Differential Poisson Sigma Models with Extended Supersymmetry

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Abstract

The induced two-dimensional topological $\mathcal{N} = 1$ supersymmetric sigma model on a differential Poisson manifold $M$ presented in \texttt{arXiv:1503.05625} is shown to be a special case of the induced Poisson sigma model on the bi-graded supermanifold $T[0, 1]M$. The bi-degree comprises the standard $\mathbb{N}$-valued target space degree, corresponding to the form degree on the worldsheet, and an additional $\mathbb{Z}$-valued fermion number, corresponding to the degree in the differential graded algebra of forms on $M$. The $\mathcal{N} = 1$ supersymmetry stems from the compatibility between the (extended) differential Poisson bracket and the de Rham differential on $M$. The latter is mapped to a nilpotent vector field $Q$ of bi-degree $(0, 1)$ on $T^*[1, 0](T[0, 1]M)$, and the covariant Hamiltonian action is $Q$-exact. New extended supersymmetries arise as inner derivatives along special bosonic Killing vectors on $M$ that induce Killing supervector fields of bi-degree $(0, -1)$ on $T^*[1, 0](T[0, 1]M)$.

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### 1 Introduction

A differential Poisson manifold is a Poisson manifold whose bracket admits an extension from the algebra of functions to the differential graded algebra of forms. An ordinary Poisson structure induces a two-dimensional topological sigma model, due to Ikeda, Schaller and Strobl
for which Cattaneo and Felder \cite{7} devised an Alexandrov-Kontsevich-Schwarz-Starobinsky \cite{8} path integral quantization scheme that reproduces Kontsevich’s star product formula in trivial topology \cite{9}. As shown in \cite{10}, a differential Poisson structure induces a fermionic extension of the Ikeda–Schaller–Strobl model, referred to as the differential Poisson sigma model, which couples to the Poisson bi-vector as well as a compatible connection\cite{11} and that has a rigid supersymmetry corresponding to the target space de Rham differential.

It remains to be seen whether the path integral quantization of the model yields a covariant star product for differential forms\cite{2} that is compatible with a suitable deformation of the de Rham differential. Indeed, McCurdy and Zumino \cite{4} have provided a covariant deformation of the wedge product along the differential Poisson bracket that is associative but incompatible with the undeformed de Rham differential at order $\hbar^2$, suggesting that compatibility requires a deformed differential\cite{3}. It may turn out, however, that the quantization on general background will actually violate associativity up to homotopies, as is often the case with BRST-like symmetries in field theory.

As the model’s local degrees of freedom are confined to boundaries, it describes quantum mechanical particles on the Poisson manifold that are entangled via topological fields inside the worldsheet. The Poisson sigma model was originally developed in response to deformation quantum mechanics \cite{12,13,14}, in which the physical state of a quantum mechanical system is represented by a density matrix obeying an evolution equation. The prospect of the Liouville equation being a linearization of a string field equation based on a suitably gauged Poisson sigma model, which might lead to a microscopic description of quantum entanglement and dynamically generated collapses of quantum states, provides a basic physical motivation behind the current work. In other words, topological strings not only implement the correspondence principle, whereby classical functions are deformed into operators, but also provide a nonlinear quantum mechanical evolution equation.

To the above end, one needs to distinguish between the gauging of Killing symmetries with \cite{15} and without Hamiltonians. The latter category contains the rigid supersymmetry, which

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1. A differential Poisson manifold also contains an additional tensorial one-form, referred to in \cite{10} as the $S$-structure.
2. On general grounds, the covariant star product must be gauge equivalent in the zero-form sector to Kontsevich’s canonical product.
3. Deformed differentials play an important role in Fedosov’s construction of covariant star products of functions on symplectic manifolds \cite{11}.
corresponds to the de Rham differential that is supposed to give the kinetic term in the (nonlinear) Liouville equation. As for its gauging, there are two methods available: treating supersymmetry as a fermionic Killing vector in target space, it can be gauged at the level of the classical action, as was done in [16]. Alternatively, treating the supersymmetry as the $Q$-structure of an integrable $QP$-structure, which means that it can be extended naturally to homotopy Poisson manifolds, requires the full AKSZ machinery, and we hope to report on it elsewhere.

A related motivation is the proposal made in [17] that topological open strings and related chiral Wess–Zumino–Witten models on Dirac cones [18] are dual to conformal field theories. Compared to earlier holography proposals [19, 20, 21], in which the bulk side is a (string) field theory containing higher spin gravity of Vasiliev type [22, 23], the proposal of [17] was based on the observation that tensionless strings in anti-de Sitter spacetime behave as collections of conformal particles, that is, on a first-quantized description of the bulk physics. Subsequent tests at the level of on-shell amplitudes in $AdS_4$, have shown that traces of Gaussian density matrices of the the quantum mechanical harmonic oscillator in two dimensions describing unfolded boundary-to-bulk propagators of massless fields [24], which one may think of as vertex operators on conformal particle worldlines in phase space, or, boundary vertex operators on the corresponding topological open string, indeed reproduce the correlation functions for bilinear operators in free conformal field theories in three dimensions [25, 26, 27, 28]. The current, slightly refined, working hypothesis is that there exists a variant of Witten’s realization of Chern-Simons theory as a topological A model [29] (see also [30]), whereby a gauging of the differential Poisson sigma model with 3-graded Chan-Paton factors [31] (see also [32, 33]) yields the Frobenius-Chern-Simons formulation [34, 35] of higher spin gravity.

In this paper, we shall induce the supersymmetric sigma model on a differential Poisson manifold using a geometric approach that yields a covariant Hamiltonian action with canonical kinetic terms. The method also applies to extended differential Poisson manifolds, whose brackets contain components with strictly positive intrinsic form degree. We shall use the resulting manifest target space superdiffeomorphism covariance to finding new extended supersymmetries.

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4. See equation Eq. (3.161) in [17].

5. The on-shell correspondence takes the form $\langle O_1...O_n\rangle_{CFT} = \sum_{\text{crossings}} \langle V_1...V_n\rangle_{PSM}$, where $O_i$ is a CFT operator at a point $x_i$ with conformal label $L_i$ and $V_i$ is a Gaussian PSM vertex operator of the form $V_i = \exp(Y^T A_i Y + Y^T B_i)$ where the matrix $A_i$ depends on $x_i$ and $B_i$ depends on $L_i$. As the PSM is topological, the right-hand side does not depend on the insertion points of the $V_i$ (along the boundary of a disc).

6. The manifest superdiffeomorphism covariance is also useful in gauging Killing supersymmetries, as we hope
corresponding to inner derivatives of the differential graded algebra of forms along special bosonic Killing vectors.

The paper is organized as follows: In Section 2 we review the induced sigma models on ordinary and differential Poisson manifolds. In Section 3 we give the geometric construction of the supercovariant Hamiltonian action on the phase space $T^* \mathcal{I} \mathcal{M}$ of an extended differential Poisson manifold $\mathcal{M}$. In particular, the geometric origin of the new Killing supersymmetries of negative degree is explained in Section 3.5. In Section 4 we show the equivalence to the original action in the unextended case with vanishing $S$-structure, and give the component form of the new Killing supersymmetries. We conclude and point to future directions for investigations in Section 5. Appendix A contains conventions and notation for affine connections, the Nijenhuis–Schouten bracket and parity shifted bundles. Appendix B spells out the extension of the Cartan algebra to vector field valued differential forms on a real manifold $\mathcal{M}$, which is mapped in Section 3.1 to the Cartan algebra on $T[1,0] \mathcal{M}$. Finally, in Appendix C we provide a manifestly $\text{Diff}(\mathcal{M})$ covariant formula for the cohomology of the Lie derivative of nilpotent vector field $Q$ on $T^*[1,0](T[0,1] \mathcal{M})$, which uplift representing to $T^*[1,0](T[0,1] \mathcal{M})$ of the de Rham differential on $\mathcal{M}$.

2 Induced two-dimensional Poisson sigma models

In this section, we review the interplay between (differential) Poisson algebras and two-dimensional (supersymmetric) Poisson sigma models.

2.1 Poisson algebra of functions

The Poisson algebra of functions on a real smooth manifold $\mathcal{M}$ is the space $C^\infty(\mathcal{M})$ equipped with its pointwise product, which turns it into a commutative associative algebra, and a second bilinear antisymmetric product

$$\{ \cdot, \cdot \} : C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) , \quad (2.1)$$

to report on in a future work.
referred to as the Poisson bracket, acting as a differential in each slot and obeying the Jacobi identity. That is, if \( f, g, h \in C^\infty(M) \) then

\[
\{ f, g + h \} = \{ f, g \} + \{ f, h \} , \tag{2.2}
\]
\[
\{ f, g \} = -\{ g, f \} , \tag{2.3}
\]
\[
\{ f, gh \} = g\{ f, h \} + h\{ f, g \} , \tag{2.4}
\]
\[
\{ f, \{ g, h \} \} + \{ h, \{ f, g \} \} + \{ g, \{ h, f \} \} = 0 . \tag{2.5}
\]

The Poisson bracket corresponds to an antisymmetric bivector field \( \Pi \) defined by

\[
\Pi(df, dg) := \frac{1}{2} \{ f, g \} , \tag{2.6}
\]

obeying the Poisson bivector condition

\[
\{ \Pi, \Pi \}_{\text{S.N.}} = 0 , \tag{2.7}
\]

where \( \{ \cdot, \cdot \}_{\text{SN}} \) denotes the Schouten–Nijenhuis bracket for antisymmetric polyvector fields, which is equivalent to the Jacobi identity (2.5). In a coordinate basis, we may expand

\[
\Pi = \frac{1}{2} \Pi^{\alpha\beta} \partial_\alpha \wedge \partial_\beta , \tag{2.8}
\]

so that

\[
\{ f, g \} = \Pi^{\alpha\beta} \partial_\alpha f \partial_\beta g , \tag{2.9}
\]

and

\[
\{ \Pi, \Pi \}_{\text{S.N.}} = \Pi^{\delta[\alpha} \partial_\beta \Pi^{\gamma]} \partial_\alpha \wedge \partial_\beta \wedge \partial_\gamma . \tag{2.10}
\]

Thus, the Poisson bivector condition reads

\[
\Pi^{\delta[\alpha} \partial_\beta \Pi^{\gamma]} = 0 . \tag{2.11}
\]

### 2.2 Bosonic sigma model

A Poisson manifold \( M \) induces a two-dimensional topological field theory with configuration space given by the space of maps

\[
\varphi : \Sigma \rightarrow M , \tag{2.12}
\]
sending a two-dimensional compact surface $\Sigma$, with or without boundary, into $M$. Its classical
dynamics is governed by Ikeda-Schaller-Strobl action \[ S[\phi, \eta] = \int_{\Sigma} \phi^* (\eta_\alpha d\phi^\alpha + \frac{1}{2} \Pi^{\alpha\beta} \eta_\alpha \eta_\beta) \]
where $\phi^\alpha$ and $\eta_\alpha$ coordinatize the base and fiber of the parity shifted cotangent bundle $T^*[1]M$
over $M$. The space $\Omega(T^*[1]M)$ of forms on $T^*[1]M$ is a differential graded algebra whose elements
have a bi-degree given by the standard form degree and a degree (preserved by $\varphi$) dictated by
\[ \deg(\phi^\alpha, \eta_\alpha; d) = (0, 1; 1) . \] (2.14)
Likewise, the space $\Omega(\Sigma)$ of forms on $\Sigma$ is a differential graded algebra with degree map given by
the form degree on $\Sigma$. The sigma model map is assumed to have vanishing intrinsic degree in the
sense that
\[ \varphi^* : \Omega(T^*[1]M) \to \Omega(\Sigma) , \] (2.15)
is a degree preserving homeomorphism of differential graded algebras, i.e. a $p$-form of degree $n$
on $T^*[1]M$ is sent to an $n$-form on $\Sigma$ and $\varphi^*d = d\varphi^*; in particular$
\[ \varphi^*(T^*[1]M) = \varphi^*(T^*M) \otimes T^*\Sigma . \] (2.16)
The action $S$ is well-defined provided that $(\varphi^*\phi^\alpha, \varphi^*\eta_\alpha)$ belong to a globally defined section over
$\Sigma$, which is what we shall assume henceforth, together with suppressing the symbol $\varphi^*$ whenever
no ambiguity can arise.

By virtue of the Poisson condition (2.11), the action (2.13) is invariant under the gauge
transformations
\[ \delta_\varepsilon \phi^\alpha = -\Pi^{\alpha\beta} \varepsilon_\beta , \] (2.17)
\[ \delta_\varepsilon \eta_\alpha = d\varepsilon_\alpha + \partial_\alpha \Pi^{\beta\gamma} \eta_\beta \varepsilon_\gamma , \] (2.18)

\[ 7 \text{The space } T^*[1]M \text{ is obtained from } T^*M \text{ by replacing its fiber, that is, the real vector space } \mathbb{R}^m, \text{ where } m = \dim(M), \text{ by the vector space } \mathbb{R}[1]^m \text{ consisting of } m\text{-tuples of elements in the space } \mathbb{R}[1] \text{ of real numbers of parity one. In general, on } \mathbb{Z}^k\text{-graded manifold } M \text{ with coordinates } Z^i \text{ with degrees } \deg Z^i \in \mathbb{Z}^k, \text{ the bundles } T[\vec{m}]M \text{ and } T^*[\vec{n}]M \text{ are obtained from } TM := T[0]M \text{ and } T^*M := T^*[0]M, \text{ respectively, by replacing their fibers by vector spaces of the same dimension with vectorial coordinates } V^i \text{ and } P_i \text{ with } \mathbb{Z}^k\text{-valued degrees given by } \deg V^i = \deg Z^i + \vec{n} \text{ (shifts) and } \deg P_i = \vec{n} - \deg Z^i \text{ (dual shifts). In the bosonic and supersymmetric Poisson sigma models formulated on Poisson and differential Poisson manifolds, respectively, we have } k = 1 \text{ and } k = 2. \]
provided that $\varepsilon^\alpha$ is globally defined on $\Sigma$, and the equations of motion, \textit{viz.}

\[ R^\alpha := d\phi^\alpha + \Pi^{\alpha\beta} \eta_\beta \approx 0 , \tag{2.19} \]
\[ R_\alpha := d\eta_\alpha + \frac{1}{2} \partial_\alpha \Pi^{\beta\gamma} \eta_\beta \wedge \eta_\gamma \approx 0 , \tag{2.20} \]

define a universally Cartan integrable system, \textit{i.e.} $dR^\alpha \approx 0$ and $dR_\alpha \approx 0$ independently of the dimension of $\Sigma$. More precisely, the gauge invariance of the action and the integrability of (2.19) are equivalent to (2.11), while the integrability of (2.20) only requires the derivative of (2.11).

Finally, if $\Sigma$ has a boundary then it is also assumed that

\[ \eta_\alpha |_{\partial \Sigma} = 0 , \quad \varepsilon_\alpha |_{\partial \Sigma} = 0 , \tag{2.21} \]

as to ensure gauge invariance and that the action is stationary on-shell including boundary terms.

Strictly speaking, in the classical theory, the variation principle only implies that the boundary condition on $\eta_\alpha$ must hold on-shell, whereas it must be imposed off-shell in the path integral in order for the AKSZ master action to obey the BV master equation.

### 2.3 Poisson algebra of differential forms

A differential Poisson algebra on a manifold $M$ \cite{1, 2, 3, 4} is an extension of a Poisson algebra on $M$ from $C^\infty(M)$ to the algebra $\Omega(M)$ of differential forms on $M$, whereby the pointwise product on $C^\infty(M)$ is replaced by the graded commutative wedge product on $\Omega(M)$, and the Poisson bracket on $C^\infty(M)$ is extended to a graded skew-symmetric map

\[ \{ \cdot , \cdot \} : \Omega(M) \otimes \Omega(M) \longrightarrow \Omega(M) , \tag{2.22} \]

referred to as differential Poisson bracket, assumed to be compatible with the de Rham differential and obeying the graded Leibniz rule, that is

\[ \deg_M(\{\omega_1 , \omega_2\}) = \deg_M(\omega_1) + \deg_M(\omega_2) , \tag{2.23} \]
\[ \{\omega_1 , \omega_2 + \omega_3\} = \{\omega_1 , \omega_2\} + \{\omega_1 , \omega_3\} , \tag{2.24} \]
\[ \{\omega_1 , \omega_2\} = (-1)^{1+\deg_M(\omega_1)\deg_M(\omega_2)}\{\omega_2 , \omega_1\} , \tag{2.25} \]
\[ d\{\omega_1 , \omega_2\} = \{d\omega_1 , \omega_2\} + (-1)^{\deg_M(\omega_1)}\{\omega_1 , d\omega_2\} , \tag{2.26} \]
\[ \{\omega_1 , \omega_2 \wedge \omega_3\} = \{\omega_1 , \omega_2\} \wedge \omega_3 + (-1)^{\deg_M(\omega_1)\deg_M(\omega_2)}\omega_2 \wedge \{\omega_1 , \omega_3\} , \tag{2.27} \]
and that satisfy the graded Jacobi identity

\[ \{ \omega_1, \{ \omega_2, \omega_3 \} \} + (-1)^{\deg_M(\omega_1)(\deg_M(\omega_2)+\deg_M(\omega_3))} \{ \omega_2, \{ \omega_3, \omega_1 \} \} + (-1)^{\deg_M(\omega_3)(\deg_M(\omega_1)+\deg_M(\omega_2))} \{ \omega_3, \{ \omega_1, \omega_2 \} \} = 0, \]  

(2.28)

where \( \omega_i \in \Omega(M) \) and \( \deg_M \) denotes the form degree on \( \Omega(M) \).

Besides a Poisson bi-vector, a differential Poisson bracket entails a connection one-form \( \tilde{\Gamma}^{\alpha \beta} = d\phi^\gamma \Gamma^{\alpha \beta}_\gamma \) and a tensorial one-form

\[ S = \frac{1}{2} d\phi^\alpha S^\beta_\alpha \partial_\beta \odot \partial_\gamma, \]  

(2.29)

defined through

\[ \{ \phi^\alpha, d\phi^\beta \} = \frac{1}{2} \tilde{\nabla} \Pi^{\alpha \beta} + S^{\alpha \beta} - \Pi^{\alpha \gamma} \tilde{\Gamma}^{\beta \gamma}. \]  

(2.30)

Choosing the connection such that

\[ \tilde{\nabla}_\gamma \Pi^{\alpha \beta} = 0, \]  

(2.31)

and imposing the compatibility between the bracket and the de Rham differential, it follows that, for any two differential forms \( \omega_1 \) and \( \omega_2 \), one has

\[ \{ \omega_1, \omega_2 \} = \Pi^{\alpha \beta} \nabla_\alpha \omega_1 \wedge \nabla_\beta \omega_2 + S^{\alpha \beta} \wedge \left[ (-1)^{\deg_M(\omega_1)} \nabla_\alpha \omega_1 \wedge i_\beta \omega_2 - i_\alpha \omega_1 \wedge \nabla_\beta \omega_2 \right] + (-1)^{\deg_M(\omega_1)} \left( \tilde{\nabla}\Pi^{\alpha \beta} - \tilde{\nabla}S^{\alpha \beta} \right) \wedge i_\alpha \omega_1 \wedge i_\beta \omega_2, \]  

(2.32)

where \( \nabla_\alpha \) is constructed from the connection coefficients

\[ \Gamma^{\alpha \beta}_\gamma = \tilde{\Gamma}^{\alpha \beta}_\gamma, \]  

(2.33)

which implies

\[ \nabla_\alpha \Pi^{\beta \gamma} = \tilde{\nabla}_\alpha \Pi^{\beta \gamma} - 2T^{\beta \gamma}_{\alpha \delta} \Pi^{\delta \gamma}, \]  

(2.34)

and where we have defined

\[ \tilde{\nabla} = \nabla - 2T^{\beta \gamma}_{\alpha \delta} \Pi^{\delta \gamma}. \]  

(2.35)
where $\tilde{R}^\alpha_{\beta}$ is the curvature two-form of $\tilde{\Gamma}^\alpha_{\beta}$. Turning to the graded Jacobi identity (2.28), using the properties of the differential Poisson bracket, it holds for all $\omega_i$ if it holds for $\text{deg}_M(\omega_1, \omega_2, \omega_3) \in \{(0, 0, 0), (0, 0, 1), (0, 1, 1)\}$. If the tensorial structure $S = 0$, these conditions are equivalent to

\[ J^{\alpha\beta\gamma}_{(0,0,0)} := \Pi^\delta[\alpha]_\gamma^{\delta\epsilon} \Pi^\gamma_{\delta\epsilon} = 0 , \tag{2.36} \]

\[ J^{\alpha\beta\gamma}_{(0,0,1)} := \Pi^\delta[\alpha]_\gamma^{\delta\epsilon} R^\gamma_{\delta\epsilon} = 0 , \tag{2.37} \]

\[ J^{\alpha\beta\gamma}_{(0,1,1)} := \Pi^\delta[\alpha]_\gamma^{\delta\epsilon} \tilde{\Gamma}^\gamma_{\delta\epsilon} = 0 , \tag{2.38} \]

of which the constraints on $J^{\alpha\beta\gamma}_{(0,0,0)}$, $J^{[\alpha,\beta,\gamma]}_{(0,0,1)}$ and $J^{[\alpha,\beta,\gamma]}_{(0,1,1)}$ are independent, whereas the remainder follows by covariant differentiation. Moreover, the exterior derivative of the graded Jacobi identity for $\text{deg}_M(\omega_1, \omega_2, \omega_3) = (0, 1, 1)$ yields that for $\text{deg}_M(\omega_1, \omega_2, \omega_3) = (1, 1, 1)$, which reads

\[ J^{\alpha\beta\gamma}_{(1,1,1)} := \tilde{R}^{\rho}_{\alpha\beta} \tilde{R}^{\rho}_{\gamma\delta} = 0 . \tag{2.39} \]

A basic example [2] consists of the algebra of functions on a Lie group, which is deformed, at the quantum level, by the canonical Poisson bi-vector into the group algebra, and yet further by the connection into the quantum group algebra, for which Eq. (2.39) provides the Yang-Baxter equation [4].

### 2.4 $\mathcal{N} = 1$ supersymmetric Poisson sigma model

A differential Poisson manifold induces an $\mathcal{N} = 1$ supersymmetric extension of the Ikeda–Schaller–Strobl sigma model [10], obtained by adding fermionic partners $(\theta^\alpha, \chi_\alpha)$ of form degrees zero and one, respectively, to the original bosonic fields in the action (2.13). The classical action is given by

\[
S[\phi, \eta, \theta, \chi] = \int \Sigma \left[ \eta_\alpha d\phi^\alpha + \frac{1}{2} \Pi^\alpha \eta_\alpha \eta_\beta + \chi_\alpha \nabla \theta^\alpha + \frac{1}{4} \tilde{R}^{\gamma\delta}_{\alpha\beta} \chi_\alpha \chi_\beta \theta^\gamma \theta^\delta \right] = \int \Sigma \left[ (\phi^\ast \eta_\alpha) \wedge d(\phi^\ast \phi^\alpha) + \frac{1}{2} (\phi^\ast \Pi^\alpha_\beta) (\phi^\ast \eta_\alpha) \wedge (\phi^\ast \eta_\beta) + (\phi^\ast \chi_\alpha) \wedge \nabla (\phi^\ast \theta^\alpha) + \frac{1}{4} (\phi^\ast \tilde{R}^{\gamma\delta}_{\alpha\beta}) (\phi^\ast \chi_\alpha) \wedge (\phi^\ast \chi_\beta) (\phi^\ast \theta^\gamma)(\phi^\ast \theta^\delta) \right],
\]

where the objects are assigned target space degrees in $\mathbb{N}$, denoted by $\text{deg}$, and fermion numbers in $\mathbb{Z}$, denoted by $n_f$, as follows:

|     | $\phi^\alpha$ | $\theta^\alpha$ | $\eta_\alpha$ | $\chi_\alpha$ | $d$ |
|-----|---------------|-----------------|----------------|--------------|-----|
| $\text{deg}$ | 0             | 0               | 1              | 1            | 1   |
| $n_f$  | 0             | 1               | 0              | -1           | 0   |
The target space thus consists of a bi-graded fiber bundle

\[ \mathcal{E} = T^*[1,0](T[0,1]M) \equiv T^*[1,0]M \oplus T^*[1,1]M \oplus T[0,1]M , \]  

(2.42)

with base and fiber coordinatized by \( \phi^\alpha \) and \((\eta_\alpha, \chi_\alpha, \theta^\alpha)\), respectively. The sigma model map \( \varphi : \Sigma \to M \) is now assumed to be bi-degree preserving, where the bi-degree on \( \Sigma \) is given by \((\text{deg}_\Sigma, n_f)\), that is, the pull back operation \( \varphi^* \) converts target space degree into form degree on \( \Sigma \) (just as in the case of the bosonic sigma model) and preserves the fermion number\(^{10}\). Thus, \( \varphi^* \) induces the following bundle structure over \( \Sigma \):

\[
\varphi^*(T^*[1,0]M \oplus T^*[1,1]M \oplus T[0,1]M) = \left( \varphi^*(T^*[0,0]M) \otimes T^*\Sigma \right) \oplus \left( \varphi^*(T^*[0,1]M) \otimes T^*\Sigma \right) \oplus \left( \varphi^*(T[0,1]M) \otimes C^\infty(\Sigma) \right),
\]

(2.43)

whose sections we shall assume are globally defined on \( \Sigma \). Moreover, in case \( \Sigma \) has a boundary, then we assume that

\[
\eta_\alpha|_{\partial \Sigma} = 0, \quad \chi_\alpha|_{\partial \Sigma} = 0.
\]

(2.44)

The kinetic term for the fermions contains the covariant derivative

\[
\nabla \theta^\alpha := d\theta^\alpha + d\phi^\beta \Gamma^\alpha_{\beta\gamma} \theta^\gamma.
\]

(2.45)

The resulting symplectic potential \( \vartheta \) on \( \mathcal{E} \) is given by the sum of the tautological one-form on \( \mathcal{E} \) and an extra term containing the connection, \textit{viz.}

\[
\vartheta = (\eta_\alpha - \Gamma^\gamma_{\alpha\beta} \chi_\gamma \theta^\beta) d\phi^\alpha + \chi_\alpha d\theta^\alpha.
\]

(2.46)

Finally, we use the following Koszul sign convention\(^{11}\).

\[
\mathcal{F}\mathcal{F}' = (-1)^{|\mathcal{F}||\mathcal{F}'|}\mathcal{F}'\mathcal{F}, \quad |\mathcal{F}| := \text{deg}(\mathcal{F}) + n_f(\mathcal{F})
\]

(2.47)

where \( \mathcal{F} \) and \( \mathcal{F}' \) are functions of \((\phi^\alpha, \theta^\alpha; \eta_\alpha, \chi_\alpha)\). We refer to \( |\mathcal{F}| \) as the total degree of \( \mathcal{F} \).

---

\(^{10}\)Put into equations, \( \varphi^* : \mathbb{R}[m,n] \to \mathbb{R}[0,n] \otimes \Omega_{t\alpha}(\Sigma) \), that is, \( \varphi^* \) converts the first entry of the bi-degree into form degree on \( \Sigma \) while preserving the second entry, given by the fermion number.

\(^{11}\)This convention, which differs from that used in [10], is motivated by the fact that it admits a direct extension to generalized Poisson sigma models in higher dimensions with target spaces given by \( \mathbb{N} \)-graded manifolds \([36, 37]\).
The equations of motion of the supersymmetric Poisson sigma model read\(^{12}\)

\[
\mathcal{R}^\phi_\alpha := d\phi^\alpha + \Pi^\alpha_\beta \eta_\beta \approx 0 , \tag{2.48}
\]
\[
\mathcal{R}^{\theta}_\alpha := \nabla \theta^\alpha + \frac{1}{2} \tilde{R}^\beta_\gamma \delta^\alpha_\beta \chi_\gamma \theta^{\delta} \approx 0 , \tag{2.49}
\]
\[
\mathcal{R}^{\eta}_\alpha := \nabla \eta_\alpha + R_{\alpha\beta\gamma} \delta^\gamma_\delta \phi^\beta \wedge \chi_\gamma \theta^{\delta} + \frac{1}{4} \nabla_\alpha \tilde{R}^\delta_\gamma \delta^\epsilon_\delta \chi_\epsilon \wedge \chi_\gamma \theta^{\delta} \approx 0 , \tag{2.50}
\]
\[
\mathcal{R}^{\chi}_\alpha := \nabla \chi_\alpha + \frac{1}{2} \tilde{R}^{\beta\gamma}_\alpha \chi_\beta \wedge \chi_\gamma \theta^{\delta} \approx 0 , \tag{2.51}
\]

after a dueful suppression of the sigma model map, and where we use (2.45), \textit{idem} for \(\nabla \eta_\alpha\) and \(\nabla \chi_\alpha\). These equations form a universally Cartan integrable system by virtue of (2.36)-(2.39), \textit{i.e.} provided that the differential Poisson bracket (2.32) obeys the graded Jacobi identity. These identities also ensure the gauge invariance of the action\(^{13}\), up to boundary terms that vanish provided the gauge parameters vanish at the boundary of \(\Sigma\).

Under the isomorphism

\[
C^\infty(T[0,1]M) \cong \Omega(M) , \tag{2.52}
\]

the de Rham operator on \(M\) is sent to the nilpotent fermionic vector field

\[
q_t := \theta^\alpha \partial_\alpha , \tag{2.53}
\]

on \(T[0,1]M\). As found in \textbf{[10]}, its action on \((\phi^\alpha, \theta^\alpha)\) can be extended into a (rigid) nilpotent supersymmetry \(\delta_t\) of the action (2.40), given by\(^{14}\)

\[
\begin{align*}
\delta_t \phi^\alpha &= \theta^\alpha , \\
\delta_t \theta^\alpha &= 0 , \\
\delta_t \eta_\alpha &= \frac{1}{2} \tilde{R}^\beta_\gamma \delta^\alpha_\beta \chi_\delta \theta^\beta \wedge \chi_\gamma \theta^{\delta} - \Gamma^\gamma_\alpha_\beta \eta_\gamma \theta^\beta , \\
\delta_t \chi_\alpha &= -\eta_\alpha + \Gamma^{\gamma\beta}_\alpha \chi_\gamma \theta^\beta . \tag{2.54}
\end{align*}
\]

This symmetry can be made manifest by writing

\[
S = \delta_t \int_{\Sigma} \mathcal{V} , \quad \mathcal{V} = -\chi_\alpha \wedge \left( d\phi^\alpha + \frac{1}{2} \Pi^\alpha_\beta \eta_\beta \right) , \tag{2.55}
\]

including total derivatives.

\(^{12}\)Some signs in Eqs. (2.48)-(2.51) differ from those in the corresponding equations in \textbf{[10]}, where a different Koszul sign convention was adopted.

\(^{13}\)The fact that the symplectic potential (2.46) is non-canonical implies that the off-shell gauge transformations differ from the on-shell ones by terms proportional to the Cartan curvatures \textbf{[10]}.

\(^{14}\)The coefficient four-fermi coupling in (2.40) is fixed by the rigid supersymmetry but not the local symmetries.
3 Superspace formulation

In this section, we use the isomorphism between the differential form algebra on $M$ and the algebra of functions on $T[0, 1]M$ to map the differential Poisson bracket on $M$ to a supersymmetric Hamiltonian zero-form on $T^*[1, 0](T[0, 1]M)$. The pullback of it and the tautological one-form to the worldsheet induces a two-dimensional $\mathcal{N} = 1$ supersymmetric topological sigma model. This geometric approach can be extended to manifolds equipped with Poisson brackets that have components with positive intrinsic form degree. It also permits a uniform treatment of Killing supersymmetries, including the rigid supersymmetry generated by the de Rham differential on $M$ as well as new supersymmetries generated by inner derivatives along ordinary bosonic Killing vectors on $TM$.

3.1 Mapping forms on $M$ to functions on $T[0, 1]M$

The construction of the supersymmetric Poisson sigma model in the previous section makes use of the isomorphism \(\mathbb{V}\) of differential graded algebras, i.e. the bijection

$$V : \Omega_{[n]}(M) \rightarrow \Omega_{[0|(0,n)]}(T[0, 1]M) ,$$

(3.1)

sending $n$-forms $\omega$ on $M$ to functions $V(\omega) \equiv V_\omega$ on $T[0, 1]M$ that are $n$th order in the fiber coordinates while preserving the associative algebra structure, and intertwining the de Rham differential on $M$ with a nilpotent vector field $q_t$ on $T[0, 1]M$, viz.

$$V(\omega_1 \wedge \omega_2) = V_{\omega_1}V_{\omega_2} , \quad V \circ d|_M = q_t \circ V .$$

(3.2)

On $T[0, 1]M$, the bi-degrees of $q_t$ and the de Rham differential are given by

$$\text{bideg}(q_t) = (0, 1) , \quad \text{bideg}(d|_{T[0, 1]M}) = (1, 0) .$$

(3.3)

Correspondingly, the algebra of forms on $T[0, 1]M$ decomposes as follows:

$$\Omega(T[0, 1]M) = \bigoplus_{n,p} \Omega_{(p,n)}(T[0, 1]M) , \quad \text{bideg}(\Omega_{(p,n)}(T[0, 1]M)) = (p, n) ,$$

(3.4)

where $\Omega_{(p,n)}(T[0, 1]M)$ thus consists of $p$-forms on $T[0, 1]M$ that are $n$th order in the fiber coordinates and their line elements.

\[\text{15}\]The pull-backs $\varphi^*V_\omega$ to $\Sigma$ by the sigma model map $\varphi$ are vertex operators of form degree zero on $\Sigma$. 


The isomorphism $V$ intertwines the derivations of $\Omega(M)$, i.e. combinations $\iota_\Xi + \mathcal{L}_\Xi$ of inner and Lie derivatives along vector field valued forms $\Xi$ and $\Xi'$ on $M$ (see Appendix B), with those of $C^\infty(T[0,1]M)$, i.e. the vector fields on $T[0,1]M$. The induced linear map

$$V : \text{Der}(\Omega(M)) \to \Gamma(T[0,1]M, T(T[0,1]M)),$$

is characterized by

$$V \circ \iota_\Xi = V(\Xi) \circ V, \quad V(\omega \wedge \Xi) = V_\omega V(\Xi),$$

from which it follows via the Cartan relation $\mathcal{L}_{\Xi'} = [d, \iota_{\Xi'}]$ and (3.2) that

$$V \circ \mathcal{L}_{\Xi'} = [q_{\xi}, V(\Xi')] \circ V.$$

### 3.2 Extended differential Poisson bracket

The isomorphism also intertwines extended differential Poisson brackets $\{\cdot, \cdot\}$ on $M$, which by their definition obey (2.23)–(2.28) and

$$\text{deg}_M(\{\cdot, \cdot\}) = 0 \mod 2,$$

with Poisson superbrackets $\{\cdot, \cdot\}_f$ on $T[0,1]N$ that are compatible with $q_f$ and have intrinsic bi-degrees

$$\text{bideg}(\{\cdot, \cdot\}_f) = (0,0) \mod (0,2).$$

In other words,

$$V \circ \{\omega, \eta\} = \{V_\omega, V_\eta\}_f, \quad \{V_1, V_2\}_f \equiv \Pi_f(dV_1, dV_2),$$

for $V_i \in C^\infty(T[0,1]M)$, where the Poisson bi-supervector field $\Pi_f$ on $T[0,1]M$ obeys

$$\{\Pi_f, \Pi_f\}_{\text{S.N.}} = 0, \quad \mathcal{L}_{q_f} \Pi_f = 0, \quad \text{bideg}(\Pi_f) = (-2,0) \mod (0,2),$$

using the Schouten–Nijenhuis superbracket $\{\cdot, \cdot\}_{\text{S.N.}}$ for graded antisymmetric polysupervector fields on $T[0,1]M$.

---

16 An isomorphism $V : A \to \bar{A}$ between associative algebras induces the isomorphism $V : \text{Der}(A) \to \text{Der}(\bar{A})$ defined by $V(\delta(a)) = V(\delta)V(a)$ for all $a \in A$ and derivations $\delta \in \text{Der}(A)$. 

13
3.3 Covariant Hamiltonian action on $T^*[1, 0](T[0, 1]M)$

The Schouten–Nijenhuis superbracket $\{\cdot, \cdot\}_{\text{S.N.}}$ on $T[0, 1]M$ can be mapped to the canonical Poisson bracket $\{\cdot, \cdot\}_{(-1, 0)}$ on the parity shifted cotangent bundle. \[ T^*[1, 0](T[0, 1]M) \equiv (\mathbb{R}^{m}[1, 0] \oplus \mathbb{R}^{m}[1, -1]) \rightarrow \mathcal{E} \xrightarrow{\pi} T[0, 1]M, \] where $m = \dim M$. To this end, we start from the tautological one-form $\vartheta$ on $T^*[1, 0](T[0, 1]M)$, which obeys \[ \text{bideg}(\vartheta) = (2, 0), \quad \iota_v \vartheta = 0 \quad \text{for all } v \in \ker \pi_*, \] where $\pi$ is the projection map of $T^*[1, 0](T[0, 1]M)$. The canonical two-form \[ \mathcal{O} := d\vartheta, \quad \text{bideg}(\mathcal{O}) = (3, 0), \] as $\text{bideg}(d|_{\mathcal{E}}) = (1, 0)$. The resulting canonical bracket on $\mathcal{E}$ has intrinsic bi-degree \[ \text{bideg}(\{\cdot, \cdot\}_{(-1, 0)}) = (-1, 0), \] and is graded antisymmetric and obey the graded Leibniz' rule and Jacobi identity, viz.

\[
\{\mathcal{F}_1, \mathcal{F}_2\}_{(-1, 0)} = (-1)^{1 + (|\mathcal{F}_1|+1)(|\mathcal{F}_2|+1)}\{\mathcal{F}_2, \mathcal{F}_1\}_{(-1, 0)}, \\
\{\mathcal{F}_1, \mathcal{F}_2\mathcal{F}_3\}_{(-1, 0)} = \{\mathcal{F}_1, \mathcal{F}_2\}_{(-1, 0)}\mathcal{F}_3 + (-1)^{|\mathcal{F}_2|(|\mathcal{F}_1|+1)}\mathcal{F}_2\{\mathcal{F}_1, \mathcal{F}_3\}_{(-1, 0)},
\]

and

\[
\{\{\mathcal{F}_1, \mathcal{F}_2\}_{(-1, 0)}, \mathcal{F}_3\}_{(-1, 0)} + (-1)^{(|\mathcal{F}_2|+|\mathcal{F}_3|)(|\mathcal{F}_1|+1)}\{\{\mathcal{F}_2, \mathcal{F}_3\}_{(-1, 0)}, \mathcal{F}_1\}_{(-1, 0)} \\
+ (-1)^{|\mathcal{F}_1|+|\mathcal{F}_2|(|\mathcal{F}_3|+1)}\{\{\mathcal{F}_3, \mathcal{F}_1\}_{(-1, 0)}, \mathcal{F}_2\}_{(-1, 0)} = 0,
\]

where $\mathcal{F}_i$ are functions on $\mathcal{E}$. These can be expanded as \[ \mathcal{F} = \sum_{n=0}^{\infty} \mathcal{F}^{(n)} , \quad \mathcal{F}^{(n)} = P^{(n)}(\vartheta \wedge^n) , \quad \text{bideg}(P^{(n)}) = \text{bideg}(\mathcal{F}^{(n)}) - (n, 0), \] where $P^{(n)}$ are rank $n$ graded antisymmetric polyvector fields on $T^*[1, 0](T[0, 1]M)$ defining equivalence classes \[ \left[P^{(n)}\right] = \left[P^{(n)} + P^{(n-1)} \wedge v\right], \quad v \in \ker \pi_*. \]  

---

\[ 17 \] This map is a classical counterpart of a map used in the AKSZ procedure \[ 3 \] for constructing covariant Hamiltonian BV master actions from integrable polyvector field structures; see also \[ 36, 37 \].

\[ 18 \] On the total space $\mathcal{E}$ of $T^*[1, 0](T[0, 1]M)$, the space of forms $\Omega(\mathcal{E}) = \bigoplus_{m, n, p} \Omega_{[p]}(m, n)|\mathcal{E}$ where $\deg_{\mathcal{E}}(\Omega_{[p]}(m, n)|\mathcal{E}) = p$ and $\text{bideg}(\Omega_{[p]}(m, n)|\mathcal{E}) = (m, n)$. 

---

14
in view of (3.13), whose projections to the base define (distinct) graded antisymmetric rank \( n \) polysupervector fields

\[ P^{(n)}_f = \pi_* P^{(n)} , \tag{3.22} \]
on \( T[0,1]M \). Conversely, one has a bijective uplift \( \rho \) as follows:

\[ [P^{(n)}] = \rho(P^{(n)}_f) , \quad \pi_* \circ \rho = \text{id} , \tag{3.23} \]

which thus has \( \ker \rho = 0 \). Writing \( [P^{(n)}] (\vartheta \wedge^n) \equiv P^{(n)}(\vartheta \wedge^n) \), the relation between the canonical Poisson bracket on \( \mathcal{E} \) and the Schouten–Nijenhuis superbracket \( \{ \cdot, \cdot \}_\text{S.N.} \) on \( T[0,1]M \) takes the following form:

\[ \{ \mathcal{F}^{(n_1)}_1, \mathcal{F}^{(n_2)}_2 \}_{(-1,0)} \equiv \{ \rho(P^{(n_1)}_{f,1})(\vartheta \wedge^{n_1}), \rho(P^{(n_2)}_{f,2})(\vartheta \wedge^{n_2}) \}_{(-1,0)} = \rho(\{P^{(n_1)}_{f,1}, P^{(n_2)}_{f,2}\}_\text{S.N.})(\vartheta \wedge^{(n_1+n_2-1)}) . \tag{3.24} \]

Next, the vector field \( q_f \) on \( T[0,1]M \) is uplifted to a nilpotent vector field \( Q \) on \( \mathcal{E} \) defined by

\[ \mathcal{L}_Q \vartheta = 0 , \quad \pi_* Q = q_f , \quad \text{bideg}(Q) = (0,1) . \tag{3.25} \]

Thus, from \( \pi_*(\mathcal{L}_Q \rho(P^{(n)}_f)) = \mathcal{L}_{q_f} P^{(n)}_f \) and (3.23) it follows that

\[ \mathcal{L}_Q \mathcal{F}^{(n)} = \mathcal{L}_Q \left( \rho(P^{(n)}_f)(\vartheta \wedge^n) \right) = \mathcal{L}_Q \rho(P^{(n)}_f) (\vartheta \wedge^n) + (-1)^{\mathcal{F}^{(n)} | n} \rho(P^{(n)}_f) ((\mathcal{L}_Q \vartheta) \wedge^n \vartheta \wedge^{n-1}) = \rho(\mathcal{L}_{q_f} P^{(n)}_f)(\vartheta \wedge^n) . \tag{3.26} \]

Turning to the supersymmetric Hamiltonian function \( \mathcal{H} \) on \( \mathcal{E} \), by definition it obeys \[ 19 \]
\[ \mathcal{Q} \mathcal{H} = 0 , \quad \{ \mathcal{H}, \mathcal{H} \}_{(-1,0)} = 0 , \quad \text{bideg}(\mathcal{H}) = (2,0) \mod (0,2) , \tag{3.27} \]
from which it follows that

\[ \mathcal{H} = \rho(\Pi_f^{(2)})(\vartheta \wedge^2) , \tag{3.28} \]
where \( \Pi_f^{(2)} \) obeys (3.11). Furthermore, as demonstrated in Appendix C locally on \( \mathcal{E} \) there exist a one-form \( \mathcal{G} \) and a function \( \mathcal{W} \) such that

\[ \vartheta = \mathcal{L}_Q \mathcal{G} , \quad \mathcal{H} = \mathcal{Q} \mathcal{W} , \quad \text{bideg} (\mathcal{G}) = \text{bideg} (\mathcal{W}) = (2, -1) . \tag{3.29} \]

\[ ^{19} \text{The condition on the bi-degree implies that } \mathcal{H} \text{ is quadratic in momenta. In the AKSZ approach, this implies that } \mathcal{H} \text{ vanishes on the trivial section as required by the boundary conditions following from the BV master equation.} \]
The resulting covariant Hamiltonian action of the classical theory reads

\[ S = \int_\Sigma \varphi^*(\vartheta + \mathcal{H}) = \delta t \int_\Sigma \varphi^* \mathcal{V}, \quad \mathcal{V} := \mathcal{G} + \mathcal{W}, \]  

(3.30)
where the sigma model map \( \varphi : \Sigma \to T[0,1]M \) is assumed to have vanishing intrinsic bi-degree, \( \text{i.e.} \)

\[ \text{bideg}(\varphi) = (0, 0), \]

(3.31)
and obey the boundary condition

\[ \varphi : \partial \Sigma \to T[0,1]M, \]

(3.32)
that is, the boundary of \( \Sigma \) is sent to the trivial section on \( T^*[1,0](T[0,1]M) \), \( \text{i.e.} \) the momenta vanishes at \( \partial \Sigma \).

### 3.4 Formulation in local coordinates

We coordinatize \( \mathcal{E} \) using \( (i = 0, 1) \)

\[ \Phi_i^\alpha = (\Phi_i^\alpha, \Phi_i^\beta) \equiv (\phi^\alpha, \theta^\beta), \quad \text{bideg}(\Phi_i^\alpha) = (0, i), \]

(3.33)
\[ H_i^\alpha = (H_i^0, H_i^1) \equiv (\omega_\alpha, \chi_\alpha), \quad \text{bideg}(H_i^\alpha) = (1, -i). \]

(3.34)
where \( H_i^\alpha \) and \( \Phi_i^\alpha \) are coordinates of the fiber and the base of \( T^*[1,0](T[0,1]M) \), respectively, and \( \phi^\alpha \) and \( \theta^\beta \) are coordinates of the base and fiber of \( T[0,1]M \), respectively. The Koszul sign convention \( (2.47) \) yields the following graded commutativity relations:

\[ \Phi_i^\alpha \Phi_j^\beta = (-1)^{ij} \Phi_j^\beta \Phi_i^\alpha, \quad \Phi_i^\alpha H_j^\beta = (-1)^{(1+i)(1+j)} H_j^\beta \Phi_i^\alpha, \quad H_i^\alpha H_j^\beta = (-1)^{(1+i)(1+j)} H_j^\beta H_i^\alpha. \]

(3.35)
The nilpotent vector field \( q_t \) is given by

\[ q_t = \theta^\alpha \frac{\partial}{\partial \phi^\alpha} \equiv q_t^i \Phi_i^\alpha \partial_\alpha^i, \quad q_t^i = \delta_t^0 \delta_t^i, \quad \partial_\alpha^i := \frac{\partial}{\partial \Phi_i^\alpha}. \]

(3.36)
As for the canonical two-form and tautological one-form, we take

\[ \mathcal{O} = dH_i^\alpha \wedge d\Phi_i^\alpha, \quad \vartheta = H_i^\alpha d\Phi_i^\alpha. \]

(3.37)
It follows that the uplift \( \mathcal{Q} \) to \( \mathcal{E} \) of \( q_t \) on \( T[0,1]M \) is given by

\[ \mathcal{Q} = q_t^i (\Phi_j^\beta \partial_\alpha^j - H_\alpha^i \partial_\alpha^j) = q_t + \tilde{q}_t, \quad \tilde{q}_t = -\omega_\alpha \frac{\partial}{\partial \chi_\alpha}, \]

(3.38)
obeying

\[(q_t)^2 = \{q_t, \tilde{q}_t\} = (\tilde{q}_t)^2 = 0 .\]  

(3.39)

Expanding the Poisson bi-supervector on \(T[0, 1]M\) as

\[\Pi_f = \Pi_{ij}^{\alpha\beta} \partial^i_{\alpha} \wedge \partial^j_{\beta} , \quad \partial^i_{\alpha} \wedge \partial^j_{\beta} = -(1)^{ij} \partial^j_{\beta} \wedge \partial^i_{\alpha} , \]

(3.40)

where thus

\[\Pi_{ij}^{\alpha\beta} = -(1)^{ij} \Pi_{ji}^{\beta\alpha} , \quad \text{deg}(\Pi_{ij}^{\alpha\beta}) = 0 , \quad n_f(\Pi_{ij}^{\alpha\beta}) = i + j , \]

(3.41)

and using \(\text{bideg}(d\Phi^\alpha) = (1, i)\), the coordinate form of the superbracket (3.10) reads

\[\{f_1, f_2\}_f = (-1)^{i+j(|f_1|+1)} \Pi_{ij}^{\alpha\beta} \partial_i \alpha f_1 \partial_j \beta f_2 .\]  

(3.42)

The Poisson bi-supervector and \(q_f\)-compatibility conditions take the following form:

\[\Pi_{il}^{\alpha\beta} \partial_l^i \delta_{ik} \partial^j_{\beta} \wedge \partial^k_{\gamma} = 0 , \]

(3.43)

\[\left( q_{i} \Pi_{kl}^{\beta\gamma} + 2 \Pi_{ij}^{\beta\gamma} q_{k} \right) \partial^k_{\beta} \wedge \partial^l_{\gamma} = 0 . \]

(3.44)

The corresponding supersymmetric Hamiltonian function on \(E\) is given by

\[\mathcal{H} = \rho(\Pi_f)(\partial \otimes \partial) = \frac{1}{2} \Pi_{ij}^{\alpha\beta} H^i_{\alpha} H^j_{\beta} . \]

(3.45)

The explicit form of the resulting covariant Hamiltonian action (3.30) takes the form

\[S = \int \varphi^*(H^i_{\alpha} d\Phi^\alpha + \frac{1}{2} \Pi_{ij}^{\alpha\beta} H^i_{\alpha} H^j_{\beta}) , \]

(3.46)

where the form degrees on \(\Sigma\) and fermion numbers of the pulled back fields are given by

\[
\begin{array}{c|cccc}
\varphi^*\Phi^\alpha_{_0} & \varphi^*\Phi^1_{_0} & \varphi^*H^0_{\alpha} & \varphi^*H^1_{\alpha} & d \\
\hline
deg_\Sigma : & 0 & 0 & 1 & 1 \\
n_f & 0 & 1 & 0 & -1 \\
\end{array}
\]

(3.47)

Suppressing \(\varphi^*\), the equations of motion read

\[\mathcal{R}^\alpha_i := d\Phi^\alpha_i + (-1)^{i+j} \Pi_{ij}^{\alpha\beta} H^j_{\beta} \approx 0 , \]

(3.48)

\[\mathcal{R}^i_{\alpha} := dH^i_{\alpha} + (-1)^j \frac{1}{2} \partial^i_{\alpha} \Pi_{jk}^{\beta\gamma} H^j_{\beta} \wedge H^k_{\gamma} \approx 0 , \]

(3.49)
which form a universally Cartan integrable system by virtue of (3.43) and (3.44). The rigid nilpotent supersymmetry transformation, \( \delta_{f} \),
\[
\delta_{f} \Phi_{i}^{\alpha} = q_{i}^{j} \Phi_{j}^{\alpha}, \quad \delta_{f} H_{i}^{\alpha} = -q_{j}^{i} H_{j}^{\alpha},
\]
leaves the action invariant, as follows from
\[
S = \delta_{f} \int_{\Sigma} \mathcal{V}, \quad \mathcal{V} = -H_{i}^{\alpha} \wedge (q^{T})_{i}^{j} \left( d\Phi_{j}^{\alpha} + \frac{1}{2} \Pi_{jk}^{\alpha\beta} H_{k}^{\beta} \right), \quad (q^{T})_{i}^{j} = \delta_{i}^{j}\delta_{0}^{l},
\]
as can be seen by using (3.44), that is,
\[
\delta_{f} \Pi_{ij}^{\alpha\beta} = (-1)^{j+k} q_{i}^{k} \Pi_{kj}^{\alpha\beta} - (-1)^{k} q_{j}^{k} \Pi_{ik}^{\alpha\beta},
\]
or more explicitly
\[
\delta_{f} \Pi_{00}^{\alpha\beta} = -2 \Pi_{01}^{[\alpha\beta]} , \quad \delta_{f} \Pi_{01}^{\alpha\beta} = \Pi_{11}^{\alpha\beta} , \quad \delta_{f} \Pi_{11}^{\alpha\beta} = -\Pi_{11}^{\alpha\beta} , \quad \delta_{f} \Pi_{11}^{\alpha\beta} = 0.
\]

As can be seen from Tables (2.41) and Table (3.47), the spectra of fields in the supersymmetric actions (2.40) and (3.46) agree. Indeed, in the case of a differential Poisson manifold, there exists a simple field redefinition that maps (2.40) to (3.46) (without the need to add any total derivative), as will be spelled out in detail in Section 4.

3.5 Rigid supersymmetries from Killing supervectors

The notion of a symmetry of a Poisson algebra of functions refers to a vector field \( K \) on \( M \) whose Lie derivative annihilates the Poisson bi-vector field, \( \delta_{K} \).
\[
\mathcal{L}_{K} \Phi = 0 , \quad \mathcal{L}_{K} H = 0 ,
\]
as this is equivalent to that
\[
\mathcal{L}_{K} \{ f, g \} = 2 \mathcal{L}_{K} (\Pi(df, dg)) \equiv 2 ((\mathcal{L}_{K} \Pi)(df, dg) + \Pi(\mathcal{L}_{K} df, g) + \Pi(f, \mathcal{L}_{K} dg))
\]
\[
= 2 (\Pi(d\mathcal{L}_{K} f, g) + \Pi(f, d\mathcal{L}_{K} g)) = \{ \mathcal{L}_{K} f, g \} + \{ f, \mathcal{L}_{K} g \} ,
\]
for all \( f, g \in C^{\infty}(M) \). Such a vector, which is often referred to as a Killing (or fundamental) vector, induces a rigid symmetry of the Ikeda–Scheller–Strobl sigma model.

The above notions have natural generalizations to the context of differential Poisson algebras and their induced supersymmetric sigma models. Thus, a symmetry of an extended differential Poisson algebra refers to a vector field valued \( p \)-form \( K \) (see Appendix B) obeying
\[
\mathcal{L}_{K} \{ \omega_{1}, \omega_{2} \} = \{ \mathcal{L}_{K} \omega_{1}, \omega_{2} \} + (-1)^{\text{deg}_{M}(\omega_{1})\text{deg}_{M}(K)} \{ \omega_{1}, \mathcal{L}_{K} \omega_{2} \} ,
\]
for all $\omega_1, \omega_2 \in \Omega(M)$.

A general vector field valued $p$-form $\Xi$ on $M$ is sent by the isomorphism $V$ in (3.1) to the supervector field $\Xi_\ell = V(\Xi)$ on $T[0,1]M$ defined by (3.7), which in its turn induces a supervector field $X_\Xi$ on $E$ defined by

$$\pi_* X_\Xi = \Xi_\ell \, , \quad \mathcal{L}_{X_\Xi} \theta = 0 \, , \quad \text{bideg}(X_\Xi) = \text{bideg}(\Xi_\ell) \, .$$

(3.57)

Thus, in a coordinate basis where $\Xi_\ell = \Xi_\ell^i \partial^i$, we have

$$\delta \Xi_\ell \Phi^i_\alpha \equiv X_\Xi (\Phi^i_\alpha) = \Xi_\ell^i \, , \quad \delta \Xi_\ell H^i_\alpha \equiv X_\Xi (H^i_\alpha) = -(-1)^{i+j+|\Xi_\ell|} \partial^i_\alpha \Xi_\ell^j H^j_\beta \, .$$

(3.58)

It follows that

$$\delta \Xi_\ell S[\Phi, H; \Pi_\ell] = \mathcal{L}_{\Xi_\ell} S[\Phi, H; \Pi_\ell] \, ,$$

(3.59)

where thus $\delta \Xi_\ell$ acts only on the fields and $\mathcal{L}_{\Xi_\ell}$ acts only on the background field.

Thus, the vector field valued $p$-form $K$ of Killing type is mapped to supervector fields $K_\ell = V(K)$ on $T[0,1]M$ and $X_K$ on $E$, obeying

$$\mathcal{L}_{K_\ell} \Pi_\ell = 0 \, , \quad X_K \mathcal{H} = 0 \, ,$$

(3.60)

and hence it generates a global symmetry, viz.

$$\delta_{K_\ell} S[\Phi, H; \Pi_\ell] = 0 \, .$$

(3.61)

In this language, the compatibility between the extended differential Poisson bracket and the de Rham differential amounts to that the vector field valued one-form $I = d\phi^\alpha \partial_\alpha$ on $M$ is of Killing type. It is mapped by $V$ to $q_\ell = V(I) = q_\ell^i \Phi^\alpha_\ell \partial^i_\alpha$, hence inducing the rigid nilpotent supersymmetry transformation (3.50). An ordinary Killing vector field $K = K^\alpha \partial_\alpha$ on $M$, on the order hand, induces the Killing supervector

$$K_\ell = K_\ell^i \partial^i_\alpha \, , \quad V \circ \mathcal{L}_K = \mathcal{L}_{K_\ell} \circ V \, , \quad \text{bideg}(K_\ell) = (0,0) \, ,$$

(3.62)

on $T[0,1]M$ with components given by

$$K_0^\alpha = K^\alpha_0 \, , \quad K_1^\alpha = -\theta^\beta \partial_\beta K^\alpha \, .$$

(3.63)

The inner derivative of forms on $M$ along an ordinary Killing vector $K$, which is mapped by $V$ to

$$\tilde{K}_\ell = \tilde{K}_\ell^i \partial^i_\alpha \, , \quad V \circ \iota_K = \mathcal{L}_{\tilde{K}_\ell} \circ V \, , \quad \text{bideg}(\tilde{K}_\ell) = (0, -1) \, ,$$

(3.64)

20Conversely, if $K$ is an ordinary vector field on $M$ and $\iota_K$ is a symmetry of the (extended) differential Poisson bracket then $K$ must be an ordinary Killing vector field.
on $T[0, 1]M$, is a symmetry as well provided that $\iota_K$ commutes with the extended Poisson bracket, or equivalently, that

$$\mathcal{L}_{\tilde{K}} \Pi = 0,$$

whose component form will be derived below in Section 4.2 in the differential Poisson case.

## 4 Component formulation

In this section, we identify the supersymmetric model in Section 2 as the special case of the model in Section 3 that arises on differential Poisson manifolds with vanishing $S$-tensor. We shall also include non-vanishing $S$-tensors, and derive the supplementary conditions on a Killing vector on $M$ for it to yield an extra supersymmetry of bi-degree $(0, -1)$.

### 4.1 Action and equations of motion

In order to obtain the action (2.40) from (3.46), we take

$$\Phi^i = (\phi^\alpha, \theta^\alpha), \quad H^i_\alpha = (\eta_\alpha - \Gamma^\gamma_{\alpha\beta} \chi^\beta, \chi^\alpha),$$

where $\Gamma^\gamma_{\alpha\beta}$ are the coefficients of the connection one-form of the differential Poisson algebra. In the unextended case, we have

$$\Pi^{\alpha\beta}_{00} = \Pi^{\alpha\beta},$$

$$\Pi^{\alpha\beta}_{01} = -\left(S^\alpha_{\gamma} + \Pi^{\alpha\delta} \Gamma^\delta_{\gamma}\right) \theta^\gamma,$$

$$\Pi^{\alpha\beta}_{10} = \Pi^{\beta\alpha}_{01},$$

$$\Pi^{\alpha\beta}_{11} = \left(\frac{1}{2} \Upsilon^\alpha_{\gamma\delta} - 2 S^\alpha_{\gamma} \Gamma^\beta_{\epsilon\gamma} + \Pi^{\epsilon\lambda} \Gamma^\alpha_{\epsilon\gamma} \Gamma^\beta_{\lambda\delta}\right) \theta^\gamma \theta^\delta,$$

where we have separated the combination

$$\Upsilon^\alpha_{\gamma\delta} := \tilde{R}^\alpha_{\gamma\delta} - 2 \tilde{\nabla}_{[\gamma} S^\alpha_{\delta]} + T^\epsilon_{\gamma\delta} S^\alpha_{\epsilon}.$$

Plugging (4.1) and (4.2) back into the action (3.46), we obtain

$$S = \int_{\Sigma} \left[ \eta_\alpha \wedge d\phi^\alpha + \chi_\alpha \wedge \nabla \theta^\alpha + \frac{1}{2} \Pi^{\alpha\beta} \eta_\alpha \wedge \eta_\beta + S^\alpha_{\gamma} \eta_\alpha \wedge \chi_\beta \theta^\gamma + \frac{1}{4} \Upsilon^\alpha_{\gamma\delta} \chi_\alpha \wedge \chi_\beta \theta^\gamma \theta^\delta \right],$$

\[21\] In the extended case, the Lagrangian contains additional terms of bi-degrees $(2, 2k)$ for $k \geq 1$. Fitting these models into the minimal AKSZ geometry requires working with actions whose degree vanishes mod two.
which reduces to (2.40) upon setting $S^\alpha_\gamma = 0$. Varying the action with respect to $(\eta, \chi, \theta)$, respectively, yields

\begin{align*}
\mathcal{R}^\phi_\alpha := &\ d\phi^\alpha + \Pi^\alpha_\beta \eta_\beta + S^\alpha_\gamma \chi_\beta \theta^\gamma \approx 0 , \\
\mathcal{R}^\eta_\alpha := &\ \nabla \theta^\alpha + S^\alpha_\gamma \eta_\beta \theta^\gamma + \frac{1}{2} \Upsilon^\gamma_\delta \chi_\beta \theta^\gamma \theta^\delta \approx 0 , \\
\mathcal{R}^{\chi}_\alpha := &\ \nabla \chi_\alpha - S^\alpha_\beta \eta_\beta \wedge \chi_\gamma + \frac{1}{2} \Upsilon^\gamma_\delta \chi_\beta \wedge \chi_\delta \theta^\beta \approx 0 ,
\end{align*}

while its variation with respect to $\phi$ yields

\begin{align*}
\mathcal{R}^\eta_\alpha := &\ d\eta_\alpha + \frac{1}{2} \partial_\alpha \Pi^\beta_\gamma \eta_\beta \wedge \eta_\gamma + \left( \partial_\alpha S^\gamma_\delta \eta_\gamma \wedge \chi_\delta - \Gamma^\gamma_\alpha \beta \eta_\gamma \wedge d\phi^\beta \right) \theta^\beta \\
&\ - \Gamma^\gamma_\alpha \beta \chi_\gamma \wedge d\theta^\beta + \frac{1}{4} \partial_\alpha \Upsilon^\gamma_\delta \chi_\delta \wedge \chi_\epsilon \theta^\beta \theta^\gamma \approx 0 ,
\end{align*}

which can be rewritten in a manifestly covariant form by using the compatibility condition $\partial_\alpha \Pi^\beta_\gamma = 2 \Gamma^\beta_\delta_\alpha \Pi^\gamma_\delta$ and (4.5)–(4.7), with the result

\begin{align*}
\mathcal{R}^\eta_\alpha := &\ \nabla \eta_\alpha + R^\alpha_\beta \gamma_\delta \phi^\beta \wedge \chi_\gamma \theta^\delta + \left( \nabla \alpha S^\gamma_\delta - T^\gamma_\alpha \delta \epsilon \right) \eta_\gamma \wedge \chi_\delta \theta^\beta + \frac{1}{4} \nabla \alpha \Upsilon^\gamma_\delta \chi_\delta \wedge \chi_\epsilon \theta^\beta \theta^\gamma .
\end{align*}

4.2 Supersymmetries

$\mathcal{N} = 1$ supersymmetry (de Rham operator): By construction, the action (4.4) can be written on the manifestly globally supersymmetric form (3.51), viz.

\begin{align*}
S = &\ \delta_f \int_\Sigma \mathcal{V} , \quad \mathcal{V} = -\chi_\alpha \wedge \left( d\phi^\alpha + \frac{1}{2} \Pi^\alpha_\beta \eta_\beta + \frac{1}{2} S^\alpha_\gamma \chi_\beta \theta^\gamma \right)
\end{align*}

where the rigid nilpotent supersymmetry transformation, which is given in general by (3.50), takes the following form in terms of the component fields $(\phi^\alpha, \theta^\alpha; \eta_\alpha, \chi_\alpha)$:

\begin{align*}
\delta_\ell \phi^\alpha &= \theta^\alpha , \\
\delta_\ell \theta^\alpha &= 0 , \\
\delta_\ell \eta_\alpha &= \frac{1}{2} \tilde{R}_\beta \gamma^\delta \alpha \chi_\delta \theta^\beta \theta^\gamma - \Gamma^\gamma_\alpha \beta \eta_\gamma \theta^\beta , \\
\delta_\ell \chi_\alpha &= -\eta_\alpha + \Gamma^\gamma_\alpha \beta \chi_\gamma \theta^\beta .
\end{align*}

Its nilpotency can be verified using the compatibility condition $\tilde{\nabla}_\alpha \Pi^\beta_\gamma = 0$ and the Bianchi identity $\tilde{\nabla}_\alpha [\tilde{R}_\beta \gamma]_\delta^\epsilon - \tilde{T}_\alpha \beta^\lambda \gamma \lambda \delta^\epsilon = 0$. 

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Extended supersymmetry (inner derivatives): Let us demonstrate at the level of components that a Killing vector $K = K^\alpha \partial_\alpha$ obeying (3.65) yields an additional nilpotent rigid supersymmetry given by (3.57) and (3.64), i.e.

$$\delta_{t,K} \phi^\alpha = 0 ,$$  \hspace{1cm} (4.12)
$$\delta_{t,K} \theta^\alpha = K^\alpha ,$$  \hspace{1cm} (4.13)
$$\delta_{t,K} \eta_\alpha = \chi_\beta \nabla_\alpha K^\beta ,$$  \hspace{1cm} (4.14)
$$\delta_{t,K} \chi_\alpha = 0 ,$$  \hspace{1cm} (4.15)

which thus acts non-trivially only on the fields with odd total degree. Indeed, by (3.57) the variation of the kinetic term under $\delta_{t,K}$, given by the pull back of the symplectic potential, vanishes identically (without the need to use the Killing vector property), whereas the variation of the Hamiltonian term reads

$$\delta_{t,K} S = \int_\Sigma \left[ (\Pi^{\gamma \alpha} \nabla_\gamma K^\beta - K^\gamma S^\alpha_{\gamma \beta} ) \eta_\alpha \wedge \chi_\beta + \left( \frac{1}{2} K^\gamma \Upsilon^\alpha_{\gamma \beta} + S^\beta_{\delta} \nabla_\gamma K^\alpha \right) \chi_\alpha \wedge \chi_\beta \theta^\delta \right] ,$$  \hspace{1cm} (4.16)

that vanishes iff the two terms vanish separately. We recall that

$$L_K \Gamma^\gamma_{\alpha \beta} \equiv \partial_\alpha \partial_\beta K^\gamma - \partial_\gamma K^\delta \Gamma^\gamma_{\delta \beta} - \partial_\beta K^\delta \Gamma^\gamma_{\alpha \delta} + \partial_\delta K^\gamma \Gamma^\delta_{\alpha \beta} = 0 ,$$  \hspace{1cm} (4.17)
$$L_K \Pi^{\alpha \beta} \equiv K^\gamma \partial_\gamma \Pi^{\alpha \beta} + 2 \partial_\gamma K^{[\alpha} \Pi^{\beta] \gamma} = 0 ,$$  \hspace{1cm} (4.18)
$$L_K S^\alpha_{\gamma \beta} \equiv K^\delta \nabla_\delta S^\alpha_{\gamma \beta} + S^\alpha_{\delta} \nabla_\gamma K^\delta - 2 S^\delta (\alpha \nabla_\gamma K^\beta ) = 0 .$$  \hspace{1cm} (4.19)

The first equation can equivalently be written as

$$\nabla_\alpha \nabla_\beta K^\gamma = K^\delta \nabla_\delta K^\gamma \beta ,$$  \hspace{1cm} (4.20)

and the second equation combined with the compatibility condition $\partial_\gamma \Pi^{\alpha \beta} = 2 \Gamma^{[\alpha}_{\delta \gamma} \Pi^{\beta] \delta}$ yields

$$\Pi^{[\alpha \nabla_\gamma K^\beta ] = 0 .$$  \hspace{1cm} (4.21)

Thus, first term in (4.16) vanishes iff the following stronger version of (4.21) holds:

$$\Pi^\gamma [\nabla_\gamma K^\beta ] = 0 .$$  \hspace{1cm} (4.22)

As for the second term in (4.16), using the $\nabla$-derivative of (4.22) together with $\nabla_\gamma \Pi^{\alpha \beta} = 0$, (4.19) and (4.20), it can be rewritten as

$$\delta_K S = 2 \int_\Sigma K^\gamma \nabla_\gamma \phi^{[\alpha \beta} \chi_\alpha \wedge \chi_\beta \theta^\delta ,$$  \hspace{1cm} (4.23)
whose vanishing requires

\[ K^\gamma \tilde{R}_{\delta \gamma} \alpha = 0 . \] (4.24)

Thus, in summary, a Killing vector \( K \) induces an extended supersymmetry of bidegree \((0,-1)\) if it obeys the additional conditions (4.22) and (4.24), which can be shown to be equivalent to (3.65). Moreover, from the Cartan relations \( \{ \iota_K, d \} = \mathcal{L}_K \) and \((\iota_K)^2 = 0\) it follows that

\[ \{ \delta_{\iota_K}, \delta_{\iota}_K \delta_{\iota} \} = \delta_K , \quad \{ \delta_K, \delta_K \} = 0 , \] (4.25)

where \( \delta_K \) denotes the action of the ordinary Killing vector \( K \) on the fields, viz.

\[ \begin{align*}
\delta_K \phi^\alpha &= K^\alpha \\
\delta_K \theta^\alpha &= \theta^\beta \partial_\beta K^\alpha \\
\delta_K \omega_\alpha &= \partial_\alpha \partial_\beta K^\gamma \chi^\beta - \omega_\beta \partial_\alpha K^\beta \\
\delta_K \chi_\alpha &= -\chi_\beta \partial_\alpha K^\beta .
\end{align*} \] (4.26)

5 Conclusion and outlook

We have reformulated the supersymmetric extension [10] of the Ikeda–Schaller–Strobl model [5, 6] induced on a differential Poisson manifold \( M \) as a special case of the induced sigma model on the bi-graded supermanifold \( T[0,1]M \) equipped with a Poisson superbracket corresponding to an extended differential Poisson bracket on the differential graded algebra of forms on \( M \) with intrinsic form degree valued in \( \{0,2,\ldots\} \). The resulting covariant Hamiltonian action on the phase space \( T^*[1,0](T[0,1])M \) is manifestly \( \text{Diff}(T[0,1]) \) covariant. Consequently, it is manifestly invariant under global symmetries generated by Killing supervector fields on \( T[0,1]M \), which we have used to give new extended supersymmetries associated to inner derivatives along special Killing vectors fields on \( M \).

The current model is a special case of the more general sigma model with canonical action

\[ S_{\text{can}} = \int_{\Sigma} \left( H_\alpha \wedge d\Phi^\alpha + \frac{1}{2} \mathcal{P}^{\alpha\beta} H_\alpha \wedge H_\beta \right) + \oint_{\partial \Sigma} H_\alpha \mathcal{B}^\alpha , \] (5.1)

where \( \mathcal{P} := \mathcal{P}_\alpha \beta \partial_\alpha \wedge \beta \) is a Poisson bi-supervector on target supermanifold of type \((m|m')\) coordinatized by \( \Phi^\alpha = (\Phi^\alpha, \Phi'^\alpha) \), \( \alpha = 1,\ldots,m \) and \( \alpha' = 1,\ldots,m' \). The case \( m = m' \) is distinguished, however, by the facts that it can be made to exhibit the de Rham-like supersymmetry and that
it is possible to quantize the model using a minimal AKSZ gauge fixing procedure without additional trivial pairs (instead of the direct extension of the Cattaneo–Felder scheme), as we shall report on in a separate work. Whether this is tied to the subtleties of the work of McCurdy and Zumino remains to be seen.

Another advantage of the canonical form of the action is that it facilitates the gauging of Killing (super)symmetries using the direct supersymmetrization of Zucchini’s bosonic formalism [15]. Concerning the gauging of the original rigid supersymmetry, a subtlety arises as there are two approaches available, depending on whether it is gauged as a Killing supersymmetry as in [16], or if it treated together with $\mathcal{P}$ as an integrable $QP$-structure, as we shall present in a forthcoming work.

Concerning the perturbative quantization of the model, it remains to be investigated whether it yields a differential graded associative algebra or if quantum corrections will induce a homotopy associative structure. As shown in [2], the former structure arises for special differential Poisson geometries related to quantum groups. However, the results of [4] indicate that on more general manifolds there is an incompatibility between the “canonical” associative star product and the de Rham differential at order $\hbar^2$, though there analysis did not exclude the possibility that compatibility can be restored by deforming the differential. Formally, the argument that the star product is compatible with a deformed differential goes as follows: The $\text{Diff}(T[0,1]M)$ covariance of the classical action is broken down to $\text{Diff}(M)$ by means of the minimal gauge fixing procedure and the background field method (using covariant derivatives on $M$ for the Taylor expansion of the background fields) Assuming that there exists a generalization of the Cattaneo–Felder subtraction scheme that yields an associative binary product map $\text{mult}_2$, the action of $\varphi \in \text{Diff}(T[0,1]M)$ on it is equivalent to a Kontsevich-style supergauge transformation $G$, viz

$$((\varphi)^{-1})^*\text{mult}_2(\varphi^*V_\omega, \varphi^*V_\eta; (\varphi^{-1})_*\Pi_f) = G^{-1}\text{mult}_2(GV_\omega, GV_\eta; \Pi_f) , \quad (5.2)$$

using the notation of Section 3 wherein $V_\omega$ is the function on $T[0,1]M$ corresponding to the form $\omega$ on $M$. Since the background is $q_\ell$-invariant it follows that

$$Q_\ell \circ \text{mult}_2 = \text{mult}_2 \circ (Q_\ell \otimes 1 + 1 \otimes Q_\ell) , \quad Q_\ell = q_\ell + A_\ell , \quad (5.3)$$

where $A_\ell = \sum_{n \geq 1} \hbar^n A_\ell^{(n)}$ is the multi-differential operator generating the supergauge transformation induced by $q_\ell$. Thus, assuming that the subtraction scheme does not yield any anomalies in

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22The resulting BRST differential will have a total degree in $\{1,3,\ldots\}$ unless $\mathcal{P}$ is taken to be unextended.

23The analyses of [2] and [4] did not include the tensorial one-form $S$ defined in Eq. (2.29).
the conservation law for the current of $Q_f$, then one has the flatness condition

$$Q_f \circ Q_f = 0 ,$$

(5.4)

of a differential graded associative algebra. We leave it for future work to settle the above subtleties in more detail and whether there will be a need for homotopies in the associativity rule.

Alternatively, higher products can be introduced already at the semiclassical level by considering homotopy Poisson algebras given by sets of $n$-ary brackets, for $n \geq 2$, obeying Jacobi identity up to homotopies. To our best understanding, corresponding induced homotopy Poisson sigma models have not been studied in the literature, and we plan to address them in a future publication.

As pointed out in [35], quantum homotopy associative algebras can be used to extend the cubic Frobenius–Chern–Simons gauge theory by employing the tensor constructions of [32, 33]. Thus, drawing further on [29, 30], we expect that the off-shell formulation of higher spin gravity on general backgrounds requires a deformation of the Chern-Simons-like cubic action found in [34] consisting of simultaneously i) adding quadratic and higher order terms to its Hamiltonian function leading to an “internal” homotopy associative algebra generated by generalized Chan-Paton-like factors corresponding to “discrete” degrees of freedom of an induced homotopy Poisson sigma model; and ii) replacing the differential graded associative algebra of forms on the base manifold valued in the higher spin associative algebra by an “external” homotopy associative algebra corresponding to “continuous” degrees of freedom of the sigma model. The resulting topological string field would thus be valued in the direct product [32, 33] of two first-quantized homotopy associative algebras. One may speculate that such topological open string field theories on target spaces with boundaries may lead to realizations of mirror symmetry transformations as nontrivial transitions between topologically inequivalent boundary states.

Finally, let us point to a few interesting direction for future research: First of all, the results that we have accumulated so far are supportive of the working hypothesis that higher spin gravity on a noncommutative manifold $M$ i) is dual to first-quantized open strings; ii) has a formulation as a second-quantized topological theory on $M$; iii) admits a sum over topologies of $M$ that is dual to a third-quantized theory, which is supported in part by the fact that the kinetic terms make up infinite-dimensional abelian $p$-form systems [38, 39, 40, 41] for which there are cancellations
leading to a well-defined partition functions at one-loop. The goal of these investigations is to establish that the above types of dualities provide a good “quantum gauge principle” for fundamental interactions in nature. A related idea, also mentioned in the Introduction, is that that topological open string fields of suitably gauged models contains zero-forms identifiable with density matrices obeying nonlinear quantum mechanical evolution equations.

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A Conventions and notation

Affine connections. Given a manifold \( M \) with tangent bundle \( TM \) and tensor bundle \( \mathcal{T} = \bigoplus_{m,n \in \mathbb{M}} TM^\otimes m \otimes T^* M^\otimes n \), which is an associative algebra with product \( \otimes \), an affine connection is a \( C^\infty(M) \)-linear map \( \nabla : TM \to \text{Der}(\mathcal{T}) \) that commutes to the diffeomorphism invariant subalgebra of \( \text{End}(\mathcal{T}) \), which is generated by contractions and insertions of the identity tensor. The connection \( \nabla \) is normalized such that if \( X \) is a vector field and \( \phi \in C^\infty(M) \) then

\[
\nabla_X(\phi) = X(\phi) ,
\]

(A.1)

using the notation \( \nabla_X \equiv \nabla(X) \). Thus, \( \nabla_X : TM^\otimes m \otimes T^* M^\otimes n \to TM^\otimes m \otimes T^* M^\otimes n \), and if \( T, T' \in T \) then

\[

\nabla_{\phi X} T = \phi \nabla_X T ,
\]

(A.2)

\[
\nabla_X (T \otimes T') = (\nabla_X T) \otimes T' + T \otimes \nabla_X T' .
\]

(A.3)

\[24\]See discussion of Eqs. (7.1) and (7.2) in [34].
From the compatibility between the connection and the contraction map it follows that if $V$ is a vector field and $\omega$ is a one-form then

$$\nabla_X(\omega(V)) = (\nabla_X \omega)(V) + \omega(\nabla_X V). \tag{A.4}$$

In terms of local coordinates $\phi^\alpha$, the connection is characterised by the one-form

$$\Gamma^\alpha_{\beta\gamma} = d\phi^\gamma \Gamma^\alpha_{\gamma\beta}, \tag{A.5}$$

where the connection coefficients are defined by

$$\nabla_{\partial_\alpha} \partial_\beta = \Gamma^\gamma_{\alpha\beta} \partial_\gamma. \tag{A.6}$$

From (A.4) and $d\phi^\alpha(\partial_\beta) = \delta^\alpha_\beta$ it follows that

$$\nabla_{\partial_\alpha} d\phi^\beta = -\Gamma^\beta_{\alpha\gamma} d\phi^\gamma. \tag{A.7}$$

For notational simplicity, when acting on a tensor $T = T_{\alpha\beta\gamma...\delta}$, we write

$$\nabla_{\partial_\alpha} T \equiv \nabla_\alpha T \equiv (\nabla_\alpha T_{\beta\gamma...\delta}) d\phi^\beta \otimes \cdots \otimes \partial_\gamma \otimes \cdots, \tag{A.8}$$

where thus

$$\nabla_\alpha T_{\beta\gamma...\delta} := \partial_\alpha T_{\beta\gamma...\delta} - \Gamma^\delta_{\alpha\beta} T_{\gamma...\delta} - \cdots + \Gamma^\gamma_{\alpha\delta} T_{\beta\gamma...\delta} + \cdots. \tag{A.9}$$

Given a pair $(X, Y)$ of vector fields, the torsion and Riemann two-forms $T \in TN$ and $R \in \text{End}(\mathcal{T})$, respectively, are defined by the decomposition

$$[\nabla_X, \nabla_Y] = \nabla_{\nabla_X Y - \nabla_Y X - T(X,Y)} + R(X,Y). \tag{A.10}$$

Thus, upon expanding

$$T = \frac{1}{2} d\phi^\alpha \wedge d\phi^\beta T_{\alpha\beta}, \quad T_{\alpha\beta} = T^\gamma_{\alpha\beta} \partial_\gamma, \tag{A.11}$$

$$R = \frac{1}{2} d\phi^\alpha \wedge d\phi^\beta R_{\alpha\beta}, \quad R_{\alpha\beta} \partial_\gamma = R^\delta_{\alpha\beta} \partial_\gamma, \tag{A.12}$$

it follows from

$$[\nabla_\alpha, \nabla_\beta] \partial_\gamma = \nabla_\alpha (\Gamma^\delta_{\beta\gamma} \partial_\delta) - (\alpha \leftrightarrow \beta), \tag{A.13}$$

that

$$R^\gamma_{\alpha\beta\delta} = 2 \partial_{[\alpha} \Gamma^\gamma_{\beta]\delta} + 2 \Gamma^\gamma_{[\alpha\epsilon} \Gamma^\epsilon_{\beta]\delta}, \quad T^\gamma_{\alpha\beta} = 2 \Gamma^\gamma_{[\alpha\beta]} \tag{A.14}.$$
Alternatively, in terms of the covariant derivatives defined in (A.9) one has

\[ [\nabla_\alpha, \nabla_\beta] V^\gamma = -T^\delta_{\alpha\beta} \nabla_\delta V^\gamma + R^\gamma_{\alpha\beta\delta} V^\delta . \]  

(A.15)

If \( T, T' \in T \) and \( X \) is a vector field, we define

\[ \nabla_{T \otimes X} T' = T \otimes \nabla_X T' . \]  

(A.16)

It follows that if \( \omega \) and \( \omega' \) are forms then

\[ \nabla_{\omega \otimes X} (\omega' \otimes X') = \omega \wedge (\nabla_X \omega' \otimes T' + \omega' \otimes \nabla_X T') . \]  

(A.17)

Introducing the vector field valued one-form \( I = d\phi^\alpha \partial_\alpha \), we define, in a slight abuse of notation, the exterior covariant derivative \( \nabla : \Omega(M) \otimes T \rightarrow \Omega(M) \otimes T \) as follows

\[ \nabla (\omega \otimes T) = \nabla I (\omega \otimes T) , \]  

(A.18)

whose action thus take the following form in components:

\[ \nabla (\omega \otimes T) = d\omega \otimes T + d\phi^\alpha \wedge \omega \otimes \nabla_\partial_\alpha T . \]  

(A.19)

In particular, acting on a \( p \)-form \( \omega = \frac{1}{p!} d\phi^{\alpha_1} \wedge \cdots \wedge d\phi^{\alpha_p} \omega_{\alpha_1 \cdots \alpha_p} \) one has

\[ \nabla \omega = d\omega = \frac{1}{p!} d\phi^{\alpha_1} \wedge \cdots \wedge d\phi^{\alpha_{p+1}} \left( \nabla_{\alpha_1} \omega_{\alpha_2 \cdots \alpha_{p+1}} + \frac{p}{2} T^\beta_{\alpha_1 \alpha_2} \omega_{\beta \alpha_3 \cdots \alpha_{p+1}} \right) . \]  

(A.20)

Finally, the action of the exterior covariant derivatives of components is defined via

\[ \nabla (V^\alpha \partial_\alpha) = (\nabla V^\alpha) \partial_\alpha , \quad \nabla (d\phi^\alpha \omega_\alpha) = -d\phi^\alpha \nabla \omega_\alpha , \]  

(A.21)

where thus

\[ \nabla V^\alpha = dV^\alpha + \Gamma^\alpha_{\beta} V^\beta = d\phi^\beta \nabla_\alpha V^\beta , \quad \nabla \omega_\alpha = d\omega_\alpha - \Gamma^\beta_{\alpha} \omega_\beta = d\phi^\beta \nabla_\beta \omega_\alpha . \]  

(A.22)

Defining the torsion and curvature two-forms

\[ T^\alpha = \frac{1}{2} d\phi^\gamma \wedge d\phi^\delta T^\alpha_{\gamma\delta} , \quad R^\alpha_{\beta \gamma} = \frac{1}{2} d\phi^\gamma \wedge d\phi^\delta R^\alpha_{\gamma \delta \beta} , \]  

(A.23)

which can be rewritten as

\[ T^\alpha = d\phi^\beta \wedge d\phi^\gamma \Gamma^\alpha_{\beta \gamma} = \Gamma^\alpha_{\beta} \wedge d\phi^\beta = \nabla d\phi^\alpha , \quad R^\alpha_{\beta} = d\Gamma^\alpha_{\beta} + \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta} . \]  

(A.24)
one has the Ricci identities
\[ \nabla^2 V^\alpha = R^\alpha_{\beta \gamma} V^\gamma, \quad \nabla^2 \omega_\alpha = -R^\beta_{\alpha \gamma} \omega_\gamma, \]
and the Bianchi identities
\[ \nabla T^\alpha = R^\alpha_{\beta \gamma} \wedge d\phi^\beta, \quad \nabla R^\alpha_{\beta \gamma} = 0, \]
or, in components,
\[ R[\alpha \beta \gamma \delta] = \nabla[\alpha \beta \gamma] - T[\alpha \beta \gamma \epsilon] \wedge \nabla^\epsilon \delta, \quad \nabla[\alpha R_{\beta \gamma} \delta] - T[\alpha \beta \gamma] \lambda \delta = 0. \]

**Polyvector fields and Schouten–Nijenhuis brackets.** The graded spaces
\[ \text{Poly}^{(\pm)}(M) = \bigoplus_{n \in \mathbb{Z}} \text{Poly}^{(\pm)}_n(M) \]
where \( \text{Poly}^{(\pm)}_n(M) := 0 \) for \( n \leq -2 \) and \( \text{Poly}^{(\pm)}_{[-1]}(M) := C^\infty(M) \), and where for \( n \geq 0 \)
\[ \text{Poly}^{(-)}_n(M) := T M^{\wedge(n+1)}, \quad \text{Poly}^{(0)}_n(M) := T M^{\odot(n+1)}, \]
have degree map \( \text{deg} \text{(Poly}^{(\pm)}_n(M)) := n \) and degree preserving Schouten–Nijenhuis brackets \( \{\cdot, \cdot\}^{(\pm)}_{\text{S.N.}} \), defined by
\[ \{A, B\}^{(\pm)}_{\text{S.N.}} = -(\pm 1)^{\text{deg}(A)+1)(\text{deg}(B)+1)}\{B, A\}^{(\pm)}_{\text{S.N.}}, \]
and obey the Leibniz’ rule
\[ \{A, B \wedge C\}^{(-)}_{\text{S.N.}} = \{A, B\}^{(\pm)}_{\text{S.N.}} \wedge C + (\pm 1)^{\text{deg}(A)+1}\text{deg}(B)} B \wedge \{B, C\}^{(\pm)}_{\text{S.N.}}, \]
\[ \{A, B \odot C\}^{(\pm)}_{\text{S.N.}} = \{A, B\}^{(\pm)}_{\text{S.N.}} \odot C + B \odot \{B, C\}^{(\pm)}_{\text{S.N.}}, \]
where \( \{A, B\}^{(\pm)}_{\text{S.N.}} := A B \) if \( \text{deg}(A, B) = (0, -1) \) and \( \{A, B\}^{(\pm)}_{\text{S.N.}} := [A, B] \) if \( \text{deg}(A, B) = (0, 0) \).

**Parity shifted bundles.** We recall that an element \( u_p \) of \( T^* M \) over a point \( p \in M \) can be expanded as \( u_p = u_\alpha(p) e^\alpha_p \) where \( u_\alpha(p) \) are real numbers and \( e^\alpha_p \) is a basis for the fiber over \( p \), i.e. \( \pi(e^\alpha_p) = p \), and that a section \( u \) of \( T^* M \) is a map from \( M \) to \( T^* M \) such that \( u : p \mapsto u_p \), i.e. \( \pi \circ u = \text{Id}_M \). Moreover, a section \( \varphi^* u \) of the pulled back bundle \( \varphi^* T^* M \) over \( \Sigma \) obeys \( (\varphi^* u)_x = u_{\varphi(x)} = u_\alpha(\varphi(x)) e^\alpha_{\varphi(x)} \) for all \( x \in \Sigma \). To apply \( \varphi^* \) to \( T^*[n] M \) we first define \( \varphi^* (\mathbb{R}[n] \varphi(x)) \in (T^*_x \Sigma)^\wedge n \), that is, a real number \( r[n] \) of degree \( n \) of a fiber space over a point \( p \) in the image of \( \varphi \) that is mapped by \( \varphi^* \) to an \( n \)-form on \( \Sigma \). Thus, as an element of \( T^*[n] M \) is of
the form $u[n]_\rho = u_\alpha[n](\rho) e_\rho^\alpha$ where $u_\alpha[n](\rho) \in \mathbb{R}[n]$ and $e_\rho^\alpha$ is the basis of the fiber of $T^*M$, the application of $\varphi^*$ to $T^*[n]_M$ yields the right hand side of Eq. (2.16) for $n = 1$. Alternatively, one can redefine the notation, and take $\varphi : T[1][\Sigma] \to T^*[1]M$ to be a map of vanishing intrinsic degree, such that $\eta_\alpha \circ \varphi$ is a linear function in the fiber coordinate of $T[1][\Sigma]$, and use the canonical definition of $\varphi^* : \Omega(T^*[1]M) \to \Omega(T[1][\Sigma])$. The Lagrangian can then be obtained by composing $\varphi^*$ with the homeomorphism $\mu : \Omega(T[1][\Sigma]) \to \Omega(\Sigma)$ of differential graded algebras, defined in local coordinates $(x^\mu, \theta^\mu)$ on $T[1][\Sigma]$ by $\mu : (x^\mu, \theta^\mu; dx^\mu, d\theta^\mu) \mapsto (x^\mu, dx^\mu; dx^\mu, 0)$ (such that $\mu \circ d = d \circ \mu$), viz. $S = \int_\Sigma \mu \circ \varphi^* \left( \eta_\alpha d\phi^\alpha + \frac{1}{2} \Pi^{\alpha\beta} \eta_\alpha \eta_\beta \right)$.

**B Graded Lie algebra of derivations of the differential form algebra**

The graded Lie algebra $\text{Der}(\Omega(M))$ of derivations of the differential graded associative algebra $\Omega(M)$ of forms on $M$ consists of maps

$$\delta : \Omega[p](M) \to \Omega[p + \deg(\delta)](M) , \quad \deg(\delta) \in \{-1, 0, 1, \ldots\} ,$$

(B.1)

obeying the graded Leibniz’ rule

$$\delta(\omega \wedge \eta) = (\delta\omega) \wedge \eta + (-1)^{\deg(\omega)\deg(\delta)} \omega \wedge (\delta\eta) , \quad \omega, \eta \in \Omega(M) .$$

(B.2)

The graded commutator of $\delta_1, \delta_2 \in \text{Der}(\Omega(M))$ is the element in $\text{Der}(\Omega(M))$ defined by

$$[\delta_1, \delta_2] := \delta_1 \delta_2 - (-1)^{\deg(\delta_1)\deg(\delta_2)} \delta_2 \delta_1 .$$

(B.3)

A general element $\delta \in \text{Der}(\Omega(M))$ can be decomposed into a Lie derivative $\mathcal{L}_K$ and an inner derivative $\iota_{\Xi'}$, viz.

$$\delta = \mathcal{L}_\Xi + \iota_{\Xi'} ,$$

(B.4)

where $\Xi$ and $\Xi'$ are vector field valued forms on $M$. If $\Xi = \omega \otimes X$, where $\omega \in \Omega[k](M)$ and $X$ is a vector field on $M$, then we define

$$\deg(\Xi) = k ,$$

(B.5)

and

$$\iota_{\Xi} \eta := \omega \wedge (\iota_X \eta) , \quad \deg(\iota_{\Xi}) := \deg(\Xi) - 1 ,$$

(B.6)
for all $\eta \in \Omega(M)$. The Lie derivative
\[
\mathcal{L}_\Xi := [i_\Xi, d] \equiv i_\Xi d + (-1)^k d i_\Xi , \quad \deg(\mathcal{L}_\Xi) = \deg(\Xi) ,
\] (B.7)

where $d$ denotes the exterior derivative on $M$, which is itself a derivation of $\Omega(M)$ of degree one that can be represented as a Lie derivative, viz.
\[
d \equiv \mathcal{L}_I , \quad \deg(d) = 1 ,
\] (B.8)

where $I$ is the vector field valued one-form defined by
\[
\iota_I \omega = \deg(\omega) \omega ,
\] (B.9)

for all $\omega \in \Omega(M)$; in a coordinate basis, we have
\[
I = d\phi^\alpha \partial_\alpha , \quad d = d\phi^\alpha \mathcal{L}_{\partial_\alpha} .
\] (B.10)

It follows that exterior and Lie derivatives commute in the graded sense, that is
\[
[d, \mathcal{L}_\Xi] \equiv d \mathcal{L}_\Xi - (-1)^{\deg(\Xi)} \mathcal{L}_\Xi d = 0 .
\] (B.11)

The inner and Lie derivatives form two subalgebras of $\text{Der}(\Omega(M))$, viz.
\[
[i_\Xi, i_{\Xi'}] = i_{[\Xi, \Xi'][-1]} , \quad [\mathcal{L}_\Xi, \mathcal{L}_{\Xi'}] = \mathcal{L}_{[\Xi, \Xi']}|_0 ,
\] (B.12)

while their mutual graded commutator
\[
[\mathcal{L}_\Xi, i_{\Xi'}] = i_{[\Xi, \Xi']}|_0 - (-1)^{\deg(\Xi)(\deg(\Xi') + 1)} \mathcal{L}_{i_{\Xi'} \Xi} ,
\] (B.13)

where the induced brackets, whose subscripts indicate their intrinsic degrees, are the Nijenhuis–Richardson bracket
\[
[\Xi, \Xi']_{[-1]} = i_{\Xi \Xi'} - (-1)^{\deg(\Xi)(\deg(\Xi') + 1)} \mathcal{L}_{i_{\Xi'} \Xi} ,
\] (B.14)

where, for $\Xi = \omega \otimes X$ and $\Xi' = \omega' \otimes X'$, we have defined
\[
\iota_{\omega \otimes X} (\omega' \otimes X') := (\omega \wedge i_X \omega') \otimes X' ,
\] (B.15)

and the Frölicher–Nijenhuis bracket
\[
[\omega \otimes X, \omega' \otimes X']|_0 = \omega \wedge \omega' \otimes [X, X'] + (\omega \wedge \mathcal{L}_X \omega' + (-1)^{\deg(\omega)} d\omega \wedge i_X \omega') \otimes X' - (\mathcal{L}_{X'} \omega \wedge \omega' - (-1)^{\deg(\omega)} i_{X'} \omega \wedge d\omega') \otimes X .
\] (B.16)
The Nijenhuis–Richardson bracket follows readily, while the Frölicher–Nijenhuis bracket can be obtained by combining (B.7), (B.11) and (B.13) as follows:

\[ [\mathcal{L}_\Xi, \mathcal{L}_\Xi'] = [\mathcal{L}_\Xi, [\iota_{\Xi'}, d]] = [[\mathcal{L}_\Xi, \iota_{\Xi'}], d] = \mathcal{L}_{[\Xi, \Xi']} . \]  

(B.17)

It remains to show (B.13), for which it suffices to verify that it holds when acting on zero-forms, say \( \phi \), and one-forms, say \( \lambda \), for \( \Xi = \omega \otimes X \) and \( \Xi' = \omega' \otimes X' \) with \( \omega \) and \( \omega' \) being even forms (after which the general case follows by applying graded degree shifts to \( \Xi \) and \( \Xi' \) and making use of the derivation property). To this end, acting on \( \phi \), we have

\[ [\mathcal{L}_{\omega \otimes X}, \iota_{\omega' \otimes X'}] \phi = -\omega' \wedge \iota_X (\omega_X d\phi) \]

\[ = -(\omega' \wedge \iota_X \omega) \iota_X d\phi = -\iota_{\omega' \wedge X} \omega_X d\phi = -\mathcal{L}_{\omega \otimes X, \omega' \otimes X} \phi \, , \]  

(B.18)

in immediate agreement with (B.13) (as \( \Xi \) has been assumed to be even). Acting on \( \lambda \) and using \( \iota_X \iota_{X'} \lambda = 0 \), we have

\[ [\mathcal{L}_{\omega \otimes X}, \iota_{\omega' \otimes X'}] \lambda = d(\omega \wedge \iota_X (\omega' \iota_{X'} \lambda)) + \omega \wedge \iota_X (d(\omega' \iota_{X'} \lambda)) - \omega' \wedge \iota_X (d(\omega_X \lambda) + \omega \wedge \iota_X d\lambda) \]

\[ = (\iota_{\omega' \otimes X'} \iota_X \omega' + \omega \wedge (d \iota_X \omega') \iota_X + \omega \wedge (\iota_X d \omega') \iota_{X'} + \omega \wedge \omega' \iota_X d \iota_{X'}) \]

\[ - \omega' \wedge (\iota_{X'} d \omega) \iota_X + (\iota_{X'} \omega) \wedge d \iota_X + \omega \wedge \iota_{X'} d \iota_X + (\iota_{X'} \omega) \wedge \iota_X d + \omega_X X d) \lambda \]

\[ = (\iota_{(\omega \wedge \iota_X \omega' + \omega \wedge \mathcal{L}_{X, X'}) \otimes X' - \omega' \wedge \iota_X d \omega} \otimes X) + \omega \wedge \omega' [\mathcal{L}_X, \iota_{X'}] - \omega' \wedge \iota_{X'} \omega \wedge \mathcal{L}_X \lambda \]  

(B.19)

where \( \mathcal{L}_X = \iota_X d + d \iota_X \). The last two terms can be rewritten using

\[ [\mathcal{L}_X, \iota_{X'}] = \eta_{[X, X']}, \]  

(B.20)

which follows by evaluating both sides on forms of degree zero and one, and using the fact that if \( \eta \) is an odd form then

\[ \eta \wedge \mathcal{L}_X = \mathcal{L}_{\eta \otimes X} + d \eta \wedge \iota_X \, , \]  

(B.21)

that we then apply for \( \eta = \omega' \wedge \iota_{X'} \omega \). Thus,

\[ [\mathcal{L}_{\omega \otimes X}, \iota_{\omega' \otimes X'}] \lambda = \iota_{\omega \wedge \omega' \otimes [X, X'] + (d \omega \wedge \iota_X \omega' + \omega \wedge \mathcal{L}_{X, X'}) \otimes X' - (\omega' \wedge \iota_X d \omega + d(\omega' \wedge \iota_X \omega)) \otimes X} \lambda - \mathcal{L}_{\omega' \wedge \iota_{X'} \omega \otimes X} \lambda \, , \]  

(B.22)

in agreement with (B.13) (again under the assumption that \( \Xi \) is even).
C Covariant local cohomology of $\mathcal L_Q$ on $T^*[1,0](T[0,1]M)$

The construction of the supersymmetric covariant Hamiltonian action makes use of differential forms $\mathcal F$ on $T^*[1,0](T[0,1]M)$ that are at least linear in momenta and annihilated by $\mathcal L_Q$ where

$$Q := q_t + \tilde q_t, \quad q_t = \theta^\alpha \frac{\partial}{\partial \phi^\alpha}, \quad \tilde q_t = -\omega_\alpha \frac{\partial}{\partial \chi^\alpha}, \quad (C.1)$$

is the nilpotent vector field on $T^*[1,0](T[0,1]M)$ of bi-degree $(0,1)$ that is the uplift of the de Rham differential on $M$. Such forms are $\mathcal L_Q$ exact and admit potentials that are $\text{Diff}(M)$ covariant. To demonstrate this, let us study the problem

$$\mathcal L_Q \mathcal F = \mathcal J, \quad \mathcal L_Q \mathcal J = 0. \quad (C.2)$$

Introducing the nilpotent vector field

$$\overline{Q} := \overline{q}_t, \quad \overline{q}_t = -\chi_\alpha \frac{\partial}{\partial \omega_\alpha}, \quad (C.3)$$

of bideg$(\overline{Q}) = (0,-1)$, we can define the homotopy contracting vector field

$$\mathcal N := \{ Q, \overline{Q} \} = \omega_\alpha \frac{\partial}{\partial \omega_\alpha} + \chi_\alpha \frac{\partial}{\partial \chi_\alpha}, \quad (C.4)$$

without breaking the $\text{Diff}(M)$ covariance. If we expand

$$\mathcal F = \sum_{p,q,r} \mathcal F_{[p,q,r]}, \quad \text{deg}_\mathcal E(\mathcal F_{[p,q,r]}) = p, \quad \text{bideg}(\mathcal F_{[p,q,r]}) = (q,r), \quad (C.5)$$

idem $\mathcal J$, where thus

$$q \geq p \geq 0. \quad (C.6)$$

then it follows from

$$[Q, \mathcal N] = 0 = [\overline{Q}, \mathcal N], \quad (C.7)$$

and

$$\mathcal L_N \mathcal F_{[p,q,r]} = q \mathcal F_{[p,q,r]}, \quad (C.8)$$

that we can fix the value of $q$. Moreover, from the assumption on the the dependence of $\mathcal F$ on the momentum variables $(\omega_\alpha, \chi_\alpha)$ it follows that

$$q \geq 1. \quad (C.9)$$

Hence,

$$\mathcal F = (\mathcal L_N)^{-1} \mathcal L_{\overline{Q}} \mathcal J + \mathcal L_Q \mathcal G, \quad (C.10)$$
whose decomposition into definite quantum numbers read
\[ \mathcal{F}_{[p|q,r]} = q^{-1} \mathcal{L}_q \mathcal{F}_{[p|q,r+1]} + \mathcal{L}_q \mathcal{G}_{[p|q,r-1]} \]  
and we can choose \( \mathcal{G} \) such that
\[ \mathcal{L}_q \mathcal{G} = 0 \]  
\( \text{Zero-form} \ (p = 0) \). A function \( \mathcal{F} \) on \( T^*[1,0](T[0,1]M) \) that is at least linear in momenta can be expanded as
\[ \mathcal{F} = \sum_{q \geq 1} \mathcal{F}_q, \quad \mathcal{F}_q = \sum_{m+n=q} \mathcal{F}_{m,n}, \quad \mathcal{F}_{m,n} = \frac{1}{m!n!} \omega^m_{\alpha[m]} \lambda^n_{\beta(n)} \mathcal{F}^\alpha_{(m,n)} \]  
using a notation in which \( \alpha[m] \) and \( \beta(n) \) stand for \( m \) antisymmetric and \( n \) symmetric indices, respectively. Thus, if \( \mathcal{Q} \mathcal{F} = 0 \), then \( \mathcal{F}_q = \mathcal{Q} \mathcal{G}_q \) where one can choose \( \mathcal{Q} \mathcal{G}_q = 0 \), which in particular implies that \( \mathcal{G}_q |_{\lambda = 0} = 0 \). It is instructive to arrive at this result by instead relying on the local Poincarè lemma on \( M \). To this end, in the simplest case, one has
\[ \mathcal{F}_1 = \mathcal{F}_{(0,1)} + \mathcal{F}_{(1,0)} = \omega_\alpha \mathcal{F}^\alpha_{(0,1)} + \chi_\alpha \mathcal{F}^\alpha_{(1,0)}. \]  
It follows that
\[ \mathcal{Q} \mathcal{F}_1 = -\omega_\alpha q_t \mathcal{F}^\alpha_{(1,0)} + \chi_\alpha q_t \mathcal{F}^\alpha_{(0,1)} + \omega_\alpha \mathcal{F}^\alpha_{(0,1)} \]  
and thus, imposing \( \mathcal{Q} \mathcal{F}_1 = 0 \), one has the conditions
\[ \mathcal{F}^\alpha_{(0,1)} - q_t \mathcal{F}^\alpha_{(1,0)} = 0, \quad q_t \mathcal{F}^\alpha_{(0,1)} = 0, \]  
whose general solution is given by
\[ \mathcal{F}^\alpha_{(0,1)} = q_t \mathcal{G}^\alpha_{(0,1)}, \quad \mathcal{F}^\alpha_{(1,0)} = \mathcal{G}^\alpha_{(0,1)} + q_t \mathcal{G}^\alpha_{(1,0)}. \]  
Plugging it back into \( \text{(C.14)} \) one obtains \( \mathcal{F}_1 = \mathcal{Q} \mathcal{G}_1 \), with
\[ \mathcal{G}_1 = \omega_\alpha \mathcal{G}^\alpha_{(1,0)} + \chi_\alpha \mathcal{G}^\alpha_{(0,1)}. \]  
The symmetry transformations
\[ \delta \mathcal{G}^\alpha_{(0,1)} = q_t \mathcal{L}^\alpha_{(0,1)}, \quad \delta \mathcal{G}^\alpha_{(1,0)} = \mathcal{L}^\alpha_{(1,0)} + q_t \mathcal{L}^\alpha_{(1,0)}, \]  
can be used to set \( \mathcal{G}^\alpha_{(1,0)} = 0 \), and one concludes that
\[ \mathcal{G}_1 = \chi_\alpha \mathcal{G}^\alpha_{(0,1)}, \quad \mathcal{Q} \mathcal{G}_1 = 0. \]
Turning to the case of \( q = 2 \), which concerns the Hamiltonian function, we insert
\[
\mathcal{F}(2) = \frac{1}{2} \chi_{(2)}^2 \mathcal{F}_{(0,2)}^{\beta} + \omega_\alpha \chi_\beta \mathcal{F}_{(1,1)}^{\alpha \beta} + \frac{1}{2} \omega_\alpha^{[2]} \mathcal{F}_{(2,0)}^{\alpha [2]} \tag{C.21}
\]

into \( Q\mathcal{F}(2) = 0 \), which yields the Cartan integrable system
\[
q_t \mathcal{F}_{(0,2)}^{\alpha (2)} = 0 , \tag{C.22}
\]
\[
q_t \mathcal{F}_{(1,1)}^{\alpha \beta} + \mathcal{F}_{(0,2)}^{\alpha (2)} = 0 , \tag{C.23}
\]
\[
q_t \mathcal{F}_{(2,0)}^{\beta [2]} + 2 \mathcal{F}_{(1,1)}^{\beta [2]} = 0 , \tag{C.24}
\]
with general solution
\[
\mathcal{F}_{(0,2)}^{\alpha (2)} = q_t \mathcal{G}_{(0,2)}^{\alpha (2)} , \tag{C.25}
\]
\[
\mathcal{F}_{(1,1)}^{\alpha \beta} = -q_t \mathcal{G}_{(1,1)}^{\alpha \beta} - \mathcal{G}_{(0,2)}^{\alpha (2)} , \tag{C.26}
\]
\[
\mathcal{F}_{(2,0)}^{\beta [2]} = q_t \mathcal{G}_{(2,0)}^{\beta [2]} + 2 \mathcal{G}_{(1,1)}^{\beta [2]} . \tag{C.27}
\]

It follows that \( \mathcal{F}_2 = Q\mathcal{G}_2 \), with
\[
\mathcal{G}_2 = \frac{1}{2} \chi_{(2)}^2 \mathcal{G}_{(0,2)}^{\beta (2)} + \omega_\alpha \chi_\beta \mathcal{G}_{(1,1)}^{\alpha \beta} + \frac{1}{2} \omega_\alpha^{[2]} \mathcal{G}_{(2,0)}^{\alpha [2]} , \tag{C.28}
\]
modulo symmetry transformations. Using the parameters of \( \mathcal{F}_{(0,2)}^{\alpha (2)} \) and \( \mathcal{F}_{(1,1)}^{\alpha \beta} \) to eliminate \( \mathcal{G}_{(1,1)}^{(1,1)} \) and \( \mathcal{G}_{(0,2)}^{(0,2)} \), respectively, we arrive at
\[
\mathcal{G}_2 = \frac{1}{2} \chi_{(2)}^2 \mathcal{G}_{(0,2)}^{\beta (2)} + \omega_\alpha \chi_\beta \mathcal{G}_{(1,1)}^{\alpha \beta} , \quad Q\mathcal{G}_2 = 0 . \tag{C.29}
\]
The general case follows the same pattern, such that if \( Q\mathcal{F}(q) = 0 \) then we can write \( \mathcal{F}(q) = Q\mathcal{G}(q) \) where
\[
\mathcal{G}(q) = \sum_{m=0}^{q-1} \frac{1}{m!(q-m)!} \omega_\alpha^{m} \chi_{\alpha}^{q-m} \mathcal{G}_{(m,q-m)}^{[\alpha[m],\beta(q-m-1)} , \quad Q\mathcal{G}(q) = 0 . \tag{C.30}
\]

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