Quantum vs Classical Integrability in Ruijsenaars-Schneider Systems

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Abstract

The relationship (resemblance and/or contrast) between quantum and classical integrability in Ruijsenaars-Schneider systems, which are one parameter deformation of Calogero-Moser systems, is addressed. Many remarkable properties of classical Calogero and Sutherland systems (based on any root system) at equilibrium are reported in a previous paper (Corrigan-Sasaki). For example, the minimum energies, frequencies of small oscillations and the eigenvalues of Lax pair matrices at equilibrium are all “integer valued”. In this paper we report that similar features and results hold for the Ruijsenaars-Schneider type of integrable systems based on the classical root systems.

1 Introduction

If a many-body dynamical system is (Liouville) integrable at both classical and quantum levels, the classical system appears to share many ‘quantum’ features. For example, the frequencies of small oscillations near equilibrium are ‘quantised’ together with the eigenvalues of the associated Lax matrices at the equilibrium. This phenomenon of close relation/contrast between quantum and classical integrability has been explored extensively by
Corrigan-Sasaki [1] (this paper will be referred to as I hereafter) for Calogero-Moser (C-M) systems [2, 3, 4] based on any root system [5, 6, 7]. In this paper we show that similar phenomenon occurs in Ruijsenaars-Schneider [8, 9, 10, 11] systems, which are one parameter deformation of C-M systems. The equations determining equilibrium can be presented in a form similar to Bethe ansatz equations. The equilibrium positions are described as the zeros of certain ‘deformed’ classical polynomials, which tend to the Hermite, Laguerre and Jacobi polynomials in the C-M limit [1]. The frequencies of small oscillations at equilibrium are either integers or a sort of ‘deformed’ integers according to the potential is rational or trigonometric, respectively. In the C-M limits, these frequencies tend to those presented in I, which provide non-trivial support for the current results.

This paper is organised as follows. In section 2, the Ruijsenaars-Schneider (R-S) systems [8] are briefly introduced and the formulas for evaluating the frequencies of small oscillations at equilibrium are derived. In most cases, the evaluation of the frequencies is done numerically. In section 3, the rational Ruijsenaars-Schneider systems with two types of confining potentials for the classical root systems $A$, $B$, $C$, $BC$ and $D$ [9, 10] are recapitulated. The formulas of the frequencies of the small oscillations are presented and compared with the corresponding results in Calogero systems [2] given in I. In section 4 the trigonometric R-S systems for the classical root systems are reviewed briefly. The formulas of the frequencies of small oscillations at equilibrium are presented and compared with the results of the Sutherland systems [3] given in I.

2 Ruijsenaars-type systems

The Ruijsenaars-Schneider systems are “discrete” versions of the C-M systems [2, 3, 4], that is the momentum variables appear in the Hamiltonian not as polynomials but as exponential (hyperbolic) functions. In quantum theoretical setting this would mean that the operator (an “analytic difference operator”, according to S. Ruijsenaars) $e^{\pm \beta p} = e^{\mp i\beta \hbar \partial / \partial q}$ causes a shift of the wavefunction by a finite unit $\beta \hbar$ in the imaginary direction, i.e. $\psi(q)$ to $\psi(q \mp i\beta \hbar)$. The parameter $\beta$ has the dimensions of a momentum$^{-1}$ and can be expressed as $\beta = 1/mc$, in which $m$ and $c$ are constants of the dimensions of the mass and the velocity, respectively. The R-S systems can also be considered as a one-parameter ($c$) deformation of the C-M systems (which correspond to the $c \to \infty$ limit) and are sometimes referred to, somehow misleadingly, as “relativistic” Calogero-Moser systems. See [12] for comments on this point.
The dynamical variables of the classical Ruijsenaars-Schneider model are the coordinates \( \{ q_j \mid j = 1, \ldots, r \} \) and their canonically conjugate momenta \( \{ p_j \mid j = 1, \ldots, r \} \), with the Poisson bracket relations:

\[ \{ q_j, p_k \} = \delta_{jk}, \quad \{ q_j, q_k \} = \{ p_j, p_k \} = 0. \]

These will be denoted by vectors in \( \mathbb{R}^r \)

\[ q = (q_1, \ldots, q_r), \quad p = (p_1, \ldots, p_r), \]

in which \( r \) is the number of particles and it is also the rank of the underlying root system \( \Delta \). In this paper we will discuss those models associated with the classical root systems, namely the \( A, B, C, D \) and \( BC \). The fact that all the roots are neatly expressed in terms of the orthonormal basis of \( \mathbb{R}^r \) makes formulation much simpler than those systems based on exceptional root systems. Throughout this paper we adopt the convention of \( \beta \equiv 1 \) for convenience.

Following Ruijsenaars-Schneider [8] and van Diejen [9], let us start with the following Hamiltonian

\[ H(p, q) = \sum_{j=1}^{r} \left[ 2 \cosh p_j \sqrt{V_j(q)V_j^*(q)} - (V_j(q) + V_j^*(q)) \right], \tag{2.1} \]

in which \( V_j^* \) is the complex conjugate of \( V_j \). The form of the function \( V_j = V_j(q) \) is determined by the root system \( \Delta \) as:

\[ A : \quad V_j(q) = w(q_j) \prod_{k \neq j} v(q_j - q_k), \quad j = 1, \ldots, r + 1, \tag{2.2} \]

\[ B, C, BC, \ & D : \quad V_j(q) = w(q_j) \prod_{k \neq j} v(q_j - q_k)v(q_j + q_k), \quad j = 1, \ldots, r. \tag{2.3} \]

The elementary potential functions \( v \) and \( w \) depend on the nature of interactions (rational, trigonometric, etc) and the root system \( \Delta \).

2.1 Equilibrium position and frequencies of small oscillations

It is easy to see that the system has a stationary solution

\[ p = 0, \quad q = \bar{q}, \tag{2.4} \]
of the canonical equations of motion:

\[ \dot{q}_j = \frac{\partial H(p,q)}{\partial p_j} = \sum_{j=1}^{r} 2 \sin p_j \sqrt{V_j V_j^*}, \quad \dot{p}_j = -\frac{\partial H(p,q)}{\partial q_j}, \]

in which \( \bar{q} \) satisfies

\[ \frac{\partial H(0,q)}{\partial q_j} \bigg|_{\bar{q}} = 0, \quad j = 1, \ldots, r. \]  

By expanding the Hamiltonian around the stationary solution \( 2.4 \), we obtain

\[ H(p,q) = K(p) + P(q) + \text{higher order terms in } p, \]

in which the kinetic part \( K \) is quadratic in \( p \)

\[ K(p) = \sum_{j=1}^{r} (p_j)^2 a_j, \quad a_j \equiv |V_j(\bar{q})|, \]

and the ‘potential’ \( P \) is given by

\[ P(q) = \sum_{j=1}^{r} \left( 2\sqrt{V_j(q)} V_j^*(q) - (V_j(q) + V_j^*(q)) \right) = -\sum_{j=1}^{r} \left( \sqrt{V_j(q)} - \sqrt{V_j^*(q)} \right)^2. \]

This should be compared with the classical potential in C-M systems, \( V_C = \sum_{j=1}^{r}(\partial W/\partial q_j)^2 \), in which \( W \) is the prepotential \( 7 \). It is obvious that the equilibrium is achieved at the point(s) in which all the functions \( V_j \) become real and positive:

\[ V_j(\bar{q}) = V_j(\bar{q})^* > 0, \quad j = 1, 2, \ldots, r. \]

Therefore the “minimal energy” \( P(\bar{q}) \) is always 0 in contrast to the C-M cases.

Let us define the Hessian of the ‘potential’ \( P \) at equilibrium as \( B_{jk} \):

\[ B_{jk} \equiv \frac{\partial^2 P(q)}{\partial q_j \partial q_k} \bigg|_{\bar{q}}, \quad j, k = 1, \ldots, r. \]

It is easy to verify that

\[ B_{jk} = \frac{1}{2} \sum_{l=1}^{r} \frac{1}{a_l} \frac{\partial IV_l}{\partial q_j} \bigg|_{\bar{q}} \frac{\partial IV_l}{\partial q_k} \bigg|_{\bar{q}}, \quad IV_l(q) \equiv -i(V_l(q) - V_l^*(q)), \quad j, k = 1, \ldots, r. \]

Thus the small oscillations near the stationary point \( 2.4 \) are described by the effective quadratic Hamiltonian in \( p \) and \( q - \bar{q} \):

\[ H_{eff}(p,q) = K(p) + \frac{1}{2} \sum_{j,k=1}^{r} B_{jk}(q - \bar{q})_j(q - \bar{q})_k. \]
In terms of a canonical transformation

\[ p'_j = \sqrt{2a_j} p_j, \quad q'_j = \frac{1}{\sqrt{2a_j}} (q - \bar{q})_j, \quad j = 1, \ldots, r; \quad (2.14) \]

the quadratic Hamiltonian reads

\[ H_{\text{eff}}(p', q') = \frac{1}{2} \sum_{j=1}^{r} (p'_j)^2 + \frac{1}{2} \sum_{j,k=1}^{r} B'_{jk} q'_j q'_k, \quad (2.15) \]

\[ B'_{jk} \equiv (W^2)_{jk}, \quad (2.16) \]

in which a real symmetric \( r \times r \) matrix \( W \) is defined by

\[ W_{jk} \equiv \sqrt{a_j} \left. \frac{\partial IV_k}{\partial q_j} \right|_{\bar{q}} \frac{1}{\sqrt{a_k}}. \quad (2.17) \]

The frequencies of small oscillations at the equilibrium are given simply by the eigenvalues of a matrix \( \tilde{W} \):

\[ \tilde{W}_{jk} \equiv \left. \frac{\partial IV_k}{\partial q_j} \right|_{\bar{q}}, \quad (2.18) \]

which are relatively easy to evaluate.

### 3 Ruijsenaars-Calogero systems

Here we will discuss the discrete analogue of the Calogero systems [2], to be called Ruijsenaars-Calogero systems, which were introduced by van Diejen for classical root systems only [9, 10]. (For the definition of C-M systems based on any root system and the associated Lax representation, etc, see [5, 6, 7].) The original Calogero systems [2] have the rational \( 1/q^2 \) potential plus the harmonic \( q^2 \) confining potential, having two coupling constants \( g \) and \( \omega \) for the systems based on the simply-laced root systems, \( A \) and \( D \), and three couplings \( \omega \) and \( g_L \) for the long roots and \( g_S \) for the short roots in \( B (C) \) root system. (For the rational Calogero systems, those based on \( B \) and \( C \) root systems are equivalent.)

The discrete Calogero systems have two varieties (deformations), according to the number of independent coupling constants. The first has two (three for the non-simply-laced root systems) coupling constants \( g \) (\( g_L \) and \( g_S \)) and \( a \) which corresponds to \( \omega \) in the Calogero systems. This can be called a “minimal” discretisation of the Calogero systems. The second

\[^1\text{This property stems from the structure of the functions } V_j, a, b \text{ and from the even nature of } v'(x).\]
is introduced by van Diejen \cite{9,10} having three (four for the non-simply-laced root systems) coupling constants $g$ ($g_L$ and $g_S$) and $a$, $b$ both of which correspond to $\omega$. In this case the $B$ and $C$ systems are not equivalent in contrast to the Calogero systems. The integrability (classical and quantum) of the latter was discussed by van Diejen in some detail \cite{9,10}. Whereas, the former (the minimal discretisation) is new and its integrability has not been discussed to the best of our knowledge. As we will show in the next subsection, the very orderly spectrum of the small oscillations would give strong evidence for its integrability.

### 3.1 Linear confining potential case

Let us first write down the explicit forms of the elementary potential functions $v$ and $w$. For the simply-laced root systems $A$ and $D$ the elementary potential functions are:

\begin{align}
A, \quad D : \quad v(x) &= 1 - ig/x, \quad w(x) = a + ix, \quad (3.1)
\end{align}

in which $a$ and $g$ are real coupling constants assumed to be positive. For the non-simply-laced root systems $B$ ($C$), $BC$, they are:

\begin{align}
B, \quad (C) : \quad v(x) &= 1 - ig_L/x, \quad w(x) = (a + ix)(1 - ig_S/x), \quad (3.2)
BC : \quad v(x) &= 1 - ig_M/x, \quad w(x) = (a + ix)(1 - ig_S/x)(1 - ig_L/2x), \quad (3.3)
\end{align}

in which $g_L$, $g_M$ and $g_S$ are the independent positive coupling constants for the long, middle and short roots, respectively. As in the Calogero case, those based on $B$ and $C$ systems are equivalent. In all these cases the ‘potential’ $P$ \eqref{2.9} grows linearly in $|q|$ as $|q| \to \infty$. Except for the $BC$ case, there are simple identities:

\begin{align}
\sum_j \{ V_j(q) + V_j(q)^* \} = \text{const.} \quad (3.4)
\end{align}

Thus the Hamiltonian \eqref{2.1} can be replaced by a simpler one

\begin{align}
H'(p, q) = 2 \sum_{j=1}^r \cosh p_j \sqrt{V_j(q) V_j^*(q)}, \quad (3.5)
\end{align}

which is usually used as a starting point for the trigonometric (hyperbolic) interaction theory, see section \ref{4}. To be more precise, the identities are:

\begin{align}
A_r : \quad \sum_{j=1}^{r+1} \{ V_j(q) + V_j(q)^* \} = 2(r+1)a + r(r+1)g, \quad (3.6)
\end{align}
and
\[ B_r (C_r) \& D_r : \sum_{j=1}^{r} \{ V_j(q) + V_j(q)^* \} = 2r (a + (r - 1)g_L + g_S). \tag{3.7} \]

(For \( D_r, g_L \to g \) and \( g_S \equiv 0 \).)

It is interesting to note that the equations determining the equilibrium (2.10), in general, can be cast in a form which looks similar to the Bethe ansatz equation. For example, for the elementary potential (3.1)–(3.2), (2.10) read
\[ A_r : \prod_{k \neq j} \frac{q_j - q_k - ig}{q_j - q_k + ig} = \frac{a - i\bar{q}_j}{a + i\bar{q}_j}, \quad j = 1, \ldots, r + 1, \tag{3.8} \]
\[ D_r : \prod_{k \neq j} \frac{q_j - q_k - ig q_j + q_k - ig}{q_j - q_k + ig q_j + q_k + ig} = \frac{a - i\bar{q}_j}{a + i\bar{q}_j}, \quad j = 1, \ldots, r, \tag{3.9} \]
\[ B_r : \prod_{k \neq j} \frac{q_j - q_k - ig_L q_j + q_k - ig_L}{q_j - q_k + ig_L q_j + q_k + ig_L} = \frac{a - i\bar{q}_j + i\alpha g_S}{a + i\bar{q}_j - i\alpha g_S}, \quad j = 1, \ldots, r. \tag{3.10} \]

They determine the zeros of deformed Hermite and Laguerre polynomials. For \( A_r \), let us define
\[ \bar{q}_j = \sqrt{ag} y_j, \quad \delta = g/a, \tag{3.11} \]
and introduce a degree \( r + 1 \) polynomial
\[ H_{r+1}^{(\delta)}(x) = 2^{r+1} \prod_{j=1}^{r+1} (x - y_j), \tag{3.12} \]
which is a deformation of the Hermite polynomial \( H_{r+1}(x) \). For lower \( n \), \( H_n^{(\delta)}(x) \) are:
\[ H_n^{(\delta)}(x) = H_n(x), \quad n = 0, 1, 2, \quad H_3^{(\delta)}(x) = H_3(x) - 4x\delta, \]
\[ H_4^{(\delta)}(x) = H_4(x) - 32x^2\delta + 12\delta, \quad H_5^{(\delta)}(x) = H_5(x) - 160x^3\delta + 200x\delta + 48x^2\delta, \]
\[ H_6^{(\delta)}(x) = H_6(x) - 640x^4\delta + x^2(736\delta^2 + 1680\delta) - 240\delta^2 - 360\delta. \tag{3.13} \]

For \( B_r \) let us define
\[ \bar{q}_j = \sqrt{ag_L} y_j, \quad \alpha = g_S/g_L - 1, \quad \delta = g/a, \tag{3.14} \]
and introduce a degree \( r \) polynomial
\[ L_r^{(\delta, \alpha)}(x) = (-1)^r \prod_{j=1}^{r} (x - y_j^2)^2/r!, \tag{3.15} \]
which is a deformation of the Laguerre polynomial $L_{\alpha}^r(x)$. For lower $n$, we have

$$L_{n}^{(\delta,\alpha)}(x) = L_{n}^{(\alpha)}(x), \quad n = 0, 1, \quad L_{2}^{(\delta,\alpha)}(x) = L_{2}^{(\alpha)}(x) - \frac{\delta}{2} (-2 + 3x - 3\alpha + 2x\alpha - \alpha^2), \quad (3.16)$$

$$L_{3}^{(\delta,\alpha)}(x) = L_{3}^{(\alpha)}(x) - \frac{\delta}{6} (-18 + 45x - 13x^2 - 33\alpha + 42x\alpha - 6x^2\alpha - 18\alpha^2 + 9x\alpha^2 - 3\alpha^3$$
$$-12\delta + 22x\delta - 22\alpha\delta + 24x\alpha\delta - 12\alpha^2\delta + 6x\alpha^2\delta - 2\alpha^3\delta), \quad (3.17)$$

$$L_{4}^{(\delta,\alpha)}(x) = L_{4}^{(\alpha)}(x) - \frac{\delta}{24} (-144 + 504x - 280x^2 + 34x^3 - 300\alpha + 582x\alpha - 190x^2\alpha$$
$$+ 12x^3\alpha - 210\alpha^2 + 210x\alpha^2 - 30x^2\alpha^2 - 60\alpha^3 + 24x\alpha^3 - 6\alpha^4$$
$$-264\delta + 760x\delta - 241x^2\delta - 550x\alpha\delta + 950x\alpha\delta - 192x^2\alpha\delta$$
$$-385\alpha^2\delta + 366x\alpha^2\delta - 36x^2\alpha^2\delta - 110\alpha^3\delta + 44x\alpha^3\delta - 11\alpha^4\delta$$
$$-144\delta^2 + 300x\delta^2 - 300\alpha\delta^2 + 420x\alpha\delta^2 - 210\alpha^2\delta^2 + 180x\alpha^2\delta^2$$
$$-60\alpha^3\delta^2 + 24x\alpha^3\delta^2 - 6\alpha^4\delta), \quad \ldots \quad (3.18)$$

They are not the so-called “$q$-deformed” Hermite or Laguerre polynomials [13]. As in the Calogero systems, the $D_r$ is a special case $g_S = 0$ of the $B_r$ theory described by $L_{(\delta,-1)}^{(\alpha)}(x)$, which has a zero at $x = 0$ for all $r$, see (1.4.20).

It is remarkable that the spectrum of the small oscillations at equilibrium is completely independent of the coupling constant $g$, $a$, $g_L$ or $g_S$. In other words, the spectrum is the topological invariant of the theory. It is solely determined by the root system. In fact, the spectrum is

$$2(1 + e_j), \quad j = 1, \ldots, r, \quad (3.19)$$

in which $e_j$ is the $j$-th exponent of the root system $\Delta$. Explicitly, the spectrum is

$$A : \quad 2 \times (1, 2, \ldots, r + 1), \quad (3.20)$$

$$B (C) & BC : \quad 2 \times (2, 4, \ldots, 2r), \quad (3.21)$$

$$D : \quad 2 \times (2, 4, \ldots, 2r - 2, r). \quad (3.22)$$

(The lowest frequency of $A$ series is due to the center of mass motion which is not described by the data of the root system.) The situation is essentially the same as in the Calogero systems, in which the frequencies of the small oscillations are proportional to the above formula (3.19) and are independent of the coupling constant(s) of the rational potential [1].

We strongly believe that this very orderly spectrum is a good evidence for the integrability of this type of models, as is the case for the Calogero systems.
3.2 Quadratic confining potential case

Let us first write down the explicit forms of the elementary potential functions \( v \) and \( w \). For the simply-laced root systems \( A \) and \( D \), the elementary potential functions are:

\[
v(x) = 1 - ig/x, \quad w(x) = (a + ix)(b + ix), \quad a, b, g > 0.
\]  

(3.23)

The elementary potential functions are \((a, b, g_L, g_M, g_S > 0)\):

\[
B : \quad v(x) = 1 - ig_L/x, \quad w(x) = (a + ix)(b + ix)(1 - ig_S/x),
\]  

(3.24)

\[
C : \quad v(x) = 1 - ig_S/x, \quad w(x) = (a + 2ix)(b + 2ix)(1 - ig_L/2x),
\]  

(3.25)

\[
BC : \quad v(x) = 1 - ig_M/x, \quad w(x) = (a + ix)(b + ix)(1 - ig_L/2x)(1 - ig_S/x). \quad (3.26)
\]

The \( C \) system is slightly different from the one given by van Diejen [10], since the latter is the quantum theory. In the limit \( \hbar \to 0 \), (3.25) is the same as van Diejen’s. Here again those based on \( B \) and \( C \) systems are equivalent in terms of the overall scaling of the potential and scaling of the coupling constants. This fact is reflected in their spectra (3.29), (3.30). In all these cases the ‘potential’ \( P \) (2.9) grows quadratically in \(|q|\) as \(|q| \to \infty\). In the present case the identities (3.4) are replaced by

\[
\sum_j \{V_j(q) + V_j(q)^*\} = \left\{ \begin{array}{ll} -2(r + 1)q^2 + const, & A_r \\ -2rq^2 + const, & B_r, (C_r), D_r \end{array} \right. \]  

(3.27)

except for the \( BC \) case.

Again the equations determining the equilibrium (2.10) are *Bethe ansatz* -like equations which are expected to determine two-parameter deformation of the Hermite and Laguerre polynomials. These will be discussed elsewhere. The spectrum of the small oscillations at equilibrium has a very simple form. Explicitly, the spectrum is

\[
A : \quad 2j [a + b + g (r - (j - 1)/2)], \quad j = 1, 2, \ldots, r + 1,
\]  

(3.28)

\[
B : \quad 4j [a + b + gs + gl(2r - j - 1)], \quad j = 1, 2, \ldots, r,
\]  

(3.29)

\[
C : \quad 8j [a + b + gl + 2gs(2r - j - 1)], \quad j = 1, 2, \ldots, r,
\]  

(3.30)

\[
BC : \quad 4j [a + b + gl/2 + gs + gm(2r - j - 1)], \quad j = 1, 2, \ldots, r,
\]  

(3.31)

\[
D : \quad 4j [a + b + g(2r - j - 1)], \quad j = 1, 2, \ldots, r - 1,
\]  

and \( 2r(a + b + g(r - 1)) \).  

(3.32)
The Calogero models are obtained in the singular limit, \(a, b \to \infty\) and by division by \(ab\). In this limit, the above spectrum of small oscillations, (3.28)–(3.32), will be proportional to those of the Calogero models, i.e. (3.19), (3.20)–(3.22), as expected.

It is interesting to compare the above spectrum of small oscillations with the quantum energy eigenvalues. The quantum spectrum is given by van Diejen [9]:

\[
A: \quad E_{\vec{n}} = \sum_{1 \leq j \leq r+1} n_j \left[ n_j + 2(a + b) - 1 + 2g(r + 1 - j) \right], \tag{3.33}
\]

\[
B: \quad E_{\vec{n}} = 4 \sum_{1 \leq j \leq r} n_j \left[ n_j + a + b - 1 + g_S + 2g_L(r - j) \right], \tag{3.34}
\]

\[
C: \quad E_{\vec{n}} = 8 \sum_{1 \leq j \leq r} n_j \left[ n_j + a + b - 1 + g_L + 4g_S(r - j) \right], \tag{3.35}
\]

\[
BC: \quad E_{\vec{n}} = 4 \sum_{1 \leq j \leq r} n_j \left[ n_j + a + b - 1 + g_L/2 + g_S + 2g_M(r - j) \right], \tag{3.36}
\]

in which \(\vec{n} = (n_1, \ldots, n_r, (n_{r+1}))\) is a set of ‘quantum numbers’ parametrising the eigenstates. They are non-increasing, non-negative integers \((n_1 \geq n_2 \geq \cdots \geq n_r \geq (n_{r+1}) \geq 0)\). In fact, the \(B\) and \(C\) formulas are special cases of the \(BC\) formula. However, the \(D\) formula needs yet to be derived. The \(r, (r + 1)\) independent ‘lowest lying’ modes corresponding to the quantum numbers

\[
\vec{n} = (1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1, 0), (1, 1, \ldots, 1),
\]

have energies,

\[
A: \quad 2j \left[ a + b + g(r - (j - 1)/2) \right], \quad j = 1, \ldots, r + 1, \tag{3.38}
\]

\[
B: \quad 4j \left[ a + b + g_S + g_L(2r - 1 - j) \right], \quad j = 1, \ldots, r, \tag{3.39}
\]

\[
C: \quad 8j \left[ a + b + g_L + 2g_S(2r - 1 - j) \right], \quad j = 1, \ldots, r, \tag{3.40}
\]

\[
BC: \quad 4j \left[ a + b + g_L/2 + g_S + g_M(2r - 1 - j) \right], \quad j = 1, \ldots, r, \tag{3.41}
\]

which are exactly the same as (3.28)–(3.31).

4 Ruijsenaars-Sutherland systems

The discrete analogue of the Sutherland systems \(\mathbb{R}\), to be called Ruijsenaars-Sutherland systems, was introduced originally by Ruijsenaars and Schneider \(\mathbb{R}\) for the \(A\) type root system. The quantum eigenfunctions of the \(A\) type Ruijsenaars-Sutherland systems are called
Macdonald polynomials [15], which are a one-parameter deformation of the Jack polynomials [16]. Here we will discuss Ruijsenaars-Sutherland systems for all the classical root systems, $A$, $B$, $C$, $D$ and $BC$ [10]. The structure of the functions $\{V_j(q)\}$, (2.2) and (2.3) are the same as in the Ruijsenaars-Calogero systems, but the elementary potential functions $v$ and $w$ are trigonometric instead of rational. Because of the identity $\sum_{j=1}^r \{V_j(q) + V_j(q)^*\} = \text{const.}$, (3.4), the Hamiltonian (2.1) can be replaced by a simpler one

$$H'(p, q) = 2 \sum_{j=1}^r \cosh p_j \sqrt{V_j(q) V_j^*(q)}, \quad (4.1)$$

which is obviously positive definite. This is also valid for the $BC$ case in contrast to the rational potential cases discussed in the preceding section 3.

The elementary potential functions $v$ and $w$ are:

- **$A$, $D$**: 
  
  $v(x) = \cosh \gamma - i \sinh \gamma \cot x, \quad w(x) = 1, \quad (4.2)$

- **$B$**: 
  
  $v(x) = \cosh \gamma_L - i \sinh \gamma_L \cot x, \quad w(x) = \cosh \gamma_S - i \sinh \gamma_S \cot x, \quad (4.3)$

- **$C$**: 
  
  $v(x) = \cosh \gamma_S - i \sinh \gamma_S \cot x, \quad w(x) = \cosh \gamma_L - i \sinh \gamma_L \cot 2x, \quad (4.4)$

- **$C'$**: 
  
  $v(x) = \cosh \gamma_S - i \sinh \gamma_S \cot x, \quad w(x) = (\cosh \gamma_L - i \sinh \gamma_L \cot 2x)^2, \quad (4.5)$

- **$BC$**: 
  
  $v(x) = \cosh \gamma_M - i \sinh \gamma_M \cot x, \quad w(x) = (\cosh \gamma_S - i \sinh \gamma_S \cot x)(\cosh \gamma_L - i \sinh \gamma_L \cot 2x), \quad (4.6)$

in which $\gamma_L$, $\gamma_M$ and $\gamma_S$ are the positive coupling constants for the long, middle and short roots, respectively. Both $C$ and $C'$ and $BC$ cases are special cases of the most general integrable interactions including the long roots ($\alpha^2_L = 4$) introduced by van Diejen [11]. Note that in our paper only the classical dynamics $\hbar \to 0$ is discussed. That is, van Diejen’s constant $\gamma = i \beta \hbar/2$ (eq.(2.7) in [11]) is treated as vanishing.

For the systems based on $A$ type root system, the above identity (3.4)

$$A_r : \quad \sum_{j=1}^{r+1} V_j(q) = \sinh[(r + 1)\gamma]/\sinh \gamma \quad (4.7)$$

is known in a different context [14]. For the other root systems the identity (3.4) reads

$$\sum_{j=1}^r \{V_j(q) + V_j(q)^*\} = \begin{cases} 
2 \sinh[r\gamma_L] \cosh[(r - 1)\gamma_L + \gamma_S]/\sinh \gamma_L, & B_r, \\
2 \sinh[r\gamma_S] \cosh[(r - 1)\gamma_S + \gamma_L]/\sinh \gamma_S, & C_r, \\
\sinh[(2r - 1)\gamma]/\sinh \gamma + 1, & D_r.
\end{cases} \quad (4.8)$$

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It is easy to verify that the $B_r$ formula reduces to the $D_r$ one for $\gamma_L \rightarrow \gamma$ and $\gamma_S = 0$. Had we started from the simplified Hamiltonian (4.11) instead of the original one (2.1), the above constants (4.7)–(4.8) would give the minimal energies. In the simply-laced $A$ and $D$ cases, the r.h.s of (4.7) and (4.8) can be interpreted as “$q$-deformed integer” version of the dimensionality of $A_r$ and $D_r$ vector representations, $[r + 1]_q$ and $1 + [2r - 1]_q$.

The equilibrium of the $A$ type theories is achieved at “equally-spaced”

$$\bar{q} = \pi(0, 1, \ldots, r - 1, r)/(r + 1) + \xi(1, 1, \ldots, 1), \quad \xi \in \mathbb{R}: \text{arbitrary}, \quad (4.9)$$
configuration, as in the original Sutherland systems. In all the other cases, the equilibrium and the frequencies of the small oscillations are determined by solving the Bethe ansatz-like equations (2.10) numerically. Certain one-parameter deformation of the Jacobi polynomials [1] is expected to describe the equilibrium for $B$, $C$, $D$ and $BC$ systems, which will be discussed elsewhere. The spectrum of the small oscillations at equilibrium has a very simple form. Explicitly, the spectrum is

$$A: \quad 4 \sinh[(r + 1 - j)\gamma]\sinh[j\gamma]/\sinh \gamma, \quad j = 1, 2, \ldots, r + 1, \quad (4.10)$$
$$B: \quad 4 \sinh[(2r - 1 - j)\gamma_L + \gamma_S]\sinh[j\gamma_L]/\sinh \gamma_L, \quad j = 1, 2, \ldots, r - 1, \quad 2 \sinh[(r - 1)\gamma_L + \gamma_S]\sinh[r\gamma_L]/\sinh \gamma_L, \quad (4.11)$$
$$C: \quad 4 \sinh[(2r - 1 - j)\gamma_S + \gamma_L]\sinh[j\gamma_S]/\sinh \gamma_S, \quad j = 1, 2, \ldots, r, \quad (4.12)$$
$$C': \quad 4 \sinh[(2r - 1 - j)\gamma_S + 2\gamma_L]\sinh[j\gamma_S]/\sinh \gamma_S, \quad j = 1, 2, \ldots, r, \quad (4.13)$$
$$BC: \quad 4 \sinh[(2r - 1 - j)\gamma_M + 2\gamma_L + \gamma_S]\sinh[j\gamma_M]/\sinh \gamma_M, \quad j = 1, 2, \ldots, r, \quad (4.14)$$
$$D: \quad 4 \sinh[(2r - 1 - j)\gamma]\sinh[j\gamma]/\sinh \gamma, \quad j = 1, 2, \ldots, r - 2, \quad 2 \sinh[(r - 1)\gamma]\sinh[r\gamma]/\sinh \gamma, \quad \text{twofold degenerate}. \quad (4.15)$$

The spectrum of the $A$ system (4.10)\(^2\) reflects the symmetry of the Dynkin diagram $j \leftrightarrow r + 1 - j$. The twofold degeneracy of the $D$ spectrum (4.15) also reflects the symmetry of the $D$ Dynkin diagram.

The original Sutherland models are obtained in the singular limit in which all the coupling constant(s) become infinitesimally small: $0 < \gamma, \gamma_L, \gamma_M, \gamma_S \ll 1$. In this limit, the spectrum of small oscillations at equilibrium will be linear in the coupling constant(s). In these limits, the known spectrum of small oscillations obtained in I is reproduced. As for $A_r$, the spectrum

\(^2\)This formula was known to S. Ruijsenaars [17].
becomes

$$A: \quad 4(r + 1 - j)j\gamma, \quad j = 1, \ldots, r + 1,$$

which is eq.(5.16) of Corrigan-Sasaki [1], to be referred to as (I.5.16). For $B_r$, the spectrum (4.11) in the limit is

$$B_r: \quad 4[(2r - 1 - j)\gamma_L + \gamma_S]j, \quad j = 1, \ldots, r - 1, \quad 2[(r - 1)\gamma_L + \gamma_S]r, \quad [2],$$

which is (I.5.74). For $C'_r$, the spectrum (4.13) in the limit is

$$C': \quad 4[(2r - 1 - j)\gamma_S + 2\gamma_L]j, \quad j = 1, \ldots, r,$$

which is (I.5.81). For $D_r$, the spectrum (4.15) in the limit is

$$D: \quad 4(2r - 1 - j)j\gamma, \quad j = 1, \ldots, r - 2, \quad 2r(r - 1)\gamma, \quad [2],$$

which is (I.5.60). The limiting spectra provide non-trivial supporting evidences for the formulas (4.10)–(4.15).

It is a challenge to understand the connection between the quantum spectrum of the Ruijsenaars-Sutherland systems, eq. (5.17) of [15] and the above spectrum of small oscillations (4.10)–(4.15). It would be interesting to evaluate the eigenvalues of the Lax matrices at equilibrium as shown for the C-M systems [1]. However, the knowledge of the Lax pairs for the Ruijsenaars-Schneider systems is still quite limited [18].

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References

[1] E. Corrigan and R. Sasaki, “Quantum vs Classical Integrability in Calogero-Moser Systems”, J. Phys. A 35 (2002) 7017-7062, hep-th/0204039; S. Odake and R. Sasaki, “Polynomials Associated with Equilibrium Positions in Calogero-Moser Systems,” J. Phys. A 35 (2002) 8283-8314, hep-th/0206172.

[2] F. Calogero, “Solution of the one-dimensional $N$-body problem with quadratic and/or inversely quadratic pair potentials”, J. Math. Phys. 12 (1971) 419-436.
[3] B. Sutherland, “Exact results for a quantum many-body problem in one-dimension. II”, Phys. Rev. A5 (1972) 1372-1376.

[4] J. Moser, “Three integrable Hamiltonian systems connected with isospectral deformations”, Adv. Math. 16 (1975) 197-220; J. Moser, “Integrable systems of non-linear evolution equations”, in Dynamical Systems, Theory and Applications; J. Moser, ed., Lecture Notes in Physics 38 (1975), Springer-Verlag; F. Calogero, C. Marchioro and O. Ragnisco, “Exact solution of the classical and quantal one-dimensional many body problems with the two body potential \( V_a(x) = g^2 a^2 / \sinh^2 ax \)”, Lett. Nuovo Cim. 13 (1975) 383-387; F. Calogero, “Exactly solvable one-dimensional many body problems”, Lett. Nuovo Cim. 13 (1975) 411-416.

[5] M. A. Olshanetsky and A. M. Perelomov, “Completely integrable Hamiltonian systems connected with semisimple Lie algebras”, Inventions Math. 37 (1976), 93-108; “Classical integrable finite-dimensional systems related to Lie algebras”, Phys. Rep. C71 (1981), 314-400.

[6] E. D’Hoker and D. H. Phong, “Calogero-Moser Lax pairs with spectral parameter for general Lie algebras”, Nucl. Phys. B530 (1998) 537-610, [hep-th/9804124]; A. J. Bordner, E. Corrigan and R. Sasaki, “Calogero-Moser models I: a new formulation”, Prog. Theor. Phys. 100 (1998) 1107-1129, [hep-th/9805106]; “Generalized Calogero-Moser models and universal Lax pair operators”, Prog. Theor. Phys. 102 (1999) 499-529, [hep-th/9905011].

[7] A. J. Bordner, N. S. Manton and R. Sasaki, “Calogero-Moser models V: Supersymmetry and Quantum Lax Pair”, Prog. Theor. Phys. 103 (2000) 463-487, [hep-th/9910033]; S. P. Khastgir, A. J. Pocklington and R. Sasaki, “Quantum Calogero-Moser Models: Integrability for all Root Systems”, J. Phys. A33 (2000) 9033-9064, [hep-th/0005277].

[8] S. N. Ruijsenaars and H. Schneider, “A New Class Of Integrable Systems And Its Relation To Solitons,” Annals Phys. 170 (1986) 370-405; “Complete Integrability of Relativistic Calogero-Moser Systems And Elliptic Function Identities,” Commun. Math. Phys. 110 (1987) 191.
[9] J. F. van Diejen, “The relativistic Calogero model in an external field,” solv-int/9509002; “Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems”, J. Phys. A28 (1995) L369-L374.

[10] J. F. van Diejen, “Difference Calogero-Moser systems and finite Toda chains”, J. Math. Phys. 36 (1995) 1299-1323.

[11] J. F. van Diejen, “Integrability of difference Calogero-Moser systems”, J. Math. Phys. 35 (1994) 2983-3004.

[12] H. W. Braden and R. Sasaki, “The Ruijsenaars-Schneider Model,” Prog. Theor. Phys. 97 (1997) 1003-1018, hep-th/9702182.

[13] G. E. Andrews, R. Askey and R. Roy, “Special Functions”, Encyclopedia of mathematics and its applications, Cambridge, (1999).

[14] F. Calogero, “Classical many-body problems amenable to exact treatments”, Springer, Berlin (2001).

[15] I. G. Macdonald, “Orthogonal polynomials associated with root systems”, preprint 1988, Séminaire Lotharingien de Combinatoire, 45 (2000), Article B45a.

[16] H. Jack, “A class of symmetric polynomials with a parameter”, Proc. R. Soc. Edinburgh (A), 69 (1979) 1-18.

[17] S. N. Ruijsenaars, “Action-angle maps and scattering theory for some finite-dimensional integrable systems. III. Sutherland type systems and their duals”, Publ. Res. Inst. Math. Sci. 31 (1995) 247–353.

[18] K. Chen, B.-Y. Hou and W.-L. Yang, “Integrability of the $C_n$ and $BC_n$ Ruijsenaars-Schneider models,” J. Math. Phys. 41 (2000) 8132-8147, hep-th/0006004. K. Chen, B.-Y. Hou and W.-L. Yang, “The Lax pairs for elliptic $C_n$ and $BC_n$ Ruijsenaars-Schneider models and their spectral curves,” J. Math. Phys. 42 (2001) 4894-4914, hep-th/0011145. K. Chen and B.-Y. Hou, “The $D_n$ Ruijsenaars-Schneider model,” J. Phys. A 34 (2001) 7579-7589, hep-th/0102036.