Gelfand models for classical Weyl groups

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Abstract

In a recent preprint Kodiyalam and Verma give a particularly simple Gelfand model for the symmetric group that is built naturally on the space of involutions. In this manuscript we give a natural extension of Kodiyalam and Verma’s model to a Gelfand model for Weyl groups of type $B_n$ and $D_{2n+1}$. Then we define an explicit isomorphism between this Gelfand model and the polynomial model for the groups $W(A_n)$ and $W(B_n)$ using a technique that we call telescopic decomposition.

1 Introduction

A Gelfand model for a finite group is a complex representation that decomposes into a multiplicity-free sum of all the irreducible complex representations. The terminology was introduced in [15] and alludes to seminal work by Bernstein, Gelfand and Gelfand [7] where models for connected compact Lie group are constructed using induced representations. Continuing along these lines, Klyachko [13] constructed models for general linear groups over finite fields using sums of induced representations. A body of recent work constructs natural Gelfand models for other kinds of groups, focusing in particular on the case of finite reflection groups.

Two types of Gelfand models have emerged in the literature. The first type is an involution model, inspired by Klyachko’s work and studied, for example, in [8], [9] and [11]. The models for the symmetric group in [1] and the generalized symmetric group in [14] are versions of this kind of model. A general result about the existence of involution models for finite Coxeter groups is treated in [10]. For a finite group it is known that the dimension of a Gelfand model is equal to the number of involutions if and only if the irreducible representations can be realized over the real numbers. Hence criteria for the existence of generalized involution models are studied in [14] to treat the case of complex reflection groups.

A second type of model, the polynomial model, was introduced in [2] and used to construct a Gelfand model for the symmetric group. This second type of model is associated to a finite subgroup of the complex general linear group, and is shown to be a Gelfand model for reflection groups of type $B_n$, $D_{2n+1}$, $I_2(n)$ and $G(m,1,n)$ in [4], [5] and [6]. Garge and Oesterlé [10] study the polynomial model in a more general context and give a criteria for when it is a Gelfand model for a finite Coxeter group.

In a recent preprint [12] Kodiyalam and Verma give a particularly simple Gelfand model for the symmetric group that is built naturally on the space of involutions. They raise the prospect of extending their model to other Weyl groups and of finding an explicit relationship to the polynomial model. In the second section of this manuscript we give natural extensions of Kodiyalam and Verma’s model to representations for the Weyl groups of type $B_n$ and type $D_n$. We prove these extensions are Gelfand models for $W(B_n)$ and for $W(D_{2n+1})$. In the third section of this
manuscript, we give an explicit isomorphism between the Gelfand models constructed for $W(A_n)$ and $W(B_n)$ and the polynomial model for these groups by using a technique we call telescopic decomposition. Our main result is Theorem 3.9 at the end of the manuscript.

2 A Gelfand model

In this section we construct natural extensions of Kodiyalam and Verma’s Gelfand model for a Weyl group of type $A_{n-1}$ to representations for the Weyl groups of type $B_n$ and $D_n$. We prove the representation for $W(B_n)$ is a Gelfand model and that representation for $W(D_n)$ is a Gelfand model when $n$ is odd. In what follows, $W$ will denote a Weyl group of type $A_{n-1}$, $B_n$ or $D_n$. Let $\mathfrak{S}_n$ denote the permutation group for the set of indices $I_n = \{1, 2, \ldots, n\}$. We introduce the group $W(B_n)$ as the semidirect product:

$$C_2 \rtimes \mathfrak{S}_n$$

where we think of $C_2 = \{\pm 1\}$ as subgroup of $C^*$. There are natural inclusions:

$$W(A_{n-1}) \simeq \mathfrak{S}_n \subseteq W(D_n) \subseteq W(B_n)$$

Consider the decomposition of $P = \mathbb{C}[x_1, \ldots, x_n]$ into homogeneous components

$$P = \bigoplus_{n \geq 0} \mathcal{P}_n$$

The Weyl group $W$ acts naturally on $P$ by extending the linear action of $W$ on the homogeneous component $\mathcal{P}_1$ of $P$ given by

$$(\zeta, \pi) \cdot x_i = \zeta_{\pi(i)} x_{\pi(i)}$$

for $i \in I_n$.

Let $I$ be the set of involutions in $W$. An element $\tau = (\zeta, \pi) \in I$ divides the set $I_n$ into four subsets $L_1^\tau$, $C_1^\tau$, $L_2^\tau$ and $C_2^\tau$ defined in the following way:

$$L_1^\tau \cup C_1^\tau = \{i : \pi(i) = i\} \quad \text{and} \quad L_2^\tau \cup C_2^\tau = \{i : \pi(i) \neq i\}$$

and

$$\tau(x_i) = \begin{cases} -x_i & \text{if } i \in L_1^\tau \\ x_i & \text{if } i \in C_1^\tau \\ -x_{\pi(i)} & \text{if } i \in L_2^\tau \\ x_{\pi(i)} & \text{if } i \in C_2^\tau \end{cases}$$

We introduce the $W$-module

$$V = \mathcal{P} \oplus (\mathcal{P} \wedge \mathcal{P})$$

and let $S(V)$ denote the symmetric algebra of $V$. To each involution $\tau = (\zeta, \pi) \in I$ we associate the element $e_\tau \in S(V)$ defined by

$$e_\tau = \left( \prod_{k \in L_1^\tau} x_k \right) \left( \prod_{m \in C_1^\tau} (x_m \wedge x_{\pi(m)}) \right) \left( \prod_{l \in C_2^\tau} (x_{\pi(l)}^2 \wedge x_{\pi(l)}^2) \right)$$

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The vectors $e_{\tau}$ transform nicely under the action of elements from $W$. In particular, if $\omega = (\varepsilon, \eta) \in W$ then
\[
\omega e_{\tau} = \pm \prod_{k \in \mathcal{L}^1} x_{\eta^{-1}(k)} \times \prod_{m \in I^1} (x_{\eta^{-1}(m)} \wedge x_{\eta \pi(m)}) \times \prod_{i \in I^2} \left( x_{\eta^{-1}(i)}^2 \wedge x_{\eta^{-1}(i)}^{2, -1} \right) \\
= \pm e_{\eta^{-1} \tau}
\]
Let $\mathcal{M}$ be the subspace of $S(V)$ generated by the elements $e_{\tau}$ with $\tau \in \mathcal{I}$. The action of $W$ on $V$ extends naturally to an action on $\mathcal{M}$. For $\tau \in \mathcal{I}$, we let $\mathcal{M}_\tau \subseteq \mathcal{M}$ be the $W$-submodule of $\mathcal{M}$ generated by the $W$-orbit of $\tau$.

**Remark 2.1** Observe that if $\tau = (\zeta, \pi)$ is an involution in $W$, then $\pi$ is an involution in $\mathfrak{S}_n$ and so the cardinalities $|C_1^\tau|$ and $|C_2^\tau|$ are both even numbers, because the elements in $C_1^\tau$ and $C_2^\tau$ are paired by $\pi$.

**Proposition 2.2** Let $\mathcal{M}$ be the previously defined $W$-module

i) $\dim_{\mathbb{C}}(\mathcal{M}) = |\mathcal{I}|$.

ii) $\mathcal{M}_\tau = \langle \pi e_{\tau} : \pi \in \mathfrak{S}_n \rangle$.

iii) $\mathcal{M}_\tau = \mathcal{M}_\mu$ if and only if $\tau$ and $\mu$ are conjugate under the action of $\mathfrak{S}_n$.

iv) If $\mathcal{R}$ is a system of representatives of the $\mathfrak{S}_n$-orbit in $\mathcal{I}$, then $\mathcal{M} = \bigoplus_{\rho \in \mathcal{R}} \mathcal{M}_\rho$.

**Proof.**

i) For $i < j$ the factors $x_i, x_i^2 \wedge x_j^2$ and $x_i \wedge x_j$ are linearly independent in $V$, thus it follows that the elements $e_{\tau}$ with $\tau \in \mathcal{I}$ form a basis of $\mathcal{M}$.

ii) and iii) follow from the way elements in $W$ transform the basis vectors $e_{\tau}$ and $e_{\mu}$.

iv) follows from i). $\blacksquare$

Since the dimension of a Gelfand model for $W$ coincides with the number of involutions, it follows from the previous proposition that $\mathcal{M}$ is a Gelfand model if and only the module is multiplicity-free. In order to prove this, in what follows, we will show $\text{End}_W(\mathcal{M}, \mathcal{M})$ is a commutative ring except for $W = W(D_{2n})$.

Given $\tau, \mu \in \mathcal{I}$ we define an equivalence relation $\tau \sim_{\mathfrak{S}_n} \mu$ if $\tau$ and $\mu$ are conjugate under the action of $\mathfrak{S}_n$.

**Proposition 2.3** If $\tau, \mu \in \mathcal{I}$, then $\tau \sim_{\mathfrak{S}_n} \mu$ if, and only if $|L_1^j| = |L_2^\mu|$ and $|C_1^j| = |C_2^\mu|$ for each $j = 1, 2$.

**Proof.** Suppose $\pi \in \mathfrak{S}_n$ and that $\tau = \pi \mu \pi^{-1}$. Then it follows that $L_1^j = \pi(L_1^\mu)$ and $C_1^j = \pi(C_1^\mu)$ for $j = 1, 2$. Conversely, if the cardinalities of the given sets are the same, then it clear there is a permutation $\pi \in \mathfrak{S}_n$ such that $\tau = \pi \mu \pi^{-1}$ $\blacksquare$

**Lemma 2.4** Given $\tau, \mu \in \mathcal{I}$ such that $I_1 = L_1^1 \cup C_1^1 = L_2^\mu \cup C_1^\mu$ or $I_2 = L_1^2 \cup C_1^2 = L_2^\mu \cup C_2^\mu$, then there exists an transposition $\omega \in \mathfrak{S}_n$ such that $\omega e_{\tau} = \pm e_{\tau}$ and $\omega e_{\mu} = \mp e_{\mu}$ or else there exists an involution $\omega \in \mathfrak{S}_n$ such that $\omega e_{\tau} = \pm e_{\mu}$.

**Proof.** Letting $\tau = (\zeta, \pi)$ and $\omega = (\xi, \mu)$ we will work on the involutions $\pi$ and $\mu$ in $\mathfrak{S}_n$. Suppose that $I_1 = L_1^1 \cup C_1^1 = L_1^\mu \cup C_1^\mu$. 

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First we treat the case where \( C_1^\mu \cap L_1^\tau \neq \emptyset \). Let \( i \in C_1^\mu \cap L_1^\tau \) be. Starting with \( i \), we build a sequence by applying first \( \mu \) and then \( \pi \) alternately. Let \( k \) be the greatest natural number such that this sequence is injective. We have

\[
i = i_1 \xrightarrow{\mu} i_2 \xrightarrow{\pi} i_3 \xrightarrow{\mu} i_4 \xrightarrow{\pi} \cdots \xrightarrow{\pi} i_k \quad \text{if} \quad k \text{ is odd}
\]

\[
i = i_1 \xrightarrow{\mu} i_2 \xrightarrow{\pi} i_3 \xrightarrow{\mu} i_4 \xrightarrow{\pi} \cdots \xrightarrow{\mu} i_k \quad \text{if} \quad k \text{ is even}
\]

Suppose \( k = 2l + 1 \) is an odd number. The value of \( \mu(i_k) \) must match one of the elements \( i_1, i_2, \ldots, i_k \). From the above table it follows that the product of transpositions

\[(i_1, i_2) (i_3, i_4) \cdots (i_{k-2}, i_{k-1})\]

is part of the cyclic decomposition of \( \mu \). Therefore \( \mu(i_k) = i_k \). Let \( \pi_1, \mu_1 \) and \( \sigma \) be the involutions given by

\[
\pi_1 = (i_1) (i_2, i_3) (i_4, i_5) \cdots (i_{k-1}, i_k) \\
\mu_1 = (i_1, i_2) (i_3, i_4) \cdots (i_{k-2}, i_{k-1}) (i_k) \\
\sigma = (i_1, i_k) (i_2, i_{k-1}) \cdots (i_l, i_{l+1})
\]

Thus \( \pi_1 \) and \( \mu_1 \) are part of the cyclic decompositions of \( \pi \) and \( \mu \) respectively, and satisfy the condition \( \sigma \pi_1 \sigma^{-1} = \mu_1 \). Hence

\[
\sigma \left( (x_{i_1} \land x_{i_2}) (x_{i_3} \land x_{i_4}) \cdots (x_{i_{k-2}} \land x_{i_{k-1}}) x_{i_k} \right) = (-1)^{l} (x_{i_1} \land x_{i_2}) (x_{i_3} \land x_{i_4}) \cdots (x_{i_{k-2}} \land x_{i_{k-1}}) x_{i_k}
\]

so that \( \sigma_{e_\tau} = \pm e_\omega \).

Suppose \( k = 2l \) is an even number. Arguing as in the previous case, we obtain \( \pi(i_k) = i_k \). This time the involutions \( \pi_1, \mu_1 \) and \( \sigma \) are given by

\[
\pi_1 = (i_1) (i_2, i_3) (i_4, i_5) \cdots (i_{k-1}, i_k) \\
\mu_1 = (i_1, i_2) (i_3, i_4) \cdots (i_{k-2}, i_{k-1}) (i_k) \\
\sigma = (i_1, i_k) (i_2, i_{k-1}) \cdots (i_l, i_{l+1})
\]

and verify the condition \( \sigma \pi_1 \sigma^{-1} = \pi_1, \sigma \mu_1 \sigma^{-1} = \mu_1 \). This time we have

\[
\sigma \left( (x_{i_1} \land x_{i_2}) (x_{i_3} \land x_{i_4}) \cdots (x_{i_{k-2}} \land x_{i_{k-1}}) x_{i_k} \right) = (-1)^{l+1} (x_{i_1} \land x_{i_2}) (x_{i_3} \land x_{i_4}) \cdots (x_{i_{k-2}} \land x_{i_{k-1}}) x_{i_k}
\]

so that \( \sigma e_\tau = \pm e_\omega \) and \( e_\omega = \mp e_\omega \).

Suppose now \( C_1^\mu \cap L_1^\tau = \emptyset \), or equivalently \( C_1^\mu = C_1^\tau \). It is clear that if \( C_1^\mu = \emptyset \) then \( e_\tau = e_\mu \). Otherwise, as before, we construct a sequence \( i_1, i_2, \ldots, i_k \). In this case, \( k = 2l \) is necessarily an even number, since from a table of type

\[
i_1 \xrightarrow{\mu} i_2 \xrightarrow{\pi} i_3 \xrightarrow{\mu} i_4 \xrightarrow{\pi} \cdots \xrightarrow{\pi} i_k
\]

it follows that \( \mu(i_k) = i_k \). Since \( i_k \in C_1^\mu \cap C_1^\tau \), this is impossible.

Therefore the table must be of the type

\[
i_1 \xrightarrow{\mu} i_2 \xrightarrow{\pi} i_3 \xrightarrow{\mu} i_4 \xrightarrow{\pi} \cdots \xrightarrow{\pi} i_k
\]

and \( \pi(i_k) = i_1 \). As in previous cases, we have the involutions \( \pi_1, \mu_1 \) and \( \sigma \) given by

\[
\pi_1 = (i_2, i_3) (i_4, i_5) \cdots (i_{k-2}, i_{k-1}) (i_k, i_1) \\
\mu_1 = (i_1, i_2) (i_3, i_4) \cdots (i_{k-1}, i_k) \\
\sigma = (i_2, i_k) (i_3, i_{k-1}) \cdots (i_l, i_{l+1})
\]
which verify $\sigma_1 \sigma_1^{-1} = \mu_1$. Hence

$$\sigma \left( (x_{i_2} \wedge x_{i_3}) (x_{i_4} \wedge x_{i_5}) \cdots (x_{i_{k-2}} \wedge x_{i_{k-1}}) (x_{i_1} \wedge x_{i_k}) \right) = (-1)^{k} (x_{i_1} \wedge x_{i_2}) (x_{i_3} \wedge x_{i_4}) \cdots (x_{i_{k-1}} \wedge x_{i_k})$$

and it follows that $\sigma \sigma_\tau = \pm e_{\omega}$. The proof for the case $\mathbb{I}_n = L_2 \cup C_2 = L_2^1 \cup C_2^1$ is similar. $lacksquare$

**Remark 2.5** When $W$ is the symmetric group $S_n$, this is precisely the case $\mathbb{I}_n = L_2 \cup C_2 = L_2^1 \cup C_2^1$ and all the factors involved have the form $x_i^1 \wedge x_j^1$. Moreover, in the case where $W = W(D_n)$ then $|L_1^1|$ is an even number for every involution $\tau \in W(D_n)$.

We find it useful to introduce the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $M$ with orthonormal basis $\{e_\tau : \tau \in \mathcal{I}\}$. Using the formula for the way elements of $W$ act on the basis vectors of $M$, it follows that the form is invariant under $W$. In particular

$$\langle \sigma \sigma_\tau, e_e \rangle = \langle e_\tau, e_e \rangle = \delta_{\tau, e} \quad \forall \tau, e \in \mathcal{I}, \forall \sigma \in W$$

**Lemma 2.6** If $W$ is a Weyl group of type $A_{n-1}$, $B_n$ or $D_n$ with $n$ odd in the last case, then $\text{End}_W (M, M)$ is a commutative ring.

**Proof.** Fix $\phi \in \text{End}_W (M, M)$. We will show that $\phi$ is a complex symmetric operator with respect to the invariant form defined above. In particular we will show that

$$\langle \phi e_\tau, e_e \rangle = \langle e_\tau, \phi e_e \rangle \quad \forall \tau, e \in \mathcal{I}$$

Let $\sigma_i$ be the reflection in $W(B_n)$ defined as $\sigma_i x_j = (1 - 2\delta_{ij}) x_j$, where $\delta$ is the Kronecker function. We denote by $\zeta_i$ the element in $W$ given by

$$\zeta_i = \begin{cases} 
\sigma_i & \text{if } W = W(B_n) \\
-\sigma_i & \text{if } W = W(D_n)
\end{cases}$$

This last point is where we use the fact that $n$ is odd for the group $W(D_n)$. When $n$ is even then $\zeta_i$ does not belong to $W(D_n)$. If $\tau \in \mathcal{I}$ and $i \in \mathbb{I}_n$ we claim that

$$\zeta_i e_\tau = \begin{cases} 
e_\tau & \text{if } i \notin L_1^1 \cup C_1^1 \\
e_e & \text{if } i \in L_1^1 \cup C_1^1 
\end{cases}$$

This is clear if $W = W(B_n)$ and in the case that $W = W(D_n)$ we have

$$\zeta_i e_\tau = \begin{cases} (-1)^{|L_1^1 \cup C_1^1|} e_\tau & \text{if } i \notin L_1^1 \cup C_1^1 \\
(-1)^{|L_1^1 \cup C_1^1| - 1} e_\tau & \text{if } i \in L_1^1 \cup C_1^1 
\end{cases}$$

Thus the assertion follows from the fact that $|L_1^1|$ and $|C_1^1|$ are even numbers and the condition $L_1^1 \cap C_1^1 \neq \emptyset$.

Let $\tau, \omega \in \mathcal{I}$ be and suppose $L_1^1 \cup C_1^1 \neq L_2 \cup C_2$. Suppose $i \in (L_1^1 \cup C_1^1) \setminus (L_2 \cup C_2)$. Then we have

$$\zeta_i e_\tau = -e_\tau \quad \text{and} \quad \zeta_i e_\omega = e_\omega.$$ 

Hence

$$\langle e_\tau, \phi e_\omega \rangle = \langle \zeta_i e_\tau, \zeta_i \phi e_\omega \rangle = \langle \zeta_i e_{\tau}, \phi \zeta_i e_\omega \rangle = -\langle e_\tau, \phi e_\omega \rangle$$
that is \(<e, \phi e, e_\omega> = 0\) and similarly \(<\phi e, e_\omega, e_\tau> = 0\). In the case that \(L_1 \cup C_1 = L_2 \cup C_2\) we let \(J\) denote this union and also write \(L_1 \cup C_1 = L_2 \cup C_2 = K\). Let \(W_J\) and \(W_K\) be the subgroups of \(W\) defined by:

\[
W_J = \{ \sigma \in W : \sigma x_k = x_k, \forall k \in K \} \\
W_K = \{ \sigma \in W : \sigma x_j = x_j, \forall j \in J \}
\]

Then \(W_J\) and \(W_K\) are both Weyl groups of the same type as \(W\). The involutions \(\tau\) and \(\omega\) can be factored as:

\[
\tau = \tau_J \tau_K \quad \text{and} \quad \omega = \omega_J \omega_K
\]

in \(W_J \times W_K\). Applying Lemma 2.4 to the pairs \(e_\tau, e_\omega\) and \(e_\tau, e_\omega\), we conclude that there is an involution \(\omega\) such that:

\[
\omega e_\tau = -e_\tau \quad \text{and} \quad \omega e_\omega = e_\omega \quad \text{or} \quad \omega e_\tau = \pm e_\omega
\]

In the first case, as before, is \(<e, \phi e_\omega> = 0 = <e_\omega, \phi e_\omega>\) and in the second case we have:

\[
<e, \phi e_\omega> = <\omega e_\tau, \omega \phi e_\omega> = <\pm e_\omega, \pm \phi e_\tau> = <e_\omega, \phi e_\tau> = <\phi e_\tau, e_\omega>
\]

Hence \(\text{End}(\mathcal{M}, \mathcal{M})\) is an algebra of operators contained in the space of symmetric operators. It follows that \(\text{End}_W (\mathcal{M}, \mathcal{M})\) is commutative.

**Theorem 2.7** Suppose \(W\) is a Weyl group of type \(A_n, B_n\) or \(D_{2n+1}\), then \(\mathcal{M}\) is a Gelfand Model for \(W\).

**Proof.** By Lemma 2.6 \(\mathcal{M}\) is a multiplicity-free \(W\)-module. On the other hand, the dimension of \(\mathcal{M}\) coincides with the number of involutions. As mentioned above, this is known to be the dimension of a Gelfand model for a Weyl group.

**Remark 2.8** Since Baddeley has shown in his doctoral thesis that a Weyl group of type \(D_{2n}\) does not have an involution model (see, for example [14]), one would expect that the representation defined in this section is not a Gelfand model for \(W(D_{2n})\). We remark that it is not hard to see that \(\mathcal{M}\) is not multiplicity-free in this case.

### 3 Relation to the polynomial model

Suppose \(G \subseteq GL_n(\mathbb{C})\) is a finite subgroup. The group \(G\) acts on the space of complex valued polynomials \(\mathcal{P} = \mathbb{C}[x_1, \ldots, x_n]\) according to the natural action of \(GL_n(\mathbb{C})\). The polynomial model for the group \(G\) is based on subspace \(\mathcal{N}_G \subseteq \mathcal{P}\) determined as the zeros of an associated subset of the \(G\)-invariant invariant differential operators defined on \(\mathcal{P}\). In particular, let \(\mathcal{D}\) denote the Weyl algebra of differential operators with polynomial coefficients. Then \(G\) acts naturally on \(\mathcal{D}\) by

\[
g \cdot D = gDg^{-1} \quad g \in G, D \in \mathcal{D}
\]

where both \(g\) and \(D\) are taken as elements in \(\text{End}_C(\mathcal{P})\). Let \(\mathcal{D}^G\) denote the centralizer of \(G\) in \(\mathcal{D}\). We will define a \(G\) invariant subset of elements of negative degree in \(\mathcal{D}^G\). Let \(I_n = \{1, 2, \ldots, n\}\)
be the set of indices and let $M = \{ \alpha : I_n \to \mathbb{N}_0 \}$ be the set of multi-indices in $I_n$. Given $\alpha \in M$ we put:

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$ 

Each element in $\mathcal{D}$ can be written uniquely in the form

$$\sum_{\alpha, \beta \in M} a_{\alpha, \beta} x^\alpha \partial^\beta.$$ 

We define the subset $\mathcal{D}_G^-$ of elements of negative degree in $\mathcal{D}_G$ as follows

$$\mathcal{D}_G^- = \left\{ D \in \mathcal{D}_G : D = \sum_{|\beta| > |\alpha|} a_{\alpha, \beta} x^\alpha \partial^\beta \right\}.$$ 

The space $\mathcal{N}_G$ is defined by

$$\mathcal{N}_G = \{ P \in \mathcal{P} : D(P) = 0, \forall D \in \mathcal{D}_G^- \}.$$ 

In general one knows that $\mathcal{N}_G$ contains a Gelfand model for $G$ [3]. As mentioned in the introduction, when $G$ is a Weyl group of type $A_n$, $B_n$, or $D_{2n+1}$ realized in linear form by its geometric representation then $\mathcal{N}_G$ is a Gelfand model for $G$.

In this section we give a specific relation between Gelfand model defined in the previous section and the polynomial model for the Weyl groups $W = \mathfrak{S}_n$ or $W = W(B_n)$. The treatment for a Weyl of type $D_{2n+1}$ proceeds in somewhat distinct manner and will not be not included.

We continue to let $I$ denote the set of involutions in $W$. Observe that the symmetric group $\mathfrak{S}_n$ acts on the set of multi-indices $M$ in $I_n$ by

$$\pi \cdot \alpha = \alpha \circ \pi^{-1} \quad \alpha \in M, \pi \in \mathfrak{S}_n.$$ 

To avoid ambiguity, in all that follows, we assume that the elements of the subsets $I = \{i_1, \ldots, i_k \}$ of $I_n = \{1, 2, \ldots, n\}$ are arranged in increasing order. Given a multi-index $\alpha : I \to \mathbb{N}_0$ we let $V(\alpha, I)$ denote the polynomial defined by the determinant of the matrix $[x_{i_j}^{\alpha_i}]$. Hence

$$V(\alpha, I) = \det [x_{i_j}^{\alpha_i}]$$

Recall the action of $W$ on the space of polynomials $\mathcal{P}$ defined in the previous section. Then for each permutation $\eta \in \mathfrak{S}_n$ we have

$$\eta V(\alpha, I) = \det [x_{\eta(i_j)}^{\alpha_i}] = V(\alpha \eta^{-1}, I).$$

The polynomial $V(\alpha, I)$ is said to have positive orientation if it coincides with $V(\tilde{\alpha}, I)$ where $\tilde{\alpha}$ is obtained by rearranging $\alpha$ in increasing order. Otherwise, we say that $V(\alpha, I)$ has negative orientation.

Consider the symmetric group $\mathfrak{S}_{2k}$ of the set $I_{2k} = \{i_1, i_2, \ldots, i_{2k-1}, i_{2k} \}$ and the symmetric group $\mathfrak{S}_k$ of the set $I_k = \{i_1, i_3, \ldots, i_{2k-1} \}$. Every permutation $\pi \in \mathfrak{S}_k$ can be extended to a permutation $\tilde{\pi}$ in $\mathfrak{S}_{2k}$ defined as $\tilde{\pi}(i_{2j}) = i_{k+1}$ if $\pi(i_{2j-1}) = i_k$. The permutations in $\mathfrak{S}_{2k}$ obtained in this way, will be called double permutations.
Proposition 3.1. Suppose \( \alpha : I_{2k} \to \mathbb{N}_0 \) is a multi-index and \( F \subseteq I_{2k} \) is a subset that satisfies \( i_{2j-1} \in F \Leftrightarrow i_{2j} \in F \). Then \( V(\alpha, F) \) and \( \pi V(\alpha, F) \) have the same orientation for every double permutation \( \pi \).

Proof. Suppose that \( V(\alpha, F) \) has positive orientations. By exchanging rows (or perhaps columns) in the matrix 
\[
\begin{bmatrix}
\alpha_{i_{ij}}^{-1}
\end{bmatrix}
\]
associated to the polynomial \( V(\alpha^{-1}, F) \), we can arrange that the terms of \( \alpha_{i_{ij}}^{-1} \) appear in increasing order. Since \( \pi \) is a double permutation, these exchanges can be done in pairs. Therefore \( V(\alpha^{-1}, F) \) has positive orientation. \( \blacksquare \)

Recalling notation from the previous section, for every \( \tau \in \mathcal{I} \) we put \( C^\tau = C^\tau_1 \cup C^\tau_2 \) and \( L^\tau = L^\tau_1 \cup L^\tau_2 \). Note that \( \tau \) can be factored uniquely as \( \tau = \tau^+ \times \tau^- \) where \( \tau^+ \) is the product of the positive cycles that decompose \( \tau \) and \( \tau^- \) is the product of the negative cycles that decompose \( \tau \).

To each subgroup \( \mathfrak{R} \subseteq \mathfrak{S}_n \) we associate the operator \( \Omega_\mathfrak{R} \) defined as
\[
\Omega_\mathfrak{R} = \sum_{\kappa \in \mathfrak{R}} sg(\kappa) \kappa
\]
Observe that this operator can be applied to elements of any \( \mathfrak{S}_n \)-module.

We can write
\[
L^\tau_2 = \{i_1, j_1, \ldots, i_r, j_r\} \quad \text{and} \quad C^\tau_2 = \{k_1, l_1, \ldots, k_s, j_s\}
\]
where the sequences \( i_1, \ldots, i_r \) and \( k_1, \ldots, k_s \) are strictly increasing and where \( i_p < j_p = \tau(i_p) \), \( k_q < l_q = \tau(k_q) \) for \( 1 \leq p \leq r, 1 \leq q \leq s \) and define a corresponding multi-index \( \alpha_\tau \in M \) in the following manner
\[
\alpha_\tau(i) = \begin{cases}
1 & \text{if } i \in L^\tau_1 \\
0 & \text{if } i \in C^\tau_1
\end{cases}
\]
and
\[
\alpha_\tau(j_p) = 2p - 1 \quad \alpha_\tau(k_q) = 2q
\]
\( \alpha_\tau \) (1)

To an involution \( \tau \in W \) we associate the polynomial
\[
P_\tau = \Omega_\mathfrak{R} x^{\alpha_\tau}
\]
where \( \vartheta_\tau = \vartheta_\tau^+ \times \vartheta_\tau^- \) with \( \vartheta_\tau^+ \) and \( \vartheta_\tau^- \) the centralizers of \( \tau^+ \) and \( \tau^- \) in \( \mathfrak{S}(C^\tau_2) \) and \( \mathfrak{S}(L^\tau_2) \), respectively.

We let \( \mathcal{M} \) denote the \( W \)-module generated by \( P_\tau \) with \( \tau \in \mathcal{I} \).

Remark 3.2. Given \( \pi \in \mathfrak{S}_n, \mathfrak{R} \subseteq \mathfrak{S}_n \) and \( \Omega_\mathfrak{R} \) as defined before, note that:
\[
\pi \Omega_\mathfrak{R} \pi^{-1} = \Omega_{\pi \mathfrak{R} \pi^{-1}}
\]
(3)
\[
\pi \Omega_\mathfrak{R} = \Omega_\mathfrak{R} \pi = sg(\pi) \Omega_\mathfrak{R} \text{ if } \pi \in \mathfrak{R}
\]

Proposition 3.3. The application \( \phi : \mathcal{M} \to \mathcal{M} \) defined on the basis vectors by the formula \( \phi(e_\tau) = P_\tau \), for \( \tau \in \mathcal{I} \), is morphism of \( W \)-modules.

Proof. Suppose \( \tau \in \mathcal{I} \) and let \( \sigma = (\zeta, 1) \in W \). From our description (in the previous section) of the action of \( W \) on the basis vectors in \( \mathcal{M} \) it follows that
\[
\sigma e_\tau = \left( \prod_{j \in L^\tau_1 \cup C^\tau_1} \zeta_j \right) e_\tau
\]
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Decomposing the operator $\Omega_{\vartheta_r} = \Omega_{\vartheta_r} \times \Omega_{\vartheta_r}$ and applying this to the polynomial

$$x^{\alpha_r} = \left( \prod_{j \in L_1} x_j \right) \left( \prod_{j \in L_2} x_j^{(\alpha_r)_j} \right) \left( \prod_{j \in C_2} x_j^{(\alpha_r)_j} \right)$$

it follows that

$$P_r = \left( \prod_{j \in L_1} x_j \right) \times \Omega_{\vartheta_r} \left( \prod_{j \in L_2} x_j^{(\alpha_r)_j} \right) \times \Omega_{\vartheta_r} \left( \prod_{j \in C_2} x_j^{(\alpha_r)_j} \right).$$

Using the definition of $\alpha_r$ in (1) we obtain

$$\sigma P_r = \left( \prod_{j \in L_1 \cup C_2} \zeta_j \right) P_r.$$

Therefore $\phi(\sigma e_r) = \sigma \phi(e_r)$.

Now consider an element in $W$ of the form $(1, \mu)$. Then

$$\mu e_r = (-1)^{\varepsilon} e_{\mu r - 1},$$

where $\varepsilon$ is the number of transpositions $(i, j)$ which are factors of $\tau$ with $\mu(i) > \mu(j)$. Using the formula in (3) we obtain

$$\mu P_r = \mu \Omega_{\vartheta_r} \mu^{-1} \mu x^{\alpha_r} = \Omega_{\mu \vartheta_r \mu^{-1}} \mu x^{\alpha_r}.$$ From (1) it follows there exists $\varphi \in \mu \vartheta_r \mu^{-1}$ such that:

$$\varphi \mu x^{\alpha_r} = x^{\alpha_{\mu r - 1}}.$$ The element $\varphi$ can be factored as $\varphi = \kappa \omega$ where $\omega$ is a double permutation and

$$\kappa = \prod_{\mu(i_p) > \mu(j_p)} (\mu(i_p), \mu(j_p)) \times \prod_{\mu(k_q) > \mu(l_q)} (\mu(k_q), \mu(l_q)).$$

Thus we obtain

$$\mu P_r = \sigma g(\varphi) \Omega_{\mu \vartheta_r \mu^{-1}} x^{\alpha_{\mu r - 1}} = \sigma g(\kappa) P_{\mu \vartheta_r \mu^{-1}} = (-1)^{\varepsilon} P_{\mu \vartheta_r \mu^{-1}}.$$ Therefore $\phi(\mu e_r) = \mu \phi(e_r)$. Hence the proposition follows from the fact that $W$ can be generated by the elements of the form $\sigma$ and $\mu$.

**Definition 3.4 Telescopic decomposition**

Let $U$ be a $G$-module and suppose $N$ is a semisimple $G$-submodule of $U$. If $\vartheta : U \to U$ is a nilpotent $G$-morphism then $\vartheta$ determines an isomorphism of $N$ with a submodule of $U$ that decomposes as a direct sum

$$N \simeq \oplus_{k \geq 0} N_k \subseteq U$$

where

$$N_0 = N \cap \text{Ker}(\vartheta) \quad \text{and if } k \geq 1 \quad N_k = \{ m \in \vartheta^k(N) : \vartheta(m) = 0 \}.$$ This $G$-submodule of $U$ will be called the telescopic decomposition of $N$ respect to $\vartheta$ and will be denoted as:

$$D_\vartheta(N) = \oplus_{k \geq 0} N_k$$

Note that

$$N \simeq D_\vartheta(N) \subseteq U$$
From this point our basic strategy is as follows. So far we have defined a morphism of $W$-modules
\[ \phi : M \to \mathcal{M} \subseteq P \]
where $M$ is the Gelfand model from the previous section and where $\varphi(M) = \mathcal{M}$ is the image of $\phi$ in $\mathcal{P}$. We introduce the $W$-invariant differential operator $\partial$ defined by
\[ \partial = \begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} & \text{if } W \simeq \mathfrak{S}_n \\ \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} & \text{if } W = W(B_n) \end{cases}. \]

The remainder of this section will be spent proving that the polynomial model $N_W$ is the telescopic decomposition of $\mathcal{M}$ with respect to $\partial$.

A $\mathfrak{S}_n$-orbit $\gamma$ in the set of multi-indices $M$ is said $W$-minimal if for each $\alpha \in \gamma$ we have
\[ \alpha^{-1}(i) \geq \alpha^{-1}(i+1) \quad \text{if} \quad W \simeq \mathfrak{S}_n \\
\alpha^{-1}(i) \geq \alpha^{-1}(i+2) \quad \text{if} \quad W = W(B_n). \]

Given a $W$-minimal orbit $\gamma$ in $M$ we define a corresponding $W$-submodule of $\mathcal{P} = \{ \}$ by
\[ S_\gamma = \left\{ \sum_{\alpha \in \gamma} \lambda_\alpha x^\alpha : \lambda_\alpha \in \mathbb{C} \right\}. \]
and let $S_\gamma^0$ be the $W$-submodule of $S_\gamma$ defined by
\[ S_\gamma^0 = \{ P \in S_\gamma : \partial(P) = 0 \}. \]
The following theorem is established in [2] and [4].

**Theorem 3.5** If $\gamma$ is $W$-minimal then representation $S_\gamma^0$ is irreducible and
\[ N_W = \bigoplus_{\gamma \text{W-minimal}} S_\gamma^0. \]

After some initial preparation we will show when $\gamma$ is $W$-minimal then $S_\gamma^0 \cap D_\partial(\mathcal{M}) \neq \{0\}$. Let $I = \{i_1, i_2, \ldots, i_k\} \subseteq \mathbb{N}_n$ and define multi-indices $\iota, \varphi$ and $\kappa : I \to \mathbb{N}_0$ by
\[ \iota_{ij} = j - 1, \quad \varphi_{ij} = 2(j - 1) \quad \text{and} \quad \kappa_{ij} = 2j - 1. \]

We introduce the groups $\mathfrak{S}_k = \mathfrak{S}(I)$ and $W(B_k) = C_2^k \rtimes \mathfrak{S}(I)$. Note that $\iota$ is $\mathfrak{S}_k$-minimal and that $\varphi$ and $\delta$ are $W(B_k)$-minimal. We also introduce the restriction $\partial_I$ of $\partial$ to $\mathbb{C}[x_{i_1}, \ldots, x_{i_k}]$ given by
\[ \partial_I = \begin{cases} \sum_{j=1}^{k} \frac{\partial}{\partial x_{i_j}} & \text{if} \quad W \simeq \mathfrak{S}_n \\ \sum_{j=1}^{n} \frac{\partial^2}{\partial x_{i_j}^2} & \text{if} \quad W = W(B_n) \end{cases}. \]

**Lemma 3.6** i) Suppose $\alpha \in M$ is injective on $I$ and, if $W = W(B_n)$, $\alpha$ takes the same parity on $I$. In that case, when $V(\alpha, I)$ has positive orientation then a power of $\partial_I$ transforms $V(\alpha, I)$ in a positive scalar multiple of either $V(\iota, I), V(\varphi, I)$ or $V(\kappa, I)$.

ii) $\partial_I(V(\iota, I)) = 0$ and, if $W = W(B_n)$, $\partial_I(V(\kappa, I)) = 0 = \partial_I(V(\kappa, I))$. 

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Proof. i) Assume $V(\alpha, I)$ has positive orientation. We write $\alpha^i = (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_k)$ in the case $W = W(B_n)$ and $\alpha^i = (\alpha_1, \ldots, \alpha_i - 2, \ldots, \alpha_k)$ in the case $W = \mathcal{S}_n$. Since $\partial_I$ is $W_I$-invariant, we have

$$\partial_I (V(\alpha, I)) = \partial_I \sum_{\pi \in \mathcal{S}_k} s g(\pi) \pi \times x^\alpha = \sum_{\pi \in \mathcal{S}_k} s g(\pi) \pi \partial_I (x^\alpha)$$

$$= \sum_{\pi \in \mathcal{S}_k} s g(\pi) \pi \left( \sum_{i=1}^k \lambda_i x^\alpha_i \right)$$

$$= \sum_{i=1}^k \lambda_i \sum_{\pi \in \mathcal{S}_k} s g(\pi) \pi \times x^\alpha_i$$

$$= \sum_{i=1}^k \lambda_i V(\alpha_i, I)$$

where $\lambda_i = \alpha_i (\alpha_i - 1)$ if $W = W(B_n)$ or $\lambda_i = \alpha_i$ if $W = \mathcal{S}_n$. In the terms where $V(\alpha^i, I_k) \neq 0$, it is clear that $V(\alpha^i, I_k)$ has positive orientation and $|\alpha^i| = |\alpha| - 1$ if $W = \mathcal{S}_n$ or $|\alpha^i| = |\alpha| - 2$ if $W = W(B_n)$. Now the proof follows by induction on the $|\alpha| = \sum_{i=1}^k \alpha_i$.

ii) By i) we have:

$$\partial_I (V(\alpha, I)) = \sum_{i=1}^k \beta_i V_{\alpha^i}$$

but $\alpha_i$ is not injective when $\alpha = \iota$, $\varphi$ or $\kappa$ so $V_{\alpha^i} = 0$ ($1 \leq i \leq k$) in these cases. 

Corollary 3.7 Let $I_0 = \bigcup_{j=1}^h I_j$ be a partition of $I_0$, and take $\alpha \in M$ injective. If $\alpha_j$ denotes the restriction of $\alpha$ to the set $I_j$, then there exists $m \in \mathbb{N}_0$ such that:

$$\partial^m \left( \prod_{j=1}^h V(\alpha_j, I_j) \right) = q \prod_{j=1}^h V(I_j)$$

where $q \in \mathbb{N}$.

Proof. The operator $\partial$ can be decomposed as $\partial = \sum_{j=1}^h \partial_j$ with

$$\partial_j = \begin{cases} \sum_{k \in I_j} \frac{\partial}{\partial x_k} & \text{if } W = \mathcal{S}_n \\ \sum_{k \in I_j} \frac{\partial}{\partial x_k} & \text{if } W = W(B_n) \end{cases}$$

Let $m_j \in \mathbb{N}_0$ be such that $\partial^m_j V(\alpha_j, I_j) = q_j V(\beta_j, I_j)$ where $\beta_j = \iota$, $\varphi$ or $\kappa$ defined on $I_j$ as before. Let $m = \sum_{j=1}^h m_j$. Then we have

$$\partial^m = \sum_{|S|=m} c_S \prod_{j=1}^h \partial_j^{m_j}$$

so that

$$\partial^m \left( \prod_{j=1}^h V(\alpha_j, I_j) \right) = \sum_{|S|=m} c_S \left( \prod_{j=1}^h \partial_j^{m_j} V(\alpha_j, I_j) \right)$$
Take \( \delta \) such that \( |\delta| = m \). If \( \delta_j > m_j \) for some index \( j \), then \( \partial_j^{\partial_j} V(\alpha_j, I_j) = 0 \). If \( \delta_j < m_j \) for some index \( j \) then there exists \( i \) such that \( \delta_i > m_i \) and \( \partial_i^{\partial_i} V(\alpha_i, I_i) = 0 \). It follows that the only non-zero term in the right-hand side of the previous equality corresponds to \( \delta = (m_1, \ldots, m_h) \). Hence

\[
\partial^m \left( \prod_{j=1}^h V(\alpha_j, I_j) \right) = q \left( \prod_{j=1}^h V(\beta_j, I_j) \right).
\]

**Lemma 3.8** Let \( \gamma \) be a \( W \)-minimal orbit in \( M \). Then \( S_\gamma^0 \subseteq D_\varnothing (\mathfrak{M}) \).

**Proof.** The idea of the proof is to produce a nontrivial element in \( S_\gamma^0 \cap D_\varnothing (\mathfrak{M}) \). Suppose \( \alpha \in \gamma \).

There is a decomposition of \( \mathbb{I}_n = \bigcup_{j=1}^h I_j \) such that

i) The restrictions \( \alpha_j \) of \( \alpha \) to \( I_j \) are injective;

ii) If \( W = W(B_n) \), \( \alpha_j \) takes the same parity on \( I_j \);

iii) Each \( I_j \) is maximal with the properties i) and ii).

Since \( \gamma \) is a \( W \)-minimal orbit, it follows that every \( \alpha_j \) is of the form \( \iota, \varphi \) or \( \kappa \), so by Lemma 3.6 and Corollary 3.7 we have

\[
\prod_{j=1}^h V(\alpha_j, I_j) \in S_\gamma^0.
\]

We will show that the polynomial \( \prod_{j=1}^h V(\alpha_j, I_j) \) is in the telescopic decomposition. Since \( \prod_{j=1}^h V(\alpha_j, I_j) \) is in the space \( S_\gamma^0 \) this means it is in the kernel of \( \partial \). Therefore it suffices to prove that the polynomial in question is in the image of power of \( \partial \). Put

\( I_j = \{ i_1, i_2, \ldots, i_{m_j} \} \). When \( W = W(B_n) \) define

\[
\tau_j = \begin{cases} 
(i_1, i_2) \cdots (i_{m_j-1}, i_{m_j}) & \text{if } m_j \text{ is even and } \alpha \text{ is even on } I_j \\
(i_1, i_2) \cdots (i_{m_j-1}, i_{m_j})^- & \text{if } m_j \text{ is even and } \alpha \text{ is odd on } I_j \\
(i_1, i_2) \cdots (i_{m_j-2}, i_{m_j}) & \text{if } m_j > 1 \text{ is odd and } \alpha \text{ is even on } I_j \\
(i_1, i_2)^- \cdots (i_{m_j-2}, i_{m_j})^- & \text{if } m_j > 1 \text{ is odd and } \alpha \text{ is odd on } I_j
\end{cases}
\]

and

\[
\tau_j = \begin{cases} 
1 & \text{if } m_j = 1 \text{ and } \alpha \text{ is even on } I_j \\
(i_1)^- & \text{if } m_j = 1 \text{ and } \alpha \text{ is odd on } I_j
\end{cases}
\]

where the involution \( (ij)^- \in C_{\mathfrak{G}_n} \times \mathfrak{G}_n \) has the form \( (\zeta, (ij)) \) with \( \zeta \in C_n^2 \) being the \( n \)-tuple that takes on the value \(-1\) only at points \( i \) and \( j \). In case \( W = \mathfrak{G}_n \) define

\[
\tau_j = \begin{cases} 
(i_1, i_2) \cdots (i_{m_j-1}, i_{m_j}) & \text{if } m_j \text{ is even} \\
(i_1, i_2) \cdots (i_{m_j-2}, i_{m_j}) & \text{if } m_j > 1 \text{ is odd}
\end{cases}
\]

and

\( \tau_j = 1 \) if \( m_j = 1 \)

Let \( \tau \) be the product of the involutions \( \tau_j \). We can write \( \tau = \tau^- \tau^+ \) by grouping the positive and negative cycles of \( \tau \). Let \( \partial_j \) be the normalizer of \( \tau_j^\pm \) in \( \mathfrak{G}_j = \mathfrak{G}(I_j) \) and let be \( T_j \) a system of
representatives of right cosets of \( \vartheta_j \) in \( S_j \). We put

\[ S_{\gamma} = S_1 \times \cdots \times S_m \]
\[ \vartheta_{\gamma} = \vartheta_1 \times \cdots \times \vartheta_m \]
\[ T_{\gamma} = T_1 \times \cdots \times T_m \]

Let \( \vartheta_\tau = \vartheta_\tau^+ \times \vartheta_\tau^- \) as before and let \( \mathcal{H} \) denote a set of doubles permutations that give a system representatives of left cosets of \( \vartheta_{\gamma} \) in \( \vartheta_\tau \). According to the definition of \( P_\tau \) in (2) we write

\[ \Omega_{T_{\gamma}} P_{\tau} = \Omega_{T_{\gamma}} \vartheta_\tau x^{\alpha_\tau} \]

Decomposing

\[ \Omega_{\vartheta_\tau} = \sum_{\eta \in \mathcal{H}} \Omega_{\vartheta_\tau} \eta \]

we have

\[ \Omega_{T_{\gamma}} P_{\tau} = \Omega_{T_{\gamma}} \left( \sum_{\eta \in \mathcal{H}} \Omega_{\vartheta_\tau} \eta \right) x^{\alpha_\tau} \]
\[ = \sum_{\eta \in \mathcal{H}} \Omega_{\vartheta_\tau} \eta x^{\alpha_\tau} \]
\[ = \sum_{\eta \in \mathcal{H}} \prod_{j=1}^n V \left( \left( \alpha_\tau \eta^{-1} \right)_j, I_j \right) \]

Since the permutations \( \eta \) are double permutations all factors \( V \left( \left( \alpha_\tau \eta^{-1} \right)_j, I_j \right) \) have positive orientation. Now, by Corollary 3.7 there is a power \( m \) of \( \vartheta \) such that

\[ \vartheta^m \left( \sum_{\eta \in \mathcal{H}} \prod_{j=1}^n V \left( \left( \delta_\tau \eta^{-1} \right)_j, I_j \right) \right) = \vartheta^h \prod_{j=1}^n V \left( \alpha_j, I_j \right) \]

where \( q \in \mathbb{N} \). ■

**Theorem 3.9** Suppose \( W = \mathfrak{S}_n \) or \( W = W(B_n) \) and \( \mathcal{M} \) is the Gelfand model defined in the previous section. Let \( \phi : \mathcal{M} \to \mathcal{P} = \mathbb{C}[x_1, \ldots, x_n] \) be the \( W \)-morphism defined by

\[ e_\tau \mapsto P_{\tau} \]

and put \( \mathfrak{M} = \phi(\mathcal{M}) \). Let

\[ \vartheta : \mathcal{P} \to \mathcal{P} \]

be the \( W \)-invariant differential operator defined in this section. Then \( \mathcal{N}_W = \mathcal{D}_\vartheta(\mathfrak{M}) \), i.e. the polynomial model for \( W \) is the telescopic decomposition of \( \mathfrak{M} \) with respect to \( \vartheta \). In particular we have isomorphisms

\[ \mathcal{M} \cong \mathfrak{M} \cong \mathcal{D}_\vartheta(\mathfrak{M}) = \mathcal{N}_W \]

**Proof.** By the Lemma 3.8 it follows that \( S_0^1 \subseteq \mathfrak{M} \) for every \( W \)-minimal \( \gamma \in \mathcal{M} \). From Theorem 3.3 it follows that \( \mathcal{N}_W \subseteq \mathfrak{M} = \phi(\mathcal{M}) \). But \( \dim_{\mathbb{C}}(\mathcal{N}_W) = \dim_{\mathbb{C}}(\mathcal{M}) \) because both are Gelfand models. This proves our main theorem. ■
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