Estimate for norm of a composition operator on the Hardy-Dirichlet space

Perumal Muthukumar, Saminathan Ponnusamy and Hervé Queffélec

Abstract. By using the Schur test, we give some upper and lower estimates on the norm of a composition operator on $H^2$, the space of Dirichlet series with square summable coefficients, for the inducing symbol $\varphi(s) = c_1 + c_q s^{-q}$ where $q \geq 2$ is a fixed integer. We also give an estimate on the approximation numbers of such an operator.

Mathematics Subject Classification (2010). Primary: 47B33, 47B38; Secondary: 11M36, 37C30.

Keywords. Composition operator, Hardy space, Dirichlet series, Schur test, zeta function.

1. Introduction
Let $\Omega$ be a domain in the complex plane $\mathbb{C}$. For a given analytic self map $\varphi$ of $\Omega$, the corresponding composition operator $C_\varphi$ induced by the symbol $\varphi$ is defined by $C_\varphi(f) = f \circ \varphi$ for every analytic function $f$ on $\Omega$. In the classical case, $\Omega$ is taken as the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and the operator $C_\varphi$ is considered on various analytic function spaces on $\mathbb{D}$ such as the Hardy spaces $H^p$, the Bergman spaces $A^p$ and the Bloch space $B$.

For a real number $\theta$, we set $C_\theta = \{ s \in \mathbb{C} : \Re s > \theta \}$. In this article, $\Omega$ will be taken to be the half plane $\mathbb{C}_{1/2}$, the map $\varphi$ to be the analogue of affine map in the classical case and the composition operator $C_\varphi$ is considered on the Hardy-Dirichlet space $\mathcal{H}^2$, which is a Dirichlet series analogue of the classical Hardy space.

Determining the value of the norm of composition operators is not an easy task and hence, not much is known on this problem even in the case of classical Hardy space except for some special cases. For example, the norm of a composition operator on $H^2$ induced by the simple affine mapping of $\mathbb{D}$ is complicated (see [7, Theorem 3]). Not to speak of the approximation

The second author is currently at ISI Chennai Centre.
numbers of $C_\varphi$, even though the latter were computed in [6]. In case of the space $H^2$ of Dirichlet series with square-summable coefficients, there are no good lower and upper bounds even for the norm of such operators except for some special cases. As a first step, in this paper, we give some upper and lower estimates on the norm of a composition operator on $H^2$, for the inducing symbol $\varphi(s) = c_1 + c_q q^{-s}$ with $q \in \mathbb{N}$, $q \geq 2$. Here $\mathbb{N}$ denotes the set of all natural numbers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Without loss of generality, we will assume that $q = 2$. One significant difference is that some properties of the Riemann zeta function, be it only in the half-plane $\mathbb{C}_1$, are required.

The article is organized as follows. In Section 2, definition and some important properties of Hardy-Dirichlet space $H^2$ are recalled. Also, the boundedness of composition operators on $H^2$ is discussed. In Section 3, motivation for this work and estimates for the norm of $C_\varphi$ for the affine-like inducing symbols are given. Finally, in Section 4, we give an estimate for approximation numbers of a composition operators in our $H^2$ setting.

One may refer to [20] for basic information about analytic function spaces of $\mathbb{D}$ and operators on them. Basic issues on composition operators on various function spaces on $\mathbb{D}$ may be obtained from [8]. See also [14] for results related to analytic number theory.

2. Composition operators on the Hardy space of Dirichlet series

The Hardy-Dirichlet space $H^2$ is defined by

$$H^2 = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \|f\|^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}. \tag{2.1}$$

The space $H^2$ has been used in [12] for the study of completeness problems of a system of dilates of a given function. The following properties are obvious:

- If $f \in H^2$, then the Dirichlet series in (2.1) converges absolutely in $\mathbb{C}_{1/2}$, and therefore $H^2$ is a Hilbert space of analytic functions on $\mathbb{C}_{1/2}$.
- The functions $\{e_n\}$ defined on $\mathbb{C}_{1/2}$ by $e_n(s) = n^{-s}, n \geq 1$, form an orthonormal basis for $H^2$.
- Accordingly, the reproducing kernel $K_a$ of $H^2$ ($f(a) = \langle f, K_a \rangle$ for all $f \in H^2$) is given by

$$K_a(s) = \sum_{n=1}^{\infty} e_n(s) \overline{e_n(a)} = \zeta(s + \overline{a}), \text{ with } a, s \in \mathbb{C}_{1/2},$$

where $\zeta$ denotes the Riemann zeta function.

Let $H(\Omega)$ denote the space of all analytic functions defined on $\Omega$. If $\varphi: \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ is analytic, then the composition operator

$$C_\varphi : H^2 \to H(\mathbb{C}_{1/2}), \quad C_\varphi(f) = f \circ \varphi,$$
is well defined and we wish to know for which “symbols” \( \varphi \) this operator maps \( \mathcal{H}^2 \) to itself. Then, \( C_{\varphi} \) is a bounded linear operator on \( \mathcal{H}^2 \) by the closed graph theorem. A complete answer to this fairly delicate question was obtained in [9]. A slightly improved version of the same may be stated in the following form, as far as uniform convergence on all half-planes \( \mathbb{C}_\varepsilon \) is concerned. See [19] for details.

**Theorem A.** The analytic function \( \varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2} \) induces a bounded composition operator on \( \mathcal{H}^2 \) if and only if

\[
\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} =: c_0 s + \psi(s),
\]

where \( c_0 \in \mathbb{N}_0 \) and the Dirichlet series \( \sum_{n=1}^{\infty} c_n n^{-s} \) converges uniformly in each half-plane \( \mathbb{C}_\varepsilon \), \( \varepsilon > 0 \). Moreover, \( \psi \) has the following mapping properties:

1. If \( c_0 \geq 1 \), then \( \psi(C_0) \subset C_0 \) and so \( \varphi(C_0) \subset C_0 \).
2. If \( c_0 = 0 \), then \( \psi(C_0) = \varphi(C_0) \subset C_{1/2} \).

In addition to the above formulation, it is worth to mention that \( \|C_{\varphi}\| \geq 1 \) and

\[
\|C_{\varphi}\| = 1 \iff c_0 \geq 1.
\]

This result follows easily from the fact that \( C_{\varphi} \) is contractive on \( \mathcal{H}^2 \) if \( c_0 \geq 1 \) (See [9]).

### 3. A special, but interesting case

To our knowledge, except the recent work of Brevig [4] in a slightly different context, no result has appeared in the literature on sharp evaluations of the norm of \( C_{\varphi} \) when \( c_0 = 0 \). The purpose of this work is to make some attempt, in the apparently simple-minded case

\[
\varphi(s) = c_1 + c_2 2^{-s} \quad \text{with} \quad \text{Re} \, c_1 \geq \frac{1}{2} + |c_2|.
\]

The condition on \( c_1 \) and \( c_2 \) in (3.1) is the exact translation of the mapping conditions of “affine map” to be a map of \( \mathbb{C}_0 \) into \( \mathbb{C}_{1/2} \).

We should point out the fact that, even though the symbol \( \varphi \) is very simple, the boundedness of \( C_{\varphi} \), and its norm, are far from being clear. This is already the case for affine maps \( \varphi(z) = az + b \) from \( \mathbb{D} \to \mathbb{D} \) whose exact norm has a complicated expression first obtained by Cowen [7] and then by the third-named author of this article (see [18]) with a simpler approach based on an adequate use of the Schur test, which we recall in Lemma 3.1 below, under an adapted form.

Finally, we would like to mention the following: In [15], Hurst obtained the norm of \( C_{\varphi} \) on weighted Bergman spaces for the affine symbols whereas in [11], Hammond obtained a representation for the norm of \( C_{\varphi} \) on the Dirichlet space for such affine symbols.
Lemma 3.1. [10] page 24] Let $A = (a_{i,j})_{i \ge 0, j \ge 1}$ be a scalar matrix, formally defining a linear map $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}_0)$ by the formula $A(x) = y$ with $y_i = \sum_{j=1}^{\infty} a_{i,j} x_j$. Assume that there exist two positive numbers $\alpha$ and $\beta$ and two sequences $(p_i)_{i \ge 0}$ and $(q_j)_{j \ge 1}$ of positive numbers such that

$$\sum_{i=0}^{\infty} |a_{i,j}| q_i \le \alpha p_j \quad \text{for all } j \ge 1 \quad (3.2)$$

and

$$\sum_{j=1}^{\infty} |a_{i,j}| p_j \le \beta q_i \quad \text{for all } i \ge 0. \quad (3.3)$$

Then $\|A\| \le \sqrt{\alpha \beta}$.

Remark 3.2. Let $\varphi$ be a map as in (3.1). Then $C_{\varphi}$ is compact operator on $\mathcal{H}_2$ if and only if $\text{Re} c_1 > \frac{1}{2} + |c_2|$ (see [2, Corollary 3]). Also the spectrum of $C_{\varphi}$ is

$$\sigma(C_{\varphi}) = \{0, 1\} \cup \{[\varphi'(\alpha)]^k : k \in \mathbb{N}\},$$

where $\alpha$ is the fixed point in $C_{1/2}$ (see [2, Theorem 4]). Since the spectrum $\sigma(C_{\varphi})$ is compact, we have $|\varphi'(\alpha)| < 1$ and thus the spectral radius

$$r(C_{\varphi}) := \sup\{ |\lambda| : \lambda \in \sigma(C_{\varphi}) \}$$

is equal to 1.

In [13], Hedenmalm asked for estimate from above for the norm $\|C_{\varphi}\|$ in terms of $\varphi(+\infty)$, that is, $c_1$ for the map $\varphi(s) = c_1 + c_2 2^{-s}$. We give a partial answer to his question at least for this special choice of $\varphi$. To do this, we list below some useful lemmas here.

Lemma 3.3. Let $s > 1$. Then, we have

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{s}{s-1}. \quad (3.4)$$

Proof. The result follows, by comparison with an integral, from the fact that $x \mapsto x^{-s}$ is decreasing for $s > 1$. See for instance, [17, p. 299]. Indeed for $f(x) = x^{-s} = e^{-s \ln x}$, we have

$$\int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} f(k) \leq f(1) + \int_1^{\infty} f(x)dx,$$

from which one can obtain the desired inequality, since $\int_1^{\infty} f(x)dx = \frac{1}{s-1}$. \qed

Lemma 3.4. For all $s > 1$, we have

$$\frac{1}{s-1} + \left( \frac{s-1}{s} \right) \frac{1}{\sqrt{2\pi}} \leq \zeta(s). \quad (3.4)$$
Proof. Let
\[ h(s) = \frac{1}{s-1} + \left( \frac{s-1}{s} \right) \frac{1}{\sqrt{2\pi}}. \]
Then, we observe that both \( h \) and \( \zeta \) are decreasing functions on \((1, \infty)\). Thus,
\[ h(s) \leq h(3) = \frac{1}{2} + \frac{1}{3} \frac{1}{\sqrt{2\pi}} < \frac{1}{2} + \frac{1}{3} = 1 < \zeta(s) \quad \text{for all } s \geq 3. \]
This shows that the inequality (3.4) is true for \( s \geq 3 \). Now we need to verify the inequality (3.4) only for \( 1 < s < 3 \). By setting \( s = x + 1 \), it is enough to prove that
\[ h(x + 1) = \frac{1}{x} + f(x) \leq \zeta(x + 1) \quad \text{for } 0 < x < 2, \]
where
\[ f(x) = \frac{1}{\sqrt{2\pi}} \left( \frac{x}{x+1} \right). \]
Clearly, \( f \) is an increasing function on \( x > 0 \). From [4, Lemma 10], we have
\[ \frac{1}{x} + g(x) \leq \zeta(1 + x) \quad \text{for } x > 0, \]
where
\[ g(x) = \frac{1}{2} + \frac{x+1}{12} - \frac{(x+1)(x+2)(x+3)}{6!} = \frac{1}{6!} (414 + 49x - 6x^2 - x^3). \]
In view of [4, Lemma 10], it suffices to show that \( f(x) \leq g(x) \) on \((0, 2)\). For \( 0 < x < 2 \),
\[ g'(x) = \frac{1}{6!} (49 - 3x(x+4)) > 0, \]
which shows that \( g \) is increasing on \((0, 2)\). Since
\[ f(2) = \frac{1}{3} \frac{\sqrt{2}}{\pi} < \frac{1}{3} < g(0) = \frac{23}{40}, \]
we have \( f(x) \leq f(2) \leq g(0) \leq g(x) \) for all \( 0 < x < 2 \). This proves the claim for \( 0 < x < 2 \), i.e., \( 1 < s < 3 \). In conclusion, the inequality (3.4) is verified for all \( s > 1 \). \( \square \)

Remark 3.5. Consider the functions \( f \) and \( g \) as in Lemma 3.4. Thus, \( \frac{1}{x} + f(x) \) and \( \frac{1}{x} + g(x) \) both forms a lower bound for \( \zeta(1 + x) \) for \( x > 0 \). For \( x > 3 \), we have
\[ g'(x) = -\frac{1}{6!} (3x(x+4) - 49) < 0, \]
which shows that \( g \) is decreasing on \((3, \infty)\) and therefore, \( g(x) \leq f(x) \) for all \( x > s_2 \approx 6.2102 \), where \( s_2 \) is the unique positive root of the equation given by \( f(x) = g(x) \), i.e.,
\[ \left( \frac{x}{x+1} \right) \frac{1}{\sqrt{2\pi}} = \frac{1}{2} + \frac{x+1}{12} - \frac{(x+1)(x+2)(x+3)}{6!}. \]
Figure 1. The range for $x$ varies from 0.1 to 10

It follows that Lemma 3.4 is an improved version of [4, Lemma 10] for $x \geq s_2$. For a quick comparison with the zeta function, in Figure 1, we have drawn the graphs of $(1/x) + f(x)$, $(1/x) + g(x)$ and $\zeta(x + 1)$.

Remark 3.6. Before seeing the work of [4], we made use of a result of Lavrik [16]: For $1 < s < 3$,

$$\zeta(s) - \frac{1}{s-1} - \gamma = \sum_{n=1}^{\infty} \frac{\gamma_n}{n!}(s-1)^n,$$

where $\gamma$ is the Euler constant and $|\gamma_n| \leq \frac{n!}{2^n + 1}$. We thus obtained an alternative proof of (3.4).

Lemma 3.7. If $s > 1$, $i \geq 1$ is an integer, and $f(x) = \left(\frac{\log x}{x^i}\right)^s$, then one has

$$\sum_{k=1}^{\infty} f(k) \leq \frac{i!}{(s-1)^{i+1}}\zeta(s).$$

Proof. The function $f$ increases for $x \leq e^{i/s}$ and then decreases for $x \geq e^{i/s}$. By a simple change of variables, we have

$$I = \int_1^{\infty} f(x)dx = \frac{i!}{(s-1)^{i+1}}.$$

Let $N \geq 1$ be the integral part of $e^{i/s}$, so that $N \leq e^{i/s} < N + 1$. Computations give, with help of Stirling’s inequality $(i/e)^i \leq \frac{i!}{\sqrt{2\pi i}}$:

$$\sum_{k=1}^{N-1} f(k) \leq \int_1^{N} f(x)dx$$
and
\[ \sum_{k=N+2}^{\infty} f(k) \leq \int_{N+1}^{\infty} f(x) dx. \]

It follows that
\[ \int_{N}^{N+1} f(x) dx \geq \begin{cases} f(N) & \text{if } f(N) \leq f(N+1) \\ f(N+1) & \text{otherwise,} \end{cases} \]
and therefore,
\[ f(N) + f(N+1) - \int_{N}^{N+1} f(x) dx \leq f(e^{i/s}) = \frac{(i/s)^i}{e^i} \leq \frac{i!}{\sqrt{2\pi is}}. \]

From the above three inequalities, we get that
\[ \sum_{k=1}^{\infty} f(k) \leq I + f(e^{i/s}) \leq \frac{i!}{(s-1)^{i+1} + \sqrt{2\pi is}} \leq \frac{i!}{(s-1)^i} \zeta(s). \]

The third and the fourth inequalities follow from \( s-1 < 1 \) and Lemma 3.4, respectively. This completes the proof of the lemma.

Our next result provides bounds for the norm estimate of \( C_{\varphi} \) on both sides.

**Theorem 3.8.** Let \( \varphi(s) = c_1 + c_2 2^{-s} \) with \( \text{Re } c_1 \geq \frac{1}{2} + |c_2| \) and \( c_2 \neq 0 \), thus inducing a bounded composition operator \( C_{\varphi} : \mathcal{H}^2 \rightarrow \mathcal{H}^2 \). Then, we have
\[ \zeta(2\text{Re } c_1) \leq \|C_{\varphi}\|^2 \leq \zeta(2\text{Re } c_1 - r|c_2|), \]
where \( r \leq 1 \) is the smallest positive root of the quadratic polynomial
\[ P(r) = |c_2|r^2 + (1 - 2\text{Re } c_1)r + |c_2|. \]

**Remark 3.9.** Observe that \( P \) has two positive roots with product 1, so one of them is less than or equal to 1 (because \( P(0) > 0 \) and \( P(1) \leq 0 \)) and by our assumption \( 2\text{Re } c_1 - r|c_2| \geq 2\text{Re } c_1 - |c_2| \geq 1 + |c_2| > 1 \), so that \( \zeta(2\text{Re } c_1 - r|c_2|) \) is well defined.

**Proof of Theorem 3.8.** Without loss of generality, we can assume that \( c_1 \) and \( c_2 \) are positive. Indeed, in the general case, for \( \varphi(s) = c_1 + c_2 2^{-s} \), we set \( c_1 = \sigma_1 + it_1 \) and \( c_2 = |c_2|2^{i\varphi_2} \). Note that \( \text{Re } c_1 = \sigma_1 > 0 \) by our assumption of the theorem. Consider the two vertical translations \( T_1 \) and \( T_2 \) defined respectively by \( T_1(s) = s + it_1 \) and \( T_2(s) = s - i\varphi_2 \), and set \( \psi(s) = \sigma_1 + |c_2|2^{-s} \). Then, one has \( \varphi = T_1 \circ \psi \circ T_2 \) whence
\[ C_{\varphi} = C_{T_2} \circ C_{\psi} \circ C_{T_1}. \]
where $C_{T_2}$ and $C_{T_1}$ are unitary operators.

Note that $C_{\varphi}(1) = 1$. Now for $j > 1$, we see that

$$C_{\varphi}(j^{-s}) = j^{-c_1} \exp(-c_2 2^{-s} \log j) = j^{-c_1} \sum_{i=0}^{\infty} \frac{(-c_2 \log j)^i}{i!} (2^i)^{-s}.$$ 

In other terms, considering the orthonormal system $\{2^i\}^{-s}_{i \geq 0}$ as the canonical basis of the range of $C_{\varphi}$ and the orthonormal system $\{j^{-s}\}_{j \geq 1}$ as the canonical basis of $H^2$, $C_{\varphi}$ can be viewed as the matrix $A = (a_{i,j})_{i \geq 0, j \geq 1}: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}_0)$ with

$$a_{i,1} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases},$$

and

$$a_{i,j} = j^{-c_1} \frac{(-c_2 \log j)^i}{i!} \text{ for } i \geq 0, j > 1.$$ 

By Theorem A, we already know that $A$ is bounded. We will give a direct proof of this fact, and moreover an upper and lower estimates of its norm. To that effect, we apply the Schur test with the following values of the parameters

$$\alpha = 1, \quad \beta = \zeta(2c_1 - rc_2), \quad p_j = j^{rc_2-c_1} \quad \text{and} \quad q_i = r^i.$$ 

Now, we can check the assumptions of Schur’s lemma. Equality holds trivially in the inequality (3.2) for the case of $j = 1$. For $j > 1$,

$$\sum_{i=0}^{\infty} |a_{i,j}| q_i = \sum_{i=0}^{\infty} j^{-c_1} \frac{(c_2 \log j)^i}{i!} r^i = j^{rc_2-c_1} = \alpha p_j$$

Thus, the inequality (3.2) is verified. Now, we verify the inequality (3.3). For the case $i = 0$, we have

$$\sum_{j=1}^{\infty} |a_{0,j}| p_j = \sum_{j=1}^{\infty} j^{-2c_1-rc_2} = \zeta(2c_1 - rc_2) \leq \beta q_0.$$ 

Finally, for $i \geq 1$, with the help of Lemma 3.7, we have

$$\sum_{j=1}^{\infty} |a_{i,j}| p_j = \frac{c_2}{i!} \sum_{j=2}^{\infty} \frac{(\log j)^i}{j^{2c_1-rc_2}} \leq \frac{c_2}{i!} \frac{i!}{(2c_1 - rc_2 - 1)} \zeta(2c_1 - rc_2) = \beta q_i,$$

where $\frac{c_2}{2c_1-rc_2-1} = r$, that is, $P(r) = 0$. The assumptions of the Schur lemma with the claimed values are thus verified, and the upper bound ensues.

For the lower bound, we use reproducing kernels as usual (recall that $C_{\varphi}^*(K_a) = K_{\varphi(a)}$):

$$\|C_{\varphi}\|^2 \geq (S_{\varphi}^*)^2 := \sup_{a \in C_{1/2}} \frac{\|K_{\varphi(a)}\|^2}{\|K_a\|^2} = \sup_{a \in C_{1/2}} \frac{\zeta(2 \operatorname{Re} \varphi(a))}{\zeta(2 \operatorname{Re} a)} = \sup_{x > 1/2} \frac{\zeta(2c_1 - 2c_2 2^{-x})}{\zeta(2x)}.$$ 

The last equality in the above is obtained from basic trigonometry and the fact that $\zeta(s)$ is a decreasing function on $(1, \infty)$. Now by letting $x \to \infty$, we get the lower bound for $\|C_{\varphi}\|$.

$\square$
**Corollary 3.10.** Let \( \varphi(s) = c_1 + c_22^{-s} \) with \( \text{Re} c_1 = \frac{1}{2} + |c_2| \) and \( c_2 \neq 0 \). Then, for the inducing composition operator \( C_\varphi : \mathcal{H}^2 \to \mathcal{H}^2 \), we have
\[
\zeta(2\text{Re} c_1) = \zeta(1 + 2|c_2|) \leq \|C_\varphi\|^2 \leq \zeta(1 + |c_2|) = \zeta(2\text{Re} c_1 - |c_2|).
\]

**Proof.** It suffices to observe that \( r = 1 \) in Theorem 3.8 when \( \text{Re} c_1 = \frac{1}{2} + |c_2| \). \( \square \)

**Remark 3.11.** From the proof of Theorem 3.8, it is evident that the lower bound of \( \|C_\varphi\| \) continues to hold for any composition operator \( C_\varphi \) with \( c_0 = 0 \) in (2.2), namely, for any \( \varphi(s) = \sum_{n=1}^\infty c_n n^{-s} \).

**Remark 3.12.**
(a) Note that, if \( c_2 = 0 \), then \( \varphi \) becomes a constant map and the induced composition operator \( C_\varphi \) is the evaluation map at \( c_1 \). Also it is known that
\[
\|C_\varphi\|^2 = \zeta(2\text{Re} c_1).
\]
(b) Let \( \varphi \) be a map as in (3.1). Then \( C_\varphi \) cannot be a normal operator. More generally, it cannot be a normaloid operator because,
\[
r(C_\varphi) = 1 < \sqrt{\zeta(2\text{Re} c_1) \leq \|C_\varphi\|}.
\]
(see Remark 3.2 and Theorem 3.8).

### 4. Approximation numbers

Recall that the \( N^{th} \) approximation number \( a_N(T) \), \( N = 1, 2, \ldots \), of an operator \( T : H \to H \), where \( H \) is a Hilbert space, is the distance (for the operator norm) of \( T \) to operators of rank \( < N \). We refer to [5] for the definition and basic properties of those numbers. In the case \( \varphi(z) = az + b \) on \( H^2 \) with \( |a| + |b| \leq 1 \), Clifford and Dabkowski [6] computed exactly the approximation numbers \( a_N(C_\varphi) \). In the compact case \( |a| + |b| < 1 \), they [6] showed in particular that
\[
a_N(C_\varphi) = |a|^{N-1}Q^{N-1/2} \quad \text{for all } N \geq 1,
\]
where
\[
Q = \frac{1 + |a|^2 - |b|^2 - \sqrt{\Delta}}{2|a|^2}
\]
and where \( \Delta > 0 \) is a discriminant depending on \( a \) and \( b \).

It is natural to ask whether we could get something similar for \( \varphi(s) = c_1 + c_22^{-s} \) and the associated \( C_\varphi \) acting on \( \mathcal{H}^2 \). We have here the following upper bound, in which \( 2\text{Re} c_1 - 2|c_2| - 1 \) is assumed to be positive which is indeed a necessary and sufficient condition for the compactness of \( C_\varphi \).

**Theorem 4.1.** Assume that \( 2\text{Re} c_1 - 2|c_2| - 1 > 0 \). Then the following exponential decay holds:
\[
a_{N+1}(C_\varphi) \leq \sqrt{\frac{(2\text{Re} c_1 - 1)(2\text{Re} c_1)}{(2\text{Re} c_1 - 1)^2 - (2|c_2|)^2}} \left(\frac{2|c_2|}{2\text{Re} c_1 - 1}\right)^N.
\]
Proof. Without loss of generality, we can assume that $c_1$ and $c_2$ are non-negative. Let $f(s) = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{H}^2$. Then

$$C_\varphi f(s) = \sum_{n=1}^{\infty} b_n n^{-c_1} \exp(-c_2 2^{-s} \log n)$$

$$= \sum_{k=0}^{\infty} \frac{(-c_2)^k}{k!} \left( \sum_{n=1}^{\infty} b_n n^{-c_1} (\log n)^k \right) 2^{-ks}.$$ 

Thus, designating by $R$ the operator of rank $\leq N$ defined by

$$Rf(s) = \sum_{k=0}^{N-1} \frac{(-c_2)^k}{k!} \left( \sum_{n=1}^{\infty} b_n n^{-c_1} (\log n)^k \right) 2^{-ks},$$

we obtain via the classical Cauchy-Schwarz inequality that

$$\|C_\varphi(f) - R(f)\|^2 = \sum_{k=N}^{\infty} \frac{c_2^{2k}}{k!^2} \left( \sum_{n=1}^{\infty} b_n n^{-c_1} (\log n)^k \right) \left( \sum_{n=1}^{\infty} \frac{(\log n)^{2k}}{n^{2c_1}} \right).$$

By Lemma 3.7, the latter sum is nothing but

$$\sum_{n=1}^{\infty} \frac{(\log n)^{2k}}{n^{2c_1}} = \zeta(2k)(2c_1)$$

$$\leq \frac{(2k)!}{(2c_1 - 1)^{2k}} \zeta(2c_1)$$

$$\leq \frac{(2k)!(2c_1)}{(2c_1 - 1)^{2k+1}}.$$ 

The last inequality follows by the simple fact that $\zeta(s) \leq \frac{s}{s-1}$ (see Lemma 3.3). Since

$$\sum_{n=1}^{\infty} |b_n|^2 = \|f\|^2 \quad \text{and} \quad \frac{(2k)!}{(k!)^2} \leq \sum_{j=0}^{2k} \binom{2k}{j} = 4^k,$$

we get the following:

$$\|C_\varphi - R\|^2 \leq \sum_{k=N}^{\infty} \frac{c_2^{2k}}{(k!)^2} \frac{(2k)!(2c_1)}{(2c_1 - 1)^{2k+1}}$$

$$\leq \sum_{k=N}^{\infty} \left( \frac{2c_2}{2c_1 - 1} \right)^{2k} \frac{2c_1}{2c_1 - 1}$$

$$= \frac{2c_1(2c_1 - 1)}{(2c_1 - 1)^2 - (2c_2)^2} \left( \frac{2c_2}{2c_1 - 1} \right)^{2N}.$$ 

Thus, we complete the proof. \qed
5. Comments and questions

- Is there a symbol $\varphi$ for which the strict inequalities
  $$\|C_\varphi\| > S^*_\varphi > S_\varphi$$
  hold for $C_\varphi$ on $H^2$? (refer to [1] for similar problem in the case of classical Hardy space $H^2$). In the case $\varphi(s) = c_1 + c_2 2^{-s}$, we probably have
  $$\|C_\varphi\| = S^*_\varphi = S_\varphi,$$
  but this still needs a proof. Also observe that this $\varphi$ is not injective on $\mathbb{C}_{1/2}$.
- What can be said about $\|C_\varphi\|$ acting on $H^2(\Omega)$, where $\Omega$ is the ball $B_d$, or the polydisk $D_d$, when $\varphi(z) = A(z) + b$ with $A : \mathbb{C}^d \to \mathbb{C}^d$ a linear operator, i.e. when $\varphi$ is an affine map such that $\varphi(\Omega) \subset \Omega$? This might be difficult [3], but interesting.

Acknowledgement

The authors thank the referee for many useful comments. The first author thanks the Council of Scientific and Industrial Research (CSIR), India, for providing financial support in the form of a SPM Fellowship to carry out this research. The third author thanks the Indian Statistical Institute of Chennai for providing good and friendly working conditions in December 2015, when this collaboration was initiated.

References

[1] M. J. Appel, P. S. Bourdon and J. J. Thrall, *Norms of composition operators on the Hardy space*, Experiment. Math. 5(2)(1996), 111–117.
[2] F. Bayart, *Compact composition operators on a Hilbert space of Dirichlet series*, Illinois J. Math. 47(3)(2003), 725–743.
[3] F. Bayart, *Composition operators on the polydisk induced by affine maps*, J. Funct. Anal. 260(7)(2011), 1969–2003.
[4] O. F. Brevig, *Sharp norm estimates for composition operators and Hilbert-type inequalities*, Bull. Lond. Math. Soc. 49(6)(2017), 965–978.
[5] B. Carl and I. Stephani, *Entropy, compactness and the approximation of operators*, Cambridge University Press, Cambridge, 1990.
[6] J. H. Clifford and M. G. Dabkowski, *Singular values and Schmidt pairs of composition operators on the Hardy space*, J. Math. Anal. Appl. 305(1)(2005), 183–196.
[7] C. C. Cowen, *Linear fractional composition operators on $H^2$*, Integral Equations Operator Theory 11(2)(1988), 151–160.
[8] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Florida, 1995.
[9] J. Gordon and H. Hedenmalm, *The composition operators on the space of Dirichlet series with square summable coefficients*, Michigan Math. J. 46(2)(1999), 313–329.
[10] P. R. Halmos, *A Hilbert space problem book*, Second edition, Springer-Verlag, New York-Berlin, 1982.

[11] C. Hammond, *The norm of a composition operator with linear symbol acting on the Dirichlet space*, J. Math. Anal. Appl. **303**(2)(2005), 499–508.

[12] H. Hedenmalm, P. Lindqvist and K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$*, Duke Math. J. **86**(1)(1997), 1–37.

[13] H. Hedenmalm, *Dirichlet series and functional analysis*, The legacy of Niels Henrik Abel, pp. 673–684, Springer, Berlin, 2004.

[14] E. Hlawka, J. Schoissengeier and R. Taschner, *Geometric and analytic number theory*, Universitext Springer-Verlag, Berlin, 1991.

[15] P. R. Hurst, *Relating composition operators on different weighted Hardy spaces*, Arch. Math. (Basel) **68**(6)(1997), 503–513.

[16] A. F. Lavrik, *On the main term of the divisor’s problem and the power series of the Riemann’s zeta function in a neighbourhood of its pole* (in Russian), Trudy Mat. Inst. Akad. Nauk. SSSR **142**(1976), 165–173.

[17] S. Ponnusamy, *Foundations of mathematical analysis*, Birkhäuser/Springer, New York, 2012.

[18] H. Queffélec, *Norms of composition operators with affine symbols*, J. Anal. **20**(2012), 47–58.

[19] H. Queffélec and K. Seip, *Approximation numbers of composition operators on the $H^2$ space of Dirichlet series*, J. Funct. Anal. **268**(6)(2015), 1612–1648.

[20] K. Zhu, *Operator Theory in Function Spaces*, Second edition, Mathematical Surveys and Monographs, Vol. 138, American Mathematical Society, Providence, RI, 2007.

Perumal Muthukumar  
Stat-Math Unit, Indian Statistical Institute (ISI), Chennai Centre, 110, Nelson Manickam Road, Aminjikarai, Chennai, 600 029, India.  
e-mail: pmuthumaths@gmail.com

Saminathan Ponnusamy  
Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India  
e-mail: samy@iitm.ac.in, samy@isichennai.res.in

Hervé Queffélec  
Univ Lille Nord de France F-59, 000 Lille, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524, F-59 655 Villeneuve D’ascq Cedex, France.  
e-mail: herve.queffelec@univ-lille1.fr