ANALOGUES OF THE BALOG–WOOLEY DECOMPOSITION
FOR SUBSETS OF FINITE FIELDS AND CHARACTER SUMS
WITH CONVOLUTIONS

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Abstract. Balog and Wooley have recently proved that any subset $A$ of either real numbers or of a prime finite field can be decomposed into two parts $U$ and $V$, one of small additive energy and the other of small multiplicative energy. In the case of arbitrary finite fields, we obtain an analogue that under some natural restrictions for a rational function $f$ both the additive energies of $U$ and $f(V)$ are small. Our method is based on bounds of character sums which leads to the restriction $|A| > q^{1/2}$ where $q$ is the field size. The bound is optimal, up to logarithmic factors, when $|A| \geq q^{9/13}$. Using $f(X) = X^{-1}$ we apply this result to estimate some triple additive and multiplicative character sums involving three sets with convolutions $ab + ac + bc$ with variables $a, b, c$ running through three arbitrary subsets of a finite field.

1. Introduction

1.1. Background. Let $\mathbb{F}_q$ denote the finite field of $q$ elements of characteristic $p$.

Given two sets $U, V \subseteq \mathbb{F}_q$, as usual, we define their sum and product sets as

$U + V = \{ u + v : u \in U, v \in V \}$ and $U \cdot V = \{ uv : u \in U, v \in V \}$.

The sum-product problem is concerned with proving that, for a given set $U$ in a field $\mathbb{F}$, at least one of $U + U$ and $U \cdot U$ has cardinality significantly larger than the original set $U$. This problem has been widely studied in the finite field setting in recent years, originating from the work of Bourgain, Katz and Tao [8] and subsequently Bourgain,
Glibichuk and Konyagin [7] in proving that for some absolute constants $c, \varepsilon > 0$, the bound
\begin{equation}
\max \{ \#(U + U), \#(U \cdot U) \} \geq c(\#U)^{1+\varepsilon}
\end{equation}
holds for all $U \subseteq \mathbb{F}_p$, subject to certain necessary restrictions on $\#U$. See [17] for the best estimates for this problem and for further background on sum-product estimates.

A basic tool in sum-product estimates is the notion of different kinds of energy. The additive energy $E(U)$ of the set $U \subseteq \mathbb{F}_q$ is defined as
\[
E(U) = \# \{ (u_1, u_2, u_3, u_4) \in U^4 : u_1 + u_2 = u_3 + u_4 \}.
\]
Note that the multiplicative energy, denoted by $E^*(U)$, is defined similarly with respect to the equation $u_1u_2 = u_3u_4$. It follows from a straightforward application of the Cauchy-Schwarz inequality that
\[
E(U) \geq \frac{(\#U)^4}{\#(U + U)},
\]
and so good upper bounds on the additive energy of $U$ translate into good lower bounds for the size of the sum set of $U$. Similarly, good upper bounds on the multiplicative energy of $U$ translate into good lower bounds for the size of the product set of $U$.

In the spirit of the sum-product problem, one may naively expect that an analogue of the inequality (1.1) holds, and that at least one of $E(U)$ and $E^*(U)$ must be small. This is not true, as can be seen by taking $U$ to be the union of an arithmetic progression and a geometric progression of the same size. However, Balog and Wooley [2] have shown something of this nature when they proved (in both the Euclidean and finite field setting) that the set $U$ can be written as a union of $V$ and $W$ such that $E(V)$ and $E^*(W)$ are both small. These results were improved quantitatively in [13] and [19].

1.2. An analogue of the Balog-Wooley Theorem. Our main result is a generalisation of the Balog-Wooley decomposition [2, Theorem 1.3].

For real $Z > 1$ we define the quantity
\begin{equation}
M(Z) = \min \left\{ \frac{q^{1/2}}{Z^{1/2}(\log Z)^{11/4}}, \frac{Z^{4/5}}{q^{2/5}(\log Z)^{31/10}} \right\}.
\end{equation}

As usual, we use the expressions $F \ll G$, $G \gg F$ and $F = O(G)$ to mean $|F| \leq cG$ for some constant $c > 0$. If the constant $c$ depends on a parameter $k$, we write $F = O_k(G)$ or $F \ll_k G$. We also write
\( F(x) = o(G(x)) \) as an equivalent to \( \lim_{x \to \infty} F(x)/G(x) = 0 \). Throughout the paper, we always use:
\[
\#A = A, \quad \#B = B, \quad \#C = C.
\]
We denote by \( p \) the characteristic of \( \mathbb{F}_q \).

**Theorem 1.1.** For any set \( A \subseteq \mathbb{F}_q \) and any rational function \( f \in \mathbb{F}_q(X) \) of degree \( k \) which is not of the form \( g(X)^p - g(X) + \lambda X + \mu \), there exist disjoint sets \( S, T \subseteq A \) such that \( A = S \cup T \) and with
\[
\max \{ E(S), E(f(T)) \} \ll_p A^3 \frac{A^3}{M(A)}.
\]

One may check that Theorem 1.1 is non-trivial when \( A \geq q^{1/2+\varepsilon} \), for any fixed \( \varepsilon > 0 \). For comparison, note that the decomposition results in [2] and [19] are applicable below this range. This is because the main tool in [2] and [19] is a strong new point-plane incidence bound of Rudnev [18], whereas our main tool is the Weil bound. It may be within reach to obtain a version of Theorem 1.1 which is non-trivial for smaller sets by finding a way to apply new results in incidence theory, but we have been unable to do this in the present paper.

Theorem 1.1 covers some particularly natural choices of functions such as \( f(X) = X^{-1} \) and \( f(X) = X^2 \) for odd \( q \) which have been seen in sum-product literature before. For example, for these two functions, it is known (see [1, Propositions 12 and 14]) that
\[
\#\{ u + v^2 : u, v \in U \} \gg (\#U)^{11/10} \quad \text{if} \quad \#U < p^{3/5}
\]
and
\[
\#\{ u + v^{-1} : u, v \in U \} \gg (\#U)^{31/30} \quad \text{if} \quad \#U < p^{5/8}
\]
(we remark that the size of \( U \) is bounded in terms of the characteristic \( p \) rather than of \( q \)). The moral here is that a non-linear function \( f \) destroys any additive structure that originally exists in a set. A version of Theorem 1.1 with \( A \subseteq \mathbb{C} \) and \( f \in \mathbb{C}(X) \) defined by \( f(X) = X^{-1} \) has been given in [19, Theorem 9].

Note that the bounds (1.3) and (1.4) hold for smaller sets. This gives another hint that it may be possible to obtain a version of Theorem 1.1 that gives a non-trivial bound below the square root threshold for certain special functions \( f \).

The proof of Theorem 1.1 is partly based on the work of Rudnev, Shkredov and Stevens [19]. We believe that it is of independent interest and may have several other applications.
1.3. The tightness of the bound and conditions of Theorem 1.1. Theorem 1.1 becomes increasingly accurate as \( A \) gets larger, and in fact is optimal up to logarithmic factors when \( A \geq q^{9/13} \), as the following example over the prime field \( \mathbb{F}_p \) illustrates. We consider the case when \( f(X) = X^{-1} \), although a natural adaptation of the construction works for any rational function \( f \) of degree \( k \), with the construction getting slightly worse as \( k \) increases. This is an adaptation of a construction from finite field sum-product theory, see Garaev [9, page 2736] for a presentation.

Let \( \mathcal{J} \) be the interval \( \{1, 2, \ldots, \lambda\} \), for an integer parameter \( \lambda < p \) to be chosen later, and then cover \( \mathbb{F}_p \) by \([p/\lambda] \ll p/\lambda\) disjoint intervals of size at most \( \lambda \). By the pigeonhole principle, one of these intervals \( \mathcal{J}_0 \) has the property that \( \#(\mathcal{J}_0 \cap f(\mathcal{J})) \gg \lambda^2/p \). Define \( \mathcal{A} = \mathcal{J}_0 \cap f(\mathcal{J}) \). Then it follows that \( \#(\mathcal{A} + \mathcal{A}) \ll \lambda \ll (Ap)^{1/2} \) and \( \#(f(\mathcal{A}) + f(\mathcal{A})) \ll \lambda \ll (Ap)^{1/2} \). For an arbitrary function \( f \) one has to work with the preimage \( f^{-1}(\mathcal{J}) \) of \( \mathcal{J} \) (note that \( f(X) = X^{-1} \) we have \( f^{-1} = f \) and thus \( f(\mathcal{J}) = f^{-1}(\mathcal{J}) \)).

Now, we can apply Theorem 1.1 to this set, obtaining a decomposition \( \mathcal{A} = \mathcal{S} \cup \mathcal{T} \). One of these sets has cardinality at least \( A/2 \), and without loss of generality we assume that \( \mathcal{S} = \#\mathcal{S} \geq A/2 \). Then, by the Cauchy-Schwarz inequality,

\[
A^4 \ll \mathcal{S}^4 \leq \#(\mathcal{S} + \mathcal{S})E(\mathcal{S}) \leq \#(\mathcal{A} + \mathcal{A})E(\mathcal{S}) \ll (Ap)^{1/2}E(\mathcal{S}),
\]

and so

\[
E(\mathcal{S}) \gg \frac{A^3}{(p/A)^{1/2}}.
\]

When \( A \geq p^{9/13} \) (that is, for any choice of \( \lambda \geq p^{11/13} \)) we have \( M(A) \gg (p/A)^{1/2}(\log A)^{-11/4} \), hence this lower bound matches the upper bound given by Theorem 1.1, up to logarithmic factors.

The following example shows that the condition on \( f \) in Theorem 1.1 is needed: let \( \mathcal{A} \) be any subset of \( \mathbb{F}_q \) of additive energy \( E(\mathcal{A}) \gg A^3 \) such as an arithmetic progression or an additive subgroup. Then by the forthcoming Lemma 2.4 we have \( \max\{E(\mathcal{S}), E(\mathcal{T})\} \gg A^3 \) for any decomposition \( \mathcal{A} = \mathcal{S} \cup \mathcal{T} \) of \( \mathcal{A} \). Now if \( f(X) = \sum a_iX^{p^i} \in \mathbb{F}_q[X] \) is a linearized permutation polynomial, and thus of the form \( g(X)^p - g(X) + \lambda X + \mu \), we have \( f(a) + f(b) = f(a + b) \) and so

\[
\max\{E(\mathcal{S}), E(f(\mathcal{T}))\} = \max\{E(\mathcal{S}), E(\mathcal{T})\} \gg A^3.
\]

1.4. Applications to character sums. We use \( \Psi \) and \( \mathcal{X} \) to denote, respectively, the sets of additive and multiplicative characters in \( \mathbb{F}_q \),
see [12, 14] for some background on characters. Furthermore, we use \( \Psi^* \) and \( \mathcal{X}^* \) to denote the sets of nontrivial characters.

Given three sets \( A, B, C \subseteq \mathbb{F}_q \) we define the following sums of additive and multiplicative characters

\[
S_\psi(A, B, C) = \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \psi(ab + ac + bc), \quad \psi \in \Psi,
\]

and

\[
T_\chi(A, B, C) = \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \chi(ab + ac + bc), \quad \chi \in \mathcal{X}.
\]

Our interest to these sums is motivated by recent progress in bounds of additive and multiplicative character sums involving three sets, see [6, 15] and [2, 11, 20], respectively.

For \( \psi \in \Psi^* \) and \( \chi \in \mathcal{X}^* \) we have the classical bounds of double sums

\[
\max \left\{ \left| \sum_{b \in B, c \in C} \psi(bc) \right|, \left| \sum_{b \in B, c \in C} \chi(bc + 1) \right| \right\} = O(\sqrt{BCq}),
\]

see, for example, [10, Corollaries 1 and 5], where the constant in the symbol “\( O \)” is absolute and can be easily evaluated, (in fact it can be taken as 1 for additive character sums and also for multiplicative character sums if \( B \subseteq \mathbb{F}_q^* \) or \( C \subseteq \mathbb{F}_q^* \)). These immediately yield the bounds

\[
S_\psi(A, B, C) = O\left( A\sqrt{BCq} \right)
\]

and, provided \( 0 \notin A, \)

\[
T_\chi(A, B, C) = O\left( A\sqrt{BCq} \right).
\]

Note that (1.5) and (1.6) are best possible in general. For example, take \( q = r^2 \), for a prime power \( r \), then it is easy to check that with \( A = \mathbb{F}_r \), resp. \( A = \mathbb{F}_r^* \), \( B = C = \mathbb{F}_r \) and any \( \psi \in \Psi^* \) and \( \chi \in \mathcal{X}^* \) which are trivial on \( \mathbb{F}_r \), these bounds are attained.

For additive character sums we also provide an example when \( q = p \) is prime. Take \( A = B = C = \{0, 1, 2, \ldots, \lfloor 0.1p^{1/2} \rfloor \} \). Then we have \( 0 \leq ab + ac + bc \leq 0.03p \) for any \( a, b, c \in A \times B \times C \) and thus for the additive canonical character \( \psi(x) = \cos(2\pi x/p) + i \sin(2\pi x/p) \) of \( \mathbb{F}_p \) we get \( |S_\psi(A, B, C)| \geq ABC \cos(0.06\pi) \geq 0.98ABC \) which is of the same order of magnitude \( p^{3/2} \) as \( A\sqrt{BCp} \).
However, if, say, $C$ is a sufficiently large structured set, we can get improvements. For example, if $C$ is an additive subgroup of $\mathbb{F}_q$, we get
\[
S_x(A, B, C) \leq A \sum_{b \in \mathbb{F}_q} \left| \sum_{c \in C} \psi(bc) \right| \leq Aq
\]
by [22, Lemma 3.4], which improves (1.5) if $q = o(BC)$, as well as,
\[
T_x(A, B, C) \leq \min\{A, B\}C + ABq^{1/2}
\]
by [22] (Remark (iii) below Theorem 3.7), which improves (1.6) if $B = o(C)$. Similar results can be obtained for other structured sets $C$ such as arithmetic or geometric progressions. If the sets $A$ and $B$ are also structured, one can take further advantage of this.

In passing, we note that, sum-product and incidence theory can be used to show that
\[
C \cup A, B, C = \{ab + ac + bc : a \in A, b \in B, c \in C\}
\]
is always large. A recent result of Pham, Vinh and de Zeeuw [16] gives a lower bound if $A = B = C$,
\[
\#C(A, B, C) \gg \min\{p, A^{3/2}\}.
\]
There is little doubt that it can be extended to cover the case when the variables come from sets $A, B, C$ of different sizes.

Although we have not been able to improve (1.5) and (1.6), we obtain several related results about the structure of the set $C(A, B, C)$. For example, we use Theorem 1.1 with $f(x) = x^{-1}$, to show that any sufficiently large set $B \subseteq \mathbb{F}_q$ contains a large subset $W$ such that for any set $A \subseteq \mathbb{F}_q$ at least one of $S_x(A, W, W)$ or $T_x(A, W, W)$ can be estimated nontrivially.

**Theorem 1.2.** For any sets $A, B \subseteq \mathbb{F}_q^*$ there exists a subset $W \subseteq B$ of cardinality $W \geq B/2$ such that for any characters $(\chi, \psi) \in \Psi^* \times \chi^*$ we have
\[
\min \{ |S_x(A, W, W)|, |T_x(A, W, W)| \} \ll A^{1/2}B^{3/2}q^{1/2}M(B)^{-1/2},
\]
where $M(Z)$ is defined by (1.2).

We can prove a weaker bound for three possibly different sets $A, B, C$.

**Theorem 1.3.** For any sets $A, B, C \subseteq \mathbb{F}_q^*$ there exist subsets $W_1 \subseteq B$ and $W_2 \subseteq C$ of cardinalities $W_1 \geq B/2$ and $W_2 \geq C/2$ such that for any characters $(\chi, \psi) \in \Psi^* \times \chi^*$ we have
\[
\min \{ |S_x(A, W_1, W_2)|, |T_x(A, W_1, W_2)| \} \ll \frac{A^{1/2}(BC)^{3/4}q^{1/2}}{\max\{M(B), M(C)\}^{1/4}},
\]
where $M(Z)$ is defined by (1.2).

Finally, we present a bound for a mixed sum of multiplicative and additive characters. Define
\[ \mathcal{S}_{\chi,\psi}(A, B, C) = \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \chi(ab + ac + bc)\psi(ab + ac + bc), \quad (\chi, \psi) \in \Psi \times \mathcal{X}. \]

In the case when both $\chi$ and $\psi$ are nontrivial, we obtain a bound which makes an effective use of all three variables.

**Theorem 1.4.** For any sets $A, B, C \subseteq \mathbb{F}_q^*$ and characters $(\chi, \psi) \in \Psi^* \times \mathcal{X}^*$ we have
\[ \mathcal{S}_{\chi,\psi}(A, B, C) \ll (ABCq)^{1/2} + A^{1/2}BCq^{1/4}. \]

Note that Theorem 1.4 is non-trivial provided that
\[ q = o(\min\{ABC, A^2\}), \]
and takes the form \( \mathcal{S}_{\chi,\psi}(A, B, C) \ll (ABCq)^{1/2} \) for $BC \leq q^{1/2}$.

We also give an application of Theorem 1.1 to bilinear sums with incomplete Kloosterman sums over arbitrary sets. In fact this result is motivated by, and somewhat mimics, the argument of Balog and Wooley [2, Section 6], which in turn is based on a low energy decomposition [2, Theorem 1.3]. Namely, given sets $A, B, C \subseteq \mathbb{F}_q^*$ and three sequences of complex weights $\alpha = (\alpha_a)_{a \in A}$, $\beta = (\beta_b)_{b \in B}$ and $\gamma = (\gamma_c)_{c \in C}$, we define
\[ K(A, B, C; \alpha, \beta, \gamma) = \sum_{a \in A} \sum_{b \in B} \alpha_a \beta_b \left\| \sum_{c \in C} \gamma_c \psi(ac + bc) \right\|^2. \]

As usual we use $\|\alpha\|_\sigma$ to denote the $L_\sigma$-norm of the weights $\alpha$, see (1.7) below.

**Theorem 1.5.** For any sets $A, B, C \subseteq \mathbb{F}_q^*$, complex weights $\alpha = (\alpha_a)_{a \in A}$, $\beta = (\beta_b)_{b \in B}$, $\gamma = (\gamma_c)_{c \in C}$ and a character $\psi \in \Psi^*$, we have
\[ K(A, B, C; \alpha, \beta, \gamma) \ll (\|\alpha\|_1\|\beta\|_2 + \|\alpha\|_2\|\beta\|_1) \|\gamma\|_\infty^2 q^{1/2}C^{3/2}M(C)^{-1/2}, \]
where $M(Z)$ is defined by (1.2).

Writing the bound of Theorem 1.5 in terms of $\|\alpha\|_\infty$ and $\|\beta\|_\infty$, we derive
\[ K(A, B, C; \alpha, \beta, \gamma) \ll \|\alpha\|_\infty\|\beta\|_\infty\|\gamma\|_\infty^2 q^{1/2}(AB^{1/2} + A^{1/2}B)C^{3/2}M(C)^{1/2}. \]
This is nontrivial provided
\[ \min\{A, B\}C \geq \frac{q^{1+\varepsilon}}{M(C)} \]
for some fixed \( \varepsilon > 0 \), which improves the range \( \max\{A, B\}C \geq q^{1+\varepsilon} \) which can be achieved by the bound \( \|\alpha\|_x \|\beta\|_x \|\gamma\|_x^2 \min\{A, B\}Cq \) obtained via the standard approach if \( A \) and \( B \) are of the same order of magnitude and \( C \) satisfies the above inequality.

We prove these bounds in Section 4.

1.5. Notation. Usually we use capital letters in italics to denote sets and the same capital letters in roman to denote their cardinalities, as in the following example \( \#X = X \). In particular, as we have mentioned, we always do this for the sets \( A, B, C \).

The notation \( A = U \cup V \) is used for the partition of \( A = U \cap V \) in a union of disjoint sets \( U \cap V = \emptyset \).

Let \( f \neq 0 \) be a rational function on \( \mathbb{F}_q \). We can express \( f \) as a quotient \( f = g/h \), where \( g \) and \( h \neq 0 \) are coprime polynomials. The degree of \( f \) is defined as \( \max\{\deg g, \deg h\} \). Note that, for a rational function \( f \) of degree \( k \geq 1 \) and any \( c \in \mathbb{F}_q \), we have \( \#\{x : f(x) = c\} \leq k \).

Given two sets \( U, V \subseteq \mathbb{F}_q^* \), a rational function \( f \in \mathbb{F}_q(X) \) and an element \( a \in \mathbb{F}_q \), we use \( r_{U,V}(f; a) \) to denote the number of solutions to the equation \( f(u) + f(v) = a \), \((u, v) \in U \times V\). Furthermore, we simplify it in two special cases by writing \( r_U(f; a) \) if \( U = V \) and \( r_{U,V}(a) \) if \( f(X) = X \); and use \( r_U(a) \) when both. This notation is used with flexibility; for example \( r_{U,-V}(a) \) denotes the number of solutions to the equation \( u - v = a \) with \((u, v) \in U \times V\).

The letters \( k, m, \) and \( n \) (in both the upper and lower cases) denote positive integer numbers.

We define the norms of a complex sequence \( \alpha = (\alpha_m)_{m \in I} \) for some finite set \( I \) of indices by
\[ \|\alpha\|_\infty = \max_{m \in I} |\alpha_m| \quad \text{and} \quad \|\alpha\|_\sigma = \left( \sum_{m \in I} |\alpha_m|^\sigma \right)^{1/\sigma}, \]
where \( \sigma > 0 \).

2. Preliminary results

We need the following bound on mixed character sums, which we derive from the Weil bound.
Lemma 2.1. Take characters \((\chi, \psi) \in \mathcal{X} \times \Psi^*\). For any rational function \(f \in \mathbb{F}_q(X)\) of degree \(k\) and not of the form \(f(X) = g(X)^p - g(X) + \lambda X + \mu\) if \(\chi\) is trivial, we have
\[
\sum_{u \in U} \sum_{v \in V} \chi(u + v) \psi(f(u + v)) \ll_k \sqrt{UVq},
\]
where we use the convention that \(\psi(f(x)) = 0\) if \(x\) is a pole of \(f\).

Proof. Denote
\[
\Sigma = \sum_{u \in U} \sum_{v \in V} \chi(u + v) \psi(f(u + v)).
\]
By the orthogonality of additive characters
\[
\Sigma = \sum_{x \in \mathbb{F}_q} \chi(x) \psi(f(x)) \frac{1}{q} \sum_{\lambda \in \mathbb{F}_q} \sum_{u \in U} \sum_{v \in V} \psi(\lambda(u + v - x))
\]
\[
= \frac{1}{q} \sum_{x \in \mathbb{F}_q} \sum_{\lambda \in \mathbb{F}_q} \chi(x) \psi(f(x) - \lambda x) \sum_{u \in U} \psi(\lambda u) \sum_{v \in V} \psi(\lambda v).
\]
By the Weil bound, see [21, Appendix 5, Example 12], and the assumption of non-linearity of \(f\), the sum over \(x\) is \(O_k(q^{1/2})\). Thus
\[
\Sigma \ll_k q^{-1/2} \sum_{\lambda \in \mathbb{F}_q} \left| \sum_{u \in U} \psi(\lambda u) \right| \left| \sum_{v \in V} \psi(\lambda v) \right|.
\]
Using the Cauchy-Schwarz inequality and then again orthogonality of additive characters, we derive
\[
\sum_{\lambda \in \mathbb{F}_q} \left| \sum_{u \in U} \psi(\lambda u) \right| \left| \sum_{v \in V} \psi(\lambda v) \right| \leq \left( \sum_{\lambda \in \mathbb{F}_q} \left| \sum_{u \in U} \psi(\lambda u) \right|^2 \right)^{1/2} \left( \sum_{\lambda \in \mathbb{F}_q} \left| \sum_{v \in V} \psi(\lambda v) \right|^2 \right)^{1/2}
\]
\[
= (q^2UV)^{1/2}
\]
and the result follows.

We need the following result, which bounds the number of solutions to certain equations over \(\mathbb{F}_q\).

Lemma 2.2. Suppose that \(\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathbb{F}_q\). For any rational function \(f \in \mathbb{F}_q(X)\) of degree \(k\) and not of the form \(f(X) = g(X)^p - g(X) + \lambda X + \mu\), the number \(J\) of solutions to the equation
\[
f(w + x) = y + z \quad (w, x, y, z) \in \mathcal{W} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}
\]
satisfies the bound

\[ J \leq \frac{WXYZ}{q} + O_k((WYZq)^{1/2}). \]

**Proof.** Using the orthogonality of additive characters, we write

\[ J = \sum_{(w,x,y,z) \in W \times X \times Y \times Z} \frac{1}{q} \sum_{\psi \in \Psi} \psi(f(w + x) - y - z). \]

Rearranging the terms and separating the contribution from the trivial character, we obtain

\[ J \leq WXYZq \left( \frac{1}{\sqrt{WYXq}} \sum_{\psi \in \Psi^*} \left| \sum_{y \in Y} \psi(y) \right| \left| \sum_{z \in Z} \psi(z) \right| \right). \]

By Lemma 2.1 with the trivial multiplicative character, we have

\[ J \leq WXYZq \left( \frac{1}{\sqrt{WYXq}} \sum_{\psi \in \Psi^*} \left| \sum_{y \in Y} \psi(y) \right| \left| \sum_{z \in Z} \psi(z) \right| \right). \]

By the Cauchy-Schwarz inequality as in the proof of Lemma 2.1 we obtain

\[ \sum_{\psi \in \Psi^*} \left| \sum_{y \in Y} \psi(y) \right| \left| \sum_{z \in Z} \psi(z) \right| \leq (q^2YZ)^{1/2} \]

and the result follows. \(\Box\)

We now use Lemma 2.2 to study the multiplicities of elements in the sum set of values of \(f\). The names of the variables in Lemma 2.3 are chosen to match those in Lemma 2.5, where it is applied.

**Lemma 2.3.** Let \(A, S, U \subseteq \mathbb{F}_q^*\). Let \(u > 0\) be such that \(r_{S-A}(x) \geq u\) for all \(x \in U\). Let \(k\) be a fixed positive integer and suppose also that

\[ \tau \geq \frac{kASU}{u^q}. \]

Then, for any rational function \(f \in \mathbb{F}_q(X)\), of degree \(k\) and not of the form \(f(X) = g(X)^p - g(X) + \lambda X + \mu\), we have

\[ \#\{x \in \mathbb{F}_q : r_U(f; x) \geq \tau\} \leq k \frac{ASUq}{u^2 \tau^2}. \]

**Proof.** Define

\[ \mathcal{R} = \{x \in \mathbb{F}_q : r_U(f; x) \geq \tau\}. \]
Note that for $R = \#\mathcal{R}$ we have

$$\tau R \leq \sum_{x \in \mathcal{R}} r_{U}(f; x) = \#\{(x, y, z) \in \mathcal{R} \times \mathcal{U} \times \mathcal{U} : x = f(y) + f(z)\}.$$ 

On the other hand, since $r_{S,-A}(z) \geq u$ for $z \in \mathcal{U}$, we have

$$\#\{(x, y, z) \in \mathcal{R} \times \mathcal{U} \times \mathcal{U} : x = f(y) + f(z)\} \leq u^{-1}\#\{(v, w, x, y) \in \mathcal{S} \times \mathcal{A} \times \mathcal{R} \times \mathcal{U} : x = f(y) + f(v - w)\}.$$ 

Therefore,

$$\tau u R \leq \#\{(v, w, x, y) \in \mathcal{S} \times \mathcal{A} \times \mathcal{R} \times \mathcal{U} : x = f(y) + f(v - w)\} \leq k \cdot \#\{(v, w, x, \tilde{y}) \in \mathcal{S} \times \mathcal{A} \times \mathcal{R} \times f(\mathcal{U}) : x = \tilde{y} + f(v - w)\}.$$ 

Applying Lemma 2.2, it follows that

$$\tau u R \leq \frac{kARSU}{q} + O_k((ARSUq)^{1/2}).$$ 

The assumed lower bound on $\tau$ then implies that

$$\tau u R \ll_k (ARSUq)^{1/2}$$

and the result follows. \hfill \Box

We need the following result [19, Lemma 17]. We include a short proof for the convenience of the reader, which can be easily extended to finite sets in any group.

**Lemma 2.4.** Let $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathbb{F}_q$. Then

$$E\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right) \leq \left(\sum_{i=1}^{n} E^{1/4}(\mathcal{A}_i)\right)^4.$$
Proof. We can assume that the sets $A_1, \ldots, A_n$ are disjoint. Then we have
\[
E \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i,j,k,\ell=1}^{n} r_{A_i,A_j}(x)r_{A_k,A_\ell}(x)
\]
\[
\leq \sum_{i,j,k,\ell=1}^{n} \left( \sum_{x \in \mathbb{F}_q} r_{A_i,A_j}(x)^2 \right)^{1/2} \left( \sum_{x \in \mathbb{F}_q} r_{A_k,A_\ell}(x)^2 \right)^{1/2}
\]
\[
= \left( \sum_{i,j=1}^{n} \left( \sum_{x \in \mathbb{F}_q} r_{A_i,A_j}(x)^2 \right) \right)^{1/2}.
\]
\[
= \left( \sum_{i,j=1}^{n} \left( \sum_{x \in \mathbb{F}_q} r_{A_i,-A_i}(x)r_{A_j,-A_j}(x) \right) \right)^{1/2}
\]
\[
\leq \left( \sum_{i,j=1}^{n} \left( \sum_{x \in \mathbb{F}_q} r_{A_i,-A_i}(x)^2 \right)^{1/4} \left( \sum_{x \in \mathbb{F}_q} r_{A_j,-A_j}(x)^2 \right)^{1/4} \right)^2
\]
\[
= \left( \sum_{i=1}^{n} \left( \sum_{x \in \mathbb{F}_q} r_{A_i,-A_i}(x)^2 \right)^{1/4} \right)^4 = \left( \sum_{i=1}^{n} E(A_i)^{1/4} \right)^4.
\]

via two consecutive applications of the Cauchy-Schwarz inequality. \qed

We now formulate and prove our main technical tool.

**Lemma 2.5.** Let $A \subseteq \mathbb{F}_q$. Then for any rational function $f \in \mathbb{F}_q(X)$ of degree $k$ and not of the form $f(X) = g(X)^p - g(X) + \lambda X + \mu$, there exists $U \subseteq A$ of cardinality $U$ such that
\[
U \gg \frac{E^{1/2}(A)}{A^{1/2}(\log A)^{7/4}}
\]
and
\[
E(f(U)) \ll_k \frac{AU^6q^{-1}(\log A)^{11/2} + AU^3q(\log A)^6}{E(A)}.
\]

**Proof.** The additive energy of a set $A \subseteq \mathbb{F}_q$ can be written in the form
\[
E(A) = \sum_{x \in A+A} r_{A,A}^2(x).
\]
Dyadically decompose this sum, and deduce that there is a popular dyadic set
\[
S = \{x \in A + A : \rho \leq r_{A,A}(x) < 2\rho\}
\]
with some integer $1 \leq \rho \leq |A|$ where $\rho$ is a power of 2, and such that
\[ \rho^2 \# S \gg \frac{E(A)}{\log A}. \]

Consider the point set
\[ \mathcal{P} = \{(a, b) \in A \times A : a + b \in S\}. \]

Following our standard convention, we denote
\[ P = \# \mathcal{P} \quad \text{and} \quad S = \# S \]
and note that
\[ \rho S \leq P < 2\rho S. \]

We then make a second dyadic decomposition of this point set to find a large subset supported on vertical lines with approximately the same richness.

To be precise, for any $x \in \mathbb{F}_q$, define
\[ \mathcal{A}_x = \{ y : (x, y) \in \mathcal{P} \} \quad \text{and} \quad A_x = \# \mathcal{A}_x. \]

Note that
\[ \sum_{x \in A} A_x = P. \]

Therefore, for some $s$ there exists a dyadic set
\[ \mathcal{V} = \{ x \in A : s \leq A_x < 2s \} \]
such that, recalling (2.2), for $V = \# \mathcal{V}$ we have
\[ V \gg \frac{P}{\log A} \gg \frac{\rho S}{\log A}. \]

We now separate into two cases
\[ V \geq \frac{s}{(\log A)^{1/2}} \quad \text{and} \quad V < \frac{s}{(\log A)^{1/2}}. \]

**Case I: $V \geq s(\log A)^{-1/2}$**. Note for any $x \in \mathcal{V}$, there exist
\[ y_1, y_2, \ldots, y_s \in \mathcal{A}_x \subseteq A \]
such that $(x, y_i) \in \mathcal{P}$ for all $1 \leq i \leq s$. Therefore
\[ x + y_1, x + y_2, \ldots, x + y_s \in S. \]

It follows that $r_{\# \mathcal{A}}(x) \geq s$ for every $x \in \mathcal{V}$ and in this case we define
\[ (2.4) \quad \mathcal{U} = \mathcal{V} \quad \text{and} \quad u = s. \]
Case II: $V < s(\log A)^{-1/2}$. In this case, consider the point set 
\[ Q = \{(x, y) \in P : x \in V\} \]
of cardinality $Q = \#Q$. As before, we note that for any $x \in V$, there 
exist at least $s$ values of $y \in A_x \subseteq A$ with $(x, y) \in P$. Hence $Q \geq Vs$.

Now, for any $y \in \mathbb{F}_q$, define 
\[ B_y = \{x : (x, y) \in Q\} \quad \text{and} \quad B_y = \#B_y. \]
Note that 
\[ \sum_{y \in A} B_y = Q. \]
Therefore, for some $t$ there exists a dyadic set 
\[ W = \{y \in A : t \leq B_y < 2t\} \]
such that for $W = \#W$ we have
\[ Wt \gg \frac{Q}{\log A} \geq \frac{Vs}{\log A}. \tag{2.5} \]
Note that since $Q \subseteq V \times A$ we also have $t \leq V$. It follows from (2.5) and the assumption that $s > V(\log A)^{1/2}$ that
\[ WV \geq Wt \geq \frac{Vs}{\log A} > \frac{V^2}{(\log A)^{1/2}} \]
and thus
\[ W \gg V(\log A)^{-1/2} \geq t(\log A)^{-1/2}. \tag{2.6} \]
Also, by (2.5) and (2.3),
\[ Wt \gg \frac{Vs}{\log A} \gg \frac{\rho S}{(\log A)^2}. \tag{2.7} \]

Now, let $y \in W$. So, there exist $x_1, x_2, \ldots, x_t \in A$ such that $(x_i, y) \in P$ for all $1 \leq i \leq t$. Therefore 
\[ x_1 + y, x_2 + y, \ldots, x_t + y \in S. \]
It follows that $r_{S, -A}(y) \geq t$ for every $y \in W$.

In this case we take
\[ U = W \quad \text{and} \quad u = t. \tag{2.8} \]

One now verifies that for both choices (2.4) and (2.8) we have $U \subseteq A$ 
of cardinality $U$ with
\[ U \gg u(\log A)^{-1/2} \tag{2.9} \]
and
\( uU \gg \frac{\rho S}{(\log A)^2} \)

and such that
\( r_{S-A}(x) \geq u, \quad \forall x \in \mathcal{U}. \)

Indeed in Case I, the inequality (2.9) is by the assumption, while in Case II this follows from (2.6). Furthermore, in Case I, the inequality (2.10) is weaker than (2.3), while in Case II this follows from (2.7).

Note also that, multiplying (2.9) and (2.10) and then using (2.1) together with the fact that \( \rho \leq A \), we obtain
\( U^2 \gg \frac{\rho S}{(\log A)^{5/2}} = \frac{\rho^2 S}{\rho(\log A)^{5/2}} \gg \frac{E(A)}{A(\log A)^{7/2}}. \)

This \( \mathcal{U} \) is the desired set. It remains to estimate the energy \( E(f(\mathcal{U})) \) of \( f(\mathcal{U}) \).

We have
\( E(f(\mathcal{U})) = \sum_{x \in \mathbb{F}_q} r_{f(\mathcal{U})}(x)^2 \leq \sum_{x \in \mathbb{F}_q} r_{U}(f;x)^2. \)

Define the set
\( R_0 = \left\{ x \in \mathbb{F}_q : r_{U}(f;x) \leq 2 \frac{kASU}{uq} \right\} \)

and then for \( J = \lfloor \log A/\log 2 \rfloor \), the sets
\( R_j = \left\{ x \in \mathbb{F}_q : 2^j \frac{kASU}{uq} < r_{U}(f;x) \leq 2^{j+1} \frac{kASU}{uq} \right\}, \quad j = 1, \ldots, J. \)

Since
\( \sum_{x \in \mathbb{F}_q} r_{U}(f;x) = U^2 \)

the contribution from \( x \in R_0 \) is bounded by
\( \sum_{x \in R_0} r_{U}(f;x)^2 \leq 2 \frac{kASU}{uq} \sum_{x \in \mathbb{F}_q} r_{U}(f;x) \ll \frac{kASU^3}{uq}. \)

For \( j = 1, \ldots, J \), we apply Lemma 2.3 with
\( \tau = 2^j \frac{kASU}{uq} \)

to derive
\( \sum_{x \in R_j} r_{U}(f;x)^2 \leq (2\tau)^2 \# R_j \ll_k \frac{ASUq}{u^2}. \)
Substituting the bounds (2.13) and (2.14) in (2.12) and using the fact that $J \ll \log A$, we obtain

$$E(f(U)) \ll_k \frac{ASU^3}{uq} + \frac{ASUq}{u^2} \log A. \quad (2.15)$$

We now deal with these two terms in (2.15) one at a time.

Firstly, multiplying (2.10) with the first inequality in (2.11), and then recalling (2.1), we obtain

$$uU^3 \gg \frac{\rho^2 S^2}{(\log A)^{9/2}} \gg \frac{SE(A)}{(\log A)^{11/2}}$$

which we rewrite as

$$S \ll \frac{U^3(\log A)^{11/2}}{E(A)}.$$

Hence, for the first term in (2.15) we have

$$\frac{ASU^3}{uq} \ll \frac{AU^6(\log A)^{11/2}}{E(A)q}. \quad (2.16)$$

For the second term, squaring (2.10) and then using (2.1) again, we obtain

$$u^2 U^2 \gg \frac{\rho^2 S^2}{(\log A)^4} \gg \frac{SE(A)}{(\log A)^5}$$

which we rewrite as

$$S \ll \frac{U^2(\log A)^5}{E(A)}.$$

Hence, for the second term in (2.15) we have

$$\frac{ASUq}{u^2} \log A \ll \frac{AU^3q(\log A)^6}{E(A)}. \quad (2.17)$$

Substituting (2.16) and (2.17) in (2.15), we conclude the proof. □

3. Proof of Theorem 1.1

3.1. Partition procedure. Below, we use Lemma 2.5 to construct a nested sequence of sets

$$\emptyset = U_1 \subseteq U_2 \subseteq \ldots \subseteq U_m$$

and a corresponding sequence

$$\mathcal{V}_m \subseteq \mathcal{V}_{m-1} \subseteq \ldots \subseteq \mathcal{V}_1 = \mathcal{A}$$

where $U_i \uplus \mathcal{V}_i = \mathcal{A}$, $i = 1, \ldots, m$. 
When

\[(3.1) \quad E(V_m) \leq A^3/M(A)\]

we terminate this process and take \(S = V_m\) and \(T = U_m\) (as we explain below, (3.1) is satisfied for some \(m\)).

The nested sequence is defined as follows. Suppose that

\[(3.2) \quad E(V_i) > A^3/M(A).\]

Then, by Lemma 2.5, there exists \(Q_i \subseteq V_i\) such that for \(Q_i = \#Q_i\)

\[(3.3) \quad Q_i \gg \frac{E^{1/2}(V_i)}{A^{1/2} \log^{7/4} A} > \frac{A}{M(A)^{1/2} \log^{7/4} A}\]

and using (3.2) we derive

\[(3.4) \quad E(f(Q_i)) \ll \frac{V_i Q_i^6 q^{-1}(\log A)^{11/2} + V_i Q_i^3 q(\log A)^6}{E(V_i)}\]

\[< \frac{M(A)}{A^3} \left( \frac{V_i Q_i^6 (\log A)^{11/2}}{q} + V_i Q_i^3 q(\log A)^6 \right).\]

In particular, inequality (3.3) implies that

\[(3.5) \quad V_i \ll Q_i M(A)^{1/2} \log^{7/4} A.\]

Then, define \(V_{i+1} = V_i \setminus Q_i\). This automatically defines \(U_{i+1} = U_i \cup Q_i\). This iterative construction in fact gives

\[(3.6) \quad U_{i+1} = \bigsqcup_{j=1}^{i} Q_j.\]

Note that this process certainly terminates, since we have a uniform lower bound on the cardinality of \(Q_i\) and so the cardinality \(V_i\) is monotonically decreasing, thus we eventually reach the termination condition (3.1).

### 3.2. Final estimate.

Recall that by (3.6) we have

\[T = U_m = \bigsqcup_{j=1}^{m-1} Q_j.\]
Therefore, Lemma 2.4 and the inequality (3.4) imply that
\[ \mathbb{E}^{1/4}(\mathcal{F}(\mathcal{T})) = \left( \mathbb{E} \left( \bigcup_{i=1}^{m-1} f(Q_i) \right) \right)^{1/4} \leq \sum_{i=1}^{m-1} \mathbb{E}^{1/4}(\mathcal{F}(\mathcal{Q}_i)) \]
\[ \ll_{k} \sum_{i=1}^{m-1} \left( \frac{M(A)}{A^3} \left( \frac{V_i Q_i^6 (\log A)^{11/2}}{q} + V_i Q_i^3 q (\log A)^6 \right) \right)^{1/4}. \]

Now the inequality (3.5) yields
\[ \mathbb{E}^{1/4}(\mathcal{F}(\mathcal{T})) \]
\[ \ll_{k} \sum_{i=1}^{m-1} \left( \frac{M(A)}{A^3} \left( \frac{V_i Q_i^6 (\log A)^{11/2}}{q} + M(A)^{1/2} Q_i^4 q (\log A)^{31/4} \right) \right)^{1/4}. \]

Using that \( A \geq V_i \geq Q_i \) and thus replacing \( V_i Q_i^6 \) with \( A^3 Q_i^4 \) in the first term, we now obtain
\[ \mathbb{E}^{1/4}(\mathcal{F}(\mathcal{T})) \]
\[ \ll_{k} \left( \frac{M(A)(\log A)^{11/2}}{q} + A^{-3} M(A)^{3/2} q (\log A)^{31/4} \right)^{1/4} \sum_{i=1}^{m-1} Q_i. \]

Using that
\[ \sum_{i=1}^{m-1} Q_i \leq A \]
we obtain
\[ (3.7) \quad \mathbb{E}(\mathcal{F}(\mathcal{T})) \ll_{k} \frac{M(A) A^4 (\log A)^{11/2}}{q} + M(A)^{3/2} A q (\log A)^{31/4}. \]

We now see that the choice (1.2) balances between (3.1) and (3.7) and leads to the inequality
\[ \mathbb{E}(\mathcal{F}(\mathcal{T})) \ll_{k} A^3 / M(A) \]
implied by our choice \( S = \mathcal{V}_m \) and the bound (3.1). This completes the proof. \( \square \)

4. Proofs of Theorems 1.2, 1.3, 1.4 and 1.5

4.1. Character sums and energy. First we give some basic bounds of the sums \( S_{\psi}(A, B, C) \) and \( T_{\chi}(A, B, C) \) in terms of energies of various sets.

**Lemma 4.1.** For any sets \( A, B, C \subseteq \mathbb{F}_q \) and additive character \( \psi \in \Psi^* \) we have
\[ S_{\psi}(A, B, C) \leq A^{1/2} E(B)^{1/4} E(C)^{1/4} q^{1/2}. \]
Proof. By the Cauchy-Schwarz inequality we have

\[ |S_\psi(A, B, C)|^2 \leq A \sum_{a \in A} \left| \sum_{b \in B} \sum_{c \in C} \psi(ab + ac + bc) \right|^2 \]

\[ \leq A \sum_{a \in \mathbb{F}_q} \left| \sum_{b \in B} \sum_{c \in C} \psi(ab + ac + bc) \right|^2 \]

\[ = A \sum_{b_1, b_2 \in B} \sum_{c_1, c_2 \in C} \psi(b_1 c_1 - b_2 c_2) \sum_{a \in \mathbb{F}_q} \psi(a(b_1 + c_1 - b_2 - c_2)) . \]

By the orthogonality of additive characters we obtain

\[ S_\psi(A, B, C) \leq \sqrt{AE(B, C)q}, \]

where

\[ E(B, C) = \# \{(b_1, b_2, c_1, c_2) \in B \times B \times C \times C : b_1 + c_1 = b_2 + c_2 \}. \]

Finally, it follows from the Cauchy-Schwarz inequality that

\[ E(B, C) \leq E(B)^{1/2}E(C)^{1/2}. \]

Indeed,

\[ E(B, C) = \sum_{x \in \mathbb{F}_q} r_{B,-B}(x)r_{C,-C}(x) \]

\[ \leq \left( \sum_{x \in \mathbb{F}_q} r_{B,-B}^2(x) \right)^{1/2} \cdot \left( \sum_{x \in \mathbb{F}_q} r_{C,-C}^2(x) \right)^{1/2} \]

\[ = E(B)^{1/2}E(C)^{1/2}, \]

which completes the proof. \( \square \)

We also have an analogue of Lemma 4.1 for multiplicative characters.

**Lemma 4.2.** For any sets \( A, B, C \subseteq \mathbb{F}_q^* \) and character \( \psi \in \Psi^* \) we have

\[ T_\chi(A, B, C) \ll A^{1/2}E(B^{-1})^{1/4}E(C^{-1})^{1/4}q^{1/2} + A^{1/2}BC. \]
Proof. By the Cauchy-Schwarz inequality we have

\[
|T_\chi(A, B, C)|^2 \leq A \sum_{a \in A} \left| \sum_{b \in B} \sum_{c \in C} \chi(ab + ac + bc) \right|^2
\]

\[
\leq A \sum_{a \in \mathbb{F}_q^*} \left| \sum_{b \in B} \sum_{c \in C} \chi(ab + ac + bc) \right|^2
\]

\[
= A \sum_{b_1, b_2 \in B} \sum_{c_1, c_2 \in C} \chi \left( \frac{b_1 c_1}{b_2 c_2} \right) \sum_{a \in \mathbb{F}_q^*} \chi \left( a \left( b_1^{-1} + c_1^{-1} \right) + 1 \right) \chi \left( a \left( b_2^{-1} + c_2^{-1} \right) + 1 \right)
\]

\[
= A \sum_{b_1, b_2 \in B} \sum_{c_1, c_2 \in C} \chi \left( \frac{b_1 c_1}{b_2 c_2} \right) \sum_{a \in \mathbb{F}_q^*} \chi \left( a^{-1} + b_1^{-1} + c_1^{-1} \right) \chi \left( a^{-1} + b_2^{-1} + c_2^{-1} \right),
\]

where \( \chi \) is the complex conjugate character. Making the change of variable \( a \to a^{-1} \) in the sum over \( a \in \mathbb{F}_q^* \) we arrive to

\[
|T_\chi(A, B, C)|^2 \leq A \sum_{b_1, b_2 \in B} \sum_{c_1, c_2 \in C} \chi \left( \frac{b_1 c_1}{b_2 c_2} \right) \sum_{a \in \mathbb{F}_q^*} \chi \left( a + b_1^{-1} + c_1^{-1} \right) \chi \left( a + b_2^{-1} + c_2^{-1} \right).
\]

By the “approximate” orthogonality of multiplicative characters, that is, by

\[
\sum_{a \in \mathbb{F}_q^*} \chi(a + u) \overline{\chi}(a + v) \ll \begin{cases} 1 & \text{if } u \neq v, \\ q & \text{if } u = v, \end{cases}
\]

we obtain

\[
T_\chi(A, B, C) \ll \sqrt{AE(B^{-1}, C^{-1})q} + A^{1/2} BC,
\]

where

\[
E(B^{-1}, C^{-1}) = \# \{(b_1, b_2, c_1, c_2) \in B \times B \times C \times C : b_1^{-1} + c_1^{-1} = b_2^{-1} + c_2^{-1}\}.
\]

As in the proof of Lemma 4.1 we obtain

\[
E(B^{-1}, C^{-1}) \leq \sqrt{E(B^{-1})E(C^{-1})}
\]

and the result follows. \( \square \)
4.2. **Concluding the proofs.** We are now ready to establish the desired results.

**Proof of Theorem 1.2.** By Theorem 1.1 there exist \( S \) and \( T \) with additive energies

\[
\max \{ E(S), E(T^{-1}) \} \ll \frac{B^3}{M(B)}
\]

and \( B = S \cup T \). Hence, one of these sets contains at least \( B/2 \) elements. Let \( \mathcal{W} \) be this set. Lemmas 4.1 and 4.2 complete the proof. Note that the second summand in Lemma 4.2 is smaller than the bound of Theorem 1.2.

**Proof of Theorem 1.3.** Similarly to the proof of Theorem 1.2, there exist \( S_1, S_2, T_1, T_2 \) with

\[
\max \{ E(S_1), E(T_1^{-1}) \} \ll \frac{B^3}{M(B)};
\]

\[
\max \{ E(S_2), E(T_2^{-1}) \} \ll \frac{C^3}{M(C)};
\]

\( B = S_1 \cup T_1 \) and \( C = S_2 \cup T_2 \). Let \( \mathcal{W}_i, i = 1, 2 \), be the larger set of \( S_i \) and \( T_i \). Then the result follows again by Lemmas 4.1 and 4.2, which completes the proof of Theorem 1.3.

**Proof of Theorem 1.4.** The proof follows a combination of the calculations of Lemmas 4.1 and 4.2. By the Cauchy-Schwarz inequality we have

\[
|\mathcal{G}_{\chi, \psi}(A, B, C)|^2 \leq A \sum_{a \in A} \left| \sum_{b \in B} \sum_{c \in C} \chi(ab + ac + bc) \psi(ab + ac + bc) \right|^2
\]

\[
\leq A \sum_{a \in \mathbb{F}_q^*} \left| \sum_{b \in B} \sum_{c \in C} \chi(ab + ac + bc) \psi(ab + ac + bc) \right|^2
\]

\[
= A \sum_{b_1, b_2 \in B} \sum_{c_1, c_2 \in C} \chi \left( \frac{b_1 c_1}{b_2 c_2} \right) \psi \left( b_1 c_1 - b_2 c_2 \right) \sum_{a \in \mathbb{F}_q^*} \chi \left( a^{-1} + b_1^{-1} + c_1^{-1} \right) \chi \left( a^{-1} + b_2^{-1} + c_2^{-1} \right) \psi \left( a \left( b_1 + c_1 - b_2 - c_2 \right) \right).
\]

Since both \( \chi \) and \( \psi \) are nontrivial characters we see that the inner sum over \( a \) satisfies the Weil bound, see [21, Appendix 5, Example 12],
unless
\[ b_1^{-1} + c_1^{-1} = b_2^{-1} + c_2^{-1} \quad \text{and} \quad b_1 + c_1 = b_2 + c_2. \]
Clearly, when \((b_2, c_2) \in \mathcal{B} \times \mathcal{C}\) are fixed (in \(BC\) ways) the pair \((b_1, c_1)\) can take at most two values. Hence, we obtain
\[ |\mathcal{S}_{\chi, \psi}(\mathcal{A}, \mathcal{B}, \mathcal{C})|^2 \ll A(B^2C^2q^{1/2} + BCq) \]
and the result follows. \(\Box\)

**Proof of Theorem 1.5.** We now partition the set \(\mathcal{C}\) into two sets \(\mathcal{S}\) and \(\mathcal{T}\) as in Theorem 1.1 with respect to the function \(f(X) = X^{-1}\). Thus
\[ \max\{E(\mathcal{S}), E(\mathcal{T}^{-1})\} \ll \frac{C^3}{M(C)}. \]
Since \((x + y)^2 \leq 2(x^2 + y^2)\) for any real \(x\) and \(y\), we obtain
\[ K(\mathcal{A}, \mathcal{B}, \mathcal{C}; \alpha, \beta, \gamma) \leq 2 \left( \sum_{a \in \mathcal{A}} |\alpha_a| U_a + \sum_{b \in \mathcal{B}} |\beta_b| V_b \right), \]
where
\[ U_a = \sum_{b \in \mathcal{B}} |\beta_b| \left| \sum_{c \in \mathcal{T}} \gamma_c \psi \left( ac + bc^{-1} \right) \right|^2, \quad V_b = \sum_{a \in \mathcal{A}} |\alpha_a| \left| \sum_{c \in \mathcal{S}} \gamma_c \psi \left( ac + bc^{-1} \right) \right|^2. \]

By the Cauchy-Schwarz inequality, as before, for every \(a \in \mathcal{A}\) we derive
\[ U_a^2 \ll \|\beta\|_2^2 \sum_{b \in \mathcal{B}} \left| \sum_{c \in \mathcal{T}} \gamma_c \psi \left( ac + bc^{-1} \right) \right|^4 \]
\[ \leq \|\beta\|_2^2 \sum_{b \in \mathcal{B}} \left| \sum_{c \in \mathcal{T}} \gamma_c \psi \left( ac + bc^{-1} \right) \right|^4 \]
\[ \leq \|\beta\|_2^2 \|\gamma\|_x^4 qE(\mathcal{T}^{-1}) \ll \|\beta\|_2^2 \|\gamma\|_x^4 qC^3M(C)^{-1}. \]
Similarly, for every \(b \in \mathcal{B}\), we have
\[ V_b^2 \ll |\alpha|_2^2 \|\gamma\|_x^4 qE(\mathcal{S}) \ll |\alpha|_2^2 \|\gamma\|_x^4 qC^3M(C)^{-1}. \]
Substituting these bounds in (4.1) we conclude the proof. \(\Box\)
5. Comments

It is easy to verify that at the cost of merely typographical changes one can obtain a full analogue of Theorem 1.1 for
\[ E(f(T), g(T)) = \#\{t_1, t_2, t_3, t_4 \in T^4 : f(t_1) + g(t_2) = f(t_3) + g(t_4)\} \]
Most likely, using multiplicative character sums instead of additive character sums, one can obtain similar results for \( \max\{E^*(S), E(f(T))\} \) or even \( \max\{E^*(S), E(f(T), g(T))\} \). However the following question appears to require new ideas.

Question 5.1. Given two bivariate polynomials \( F(X, Y), G(X, Y) \in \mathbb{F}_q[X, Y] \), satisfying some natural conditions, show that any set \( A \subseteq \mathbb{F}_q \) can be partitioned as \( A = S \cup T \) in such a way that the two sets
\[ \{(s_1, s_2, s_3, s_4) \in S^4 : F(s_1, s_2) = F(s_3, s_4)\} \]
and
\[ \{(t_1, t_2, t_3, t_4) \in T^4 : G(t_1, t_2) = G(t_3, t_4)\} \]
are both of small cardinality.

Finally, we note that Theorem 1.5 can be extended to the sums
\[ \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \sum_{d \in D} \psi(a(c + d) + b(f(c) + g(d))) \tag{5.1} \]
over sets \( A, B, C, D \subseteq \mathbb{F}_q \) and rational functions \( f, g \in \mathbb{F}_q(X) \). Within our approach, one can show that, under some natural conditions on \( f \) and \( g \), for any \( \varepsilon > 0 \) there are some \( \delta > 0 \) and \( \kappa > 0 \), such that as long as \( A(CD)^{1/2}, B(CD)^{1/2} \geq q^{1-\delta} \) and \( C, D \geq q^{1/2+\varepsilon} \) the sums (5.1) are of order at most \( ABCDq^{-\kappa} \). Despite a somewhat exotic shape of the sums (5.1), they may be used in the theory of randomness extractors in arbitrary finite fields where the theory falls far below its counterpart in prime fields, see [3-5] for more details and further references. One can also introduce weights of the form \( \alpha_a, \beta_b \) and \( \gamma_{c,d} \) in the sums (5.1).

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