CONNECTED OBJECTS IN CATEGORIES OF $S$-ACTS

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Abstract. The main goal of the paper is description of connected and projective objects of classes of categories that include categories of acts along with categories of pointed acts. In order to establish a general context and to unify the approach to both of the categories of acts, the notion of a concrete category with a unique decomposition of objects is introduced and studied. Although these categories are not extensive in general, it is proved in the paper that they satisfy a version of extensivity, which ensures that every noninitial object is uniquely decomposable into indecomposable objects.

1. Introduction

While the great impact of category theory on theory of rings and modules is well known, the analogous concept in the context of theory of monoids and acts over monoids on sets is significantly less studied, however it seems to be promising and fruitful as it is demonstrated in the monograph [20]. The aim of this paper is to introduce and study a class of categories including the most important categories of acts, which are given by acting of a general monoid on sets and by acting of a monoid with zero on pointed sets. However a category of pointed acts over a monoid with zero seems to be closer to a module category then a category of standard acts, our main goal is to develop tools that can apply simultaneously to both cases.

Recall that an object $c$ of an abelian category closed under coproducts and products is said to be compact if the corresponding covariant functor $\text{Hom}(c, -)$ commutes with arbitrary coproducts i.e. there is a canonical isomorphism in the category of abelian groups $\text{Hom}(c, \coprod D) \cong \coprod \text{Hom}(c, D)$ for every family of objects $D$, where $\coprod$ denotes a coproduct [12, 34]. Compact objects are called small in categories of (right $R$-)modules [4, 9, 11]. The notion of a compact object in non-abelian categories has different meaning; it is usually defined as an object such that the corresponding covariant hom-functors commute with filtered colimits, nevertheless, it can be proved that that this notion is stronger [14]. In compliance with convention, an object $c$ such that $\text{Mor}(c, -)$ commutes with all coproducts is called connected in this paper.

The main motivation of the present paper, reflected by its title, is an issue of translating description of connectedness from abelian categories to a more general context. The constitutive example of such a generalization is provided by the analogy between (abelian) categories of modules over rings and (non-abelian) categories of acts over monoids (cf. also the corresponding description of connectedness in the case of Ab5 categories [12]). Moreover, note that both the categories of acts and pointed acts are quasi pointed categories (see [6]).

The list of works dedicated to the research of connected and compact objects in various categories is long. Let us mention only those related to our concept of linking (self)small modules, (auto)compact objects in abelian categories and (auto)connected objects in non-abelian context. However, the notion of autocompactness of modules [1 Proposition 1.1] was generalized to Grothendieck categories in [13]. The main motivation for the study of connected objects in abelian categories comes from the context of representable equivalences of module categories [8, 9]. Moreover, note that both the categories of acts and pointed acts are quasi pointed categories (see [9]).

Note that compact objects play also an important role in triangulated categories [24].
are compactly generated, in particular, for the description of Brown representability \[22\]. Although locally presentable and accessible categories deal with connectedness in the narrow sense \[2, 24\], the motivation and application of the notion is close to the abelian case.

Turning the attention towards the categories of modules, as is shown in \[5\] and in \[27, 1\]^o, small modules can be structurally described in a natural way by the language of families of submodules:

**Lemma 1.1.** \[5, 27\] The following conditions are equivalent for a module \(M\):

1. \(M\) is small,
2. if \(M = \bigcup_{i<\omega} M_n\) for an increasing chain of submodules \(M_n \subseteq M_{n+1} \subseteq M\), then there exists \(n\) such that \(M = M_n\),
3. if \(M = \sum_{i<\omega} M_n\) for a family of submodules \(M_n \subseteq M\), \(n < \omega\), then there exists \(k\) such that \(M = \sum_{i<k} M_n\).

Note that the condition (2) implies immediately that every finitely generated module is small and (3) shows that there are no countably infinitely generated small modules. On the other hand, there are natural constructions of infinitely generated small modules:

**Example 1.2.**
1. A union of a strictly increasing chain of length \(\kappa\), for an arbitrary cardinal \(\kappa\) of uncountable cofinality, consisting of small (in particular finitely generated) submodules is small.
2. Every \(\omega_1\)-generated uniserial module is small.

A ring over which the class of all small right modules coincides with the class of all finitely generated ones is called **right steady**. Note that the class of all right steady rings is closed under factorization \[9, Lemma 1.9\], finite products \[30, Theorem 2.5\], and Morita equivalence \[13, Lemma 1.7\]. However, a ring theoretical characterization of steadiness remains an open problem with partial results concerning right steadiness of certain natural classes of rings including right noetherian \[27, 70\], right perfect \[9, Corollary 1.6\], right semisimple of finite socle length \[31, Theorem 1.5\] countable commutative \[27, 11^o\], and abelian regular rings with countably generated ideals \[32, Corollary 7\].

The main task of the first half of the paper is presenting two variants of categories of acts over monoids, namely acts and pointed acts (i.e. acts with a zero element), via a joint general categorial language, in particular, the notion of a \(UD\)-category is introduced. Section 3 deals with the crucial issue of decompositions in \(UD\)-categories. Although categories of pointed acts are not extensive, it is proved that \(UD\)-categories satisfy weak version of extensivity (Proposition 3.12), which ensures uniqueness of decomposition (Proposition 3.13). Since \(UD\)-categories contain enough indecomposable objects, their necessary basic properties follow, with Theorem 3.15 formulating the existence and uniqueness of indecomposable decomposition. A general composition theory of projective objects in a \(UD\)-category is built in the next section, where the main result of the section, Theorem 4.3 characterizes projective objects as coproducts of indecomposable projective objects. Section 5 lists general properties of connected objects in a \(UD\)-category, in particular, Theorem 5.13 presents a general criterion for connectedness of objects in a \(UD\)-category. Finally, as an application of this theory the characterization of (auto)connected objects in categories of \(S\)-acts is also provided (Theorem 6.11, Proposition 6.15).

2. AXIOMATIC DESCRIPTION OF CATEGORIES OF ACTS

Before we start the study of common categorial properties of classes of acts over monoids, let us recall some necessary terminology and notation.

Let \(C\) be a category. Denote by \(\text{Mor}_C(A, B)\) the class of all morphisms \(A \to B\) in \(C\) for every pair of objects \(A, B\) of \(C\); in case \(C\) is clear from the context, the subscript will be omitted. A monomorphism (epimorphism) in \(C\) is a left (right)-cancellable morphism, i.e., a morphism \(\mu\) such that \(\mu \alpha = \mu \beta\) \((\alpha \mu = \beta \mu)\) implies \(\alpha = \beta\). A morphism is a bimorphism, if it is both mono- and epimorphism. A category is balanced, if bimorphisms are isomorphisms (the reversed inclusion holds in general). An object \(\theta\) is called initial provided \(|\text{Mor}(\theta, A)| = 1\) for each object \(A\). The category is (co)product complete if the class of objects is closed under all (co)products. Note that any coproduct complete category contains an initial object, which is isomorphic to \(\coprod \emptyset\). A pair
$(C, U)$ is said to be a concrete category over the category of sets, which is denoted Set in the whole paper, if $C$ is a category and $U : C \to \text{Set}$ is a faithful functor. Finally, a family of objects means any discrete diagram and the phrase the universal property of a coproduct refers to the existence of unique morphism from a coproduct.

Let $S = (S, \cdot, 1)$ be a monoid and $A$ a nonempty set. If there is a mapping $\mu : S \times A \to A$ satisfying the following two conditions: $\mu(1, a) = a$ and $\mu(s_2, \mu(s_1, a)) = \mu(s_2 \cdot s_1, a)$ then the pair $(A, \mu)$ of a set $A$ along with the left action $\mu$ is said to be a left $S$-act. For simplicity, $\mu(s, a)$ is often written as $s \cdot a$ or $sa$ and an act and the act $(A, \mu)$ is denoted $sA$. A mapping $f : sA \to sB$ is a homomorphism of $S$-acts, or an $S$-homomorphism provided $f(sa) = sf(a)$ holds for any $s \in S, a \in A$. We denote by $S$-Act the category of all left $S$-acts with homomorphisms of $S$-acts.

Let us point out that we consider the empty $S$-act $s\emptyset$ to be an (initial) object of $S$-Act. Henceforth this category corresponds to the category $S$-$\text{Act}$ in [20].

Let $S$ be a monoid containing a (necessarily unique) zero element $0$. Then an $S$-act $A$ containing a fixed element $0_A$ which satisfies the axiom $0a = 0_A$ for each $a \in A$ is called a pointed left act and then the category of all pointed left $S$-acts with homomorphisms of $S$-acts compatible with zero as morphisms will be denoted by $S$-$\text{Act}_0$. Observe that $\{0\}$ is an initial object of the category $S$-$\text{Act}_0$.

Recall that both categories $S$-Act and $S$-$\text{Act}_0$ are complete and cocomplete [20] Remarks II.2.11, Remark II.2.22, in particular, a coproduct of a family of objects $(A_i, i \in I)$ is

\begin{enumerate}
  \item a disjoint union $\coprod_{i \in I} A_i = \bigcup A_i$ in $S$-Act by [20] Proposition II.1.8 and \(\prod_{i \in I} A_i = \{(a_i) \in \prod_{i \in I} A_i | \exists j : a_i = 0 \forall i \neq j\} \text{ in } S$-$\text{Act}_0 \text{ by [20] Remark II.1.16.} \)
\end{enumerate}

Furthermore, if we denote the natural forgetful functor from $S$-Act or $S$-$\text{Act}_0$ into Set by $U$ (which maps an act to the underlying set of elements and an $S$-homomorphism to the corresponding mapping between sets) both $(S$-$\text{Act}_0, U)$ and $(S$-Act, $U$) are concrete categories over Set. Note that an act can be defined as a functor from a single-object category to the category of pointed sets and a pointed act can be defined as a point-preserving functor from a single-object pointed category to the category of pointed sets where morphisms are natural transformations. In consequence, the category of acts is a presheaf category, while the category of pointed acts forms a point-enriched version of a presheaf category, in the sense of enriched category theory.

Let $C$ be a coproduct complete category with an initial object $\theta \cong \coprod \emptyset$. An object $A \in C$ is called indecomposable if it is not isomorphic to an initial object nor to a coproduct of two non-initial objects. Note that cyclic acts present natural examples of indecomposable objects in both categories $S$-Act and $S$-$\text{Act}_0$. Nevertheless, the class of indecomposable acts can be much larger, e. g. the rational numbers form a non-cyclic indecomposable ($\mathbb{Z}$, $\cdot$)-act.

As we have declared, the main motivation of the present paper is to describe and investigate connectedness properties of categories of acts over monoids in the general categorial language. In particular, we focus on the categories $S$-Act and $S$-$\text{Act}_0$. The key feature of both of these categories is the existence of a unique decomposition of every object into indecomposable objects, which is proved in [20] Theorem I.5.10] for the case of the category $S$-Act.

First of all, we list several natural categorial properties which ensure an easy handling of the category, the uniqueness of decomposition and provide the existence condition as well. Recall that a pair $(S, \nu)$ is said to be a subobject of an object $A$ if $S$ is an object and $\nu$ of $A$ is a monomorphism.

We say that a concrete category $(C, U)$ over the category Set is a UD-category (unique decomposition) if the following conditions hold:

\begin{enumerate}
  \item [(UD1)] $C$ is a coproduct complete balanced category with an initial object $\theta \cong \coprod \emptyset$, for which each morphism $\theta \to A$ is a monomorphism and there is at most one morphism $A \to \theta$.
  \item [(UD2)] For any morphism $f \in \text{Mor}(A, B)$ in $C$, there exists a subobject $(A', i)$ of $B$ such that $U(A') = U(f)(U(A)) \subseteq U(B)$ and $U(i) \in \text{Mor}(U(A'), U(B))$ is the subset inclusion map.
  \item [(UD3)] For each morphism $f \in \text{Mor}(A, B)$ and every subobject $(S, \nu)$ of $B$ such that $U(f)(U(A)) \subseteq U(\nu)(U(S))$, there exists a morphism $g \in \text{Mor}(A, S)$ such that $f = \nu g$.
\end{enumerate}
(UD4) for every family \((A_i, \nu_i)_{i \in I}\) of subobjects of an object \(A\), there exist subobjects denoted by \(\bigcap_i A_i^{\nu_i}, \iota_{\nu_i}\) and \(\bigcup_i A_i^{\nu_i}, \iota_{\nu_i}\) such that

\[
U \left( \bigcap_i A_i^{\nu_i} \right) = \bigcap \left( U(\nu_i)U(A_i) \right) = \bigcap_i U(A_i^{\nu_i}), \quad U \left( \bigcup_i A_i^{\nu_i} \right) = \bigcup \left( U(\nu_i)U(A_i) \right) = \bigcup_i U(A_i^{\nu_i})
\]

and both \(U(\iota_{\nu_i}), U(\iota_{\nu_i})\) are the corresponding subset inclusion mappings,

(UD5) if \((A, (\nu_0, \nu_1))\) is a coproduct of a pair of objects \((A_0, A_1)\), then \(\nu_0\) and \(\nu_1\) are monomorphisms and \(\bigcap_{i=1}^2 A_i^{\nu_i}\) is isomorphic to \(\theta\),

(UD6) for every object \(A\) and every \(x \in U(A)\) there exists an indecomposable subobject \((B, \nu)\) of \(A\) such that \(x \in U(B^\nu) = U(\nu)(U(B)) \subseteq U(A)\).

Any monomorphism \(\iota : A \to B\) such that \(U(\iota)\) is the subset inclusion map is called inclusion morphism. Note that we will use the notation \((A^\iota, \iota)\) from (UD2) and \(\bigcup_i A_i^{\nu_i}, \iota_{\nu_i}\) from (UD4) freely without other explanations. Moreover, we will write \(I_0^{A_0} \cap A_1^{\nu_1}\) instead of \(\bigcap_{i=1}^2 A_i^{\nu_i}\) (\(\bigcup_{i=1}^2 A_i^{\nu_i}\) respectively).

First we make an elementary but frequently used (sometimes without reference) observation:

**Lemma 2.1.** Let \(\psi\) be a morphism of a UD-category \((C, U)\).

1. If \(U(\psi)\) is injective, then \(\psi\) is a monomorphism.
2. If \(U(\psi)\) is surjective, then \(\psi\) is an epimorphism.
3. \(\psi\) is an isomorphism if and only if \(U(\psi)\) is a bijection.

**Proof.** Since injective maps are monomorphisms and surjective maps are epimorphisms in Set, (1) and (2) follow immediately from the hypothesis that \(U\) is a faithful functor.

(3) The direct implication follows from the well-known fact that any functor preserves isomorphisms and the reverse one follows from (1) and (2) since \(C\) is a balanced category. \(\square\)

As an easy consequence we obtain a natural property of subobjects in UD-categories:

**Lemma 2.2.** Let \((B, \nu)\) be a subobject of an object \(A\) in a UD-category \((C, U)\). If \((B^\nu, \iota)\) is a subobject with the inclusion morphism \(\iota\) from (UD2) and \(\nu \in \text{Mor}(B, B^\nu)\) from (UD3) satisfying \(\nu = \nu\), then \(\nu\) is an isomorphism and so \(U(\nu)\) is injective.

**Proof.** Since \(U(\iota)(U(B)) = U(\iota)U(\nu)(U(B)) = U(\nu)(U(B)) = U(B^\nu)\) by (UD2), the morphism \(\nu\) is an epimorphism by Lemma 2.1(2). As \(\nu\) is a monomorphism and \(C\) is a balanced category, \(\nu\) is an isomorphism. Since \(U(\iota)\) is a bijection by Lemma 2.1(3), \(U(\nu) = U(\iota)U(\nu)\) is injective. \(\square\)

Let us note that both categories of acts treated in this paper satisfy the previous axioms:

**Example 2.3.** (1) Let \(S = (S, \cdot, 1)\) be a monoid. We show that all conditions (UD1)–(UD6) are satisfied by \((S, \text{Act}, U)\) for the natural forgetful functor \(U : S, \text{Act} \to \text{Set}\), hence it is a UD-category.

We have already mentioned that \(S, \text{Act}\) is a coproduct complete category and that \((S, \text{Act}, U)\) is a concrete category over Set. Furthermore, the empty act \(0\) with the empty mapping represents an initial object and the empty map is a monomorphism, since there is no morphism of a nonempty act into \(0\). Since monomorphisms are exactly injective morphisms, epimorphisms are surjective morphisms and isomorphisms are bijections, \(S, \text{Act}\) is (epi,mono)-structured hence a balanced category (cf. [1], Section 14)), which proves (UD1). Let us put \(A_f = f(A)\) for every morphism \(f : A \to B\) and note that intersections and unions of subacts forms subacts as well, then the conditions (UD2), (UD3), (UD4) and (UD5) follow either immediately from the definition of an act or from well-known basic properties (cf. [20]), and (UD6) holds true since cyclic acts are indecomposable.

(2) Let \(S_0 = (S_0, \cdot, 1)\) be a monoid with a zero element \(0\). Then \(S_0, \text{Act}_0\), similarly as in (1) is also a coproduct complete category and \((S_0, \text{Act}_0, U)\) is a concrete category over Set, where \(U\) is the forgetful functor. Clearly, the zero object \(\{0\}\) with the zero (mono)morphism forms an initial object of the category \(S_0, \text{Act}_0\). Since there is exactly one (zero) morphism from an arbitrary object to the zero object, (UD1) holds true. A similar argumentation as in (1) shows that \((S_0, \text{Act}_0, U)\) satisfies also the conditions (UD2)–(UD6), i.e., it is a UD-category.
Example 2.4. The concrete category $\text{Set}, \text{id}_{\text{Set}}$ is a trivial example of a UD-category: conditions (UD1)-(UD5) are clearly satisfied and for (UD6) note that singletons are indecomposable objects.

Example 2.5. Observe that the faithful functor $U$ from the definition of UD-category need not preserve coproducts. While coproducts of the category $S\text{-Act}$ are precisely disjoint unions, which are coproducts also in $\text{Set}$, and so the forgetful functor $U$ preserves coproducts here, coproducts in $S_0\text{-Act}_0$ glue together zero elements, so they do not coincide with coproducts of the category of all sets, hence the forgetful functor $U$ does not preserve coproducts of $S_0\text{-Act}_0$.

3. Decompositions

Our first step in describing UD-categories consists in showing the existence of a unique decomposition into coproduct of indecomposable objects. Categories satisfying some version of unique decomposition property are widely studied. Abelian categories with unique finite coproduct decomposition into indecomposable objects are called Krull-Schmidt categories (see e.g. [23]) and they play distinctive role in structural module theory.

Recall an important class of categories, non-abelian in general, in which all objects possess a coproduct decomposition, the extensive categories, which can be characterized as categories $B$ with (finite) coproducts which have pullbacks along colimit structural morphisms and in every commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Z \\
\downarrow & & \downarrow \\
A & \longrightarrow & A \sqcup B
\end{array}
\]

the squares are pullbacks if and only if the top row is a coproduct diagram in $B$ (see [10] Proposition 2.2, cf.also [33]). An extensive category $B$ is said to be infinitary extensive if the coproduct diagram above is considered also for infinite coproducts. Note that extensive categories have a wide range of applicability spanning from being a starting point for construction of distributive categories, which seem to be the correct setting for study of acyclic programs in computer science (cf. [7], [10]), to the theory of elementary topos.

Proposition 3.1. The category $S\text{-Act}$ is infinitary extensive.

Proof. Since $S\text{-Act}$ is a complete category, there exists any pullback.

Let us for $i \in I$ consider the following diagram in $S\text{-Act}$ with pullback squares

\[
\begin{array}{ccc}
A_i & \overset{\alpha_i}{\longrightarrow} & A \\
\downarrow & & \downarrow f \\
B_i & \overset{\nu_i}{\longrightarrow} & \coprod_{i \in I} B_i
\end{array}
\]

and let us prove $A$ is a coproduct of $A_i$’s. Since we can by [20] 2.2, 2.5] consider

\[A_i = \{(b, a) \in B_i \times A \mid \nu_i(b) = f(a)\} \quad \text{and} \quad \coprod B_i = \underset{i \in I}{\bigcup} \nu_i(B_i),\]

then for each $a$ there exist $i \in I, b \in B_i$ with $\nu_i(b) = f(a)$, so $\alpha_i((b, a)) = a$. In consequence $A = \bigcup \alpha_i(A_i)$. Assume $a \in \alpha_{i_0}(x_{i_0}) \cap \alpha_{j_0}(x_{j_0})$; then $f(a) = f \alpha_{i_0}(x_{i_0}) \in \nu_{i_0}(B_{i_0})$, hence $f(a) \in \nu_{i_0}(B_{i_0}) \cap \nu_{j_0}(B_{j_0}) = \emptyset$, so $\alpha_i(A_i) \cap \alpha_j(A_j) = \emptyset$ for $j \neq i$ and $A \cong \prod_{i \in I} A_i$. \hfill \Box

Example 3.2. (1) Consider the monoid of integers $Z = (\mathbb{Z}, \cdot, 1)$. Then the $\mathbb{Z}$-act $A = 2\mathbb{Z} \cup 3\mathbb{Z}$ shows that the category $S\text{-Act}_0$ is not extensive (and so it is not infinitary extensive). See the following commutative diagram with pullback squares where the top row is not a coproduct diagram:

\[
\begin{array}{ccc}
A_3 & \overset{\alpha_3}{\longrightarrow} & 2\mathbb{Z} \cup 3\mathbb{Z} \\
\downarrow & & \downarrow \tau \\
Z_3 & \overset{\nu_3}{\longrightarrow} & Z_3 \cup Z_2
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
\downarrow & & \downarrow \\
& & \\
A_2 & \overset{\alpha_2}{\longrightarrow} & A_2 \\
\downarrow & & \downarrow \\
& & \\
Z_2 & \overset{\nu_2}{\longrightarrow} & Z_2
\end{array}
\]
where \( A_2 = \{(0, a) \mid a \in 6\mathbb{Z}\} \cup \{(1, a) \mid a \in 3 + 6\mathbb{Z}\}, \ A_3 = \{(0, a) \mid a \in 6\mathbb{Z}\} \cup \{(1, a) \mid a \in 2 + 6\mathbb{Z}\} \cup \{(2, a) \mid a \in 4 + 6\mathbb{Z}\}, \) \( \alpha_i \) denotes projections on the second coordinate, while \( \pi, \pi \) on the first one. If we put \( Z_2 \sqcup Z_2 = \{(a, 0) \mid a \in Z_2\} \cup \{(0, b) \mid b \in Z_2\}, \) the morphism \( \pi \) can be described by the conditions \( \pi^{-1}(a, 0) = 2a + 6\mathbb{Z} \) and \( \pi^{-1}(0, b) = 3b + 6\mathbb{Z}. \)

(2) The category \( \text{Top} \) of topological spaces with continuous maps is extensive, but it is not UD, since it does not satisfy the condition (UD1): the inverse of a continuous bijection need not be continuous, hence \( \text{Top} \) is not balanced.

Although UD-categories are not extensive in general, we show that all objects of a UD-category satisfy dual version of the mono-coextensivity condition introduced in the paper [14], which allows us to prove that every object can be uniquely decomposed.

We suppose in the sequel that \( (\mathcal{C}, U) \) is a UD-category, i.e., a concrete category over \( \text{Set} \) satisfying all axioms (UD1)–(UD6) and the notions of objects and morphisms refer to objects and morphisms of the underlying category \( \mathcal{C}. \) First, we prove a key observation that the description of coproducts in both categories of acts [20, Proposition II.1.8, Remark II.1.16] can be easily unified within the context of UD-categories.

If \( \mathcal{A} \) is a family of objects, the corresponding coproduct will be designated \( (\coprod \mathcal{A}, (\nu_i)_{A_i \in \mathcal{A}}) \) where \( \nu_A \) is said to be the \textit{structural morphisms} of a coproduct for each \( A \in \mathcal{A}. \)

\textbf{Proposition 3.3.} Let \( \mathcal{A} = (A_i)_{i \in I} \) be a set of objects and \( (\coprod_i A_j, (\nu_i)_{i \in I}) \) be a coproduct of \( \mathcal{A}. \) Then there exist an inclusion morphism \( \iota_{ij} \in \text{Mor}(\coprod_i A_i, \coprod_i A_j) \) and morphisms \( \mu_j \in \text{Mor}(A_j, \coprod_i A_i) \) satisfying \( \iota_{ij} \mu_j = \nu_j \) for each \( j. \) Furthermore, \( (\coprod_i A_i, \iota_{ij}, \mu_j) \) is a coproduct of \( \mathcal{A} \) and \( U(\coprod_i A_i) = \bigcup_{i \in I} U(A_i). \)

\textit{Proof.} Note that the inclusion morphism \( \iota_{ij} \in \text{Mor}(\coprod_i A_i, \coprod_i A_j) \) exists by (UD4) and morphisms \( \mu_j \in \text{Mor}(A_j, \coprod_i A_i) \) with \( \iota_{ij} \mu_j = \nu_j \) exist by (UD3) for each \( j. \) Using the universal property of a coproduct, we obtain a morphism \( \varphi \) such that \( \varphi \iota_{ij} = \mu_j \) for every \( j, \) i.e. the left square of the diagram

\[
\begin{array}{ccc}
A_j & \rightarrow & \coprod_i A_i \\
\iota_{ij} \downarrow & & \varphi \downarrow \\
A_j & \rightarrow & \coprod_i A_i
\end{array}
\]

commutes in \( \mathcal{C} \) and we will show that the right square commutes as well.

Since \( \iota_{ij} \varphi \iota_{ij} = \iota_{ij} \mu_j = \nu_j \) for each \( j, \) we get again by the colimit universal property that \( \varphi \iota_{ij} = \iota \coprod A_i. \) Since \( \varphi \) is a left inverse which is a monomorphism, both \( \iota_{ij} \) and \( \varphi \) are isomorphisms. Hence \( U(\coprod_i A_i) = U(\coprod_i A_i) = U(\coprod_i A_i) \) and \( (\coprod_i A_i, (\iota_{ij}), (\mu_j)) = (\coprod_i A_i, (\varphi \iota_{ij})) \) is a coproduct of the family \( \mathcal{A}. \)

Let \( A \) be an object, and \( (A_j, \iota_j) \) be subobjects such that \( \iota_j \) is the inclusion morphism for each \( j \in J. \) We say that \( ((A_j, \iota_j), j \in J) \) is a \textit{decomposition} of \( A \) if \( (A_j, (\iota_j)_{j \in J}) \) is a coproduct of the family \( (A_j, j \in J). \) Note that we have defined the decomposition for sets of subobjects with inclusion morphisms mainly for clarity of exposition: had we defined it for general subobjects, we would have got the same result using (UD2) and Lemma 2.2 afterward.

The following easy consequence of Proposition 3.3 describes a natural decomposition of a coproduct in \( \mathcal{C}. \)

\textbf{Corollary 3.4.} Let \( \mathcal{A} = (A_j, j \in J) \) be a family of objects and \( (A_j, (\nu_j)_j) \) be a coproduct of \( \mathcal{A}. \) Then for each \( j \in J \) there exists an inclusion morphism \( \iota_j \in \text{Mor}(A_j, A) \) such that \( ((A_j, \iota_j), j \in J) \) forms a decomposition of \( A. \)

It is well-known that there is a canonical isomorphism \( \coprod_i (\coprod_i A_i) \cong \coprod_i (\bigcup_{i \in I} A_i) \) for every family of sets of objects \( A_i, \) \( i \in I \) in any coproduct-complete category with an initial object \( \theta, \) hence \( \theta \sqcup A \cong A. \) Furthermore, let us formulate elementary but useful property of decompositions.

\textbf{Corollary 3.5.} Let \( A \) be an object and \( (A_i, i \in I) \) a family of sets of subobjects \( (\mathcal{C}, \iota_i) \) of \( A \) such that \( \iota_{ij} \) is the inclusion morphism and let \( B_i = \bigcup_{(\mathcal{C}, \iota_i) \in \mathcal{A}_i} \mathcal{C} \iota_i \) and \( \iota_i \) be the inclusion morphism ensured by (UD4) for each \( i \in I. \) The following conditions are equivalent:

\( \star \)

\( \star \)

\( \star \)

\( \star \)
(1) For each \( i \in I \), \( A_i \) forms a decomposition of the object \( B_i \) and \( ((B_i, \iota_i), i \in I) \) is a decomposition of \( A \).

(2) \( \bigcup_{i \in I} A_i \) is a decomposition of \( A \).

As the morphism of an initial object to an arbitrary object is a monomorphism by (UD1), for every object \( A \), there exists a subobject denoted by \( (\theta_A, \vartheta_A) \) with the inclusion morphism \( \vartheta_A \) and \( \theta_A \cong \theta \). Note that an initial object \( \theta \) has no proper subobject.

It is not a priori clear that the existence of different (even though necessarily isomorphic) initial objects will not cause obstacles. However, the following observations, that will also turn out to be useful for further dealing with decompositions of objects in a general UD-category, show that the situation is favourable.

**Lemma 3.6.** If \( A \) is an object and \((S, \iota)\) is a subobject of \( A \) with the inclusion morphism \( \iota \), then

1. \( U(\theta_S) = U(\theta_A) \),
2. \( S \cong \theta \) if and only if \( U(S) = U(\theta_A) \) if and only if \( U(S) \subseteq U(\theta_A) \).

**Proof.** (1) Since there exists the unique isomorphism \( \tau : \theta_S \to \theta_A \) and both \( \iota \theta_S \tau = \vartheta_A \) and \( \iota \theta_S \tau = \vartheta_A \iota \tau^{-1} \) are inclusion morphisms, we get the equality \( U(\theta_S) = U(\theta_A) \).

(2) If \( S \cong \theta \), then \( U(S) = U(\theta_S) = U(\theta_A) \). The implication \( U(S) = U(\theta_A) \Rightarrow U(S) \subseteq U(\theta_A) \) is clear and, proving indirectly, suppose that \( S \) is not isomorphic to \( \theta \). Then \( \theta_S \) is a monomorphism by (UD1) which is not an epimorphism. Hence \( U(\theta_S) \) is not surjective by Lemma 2.1(2) and so \( U(\theta_A) = U(\theta_S) \not\subseteq U(S) \) by (1). We have proved that \( U(S) \not\subseteq U(\theta_A) \). \( \square \)

Now, we formulate a natural description of decompositions of objects and of its subobjects.

**Proposition 3.7.** Let \( A \) be an object and \((A_j, \iota_j)\) be a subobject of \( A \) with the inclusion morphism \( \iota_j \) into \( A \) for every \( j \in J \). Then \( ((A_j, \iota_j), j \in J) \) is a decomposition of \( A \) if and only if \( U(A) = \bigcup_j U(A_j) \) and \( U(A_i) \cap \bigcup_{j \neq i} U(A_j) = U(\theta_A) \) for each \( i \in J \).

**Proof.** Let \( ((A_j, \iota_j), j \in J) \) be a decomposition of \( A \). Then \( U(A) = \bigcup_j U(A_j) \) and \( U(A_j) \cap \bigcup_{j \neq i} U(A_j) = U(\theta_A) \) for each \( j \in J \) and so \( U(A_j) \) is a subobject of \( A \) with the inclusion morphism \( \theta_j \). Then there exists a morphism \( \varphi \in \text{Mor}(\prod_j A_j, \theta) \) such that \( \phi \iota_j = \iota \tau_j \) for all \( j \) by the universal property of a coproduct. Since \( U(A) = \bigcup_j U(A_j) \subseteq U(\varphi(U(\prod_j A_j))) \), the mapping \( U(\varphi) \) is surjective. Let \( U(\varphi)(a) = U(\varphi)(b) \) for elements \( a, b \in U(\prod_j A_j) \). Then there are indexes \( j_0, j_1 \in J \) and elements \( \tilde{a} \in U(A_{j_0}) \), \( \tilde{b} \in U(A_{j_1}) \) for which \( a = U(\nu_{j_0})(\tilde{a}) \), \( b = U(\nu_{j_1})(\tilde{b}) \) by Proposition 3.8, hence \( \tilde{a} = \tilde{b} \) and \( U(\nu_{j_0})(\tilde{a}) = U(\nu_{j_1})(\tilde{b}) \). Since \( \tilde{a} = \tilde{b} \) and \( U(\nu_{j_0})(\tilde{a}) = U(\nu_{j_1})(\tilde{b}) \), we have \( U(\nu_{j_0})(a) = U(\nu_{j_1})(b) \) and \( U(\nu_{j_0})(b) = U(\nu_{j_1})(a) \). Since \( U(\varphi) \) is surjective and injective, \( \varphi \) is an isomorphism by Lemma 2.1(3). Thus \( (A_j, \iota_j) \) is a decomposition of \( A_j \). Note that the argument of the reverse implication depends strongly on the fact that \( (C, U) \) is a concrete category over \( Set \).

**Lemma 3.8.** Let \( A \) be an object, \((B, \mu)\) its subobject with inclusion morphism \( \mu \) and let \( ((A_j, \iota_j), j \in J) \) be a decomposition of \( A \). Then there exists a decomposition \( (A_j^i \cap B^i, \mu_j^i), j \in J \) of \( B \) with inclusion morphisms \( \mu_j^i \), \( j \in J \), such that \( U(B_j) = U(B) \cap U(A_j) \) for each \( j \in J \).
Proof. We put \( B_j = A_j^\gamma \cap B^\delta \) and it is enough to take a morphism \( \mu_j \in \text{Mor}(B_j, B) \) such that \( \mu \mu_j \) is the inclusion morphism \( B_j \to A \) and \( U(B_j) = U(B) \cap U(A_j) \) for each \( j \in J \), which exists by (UD4) and (UD3). Since \( U(\mu \mu_j) = U(\mu)U(\mu_j) \) and \( U(\mu) \) are inclusions, \( \mu_j \) is an inclusion morphism for all \( j \in J \). Since \( \bigcup_j U(B_j) = \bigcup_j U(B) \cap U(A_j) = U(B) \) and

\[
U(\theta_A) \subseteq U(B_i) \cap \bigcup_{j \neq i} U(B_j) = U(B) \cap U(A_i) \cap \bigcup_{j \neq i} U(A_j) = U(\theta_A)
\]

for each \( i \in J \), we obtain that \( ((B_j, \mu_j), j \in J) \) is a decomposition by Proposition 3.7. \qed

Let us recall the dual version of terminology from \[16\]. If \( \mathcal{M} \) is a class of morphisms, we say that a commutative square

\[
P \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\delta} M.
\]

is \( \mathcal{M} \)-pull-back provided it is a pull-back diagram with \( \alpha, \beta \in \mathcal{M} \).

We denote by \( \mathcal{M} \) the class of all coproduct structural morphisms in the rest of the section. Remark that all coproduct structural morphisms are monomorphisms by (UD5).

An object \( M \) is said to be mono-extensive if for each pair of morphisms \( \gamma, \delta \in \mathcal{M} \) with codomain \( M \) there exist an object \( P \) and morphisms \( \alpha, \beta \in \mathcal{M} \) with domain \( P \) such that

\[
P \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\delta} M.
\]

is \( \mathcal{M} \)-pull-back and in every commutative diagram

\[
P_1 \xrightarrow{} A \xleftarrow{} P_2
\]

\[
\xrightarrow{} B_1 \xrightarrow{} B_1 \sqcup B_2 \xleftarrow{} B_2
\]

where \( M \cong B_1 \sqcup B_2 \), the bottom row is a coproduct diagram, and the vertical morphisms belong to \( \mathcal{M} \), the top row is coproduct if and only if both the squares are \( \mathcal{M} \)-pull-backs. Finally, a category is called mono-extensive if all objects are mono-extensive.

Lemma 3.9. If \( \gamma \in \text{Mor}(A, M) \) and \( \delta \in \text{Mor}(B, M) \) are monomorphisms, then there exist monomorphisms \( \alpha \in \text{Mor}(A^\gamma \cap B^\delta, A) \) and \( \beta \in \text{Mor}(A^\gamma \cap B^\delta, B) \) such that \( \gamma \alpha = \delta \beta = \iota_\cap \) for the inclusion morphism \( \iota_\cap \) and

\[
A^\gamma \cap B^\delta \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\delta} M.
\]

is a pull-back diagram.

Proof. Since the existence of morphisms \( \alpha \in \text{Mor}(A^\gamma \cap B^\delta, A) \) and \( \beta \in \text{Mor}(A^\gamma \cap B^\delta, B) \) follows from (UD3) and (UD4) and both are monomorphisms as \( \gamma \alpha = \delta \beta = \iota_\cap \) is a monomorphism, it is enough to check the pull-back universal property. If

\[
P \xrightarrow{\tilde{\alpha}} A
\]

\[
\xrightarrow{\tilde{\beta}} \xrightarrow{} B \xrightarrow{\delta} M.
\]

is a commutative diagram, then there exists \( \tau \in \text{Mor}(P, A^\gamma \cap B^\delta) \) satisfying

\[
\gamma \alpha \tau = \gamma \tilde{\alpha} = \delta \beta = \delta \beta \tau
\]
again by (UD3), which implies that $\alpha \tau = \tilde{\alpha}, \beta \tau = \tilde{\beta}$ as $\gamma$ and $\delta$ are monomorphisms. Since $\alpha$ and $\beta$ are monomorphisms as well, the same argument gives the uniqueness of $\tau$. \hfill \Box

Let us formulate a description of pull-back diagrams along inclusion morphisms.

**Lemma 3.10.** Let all morphisms of a commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & M.
\end{array}
$$

be inclusion morphisms. Then it is a pull-back diagram if and only if $U(P) = U(A) \cap U(B)$.

**Proof.** The direct implication follows from Lemma 3.9 and (UD4), since $P$ and $A^\gamma \cap B^\delta$ are isomorphic.

To prove the reverse implication, let us suppose that the commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & M,
\end{array}
$$

where all morphisms are inclusion morphisms, satisfies the condition $U(P) = U(A) \cap U(B)$. Since we have a pull-back diagram

$$
\begin{array}{ccc}
A^\gamma \cap B^\delta & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & M,
\end{array}
$$

by Lemma 3.9 there exists a morphism $\tau \in \text{Mor}(P, A^\gamma \cap B^\delta)$ satisfying $\alpha \tau = \tilde{\alpha}, \beta \tau = \tilde{\beta}$. Since both mappings $U(\alpha)$ and $U(\tilde{\alpha})$ are inclusions and $U(\alpha)U(\tau) = U(\tilde{\alpha}), U(\tau)$ is an inclusion as well. By the hypothesis $U(\tau)$ is surjective, hence $U(\tau)$ is a bijection and so $\tau$ is an isomorphism by Lemma 2.13. \hfill \Box

**Lemma 3.11.** The class $\mathcal{M}$ of all coproduct structural morphisms is closed under composition, coproducts and pull-back morphisms.

**Proof.** Applying Lemma 2.2 we may suppose without loss of generality that all considered coproduct structural morphisms are inclusion morphisms.

Let $\phi_0 \in \text{Mor}(A_0, B)$ and $\psi \in \text{Mor}(B, C)$ be coproduct structural morphisms. Then there exist $\phi_1 \in \text{Mor}(A_1, B)$ and $\phi_2 \in \text{Mor}(A_2, C)$ such that $((A_0, \phi_0), (A_1, \phi_1))$ is a decomposition of $B$ and $((B, \psi), (A_2, \phi_2))$ forms a decomposition of $C$. Now it remains to apply Proposition 3.7. Since $U(A_0) \cup U(A_1) \cup U(A_2) = U(B) \cup U(A_2) = U(C)$ and $U(A_i) \cap (U(A_j) \cup U(A_k)) = U(\emptyset)$ for all $i \neq j \neq k \neq i$, we can see that $((A_0, \psi \phi_0), (A_1, \psi \phi_1), (A_2, \phi_2))$ forms a decomposition of $C$, hence $\psi \phi_0 \in \mathcal{M}$.

The argument for a coproduct morphism is similar; if $\phi_i \in \text{Mor}(A_i, B_i), i \in I$, are coproduct structural morphisms, then there exists $\phi_i \in \text{Mor}(A_i, B_i)$ such that $((A_i, \phi_i), (A_i, \phi_i))$ forms a decomposition of $B_i$ for each $i \in I$. Then $((\prod_i A_i, \prod_i \phi_i), (\prod_i A_i, \prod_i \phi_i))$ is a decomposition of $\prod_i B_i$ and since $U(A_i) \cup \bigcup_i A_i = U(\prod_i B_i)$ and $U(A_i) \cap \bigcup_i A_i = U(\emptyset)$, $\prod_i \phi_i$ is a coproduct structural morphism by Proposition 3.7.
Finally, let

\[
P \xrightarrow{\alpha} A \\
\beta \downarrow \qquad \downarrow \gamma \\
B \xrightarrow{\delta} M.
\]

be a pull-back diagram with \(\gamma, \delta \in \mathcal{M}\). Then \(U(P) = U(A) \cap U(B)\) by Lemma 3.10 and there exists a decomposition \(((A, \gamma), (\tilde{A}, \tilde{\gamma}))\) of \(M\). Thus \((P, \beta)\) and symmetrically \((P, \alpha)\) are members of decompositions of \(M\), hence \(\alpha\) (\(\beta\), resp.) are coproduct structural morphisms by Lemma 3.8. \(\square\)

**Proposition 3.12.** Any UD-category is mono-extensive.

**Proof.** The existence of \(\mathcal{M}\)-pull-backs follows immediately from Lemma 3.9 since all coproduct structural morphisms are monomorphisms by (UD5). Let

\[
P_1 \longrightarrow P_1 \sqcup P_2 \longleftarrow P_2 \\
\downarrow \qquad \qquad \qquad \downarrow \\
B_1 \longrightarrow B_1 \sqcup B_2 \longleftarrow B_2
\]

be a commutative diagram with coproduct diagrams on both the rows where all morphisms belong to \(\mathcal{M}\). We may suppose without loss of generality by Lemma 2.2 that all morphisms are inclusion morphisms and we will show that both the squares are \(\mathcal{M}\)-pull-backs by applying the criterion Lemma 3.10. Since, \(U(P_i) \subseteq U(B_i)\) for \(i = 1, 2\) and \(U(B_1) \cap U(B_2) = U(\emptyset)\) we obtain

\[
U(P_1) \subseteq U(P_1 \sqcup P_2) \cap U(B_1) = (U(P_1) \cup U(P_2)) \cap U(B_1) = \\
= (U(P_1) \cap U(B_1)) \cup (U(P_2) \cap U(B_1)) = U(P_1) \cap U(\emptyset) = U(P_1),
\]

hence \(U(P_1) = U(P_1 \sqcup P_2) \cap U(B_1)\) and the left square is a pull-back. The argument for the right square is symmetric.

Conversely, let

\[
P_1 \longrightarrow A \longleftarrow P_2 \\
\downarrow \qquad \qquad \downarrow \\
B_1 \longrightarrow B_1 \sqcup B_2 \longleftarrow B_2
\]

be a commutative diagram with all vertical morphisms belonging to \(\mathcal{M}\) and with a coproduct diagram in the bottom row such that both the squares are \(\mathcal{M}\)-pull-backs. By Lemma 3.11 we have that all morphisms of the diagram belong to \(\mathcal{M}\). We may suppose that all morphisms are inclusion morphisms again and we need to prove that the top row forms a decomposition of the object \(A\). Since \(U(P_i) = U(A) \cap U(B_i)\) for \(i = 1, 2\) by Lemma 3.11, we can easily compute that

\[
U(P_1) \cap U(P_2) = (U(A) \cap U(B_1)) \cap (U(A) \cap U(B_2)) = U(A) \cap (U(B_1 \cap U(B_2)) = U(\emptyset),
\]

\[
U(P_1) \cup U(P_2) = (U(A) \cap U(B_1)) \cup (U(A) \cap U(B_2)) = U(A) \cup (U(B_1 \cup U(B_2)) = U(A),
\]

by Proposition 3.7, which implies that the top row is a coproduct diagram. \(\square\)

Recall that an object of a UD-category \((\mathcal{C}, U)\) is indecomposable provided it is indecomposable object of \(\mathcal{C}\) and let us say that an object \(A\) is *uniquely decomposable* if there exist a family of indecomposable objects \((A_j, j \in J)\) such that \(A \cong \coprod_{j \in J} A_j\) and for each family \((\tilde{A}_j, j \in J)\) satisfying \(A \cong \coprod_{j \in J} \tilde{A}_j\) there exists a bijection \(b : J \rightarrow \tilde{J}\) such that \(A_j \cong \tilde{A}_{b(j)}\) for each \(j \in J\).

**Proposition 3.13.** Every object of a mono-extensive category possessing a decomposition into indecomposable objects is uniquely decomposable.

**Proof.** It follows from the dual version of [16, Remark 2.3] using the dual version of [16, Theorem 2.1] which can be applied by Lemma 3.11. \(\square\)

The next assertion presents a natural construction of indecomposable objects in a UD-category.
Lemma 3.14. Let \((A_i, \nu_i), i \in I\) be a family of subobjects of an object \(A\) such that \(A_i\) is indecomposable for each \(i \in I\). If \(\bigcap_{i \in I} U(A_i^\nu_i) \neq U(\theta_A)\), then there exists an inclusion morphism \(\nu \in \text{Mor}(U(A_i^\nu_i), A)\) such that \((\bigcup_{i \in I} A_i^\nu_i, \nu)\) is an indecomposable subobject of \(A\).

Proof. Put \(A' = \bigcup_{i \in I} A_i^\nu_i\) and let \(\nu \in \text{Mor}(A', A)\) be the inclusion morphism ensured by (UD4). Since \((A', \nu)\) is a subobject, we may suppose without loss of generality that \(A = A'\). Remark that the proof repeats the argument of the proof of \([20, \text{Lemma I.5.9}].\)

Assume that \(((B_0, \iota_0), (B_1, \iota_1))\) is a decomposition of \(A\) such that \(U(B_i) \neq U(\theta_A)\) for both \(i = 0, 1\). Since \(\bigcap_{i \in I} U(A_i^\nu_i) \neq U(\theta_A)\) and \(U(B_0) \cap U(B_1) = U(A)\) by Proposition 3.7, there exists \(j\) for which \(U(B_j) \cap U(A_i^\nu_i) \neq U(\theta_A)\), we may w.l.o.g. assume that \(j = 0\). Moreover, there exists \(i\) such that \(U(B_j) \cap U(A_i) \neq U(\theta_A)\). Thus \(U(B_j) \cap U(A_i) \neq U(\theta_A)\) for both \(j = 0, 1\). Then by Lemma 3.8 there exists a decomposition \(((\hat{B}_0, \iota_0), (\hat{B}_1, \iota_1))\) of \(A\) such that \(U(B_j) \neq U(\theta_A)\) for both \(j = 0, 1\). Hence we obtain by Proposition 3.7 a contradiction with the hypothesis that \(A_i\) is indecomposable.

Now we can formulate a version of \([20, \text{Theorem I.5.10}]\) valid in a general UD-category:

Theorem 3.15. Every noninitial object of a UD-category is uniquely decomposable.

Proof. First, we prove the existence of indecomposable decomposition of a noninitial object \(A\). For \(a \in U(A) \setminus U(\theta_A)\), which exists by Lemma 3.10, consider the set

\[ I_a = \{C \mid (C, \nu_C) \text{ is an indecomposable subobject of } A \text{ and } a \in U(C)\} \]

and let \((A_a, \iota_a)\) be a subobject with the inclusion map, where \(A_a = \bigcup_{(C, \nu_C) \in I_a} C^{\nu_C}\), which exists by (UD4). Then \((A_a, \iota_a)\) is an indecomposable subobject of \(A\) by Lemma 3.14.

Furthermore, if \(a = b\) then either \(U(A_a) = U(A_b)\), or \(U(A_a) \cap U(A_b) = U(\theta_A)\). Indeed, let \(U(A_a) \cap U(A_b) \neq U(\theta_A)\), take \(z \in (U(A_a) \cap U(A_b)) \setminus U(\theta_A)\), which exists by and (UD4) and consider the indecomposable object \(A_z\). Since \(z \in U(A_a)\), we have \((A_a, \iota_a) \in I_a\), hence \((A_a, \iota_a)\) is a subobject of \(A_z\), similarly for \(b\) and vice versa. Therefore \(U(A_z) = U(A_a) = U(A_b)\).

Note that for each \(a \in U(A)\) there exists an indecomposable subobject \((C, \nu_C)\) of \(A\) such that \(U(C)\) contains \(a\) by (UD6), hence \(a \in A_z\). Moreover, as \(A\) is not isomorphic to \(\theta\), we get that \(U(A) = \bigcup_{a \in U(A) \cup U(\theta_A)} U(A_a)\), and we have proved that the representative set of subobjects of the form \((A_x, \iota_x)\) is the desired decomposition.

The uniqueness follows from Propositions 3.12 and 3.16. \(\square\)

The following example shows that objects of a mono-extensive category need not be decomposable in general, which illustrates that the existence part of the assertion of Theorem 3.15 depends strongly on the axiom (UD5).

Example 3.16. Consider a category whose objects are infinite pointed sets (i.e. sets with one base point \(\bullet\)) and an initial one-element pointed set \(\{\bullet\}\) where morphisms are exactly injective maps compatible with the point \(\bullet\).

Then coproducts and pull-backs can be described similarly as in the category of pointed acts: coproducts are disjoint unions with their base points glued together and pull-back diagrams are either of the form

\[
\begin{array}{ccc}
\gamma(A) \cap \delta(B) & \longrightarrow & A \\
\downarrow & & \downarrow \gamma \\
B & \delta \quad \longrightarrow & M,
\end{array}
\]

depending on infiniteness or finiteness of the set \(\gamma(A) \cap \delta(B)\) (cf. Lemma 3.9). Thus using similar arguments as in the proof of Proposition 3.12 we can see that the category is mono-extensive with arbitrary coproducts.

However, since every infinite pointed set can be expressed as a coproduct of two infinite pointed subsets, the category contains no indecomposable object.
4. Projective objects

Recall that \((\mathcal{C}, U)\) is supposed to be a UD-category. We say that an object \(P \in \mathcal{C}\) is projective, if for any pair of objects \(A, B \in \mathcal{C}\) and any pair of morphisms \(\pi : A \to B, \alpha : P \to B\), where \(\pi\) is an epimorphism, there exists a morphism \(\alpha\) such that \(\alpha = \pi\), i.e. any diagram

\[
\begin{array}{c}
\xrightarrow{\pi} \\
A \\
\downarrow \alpha \\
B
\end{array}
\]

in \(\mathcal{C}\) with \(\pi\) an epimorphism, can be completed into a commutative diagram

\[
\begin{array}{c}
P \\
\downarrow \pi \\
A \\
\xrightarrow{\pi} \\
B
\end{array}
\]

\[
P \\
\downarrow \alpha \\
P
\]

Note that the notion of projectivity is one of basic tools of category theory and issue of description of projective objects seems to be important task in research of any (concrete) category (see e.g. [1 Chapter 9] or [20 Section III.17]). The main goal of the section is to confirm that the structure of projective objects of the underlying category \(\mathcal{C}\) of a UD-category \((\mathcal{C}, U)\) can be described as a coproduct of indecomposable projective objects in accordance with the case of categories of acts.

**Lemma 4.1.** A coproduct of a family \((P_i, i \in I)\) of projective objects is projective.

**Proof.** It is well-known, see e.g. [1 dual version of 10.40]. \(\square\)

**Lemma 4.2.** If a coproduct of objects \(P = \coprod_{i \in I} P_i\) is projective, then each object \(P_i, i \in I\), is projective.

**Proof.** As \(\coprod_{i \in I} P_i \cong P \sqcup \left( \coprod_{i \notin I} P_i \right)\), it is enough to prove that for any pair of objects \(P_0, P_1\), if \(P_0 \sqcup P_1\) is projective, then \(P_0\) is projective.

Let the projective situation

\[
\begin{array}{c}
P_0 \\
\downarrow \alpha \\
A \\
\xrightarrow{\pi} \\
B
\end{array}
\]

be given and let \(\lambda_X : X \to P_0 \sqcup P_1, \mu_X : X \to A \sqcup P_1\) for \(X \in \{P_0, P_1\}\) be structural coproduct morphisms, which all are monomorphisms by (UD5). Denote by \(\tilde{\alpha} : P_0 \sqcup P_1 \to B \sqcup P_1\) a coproduct of morphisms \(\alpha : P_0 \to B\) and \(1_{P_1} : P_1 \to P_1\) which is uniquely determined by the universal property of the coproduct \(P_0 \sqcup P_1\), i.e. \(\tilde{\alpha}\lambda_{P_0} = \mu_B\alpha\) and \(\tilde{\alpha}\lambda_{P_1} = \nu_{P_1}\). Similarly, denote by \(\tilde{\pi} : A \sqcup P_1 \to B \sqcup P_1\) a coproduct of morphisms \(\pi : A \to B\) and \(1_{P_1} : P_1 \to P_1\), which means that \(\tilde{\pi}\mu_A = \nu_B\pi\) and \(\tilde{\pi}\mu_{P_1} = \nu_{P_1}\). It is easy to compute applying the universal property of the coproduct \(B \sqcup P_1\) that \(\tilde{\pi}\) is an epimorphism since both \(\pi\) and \(1_{P_1}\) are epimorphisms.

Hence we obtain another projective situation:

\[
\begin{array}{c}
P_0 \sqcup P_1 \\
\downarrow \tilde{\alpha} \\
A \sqcup P_1 \\
\xrightarrow{\tilde{\pi}} \\
B \sqcup P_1
\end{array}
\]

By the assumption, there exists a morphism \(\varphi \in \text{Mor}(P_0 \sqcup P_1, A \sqcup P_1)\) such that \(\tilde{\pi}\varphi = \tilde{\alpha}\). Let us show that \(U(\varphi_{\lambda_{P_0}})(U(P_0)) \subseteq U(\mu_A)(U(A))\).

By (UD2), (UD3) and (UD4) there exists a subobject \((S, i)\) of \(A \sqcup P_1\) with the inclusion morphism \(i\) satisfying \(U(S) = U(\varphi_{\lambda_{P_0}})(U(P_0)) \cap U(\mu_{P_1})(U(P_1))\). Since \(\mu_{P_1}\) is a monomorphism, there exists a monomorphism \(\sigma : S \to P_1\), such that \(\mu_{P_1}\sigma = i\). Then \(\tilde{\pi}i = \tilde{\pi}\mu_{P_1}\sigma = \nu_{P_1}\sigma\) is a monomorphism. Since

\[
U(\tilde{\pi})U(S) \subseteq U(\tilde{\pi})U(\varphi_{\lambda_{P_0}})(U(P_0)) = U(\tilde{\alpha}\lambda_{P_0})(U(P_0)) = U(\nu_B\alpha)(U(P_0)) \subseteq U(\nu_B)(U(B))
\]
We generalize arguments of [20, Propositions III.17.4 and III.17.7].

by Proposition 3.7 and Proposition 3.3 we get that $U(\varphi \lambda \rho_0)(U(P_0)) \subseteq U(\mu_A)(U(A)) \cup U(\mu_P)(U(P_1))$. In consequence, by (UD3) there exists a morphism $\tau : P_0 \to A$ such that $\mu_A \tau = \varphi \lambda \rho_0$; therefore $\tilde{\pi} \varphi \lambda \rho_0 = \tilde{\pi} \mu_A \tau = \nu_B \pi \tau$ and on the other hand $\tilde{\pi} \varphi \lambda \rho_0 = \tilde{\alpha} \lambda \rho_0 = \nu_B \alpha$. Finally, as $\nu_B \pi \tau = \nu_B \alpha$ and the morphism $\nu_B$ is a monomorphism by (UD5), we have $\pi \tau = \alpha$.

Now we are ready to characterize projective objects of UD-categories:

**Theorem 4.3.** An object of a UD-category is projective if and only if it is isomorphic to a coproduct of indecomposable projective objects.

**Proof.** If an object is projective, it possesses a decomposition by Theorem 3.15 which consists of projective objects by Lemma 3.14. The reverse implication follows immediately from Lemma 3.11.

Let $A$ and $B$ be a pair of objects. Recall that $B$ is a retract of $A$ if there are morphisms $f \in \text{Mor}(A,B)$ and $g \in \text{Mor}(B,A)$ such that $fg = 1_B$. The morphism $f$ is then called retraction and $g$ coretraction. Note that each retraction is an epimorphism and each coretraction is a monomorphism. An object $G$ of a category is said to be generator if for any object $A \in C$ there exists an index set $I$ and an epimorphism $\pi : \prod_{i \in I} G_i \to A$ where $G_i \simeq G$.

**Lemma 4.4.** If $C$ contains a generator $G$, then every indecomposable projective object is a retract of $G$.

**Proof.** We generalize arguments of [20] Propositions III.17.4 and III.17.7.

Let $P$ be an indecomposable projective object. Since $G$ is a generator, there are coproducts $\prod_{i \in I} G_i$ of objects $G_i \cong G$ with structural morphisms $\nu_i \in \text{Mor}(G_i, \prod_{i \in I} G_i)$ and an epimorphism $\pi \in \text{Mor}(\prod_{i \in I} G_i, P)$. Moreover, there exists a (mono)morphism $\gamma \in \text{Mor}(P, \prod_{i \in I} G_i)$ for which $\pi \gamma = 1_P$ due to the projectivity of $P$. Note that $P \cong P^\gamma$ by Lemma 2.1 and there exists a decomposition $((H_i, \mu_i), i \in I)$ of $P^\gamma$ for which $U(H_i) = U(P^\gamma) \cap U(G_i^{\mu_i}) = U(\gamma)(U(P)) \cap U(\nu_i)(U(G_i))$ for each $i \in I$ by Lemma 3.3. As $P^\gamma$ is indecomposable, there exists an $i \in I$ such that $U(\gamma)(U(P)) = U(P^\gamma) \subseteq U(\nu_i)(U(G_i))$ by Proposition 3.7 hence there exists a morphism $\varphi \in \text{Mor}(P,G_i)$ such that $\nu_i \varphi = \gamma$ by (UD3). Thus $\nu_i \varphi = \gamma = 1_P$ which shows that $\nu_i \varphi$ is the desired retraction.

5. **Connected objects**

In this section we describe connected objects in a UD-category $(C,U)$.

Let $C$ be an object, $A = (A_i, i \in I)$ a family of objects and $(\prod_{i \in I} A_i, \{\nu_i\}_{i \in I})$ a coproduct of the family $A$. Using the covariant functor $\text{Mor}(C,-)$ from $C$ to Set, we define a natural morphism in the category $\Psi^C_A : \prod_{i \in I} \text{Mor}(C,A_i) \to \text{Mor}(C,\prod_{i \in I} A_i)$ which is the unique morphism such that the following square is commutative for all $i \in I$

\[
\begin{array}{ccc}
\text{Mor}(C,A_i) & \xrightarrow{\mu_i} & \prod_{i \in I} \text{Mor}(C,A_i) \\
\text{Mor}(C,A_i) \downarrow & & \Psi^C_A \downarrow \\
\text{Mor}(C,\prod_{i \in I} A_i) & \xrightarrow{\text{coproduct structural inclusion}} & \text{Mor}(C,\prod_{i \in I} A_i)
\end{array}
\]

where $\mu_i : \text{Mor}(C,A_i) \to \prod_{i \in I} \text{Mor}(C,A_i)$ is a coproduct structural inclusion in Set. Since coproducts of objects in Set are isomorphic to disjoint unions of the corresponding objects, we have $\prod_{i \in I} \text{Mor}(C,A_i) = \bigcup \text{Mor}(C,A_i)$ and we can describe $\Psi^C_A$ explicitly as $\Psi^C_A(\alpha) = \nu_i \alpha$ for each index $i$ satisfying $\alpha \in \text{Mor}(C,A_i)$. 

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It is worth mentioning that it is natural to consider morphisms
\[ \hat{\Psi}_A^C : \prod_{i \in I} \text{Mor}(U(C), U(A_i)) \to \text{Mor}(U(C), U \bigcup_{i \in I} U(A_i)) \]
as we deal with concrete category. Since \( U \) is a faithful functor, such a concept is equivalent to original one, however it seems to be technically more difficult.

**Lemma 5.1.** Let \( C \) be an object, \( I \) an index set consisting of at least two elements, \( A = \{ A_i \mid i \in I \} \) a family of objects. Suppose that \( (A, \{ \nu_i \}_{i \in I}) \) is a coproduct of the family \( A \) and \( \alpha, \beta \in \prod_{i \in I} \text{Mor}(C, A_i) = \bigcup_{i \in I} \text{Mor}(C, A_i) \).

1. If \( \alpha \neq \beta \) and \( \Psi_A^C(\alpha) = \Psi_A^C(\beta) \), then there exist \( i \neq j \) such that \( \alpha \in \text{Mor}(C, A_i), \beta \in \text{Mor}(C, A_j) \) and \( U(\nu_i\alpha)(U(C)) = U(\theta_A) = U(\nu_j\beta)(U(C)) \).
2. If \( i, j \in I \) and \( \alpha \in \text{Mor}(C, A_i) \) and \( \beta \in \text{Mor}(C, A_j) \) such that \( U(\alpha)(U(C)) = U(\theta_A) \) and \( U(\beta)(U(C)) = U(\theta_A) \), then \( \Psi_A^C(\alpha) = \Psi_A^C(\beta) \).
3. \( \Psi_A^C \) is injective (i.e. it is a monomorphism in the category \( \text{Set} \)) if and only if \( \text{Mor}(C, \theta) = \emptyset \).

**Proof.** (1) If there exists an \( i \) for which \( \alpha, \beta \in \text{Mor}(C, A_i) \), then \( \nu_i\alpha = \nu_i\beta \), hence \( \alpha = \beta \) as \( \nu_i \) is a monomorphism by (UD5). In consequence, the hypotheses \( \alpha \neq \beta \) and \( \Psi_A^C(\alpha) = \Psi_A^C(\beta) \) imply that there exists \( i \neq j \) such that \( \alpha \in \text{Mor}(C, A_i), \beta \in \text{Mor}(C, A_j) \), so we get
\[ U(\nu_i\alpha)(U(C)) = U(\Psi_A^C(\alpha))(U(C)) = U(\Psi_A^C(\beta))(U(C)) = U(\nu_j\beta)(U(C)). \]
Since \( U(\nu_i)(U(A_i)) \cap U(\nu_j)(U(A_j)) = U(\theta_A) \) by Corollary 3.6 and Proposition 3.7, and since
\[ U(\nu_i\alpha)(U(C)) = U(\nu_j\beta)(U(C)) \leq U(\nu_i)(U(A_i)) \cap U(\nu_j)(U(A_j)) = U(\theta_A), \]
we get that \( U(\nu_i\alpha)(U(C)) = U(\nu_j\beta)(U(C)) = U(\theta_A) \) by Lemma 3.9.

(2) Since \( U(\nu_i)(U(\theta_A)) = U(\theta_A) = U(\nu_j)(U(\theta_A)) \), we have
\[ U(\Psi_A^C(\alpha))(U(C)) = U(\nu_i\alpha)(U(C)) = U(\nu_j\beta)(U(C)) = U(\Psi_A^C(\beta))(U(C)) = U(\theta_A) \]
again by the same argument as in (1) using Corollary 3.4, Lemma ZeroSubobj, and Proposition 3.7.

As \( U \) is a faithful functor, both morphisms \( \Psi_A^C(\alpha), \Psi_A^C(\beta) \) can be viewed as elements of \( \text{Mor}(C, \theta_A) \).

Since \( |\text{Mor}(C, \theta_A)| = |\text{Mor}(C, \theta)| \leq 1 \) by (UD1), we get the required equality \( \Psi_A^C(\alpha) = \Psi_A^C(\beta) \).

(3) If \( \text{Mor}(C, \theta) \neq \emptyset \), there exists \( \alpha_i \in \text{Mor}(C, A_i) \) such that \( U(\alpha_i)(U(C)) = U(\theta_A) \) for all \( i \in I \) by Lemma 3.8 again.

Thus \( \Psi_A^C(\alpha_i) = \Psi_A^C(\alpha_j) \) for all \( i, j \in I \) by (2), which implies that \( \Psi_A^C \) is not injective.

On the other hand, if \( \Psi_A^C \) is not injective, then there exists \( i \) and \( \alpha \in \text{Mor}(C, A_i) \) such that \( U(\nu_i\alpha)(U(C)) = U(\theta_A) \) by (1). As \( \theta_A = \theta \), there exists a morphism in \( \text{Mor}(C, \theta) \) by (UD3).

Since there is no morphism of a nonempty act \( C \) into the empty act \( \emptyset \), all mappings \( \Psi_A^C \) are injective in the category \( \text{S-Act} \) by Lemma 5.1(3), similarly to the case of abelian categories (cf [18, Lemma 1.3]). Applying the same assertion, we can see that it is not the case of the category \( \text{S}_0-\text{Act}_0 \).

**Example 5.2.** If \( \text{S}_0 = (\text{S}_0, \cdot, 1) \) is a monoid with a zero element (for example \((\mathbb{Z}, \cdot, 1)\)), \( C \) is a right \( \text{S}_0 \)-act and \( A = \{ A_i \mid i \in I \} \) is a family of \( \text{S}_0 \)-acts contained in the category \( \text{S}_0-\text{Act}_0 \) satisfying \( |A| \geq 2 \), then the mapping \( \Psi_A^\text{C} \) is not injective by Lemma 5.1(3).

In particular, if we put \( \bar{C} = A_i = \{ 0 \} \) for every \( i \in I \), then \( \prod_i \text{Mor}(C, A_i) = |I| \) and \( |\text{Mor}(C, \prod_i A_i)| = 1 \), so the mapping \( \Psi_A^\text{C} \) glues together all morphisms of the arbitrarily large set \( \prod_i \text{Mor}(C, A_i) \).

Using the notation of the mapping \( \Psi_A^\text{C} \) we are ready to generalize abelian-category definition of a connected object to UD-categories.

We say that an object \( C \) is \( \mathcal{D} \)-connected (or connected with respect to \( \mathcal{D} \)), if the morphism \( \Psi_A^\text{C} \) is surjective for each family \( A \) of objects from the class \( \mathcal{D} \) and \( C \) is connected if it is \( O_C \)-connected for the class \( O_C \) of all objects of the category \( C \). Finally, an object \( C \) is called autoconnected, if it is \( \{ C \} \)-connected. Observe that every connected object is \( \mathcal{D} \)-connected for an arbitrary class \( \mathcal{D} \) of objects, in particular, it is autoconnected.
Let \( \mathcal{D} \) be a class of objects of the category \( \mathcal{C} \) and denote by \( \mathcal{D}^{\square} = \{ \coprod_i D_i \mid D_i \in \mathcal{D} \} \) the class of all coproducts of all families of objects of \( \mathcal{D} \).

Let us formulate a non-abelian version of [18, Proposition 2.1] (cf. also [15, Theorem 2.5]):

**Theorem 5.3.** The following conditions are equivalent for an object \( C \) and a class of objects \( \mathcal{D} \):

1. \( C \) is \( \mathcal{D} \)-connected,
2. for each pair of objects \( A_1 \in \mathcal{D} \) and \( A_2 \in \mathcal{D}^{\square} \) and each morphism \( f \in \text{Mor}(C, A_1 \sqcup A_2) \) there exists \( i \in \{1, 2\} \) such that \( f \) factorizes through \( \nu_i \),
3. for each pair of objects \( A_1 \in \mathcal{D} \) and \( A_2 \in \mathcal{D}^{\square} \) and each morphism \( f \in \text{Mor}(C, A_1 \sqcup A_2) \) there exists \( i \in \{1, 2\} \) such that \( U(f)(U(C)) \subseteq U(\nu_i)(U(A_i)) \),
4. for each family \( \{A_i, i \in I\} \) of objects of \( \mathcal{D} \) and each morphism \( f \in \text{Mor}(C, \coprod_{i \in I} A_i) \) there exists \( i \in I \) such that \( f \) factorizes through \( \nu_i \),
5. for each family \( \{A_i, i \in I\} \) of objects of \( \mathcal{D} \) and each morphism \( f \in \text{Mor}(C, \coprod_{i \in I} A_i) \) there exists \( i \in I \) such that \( U(f)(U(C)) \subseteq U(\nu_i)(U(A_i)) \),

where \( \nu_i \) denotes the structural morphism of a corresponding coproduct \( A_1 \sqcup A_2 \) or \( \coprod_{i \in I} A_i \).

**Proof.** (1)\( \Rightarrow \) (4) Let \( \mathcal{A} = \{A_i \mid i \in I\} \) be a family of objects of the class \( \mathcal{D} \) and \( f \in \text{Mor}(C, \coprod_{i \in I} A_i) \). Since \( C \) is \( \mathcal{D} \)-connected, the mapping \( \Psi_{\mathcal{D}}^C \) is surjective by definition, hence there exists \( i \) and \( \alpha \in \text{Mor}(C, A_i) \) such that \( f = \nu_i \alpha \).

(4)\( \Rightarrow \) (1) Let \( f \in \text{Mor}(C, \coprod_{i \in I} A_i) \) for a family \( \mathcal{A} = \{A_i, i \in I\} \subseteq \mathcal{D} \). Then there exists \( i \in I \) and \( \tilde{f} \in \text{Mor}(C, A_i) \) such that \( f = \nu_i \tilde{f} \); hence \( \Psi_{\mathcal{D}}^C(\tilde{f}) = f \).

(4)\( \Rightarrow \) (5) Since there exists \( i \in I \) and \( \tilde{f} \in \text{Mor}(C, A_i) \) for which \( f = \nu_i \tilde{f} \) we get

\[
U(f)(U(C)) = U(\nu_i)U(\tilde{f})(U(C)) \subseteq U(\nu_i)(U(A_i)).
\]

(5)\( \Rightarrow \) (4) It is a direct consequence of (UD3).

The equivalence (2)\( \Leftrightarrow \) (3) is a special case of (4)\( \Leftrightarrow \) (5).

(3)\( \Rightarrow \) (5) Let \( f \in \text{Mor}(C, \coprod_{i \in I} A_i) \) for a family \( \{A_i, i \in I\} \) of objects of \( \mathcal{D} \), put \( \mathcal{A} = \coprod_{i \in I} A_i \) and assume to the contrary that \( U(f)(U(C)) \not\subseteq U(\nu_i)(U(A_i)) \) for all \( i \in I \). Then by (3) \( U(f)(U(C)) \subseteq \bigcup_{i \in I} U(A_i^{\nu_i}) \) for each \( j \in I \), hence by Proposition 3.17

\[
U(f)(U(C)) \subseteq \bigcap_{j \in I \neq j} U(A_i^{\nu_i}) = U(\theta_A) \cup \bigcap_{j \in I} U(A) \setminus U(A_j^{\nu_j}) = U(\theta_A),
\]

a contradiction.

The implication (4)\( \Rightarrow \) (2) is clear, since \( A \in \mathcal{D}^{\square} \) if and only if there exists a family \( \mathcal{A} = \{A_i \mid i \in I\} \) of objects of \( \mathcal{D} \) satisfying \( A = \coprod_{i \in I} A_i \).

Let us reformulate the Theorem 5.3 for the particular (but important) case of connectedness:

**Corollary 5.4.** The following conditions are equivalent for an object \( C \):

1. \( C \) is connected,
2. for every pair of objects \( A_1 \) and \( A_2 \) and each morphism \( f \in \text{Mor}(C, A_1 \sqcup A_2) \) there exists \( i \in \{1, 2\} \) such that \( f \) factorizes through the structural coproduct morphism \( \nu_i \),
3. for every pair of objects \( A_1 \) and \( A_2 \) and each morphism \( f \in \text{Mor}(C, A_1 \sqcup A_2) \) there exists \( i \in \{1, 2\} \) such that \( U(f)(U(C)) \subseteq U(\nu_i)(U(A_i)) \) or \( U(f)(U(C)) \subseteq U(\nu_2)(U(A_2)) \), where \( \nu_i, i = 1, 2 \), is the structural coproduct morphism.

The following description of autoconnectedness presents another consequence of Theorem 5.3:

**Corollary 5.5.** The following conditions are equivalent for an object \( C \):

1. \( C \) is autoconnected,
2. for each morphism \( f \in \text{Mor}(C, \coprod_{i \in I} C_i) \), where \( C_i \simeq C \) for all \( i \in I \), there exists an index \( i \) such that \( U(f)(U(C)) \subseteq U(\nu_i)(U(C_i)) \),
3. for each morphism \( f \in \text{Mor}(C, \coprod_{i \in I} C_i) \), where \( C_i \simeq C \) for all \( i \in I \), there exists an index \( i \) such that \( U(f)(U(C)) \cap U(\nu_j)(U(C_j)) = U(\theta_{\prod_{i \neq j} C_i}) \) for each \( j \neq i \),

where \( \nu_i \) are the structural morphism of a coproduct \( \coprod_{i \in I} C_i \).
In order to obtain useful characterization of connected objects in a general UD-category we say that an object $B$ in an image of an object $A$ if there is a morphism $\pi \in \text{Mor}(A,B)$ with $\pi(\alpha)$ surjective. Observe that connected objects in the category $C$ are precisely objects whose every image is indecomposable.
of a cyclic act is cyclic, and so indecomposable, the Proposition 5.6 gives the result in the category
shall deal with all factors of an act, namely, connected objects in the category
Proof. By Corollary 6.2 we only need to prove the claim for the category $\mathsf{S}$-Act. Since any factor of a cyclic act is cyclic, and so indecomposable, the Proposition 5.6 gives the result in the category $\mathsf{S}$-Act$\circ$.

6. Categories of $S$-acts

Let $S = (S, \cdot, 1)$ be a monoid (possibly with zero 0) through the whole section. Recall that for $S$ both categories $\mathsf{S}$-Act$\circ$ and $\mathsf{S}$-Act of $S$-acts are UD-categories by Example 2.9. We will use basic properties of these categories summarized in the axiomatics (UD1)–(UD6) freely in the sequel. For standard terminology concerning the theory of acts we refer to the monograph [20].

6.1. Connected acts. The following consequence of Corollary 5.4 shows that the reverse implication of [20] Lemma I.5.36 holds true.

Lemma 6.1. Connected objects in the category $S$-Act are precisely indecomposable objects.

Proof. The assertion follows from [20] Lemma I.5.36 and Corollary 5.4. □

Since the category $S$-Act is infinitary extensive by Proposition 3.1, the previous lemma also follows from [20] Theorem 3.3.

Recall that a left $S$-act $A$ is called cyclic if there exists $a \in A$ for which $Sa = \{ sa \mid s \in S \} = A$, and $A$ is called locally cyclic if for any pair $a,b \in A$ there exists $c \in A$ such that $a,b \in Sc$. Since cyclic acts are locally cyclic and locally cyclic acts are indecomposable, we obtain an immediate consequence of Lemma 6.1.

Corollary 6.2. Every locally cyclic left act is connected in the category $S$-Act.

Furthermore, we prove a sufficient condition of connectedness for both considered categories of acts.

Proposition 6.3. Every cyclic left act is connected in both categories $\mathsf{S}$-Act and $\mathsf{S}$-Act$\circ$.

Proof. By Corollary 6.2 we only need to prove the claim for the category $\mathsf{S}$-Act$\circ$. Since any factor of a cyclic act is cyclic, and so indecomposable, the Proposition 5.6 gives the result in the category $\mathsf{S}$-Act$\circ$.

The corresponding variant of Lemma 6.1 as a criterion of connectedness in the category $\mathsf{S}$-Act$\circ$ shall deal with all factors of an act, namely, connected objects in the category $\mathsf{S}$-Act$\circ$ are precisely objects whose every image is indecomposable by Proposition 5.6.

The following example shows that in the case of the category $\mathsf{S}$-Act$\circ$ the implication in Proposition 5.3 cannot be inverted in general:

Example 6.4. Let $Z = (\mathbb{Z}, \cdot, 1)$ be a monoid with zero.

(1) Consider again $Z$-act $A = 2\mathbb{Z} \cup 3\mathbb{Z}$ from Example 5.7. Then $A$ is an indecomposable act which is not connected in the category $\mathsf{S}$-Act$\circ$. Indeed, if we consider the morphism $f_6 : A \to \mathbb{Z}_6$ given by $f_6(a) = a \mod 6$, then the image $f_6(A) = \{ 0, 2, 4 \} \cup \{ 0, 3 \}$ decomposes, hence it is not connected by Proposition 5.6.

(2) Every abelian group is connected in the category $Z-\mathsf{Act}$ since every $Z$-subact contains 0. More generally, for a monoid $S$ with zero, any $A \in S$-Act$\circ$ can be recognized as an object of $S$-Act and it becomes indecomposable in $Z-\mathsf{Act}$, hence connected by Lemma 6.1.
In compliance with [21, Definition 4.20] recall that for a subact \( B \) of an act \( A \) the Rees congruence \( \rho_B \) on \( A \) is defined by setting \( a_1\rho_2 \) if \( a_1 = a_2 \) or \( a_1, a_2 \in B \). The corresponding factor act \( A/B \) is called Rees factor of \( A \) by \( B \) then.

**Lemma 6.5.** Let \( A \in S\text{-Act}_0 \) and \( A_1 \) and \( A_2 \) be its proper subacts. If \( A = A_1 \cup A_2 \) and \( A_i \setminus (A_1 \cap A_2) \neq \emptyset \) for both \( i = 1, 2 \), then \( A \) is not connected in \( S\text{-Act}_0 \).

**Proof.** Consider the projection \( \pi \) of \( A \) onto the Rees factor \( A/\langle A_1 \cap A_2 \rangle \), which is decomposable into \( \pi(A_1) \cup \pi(A_2) \). Now use Corollary 5.4. \( \square \)

Note that a subact \( B \) of an act \( A \) can be viewed as an subobject \( (B, i) \) of \( A \) with the inclusion morphism \( i \) and recall that a subact \( B \) of a left \( S \)-act \( A \) (in \( S\text{-Act} \) or \( S\text{-Act}_0 \)) is called superfluous if \( B \cup C \neq A \) for any proper subact \( C \) of \( A \) (see [21, Definition 2.1]). An act is called hollow if each of its proper subacts is superfluous (see [21, Definition 3.1]). Note that the situation of Lemma 6.5 is precisely that of non-hollow acts.

**Proposition 6.6.** An \( S \)-act \( A \) is connected in the category \( S\text{-Act}_0 \) if and only if it is hollow.

**Proof.** Suppose \( A \) is hollow and it is not connected, i.e., there is a decomposable factor \( \pi(A) = A_1 \cup A_2 \) by Proposition 6.6. Then the preimages \( \pi^{-1}(A_1) \) and \( \pi^{-1}(A_2) \) form subacts of the act \( A \) such that \( A = \pi^{-1}(A_1) \cup \pi^{-1}(A_2) \), but neither of \( \pi^{-1}(A_i) \) equals \( A \). Since the decomposition is proper, we get a contradiction.

On the other hand, if \( A \) is not hollow, use the construction of Lemma 6.5. \( \square \)

6.2. **Steady monoids.** In accordance with the definition of steady rings (cf. [11, 13, 31]) we say that a monoid (resp. monoid with zero element) \( S \) is left steady (resp. left 0-steady) provided every connected left act in the category \( S\text{-Act} \) (resp. \( S\text{-Act}_0 \)) is necessarily cyclic. Note that every cyclic act is connected by Proposition 6.3.

**Example 6.7.** (1) If \( S \) is a group, then it is easy to see that indecomposable \( S \)-acts are cyclic. Hence connected \( S \)-acts are precisely cyclic ones by [21, Theorem I.5.10] (cf. Proposition 5.6) thus groups are (left) steady monoids.

(2) The Prüfer group \( \mathbb{Z}_{p^\infty} \) is a connected act over the monoid \( (\mathbb{N}, +, 0) \). Clearly, it is not a cyclic \( \mathbb{N} \)-act, as it is not a cyclic \( \mathbb{Z} \)-act. Hence \( (\mathbb{N}, +, 0) \) is not steady.

The following assertion presents an analogy of the description of connected projective objects in categories of modules.

**Proposition 6.8.** Let \( C \) be either \( S \text{-Act} \) or \( S\text{-Act}_0 \). Then a projective left act is connected in \( C \) if and only if it is cyclic.

**Proof.** For the direct implication note that, by Theorem 4.20 any projective act has a decomposition into indecomposable projective subacts, since both \( S\text{-Act} \) and \( S\text{-Act}_0 \) are UD-categories. As it is connected, it is indecomposable by Proposition 6.3. Now the result follows from Lemma 1.4 since \( S \) generates both of the categories \( S\text{-Act} \) and \( S\text{-Act}_0 \).

The reverse implication is a consequence of Proposition 6.3. \( \square \)

A monoid \( S \) is called left perfect (left 0-perfect) if each \( A \in S\text{-Act} \) (\( A \in S\text{-Act}_0 \)) has a projective cover, i.e., there exists (up to isomorphism unique) a projective \( S \)-act \( P \) and an epimorphism \( f : P \to A \) such that for any proper subact \( P' \subset P \) the restriction \( f|_{P'} : P' \to A \) is not an epimorphism (cf. [17, 19]).

Analogously to the case of perfect rings, which are known to be steady, we prove that 0-perfect monoids are 0-steady.

**Proposition 6.9.** Let \( S \) be a monoid with zero. If \( S \) is left 0-perfect, then connected objects of \( S\text{-Act}_0 \) are precisely cyclic acts. Hence \( S \) is left 0-steady.

**Proof.** Let \( A \) be a connected \( S \)-act and \( \pi \in \text{Mor}(P, A) \) be a projective cover of \( A \). Assume that \( P \) is not indecomposable with a nontrivial decomposition \((P_0, P_1)\). Then neither \( \pi(P_0) \) nor \( \pi(P_1) \) is not equal to \( A \) and \( B = \pi(P_0) \cap \pi(P_1) \) is a subact of \( A \). Then \((\pi(P_0)/B, \pi(P_1)/B)\) forms a
decomposition of the Rees factor $A/B$. Note that it is non-trivial, otherwise $\pi(P_0) \subseteq \pi(P_1)$ or $\pi(P_1) \subseteq \pi(P_0)$ which contradicts to the fact that $\pi(P_0) \neq \pi(P_1)$. Since every factor of $A$ is indecomposable by Lemma 6.1 we obtain a contradiction. \hfill \Box

6.3. Autoconnected acts. Let us formulate a direct consequence of Lemma 5.7 and Proposition 5.8.

Lemma 6.10. Let $C$ be an a autoconnected object in either $S$-$\text{Act}_0$ or $S$-$\text{Act}$ and let $\varphi$ be an endomorphism of $C$. Then $\varphi(C)$ is autoconnected and indecomposable, in particular, $C$ is indecomposable.

Now we can formulate a criterion of autoconnectedness in $S$-$\text{Act}$ (cf. [24, Lemma 4.1]):

Theorem 6.11. The following conditions are equivalent for an act $C \in S$-$\text{Act}$:

1. $C$ is autoconnected,
2. $C$ is connected,
3. $C$ is indecomposable.

Proof. The implication $(2) \Rightarrow (1)$ is clear, the implication $(1) \Rightarrow (3)$ follows from Lemma 6.10 and the equivalence $(2) \Rightarrow (3)$ is proved in Lemma 6.11. \hfill \Box

Example 6.12. Consider the monoid $\mathbb{Z} = (\mathbb{Z}, \cdot, 1)$ and the $\mathbb{Z}$-act $A = 2\mathbb{Z} \cup 3\mathbb{Z}$ from Examples 6.4 and 3.2. Then $A$ is autoconnected in $S$-$\text{Act}_0$, since for any morphism $A \to \bigsqcup_{i \in I} A_i$ with $A_i \cong A$, the component in which the image lies is determined by the image of the element 6.

The previous example shows that within the category $S$-$\text{Act}_0$ the class of autoconnected acts is in general strictly larger than the class of connected acts; whereas the following example will show that the class of autoconnected acts is in general strictly smaller than that of indecomposable objects, even for left perfect monoids.

Example 6.13. Consider the commutative monoid $S = (\{0, 1, s, s^2\}, \cdot, 1)$ (which could be embedded into the multiplicative monoid of the factor ring $\mathbb{Z}[s]/(s^3)$) with the following multiplication table:

|    | 0   | 1   | s   | s^2 |
|----|-----|-----|-----|-----|
| 0  | 0   | 0   | 0   | 0   |
| 1  | 0   | 1   | s   | s^2 |
| s  | 0   | s   | 0   | s^2 |
| s^2| 0   | s^2 | 0   | 0   |

Then consider the $S$-act $A = \{x, y, z, t, \theta\}$ with the action of $S$ given as follows:

\begin{align*}
0 \cdot a &= \theta, \quad 1 \cdot a = a \text{ for any } a \in A, \\
 s \cdot x &= s \cdot y = z, \quad s \cdot z = t, \quad s \cdot t = \theta.
\end{align*}

Then $A$ is indecomposable, while the Rees factor $A/\langle z \rangle$ decomposes into two isomorphic components (so $A$ is not connected), each of which can be mapped onto $\langle t \rangle \leq A$, hence $A$ is not autoconnected.

One can furthermore prove that $S$ is left perfect using [17, Theorem 1.1].

For $S$-acts $A_1, A_2 \in S$-$\text{Act}_0$ denote by $\pi_i : A_1 \sqcup A_2 \to A_i, i = 1, 2$ the canonical projections and note that any canonical projection is a correctly defined morphism in the category $S$-$\text{Act}_0$.

Lemma 6.14. Let $C, C_1, C_2 \in S$-$\text{Act}_0$ and $C \cong C_1 \cong C_2$. Then $C$ is autoconnected if and only if for each morphism $f : C \to C_1 \sqcup C_2$ there exists $i$ such that $\pi_i f(C) = \theta$.

Proof. The direct implication follows immediately from Corollary 6.5.

If $C$ is not autoconnected, then by Corollary 6.5(3) there exists a morphism $g : C \to \bigsqcup_{i \in I} C_i$ where $C_i \cong C$ and for each $i \in I$ there exists $j \neq i$ such that $g(C) \cap \nu_j(C_j) \neq \theta$, which implies that there are two distinct $j_1, j_2 \in I$ such that $g(C) \cap \nu_{j_k}(C_{j_k}) \neq \theta$ for both $k = 1, 2$. Now the composition of $g$ with the canonical projection to $C_i \sqcup C_j$ presents an example of a morphism $f : C \to C_{j_1} \sqcup C_{j_2}$ satisfying $\pi_{j_k} f(C) \neq \theta$ for $k = 1, 2$. \hfill \Box
For a pair $B_1, B_2$ of subacts of a left $S$-act $A$ with inclusions $i_i : B_i \to A$ denote by $\rho_{B_1B_2} : B_1 \sqcup B_2 \to A$ the unique morphism satisfying $\rho_{B_1B_2} i_i = i_i$ for $i = 1, 2$, where $i_i$ denotes the coproduct structural morphism. We finish the paper by a characterization of non-autoconnected $S$-acts in the category $S$-Act$_0$, which can by provided by narrowing the class of non-hollow (i.e., non-connected) acts by

**Proposition 6.15.** The following conditions are equivalent for a triple of isomorphic acts $A, A_1, A_2$ in the category $S$-Act$_0$:

1. $A$ is not autoconnected in $S$-Act$_0$.
2. there exists a pair $B_1, B_2$ of proper subacts of $A$ satisfying $A = B_1 \sqcup B_2$ and there exists a morphism $f : B_1 \sqcup B_2 \to A_1 \sqcup A_2$ such that $\pi_i f(B_1 \sqcup B_2) \neq \theta_{A_i}$ for $i = 1, 2$ and $\ker \rho_{B_1B_2} \subseteq \ker f$.

**Proof.** Sufficiency follows from the Homomorphism Theorem [20, Theorem 4.21], which ensures the existence of a morphism $f' : A \to A_1 \sqcup A_2$:

$$
\begin{align*}
B_1 \sqcup B_2 & \xrightarrow{f} A_1 \sqcup A_2 \\
\rho_{B_1B_2} & \downarrow \\
A & \xrightarrow{f'} A
\end{align*}
$$

which turns to be the witnessing morphism for non-autoconnectedness thanks to the property $\pi_i f(B_1 \sqcup B_2) \neq \theta_{A_i}$ for both $i = 1, 2$.

Let $g : A \to A_1 \sqcup A_2$ be the morphism witnessing non-autoconnectedness by Lemma [6.14] hence $\pi_i g(A) \neq \theta_{A_i}$ for both $i = 1, 2$. Let $\nu_i : A_i \to A_1 \sqcup A_2$ denote the coproduct structural morphism and set $B_i = g^{-1}(g(A) \cap \nu_i(A_i))$; then clearly $A = B_1 \sqcup B_2$. Set now $f = g\rho_{B_1B_2}$. \qed

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