Applications of a Feynman Path Integral for a Position Dependent Planck’s Constant

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Abstract: There is controversial evidence that Planck’s constant shows unexpected variations with altitude above the earth due to Kentosh and Mohageg, and yearly systematic changes with the orbit of the earth about the sun due to Hutchin. Many others have postulated that the fundamental constants of nature are not constant either locally, or universally. This work is a mathematical study, examining the impact of a position dependent Planck’s constant on the Feynman path integral, using results from prior papers by the author. A derivation is shown for how the integrand in the path integral exponent becomes $L_c/\hbar(r)$, where $L_c$ is the classical action. The path that makes stationary the integral in the exponent is termed the "dominant" path, and deviates from the classical path systematically due to the position dependence of $\hbar$. The changes resulting in the Euler-Lagrange equation, Newton’s first and second laws, Newtonian gravity, and the Friedmann equation with a Cosmological Constant for the dominant path are shown and discussed. An attempt is made to explain the anomalous energy changes in the Earth flybys of six satellites in published data from the NASA Jet Propulsion Laboratory, which is considered to be an unsolved problem in physics, and a reasonable agreement is found with the model developed here for the Earth. The reported results of an experiment with tunnel diodes looking for Planck’s constant variations is discussed in the context of the model, and an attempt is made to calibrate the model for the sun based on the latter. Parameters based on cosmological to planetary length scales are discussed, and also those based on the inspiral of masses emitting gravitational radiation.

Key Words: Planck’s constant, Variable Planck’s constant, non-Hermitian operators, Schrödinger Equation, Path Integral, Friedmann equation, Flyby Anomaly, Gravitational Waves.

1. Introduction

The possibility of the variation of fundamental constants would impact all present physical theory, while all reported variations or interpretations of data concluding a constant has varied are extremely controversial. Examples of work in this area include Dirac’s Large Number Hypotheses [1], the Oklo mine from which could be extracted a variation of the fine structure constant [2,3], and the observations of quasars bounding the variation of the latter per year to one part in $10^{17}$ [4-6]. Recent theoretical work includes the impact of time dependent stochastic fluctuations of Planck’s constant [7], and the changes with Planck’s constant on mixed quantum states [8]. An authoritative review of the status of the variations of fundamental constants is given in [9].

Most physicists hold the position that any attempt to measure variations in dimensionful fundamental constants in isolation is physically meaningless. The succinct reasoning for
examining a variable $\hbar$, here treated in isolation, is that its variation leads to forces that may be compared to standard theory and noted measurement anomalies. Specifically, data pertaining to the Earth flybys of six satellites showing anomalous energy changes will be analyzed [34]. If an anomaly can be connected to the variation of a constant, then the constant is no longer by definition a constant, and it becomes meaningful to measure its change, replace it with new, correct constants that describe its variation, and augment physical theory based on the new phenomena. The anomalies will be treated as non-artifactual, real examples of energy non-conservation.

It is understood that the flyby anomaly has not been observed again in the very few flybys that have occurred since the publication of [34] in 2008. However, a rationale is given for why a variation in Planck’s constant may not be detectable in experiments with clocks to first order in [25], making the flyby anomaly a candidate to apply to the model of this paper to for calibration, and to infer a variation of Planck’s constant with, despite that the anomaly is not observed routinely. The flyby anomaly has not been explained in the dual sense of why it occurred in the first place, and then also why it is no longer being seen [38], leading to some to dismiss any work on the subject out of hand. An analysis has recently been conducted on the flyby of Juno past Jupiter, suggesting another possible anomaly observation [40].

Publicly available Global Positioning System (GPS) data was used to attempt to confirm the Local Position Invariance (LPI) of Planck’s constant under General Relativity [10-11]. LPI is a concept from General Relativity, where all local non-gravitational experimental results in freely falling reference frames should be independent of the location that the experiment is performed in. That foundational rule should hold when the fundamental physical constants are not dependent on the location. If the fundamental constants vary universally, but their changes are only small locally, then it is the form of the physical laws that should be the same in all locations.

The LPI violation parameter due to variations in Planck’s constant is called $\beta_{h}$. The fractional variation of Planck’s constant is proportional to the gravitational potential difference and $\beta_{h}$. The value found in [10] for variations in Planck’s constant was $|\beta_{h}|<0.007$. This parameter is not zero, and is the largest of the violation parameters extracted in the study. The study did not report on the altitude dependence of Planck’s constant above the earth. A very recent study involving the Galileo satellites found that GR could explain the frequency shift of the onboard hydrogen maser clocks to within a factor of $(4.5\pm3.1)\times10^{-5}$ [12], improved over Gravity Probe A in 1976 of $\sim1.4\times10^{-4}$, these are the $\alpha_{rs}$ redshift violation values that may be compared to $\beta_{h}$.

Consistent sinusoidal oscillations in the decay rate of a number of radioactive elements with periods of one year taken over a 20 year span has been reported [13-18]. These measurements were taken by six organizations on three continents. As both the strong and weak forces were involved in the decay processes, and might be explainable by oscillations of $\hbar$ influencing the probability of tunneling, an all electromagnetic experiment was conducted, designed specifically to be sensitive to Planck’s constant variations [19]. Consistent systematic sinusoidal oscillations of the tunneling voltage of Esaki diodes with periods of one year were monitored for 941 days. The tunnel diode oscillations were attributed to the combined effect of changes in the WKB tunneling exponent going as $\hbar^{-1}$, and changes in the width of the barrier going as $\hbar^{2}$. The
electromagnetic experiment voltage oscillations were correctly predicted to be 180 degrees out of phase with the radioactive decay oscillations. This data can be made available for independent analysis by requesting it from the author of [19].

It is reasonable to suspect that the oscillations of decay rates and tunnel diode voltage are related to the relative position of the sun to the orbiting earth, and that there are resulting oscillations in Planck’s constant due to position dependent gravitational effects, or effects with proximity to the sun. It should be mentioned that there have been studies in which it was concluded there was no gravitational dependence to the decay rate oscillations [20-21]. There is also dispute in the literature concerning the reality of the decay rate oscillations [22-24].

Either way, whether by gravitation or by some other mechanism, for the work to be presented, all that matters is that there be a position dependent ħ, and it would be of value to understand the impact on the familiar formulations of the Feynman path integral, the Euler-Lagrange equation, Newton’s laws, and Newtonian gravity. Findings from reference [25] will be used in the derivation, where frequency conservation in the Schrödinger equation as a means to retain Hermitivity was examined.

For the treatment of ħ in this paper, and also in [25], it is important to emphasize is not as a dynamical field, and this leads to energy non-conservation. In another paper by this author, variations in ħ are treated as a scalar dynamical field, coupling to fields through the derivative terms in the Lagrangian density [26], and the energy is shared between the fields. One of the solutions of [26] suggests that frequency may be a more fundamental dynamical variable than energy, leading to the idea of frequency conservation in [25], where it arises quite naturally. This paper concerns issues specific to the formalisms mentioned in a single-particle, non-field theoretic framework, however.

Variations in ħ or any fundamental constant may be explainable by treatment as dynamical fields, but, they may not be, especially where the spatial dependence is concerned because there is so little experimental data on the subject. Noone presently knows whether they actually are dynamical fields, or fixed background fields, or something else entirely, though much work has been done representing some of them as dynamical fields, with an emphasis on solving problems in cosmology: the Jordan-Brans-Dicke scalar-tensor theory with variable G developed in the late 1950’s and early 1960’s; and note that G is dimensionful; Bekenstein models with variable fine structure constant, specifically the squared charge is varied, introduced in 1982 [27-28] where the emphasis was on the electromagnetic sector; the Cosmon of Wetterich with a field dependent pre-factor to the dynamical terms functioning somewhat like Planck’s constant, falling to a constant value at high fields [29-30]; the investigations of Albrecht, Magueijo, Moffat, and Barrow on variable c used towards the explanation of the flatness, horizon, homogeneity, and cosmological constant problems [31-32, 41-42], where c^4 is made a dynamical field and is dimensioned.

Take for example,
Equation (IIa-e) shows in a single form an amalgam of possible couplings including a Jordan-Brans-Dicke-like scalar-tensor theory of alternative General Relativity with variable \( G \), an Albrecht-Magueijo-Barrow-Moffat-like field for \( c \), a field for \( \hbar \) like that of [26], which is different than the form of Bekenstein\( \phi \) for variable \( e^2 \) whose representative field squared divided the derivative terms. There is also the field theory of Modified Gravity (MOG) of Moffat, and the Tensor-Vector-Scalar (TeVeS) gravity of Bekenstein. There are many ways all the constants might be represented as fields, and many ways they might be coupled. Coupling fields together in this way is the accepted approach for the treatment of a constant, but is not the only possible approach, and here, something different will be tried.

As mentioned, the objective here is to formulate what the changes are to the most basic and familiar dynamical expressions in physics when \( \hbar \) varies spatially, conserving total frequency, not energy, following a logical course. The aforementioned anomalies will be examined [34], and relative to testing cosmological theories, the models of this paper would be experimentally testable by intentionally setting up more flyby orbits to be analyzed, or at least analyzing newly generated datasets from flybys as they become available, and by analyzing any orbital discrepancies of binary mergers emitting gravitational radiation.

2. The Altered Path Integral

The development to follow will depend on some prior results found in [25]. Equation (1) is the anticommutator-symmetrized Hermitian frequency-conserving operator controlling unitary time-evolution,

\[
\frac{\partial}{\partial x_\alpha} = \frac{1}{c_0 \mathcal{C}} \frac{\partial}{\partial t}
\]

\( c = c_0 \mathcal{C} \)

\( \hbar = \hbar_0 \mathcal{C} \)

\( G = G_0 / \xi \)

\[
S_{GR} = \int \left\{ \frac{\left( c_0 \mathcal{C}\right)^4}{16 G_0 \pi} \xi R + \left( \frac{\hbar_0 \mathcal{C}}{2} \right)^2 g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi \right. + \frac{\left( \hbar_0 \mathcal{C} \right)^2}{2} \xi g^{\mu\nu} \xi \nabla_\mu \xi \nabla_\nu \xi + \frac{\left( \hbar_0 \mathcal{C} \right)^2}{2} g^{\mu\nu} \xi \nabla_\mu \psi \nabla_\nu \psi \right. \\
+ \lambda \xi \psi \nabla_\mu \xi + L_m \{ \hbar, c, G \} \left. \right\} \sqrt{-g} d^4 x
\]

\[
\hat{E}_h = -\frac{1}{2 m} \left\{ \hbar(\bar{\mathcal{F}}), \nabla^2 \right\} + \frac{V(\bar{\mathcal{F}})}{\hbar(\bar{\mathcal{F}})} = \frac{\hat{E}_h}{2m} + V_h(\bar{\mathcal{F}}) \quad (1)
\]

and the analog of momentum with units of \([\text{kg} \cdot \text{m/s/} \hbar^{1/2}]\) is,
\[
\hat{\mathbf{E}}_n = \frac{1}{i} \sqrt{\frac{1}{2}\{\hbar(\bar{r}), \nabla^2\}}
\]  

(2)

Equations (1-2) were derived for the condition that \( \hbar \) had no explicit time dependence. The completeness operator in the position representation that will result in summation over every possible path at each time-slice is,

\[
\int dx_j(t_j) | x_j, t_j \rangle \langle x_j, t_j | = 1
\]

(3)

Using (3) by repeated insertion \( N \) times (for \( N \) time slices) between the brackets of the transition amplitude for a particle initially at \((x_i, t_i)\) to be found at \((x_f, t_f)\), one may write for the amplitude,

\[
\langle x_f, t_f | x_i, t_i \rangle = \int \prod_{j=1}^{N-1} dx_j(t_j) \prod_{k=0}^{N-1} \langle x_{k+1}, t_{k+1} | x_k, t_k \rangle
\]

(4)

The completeness operator will be needed for the equivalent of the momentum representation to be used,

\[
\int dp_n | p_n \rangle \langle p_n | = 1
\]

(5)

A general amplitude in (4) will now be examined. Using the time evolution operator followed by approximation to first order,

\[
\langle x_1, t_o + \Delta t | x_o, t_o \rangle = \langle x_1 | \exp\left(-i\mathbf{\hat{E}}_n \Delta t\right) | x_o \rangle \\
\approx \langle x_1 | \exp\left(-i\frac{\mathbf{\hat{E}}_n}{2m} \Delta t\right) \exp(-iV_n(\mathbf{\hat{E}}) \Delta t) | x_o \rangle
\]

(6)

Inserting (5) into (6) and acting with the operators,

\[
\langle x_1, t_o + \Delta t | x_o, t_o \rangle = \int dp_n \langle x_1 | \exp\left(-i\frac{\mathbf{\hat{E}}_n}{2m} \Delta t\right) | p_n \rangle \langle p_n | \exp\left(-i\frac{V_n(\mathbf{\hat{E}})}{\hbar} \Delta t\right) | x_o \rangle \\
= \int dp_n \langle x_1 | p_n \rangle \langle p_n | x_o \rangle \exp\left(-i\frac{p_n^2}{2m} \Delta t\right) \exp\left(-i\frac{V_n(x_o)}{\hbar} \Delta t\right)
\]

(7)

The two needed \( p_n \) eigenfunctions, with factors of \( \hbar^{1/2} \) dividing the exponent to produce the right units for the approximate basis become,
\[ \langle p_h | x_o \rangle \approx \frac{\exp\left(-ip_h x_o / \sqrt{\hbar(x_o)}\right)}{\sqrt{2\pi \hbar(x_o)}} \]
\[ \langle x_i \mid p_h \rangle \approx \frac{\exp\left(ip_h x_i / \sqrt{\hbar(x_i)}\right)}{\sqrt{2\pi \hbar(x_i)}} \approx \frac{\exp\left(ip_h x_i / \sqrt{\hbar(x_o)}\right)}{\sqrt{2\pi \hbar(x_o)}} \]

(8a-b)

The approximate basis above is justified in [25] where it was shown that for a mild enough gradient in \( \hbar \), the free-particle wavefunctions are approximately plane waves, and the Ehrenfest theorem relating the position expectation value time derivative to the momentum is exactly retained. The approximation will break down if the gradient becomes too large, and will become important in the analysis to follow of the sun. Substituting (8a-b) into (7) and (6) and integrating,

\[ \langle x_i, t + \Delta t \mid x_o, t_o \rangle = \int dp_h \exp\left(-i \left\{ \frac{p_h(x_o-x_i)}{\sqrt{\hbar(x_o)}} + \left( \frac{p_h^2}{2m} + \frac{V(x_o)}{\hbar(x_o)} \right) \Delta t \right\} \right) \]
\[ = -e^{-\frac{3\pi}{4}} \sqrt{\frac{m}{2\hbar(x_o)\Delta t}} \exp\left\{ \frac{i}{\hbar(x_o)} \left( \frac{1}{2} m \left( \frac{x_o-x_i}{\Delta t} \right)^2 - V(x_o) \right) \Delta t \right\} \]

(9)

The term in the curly brackets of (9) is the classical Lagrangian \( L_c \), now divided by the position dependent \( \hbar \),

\[ \langle x_i, t_1 \mid x_o, t_o \rangle = \frac{A}{\sqrt{\hbar(x_o)}} \exp\left( \frac{i}{\hbar(x_o)} \Delta t \right) \]

(10)

Now substituting (10) into (4), taking the limit \( N \to \infty \), and passing the resulting sum in the exponent to an integral, the new form of the path integral is,

\[ \langle x_f, t_f \mid x_i, t_i \rangle = \int D_h x(t) \exp\left( i \int \frac{dt}{\hbar(x(t))} L_c(x(t), \dot{x}(t)) \right) \]

(11a-b)

\[ D_h x(t) = \lim_{N \to \infty} \prod_{j=1}^{N-1} A \frac{dx(t_j)}{\sqrt{\hbar(x(t_j))}} \]

Normally, the \( \hbar \) that appears in the denominator of the exponent is a constant, but in (11) it is not, and is being integrated. The complication of the product of the root of \( \hbar(x) \) in the pre-factor of (11) appears in \( D_h x(t) \), but is not important in what follows.

Equation (11a) has been written to provide an interpretation for what a variation in Planck’s constant means it is related to variations in the rate of time passage over the classical path,
which must be made stationary in combination with the Lagrangian as part of the action. Once made stationary, there are resulting detectable forces.

3. **Euler-Lagrange Equation for Dominant Path and Total Frequency Conservation**

The usual argument is to say that the classical path is the one that makes the classical action in the exponent of the path integral stationary, all other paths cancelling by rapid oscillations in the limit that the constant $\hbar \to 0$. That will still be the case if the entire function $\hat{h}$ in (11) goes to zero.

To be more precise, the classical trajectory is recovered from the path integral when the classical action is much larger than the constant $\hbar$, due to mass or energies becoming large.

Let a different question be posed. The "dominant path" will be used to refer to one that makes the integral in the exponent of (11) stationary. The trajectories around the dominant path are systematically different from the true classical path due to the variation of $\hbar$. What may be expected?

To answer that, the condition for a stationary exponent in (11) in terms of generalized coordinates is given by a modified form of the Euler-Lagrange equation,

\[
L = L(q_i(t), q_2(t), q_3(t), \dot{q}_i(t), \dot{q}_2(t), \dot{q}_3(t))
\]

\[
\hat{h} = h(q_i(t), q_2(t), q_3(t))
\]

\[
\frac{d}{dt} \left( \frac{\partial (L_c / \hat{h})}{\partial \dot{q}_i} - \frac{\partial (L_c / h)}{\partial \dot{q}_i} \right) = 0
\]

\[
\frac{d}{dt} \frac{\partial (L_c / \hat{h})}{\partial \dot{q}_i} = \frac{d}{dt} \left( \frac{\partial L_c}{\partial q_i} - \frac{L_c}{\hbar^2} \frac{\partial h}{\partial q_i} \right) = \frac{1}{\hbar} \frac{d}{dt} \frac{\partial L_c}{\partial \dot{q}_i} - \frac{1}{\hbar^2} \frac{\partial L_c}{\partial q_i} \sum_j \frac{\partial h}{\partial q_j} q_j
\]

\[
\frac{\partial (L_c / \hat{h})}{\partial q_i} = \frac{1}{\hbar} \frac{\partial L_c}{\partial q_i} - \frac{L_c}{\hbar^2} \frac{\partial h}{\partial q_i}
\]

\[
\frac{d}{dt} \frac{\partial L_c}{\partial \dot{q}_i} - \frac{\partial L_c}{\partial q_i} = \left( \nabla \ln \hat{h} \cdot \dot{q} \right) \frac{\partial L_c}{\partial q_i} - L_c \frac{\partial \ln \hat{h}}{\partial q_i}
\]

The left side of (12f) are the usual Euler-Lagrange terms, but the right that is usually zero is no longer. The classical equation of motion is recovered when the logarithmic derivative of $\hat{h}$ vanishes. In the absence of an external potential, classical conjugate momentum $p_i$ and modified conjugate momentum $\tilde{p}_i$ are not conserved due to the position dependence of $\hat{h}$.
Equation (13) shows that the total frequency $W = H_c / \hbar$ is conserved and not the total classical energy. Taking the total time derivative of $L_c / \hbar$ and using (12c),

$$- \frac{\partial (L_c / \hbar)}{\partial t} = \frac{d}{dt} \left( \sum \dot{q}_i \frac{\partial (L_c / \hbar)}{\partial \dot{q}_i} - L_c / \hbar \right) = \frac{d}{dt} \frac{H_c}{\hbar} = \frac{dW}{dt} = 0$$

(13)

For a conserved $W$, on any trajectory in which the value of $\hbar$ is equal at the start and end, the total energy is restored, though not conserved.

4. Newton’s First and Second Law for Dominant Path

From (12f), the equation of motion for the path that makes (11) stationary is a modified Newton’s second law,

$$m \ddot{x} = -\nabla V_c + m \left( \nabla \ln \hbar \cdot \dot{x} \right) \dot{x} - \left( \nabla \ln \hbar \right) \left( \frac{1}{2} m \dot{x}^2 - V_c \right)$$

$$m \ddot{x} = F_c + \frac{\partial \ln \hbar}{\partial x} H_c$$

(14a-b)

where (14a-b) are in 3-D and 1-D, respectively. $H_c = T_c + V_c$ is the classical total energy, and can be written as (14b) only in 1-D. With no external potential, the equation (14b) becomes,

$$\ddot{x} = \frac{\partial \ln \hbar}{\partial x} \left( \frac{1}{2} \dot{x}^2 \right)$$

(15)

From Equation (15), if the particle is at rest, it stays at rest, per the first half of Newton’s first law. For the second half of Newton’s first law, it is found that a particle in motion tends to accelerate or decelerate, depending on the functional form of $\hbar$. Therefore, momentum is not conserved.

Assuming the logarithmic derivative of $\hbar$ is a constant $k$, one finds,

$$\dot{x}(t) = -\frac{2}{k} \ln \left( \frac{c_1 + kt}{c_2} \right)$$

$$\ddot{x}(t) = -\frac{2}{c_1 + kt}$$

(16a-d)

$$\dddot{x}(t) = \frac{2k}{(c_1 + kt)^2}$$
From (16c), one sees that once in motion, the particle can never be at rest unless an infinite amount of time has passed. The acceleration may increase, or decrease depending on the sign of $k$. In Figure 1, Equations (16b-c) are plotted for $k=\pm 1$, and $c_1 = c_2 = 1$ arbitrary unit.

An object initially moving in the direction of lower Planck's constant decelerates asymptotically to zero velocity, but never fully stops. If it is initially moving towards higher Planck's constant, it accelerates in that direction to infinite velocity, where after the position becomes undefined.

Clearly, the classical energy is no longer conserved, as there is a tendency for matter to receive an added push through space in the direction of increasing $\hbar$ at the gentlest disturbance from rest in that direction, for large $|k|$. 

**Figure 1a.** (Left) Velocity as a function of time. (Right) Position as a function of time.

### 5. Newtonian Gravity for Dominant Path (NGDP)

From (14a), for a mass $m$ in a gravitational potential caused by $M$,

\[
m\ddot{x} = -\frac{GMm}{|x|^2} \dot{x} + m \left(\nabla \ln \hbar \cdot \dot{x} - \left(\nabla \ln \hbar\right) \ddot{x}\right) \left(\frac{1}{2} m \dot{x}^2 - \frac{GMm}{|x|}\right)
\]

\[
m\ddot{r} = F = -\frac{GMm}{r^2} + \partial_r \ln \hbar \left(\frac{1}{2} m\dot{r}^2 - \frac{GMm}{r}\right)
\]

\[
r_o = -\frac{1}{\partial_r \ln \hbar(r_o)} > 0
\]

\[
\frac{y_o^2}{2} = -\frac{1}{\partial_r \ln \hbar} \frac{GMm}{r^2} = \frac{GMm}{r}
\]

where (17b) is for the situation with no velocity other than radial, and a radially dependent $\hbar$. It is possible now for the particle to remain at rest in the gravitational field, which is normally not possible except infinitely far away from $M$. It will be so if placed at zero velocity at a radius.
equal to (17c). For the solutions presented, Equation (26b), it can be shown that \( r_o = 0 \) and \( \infty \) are the only values admitted. There is also a tangential velocity to the gradient at which the total radial force goes to zero, \( v_o \), Equation (17d).

6.0 Model Parameterization at Different Scales

The NGDP model offers a new degree of freedom that may be applied to behavior at varying length scales. There may be interactions between the behaviors at varying length scales, where that of a longer length scale sets an overall trend that the smaller length scale behavior is superimposed on.

6.1 Galactic Scale Parameters: Galaxy Rotation Curves and Dark Matter

It is possible to find a profile of \( h \) that would lead to the flattening of a galaxy rotation curve without dark matter with NGDP, although it requires a large variation in \( h \), and this will be shown. From (14a) for a radially dependent \( h \), a general radially dependent classical potential energy \( V_c \) and a velocity \( v \) perpendicular to the gradient, the total force is radial. \( V_c \) is general, and it can contain the potential energy of multiple distributions of matter (luminous, dark, and point-like). The term \( v_o \) is the net velocity from all matter, and \( v \) is the measured velocity. Set equal to the centripetal force for a circular orbit, the velocity may be solved for, producing (18a-d). Setting the velocity to be a constant independent of radius per a perfectly flat rotation curve of infinite range, one may solve for the profile using (18a-d). To find values to insert, the rotation curve is computed for a dark matter containing galaxy, using the values of \( M = 1.3 \times 10^{11} \) solar masses for all the visible stars in the galaxy, and a dark matter halo of \( 1.5 \times 10^{12} \) solar masses, where at a distance of 60 kpc the velocity is about 150 km/s, and flattening out. No point mass for the black hole is used, as it does not impact the rotation curve in the outer points of the galaxy. Then, using the latter values for \( v \), and \( M \) with \( \phi_c = -GM/r \) but without the mass of the dark matter, it is found that in order to maintain the constant velocity, \( h \) would have to decrease by a factor of 0.755 over 10-60 kpc reaching a minimum mid-range, using (18d). The required mid-range minimum becomes a factor of 0.9 for \( M = 9 \times 10^{10} \) solar masses over 10-30 kpc. This is the simplest approximation for a star near the edge of the galaxy, and illustrates that a minimum can result.

While more realistic simulations with actual non-dark matter density profiles and real rotation curves may reduce the required change in \( h \), it is not likely to be small, and would be problematic for fusion in most of the stars in the flat velocity region. A reason was given in [25] for why experiments with clocks (and hence the frequency of light) would not, to first order, reveal a change in \( h \), and it is due to the proposed conservation of total frequency instead of total energy. The spatial variation would be revealed in the motion of gravitationally interacting bodies and light lensing in that case. However, the large variation needed, and the issue with fusion in the stars, prompts the conclusion that the galaxy rotation curves could not be explained by this mechanism directly and alone. The needed variation in \( h \) can be reduced if the impact of a varying \( h \) works in tandem with dark matter, shown in Equations (18c-d) and Figure 1b.

Consider an alternative starting point for the explanation of the nature of dark matter. The stars are seen because of a combination of gravitational heating, and fusion via tunneling, which
becomes more difficult if $\hbar$ is smaller, but also, the strengths of all the known forces dependent on $\hbar$ would be altered. If the galactic-scale trend in Planck's constant suppressed fusion and other normal quantum mechanical interactions via known forces in most of the matter in the galaxy, most of it would be dark (or at least less luminous), but would be gravitationally active. The galactic-scale variation of Planck's constant would then be limited to that necessary to prevent fusion and the other expected interactions. Then, local variations near stars could raise Planck's constant so some matter may fuse and normally interact, and that is what is luminous. The combination of the non-fusing matter and the added force from the derivative in Planck's constant could together explain the rotation curves, and simultaneously explain why most of the matter is dark, and why it can only interact gravitationally.

Figure 1b. Depression in Planck's constant in the vicinity of a galaxy. The numerical integration of the net velocity curve to produce $\phi$ is good up to a constant of integration $\phi_o$, which must be negative. The $r_1$ value where the fraction of Planck's constant falls from unity is set at 1 kpc.

For NGDP, there would not be a MOG-, TeVeS-, or MoND-like explanation of the rotation curves without dark matter, as it is still needed, working in concert with $\partial \ln \hbar$. The analysis of the flyby anomaly will show that $\hbar$ needs to increase approaching the focus of an orbit, and provides a rationale for why there could be local spikes in $\hbar$. It is also pointed out in [25] that the wavefunctions of particles increase in amplitude in regions of lower $\hbar$, so particles with mass can gather probabilistically, providing an alternative rationale for why there are massive particles accumulating around galaxies forming a halo. As an exaggerated example, consider Figure 1b, where an ideally flat rotation curve is sought. With a star matter density profile, reduced dark
matter density profile, and a black hole mass, it is possible to find a solution in which a minimum forms in $\hbar$, which per [25] would increase the amplitude of single particle wavefunctions in the vicinity.

$$v^2(r) = \frac{-\partial_r V_c + V_c \partial_r \ln \hbar}{-\frac{m}{r} + \frac{m}{2} \partial_r \ln \hbar}$$

$$\partial_r \varphi_c = \frac{v_n^2(r)}{r}$$

$$\varphi_c = \int \frac{v_n^2(r)}{r} dr + \varphi_o \leq 0$$

$$\partial_r \ln \hbar = -\frac{r}{2} \frac{v^2(r) - \partial_r \varphi_c}{1 - v^2(r) - \varphi_c}$$

(18a-d)

The inclusion of more dark matter necessitates a smaller change in $\hbar$. Both the probability of fusion, and the residual strong force, contain factors of $\exp(-C/\hbar)$. The following must be elucidated in a mutually consistent way that constrains the possible solutions:

1. How much smaller $\hbar$ needs to become to shunt fusion, and other forces.
2. Whether the concentration of the wavefunctions of massive particles in the depression in $\hbar$ can be made consistent with the required amount of dark matter.
3. Finding unique solutions of $\hbar$. With conditions (1) and (2) not participating, for any fixed set of matter distributions, the solutions for $\hbar$ are non-unique, as the radius $r_1$ set for the fraction of $\hbar$ to vary from unity, and the constant $\varphi_0$ both alter the $\hbar$ profiles, but still produce the target curve.

6.2 Cosmological Scale Parameters: Newtonian Derivation of Friedmann Equation for Dominant Path and the Cosmological Constant

A connection of the path integral derivation above has not been made with general relativity, although its form may be that of Equation (32). In the absence of a higher theory for a total frequency-conserving version of the Einstein field equations, it is possible to derive an expression for the Friedmann equation using Newtonian gravity, following a procedure outlined in Liddle [33], adapted to include features of NGDP. Since the Lagrangian $L_c/\hbar$ is in terms of frequency and not energy, the Hamiltonian is also, is total frequency conserving as there is no explicit time dependence, and was called $W$,

$$W = \frac{H_c(r)}{\hbar(r)} = \frac{1}{\hbar(r)} \left( \frac{1}{2} m r^2 - \frac{G M m}{r} \right) = \frac{1}{\hbar(r)} \left( \frac{1}{2} m r^2 - \frac{4 \pi G \rho r^2}{3} \right)$$

(19)
Equation (19) describes the frequency of a particle of mass \( m \) at a radius \( r \) from the origin of a homogeneous mass distribution of density \( \rho \). Changing to the co-moving coordinates \( x \) in terms of the scale factor \( a \),

\[
r(t) = a(t)x
\]  

(20)

Making this substitution, multiplying both sides by \( 2/ma^2x^2 \), it is found that,

\[
\left( \frac{a}{\dot{a}} \right)^2 = \frac{8\pi G}{3} \rho + \frac{2W}{ma^2x^2}h(ax)
\]  

(21)

The second term of (21) must be independent of \( x \) in order to maintain homogeneity. There are a number of ways this might be accomplished. One way will be examined that produces a term like the cosmological constant. In the usual derivation, the energy of a particle is constant, but changes with separations as \( U \propto x^2 \), to allow a connection to be made to the curvature \( k \), and to arrive at the same form of expression derived from general relativity. Let it be assumed this persists in the conserved frequency as \( W \propto x^2 \), and that the variation of \( \hbar \) has a similar dependence. Since \( \hbar \) must have no explicit time dependence at a separation \( r \), using one of the solutions found in [26] for the functional form of \( \hbar \) shown in Equation (26) below,

\[
h = h_o + f_h(ax) = h_o + b(ax)^2
\]  

(22)

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{2Wh_o}{ma^2x^2} + \frac{2Wh_f(ax)}{ma^2x^2}
\]  

(23)

Rewriting (23) using \( kc^2 = -2Wh_o/mx^2 \),

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2} - \frac{kc^2f_h(ax)}{h_o a^2}
\]  

(24)

It is seen from (24) that another term enters the usual Friedmann equation. Provided that \( f_h/a^2 \) is sufficiently constant and negative, the additional term could serve as the cosmological constant. The factor \( f_h \) is not written as \( f_h(a) \) without \( x \), because then it would have explicit time dependence, which was not the condition under which (1), (2) and (12) were derived.

From (22) the universe is still isotropic, as a quadratic change in \( \hbar \) is seen in every direction. As there is no single origin of \( x \), the universe is still homogeneous in the sense that at one specific position an observer concludes that \( \hbar \) is multi-valued, seeing the same distribution of \( \hbar \) values from every other position when treated as the origin. The issue is not homogeneity, but the multiple values. The problem is mitigated if what one actually observes is the average of all possible values. Averaging (22) and (24) over the co-moving coordinates, only the last term is affected by the averaging, from which follows,
\[ \langle x^2 \rangle_x = \frac{\pi^2 - 4}{2} x_R^2 - \frac{\pi^2 - 4}{2} \frac{1}{k} \]

\[ \langle h \rangle_x - h_o = \langle f_k(ax) \rangle_x = \langle ba^2 x^2 \rangle_x = ba^2 \langle x^2 \rangle_x \sim \frac{\pi^2 - 4}{2} \frac{ba^2}{k} \]

\[ \frac{\Lambda}{3} = -\frac{k c^2}{\hbar_o} b \langle x^2 \rangle_x \sim -\frac{\pi^2 - 4}{2} \frac{bc^2}{\hbar_o} \]

\[ \hbar \sim \hbar_o \left( 1 - \frac{2\Lambda}{3(\pi^2 - 4)c^2 r^2} \right) \]

\[ \langle h \rangle_x = \hbar_o \left( 1 - \frac{\Lambda}{3} \frac{a^2}{kc^2} \right) \]

(25a-e)

\[ \Lambda = 9.95 \times 10^{-36} [1 / s^2] \]

\[ |b| \leq 1.324 \times 10^{-87} [Js / m^2] \]

\[ r_o \sim 2.822 \times 10^{26} [m] \]

From the second term of (25c), in order for there to be a non-zero and positive cosmological constant, \( k \neq 0 \) so that the universe could not be perfectly flat, but instead open or closed. If it is closed, the average value of \( x^2 \) over \( x \) would be finite and also be a constant, and then necessarily \( b < 0 \).

To go farther than this, another identification must be made. Evaluating the average (25a) using a spherical surface of radius \( x_R \), there is a resulting factor of \( x_R^2 \) by that example, and so the average of \( x^2 \) is identified with \( 1/k \), from which the last terms of (25a,b,c) are derived. The geometrical factor that results will be used for the order of magnitude estimates, though (25e) is independent of it. One sees explicitly from the last term of (25b) that since \( b \) must be zero if the universe is perfectly flat, the cosmological constant would be zero. From the last term of (25a), the \( \hbar \) that is measured falls with time and can eventually become zero, and also negative.

The measured value \( < \hbar >_x \leq \hbar_o \), so an upper limit on the absolute value of \( b \) can be estimated from (25c) using the measured value. From (25d) the separation \( r_o \) is that where Planck's constant would fall to zero before averaging - remarkably on the same order as the radius of the present-day observable universe.

### 6.3 Planetary Scale Parameters: The Earth Flyby Anomaly

The behavior expressed in the most fundamental equations in physics for the dominant path certainly does not describe behavior witnessed every day. None of the equations here conserve energy, rather, total frequency. Evidence of non-conservation of energy in a classical system may be relatable to the effects described here, and more examples need to be found.
Consider Equation (13) and conservation of the total frequency $W$. If infinitely far from a potential center, should $\hbar$ fall to a constant and the potential to zero, then a moving body total energy would be equal at the extreme distances, but not between when closer to the center. That is, energy would be restored but not conserved. Therefore, when a measurement is made of the moving bodies velocity and position when not infinitely far from the center on a hyperbolic trajectory, a non-energy conserving anomaly will be shown in the extracted osculating excess hyperbolic velocity $V(\infty)$, and it can only be seen if the analysis is done on data specifically not taken at effectively infinite ranges, with a radial asymmetry about the center. It is emphasized the latter is not seen in normal Newtonian gravity.

In NGDP on a hyperbolic trajectory, the greater the range, the greater the percentage of restored energy, and the less anomalous the energy change will appear. Also, the more symmetric the analysis about the center, the less anomalous the energy difference will appear.

Parameters for the Earth for the variation of Planck’s constant will now be extracted from an analysis of the flyby anomaly, with all trajectories starting and terminating at equal time. It is understood that the range chosen for the analysis will impact the extracted parameters. Though an analysis may show energy conservation at symmetric terminal points along a trajectory, or at effectively infinite range, the position as a function of time will differ from Newtonian gravity, and this is why the full trajectory dataset is needed for a proper parameter extraction. All these points will be demonstrated. That the anomaly can be made to go away depending on when the analysis along the trajectory is done offers an explanation as to why the anomaly is no longer seen, or may not been seen routinely.
Figure 2. Trajectory of NGDP (blue) per Equations (17a) and (26d). Normal gravity is also shown (red). The total velocity is fixed at 9 km/s, and the x-velocity ranges from −0.5 to −1.42 km/s producing the trajectory fan. The Earth is the green circle, the equator is the x-axis. The leftmost trajectory has a starting x-velocity of −1.42 km/s, for which the asymptotic velocity change is zero.

The flyby anomaly is described in a paper from the Jet Propulsion Laboratory. It describes anomalous changes in the orbital energy in the Earth flybys of six satellites [34]. The energy non-conservation was on the order of one part in $10^6$, and was fit by an empirical formula. No physical explanation was found in the investigation. If the detailed orbital trajectory data as a function of position and time were available, and it is not presently, Equation (14a) or (17a) could be used for trajectory analysis. Using Equation (17a) and (26a-e) derived in [25-26], one can compute the trajectories of masses in flyby orbits. The spatial solution for $h$ has an infinite number of terms and was derived from principles outlined in [26], and they are the classical
vacuum solutions of a massless zero-momentum field. There is no mass or gravity involved in the derivations. The solutions were,

\[
\left(\frac{h(r)}{h_\infty}\right)^2 = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \left( A_l r^l + B_l r^{-l-1} \right) P_l^m (\cos \varphi) \left( S_m \sin(m\theta) + C_m \cos(m\theta) \right)
\]

(26a-b)

The forms used for fitting will be based on (26a-b), with some additional features, and with the understanding that the values of the function are very close to unity,

\[
\frac{h(r)}{h_\infty} \rightarrow \begin{cases} 
\sqrt{1+ \left(\frac{b_4 / r_3}{r} + \sum_{l=1}^{\infty} B_l r^{-2l-1}\right) f(r)} & r \geq r_h \\
\frac{h(r_0)}{h_\infty} & r < r_h 
\end{cases}
\]

\[
\partial_{x,y,z} \ln h = \begin{cases} 
\frac{1}{2} \left(\frac{x, y, z}{h(r)}\right)^2 \left(\frac{f(r)}{r^2} - \frac{3 B_2}{r^4} - 5 B_2 - \frac{7 B_3}{r^5} \right) & r \geq r_h \\
0 & r < r_h 
\end{cases}
\]

(26c-e)

\[
f(r) = \frac{1}{e^{(r-a)/\alpha} + 1}
\]

Although not a set of orthogonal functions forming a basis, a large range of profiles, symmetries or lack thereof can be addressed. Averaging over angular dependences may be performed to develop a pure radial dependence, as there are no constraints pinning the solution to any particular orientation. For example, with azimuthal symmetry \(m=0\), the Associated Legendre Polynomials become just the Legendre Polynomials \(P_l\), and averaging from \(\varphi = 0\) to \(\pi\) will leave the even \(l\) terms non-zero. This is the same as eliminating the angular dependence per (26b), written so that \(l\) can be odd or even. Also, only the solutions going to zero at infinite range are kept. An ad-hoc decay function \(f(r)\) is added to the expressions to prevent interference with the orbits of other bodies at greater range. For example, a function resembling the Fermi-Dirac distribution, per (26e) might be used. Neither the clipping at close range below \(r_h\), or the decay at greater range were derived from a theory.
At this time the author is not able to numerically simulate all of the very fine deviations and
details in the orbital trajectories reported in [34] to a precision matching the capability of the Jet
Propulsion Laboratory, which includes general relativity and many other effects. However,
Equations (17a) with various terms of (26d) are numerically integrated in Matlab in what is to
follow. All of the Matlab scripts and functions will be made available upon request from this
author for independent analysis.

For satellite trajectories with velocities and distances about the Earth similar to those reported in
[34], Equations (17a) and (26d) are capable of producing energy changes on the flyby. The
calculation done here is very simple, and does not use the exact orbits of [34]. The intention is to
demonstrate generally how such behavior as noted by Anderson can come about. The flyby
orbits of Figure 2 are set up, the x-velocity of the starting point is swept over a range, and the
orbit is computed numerically for each of the starting x-velocities, with all other parameters
fixed. The initial and final velocities at the start and end of the trajectories \( V(r) \) are converted to
the osculating element asymptotic velocities \( V(\infty) \) using (27d) for a hyperbolic orbit, and the
fractional differences between \( \Delta V(\infty)/V(\infty) \) are computed, along with the incoming and outgoing
orbital velocity declination angles at the start and ends of the trajectory approximating infinity as
\( \delta^{i,o} = \tan^{-1}(V_{y,i}^{i,o} / V_{x,i}^{i,o}) \). With no Earth rotation in the model, the initial and final asymptotic
velocities should be equal with normal gravity due to energy conservation, and there should be a
delta with the NGDP due to energy non-conservation. For the negative values of the delta, the
simulation is run from the ending point of the forward simulation, but with negative velocities
(or backward), which proved to produce results very close to the negative values of the forward
simulation for several trial points. Therefore, the backward simulation is just the negative of the
forward simulation, and that is what is shown in Figure 3a-b. The calculation is repeated for
normal gravity for each x-velocity, and changes in the relative asymptotic velocity delta are
subtracted from the curve for the NGDP. That is, the delta in relative asymptotic velocity
resulting when there is normal gravity is the numerical background error that is subtracted off of
the NGDP delta, Figure 3e. A sensitivity analysis was performed, reducing the time step by
factors of ten in the numerical solution of the differential equations, until showing: 1) solution
curves that overlay the data of [34] without changing; 2) a nearly flat numerical background.

The calculation has also been done without the conversion to the osculating element \( V(D) \) using
(27d) at the beginning and end of the trajectory. This had little impact on the agreement of the
model with the Anderson data, but it does increase the numerical background greatly, and so the
conversion is employed.

There is no Earth rotation, and therefore no equator inherent in the model. The equivalent of the
equator in this model is found by changing the x-velocity in normal gravity, until the velocity
delta is minimized, and incoming and outgoing declination angles are symmetric about some
axis. That axis turns out just to be parallel to the x-axis bisecting the Earth, Figure 2.

The authors of [34] fit their data to an empirical function of the form,
\[
\Delta V(\infty) = K \left( \cos \delta^i - \cos \delta^o \right) \\
\vert \delta^i \vert \leq \vert \delta^o \vert \to \Delta V(\infty) \geq 0 \\
K = 3.099 \times 10^{-6} = \frac{2v_e}{c} \\
V(\infty) = \sqrt{V^2(r) - 2GM/r}
\] (27a-d)

where \( V(\infty) \) is the asymptotic velocity expected on a hyperbolic trajectory, \( \Delta \theta(\infty) \) is the anomalous additional velocity, and \( \delta^i \) and \( \delta^o \) are the incoming and outgoing asymptotic velocity declinations of the flyby. The equatorial velocity of the Earth is \( v_E = \omega_E R_E \), and the constant is found empirically, not from a theory, though has the form of twice the factor appearing in the low-speed Doppler shift \( \Delta f/f \approx v/c \). Mbelek provides a derivation of this formula based on both the transverse Doppler effect and time dilation in special relativity, and concludes that the reported anomaly is not an actual energy gain of the craft, rather only appears as one due to the unaccounted for effect in the ground-based tracking [37]. Equation (27d) is the asymptotic velocity at infinity extracted from a known velocity and radius for a hyperbolic trajectory. The value of \( K \) when plotting the Anderson data was found by this author to be 3.14 \times 10^{-6} with an \( R^2 = 0.998 \), not the value reported of 3.099 \times 10^{-6} reported.

Figure 3a shows all of the data of reference [34] plotted against the modeling results of a six-parameter fit. In order to be consistent with the definition of a declination angle restricted to \( \pm 90^\circ \) above and below the x-axis and to get the sign of the slope correct, positive values of the leading term \( b_4/b_3 \) must be used. The positive values mean that \( \delta \) must increase approaching the orbit focus. The total fixed velocity of the simulations used was 9 km/s, and from \(-0.5 \) to \(-1.42 \) km/s in the x-direction, in the range of [34]. The six-parameter representation of \( \delta \) using [ \( b_4/b_3, B_1, B_2, B_3, B_4, r_h \) ] = [ \( 6 \times 10^4 \) m, \(-6 \times 10^{18} \) m\(^3\), \( 1.5 \times 10^{32} \) m\(^4\), \(-4 \times 10^{45} \) m\(^5\), \( 2.5 \times 10^{59} \) m\(^6\), \( 1.628 \times 10^7 \) m] is shown in Figure 3c, optimized to the linear function (27a-c) in the displayed range. Most of the behavior is dominated by the leading term \( b_4/b_3 \), and will appear linear only if an infinite number of terms are used. With five terms, the optimization routine was finding solutions where it appeared to flatten the function for \( \delta \) below a certain radius, which would also require an infinite number of terms. In order to be able to fit the data for the larger range of declination angle differences with a smaller number of terms, the function for \( \delta \) was made a constant (or clipped) below a threshold radius, \( r_h \), which was then optimized.

Figure 3a also shows the result of analyzing the flyby at a starting and ending range 10 times greater than that of Figure 2 for the same set of six parameters, and the curve is lower in slope. The result appears increasingly more energy conserving, and in order to re-fit the data, the parameters would have to change, namely, \( b_4/b_3 \), would have to increase with the range of the analysis.

In Figure 3b, the end times of each trajectory are adjusted by a point-by-point, nearly-quadratic factor to bring the data into closer agreement with Anderson\( \delta \), and the adjustment factors range from 1 to 1.002. A time adjustment factor cannot be found for normal gravity (the numerical background) that would make it agree with the Anderson data, as the profile is very insensitive to
the time factors at these ranges. Therefore, in normal gravity, the time of conversion to the osculating element $V(\mathcal{D})$ is much less important than in the NGDP. The time adjustments do not have a physical interpretation, only a numerical one, which is that they are small, express the impact of the use of a limited number of terms from (26d), and again show the sensitivity of the timing and range of the analysis. Figure 3b also shows that a uniform reduction of the duration of the trajectory causes an offset without a change in shape, but has no impact on the numerical background.

The starting x-velocity of the fit in Figures 3a-b that produces zero asymptotic velocity change on the flyby is $-1.42$ km/s, however, the entire trajectory is different from what is expected of Newtonian gravity, shown in Figure 3d. Thus, there should be anomalous deviations from normal gravity throughout the entire trajectory even if the velocity change measured is zero at the terminal points. Note from Figure 3d that the difference in radial distance from the Earth between normal and NGDP for this example about 70 km. It is not publicly known whether it really was, and is why the model needs the full trajectory data for a proper calibration.
Figure 3a. Six-parameter NGDP modeling results (green and blue), against all the data of reference [34] on the flyby anomaly (red) and its reported linear fit (27a-c). Parameter values are given in the text. Equations (17a) and (26d) are used. The higher slope curves are the trajectories of Figure 2, and the lower slope curves have starting and ending range greater by a factor of 10, all other parameters fixed. The slope is going to zero with range for the same parameter set.
Figure 3b. Six-parameter NGDP modeling results (green and blue), against all the data of reference [34] red. The data conversion to the osculating element $V(D)$ occurs at the beginning and the end of trajectories, whose end times are adjusted by factors of 1 to 1.002 using a quadratic function, relative to Figure 3a. The lines offset from the red flyby data result from a uniform duration reduction of the trajectories, but not a change in shape or a change of the background.
Figure 3c. Six-parameter modeling result for Planck's constant using the first five terms of (26d) (green and blue). Below \( r_h = 1.628 \times 10^7 \) m, the Planck's constant becomes constant (or clipped). The lowest value plotted is the radius of the Earth.
Figure 3d. For the six-parameter NGDP fit of Figure 3a, the starting x-velocity that produces an asymptotic velocity change of zero on the flyby is \( -1.42 \) km/s. The entire trajectory differs from that of Newtonian gravity, shown above as a difference in orbital radius versus time, emphasizing the orbital data as a function of time is needed to develop the true parameters, and because the energy changes inferred at finite range depend on the range of the analysis.
Figure 3e. The numerical background (green) of normal gravity that is subtracted from the signal+background of NGDP (red) to produce the simulation curve of Figure 3a.

In [25], a function for the variation of Planck's constant was developed from a general relativistic argument of the form,

\[
\frac{h(r)}{h_\infty} = \left(1 - \beta_h \frac{R_S}{r}\right)^{\frac{1}{2}} = \left(1 + \frac{b_4}{b_3} \frac{r}{R_S}\right)^{\frac{1}{2}} = -\beta_h \Delta R_S = \beta_h \frac{2GM}{c^2}
\]

(28a-b)
where $\beta_h$ is the Local Position Invariance (LPI) violation parameter, $R_S = 2GM/c^2$ is the Schwarzschild radius. Table 1 shows the extracted $\beta_h$ for Earth is many orders of magnitude larger than any violation ever observed, compared to the redshift violations $\alpha_{3b}$ of Gravity Probe A, the GPS satellites, and the Galileo satellites. The conclusion is therefore that $b_4/b_3$ is independent of mass, and is representative of an effect far stronger than the redshift. The latter is discussed further in [25].

6.4 Solar System Scale Parameters: The Sun based on the Diode Experiment

The extraction of parameters for the sun is somewhat problematic. The diode experiment of [19] was essentially an Earth-bound astrophysical experiment, taking over three years to set up, collect the data, and to analyze it. Precautions were taken to remove artifacts, and divorce it from anything that might cause a systematic artifactual variation - high precision power regulation to prevent systematic drift from things like air conditioning loads in summer versus winter, and also temperature monitoring and regulation with impact on the diodes tracked and calibrated out. It was based on competing factors of Planck $\hbar$ constant in the tunneling exponent on which the operation of Esaki diodes is based. One factor is $\hbar^2$ coming from the barrier width, the other factor is $\hbar^{-1}$ in the WKB tunneling approximation. It was the first experiment specifically devised to be sensitive to changes in $\hbar$ $i$ not like the high precision measurements that do not have this specific intentional feature. The signal-to-noise ratio is beyond dispute. It was correctly predicted to show a phase shift in voltage 180 degrees relative to the highly disputed radioactive decay oscillations in the literature, and when closer to the sun as the Earth orbits it, $\hbar$ increases. There is a year-period sinusoidal signal whose analysis was published in the optics journal [19]. There is a daily signal that has gone unanalyzed, with data every 10 seconds for 941 days. The pitfall of such an experiment is that it attempts to measure the variation in a dimensioned constant. The result is potentially very important, however, because it does not attempt to extract a specific value of the constant, rather its fractional change, and because of the predicted difference in phase from the decay rates.

This is the only data of its kind, and allows an attempt at calibration of the model for the sun, for which the parameter extraction was done in [26], and is shown in Table 1. The peak-to-peak swing $\delta h/\hbar$ is reported to be 21 ppm [19]. These diode measurements show variations higher than that of NIST measurements of Planck $\hbar$ constant with a precision of $\sim 10^8$ [35], where the latter measurements are made at the Earth $\hbar$ surface, but were not specifically designed to be sensitive to variations in Planck $\hbar$ constant, such as the diode experiment was with intentionally competing factors of $\hbar$.

A notable result shown in Table 1 is that the $b_4/b_3$ of the Earth based on the flyby is about the same as that extracted from the uncertainty of the NIST measurement of $\hbar$ used as if it was equal to the swing $\delta h/\hbar$, and both are less than the sun $\hbar$ by several orders based on the diodes. One might conclude that $b_4/b_3$ increases with $M$, but it will be explained why this is not so in the discussion, and it is essentially because it leads to problems with the orbits of a Hulse-Taylor-like binary, the Earth, and Mercury, the latter being rectifiable even for the diode swings with more terms in the series, but not the former.
Table 1 also shows that the $\beta_h$ values from the diode experiment greatly exceed the redshift violation parameters $\alpha_{rs}$ of Gravity Probe A, the GPS satellites, and the Galileo satellites, where again it is concluded that $b_4/b_3$ is independent of mass, and is representative of an effect far stronger than the redshift.

The gain in the diode experiment may be different than was computed in [19], where,

$$I = C(h)V \exp \left( -\frac{8\pi}{3} x_b \frac{h}{h_o} \sqrt{2m_e(V_s(h) - V)} \right)$$

$$\delta V / V = f \cdot \delta h / h$$

(29a-b)

If one assumes that the only influence on a change in $\hbar$ is the change itself in the exponent, and the diode barrier width $x_b = x_{bo}(\hbar/\hbar_o)^2$, and nothing else, one finds the result of [19] that the gain in constant current mode was $f = -40.8$. However, if the length can change with $\hbar$ in all of the rest of the metrology (resistors, diodes, meters, sources) encapsulated in $C(h)$, and also the built-in voltage of the diode $V_b(h)$, then the gain will be different than what was computed. The true diode characteristics (I-V curves) were also not used, only the basic physical argument. The metrology of such an experiment needs much deeper analysis, as there are many neglected terms contributing to the gain.

Near the sun surface, the $\hbar/h_o$ extracted would also be unphysically high. As will be explained, this is rectified by the use of more terms in the expansion of (26d) to reduce it, without impacting the results at greater distances, or to clip it, as was done for the flyby.

7. Discussion and Impact on Gravitational Radiation

The model presented that seems to fit the flyby anomaly data fairly well is truly non-energy conserving. There is no additional field in the model for gravity to exchange energy with causing it to simply appear that energy was not conserved because the exchange is unaccounted for in the motion of the spacecraft. The starting point for the development in this paper could not be derived from an energy-conserving field theory [26]. In this model, $W$ is conserved, the frequency of the dominant path. It has been recently confirmed that the flyby anomaly observation has gone unresolved to this day, while also the very few satellites since that have been examined did not exhibit the anomaly [36]. A detailed modeling effort using the actual orbital data as a function of time, if it could be made available, would be worthwhile to undertake. The effects of the model here could be investigated, and also of a field-based model, where the exchange of energy is possible, both with and without a field representative of Planck's constant that was formatively derived in [26].
| Body         | Obs.   | $\delta h/h^{(2)}$ | $M$ (kg) | $b_4/b_3$ (m $^{(2)}$) | $\Omega_{PE}\,^{(8)}$ | $\Omega_{PM}\,^{(9)}$ | $R$ (m $^{(4)}$) | $\beta_h\,^{(5,3)}$ | $\Theta (R)/h_\infty$ | Ref. |
|--------------|--------|-------------------|---------|------------------------|------------------------|------------------------|----------------|----------------|-------------------|------|
| Earth        | Flyby  | 5.97x10$^{-24}$   | 6x10$^4$|                        |                        |                        | 6.38x10$^6$   | -6.67x10$^6$ | 4.69x10$^{-3}$   | [34] |
| Sun          | Diode  | 21x10$^{-6}$      | 1.99x10$^{6}$ | 1.62x10$^{8}$       | -1411                  | -3751                 | 6.96x10$^{8}$ | -5.43x10$^4$ | 1.09x10$^{-1}$   | [19],[26] |
| Sun/Earth$^{(1)}$ |        |                   |         |                        |                        |                        |                | 3.3x10$^4$  | 2.70x10$^{1}$   |      |
| Sun          | NIST$^{(11)}$ | <10$^8$          | 1.99x10$^{6}$ | <7.70x10$^{8}$       | -0.996                 | -2.599                | 6.96x10$^{8}$ | < -26.1      | <5.53x10$^{-3}$   | [35],[26] |
| Sun          | Small$^{(7)}$ | 1.99x10$^{6}$     | 100     | -0.001                 | -0.0034                |                        |                |               |                   |      |
| 1.4xSun$^{(10)}$ | Binary | $\pm$ 4.5x10$^3$ |         |                        |                        |                        |                |               |                   | [38] |
| Satellite    |        |                   |         |                        |                        |                        |                |               |                   |      |
| Galileo      |        |                   |         |                        |                        |                        |                |               |                   | [12] |
| GP- A        |        |                   |         |                        |                        |                        |                |               |                   | [12] |
| GPS          |        |                   |         |                        |                        |                        |                | $\beta_h < 0.007$ | [10],[11] |

Table 1. Use of equations (17a), (26), and (28a-b) to extract parameters for Planck\(\Theta\) constant using the leading order one-parameter fit. The reference for the data used in the calculation is indicated in the last column. Note the \(b_4/b_3\) values extracted for the sun are greater than the Earth\(\Theta\).

1 Ratios are those of sun-Diode/Earth-Flyby

2 Procedure to extract $b_4/b_3$ from the swing $\delta h/h$ is described in [26]. The evaluation radius is the sun-Earth orbit 1 AU.

3 Compare the redshift violation parameter $\alpha_{rs}$ to Planck\(\Theta\) constant LPI violation parameter $\beta_h$.

4 Surface radius of the body.

5 Inequalities refer to absolute values. $\beta_h = (b_4/b_3)/R_S$ is used to calculate the top three table elements. The others are from literature.

6 GR could explain clock frequency shifts up to this redshift factor $\Delta f/f = (1 + \alpha_{rs}) \Delta U/c^2$.

7 Investigating the "small value" of 100 m, evidencing that the extracted value for the sun must fall off with range much faster.

8 Perihelion precession of Earth in arcseconds per orbital period.

9 Perihelion precession of Mercury in arcseconds per orbital period. For comparison the GR result for this is +0.104 arcseconds per period.

10 Value needed to alter the rate of orbital period change by $\mp 0.13\%$ of a 1.4 solar mass Hulse-Taylor-like binary.

11 The uncertainty of the NIST measurement of $h$ is used as if equal to the swing $\delta h/h$ at the Earth caused by the Sun\(\Theta\) field.

Such actions for interacting fields could be of the form (IIa-e), or
\[ S_N = \int \left\{ -\frac{1}{8G_o \pi} \xi \eta^{\mu \nu} \partial_\nu \phi \partial_\nu \phi + \rho \phi + \frac{(\hat{h}_o \nu)^2}{2} \eta^{\mu \nu} \partial_\nu \phi \partial_\nu \phi + \frac{(\hat{h}_o \nu)^2}{2} \xi \eta^{\mu \nu} \partial_\nu \phi \partial_\nu \xi \nu + \lambda \xi \phi \nu \partial_\nu \right\} d^4 x \]

\[ \hat{h} = h_o \nu \]

\[ G = G_o / \xi \]

or, considering actions of the form of (32) where \( \hat{h} \) may or may not have time dependence, and the dynamical term for Planck\( \xi \) constant may or may not appear,

\[ \int dt \frac{L_c(x(t), \dot{x}(t))}{\hat{h}(x(t), t)} \rightarrow \int \left\{ \frac{c^4}{16G \pi} R + \frac{(\hat{h}_o \nu)^2}{2} g^{\mu \nu} \nabla_\nu \phi \nabla_\mu \phi + L_m(\hat{h}) \right\} \sqrt{-g} d^4 x \]

Equation (31a) is a special relativistic version in flat spacetime, with gravitational potential \( \nu \).

There are many such couplings and combinations that may be tried. Variations in \( c \) or \( c^4 \) might be added, per (I1a-e) and [31-32]. Equation (32) is motivated by the path integral result.

The couplings of the fields would have to be orchestrated in a way that a test mass can acquire additional velocity in an energy conserving manner, where for example, \( G \) falls in time or position while the spacecraft is passing Earth, leading to a higher than expected final velocity, because it took less energy to escape the Earth compared to when the test mass entered. As for positional dependences of \( G \) alone with no time dependence, they would have to be asymmetric about the Earth in some way to appear non-conservative, and while that is possible to arrange mathematically, it seems physically unlikely since the asymmetric arrangement must generate just the right boosts for all the different satellite orbits. That leaves a time variation as a possibility that would have to be exquisitely timed in order to coincide with the when and how of all six satellite flybys, and so is also unlikely.

That leaves artifacts. Since the very few flybys since the Anderson paper [34] have not shown the anomaly [36], it may be that an undiscovered systematic problem was corrected in the tracking improvements since 2008, and whether that is so is an unknown and may remain so. It may not be sufficient to simply analyze the asymptotic velocity differences, rather, the data of the entire trajectory should be examined because it may show other anomalies. Such data needs to be made available. The flyby anomaly is simply not something that is routinely observed every time, for example, the Juno spacecraft passing near Earth in 2013 [38].

The value of \( K \) and the form found empirically for it shown in (27a) resembles twice the Doppler shift from the velocity of the rotating Earth, multiplied by an angular dependence. The derivation in [37] based on the transverse Doppler effect results in Equation (27a), the change in velocity is concluded to be an artifact, and one wonders why the flyby anomaly is still listed as an unsolved problem in physics in light of this result – the theory of [37] has not been directly proven by a detailed analysis of the flyby data throughout the entire trajectory, however. The models
developed here do not address the empirically determined constant involving the Earth rotation, which may be fortuitous.

The model developed here must be calibrated so as to not be at odds with well-observed phenomena. Recall the path integral derivation was for mild gradients in $\hbar$, both in the approximate basis used, and also due to the findings in [25] on the Schrödinger equation for when the gradient is small, that planewave solutions for free particles hold approximately, and the Ehrenfest theorem relating the time derivative of the expectation value of position to the momentum is exactly reproduced. The latter is maintained when $b_4/b_3 \ell r \ll 1$.

Consider the impact of the flyby parameters on the moon $\mathcal{A}$ orbit, where at the radius of the lunar-Earth orbit, $\varphi \hbar /\hbar_0 = 8.269 \times 10^{-5}$. A numerical calculation performed for the moon $\mathcal{A}$ orbit about a fixed-position Earth, using (17a) and the six-parameter fit resulted in small changes that were computationally challenging. The apogee would increase by about 94 km compared to normal gravity, when the orbit is begun at the same perigee and orbital speed as is measured. The actual perigee is known to vary over a range of about 14,000 km, and the apogee over 2700 km, so if the effect was active, this aspect of it may be masked. The apogee increase is insensitive to the time increment used in the simulation, while the orbital radius change with time is. The orbital radius at apogee falls by -3700 m/yr at 5000 timesteps per orbital period, -926 m/yr at 20,000 timesteps per period, and -308 m/yr at 60,000 timesteps per period, converging as $-2 \times 10^7/N^{1.002}$ m/yr, where $N$ is the number of time divisions per orbital period. The numerical solution is approaching zero. These calculations are pushing the limits of the computational resources available to the author at the time of this writing. Lunar Laser Ranging puts the moon $\mathcal{A}$ orbit to be increasing in radius by +3 cm/yr. The 6-parameter fit also predicts an additional -201 arcsecond/period apsidal precession, small compared to the observed one, ~ +11,000 arcsecond/period. If these are not numerical artifacts, the decay function (26e) could be utilized. Putting $a \sim s \sim 10^8$ m rectifies all issues with the moon while not affecting the ability to fit the flyby, and without introducing sharp transitions in the function for $\hbar$.

For the sun-to-Earth radius based on the diode experiment, Gravity Probe A, and the Galileo satellites, at 1 AU $\varphi \hbar /\hbar_0 \geq 5.41 \times 10^3$, and the extracted value $\varphi \hbar /\hbar_0 \geq 0.11$ or higher at the sun surface (Table 1) would negate fusion - however, the approximation of the mild gradient is now breaking down. The $b_4/b_3$ value extracted from it for the sun would affect the orbit of the Earth and Mercury. A numerical study performed for the orbits shows that if the diode parameter values of Table 1 are used for the sun along with only the leading term of (28), the apsidal precession becomes large and negative, but that is what has to happen to maintain the 21 ppm swing $\delta h/\hbar$ reported in the diode experiment, and adding terms to lower the gradient there will reduce the predicted variation, taking it out of agreement. The precession and the measured reported variation in $\delta h/\hbar$ are boundary conditions in opposition, and the problem cannot be solved this way, but could if the variation were just shown to be smaller from neglected gain terms. The diode data therefore remains as a qualitative example of a dimensioned constant that might be varying, awaiting more analysis, especially of the measurement system for superimposed signals or unaccounted-for amplification factors. The data can be made available by the author of [19].
Table 1 shows that if the swing in $\delta h/h$ at the Earth over the sun-Earth orbit was not 21 ppm, but set as if $1 \times 10^{-8}$ per the precision of the NIST measurements or the flyby $b_3/b_3$ value, the $\zeta h/h_\infty$ at the sun surface $\sim 10^{-5}$, so there is no fusion problem, but the precessions are still too high.

It will be illustrated how the use of additional terms can rectify some problems while not disturbing other results. Assume that the 21 ppm swing of the diode test is real, and must be maintained in the model. The issue at the sun surface a distance from its center $R$ with the overly large change in $\hat{h}$ is rectified through the use of the $l=1$ term in (26d) with $B_1 = -7.85 \times 10^{16}$ km$^3$. With this, the $\zeta h/h_\infty$ at the sun surface goes to zero, yet the extra term is too small to alter the 21 ppm variation at the Earth's orbital radius. The problem with the precession of the orbit of Mercury can also be resolved by adding another term, and generally problems $\tilde{f}_{\text{interior}}$ to the leading term are fixed with more terms. The problem with the predicted orbit of the Earth is not resolved if the swing remains 21 ppm, of course.

If the diode data were used at face-value, it might have been concluded that the effects increase proportionally with mass. That conclusion cannot be made at this time. An effect increasing proportionally with mass may have a detectable impact on systems like the Hulse-Taylor binary pulsar, and for that case, the evolution through the radiation of gravitational waves matches the predictions of general relativity to within 0.13% [38]. A system may be found in the future where it is not as well described.

An orbit for a Hulse-Taylor-like binary for a highly-exaggerated effect is computed using only the model of this paper and the parameter extracted from the diode experiment, shown in Figures 4-5. While the orbital period does not change with time, the orbital period increases from 8.1 hours for the Newtonian calculation shown in Figure 4, to 11.8 hours in this paper's model, with a pronounced negative apsidal precession shown in Figure 5, for all other parameters fixed.
Figure 4. Use of Equations (17a) and (26d-e) to numerically compute the orbit of a Hulse-Taylor-like binary. Each mass is 1.4 solar masses placed at 746,000 km at periastron and launched in opposite directions vertically at 450 km/s. The Newtonian result is shown and has a period of 8.1 hours.
Figure 5. Use of Equations (17a) and (26d-e) to numerically compute the orbit of a Hulse-Taylor-like binary for NGDP. Each mass is 1.4 solar masses placed at 746,000 km at periastron and launched in opposite directions vertically at 450 km/s. Using the diode $b_4/b_3$ value for the sun from Table 1 multiplied by 1.4, the resulting orbit is shown, and now has a period of 11.78 hours, and a negative apsidal precession (opposite the orbital direction) of -104,000 arcseconds per period.

For the example of the binary orbit being considered, its rate of period reduction by emission of gravitational waves should be per [39] with $e=0.62$ and Equation (33),

$$\dot{P}_{GW} = -\frac{192\pi G^{5/3}}{5c^5} \left(\frac{P}{2\pi}\right)^{-5/3} (1-e^2)^{-7/2} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \frac{m_p m_e}{(m_p + m_e)^{\frac{2}{3}}}$$

(33)
the rate of period reduction is $-2.25 \times 10^{-12}$ s/s for the binary of normal gravity, and $-1.20 \times 10^{-12}$ s/s for the NGDP calibrated with the diode data multiplied by 1.4. Calibrated with the flyby parameters (or assumed NIST parameters), the precession of the periastron is predicted to be about 13 deg/year. In either case, it was concluded that there is no change in the orbital period as a function of time. A reduction in the period as a function of time was actually computed, but this period reduction rate was linear in the time increment of the numerical simulation tending towards zero, suggesting a numerical artifact, while the overall shape and precession of the orbit was insensitive to the time increment.

Therefore, the impact of such an additional force on the radiation of gravitational waves is to cause a negative apsidal precession that would not normally occur, and to have larger than expected orbital periods, resulting in slower than expected period decrease rate due to decreased gravitational wave emission. Using Equation (33) and the leading term only in the numerical simulation of the binary under discussion, a $\pm 0.13\%$ discrepancy in the rate of period reduction could be explained with a $b_4/b_3 = \mp 4.5 \times 10^3$ m, a surprisingly large number, and an apsidal precession results of $-90.44$ deg/year, too large, and is just further evidence that the parameter cannot be this large, or must decay with range faster than is modeled.

The diode scenario considered for the sun in Table 1 all has upper-limit $b_4/b_3$ values significantly higher than that extracted for the Earth, but it is concluded that the parameter is not mass dependent owing to the problems that arise if it were, and because the derivation of it in [26] does not involve mass. Since it is not in direct proportion to mass, which the definition of the parameter $\beta h$ depends on, this explains the high values extracted for it. The value of $b_4/b_3 = 60,000$ m is consistent with the flybys, but to be consistent with the observed orbits of the planets, possibly the moon, and Hulse-Taylor binary, must be less than this, and/or decay faster with range than $1/r$. Additional terms in the function for Planck\$ constant are available for the latter, but, to extract the parameters more meaningfully, the entire trajectory really needs to be fit, and that data is presently not available.

The constant $b_4/b_3$ value is arrived at from fitting, and whose value cannot be explained anymore than that of Planck\$ constant, or any other. This constant arises in a classical field theory for the vacuum, whose center is made coincident with large masses, but is independent of their mass. It is associated with the leading term, and is the active term when $b_4/b_3 \ll r$. When this condition is not satisfied at closer ranges, other terms become important in reducing the $\alpha \hbar/h_c$.

In [25], it was found that there was little simpatico between the Ehrenfest theorems and classical mechanics for a position dependent $\hbar$ in the frequency-conserving Schrödinger equation, and Equation (14b) clarifies why this is so - Newtonian dynamics changes.

Other accepted non-energy conserving events were, or are at work, such as the Big Bang, and the accelerated expansion of the universe driven by the cosmological constant. The additional term in the Friedmann equation (24) from a spatial variation in $\hbar$ may be relatable to the latter, and while total energy in the universe is still not conserved in the expansion, total frequency would be. An alternative path, a starting point, to an explanation of the nature of dark matter is offered – a spatial variation in $\hbar$ causes a macroscopic force assisting the dark matter in holding galaxies together, while suppressing the normal known forces dependent on $\hbar$, and unable to quantum
mechanically interact, it cannot lose energy (or frequency) by normal mechanisms, and remains dark.

8. Conclusions

A modified form of the path integral for a position dependent Planck\(\hbar\) constant is derived for the limit of a mild spatial gradient. The variations in Planck\(\hbar\) constant are not treated as an energy conserving dynamical field here \(\tilde{I}\) that may be found in [26]. A path termed as \(\tilde{\Omega}\) is identified that makes the integral in the exponent stationary. Modifications to Newton\(\hbar\) Laws, Newtonian gravity, and the Friedmann equation are found. An initial attempt is made to address anomalous energy changes in satellite flyby orbits noted by the Jet Propulsion Laboratory, and a good, quantitative agreement is found for the energy changes, but requires detailed orbital data as a function of time, and greater computational precision than was available at the time of this writing to do better. The anomaly is here explained to stem from a combination of the lack of energy conservation at close range, and the inability to collect and analyze data at true infinite range. The detailed trajectory data for the whole flyby would need to be brought out of archive, and then also formatted so that it can be used by independent investigators, if the information still even exists [36]. The \(\beta_h\) parameter values extracted for the Earth for the flybys are very large, as are those extracted for the sun from the diode experiment, neither of which are experiments with clocks and light. The Gravity Probe A, GPS, and the Galileo satellite redshift violation parameters \(a_\alpha\) and \(\beta_\hbar\) are orders lower, are three experiments with clocks, and it was rationalized that a variation in \(\hbar\) may be undetectable with clock and light measurements at different altitudes if total frequency is conserved [25]. It is concluded the leading parameter \(b_4/b_3\) does not depend on mass and describes an effect much stronger than the redshift [25]. The high \(b_4/b_3\) parameter values extracted for the sun from the diode experiment cause a number of orbital problems, and so it is concluded the diode experiment needs repeating by independents, and more in-depth analysis of what its measured variations mean. The parameter \(b_4/b_3 \leq 60,000\) m, and/or \(\hbar\) must decay with range faster than \(1/r\).

Methods to test the model\(\hbar\) concrete predictions would be: 1) the intentional set up and analysis of the entire trajectories of satellite flyby orbits that maximize the effect; 2) monitoring of gravitational wave emission of binary mergers that do not match expected theory, with longer periods than the masses involved predict, and slower orbital period reductions; 3) observation of unexpected negative apsidal precession in bound orbits.

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Planck’s Constant as a Dynamical Field

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Abstract. The constant $\hbar$ is elevated to a dynamical field, coupling to other fields, and itself, through the Lagrangian density derivative terms. The spatial and temporal dependence of $\hbar$ falls directly out of the field equations themselves. Additional constants are necessary to set up the equations, followed by yet more that are generated in the differential equation solutions, and one may not escape the ultimate need of a true constant of some sort. Three solutions are found, two are quantizable, and the third is not quantizable. The third corresponds to a zero-momentum classical field that naturally decays spatially to a constant with no ad-hoc terms added to the Lagrangian. An attempt is made to calibrate the constants in the third solution based on experimental data. The three fields are referred to as actons. It is tentatively concluded that the acton origin coincides with a massive body. An expression for the positional dependence of Planck’s constant is derived from a field theory in this work that matches in functional form that of one derived from considerations of LPI violation in GR in another paper by this author.

1. Introduction

Constants of nature are measured carefully, seem to be only very weakly dependent on position, and if time varying, this time variation must be very slow. A solution is sought that is consistent with those observations that still might explain how a constant can come into being from some sort of field, that can describe energies associated with the generation of the constant, and variations with time or position, if any. One must also explain why the constants do not seem to be limited by the need to propagate at a speed $c$ or lower to have an effect—they are infinite in range, everywhere at all times, nearly equal in magnitude in all locations, persistent in duration, and operate seemingly without any mechanism at all, so there is by definition no “spooky action at a distance” where they are concerned. That, however, is in itself somewhat spooky. Constants therefore would not seem to be described by standard model particle-field interactions. The term acton may serve as an appropriate distinction for a field related to a physical constant.

The possibility of the variation of fundamental constants would impact all present physical theory, while all reported variations or interpretations of data concluding a constant has varied are extremely controversial. Examples of work in this area include Dirac’s Large Number Hypotheses [1], the Oklo mine from which could be extracted a variation of the fine structure constant [2,3], and the observations of quasars bounding the variation of the latter per year to one part in $10^{17}$ [4-6]. Recent theoretical work includes the impact of time dependent stochastic fluctuations of Planck’s constant [7], and the effect of a varying Planck’s constant on mixed quantum states [8]. An authoritative review of the status of the variations of fundamental constants is given in [9].

Publicly available Global Positioning System (GPS) data was used to attempt to confirm the Local Position Invariance (LPI) of Planck’s constant under General Relativity [10-11]. LPI is a
concept from General Relativity, where all local non-gravitational experimental results in freely falling reference frames should be independent of the location that the experiment is performed in. That foundational rule should hold when the fundamental physical constants are not dependent on the location. If the fundamental constants vary universally, but their changes are only small locally, then it is the form of the physical laws that should be the same in all locations.

The LPI violation parameter due to variations in Planck's constant is called $\beta_h$. The fractional variation of Planck's constant is proportional to the gravitational potential difference and $\beta_h$. The value found in [10] for variations in Planck's constant was $|\beta_h|<0.007$. This parameter is not zero, and is the largest of the violation parameters extracted in the study. The study did not report on the altitude dependence of Planck's constant above the earth. A very recent study involving the Galileo satellites found that GR could explain the frequency shift of the onboard hydrogen maser clocks to within a factor of $(4.5\pm3.1)\times 10^{-5}$ [12], improved over Gravity Probe A in 1976 of $\sim 1.4\times 10^{-4}$, these are the $\alpha_{rs}$ redshift violation values that may be compared to $\beta_h$.

Consistent sinusoidal oscillations in the decay rate of a number of radioactive elements with periods of one year taken over a 20 year span has been reported [13-18]. These measurements were taken by six organizations on three continents. As both the strong and weak forces were involved in the decay processes, and might be explainable by oscillations of $\hbar$ influencing the probability of tunneling, an all electromagnetic experiment was conducted, designed specifically to be sensitive to Planck's constant variations [19]. Consistent systematic sinusoidal oscillations of the tunneling voltage of Esaki diodes with periods of one year were monitored for 941 days. The tunnel diode oscillations were attributed to the combined effect of changes in the WKB tunneling exponent going as $\hbar^{-1}$, and changes in the width of the barrier going as $\hbar^2$. The electromagnetic experiment voltage oscillations were correctly predicted to be 180 degrees out of phase with the radioactive decay oscillations. This data can be made available for independent analysis by requesting it from the author of [19].

It is reasonable to suspect that the oscillations of decay rates and tunnel diode voltage are related to the relative position of the sun to the orbiting earth, and that there are resulting oscillations in Planck's constant due to position dependent gravitational effects, or effects with proximity to the sun. It should be mentioned that there have been studies in which it was concluded there was no gravitational dependence to the decay rate oscillations [20-21]. There is also dispute in the literature concerning the reality of the decay rate oscillations [22-24].

Either way, whether by gravitation or by some other mechanism, for the work to be presented, all that matters is that there be a position and time dependent scalar $\hbar$ field, and it would be of value to understand the impact on the fundamentals of field theory under such a condition.

At this time, it is not known conclusively whether the variation of a constant signifies that it is assuredly a dynamical field, or not, or is something else entirely. In a separate paper by this author, issues specific to the Schrödinger equation in a single-particle, non-relativistic, non-field theoretic framework for a position dependent $\hbar$ that is not treated as a dynamical field were examined [25]. That work is relevant to the present paper for two reasons. The first reason is that the positional variation of $\hbar$ is derived in [25], from a completely different starting point, that
results in the same in functional form as one derived in this paper. In this paper, variations in $\hbar$ will be treated as a scalar dynamical field, coupling fields through the derivative terms in the Lagrangian density. The second reason is that one of the results to follow in this paper suggests that frequency may be a more fundamental parameter than energy, and the Schrödinger equation of [25] is frequency-conserving and Hermitian.

The scope of much work with variable constants as dynamical fields has been to address unsolved problems in cosmology. Consider the Cosmon of Wetterich [26-27], using a field dependent prefactor to the derivative term in the scalar Lagrangian that decays to a constant value for large field values [27]. This prefactor does appear to play a similar role as those to follow in the present work, although the functional forms of the prefactors are different. Existing theories with varying constants as fields are the Jordan-Brans-Dicke scalar-tensor theory developed in the late 1950s with variable $G$, and Bekenstein variable fine structure constant theory developed in 1982 [28-29], though this model did not contain gravity, and was concerned with the electromagnetic sector. Albrecht, Magueijo and Moffat, examined a variable $c^4$ to attempt to explain the flatness and horizon problems, the cosmological constant problem, and homogeneity problem [30]. Cosmologies of varying $c$ were examined by Barrow [31] and Moffat [32-33].

Equation (I1a-e) shows in a single form an amalgam of possible couplings including a Jordan-Brans-Dicke-like scalar-tensor theory of alternative General Relativity with variable $G$, an Albrecht-Magueijo-Barrow-Moffat-like field for $c$, a field for $\hbar$ like that of [26], which is different than the form of Bekenstein for variable $e^2$ whose representative field squared divided the derivative terms. There is also the field theory of Modified Gravity (MOG) of Moffat, and the Tensor-Vector-Scalar (TeVeS) gravity of Bekenstein. There are many ways all the constants might be represented as fields, and many ways they might be coupled. Coupling fields together in this way is the accepted approach for the treatment of a constant, but is not the only possible approach, and here, something different will be tried.
The work to follow is certainly related, but the Lagrangian densities and actions, even the intended scope, differ from the above, and the number of couplings is simplified in flat spacetime to understand how scalar fields couple through the derivative terms, and to itself. The fields in the prefactors to follow are literally assigned to represent Planck’s constant, distinct from c, and e is not present. This work is more of an exploration of what a constant actually is, of required energy to sustain a constant, the prospect and problems of trying to eliminate constants, calibration of the new constants that represent the variations, and quantization of the constants.

2. Classical Field Lagrangian Densities

Consider two symmetrically coupled scalar fields of mass m and M with coupling constant g, showing the unburied location of the constants h and c in the Lagrangian density L,

\[
L = \frac{1}{2} \frac{\hbar^2}{\alpha} \frac{\partial}{\partial x} \phi \frac{\partial}{\partial x} \phi - \frac{1}{2} m^2 c^2 \phi^2 + \frac{1}{2} \frac{\hbar^2}{\alpha} \frac{\partial}{\partial x} \psi \frac{\partial}{\partial x} \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \phi
\]  

(1a)

The Lagrangian density of Equation (1) produces the normal coupled Klein-Gordon equations for the fields (or particles) \( \psi \) and \( \phi \). The constant \( \hbar \) might be associated with the fields directly (though this does not matter if \( \hbar \) is just a constant). If so, as a prelude to making \( \hbar \) a field, it is shown as operated on by the derivatives,

\[
L = \frac{1}{2} \frac{\partial}{\partial x} h \frac{\partial}{\partial x} \phi \frac{\partial}{\partial x} \phi - \frac{1}{2} m^2 c^2 \phi^2 + \frac{1}{2} \frac{\partial}{\partial x} h \frac{\partial}{\partial x} \psi \frac{\partial}{\partial x} \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \phi
\]

(1b)

Whether the latter is necessary or not is an unknown, but is an option to investigate.

Let it now be supposed that \( \hbar \) is itself a dynamical field, which would be a natural way to introduce it as a non-constant. If \( \hbar \) is to have the usual units of J s, the unit keeping then necessitates the introduction of yet another constant \( \beta \). One must also consider that \( \hbar \) may be proportional to a field raised to a power, where \( \hbar=\beta \psi^2 \),

\[
L = \frac{1}{2} \left( \beta \psi \right)^2 \frac{\partial}{\partial x} \phi \frac{\partial}{\partial x} \phi - \frac{1}{2} m^2 c^2 \phi^2 + \frac{1}{2} \left( \beta \psi \right)^2 \frac{\partial}{\partial x} \psi \frac{\partial}{\partial x} \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \phi
\]

(2a)

or, owing to the unknowns surrounding how to associate the field \( \hbar \) to another field as far as the derivatives,

\[
L = \frac{1}{2} \frac{\partial}{\partial x} \left( \beta \psi \right)^2 \phi \frac{\partial}{\partial x} \left( \beta \psi \right)^2 \phi - \frac{1}{2} m^2 c^2 \phi^2 + \frac{1}{2} \frac{\partial}{\partial x} \left( \beta \psi \right)^2 \psi \frac{\partial}{\partial x} \left( \beta \psi \right)^2 \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \phi
\]

(2b)

The third term of the Lagrangian (2a) differs from the variable fine structure constant introduced by Bekenstein, where the charge is represented by a field and the square of this field divides the denominator of the derivative terms \[28-29\]. That difference makes possible the manipulations to follow in later sections of this paper.

Now the association of \( \hbar \) matters, as it makes the derivates more or less complicated. The latter equation of motion will have more terms than the former.
The last two equations (2a) and (2b) imply that there would be something "special" about the field $\psi$ as the only generator of $\hbar$, with $\psi$ imposing the quantum of action on both itself and on $\phi$. Since it is difficult to rationalize why only one field would have this special property over any other, consider also that any field has the ability to function as the generator of the quantum of action of any other, giving,

$$L = \frac{1}{2} (\beta \psi)^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} (\beta \varphi)^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(3a)

or,

$$L = \frac{1}{2} \partial_u (\beta \psi)^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} \partial_u (\beta \varphi)^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(3b)

If the fields act onize others and themselves symmetrically,

$$L = \frac{1}{2} (\beta \psi \psi)^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} \partial_u (\beta \varphi \psi)^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(4a)

or,

$$L = \frac{1}{2} \partial_u (\beta \psi \psi)^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} \partial_u (\beta \varphi \psi)^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(4b)

where the tilde over the constants in (4b) will be explained momentarily.

Since it is not known whether all fields have an equal ability to act onize another, constants $\beta_{XY}$ are introduced, where the subscript is understood to be read as "field X's ability to act onize field Y",

$$L = \frac{1}{2} (\beta_{XY})^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} \partial_u (\beta_{XY})^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(5a)

or,

$$L = \frac{1}{2} \partial_u (\beta_{XY})^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} \partial_u (\beta_{XY})^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(5b)

If the field $\psi$ is special as the act onizer of itself and all others, then one finds,

$$L = \frac{1}{2} (\beta_{\psi \psi})^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} (\beta_{\psi \psi})^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(6a)

or,

$$L = \frac{1}{2} \partial_u (\beta_{\psi \psi})^2 \partial_u \phi \partial^u \phi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{2} \partial_u (\beta_{\psi \psi})^2 \partial_u \psi \partial^u \psi - \frac{1}{2} M^2 c^2 \psi^2 - g \psi \varphi$$

(6b)

Such a distinction among the constants is not necessary in Equation (4a-b), as the prefactors are the same products of the individual fields, and require different units than the constants in the other equations, explaining the $\beta$-tilde.
So, to describe one constant $\hbar$ requires many additional constants, $\beta_{\psi\phi}$, $\beta_{\psi\hbar}$, $\beta$, $\beta$-tilde, and $z$.

Equations (2a) and (6a) will be the subject of the remainder of this paper. Yet more new constants will be generated in the solutions. All of the Lagrangian densities discussed can be shown to be Lorentz invariant, as none of them are linear in the time derivatives, and all derivatives are contracted with the Minkowski metric. The Lorentz invariance persists due to the constancy of $c$, but, suppose $c$ were a field also, one finds for example, from (2a),

$$c = \tilde{c}C$$

$$L = \frac{1}{2} (\beta_{\psi\phi})^2 \left( \left( \frac{1}{\tilde{c}C} \right)^2 (\partial_{\phi}^2 - (\nabla \phi)^2) - \frac{1}{2} m^2 (\tilde{c}C)^2 \phi^2 \right)$$

$$+ \frac{1}{2} (\beta_{\psi\hbar})^2 \left( \left( \frac{1}{\tilde{c}C} \right)^2 (\partial_{\hbar}^2 - (\nabla \hbar)^2) - \frac{1}{2} M^2 (\tilde{c}C)^2 \hbar^2 \right)$$

$$+ \frac{1}{2} (\beta_{\psi\phi})^2 \left( \left( \frac{1}{\tilde{c}C} \right)^2 (\partial_{\phi}^2 - (\nabla \phi)^2) - \frac{1}{2} \mathcal{M}^2 (\tilde{c}C)^4 - g_{\phi\phi} \right)$$

In the Equations (6c-d) it can be seen that the fields $\hbar$ and $c$ are distinct, and how the two fields support themselves and the supported field $\phi$. This paper will concern the situation when $c$ is constant.

Used throughout, for an arbitrary field $\phi$, the equations of motion and Hamiltonian density are derived using Equations (7a) to (7f), shown to clarify the compact notation, as it will affect the taking of derivatives of fields for non-constant $\hbar$.

$$\partial_u \left( \frac{\partial L}{\partial (\partial_u \phi)} \right) - \frac{\partial L}{\partial \phi} = \left( \frac{\partial}{\partial x_o} \right) \cdot \nabla_T \left( \frac{\partial L}{\partial \phi_{\xi}} + \frac{\partial L}{\partial \nabla \phi_T} \right) - \frac{\partial L}{\partial \phi} = 0$$

$$x_o = ct$$

$$\phi = \frac{\partial \phi}{\partial x_o}$$

$$\phi \partial_u \phi = \phi^2 - (\nabla \phi)^2$$

$$\pi_\phi = \frac{\partial L}{\partial \phi_{\xi}}$$

$$H = \sum_i \pi_i \phi_i - L$$

3.0 Planck’s Constant as a Self-Actionizing Field in Isolation

Consider first a much more standard form of coupling. Take Equation (1a) with $M=m=0$, describing two massless fields interacting through a coupling term that is separate from the
derivative term. Then let the two fields be the same field, that is, let \( \varphi = \psi \). Equation (1a) then becomes,

\[
L = \hbar^2 \partial_y \partial^\dagger \psi - g \psi^2
\]  

(8)

The resulting equation of motion is,

\[
\partial_i^2 \psi - c^2 \nabla^2 \psi + \frac{gc^4}{\hbar^2} \psi = 0
\]  

(9)

From Equation (8) and (9) it is seen that a fields ability to couple to itself separately from the derivative term, that is it self-interacts, produces the equivalent of a mass equal to \( m^2 = g \).

### 3.1 The Squared Field \( \chi \)

The latter observation prompts questions of what may happen when a field can self-couple through the derivative term. This will next be examined. An interpretation of the results will be given in a later section, and also discussed.

Noting Equation (6a) for \( z = 1 \), when the special field \( \psi \) is in isolation, and providing its own action, the Lagrangian density is,

\[
L = \frac{1}{2} \left( \beta_{\nu\mu} \psi_\nu \right)^2 \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} M^2 c^2 \psi^2 = \frac{1}{2} \left( \beta_{\nu\mu} \partial_\nu \psi \partial^\mu \psi \right) \psi^2 - \frac{1}{2} M^2 c^2 \psi^2
\]

\[
\frac{\partial}{\partial \chi_\nu} (\psi^2) = 2 \psi \psi_\nu
\]

\[
\nabla (\psi^2) = 2 \psi \nabla \psi
\]

\( \chi = \psi^2 \)

\[
L_\chi = \frac{1}{8} \beta_{\nu\mu} \partial_\nu \chi \partial^\mu \chi - \frac{1}{2} M^2 c^2 \chi
\]

\[
\dot{\chi} - \nabla^2 \chi + 4 \frac{M^2 c^2}{\beta_{\nu\mu}} = 0
\]

A quantity referred to as dynamical mass \( M_D \) has been defined in the underbracket of Equation (10a). Using Equations (10b-c), Equation (10a) can be written in terms of a new field \( \chi = \psi^2 \) to produce Equations (10e-f) using (7a-f). Note that the mass term of (10e) is linear in \( \chi \), and \( \chi \) is not in the mass term of (10f). As such, the term no longer represents a mass and needs to be eliminated, but also (10f) will be easier to quantize with \( M = 0 \). If \( M \neq 0 \), the profile of the static solution diverges as \( r^2 \), representative of an attractive force that increases as \( r \). For \( M = 0 \) the resulting equation of motion is linear and homogeneous, and is just the massless Klein-Gordon equation for \( \chi \). Its solution by Fourier expansion in terms of wavevector \( p \) is Equation (10k), derived as follows,
The additional factors in (10k) allow canonical quantization. \( \chi \) can be quantized as any other free scalar field, representing spinless, massless, momentum carrying particles propagating at the speed of light, with occurrences of 2 replaced by 8. In making the transition to a quantum mechanical field indexed by 3-momentum, the time dependence is removed from the fields so as to be consistent with the Schrödinger picture, the coefficients become operators, and the form of the pre-factors in the canonically conjugate variables is guided by the classical field in the taking of the derivatives. From the Hamiltonian density,

\[
H_\chi = \frac{1}{8} \beta_{\psi \psi}^2 \dot{\chi}^2 + \frac{1}{8} \beta_{\psi \psi}^2 (\nabla \chi)^2
\]  

the total energy is, using the standard commutation relationships between creation and annihilation operators,

\[
E_\chi = \frac{1}{4} \beta_{\psi \psi}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{\omega_p}{c} \left( a_p^\dagger a_p + \frac{(2\pi)^3}{2} \delta^{(3)}(0) \right)
\]  

The vacuum term has been left intentionally in (12). The result resembles that of other free fields save for the pre-factors, and that the operators have been left with dimensions.

It is now natural to ask whether \( \beta_{\psi \psi} \), the new constant, should also be represented by a field. By repeated application of the procedure, one may show there is no end to the generation of new constants, new fields, and new vacuums. One finds for the \( n^{th} \) repeated application,
\[ L = \frac{1}{2} \frac{1}{4^n} \left( \beta^{(n)}_{\Psi\Psi} \right)^2 \partial_{\alpha} \psi^{(n+1)} \partial^{\alpha} \psi^{(n+1)} \]
\[ \beta^{(n-1)}_{\Psi\Psi} = \beta^{(n)}_{\Psi\Psi} \psi^{(n)} \]
\[ \psi^{(n)} = \left( \psi^{(n-1)} \right)^2 = \left( \psi^{(1)} \right)^{2^n} \]
\[ h = \beta^{(n)}_{\Psi\Psi} \prod_{i=1}^{n} \psi^{(i)} \]

(13a-f)

\[ E^{(n+1)} = \frac{1}{4^n} \left( \beta^{(n)}_{\Psi\Psi} \right)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{a^{(n+1)}_{\Psi\Psi}}{c} \left( a^{(n+1)}_{\Psi\Psi} a^{(n+1)}_{\Psi\Psi} + \left( \frac{(2\pi)^3}{2} \sigma^{(3)}(0) \right)^{(n+1)} \right) \]
\[ E^{(\infty)} = \frac{1}{4^n} \left( \beta^{(\infty)}_{\Psi\Psi} \right)^2 \int \frac{d^3 p}{(2\pi)^3} \frac{a^{(\infty)}_{\Psi\Psi}}{c} \left( (2\pi)^3 \sigma^{(3)}(0) \right)^{(\infty)} \]

One may go infinitely deep with the process, and never get to a definitive answer of what a constant actually is, or eliminate the need of one. However, for \( n=\infty \), and for a finite number of particles and finite frequency, the term of (13e) that has a prospect of remaining non-zero is the vacuum term, written in (13f). Without a formal mathematical investigation of the infinities, one may tentatively conclude that the energy that sustains a constant is that of a vacuum.

Also, for each new field, just as quantizable as the last, there is a new vacuum. With the goal of being practical the discussion will continue as if \( n=1 \) is sufficient, which corresponds to the Equations (10a-n). Note that no matter the level \( n \), the most fundamental physical parameter remains the frequency. This was the result mentioned in the introduction that rationalizes the investigations made of a frequency-conserving modified form of the Schrödinger equation in [25].

3.2 Limit of Zero Momentum Taken Last for the Squared Field \( \chi \)

Perhaps the reason there is no propagation required for a constant to have an effect is because the field associated with it has zero momentum, that is, the occupation number of the particles is zero, leaving only the vacuum. Equation (10k) will be derived in a different way without using a Fourier transform, to highlight aspects of the order of operations when looking for solutions. From (10f) with \( M=0 \), write,
\[ \chi = S_\chi(p) \varphi_\chi(x_o) \]
\[ \frac{\ddot{\varphi}_\chi}{\varphi_\chi} = \nabla^2 S = -\frac{\omega^2}{c^2} = -\vec{p} \cdot \vec{p} = -p^2 \]
\[ \varphi_{p\chi} = A_p e^{i\frac{\omega p}{c} x_o} + B_p e^{-i\frac{\omega p}{c} x_o} \]
\[ S_{p\chi} = C_p e^{i\varphi} + D_p e^{-i\varphi} \]
\[ D_p = A_p = 0 \]
\[ \chi_p = C_p B_p e^{i\varphi} + c.c. = \tilde{a}_p e^{i\varphi} + \tilde{a}_p^* e^{-i\varphi} \]
\[ \chi \xrightarrow{\text{quantize}} \frac{1}{(2\pi)^3} \int \sqrt{\frac{c}{8\omega_p}} \left( a_p e^{i\varphi} + a_p^* e^{-i\varphi} \right) \]
\[ \lim_{p \to 0} \chi_p = \chi_o = (\tilde{a}_o + \tilde{a}_o^*) \]

The equation of motion for \( \chi \) is linear and homogeneous, allowing the sum over the solutions \( \chi_p \).

In taking the limit \( p \to 0 \) after the solution of the differential equation, it is seen that the zeroth component \( \chi_o \) is constant, so \( (\beta_\psi)^2 \chi_o \) is constant and real. The latter has units of \( \hbar^2 \), but what of the other components of \( \chi \) that do not have zero momentum still coupling to the derivative term of \( \phi \) in Equation (6a)? If it is stipulated that the occupation numbers of all states are normally zero for \( p \neq 0 \), then there is a constant \( \hbar^2 \chi_o \) that becomes the derivative prefactor in Equation (6a). The \( \psi \) derivatives become zero and \( g = M = 0 \) already. For \( \beta_\psi = \beta_\psi^0 \) Equation (6a) becomes Equation (1a) with all instances of \( \psi \) removed, and the energies (11) and (12) are those of the vacuum. Therefore, the energy of the vacuum of \( \chi = \psi^2 \) may once again be associated with the constant \( \hbar \). There is the opportunity for interesting effects to occur if the unoccupied states become occupied, but that is a much harder problem.

### 3.3 Limit of Zero Momentum Taken First for Squared Field \( \chi \) and Unsquared \( \psi \)

The latter is a handy, physical feeling solution, but is also not the only solution. Consider taking the limit \( p \to 0 \) before solving the differential equation. Mathematically, it is known the limit of the solution will not equal the solution of the limit, yet, there is no reason physically to exclude either as a solution in the interest of finding new explanations for variations of constants, which are not yet experimentally well-understood. Such spatial solutions are those of Laplace's equation, the same as the equation of the Newtonian gravitational potential in vacuum, or the electric potential in vacuum. The solutions for the zero-momentum field representing \( \hbar^2 \) are therefore the allowed classical vacuum solutions. From (14b) with first taking \( p \to 0 \),

\[ S_{\chi} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \left( A_l r^l + B_l r^{-l-1} \right) p^m \cos (\phi) \left( S_m \sin (m\theta) + C_m \cos (m\theta) \right) \]
\[ \varphi_{\chi}(x_o) = b_1 + b_2 x_o \]
where the underline means the limit $p \to 0$ is taken before solving the differential equation, and with the spatial form selected so as to be real. Of particular interest is the solution for $l=m=0$ with all other coefficients zero,

$$S_{\omega x}(\vec{r})_0 = b_1 + \frac{b_1}{r}$$  \hspace{1cm} (16)

The reason for the interest is that (16) may be always positive, and decays to a constant at $r=\infty$. This way, very far from the origin of the acton, which is to represent a physical constant, there is no asymmetry of behavior in space. The solution (15a) and (16) cannot be quantized. There is, however, now a classical energy associated with the solution (15b) and (16), despite that momentum is zero. The Hamiltonian density is from (11), (15b) and (16),

$$H_{\omega x}^0 = \frac{\beta_{\psi\psi}^2}{8} \left( b_2 + b_2 + \frac{b_2}{r^2} \right) + \left( \frac{b_2 b_2 + b_2}{r^2} \right) \left( \frac{b_2 b_2 + b_2}{r^2} \right)$$  \hspace{1cm} (17)

and Planck's constant becomes,

$$\hbar_{\omega x}^0 = \beta_{\psi\psi} \sqrt{b_1 + b_2 + \frac{b_2}{r}}$$  \hspace{1cm} (18)

It can be seen from (17) and (18) that Planck's constant, per this model, can change as a function of time and position, and, that there is an energy density associated with its existence that also changes as a function of time and position. From (15a) there are many profiles that might be explored. Equation (18) may be compared to the results of [25].

Consider now another related solution path, where the equation of motion for $\psi$ is derived from Equation (10a), and not its square $\chi$. For the equations of motion, showing the derivatives explicitly from Equations (7a) to (7f),

$$\frac{\partial L}{\partial \dot{\psi}_\xi} = \left( \beta_{\psi\psi} \psi \right)^2 \dot{\psi}_\xi = \pi_\psi$$

$$\frac{\partial L}{\partial \nabla \psi_\tau} = - \left( \beta_{\psi\psi} \psi \right)^2 \nabla \psi_\tau$$

\(19a-d\)

$$\partial_\mu \left( \frac{\partial L}{\partial (\partial_\nu \psi)} \right) = \left( \beta_{\psi\psi} \psi \right)^2 \dot{\psi} + 2 \beta_{\psi\psi} \psi^2 \dot{\psi} - \left( \beta_{\psi\psi} \psi \right)^2 \nabla^2 \psi - 2 \beta_{\psi\psi} \left( \nabla \psi \right)^2 \psi$$

$$\frac{\partial L}{\partial \psi} = \beta_{\psi\psi} \left( \psi^2 - \left( \nabla \psi \right)^2 \psi - M^2 c^2 \psi \right)$$

the resulting non-linear homogeneous equation of motion is,
\[
\begin{align*}
(\beta_{\nu\nu}\psi)\ddot{\psi} + (\beta_{\nu\nu}\psi)^2\psi^2\psi - (\beta_{\nu\nu}\psi)^2\nabla^2\psi - (\beta_{\nu\nu}\psi)^2\psi (\nabla \psi)^2 + M^2 c^2 \psi = 0 \\
(\beta_{\nu\nu}\psi)^2\ddot{\psi} - (\beta_{\nu\nu}\psi)^2\nabla^2\psi + M^2 c^2 \psi = - (\beta_{\nu\nu}\psi)^2\psi^2\psi + (\beta_{\nu\nu}\psi)^2\psi (\nabla \psi)^2
\end{align*}
\] (20a-b)

The underbrackets of Equation (20) show what would be the normal Klein-Gordon terms if \(\hbar\) were constant. Setting the time derivatives to zero for the static case, one sees the field is its own source. Equation (20) is separable into a time dependent \((x_o\) dependent) factor and spatially dependent factor if and only if \(M=0\), which will be the case examined in the interest of deriving analytical expressions. Writing \(\psi = S_{\nu}(r)\phi_{\nu}(x_o)\), the equation of motion becomes, for a specific wavenumber \(p\) or frequency \(\omega_p\),

\[\frac{\ddot{\psi}_{\nu}}{S_{\nu}} + \frac{\phi_{\nu}^2}{S_{\nu}} = \frac{\nabla^2 S_{\nu}}{S_{\nu}^2} + \left(\frac{\nabla S_{\nu}}{S_{\nu}^2}\right)^2 = - \frac{\omega_p^2}{c^2} = - p^2\] (21)

The limit \(p \to 0\) is again taken first, now stipulating a radial dependence only, (so using the radial term of the spherical Laplacian),

\[S_{\nuw} (r) = \left(b_1 + b_2 \frac{r}{r}\right)^{1/2}\] (22a-b)

\[\phi_{\nuw} (x_o) = \left(b_1 + b_2 x_o\right)^{1/2}\]

Squaring (22a-b), and comparing to (15a) and (16), one sees they are identical. Enforcing a radial dependence in \(\psi\) produces the same solution as \(l=m=0\) for \(\chi=\psi^2\). One finds,

\[\psi_{\nuw} = \sqrt{Z_{\nuw}} \rho_0\]

\[b_{\nuw} = b_{\nuw}^0\]

\[H_{\nuw} = H_{\nuw}^0\] (23a-c)

which is very reassuring. The Hamiltonian density used for (23c) was,

\[H = \frac{1}{2} (\beta_{\nu\nu}\psi)^2 \psi^2 + \frac{1}{2} (\beta_{\nu\nu}\psi)^2 (\nabla \psi)^2\] (24)

### 3.4 Solution for the Unsquared Field \(\psi\) then Squared

Now, an unusual step will be taken pertaining to Equations (20) and (21), the non-linear differential equations, where the order of solution operations also matter. The step is in keeping with finding solutions that far from the acton origin do not show a spatial asymmetry. This is accomplished by stipulating that the solutions \(\psi\) have spherical symmetry with only a radial momentum \(p_r\), so \(\psi\) has no angular dependence. The step produces localized standing wave solutions that do not decay in time. There are no resulting dot products of \(p\) with position \(r\) in the solutions, only the scalar product \(p_r \cdot r\). To clarify the difference between the radial component of
momentum, and radial momentum, the quantum mechanical operators for the two are shown, respectively,

\[
p'_r = \frac{\hbar}{i} \frac{\partial}{\partial r} \\
p_r = \frac{\hbar}{i} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right)
\]  

(25a-b)

Here, the analog of (25b) is being used.

Continuing, the solutions to the non-linear equation of motion \(\psi_p\) will then be squared to produce \(\psi_p^2\), analogous to \(\chi_p\) which one recalls is derived from a linear and homogeneous equation of motion, allowing solution summation. The \(\psi_p^2\) will then be summed to form \(\psi^2\), followed by quantization of \(\psi^2\), not \(\psi\). Proceeding in that order produces a different solution than those of \(\chi\) that are interpretable to this author.

Returning to Equation (21) for \(p=p_r \neq 0\), again, where there is only a radial dependence, one finds,

\[
S_{\omega p} (r) = d_2 \left( \frac{\cos( \sqrt{2} p r + d_1) }{ \sqrt{2} p r } \right)^{1/2} \\
\varphi_{\omega p} (x_o) = d_4 \left( \cos( \sqrt{2} p (d_3 + x_o) ) \right)^{1/2}
\]  

(26a-b)

where the subscript \(r\) on \(p_r\) has been dropped. The factor \(pr\) is simply a scalar product, not a dot product between vectors. When squared, the solutions (26a-b) are that of a localized spherical standing wave, not a traveling wave. This field on its own would not seem to represent a constant like \(\hbar\) very well, since (26a) can be either imaginary, or when squared, negative, and in either case, falling to zero at \(r=\infty\). The existence of the field (26a-b) may be a related byproduct of the existence of the component of the field that other fields do couple to. The field (26a-b) will be quantized in the next section, despite these problems.

The profile of the field \(S_{\omega p}\) from (22a) is shown in Figure 1a for various values of the constants, and the profile of \(S_{\omega p}^2\) from (26a) is shown in Figure 1b.
3.5 Canonical Quantization of the Self-Actionizing Field $\psi^2$ in Isolation

What may the $p=p_r \neq 0$ solutions, Equations (26a-b), correspond to in a quantum field theory? The endeavor will be to attempt their canonical quantization, following the unusual steps outlined in the last section. Observe that the standing wave field $\psi_p = \mathcal{S}_{\psi\psi}^2$ is complex, but $\psi_p^2$ is real, spatially oscillating and decaying with an envelope $1/r$, and also oscillating with an overall amplitude in time,

$$\psi_p^2 = (d_2d_4)^2 \left( \frac{\cos \left( \sqrt{2} pr + d_1 \right)}{\sqrt{2} pr} \right) \left( \cos \left( \sqrt{2} p (d_3 + x_o) \right) \right)$$  \hspace{1cm} (28)

Expressing (28) in exponential form, and setting $d_1$ and $d_3$ to zero,

$$\psi_p^2 = \left( \frac{1}{2} \right)^2 (d_2d_4)^2 \frac{1}{\sqrt{2} pr} \left( e^{i\sqrt{2} pr} + e^{-i\sqrt{2} pr} \right) \left( e^{-i\sqrt{2} pr} + e^{i\sqrt{2} pr} \right)$$  \hspace{1cm} (29)

In order to make the transition from (29) to an expansion, the expansion coefficients will be associated only with the time ($x_o$) components only as is normally done, and formulated so as to obey a reality condition,

$$\psi_p^2 = \left( \frac{1}{2} \right)^2 (d_2d_4)^2 \frac{1}{\sqrt{2} pr} \left( e^{i\sqrt{2} pr} + e^{-i\sqrt{2} pr} \right) \left( \tilde{a}_p e^{-i\sqrt{2} pr} + \tilde{a}^*_p e^{i\sqrt{2} pr} \right)$$  \hspace{1cm} (30)

An integration over wavenumber is performed to form the complete solution as a superposition. Note that the wavenumber integral is one dimensional, since the momentum is only radial.

$$\psi^2 = \left( \frac{1}{2} \right)^2 (d_2d_4)^2 \int dp \frac{1}{(2\pi) \sqrt{2} pr} \left( e^{i\sqrt{2} pr} + e^{-i\sqrt{2} pr} \right) \left( \tilde{a}_p e^{-i\sqrt{2} pr} + \tilde{a}^*_p e^{i\sqrt{2} pr} \right)$$  \hspace{1cm} (31)
As for the derivatives,

\[
\begin{align*}
\partial_x \psi^2 &= \left(\frac{1}{2}\right) \left( d_s \bar{d}_s \right)^2 \int \frac{dp}{(2\pi)} \left( -i \right) \frac{\sqrt{\omega_p / c}}{r} \left( e^{i\sqrt{2}pr} + e^{-i\sqrt{2}pr} \right) \left( \hat{a}_p e^{-i\sqrt{2}p_x} - \hat{a}_p^* e^{i\sqrt{2}p_x} \right) \frac{1}{2 \cos(\sqrt{2}pr)} \\
\nabla \psi^2 &= \left(\frac{1}{2}\right) \left( d_s \bar{d}_s \right)^2 \int \frac{dp}{(2\pi)} \left( i \right) \frac{\sqrt{\omega_p / c}}{r} \left( e^{i\sqrt{2}pr} - e^{-i\sqrt{2}pr} \right) \left( \hat{a}_p e^{-i\sqrt{2}p_x} + \hat{a}_p^* e^{i\sqrt{2}p_x} \right) \frac{1}{2 \sin(\sqrt{2}pr)} 
\end{align*}
\]  

(32a-b)

Collecting factors carefully, and eliminating the time dependence on quantization, the fields become,

\[
\begin{align*}
\psi^2 &= \left(\frac{1}{2}\right) \left( d_s \bar{d}_s \right)^2 \int \frac{dp}{(2\pi)} \frac{1}{\sqrt{2 \cdot 2\omega_p / cr}} \left( a_p + a_p^\dagger \right) \cos(\sqrt{2}pr) \\
\partial_x \psi^2 &= \left(\frac{1}{2}\right) \left( d_s \bar{d}_s \right)^2 \int \frac{dp}{(2\pi)} \left( -i \right) \frac{\sqrt{\omega_p / 8c}}{r} \left( a_p - a_p^\dagger \right) \cos(\sqrt{2}pr) \\
\nabla \psi^2 &= \left(\frac{1}{2}\right) \left( d_s \bar{d}_s \right)^2 \int \frac{dp}{(2\pi)} \left( -i \right) \frac{\sqrt{\omega_p / 8c}}{r} \left( a_p + a_p^\dagger \right) \sin(\sqrt{2}pr) 
\end{align*}
\]  

(33a-c)

The Hamiltonian density is,

\[
H = \frac{1}{8} \beta_{\psi \psi} \left( \partial_x \psi^2 \right)^2 + \frac{1}{8} \beta_{\psi \psi} \left( \nabla \psi^2 \right)^2 
\]  

(34)

and the total energy,

\[
E = \frac{1}{8^2} \beta_{\psi \psi} \left( d_s \bar{d}_s \right)^4 \left(\frac{1}{2}\right)^2 \int d^3r \frac{dpdk}{(2\pi)^2} \frac{\sqrt{\omega_p \omega_k}}{cr^2} \times \\
\left\{ (-1) \cos(\sqrt{2}pr) \cos(\sqrt{2}kr) \left( a_p - a_p^\dagger \right) \left( a_k - a_k^\dagger \right) \right. \\
\left. + \left( +1 \sin(\sqrt{2}pr) \sin(\sqrt{2}kr) \left( a_p + a_p^\dagger \right) \left( a_k + a_k^\dagger \right) \right) \right\} 
\]  

(35)

Multiplying out (35), rearranging terms, integrating over spatial angles, and cancelling the \( r^2 \) from the denominator stemming from the spatial integral,
The interpretation of the formally non-convergent integral is shown in the under-brackets of (36). Consider the integrands of,

\[ \int dr \left( \sin(\sqrt{2} pr) \sin(\sqrt{2} kr) \pm \cos(\sqrt{2} pr) \cos(\sqrt{2} kr) \right) \]

\[ = \pm \int \cos((k \mp p)r) dr = \pm (2\pi) \delta(k \mp p) \]  

The \( \hat{\Phi} \hat{\delta} \) state integrand is equal to \( \pm 1 \) for all \( r \) when \( k=\pm p \) where the integral formally diverges. For \( k \neq \pm p \) the integrand oscillates symmetrically above and below the abscissa, so the integral is zero. This is precisely the same behavior as in the complex exponential representation of the delta function used throughout field theory.

Continuing,

\[ E = \frac{\pi}{8^2} \beta_{\psi'}^2 \left( d_2 d_4 \right)^4 \int \frac{dp}{(2\pi)} \frac{\omega_p}{c} \left( a_p a^\dagger_p + a_p^\dagger a_p^\dagger \right) \left( a_p a_{-p} + a_{-p}^\dagger a_{-p}^\dagger \right) \]

\[ E = \frac{\pi}{32} \beta_{\psi'}^2 \left( d_2 d_4 \right)^4 \int \frac{dp}{(2\pi)} \frac{\omega_p}{c} \left( a_p^\dagger a_p + \frac{(2\pi)}{2} \delta^{(1)}(0) \right) \]  

\[ : E : = \frac{\pi}{32} \beta_{\psi'}^2 \left( d_2 d_4 \right)^4 \int \frac{dp}{(2\pi)} \frac{\omega_p}{c} a_p^\dagger a_p \]

Quantization immediately follows if the integrals are restricted to range from \( p=0 \) to \( p=+D \), where equivalently the creation and annihilation operators for states with subscripts that are negative are taken to be zero in (38a), so (38b) results. One may simply take this as a condition necessary to achieve quantization with no further explanation, however, a physical rationale will be offered.

Standing waves are formed from oppositely directed traveling waves. The classical solution (28) does not rely on the \( \hat{\Gamma} p \) solutions to combine with the \( +p \) solutions to form the standing wave \( \hat{\Gamma} \) it is a steady state spherical standing wave in and of itself already, describing the situation long after any transient behavior is over. Now visualize the early transient behavior whereby the standing wave comes about. The acton emits a traveling spherical wave from its origin that
propagates outward into space, and to develop the spherical standing wave (28), this emitted wave would have to somehow reflect off of a perfectly symmetrical spherical boundary and be directed back to the origin to interfere with the emitted wave. However, there is no physical boundary to provide reflection, so how can this occur? Impose periodic boundary conditions with allowed wavevectors incremented by \(2\pi/L\), where \(L\) is the period of the boundary taken to the limit \(L=\infty\). What happens is that each point on the traveling spherical wavefront emitted from the origin at \(r=0\) travels full circle, returning from \(r=\infty\) to the origin from the opposite side it was emitted from, and then interferes with the emitted wave. So, for the \(+p\) states, which are initially outgoing waves emitted from the origin,

\[
\frac{e^{i\sqrt{3}/2|p| r}}{2^{3/2}|p|^r_{\text{emitted, }r=0}} + \frac{e^{-i\sqrt{3}/2|p| r}}{2^{3/2}|p|^r_{\text{returning, }r=\infty}} \propto \frac{\cos(\sqrt{2}|p|r)}{\sqrt{2}|p|r} \cos(\sqrt{2}\omega_p t)
\]  

(39)

and (39) is becomes the standing spherical wave (28) for \(+p\) after taking the real part.

The \(-p\) states must then be initially inbound waves emitted from a spherical boundary at \(r=\infty\) traveling to the origin at \(r=0\), and forming the standing wave with the wave also traveling inbound from infinity that passed through the origin from the opposite side,

\[
\frac{e^{-i\sqrt{3}/2|p| r}}{2^{3/2}|p|^r_{\text{emitted, }r=\infty}} + \frac{e^{i\sqrt{3}/2|p| r}}{2^{3/2}|p|^r_{\text{pass through, }r=0}} \propto \frac{\cos(\sqrt{2}|p|r)}{\sqrt{2}|p|r} \cos(\sqrt{2}\omega_p t)
\]  

(40)

So, it can be seen that the real part of (40) is the solution (28) for \(i|p|\).

However, the case of a distant boundary of sources miraculously located for simultaneous convergence on the acton center feels as unphysical as a distant reflective boundary, even though mathematically the solution is allowed and the steady state behavior differs by only a phase of \(\pi\) between \(+p\) and \(i|p|\). So, the acton origin is thought of as the source of the activity, \(i|p|\) states are unphysical, the integrals \(dp\) range from \(p=0\) to \(\infty\), the operators with negative subscripts are zero, and the second term of (38a) is zero.

Then normal ordering eliminates the vacuum term in (38b) giving (38c).

The commutation relationships used were,

\[
\begin{align*}
\left[a_p, a^\dagger_k\right] &= (2\pi)\delta^{(i)}(p-k) \\
\left[a^\dagger_p, a^\dagger_k\right] &= \left[a_p, a_k\right] = 0
\end{align*}
\]  

(41a-b)

from which may be derived the relationship between conjugate variables,

\[
\begin{align*}
\left[y^2(r), y^2(r')\right] &= \left[\partial_{x^i} y^2(r), \partial_{x^i} y^2(r')\right] = 0 \\
\left[y^2(r), \partial_{x^i} y^2(r')\right] &= i \left(\frac{d_x d_y}{16\sqrt{2}}\right) \delta^{(i)}(r-r')
\end{align*}
\]  

(42a-b)
where (41b) is arrived at in interpreting the following integral,

\[
\int dp \frac{\cos(\sqrt{2}pr \cos(\sqrt{2}pr')}{rr'} = (2\pi)\delta^{(1)}(r-r')
\]

(43)

Equation (43) is another non-convergent integral whose integrand beats symmetrically positive and negative about the \( p \) abscissa making the integral zero unless \( r=r' \), in which case the integrand is positive and the integral diverges. Again, this is the same behavior as in the complex exponential representation of the delta function.

The final result (38c) resembles those of normal free-fields, save for the pre-factors, and that the operators are left dimensioned.

4. Calibration of Coefficients

It is conjectured here that the field that standard particle fields couple to in the derivative terms is, from (22b) or (18), the square of (44a). This is because it is the only solution for \( h \) that is positive and real at all times. From (44a) follow other relations to be used in the calibration,

\[
\frac{\partial h_{\nu\psi}(r,t)}{h_{\nu\psi}(r,t)} = \frac{1}{2} \left( \frac{c}{b_1/b_2+ct} \right) = \frac{\partial \alpha}{\alpha} = 4.73 \times 10^{-18}[\text{yr}^{-1}]
\]

(44a-d)

The value that is measured is \( h_m \) where \( t_H \) is the Hubble time. The position dependence of \( h_{\nu\psi} \) is thought provoking, see Equation (67) of reference [25]. In (44d), \( \alpha \) is the fine structure constant, and the value for its time dependence comes from references [4-6]. For \( t=t_H \), one then finds,

\[
b_1/b_2 = 10^{33}[m]
\]

(45)

From (44b-c), again putting \( t=t_H \), one finds,

\[
\beta_{\nu\psi}^2 b_2 b_3 = \frac{h_m^2}{(b_1/b_2+ct_H)} \sim 1.11 \times 10^{-10}[h^2/m]
\]

(46)

It was conjectured in Section 3.1 and 3.2 that the energy density associated with the constant \( h \) when the field has no momentum is that of the vacuum. Now that will be followed up on. From (17), very far from the action origin, equating to the gravitational cosmological constant,

\[
H_{\nu\psi}^0 (r=\infty) = \frac{\beta_{\nu\psi}^2 (b_2 b_3)^2}{8} = \Lambda = 5.63 \times 10^{-10}[J/m^3]
\]

(47)

Dividing by (46),
\[ b_2 b_3 = 3.86 \times 10^{92} \left[ J^{-1} m^{-2} s^{-2} \right] \]
\[ \beta_{\nu \nu} = 1.70 \times 10^{-97} \left[ kg^{3/2} m^{7/2} / s \right] \]
\[ b_4 b_3 = 3.86 \times 10^{125} \left[ J^{-1} m^{-1} s^{-2} \right] \] (48a-c)

From (44a),
\[
\frac{\dot{\mathbf{r}} \cdot \mathbf{h}_{\nu \nu}}{\mathbf{h}_{\nu \nu}} = \frac{1}{2} \frac{b_4}{b_3} \frac{1}{r^2} \left( \frac{1}{1 + \frac{b_4}{b_3}} \right) \ \approx \ \frac{\delta \mathbf{h}}{\mathbf{h}} \frac{1}{\Delta R_o} = -3.5 \times 10^{-15} [m^{-1}] 
\] (49a-b)

\[ b_4 / b_3 = 1.62 \times 10^8 [m] \]

The numerical value in (49) comes from the work of Hutchin [19] where it was surmised \( \hbar \) varies by 21 ppm across the Earth's orbit of radius \( R_o \sim 1.52 \times 10^{11} \) m and the difference between the maximum and minimum radii \( \Delta R_o = 6 \times 10^9 \) m. In [25] an expression was derived for the variation of \( \hbar \) as a function of position, from completely different starting assumptions, that has the same form as (44a), and single particle wavefunctions were found to be concentrated in regions of lower \( \hbar \).

From Equation (10a) where the dynamical mass was defined, and using (22a-b),
\[
-M^2 c^2 = \frac{\beta_{\nu \nu}^2}{4} \left( \frac{b_4}{b_3} \left( \frac{b_4}{x_0} \right) \frac{x_0}{r} - \frac{b_3}{r^4} \left( \frac{b_4}{x_0} \right) \right) = -M^2 c^2 - M^2 e^2 
\] (50a-b)

\[ M \cdot c^2 |_{x=\infty} = M^2 c^2 \sim i \cdot 10^{-40} [eV] \]

Only the first term of (50) can be evaluated. The \( \mathbf{\tilde{n}}^* \mathbf{\hat{\partial}} \) dynamical mass is imaginary and incredibly small. For sufficiently large \( r \) the \( \mathbf{\tilde{n}}^* \mathbf{\hat{\partial}} \) dynamical mass is zero.

5. Coupling to Other Fields

The equations of motion for the fields coupled by the dynamical terms can only be done for the simplest of cases analytically, but that shall be the goal here. From Equations (2a), (6a) and (10e), putting \( \beta = \beta_{\nu \nu} \), the Lagrangian density becomes in terms of \( \chi \) and \( \phi \) for both fields taken as massless,
\[
L = \frac{1}{2} \beta^2 \chi \partial_\nu \phi \partial^\nu \phi + \frac{1}{8} \beta^2 \partial_\nu \chi \partial^\nu \chi 
\] (51)

The equations of motion for \( \chi \) and \( \phi \) are, respectively,
\[
\frac{1}{4} (\dddot{\varphi} - \nabla^2 \chi) - \frac{1}{2} (\varphi^2 - (\nabla \varphi)^2) = 0 \\
(\dddot{\varphi} + \chi^2 \varphi) - (\nabla \chi \cdot \nabla \varphi + \chi \nabla^2 \varphi) = 0 
\] (52a-b)

The situations when the second term of (52a) is zero are the easiest to solve for. Then \(\varphi\) is close to the form of a plane wave, \(\varphi\) does not influence the form of \(\chi\), and solutions for the latter already developed may be used, namely, those of (15b) and (16). Therefore, the equation of motion (52b) can be solved with \(\chi\) as an input from its solution in isolation for the limit \(p=0\) taken first.

The following will be needed,

\[
\zeta_0 = (b_1 + b_x x_o) \left( b_2 + \frac{b_4}{r} \right)
\]

\[
\ddot{\zeta}_0^0 = b_2 \left( b_2 + \frac{b_4}{r} \right)
\]

\[
\nabla \zeta_0^0 = -b_4 \left( b_3 + b_3 x_o \right) \frac{1}{r^2}
\]

\[
\frac{\ddot{\zeta}_0^0}{\zeta_0^0} = b_2 \left( b_3 + b_3 x_o \right)
\]

\[
\nabla \zeta_0^0 = -b_4 \left( b_2 r^2 + b_2 r \right)
\]

For \(r=\mathcal{D}\), the gradient of \(\zeta\) is zero, and (52b) becomes,

\[
\frac{\dot{\chi}}{\chi} \frac{\ddot{\varphi} + \dot{\varphi}}{-\nabla^2 \varphi} = 0
\]

(54)

For very early times \(x_o=0\), and from (53d), (54) becomes, writing \(\varphi = T(t)S(r)\) and separating variables,

\[
\frac{b_2}{b_2} c \frac{\partial}{\partial t} T + \frac{\partial^2}{\partial r^2} T = -\omega_p^2 T
\]

\[
\nabla^2 S = -p^2 S
\]

\[
\varphi = (A e^{\eta \tau} + B e^{-\eta \tau}) \left( C^{-\frac{1}{2}} (\sqrt{\eta^2 + 4 \omega_p^2 - \eta^2}) + D^{-\frac{1}{2}} (\sqrt{\eta^2 + 4 \omega_p^2 - \eta^2}) \right)
\]

\[
k \sim 10^{-25} \text{ [s}^{-1}\text{]}
\]

Based on the calibration of the constants, even at a time of \(t_{H} \sim 10^{17} \text{ [s]}\), the solution is almost that of a plane wave for any reasonable value of frequency, but with a very small decay in time.
The plane wave condition for the second term of (52a) being approximately zero is therefore satisfied, and for \(kt\) so small, it is clear how to quantize (55c) for early times \(\bar{t}\) simply neglect \(kt\) and proceed with canonical quantization.

Many Hubble times from now, from (53d), (54) becomes, again separating variables,

\[
\frac{1}{t}\partial_t T + \partial_t^2 T = -\omega_p^2 T \\
\nabla^2 S = -p^2 S \\
\phi = \left(Ae^{\omega_p t} + Be^{-\omega_p t}\right)\left(CJ_o(\omega_p t) + DY_o(\omega_p t)\right)
\]

(56a-c)

where \(J_o\) and \(Y_o\) and the Bessel functions of the first and second kind, oscillating, decaying functions of time. At this extreme, Planck’s constant has become very large, and the plane wave approximation is not satisfied, so (56) actually pertains to enforcing \(\chi\) be a fixed field in isolation, with the second term of (52a) removed from the equation. The net result is that fields \(\phi\) have lifetimes \(\tau \sim 1/\omega_p\). How to quantize the classical solution (56c) is not presently clear.

If the constant \(b_2\) is zero, then the time derivative of \(\chi\) is zero, and (52b) becomes,

\[
-\frac{\nabla \chi}{\chi} \cdot \nabla \phi + \dot{\phi} - \nabla^2 \phi = 0
\]

(57)

From (53e) for limits of very large and small \(r\) is found, respectively,

\[
\frac{b_4}{r^3} \nabla \phi + \dot{\phi} - \nabla^2 \phi = 0
\]

(58a-b)

\[
\frac{\ddot{\mathbf{E}}}{r} \cdot \nabla \phi + \dot{\phi} - \nabla^2 \phi = 0
\]

In the limit of infinite \(r\) Equation (58a) reduces to the massless Klein-Gordon equation. The solution of the latter two equations and the other types of coupling involving fields related to those of Sections 3.1, 3.2, and 3.4 will be taken up in future studies.

6. Discussion

Obviously, this is a very complex problem, to describe a physical constant as a field, and actually quantize it. The attempt to do so then requires the generation of a large number of new constants to describe the one constant’s attributes, or an infinite number of fields. Then, one wonders if all the new constants (and old) are in fact fields themselves, requiring yet more new constants and fields, and the generation of new ones proceeds ad infinitum.

The mathematical machinery of perturbation theory does not seem to apply very well here. The derivative terms have been coupled, and so the perturbation is not small. The resulting equations of motion of fields coupled in this way contain additional sources, mixed field terms, and complexities beyond the simple harmonic oscillator, and so new approaches may need to be discovered, and/or numerical work undertaken.
An object called an acton has been so named to make referring to it easier. It is possible only one is necessary to exist to generate Planck's constant.

The name encompasses both its classical zero momentum state, the quantum mechanical vacuum state, and its particle excitations. Reference to the coordinate origin of the acton was made several times in the various sections. Solutions were found for behaviors when coupling between fields in locations very far from the acton origin that were not very different from normal free fields. Close to the origin, the behavior will become very different, and the mathematics harder.

An interesting classical solution was found, where the momentum of the field was *a priori* set to zero, and it will be referred to as the supporting field. The spatial part of the supporting solution decays to a constant far from the origin, and its functional form occurs completely naturally, with no ad-hoc terms to bring the form about. It has an ever increasing or decreasing energy density in time near the origin, though very far from the origin, the energy density is constant, also. The supporting field has a dynamical mass that is a complex number. The imaginary part is exceedingly small. The real part of the dynamical mass is zero far from the origin. When coupled to another field (the supported field), for very early times far from the origin, the solution for the supported field is virtually a plane wave, but with a very small decay as a function of time. At later times, many times the age of the universe, the supported field decay time is roughly the inverse of its frequency, and it becomes impossible to sustain.

The supporting field solution also resembles the position dependent Planck's constant arrived at in reference [25] using a general relativistic argument shown in (59a), to which can be compared the result from field theory of this work in (59b),

\[ \frac{h(r)}{h_\infty} = \left(1 - \beta_h \frac{R_S}{r}\right)^{1/2} \]
\[ \frac{h_{s\nu}(r,t)}{h_{s\nu}(\infty,t)} = \left(1 + \frac{b_3}{b_5} \frac{t}{r}\right)^{1/2} \]

In (59a), \( \beta_h \) is the LPI violation parameter for Planck's constant, \( R_S \) is the Schwarzschild radius. The expressions come from completely different starting points. Comparing (59a-b), one is prompted to conclude the origin of the acton coincides with the origin of a massive body. The word *coincide* was used, since the acton here is actually massless. Using the mass of the sun and (49b), \( \beta_h = -5.43 \times 10^4 \), larger than any LPI violation ever measured, so it may coincide with mass, but \( b_4/b_5 \) is not dependent on mass, and describes a variation far stronger than the redshift.

In addition to the above, there was a field solution represented by \( \chi \) that classically had a plane wave solution and was quantizable much like a free field. A massless spin zero boson may be associated with Planck's constant in this model.

Another solution was found where spherical symmetry was enforced that lead to a standing spherical wave solution that spatially decays to zero far from its origin. It could be quantized, showing energy similar in form to a free field. The traveling waves that comprise the standing wave were interpreted to circulate continually through the origin, to infinity and back around to the other side, due to imposed periodic boundary conditions. One may wonder how this is physically possible in reality, as the waves must be moving at \( c \) and returning from regions of
space that are not observable because the expansion of the universe is faster than $c$. Neither expansion or closed curvature is accounted for in the equations from which the result was derived. Perhaps this is reason to reject this solution, as an artifact of inherent periodic boundary conditions. The author is admittedly unsure.

Whether the latter two fields participate in coupling to other fields, or whether the three fields coexist are all unknowns.

The two quantizable fields have their own vacuum energy densities, and energies of their excitations. The classical field has its own energy density despite having a momentum of zero. All of these energies are associated with the physical constant $\hbar$ that they underlie, and there is a rationale for requiring energy to generate a physical constant.

7. Conclusions

The equations (2a) to (6b) number ten expressions, with five-terms each. Each is a candidate Lagrangian density that may be representative of Planck's constant. Most of the effort comprising this paper focused essentially on only the third term of Equation (2a), with a small effort on the interacting first and third terms of (2a) for what could be analytically solved. It is tentatively concluded that the existence of the acton depends on a violation of Local Position Invariance, and that the origin of the acton coincides with a massive body.

What non-relativistic, single-particle quantum mechanics becomes should be reducible from the Lagrangian density. It is important to know this because it will affect the time evolution of states and the form of the perturbative expansion. This needs to be worked out fully and consistently.

There are many more options to investigate. This a fertile area of research related to the cosmon, inflation, the inflaton, the cosmological constant, the origin of physical constants, quintessence, accelerated expansion, and "dark entities".

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Abstract: There is controversial evidence that Planck’s constant shows unexpected variations with altitude above the earth due to Kento and Mohageg, and yearly systematic changes with the orbit of the earth about the sun due to Hutchin. Many others have postulated that the fundamental constants of nature are not constant either locally or universally. This work is a mathematical study, examining the impact of a position dependent Planck’s constant in the Schrödinger equation. With no modifications to the equation, the Hamiltonian becomes a non-Hermitian radial frequency operator. The frequency operator does not conserve normalization, time evolution is no longer unitary, and frequency eigenvalues can be complex. The wavefunction must continually be normalized at each time in order that operators commuting with the frequency operator produce constants of the motion. To eliminate these problems, the frequency operator is replaced with a symmetrizing anti-commutator so that it is once again Hermitian. It is found that particles statistically avoid regions of higher Planck’s constant in the absence of an external potential. Frequency is conserved, and the total frequency equals kinetic frequency plus potential frequency. No straightforward connection to classical mechanics is found, that is, the Ehrenfest’s theorems are more complicated, and the usual quantities related by them can be complex or imaginary. Energy is conserved only locally with small gradients in Planck’s constant. Two Lagrangian densities are investigated to determine whether they result in a classical field equation of motion resembling the frequency-conserving Schrödinger equation. The first Lagrangian is the ’energy squared’ form, the second is a ’frequency squared’ form. Neither reproduces the target equation, and it is concluded that the frequency-conserving Schrödinger equation may defy deduction from field theory. An expression for the positional dependence of Planck’s constant is derived from considerations of LPI violations in GR in this paper that matches in functional form that of one derived in another paper from field theory by this author.

Key Words: Planck’s constant, Variable Planck’s constant, non-Hermitian operators, Schrödinger Equation

1. Introduction

The possibility of the variation of fundamental constants would impact all present physical theory, while all reported variations or interpretations of data concluding a constant has varied are extremely controversial. Examples of work in this area include Dirac’s Large Number Hypotheses [1], the Oklo mine from which could be extracted a variation of the fine structure constant [2,3], and the observations of quasars bounding the variation of the latter per year to one part in $10^{17}$ [4-6]. Recent theoretical work includes the impact of time dependent stochastic fluctuations of Planck’s constant [7], and the changes with Planck’s constant on mixed quantum
states [8]. An authoritative review of the status of the variations of fundamental constants is
given in [9].

Publicly available Global Positioning System (GPS) data was used to attempt to confirm the
Local Position Invariance (LPI) of Planck's constant under General Relativity [10-11]. LPI is a
concept from General Relativity, where all local non-gravitational experimental results in freely
falling reference frames should be independent of the location that the experiment is performed
in. That foundational rule should hold when the fundamental physical constants are not
dependent on the location. If the fundamental constants vary universally, but their changes are
only small locally, then it is the form of the physical laws that should be the same in all
locations.

The LPI violation parameter due to variations in Planck's constant is called $\beta_h$. The fractional
variation of Planck's constant is proportional to the gravitational potential difference and $\beta_h$. The
value found in [10] for variations in Planck's constant was $|\beta_h| < 0.007$. This parameter is not zero,
and is the largest of the violation parameters extracted in the study. The study did not report on
the altitude dependence of Planck's constant above the earth. A very recent study involving the
Galileo satellites found that GR could explain the frequency shift of the onboard hydrogen maser
clocks to within a factor of $(4.5 \pm 3.1) \times 10^{-5}$ [12], improved over Gravity Probe A in 1976 of ~
$1.4 \times 10^{-4}$, these are the $\alpha_{rs}$ redshift violation values that may be compared to $\beta_h$.

Consistent sinusoidal oscillations in the decay rate of a number of radioactive elements with
periods of one year taken over a 20 year span has been reported [13-18]. These measurements
were taken by six organizations on three continents. As both the strong and weak forces were
involved in the decay processes, and might be explainable by oscillations of $\hbar$ influencing the
probability of tunneling, an all electromagnetic experiment was conducted, designed specifically
to be sensitive to Planck's constant variations [19]. Consistent systematic sinusoidal oscillations
of the tunneling voltage of Esaki diodes with periods of one year were monitored for 941 days.
The tunnel diode oscillations were attributed to the combined effect of changes in the WKB
tunneling exponent going as $\hbar^{-1}$, and changes in the width of the barrier going as $\hbar^2$. The
electromagnetic experiment voltage oscillations were correctly predicted to be 180 degrees out of
phase with the radioactive decay oscillations. This data can be made available for independent
analysis by requesting it from the author of [19].

It is reasonable to suspect that the oscillations of decay rates and tunnel diode voltage are related
to the relative position of the sun to the orbiting earth, and that there are resulting oscillations in
Planck's constant due to position dependent gravitational effects, or effects with proximity to the
sun. It should be mentioned that there have been studies in which it was concluded there was no
gravitational dependence to the decay rate oscillations [20-21]. There is also dispute in the
literature concerning the reality of the decay rate oscillations [22-24].

Either way, whether by gravitation or by some other mechanism, for the work to be presented, all
that matters is that there be a position dependent $\hbar$, and it would be of value to understand the
impact on the fundamentals of quantum mechanics and the Schrödinger equation under such a
condition, and where conservation of frequency as opposed to energy will be explored as a
means to retain Hermitivity.
For the treatment of $\hbar$ in this paper, it is important to emphasize is not as a dynamical field, and leads to energy non-conservation. In another paper by this author, variations in $\hbar$ are treated as a scalar dynamical field, coupling to fields through the derivative terms in the Lagrangian density [25], and the energy is shared between the fields. One of the solutions of [25] suggests that frequency may be a more fundamental dynamical variable than energy, leading to the idea of frequency conservation in this paper, where it arises quite naturally. This paper concerns issues specific to the Schrödinger equation in a single-particle, non-field theoretic framework, however. In the Appendix of this paper, an attempt will be made to derive a classical field equation of motion (the Schrödinger field) resembling the frequency conserving Schrödinger wavefunction equation developed in the body of the paper, from two Lagrangian densities. The attempt will not be successful.

Variations in $\hbar$ or any fundamental constant may be explainable by treatment as dynamical fields, but, they may not be, especially where the spatial dependence is concerned because there is so little experimental data on the subject. Noone presently knows whether they actually are dynamical fields or not, though much work has been done representing some of them as dynamical fields: Jordan-Brans-Dicke scalar-tensor theory with variable $G$ developed in the late 1950s and early 1960s and note that $G$ is dimensionful; Bekenstein models with variable fine structure constant introduced in 1982 [26-27]; the Cosmon of Wetterich with a field dependent pre-factor to the dynamical terms functioning somewhat like Planck's constant [28-29], falling to a constant value at high fields; the investigations of Albrecht, Magueijo, Moffat, and Barrow on variable $c$ used towards the explanation of the flatness, horizon, homogeneity, and cosmological constant problems [30-31, 39-40]. For example,

\begin{equation}
S_{mn} = \left[ \frac{(c_o \mathcal{C})^4}{16G_o \pi} \xi R + \frac{(\hbar \omega)^2}{2} g^{\mu\nu} \nabla_{\mu} \psi \nabla_{\nu} \psi + \frac{\hbar \omega^2}{2} \xi g^{\mu\nu} \nabla_{\mu} \xi \nabla_{\nu} \xi + \frac{(\hbar \omega)^2}{2} g^{\mu\nu} \nabla_{\mu} \mathcal{C} \nabla_{\nu} \mathcal{C} \right] \sqrt{-g} d^4 x + \lambda \xi \psi \mathcal{C} + L_m \{h, c, G\}
\end{equation}

\begin{equation}
\frac{\partial}{\partial x_o} = \frac{1}{c_o \mathcal{C}} \frac{\partial}{\partial t}
\end{equation}

Equation (I1a-e) shows in a single form an amalgam of possible couplings including a Jordan-Brans-Dicke-like scalar-tensor theory of alternative General Relativity with variable $G$, an Albrecht-Magueijo-Barrow-Moffat-like field for $c$, a field for $\hbar$ like that of [26], which is different than the form of Bekenstein's for variable $e^2$ whose representative field squared divided the derivative terms. There is also the field theory of Modified Gravity (MOG) of Moffat, and
the Tensor-Vector-Scalar (TeVeS) gravity of Bekenstein. There are many ways all the constants might be represented as fields, and many ways they might be coupled. Coupling fields together in this way is the accepted approach for the treatment of a constant, but is not the only possible approach, and here, something different will be tried.

What is to follow serves as a starting point for investigating what happens to the most familiar equations in physics, if Planck's constant variations are that of a fixed-background parameter and not a field, and so there is no energy exchange between fields conserving the total. Instead, frequency conservation is explored, and energy is intentionally not conserved. In [38] it will be shown that energy non-conservation leads to a possible explanation of the NASA Flyby Anomaly.

2. Derivation of the Expectation Value Time Derivative

The time derivative of expectation values for a position dependent Planck's constant will be derived. No modification will be made to the form of the Schrödinger equation in this section, and the purpose is to make clear the difficulties that arise, and the special conditions that would have to be imposed on the wavefunction and Planck's constant to maintain the basic framework of quantum mechanics. After, a modification will be suggested.

Begin with the time-dependent Schrödinger equation in which Planck's constant is allowed to be position dependent, and real,

\[ i\hbar(\bar{r}) \frac{\partial \psi_u(\bar{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi_u(\bar{r}, t) + V(\bar{r}) \psi_u(\bar{r}, t) \]  

(1)

The subscript \( u \) indicates that the wavefunctions are not normalized over space at any given time. To separate the time and position variables, divide both sides by \( \hbar \),

\[ i \frac{\partial \psi_u(\bar{r}, t)}{\partial t} = -\frac{\hbar}{2m} \nabla^2 \psi_u(\bar{r}, t) + \frac{V(\bar{r})}{\hbar(\bar{r})} \psi_u(\bar{r}, t) \]  

(2)

Let,

\[ \psi_u(\bar{r}, t) = S_u(\bar{r}) \phi(t) \]  

(3)

Substituting (3) into (2) and dividing both sides by (3) gives,

\[ \frac{i}{\phi(t)} \frac{\partial \phi(t)}{\partial t} = -\frac{\hbar}{2m S_u(\bar{r})} \nabla^2 S_u(\bar{r}) + \frac{V(\bar{r})}{\hbar(\bar{r})} = \omega \]  

(4)

where \( \omega \) is the constant of separation with units of frequency. The left-hand side of (4) has the solution,
\[ \phi(t) = e^{-iat} \]  

and the right-hand side of (4) becomes,

\[ -\frac{\hbar(\vec{r})}{2m} \nabla^2 S_u(\vec{r}) + \frac{V(\vec{r})}{\hbar(\vec{r})} S_u(\vec{r}) = aS_u(\vec{r}) \]  

(6)

Defining the frequency operator \( F \),

\[ \tilde{E} = -\frac{\hbar(\vec{r})}{2m} \nabla^2 + \frac{V(\vec{r})}{\hbar(\vec{r})} \]  

(7)

Switching to the Dirac notation, Equation (2) becomes,

\[ i \frac{\partial \psi_u(\vec{r}, t)}{\partial t} = \tilde{E} \psi_u(\vec{r}, t) \rightarrow \]  

(8)

Taking the complex conjugate of (8),

\[ -i \frac{\partial \psi_u^*(\vec{r}, t)}{\partial t} = (\tilde{E} \psi_u(\vec{r}, t))^* = \tilde{E} \psi_u^*(\vec{r}, t) = \tilde{E} \psi_u^*(\vec{r}, t) \rightarrow \]  

(9)

where the superscript \( \hat{\cdot} \) designates the adjoint operator acting to the right. The frequency operator is not Hermitian, noted from writing out in integral form the problematic part,

\[ \int (\hbar(\vec{r}) \nabla^2 \psi)^* \psi d^3r = \int \nabla^2 \psi^* \psi \hbar(\vec{r}) d^3r = \int \psi^* \nabla^2 (\psi \hbar(\vec{r})) d^3r \]  

(10)

where the lower \( \hat{\cdot} \) indicates where the operator stops operating. The Hermiticity of the Laplacian has been used in (10), derivable by the use of Green's second identity in the second to the third step, as long as products of \( \hbar \psi \) and \( \psi \) vanish at the boundary at infinity. The fourth step is what the answer would need to be in order to be Hermitian. Therefore, the frequency operator is non-Hermitian,
†

As a result, the normalization will not be conserved, and the frequency eigenvalues may be complex or imaginary. The rate of change of expectation values can now be derived using (8) and (9). The expectation value of an operator is,

\[
\langle \hat{\mathcal{F}} \rangle = \frac{\langle \psi_u (\bar{\mathcal{F}}, t) | \hat{\mathcal{F}} | \psi_u (\bar{\mathcal{F}}, t) \rangle}{\langle \psi_u (\bar{\mathcal{F}}, t) | \psi_u (\bar{\mathcal{F}}, t) \rangle} = \frac{\langle \hat{\mathcal{F}} \rangle_u}{\langle 1 \rangle_u}
\]

(12)

where the denominator is the normalization, and normalization is redone continually for all times. Differentiating (12) with respect to time,

\[
\frac{d}{dt} \frac{\langle \hat{\mathcal{F}} \rangle}{\langle 1 \rangle} = \frac{\langle 1 \rangle_u \partial_t \langle \hat{\mathcal{F}} \rangle_u - \partial_t \langle 1 \rangle_u \langle \hat{\mathcal{F}} \rangle_u}{\langle 1 \rangle_u^2}
\]

(13)

Working out the numerator of (13) and then using (8) and (9),

\[
\partial_t \langle \hat{\mathcal{F}} \rangle_u = \partial_t \langle \psi_u (\bar{\mathcal{F}}, t) | \hat{\mathcal{F}} | \psi_u (\bar{\mathcal{F}}, t) \rangle + \langle \psi_u (\bar{\mathcal{F}}, t) | \hat{\mathcal{F}} | \psi_u (\bar{\mathcal{F}}, t) \rangle + \langle \partial_t \psi_u (\bar{\mathcal{F}}, t) | \hat{\mathcal{F}} | \psi_u (\bar{\mathcal{F}}, t) \rangle
\]

(14)

Therefore, from (14), the rate of change of the normalization is,

\[
\partial_t \langle 1 \rangle_u = \frac{1}{i} \langle \psi_u (\bar{\mathcal{F}}, t) | \hat{\mathcal{F}} | \psi_u (\bar{\mathcal{F}}, t) \rangle + \langle \partial_t \psi_u (\bar{\mathcal{F}}, t) | \hat{\mathcal{F}} | \psi_u (\bar{\mathcal{F}}, t) \rangle
\]

(15)

Substituting (14) and (15) into (13) would give the full time dependence of the operator \( A \), but this can be written in a cleaner way showing the extra terms that do not show up in normal quantum mechanics. To that end, remembering that \( F \) is real,

\[
\hat{\mathcal{F}} = \mathcal{F}
\]

(16)

so (14) in integral form is,

\[
\partial_t \langle \hat{\mathcal{F}} \rangle_u = -\frac{1}{i} \int \bar{\mathcal{F}} \psi_u^* \cdot \hat{\mathcal{F}} \psi_u - \psi_u^* \cdot \hat{\mathcal{F}} \psi_u \, d^3r + \langle \partial_t \hat{\mathcal{F}} \rangle_u
\]

(17)

Writing out the first term of (17), and of that, only the part containing the non-Hermitian portion of the frequency operator,
\[
\int h(\vec{r}) \nabla^2 \psi^*_u \vec{F} \psi_u d^3r = \int \nabla^2 \psi^*_u h(\vec{r}) \vec{F} \psi_u d^3r = \int \psi^*_u \nabla^2 \left( h(\vec{r}) \vec{F} \psi_u \right) d^3r
\]  
\tag{18}
\]

where on going from the second to the third part in (18), Green's second identity was used again with \( \psi \) and \( hA\psi \) vanishing at the boundary at infinity. Note,

\[
\nabla^2 \left( h(\vec{r}) \vec{F} \psi_u \right) = h(\vec{r}) \nabla^2 (\vec{F} \psi_u) + (\vec{F} \psi_u) \cdot \nabla^2 h(\vec{r}) + 2 \nabla h(\vec{r}) \cdot \nabla (\vec{F} \psi_u)
\]
\tag{19}
\]

where the large dot between the gradients is the vector dot product. Equation (19) allows (17) to be written as a commutation relationship with extra terms. Defining the functional,

\[
I[\vec{F}] = -\frac{i}{2m} \int \psi^*_u \left\{ \vec{F} \psi_u \cdot \nabla^2 h(\vec{r}) + 2 \nabla h(\vec{r}) \cdot \nabla (\vec{F} \psi_u) \right\} d^3r
\]
\tag{20}
\]

\[
\partial_t \langle \vec{F} \rangle_u = \langle i[\vec{F}, \vec{F}] + \partial_j \vec{F} \rangle_u + I[\vec{F}]
\]
\tag{21}
\]

and from (20) follows the time dependence of the normalization,

\[
\partial_t \langle 1 \rangle_u = I[1] = -\frac{i}{2m} \int \psi^*_u \left\{ \psi_u \cdot \nabla^2 h(\vec{r}) + 2 \nabla h(\vec{r}) \cdot \nabla (\psi_u) \right\} d^3r
\]
\tag{22}
\]

Combining (20), (21) and (13), the result is,

\[
\frac{d\langle \vec{F} \rangle}{dt} = \frac{\langle i[\vec{F}, \vec{F}] + \partial_j \vec{F} \rangle_u + \langle 1 \rangle_u I[\vec{F}] - \langle \vec{F} \rangle_u I[1]}{\langle 1 \rangle_u^2}
\]
\tag{23}
\]

The second term of (23) appears because \( F \) is not Hermitian, and were it not there, (23) would look like the result of normal quantum mechanics.

3. Time Evolution Operator under \( F \)

Time evolution is no longer unitary. From (5) it is inferred that the time evolution operator is,

\[
\vec{E} = \exp(-i\vec{F}t)
\]
\tag{24}
\]

and its adjoint is

\[
\vec{E}^\dagger = \exp(i\vec{E}^\dagger t)
\]
\tag{25}
\]

Therefore,
\( \langle \hat{F}_u \rangle = \langle \psi_u(\bar{r},t) | \hat{F} | \psi_u(\bar{r},t) \rangle = \langle \psi_u(\bar{r},0) | t_{\hat{E}} U \hat{E} | \psi_u(\bar{r},0) \rangle \) \tag{26} \\

and for the normalization,

\( \langle 1 \rangle_u = \langle \psi_u(\bar{r},t) | \psi_u(\bar{r},t) \rangle = \langle \psi_u(\bar{r},0) | t_{\hat{E}} U \hat{E} | \psi_u(\bar{r},0) \rangle \) \tag{27} \\

Since \( F \neq F^\dagger \), it is seen that \( U^\dagger \neq U^{-1} \), the normalization is not conserved noting (27), and from (26) for the non-normalized wavefunctions, the expectation values of \( A \) are not constants of the motion even if \( A \) commutes with \( F \) (and therefore \( U \)).

4. Result for Expectation Values of Operators Commuting with the Frequency Operator \( F \)

If \( A \) commutes with \( F \) then from (20),

\( J[\hat{F}] = a J[1] \) \tag{28} \\

and from (26) and (27),

\( \langle \hat{A}_u \rangle = a \langle 1 \rangle_u \) \tag{29} \\

and substitution of (28) and (29) into (23) gives that the expectation value time derivative of the operator \( A \) is zero. For the non-Hermitian \( F \) operator, this result only holds because of the continual normalization procedure at each time.

5. Symmetrized Hermitian Frequency Operator \( F_h \) and modified Schrödinger Equation

The basic framework of quantum mechanics is disturbed without modification to the Schrödinger equation for a position dependent \( \hbar \), or imposing special conditions of some sort. Inspecting (20) and (22), one might consider special conditions on the forms of \( \hbar \) or \( \psi \) so the additional terms are zero, and the operator becomes effectively Hermitian. It is worth mentioning there is ongoing work on non-Hermitian and complex Hamiltonians being used to describe dissipative and open systems \([32-33]\). There is also work on complex non-Hermitian Hamiltonians with \( PT \)-symmetry that produce real eigenvalues \([34-35]\).

Looking at (14), unusual symmetries or operators such that \( AF = F^\dagger A \) might also be tried. It was shown in \([36]\) that such a symmetry results in an expectation value that changes with time in inverse proportion to the wavefunction normalization, while the latter is not conserved noting (15).

Instead, to rectify the problems thus far mentioned, without exotic conditions or symmetries, to retain the property that a dynamical variable is a constant of the motion when its operator
commutes with the frequency operator, and that normalization be conserved so the wavefunction has a probabilistic interpretation, a modified symmetrical form of $F$ is proposed.

For Hermitian operators $P$ and $Q$ the product operator $PQ$ is not Hermitian unless they commute. However, two symmetrized operators that are Hermitian, are,

$$i[\hat{F}, \hat{G}]$$

$$\hat{F}\hat{G} + \hat{G}\hat{F} = \{\hat{F}, \hat{G}\}$$

Of the two candidates for symmetrizing the non-Hermitian product of Hermitian operators $\hbar(r)$ and $\nabla^2$,

$$\frac{\hbar(\vec{r})\nabla^2 + \nabla^2\hbar(\vec{r})}{2} = \frac{1}{2}\{\hbar(\vec{r}), \nabla^2\}$$

is the one reducing to the standard Schrödinger equation for constant $\hbar$. Therefore, the symmetrized equation proposed is,

$$\frac{i}{\hbar} \frac{\partial \psi_u(\vec{r}, t)}{\partial t} = -\frac{1}{2m} \left\{ \frac{\hbar(\vec{r}), \nabla^2}{2} \right\} \psi_u(\vec{r}, t) + \frac{V(\vec{r})}{\hbar(\vec{r})} \psi_u(\vec{r}, t) = \hat{E}_h \psi_u(\vec{r}, t)$$

$$\hat{E}_h = -\frac{1}{2m} \left\{ \frac{\hbar(\vec{r}), \nabla^2}{2} \right\} + \frac{V(\vec{r})}{\hbar(\vec{r})}$$

The time dependence of the wavefunction is still given by (5), and the spatial component on separation becomes,

$$\hat{E}_h S_u(\vec{r}) = \alpha S_u(\vec{r})$$

The general principles and framework of quantum mechanics is then restored, with the difference being the Hamiltonian is replaced with the symmetrized frequency operator. The previously problematic relations become much more like normal quantum mechanics, namely

$$\frac{d}{dt} \langle \hat{F} \rangle_u = \langle i[\hat{E}_h, \hat{F}] \rangle_u + \langle \partial_i \hat{F} \rangle_u$$

$$\partial_i \langle 1 \rangle_u = 0$$

$$\frac{d}{dt} \langle \hat{E} \rangle_u = \frac{\partial_i \langle \hat{F} \rangle_u}{\langle 1 \rangle_u}$$
\[ i\hat{E}_h = \exp(-i\hat{E}_h t) \]  

(39) \[ i\hat{F}_h = \exp(i\hat{F}_h t) \]  

(40)

Since \( F_h = F_h^\dagger \), it is seen that \( U_h^\dagger = U_h^{-1} \), time evolution is unitary, and the normalization is now again conserved,

\[ \langle 1 \rangle_u = \langle \psi_u(\bar{r}, t) | \psi_u(\bar{r}, t) \rangle = \langle \psi_u(\bar{r}, 0) | (\hat{F}_h^\dagger \hat{E}_h) | \psi_u(\bar{r}, 0) \rangle = \langle \psi_u(\bar{r}, 0) | \psi_u(\bar{r}, 0) \rangle \]  

(41)

6. Free Particles under \( F_h \)

Since,

\[ -\frac{1}{2m} \frac{1}{2} \{ \hbar(\bar{r}), \nabla^2 \} S(\bar{r}) = \hat{W}_h S(\bar{r}) = \omega S(\bar{r}) \]  

(42)

the spatial part of the free particle wavefunction depends explicitly on the attributes of Planck’s constant. The free particle frequency operator \( W_h \) is introduced. The wavefunction time dependence is still given by (5), however, the spatial wavefunction of a free particle is not of the usual form,

\[ \exp(i\vec{k} \cdot \bar{r}) \]  

(43)

A simple but illustrative case will demonstrate the interesting feature that the particle tends to statistically avoid regions of higher \( \hbar \). Consider a slight linear gradient in \( \hbar \). In one dimension, the free particle wave equation with \( V=0 \) becomes,

\[ -\frac{1}{2m} \left[ \hbar \hat{\partial}_x^2 + \frac{1}{2} (\hat{\partial}_x^2 \hbar) + (\partial_x \hbar) \hat{\partial}_x \right] S(x) = \omega S(x) \]  

(44)

where the parentheses \( \hat{\partial}() \) indicate that the enclosed derivative operations stop on \( \hbar \) and do not operate on \( S(x) \). For the simplest position dependent Planck’s constant,

\[ \hbar(x) = \hbar_o + \eta x \]  

(45)

it is found that,

\[ (\hbar_o + \eta x) \hat{\partial}_x^2 S + \eta \hat{\partial}_x S + 2m\omega S = 0 \]  

(46)

The interest is in solutions for \( \eta > 0 \), and for simplicity in regions where \( \eta x / \hbar_o << 1 \), so the \( \eta x \) in the first term of (46) can be dropped. An oscillating solution will be investigated. The result is a second order homogeneous differential equation with solution,
\begin{equation}
S(x) = \exp\left(\frac{-\eta x}{2\hbar_o}\right)(c_1 \exp(ikx) + c_2 \exp(-ikx))
\end{equation}

(47)

where,

\begin{equation}
k = \frac{\sqrt{\eta^2 - 8m\omega\hbar_o}}{2\hbar_o}
\end{equation}

(48)

and \(\eta^2 < 8m\omega\hbar_o\), where the exponential terms can sum to \(\cos(kx)\) or \(\sin(kx)\) depending on the boundary conditions, resulting in quantization of frequency in the usual way, by restriction of the allowed values of \(k\).

One sees from (47) that for very small gradients in \(\hbar\) the normal free particle solution \(\exp(ikx)\) is approximated. The wavefunction is concentrated near the region of smaller \(\hbar\). A well-defined wavenumber appears, but only as a consequence of the small gradient in \(\hbar\). Even though there is no external potential, the particle is not \(\text{free}\) in the usual sense, since the gradient in \(\hbar\) plays a role in positioning it. If the particle energy can still be defined as \(E = \hbar\omega\), the particle is most likely to be found in regions where its energy is lowest.

The full general solution, retaining the \(\eta x\) so that the changes in \(\hbar\) can become larger is,

\begin{equation}
S(x) = c_1 \left[ \frac{2}{\sqrt{\eta}} I_o \left( i \sqrt{\frac{8m\omega(\hbar_o + \eta x)}{\eta^2}} \right) + c_2 \sqrt{\frac{2}{\eta}} K_o \left( i \sqrt{\frac{8m\omega(\hbar_o + \eta x)}{\eta^2}} \right) \right]
\end{equation}

(49)

where \(I_o\) and \(K_o\) are the modified Bessel functions of the first and second kind, oscillating functions with a decay envelope. The first term of (49) is the relevant one, as it has no divergences. Noting the square root in the argument containing \(x\), there is not a clearly definable constant wavenumber despite that the particle is \(\text{free}\). Using \(I_o(iz^{1/2}) = J_o(z^{1/2})\) is found the Bessel function of the first kind. For a particle in a box, the infinite sidewall positions must be located such that \(L_{1,2} \geq -\hbar_o /\eta\), so that \(\hbar\) is positive. The wavefunctions are then concentrated on the low Planck\(\hbar\) constant side of the box, decaying to the right of the leftmost sidewall. For quantization, the relation between the frequency and the two of the zeroes of the Bessel function \(Z[J_o]\) is,

\begin{equation}
\omega_o = \frac{\eta^2 Z_o^2 [J_o]^{1,2}}{8m(\hbar_o + \eta L_{1,2})}
\end{equation}

(50)

which must be solved numerically. The overall form of (49) is shown in Figure 1.
Figure 1. Plot of the overall form Equation (49), demonstrating that the wavefunction amplitude increases when Planck's constant is lower. Planck's constant increases with increasing x position.

7. Lack of Conservation of Energy and Ehrenfest’s Theorems under $F_h$

Using (36), (38) and writing $V/\hbar = F_h - W_h$ one sees that,

$$
\frac{d\langle V(\bar{r})/\hbar(\bar{r}) \rangle}{dt} = \langle i\hbar \tilde{\mathbf{E}}_h, V/\hbar \rangle = -\frac{d\langle \tilde{\mathbf{F}}_h(\bar{r}) \rangle}{dt} \tag{51a}
$$

So that $V/\hbar$ is $\tilde{\mathbf{F}}_h$ potential frequency and $W_h$ is $\tilde{\mathbf{F}}_h$ kinetic frequency acting together to conserve total frequency as the particle moves. Energy is not conserved now, and in addition, even if the particle is free, the wavenumber is also not conserved, both changing value with position in the absence of an external potential. Frequency, however, is conserved. Changes in $V/\hbar$ from a starting to an ending position is the frequency equivalent of work done on or by the system.

On examining the free particle operator $W_h$, this author is unable to identify a simple operator for momentum or wavenumber. Perhaps,

$$
\tilde{\mathbf{E}}_h = \frac{1}{i} \sqrt{\frac{1}{2} \{h(\bar{r}), \nabla^2 \}} = -\frac{\tilde{\mathbf{F}}}{\sqrt{h(\bar{r})}} \tag{51b}
$$

although the square root operator is difficult to work with, it could be definable in terms of Fourier transforms. Lacking an operator for the wavenumber or momentum means there is no relation equivalent to Newton’s first and second laws between expectation values as there is in normal quantum mechanics with Ehrenfest’s theorems. An attempt at a connection with normal quantum mechanics is made by borrowing its momentum operator, but now with a position dependent $\hbar$. 
\[ \tilde{\mathbf{E}}_x = \frac{\hat{\mathbf{h}}(x)}{i} \partial_x \]  

(52)

from which can be defined a wavenumber operator,

\[ \tilde{\mathbf{k}}_x = \frac{1}{i} \partial_x \]  

(53)

An infinitesimal displacement operator can be defined as,

\[ D_x = 1 + i \partial \tilde{\mathbf{E}}_x \]  

(54)

By inspection, the free particle operator \( W_h \) is not generally invariant to the infinitesimal displacements owing to \( \hat{\mathbf{h}}(x) \), therefore,

\[ [\hat{W}_h, \tilde{\mathbf{E}}_x] = i\epsilon [\hat{W}_h, \tilde{\mathbf{k}}_x] \neq 0 \]  

(55)

so neither momentum or wavenumber is conserved by the definitions of normal quantum mechanics by this symmetry argument for a free particle.

Also,

\[ \frac{d}{dt} \langle \hat{\mathbf{E}}_x \rangle = \langle i [\hat{W}_h, \tilde{\mathbf{E}}_x] \rangle = \frac{\langle \tilde{\mathbf{E}}_x \rangle}{m} + \frac{\langle (\partial_x \hat{\mathbf{h}}) \rangle}{2mi} \]  

(56)

While (56) looks simple enough, the first term is complex, and the second term is always imaginary. It has not been shown whether the imaginary parts of (56) generally exactly cancel for any arbitrary choice of \( \hat{\mathbf{h}} \). For the free particle of (47) with its very mild \( \hat{\mathbf{h}} \) gradient all is per the norm, as the imaginary terms that result in (56) do exactly cancel and the right side equals \( \hbar \omega/k/m \). For the particle in a box with a slight \( \hat{\mathbf{h}} \) gradient of (49) and full solution, it has not been shown that all eigenstates lead to a real result for (56).

For forces,

\[ \frac{d}{dt} \langle \hat{\mathbf{E}}_x \rangle = \langle i [\hat{W}_h, \tilde{\mathbf{E}}_x] \rangle = \langle \hat{W}_h \tilde{\mathbf{E}}_x \rangle - \langle h(\partial_x (V / \hat{\mathbf{h}})) \rangle = \frac{d}{dt} \langle \tilde{\mathbf{E}}_x \rangle \bigg|_{\text{free}} - \langle h(\partial_x (V / \hat{\mathbf{h}})) \rangle \]  

(57)

Equations (56) and (57) reduce to the normal Ehrenfest\( \hat{\mathbf{h}} \) theorems for constant \( \hat{\mathbf{h}} \), but do not appear like them, otherwise.

So, while particle frequencies are conserved, and local energies, probabilities of particle location, and average values of quantities can all be computed and are real, there seems to be no assured connection with classical dynamics. The position expectation value time derivative being
complex or imaginary is difficult to interpret. Consider an analogy in classical mechanics, where a particle sits at the bottom of the harmonic oscillator potential with zero energy and velocity. Integrating the equations of motion, one finds for the velocity \( v = \left( -\frac{kx^2}{m} \right)^{1/2} \). If the particle is then suddenly found at any position other than \( x = 0 \) with no source of energy, the particle velocity is imaginary, and the magnitude of the imaginary velocity tells you the extent of the energy non-conservation.

This model may be producing complex position expectation value time derivatives, generally. With a conserved frequency and a position dependent \( \hbar \), this suggests \( \hbar \) is a minimum at some position in space that serves as the reference of lowest energy, meanwhile the particle wavefunctions may extend to locations where \( \hbar \) and energy are larger. Then the particle has a finite probability to be observed in both high and low energy locations. Complex values of (56) and (57) tell you that the particle is forbidden to be there in classical mechanics and normal quantum mechanics, but is there anyway.

The lack of conservation of energy, while something that is difficult to accept based on the heritage of its use as a guiding law, is not yet a reason to abandon a model, as the uncertainty principle, virtual mediating particles, and the cosmological constant attest.

8. Average Value of \( \hbar \) under \( F_h \)

\[
\langle \hbar \rangle = \langle S(\mathbf{r}) \mid \hbar(\mathbf{r}) \mid S(\mathbf{r}) \rangle
\]  

(58)

This equation underscores the importance of the position dependence of Planck's constant only over the extent of the substantially non-zero areas of the wavefunction. If Planck's constant does not vary greatly over this region, it may be treated as a constant.

9. Time Dependence of the Expectation Value of \( \hbar \) under \( F_h \)

As \( V/\hbar \) and \( W/\hbar \) take up total conserved frequency between them, it is interesting to see if there is a simple quantity taken up by \( \hbar \) distinctly. That is, what quantity is stored in \( \hbar \)? Since \( F_h \) and \( \hbar \) do not commute,

\[
\frac{d}{dt} \langle \hbar \rangle = \langle i [\hat{F}_h, \hbar] \rangle = \langle (-i/4m)[\nabla^2, \hbar^2] \rangle \neq 0
\]  

(59)

The spatial dependence of Planck's constant would give rise to a temporal dependence as the particle moves through the \( \hbar \) field, but there is no simple quantity working in tandem with \( \hbar \) to conserve another constant of the motion, generally.

However, in the case where the external potential is constant and non-zero, (51) shows that \( \hbar^{-1} \) becomes the \( \hat{F} \) potential frequency\( \hat{\omega} \).

10. Indeterminacy of \( \hbar \) under \( F_h \)

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For non-commuting Hermitian operators $P$ and $Q$, the indeterminacy relationship between them is,

$$\Delta P \Delta Q \geq \frac{1}{2} |\langle [-i\hat{F}_P, \hat{F}_Q] \rangle| \quad (60)$$

Since $F_h$ and $\hbar$ do not commute but are Hermitian,

$$\Delta F_h \Delta \hbar \geq \frac{1}{2} |\langle [-i\hat{F}_h, \hbar] \rangle| = \frac{1}{2} |\langle (i/4m)[\nabla^2, \hbar^2] \rangle| \neq 0 \quad (61)$$

Our ability to know the frequency of the particle and the Planck constant experienced by it simultaneously is mutually limited.

11. Uncertainty under $F_h$

Using (60), since it can be shown $[x, p_x] = i\hbar(x)$, it is found that $\Delta p_x \Delta x \geq \frac{1}{2} |\langle h(x) \rangle| / 2$. Note that there is an integration over the spatial domain in the latter being performed, or the average of $\hbar$. For frequency and time, it can be seen from the same arguments applied in normal quantum mechanics that,

$$\Delta F_h \Delta Q \geq \frac{1}{2} \left| \frac{d\langle \hat{F}_h \rangle}{dt} \right| \quad (62)$$

Multiplying by a time increment,

$$\Delta t \Delta F_h \Delta Q \geq \frac{1}{2} \Delta t \left| \frac{d\langle \hat{F}_h \rangle}{dt} \right| \geq \frac{1}{2} \Delta Q \quad (63)$$

is found the uncertainty relationship,

$$\Delta F_h \Delta t \geq \frac{1}{2} \quad (64)$$

Multiplying (64) by a position dependent $\hbar$ gives the more familiar relationship in terms of energy and time, and, there is no averaging of $\hbar$.

12. Discussion

Many persons hold the position that the measurement of a single dimensionful constant in isolation is not physically meaningful, and the only meaningful measurements to be made are those of dimensionless products of the isolated dimensionful constants. The reason given is that the dimensionless constants are free of units that rely on arbitrary standards, and on calibrations.
of the metrology tools based on them. Both may be influenced by the variation of the constant itself, and also the measurement always involves multiple mechanisms with which other constants are convolved.

The above philosophy is sound when the metrology tools are located in the same place that the physical constants may be varying in, and only one technique is used for the measurement, and that single device-type is changed by the variation itself, and the standards on which the calibrations are based are in flux. However, it has not yet been experimentally borne out whether multiple techniques used in coordinated concert in the same location could or could not attribute the results of all the techniques to a single isolated dimensionful constant changing. It is also possible that a specific experiment could be devised at some point that is sensitive to only one dimensionful constant, and designed not to be disturbed by the constant’s variations.

An extreme example is to ask what would happen if Planck’s constant doubled in the sun, but not on the earth? Would there then be a discernable effect or not, would it be detectable from the earth, could it be determined that it was Planck’s constant that was the single constant that had changed, and would it then be worthwhile to attempt to measure the change in isolation?

Suppose that the spatial dependence change of a physical constant is very gradual, so that locally, it is as if the dimensionful constant were approximately constant, such as the case developed in Section 6 of this work. Then the form of the local physical laws would be the same in the two remote locations X and Y, but the dimensionful physical constants would be different. Experimenters in location Y could make observations on emissions from X with their metrology, exploiting invariants, and communicating results to one another.

Particles emitted from X with local energy $E_X$ traverse the mild $\hbar$-gradient to Y with fixed frequency $\omega$ where its local energy $E_Y$ can be measured. With no external potential active in the traversal (or the impact subtracted out if there is one), there will be an energy change $\Delta E_{YX} = \Delta \hbar \omega$ due to the $\hbar$ gradient. If experimenters in X and Y communicate and both agree on the frequency and report the local energies, the differences measured in $\hbar$ in X and Y could be confirmed. While this may be difficult to arrange, in principle it can be tested.

According to the model developed here, a particle conserves energy and momentum and obeys Newton-like laws only locally for a small enough gradient in $\hbar$. This limit is consistent with the tenet that the laws of physics be the same in all locations. Energy conservation and free-particle momentum conservation would become local laws, but would not be universally upheld. Universally, energy and momentum are definable artificially in terms of the normal quantum mechanical operators. For a sufficiently mild $\hbar$ gradient, quantum mechanics becomes locally per the norm, energy is conserved, frequencies can change, redshifts can occur, position expectation value time derivatives are real, and momentum is an entity. Though energy would not be universally conserved, X cannot benefit by any energy gain at Y, since returning the particle from Y back to X returns it to its original local energy.

There is the result from this model that free particles have a higher probability to be found in regions of lower $\hbar$. If it were found that $\hbar$ were lower near large masses, then in the absence of an
external potential there would be a quantum mechanical reason for mass to tend to locate near other mass. If the opposite, there would be a quantum mechanical reason for mass to avoid mass.

The model still requires the definition of a local potential energy $V$ to be put into the frequency operator, and, there are difficulties with Ehrenfest theorems, as far as identifying a straightforward relationship with classical mechanics. It was rationalized there is some reference point in space in which $\hbar$ is a minimum. The latter is the classical limit, or more precisely, the limit when the classical action $S_c \gg \hbar$. In the latter, it is not that $\hbar$ is actually going to zero, rather, masses and kinetic energies are getting very large, and the classical behavior is recovered. In the model of this paper, it is suggested that should $\hbar$ be found to vary spatially anywhere, then somewhere else $\hbar$ is minimum. Recall in the result of Section 6 of this paper, that wavefunctions are concentrated in areas of lower $\hbar$–particles would want to collect in those regions, for reasons beyond gravity, and in collecting, also approach the classical limit.

It would be desirable to find some physical system in which $\hbar$ depended on position to test the model, and is taken up in [38] in the analysis of the flyby anomaly, and Hulse-Taylor-like binaries. There, the effects of a position-dependent $\hbar$ are apparent over large, non-local distances.

An argument will be offered for how a position dependent Planck's constant would appear to not violate local position invariance, and how it would appear to be consistent with the Einstein Equivalence Principle, based on local experiments with clocks and light. The argument comes by way of an often-seen pedagogical derivation of the gravitational redshift, and is used here, because at present, there is no higher theory for frequency-conserving Einstein field equations, and this prescription to follow leads to the correct answer, to first order. Consider a photon falling into a gravitational potential due to its "gravitational mass" $\hat{m}(r)=\hbar(r) \omega(r)/c^2$, analyzed as if conserving total frequency $\Omega_{TOT}=\omega_\infty$, not total energy. The Newtonian field for a spherical mass of $g=-GM/r^2$ is integrated from $\infty$ to $r$ to produce the gravitational potential $\phi=-GM/r$, which is then multiplied by the gravitational photon mass, but without inclusion in the prior integration. This approach produces the result of GR for the gravitational frequency shift to first order. So, with no higher theory of a total frequency-conserving stress tensor, the sum of the kinetic frequency and potential frequency are per (65b), from which (65c) follows. Kinetic frequency is what is measured. Note that $\hbar(r)$ has cancelled in (65c), and is precisely the same expression derived when $\hbar$ is constant. Equation (65a) is the usual expression from GR for a constant $\hbar$, conserving total energy.

If the falling photon is analyzed as conserving total energy $E_{TOT}=\hbar \omega_\infty$ with a position dependent $\hbar$ then (65d-e) results. A functional form of the LPI violation for $\hbar \omega_\infty$ is chosen to resemble (65a) written with the Schwarzschild radius $R_S=2GM/c^2$. If total frequency is actually conserved and not total energy, the value of the LPI violation parameter $\beta_\hbar$ returned will be zero even if $\hbar$ is not constant (at least to first order).
\[ \omega_{GR}(r) = \omega_\infty \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \]

\begin{align*}
\Omega_{TOT} &= \omega_\infty = \omega(r) - \frac{GMm}{r} \frac{1}{\dot{h}(r)} \\
\omega(r) &= \omega_\infty \left(1 + \frac{GM}{rc^2}\right) \approx \omega_{GR}(r)
\end{align*}

\[ E_{TOT} = \hbar_\infty \omega_\infty = \hbar(r) \omega(r) - \frac{GMm}{r} \]

\begin{align*}
\omega(r) &= \omega_\infty \left(1 + \frac{GM}{rc^2}\right) \frac{\hbar_\infty}{\hbar(r)} \approx \omega_{GR}(r) \frac{\hbar_\infty}{\hbar(r)}
\end{align*}

Equation (66) reduces to the expression for small changes seen in [10]. A systematic dependence of \( \hbar \) on altitude was not developed there, only that there was a variation with a range per (66) with \(|\beta_\hbar|<0.007\).

\[ \frac{\hbar(r)}{\hbar_\infty} = \left(1 - \beta_\hbar \frac{R_S}{r}\right)^{1/2} \]

Table 1 summarizes the findings of (65a-e), and it is concluded it may be difficult to detect the variation \( \hbar(r) \) using falling light or clocks at different altitudes if total frequency is actually conserved, even if \( \hbar \) truly varies. Tests such as the Pound-Rebka experiment, and observations with clocks on satellites at different altitudes would be completely insensitive to the variation, as such.

Consider the following thought experiment. Bob is inside a closed elevator in the vicinity of a blackhole, held on a rope by an immobilized Alice, above. They both have a clock, which is a sourceless, perfectly interior-reflecting box of trapped light of frequency \( \omega \) that they can each measure. If total frequency is conserved, by Table 1, whether \( \hbar \) varies or not, or whether Alice slows Bob, or cuts the rope and allows him to freefall, he will register no change in the frequency of his own clock, and since he cannot see emissions from Alice’s clock, he registers no perception of any difference. Thus, the Einstein Equivalence Principle would be apparently consistent, as would the local position invariance of \( \hbar \), since when the two clock readings are compared later when Alice and Bob communicate, they will show only the differences predicted by normal GR, whether \( \hbar \) varies or not. Now, let Bob kick the box with a known force parallel to the floor of the elevator at several different altitudes in a gradient in \( \hbar \). Though the frequency of light in the box to Bob is fixed, the energy of the light in the box is not, hence its gravitational mass changes, as does the result of the kicking experiment as a function of his altitude. Since the result of the kicking experiment varies with his position in spacetime, and the experiment is not gravitational, both the Einstein and Strong Equivalence principles do not actually hold (unless the former is interpreted to hold, if the kicking of the box is interpreted to be a gravitational experiment, since it measures the gravitational mass and inertial mass simultaneously, or if the experiment is considered to be non-local, as he must kick when he knows he is in a different spacetime position to register a difference). The Weak Equivalence Principle will still hold,
Despite that the mass of any object becomes position dependent due to the variation of \( \hbar \), and despite that different substances will show different ratios of mass change, gravitational and inertial mass are still equal.

Reference [38] shows that objects in non-circular orbits, or elliptical orbits, will enhance the effect of a position dependent Planck constant, especially a flyby orbit, as a hyperbolic orbit cuts through the isocontours of Planck constant maximally. The analysis of an entire orbit is a non-local experiment, from which the variation in Planck constant can be detected.

| Conserved Quantity | \( h \) dependence | \( \omega(r) \) |
|--------------------|---------------------|-----------------|
| \( E_{TOT} = h\omega + \omega\infty \) | constant | \( \omega_{GR}(r) \) |
| \( E_{TOT} = h\omega + \omega\infty \) | position dependent | \( \omega_{GR}(r) \frac{\hbar}{\hbar(r)} \) |
| \( \Omega_{TOT} = \omega_{\infty} \) | constant | \( \omega_{GR}(r) \) |
| \( \Omega_{TOT} = \omega_{\infty} \) | position dependent | \( \omega_{GR}(r) \) |

Table 1. The case for a photon (or clock) changing position in gravity radially that would register a detectable change in frequency deviating from GR is when total energy is conserved and Planck constant is position dependent. It is concluded that a variable Planck constant would show an apparent consistency with the Einstein Equivalence Principle, to first order, for total conserved frequency, in experiments with clocks and light.

The discussion will continue as if energy is conserved. Using \( |\beta_h| = 0.007 \) and the mass and radius of the Earth, (66) results in very small fractional changes near the surface of the Earth relative to infinity, on the order of one part in \( 10^{12} \). The form (66) does not persist beneath the Earth surface due to volume filling matter. The same order of magnitude for the fractional change is found in the ratio of \( \hbar \) at the maximum and minimum radii of the Earth's orbit around the sun. These variations are four orders of magnitude lower than the very best terrestrial laboratory measurement capability, achieving on the order of \( 10^{-8} \) relative uncertainty using the superconducting Watt balance [37]. Therefore, the authors of [10] used the GPS data to attempt to measure changes four orders of magnitude smaller than the capability of the very best earthbound metrology.

The variations taken from (66) are much smaller than the 21 ppm peak-to-peak \( \hbar \) variation extracted from the electromagnetic experiment (850 ppm peak-to-peak annual diode voltage variation) in [19], and the 1000-3000 ppm peak-to-peak annual variations of the decay rates in [12-17]. Either completely different mechanisms are at work in [12-17] and [19] than the gravitational one expressed by (66), or \( \beta_h \) would have to be 7 to 9 orders larger to account for the difference. At the surface of the sun using \( |\beta_h| = 0.007 \) the fractional change in \( \hbar \) is 1 part in \( 10^{8} \), getting closer to the relative uncertainty of the best terrestrial measurement.

For more dense bodies, the fractional change would increase greatly. For a black hole (assuming there is no volume filling interior matter below the event horizon), for \( \beta_h < 0 \) equation (66) decreases with \( r \) to unity asymptotically, while for \( \beta_h > 0 \) and \( r < \beta_h R_S \) one sees that \( \hbar \) is pure imaginary, where after it rises from zero to unity asymptotically with increasing \( r \). At the Schwarzschild radius,
\[ \frac{\hbar(R_3)}{\hbar_{\infty}} \bigg|_{BH} = (1 - \beta_h)^{1/2} \]  

where the fractional change at the event horizon is 3500 ppm.

In reference [25] using a field theoretical coupling argument without gravity and for a classical vacuum, it was derived that,

\[ \frac{\hbar(r)}{\hbar_{\infty}} = \left(1 + \frac{b_4/b_3}{r}\right)^{1/2} \]  

(68)

It was surmised, based on an attempt to calibrate the various new constants, that $b_4/b_3$ is positive for the sun and Earth. The similarity between (68) and (66) is worth noting, as (68) was derived without general relativity. The relationship between Equations (68) and (66) is explored in [38], where owing to the extremely large extracted values of $\beta_h$ there and also here, it was concluded that $b_4/b_3$ is not dependent on mass, but the $\hbar$ field is coincident with massive bodies, allowing an explanation of the flyby anomaly.

Strong $\hbar$ positional dependence happening near very dense bodies or black hole event horizons might be observed and probed. A reanalysis of the GPS data per [10] up to the current date to refine $\beta_h$ and look specifically for a systematic change in $\hbar$ with altitude may be worthwhile. An independent analysis of the data of the diode experiment in reference [19] along with an analysis of any issues with the theory of the measurement is also needed, along with repeats of the experiment by independent investigators.

13. Conclusions

A mathematical study was undertaken concerning how the Schrödinger equation would have to be changed if Planck’s constant was position dependent. Notable departures from normal quantum mechanics are described. A frequency operator results, and to make it Hermitian, is augmented with an anti-commutator of the non-Hermitian part, which is the simplest alteration. While total frequency is a constant of the motion, total energy is not, and momentum becomes a non-entity except in regions where Planck’s constant has a very small gradient. There are quantities now named kinetic frequency and potential frequency which together conserve total frequency between them. Wavefunctions are concentrated in regions of lower Planck’s constant even in the absence of an external potential. A functional form of Planck’s constant near massive bodies is alluded to, based on this authors speculation on [10], and another analysis of the GPS data associated with it might be valuable. Further work might entail solving the symmetrized Schrödinger equation for the specific position dependent Planck’s constant of (68) with and without an external gravitational potential, approximate or exact quantum harmonic oscillator solutions (the author has derived this by two means in 1-D in a linear $\hbar$ gradient, unpublished, and the wavefunctions are those of the normal oscillator multiplied by the same exponential factor as Equations (45) and (47), concentrating the wavefunctions on the lower $\hbar$ side ), and working out how to incorporate a position dependent Planck’s constant into a canonically quantized field theory (done in [25]). Fuller investigations of the symmetries...
resulting from those cases could be made. The latter would help determine other dynamical variable operators commuting with the frequency operator, as so far, the only one found is itself. It may also be possible to arrive at the modified Schrödinger equation from a modified Feynman path integral with a position and/or time dependent $\hbar$ (partially examined in [38]). Observations of phenomena near massive objects combined with model predictions are needed.

A.1. Classical Field Equation of Motion Compared to Frequency-Conserving Schrödinger Equation

It is known that the Lagrangian density that produces a classical field equation of motion (the Schrödinger field) in the non-relativistic limit, that is the same in functional form as the single-particle Schrödinger wavefunction equation, when that field in the Hamiltonian density is then quantized, it will give the correct description of the non-relativistic single- and multi-particle states. Here it will be determined if the Lagrangian density of the Planck constant field developed in [25] leads to a classical Schrödinger field equation of motion resembling the frequency-conserving Schrödinger wavefunction equation for which $\hbar$ is not a dynamical field, but is position dependent.

The Schrödinger equation referred to features a Hermitian frequency-conserving Hamiltonian when Planck's constant is position dependent only,

$$i \frac{\partial \varphi}{\partial t} = -\frac{1}{2m} \frac{1}{2} \left\{ h(\vec{r}), \nabla^2 \right\} \varphi$$ (A1)

where the curly bracket signifies an anticommutator. The $\varphi$ in (A1) is the single-particle wavefunction with a probability interpretation. The classical fields will not have a probability interpretation.

The goal now is to try to arrive at (A1) as a supported field $\varphi$, using a supporting Planck constant field $\hbar = \beta \psi$, the latter being real.

A.2. Energy Squared Lagrangian

The Lagrangian density is usually written in terms of the squares of energies, and the resulting equations are energy conserving. The Lagrangian and Hamiltonian density when Planck's constant is a dynamical field $\hbar = \beta \psi = \beta \chi^{1/2}$ supporting the field $\varphi$ is from [25],

$$L_{\varphi} = \frac{1}{2} \beta^2 \chi \partial_u \varphi \partial^u \varphi - \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{8} \beta^2 \partial_u \chi \partial^u \chi$$ (A2a)

$$H_{\varphi} = \frac{1}{2} \beta^2 \chi \left( \dot{\varphi}^2 + (\nabla \varphi)^2 \right) + \frac{1}{2} m^2 c^2 \varphi^2 + \frac{1}{8} \beta^2 \left( \dot{\chi}^2 + (\nabla \chi)^2 \right)$$ (A2b)

where $\varphi$ is the supported field. The energy is shared between the two fields per (A2b). The resulting equations of motion for the coupled fields $\varphi$ and $\chi$ are, respectively,
Replacing $\chi = \psi^2$, then multiplying (A3a) by $\phi / \psi$ and adding to (A3b) after division by $\psi$ in the latter, one finds for the equation of motion for the combined fields,

$$\psi \frac{1}{2} \{ \psi, \nabla^2 \} \phi + \frac{m^2 c^2}{\beta^2} \frac{\phi}{\psi} = 0$$

Re-writing (A3a-b)

\[
\begin{align*}
\frac{1}{4} (\dot{\chi} - \nabla^2 \chi) - \frac{1}{2} (\dot{\phi}^2 - (\nabla \phi)^2) &= 0 \\
(\dot{\chi} \phi + \dot{\chi} \phi) - (\nabla \chi \cdot \nabla \phi + \chi \nabla^2 \phi) + \frac{m^2 c^2}{\beta^2} \phi &= 0
\end{align*}
\]

The field $\phi$ will be decomposed as Equation (A5a-b),

$$\frac{\omega_m}{c} = \frac{1}{c} \sqrt{\frac{m^2 c^4}{\beta^2 \psi^2}}$$

$$\phi = \tilde{\phi} \exp \left\{ -i \frac{mc}{\beta \psi} x_o \right\} = \tilde{\phi} E$$

with which the following derivatives are computed,

\[
\begin{align*}
\frac{\dot{\phi}}{E} &= \tilde{\phi} + \tilde{\phi} \left( -i \frac{mc}{\beta} \right) \frac{1}{\psi} \nabla \left( \frac{1}{\psi^2} x_o \right) \\
\frac{\ddot{\phi}}{E} &= \tilde{\phi} + 2 \tilde{\phi} \left( -i \frac{mc}{\beta} \right) \frac{1}{\psi} \nabla \left( \frac{1}{\psi^2} x_o \right) + \tilde{\phi} \left[ \left( -i \frac{mc}{\beta} \right) \left( \frac{1}{\psi} \nabla \left( \frac{1}{\psi^2} x_o \right) \right) \right]^2 \\
&+ \frac{\tilde{\phi}}{E^2} \left\{ \left( -i \frac{mc}{\beta} \right) \nabla \left( \frac{1}{\psi^2} x_o \right) \right\} \left( -i \frac{mc}{\beta} \right) \left( \frac{1}{\psi} \nabla \left( \frac{1}{\psi^2} x_o \right) \right) \\
\frac{\nabla \phi}{E} &= \nabla \tilde{\phi} + \tilde{\phi} \left( i \frac{mc}{\beta} \nabla \left( \frac{1}{\psi^2} x_o \right) \right)
\end{align*}
\]
\[ \frac{\nabla^2 \phi}{E} = \nabla^2 \phi + 2 \nabla \phi \left( \frac{\text{imc} \nabla \psi}{\beta \psi^2 x_o} \right) + \phi \left( \frac{\text{imc} \nabla \psi}{\beta \psi^2 x_o} \right)^2 + \left( \frac{\text{imc}}{\beta} \right) \left( \nabla^2 \frac{\psi^2}{\beta^2} x_o - 2 \left( \frac{\nabla \psi}{\beta^2} x_o \right)^2 \right) \]  \tag{A9}

In Equation (A7), the first term in the underbracket will cancel the mass term of Equation (A4).

If (A6) to (A9) were substituted into (A4), the resulting equation of motion would have a very large number of terms.

Since Equation (A1) was derived with no \( \hbar \) time dependence, the time derivative of \( \psi \) will be set to zero. Then from (A4), (A6), and (A7) follows,

\[
\psi \frac{\dot{\phi}}{E} - \frac{1}{2} \left\{ \psi, \nabla^2 \right\} \phi + \frac{m^2 c^2}{\beta^2} \frac{\phi}{\psi} + \frac{1}{2} \left( \frac{\nabla \psi^2}{\psi} - \left( \frac{\phi^2}{\psi} \nabla \phi \right)^2 \right) - \nabla \psi \cdot \nabla \phi = 0 \tag{A10}
\]

\[
\phi \frac{\ddot{\phi}}{E} = \frac{\dot{\phi}}{2} + \phi \left( - \frac{\text{imc}}{\beta \psi} \right) \tag{A11}
\]

\[
\phi \frac{\dddot{\phi}}{E} = \frac{\ddot{\phi}}{2} + 2 \phi \left( - \frac{\text{imc}}{\beta \psi} \right) - \frac{m^2 c^2}{\beta^2} \frac{\dot{\phi}}{\psi^2} \tag{A12}
\]

To approach Equation (A1) some additional conditions must be imposed. The classical limit implies the kinetic energy is much less than the rest energy, therefore

\[
\dot{\phi} \gg \nabla \phi \tag{A13a}
\]

\[
\ddot{\phi} \ll m \ddot{\phi} \tag{A13b}
\]

\[
\dddot{\phi} \ll m \dddot{\phi} \tag{A13c}
\]

so the first derivative in (A11) and second derivative in (A12) can be dropped, and also \((\nabla \phi)^2\) in the large bracketed term of (A10). In Equations (A8) and (A9) occurrences of \(\nabla \psi\) or \(\nabla^2 \psi\) that are either second order and/or multiplied by \(x_o\) are assumed to be negligible, and only the first terms of (A8) and (A9) remain. This condition implies a combination of an early epoch and/or second order spatial changes. Then substituting the resulting equations (A11), (A12), (A8) and (A9) into (A10),

\[
i \frac{\partial \tilde{\phi}}{\partial t} = -\frac{1}{2m} \left\{ \beta \psi, \nabla^2 \right\} \tilde{\phi} - \frac{1}{2m} \nabla \beta \psi \cdot \nabla \tilde{\phi} + \frac{m c^2}{4 \beta} \left( \frac{\tilde{\phi}}{\psi} \right)^3 E(t, \psi(\vec{r}))^2 \tag{A14}
\]

Equation (A14) resembles Equation (A1) with no potential, but with two additional terms that cannot easily be explained away. The extra term \(B\) may be argued to vanish if the mass is very large, to the extent that the frequency \(\omega_m\) is very much larger than the frequency of the non-
relativistic field, so that over much less than one cycle of the latter, the term $B$ in (A14) would average to zero. That still leaves the problematic term $A$.

### A3. Frequency Squared Lagrangian

Here, it will be determined if a squared-frequency Lagrangian will produce an equation of motion conserving frequency in the non-relativistic limit with the form of (A1). To produce this Lagrangian, Equation (A2a) is divided by $\hbar^2 = (\beta \psi)^2$ and the latter is absorbed into $L$, but appears in the denominator of the mass term,

$$L_{\phi} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} \frac{m^2 c^2 \phi^2}{\beta^2 \psi^2} + \frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \psi$$

$$H_{\phi} = \frac{1}{2} \left( \phi^2 + (\nabla \phi)^2 \right) + \frac{1}{2} \frac{m^2 c^2 \phi^2}{\beta^2 \psi^2} + \frac{1}{2} \left( \psi^2 + (\nabla \psi)^2 \right)$$

From (A15b) it is seen that the frequency is shared between the two fields. The equations of motion for fields $\phi$ and $\psi$ are, respectively,

$$\ddot{\phi} - \nabla^2 \phi + \frac{m^2 c^2 \phi}{\beta^2 \psi^2} = 0$$

$$\ddot{\psi} - \nabla^2 \psi - \frac{m^2 c^2 \psi^2}{\beta^2 \psi^2} = 0$$

Note that the frequency-squared Lagrangian reproduces with (A16a) the Klein-Gordon equation with a variable $\hbar^2$ in the denominator of the mass term, and that the fields are uncoupled if $m=0$. Multiplying (A16a) by $\psi$ and (A16b) by $\phi/2$ and adding them produces,

$$\psi \ddot{\psi} - \frac{1}{2} \left( \psi, \nabla^2 \right) + \nabla \psi \cdot \nabla \phi + \frac{\phi}{2} \psi + \frac{m^2 c^2}{\beta^2} \left( \frac{\phi}{\psi} - \left( \frac{\phi}{\psi} \right)^3 \right) = 0$$

Continuing the procedure as before with the field decomposition (A5a-b) also produces a large number of terms, and the approximations eliminate all but the extra term in the underbracket of (A17), the same term as in (A14). The final equation looks like (A14) without the $B$ term.

### A.4. Discussion

Equation (A1) was not derived from field theory, rather, it was found by making the leap that frequency could be a constant of the motion if the Hamiltonian remained Hermitian by the addition of terms to the Schrödinger equation, in face of the fixed background of a position dependent $\hbar$. The expense is that energy and momentum are no longer conserved, even for a free-particle.

The plausibility of Equation (A1) depends on how plausible it is for energy conservation to be an inviolable law. Important quantities and events involved in our present physical understanding seem to violate energy conservation, such as the cosmological constant driving the accelerated expansion of the universe, and the occurrence of the big bang. Also, can the infinite energies of
the vacuum state, or states, be said to be conserved? Those observations, in combination with not truly knowing whether constants are inconstant, or if they actually are dynamical fields, or fixed background fields, or neither and something else entirely, make Equation (A1) viable to contemplate, and there may yet be a specific form of the action that leads to it, other than

\[ L = \phi \left( i \frac{\partial \phi}{\partial t} + \frac{1}{2m} \frac{1}{2} \left\{ h(\Sigma), \nabla^2 \right\} \phi \right) \tag{A18} \]

in which \( h \neq \beta \psi \) and is not a supporting dynamical field. The form of Equation (A1) is extremely simple, relative to the equations that result from coupling through the derivative terms in field theory.

A.5. Conclusions

A form of the action leading to a classical, non-relativistic Schrödinger field equation of motion matching the form of the frequency-conserving Schrödinger wavefunction equation is still sought. Thus far, the latter could not be derived from field theory when there is a supporting dynamical field \( h=\beta \psi \).

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