The status of the strong coupling from tau decays in 2016

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Abstract

While the idea of using the operator product expansion (OPE) to extract the strong coupling from hadronic $\tau$ decay data is not new, there is an ongoing controversy over how to include quark-hadron “duality violations” (i.e., resonance effects) which are not described by the OPE. One approach attempts to suppress duality violations enough that they might become negligible, but pays the price of an uncontrolled OPE truncation. We critically examine a recent analysis using this approach and show that it fails to properly account for non-perturbative effects, making the resulting determination of the strong coupling unreliable. In a different approach duality violations are taken into account with a model, avoiding the OPE truncation. This second approach provides a self-consistent determination of the strong coupling from $\tau$ decays.

1. Introduction

The recently revised ALEPH data $^1$ for the vector (V) and axial (A) non-strange hadronic spectral functions extracted from $\tau$ decays led to renewed efforts to extract the strong coupling $\alpha_s(m_\tau^2)$ from these data $^2$. These updates employ different methods: while perturbation theory and the operator product expansion (OPE) are central to both, they differ in the way they treat resonance effects, or, equivalently, violations of quark-hadron duality (DVs) $^3$. The method employed in Refs. $^1$ (the “truncated-OPE-model” strategy) aims to suppress DVs sufficiently to be able to ignore them altogether, while the method of Ref. $^2$ (the “DV-model” strategy) models DVs explicitly. These two approaches lead to values for $\alpha_s(m_\tau^2)$ which differ by about 8%, while the errors claimed by each method are significantly smaller than that.

Both approaches start from finite-energy sum rules (FESRs) $^4$, relating weighted integrals of the experimental spectral function to a representation of the theory in terms of $\alpha_s(m_\tau^2)$ and non-perturbative effects, captured by higher-dimension terms in the OPE, and, in the case of the DV-model strategy, a representation of DVs. The truncated-OPE-model strategy uses weight functions that suppress DVs and restricts the FESRs to the largest energy available, i.e., $m_\tau$, but requires terms

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$^1$Speaker

$^2$For the original ALEPH data, see Ref. $^2$.
in the OPE up to dimension 16. As we will see below, this necessitates an ad hoc truncation of the OPE in order to be able to fit the data provided by the spectral integrals, a truncation for which there is no basis in QCD. The DV-model strategy models DVs explicitly, and makes use of the energy dependence of the sum rules, requiring the OPE only up to dimension 8, with no truncation necessary. Both strategies need to be tested for consistency, because both make assumptions in order to handle non-perturbative effects.

Here we provide a critical appraisal of the truncated-OPE strategy, focussing on the most recent and very extensive application \[14\] of this strategy to the ALEPH data. We focus on the analysis of the \( V + A \) non-strange spectral data, as these have been advertised as being least affected by duality violations, the neglect of which is central to this strategy. We find that this strategy is fundamentally flawed, as a number of tests show unambiguously that the value for \( \alpha_s(m^2) \) through this strategy is afflicted by uncontrolled systematic errors related to problems with its treatment of non-perturbative physics.

This brief writeup summarizes only some of the highlights of our analysis. An extensive account can be found in Ref. \[10\]. For a detailed account of the DV-model strategy as applied to the determination of \( \alpha_s(m^2) \) from the ALEPH data, including a number of self-consistency tests of the assumptions underlying this approach, we refer to Ref. \[3\]. For the application of the same strategy to the OPAL data \[11\] we refer to Refs. \[12\] \[13\]. We note that Ref. \[10\], as a by-product, provides further support for the DV-model strategy as well.

2. Theory

The extraction of \( \alpha_s(m^2) \) from the non-strange \( V \) or \( A \) spectral functions \( \rho_{V/A}^{(1+0)}(s) \) starts from the sum rule

\[
\frac{1}{x_0} \int_{x_0}^{s_0} ds w(s/s_0) \rho_{V/A}^{(1+0)}(s) = \frac{1}{2\pi i x_0} \int_{|s| = x_0} ds w(s/s_0) \Pi_{OPE,V/A}^{(1+0)}(s) - \frac{1}{x_0} \int_{s_0}^{s} ds w(s/s_0) \frac{1}{2\pi i x_0} \text{Im} \Delta_{V/A}(s),
\]

in which \( \Pi_{OPE,V/A}^{(1+0)}(s) \) is the OPE approximation to the exact \( V \) or \( A \) current two-point function, given by

\[
\Pi_{OPE,V/A}^{(1+0)}(s) = \sum_{k=0}^{\infty} C_{2k}(s) \left( -s \right)^k,
\]

\( \Delta_{V/A}(s) \) represents the part not captured by the OPE, the so-called “duality violations” (DVVs), and \( w(x = s/s_0) \) is a polynomial weight. (For a derivation of the sum rule, and the precise meaning of all quantities, see Ref. \[10\].)

An important observation for the analysis to follow is that a monomial of degree \( n \) in the weight \( w(s/s_0) \) picks out the term of order \( 2k = 2(n+1) \) in the OPE, because

\[
\frac{1}{2\pi i x_0} \int_{|s| = x_0} ds \left( \frac{s}{s_0} \right)^n C_{2k}(-s)^k = (-1)^{n+1} \frac{C_{2(n+1)}}{s_0^{n+1}} \delta_{k,n+1}.
\]

The \( k = 0 \) term in the OPE represents the mass-independent, purely perturbative contribution, which is available to \( O(\alpha_s^4) \) \[14\]. In the literature, two different resummation schemes, “fixed-order” (FO or FOPT) and “contour-improved” (CI or CIPT) \[15\] have been employed. Following Ref. \[4\], we have considered both.

3. The truncated-OPE strategy

The analysis of Ref. \[4\] starts with the weights \[16\]

\[
w_{2/}(x) = (1 - x + 2x^2)(1 + 2x), \tag{4}
\]

where \((kl) \in \{00,10,11,12,13\}\), and takes \( s_0 = m_0^2 \) in the sum rule \[1\]. Since \( C_2 \) is negligibly small, the parameters in the fit to these 5 weighted integrals would be \( \alpha_s \) and \( C_{4...16} \), because \( w_{13}(x) \) has degree 7 (cf. Eq. \[5\]). Thus, in order to make a fit possible, one chooses \( C_{2k \geq 10} = 0 \). The hope is that the suppression of the vicinity of the contribution near the real axis at \( s = s_0 \) by the double or triple zero in \( w_{2/}(s/s_0) \) is sufficient to ignore the DV term in Eq. \[1\]. Ref. \[4\] carried out tests using other weights, advocating the “optimal” weights

\[
w_{2/\phi}^{opt} = 1 - (n + 2)x^{n+1} + (n + 1)x^{n+2} \tag{5}
\]

in particular. In Tables 1 and 2 we reproduce these two fits, for both FO and CI.

\begin{table}[h]
| \( \alpha_s(m^2) \) | \( C_4 \) (GeV\(^4\)) | \( C_6 \) (GeV\(^6\)) | \( C_8 \) (GeV\(^8\)) |
|---|---|---|---|
| FO | 0.316(3) | -0.0006(3) | 0.0012(3) | -0.0008(3) |
| CI | 0.336(4) | -0.0026(4) | 0.0009(3) | -0.0010(4) |
\end{table}

Table 1: Reproduction of the \( V + A \) fits of Table 1 of Ref. \[4\], based on the weights \[4\]. By assumption, \( C_{10} = C_{12} = C_{14} = C_{16} = 0 \). Errors are statistical only.

\begin{table}[h]
| \( \alpha_s(m^2) \) | \( C_6 \) (GeV\(^6\)) | \( C_8 \) (GeV\(^8\)) | \( C_{10} \) (GeV\(^{10}\)) |
|---|---|---|---|
| FO | 0.317(3) | 0.0014(4) | -0.0010(5) | 0.0004(3) |
| CI | 0.336(4) | 0.0010(4) | -0.0011(5) | 0.0003(3) |
\end{table}

Table 2: Reproduction of the \( V + A \) fits of Table 7 of Ref. \[4\], based on the five “optimal” weights \[4\] with \( n = 1, \ldots, 5 \). By assumption, \( C_{12} = C_{14} = C_{16} = 0 \). Errors are statistical only.
The OPE truncation is not based on QCD, and one might equally well pick another rather arbitrary, but reasonable set of values, such as

\[
\begin{align*}
C_{10} & = -0.0832 \text{ GeV}^{10}, \\
C_{12} & = 0.161 \text{ GeV}^{12}, \\
C_{14} & = -0.17 \text{ GeV}^{14}, \\
C_{16} & = -0.55 \text{ GeV}^{16}.
\end{align*}
\]

Using these values instead of $C_{10-16} = 0$ in the fits leads to the results of Tables 3 and 4. All fits shown in these tables are good fits [10], as good, or better, in fact, than those of Tables 1 and 2, but the alternative results of Tables 3 and 4 are not compatible with those of Tables 1 and 2. Clearly, the truncated-OPE strategy allows several, significantly different solutions. Which of these solutions gets selected by the fits is determined by the arbitrary choice made on the putative values of the higher-order $C_{2k}$.

### 4. Fake data test

In order to investigate this problem in more detail, we applied the truncated-OPE strategy to a constructed “fake” data set, in which the value of $\alpha_s$ and the size of the DV’s is known, with central values generated from a model that represents the real-world data very well, and using real-world covariances [10]. The model-solution for the fit parameters of Tables 1 and 2 is given by

\[
\begin{align*}
\alpha_s(m_\tau^2) & = 0.312, \\
C_4 & = 0.0027 \text{ GeV}^4, \\
C_6 & = -0.013 \text{ GeV}^6, \\
C_8 & = 0.035 \text{ GeV}^8, \\
C_{10} & = -0.083 \text{ GeV}^{10}.
\end{align*}
\]

We then apply the fits of the truncated-OPE strategy to these fake data, and find the results shown in Tables 5 and 6. Since the fake-data model was constructed using CIPT, these tables only show CIPT fits. (A similar test can be carried out using FOPT.) Even though the results in Tables 5 and 6 are consistent with one another, and this stability might perhaps lead one to believe in the robustness of these results, clearly, the truncated-OPE strategy fails to find to correct solution, and, in particular, the value of $\alpha_s(m_\tau^2)$ is off by more than 5 $\sigma$. The values of the OPE coefficients are also far too small.

### 5. Duality violations

![Figure 1: Blow-up of the large-s region of the V + A non-strange spectral function. Dashed (black) line: the perturbative (CIPT) representation of the model. Solid (blue) curve: full model representation, including DVs. Dot-dashed (blue) curves: separate V and A parts of the model spectral function.](image-url)

It is interesting to consider why the truncated-OPE strategy fails the fake data test. Nominally, it is the uncontrolled truncation of the OPE that is responsible, but the failure of the OPE truncation employed is also intimately connected with the presence of DVs [3,12]. In

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**Table 3: Fits as in Table 1, but with C_{10}, C_{12}, C_{14} and C_{16} as given in Eq. 8. Errors are statistical only.**

| $\alpha_s(m_\tau^2)$ | $C_4$ (GeV^4) | $C_6$ (GeV^6) | $C_8$ (GeV^8) |
|----------------------|----------------|----------------|----------------|
| FO                   | 0.295(3)       | -0.0128(3)     | 0.0355(3)      |
| CI                   | 0.308(4)       | -0.0129(3)     | 0.0354(3)      |

**Table 4: Fits as in Table 2, but with C_{12}, C_{14} and C_{16} as given in Eq. 8. Errors are statistical only.**

| $\alpha_s(m_\tau^2)$ | $C_4$ (GeV^4) | $C_6$ (GeV^6) | $C_8$ (GeV^8) | $C_{10}$ (GeV^{10}) |
|----------------------|----------------|----------------|----------------|--------------------|
| FO                   | -0.0130(4)     | 0.0356(5)      | -0.0836(3)     |
| CI                   | -0.0130(4)     | 0.0355(5)      | -0.0836(3)     |

**Table 5: CIPT fits employing the truncated-OPE strategy on the fake data, based on the weights [10]. By assumption, C_{10} = C_{12} = C_{14} = C_{16} = 0. Errors are statistical only.**

| $\alpha_s(m_\tau^2)$ | $C_4$ (GeV^4) | $C_6$ (GeV^6) | $C_8$ (GeV^8) | $C_{10}$ (GeV^{10}) |
|----------------------|----------------|----------------|----------------|--------------------|
| 0.334(4)             | -0.0023(4)     | 0.0007(3)      | -0.0008(4)     |

**Table 6: CIPT fits employing the truncated-OPE strategy on the fake data, based on the “optimal” weights [10] with m = 1 and n = 1, . . . , 5. By assumption, C_{12} = C_{14} = C_{16} = 0. Errors are statistical only.**

| $\alpha_s(m_\tau^2)$ | $C_4$ (GeV^4) | $C_6$ (GeV^6) | $C_8$ (GeV^8) | $C_{10}$ (GeV^{10}) |
|----------------------|----------------|----------------|----------------|--------------------|
| 0.334(4)             | 0.0008(4)      | -0.0008(5)     | 0.0001(3)      |
Fig. 1, we show the large-$s$ region of the ALEPH data for the $V+A$ non-strange spectral function $\rho_{V+A}$ [1]. The thick blue curve (which is the sum of the two dot-dashed curves) shows the $V + A$ model underlying the fake data set, and it is perfectly consistent with the ALEPH data. The thin dashed black curve shows the perturbative part of the spectral function. Two observations are notable: first, the dynamics of QCD, which corresponds to the difference of the data and the horizontal line at $2\pi^2 \rho_{V+A}(s) = 1$ has a large duality-violating contribution in comparison to perturbation theory, as evidenced by the clearly visible resonance oscillations in the data. Second, the model curve shows a large DV at $s = m_\tau^2$, larger, in fact, than at any other value of $s$ above 1.7 GeV$^2$, even though the DVs are exponentially damped toward large $s$. This large effect leads to a significant upward shift in the fitted value of $\alpha_s$ away from the exact model value if DVs are entirely ignored, as we found in Sec. 4.

6. Conclusion

In Ref. [10] we conducted an extensive study of the truncated-OPE strategy for obtaining $\alpha_s(m_\tau^2)$ from hadronic $\tau$ decay data, taking the most recent such analysis of Ref. [2] as our starting point.

A key component of the truncated-OPE strategy is the arbitrary truncation of the OPE. In addition, this strategy claims that it is safe to neglect DVs, employing weighted spectral integrals which suppress their contribution. We demonstrated in Secs. 3 and 4 that these assumptions cause the truncated-OPE strategy to fail. It is not capable of detecting residual DVs, and, together with the unavoidable, but uncontrolled truncation of the OPE, it leads to values for $\alpha_s(m_\tau^2)$ about 8% too high, even though the error is claimed to be less than 4%. This difference is larger than the long-standing disagreement between FOPT- and CIPT-based results.

Here we only had space to summarize some of our findings, and we refer to Ref. [10] for full details, and more discussion comparing the truncated-OPE and DV-model strategies. In particular, Ref. [10] also contains a refutation of the criticism of the DV-model strategy contained in Ref. [4], showing that in fact this “criticism” provides further support for the consistency of the DV-model strategy.

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