The Elliptic Function in Statistical Integrable Models II

Tezukayama University, Tezukayama 7, Nara 631, Japan
Kazuyasu Shigemoto

Abstract

Two dimensional statistical integrable models, such as the Ising model, the chiral Potts model and the Belavin model, becomes integrable. Because of the $SU(2)$ symmetry of these models, these models become integrable. The integral models are often parameterized by the elliptic function or the elliptic theta function. In this paper, we study the Ising model, and we show that the Yang-Baxter equation of the Ising model can be written as the integrability condition of $SU(2)$, and we gives the natural explanation why the Ising model can be parameterized by the elliptic function by connecting the spherical trigonometry relation and the elliptic function. Addition formula of the elliptic function is the secret of the exact solvability of the Ising model.

1 Introduction

There are many two dimensional integrable statistical models, which are classified into the spin model, the vertex model and the face model[1].

First and famous integrable and exactly solvable model is the Ising model[2]. The Ising model is the 2-state model, and the generalized $N$-state integrable model is the chiral Potts model[3]. The structure of the integrability condition of the Ising model is $SU(2)$. While that of the chiral Potts model is the cyclic $SU(2)$[4, 5]. The Boltzmann weight of the Ising model can be parameterized by the elliptic function.

While the $2 \times 2$-state integrable vertex model is the 8-vertex model[6], and the generalized $N \times N$-state integrable model is the Belavin model[7]. In the Belavin model, the structure of the integrability condition is the cyclic $SU(2)$[8, 9]. The Boltzmann weight for the 8-vertex

1E-mail address: shigemot@tezukayama-u.ac.jp
model is parameterized by the elliptic function, and that for the Belavin model by the elliptic theta function with characteristics.

Baxter’s hard hexagon model[10] is the first integrable 2-state face model, which is obtained from the 8-vertex model by the vertex-face correspondence, and the generalized integrable model is $A_{N-1}^{(1)}$ model[11]. The Boltzmann weight for these models are parameterized by the elliptic theta function with characteristics.

Then the origin of the integrability condition of the two dimensional statistical model comes from the $SU(2)$ symmetry or the variant of $SU(2)$ symmetry. And we have the elliptic representation of the Boltzmann weight in many important cases. Then we expect the correspondence between the $SU(2)$ symmetry and the elliptic function. In other words, we expect that the symmetry of the elliptic function is the $SU(2)$ symmetry, and the origin of the addition formulae of the elliptic function comes from the $SU(2)$ group structure. This is just the same as the origin of the trigonometric addition formulae comes from the $U(1)$ group structure. For the trigonometric function, we make the connection with the circle, which has $U(1)$ symmetry. For the elliptic function, we make the connection with the surface of the sphere, which has $SU(2)$ symmetry.

2 Integrability condition and exactly solvable condition

Let’s consider the general Yang-Baxter equation of the spin model, and we explain the difference between the integrability condition and the exactly solvable condition. The Yang-Baxter equation is given by

$$V(u_3)U(u_3, u_1)V(u_1) = U(u_1)V(u_1, u_3)U(u_3). \quad (2.1)$$

This type of Yang-Baxter equation is called the integrability condition. The meaning of the integrability condition is that the model has the group structure so that the model has nice symmetry. As the Yang-Baxter equation of this type says that the product of three group action for two different path gives the same group action, the group structure of the model is expected to be $SU(2)$ or the variation of $SU(2)$.

Furthermore if the Yang-Baxter equation satisfies the difference property such as

$$V(u_3)U(u_1 + u_3)V(u_1) = U(u_1)V(u_1 + u_3)U(u_3), \quad (2.2)$$
we call this type of equation as the exactly solvable condition. This is called the exactly solvable condition because the non-Abelian group action can be realized by the addition of the parameter $u_i$. Then the non-Abelian model is parameterized by the Abelian parameters $u_i$, which makes the model exactly solvable. The non-linear addition formula of $u_i$ is the key relation that the non-Abelian integrable model becomes exactly solvable.

The Ising model is one of the examples of the exactly solvable model. Then we pick up this Ising Model and explain the parameterization of the Ising model, which has the difference property, in a quite explicite way. The star-triangle relation of the Ising model can be written in the form \[4, 5\]

$$\exp(L_1^*\sigma_x)\exp(K_2\sigma_z)\exp(L_3^*\sigma_x) = \exp(K_1\sigma_z)\exp(L_2^*\sigma_x)\exp(K_3\sigma_z).$$ \tag{2.3}$$

In the previous paper\[5\], we give the parameterization of the Ising model by the elliptic function in the form

$$\cosh 2K_i = \frac{1}{\cn(u_i)}, \quad \sinh 2K_i = \frac{\sn(u_i)}{\cn(u_i)}, \quad (i = 1, 2, 3),$$ \tag{2.4}$$

$$\cosh 2L_i^* = \frac{1}{\sn(K - u_i)}, \quad \sinh 2L_i^* = \frac{\cn(K - u_i)}{\sn(K - u_i)}, \quad (i = 1, 2, 3).$$ \tag{2.5}$$

Comparing Eq.(2.1) with Eq.(2.3), we have

$$U(u_i) = \exp(K_i\sigma_z), \quad V(u_i) = \exp(L_i^*\sigma_x),$$ \tag{2.6}$$

and the Yang-Baxter equation comes to have the difference property, that is, we have the exactly solvable condition in the form

$$V(u_3)U(u_1 + u_3)V(u_1) = U(u_1)V(u_1 + u_3)U(u_3),$$ \tag{2.7}$$

by the parameterization of the elliptic function. We will explain why the Ising model can be parameterized by the elliptic function, and the Ising model comes to have the difference property.

3 Differential equation for the angle and the arc in the spherical triangle

The connection between the addition formula of the elliptic function and the addition formula of the spherical triangle has the long history, which starts from Lagrange and Legendre\[12, 13\]
and nice review article is given by Greenhill[14]. We briefly review how to parameterize spherical angle and spherical arc by the elliptic function, which give the parameterization of the Yang-Baxter relation by the elliptic function.

### 3.1 First differential equation

In this subsection, we will give the differential equation for the angle on the sphere. Various spherical trigonometry formulae are given by Todhunter[17]. The proof of the necessary formula we use here is given in the Appendix.

The first law of cosine is given by

\[
\cos(a_1) = \cos(a_2) \cos(a_3) + \cos(A_1) \sin(a_2) \sin(a_3),
\]

\( (3.1) \)

\[
\cos(a_2) = \cos(a_3) \cos(a_1) + \cos(A_2) \sin(a_3) \sin(a_1),
\]

\( (3.2) \)

\[
\cos(a_3) = \cos(a_1) \cos(a_2) + \cos(A_3) \sin(a_1) \sin(a_2),
\]

\( (3.3) \)

and the law of sine is given by

\[
\frac{\sin(A_1)}{\sin(a_1)} = \frac{\sin(A_2)}{\sin(a_2)} = \frac{\sin(A_3)}{\sin(a_3)} = k.
\]

\( (3.4) \)

We keep \( k \) to be constant, and we fix \( a_3 \), which means that we fix \( A_3 \) through Eq.(3.4).

Next we differentiate Eq.(3.3) and we have

\[
0 = -\sin(a_1) \cos(a_2) da_1 - \cos(a_1) \sin(a_2) da_2
+ \cos(A_3) \cos(a_1) \sin(a_2) da_1 + \cos(A_3) \sin(a_1) \cos(a_2) da_2
\]

\[= ( -\sin(a_1) \cos(a_2) + \cos(A_3) \cos(a_1) \sin(a_2) ) da_1 \]

\[+ ( -\cos(a_1) \sin(a_2) + \cos(A_3) \sin(a_1) \cos(a_2) ) da_2. \]

\( (3.5) \)

Substituting expressions

\[
\cos(A_1) = \frac{\cos(a_1) - \cos(a_2) \cos(a_3)}{\sin(a_2) \sin(a_3)},
\]

\[
\cos(A_2) = \frac{\cos(a_2) - \cos(a_3) \cos(a_1)}{\sin(a_3) \sin(a_1)},
\]

\[
\cos(A_3) = \frac{\cos(a_3) - \cos(a_1) \cos(a_2)}{\sin(a_1) \sin(a_2)},
\]

\( (3.6) \)
into the first term of Eq.(3.5), we have
\[
(\text{First term}) = (-\sin(a_1) \cos(a_2) + \cos(A_3) \cos(a_1) \sin(a_2)) da_1 \\
= -\sin^2(a_1) \cos(a_2) + \cos(a_1) \cos(a_3) - \cos^2(a_1) \cos(a_2) da_1 \\
= -\cos(a_2) + \cos(a_1) \cos(a_3) da_1 \\
= -\cos(A_2) \sin a_3 da_1. \quad (3.7)
\]

Similarly, for the second term of Eq.(3.5), we have
\[
(\text{Second term}) = (-\cos(a_1) \sin(a_2) + \cos(A_3) \sin(a_1) \cos(a_2)) da_2 \\
= -\cos(A_1) \sin a_3 da_2. \quad (3.8)
\]

Combining Eq.(3.5), Eq.(3.7), Eq.(3.8), we have
\[
\cos(A_1) da_1 + \cos(A_2) da_2 = 0. \quad (3.9)
\]

Using the relation $\sin(A_1) = k \sin(a_1)$, we have $\cos(A_1) = \sqrt{1 - k^2 \sin^2(a_1)}$ where we assume $A_1$ is acute. Then Eq.(3.9) can be written in the form
\[
\frac{da_1}{\sqrt{1 - k^2 \sin^2(a_1)}} + \frac{da_2}{\sqrt{1 - k^2 \sin^2(a_2)}} = 0. \quad (3.10)
\]

We put $x = \sin(a_1)$, $y = \sin(a_2)$, then we have $dx = \cos(a_1) da_1 = \sqrt{1 - x^2} da_1$, which gives $da_1 = \frac{1}{\sqrt{1 - x^2}}$. Then Eq.(3.10) is written in the form of the differential equation of the elliptic function
\[
\frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} + \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} = 0. \quad (3.11)
\]

Up to now, we consider $a_3$ to be fixed. For the general case, we put $z = \sin(a_3)$ and obtain the general differential equation of the form
\[
\frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} + \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}} + \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} = 0. \quad (3.12)
\]
where we assume that three angels $A_1, A_2, A_3$ are all acute. If we put

$$u_1 = \int^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},$$  \hspace{1cm} (3.13)$$

$$u_2 = \int^y \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}},$$  \hspace{1cm} (3.14)$$

$$u_3 = \int^z \frac{dy}{\sqrt{(1 - z^2)(1 - k^2 z^2)}},$$  \hspace{1cm} (3.15)$$

$$u_1 + u_2 + u_3 = \text{Const.},$$  \hspace{1cm} (3.16)$$

we have $x = \sin(a_1) = \sin(u_1, k)$, that is, $a_1 = \text{am}(u_1, k)$. Similarly we have $a_2 = \text{am}(u_2, k)$, $a_3 = \text{am}(u_3, k)$.

### 3.2 Second differential equation

In this subsection, we will give the differential equation for the arc on the sphere. We start from the second law of cosine

$$-\cos(A_1) = \cos(A_2) \cos(A_3) - \cos(a_1) \sin(A_2) \sin(A_3),$$

$$-\cos(A_2) = \cos(A_3) \cos(A_1) - \cos(a_2) \sin(A_3) \sin(A_1),$$

$$-\cos(A_3) = \cos(A_1) \cos(A_2) - \cos(a_3) \sin(A_1) \sin(A_2),$$  \hspace{1cm} (3.17)$$

and the law of sine

$$\frac{\sin(a_1)}{\sin(A_1)} = \frac{\sin(a_2)}{\sin(A_2)} = \frac{\sin(a_3)}{\sin(A_3)} = \frac{1}{k}.$$  \hspace{1cm} (3.18)$$

The differential equation for this system is obtained by replacing $a_i \rightarrow \pi - A_i (i = 1, 2, 3)$ and $k \rightarrow 1/k$ in Eq.(3.10). Then we put $X = \sin(A_1)$, $Y = \sin(A_2)$, $Z = \sin(A_3)$, and we obtain

$$\frac{dX}{\sqrt{(1 - X^2)(1 - x^2/k^2)}} + \frac{dY}{\sqrt{(1 - Y^2)(1 - Y^2/k^2)}} + \frac{dZ}{\sqrt{(1 - Z^2)(1 - Z^2/k^2)}} = 0,$$  \hspace{1cm} (3.19)$$
where we assume that three length of arc $a_1$, $a_2$, $a_3$ are all acute. If we put

\[ U_1 = \int_0^X \frac{dX}{\sqrt{(1 - X^2)(1 - X^2/k^2)}}, \quad (3.20) \]

\[ U_2 = \int_0^Y \frac{dY}{\sqrt{(1 - Y^2)(1 - Y^2/k^2)}}, \quad (3.21) \]

\[ U_3 = \int_0^Z \frac{dZ}{\sqrt{(1 - Z^2)(1 - Z^2/k^2)}}, \quad (3.22) \]

\[ U_1 + U_2 + U_3 = \text{Const.}, \quad (3.23) \]

we have $A_i = \text{am}(U_i, 1/k)$ ($i = 1, 2, 3$). We also have another expression $X = \sin(A_i) = \text{sn}(U_i, 1/k) = k\text{sn}(U_i/k, k)$. Using the law of sine, we have $k \sin(a_i) = \sin(A_i)$, and $\sin(a_i) = \text{sn}(u_i, k)$, $\sin(A_i) = k\text{sn}(U_i/k, k)$, we can connect $u_i$ and $U_i$ through $\text{sn}(u_i, k) = \text{sn}(U_i/k, k)$, so that we have $U_i = ku_i$, that is,

\[ \sin(a_i) = \text{sn}(u_i, k) \quad (3.24) \]

\[ \sin(A_i) = k \sin(a_i) = \text{sn}(ku_i, 1/k) = k\text{sn}(u_i, k). \quad (3.25) \]

We can obtain the addition formula of the elliptic function from the addition formula of the spherical cosine formula. From Eq.(3.2) and Eq.(3.17), we have

\[ \cos(a_2) = \cos(a_3) \cos(a_1) + \cos(A_2) \sin(a_3) \sin(a_1), \quad (3.26) \]

\[ - \cos(A_2) = \cos(A_3) \cos(A_1) - \cos(a_2) \sin(A_3) \sin(A_1), \quad (3.27) \]

and eliminating $\cos(A_2)$ or $\cos(a_2)$, we have

\[ \cos(a_2) = \frac{\cos(a_1) \cos(a_3) - \sin(a_1) \cos(A_1) \sin(a_3) \cos(A_3)}{1 - \sin(a_1) \sin(A_1) \sin(a_3) \cos(A_3)}, \quad (3.28) \]

\[ - \cos(A_2) = \frac{\cos(A_1) \cos(A_3) - \cos(a_1) \sin(A_1) \cos(a_3) \sin(A_3)}{1 - \sin(a_1) \sin(A_1) \sin(a_3) \cos(A_3)}, \quad (3.29) \]

From here, we take $A_2$ to be obtuse to find the addition formula of the elliptic function in the convenient form. In this case, we have $u_2 = u_1 + u_3$, and $\sin(a_i) = \text{sn}(u_i, k)$, $\sin(A_i) = k \sin(a_i)$, $\cos(a_i) = \text{cn}(u_i, k)$ ($i = 1, 2, 3$), $\cos(A_1) = \text{dn}(u_1, k)$, $\cos(A_2) = -\text{dn}(u_2, k)$, $\cos(A_3) = \text{dn}(u_3, k)$. Then Eq.(3.28) and Eq.(3.29) gives

\[ \text{cn}(u_2, k) = \text{cn}(u_1 + u_3, k) \]
\[ \frac{\text{cn}(u_1, k)\text{cn}(u_3, k) - \text{sn}(a_1, k)\text{dn}(u_1, k)\text{sn}(u_3, k)\text{dn}(u_1, k)}{1 - k^2\text{sn}^2(u_1, k)\text{sn}^2(u_3, k)}, \]  

\[ (3.30) \]

\[ \text{dn}(u_2, k) = \text{dn}(u_1 + u_3, k) \]

\[ = \frac{\text{dn}(u_1, k)\text{dn}(u_3, k) - k^2\text{sn}(u_1, k)\text{cn}(u_1, k)\text{sn}(u_3, k)\text{cn}(u_3, k)}{1 - k^2\text{sn}^2(u_1, k)\text{sn}^2(u_3, k)}, \]  

\[ (3.31) \]

and using \( \text{sn}(u_3, k) = \sqrt{1 - \text{cn}^2(u_3, k)} \), we have

\[ \text{sn}(u_2, k) = \text{sn}(u_1 + u_3, k) \]

\[ = \frac{\text{sn}(u_1, k)\text{cn}(u_3, k)\text{dn}(u_3, k) + \text{cn}(u_1, k)\text{dn}(u_1, k)\text{sn}(u_3, k)}{1 - k^2\text{sn}^2(u_1, k)\text{sn}^2(u_3, k)}. \]  

\[ (3.32) \]

These are convenient form of addition formula of the elliptic function.

### 4 Parameterization by the elliptic function

The spherical trigonometry relation such as laws of the first and the second cosine and the law of sines are obtained from the group theoretical relation of \( SU(2) \) in the form

\[ \exp\{iA_1J_x\} \exp\{ia_2J_z\} \exp\{iA_3J_x\} = \exp\{ia_3J_z\} \exp\{i(\pi - A_2)J_x\} \exp\{ia_1J_z\}, \]  

\[ (4.1) \]

where \( J_x, J_y, J_z \) are \( SU(2) \) generators with \([J_x, J_y] = iJ_z, [J_y, J_z] = iJ_x, [J_z, J_x] = iJ_y\). For \( J = 1 \) case, Eq.(4.1) becomes

\[ \{1 - (1 - \cos(A_1))J_x^2 + i\sin(A_1)J_z\} \{1 - (1 - \cos(a_2))J_z^2 + i\sin(a_2)J_x\} \]

\[ \{1 - (1 - \cos(A_3))J_x^2 + i\sin(A_3)J_z\} = \{1 - (1 - \cos(a_3))J_z^2 + i\sin(a_3)J_x\} \]

\[ \{1 - (1 - \cos(\pi - A_2))J_x^2 + i\sin(\pi - A_2)J_z\} \{1 - (1 - \cos(a_1))J_z^2 + i\sin(a_1)J_x\}, \]

\[ J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
This relation is understood as the integrability condition on the sphere. We consider 3 points on the sphere $P_1$, $P_2$, $P_3$. Starting from the vector $\mathbf{n}_1$ at point $P_3$. Then the vector parallel transports along the tangential direction of the large circle or rotates around the point $P_i (i = 1, 2, 3)$. The following group element corresponds to the parallel transport or the rotation around the point $P_i (i = 1, 2, 3)$ of the vector in the following way

$$\exp \{ iA_3 J_x \} : \mathbf{n}_1 \rightarrow \mathbf{n}_5,$$
$$\exp \{ ia_2 J_z \} : \mathbf{n}_5 \rightarrow \mathbf{n}_6,$$
$$\exp \{ iA_1 J_x \} : \mathbf{n}_6 \rightarrow \mathbf{n}_4,$$
$$\exp \{ ia_1 J_z \} : \mathbf{n}_1 \rightarrow \mathbf{n}_2,$$
$$\exp \{ i(\pi - A_2) J_x \} : \mathbf{n}_2 \rightarrow \mathbf{n}_3,$$
$$\exp \{ ia_3 J_z \} : \mathbf{n}_3 \rightarrow \mathbf{n}_4$$

Therefore Eq.(4.1) is the integrability condition that two different ways to change the vector from $\mathbf{n}_1$ to $\mathbf{n}_4$, that is, path 1:{$\mathbf{n}_1 \rightarrow \mathbf{n}_5 \rightarrow \mathbf{n}_6 \rightarrow \mathbf{n}_4$} and path 2:{$\mathbf{n}_1 \rightarrow \mathbf{n}_2 \rightarrow \mathbf{n}_3 \rightarrow \mathbf{n}_4$}, gives the same group action, which is nothing but the integrability condition of $SU(2)$. While the star-triangle relation can be written in the form

$$\exp \{ 2L_1^* J_x \} \exp \{ 2K_2 J_z \} \exp \{ 2L_3^* J_x \} = \exp \{ 2K_3 J_z \} \exp \{ 2L_2^* J_x \} \exp \{ 2K_1 J_z \}. \quad (4.2)$$

In the previous paper, we parameterize the Ising model with the elliptic function in the form

$$\cosh(2K_i) = \frac{1}{\text{cn}(v_i, k')}, \quad \sinh(2K_i) = \frac{\text{sn}(v_i, k')}{\text{cn}(v_i, k')}, \quad (4.3)$$
$$\cosh(2L_i^*) = \frac{\text{dn}(v_i, k')}{\text{cn}(v_i, k')}, \quad \sinh(2L_i^*) = \frac{k\text{sn}(v_i, k')}{\text{cn}(v_i, k')}, \quad (4.4)$$
\[ v_2 = v_1 + v_3, \quad (i = 1, 2, 3). \]  

(4.5)

By using the Jacobi’s imaginary transformation, we put \( v_i = iu_i \) and we have

\[
\begin{align*}
\sn(v_i, k') &= i \frac{\sn(u_i, k)}{\cn(u_i, k)}, \\
\cn(v_i, k') &= \frac{1}{\cn(u_i, k)}, \\
\dn(v_i, k') &= \frac{\dn(u_i, k)}{\cn(u_i, k)}.
\end{align*}
\]

(4.6)

then we have

\[
\begin{align*}
\cosh(2K_i) &= \cn(u_i, k), \quad \sinh(2K_i) = isn(u_i, k), \quad \text{(4.7)} \\
\cosh(2L_i^*) &= \dn(u_i, k), \quad \sinh(2L_i^*) = i\sn(u_i, k), \quad \text{(4.8)} \\
u_2 &= u_1 + u_3, \quad (i = 1, 2, 3). \quad \text{(4.9)}
\end{align*}
\]

Using the relation

\[
\begin{align*}
\sn(u, 1/k) &= k\sn(u/k, k), \\
\cn(u, 1/k) &= \dn(u/k, k), \\
\dn(u, 1/k) &= \cn(u/k, k), \quad \text{(4.10)}
\end{align*}
\]

and putting \( K_i = i\hat{K}_i, \quad L_i^* = i\hat{L}_i^* \), we have

\[
\begin{align*}
\cos(2\hat{K}_i) &= \cn(u_i, k), \quad \sin(2\hat{K}_i) = \sn(u_i, k), \quad \text{(4.11)} \\
\cos(2\hat{L}_i^*) &= \cn(ku_i, 1/k), \quad \sin(2\hat{L}_i^*) = \sn(ku_i, 1/k), \quad \text{(4.12)} \\
u_2 &= u_1 + u_3, \quad (i = 1, 2, 3). \quad \text{(4.13)}
\end{align*}
\]

For the spherical triangle, we take \( A_2 \) as obtuse, and \( A_1 \) and \( A_3 \) are acute. In this case

\[
\frac{dA_2}{\sqrt{1 - \sin^2(A_2)/k^2}} = \frac{dA_1}{\sqrt{1 - \sin^2(A_1)/k^2}} + \frac{dA_3}{\sqrt{1 - \sin^2(A_3)/k^2}},
\]

(4.14)

and we take \( U_2 = U_1 + U_3 \), which gives \( u_2 = u_1 + u_3 \) by using \( U_i = ku_i(i = 1, 2, 3) \). While the spherical triangle is parameterized by

\[
\begin{align*}
\cos(a_i) &= \cn(u_i, k), \quad \sin(a_i) = \sn(u_i, k), \quad \text{(4.15)} \\
\cos(A_i) &= \cn(U_i, 1/k) = \cn(ku_i, 1/k), \quad \sin(A_i) = \sn(ku_i, 1/k), \quad \text{(4.16)} \\
u_2 &= u_1 + u_3, \quad (i = 1, 2, 3). \quad \text{(4.17)}
\end{align*}
\]

Then we have

\[
2\hat{K}_i = a_i = am(u_i, k), \quad 2\hat{L}_i^* = A_i = am(ku_i, 1/k), \quad \text{(4.18)}
\]

\]
where we must notice that \( A_2 \) is obtuse, so that \( \pi - A_2 \) is acute. Then the Yang-Baxter equation is written in the form
\[
\exp\{i\alpha(ku_1, 1/k)J_x\} \exp\{i\alpha(u_2, k)J_z\} \exp\{i\alpha(ku_3, 1/k)J_x\} = \exp\{i\alpha(u_3, k)J_z\} \exp\{i\alpha(ku_2, 1/k)J_x\} \exp\{i\alpha(u_1, k)J_z\}.
\] (4.19)

5 Abel’s addition theorem

The addition theorem of the algebraic function, that is, the Abel’s addition theorem is the essential key relation to solve the statistical model. Then we consider the old problem of the Abel’s addition theorem here. We put
\[
f_4(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4,
\]
and consider the \( g = 1 \) algebraic curve \( y^2 = f_4(x) \), then Jacobi or Richelot\(^{15, 16}\) tried to find the conserved quantity of the differential equation
\[
\frac{dx_1}{\sqrt{f_4(x_1)}} + \frac{dx_2}{\sqrt{f_4(x_2)}} = 0.
\] (5.1)

While we consider here the Abel’s addition theorem of the standard type, that is,
\[
\frac{dx_1}{\sqrt{f_4(x_1)}} + \frac{dx_2}{\sqrt{f_4(x_2)}} + \frac{dx_3}{\sqrt{f_4(x_3)}} = 0.
\] (5.2)

In general, we take
\[
f_{2n-2}(x) = A_0 + A_1x + A_2x^2 + \cdots + A_{2n-3}x^{2n-3} + A_{2n-2}x^{2n-2},
\] (5.3)
and consider the differential equation of the form
\[
\frac{dx_1}{\sqrt{f_{2n-2}(x_1)}} + \frac{dx_2}{\sqrt{f_{2n-2}(x_2)}} + \cdots + \frac{dx_n}{\sqrt{f_{2n-2}(x_n)}} = 0,
\]
\[
\frac{x_1dx_1}{\sqrt{f_{2n-2}(x_1)}} + \frac{x_2dx_2}{\sqrt{f_{2n-2}(x_2)}} + \cdots + \frac{x_n dx_n}{\sqrt{f_{2n-2}(x_n)}} = 0,
\]
\[
\cdots
\]
\[
\frac{x_1^{n-3}dx_1}{\sqrt{f_{2n-2}(x_1)}} + \frac{x_2^{n-3}dx_2}{\sqrt{f_{2n-2}(x_2)}} + \cdots + \frac{x_n^{n-3}dx_n}{\sqrt{f_{2n-2}(x_n)}} = 0.
\] (5.4)
We define $F(x)$ by

$$ F(x) = (x - x_1)(x - x_2)\cdots(x - x_{n-1})(x - x_n), \quad (5.5) $$

then we have

$$ F'(x_1) = (x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_{n-1})(x_1 - x_n). \quad (5.6) $$

e tc. Then if $x_i$, which depend on $t$, satisfies the following equation

$$ \frac{dx_i}{dt} = \frac{x_i\sqrt{f_{2n-2}(x_i)}}{F'(x_i)}, \quad (5.7) $$

we can show that Eq.(5.4) is satisfied. Here we use the following theorem for $F(x)$

$$ \frac{x^k}{F(x)} = \sum_{i=1}^{n} \frac{x_i^k}{F'(x_i)} \frac{1}{(x - x_i)}, \quad (k = 0, 1, \ldots, n - 1). \quad (5.8) $$

The proof is the followings: As $F(x)$ is the $n$-th polynomial and it is much higher polynomial than $x^k$, we can write

$$ \frac{x^k}{F(x)} = \sum_{i=1}^{n} \frac{a_i}{(x - x_i)}, \quad (k = 0, 1, \ldots, n - 1), \quad (5.9) $$

then we multiply $(x - x_j)$ and take the limit $x \to x_j$, we have

$$ \frac{x^k_j}{F'(x_j)} = \sum_{i=1}^{n} a_i \delta_{i,j} = a_j, \quad (5.10) $$

which gives Eq.(5.8). By multiplying $x$ and taking $x \to \infty$ in Eq.(5.8), we have

$$ \sum_{i=1}^{n} \frac{x_i^k}{F'(x_i)} = \delta_{k,n-1}, \quad (k = 0, 1, \ldots, n - 1). \quad (5.11) $$

Then the differential equation of the problem is satisfied in the following way

$$ \sum_{i=1}^{n} \frac{x_i^kdx_i}{\sqrt{f_{2n-2}(x_i)}} = \sum_{i=1}^{n} \frac{x_i^{k+1}\sqrt{f_{2n-2}(x_i)}}{F'(x_i)\sqrt{f_{2n-2}(x_i)}} = \sum_{i=1}^{n} \frac{x_i^{k+1}dt}{F'(x_i)} = 0, \quad (5.12) $$

$(k = 0, 1, \cdots, n - 3)$
5.1 Abel’s addition theorem I: one conserved quantity

We will integrate this differential equation and find the conserved quantity. Starting from Eq. (5.4), we have

\[
\frac{d^2 x_1}{dt^2} = \frac{dx_1}{dt} \frac{d}{dx_1} \left( \frac{x_1 \sqrt{f_{2n-2}(x_1)}}{F'(x_1)} \right) + \frac{dx_2}{dt} \frac{d}{dx_2} \left( \frac{x_1 \sqrt{f_{2n-2}(x_1)}}{F'(x_1)} \right) + \ldots
\]

\[
+ \frac{dx_n}{dt} \frac{d}{dx_n} \left( \frac{x_1 \sqrt{f_{2n-2}(x_1)}}{F'(x_1)} \right)
\]

\[
= \frac{x_1 \sqrt{f_{2n-2}(x_1)}}{F'(x_1)} \frac{d}{dx_1} \left( \frac{x_1 \sqrt{f_{2n-2}(x_1)}}{F'(x_1)} \right) + \frac{x_1 x_2 \sqrt{f_{2n-2}(x_1)f_{2n-2}(x_2)}}{F'(x_1)F'(x_2)} \frac{1}{(x_1 - x_2)} + \ldots
\]

\[
+ \frac{x_1 x_n \sqrt{f_{2n-2}(x_1)f_{2n-2}(x_n)}}{F'(x_1)F'(x_n)} \frac{1}{(x_1 - x_n)}
\]

\[
= \frac{1}{2} \frac{d}{dx_1} \left( \frac{x_1^2 f_{2n-2}(x_1)}{(F'(x_1))^2} \right) + \sum_{i=1}^{n} \frac{x_1 x_i \sqrt{f_{2n-2}(x_1)f_{2n-2}(x_i)}}{F'(x_1)F'(x_i)} \frac{1}{(x_1 - x_i)}.
\]

By taking the sum, we have

\[
\frac{d^2}{dt^2} \sum_{i=1}^{n} x_i = \frac{1}{2} \sum_{i=1}^{n} \frac{d}{dx_i} \left( \frac{x_i^2 f_{2n-2}(x_i)}{(F'(x_i))^2} \right).
\] (5.16)

While we use the formula

\[
\frac{x^2 f_{2n-2}(x)}{F(x)^2} - A_{2n-2} = \sum_{i=1}^{n} \left( \frac{x_i^2 f_{2n-2}(x_i)}{(F'(x_i))^2(x - x_i)^2} + \frac{d}{dx_i} \left( \frac{x_i^2 f_{2n-2}(x_i)}{(F'(x_i))^2} \right) \frac{1}{(x - x_i)} \right).
\] (5.17)

The proof of the above is the following: We can write

\[
\frac{x^2 f_{2n-2}(x)}{F(x)^2} - A_{2n-2} = \sum_{i=1}^{n} \left( \frac{a_i}{(x - x_i)^2} + \frac{b_i}{(x - x_i)} \right).
\] (5.18)

We multiply \((x - x_j)^2\) and take the limit \(x \rightarrow x_j\), the we have

\[
\frac{x_j^2 f_{2n-2}(x_j)}{F'(x_j)^2} = \sum_{i=1}^{n} a_i \delta_{i,j} = a_j.
\] (5.19)
Next we multiply \((x - x_j)^2\) and further multiply \(\frac{d}{dx}\) and take the limit \(x \to x_j\), we have

\[
\lim_{x \to x_j} \frac{d}{dx} \left( \frac{(x - x_j)^2 x^2 f_{2n-2}(x)}{F(x)^2} \right) = \frac{d}{dx_j} \left( \frac{x^2 f_{2n-2}(x_j)}{F'(x_j)^2} \right) = \sum_{i=1}^{n} b_i \delta_{i,j} = b_j.
\]  

(5.20)

From Eq.(5.18), Eq.(5.19), Eq.(5.20), we have Eq.(5.17). While by using the relation

\[
\frac{x^2 f_{2n-2}(x)}{F(x)^2} - A_{2n-2} = \left( A_{2n-3} + 2A_{2n-2} \sum_{i=1}^{n} x_i \right) \frac{1}{x} + O \left( \frac{1}{x^2} \right),
\]

and by comparing the coefficient of \(O \left( \frac{1}{x} \right)\) in Eq.(5.17), we have

\[
A_{2n-3} + 2A_{2n-2} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{d}{dx_i} \left( \frac{x^2 f_{2n-2}(x_i)}{F'(x_i)^2} \right) = 2 \frac{d^2}{dt^2} \sum_{i=1}^{n} x_i,
\]

(5.22)

where we use Eq.(5.16). We put \(p = \sum_{i=1}^{n} x_i\), then we have the differential equation of the form

\[
2 \frac{d^2 p}{dt^2} = A_{2n-3} + 2A_{2n-2}p.
\]

(5.23)

Multiplying \(\frac{dp}{dt}\) in both side of Eq.(5.23), we have

\[
\frac{d}{dt} \left( \left( \frac{dp}{dt} \right)^2 - A_{2n-3}p - A_{2n-2}p^2 \right) = 0.
\]

(5.24)

Then we have one conserved quantity

\[
\left( \sum_{i=1}^{n} \frac{x_i \sqrt{f_{2n-2}(x_i)}}{F'(x_i)} \right)^2 = A_{2n-3} \sum_{i=1}^{n} x_i + A_{2n-2} \left( \sum_{i=1}^{n} x_i \right)^2 + C_1.
\]

(5.25)

### 5.2 Abel’s addition theorem II: another conserved quantity

As Richelot[16] showed, we transform the original differential equation of \(x_i\) into that of \(\xi_i = \frac{1}{x_i}\). In this variable, we have

\[
dx_i = - \frac{d\xi_i}{\xi_i^2},
\]

(5.26)
\[
\frac{1}{\sqrt{f_{2n-2}(x_i)}} = \frac{1}{\sqrt{f_{2n-2}(1/\xi_i)}} = \frac{\xi_i^{n-1}}{\sqrt{g_{2n-2}(\xi_i)}},
\]

where
\[
g_{2n-2}(\xi_i) = A_{2n-2} + A_{2n-3}\xi + \cdots + A_1\xi^{2n-3} + A_0\xi^{2n-2},
\]

which gives
\[
\frac{x_i^k dx_i}{\sqrt{f_{2n-2}(x_i)}} (k = 0, 1, \ldots, n - 3)
\]
\[
= -\frac{\xi_i^{k'} d\xi_i}{\sqrt{g_{2n-2}(\xi_i)}} (k' = n - 3 - k = 0, 1, \ldots, n - 3).
\]

Therefore the differential equation is transformed into the form
\[
\sum_{i=1}^{n} \frac{\xi_i^k d\xi_i}{\sqrt{g_{2n-2}(\xi_i)}} = 0, \quad (k = 0, 1, \ldots, n - 3).
\]

Defining \( F_1(\xi) = \prod_{i=1}^{n}(\xi - \xi_i) \) and comparing with Eq.(5.25), we have another conserved quantity
\[
\left( \sum_{i=1}^{n} \frac{\xi_i \sqrt{g_{2n-2}(x_i)}}{F_1(\xi_i)} \right)^2 = A_1 \sum_{i=1}^{n} \xi_i + A_0 (\sum_{i=1}^{n} \xi_i)^2 + C_2.
\]

Using
\[
F_1'(\xi_i) = \frac{(-1)^{n-1} F'(x_i)}{x_1 x_2 \cdots x_n x_i^{n-2}},
\]

we have
\[
\left( \sum_{i=1}^{n} \frac{\sqrt{f_{2n-2}(x_i)}}{x_i^2 F'(x_i)} \right)^2 (x_1 x_2 \cdots x_n)^2 = A_1 \sum_{i=1}^{n} \frac{1}{x_i} + A_0 (\sum_{i=1}^{n} \frac{1}{x_i})^2 + C_2.
\]
5.3 Simple $n = 3$ case: Abel’s addition theorem for the elliptic function

For $n = 3$ case with $f_4(x) = (1 - x^2)(1 - k^2 x^2)$, we numerically checked by Maxima the following addition formula

$$
\sum_{i=1}^{3} x_i \sqrt{\frac{f_4(x_i)}{F'(x_i)}} + k \sum_{i=1}^{3} x_i = 0,
$$

\[ (x_1 x_2 x_3) \sum_{i=1}^{3} \frac{\sqrt{f_4(x_i)}}{x_i^2 F'(x_i)} - \sum_{i=1}^{3} \frac{1}{x_i} = 0, \]

where

$$
x_i = \text{sn}(u_i), \quad \sqrt{f_4(x_i)} = \text{cn}(u_i)\text{dn}(u_i), \quad u_1 + u_2 + u_3 = 0.
$$

6 Summary and discussion

The integrability condition, which is called the Yang-Baxter equation, in two dimensional statistical models means that such integrable models have the group structure. The Boltzmann weight of such integrable models often can be parameterized by the elliptic function or the elliptic theta function. Furthermore if the Yang-Baxter equation have the nice difference property, we can exactly solve the model. Such exactly solvable models often can be parameterized by the elliptic function. If we notice that the Yang-Baxter relation is the relation of the products of the three group action, we expect that the integrable model has the $SU(2)$ or $SU(2)$ variant group structure. Furthermore if the model is exactly solvable, we expect that the exactly solvable model can be parameterized by the elliptic function. As the Ising model is the exactly solvable model, the above expectation is realized by rewriting the Yang-Baxter relation in the form of the integrability condition of $SU(2)$ group. Furthermore we can parameterize the Boltzmann weight by the elliptic function, which comes to have the difference property, by making the connection between the $SU(2)$ group element and the elliptic function.

In this way, we can understand the reason why the Ising model can be parameterized by the elliptic function in a quite natural way, where the connection between the $SU(2)$ and the elliptic function is essential. Addition formula of the elliptic function is the secret of the exact solvability of the Ising model.

Finally, we find some of the conserved quantity for the hyperelliptic differential equation, which gives the explicit form of the Abel’s addition theorem. This will will be useful to find
the group structure of such hyperelliptic function.

References

[1] R. J. Baxter, ”Exactly Solved Models in Statistical Mechanics”, (Academic, New York), 1982.

[2] L. Onsager, Phys. Rev., 60 (1944), 117.

[3] R. J. Baxter, J. H. H. Perk and H. Au-Yang, Phys. Lett., A128 (1973), 138.

[4] M. Horibe and K. Shigemoto, Nuovo Cimento, 116B (2001), 1017.

[5] K. Shigemoto, Tezukayama Academic Review, No.17 (2011), 15.

[6] R. J. Baxter, Ann. Phys., 76 (1973), 1.

[7] A. A. Belavin, Nucl. Phys., 180 (1981), 189.

[8] C. A. Tracy, Physica, D16 (1985), 203.

[9] I. V. Cherednik, Sov. J. Nucl. Phys., 36 (1982), 320.

[10] R. J. Baxter, Ann. Phys., 76 (1973), 25.

[11] M. Jimbo, T. Miwa and M. Okado, Nucl. Phys., B300 [FS22] (1988), 74.

[12] J.L. Lagrange, ”Théorie des Fonctions Analytiques”, (1801), p.85.

[13] A.M. Legendre, ”Traité des Fonctions Elliptiques”, I (1825), p.19.

[14] G. Greenhill, ”The Application of Elliptic Functions”, (London, Macmillan), (1892), p.131.

[15] C.G.J. Jacobi, ”Demonstratio Nova Theorematis Abelianii”, J. Reine Angew. Math., 24 (1842), 28-35.
[16] F. Richelot, "Ueber die Integration eines merkwürdigen Systems Differentialgleichungen", *J. Reine Angew. Math.*, 23 (1842), 354-369.

[17] I. Todhunter, "Spherical Trigonometry", (London, Macmillan), (1919), p.24.
A Spherical trigonometry formulae

In this appendix, we give the brief proof of various spherical trigonometry relations. We consider the sphere with unit radius and we put three points \( P_1, P_2, P_3 \) on the sphere. We consider three unit vectors which go from the origin to points \( P_1, P_2, P_3 \), and we denote such unit vectors as \( \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \). Further, we denote the length of the arc connecting \( \mathbf{n}_2 \) and \( \mathbf{n}_3 \) as \( a_1 \), and that connecting \( \mathbf{n}_3 \) and \( \mathbf{n}_1 \) as \( a_2 \), and that connecting \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) as \( a_3 \).

Then we have

\[
\cos(a_1) = \mathbf{n}_2 \cdot \mathbf{n}_3, \quad \cos(a_2) = \mathbf{n}_3 \cdot \mathbf{n}_1, \quad \cos(a_3) = \mathbf{n}_1 \cdot \mathbf{n}_2. \tag{A.1}
\]

We make the dual unit vectors perpendicular to \( \mathbf{n}_i \) and \( \mathbf{n}_j \) \( (i, j = 1, 2, 3), (i \neq j) \) in the following way

\[
\mathbf{n}_i^* = \frac{\mathbf{n}_2 \times \mathbf{n}_3}{|\mathbf{n}_2 \times \mathbf{n}_3|}, \quad \mathbf{n}_2^* = \frac{\mathbf{n}_3 \times \mathbf{n}_1}{|\mathbf{n}_3 \times \mathbf{n}_1|}, \quad \mathbf{n}_3^* = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{|\mathbf{n}_1 \times \mathbf{n}_2|}. \tag{A.2}
\]

For planes perpendicular to \( \mathbf{n}_i \) and \( \mathbf{n}_j \), we denote \( l_{ij}(i, j = 1, 2, 3), (i \neq j) \), which are given by

\[
l_{23} : \mathbf{n}_1^* \cdot \mathbf{x} = 0, \quad l_{31} : \mathbf{n}_2^* \cdot \mathbf{x} = 0, \quad l_{12} : \mathbf{n}_3^* \cdot \mathbf{x} = 0. \tag{A.3}
\]

Fig. 2 Angles and arcs of the spherical triangle
We denote the rotation angle of the spherical triangle around vector \( \mathbf{n}_1 \) as \( A_1 \) as is shown in Fig. 2, then we have \( \cos(A_1-\pi) = \mathbf{n}_2^* \cdot \mathbf{n}_3^* \) from geometrical consideration. Similarly we have

\[
\cos(A_1 - \pi) = \mathbf{n}_2^* \cdot \mathbf{n}_3^*, \quad \cos(A_2 - \pi) = \mathbf{n}_3^* \cdot \mathbf{n}_1^*, \quad \cos(A_3 - \pi) = \mathbf{n}_1^* \cdot \mathbf{n}_2^*. \tag{A.4}
\]

Using the formula

\[
(a \times b) \times (c \times d) = |a, c, d|b - |b, c, d|a = |a, b, d|c - |a, b, c|d,
\]

where

\[
|a, b, c| = a \cdot (b \times c) = (a \times b) \cdot c = \text{(determinant of } a, b, c),
\]

we have

\[
\mathbf{n}_1 = \frac{\mathbf{n}_2^* \times \mathbf{n}_3^*}{|\mathbf{n}_2^* \times \mathbf{n}_3^*|}, \quad \mathbf{n}_2 = \frac{\mathbf{n}_3^* \times \mathbf{n}_1^*}{|\mathbf{n}_3^* \times \mathbf{n}_1^*|}, \quad \mathbf{n}_3 = \frac{\mathbf{n}_1^* \times \mathbf{n}_2^*}{|\mathbf{n}_1^* \times \mathbf{n}_2^*|}. \tag{A.6}
\]

### A.1 Law of cosines

Using the theorem

\[
(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c), \tag{A.7}
\]

we have

\[
\cos(\pi - A_1) = \mathbf{n}_2^* \cdot \mathbf{n}_3^* = \frac{(\mathbf{n}_3 \times \mathbf{n}_1)}{|\mathbf{n}_3 \times \mathbf{n}_1|} \cdot \frac{(\mathbf{n}_1 \times \mathbf{n}_2)}{|\mathbf{n}_1 \times \mathbf{n}_2|} = \frac{(\mathbf{n}_1 \cdot \mathbf{n}_3)(\mathbf{n}_1 \cdot \mathbf{n}_2) - (\mathbf{n}_2 \cdot \mathbf{n}_3)}{|\mathbf{n}_3 \times \mathbf{n}_1||\mathbf{n}_1 \times \mathbf{n}_2|} = \frac{\cos(a_2) \cos(a_3) - \cos(a_1)}{\sin(a_2) \sin(a_3)}. \tag{A.8}
\]

Then we have the first law of cosine in the form

\[
\cos(a_1) = \cos(a_2) \cos(a_3) + \cos(A_1) \sin(a_2) \sin(a_3),
\]

\[
\cos(a_2) = \cos(a_3) \cos(a_1) + \cos(A_2) \sin(a_3) \sin(a_1),
\]

\[
\cos(a_3) = \cos(a_1) \cos(a_2) + \cos(A_3) \sin(a_1) \sin(a_2). \tag{A.9}
\]

While from the following relation

\[
\cos(a_1) = \mathbf{n}_2 \cdot \mathbf{n}_3 = \frac{(\mathbf{n}_3 \times \mathbf{n}_1^*)}{|\mathbf{n}_3 \times \mathbf{n}_1^*|} \cdot \frac{(\mathbf{n}_1^* \times \mathbf{n}_2^*)}{|\mathbf{n}_1^* \times \mathbf{n}_2^*|} = \frac{(\mathbf{n}_1^* \cdot \mathbf{n}_3^*)(\mathbf{n}_1^* \cdot \mathbf{n}_2^*) - (\mathbf{n}_2^* \cdot \mathbf{n}_3^*)}{|\mathbf{n}_3 \times \mathbf{n}_1^*||\mathbf{n}_1^* \times \mathbf{n}_2^*|} = \frac{\cos(\pi - A_2) \cos(\pi - A_3) - \cos(\pi - A_1)}{\sin(\pi - A_2) \sin(\pi - A_3)}. \tag{A.10}
\]
Then we have the second law of cosines in the following form

\[
- \cos(A_1) = \cos(A_2) \cos(A_3) - \cos(a_1) \sin(A_2) \sin(A_3), \\
- \cos(A_2) = \cos(A_3) \cos(A_1) - \cos(a_2) \sin(A_3) \sin(A_1), \\
- \cos(A_3) = \cos(A_1) \cos(A_2) - \cos(a_3) \sin(A_1) \sin(A_2).
\] (A.11)

### A.2 Law of sines

Using the relation

\[
|n^* \times n^*_3| = \frac{|n_3 \times n_1|}{|n_3 \times n_1|} \times \frac{|n_1 \times n_2|}{|n_1 \times n_2|} \cdot n_1 = \frac{|n_1, n_2, n_3|}{|n_1 \times n_2|} \cdot \frac{|n_1 \times n_2|}{|n_3 \times n_1|} \cdot n_1,
\] (A.12)

we have

\[
\frac{|n^* \times n^*_3|}{|n_2 \times n_3|} = \frac{|n^*_3 \times n^*_1|}{|n_3 \times n_1|} = \frac{|n^*_1 \times n^*_2|}{|n_1 \times n_2|} \\
= \frac{|n_1, n_2, n_3|}{|n_1 \times n_2|} \cdot \frac{|n_1 \times n_2|}{|n_3 \times n_1|} = \frac{|n^*_1, n^*_2, n^*_3|}{|n_1, n_2, n_3|}.
\] (A.13)

where we use

\[
|n^*_1, n^*_2, n^*_3| = (n^*_1 \times n^*_2) \cdot n^*_3 = \frac{|n_1, n_2, n_3|}{|n_2 \times n_3|} \cdot \frac{|n_3 \times n_1|}{|n_1 \times n_2|} \cdot (n_1 \times n_2) \\
= \frac{(|n_1, n_2, n_3|)^2}{|n_2 \times n_3| |n_3 \times n_1| |n_1 \times n_2|}.
\] (A.14)

From Eq.(A.13), we have the law of sines in the following form

\[
\frac{\sin(\pi - A_1)}{\sin(a_1)} = \frac{\sin(\pi - A_2)}{\sin(a_2)} = \frac{\sin(\pi - A_3)}{\sin(a_3)},
\]
or

\[
\frac{\sin(A_1)}{\sin(a_1)} = \frac{\sin(A_2)}{\sin(a_2)} = \frac{\sin(A_3)}{\sin(a_3)}.
\] (A.15)

We use Eq.(A.9), Eq.(A.11), Eq.(A.15) in the body of the text.