Ground state and nodal solutions for fractional Orlicz problems without the Ambrosetti-Rabinowitz condition

Hlel Missaoui* and Hichem Ounaies†

Mathematics Department, Faculty of Sciences, University of Monastir, 5019 Monastir, Tunisia

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Abstract

We consider a non-local Schrödinger problem driven by the fractional Orlicz g-Laplace operator as follows

\[-\Delta_g^\alpha u + g(u) = K(x)f(x,u), \quad \text{in } \mathbb{R}^d,\]

where \(d \geq 3\), \((-\Delta_g)^\alpha\) is the fractional Orlicz g-Laplace operator, \(f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) is a measurable function and \(K\) is a positive continuous function. Employing the Nehari manifold method and without assuming the well-known Ambrosetti-Rabinowitz and differentiability conditions on the non-linear term \(f\), we prove that the problem \((P)\) has a ground state of fixed sign and a nodal (or sign-changing) solutions. Indicate that the result of ground state and nodal solution is new for the fractional p-Laplace operator.

Keywords: Ground state, Nodal solutions, Fractional Orlicz-Sobolev spaces, Nehari method, Generalized subdifferential.

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1 Introduction

Recently, much attention has been focused on the study of non-linear problems involving non-local operators. These types of operators arise in several areas such as the description of many physical phenomena (see [29, Part II - Chapters 12 and 13]).

In this paper we consider the following non-local Schrödinger equation

\[-\Delta_g^\alpha u + g(u) = K(x)f(x,u), \quad \text{in } \mathbb{R}^d,\]

where \(d \geq 3\), \(f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) is a measurable function, \(K\) is a positive continuous function, and \((-\Delta_g)^\alpha\) is the fractional Orlicz g-Laplace operator introduced in [13] and defined as

\[(-\Delta_g)^\alpha u(x) = \text{p.v.} \int_{\mathbb{R}^d} g\left(\frac{|u(x) - u(y)|}{|x-y|^{d+\alpha}}\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{dy}{|x-y|^{d+\alpha}}, \quad (1.1)\]

for every \(u : \mathbb{R}^d \to \mathbb{R}\) sufficiently smooth, where p.v. being a commonly used abbreviation for "in the principle value sense", \(\alpha \in (0,1)\) and \(G\) is an N-function such that \(g = G'\). We recall the definition of N-function and its properties later in Section 2. The variational setting for the fractional g-Laplace operator is the fractional Orlicz-Sobolev space \(W^{\alpha,G}(\mathbb{R}^d)\), which is introduced in [13]. For more details on the fractional Orlicz-Sobolev spaces, we refer the reader to [4, 4, 4, 14, 14, 21] and the references therein.

Observe that when \(\alpha = 1\) and \(g(t) = |t|^{p-2}t, p > 1\), the problem \((P)\) turns into the following classical Sobolev problems

\[-\text{div}(|\nabla u|^{p-2}\nabla u) = K(x)f(x,u).\]
When $\alpha = 1$ and $g(t) = a(|t|)t$, the problem \( P \) transmute into the following Orlicz-Sobolev problems
\[
-\text{div}(a(|\nabla u|)u) = K(x)f(x,u), \quad \text{where } \text{div}(a(|\nabla u|)u) \text{ is the Orlicz g-Laplace operator.}
\]

When $\alpha \in (0,1)$ and $g(t) = |t|^{p-2}t$, the problem \( P \) transformed into the following order fractional Sobolev problems
\[
-(\Delta_p)^\alpha u = K(x)f(x,u), \quad \text{where } (\Delta_p)^\alpha u \text{ is the fractional } p\text{-Laplace operator.}
\]

In the last decades, the existence of ground state and nodal solutions for the above problems (classical, Orlicz and fractional Sobolev problems) have been studied extensively. We do not intend to review the huge bibliography of ground state and nodal solutions, we just emphasize that the Nehari method is a very effective method for proving the existence of ground state and sign-changing solutions, see [1, 2, 7, 8, 9, 11, 12, 16, 17, 21, 22, 23, 27, 30, 34] and the references therein.

In [8], S. Barile and G. M. Figueredo have studied the following equation
\[
-\text{div}(a(|\nabla u|^p)\nabla u^{p-2}\nabla u) + V(x)b(|u|^p)|u|^{p-2} = K(x)f(u), \quad \text{in } \mathbb{R}^d, \quad (P_1)
\]
where $d \geq 3$, $2 \leq p < d$, $a$, $b$, are $C^1$ real functions and $V$, $K$ are continuous positives functions. By assuming the well-Known Ambrosetti-Rabinowitz (AR for short) and differentiability ($f \in C^1$) conditions on the non-linear term $f$ and using a minimization argument coupled with a quantitative deformation lemma, they proved the existence of a least energy sign-changing solution for equation \( P_1 \) with two nodal domains.

In [21], G. M. Figueredo considered the following equation
\[
-M\left(\int_\Omega g(|\nabla u|)dx\right)\Delta_g u = f(u), \quad \text{in } \Omega, \quad (P_2)
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^d$, $M$ is $C^1$ function and $\Delta_g u := \text{div}(a(|\nabla u|)\nabla u)$ is the Orlicz g-Laplace operator ($g(t) = a(|t|)t$). By considering the same hypotheses on the non-linear term $f$ and using the same approaches as in [8], he proved the existence of a least energy nodal solution for the equation \( P_2 \).

In [2], V. Ambrosio and T. Isernia have studied the following fractional equation
\[
-(\Delta)^\alpha u + V(x)u = K(x)f(u), \quad \text{in } \mathbb{R}^d, \quad (P_3)
\]
where $\alpha \in (0,1)$ and $d > 2\alpha$, $(\Delta)^\alpha u$ is the fractional Laplace operator and $V, K$ are continuous functions. Under the (AR) and differentiability conditions on the non-linear term $f$ and using a minimization argument and a quantitative deformation lemma, the authors proved the existence of a least energy nodal solution.

In [7], we extended the result obtained in [2] to a non-local generalized fractional Orlicz equations of type \( P \) (under the same hypotheses on the non-linear term as in [2, 8, 21]).

In [22] without considering the (AR) or differentiability conditions on the non-linear term $f$, G. M. Figueiredo and J. R. Santos Júnior established the existence of least energy sign-changing solutions for problem \( P_2 \). Note that, the authors used different approaches than [2, 7, 8, 21]. In [1], by the same hypotheses and approaches as in [22], V. Ambrosio et al. proved the same result for the equation \( P_3 \).

The goal of this paper is to prove the existence of ground state and nodal solutions for problem \( P \) without the (AR) or differentiability conditions on the non-linear term $f$. Our approaches will be based on the Nehari method and the generalized subdifferential. The main features and difficulties involved in the study of the ground state and nodal solutions of problem \( P \) are listed as follow:

1. The non-local character of the fractional Orlicz g-Laplacian. More precisely, in contrast with the classic case \[3, 9, 11, 12, 17\], in the fractional Orlicz framework we do not have the following decompositions
\[
J(u) = J(u^+) + J(u^-), \quad \text{and } J'(u^-)u^- = J'(u^+)u^+ = 0,
\]
for all $u$ belonging to a subset $\mathcal{M}$ of Nehari manifold $\mathcal{N}$ of $J$ (the definitions of $u^+, u^-$, the sets $\mathcal{N}, \mathcal{M}$ and the functional $J$ will be specified later in Section 2), in which we look for nodal solutions of \( P \). Such facts make the use of the minimization arguments more complicated. In [11, 2, 7, 10, 21, 22, 23, 27, 30, 31, 34], the authors have a similar difficulty. In our case, we use other new estimates which are inspired by the work [32].
(2) Unlike with the hypotheses on the non-linear term in [2, 7, 31, 32], in this paper, we do not require the differentiability of \( f \) (see hypotheses \((H_f)\) Section 2). So, we do not hope by the existence of a differentiable structure in the sets \( \mathcal{N} \) and \( \mathcal{M} \), for more details about this subject, we refer the reader to [10]. To overcome the lack of differentiability, we use the same idea as in [25]. Which is based on the non-smooth multiplier rule of Clarke [18, Theorem 10.47, p. 221].

(3) Since we do not assume the classical (AR) condition on \( f \), we use another technique to prove that the minimizing sequences are bounded. Different with other papers [1, 22, 23, 27], which use the same approach to obtain the existence of nodal solutions [1, 22, 23, 27], we invoke Miranda’s theorem [28] to prove that \( \mathcal{M} \neq \emptyset \).

To the best of our knowledge, there is only two papers devoted for the existence and multiplicity of sign-changing solutions for fractional Orlicz problems appear in the literature see [7, 31]. Add to that, in these two last papers, the authors assumed the (AR) condition. Motivated by the above-mentioned works, in present paper, we study the existence of ground state and nodal (or sign-changing) weak solutions for problem (P) without the (AR) or differentiability conditions on the non-linear term \( f \).

The paper is organized as follows. In Section 2 we consider two subsections. In the first, we construct the variational framework of the problem (P) and we establish some properties about N-functions. In the second, we present the definition of the generalized subdifferential. In Section 3, we give the hypotheses on the weight function \( K \) and the non-linear term \( f \), and we state the main result (Theorem 3.2). In Section 4, we present the energy functional associated to problem (P) and we prove some technical lemmas. In Section 5, we show the existence of a ground state solution for problem (P). Finally, we prove the existence of a nodal weak solution and we give the proof of main result.

### 2 Mathematical preliminaries

#### 2.1 Framework setting: Fractional Orlicz-Sobolev Spaces

In this subsection, we recall some necessary properties about N-functions and the fractional Orlicz-Sobolev spaces. for more details we refer the reader to [5, 6, 13, 33].

In order to construct a fractional Orlicz-Sobolev space setting for problem (P), we impose a class of assumptions on \( G \) and \( g \) as follow:

\( (H_G) \) \( g : \mathbb{R} \rightarrow \mathbb{R} \) is an odd, continuous and non-decreasing function and \( G : \mathbb{R} \rightarrow \mathbb{R}^+ \) defined by

\[
G(t) = \int_0^t g(s) \, ds,
\]

such that \( G \) and \( g \) satisfy the following assumptions

\( (g_1) \) \( g(t) > 0, \) for all \( t > 0, \) \( g(0) = 0 \) and \( \lim_{t \to +\infty} g(t) = +\infty. \)

\( (g_2) \) There exist \( g^- \), \( g^+ \) \( \in (1, d) \) such that

\[
g^- \leq \frac{g(t)t}{G(t)} \leq g^+, \quad \text{for all } t > 0.
\]

\( (g_3) \) \( g \in C^1(\mathbb{R}_+^*) \) and \( g^- - 1 \leq \frac{g'(t)t}{g(t)} \leq g^+ - 1, \) for all \( t > 0. \)

\( (g_4) \) \( \int_0^1 \frac{G^{-1}(t)}{t^\alpha} \, dt < \infty \) and \( \int_1^{+\infty} \frac{G^{-1}(t)}{t^\alpha} \, dt = \infty. \)

From the definition (2.1) and assumption \((g_1)\), we infer that \( G \) is an N-function (see [33]). Then, \( G \) is even, positive, continuous and convex function. Moreover \( \frac{G(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{G(t)}{t} \to +\infty \) as \( t \to +\infty. \)

The conjugate N-function of \( G \) denoted \( \tilde{G} \) and defined by

\[
\tilde{G}(t) = \int_0^t \tilde{g}(s) \, ds,
\]
where \( \tilde{g} : \mathbb{R} \to \mathbb{R} \) is given by \( \tilde{g}(t) = \sup \{ s : g(s) \leq t \} \). In our case, since \( g \) is a continuous function, it comes that \( \tilde{g}(-) = g^{-1}(-) \).

The N-functions \( G \) and \( \tilde{G} \) are complementary. Involving the functions \( G \) and \( \tilde{G} \), we have the Young’s inequality given by

\[
st t \leq G(s) + \tilde{G}(t).
\]

We say the \( G \) satisfies the \( \triangle_2 \)-condition, if there exists \( C > 0 \) such that

\[
g(2t) \leq CG(t), \quad \text{for all } t > 0.
\]

Another important function related to function \( G \), is the Sobolev conjugate N-function of \( G \) denoted \( G^* \) and defined by

\[
G^* -1(t) = \int_0^t \frac{G^{-1}(s)}{s^{1+}} ds, \quad t > 0.
\]

As a consequence, from assumption \((g_2)\), we have

\[
g^- \leq \frac{g^*_-(t)}{G^*_+(t)} \leq g^+_*, \quad \text{for all } t > 0
\]

where \( G^*_+(t) = \int_0^t g^*_+(s) ds \) and \( g^+_* \) is the Gagliardo semi-norm defined by

\[
\|u\|_{(\alpha,G)} = \inf \{ \lambda > 0 : \tilde{\rho}(\frac{u}{\lambda}) \leq 1 \}.
\]

Next, we introduce the fractional Orlicz-Sobolev space.

We denote by \( W^{\alpha,G}(\mathbb{R}^d) \) the fractional Orlicz-Sobolev space defined by

\[
W^{\alpha,G}(\mathbb{R}^d) = \left\{ u \in L^G(\mathbb{R}^d) : \bar{\rho}(\alpha; u) < \infty \right\},
\]

where \( \bar{\rho}(\alpha; u) : = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x-y|^{\alpha}} \right) \frac{|x-y|}{|x-y|^d} dy dx \).

The space \( W^{\alpha,G}(\mathbb{R}^d) \) is equipped with the norm,

\[
\|u\|_{\alpha,G} = \|u\|_{(G)} + [u]_{(G)}.
\]

where \([\cdot]_{(\alpha,G)}\) is the Gagliardo semi-norm defined by

\[
[u]_{(\alpha,G)} = \inf \left\{ \lambda > 0 : \bar{\rho}(\alpha; \frac{u}{\lambda}) \leq 1 \right\}.
\]

Since \( G \) and \( \tilde{G} \) satisfy the \( \triangle_2 \)-condition, so the fractional Orlicz-Sobolev space \( W^{\alpha,G}(\mathbb{R}^d) \) is a separable and reflexive Banach space. Moreover, \( C_0^{\infty}(\mathbb{R}^d) \) is dense in \( W^{\alpha,G}(\mathbb{R}^d) \) (see [13, Proposition 2.10]).
Lemma 2.3. \[ \|u\| \text{ is an equivalent norm to } \|u\|_{\alpha,G} \text{ with the relation} \]
\[ \frac{1}{2} \|u\|_{\alpha,G} \leq \|u\| \leq 2 \|u\|_{\alpha,G}, \text{ for all } u \in W^{\alpha,G}(\mathbb{R}^d). \]

In the sequel we will use \( \| \cdot \| \) as a norm for the space \( W^{\alpha,G}(\mathbb{R}^d) \).

As we mentioned in Section 1, the fractional g-Laplace operator is defined by
\[ \begin{align*}
(-\triangle_g)^{\alpha} u(x) &= \text{p.v.} \int_{\mathbb{R}^d} g\left(\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right) \frac{u(x) - u(y)}{|x - y|^{d+\alpha}} dy \\
&= \text{p.v.} \int_{\mathbb{R}^d} g\left(\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right) \frac{dy}{|x - y|^{d+\alpha}} \text{ (since } g \text{ is odd).}
\end{align*} \]

This operator is well defined between \( W^{\alpha,G}(\mathbb{R}^d) \) and its topological dual space \( (W^{\alpha,G}(\mathbb{R}^d))^* = W^{-\alpha,G}(\mathbb{R}^d) \).

In fact, in [13] Theorem 6.12, the following representation formula is provided
\[ \langle (-\triangle_g)^{\alpha} u, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left(\frac{|u(x) - u(y)|}{|x - y|^{\alpha}}\right) \frac{v(x) - v(y)}{|x - y|^{\alpha}} \frac{dxdy}{|x - y|^{d}}, \]
for all \( u, v \in W^{\alpha,G}(\mathbb{R}^d) \). Where \( \langle \cdot, \cdot \rangle \) is duality brackets for the pair \( (W^{\alpha,G}(\mathbb{R}^d), W^{-\alpha,G}(\mathbb{R}^d)) \).

In what follows, we give some theorems and lemmas related to the fractional Orlicz-Sobolev space.

**Theorem 2.1.** (see [2])
Under the assumptions \( (H_G) \), we have the following continuous embedding
\[ W^{\alpha,G}(\mathbb{R}^d) \hookrightarrow L^B(\mathbb{R}^d). \]

Where \( B \) is an N-function satisfies the \( \triangle_2 \)-condition and \( B \prec G_\ast \).

Under the assumptions \( (g_1) - (g_3) \), some elementary inequalities and properties listed in the following lemmas are valid. For the proofs, see [3, 4, 21, 23].

**Lemma 2.2.** If \( G \) is an N-function, then
\[ G(a + b) \geq G(a) + G(b), \text{ for all } a, b \geq 0. \]

**Lemma 2.3.** Under the assumptions \( (g_1) - (g_3) \), the functions \( G \) and \( \tilde{G} \) satisfy the following inequalities
\[ \tilde{G}(g(t)) \leq G(2t) \text{ and } \frac{\tilde{G}(G(t))}{t} \leq G(t) \forall t \geq 0. \]

**Lemma 2.4.** Assume that the assumptions \( (g_1) - (g_3) \) hold, then
(1) \( \min\{a^{g^{-}}\}, a^{g^{+}} \} G(t) \leq G(at) \leq \max\{a^{g^{-}}\}, a^{g^{+}} \} G(t), \text{ for all } a, t \geq 0. \)
(2) \( \min\{a^{g^{-}}-1, a^{g^{+}}-1 \} g(t) \leq g(at) \leq \max\{a^{g^{-}}-1, a^{g^{+}}-1 \} g(t), \text{ for all } a, t \geq 0. \)
(3) \( \min\{a^{g^{-}}\}, a^{g^{+}} \} G_\ast(t) \leq G_\ast(at) \leq \max\{a^{g^{-}}\}, a^{g^{+}} \} G_\ast(t), \text{ for all } a, t \geq 0. \)
(4) \( \min\{a^{g^{-}}-1, a^{g^{+}}-1 \} \tilde{G}(t) \leq \tilde{G}(at) \leq \max\{a^{g^{-}}-1, a^{g^{+}}-1 \} \tilde{G}(t), \text{ for all } a, t \geq 0. \)

**Lemma 2.5.** Assume that the assumptions \( (g_1) - (g_3) \) hold, then
(1) \( \min\{\|u\|_{(G)}^{-}, \|u\|_{(G)}^{+}\} \leq \tilde{\rho}(u) \leq \max\{\|u\|_{(G)}^{-}, \|u\|_{(G)}^{+}\}, \text{ for all } u \in L^G(\mathbb{R}^d). \)
(2) \( \min\{\|u\|_{(G)}^{-}, \|u\|_{(G)}^{+}\} \leq \tilde{\rho}(u) \leq \max\{\|u\|_{(G)}^{-}, \|u\|_{(G)}^{+}\}, \text{ for all } u \in L^{G_\ast}(\mathbb{R}^d). \)
(3) \( \min\{\|u\|_{(G)}^{-}, \|u\|_{(G)}^{+}\} \leq \tilde{\rho}(u) \leq \max\{\|u\|_{(G)}^{-}, \|u\|_{(G)}^{+}\}, \text{ for all } u \in W^{\alpha,G}(\mathbb{R}^d). \)
(4) \( \min\{\|u\|^{-}, \|u\|^{+}\} \leq \rho_\alpha(u) \leq \max\{\|u\|^{-}, \|u\|^{+}\}, \text{ for all } u \in W^{\alpha,G}(\mathbb{R}^d). \)
2.2 The ”generalized subdifferential”

A main tool used in our paper is the subdifferential theory of Clark [18, 19] for locally Lipschitz functionals. Let \( X \) be a Banach space, \( X^* \) its topological dual, and let \( (\cdot, \cdot)_X \) denote the duality brackets for the pair \((X, X^*)\).

**Definition 2.6.** Let the functional \( \phi : X \to \mathbb{R} \). We say that \( \phi \) is locally Lipschitz if, for every \( x \in X \), we find an open neighborhood \( U(x) \) of \( x \) and \( k_x > 0 \) such that

\[
|\phi(u) - \phi(v)| \leq k_x \|u - v\|_X \quad \text{for all } u, v \in U(x).
\]

It’s clear that, if \( \phi : X \to \mathbb{R} \) is Lipschitz continuous on every bounded set in \( X \), then \( \phi \) is locally Lipschitz. Moreover, if \( \phi : X \to \mathbb{R} \) is continuous convex or if \( \phi \in C^1(X, \mathbb{R}) \), then \( \phi \) is locally Lipschitz.

Given a locally Lipschitz function \( \phi : X \to \mathbb{R} \), the ”generalized directional derivative” of \( \phi \) at \( u \in X \) in the direction \( v \in X \), denoted by \( \partial \phi(u; v) \), is defined by

\[
\partial \phi(u; v) = \lim_{t \to 0} \frac{\phi(u + tv) - \phi(u)}{t}.
\]

**Definition 2.7.** The ”generalized subdifferential” of \( \phi \) at \( u \in X \) is the set \( \partial \phi(u) \subseteq X^* \) defined by

\[
\partial \phi(u) = \{ \phi^* \in X^* : (\phi^*, v)_X \leq \tilde{\phi}(u; v) \quad \text{for all } v \in X \}.
\]

The Hahn-Banach theorem implies that \( \partial \phi(u) \neq \emptyset \) for all \( u \in X \), and it is convex and \( w^*\)-compact (in weak topology sense). If \( \phi \) is also convex, then it coincides with the subdifferential in the sense of convex functionals (see [20]). If \( \phi \in C^1(X, \mathbb{R}) \), the \( \partial \phi(u) = \{ \phi'(u) \} \). Note that the generalized subdifferential has a remarkable calculus, similar to that in the classical derivative (see [18, 19, 20]).

In the following section, we state our hypotheses (on \( f \) and \( K \)) and main result.

3 The hypotheses and main result

Next, we introduce the hypotheses on the weight function \( K(x) \) and the reaction function \( f(x, t) \).

\((H_K)\) \( K : \mathbb{R}^d \to \mathbb{R} \) is a continuous function and satisfies

\( (K_1) \) \( K(x) > 0 \), for all \( x \in \mathbb{R}^d \) and \( K \in L^\infty(\mathbb{R}^d) \).

\( (K_2) \) If \( \{A_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d \) is a sequence of Borel sets such that the Lebesgue measure \( \text{mes}(A_n) \leq R \), for all \( n \in \mathbb{N} \) and some \( R > 0 \), then

\[
\lim_{r \to +\infty} \int_{A_n \cap B_r(0)} K(x) \, dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.
\]

\((H_f)\) \( f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) is a measurable function such that for a.a. \( x \in \mathbb{R}^d \), \( f(x, 0) = 0 \), \( f(x, \cdot) \) is locally Lipschitz and

\( (f_1) \) \( \lim_{|s| \to 0^+} \frac{f(x, s)}{g(s)} = 0 \) uniformly in \( x \in \mathbb{R}^d \).

\( (f_2) \) \( \lim_{|s| \to +\infty} \frac{f(x, s)}{g^*(s)} = 0 \) uniformly in \( x \in \mathbb{R}^d \).

\( (f_3) \) \( \lim_{s \to +\infty} \frac{F(x, s)}{|s|^{p^*}} = +\infty \) uniformly in \( x \in \mathbb{R}^d \), where \( F(x, s) = \int_0^s f(x, t) \, dt \).

\( (f_4) \) \( 0 < (g^+ - 1)f(x, s)s < f^*(x, s)s^2 \), for a.a. \( x \in \mathbb{R}^d \), all \( f^*(x, s) \in \partial_s f(x, s) \), and all \( |s| > 0 \).
Remark 3.1. • Hypothesis $(f_{4})$ and the generalized subdifferential calculus of Clarke [19, p. 48] imply that, for a.a. $x \in \mathbb{R}^d$, we have

\[ s \mapsto f(x,s) \] is increasing on $(0, +\infty)$ and on $(-\infty, 0)$ \hspace{2cm} (3.1)

and

\[ s \mapsto f(x,s) - g^+ F(x,s) \] is increasing on $[0, +\infty)$ and decreasing on $(-\infty, 0]$. \hspace{2cm} (3.2)

• Note that, we do not employ the Ambrosetti-Rabinowitz (AR for short) condition. Indeed, the following function (for the sake of simplicity, we drop the $x$-dependence):

\[ f(s) = |s|^{\gamma^+ - 2} s \ln(1 + |s|) \]

satisfies the hypotheses $(H_f)$ but not the (AR) condition.

• From hypothesis $(f_{4})$ and the fact that $f(x,0) = 0$, for a.a. $x \in \mathbb{R}^d$, we get

\[ f(x,s) \geq 0 \quad (0 \leq s \leq 0), \] for a.a. $x \in \mathbb{R}^d$ and all $s \geq 0$ ($s \leq 0$).

Therefore,

\[ F(x,t) = \int_0^t f(x,s)ds \geq 0, \quad \text{for a.a.} \quad x \in \mathbb{R}^d \quad \text{and all} \quad t \geq 0. \]

On the other hand, if $t < 0$, by [19], for a.a. $x \in \mathbb{R}^d$, we have

\[
F(x,t) = \int_0^t f(x,s)ds = \int_0^t \frac{f(x,s)}{|s|^{\gamma^+ - 1}} |s|^{\gamma^+ - 1} ds \\
\geq \frac{f(x,t)}{|t|^{\gamma^+ - 1}} \int_0^t |s|^{\gamma^+ - 1} ds = \frac{1}{g^+} f(x,t) t \\
\geq 0 \quad (\text{since} \quad t \leq s < 0 \quad \text{and} \quad f(t) < 0).
\]

Thus,

\[ F(x,t) \geq 0, \quad \text{for a.a.} \quad x \in \mathbb{R}^d \quad \text{and all} \quad t \in \mathbb{R}. \] \hspace{2cm} (3.3)

Now, we can state the following multiplicity theorem for problem [19].

**Theorem 3.2.** If the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ hold, then problem [19] has a ground state solution \( \hat{u} \in W^{\alpha,G}(\mathbb{R}^d) \) of fixed sign and a nodal weak solution \( \tilde{u} \in W^{\alpha,G}(\mathbb{R}^d) \).

In the following section, we give the energy functional associated to problem [19] and some technical lemmas.

### 4 The energy functional and technical lemmas

We start by given the energy functional associated to problem [19].

Let $J : W^{\alpha,G}(\mathbb{R}^d) \to \mathbb{R}$ be the energy functional for problem [19] defined by

\[
J(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right) \frac{dx dy}{|x - y|^d} + \int_{\mathbb{R}^d} G(u) \ dx - \int_{\mathbb{R}^d} K(x) F(x,u) \ dx, \quad \text{for all} \ u \in W^{\alpha,G}(\mathbb{R}^d).
\]

In view of hypotheses $(H_G)$, $(H_f)$ and $(H_K)$, we have that $J \in C^1(W^{\alpha,G}(\mathbb{R}^d))$ and

\[
\langle J'(u), v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right) \frac{v(x) - v(y)}{|x - y|^{\alpha + d}} \ dx dy + \int_{\mathbb{R}^d} g(u) v \ dx - \int_{\mathbb{R}^d} K(x) f(x,u) v \ dx,
\]

for all $u, v \in W^{\alpha,G}(\mathbb{R}^d)$.  

As we mentioned, our approach is based on the Nehari method. We introduce the Nehari manifold for $J$, defined by

$$\mathcal{N} = \left\{ u \in W^{\alpha,G}({\mathbb{R}}^d) : u \neq 0, \quad \langle J'(u), u \rangle = 0 \right\}$$

where $\langle.,.\rangle$ is the duality brackets for the pair $((W^{\alpha,G}({\mathbb{R}}^d))^*, W^{\alpha,G}({\mathbb{R}}^d))$. We want to remark that all the non-trivial weak solutions of $\mathbb{P}$ belongs to $\mathcal{N}$. Since, our aim is to produce nodal solutions, we consider the following set

$$\mathcal{M} = \left\{ w \in W^{\alpha,G}({\mathbb{R}}^d) : w^+ \neq 0, w^- \neq 0, \langle J'(w), w^+ \rangle = \langle J'(w), w^- \rangle = 0 \right\}.$$  

Recall that $w^+ = \max\{w, 0\}$, $w^- = \min\{w, 0\}$ for $w \in W^{\alpha,G}({\mathbb{R}}^d)$. Evidently, we have $w^+, w^- \in W^{\alpha,G}({\mathbb{R}}^d)$ and $w = w^+ + w^-, |w| = |w^+| + |w^-|$.

In what follows, we give some technical lemmas and results which are crucial in the proof of Theorem [1,2].

**Lemma 4.1.** Assume that the hypothesis $(g_3)$ holds. Then, the function defined by

$$s \mapsto G(s) - \frac{1}{g^+} g(s)s$$

is non-decreasing on $(0, +\infty)$ and non-increasing on $(-\infty, 0)$.

**Proof.** From $(g_3)$, we have

$$g(s) - \frac{1}{g^+} g'(s)s - \frac{1}{g^+} g(s) \geq \left( \frac{g^+ - 1}{g^+} - \frac{g^- - 1}{g^+} \right) g(s) \geq 0, \quad \text{for all } s \in (0, +\infty).$$

Which gives us that $$s \mapsto G(s) - \frac{1}{g^+} g(s)s$$ is non-decreasing on $(0, +\infty)$. Exploiting the fact that $G(s) - \frac{1}{g^+} g(s)s$ is even on $\mathbb{R}$, we deduce that $$s \mapsto G(s) - \frac{1}{g^+} g(s)s$$ is non-increasing on $(-\infty, 0)$.

This ends the proof. \[\square\]

**Lemma 4.2.** [3, Lemma 3.2]
Assume that the hypotheses $(f_1) - (f_2)$ and $(H_K)$ hold. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence such that $u_n \to u$ in $W^{\alpha,G}({\mathbb{R}}^d)$, then, up to a sub-sequence, one has

1. \[\lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x)F(x, u_n)dx = \int_{\mathbb{R}^d} K(x)F(x, u)dx.\]
2. \[\lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x)f(x, u_n)u_n dx = \int_{\mathbb{R}^d} K(x)f(x, u)u dx.\]
3. \[\lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x)f(x, u_n^+)u_n^+ dx = \int_{\mathbb{R}^d} K(x)f(x, u^+)u^+ dx.\]

**Lemma 4.3.** [4, Lemma 4.3]
Assume that the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ hold. If $w \in \mathcal{M}$, then

$$\langle J'(w^\pm), w^\pm \rangle \leq \langle J'(w), w^\pm \rangle.$$ 

**Lemma 4.4.** Assume that the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ hold. Then, for all $w \in W^{\alpha,G}({\mathbb{R}}^d)$ such that $w^\pm \neq 0$, we have

$$J(w^+) + J(w^-) < J(w).$$
Proof. Let \( w \in W^{\alpha,G}(\mathbb{R}^d) \) such that \( w^\pm \neq 0 \). We argue by contradiction, suppose that
\[
J(w) \leq J(w^+) + J(w^-).
\] (4.1)

Firstly, we observe that
\[
J(w) = J(w^+) + J(w^-) + \int_{\text{supp}(w^-)} \int_{\text{supp}(w^+)} G \left( \frac{w^+(x) - w^-(y)}{|x-y|^{\alpha}} \right) \frac{dx dy}{|x-y|^2} + \int_{\text{supp}(w^+)} \int_{\text{supp}(w^-)} G \left( \frac{w^-(x) - w^+(y)}{|x-y|^{\alpha}} \right) \frac{dx dy}{|x-y|^2}.
\]

By (4.1) and the fact that \( G(\cdot) \) is an even positive function, then
\[
\begin{cases}
\int_{\text{supp}(w^-)} \int_{\text{supp}(w^+)} G \left( \frac{w^+(x) - w^-(y)}{|x-y|^{\alpha}} \right) \frac{dx dy}{|x-y|^2} = 0,
\end{cases}
\] (4.2)

Thus, one has
\[
w^+(x) = w^-(y) \quad \text{and} \quad w^-(x) = w^+(y), \quad \text{for a.a. } x, y \in \mathbb{R}^d.
\]

Therefore, \( w^+ = 0 \) in \( \mathbb{R}^d \) (since \( \text{supp}(w^+) \cap \text{supp}(w^-) = \emptyset \)). Which is a contradiction with the hypothesis of lemma \( (w^\pm \neq 0) \).

This ends the proof. \( \square \)

5 Ground state solutions

In this section, we prove the existence of a ground state solution for our problem [13].

**Proposition 5.1.** Assume that the hypotheses \( (H_f), (H_G) \) and \( (H_K) \) hold, then for all \( u \in W^{\alpha,G}(\mathbb{R}^d) \setminus \{0\} \) there exists a unique \( t_u > 0 \) such that \( t_u u \in \mathcal{N} \).

**Proof.** Let \( u \in W^{\alpha,G}(\mathbb{R}^d) \setminus \{0\} \), consider the following fibering map \( h_u : (0, +\infty) \rightarrow \mathbb{R} \) defined by
\[
h_u(t) = \langle J'(tu), tu \rangle \quad \text{for all } t > 0.
\]

From hypotheses \( (f_1) \) and \( (f_2) \), for all \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
f(x,s) s \leq \varepsilon g(s)s + C_\varepsilon g^*_s(s), \quad \text{for a.a. } x \in \mathbb{R}^d \text{ and all } s \in \mathbb{R}.
\] (5.1)

Using [5.1], Lemma [2.4] and assumptions \( (g_2) \) and \( (K_1) \), we get
\[
\begin{align*}
h_u(t) & \geq g^{-t 9^+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x-y|^{\alpha}} \right) \frac{dx dy}{|x-y|^2} + \int_{\mathbb{R}^d} g(tu) t ud x \\
& \quad - \varepsilon \|K\|_\infty \int_{\mathbb{R}^d} g(tu) t ud x - C_\varepsilon \|K\|_\infty \int_{\mathbb{R}^d} g^*_s(tu) t ud x \\
& \geq g^{-t 9^+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x-y|^{\alpha}} \right) \frac{dx dy}{|x-y|^2} + (1 - \varepsilon \|K\|_\infty) g^{-t 9^+} \int_{\mathbb{R}^d} G(u) d x \\
& \quad - g^*_t 9^{-} \int_{\mathbb{R}^d} G^*_s(u) d x \\
& = g^{-t 9^+} \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x-y|^{\alpha}} \right) \frac{dx dy}{|x-y|^2} + (1 - \varepsilon \|K\|_\infty) \int_{\mathbb{R}^d} G(u) d x \right] \\
& \quad - g^*_t 9^{-} \int_{\mathbb{R}^d} G^*_s(u) d x.
\end{align*}
\] (5.2)
Choosing \( \varepsilon \in (0, \frac{1}{\|K\|_\infty}) \) in (3.2) and using the fact that \( g^+ < g^- \), we find \( t_0 > 0 \) small enough such that
\[
 h_u(t) > 0, \quad \forall \, t \in (0, t_0).
\]

Let \( A \subset \text{supp}(u) \) such that the Lebesgue measure of \( A \) is positive.

Using (2.2, 2.3, Lemmas 2.4, 2.5 and assumptions (g), (K)), for \( t \) large, we find
\[
\frac{h_u(t)}{t^{\varphi^+}} \leq g^+ \max \left\{ \|u\|^\varphi^-, \|u\|^\varphi^+ \right\} - \int_{\mathbb{R}^d} K(x) \frac{f(x, tu)}{t^{\varphi^+}} dx
\leq g^+ \left[ \max \left\{ \|u\|^\varphi^-, \|u\|^\varphi^+ \right\} - \int_{\mathbb{R}^d} K(x) \frac{F(x, tu)}{t^{\varphi^+}} dx \right]
\leq g^+ \left[ \max \left\{ \|u\|^\varphi^-, \|u\|^\varphi^+ \right\} - \int_{A} K(x) \frac{F(x, tu)}{|tu|^\varphi^+} |u|^\varphi^+ dx \right].
\]

(5.3)

In light of hypothesis (f3), we see that
\[
\lim_{t \to +\infty} \frac{F(x, tu)}{|tu|^\varphi^+} |u|^\varphi^+ = +\infty, \quad \text{uniformly for all } x \in A.
\]

(5.4)

It follows, by assumption (K) and Fatou’s lemma, that
\[
\int_{A} K(x) \frac{F(x, tu)}{|tu|^\varphi^+} |u|^\varphi^+ dx \to +\infty \quad \text{as } t \to +\infty.
\]

(5.5)

From (5.3) and (5.5), we get
\[
\limsup_{t \to +\infty} \frac{h_u(t)}{t^{\varphi^+}} \leq -\infty.
\]

Therefore, there is \( t_1 > 0 \) large enough such that
\[
h_u(t) < 0, \quad \text{for all } t \in (t_1, +\infty).
\]

Then by Bolzano’s theorem, we can find \( t_u > 0 \) such that \( h_u(t_u) = 0 \). Hence, we have \( \langle J'(t_u u), t_u u \rangle = 0 \), that is, \( t_u u \in \mathcal{N} \).

In what follows, we prove the uniqueness of \( t_u > 0 \). Let \( t_1, t_2 \) be the two different positive number such that \( t_i u \in \mathcal{N}, \quad i = 1, 2 \).

Firstly, we consider the case that \( u \in \mathcal{N} \). Then, without loss of generality, we may take \( t_1 = 1 \) and \( t_1 \neq t_2 \).

Thus,
\[
\langle J'(u), u \rangle = 0
\]

(5.6)

and
\[
\langle J'(t_2 u), u \rangle = 0.
\]

(5.7)

From (5.6), we have
\[
\int_{\mathbb{R}^d} K(x) \frac{f(x, u)}{|u|^\varphi^+-1} |u|^\varphi^- dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{u(x) - u(y)}{|x - y|^{\alpha + \varphi^+}} dxdy
+ \int_{\mathbb{R}^d} g(u) dx.
\]

(5.8)

If \( t_2 < t_1 = 1 \), then, by (5.7) and Lemma 2.3, we obtain
\[
\int_{\mathbb{R}^d} K(x) \frac{f(x, t_2 u)}{|t_2 u|^\varphi^+-1} |t_2 u|^\varphi^- dx \geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{u(x) - u(y)}{|x - y|^{\alpha + \varphi^+}} dxdy
+ \int_{\mathbb{R}^d} g(u) dx.
\]

(5.9)
Subtracting (5.9) from (5.8), using (3.1) and hypothesis $(K_1)$, we infer that

\[0 < \int_{\mathbb{R}^d} K(x)|u|^{q^+ - 1}u \left[ \frac{f(x,u)}{|u|^{q^+ - 1}} - \frac{f(x,t_2u)}{|t_2u|^{q^+ - 1}} \right] \, dx \leq 0. \quad (5.10)\]

Which is a contradiction.

If $t_2 > t_1 = 1$, then by (5.7) and Lemma 2.4 we get

\[\int_{\mathbb{R}^d} K(x) \frac{f(x,t_2u)}{|t_2u|^{q^+ - 1}}|u|^{q^+ - 1} \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{|u(x) - u(y)|}{|x - y|^{\alpha + d}} \right) u(x) - u(y) \, dx \, dy
+ \int_{\mathbb{R}^d} g(u) \, dx. \quad (5.11)\]

Subtracting (5.11) from (5.10), using (3.1) and hypothesis $(K_1)$, we deduce that

\[0 \leq \int_{\mathbb{R}^d} K(x)|u|^{q^+ - 1}u \left[ \frac{f(x,u)}{|u|^{q^+ - 1}} - \frac{f(x,t_2u)}{|t_2u|^{q^+ - 1}} \right] \, dx < 0. \quad (5.12)\]

Which is a contradiction too. Therefore $t_1 = t_2 = 1$.

Secondly, for the case that $u \notin \mathcal{N}$. Let $v = t_1u \in \mathcal{N}$, so $t_1 \neq 1$. Hence, we have

\[t_2u = \frac{t_2}{t_1} t_1u = \frac{t_2}{t_1} v \in \mathcal{N}.\]

Applying the same arguments as above, we prove that $\frac{t_2}{t_1} = 1$.

This ends the proof. \qed

**Corollary 5.2.** Under the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$, we have $\mathcal{N} \neq \emptyset$.

**Proposition 5.3.** Assume that the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ hold. Then for all $u \in \mathcal{N}$,

\[J(tu) \leq J(u), \text{ for all } t > 0.\]

**Proof.** Let $u \in \mathcal{N}$ and consider the fibering map $k_u : (0, +\infty) \rightarrow \mathbb{R}$ defined by

\[k_u(t) = J(tu) \text{ for all } t > 0.\]

By the hypotheses $(f_1)$ and $(f_2)$, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

\[F(x,s) \leq \varepsilon G(s) + C_\varepsilon G_*(s), \text{ for a.a. } x \in \mathbb{R}^d \text{ and all } |s| > 0. \quad (5.13)\]

Using (5.13), Theorem 2.1, Lemmas 2.3 and hypothesis $(K_1)$, then

\[k_u(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{tu(x) - tu(y)}{|x - y|^\alpha} \right) \, dx \, dy \, dx - \int_{\mathbb{R}^d} G(tu) \, dx
- \int_{\mathbb{R}^d} K(x)F(x,tu) \, dx
+ \frac{t^q}{\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \, dx \, dy
+ (1 - \varepsilon \|K\|_\infty) \int_{\mathbb{R}^d} G(tu) \, dx
- C_\varepsilon k_u(t) \, dx
\geq t^q (1 - \varepsilon \|K\|_\infty) \rho(\alpha;u) - t^q C_\varepsilon \int_{\mathbb{R}^d} G_*(u) \, dx.\]

Taking $\varepsilon \in (0, \frac{1}{\|K\|_\infty})$, having in mind that $g^+ < g^-$, then, there is $t_0 > 0$ sufficiently small such that

\[0 < k_u(t), \text{ for all } t \in (0,t_0). \quad (5.14)\]
Let $A \subset \text{supp}(u)$ such that the Lebesgue measure of $A$ is positive. It’s clear that, for $t$ large enough, we have

$$\frac{k_u(t)}{t^{g_+}} \leq g^+ \max \left\{ \|u\|^{g_-}, \|u\|^{g_+} \right\} - \int_A K(x) \frac{F(x, tu)}{|tu|^{g_+}} |u|^{g_+} dx. \quad (5.15)$$

Under (5.5), it follows that

$$\limsup_{t \to +\infty} k_u(t) \leq -\infty.$$ 

Therefore, the map $k_u(.)$ has a global maximum $t_u > 0$. So, $t_u$ is a critical point for $k_u(.)$:

$$\langle J'(t_u u), u \rangle = 0,$$

which gives that $t_u \in N$. By Proposition 5.1 and the fact that $u \in N$, we deduce that $t_u = 1$.

Hence,

$$J(tu) \leq J(u), \quad \text{for all } t > 0.$$ 

This completes the proof. \hfill \Box

In order to prove the existence of a ground state solution for problem $[\mathcal{M}]$, we consider the following minimization problem

$$m_0 := \inf_{A} J.$$ 

Proposition 5.4. Suppose that the hypotheses $(H_f)$, $(H_G)$ and $(H_K)$ are satisfied, then $0 < m_0$.

Proof. Let $u \in W^{\alpha, G}(\mathbb{R}^d) \setminus \{0\}$ such that $\|u\| \leq 1$ and $0 < \varepsilon$. Using assumption $(K_1)$, (5.13), Lemma 2.4 and Theorem 2.1 we get

$$J(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^{d}} + \int_{\mathbb{R}^d} G(u) \ dx - \int_{\mathbb{R}^d} K(x)F(x, u) \ dx$$

$$\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G \left( \frac{u(x) - u(y)}{|x - y|^{\alpha}} \right) \frac{dxdy}{|x - y|^{d}} + (1 - \varepsilon \|K\|_{\infty}) \int_{\mathbb{R}^d} G(u) \ dx$$

$$- C_{\varepsilon} \|K\|_{\infty} \int_{\mathbb{R}^d} G_*(u) \ dx$$

$$\geq (1 - \varepsilon \|K\|_{\infty}) \rho(\alpha; u) - C_{\varepsilon} \|K\|_{\infty} \int_{\mathbb{R}^d} G_*(u) \ dx$$

$$\geq (1 - \varepsilon \|K\|_{\infty}) \|u\|^{g_+} - C_{\varepsilon} \|K\|_{\infty} \max \left\{ \|u\|_{(G_{\ast})}, \|u\|^{g_+}_{(G_{\ast})} \right\}$$

$$\geq (1 - \varepsilon \|K\|_{\infty}) \|u\|^{g_+} - C_{\varepsilon} \|K\|_{\infty} C_1 \|u\|^{g_+}.$$ 

Taking $\varepsilon \in (0, \frac{1}{\|K\|_{\infty}})$. Since $g^+ < g^-$, we find $\varrho \in (0, 1)$ small enough and $\eta > 0$ such that

$$J(u) \geq \eta, \quad \text{for all } \|u\| = \varrho.$$ 

Let $u \in N$, choosing $t_u > 0$ such that $\|t_u u\| = \varrho$. By Proposition 5.3 we conclude that

$$J(u) \geq J(t_u u) \geq \eta > 0.$$ 

Therefore, $m_0 > 0$. Thus the proof. \hfill \Box

In the following proposition, we prove that the minimization problem $[\mathcal{M}]$ admits a solution.

Proposition 5.5. Under hypotheses $(H_f)$, $(H_G)$ and $(H_K)$, there exists $\tilde{u} \in N$ such that $J(\tilde{u}) = m_0$.  

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Proof. Let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N} \) such that
\[
J(u_n) \rightarrow m_0. \tag{5.16}
\]

Firstly, let prove that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{\alpha,G}(\mathbb{R}^N) \). We argue by contradiction, assume that there exists a subsequence, denoted again by \( \{u_n\}_{n \in \mathbb{N}} \) such that
\[
\|u_n\| \rightarrow +\infty \text{ as } n \rightarrow +\infty,
\]
and define
\[
v_n := \frac{u_n}{\|u_n\|} \text{ for all } n \in \mathbb{N}. \tag{5.17}
\]

Since \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{\alpha,G}(\mathbb{R}^d) \), which is a reflexive space. Then, there exists \( v \in W^{\alpha,G}(\mathbb{R}^d) \) such that
\[
v_n \rightharpoonup v \text{ in } W^{\alpha,G}(\mathbb{R}^d),
\]
and
\[
v_n(x) \rightarrow v(x) \text{ as } n \rightarrow +\infty, \text{ for a.a. in } \mathbb{R}^d. \tag{5.18}
\]

Let prove that \( v \neq 0 \). Since \( u_n \in \mathcal{N} \), according to Proposition 5.3 and Lemma 2.5 we get, for all \( t \geq 0 \)
\[
J(u_n) = J(\|u_n\|v_n) \geq J(tv_n)
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G\left(\frac{tv_n(x) - tv_n(y)}{|x - y|^\alpha}\right) \frac{dxdy}{|x - y|^d} + \int_{\mathbb{R}^d} G(tv_n)dx
\]
\[
- \int_{\mathbb{R}^d} K(x)F(x, tv_n)dx
\]
\[
\geq \min\left\{\|tv_n\|^g, \|tv_n\|^g\right\} - \int_{\mathbb{R}^d} K(x)F(x, tv_n)dx
\]
\[
= \min\left\{t^g, t^g\right\} - \int_{\mathbb{R}^d} K(x)F(x, tv_n)dx. \tag{5.19}
\]

Suppose that \( v_n \rightharpoonup v = 0 \), by Lemma 4.4, we obtain
\[
\int_{\mathbb{R}^d} K(x)F(x, tv_n)dx \rightarrow 0, \text{ for all } t > 0.
\]

Passing to the limit in (5.19) as \( n \rightarrow +\infty \), we get
\[
+\infty > m_0 \geq \min\left\{t^g, t^g\right\}, \text{ for all } t > 0.
\]

Thus the contradiction, therefore, \( v \neq 0 \).

Using (5.17) and Lemma 2.5 we see that
\[
J(u_n) = J(\|u_n\|v_n)
\]
\[
\leq g^+ \max\left\{\|u_n\|^g, \|u_n\|^g\right\} - \int_{\mathbb{R}^d} K(x)F(x, \|u_n\|v_n)dx
\]
\[
\leq g^+ \max\left\{\|u_n\|^g, \|u_n\|^g\right\} - \int_{\mathbb{R}^d} K(x)F(x, \|u_n\|v_n)dx,
\]
which is equivalent to
\[
\frac{J(u_n)}{\max\left\{\|u_n\|^g, \|u_n\|^g\right\}} \leq g^+ - \int_{\mathbb{R}^d} K(x)\frac{F(x, \|u_n\|v_n)}{\max\left\{\|u_n\|^g, \|u_n\|^g\right\}}dx. \tag{5.20}
\]
Exploiting hypotheses \((f_3), (K_1)\), Fatou’s lemma and the fact that \(v \neq 0\), we obtain
\[
\liminf_{n \to +\infty} \int_{\mathbb{R}^d} K(x) \frac{F(x, ||u_n||v_n)}{\max \{ ||u_n||^{\alpha}, ||u_n||^{\alpha+1} \}} \, dx = \liminf_{n \to +\infty} \int_{\mathbb{R}^d} K(x) \frac{F(x, ||u_n||v_n)}{||u_n||^{\alpha+1}} \, dx
\]
\[
= \liminf_{n \to +\infty} \int_{\mathbb{R}^d} K(x) \frac{F(x, ||u_n||v_n)}{||u_n||^{\alpha+1}} \, dx
\]
\[
= +\infty. \tag{5.21}
\]
By \(5.21\), comes that
\[
\frac{J(u_n)}{\max \{ ||u_n||^{\alpha}, ||u_n||^{\alpha+1} \}} \to -\infty \text{ as } n \to +\infty,
\]
which leads to a contradiction with \(5.16\).
Therefore, \(\{u_n\}_{n \in \mathbb{N}}\) is bounded in \(W^{\alpha,G}(\mathbb{R}^d)\). Up to a subsequence, then
\[
u_n \to \hat{u} \text{ in } W^{\alpha,G}(\mathbb{R}^d),
\]
and
\[
u_n(x) \to \hat{u}(x) \text{ as } n \to +\infty, \text{ for a.a. } x \in \mathbb{R}^d. \tag{5.22}
\]
Suppose that \(\hat{u} = 0\), then
\[
0 < m_0 \leq J(u_n) \xrightarrow{n \to +\infty} J(0) = 0,
\]
which is a contradiction. Thus, \(\hat{u} \neq 0\). According to Proposition \(5.1\) there is a unique \(t_{\hat{u}} > 0\) such that
\[
t_{\hat{u}} \hat{u} \in \mathcal{N}. \tag{5.23}
\]
By \(5.22\), Proposition \(5.3\), Lemma \(4.2\) and Fatou’s lemma, it follows that
\[
m_0 = \lim_{n \to +\infty} J(u_n) \geq \liminf_{n \to +\infty} J(t_{\hat{u}}u_n) \geq J(t_{\hat{u}}\hat{u}) \geq m_0. \tag{5.24}
\]
Therefore, \(m_0 = J(t_{\hat{u}}\hat{u}) = \inf_{\mathcal{N}} J\).
Next, we shall prove that \(t_{\hat{u}} = 1\). Since \(\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{N}\), then
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{u_n(x) - u_n(y)}{|x - y|^\alpha} \right) \frac{u_n(x) - u_n(y)}{|x - y|^{\alpha+1}} \, dx \, dy + \int_{\mathbb{R}^d} g(u_n)u_n \, dx = \int_{\mathbb{R}^d} K(x)f(x, u_n)u_n \, dx,
\]
for all \(n \in \mathbb{N}\).
By \(5.22\), Lemma \(4.2\) and Fatou’s lemma, it comes that
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{\hat{u}(x) - \hat{u}(y)}{|x - y|^\alpha} \right) \frac{\hat{u}(x) - \hat{u}(y)}{|x - y|^{\alpha+1}} \, dx \, dy + \int_{\mathbb{R}^d} g(\hat{u})\hat{u} \, dx \leq \int_{\mathbb{R}^d} K(x)f(x, \hat{u})\hat{u} \, dx, \tag{5.25}
\]
Suppose that \(t_{\hat{u}} > 1\). From \(5.23\) and Lemma \(2.4\) one has
\[
\int_{\mathbb{R}^d} K(x)f(x, t_{\hat{u}}\hat{u})|\hat{u}|^{\alpha+1} \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{\hat{u}(x) - \hat{u}(y)}{|x - y|^\alpha} \right) \frac{\hat{u}(x) - \hat{u}(y)}{|x - y|^{\alpha+1}} \, dx \, dy + \int_{\mathbb{R}^d} g(\hat{u})\hat{u} \, dx. \tag{5.26}
\]
Putting together \(5.25\) and \(5.26\), using \(5.1\) and hypothesis \((K_1)\), we find that
\[
0 \leq \int_{\mathbb{R}^d} K(x)|u|^{\alpha+1} \left[ \frac{f(x, u)}{|u|^{\alpha+1}} - \frac{f(x, t_n u)}{|t_n u|^{\alpha+1}} \right] \, dx < 0. \tag{5.27}
\]
Thus the contradiction. Therefore, \(0 < t_u \leq 1\).

Suppose that \(t_u \neq 1\). Using (3.2), Lemmas 4.1, 4.2, Fatou’s lemma and hypothesis \((K_1)\), we see that

\[
m_0 = J(t_u) = J(t_u) - \frac{1}{g^+}(J'(t_u), t_u)
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ G\left( \frac{t_u(x) - t_u(y)}{|x - y|^\alpha} \right) \frac{t_u(x) - t_u(y)}{|x - y|^\alpha} \right] dx dy \\
+ \int_{\mathbb{R}^d} G(t_u) - \frac{1}{g^+} g(t_u) t_u dx + \int_{\mathbb{R}^d} K(x) \left[ \frac{1}{g^+} f(x, t_u) t_u - F(x, t_u) \right] dx
\]

which is a contradiction. Thus, \(t_u = 1\). Hence,

\[
m_0 = J(1) = \inf_{N} J.
\]

This completes the proof. \(\square\)

In the following we prove that \(\hat{u}\) is a critical point of the functional \(J\).

**Proposition 5.6.** Assume that the hypotheses \((H_f)\), \((H_G)\) and \((H_K)\) hold. Then, \(\hat{u}\) is a critical point of \(J\). Hence, it is a weak solution of problem \(P\).

**Proof.** Let consider the functional \(\varphi : W^{\alpha,G}(\mathbb{R}^d) \to \mathbb{R}\) defined by

\[
\varphi(u) = (J'(u), u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{u(x) - u(y)}{|x - y|^\alpha} \frac{u(x) - u(y)}{|x - y|^\alpha} dx dy \\
+ \int_{\mathbb{R}^d} g(u) u dx - \int_{\mathbb{R}^d} K(x) f(x, u) u dx.
\]

By hypotheses \((H_f)\), \(\varphi\) is locally Lipschitz (see [19] Theorem 2.7.2, p. 221]).

Let \(u \in W^{\alpha,G}(\mathbb{R}^d)\), for all \(\varphi^*_u \in \partial \varphi(u)\), there is \(f^*(x, u) \in \partial u f(x, u)\) such that

\[
\langle \varphi^*_u, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left( \frac{u(x) - u(y)}{|x - y|^\alpha} \right) \frac{v(x) - v(y)}{|x - y|^\alpha} \frac{v(x) - v(y)}{|x - y|^\alpha} dx dy \\
+ \int_{\mathbb{R}^d} g(u) v dx - \int_{\mathbb{R}^d} K(x) f(x, u) v dx
\]

for all \(v \in W^{\alpha,G}(\mathbb{R}^d)\).

Moreover, \(J \in C^1(W^{\alpha,G}(\mathbb{R}^d))\), hence it is locally Lipschitz too.

From Proposition 5.5 we have

\[
J(\hat{u}) = m_0 = \inf \{ J(u) : \varphi(u) = 0, u \in W^{\alpha,G}(\mathbb{R}^d) \setminus \{0\} \}.
\]
Then, according to the non-smooth multiplier rule of Clarke [18, Theorem 10.47, p. 221], there exists $\lambda_0 \geq 0$ such that

$$0 \in \partial(J + \lambda_0 \varphi)(\tilde{u}).$$

It follows, by the subdifferential calculus of Clarke [19, p. 48], that

$$0 \in \partial J(\tilde{u}) + \lambda_0 \partial \varphi(\tilde{u}).$$

Thus,

$$0 = J'(\tilde{u}) + \lambda_0 \varphi^*_a \ \text{in} \ (W^{\alpha,G}(\mathbb{R}^d))^*,\ \text{for all} \ \varphi^*_a \in \partial \varphi(\tilde{u}). \quad (5.29)$$

Since $\tilde{u} \in \mathcal{N}$, we have

$$0 = \langle J'(\tilde{u}), \tilde{u} \rangle + \lambda_0 (\varphi^*_a, \tilde{u}) = \lambda_0 (\varphi^*_a, \tilde{u}), \ \text{for all} \ \varphi^*_a \in \partial \varphi(\tilde{u}) \quad (5.30)$$

Using (5.29), hypotheses (f4), (g3) and the fact that $\tilde{u} \in \mathcal{N}$, we get

$$\langle \varphi^*_a, \tilde{u} \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g'(\tilde{u}(x) - \tilde{u}(y)) \frac{[\tilde{u}(x) - \tilde{u}(y)]^2}{|x-y|^{2\alpha}} \frac{dxdy}{|x-y|^2}$$

$$+ \int_{\mathbb{R}^d} g'(\tilde{u})\tilde{u}^2dx - \int_{\mathbb{R}^d} K(x)f^*(x, \tilde{u})\tilde{u}^2dx$$

$$\leq |g^+ - 1| \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^\alpha} \right) \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^\alpha} \frac{dxdy}{|x-y|^2} \right]$$

$$+ \int_{\mathbb{R}^d} g(\tilde{u})\tilde{u}^2dx - \int_{\mathbb{R}^d} K(x)f^*(x, \tilde{u})\tilde{u}^2dx$$

$$= \int_{\mathbb{R}^d} K(x) (|g^+ - 1| f(x, \tilde{u})\tilde{u} - f^*(x, \tilde{u})\tilde{u}^2) dx$$

$$< 0 \quad (5.31)$$

It follows, from (5.30), that $\lambda_0 = 0$. Therefore, by (5.29), we deduce that

$$J'(\tilde{u}) = 0 \ \text{in} \ (W^{\alpha,G}(\mathbb{R}^d))^*. \quad \square$$

Hence, $\tilde{u}$ is a critical point of $J$, so, it is a weak solution of problem (P). Thus the proof.

6 Ground state nodal solutions

In this section, we establish the existence of least energy nodal solution for problem (P) and we prove that the ground state solution $\tilde{u}$, obtained in Section 3, is of fixed sign and we give the proof of Theorem 3.2.

In order to find a least energy nodal solution for problem (P), we must first look for a minimizer of energy functional $J$ with the constraint $\mathcal{M}$. We consider the following minimization problem

$$m_1 = \inf_{\mathcal{M}} J. \quad (M-1)$$

**Proposition 6.1.** Assume that the hypotheses $(H_f), (H_G)$ and $(H_K)$ hold. If $w \in W^{\alpha,G}(\mathbb{R}^d)$ such that $w^\pm \neq 0$, then there exist a unique pair $t_{w^+}, s_{w^-} > 0$ such that

$$t_{w^+}w^+ + s_{w^-}w^- \in \mathcal{M}. \quad \text{Proof.} \quad \text{Let} \ \xi : (0, +\infty) \times (0, +\infty) \to \mathbb{R}^2 \ \text{be a continuous vector field given by}$$

$$\xi(t, s) = (\xi_1(t, s), \xi_2(t, s)), \ \text{for all} \ t, s \in (0, +\infty) \times (0, +\infty)$$

where

$$\xi_1(t, s) = \langle J'(tw^+ + sw^-), tw^+ \rangle \ \text{and} \ \xi_2(t, s) = \langle J'(tw^+ + sw^-), sw^- \rangle.$$
Using the same techniques as in Proposition 5.1, we can see that there exist \( r_1 > 0 \) small enough and \( R_1 > 0 \) large enough such that
\[
\xi_1(t, t) > 0, \quad \xi_2(t, t) > 0, \quad \text{for all } t \in (0, r_1),
\]
\[
\xi_1(t, t) < 0, \quad \xi_2(t, t) < 0, \quad \text{for all } t \in (R_1, +\infty).
\] (6.1)

Note that \( \xi_1(t, s) \) is non-decreasing in \( s \) on \((0, +\infty)\) for fixed \( t > 0 \) and \( \xi_2(t, s) \) is non-decreasing in \( t \) on \((0, +\infty)\) for fixed \( s > 0 \) (see [7] Proof of Lemma 4.7). It follows that there are \( r > 0, R > 0 \) with \( r < R \) such that
\[
\xi_1(r, s) > 0, \quad \xi_1(R, s) < 0, \quad \text{for all } s \in (r, R],
\]
\[
\xi_2(t, r) > 0, \quad \xi_2(t, R) < 0, \quad \text{for all } t \in (r, R].
\]

By applying the Miranda’s theorem [28] on \( \xi \), there exist some \( t_{w^+}, s_{w^-} \in (r, R] \) such that \( \xi_1(t_{w^+}, s_{w^-}) = \xi_2(t_{w^+}, s_{w^-}) = 0 \). Which implies that \( t_{w^+}w^+ + s_{w^-}w^- \in \mathcal{M} \).

For the uniqueness of the pair \((t_{w^+}, s_{w^-})\), we argue by contradiction. Suppose that there exist two different pair \((t_1, s_1)\) and \((t_2, s_2)\) such that
\[
t_1w^+ + s_1w^- \in \mathcal{M} \quad \text{and} \quad t_2w^+ + s_2w^- \in \mathcal{M}.
\]

We distinguish two cases:

(A): If \( w \in \mathcal{M} \). Without loss of generality, we may take \((t_1, s_1) = (1, 1)\) and assume \( t_2 \leq s_2 \). Since \( w \in \mathcal{M} \), we have
\[
\int_{\mathbb{R}^d} K(x)f(x, w^+)w^+dx = A^+(w)
\] (6.2)

and
\[
\int_{\mathbb{R}^d} K(x)f(x, w^-)w^-dx = A^-(w).
\] (6.3)

Where
\[
A^+(w) = \int_{\text{supp}(w^+)} \int_{\text{supp}(w^+)} g \left( \frac{w^+(x) - w^+(y)}{|x - y|^\alpha} \right) \frac{w^+(x) - w^+(y)}{|x - y|^\alpha + d} dxdy
\]
\[
+ \int_{\text{supp}(w^-)} \int_{\text{supp}(w^+)} g \left( \frac{w^+(x) - w^-(y)}{|x - y|^\alpha} \right) \frac{w^+(x)}{|x - y|^\alpha + d} dxdy
\]
\[
+ \int_{\text{supp}(w^+)} \int_{\text{supp}(w^-)} g \left( \frac{w^-(x) - w^+(y)}{|x - y|^\alpha} \right) \frac{-w^+(y)}{|x - y|^\alpha + d} dxdy
\]
\[
+ \int_{\mathbb{R}^d} g(w^+)w^+dx
\] (6.4)

and
\[
A^-(w) = \int_{\text{supp}(w^-)} \int_{\text{supp}(w^-)} g \left( \frac{w^-(x) - w^-(y)}{|x - y|^\alpha} \right) \frac{w^-(x) - w^-(y)}{|x - y|^\alpha + d} dxdy
\]
\[
+ \int_{\text{supp}(w^-)} \int_{\text{supp}(w^+)} g \left( \frac{w^+(x) - w^-(y)}{|x - y|^\alpha} \right) \frac{-w^-(y)}{|x - y|^\alpha + d} dxdy
\]
\[
+ \int_{\text{supp}(w^+)} \int_{\text{supp}(w^-)} g \left( \frac{w^-(x) - w^+(y)}{|x - y|^\alpha} \right) \frac{w^-(x)}{|x - y|^\alpha + d} dxdy
\]
\[
+ \int_{\mathbb{R}^d} g(w^-)w^-dx.
\] (6.5)
Since \( s \mapsto g(s) \) is non-decreasing on \((0, +\infty)\) and on \((\infty, 0)\) and \(t_2 \leq s_2\), we infer that

\[
\begin{align*}
& g \left( \frac{t_2 w^+(x) - s_2 w^-(y)}{|x - y|^\alpha} \right) t_2 w^+(x) \geq g \left( \frac{t_2 w^+(x) - t_2 w^-(y)}{|x - y|^\alpha} \right) t_2 w^+(x), \\
& g \left( \frac{s_2 w^-(x) - t_2 w^+(y)}{|x - y|^\alpha} \right) (-t_2 w^+(y)) \geq g \left( \frac{t_2 w^+(x) - t_2 w^+(y)}{|x - y|^\alpha} \right) (-t_2 w^+(y)), \\
& g \left( \frac{t_2 w^+(x) - s_2 w^-}{|x - y|^\alpha} \right) (-s_2 w^-) \leq g \left( \frac{s_2 w^+(x) - s_2 w^-}{|x - y|^\alpha} \right) (-s_2 w^-), \\
& g \left( \frac{s_2 w^-(x) - t_2 w^+(y)}{|x - y|^\alpha} \right) s_2 w^- (x) \leq g \left( \frac{s_2 w^-(x) - s_2 w^-}{|x - y|^\alpha} \right) s_2 w^- (x), \\
\end{align*}
\]

for a.a. \( x, y \in \mathbb{R}^d \).

By Lemma 2, the fact that \( t_1 w^+ + s_1 w^- \in \mathcal{M} \) and \( t_2 w^+ + s_2 w^- \in \mathcal{M} \), it follows that

\[
\int_{\mathbb{R}^d} K(x) \frac{f(x, t_2 w^+) t_2 w^+}{\min\{t_2^-, t_2^+\}} \, dx \geq A^+(w) \tag{6.7}
\]

and

\[
\int_{\mathbb{R}^d} K(x) \frac{f(x, s_2 w^-) s_2 w^-}{\max\{s_2^-, s_2^+\}} \, dx \leq A^-(w). \tag{6.8}
\]

In what follows, we will show that the following five cases cannot happen:

1. \( t_2 < s_2 = 1 \).
2. \( s_2 > t_2 = 1 \).
3. \( 0 < t_2 \leq s_2 < 1 \).
4. \( 1 < t_2 \leq s_2 \).
5. \( 0 < t_2 < 1 < s_2 \).

Suppose that one of the cases (1), (3) or (5), holds. According to (3.1), (6.2), (6.7) and hypothesis (K₁), we infer that

\[
0 \leq \int_{\mathbb{R}^d} K(x) |w^+|^\gamma - 1 w^+ \left[ \frac{f(x, t_2 w^+)}{|t_2 w^+|^\gamma - 1} - \frac{f(x, w^+)}{|w^+|^\gamma - 1} \right] \, dx < 0.
\]

Thus the contradiction. Then, the cases (1), (3) and (5) cannot be realized.

Suppose that case (2) or (4) holds. According to (3.1), (6.3), (6.8) and hypothesis (K₁), one has

\[
0 \leq \int_{\mathbb{R}^d} K(x) |w^-|^\gamma - 1 w^- \left[ \frac{f(x, w^-)}{|w^-|^\gamma - 1} - \frac{f(x, s_2 w^-)}{|s_2 w^-|^\gamma - 1} \right] \, dx < 0.
\]

Which is a contradiction too. Then, the cases (2) and (4) cannot be realized. Therefore, we deduce that \((t_1, s_1) = (1, 1) = (t_2, s_2)\).

(B): If \( w \notin \mathcal{M} \). Let \( v = t_1 w^+ + s_1 w^- \in \mathcal{M} \), \( v^+ = t_1 w^+ \) and \( v^- = s_1 w^- \), so \((t_1, s_1) \neq (1, 1)\). It is clear that

\[
t_2 w^+ + s_2 w^- = \frac{t_2}{t_1} t_1 w^+ + \frac{s_2}{s_1} s_1 w^- = \frac{t_2}{t_1} v^+ + \frac{s_2}{s_1} v^- \in \mathcal{M}.
\]

Arguing as in the case (A), we deduce that

\[
\frac{t_2}{t_1} = \frac{s_2}{s_1} = 1.
\]

This completes the proof.
Corollary 6.2. Suppose that hypotheses \((H_f), (H_G)\) and \((H_K)\) are satisfied, then, \(\mathcal{M} \neq \emptyset\).

Proposition 6.3. Suppose that hypotheses \((H_f), (H_G)\) and \((H_K)\) are satisfied, then

\[ 0 < m_1 := \inf_{\mathcal{M}} J. \]

Proof. By Proposition 5.4 and the fact that \(\mathcal{M} \subset \mathcal{N}\), we get

\[ m_1 = \inf_{\mathcal{M}} J \geq \inf_{\mathcal{N}} J = m_0 > 0. \]

Thus the proof.

Proposition 6.4. Assume that the hypotheses \((H_f), (H_G)\) and \((H_K)\) hold. Then, for all \(w \in \mathcal{M}\), we have

\[ J(tw^+ + sw^-) \leq J(w), \quad \text{for all } t, s > 0. \]

Proof. Let \(w \in \mathcal{M}\) and consider the fibering map \(\mu_w : (0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}\) defined by

\[ \mu_w(t, s) = J(tw^+ + sw^-) \quad \text{for all } t, s > 0. \]

In light of Proposition 5.1 we have

\[ \mu_w(0, 0) = J(0) = 0 < m_1 \leq \mu_w(1, 1) = J(w). \] (6.9)

Let \(t, s > 0\) large enough, using Lemmas 2.3 and 2.5 we obtain

\[
\mu_w(t, s) \leq \max \left\{ \|tw^+ + sw^-\|^g, \|tw^+ + sw^-\|^g \right\} - \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx
\]

\[
\leq 2^{g^+-1} \max \left\{ \|t\|^g \|w^+\|^g + \|s\|^g \|w^-\|^g, \|t\|^g \|w^+\|^g + \|s\|^g \|w^-\|^g \right\}
\]

\[
- \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx
\]

\[
\leq 2^{g^+-1} \max \left\{ \max\{t\|^g, |s\|^g\} \left( \|w^+\|^g + \|w^-\|^g \right), \max\{t\|^g, |s\|^g\} \left( \|w^+\|^g + \|w^-\|^g \right) \right\}
\]

\[
- \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx
\]

\[
\leq 2^{g^+-1} \max\{t\|^g, |s\|^g\} \max \left\{ \|w^+\|^g + \|w^-\|^g, \|w^+\|^g + \|w^-\|^g \right\}
\]

\[
- \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-)dx. \] (6.10)

It follows that

\[
\frac{\mu_w(t, s)}{\max\{t\|^g, |s\|^g\}} \leq 2^{g^+-1} \max \left\{ \|w^+\|^g + \|w^-\|^g, \|w^+\|^g + \|w^-\|^g \right\}
\]

\[
- \int_{\mathbb{R}^d} K(x)F(x, tw^+ + sw^-) \max\{t\|^g, |s\|^g\} dx. \] (6.11)

By assumption \((f_3)\) and the fact that \(\text{supp}(w^+) \cap \text{supp}(w^-) = \emptyset\), we infer that

\[
\lim_{\|t, s\| \to +\infty} \frac{F(x, tw^+ + sw^-)}{\max\{t\|^g, |s\|^g\}} = +\infty, \quad \text{for a.a. } x \in \mathbb{R}^d. \] (6.12)

Applying (6.11) and (6.12), we deduce that

\[
\limsup_{\|t, s\| \to +\infty} \mu_w(t, s) \leq -\infty. \]
Using (6.9), we deduce that the map $\mu_w(\cdot, \cdot)$ has a global maximum $(t_{w+}, s_{w-}) \in (0, +\infty) \times (0, +\infty)$. Therefore, $(t_{w+}, s_{w-})$ is a critical point for $\mu_w(\cdot, \cdot)$, that is,

$$\langle J'(t_{w+}w^+ + s_{w-}w^-), w^+ \rangle = 0$$

and

$$\langle J'(t_{w+}w^+ + s_{w-}w^-), w^- \rangle = 0.$$

By Proposition 6.4 and the fact that $w \in \mathcal{M}$, it follows that

$$(t_{w+}, s_{w-}) = (1, 1).$$

Hence,

$$J(tw^+ + sw^-) \leq J(t_{w+}w^+ + s_{w-}w^-) = J(w), \text{ for all } t, s > 0.$$

This ends the proof. \hfill \Box

**Proposition 6.5.** Assume that hypotheses $(H_f), (H_G)$ and $(H_K)$ hold. Let $\{w_n\}_n \subset \mathcal{M}$ such that $w_n \to w$ in $W^{\alpha, G}(\mathbb{R}^d)$, then $w^\pm \not\equiv 0$.

**Proof.** Let us observe that there is $\varrho > 0$ such that

$$\varrho \leq \|w^\pm\|, \text{ for all } v \in \mathcal{M}. \tag{6.13}$$

Indeed, if $w \in \mathcal{M}$, using Lemma 4.8, we deduce that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left( \frac{w^\pm(x) - w^\pm(y)}{|x - y|^\alpha} \right) \frac{w^\pm(x) - w^\pm(y)}{|x - y|^\alpha + d} dxdy + \int_{\mathbb{R}^d} g(w^\pm)dx \leq \int_{\mathbb{R}^d} K(x)f(x, w^\pm)dx.$$

Exploiting (2.4), (5.1), hypotheses $(g_2)$ and $(K_1)$, we get for all $\varepsilon > 0$

$$[g^- - g^+\varepsilon\|K\|_\infty] \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G\left( \frac{w^\pm(x) - w^\pm(y)}{|x - y|^\alpha} \right) \frac{dx dy}{|x - y|^d} \right] \leq g^*_+ C\varepsilon\|K\|_\infty \int_{\mathbb{R}^d} G_w(\|w^\pm\|)dx.$$

Without lose of generality, we may assume that $0 \neq \|u\| < 1$. Then, by Lemma 2.5 and Theorem 2.1 we deduce that

$$|g^- - g^+\varepsilon\|K\|_\infty\|w^\pm\| \leq g^*_+ \tilde{C}\varepsilon\|K\|_\infty\|w^\pm\|^2. \tag{6.14}$$

Hence, by choosing $\varepsilon$ small enough, we obtain

$$\left( \frac{C_1}{C_2} \right)^{\frac{1}{\alpha^2 - \alpha + 1}} \leq \|w^\pm\|,$$

where $C_1 = g^- - g^+\varepsilon\|K\|_\infty > 0$ and $C_2 = g^*_+ \tilde{C}\varepsilon\|K\|_\infty > 0$. Consequently, there exists a positive radius $\varrho > 0$ such that $\|w^\pm\| \geq \varrho$, with $\varrho = \left( \frac{C_1}{C_2} \right)^{\frac{1}{\alpha^2 - \alpha + 1}}$.

By (6.13), if $\{w_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$, we have

$$\|w^\pm_n\| \geq \varrho, \text{ for all } n \in \mathbb{N}. \tag{6.15}$$

According to Lemma 4.8 and the fact that $w_n \in \mathcal{M}$, we have

$$\langle J'(w^\pm_n), w^\pm_n \rangle \leq \langle J'(w_n), w^\pm_n \rangle = 0.$$

By assuming $(g_2)$ and by Lemma 2.3 we get

$$g^- \min\{\|w^\pm_n\|, \|w^\pm_n\| \leq \|w^\pm_n\| \leq \int_{\mathbb{R}^d} K(x)f(x, w^\pm_n)dx. \tag{6.16}$$
Putting together (6.15) and (6.16), we deduce that
\[ g^- \min \{g^-, g^+\} \leq g^- \min \{\|w_n^+\|^\alpha, \|w_n^-\|^\alpha\} \leq \int_{\mathbb{R}^d} K(x) f(x, w_n^+) \, dx. \tag{6.17} \]

On the other hand, in light of Lemma 4.2, one has
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^d} K(x) f(x, w_n^+) \, dx = \int_{\mathbb{R}^d} K(x) f(x, w^+) \, dx. \tag{6.18} \]
Combining (6.17) with (6.18), we get
\[ 0 < g^- \min \{g^-, g^+\} \leq \int_{\mathbb{R}^d} K(x) f(x, w^+) \, dx, \]
thus, we conclude that \( w^+ \neq 0 \).
This ends the proof. \( \square \)

In the following proposition, we prove that the minimization problem (M-1) admits a solution.

**Proposition 6.6.** Assume that the hypotheses \((H_f), (H_G)\) and \((H_K)\) hold. Then, there exists \( \hat{w} \in \mathcal{M} \) such that \( J(\hat{w}) = m_1 \).

**Proof.** Let \( \{w_n\}_{n \in \mathbb{N}} \subset \mathcal{M} \) such that
\[ J(w_n) \overset{n \to +\infty}{\longrightarrow} m_1. \]

Arguing as in Proposition 5.5, we prove that \( \{w_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{\alpha, G}(\mathbb{R}^d) \).
By passing to a subsequence if necessary we may assume that
\[ w_n \to \hat{w} \text{ in } W^{\alpha, G}(\mathbb{R}^d), \]
\[ w_n(x) \to \hat{w}(x) \text{ as } n \to +\infty, \text{ for a.a. } x \in \mathbb{R}^d \tag{6.19} \]
and
\[ w_n^\pm(x) \to \hat{w}^\pm(x) \text{ as } n \to +\infty, \text{ for a.a. } x \in \mathbb{R}^d. \tag{6.20} \]
By Proposition 6.5, it follows that \( \hat{w}^\pm \neq 0 \).
According to Proposition 6.1, there is a unique pair \( t_{\hat{w}^+}, s_{\hat{w}^-} > 0 \) such that \( t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^- \in \mathcal{M} \), that is,
\[ \langle J'(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-), \hat{w}^+ \rangle = 0 \text{ and } \langle J'(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-), \hat{w}^- \rangle = 0. \tag{6.21} \]
Since \( \{w_n\}_{n \in \mathbb{N}} \subset \mathcal{M} \), by (6.20), Proposition 6.4, Lemma 1.2 and Fatou’s lemma, we deduce
\[ m_1 = \lim_{n \to +\infty} J(w_n) \geq \liminf_{n \to +\infty} J(t_{\hat{w}^+} w_n^+ + s_{\hat{w}^-} w_n^-) \]
\[ \geq J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-) \]
\[ \geq m_1. \tag{6.22} \]
Therefore,
\[ m_1 = \inf_{\mathcal{M}} J = J(t_{\hat{w}^+} \hat{w}^+ + s_{\hat{w}^-} \hat{w}^-). \tag{6.23} \]
Next, let show that \( t_{\hat{w}^+} = s_{\hat{w}^-} = 1 \). We do this in two steps.
Firstly, let prove that \( 0 < t_{\hat{w}^+}, s_{\hat{w}^-} \leq 1 \).
Using (6.20), Lemma 1.2 and Fatou’s lemma, we find that
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(\hat{w}(x) - \hat{w}(y)) \frac{\hat{w}^+(x) - \hat{w}^+(y)}{|x - y|^\alpha + d} \, dx dy + \int_{\mathbb{R}^d} g(\hat{w}^\pm) \hat{w}^\pm dx \leq \int_{\mathbb{R}^d} K(x) f(x, \hat{w}^\pm) \hat{w}^\pm dx. \tag{6.24} \]
From (6.21), we have

\[
\int_{\mathbb{R}^d} K(x) f(x, t_{\hat{\omega}^+}) t_{\hat{\omega}^+} dx = \int_{\text{supp}(\hat{\omega}^+)} \int_{\text{supp}(\hat{\omega}^+)} g \left( \frac{t_{\hat{\omega}^+} \hat{\omega}^+(x) - t_{\hat{\omega}^+} \hat{\omega}^+(y)}{|x - y|^{\alpha}} \right) t_{\hat{\omega}^+} \hat{\omega}^+(x) - t_{\hat{\omega}^+} \hat{\omega}^+(y) \frac{dxdy}{|x - y|^{\alpha+d}}
\]

\[+ \int_{\text{supp}(\hat{\omega}^-)} \int_{\text{supp}(\hat{\omega}^+)} g \left( \frac{t_{\hat{\omega}^+} \hat{\omega}^+(x) - s_{\hat{\omega}^-} \hat{\omega}^- (y)}{|x - y|^{\alpha}} \right) t_{\hat{\omega}^+} \hat{\omega}^+(x) - s_{\hat{\omega}^-} \hat{\omega}^- (y) \frac{dxdy}{|x - y|^{\alpha+d}}
\]

\[+ \int_{\text{supp}(\hat{\omega}^+)} \int_{\text{supp}(\hat{\omega}^-)} g \left( \frac{s_{\hat{\omega}^-} \hat{\omega}^- (x) - t_{\hat{\omega}^+} \hat{\omega}^+ (y)}{|x - y|^{\alpha}} \right) -t_{\hat{\omega}^+} \hat{\omega}^+ (y) \frac{dxdy}{|x - y|^{\alpha+d}}
\]

\[+ \int_{\mathbb{R}^d} g(t_{\hat{\omega}^+} \hat{\omega}^+) t_{\hat{\omega}^+} \hat{\omega}^+ dx.
\]

Without loss of generality, we can suppose that \(t_{\hat{\omega}^+} \geq s_{\hat{\omega}^-}\). By (6.25), Lemma 2.4 and the fact that \(s \mapsto g(s)\) is non-decreasing function on \(\mathbb{R}\), we get

\[
\int_{\mathbb{R}^d} K(x) f(x, t_{\hat{\omega}^+}) t_{\hat{\omega}^+} dx \leq \max\{t_{\hat{\omega}^+}^-, t_{\hat{\omega}^+}^+\} \sigma,
\]

where

\[
0 \leq \sigma = \int_{\text{supp}(\hat{\omega}^+)} \int_{\text{supp}(\hat{\omega}^+)} g \left( \frac{\hat{\omega}^+(x) - \hat{\omega}^+(y)}{|x - y|^{\alpha}} \right) \hat{\omega}^+(x) - \hat{\omega}^+(y) \frac{dxdy}{|x - y|^{\alpha+d}}
\]

\[+ \int_{\text{supp}(\hat{\omega}^-)} \int_{\text{supp}(\hat{\omega}^+)} g \left( \frac{\hat{\omega}^+(x) - \hat{\omega}^- (y)}{|x - y|^{\alpha}} \right) \hat{\omega}^+(x) - \hat{\omega}^- (y) \frac{dxdy}{|x - y|^{\alpha+d}}
\]

\[+ \int_{\text{supp}(\hat{\omega}^+)} \int_{\text{supp}(\hat{\omega}^-)} g \left( \frac{\hat{\omega}^- (x) - \hat{\omega}^+ (y)}{|x - y|^{\alpha}} \right) -\hat{\omega}^+ (y) \frac{dxdy}{|x - y|^{\alpha+d}}
\]

\[+ \int_{\mathbb{R}^d} g(\hat{\omega}^+) \hat{\omega}^+ dx
\]

\[= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{\hat{\omega} (x) - \hat{\omega} (y)}{|x - y|^{\alpha}} \right) \hat{\omega}^+(x) - \hat{\omega}^+(y) \frac{dxdy}{|x - y|^{\alpha+d}}
\]

\[+ \int_{\mathbb{R}^d} g(\hat{\omega}^+) \hat{\omega}^+ dx.
\]

Suppose that \(t_{\hat{\omega}^+} > 1\), then

\[
\int_{\mathbb{R}^d} K(x) \frac{f(x, t_{\hat{\omega}^+} \hat{\omega}^+)}{t_{\hat{\omega}^+}^+} t_{\hat{\omega}^+} dx \leq \sigma.
\]

Putting together (6.24) and (6.26), using (K1) and (3.1), we obtain

\[
0 \leq \int_{\mathbb{R}^d} K(x)(\hat{\omega}^+)^{\sigma} \left[ \frac{f(x, \hat{\omega}^+)}{(\hat{\omega}^+)^{\sigma+1}} - \frac{f(x, t_{\hat{\omega}^+} \hat{\omega}^+)}{(t_{\hat{\omega}^+} \hat{\omega}^+)^{\sigma+1}} \right] dx < 0
\]

which is a contradiction. Therefore, \(0 < t_{\hat{\omega}^+}, s_{\hat{\omega}^-} \leq 1\).

Secondly, Let prove that \(t_{\hat{\omega}^+} = s_{\hat{\omega}^-} = 1\). We argue by contradiction, suppose that \((t_{\hat{\omega}^+}, s_{\hat{\omega}^-}) \neq (1,1)\).
It follows, by (3.2), (6.19), (6.20), Lemma 4.1 and Fatou’s lemma, that
\[
m_1 \leq J(t_{\widehat{\omega}^+} \widehat{w}^+ + s_{\widehat{\omega}^-} \widehat{w}^-) \quad \text{(since } t_{\widehat{\omega}^+} \widehat{w}^+ + s_{\widehat{\omega}^-} \widehat{w}^- \in \mathcal{M}) \]
\[
= J(t_{\widehat{\omega}^+} \widehat{w}^+ + s_{\widehat{\omega}^-} \widehat{w}^-) - \frac{1}{g^+} \langle J'(t_{\widehat{\omega}^+} \widehat{w}^+ + s_{\widehat{\omega}^-} \widehat{w}^-), t_{\widehat{\omega}^+} \widehat{w}^+ + s_{\widehat{\omega}^-} \widehat{w}^- \rangle 
\leq J(\widehat{w}) - \frac{1}{g^+} \langle J'(\widehat{w}), \widehat{w} \rangle
\leq \liminf_{n \to +\infty} \left[ J(w_n) - \frac{1}{g^+} \langle J'(w_n), w_n \rangle \right]
= \liminf_{n \to +\infty} J(w_n) \quad \text{(since } w_n \in \mathcal{M})
= m_1.
\]

Which is a contradiction, thus, \( t_{\widehat{\omega}^+} = s_{\widehat{\omega}^-} = 1 \). Hence, by (6.23), we comes that
\[
m_1 = \inf_{\mathcal{M}} J = J(t_{\widehat{\omega}^+} \widehat{w}^+ + s_{\widehat{\omega}^-} \widehat{w}^-) = J(\widehat{w}).
\]

This ends the proof. \( \square \)

**Proposition 6.7.** Under the hypotheses \((H_f), (H_G)\) and \((H_K)\), \( \widehat{w} \) is a critical point for \( J \), hence, it is a weak nodal solution of problem 17.

**Proof.** We consider the functionals \( \varphi_{\pm} : W^{\alpha,G}(\mathbb{R}^d) \to \mathbb{R} \) defined by
\[
\varphi_{\pm}(w) = \langle J'(w), w^\pm \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{w(x) - w(y)}{|x - y|^\alpha} \right) \frac{w^\pm(x) - w^\pm(y)}{|x - y|^{\alpha + d}} \, dx \, dy
+ \int_{\mathbb{R}^d} g(w) w^\pm \, dx
- \int_{\mathbb{R}^d} K(x) f(x, w^\pm) w^\pm \, dx.
\]

From hypotheses \((H_f)\), \( \varphi_{\pm} \) is locally Lipschitz (see 19 Theorem 2.7.2, p. 221)).

Let \( w \in W^{\alpha,G}(\mathbb{R}^d) \), for all \( \varphi_{w^\pm}^* \in \partial \varphi_{\pm}(w) \), there is \( f^*(x, w^\pm) \in \partial w^\pm f(x, w^\pm) \) such that
\[
\langle \varphi_{w^\pm}^*, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{w(x) - w(y)}{|x - y|^\alpha} \right) \frac{v^\pm(x) - v^\pm(y)}{|x - y|^{\alpha + d}} \, dx \, dy
+ \int_{\mathbb{R}^d} g(w) v^\pm \, dx
- \int_{\mathbb{R}^d} K(x) f^*(x, w^\pm) v^\pm \, dx
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g' \left( \frac{w(x) - w(y)}{|x - y|^\alpha} \right) \left[ v(x) - v(y) \right] \left[ \frac{w^\pm(x) - w^\pm(y)}{|x - y|^{2\alpha + d}} \right] \, dx \, dy
+ \int_{\mathbb{R}^d} g'(w^\pm)(v^\pm)^2 \, dx
- \int_{\mathbb{R}^d} K(x) f^*(x, w^\pm)(v^\pm)^2 \, dx,
\]
for all \( v \in W^{\alpha,G}(\mathbb{R}^d) \).

From Proposition 6.6 we know that
\[
J(\widehat{w}) = m_1 = \inf \left\{ J(w) : w \in W^{\alpha,G}(\mathbb{R}^d), \ w^\pm \neq 0, \ \varphi_{\pm}(w) = \varphi_{\pm}(0) = 0 \right\}.
\]

Thus, according to the non-smooth multiplier rule of Clarke 18 Theorem 10.47, p. 221], we can find \( \lambda_+, \lambda_- \geq 0 \) such that
\[
0 \in \partial(J + \lambda_+ \varphi_+ + \lambda_- \varphi_-)(\widehat{w}).
\]

It follows, by the subdifferential calculus of Clarke 19 p. 48, that
\[
0 \in \partial J(\widehat{w}) + \lambda_+ \partial \varphi_+(\widehat{w}) + \lambda_- \partial \varphi_-(\widehat{w}).
\]

Then,
\[
0 = J'(\widehat{u}) + \lambda_+ \varphi_{w^+}^* + \lambda_- \varphi_{w^-}^* \quad \text{in } (W^{\alpha,G}, (\mathbb{R}^d))^*, \quad \text{for all } \varphi_{w^+}^* \in \partial \varphi_+(\widehat{w}) \text{ and all } \varphi_{w^-}^* \in \partial \varphi_-(\widehat{w}). \quad (6.28)
\]
Since \( \hat{w} \in \mathcal{M} \), we have
\[
0 = (J'(\hat{w}), \hat{w}) + \lambda_+ \langle \varphi_{\hat{w}+}^*, \hat{w} \rangle + \lambda_- \langle \varphi_{\hat{w}-}^*, \hat{w} \rangle = \lambda_+ \langle \varphi_{\hat{w}+}^*, \hat{w} \rangle + \lambda_- \langle \varphi_{\hat{w}-}^*, \hat{w} \rangle, \tag{6.29}
\]
for all \( \varphi_{\hat{w}+}^* \in \partial \varphi_+ (\hat{w}) \) and all \( \varphi_{\hat{w}-}^* \in \partial \varphi_- (\hat{w}) \).

Let us observe that
\[
\text{sign} (\hat{w}(x) - \hat{w}(y)) = \text{sign} (\hat{w}^+(x) - \hat{w}^+(y)), \quad \text{for a.a. } x, y \in \mathbb{R}^d. \tag{6.30}
\]

Using (6.27), (6.30), assumption (g₃) and the fact that \( \hat{w} \in \mathcal{M} \), we obtain
\[
\langle \varphi_{\hat{w}+}^*, \hat{w} \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g' \left( \frac{\hat{w}(x) - \hat{w}(y)}{|x-y|^{\alpha}} \right) \frac{[\hat{w}(x) - \hat{w}(y)][\hat{w}^+(x) - \hat{w}^+(y)]}{|x-y|^{2\alpha+d}} \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^d} g' (\hat{w}^+) (\hat{w}^+) \, dx - \int_{\mathbb{R}^d} K(x) f^* (x, \hat{w}^+) (\hat{w}^+) \, dx
\]
\[
\leq [g^+ - 1] \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g \left( \frac{\hat{w}(x) - \hat{w}(y)}{|x-y|^{\alpha}} \right) \frac{\hat{w}^+(x) - \hat{w}^+(y)}{|x-y|^{\alpha+d}} \, dx \, dy \right]
\]
\[
+ \int_{\mathbb{R}^d} g (\hat{w}^+) \hat{w}^+ \, dx - \int_{\mathbb{R}^d} K(x) f^* (x, \hat{w}^+) (\hat{w}^+) \, dx
\]
\[
= \int_{\mathbb{R}^d} K(x) \left( [g^+ - 1] f(x, \hat{w}^+) \hat{w}^+ - f^* (x, \hat{w}^+) (\hat{w}^+) \right) \, dx. \tag{6.31}
\]

From (6.31), hypotheses (f₄) and (K₁), we infer that
\[
\langle \varphi_{\hat{w}+}^*, \hat{w} \rangle < 0 \quad \text{and} \quad \langle \varphi_{\hat{w}-}^*, \hat{w} \rangle < 0. \tag{6.32}
\]

It follows, by (6.29), that
\[
\lambda_{\pm} = 0. \]

Therefore, by (6.28), we deduce that \( \hat{w} \) is a critical point of \( J \). Hence, \( \hat{w} \) is a nodal weak solution for problem \( P \).

This completes the proof. \( \square \)

**Proposition 6.8.** Assume that the hypotheses \( (H_f) \), \( (H_G) \) and \( (H_K) \) hold. Then, the ground state solution \( \hat{u} \) of problem \( P \) has a fixed sign. Moreover,
\[
m_0 = J(\hat{u}) = \inf_{\mathcal{N}} J < \inf_{\mathcal{M}} J = J(\hat{w}) = m_1.
\]

**Proof.** We argue by contradiction. Suppose that \( \hat{w}^\pm \neq 0 \), then
\[
m_0 = \inf_{\mathcal{N}} J \geq \inf_{\mathcal{M}} J = m_1. \tag{6.33}
\]

Since \( \mathcal{M} \subset \mathcal{N} \), we have
\[
m_0 = \inf_{\mathcal{N}} J \leq \inf_{\mathcal{M}} J = m_1. \tag{6.34}
\]

Putting together (6.33) and (6.34), we get
\[
m_0 = J(\hat{u}) = \inf_{\mathcal{N}} J = \inf_{\mathcal{M}} J = J(\hat{w}) = m_1. \tag{6.35}
\]

On the other hand, since \( \hat{w} \in \mathcal{M} \), we have \( \hat{w}^\pm \neq 0 \). Then, from Proposition 5.1 there is a unique pair \( t_{\hat{w}+}, s_{\hat{w}-} > 0 \) such that
\[
t_{\hat{w}+} \hat{w}^+ \in \mathcal{N} \quad \text{and} \quad s_{\hat{w}-} \hat{w}^- \in \mathcal{N}.
\]

It follows, by Lemma 4.4 and Proposition 6.4 that
\[
2m_0 \leq J(t_{\hat{w}+} \hat{w}^+) + J(s_{\hat{w}-} \hat{w}^-)
\]
\[
< J(t_{\hat{w}+} \hat{w}^+ + s_{\hat{w}-} \hat{w}^-)
\]
\[
\leq J(\hat{w}) = \inf_{\mathcal{M}} J = m_1. \tag{6.36}
\]
Which is a contradiction with (6.35). Therefore, \( \hat{u} \) has a fixed sign. Hence,

\[
m_0 = J(\hat{u}) = \inf_N J < \inf_M J = J(\hat{w}) = m_1.
\]

Thus the proof. \( \square \)

**Proof of Theorem 3.2**: The proof of Theorem 3.2 is concluded from the Propositions 6.6, 6.7 and 6.8. \( \square \)

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