On the Failure of Spin-Statistics Connection in Quantum Gravity

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Abstract

Many years ago Friedman and Sorkin [1] established the existence of certain topological solitonic excitations in quantum gravity called (topological) geons. Geons can have quantum numbers like charge and can be tensorial or spinorial having integer or half-odd integer spin. A striking result is that geons can violate the canonical spin-statistics connection [2, 3]. Such violation induces novel physical effects at low energies. The latter will be small since the geon mass is expected to be of the order of Planck mass. Nevertheless, these effects are very striking and include CPT and causality violations and distortion of the cosmic microwave spectrum. Interesting relations of geon dynamics to supersymmetry are also discussed.

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1 Introduction

The spin-statistics connection asserts that tensorial particles (those with integral spin) obey Bose statistics and spinorial ones obey Fermi statistics. It has a central role in determining properties of matter including its stability and is generally regarded as a fundamental result of quantum physics. It has a counterpart in (2+1) dimensions where particles of fractional spin \( \theta \) are asserted to obey fractional statistics with the same \( \theta \).

Our understanding of this connection however is not perfect. It has been proved using the axioms of relativistic local quantum field theories (RQFT’s) [4]. It has also been established for Skyrme-like solitons and ‘tHooft-Polyakov monopoles [5], and particles of fractional spin in (2+1) dimensions [6], apparently using physical principles which are mutually different and different too from those of RQFT’s. In the literature, we also encounter proofs of this connection using yet other considerations [7]. In addition, no such theorem can be established in conventional nonrelativistic physics. The standard spin-statistics connection cannot also be established in generality for the gravitational topological excitations known as geons [2, 3, 8, 9, 10], although certain novel spin-statistics connections can be proved for them [8].

The lack of spin-statistics connection in quantum gravity attracts attention. Nonrelativistic physics is a limiting form of RQFT’s and therefore its loss there is attributed to an imperfect limiting procedure. But this escape route is not available for quantum gravity with its enormous energy scales where RQFT loses its validity. It is rather the latter which is a limiting form of a unified model for gravity and elementary particles.

Studies of the common principles underlying the different approaches to this connection suggest that it needs the possibility of creation-annihilation processes. Nonrelativistic models incorporating such events have been devised (see Balachandran et al. papers in [3]), they also naturally correlate spin and statistics. For geons, these processes can occur only with topology change, but even in their presence, the desired relation can be recovered but imperfectly, for a limited class of geons [11]. An alternative algebraic approach to quantum gravity and geon statistics has also been devised with physical inputs like cluster decomposition [8]. It predicts definite spin-statistics connection in (2+1) dimensions [with its probable extension to higher dimensions], which however does not necessarily assert that spinoral geons are fermions or tensorial geons are bosons.

In summary then, there are strong indications that the canonical spin-statistics connection fails in quantum gravity. We can then enquire how this failure percolates to interactions of elementary particles. We initiate the study of this issue in this paper. Quantum gravity effects cannot be important for low energy phenomenology unless they are enhanced by coherent processes involving large numbers, experiments are very accurate or proposals of “large extra dimensions” [11] are intimations of reality. But their study is important even if they are tiny as they challenge concepts of traditional quantum physics. If quantum gravity and string physics are judged by their verifiable predictions, there is no reason to pursue those enterprises [12]. An added reason for our work here that it lets us model spin - statistics violation in a particular way and derive bound on the violation parameters.
Geons are discoveries of Friedman and Sorkin [1]. Their existence has far-reaching implications for quantum gravity. We begin with a brief introduction to geons and their spin-statistics properties in Section 2 and follow it up in Section 3 with the effective interaction they generate among (say) standard model particles. They can be written down using guess work on operator product expansions, but we go a bit beyond that by identifying processes that fix their coefficients. The leading interaction is simple. Geons can be charged or neutral, spinorial or tensorial. Let the particle symbols also denote their fields. A spin-1/2 charged geon can then interact with the electron via the coupling $L'(x) = \eta(e^\dagger G + G^\dagger e)(x)$. $G$ here is a Bose field: otherwise this interaction is not very striking. Similar interactions can happen between a tensorial fermion $G$ and standard particles.

An interesting consequence of these interactions is that it can lead to effective supersymmetry and quantum Hall effect-like phenomena at low energies.

Interactions of this sort, or more generally even the existence of these exotic $G$ fields, are not compatible with RQFT. We provisionally take the following stand about this point. Geons are massive, with mass of the order of Planck mass, and we look only at low energy processes where they can be handled nonrelativistically. There is then no inconsistency. At energies where relativity is important, we presume that new effects enter the picture, perhaps dictated by the extended structure of geons. Our models are no good for these energies.

Interactions such as $H'(x)$ have physical consequences. These are briefly outlined in Sections 4 and 5. In particular, we discuss level distortions and black body spectrum. There are in addition violations of causality and CPT which are also pointed out. The three Appendices are devoted to technical calculations.

Summarizing, the main results of the paper are conceptual and concern the above strikingly novel interaction between geons and standard elementary particles. These interactions, studied in the non-relativistic approximation and low energies here, arise from the mediation of black holes. They violate non-relativistic causality which requires energy densities at spatially separated points to commute at a fixed time. They are not CPT invariant either. Nevertheless the emergent nonrelativistic physics has energy levels bounded below and no obvious inconsistency. The influences of such an interaction on energy levels and black body radiation are investigated, but unfortunately no convincing signal characteristic of the new interaction and large enough to be detected has been found. Causality violations can have a sensitive impact on dispersion relations [19] and latter can possibly detect the novel interactions if the Planck scale is in the TeV. range.

2 What are Geons

Elementary approaches to gravity work with spacetimes $X \times \mathbb{R}$ with $\mathbb{R}$ accounting for time $t$, and the spatial slice $X \times \{t\} \approx X$ being $\mathbb{R}^D$, except during treatment of black holes.

In the 70’s, Friedman and Sorkin [1] initiated studies of asymptotically flat spatial slices (diffeomorphic to) $X$ different from $\mathbb{R}^D$. They pointed out that there are classes of manifolds $X$ called prime manifolds which are perfect infrastructures for describing elementary solitonic
excitations in quantum gravity. There is only one such orientable manifold for $D = 2$ and that is the plane with a handle. It leads to the $2D$ geon. A plane with $n$ handles then gives the excitation of $n$ $2D$ geons. For $D = 3$, there are an infinity of basic manifolds (connected sums of $\mathbb{R}^3$ and closed prime manifolds) and an infinity of geons. A deep result of Friedman and Sorkin was that quantisation of geons, just like the quantisation of two-flavour Skyrmions [13], is not unique, and a certain class of geons can be quantized to give spinorial particles. The underlying primes are known as spinorial primes. Don Witt [14] later extended this work by an exhaustive study of spinorial primes. Later studies [3, 15] revealed that in $2D$, the geon for a plane with a handle can be quantized to have any spin.

Meanwhile Sorkin [2] studied the spin-statistics connection for geons and argued that no such relation can exist in the absence of topology change. This result was elaborated by Sorkin and coworkers [3] and others [8, 9] and the generic failure of the spin-statistic connection in quantum gravity was firmly established. As indicated earlier, a correlation between spin and statistics can be shown with enough physical inputs, but still as a rule it fails to be conventional.

3 Effective Interactions Mediated by Geons

Geons of pure gravity can be tensorial or spinorial, but will be a singlet under the standard model group. Geons with nontrivial standard model quantum numbers can occur when the standard model interactions are also included. In what follows, we have in mind geons of this enhanced theory which violate the spin-statistics connection.

We are after novel interactions among elementary particles induced by this violation in quantum gravity. Processes leading to such couplings are not abundant. Those involving black holes seem to be the sole mediations for this purpose. Black hole processes conserve quantum numbers like charge and angular momentum expressible as flux integrals over a sphere at infinity. But they need not conserve statistics. For this reason the following transition can occur. If $G$ is a spinorial boson with the same charge as the electron $e$, a black hole can absorb $e$ and emit $G$ as Fig.1 illustrates, and vice versa. This process leads to a direct $e - G$ coupling because of vacuum fluctuations involving the creation and annihilation of black holes. A virtual black hole can thus mediate $e - G$ mixing (This process could be especially important should the scenarios presented in the last two references of [11] prove relevant).

Several calculations along these lines exist for gravity-induced proton decay [15, 16] where a proton for example is converted by a black hole into $e$ plus tensorial particles like photons. Accurate calculations are not possible because of lack of control of quantum gravity. The importance of such research for the present work is to show that geons will certainly mix with standard model particles and suggest estimates for the coefficients in operator product expansions.

We conclude that black hole fluctuations in the vacuum induce $G - e$ couplings with the leading term $\lambda e^\dagger G + \lambda^* G^\dagger e$ at low energies. That is for a charged spin-$\frac{1}{2}$ geon. A neutral spin-1 geon with real field $G_\mu$ can likewise couple to the photon field $A_\mu$ by the term constant $\times (\partial_\mu G_\nu - \partial_\nu G_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$. We can write other similar quadratic couplings of
geons and low energy excitations.

We can assume $\lambda$ to be $> 0$ in $G - e$ coupling by writing $\lambda = |\lambda|e^{i\chi}$ and absorbing the phase $\chi$ into the definition of $G$. In Fig.1 we can exchange the particles, so that $\lambda$ has to be a symmetrical function of $m_e$ and $m_G$ where $m_A$ is the mass of particle $A$. We assume that $m_e << m_G$ and keep its leading term in $m_e/m_G$. The symmetry is lost in this approximation. Dimensional considerations then show that $\lambda = m_e f(m_e/m_G)$ where we retain dependence of $f$ only on the single mass ratio $m_e/m_G$, ignoring other elementary particles. According to [15] (see also [16]), $f\left(\frac{m_e}{m_G}\right) = \left(\frac{m_e}{m_G}\right)^K$ (times a factor of order 1), where the integer $K = 2$ (spin of $G$) = 1. We let $K$ be free for caution.

The conventional choice for $m_G$ is Planck mass $m_{pl} \sim 10^{19}$Gev. That gives $\frac{m_e}{m_G} \approx 10^{-22}$. In models with “large extra dimensions” [11], $m_G$ can be low. For the Tev scale gravity the same ratio becomes $\frac{m_e}{m_G} \approx 10^{-12}$.

More favorable values of $\lambda$ can be got by changing $e$ to a heavier particle. Already with a neutron we gain a factor of $10^3$:

$$m_n/m_G \approx 10^{-19} \quad \text{if} \quad m_G \sim m_{pl},$$

$$m_n/m_G \approx 10^{-9} \quad \text{if} \quad m_G \sim 1 \text{ Tev.}$$

In this case, if $K = 1$ or 2, the effects studied below are within experimental reach [17, 18].

$\textbf{4 Level Distortions}$

In this section, we explore the effects of the interaction in section 3 on energy levels. They get shifted as is to be expected. This effect is illustrated using the harmonic oscillator system. It is a simple, but basic system where the new physics can be understood with relative transparency and then applied to other situations. It is also an approximation to quantum field theory where we retain only one mode each of a geon and a standard model field and only terms in the Lagrangian density quadratic in these fields (see below).

$\textit{i) A Boson and a Fermion}$

There are new important features encountered when more than one fermion or boson is considered in the Hamiltonian. We will therefore study them later.

Let us consider a system that has only two degrees of freedom, represented by creation operators ($b^\dagger, f^\dagger$) an annihilation operators ($b, f$). Commutation (anticommutation) relations
are taken to be
\[ [b, b^\dagger] = 1, \quad \{f, f^\dagger\} = 1 \] \quad (2)
where curly brackets mean anti-commutators as usual. Assuming the existence of the common vacuum, \( |0\rangle \), which is annihilated by both \( f \) and \( b \), the Hilbert space is spanned by the linear combinations of the following states:
\[ |n\rangle = \frac{1}{\sqrt{n!}} (b^\dagger)^n |0\rangle, \quad f^\dagger |n\rangle = \frac{1}{\sqrt{n!}} (b^\dagger)^n f^\dagger |0\rangle \]
\[ b |0\rangle = f |0\rangle = 0 \] \quad (3)

As has already been mentioned in the introduction, a relativistic particle with half-integer (integer) spin can only be successfully described in a local field theoretic formalism if it has fermionic (bosonic) commutation rules. Since the excitations that we want to study are very heavy \( (m \approx M_{pl}) \), they should admit a non-relativistic description. In this case there is no connection between spin and statistics and we will therefore assume that one of the operators (it does not matter for the moment which one) represents the excitation with the “wrong” statistics (e.g. either \( b \)’s are spin half or \( f \)’s are spin 0,1 etc.). [The spin degrees of freedom are being ignored.]

At this point, a few comments are in order. The Hamiltonian of the model is
\[ H = \omega b^\dagger b + \omega f^\dagger f + gb^\dagger f + gf^\dagger b \] \quad (4)
where we can assume that \( g > 0 \) as pointed out earlier.

We can obtain (4) for example from a Hamiltonian density \( \hat{\mathcal{H}} \) with standard free field terms for \( G \) and \( e \) and an additional interaction \( \hat{\mathcal{H}}' \):
\[
\hat{\mathcal{H}} = -\frac{1}{2m} G^\dagger \nabla^2 G - e^\dagger (\bar{\alpha} \cdot \vec{p} + \beta m_e) e + \hat{\mathcal{H}}',
\]
\[
\hat{\mathcal{H}}' = \eta (G^\dagger e + e^\dagger G).
\]

To get (4) we then mode expand \( G \) and \( e \) so as to diagonalize the free field terms. On retaining just one mode in these expansions (pretending that they are discrete) and including \( \hat{\mathcal{H}}' \), we get (4).

Below we will diagonalize (4). The generic eigenstates of \( H \) are not created from the vacuum by simple linear expressions in \( b^\dagger \) and \( f^\dagger \) and their powers. Rather they are created from the vacuum by complicated expressions involving \( b^\dagger \) and \( f^\dagger \). For this reason, the mode expansion diagonalizing the Hamiltonian \( \int d^3x \, \mathcal{H}(x) \) is unknown to us.

The problem can be seen in yet another manner. The Hamiltonian for the \( G,e \) fields has the form
\[ \int d^3x \left( G^\dagger + e^\dagger \right)(x) \hat{\mathcal{H}} \left( \begin{array}{c} G \\ e \end{array} \right)(x), \]
\[ \hat{\mathcal{H}} = \left( \begin{array}{cc} -\frac{1}{2m} \nabla^2 & \eta \\ \eta & \bar{\alpha} \cdot \vec{p} + \beta m_e \end{array} \right) \]
where $G$ and $e$ are say Bose and Fermi fields and $\hat{H}$ is the “single particle Hamiltonian”. Normally we would expand $G$ and $e$ in terms of eigenstates of $\hat{H}$ with creation and annihilation operators of appropriate statistics as coefficients. But that does not work now. The eigenstates are given by $\hat{H}\Psi_n = E_n\Psi_n$ and have the form

$$\Psi_n = \begin{pmatrix} \beta_n \\ \phi_n \end{pmatrix}.$$  

They mix Bose and Fermi modes, $(\beta_n,0)$ and $(0,\phi_n)$ not being eigenstates. But then, we do not know what statistics to assign to $a_n$ in the expansion

$$\begin{pmatrix} G \\ e \end{pmatrix} = \sum a_n \Psi_n.$$  

Let us return to study of the spectrum of this Hamiltonian. It is easy to construct the exact eigenstates of this model (see Appendix A). The Schrödinger equation is easily solved by using the ansatz

$$|\psi_n\rangle = (\alpha_n(b^\dagger)f^\dagger + \phi_n(b^\dagger))|0\rangle$$

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$  

The spectrum is given by two series of states labeled by the non-negative integer $n$ with $\alpha_n(x) \sim x^n$ and $\phi_n(x) \sim x^{n+1}$ (plus the vacuum state, $|0\rangle$ which remains an eigenstate even for $g \neq 0$). Energies of the pair of $n^{th}$ states are (see (A.9))

$$E_n^\pm = \frac{1}{2} \left( \omega_b(2n + 1) + \omega_f \pm \sqrt{(\omega_b - \omega_f)^2 + 4g^2(n + 1)} \right).$$

This is the complete spectrum of the system. As $g$ tends to 0 each state smoothly goes into one eigenstate of the unperturbed Hamiltonian. It is interesting that perturbative ground state may or may not remain as such depending on the value of $g$. The condition for the existence of negative eigenvalues in the spectrum is

$$n\omega_b^2 + \omega_f^2 < |g|^2$$

The minimal value of $|g|^2$ when this happens is therefore $|g| = \sqrt{\omega_b\omega_f}$ while the approximate number of negative eigenvalues is $n_\pm = (|g|^2 - \omega_f^2)/\omega_b^2$. It is worth mentioning that a similar Hamiltonian with both $f$ and $b$ considered to be bosons will not have a ground state once $g$ is sufficiently large ($|g| > \sqrt{\omega_b\omega_f}$).

A striking feature of these levels is that they are $b-f$ mixtures. For a small $g$ and $\omega_f > \omega_b$, ...
we find the following behaviour:

\[
\alpha_n^+ = \left(1 - \frac{g^2 (n + 1)}{2(\omega_b - \omega_f)^2} + \cdots\right) \frac{1}{\sqrt{n!}}
\]

(9)

\[
\phi_n^+ = \frac{g}{(\omega_f - \omega_b) \sqrt{n!}}
\]

(10)

\[
\alpha_n^- = -\frac{g}{(\omega_f - \omega_b) \sqrt{n!}}
\]

(11)

\[
\phi_n^- = \left(1 - \frac{g^2 (n + 1)}{2(\omega_b - \omega_f)^2} + \cdots\right) \frac{1}{\sqrt{(n + 1)!}}
\]

(12)

(In case of \(\omega_f < \omega_b\), one makes the interchanges \(\alpha_n^+ \leftrightarrow \alpha_n^-, \phi_n^+ \leftrightarrow \phi_n^-\)).

A point worthy of attention is that level degeneracy for the generic level is not affected as \(g\) is turned on (with the exception of the occasional coincidence of energies for some special values of \(g\)). As it is increased from zero, each level adiabatically and smoothly evolves. Degeneracy of the levels is therefore not affected when \(g\) becomes non-zero.

However, there is one special case \(g = g_s = \sqrt{\omega_b \omega_f}\) which makes (8) into an equality. In this case \(E^{-}_0 = 0\) as it is an eigenvalue for the perturbative vacuum \(|0\rangle\) (remember that it is still an eigenstate). So in this case the “ground” state becomes degenerate, with two states of the same energy 0 being

\[
|0\rangle, \quad \left(-\sqrt{\frac{\omega_b}{\omega_f + \omega_b}} f^\dagger + \sqrt{\frac{\omega_f}{\omega_f + \omega_b}} b^\dagger\right) |0\rangle.
\]

(13)

ii) Connection to Supersymmetry and QHE.

It has been pointed out to us by Joseph Samuel that the Hamiltonian (11) is an element of the graded algebra of a supergroup, with \(SO(2)\) as its underlying classical group. One sees this from the anticommutator

\[
\left[b^\dagger f, f^\dagger b\right]_+ = b^\dagger b + f^\dagger f.
\]

(14)

The graded algebra has \(b^\dagger b\) and \(f^\dagger f\) as even generators and \(b^\dagger f\) and its adjoint as odd generators.

There is also an interesting connection of (11) to the Dirac Hamiltonian in a plane with a perpendicular uniform magnetic field as has also been pointed out to us by Samuel. In this case the three-dimensional zero-mass Dirac Hamiltonian in a magnetic field, \(\vec{\alpha} \cdot \vec{\pi}\), becomes \(\alpha^+ \pi^- + \alpha^- \pi^+, \{\alpha^+, \alpha^-\} = 1, [\pi^+, \pi^-] = eB\) where \(B\) is the magnetic field along the perpendicular direction. This Hamiltonian can be identified with the last two terms in (11).

iii) One Boson and Two Fermions.

In the hydrogen atom, there are two levels with principal quantum number \(n = 1\) corresponding to spin up and spin down. Their creation operators \(f^\dagger_i\) \((i = 1, 2)\) in the second-quantized formalism anticommute. Similarly we can associate fermionic oscillators to bound state levels of any spinorial particle.

This association is in conventional physics in the absence of the disturbing presence of geons. With geons in the spectrum there are additional interactions which spoil such simple
associations. The simplest model that we can consider is a generalization of the Hamiltonian \( H \) to the two fermionic modes \( f_{1,2} \) interacting with a single bosonic geon \( b \):

\[
H = \omega_1 f_1^\dagger f_1 + \omega_2 f_2^\dagger f_2 + \Omega b^\dagger b + g_1(f_1^\dagger b + b^\dagger f_1) + g_2(f_2^\dagger b + b^\dagger f_2)
\]  

(15)

This Hamiltonian is analyzed in Appendix B. One can show that for the most interesting case when both fermions are degenerate, i.e. \( \omega_1 = \omega_2 \) it is possible to find the eigenfunctions and exact spectrum of the model. We shall show that the operator for the level corresponding to two fermionic excitations has the form (B.3):

\[
|\psi\rangle = \left\{ \alpha(b^\dagger) f_1^\dagger f_2^\dagger + \phi_1(b^\dagger) f_1^\dagger + \phi_2(b^\dagger) f_2^\dagger + \Psi(b^\dagger) \right\} |0\rangle
\]  

(16)

It goes over to just \( f_1^\dagger f_2^\dagger \) as the interaction with the geon is switched off.

Let us next consider the case where \( \omega_1 = \omega_2 \) and \( g_1 = g_2 = g \). In this case, (15) is invariant under the exchange of \( f_1 \) with \( f_2 \). In order to study the possible Pauli principle violation in this case, one should consider what happens when the two fermionic operators are exchanged. Then \( |\psi\rangle \) becomes \( |\psi'\rangle \), where

\[
|\psi'\rangle = \left\{ \alpha(b^\dagger) f_2^\dagger f_1^\dagger + \phi_1(b^\dagger) f_2^\dagger + \phi_2(b^\dagger) f_1^\dagger + \Psi(b^\dagger) \right\} |0\rangle.
\]  

(17)

Thus \( |\psi\rangle \) will be an eigenstate of the permutation operator with -1 eigenvalue only if \( \Psi = 0 \) and \( \phi_1 = -\phi_2 \). In general, if \( \omega_1 \neq \omega_2 \) and/or \( g_1 \neq g_2 \), this is not the case. However, we have checked that for \( \omega_1 = \omega_2 \) and \( g_1 = g_2 = g \), for the eigenvalue series that goes to \( E_3 = 2\omega + n\Omega \) in the limit \( g_{1,2} \to 0 \), this is precisely the case: the eigenvectors of the (B.3) matrix have the structure \( (X_n(b), Y_n(b), -Y_n(b), 0) \). As we expect this result to be generally true, we conclude that there is no apparent Pauli principle violation in this model.

Actually we can argue that models like this cannot violate Pauli principle unless level degeneracy is affected as \( g (= g_1 = g_2) \) becomes nonzero. That is because (15) for \( \omega_1 = \omega_2 \) and \( g_1 = g_2 = g \) is symmetric under exchange of \( f_1 \) and \( f_2 \) and hence its eigenstates can be organized in irreducible representations (IRR’s) of the permutation group \( S_2 \). At \( g = 0 \), the energy eigenstates with two fermions change sign under their exchange: they transform by the nontrivial IRR of \( S_2 \). By continuity, this IRR will persist if \( g \) is made nonzero. New effects can arise if level degeneracy is changed when \( g \) becomes nonzero, so that there is a state symmetric under \( S_2 \) degenerate with this IRR. But that does not happen in our model.

5 Black Body Spectrum

As it is written now, either of the frequencies \( \omega_{b,f} \) may be taken to correspond to a geon, the actual choice is dictated by the low energy effect that one wants to study. In the case of effects in atomic systems, one assumes that \( \omega_1 \sim M_{pl} \) and \( \omega_f = \sqrt{m_e^2 + k^2}, \omega_f << \omega_b \). In this case the geon is a spin half bosonic excitation in gravity. From the cosmological point of view, however, it is very interesting to look at the case of the microwave background radiation,
where the geon must be the spin 1 excitation and hierarchy of scales is opposite: \( \omega_b \sim k_{\text{photon}} \) and \( \omega_b \ll \omega_f \sim M_{\text{pl}} \).

In order to determine the effect of geons on the Planck spectrum one has to determine the correction to the thermal distribution function for the photons. Since the spectrum of the Hamiltonian \( (\text{I}) \) is known, the task reduces to the computation of the proper partition function (see Appendix C for details), which can be done perturbatively in \( |g|^2 \). The first non-trivial correction to the distribution function turns out to be given by \( (C.7) \),

\[
n(\omega) = n_0(\omega) - n_0(\omega) \frac{|g|^2}{M_{\text{pl}}} + \frac{e^{\beta \omega}}{e^{\beta \omega} - 1} \frac{\beta |g|^2}{M_{\text{pl}}} + \cdots , \quad n_0(\omega) = \frac{1}{e^{\beta \omega} - 1} \tag{18}
\]

where \( n_0 \) is the free bosonic Planck distribution.

One can immediately see that the expansion of \( (18) \) in powers of \( |g|^2 \) is singular as \( \omega_b \) goes to 0: the first correction diverges as \( 1/\omega_b^2 \). This is the same problem that plagues any theory that has massless modes at non-zero temperature. However, in this case a more careful treatment is needed. The reason for that is that the proliferation of the soft photons in the usual, free case does not lead to a divergent energy density: the phase space volume scales like \( \omega^3 d\omega \) and the denominator of the bosonic distribution produces a \( 1/\omega \) factor thus keeping the product finite. For us, the next order correction produces terms like \( 1/\omega^k \) where \( k \) is roughly proportional to the order of the perturbative expansion. One possible solution to this problem is the following: in the case of the photon, due to dimensional considerations one should consider \( g \sim k^2/M_{\text{pl}} \), which will keep the answer finite in the limit \( k = \omega_b \to 0 \). This behaviour of \( g \) is supported by the fact that the leading gauge invariant geon-photon coupling involves derivatives being \( \text{constant} \times (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho G_\nu - \partial_\nu G_\rho) \) for a spin 1 geon as indicated earlier. Numerically, one can stop at the first order if \( (\beta \omega_b)^2 >> \beta |g|^2/\omega_f \). Using the suggestion above this condition reduces to just \( T << M_{\text{pl}} \) which is obviously satisfied.

The first correction to \( n_0 \) in the formula \( (18) \) is just a “grey body” factor, while the second one is more important at low frequencies of the photon. However, at this point one has to use the expression \( \sim k^2/M_{\text{pl}} \) for \( g \), which makes the corrections look like

\[
\Delta n(\omega) = -n_0(\omega) \frac{k^4}{M_{\text{pl}}^4} + \frac{e^{\beta \omega}}{e^{\beta \omega} - 1} \frac{\beta k^4}{M_{\text{pl}}^3} \tag{19}
\]

In the limit \( k = \omega \to 0 \) the second term becomes \( k^2/(\beta M_{\text{pl}}^3) \) and it is clearly insignificant for small \( k \). It seems then, that even though there are some corrections to the “background radiation” following from the Hamiltonian described above they are too tiny to be detected in the experimentally accessible region.

It is important to make sure that whatever correction that the observed distribution function gets is a signature of the effect in hand and not of some other origin. While it is almost certainly impossible to establish this rigorously, one elementary test is possible here - what if the “mixed” mode is a boson as well? Now, for comparison let us consider the similar situation in the case when we have two bosonic operators coupled the same way as in the Hamiltonian \( (\text{I}) \),

\[
H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g a_1^\dagger a_2 + g^* a_2^\dagger a_1 \tag{20}
\]
Here both $a_1$ and $a_2$ are bosons.

The correction to order $|g|^2$ to the distribution function can be calculated from (C.10), the result is identical to that of (C.7) to order $|g|^2$, with the same “grey body” factor and correction to the low energy behaviour. Thus, unfortunately, the difference between (C.7) and (C.10), therefore, comes only in the next order of perturbation theory. While the full expressions are given by (C.11), this difference in powers of $|g|^2$ is

$$\Delta n(\omega)_{b+g} - \Delta n(\omega)_{b+b} = \left\{ \frac{\beta}{M_{Pl}^4} \left( \frac{2 - 6e^{\beta\omega}}{(e^{\beta\omega} - 1)^3} \right) + \frac{6}{M_{Pl}^4} \left( \frac{e^{\beta\omega}}{(e^{\beta\omega} - 1)^2} \right) \right\} g^4 + \cdots \quad (21)$$

where $\Delta n(\omega)_{b+g}$ is $n(\omega) - n_0(\omega)$ in (18), while $\Delta n(\omega)_{b+b}$ is given in (C.11). These corrections become identical in the ultraviolet limit $\beta\omega \to \infty$ and the only difference comes in the sub-leading order in $1/M_{Pl}$. For that reason, it is difficult to propose an experimental test which would be able to see these corrections.

### 6 Final remarks: CPT, Causality

**i) CPT.**

In the course of proving the CPT theorem, anti-commutativity of fermionic fields and commutativity of tensorial ones are explicitly used [4]. But this feature need not hold for geons. CPT thus can fail in the presence of geons. The failure will be by small numbers like $10^{-19}, 10^{-9}$ or its powers [cf.Eq. (1)]. Detailed calculations may be possible by allowing for mixing of quarks and leptons for instance with geons and integrating out the latter, but we have not done this work.

**ii) Causality.**

If $\Psi$ is a spinorial fermion field and $G$ a spinorial boson field, for example, the term $\mathcal{H}_I(x) = \lambda (G^\dagger \Psi + \Psi^\dagger G)(x)$ in the Hamiltonian density $\mathcal{H}(x)$ does not commute for the space-like separations: $[\mathcal{H}(x), \mathcal{H}(y)] \neq 0$, $(x - y)^2 > 0$. As $\mathcal{H}(x) = \mathcal{H}_0(x) + \mathcal{H}_I(x)$ where $\mathcal{H}_0(x)$ commutes for spacelike separations, $\mathcal{H}(x)$ and $\mathcal{H}(y)$ neither commute nor anti-commute for space-like separations. Thus Hamiltonian density, an observable, violates local causality.

Our model is in reality valid only nonrelativistically, so that we have to interpret this statement as asserting that the Hamiltonian densities at distinct spatial points at the same time do not commute. (Hence they cannot be “simultaneously measured”.)

The implications of this microscopic violation of causality are not adequately clear to us. It does have a phenomenological implication: forward dispersion relations will not be correct. Such violations of causality also occur in noncommutative geometry, in particular of D-branes in string physics [21]. Basically, this causality violation in our model is controlled by the intrinsic non-locality of the geons, and this non-locality is similar to having a fundamental length $l_f \sim 1/M_{Pl}$ in the theory. There are some indications [19] that forward dispersion relations can be a sensitive probe of $l_f$ provided it is not too small, say if $1/l_f$ is in the TeV range. Nonlocality will also spoil the analyticity of scattering amplitudes and its implications by small corrections.
Investigations of these effects would be of great interest, being characteristic manifestations of intrusions of quantum gravity or string physics into elementary particle theory.

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Appendix A

In this appendix we construct eigenstates and find eigenvalues for the Hamiltonian (4). We start by writing an arbitrary eigenstate

\[ |\psi\rangle = (\alpha(b^\dagger) f^\dagger + \phi(b^\dagger)) |0\rangle, \]

\[ H|\psi\rangle = E |\psi\rangle \] (A.1)

Using commutation relations (2) one has

\[ [b, \phi(b^\dagger)] = \partial\phi(b^\dagger) \frac{\partial}{\partial b^\dagger}, \quad [b, \alpha(b^\dagger)] = \partial\alpha(b^\dagger) \frac{\partial}{\partial b^\dagger} \] (A.3)

and equation (A.2) becomes

\[ \left\{ \omega_f \alpha(b^\dagger) f^\dagger + \omega_b \frac{\partial\alpha(b^\dagger)}{\partial b^\dagger} b^\dagger f^\dagger + g \alpha(b^\dagger) b^\dagger + \omega_b \frac{\partial\phi(b^\dagger)}{\partial b^\dagger} b^\dagger + g \frac{\partial\phi(b^\dagger)}{\partial b^\dagger} f^\dagger \right\} |0\rangle = E \left\{ \alpha(b^\dagger) f^\dagger + \phi(b^\dagger) \right\} |0\rangle. \] (A.4)

By rewriting this in the matrix form one gets:

\[ \omega_f \alpha(b^\dagger) + \omega_b \frac{\partial\alpha(b^\dagger)}{\partial b^\dagger} b^\dagger + g \frac{\partial\phi(b^\dagger)}{\partial b^\dagger} b^\dagger = E \alpha(b^\dagger), \] (A.5)

\[ g \alpha(b^\dagger) b^\dagger + \omega_b \frac{\partial\phi(b^\dagger)}{\partial b^\dagger} b^\dagger = E \phi(b^\dagger). \] (A.6)

At this point let us assume the following behaviour of the functions \( \alpha \) and \( \phi \):

\[ \alpha(b^\dagger) = \alpha_n (b^\dagger)^n, \quad \phi(b^\dagger) = \phi_n (b^\dagger)^{(n+1)}. \] (A.7)

Here \( \alpha_n \) and \( \phi_n \) are (complex) numbers. It is easy to see that with this choice equations (A.6) reduce to a simple eigenvalue problem for a \( 2 \times 2 \) matrix:

\[ \begin{pmatrix} \omega_f + n \omega_b & g(n+1) \\ g & (n+1)\omega_b \end{pmatrix} \begin{pmatrix} \alpha_n \\ \phi_n \end{pmatrix} = E_n \begin{pmatrix} \alpha_n \\ \phi_n \end{pmatrix} \] (A.8)

The eigenvalue equation is quadratic, and yields the following two sets of solutions:

\[ E_{n} = \frac{1}{2} \left( \omega_b(2n + 1) + \omega_f \pm \sqrt{(\omega_b - \omega_f)^2 + 4g^2(n+1)} \right). \] (A.9)
With these values of energy the coefficients $\alpha_n$ and $\phi_n$ can be determined from the normalization condition $n!|\alpha_n|^2 + (n + 1)!|\phi_n|^2 = 1$ and equation (A.8). They are found (upto an over-all phase) to be

$$\alpha_n^\pm = \frac{\Delta \omega \pm \sqrt{\Delta \omega^2 + 4g^2(n + 1)}}{\left(\Delta \omega \pm \sqrt{\Delta \omega^2 + 4g^2(n + 1)}\right)^2 + 4g^2(n + 1)} \frac{1}{\sqrt{n!}}$$  \hspace{2cm} (A.10)

$$\phi_n^\pm = \frac{2g}{\left(\Delta \omega \pm \sqrt{\Delta \omega^2 + 4g^2(n + 1)}\right)^2 + 4g^2(n + 1)} \frac{1}{\sqrt{n!}}$$  \hspace{2cm} (A.11)

$$\Psi_n = \sqrt{n!} |\alpha_n^\pm, \phi_n^\pm\rangle$$  \hspace{2cm} (A.12)

where $\Delta \omega = \omega_f - \omega_b$. These expressions characterize the complete spectrum of the model.

In the limit $g \to 0$, this smoothly goes to the unperturbed spectrum for which $g = 0$. This ensures that we have found all of the eigenstates of the system.

**Appendix B**

A similar treatment can be applied to the case of two fermionic modes coupled to a boson as in the Hamiltonian $H$. The most generic ansatz for an energy eigenstate is

$$|\psi\rangle = \left\{ \alpha(b^\dagger) f_1^\dagger f_2^\dagger + \phi_1(b^\dagger) f_1^\dagger + \phi_2(b^\dagger) f_2^\dagger + \Psi(b^\dagger) \right\} |0\rangle.$$  \hspace{2cm} (B.1)

Applying Hamiltonian $H$ and using relations similar to those of (A.2) and (A.3), we get

$$\begin{align*}
(\omega_1 + \omega_2)\alpha(b^\dagger) + \Omega_1 \frac{\partial \alpha(b^\dagger)}{\partial b^\dagger} b^\dagger + -g_1 \frac{\partial \phi_1(b^\dagger)}{\partial b^\dagger} + g_1 \frac{\partial \phi_2(b^\dagger)}{\partial b^\dagger} &= E \alpha(b^\dagger), \\
-g_2 \alpha(b^\dagger) b^\dagger + \omega_1 \phi_1(b^\dagger) + \Omega_1 \frac{\partial \phi_1(b^\dagger)}{\partial b^\dagger} b^\dagger + g_1 \frac{\partial \Psi(b^\dagger)}{\partial b^\dagger} &= E \phi_1(b^\dagger), \\
+g_1 \alpha(b^\dagger) b^\dagger + \omega_2 \phi_2(b^\dagger) + \Omega_1 \frac{\partial \phi_2(b^\dagger)}{\partial b^\dagger} b^\dagger + g_2 \frac{\partial \Psi(b^\dagger)}{\partial b^\dagger} &= E \phi_2(b^\dagger), \\
g_1 \phi_1(b^\dagger) b^\dagger + g_2 \phi_2(b^\dagger) b^\dagger + \Omega_1 \frac{\partial \Psi(b^\dagger)}{\partial b^\dagger} b^\dagger &= E \Psi(b^\dagger). 
\end{align*}$$  \hspace{2cm} (B.2)

As in the previous case we first assume the power-law behaviour of coefficients $\alpha, \phi_1, \phi_2, \Psi$ and write

$$\alpha(b^\dagger)_n = \alpha_n (b^\dagger)^n, \quad \phi_1(b^\dagger)_n = \phi_1_n (b^\dagger)^{n+1}, \quad \phi_2(b^\dagger)_n = \phi_2_n (b^\dagger)^{n+1}, \quad \Psi(b^\dagger)_n = \Psi_n (b^\dagger)^{n+2}.$$  \hspace{2cm} (B.3)

The corresponding $(4 \times 4)$ matrix equation can be read off the equations (B.2):

$$\begin{pmatrix}
\omega_1 + \omega_2 + \Omega n & -g_2(n + 1) & 0 & g_1(n + 1) \\
-g_2 & \omega_1 + \Omega(n + 1) & 0 & g_1(n + 2) \\
+g_1 & 0 & \omega_2 + \Omega(n + 1) & g_2(n + 2) \\
0 & g_1 & g_2 & \Omega(n + 2)
\end{pmatrix} \begin{pmatrix}
\alpha_n \\
\phi_1_n \\
\phi_2_n \\
\Psi_n
\end{pmatrix} = E \begin{pmatrix}
\alpha_n \\
\phi_1_n \\
\phi_2_n \\
\Psi_n
\end{pmatrix}$$  \hspace{2cm} (B.4)

The eigenvalues can be obtained from the fourth order secular equation and are quite complicated for generic values of the frequencies and coupling constants. Nevertheless for the
physically interesting case of degenerate ($\omega_1 = \omega_2 = \omega$) fermions coupled to a bosonic “geon”,
the eigenvalue equation for the matrix (B.4) splits into the product of two quadratic ones and yields

\begin{align*}
(\omega + \Omega(n + 1) - E)(\Omega(n + 2) - E) - (g_1^2 + g_2^2)(n + 2) &= 0, \\
(2\omega + \Omega n - E)(\omega + \Omega(n + 1) - E) - (g_1^2 + g_2^2)(n + 1) &= 0.
\end{align*}

(B.5) (B.6)

The corresponding energy eigenvalues are

\begin{align*}
E_{1,2} &= \frac{1}{2} \left\{ \omega + \Omega(2n + 3) \pm \sqrt{(\omega - \Omega)^2 + 4g^2(n + 2)} \right\} \\
E_{3,4} &= \frac{1}{2} \left\{ 3\omega + \Omega(2n + 1) \pm \sqrt{(\omega - \Omega)^2 + 4g^2(n + 1)} \right\}
\end{align*}

(B.7)

But these do not exhaust all the energy eigenstates. Thus, if we look at the limiting case
of $g_{1,2} \to 0$, we find

\begin{align*}
E_1 &= \omega + \Omega(n + 1), \quad E_2 = \Omega(n + 2), \\
E_3 &= 2\omega + \Omega n, \quad E_4 = \omega + \Omega(n + 1).
\end{align*}

(B.8)

Taking into account that vacuum state $|0\rangle$ remains an eigenstate with energy 0, we see that
there are three eigenvalues that are missing in the above sets, namely $\Omega, \omega_1, \omega_2$. This has
happened because while solving (B.2) we assumed (B.3), which is not the only possibility. Assuming that $\alpha(b^\dagger) = 0$, one can show that the following equations for $\phi_1, \phi_2$ and $\Psi$ result:

\[ g_2\phi_1(b^\dagger) = g_1\phi_2(b^\dagger). \]

These equations bring in the “missing” energies.

**Appendix C**

In this appendix we give detailed calculations of the influence of the boson-fermion mixing
on the black body radiation spectrum (microwave background).

The standard way to do so is to introduce chemical potentials $\mu_f, \mu_b$ for fermionic and
bosonic excitations of the model. Then

\[ n_b(\omega_b) = \frac{\partial \Omega}{\partial \mu_b}|_{\mu_b=0} \]

where $n_A(\omega_A)$ is the mean number of particles of type A with energy $\omega_A$, and the potential $\Omega$
is given by

\[ \Omega = -\frac{1}{\beta} \ln Z, \quad Z = \text{tr} e^{-\beta(H - \mu_b a^\dagger a - \mu_f b^\dagger b)}. \]

(C.2)

Because our initial $H$ is quadratic, the addition of number operators leads just to the effective
changes $\omega_b \to \omega_b - \mu_b$ and $\omega_f \to \omega_b - \mu_f$. Since the trace can be computed over any set of
complete states, one can use these new values in the expression for the spectrum (7) and just sum over $n$:

$$Z = 1 + \sum_{n=0}^{\infty} \{e^{-\beta E_n^+} + e^{-\beta E_n^-}\}$$

$$= 1 + 2 \sum_{n=0}^{\infty} e^{-\frac{\beta}{2} (2n+1)\omega_b+\omega_f} \text{ch} \left( \frac{\beta}{2} \sqrt{(\omega_b - \omega_f)^2 + 4|g|^2(n+1)} \right)$$

As the coupling parameter $g$ goes to 0, the above expression tends to

$$1 + \sum_{n=0}^{\infty} \{e^{-\beta(n+1)\omega_b} + e^{-\beta(n\omega_b+\omega_f)}\}$$

$$= \frac{1 + e^{-\beta\omega_f}}{1 - e^{-\beta\omega_b}} = Z_f \times Z_b$$

where $Z_f$ and $Z_b$ are free fermionic and free bosonic partition functions. Expanding in powers of $|g|^2$, one gets

$$Z = Z_f Z_b + 2 \frac{e^{-\frac{\beta}{2}(\omega_f+\omega_b)} Z_b \beta \text{sh} \left( \frac{-\beta}{2} |\omega_b - \omega_f| \right)}{(1 - e^{-\beta\omega_b})^2} |g|^2 + \cdots.$$  

Rewriting the expression for the partition function as

$$Z = Z_f Z_b \left( 1 + 2 e^{-\frac{\beta}{2}(\omega_f+\omega_b)} \frac{Z_b \beta \text{sh} \left( \frac{-\beta}{2} |\omega_b - \omega_f| \right)}{Z_f \omega_b - \omega_f^2} |g|^2 + \cdots \right)$$

and assuming that $\omega_f \sim M_{Pl}$, the effective correction to the distribution function is

$$\Omega = \Omega_0 - \frac{e^{-\beta\omega_b} Z_b}{(M_{Pl} - \omega_b)} |g|^2 + \cdots = \Omega_0 - \frac{n_b(\omega_b)}{(M_{Pl} - \omega_b)} |g|^2 + \cdots$$

$$n(\omega) - n_0(\omega) = -n_0(\omega) \frac{|g|^2}{M_{Pl}^2} + \frac{e^{\beta\omega} |g|^2}{(e^{\beta\omega} - 1)^2 M_{Pl}}$$

Here $n_0$ is the free bosonic distribution function.

Consider next the case of two bosons coupled in the same way as in the Hamiltonian (4):

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g a_1^\dagger a_2 + g^* a_2^\dagger a_1$$

Here both $a_1$ and $a_2$ are bosons. Contrary to the previous case this model is exactly solvable and the spectrum is

$$E_{n_1, n_2} = \omega_+ n_1 + \omega_- n_2, \quad n_1, n_2 \in N,$$

$$\omega_\pm = \frac{1}{2} \left\{ (\omega_1 + \omega_2) \pm \sqrt{(\omega_1 - \omega_2)^2 + 4g^2} \right\},$$

$$Z = \frac{1}{1 - e^{-\omega_+ \beta}} \frac{1}{1 - e^{-\omega_- \beta}}.$$
Here we take \( \omega_1 = \omega \) to be the “photon” frequency and \( \omega_2 \) to be that of the mixing bosonic mode. The distribution function for the photon is

\[
n(\omega) = \frac{1}{e^{\beta \omega} - 1} \frac{\partial \omega}{\partial \omega} + \frac{1}{e^{\beta \omega} - 1} \frac{\partial \omega}{\partial \omega} + \frac{1}{e^{\beta \omega} - 1} \frac{\partial \omega}{\partial \omega} \quad (C.10)
\]

If one expands this expression in coupling constant \( g \) in the limit \( \omega = \omega_1 \ll \omega_2 \sim M_{Pl} \), the result is identical to that of (C.7) to order \(|g|^2\), with the same “grey body” factor and correction to the low energy behaviour. Unfortunately, the difference between (C.7) and (C.10), therefore, comes only in the next order of perturbation theory. After some algebra one finds

\[
\Delta n(\omega)_{b+g} = \left\{ -\frac{1}{M_{Pl}^2 (e^{\beta \omega} - 1)} + \frac{1}{M_{Pl} (e^{\beta \omega} - 1)^2} \right\} g^2 + \frac{\beta}{M_{Pl}^2 (e^{\beta \omega} - 1)} \left( \frac{e^{2 \beta \omega} + e^{\beta \omega}}{(e^{\beta \omega} - 1)^3} \right) + \frac{\beta}{M_{Pl} (e^{\beta \omega} - 1)^2} \left( \frac{2 - 4e^{2 \beta \omega} - 6e^{\beta \omega}}{(e^{\beta \omega} - 1)^3} \right) + \frac{6}{M_{Pl}^4} \left( \frac{e^{\beta \omega} + 1}{(e^{\beta \omega} - 1)^2} \right) \right\} g^4 + \cdots
\]

\[
\Delta n(\omega)_{b+b} = \left\{ -\frac{1}{M_{Pl}^2 (e^{\beta \omega} - 1)} + \frac{1}{M_{Pl} (e^{\beta \omega} - 1)^2} \right\} g^2 + \frac{\beta}{M_{Pl}^2 (e^{\beta \omega} - 1)} \left( \frac{e^{2 \beta \omega} + e^{\beta \omega}}{(e^{\beta \omega} - 1)^3} \right) + \frac{\beta}{M_{Pl} (e^{\beta \omega} - 1)^2} \left( \frac{2 - 4e^{2 \beta \omega} - 6e^{\beta \omega}}{(e^{\beta \omega} - 1)^3} \right) + \frac{6}{M_{Pl}^4} \left( \frac{1}{(e^{\beta \omega} - 1)} \right) \right\} g^4 + \cdots \quad (C.11)
\]

where \( \Delta n(\omega)_{b+g} \) is the left-hand side of (C.7) while \( \Delta n(\omega)_{b+b} \) is the difference of (C.10) and \( n_0(\omega) \). These corrections become identical in the ultraviolet limit \( \beta \omega \to \infty \) and the only difference comes in the subleading order in \( 1/M_{Pl} \).

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