Abstract

The Langmuir fluctuations trapped and enhanced inside a cavity of the electron concentration in a plasma are considered. Stationary fluctuation spectrum is calculated with regard to finite collisional damping of the Langmuir waves and nonequilibrium fluctuation source. It is shown that the fluctuation spectrum obtained exhibits a frequency modulation due to the reflection and superposition of the elementary Langmuir waves in the irregularity. The modulation depth and period as a function of the irregularity size and the electron–ion collision frequency is calculated. These results may help to find from experimental observations such important plasma parameters as the irregularly size and lifetime, as well as the electron collision frequency.

1. Introduction

Calculation of plasma fluctuations is important for understanding the plasma properties [1–10] and for the plasma diagnostics using Thomson scattering of electromagnetic waves [11–20]. For the past decades, the plasma fluctuation theory has concentrated deeply on nonequilibrium plasma states of various kinds, including non-Maxwellian particle distributions, space and time inhomogeneity of the plasma density, powerful external impact on the plasma, etc. The inhomogeneity of the plasma density is one of the main factors affecting the fluctuation spectra [5, 7, 15, 21]. The influence of the inhomogeneity is even more significant if the fluctuations become captured in finite regions and exhibit a drastic change in the spectrum shape [22–26].

Spatial localization can be pronounced particularly in case of the Langmuir fluctuations trapped inside plasma density cavities. The Langmuir fluctuations constitute a set of electrostatic plasma waves with a frequency close to the Langmuir frequency. If the damping of these waves is small enough they can propagate in a cavity. But there exists a frequency interval when the waves reflect at the turning points and return back to the cavity. In this case, if a steady source drives the fluctuations, their intensity may be enhanced significantly. The behavior of fluctuations depends on the collision frequency of electrons with heavy particles. Small collisional damping of the Langmuir waves creates conditions for the modulational instability of the Langmuir waves [27]. This kind of instability results in development of the strong Langmuir turbulence in which the fluctuation level is determined by nonlinear processes.

At the same time, collisions can play an important role in achieving steady state fluctuations. As is shown in [28], due to the collisions, an instability threshold for the plasma wave arises both in laser and space plasmas. For example, in the ionosphere, the collisions become important under a wide set of conditions when the fluctuation electric field is of the order or less than 1 mV m$^{-1}$ [28] and the cavity size is more than 10 m [22] (i.e. for mid- and large scale irregularities). Note that only such irregularities can be resolved by incoherent scatter radars in the experiments on plasma diagnostics with the help of the Thomson scattering.

In this paper we calculate stationary amplitudes and spectrum of the Langmuir fluctuations captured in a parabolic cavity of the plasma density in the presence of the electron–ion collisions. Our results are applicable for arbitrary distribution functions of the electrons, provided the parametric instabilities are not excited. The fluctuation spectrum as a function of the collision frequency, the irregularity parameters, and the fluctuation...
source is presented. It is shown that collisions restrict fluctuations intensity and smear the discrete spectrum which is obtained in the collisionless approximation [22, 24, 25, 29].

Earlier the Langmuir wave trapping in the presence of the collisional damping was treated for the Langmuir waves generated by the resonant mixing of 2 transverse electromagnetic waves [30]. In the paper [30] only regular Langmuir wave with a fixed frequency and wavenumber was analyzed. Such an analysis is insufficient for the fluctuation problem, as it does not take into account a set of the multiple Langmuir waves and their correlation mechanism.

In the present work we consider another generation mechanism which is appropriate to the real fluctuation problem. The mechanism uses a fluctuation source calculated in [31, 32] which depends on nonequilibrium plasma parameters and has a broad set of wavenumbers and frequencies exciting the fluctuations. This source does not use some additional assumptions about specific forms of the collision integrals (e.g., so-called model collision integrals [5, 33, 34]) which restrict the generality of the results and do not ensure explicit formulation of the application conditions for the results obtained.

The fluctuation source found in [31, 32] is applicable for arbitrary (not only maxwellian) distribution functions of particles and takes into account collective effects in a plasma. This source was used in [31, 32] to calculate fluctuation spectra in the case of a homogeneous plasma. In the present paper, we show that the same fluctuation source can be used to solve an inhomogeneous problem.

2. Excitation of the Langmuir waves by the fluctuation source

Consider a plasma layer where the electron concentration \( n(x) \) changes in the direction \( x \). Let the ambient electron density \( n(x) \) depend on the coordinate \( x \) in the following way

\[
n(x) = n_0 + \frac{1}{2} n_0 b^2 x^2,
\]

where \( n_0 \) is the density at the minimum point. The parameter \( b \) determines the irregularity size. For example, if the electron density in the ionosphere changes 10% at a distance of 100 m then \( b = 4.47 \times 10^{-5} \text{ cm}^{-1} \). If this distance is 1 km, \( b = 4.47 \times 10^{-8} \text{ cm}^{-1} \). The parabolic model (1) is a good approximation for real irregularities, as the most intensive fluctuations are formed in the vicinity of the density minimum.

The total electron density variation \( N(x, t) \) consists of the unperturbed density \( n(x) \) and small amplitude fluctuations \( \delta n(x, t) \)

\[
N(t, x) = n(x) + \delta n(t, x), \quad |\delta n| \ll n.
\]

The Langmuir fluctuations electric field \( \delta E \) can be described by the equation derived in [21]

\[
\begin{align*}
\hat{L}(\omega, x) \delta E &= \frac{\omega_p^2}{3v_{xe}^2} \int dk_x \frac{|k_x|}{k_x} \exp(-ik_x x) S(k_x, k_x), \\
\hat{L}(\omega, x) &= \frac{\partial^2}{\partial x^2} + \frac{\omega_p^2}{3v_{xe}^2} \varepsilon_0(\omega, x) = \frac{4\pi i \omega_p^2}{3v_{xe}^2} \varepsilon_0 \omega, \\
\varepsilon_0(\omega, x) &= 1 - \frac{\omega_p^2(x)}{\omega^2},
\end{align*}
\]

where \( S(k_x, k_x) \) is a random function (fluctuation source) calculated in [31, 32] on the basis of the Klimontovich equations [4]. The operator \( \hat{L}(\omega, x) \) describes propagation of the Langmuir wave in the presence of the collisional damping which is represented by the conductivity \( \sigma \). This term is related to the efficient collision frequency of electrons \( \nu_{ct} \)

\[
\sigma(\omega, x) = \frac{1}{4\pi} \frac{\omega_p^2(x)}{\omega^2} \nu_{ct}.
\]

The electron Langmuir frequency \( \omega_p(x) \) depends on \( x \) via the electron density

\[
\begin{align*}
\omega_p(x) &= \sqrt{\frac{4\pi e^2 n(x) / m_e}{3}}^{1/2}, \\
\omega_p(0) &= \omega_p(0) = \sqrt{\frac{4\pi e^2 n_0 / m_e}{3}}^{1/2}.
\end{align*}
\]

where \( e \) and \( m_e \) are the electron charge and mass, respectively, and \( \omega_p(0) \) is the Langmuir frequency at the minimum density point. \( v_{xe} \) in (3) is the root-mean-square velocity of electrons in the \( x \)-direction.

\( \varepsilon_0(\omega, x) \) is the part of the longitudinal dielectric function of the plasma without the spatial dispersion and collisional terms (both are taken into account by the operator \( \hat{L} \)). The electric field fluctuation \( \delta E \) in (3) is a function of \( \omega \) and \( x \). In the stationary case it is convenient to apply the Laplace transform to the time-dependent function \( \delta E(t, x) \) [4].
\[ \delta E(\omega, x) = \frac{1}{2\pi} \int_0^\infty dt \exp[-i(\omega - i\Delta)t] \delta E(t, x). \]  
(6)

In this equation \( \Delta > 0 \), and the limit \( \Delta \to 0 \) should be taken in the final results. The inverse transform for (6) is as follows

\[ \delta E(t, x) = \int_{-\infty}^{\infty} d\omega \exp(i\omega t) \delta E(\omega, x). \]  
(7)

For simplicity, we will present results only for the frequencies \( \omega > 0 \). The case of the negative frequencies is quite similar.

As mentioned above, the most intensive fluctuations are formed near the density minimum, i.e., in the vicinity of the point \( x = 0 \). This means that the frequency \( \omega \) of the fluctuations is close to the electron Langmuir frequency \( \omega_{pe} \) and varies within narrow limits:

\[ |\omega - \omega_{pe}| \ll \omega_{pe}. \]  
(8)

In addition to that, the Landau damping in our case is negligible (see below). Under these conditions the \( \omega \)- and \( x \)-dependence of all the terms except \( e^x(\omega, x) \) can be neglected. Note that the \( \omega \) and \( x \) dependence of \( S \) remains weak and it will have no effect on the amplitude calculations. We retain this dependence here just for the obtaining correct relationship between the scattered spectrum and the statistical moment of the fluctuation source \( S \) (see below).

The right-hand side of (3) contains integration with respect to the wavenumber \( k_x \). Since equation (3) is linear, we can solve it for a fixed \( k_x \) and then make the integration in the final results:

\[ \tilde{L}(\omega, x)\delta E = \frac{\omega_{pe}^2 |k_x|}{3\nu_{ce}^2} \exp(-ik_x x)S(\omega, k_x). \]  
(9)

This is a one-dimensional equation for the elementary wave driven by the source in the right-hand side with a fixed \( k_x \) and \( \omega \). Equation (9) will be solved with the help of the WKB approximation.

Represent the fluctuation field \( \delta E \) in the form

\[ \delta E(\omega, x, k_x) = \frac{Q(\omega, x, k_x)}{\sqrt{\psi(\omega, x)}} \exp\left[-i\int_0^x \psi(\omega, x')dx'\right], \]  
(10)

with the wavenumber \( \psi(\omega, x') > 0 \). Along with the presentation of the electric field (7) the minus sign in this expression corresponds to the wave propagation in the positive direction when \( \omega > 0 \). The exponential containing the phase changes with \( x \) much faster than \( Q \) and \( \psi \). Substitute the relationship (10) into (9). As a result we find the zero-order approximation for the solution of equation (9)

\[ \psi(\omega, x) = \psi(\omega, 0)(1 - x^2/d^2)^{1/2}, \]

\[ \psi(\omega, 0) = [2\omega_{pe}(\omega - \omega_{pe})/3\nu_{ce}^2]^{1/2}, \]

\[ d = 2b[1(\omega - \omega_{pe})/\omega_{pe}]^{1/2}. \]

Note that \( \psi(\omega, x) \) is the wavenumber of the Langmuir wave. At the points \( x_{1,2} = \pm d \) the value \( \psi(\omega, x) \) is equal to 0. Thus, the wave going outside the irregularity reflects here and returns back, executing a cyclic propagation. It is seen from (11) that the solution (10) is valid when \( \omega > \omega_{pe} \). Moreover, the backscattering of the probe wave by the Langmuir fluctuations is possible only when \( \psi(\omega, 0) \) is equal or greater than the twice wavenumber of the probe wave (see below). Hence, we will consider the fluctuations with a finite frequency shift \( \omega - \omega_{pe} \) satisfying this condition. In such a case the distance \( 2d \) between the turning points \( x_{1,2} \) is much more than the Langmuir wavelength \( 2\pi/\psi(\omega, x) \) anywhere inside the irregularity except a relatively small vicinity of the turning points. This makes the WKB solution accurate enough in the most part of the irregularity. As for the turning points, their influence will be discussed later.

The equation for the first order solution of (9) is as follows

\[ \frac{\partial Q}{\partial x} + \gamma(\omega, x)Q = \frac{i}{2} \frac{\omega_{pe}^2 |k_x|}{3\nu_{ce}^2} \frac{1}{\sqrt{\psi(\omega, x)}} \exp[-iT(x)]S(\omega, k_x), \]

\[ T(x) = \int_0^x [k_x - \psi(\omega, x')]dx', \]  
(12)

where \( \gamma(\omega, x) \) is the plasma wave damping rate in space

\[ \gamma(\omega, x) = \frac{2\pi}{\sqrt{3} \nu_{ce}} \frac{\sigma}{[e^x(\omega, x)]^{3/2}}. \]  
(13)

As the plasma wavelength is significantly less than the distance between the turning points \( 2d \), the exponential term in (12) is a fast-oscillation function except the points where the derivative of the phase is equal...
to zero

\[ T'(x_{3,4}) = k_x - \psi'(\omega, x_{3,4}) = 0, \]
\[ x_{3,4} = \pm d \left[ 1 - k_x^2 / \psi^2(\omega, 0) \right]^{1/2}, \]
\[ T(x_3) = \pm T(x_3), \]
\[ T(x_3) = k_x x_3 - 2 \frac{\sqrt{2}}{\sqrt{3}} \frac{(\omega - \omega_p)}{b v_{ce}} \left[ \frac{1}{2} \arcsin \frac{x_3}{d} + \frac{x_3}{2d} \sqrt{1 - x_3^2 / d^2} \right]. \] (14)

It is seen from this equation that a parabolic layer has 2 such points \( x_3 \) and \( x_4 \) when

\[ |k_x| < \psi'(\omega, 0). \] (15)

We call these points stationary or plasma wave generation points.

It is seen from equation (14) that the coordinate of a stationary point \( x_{3,4} \) satisfies the condition \( k_x = \psi'(\omega, x_{3,4}) \). In fact, this is the synchronism condition meaning that the plasma wave is generated in a small region where the wavenumber of this wave \( \psi \) is equal to that of the fluctuation source \( k_x \). Note that \( \omega \) is the frequency of the fluctuation source component playing a role of external driver for the plasma wave. It is interesting to note that in our case the system ‘Langmuir wave + fluctuation source’ behaves like a mechanical vibratory system driven by an external harmonic force [30].

The key feature of the fluctuation problem considered here is that we have a set of Langmuir waves with different frequencies \( \omega \) and wavenumbers \( \psi \). The stationary point position depends on \( \omega \) and the fluctuation source wavenumber \( k_x \). As the fluctuation source contains a broad set of \( k_x \) and \( \omega \) (see below, section 4) we need to summarize the effect of all the stationary points distributed over the interval determined by the second line of equation (14) and the condition equation (15). Thus, the stationary points giving contribution to the total fluctuation spectrum turn to be distributed in the interval \((-d, +d)\), i.e. between the turning points \( x_3 \) and \( x_2 \).

Equation (12) can be solved using the stationary phase approximation. Expand \( T(x) \) in the vicinity of the stationary points as a Taylor series

\[ T(x) = T(x_{3,4}) + \frac{1}{2} T''(x_{3,4})(x - x_{3,4})^2, \]
\[ T''(x_{3,4}) = \frac{\omega_p^2}{6v_{ce}^2} b^2 x_{3,4}. \] (16)

Thus, near the stationary points the solution of equation (12) can be found with the help of the expansion (16). Far from the stationary points the phase \( T(x) \) varies fast enough and causes short-period oscillations of the rhs in (12). Such oscillations suppress any substantial effect of the rhs on \( Q(\omega, x) \). In view of that, outside the vicinity of the stationary points the equation (12) reduces to the homogeneous form

\[ \frac{\partial Q}{\partial x} + \gamma(\omega, x) Q = 0 \] (17)

with the solution

\[ Q(\omega, x, k_x) = Q_{3,4} \exp \left[ -f(x_{3,4}, x) \right], \]
\[ f(x_3, x_4) = \int_{x_3}^{x_4} \gamma(\omega, x) dx = \frac{\Gamma}{2\pi} \left( \arcsin \frac{x_4}{d} - \arcsin \frac{x_3}{d} \right), \]
\[ \Gamma = 2 \int_{x_3}^{x_4} \gamma(\omega, x) dx = \pi \frac{2}{3} \frac{v_{ce}}{b v_{ce}}. \] (18)

This equation describes a plasma wave propagating freely after it leaves the generation region localized near the point \( x_3 \) or \( x_4 \). The wave exhibits collisional damping represented by the exponential argument. Here \( f(x_{3,4}, x) \) determines the damping magnitude on the way from a generation point to the point \( x \). The damping of the wave passing a full cycle back and forth inside the irregularity is given by the exponent \( \Gamma \).

In order to find the factor \( Q_{3,4} \) in (18) we go back to equation (12). In a small vicinity of the turning points, where the plasma wave is generated, the damping effect is negligible. Hence in calculating \( Q_{3,4} \) we can omit the term with \( \gamma(\omega, x) \) in the rhs of (12). In this case the solution of (12) reduces to a simple integration of the rhs of this equation

\[ Q_{3,4} = \frac{i}{2} \frac{\omega_p^2}{3v_{ce}^2} \frac{\bar{S}(\omega, k_x)}{k_x} \exp \left[ -iT(x_{3,4}) \right] \times \int_{-\infty}^{+\infty} dx \exp \left[ -\frac{1}{2} T''(x_{3,4})(x - x_{3,4})^2 \right]. \] (19)
Here the integration limits are extended to ±∞. In fact, the convergence of the integral to it is limit takes place when the integration interval Δx meets the condition \(|T^n(x_3)\) (Δx)² > 1. This relationship gives an estimation for the generation interval of the plasma wave at the stationary points

\[ \Delta x_q \sim |T^n(x_3)|^{-1/2} \]  

(20)

The integral in (19) reduces to the table integrals [35]

\[ \int_{-\infty}^{+\infty} \cos ax^2 \, dx = \sqrt{\frac{\pi}{2|a|}}, \]

\[ \int_{-\infty}^{+\infty} \sin ax^2 \, dx = \sqrt{\frac{\pi}{2|a|}} \text{sign}(a), \]  

(21)

and the factor \(Q_{3,4}\) takes the form

\[
Q_{3,4}(\omega, k_x) = i \frac{\pi}{3|x_{3,4}|} \frac{\omega_p}{b \nu_{se}} k_x \left[ \int_0^\infty \frac{w P}{k_x} \exp \left[ -iT(x_{3,4}) - \frac{i\pi}{4} \text{sign}(x_{3,4}) \right] \right].
\]  

(22)

The expression (22) has a divergence when \(|x_{3,4}| \to 0\) or \(|k_4| \to \psi(\omega, 0)\). This means that the stationary phase approximation brakes down when the stationary points are close to the density minimum point. Nevertheless, due to the subsequent integration with respect to \(k_4\) the divergence becomes integrable and has no effect on the total result. Numerical integration shows that the total plasma wave amplitude gets the main contribution almost uniformly from all the permissible wave numbers \(k_x < \psi(\omega, 0)\) (see (15)). This enables us to estimate approximately the generation interval \(\Delta x_q\) (20) by taking the difference \(\psi(\omega, 0) - k_3 \sim \psi(\omega, 0)\). As a result we obtain

\[ \Delta x_q \sim \left[ \nu_{se}/(b \omega_p) \right]^{1/2}. \]  

(23)

Evaluation of this expression for the ionospheric parameters demonstrates that \(\Delta x_q\) is much less than the distance between the turning points \(2d\). This justifies the stationary phase approximation employed here.

3. Spatial evolution of the waves in the parabolic irregularity in the presence of collisional damping

In order to calculate the scattered spectrum of the probe wave we need to have the amplitude of the electron density fluctuations \(\delta n\). This amplitude can be found from \(\delta E\) by use of the Poisson equation \(\nabla \cdot \mathbf{E} = 4\pi \rho_e\) where \(\rho_e\) is the electron charge density. In the one-dimensional case the Poisson equation reduces to

\[ \frac{d}{dx} \delta E = 4\pi \delta n, \]

\[ \delta n = \frac{1}{4\pi e} \frac{d}{dx} \delta E = -\frac{i}{4\pi e} \psi(\omega, x) \delta E. \]  

(24)

To obtain the last relationship in (24) we used the expression (10) for \(\delta E\) and took the derivative of the exponential only, as it is the fastest varying term.

For simplicity of the presentation, we will give analytical expressions only for the amplitude \(\delta n\) coming out of the generation point \(x_3\) in the positive direction. Other cases are quite similar. Combining the equations (10), (18), (22) and (24) we find

\[ \delta n_{3,(+)}^{(1)}(\omega, x_3, x < x < x_3, k_x) = A_{3,(+)}^{(1)}(x_3) \exp \left[ -i \int_0^x \psi(\omega, x') \, dx' \right], \]

\[ A_{3,(+)}^{(1)}(x_3) = -\frac{i}{4\pi e} \sqrt{\psi(\omega, x_3)} Q_3 \exp \left[ -J(x_3, x) \right], \]  

(25)

where the subscript ‘\(x_3\), (+)’ denotes the wave generated in the vicinity of the stationary point \(x_3\) and coming out of this vicinity in the positive direction. The superscript (1) stands for the wave passing it’s first cycle inside the irregularity.

Propagation of the waves of the electron density inside the irregularity is illustrated schematically in figure 1. Here all the characteristic points: \(x_{1,2}\) and \(x_{3,4}\), as well as the electron density profile are depicted. The figure shows 4 waves. So far we considered only 2 of them, which are generated at the points \(x_3\) and \(x_4\) by the fluctuation source with the wavenumber \(k_x > 0\) (panels I and II in the figure). These waves come out of the generation regions in the positive direction. Another pair of waves with the same frequency \(\omega\) is excited in the vicinity of the
points $x_3, x_4$ by the source component with $k_x < 0$ (panels III and IV). These two waves leave the generation point in the negative direction.

The amplitude (25) is valid until the wave reaches the turning point and reflects back. The full first-cycle amplitude is a sum of four amplitudes

$$A^{(1)} = A_{x_{3+}}^{(1)} + A_{x_{3-}}^{(1)} + A_{x_{4+}}^{(1)} + A_{x_{4-}}^{(1)},$$

$$\delta \tilde{A}^{(1)} = A^{(1)} \exp \left[ -i \int_0^x \psi(\omega, x') \, dx' \right],$$

where the subscripts in the first line denote generation points ($x_3$ or $x_4$) and the direction in which the plasma wave leaves the vicinity of the stationary point (+ or −).

Figure 1 demonstrates that, as the waves propagate in the irregularity, their normalized amplitude $\tilde{A}^{(1)} = A^{(1)} / \sqrt{\psi(\omega, x)}$ decreases because of the collisional damping. Due to the turning points $x_{1,2}$ the waves cannot leave the irregularity, and they pass a number of cycles until the full damping occurs. The solid lines in figure 1 denote the propagation of all the waves in the positive direction. The propagation in the negative direction can be treated exactly in the same way due to the symmetry of the problem.

After the reflection at the point $x_1$ the wave with the amplitude $A_{x_{3+}}^{(1)}$ (panel I in figure 1) travels in the negative direction (dashed curve) reflects at $x_2$, and then goes in the positive direction (solid curve). The amplitude of the wave for the last stage at the interval $x_2 < x < x_3$ is given by

$$A_{x_{3+}}^{(1)}(\omega, x, x_2 < x < x_3, k_x) = \frac{-i}{4\pi e_x} \sqrt{\psi(\omega, x)} Q_3 \exp(-i\Phi) \times \exp \left[ -f(x_3, x_2) - f(x_2, x_1) - f(x_1, x) \right].$$

The solutions (25) and (27) represent the first-cycle amplitude at the stages of the positive propagation of the plasma wave which is initially generated at the point $x_3$ and comes out of the generation interval in the positive direction. The procedure for calculating the three other amplitudes $A_{x_{3-}}^{(1)}$, $A_{x_{4+}}^{(1)}$, and $A_{x_{4-}}^{(1)}$ shown in figure 1 is quite similar. Taking the sum of the amplitudes as in (26) we obtain the full first-cycle amplitude $A^{(1)}$ formed by all the waves.

The total amplitude of each wave is the sum of the amplitudes from all the cycles [30]. In order to obtain the total amplitude we should find the first-cycle amplitude and then multiply this by the factor calculated in [30]

$$A(x, \omega, k_x) = \frac{A^{(1)}(x, \omega, k_x)}{1 - \exp(-i\Phi(\omega) - \Gamma)},$$

where $\Gamma$ determines the one-cycle damping (see (18)), and $\Phi(\omega)$ is the full phase shift of the wave passed one cycle.

Figure 1. Plasma waves generated at the stationary points $x_3$ and $x_4$ by the fluctuation source with a fixed frequency $\omega$ and wave number $k_x$. Only one cycle of their propagation between the turning points $x_1$ and $x_3$ is shown here. $\tilde{A}^{(1)} = A^{(1)} / \sqrt{\psi(\omega, x)}$ stand for the first-cycle amplitudes normalized to the factor $\sqrt{\psi(\omega, x)}$ (see (25)), $n(x)$ is the ambient electron density. The solid lines denote the stage of the propagation in the positive direction for each wave.
\[ \Phi(\omega) = 2 \int_{t_1}^{t_2} \psi(\omega, x) \, dx + \pi = \frac{4\pi}{\sqrt{b}} \frac{(\omega - \omega_p)}{b} + \pi. \]  

Additional phase shift \( \pi \) here arises from the solution of the equation (3) near the turning points in terms of the Airy function.

Then the total amplitude \( A \) taking into account the plasma wave circulation inside the irregularity can be easily found by (28). Finally we arrive at the expression for the electron density fluctuations trapped in the irregularity

\[ \delta n(\omega, x, k_x) = A(\omega, x, k_x) \exp \left[ -i \int_0^x \psi(\omega, x') \, dx' \right] . \]  

The amplitude \( A \) is proportional to the fluctuation source via the factors \( Q_2 \) and \( Q_4 \), i.e., we can write

\[ A(\omega, x, k_x) = B(\omega, x, k_x)S(\omega, k_x), \]
\[ A^{(1)}(\omega, x, k_x) = B^{(1)}(\omega, x, k_x)S(\omega, k_x), \]
\[ B(\omega, x, k_x) = \frac{B^{(1)}(\omega, x, k_x)}{1 - \exp[-i\Phi(\omega) - \Gamma}], \]

where \( B \) is the regular (non-random) factor of the amplitude \( A \). The amplitude \( A \) (or \( B \)) is fully determined by the first-cycle amplitudes of the type (25), (27) and the equations (26) and (28). The amplitudes \( B^{(1)} \) related to the full amplitudes \( A^{(1)} \) and eventually to \( A \) are presented in appendix A for all the intervals \( x \) and both directions of the source wavenumber \( k_x \).

Equation (30) is the Langmuir wave excited by the fluctuation source with a definite wavenumber \( k_x \). As noted above, in order to find the total fluctuations \( \delta n(\omega, x) \) produced by all the permissible \( k_x \), we must integrate \( \delta n(\omega, x, k_x) \) with respect to \( k_x \)

\[ \delta n(\omega, x) = A(\omega, x) \exp \left[ -i \int_0^x \psi(\omega, x') \, dx' \right] , \]
\[ A(\omega, x) = \int_{-\psi(\omega, 0)}^{+\psi(\omega, 0)} A(\omega, x, k_x) \, dk_x, \]

where the integration interval \([-\psi(\omega, 0), \psi(\omega, 0)]\) results from the condition of the existence of stationary points (15). Thus, the procedure presented here makes it possible to find the stationary amplitude of the Langmuir fluctuations trapped in the irregularity in the presence of a finite collisional damping.

### 4. Spectrum of Langmuir fluctuations in the parabolic irregularity

In order to calculate the frequency and wavenumber spectrum let us consider the second moment of the electron density fluctuations

\[ \langle \delta n(t_1, x_1) \delta n(t_2, x_2) \rangle = \langle \delta n \delta n \rangle_{t_1-t_2,x_1-x_2} , \]

where the angle brackets denote the average. The moment \( \langle \delta n \delta n \rangle \) depends only on the difference of the time variables \( t_1 - t_2 \) due to the stationary character of the problem considered. According to the Laplace–Fourier transform (6) and (7), the frequency spectrum is defined as

\[ \langle \delta n \delta n \rangle_{\omega, k_x} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\tau \exp(-i\omega\tau) \langle \delta n \delta n \rangle_{\tau, k_x} , \]

where the time difference \( t_1 - t_2 \) is replaced by \( \tau \).

It is shown in [4] that the spectrum (34) is related to the fluctuations \( \delta n(\omega, x, k_x) \) by the formula

\[ \lim_{\Delta \to 0} \Delta \langle \delta n(\omega, x) \delta n^*(\omega, x) \rangle = \frac{1}{4\pi} \langle \delta n \delta n \rangle_{\omega, k_x} . \]

Thus, we can calculate the frequency spectrum using the equations for the fluctuation amplitudes found in the previous section. To do so, let us take into account equation (31) where the fluctuation source is shown explicitly. Combination of equations (30)–(32) and (35) yields

\[ \langle \delta n \delta n \rangle_{\omega, k_x} = \frac{4\pi}{|1 - \exp[-i\Phi(\omega) - \Gamma]|^2} \exp \left[ -i \int_{x_1}^{x_2} \psi(\omega, x') \, dx' \right] \]
\[ \times \int_{-\psi(\omega, 0)}^{+\psi(\omega, 0)} \mathrm{d}k_x \int_{-\psi(\omega, 0)}^{+\psi(\omega, 0)} \mathrm{d}k_x \hat{B}(\omega, x_1, k_x, 1)[\hat{B}^{(1)}(\omega, x_2, k_x, 2)]^* \]
\[ \times \lim_{\Delta \to 0} \Delta \langle \hat{S}(\omega, k_x, 1) \hat{S}^*(\omega, k_x, 2) \rangle . \]  

(36)
The second moment of the fluctuation source here can be found in a way similar to that used in equation (35) (see [4])

\[
\lim_{\Delta \to 0} \Delta \langle \hat{S}(\omega, k_{x,1})S^*(\omega, k_{x,2}) \rangle = \frac{1}{4\pi} \delta(k_{x,1} - k_{x,2}) \langle S^2 \rangle_{\omega, k_{x,1}}, \tag{37}
\]

where \( \langle S^2 \rangle_{\omega, k_{x,1}} \) is the spectral function of the fluctuation source calculated in \([31, 32]\)

\[
\langle S^2 \rangle_{\omega, k} = \frac{4e^2 p}{\pi \omega^2 m^2} \int \frac{d\mathbf{k}d\mathbf{p}}{m^2} \frac{f_{\mathbf{p}}(\omega + \mathbf{k} \cdot \mathbf{v} - \mathbf{k}' \cdot \mathbf{v})}{f_{\mathbf{k}}(\mathbf{k})^2} \left[ \int (1 + e_x(\omega, \mathbf{k}'))^2 \sum_i e_i^2 n_i f_i(\mathbf{p}_i) + e_i^2 n_i f_i(\mathbf{p}_i) \right] \times \left[ \sum_i e_i(\omega + \mathbf{k} \cdot \mathbf{v} - \mathbf{k}' \cdot \mathbf{v})^2 \right]. \tag{38}
\]

Here \( f_{\mathbf{x}}(\mathbf{p}) \) are the electron and ion distribution functions, \( \mathbf{p} \) is the pulse, \( \mathbf{v} = \mathbf{p}/m_e \) is the velocity. The functions \( e_x, i \) are the contributions of each particle species to the longitudinal dielectric function \( \varepsilon = 1 + e_x + \sum_i e_i \) [4].

The equation (38) is applicable for arbitrary distribution functions \( f_{\mathbf{x}} \), which can differ from the Maxwell distributions due to various nonequilibrium processes in a plasma. In this paper, we will not consider specific mechanisms of generation of the Langmuir fluctuations. Instead, we concentrate on the effect of a parabolic irregularity on the fluctuation spectrum. It is shown in [31] that the spectral function of the fluctuation source changes only slightly with a small variation of the electron density, as it takes place in the vicinity of the density minimum point (1). That’s why we use the expression for the fluctuation source (37) valid in the homogeneous case.

Substitute equation (37) into (36) and integrate the \( \delta \)-function with respect to \( k_{x,2} \)

\[
(bn \delta n)_{\omega, x_1, x_2} = \frac{1}{|1 - \exp[-i\Phi(\omega)]|} \exp \left[ -i \int_{x_2}^{x_1} \psi(\omega, x') dx' \right] \times \int_{-\psi(\omega,0)}^{\psi(\omega,0)} d\psi_{\omega, x_1, x_2} B^{(1)}(\omega, x_1, k_{x,1}) B^{(1)}(\omega, x_2, k_{x,1}) \langle S^2 \rangle_{\omega, k_{x,1}} . \tag{40}
\]

Numerical calculations presented below show that the integration with respect to \( k_{x,1} \) in (40) gives a result slowly varying at a distance of the Langmuir wavelength. At the same time, the exponential in the equation (40) is a rapidly oscillating function of \( x_1 - x_2 \) with the characteristic scale of the order of the Langmuir wavelength (remind that \( \psi \) is the Langmuir wavenumber). Therefore, the exponential factor will give the main contribution to the spectrum of the Langmuir fluctuations in the wavenumber space.

Introduce new variables \( \chi = x_1 - x_2 \), \( x = (x_1 + x_2)/2 \) which represent the small and large scales in space, respectively. Then the full frequency and wavenumber spectrum of the Langmuir fluctuations can be found in the following way

\[
(bn \delta n)_{\omega, x_1, x_2} = \frac{1}{2\pi} \int d\chi \exp \left( i\xi \chi \right) (bn \delta n)_{\omega, x_1, x_2} \tag{41}
\]

where \( \xi \) is the wavenumber variable in the spectrum, and the slow dependence on \( x \) is retained. The exponential in equation (40) is the only factor depending on the ‘fast’ variable \( \chi \); the integral in this exponential can be reduced to the form

\[
\int_{x_2}^{x_1} \psi(\omega, x') dx' \approx \chi \psi(\omega, x) . \tag{42}
\]

We have replaced here \( \psi(\omega, x') \) with \( \psi(\omega, x) \) because \( \psi(\omega, x') \) changes negligibly at a distance \( \chi \) being of the order of the Langmuir wavelength. Then the integral in equation (41) with respect to \( \chi \) can be expressed in terms of the \( \delta \)-function

\[
\frac{1}{2\pi} \int d\chi \exp \left[ i\xi \left[ \psi(\omega, x) \right] \right] = \delta[\xi - \psi(\omega, x)]. \tag{43}
\]

Substitute equations (40), (43) in (41) and use the expressions for the amplitudes \( B^{(1)} \) given in appendix A. As a result we obtain the spectrum of the Langmuir fluctuations

\[
(bn \delta n)_{\omega, x_1, x_2} = \psi(\omega, x) \delta[\xi - \psi(\omega, x)] \langle S^2 \rangle_{\omega, k_{x,1}, k_{x,1}} . \tag{44}
\]
\[ P_0(\omega, x) = \frac{\omega_p^2}{b_{\omega}^2 v_{\omega}} P_1(\omega) \psi(\omega, x) G(\omega, x), \]
\[ P_1(\omega) = \frac{1}{1 - \exp[-i\Phi(\omega) - \Gamma]]^2} \]
\[ = \frac{1}{1 - 2\cos[\Phi(\omega)]\exp(-\Gamma) + \exp(-2\Gamma)}, \]

where \( G(\omega, x) \) is the function presented in appendix B.

5. Discussion of the results

The formula (44) is the frequency and wavenumber spectrum of the Langmuir fluctuations. According to one-dimensional approximation considered here, the y- and z- components of the wavenumber are omitted in the expression for \( (dn_e, dn_i)_{x, x} \). The spectrum (44) is proportional to the spectral function of the fluctuation source (38).

As we pointed out above, the fluctuation source used here is applicable for non-maxwellian distribution functions. In this paper, we do not concentrate on the role of various non-equilibrium processes in the generation of fluctuations. Here we only note that non-maxwellian distributions can increase the fluctuation source intensity, and make it is dependence on \( \omega \) and \( k \) more complicated. An example of the ionospheric plasma in which the electron distribution function is anisotropic (in contrast to the Maxwell distribution) is given in [31]. It is shown in this paper that such an anisotropy results actually in an enhancement of the spectrum intensity and makes the spectrum anisotropic, as well.

As a rule, this is a nonresonant function in the frequency range of the Langmuir waves \( \omega \sim \omega_p0 \). That's why we can take \( \omega \approx \omega_p0 \) in the frequency argument of \( \langle S^2 \rangle \) and omit the dependence on \( x \) in this spectral function. The condition \( \xi \ll k_0 \) corresponds to a small Landau damping of the Langmuir waves \( (k_0 = 2\pi/\lambda_D \text{ where } \lambda_D \text{ is the Debye length}) \). For illustration, let us give the source spectral function in a particular case of thermal equilibrium [31]

\[ \langle S^2 \rangle_{\omega \sim \omega_p0, \xi \ll k_0} = \frac{T_{\text{e}f}}{2\pi^2 \omega_p0}, \]

where \( T \) is the electron temperature. The rhs of this equation is proportional to the electron collision frequency \( \nu_{\text{ef}} \), i.e. to the damping rate of the Langmuir waves in the case \( \xi \ll k_0 \). This is in line with the fluctuation–dissipation theorem [34]. When the plasma is nonequilibrium, equation (46) becomes invalid, and the formula (38) should be used instead. In this paper we do not concentrate on effects of non-equilibrium distribution functions on the spectrum properties and consider only the spectrum peculiarities caused by the electron density irregularity.

The presence of the \( \delta \)-function in the expression (44) means that the spectrum component with a given wavenumber \( \xi \) is different from zero at the point where the wavenumber of the Langmuir wave \( \psi(\omega, x) \) is equal to \( \xi \). The formula (44) can be used to obtain the frequency spectrum of an electromagnetic wave scattered by the parabolic plasma cavity. In the homogeneous plasma the scattered spectrum intensity is proportional to the electron density fluctuation spectrum multiplied by the plasma volume (see e.g. [34]). In our case of one-dimensional plasma irregularity the scattered spectrum can be calculated by integrating the expression (44) with respect to the variable \( x \) [21]. This integration can be easily carried out due to the presence of the \( \delta \)-function in the expression for the spectrum. Thus, the subsequent analysis of the frequency spectrum modulation represented by the factor \( P_0(\omega, x) \) is applicable to both the electron density fluctuation spectrum and the spectrum of a scattered electromagnetic wave.

Numerical calculation shows that the function \( G(\omega, x) \) changes only by a few percent in the frequency interval of interest. Thus, the behavior of the spectrum as a function of \( \omega \) and \( x \) is determined in principal by three factors: \( \delta \)-function, \( \psi(\omega, x) \) (the Langmuir wavenumber) and \( P_1(\omega) \).

The factor \( P_1(\omega) \) has the maximum value \( P_1^{\text{max}} = [1 - \exp(-\Gamma)]^{-2} \) when the phase shift \( \Phi(\omega) \) (29) of the Langmuir waves passed a full cycle inside the inhomogeneity is equal to \( 2\pi m \text{ (} m \text{ is a positive integer)} \)

\[ \Phi(\omega) = \frac{4\pi (\omega - \omega_p0)}{\sqrt{6}b_{\omega}v_{\omega}} + \pi = 2\pi m. \]

This condition is equivalent to that for the position of discrete spectrum lines discussed in [22, 24, 25, 29] for the case of a collisionless approximation. Unlike [22, 24, 25, 29] in our case of a collisional plasma the spectrum lines have finite width and maximum values. It is seen that these lines become less distinct with an increase in \( \Gamma \) due to a decrease in the maximum value \( P_1^{\text{max}} = [1 - \exp(-\Gamma)]^{-2} \).
The factor $P(\omega, x)$ determines $\omega$- and $x$-dependence of the spectrum part proportional to the $\delta$-function. It is seen from (45) that the spectrum intensity increases when the parameter $b$ is reduced which means the irregularity scale increase.

Figure 2 shows the factor $P(\omega, x)$ as a function of the coordinate $x$ for 3 fixed frequencies $\omega$. These frequencies are taken so that they match the condition (47) for the local maxima in the spectrum. It is seen from this picture that the Langmuir fluctuations occupy larger region in the irregularity when the relative frequency shift becomes greater. The zero points for $P_0$ in this picture coincide with the turning points for a chosen frequency $\omega$.

It is necessary to emphasize that the relative frequency shift $\delta\omega = (\omega - \omega_p)/\omega_p$ should be small enough so that we could neglect the Landau damping of the Langmuir waves along the path between the turning points. For the parameters used in the calculation of figure 2 this condition is satisfied when $\psi(\omega, 0) \gamma_{dx} < 0.2$ or $\delta\omega < 0.06$.

The frequency dependence of $P_0(\omega, x)$ at the point $x = 0$ is shown in figure 3. It is seen from this figure that the spectrum modulation depth increases rapidly with decreasing the collision frequency. This kind of modulation represents a fine structure of the fluctuation spectrum caused by the Langmuir waves trapping inside the irregularity. The position of the local maxima in figure 3 can be obtained from equation (47)

$$[\delta\omega]_m = \frac{\sqrt{6} \, b v_{ce}}{4 \, \omega_p} (2m - 1), \ m = 1, 2, 3 \ldots$$  

Experimental observations of the spectrum modulation shown in figure 3 may give important information about some plasma parameters. Such observations can be implemented by means of electromagnetic wave scattering from the fluctuations. According to (45), the modulation depth (the ratio between neighboring local maximum $P_{1}^{\text{max}}$ and minimum $P_{1}^{\text{min}}$) is determined by the only parameter $\Gamma$ (18)

$$\eta = \frac{P_{1}^{\text{max}}}{P_{1}^{\text{min}}} = \frac{[1 + \exp(-\Gamma)]^2}{[1 - \exp(-\Gamma)]^2}.$$  

Therefore measuring the ratio $P_{1}^{\text{max}}/P_{1}^{\text{min}}$ we can find the parameter $\Gamma$ which is directly proportional to the electron collision frequency and the irregularity size (represented by the value $b^{-1}$).

Another characteristic which can be measured experimentally, is the interval between neighboring maximum positions. From (48) we can find for this interval...
\[
[b(\delta \omega)]_{m+1} - [b(\delta \omega)]_m = \frac{\sqrt{6}}{2} \frac{b \nu_w}{\omega_{p0}} \\
\]

which is proportional to the parameter \(b\), i.e. inversely proportional to the irregularity size.

The electron collision frequency and the irregularity size can be found from the Thomson scattering experiment with the help of the relationships (49) and (50). When the electron distribution function is close to the maxwellian distribution, the ratio \(\nu_w/\omega_{p0}\) is equal to the electron Debye length \(\lambda_D = [\Sigma/(4\pi \varepsilon^2 n)]^{1/2}\). The Debye length for the ionospheric plasma can be found using the International Reference Ionosphere model [36]. In particular, for the ionospheric F-layer the Debye length is usually of the order of 1 cm. Knowing the Debye length we can directly obtain the irregularity parameter \(b\) from the measurements of the interval \([b(\delta \omega)]_{m+1} - [b(\delta \omega)]_m\) between neighboring maximum positions in the fluctuation spectrum

\[
b = \frac{2}{\sqrt{6}} \lambda_D \{ [b(\delta \omega)]_{m+1} - [b(\delta \omega)]_m \}.
\]

Another parameter which can be obtained directly from the scattering spectrum is the modulation depth \(\eta\) (49). Combining this relationship with the expression for \(\Gamma\) (18) we find

\[
\frac{\nu_{cf}}{\omega_{p0}} = \frac{1}{\pi} \frac{1}{b} \lambda_D \ln \left[ \frac{\eta^{1/2} + 1}{\eta^{1/2} - 1} \right].
\]

The parameters \(b\) and \(\lambda_D\) in the right-hand side of this relationship are already known, and the modulation depth \(\eta\) is obtained from the experiment. Thus, equation (52) permits the ratio \(\nu_{cf}/\omega_{p0}\) to be found experimentally.

Note that the Langmuir frequency \(\omega_{p0}\) is easily found in the Thomson scattering experiment, as it is the difference between the frequencies of the scattered and probe waves. This enables one to find the collision frequency \(\nu_{cf}\).

The spectrum modulation shown in figure 3 may be detected experimentally if the change in the local maxima position during the detection time is less than the distance between the maxima \([b(\delta \omega)]_{m+1} - [b(\delta \omega)]_m\). For example, the change in a maximum position depends on the variation of the irregularity parameter \(b\) due to the diffuse spreading of the irregularity:

\[
\Delta[b(\delta \omega)]_m = \Delta b \frac{\sqrt{6}}{4} \frac{\nu_w}{\omega_{p0}} (2m - 1).
\]

Therefore, the condition of small frequency shift for a local maximum position \(\Delta[b(\delta \omega)]_m < [b(\delta \omega)]_{m+1} - [b(\delta \omega)]_m\) reduces to

\[
\Delta b (2m - 1) < b.
\]

It is clear from equation (54) that the first maximum with \(m = 1\) can be observed if during the detection time the relationship \(\Delta b < b\) holds valid, i.e. the diffuse increase in the irregularity size is less than the size itself. With an increase in the local maximum number \(m\) the condition (54) becomes more severe. Thus, the irregularity variation may reduce the number of maxima observed at a fixed detection interval \(\tau_d\). If \(m_0\) is the number of the last observable maximum then from equation (54) we obtain that at the interval \(\tau_d\) the irregularity size change is as follows \(\Delta b = b/(2m_0 - 1)\). This means that the full time for the irregularity diffuse spread (the irregularity lifetime) \(\eta\) will be

\[
\eta = \tau_d (2m_0 - 1).
\]

Hence, experimental observation of the Langmuir fluctuation spectrum fine structure may give an estimation for the lifetime of the parabolic plasma irregularity.

The main features of the Langmuir fluctuation spectrum discussed in this paper may be applicable for various kinds of plasmas where the electron density irregularities and electron collisions are important. For example, rocket experiments carried out in the ionospheric plasma (see e.g. [37]) show complex structure of Langmuir oscillations which can be attributed to the plasma wave trapping inside the irregularities [24]. Such irregularities could be investigated by the Thomson scattering technique. The most suitable location for such an experiment is the Arecibo Observatory, Puerto Rico where the 430 MHz incoherent scattering radar is in operation [38]. This radar is the most sensitive radar in the world. At its location, the ionosphere is well horizontally stratified at the F-layer altitudes. If the dependence of the electron density on the vertical coordinate has a local minimum with a characteristic size 100 m or more, this geometry is relevant to our results. At the same time the collisional damping of Langmuir waves in such irregularities becomes important.

It is interesting to note one more possibility to conduct the research of Langmuir fluctuations in experiments on laboratory modeling of the ionospheric plasma [39]. In such experiments (with a moderate collision frequency), electron density irregularities can be deliberately created, and the distribution of the Langmuir waves in them might be observed.
It is important that the Thomson scattering process selects only the fluctuations with the wave vectors parallel to the scattering vector [34]. Thus, if the scattering vector is parallel to the density gradient of the irregularity, the observable scattering spectrum will depend only on the fluctuations propagating along the density gradient, which is the case considered in this paper. Such a geometry is appropriate for observing the most pronounced effects of plasma irregularities. If the scattering vector makes a nonzero angle with the density gradient, we will observe the fluctuations obliquely propagating in the irregularity. In this case the Langmuir wave amplification due to the superposition of incident and reflected waves inside the irregularity decreases, and the frequency spectrum of the scattered wave will be not modulated.

The one-dimensional model considered in this paper can be valid for real 3D plasma if the distance $2d$ between the turning points (11) is sufficiently less than the radius of curvature for the surface where the electron density $n_e(x)$ is constant. For a 100 m irregularity ($b = 4.47 \times 10^{-5}$ cm$^{-1}$, see the beginning of section 2) and the relative frequency shift $\delta f = 10^{-2}$ we obtain that $2d$ is of the order of the irregularity size $= 100$ m. Then the radius of curvature should be more than 100 m. Such flat-layered irregularities are present both in the ionosphere [40, 41] and in the laboratory [23].

If the flat-layer approximation is not valid, we should consider excitation of Langmuir fluctuations in a 3D plasma cavity. The solution of the problem in this case is much more complicated but in a qualitative sense the main conclusions presented above remain valid. Similarly to 1D case, the plasma waves trapped in the cavity will be represented by a set of eigenfunctions satisfying the quantization conditions. This will result in the frequency modulation of the electron density spectrum. The modulation depth will depend on the collision frequency in a way similar to that shown in figure 3, i.e. the modulation depth will decrease with an increase in the collision frequency. In order for the frequency modulation to be observed in Thomson scattering experiments, the scattering volume should normally include only one plasma cavity. In the opposite case the frequency maxima in the observable spectrum formed by many cavities will be blurred. At the same time, the total intensity of the spectrum will be enhanced when the collision frequency decreases.

6. Conclusions

We have calculated the spectrum of Langmuir fluctuations trapped in a parabolic irregularity of plasma density in the presence of collisional damping. Stationary fluctuation amplitudes are expressed in terms of the fluctuation source which is applicable to non-maxwellian distribution functions of particles. The source contains a broad range of wavenumbers driving a set of elementary waves which form the full fluctuation pattern.

The explicit expression for the fluctuation spectrum is modulated with respect to the frequency variable. This modulation results from reflection of the elementary Langmuir waves at the turning points and superposition of these waves. The depth and the period of the modulation depend on the electron collision frequency and the irregularity size. Observing these modulation characteristics in the experiments on the Langmuir wave scattering by the fluctuations may provide a possibility to find such important plasma parameters as the electron collision frequency, the size and the lifetime of the irregularity.

Appendix A. First-cycle amplitudes $B^{(1)}$

The amplitude $B^{(1)}$ is the regular part of the Langmuir wave amplitude $A^{(1)}$ (see (31)) where the fluctuation source is omitted. The explicit form of $B^{(1)}$ depends on the intervals determined by the points $x_3 - x_4$ and the sign of the source wavenumber $k_x$. The following equations are obtained according to the procedure described in the sections 2 and 3.

\begin{align}
B^{(1)}(x_3 < x < x_4, k_x > 0) &= Y(x, k_x) \times \exp\left\{-i\Phi/2 - J(x_4, x) - J(x_3, x) - i\Phi\right\}, \quad (A.1) \\
B^{(1)}(x_3 < x < x_4, k_x < 0) &= -Y(x, k_x) \times \exp\left\{-J(x_4, x) - J(x_3, x) - i\Phi/2\right\}, \quad (A.2) \\
B^{(1)}(x_4 < x < x_3, k_x > 0) &= Z(x, k_x) \times \exp\left\{-J(x_4, x)\right\}, \quad (A.3) \\
B^{(1)}(x_4 < x < x_3, k_x < 0) &= -Y(x, k_x) \times \exp\left\{-2J(x_4, x) - J(x_3, x) - i\Phi/2\right\}, \quad (A.4) \\
B^{(1)}(x_3 < x < x_4, k_x > 0) &= Y(x, k_x) \times \exp\left\{-J(x_4, x)\right\}, \quad (A.5) \\
B^{(1)}(x_3 < x < x_4, k_x < 0) &= -Y(x, k_x) \times \exp\left\{-2J(x_4, x) - J(x_3, x) - i\Phi/2\right\}, \quad (A.6)
\end{align}
where
\[ Y(x, k_p) = \sqrt{\frac{3\psi(\omega, x)}{\pi x_3}} \frac{\omega_p}{12c_e b_{\text{neq}}} \times \left\{ \exp \left[ -iT(x_3, |k_x|) + \frac{i\pi}{4} - J(x_4, x_3) \right] + \exp \left[ -iT(x_3, |k_x|) - \frac{i\pi}{4} \right] \right\}. \]  
(A.7)
\[ Z(x, k_p) = \sqrt{\frac{3\psi(\omega, x)}{\pi x_3}} \frac{\omega_p}{12c_e b_{\text{neq}}} \left\{ \exp \left[ -iT(x_3, |k_x|) + \frac{i\pi}{4} \right] + \exp \left[ -iT(x_3, |k_x|) - \frac{i\pi}{4} - i\Phi - \Gamma + J(x_4, x_3) \right] \right\}. \]  
(A.8)

The equations (A.1)–(A.8) determine in full the function \( B^{(1)} \) which enters into the expression for the electron density fluctuation spectrum (36). Recall that \( x_{1,2} \) and \( x_{3,4} \) are the turning points and the generation points, respectively (see figure 1); \( \Gamma \) determines the one-cycle damping of the Langmuir wave (18); \( \Phi(\omega) \) is the full phase shift of the wave passed one cycle (29); \( \psi(\omega, x) \) is the wavenumber of the Langmuir wave depending on the frequency \( \omega \) and the coordinate \( x \) (11), and \( T(x_3, |k_x|) \) is the phase function given by (14).

Appendix B. The function \( G(\omega, x) \)

The function \( G(\omega, x) \) enters into the final equation for the electron density fluctuation spectrum (45). This function is determined by the amplitudes of the Langmuir waves given by appendix A. Numerical calculation shows that the function \( G(\omega, x) \) changes only by a few percent in the frequency interval where the fluctuation level is the highest. That is why the main features of the fluctuation spectrum are represented by simple functions \( P_1(\omega), \psi(\omega, x), \) and \( \delta[\xi - \psi(\omega, x)] \). The function \( G(\omega, x) \) has the form
\[ G(\omega, x) = \frac{1}{24\pi^2\sqrt{6}} \int_0^1 \frac{du}{(2 - u^2)^{1/2}} \times [H(\omega, x, k_x) + H(\omega, x, -k_x)]|_{k_x=(1-u^2)^{1/2}}, \]  
(B.1)
with
\[ H(\omega, x, x_2 < x < x_4, k_x > 0) = F(\omega, k_x) \times \exp [-\Gamma - 2J(x_3, x_4) - 2J(x_3, x_3)], \]  
(B.2)
\[ H(\omega, x_2 < x < x_4, k_x < 0) = F(\omega, k_x) \times \exp [-2J(x_3, x_4) - 2J(x_3, x_3)], \]  
(B.3)
\[ H(\omega, x_4 < x < x_4, k_x > 0) = U(\omega, k_x) \exp [-2J(x_3, x_4)], \]  
(B.4)
\[ H(\omega, x_4 < x < x_4, k_x < 0) = F(\omega, k_x) \times \exp [-2J(x_3, x_4) - 4J(x_3, x_3)], \]  
(B.5)
\[ H(\omega, x_3 < x < x_4, k_x > 0) = F(\omega, k_x) \times \exp [-2J(x_3, x_4) - 4J(x_3, x_4)], \]  
(B.6)
and the functions \( F \) and \( U \) are as follows
\[ F(\omega, k_x) = 1 - 2 \exp [-J(x_4, x_3)] \times \sin [2T(x_3, |k_x|)] + \exp [-2J(x_4, x_3)], \]  
(B.8)
\[ U(\omega, k_x) = 1 - 2 \exp [-\Gamma + J(x_4, x_3)] \times \sin [2T(x_3, |k_x|) + \Phi] + \exp [-2\Gamma + 2J(x_4, x_3)]. \]  
(B.9)
All other notations are summarized in the last paragraph of appendix A.

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