Smooth Tests for Normality in ANOVA

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Abstract

The normality assumption for errors in the Analysis of Variance (ANOVA) is common when using ANOVA models. But there are few people to test this normality assumption before using ANOVA models, and the existent literature also rarely mentions this problem. In this article, we propose an easy-to-use method to testing the normality assumption in ANOVA models by using smooth tests. The test statistic we propose has asymptotic chi-square distribution and our tests are always consistent in various different types of ANOVA models. Discussion about how to choose the dimension of the smooth model (the number of the basis functions) are also included in this article. Several simulation experiments show the superiority of our method.

Keywords: ANOVA, estimation effect, smooth tests.

1. Introduction

Analysis of Variance (ANOVA) is a classic model with a long history. While the analysis of variance reached fruition in the 20th century, antecedents extend centuries into the past according to [1], and this model is also quite common when dealing with practical problems now. As a base for F-test, ANOVA usually requires normality assumption, see [2], Chapter 10 in [1], and Chapter 4 of [3]. Although there are some researchers already have few articles regarding the non-normal ANOVA, their results are restricted and cannot be applied in practice widely (for example, [4] points...
out the kurtoses of the distribution must be known for $F$ test if the distribution is not normal). As a consequence, determining whether a set of data satisfying the normality assumption is anything but trivial. Traditionally, researchers usually use the residuals to run the Jarque-Bera test (see [5]), quantile-quantile plot (QQ plot), Kolmogorov-Smirnov test, and so on (see [6]). These methods are either not robust or too involved, and there is no specific method for normality tests in ANOVA until now to the best of our knowledge. In this article, we propose a specialized smooth test for the normality assumption of ANOVA in the spirits of [7], and discuss how to use our smooth tests in different types of ANOVA in detail.

Indeed, the smooth test often serves as a powerful tool in testing whether a distribution coincides with a known distribution. Its basic idea is that: if the random variable $Z$ is distributed as $F$, then $F(Z)$ is uniformly distributed on $[0, 1]$. Therefore, when the true distribution function of $Z$ is not $F$, the $F(Z)$ is not uniformly distributed anymore. Hence, any type of smooth tests is equivalent to testing whether a distribution is uniformly distributed on $[0, 1]$. Following this way, we construct the test statistic, under the null hypothesis, that is the error items are independently distributed as some Gaussian distribution, is the likelihood ratio statistic. This implies our test is the optimal test.

Our contributions are as follows: (i) We propose different smooth test statistics for different types of one-way fixed effects ANOVA models, and derive their asymptotic properties. These contents are discussed in Section 2 and 3 in detail; (ii) For the tuning parameter, the number of the basis functions, we apply a modification of Schwarz’s selection rule in Section 4 to choosing them reasonably; (iii) Other common types of ANOVA models in Section 6, such as random effects models or two-way models, are showed that can be reduced to the cases in Section 4. The simulation results for numerically exploring the robustness and consistency of our tests in Section 5 also verify our theory.

2. Smooth Test Statistic

It is a standard assumption of errors is that we assume that observation errors (or measurement errors) $\varepsilon_i = \sigma e_i$ i.i.d. $N(0, \sigma^2)$ for some positive constant $\sigma^2$. To motivate our test, consider a transformed random variable $Z = \Phi(e)$, with $\Phi(\cdot)$ denoting the CDF of standard normal variable. Note
that the CDF of $Z$ under the null hypothesis of normality is given by

$$H(z) = P(Z \leq z) = P(e \leq \Phi^{-1}(z)) = \Phi(\Phi^{-1}(z)) = z.$$ 

Or, equivalently, the probability density function (PDF) of $Z$ under $H_0$ is given by $h(z) = 1$ for $z \in [0, 1]$. That is, $Z \sim U[0,1]$ under $H_0$. Under the alternative hypothesis $H_1$, the PDF $h(z)$ differs from 1. The distinct behavior of $Z$ under $H_0$ and $H_1$ provides the very basis for our Neyman’s type smooth test. 

[7] considered the following smooth alternative to the uniform density:

$$h(z) = c(\theta) \exp \left[ \sum_{k=1}^{K} \theta_k \pi_k(z) \right], \quad 0 < z < 1, \quad (1)$$

where $c(\theta)$ is a constant of integration depending on $\{\theta_1, \theta_2, \ldots, \theta_K\}$, and $\{\pi_k(z)\}_{k=0}^\infty$ is an orthonormal system in $L_2[0,1]$ with $\pi_0(z) = I_{[0,1]}$ and

$$\int_0^1 \pi_k(z) \pi_l(z) \, dz = \delta_{kl}, \quad \text{where} \quad \delta_{kl} = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases} \quad (2)$$

The null hypothesis that $Z \sim U[0,1]$ can then be tested by testing $\theta_1 = \theta_2 = \cdots = \theta_K = 0$ in (1).

If $\{\varepsilon_i\}$ and $\sigma$ were both known, the smooth test statistic for testing $H_0$ has the following quadratic form:

$$\Psi^2_K = \sum_{k=1}^{K} \left( \frac{1}{n} \sum_{i=1}^{n} \pi_k(Z_i) \right)^2, \quad (3)$$

where $K$ is some given positive integer and $Z_i = \Phi(e_i)$. Under $H_0$, using the properties of $\pi_k(\cdot)$, it is straightforward to show that the scaled infeasible smooth test statistic $N \Psi^2_K$ converges asymptotically to a $\chi^2$-distribution with $K$ degrees of freedom (denoted as $\chi^2_K$). Furthermore, if the parameter space $\Theta$ for $\theta$ is tight and big enough, by Cauchy-Schwartz inequality, the likelihood
function $L(\theta)$ satisfies
\[
-2 \log \left[ \frac{L(0)}{\sup_{\theta \in \Theta} L(\theta)} \right] = 2n \sup_{\theta \in \Theta} \sum_{k=1}^{K} \theta_k \left( \frac{1}{n} \sum_{i=1}^{n} \pi_k(Z_i) \right)
\]
\[
= 2n \sup_{\theta \in \Theta} \theta_k / \sum_{i=1}^{n} \pi_k(Z_i) \left[ \sum_{k=1}^{K} \theta_k^2 \right]^{1/2} \left[ \sum_{k=1}^{K} \left( \frac{1}{n} \sum_{i=1}^{n} \pi_k(Z_i) \right)^2 \right]^{1/2}
\]
\[
\propto \sum_{k=1}^{K} \left( \frac{1}{n} \sum_{i=1}^{n} \pi_k(Z_i) \right)^2 = \Psi_K^2.
\]
Thus, it is not hard to see this test statistic is a likelihood ratio test statistic, hence it is the optimal test.

In most cases, however, we do not know the value of $\epsilon_i$ exactly. So the estimations of them should be used. If we obtain reasonable estimators $\hat{\epsilon}_i$ and $\hat{\sigma}$, then we can use the approximate value of $Z_i$, i.e.
\[
\hat{Z}_i = \Phi(\hat{\epsilon}_i) = \Phi \left( \frac{\hat{\epsilon}_i}{\hat{\sigma}} \right)
\]
With $\{\hat{Z}_i\}$ at hand, it is tempting to construct a feasible smooth test statistic of the following form:
\[
\sum_{k=1}^{K} \left( \frac{1}{n} \sum_{i=1}^{n} \pi_k(\hat{Z}_i) \right)^2.
\]
Nonetheless, due to the estimation of $\epsilon_i$, the transformed random variable $\{\hat{Z}_i\}$ may neither independent nor identically distributed. Therefore, it is crucial to study the asymptotic representation of $n^{-1} \sum_{i=1}^{n} \pi_k(\hat{Z}_i)$ in terms of the original sequence $\{\epsilon_i\}$. In sharp comparison with $\Psi_K^2$ in (3), using $\hat{Z}_i$ has non-negligible estimation effects, as demonstrated in the later Theorems. Consequently, the presence of estimation effect invalidates the above form of test statistic and requires to use a different normalizing matrix to restore the $\chi^2_K$ distribution.

3. One-Way Fixed Effects Models

The one-way fixed effects model is the most common model in ANOVA, so it will be discussed by us in detail. Specifically, it can be considered in three cases.
3.1. The same mean and the same variance

In this case, we observe
\[ Y_{ij} = \mu + \varepsilon_{ij} = \mu + \sigma e_{ij}, \quad i = 1, \ldots, N_j, \quad j = 1, \ldots, J, \] (4)

with \( e_{ij} \) i.i.d. \( F(0, 1) \) where \( \mu \) is the common mean of each group, \( N_j \) represents the number of individuals for each group \( 1 \leq j \leq J \), with \( J \) representing the total number of groups, and \( \varepsilon_{ij} = \sigma e_{ij} \) represents the independent and identically distributed random errors. Denote \( N = \sum_{j=1}^{J} N_j \).

We have to estimate \( \mu \) and \( \sigma^2 \). Note that under the null, \( Y_{ij} \) i.i.d. \( N(\mu, \sigma^2) \).

Let \( \hat{\mu} = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} Y_{ij} \) and \( \hat{\sigma}^2 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (Y_{ij} - \hat{\mu})^2 \) denote the sample mean and standard deviation of \( Y \). Let \( \hat{e}_{ij} = (Y_{ij} - \hat{\mu})/\hat{\sigma} \) and define
\[ \hat{Z}_{ij} = \Phi(\hat{e}_{ij}), \]
then we can analyze the asymptotic linear expression of \( N^{-1} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) \).

For convenience of our asymptotic analysis, we impose the following assumptions.

A.1 \( \{e_{ij}\} \) are independent and identically distributed with the continuous CDF \( F(x) \).

A.2 For \( k = 1, 2, \ldots, K \), \( \pi_k(\cdot) \) are two times differentiable with derivatives \( \pi'_k(\cdot) \) and \( \pi''_k(\cdot) \), and they are both bounded.

Introduce two constants
\[ c_{1k} = \int_{0}^{1} \pi_k(z) \Phi^{-1}(z) \, dz, \quad c_{2k} = \int_{0}^{1} \pi_k(z) (\Phi^{-1}(z))^2 \, dz. \]

**Theorem 1.** Suppose Assumptions A.1 and A.2 hold. Then, under the null hypothesis \( H_0 : e \sim \Phi(x) \), for \( k = 1, \ldots, K \),
\[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left\{ \pi_k(Z_{ij}) - c_{1k} e_{ij} - \frac{c_{2k}}{2} (e_{ij}^2 - 1) \right\} + R_N, \]
with \( R_N = o_p \left( \frac{1}{\sqrt{N}} \right) \) as \( N \to \infty \).
It is clear from the proof of Theorem 1 that, in the decomposition of \( N^{-1} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) \), the term
\[
c_1 k \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} e_{ij}
\]
represents the estimation effect due to estimating \( \mu \), while the term
\[
c_2 k \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \frac{1}{2} (e_{ij}^2 - 1)
\]
represents the estimation effect due to estimating \( \sigma^2 \). Both terms are non-negligible and have to be considered in recovering the \( \chi^2_K \) distribution of feasible smooth test based on \( N^{-1} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) \) for \( k = 1, \ldots, K \).

With the assistance of Theorem 1, we can construct asymptotic normality of the \( K \)-dimensional vector
\[
\frac{1}{\sqrt{N}} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) = \sqrt{N} \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_1(\hat{Z}_{ij}), \ldots, \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_K(\hat{Z}_{ij}) \right]^{\top}
\rightarrow_d N(v, \Sigma),
\]
where the components of \( v \) and \( \Sigma \) are
\[
v_k = \text{E} \left\{ \pi_k(Z) - c_{1k} e - \frac{C_{2k}}{2} (e^2 - 1) \right\} = 0,
\]
\[
\sigma_{kl} = \text{E} \left\{ \pi_k(Z) - c_{1k} e - \frac{C_{2k}}{2} (e^2 - 1) \right\} \left\{ \pi_l(Z) - c_{1l} e - \frac{C_{2l}}{2} (e^2 - 1) \right\} = \delta_{kl} - c_{1k} c_{1l} - \frac{1}{2} c_{2k} c_{2l}
\]
respectively. Therefore, our proposed feasible smooth test statistic is given by
\[
\hat{\Psi}_K^2 = \left( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \right)^{\top} \Sigma^{-1} \left( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \right).
\]
By the asymptotic normality of \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \), we can give the good asymptotic propriety of our test statistic.
Corollary 1. Suppose Assumptions A.1-A.2 hold. Then, under the null hypothesis,

\[ N \hat{\Psi}_K^2 \to_d \chi_K^2, \]

as \( N \to \infty \).

By the result of Corollary 1, an asymptotic \( \chi^2 \) test for normality in ANOVA can be constructed using the smooth statistic \( N \hat{\Psi}_K^2 \). Specifically, an asymptotic level \( \alpha \) test for the null hypothesis is obtained by rejecting \( H_0 \) whenever

\[ \hat{\Psi}_K^2 > N^{-1} \chi_{K,1-\alpha}^2, \]

where \( \chi_{K,1-\alpha}^2 \) denotes the \((1 - \alpha)\)-th quantile of the \( \chi_K^2 \) distribution.

Under the alternative hypothesis \( H_1 \) that \( e \) follows a non-normal distribution with CDF given by \( F(x) \), with \( F(\cdot) \) being different from \( \Phi(\cdot) \), the term \( N^{-1} \sum_{j=1}^J \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) \) is not centered at zero. In the following theorem, we obtain the asymptotic representation of the latter term under \( H_1 \). To this end, we assume that under \( H_1 \), \( F(\cdot) \) admits a density \( f(\cdot) \). Denote

\[ d_{1k} = E[\hat{\pi}_k(Z)\phi(e)] = \int_{-\infty}^{\infty} \hat{\pi}_k(\Phi(x)) \phi(x) f(x) \, dx, \]

and

\[ d_{2k} = E[\hat{\pi}_k(Z)\phi(e)e] = \int_{-\infty}^{\infty} \hat{\pi}_k(\Phi(x)) \phi(x) xf(x) \, dx. \]

Theorem 2. Suppose Assumptions A.1-A.2 hold. Then, under the alternative hypothesis \( H_1 : e \sim F(x) \), for \( k = 1, \ldots, K \),

\[
\frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) = \frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \left\{ (\pi_k(Z_{ij}) - E[\pi_k(Z)]) - d_{1k} - \frac{d_{2k}}{2} (e_{ij}^2 - 1) \right\}
\]

\[ + E[\pi_k(Z)] + o_p \left( \frac{1}{\sqrt{N}} \right), \]

as \( N \to \infty \). Furthermore,

\[ \hat{\Psi}_K^2 \to_p a^\top \Sigma^{-1} a \]

and

\[ \sqrt{N} \left( \hat{\Psi}_K^2 - a^\top \Sigma^{-1} a \right) \to_d N(0, a^\top \Sigma^{-1} \Xi \Sigma^{-1} a), \]

as \( N \to \infty \).
where \( a = (E\pi_1(Z), \cdots, E\pi_K(Z))^\top \) and the elements in \( \Xi \) are:

\[
\xi_{kl} = E[\pi_k(Z)\pi_l(Z)] - d_{1k}d_{1l} + \frac{1}{2} [a_k d_{2l} + a_l d_{2k}]
+ \frac{1}{2} [d_{1k}d_{2l} + d_{1l}d_{2k}] E e^3 - \frac{d_{2k}d_{2l}}{4} [5 - E e^4].
\]

Note that \( a_k = E[\pi_k(Z)] = \int_{-\infty}^\infty \pi_k(\Phi(z)) f(z) dz = \int_0^1 \pi_k(z) \frac{f(\Phi^{-1}(z))}{\phi(\Phi^{-1}(z))} dz \)
and the elements in \( \Xi \) are:

\[
\xi_{kl} = E[\pi_k(Z)\pi_l(Z)] - d_{1k}d_{1l} + \frac{1}{2} [a_k d_{2l} + a_l d_{2k}]
+ \frac{1}{2} [d_{1k}d_{2l} + d_{1l}d_{2k}] E e^3 - \frac{d_{2k}d_{2l}}{4} [5 - E e^4].
\]

As long as \( f(\cdot) \neq \phi(\cdot) \), then \( a_k = 0 \), thus \( a^\top \Sigma^{-1} \Xi \Sigma^{-1} a \neq 0 \), it is seen that
the power can be made arbitrarily close to 1 as \( N \to \infty \). So this test is consistent.

### 3.2. Different Means and the Same Variance

We can consider the following as an extension of standard the fixed effects one-way ANOVA

\[
Y_{ij} = \mu_j + \epsilon_{ij}, \quad i = 1, \ldots, N_j, \quad j = 1, \ldots, J,
\]

where \( \mu_j \) represents the (potentially different) means of each group. Under
the null, \( Y_{ij} \) i.i.d. \( N(\mu_j, \sigma^2) \). Let \( \hat{\mu}_j = N_j^{-1} \sum_{i=1}^{N_j} Y_{ij} \). Then, \( Z_{ij} = \Phi(\hat{e}_{ij}) \)
is defined as before except now \( \hat{\sigma}^2 = N^{-1} \sum_{j=1}^J \sum_{i=1}^{N_j} (Y_{ij} - \hat{\mu}_j)^2 \) and \( \hat{e}_{ij} = (Y_{ij} - \hat{\mu}_j)/\hat{\sigma} \).

**Theorem 3.** Suppose Assumptions A.1-A.2 hold. Then, under the null hypothesis \( H_0 : e \sim \Phi(x) \), for \( k = 1, \ldots, K \),

\[
\frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \pi_k(Z_{ij}) = \frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \left\{ \pi_k(Z_{ij}) - c_{1k} e_{ij} - \frac{c_{2k}}{2} (e_{ij}^2 - 1) \right\} + \mathcal{R}_N,
\]

with \( \mathcal{R}_N = o_p \left( \frac{1}{\sqrt{N}} \right) \) as \( \min\{N_1, \ldots, N_J\} \to \infty, J = o(N^{1/2}) \), and for any \( j, \ell \in \{1, \ldots, J\} \),

\[
N_j/N_\ell \to p_{j,\ell} \in (0, \infty).
\]
Obviously, the asymptotic normality of vector \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \) is totally same as that in Theorem 1. So the test statistic is also \( \hat{\Psi}_K^2 \), and Corollary 1 is guaranteed again. Similarly, we give the following theorem corresponding to theorem 2, which establishes the asymptotic decomposition of \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \) under the alternative hypothesis when \( \mu_j \) are not all equal.

**Theorem 4.** Suppose Assumptions A.1-A.2 hold. Then, under the null hypothesis \( H_1 : \varepsilon \sim F(x) \), for \( k = 1, \ldots, K \),

\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left\{ \left( \pi_k(Z_{ij}) - E[\pi_k(Z)] \right) - d_{1k}e_{ij} - \frac{d_{2k}}{2} \left( e_{ij}^2 - 1 \right) \right\} + E[\pi_k(Z)] + o_p \left( \frac{1}{\sqrt{N}} \right),
\]

as \( \min\{N_1, \ldots, N_J\} \to \infty \), \( J = o \left( N^{1/2} \right) \), for any \( j, \ell \in \{1, \ldots, J\} \), \( N_j/N_\ell \to p_{j,\ell} \in (0, \infty) \). Furthermore,

\[
\hat{\Psi}_K^2 \to_p a^\top \Sigma^{-1} a
\]

and

\[
\sqrt{N} \left( \hat{\Psi}_K^2 - a^\top \Sigma^{-1} a \right) \to_d N(0, a^\top \Sigma^{-1} \Xi \Sigma^{-1} a),
\]

where \( a = (E\pi_1(Z), \cdots, E\pi_K(Z))^\top \) and the elements in \( \Xi \) are:

\[
\xi_{kl} = E[\pi_k(Z)\pi_l(Z)] - d_{1k}d_{1l} + \frac{1}{2} \left[ a_{k} d_{2l} + a_{l} d_{2k} \right] + \frac{1}{2} \left[ d_{1k}d_{2l} + d_{1l}d_{2k} \right] E e_3 - \frac{d_{2k}d_{2l}}{4} \left[ 5 - E e_4 \right].
\]

From Theorem 3 and Theorem 4, the asymptotic properties of our test is exactly the same as that in Section 3.1. Our test is always consistent no matter the true mean in each group are equal or not. However, it is worthy to note that the same decomposition requires that sample size for each group increases to infinity, while for the equal mean case only the total sample size going to infinity suffices.
3.3. the Same Mean and Different Variances

Now the model follows:

\[ Y_{ij} = \mu + \varepsilon_{ij} = \mu + \sigma_j e_{ij}, \quad i = 1, \ldots, N_j, \quad j = 1, \ldots, J. \]

Under the null, \( Y_{ij} \) i.i.d. \( N(\mu, \sigma_j^2) \) for all \( j = 1, \ldots, J \). At this time, we let:

\[ \hat{\mu} = \frac{1}{J} \sum_{j=1}^{J} \frac{1}{N_j} \sum_{i=1}^{N_j} Y_{ij}, \quad \hat{\sigma}_j^2 = N_j^{-1} \sum_{i=1}^{N_j} (Y_{ij} - \hat{\mu})^2 \]

Furthermore, let \( \hat{e}_{ij} = (Y_{ij} - \hat{\mu})/\hat{\sigma}_j \) and define \( \hat{Z}_{ij} = \Phi(\hat{\varepsilon}_{ij}/\hat{\sigma}_j) \). The estimator of mean is different from that in Section 3.1 is due to the structures in different group are not the same though the means are identical.

Theorem 5. Suppose Assumptions A.1-A.2 hold. Then, under the null hypothesis \( H_0 : e \sim \Phi(x) \), for \( k = 1, \ldots, K \),

\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left\{ \pi_k(Z_{ij}) - \left[ \sum_{\ell=1}^{J} \frac{c_{1k}p\ell}{\sigma_\ell} \frac{\sigma_j e_{ij}}{q_j} - \frac{c_{2k}}{2} (e_{ij}^2 - 1) \right] \right. \\
+ \mathcal{R}_N,
\]

with \( \mathcal{R}_N = o_p\left(\frac{1}{\sqrt{N}}\right) \) as \( \min\{N_1, \ldots, N_J\} \to \infty, J = o\left( N^{1/2} \right) \), and for any \( j, \ell \in \{1, \ldots, J\} \),

\[ N_j/N_\ell \to p_{j,\ell} \in (0, \infty), \]

where \( N_j/N \to p_j \in [0, \infty) \), and \( JN_j/N \to q_j \in (0, \infty) \).

It is valuable to note that the decomposition in Theorem 5 is slightly different from that in previous theorems, so is the asymptotic normality of the vector \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \), which means we should use a different test statistic. As a matter of fact, in this case, by central limit theorem,

\[
\frac{1}{\sqrt{N}} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) = \sum_{j=1}^{J} \frac{\sqrt{p_j}}{\sqrt{N_j}} \sum_{i=1}^{N_j} \left\{ \pi_k(Z_{ij}) - \left[ \sum_{\ell=1}^{J} \frac{c_{1k}p\ell}{\sigma_\ell} \frac{\sigma_j e_{ij}}{q_j} - \frac{c_{2k}}{2} (e_{ij}^2 - 1) \right] \right. \\
+ o_p\left(\frac{1}{\sqrt{N}}\right) \to_d \sum_{j=1}^{J} \sqrt{p_j}Z_j,
\]
where $Z_1, \ldots, Z_J$ are independently distributed as

$$Z_j \sim N \left(0, \Omega^{(j)}\right).$$

And using $Ee^3 = Ee = 0$ under $H_0$, it is not hard to give the components in $\Omega^{(j)}$ are

$$\omega_{kl}^{(j)} = E\left\{ \pi_k(Z) - \left[ \sum_{\ell_1=1}^J \frac{c_{1k}p_{\ell_1}}{\sigma_{\ell_1}} \right] \frac{\sigma_{j}e}{q_j} - \frac{c_{2k}}{2} (e^2 - 1) \right\}$$

$$= \delta_{kl} - \frac{2c_{1k}c_{1l}p_{\ell}q_j}{q_j} \sum_{\ell=1}^J \frac{p_{\ell}p_{\ell}}{\sigma_{\ell}^2} + \frac{c_{1k}c_{1l}p_{\ell}^2}{q_j^2} \left( \sum_{\ell=1}^J \frac{p_{\ell}^2}{\sigma_{\ell}^2} \right)^2 - \frac{c_{2k}c_{2l}}{2},$$

and here we also allow $J$ goes to infinity. And the independence of $Z_1, \ldots, Z_J$ implies

$$\frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \rightarrow_d N \left(0, \sum_{j=1}^J p_j \Omega^{(j)}\right).$$

Replacing $p_j$ and $q_j$ by $\hat{p}_j = N_j/N$ and $\hat{q}_j = JN_j/N$ respectively, we can get an estimate of $\Omega^{(j)}$. Denote it as $\hat{\Omega}^{(j)}$. In this case, the proper statistic can be proposed as follows

$$\hat{\Psi}_K^2 := \left( \frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \right)^\top \left[ \sum_{j=1}^J p_j \hat{\Omega}^{(j)} \right]^{-1} \left( \frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \right).$$

It is easy to see, in this case, Corollary 1 also holds. Similarly, it can be shown that

$$\sqrt{N} \left[ \hat{\Psi}_K^2 - a^\top \left( \sum_{j=1}^J \hat{p}_j \hat{\Omega}^{(j)} \right)^{-1} a \right]$$

weakly converges to a normal distribution, and the power of our statistic goes to one again.
3.4. the Different Mean and Variance

In the end, we would like to discuss one-way ANOVA model with different mean and variance sightly. The model is as follows

\[ Y_{ij} = \mu_j + \varepsilon_{ij} = \mu_j + \sigma_j \epsilon_{ij} \quad i = 1, \ldots, N_j, \quad j = 1, \ldots, J, \]

and we estimate the mean and variance for each group as previous, i.e.

\[ \hat{\mu}_j = \frac{1}{N_j} \sum_{i=1}^{N_j} Y_{ij}, \quad \hat{\sigma}_j^2 = \frac{1}{N_j} \sum_{i=1}^{N_j} (Y_{ij} - \hat{\mu}_j)^2. \]

If we use \( \hat{e}_{ij} = (Y_{ij} - \hat{\mu}_j)/\hat{\sigma}_j \) and \( \hat{Z}_{ij} = \Phi(\hat{e}_{ij}) \), and then the statistic and test share the same properties in Section 3.1 as long as \( J \) is finite. In fact, for the situation, as all the estimations of one group use the data in that group only, there is no need to combining them and doing the test. We can just test each group one by one.

4. the Choice of \( K \)

In this section, we will give a glimpse on how to choose a proper \( K \) when applying our method. Although the result in Section 3 implies that for any \( K \), our test is always theoretically correct and consistent. Nonetheless, in application, the proper choice of \( K \) plays an important role. Indeed, if we choose \( K \) too large, we test against a superfluously complex alternative. It contains redundant covariates which do not contribute to the test statistic markedly but increase the number of degrees of freedom (and, hence, critical values). This causes a loss of power. (see Section 4 in [8] and [9]). So choose a reasonable data-driven \( K = K^* \) is anything but meaningless. Although [10] propose a general framework to choose the optimal \( K \). This method may be cumbersome in our setting since it requires us to calculate the exact likelihood of the data for each possible \( K \), but this step is obviously redundant for us.

Denote \( \hat{C}_H K = N \hat{\Psi}_K^2 \). From Section 3, we know that, under \( H_0 \)

\[ \hat{C}_H K \rightarrow_d \text{CH}_K \overset{d}{=} \chi^2_K. \]

In spirit of [11] and [12], we propose the following method that can choose the the most proper \( K^* \) under \( H_0 \) by Schwarz’s rule.

\[ K^* := \min_{1 \leq K \leq D(N)} \arg \max \left( \hat{C}_H K - K \log N \right) \]

where \( D(N) \) is a sequence of numbers tending to infinity as \( N \to \infty \).
Theorem 6. Suppose Assumption A.1-A.2, and \( H_0 : e \sim \Phi(x) \) hold. If \( D(N) = o(\log^2 N) \), then \( K^* \to_{a.s.} 1 \).

So under the null hypothesis, this method selects out the optimal \( K \) under \( H_0 \). Consequently, the following result holds.

Corollary 2. Suppose Assumption A.1-A.2. If \( D(N) = o(\log^2 N) \), then \( N\hat{\Psi}^2_{K^*} \to_d \chi^2_1 \) under \( H_0 \) and \( \sqrt{N}(\hat{\Psi}^2_{K^*} - a^\top (P\pi\pi^\top)^{-1} a) \) converge to some normal distribution weakly under \( H_1 \), thus the power goes to 1 as \( N \to \infty \).

Unfortunately, due to finite sample in practice, this corollary may not be perfect. \[12\] point out that the \( \chi^2_1 \) approximation of the null distribution of \( N\hat{\Psi}^2_{K^*} \) is often inaccurate. Typically, when this approximation is used, the test considerably exceeds its prescribed nominal level. So following \[11\] and \[9\], we use the following revised approximation for the distribution of \( N\hat{\Psi}^2_{K^*} \).

\[
H(x) := P(N\hat{\Psi}^2_{K^*} \leq x) \\
= P(N\hat{\Psi}^2_1 \leq x, K^* = 1) + P(N\hat{\Psi}^2_2 \leq x, K^* = 2) + P(N\hat{\Psi}^2_{K^*} \leq x, K^* \geq 3) \\
\approx \begin{cases} 
[2\Phi(\sqrt{x}) - 1][2\Phi(\sqrt{\log N}) - 1], & x \leq \log N \\
H(\log N) + \frac{x - \log N}{\log n}[H(2\log N) - H(\log N)], & \log N < x < 2\log N \\
[2\Phi(\sqrt{x}) - 1][2\Phi(\sqrt{\log N}) - 1] + 2[1 - \Phi(\sqrt{\log N})], & x \geq 2\log N.
\end{cases}
\]

In the simulation, we will see the usefulness of \( K^* \) and this revised approximation.

5. Simulation

Here we perform a Monte Carlo experiment to study the performance of our method in Section 3 and 4. Following \[7\] and \[13\], we consider the orthogonal Legendre polynomials on \([0, 1]\) for \( \{\pi_k\}_{k=0}^\infty \). In this case, since

\[
c_{1k} = \frac{\sqrt{2k + 1}}{2^k} \sum_{j=0}^{[k/2]} (-1)^j \binom{k}{j} \binom{2(k - j)}{k} \int_0^1 (2z - 1)^k 2^{k-2j} \Phi^{-1}(z) \, dz \\
= \sqrt{2k + 1} \sum_{j=0}^{[k/2]} \left(-\frac{1}{4}\right)^j \binom{k}{j} \binom{2(k - j)}{k} \int_{-1/2}^{1/2} x^{k-2j} \Phi^{-1} \left( x + \frac{1}{2} \right) \, dx,
\]

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and
\[ c_{2k} = \sqrt{2k+1} \sum_{j=0}^{[k/2]} \left( \frac{-1}{4} \right)^j \binom{k}{j} \binom{2(k-j)}{k} \int_{-1/2}^{1/2} x^{k-2j} \left[ \Phi^{-1} \left( x + \frac{1}{2} \right) \right]^2 \, dx, \]

it is not hard to see that \( c_{1k} \) is non-zero and \( c_{2k} \) is zero for odd \( k \), and \( c_{2k} \) is non-zero for even \( k \) due to \( \Phi^{-1} \left( x + \frac{1}{2} \right) \) is an odd function. Therefore, when calculating \( \Sigma \) or \( \Omega^{(j)} \), there is only one half elements in these matrices need to compute. As Section 3.1 is only a special case of Section 3.2 or Section 3.3, we only verify the our method in Section 3.2 and Section 3.3.

First, we generate the data as follows
\[ Y_{ij}^{(1)} \sim N(5j,4), \quad Z_{ij}^{(1)} \sim \chi_2^2 + (5j - 2), \quad i = 1, \ldots, jm, \quad j = 1, \ldots, 5, \]
where \( Y_{ij}^{(1)} \) and \( Z_{ij}^{(1)} \) are independently distributed. Then \( EY_{ij}^{(1)} = EZ_{ij}^{(1)} = 5j \) and \( \text{var}(Y_{ij}^{(1)}) = \text{var}(Z_{ij}^{(1)}) = 4 \). We apply our method in Section 3.2 and Section 4 with \( \alpha = 0.05 \) with the order of the Legendre polynomial \( K^* \leq 5 \), increase the value of \( m \) from 10 to 150 with increments 10, and perform 500 replications, and the result can be seen in table 1 and figure 1. We also generate the data with the same mean and different variance as follows
\[ Y_{ij}^{(2)} \sim N(8, j^2), \quad Z_{ij}^{(2)} \sim U[8 - 2\sqrt{3}j, 8 + 2\sqrt{3}j], \quad i = 1, \ldots, jm, \quad j = 1, \ldots, 5, \]
where \( Y_{ij}^{(2)} \) and \( Z_{ij}^{(2)} \) are independently distributed and satisfying \( EY_{ij}^{(2)} = EZ_{ij}^{(2)} = 8 \) and \( \text{var}(Y_{ij}^{(2)}) = \text{var}(Z_{ij}^{(2)}) = j^2 \). By applying the same setting in the previous stimulation, the result of this case is shown in table 2 and figure 2. It is clearly seen that our method can achieve a good performance even \( m \) is relatively small (\( m = 15 \) and in this case the totally number of our sample is \( N = 225 \)).

With a glimpse of table 1 and 2 one can immediately find our method does achieve a good performance even for a relative small size of a set of data. Indeed, in the first data set, \( m = 10 \) (the totally size \( N = 150 \)) can let the reject rate is approximately 0.05 under \( H_0 \) and exactly 1 under \( H_1 \) for any fixed \( K \), data-driven \( K^* \), and revised data-driven test \( H(\cdot) \). However, in the second data set, where the difference of distributions between \( H_1 \) and \( H_2 \) is not as large as that in the first data set, the usefulness of \( K^* \) and \( H(\cdot) \) appears: for fixed \( K \), a bigger \( K \) often causes a slightly higher rejection rate than the specified significant level while a too small \( K \) accepts too much \( H_1 \).
Table 1: Empirical reject rate for $Y^{(1)}$ and $Z^{(1)}$ when applying our chi-square tests under fixed $K (K = 1, \ldots, 5)$, original data-driven chi-square test ($K^*$), and revised data-driven chi-square test ($H(x)$).

| $m$  | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ | $H(x)$ | $K = 1, \ldots, 5, K^* \& H(x)$ |
|------|---------|---------|---------|---------|---------|--------|----------------------------------|
| 10   | 0.050   | 0.046   | 0.048   | 0.048   | 0.052   | 0.066  | 0.044                           |
| 20   | 0.072   | 0.060   | 0.050   | 0.046   | 0.050   | 0.092  | 0.072                           |
| 30   | 0.054   | 0.046   | 0.054   | 0.038   | 0.056   | 0.060  | 0.044                           |
| 40   | 0.046   | 0.048   | 0.060   | 0.054   | 0.048   | 0.056  | 0.048                           |
| 50   | 0.044   | 0.060   | 0.052   | 0.052   | 0.048   | 0.062  | 0.054                           |
| 60   | 0.066   | 0.044   | 0.048   | 0.052   | 0.058   | 0.076  | 0.062                           |
| 70   | 0.044   | 0.048   | 0.038   | 0.046   | 0.048   | 0.054  | 0.042                           |
| 80   | 0.050   | 0.044   | 0.048   | 0.038   | 0.042   | 0.060  | 0.056                           |
| 90   | 0.040   | 0.060   | 0.046   | 0.054   | 0.046   | 0.054  | 0.046                           |
| 100  | 0.062   | 0.056   | 0.042   | 0.050   | 0.048   | 0.066  | 0.052                           |
| 110  | 0.060   | 0.046   | 0.066   | 0.052   | 0.050   | 0.070  | 0.060                           |
| 120  | 0.034   | 0.044   | 0.038   | 0.038   | 0.036   | 0.042  | 0.036                           |
| 130  | 0.038   | 0.050   | 0.042   | 0.052   | 0.050   | 0.054  | 0.046                           |
| 140  | 0.038   | 0.050   | 0.048   | 0.042   | 0.044   | 0.046  | 0.044                           |
| 150  | 0.064   | 0.056   | 0.048   | 0.058   | 0.058   | 0.068  | 0.062                           |

Figure 1: The choice of $K^*$ with the error bars when the size of the sample increasing for $Y^{(1)}$ and $Z^{(1)}$.

Figure 2: The choice of $K^*$ with the error bars when the size of the sample increasing for $Y^{(2)}$ and $Z^{(2)}$. 
Table 2: Empirical reject rate for $Y^{(2)}$ and $Z^{(2)}$ when applying our chi-square tests under fixed $K$ ($K = 1, \ldots, 5$), original data-driven chi-square test ($K^*$), and revised data-driven chi-square test ($H(x)$).

|       | $H_0$       | $H_1$       |
|-------|-------------|-------------|
|       | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ | $K^*$ | $H(x)$ | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ | $K^*$ | $H(x)$ |
| $m = 10$ | 0.034     | 0.044    | 0.040   | 0.030   | 0.028   | 0.062  | 0.044  | 0.040  | 0.998   | 0.998   | 0.996   | 0.994   | 0.998  | 0.998  |
| $m = 20$ | 0.062     | 0.064   | 0.056   | 0.058   | 0.078   | 0.066  | 0.034  | 0.062  | 0.064   | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 30$ | 0.040     | 0.040   | 0.038   | 0.042   | 0.048   | 0.048  | 0.042  | 0.062  | 0.040   | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 40$ | 0.038     | 0.042   | 0.044   | 0.028   | 0.032   | 0.046  | 0.036  | 0.042  | 0.04     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 50$ | 0.044     | 0.042   | 0.042   | 0.044   | 0.050   | 0.038  | 0.036  | 0.042  | 0.04     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 60$ | 0.062     | 0.080   | 0.052   | 0.064   | 0.074   | 0.068  | 0.016  | 0.062  | 0.066   | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 70$ | 0.062     | 0.066   | 0.060   | 0.050   | 0.062   | 0.076  | 0.070  | 0.050  | 0.04     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 80$ | 0.046     | 0.048   | 0.042   | 0.058   | 0.036   | 0.048  | 0.044  | 0.042  | 0.04     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 90$ | 0.060     | 0.058   | 0.050   | 0.052   | 0.046   | 0.066  | 0.058  | 0.032  | 0.03     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 100$ | 0.052    | 0.038   | 0.046   | 0.058   | 0.042   | 0.058  | 0.048  | 0.018  | 0.03     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 110$ | 0.040   | 0.034   | 0.038   | 0.032   | 0.040   | 0.042  | 0.038  | 0.032  | 0.03     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 120$ | 0.056   | 0.044   | 0.066   | 0.046   | 0.046   | 0.062  | 0.056  | 0.040  | 0.04     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 130$ | 0.052   | 0.058   | 0.054   | 0.056   | 0.050   | 0.056  | 0.052  | 0.028  | 0.04     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 140$ | 0.050   | 0.060   | 0.054   | 0.042   | 0.040   | 0.052  | 0.046  | 0.028  | 0.03     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
| $m = 150$ | 0.056   | 0.064   | 0.054   | 0.058   | 0.060   | 0.058  | 0.054  | 0.028  | 0.03     | 0.034   | 0.034   | 0.042   | 0.034  | 0.034  |
which even attends to 1! In contrast to the fixed $K$, data-driven $K^*$ and revised $H(x)$ renders near correct rejection rate under $H_0$ (0.05) and totally right rejection rate under $H_1$ (1). And the stability of revised $H(x)$ is also better than that of data-driven $K^*$ chi-square test. Figure 1 and 2 verfies Theorem 6.

6. Other Situations

6.1. One-Way Random Effects Models

First, we consider the one-Way random effects models that all error components in different groups share with the same variance, in which we observe that

$$Y_{ij} = \mu_j + \varepsilon_{ij} = \mu + A_j + \varepsilon_{ij} = \mu + \sigma_a a_j + \sigma e_{ij}, \quad i = 1, \ldots, N_j, \quad j = 1, \ldots, J,$$

where the $a_i$ are i.i.d. the $e_{ij}$ are i.i.d., the $a_j$ are independent of the $e_{ij}$, and

$$E a_j = E e_{ij} = 0, \quad \text{var}(a_j) = \text{var}(e_{ij}) = 1.$$

Then we can construct the following estimators:

$$\hat{\varepsilon}_{ij} = Y_{ij} - \bar{Y}_j, \quad \hat{\sigma}^2 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (Y_{ij} - \bar{Y}_j)^2 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\varepsilon}_{ij}^2.$$

The expression of the estimator is totally the same as that in section 3.2. In fact, under this situation, the asymptotic properties of our statistic $\hat{\Psi}_K^2$ is totally the same as these in Section 3.2 due to

$$\hat{\varepsilon}_{ij} = Y_{ij} - \bar{Y}_j = \varepsilon_{ij} - \bar{\varepsilon}_j$$

is free from another random component $A_j$. The other situation (different variance) is also the same as Section 3.4.

6.2. Two-way Models

In a two-way fixed effects model, what we observe is $Y_{ij\ell} = \mu_{jk} + \varepsilon_{ij\ell} = (\mu + \alpha_j + \beta_{\ell} + \gamma_{j\ell}) + \sigma e_{ij\ell}$ with the same variance, where

$$\sum_{j=1}^{J} \alpha_j = \sum_{\ell=1}^{L} \beta_{\ell} = \sum_{j=1}^{J} \gamma_{j\ell} = \sum_{\ell=1}^{L} \gamma_{j\ell} = 0.$$
Note that under this circumstance, the estimation of the mean and residual of each group is exactly the same as that in Section 3 except that there is an extra subscript. So the result of 3 can be completely parallel to this section.

Similarly, in a two-way random effects model, we observe that

\[ Y_{ij\ell} = \mu_{jk} + \varepsilon_{ij\ell} = (\mu + A_j + B_{\ell} + D_{j\ell}) + \varepsilon_{ij\ell} = (\mu + \sigma_a a_j + \sigma_b b_{\ell} + \sigma_d d_{j\ell}) + \sigma e_{ij\ell}, \]

where \( a_j, b_{\ell}, d_{j\ell}, \) and \( e_{ij\ell} \) are all i.i.d. random variables with zero mean and one variance value, and independent from each other. Since we do not care about \( \sigma_a, \sigma_b, \) and \( \sigma_d, \) we can treat \( A_j + B_{\ell} + D_{j\ell} \) as a single random variable which are also i.i.d. and has zero mean. So the case is same as the situation in Section 6.1. Evidently, there is no point in discussing higher-order models.

Two-way Models with different variance can be discussed as previous.

Appendix

We provide the proofs for our main theoretical results in this Appendix. For proof convenience, we will use \( \hat{\varepsilon}_{ij} = Y_{ij} - \hat{\mu} \) (or \( \hat{\varepsilon}_{ij} = Y_{ij} - \hat{\mu}_j \)), then \( \hat{\varepsilon}_{ij} = \hat{\sigma} \hat{\varepsilon}_{ij} \) (or \( \hat{\varepsilon}_{ij} = \hat{\sigma}_j \hat{\varepsilon}_{ij} \)).

**Proof of Theorem 1.**

Note that the Taylor expansion yields

\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(Z_{ij}) = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(Z_{ij}) + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \left( \hat{Z}_{ij} - Z_{ij} \right) + R_{N1}.
\]

(1)

where

\[
R_{N1} = \frac{1}{2N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \left( \hat{Z}_{ij} - Z_{ij} \right)^2 = o_p \left( \frac{1}{\sqrt{N}} \right),
\]

(2)

where \( \hat{Z}_{ij} \) lies between \( \hat{Z}_{ij} \) and \( Z_{ij} \).

Recall that \( \hat{Z}_{ij} = \Phi(\hat{\varepsilon}_{ij}/\hat{\sigma}) \) and \( Z_{ij} = \Phi(\varepsilon_{ij}/\sigma) \). Then by Taylor expansion again, the second term in (1) can be further decomposed as

\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi(e_{ij}) \left( \frac{\hat{\varepsilon}_{ij}}{\hat{\sigma}} - \frac{\varepsilon_{ij}}{\sigma} \right) + R_{N2}.
\]
We also need to prove
\[ R_{N2} = \frac{1}{2N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( \frac{\hat{\xi}_{ij}}{\hat{\sigma}} \right) \left( \frac{\hat{\xi}_{ij} - \varepsilon_{ij}}{\sigma} \right)^2 = o_p \left( \frac{1}{\sqrt{N}} \right), \tag{3} \]
where, with abuse of notations, \( \hat{\xi}_{ij}/\hat{\sigma} \) lies between \( \hat{\xi}_{ij}/\hat{\sigma} \) and \( \varepsilon_{ij}/\sigma \).

We will prove (2) and (3) at last. Now, on the other hand, we can show
\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi (e_{ij}) \left( \frac{\hat{\xi}_{ij} - \varepsilon_{ij}}{\sigma} \right)
= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi (e_{ij}) \frac{\hat{\xi}_{ij} - \varepsilon_{ij}}{\sigma}
- \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi (e_{ij}) \frac{\varepsilon_{ij}(\hat{\sigma}^2 - \sigma^2)}{\sigma(\hat{\sigma} + \sigma)}
= - \frac{\hat{\mu} - \mu}{\sigma} \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi (e_{ij}) - \frac{\hat{\sigma}^2 - \sigma^2}{2\sigma^2} \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi (e_{ij}) e_{ij}
+ o_p \left( \frac{\hat{\mu} - \mu}{\sigma} + \frac{\hat{\sigma}^2 - \sigma^2}{2\sigma^2} \right), \tag{4}
\]
where the last step follows due to law of large numbers for \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi (e_{ij}) \)
and \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi (e_{ij}) e_{ij} \) as \( N \to \infty \) as well as the consistency of \( \hat{\sigma} \) to \( \sigma \). Note that under the null, by integration by parts
\[ E [\hat{\pi}_k(Z) \phi (e)] = \int_0^1 \pi_k(z) \Phi^{-1}(z) dz, \]
and
\[ E [\hat{\pi}_k(Z) \phi (e) e] = \int_0^1 \pi_k(z) \left[ (\Phi^{-1}(z))^2 - 1 \right] dz = \int_0^1 \pi_k(z) (\Phi^{-1}(z))^2 dz \]
where we use the fact that
\[ d\Phi^{-1}(z) = \frac{dz}{\phi (\Phi^{-1}(z))}, \]
and \( \int_0^1 \pi_k(z) dz = 0 \). Note that
\[ \hat{\mu} - \mu = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (Y_{ij} - \mu) = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \varepsilon_{ij} = o_p \left( \frac{1}{\sqrt{N}} \right), \]
where the last step follows due to the central limit theorem, and

\[
\hat{\sigma}^2 - \sigma^2 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (Y_{ij} - \hat{\mu})^2 - \sigma^2
\]

\[
= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma^2) - (\hat{\mu} - \mu)^2
\]

\[
= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma^2) + O_p\left(\frac{1}{N}\right)
\]

\[
= O_p\left(\frac{1}{\sqrt{N}}\right)
\]

where the last step follows due to the central limit theorem, we also have

\[
\hat{\sigma}^2 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \varepsilon_{ij}^2 + O_p\left(\frac{1}{N}\right)
\]

from the 3rd =. By plugging these expressions into the decomposition \((4.4)\) and combing with all previous results, we obtain the equation in Theorem 1.
In the end, we give the detailed proof for (2) and (3). In fact,

\[ |R_{N1}| \leq \frac{1}{2N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left\| \hat{\pi}_k \right\|_{\infty} \left[ \Phi \left( \frac{\hat{\beta}_{ij}}{\hat{\sigma}} \right) - \Phi \left( \frac{\beta_{ij}}{\sigma} \right) \right]^2 \]

\[ \leq \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \phi^2 \left( \frac{\hat{\beta}_{ij}}{\hat{\sigma}} \right) \left( \frac{\hat{\beta}_{ij}}{\hat{\sigma}} - \frac{\beta_{ij}}{\sigma} \right)^2 \]

\[ \leq \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\beta}_{ij} - \beta_{ij}}{\sigma} \right)^2 \left( \sigma \frac{\hat{\sigma} \hat{\sigma} - \sigma \sigma}{\sigma \sigma} \right)^2 \]

\[ \cong (\hat{\mu} - \mu)^2 + (\hat{\sigma}^2 - \sigma^2)^2 \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\beta}_{ij}^2 \]

\[ = O_p \left( N^{-1} \right) + O_p \left( N^{-1} \right) (\sigma^2 + o_p(1)) = O_p \left( \frac{1}{\sqrt{N}} \right) , \]

and

\[ |R_{N2}| = \left| \frac{1}{2N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k (Z_{ij}) \left[ -\hat{\beta}_{ij} \phi \left( \frac{\hat{\beta}_{ij}}{\hat{\sigma}} \right) \right] \left( \frac{\hat{\beta}_{ij}}{\hat{\sigma}} - \frac{\beta_{ij}}{\sigma} \right)^2 \right| \]

\[ \leq \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left\| \hat{\pi}_k \right\|_{\infty} \left\| \phi \right\|_{\infty} \left| \frac{\hat{\beta}_{ij}}{\hat{\sigma}} \right| \left( \frac{\hat{\beta}_{ij}}{\hat{\sigma}} - \frac{\beta_{ij}}{\sigma} \right)^2 \]

\[ \leq \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \frac{\hat{\beta}_{ij}^2}{\sigma^2} \right]^{1/2} \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\beta}_{ij}}{\hat{\sigma}} - \frac{\beta_{ij}}{\sigma} \right)^4 \right]^{1/2} \]

where the first "=" is due to \( \dot{\phi}(x) = -x \phi(x) \). Similar to the technique in
previous, we can obtain

\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\varepsilon}_{ij}}{\bar{\sigma}} - \varepsilon_{ij} \right)^4 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\varepsilon}_{ij} - \varepsilon_{ij}}{\bar{\sigma}} - \varepsilon_{ij} \frac{(\hat{\sigma}^2 - \sigma^2)}{\bar{\sigma}(\bar{\sigma} + \sigma)} \right)^4
\]

\[
\leq 2^3 \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\varepsilon}_{ij}}{\bar{\sigma}} \right)^4 + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \varepsilon_{ij} \frac{(\hat{\sigma}^2 - \sigma^2)}{\bar{\sigma}(\bar{\sigma} + \sigma)} \right)^4 \right]
\]

\[
\simeq (\hat{\mu} - \mu)^4 + (\hat{\sigma}^2 - \sigma^2)^4 \left( E\varepsilon^4 + o_p(1) \right) = O_p \left( \frac{1}{N^2} \right),
\]

and

\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \frac{\hat{\varepsilon}_{ij}^2}{\bar{\sigma}^2} \leq \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\varepsilon}_{ij}}{\bar{\sigma}} \right)^2 + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\varepsilon_{ij}}{\sigma} \right)^2
\]

\[
= \frac{1}{\sigma^2(1 + o_p(1))} \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\varepsilon}_{ij}^2 + (1 + o_p(1)) = O_p(1).
\]

The inequalities above imply \( |R_{N2}| = O_p(N^{-1}) \), hence \( |R_{N1}|, |R_{N2}| = o_p \left( \frac{1}{\sqrt{N}} \right) \).

\[\square\]

**Proof of Theorem 2.**

The proof of equation (8) is totally similar to the proof of Theorem 1. And from (6), we know that:

\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi_k(\hat{Z}_{ij}) = E\pi_k(Z) + o_p(1),
\]

which implies \( \hat{\Psi}^2_k \rightarrow_p a^\top \Sigma^{-1} a \). If we denote the \((k, m)\)-th component in \( \Sigma^{-1} \)
is \( \omega_{km} \), then

\[
\sqrt{N} \left( \hat{\Psi}_k^2 - a^\top \Sigma^{-1} a \right) = \sqrt{N} \left\{ [a + o_p(1)]^\top \Sigma^{-1} \left[ \frac{1}{N} \sum_{j=1}^J \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) \right] - a^\top \Sigma^{-1} a \right\}
\]

\[
= a^\top \Sigma^{-1} \frac{1}{\sqrt{N}} \sum_{j=1}^J \sum_{i=1}^{N_j} \left( \pi(\hat{Z}_{ij}) - a \right)
\]

\[
+ o_p \left( \frac{1}{\sqrt{N}} \sum_{j=1}^J \sum_{i=1}^{N_j} \left\| \pi(\hat{Z}_{ij}) - a \right\|_\infty \right).
\]

Since \( \frac{1}{\sqrt{N}} \sum_{j=1}^J \sum_{i=1}^{N_j} \left( \pi(\hat{Z}_{ij}) - a \right) \) is asymptotically normal. Indeed, the \( k \)-th component in it is

\[
\frac{1}{\sqrt{N}} \sum_{j=1}^J \sum_{i=1}^{N_j} \left( \pi_k(\hat{Z}_{ij}) - a_k \right) = \frac{1}{\sqrt{N}} \sum_{j=1}^J \sum_{i=1}^{N_j} \left( \zeta_k(e_{ij}) - E \zeta_k(e_{ij}) \right) + o_p(1),
\]

where

\[
\zeta_k(e) = \pi_k(\Phi(e)) - d_{1k}e - \frac{d_{2k}}{2} (e^2 - 1).
\]

By the multivariate center central limit theorem,

\[
\frac{1}{\sqrt{N}} \sum_{j=1}^J \sum_{i=1}^{N_j} \left( \pi(\hat{Z}_{ij}) - a \right) \rightarrow_d N(0, \Xi),
\]

and the \( (k, l) \)-the element in the asymptotic variance \( \Xi \) is

\[
\xi_{kl} = E \left[ \zeta_k(e) \zeta_l(e) \right]
\]

\[
= E \left[ \pi_k(Z) \pi_l(Z) \right] - d_{1k}d_{1l} + \frac{1}{2} \left[ a_k d_{2l} + a_l d_{2k} \right]
\]

\[
+ \frac{1}{2} \left[ d_{1k}d_{2l} + d_{1l}d_{2k} \right] E e^3 - \frac{d_{2k}d_{2l}}{4} \left[ 5 - 4 E e^4 \right],
\]

which gives the asymptotic normality in Theorem 2.

Proof of Theorem 3 and Theorem 4. ☐

Proof of Theorem 3 and Theorem 4.
The proof is similar to that of Theorem 1 and 2. The major difference is now
\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( \frac{\varepsilon_{ij}}{\sigma} \right) \left( \frac{\varepsilon_{ij}}{\hat{\sigma}} - e_{ij} \right)
\]
\[
= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( e_{ij} \right) \frac{\hat{\varepsilon}_{ij} - \varepsilon_{ij}}{\hat{\sigma}} - \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( e_{ij} \right) \frac{\varepsilon_{ij}(\hat{\sigma}^2 - \sigma^2)}{\hat{\sigma}\sigma(\hat{\sigma} + \sigma)}
\]
\[
= -\frac{1}{\hat{\sigma}} \frac{1}{N} \sum_{j=1}^{J} N_j (\hat{\mu}_j - \mu_j) \left\{ \frac{1}{N_j} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( e_{ij} \right) \right\} - \frac{\hat{\sigma}^2 - \sigma^2}{\hat{\sigma}(\hat{\sigma} + \sigma)} \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( e_{ij} \right) e_{ij}
\]
\[
= -E[\hat{\pi}_k(Z)\phi(e)] \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} e_{ij} - \frac{1}{2\sigma^2} E[\hat{\pi}_k(Z)\phi(e)] e \left( \sigma^2 - \sigma^2 \right)
\]
\[
+ o_p \left( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} e_{ij} \right) + o_p \left( \hat{\sigma}^2 - \sigma^2 \right) ,
\]
(5)
where the last step follows due to law of large numbers for \( \frac{1}{N_j} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( e_{ij} \right) \) as \( N_j \to \infty \) for each \( 1 \leq j \leq J \) and \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi \left( e_{ij} \right) e_{ij} \) as \( N \to \infty \) as well as the consistency of \( \hat{\sigma}^2 \) to \( \sigma^2 \). Note that all \( \varepsilon_{ij} \) are all i.i.d., so
\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} e_{ij} = O_p \left( \frac{1}{\sqrt{N}} \right)
\]
as \( \min \{ N_1, \cdots, N_J \} \to \infty \) (then \( N \to \infty \)). On the other hand, since \( J = \)
we find that
\[
\hat{\sigma}^2 - \sigma^2 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( (Y_{ij} - \hat{\mu}_j)^2 - \sigma^2 \right)
\]
\[
= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma^2) - \frac{1}{N} \sum_{j=1}^{J} N_j (\mu_j - \hat{\mu}_j)^2
\]
\[
= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma^2) + \frac{1}{N} \sum_{j=1}^{J} \sigma_n(1)
\]
\[
= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma^2) + o_p \left( \frac{1}{\sqrt{N}} \right)
\]
\[
= O_p \left( \frac{1}{\sqrt{N}} \right),
\]
with the last step following by the central limit theorem. Besides, we also have:
\[
\hat{\sigma}^2 = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \varepsilon_{ij}^2 + O_p \left( \frac{1}{N} \right).
\]

Then, plugging all these results into (5) concludes the proof of Theorem 3 and Theorem 4 as long as \( R_{N1} \) and \( R_{N2} \) are \( o_p \left( \frac{1}{\sqrt{N}} \right) \) too. In fact, from the proof of Theorem 1 and \( J = o(N^{1/2}) \), we know that:
\[
|R_{N1}| \lesssim \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\varepsilon_{ij} - \hat{\varepsilon}_{ij}}{\hat{\sigma}} \right)^2 + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \hat{\sigma}^2 - \sigma^2 \right)^2, \]
\[
\lesssim \frac{1}{N\sigma^2} \sum_{j=1}^{J} N_j (\hat{\mu}_j - \mu_j)^2 + (\hat{\sigma}^2 - \sigma^2)^2 \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \varepsilon_{ij}^2
\]
\[
\lesssim \frac{1}{N} \sum_{j=1}^{J} O_p(1) + O_p \left( \frac{1}{N} \right) \sigma^2 + o_p(1) = o_p \left( \frac{1}{\sqrt{N}} \right).
\]

And from the condition for any \( j, k \in \{1, \ldots, J\} \), \( N_j/N_k \to p_{j,k} \in (0, \infty) \),
for $R_{N2}$, as

$$\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\epsilon}_{ij} - \tilde{\epsilon}_{ij}}{\hat{\sigma}} \right)^4 \leq 2^3 \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\epsilon}_{ij}}{\hat{\sigma}} \right)^4 + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\tilde{\epsilon}_{ij}(\hat{\sigma}^2 - \sigma^2)}{\hat{\sigma}\sigma(\hat{\sigma} + \sigma)} \right)^4 \right]$$

$$\lesssim \frac{1}{N} \sum_{j=1}^{J} N_j (\hat{\mu}_j - \mu_j)^4 + (\hat{\sigma}^2 - \sigma^2)^4 (E\tilde{\epsilon}^4 + o_p(1))$$

$$\lesssim \frac{1}{N} \sum_{j=1}^{J} N_j O_p \left( \frac{1}{N_j^2} \right) + O_p \left( \frac{1}{N^2} \right)$$

$$\asymp \sum_{j=1}^{J} O_p \left( \frac{J}{N_j^2} \right) + O_p \left( \frac{1}{N^2} \right) = o_p \left( \frac{1}{N} \right),$$

and $\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\epsilon}_{ij}^2 = o_p(1)$ too, so

$$|R_{N2}| \leq \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\epsilon}_{ij}^2 \right]^{1/2} \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\epsilon}_{ij} - \tilde{\epsilon}_{ij}}{\hat{\sigma}} \right)^4 \right]^{1/2} = o_p \left( \frac{1}{\sqrt{N}} \right).$$

Consequently, we concludes the two theorems. □

**Proof of Theorem 5**

The technique is still same as previous. We also tackle the following decomposition and $R_{N1}$ and $R_{N2}$ then. First,

$$\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k (Z_{ij}) \phi \left( \frac{\hat{\epsilon}_{ij}}{\hat{\sigma}} \right) \left( \frac{\hat{\epsilon}_{ij} - \tilde{\epsilon}_{ij}}{\hat{\sigma}} \right) = D_{N1} - D_{N2}$$

where

$$D_{N1} = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k (Z_{ij}) \phi \left( e_{ij} \right) \frac{\hat{\epsilon}_{ij} - \tilde{\epsilon}_{ij}}{\hat{\sigma}^j},$$

and

$$D_{N2} = \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\pi}_k (Z_{ij}) \phi \left( e_{ij} \right) \frac{\hat{\epsilon}_{ij}(\hat{\sigma}^2 - \sigma^2)_{ij}}{\hat{\sigma}^j\sigma_j(\hat{\sigma}^j + \sigma^j)}.$$
For $D_{N1}$, we have:

$$D_{N1} = -\frac{\hat{\mu} - \mu}{N} \sum_{j=1}^{J} \frac{N_j}{\sigma_j} \left\{ \frac{1}{N_j} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi(e_{ij}) \right\}$$

$$= -\frac{1}{N} \left( \sum_{j=1}^{J} \frac{N}{JN_j} \sum_{i=1}^{N_j} \varepsilon_{ij} \right) \left( \sum_{j=1}^{J} \frac{N_j}{\sigma_j} \left\{ \frac{1}{N_j} \sum_{i=1}^{N_j} \hat{\pi}_k(Z_{ij}) \phi(e_{ij}) \right\} \right)$$

$$= -\frac{1}{N} \left( \sum_{j=1}^{J} \sum_{i=1}^{N_j} \varepsilon_{ij} q_j \right) \left( \sum_{j=1}^{J} \frac{p_j}{\sigma_j} \left\{ c_{1k} + O_p \left( \frac{1}{\sqrt{N_j}} \right) \right\} \right)$$

$$= -\frac{1}{N} \left[ \sum_{j=1}^{J} \frac{c_{1k}p_j}{\sigma_j} \right] \sum_{j=1}^{J} \sum_{i=1}^{N_j} \varepsilon_{ij} q_j + O_p \left( \frac{1}{N \sqrt{N_j}} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \varepsilon_{ij} q_j \right).$$

Since

$$\frac{1}{N \sqrt{N_j}} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \frac{\varepsilon_{ij}}{q_j} = \frac{1}{N} \sum_{j=1}^{J} \frac{1}{q_j \sqrt{N_j}} \sum_{i=1}^{N_j} \varepsilon_{ij}$$

$$= \frac{1}{N} \sum_{j=1}^{J} \frac{1}{q_j} O_p(1) = \sum_{j=1}^{J} \frac{1}{JN_j} O_p(1) = O_p \left( \frac{J}{N} \right) = o_p \left( \frac{1}{\sqrt{N}} \right),$$

we obtain

$$D_{N1} = -\frac{1}{N} \left[ \sum_{j=1}^{J} \frac{c_{1k}p_j}{\sigma_j} \right] \sum_{j=1}^{J} \sum_{i=1}^{N_j} \frac{\varepsilon_{ij}}{q_j} + O_p \left( \frac{1}{\sqrt{N}} \right).$$
Similarly, for $D_{N2}$,

$$D_{N2} = \frac{1}{N} \sum_{j=1}^{J} \left( \frac{\sigma_j^2 - \sigma_j^2}{\sigma_j^2} \right) N_j \left\{ \frac{1}{N_j} \sum_{i=1}^{N_j} \hat{\pi}_k (Z_{ij}) \phi(e_{ij}) \varepsilon_{ij} \right\}$$

$$= \frac{1}{N} \sum_{j=1}^{J} \frac{1}{2\sigma_j^2} N_j (\hat{\sigma}_j^2 - \sigma_j^2) \left\{ c_{2k} + O_p \left( \frac{1}{\sqrt{N_j}} \right) \right\}$$

$$= \frac{c_{2k}}{N} \sum_{j=1}^{J} \frac{1}{2\sigma_j^2} \left[ \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma_j^2) + o_p \left( \frac{N_j}{N^{1/2}} \right) \right] + O_p \left( \sum_{j=1}^{J} \frac{\sqrt{N_j}}{N} (\hat{\sigma}_j^2 - \sigma_j^2) \right)$$

$$= \frac{1}{N} \sum_{j=1}^{J} N_j \sum_{i=1}^{N_j} \frac{c_{2k}}{2} \left( \frac{\varepsilon_{ij}^2}{\sigma_j^2} - 1 \right) + o_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \sum_{j=1}^{J} \frac{\sqrt{N_j}}{N} (\hat{\sigma}_j^2 - \sigma_j^2) \right)$$

$$= \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \frac{c_{2k}}{2} \left( \frac{\varepsilon_{ij}^2}{\sigma_j^2} - 1 \right) + o_p \left( \frac{1}{\sqrt{N}} \right),$$

where we use the fact that:

$$\hat{\mu} - \mu = \frac{1}{J} \sum_{j=1}^{J} \frac{1}{N_j} \sum_{i=1}^{N_j} \varepsilon_{ij} = \frac{1}{J} \sum_{j=1}^{J} O_p \left( \frac{1}{\sqrt{N_j}} \right) = O_p \left( \sqrt{\frac{J}{N}} \right) = o_p \left( N^{-1/4} \right),$$

and

$$\hat{\sigma}_j^2 - \sigma_j^2 = \frac{1}{N_j} \sum_{i=1}^{N_j} (Y_{ij} - \hat{\mu})^2 - \sigma_j^2 = \frac{1}{N_j} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma_j^2) - (\hat{\mu} - \mu)^2$$

$$= \frac{1}{N_j} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma_j^2) + o_p \left( N^{-1/2} \right),$$

with

$$\sum_{j=1}^{J} \sqrt{\frac{N_j}{N}} (\hat{\sigma}_j^2 - \sigma_j^2) = \sum_{j=1}^{J} \frac{\sqrt{N_j}}{N} \left( O_p \left( \frac{1}{\sqrt{N_j}} \right) + o_p \left( \frac{1}{\sqrt{N}} \right) \right)$$

$$= O_p \left( \frac{J}{N} \right) + o_p \left( \sqrt{\frac{J}{N}} \right) = o_p \left( \frac{1}{\sqrt{N}} \right)$$

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in turn. Furthermore, in this case, we still have $R_{N_1}$ and $R_{N_2}$ are $o_p\left(\frac{1}{\sqrt{N}}\right)$.

Similar to the proof of Theorem 3 and Theorem 4,

$$|R_{N_1}| \lesssim \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\varepsilon}_{ij} - \varepsilon_{ij}}{\hat{\sigma}_j} \right)^2 + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\varepsilon_{ij}(\hat{\sigma}_j^2 - \sigma_j^2)}{\sigma_j^2} \right)^2 := R_{N_1}^A + R_{N_1}^B.$$ 

In the above display, we can find that:

$$R_{N_1}^A = \frac{(\hat{\mu} - \mu)^2}{N} \sum_{j=1}^{J} \frac{N_j}{\hat{\sigma}_j^2} \approx \frac{1}{N} o_p\left(N^{-1/2}\right) O_p(N) = o_p\left(\frac{1}{\sqrt{N}}\right).$$

And from (6), it can be checked that

$$\sqrt{N_j} (\hat{\sigma}_j^2 - \sigma_j^2) = \frac{1}{\sqrt{N_j}} \sum_{i=1}^{N_j} (\varepsilon_{ij}^2 - \sigma_j^2) + o_p\left(N_j^{1/2} N^{-1/2}\right) = O_p(1) + o_p\left(\frac{1}{\sqrt{J}}\right),$$

so

$$R_{N_1}^B \lesssim \frac{1}{N} \sum_{j=1}^{J} N_j (\hat{\sigma}_j^2 - \sigma_j^2)^2 \left\{ \frac{1}{N_j} \sum_{i=1}^{N_j} \varepsilon_{ij}^2 \right\}$$

$$= \frac{1}{N} \sum_{j=1}^{J} \left[ O_p(1) + o_p\left(\frac{1}{\sqrt{J}}\right)\right]^2 \left[ \sigma_j^2 + o_p(1) \right]$$

$$\asymp \frac{1}{N} \sum_{j=1}^{J} \left[ O_p(1) + o_p\left(\frac{1}{\sqrt{J}}\right)\right] = O_p\left(\frac{J}{N}\right) + o_p\left(\frac{\sqrt{J}}{N}\right) = o_p\left(\frac{1}{\sqrt{N}}\right),$$

where the last "=" is due to $J = o(N^{1/2})$. Hence, $R_{N_1} = o_p\left(\frac{1}{\sqrt{N}}\right)$. On the other hand, for $R_{N_2}$, we first note that from (7),

$$(\hat{\sigma}_j^2 - \sigma_j^2)^4 = \left[ O_p\left(\frac{1}{\sqrt{N_j}}\right) + o_p\left(\frac{1}{\sqrt{JN_j}}\right)\right]^4 = O_p\left(\frac{1}{N_j^2}\right),$$

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then
\[
\frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\epsilon}_{ij} - \hat{\sigma}_j}{\hat{\sigma}_j} \right)^4 \leq \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\epsilon}_{ij}}{\hat{\sigma}_j} \right)^4 + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\epsilon_{ij} (\hat{\sigma}_j^2 - \sigma_j^2)}{\hat{\sigma}_j \sigma_j (\hat{\sigma}_j + \sigma_j)} \right)^4 \\
\asymp (\hat{\mu} - \mu)^4 + \frac{1}{N} \sum_{j=1}^{J} N_j (\hat{\sigma}_j^2 - \sigma_j^2)^4 \left\{ \frac{1}{N_j} \sum_{i=1}^{N_j} \hat{\epsilon}_{ij} \right\}
\asymp o_p \left( \frac{1}{N} \right) + \frac{1}{N} \sum_{j=1}^{J} N_j O_p \left( \frac{1}{N_j} \right)
= o_p \left( \frac{1}{N} \right) = o_p \left( \frac{1}{\sqrt{N}} \right).
\]

Since \( \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\epsilon}_{ij}^2 = O_p(1) \),
\[
|R_N| \leq \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \hat{\epsilon}_{ij}^2 \right]^{1/2} \left[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left( \frac{\hat{\epsilon}_{ij} - \epsilon_{ij}}{\hat{\sigma}_j} \right)^4 \right]^{1/2} = o_p \left( \frac{1}{\sqrt{N}} \right),
\]
i.e. \( R_N = o_p \left( \frac{1}{\sqrt{N}} \right) \) too. Up to now, the decomposition formula in Theorem 5 is verified. \( \square \)

**Proof of Theorem 6.**

We only prove the argument in Section 3.2 because other situations in Section 3 are totally the same. The theorem is for \( H_0 \), so in this proof, we also assume \( e \sim \Phi(x) \). Denote \( \mathcal{A}_N := \{ K^* = 1 \text{ under } D(N) \} \). First, we will deal with \( P \left( \mathcal{A}_N \right) \)
\[
P \left( \mathcal{A}_N \right) = \sum_{k=2}^{D(N)} P \left( K^* = k \right) \leq \sum_{k=2}^{D(N)} P \left( \hat{\mathcal{H}}_k - k \log N \geq \hat{\mathcal{H}}_1 - \log N \right)
\]
\[
= \sum_{k=2}^{D(N)} P \left( \hat{\mathcal{H}}_k - k \log N \geq - \log N \right) = \sum_{k=2}^{D(N)} P \left( \hat{\mathcal{H}}_k \geq a_{k,N} \right),
\]
where \( a_{k,N} := (k - 1) \log N \). Since
\[
P \left( \left| \mathcal{H}_k \right| \geq \frac{1}{2} a_{k,N} \right) \leq \frac{E \mathcal{H}_k^2}{(a_{k,N}/2)^2} \leq \frac{4(k^2 + 2k)}{(k - 1)^2 \log^2 N},
\]

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we have
\[ \sum_{k=2}^{D(N)} P\left( |\hat{CH}_k| \geq \frac{1}{2} a_{k,N} \right) \leq \sum_{k=2}^{D(N)} \frac{4(k^2 + 2k)}{(k - 1)^2 \log^2 N} \leq \frac{4}{\log^2 N} \left( D(N) + 4 \log D(N) + 3 \pi^2 / 6 \right). \]

On the other hand, from Theorem 3, we know
\[ \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} \pi(\hat{Z}_{ij}) = E \pi(Z) + \frac{1}{N} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\pi(Z_{ij}) - E\pi(Z_{ij})) + R_N \]
where \( R_N = o_p\left( \frac{1}{\sqrt{N}} \right) \). Then
\[ \hat{CH}_k = CH_k + 2 \sqrt{\frac{2}{N}} (E\pi(Z))^\top \Sigma^{-1} \left[ \frac{1}{\sqrt{N}} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\pi(Z_{ij}) - E\pi(Z_{ij})) \right] + o_p\left( \frac{1}{\sqrt{N}} \right). \]

Consequently, for \( N \to \infty \),
\[ P\left( |\hat{CH}_k - CH_k| \geq \frac{1}{2} a_{k,N} \right) = P\left( (E\pi(Z))^\top \Sigma^{-1} \left[ \frac{1}{\sqrt{N}} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\pi(Z_{ij}) - E\pi(Z_{ij})) \right] \geq \sqrt{N} a_{k,N} / 4 \right). \]

If we denote \( \xi = \Sigma^{-1} E\pi(Z) \in \mathbb{R}^k \) and \( \mathcal{G}_N \pi_m := \frac{1}{\sqrt{N}} \sum_{j=1}^{J} \sum_{i=1}^{N_j} (\pi_m(Z_{ij}) - E\pi_m(Z_{ij})) \), it gives
\[ P\left( |\hat{CH}_k - CH_k| \geq \frac{1}{2} a_{k,N} \right) \leq \sum_{m=1}^{k} P\left( \xi_m \mathcal{G}_N \pi_m \geq \frac{\sqrt{N}}{4k} a_{k,N} \right). \]

Using Bernstein’s inequality (see Lemma 19.32 in [14]),
\[ P\left( \xi_m \mathcal{G}_N \pi_m \geq \frac{\sqrt{N}}{4k} a_{k,N} \right) \leq 2 \exp \left\{ - \frac{1}{2} \frac{N a_{k,N}^2 / (16k^2 \xi_m^2)}{\mathcal{E} \pi_m^2(Z) + 1/3 \|\pi_m\|_\infty a_{k,N} / (4k \xi_m)} \right\} \]
\[ = 2 \exp \left\{ - \frac{(k - 1)^2 N \log^2 N / k^2}{c_1 \xi_m^2 + c_2 \xi_m (k - 1) \log N / k} \right\} \]
\[ \leq c_3 \exp \left\{ -(k - 1) N \log N / k \right\} = \frac{c_3}{N^{N(1-1/k)}}, \]

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where $c_1$, $c_2$ and $c_3$ are strictly positive constants free of $k$ and $N$. This implies
\[
\sum_{k=2}^{D(N)} P\left( \left| \hat{C}_k - C_k \right| \geq \frac{1}{2} a_{k,N} \right) \leq \sum_{k=2}^{D(N)} \frac{c_3 k^k}{N^{N(1-1/k)}} \leq c_4 \frac{D(N)^2}{N^2}
\]
with $0 < c_4 < 1$. Hence, when $N \to \infty$, we obtain
\[
P(\mathcal{A}) \geq 1 - \frac{4}{\log^2 N} \left( D(N) + 4 \log D(N) + \frac{\pi^2}{2} \right) - c_4 \frac{D(N)^2}{N^2}.
\]
Immediately, it gives $\sum_{N=1}^{\infty} P(\mathcal{A}) = \infty$. Note that $\mathcal{A}_{N+1} \subseteq \mathcal{A}_N$, using the second Borel-Cantelli lemma (see [15] for instance),
\[
\limsup_{M \to \infty} \frac{\left( \sum_{N=1}^{M} P(\mathcal{A}_N) \right)^2}{\sum_{N_1,N_2=1}^{M} P(\mathcal{A}_{N_1} \cap \mathcal{A}_{N_2})} = \infty,
\]
which implies $P(\limsup \mathcal{A}_N) = 1$. □

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