Tunnelling Times: An Elementary Introduction

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Abstract

In this paper we examine critically and in detail some existing definitions for the tunnelling times, namely: the phase-time; the centroid-based times; the Büttiker and Landauer times; the Larmor times; the complex (path-integral and Bohm) times; the dwell time; and finally the generalized (Olkhovsky and Recami) dwell time, by adding also some numerical evaluations. Then, we pass to examine the equivalence between quantum tunnelling and “photon tunnelling” (evanescent waves propagation), with particular attention to tunnelling with Superluminal group-velocities (“Hartman effect”). At last, in an Appendix, we add a bird-eye view of all the experimental sectors of physics in which Superluminal motions seem to appear.

(†) Work partially supported by INFN and MURST (Italy); and by I.N.R., Kiev, and S.K.S.T. (Ukraine).
1 Introduction

Let us consider a freely moving particle which, at a given time, meets a potential barrier higher than its energy. As it is known, quantum mechanics implies a non-vanishing probability for the particle to cross the barrier (i.e., the tunnel effect). We may actually ask ourselves if it is possible to define a time duration, and therefore an average speed, for the barrier crossing; in that case we might try to calculate and measure those quantities. It seems surprising that no answer to such a question, so (apparently) straightforward, has gained general acceptance yet. The problem was first pointed out in 1931 by Condon[1]; an early attempt to solve it was due to MacColl[2] a year later. Afterwards, this subject was almost ignored up to the fifties, apart from the introduction of a quantum-theoretical observable for Time in scattering theory (For a simple introduction in quantum mechanics of a non-selfadjoint but hermitian time-operator, see refs.[3]). However, the interest on the subject has grown up, mostly during the last twentyfive years, for the increasing use of high-speed electronic devices (based on tunnelling processes) as well as for the acknowledged importance of the tunnel effect in nuclear fission and fusion below threshold. As a consequence, after 1987, several[4-6] theoretical reviews about tunnelling times appeared in the literature. Unfortunately on the experimental side it has always been rather difficult to perform measurements on particles, such as electrons, in order to check the theoretical predictions. Only in recent years various measurements of the sub-barrier “transmission times” (for microwaves and optical photons) have been performed, by exploiting the mathematical equivalence between the behaviour of (classical, relativistic) “evanescent” electromagnetic waves[7] and the (quantum, non-relativistic) tunnelling of a particle (or photon). We shall come back to the mentioned mathematical equivalence when dealing with the relevant experiments.

Tunnel effects can be met in various physical processes, as scattering, disexcitation of metastable states, fission and fusion above threshold, and so on. Anyway, we shall analyze only the one-dimensional tunnelling of a particle through a constant potential barrier\#1 $V_0$, localized in the interval $[0,d]$ (see Fig.1): a case that represents well enough the currently employed experimental setups. Despite the large quantity of theoretical papers on the subject, a universally accepted approach to tunnelling time does not seem

\#1 Only in the case of the Büttiker-Landauer time, the barrier will be a function of time too.
to exist, till now. We shall therefore group the existing approaches into four classes.

In the first group we consider all times built by “following” the entering packet while it crosses the barrier. After the choice of a particular feature of the packet, e.g., its central peak, one can compare the incoming and the outgoing peak, obtaining their time correlation. Various authors, alternatively, have considered the “centroid” (that is, the wavepacket center-of-mass), or the sharp wave-front of a “step-like” wave. To this class of times it belongs the phase time, which is obtained, on employing a definition of group-velocity, through the “stationary-phase approximation”. It appears to agree rather well with the so far available experimental data. Among the more common critical remarks to this approach, let us recall the following: The outgoing peak might not always correspond to the incoming one, because of the presence, among the Fourier components of the wavepacket, of frequencies close or above the barrier energy. These high frequencies would reach the barrier earlier, since they travel towards the barrier less distorted by the reflected waves with respect to the low-frequency components. As a consequence, besides the dispersion effects (often present also while approaching the barrier), there appears an effective acceleration of the wavepacket. The further known effects taking place in the barrier region lead to the conclusion that the packet may even seem to exit the barrier before the main peak has entered it.

A second approach consists in assuming the existence of some degrees of freedom in the barrier-particle system, in order to be able to define an “internal clock” which yield the time spent by the particle inside the barrier. Through the effect of the “barrier clock” on the particle or, viceversa, through the effect of the barrier on the “particle clock”, we infer the duration of their interaction during the barrier crossing. Büttiker and Landauer, for example, tried to deduce the tunnelling time by calculating the energy quanta exchanged by a particle crossing a square barrier endowed with a time-varying height (Büttiker-Landauer time). Another example can be the measurement of the electron spin “flip” when inside the barrier a uniform magnetic field is present (Larmor times). This “clock approach” allows a wide choice in the degrees of freedom of the system used as a clock; furthermore, it yields some experimental procedures in order to check the theor-
ical predictions. Nevertheless, it has been criticized by some authors, since not all clocks are equivalent, and, even more, since such clocks, besides changing the number of the degrees of freedom of the system, involve invasive processes, which affects the experimental outputs. We cannot be sure that the duration of the mentioned interaction (during which the state of the system suffers a perturbation, anyway) actually coincides with the crossing time or with the reflection time.

The third approach attributes to the below-barrier motion of a particle a set of “semiclassical” trajectories, with respect to which an average tunnelling time can be evaluated. The paths can be built up in various ways, e.g., through the Feynman path-integrals, the Bohm mechanics, or the Wigner distribution. Of course, each of these methods carries a distribution of crossing times. One of the inconveniences of this approach is the complex nature of the computed time quantities. Nevertheless, we can extract from these times some real quantities—the magnitude, the real part, the imaginary part—which result to be strictly related with some of the above definitions of tunnelling times. Just for this reason the physical interpretation of the semiclassical times appears rather interesting.

The last class of tunnelling times we are going to consider is the one which starts from the definition of dwell time. Such a quantity is defined as follows:

$$\tau^D(x_1, x_2; k) = j^{-1}_{\text{in}} \int_{x_1}^{x_2} |\psi(x, k)|^2 dx,$$

that is, it is the ratio between the probability density in the tunnelling region and the incoming flux entering the barrier $j_{\text{in}}$. The inconvenience of this definition is that it actually yields the dwell time inside the barrier, but without distinguishing between the transmission and the reflection channels. In this regard, an equation—often considered obvious even if not universally accepted—expected to link the times of the above two channels is the following:

$$\tau^D = |T(k)|^2 \tau_T + |R(k)|^2 \tau_R.$$

This equation, even if correct, seems to be not sufficient to determine uniquely $\tau_T$ and $\tau_R$. To overcome this difficulty, some authors introduced the so-called “space approaches”. Those pictures link the time spent inside the barrier with the reflection and transmission
times, and are built up through the standard quantum probabilistic interpretation of the reflection and transmission “currents”.

Let us anticipate that some definitions of tunnelling time we are going to analyze are sometimes regarded to be more suitable than others: in fact, they do not involve the so-called Hartman effect (see Sect.12) and do agree with eq.(1.2). Nevertheless, they result to disagree with the experimental data. We shall actually see that the really (physically) reasonable definitions of tunnelling time do all imply, among the others, the existence of Hartman-type effects, which have been confirmed by experiments.

This review is meant to be an elementary one; for further details see ref.[8].

2 Assumptions and notations

Let us suppose to have solved exactly the stationary case. Then, for each given energy $E = \hbar^2 k^2 / 2m$, we have:

$$
\psi(x; k) = \begin{cases} 
\psi_I = e^{ikx} + R(k)e^{-i(kx-\beta)} & x \leq 0 \\
\psi_H = \chi(x; k) & 0 \leq x \leq d \\
\psi_{III} = T(k)e^{i(kx+\alpha)} & x \geq d
\end{cases}
$$

(2.1)

where $R(k)$ and $T(k)$ are the reflection and transmission amplitudes, respectively, so that

$$R(k) = \sqrt{1 - T(k)^2},$$

quantities $\beta = \beta(k)$, $\alpha = \alpha(k)$ being the respective phase delays. In the particular case of a square barrier we have

$$V(x) = \begin{cases} 
V_0 & 0 \leq x \leq d \\
0 & \text{elsewhere,}
\end{cases}$$

while $\chi(x; k)$, $R(k)$, $T(k)$, $\alpha(k)$, $\beta(k)$ are analytically known. Therefore, we shall write

$$\chi(x; k) = \begin{cases} 
A(k)e^{-ikx} + B(k)e^{ikx} & E < V_0 \\
A(k)e^{-ikx} + B(k)e^{ikx} & E > V_0
\end{cases},
$$

(2.2)

with

$$\kappa = \begin{cases} 
\sqrt{2m(V_0 - E)}/\hbar, & E < V_0 \\
\sqrt{2m(E - V_0)}/\hbar, & E > V_0
\end{cases}$$
In the following we shall not consider simple stationary waves, but the wavepackets 
\[ \Psi(x; t) \sim \int dk \ f(k - k_0) \ \psi(x; k) \ e^{-iE(k)\varpi} = \int dE \ g(E - E_0) \ \psi(x; k(E)) \ e^{-iE\varpi}. \] (2.7)

We get also:
\[ \rho = |\psi(x)|^2 \quad j = \text{Re} \left\{ \frac{ih}{2m} \psi(x) \frac{\partial}{\partial x} \psi(x) \right\}. \] (2.8)

In the electromagnetic framework (we always suppose photons and microwaves in the T.E. or T.M. modes), \( \psi \) represents the scalar part of one of the two components of the field.

Finally, let us define the equivalent time, \( \tau_{eq}^T \), namely the time which the particle would spend in crossing the barrier region in the absence of the barrier: i.e., \( \tau_{eq}^T = md/\hbar k \).

Let us now review the definitions of some above-mentioned times.

### 3 Phase time

Let us consider a very narrow wavepacket around a wave-number \( k_0 \). The picture of its time evolution is often very difficult because of its own dispersive nature. Anyway, under suitable conditions, it is possible to follow the position of the peak of a symmetric
packet with a good precision, neglecting the dispersion effects[5,8]. We shall try therefore
to identify the packet taking the peak as reference point. To this end we shall use the
method of the stationary phase.

The peak of the packet is formed by those Fourier components for which the phase
variation in the surrounding of \( k_0 \) is reduced enough, so much that they do not interfere
destructively. Also the transmitted and reflected packets will be described by wave-
functions, corresponding to a small range of frequencies:

\[
\psi(x; k) \sim \exp \left\{ i \left[ kx - \frac{E(k)t}{\hbar} + \alpha(k) \right] \right\}.
\]

If we want to follow the position \( x_p(t) \) of the peak, we must look for which values of \( x_p(t) \)
the phase is stationary (maximum) at a given time \( t \). Then we must have:

\[
\frac{d}{dk} \left( kx_p(t) - \frac{E(k)t}{\hbar} + \alpha(k) \right) = 0 \quad \Rightarrow \quad x_p(t) = \frac{1}{\hbar} \frac{dE}{dk}t - \frac{d\alpha}{dk}.
\]

Quantity \( \alpha'(k_0) = (d\alpha/dk)k_0 \) represents the space delay \( \delta x \) caused by the tunnelling
process. Dividing by \( v_g \) (group-velocity of the wavepacket) we obtain the time delay

\[
\delta \tau_T = \frac{\delta x}{v_g} = (v_g)^{-1} \frac{d\alpha}{dk} = \left( \frac{1}{\hbar} \frac{dE}{dk} \right)^{-1} \frac{d\alpha}{dk} = \hbar \frac{d\alpha}{dE}.
\]

Notice that we are computing \( v_g \) and the other quantities at \( k = k_0 \) (we’ll see later on
whether and when this procedure may be considered correct).

For reasons which we shall later explain, we define as phase time the total time
\( \tau_T^\varphi(x_1, x_2; k) \), spent by a particle between two points, \( x_1 \) and \( x_2 \), external to the barrier
and sufficiently distant from it, i.e., such that \( x_1 \ll 0 \) and \( x_2 \gg d \). Then

\[
\tau_T^\varphi(x_1, x_2; k) = \frac{1}{v_g} (x_2 - x_1 + \alpha'(k)),
\]

and analogously, for the reflected particles,

\[
\tau_R^\varphi(x_1, x_2; k) = \frac{1}{v_g} (-2x_1 + \beta'(k)).
\]

\(^{\#2}\text{Applying the same reasoning to the incident part, we’ll have: } x_p(t) = \hbar^{-1} (dE/dk) t = (d\omega/dk) t, \text{ where } (d\omega/dk) \text{ is just, from its own definition, the group-velocity of the incoming packet. Moreover we notice that } v_g = (d\omega/dk) \text{ will depend on the dispersion relation } \omega(k) \text{ of the medium in which the packet propagates.} \)
Since both $\tau_T^\varphi$ and $\tau_R^\varphi$ depend linearly on $x_1$ and $x_2$, we might try to extrapolate the crossing and reflection times directly from them: by letting $x_1$ and $x_2$ approach 0 and $d$, respectively. Yet, this is not correct. In fact, since we have supposed the components of the packet tightly distributed around the wave-number $k_0$, the space spreading of the packet is the order of $\sigma^{-1}$ ($\sigma \equiv \Delta k$). Therefore the packet will be the more spread, the more it is peaked around $k_0$. Thus the incident part and the reflected part of the wavefunction will be able to interfere with each other also at a certain distance ($\sim \sigma^{-1}$) from the barrier (see Fig.2). Moreover, since we are employing a stationary approximation, we do not really follow the packet in its time evolution, but only observe (asymptotically) the phase-delay corresponding to the wave-number $k_0$. Let us define

$$\Delta \tau_T^\varphi = \frac{1}{v_g} [d + \alpha'(k)],$$

(3.5)

$$\Delta \tau_R^\varphi = \frac{1}{v_g} [\beta'(k)],$$

(3.6)

that we call extrapolated phase times. We have always to keep in mind the purely asymptotic character of such definitions. For a square barrier we have:

$$\Delta \tau_T^\varphi(0, d; k) = \Delta \tau_R^\varphi(0, d; k) = \frac{m \hbar k \kappa}{2} \left( \frac{2 \kappa d k^2 (\kappa^2 - k^2) + \varepsilon^4 \sinh(2\kappa d)}{4 \kappa^2 k^2 + \varepsilon^4 \sinh^2(\kappa d)} \right),$$

(3.7)

with $\varepsilon = 2mV_0/\hbar$. As $d$ increases the term in brackets approaches 2, and then $\Delta \tau_T^\varphi$ and $\Delta \tau_R^\varphi$ approach $2m/\hbar k \kappa = 2/v_g \kappa$. Such a value does not depend at all on the barrier thickness. Therefore, by increasing the thickness, one can meet arbitrarily large tunnelling velocities $d/\tau_T^\varphi$: i.e., one finds the so-called Hartman–Fletcher effect[9] (or simply Hartman effect). Let us remark that such an effect—which has been actually observed in all the previously mentioned experiments—implies Superluminal group-velocities for thick (opaque) barriers. This seeming disagreement with the so-called Einstein causality is justified by many authors[10] in terms of reshaping\footnote{One has a reshaping of a packet when, in crossing the barrier, the lower energy components are, in general, transmitted less efficiently than those of higher energy. All that can involve a modification of the form of the packet in the $k$-space with an effective acceleration.} of the packet; for a deeper analysis, see ref.[8] and refs. therein, as well as below.

By starting from the phase time, one can study the behaviour of the various frequencies composing a wavepacket which crosses the barrier. One purpose is to understand if, and
in which cases, reshaping effects can actually appear, and whether it may be possible to avoid them. In Fig.3 we represent the values of $T(k)$, computed for different values of $d$ as a function of $\varepsilon$. This figure shows also the distribution function of a gaussian packet peaked around $k_0 = 0.7\varepsilon$, namely $f(k - k_0) = \exp[-(k - k_0)^2/2(\Delta k)^2]$, with $\Delta k = 0.1 k_0$. The weight of each Fourier component in the transmitted packet will be given by the product $T(k) f(k - k_0)$. For very thin barriers, $(d\varepsilon \ll 1)$ $T(k)$ is quasi-constant and very close to 1, except for very small $k$. Therefore, if $k_0/\varepsilon$ is not too close to 0, the function $T(k) f(k - k_0)$ will always reach its maximum at $k_0$ and the outgoing packet will present no distortions. In this case the transmission time, still computed as a phase time, results longer than the equivalent time, and no problems arise. On increasing the barrier thickness, however, the lower energy components of the packet will be transmitted worse and worse. In fact, for sufficiently large values of $d$, quantity $T(k)$ is very small except when $k/\varepsilon$ is very close to 1 (in which case it increases, instead, very quickly). For this reason, the packet is expected by many authors to be accelerated, since the maximum of the transmitted packet will result to be situated at $k_0' > k_0$. According to those authors, all happens as if only a subset of frequencies crossed the barrier: namely, the higher frequencies, propagating at a velocity larger than $v_g$ (also before reaching the barrier itself!). Anyway, by choosing suitable initial conditions, all these reshaping effects can be avoided (and nevertheless the Superluminal group-velocities do not disappear). Let us see how.

First of all, notice that the barrier cannot have any amplifying effect on each component of the packet, but it only acts as a filter for some of them. Then $T(k)$ is limited and can increase only up to a maximum value equal to 1 (as we can see also from Fig.3). Let us also notice that, for $k < \varepsilon$, function $T(k)$ is monotonically increasing and, for sufficiently thick barriers, grows very quickly only for values of $k$ close to $\varepsilon$, where it shows an “oblique flexus” going on from concavity to convexity. We have

$$\lim_{k \to \varepsilon} T(k) = \frac{1}{\sqrt{1 + \frac{\varepsilon^2 d^2}{4}}}$$

and

$$\lim_{k \to \varepsilon} T'(k) = \frac{2d^2 \varepsilon (3 + d^2 \varepsilon^2)}{3(1 + d^2 \varepsilon^2)^{\frac{3}{2}}}$$

(3.8)

(3.9)
holding also for $k > \varepsilon$. In the vicinity of $\varepsilon$ we may write

$$T(k) \simeq \frac{1}{\sqrt{1 + \frac{\varepsilon^2 d^2}{4}}} + \frac{2d^2\varepsilon(3 + d^2\varepsilon^2)}{3(1 + d^2\varepsilon^2)^{\frac{3}{2}}} (k - \varepsilon)$$  \hspace{1cm} (3.10)

and therefore, even close to $k = \varepsilon$, quantity $T(k)$ does not increase faster than $T'(k)(k-\varepsilon)$.

By contrast, with regard to $f(k - k_0)$, we have

$$f(k - k_0) = \exp\left[-\frac{(k - k_0)^2}{2(\Delta k)^2}\right].$$

The argument of the exponential is smaller than 1 for $(k - k_0) < \Delta k$ and larger than 1 if $(k - k_0) > \Delta k$: in the latter case, $f(k - k_0)$ decreases exponentially with $[(k - k_0)/\Delta k]^2$.

Except for very thin barriers, for which we have already seen that $T(k) \simeq 1$, we expect that, if $\varepsilon - k_0 \sim \Delta k$, a large part of the tunnelling may also be due to transmission of the higher energy frequencies. Then we expect that even if the $k$-peak moves forward, it will always be $k'_0 - k_0 \sim \Delta k$. On the contrary, by decreasing the energy of the incident packet, i.e., by taking $k_0$ smaller, the contribute of the higher energies to the tunnelling decreases considerably, since $f(k - k_0)$ decreases exponentially with $[(k - k_0)/\Delta k]^2$: And all that happens still faster for increasing $|\varepsilon - k_0|$. To make possible a reshaping of the packet (in order to get a forward shifting, or a second peak), it is then necessary that, in a certain interval, $T(k)$ grows very quickly, and at a larger extent than the decrease of $f(k - k_0)$. Namely

$$\frac{d}{dk}[T(k)f(k)] = T'(k)f(k - k_0) + T(k)f'(k - k_0) > 0,$$  \hspace{1cm} (3.11)

which is a constraint very difficult to be studied because of the rather complicate form of $T'(k)$. But, since $f'(k - k_0) = -\frac{(k - k_0)}{(\Delta k)^2} f(k - k_0)$, and moreover it is always $f(k - k_0) > 0$, condition (3.11) reduces to:

$$T'(k) - \frac{(k - k_0)}{(\Delta k)^2} T(k) > 0.$$  \hspace{1cm} (3.12)

One can therefore remark that, as expected, for $k < k_0$ equation (3.12) is always satisfied. For $k > k_0$, on the contrary, in order that $T(k)f(k - k_0)$ be increasing, one has to require

$$T'(k) > T(k)\frac{(k - k_0)}{(\Delta k)^2}.$$  \hspace{1cm} (3.13)
But, as already seen, \( T'(k) \) is limited, and for sufficiently thick barriers reaches its maximum at \( k = \varepsilon \). Then, as far as quantity \( T(k) \) happens to be small, we can always find some values of \( \Delta k \) such that eq. (3.13) does not hold any more: thus the peak remains at \( k_0 \). This holds also for very thick barriers,\(^4\) for which the tunnelling probability becomes infinitesimal. It is sufficient that: 1) \( k_0 \) be not too close to \( \varepsilon \), and 2) the \( k \)-distribution be narrow enough around \( k_0 \).

In Fig. 4 the plots of \( T(k) f(k-k_0) \) are shown for different values of the barrier thickness and of \( k_0 \). Only for values of \( k_0/\varepsilon \) very close to 1 (\( k_0/\varepsilon \simeq 0.9 \)), the tunnel occurs mostly for the above-barrier energy components.

Let us finally notice that, even though the peak should move forward in the \( k \)-space, this would not have any direct influence on its position in the \( x \)-space; actually, in that case, either \( \tau^\varphi \) or \( \Delta \tau^\varphi \) would lose any physical sense, since they have been evaluated for \( k = k_0 \). This ensures that the transmitted packet does not suffer sensible distortions with respect to the incident one.

4 Times built-up following the centroid method

Some authors, rather than following the peak through the stationary phase method, prefer to refer to the centroid of the packet. This because, when applying the stationary phase method, we are forced to employ packets very narrow in \( k \) (and then very extended in \( x \)). And also because, in so doing, we can better evaluate the effects of the possible acceleration caused by the barrier crossing.

Suppose the packet be initially \((t \leq 0)\) located at a certain distance from the barrier, so that \( \int_0^\infty |\psi(x,0)|^2 dx \simeq 0 \), that is, the probability is negligible that the particle lies on the right of 0. We identify the “particle position” at time \( t = 0 \) with the position of its center-of-mass (the average space-coordinate):

\[
\bar{x}(0) = \frac{\int_{-\infty}^{\infty} x |\psi(x,0)|^2 dx}{\int_{-\infty}^{\infty} |\psi(x,0)|^2 dx}.
\]

\(^4\) For \( \kappa d = (\varepsilon^2 - k^2)^{1/2} d \gg 1 \), we have \( T(\kappa) = T((\varepsilon^2 - k^2)^{1/2}) \sim \frac{4k\kappa}{\varepsilon^2} e^{-\kappa d} \), but in such a case it is also \( T'(k) \sim e^{-\kappa d} \).
It being
\[ f(k) = f(k, 0) \frac{1}{\sqrt{2\pi}} \int dx \psi(x, 0) e^{-ikx} = |f(k)| e^{i\xi(k)}, \] (4.2)
it is possible to show that
\[ x_0 = \bar{x}(0) = \frac{-\int_0^\infty dk |f(k)|^2 \frac{d\xi}{dk}}{\int_0^\infty dk |f(k)|^2} = -\langle \xi'(k) \rangle. \] (4.3)

We’ll have
\[ \bar{x}_{in}(t) = x_0 + \frac{\hbar}{m} \langle k \rangle_{in} t, \quad (t \to 0); \] (4.4)
\[ \bar{x}_{T}(t) = x_0 + \frac{\hbar}{m} \langle k \rangle_{T} t - \langle \alpha' \rangle_{T}, \quad (t \to \infty); \] (4.5)
\[ \bar{x}_{R}(t) = x_0 + \frac{\hbar}{m} \langle k \rangle_{R} t - \langle \beta' \rangle_{R}, \quad (t \to \infty); \] (4.6)

where
\[ \langle k \rangle_{in} \equiv \frac{\int_{-\infty}^{\infty} dk k |f(k)|^2}{\int_{-\infty}^{\infty} dk |f(k)|^2}, \quad \langle k \rangle_{T} \equiv \frac{\int_{-\infty}^{\infty} dk k |T|^2}{\int_{-\infty}^{\infty} dk |T|^2}, \quad \langle k \rangle_{R} \equiv \frac{\int_{-\infty}^{\infty} dk k |R|^2}{\int_{-\infty}^{\infty} dk |R|^2}, \]
\[ \langle \alpha' \rangle_{T} \equiv \frac{\int_{-\infty}^{\infty} dk \langle \alpha' \rangle |T|^2}{\int_{-\infty}^{\infty} dk |T|^2}, \quad \langle \beta' \rangle_{R} \equiv \frac{\int_{-\infty}^{\infty} dk \langle \beta' \rangle |R|^2}{\int_{-\infty}^{\infty} dk |R|^2}. \]

On shifting \( \bar{x}_{in}(t) \) and \( \bar{x}_{R}(0) \) forward in time, and \( \bar{x}_{T}(t) \) backwards, it is possible to extrapolate \( t_{in}(0) \), \( t_{R}(0) \) and \( t_{T}(d) \), respectively, to times at which the centroid passes through 0 and \( d \). Then the transmission and reflection times will be given by
\[ \tau_{T}^{C} = t_{T}(d) - t_{in}(0) = \frac{m}{\hbar} \left[ \frac{d - x_0 + \langle \alpha' \rangle_{T}}{\langle k \rangle_{T}} + \frac{x_0}{\langle k \rangle_{in}} \right]; \] (4.7)
\[ \tau_{R}^{C} = t_{R}(0) - t_{in}(0) = \frac{m}{\hbar} \left[ \frac{-x_0 + \langle \beta' \rangle_{R}}{\langle k \rangle_{R}} + \frac{x_0}{\langle k \rangle_{in}} \right]. \] (4.8)

Leavens and Aers[12] showed that, for \( \Delta k \to 0 \), we have \( \tau_{T}^{C} \to \Delta \tau_{T}^{\varphi} \) and \( \tau_{R}^{C} \to \Delta \tau_{R}^{\varphi} \), while the possible corrections are of the first order in \( \Delta k \), according to our previous qualitative reasonings.

Analogous results where achieved by Martin and Landauer[13], who performed the calculations in the electromagnetic framework, and by Collins, Lowe and Barker[4] who followed the time evolution of a gaussian packet obeying the time-dependent Schrödinger

\[ #5 Notice that we are still using asymptotic forms of the wave-function, always neglecting, then, the self-interference effects near the barrier. \]
equation. Nevertheless, they also got, even by explicit computations, the time spent by the centroid to leave the barrier (there are no self-interferences in the wave-function for $x \geq d$), by *extrapolating* the time spent to reach the barrier.[6]

Let us go on now to study some tunnelling times defined by means of suitable “clocks”.

5 Büttriker and Landauer times

In order to determine $\tau_T$, Büttriker and Landauer[14-16] in 1982 proposed to consider a time-oscillating square-barrier, and supposed the crossing time to equal the duration of the interaction between the particle and the oscillating potential.

Let us consider a square potential with height $V_0$, upon which an oscillating potential $\delta V \cos \omega t$ is superimposed. At very low frequencies, the potential varies very slowly; therefore the particle, during the crossing, will feel the effects of only a part of the modulation cycle. So, as long as the period corresponding to the oscillation is long compared to the crossing time, the particle will interact with a quasi-static potential. At higher frequencies ($\omega \gg 1/\tau_T$), the particle will undergo the effect of several cycles of oscillation and will absorb or release energy quanta equal to $\hbar \omega$. The transition frequency between the adiabatic behaviour, typical of the low frequencies, and the non-adiabatic behaviour, yields an approximate measure of the duration of the particle interaction with the barrier.

At the first order in $\delta V$, only the two bands $E \pm \hbar \omega$ will appear. Moreover, the particles belonging to the higher energy band will have a larger crossing-probability, with respect to the low energy ones. Büttriker and Landauer showed that, for thick barriers and not too high frequencies ($\hbar \omega$ small compared with either $E$ or $V_0 - E$), the relative intensity of the two bands will be given by:

$$I^T_\pm(\omega) = \left| \frac{T_\pm(\omega)}{T(\omega)} \right|^2 = \left( \frac{\delta V \tau_T^{\text{BL}}}{2\hbar \omega} \right)^2 \left[ e^{\pm \omega \frac{md}{\hbar \kappa}} - 1 \right]^2. \tag{5.1}$$

Then those two authors identified the crossing time with $\tau_T^{\text{BL}} = md/(\hbar \kappa)$. In the very small frequency limit, eq.(5.1) reduces to

$$\left| \frac{T_\pm(\omega)}{T(\omega)} \right|^2 = \left( \frac{\delta V \tau_T^{\text{BL}}}{2\hbar} \right)^2. \tag{5.2}$$
As expected, the number of particles that will have absorbed or released energy is, in this case, quite independent of $\omega$. Again, from eq.(5.1), we have:

$$\frac{T_+(\omega) - T_-(\omega)}{T_+(\omega) + T_-(\omega)} = \tanh(\omega \tau_{BL}^T).$$

(5.3)

This shows that just $\tau_{BL}^T$ determines the transition from the adiabatic behaviour at low frequencies, when $T_+ \approx T_-$, to the high frequencies behaviour, when $T_+ \gg T_-$. With regard to the reflected particles, always within the limits $\hbar \omega \ll E$ and $\hbar \omega \ll V_0 - E$, Büttiker and Landauer found that

$$|R_\pm|^2 = \left( \frac{\delta V \tau_{BL}^R}{2\hbar} \right)^2,$$

(5.4)

with $\tau_R = \hbar k/(V_0 \kappa)$. Notice that also eq.(5.4) is independent of $\omega$.

The low frequencies behaviour of eqs. (5.2) and (5.4) is typical of a system characterized by two-states, $|1\rangle$ and $|2\rangle$, endowed with energy $E$ and $E \pm \hbar \omega$ respectively, brought to resonance by a perturbation $V_1 \cos \omega t$. If for $t = 0$ the whole population of the system is in the state $|1\rangle$, the population of the state $|2\rangle$ increases initially as $(V_1 t/2\hbar)^2$. The same happens if the energies of the two levels $E_1$ and $E_2$ are equal. Thus, in eqs.(5.2) and (5.4), quantities $\tau_{BL}^T$ and $\tau_{BL}^R$ actually play the role of interaction times of the particle-barrier system.

## 6 Larmor times

In 1966 Baz’\cite{17,18} proposed to exploit the Larmor precession, caused by the presence of a magnetic field on particles endowed with spin, to measure the collision times of these particles. In the same year, Rybachenko\cite{19} applied this method to compute the tunnelling times for a one-dimensional square barrier. Let us consider a beam of spin-$\frac{1}{2}$ particles, polarized along $\hat{x}$, with mass $m$ and kinetic energy $E$, moving along $\hat{y}$ (see Fig.5; notice that, following the original paper, also in this figure the propagation axis has been named $y$). Let us furthermore suppose that a weak, homogeneous magnetic field, $B_0$, directed along the $\hat{z}$-axis, is present in the barrier zone, overlapping the barrier potential. Following Rybachenko, the particles which enter the barrier, on crossing the magnetic field will undergo a Larmor precession with frequency $\omega_{La} = g\mu B_0/\hbar$, where $g$ is
the gyromagnetic ratio, and \( \mu \) the magnetic moment. The precession will stop just when the particle will come out from one of the two sides of the barrier. Since we have\[20\]

\[ \langle S_x \rangle_T = \frac{\hbar}{2} \cos \omega_{La} \tau_{La} , \]

\[ \langle S_y \rangle_T = -\frac{\hbar}{2} \sin \omega_{La} \tau_{La} , \]

we can write, in the weak magnetic field limit,

\[ \langle S_x \rangle_T \simeq \frac{\hbar}{2} , \]

\[ \langle S_y \rangle_T \simeq -\frac{\hbar}{2} \omega_{La} \tau_{La} . \]

Thus the nonzero spin component along \( \hat{y} \) will be proportional to the dwell time inside the barrier. As a consequence:

\[ \tau_{yT}^{La} = \lim_{\omega_{La} \to 0} \frac{\langle S_y \rangle_T}{-\frac{\hbar}{2} \omega_{La} \tau_{yT}} . \]

Rather strangely, Rybachenko did not consider the main effect of the field on the particles, namely the spin alignment. As a matter of fact, after having left the barrier, the spin will have a component along \( \hat{z} \) equal to \( \pm \hbar/2 \). While outside the barrier the particle energy is independent of the spin, on the contrary inside the barrier its energy will depend also on the spin \( z \)-component (because of the Zeeman effect). The difference between the spin-up and spin-down energies is \( \pm \hbar \omega_{La}/2 \). Let us state:

\[ \psi_{in} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{iky} \]

\[ \psi_T = (|D_+|^2 + |D_-|^2)^{-1/2} \left( \frac{D_+}{D_-} \right) e^{iky} \]

with

\[ D_\pm = T(\kappa_\pm) e^{\alpha} e^{-i\kappa_\pm d} , \]

where \( \kappa_\pm \) are the \( \kappa \)-values corresponding to \( E \pm \hbar/2 \omega_{La} \). If

\[ \langle S_i \rangle_T = \frac{\hbar}{2} \langle \psi | \hat{\sigma}_i | \psi \rangle , \]

we obtain

\[ \langle S_z \rangle_T = \frac{\hbar}{2} \frac{|T_+|^2 - |T_-|^2}{|T_+|^2 + |T_-|^2} , \]

(6.1a)
\[ \langle S_y \rangle_T = -\hbar \sin(\alpha_+ - \alpha_-) \frac{|T_+ T_-|}{|T_+|^2 + |T_-|^2}, \]  \hspace{2cm} (6.1b)  
\[ \langle S_x \rangle_T = \hbar \cos(\alpha_+ - \alpha_-) \frac{|T_+ T_-|}{|T_+|^2 + |T_-|^2}. \]  \hspace{2cm} (6.1c)

Analogous expressions are found for the reflected particles: it is sufficient to replace \( T_\pm \) by \( R_\pm \). Furthermore, it can be shown that

\[ \langle S_z \rangle_R = \frac{-\hbar}{2} \omega_{La} \tau_{zT}, \]  \hspace{2cm} (6.2a)

\[ \langle S_y \rangle_R = -\frac{\hbar}{2} \omega_{La} \tau_{yT}, \]  \hspace{2cm} (6.2b)

\[ \langle S_x \rangle_R = \frac{\hbar}{2} \omega_{La} \tau_{xT}. \]  \hspace{2cm} (6.2c)

For a weak magnetic field we have \( \kappa_\pm \simeq \kappa \mp m \omega_{La}/\hbar \), and also

\[ |T_+|^2 - |T_-|^2 \sim -\frac{m \omega_{La} \partial T}{\hbar \kappa} \partial \ln \kappa. \]  \hspace{2cm} (6.3)

In (6.3) the term \(-m(\hbar \kappa)^{-1}(\partial T/\partial \kappa)\), which multiplies \( \omega_{La}, \) has, of course, the dimensions of a time. In 1983 Büttiker suggested\[21\] the introduction of three times, namely \( \tau_{zT}^{La} \), \( \tau_{yT}^{La} \) and \( \tau_{xT}^{La} \), in the following way. He assumed

\[ \langle S_z \rangle_T = (\hbar/2) \omega_{La} \tau_{zT}^{La}, \]  \hspace{2cm} (6.4a)

\[ \langle S_y \rangle_T = -(\hbar/2) \omega_{La} \tau_{yT}^{La}, \]  \hspace{2cm} (6.4b)

\[ \langle S_x \rangle_T = (\hbar/2)[1 - (\omega_{La}^2 \tau_{xT}^{2La})/2], \]  \hspace{2cm} (6.4c)

and then:

\[ \tau_{zT}^{La} = \lim_{\omega_{La} \to 0} \frac{\langle S_z \rangle_T}{\hbar/2 \omega_{La}} = -\frac{m}{\hbar \kappa} \frac{\partial \ln T}{\partial \kappa}, \]  \hspace{2cm} (6.5a)

\[ \tau_{yT}^{La} = \lim_{\omega_{La} \to 0} \frac{\langle S_y \rangle_T}{\hbar/2 \omega_{La}} = -\frac{m}{\hbar \kappa} \frac{\partial \alpha}{\partial \kappa}, \]  \hspace{2cm} (6.5b)
$$\tau_{xT}^{La} = \lim_{\omega_{La} \to 0} \frac{\langle S_x \rangle_T}{2\omega_{La}} = \frac{m}{\hbar\kappa} \left[ \left( \frac{\partial\alpha}{\partial\kappa} \right)^2 + \left( \frac{\partial \ln T}{\partial\kappa} \right)^2 \right]. \quad (6.5c)$$

On developing the calculations, Böttiker obtained

$$\tau_{xT}^{La} = \frac{m\varepsilon^2 (k^2 - k^2) \sinh^2 (kd) + (kd\varepsilon^2/2) \sinh (2kd)}{4k^2\kappa^2 + \varepsilon^4 \sinh^2 (kd)} \quad (6.6a)$$

$$\tau_{yT}^{La} = \frac{mk (k^2 - k^2) + \varepsilon^2 \sinh (2kd)}{4k^2\kappa^2 + \varepsilon^4 \sinh^2 (kd)} \quad (6.6b)$$

$$\tau_{xT}^{La} = \sqrt{\tau_{zT}^{La} + \tau_{yT}^{La}} \quad (6.6c)$$

For sufficiently thick barriers he got

$$\tau_{zT}^{La} \approx \frac{md}{\hbar\kappa}, \quad \tau_{yT}^{La} \approx \frac{2mk}{\hbar\varepsilon^2\kappa}. \quad (6.7)$$

In such a way, in the thick barriers limit, $\tau_{xT}^{La} = \tau_T^{BL}$. After all, we are not confronting a real spin precession, but just a spin “flip” together with a splitting of the energy levels. Böttiker himself[21] showed that for $\tau_{xT}^{La}$ there hold considerations analogous to the ones holding for $\tau_T^{BL}$. In connection with $\tau_{xT}^{La}$, it is even more difficult to recognize a physical meaning in it. Indeed, if we think that also the spin $x$-component precedes around the $\hat{z}$-axis, therefore to such a precession it should correspond an average spin $x$-component equal to

$$\langle S_x \rangle_T = \langle \hbar/2 \rangle \left[ 1 - \left( \omega_{La}^2 \tau_{yT}^{La}^2 \right)^2 \right],$$

and not to

$$\langle S_x \rangle_T = \langle \hbar/2 \rangle \left[ 1 - \left( \omega_{La}^2 \tau_{xT}^{La}^2 \right)^2 \right]. \quad (6.4c)$$

For the previous considerations about $\tau_{yT}^{La}$, quantity $\tau_{xT}^{La}$ could be regarded, at most, as an average of $\tau_{yT}^{La}$ and $\tau_{xT}^{La}$; and, in fact, some authors introduce directly a time $\tau_T^{B} = \sqrt{\tau_{yT}^{La}^2 + \tau_{xT}^{La}^2}$. In the thick barriers limit, we have $\tau_T^{B} \approx \tau_{xT}^{La}$. Therefore, the only one of the three Larmor times which seems endowed with a clear physical meaning is $\tau_{yT}^{La}$. Associated with this quantity, Falk and Hauge[22] found, in 1988, the two relations:

$$\tau_{yT}^{La} = \frac{m}{\hbar\kappa} (x_2 - x_1 + \alpha') + \frac{mR}{2\hbar k^2} [ \sin (\beta - 2kx_1) - \sin (2\alpha - \beta + 2kx_2) ], \quad (6.8a)$$

as well as, for the reflected part,

$$\tau_{yT}^{La} = \frac{m}{\hbar\kappa} (x_2 - x_1 + \alpha') + \frac{mR}{2\hbar k^2} [ \sin (\beta - 2kx_1) - \sin (2\alpha - \beta + 2kx_2)] +$$
\[ + \frac{m}{2 \hbar k^2 R} \left[ \sin(\beta - 2kx_1) + \sin(2\alpha - \beta + 2kx_2) \right], \] 

(6.8b)

where \(x_1\) and \(x_2\) are any pair of points outside the barrier (one on its right and one on its left), and the magnetic field, rather than limited to the mere barrier zone, covers the whole range \((x_1, x_2)\). It can be easily seen that eqs. (6.7) are related to the phase time much more than the oscillating terms, whose amplitudes increase when the incident energy decreases. For a square barrier we obtain

\[ \tau_{yT}^{La}(d) = \tau_{yR}^{La}(d) = \tau^D(d), \]

where \(\tau^D\) is nothing but the \textit{dwell time}, which will be introduced in the following.

## 7 Complex times: path-integrals

The introduction of complex times follows initially from the idea that for above-barrier energies we can write \(v = \hbar \kappa = \hbar \sqrt{k^2 - \varepsilon^2}\) and, therefore, \(\tau_T = d/v = md/\hbar \kappa\). For \(E < V_0\), on the contrary, the wave-vector becomes imaginary. Nevertheless, let us imagine the below-barrier motion of a particle to occur along a classical trajectory, but with \textit{imaginary velocity} and \textit{time}. We can realize that the trajectories have to be complex, whilst times and velocities may be assumed as real, by considering the quantities:

\[ v^S = \frac{\hbar |\kappa|}{m}, \]

\[ \tau^S_T = \frac{d}{v} = \frac{md}{\hbar \kappa}, \]

where label S means “semiclassical”. Of course, also \(\tau^S_T\) is not physically meaningful, since it diverges for \(k = \varepsilon\). Anyhow, notice that in the case of thick barriers also \(\tau_T^{BL}\), and of course \(\tau_T^{La}\), approach \(\tau^S_T\). Let us recall that in 1992 Hagmann[23] proposed to consider the case of a particle that, in order to cross the barrier, receives a certain energy \(\Delta E\), during a time interval \(\Delta t\). Even if that procedure seems to lack physical meaning, his result (obtained by applying the uncertainty principle) is correct: \(\Delta t = \tau^S_T = md/\hbar \kappa\).

Going back to the complex times, in 1987 Sokolovski and Baskin[24] set forward a generalization of the classical concept of time to quantum mechanics and hence applied their method to the tunnel effect. Let us consider a particle that, emitted at the point
\( r_1 \) at time \( t_1 \), is detected at the point \( r_2 \) at time \( t_2 \). Moreover, suppose that the particle, moving along the trajectory \( r(t) \) inside a potential \( V(r) \), has crossed a certain space region \( \Omega \). Then the time spent by the particle in that space region will be given by

\[
\tau_{\text{cl}}^\Omega = \int_{t_1}^{t_2} dt \Theta_\Omega(r(t)),
\]

where quantity \( \Theta_\Omega(r(t)) \) is 1 if \( r(t) \) belongs to \( \Omega \), and 0 otherwise. In the one-dimensional case, we’ll have

\[
\tau_{\text{cl}}^\Omega = \int_{t_1}^{t_2} dt \int_0^d dx \delta(x - x(t)).
\]

If we then use the Feynman path-integral method to build-up trajectories along which to perform the time averages, we get

\[
\tau^\Omega(x_1, t_1; x_2, t_2; k) = \langle \tau_{\text{cl}}^\Omega(x(\cdot)) \rangle_{\text{path}},
\]

where \( x(\cdot) \) is an arbitrary path (in the phase-space) between \((x_1, t_1)\) and \((x_2, t_2)\). In general \( \tau^\Omega \) will be complex. Sokolovski and Baskin found

\[
\tau^\Omega_T = i \hbar \int_0^d dx \frac{\delta \ln A}{\delta V(x)},
\]

\[
\tau^\Omega_R = i \hbar \int_0^d dx \frac{\delta \ln B}{\delta V(x)},
\]

with \( A = Te^{i\alpha} \), \( B = Re^{i\beta} \). In the same paper, those authors wrote between \( \tau^\Omega \) and the Larmor times the relations

\[
\text{Re } \tau^\Omega_T = \tau^\text{La}_{yT},
\]

\[
\text{Im } \tau^\Omega_T = \tau^\text{La}_{zT},
\]

\[
|\tau^\Omega_T| = -\tau^\text{La}_{xT}.
\]

Analogously, for the reflected part:

\[
\text{Re } \tau^\Omega_R = \tau^\text{La}_{yR},
\]

\[
\text{Im } \tau^\Omega_R = \tau^\text{La}_{zR},
\]

\[
|\tau^\Omega_R| = \tau^\text{La}_{xR}.
\]
In spite of such a surprising correlation between $\tau^\Omega$ and the Larmor times, it is difficult to physically interpret such results. A possible interpretation is the one given by Hänggi\cite{25} in 1993. According to that author, the tunnelling time would be characterized by two time scales. However, he found it difficult to justify his approach from the physical point of view. Sokolovski and Connor\cite{26}, the same year, criticized that theory: they regarded the existence of two different crossing times as physically doubtful; and concluded that what is to be considered as crossing time is the magnitude of $\tau^\Omega_T$.

Let us go back to Hänggi’s hypothesis. If we look at the transmitted wave-form, we can point out that

$$\psi_T = \psi_{III}(x,k) = T(k)e^{i\alpha}e^{ikx} = e^{\ln T(k)}e^{i\alpha}e^{ikx}.$$  \hfill (7.6)

Notice that, when applying the phase time definition to the above equation, one finds two distinct components: one proportional to $d\alpha/dE$, and one to $d(\ln T)/dE$. We can then imagine the latter to be the time needed to damp the signal during the crossing. It appears strange the dependence of such a time on $d$, since $\tau^\Omega_T$, rather than decreasing when the barrier thickness increases, does increase proportionally to it. It should be better understood how such a time may be linked to the level transitions, both in the Büttiker-Landauer theory, and in the spin-flip case.

8 Complex times: Bohm’s method

Still in the complex-times approach, Leavens and Aers\cite{27} in 1993, starting from the same operator introduced by Sokolowski and Baskin ($\tau^\Omega_{cl}$), suggested to have recourse to Bohm mechanics. Bohm mechanics yields “semiclassical” trajectories, which can be employed to compute the average tunnelling time. The Bohm method provides us with a couple of equations fully equivalent to the Schrödinger equation, simultaneously allowing a rather “classical” interpretation of quantum theory. Let us resume it shortly, following previous work of ours.

\#\# Apparently meaningless, but probably not that much as it seems; recall, e. g., the Caldirola chronon: cf. R.H.A.Farias and E.Recami, “Introduction of a quantum of Time (‘chronon’) and its consequences for quantum mechanics”, LANL Arhives e-print # quant-ph/9706059.
The most general scalar wave-function $\psi \in \mathbb{C}$ may be factorized as follows:

$$\psi = \sqrt{\rho} \exp\left(\frac{i\varphi}{\hbar}\right), \quad (8.1)$$

where $\rho(x, t), \varphi(x, t) \in \mathbb{R}$. Applying this assumption in the Schrödinger equation, and separating the real from the imaginary part, we easily get the two well-known equations\[28\] for the so-called Madelung probabilistic fluid (which, taken together, are equivalent to the Schrödinger equation), i.e.:

$$\partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] + U = 0, \quad (8.2)$$

where

$$\frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] \equiv -\frac{\hbar^2}{2m} \frac{\Delta |\psi|}{|\psi|}, \quad (8.3)$$

is often called “quantum potential”; and

$$\partial_t \rho + \nabla \cdot (\rho \nabla \varphi/m) = 0. \quad (8.4)$$

Equations (8.2), (8.4) are the Hamilton–Jacobi and the continuity equations for the non-relativistic probabilistic fluid, respectively, and constitute the “hydrodynamic” formulation of the Schrödinger theory. Once chosen the initial and the boundary conditions, the solution of the Madelung equation system yields the semiclassical phase $\varphi$ and the probability density $\rho$. Matching the quantum phase with the classical action, and neglecting the presence of spin, Bohm assumed for the particle “local” velocity the expression

$$v(x, t) = \frac{p(x, t)}{m} = \frac{\nabla \varphi(x, t)}{m}. \quad (8.5)$$

Therefore, by integrating the above velocity field, we can compute the semiclassical Bohm trajectories, and from them derive semiclassical tunnelling times.

9 Dwell time

The dwell time was introduced first by Smith[29] in 1960, in order to estimate the average duration of a collision process without distinguishing among the various channels. As
already said, it is defined as the ratio between the probability that the particle is in a certain region of space and the flux $j_{in}$ entering that same region, without taking into account if the particle is reflected or transmitted:

$$\tau_D(x_1, x_2; k) = j_{in}^{-1} \int_{x_1}^{x_2} |\psi(x, k)|^2 dx = \frac{1}{v_g} \int_{x_1}^{x_2} |\psi(x, k)|^2 dx. \quad (1.1)$$

For a square barrier, we have

$$\tau_D(x_1, x_2; k) = \frac{mk}{\hbar} \frac{2\kappa d (\kappa^2 - k^2) + \varepsilon^2 \sinh(2\kappa d)}{4k^2\kappa^2 + 4\sinh^2(\kappa d)}, \quad (9.1)$$

which, for $\kappa d \gg 1$, becomes

$$\tau_D = \frac{\hbar k}{V_0 \kappa} = \frac{2mk}{\hbar \varepsilon^2 \kappa}. \quad (9.2)$$

Thus, for sufficiently thick barriers, also $\tau_D$, like $\tau_{T,R}^\Omega$ and $\Delta \tau_{T,R}^\Omega$, turns out to be independent of the thickness, while it decreases for increasing $k$, and vanishes for $k = 0$. Assuming that transmission and reflection are situations excluding each other, almost all the authors concluded that any dwell time must necessarily satisfy the relation

$$\tau_D = |R(k)|^2 \tau_R + |T(k)|^2 \tau_T. \quad (1.2)$$

Such a relation is, for instance, satisfied by $\tau^\Omega$. Indeed Sokolovski and Baskin[24], in 1987, found that

$$\tau_D = |R|^2 \tau_R^\Omega + |T|^2 \tau_T^\Omega. \quad (1.2)$$

From the above equation, by separating the real from the imaginary part, we get

$$\left\{ \begin{array}{l}
\tau_D = |R|^2 \tau_{y,R}^\Omega + |T|^2 \tau_{y,T}^\Omega \\
|R|^2 \tau_{z,R}^\Omega + |T|^2 \tau_{z,T}^\Omega = 0.
\end{array} \right. \quad (9.3)$$

The former equation was obtained independently by Falck and Hauge[22] one year later. The latter equation, instead, is nothing but the conservation law of an angular momentum: in the present case, of the spin $z$-component. Equation (9.1) does not appear to be satisfied for the phase time and, a fortiori, for the extrapolated phase times. Actually, Hauge et al.[11] found the relation

$$\tau_D(x_1, x_2; k) = |T(k)|^2 \tau_{T}^\varphi(x_1, x_2; k) + |R(k)|^2 \tau_{R}^\varphi(x_1, x_2; k) + \frac{mR}{\hbar k^2} \sin(\beta - 2kx_1). \quad (9.3)$$
Yet, those two authors showed also that, if one considers that any packet has a certain spread in $k$ and applies eq.(9.3) to the whole packet, one gets\(^7\)

$$
\langle \tau^D(x_1, x_2; k) \rangle \approx \langle |T(k)|^2 \tau_T^\varphi(x_1, x_2; k) \rangle + \langle |R(k)|^2 \tau_R^\varphi(x_1, x_2; k) \rangle + \frac{mR}{\hbar k^2} \sigma^{-1} \int dk \sin(\beta - 2kx_1) + O(\sigma). \tag{9.4}
$$

Thus, if $|x_1| \gg \sigma^{-1}$, the argument of the integral will oscillate quickly enough to make it negligible, and then we can write

$$
\langle \tau^D(x_1, x_2; k) \rangle = \langle |T(k)|^2 \tau_T^\varphi(x_1, x_2; k) \rangle + \langle |R(k)|^2 \tau_R^\varphi(x_1, x_2; k) \rangle. \tag{9.5}
$$

This is an equation agreeing with eq.(1.2) up to $O(\sigma)$. Actually eq.(9.5) does not hold for the extrapolated phase times, in such a way showing once more the purely asymptotic nature of the phase time.

Let us analyze a little better eq.(9.3). According to Hauge and Støvngen\[^4\], this equation shows that $\tau^D$ represents just the exact time spent by the particles inside the barrier, while the term $mR(\hbar k^2)^{-1} \sin(\beta - 2kx_1)$ represents a $\Delta \tau$ caused by self-interference effects. In fact, the dwell time computed in the interval $(-L, x_1)$ does diverge when $L$ grows to infinity. When subtracting the dwell time in $(-L, x_1)$ computed only for the incident part of the wavepacket, we obtain for\(^8\) $\Delta \tau^D(x < x_1; k)$, after some algebra:

$$
\Delta \tau^D(x < x_1; k) = -\frac{mR}{\hbar k} \sin(\beta - 2x_1). \tag{9.6}
$$

Equation (9.3) may be then re-written as

$$
|T(k)|^2 \tau_T^\varphi(x_1, x_2; k) + |R(k)|^2 \tau_R^\varphi(x_1, x_2; k) = \tau^D(x_1, x_2; k) + \Delta \tau^D(x < x_1; k) = \tau^D(x_1, x_2; k) - \frac{mR}{\hbar k^2} \sin(\beta - 2kx_1). \tag{9.7}
$$

To support such a reasoning, those two authors stressed the fact that the aforesaid self-interference term is completely independent of $T(k)$ and of $\alpha(k)$, just because there is not interference for $x > d$. Furthermore, they treated two particular cases as examples.

\(^7\) Of course, in eq.(9.4) we should consider $\langle |T(k)|^2 \tau_T^\varphi(x_1, x_2; k) \rangle$ and $\langle |R(k)|^2 \tau_R^\varphi(x_1, x_2; k) \rangle$, rather than $\langle |T(k)|^2 \tau_T^\varphi(x_1, x_2; k) \rangle$ and $\langle |R(k)|^2 \tau_R^\varphi(x_1, x_2; k) \rangle$. But it can be proved\[^4\] that the error made by using eq.(9.5) is simply of the same order as $\sigma$.

\(^8\) A similar approach, in which positive and negative fluxes are however evaluated separately, was later adopted by Olkhovsky and Recami in order to generalize the definition of dwell time: cf. refs.[5], and [8].
The first one regards an infinitely thick barrier (a step). Being the barrier infinitely thick, there will be no transmitted particles and all the particles will be reflected: $R = 1$. It is therefore easy to prove that

$$\Delta \tau_R^\phi = \frac{2}{\kappa v} = \frac{2m}{\hbar \kappa},$$

$$\tau_D = \frac{E}{V_0} \Delta \tau_R^\phi,$$

$$\Delta \tau_D = \frac{E - V_0}{V_0} \Delta \tau_R^\phi.$$

In this case no contrast arises between the extrapolated phase time, which increases with $k^{-1}$ when $k \to 0$, and the dwell time that, on the contrary, goes to 0 with $E/v_\kappa \sim k$. In fact, if $\tau_D$ is the time spent inside the barrier and $\Delta \tau_D$ the delay (or the advance) due to the self-interference, the latter term will be the larger one. This because, the more the incident energy decreases, the less the particle penetrates inside the barrier.

The second example concerns, instead, a Dirac $\delta$ barrier: a case that is one of the first to be treated and solved in the literature about the subject. Of course, in such a case the dwell time has to be zero; nevertheless we get: $^9$

$$\Delta \tau_R^\phi = \Delta \tau_T^\phi = T(k) \frac{V_0 d}{m \nu^3}$$

with:

$$T(k) = \frac{1}{1 + \left(\frac{V_0 d}{\hbar k}\right)^2}.$$

This means that in this case the tunnelling time can originate only from the self-interference term.

### 10 Generalization of the dwell time

As already said, not all the authors agree about the importance attributed till now to the dwell time, also [but not only] because of eq.(1.2). In fact such a relation, claimed to be a

---

$^9$In the present case, $d$ has the role of a “parameter” only, since the Dirac $\delta$ is got as the limit of narrower and narrower, and at the same time higher and higher, barriers (while the area $V_0 d$ is kept constant).
consequence of the superposition principle and of the “complementarity” of transmission
and reflection, would imply that

\[ \int_{\text{Barrier}} |\psi(x,k)|^2 dx = j \left( |T(K)|^2 \tau_T + |R(k)|^2 \tau_R \right), \]  

(10.1)

which does a priori require at least that \( \tau_T = \tau_R \), independently of the form of the potential
barrier.

Besides that, eq.(1.1) had been obtained by Büttiker[21] in 1983 following a method
which raised some critical comments[5,30,31]: In fact, even if expressed in terms of \( \psi(x,t) \),
such a definition (1.1) does not appear to fully account for the time evolution of the
wavepacket. Moreover, except for relation (1.2), the mentioned definition does not sug-

\[ \tau_T = \int_{x_i}^{x_f} dt J^\text{III}_T(x,t) - \int_{x_i}^{x_f} dt J^\text{in}(x,t) = \int_{0}^{\infty} dE v|g(E)T|^2 \left( \langle v^{-1} \rangle_T + \langle \delta T \rangle_T \right), \]  

(10.2a)

and

\[ \tau_R = \int_{x_i}^{x_f} dt J^\text{I}_R(x,t) - \int_{x_i}^{x_f} dt J^\text{in}(x,t) = \int_{0}^{\infty} dE v|g(E)R|^2 \left( \langle v^{-1} \rangle_R + \langle \delta R \rangle_R \right). \]  

(10.2b)

In fact, since \( J(x,t)dt \) represents the probability density for a particle to pass through
the point \( x \) during the time interval \( (t, t + dt) \), in order to determine the average time at
which a wavepacket \( \Psi(x,t) \) reaches the point \( x \) we have to perform a weighted average
over the variable \( t \) by means of

\[ w(x,t) = \frac{J(x,t)}{\int_{-\infty}^{\infty} J(x,t)dt}. \]  

(10.3)

Soon after, however, the same authors[5] noticed definitions (10.2) to hold only when
the incident and transmitted wavepackets are totally separated both in space and in
time. Indeed, when \( x_i \) and \( x_f \) are not far enough from the barrier walls, it is possible
to have interference effects between the incident and the reflected part. Moreover, the sign of the current density \( J(x, t) \) can change during the time evolution of the packet (for instance when the peak of the incident wave reaches the front-edge of the barrier). As a consequence, the integral \( \int_{-\infty}^{\infty} dt \, t J(x, t) \), which represents the algebraic sum of positive and negative quantities (fluxes), as well as the probability densities \( w(x, t) \), may be no longer positive-definite quantities: And each probability density would be endowed with a physical meaning only during the time intervals in which the relevant current does not invert its direction. Therefore, it appears necessary to break the mentioned integral into several integrals, each of them taken over a time interval during which the sign of \( J(x, t) \) is only positive or only negative. In such a way we’ll obtain probability densities everywhere positive-definite:

\[
    w_+(x, t) = \frac{J_+(x, t) dt}{\int_{-\infty}^{\infty} dt J_+(x, t)} , \quad w_-(x, t) = \frac{J_-(x, t) dt}{\int_{-\infty}^{\infty} dt J_-(x, t)} ,
\]

where \( J_+ \) and \( J_- \) represent the positive and negative values of \( J(x, t) \), respectively. Taking such considerations into account, Olkhovsky and Recami\[5\] were led to propose as average transmission and reflection times the new expressions

\[
    \tau_T = \frac{t(x_f)_+ - t(x_i)_+}{\int_{-\infty}^{\infty} dt J_+(x_i, t)} - \frac{t(x_f)_+ - t(x_i)_+}{\int_{-\infty}^{\infty} dt J_+(x_i, t)} = \frac{\int_{-\infty}^{\infty} dt t J_+(x_i, t)}{\int_{-\infty}^{\infty} dt J_+(x_i, t)} (10.4a)
\]

and

\[
    \tau_R = \frac{t(x_i)_+ - t(x_i)_+}{\int_{-\infty}^{\infty} dt J_-(x_i, t)} - \frac{t(x_i)_+ - t(x_i)_+}{\int_{-\infty}^{\infty} dt J_-(x_i, t)} = \frac{\int_{-\infty}^{\infty} dt t J_-(x_i, t)}{\int_{-\infty}^{\infty} dt J_-(x_i, t)} . (10.4b)
\]

Before going on, it is important to stress that, starting only from the continuity equation

\[
    \frac{\partial \rho(x, t)}{\partial t} + \frac{\partial J(x, t)}{\partial x} = 0
\]

and from the standard quantum-mechanical probabilistic interpretation of \( \rho(x, t) \), one can easily prove that the above quantities \( w_\pm(x, t) \) correspond just to the probability that our particle (moving forwards or coming backwards, respectively) is located at point \( x \) during the time interval \((t, t + dt)\). Actually, during each time interval in which it is either \( J = J_+ \) or \( J = J_- \), we can apply the continuity equation [which is always valid] to \( J_\pm \):

\[
    \frac{\partial \rho_+(x, t)}{\partial t} = - \frac{\partial J_+(x, t)}{\partial x} , \quad \frac{\partial \rho_-(x, t)}{\partial t} = - \frac{\partial J_-(x, t)}{\partial x} , \quad (10.5a)
\]

and

\[
    \frac{\partial \rho_+(x, t)}{\partial t} = - \frac{\partial J_+(x, t)}{\partial x} , \quad \frac{\partial \rho_-(x, t)}{\partial t} = - \frac{\partial J_-(x, t)}{\partial x} , \quad (10.5b)
\]
obtaining in such a way the two quantities $\partial \rho_>(x, t) / \partial t$ and $\partial \rho_<(x, t) / \partial t$. On integrating with respect to time over the interval $(-\infty, t)$, we can then define:

\[ \rho_>(x, t) = - \int_{-\infty}^{t} \frac{\partial J_+(x, t)}{\partial x} dt', \quad (10.6a) \]

\[ \rho_<(x, t) = - \int_{-\infty}^{t} \frac{\partial J_-(x, t)}{\partial x} dt'. \quad (10.6b) \]

Let us also impose the constraints $\rho_>(x, -\infty) = 0$ and $\rho_<(x, -\infty) = 0$: Let us suppose, in other words, that initially the particle (or wavepacket) is infinitely far from $x$. By integrating now with respect to $x$ we obtain two more quantities which we’ll call $N_>(x, \infty; t)$, $N_<(\infty, x; t)$ and for which one has

\[ N_>(x, \infty; t) = \int_{-\infty}^{x} \rho_>(x', t) dx' = \int_{-\infty}^{t} J_+(x, t') dt' > 0, \quad (10.7a) \]

\[ N_<(\infty, x; t) = \int_{-\infty}^{x} \rho_<(x', t) dx' = - \int_{-\infty}^{t} J_-(x, t') dt' > 0. \quad (10.7b) \]

The last two expressions will give us the probability —as a function of the current densities $J_\pm(x, t)$— that our particle, moving forward or backwards, be located at time $t$ on the right or on the left of $x$, respectively. Let us notice that the constraints $\rho_>(x, -\infty) = 0$, $\rho_<(x, -\infty) = 0$, which we imposed before integrating, are equivalent now to $J_\pm(-\infty, t) = 0$. Finally, on differentiating again eq.(10.6) (now with respect to time), one gets

\[ J_+(x, t) = \frac{\partial}{\partial t} N_>(x, \infty; t) > 0, \quad (10.8a) \]

\[ J_-(x, t) = \frac{\partial}{\partial t} N_<(\infty, x; t) > 0, \quad (10.8b) \]

and then

\[ w_+(x, t) = \frac{\frac{\partial}{\partial t} N_>(x, \infty; t)}{N_>(x, -\infty, \infty)}, \quad (10.9a) \]

\[ w_-(x, t) = \frac{\frac{\partial}{\partial t} N_<(\infty, x; t)}{N_<(\infty, -\infty, x)} \quad (10.9b) \]

Such relations are sufficient for justifying the quantum-mechanical probabilistic interpretation of $w_+(x, t)$ and $w_-(x, t)$.
At this point we can define the *average value* of the time at which the particle is at the point \( x \), while moving in the positive or negative direction along the chosen axis:

\[
\overline{t_+ (x)} = \frac{\int_{-\infty}^{\infty} t J_+ (x, t) dt}{\int_{-\infty}^{\infty} J_+ (x, t) dt}, \quad (10.10a)
\]

\[
\overline{t_- (x)} = \frac{\int_{-\infty}^{\infty} t J_- (x, t) dt}{\int_{-\infty}^{\infty} J_- (x, t) dt}. \quad (10.10b)
\]

We are endowed now with all the means needed to define even the *variances* of the distributions related to the above-mentioned times:

\[
\sigma^2 (t_+ (x)) = \frac{\int_{-\infty}^{\infty} t^2 J_+ (x, t) dt}{\int_{-\infty}^{\infty} J_+ (x, t) dt} - \overline{(t_+ (x))^2}, \quad (10.11a)
\]

\[
\sigma^2 (t_- (x)) = \frac{\int_{-\infty}^{\infty} t^2 J_- (x, t) dt}{\int_{-\infty}^{\infty} J_- (x, t) dt} - \overline{(t_- (x))^2}. \quad (10.11b)
\]

We then succeeded in constructing a formalism which allows us obtaining both the average values and the variances (and other possible higher-order moments) for the “time distributions” of all the possible processes relevant to one-dimensional tunnelling. The same definitions, anyway, can be extended to any other collision processes, even different from tunnelling and in the presence of any kind of potentials. As we have already seen, for the tunnelling and reflection times one has

\[
\overline{\tau_{T}} (x_i, x_f) = \overline{t (x_f)}_+ - \overline{t (x_i)}_+ = \tau_{Tun} (0, d) = \tau_{Pen} (0, d).
\]

with \(-\infty < x_i < 0 \) and \( d < x_f < \infty \) and, according to eq.(10.11),

\[
\sigma^2 (\tau_{T} (x_i, x_f)) = \sigma^2 (t_+ (x_f)) + \sigma^2 (t_+ (x_i)). \quad (10.12)
\]

Moreover, in the case \( x_i = 0, \) \( x_f = d \), we may write

\[
\begin{align*}
\tau_{Tun} (0, d) &= \overline{t (d)}_+ - \overline{t (0)}_+ \\
\sigma^2 (\tau_{Tun} (0, d)) &= \sigma^2 (t_+ (d)) + \sigma^2 (t_+ (0)).
\end{align*}
\]

(10.13)

Taking as an example \( x_i = 0 \) and \( 0 < x_f < d \), we can obtain the penetration times inside the barrier as

\[
\tau_{Pen} (0, x_f) = \overline{t (x_f)}_+ - \overline{t (0)}_+.
\]

(10.14)
or, analogously, for $0 < x < d$, the times

$$
\tau_{\text{Ret}}(x, x) = t(x)_- - t(x)_+
$$

while, for $-\infty < x_i < d$, it will be

$$
\tau_R(X_i, x_i) = t(x_i)_+ - t(X_i)_+.
$$

At last, let us re-examine, on the basis of the definitions reported above, the previous definitions of phase time and dwell time. As far as the former is concerned, it appears once again its merely asymptotic character. Indeed, being the phase time deduced within to an explicitly stationary context, on the basis of our previous results it can get a physical meaning only when $x_i \to \infty$; that is, when $J_+ (x, t)$ is the current density of the initial packet in the absence of any interference (between the transmitted and the reflected part) due to reflected waves. Similarly, the dwell time, represented by the equivalent expression\cite{32,33}

$$
\tau^D(x_i, x_f) = \left[ \int_{-\infty}^{\infty} t J(x_f, t) \, dt - \int_{-\infty}^{\infty} t J(x_i, t) \, dt \right] \left[ \int_{-\infty}^{\infty} J_{\text{in}}(x_i, t) \, dt \right]^{-1} ,
$$

with $-\infty < x_i < 0$, and $x_f > d$, is not, in general, physically meaningful: In fact, the weight in the time averages is positive-definite, and normalized to 1, only in the rare cases in which $x_i \to -\infty$ and $J_{\text{in}} = J_{\text{III}}$ (i.e., when the barrier is “transparent”).

### 11 Penetration and return times: numerical results

The penetration and return times will be extensively analyzed elsewhere\cite{8,31}; here they will be studied only briefly.

Equations (10.3) and (10.12-15) do not allow an easy analytical evaluation of expressions for the tunnelling times (namely for penetration, return and reflection), not even in the rather simple case of a square barrier. Let us then present the results of numerical calculations for the average duration of several penetration and return processes for gaussian packets inside a square barrier, performed by Olkhovsky et al. in the second one of refs.\cite{5}. Such calculations confirmed the existence of the Hartman effect, and seem to be in agreement (due to theoretical connection between tunnelling and evanescent-wave
propagation) with the experimental data of Cologne, Berkeley, Florence, Vienna, Orsay, Rennes, etc.

Let us remember that, following the previous notations, it holds

$$\Psi_{in}(x,t) = \int_{0}^{\infty} C f(k - \bar{k}) \exp[ikx - iEt/\hbar] \, dk,$$

where

$$f(k - \bar{k}) = \exp \left[ -\frac{(k - \bar{k})^2}{2(\Delta k)^2} \right],$$

$$E = \hbar^2 k^2 / 2m,$$  

$C$ is a normalization constant, and $m$ is, in this case, the electron mass. The penetration lengths will be expressed in angstroms, and the penetration times in seconds.

In Fig.6a the plots are shown of $\tau_{Pen}(0, x)$, with $0 < x < d$, corresponding to $d = 5 \, \text{Å}$, for $\Delta k = 0.02 \, \text{Å}^{-1}$ and $0.01 \, \text{Å}^{-1}$. Notice right now that the penetration time $\tau_{Pen}(0, x)$ does always show a clear tendency to saturation. In Fig.6b we depict, instead, the plot corresponding to $d = 10 \, \text{Å}$ and $\Delta k = 0.01 \, \text{Å}^{-1}$. It is interesting to observe that, for constant $\Delta k$, the values of the total penetration time $\tau_{Pen} \equiv \tau_{Pen}(0, d)$ remain practically unchanged, when going on from $d = 5 \, \text{Å}$ to $d = 10 \, \text{Å}$: a result which brings more evidence, once again, in support of the so-called Hartman effect. Analogous results have been obtained even for $d > 10 \, \text{Å}$, by varying the parameter $\Delta k$ between $0.005$ and $0.050 \, \text{Å}^{-1}$, and the energy $E$ in the range $1$ to $10 \, \text{eV}$.

In Figs. 7, 8 and 9 the behaviour is shown of the average durations of the penetration and return processes, as a function of the penetration length (with $x_i = 0$, and $0 \leq x_f \equiv x \leq d$), for barriers with height $V_0 = 10 \, \text{eV}$, and width $d = 5 \, \text{Å}$ or in some cases $10 \, \text{Å}$. In particular:

— In Fig.7, the plots are presented of $\tau_{Pen}(0, x)$, corresponding to different values of the average kinetic energy: $\bar{E} = 2.5$, $5$ and $7.5 \, \text{eV}$ with $\Delta k = 0.02 \, \text{Å}^{-1}$ (lines 1, 2 and 3); and $\bar{E} = 5 \, \text{eV}$ with $\Delta k = 0.04 \, \text{Å}^{-1}$ (line 4). In all the four cases it is $d = 5 \, \text{Å}$.

— In Fig.8 we show the plots of $\tau_{Pen}(0, x)$ corresponding to: $d = 5 \, \text{Å}$, with $\Delta k = 0.024$ and $0.04 \, \text{Å}^{-1}$ (lines 1 and 2); and to $d = 10 \, \text{Å}$, with $\Delta k = 0.02$ and $0.04 \, \text{Å}^{-1}$ (lines 3 and 4). The average kinetic energy $\bar{E}$ is $5 \, \text{eV}$, i.e., half the barrier energy $V_0 = 10 \, \text{eV}$.

— Finally, in Fig.9 we present some plots of $\tau_{Ret}(x, x)$. The lines 1, 2 and 3 correspond
to: $E = 2.5$, 5 and 7.5 eV, respectively, with $\Delta k = 0.02 \text{Å}^{-1}$ and $d = 5 \text{Å}$. The lines 4, 5 and 6 correspond, instead, to: $E = 2.5$, 5 and 7.5 eV, with $\Delta k = 0.04 \text{Å}^{-1}$ and $d = 5 \text{Å}$; while the lines 7 and 8 correspond to $\Delta k = 0.02$ and 0.04 $\text{Å}^{-1}$, respectively, with $E = 5$ eV, and, this time, $d = 10\text{Å}$.

With regard to the employed numerical methods, those authors remarked the integration in $dt$ to have been performed using the time interval $[-10^{-13}, +10^{-13}]$ s, symmetric with respect to $t = 0$: An interval three orders of magnitude larger than the temporal width of the wavepacket, which, in its turn, is of the order of $1/(\bar{v} \Delta k) = (\Delta k \sqrt{2E/m})^{-1} \sim 10^{-16}$ s. This is equivalent to consider the evolution of the wavepacket along the time interval $[-\infty, +\infty]$, without assigning to it any finite starting time $t$; in agreement with the relations $J_\pm(-\infty, t) = 0$ and, equivalently, with $\rho_\pm(x, -\infty) = 0$. Let us moreover remark that the considered packet has been constructed in order that its centroid arrives at $x_0$ at time $t = 0$.

From the above Figures 7)–9) it can be inferred that:

1) the average duration of a tunnelling process $\tau_{\text{Tun}}(0, d)$ does not depend on the width $d$ of the barrier (Hartman effect);

2) the penetration time increases quickly only at the beginning of the barrier, namely, in the barrier region close to $x = 0$;

3) $\tau_{\text{Pen}}(0, x)$ tends to a saturation value in the final part of the barrier, that is, for $x \to d$.

With regard to the effects mentioned at points 1)-3) above, they could be caused, according to the same authors, by interference inside the barrier between the forward propagating [sometimes called, loosely speaking, ”incoming” or ”entering” or penetrating] and the backwards propagating [”returning” or reflected] waves, whose superposition produces the $J_+$ and $J_-$ fluxes. See also the pictures (in particular Fig.3, p.351) in the second one of refs.[5].

Eventually, in connection with the plots of $\tau_{\text{Ret}}(x, x)$ as a function of $x$, shown in Fig.9, we notice that:
4) the average duration of the reflection, $\overline{\tau_R}(0,0) \equiv \overline{\tau_{\text{Ret}}}(0,0)$, does not depend on the barrier width $d$;

5) between 0 and $\sim 0.6d$ the value of $\overline{\tau_{\text{Ret}}}(0,x)$ is approximately constant;

6) the value of $\overline{\tau_{\text{Ret}}}(0,x)$ increases with $x$ only in the region close to $x = d$ (even if, it was stressed by the mentioned authors, the calculations in such a region are not very accurate, because quantity $\int_{-\infty}^{\infty} J_{-}(x,t)dt$ takes on very small values in correspondence to it).

Notice that point 4) —a result expected, like point 1), for quasi-monochromatic particles also on the basis of Dumont and Marchioro’s paper[34]— agrees with the data obtained by Steinberg et al.[35] for arbitrary wavepackets. As before, also points 5) and 6) can be due to interference phenomena inside the barrier. Actually, if the returning wavepacket is almost totally damped down near $x = d$ by the penetrating wavepacket, only a negligible part of its back-tail (composed by the slower components) will survive. When $x$ decreases ($x \to 0$), the non-damped part of the returning packet is larger (including the faster components), so that the difference $\overline{\tau_{\text{Ret}}}(0,x) - \overline{\tau_{\text{Pen}}}(0,x)$ remains approximately constant. Moreover, the interference between the incident and the reflected waves in the region $x \leq 0$ causes $\overline{t_{-}}(0)$ to be larger than $\overline{t_{+}}(0)$: This could explain, going back to our terminology, the larger values of $\overline{\tau_R}(0,0)$ with respect to $\overline{\tau_{\text{Tun}}}(0,d)$. Let us recall, and stress, that the mentioned interference between reflected and incoming waves does cause an acceleration of the wavepacket peak, so that in general its speed becomes Superluminal even before (near) the barrier.

12 On the general validity of the Hartman effect

Let us recall that we named Hartman Effect (HE) the independence of the mean tunnelling time from the barrier width, so that for large (opaque) barriers the effective tunnelling–velocity can become arbitrarily large. Such effect has been analyzed in the first part of Sect.3. Now we shall briefly discuss the validity of the HE for all the other theoretical expressions proposed for the mean tunnelling times. Let us first consider the mean dwell time, the mean Larmor time, and the real part of the complex tunnelling time obtained by averaging over the Feynman paths. All of them, in the case of quasi-monochromatic
particles and opaque rectangular barriers, become equal to $\hbar k/(\kappa V_0)$: And one immediately verifies\textsuperscript{#11} that also for all such mean tunnelling times there is no dependence on the barrier width, and consequently the HE is valid. The validity of the HE for the mean tunnelling time has been proved in 1992, as we know, in the nonrelativistic approach by Olkhovskiy and Recami, an approach developed in refs.\textsuperscript{[5,30,31]} and moreover confirmed by the numerical simulations performed in the same set of papers (for various cases of gaussian wavepackets).

By contrast, the “second Larmor time” $\tau^{La}_T$, as well as the Büttiker-Landauer time $\tau^{BL}_T$ and the imaginary part Im $\tau^{\Omega}_T$ of the complex tunnelling time obtained within the Feynman approach, which too are equal to $\tau^{La}_T$, yield the result $md/(\hbar \kappa)$ in the opaque rectangular-barrier limit\textsuperscript{[30,31]}: That is, they all are proportional to the barrier width $d$, so that the HE is not valid for them! However, it has been shown in refs.\textsuperscript{[5,8]} that\textsuperscript{#12} such last three times are not mean times, but rather standard deviations (or “mean square fluctuations”) of the tunnelling-time distributions, In conclusion, the latter three times are not connected with the peak (or group) velocity of the tunnelling particles, but with the spread of the tunnelling velocity distributions.

All the results have been obtained for transparent media (without absorption or dissipation). As it was theoretically demonstrated in ref.\textsuperscript{[36]} within nonrelativistic quantum mechanics, the HE vanishes for barriers with high enough absorption. This was confirmed experimentally for electromagnetic (microwave) tunnelling in ref.\textsuperscript{[37]}.

Let only add a comment. From some recent papers\textsuperscript{[38]}, it seems that the integral penetration time, needed to cross a portion of a barrier, in the case of a very long barrier starts to increase again —after the plateau corresponding to infinite speed— proportionally to the distance. This is due to the contribution of the above-barrier frequencies contained in the considered wavepackets, which become more and more important as the tunnelling components are progressively damped down. In this paper, however, we refer to the behaviour of the tunnelling (or, in the classical case, of the evanescent) waves only.

Actual deviations from the “Hartman effect”-behaviour, however, exist. They are very interesting: but will be considered in ref.\textsuperscript{[8]}, and elsewhere; here, we shall just mention some of them in our Sect.14.

\textsuperscript{#11}See refs.\textsuperscript{[47]} below, besides refs.\textsuperscript{[31]}.  
\textsuperscript{#12}See also refs.\textsuperscript{[47]}, below.
13 Optical equivalence of tunnelling

As already mentioned, several experimental evidences of the Hartman effect have been obtained during the last ten years, or so, in a series of measurements made in Cologne,[39] Berkeley,[40] Florence,[41], Vienna[42], Rennes, Orsay, etc. However, those measurements regarded the transmission times of electromagnetic waves (including optical photons) by examining the propagation of evanescent modes inside a “classical barrier”, like a segment of waveguide for “below cut-off” frequencies, or “frustrated refraction” regions, respectively. Various experiments employing quantum particles, such as, e.g., electrons, have been also proposed: But such measurements seem to be still difficult to perform, particularly because of the very small times involved in the tunnelling processes. For a Josephson junction, for instance, such times result to be of about 10 fs, and maybe of the order of 1 fs in other solid-state devices. For optical systems, instead, such times are already of the order of some ps, for frequencies in the visible range, and reach the ns in some of the experiments made in Florence and Cologne with microwaves.

Even without reviewing here such celebrated[43] experiments, and the results by them obtained, let us however recall—in connection with them—the equivalence between the classical transmission of evanescent electromagnetic modes and the quantum tunnelling of particles. As an example of systems with classical barriers, we shall fix our attention on waveguides. Let us consider a particle with mass \( m \) and kinetic energy \( E = \frac{\hbar^2 k^2}{2m} \).

For (one-dimensional) quantum propagation in the presence of a uniform potential \( V_0 \), the Schrödinger equation for such a particle writes:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0.
\]  

(12.1)

If we define

\[
\kappa^2 \equiv \frac{2m}{\hbar^2}(E - V_0),
\]  

(12.2)

Equation (12.1) results to be formally identical to the Helmholtz equation for any (electric or magnetic) component of an e.m. field propagating through a dispersive medium:

\[
\frac{\partial^2 \psi}{\partial x^2} + \kappa^2 \psi = 0.
\]  

(12.3)

with

\[
\kappa = \frac{2\pi}{\lambda_m} = \frac{2\pi}{\lambda} n,
\]
\(\lambda_m\) being the wave-length inside the medium, \(\lambda\) the wave-length in vacuum, and \(n\) the refraction index of the medium in which the field propagates. The comparison between the two equations suggests the obvious correspondence:

\[
\sqrt{\frac{2m}{\hbar \pi}} (E - V_0) \to \frac{2\pi}{\lambda} n.
\]

For a “square” waveguide, with dimensions \(a \times b\) (\(a < b\), and with perfectly conducting walls, we know that:

\[
\kappa = \frac{2\pi}{\lambda} \sqrt{1 - \left(\frac{\lambda}{2b}\right)^2} = \frac{2\pi}{\lambda} \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}, \tag{12.4}
\]

where \(\lambda_c = 2b\) is the “cut-off” wave-length above which the square-root term becomes negative and, as a consequence, \(\kappa\) turns out to be imaginary. Since \(\lambda = c/\nu = 2\pi c/\omega\), one has

\[
\kappa = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{b^2}} = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}, \tag{12.5}
\]

with \(\omega_c = \pi c/b\) (\(\nu_c = c/2b\)), that yields, in its turn, the cut-off frequency below which \(\kappa\) becomes imaginary [as it is known to happen for a tunnelling particle too].

Notice that the dispersion relation for a square waveguide is surprisingly similar[44] (as noticed also in the “Feynman Lectures”[7]) to the one of a relativistic particle, once we make the substitution

\[
\frac{\pi}{b} = \frac{\omega_c}{c} = \frac{mc}{\hbar}.
\]

In fact, on multiplying eq.(12.5) by \(\hbar\), we obtain

\[
\hbar \kappa = \sqrt{\frac{(\hbar \omega)^2}{c^2} - \frac{(2\hbar \pi)^2}{b^2}} \to p^2 = \sqrt{\frac{E^2}{c^2} - mc^2}.
\]

Let us emphasize that a comparison is being made between the Helmholtz equation, which is relativistic and classical [it is obtained from Maxwell equations], and the Schrödinger’s one, which is a non-relativistic and quantum equation. Let us remark that, on the basis of the Maxwell equations only, computer simulations have been carried on[45], which numerically verified the Superluminality of the evanescent waves along “under-sized” (sub-critical) waveguides; further computers simulations have been performed more extensively[46], with analogous results.
Let us go back to our eqs.(12.1) and (12.3). By differentiating eqs.(12.2) and (12.5), we obtain
\[ v_{\text{particle}}^{\text{group}} = \frac{d\omega}{d\kappa} = \frac{\hbar}{m\kappa}; \quad v_{\text{e.m.\,wave}}^{\text{group}} = \frac{d\omega}{d\kappa} = \frac{c^2}{\omega} \kappa. \]

Through the replacement
\[ \frac{\hbar}{m} \rightarrow \frac{c^2}{\omega} = \frac{c}{2\pi\nu}, \tag{12.6} \]
one can observe a mathematical identity between the electromagnetic results, for transmission of microwaves through a waveguide, and the quantum-theoretical predictions for one-dimensional motion of a particle in the presence of a uniform potential. Actually, in both cases the solutions of the two equations are given by linear combinations of the wave-functions
\[ \psi(x, t) = e^{\pm i\kappa x} e^{i\omega t}. \]

In particular, as already mentioned, when the particle energy results smaller than \( V_0 \) or when, in electromagnetic case, the angular frequency \( \omega \) becomes smaller than \( \omega_c \), quantity \( \kappa \) becomes imaginary and the wavepacket exponentially decrease as \( e^{-|\kappa|x} \) (tunnelling, or evanescent wave propagation, respectively). Notice that the field (or, analogously, the particle wave-function) penetrate in a similar way along an undersized segment, i.e., the forbidden region, of the waveguide (or inside the quantum barrier, for particles), up to a distance of the order of \( |\kappa|^{-1} \). Obviously, despite of the formal analogies, there are physical differences between the tunnelling of electrons and the propagation of microwaves below cut-off in waveguides. In fact, as already said, the Helmholtz wave-equation and the Schrödinger particle-equation are mathematically the same equation; but, whilst in the first case what propagates is a field component (it is the field itself), in the second case, instead, it is the particle wave-function. Nevertheless, since in both cases we have to do with the time-evolution of wavepackets, nothing prevents us from interpreting the results of experiments with electromagnetic waves (microwaves, e.g.) as “physical simulations” of electron tunnelling. The results of such “simulations”\[39-43\] reproduce quite well also the quantum-theoretical predictions, so that the equivalence between the two cases is verified.

More subtle is the circumstance that in the time-dependent case the Schroedinger equation (in the presence of barrier) and the Helmholtz equation (for electromagnetic

\[^{#13}\text{Notice that eq.(12.6) is equivalent to the relation } \hbar \omega = mc^2.\]
waves in a waveguide) are no more mathematically equivalent, since the time-derivative is of the first order in the former case and of the second order in the latter case. Anyway, it can be seen[47,31,8] that those equations still admit analogous classes of solutions, different only for their “spreading” properties.

Notice explicitly that all the same can be repeated, of course, for other classical barriers, as any band-gap filters (cf., e.g., refs.[35,40-42]).

We cannot skip mentioning, at last, the surprising results of one-dimensional non-resonant tunnelling through two successive opaque potential barriers (see the last four refs.[31]), separated by an intermediate free region $R$, by analyzing the relevant solutions to the Schroedinger equation. In Olkhovsky, Recami and Salesi[31], it has been found that the total traversal time does not depend not only on the opaque barrier widths (“Hartman effects”), but also on the width $R$ of region $R$: so that the effective velocity in the region $R$, between the two barriers, can be regarded as infinite. This agrees with the results known from the corresponding metallic waveguide experiments[39], which simulated the mentioned tunnelling experiment just because of the known mathematical identity between Schroedinger and Helmholtz equation. It is worth mentioning that the above prediction of quantum mechanics have been theoretically confirmed and generalized (and explained in terms of “superoscillations”) by Aharonov et al.[31]: Indeed, the claim of those authors is that, according to QM, a wavepacket can travel, in zero time and negligible distortion, a distance arbitrarily larger than the width of the wavepacket. From the experimental point of view, Olkhovsky, Recami and Salesi’s[31] prediction has been re-verified by Longhi et al.[31] on using as (classical) barriers two gratings in an optical fiber. We shall spend a few more words about these question in the Appendix.

14 A brief mention of some experimental results

As we have seen, the problem stated at the beginning —how much time does a particle spend to cross a potential barrier— even though substantially solved, still remains debated. On the one hand, it actually exists experimental evidence supporting, e.g., the simple definition of phase time. On the other hand, this very definition, and the same experimental results, force us to accept, in certain cases, the arising of group-velocities
larger than the speed of light in vacuum (Hartman effect, as it was called by Olkhovsky and Recami). It must be noticed however that the appearance of Superluminal, or even of negative group-velocities in classical optics is not a new phenomenon: It was studied and experimentally observed in works\cite{48,49} that only recently received the attention they deserved. As a matter of fact, in a dispersive linear non-absorbing medium

\[ v_g = \frac{d\omega}{dk} = \frac{c}{n(\omega) + \omega n'(\omega)}, \]

and in regions with strong anomalous dispersion (near a resonance), the group-velocity can exceed \( c \) or even, as already said, become negative. Various authors have proven or proposed that in those cases the group-velocity may lose its ordinary physical meaning, so that no signal (no information) can indeed be transmitted by the medium with velocity larger than \( c \) [we are referring ourselves to the mathematical formulations, and interpretations, in Chap.7 of Jackson’s Classical Electrodynamics, or in Chap.3 of Sommerfeld’s book]. In those cases, moreover, the phenomenon appears to be easily explained in terms of the reshaping produced by the attenuation of the less energetic, and slower, components. The outgoing signal, therefore, may seem to have travelled at velocities larger than the velocity at which its more energetic components did actually travel.

In the tunnelling case, however, we have seen in Sect.3 that (both for particles and for e.m. waves) we can always be able to avoid the effects due to possible reshaping, or to the transmission of initially faster particles. This is experimentally supported also from the fact that, as it has been seen, e.g., in the experiments by Enders and Nimtz, the width of the “signals” remains unaffected (something relevant when thinking of a transmission by Morse’s alphabet), even if their amplitude decreases. One has to notice that the barrier-crossing of particles is a statistical process, in the sense that one cannot know a priori which particle will pass through the barrier. This is true, of course; but the weight of such a consideration becomes lower when it grows the number of the particles at our disposal for attempting a “signal” transmission: To remain within the Morse alphabet example, one can send out dots and dashes by emitting pulses of, say, one thousand and ten thousand particles each, respectively; the dot and dashes will then be recognized also after the tunnelling. This becomes even more meaningful when one approaches the classical limit. The claim that Superluminal tunnelling cannot be used to transmit any information is in need, therefore, for further discussion; more details can be found in
In tunnelling, moreover, the appearance of Superluminal group-velocities takes place in general when the probability associated with such an event is rather low. This circumstance is at the basis of the interpretation put forward by Steinberg et al.[10]. According to them, in the Superluminal tunnelling processes the corresponding crossing times should be regarded as weak values, coming from weak measurements. The concept of weak measure had been introduced in 1988 by Aharonov et al.[52] starting from the “classical” measurement theory by von Neumann[53]. According to the authors of ref.[10], when we make a measurement on a subset associated with low probability (as, e.g., the one composed by the transmitted particles), and this subset belongs to a set on which it has been made a weak measurement (that is, a measurement with a large uncertainty, that leaves quasi-undisturbed the whole set), it is possible to obtain as the result of the measurement on the subset a value completely different from all the eigenvalues accessible to the system. Such a value would not be however a value really assumed by the system, corresponding to a wave-function that is not an eigenstate.

According to other authors, on the contrary, the Superluminal velocities associated with tunnelling would be actually real; while what is to be re-interpreted is the causality principle.[54-56,50] To this purpose, let us remember that Special Relativity can be extended —without dropping the ordinary Postulates— so as to include Superluminal motions. Such an “extended relativity”[55], in other words, can incorporate tachyons without destroying Einstein’s relativity, but only extending it to the new velocity realm. In particular, it is possible to solve the causal problems and the so-called causal paradoxes.[56,50] It is also worthwhile recalling that extended relativity itself predicts, on the basis of simple classico-geometrical considerations, the transition for any Superluminal object (wave pulse, or particle) from positive to negative group-velocities (whenever the object “overcomes” the infinite speed): see refs.[55,57]. The fact that to negative speeds there correspond negative crossing times[57], becomes interesting in the light of many subsequent, more or less recent, experimental results[58].

We shall come back to such problems in the Appendix, where we present a bird’s-eye view of all the sectors of physics in which Superluminal motions seem to appear (cf. also ref.[59]).
ACKNOWLEDGEMENTS

The Authors are grateful to F.Bassani, A.Paoletti, R.A.Ricci and to C.Vasini for stimulating discussions and their kind interest. Thanks are also due for scientific collaboration to A.Agresti, V.Abate, M.Baldo, J.D.Bekenstein, N.Ben-Amots, A.Bertin, G.Bonera, R.Bonifacio, L.Bosi, M.Brambilla, G.Brown, D.Campbell, G.Cavalleri, R.Chiao, C.Cocca, C.Conti, C.A.Dartora, G.Degli Antoni, S.Esposito, J.R.Fanchi, F.Fontana, R.Garavaglia, A.Gigli Berzolari, L.Horwitz, H.E.Hernández, L.C.Kretly, G.Kurizki, J.Jakiel, J.-y.Lu, G.D.Maccarrone, A.van der Merwe, D.Mugnai, G.Nimtz, K.Z.Nóbrega, V.Petrillo, M.Pernici, A.Ranfagni, F.Raciti, B.Reznik, A.Shaarawi, P.Saari, D.Stauffer, A.M.Steinberg, M.T.Vasconselos, M.Villa, A.Vitale, A.K.Zaichenko and M.Zamboni-Rached.
Captions of the Figures (of the Text)

Fig.1 – One-dimensional tunnelling of a particle through a rectangular potential barrier of height $V_0$ and width $d$. (The propagation axis is considered to be the $x$-axis).

Fig.2 – Time evolution of a minimum-uncertainty wavepacket (at $t = 0$) incident on an infinitely high rectangular barrier which extends on the right of $z = 0$. The initial width and centroid of the packet are $3 \text{Å}$ and at $-20 \text{ Å}$, respectively. The average energy of the incident packet is $4 \text{ eV}$. (In this figure, the propagation axis has been called $z$).

Fig.3 – Transmission coefficients $T(k) \equiv A(k)$ calculated for different values of the thickness $d$ of a barrier consisting in a square potential with height $V_0 = \hbar^2 \varepsilon^2 / 2m$. In this figure it is moreover presented the plot of $f(k - k_0) = e^{-(k-k_0)^2 / 2(\Delta k)^2}$, with $k_0 = 0.7 \varepsilon$ and $\Delta k = 0.1 k_0$.

Fig.4 – Plots of $T(k) f(k - k_0)$, for below-barrier energies, as a function of the wave-number $k$, for different values of $k_0$ and different values of the barrier thickness $d \equiv a$ (Fig.4a: $k_0 = 0.3 \varepsilon$; Fig.4b: $k_0 = 0.5 \varepsilon$; Fig.4c: $k_0 = 0.7 \varepsilon$; Fig.4d: $k_0 = 0.9 \varepsilon$), with $\Delta k = 0.1 k_0$; and, from up to down: $a = 1/\varepsilon$, $a = 3/\varepsilon$, $a = 6/\varepsilon$, $a = 10/\varepsilon$, $a = 15/\varepsilon$, $a = 20/\varepsilon$, $a = 25/\varepsilon$. Let us recall (see the text) that $\varepsilon \equiv 2mV_0/\hbar$. Notice that the line corresponding to the last value does not appear in Fig.4a, because of the strong attenuation of the peak.

Fig.5 – The quantum clock of Baz’ and Rybachenko: a particle entering the barrier starts to suffer a Larmor precession in the presence of a magnetic field confined inside the barrier. The spin of a tunnelling particle is turned parallel to the direction of the field. Notice that, following the original paper, in this figure the propagation axis has been called $y$.

Figs.6 – Fig. a): Behaviour of the average penetration time, $\bar{\tau}_{\text{pen}}(0, x)$ (expressed in seconds), as a function of the penetration length $x_t = x$ (expressed in angstrom) for a
square barrier with width $d = 5 \, \text{Å}$, with $\Delta k = 0.02 \, \text{Å}^{-1}$ (broken line) and $\Delta k = 0.01 \, \text{Å}^{-1}$ (continuous line). It is worth noticing that $\tau_{\text{Pen}}(0, x)$ increases quickly for the first initial angstroms ($\sim 2.5 \, \text{Å}$), and afterwards approaches a "saturation" value. This supports the existence of the Hartman effect. Fig. b): As in 6a), with $\Delta k = 0.01 \, \text{Å}^{-1}$, but with a barrier endowed with a double thickness, $d = 10 \, \text{Å}$ . Notice that the numerical values of the total tunnelling time $\tau_{\text{Tun}}(0, d)$ remain practically unchanged when we go on from $d = 5 \, \text{Å}$ to $d = 10 \, \text{Å}$, as a further evidence of the appearance of the Hartman effect.

Fig. 7 – Behaviour of $\tau_{\text{Pen}}(0, x)$ (expressed in seconds) as a function of $x$ (in angstroms), relative to tunnelling through a barrier of thickness $d = 5 \, \text{Å}$ and for several values of $\overline{E}$ and of $\Delta k$: line 1: $\Delta k = 0.02 \, \text{Å}^{-1}$ and $\overline{E} = 2.5 \, \text{eV}$; line 2: $\Delta k = 0.02 \, \text{Å}^{-1}$ and $\overline{E} = 5.0 \, \text{eV}$; line 3: $\Delta k = 0.02 \, \text{Å}^{-1}$ and $\overline{E} = 7.5 \, \text{eV}$; line 4: $\Delta k = 0.04 \, \text{Å}^{-1}$ and $\overline{E} = 5.0 \, \text{eV}$.

Fig. 8 – Behaviour of $\tau_{\text{Pen}}(0, x)$ (in seconds) as a function of $x$ (in angstroms) for $\overline{E} = 5 \, \text{eV}$, and for different values of $d$ and $\Delta k$: line 1: $d = 5 \, \text{Å}$, $\Delta k = 0.02 \, \text{Å}^{-1}$; line 2: $d = 5 \, \text{Å}$, $\Delta k = 0.04 \, \text{Å}^{-1}$; line 3: $d = 10 \, \text{Å}$, $\Delta k = 0.02 \, \text{Å}^{-1}$; line 4: $d = 10 \, \text{Å}$, $\Delta k = 0.04 \, \text{Å}^{-1}$.

Fig. 9 – Behaviour of $\tau_{\text{Ret}}(x, x)$ (in seconds) as a function of $x$ (in angstroms) for different values of $d$, $\overline{E}$ and $\Delta k$; namely: line 1: $d = 5 \, \text{Å}$, $\overline{E} = 2.5 \, \text{eV}$ and $\Delta k = 0.02 \, \text{Å}^{-1}$; line 2: $d = 5 \, \text{Å}$, $\overline{E} = 5.0 \, \text{eV}$ and $\Delta k = 0.02 \, \text{Å}^{-1}$; line 3: $d = 5 \, \text{Å}$, $\overline{E} = 7.5 \, \text{eV}$ and $\Delta k = 0.02 \, \text{Å}^{-1}$; line 4: $d = 5 \, \text{Å}$, $\overline{E} = 2.5 \, \text{eV}$ and $\Delta k = 0.04 \, \text{Å}^{-1}$; line 5: $d = 5 \, \text{Å}$, $\overline{E} = 5.0 \, \text{eV}$ and $\Delta k = 0.04 \, \text{Å}^{-1}$; line 6: $d = 5 \, \text{Å}$, $\overline{E} = 7.5 \, \text{eV}$ and $\Delta k = 0.04 \, \text{Å}^{-1}$; line 7: $d = 10 \, \text{Å}$, $\overline{E} = 5.0 \, \text{eV}$ and $\Delta k = 0.02 \, \text{Å}^{-1}$; line 8: $d = 10 \, \text{Å}$, $\overline{E} = 5.0 \, \text{eV}$ and $\Delta k = 0.04 \, \text{Å}^{-1}$.
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APPENDIX

A1. - Introduction.

The question of Super-luminal ($V^2 > c^2$) objects or waves has a long story, starting perhaps in 50 b.C. with Lucretius’ *De Rerum Natura* (cf., e.g., book 4, line 201). Still in pre-relativistic times, one meets various related works, from those by J.J.Thomson to the papers by the great A.Sommerfeld. With Special Relativity, however, since 1905 the conviction spread over that the speed $c$ of light in vacuum was the upper limit of any possible speed. For instance, R.C.Tolman in 1917 believed to have shown by his “paradox” that the existence of particles endowed with speeds larger than $c$ would have allowed sending information into the past. Such a conviction blocked for more than half a century —apart from an isolated paper (1922) by the Italian mathematician G.Somigliana— any research about Superluminal speeds. Our problem started to be tackled again essentially in the fifties and sixties, in particular after the papers[A1] by E.C.George Sudarshan et al., and later on[A2] by E.Recami, R.Mignani, et al. [who rendered the expressions subluminal and Superluminal of popular use by their works at the beginning of the Seventies], as well as by H.C.Corben and others (to confine ourselves to the theoretical researches). The first experiments looking for tachyons were performed by T.Alvåger et al. For references, one can check pages 162-178 in ref.[A1], where about 600 citations are listed; pages 285-290 in ref.[A3]; pages 592-597 of ref.[A4] or pages 295-298 of ref.[A5]; as well as the large bibliographies by V.F.Perepelitsa[A6] and as the book in ref.[A7]. In particular, for the causality problems one can see refs.[A1,A8] and references therein, while for a model theory for tachyons in two dimensions one can be addressed to refs.[A1,A9]. The first experiments looking for tachyons were performed by T.Alvåger et al.; some citations about the early experimental quest for Superluminal objects may be found e.g. in refs.[A1,A10].

In recent years, terms as “tachyon” and “Superluminal” fell unhappily into the (cunning, rather than crazy) hands of pranotherapists and mere cheats, who started squeezing money out of simple-minded people; for instance by selling plasters (!) that should cure various illnesses by “emitting tachyons”... We are dealing with them here,
however, since at least four different experimental sectors of physics seem to indicate the actual existence of Superluminal motions, thus confirming some long-standing theoretical predictions[A3]. In this rapid informative Appendix, after a sketchy theoretical introduction, we are going to set forth a reasoned outline of the experimental state-of-the-art: brief, but accompanied by a bibliography sufficient in some cases to provide the interested readers with coherent, adequate information; and without forgetting to call attention—at least in the two sectors more in fashion today—to some other worthy experiments.

A2. Special and Extended Relativity.

Let us state first that special relativity (SR), abundantly verified by experience, can be built on two simple, natural Postulates: 1) that the laws (of electromagnetism and mechanics) be valid not only for a particular observer, but for the whole class of the “inertial” observers: 2) that space and time are homogeneous and space is moreover isotropic. From these Postulates one can theoretically deduce that one, and only one, invariant speed exists: and experience tells us such a speed to be that, $c$, of light in vacuum; in fact, light possesses the peculiar feature of presenting always the same speed in vacuum, even when it runs towards or away from the observer. It is just that feature, of being invariant, that makes quite exceptional the speed $c$: no bradyons, and no tachyons, can enjoy the same property.

Another (known) consequence of our Postulates is that the total energy of an ordinary particle increases when its speed $v$ increases, tending to infinity when $v$ tends to $c$. Therefore, infinite forces would be needed for a bradyon to reach the speed $c$. This fact generated the popular opinion that speed $c$ can be neither achieved nor overcome. However, as speed-$c$ photons exist, which are born live and die always at the speed of light (without any need for accelerating from rest to the light speed), so particles can exist —tachyons[A4]— always endowed with speeds $V$ larger than $c$ (see Fig.A1). This circumstance has been picturesquely illustrated by George Sudarshan (1972) with reference to an imaginary demographer studying the population patterns of the Indian subcontinent: "Suppose a demographer calmly asserts that there are no people North of the Himalayas, since none could climb over the mountain ranges! That would be an absurd
conclusion. People of central Asia are born there and live there: they did not have to be born in India and cross the mountain range. So with faster-than-light particles. Let us add that, still starting from the above two Postulates (besides a third one, even more obvious), the theory of relativity can be generalized in such a way to accommodate also Superluminal objects; such an extension is largely due to a series of works performed mainly in the Sixties–Seventies. Also within such an “Extended Relativity” the speed \( c \), besides being invariant, is a limiting velocity: but every limiting value has two sides, and one can a priori approach it both from the left and from the right.

Actually, the ordinary formulation of SR is too much restricted. For instance, even leaving tachyons aside, it can be easily so widened as to include antimatter. Then, one finds space-time to be a priori populated by normal particles \( P \) (which travel forward in time carrying positive energy), and by dual particles \( Q \) “which travel backwards in time carrying negative energy”. The latter shall appear to us as antiparticles, i.e., as particles—regularly travelling forward in time with positive energy, but—with all their “additive” charges (e.g., the electric charge) reversed in sign: see Fig. A2. To clarify this point, let us recall that we, macroscopic observers, have to move in time along a single, well-defined direction, to such an extent that we cannot even see a motion backwards in time; and every object like \( Q \), travelling backwards in time (with negative energy), will be necessarily reinterpreted by us as an anti-object, with opposite charges but travelling forward in time (with positive energy).

But let us forget about antimatter and go back to tachyons. A strong objection against their existence is based on the opinion that by tachyons it should be possible to send signals into the past, owing to the fact that a tachyon \( T \) which—say—appears to a first observer \( O \) as emitted by \( A \) and absorbed by \( B \), can appear to a second observer \( O' \) as a tachyon \( T' \) which travels backwards in time with negative energy. However, by applying (as it is obligatory to do) the “reinterpretation rule” or switching procedure seen above, \( T' \) will appear to the new observer \( O' \) just as an antitachyon \( \bar{T} \) emitted by \( B \) and absorbed by \( A \), and therefore travelling forward in time, even if in the contrary space direction. In such a way, every travel towards the past, and every negative energy, do disappear.

Starting from this observation, it is possible to solve the so-called causal paradoxes
associated with Superluminal motions: paradoxes which result to be the more instructive and amusing, the more sophisticated they are; but that cannot be re-examined here (some of them having been proposed by R.C.Tolman, J.Bell, F.A.E.Pirani, J.D.Edmonds and others).[A6,A3] Let us only mention here the following. The reinterpretation principle —according to which, in simple words, signals are carried only by objects which appear to be endowed with positive energy— does eliminate any information transfer backwards in time, but this has a price: That of abandoning the ingrained conviction that the judgement about what is cause and what is effect is independent of the observer. In fact, in the case examined above, the first observer $O$ considers the event at A to be the cause of the event at B. By contrast, the second observer $O'$ will consider the event at B as causing the event at A. All the observers will however see the cause to happen chronologically before its own effect.

Taking new objects or entities into consideration always forces us to a criticism of our prejudices. If we require the phenomena to obey the law of (retarded) causality with respect to all the observers, then we cannot demand also the description “details” of the phenomenon to be invariant too: Namely, we cannot demand in that case also the invariance of the “cause” and “effect” labels.[A6,A2] To illustrate the nature of our difficulties in accepting that e.g. the parts of cause and effect depend on the observer, let us cite an analogous situation that does not imply present-day prejudices: <<For ancient Egyptians, who knew only the Nile and its tributaries, which all flow South to North, the meaning of the word “south” coincided with the one of “upstream”, and the meaning of the word “north” coincided with the one of “downstream”. When Egyptians discovered the Euphrates, which unfortunately happens to flow North to South, they passed through such a crisis that it is mentioned in the stele of Thutmose I, which tells us about that inverted water that goes downstream (i.e. towards the North) in going upstream>> (Csonka, 1970).

The last century theoretical physics led us in a natural way to suppose the existence of various types of objects: magnetic monopoles, quarks, strings, tachyons, besides black-holes: and various sectors of physics could not go on without them, even if the existence of none of them is certain (also because attention has not yet been paid to some links existing among them: e.g., a Superluminal electric charge is expected to behave as
a magnetic monopole; and a black-hole a priori can be the source of tachyonic matter). According to Democritus of Abdera, everything that was thinkable without meeting contradictions had to exist somewhere in the unlimited universe. This point of view — which was given by M.Gell-Mann the name of “totalitarian principle”— was later on expressed (T.H.White) in the humorous form “Anything not forbidden is compulsory”. Applying it to tachyons, Sudarshan was led to claim that <<if tachyons exist, they must be found; if they do not exist, we must be able to say clearly why...>>

A3. The experimental state-of-the-art.

Extended Relativity can allow a better understanding of many aspects also of ordinary relativistic physics, even if tachyons would not exist in our cosmos as asymptotically free objects. As already said, we are dealing with them — however — since their topic is presently returning in fashion, especially because of the fact that at least four different experimental sectors of physics seem to suggest the possible existence of faster-than-light motions. We wish to put forth in the following some information (mainly bibliographical) about the experimental results obtained in each one of those different physics sectors.

A) Neutrinos – First: A long series of experiments, started in 1971, seems to show that the square $m_0^2$ of the mass $m_0$ of muonic neutrinos, and more recently of electronic neutrinos too, is negative; which, if confirmed, would mean that (when using the naïve language, commonly adopted) such neutrinos possess an “imaginary mass” and are therefore tachyonic, or mainly tachyonic.[A7,A3] [In Extended Relativity, the dispersion relation for a free tachyon becomes $E^2 - p^2 = -m_0^2$, and there is no need therefore for imaginary masses].

B) Galactic Micro-quasars – Second: As to the apparent Superluminal expansion observed in the core of quasars[A8] and, recently, in the so-called galactic micro-quasars[A9], we shall not deal here with that problem, too far from the other topics of
this paper: without mentioning that for those astronomical observations here exist orthodox interpretations, based on ref.[A10], that are accepted by the astrophysicists’ majority (even if hampered by statistical considerations). For a theoretical discussion —considering all the possible explanations, the “Superluminal” ones included—, see ref.[A11]. Here, let us mention only that simple geometrical considerations in Minkowski’s space show that a single Superluminal light source would look[A11,A3]: (i) initially, in the “optical boom” phase (analogous to the acoustic “boom” produced by a plane travelling with constant supersonic speed), as an intense source which suddenly appears, and later on becomes weaker; and that (ii) afterwards seems to split into TWO objects receding one from the other with speed $V > 2c$.

C) Evanescent waves and “tunnelling photons” – Third: Within quantum mechanics (and precisely in the tunnelling processes), it had been shown that the tunnelling time —firstly evaluated as a simple “phase time” and later on calculated through the analysis of the wavepacket behaviour— does not depend on the barrier width in the case of opaque barriers (“Hartman effect”) [A12]: which implies Superluminal and arbitrarily large (group) velocities $V$ inside long enough barriers: see Figs.6 of the text. Experiments that may verify this prediction by, say, electrons are difficult. Luckily enough, however, the Schroedinger equation in the presence of a potential barrier is mathematically identical to the Helmholtz equation for an electromagnetic wave propagating, e.g., down a metallic waveguide along the $x$-axis: and a barrier height $U$ bigger than the electron energy $E$ corresponds (for a given wave frequency) to a waveguide transverse size lower than a cut-off value. A segment of undersized guide does therefore behave as a barrier for the wave (photonic barrier)[A13]: So that the wave assumes therein —like an electron inside a quantum barrier— an imaginary momentum or wave-number and gets, as a consequence, exponentially damped along $x$. In other words, it becomes an evanescent wave (going back to normal propagation, even if with reduced amplitude, when the narrowing ends and the guide returns to its initial transverse size). Thus, a tunnelling experiment can be simulated[A13] by having recourse to evanescent waves (for which the concept of group velocity can be properly extended[A14]). And the fact that evanescent waves travel with Superluminal speeds has been actually verified in a series of famous experiments (cf.
Namely, various experiments —performed since 1992 onwards by G.Nimtz at Cologne [A15], by R.Chiao’s and A.Steinberg’s group at Berkeley [A16], by A.Ranfagni and colleagues at Florence [A17], and by others at Vienna, Orsay, Rennes [A17]— verified that “tunnelling photons” travel with Superluminal group velocities. Such experiments roused a great deal of interest [A18], also within the non-specialized press, and were reported by Scientific American, Nature, New Scientist, and even Newsweek, etc. Let us add that also Extended Relativity had predicted [A19] evanescent waves to be endowed with faster-than-c speeds; the whole matter appears to be therefore theoretically selfconsistent.

The debate in the current literature does not refer to the experimental results (which can be correctly reproduced by numerical elaborations [A20,A21] based on Maxwell equations only), but rather to the question whether they allow, or do not allow, sending signals or information with Superluminal speed [A21,A14].

Let emphasize that the most interesting experiment of this series is the one with two “barriers” (e.g., with two segments of undersized waveguide separated by a piece of normal-sized waveguide: Fig.A4). For suitable frequency bands —i.e., for “tunnelling” far from resonances—, it was found that the total crossing time does not depend on the length of the intermediate (normal) guide: namely, that the beam speed along it is infinite [A22]. This agrees with what predicted by Quantum Mechanics for the non-resonant tunnelling trough two successive opaque barriers (the tunnelling phase time, which depends on the entering energy, has been shown by us to be independent of the distance between the two barriers [A23]). Let us here repeat that the above prediction of quantum mechanics, even if rather surprising, has been theoretically confirmed and generalized by Aharonov et al. [A23]: Indeed, those authors have found that, according to QM, a wavepacket can travel, in zero time and negligible distortion, a distance arbitrarily larger than the width of the wavepacket. From the experimental point of view, our prediction [A23] has been re-verified by Longhi et al. [A23] on using as (classical) barriers two gratings in an optical fiber. Such important experiments could and should be repeated, taking advantage also of the circumstance that quite interesting evanescence regions can be easily constructed in the most varied manners, by several “photonic band-gap” filters (including photonic crystals). In any case, both this “extended Hartman effect”, and the Hartman effect itself,
can result to be important for *applications*—various of which can be easily imagined—, even more than for theory.

We cannot skip a further topic—which, being delicate, should not appear in a brief review like this one—since one of the very last experimental contribution to it (performed at Princeton in 2000 by J. Wang et al.) raised a lot of interest. Even if in Extended Relativity all the ordinary causal paradoxes seem to be solvable[A3,A6], nevertheless one has to bear in mind that (whenever it is met an object, \( \mathcal{O} \), travelling with Superluminal speed) one may have to deal with *negative contributions* to the tunnelling times[A24]: and this ought not to be regarded as unphysical. In fact, whenever an “object” (particle, electromagnetic pulse) \( \mathcal{O} \) *overcomes* the infinite speed[A3,A6] with respect to a certain observer, it will afterwards appear to the same observer as the “anti-object” \( \overline{\mathcal{O}} \) travelling in the opposite *space* direction[A3,A6]. For instance, when going on from the lab to a frame \( \mathcal{F} \) moving in the *same* direction as the particles or waves entering the barrier region, the object \( \mathcal{O} \) penetrating through the final part of the barrier (with almost infinite speed[A12,A21,A23], like in Figs. 6 of the text) will appear in the frame \( \mathcal{F} \) as an anti-object \( \overline{\mathcal{O}} \) crossing that portion of the barrier in the *opposite space–direction* [A3,A6]. In the new frame \( \mathcal{F} \), therefore, such anti-object \( \overline{\mathcal{O}} \) would yield a *negative* contribution to the tunnelling *time*: which could even result, in total, to be negative. For any clarification or explanation, see refs.[A18]. What we want to stress here is that the appearance of such negative times is predicted by Relativity itself, on the basis of the ordinary postulates[A3,A6,A24,A12,A21]. (In the case of a non-polarized beam, the anti-packet coincides with the initial wave packet; if a photon is however endowed with helicity \( \lambda = +1 \), the anti-photon will bear the opposite helicity \( \lambda = -1 \).) From the theoretical point of view, besides refs.[A24,A12,A21,A6,A3], see refs.[A25]. On the (quite interesting!) experimental side, see papers [A26], the last one having already been mentioned above.

Let us *add* here that, via quantum interference effects in three-levels atomic systems, it is possible to obtain dielectrics with refraction indices very rapidly varying as a function of frequency, with almost complete absence of light absorption (i.e., with quantum induced transparency) [A27]. The group velocity of a light pulse propagating in such a medium can decrease to very low values, either positive or negatives, with *no*
pulse distortion. It is known that experiments were performed both in atomic samples at room temperature, and in Bose-Einstein condensates, which showed the possibility of reducing the speed of light to few meters per second. Similar, but negative group velocities —implying a propagation with Superluminal speeds thousands of time higher than the previously mentioned ones— have been recently predicted, in the presence of such an “electromagnetically induced transparency”, for light moving in a rubidium condensate[A28], while corresponding experiments are being done, e.g., at the Florence European laboratory “LENS”.

Finally, let us emphasize that faster-than-c propagation of light pulses can be (and was, in some cases) observed also by taking advantage of anomalous dispersion near an absorbing line, or nonlinear and linear gain lines, or nondispersive dielectric media, or inverted two-level media, as well as of some parametric processes in nonlinear optics (cf. G.Kurizki et al.’s work).

D) Superluminal Localized Solutions (SLS) to the wave equations. The “X-shaped waves” – The fourth sector (to leave aside the others) is not less important. It returned in fashion when some groups of scholars in engineering (for sociological reasons, most physicists had abandoned the field) rediscovered by a series of works that any wave equation —to fix the ideas, let us think of the electromagnetic case— admit also solutions so much sub-luminal as Super-luminal (besides the ordinary waves endowed with speed $c/n$). Let us recall that, starting with the pioneering work by H.Bateman, it had slowly become known that all wave equations (in a general sense: scalar, electromagnetic, spinorial) admit wavelet-type solutions with sub-luminal group velocities[A29]. Subsequently, also Superluminal solutions started to be written down, in refs.[A30] and, independently, in refs.[A31] (in one case just by the mere application of a Superluminal Lorentz “transformation”[A3,A32]).

A quite important feature of some new solutions of these is that they propagate as localized, non-diffractive pulses: namely, according to the Courant and Hilbert’s[A29] terminology, as “undistorted progressive waves”. It is easy to realize the practical importance, for instance, of a radio transmission carried out by localized beams, independently of their being sub- or Super-luminal. But non-dispersive wave packets can be of use also
in theoretical physics for a reasonable representation of elementary particles[A33].

Within Extended Relativity since 1980 it had been found[A34] that—whilst the simplest subluminal object conceivable is a small sphere, or a point as its limit—the simplest Superluminal objects results by contrast to be (see refs.[A34], and Figs. A5 and A6) an “X-shaped” wave, or a double cone as its limit, which moreover travels without deforming—i.e., rigidly—in a homogeneous medium[A3]. It is worth noticing that the most interesting localized solutions happened to be just the Superluminal ones, and with a shape of that kind. Even more, since from Maxwell equations under simple hypotheses one goes on to the usual scalar wave equation for each electric or magnetic field component, one can expect the same solutions to exist also in the field of acoustic waves, and of seismic waves (and perhaps of gravitational waves too). Actually, such beams (as suitable superpositions of Bessel beams) were mathematically constructed for the first time, in acoustics, by Lu et al.[A35], and were then called “X-waves” or rather X-shaped waves.

It is more important for us that the X-shaped waves have been in effect produced in experiments both with acoustic and with electromagnetic waves; that is, X-beams were produced that, in their medium, travel undistorted with a speed larger than sound, in the first case, and than light, in the second case. In Acoustics, the first experiment was performed by Lu et al. themselves[A36] in 1992, at the Mayo Clinic (and their papers received the first 1992 IEEE award). In the electromagnetic case, certainly more “intriguing”, Superluminal localized X-shaped solutions were first mathematically constructed (cf., e.g., Fig.A7) in refs.[A37], and later on experimentally produced by Saari et al.[A38] in 1997 at Tartu by visible light (Fig.A8), and more recently by Mugnai, Ranfagni and Ruggeri at Florence by microwaves[A39]. Further experimental activity is in progress, while in the theoretical sector the activity is even more intense: in order to build up—for example—new analogous solutions with finite total energy or more suitable for high frequencies[A40], and localized solutions Superluminally propagating even along a normal waveguide[A41], or in dispersive media[A42]; without forgetting the aim of focusing X-shaped waves at a given space-point[A43], and so on.

Let us eventually touch the problem of producing an X-shaped Superluminal wave like the one in Fig.A6, but truncated—of course—in space and in time (by the use
of a finite, dynamic antenna, radiating for a finite time): in such a situation, the wave will keep its localization and Superluminality only along a certain “depth of field”, decaying abruptly afterwards[A35,A37,A44]. We can become convinced about the possibility of realizing it, by imaging the simple ideal case of a negligibly sized Superluminal source S endowed with speed $V > c$ in vacuum and emitting electromagnetic waves $W$ (each one travelling with the invariant speed $c$). The electromagnetic waves will result (cf. Fig.A6) to be internally tangent to an enveloping cone $C$ having $S$ as its vertex, and as its axis the propagation line $x$ of the source[A45,A34,A3]. This is analogous to what happens for a plane that moves in the air with constant supersonic speed. The waves $W$ interfere negatively inside the cone $C$, and constructively only on its surface. We can place a plane detector orthogonally to $x$, and record magnitude and direction of the $W$ waves that hit on it, as (cylindrically symmetric) functions of position and of time. It will be enough, then, to replace the plane detector with a plane antenna which emits — instead of recording— exactly the same (axially symmetric) space-time pattern of waves $W$, for constructing a cone-shaped electromagnetic wave $C$ that will propagate with the Superluminal speed $V$ (of course, without a source any longer at its vertex): even if each wave $W$ travels with the invariant speed $c$. For further details, see the first one of refs.[A37], and refs.[A45,A34]. Here let us only add that such localized Superluminal waves appear to keep their good properties only as long as they are fed by the waves arriving (with speed $c$) from the dynamic antenna: taking the time needed for their generation into account, these waves seem therefore unable to transmit information faster than $c$; however, they have nothing to do with the illusory “scissors effect”, since they certainly carry energy-momentum Superluminally along their field depth (for instance, they can get two detectors at a distance $L$ to click after a time smaller than $L/c$). And some authors started recently considering the possibility that the Superluminal localized solutions to the wave equations be actually tachyonic: see. refs.[A46,A14], as well as the older refs.[A34,A6,A3].

We cannot end without calling attention to some recent, interesting experimental results, presented in refs.[A47], which regard the production of X-shaped waves in non-linear materials.

As we mentioned above, the existence of all these X-shaped Superluminal (or
“Super-sonic”) seem to constitute at the moment—together, e.g., with the Superluminal-"nality of evanescent waves— one confirmation of Extended Relativity. But, at present, the existence of localized (non-diffractive) pulses or wave-trains, even if rather interesting for theory, appear to be even more important for their possible applications: One of the first applications of such X-waves (that takes advantage just of their propagation without deformation) is in advanced progress in the field of medicine, and precisely of ultrasound scanners[A48].
Captions of the Figures of the Appendix

Fig.A1 – Energy of a free object as a function of its speed[A2-A4].

Fig.A2 – Depicting the “switching rule” (or reinterpretation principle) by Stueckelberg-Feynman-Sudarshan-Recami[A3-A5]: particle \( Q \) will appear as the antiparticle of \( P \). See the text.

Fig.A3 – Simulation of tunnelling by experiments with classical evanescent waves (see the text), which were predicted to be Superluminal also on the basis of Extended Relativity[A3-A4]. The figure shows one of the experimental results in refs.[15]: that is, the average speed of the wave while crossing the evanescence region, as a function of its length. In the present case, the classical “barrier” is a segment of undersized waveguide. As theoretically predicted[A19,A12], such an average speed exceeds \( c \) for long enough barriers. (In this figure, the barrier width has been called \( a \), instead of \( d \), at variance with our choices).

Fig.A4 – The very interesting experiment having recourse to a metallic waveguide with TWO “barriers” (undersized guide segments), i.e., with two evanescence regions[A22]. See the text.

Fig.A5 – An intrinsically spherical (or pointlike, in the limiting case) object appears in the vacuum as an ellipsoid contracted along the motion direction —due to Lorentz contraction— when endowed with a speed \( v < c \). By contrast, if endowed with a speed \( V > c \) (even if the \( c \)-speed barrier cannot be crossed, neither from the left nor from the right), it would appear no longer as a particle, but rather as an “X-shaped” wave[A34] travelling rigidly (namely, as occupying the region delimited by a double cone and a two-sheeted hyperboloid—or as a double cone, in the considered limiting case—, moving Superluminally and without distortion in the vacuum, or in a homogeneous medium).

Fig.A6 – Here the intersections are shown of an “X-shaped wave”[A34] with planes or-
thogonal to its motion line, according to Extended Relativity\textsuperscript{A2-A4}. The examination of this figure suggests how to construct a simple dynamic antenna for generating such localized Superluminal waves (such an antenna was in fact adopted, independently, by Lu et al.\textsuperscript{A36} for the production of non-diffractive beams)

Fig.A7 – Theoretical prediction of the Superluminal localized “X-shaped” waves for the electromagnetic case (from Lu, Greenleaf and Recami\textsuperscript{A37}, and Recami\textsuperscript{A37}).

Fig.A8 – Scheme of the experiment by Saari et al.\textsuperscript{A38}, who announced (PRL of 24 Nov.1997) the production in optics of the beams depicted in Fig.8: In this figure one can see what shown by the experiment: i.e., the Superluminal “X-shaped” waves, which run after and catch up with the plane waves (the latter regularly travelling with speed $c$). An analogous experiment has been later on (PRL of 22 May 2000) performed with microwaves at Florence by Mugnai, Ranfagni and Ruggeri\textsuperscript{A39}. 
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