SPECTRAL PROPERTIES OF LANDAU HAMILTONIANS WITH NON-LOCAL POTENTIALS

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Abstract. We consider the Landau Hamiltonian $H_0$, self-adjoint in $L^2(\mathbb{R}^2)$, whose spectrum consists of an arithmetic progression of infinitely degenerate positive eigenvalues $\Lambda_q$, $q \in \mathbb{Z}_+$. We perturb $H_0$ by a non-local potential written as a bounded pseudo-differential operator $\text{Op}^w(\mathcal{V})$ with real-valued Weyl symbol $\mathcal{V}$, such that $\text{Op}^w(\mathcal{V})H_0^{-1}$ is compact. We study the spectral properties of the perturbed operator $H_V = H_0 + \text{Op}^w(\mathcal{V})$. First, we construct symbols $\mathcal{V}$, possessing a suitable symmetry, such that the operator $H_V$ admits an explicit eigenbasis in $L^2(\mathbb{R}^2)$, and calculate the corresponding eigenvalues. Further, for more general symbols $\mathcal{V}$, we study the asymptotic distribution of the eigenvalues of $H_V$ adjoining any fixed $\Lambda_q$. We show that the effective Hamiltonian which governs the spectral asymptotics of $H_V$ near $\Lambda_q$ is the compact Toeplitz operator $T_q(\mathcal{V}) = p_q \text{Op}^w(\mathcal{V})p_q$, $p_q$ being the orthogonal projection onto $\text{Ker}(H_0 - \Lambda_q I)$, and investigate the eigenvalue distribution of $T_q(\mathcal{V})$.

AMS 2010 Mathematics Subject Classification: 35P20, 81Q10

Keywords: Landau Hamiltonian, non-local potentials, Weyl pseudo-differential operators, eigenvalue asymptotics

1. Introduction

We consider the Landau Hamiltonian $H_0$, i.e. the 2D Schrödinger operator with constant scalar magnetic field $b > 0$, self-adjoint in $L^2(\mathbb{R}^2)$. We have

\begin{equation}
H_0 = \left(-i \frac{\partial}{\partial x} + \frac{by}{2}\right)^2 + \left(-i \frac{\partial}{\partial y} - \frac{bx}{2}\right)^2, \quad (x, y) \in \mathbb{R}^2.
\end{equation}

As is well known, the spectrum $\sigma(H_0)$ of the operator $H_0$ consists of eigenvalues of infinite multiplicity

\begin{equation}
\Lambda_q := b(2q + 1), \quad q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\},
\end{equation}

called the Landau levels (see [19, 27]).

Let $\text{Op}^w(\mathcal{V})$ be a bounded pseudo-differential operator ($\Psi$DO) with real-valued Weyl symbol $\mathcal{V}$; then $\text{Op}^w(\mathcal{V})$ is self-adjoint in $L^2(\mathbb{R}^2)$. We assume moreover that $\text{Op}^w(\mathcal{V})$ is relatively compact with respect to $H_0$, i.e that the operator $\text{Op}^w(\mathcal{V})H_0^{-1}$ is compact. Proposition 3.3 below contains simple sufficient conditions which guarantee the validity of these general assumptions on $\text{Op}^w(\mathcal{V})$. We will study the spectral properties of the perturbed operator

\begin{equation}
H_V := H_0 + \text{Op}^w(\mathcal{V}).
\end{equation}
By the Weyl theorem on the invariance of the essential spectrum under relatively compact perturbations, we have

$$\sigma_{\text{ess}}(H_V) = \sigma_{\text{ess}}(H_0) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}.$$ 

We will be interested, in particular, in the asymptotics of the discrete spectrum of $H_V$ near any fixed Landau level $\Lambda_q$, $q \in \mathbb{Z}_+$. As we will see, the effective Hamiltonian which governs this asymptotic behavior is the Toeplitz-type operator

$$(1.2) \quad T_q(V) := p_q \text{Op}^w(V) p_q,$$

considered as an operator in $\text{Ran} \, p_q = p_q L^2(\mathbb{R}^2)$, where $p_q$ is the orthogonal projection onto $\ker (H_0 - \Lambda_q I)$.

Let us explain briefly our motivation to study the spectral properties of the operator $H_V$. During the last three decades, there have been published several works on the spectral asymptotics for various perturbations of $H_0$. For example, electric perturbations, i.e. perturbations of $H_0$ by a multiplier $V$, playing the role of an electric potential, were considered in [33, 23, 34, 18], magnetic perturbations which involve a first-order differential operator, were investigated in [23, 35], and metric perturbations by a second-order operator, were studied in [23, 29]. The perturbations considered in [33, 23] are of power-like decay at infinity, while those studied in [34, 18, 35, 29] are of exponential decay or compact support. Recently, several articles [32, 30, 13] treated the eigenvalue asymptotics for the Landau Hamiltonian defined on the complement of a compact in $\mathbb{R}^2$, and equipped with Dirichlet, Neumann, or Robin boundary conditions. In this geometric setting, the effective Hamiltonian which governs the eigenvalue behavior near the Landau level $\Lambda_q$ is an integral operator sandwiched between the projections $p_q$, quite similar to the Toeplitz operator $T_q(V)$ introduced in (1.2).

Further, the so-called non-local potentials defined by appropriate integral operators play an important role in the nuclear physics (see e.g. [15, 16, 36]). Let us recall that any integral operator in $L^2(\mathbb{R}^n)$, which has a reasonable integral kernel can be represented as a Weyl $\Psi$DO (see e.g. [37, Eq. (23.39)]). Finally, in the mathematical physics literature there is a persistent interest in Schrödinger operators with non-local, in particular, pseudo-differential potentials (see e.g. [25, 2, 17]).

All these reasons are the source of our motivation to think of a unified approach to the spectral theory for pseudo-differential perturbations of magnetic quantum Hamiltonians. We believe that our present work could be a small but useful step in this direction. In particular, we are oriented to the possible applications of our results in the study of the threshold singularities of the spectral shift function arising in 3D exterior problems for magnetic quantum Hamiltonians; related results in the case of the perturbation by a decaying electric potential could be found in [14].

The article is organized as follows. In Section 2 we summarize the necessary facts from the general theory of $\Psi$DOs with Weyl and anti-Wick symbols. Section 3 contains the description of several unitary operators which map $H_V$ to operators which, in a way, are more accessible and easier to investigate. In particular, in Corollary 3.7 we
show that the Toeplitz operator $T_q(V)$, $q \in \mathbb{Z}_+$, is unitarily equivalent to a Weyl ΨDO $\text{Op}^w(v_{b,q}) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ whose symbol is a certain restriction of $V$ (see (3.41)).

In Section 4 we deal with Weyl and anti-Wick ΨDOs with radial symbols, and obtain explicit formulas for their eigenvalues and eigenfunctions. Some of these results are known, others, to our best knowledge, are new and could be of independent interest.

Further, in Section 5 we consider the eigenvalue asymptotics near the Landau level $\Lambda_q$, $q \in \mathbb{Z}_+$, for the operator $H_V$. First, we construct an explicit example of a radial symbol $v_{b,q}$ in which case $\text{Op}^w(v_{b,q})$ admits an anti-Wick symbol $\tilde{v}_{b,q}$ which has a compact support or decays exponentially at infinity. We examine the eigenvalue asymptotics for the operators $H_{\pm V}$ near any $\Lambda_q$, using the results and the methods of [18] and [29].

Finally, we assume that $v_{b,q}$ has a power-like decay, and find the main asymptotic term of the local eigenvalue counting function as the energy approaches the Landau level $\Lambda_q$, $q \in \mathbb{Z}_+$. Here we apply certain techniques developed in [33] and [10].

2. WEYL AND ANTI-WICK ΨDOs

In this section we recall briefly some basic facts from the theory of ΨDOs with Weyl and anti-Wick symbols, assuming that the dimension $n \geq 1$. We will use the following notations. Let $X$ be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_X$, linear with respect to the first factor, and norm $\| \cdot \|_X$. By $\mathfrak{B}(X)$ (resp., by $\mathfrak{S}_p(X)$) we will denote the space of linear bounded (resp., compact) operators in $X$, and by $\mathfrak{S}_p(X)$, $p \in [1, \infty)$, the $p$th Schatten-von Neumann space of operators $T \in \mathfrak{S}_\infty(X)$ for which the norm $\|T\|_p := (\text{Tr} (T^*T)^{p/2})^{1/p}$ is finite. In particular, $\mathfrak{S}_1(X)$ is the trace class, and $\mathfrak{S}_2(X)$ is the Hilbert-Schmidt class.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class over $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ be its dual class. For $F \in \mathcal{S}(\mathbb{R}^{2n})$ we define the Weyl ΨDO as the operator with integral kernel

$$K(x, x') = (2\pi)^{-n} \int_{\mathbb{R}^n} F \left( \frac{x + x'}{2}, \xi \right) e^{i(x - x') \cdot \xi} \, d\xi, \quad x, x' \in \mathbb{R}^n.$$  

Let $u, v \in \mathcal{S}(\mathbb{R}^n)$. Define the Wigner transform $W(u, v)$ of the pair $(u, v)$ by

$$(W(u, v))(x, \xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} u(x - x'/2) v(x + x'/2) \, dx', \quad (x, \xi) \in \mathbb{R}^{2n}.$$  

Then $W(u, v) \in \mathcal{S}(\mathbb{R}^{2n})$ and we have $W(v, u) = \overline{W(u, v)}$. Moreover, the Wigner transform extends to $u, v \in L^2(\mathbb{R}^n)$ in which case

$$\|W(u, v)\|_{L^2(\mathbb{R}^{2n})}^2 = (2\pi)^{-n} \|u\|_{L^2(\mathbb{R}^n)}^2 \|v\|_{L^2(\mathbb{R}^n)}^2.$$  

By [37], Eq. (23.39)], the function $(2\pi)^n W(u, v)$ coincides with the Weyl symbol of the operator with integral kernel $u(x)v(x')$, $x, x' \in \mathbb{R}^n$. Note that if $F \in \mathcal{S}(\mathbb{R}^{2n})$ and $u, v \in \mathcal{S}(\mathbb{R}^n)$, then

$$\langle \text{Op}^w(F)u, v \rangle_{L^2(\mathbb{R}^n)} = \langle F, W(v, u) \rangle_{L^2(\mathbb{R}^{2n})}.$$  

Therefore, if $\mathcal{F} \in \mathcal{S}'(\mathbb{R}^{2n})$, then (2.2) defines a linear continuous mapping $\text{Op}^w(\mathcal{F}) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Let us now introduce the Fourier transform

$$(\Phi u)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx, \quad \xi \in \mathbb{R}^N,$$

for $u \in \mathcal{S}(\mathbb{R}^N)$, $N \geq 1$, and then extend it to $\mathcal{S}'(\mathbb{R}^N)$. In particular, $\Phi$ extends to a unitary operator in $L^2(\mathbb{R}^N)$.

If $\mathcal{F} \in \mathcal{S}(\mathbb{R}^{2n})$, then the integral kernel of the operator $\text{Op}^w(\mathcal{F})$ can be written not only as (2.1) but also as

$$(2.3) \quad K(x, x') = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\mathcal{F}}(\xi, x' - x) e^{i\frac{(x + x')}{2} \cdot \xi} \, d\xi, \quad x, x' \in \mathbb{R}^n.$$ 

Let $\Gamma_w(\mathbb{R}^{2n})$, $n \geq 1$, denote the set of functions $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{C}$ such that

$$\|\mathcal{F}\|_{\Gamma_w(\mathbb{R}^{2n})} := \sup_{\{\alpha, \beta \in \mathbb{Z}_+^n \mid |\alpha|, |\beta| \leq \frac{3}{2}\}} \sup_{(x, \xi) \in \mathbb{R}^{2n}} |D_x^\alpha D_\xi^\beta \mathcal{F}(x, \xi)| < \infty.$$ 

Note that $\Gamma_w(\mathbb{R}^{2n}) \subset \mathcal{S}'(\mathbb{R}^{2n})$.

**Proposition 2.1.** ([6], [8], [11 Corollary 2.5 (i)]) Let $\mathcal{F} \in \Gamma_w(\mathbb{R}^{2n})$. Then $\text{Op}^w(\mathcal{F})$ extends to an operator bounded in $L^2(\mathbb{R}^n)$. Moreover, there exists a constant $c_0$ independent of $\mathcal{F}$, such that

$$\|\text{Op}^w(\mathcal{F})\| \leq c_0 \|\mathcal{F}\|_{\Gamma_w(\mathbb{R}^{2n})}.$$ 

**Remark:** We will consider Weyl $\Psi$DOs $\text{Op}^w(\mathcal{F})$ acting in $L^2(\mathbb{R}^n)$, under the generic assumption $\mathcal{F} \in \Gamma_w(\mathbb{R}^{2n})$; then, by Proposition 2.1, we have $\text{Op}^w(\mathcal{F}) \in \mathcal{B}(L^2(\mathbb{R}^n))$. However, many assertions in the sequel remain valid under more general assumptions about $\mathcal{F}$.

Further, for $m \in \mathbb{R}$ and $\varrho \in (0, 1]$, introduce the Hörmander–Shubin class

$$\mathcal{S}^m_\varrho(\mathbb{R}^N) := \{ u \in C^\infty(\mathbb{R}^N) \mid |D^\alpha u(x)| \leq C_\alpha(x)|m - \varrho|, \quad x \in \mathbb{R}^N, \quad \alpha \in \mathbb{Z}_+^n \}.$$ 

**Proposition 2.2.** ([37] Problem 24.9) Let $\mathcal{F} \in \mathcal{S}^0_\varrho(\mathbb{R}^{2n})$ with $\varrho \in (0, 1]$. Assume that

$$\lim_{|w| \to \infty} \mathcal{F}(w) = 0.$$ 

Then $\text{Op}^w(\mathcal{F}) \in \mathcal{S}_\infty(L^2(\mathbb{R}^n))$.

**Proposition 2.3.** Let $\mathcal{F} \in L^2(\mathbb{R}^{2n})$. Then $\text{Op}^w(\mathcal{F})$ extends to a Hilbert-Schmidt operator in $L^2(\mathbb{R}^n)$, and

$$(2.4) \quad \|\text{Op}^w(\mathcal{F})\|^2_2 = (2\pi)^{-n} \|\mathcal{F}\|^2_{L^2(\mathbb{R}^{2n})}.$$ 

**Proof.** Identity (2.4) follows immediately from (2.1) and

$$\|\text{Op}^w(\mathcal{F})\|^2_2 = \int_{\mathbb{R}^{2n}} |K(x, x')|^2 \, dx \, dx'.$$ 

\qed
Next, we describe the **metaplectic unitary equivalence** of Weyl ΨDOs whose symbols are mapped into each other by a linear symplectic transformation.

**Proposition 2.4.** [11] Chapter 7, Theorem A.2] Let \( \kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \), \( n \geq 1 \), be a linear symplectic transformation. Assume that \( F \in \Gamma_w(\mathbb{R}^{2n}) \). Then there exists a unitary operator \( M_\kappa : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) such that

\[
(2.5) \quad Op^w(F \circ \kappa) = M_\kappa^* Op^w(F) M_\kappa.
\]

Remark: (i) Proposition \[22\] remains valid for a considerably wider class of symbols including the linear and the quadratic ones.

(ii) The operator \( M_\kappa \) is called the **metaplectic operator** generated by the linear symplectomorphism \( \kappa \).

Further, we discuss the anti-Wick ΨDOs. Let at first \( F \in \Gamma_w(\mathbb{R}^{2n}) \). Set

\[
(2.6) \quad G_n(w) := \pi^{-n} e^{-|w|^2}, \quad w \in \mathbb{R}^{2n},
\]

and define the ΨDO

\[
Op^aw(F) := Op^w(F * G_n).
\]

Then we will say that \( Op^aw(F) \) is a ΨDO with anti-Wick symbol \( F \). If \( F \in S(\mathbb{R}^{2n}) \) and \( u, v \in S(\mathbb{R}^n) \), then, similarly to \( (2.2) \), we have

\[
(2.7) \quad \langle Op^aw(F) u, v \rangle_{L^2(\mathbb{R}^n)} = \langle F, G_n * W(v, u) \rangle_{L^2(\mathbb{R}^{2n})}
\]

where \( G_n * W(u, v) \in S(\mathbb{R}^{2n}) \) is the **Husimi transform** of \((u, v)\). Therefore, if \( F \in S^1(\mathbb{R}^{2n}) \), then \( (2.7) \) defines, similarly to \( (2.2) \), a linear continuous mapping \( Op^aw(F) : S(\mathbb{R}^n) \to S^1(\mathbb{R}^n) \).

Since the convolution with the Gaussian function \( G_n \) may improve the regularity and the decay rate of the symbol \( F \), the definition of the anti-Wick ΨDOs can be extended to a class of symbols, considerably larger than \( \Gamma_w(\mathbb{R}^{2n}) \). In particular, we have the following

**Proposition 2.5.** [31] Lemma 2.5] (i) Let \( F \in L^\infty(\mathbb{R}^{2n}) \). Then \( Op^aw(F) \in B(L^2(\mathbb{R}^n)) \), and

\[
\|Op^aw(F)\| \leq \|F\|_{L^\infty(\mathbb{R}^{2n})}.
\]

(ii) Let \( F \in L^p(\mathbb{R}^{2n}) \), \( p \in [1, \infty) \). Then \( Op^aw(F) \in \mathcal{S}_p(L^2(\mathbb{R}^n)) \), and

\[
\|Op^aw(F)\|_p \leq (2\pi)^{-n} \|F\|_{L^p(\mathbb{R}^{2n})}^p.
\]

Set

\[
\Gamma^{aw}(\mathbb{R}^{2n}) := L^1(\mathbb{R}^{2n}) + L^\infty(\mathbb{R}^{2n}).
\]

Our generic assumption concerning anti-Wick ΨDOs \( Op^aw(F) \) will be \( F \in \Gamma^{aw}(\mathbb{R}^{2n}) \). As in the case of Weyl ΨDOs, many assertions in the sequel hold true under wider assumptions.

Note that if \( F \in \Gamma^{aw}(\mathbb{R}^{2n}) \), then \( F * G_n \in \Gamma(\mathbb{R}^{2n}) \).

Let us give an alternative definition of the anti-Wick ΨDO \( Op^aw(F) \) with \( F \in \Gamma^{aw}(\mathbb{R}^{2n}) \). For \((x, \xi) \in \mathbb{R}^{2n}\) set

\[
\phi_{x, \xi}(y) := \pi^{-n/4} e^{i\xi \cdot y} e^{-\frac{|x-y|^2}{2}}, \quad y \in \mathbb{R}^n,
\]

where

(2.10) \( \phi_{x, \xi}(y) := \pi^{-n/4} e^{i\xi \cdot y} e^{-\frac{|x-y|^2}{2}}, \quad y \in \mathbb{R}^n \).
and introduce the rank-one orthogonal projection

\[ P_{\phi, \xi} := \langle \cdot, \phi_{\phi, \xi} \rangle_{L^2(\mathbb{R}^n)} \phi_{\phi, \xi}. \]

Then we have

\[ \text{Op}^{aw}(F) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} F(x, \xi) P_{x, \xi} \, dx \, d\xi, \]

the integral being understood in the weak sense. Identity (2.8) implies the monotonicity of \( \text{Op}^{aw}(F) \) with respect to the symbol \( F \). Namely, we have the following important

**Proposition 2.6.** Assume that \( F \in \Gamma_{aw}(\mathbb{R}^{2n}) \), and \( F(x, \xi) \geq 0 \) for almost every \( (x, \xi) \in \mathbb{R}^{2n} \). Then \( \text{Op}^{aw}(F) \geq 0 \).

**Remark:** Not every Weyl \( \Psi \text{DO} \) \( \text{Op}^{w}(F) \) admits an anti-Wick symbol \( \tilde{F} \in S'(\mathbb{R}^{2n}) \). If \( F \) is a given Weyl symbol, then in order to find the corresponding anti-Wick symbol \( \tilde{F} \), we have to solve the equation

\[ F = \tilde{F} \ast G_n, \]

i.e. to invert the so called Weierstrass transform, or, which is equivalent, to solve the inverse heat equation (see [37, Remark 24.2]). For example, if \( F \in C_0^\infty(\mathbb{R}^{2n}) \), then there exists no \( \tilde{F} \in S'(\mathbb{R}^{2n}) \) such that (2.9) holds true. On the other hand, if \( F \in \Phi^* C_0^\infty(\mathbb{R}^{2n}) \), then \( \text{Op}^{w}(F) \) admits an anti-Wick symbol \( \tilde{F} \in S(\mathbb{R}^{2n}) \) given by

\[ \tilde{F}(w) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i \langle u, w \rangle} \frac{1}{|u|^2/4} \tilde{F}(u) \, du, \quad w \in \mathbb{R}^{2n}. \]

### 3. Unitary equivalences of the operators \( H_{V} \)

In this section we establish unitary equivalences for the Landau Hamiltonian \( H_0 \) and its perturbation \( \text{Op}^{w}(V) \). First, we describe a suitable spectral representation of \( H_0 \). Let \( \psi(x, y) := \frac{b(x^2 + y^2)}{4} \), so that \( \Delta \psi = b \). Introduce the magnetic creation operator

\[ a^* = -2i e^\psi \frac{\partial}{\partial z} e^{-\psi}, \quad z = x + iy, \]

and the magnetic annihilation operator

\[ a = -2i e^{-\psi} \frac{\partial}{\partial \bar{z}} e^\psi, \quad \bar{z} = x - iy. \]

The operators \( a \) and \( a^* \) are closed on \( \mathfrak{D}(a) = \mathfrak{D}(a^*) = \mathfrak{D}(H_0^{1/2}) \), and are mutually adjoint in \( L^2(\mathbb{R}^2) \). Moreover,

\[ [a, a^*] = 2b \, I, \]

and

\[ H_0 = a^*a + bI = aa^* - bI. \]

Therefore,

\[ \text{Ker}(H_0 - \Lambda_q I) = (a^*)^q \text{Ker} a, \quad q \in \mathbb{Z}_+, \]
and, by \((3.2)\), we have

\[ \operatorname{Ker} a = \left\{ u \in L^2(\mathbb{R}^2) \mid u = e^{-\varphi} g, \quad \frac{\partial g}{\partial \bar{z}} = 0 \right\}. \]

Thus, up to the unitary mapping \( g \mapsto e^{-\varphi} g \), \( \operatorname{Ker} a \) coincides with \textit{Fock-Segal-Bargmann space} of holomorphic functions (see e.g. [21, Section 3.2]).

Next, we recall that \( H_0 \) is unitarily equivalent under an appropriate metaplectic mapping to the operator \((b \hbar) \otimes I_y\), where

\( \hbar := -\frac{d^2}{dx^2} + x^2 \)

is the harmonic oscillator, self-adjoint in \( L^2(\mathbb{R}_x) \), and essentially self-adjoint on \( C_0^{\infty}(\mathbb{R}) \), while \( I_y \) is the identity in \( L^2(\mathbb{R}_y) \). Let us recall the spectral properties of \( \hbar \). We have

\[ \hbar = \alpha^* \alpha + I = \alpha \alpha^* - I, \]

where

\[ \alpha := -i \frac{d}{dx} - ix, \quad \alpha^* := -i \frac{d}{dx} + ix, \]

are the annihilation and creation operators respectively, closed on \( \mathcal{D}(\alpha) = \mathcal{D}(\alpha^*) = \mathcal{D}(\hbar^{1/2}) \), and mutually adjoint in \( L^2(\mathbb{R}) \). Moreover, they satisfy the commutation relation

\[ [\alpha, \alpha^*] = 2I. \]

Therefore,

\[ \sigma(\hbar) = \bigcup_{q \in \mathbb{Z}_+} \{2q + 1\}, \]

\[ \operatorname{Ker} (\hbar - (2q + 1)I) = (\alpha^*)^q \operatorname{Ker} \alpha, \quad q \in \mathbb{Z}_+. \]

Since

\[ \operatorname{Ker} \alpha = \left\{ u \in L^2(\mathbb{R}) \mid u(x) = ce^{-x^2/2}, x \in \mathbb{R}, \quad c \in \mathbb{C} \right\}, \]

we have

\[ \dim \operatorname{Ker} (\hbar - (2q + 1)I) = 1, \quad q \in \mathbb{Z}_+. \]

Denote by \( \pi_q \) the orthogonal projection onto \( \operatorname{Ker} (\hbar - (2q + 1)I), \quad q \in \mathbb{Z}_+. \) Set

\[ \tilde{\psi}_q(x) := \left(-\frac{d}{dx} + x\right)^q e^{-x^2/2} = (-i)^q (\alpha^*)^q e^{-x^2/2}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+. \]

Then the functions \( \tilde{\psi}_q \) satisfy

\[ \hbar \tilde{\psi}_q = (2q + 1) \tilde{\psi}_q, \quad q \in \mathbb{Z}_+, \]

and form an orthogonal eigenbasis in \( L^2(\mathbb{R}) \). A simple calculation shows that

\[ \psi_q := \tilde{\psi}_q / \| \tilde{\psi}_q \| = \frac{H_q(x)e^{-x^2/2}}{(\sqrt{\pi}2^q q!)^{1/2}}, \quad x \in \mathbb{R}, \]

where

\[ H_q(x) := e^{x^2/2} \left(-\frac{d}{dx} + x\right)^q e^{-x^2/2} = (-1)^q e^{x^2} \frac{d^q}{dx^q} e^{-x^2}, \quad x \in \mathbb{R}, \]
is the Hermite polynomial of degree $q$ (see e.g. [20]). Thus, the functions $\psi_q$, $q \in \mathbb{Z}_+$, form an orthonormal basis in $L^2(\mathbb{R})$. Introduce the Wigner functions

$$\Psi_{j,k} := W(\psi_j, \psi_k), \quad j, k \in \mathbb{Z}_+. \quad (3.8)$$

If $j = k$, we will write

$$\Psi_k = \Psi_{k,k}, \quad k \in \mathbb{Z}_+. \quad (3.9)$$

Lemma 3.1 below contains explicit expressions for $\Psi_{k,\ell}$, $k, \ell \in \mathbb{Z}_+$. In order to formulate it, we introduce the (generalized) Laguerre polynomials

$$L_q^{(\nu)}(\xi) := \frac{e^{-\nu \xi}}{q!} \frac{d^n}{d\xi^n} \left( \xi^q e^{-\xi} \right), \quad \xi > 0, \quad \nu \in \mathbb{R}, \quad q \in \mathbb{Z}_+. \quad (3.10)$$

As usual, we will write

$$L_q(\xi) := L_q^{(0)}(\xi) = \frac{e^\xi}{q!} \frac{d^n}{d\xi^n} \left( \xi^q e^{-\xi} \right) = \sum_{\ell=0}^{q} \binom{q}{\ell} \frac{(-\xi)^\ell}{\ell!}, \quad \xi \in \mathbb{R}. \quad (3.11)$$

**Lemma 3.1.** Let $k, \ell \in \mathbb{Z}_+$. Then for $(x, \xi) \in \mathbb{R}^2$ we have

$$\Psi_{k,\ell}(x, \xi) =$$

$$\begin{cases} \frac{1}{\pi} (-1)^\ell 2^{\frac{k-\ell}{2}} \left( \frac{q}{2} \right)^{1/2} (x + i\xi)^k \ell \frac{1}{2} \left( 2x^2 + \xi^2 \right) e^{-(x^2 + \xi^2)} & \text{for } \ell \geq k, \\
\frac{1}{\pi} (-1)^k 2^{\frac{k+\ell}{2}} \left( \frac{q}{2} \right)^{1/2} (x - i\xi)^\ell \frac{1}{2} \left( 2x^2 + \xi^2 \right) e^{-(x^2 + \xi^2)} & \text{for } \ell \leq k. \end{cases} \quad (3.12)$$

In particular,

$$\Psi_{k,\ell}(r \cos \theta, r \sin \theta) = e^{i(k-\ell)\theta} \Phi_{k,\ell}(r), \quad k, \ell \in \mathbb{Z}_+, \quad \theta \in [0, 2\pi), \quad r \in [0, \infty), \quad \text{where } \Phi_{k,\ell}(r) \text{ is a symmetric real valued matrix. Moreover,}$$

$$\Psi_k(x, \xi) = \frac{1}{\pi} (-1)^k L_k(2(x^2 + \xi^2)) e^{-(x^2 + \xi^2)}, \quad k \in \mathbb{Z}_+, \quad (x, \xi) \in \mathbb{R}^2. \quad (3.14)$$

**Proof.** An elementary calculation taking into account the parity of the Hermite polynomials easily yields

$$\Psi_{k,\ell}(x, \xi) = \frac{(-1)^k}{(2\pi)^{1/2} \left( k! \ell! \right)^{1/2} 2^{\frac{k+\ell}{2}}} e^{-(x^2 + \xi^2)} \int_{\mathbb{R}} e^{-\left( \frac{y}{2} + i\xi \right)^2} H_k \left( \frac{y}{2} - x \right) H_\ell \left( \frac{y}{2} + x \right) dy. \quad (3.15)$$

Changing the variable $\frac{y}{2} + i\xi = t$, and applying a standard complex-analysis argument showing that we can replace the interval of integration $\mathbb{R} + i\xi$ by $\mathbb{R}$, we find that

$$\int_{\mathbb{R}} e^{-\left( \frac{y}{2} + i\xi \right)^2} H_k \left( \frac{y}{2} + x \right) H_\ell \left( \frac{y}{2} - x \right) dy = 2 \int_{\mathbb{R}} e^{-t^2} H_k (t - x - i\xi) H_\ell (t + x - i\xi) dt. \quad (3.16)$$

By [20, Eq. (7.377)],

$$\int_{\mathbb{R}} e^{-t^2} H_k (t - x - i\xi) H_\ell (t + x - i\xi) dt =$$

$$= \frac{(-1)^k}{(2\pi)^{1/2} \left( k! \ell! \right)^{1/2} 2^{\frac{k+\ell}{2}}} e^{-(x^2 + \xi^2)} \int_{\mathbb{R}} e^{-\left( \frac{y}{2} + i\xi \right)^2} H_k \left( \frac{y}{2} - x \right) H_\ell \left( \frac{y}{2} + x \right) dy. \quad (3.17)$$
\[(3.17)\]
\[
\begin{aligned}
2^k \sqrt{\ell!}((-x - i\xi)^{k+\ell} L_k^{(k+\ell)}(2(x^2 + \xi^2)), & \quad k \geq \ell, \\
2^\ell \sqrt{\ell^!}((-x - i\xi)^{\ell-k} L_k^{(\ell-k)}(2(x^2 + \xi^2)), & \quad k \leq \ell.
\end{aligned}
\]
Putting together (3.15), (3.16), and (3.17), we obtain (3.12). □

Remark: By (3.14) with \(q = 0\), we have
\[(3.18)\]
\[
\Psi_0(x, \xi) = \frac{1}{\pi} e^{-(x^2 + \xi^2)} = G_1(x, \xi), \quad (x, \xi) \in \mathbb{R}^2,
\]
\(G_1\) being defined in (2.6).

For \(x = (x, y) \in \mathbb{R}^2\), \(\xi = (\xi, \eta) \in \mathbb{R}^2\), set
\[(3.19)\]
\[
\kappa_b(x, \xi) := \left( \frac{1}{\sqrt{b}}(x - \eta), \frac{1}{\sqrt{b}}(\xi - y), \frac{\sqrt{b}}{2}(\xi + y), -\frac{\sqrt{b}}{2}(\eta + x) \right).
\]
Evidently, the mapping \(\kappa_b\) is linear and symplectic. Introduce the Weyl symbol
\[(3.20)\]
\[
H_0(x, \xi) = \left( \xi + \frac{1}{2}b y \right)^2 + \left( \eta - \frac{1}{2}b x \right)^2, \quad (x, y) \in \mathbb{R}^2, \quad \xi = (\xi, \eta) \in \mathbb{R}^2,
\]
of the operator \(H_0\) defined in (1.1). Then we have
\[(3.21)\]
\[
(H_0 \circ \kappa_b)(x, \xi) = b(x^2 + \xi^2), \quad (x, \xi) \in T^* \mathbb{R}^d.
\]
Note that the function on the r.h.s. of (3.21) coincides with the Weyl symbol of the operator \((b\hbar) \otimes I_y\). Next, define the unitary operator \(U_b : L^2(\mathbb{R}^2_{x,y}) \rightarrow L^2(\mathbb{R}^2_{x,y})\) by
\[(3.22)\]
\[
(U_b u)(x, y) := \frac{\sqrt{b}}{2\pi} \int_{\mathbb{R}^2} e^{i\phi_b(x, y; x', y')} u(x', y') dx' dy'
\]
where
\[
\phi_b(x, y; x', y') := b \frac{xy}{2} + b^{1/2}(xy' - yx') - x'y'.
\]
Writing \(\kappa_b\) as a product of elementary linear symplectic transformations (see e.g. [22 Lemma 18.5.8]), and composing the corresponding elementary metaplectic operators, we easily check that \(U_b\) is a metaplectic operator generated by the symplectic mapping \(\kappa_b\) in (3.19).

Proposition 3.2. We have
\[(3.23)\]
\[
U_b^* H_0 U_b = (b\hbar) \otimes I_y,
\]
\[(3.24)\]
\[
U_b^* a U_b = (\sqrt{b}a) \otimes I_y, \quad U_b^* a^* U_b = (\sqrt{b}a^*) \otimes I_y.
\]
Moreover, if \(V \in \Gamma_w(\mathbb{R}^4)\), then
\[(3.25)\]
\[
U_b^* \text{Op}^w(V) U_b = \text{Op}^w(V_b)
\]
where
\[(3.26)\]
\[
V_b := V \circ \kappa_b.
\]
Therefore, the functions form an orthonormal basis of $\text{Ran} p$. The commutation relation (3.3) easily implies
\[ \lim_{x^2+y^2+\xi^2+\eta^2 \to \infty} V_b(x,y,\xi,\eta) (x^2 + \xi^2)^{-1} = 0. \]
Then, $\text{Op}^w(V)$ is bounded, and $\text{Op}^w(V)H_0^{-1}$ is compact in $L^2(\mathbb{R}^2)$.

**Proof.** Since $\mathcal{S}_q^0(\mathbb{R}^4) \subseteq \Gamma_w(\mathbb{R}^4)$, the boundedness of $\text{Op}^w(V)$ follows from Proposition 2.1. By Proposition 3.2, we have
\[ \mathcal{U}_b^* \text{Op}^w(V)H_0^{-1} \mathcal{U}_b = \text{Op}^w(V_b)((bh)^{-1}\otimes I_y). \]
By the pseudo-differential calculus, we easily find that the Weyl symbol of the operator $\text{Op}^w(V_b)((bh)^{-1}\otimes I_y)$ is in the class $\mathcal{S}_q^0(\mathbb{R}^4)$, while (3.27) guarantees that this symbol decays at infinity. Then Proposition 2.2 implies that $\text{Op}^w(V_b)((bh)^{-1}\otimes I_y) \in \mathcal{S}_\infty(L^2(\mathbb{R}^2))$, and by (3.28) we find that the operator $\text{Op}^w(V)H_0^{-1}$ is compact as well.

Our next goal is to establish the unitary equivalence between $\text{Op}^w(V)$ and an operator $M : \ell^2(\mathbb{Z}_+^2) \rightarrow \ell^2(\mathbb{Z}_+^2)$. Similarly, we will establish the unitary equivalence between the Toeplitz operator $T_q(V)$ with fixed $q \in \mathbb{Z}_+$, defined in (1.2), and an operator $M_q : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$. To this end, we need the canonical basis $\{\varphi_{k,q}\}_{k \in \mathbb{Z}_+}$ of $\text{Ran} p_q$, $q \in \mathbb{Z}_+$.

Let at first $q = 0$. Then the functions
\[ \tilde{\varphi}_{k,0}(x) = z^k e^{-b|x|^2/4}, \quad x = (x,y) \in \mathbb{R}^2, \quad z = x + iy, \quad k \in \mathbb{Z}_+, \]
form a natural orthogonal basis of $\text{Ker} a = \text{Ran} p_0$ (see e.g. [21, Sections 3.1 – 3.2]). Normalizing, we obtain the following orthonormal basis of $\text{Ran} p_0$:
\[ \varphi_{k,0}(x) := \frac{\tilde{\varphi}_{k,0}(x)}{\|\tilde{\varphi}_{k,0}\|_{L^2(\mathbb{R}^2)}} = \sqrt{\frac{b}{2\pi}} \sqrt{\frac{1}{k!}} \left( \sqrt{\frac{b}{2}} z \right)^k e^{-b|x|^2/4}, \quad x \in \mathbb{R}^2, \quad k \in \mathbb{Z}_+. \]

Let now $q \geq 1$. Set
\[ \tilde{\varphi}_{k,q} = (a^*)^q \varphi_{k,0}, \quad k \in \mathbb{Z}_+. \]
The commutation relation (3.3) easily implies
\[ \langle \tilde{\varphi}_{k,q}, \tilde{\varphi}_{\ell,q} \rangle_{L^2(\mathbb{R}^2)} = (2b)^q q! \delta_{k\ell}, \quad k, \ell \in \mathbb{Z}_+. \]
Therefore, the functions
\[ \varphi_{k,q} := \frac{\tilde{\varphi}_{k,q}}{\|\tilde{\varphi}_{k,q}\|_{L^2(\mathbb{R}^2)}} = \frac{\tilde{\varphi}_{k,q}}{\sqrt{(2b)^q q!}}, \quad k \in \mathbb{Z}_+, \]
form an orthonormal basis of $\text{Ran} p_q$, $q \in \mathbb{N}$.

**Remark:** The functions $\varphi_{k,q}$ admit an explicit expression, namely
\[ \varphi_{k,q}(x) = \ldots \]
\[
\frac{1}{i^q} \sqrt{\frac{b}{2\pi}} \sqrt{\frac{q!}{k!}} \left( \sqrt{\frac{b}{2}} z \right)^{k-q} L_q^{(k-q)} \left( \frac{b|x|^2}{2} \right) e^{-b|x|^2/4}, \quad x \in \mathbb{R}^2, \ k, q \in \mathbb{Z}_+,
\]
the Laguerre polynomials being defined in (3.10).

Let \( V \in \Gamma_w(\mathbb{R}^4) \). Set

\[
m_{k,\ell,q,r}(V) := \langle \text{Op}^w(V) \varphi_{\ell,r}, \varphi_{k,q} \rangle_{L^2(\mathbb{R}^2)}, \quad m_{k,\ell,q}(V) := m_{k,\ell,q,q} \quad k, \ell, q, r \in \mathbb{Z}_+.
\]

The facts that \( \{\varphi_{k,q}\}_{(k,q) \in \mathbb{Z}_+^2} \) is an orthonormal basis in \( L^2(\mathbb{R}^2) \), while \( \{\varphi_{k,q}\}_{k \in \mathbb{Z}_+} \) is an orthonormal basis of \( \text{Ran} p_q \) with fixed \( q \in \mathbb{Z}_+ \), imply immediately the following elementary

**Proposition 3.4.** Let \( V \in \Gamma_w(\mathbb{R}^4) \).

(i) The operator \( \text{Op}^w(V) \) is unitarily equivalent to \( \mathcal{M} : \ell^2(\mathbb{Z}_+^2) \rightarrow \ell^2(\mathbb{Z}_+^2) \) defined by

\[
(\mathcal{M}c)_{k,q} := \sum_{(\ell,r) \in \mathbb{Z}_+^2} m_{k,\ell,q,r} c_{\ell,r}, \quad (k,q) \in \mathbb{Z}_+^2, \quad c = \{c_{\ell,r}\}_{(\ell,r) \in \mathbb{Z}_+^2} \in \ell^2(\mathbb{Z}_+^2).
\]

(ii) Fix \( q \in \mathbb{Z}_+ \). Then the operator \( \mathcal{T}_q(V) \) is unitarily equivalent to \( \mathcal{M}_q : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+) \) defined by

\[
(\mathcal{M}_q c)_k := \sum_{\ell \in \mathbb{Z}_+} m_{k,\ell,q} c_{\ell}, \quad k \in \mathbb{Z}_+, \quad c = \{c_{\ell}\}_{\ell \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+).
\]

We would like to give a more explicit form of the matrices defining the operators \( \mathcal{M} \) and \( \mathcal{M}_q, q \in \mathbb{Z}_+ \). To this end we need the following

**Lemma 3.5.** We have

\[
\mathcal{U}_b^* \varphi_{k,q} = i^q k \psi_q \otimes \psi_k, \quad k, q \in \mathbb{Z}_+,
\]
the Hermite functions \( \psi_q, q \in \mathbb{Z}_+ \), being defined in (3.1).

*Proof.* By (3.29) – (3.31), and (3.26), we get

\[
(3.36) \quad \mathcal{U}_b^* \varphi_{k,q} = \sqrt{\frac{b^{k+1}}{\pi 2^{k+q+1} k! q!}} (\alpha^q \otimes I_g) \mathcal{U}_b^* u_k
\]

where

\[
u_k(x,y) = (x + iy)^k e^{-b(x^2 + y^2)/4}, \quad (x,y) \in \mathbb{R}^2.
\]

Using (3.22), we easily find that

\[
(3.37) \quad (\mathcal{U}_b^* u_k)(x,y) = \frac{1}{2\pi \sqrt{b}} \left( \frac{2}{\sqrt{b}} \right)^k e^{i xy} \left( \frac{\partial}{\partial z} \right)^k J(x,y)
\]

where

\[
J(x,y) := \int_{\mathbb{R}^2} e^{-i(ty-sx)} e^{-its/2} e^{-(t^2+s^2)/4} dt ds, \quad (x,y) \in \mathbb{R}^2.
\]

Calculating, we find that

\[
(3.38) \quad J(x,y) = \sqrt{2(2\pi)} e^{-i xy} e^{-((x^2+y^2)/2)}.
\]
Inserting (3.38) into (3.37), we get
\[
(U_0^* u_k)(x, y) = \sqrt{\frac{2}{j^{k+1}}} e^{-x^2/2}(-1)^k (\alpha^*)^k e^{-y^2/2}.
\]

Inserting (3.39) into (3.36), we obtain (3.35). □

**Remark**: By (3.35), we have
\[
U_b f = i^{k-q} \sum_{(k,q) \in \mathbb{Z}^2_+} \langle f, \psi_q \otimes \psi_k \rangle L^2(\mathbb{R}^2) \varphi_{k,q}, \quad f \in L^2(\mathbb{R}^2).
\]

Fix \( q \in \mathbb{Z}_+ \). Let \( \mathcal{V} \in \Gamma_w(\mathbb{R}^4) \). Set
\[
v_{b,q}(y, \eta) := \int_{\mathbb{R}^2} \mathcal{V}_b(x, y, \xi, \eta) \Psi_q(x, \xi) \, dx \, d\xi, \quad (y, \eta) \in \mathbb{R}^2,
\]
the symbol \( \mathcal{V}_b \) being defined in (3.20), and the Wigner function \( \Psi_q \) being defined in (3.9).

**Proposition 3.6.** Let \( \mathcal{V} \in \Gamma_w(\mathbb{R}^4) \). Then we have
\[
m_{k,\ell,q,r}(\mathcal{V}) = i^{k-\ell-q+r} \langle \mathcal{V}_b, \Psi_{q,r} \otimes \Psi_{k,\ell} \rangle L^2(\mathbb{R}^4)
\]
where
\[
(\Psi_{q,r} \otimes \Psi_{k,\ell})(x, y, \xi, \eta) := \Psi_{q,r}(x, \xi) \Psi_{k,\ell}(y, \eta), \quad k, \ell, q, r \in \mathbb{Z}_+, \quad (x, y, \xi, \eta) \in \mathbb{R}^4.
\]
In particular,
\[
m_{k,\ell,q}(\mathcal{V}) = i^{k-\ell} \langle \mathcal{V}_b, \Psi_q \otimes \Psi_{k,\ell} \rangle L^2(\mathbb{R}^2) = i^{k-\ell} \langle v_{b,q}, \Psi_{k,\ell} \rangle L^2(\mathbb{R}^2).
\]

**Proof.** By (3.32), (3.25), and (3.35), we have
\[
m_{k,\ell,q,r}(\mathcal{V}) = \langle \text{Op}^w(\mathcal{V}) \varphi_{\ell,r}, \varphi_{k,q} \rangle L^2(\mathbb{R}^2)
\]
\[
= \langle U_0^* \text{Op}^w(\mathcal{V}) U_0^* \varphi_{\ell,r}, U_0^* \varphi_{k,q} \rangle L^2(\mathbb{R}^2)
\]
\[
= i^{k-\ell-q+r} \langle \text{Op}^w(\mathcal{V}_b) \psi_r \otimes \psi_{\ell}, \psi_q \otimes \psi_k \rangle L^2(\mathbb{R}^2)
\]
\[
= i^{k-\ell-q+r} \langle \mathcal{V}_b, W(\psi_q \otimes \psi_k, \psi_r \otimes \psi_{\ell}) \rangle L^2(\mathbb{R}^4)
\]
\[
= i^{k-\ell-q+r} \langle \mathcal{V}_b, W(\psi_q, \psi_k) \otimes W(\psi_r, \psi_{\ell}) \rangle L^2(\mathbb{R}^4)
\]
\[
= i^{k-\ell-q+r} \langle \mathcal{V}_b, \Psi_{q,r} \otimes \Psi_{k,\ell} \rangle L^2(\mathbb{R}^4).
\]

Let \( q \in \mathbb{Z}_+ \). By analogy with (3.40), define the operator \( U_{b,q} : L^2(\mathbb{R}) \to \text{Ran} p_q \) by
\[
U_{b,q} f = i^k \sum_{k \in \mathbb{Z}_+} \langle f, \psi_k \rangle L^2(\mathbb{R}) \varphi_{k,q}, \quad f \in L^2(\mathbb{R}).
\]

**Corollary 3.7.** Let \( q \in \mathbb{Z}_+, \mathcal{V} \in \Gamma_w(\mathbb{R}^4) \). Then we have
\[
U_{b,q}^* \mathcal{V}(\mathcal{V}) U_{b,q} = \text{Op}^w(v_{b,q}),
\]
v\(_{b,q}\) being defined in (3.41).
Proof. By (3.42), we have
\[ \langle T_q(\mathcal{V})\varphi_{\ell,q}, \varphi_{k,q}\rangle_{L^2(\mathbb{R}^2)} = m_{k,\ell,q}(\mathcal{V}) = i^{k-\ell}\langle \psi_{k,\ell}, \psi_{\ell,q}\rangle_{L^2(\mathbb{R}^2)} = i^{k-\ell}\langle \operatorname{Op}^w(v_{b,q})\psi_{\ell}, \psi_{k}\rangle_{L^2(\mathbb{R}^2)}, \]
which implies (3.45). \qed

At the end of this section we consider the important case where the operator \( \operatorname{Op}^w(v_{b,q}) \) admits an anti-Wick symbol \( \tilde{v}_{b,q} \in \Gamma_{aw}(\mathbb{R}^2) \). Set
\begin{equation}
(3.46) \quad \omega_{b,q}(x,y) := \tilde{v}_{b,q}(-b^{1/2}y, -b^{1/2}x), \quad (x,y) \in \mathbb{R}^2.
\end{equation}
Then, of course, \( \omega_{b,q} \in \Gamma_{aw}(\mathbb{R}^2) \).

**Corollary 3.8.** Let \( q \in \mathbb{Z}_+ \), \( \nu \in \Gamma_{aw}(\mathbb{R}^4) \). Assume that the operator \( \operatorname{Op}^w(v_{b,q}) \) has an anti-Wick symbol \( \tilde{v}_{b,q} \in \Gamma_{aw}(\mathbb{R}^2) \). Then,
\begin{equation}
(3.47) \quad U^*_{b,0}p_0\omega_{b,q}p_0U_{b,0} = \operatorname{Op}^w(v_{b,q}) = \operatorname{Op}^{aw}(\tilde{v}_{b,q}),
\end{equation}
\( U_{b,0} \) being defined in (3.44), and \( \omega_{b,q} \) being defined in (3.46).

Proof. Assume at first that \( \omega_{b,q} \in C_0^\infty(\mathbb{R}^2) \). Then, by Corollary 3.7 the operator \( p_0\omega_{b,q}p_0 \) is unitarily equivalent under the operator \( U_{b,0} \) to a \( \Psi \)DO with Weyl symbol
\[ \int_{\mathbb{R}^2} (\omega_{b,q} \circ \kappa_b)(x,y,\xi,\eta)\Psi_0(x,\xi)dxd\xi = \int_{\mathbb{R}^2} \omega_{b,q}(b^{-1/2}(x-\eta), b^{-1/2}(\xi-y))\Psi_0(x,\xi)dxd\xi = \frac{1}{\pi} \int_{\mathbb{R}^2} \tilde{v}_{b,q}(y-\xi,\eta-x)e^{-(x^2+\xi^2)}dxd\xi = (\tilde{v}_{b,q} \ast \mathcal{G}_1)(y,\eta), \quad (y,\eta) \in \mathbb{R}^2, \]
where we have taken into account (3.18). Thus we get (3.47) for \( \omega_{b,q} \in C_0^\infty(\mathbb{R}^2) \). The result for general \( \omega_{b,q} \in \Gamma_{aw}(\mathbb{R}^2) \) is obtained by an approximation argument similar to the one applied in the proof of [31, Theorem 2.11]. \qed

The operator \( p_0\omega_{b,q}p_0 \) admits a further useful unitary equivalence. For \( r \in \mathbb{Z}_+ \) set
\begin{equation}
(3.48) \quad D_{b,r} := L_r \left( -\frac{\Delta}{2b} \right),
\end{equation}
the Laguerre polynomial \( L_r \) being defined in (3.11). Thus, if \( r = 0 \), we have \( D_{b,0} = I \), and if \( r \geq 1 \), then \( D_{b,r} \) is a partial differential operator with constant coefficients of order \( 2r \).

**Corollary 3.9.** Assume that \( \omega \in \Gamma_{aw}(\mathbb{R}^2) \), and there exist \( r \in \mathbb{N} \) and \( \zeta \in \mathcal{S}'(\mathbb{R}^2) \), such that
\begin{equation}
(3.49) \quad \omega = D_{b,r} \zeta.
\end{equation}
Then the operator \( p_0\omega p_0 : \text{Ran} \ p_0 \to \text{Ran} \ p_0 \) is unitarily equivalent to the operator \( p_r \zeta p_r : \text{Ran} \ p_r \to \text{Ran} \ p_r \).

Proof. By [5, Lemma 3.1] and (3.49), we have
\begin{equation}
(3.50) \quad \langle \omega \varphi_{\ell,0}, \varphi_{k,0}\rangle_{L^2(\mathbb{R}^2)} = \langle (D_{b,r}\zeta) \varphi_{\ell,0}, \varphi_{k,0}\rangle_{L^2(\mathbb{R}^2)} = \langle \zeta \varphi_{\ell,r}, \varphi_{k,r}\rangle_{L^2(\mathbb{R}^2)}, \quad k, \ell \in \mathbb{Z}_+.
\end{equation}
Let \( u \in \text{Ran} \, p_0 \). Then \( u = \sum_{k \in \mathbb{Z}_+} c_k \varphi_{k,0} \) with \( \{c_k\}_{k \in \mathbb{Z}_+} \in l^2(\mathbb{Z}_+) \). Define the unitary operator \( U_r : \text{Ran} \, p_0 \to \text{Ran} \, p_r \) by \( U_r u := \sum_{k \in \mathbb{Z}_+} c_k \varphi_{k,r} \). Then (3.50) implies that
\[
p_0 \omega_{b,q} p_0 = U_r^* \omega p_r \zeta p_r U_r.
\]

4. Spectral properties of Weyl \( \Psi \)DOs with radial symbols

In this section we recall the fact that the Weyl \( \Psi \)DOs \( \text{Op}^w(\mathcal{V}) \) with radial symmetric symbols \( \mathcal{V} \) are diagonalizable in the basis formed by Hermite functions, and obtain explicit expressions for the eigenvalues of the operators \( \text{Op}^w(\mathcal{V}) \) and \( \text{Op}^{\text{aw}}(\mathcal{V}) \). As a corollary, we find explicitly the eigenvalues and the eigenfunctions of \( H_\mathcal{V} \) under the assumption that the symbol \( \mathcal{V}_b = \mathcal{V} \circ \kappa_b \) is radial.

Let \( n \geq 1 \). We will say that the symbol \( \mathcal{F} \in \mathcal{S}(\mathbb{R}^{2n}) \) is radial if there exists a function \( \mathcal{R}_\mathcal{F} : \mathbb{R}^n_+ \to \mathbb{C} \) with \( \mathbb{R}^n_+ := [0, \infty) \), such that
\[
\mathcal{F}(x, \xi) = \mathcal{F}(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = \mathcal{R}_\mathcal{F}(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2), \quad (x, \xi) \in \mathbb{R}^{2n}.
\]
We will say that \( \mathcal{F} \in \mathcal{S}'(\mathbb{R}^{2n}) \) is radial if for each \( \mathcal{Q} \in \mathcal{S}(\mathbb{R}^{2n}) \) there exists a radial \( \mathcal{R} \in \mathcal{S}(\mathbb{R}^{2n}) \) such that
\[
(\mathcal{F}, \mathcal{Q}) = (\mathcal{F}, \mathcal{R}),
\]
\((\cdot, \cdot)\) being the usual pairing between \( \mathcal{S} \) and \( \mathcal{S}' \). Note that if \( \mathcal{F} \in \mathcal{S}'(\mathbb{R}^n) \) is radial, then its Fourier transform \( \hat{\mathcal{F}} \) is radial as well. Moreover, if the radial symbol \( \mathcal{F} \) is real-valued, then \( \hat{\mathcal{F}} \) is real-valued as well.

Set
\[
\mathcal{L}_k(t) = \prod_{j=1}^n (L_{k_j}(t_j) e^{-t_j/2}), \quad t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+, \quad k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+.
\]
As is well known, \( \{\mathcal{L}_k\}_{k \in \mathbb{Z}^n_+} \), and hence \( \{(-1)^{|k|} \mathcal{L}_k\}_{k \in \mathbb{Z}^n_+} \), are orthonormal bases in \( L^2(\mathbb{R}^n_+) \). Note that the corresponding Fourier coefficients are defined not only for functions in \( L^2(\mathbb{R}^n_+) \) but also for elements of \( L^1(\mathbb{R}^n_+) + \mathcal{L}(\mathbb{R}^n_+) \), as well as for more general distributions (see e.g. [12, 24]).

**Proposition 4.1.** (i) Let \( \mathcal{F} \in \Gamma_w(\mathbb{R}^{2n}) \) be a radial symbol. Then the operator \( \text{Op}^w(\mathcal{F}) \) has eigenfunctions \( \{\psi_k\}_{k \in \mathbb{Z}^n_+} \) with
\[
\psi_k(x) = \prod_{j=1}^n \psi_{k_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+.
\]
the Hermite functions \( \{\psi_q\}_{q \in \mathbb{Z}_+} \) being defined in (3.7). The eigenfunctions \( \psi_k \) with \( k \in \mathbb{Z}^n_+ \) correspond to eigenvalues
\[
\mu_k^w(\mathcal{F}) = \frac{(-1)^{|k|}}{2^n} \int_{\mathbb{R}^n_+} \mathcal{R}_\mathcal{F}(t/2) \mathcal{L}_k(t) \, dt
\]
(4.1)
\[
= \int_{\mathbb{R}^n_+} \mathcal{R}_\hat{\mathcal{F}}(2t) \mathcal{L}_k(t) \, dt.
\]
(4.2)
(ii) Let \( \mathcal{F} : \mathbb{R}^{2n} \rightarrow \mathbb{C} \) be an admissible radial anti-Wick symbol. Then the eigenfunctions \( \{ \psi_k \}_{k \in \mathbb{Z}_n^+} \) of the operator \( \text{Op}^{aw}(\mathcal{F}) = \text{Op}^w(\mathcal{F} \ast \mathcal{G}_n) \) correspond to eigenvalues 

\[
\mu_k^{aw}(\mathcal{F}) = \int_{\mathbb{R}^n_+} \mathcal{R}_\mathcal{F}(2t) \prod_{j=1}^n \left( \frac{t_j^{k_j}}{k_j!} \right) dt, \quad k \in \mathbb{Z}_n^+. 
\]

**Remark:** In view of (2.3), it is not unnatural to express the eigenvalues of \( \text{Op}^w(\mathcal{F}) \) in terms of the Fourier transform of the symbol \( \mathcal{F} \) of \( \mathcal{F} \), as in (4.2).

**Proof of Proposition 4.4.** We have 

\[
\langle \text{Op}^w(\mathcal{F}) \psi_1, \psi_k \rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{F}, W(\psi_k, \psi_1) \rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{F}, \otimes_{j=1}^n \psi_{k_j, \ell_j} \rangle_{L^2(\mathbb{R}^n)}.
\]

Due to the radial symmetry of \( \mathcal{F} \) and (3.13), we find that 

\[
\langle \mathcal{F}, \otimes_{j=1}^n \psi_{k_j, \ell_j} \rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{F}, \otimes_{j=1}^n \psi_k \rangle_{L^2(\mathbb{R}^n)} \prod_{j=1}^n \delta_{k_j, \ell_j}.
\]

By (3.14), 

\[
\langle \mathcal{F}, \otimes_{j=1}^n \psi_k \rangle_{L^2(\mathbb{R}^n)} = \frac{(-1)^{k_j} \pi^n}{\pi^n} \int_{\mathbb{R}^{2n}} \mathcal{R}_{\mathcal{F}}(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2) \prod_{j=1}^n \left( L_{k_j}(2(x_j^2 + \xi_j^2)) e^{-(x_j^2 + \xi_j^2)/4} \right) dx d\xi.
\]

Changing the variables \( x_j = r_j \cos \theta_j, \xi_j = r_j \sin \theta_j \), and then \( t_j = 2r_j^2, j = 1, \ldots, n \), we obtain (4.1). In order to check (4.2), we first note that by the Parseval identity, 

\[
\langle \mathcal{F}, \otimes_{j=1}^n \psi_k \rangle_{L^2(\mathbb{R}^n)} = \langle \hat{\mathcal{F}}, \otimes_{j=1}^n \hat{\psi}_k \rangle_{L^2(\mathbb{R}^n)}.
\]

By [31], Eq. (3.6)], we have 

\[
\hat{\psi}_k(w) = \frac{(-1)^k}{2} \psi_k(w/2), \quad k \in \mathbb{Z}_+, \quad w \in \mathbb{R}^2.
\]

Therefore, 

\[
\langle \hat{\mathcal{F}}, \otimes_{j=1}^n \hat{\psi}_k \rangle_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \mathcal{R}_{\hat{\mathcal{F}}}(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2) \prod_{j=1}^n \left( L_{k_j}((x_j^2 + \xi_j^2)/2) e^{-(x_j^2 + \xi_j^2)/4} \right) dx d\xi,
\]

which implies (4.2). Let us now handle the anti-Wick case. Similarly to (4.4) - (4.5), we have 

\[
\langle \text{Op}^{aw}(\mathcal{F}) \psi_1, \psi_k \rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{F} \ast \mathcal{G}_n, \otimes_{j=1}^n \psi_k \rangle_{L^2(\mathbb{R}^n)} \prod_{j=1}^n \delta_{k_j, \ell_j}.
\]

A simple calculation yields 

\[
\langle \mathcal{F} \ast \mathcal{G}_n, \otimes_{j=1}^n \psi_k \rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{F}, \mathcal{G}_n \ast (\otimes_{j=1}^n \psi_k) \rangle_{L^2(\mathbb{R}^n)} = \frac{(-1)^k}{4^n} \int_{\mathbb{R}^n_+} \mathcal{R}_\mathcal{F}(\tau/2) \prod_{j=1}^n \left( e^{-\tau_j/2} \int_0^\infty g(s \tau_j) L_{k_j}(s) e^{-s} ds \right) d\tau
\]
where
\[ g(y) := (2\pi)^{-1} \int_0^{2\pi} e^{\sqrt{y} \cos \theta} d\theta, \quad y \geq 0. \]

The function \( g \) extends to an entire function satisfying
\[ g(z) = \sum_{j=0}^{\infty} \frac{1}{(j!)^2} \left( \frac{z}{4} \right)^j, \quad z \in \mathbb{C}. \]

Using the first representation of the Laguerre polynomials in (3.11), we get
\[ (4.9) \quad e^{-\tau/2} \int_0^{\infty} g(s\tau) L_k(s) e^{-s} ds = (-1)^k \frac{\tau^k}{k!} \left( \frac{\tau}{4} \right)^k, \quad \tau \geq 0, \quad k \in \mathbb{Z}_+. \]

Inserting (4.9) into (4.8), changing the variables \( \tau = 4t \), and then inserting (4.8) into (4.7), we get (4.3).

Remark: Relation (4.3) is equivalent to the fact that the Husimi function \( G_1 * \Psi_k \) satisfies
\[ (4.10) \quad (G_1 * \Psi_k)(x,\xi) = \frac{1}{(2\pi)^k} \left( \frac{x^2 + \xi^2}{2} \right)^k e^{-(x^2+\xi^2)/2}, \quad (x,\xi) \in \mathbb{R}^2, \quad k \in \mathbb{Z}_+. \]

Possibly, (4.10) is known to the experts but since we couldn’t find it in the literature, we include a proof.

**Corollary 4.2.** Let \( F \in \Gamma_w(\mathbb{R}^{2n}) \) be a radial symbol.

(i) Then \( \text{Op}_w^w(F) \geq 0 \) if and only if the Fourier coefficients of the function \( 2^{-n} R_F(t/2) \), \( t \in \mathbb{R}^n_+ \), with respect to the system \( \{ (-1)^k L_k \}_{k \in \mathbb{Z}_+^n} \), are positive.

(ii) Equivalently, we have \( \text{Op}_w^w(F) \geq 0 \) if and only if the Fourier coefficients of the function \( R_F(2t) \), \( t \in \mathbb{R}^n_+ \), with respect to the system \( \{ L_k \}_{k \in \mathbb{Z}_+^n} \) are positive.

**Proof.** The first part follows from (4.11), and the second one from (4.2).

Remark: The criterion in the first part of Corollary 4.2 has been established in [39] for the one-dimensional case \( n = 1 \), and in [24] for the multidimensional case. Presumably, at heuristic level, these facts have been known since long ago.

**Corollary 4.3.** Let \( F \in \Gamma_{aw}(\mathbb{R}^{2n}) \) be a radial symbol. Then \( \text{Op}_{aw}^w(F) \geq 0 \) if and only if
\[ \int_{\mathbb{R}^n_+} R_F(2t) \prod_{j=1}^n \left( t_j^k e^{-t_j} \right) dt \geq 0, \quad k \in \mathbb{Z}_+^n. \]

**Proof.** The claim follows from (4.3).

In the case \( n = 1 \), Proposition 4.1 tells us that the matrix \( \{ \langle \text{Op}_{aw}^w(F) \psi_k, \psi_j \rangle_{L^2(\mathbb{R})} \}_{j,k \in \mathbb{Z}_+} \) is diagonal, provided that the symbol \( F \) is radial. This fact admits an obvious generalization to the case where \( F(r \cos \theta, r \sin \theta) \) has a finite Fourier series with respect to the angle \( \theta \):
Proposition 4.4. Let $F \in \Gamma_w(\mathbb{R}^2)$. Assume that there exists $K \in \mathbb{Z}_+$ such that

$$F(r \cos \theta, r \sin \theta) = \sum_{k=-K}^{K} F_k(r) e^{ik\theta}, \quad r \in [0, \infty), \quad \theta \in [0, 2\pi).$$

Then the matrix $\{ (\text{Op}^w(F)\psi_\ell, \psi_j)_{L^2(\mathbb{R})} \}_{j,\ell \in \mathbb{Z}_+}$ is $(2K + 1)$-diagonal.

Of course, Proposition 4.4 admits an immediate extension to any dimension $n \geq 1$. Proposition 4.1 allows us to calculate explicitly the spectrum of the perturbed Landau Hamiltonian $H_V = H_0 + \text{Op}^w(V)$ provided that the symbol $V_b$ is radial.

Corollary 4.5. Let $V \in \Gamma_w(\mathbb{R}^4)$. Assume that the symbol $V_b = V \circ \kappa_b$ is radial. Then the operator $H_V$, normal on the domain $\mathcal{D}(H_0)$, has eigenfunctions $\{ \varphi_{k,q} \}_{(k,q) \in \mathbb{Z}_+^2}$ which correspond to eigenvalues

$$\Lambda_q + \mu_{(q,k)}(V_b), \quad (q, k) \in \mathbb{Z}_+^2.$$

Proof. By Proposition 3.2 we have

$$H_V = U_b \left( ((b^2) \otimes I_y) + \text{Op}^w(V_b) \right) U_b^*,$$

while Lemma 3.5 and Proposition 4.1 imply

$$U_b \left( ((b^2) \otimes I_y) + \text{Op}^w(V_b) \right) U_b^* \varphi_{k,q} = (\Lambda_q + \mu_{(q,k)}(V_b)) \varphi_{k,q}, \quad (k, q) \in \mathbb{Z}_+^2.$$

5. Eigenvalue distribution for the operator $H_V$

5.1. Main results. In this section we study the eigenvalue asymptotics near a fixed Landau level $\Lambda_q$, $q \in \mathbb{Z}_+$, for the perturbed Landau Hamiltonian $H_V = H_0 + \text{Op}^w(V)$ with appropriate symbol $V$ such that $\text{Op}^w(V)$ is bounded, self-adjoint in $L^2(\mathbb{R}^2)$, and relatively compact with respect to $H_0$. Proposition 5.1 below shows, in particular, that the eigenvalues of $H_V$ with $V \in \mathcal{S}(\mathbb{R}^4)$, adjoining the Landau levels $\Lambda_q$, $q \in \mathbb{Z}_+$, may have quite arbitrary asymptotic behavior; they may not accumulate at a given $\Lambda_q$, or may accumulate at any prescribed sufficiently fast accumulation rate.

Let $T$ be an operator, self-adjoint in a given Hilbert space, and $(\mu_1, \mu_2)$ be an open interval with $-\infty \leq \mu_1 < \mu_2 \leq \infty$. Set

$$N_{(\mu_1, \mu_2)}(T) := \text{Tr} \mathbf{1}_{S(\mu_1, \mu_2)}(T).$$

Here and in the sequel $\mathbf{1}_S$ denotes the characteristic function of the set $S$. Thus, $\mathbf{1}_{(\mu_1, \mu_2)}(T)$ is just the spectral projection of $T$ corresponding to the interval $(\mu_1, \mu_2)$. If $(\mu_1, \mu_2) \cap \sigma_{\text{ess}}(T) = \emptyset$, then $N_{(\mu_1, \mu_2)}(T)$ is the number of the eigenvalues of $T$, lying on $(\mu_1, \mu_2)$ and counted with the multiplicities. If, moreover, $(\mu_1, \mu_2) \cap \sigma_{\text{ess}}(T) = \emptyset$, then $N_{(\mu_1, \mu_2)}(T) < \infty$. 

Proposition 5.1. Let \( \{m_q\}_{q \in \mathbb{Z}^+} \) be a given sequence with \( m_q \in \mathbb{Z}_+ \cup \{\infty\}, \, q \in \mathbb{Z}_+ \).
Then there exists a real-valued radial symbol \( \mathcal{V} \in \mathcal{S}(\mathbb{R}^4) \) such that \( \text{Op}^w(\mathcal{V}) \leq 0 \), and
\[
(5.1) \quad N_{(\Lambda_{q-1}, \Lambda_q)}(H_\mathcal{V}) = m_q, \quad q \in \mathbb{Z}_+.
\]

Proof. Set
\[
\mathcal{Z} := \{q \in \mathbb{Z}_+ \mid m_q \neq 0\}.
\]
If \( \mathcal{Z} = \emptyset \), it suffices to take \( \mathcal{V} = 0 \). Assume \( \mathcal{Z} \neq \emptyset \). Let \( \{c_{1,q}\}_{q \in \mathcal{Z}} \) be a decreasing set of numbers \( c_{1,q} \in (0, 2b) \); if \( 0 \in \mathcal{Z} \), we can omit the condition \( c_{1,0} < 2b \). If \( \#\mathcal{Z} = \infty \), we assume that \( \lim_{q \to \infty} q^m c_{1,q} = 0 \) for any \( m \in \mathbb{N} \). Fix \( q \in \mathcal{Z} \). Let \( \{c_{2,k}\}_{k=0}^{m_q-1} \) be a decreasing set of numbers \( c_{2,q} \in (0,1) \); if \( q = 0 \), we can omit the condition \( c_{2,0} < 1 \). If \( m_q = \infty \), we assume that \( \lim_{k \to \infty} k^m c_{2,k} = 0 \) for any \( m \in \mathbb{N} \). Now put
\[
C_{k,q} := -c_{1,q} c_{2,k}, \quad k = 0, \ldots, m_q - 1, \quad q \in \mathcal{Z},
\]

(5.2) \[
\mathcal{V} := (2\pi)^2 \left( \sum_{q \in \mathcal{Z}} \sum_{k=0}^{m_q-1} C_{k,q} \Psi_q \otimes \Psi_k \right) \circ \kappa_b^{-1}.
\]

Then, \( \mathcal{V} \in \mathcal{S}(\mathbb{R}^4) \) (see [12 Theorem 2.5 (a)]), and, evidently, \( \mathcal{V}_b = \mathcal{V} \circ \kappa_b \) is real-valued and radial. Moreover, by Corollary [4.5], \( \text{Op}^w(\mathcal{V}) \leq 0 \) and

\[
(5.3) \quad \sigma(H_0 + \text{Op}^w(\mathcal{V})) \cap (\Lambda_{q-1}, \Lambda_q) = \left\{ \begin{array}{ll}
\emptyset & \text{if } q \notin \mathcal{Z}, \\
\bigcup_{k=0}^{m_q-1} \{\Lambda_q + C_{k,q}\} & \text{if } q \in \mathcal{Z}.
\end{array} \right.
\]

By construction, all the eigenvalues \( \Lambda_q + C_{k,q}, \, k = 0, \ldots, m_q - 1 \), lying in \( (\Lambda_{q-1}, \Lambda_q) \) with \( q \in \mathcal{Z} \), are simple. Therefore, (5.1) holds true. \( \square \)

Remarks: (i) The proof of Proposition 5.1 contains an explicit construction of a negative compact perturbation of \( H_0 \) so that the eigenvalues \( H_\mathcal{V} \) may accumulate to \( \Lambda_q \) only from the left. Of course, it is possible to construct positive compact perturbations whose eigenvalues may accumulate to \( \Lambda_q \) only from the right, or self-adjoint compact perturbation with non-trivial positive and negative parts whose eigenvalues may accumulate to \( \Lambda_q \) both from the right and from the left.

(ii) It is easy to check that if for some \( q \in \mathbb{Z}_+ \) we have \( m_q < \infty \), then the Landau level remains an eigenvalue of infinite multiplicity of \( H_\mathcal{V} \). In contrast to this situation, it was shown in [26] that if \( \text{Op}^w(\mathcal{V}) = \mathcal{V} \) is a multiplier, i.e. if \( \mathcal{V} = \mathcal{V}(x,y), \, (x,y) \in \mathbb{R}^2 \), and \( \mathcal{V} \leq 0 \), \( \|\mathcal{V}\|_{L^\infty(\mathbb{R}^2)} < 2b \), then \( \text{Ker} \left( H_\mathcal{V} - \Lambda_q I \right) = 0 \).

(iii) In the proof of Proposition 5.1, we have assumed that the sequences \( \{c_{1,q}\} \) and \( \{c_{2,k}\} \) decay rapidly in order to obtain a Schwartz-class symbol \( \mathcal{V} \). If, for example, we assume instead that we have only
\[
\sum_{q \in \mathcal{Z}} \sum_{k=0}^{m_q-1} c_{1,q}^2 c_{2,k}^2 < \infty,
\]
then the symbol defined in (5.2) generates by Proposition 2.3 a Hilbert-Schmidt operator. In this case, (5.3) still holds true, just the eigenvalues of \( H_\mathcal{V} \) lying in a given gap
We will study the eigenvalue asymptotics for the operators $H_{\omega}$. In the following two theorems we assume that the operator $H_{\omega}$ satisfies several general assumptions. The first one is:

**H$_1$:** The operator $\text{Op}^w(\mathcal{V})$ is bounded, self-adjoint and positive in $L^2(\mathbb{R}^2)$. Moreover, $\text{Op}^w(\mathcal{V}) H_0^{-1} \in \mathcal{S}_\infty(L^2(\mathbb{R}^2))$.

We will study the eigenvalue asymptotics for the operators $H_{\pm\mathcal{V}} = H_0 \pm \text{Op}^w(\mathcal{V})$. Since $\text{Op}^w(\mathcal{V}) \geq 0$ by Assumption H$_1$, we recall that the eigenvalues of the operator $H_0 + \text{Op}^w(\mathcal{V})$ may accumulate to given Landau level $\Lambda_q$ only from the right, while the eigenvalues of $H_0 - \text{Op}^w(\mathcal{V})$ may accumulate to $\Lambda_q$ only from the left. Let $\{\lambda_{k,q}^+\}$ (resp., $\{\lambda_{k,q}^-\}$) be the eigenvalues of the operator $H_0 + \text{Op}^w(\mathcal{V})$ (resp., $H_0 - \text{Op}^w(\mathcal{V})$) lying in the interval $(\Lambda_q, \Lambda_{q+1})$ (resp., $(\Lambda_{q-1}, \Lambda_q)$), counted with the multiplicities and enumerated in decreasing (resp., increasing) order.

Let us now formulate our second general assumption:

**H$_2,q,r$:** Let $q \in \mathbb{Z}_+$. Then the operator $\text{Op}^w(v_{b,q})$, $v_{b,q}$ being defined in (3.41), has an anti-Wick symbol $\tilde{v}_{b,q} \in \Gamma_{aw}(\mathbb{R}^2)$. Moreover, there exists $r \in \mathbb{Z}_+$ and $0 \leq \zeta_{b,q,r} \in L^\infty(\mathbb{R}^2)$ such that, in the distributional sense,

$$\omega_{b,q} = D_{b,r} \zeta_{b,q,r},$$

$\omega_{b,q}$ being defined in (3.46), and $D_{b,r}$ being defined in (3.48).

**Remarks:** (i) In what follows we will write $\zeta$ instead of $\zeta_{b,q,r}$.

(ii) It is easy to check that for each $q,r \in \mathbb{Z}_+$ there exist symbols $\mathcal{V} \in \Gamma_w(\mathbb{R}^4)$ satisfying Assumptions H$_1$ and H$_2,q,r$. A simple example can be constructed as follows. Pick $0 \leq \zeta \in C^\infty(\mathbb{R}^2)$, bounded together with all its derivatives. Set $\omega = D_{b,r} \zeta$, $\tilde{v}(x,y) := \omega(-b^{-1/2}y, b^{-1/2}x)$, $(x,y) \in \mathbb{R}^2$, $v := \tilde{v} \ast G_1$, and

$$\mathcal{V} := 2\pi (\Psi_q \otimes v) \circ \kappa_b^{-1}.$$ 

Then, according to (3.41), we have $\tilde{v}_{b,q} = v$, and hence $\mathcal{V}$ satisfies H$_1$ and H$_2,q,r$. However, if we consider the operator defined in (5.2), and assume that for a certain $q \in \mathbb{Z}$ we have $m_q < \infty$, then the corresponding $\text{Op}^w(v_{b,q})$ does admit an anti-Wick symbol $\tilde{v}_{b,q} \in \Gamma_{aw}(\mathbb{R}^2)$. Indeed, in this case we have

$$\text{Op}^w(v_{b,q}) = 2\pi \sum_{k=0}^{m_q-1} C_{k,q} \Psi_k.$$ 

If $v_{b,q} = \tilde{v}_{b,q} \ast G_1$ with $\tilde{v}_{b,q} \in S'(\mathbb{R}^2)$, then (4.6) easily implies that the Fourier transform of $\tilde{v}_{b,q}$ is a polynomial so that $\tilde{v}_{b,q} \in S'(\mathbb{R}^2)$ with $\text{supp}(\tilde{v}_{b,q}) = \{0\}$.

(iii) As we will see in the proof of Proposition 5.5 below, the Toeplitz operator $T_q(\mathcal{V})$ is the effective Hamiltonian which governs the eigenvalue asymptotics of $H_{\pm\mathcal{V}}$ near the Landau level $\Lambda_q$, $q \in \mathbb{Z}$. The operator $T_q(\mathcal{V}) = p_q \text{Op}^w(\mathcal{V}) p_q$ is an appropriate restriction.
of non-local \( \Psi \text{DO } \text{Op}^w(\mathcal{V}) \), and is unitarily equivalent by Corollary 3.7 to \( \text{Op}^w(v_{b,q}) \). By our assumption, \( \text{Op}^w(v_{b,q}) \) admits an anti-Wick symbol \( \tilde{v}_{b,q} \) and, hence, by Corollary 3.8 it is unitarily equivalent to \( p_0 \omega_{b,q} p_0 \), a restriction of the local multiplier \( \omega_{b,q} \). Thus, the existence of an anti-Wick symbol \( \tilde{v}_{b,q} \) of \( \text{Op}^w(v_{b,q}) \) allows us to replace, in a certain sense, the non-local operator \( \text{Op}^w(\mathcal{V}) \) by the local one \( \omega_{b,q} \) in the asymptotic analysis of the eigenvalue distribution of \( H_{\pm V} \) near \( \Lambda_q \). In different situations (in particular, in the absence of magnetic fields), similar replacements of non-local potentials by local potentials have been considered in the physics literature (see e.g. [9, 38, 7]).

(iv) We introduce the passage from \( \omega_{b,q} \) to \( \zeta \) in (5.4) in particular due to our requirement that \( \zeta \) is non-negative: it may happen that \( \zeta \geq 0 \) while \( \omega_{b,q} \) is not sign-definite.

For the formulation of our first theorem we need the notion of a logarithmic capacity \( \mathcal{C}(\mathcal{R}) \) of a compact set \( \mathcal{R} \subset \mathbb{R}^2 \). Let \( M(\mathcal{R}) \) denote the set of probability measures on \( \mathcal{R} \). Then we have \( \mathcal{C}(\mathcal{R}) := e^{-\mathcal{I}(\mathcal{R})} \) where

\[
\mathcal{I}(\mathcal{R}) := \inf_{\mu \in M(\mathcal{R})} \int_{\mathcal{R} \times \mathcal{R}} \ln |x - y|^{-1} d\mu(x) d\mu(y).
\]

If \( \mathcal{R} \) is simply connected, then \( \mathcal{C}(\mathcal{R}) \) is equal to the conformal radius of \( \mathcal{R} \) (see e.g. [28 Chapter II, Section 4]).

**Theorem 5.2.** Let \( H_1 \) and \( H_{2,q,r} \) with fixed \( q, r \in \mathbb{Z}_+ \), hold true. Assume that \( \zeta \in C(\mathbb{R}^2) \), \( \text{supp } (\zeta) = \overline{\Omega} \) where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with boundary \( \partial \Omega \in C^2 \), and \( \zeta > 0 \) on \( \Omega \). Then we have

\[
\ln \left( \pm (\lambda_{k,q}^\pm - \Lambda_q) \right) = -k \ln k + \left( 1 + \ln \left( \frac{b \mathcal{C}(\overline{\Omega})^2}{2} \right) \right) k + o(k), \quad k \to \infty.
\]

**Remark:** Set

\[
\mathcal{N}_q(\lambda; \pm \text{Op}^w(\mathcal{V})) = \# \left\{ k \in \mathbb{Z}_+ \mid \pm (\lambda_{k,q}^\pm - \Lambda_q) > \lambda \right\}.
\]

Then a less precise version of (5.5) could be written in terms of the eigenvalue counting functions:

\[
\mathcal{N}_q(\lambda; \pm \text{Op}^w(\mathcal{V})) = \frac{\ln \lambda}{\ln |\ln \lambda|} (1 + o(1)), \quad \lambda \downarrow 0.
\]

Our next theorem concerns the case where \( \zeta \) decays exponentially at infinity. More precisely, we assume that there exist \( \beta > 0 \) and \( \gamma > 0 \) such that

\[
\ln \zeta(x) = -\gamma |x|^{2\beta} + O(\ln |x|), \quad |x| \to \infty.
\]

Set \( \mu := \gamma(2/b)^\beta \), \( b > 0 \) being the constant scalar magnetic field.

**Theorem 5.3.** Let \( H_1 \) and \( H_{2,q,r} \) with fixed \( q, r \in \mathbb{Z}_+ \), hold true. Assume that \( \zeta \) satisfies (5.7). Then:

(i) If \( \beta \in (0, 1) \), then there exist constants \( f_j = f_j(\beta, \mu) \), \( j \in \mathbb{N} \), with \( f_1 = \mu \), such that

\[
\ln \left( \pm (\lambda_{k,q}^\pm - \Lambda_q) \right) = -\sum_{1 \leq j < \frac{1}{1 - \beta}} f_j k^{(\beta - 1)j + 1} + O(\ln k), \quad k \to \infty.
\]

(ii) If \( \beta \in [1, 2) \), then there exist constants \( f_j = f_j(\beta, \mu) \), \( j \in \mathbb{N} \), with \( f_1 = \mu \), such that

\[
\ln \left( \pm (\lambda_{k,q}^\pm - \Lambda_q) \right) = -\sum_{1 \leq j < \frac{1}{1 - \beta}} f_j k^{(\beta - 1)j + 1} + O(\ln k), \quad k \to \infty.
\]
(ii) If \( \beta = 1 \), then
\[
\ln \left( \pm \left( \lambda_{k,q}^\pm - \Lambda_q \right) \right) = - (\ln (1 + \mu)) k + O(\ln k), \quad k \to \infty.
\]

(iii) If \( \beta \in (1, \infty) \), then there exist constants \( g_j = g_j(\beta, \mu) \), \( j \in \mathbb{N} \), such that
\[
\ln \left( \pm \left( \lambda_{k,q}^\pm - \Lambda_q \right) \right) =
\]
\[
\frac{\beta - 1}{\beta} k \ln k + \left( \frac{\beta - 1 - \ln (\mu \beta)}{\beta} \right) k - \sum_{1 \leq j < \frac{\beta}{\beta - 1}} g_j k^{\frac{1}{\beta} - 1)j + 1} + O(\ln k), \quad k \to \infty.
\]

Remarks: (i) The coefficients \( f_j \) and \( g_j \), \( j \in \mathbb{N} \), appearing in (5.8) and (5.10), are described explicitly in [29, Theorem 2.2]. For the completeness of the exposition, we reproduce this description here.

Assume at first \( \beta \in (0, 1) \). For \( s > 0 \) and \( \epsilon \in \mathbb{R}, |\epsilon| \ll 1 \), introduce the function
\[
F(s; \epsilon) := s - \ln s + \epsilon \mu s^\beta.
\]
Denote by \( s_<(\epsilon) \) the unique positive solution of the equation \( s = 1 - \epsilon \mu s^\beta \), so that \( \frac{\partial F}{\partial s}(s_<(\epsilon); \epsilon) = 0 \). Set
\[
f(\epsilon) := F(s_<(\epsilon); \epsilon).
\]
Note that \( f \) is a real analytic function for small \( |\epsilon| \). Then \( f_j := \frac{1}{j!} \frac{\partial^j f}{\partial \epsilon^j}(0), j \in \mathbb{N} \).

Let now \( \beta \in (1, \infty) \). For \( s > 0 \) and \( \epsilon \in \mathbb{R}, |\epsilon| \ll 1 \), introduce the function
\[
G(s; \epsilon) := \mu s^\beta - \ln s + \epsilon s.
\]
Denote by \( s_>(\epsilon) \) the unique positive solution of the equation \( \beta \mu s^\beta = 1 - \epsilon s \) so that \( \frac{\partial G}{\partial s}(s_>(\epsilon); \epsilon) = 0 \). Define
\[
g(\epsilon) := G(s_>(\epsilon); \epsilon),
\]
which is a real analytic function for small \( |\epsilon| \). Then \( g_j := \frac{1}{j!} \frac{\partial^j g}{\partial \epsilon^j}(0), j \in \mathbb{N} \).

(ii) Similarly to (5.6), less precise versions of (5.8) – (5.10), can be written in terms of the eigenvalue counting functions:
\[
N_q(\lambda; \pm \text{Op}^w(\mathcal{V})) = \begin{cases}
\mu^{-1/\beta} |\ln \lambda|^{1/\beta}(1 + o(1)) & \text{if } \beta \in (0, 1), \\
\frac{1}{\ln(1 + \rho)} |\ln \lambda|(1 + o(1)) & \text{if } \beta = 1, \quad \lambda \downarrow 0,
\end{cases}
\]
\[
\frac{\beta - 1}{\beta - 1 - \ln \lambda} |\ln \lambda|(1 + o(1)) & \text{if } \beta \in (1, \infty),
\]

In our next theorem we deal with the case where \( v_{b,q} \) admits a power-like decay at infinity. Our general assumption concerning the perturbation \( \text{Op}^w(\mathcal{V}) \) is:

**H3** The symbol \( \mathcal{V} \) is real-valued and satisfies the hypotheses of Proposition 3.3.

We recall that under Assumption **H3** the operator \( \text{Op}^w(\mathcal{V}) \) is self-adjoint and bounded in \( L^2(\mathbb{R}^2) \), and \( \text{Op}^w(\mathcal{V}) H_0^{-1} \) is compact. However, we do not suppose now that \( \text{Op}^w(\mathcal{V}) \) has a definite sign.
Further, under Assumption $H_3$, there exists a symbol $W \in \mathcal{S}_v^0(\mathbb{R}^4)$ such that $\text{Op}^w(V)^2 = \text{Op}^w(W)$. By analogy with (3.41), set
\[
w_{b,q}(y, \eta) := \int_{\mathbb{R}^2} (W \circ \kappa_b)(x, \xi, y, \eta)\Psi_q(x, \xi)d\xi, \quad (y, \eta) \in \mathbb{R}^2, \quad q \in \mathbb{Z}_+.
\]

Our next assumption concerns the decay of the symbols $v_{b,q}$ and $w_{b,q}$ at infinity:

**$H_{4, q, \gamma}$** Let $q \in \mathbb{Z}_+$. Then there exist $\gamma > 0$ and $\varrho \in (0, 1]$ such that $v_{b,q} \in \mathcal{S}_v^{-\gamma}(\mathbb{R}^2)$ and $w_{b,q} \in \mathcal{S}_v^{-2\gamma}(\mathbb{R}^2)$.

**Remark:** A simple sufficient condition which guarantees the fulfillment of $H_3$ and $H_{4, q, \gamma}$ is that $V \in \mathcal{S}_v^{-\gamma}(\mathbb{R}^4)$ with some $\gamma > 0$ and $\varrho \in (0, 1]$. In this case, the operator $\text{Op}^w(V)$ is not only bounded but also compact. Another condition which implies the validity of $H_3$ and $H_{4, q, \gamma}$ is that $\text{Op}^w(V) = V$ is a multiplier, and $V \in \mathcal{S}_v^{-\gamma}(\mathbb{R}^2)$.

This case corresponds to an electric perturbation of $H_0$ and was considered in [33, 23].

It is more convenient to formulate Theorem 5.4 below in the terms of eigenvalue counting functions. For $V = V^* \in \mathcal{B}(L^2(\mathbb{R}^2))$, $V H_{0}^{-1} \in \mathcal{S}_v(L^2(\mathbb{R}^2))$, and $q \in \mathbb{Z}_+$, set
\[
\mathcal{N}_{q}^{\geq}(\lambda; V) := N_{(\Lambda_q + \lambda, \Lambda_q + b)}(H_0 + V), \quad \lambda \in (0, b),
\]
and for $q \in \mathbb{N}$ put
\[
\mathcal{N}_{q}^{<}(\lambda; V) := N_{(\Lambda_q - b, \Lambda_q - \lambda)}(H_0 + V), \quad \lambda \in (0, b).
\]
If $q = 0$, then
\[
\mathcal{N}_{0}^{<}(\lambda; V) := N_{(-\infty, \Lambda_q - \lambda)}(H_0 + V), \quad \lambda > 0.
\]
Let $f : (0, \infty) \to [0, \infty)$ be a non-increasing function. We will say that $f$ satisfies the condition $C$ if there exists $\lambda_0 \in (0, \infty)$ such that:

- $f$ is derivable on $(0, \lambda_0)$;
- there exist numbers $0 < \gamma_1 < \gamma_2 < \infty$ such that for any $\lambda \in (0, \lambda_0)$ we have
\[
\gamma_1 f(\lambda) < -\lambda f'(\lambda) < \gamma_2 f(\lambda).
\]
Let $n \in \mathbb{N}$. For a Lebesgue-measurable function $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}$ set
\[
\mathcal{M}^{\pm}_n(\lambda; \mathcal{F}) := (2\pi)^{-n} \left| \left\{ (x, \xi) \in \mathbb{R}^{2n} \mid \pm \mathcal{F}(x, \xi) > \lambda \right\} \right|, \quad \lambda > 0,
\]
where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^{2n}$.

**Theorem 5.4.** Assume that $V$ satisfies $H_3$ and $H_{4, q, \gamma}$ with $q \in \mathbb{Z}_+$ and $\gamma > 0$. Assume that the functions $\mathcal{M}^{+}_n(\lambda; v_{b,q})$, $v_{b,q}$ being defined in (3.41), satisfy the condition $C$. If $\liminf_{\lambda \downarrow 0} \lambda^{2/\gamma} \mathcal{M}^{+}_1(\lambda; v_{b,q}) > 0$ (resp., if $\liminf_{\lambda \downarrow 0} \lambda^{2/\gamma} \mathcal{M}^{-}_1(\lambda; v_{b,q}) > 0$), then we have
\[
\mathcal{N}_{q}^{\geq}(\lambda; \text{Op}^w(V)) = \mathcal{M}^{+}_1(\lambda; v_{b,q})(1 + o(1)), \quad \lambda \downarrow 0,
\]
or, respectively,
\[
\mathcal{N}_{q}^{<}(\lambda; \text{Op}^w(V)) = \mathcal{M}^{-}_1(\lambda; v_{b,q})(1 + o(1)), \quad \lambda \downarrow 0.
\]
Remark: It is easy to show that there exists \( \delta > 0 \) such that we can replace \( o(1) \) by \( \mathcal{O}(\lambda^{\delta}) \) in the remainder estimates in (5.18) - (5.19). Since anyway these remainder estimates wouldn’t be sharp, we omit the tedious technical details.

5.2. Proofs of Theorems 5.2 and 5.3. Let \( T = T^* \) be a compact operator in a Hilbert space. Denote by \( \{ \nu_k(T) \}_{k=0}^{\text{rank} T} \) the non-increasing set of the positive eigenvalues of \( T \). Assume that \( X \) is a Hilbert space, and \( T = T^* \in \mathfrak{S}_\infty(X) \). For \( s > 0 \) set

\[
n_\pm(s; T) := N_{(s, \infty)}(\pm T).
\]

Then the Weyl inequalities

\[
n_\pm(s_1 + s_2; T_1 + T_2) \leq n_\pm(s_1; T_1) + n_\pm(s_2; T_2)
\]

hold for \( T_j = T_j^* \in \mathfrak{S}_\infty(X) \) and \( s_j > 0, \ j = 1, 2 \), (see e.g. [3] Theorem 9, Section 9.2]).

Proposition 5.5. Assume \( H_1 \) and \( H_{2, q, r} \) with \( q, r \in \mathbb{Z}_+ \). Then, for each \( \varepsilon \in (0, 1) \) there exists \( k_0 \in \mathbb{Z}_+ \) such that for sufficiently large \( k \in \mathbb{N} \) we have

\[
\frac{1}{1 + \varepsilon} \nu_k + k_0(p_r \zeta p_r) \leq \pm(\lambda_{k, q}^\pm - \Lambda_q) \leq \frac{1}{1 - \varepsilon} \nu_k - k_0(p_r \zeta p_r).
\]

Proof. By the generalized Birman-Schwinger principle (see e.g. [1] Theorem 1.3]),

\[
N_q(\lambda; \pm \text{Op}^w(\mathcal{V})) = n_\pm(1; \text{Op}^w(\mathcal{V})^{1/2}(H_0 - \Lambda_q \mp \lambda)^{-1}\text{Op}^w(\mathcal{V})^{1/2}) + \mathcal{O}(1).
\]

Writing

\[
(H_0 - \Lambda_q \mp \lambda)^{-1} = \mp \lambda^{-1} p_q + (I - p_q)(H_0 - \Lambda_q \mp \lambda)^{-1},
\]

bearing in mind that the operator \( (I - p_q)(H_0 - \Lambda_q \mp \lambda)^{-1} \) admits a uniform limit as \( \lambda \downarrow 0 \), and applying the Weyl inequalities (5.20), we easily find that for each \( \varepsilon \in (0, 1) \) we have

\[
n_+(1 + \varepsilon) \lambda; \text{Op}^w(\mathcal{V})^{1/2} p_q \text{Op}^w(\mathcal{V})^{1/2}) + \mathcal{O}_{\varepsilon, q}(1) \leq n_+(1; \text{Op}^w(\mathcal{V})^{1/2}(H_0 - \Lambda_q \mp \lambda)^{-1}\text{Op}^w(\mathcal{V})^{1/2}) \leq n_+(1 - \varepsilon) \lambda; \text{Op}^w(\mathcal{V})^{1/2} p_q \text{Op}^w(\mathcal{V})^{1/2}) + \mathcal{O}_{\varepsilon, q}(1),
\]

as \( \lambda \downarrow 0 \). Further, by Corollaries 5.7, 5.8 and 5.9 we have

\[
n_+(s; \text{Op}^w(\mathcal{V})^{1/2} p_q \text{Op}^w(\mathcal{V})^{1/2}) = n_+(s; p_q \text{Op}^w(\mathcal{V})) = n_+(s; \text{Op}^w(\mathcal{V})) = n_+(s; \text{Op}^w(\mathcal{V}))(1) \leq n_+(s; p_q \text{Op}^w(\mathcal{V}))p_q = n_+(s; \text{Op}^w(\mathcal{V}))(1)
\]

Putting together (5.22), (5.23), and (5.24), we get

\[
n_+(1 + \varepsilon) p_r \zeta p_r) + \mathcal{O}_{\varepsilon, q}(1) \leq N_q(\lambda; \pm \text{Op}^w(\mathcal{V})) \leq n_+(1 + \varepsilon) p_r \zeta p_r) + \mathcal{O}_{\varepsilon, q}(1)
\]

which easily yields (5.21). □
Proof of Theorem 5.2. Pick a decreasing positive sequence \( \{ \varepsilon_j \}_{j \in \mathbb{N}} \) such that the sets

\[ \Omega_j := \{ x \in \Omega \mid \text{dist} (x, \partial \Omega) > \varepsilon_j \}, \quad j \in \mathbb{N}, \]

are well defined domains with boundaries \( \partial \Omega_j \in C^2 \). Set

\[ m_j^- := \inf_{x \in \Omega_j} \zeta(x), \quad j \in \mathbb{N}, \quad m^+ := \sup_{x \in \Omega} \zeta(x). \]

Evidently, \( 0 < m_j^- \leq m^+ < \infty \). Moreover,

\[ m_j^- \mathbb{1}_{\Omega_j}(x) \leq \zeta(x) \leq m^+ \mathbb{1}_{\Omega}(x), \quad x \in \mathbb{R}^2, \quad j \in \mathbb{N}. \]

By the mini-max principle, estimate (5.25) implies

\[ (5.26) \quad m_j^- \nu_k(p_r \mathbb{1}_{\Omega_j} p_r) \leq \nu_k(p_r \zeta p_r) \leq m^+ \nu_k(p_r \mathbb{1}_{\Omega} p_r), \quad j, k \in \mathbb{N}. \]

By [18, Lemma 2], we have

\[ \ln \nu_k(p_r \mathbb{1}_K p_r) = -k \ln k + \left( 1 + \ln \left( \frac{b \mathcal{C}(K)^2}{2} \right) \right) k + o(k), \quad k \to \infty, \]

for any compact set \( K \subset \mathbb{R}^2 \) with Lipschitz boundary. Therefore, (5.25) implies

\[ (5.27) \quad \limsup_{k \to \infty} \frac{\ln \nu_k(p_r \zeta p_r) + k \ln k}{k} \leq 1 + \ln \left( \frac{b \mathcal{C}(\Omega)^2}{2} \right), \quad j \in \mathbb{N}. \]

It is easy to check that under our assumptions

\[ (5.28) \quad \lim_{j \to \infty} \mathcal{C}(\overline{\Omega}_j) = \mathcal{C}(\overline{\Omega}). \]

Combining (5.27) and (5.28), we get

\[ (5.29) \quad \ln \nu_k(p_r \zeta p_r) = -k \ln k + \left( 1 + \ln \left( \frac{b \mathcal{C}(\Omega)^2}{2} \right) \right) k + o(k), \quad k \to \infty. \]

Now (5.3) follows from (5.21) and (5.29). \( \square \)

Proof of Theorem 5.3. For \( \delta \in \mathbb{R}, \ c_0 > 0, \ c_1 \in \mathbb{R}, \) and \( R > 0 \), set

\[ \chi_{\delta, c_0, c_1, R}(x) := c_0 |x|^\beta e^{-\gamma |x|^2} \mathbb{1}_{\mathbb{R}^2 \setminus B_R}(x) + c_1 \mathbb{1}_{B_R}(x), \quad x \in \mathbb{R}^2, \]

where \( B_R := \{ y \in \mathbb{R} \mid |x - y| < R \} \), and \( \beta > 0, \ \gamma > 0 \), are the parameters introduced in the statement of the theorem. Arguing as in the proof of [29, Theorem 2.2], we can show that there exist \( \delta_\leq \leq \delta_\geq \in \mathbb{R}, \) \( 0 \leq c_{0, \leq} \leq c_{0, \geq}, \ c_{1, \leq} \leq c_{1, \geq} \in \mathbb{R} \) and \( R > 0 \), such that

\[ (5.30) \quad \nu_k(p_0 \chi_{\delta_\leq, c_{0, \leq}, c_{1, \leq}, R} p_0) \leq \nu_k(p_r \zeta p_r) \leq \nu_k(p_0 \chi_{\delta_\geq, c_{0, \geq}, c_{1, \geq}, R} p_0)), \quad k \in \mathbb{Z}_+. \]
Since the functions $\chi_{s, c_0, c_1, R}$ are radial, we easily check that the eigenvalues of the operator $p_0 \chi_{s, c_0, c_1, R} : \text{Ran} p_0 \to \text{Ran} p_0$ coincide with the numbers

$$(\chi_{s, c_0, c_1, R} \varphi_{k, 0}, \varphi_{k, 0})_{L^2(\mathbb{R}^2)} = \frac{1}{k!} \left( \frac{2}{b} \right)^{\delta/2} \sum_{\nu} \int_{0}^{\infty} t^{k+\delta/2} e^{-\mu t} dt + c_1 \int_{0}^{\rho} e^{-t} dt, \quad k \in \mathbb{Z}_+,$$

with $\mu = (2/b)^{\delta} \gamma$ and $\rho = bR^2/2$. Applying (5.30) and [29, Lemma 5.3], we find that

$$(5.31) \quad \ln \nu_k(p_\nu, \zeta p_r) = \begin{cases} -\sum_{1 \leq j < \frac{\mu}{\beta}} f_j k^{(\beta-1)j+1} + O((\ln k) \ln k) & \text{if } \beta \in (0, 1), \\ -(\ln (1 + \mu)) \frac{k}{\beta} + O((\ln k)^2) & \text{if } \beta = 1, \\ -\frac{\beta-1}{\beta} k \ln k + k^{(\beta-1)-\ln(\mu^\beta)} & \text{if } \beta \in (1, \infty), \\ -\sum_{1 \leq j < \frac{\mu}{\beta}} g_j k^{(\beta-1)j+1} + O((\ln k) \ln k) & \text{if } \beta \in (1, \infty), \end{cases}$$

as $k \to \infty$, the coefficients $f_j$ and $g_j$ being introduced in the statement of Theorem 5.3.

Now asymptotic relations (5.8) – (5.10) follow from (5.21), and (5.31).

### 5.3. Proof of Theorem 5.4

Our first step, Proposition 5.6 below, reduces the asymptotic analysis of $N_0^2(\lambda; \text{Op}^w(V))$ and $N_q^2(\lambda; \text{Op}^w(V))$ as $\lambda \downarrow 0$, to the eigenvalue asymptotics for the Toeplitz operator $T_q(V)$, $q \in \mathbb{Z}_+$. In fact, we formulate Proposition 5.6 in a more general setting.

**Proposition 5.6.** Let $V = V^* \in \mathfrak{B}(L^2(\mathbb{R}^2))$ such that the operator $VH_{0}^{-1}$ is compact. Let $T := \Lambda_q \left( |\text{Re} (VH_{0}^{-1})| + |\text{Im} (VH_{0}^{-1})| \right)$. Then for any $q \in \mathbb{Z}_+$ and $\varepsilon > 0$ we have

$$(5.32) \quad n_+(\lambda; p_q(V - \varepsilon |T|)p_q) + O_{q, \varepsilon}(1) \leq N_0^q(\lambda; V) \leq n_+(\lambda; p_q(V + \varepsilon |T|)p_q) + O_{q, \varepsilon}(1),$$

$$(5.33) \quad n_-(\lambda; p_q(V + \varepsilon |T|)p_q) + O_{q, \varepsilon}(1) \leq N_q^{-\varepsilon}(\lambda; V) \leq n_-(\lambda; p_q(V - \varepsilon |T|)p_q) + O_{q, \varepsilon}(1),$$

as $\lambda \downarrow 0$.

We omit the standard proof which follows the general lines of [33, Section 5].

Now note that under the hypotheses of Proposition 5.6 the Weyl inequalities (5.20) and the mini-max principle easily imply

$$(5.34) \quad n_+(\lambda; p_q(V \mp \varepsilon |T|)p_q) \geq n_+(\lambda (1 + \eta); p_q V p_q) - 2n_+(\lambda^2 \eta^2 \varepsilon^2; (1 + \Lambda_0^{-1} \Lambda_q^2) p_q V p_q),$$

$$(5.35) \quad n_+(\lambda; p_q(V \mp \varepsilon |T|)p_q) \leq n_+(\lambda (1 - \eta); p_q V p_q) + 2n_+(\lambda^2 \eta^2 \varepsilon^2; (1 + \Lambda_0^{-1} \Lambda_q^2) p_q V p_q),$$

for any $\lambda > 0$, $\varepsilon > 0$, and $\eta \in (0, 1)$. Combining (5.32) - (5.33) and (5.34) - (5.35) with $V = \text{Op}^w(V)$, and bearing in mind Corollary 3.7 we obtain

**Corollary 5.7.** Under the hypotheses of Theorem 5.4 there exists a constant $C_1 > 0$ such that

$$(5.36) \quad n_-(\lambda (1 + \eta); \text{Op}^w(vb_{b,q})) - 2n_+(C_1 \lambda^2 \eta^2 \varepsilon^2; \text{Op}^w(wb_{b,q})) + O_{q, \varepsilon}(1) \leq N_q^{-\varepsilon}(\lambda),$$

$$(5.37) \quad n_-(\lambda (1 - \eta); \text{Op}^w(vb_{b,q})) + 2n_+(C_1 \lambda^2 \eta^2 \varepsilon^2; \text{Op}^w(wb_{b,q})) + O_{q, \varepsilon}(1).$$
Our next goal is to study the asymptotics of $n_+(\lambda; \text{Op}^w(v_{b,q}))$ and $n_+(\lambda; \text{Op}^w(w_{b,q}))$ as $\lambda \downarrow 0$. To this end, we will apply the approach developed in [10].

**Proposition 5.8.** Let $F = \mathcal{F} \in S^{-\gamma}(\mathbb{R}^{2n})$ for some $n \in \mathbb{N}$, $\gamma > 0$, and $\rho \in (0,1]$. Assume that the functions $\mathfrak{V}^+_\rho(\cdot; F)$ satisfy the condition $C$. If $\liminf_{\lambda \downarrow 0} \lambda^{2n/\gamma} \mathfrak{V}^+_\rho(\lambda; F) > 0$ (resp., if $\liminf_{\lambda \downarrow 0} \lambda^{2n/\gamma} \mathfrak{V}^-_{\rho}(\lambda; F) > 0$), then there exists $\delta > 0$ such that

$$n_+(\lambda; \text{Op}^w(F)) = \mathfrak{V}^+_n(\lambda; F)(1 + O(\lambda^\delta)), \quad \lambda \downarrow 0,$$

or, respectively,

$$n_-(\lambda; \text{Op}^w(F)) = \mathfrak{V}^-_n(\lambda; F)(1 + O(\lambda^\delta)), \quad \lambda \downarrow 0.$$

Proposition 5.8 follows from the main theorem of [10] with $\varphi(x, \xi) = \phi(x, \xi) := (1 + |x|^2 + |\xi|^2)^{\rho/2}$ and $m(x, \xi) := (1 + |x|^2 + |\xi|^2)^{-\gamma/2}$, $(x, \xi) \in \mathbb{R}^{2n}$.

Now we are in position to prove Theorem 5.4. For definiteness, let us handle (5.18). By Proposition 5.8

$$n_+(s; \text{Op}^w(v_{b,q})) = \mathfrak{V}^+_1(s; v_{b,q})(1 + o(1)), \quad s \downarrow 0. \tag{5.38}$$

Since $\mathfrak{V}^+_1(\cdot; v_{b,q})$ satisfies by assumption condition $C$, we find that

$$(1 + \eta)^{-\gamma_1} \mathfrak{V}^+_1(\lambda; v_{b,q}) \leq (1 + \eta)\lambda \mathfrak{V}^+_1((1 + \eta)\lambda; v_{b,q}), \quad \mathfrak{V}^+_1((1 - \eta)\lambda; v_{b,q}) \leq (1 - \eta)^{-\gamma_2} \mathfrak{V}^+_1(\lambda; v_{b,q}), \tag{5.39}$$

for any $\eta \in (0, 1)$ and $\lambda > 0$. It is easy to check that our assumption $w_{b,q} \in S^{-2\gamma}_\rho(\mathbb{R}^2)$ implies the existence of constant a constant $C_2$ such that

$$n_+(s; \text{Op}^w(w_{b,q})) \leq C_2 s^{-1/\gamma} \tag{5.40}$$

for $s > 0$ small enough. Putting together (5.30) - (5.37) and (5.38) - (5.40), we find that there exists a constant $C_3$ such that for any $\eta \in (0, 1)$ and $\varepsilon > 0$ we have

$$(1 + \eta)^{-\gamma_2} - C_3(\eta^2 \varepsilon^{-2})^{-1/\gamma} \leq \liminf_{\lambda \downarrow 0} \frac{\mathcal{N}_{\rho}(\lambda; \text{Op}^w(\mathcal{V}))}{\mathfrak{V}^+_1(\lambda; v_{b,q})} \leq \limsup_{\lambda \downarrow 0} \frac{\mathcal{N}_{\rho}(\lambda; \text{Op}^w(\mathcal{V}))}{\mathfrak{V}^+_1(\lambda; v_{b,q})} \leq (1 - \eta)^{-\gamma_1} + C_3(\eta^2 \varepsilon^{-2})^{-1/\gamma}.$$

Choosing $\eta = \sqrt{\varepsilon}$ and letting $\varepsilon \downarrow 0$, we obtain (5.18). The proof of (5.19) is completely analogous.

**Acknowledgements.** The authors thank Dimiter Balabanski (ELI - NP, Bucharest) and Hajo Leschke (University of Erlangen-Nuremberg) for useful comments on the applications of non-local potentials in physics. The partial support of the Chilean Scientific Foundation *Fondecyt* under Grant 1170816 is gratefully acknowledged.
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