Learning from Julius’ star, *, *

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Abstract

While collecting some personal memories about Julius Wess, I briefly describe some aspects of my recent work on many particle quantum mechanics and second quantization on noncommutative spaces obtained by twisting, and their connection to him.

Keywords: Noncommutative spaces; Drinfel’d twist; second quantization.
PACS numbers: 03.70.+k, 02.40.Gh, 02.20.Uw

1 Introduction

I’ll try to sketch Julius’ direct or indirect impact on my life and scientific activity. I have learnt a lot from his work (I wish I could have learnt more! Reading his papers I still learn), especially on some issues he was never tired of emphasising, such as the importance of symmetries (groups, supergroups, quantum groups,...) and conservation laws in physics, but also his guiding idea that fundamental physical laws should be coincisely expressible in algebraic form (sometimes he joked: “At the Very Beginning There Was the Algebra...”). I have learnt also from him through his scientific and human qualities (they often overlapped). Among them I would certainly mention: physical intuition and “exploration sixth sense”; open-minded, independent and creative thinking; search for beauty and simplicity; hierarchy of arguments, conciseness, clarity; concreteness, honesty, humbleness; ambition, courage.

*Talk given at the SEENET-MTP Workshop “Scientific and Human Legacy of Julius Wess” (JW2011), Donji Milanovac (Serbia), 27-28 August 2011, within the Balkan Summer Institute 2011. To appear in the proceedings.
to dare; coherence, rigour, determination; familiarity, cosiness; kindness, elegance, sense of humour.

I first learned about Prof. J. Wess at Naples University during my degree thesis on BRST quantization of gauge theories, when I met the Wess-Zumino consistency equation for the anomaly. After the degree I read with enthusiasm his seminal paper Ref. [1] with Zumino, which marked the beginning of their work on quantum spaces and quantum groups. Spurred by my advisor M. Abud, in early 1990 I sent him a letter asking whether I could do my PhD under his guide. As customary, he answered he would accept me, but could not provide financial support. Later that year I won a PhD grant at SISSA and started my PhD there. It was a very stimulating environment, nevertheless I kept the eye on what Julius and his group were doing, and in 1991 my Master and PhD supervisor L. Bonora accepted that I would do my theses on the same topics. Soon I succeeded in finding a sensible definition of integration over the so-called quantum Euclidean space $\mathbb{R}^n_q$ and in solving the eigenvalue problem for the harmonic oscillator Hamiltonian on $\mathbb{R}^n_q$, which Wess had proposed to me; later I succeeded in realizing $U_q so(n)$ (the deformed infinitesimal rotations) by differential operators on $\mathbb{R}^n_q$ (the analog of the angular momentum components)[2]. In summer 1993 I wished to update him about the progresses, but he was very busy and difficult to meet. I remember that as a last year Sissa student I could participate to one conference outside Europe, and felt a strong appeal towards the “First Caribbean School of Mathematics and Theoretical Physics Saint-François, Guadeloupe”. But I decided to go to the Workshop “Interface between physics and mathematics” in Hangzhou, China, after noticing Julius among the invited speakers. It was an interesting conference and a marvellous trip, but Julius was at some other conference. I thus learned what being a scientific “Star” like Julius meant: I soon realized that the

| Blabla Conference       | Blabla Symposium |
|-------------------------|------------------|
| 9-13/9/yyyy            | 10-15/9/yyyy     |
| Speakers include:      | Speakers include:|
| - Julius Wess*         | - Julius Wess*   |
| - …                    | - …              |
| - …                    | - …              |
| - …                    | - …              |

* To be confirmed

I still don’t know the right answer. After my PhD I was hosted by Julius as a A. v. Humboldt post-doc at the Ludwig-Maximilian-Universität in Munich. At the time the symbol “*” recurred obsessively in our computations for a different reason: we were struggling with $*$-structures (i.e. the algebraic formulations of hermitean conjugations). Julius and his group we could not find (for real deformation parameters $q$) $*$-structures compatible with the $q$-Poincaré (nor the $q$-Euclidean) quantum group without doubling the generators of translations and adding dilatations. I appreciated first his rigorous and hard working, finally his intellectual honesty in admitting that
fact made those deformations unsatisfactory and to quit. Ever since I have worried about implementing ∗-structures in the noncommutative world.

In Munich I also learnt (especially thanks to his student R. Engeldinger) about the construction of quantum groups from groups using Drinfel’d twists $\mathcal{F}$ [3]; among other things $\mathcal{F}$ intertwined between the group and the quantum group actions on tensor product representations. In the joint papers Ref. [4] with Peter Schupp we pointed out that the unitary transformation $\mathcal{F}$ (and its descendants) intertwines also between the conventional and an unconventional realization of the permutation group [and therefore of (anti)symmetrization] on tensor products, and therefore that quantum group transformations were compatible with Bose and Fermi statistics. Here I would like to sketch some related, more recent results [5] illustrating the crucial role of twists not only in deforming spaces but also in quantizing (for simplicity scalar) fields on the latter. This will clarify also the last symbol ∗ in the title.

A rather general way to deform an (associative) algebra $\mathcal{A}$ (over $\mathbb{C}$, say) into a new one $\mathcal{A}_\star$ is by deformation quantization [9]. Calling $\lambda$ the deformation parameter, this means that the two have the same vector space over $\mathbb{C}[[\lambda]]$, $V(\mathcal{A}_\star) = V(\mathcal{A})[[\lambda]]$, but the product $\star$ in $\mathcal{A}_\star$ is a deformation of the product $\cdot$ in $\mathcal{A}$. On the algebra $\mathcal{X}$ of smooth functions on a manifold $\mathcal{X}$, and on the algebra $\mathcal{D} \supset \mathcal{X}$ of differential operators on $\mathcal{X}$, $f \star h$ can be defined applying to $f \otimes h$ first a suitable bi-pseudodifferential operator $F$ (depending on the deformation parameter $\lambda$ and reducing to the identity when $\lambda = 0$) and then the pointwise multiplication $\cdot$. The simplest example is probably the Grönewold-Moyal-Weyl (Moyal, for brevity) $\star$-product on $\mathcal{X} = \mathbb{R}^m$:

$$a(x) \star b(x) := a(x) \exp \left[ \frac{i}{2} \sum_{h, k} \lambda \vartheta_{hk} \partial_h \partial_k \right] b(x) = \cdot \left[ \mathcal{F}(\triangleright \otimes \triangleleft)(a \otimes b) \right],$$  

where $P_a$ are the generators of translations (on $\mathcal{X}$ $P_a$ can be identified with $-i\partial_a := -i\partial/\partial x^a$), and $\vartheta_{hk}$ is a fixed real antisymmetric matrix; as recalled below, definition (1) can be made non-formal in terms of Fourier transforms. $\mathcal{X}_\star, \mathcal{X}$ have the Poincaré-Birkhoff-Witt (PBW) property, i.e. the subspaces of $\star$-polynomials and $\cdot$-polynomials of any fixed degree in $x^h$ coincide. One can define a linear map $\wedge: f \in \mathcal{X}[[\lambda]] \to \hat{f} \in \mathcal{X}_\star$ (the Weyl map) by requiring that it reduces to the identity on the vector space $V(\mathcal{X}_\star) = V(\mathcal{X})[[\lambda]]$: $\hat{f}(x\star) = f(x)$. One finds

$$\wedge(x^h) = x^h, \quad \wedge(x^h x^k) = x^h \star x^k - \frac{i}{2} \vartheta_{hk}, \quad \Rightarrow \quad [x^h \star x^k] = 1i\vartheta_{hk},$$  

and so on (again, this can be extended to non-polynomial functions through Fourier transforms). In other words, by $\wedge$ one expresses functions of $x^h$ as functions of $x^h \star$.

As $\mathcal{X}$, also $\mathcal{X}_\star$ can be defined purely through generators and relations: the coordinates $x^h$ and $1$ are the generators of both, and fulfill $[x^h, x^k] = 0$ in $\mathcal{X}_\star$. Similarly one deforms $\mathcal{D}$ into $\mathcal{D}_\star$; however for the twist (1) $[\partial_a \star \cdot] = [\partial_a, \cdot]$. 

...
Replacing all \( \cdot \) by \( \star \)'s e.g. in the Schrödinger equation of a particle with charge \( q \)

\[
\hbar \psi(x) = i\hbar \partial_t \psi(x), \quad \hbar := \frac{-\hbar^2}{2m} D_a \star D_a + V, \quad D_a = \partial_a + i \frac{q}{\hbar c} a, \quad (3)
\]

we obtain a pseudodifferential equation and therefore introduce some (quite special) non-local interaction, which might e.g. give an effective description of a complicated background. The use of noncommutative coordinates may then help to solve the dynamics: if we express \( V(x) \star, A_a(x) \star \) and \( \psi(x) \) as their Weyl map images \( \hat{V}(x) \), \( \hat{A}_a(x) \) and \( \hat{\psi}(x) \), then (3) becomes a second order \( \star \)-differential equation (i.e. of second degree in \( \partial_{\hbar} \)) where the unknown is now a function \( \hat{\psi}(x) \) of \( x \):

\[
i\hbar \partial_t \hat{\psi}(\hat{x}) = \frac{-\hbar^2}{2m} \hat{D}_a \hat{D}_a \hat{\psi}(\hat{x}) + \hat{V}(\hat{x}) \hat{\psi}(\hat{x});
\]

here we have made the notation lighter by denoting \( x^a \star, \partial_a \) as \( \hat{x}^a, \hat{\partial}_a \). (More generally, we often change notation as follows: \( X \star \rightarrow \hat{X}, \ D \star \rightarrow \hat{D}, \ x^i \star \rightarrow \hat{x}^i, \ \partial_i \star \rightarrow \hat{\partial}_i, \ a_i \star \rightarrow \hat{a}_i, \) etc.). Nonetheless, in a conservative approach we still measure the position of a particle using the observables \( x^a \) of commutative space. In a radical one we will rather use the noncommutative observables \( \hat{x}^a \) for the latter purpose.

Eq. (1-2) on Minkowski (resp. Euclidean) space are not covariant under the Poincaré (resp. Euclidean) group \( G^e \), or equivalently under the associated universal enveloping algebra \( U^e \). Julius and coworkers in Ref. [12, 13], simultaneously to Ref. [14], realized that \( F := F^{-1} \) could be used to twist \( U^e \) into the symmetry Hopf algebra of \( \hat{U}^e \) of the \( \star \)-product itself: one could recover Poincaré covariance in a deformed form! In the joint paper [10] with Julius we pointed out that in fact \( \hat{U}^e \)-covariance implied as \( \star \)-commutation relations for \( n \) copies of \( X \),

\[
[x^\mu_i, x^\nu_j] = i\theta^{\mu\nu} \quad \iff \quad [\hat{x}^\mu_i, \hat{x}^\nu_j] = i\theta^{\mu\nu} \quad i = 1,2,\ldots, n, \quad (4)
\]

and not the ones with \( i\delta^\mu_i \theta^{\mu\nu} \) at the rhs: so for \( i \neq j \) the rhs is not automatically zero. That has important consequences for both multiparticle quantum mechanics \( [x_i, \) denoting the space(time) coordinates of the \( i \)-th particle] and quantum field theory (products of fields evaluated at \( n \) different spacetime points \( x_i \)). In Ref. [10, 11] we found the surprising result that a translation-invariant Lagrangian implied that the field commutators and Green functions, as functions of the coordinates’ differences, remained as those of the undeformed theory (at least for scalar fields). To put the field quantization prescription on a firmer ground, in Ref. [5] I have re-derived it by a second quantization procedure from \( n \)-particle wavefunctions preserving Bose/Fermi statistics (i.e. the rule to compute the number of allowed states of \( n \) identical bosons/fermions): not only the function algebra, but also that of creation and annihilation operators and their tensor product are \( \star \)-deformed. [Following Julius, to guess the deformed analog of a known theory (be it noncommutative gravity[6, 7] or gauge field theory[8] before quantization, or QFT) we should translate all commutative notions into their noncommutative analogs by just expressing all products
2 Preliminaries

2.1 Twisting $H = U_g$ to a noncocommutative Hopf algebra $\hat{H}$.

The Universal Enveloping $*$-Algebra (UEA) $H := U_g$ of the Lie algebra $g$ of any Lie group $G$ is a Hopf $*$-algebra. We briefly recall first what this means. Let

$$\varepsilon(1) = 1, \quad \Delta(1) = 1 \otimes 1, \quad S(1) = 1,$$
$$\varepsilon(g) = 0, \quad \Delta(g) = g \otimes 1 + 1 \otimes g, \quad S(g) = -g, \quad \text{if } g \in g;$$

$\varepsilon, \Delta$ are extended to all of $H$ as $*$-algebra maps, $S$ as a $*$-antialgebra map:

$$\varepsilon : H \to \mathbb{C}, \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(a^*) = [\varepsilon(a)]^*,$$
$$\Delta : H \to H \otimes H, \quad \Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(a^*) = [\Delta(a)]^*, \quad (5)$$
$$S : H \to H, \quad S(ab) = S(b)S(a), \quad S\{[S(a^*)]^\ast\} = a.$$  

The extensions of $\varepsilon, \Delta, S$ are unambiguous, as $\varepsilon(g) = 0$, $\Delta([g, g']) = [\Delta(g), \Delta(g')]$, $S([g, g']) = [S(g'), S(g)]$ if $g, g' \in g$. The maps $\varepsilon, \Delta, S$ are the abstract operations by which one constructs the trivial representation, the tensor product of any two representations and the contragredient of any representation, respectively; $H$ equipped with $*, \varepsilon, \Delta, S$ is a Hopf $*$-algebra. One can deform this Hopf algebra using a twist [3] (see also [15]), i.e. an element $F \in (H \otimes H)[[\lambda]]$ fulfilling

$$F = 1 \otimes 1 + O(\lambda), \quad (\varepsilon \otimes \text{id})F = (\text{id} \otimes \varepsilon)F = 1,$$
$$F \otimes 1)((\Delta \otimes \text{id})(F)) = (1 \otimes F)((\text{id} \otimes \Delta)(F)) =: F^3. \quad (7)$$

Let $H_s \subseteq H$ be the smallest Hopf $*$-subalgebra such that $F \in (H_s \otimes H_s)[[\lambda]],$

$$F^\alpha \otimes F_\alpha := F, \quad F^\ast \otimes F^\ast_\alpha := F^{-1}, \quad \beta := F^\ast S(F_\alpha) \in H_s[[\lambda]] \quad (8)$$

(sum over $\alpha$) be the tensor decompositions of $F, F^{-1}$, and $\hat{H} = H[[\lambda]]$. We assume $\lambda$ real and $F$ unitary ($F^{\ast\ast} = F^{-1}$), implying $\beta^* = S(\beta^{-1})$. Extending the product,
*, Δ, ε, S linearly to the formal power series in λ and setting

\[ \hat{\Delta}(g) := F \Delta(g) F^{-1}, \quad \hat{S}(g) := \beta S(g) \beta^{-1}, \quad \mathcal{R} := \tau(\mathcal{F}) \mathcal{F}^{-1}, \] (9)

one finds that the analogs of conditions (5) are satisfied and therefore \( \hat{H}, *, \hat{\Delta}, \varepsilon, \hat{S} \) is a Hopf *-algebra deformation of the initial one. While \( H \) is cocommutative, i.e. \( \tau \circ \Delta(g) = \Delta(g) \) where \( \tau \) is the flip operator \([\tau(a \otimes b) = b \otimes a]\), \( \hat{H} \) is triangular non-cocommutative, i.e. \( \tau \circ \hat{\Delta}(g) = \mathcal{R} \hat{\Delta}(g) \mathcal{R}^{-1} \), with unitary triangular structure \( \mathcal{R} \) (i.e. \( \mathcal{R}^{-1} = \mathcal{R}_{21} = \mathcal{R}^\otimes \)). Correspondingly, \( \hat{\Delta}, \hat{S} \) replace \( \Delta, S \) in the construction of the tensor product of any two representations and the contragredient of any representation, respectively. Drinfel’d has shown[3] that any triangular deformation of the Hopf algebra \( \hat{H} \) can be obtained in this way (up to isomorphisms).

Eq. (7), (9) imply the generalized intertwining relation \( \Delta^{(n)}(g) = F^n \Delta^{(n)}(g)(F^n)^{-1} \) for the iterated coproduct. By definition

\[
\hat{\Delta} : \hat{H} \to \hat{H}^{\otimes n}, \quad \Delta^{(n)} : H[[\lambda]] \to (H[[\lambda]])^\otimes n, \quad F^n \in (H_s)^\otimes n[[\lambda]]
\]

reduce to \( \hat{\Delta}, \Delta, F \) for \( n = 2 \), whereas for \( n > 2 \) they can be defined recursively as

\[
\Delta^{(n+1)} = (\mathrm{id} \otimes (n-1) \otimes \Delta) \circ \Delta^{(n)}, \quad \Delta^{(n+1)} = (\mathrm{id} \otimes (n-1) \otimes \Delta) \circ \Delta^{(n)},
\]

\[
F^{n+1} = (1 \otimes (n-1) \otimes F)[(\mathrm{id} \otimes (n-1) \otimes \Delta)F^n].
\] (10)

The iterated definitions (10) do not change if we resp. apply \( \hat{\Delta}, \Delta, F \hat{\Delta} \) to different tensor factors \([\text{coassociativity of } \hat{\Delta}; \text{this follows from the coassociativity of } \Delta \text{ and the cocycle condition (7)}]\); for instance, for \( n = 3 \) this amounts to (7) and \( \hat{\Delta}^{(3)} = (\Delta \otimes \mathrm{id}) \circ \hat{\Delta} \).

For any \( g \in H[[h]] = \hat{H} \) we shall use the Sweedler notations

\[
\Delta^{(n)}(g) = \sum_I g_1^I \otimes g_2^I \otimes \ldots \otimes g_n^I, \quad \hat{\Delta}^{(n)}(g) = \sum_I g_1^I \otimes g_2^I \otimes \ldots \otimes g_n^I.
\]

Deforming the Euclidean group \( U \mathfrak{g}^e \) by the twist (1) one finds \( \beta = 1, \hat{S} = S \),

\[
\hat{\Delta}(P_a) = P_a \otimes 1 + 1 \otimes P_a = \Delta(P_a),
\]

\[
\hat{\Delta}(M_\omega) = M_\omega \otimes 1 + 1 \otimes M_\omega + ([\omega, \theta])^{ab} P_a \otimes P_b
\]

where \( M_\omega = \omega^{ab} M_{ab} \) and \( M_{ab} \) are the generators of \( \mathfrak{so}(m) \); the Hopf subalgebra of translations is not deformed. Similarly one deforms Poincaré transformations[14] [12] [13].

2.2 Twisting \( H \)-module *-algebras. \( \) A left \( H \)-module *-algebra \( \mathcal{A} \) is a *-algebra equipped with a left action, i.e. a \( \mathbb{C} \)-bilinear map \( (g, a) \in H \times \mathcal{A} \to g \triangleright a \in \mathcal{A} \) such that

\[
(gg') \triangleright a = g \triangleright (g' \triangleright a), \quad (g \triangleright a)^* = S(g)^* \triangleright a^*, \quad g \triangleright (ab) = \sum_I g_1^I \triangleright a \cdot g_2^I \triangleright b. \] (11)
Given such an \( \mathcal{A} \), let \( V(\mathcal{A}) \) the vector space underlying \( \mathcal{A} \). \( V(\mathcal{A})[\![\lambda]\!] \) becomes a \( H \)-module \(*\)-algebra \( \mathcal{A} \ast \) when endowed with the product and \(*\)-structure
\[
a \ast a' := (\mathcal{F}^{\ast} a) (\mathcal{F}_{a} a'), \quad a^{\ast\ast} := S(\beta) a^*.
\]
In fact, \(*\) is associative by (7), fulfills \((a \ast a')^\ast = a'^\ast \ast a^\ast\) and
\[
(g \triangleright a)^\ast = \hat{S}(g) a^\ast, \quad g \triangleright (a \ast a') = \sum_i g_i^1 \triangleright a \ast g_i^2 \triangleright a'.
\]
For the Moyal twist (1) \( \beta = 1, \ast = \ast \). The \(*\) is ineffective if \( a \) or \( a' \) is \( H_s \)-invariant:
\[
g \triangleright a = \epsilon(g) a \quad \text{or} \quad g \triangleright a' = \epsilon(g) a' \quad \forall g \in H_s \quad \Rightarrow \quad a \ast a' = a a'.
\]
Given \( H \)-module \(*\)-algebras \( \mathcal{A}, \mathcal{B}, \) also \( \mathcal{A} \otimes \mathcal{B} \) is, so (12) makes \( V(\mathcal{A} \otimes \mathcal{B}) \) into a \( \hat{H} \)-module \(*\)-algebra \( (\mathcal{A} \otimes \mathcal{B}) \ast \). Denoting \( a \otimes \ast b = (a \otimes 1_{B} \ast (1_{A} \otimes b), \mathcal{R} = \mathcal{R} \otimes \mathcal{R} \), one finds
\[
(a \otimes b) \ast (a' \otimes b') = a \ast (\mathcal{R} \triangleright a') \otimes (\mathcal{R} \triangleright b) \ast b',
\]
so \( \otimes \) is the braided tensor product associated to \( \mathcal{R} \), and \( (\mathcal{A} \otimes \mathcal{B}) \ast = \mathcal{A} \ast \otimes \mathcal{B} \ast \).

If \( \mathcal{A} \) is defined by generators \( a_i \) and relations, then so is \( \mathcal{A} \ast \), and fulfills PBW[5]. The generalized Weyl map is the linear map \( \wedge : f \in \mathcal{A} \rightarrow \hat{f} \in \mathcal{A} \), defined by
\[
f(a_1, a_2, \ldots) \ast = \hat{f}(a_1 \ast, a_2 \ast, \ldots) \quad \text{in } V(\mathcal{A}) = V(\mathcal{A} \ast),
\]
generalizing (2). It fulfills \( \wedge(f f') = (\mathcal{F} \triangleright f) \ast (\mathcal{F} \triangleright f') \).

### 2.3 Deformation of the Heisenberg, Clifford algebra \( \mathcal{A} \pm \)

Quantum mechanics on \( \mathbb{R}^n \) is covariant w.r.t. the Lie group \( G^e \) of Euclidean - or, more generally, Galilei - transformations (here we consider them as active transformations). This implies that the Heisenberg algebra \( \mathcal{A}^+ \) (resp. Clifford algebra \( \mathcal{A}^- \)) associated to a species of bosons (resp. fermions) is a \( U g^e \)-module \(*\)-algebra. As the \( G^e \)-action is unitarily implemented on the Hilbert spaces of the systems, that of \( H = U g^e \) is defined on dense subspaces, in particular on a pre-Hilbert space \( \mathcal{H} \) of the one-particle sector, on which it will be denoted as \( \rho \): \( g \triangleright =: \rho(g) \in \mathcal{O} := \text{End}(\mathcal{H}) \).

The pre-Hilbert space of \( n \) bosons (resp. fermions) is described by the completely symmetrized (resp. antisymmetrized) tensor product \( \mathcal{H}^{\otimes n} \) (resp. \( \mathcal{H}^{\ominus n} \)), which is a \( H \)-\(*\)-module of \( \mathcal{H}^{\otimes n} \). Assuming a unique, invariant vacuum state \( \Psi_0 \), the bosonic (resp. fermionic) Fock space is defined as the closure \( \mathcal{H}^{\otimes n}_\pm \) of
\[
\mathcal{H}^{\otimes n}_\pm := \{ \text{finite sequences } (s_0, s_1, s_2, \ldots) \in C \Psi_0 \oplus \mathcal{H} \oplus \mathcal{H}^2_\pm \oplus \ldots \}
\]
(finite means that there exists an integer \( l \geq 0 \) such that \( s_n = 0 \) for all \( n \geq l \)). The creation, annihilation operators \( a^+_i, a^i \) associated to an orthonormal basis \( \{ e_i \}_{i \in \mathbb{N}} \) of \( \mathcal{H} \) fulfill the Canonical (anti)Commutation Relations (CCR)
\[
a^i a^j = \pm a^j a^i, \quad a^+_i a^+_j = \pm a^+_j a^+_i, \quad a^+_i a^+_j + a^+_j a^+_i = \delta^i_j 1_{\mathcal{A}}.
\]
(\(\pm\) for bosons, \(-\) for fermions). \(a^+_i, a^i\) resp. transform as \(e_i = a^+_i \Psi_0\) and \(\langle e_i, \cdot \rangle\):
\[
g \triangleright e_i = \rho^i_j (g) e_j, \quad g \triangleright a^+_i = \rho^i_j (g) a^+_j, \quad g \triangleright a^i = \rho^i_j (g) a^j = \rho^j_i [S(g)] a^j \quad (18)
\]
(\(\rho^i = \rho \circ S\) is the contragredient of \(\rho\). \(A^\pm\) are \(H\)-module \(*\)-algebras because the \(g^*\)-action (extended to products as a derivation) is compatible with \(\rho\).

Applying the deformation procedure one obtains \(\hat{H}\)-module \(*\)-algebras \(\hat{A}^\pm\). The generators \(a^+_i, a^i := a^+_i * a^i = \rho^j_i (\beta) a^j\) fulfill the \(*\)-commutation relations
\[
a^i * a^j = \pm R^i_j a^m * a^v, \quad \hat{a}^i \hat{a}^j = \pm R^i_j \hat{a}^m \hat{a}^v; \quad \hat{a}^i \hat{a}^+_j = \pm R^i_j \hat{a}^+_m \hat{a}^v, \quad \hat{a}^i \hat{a}^+_j = \delta^i_j \mathbf{1}_A \pm R^i_j \hat{a}^+_m \hat{a}^v, \quad \hat{a}^i \hat{a}^j = \delta^i_j \mathbf{1}_A \pm R^i_j \hat{a}^+_m \hat{a}^v, \quad (19)
\]
where \(R := (\rho \otimes \rho)(\mathcal{R})\). The \(a^i\) transform according to the rule of the \textit{twisted} contragredient representation: \(g \triangleright a^i = \rho^j_i [S(g)] a^j\). Equivalently, \(\hat{A}^\pm \sim A^\pm\) has generators \(\hat{a}^i, \hat{a}^i\) fulfilling \(\hat{a}^i \hat{a}^+ = \mathbf{1}\) and the rhs \([19],[16]\).

Is there a Fock-type representations of \(\hat{A}^\pm\)? Yes, only one, on the \textit{undeformed} Fock space of bosons/fermions \([4]\). The important consequence is that \(\hat{A}^\pm\) are \textbf{compatible with Bose/Fermi statistics} \([4]\). In fact one can \textit{realize} \([16]\) \(\hat{a}^i, \hat{a}^i\) as ‘dressed’, hermitean conjugate elements \(\hat{a}^i, \hat{a}^i\) in \(A^\pm[[\lambda]]\) fulfilling \([19]\):
\[
\hat{a}^i = (\mathcal{F} \circ a^i) \sigma (\mathcal{F}_a), \quad \hat{a}^i = (\mathcal{F} \circ a^i) \sigma (\mathcal{F}_a). \quad (20)
\]
\(\sigma\) is the generalized Jordan-Schwarizer realization of \(U_g\), i.e. the \(*\)-algebra map \(\sigma: H[[\lambda]] \to A^\pm[[\lambda]]\) such that \(\sigma(1) = 1, \sigma(g) = (g \triangleright a^i) a^j\) if \(g \in g\); it fulfills
\[
g \triangleright a = \sum_i \sigma (g^i_1) \ a \ \sigma ([S(g^i_1)]) \quad \forall g \in H, \ a \in A^\pm.
\]
For \(G = G^e\), and the Moyal twist let \(a^+_p, a^p\) be the creation, annihilation operators associated to the joint, generalized eigenvectors of the \(P_a, P_a e_p = p_a e_p\) \((p \in \mathbb{R}^m)\); in that basis \(P^{\pm\pm}_{q'q} = e^{i \theta q^p \delta (p-q) \delta (p' - q')}\), where we abbreviate \(p \theta q := p_a \theta_{ab} p_b\), so that e.g. \([19],[3]\) becomes \(\hat{a}^p \hat{a}^+_q = \delta (p-q) \mathbf{1}_A \pm e^{i \theta q} \hat{a}^+_q \hat{a}^p\), and \(\hat{a}^i = (\mathcal{F} \circ a^i) \sigma (\mathcal{F}_a)\) becomes
\[
\hat{a}^+_p = a^+_p e^{-i \theta \sigma (P)}, \quad \hat{a}^p = a^p e^{i \theta \sigma (P)}, \quad \sigma (P_a) := \int d^m p \ p_a a^+_p a^p.
\]

\[2.4\] \textbf{Moyal-deforming functions, differential, integral calculi on} \(\mathbb{R}^m\). We denote as \(\mathcal{D}_p\) the Heisenberg algebra on \(X = \mathbb{R}^m\). The \(*\)-structure and the \(*\)-commutation relations of the \(H\)-module \(*\)-algebra \(\mathcal{D}^m_{\tilde{p}}\) are as the undeformed ones except \([1]\), where we have abbreviated \(x^h, x^h, \ldots\) for \(x^h \otimes 1 \otimes \ldots, 1 \otimes x^h \otimes \ldots\) and \(\partial x^i = \partial / \partial x^i, \partial x^i = \partial / \partial x^i\). \(1, x^i_1, x^i_2\) generate the \(H\)-module \(*\)-subalgebra \(\mathcal{A}_{\tilde{p}}\). The latter is ‘too small’ for physical purposes, but one can extend \(*\) and the Weyl map
∧ to other $H$-module $*$-algebras, e.g. the Schwarz space $\mathcal{X} := \mathcal{S}(\mathbb{R}^m)$, the distribution space $\mathcal{X}'$ and the algebra of $\mathcal{D} \supset \mathcal{D}_p$ of smooth differential operators on $X$, replacing the (discrete) set of polynomials in $x^i$ by the (continuous) set of exponentials $e^{i \Sigma_i h_i x_i}$ (labelled by $n$ indices $h_i \in \mathbb{R}^m$) as a basis: since the $\theta$-power series expansion for the $*$-product of two exponentials converges, giving in particular
\[
e^{ihx_i} \ast e^{ikx_j} = e^{i(hx_i + kx_j - \frac{h \cdot k}{2})},
\] (21)

it suffices to express the $a, b \in \mathcal{X}, \mathcal{X}'$ through their Fourier transforms $\tilde{a}, \tilde{b}$ and define
\[
a(x_i) \ast b(x_j) := \int \mathcal{D}^m h \int \mathcal{D}^m k \ e^{i(hx_i + kx_j - \frac{h \cdot k}{2})} \tilde{a}(h) \tilde{b}(k).
\] (22)

The Moyal $\ast$-product fulfills the cyclic property w.r.t. Riemann integration:
\[
\int dx a(x) \ast b(x) = \int dx a(x) b(x) = \int dx b(x) \ast a(x)
\] (23)

(this is modified when $\beta \neq 1$); the same applies for multiple integrations, after having reordered through (15) all the functions depending on the same argument beside each other, e.g. $f(x_i) \ast f'(x_j) \ast f''(x_i) = \mathcal{R}^a \mathcal{R}^{a'} \mathcal{R}^{a''}[f'(x_j)] \ast [f(x_i)] \ast [f''(x_i)]$. One can define also a $\hat{H}$-invariant “integration over $\hat{X}$” $\int \hat{dx}$ such that for each $f \in \mathcal{X}$
\[
\int \hat{dx} f(\hat{x}) = \int dx f(x).
\] (24)

We shall call $\wedge^n$ the analogous maps $\wedge^n : f \in \mathcal{X}^\otimes_n[[\lambda]] \to \hat{f} \in (\mathcal{X}^\otimes_n)_\wedge$. One finds $\wedge(e^{ihx_i}) = e^{ihx_i} \ast \sim a e^{ihx_i}$. Eq. (23, 24) generalize to integration over $n$ independent variables. The differences $\xi^i_i := x^i_1 - x^i_{n+1}$, $i = 1, \ldots, n-1$, are translation invariant, so by (14) $f(x) \ast h(\xi) = f(x)h(\xi) = h(\xi) \ast f(x)$ for all $f, h$.

3 Twisting non-relativistic second quantization

In the wave-mechanical description of a system of $n$ bosons/fermions on $\mathbb{R}^m$ we describe any abstract state vector (ket) $s \in \mathcal{H}^\otimes_n$ as a smooth wavefunction $\psi_s \in \mathcal{X}^\otimes_n$ of $x_1, \ldots, x_n$. By the Weyl map we can describe the same state $s$ also as the noncommutative wavefunction $\wedge(\psi_s) := \hat{\psi}_s \in \mathcal{X}^\otimes_n$ of $\hat{x}_1, \ldots, \hat{x}_n$. The maps $s \mapsto \psi_s \mapsto \hat{\psi}_s$ are unitary. Differential operators $\hat{D} \in \mathcal{D}^\otimes_n$ acting on $\mathcal{X}^\otimes_n$ are mapped into differential operators $\hat{\hat{D}} \in \mathcal{D}^\otimes_n$ acting on $\hat{\mathcal{X}}^\otimes_n$. The action of the symmetric group $S_n$ on $\mathcal{X}^\otimes_n$ is obtained by “pull-back” from that on $\mathcal{X}^\otimes_n$: a permutation $\tau \in S_n$ is represented on $\mathcal{X}^\otimes_n, \hat{\mathcal{X}}^\otimes_n$ resp. by the permutation operator $\mathcal{P}_\tau$ and the “twisted permutation operator” $\hat{\mathcal{P}}^\tau = \wedge^n \mathcal{P}_\tau [\wedge^n]^{-1}$. Thus, $\hat{\mathcal{X}}^\otimes_n$ are (anti)symmetric up to the similarity transformation $\wedge^n$ (cf. Ref. [4]). $H$-equivariance of the commutative description
translates into $\hat{H}$-equivariance of the noncommutative one. Given a basis $\{e_i\}_{i \in \mathbb{N}} \subset \mathcal{H}$, let $\varphi_i = \kappa(e_i)$, $\hat{\varphi}_i = \wedge(\varphi_i)$; we illustrate how $\wedge^2$ transforms the (anti)symmetrized tensor product basis:

$$\varphi_i(x_1)\varphi_j(x_2) \pm \varphi_j(x_1)\varphi_i(x_2) \xrightarrow{\wedge^2} F_{ij}^{hk} \hat{\varphi}_h(x_1)\hat{\varphi}_k(x_2) \pm F_{ji}^{hk} \hat{\varphi}_h(x_1)\hat{\varphi}_k(x_2)$$

(25)

where $F := (\rho \otimes \rho)(\mathcal{F})$. To make at most the matrix $R := (\rho \otimes \rho)(\mathcal{R})$ appear at the rhs one should rather use at the lhs as vectors of a (non-orthonormal) basis of $(\mathcal{X} \otimes \mathcal{X})^\pm$

$$\mathcal{F}_{ij}^{hk}[\varphi_h(x_1)\varphi_k(x_2) \pm \varphi_k(x_1)\varphi_h(x_2)] \xrightarrow{\wedge^2} \hat{\varphi}_j(x_1)\hat{\varphi}_j(x_2) \pm R_{ij}^{hk} \hat{\varphi}_h(x_1)\hat{\varphi}_k(x_2) = \hat{\varphi}_i(x_1)\hat{\varphi}_j(x_2) \pm \hat{\varphi}_i(x_2)\hat{\varphi}_j(x_1).$$

(26)

Their form (27) is closer than (26) to the undeformed counterpart. Both generalize to $n > 2$. The generalization of (27) for $n$ fermions is the Slater determinant

$$\psi^{(n)}_{-i_1...i_n}(\hat{x}_1, ..., \hat{x}_2) = ... \begin{vmatrix} \hat{\varphi}_{i_1}(\hat{x}_1) & \hat{\varphi}_{i_2}(\hat{x}_1) & ... & \hat{\varphi}_{i_n}(\hat{x}_1) \\ \hat{\varphi}_{i_1}(\hat{x}_2) & \hat{\varphi}_{i_2}(\hat{x}_2) & ... & \hat{\varphi}_{i_n}(\hat{x}_2) \\ ... & ... & ... & ... \\ \hat{\varphi}_{i_1}(\hat{x}_n) & \hat{\varphi}_{i_2}(\hat{x}_n) & ... & \hat{\varphi}_{i_n}(\hat{x}_n) \end{vmatrix},$$

(28)

provided we keep the order of the wavefunctions and permute the $\hat{x}_h$: to the permutation $(h_1, h_2, ..., h_n)$ there corresponds the term $\pm \hat{\varphi}_{i_1}(\hat{x}_{h_1})\hat{\varphi}_{i_2}(\hat{x}_{h_2})...\hat{\varphi}_{i_n}(\hat{x}_{h_n})$.

The **nonrelativistic field operator** and its hermitean conjugate (in the Schrödinger picture)

$$\varphi(x) := \varphi_i(x)a_i^\dagger, \quad \varphi^*(x) = \varphi_i^+(x)a_i$$

(29)

(infinite sum over $i$) are operator-valued distributions fulfilling the CCR

$$[\varphi(x), \varphi(y)]_\mp = \text{h.c.} = 0, \quad [\varphi(x), \varphi^*(y)]_\mp = \varphi_i(x)\varphi_i^*(y) = \delta(x-y)$$

(30)

($\mp$ for bosons/fermions). The **field $*$-algebra** $\Phi$ is spanned by all monomials

$$\varphi^*(x_1)...\varphi^*(x_m)\varphi(x_{m+1})...\varphi(x_n)$$

(31)

$x_1, ..., x_n$ are independent points). So $\Phi \subset \Phi^e := \mathcal{A}^\pm \otimes (\bigotimes_{i=1}^{\infty} \mathcal{X}^\prime)$. Here the 1st, 2nd,.. tensor factor $\mathcal{X}^\prime$ is the space of distributions depending on $x_1, x_2, ...$; the dependence of (31) on $x_h$ is trivial for $h > n$. $\Phi^e$ is a huge $H$-module $*$-algebra: $a_i^\dagger, \varphi_i$ transform as $e_i$, and $a_i^\dagger, \varphi_i^*$ transform as $\langle e_i, \cdot \rangle$. The CCR (17) of $\mathcal{A}^\pm$ are the only nontrivial commutation relations in $\Phi^e$. A key property is that $\varphi, \varphi^*$ are basis-independent, i.e. **invariant under the group $U(\infty)$ of unitary transformations of $\{e_i\}_{i \in \mathbb{N}}$**, in particular under the subgroup $G^e$ of Euclidean transformations (transformations of the states $e_i$ obtained by translations or rotations of the 1-particle system), or (in infinitesimal form) **under $U^e$: $g \cdot \varphi(x) = \epsilon(g)\varphi(x)$**. By (14), deforming $U^e \rightarrow \widehat{U^e}$, $Uu(\infty) \rightarrow \widehat{Uu(\infty)}$ and $\Phi^e \rightarrow \Phi^e$, we still find for all $\omega \in V(\Phi^e)[[\lambda]]$

$$\varphi(x) \ast \omega = \varphi(x)\omega, \quad \omega \ast \varphi(x) = \omega \varphi(x), \quad \& \text{ h. c.}$$

(32)
Since $\epsilon(\beta) = 1$ and the definition $a^\mu := a_i^+* = S(\beta) \triangleright a^i$ imply
\[ \varphi(x) = \varphi_i(x) \ast a^i, \quad \varphi^*(x) = \varphi_i^*(x) = a^+_i \ast \varphi^*_i(x), \] (33)
and $\varphi_i(x)\varphi^*_i(y) = \varphi_i(x) \ast \varphi^*_i(y)$, in $\Phi^e_\ast$ the CCR (30) become
\[ [\varphi(x) \ast \varphi(y)]_\pm = h.c. = 0, \quad [\varphi(x) \ast \varphi^*(y)]_\pm = \varphi_i(x) \ast \varphi^*_i(y) \] (34)
(here $[A;B]_\pm := A \ast B - B \ast A$). $\Phi^e_\ast$ is a huge $U^e\ast$- and $U(\infty)$ module $\ast$-algebra.

The unitary map $\hat{\kappa}^\alpha = \wedge \circ \kappa^\alpha : s \in \mathcal{H}_\pm^\infty \leftrightarrow \hat{\psi}_s \in \hat{\mathcal{X}}^\infty_\pm$ and its inverse can be expressed using the field, as in the undeformed theory [3]. For instance, $\hat{\kappa}(e_j) = \hat{\varphi}_i(\hat{x}) = \langle \Psi_0, \hat{\varphi}(\hat{x}) \hat{a}^+_i \Psi_0 \rangle$ and the wavefunction (27) equals $\langle \Psi_0, \hat{\varphi}(\hat{x}_2) \hat{\varphi}(\hat{x}_1) \hat{a}^+_i \hat{a}^+_j \Psi_0 \rangle$.

Assume the $n$-particle wavefunction $\psi^{(n)}$ fulfills the Schrödinger eq. (3) if $n=1$, and
\[ ih\frac{\partial}{\partial t}\psi^{(n)} = H^{(n)} \psi^{(n)}, \quad H^{(n)} := \sum_{h=1}^n h(x_h, \partial_h, t) \ast + \sum_{h<k} W(|x_h - x_k|) \ast \] (35)
if $n \geq 2$; here we keep the time coordinate $t$ “commuting”. $H^{(n)}$ is hermitean if $h$ is and $\beta h = h$, as we shall assume. In general, (35) is a $\ast$-differential, pseudodifferential equation preserving the (anti)symmetry of $\psi^{(n)}$. The Fock space Hamiltonian
\[ H = \int d\hat{x} \varphi^* \ast (\varphi) \ast h \ast (\varphi) \ast + \int d\hat{x} d\hat{y} \varphi^* \ast (\varphi) \ast W(|\hat{x} - \hat{y}|) \ast \] (36)
annihilates the vacuum, commutes with the number-of-particles operator $n := a_i^+ \ast a^i$ and its restriction to $\mathcal{H}_\pm^\infty$ coincides with $H^{(n)}$ up to the unitary transformation $\hat{\kappa}^\alpha$. As in the undeformed theory, formulating the dynamics on the Fock space allows to consider also more general Hamiltonians $H_n$, which do not commute with $n$. Replacing $\hat{V}(x\ast, t) = V(x, t) \ast, \hat{A}(x\ast, t) = A(x, t) \ast, \hat{\varphi}_i(x\ast) = \varphi_i(x) \ast$ we can reformulate the previous equations within $\hat{\Phi}^e, \hat{\Phi}$ using only $\ast$-products, or equivalently, dropping $\ast$-symbols and using only “hatted” objects:
\[ \hat{\varphi}(\hat{x}) = \hat{\varphi}_i(\hat{x}) \hat{a}^i, \quad \varphi^*(\hat{x}) = \hat{a}^+_i \hat{\varphi}_i^*(\hat{x}) \]
\[ [\hat{\varphi}(\hat{x}), \hat{\varphi}(\hat{y})]_\pm = h.c. = 0, \quad [\hat{\varphi}(\hat{x}), \hat{\varphi}^*(\hat{y})]_\pm = \hat{\varphi}_i(\hat{x}) \hat{\varphi}^*_i(\hat{y}), \]
\[ ih\frac{\partial}{\partial t}\hat{\psi}^{(n)} = \hat{H}^{(n)} \hat{\psi}^{(n)}, \quad \hat{H}^{(n)} = \sum_{h=1}^n \hat{h}(\hat{x}_h, \hat{\partial}_h, t) + \sum_{h<k} \hat{W}(|\hat{x}_h - \hat{x}_k|), \] (37)
\[ \hat{H} = \int d\hat{x} \varphi^*(\hat{x}) \hat{h}(\hat{x}, t) \hat{\varphi}(\hat{x}) + \int d\hat{x} d\hat{y} \varphi^*(\hat{x}) \hat{\varphi}^*(\hat{y}) \varphi(\hat{x}) W(|\hat{x} - \hat{y}|) \hat{\varphi}(\hat{x}) \hat{\varphi}(\hat{y}). \]

As in the undeformed case, the field in the Heisenberg picture fulfills the equation of motion $ih\frac{\partial}{\partial t}\hat{\varphi}_\mu = [\hat{\varphi}^\mu, \hat{H}]$ and (at equal times) the (anti)commutation relations (37), where the rhs is a “c-number” distribution. Formulæ (37), with the related ones for the Heisenberg field $\hat{\varphi}_H$, summarize our framework for a $\mathcal{H}$-covariant nonrelativistic
field quantization on the noncommutative spacetime $\mathbb{R}^m_\theta$ compatible with the axioms of quantum mechanics, including Bose and Fermi statistics.

If $h, W$ are $H_s$-invariant then $H^{(n)}_s = H^{(n)}$, $H_s = H$, i.e. the dynamics is not deformed, and the total energy is additive if $W \equiv 0$. As we now show, if $W \equiv 0$ but $h$ is not $H_s$-invariant then additivity fails, i.e. $H^{(n)}_s \neq \sum_{h=1}^n h_s(x_h, \partial_h, t)$: so to say, a mutual interaction among the particles is built in due to the $\star$-products.

We just consider a system of 2 bosons/fermions in a perpendicular magnetic field $B = \text{const}$ on the 2-dimensional Moyal space; this means that $\theta_{ab} = \theta_{\epsilon ab}$ and that in the Hamiltonian $h_s$ of (3) $V \equiv 0$ and, choosing the symmetric gauge for the vector potential, $A^a(x) = -B \epsilon^{ab}x^b/2$, implying $D_a = \partial_a - ib \epsilon^{ab}x^b$, $b := \frac{qB}{2\hbar c}$. One finds

$$h_s(x, \partial_x) = -\frac{\hbar^2}{2m}D_a \star D_a^* = \frac{\hbar^2}{2m} \left[ -\left(1 + \frac{b\theta}{2}\right) \Delta + b^2x^2 - 2b \left(1 + \frac{b\theta}{2}\right) l \right]$$

where $l = -i\epsilon^{ab}x^a \partial_b = i(x^2 \partial_1 - x^1 \partial_2)$ is the angular momentum in dimensionless units. So $h_s$ can be obtained from $h$ rescaling the coordinates by $\sqrt{1 + \frac{b\theta}{2}}$ and multiplying the result by $1 + \frac{b\theta}{2}$. $h_s$ has Landau-type levels and eigenfunctions, more easily expressible\[5\] using the $\hat{x}$. If $W = 0$ we find instead by an easy computation

$$H^{(2)}_s = h_s(x_1, \partial_{x_1}) + h_s(x_2, \partial_{x_2}) + \frac{\hbar^2b\theta}{2m} \left[ ib\epsilon^{ab}(x_1^a \partial_{x_2}^b + x_2^a \partial_{x_1}^b) - (2 + b\theta) \partial_{x_1}^a \partial_{x_2}^a \right].$$

The last term breaks the additivity of $H^{(2)}_s$, as claimed.

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