Top-Down Synthesis of Multi-Agent Formation Control:
An Eigenstructure Assignment based Approach

Takatoshi Motoyama and Kai Cai

Abstract—We propose a top-down approach for formation control of heterogeneous multi-agent systems, based on the method of eigenstructure assignment. Given the problem of achieving scalable formations on the plane, our approach globally computes a state feedback control that assigns desired closed-loop eigenvalues/eigenvectors. We design special (sparse) topologies such that the synthesized control may be implemented locally by the individual agents. Moreover, we apply the proposed approach to achieve rigid formation, and present a hierarchical synthesis procedure that significantly improves computational efficiency. Finally, we illustrate these results by simulation examples.

I. INTRODUCTION

Cooperative control of multi-agent systems has been an active research area in the systems control community [1], [2], [3]. Among many problems, formation control has received much attention [4] owing to its wide applications such as satellite formation flying, search and rescue, terrain exploration, and foraging. A main problem studied is stabilization to a rigid formation, where the goal is to steer the agents to achieve a formation with a specified size and only freedoms of translation and rotation. Several control strategies have been proposed: affine feedback laws [5], [6], nonlinear gradient-based control [7], [8], and angle-based algorithms [9]. Achieving a scalable formation with unspecified size has also been studied [10], [11]; a scalable formation may allow the group to adapt to unknown environment with obstacles or targets. In addition, [12], [13] have presented methods of controlling formations in motion.

These different methods for formation control have a common feature in design: namely bottom-up. Specifically, the inter-agent communication topology is given a priori, which defines the neighbors for each agent. Then based only on the neighborhood information, local control strategies are designed for the individual agents. The properties of the designed local strategies are finally analyzed at the systemic (i.e. global) level, and correctness is proved under certain graphical conditions on the communication topology. This bottom-up design is indeed the mainstream approach for cooperative control of multi-agent systems that places emphasis on distributed control.

In this paper, we propose a distinct, top-down approach for formation control, based on a known method called eigenstructure assignment [14], [15]. Different from the bottom-up approach, here there need not be any communication topology imposed a priori (in fact the agents are typically assumed independent, i.e. uncoupled), and no design will be done at the local level. Indeed, given a multi-agent formation control problem characterized by specific eigenvalues and eigenvectors (precisely defined in Section II), our approach constructs on the global level a feedback matrix (if it exists) that renders the closed-loop system to possess those desired eigenvalues/eigenvectors, thereby achieving desired formations. Moreover, the synthesized feedback matrix (its off-diagonal entries being zero or nonzero) defines the communication topology, and accordingly the computed feedback control may be implemented by individual agents. Thus our approach features “compute globally, implement locally”.

The inter-agent communication topology is a result of control synthesis, rather than given a priori. We show that by appropriately choosing desired eigenvalues and the corresponding eigenvectors, special topologies (star, cyclic, line) can be designed, and the computed feedback control may be implemented locally over these (sparse) topologies.

Although our method requires centralized computation of control gain matrices, we show that a straightforward extension of the approach to a hierarchical synthesis procedure significantly reduces computation time. Empirical evidence is provided to show the efficiency of the proposed hierarchical synthesis procedure; in particular, computation of a feedback control for a group of 1000 agents needs merely a fraction of a second, which is likely to suffice for many practical purposes.

The main advantage of our top-down approach is that it is systematic, in the sense that it treats heterogeneous agent dynamics and different cooperative control specifications (characterizable by desired eigenstructure) by the same synthesis procedure. We show that scalable formation and rigid formation can both be addressed using the same method. Although not studied in this paper, this approach is naturally suited to problems with higher-order heterogeneous agent dynamics, possibly not even individually stabilizable, which will be addressed in our future research.

We first proposed this eigenstructure assignment based approach in [16], where we applied it to solve the consensus problem. This work extends [16] to solve scalable formation and rigid formation, and proposes a hierarchical synthesis procedure to significantly shorten the computation time. We note that [17] also proposed an eigenstructure assignment method and applied it to the multi-agent consensus problem. Their approach is bottom-up: first a communication topology
is imposed among the agents, then local control strategies are designed based on eigenstructure assignment respecting the topology, and finally the correctness of the proposed strategies is verified at the global level. By contrast, our approach is top-down: no topology is imposed \textit{a priori}, and topology is a result of control synthesis. Moreover, we study the relation between topology and eigenstructure, and design special topologies by selecting special eigenstructures. In addition, the problems addressed in the paper are distinct, namely scalable/rigid formation, which involve complex eigenvalues and eigenvectors. The rest of the paper is organized as follows. In Section II we review the basics of eigenstructure assignment and formulate the multi-agent formation control problem. In Section III we solve the problem by eigenstructure assignment, and design special topologies to facilitate distributed implementation; moreover, we extend the method to achieve rigid formation. In Section IV we present a hierarchical synthesis procedure to reduce computation time. Simulation examples are given in Section V and our conclusions stated in Section VI.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Preliminaries on Eigenstructure Assignment

First, we review the basics of eigenstructure assignment [14]. Consider a linear time-invariant finite-dimensional system modeled by

\[ \dot{x} = Ax + Bu \]

where \( x \in \mathbb{C}^n \) is the state vector, \( u \in \mathbb{C}^m \) the input vector, and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \).

Suppose we modify (1) by state feedback \( u = Fx \). It is well-known that \( F \) may be chosen to assign any (self-conjugate) set of closed-loop eigenvalues for \( \dot{x} = (A + BF)x \) if and only if \((A, B)\) is controllable. Unless \( m = 1 \) (single input), however, \( F \) is not uniquely determined by a set of closed-loop eigenvalues. Indeed, with state feedback \( F \) one has additional freedom to assign certain sets of closed-loop eigenvectors. Simultaneously assigning both eigenvalues and eigenvectors is referred to as \textit{eigenstructure assignment}.

Let \( \lambda \in \mathbb{C} \). It is shown in [14] that if \((A, B)\) is controllable, then there exists

\[ N(\lambda) := \begin{bmatrix} N_1(\lambda) \\ N_2(\lambda) \end{bmatrix} \in \mathbb{C}^{(n+m) \times m} \]

with linearly independent columns such that

\[ [\lambda I - A \ B] \begin{bmatrix} N_1(\lambda) \\ N_2(\lambda) \end{bmatrix} = 0. \]

Thus the columns of \( N(\lambda) \) form a basis of \( \text{Ker}[\lambda I - A \ B] \); \text{Ker} denotes \textit{kernel}. Also, we will use \( \text{Im} \) to denote \textit{image}.

\textbf{Lemma 1:} ([14]) Consider the system (1) and suppose that \((A, B)\) is controllable and \( \text{Ker}B = 0 \). Let \( \{\lambda_1, \ldots, \lambda_n\} \) be a set of distinct complex numbers, and \( \{v_1, \ldots, v_n\} \) a set of linearly independent vectors in \( \mathbb{C}^n \). Then there is a unique \( F \) such that for every \( i \in [1, n] \), \((A + BF)v_i = \lambda_i v_i \) if and only if

\[ (\forall i \in [1, n]) v_i \in \text{Im} N_1(\lambda_i) \]

where \( N_1(\cdot) \) is in (2).

Lemma 1 provides a necessary and sufficient condition of eigenstructure assignment. When the condition holds and \( F \) exists for assigning distinct complex eigenvalues \( \lambda_i \) and the corresponding eigenvectors \( v_i \) (\( i \in [1, n] \)), \( F \) may be constructed by the following procedure [14].

(i) For each \( \lambda_i \) compute an arbitrary basis of \( \text{Ker}[\lambda_i I - A \ B] \). Stack the basis vectors to form \( N(\lambda_i) \) in (2); partition \( N(\lambda_i) \) properly to get \( N_1(\lambda_i) \) and \( N_2(\lambda_i) \).

(ii) Find \( w_i = -N_2(\lambda_i)k_i \), where \( k_i \in \mathbb{C} \) is such that \( N_1(\lambda_i)k_i = v_i \) (the condition \( \text{Ker}B = 0 \) in Lemma 1 ensures that \( N_1(\lambda_i) \) has independent columns; thus \( k_i \) may be uniquely determined).

(iii) Compute \( F \) by

\[ F = [w_1 \cdots w_n][v_1 \cdots v_n]^{-1}. \]

Note that the entries of \( F \) may include complex numbers in general. If \( \{\lambda_1, \ldots, \lambda_n\} \) is a self-conjugate set of distinct complex numbers, and \( v_i = v_i^* \) wherever \( \lambda_i = \lambda_i^* \) (\( \ast \) denotes complex conjugate), then all entries of the \( F \) are real numbers.

The procedure (i)-(iii) of computing \( F \) has complexity \( O(n^3) \), inasmuch as the calculations involved are solving systems of linear equations, matrix inverse and multiplication.

We note that the above eigenstructure assignment result may be extended to the case of repeated eigenvalues with generalized eigenvectors. For details refer to [15].

B. Problem Formulation

Consider a heterogeneous multi-agent system where each agent is modeled by a first-order ODE:

\[ \dot{x}_i = a_i x_i + b_i u_i, \quad i = 1, \ldots, n. \]

Here \( x_i \in \mathbb{C} \) is the state variable, \( u_i \in \mathbb{C} \) the control variable, \( a_i \in \mathbb{R} \) and \( b_i (\neq 0) \in \mathbb{R} \) are constant parameters. Thus each agent is a point mass moving on the complex plane, with possibly stable \( (a_i < 0) \), semistable \( (a_i = 0) \) or unstable \( (a_i > 0) \) dynamics. The requirement \( b_i \neq 0 \) is to ensure controllability of \( (a_i, b_i) \); thus each agent is controllable. Note that represented by (6), the agents are independent (i.e. uncoupled) and no inter-agent topology is imposed at this stage.

In vector-matrix form, the system of \( n \) independent agents is

\[ \dot{x} = Ax + Bu \]

where \( x := [x_1 \cdots x_n]^T \in \mathbb{C}^n, u := [u_1 \cdots u_n]^T \in \mathbb{C}^n \), \( A := \text{diag}(a_1, \ldots, a_n) \) and \( B := \text{diag}(b_1, \ldots, b_n) \); here
$\text{diag}(\cdot)$ denotes a diagonal matrix with the specified diagonal entries. Consider modifying (7) by a state feedback $u = Fx$ and thus the closed-loop system is

$$\dot{x} = (A + BF)x.$$  

Straightforward calculation shows that the diagonal entries of $A + BF$ are $a_i + b_i F_{ii}$, and the off-diagonal entries $b_i F_{ij}$. Since $b_i \neq 0$, the off-diagonal entries $(A + BF)_{ij} \neq 0$ if and only if $F_{ij} \neq 0 (i \neq j)$.

In view of the structure of $A + BF$, we can define a corresponding directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows: the node set $\mathcal{V} := \{1, \ldots, n\}$ with node $i \in \mathcal{V}$ standing for agent $i$ (or state $x_i$); the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ with edge $(j, i) \in \mathcal{V}$ if and only if $F$'s off-diagonal entry $F_{ij} \neq 0$. Since $F_{ij} \neq 0$ implies that $x_i$ uses $x_j$ in its state update, we say for this case that agent $j$ communicates its state $x_j$ to agent $i$, or $j$ is a neighbor of $i$. The graph $\mathcal{G}$ is therefore called a communication network among agents, whose topology is decided by the off-diagonal entries of $F$. Thus the communication topology is not imposed a priori, but emerges as the result of applying the state feedback control $u = Fx$.

Now we define the formation control problem of the multi-agent system (7).

**Problem 1:** Consider the multi-agent system (7) and specify a vector $f \in \mathbb{C}^n (f \neq 0)$. Design a state feedback control $u = Fx$ such that for every initial condition $x(0) \in \mathbb{C}^n$, $\lim_{t \to \infty} x(t) = cf$ for some constant $c \in \mathbb{C}$.

In Problem 1, the specified vector $f$ represents a desired formation configuration on the complex plane. By formation configuration we mean that the geometric information of the formation remains when scaling and rotational effects are discarded. Indeed, by writing the constant $c \in \mathbb{C}$ in the polar coordinate form (i.e. $c = \rho e^{j\theta}$, $j = \sqrt{-1}$), the final formation $cf$ is the configuration $f$ scaled by $\rho$ and rotated by $\theta$. Note also that Problem 1 includes the consensus problem as a special case when $f = 1 := [1 \cdots 1]^\top$.

To solve Problem 1, we note the following fact.

**Proposition 1:** Consider the multi-agent system (7) and state feedback $u = Fx$. If $A + BF$ has a simple eigenvalue 0, with the corresponding eigenvector $f$, and other eigenvalues have negative real parts, then for every initial condition $x(0)$, $\lim_{t \to \infty} x(t) = cf$ for some $c \in \mathbb{C}$.

For a proof of Proposition 1, as well as proofs of all the following results, refer to [18]. In view of Proposition 1, if the specified eigenvalues and the corresponding eigenvectors may be assigned by state feedback $u = Fx$, then Problem 1 is solved. For this we resort to eigenstructure assignment.

### III. Main Result

In this section, we solve Problem 1, the formation control problem of multi-agent systems, by the method of eigenstructure assignment. The following is our main result.

**Theorem 1:** Consider the multi-agent system (7) and let $f$ be a desired formation configuration. Then there always exists a state feedback control $u = Fx$ that solves Problem 1, i.e.,

$$\lim_{t \to \infty} (x(t) - cf) = 0$$

We consider distinct eigenvalues $\lambda_i (i \in [1, n])$, and compute by (5) the control gain matrix $F$; this in turn gives rise to the agents’ communication graph $\mathcal{G}$. We note that $F$’s off-diagonal entries, which determine the topology of $\mathcal{G}$, are dependent on the choice of eigenvalues as well as eigenvectors. This is illustrated by the example below.

**Example 1:** Consider the multi-agent system (7) of 3 single integrators (that is, $a_i = 0$ and $b_i = 1, i = 1, 2, 3$).

(i) Triangular formation with $f = [1 \ 2 \ 1 - j]^\top$ ($j = \sqrt{-1}$). Let the desired closed-loop eigenvalues be $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -2$ and the corresponding eigenvectors be $v_1 = f$, $v_2 = [j \ 1 \ 0]^\top$, $v_3 = [j \ 0 \ 0]^\top$. By (5) one computes the control gain matrix $F_1$, which determines the corresponding communication graph $\mathcal{G}_1$ (see Fig. 1). Observe that $F_1$ contains complex entries, and $\mathcal{G}_1$ has a spanning tree with node 3 the root. Thus the computed feedback control $u = Fx$ can be implemented by the three agents individually.

(ii) Consensus with $f = [1 \ 1 \ 1]^\top$. Let the eigenvalues be $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -3$ and the corresponding eigenvectors be $v_1 = f$, $v_2 = [-1 \ 1 \ 0]^\top$, $v_3 = [1 \ -1 \ 1]^\top$. Again by (5) one computes the control gain matrix $F_2$ and the corresponding graph $\mathcal{G}_2$ (see Fig. 1). Note that in this case $F_2$ is real and $\mathcal{G}_2$ fully connected. But unlike the usual consensus algorithm [2], $-F_2$ is not a graph Laplacian matrix for the entries $(1, 2), (2, 1), (2, 3)$ and $(3, 2)$ are negative. Thus our eigenstructure assignment based approach may generate a larger class of consensus algorithms with negative weights.

We also remark that in our approach, the convergence speed to the desired formation configuration is assignable. This is because the convergence speed is dominated by the eigenvalue $\lambda_2$ (having the second largest real part) of the closed-loop system $\dot{x} = (A + BF)x$, and $\lambda_2$ is freely assignable. The larger the Re($\lambda_2$) is, the faster the convergence to formation occurs (at the cost of higher control gain).

As we have seen in Example 1, different sets of eigenvalues and eigenvectors may result in different inter-agent communication topologies. In general, therefore, the topology is a result of calculating $F$, thus unknown a priori.
We show in the next subsection, nevertheless, that choosing appropriate eigenstructures may result in certain special (sparse) topologies. With these topologies the synthesized control \( u = Fx \) may be implemented in a distributed fashion.

### A. Special Topologies

We show how to derive the following three types of special topologies by choosing appropriate eigenstructures.

1) **Star topology**: A directed graph \( G = (V, E) \) is a star topology if there is a single root node, say node 1, and \( E = \{ (1, i) | i \in [2, n] \} \). Thus all the other nodes receive information from, and only from, the root node 1. In terms of the total number of edges, a star topology is one of the sparsest topologies, with the least number \((n - 1)\) of edges, that contain a spanning tree. Now consider the following eigenstructure.

\[
eigenvalues: \lambda_1 = 0, \lambda_2, \ldots, \lambda_n \text{ distinct and } \Re(\lambda_2), \ldots, \Re(\lambda_n) < 0
\]

\[
eigenvectors: [v_1 \cdots v_n] = \begin{bmatrix}
f_1 & 0 & \cdots & 0 \\
f_2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
f_n & 0 & \cdots & 1
\end{bmatrix}
\] (8)

**Proposition 2**: Consider the multi-agent system (7). If the eigenstructure (8) is used in the synthesis of feedback control \( u = Fx \), then Problem 1 is solved and the resulting graph \( G \) (defined by \( F \)) is a star topology.

2) **Cyclic topology**: A directed graph \( G = (V, E) \) is a cyclic topology if \( E = \{ (1, 2), (2, 3), \ldots, (n - 1, n), (n, 1) \} \). Consider the following eigenstructure.

\[
eigenvalues: \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} = \{0, \omega - 1, \ldots, \omega^{n-1} - 1\}
\]

\[
eigenvectors: [v_1 \cdots v_n] = \begin{bmatrix}
f_1 & f_1 & \cdots & f_1 \\
f_2 & f_2 & \cdots & f_2 \\
\vdots & \vdots & \ddots & \vdots \\
f_n & f_n & \cdots & f_n
\end{bmatrix}
\] (9)

where \( \omega = e^{2\pi j/n} \) \((j = \sqrt{-1})\).

**Proposition 3**: Consider the multi-agent system (7). If the eigenstructure (9) is used in the synthesis of feedback control \( u = Fx \), then Problem 1 is solved and the resulting graph \( G \) (defined by \( F \)) is a cyclic topology.

3) **Line topology**: A directed graph \( G = (V, E) \) is a (directed) line topology if there is a single root node, say node 1, and \( E = \{ (1, 2), (2, 3), \ldots, (n - 1, n) \} \). A line topology is also one of the sparsest topologies containing a spanning tree. Now consider the following eigenstructure.

\[
eigenvalues: \lambda_1 = 0, \lambda_2 = \cdots = \lambda_n < 0
\]

\[
eigenvectors: [v_1 \cdots v_n] = \begin{bmatrix}
f_1 & 0 & \cdots & 0 \\
f_2 & 0 & \cdots & -f_2 \\
\vdots & \vdots & \ddots & \vdots \\
f_n & -f_n & \cdots & -f_n
\end{bmatrix}
\] (10)

Note that in (10) we have repeated eigenvalues \( \lambda_2 \) and the corresponding generalized eigenvectors. Thus we resort to the method of [15], which generalizes Lemma 1 and (5) for computing the control gain matrix \( F \) (for details refer to [15]).

**Proposition 4**: Consider the multi-agent system (7). If the eigenstructure (10) is used in the synthesis of feedback control \( u = Fx \), then Problem 1 is solved and the resulting \( G \) (defined by \( F \)) is a line topology.

### B. Rigid Formation

We apply our method to study the problem of achieving a rigid formation, one that has translational and rotational freedom but fixed size.

**Problem 2**: Consider the multi-agent system (7) and specify \( f \in \mathbb{C}^n \) and \( d \in \mathbb{R}(f, d \neq 0) \). Design a control \( u \) such that for every initial condition \( x(0) \), \( \lim_{t \to \infty} x(t) = c1 + df e^{i\theta} \) for some \( c \in \mathbb{C} \) and \( \theta \in [0, 2\pi) \).

In Problem 2, the goal of the multi-agent system (7) is to achieve a rigid formation \( df \), with translational freedom in \( c \), rotational freedom in \( \theta \), and fixed size \( d \).

We now present a rigid-formation synthesis procedure.

(i) Compute \( F \) by (5) such that 1) \( A + BF \) has two eigenvalues 0 with the corresponding (non-generalized) eigenvectors \( i \) and \( f \), and other eigenvalues have negative real parts; \(^1\)

2) the topology defined by \( F \) is 2-rooted\(^2\) with exactly 2 roots (say nodes 1 and 2).

(ii) Let \( f_1, f_2 \) be the first two components of \( f \), and set

\[
[\hat{x}_1] = \begin{bmatrix}
(x_2 - x_1)(||x_2 - x_1||^2 - d^2|f_2 - f_1|^2)
\end{bmatrix}, \quad \hat{x}_2 = \begin{bmatrix}
(x_1 - x_2)(||x_1 - x_2||^2 - d^2|f_1 - f_2|^2)
\end{bmatrix} =: r(x_1, x_2).
\]

(iii) Set the control

\[
u := Fx + B^{-1} \begin{bmatrix} r(x_1, x_2) \\ 0 \end{bmatrix}.
\] (11)

Our result is the following.

**Proposition 5**: Consider the multi-agent system (7) and let \( f \in \mathbb{C}^n \), \( d \in \mathbb{R} \). Then the control \( u \) in (11) synthesized by the rigid-formation synthesis procedure solves Problem 2 for all initial conditions \( x(0) \) with \( x_1(0) \neq x_2(0) \).

### IV. Hierarchical Eigenstructure Assignment

In the previous section, we have shown that a control gain matrix \( F \) can always be computed such that the multi-agent formation Problem 1 is solved. Computing such \( F \) by (5) has complexity \( O(n^3) \), where \( n \) is the number of agents. Consequently the computation cost becomes expensive as the number of agents increases.

To address this issue of centralized computation, we propose in this section a hierarchical synthesis procedure. We shall show that the control gain matrix \( F \) computed by this

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\(^1\)For repeated eigenvalues with non-generalized eigenvectors, the eigenstructure assignment result Lemma 1 and the computation of control gain matrix \( F \) in (5) remain the same as for the case of distinct eigenvalues.

\(^2\)A 2-rooted topology is one where there exist 2 nodes from which every other node \( v \) can be reached by a directed path after removing an arbitrary node other than \( v \) [11].
hierarchical procedure again solves Problem 1, which however significantly improves computational efficiency (empirical evidence provided in Section V).

Consider again the multi-agent system (7), and Problem 1 with the desired formation configuration \( f \in \mathbb{C}^n \). Partition the agents into \( f (\geq 1) \) pairwise disjoint groups. Let group \( k (\in \{1, l\}) \) have \( n_k (\geq 1) \) agents; \( n_k \) may be different and \( \sum_{k=1}^{l} n_k = n \).

Now for the configuration \( f \) and \( x, u, A, B \) in (7), write in accordance with the partition (possibly with reordering)

\[
f = \begin{bmatrix} g_1 \\ \vdots \\ g_l \end{bmatrix},
\quad x = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix},
\quad u = \begin{bmatrix} w_1 \\ \vdots \\ w_l \end{bmatrix},
\quad A = \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_l \end{bmatrix},
\quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_l \end{bmatrix}
\]

where \( g_k, y_k, w_k \in \mathbb{C}^{n_k} \) and \( A_k, B_k \in \mathbb{C}^{n_k \times n_k}, k \in \{1, l\} \). Thus for each group \( k \), the dynamics is

\[\dot{y}_k = A_k y_k + B_k w_k.\]  \hspace{1cm} (12)

For later use, also write \( g_k, y_k, e_k, u_k, x_k \) (resp. \( A_k, B_k \)) for the first component of \( g_k, y_k, w_k \) (resp. \( A_k, B_k \)) and \( g_0 := [g_{11} \cdots g_{1l}]^T, y_0 := [y_{11} \cdots y_{1l}]^T, w_0 := [w_{11} \cdots w_{1l}]^T, A_0 := \text{diag}(A_{11}, \cdots, A_{1l}), B_0 := \text{diag}(B_{11}, \cdots, B_{1l}) \).

We now present the hierarchical synthesis procedure.

(i) For each group \( k \in \{1, l\} \) and its dynamics (12), compute \( F_k \) by (5) such that \( A_k + B_k F_k \) has a simple eigenvalue 0 with the corresponding eigenvector \( g_k \), and other eigenvalues have negative real parts; moreover the topology defined by \( F_k \) has a unique root node \( y_{k1} \) (e.g. star or line by the method given in Section IIIA).

(ii) Treat \( \{y_k | k \in \{1, l\}\} \) (the group leaders) as a higher-level group, with the dynamics

\[\dot{y}_0 = A_0 y_0 + B_0 w_0.\] \hspace{1cm} (13)

Compute \( F_0 \) by (5) such that \( A_0 + B_0 F_0 \) has a simple eigenvalue 0 with the corresponding eigenvector \( g_0 \), and other eigenvalues have negative real parts.

(iii) Set the control gain matrix \( F := F_{\text{low}} + F_{\text{high}} \), where

\[F_{\text{low}} := \begin{bmatrix} F_1 \\ \vdots \\ F_l \end{bmatrix}
\]

and \( F_{\text{high}} \) is partitioned according to \( F_{\text{low}} \), with each block

\[(F_{\text{high}})_{ij} = (F_0)_{ij} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} (F_0)_{ij} \\ \vdots \\ (F_0)_{ij} \end{bmatrix}.
\]

The computational complexity of Step (i) is \( O(\tilde{n}^3) \), where \( \tilde{n} := \max\{n_1, \ldots, n_l\} \); and Step (ii) is \( O(P^3) \). Let \( n^* := \max\{\tilde{n}, l\} \). Then the complexity of the entire hierarchical synthesis procedure is \( O((n^*)^3) \). With proper group partition, this hierarchical procedure can significantly reduce computation time, as demonstrated by an empirical study in Section V.

The correctness of the hierarchical synthesis procedure is asserted in the following.

**Theorem 2:** Consider the multi-agent system (7) and let \( f \) be a desired formation configuration. Then the state feedback control \( u = Fx \) synthesized by the hierarchical synthesis procedure satisfies Problem 1, i.e.

\[\forall x(t) \in \mathbb{C}^n, \exists c \in \mathbb{C}, \lim_{t \to \infty} x(t) = cf.
\]

V. SIMULATIONS

We illustrate the eigenstructure assignment based approach by several simulation examples. For all the examples, we consider the multi-agent system (7) with 5 heterogeneous agents, where \( A = \text{diag}(1.6, 4.7, 3.0, -0.7, -4.2) \) and \( B = \text{diag}(0.2, 1.5, -0.5, -3.3, -3.7) \). First, to achieve a scalable (regular) pentagon formation, assign the following eigenstructure:

- eigenvalues: \( \{\lambda_1, \ldots, \lambda_5\} = \{0, -1, -2, -3, -4\} \)
- eigenvectors: \( [v_1 \cdots v_5] = \begin{bmatrix} e^{2\pi i x} \\ e^{2\pi i y} \\ e^{2\pi i z} \\ e^{2\pi i w} \\ e^{2\pi i t} \end{bmatrix} \)

By (5) we compute the control gain matrix

\[F = \begin{bmatrix} -10.5 - 1.8j & 2.5 - 1.8j & 0 & 0 & 0 \\ 0.3 + 0.2j & -3.5 + 0.2j & 0 & 0 & 0 \\ -0.5 - 2.4j & -0.5 - 2.4j & 10 & -2 & 0 \\ -0.4j & 1.2 - 0.4j & 0 & 0 & 0 \\ -0.4 + 0.4j & 0.1 + 0.4j & 0 & -0.5 & 0.1 \end{bmatrix} \]

Simulating the closed-loop system with initial condition \( x(0) = [1 + j 1 - 0.5j 1 j - 1 - j]^T \), the result is displayed in Fig. 2. Observe that a regular pentagon is formed, and the topology determined by \( F \) contains a spanning tree.

Next, to achieve a rigid pentagon formation, we follow the method presented in Section III B: first assign the following eigenstructure:

- eigenvalues: \( \{\lambda_1, \ldots, \lambda_5\} = \{0, 0, -1, -2, -3\} \)
- eigenvectors: \( [v_1 \cdots v_5] = \begin{bmatrix} 1 & e^{2\pi i x} & 0 & 0 & 0 \\ e^{2\pi i y} & 0 & 0 & 0 & 0 \\ e^{2\pi i z} & 1 & 0 & 0 & 0 \\ e^{2\pi i w} & 0 & 1 & 0 & 0 \\ e^{2\pi i t} & 0 & 0 & 1 & 0 \end{bmatrix} \)

and by (5) compute the control gain matrix

\[F = \begin{bmatrix} -8 & 0 & 0 & 0 & 0 \\ 0 & -3.1 & 0 & 0 & 0 \\ 0.6 + 1.9j & -2.6 - 1.9j & 8 & 0 & 0 \\ -0.3 + 0.9j & -0.3 - 0.9j & 0 & 0.4 & 0 \\ -1.1 + 0.8j & 0.3 - 0.8j & 0 & 0 & -0.3 \end{bmatrix} \]
Thus the topology determined by $F$ is 2-rooted with nodes 1 and 2 the only two roots. Then for $d = 0.05$ we obtain by (11) the control $u$. Simulating the closed-loop system with the same initial condition as above, the result is displayed in Fig. 3, where a pentagon with the specified size is formed.

Finally, we present an empirical study on the computation time. In particular, we compare the centralized synthesis by (5) and the hierarchical synthesis in Section IV; the result is listed in Table I for different numbers of agents. Here for the hierarchical synthesis, we partition the agents in such a way that the number of groups and the number of agents in each group are balanced: e.g. 100 agents are partitioned into 10 groups of 10 agents each; 500 agents are partitioned into 16 groups of 23 agents each plus 6 groups of 22 each. Observe that the hierarchical synthesis is significantly more efficient than the centralized one, and the efficiency increases as the number of agents increases. In particular, for 1000 agents only 0.525 seconds needed, the hierarchical approach might well be sufficient for many practical purposes.

**VI. CONCLUSIONS**

We have proposed a top-down, eigenstructure assignment based approach to synthesize state feedback control for multi-agent formation control. Special topologies have been designed by choosing appropriate eigenstructures, rigid formation has been achieved, and hierarchical synthesis presented that improves computational efficiency. Our ongoing work applies this top-down approach to deal with cooperative control of heterogeneous higher-order agents with possibly non-stabilizable individual dynamics.

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TABLE I

| agent # | centralized method by (5) | hierarchical method in Sec. IV |
|---------|---------------------------|-------------------------------|
| 100     | 0.398                     | 0.027                         |
| 500     | 57.308                    | 0.179                         |
| 900     | 552.8419                  | 0.394                         |
| 1000    | 1068.729                  | 0.525                         |