Massive particles in acoustic space-times
emergent inertia and passive gravity

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I. INTRODUCTION

It is well documented that the propagation of acoustic waves in inviscid, barotropic, irrotational background flows bears some enlightening resemblances to propagation of light in curved space times (see the seminal paper of Urruh[1] and many subsequent expansions; e.g., [2, 3], and the recent extensive review in [4]): The flow potential, $\vec{v}$, describing weak acoustic waves moving on a given background $D$-dimensional flow satisfies the wave equation

$$\Box \eta \equiv (-g)^{-1/2} \left[ (-g)^{1/2} g^{\mu \nu} \eta_{\mu \nu} \right]_{; \nu} = 0,$$

(1)

where $g^{\mu \nu}$ is the inverse of the matrix $g_{\mu \nu} = (\rho/c)^{2/(D-1)} q_{\mu \nu}$ with $q_{00} = -(c^2 - \vec{v}^2)$, $q_{i0} = q_{0i} = -\vec{v}_i$, $q_{ij} = \delta_{ij}$, with $\rho$ the background flow density, $\vec{v}$ its velocity, and $c$ the local speed of sound ($g = -[\rho(D+1)/c^2]^{2/(D-1)}$ is the determinant of $g_{\mu \nu}$). The matrix $g_{\mu \nu}$ is called the acoustic space-time metric because eq.(1) is identical to the massless-scalar wave equation in a curved space time described by the metric $g_{\mu \nu}$, $\Box$ being the covariant d’Alembertian. This setup is used to simulate the propagation of light in gravitational fields. The analogy is, however, anything but complete. For example, a coordinate transformation of an acoustic metric does not take us to another acoustic metric. And, in the flat-space-time analog (homogeneous background flow at rest) there is no parallel with observer independence of the speed of light. The situation is more akin to propagation of light in the old aether. Also, there is not yet an analog of the Einstein equations whereby the effective geometry is determined by its sources. Still, the analogy, where it does exist, is very useful and captures some crucial aspects of photon propagation in curved space times. For example, it elucidates the behavior of light near event horizons in “black hole” geometries.

Here I propose to extend this analogy to massive particles, also in the context of ideal fluids, in the usual hope that it might teach us about the real processes it represents. The interest in such models might be twofold. First, they provide models for emergent relativistic inertias: Starting with objects that have negligible inertia of their own, their interaction with the fluid puts a cost on their motion by endowing them with an effective kinetic action and thus with pseudo-energy and pseudo-momentum. Such models might shed light on the origins of real inertia, in the spirit of Mach’s principle. (I mean here Mach’s principle in the extended sense that inertia is not an innate attribute of bodies but emerges as a result of their interaction with some omnipresent agent, such as a field, the vacuum, the fluid in our case, or, as in the original view, the totality of other bodies in the universe.) They also permit us to study possible mechanisms for breakdown of the standard Lorentzian kinematics at
high Lorentz factors. We can also study possible departures from standard inertia when the global setup of the fluid is changed to mimic real inertia in the context of the non-trivial cosmology of our universe. In fact, my own interest in the subject stemmed originally from the wish to construct models of modified inertia that will mimic the behavior of MOND, a theory that I proposed to replace the need for dark matter in galactic systems (e.g. [7-9]).

Second, these models extend the usefulness of acoustic analogs of light propagation in curved space times to that of massive particles in gravitational fields. Interestingly, I find that for properly defined particles the same attribute that plays the role of inertial mass also plays the role of passive gravitational mass thus conforming to the weak equivalence principle. With these models we can study mechanisms for the breakdown of the weak equivalence principle, dynamics of massive particles near black hole analogs, such as the existence of a last stable orbit, etc. And, with an appropriate definition of the particle vacuum we may be able to study Hawking radiation and other quantum effects in curved space time for massive particles.

The particles I shall describe do not generally follow geodesics of the acoustic metric itself, for which the proper time is

$$\text{d}\tau = \alpha \gamma^{-1}\text{d}t,$$

(2)

where \(\gamma = \left(1 - \left[\frac{d\vec{x}}{dt} - \vec{v}(\vec{r})\right]^{2}\right)^{-1/2}\) is the “Lorentz factor” of the velocity relative to the fluid, and \(\alpha \equiv \frac{\rho c(D-2)}{\left[\rho c(D-2)\right]^{2/(D-1)}}\). In the real world Lorentz invariance dictates the above path length as the particle action; but this is not so in the fluid context. Nevertheless, it would still be useful to find analogs that have enough of the properties of real particles, in particular, relativistic, quasi-Lorentzian dynamics. This I begin to do in this paper. As I shall show, there is, in fact, a generalization of the acoustic metric in the form of a Finslerian one whose path length is the particle action and which shares the null geodesics with the acoustic metric; so, a unified description of massive and massless particles does emerge with a Finslerian acoustic metric.

I am not concerned here with the practicability of laboratory construction of such analogs. I view their usefulness mainly as theoretical laboratories for testing ideas concerning inertia and gravity.

A rather different approach towards mimicking massive particles in the context of Bose-Einstein condensates is described in [10, 11]. The acquisition of induced mass by vortices moving in superfluids has been discussed in [12] and references therein.

In section II I discuss the general idea and define the particles. Section III contains the derivation of the effective particle action in \(D\) dimensions. In section IV I discuss various aspects of the resulting dynamics of the particles, first in flat space times, then in the presence of analog gravity. Section V brings up some additional issues.

II. MASSIVE PARTICLES IN FLAT SPACE TIME

Very weak perturbations of the fluid flow itself, to wit acoustic waves, are the analog of light in the fluid context. They are described by the same degrees of freedom as the unperturbed, background flow and move with the local speed of sound relative to the fluid. Analogs of massive particles should be able to move at any “subluminal” speed relative to the fluid. They should thus be defined as regions of space where the equations of motion for the background flow break down. The exact definition of the particles, with the prerequisites they have to satisfy, is best presented in the context of “flat” space times; i.e., homogeneous fluids at rest.

A rigid body moving with constant speed in an inviscid, incompressible fluid is subject to no force; this is known as the d’Alembert paradox. It follows straightforwardly from scaling that the energy of the fluid in this configuration is proportional to the body’s velocity squared. Accelerating the body increases the energy of the fluid hence creating an effective force resisting acceleration. The force is proportional to the acceleration, giving rise to acquired Newtonian inertia: The speed of sound in incompressible fluids is formally infinite, so there is no velocity parameter to represent the speed of light. The effective inertial mass tensor is the fluid density times some geometrical volume matrix that depends on the shape of the body. (See e.g. [13] and [14 §11].)

My aim here is to extend this idea to bodies with relativistic kinematics. Naturally one then begins with a compressible background fluid whose speed of sound will play the role of the speed of light. A rigid body is however not a good model for a particle (see below) so we’ll have to find others.

A more general discussion of forces on static bodies in a class of nonlinear media, of which our simple flow is an example, can be found in [15] where I consider different possible definitions of bodies. For example, to define a body one can dictate boundary conditions on a closed rigid surface of a region that can move in the fluid. Dictating a vanishing normal component of the flow velocity defines a rigid body, for instance. Alternatively, we can define the particle as a rigid collection of sources (and sinks). Yet another way is to dictate inside the particle an external potential that couples to the flow density. And yet another is to take the particle as a small region of non-vanishing vorticity. There are more options; the choice, however, is limited by the following requirements that I think should apply in the quest of analog massive particles: a. The particle should constitute a controllably weak perturbation on the background flow. This is not just to facilitate the derivation of the particle’s dynamics but mainly to prevent the particle from probing the equation of state of the fluid at densities other than the background value. This requirement eliminates, for example, rigid bodies as candidate particles because when such bodies move with relativistic speeds they cre-
ate strong perturbations in their vicinity, no matter how small in size they are. b. An effective principle of inertia should hold: when the particle is set in motion in a homogeneous fluid it should retain a constant speed; i.e. it should not be subject to forces by the fluid. The d’Alembert paradox insures this for a rigid body in an incompressible ideal fluid. It was shown in [15] that this is true also for compressible fluids, and, in such a case, also when the body is defined as a region of dictated external potential or as a distribution of sources, provided the integrated source out-flux vanishes (this holds exactly, even when the body is not a weak perturbation). c. To limit the scope of the discussion I also require in this paper that the particle is a rigid object, with no internal degrees of freedom. It may be interesting to relax this assumption in various ways.

I shall indeed concentrate here on particles defined as either a distribution of sources or a dictated potential. The former is epitomized by a source-sink dipole such as a small pipe within which there is a pump sucking fluid at one end and ejecting it at the other, or by any arrangement of dipoles such as a dipole source layer, etc.. The second type may be realized archetypically as a set of electric charges held together rigidly by a structure that does not disturb the fluid mechanically (a rigid cage), and moving in a weakly charged fluid with constant charge-to-mass density ratio. It is best to take the total body charge as zero so as to attain a confined potential. Ideally it would be good to add an inert (static) background with the opposite charge to cancel that of the unperturbed fluid, so that only density perturbations carry net charge.

III. THE EFFECTIVE PARTICLE ACTION

One can get the equations of motion of an irrotational, inviscid, barotropic fluid in $D$ space dimensions from the effective action

$$S = -\int d^{D+1}x [\rho \phi, + \frac{1}{2} \rho (\nabla \phi)^2 + e + \rho \theta + \rho \psi + \phi s].$$

(3)

where $d^{D+1}x$ stands for $dtd^Dr$, $\rho$ is the fluid density, $\phi$ is the velocity potential: $\vec{v} = \nabla \phi$, and $e(\rho)$ is the intrinsic energy per unit volume, which is a function of $\rho$ for a barotropic fluid. The action $S$ is based on that derived in [16] to which I have added a source term with source density $s(\vec{r}, t)$, and potential terms. The potential fields $\theta(\vec{r}, t)$ and $\psi(\vec{r}, t)$ couple to the fluid density. They can be of the same type but I write them separately because they have different roles: $\theta$ is completely dictated externally, and partakes in establishing the unperturbed background flow, while $\psi$ represents a particle and so constitutes a weak perturbation confined to a very small, freely moving region of space. (I kept here the sign of the action, which is derived in [16] from the fluid action $\int \rho u^2/2 - e$; so it is clear with which sign to add actions for additional degrees of freedom. Kinetic energies are added with a plus sign, while potential energies, such as $\rho \psi$, appear with a minus sign.)

Varying the action over $\phi$ gives the continuity equation:

$$\rho, + \nabla \cdot (\rho \nabla \phi) = s. \quad (4)$$

Varying over $\rho$ gives the Bernoulli equation:

$$\phi, + \frac{1}{2} (\nabla \phi)^2 + h(\rho) + \theta + \psi = 0, \quad (5)$$

which in the barotropic irrotational case is equivalent to the Euler equation. Here $h(\rho) = e'(\rho)$ is the specific enthalpy.

To the fluid degrees of freedom we now add those of the model particle: its position $\vec{r}_s(t)$, and possibly its orientation. At this stage I want to eliminate the orientation as an unnecessary (but possibly interesting) complication. Later on I shall assume a spherically symmetric particle for which this is not an issue. For the time being I shall take an arbitrarily shaped particle but assume that its orientation is kept fixed in space (e.g. by providing it with a gyroscope), which generically gives anisotropic inertia. The particle’s dynamics will turn out to depend on its orientation with respect to its velocity relative to the fluid. If we can somehow keep this orientation fixed (e.g. by providing the body with efficacious fins) dynamics will be isotropic for any body shape.

One type of particle I treat is a small region of space, positioned around $\vec{r}_s(t)$, where a rigid arrangement of sources is dictated. This corresponds to a source distribution

$$s(\vec{r}, t) = \hat{s}[\vec{r} - \vec{r}_s(t)]. \quad (6)$$

$s$ vanishes everywhere except in a volume of diameter $\epsilon$ much smaller than any length scale characterizing the unperturbed flow and the trajectory of the particle, and corresponds to a vanishing total outflow:

$$\int \hat{s}(\vec{r}) d^Dr = 0. \quad (7)$$

A different type of particle may be represented by a small volume of diameter $\epsilon$ around $\vec{r}_s(t)$ where an external potential is dictated:

$$\psi(\vec{r}, t) = \hat{\psi}[\vec{r} - \vec{r}_s(t)] \quad (8)$$

with $\hat{\psi}$ vanishing rapidly beyond the radius of the source. While $\hat{s}(\vec{r})$ or $\hat{\psi}(\vec{r})$ are fixed and constitute the internal structure of the particle, its position $\vec{r}_s$ is free. My aim is to derive an effective action for $\vec{r}_s$ by solving for $\rho$ and $\phi$ for a given trajectory $\vec{r}_s(t)$, then substitute these back in the action to get a functional of $\vec{r}_s(t)$ that is the required effective action of the particle. I do this under the assumption that the particle is a weak perturbation on the background flow and is very small.

Let us keep the designation $\rho, \phi$ for the unperturbed, background flow attributes, and write the density and
velocity potential in the presence of such perturbations
as $\rho + \zeta$ and $\phi + \eta$, respectively. Expanding the action
to second order in $\zeta$ and $\eta$ we get a zeroth order term, which is
taken as a constant for a given background. The first order term is, after some integrations by parts,

$$S_I = -\int d^{D+1}x \{ [\zeta, \phi] + \frac{1}{2} (\nabla \phi)^2 + h(\rho) + \theta \}
- \eta [\rho, + \zeta \cdot (\rho \nabla \phi)] + (\rho \eta), + \zeta \cdot (\eta \rho \nabla \phi) \}
+ \int d^{D+1}x (\rho \psi + \phi s). \quad (9)$$

The first two terms vanish for solutions of the unperturbed field equations. The next two terms are the usual integrals of complete derivatives; they vanish if we can neglect the perturbation at space and time infinities. The second integral may engender first order effective forces on our particle and I want to eliminate it. For a potential particle this can be done by assuming a background flow of constant density, as I shall eventually assume anyway.

In this case this term becomes an immaterial constant contribution to the Lagrangian $\propto \int d^D r \psi$. If we want to permit a variable density we add to the background flow an inert background distribution of charges that cancels that of $\rho$, then $\psi$ couples only to the perturbation $\zeta$ and this first order term disappears. For a source particle this first order term is analogous to the energy of a lower order in the perturbation) and I neglect such contribution to the action because after the strength of the perturbation is set we can take the particle size as small as we wish.

Turn now to the second order action, from which the effective particle action is constructed. We have

$$S_{II} = -\int d^{D+1}x \{ [\zeta, \phi] + \frac{1}{2} (\nabla \phi)^2 + h(\rho) + \theta \}
- \eta [\rho, + \zeta \cdot (\rho \nabla \phi)] + (\rho \eta), + \zeta \cdot (\eta \rho \nabla \phi) \}
+ \int d^{D+1}x (\rho \psi + \phi s) + c^2 \zeta^2 + \zeta \psi + \eta s. \quad (10)$$

with $c$ the speed of sound: $c^2 \equiv \rho'(\rho) = \rho h'(\rho)$. It gives the first order Bernoulli equation by varying over $\zeta$:

$$\eta_s + (c^2/\rho) \zeta + \zeta \phi \cdot \nabla \eta + \psi = 0, \quad (11)$$

and the continuity equation,

$$\zeta_s + \zeta \cdot (\rho \nabla \eta) + \nabla \cdot (\zeta \nabla \phi) = s, \quad (12)$$

by varying over $\eta$. The first can be used to eliminate

$$\zeta = -(c^2/\rho^2)(\eta_s + \zeta \phi \cdot \nabla \eta + \psi), \quad (13)$$

and substituting in the second we get

$$- [\rho/c^2](\eta_s + \zeta \phi \cdot \nabla \eta) + \zeta \cdot (\rho \nabla \eta) \quad (14)$$

Rearranging gives

$$- [\rho/c^2](\eta_s + \zeta \phi \cdot \nabla \eta) + \zeta \cdot (\rho \nabla \eta) - \zeta \cdot [(\rho/c^2)(\eta_s + \zeta \phi \cdot \nabla \eta) \nabla \phi]
= s + [(\rho/c^2)\psi] + \zeta \cdot [(\rho/c^2)\psi \nabla \phi]. \quad (15)$$

I use the continuity equation for the background flow to write $s + \rho(\psi/c^2) + \rho \nabla \phi \cdot \nabla \psi/c^2$ for the right hand side. The left hand side is $(-g)^{-1/2} \sqrt{g} \eta$, where, as in eq.(1), $\Box$ is the covariant d’Alembertian corresponding to the acoustic metric $g_{\mu \nu}$. So

$$\Box \eta = \tilde{s} + (\psi/c^2) \nu J^\mu, \quad (16)$$

where $\tilde{s} \equiv (-g)^{-1/2} s$ is the covariant source density, and the current

$$J^\mu \equiv (-g)^{-1/2} \sqrt{g} \eta. \quad (17)$$

is covariantly conserved:

$$J^\mu_{\nu} \equiv (-g)^{-1/2} \sqrt{g} J^\mu_{\nu} = 0. \quad (18)$$

Now eliminate $\zeta$ from the action $S_{II}$ itself. After some algebra, one gets inside the integral (up to a total derivative)

$$S_{II} = -\int d^{D+1}x (-g)^{1/2}[\frac{1}{2} \eta \cdot \eta g_{\mu \nu} + \eta(\psi/c^2) \nu J^\mu
+ \eta \tilde{s} - \frac{1}{2 \rho c^2} \sqrt{g} \psi^2]. \quad (19)$$

Varying over $\eta$ gives the field equation (16).

The program is then as follows: for a given $\tilde{r}_s(t)$, which, together with the given $\tilde{s}$ or $\psi$, determines the source term, solve eq.(16) for $\eta(\tilde{r}_s, t)$, then substitute it in the expression for $S_{II}$ to get the value of the effective action as a functional of the trajectory; this is the particle action we are after, $\mathcal{S}(\tilde{r}_s, t)$. Equation (19) requires knowledge of the solution $\eta$ everywhere in space time. A
more manageable expression is gotten by employing the integral relation

\[ \int d^{D+1}x (-g)^{1/2} \eta \frac{\partial}{\partial t} g^{\mu\nu} + \eta (\psi/c^2)_{,\nu} J^\mu + \eta \hat{s} = 0, \]  

(20)

which holds for solutions of the field equation up to surface terms at space-time infinity. (This is a simple special case of the results of [17] and follows straightforwardly by integrating the first term by parts to give \(-1/2)\varphi \Box \eta\), then using the field equation (16).) So we can set

\[ S[\vec{r}_s(t)] = - \int d^{D+1}x (-g)^{1/2} \left[ \frac{1}{2} \eta (\psi/c^2)_{,\nu} J^\mu + \frac{1}{2} \eta \hat{s} - \frac{1}{2 \rho c} \psi^2 \right]. \]  

(21)

This expression requires knowledge of \(\eta\) only inside the particle, where either \(\psi\) or \(\hat{s}\) don’t vanish; this is very helpful.

It is impracticable to solve for \(\eta\) for an arbitrary trajectory in an arbitrary background flow. It is clear that the resulting effective action would be time non local. However, the assumed smallness of the particle permits us to approximate \(\eta\) inside the particle in a way that depends only on the instantaneous state of motion, and this will result in a local approximation of the action. For a very small particle we can assume that as it moves about, a steady state corresponding to the instantaneous conditions is reestablished within the particle on the short time scale it takes sound waves to get from one end of it to the other. We essentially separate the dependence on macroscopic coordinates and the microscopic ones within the body, where \(\eta\) changes quickly, by assuming that from eqs.(6) and (8) we can write to a very good approximation (becoming exact in the limit of infinitesimal particle size)

\[ \eta(\vec{r}, t) = \hat{\eta} (\vec{r} - \vec{r}_s(t)) \]  

(22)

to describe the fast variations of \(\eta\) around the particle’s position in space time, and where \(\hat{\eta}\) still depends on macroscopic properties such as the flow and particle velocities and the fluid density at \(\vec{r}_s(t)\). I shall discuss below the conditions for this approximation to hold.

We now calculate \(\Box \eta\) with this ansatz. I again make use of the fact that, due to the smallness of the particle, the space and time variations of \(\eta\) are dominated by those produced by the fast variations in \(\hat{\eta}(\vec{r} - \vec{r}_s(t))\). So, for example, in \(\eta, = -\vec{v} \cdot \vec{\nabla} \hat{\eta} + q\), where \(q\) represents terms coming from the implicit dependence of \(\hat{\eta}\) on macroscopic quantities and their time variation, we neglect all such terms. Then in \(\eta, \approx (\vec{v} \cdot \vec{\nabla}) \hat{\eta} - (d\vec{v}_s/\partial t) \cdot \vec{v} \hat{\eta}\) (again neglecting \(q\) terms) I further neglect the second term (by our approximation \(d\vec{v}_s/\partial t \ll \vec{v}^2/\alpha\) generically, \(a\) being the diameter of the particle). With this approximation, which leaves us only with terms with second derivatives of \(\hat{\eta}\), we get

\[ \Box \eta = \frac{c}{\rho} \{ \Delta - \left( \frac{\vec{U}}{c} \cdot \vec{\nabla} \right)^2 \} \hat{\eta}, \]  

(23)

where, \(\vec{U} \equiv \vec{v} - \vec{v}\) is the relative velocity of the particle with respect to the fluid, and \(\vec{v}, \rho, c\) are evaluated at \(\vec{r}_s(t)\). Thus

\[ \hat{\eta}_{,x} + \hat{\eta}_{,y} + \gamma^2 \hat{\eta}_{,z} = \rho^{-1} \hat{s} - (U/c^2) \hat{\psi}_{,z}, \]  

(24)

where \(\gamma\) is the relative “Lorentz factor” \(\gamma = (1 - U^2/c^2)^{-1/2}\), the \(z\) axis is in the direction of \(\vec{U}\), and where I used the fact that now \(J^\mu(\psi/c^2)_{,\mu} = (-g)^{-1/2}(\rho/c^2)(\psi, + \vec{v} \vec{\nabla} \psi) = -(\rho c)^{-1} \vec{U} \cdot \vec{\nabla} \psi\). In the coordinates \(x' = x, y' = y, z' = \gamma z\) eq.(24) becomes the Poisson equation

\[ \Delta' \hat{\eta} = \rho^{-1} \hat{s}[\vec{R}(\vec{R}') - \vec{U} c^{-2} \hat{\psi}_{,z} [\vec{R}(\vec{R}')]], \]  

(25)

provided the relative speed \(\vec{U}\) is subsonic. This appearance of the stretched Laplacian in the linearized equation for weak perturbations moving with subsonic speed in a compressible fluid is familiar, for example, from the treatment of a constant flow past a thin wing very nearly parallel to the flow (e.g. [14] §124). The perturbation there enters not through source terms as here, but through the boundary conditions on the rigid wing, leaving us with a distorted Laplace equation instead of Poisson’s as here. This equation is elliptical for subsonic speeds for which our treatment below applies, but become hyperbolic for supersonic speeds.

The effective particle action can then be written as

\[ S[\vec{r}_s(t)] = \int L \, dt, \]  

(26)

with the particle Lagrangian

\[ L = - \frac{1}{2} \int d^D r \, \hat{\eta}(\vec{r}) \hat{s}(\vec{r}) + \frac{1}{2} \frac{\rho}{c^2} \int d^D r \, \hat{\eta}(\vec{U} \cdot \vec{\nabla}) \hat{\psi} + \frac{1}{2} \frac{\rho}{c^2} \int d^D r \, \hat{\psi}^2, \]  

(27)

where (for \(D > 2\))

\[ \hat{\eta}(\vec{r}) = - \frac{1}{(D - 2)\Omega_D} \int d^D R' \frac{\rho^{-1} \hat{s} - c^{-2} (\vec{U} \cdot \vec{\nabla}) \hat{\psi}}{|\vec{r}' - \vec{R}'|^{D-2}}. \]  

(28)

is the solution of the stretched Poisson equation; \(\Omega_D\) is the solid angle in \(D\) dimensions. The \(D = 1, 2\) cases will be treated separately in Appendix A.

To recapitulate, the approximation I made amounts to the following procedure: At any given time take the local values of the velocities of the fluid and the particle and of the fluid density, calculate the steady state solution, \(\hat{\eta}\), from eq.(24) for a homogeneous fluid with these properties and an eternally constant particle velocity, then use this for the instantaneous \(\eta\) inside the particle.

I now proceed to discuss separately source and potential particles.
A. Source particles

For a pure source particle put \( \psi = 0 \); then substituting expression (28) in eq.(27) and changing to the \( \vec{r}' \) variables we get (for \( D > 2 \))

\[
L = \frac{1}{2(D-2)|\Omega_D|}\int d^DR' d^D\hat{R}' \frac{\hat{s}(\hat{R}')\hat{s}(\vec{r})}{|\vec{r}' - \hat{R}'|^D-2}. \tag{29}
\]

The integral in eq.(29) is proportional to the “electrostatic” energy of a charge distribution \( \hat{s} \) stretched by a factor \( \gamma \) in the \( z \) direction. Note that \( L \) is positive because it is proportional to \( -\int \tilde{\eta} \Delta \tilde{\eta} = \int (\nabla \tilde{\eta})^2 \) \(-\Delta \) is a positive definite operator).

We can also write the integral in terms of the \( \vec{r} \) coordinates as

\[
L = \frac{\gamma}{2(D-2)|\Omega_D|}\int d^DrdD\hat{R} \frac{\hat{s}(\hat{R})\hat{s}(\vec{r})}{|\vec{r} - \hat{R}|^{D-2}}, \tag{30}
\]

where the full dependence of \( L \) on \( \gamma \), the structure of the particle, and its orientation with respect to the relative velocity is explicit.

In the non-relativistic limit, \( U \ll c \), eq.(30) tells us that

\[
L = \frac{E_0}{\Omega_D} + \frac{1}{2}U_i m_{ij} U_j + O(U^4/c^4); \tag{31}
\]

\( E_0 = [2(D - 2)]^{-1} \int d^Drd^D\hat{R} \frac{\hat{s}(\hat{R})\hat{s}(\vec{r})}{|\vec{r} - \hat{R}|^{D-2}} \)

is the value for the unstretched configuration, and the effective mass tensor is

\[
m_{ij} = \frac{1}{2(D-2)|\Omega_D|c^2} \times \int d^Drd^D\hat{R} \frac{\hat{s}(\hat{R})\hat{s}(\vec{r})}{|\vec{r} - \hat{R}|^{D-2}} \times [\delta_{ij} - (D - 2)\frac{(\vec{r} - \hat{R})i(\vec{r} - \hat{R})j}{|\vec{r} - \hat{R}|^2}]. \tag{32}
\]

In the isotropic case (for which a cubic symmetry of the particle suffices) we get the mass of the particle

\[
m = Tr(m_{ij})/D = \frac{2E_0}{D\Omega_Dc^2}. \tag{33}
\]

When the background density is not a constant of the configuration this mass parameter is a function of space-time position through \( \rho \) and possibly \( c \). I shall still refer to it as the mass of the particle.

To insure isotropy of inertia I shall assume henceforth that our particle is spherically symmetric. In this case \( L \)

can be obtained analytically. This is done in Appendix A and yields

\[
L = L_0 F(1, \frac{1}{2}; \frac{D}{2}; \frac{U^2}{c^2}), \tag{34}
\]

where \( L_0 = E_0/\Omega_D \rho \) is the value of the effective Lagrangian for \( U = 0 \), and \( F \) is the Gauss hypergeometric function.

B. Potential particles

Consider now a pure potential particle \( (\hat{s} = 0) \). Repeating the same argumentation as before

\[
L = \frac{\rho}{2(D-2)|\Omega_D|c^4} \times \int d^Drd^D\hat{R} \frac{\tilde{\nabla} \tilde{\nabla} \hat{\psi}(\hat{R})\hat{\psi}(\vec{r})}{|\vec{r} - \hat{R}|^{D-2}} \times + \frac{1}{2} \frac{\rho}{c^2} \int \tilde{\psi}^2 d^Dr. \tag{35}
\]

Or, with derivatives with respect to \( \vec{r}' \) and \( \hat{R}' \),

\[
L = \frac{\rho\gamma}{2(D-2)|\Omega_D|c^4} \times \int d^Drd^D\hat{R} \frac{\tilde{\nabla} \tilde{\nabla} \hat{\psi}(\hat{R})\hat{\psi}(\vec{r})}{|\vec{r} - \hat{R}|^{D-2}} \times + \frac{1}{2} \frac{\rho}{c^2} \int \tilde{\psi}^2 d^Dr. \tag{36}
\]

In the non-relativistic limit

\[
L = \frac{1}{2} \frac{\rho}{c^2} \int \tilde{\psi}^2 d^Dr + \frac{1}{2} U_i m_{ij} U_j + O(U^4/c^4), \tag{37}
\]

where the mass tensor is (integrating by parts)

\[
m_{ij} = \frac{\rho}{(D-2)|\Omega_D|c^2} \times \int d^Dr d^D\hat{R} \frac{\hat{\psi} (\hat{R})\tilde{\psi}(\vec{r}) \partial_r \partial_{R_i} \partial_{R_j} |\vec{r} - \hat{R}|^{-(D-2)}}. \tag{38}
\]

In the isotropic case

\[
m = Tr(m_{ij})/D = \frac{\rho}{Dc^4} \int \tilde{\psi}^2 d^Dr, \tag{39}
\]

where I used \( -\Delta |\vec{r}^{-(D-2)}| = (D - 2)|\Omega_D|\delta^{(D-2)}(\vec{r}) \). The first integral in eq.(36) may be viewed as proportional to the energy of a polarized medium with unidirectional polarization of magnitude \( \propto U \hat{\psi}(\vec{r}'^2) \). It can be calculated
exactly for an arbitrary, spherically symmetric distribution \( \psi(r) \). The integral is calculated in Appendix B, and when added to the second integral we get

\[
L = \frac{\rho}{2c^2} \int \psi^2 d^2 r E(1, \frac{1}{2}, \frac{D}{2}; \frac{U^2}{c^2}) = \frac{D}{2} mc^2 F(1, \frac{1}{2}, \frac{D}{2}; \frac{U^2}{c^2}),
\]

with the same hypergeometric function appearing in the Lagrangian of a source particle. These identical results are obtained from rather different starting expressions, and I have not been able to find an underlying physical reason for the equality.

C. Aspherical and compound particles

A larger variety of \( \gamma \) dependences of \( L \) is afforded by considering aspherical particles. If the orientation of the particle is kept fixed in space, anisotropic inertia results generally; but, if we can somehow keep the orientation fixed with respect to \( \vec{U} \), the effective inertia is isotropic.

As an example, consider a hyper-planar, bipolar source layer (charged planar capacitor in the electrostatic analog) whose normal always makes an angle \( \Theta \) with the relative velocity vector. (Actually, because of the reservations discussed above, we need to take two, back-to-back dipole layers to annihilate the dipole moment of the particle, but this is immaterial for the results since the two layers do not interact, so I shall just continue to speak of one layer.) From eq.(29) the Lagrangian is \( \gamma^{-1} E_c \), \( E_c \) being the energy of the stretched bilayer. This energy, like that of a stretched capacitor, is \( E_c \propto Q^2/2A \), where \( Q \) is the total charge on one layer, \( A \) the area, and \( d \) the spacing. Under stretching \( Q \rightarrow \gamma Q \), \( A \rightarrow A(\cos^2 \Theta + \gamma^2 \sin^2 \Theta)^{1/2} \), and \( d \rightarrow d \gamma(\cos^2 \Theta + \gamma^2 \sin^2 \Theta)^{-1/2} \). So,

\[
L = L_0 \frac{\gamma^2}{\cos^2 \Theta + \gamma^2 \sin^2 \Theta} = \frac{L_0}{1 - (U^2/c^2)\cos^2 \Theta} \quad (41)
\]

This differs from \( L = L_0 \gamma^2 \) for \( \Theta = 0 \), as in the 1-D case, to a constant \( L = L_0 \) when the bilayer moves parallel to itself relative to the fluid. Integrating over angles with weight \( \sin D \theta d \theta \) gives back our result for the spherical case. (In the relativistic limit the contribution to the action of a spherical particle then comes from a small \( \Theta \) region near the leading point on the sphere. The corresponding area decreases with increasing dimension; hence the strong \( D \) dependence of the relativistic limit.) All the above applies to any collections of bi-layers making the same angle with the relative velocity vector, for example a cone of half-opening angle \( \gamma/2 - \Theta \) moving always along its axis relative to the fluid, like an arrowhead. In general, if we tie together several particles that are so far from each other that their mutual interactions can be neglected compared to their self interactions, the body will have an effective Lagrangian that is the sum of those of the individual components. (The inter-component distance still has to be small compared with the scale over which macroscopic properties of the flow vary).

D. Limitations and caveats

I made two types of assumptions about the particle: that it constitute a very weak perturbation on the fluid even within the particle itself, and that it is very small in size so that conditions in it very quickly take up steady state values corresponding to the momentary macroscopic conditions around it. I now discuss in more detail what these require from the strength of the potential \( \psi \), from the source density \( s \), and from the particle size \( a \). I find that in general these approximations break down at high \( \gamma \) however small \( a \) and \( \psi \) or \( s \) are. The perturbation treatment assumes that \( \zeta \ll \rho \). Equation(13) gives

\[
\zeta = -(\rho/c^2)(\eta \cdot \vec{\nabla} \eta + \psi) \\
\approx (\rho/c^2)(\vec{U} \cdot \vec{\nabla} \eta - \psi),
\]

where I used our approximation in the second equality. Consider first a potential particle. For \( \vec{U} \rightarrow 0, \eta \rightarrow 0 \) and the basic requirement is \( |\zeta|/\rho \approx |\psi|/c^2 < 1 \). We further have to insure that for the limit \( \vec{U} \rightarrow c \) we still have \( \zeta/\rho < 1 \), so we need \( |\vec{U} \cdot \vec{\nabla} \eta|/c^2 \ll 1 \) everywhere in the body. The exact constraint this puts on \( \gamma \) depends on the particle structure. We can get an estimate of this quantity by noting that what I calculated as the first term contributing to \( L \) is \((\rho/2c^2) \int d^3 r \eta(\vec{U} \cdot \vec{\nabla})\psi = -(\rho/2c^2) \int d^3 r \psi(\vec{U} \cdot \vec{\nabla})\eta \). If we take, for example, a particle of constant \( \psi \) we know the value of \((\vec{U} \cdot \vec{\nabla})\eta \) is constant inside the particle and from the results for \( L \) it is

\[
\vec{U} \cdot \vec{\nabla} \eta = -\dot{\psi}[F(1, \frac{1}{2}, \frac{D}{2}; \frac{U^2}{c^2}) - 1].
\]

So we also need

\[
(|\psi|/c^2)|F(1, \frac{1}{2}, \frac{D}{2}; \frac{U^2}{c^2}) - 1| \ll 1.
\]

In the non-relativistic limit the expression in square parentheses behaves as \( U^2/c^2 \), so no new requirement is added. In the relativistic regime we can get a validity limit on \( \gamma \). For \( D > 3 \) the \( F \) above is finite for \( U = c \) and we do not get an additional constraint; the basic one suffices for all values of \( \gamma \). For \( D = 1 \) we have to have \( \gamma^2|\psi|/c^2 \ll 1 \), for \( D = 2 \): \( \gamma|\psi|/c^2 \ll 1 \), and for \( D = 3 \): \( ln(\gamma)|\psi|/c^2 \ll 1 \). For a source particle we have, as before, to first order in the source strength \( \zeta \approx (\rho/c^2)\vec{U} \cdot \vec{\nabla} \eta \), which vanishes when the particle is at rest with respect to the fluid. Consider such a particle in a static fluid. Neglecting the variation of the external potential across the particle, the Bernoulli equation tells us that \( \zeta \) is indeed second order in the source. If we write \( \dot{s} = \rho \Delta \Phi \) we have
from the continuity equation that to first order \( \dot{\hat{\rho}} = \Phi \) (with 3rd order corrections), and so \( \zeta \approx -\left(\rho/2c^2\right)(\nabla \Phi)^2 \).

The basic requirement from \( \hat{s} \) for our approximation to hold is then \( |\nabla \Phi| \ll c \). And here too, for \( \gamma \gg 1 \) we have to have \( |U \cdot \nabla \hat{\rho}|/c^2 \ll 1 \), which casts a constraint on \( \gamma \) that may depend on the particle structure and the dimension.

Consider now the condition on the particle size. When the particle, having diameter \( a \), is moving with velocity \( \vec{U} \) relative to the fluid, it takes a sound wave time \( \delta t \approx (a/c) (1 - U/c)^{-1} = (a/c) (1 + U/c) \gamma^2 \) to move from the aft of the particle to its fore. We want \( \delta t \) to be much shorter than any time scale, \( T \), over which the environmental parameters change. The basic requirement, which should hold even at low relative velocities is then \( a \ll cT/2\gamma^2 \).

We thus see that even for very small values of \( \hat{\psi} \) or \( \hat{s} \), and of \( a \), our approximations, and thus our results, are not valid for \( \gamma \rightarrow \infty \). This is to be expected: For \( D \leq 5 \) the particle energy diverges for \( \gamma \rightarrow \infty \), which, if valid indefinitely, says that we cannot accelerate our particles to supersonic speeds; but this is clearly not true.

It would be interesting to see how our acquired dynamical properties of particles are modified when the approximations break down, as some of these may also be taking place in reality. For example, the time locality of the Lagrangian is only a result of the approximation: The effects of the particle on the fluid at one time affect, at some level, the motion of the particle at another time. Indeed they may affect other particles as well, thus creating an effective interaction between particles mediated by the fluid akin to the Cooper pairing interaction between electrons in a superconductor. We also have here some ready made mechanisms for the breakdown of Lorentzian dynamics at high \( \gamma \). All these departures are still considered in the context of inviscid, irrotational, barotropic fluids; and these attributes are also only approximations (see a discussion of these in the context of phonon propagation in [6]).

I have also neglected the goings on inside the source itself. This is after all some parallel flow (e.g. in some pipes with pumps) that move the fluid from the sinks to the sources.

IV. PARTICLE DYNAMICS

Notwithstanding the absence of true inertia in our particles, they acquire relativistic inertia through their interaction with the fluid; this is encapsulated in the kinetic action for free particles.

From now on I shall assume a position and time independent density for the background flow. This situation is rather less cumbersome to describe, captures most of the concepts I want to introduce, and insures a constant speed of sound, which after all is our analog of the speed of light (this can also be insured, without imposing a constant \( \rho \), by having a fluid equation of state of the form \( p = c^2 \rho + \text{const.} \)). It also means that \( m \) is a constant and so both types of particles have the same motion. (If the fluid density depends on position or on time we get a variable mass for the particles, which might be interesting to explore.) The freedom left in selecting a background flow—a solution of the field equations with \( s = 0 \) and \( \psi = 0 \)—is then only in choosing the velocity potential field from among the harmonic functions. We then have to impose an external potential \( \theta \) that will satisfy the Bernoulli equation for the chosen velocity field. The exact form of the equation of state is immaterial since we shall probe it only at one density value where its derivative only (speed of sound) has to be known.

For either definition of a particle the effective action can be written in the form

\[
S = \int mc^2 \ell(\gamma(t)) dt.
\]  

For a source particle \( m \) is proportional to the “electrostatic” energy of a charged medium with charge density \( s \), and for a potential particle it is proportional to the “magnetostatic” energy of a sphere with unidirectional polarization \( \psi \).

I now proceed to discuss various aspects of the resulting dynamics.

A. Flat-space-time dynamics: emergent inertia

In a flat space-time; i.e., in a homogeneous background flow at rest, the effective Lagrangian for the two types of particles discussed here is

\[
L = mc^2(D/2) \{ 1 - \frac{1}{2} \frac{D}{D^2} - \frac{v_s^2}{c^2} \},
\]  

where \( v_s \) is the particle velocity and \( m \) is constant. Note that \( D \) here is determined by the symmetry of the particle, and is not necessarily the dimension of the space in which it moves. For example, a plane symmetric particle that moves only along its normal has \( D = 1 \). More generally, a particle in \( N \) dimensions of cylindrical symmetry having the symmetry of \( S^D \otimes R^{(N-D)} \) whose velocity is in the \( S^D \) subspace corresponds to dimension \( D \).

Using the formula for the derivative of the hypergeometric function we get for the momentum

\[
\vec{p} = \frac{\partial L}{\partial \vec{v}_s} = m\vec{v}_s F(2, \frac{3}{2} \frac{D}{D^2} - \frac{v_s^2}{c^2}).
\]  

The kinetic energy

\[
E_k = -L + L_0 + \vec{v}_s \cdot \vec{p} = mc^2 \{ D \left[ 1 - F(1, \frac{1}{2} \frac{D}{D^2} - \frac{v_s^2}{c^2}) \right] + \frac{v_s^2}{c^2} F(2, \frac{3}{2} \frac{D}{D^2} - \frac{v_s^2}{c^2}) \},
\]  

where I added a constant so as to make \( E_k \) vanish for \( \vec{v}_s = 0 \).
These energy and momentum are conserved—as follows from Nöther theorem’s related to the assumed time independence and homogeneity of the fluid: If the particle is subject to a conservative force derived from a potential \( \xi \), we have to add \( \int -\xi [\vec{p},(t)]dt \) to the action we started with; so; the particle now satisfies \( \frac{d\vec{x}}{dt} = -\nabla \xi \), and \( E_k + \xi \) is conserved. And if we have some inter-particle forces \( \sum \vec{p}_i \) is conserved, and if these forces are derived from a potential again the total energy is conserved.

Of course, \( E \) and \( \vec{p} \) are not the real energy and momentum of the particles; these were assumed to have no inertia of their own so they can carry no energy and momentum. It is the stirring of the fluid by the motion of the particles that puts a real inertial cost to their motion. The rates of change of \( E_k \) and \( \vec{p} \) equal the rates of change of the energy and momentum of the fluid induced when the particle changes its velocity; they are thus equal to the external power input and the external force imparted to the particle. Such quantities are called pseudo-energy and pseudo-momentum. These are often useful in the description of motion of objects in homogeneous media with which they interact (See [18] 2.4 and the review by [19]). A direct calculation of the energy and momentum of the fluid might have also served, but it is impractical. Attempting to calculate them even for the simple, steady-state configuration, with the particle ever at constant velocity, gave me ambiguous results: When these quantities are written as integrals of fluid attributes over a volume that has to be taken to infinity, the results depend on the shape of the integration volume. This is similar to what Peierls [18] finds when trying to calculate these quantities for a sound wave. The present approach of proceeding through the action seems to be the proper way to proceed.

Since \( F(1, \frac{1}{2}; \frac{D}{2}, \frac{z}{c^2}) = 1 + D^{-1}v^2/c^2 + O(v^4/c^4) \), we have in all dimensions the non-relativistic behavior

\[
L - L_0 \approx \frac{1}{2}mv^2, \quad \vec{p} \approx mv\vec{v}, \quad E_k \approx \frac{1}{2}mv^2. \tag{49}
\]

In the highly relativistic regime the behavior depends strongly on the dimension. Dimension \( D = 3 \) is critical in some sense: Because \( F(a, b; c; z) \) is finite for \( z = 1 \) when \( c > a + b \), \( L \) is finite as \( \gamma \to \infty \) for \( D > 3 \), diverges logarithmically for \( D = 3 \), and diverges as a power of \( \gamma \) for \( D < 3 \). Dimension \( D = 5 \) is another critical dimension above which the energy and momentum remain finite for \( \gamma \to \infty \); for \( D = 5 \) itself these quantities behave as \( \ln(\gamma) \) in this limit (see below), while for \( D < 5 \) they diverge as a power of \( \gamma \).

Following are the Lagrangian, the momentum, and the energy for dimensions \( D \leq 5 \) in closed forms with their relativistic limits:

For \( D = 1 \)

\[
L = mc^2\gamma^2/2, \quad \vec{p} = mv\gamma \vec{v}, \quad E_k = mv^2\gamma^2(\gamma^2 - 1/2) \to mc^2\gamma^2. \tag{50}
\]

For \( D = 2 \)

\[
L = mc^2\gamma, \quad \vec{p} = mv^2\gamma \vec{v}, \quad E_k = mv^2\gamma^2(\gamma^2 + \gamma - 1) \to mc^2\gamma^3. \tag{51}
\]

For \( D = 3 \)

\[
L = mc^2 \frac{3}{4(v_*/c)}\ln\left(\frac{1 + v_*/c}{1 - v_*/c}\right), \tag{52}
\]

\[
\vec{p} = mv\vec{v}_* \frac{3}{2(v_*/c)^2}(\gamma^2 - \frac{1}{2(v_*/c)^2}\ln\left(\frac{1 + v_*/c}{1 - v_*/c}\right)) \to \frac{3}{2}mv\gamma^2, \tag{53}
\]

and

\[
E_k = \frac{3}{2}mc^2[1 - \frac{1}{(v_*/c)^2}\ln\left(\frac{1 + v_*/c}{1 - v_*/c}\right) + \gamma^2] \to \frac{3}{2}mc^2\gamma^2. \tag{54}
\]

The case \( D = 4 \) is particularly interesting. We can then write, using formula 9.131.2 in [20] (henceforth GR)

\[
L = 2mc^2F(1, \frac{1}{2}; 2; \frac{v^2}{c^2}) = 4mc^2[1 + \frac{1}{2}; \gamma^{-2}) - \gamma^{-1}F(1, \frac{3}{2}; 3; \gamma^{-2})]. \tag{55}
\]

However, we have generally \( F(1, b; b; z) = (1 - z)^{-1} \), which gives

\[
L = \frac{4mc^2}{1 + \gamma^{-1}}, \quad \vec{p} \approx \frac{4mv\gamma\vec{v}_*}{(1 + \gamma^{-1})^2}, \quad E_k = 4mc^2\left(\frac{\gamma - 2}{\gamma + 1} + \frac{1}{2}\right). \tag{56}
\]

The kinematics is quasi-Lorentzian and becomes Lorentzian in the limit of high \( \gamma \), with \( L \approx Mc^2(1 - \gamma^{-1}) \), \( \vec{p} \approx M\gamma\vec{v}_* \), and \( E_k \approx Mc^2\gamma \), where \( M = 4m \).

For \( D = 5 \), using formula 9.137.14 and then 9.121.1 in GR we can write

\[
L = mc^2(5/2)F(1, \frac{5}{2}; 2; \frac{v^2}{c^2}) = \frac{15}{4}mc^2(v_*/c)^{-2}[1 - \gamma^{-2}F(1, \frac{3}{2}; 3; \frac{v^2}{c^2})], \tag{57}
\]

where

\[
F(1, \frac{3}{2}; 3; \frac{v^2}{c^2}) = \frac{1}{2(v_*/c)^2}\ln\left(\frac{1 + v_*/c}{1 - v_*/c}\right) \tag{58}
\]

is, in fact, the \( D = 3 \) Lagrangian. So, for \( \gamma \to \infty \), \( \vec{p} \approx (15/2)mv\vec{v}_*\ln(\gamma), E_k \approx (15/2)mc^2\ln(\gamma) \).
B. Antiparticles and the particle vacuum

If we wish to push the analogy beyond the dynamics of isolated, ever-existing particles, and discuss pair creation and annihilation we need to define new notions. We have to identify antiparticles, and we need to have a proper definition of the rest mass of our particles. Also, unlike phonons, our particles are not an organic part of the fluid; they cannot be created out of it if not put in by hand. So, pairs will not spring out of the fluid even under energetically favorable conditions, such as near event horizons or in strong “electric” fields unless we prepare energetically favorable conditions, such as near event horizons. So, pairs will not spring out of the fluid even un

Again, no new mass parameter is introduced and it determines the rest mass.

We can thus write the complete expression for the energy of a source particle

\[
E = mc^2[D - \frac{D}{2} F(1, \frac{1}{2}, \frac{D}{2}; \frac{v^2}{c^2})] + \frac{v^2}{c^2} F(2, \frac{3}{2}, \frac{D+2}{2}; \frac{v^2}{c^2})
\]

\[
E = mc^2[D - \frac{D}{2} F(1, \frac{1}{2}, \frac{D}{2}; \frac{v^2}{c^2})] + \frac{v^2}{c^2} F(2, \frac{3}{2}, \frac{D+2}{2}; \frac{v^2}{c^2})]
\] (60)

C. Curved-space-time dynamics

When the acoustic space-time is not flat—i.e., when the background flow velocity is not constant—all our particles fall in the same way in a gravitational field thus obeying the weak equivalence principle. This is non-trivial and might have well been otherwise. For example, the first order terms that we arranged to be absent could destroy universal free fall. But barring such departures, which we saw can be avoided by properly defining the setup, it can be thought of as both the inertial mass and the passive gravitational mass of the particle. (When the background density is not constant the two types of particles see two different, but conformally related, space times.)

Our Lagrangian is of the form \( L = L(U^2/c^2) \), which is also true of the standard acoustic line element. These give the Euler-Lagrange equation

\[
\frac{d(L'U_i)}{dt} + L'U_k v_{ik} = 0.
\] (61)

Using the fact that \( \vec{v} \) is irrotational we have

\[
c^{-2} \frac{L''}{L'} U d(U^2) + \frac{dc}{dt} - \frac{1}{2} \nabla v^2 = 0.
\] (62)

Using the formula for the derivative of the hypergeometric function we have for our actions

\[
L' = \frac{1}{2} mc^2 F(2, \frac{3}{2}, \frac{D+2}{2}; \frac{U^2}{c^2}),
\] (63)

and

\[
L'' = \frac{3}{D+2} mc^2 F(3, \frac{5}{2}, \frac{D+4}{2}; \frac{U^2}{c^2}).
\] (64)

In the limit \( U \ll c \) the first term in eq.(62) is of a higher order in \( U/c \) (\( L' \) and \( L'' \) are finite there) and we are left with the standard non-relativistic equation for all \( L(U^2/c^2) \):

\[
\frac{d\vec{v}}{dt} = -\nabla \chi,
\] (65)

where \( \chi = -v^2/2 \) can be identified as the Newtonian gravitational potential. This holds when the gravitational field is weak (\( v \ll c \)) and the motions are slow (\( v \ll c \)), but also when only \( U \ll c \). This can also be gotten directly from the action, which for \( U \ll c \) is
\[ U^2 dt \] up to a constant. This limiting behavior is common to all theories with an \( L(\gamma) \) Lagrangian including the standard acoustic line element.

Because the background flow is assumed to have a constant density we see from the Bernoulli equation that \( \chi \) equals the external potential \( \theta \) used to establish the background flow. (This might point the way to introduce dynamics for the acoustic metric through that of the external potential \( \theta \).)

Consider now the null geodesics of the acoustic metric characterized by \( U = c \), or \( d\tau = 0 \). In theories for which \( L''/L' \) diverges when \( U^2/c^2 \rightarrow 1 \) the equation of motion implies that \( d(U^2)/dt = 0 \) if initially \( U = c \), so \( U \) remains constant at this value. In other words, in such theories the null world lines of the acoustic metric are solutions of the equation of motion (this can be shown to hold even in flows with variable background density). This is the case for the acoustic Lagrangian itself, but also for our Lagrangians when \( D \leq 7 \). For \( D > 7 \), \( L, L', L'' \) are finite as \( \gamma \rightarrow \infty \); for \( 5 < D < 7 \), \( L, L' \) are still finite, but \( L'' \) diverges like \( \gamma^{(7-D)} \) (for \( D = 7 L'' \) diverges logarithmically) and so does \( L''/L' \); for \( D < 5 \), \( L' \) diverges like \( \gamma^{(5-D)} \) (logarithmically for \( D = 5 \)) and \( L'' \) still as \( \gamma^{(7-D)} \), so \( L''/L' \) behaves as \( \gamma^2 \).

Note that \( \dot{\hat{U}} = 0 \) is a solution (i.e., the body just moving with the fluid). This is also a geodesic of the acoustic metric.

We saw above several instances of solutions of our field equations that are geodesics of the acoustic metric, and we shall see another in the next subsection. But in general the solutions of the Euler-Lagrange equations are not geodesics of the that metric, as the particle action is not its arc length. It may be useful, however, to generalize the acoustic proper-time interval and use our action to define a Finslerian one (defined only for intervals that are time- and null-like with respect to the acoustic metric) of whose geodesics are the solutions of our Euler-Lagrange equations. For \( D \leq 3 \), where the Lagrangian diverges for \( \gamma \rightarrow \infty \), this is well defined only for time-like elements, but for \( D > 3 \), \( d\hat{\tau} \propto [\ell(U^2/c^2) - \ell(1)]d\tau \) is well defined also for null intervals of the acoustic metric (for which it vanishes). Furthermore we saw that for \( D < 7 \) this scheme embraces both massive and massless particles.

I demonstrate this for the more interesting case \( D = 4 \): Subtract from \( L \) a constant equal to its relativistic limit:

\[ \hat{L} \equiv L - 4mc^2 = -Mc^2\gamma^{-1}\lambda(\gamma), \]  

where \( M \equiv 4m \), and

\[ \lambda(\gamma) = \gamma/(1 + \gamma). \]  

( When the background fluid density is not taken as a constant the term we subtract in eq.(66) is position dependent, we can then consider it as an external potential for the particle, which does not couple to the fluid.) We can then define a Finslerian line element

\[ d\hat{\tau}(d\hat{x}, dt) = \gamma^{-1}\lambda(\gamma)dt, \]  

defined for time- and light-like intervals, where here \( \gamma \) stands for \( \{1 - [d\hat{x}/d\tau - \hat{v}(\hat{\gamma})^2]\}^{-1/2} \). Clearly, \( d\hat{\tau}(d\hat{x}, dt) \) is homogeneous of order one as required. Our particles follow geodesics of this Finslerian metric since the action is \( S = -Mc^2 \int d\hat{\tau} \), and furthermore, phonons follow its null geodesics, since \( \delta\hat{\tau} = 0 \Leftrightarrow d\tau = 0 \), where \( d\tau \propto \gamma^{-1}dt \) is the standard acoustic line element, and we saw above that these too extremize the Finslerian arc length (because they solve the equation of motion).

### D. Circular orbits in spherically symmetric space times

Consider now circular orbit in a spherically symmetric, or axi-symmetric, configuration. Since \( U^2 \) is constant we can write from eq.(62)

\[ \frac{d\hat{v}^a}{dt} = \frac{1}{2} \nabla_v \hat{v}^2, \]  

identical to the non-relativistic equation of motion. These orbits are thus geodesics for all choices of \( L(U^2/c^2) \) including the acoustic one and all of our Lagrangians. For a circular orbit \( \frac{d\hat{v}^a}{dt} = -v^a r^{-2} \), and \( v^2 \) is a function of \( r \) (\( \hat{v} \) is not necessarily radial). So the relation between the velocity and the radius is:

\[ v_a(r) = v(r)^{1/2}, \]  

where \( v = |\hat{v}|, f = \frac{\hat{v}^a}{\hat{v}^2} \). Because I assumed a constant density the continuity equation and zero vorticity condition dictates that \( f \) is determined by the symmetry of the flow. For example, in purely radial flow in \( D \) space dimensions \( f = D - 1 \).

When \( \hat{v} \) is radial \( |\hat{U}| = v(1 + f)^{1/2} \), so subsonicity dictates for massive particles that \( v < c(1 + f)^{-1/2} \), which sets the limit for the innermost circular orbit. Equality corresponds to a “photon” orbit. In the canonical acoustic black hole configuration (e.g. [6]), where \( v = c(r_h/r)^f \) with \( r_h \) the horizon radius, this implies \( r > r_h(1 + f)^{1/2f} \) for massive particles. This is analogous to the radius occurring at \( 3m = 1.5r_h \) for real Schwarzschild black holes in \( 3 + 1 \) dimensions.

In a vortex geometry in (2+1) dimensions

\[ \frac{\hat{v}^2}{c} = -r_h f/r + (r^2 - r_h^2)^{1/2} \]  

(e.g. [6]) where \( r_e \) is the radius of the ergosphere, \( r_h \) is that of the event horizon and \( e_\theta \) is a unit vector in the azimuthal direction (I take an ingoing flow but the results below are the same for an outgoing one); so \( f = 1 \). The minimum radius of a circular orbit is

\[ r_L^\pm = [2r_e(r_e \mp \sqrt{r_e^2 - r_h^2})]^{1/2}, \]  

where \( r_L^- \) is for prograde motion and \( r_L^+ \) for retrograde one.
As in the real world \( r_c \) is a static limit since \( \vec{v}_c = 0 \) is not permitted below it lest \( U \) become supersonic. Also for \( r < r_b \), \( \vec{v}_s \) must have a component in the radial direction of the flow.

I haven’t fully checked the question of stability of the circular orbits. But note that in our constant density configurations the Newtonian potential \( \chi \) is a power law of the radius: \( \chi \propto r^{-2} \) in the \( D = 2 \) case and with a higher power in higher dimensions. This means that the effective radial potential (including the centrifugal barrier) for a particle with given angular momentum has only a maximum for \( D > 2 \). So there aren’t any stable bound orbits in the nonrelativistic case and the circular orbits are unstable. For \( D = 2 \), depending on the value of the angular momentum an orbit is either unbound or goes through the origin. This is not necessarily so in configurations with non-constant background densities, but their discussion is beyond what I wish to consider here.

\[\text{V. DISCUSSION}\]

Evidently, flow models can provide quasi-realistic analogs for relativistic inertia of massive particles and their behavior in gravitational fields. I am presenting these models in the twofold hope that they can inspire us in understanding the origins and the validity limits of genuine inertia, and that by considering how these models respond to tweaking we can learn about possible modification of standard physics in the real world. For example, looking where and how our approximations break down we can gain insight as to where and how standard dynamics may go awry. Such departures may include breakdown of standard dynamics at high \( \gamma \), time non-locality of the particle action, and fluid-mediated interactions between particles, all of which are not part of standard dynamics. Such models can also be used to enlighten us on how local dynamics might be affected by cosmology at large. Cosmological expansion may be included in the context of fluid analogs and could model, for instance, cosmological variations of particle masses (through variations in the fluid density, which enters the normalization of the masses), or variations in the speed of light. My hope in this connection is to simulate the dynamics implied by MOND, which revolves around an acceleration constant, \( a_0 \) that turns out to be of the order of the cosmic acceleration.

The fact that the more realistic models emerge for higher space dimensions than we seem to be living in is not disconcerting. Recent work on membrane universes has taught us that while most of the physical objects we deal with may be confined to sub-manifold of lower dimensions some aspects of physics, such as gravity, may be probing the higher dimensional aspects of space-time. In our models we could, for example, envisage particles moving in a fluid in \( D \) dimensions space, but somehow confined to reside in a three-dimensional sub-manifold.

This would give rise to \( D \)-dimension inertia in a lower dimensional effective space.

My main purpose in this paper is to demonstrate the concept: Instead of considering weak perturbations that are part of the background itself, and which thus move with a speed dictated by the characteristic speeds of the background, define the particle as an externally dictated perturbation that breaks the field equation of the background, but that can otherwise move freely on the background field. Various extensions and generalizations suggest themselves that are worth exploring. For example, we can define other types of particles, or permit non-rigid particles with responsive intrinsic structure. This would produce longer range interactions between particles similar in nature to van der Vals interactions between neutral charge distributions (our rigid particles interact only on contact). Such particles with dynamical, internal degrees of freedom; e.g., with the different charges in a source system connected by “springs”, may also serve as detectors for (phononic) Unruh radiation, as the internal degrees of freedom will couple to the phononic field.

And, we can generalize this idea to the whole gamut of analog models for which photon propagation can be simulated (e.g., [4]). One possibility, for example, is to look at small charge distributions (of vanishing total charge) in the context of non-linear electrodynamics where the action of the electromagnetic field is not the invariant \( F_{\mu\nu} F^{\mu\nu} \) but some function of it.

Finally note that as things now stand, our particles cannot serve as sources for a mock gravitational field through their effect on the fluid: Their mass \( m \) is not an active gravitational mass. So, they do not help towards constructing an analog of the Einstein-Hilbert action.

\[\text{APPENDIX A: CALCULATION OF THE LAGRANGIAN FOR A SOURCE PARTICLE}\]

Here I calculate the energy integral in eq.(29) for a spherical distribution of sources \( \bar{s}(r) \). Divide the distribution into concentric thin shells of radii \( r_i \) and total charges \( q_i \). The integral is twice the electrostatic energy of the system made of these shells all stretched by a factor \( \gamma \) in the \( z \) direction. The stretching is only of the geometry without thinning the density. Each spherical shell becomes a homoeoid: a shell bound by two concentric, oriented ellipsoids of the same axes ratio \( \gamma \), with the original density inside; so, the resulting homoeoid \( i \) has a total charge \( Q_i = \gamma q_i \). Write now the energy as

\[E = \sum_i E_i + \sum_{i<j} E_{ij}, \quad (A1)\]

where \( E_i \) is the self energy of shell \( i \) and \( E_{ij} \) is the interaction energy between shells \( i \) and \( j \) (\( i \) is interior to \( j \)). A homoeoid produces a constant potential inside its cavity (\( \varphi_i \) for homoeoid \( i \)) and if the homoeoid is thin, as here, this is also the potential on the shell. Thus, \( E_i = Q_i \varphi_i / 2 \) and \( E_{ij} = Q_i \varphi_j \). We can most easily calculate \( \varphi_i \) as the
value of the potential at the center of the cavity and this is simply (for $D > 2$)

$$\varphi_1(\gamma) = \frac{\varphi(1) \gamma}{\Omega_{D-1}^{D-1}} \int_0^\pi \frac{\sin^{D-2} \theta d\theta}{[1 + (\gamma^2 - 1) \cos^2 \theta]^{(D-2)/2}} = \frac{\varphi(1) \Omega_{D-1}^{D-1} 2 \gamma^3 - D}{\Omega_D} \int_0^{\pi/2} \frac{\sin^{D-2} \theta d\theta}{[1 - (U/c^2) \sin^2 \theta]^{(D-2)/2}}.$$  \hspace{1cm} (A2)

where $\Omega_D$ is the $D$-dimensional solid angle ($\Omega_D = \Omega_{D-1}^{D-1} \int_0^\pi \sin^{D-2} \theta d\theta$), and $\varphi(1) = q_i/r_i^{-2}$ is the potential for the unstretched shell. The integral can be expressed using a Gauss hypergeometric function (using formula 3.681.1 in [20], henceforth GR):

$$\int_0^{\pi/2} = \frac{1}{2} B\left(\frac{D - 1}{2}, \frac{1}{2} \right) F\left(\frac{D - 2}{2}, \frac{D - 1}{2}; \frac{D}{2}; \frac{U^2}{c^2}\right).$$ \hspace{1cm} (A3)

It can be shown that $B[(D - 1)/2, 1/2] = \Omega_D/\Omega_{D-1}^{D-1}$. Also use formula 9.131.1 in GR to further simplify and get

$$E = E_0 \gamma F(1, 1; D/2; U^2/c^2),$$ \hspace{1cm} (A4)

so for the Lagrangian one has

$$L = L_0 F(1, 1; D/2; U^2/c^2).$$ \hspace{1cm} (A5)

For $D = 2$ one finds that the dependence of the energy on $\gamma$ is of the form $E = \gamma^2 E_0 + \gamma^2 Q^2 f(U)$, where $Q$ is the total charge and $f(U) = (1/\pi) \int_0^\pi d\theta \ln[1 + (\gamma^2 - 1) \cos^2 \theta]$. Since we have to take a vanishing total charge, we are left with $E = \gamma^2 E_0$, so $L = L_0 \gamma$, which also conforms with eq.(A5) since $F(1, 1; 1; U^2/c^2) = \gamma$. Calculation for $D = 1$ is also straightforward and the result is also given by eq.(A5): $L = L_0 \gamma^2$.

\section{Appendix B: Calculation of the Lagrangian for a Potential Particle}

Here I calculate the Lagrangian for a spherical potential particle. We need the middle term in eq.(27), which can be written as

$$-\frac{\rho}{2\gamma^2} \int d^D r' \hat{\nabla}^2 \tilde{\psi}(r') q(r') = \frac{\rho}{2\gamma^2} \int d^D r' [\hat{\nabla}^2 \tilde{\psi}(r') ]^2,$$ \hspace{1cm} (B1)

with $q(r') = -e^{-2\gamma U'} \cdot \hat{\nabla}^2 \tilde{\psi}[\tilde{\psi}(r')]$ and $\hat{\nabla}'$ that solves $\Delta' \tilde{\psi} = q(r')$. The integral is twice the electrostatic energy of the distribution function $q(r')$, which is produced by a polarized body with unidirectional polarization $\vec{P}(r') = -e^{-2\gamma \tilde{\psi}[\tilde{\psi}(r')]} \vec{U}$. We start from a spherical body having some $\tilde{\psi}(r)$ and divide it into thin spherical shells of radii $r_i$ and thicknesses $dr_i$ of constant $\tilde{\psi}(r_i)$. These are stretched into thin, concentric, nested homoeoids of minor axes $r_i$ and axes ratio $\gamma$ with constant polarization $P(r_i)$ along the major axis. We need the energy of this configuration. It is well known that the field inside an ellipsoid with uniform polarization along the major axis, is constant and proportional to the polarization with the proportionality factor depending only on the axes ratio. In our case, for a constant $\tilde{\psi}$ we write $\hat{\nabla}^2 \tilde{\psi} = -\gamma^2 e^{-2\gamma \tilde{\psi}} \vec{U}$. (In the context of magnetostatics $\hat{\nabla}^2 \tilde{\psi}$ is called the demagnetizing factor.) This means that a thin homoeoid with uniform polarization $\vec{P}$ along its major axis produces a vanishing field inside it, and thus the interaction energy of two nested homoeoids such as ours vanishes. The Lagrangian produced by our stretched expression is then the sum of the contributions of the self energies of the individual thin homoeoids. Each thin homoeoid may be viewed as an infinitesimally thin dipole bilayer whose different elements thus do not interact with each other. Its energy is then the surface integral of the energy of a charged parallel-plate capacitor $E = (1/2) \int d\sigma dS$, where $\sigma$ is the surface density of the charge, and $d\sigma$ is the thickness, both dependent on the polar angle $\theta'$ (relative to the major axis) at the position of the integration point $q'$ on the homoeoid. Use instead as variable the polar angle $\theta$ at point $q$ on the spherical shell which was stretched into $q'$. we then have $\sigma = (U\gamma^2/c^2) \psi \cos^2 \theta + \gamma^2 \sin^2 \theta$$^{-1/2}$, $\delta a = \gamma dr_i \cos^2 \theta + \gamma^2 \sin^2 \theta$$^{-1/2}$, $dS = (\cos^2 \theta + \gamma^2 \sin^2 \theta$$^{-1/2}) \Omega_{D-1}^{D-1} \sin^{D-2} \theta d\theta$, where $\Omega_D$ is the $D$-dimensional solid angle $\Omega_D = \Omega_{D-1}^{D-1} \int_0^\pi \sin^{D-2} \theta d\theta$.

Thus the required contribution to $L$ of the ith shell (for $D \geq 2$) can be written as

$$dL_i = \frac{1}{2} \rho \frac{U^2 \gamma^2 \psi^2(r_i) \Omega_{D-1}^{D-1} dr_i}{\Omega_D} \int_0^\pi \frac{\cos^2 \theta \sin^{D-2} \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta.$$ \hspace{1cm} (B2)

Summing over the shells we replace $\psi^2(r_i) \Omega_{D-1}^{D-1} dr_i$ to $\int \psi^2 d^D r$

The integral over $\theta$ is

$$2 \int_0^{\pi/2} \frac{\cos^2 \theta \sin^{D-2} \theta}{1 - (1 - \gamma^2) \sin^2 \theta} d\theta,$$

which can be read off formula 3.681.1 in GR to be

$$\frac{\Gamma[(D - 1)/2] \Gamma[3/2]}{\Gamma[(D + 2)/2]} F(1, \frac{D - 1}{2}; \frac{D + 2}{2}; 1 - \gamma^2).$$

The factor in front can be shown to give $\Omega_D/\Omega_{D-1}$. I also use formula 9.131.1 in GR to transform

$$F(1, \frac{D - 1}{2}; \frac{D + 2}{2}; 1 - \gamma^2) = - \gamma^2 F(1, \frac{3}{2}; \frac{D + 2}{2}; \frac{U^2}{c^2}),$$

and then 9.137.12 in GR to write

$$F(1, \frac{3}{2}; \frac{D + 2}{2}; \frac{U^2}{c^2}) = D \left(\frac{U^2}{c^2}\right)^{-1} [F(1, \frac{1}{2}; \frac{D}{2}; \frac{U^2}{c^2}) - 1],$$

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then adding the last term in eq.(27) I get finally
\[ L = L_0 F(1, \frac{1}{2}, \frac{D}{2}, \frac{U^2}{c^2}), \]  
(B3)
with
\[ L_0 = \frac{\rho}{2c^2} \int \hat{\nu}^2 d^D r. \]  
(B4)
For \( D = 1 \) it is straightforward to solve directly for \( \eta \) from eq.(24) and substitute in eq.(27). It turns out that eq.(B3) is still valid giving \( L = L_0 \gamma^2 \). The same is true for \( D = 2 \) where we have \( L = L_0 \gamma \).

As a byproduct of the above calculation we get the expression for the \( D \) dimensional demagnetizing factor for a prolate ellipsoid magnetized along the symmetry axis. Consider a case where \( \hat{\nu} \) is constant inside the stretched ellipsoid in which \( \nabla' \hat{\eta} = -d(\gamma)c^{-2}\gamma \psi \hat{U} \). What I calculated above is the quantity
\[ \frac{\rho}{2c^2} \int d^D r \hat{\eta}(\hat{U} \cdot \nabla \hat{\psi}) = -\frac{\gamma\rho}{2c^2} \int d^D r \hat{\psi}(\hat{U} \cdot \nabla' \hat{\eta}) \]
\[ = d(\gamma)\frac{\gamma\rho}{2c^2}(U/c^2)^2 \int \hat{\psi}^2 d^D r. \]  
(B5)
This equals \( (\rho/2c^2)(\int \hat{\psi}^2 d^D r)[F(1, \frac{1}{2}, \frac{D}{2}, \frac{U^2}{c^2}) - 1] \), as we found above. Comparing the two expressions one gets
\[ d(\gamma) = (\gamma^2 - 1)^{-1}[F(1, \frac{1}{2}, \frac{D}{2}, \frac{U^2}{c^2}) - 1]. \]  
(B6)
For \( D = 3 \) this gives
\[ d(\gamma) = \frac{1}{\gamma^2 - 1} \times \]
\[ \left[ \frac{\gamma}{2(\gamma^2 - 1)^{1/2}} \ln \left( \frac{\gamma + (\gamma^2 - 1)^{1/2}}{\gamma - (\gamma^2 - 1)^{1/2}} \right) - 1 \right], \]  
(B7)
which reproduces the result found in [21].

**APPENDIX C: THE REST MASS OF A SOURCE PARTICLE**

The rest energy of a source particle is the energy difference between two configurations, one of a uniform fluid at rest, the other likewise but with the source inserted. There are subtleties involved in the determination of this difference as the two configurations have infinite mass and energy. I adopt the following scheme: Consider a container of finite volume \( V \) much larger than that of the particle and filled with a static homogeneous fluid at the reference density \( \rho \). Consider now a spherical source of vanishing total out-flux somewhere inside the container, in a steady state. Write the source density as \( \tilde{s}(r) = \rho \Delta \Phi = \rho \nabla \cdot \tilde{u} \) for some \( \tilde{u} = \nabla \Phi \); and if the support of \( \tilde{s} \) is within radius \( R_0 \) of its center we deduce from Gauss theorem that \( \Phi \) and \( \tilde{u} \) vanish everywhere outside \( R_0 \). Writing the density as \( \rho + \zeta \), the continuity equation is \( \nabla \cdot [(\rho + \zeta) \tilde{v} - \rho \tilde{u}] = 0 \), with \( \tilde{v} \) and \( \tilde{u} \) radial from the center of the source; hence
\[ \tilde{v} = (1 + \zeta/\rho)^{-1} \tilde{u} \]  
(C1)
is exact and \( \tilde{v} \) vanishes everywhere outside the source. The Bernoulli equation is
\[ \frac{1}{2} \rho \tilde{v}^2 + h(\rho + \zeta) = constant \equiv h(\tilde{\rho}). \]  
(C2)
Since outside the source \( \tilde{v} = 0 \) it follows that the density is constant there and equals \( \tilde{\rho} \). I require this configuration with the source to have the same total fluid mass as the reference configuration, so \( \int_V \zeta d^D r = 0 \) and this closes the set of algebraic equations that determines the configuration completely. The energy of the source configuration relative the reference one is then
\[ E_r = \int_V \frac{1}{2} \rho (\zeta + \rho + \zeta) \tilde{v}^2 + e(\rho + \zeta) - e(\rho))d^D r. \]  
(C3)
It is evident that neither the run of \( \zeta \) and \( \tilde{v} \) inside the source nor the value of \( \tilde{\rho} \), the constant density outside the source, depend on the position of the source inside the container, nor on the shape of the container (though its volume does enter). So far everything is exact. When the source may be considered a weak perturbation as in our case \( (\zeta \ll \rho, v \ll c) \) we can expand to lowest order in \( \zeta \): Bernoulli’s equation gives
\[ \zeta(\tilde{r}) \approx \tilde{\zeta} - \frac{\rho \tilde{v}^2(\tilde{r})}{2c^2}, \]  
(C4)
where \( \tilde{\zeta} \equiv \tilde{\rho} - \rho; \) the continuity equation gives \( \tilde{v} \approx \tilde{u} \approx \nabla \Phi \); and the preserved-total-mass constraint gives
\[ \tilde{\zeta} \approx \frac{\rho}{2c^2 V} \int_S \tilde{s}^2 d^D r, \]  
(C5)
where the integral is over the volume of the source. Clearly \( \tilde{\zeta} \) vanishes in the limit of infinite \( V \). Writing \( e(\rho + \zeta) - e(\rho) \approx h(\rho)\zeta \), the fact that \( \tilde{\zeta} \) integrates to zero means that there is no first order contribution to the intrinsic energy difference. The second order contribution \( \int \frac{\rho^2}{2} \zeta^2 d^D r \) is higher order in \( u^2/c^2 \). We are left with the dominant contribution
\[ E_r \approx \frac{1}{2} \int_S \rho \nabla \Phi \tilde{v}^2 d^D r \]
\[ = -\frac{1}{2} \int_S \tilde{\psi} d^D r = L_0, \]  
(C6)
which depends neither on the position of the source nor on the volume of the container or its shape, and we can identify it as the rest energy for large \( V \). It is seen that \( E_r = E_0/\rho \Omega_D \), so we finally get
\[ m_0 = \frac{D}{2} \tilde{m}. \]  
(C7)
One might also worry about the elastic energy of the container’s wall, which is different in the reference configuration—where the wall is subject to pressure $p(\rho)$—and the one with the source where the pressure on the wall is $p(\tilde{\rho}) \approx p(\rho) + c^2\tilde{\zeta}$. But this energy scales with the size of the container as $A^2\tilde{\zeta} \propto V^{-1/3}$ ($A$ the area of the wall), so it can be neglected for large volumes.

When more than one particle is present without overlap they do not interact and the total energy is the sum of the rest masses. (Due to the non-linearity of the energy there is interaction when particles overlap, but since $\Phi = 0$ outside particles there is no long range interaction—similar to the case of two rigid spherical charge distributions, each of vanishing total charge.)

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