Wormhole interaction in 2d Horava-Lifshitz quantum gravity

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A lattice regularization for the 2d projectable Horava-Lifshitz (HL) quantum gravity is known to be the 2d causal dynamical triangulations (CDT), and the 2d CDT can be generalized so as to include all possible genus contributions non-perturbatively. We show that in the context of HL gravity, effects coming from such a non-perturbative sum over topologies can be successfully taken into account, if we quantize the 2d projectable HL gravity with a simple bi-local wormhole interaction. This conference paper is based on the article, Phys. Lett. B 816 (2021), 136205.

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1. Introduction

Two-dimensional models of quantum gravity are useful for understanding some aspects of non-perturbative physics, since they can be examined beyond the framework of perturbation theory by analytic computations in many cases. One of such models is the two-dimensional causal dynamical triangulations (2d CDT), which is a lattice toy model of quantum gravity with a global time foliation. Global hyperbolicity imposed at the quantum level do not allow for any topology change in the CDT model, and the continuum limit can be described by quantum mechanics of a 1d universe.

Yet another toy model for quantum gravity that has a global time foliation is 2d Horava-Lifshitz (HL) quantum gravity that was introduced first in higher dimensions to resolve the issue of perturbative renormalizability by breaking the diffeomorphisms down to the foliation-preserving diffeomorphisms, and so the model has a preferred foliation structure. The projectable version of the 2d HL quantum gravity was discussed in the articles, and in particular it was shown that the canonical
quantization of the model yields exactly the same quantum Hamiltonian as the one obtained by the continuum limit of 2d CDT\(^4\). Therefore, one can interpret 2d CDT as a lattice regularization for the 2d projectable HL quantum gravity.

Although the creation of baby universes and wormholes (handles) is not allowed to occur in the original setup of CDT, one can generalize the 2d model to include such configurations in a manner consistent with the scaling limit\(^5\), and such a generalized model can be fully described by a string field theory for CDT\(^6\). Here the word “string” refers to the 1d spatial universe. Based on the string field theory, one can take into account all genera (handles) as well as all baby universes of the two-dimensional spacetime. In addition, as long as one only asks for the amplitude between the states describing connected spatial universes separated a proper time \(t\), this sum over topologies and baby universes can effectively be described by a one-body Hamiltonian\(^10,11\).

Having the equivalence between the continuum limit of 2d CDT and the 2d projectable HL quantum gravity, we discussed in the article\(^1\) what kind of effective interaction should be added to the classical Lagrangian of the 2d projectable HL gravity when canonically quantized, in order to obtain the above mentioned one-body quantum Hamiltonian that includes all wormholes and baby universes obtained in 2d CDT. The answer is that it is enough to include a simple bi-local and spatial wormhole interaction that is compatible with the foliation-preserving diffeomorphisms.

This article is organized as follows. In Section 2, we give a brief introduction to 2d CDT and 2d HL quantum gravity, and explain the relation between the two. In Section 3 the string field theory for CDT is described and we introduce the one-body quantum Hamiltonian that includes all possible topologies. In Section 4 we introduce a simple wormhole interaction to the 2d projectable HL gravity and show that one can precisely recover the one-body Hamiltonian when canonically quantizing the model. Section 5 is devoted to discussions.

2. CDT and Horava-Lifshitz gravity in two dimensions

In this Section, we review causal dynamical triangulations (CDT) and Horava-Lifshitz gravity (HL) in two dimension, and explain the relation between the two.

2.1. 2d CDT

The starting point is a globally hyperbolic manifold equipped with a global time foliation:

\[
\mathcal{M} = \bigcup_{t \in \mathbb{R}} \Sigma_t ,
\]

\(^a\)The time-independent amplitudes can be also computed by the matrix model for CDT\(^8\) and by the new scaling limit of the Hermitian one-matrix model\(^9\).
where each leaf $\Sigma_t$ is a Cauchy “surface”. 2d CDT is a model which quantizes 2d geometries with such a proper-time foliation. The geometries in the path integral are regularized by piecewise linear geometries constructed by gluing together special kinds of triangles. Each triangle consists of one space-like edge and two time-like edges such that a square of the space-like edge is positive, $a^2_s = \varepsilon^2$, while it is negative for the time-like edge, $a^2_t = -\alpha \varepsilon^2$ with positive $\alpha$.

If we prohibit a creation of baby universes, a reasonable choice of the action is the cosmological constant term that is regularized as follows:

$$S[\Sigma] = -\lambda_0 \int d^2x \sqrt{-\det(g_{ij})},$$

$$\rightarrow S_R[T; \alpha] = -\mu \varepsilon^2 \times \left( \frac{4\alpha + 1}{4} \varepsilon^2 \times n(T) \right),$$

where the bare cosmological constant $\lambda_0$ is replaced by the dimensionless cosmological constant $\mu$, and the quantity in the parentheses is the discretized area of the triangulation $T$, i.e. the area of each triangle times the number of triangles $n(T)$. The lattice action $S_R$ in eq. (2) is called the Regge action.

For computational convenience we implement a rotation to the Euclidean signature which can be done by replacing $\alpha$ with $-\alpha$:

$$iS_R[T; \alpha] \rightarrow -S_R^{(\varepsilon)}[-\alpha; T] = -\mu \frac{\sqrt{4\alpha + 1} - 1}{4} n(T) =: -\mu n(T),$$

where we have absorbed a numerical factor into the dimensionless cosmological constant. Note that this map is a bijection between individual Lorentzian and Euclidean geometries.

In CDT, the integration over diffeomorphism equivalent classes of metric $g$ keeping both initial and final geometries fixed can be regularized by the sum over “all” triangulations. Therefore, the 2d Euclidean path-integral regularized by CDT is

$$G^{(0)}_{\text{lattice}}(l_1, l_2; \tau) = \sum_{T \in \mathcal{T}(l_1, l_2; \tau)} \frac{1}{C_T} e^{-\mu n(T)},$$

where $C_T$ is the order of automorphism group of $T$, and $\mathcal{T}(l_1, l_2; \tau)$ is a set of triangulations such that the initial and final boundaries whose lengths are kept fixed to $l_1$ and $l_2$ are separated by $\tau$ Euclidean time steps.

One can compute the amplitude (4) analytically through the use of the generating function for the numbers $G^{(0)}_{\text{lattice}}(l_1, l_2; \tau)$. As in the case of lattice QCD, tuning the UV relevant coupling constant $\mu/\varepsilon^2$ to its critical value $\mu_c/\varepsilon^2$ and taking $\varepsilon \rightarrow 0$ in a correlated manner, one can transmute the dimension of the lattice spacing to the renormalized coupling constant:

$$\lambda := \lim_{\varepsilon \rightarrow 0} \frac{\mu - \mu_c}{\varepsilon^2},$$
where $\lambda$ is the renormalized cosmological constant. Introducing the renormalized quantities, boundary lengths and a proper time such that
\[
\ell_1 := \varepsilon l_1, \quad \ell_2 := \varepsilon l_2, \quad t := \varepsilon \tau,
\]
one can obtain the renormalized amplitude $G^{(0)}(\ell_1, \ell_2; t)$ that is known to satisfy the differential equation:
\[
-\frac{\partial}{\partial t}G^{(0)}(\ell_1, \ell_2; t) = H_a^{(0)}(\ell_1)G^{(0)}(\ell_1, \ell_2; t),
\]
where $H_a^{(0)}$ is the quantum Hamiltonian defined as
\[
H_0^{(0)}(\ell) = -\frac{\partial}{\partial \ell} \ell \frac{\partial}{\partial \ell} + \lambda \ell, \quad H_{-1}^{(0)}(\ell) = -\ell^2 \frac{\partial^2}{\partial \ell^2} + \lambda \ell, \quad H_{+1}^{(0)}(\ell) = -\frac{\partial^2}{\partial \ell^2} \ell + \lambda \ell.
\]
Here the label $a$ in eq. (7) specifies the ordering of the Hamiltonian. If we define a quantum state of the one-dimensional universe whose length is $\ell$ as $|\ell\rangle$, the amplitude $G^{(0)}(\ell_1, \ell_2; t)$ can be rewritten as
\[
G^{(0)}(\ell_1, \ell_2; t) = \langle \ell_2 | e^{-tH_a^{(0)}(\ell_1)} | \ell_1 \rangle.
\]
In fact, each ordering specifies the geometry of the quantum state $|\ell\rangle$, i.e., if $a = 0, -1, +1$, then the geometry of the one-dimensional universe is open, closed with a mark, and closed, respectively. Marking a point on a closed universe is analogous to introducing a coordinate. The Hamiltonian is hermitian with respect to the following inner product:
\[
\langle \phi | H_a^{(0)} | \psi \rangle = \int_{0}^{\infty} \phi^*(\ell)H_a^{(0)}(\ell)\psi(\ell) \, d\mu_\alpha(\ell), \quad \text{with} \quad d\mu_\alpha(\ell) = \ell^\alpha \ell \, d\ell.
\]
As a result, the physics of 2d CDT can be described by the quantum mechanics of a one-dimensional universe.

### 2.2. 2d projectable HL gravity

The starting point is the same as that of CDT, i.e., a globally hyperbolic manifold equipped with a global time foliation. A natural parametrization for the metric on such a geometry is given by the Arnowitt-Deser-Misner metric:
\[
g = -N^2 ds^2 + h_{11}(dx + N^1 ds)(dx + N^1 ds),
\]
where $h_{11}$ is the spatial metric on the leaf; $N$ and $N^1$ called lapse and shift functions, quantify the normal and tangential directions of the proper time to the leaf, respectively.

The 2d HL gravity is introduced as a theory that keeps the foliation structure, or in other words, it is invariant under the foliation preserving diffeomorphisms (FPD):
\[
s \to s + \xi^0(s), \quad x \to x + \xi^1(s, x).
\]
Under the FPD, the fields transform as
\[
\delta \xi^1 h_{11} = \xi^0 \partial_0 h_{11} + \xi^1 \partial_1 h_{11} + 2 h_{11} \partial_1 \xi^1 ,
\]
(13)
\[
\delta \xi^1 N_1 = \xi^\mu \partial_\mu N_1 + N_1 \partial_\mu \xi^\mu + h_{11} \partial_1 \xi^1 ,
\]
(14)
\[
\delta \xi^1 N = \xi^\mu \partial_\mu N + N \partial_0 \xi^0 ,
\]
(15)
where \( N_1 = h_{11} N_1 \).

Note that if the lapse function \( N \) is a function of time, \( N = N(t) \), it stays as a function of time under the FPD. The 2d projectable HL gravity satisfies this condition on \( N \), and it is defined by the following action:
\[
I = \frac{1}{\kappa} \int ds dx N \sqrt{h} \left( (1 - \eta) K^2 - 2 \tilde{\lambda} \right) ,
\]
(16)
where \( \eta \), \( \tilde{\lambda} \) and \( \kappa \) are a dimensionless parameter, the cosmological constant and the dimensionless gravitational coupling constant, respectively; \( h \) is the determinant of the metric \( h_{11} \), i.e. \( h = h_{11} \); \( K \) is the trace of the extrinsic curvature \( K_{11} \) defined as
\[
K_{11} = \frac{1}{2N} (\partial_0 h_{11} - 2 \nabla_1 N_1) , \quad \text{with} \quad \nabla_1 N_1 := \partial_i N_1 - \Gamma^i_{11} N_1 .
\]
(17)
Here \( \Gamma^i_{11} \) is the spatial Christoffel symbol:
\[
\Gamma^i_{11} = \frac{1}{2} h^{11} \partial_i h_{11} .
\]
(18)
In principle, one can add higher spatial derivative terms to the action \( I \), but they are not needed since 2d gravity is renormalizable without such terms and we will omit such terms.

The quantization of 2d projectable HL gravity was discussed in [4,5], and in particular, it was shown that the quantum Hamiltonian coincides with the continuum Hamiltonian of 2d CDT when the following identification of the parameters is made:
\[
\lambda = \frac{\tilde{\lambda}}{2(1 - \eta)} , \quad \eta < 1 , \quad \tilde{\lambda} > 0 , \quad \kappa = 4(1 - \eta) .
\]
(19)
where \( \lambda \) is the renormalized cosmological constant in 2d CDT [5].

Let us briefly explain how to recover the quantum Hamiltonian \( H \) from the quantization of 2d HL gravity. Introducing the conjugate momentum of \( \sqrt{h} \) as \( \pi \), we have in the canonical formalism the Poisson bracket
\[
\left\{ \sqrt{h}(s, x), \pi(s, x') \right\} = \delta(x - x') ,
\]
(20)
and corresponding to the Lagrangian \( I \) we have the Hamiltonian
\[
H = \int dx \left[ N_1 \left( - \frac{\partial_1 \pi}{\sqrt{h}} \right) + N \left( \frac{\kappa}{4(1 - \eta)} \pi^2 \sqrt{h} + \frac{2}{\kappa} \tilde{\lambda} \sqrt{h} \right) \right] .
\]
(21)
\footnote{We have set unimportant dimensionless gravitational constant as \( \kappa = 4(1 - \eta) \).}
If we solve the momentum constraint at the classical level, i.e.

\[ - \frac{\partial_1 \pi}{\sqrt{h}} = 0 , \quad \Rightarrow \quad \pi = \pi(s) , \tag{22} \]

the system reduces to a one-dimensional model with the Hamiltonian

\[ H = N(s) \left( \frac{\kappa}{4(1-\eta)} \pi^2(s) \ell(s) + \frac{2}{\kappa} \tilde{\lambda} \ell(s) \right) , \quad \text{with} \quad \ell(s) := \int dx \sqrt{h(s,x)} . \tag{23} \]

Hereafter choosing the correct sign for the kinetic term, i.e. \( \eta < 1 \), we use the parametrization (19) with positive \( \lambda \) in order to discuss the relation to 2d CDT.

The classical 1d system with the Hamiltonian (23) can be alternatively described by the following action:

\[ S = \int_0^1 ds \left( \frac{\dot{\ell}^2}{4N\ell} - \lambda N\ell \right) , \tag{24} \]

where \( \dot{\ell} := d\ell/ds \). This system is invariant under the time reparametrization, \( s \rightarrow s + \xi^0(t) \), which is ensured by the lapse function. In fact, the proper time,

\[ t := \int_0^1 ds \ N(s) , \tag{25} \]

and the length, \( \ell = \ell(t) \), are invariant under the time reparametrization, and so it makes sense to discuss the probability amplitude for a 1d universe to propagate in the proper time \( t(>0) \), starting from the state with the length \( \ell_1 \) and ending up in the one with length \( \ell_2 \). Such an amplitude can be computed based on the path-integral, and we evaluate it by a rotation to the Euclidean signature for convenience. In our foliated spacetime, for \( \eta < 1 \), we can implement this procedure by a formal rotation, \( s \rightarrow is \), which yields the amplitude:

\[ G(0)(\ell_2, \ell_1; t) = \int_{\ell(0) = \ell_1}^{\ell(1) = \ell_2} D\ell(s) e^{-SE[N(s), \ell(s)]} , \tag{26} \]

where \( SE \) is the Euclidean action given by

\[ SE = \int_0^1 ds \left( \frac{\dot{\ell}^2}{4N\ell} + \lambda N\ell \right) , \tag{27} \]

where \( \dot{\ell} := d\ell/ds \).

We set \( N = 1 \) as a gauge choice. One can show that the corresponding Faddeev-Popov determinant only gives an overall constant, which we will omit in the following. The amplitude (26) then becomes

\[ G(0)(\ell_2, \ell_1; t) = \int_{\ell(0) = \ell_1}^{\ell(1) = \ell_2} D\ell(s) \exp \left[ - \int_0^t ds \left( \frac{\dot{\ell}^2}{4\ell} + \lambda\ell \right) \right] , \tag{28} \]
which can be expressed in terms of the quantum Hamiltonian \( H \) that is unknown at the moment:

\[
G^{(0)}(\ell_2, \ell_1; t) = \langle \ell_2 | e^{-tH} | \ell_1 \rangle ,
\]

(29)

where \( |\ell\rangle \) is a quantum state of the 1d universe with the length \( \ell \). By standard methods (see e.g. the article 4), one can determine the quantum Hamiltonian from eq. (28) and eq. (29). One obtains precisely the 2d CDT quantum Hamiltonians given by eq. (8) if the integral measures are chosen as

\[
D\ell(s) = \prod_{s=0}^{s=t} d\ell(s) ,
\]

(30)

where \( a = 0, \pm 1 \). The measures (30) are consistent with eq. (10) introduced in 2d CDT.

Therefore, we can conclude that 2d CDT is a lattice regularization for the 2d projectable HL quantum gravity.

3. Sum over all genera in 2d CDT

One can generalize the 2d CDT model so as to include spatial topology changes (splitting and joining interactions of 1d universe) in keeping with the foliation structure, and this is described by promoting the 1d quantum mechanics to a field theory of 1d universes, which is called the string field theory for CDT. Here the word “string” means a 1d closed spatial universe.

We introduce an operator that creates a marked closed string with the length \( \ell \), \( \Psi^\dagger(\ell) \), and an operator that annihilates a length \( \ell \) closed string without a mark, \( \Psi(\ell) \). They satisfy the commutation relation:

\[
[\Psi(\ell), \Psi^\dagger(\ell')] = \delta(\ell - \ell') , \quad [\Psi(\ell), \Psi(\ell')] = [\Psi^\dagger(\ell), \Psi^\dagger(\ell')] = 0 .
\]

(31)

The vacuum \( |\text{vac}\rangle \) is defined by \( 0 = \Psi(\ell)|\text{vac}\rangle = \langle \text{vac}|\Psi(\ell)\rangle . \)

The CDT amplitude (9) can be written in terms of the string field Hamiltonian obtained by sandwiching the one-body Hamiltonian:

\[
G^{(0)}(\ell_1, \ell_2; t) = \langle \text{vac}|\Psi(\ell_2) e^{-t\hat{H}^{(0)}} \Psi^\dagger(\ell_1)|\text{vac}\rangle ,
\]

(32)

where

\[
\hat{H}^{(0)} = \int_0^\infty \frac{d\ell}{\ell} \Psi^\dagger(\ell) \hat{H} \Psi(\ell) .
\]

(33)

In order to incorporate topology change, one has to include suitable interactions, and the full string field Hamiltonian is given by

\[
\hat{H} = \hat{H}^{(0)} - \int_0^\infty d\ell \delta(\ell) \Psi(\ell) - g_s \int_0^\infty d\ell_1 \int_0^\infty d\ell_2 \Psi^\dagger(\ell_1) \Psi^\dagger(\ell_2) [\ell_1 + \ell_2] \Psi(\ell_1 + \ell_2)
- \alpha g_s \int_0^\infty d\ell_1 \int_0^\infty d\ell_2 \Psi^\dagger(\ell_1 + \ell_2) \ell_1 \Psi(\ell_1) \ell_2 \Psi(\ell_2) ,
\]

(34)
where the second, third and fourth terms mean a string vanishing into the vacuum, the splitting interaction with the string coupling \( g_s \), and the joining interaction with the coupling \( \alpha g_s \), respectively; \( \alpha \) is a constant introduced for counting handles. In general one can compute the following amplitude:

\[
A(\ell_1, \cdots, \ell_m; \ell'_1, \cdots, \ell'_n; t) = \langle \text{vac} | \Psi(\ell'_1) \cdots \Psi(\ell'_n) e^{-itH} \Psi^\dagger(\ell_1) \cdots \Psi^\dagger(\ell_m) | \text{vac} \rangle .
\]

(35)

From now on, we focus on the full propagator \( G(\ell_1, \ell_2; t) := A(\ell_1; \ell_2; t) \) that includes the sum over all genera (handles) and baby universes, and so we simply set \( \alpha = 1 \). In this case, somewhat miraculously, the full propagator in the multi-body system can be described by an effective one-body system:

\[
G(\ell_1, \ell_2; t) = \langle \ell_2 | e^{-tH_{-1}} | \ell_1 \rangle , \quad \text{with} \quad H_{-1} = -\ell ^2 \frac{\partial^2}{\partial \ell ^2} + \lambda \ell - g_s \ell^2 .
\]

(36)

All the contributions coming from the sum over all genera and baby universes can be effectively described by the last term \(-g_s \ell^2\). Although the Hamiltonian (36) is not bounded from below, it belongs to a class of Hamiltonians which are called “classical incomplete”, where the Hamiltonians have discrete energy spectra and square integrable eigenfunctions.

The effective one-body system given by eq. (36) can be also described by the path-integral:

\[
G(\ell_1, \ell_2; t) = \int_{\ell_0(0)=\ell_1}^{\ell_0(t)=\ell_2} D\ell(s) \exp \left[ -\int_0^t ds \left( \frac{\dot{\ell}^2(s)}{4\ell(s)} + \lambda \ell(s) - g_s \ell^2(s) \right) \right].
\]

(37)

If we choose the integral measure such that

\[
D\ell(s) = \prod_{s=0}^{n=1} \ell^n(s)d\ell(s) ,
\]

(38)

where \( \alpha = 0, \pm 1 \), all possible orderings of the full one-body Hamiltonian (36) can be realized. As explained in the article, in order for the functional integral to be well defined, the boundary conditions on \( \ell(s) \) at infinity have to be chosen such that the kinetic term counteracts the unboundedness of the potential. As we will see, the information about the boundary conditions also appears in the classical Hamiltonian constraint (42) of 2d projectable HL gravity with a wormhole interaction.

In the next section, we will show that the path-integral (37) can be obtained quantizing the 2d projectable HL gravity with a wormhole interaction.

4. Wormhole interaction in 2d projectable HL gravity

We consider 2d projectable HL gravity with a space-like wormhole interaction given by the action:

\[
I_w = \frac{1}{\kappa} \int ds dx N(s) \sqrt{h(s, x)} \left( (1 - \eta) K^2(s, x) - 2\lambda \right) + \beta \int ds N(s) \int dx_1 dx_2 \sqrt{h(s, x_1)} \sqrt{h(s, x_2)} ,
\]

(39)
where $\beta$ is a dimensionful coupling constant. One can show that the action (39) is invariant under the FPD (12) with the projectable lapse function, $N = N(t)$.

The bi-local interaction in eq. (39) relates two distinct points at an equal time. The action is a simplified version of the general bi-local action suggested in the article 14, made possible because HL gravity is invariant only under the FPD (12) and not the full set of diffeomorphisms.

Following the same procedure explained in the section 2.2, we quantize the system with the action (39). Introducing a conjugate momentum of $\sqrt{\hbar}$ as $\pi$ as before, we move on to the canonical formalism, and the Hamiltonian is

$$ H = \int dx \left[ N_1 C^1(s, x) + N C(s, x) \right], \quad (40) $$

where

$$ C^1(s, x) = -\frac{\partial_1 \pi(s, x)}{\sqrt{h(s, x)}}, \quad (41) $$

$$ C(s, x) = \frac{\kappa}{4(1-\eta)} \pi^2(s, x) \sqrt{h(s, x)} + \frac{2}{\kappa} \hat{\lambda} \sqrt{h(s, x)} - \beta \sqrt{h(s, x)} \int dx_2 \sqrt{h(s, x_2)}. \quad (42) $$

If we solve the momentum constraint (41) at the classical level, the system again reduces to the 1d system with the Hamiltonian:

$$ H = N(s) \left( \frac{\kappa}{4(1-\eta)} \pi^2(s) \ell(s) + \frac{2}{\kappa} \hat{\lambda} \ell(s) - \beta \ell^2(s) \right), \quad (43) $$

where $\ell(s) := \int dx \sqrt{h(s, x)}$. Hereafter choosing the correct sign for the kinetic term, i.e. $\eta < 1$, we use the parametrization (19) with positive $\lambda$ as before.

The classical Hamiltonian constraint (12) can be solved as

$$ \pi^2 = -\lambda + \beta \ell \geq 0, \quad (44) $$

for $\sqrt{\lambda} \ell \geq 1/\xi$ with $\xi := \beta/\lambda^{3/2}$, and otherwise, the classical Hamiltonian constraint (12) requires $\ell = 0$ on the constraint surface. As in the $\beta = 0$ case, when quantizing the system based on the path-integral, we don’t have any problem with respect to the quantization around $\ell(s) = 0$.

Simply repeating the procedure in the section 2.2, one can show that the amplitude (37) can be precisely recovered by quantizing the 2d projectable HL gravity with the bi-local interaction based on the path-integral, if $\beta = g_s$.

5. Discussions

We have canonically quantized the 2d projectable HL gravity with a simple space-like wormhole interaction, and shown that the quantum Hamiltonian is equivalent to the one-body quantum Hamiltonian that includes contributions coming from all wormholes and baby universes obtained in the string field theory for CDT, if $g_s = \beta$ and $\lambda = \lambda/(2(1-\eta))$ where $\beta > 0$, $\lambda > 0$ and $\eta < 1$. 


Let us consider the classical Hamiltonian constraint $C$ in the parameter region above. When $\sqrt{\lambda \ell} \geq 1/\xi$ where $\xi$ is a dimensionless quantity defined by $\xi := g_s/\lambda^{3/2}$, the constraint surface is given by $\pi^2 = -\lambda + g_s \ell \geq 0$. However, when $\sqrt{\lambda \ell} < 1/\xi$, the only allowed solution is $\ell = 0$. In the case of the 2d projectable HL gravity ($\beta = g_s = 0$), with the parameter region corresponding to 2d CDT, i.e. $\lambda > 0$, the only solution to the classical Hamiltonian constraint is $\ell = 0$. Therefore, when $\sqrt{\lambda \ell} < 1/\xi$, the classical solution of the 2d projectable HL with the wormhole interaction will be close to that of the 2d projectable HL gravity, if one sits in the parameter region above; they can be quite different when $\sqrt{\lambda \ell} \geq 1/\xi$. Such a relation also holds at the quantum level. As shown in the article, when $\sqrt{\lambda \ell} < 1/\xi$, the eigenfunctions of the one-body quantum Hamiltonian including all wormholes and baby universes, $H_{-1}$, can be well approximated by the eigenfunctions of the quantum Hamiltonian of 2d CDT without wormholes and baby universes, $H^{(0)}$. On the other hand, when $\sqrt{\lambda \ell} \geq 1/\xi$, their behaviors are quite different, and in this case, in order for the theory to be well-defined, the unbounded nature of the potential in $H_{-1}$ will be counteracted by the kinetic term. This balance between the kinetic and potential terms is precisely what is reflected in the classical Hamiltonian constraint. This is also consistent with the boundary conditions on $\ell$ at infinity in the path-integral.

The picture of creation and annihilation of baby universes and wormholes is conceptually straightforward in the string field theory for CDT. Nevertheless it is somewhat surprising that one from this can derive an effective one-body Hamiltonian which can describe propagation of a single spatial universe, i.e. the propagation where the spatial universe starts with the topology of a circle and at a later time $t$ has the same topology, but where it in the intermediate times is allowed to split in two and either one part disappears in the vacuum (a baby universe), or the two parts join again at a later time (then changing the spacetime topology). This process of joining and splitting can be iterated at intermediate times and using string field theory we can perform the summation of all iterations and derive the effective one-body Hamiltonian. From the point of view of unitary evolution (or using Euclidean time, semigroup evolution, to be more precise), as given in eq. (29), it is difficult to understand how a complete set of intermediate states can both be given by the one spatial universe states $|\ell\rangle$ and by the complete multi-universe Fock states of the string field theory (for a recent discussion of this issue see i.e. [15]). However, this seems to be the case by explicit calculation, and we find it even more surprising that the simplest wormhole interaction term added to the classical action as in eq. (39) leads to precisely the same single universe quantum Hamiltonian as found in the string field theory for CDT.

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