Warm alpha-nucleon matter

S. K. Samaddar and J. N. De

Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700064, India

The properties of warm dilute alpha-nucleon matter are studied in a variational approach in the Thomas-Fermi approximation starting from an effective two-body nucleon-nucleon interaction. The equation of state, symmetry energy, incompressibility of the said matter as well as the α fraction are in consonance with those evaluated from the virial approach that sets a bench-mark for such calculations at low densities.

PACS numbers: 21.65.Cd, 21.65.Ef, 21.65.Mn, 24.10.Pa
Keywords: nuclear matter, alpha-clusters, Thomas-Fermi approach, S-matrix

I. INTRODUCTION

Cold nuclear matter at subsaturation density as α matter has been subjected to a critical study for some time [1, 2]. The aim is to understand the α-clustering near the surface of heavy nuclei or the putative dilute alpha-condensate in light 4n nuclei. In astrophysical context, in following the evolution of the core-collapse supernovae, these studies have been extended to the case of warm nuclear matter [3]. The homogeneous low-density nuclear matter stabilizes as a mixture of nucleons and nucleon-clusters. It has lower free energy compared to that for nucleonic matter. The cluster composition is temperature and density dependent, with increasing temperature or decreasing density, the population of heavier clusters tends to diminish leading to a mixture of nucleons and light clusters like d, t, 3He and α [4, 5]. The properties of the clustered matter undergo a major change, e.g., the incompressibility of clustered nuclear matter is quite smaller compared to that for homogeneous nucleon matter [6]. This directly influences the collapse and bounce phase of the supernova matter. The symmetry energies of nuclear matter are also affected significantly when matter gets clusterized [7, 8]. This has an important role in a better understanding of neutrino-driven energy transfer in supernova matter [9]. The symmetry energy also influences the cluster composition in the crust of neutron stars and is thus instrumental in shaping the details of their mass, cooling and structure [10].

The equation of state (EOS) of warm dilute nuclear matter with only light clusters up to α has recently been investigated in the virial approach [7, 11]; inclusion of heavier clusters has also been made in the S-matrix (SM) framework [12]. These methods relate the calculations directly to the experimental observables like the binding energies and the phase-shifts and thus, as such, are model-independent. They are usually taken as benchmark calculations in the domain of low-density and high-temperature; they are understood to exhaust all the dynamical information concerning the strong interactions in the medium. For an interacting quantum gas, the virial expansion, however, virtually ends at the second order. Formulation of higher order virial coefficients are very involved even at the formal level [13], making it difficult to estimate the domain of validity of the virial series truncated at the second order. It may further be noted that the density should be dilute enough so that the concept of asymptotic wave functions as inherent in the virial expansion should be meaningful.

An alternate avenue could be to bypass the virial expansion altogether and take recourse to nucleation in the framework of the mean-field model with a suitably chosen effective two-nucleon interaction that inherently takes an inclusive account of the scattering effects. Unlike the virial (S-matrix) approach which has direct contact with the experimental data, this method has indirect contact but it can be applied to relatively higher densities. With increasing density, a large number of different fragment species would, however, be formed that makes the numerical calculation very lengthy. Before attempting any full-blown calculation, it may then be worthwhile as a first step, to take only α-clustering in the nuclear matter and to examine whether the model works in the low-density region where the benchmark calculations exist. The present work aims towards that end.

For the study of the so-mentioned α-nucleon (αN) matter, we have chosen the Thomas-Fermi prescription for the mean-field model and the finite range, momentum and density dependent modified Seyler-Blanchard (SBM) effective interaction [14]. The properties we explore include the EOS of the αN matter, its symmetry energy, incompressibility, α concentration etc. In Sec. II, the theoretical framework for the mean-field and the S-matrix approach is presented. Sec. III contains the results and discussions. Concluding remarks are given in Sec. IV.

II. THEORETICAL FRAMEWORK

Given an effective two-nucleon interaction, the properties of the αN matter can be evaluated by exploiting the occupation functions of the n, p and alphas obtained from minimization of the thermodynamic potential of the nuclear system. The properties of warm dilute α matter are also affected significantly when matter gets clusterized [7, 8]. This has an important role in a better understanding of neutrino-driven energy transfer in supernova matter [9]. The symmetry energies of nuclear matter are also affected significantly when matter gets clusterized [7, 8]. This has an important role in a better understanding of neutrino-driven energy transfer in supernova matter [9].
system. In Sec. II A, some details of the effective interaction used are given. In Sec II B, theoretical formulation for obtaining the occupation functions from the Thomas-Fermi (TF) approximation is presented. In Sec. II C, expressions for various observables explored are given. In Sec. II D, a brief outline of the S-matrix approach is made.

A. The effective interaction

The form of the SBM effective interaction $v$ is

$$v(r,p,\rho) = C_{1u} [v_1(r,p) + v_2(r,p)],$$

$$v_1 = -(1 - \frac{p^2}{2m}) f(r_1, r_2),$$

$$v_2 = d^2 [\rho(r_1) + \rho(r_2)]^{\kappa} f(r_1, r_2),$$

with

$$f(r_1, r_2) = \frac{e^{-|r_1 - r_2|/a}}{|r_1 - r_2|/a}.\tag{2}$$

The subscripts $l$ and $u$ to the interaction strength $C$ refer to like-pair (nn or pp) and unlike-pair (np) interactions, respectively. The range of the effective interaction is given by $a$, $b$ is the measure of the strength of the momentum dependence of the interaction. The relative separation of the interacting nucleons is $r = r_1 - r_2$ and the relative momentum is $p = p_1 - p_2$; $d$ and $\kappa$ are the two parameters governing the strength of the density dependence and $\rho(r_1)$ and $\rho(r_2)$ are the nucleon densities at the sites of the two interacting nucleons. The potential parameters are given in Table I; the details for the determination of these parameters are given in [14]. The incompressibility $K_{\infty}$ of symmetric nuclear matter is mostly governed by the parameter $\kappa$; for the potential set we have chosen, the value of $K_{\infty} = 238$ MeV.

The equation of state of symmetric nuclear matter calculated with the SBM interaction is seen to agree extremely well [13] with that obtained in a variational approach by Friedman and Pandharipande [4] with $v_{14} +$TNI interaction. This interaction also reproduces quite well the binding energies, rms charge radii, charge distributions and the giant monopole resonance energies for a host of even-even nuclei ranging from $^{16}\text{O}$ to very heavy systems [14]. Interactions of this type has been used with great success by Myers and Swiatecki [10] in the context of nuclear mass formula.

B. The occupation functions

The self-consistent occupation probabilities of nucleons and alphas in $\alpha N$ matter at temperature $T$ are obtained in the TF approximation by minimizing the thermodynamic potential of the system

$$\Omega = E - TS - \sum_{\tau} \mu_{\tau} N_{\tau} - \mu_{\alpha} N_{\alpha}.\tag{3}$$

Here $\tau$ represents the isospin index (n,p). The quantities $E, S, \mu_{\tau}, \mu_{\alpha}, N_{\tau}$ and $N_{\alpha}$ are the total internal energy, entropy, nucleon chemical potentials, $\alpha$ chemical potential, free nucleon numbers and the number of $\alpha$ particles, respectively, in the system. Chemical equilibrium in the system ensures

$$\mu_{\alpha} = 2(\mu_{\alpha} + \mu_{p}).\tag{4}$$

The total energy of the $\alpha N$ matter in TF approximation is

$$E = \sum_{\tau} \left\{ \int d^3p_1 \frac{p_1^2}{2m_{\tau}} \tilde{\eta}_{\tau} (p_1) + \frac{1}{2} \int d^3p_1 dq dq \tilde{\tau}_{\tau} (p_1, p_2) \right\}$$

$$\times [v_1(|r_1 - r_2|, |p_1 - p_2|) + v_2(|r_1 - r_2|, 2|p|)|C_{1u} \tilde{\tau}_{\tau} (p_2) + C_u \tilde{\tau}_{\tau - \tau} (p_2) | \tilde{\tau}_{\tau} (p_1) + \frac{1}{2} (C_1 + C_u) \int d^3p_1 dq dq \tilde{\tau}_{\tau} (p_1, p_2)$$

$$\times \tilde{\tau}_{\tau} (p_1) \tilde{\tau}_{\tau} (p_2) \int_{V_\alpha} d r \int d^3p_1 \tilde{\tau}_{\tau} (p_1)\tilde{\tau}_{\tau} (p_1) v_1 (R', |p_1 - (p_1^\alpha + p_2)|)$$

$$+ v_2 (R', \sum_{\tau'} \tilde{\tau}_{\tau'} (p_1^\alpha + p_2^\alpha)) \right\}$$

$$+ \int d^3p_1 \frac{p_1^2}{2m_{\alpha}} \tilde{\eta}_{\alpha} (p_1)$$

$$+ (C_1 + C_u) \int d^3p_1 dq dq \tilde{\tau}_{\alpha} (p_1, p_2) \tilde{\tau}_{\alpha} (p_1) \tilde{\tau}_{\alpha} (p_2)$$

$$\times \int_{V_\alpha} d r dq \int d^3p_1 \tilde{\tau}_{\alpha} (p_1)\tilde{\tau}_{\alpha} (p_1) \tilde{\tau}_{\alpha} (p_1) v_2 (R', |p_1^\alpha - (p_1^\alpha + p_2^\alpha)| + v_2 (R', 2|p_1^\alpha|)$$

$$- N_{\alpha} B_{\alpha} \tag{5}$$

In Eq. (5), $m_{\tau}$ and $m_{\alpha}$ are the nucleon and $\alpha$ masses, the first and fourth terms correspond to the kinetic energy of the nucleons and alphas, the second and the fifth terms refer to the interaction energy among nucleons and among alphas, respectively and the third term is the interaction energy between nucleons and alphas. The various space coordinates occurring in the third and fifth terms are shown in Figs. 1 and 2, respectively.

These terms are evaluated in the single-folding and double-folding models. The last term is the binding energy contribution from the $\alpha$ particles. Here $\tilde{\eta}_{\tau} = \frac{1}{n_{\tau}} n_{\tau}, \tilde{\eta}_{\alpha} = \frac{1}{n_{\alpha}} n_{\alpha}$ where $n_{\tau}$ and $n_{\alpha}$ are the occupation probabilities for nucleons and alphas, respectively. Similarly, $\tilde{\tau}_{\tau} (p_1) = \frac{1}{n_{\tau}} n_{\tau} (p_1)$ represents the occupation probability of the constituent nucleons in the $\tau$ particle.
\[ \int \hat{n}_\alpha(p) \, dp = N_\alpha / V = \rho_\alpha, \]
\[ \int \hat{n}_\gamma(p) \, dp = 4 / V_\alpha = \rho_\gamma^0, \] (6)

where \( V \) is the volume of the \( \alpha N \) system and \( V_\alpha = \frac{4}{3} \pi R_\alpha^3 \), with \( R_\alpha \) as the sharp-surface radius of the \( \alpha \) drop taken to be 2.16 fm obtained from experimental rms charge-radius of \( \alpha \); \( \rho_\alpha^0 \) is the density of the constituent nucleons of the \( \alpha \) particles. The total baryon density \( \rho_b \) is given by \( \rho_b = \rho + 4 \rho_\alpha \) where \( \rho = \sum_{\tau} \rho_\tau \) is the density of the free nucleons and \( \rho_\alpha \) is the \( \alpha \)-particle density.

The total entropy of the \( \alpha N \) system is
\[ S = \sum_{\tau} S_{\tau} + S_\alpha, \] (7)

where in the Landau quasi-particle approximation,
\[ S_{\tau} = \frac{2}{\hbar^3} \int \left[ n_{\tau}(p) \ln n_{\tau}(p) \right. \]
\[ + (1 - n_{\tau}(p)) \ln(1 - n_{\tau}(p))] \, dr \, dp, \] (8)

and
\[ S_\alpha = \frac{1}{\hbar^3} \int \left[ n_\alpha(p) \ln n_\alpha(p) \right. \]
\[ - (1 + n_\alpha(p)) \ln(1 + n_\alpha(p))] \, dr \, dp. \] (9)

Minimization of \( \Omega \) with respect to \( n_{\tau} \) and \( n_\alpha \), remembering that \( \delta n_{\tau}(p) \) and \( \delta n_\alpha(p) \) are separately arbitrary over the whole phase space, at the end yields
\[ \frac{p_1^2}{2 m_{\tau}} + \int d^2 r_2 d^2 p_2 \left( v_1(|r_1 - r_2|, |p_1 - p_2|) \right. \]
\[ + v_2(|r_1 - r_2|, 2 \rho) \left[ C_l \tilde{n}_{\tau}(p_2) + C_u \tilde{n}_{-\tau}(p_2) \right] \]
\[ + kd^2(2 \rho)^{\kappa - 1} \sum_{\tau'} \int d^2 p'_1 d^2 d^2 p_2 \]
\[ \times \left[ C_l \tilde{n}_{\tau'}(p_2) + C_u \tilde{n}_{-\tau'}(p_2) \right] f(r_1, r_2) \]
\[ + \frac{1}{2} (C_l + C_u) \int d^2 r_2 d^2 p_2 \tilde{n}_\alpha(p_2) \]
\[ \times \int d^2 r d^2 p_1 d^2 p_2 \tilde{n}_\alpha^0(p_1) \left[ v_1(R', |p_1 - (p_1^0 + p_2)|) \right. \]
\[ + v_2(R', \rho + \rho_\alpha^0) \right) + \frac{1}{4} (C_l + C_u) kd^2(\rho + \rho_\alpha^0)^{\kappa - 1} \]
\[ \times \sum_{\tau'} \int d^2 p'_1 d^2 p'_2 \tilde{n}_{\tau'}(p_1^0) \tilde{n}_\alpha(p_2) \rho_\alpha^0 \]
\[ \times \int d^2 r_2 \int_{V_\alpha} d^2 r \, e^{-|R' - \alpha|} \left[ |R' - \alpha| \right. \]
\[ + T \left[ \ln n_{\tau}(p_1) - \ln(1 - n_{\tau}(p_1)) \right] \left. \right] - \mu_{\tau} = 0, \] (10)

and
\[ \frac{p_1^2}{2 m_{\tau}} + 2 (C_l + C_u) \int d^2 r d^2 p_2 \tilde{n}_\alpha(p_2) \]
\[ \times \int d^2 r_2 \int_{V_\alpha} d^2 r \, e^{-|R' - \alpha|} \left[ |R' - \alpha| \right. \]
\[ + T \left[ \ln n_{\tau}(p_1) - \ln(1 - n_{\tau}(p_1)) \right] \left. \right] - \mu_{\tau} = 0, \] (10)
\[
\times \int d\mathbf{r} d\mathbf{p}_i^\alpha d\mathbf{r'} d\mathbf{p}_i'^\alpha \tilde{n}_i^\alpha (\mathbf{p}_i^\alpha) \tilde{n}_i'^\alpha (\mathbf{p}_i'^\alpha)
\times \left\{ v_1 (\mathbf{R'}, |\mathbf{p}_1 + \mathbf{p}_1'^\alpha| - |\mathbf{p}_2 + \mathbf{p}_2'^\alpha|) \right\} + v_2 (\mathbf{R'}, 2\rho_1^\alpha)
\right\}
\]
\[
+ \frac{1}{2} (C_1 + C_\alpha) \sum \int d\mathbf{p}_2 d\mathbf{p}_2'^\alpha \tilde{n}_{\tau} (\mathbf{p}_2) \tilde{n}_\alpha (\mathbf{p}_2'^\alpha)
\times \int d\mathbf{r}_2 \int d\mathbf{r} \left\{ v_1 (\mathbf{R'}, |\mathbf{p}_2 - (\mathbf{p}_1 + \mathbf{p}_1'^\alpha)|) + v_2 (\mathbf{R'}, \rho + \rho_1^\alpha) \right\} + T [\ln n_\tau (\mathbf{p}_1) - \ln (1 + n_\tau (\mathbf{p}_1))]
\end{align}
\]
\[
- (\mu_\alpha + B_\alpha) = 0. \tag{11}
\]

Without any loss of generality, \( \mathbf{r}_1 \) can be set equal to zero in Eqs. (10) and (11). The single-particle occupation functions \( n_\tau (p) \) and \( n_\alpha (p) \) for nucleons and alphas are determined from Eqs. (10) and (11), respectively. Eq. (10), after some algebraic manipulations can be written as
\[
\frac{p_\tau^2}{2m_\tau} + V_\tau^0 + p_\tau^2 V_\tau^1 + V_\tau^2
+ T [\ln n_\tau (\mathbf{p}_1) - \ln (1 + n_\tau (\mathbf{p}_1))] - \mu_\tau = 0. \tag{12}
\]

The momentum-dependent nucleon single-particle potential \( V_\tau (p) \) is given by
\[
V_\tau (p) = V_\tau^0 + p_\tau^2 V_\tau^1, \tag{13}
\]
where \( V_\tau^0 \) is the momentum-independent part. Eq. (12) leads to
\[
n_\tau (p) = \left[ 1 + \exp \left\{ \left( \frac{p^2}{2m_\tau} + V_\tau^0 + V_\tau^2 - \mu_\tau \right) / T \right\} \right]^{-1} \tag{14}
\]
where \( m_\tau^* \) is the nucleon effective mass,
\[
m_\tau^* = \left[ \frac{1}{m_\tau} + 2V_\tau^1 \right]^{-1}, \tag{15}
\]
and \( V_\tau^2 \) is the rearrangement potential coming from the density dependence of the interaction. Similarly Eq. (11) can be written as
\[
\frac{p_\alpha^2}{2m_\alpha} + V_\alpha^0 + p_\alpha^2 V_\alpha^1 + T [\ln n_\alpha (\mathbf{p}_1) - \ln (1 + n_\alpha (\mathbf{p}_1))]
- (\mu_\alpha + B_\alpha) = 0, \tag{16}
\]
which yields
\[
n_\alpha (p) = \left[ \exp \left( \left\{ \frac{p^2}{2m_\alpha} + V_\alpha^0 - (\mu_\alpha + B_\alpha) \right\} / T \right) - 1 \right]^{-1} \tag{17}
\]
where
\[
m_\alpha^* = \left[ \frac{1}{m_\alpha} + 2V_\alpha^1 \right]^{-1}, \tag{18}
\]
is the \( \alpha \) effective mass. \( V_\alpha^0 \) is the momentum-independent part of the \( \alpha \) single-particle potential \( V_{\alpha, \mathbf{R'}} (\mathbf{p}_0^\alpha, \mathbf{p}_\alpha'^\alpha) \) in the system. The nucleon and \( \alpha \) masses are renormalized due to the momentum dependence in the interaction.

The expressions for \( V_\tau^0 \) can be arrived at as,
\[
V_\tau^0 = -4\pi a^3 \left\{ 1 - d^2 (2\rho)^{\kappa} \right\} (C_\tau + C_\alpha \rho_{\tau -})
+ 16\pi^2 a^3 \left\{ C_i (2m_i^* T)^{5/2} J_{3/2} (\eta_i) + C_u (2m_u^* T)^{5/2} \right\}
\times J_{3/2} (\eta_{\tau -}) + \frac{1}{4} \left[ I (C_1 + C_\alpha) \rho_\alpha \rho_\tau \right]
\left[ < p_\alpha^2 > + < (p_\tau^2)^2 > \right.
\left. + d^2 (\rho + \rho_\alpha^0) - 1 \right]. \tag{19}
\]

The first two terms come from the interaction between free nucleons, the last term originates from the presence of alphas. In Eq. (19), \( I \) is the six-dimensional integral (see Fig. 1)
\[
I = \int_{V_\alpha} d\mathbf{r} \int d\mathbf{R} e^{-|\mathbf{r} + \mathbf{R}^\alpha|/a}, \tag{20}
\]
This integral can be evaluated analytically. The quantity \( < p_\alpha^2 > \) is the mean squared value of the \( \alpha \) momentum in \( \alpha N \) matter and \( < (p_\tau^2)^2 > \) is the mean squared value of the constituent nucleon momentum inside the \( \alpha \). The value of \( < p_\alpha^2 > \) is
\[
< p_\alpha^2 > = \left( \frac{3m_\alpha^* T}{B_{3/2} (\eta_\alpha) / B_{1/2} (\eta_\alpha)} \right) \approx \frac{3}{5} m_\alpha^* T, \tag{21}
\]
and
\[
< (p_\tau^2)^2 > \approx \frac{3}{5} (P_\tau^\alpha)^2 \tag{22}
\]
where \( P_\tau^\alpha \) is the value of the zero-temperature nucleon Fermi momentum inside \( \alpha \), taken to be \( 220.5 \text{ MeV/c} \), consistent with the \( \alpha \) sharp surface radius. The \( J_k (\eta) \) and \( B_k (\eta) \) are the Fermi and Bose integrals,
\[
J_k (\eta) = \int_{0}^{\infty} \frac{x^k dx}{e^{(\eta - x)} + 1}, \tag{23}
\]
and
\[
B_k (\eta) = \int_{0}^{\infty} \frac{x^k dx}{e^{(\eta - x)} - 1}. \tag{24}
\]
with
\[
\eta_\tau = (\mu_\tau - V_\tau^0 - V_\tau^2) / T,
\eta_\alpha = (\mu_\alpha + B_\alpha - V_\alpha^0) / T. \tag{25}
\]
The expressions for \( V_\tau^1, V_\alpha^1, V_\tau^0 \) and \( V_\alpha^0 \) are given as
\[
V_\tau^1 = \frac{4\pi a^3}{b^2} \left[ C_i (\rho_\tau + C_\alpha \rho_{\tau -}) \right] \frac{1}{b^2}, \tag{26}
\]
\[
V_\alpha^1 = \frac{4\pi a^3}{b^2} \left[ C_i (\rho_\alpha + C_\alpha \rho_{\tau -}) \right] \frac{1}{b^2}, \tag{27}
\]
\[
V_\tau^0 = \frac{4\pi a^3}{b^2} \left[ C_i (\rho_\tau + C_\alpha \rho_{\tau -}) \right] \frac{1}{b^2}, \tag{28}
\]
\[
V_\alpha^0 = \frac{4\pi a^3}{b^2} \left[ C_i (\rho_\alpha + C_\alpha \rho_{\tau -}) \right] \frac{1}{b^2}. \tag{29}
\]
\[ V_\alpha^0 = \frac{1}{4} (C_l + C_u) \rho_\alpha \{ 2 \rho_i^\alpha \rho_\alpha I_\alpha \left[ d^2(2 \rho_i^\alpha)^\kappa - 1 \right] + 3 m_s^* T + \frac{6}{5} (P_F^0)^2 \left[ \right] + I_{\rho} \left[ d^2(\rho + \rho_i^\alpha)^\kappa - 1 \right] + 3 \left( P_F^0 \right)^2 \} \right\}, \] 

and

\[ V_\tau^1 = \frac{1}{4} (C_l + C_u) \rho_\alpha \{ 2 \rho_i^\alpha \rho_\alpha I_\alpha + \rho \}\} / b^2. \] 

In both \( V_\tau^1 \) and \( V_\tau^2 \), the last term stems from the \( \alpha-N \) interaction. The effective nucleon mass in pure nucleonic matter thus gets modified due to clusterization. The integral \( I_\alpha \) occurring in Eqs. (28) and (29) is a nine-dimensional integral (see Fig. 2),

\[ I_\alpha = \int_{V_\alpha} dr \int_{V_\alpha} dr' \int dR e^{-|R+r-r'|/a} |R+r-r'|/a, \]

which can be evaluated numerically. If the alphas do not interpenetrate, the integral over \( R \) excludes the \( \alpha \) volumes.

C. Expressions for observables in TF approximation

i) Energy per baryon: The energy per baryon \( e_b \) of the \( \alpha N \) matter can be calculated from Eq. (5). It can be split into the following form,

\[ e_b = e_{NN} + e_{\alpha N} + e_{\alpha \alpha}. \]

Here \( e_{NN} \) comes from the kinetic energy of the free nucleons and the interactions among them, \( e_{\alpha N} \) arises from the interaction among the free nucleons and the alphas and \( e_{\alpha \alpha} \) stems from the kinetic energy of the alphas and the interaction among themselves. The expressions for them are

\[ e_{NN} = \frac{1}{\rho_b} \sum_\tau \rho_\tau \left[ T J_{3/2}(\eta_\tau)/J_{1/2}(\eta_\tau) \{ 1 - m_s^* V_\tau^1 \} \right] + \frac{1}{2} V_\tau^0, \]

\[ e_{\alpha N} = \frac{1}{4 \rho_b} (C_l + C_u) I_{\rho_b \rho_\alpha} \left\{ \frac{3 m_s^* T + 3/5 \left[ P_F^0 \right]^2}{b^2} \right\} \rho - 1 + d^2(\rho + \rho_i^\alpha)^\kappa \} \rho + \frac{1}{2} \left[ \int dR (2 m_s^* T)^{5/2} J_{3/2}(\eta_i) \right], \]

and

\[ e_{\alpha \alpha} = \frac{1}{\rho_b} \left[ \frac{\pi}{2 m_\alpha h^3} \left( m_s^* T \right)^{5/2} B_{3/2}(\eta_\alpha) \right. \]

\[ + \frac{1}{4} (C_l + C_u) I_{\rho_\alpha \rho_\alpha} \left\{ \left[ d^2(2 \rho_i^\alpha)^\kappa - 1 \right] + 6 m_s^* T + \frac{6}{5} (P_F^0)^2 \right\} \right]. \]

In the above equations, as stated earlier, \( \rho_b = \rho + 4 \rho_\alpha \) corresponds to the total baryon density, \( \rho \) and \( \rho_\alpha \) are the free nucleon and \( \alpha \) densities, respectively, in the \( \alpha N \) system.

ii) Entropy per baryon: The entropy per baryon \( s_b \) can be evaluated using Eqs. (8) and (9). It is additive and can be written as

\[ s_b = s_N + s_\alpha, \]

where \( s_N \) and \( s_\alpha \) are the contributions to entropy from free nucleons and alphas respectively. Their expressions reduce to

\[ s_N = \frac{1}{\rho_b} \sum_\tau \rho_\tau \left[ \frac{5}{3} J_{3/2}(\eta_\tau)/J_{1/2}(\eta_\tau) - \eta_\tau \right], \]

and

\[ s_\alpha = \frac{\rho_\alpha}{\rho_b} \left[ \frac{5}{3} B_{3/2}(\eta_\alpha)/B_{1/2}(\eta_\alpha) - \eta_\alpha \right]. \]

iii) Pressure of \( \alpha N \) matter: Once the energy and entropy of the composite system are known, the pressure can be calculated from the Gibbs-Duhem thermodynamic identity,

\[ P = \sum_\tau \rho_\tau \mu_\tau + \rho_\alpha \mu_\alpha - f_b \rho_b, \]

where \( f_b \) is the free energy per baryon, \( f_b = e_b - T s_b \).

iv) Incompressibility and the symmetry coefficients: The incompressibility \( K \) can be computed from the derivative of pressure

\[ K = \frac{dP}{d\rho}. \]

The symmetry free energy and symmetry energy coefficients \( C_F \) and \( C_E \) are calculated from

\[ C_F = \frac{1}{2} \left( \frac{\partial^2 f_b}{\partial X^2} \right)_{X=0}, \]

\[ C_E = \frac{1}{2} \left( \frac{\partial^2 e_b}{\partial X^2} \right)_{X=0}, \]

where \( X \) is the neutron-proton asymmetry of the \( \alpha N \) system. It is given as \( X = (\rho_p^\alpha - \rho_n^\alpha)/\rho_b \), where \( \rho_n^\alpha \) and \( \rho_p^\alpha \) are the total neutron and proton density, respectively.
D. The S-matrix approach

The relevant key elements of the S-matrix framework [17,12] as applied in the context of dilute nuclear matter [8] are outlined in brief below.

The grand partition function of an interacting infinite system of neutrons and protons can be written as

\[ Z = \sum_{Z,N=0}^{\infty} (\zeta_p)^Z (\zeta_n)^N \text{Tr}_{Z,N} e^{-\beta H}. \]  

(42)

where \( \zeta_p = e^{\beta p} \) and \( \zeta_n = e^{\beta n} \) are the elementary fugacities with \( \beta = 1/T \) and \( \mu \)'s are the nucleonic chemical potentials. Here \( H \) is the total Hamiltonian of the system and the trace \( \text{Tr}_{Z,N} \) is taken over states of \( Z \) protons and \( N \) neutrons. The partition function can be split into two types of terms [17]

\[ \ln Z = \ln Z_{\text{part}}^{(0)} + \ln Z_{\text{scat}}. \]  

(43)

The first term on the right hand side corresponds to contributions from stable single-particle states of clusters of different sizes including free nucleons formed in the system; the second term refers to all possible scattering states. The superscript (0) indicates that the clusters behave as an ideal quantum gas. In general, \( \ln Z_{\text{part}}^{(0)} \) contains contributions from the ground states as well as the particle-stable excited states of all the clusters. The scattering term \( \ln Z_{\text{scat}} \) may be written as a sum of scattering contributions from a set of channels, each set having total proton number \( Z_t \) and neutron number \( N_t \). Since our interest in the present work is focused on \( ^{\alpha}N \) matter, in \( \ln Z_{\text{part}}^{(0)} \), we include only the nucleons and the ground state of \( \alpha \); similarly in \( \ln Z_{\text{scat}} \), only the scattering channels \( \alpha N, \alpha N, and \alpha N \) are considered, so that

\[ \ln Z_{\text{scat}} = \ln Z_{\text{NN}} + \ln Z_{\alpha N} + \ln Z_{\alpha N}. \]  

(44)

Each of the terms in Eq. (44) can be expanded in the respective virial coefficients. Expansion up to the second-order coefficients are only considered. They are written as energy integrals of the relevant phase-shifts [6,6]. The partition function can then be explicitly as

\[ \ln Z = V \left\{ \frac{2}{\lambda_N^2} \left[ \zeta_n + \zeta_p + \frac{b_{nn}}{2} \zeta_n^2 + \frac{b_{np}}{2} \zeta_p^2 + \frac{1}{2} b_{np} \zeta_n \zeta_p + 8 \zeta_\alpha + 8 b_{\text{int}} \zeta_n^2 + 8 b_{\text{int}} \zeta_n \zeta_p \right] \right\}, \]  

(45)

where \( \lambda_N = \frac{h}{2 \pi m_N \rho_b} \) is the nucleon thermal wavelength, \( \zeta_\alpha = e^{\beta (\mu_\alpha + B_\alpha)} \), \( B_\alpha \) being the binding energy of \( \alpha \) and \( \mu_\alpha = 2(\mu_n + \mu_p) \). The \( b_{nn}, b_{np}, \text{etc.} \), are the temperature dependent virial coefficients [6,12]. The value of the virial coefficient \( b_{nn} \) has been adjusted so as to exclude the resonance formation of deuteron from n-p scattering to be consistent with our choice of the \( \alpha N \) matter.

The knowledge of the partition function allows all the relevant observables to be calculated. The pressure is given by

\[ P = T \ln Z/V. \]  

(46)

The number density \( \rho_i \) is calculated from

\[ \rho_i = \zeta_i \left( \frac{\partial \ln Z}{\partial \zeta_i} \right)_{V,T}. \]  

(47)

where \( i \) stands for n,p, or \( \alpha \). Once the pressure, densities and chemical potentials are known, the free energy can be obtained from the Gibbs-Duhem relation. The entropy per baryon is calculated from

\[ s_b = \frac{1}{\rho_b} \left( \frac{\partial P}{\partial T} \right)_i, \]  

(48)

which yields the energy per baryon as \( e_b = f_b + Ts_b \). The explicit expression for the entropy per baryon is

\[ s_b = \frac{1}{\rho_b} \left[ \frac{5 P}{2 T} - \rho_i \ln \zeta_i \right. \right. \]  

\[ \left. \left. + \frac{T}{\lambda_N^2} \left[ \zeta_n \zeta_p b'_{np} + \left( \zeta_n^2 + \zeta_p^2 \right) b'_{nn} \right. \right. \]  

\[ \left. \left. + 8 \zeta_\alpha b'_{\alpha \alpha} + 8 \zeta_\alpha (\zeta_n + \zeta_p) b'_{\alpha n} \right] \right\}. \]  

(49)

The prime on the virial coefficients denotes their temperature derivatives.

III. RESULTS AND DISCUSSIONS

In the mean-field framework, the momentum and density-dependent finite-range modified Suyler-Blanchard force as scripted in Eqs. (1) and (2) has been chosen as the effective two-nucleon interaction in our calculations. To start with, we take baryon matter at a given density \( \rho_b \) at a temperature \( T \) with an isospin asymmetry \( X \). The unknowns are the free nucleon densities \( \rho_n, \rho_p \) and the \( \alpha \) concentration in the matter. The three constraints are the conservation of the total baryon number, the total isospin and the condition of chemical equilibrium between the nucleons and alphas. Starting from a guess value for the \( \alpha \) concentration, the unknowns are determined iteratively using the Newton-Raphson method. For our calculations, the masses of neutron and proton are taken to be the same, for \( \alpha \) binding energy, the experimental value of 28.3 MeV is used. For the evaluation of the \( \alpha \)-potential, the \( \alpha \)-particles are assumed to be nuclear droplets with sharp boundary and that they do not interpenetrate.

The calculations are done up to a baryon density \( \rho_b = 0.01 \text{ fm}^{-3} \). To show the effect of temperature on different properties of the dilute matter, results are reported for temperatures \( T = 3, 5, \) and 10 MeV. In Fig. 3, the baryon fraction in \( ^{\alpha}N \) matter as a function of density
FIG. 3: (color online) The $\alpha$ fraction $Y_\alpha = 4\rho_\alpha/\rho_b$ shown as a function of baryon density $\rho_b$ in TF and SM approaches at $T=3$, 5 and 10 MeV for symmetric matter ($X=0.0$) and asymmetric matter ($X=0.2$) in panels (a) and (b), respectively.

at the three temperatures mentioned are shown for symmetric ($X=0$) and asymmetric ($X=0.2$) nuclear matter in panels (a) and (b), respectively. The black lines correspond to results obtained in the TF framework [$\alpha N$ (TF)], the red lines refer to those in the SM approach [$\alpha N$ (SM)] with consideration of only $n$, $p$ and $\alpha$ as the constituents of the baryonic matter. At low temperatures and higher densities, it is seen that alphas are the major constituents of the matter, with increasing temperature, the free nucleon fraction increases at the cost of $\alpha$ density. At moderate asymmetry $X=0.2$, the $\alpha$ population is somewhat lower compared to that for symmetric nuclear matter. In the temperature and density domain that we explore, the results from both the SM and TF approach are found to be quite close. The asymmetry dependence of $\alpha$ fraction $Y_\alpha$ is displayed in Fig. 4 at two representative densities $\rho_b=0.001$ and $0.01$ fm$^{-3}$ at the three temperatures. With increasing asymmetry, the $\alpha$ concentration decreases, the decrease is more prominent at lower temperature. At the lower density (Fig. 4(a)), results for $T=10$ MeV are not shown as $Y_\alpha$ is close to zero.

In Fig. 5, the free energy per baryon for the homogeneous nucleonic matter (denoted by $N$ (TF) and $\alpha N$ (TF)). It is clearly seen that the clustered matter has lower free energy compared to homogeneous nucleonic matter.
neous nucleonic matter. This is more prominent at lower temperatures, higher temperature tends to melt away the clusters. For comparison, results from the $S$-matrix approach are also presented. They are shown by the red lines, nearly indistinguishable from those from $\alpha N$ (TF).

Fig. 6 displays the pressure of the baryonic matter. At lower temperatures ($T=3$ and $5$ MeV), the nucleonic matter shows the rise and fall of the pressure with density leading to unphysical region. For $\alpha N$ matter, however, no such unphysical region is observed in the density region we have studied. Both the TF and the SM approaches yield nearly the same value of pressure. At high temperature the $\alpha$ concentration becomes very less, the pressure in all the three approaches are then nearly the same in this density region.

In Fig. 7, the effective masses of nucleon and $\alpha$ are shown as a function of density at the temperatures mentioned. The nucleon effective mass is calculated for both nucleonic matter (blue line) and $\alpha N$ matter (full black line) in the TF approximation. The nucleon effective mass at a given $\rho_b$ in homogeneous nucleonic matter is always lower compared to that in clustered matter. It is independent of temperature. In $\alpha N$ matter it nominally decreases with temperature. At high temperature, the nucleon effective masses calculated in the homogeneous and clustered matter are nearly degenerate, with lowering of temperature, the degeneracy is lifted due to the increase in the $\alpha$ concentration. The effective $\alpha$ mass is shown by the dashed black lines. With increasing temperature, the medium effect on the $\alpha$ mass gets strikingly enhanced. This is due to the interplay of the temperature-dependent contributions from the $\alpha N$ interactions and $\alpha N$ interactions corresponding to the first and the second term within the braces in Eq. (29).

The incompressibility of the baryonic matter as a function of density is displayed in Fig. 8 at the three temperatures. At very low density and higher temperature, the matter is mostly nucleonic in all the three approaches, so the incompressibility $K$ is $\sim 9T$; this one sees at the lower densities considered at $T=10$ MeV in panel (c) of this figure. Even at this very high temperature, however, the nucleonic interactions have their role as the density increases; this results in the reduction of the incompressibility from the ideal gas value. At the lower temperatures (panels (a) and (b)), clusterization softens the matter towards compression compared to homogeneous matter (shown in the lower density region); increasing density, however, pushes the homogeneous matter towards the unphysical region leading to negative incompressibility.

The symmetry energy coefficients $C_E$ and $C_F$ of the baryonic matter as a function of density are displayed in the left and right panels, respectively, of Fig. 9 at the three temperatures studied. The blue lines refer to calculations for the homogeneous matter, the black and red lines represent results for $\alpha N$ (TF) and $\alpha N$ (SM). Clusterized matter displays a marked increase in the symmetry coefficients noticed already earlier [6, 7]. The two approaches to clusterization lead to nearly the same values of the symmetry coefficients at lower densities, with increase in density the difference widens, more so at lower temperatures.

The results presented so far in the $\alpha N$ (TF) approach...
have been calculated with the assumption that the alphas do not overlap, they are mutually impenetrable spherical drops. This assumption relies on the fact that the alphas are very tightly bound and very hard to excite. To explore the effect of overlap in alphas, we consider a possibility of penetration with at best a 5\% overlap in volume (the value of $I_\alpha$ in Eq. (30) then changes accordingly). Calculations have been repeated with this changed condition. The so-calculated free energy per baryon, pressure and the $\alpha$ fraction $Y_\alpha$ in the baryonic matter are presented in panels (a), (b) and (c), respectively, of Fig. 10 at $T=3$ MeV (the dot-dashed black lines) and compared with those calculated with the no-overlap condition (the full black lines) and also those from the $\alpha N$(SM) approach (the red lines). There is no significant change in the free energy or in $\alpha$ fraction, but the pressure changes perceptibly, particularly at higher density. The good agreement between the no-overlap $\alpha N$(TF) calculations with those from the bench-mark $\alpha N$(SM) shows the viability of the approximation of the impenetrability of the alphas.

### IV. CONCLUDING REMARKS

Clusterization in warm dilute nuclear matter has been treated earlier in the virial approach or in the $S$-matrix framework. These are model-independent parameter-free calculations. As explained in the introduction, these methods may have limitations at relatively high densities and low temperatures. An alternate avenue for dealing...
ing with clusterized matter in a broadened density and temperature domain is suggested in the mean-field framework in the present paper. The suggested method may be lengthy at relatively higher densities where many different fragment species are formed, but it is straightforward. To explore its applicability in a wider domain, as a first step, we consider only n, p and $\alpha$ as the constituents of the matter at low densities and see how the results compare with those from the model-independent virial approach.

We have chosen the SBM interaction that nicely reproduces the bulk properties of nuclear matter and of finite nuclei. We have calculated the $\alpha$ fraction, free energy, pressure, incompressibility and the symmetry coefficients of this $\alpha$N matter in this mean-field framework and find that all these results compare extremely well with those obtained from the S-matrix method, particularly in the low-density high-temperature regime. This gives one confidence in the applicability of this mean-field approach in dealing with the EOS of warm dilute baryonic matter and the possibility of extending this method to higher densities. The price, however, is consideration of a larger number of fragment species and a numerically involved calculation.

Acknowledgments

S.K.S. and J.N.D acknowledge support of DST, Government of India.

[1] J. W. Clark and T. P. Wang, Ann. Phys. (N.Y.), 40, 127 (1966).
[2] F. Carstoiu and S. Misicu, Phys. Lett. B682, 33 (2009).
[3] D. Q. Lamb, J. M. Lattimer, C. J. Pethick, and D. G. Ravenhall, Phys. Rev. Lett. 41, 1623 (1978).
[4] B. Friedman and V. R. Pandharipande, Nucl. Phys. A 361, 502 (1981).
[5] G. Peilert, J. Randrup, H. Stocker, and W. Greiner, Phys. Lett. B 260, 271 (1991).
[6] S. K. Samaddar, J. N. De, X. Viñas, and M. Centelles, Phys. Rev. C 80, 035803 (2009).
[7] C. J. Horowitz and A. Schwenk, Nucl. Phys. A 776, 55 (2006).
[8] J. N. De, S. K. Samaddar, and B. K. Agrawal, Phys. Rev. C 82, 045201 (2010).
[9] H.-Th. Janka, K. Langanke, A. Marek, G. Martínez-Pinedo, and B. Müller, Phys. Rep. 442, 38 (2007).
[10] C. Fuchs, J. Phys.G 35, 014049 (2008).
[11] E. O’connor, D. Gazit, C. J. Horowitz, A. Schwenk, and N. Barnea, Phys. Rev. C 75, 055803 (2007).
[12] S. Mallik, J. N. De, S. K. Samaddar, and Sourav Sarkar, Phys. Rev. C 77, 032201 (R) (2008).
[13] A. Pais and G. E. Uhlenbeck, Phys. Rev. 16, 250 (1959).
[14] J. N. De, N. Rudra, Subrata Pal, and S. K. Samaddar, Phys. Rev. C 53, 783 (1996).
[15] V. S. Uma Maheswari, D. N. Basu, J. N. De and S. K. Samaddar, Nucl. Phys. A615, 516 (1997).
[16] W. D. Myers and W. J. Swiatecki, Ann. Phys. (N.Y.) 204, 401 (1990).
[17] R. Dashen, S-k. Ma and H.J. Bernstein, Phys. Rev. 187, 345 (1969).