A Study of Asymptotic Freedom like Behavior for Topological States of Matter

Ranjith Kumar R,1,2 Rahul S,1,2 Surya Narayan,3 and Sujit Sarkar1,*

1Poornaprajna Institute of Scientific Research, 4, 16th Cross, Sadashivnagar, Bengaluru - 5600-80, India.
2Manipal Academy of Higher Education, Madhava Nagar, Manipal - 576104, India.
3Raman Research Institute, C. V. Raman Avenue, 5th Cross, Sadashivanagar, Bengaluru - 5600-80, India.

(Dated: January 22, 2020)

We present results for asymptotic freedom like behavior for the topological state of the helical spin liquid system with finite proximity induced superconducting gap. We derive two different quantum Berezinskii-Kosterlitz-Thouless (BKT) equations for the two different limit of this model Hamiltonian. The common quantum phase for these two quantum BKT transitions is the helical Luttinger liquid phase where there is no evidence of asymptotic freedom. There is no evidence of superconductor-insulator transition for this asymptotic freedom study. We observe the evidence of asymptotic freedom for the two model Hamiltonian, but the character of the asymptotic phases are different. We also observe that the Luttinger liquid parameter plays a significant role to determine the asymptotic freedom but the chemical potential has no effect on it.

Keywords: Asymptotic Freedom like behaviour, Berezinskii-Kosterlitz-Thouless Transition, Topology, Quantum Phase Transition.

*E-mail - sujit.tifr@gmail.com
Introduction

In the year 1955, Landau identified the problem that in the presence of an arbitrary number of virtual particles and a real particle there will be no interaction due to screening [1]. This problem was solved by David Gross and Frank Wilczek [2, 3] and independently by David Politzer in the same year [4], by finding anti-screening effect or asymptotic behavior in nonabelian gauge theories, or Yang-Mills theories [5]. This is the origin of Quantum Chromodynamics (QCD), which is the quantum field theory of strong interactions. This theory explains that the coupling constant between two quarks increases as the length scale increases, and the coupling decreases asymptotically as the length scale decreases, making it asymptotically free theory [6, 7]. The search for individual quarks was not successful and they were always found in the bound state of quark and anti-quark or the bound states of three quarks [8–10]. This principle of confinement was contradicted by the observation of an interesting fact by the SLAC (Stanford Linear Accelerator) experiment of colliding proton with energetic photons [11].

In condensed matter field theory one often encounters the theories with divergences. Renormalization group technique was used to deal with these divergences and to get finite results for observable physical quantities in the theory [12–15]. Using renormalization theory Feynman, Schwinger, and Tomonoga wrote down the corrections due to interactions with any finite number of virtual particles in QED [16–18]. It was extended to a wider class of theories by G. ‘t Hooft and M. Veltman [19]. To simplify the renormalization group analysis, in 1970, Curtis Callan and Kurt Symanzik derived a differential equation (Callan-Symanzik equation) which determines change in the n-point correlation functions under variation of the scaling parameter [20, 21]. The running coupling constant or the correlation function which we encounter in the theories depends on the momentum scale. This happens in a well-defined way through $\beta$ function in the Callan-Symanzik equation [22]. The $\beta$ function also gives the behavior of coupling at large momentum to describe asymptotic freedom in QCD and at small momentum to describe critical phenomena. Physical systems which do not have natural intrinsic energy scale will remain neutral for any variation in the energy scales. This intrinsic energy scale for any system is the one which determines whether the system is in strong or weak interaction. Apart from these there are systems which possess characteristic energy scale which acts as the separation between strong and weak interaction regimes which
helps one to identify the asymptotic nature of the system. These asymptotic behaviors can be witnessed in condensed matter systems such as, Gross-Neveu model of polyacetylene [23], 2d $\sigma$-model [24], Kondo model [25], superinsulating state of superconductor-insulator transition in superconducting films [26] etc. In all these models one can calculate the Callan-Symanzik equation and also the $\beta$ function to know the dependence of correlation function or coupling constant on the different energy scales. The negative $\beta$ function which has been obtained in all these models implies that at low energy scales the effective coupling is large and system is strongly interacting, and at high energy scales the effective coupling is small making the system asymptotically free [14]. The asymptotic freedom for topological insulators has not been studied yet, and it is still an open problem. We will show explicitly that the concept of asymptotic freedom like behavior observed in high energy physics, can also emerge in the quantum many body condensed matter system with topological background.

Another important application of renormalization group (RG) method is in the Berezinskii-Kosterlitz-Thouless (BKT) transition [27–29]. Berezinskii [30] in 1971 and Kosterlitz and Thouless [31] in 1973 explained a topological phase transition in two dimensional XY spin model, since there is no possibility of spontaneous symmetry breaking for $d \leq 2$ (d is the dimension) [32, 33]. The study of BKT transition is crucial in quantum many body systems since many quantum mechanical two dimensional systems can be approximated to two dimensional XY model [34]. In this study, we would like to unify two different regimes of RG theory, the asymptotic freedom like behavior of high-energy physics and the BKT transition of condensed matter physics.

Here we consider an interacting helical liquid system at the edge as our model Hamiltonian. The quantum spin Hall systems are associated with the helical liquid system which describes the connection between spin and momentum. The left movers in the edge of quantum spin Hall systems are associated with down spin and right movers with up spin [35–41]. The helical liquid system possess the gapless excitation at the edge and this results in the appearance of Majorana particles at both ends of the system [42].

We present the motivation for this study below.

**Motivation and importance of this study:**

**First objective:** The physics of topological state of matter is the second revolution of quantum mechanics [43]. This important concept and new important results with high im-
pact not only bound to the general audience of different branches of physics but also creates interest for the other branches of science (Mathematics, Chemistry, Biology, Engineering and also in the Philosophy of Science). This new revolution in quantum mechanics was honored by the Nobel prize in physics in the year 2016. This is one of the fundamental motivation of this present study.

Second objective: The mathematical structure and results of the renormalization group (RG) theory are the most significant conceptual advancement in quantum field theory in the last several decades in both high-energy and condensed matter physics. The need for RG is really transparent in condensed matter physics. RG theory is a formalism that relates the physics at different length scales in condensed matter physics and the physics at different energy scales in high energy physics. In the present study, we show explicitly how the physics of topology appears at different length scales of the system and also obtain the signature of asymptotic freedom like behaviour.

Third objective: The most important success of the RG theory is the prediction of Asymptotic freedom (Physics Nobel Prize 2004) and the physics of BKT (Physics Nobel Prize 2016), although it is classical BKT (the present problem is on quantum BKT which is more subtle than classical BKT). The concept of asymptotic freedom is not only confined to high energy physics but also we show that it is present in quantum condensed matter systems. In the present work, we unified these two concepts in a single framework of topological state of matter. There are no such studies in the entire literature of topological state of matter. These new, important and original results will provide a new perspective on the study of topological state of matter.

Fourth objective: Here we mention very briefly the nature of different Luttinger liquid physics to emphasise the rich physics of helical Luttinger liquid. The physics of Luttinger liquids (LL) can be of three different forms: spinful LL, chiral LL, and helical LL. Spinful LL shows linear dispersion around the fermi level with the difference of $2k_F$, in the momentum between left and right moving branches. Chiral LL has spin degenerated, strongly correlated electrons moving in only one direction. In helical LL one can observe the Dirac point due to the crossing of left and right moving branches, also electrons with opposite spins move in opposite directions. We consider this physics for the present model Hamiltonian system. The Luttinger parameter ($K$) determines the nature of interaction. $K < 1$ and $K > 1$ char-
acterize the repulsive and attractive interactions respectively, whereas $K = 1$ characterize the non-interacting situation. The present study show the importance of $K$ for quantum phases and asymptotic freedom [44].

Model Hamiltonian

We consider the interacting helical liquid system at the edge of a topological insulator as our model system [40]. These edge states are protected by the symmetries [41]. In the edge states of helical liquid, spin and momentum are connected as the right movers are associated with the spin up and left movers are associated with spin down. One can write the total fermionic field of the system as

$$\psi(x) = e^{ik_F x} \psi_{R\uparrow} + e^{-ik_F x} \psi_{L\downarrow},$$

where $\psi_{R\uparrow}$ and $\psi_{L\downarrow}$ are the field operators corresponding to right moving (spin up) and left moving (spin down) electron at the upper and lower edges of the topological insulators [42, 45–47]. For the low energy collective excitation in one dimensional system one can write the Hamiltonian as

$$H_0 = \int \frac{dk}{2\pi \nu_F} \left[ (\psi_{R\uparrow}^\dagger (i\partial_x) \psi_{R\uparrow} - \psi_{L\downarrow}^\dagger (i\partial_x) \psi_{L\downarrow}) + (\psi_{R\downarrow}^\dagger (i\partial_x) \psi_{R\downarrow} - \psi_{L\uparrow}^\dagger (i\partial_x) \psi_{L\uparrow}) \right], \quad (1)$$

where the terms in parentheses represent Kramer’s pair at the two edges of the system.

The authors of Ref [42, 45] have mapped this Hamiltonian—with Forward and umklapp interactions and in the proximity of s-wave superconductor ($\Delta$) and the magnetic field ($B$)—to the XYZ spin-chain model (up to a constant) i.e, $H_{XYZ} = \sum_i H_i$, where

$$H_i = \sum_\alpha J_\alpha S_i^\alpha S_{i+1}^\alpha - [\mu + B(-1)^i] S_i^z. \quad (2)$$

From bosonization procedure, fermionic field of 1D quantum many body system can be expressed as, $\psi_{R/L\uparrow}(x) = \frac{1}{2\pi \alpha} n_{R/L} e^{i\sqrt{4\pi} \phi_{R/L\uparrow}(x)}$, where $n_{L/R}$ is the Klein factor to preserve the anticommutativity of the fermionic field. Here we introduce two bosonic fields, $\theta(x)$ and $\phi(x)$ which are dual to each other. The relations of these two fields are, $\phi(x) = \phi_R(x) + \phi_L(x)$ and $\theta(x) = \theta_R(x) + \theta_L(x)$. After the continuum field theory one can write the bosonised form of
Hamiltonian as,

\[ H = \int dx v^2 \left[ \frac{1}{K}((\partial_x \phi)^2 + K(\partial_x \theta)^2) \right] - \left( \frac{\mu}{\pi} \right) \int dx \partial_x \phi + \left( \frac{B}{\pi} \right) \int dx \cos(\sqrt{4\pi} \phi) \]

\[- \left( \frac{\Delta}{\pi} \right) \int dx \cos(\sqrt{4\pi} \theta) + \left( \frac{g_u}{2\pi^2} \right) \int dx \cos(4\sqrt{\pi} \phi) \]

This is our model Hamiltonian where, \( J_x = v_F + \Delta, J_y = v_F - \Delta \) and \( J_z = g_u \) are coupling constants. \( \theta(x) \) and \( \phi(x) \) are the dual fields and \( v = v_F + \frac{g_u}{2\pi} \), \( v \) is the collective velocity and \( v_F \) is the Fermi velocity with \( K = 1 - \frac{g_u^2}{2\pi v_F} \); \( K \) is the Luttinger liquid parameter of the system.

Before we begin to discuss the appearance of quantum BKT transition in our system, we discuss briefly why it is necessary to study the quantum BKT transition. Here we study two different situations for our model Hamiltonian. (i) the proximity induced superconducting gap term is absent (\( \Delta = 0 \)) and (ii) the applied magnetic field is absent (\( B = 0 \)). For both of these cases only sine-Gordon coupling term is present. Therefore, there is no competition between the two mutually non local perturbation. Therefore one can think that there is no need to study the RG to extract the quantum phases and phase boundaries. But RG method is adopted for the following reason. Each of these Hamiltonians consist of two parts. The first part is the non-interacting term where the \( \phi \) and \( \theta \) fields show the quadratic fluctuations and the other part of these Hamiltonians is the sine-Gordon coupling terms of either of \( \theta \) or \( \phi \) fields. The sine-Gordon coupling term lock the field either \( \theta \) or \( \phi \) in the minima of the potential well. Therefore the system has a competition between the quadratic part of the Hamiltonian and the sine-Gordon coupling term and this competition will govern the low energy physics of these Hamiltonians in different limit of the system.

Quantum BKT equations

We consider the model Hamiltonian \( H \) (Eq.3) with \( g_u = 0 \) since it has no effect on the topological state and also on the Ising state of the system [42],

\[ H = \int dx v^2 \left[ \frac{1}{K}((\partial_x \phi)^2 + K(\partial_x \theta)^2) \right] - \left( \frac{\mu}{\sqrt{\pi}} \right) \int dx \partial_x \phi + \left( \frac{B}{\pi} \right) \int dx \cos(\sqrt{4\pi} \phi) \]

\[- \left( \frac{\Delta}{\pi} \right) \int dx \cos(\sqrt{4\pi} \theta) \]
The Hamiltonian under the situation, proximity induced superconducting gap term is present, i.e, \( \Delta \neq 0 \) but the magnetic field is absent. We have,

\[
H_1 = \int dx \frac{v}{2} \left[ \frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right] - \left( \frac{\Delta}{\pi} \right) \int dx \cos(\sqrt{4\pi} \theta(x)).
\]  

(5)

The RG equation for this Hamiltonian can be derived as (we refer to appendix A for detailed derivation)

\[
\frac{d\Delta}{dl} = \left( 2 - \frac{1}{K} \right) \Delta, \quad \frac{dK}{dl} = \Delta^2.
\]  

(6)

Under the situation that the applied magnetic field is present, i.e, \( B \neq 0 \) but \( \Delta \) is absent. The Hamiltonian is

\[
H_2 = \int dx \frac{v}{2} \left[ \frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right] + \left( \frac{B}{\pi} \right) \int dx \cos(\sqrt{4\pi} \phi(x)).
\]  

(7)

Following the same procedure as in appendix A, one can derive another set of quantum BKT equation,

\[
\frac{dB}{dl} = (2 - K)B, \quad \frac{dK}{dl} = -B^2K^2.
\]  

(8)

Thus, we derive two sets of RG equations.

**Length Scale Dependent Study of Quantum BKT transition: Evidence of Asymptotic Freedom like behavior**

In this section, we show explicitly how the physics of topology appears at different length scales of the system to get further insight into the RG results and also obtain the signature of asymptotic freedom like behaviour.

Asymptotic freedom is a feature of QCD, the quantum field theory of the strong interaction between quarks and gluons [3, 49]. In QCD, the gauge theory of quarks and gluons are asymptotically free, i.e., the coupling vanishes at very short distance (large momentum) and grows at large distance (small momentum). This allowed us to understand why quarks seemed free inside nucleons in deep inelastic scattering and are also confined at large distance.

But the present problem is not QCD. At the same time we are not interpreting our results in terms of quark and gluon physics, rather in terms of topological quantum state of mat-
FIG. 1: This figure consists of two panels (a: $K = 0.6$ and b: $K = 0.3$). It shows the variation of $\Delta$ with $L$ for the different initial values of $\Delta$.

FIG. 2: This figure shows the variation of $B$ with $L$ for different initial values of $B$.

Therefore we should interpret our results from the length scale dependent asymptotic freedom like behavior (not the strict asymptotic freedom of QCD). Length scale dependent study brings out the concept of asymptotic freedom like behavior in the RG flow sense.

We would like to explain it explicitly through the $\beta$ function explanation. This asymptotic freedom is a feature of all RG flows with a marginally relevant perturbation. This can be written as $\beta_\lambda = C\lambda^2$, where $\lambda$ is the coupling constant and $C > 0$ is a constant (here $\beta_\lambda$ is the $\beta$ function of the RG theory from where one can predict the nature of the RG flow lines of coupling constant $\lambda$).

In quantum field theory, flow lines are defined in energy scale but here we define RG flow lines in length scale. With the model Hamiltonian we study here, we study only the low energy properties. If we are interested in the high energy behaviour of the system, how the
coupling constant behave with the larger momentum, i.e., for the small length scale. The physics of ultraviolet behaviour occurs for that case only. Present study involve the low energy physics for higher length scale.

In fig. 1a, we observe that RG flow lines flowing off to the strong coupling phase, i.e., the coupling $\Delta$ increases with the length scale. This coupling term induce the topological superconducting phase in the system as we mentioned during the introduction of the model Hamiltonian system. At the same time, increasing $\Delta$ with the length scale is the signature of asymptotic freedom of the system, as we have discussed at the beginning of this section. Therefore the topological superconducting phase appears in the asymptotic degrees of freedom.

In fig. 1b ($K = 0.3$, i.e, the system is in the more repulsive regime), we observe that the coupling $\Delta$ decreases rapidly with length scale and satisfies the condition for the absence of the Majorana modes i.e., $L \frac{\Delta}{v} << 1$, ($L$ is the length, $v$ is the collective velocity for the model Hamiltonian system) which shows the system to be in non-topological state [42]. In this phase $\Delta$ decreases rapidly with the length scale, i.e, there is no asymptotic freedom here. For this situation the system shows the gapless helical Luttinger liquid phase. Therefore this study reveals that $K$ has a significant effect on the topological state of the system, as the values of $K$ become lower, i.e, the system is in more strongly correlated phase which oppose the topological superconducting phase in the system.

Fig. 2 consists of three panels: the left, middle and right are respectively for $K = 0.3, 1$ and 2.7. The left and middle panels always show the asymptotic freedom like behaviour for the coupling constant $B$. For these situations, $B$ induces the Ising phase in the system, hence there is no topological phase in the system. The system is in the Ising phase for the asymptotic freedom regime otherwise it is gapless helical Luttinger liquid phase. Therefore it become clear from this study that $K$ has a significant effect to change the asymptotic freedom like behaviour to the non-asymptotic freedom like behaviour. The appearence of gapless helical Luttinger liquid phase for the non-asymptotic freedom like behaviour regime is the same for the two sets of RG equations (eq. 6 and eq. 8).

Thus it is clear from this study that for higher values of $K (g_2 < 0, K = 1 - \frac{g_2}{\pi v_F})$ the system is in the attractive regime. The system transits from the asymptotic freedom like behavior to gapless helical Luttinger liquid phase for the RG eq. 8, whereas the system
is in non-asymptotic freedom like behaviour for strong repulsive region. Therefore it has
come clear from this study that the concept of asymptotic freedom not only belongs to the
high energy physics but alike behavior can also be observed in topological state of quantum
matter system.

**Length Scale Dependent Study of Quantum BKT Transition for Finite \( \mu \)**

The quantum BKT equations of \( H_1 \) (eq. 5) for finite \( \mu \) are,

\[
\frac{d\Delta}{dl} = \left[ 2 - \frac{1}{K} \left( 1 + \frac{\mu}{v\pi} \right) \right] \Delta, \quad \frac{dK}{dl} = \Delta^2.
\]

(9)

After doing the following transformation, \( -y_\parallel = \left[ 2 - \frac{1}{K} \left( 1 + \frac{\mu}{v\pi} \right) \right] \), the BKT equations re-
duce to

\[
\frac{d\Delta}{dl} = -y_\parallel \Delta, \quad \frac{dy_\parallel}{dl} = -\frac{\Delta^2}{(1 - \frac{\mu}{v\pi})}.
\]

(10)

There will be no corrections in the RG equation for the Hamiltonian \( H_2 \) (eq. 7). It is because
the sine-Gordon coupling is also for \( \phi \) field. For this situation one can absorb the chemical
potential term in the sine-Gordon coupling [3].

![Graph](image)

**FIG. 3:** This figure consists of two panels (a: \( \mu = 0.5 \) and b: \( \mu = 1 \)). It shows the variation
of \( \Delta \) with \( L \) for the different initial values of \( \Delta \).

Fig. 3 shows the results for finite \( \mu \) (eq. 51). This figure consists of two panels (a and b),
left and right, for \( K = 1 \) and 0.3 respectively, for the fixed value \( \mu = 0.5 \). The different
curves in each figure are for the different initial values of \( \Delta \). We observe for fig. 3a (\( \mu = 0.5 \) &
$K = 1$), RG flow lines flowing off to the strong coupling phase with length scale and finally the system is in the topological superconducting phase. In fig.3b we observe the RG flow lines decreasing with the length scale which indicates the system is in the gapless helical Luttinger liquid phase. Thus we conclude that the presence of $\mu$ does not have any effect on the asymptotic freedom like behavior of the system.

**Summary of the new and important results of the present study:**

We have obtained three quantum phases in this length scale dependent RG flow diagram study. One is topological, i.e., topological superconducting phase, another one is the Ising phase which is non-topological and finally we have obtained gapless helical Luttinger liquid phase which is also non-topological in character. Luttinger liquid parameter $K$, has played most important role to achieve these quantum phases. We have observed that the asymptotic freedom like behaviour can be present both in topological (superconducting phase) and non-topological (Ising phase) phases. We have studied the effect of finite $\mu$ and observed the behavior of the asymptotic nature is intact. To the best of our knowledge this new and important results are absent in the literature of topological state of matter. This is the first study where we show the presence of asymptotic freedom like behavior in the area of condensed matter systems with topological background even though it was only observed in the area high energy physics.

**Plausible experimental verification:**

One can find this length scale dependent quantum phases, which we presented in the previous section from the following consideration. The first step is to quantum simulate the topological quantum state in semiconductor nanowire for different length scale, i.e, hybrid superconductor-semiconductor nanowire [50]. One has to very fine control on the proximity induced superconductivity gap ($\Delta$) and the applied magnetic field ($B$), these are the two key players for the different quantum phases either topological or non-topological in character. The other points to be noted that the $K$ and $\mu$ must have to be vary with a high precision either by using the gate voltage or by the doping manipulation. The nanowires of different length scale with variation of these four factors can achieve the all phases and transition among themselves.
Conclusion
We have done the detailed length scale dependent study of quantum BKT transition for interacting helical liquid system at the edge of topological insulator. We have found the presence of asymptotic freedom like behavior for the topological state of the system for finite proximity induced superconducting gap and also for the Ising phase of the system in the presence of magnetic field. There is no evidence of superconductor-insulator transition in this asymptotic freedom study. We have presented importance of the Luttinger liquid parameter in different phases of the study. We have observed the presence of helical Luttinger liquid phase for the two different BKT transitions as a common quantum phase. We have also presented the results for finite chemical potential. This work provides a new perspective on the study of the topological state of quantum matter.

Acknowledgment
R.K.R and R.S would like to acknowledge Mr. N. Prakash and Prof. R. Srikanth for reading the manuscript critically. R.K.R and R.S also acknowledge RRI library for the books and journals and ICTS for Lectures/seminars/workshops/conferences/discussion meetings of different aspects of physics. S.S would like to acknowledge DST (EMR/2017/000898) for the support.

[1] Lev Davidovich Landau and I Ya Pomeranchuk, On point interactions in quantum electrodynamics, In Dokl. Akad. Nauk Ser. Fiz., 102:489 (1955).
[2] Frank Wilczek, Asymptotic freedom: From paradox to paradigm, Proceedings of the National Academy of Sciences of the United States of America, 102:8403–8413 (2005).
[3] David J. Gross and Frank Wilczek, Ultraviolet behavior of non-abelian gauge theories, Phys. Rev. Lett., 30:1343–1346 (1973).
[4] H. David Politzer, Reliable perturbative results for strong interactions?, Phys. Rev. Lett., 30:1346–1349 (1973).
[5] C. N. Yang and R. L. Mills, Conservation of isotopic spin and isotopic gauge invariance, Phys.
[6] Murray Gell-Mann, A Schematic Model of Baryons and Mesons, Phys. Lett., 8:214–215 (1964).

[7] G. Zweig, An SU(3) model for strong interaction symmetry and its breaking Version 1, CERN-TH-401 (1964).

[8] Y Nambu, Proceedings of the second coral gables conference on symmetry principles at high energy (1965).

[9] Moo-Young Han and Yoichiro Nambu, Three-triplet model with double SU(3) symmetry, Phys. Rev., 139:B1006 (1965).

[10] Oscar W Greenberg, Spin and unitary-spin independence in a paraquark model of baryons and mesons, Phys. Rev. Lett., 13:598 (1964).

[11] Jerome I Friedman and Henry W Kendall, Deep inelastic electron scattering, Annual Review of Nuclear Science, 22:203–254 (1972).

[12] Alexander Altland and Ben D Simons, Condensed matter field theory, Cambridge University Press (2010).

[13] E. Fradkin, Field Theories of Condensed Matter Physics, Cambridge University Press (2013).

[14] Eduardo C Marino, Quantum Field Theory Approach to Condensed Matter Physics, Cambridge University Press (2017).

[15] T. Giamarchi, Quantum Physics in One Dimension, Clarendon Press (2003).

[16] Richard P Feynman, Relativistic cut-off for quantum electrodynamics, Phys. Rev., 74:1430 (1948).

[17] Julian Schwinger, On quantum-electrodynamics and the magnetic moment of the electron, Phys. Rev., 73:416 (1948).

[18] S. Tomonaga, On a relativistically invariant formulation of the quantum theory of wave fields, Progress of Theoretical Physics, 1:27–42 (1946).

[19] G. ’t Hooft and M. Veltman, Regularization and renormalization of gauge fields, Nuclear Physics B, 44:189–213 (1972).

[20] Curtis G Callan Jr, Broken scale invariance in scalar field theory, Phys. Rev. D, 2:1541 (1970).

[21] Kurt Symanzik, Small distance behavior in field theory and power counting, Communications in Mathematical Physics, 18:227–246 (1970).

[22] Ramamurti Shankar, Quantum Field Theory and Condensed Matter: An Introduction, Cam-
bridge University Press (2017).

[23] David J Gross and Andre Neveu, Dynamical symmetry breaking in asymptotically free field theories, Phys. Rev. D, 10:3235 (1974).

[24] M. Gell-Mann and M. Lévy, The axial vector current in beta decay, Il Nuovo Cimento, 16:705–726 (1960).

[25] Jun Kondo, Resistance minimum in dilute magnetic alloys, Progress of theoretical physics, 32:37–49 (1964).

[26] M. C. Diamantini, C. A. Trugenberger and V. M. Vinokur, Confinement and asymptotic freedom with Cooper pairs, Communications Physics, 1:77 (2018).

[27] Jorge V José, Int. Jour. of Mod. Phys. B 31, 1730001 (2017).

[28] G. Ortiz, E. Cobanera, and Z. Nussinov, Berezinskii-Kosterlitz-Thouless Transition Through the Eyes of Duality, In 40 Years of Berezinskii-Kosterlitz-Thouless Theory, World Scientific Publishing Co (2013).

[29] Wolfgang Bietenholz and Urs Gerber, Berezinskii-Kosterlitz-Thouless Transition and the Haldane Conjecture: Highlights of the Physics Nobel Prize 2016, arXiv preprint arXiv:1612.06132 (2016).

[30] V.L. Berezinskii, Sov. Phys. JETP 32, 493–500 (1971).

[31] J M Kosterlitz and D J Thouless, Journal of Physics C: Solid State Physics 6, 1181 (1973).

[32] P. C. Hohenberg, Phys. Rev. 158, 383–386 (1967).

[33] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133–1136 (1966).

[34] Lara Benfatto, Claudio Castellani and Thierry Giamarchi, Berezinskii–Kosterlitz–Thouless Transition within the Sine-Gordon Approach: The Role of the Vortex-Core Energy, In 40 Years of Berezinskii–Kosterlitz–Thouless Theory, World Scientific Publishing Co (2013).

[35] M. Z. Hasan and C. L. Kane. Rev. Mod. Phys. 82 (2010) 3045.

[36] Joel E Moore. Nature, 464(7286) (2010) 194.

[37] Hidetoshi Nishimori and Gerardo Ortiz. OUP Oxford (2010).

[38] Zheng-Cheng Gu and Xiao-Gang Wen. Phys. Rev. B, 80 (2009) 155131.

[39] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. Phys. Rev. B, 83 (2011) 035107.

[40] C. L. Kane and E. J. Mele, Z₂ Topological Order and the Quantum Spin Hall Effect, Phys. Rev. Lett., 95:146802 (2005).
A. Detailed derivation of RG equation for $H_1$ Hamiltonian

In the Bosonized model Hamiltonian $H_1$ is written as

$$H_1 = \int dx \frac{v}{2} \left[ \frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right] - \left( \frac{\Delta}{\pi} \right) \int dx \cos(\sqrt{4\pi} \theta(x)).$$ \hspace{1cm} (11)

We rescale the fields as, $\phi \rightarrow \phi' = \phi/\sqrt{K}$ and $\theta \rightarrow \theta' = \sqrt{K} \theta$. The Lagrangian for quadratic and interaction terms can be written as

$$\mathcal{L}_0 = -\frac{1}{4} \left[ v^{-1} (\partial_\tau \theta')^2 + v (\partial_x \theta')^2 \right]; \quad \mathcal{L}_\Delta = \left( \frac{\Delta}{\pi} \right) \cos(\sqrt{4\pi} \theta(x)), \hspace{1cm} (12)$$

where $\tau = it$ is the imaginary time. The Euclidean action can be written as, $S_E = -\int dr \mathcal{L} =$...
\[-\int dr (\mathcal{L}_0 + \mathcal{L}_\Delta), \text{ where } r = (\tau, x). \] The partition function in terms of Euclidean action as
\[
Z = \int \mathcal{D}\theta e^{\left[ \int dr \left( -\frac{1}{4} v^{-1} (\partial_r \theta)^2 - \frac{1}{4} v (\partial_x \theta)^2 \right) - \int dr \mathcal{L}_\Delta (\theta') \right]}. \tag{13}
\]
\[
Z = \int \mathcal{D}\theta e^{\left[ -\int_{-\Lambda/b}^{\Lambda/b} d\omega \frac{|K(\omega)|^2}{2} - \int dr \mathcal{L}_\Delta (\theta) \right]}. \tag{14}
\]
We now separate the slow and fast fields and integrate out the fast field components. The field \( \theta \) is \( \theta(r) = \theta_s(r) + \theta_f(r) \) where,
\[
\theta_s(r) = \int_{-\Lambda/b}^{\Lambda/b} d\omega e^{-i\omega\tau(\omega)} \quad \& \quad \theta_f(r) = \int_{\Lambda/b < |\omega_n| < \Lambda} d\omega e^{-i\omega\tau(\omega)}, \tag{15}
\]
here \( r = (x, \tau). \) Now the partition function can be written as,
\[
Z = \int \mathcal{D}\theta_s \mathcal{D}\theta_f e^{\left[ -S_s(\theta_s) - S_f(\theta_f) - S_\Delta(\theta_s, \theta_f) \right]}, \tag{16}
\]
The effective action can be written as cumulant expansion up to second order as
\[
S_{\text{eff}}(\theta_s) = S_s(\theta_s) - \ln \langle e^{-S_\Delta(\theta_s, \theta_f)} \rangle_f. \tag{17}
\]
\[
S_{\text{eff}}(\theta_s) = S_s(\theta_s) + \langle S_\Delta(\theta_s, \theta_f) \rangle - \frac{1}{2} \left( \langle S_\Delta^2(\theta_s, \theta_f) \rangle - \langle S_\Delta(\theta_s, \theta_f) \rangle^2 \right). \tag{18}
\]
Now we calculate the first order approximation \( \langle S_\Delta(\theta_s, \theta_f) \rangle, \)
\[
\langle S_\Delta(\theta_s, \theta_f) \rangle = \frac{\Delta}{\pi} \int d\theta f e^{-S_f(\theta_f)} \int dr \left\{ \cos(\sqrt{4\pi} \theta(r)) \right\},
\]
\[
= \frac{\Delta}{2\pi} \int dr \left\{ e^{i\sqrt{4\pi} \theta_s(r)} \int d\theta f e^{\left( j_f \frac{d\omega}{\sqrt{4\pi} e^{i\omega r} \theta_f - |\omega|^2} K(\omega) \right)} + H.c \right\}, \tag{19}
\]
\[
= \frac{\Delta}{\pi} \int dr \cos[\sqrt{4\pi} \theta_s(r)] e^{\left( -\frac{1}{\pi} j_f \frac{d\omega}{\sqrt{4\pi} e^{i\omega r} \theta_f - |\omega|^2} \right)}. \]
We write, \( \int_f \frac{d\omega}{|\omega|} = \int_{-\Lambda/b}^{\Lambda/b} \frac{d\omega}{|\omega|} = \ln \Lambda - \ln(\Lambda/b) = \ln \left( \frac{\Lambda}{\Lambda/b} \right) = \ln b. \)

\[
\langle S_\Delta(\theta_s, \theta_f) \rangle = \frac{\Delta}{\pi} \int dr \cos[\sqrt{4\pi} \theta_s(r)] e^{-\frac{1}{K} \ln b}, \\
= b^{-\frac{1}{K}} S_\Delta(\theta_s).
\] (20)

Thus the effective action upto first order cumulant expansion can be written as,

\[
S_{eff}(\theta_s) = S_s(\theta_s) + b^{-\frac{1}{K}} S_\Delta(\theta_s),
\]

\[
S_{eff}(\theta_s) = \int_{-\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} |\omega| K|\theta_s(\omega)|^2 + b^{-\frac{1}{K}} \int dr \left( \frac{\Delta}{\pi} \right) \cos[\sqrt{4\pi} \theta_s(r)].
\] (21)

Now we rescale the parameters cut-off momentum to the original momentum by considering, \( \bar{\Lambda} = \frac{\Lambda}{b} \), \( \bar{\omega} = \omega b \) and \( \bar{r} = \frac{r}{b} \). The fields will be rescaled as, \( \bar{\theta}(\bar{\omega}) = \frac{\theta(\omega)}{b} \) and we choose \( \bar{\theta}(\bar{r}) = \theta_s(r) \). Also since the system is (1+1) dimensional we have \( d^2r = b^2d^2\bar{r} \). Thus the rescaled effective action is given by,

\[
S_{eff}(\theta_s) = \int_{-\Lambda/b}^{\Lambda/b} \frac{d\bar{\omega}}{2\pi b} |\bar{\omega}| K|\bar{\theta}(\bar{\omega})|^2 + b^{2-\frac{1}{K}} \int d^2\bar{r} \left( \frac{\bar{\Delta}}{\pi} \right) \cos[\sqrt{4\pi} \bar{\theta}(\bar{r})],
\]

\[
= \int_{-\Lambda/b}^{\Lambda/b} \frac{d\bar{\omega}}{2\pi |\bar{\omega}|} K|\bar{\theta}(\bar{\omega})|^2 + b^{2-\frac{1}{K}} \int d^2\bar{r} \left( \frac{\bar{\Delta}}{\pi} \right) \cos[\sqrt{4\pi} \bar{\theta}(\bar{r})].
\] (22)

Comparing the coupling constants of rescaled effective action with the not renormalized action one can observe that, \( \Delta \rightarrow \Delta b^{2-\frac{1}{K}} \). Thus we write the RG flow equation as, \( \bar{\Delta} = \Delta b^{2-\frac{1}{K}} \). We write this equation in the differential form defining the differential of a parameter as \( d\Delta = \Delta - \Delta \) and setting \( b = e^{dl} \),

\[
\frac{d\Delta}{dl} = \left( 2 - \frac{1}{K} \right) \Delta.
\] (23)

Now we solve for the second order cumulant expansion,

\[
-\frac{1}{2} \left( \langle S_\Delta^2 \rangle - \langle S_\Delta \rangle^2 \right) = -\frac{\Delta^2}{2\pi^2} - \int dr dr' \left[ \langle \cos[\sqrt{4\pi} \theta(r)] \rangle \langle \cos[\sqrt{4\pi} \theta(r')] \rangle - \langle \cos[\sqrt{4\pi} \theta(r)] \rangle \langle \cos[\sqrt{4\pi} \theta(r')] \rangle \right].
\] (24)
First we calculate $\langle S^2_\Delta \rangle$ term.

$$\langle S^2_\Delta \rangle = \frac{\Delta^2}{\pi^2} \int drdr' \left\langle \cos[\sqrt{4\pi}\theta(r)] \cos[\sqrt{4\pi}\theta(r')] \right\rangle,$$

$$\langle S^2_\Delta \rangle = \frac{\Delta^2}{2\pi^2} \int drdr' \left[ \cos \sqrt{4\pi}[\theta_s(r) + \theta_s(r')] \left( e^{-2\pi[\langle \theta_f^2(r) \rangle + \langle \theta_f^2(r') \rangle + 2\langle \theta_f(r)\theta_f(r') \rangle]} \right) + \cos \sqrt{4\pi}[\theta_s(r) - \theta_s(r')] \left( e^{-2\pi[\langle \theta_f^2(r) \rangle - \langle \theta_f(r)\theta_f(r') \rangle]} \right) \right] + H.c \right]. \tag{25}$$

Now we calculate $\langle S_\Delta \rangle^2$.

$$\langle S_\Delta \rangle^2 = \frac{\Delta^2}{\pi^2} \int drdr' \left\langle \cos[\sqrt{4\pi}\theta(r)] \right\rangle \left\langle \cos[\sqrt{4\pi}\theta(r')] \right\rangle,$$

$$\langle S_\Delta \rangle^2 = \frac{\Delta^2}{2\pi^2} \int drdr' \left[ \cos \sqrt{4\pi}[\theta_s(r) + \theta_s(r')] e^{-2\pi\langle \theta_f^2(r) \rangle} e^{-2\pi\langle \theta_f^2(r') \rangle} + \cos \sqrt{4\pi}[\theta_s(r) - \theta_s(r')] e^{-2\pi\langle \theta_f^2(r) \rangle} e^{-2\pi\langle \theta_f^2(r') \rangle} \right] + H.c \right]. \tag{26}$$

Thus the term $(\langle S^2_\Delta \rangle - \langle S_\Delta \rangle^2)$ is,

$$\langle S^2_\Delta \rangle - \langle S_\Delta \rangle^2 = \frac{\Delta^2}{\pi^2} \int drdr' \left[ \cos \sqrt{4\pi}[\theta_s(r) + \theta_s(r')] \left( e^{-2\pi[\langle \theta_f^2(r) \rangle + \langle \theta_f^2(r') \rangle + 2\langle \theta_f(r)\theta_f(r') \rangle]} \right) + \cos \sqrt{4\pi}[\theta_s(r) - \theta_s(r')] \left( e^{-2\pi[\langle \theta_f^2(r) \rangle - \langle \theta_f(r)\theta_f(r') \rangle]} \right) \right] + H.c \right]
- \frac{\Delta^2}{2\pi^2} \int drdr' \left[ \cos \sqrt{4\pi}[\theta_s(r) + \theta_s(r')] e^{-2\pi\langle \theta_f^2(r) \rangle} e^{-2\pi\langle \theta_f^2(r') \rangle} + \cos \sqrt{4\pi}[\theta_s(r) - \theta_s(r')] e^{-2\pi\langle \theta_f^2(r) \rangle} e^{-2\pi\langle \theta_f^2(r') \rangle} + H.c \right]. \tag{27}$$

$$-\frac{1}{2}(\langle S^2_\Delta \rangle - \langle S_\Delta \rangle^2) = -\frac{\Delta^2}{4\pi^2} \int drdr' \left[ \cos \sqrt{4\pi}[\theta_s(r) + \theta_s(r')] e^{-2\pi[\langle \theta_f^2(r) \rangle + \langle \theta_f^2(r') \rangle]} \left( e^{-4\pi\langle \theta_f(r)\theta_f(r') \rangle} - 1 \right) + \cos \sqrt{4\pi}[\theta_s(r) - \theta_s(r')] e^{-2\pi[\langle \theta_f^2(r) \rangle + \langle \theta_f^2(r') \rangle]} \left( e^{-4\pi\langle \theta_f(r)\theta_f(r') \rangle} - 1 \right) \right]. \tag{28}$$

The correlation function $\langle \theta_f(r)\theta_f(r') \rangle$ for $r' \to r$ can be written as [48],

$$\langle \theta_f(r)\theta_f(r') \rangle \approx \langle \theta_f^2(r) \rangle = \frac{1}{2\pi K} \int_{\Lambda/b}^\Lambda \frac{d\omega}{\omega} = \frac{1}{2\pi K} \ln b. \tag{29}$$
We introduce the relative coordinate \( r = r - r' \) and center of mass coordinate \( T = (r + r')/2 \).

For small \( s \) cosine can be approximated to,
\[
\cos \sqrt{4\pi}[\theta_s(r) + \theta_s(r')] = \cos[4\sqrt{\pi}(\theta_s(T))]; \quad \cos \sqrt{4\pi}[\theta_s(r) - \theta_s(r')] = 1 - 2\pi(s\partial_T\theta_s(T))^2.
\]

Thus eq.28 can be written as,
\[
-\frac{1}{2} \langle S^2_{\Delta} \rangle - \langle S_{\Delta} \rangle^2 = -\frac{\Delta^2}{4\pi^2} \left( 1 - \left( \frac{1}{b} \right)^2 \right) \int_0^{b/\Lambda} ds \int dT (1 - 2\pi(s\partial_T\theta_s(T))^2). \tag{30}
\]

Here the first term turns out to be field independent term. Thus we consider only second term,
\[
-\frac{1}{2} \langle S^2_{\Delta} \rangle - \langle S_{\Delta} \rangle^2 = \frac{\Delta^2}{2\pi} \left( 1 - \left( \frac{1}{b} \right)^2 \right) \int_0^{b/\Lambda} s^2 ds \int dT (\partial_T\theta_s(T))^2,
= \frac{\Delta^2}{6\pi\Lambda^3} \left( \left( \frac{1}{b} \right)^{-3} - \left( \frac{1}{b} \right)^{\frac{2}{\pi} - 3} \right) \int_{-\Lambda/\bar{b}}^{\Lambda/\bar{b}} \frac{d\bar{\omega}}{2\pi} \left| \omega \right| \left| K[\bar{\theta}(\bar{\omega})] \right|^2. \tag{31}
\]

After rescaling the parameters and fields the equation will have the form,
\[
-\frac{1}{2} \langle S^2_{\Delta} \rangle - \langle S_{\Delta} \rangle^2 = \frac{\Delta^2}{6\pi\Lambda^3} \left( \left( \frac{1}{\bar{b}} \right)^{-3} - \left( \frac{1}{\bar{b}} \right)^{\frac{2}{\bar{\pi} - 3} - 3} \right) \int_{-\Lambda/\bar{b}}^{\Lambda/\bar{b}} \frac{d\bar{\omega}}{2\pi} \left| \bar{\omega} \right| \left( \bar{K}[\bar{\theta}(\bar{\omega})] \right)^2. \tag{32}
\]

Now the effective action can be written as,
\[
S_{eff} = \int_{-\Lambda}^{\Lambda} \frac{d\bar{\omega}}{2\pi} \left| \bar{\omega} \right| \left( \bar{K}[\bar{\theta}(\bar{\omega})] \right)^2 + b^2 - \frac{\Delta}{\pi} \int d^2\bar{r} \left( \frac{\Delta}{\pi} \right) \cos[\sqrt{4\pi\bar{\theta}(\bar{r})}]
+ \frac{\Delta^2}{6\pi\Lambda^3} \left( \left( \frac{1}{\bar{b}} \right)^{-3} - \left( \frac{1}{\bar{b}} \right)^{\frac{2}{\bar{\pi} - 3} - 3} \right) \int_{-\Lambda/\bar{b}}^{\Lambda/\bar{b}} \frac{d\bar{\omega}}{2\pi} \left| \bar{\omega} \right| \left( \bar{K}[\bar{\theta}(\bar{\omega})] \right)^2, \tag{33}
\]
\[
S_{eff} = \left[ 1 + \frac{\Delta^2}{6\pi\Lambda^3} \left( \left( \frac{1}{\bar{b}} \right)^{-3} - \left( \frac{1}{\bar{b}} \right)^{\frac{2}{\bar{\pi} - 3} - 3} \right) \right] S_s(\theta_s) + b^2 - \frac{\Delta}{\pi} S_\Delta(\theta_s). \tag{34}
\]
Comparing the rescaled effective action with the original action we obtain RG flow equation,

\[
K = K \left[ 1 + \frac{\Delta^2}{6\pi \Lambda^3} \left( \frac{1}{b} \right)^{-3} - \left( \frac{1}{b} \right)^{-3} \right].
\]  

(35)

Defining the differential of a parameter as \(dK = \bar{K} - K\) and by setting \(b = e^{dl}\) we get RG flow equation in the differential form

\[
dK = -K \frac{\Delta^2}{6\pi \Lambda^3} (e^{3dl} - e^{(3-2)dl}),
\]

\[
\frac{dK}{dl} = \left( \frac{\Delta^2}{3\pi \Lambda^3} \right).
\]  

(36)

We rescale \(\Delta \rightarrow \Delta \sqrt{\frac{1}{3\pi \Lambda^3}}\). Thus RG flow equations of \(H_1\) are given by

\[
\frac{d\Delta}{dl} = \left( 2 - \frac{1}{K} \right) \Delta, \quad \frac{dK}{dl} = \Delta^2.
\]  

(37)

Similarly one can derive quantum BKT equation for Hamiltonian \(H_2\) as

\[
\frac{dB}{dl} = (2 - K)B, \quad \frac{dK}{dl} = -B^2K^2.
\]  

(38)