing the proof.

4.2. Complex structures on cotangent Lie algebras of dimension six. Recall the Lie group counterpart of the cotangent Lie algebra. The zero section in the cotangent bundle \( T^* H \) of a Lie group \( H \), can be identified with \( H \), as well as the fibre over \((e, 0)\) with \( \mathfrak{h}^* \). As a Lie group, the cotangent bundle of \( H \) is the semidirect product of \( H \) with \( \mathfrak{h}^* \) via the coadjoint representation. The tangent space of \( T^* H \) at the identity is naturally identified with the **cotangent** Lie algebra \( T^* \mathfrak{h} := \mathfrak{h} \ltimes \text{coad} \mathfrak{h}^* \), the semidirect product of \( \mathfrak{h} \) and its dual \( \mathfrak{h}^* \) via the coadjoint action.

**Proposition 4.8.** Let \( \mathfrak{h} \) be a solvable real Lie algebra of dimension three and let \( T^* \mathfrak{h} \) denote the cotangent Lie algebra spanned by \( e_1, e_2, e_3, e_4, e_5, e_6 \). The non zero Lie brackets are listed below:

\[
\begin{align*}
T^* \mathfrak{h}_1 : & \quad [e_1, e_2] = e_3, \quad [e_1, e_6] = -e_5, \quad [e_2, e_6] = e_4 \\
T^* \mathfrak{r}_3 : & \quad [e_1, e_2] = e_2, \quad [e_1, e_3] = e_2 + e_3, \quad [e_1, e_5] = -e_5 - e_6, \quad [e_1, e_6] = -e_6, \quad [e_2, e_5] = e_4, \quad [e_3, e_6] = e_4, \quad [e_3, e_5] = e_4, \quad [e_3, e_6] = e_4 \\
T^* \mathfrak{r}_{3, \lambda} : & \quad [\lambda \leq 1] \\
& \quad [e_1, e_2] = e_2, \quad [e_1, e_3] = \lambda e_3, \quad [e_1, e_5] = -e_5 - e_6, \quad [e_1, e_6] = -e_6, \quad [e_2, e_5] = e_4, \quad [e_3, e_6] = \lambda e_4 \\
T^* \mathfrak{r}_{3, \eta} : & \quad [\eta \geq 0] \\
& \quad [e_1, e_2] = \eta e_2 - e_3, \quad [e_1, e_3] = e_2 + \eta e_3, \quad [e_1, e_5] = -\eta e_5 - e_6, \quad [e_1, e_6] = -e_4, \quad [e_2, e_5] = \eta e_4, \quad [e_2, e_6] = -e_4, \quad [e_3, e_5] = e_4, \quad [e_3, e_6] = \eta e_4 \\
\end{align*}
\]

**Theorem 4.9.** Let \( \mathfrak{h} \) denote a three dimensional solvable real Lie algebra, if \( T^* \mathfrak{h} \) admits a complex structure then \( \mathfrak{h} \) is isomorphic to one of the following Lie algebras: \( \mathfrak{h}_1, \mathbb{R} \times \text{aff}(\mathbb{R}), \mathfrak{r}_{3, 1}, \mathfrak{r}_{3, -1}, \mathfrak{r}_{3, \eta} \) for any \( \eta \geq 0 \).

The following propositions gives the proof of the theorem above.
Lemma 4.10. Let \( \mathfrak{h} \) denote a solvable real Lie algebra of dimension three. If \( \mathfrak{m} \) is a complex subalgebra of \( T^* \mathfrak{h} \) such that there exist \( V, W \in \mathfrak{m} \) satisfying \( [V, W] = W \) then \( W = 0 \).

Proof. Assume \( \mathfrak{m} \) is a complex subalgebra of \( T^* \mathfrak{h} \) with \( V, W \in \mathfrak{m} \) satisfying \( [V, W] = W \). From the Lie bracket relations in (4.8) one has \( W = c_4 e_4 \) which implies \( W = 0 \). □

Corollary 4.11. Let \( \mathfrak{h} \) denote a solvable real Lie algebra of dimension three. If \( (T^* \mathfrak{h})^\mathbb{C} \) splits as a direct sum \( (T^* \mathfrak{h})^\mathbb{C} = \mathfrak{m} \oplus \mathfrak{m}, \) where \( \mathfrak{m} \) is a complex subalgebra and \( \sigma \) is the conjugation with respect to \( T^* \mathfrak{h} \), then \( \mathfrak{m} \simeq \mathbb{C} \ltimes \mathbb{C}^2 \).

For the existence problem of complex structures it remains the study of complex subalgebras \( \mathfrak{m} \) such that \( \mathfrak{m} \simeq \mathbb{C} \ltimes \mathbb{C}^2 \), we do below.

Proposition 4.12. Every complex structure on the Lie algebra \( T^* \mathfrak{h}_1 \) is three step nilpotent.

Proof. In (2.1) we proved that \( T^* \mathfrak{h}_1 \) cannot admit an abelian complex structure. For the sake of completeness we shall see first that \( T^* \mathfrak{h}_1 \) has a complex structure. Since \( T^* \mathfrak{h}_1 \) is nilpotent, if it admits a complex structure, then the corresponding complex subalgebra \( \mathfrak{m} \) must be nilpotent, hence \( \mathfrak{m} = \text{span}\{U, V, W\} \) with \( [V, W] = 0 \), the action of \( U \) of type (2) in (3) so that \( [U, V] = 0 \) and \( [U, W] = V \). Explicitly

- \( [V, W] = 0 \) implies \( b_2 c_6 - b_6 c_2 = 0 \),
- \( [U, V] = 0 \) and \( [U, W] = V \) imply the following systems of equations:

\[
\begin{align*}
b_2 &= 0, \quad -c_6 = b_5, \\
a_2 b_6 - a_6 b_2 &= 0, \quad c_2 = b_3, \\
a_2 c_6 - a_6 c_2 &= b_4, \quad b_6 = 0.
\end{align*}
\]

Since these systems have solutions, complex subalgebras of \( (T^* \mathfrak{h}_1)^\mathbb{C} \) spanned by \( U, V, W \) of the form

\[
U = e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6, \\
V = c_2 e_3 + (a_2 c_6 - a_6 c_2) e_4 - c_6 e_5, \\
W = c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6
\]

induce complex structures if and only if the vectors \( U, V, W, \sigma U, \sigma V, \sigma W \) span a basis of \( (T^* \mathfrak{h}_1)^\mathbb{C} \).

After \( \text{Mg} \) there is only one class of complex structures, therefore anyone is equivalent to the complex structure \( J \) given by

\[
(16) \quad Je_1 = e_4, \quad Je_2 = e_6, \quad Je_5 = e_3.
\]

For this complex structure, notice that \( a_0(J) = \{0\}, \ a_1(J) = \text{span}\{e_3, e_5\}, \ a_2(J) = \text{span}\{e_1, e_3, e_4, e_5\}, \ a_3(J) = \mathfrak{g} \). Hence from lemma (2.3) we conclude that any complex structure on \( T^* \mathfrak{h}_1 \) is nilpotent.

However notice that there are complex structures which are not totally real, for instance the next one

\[
(17) \quad Je_1 = e_2 - e_4, \quad Je_2 = e_6, \quad Je_5 = e_3 + e_4.
\]

□

Remark. This Lie algebra is isomorphic to \( \mathfrak{g}_{6,3} \) in \( \text{Mg} \) and to \( \mathfrak{h}_7 \) in \( \text{CFU} \).

Proposition 4.13. The Lie algebra \( T^* \mathfrak{r}_3 \) cannot be endowed with a complex structure.
**Proof.** Assume \( m \) is a complex subalgebra of \((T^* r_3)^C\) spanned by vectors \( U, V, W \) as in (8) and such that \([V, W] = 0\). Assume \( U \) acts on \( a = span\{V, W\}\).

Case 1). We assume the action of \( U \) on is of type (1) in (9), then if \([U, V] = \nu V \) and \([U, W] = \mu W \) we have that \( U = e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6, \)
\( V = b_2 e_2 + b_4 e_4 + b_6 e_6, \)
\( W = c_2 e_2 + c_4 e_4 + c_6 e_6. \) It is not difficult to verify that \( U, V, W; \sigma U, \sigma V, \sigma W \) cannot be a basis of \((T^* r_3)^C\).

Case 2) Suppose that the action of \( U \) on \( a = span\{V, W\} \) is of type (2). From the constraint \([U, V] = \nu V \) and \([U, W] = V + \nu W \), we get that there is no possibility of choosing \( \{U, V, W, \sigma U, \sigma V, \sigma W\} \) as a linearly independent set in \((T^* r_3)^C\), hence there is no complex structure associated to such \( m \).

**Proposition 4.14.** If the Lie algebra \( T^* r_{3, \lambda} \) admits a complex structure then \( \lambda = 0, 1, -1. \)

**Proof.** Let \( m \) be a complex subalgebra of \((T^* r_{3, \lambda})^C\) spanned by the vectors \( U, V, W \) as in (8) and such that \([V, W] = 0\). The following constraint must be satisfied:
\[
b_2 c_5 - b_5 c_2 + (b_3 c_6 - b_6 c_3) \lambda = 0.
\]

The Lie brackets on \((T^* r_{3, \lambda})^C\) follow
\[
[U, V] = b_2 e_2 + \lambda b_3 e_3 + \left[ a_2 b_5 - a_5 b_2 + \lambda(a_3 b_6 - a_6 b_3) \right] e_4 - b_5 e_5 - \lambda b_6 e_6,
\]
\[
[U, W] = c_2 e_2 + \lambda c_3 e_3 + \left[ a_2 c_5 - a_5 c_2 + \lambda(a_3 c_6 - a_6 c_3) \right] e_4 - c_5 e_5 - \lambda c_6 e_6,
\]
\[
[V, W] = [b_2 c_5 - b_5 c_2 + \lambda(b_3 c_6 - b_6 c_3)] e_4
\]

Case 1) Assume the action of \( U \) on \( a = span\{V, W\} \) is of type (1) in (9). The conditions \([U, V] = \nu V \) and \([U, W] = \mu W \) shows that a subalgebra \( m \) exists if \( \lambda \in \{0, 1, -1\} \). Moreover, such an \( m \) is spanned by \( U, V, W \) as given in the following table:

| \( \lambda \) | \( U \) | \( V \) | \( W \) |
| --- | --- | --- | --- |
| 0 | \( e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 \) | \( b_2 e_2 + b_4 e_4 + b_6 e_6 \) | \( c_2 e_2 + c_4 e_4 + c_6 e_6 \) |
| 1 | \( e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 \) | \( -\frac{b_2 c_5}{c_5} e_2 - \frac{b_5 c_2}{c_2} e_4 \) | \( b_6 e_6 \) |
| -1 | \( e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 \) | \( b_2 e_2 - \frac{b_2 c_5}{c_5} e_3 - \frac{b_5 c_2}{c_2} e_5 \) | \( a_5 b_6 - a_6 b_5 \) |

In all cases, \( U, V, W, \sigma U, \sigma V, \sigma W \) turn out to be a basis of \((T^* r_{3, \lambda})^C\). We also observe that none of these complex structure is abelian.

For instance, the linear map on \( T^* r_{3,0} \) given by
\[
Je_1 = e_5 \quad Je_2 = -e_4 \quad Je_3 = e_6
\]
and such that $J^2 = -I$ defines a totally real complex structure on the cotangent Lie algebra $T^\ast \mathfrak{r}_{3,0}$, while the $J$ taken as

$$Je_1 = e_2 \quad Je_4 = e_5 \quad Je_3 = e_6$$

(19)

gives rise to a complex structure which is not totally real.

For $\lambda = -1$ the linear homomorphism such that $J^2 = -I$ given by

$$Je_1 = e_4 \quad Je_2 = e_6 \quad Je_3 = -e_5$$

(20)

gives a totally real complex structure on $T^\ast \mathfrak{r}_{3,-1}$. While the $J$ satisfying $J^2 = -I$ and

$$Je_3 = -(e_1 + e_6) \quad Je_5 = e_3 - e_4 \quad Je_6 = -(e_2 + e_4)$$

(21)

induces a non totally real complex structure on $T^\ast \mathfrak{r}_{3,-1}$.

Now for $\lambda = 1$ no complex structure on $T^\ast \mathfrak{r}_{3,1}$ is totally real. For instance

$$Je_1 = e_4 \quad Je_2 = e_3 \quad Je_5 = e_6$$

(22)

is a complex structure on $T^\ast \mathfrak{r}_{3,1}$.

□

**Proposition 4.15.** The Lie algebra $T^\ast \mathfrak{r}_{3,\eta}$ carries a complex structure for any $\eta \geq 0$.

**Proof.** The linear isomorphisms $J$ such that $J^2 = -I$ given by

$$Je_1 = \pm e_4 \quad Je_2 = e_3 \quad Je_5 = e_6$$

(23)

define complex structures on $T^\ast \mathfrak{r}_{3,\eta}$ for any $\eta \geq 0$.

Notice that on $T^\ast \mathfrak{r}_{3,0}$ one has totally real complex structures (3.8), for instance

$$Je_1 = \pm e_4 \quad Je_2 = e_6 \quad Je_3 = -e_5.$$  

(24)

□

5. **Complex structures and related geometric structures**

In these paragraphs we relate complex structures to some geometric structures. We are mainly interested on hermitian structures, symplectic and Kähler structures.

5.1. **On Hermitian complex structures.** A **metric** on a Lie algebra $\mathfrak{g}$ is a non degenerate symmetric bilinear map, $\langle \cdot , \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. It is called **ad-invariant** if

$$\langle [x,y], z \rangle + \langle y, [x,z] \rangle = 0$$

for all $x,y \in \mathfrak{g}$.

**Example 5.1.** The canonical metric on a cotangent Lie algebra $T^\ast \mathfrak{h}$ is defined by

$$\langle (x,\varphi), (x',\varphi') \rangle = \varphi'(x) + \varphi(x')$$

for all $x,x' \in \mathfrak{h}$, $\varphi,\varphi' \in \mathfrak{h}^\ast$.

It is neutral and ad-invariant.

A subspace $\mathfrak{w} \subseteq (\mathfrak{g},\langle \cdot , \cdot \rangle)$ is called **isotropic** if $\langle x,y \rangle = 0$ for all $x,y \in \mathfrak{w}$, that is, if $\mathfrak{w} \subseteq \mathfrak{w}^\perp$, where

$$\mathfrak{w}^\perp = \{ y \in \mathfrak{g} \text{ such that } \langle x,y \rangle = 0 \text{ for all } x \in \mathfrak{w} \},$$

furthermore $\mathfrak{w}$ is called **totally isotropic** whenever $\mathfrak{w} = \mathfrak{w}^\perp$.

**Example 5.2.** On $T^\ast \mathfrak{h}$ equipped with its canonical metric, both subspaces $\mathfrak{h}$ and $\mathfrak{h}^\ast$ are totally isotropic.
Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ denote a real Lie algebra equipped with a metric. An (almost) complex structure $J$ on $\mathfrak{g}$ is called Hermitian if

$$\langle Jx, Jy \rangle = \langle x, y \rangle$$

for all $x, y \in \mathfrak{g}$.

If the metric is positive definite a Hermitian complex structure is also called an orthogonal complex structure.

Notice that if $J$ is Hermitian, then $\langle x, Jx \rangle = 0$ for all $x \in \mathfrak{g}$. The non degeneracy property of $\langle \cdot, \cdot \rangle$ says that there is $y \in \mathfrak{g}$ such that $\langle x, y \rangle \neq 0$. Therefore the subspace $w = \text{span}\{x, Jx, y, Jy\} \subseteq \mathfrak{g}$ is non degenerate and $J$-invariant. Furthermore

$$\mathfrak{g} = w \oplus w^\perp$$

is an orthogonal direct sum of $J$-invariant non degenerate subspaces of $\mathfrak{g}$. A similar argument can be done in the proof of the following lemma.

**Lemma 5.3.** Let $\mathfrak{g}$ denote a real Lie algebra endowed with a metric $\langle \cdot, \cdot \rangle$ and let $J$ be an almost complex structure on $\mathfrak{g}$. Assume $v$ is a totally real and totally isotropic subspace on $\mathfrak{g}$, then

i) $\mathfrak{g}$ admits a decomposition into a direct sum of totally real and totally isotropic vector subspaces

$$\mathfrak{g} = v \oplus Jv;$$

ii) $\mathfrak{g}$ splits into an orthogonal direct sum

$$\mathfrak{g} = w_1 \oplus w_2 \oplus \cdots \oplus w_n$$

of $J$-invariant non degenerate subspaces $w_1, \ldots, w_n$, where $\dim w_i \equiv 0 \pmod{4}$.

**Corollary 5.4.** Let $\mathfrak{h}$ denote a real Lie algebra, and let $\langle \cdot, \cdot \rangle$ denote a metric on $T_\pi \mathfrak{h}$ for which $\mathfrak{h}$ is totally isotropic and assume $J$ is a totally real almost complex structure on $T_\pi \mathfrak{h}$. Then the dimension of $\mathfrak{h}$ is even.

**Definition 5.5.** A generalized complex structure on a Lie algebra $\mathfrak{h}$ is a Hermitian complex structure on $(T^* \mathfrak{h}, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ denotes the canonical metric on $T^* \mathfrak{h}$.

A Hermitian complex structure $J$ on $T^* \mathfrak{h}$ which leaves $\mathfrak{h}$ invariant is called a generalized complex structure of complex type and it corresponds to a complex structure on $\mathfrak{h}$. A Hermitian complex structure $J$ on $T^* \mathfrak{h}$ which is totally real, that is $J\mathfrak{h} = \mathfrak{h}^*$, is said a generalized complex structure of symplectic type. It corresponds to a symplectic structure on $\mathfrak{h}$.

Assume $\mathfrak{h}$ is a Lie algebra which is equipped with an ad-invariant metric $(\cdot, \cdot)$. In this case it can easily be shown that the adjoint and the coadjoint representations are equivalent. In fact if $\ell : \mathfrak{h} \to \mathfrak{h}^*$ is the linear isomorphism given by $x \to \ell(x)$ such that $\ell(x)y = (x, y)$, it is straightforward to verify that $\ell^{-1}\text{coad}(x)\ell = \text{ad}(x)$ for all $x \in \mathfrak{h}$.

Results (3.2) and (3.5) of previous sections imply that totally real complex structures $J$ on $T^* \mathfrak{h}$ correspond to non singular derivations of $\mathfrak{h}$. Explicitly, a non singular derivation $d$ on $\mathfrak{h}$ induces the map $\ell \circ d : \mathfrak{h} \to \mathfrak{h}^*$ giving rise to a complex structure on $T^* \mathfrak{h}$.

Consider now the canonical neutral metric $(\cdot, \cdot)$ on $T^* \mathfrak{h}$ defined by

$$(\langle x, \ell_y \rangle, (x', \ell_{y'})) = (x, y') + (x', y).$$

Since $\mathfrak{h}$ and $\mathfrak{h}^*$ are isotropic subspaces for $(\cdot, \cdot)$, a complex structure $J$ such that $J\mathfrak{h} = \mathfrak{h}^*$ is Hermitian if and only if

$$(y, Jx) = -\langle x, Jy \rangle$$

for $x, y \in \mathfrak{h}$.
Now, since $J$ is associated to a non singular map $j: \mathfrak{h} \rightarrow \mathfrak{h}^*$, the latter corresponding to a non singular derivation $d$ of $\mathfrak{h}$, we have the linear isomorphism $j: \mathfrak{h} \rightarrow \mathfrak{h}^*$ equals
\begin{equation}
(27) \quad j = \ell \circ d,
\end{equation}
thus both (25) and (27) imply
\begin{equation}
(x, dy) = -(dx, y) \quad \text{for all } x, y \in \mathfrak{h}
\end{equation}
this means that the Hermitian complex structures on $T^*\mathfrak{h}$, such that $J\mathfrak{h} = \mathfrak{h}^*$ correspond to non singular skew symmetric derivations of $(\mathfrak{h}, (\ , \ ))$.

The previous explanations and [Ja] derive the following.

**Corollary 5.6.** Let $\mathfrak{h}$ denote an even dimensional Lie algebra endowed with an ad-invariant metric $(\ , \ )$. The following statements are equivalent

i) $\mathfrak{h}$ admits a generalized complex structure of symplectic type;

ii) $\mathfrak{h}$ admits a symplectic structure;

iii) $\mathfrak{h}$ admits a non singular derivation which is skew symmetric for $(\ , \ )$.

Furthermore if any of these structures exists then $\mathfrak{h}$ is nilpotent.

5.2. **Complex structures and symplectic structures.** A symplectic structure on a even dimensional Lie algebra $\mathfrak{g}$ is a 2-form $\omega \in \Lambda^2(\mathfrak{g}^*)$ such that $\omega$ has maximal rank, i.e., $\Lambda^2 \dim \mathfrak{g} \omega \neq 0$ and it is closed:
\begin{equation}
(28) \quad \omega([x, y], z) + \omega([y, z], x) + \omega(z, [x, y]) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.
\end{equation}

Let $(T^*\mathfrak{h} = \mathfrak{h} \ltimes V, \omega)$ denote a semidirect product equipped with a symplectic structure. Following [CLP] we say that $T^*\mathfrak{h}$ is lagrangian, if both $\mathfrak{h}$ and $V$ are lagrangian subspaces relative to $\omega$. We also say that $\omega$ is lagrangian symplectic.

Let $T^*\mathfrak{h}$ denote a generalized tangent Lie algebra, then its dual Lie algebra is the semidirect product $T^{*\prime}\mathfrak{h} := \mathfrak{h} \ltimes_{\pi^*} V^*$, where $\pi^*$ is the dual representation
\begin{equation}
(\pi^*(x)a)(u) := -a(\pi(x)(u)) \quad x \in \mathfrak{h}, a \in V^*, u \in V.
\end{equation}

Note that the cotangent Lie algebra $T^*\mathfrak{h}$ is the dual of the tangent Lie algebra $T\mathfrak{h}$.

Suppose $T^*\mathfrak{h} = \mathfrak{h} \ltimes_{\pi} V$ is a Lie algebra equipped with a totally real complex structure $J$ (that is $J\mathfrak{h} = V$). This enables us to define on $T^*\mathfrak{h} := \mathfrak{h} \ltimes_{\pi^*} V^*$ a two-form $\omega_J$ by
\begin{equation}
\omega_J(x + u, y + v) := v(Jx) - u(Jy), \quad \text{where } x, y \text{ are in } \mathfrak{h} \text{ and } u, v \text{ are in } (J\mathfrak{h} = V)^*.
\end{equation}

Then $\omega_J$ is non-degenerate and symplectic since $J$ is integrable (see [BD] or [CLP] for instance). Furthermore the converse is also true, lagrangian symplectic structures on $T^*\mathfrak{h}$ give rise to totally real complex structures on $T^*\mathfrak{h}$. Therefore

**Theorem 5.7.** Totally real complex structures on $T^*\mathfrak{h} = \mathfrak{h} \ltimes_{\pi} V$ are in correspondence to lagrangian symplectic structures on $T^*\mathfrak{h}$.

**Corollary 5.8.** i) Let $T^*\mathfrak{h}$ denote a cotangent Lie algebra. If it admits a lagrangian symplectic structure then $\mathfrak{h}$ is nilpotent.

ii) The tangent Lie algebra $T\mathfrak{h}$ admits a lagrangian symplectic structure for any $\mathfrak{h}$ isomorphic to $\mathfrak{h}_1, \mathfrak{t}_{3, -1}, \mathfrak{t}_{3, 0}$ or $\mathbb{R} \times \text{aff} (\mathbb{R})$.

If $\mathfrak{g}$ is a Lie algebra which carries a symplectic structure and a complex structure, one searches for a compatible pair $(\omega, J)$, called a Kähler structure,
\begin{equation}
(29) \quad \omega(Jx, Jy) = \omega(x, y) \quad \text{for all } x, y \in \mathfrak{g}.
\end{equation}
Let $\psi \in Aut(\mathfrak{g})$ denote a automorphism of $\mathfrak{g}$, since $\wedge^n \psi^* \omega = \psi^* \wedge^n \omega$ and $\psi^* d\omega = d(\psi^* \omega)$, one has that the existence of a compatible symplectic for a fixed complex structure $J$ is equivalent to the existence of a compatible symplectic structure for every complex structure in the orbit of $J$ under the action of $Aut(\mathfrak{g})$.

In fact if $J$ is compatible with $\omega$ and $J' = \psi^{-1} J \psi$ then $\omega' = \psi^* \omega$ is compatible with $J'$:

$$\omega'(J'x, J'y) = \omega'(\psi^{-1} J \psi x, \psi^{-1} J \psi y) = \omega(J \psi x, J \psi y) = \omega(\psi x, \psi y) = \omega'(x, y).$$

**Lemma 5.9.** Let $\omega$ denote a two form on $\mathfrak{g}$ which is compatible with the complex structure $J$. Let $\psi \in Aut(\mathfrak{g})$ be a automorphism such that $J' = \psi^{-1} J \psi$, then $\psi^* \omega$ is compatible with $J'$.

A Kähler Lie algebra is a triple $(\mathfrak{g}, J, \omega)$ consisting of a Lie algebra equipped with a Kähler structure. The Kähler pair $(J, \omega)$ origines a Hermitian structure on $\mathfrak{g}$ by defining a metric $g$ as

$$g(x, y) = \omega(Jx, y) \quad \text{for all } x, y \in \mathfrak{g}. \quad (30)$$

This kind of Hermitian structures satisfies the parallel condition

$$\nabla J = 0$$

where $\nabla$ denotes the Levi Civita connection for $g$. The pair $(J, g)$ is called a pseudo-Kähler metric on $\mathfrak{g}$.

**Remark.** A Lie algebra $\mathfrak{h}$ equipped with an ad-invariant metric $(\cdot, \cdot)$ cannot carry a complex structure $J$ which is Hermitian and parallel with respect to the Levi Civita connection of $(\cdot, \cdot)$ (see [ABO]). However $\mathfrak{h}$ can admit a pseudo Kähler metric if one relaxes the assumption on the metric to be ad-invariant (see (5.12)).

It is our aim to investigate the existence of pseudo Kähler metrics on the Lie algebras $T\mathfrak{h}$ and $T^* \mathfrak{h}$ where $\mathfrak{h}$ denotes a three dimensional real Lie algebra.

We denote by $e^i \cdots$ the exterior product $e^i \wedge e^j \wedge \ldots$, being $e^1, \ldots, e^6$ the dual basis of $e_1, \ldots, e_6$.

**Lemma 5.10.** The following Lie algebras do not carry a symplectic structure:

i) $T^* \mathfrak{r}_{3,\lambda}$ for any $\lambda$.

ii) $T^* \mathfrak{r}_{3,\eta}$ for any $\eta \geq 0$.

**Proof.** The proof follows along the following lines. Let $\alpha_{ij} \in \mathbb{R}$ be arbitrary constants and define the generic 2-form on $T^* \mathfrak{h}$

$$\theta = \sum_{i < j} \alpha_{ij} e^i \wedge e^j \quad i = 1, \ldots, 5. \quad (31)$$

If one requires $\theta$ to be closed, the condition $d\theta = 0$ generates a system depending on the parameters $\alpha_{ij}$.

We exemplify here one case. The Maurer-Cartan equations on $T^* \mathfrak{r}_{3,\lambda}$ are given by

$$
de^1 = 0 \quad de^2 = e^{12} \quad de^3 = \lambda e^{13} \quad de^4 = e^{25} + \lambda e^{36} \quad de^5 = -e^{15} \quad de^6 = -\lambda e^{16}$$

By the expansion of this expression making use of $de^i = e^i \wedge e^j - e^j \wedge de^i$ one gets conditions on the parameters $\alpha_{ij}$. For instance in the case of the Lie algebras considered in i), one obtains that $\alpha_{ij} = 0$ for all $j = 1, \ldots, 6$, therefore a closed 2-form $\theta$ belongs to $\Lambda^2 \mathfrak{r}^*$ being $\mathfrak{r} = \text{span}\{e_1, \ldots, e_5\}$, in this implies that $\theta$ cannot be symplectic.

A similar reasoning applies on $T^* \mathfrak{r}_{3,\eta}$. \qed
On $T\mathfrak{h}_1$ if a two form $\theta$ as in (31) is closed, the constrains may satisfy
\begin{equation}
0 = \alpha_{36} = \alpha_{46} = \alpha_{56} \quad 0 = \alpha_{34} + \alpha_{16} = \alpha_{35} + \alpha_{26}.
\end{equation}
The compatibility condition with the abelian complex structure $J$ given by $J e_1 = e_2 \quad J e_3 = -e_6 \quad J e_4 = e_5 \quad (11)$ implies
\begin{equation}
\alpha_{34} = \alpha_{56} \quad \alpha_{35} = -\alpha_{46}.
\end{equation}
Now (32) and (33) amounts to $\alpha_{i6} = 0$ for $i=1,2,3,4,5$, therefore a closed two form cannot be symplectic. According to results in [Mg] any abelian complex structure is equivalent to the previous one. Following a similar argument but now searching for the compatibility condition between $\theta$ and the totally real complex structure $J_s \quad (12)$, with $s=0,1$, one obtains that any such a two form cannot be symplectic.

Consider now the complex structure $J$ given by
\begin{equation}
J e_1 = 2 e_4 \quad J e_2 = -e_5 \quad J e_3 = e_6
\end{equation}
this is totally real and it is compatible with the following closed two forms
\begin{equation}
\theta = a(e^{45} - 2e^{12}) + be^{14} + c(e^{24} - 2e^{15}) + de^{25} + e(e^{26} + e^{35}) + f e^{36}.
\end{equation}
For instance the following two forms give rise to Kähler pairs
\begin{equation}
\omega = e^{45} - 2e^{12} + \mu e^{36} \quad \mu \neq 0,
\end{equation}
\begin{equation}
\omega = e^{14} + \nu(e^{26} - e^{35}) \quad \nu \neq 0.
\end{equation}

This together with (5.9) prove the next result.

**Lemma 5.11.** The Lie algebra $T\mathfrak{h}_1$ carries several Kähler structures. However no abelian structure admits a compatible Kähler pair.

Next we see that the cotangent Lie algebra $T^*\mathfrak{h}_1$ possesses several Kähler pairs $(J, \theta)$. Let $\theta$ denote a closed two form on $T^*\mathfrak{h}_1$, thus
\begin{equation}
\theta = \sum_{i<j} \alpha_{ij} e^{ij} \quad \alpha_{34} = 0 = \alpha_{35} = \alpha_{45}, \quad \alpha_{36} = \alpha_{14} + \alpha_{25}.
\end{equation}
Consider the complex structure on $T^*\mathfrak{h}_1$ given by $J e_1 = e_4 \quad J e_2 = e_6 \quad J e_3 = -e_5 \quad (16)$. The compatibility condition between $J$ and the closed two form $\theta$ implies the following conditions on the constrains $\alpha_{ij}$:
\begin{align*}
\alpha_{12} &= \alpha_{46} \quad \alpha_{13} = -\alpha_{45} \quad \alpha_{15} = -\alpha_{34} \\
\alpha_{16} &= \alpha_{24} \quad \alpha_{23} = \alpha_{56} \quad \alpha_{25} = -\alpha_{36}
\end{align*}
therefore any two form on $\mathfrak{g}$ which is compatible with $J$ has the form
\begin{equation}
\omega = a(e^{12} + e^{46}) + b(2e^{14} - e^{25} + e^{36}) + c(e^{16} + e^{24}) + d(e^{23} + e^{56}) + e e^{26} + f e^{35}.
\end{equation}

**Lemma 5.12.** The free 2-step nilpotent Lie algebra in three generators $T^*\mathfrak{h}_1$ admits several Kähler structures.

**Lemma 5.13.** The tangent Lie algebra $T(\mathbb{R} \times \mathfrak{a}(\mathbb{R}))$ carries several Kähler structures.

**Proof.** Let $J$ denote the complex structure on $T\mathfrak{r}_{3,0}$ given by $J e_1 = e_2 \quad J e_3 = -e_6 \quad J e_4 = e_5 \quad (15)$. Canonical computations show that this complex structure $J$ is compatible with the symplectic forms
\begin{equation}
\omega_{\alpha, \beta, \gamma} = \alpha e^{12} + \beta(e^{15} - e^{24}) + \gamma e^{36} \quad \beta \gamma \neq 0
\end{equation}
therefore the pairs $(J, \omega_{\alpha, \beta, \gamma})$ amount to Kähler pairs on $T\mathfrak{r}_{3,0}$. \qed
In view of explanations before, the proof of the theorem below is straightforward.

**Theorem 5.14.** Let \( \mathfrak{h} \) denote a real Lie algebra of dimension three.

i) If \( \mathfrak{h} \) is solvable and \( T^* \mathfrak{h} \) admits a complex structure then it carries a Kähler structure.

ii) If \( T^* \mathfrak{h} \) carries a Kähler structure then \( \mathfrak{h} \) is nilpotent.

5.3. **On the geometry of some pseudo Kähler homogeneous manifolds.** The goal is the study of some geometric features on the homogeneous manifolds arising in the previous paragraphs in Lemmas (5.11) (5.12) and (5.13).

In particular we shall see that in the nilpotent case there are flat and non flat metrics. It was already proved in [FPS] that pseudo-Kähler metrics on nilmanifolds are Ricci flat.

Let \( G_1 \) denotes the simply connected Lie group whose Lie algebra is \( T \mathfrak{h}_1 \). Its underlying differentiable manifold is \( \mathbb{R}^6 \) together with the multiplication

\[
(x_1, x_2, x_3, x_4, x_5, x_6) \cdot (y_1, y_2, y_3, y_4, y_5, y_6) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1), x_4 + y_4 + \frac{1}{2}(x_2y_6 - x_6y_2), x_5 + y_5 + \frac{1}{2}(x_6y_1 - x_1y_6), x_6 + y_6).
\]

The left invariant vector fields at \( Y = (y_1, y_2, y_3, y_4, y_5, y_6) \in G_1 \) are

\[
e_1(Y) = \frac{\partial}{\partial x_1} - \frac{1}{2}y_2 \frac{\partial}{\partial x_3} + \frac{1}{2}y_5 \frac{\partial}{\partial x_6}, \quad e_2(Y) = \frac{\partial}{\partial x_2} + \frac{1}{2}y_1 \frac{\partial}{\partial x_3} + \frac{1}{2}y_4 \frac{\partial}{\partial x_6}, \quad e_3(Y) = \frac{\partial}{\partial x_3},
\]

\[
e_4(Y) = \frac{\partial}{\partial x_4} - \frac{1}{2}y_2 \frac{\partial}{\partial x_6}, \quad e_5(Y) = \frac{\partial}{\partial x_5} + \frac{1}{2}y_1 \frac{\partial}{\partial x_6}, \quad e_6(Y) = \frac{\partial}{\partial x_6}.
\]

They satisfy the Lie bracket relations of \( T \mathfrak{h}_1 \). Let \( e_i \) denote the dual basis of \( e_i \) for \( i = 1,2,3,4,5,6 \), and let \( \cdot \) denote the symmetric product. The pseudo-Kähler metric for (35) is

\[
g_{\mu} = 2e^1 \cdot e^5 + e^2 \cdot e^4 + \mu(e^3 \cdot e^3 + e^6 \cdot e^6) \quad \mu \neq 0
\]

while for (36) is

\[
g_{\nu} = 2e^1 \cdot e^5 + \frac{1}{2}e^4 \cdot e^4 + \nu (e^2 \cdot e^3 + e^5 \cdot e^6) \quad \nu \neq 0.
\]

Let \( \nabla^\mu \) and \( \nabla^\nu \) denote the corresponding Levi Civita connections for \( g_{\mu} \) and \( g_{\nu} \) respectively. From the Koszul formula, for \( X = x_i e_i \) one gets

\[
\nabla^\mu_X = \frac{1}{4} \begin{pmatrix}
-\mu x_6 & 0 & 0 & 0 & 0 & -\mu x_1 \\
0 & 2\mu x_6 & 0 & 0 & 0 & 2\mu x_2 \\
-2\mu x_2 & 2x_1 & 0 & 0 & 0 & 0 \\
-2\mu x_3 & 0 & -2\mu x_1 & -2\mu x_6 & 0 & -2\mu x_4 \\
0 & \mu x_3 & \mu x_2 & 0 & \mu x_6 & \mu x_5 \\
-2x_5 & 2x_4 & 0 & -2x_2 & 2x_1 & 0
\end{pmatrix}
\]

\[
\nabla^\nu_X = \frac{1}{2} \begin{pmatrix}
0 & x_2 & 0 & 0 & \nu x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-2x_2 & 0 & 0 & \nu x_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-2x_5 & x_4 & 0 & -x_2 & 0 & 0
\end{pmatrix}
\]

Clearly the Lie subgroup \( H_1 \) of \( G_1 \) with Lie algebra \( \mathfrak{h}_1 \subset T \mathfrak{h}_1 \) is totally geodesic for \( g_{\nu} \) for every \( \nu \) but it is not totally geodesic for \( g_{\mu} \) for any \( \mu \).
Proposition 5.15. The Lie algebra $\mathfrak{g}$ is simply connected Lie group where $\nabla$ is the Levi Civita connection $\nabla^\mu$ or $\nabla^\nu$. Notice that $\nabla_{[X,Y]}^\nu = 0$ for all $X, Y$.

By computing them one can verify:
- The pseudo Kähler metrics $g_\mu$ are non flat.
- A pseudo Kähler metric $g_\nu$ is flat if $\nu = 1$.

Proposition 5.16. The Lie algebra $T^* \mathfrak{h}_1$ carry flat and non flat pseudo Kähler metrics.

The Lie algebra $T^* \mathfrak{h}_1$ is the free 2-step nilpotent Lie algebra in three generators. Its simply connected Lie group $G_2$, lies on $\mathbb{R}^6$ with the multiplication given by
\[
(x_1, x_2, x_3, x_4, x_5, x_6) \cdot (y_1, y_2, y_3, y_4, y_5, y_6) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1), x_4 + y_4 + \frac{1}{2}(x_2y_6 - x_6y_2), x_5 + y_5 + \frac{1}{2}(x_6y_1 - x_1y_6), x_6 + y_6).
\]

The left invariant vector fields at $Y = (y_1, y_2, y_3, y_4, y_5, y_6)$ are
\[
e_1(Y) = \frac{\partial}{\partial x_1} - \frac{1}{2}y_2 \frac{\partial}{\partial x_3} + \frac{1}{2}y_6 \frac{\partial}{\partial x_5},
e_2(Y) = \frac{\partial}{\partial x_2} + \frac{1}{2}y_1 \frac{\partial}{\partial x_3} - \frac{1}{2}y_6 \frac{\partial}{\partial x_4},
e_3(Y) = \frac{\partial}{\partial x_3},
e_4(Y) = \frac{\partial}{\partial x_4},
e_5(Y) = \frac{\partial}{\partial x_5},
e_6(Y) = \frac{1}{2}y_2 \frac{\partial}{\partial x_4} - \frac{1}{2}y_1 \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6},
\]
and let $e^i$ denote the dual left invariant 1-forms for $i=1,2,3,4,5,6$. Consider the metric on $G$ given by
\[
g = 2e^1 \cdot e^1 + e^2 \cdot e^2 + 2e^3 \cdot e^3 + e^4 \cdot e^4 - e^5 \cdot e^5
\]
where $\cdot$ denotes the symmetric product. In particular
\[
T^* \mathfrak{h}_1 = \mathfrak{h}_1 \oplus J\mathfrak{h}_1
\]
denotes a orthogonal direct sum as vector spaces of totally real subalgebras.

The corresponding Levi Civita connection is given by
\[
\nabla_X = \begin{pmatrix}
0 & \frac{x_2}{2} & 0 & 0 & 0 & -\frac{x_6}{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{x_2}{2} & 0 & 0 & 0 & 0 & -x_4 \\
0 & 0 & 0 & 0 & 0 & x_6 \\
x_6 & -x_4 & 0 & x_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
for $X = \sum x_i e_i$.

One can verify that the Lie subgroup $H_1$ that corresponds to the Lie subalgebra $\mathfrak{h}_1$, spanned by $e_1, e_2, e_3$, is totally geodesic.

The corresponding curvature tensor $R(X, Y)$ is given by
\[
R(X, Y)Z = (x_2y_6 - x_6y_2) (\frac{3}{2} z_6 e_3 + \frac{1}{2} z_2 e_5).
\]

Proposition 5.16. The Lie algebra $T^* \mathfrak{h}_1$ admits non flat but Ricci flat pseudo Kähler metrics.

The simply connected Lie group $G_3$ with Lie algebra $T_{r_3,0}$ is, as a manifold, diffeomorphic to $\mathbb{R}^6$. Let $(x_1, x_2, \ldots, x_6)$ denote an arbitrary element in $\mathbb{R}^6$, then the rule
multiplication is given by
\[(x_1, x_2, x_3, x_4, x_5, x_6) \cdot (y_1, y_2, y_3, y_4, y_5, y_6) = (x_1 + y_1, x_2 + e^{x_6}y_2, x_3 + e^{x_6}y_3 + \frac{e^{x_6}}{2}(x_1y_2 - x_2y_1), x_4 + y_4, x_5 + y_5, x_6 + y_6).\]

The left invariant vector fields at \(Y = (y_1, y_2, y_3, y_4, y_5, y_6) \in G_3\) are
\[e_1(Y) = \frac{\partial}{\partial x_1} + e^{y_6}y_2 \frac{\partial}{\partial x_3}, \quad e_2(Y) = e^{y_6}(\frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial x_3}),\]
\[e_3(Y) = e^{y_6} \frac{\partial}{\partial x_3}, \quad e_4(Y) = \frac{\partial}{\partial x_4}, \quad e_5(Y) = \frac{\partial}{\partial x_5}, \quad e_6(Y) = \frac{\partial}{\partial x_6},\]

and let \(e^i\) denote the dual 1-forms for \(i=1,2,3,4,5,6\).

Consider the metric \(\langle \cdot, \cdot \rangle\) on \(G\) for which the vector fields above satisfy the non zero relations
\[g = \alpha(e_1 \cdot e_1 + e_2 \cdot e_2) + \beta(e_1 \cdot e_4 + e_4 \cdot e_5) + \gamma(e_5 \cdot e_3 + e_6 \cdot e_6) \quad \beta\gamma \neq 0.\]

This metric is clearly non definite.

The complex structure on \(G\) is defined as the linear map \(J : T_Y G \to T_Y G\) such that \(J^2 = -I\) and \(Je_1 = e_2, Je_3 = e_6, Je_4 = e_5\).

This gives a complex structure on \(\mathbb{R}^6\) which is invariant under the action of the Lie group \(G\), the action is induced from the multiplication on \(G\). Moreover the complex structure is Hermitian for the metric above and it is parallel for the corresponding Levi Civita connection \(\nabla\), which on the basis of left invariant vector fields is given by
\[\nabla_X = \begin{pmatrix} 0 & x_2 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_5 & 0 & 0 & x_2 & 0 \\ -x_5 & 0 & 0 & -x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } X = \sum_{i=1}^{6} x_i e_i.\]

Let \(\mathfrak{h}\) denote the involutive distribution spanned by \(e_2, e_6, e_5\), then it admits a complementary orthogonal distribution \(J\mathfrak{h}\), therefore \(T_Y G = \mathfrak{h} \oplus J\mathfrak{h}\) as orthogonal direct sum. At the Lie algebra level, one has the following short exact sequence
\[0 \to \mathfrak{h} \to \mathfrak{g} \to J\mathfrak{h} \to 0.\]

Notice that \(\mathfrak{h}\) is a abelian ideal while \(J\mathfrak{h}\) is a abelian subalgebra. Moreover, the complex structure \(J\) is totally real with respect to this decomposition and the representation \(\pi\) deriving from the adjoint action satisfies the conditions of Corollary 3.3.

Let \(H\) denote the Lie subgroup corresponding to the distribution \(\mathfrak{h}\) and \(JH\) the Lie subgroup corresponding to \(J\mathfrak{h}\), which is totally geodesic. In fact, making use of the formula for \(\nabla\), one verifies
\[\nabla_X Y \subseteq J\mathfrak{h}, \quad \nabla_{JX} JY \subseteq J\mathfrak{h}, \quad \text{for } X, Y \in \mathfrak{h}\]
and since \((J,g)\) is Kähler, \(\nabla_X JY = J\nabla_X Y\) and \(\nabla_{JX} Y = -J\nabla_{JX} Y\) for \(X, Y \in \mathfrak{h}\).

The curvature tensor \(R\) is given by
\[R(x, y) = -\nabla_{[x,y]}\]
which implies that \(JH\) is flat.
The Ricci tensor \( r \) follows \( r(X, Y) = 2(x_1y_1 + x_2y_2) \) for \( X = \sum x_ie_i, \ Y = \sum y_ie_i; \) therefore \( G_3 \) is neither flat nor Ricci flat.

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R. Campoamor-Stursberg: IMI y FCM, Universidad Complutense de Madrid

E-mail address: rutwig@pdi.ucm.es

G. Ovando: CONICET y ECEN-FCEIA, Universidad Nacional de Rosario

E-mail address: ovando@mate.uncor.edu