Constrained Spacecraft Attitude Optimal Control via Successive Convex Optimization

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Abstract. Rapid attitude path planning is the key technique in autonomous spacecraft operation missions. An efficient method is proposed for energy-optimal spacecraft attitude control in presence of constraints. Firstly, Gauss pseudospectral method is utilized to discretize and transcribe the primal continuous problem to a nonlinear programming problem. Then a set of convexification techniques are used to convexify the nonlinear programming problem to a series of second-order cone programming problems, which can be solved iteratively by the interior-point method. A solution to the nonlinear programming problem is obtained as the iteration converges. Numerical results show the method could obtain a valid energy-optimal attitude control plan more rapidly than traditional methods.

Keywords: Spacecraft attitude; Maneuver path planning; Second order cone programming.

1. Introduction
Attitude maneuvers of spacecraft have been frequently executed in space missions. During the maneuvering, the sensitive instruments should be protected from the light of celestial objects, which specified pointing constraints for the spacecraft. Meanwhile, since the electrical energy is limited in space missions, an energy-optimal attitude control plan can extend the lifetime of spacecraft and increase mission assurance. An efficient algorithm to obtain an optimal attitude control without violation of constraints plays an important role in the implementation of spacecraft autonomy.

The constrained spacecraft attitude optimal control problem has been extensively studied. Existing frameworks for solving the control problem include geometric algorithms, artificial potential function methods, constraint monitor algorithms, randomized algorithms, semidefinite programming(SDP) based algorithms. For example, McInnes et al.[1] described the pointing constraints with an exponential barrier potential function, and obtained the feedback control law through the Lyapunov direct method. Less computing resources are required with the method, but singularities are possible to appear for the utilization of Euler angles. Sun et al.[2] converted the primal optimal control problem into a quadratic constraint quadratic programming, which is subsequently relaxed to semidefinite programming. Iterative rank minimization(IRM) algorithm is proposed to solve the SDP problems iteratively to obtain the energy-optimal control strategy. However, due to the involvement of a new matrix variable in the process of relaxation, the algorithm has low computational efficiency.

In this paper, a convex optimization based method is proposed to solve the constrained attitude optimal control problem rapidly. First, we discretize and transcribe the constrained attitude optimal control problem into a nonlinear programming(NLP) problem by application of the Gauss pseudo-spectral method. Then a set of convexification techniques are used to approximate the nonlinear programming problem by a second-order cone programming(SOCP) problem(a subclass of convex optimization problems), which can be quickly solved by the interior-point method. Successive convex
optimization (SCO) algorithm is then proposed to solve the SOCP successively and a solution of primal NLP is obtained as the algorithm converges.

2. Optimal Control Problem Formulation

The optimal attitude control problem researched in this paper aims to obtain a control torque which can drive the spacecraft to desired attitude with minimum energy consumption while satisfying various constraints over time interval \( t \in [t_0, t_f] \). The constraints consist of dynamics, kinematic, attitude pointing and boundary. Hence the optimal control problem can be formulated as follows.

\[
\begin{align*}
\min_{u(t)} & \int_{t_0}^{t_f} u^T \, u \, dt \\
\text{s.t.} & \quad \dot{\omega} = J^{-1} \left( -\omega^T J \omega + u \right) \\
& \quad \dot{q} = \frac{1}{2} Q \omega \\
& \quad \omega(t_0) - \omega_0 = 0, \quad \omega(t_f) - \omega_f = 0 \\
& \quad q(t_0) - q_0 = 0, \quad q(t_f) - q_f = 0 \\
& \quad \|q_l\| \leq u_{\text{max}}, \quad l = 1, 2, 3 \\
& \quad \|q_l\| \leq \omega_{\text{max}}, \quad l = 1, 2, 3 \\
& \quad q^T C_n q \leq 0, \quad m = 1, 2, ..., n
\end{align*}
\]

For the rigid spacecraft, the attitude dynamic and kinematic are expressed as Eqs. (2) and (3), where \( q = [q_1, q_2, q_3, q_4]^T \) is the unit quaternion describing the attitude of spacecraft, which have no singularity and satisfy the normalization constraint \( q^T q = 1 \). \( \omega = [\omega_1, \omega_2, \omega_3]^T \) is the spacecraft angular velocity in the body frame. \( u = [u_1, u_2, u_3]^T \) is the control torque. \( J = \text{diag}(J_1, J_2, J_3) \) represents the moment of inertia matrix of the spacecraft. \( \omega^* \) is the cross-product matrix of angular velocity \( \omega \).

During implementation, the magnitude of control torque and angular velocity need to be limited in a certain range as Eqs. (5) and (6). In addition, the initial and terminal conditions for attitude and angular velocity are determined with Eqs. (4).

The attitude pointing constraints is formulated as quadratic functions in Eqs. (7), where \( n \) represents the number of pointing constraints, and

\[
C_n = \begin{bmatrix}
x_n^T y_m - \cos \theta_m & (-x_n \times y_m)^T \\
-x_n \times y_m & x_n y_n^T + y_n x_n^T - (x_n^T y_m + \cos \theta_m) I_3
\end{bmatrix}
\]

where \( y_m \) represents the direction vector of the sensitive instrument in the body coordinates, \( x_n \) represents the direction vector of the bright celestial object in the inertial coordinates. Eqs.(7) mean the angle between \( x_n \) and \( y_m \) should not be less than \( \theta_m \).

3. Problem Convexification

The optimal control problem Eqs. (1) to (7) is difficult to solve because it is a non-convex problem for the nonlinear dynamic equations and the non-convex quadratic attitude constraints.

3.1. Lossless Convexification for Pointing Constraints

Let \( \tilde{C}_n = C_n + \lambda I_4 \), according to Eqs.(7) and the attitude quaternion characteristic \( q^T q = 1 \) we have

\[
\begin{align*}
q^T (C_n + \lambda I_4) q &= q^T C_n q + \lambda q^T q \\
&= q^T C_n q + \lambda \leq \lambda \\
q^T \tilde{C}_n q &\leq \lambda
\end{align*}
\]

which means Eqs.(9) represent the same set as Eqs.(8).
Assuming $\lambda_{\min}$ is the smallest eigenvalue of the symmetric matrix $C_m$ while $\lambda_{\max}$ is the largest eigenvalue, when $\lambda > -\lambda_{\min}$, it comes $\tilde{C}_m \in S^*_+$ and a convex set is defined by Eqs.(9). The eigenvalues of $C_m$ are connected with $\theta_n$. Without loss of generality, for $\theta_n \in (0, \pi)$, one has

$$-2 < \lambda_{\min} \leq q^TC_nq \leq \lambda_{\max} < 2$$

(10)

Let $\tilde{C}_m = C_m + 2I_4$, then we obtain the convex constraints in place of Eqs.(7) without any loss of accuracy

$$q^T\tilde{C}_m q \leq 2, \quad m=1,2,\ldots,n$$

(11)

the convex quadratical constraints could be written as second-order cone constraints directly

$$\left| C_m^{1/2} q \right| \leq \sqrt{2}, \quad m=1,2,\ldots,n$$

(12)

In fact, the non-convexity is not eliminated but transfers into the normalization constraint $q^Tq = 1$ which is inside the attitude quaternion kinematics and satisfied naturally.

3.2. Linear Approximation for Dynamics and Kinematics

The spacecraft attitude dynamics equations(2) and(3) have the following general form

$$\dot{x} = f(x,u,t)$$

(13)

where $x$ is the state vector comprising attitude quaternion $q$ and angle velocity $\omega$.

Above nonlinear equation can be linear approximated at a known reference trajectory $x^r(t)$ and $u^r(t)$ by the method of Taylor expansion

$$\dot{x} \approx f_{\text{linear}}(x,u,t) = f(x^r,u^r,t) + f_x(x^r,u^r,t)(x-x^r) + f_u(x^r,u^r,t)(u-u^r)$$

(14)

where $f_x(x^r,u^r,t)$ is the derivative of $f(x,u,t)$ with respect to $x$ at $(x^r,u^r)$, $f_u(x^r,u^r,t)$ is the derivative of $f(x,u,t)$ with respect to $u$ at $(x^r,u^r)$.

4. Gauss Pseudo-spectral Transcription

There are tow main approaches to handle optimal control problem: indirect methods and direct methods. Indirect method are based on the Pontryagin’s maximum principle, which involves a multiple-point boundary-value problem. Direct methods normally construct a finite-dimensional nonlinear programming problem through proper discretization of the primal optimization problem.

Gauss pseudo-spectral method belongs to the direct methods with following advantages: 1) quasi exponential convergence of the NLP solution to the optimization control problem solution as the number of nodes increases, 2) the optimality conditions of the NLP is equivalent to the discretized optimality conditions of the continuous control problem, 3) no Runge phenomenon.

The method uses global polynomial to approximate the dynamic equations at a set of Legendre-Gauss(LG) collocation points. The time interval $t \in [t_0, t_f]$ is projected to $\tau \in [-1, 1]$ for the roots of LG polynomial is between -1 and 1

$$\tau = \frac{2\tau - t_i - t_f}{t_f - t_i}$$

(15)

Assuming there are $N$ nodes employed in the interval $\tau \in [-1, 1]$, which is defined as $\tau_i, \quad i=1,2,\ldots,N$, the middle $N/2$ nodes are LG points and $\tau_{\frac{N}{2}} = -1, \tau_{\frac{N}{2}+1} = 1$. Based on Lagrange polynomials, we approximate the state function $x(\tau)$ by the first $N-1$ nodes

$$x(\tau) \approx \mathbf{x}(\tau) = \sum_{i=1}^{N-1} x_i \mathbf{L}^{N-1}_i(\tau)$$

(16)

and approximate the control function $u(\tau)$ by the middle $N/2$ nodes
\[ u(t) \approx \overline{u}(t) = \sum_{i=1}^{N} \overline{u}(\tau_i) L_i^{N-2}(t) \]  

where \( \overline{x}(t) \) and \( \overline{u}(t) \) are fit functions, and

\[ L_i^{N-1}(r) = \prod_{j=1, j\neq i}^{N-1} \frac{r - \tau_j}{\tau_i - \tau_j}, \quad L_i^{N-2}(r) = \prod_{j=2, j\neq i}^{N-1} \frac{r - \tau_j}{\tau_i - \tau_j} \]

is the nodal basis functions. The differential equation of Eqs. (16) is

\[ x(t) = \overline{x}(t) = \sum_{i=1}^{N} \overline{u}(\tau_i) L_i^{N-1}(t) \]

where \( \overline{L}_i^{N-1}(t) \) can be represented by differential approximation matrix \( D \in \mathbb{R}^{N \times N-1} \), whose elements are

\[ D_{ik} = L_i^{N-1}(\tau_k) = \frac{1}{2} \int_{\tau_i}^{\tau_k} L_i^{N-2}(\tau) \overline{u} d\tau, \quad k = 2, 3, ..., N-1 \]

According to Eqs. (2) and (3), the attitude dynamic and kinematic constraints at LG points \( i = k, \ k = 2, 3, ..., N-1 \) is described as

\[ \frac{2}{t_k - t_0} \sum_{i=1}^{N} D_{ik} \overline{u}(\tau_i) = \overline{x}(\tau_k) = \overline{q}(\tau_k) = J^{-1}(-\overline{\omega}(\tau_i) J \overline{\omega}(\tau_i) + \overline{u}(\tau_i)) \]

\[ \frac{1}{2} \overline{q}(\tau_k) \overline{\omega}(\tau_k) \]

(21)

The terminal constraints and performance index function can be approximated as

\[ \overline{x}(\tau_s) = \overline{x}(\tau_i) + \frac{t_s - t_0}{2} \sum_{k=2}^{N} w_k \overline{x}(\tau_k) = x_i = \left[ \overline{a}_i \right] \left[ \overline{q}_i \right] \]

\[ \int_{t_0}^{t_s} u^T u dt \approx \frac{t_s - t_0}{2} \sum_{k=2}^{N} w_k \overline{u}^T (\tau_k) \overline{u} (\tau_k) \]

\[ \overline{w}_k = \int_{\tau_1}^{\tau_k} \prod_{i=1, i\neq k}^{N} \frac{\tau - \tau_i}{\tau_k - \tau_i} d\tau \]

(24)

Combining Gauss pseudo-spectral transcription with the previous convexification techniques, we transformed the optimal control problem into a NLP, which is also a SOCP problem

\[ \min_{\overline{a}(\tau_k), \overline{q}(\tau_k), \overline{u}(\tau_k)} s \]

\[ \text{s.t.} \quad \left\| \left( \begin{array}{c} \frac{t_s - t_0}{2} W \\ \overline{a} \end{array} \right) \right\|_2^2 \leq s \]

\[ \frac{2}{t_k - t_0} D_{ik} \left[ \begin{array}{c} \overline{a}_i \\ \overline{q}_0 \end{array} \right] + \frac{2}{t_k - t_0} \sum_{k=2}^{N} D_{ik} \left[ \begin{array}{c} \overline{a} \\ \overline{q}(\tau_i) \end{array} \right] = f_{\text{linear}} (\overline{x}, \overline{u}, \tau_k) \]

\[ \left[ \begin{array}{c} \overline{a}_0 \\ \overline{q}_0 \end{array} \right] + \frac{t_s - t_0}{2} \sum_{k=2}^{N} w_k f_{\text{linear}} (\overline{x}, \overline{u}, \tau_k) = \left[ \begin{array}{c} \overline{a}_i \\ \overline{q}_i \end{array} \right] \]

\[ \left\| \overline{q}(\tau_k) \right\|_2 \leq u_{\text{max}}, \quad l = 1, 2, 3 \]

\[ \left\| \overline{\omega}(\tau_k) \right\|_2 \leq \omega_{\text{max}}, \quad l = 1, 2, 3 \]

\[ \left\| \overline{C} \overline{q}(\tau_k) \right\| \leq \sqrt{2}, \quad m = 1, 2, ..., n \]

where Eqs.(25), (26) replace the primal quadratical index function(23) equivalently to adapt to SOCP frame, and
5. Successive Convex Optimization Algorithm

The second-order cone programming problem (25) to (31) could be solved by most interior-point methods instantly. However, the solution of SOCP is far from the original problem because of the linear approximation in Eqs.(14). A iterative approach which can eliminate the error from approximation is proposed as follows

1) Set \( r = 0 \), provide an initial reference trajectory \( x^{(0)}(t), u^{(0)}(t) \), project the trajectory to interval \( \tau \in [-1, 1] \) and obtain \( \bar{x}^{(0)}(\tau_k), \bar{u}^{(0)}(\tau_k) \) at each LG point.

2) Set \( \bar{x}(\tau_k) = \bar{x}^{(r)}(\tau_k), \bar{u}(\tau_k) = \bar{u}^{(r)}(\tau_k) \) to determine the \( f_{\text{linear}}(\bar{x}, \bar{u}, \tau_k) \) in Eqs.(27)and(28), solve the SOCP problem and obtain a solution \( \bar{x}^{(r+1)}(\tau_k), \bar{u}^{(r+1)}(\tau_k) \).

3) Check the following convergence condition

\[
\max_k \left\| \bar{x}^{(r+1)}(\tau_k) - \bar{x}^{(r)}(\tau_k) \right\|_2 \leq \varepsilon
\]

where \( \varepsilon \) is a user-defined accuracy. If Eqs.(33) is true, execute step 4), or set \( r = r+1 \) and back to step 2).

4) Find a solution \( \bar{x}^{(r+1)}(\tau_k), \bar{u}^{(r+1)}(\tau_k) \).

The convergence condition means the solution at current iteration is almost coincide with the reference trajectory which makes the Taylor expansion reasonable.

The method solves convex optimization problems successively, so it is called successive convex optimization(SCO) algorithm. Then we focus on the same simulation case, using various methods to solve it to test the SCO algorithm.

In the case the spacecraft is retargeting its telescope while avoiding tow attitude pointing constraints.

The simulation parameters are given in Table 1.

| Table 1. Simulation parameters. |
|---------------------------------|
| Parameter Value |  |
| \( J \) | \( \text{diag}(50, 50, 50) \text{kg} \cdot \text{m}^2 \) |
| \( [t_0, t_f] \) | \( [0, 30] \text{s} \) |
| \( u_{\text{max}} \) | 3rad/m² |
| \( o_{\text{max}} \) | 0.3rad/s |
| \( q_0 \) | \([-0.470, 0.187, 0.735, 0.450]\) |
| \( q_f \) | \([0.382, -0.592, 0.675, 0.215]\) |
| \( y_m, m = 1, 2 \) | \( [0, 1, 0]\) |
| \( x_1, O_1 \) | \( [-0.2684, 0.2520, 0.9298, 40\text{deg}] \) |
| \( x_2, O_2 \) | \( [-0.8074, 0.5866, 0.0628, 20\text{deg}] \) |

We solve the spacecraft attitude control problem by SCO algorithm, IRM algorithm, general NLP method respectively. The NLP method is solved by seqential quadratic programming(SQP) algorithm and the IRM algorithm solves a SDP problem at each iteration. A linear trajectory from initial state to terminal state is chosen to initialize the SCO algorithm and SQP algorithm. The characteristic and performance of various algorithms are contrasted in Table 2. All the simulation have been run on a
desktop having an Intel Core i5-10400F processor with a clock frequency of 2.90GHz and the MOSEK optimization tools are used to solve all convex optimization problems.

Table 2. Comparison of the algorithms.

| Algorithm | Nodes | Average time for each iteration | Number of iterations | Total time | Performance index |
|-----------|-------|---------------------------------|----------------------|------------|-------------------|
| SCO(SOCP) | 10    | 0.20s                           | 18                   | 3.60s      | 10.86             |
|           | 30    | 0.24s                           | 26                   | 6.24s      | 11.87             |
|           | 50    | 0.27s                           | 25                   | 6.75s      | 11.91             |
| IRM(SDP)  | 6     | 79.6s                           | 20                   | 1592.0s    | 11.93             |
|           | 8     | 421.4s                          | 20                   | 8428.0s    | 11.31             |
|           | 10    | 2103.3s                         | 20                   | 42066.0s   | 12.16             |
| SQP       | 10    | /                               | /                    | 5.3s       | 10.84             |
|           | 30    | /                               | /                    | 25.1s      | 11.88             |
|           | 50    | /                               | /                    | 103.3s     | 11.91             |

SQP algorithm have found similar solution to SCO algorithm’s solution and both are fast when the number of node is small. However, the SQP algorithm shows efficiency decrease as the problem dimension increases while SCO stays efficient. Meanwhile, IRM algorithm is much more slow than SCO because of the involvement of a new matrix variable enormously increasing the dimension of problem, which is not necessary for SCO algorithm.

The time histories of control torque and angle velocity are shown in Figure 1. Figure 2 presents the trajectory of the telescope pointing vector in the 2-dimensional space.

**Figure 1.** Time histories of control torque and angle velocity for SCO(left) and IRM(right).

**Figure 2.** Trajectory of telescope in the 2-dimensional space for SCO(left) and IRM(right).

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6. Conclusion
This paper proposes a method to convert the constrained spacecraft attitude optimal control problem into a second-order cone programming problem based on a reference trajectory, which is successively solved by the interior-point method to obtain the solution of the original problem. Numerical simulation shows that the method is effective on the optimal control problem and more rapid than certain traditional algorithms.

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