We shall call a polynomial map \( f: \mathbb{C}^2 \rightarrow \mathbb{C} \) a “coordinate” if there is a \( g \) such that \( (f, g): \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) is a polynomial automorphism. Equivalently, by Abhyankar-Moh and Suzuki, \( f \) has one and therefore all fibres isomorphic to \( \mathbb{C} \). Following [7] we call a polynomial \( f: \mathbb{C}^2 \rightarrow \mathbb{C} \) “rational” if the general fibres of \( f \) (and hence all fibres of \( f \)) are rational curves. The following theorem, which says that a rational polynomial map with irreducible fibres cannot be part of a counterexample to the 2-dimensional Jacobian Conjecture, has appeared in the literature several times. It appears with an algebraic proof in Razar [12]. It is Theorem 2.5 of Heitmann [4] (as corrected in the Corrigendum), and Lê and Weber, who give a geometric proof in [6], also cite the reference Friedland [3], which we have not seen.

**Theorem 1.** If \( f: \mathbb{C}^2 \rightarrow \mathbb{C} \) is a rational polynomial map with irreducible fibres and is not a coordinate then \( f \) has no jacobian partner (i.e., no polynomial \( g \) such that the jacobian of \((f, g)\) is a non-zero constant).

In this note we prove the above theorem is empty:

**Theorem 2.** There is no \( f \) satisfying the assumptions of the above theorem. That is, a rational \( f \) with irreducible fibres is a coordinate.

**Proof.** This theorem is implicit in [7]. Suppose \( f \) is rational. As in [7], [6], etc., we consider a nonsingular compactification \( Y = \mathbb{C}^2 \cup E \) of \( \mathbb{C}^2 \) such that \( f \) extends to a holomorphic map \( \overline{f}: Y \rightarrow \mathbb{P}^1 \). Then \( E \) is a union of smooth rational curves \( E_1, \ldots, E_n \) with normal crossings. An \( E_i \) is called horizontal if \( \overline{f}|_{E_i} \) is non-constant.

Let \( \delta \) be the number of horizontal curves. Then we have

\[
\delta - 1 = \sum_{a \in \mathbb{P}^1} (r_a - 1),
\]

where \( r_a \) is the number of irreducible components of \( f^{-1}(a) \). This is Lemma 1.6 of Miyanishi and Sugie [8] who attribute it to Saito [10] and Lemma 4 of Lê-Weber [6] who attribute it to Kaliman [5], corollary 2. The proof is simple arithmetic from the topological observation that on the one hand the euler characteristic of \( Y \) is \( n + 2 \) and on the other hand it is \( 4 + \sum_{a \in \mathbb{P}^1} (\tau_a - 1) \), where \( \tau_a \) is the number of components of \( \overline{f}^{-1}(a), a \in \mathbb{P}^1 \).

By this formula, if \( f \) has irreducible fibres then there is just one horizontal curve. Lemma 1.7 of [8] now says that \( f \) is a coordinate. This also follows from the following proposition, which implies that the generic fibres of \( f \) have just one point at infinity and are thus isomorphic to \( \mathbb{C} \).

This research is supported by the Australian Research Council.
Proposition 3. Let \( f : \mathbb{C}^2 \to \mathbb{C} \) be any polynomial map and \( \overline{f} : Y \to \mathbb{P}^1 \) an extension as above. Denote by \( d \) the greatest common divisor of the degrees of \( \overline{f} \) on the horizontal curves of \( Y \) and \( D \) the sum of these degrees. Then the general fibre of \( f \) has \( d \) components (so \( f = h \circ f_1 \) for some polynomials \( f_1 : \mathbb{C}^2 \to \mathbb{C} \) and \( h : \mathbb{C} \to \mathbb{C} \) with \( \deg(h) = d \)), each of which is a compact curve with \( D/d \) punctures.

Proof. Let \( E_1, \ldots, E_\delta \) be the horizontal curves and \( d_1, \ldots, d_\delta \) be the degrees of \( \overline{f} \) on these. Note that the points at infinity of a general fibre \( f^{-1}(a) \) are the points where \( f^{-1}(a) \) meet the horizontal curves \( E_i \), so there are \( d_i \) such points on \( E_i \) for \( i = 1, \ldots, \delta \). The relationship between plumbing diagram and splice diagram (cf. [9, 2]) says that the splice diagram \( \Gamma \) for a regular link at infinity for \( f \) (cf. [8]) has \( \delta \) nodes with arrows at them, and the number of arrows at these nodes are \( d_1, \ldots, d_\delta \) respectively. Let \( \Gamma_0 \) be the same splice diagram but with \( d_1/d, \ldots, d_\delta/d \) arrows at these nodes. Then a minimal Seifert surface \( S \) for the link represented by \( \Gamma \) will consist of \( d \) parallel copies of a minimal Seifert surface for the link represented by \( \Gamma_0 \), so this \( S \) has \( d \) components. But the general fibre of \( f \) is such a minimal Seifert surface ([8], Theorem 1), completing the proof. (It also follows that \( \Gamma_0 \) is the regular splice diagram for the polynomial \( f_1 \) of the proposition.)

References

[1] S. Abhyankar and T.T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math. 276 (1975), 148–166.
[2] D. Eisenbud and W.D. Neumann, “Three dimensional link theory and invariants of plane curve singularities” Ann. Math. Study 101 (Princeton University Press 1985).
[3] S. Friedland, On the plane jacobian conjecture, Preprint IHES, May 1994 (per [6]).
[4] R. Heitmann, On the Jacobian conjecture, J. Pure Appl. Algebra 64 (1990), 36–72 and Corrigendum ibid. 90 (1993), 199–200.
[5] S. Kaliman, Two remarks on polynomials in two variables, Pacific J. Math. 154 (1992), 285–295.
[6] Lê Dung Tráng and Claude Weber, Polynômes à fibres rationnelles et conjecture de jacobienne à 2 variables, C. R. Acad. Sci. Paris, 320 (1995), 581–584.
[7] M. Miyanishi and T. Sugie, Generically rational polynomials, Osaka J. Math. 17 (1980), 339–362.
[8] W.D. Neumann, Complex algebraic plane curves via their links at infinity, Inv. Math. 98 (1989), 445–489.
[9] W.D. Neumann, Irregular links at infinity of complex affine plane curves, Quarterly J. Math. (to appear).
[10] H. Saito, Fonctions entières qui se réduisent à certains polynômes. II, Osaka J. Math. 9 (1977), 649–674.
[11] M. Suzuki, Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l’espace \( \mathbb{C}^2 \), J. Math. Soc. Japan 26 (1974), 241–257.
[12] Michael Razar, Polynomial maps with constant Jacobian. Israel J. Math. 32 (1979), 97–106.

Department of Mathematics, The University of Melbourne, Parkville, Vic 3052, Australia

E-mail address: neumann@maths.mu.oz.au

Department of Mathematics, The University of Melbourne, Parkville, Vic 3052, Australia

E-mail address: norbs@maths.mu.oz.au