Competition versus Cooperation: A class of solvable mean field impulse control problems

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Abstract

We discuss a class of explicitly solvable mean field control problems/games with a clear economic interpretation. More precisely, we consider long term average impulse control problems with underlying general one-dimensional diffusion processes motivated by optimal harvesting problems in natural resource management. We extend the classical stochastic Faustmann models by allowing the prices to depend on the wood supply on the market using a mean field structure. In a competitive market model, we prove that, under natural conditions, there exists an equilibrium strategy of threshold-type and furthermore characterize the threshold explicitly. If the agents cooperate with each other, we are faced with the mean field control problem. Using a Lagrange-type argument, we prove that the optimizer of this non-standard impulse control problem is of threshold-type as well and characterize the optimal threshold. Furthermore, we compare the solutions and illustrate the findings in an example.

Keywords: mean field games, mean field problems, optimal harvesting, stochastic impulse control, diffusion processes

Subject Classifications: 91A15, 91A25, 49N25, 93E20

1. Introduction

Mean field game theory has been introduced by Lasry and Lions [32] and by Huang, Melhame, Caines [28] to study Nash equilibria of differential games with many players, where each player controls a diffusion. The main feature of these games is that the players do not interact with the others individually, but only through the distribution of all players’ states. This gives rise to apply techniques similar to mean field approximation from physics to obtain approximate equilibria for N-player games. Applications include growth models, the production of an exhaustible resource by a continuum of agents, as well as opinion dynamics [11, 25]. In the classical diffusion model the equilibria of the limiting mean field game are given by a system of nonlinear partial differential equations with partly initial, partly terminal conditions, if one considers the analytic approach to mean field games, or they are given by a coupled forward backward stochastic differential equation, if one considers the probabilistic approach (for details consider [5,12,13]). Note, however, that also these systems are mostly intractable. Only in the case of linear dynamics and quadratic costs, explicit solutions have been obtained. More recently, several other types of mean field games have been introduced, such as finite state mean field games [10,24,34] or mean field games of stopping [33,36,15].

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in which case it is sometimes possible to obtain other explicitly solvable models which
might yield a deeper understanding of the nature of mean field equilibria.

Another branch of research covers mean field type control theory, which discusses a
connected question. The difference is that in these models there is no competition be-
tween the agents, but they are assumed to cooperate. From a mathematical perspective,
this means that there is only one decision maker who chooses a control to optimize the
expected reward for the whole population. We refer to [4] for an early paper on the max-
imum principle for such problems and to [14] for a comparison of mean field games and
problems. Some real-world problems have reasonable interpretations both as a mean field
game and problem, see, e.g., [6] for an example inspired by pedestrian crowd dynamics.

In this paper, we leave the classical setting of continuous stochastic control, but
consider stochastic impulse control problems. Impulse control problems form the math-
ematical framework to study a continuous time model with interventions at adaptively
chosen discrete time points only. Such problems naturally arise whenever costs have to be
paid for each interventions, e.g., in portfolio management with constant transaction costs
([21], [7]) and control of the exchange rate by the central bank ([33]). An overview on
results for jump diffusions is given in [37], see also [31] for a survey article with financial
applications. Many of these articles are based on the fundamental connection between
impulse control problems and quasi-variational inequalities developed originally in [6], see
also [17] for more references. This formulation is straightforward for discounted prob-
lems. Some more caution is needed for long-term average formulations which, however,
play an important role as well, see [35, 26, 27, 19, 18] for some recent references.

Here we consider a well-known impulse control problem that naturally arises in nat-
ural resource management. Its deterministic form is widely used to calculate optimal
harvesting strategies and originates in the work of Martin Faustmann back in 1849.
Over the last decades, different stochastic extensions of this classical model utilizing dif-
fusions have been suggested and discussed, see [20, 43, 23, 2, 3, 41] to name just a few.
The underlying question may be formulated as follows: Assume that the volume of wood
in a forest stand is modelled as a one dimensional diffusion process and that fixed non-
zero costs occur for each time harvesting. When should the forest stand be optimally
harvested and what is the optimal value?

In this paper, we add as a new feature an interacting component of different agents
by letting the reward of the agents depend on the overall supply of wood in the market.
More precisely, we consider a mean field framework and assume that the reward depends
on the expected overall supply of wood, that in turn depends on a joint strategy all other
agents are assumed to use. In this context two problems are of interest: First, we can
assume that all agents compete with each other. In this setting every agent wants to
maximize his reward given the other agents’ strategies and we are interested in finding
equilibria of these game. Second, we can assume that all agents corporate with each
other. In this setting they jointly choose one strategy that maximizes the reward of all
agents together. The first problem is a mean field game, the second problem is a mean
field control problem.

We prove, for a natural set of assumptions, that for both problems optimal strate-
gies of threshold-type exist and characterize these thresholds explicitly. We obtain that
uniqueness of this threshold strategy cannot be expected in general. Moreover, again
under natural assumptions, we obtain that the thresholds for the mean field game are
larger than the thresholds for the mean field control problem. We illustrate the findings
in a classical example where the dynamics of the forest stand are described by a logistic
SDE.

The remainder of the article is structured as follows: In Section 2 we introduce the
full model and two sets of standing assumptions. The first set is more general and arises
from mathematical necessity, the second set contains stronger assumptions with a clear
economic interpretation. In Section 3 we describe several preliminary results including
the necessary results for the underlying impulse control problems, which are proven in the
Appendix A Section 4 presents the analysis of the mean field game, Section 5 presents the treatment of the mean field control problem. Section 6 then provides the comparison result regarding the thresholds for the game and the control problem. Finally, Section 7 illustrates the theoretical findings in two examples, one with a unique equilibrium, one with three equilibria.

2. Model and Standing Assumptions In this section we introduce the model as well as the necessary assumptions used in this paper. In the first subsection we describe and motivate the general model. In the second subsection we then formulate the standing assumptions.

2.1. The Model To model the natural resource under consideration, e.g., in the original model the volume of wood in the forest stand, we use a stochastic process \((X_t)_{t \geq 0}\) that is a regular one-dimensional Itô-diffusion on \(\mathbb{R}_+ = (0, \infty)\), whose dynamics is described by
\[
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t
\]
for a standard Brownian motion \(W\) and continuous functions \(\mu : \mathbb{R}_+ \to \mathbb{R}\), \(\sigma : \mathbb{R}_+ \to \mathbb{R}_+\) that are sufficiently regular to guarantee a unique (strong) solution to the stated SDE. As usual, we denote by \(\mathbb{P}_x\) probabilities for the process conditioned to start in the initial state \(x\) and write \(E_x\) for the corresponding expectation operator. We furthermore denote the speed measure by \(M\) and the scale function by \(S\). By assumption, their densities \(m\) and \(s\), resp., are given by (see e.g. [10])
\[
m(x) = \frac{2}{\sigma^2(x)} \exp\left(\int_0^x \frac{2\mu(y)}{\sigma^2(y)} dy\right), \quad s(x) = \exp\left(-\int_0^x \frac{2\mu(y)}{\sigma^2(y)} dy\right).
\]
The forest owner can at any time decide to cut the forest, in which case the process is restarted at an externally given level \(y_0 > 0\). Formally, the cutting decision is modelled by a sequences of stopping times \(R = (\tau_n)_{n \in \mathbb{N}}\) satisfying \(\lim_{n \to \infty} \tau_n = \infty\) a.s. We just consider strategies \(R = (\tau_n)_{n \in \mathbb{N}}\) such that the controlled process fulfills \(X^R_t \geq y_0\) for all \(n \in \mathbb{N}\), which we call admissible strategies in the following. Here, by \(X^R\) we denote the solution to
\[
X^R_t = X^R_0 + \int_0^t \mu(X^R_s) \, ds + \int_0^t \sigma(X^R_s) \, dW_s - \sum_{n; \tau_n \leq t} (X^R_{\tau_n} - y_0).
\]
The most interesting strategies in practice as well as in our following discussions are threshold strategies. These are strategies \(R = (\tau_n)_{n \in \mathbb{N}}\) such that for a given threshold \(y > y_0\) we have (with \(\tau_0 = 0\)):
\[
\tau_n = \inf\{t \geq \tau_{n-1} : X^R_t \geq y\}, \quad n \in \mathbb{N}.
\]
We denote such a threshold strategy for a threshold \(y > y_0\) by \(R(y)\). In Figure 4 we present a simulated sample path for such a strategy.

In the classical version of this famous impulse control problem, the forest owner chooses an admissible strategy in order to maximize his long term average reward
\[
\liminf_{T \to \infty} \frac{1}{T} E_x \left( \sum_{n; \tau_n \leq T} (\gamma(X^R_{\tau_n}) - K) \right),
\]
where \(K > 0\) denotes the fixed costs for each cutting decision and \(\gamma(X^R_{\tau_n})\) can be interpreted as the income earned for selling the available wood at time \(\tau_n\).

However, this classical impulse control problem lacks one – often crucial – feature. It ignores that there might be other forest owners who also grow and sell their wood at
the same time and therefore influence the prices. Here we introduce these other agents in a mean field type fashion. More precisely, we assume that there is a market for wood consisting of a continuum of agents with the same structure (dynamics and reward functional) as the decision maker under consideration. We remark that considering a continuum of agents is the natural way to formalize agents with negligible individual influence on the market outcome.

If all other agents use a joint strategy $Q = (\sigma_n)_{n \in \mathbb{N}}$, then a representative agent’s forest stand is described by a process $\hat{X}^Q$ with the same dynamics as $X^Q$ and some initial distribution representing the collective distribution of the forest stands at time 0. It is natural to assume that the wood price at time $t$ depends on the volume of wood available at that time, which is described by the mean $E[\hat{X}^Q_t]$ depending on the initial distribution of $\hat{X}^Q_0$. As we consider a long-term reward structure, it is reasonable to assume that the process $\hat{X}^Q$ has an invariant distribution $\Pi^Q$ and $\hat{X}^Q$ is started from this. We call strategies satisfying this assumption invariant admissible strategies. In this setting, $E[\hat{X}^Q_t] = \int x \Pi^Q(dx) =: E[\hat{X}^Q_\infty]$ is independent of $t$ and represents the supply of wood in our setting. We remark that concentrating on such collective strategies is not crucial for the following considerations, but we use it as it is meaningful for the real world problem and the formulation of the results and the arguments become less technical.

Now, to model our reward, we define for each pair of admissible strategies $R, Q$, with $Q$ invariant, the expected reward function

$$J_x(R, Q) := \liminf_{T \to \infty} \frac{1}{T} E_x \left( \sum_{n: \tau_n \leq T} (\gamma(X^{R}_{\tau_n-}, E[\hat{X}^Q_\infty]) - K) \right),$$

with the interpretation that $J_x(R, Q)$ is the long term average reward obtained by an individual player starting in state $x$ playing according to strategy $R$, while the whole population plays according to strategy $Q$. Here, $\gamma(x, z)$ is the payoff function that models the reward the decision maker gets each time harvesting, which we assume in our model to depend on the average amount of wood $E[\hat{X}^Q_\infty]$ that the other market participant have in their forest, i.e., on the supply of wood.

In this setting two optimization problems naturally arise. First, we could consider the behaviour of agents in a competitive market. In this setting all agents would – given the choice of the others which they cannot influence – maximize their reward. The central interest now lies in finding equilibrium strategies for this problem, that is finding strategies such that if the whole population of agents plays according to such an equilibrium strategy, then the individual agent has no incentive to deviate from doing
so. Formally, we search for invariant admissible strategies such that

\[ J_x(Q, Q) = \max_{R \text{ admissible}} J_x(R, Q) \text{ for all } x. \]

The second situation of interest is the case when all agents cooperate, that is all agents choose together a strategy in order to maximize the overall reward. Mathematically speaking, we search for an invariant admissible strategy such that

\[ J_x(Q, Q) = \max_{R \text{ admissible}} J_x(R, R) \text{ for all } x \]

over all invariant admissible strategies. The first problem is a classical problem in the theory of mean field games, the second is a classical problem in mean field type control theory, which both up to now have not been considered in the context of impulse control problems. As in the standard theory, mean field equilibria are usually not solutions of the mean field type control problem and vice versa. Indeed, it might be societal beneficial to coordinate on some strategy where unilateral deviations are still possible and it might also happen that unilateral deviations are not profitable, whereas collective deviations are.

2.2. Assumptions

**Assumption 2.1.** The process \( X \) is positively recurrent with integrable stationary distribution. The stationary distribution is, with a slight abuse of notation, denoted by \( P(X_\infty \in dx) \). Furthermore, we assume that 0 is an entrance-boundary.

In terms of the speed measure, the existence of a stationary distribution means that \( M(\mathbb{R}_+) < \infty \). Then, up to standardization, \( M \) is the stationary distribution of \( X \) and we have

\[ E[X_\infty] = \int_{\mathbb{R}_+} x \frac{m(x)}{M(\mathbb{R}_+)} \, dx < \infty. \]

The assumption that \( M(\mathbb{R}_+) < \infty \) implies that the boundary \( \infty \) is natural and 0 is either entrance or natural (see [30, p.234]). In analytical terms, our additional assumption that 0 is not natural reads, for arbitrary \( x > 0 \), as

\[ \int_0^x (S(x) - S(y)) M(dy) < \infty. \]

Intuitively, it means that the process can reach an interior point from the state 0, but cannot reach the state 0 from an interior point.

**Remark 2.2.** Assumption [2.1] means that for all \( x, y \in \mathbb{R}_+ \) we have \( E_x(\hat{\tau}_y) < \infty \) for the hitting time \( \hat{\tau}_y = \inf\{t \geq 0 : X_t = y\} \), see [10, II.12]. Hence we can define the function

\[ \xi : \mathbb{R}_+ \to [0, \infty); \ y \mapsto E_y(\tau_y) \]

for the threshold time \( \tau_y = \inf\{t \geq 0 : X_t \geq y\} \). We then obtain \( \xi(x) \) for \( x \geq y_0 \) as

\[ \xi(x) = \int_{y_0}^x (S(x) - S(y)) m(y) \, dy + (S(x) - S(y_0)) M[0, y_0] \]

see [20]. In particular, \( \xi \in C^2 \) on \([y_0, \infty)\).

Before we formulate the remaining assumptions we prove the following technical lemma, which allows us to restrict our attention to a compact interval \([z_1, z_2]\) of expected values \( E[X_\infty^R] \):
Lemma 2.3. Let $R$ be an admissible impulse control strategy. Then

$$z_1 \leq \lim \inf \frac{E[X_T^R]}{T \to \infty} \leq \lim \sup \frac{E[X_T^R]}{T \to \infty} \leq z_2,$$

where $z_1 := E[X_\infty^\omega]$ for the diffusion process $X^r$ reflected downwards in $y_0$ and $z_2 := E[X_\infty^\omega]$ for the uncontrolled diffusion process $X$.

Proof. First note that the expectations $z_1, z_2$ exist by Assumption 2.1 and as $X^r$ is positively recurrent. The inequalities can be proved using an easy (partial) coupling argument:

We construct a version of $X^r$ by letting it run coupled with $X^R$ until a state $\geq y_0$ is reached. Then, we reflect $X^r$ in $y_0$ downwards and let both processes run following their dynamics with respect to the same Brownian motion until the first time the two paths meet again. Then, we couple the paths and follow this rule. Consequently, for each $t$ and each $\omega$, we have $X_t^r(\omega) \leq X_t^R(\omega)$, proving the first inequality.

Similarly, we construct a version of the uncontrolled diffusion $X$ by running coupled with $X^R$ until the first impulse time. Then we let both processes run following their dynamics with respect to the same Brownian motion until we couple them the next time the two paths meet and so on. Again, for each $t$ and each $\omega$, we have $X_t^R(\omega) \leq X_t(\omega)$.

With this preparation we are in the position to state the general assumptions necessary for our discussion of the mean field game and the mean field type control problem.

Assumption 2.4. 1. The function $\gamma : [y_0, \infty) \times [z_1, z_2] \to \mathbb{R}$ is increasing as well as twice continuously differentiable in the first argument and continuously differentiable as well as decreasing in the second argument.

2. $\gamma(y_0, z) = 0$ for all $z$, i.e., selling 0 volume of wood yields a net profit of $-K$.

3. $\min_{y \in [z_1, z_2]} \sup_{y > y_0} \gamma(y, z) > K$, i.e., selling a suitable large volume of wood yields a positive net profit.

4. For every $z \in [z_1, z_2]$ there is a unique critical point $y = y_z \in (y_0, \infty)$ of $\frac{\gamma(y, z) - K}{\xi(y)}$ and this is a global maximum.

5. For all pairs $(y, z)$ describing a critical point as given before, it holds

$$\frac{\partial^2 \gamma(y, z) - K}{\partial y^2} \xi(y) < 0.$$

Remark 2.5. Assumption 2.4 implies that the set

$$\{y \in [y_0, \infty) : \exists z \in [z_1, z_2] : (y, z) \text{ is a critical point of } (\gamma(y, z) - K)/\xi(y)\}$$

is compact and has a minimal and a maximal element, which we denote by $y_\underline{\gamma}$ and $y_\bar{\gamma}$, resp., in the following.

Assumption 2.4 is rather general, but not very handy. Thus, we also consider the following more restrictive assumptions which have a clear economic interpretation.

Assumption 2.6. 1. The function $\gamma$ is of the form $\gamma(y, z) = (y - y_0)\varphi(z)$ for a continuously differentiable and strictly decreasing function $\varphi : [z_1, z_2] \to \mathbb{R}_+$.

2. There is a $y_1 \geq y_0$ such that the drift function $\mu$ is strictly decreasing on $(y_1, \infty)$ and increasing on $(0, y_1]$.

3. $\lim_{x \to \infty} s(x) = \infty$. 

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Indeed, in our setting of a continuum of players, the payoff function “quantity times price” is the only sensible formulation of the fact that agents have a negligible impact on the market price. The assumption on the speed density $s$ could be relaxed, see the proofs below, but holds in the practically relevant examples. The assumption on the drift $\mu$ intuitively means that if the quantity of available resources is low enough – in our main example this means, young, new trees just got planted – there is a random, yet on average positive growth of the drift. But at some point a level of saturation, for example due to the limited available space, is reached and there is no more room for further growth. Thus the average growth rate starts shrinking. This assumption not only seems practical on the intuitive level, it also includes the major growth models used for modelling natural resources. So it is satisfied the dynamics of the non-random Richards curve, which is a standard deterministic model for plant growth ([40, 39]) and also one of the most widely used models to describe biologic growth in a random environment, namely the generalized logistic (Perihurst-Vearl) -diffusion satisfies this assumption. This model is given by the SDE

$$dX_t = aX_t(1-bX_t)dt + \sigma(X_t)dW_t,$$

where $a, b > 0$ and $\sigma$ is a positive function (when $\sigma$ is linear, this becomes the (standard) logistic diffusion) and it will be analysed in our example in Section 7.

**Lemma 2.7.** If Assumption 2.6 is satisfied, then also Assumption 2.4 holds.

In order to show this statement we first prove an auxiliary result, which is perhaps of more general interest. See also [41] for a related result.

**Lemma 2.8.** Assume that there exists $y_1 \geq y_0$ such that $\mu$ is (strictly) decreasing on $[y_1, \infty)$ and increasing on $[0, y_1]$. Then, there exists $y_2 \in [y_1, \infty]$ such that $\xi$ is (strictly) convex on $(y_2, \infty)$ and concave on $[y_0, y_2]$.

**Proof.** For all $x, b \in \mathbb{R}_+$ with $x < b$ using Dynkin’s formula and standard diffusion theory, see e.g. [30] Chapter 15.3, it holds that

$$b = \mathbb{E}_x[X_{\tau_b}] = x + \mathbb{E}_x \left[ \int_0^{\tau_b} \mu(X_t)dt \right]$$

$$= x + \int_x^b (S(b) - S(y))\mu(y)M(dy) + (S(b) - S(x)) \int_x^b \mu(y)M(dy).$$

Differentiating with respect to $x$ yields

$$1 = s(x) \int_0^x \mu(y)m(y)dy.$$

Now, we differentiate twice in the formula for $\xi$ in Remark 2.2 and obtain

$$\xi'(x) = s(x)M[0, x], \quad \xi''(x) = s'(x)M[0, x] + s(x)m(x).$$

Taking $s'(x) = \frac{2a(x)}{\sigma^2(x)}s(x), m(x) = \frac{2}{s(x)}\sigma^2(x)$ and (41) into account, we obtain

$$\xi''(x) = -\frac{\mu(x)}{\sigma^2(x)}s(x)M[0, x] + \frac{1}{\sigma^2(x)/2}$$

$$= \frac{2s(x)}{\sigma^2(x)} \int_0^x (\mu(y) - \mu(x))m(y) dy$$

$$= \frac{2s(x)}{\sigma^2(x)} I(x).$$

Noting that $I'(x) = -\mu'(x)M[0, x]$, we see that $I$ is increasing if $\mu$ is decreasing and vise versa. Therefore, under our assumptions, $\xi''$ changes sign at most once and from negative to positive. \qed
Proof of Lemma 2.7 Let Assumption 2.6 hold. We directly see that 2.4 1. and 2. are satisfied. By choosing choose \( y = \frac{K}{\varphi(y)} + y_0 + 1 \), we also see that Assumption 2.4 3. holds. Now fix \( z \in [z_1, z_2] \). By the previous observation, the function \( k : [y_0, \infty) \to \mathbb{R}, \ y \mapsto \frac{\gamma(y, z) - K}{\xi(y)} \) fulfils \( k(y_0) < 0 \). For large enough \( y \in \mathbb{R} \) we have \( k(y) > 0 \) and by Assumption 2.6 3.:

\[
\lim_{y \to \infty} \frac{\varphi(z)(y - y_0) - K}{\xi(y)} = \lim_{y \to \infty} \frac{\varphi(z)}{\xi'(y)} = \lim_{y \to \infty} \frac{\varphi(z)}{2s(y)M[0, y]} = 0.
\]

A maximum point therefore exists and is given by a critical point, i.e., by a root of

\[
\hat{F}(y, z) = \left( \frac{\partial}{\partial y} \gamma(y, z) \right) \xi(y) - (\gamma(y, z) - K) \xi'(y)
\]

\[
= \varphi(z)\xi(y) - (\varphi(z)(y - y_0) - K) \xi'(y).
\]

Define \( \hat{y} := \hat{g}(z) := \frac{K}{\varphi(z)} + y_0 \). Then we have

\[
\hat{F}(y, z) \geq \varphi(z)\xi(y) > 0 \text{ for all } y \leq \hat{y}
\]

since \( \varphi, \xi \) and \( \xi' \) are positive functions. Thus, \( k \) is increasing on \([y_0, \hat{y}]\). Using

\[
\frac{\partial}{\partial y} \hat{F}(y, z) = -\left( \varphi(z)(y - y_0) - K \right) \xi''(y)
\]

yields that for all \( y \) in the (possibly empty) interval \((\hat{y}, y_2)\) \((y_2 \text{ as in Lemma 2.8})\) we have

\[
\frac{\partial}{\partial y} \hat{F}(y, z) = \left( \varphi(z)(y - y_0) - K \right) \xi''(y) \geq 0,
\]

yielding that \( k \) does not have a critical point in \([y_0, \max\{\hat{y}, y_2\}]\). For all \( y > \max\{\hat{y}, y_2\} \) we have

\[
\frac{\partial}{\partial y} \hat{F}(y, z) = -\left( \varphi(z)(y - y_0) - K \right) \xi''(y) < 0.
\]

In total, we obtain that the function \( \hat{F}(\cdot, z) \) has a positive value for \( \hat{y} \). Moreover, if \( \hat{y} < y_2 \) it is increasing until \( y_2 \) and strictly decreasing after \( \max\{\hat{y}, y_2\} \). Therefore, it has exactly one root lying in \([\hat{y}, \infty)\), which implies that \( k \) has exactly one maximum point in \([\hat{y}, \infty)\), proving Assumption 2.4 4.

Furthermore, we have for any pair \((y, z)\) such that \( y \) is a critical point for \( z \)

\[
\left( \frac{\partial}{\partial y} \right)^2 \frac{\gamma(y, z) - K}{\xi(y)} = \left( \frac{\partial}{\partial y} \right)^2 \frac{\varphi(z)(y - y_0) - K}{\xi(y)}
\]

\[
= \frac{\partial}{\partial y} \varphi(z)\xi(y) - (\varphi(z)(y - y_0) - K) \xi'(y)
\]

\[
= \frac{\partial}{\partial y} \varphi(z)(y - y_0)\xi''(y) + 2\xi(y)\xi'(y) \left( \varphi(z)\xi(y) - \varphi(z)(y - y_0)\xi'(y) \right)
\]

\[
= -\frac{\varphi(z)(y - y_0)\xi''(y)}{\xi(y)^4} < 0,
\]

which proves Assumption 2.4 5. \( \square \)

3. Preliminaries This section collects preliminary results necessary to discuss the mean field game and the mean field control problem in the subsequent sections. Throughout the whole article we from now on assume that Assumptions 2.1 and 2.4 hold.
3.1. The stationary distributions of the controlled processes  

We now introduce an economically reasonable class of strategies leading to a stationary controlled process:

**Definition 3.1.** An impulse control strategy \( R = (\tau_n)_{n \in \mathbb{N}} \) is called admissible stationary strategy if there exists a stopping time \( \tau \) such that

\[
\tau_{n+1} = \tau \circ \theta_{\tau_n} + \tau_n,
\]

\( X_t^R \geq y_0 \) and \( \tau \) is non-lattice with finite mean under \( \mathbb{P}_{y_0} \). Here, \( \theta \) denotes the shift operator.

**Remark 3.2.** Note that threshold strategies \( R(y) \) for \( y > y_0 \) are admissible stationary strategies \( (\tau = \tau_y := \inf\{t \geq 0 : X_t \geq y\}) \). The integrability holds by general theory for diffusion processes as \( \mathbb{E}_{y_0}[\tau_y] = \xi(y) \), where \( \xi \) is given in Remark 2.3.

Given an admissible stationary strategy \( R \), we are interested in the asymptotic behaviour of \( X_t^R \) for \( t \to \infty \).

**Proposition 3.3.** For each admissible stationary strategy \( R \) with corresponding stopping time \( \tau \), the stationary distribution for the process \( X_t^R \), denoted by \( \Pi \), exists and is given by

\[
\int f(x)\Pi(dx) = \frac{1}{\mathbb{E}_{y_0}[\tau]} \mathbb{E}_{y_0} \left[ \int_0^\tau f(X_s)ds \right].
\]

**Proof.** It is immediately seen that \( X_t^R \) is a regenerative processes in the sense of Chapter VI. Therefore, the result holds by ibid, Theorem 1.2 on p.170.

In the case that \( R = R(y) \) is a threshold strategy, meaning that \( \tau \) is a threshold time, a more explicit description of the limiting distribution is possible. By standard diffusion theory, see e.g. [26], Proposition 3.1, we have that \( X_t^R(y) \) has a stationary distribution with density

\[
\pi_{y_0,y}(x) = \begin{cases} 
0, & x > y \\
nm(x)S(x,y], & x \in [y_0,y], \\
nm(x)S[y_0,y] & x \leq y_0
\end{cases}
\]

where \( S(x,y) := S(y) - S(x) \) denotes the Stieltjes measure and

\[
\kappa = \left( \int_{y_0}^\mu S[w,y]M(dw) + S[y_0,y]M[0,y_0] \right)^{-1}.
\]

**Lemma 3.4.** For all \( y_1 < y_2 \) it holds that \( X_t^{R(y_1)} <_{st} X_t^{R(y_2)} \), where \( <_{st} \) denotes the stochastic ordering.

In particular, \( \mathbb{E}[X_t^{R(y)}] \) is increasing in the threshold level \( y \).

**Proof.** We show that given an arbitrary pair \( y_1 < y_2 \) there is a switching point \( z \in [y_0,y_1] \) such that for the corresponding densities it holds that

\[
\pi_{y_0,y_1}(x) > \pi_{y_0,y_2}(x), x < z, \quad \pi_{y_0,y_1}(x) \leq \pi_{y_0,y_2}(x), x \geq z.
\]

This immediately yields the statement. To this end, we first prove that for fixed \( y_0 \) and \( x \leq y_0 \) the density \( \pi_{y_0,y}(x) \) is decreasing in \( y \). We have

\[
\pi_{y_0,y}(x) = m(x) \frac{g_1(y)}{f_1(y) + f_2(y)}
\]

with

\[
f_1(y) = \int_{y_0}^\mu S[w,y]dM(w), \quad f_2(y) = S[y_0,y]M[0,y_0], \quad g_1(y) = S[y,y]
\]
Using $f'_1(y) = s(y)M[y_0, y], \quad f'_2(y) = s(y)M[0, y_0], \quad g'_1(y) = s(y)$ we obtain

$$\frac{\partial}{\partial y} \pi_{y_0, y}(x) = m(x) \frac{g'_1(y)(f'_1(y) + f'_2(y)) - g'_1(y)(f'_1(y) + f'_2(y))}{(f'_1(y) + f'_2(y))^2}$$

$$= \frac{m(x)}{(f'_1(y) + f'_2(y))^2} \left[ \int_{y_0}^{y} S[w, y]dM(w) + S[y_0, y]M[0, y_0] \right. - S[y_0, y]M[0, y_0] - S[y_0, y]M[0, y_0]$$

$$= \frac{m(x)}{(f'_1(y) + f'_2(y))^2} \left[ \int_{y_0}^{y} (S[w, y] - S[y_0, y])dM(w) \right] < 0.$$  

This yields $\pi_{y_0, y_2}(x) > \pi_{y_0, y_1}(x)$ for all $x \leq y_0$. It remains to consider the case $x > y_0$. We first show that for $y \in [y_0, y_1]$ and $x \in [y_0, y]$ the derivative $\frac{\partial}{\partial y} \pi_{y_0, y}(x)$ may be decomposed as follows:

$$\frac{\partial}{\partial y} \pi_{y_0, y}(x) = m(x)h(x, y),$$

where $h(x, y)$ is increasing in $x$. Indeed, for all $x \in [y_0, y]$ using the notation $g_2(y) = S[x, y]$ we get

$$\frac{\partial}{\partial y} \pi_{y_0, y}(x) = m(x) \frac{g'_2(y)(f'_1(y) + f'_2(y)) - g'_2(y)(f'_1(y) + f'_2(y))}{(f'_1(y) + f'_2(y))^2}$$

$$= \frac{m(x)}{(f'_1(y) + f'_2(y))^2} \left[ \int_{y_0}^{y} S[w, y]dM(w) + S[y_0, y]M[0, y_0] \right.$$  

$$- S[y_0, y]M[0, y_0] - S[y_0, y]M[0, y_0]$$

$$= \frac{m(x)}{(f'_1(y) + f'_2(y))^2} \left( \int_{y_0}^{y} S[w, x]dM(w) + S[y_0, x]M[0, y_0] \right)$$

$$= m(x)h(x, y),$$

where $h(x, y)$ is indeed obviously increasing in $x$. This decomposition is sufficient as it yields that

$$\pi_{y_0, y_2}(x) - \pi_{y_0, y_1}(x) = m(x) \int_{y_1}^{y_2} h(x, y)dy$$

changes sign just once. Hence, using $\pi_{y_0, y_1}(x) > \pi_{y_0, y_2}(x)$ for $x \leq y_0$ and $\pi_{y_0, y_1}(x) = 0 < \pi_{y_0, y_2}(x)$ for $x \in (y_1, y_2)$, there exists some $z \in [y_0, y_1]$ satisfying above conditions.

\[\square\]

**Lemma 3.5.** The mapping $\mathbb{R}_+ \to \mathbb{R}, \quad y \mapsto \mathbb{E}[X_{\infty}^{R(y)}]$ is continuous and increasing. Furthermore, for each $z \in (z_1, z_2)$, there exists a unique threshold $y = y_z$ such that $z = \mathbb{E}[X_{\infty}^{R(y)}]$.

**Proof.**

$$\mathbb{E}[X_{\infty}^{R(y)}] = \int_{-\infty}^{y} x\pi_{y_0, y}(x)dx$$

yields that the mapping is continuous. Moreover, it is easily seen that

$$\lim_{y \to y_0} \mathbb{E}[X_{\infty}^{R(y)}] = z_1, \quad \lim_{y \to \infty} \mathbb{E}[X_{\infty}^{R(y)}] = z_2,$$

yielding the existence. The uniqueness holds by the monotonicity, see Lemma 3.4. \[\square\]
3.2. Control Problem Toolbox

For fixed $z \in [z_1, z_2]$ the agent faces a classical impulse control problem. Here we analyse this problem as well as a more general version necessary for the discussion of the mean field control problem. More precisely, we additionally consider a continuous cost function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the linear growth condition $h(x) \leq c(1 + x)$ for some $c > 0$. The agent now maximizes

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n: \tau_n \leq T} (\gamma(X_{\tau_n-}, z) - K) - \int_0^T h(X_s) ds \right]$$

among all admissible strategies. To analyse the control problem define for all $x \in \mathbb{R}_+$ the auxiliary value function

$$v^h(x, z) := \sup_{Q} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n: \tau_n \leq T} (\gamma(X_{\tau_n-}, z) - K) - \int_0^T h(X_s) ds \right]$$

for $Q = (\tau_n)_n$ and $X = X^Q$. We often suppress the dependence on $Q$ in the following.

Proposition 3.6. Assume there is a maximizer $y^* \in [y_0, \infty)$ of

$$y \mapsto \gamma(y, z) - K - \mathbb{E}_{y_0} \left[ \int_0^\infty h(X_s) ds \right] \xi(y),$$

then for all $x \in \mathbb{R}_+$ we have

$$v^h(x, z) = \sup_{y > y_0} \frac{\gamma(y, z) - K - \mathbb{E}_{y_0} \left[ \int_0^\infty h(X_s) ds \right]}{\xi(y)}.$$

Furthermore, the threshold strategy $R(y^*)$ is optimal for all $x \in \mathbb{R}_+$ and $R(y^*)$ is the unique optimizer amongst threshold strategies, if $y^*$ is the unique maximizer of (4).

The proof of this result is presented in Appendix A.

Remark 3.7. The assumption that a unique maximizer of (4) exists is not necessary in the formulation of Proposition 3.6. However, in the following, for both the mean field game and the mean field type control problem, the existence of a unique threshold strategy will be necessary in our approach as we need continuous dependence of the threshold on certain parameters (the expected amount of wood in the case of the game, an auxiliary parameter in the case of the control problem). By Assumption 2.4 we directly obtain a unique maximizer for the case $h = 0$ which additionally satisfies $y^* > y_0$.

4. The Mean Field Game

In this section we prove that Assumptions 2.1 and 2.4 imply the existence of a mean field equilibrium in threshold strategies and provide a criterion ensuring the uniqueness of such an equilibrium. By definition, a mean field equilibrium in threshold strategies satisfies

$$R(y^g) \in \arg \max_{R \text{ admissible}} J(R, R(y^g)).$$

The central idea is that any mean field equilibrium in threshold strategies is a fixed point of the following continuous function

$$\Phi : [y_0, \bar{y}] \rightarrow [y_0, \bar{y}]; \ y \mapsto \arg \max_{\bar{y} \in [y_0, \bar{y}]} \frac{\gamma(\bar{y}, \mathbb{E}[X_{\infty}^{R(\bar{y})}]) - K}{\xi(\bar{y})}.$$
and any fixed point of this map is a mean field equilibrium in threshold strategies. Indeed, let $y$ be a threshold such that $R(y)$ is an equilibrium. By Proposition 3.6 and Assumption 2.4 the threshold $y$ maximizes

$$\tilde{y} \mapsto \frac{\gamma(\tilde{y}, \mathbb{E}[X_{\infty}^R(y)]) - K}{\xi(\tilde{y})}$$

and satisfies $y_0 < y < \bar{y}$. If, on the other hand, $y \in [y_0, \bar{y}]$ is a fixed point of (5), then $y$ maximizes

$$\frac{\gamma(\tilde{y}, \mathbb{E}[X_{\infty}^R(y)]) - K}{\xi(\tilde{y})},$$

which by Proposition 3.6 and Assumption 2.4 means that $R(y)$ is the unique optimal threshold strategy for $z = \mathbb{E}[X_{\infty}^R(y)]$, which is the defining property of a mean field equilibrium.

**Theorem 4.1.** Let Assumptions 2.1 and 2.4 hold. Then, there is a fixed point of $\Phi$, which means that a mean field equilibrium in threshold strategies exists.

**Proof.** By Brouwer’s fixed point theorem it suffices to prove that $\Phi$ is continuous. For this we note that $\Phi = g \circ f$, where

$$f : [y_0, \bar{y}] \rightarrow [z_1, z_2]; \ y \mapsto \mathbb{E}_x[X_{\infty}^R(y)]$$

and

$$g : [z_1, z_2] \rightarrow [y_0, \bar{y}]; \ z \mapsto \arg \max_{y \in [y_0, \bar{y}]} \frac{\gamma(y, z) - K}{\xi(y)}.$$

$f$ is continuous by Lemma 3.5. It remains to prove that $g$ is continuous, for which we use the implicit function theorem. By Assumption 2.4 there exists for a fixed $z \in [z_1, z_2]$ a unique maximum, which is moreover the solution to

$$\tilde{F}(y, z) := \left( \frac{\partial}{\partial y} \gamma(y, z) \right) \xi(y) - (\gamma(y, z) - K)\xi'(y) = 0. \quad (6)$$

More precisely, the first order condition reads

$$0 = \left( \frac{\partial}{\partial y} \gamma(y, z) \right) \xi(y) - (\gamma(y, z) - K)\xi'(y) =: F(y, z),$$

but as the denominator is always positive, we can restrict our attention to solving equation (6). Moreover, we have

$$\frac{\partial}{\partial y} F(y, z) = \left( \frac{\partial}{\partial y} \right)^2 \frac{\gamma(y, z) - K}{\xi(y)} < 0$$

by Assumption 2.4 [5]. Thus, we can apply the implicit function theorem and obtain that $y = \psi(z)$ for a continuous function $\psi$ defined on an open neighbourhood $U$ of $z$. 

A relevant question in game theory is under which conditions an equilibrium is unique. As in most situations, also here the uniqueness result relies on monotonicity of the map that characterizes the equilibria. More precisely, whenever the map $\Phi$ is strictly decreasing we obtain a unique mean field equilibrium. Relying again on the implicit function theorem we obtain the following rather abstract criterion.

**Proposition 4.2.** Assume that $\frac{\partial}{\partial z \xi} \tilde{F}(g(z), z) < 0$ for all $z \in [z_1, z_2]$, where $\tilde{F}$ is defined in (6). Then there is a unique mean field equilibrium in threshold strategies.

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Proof. In the setting of the proof of Theorem 4.1 we have seen that the map \( \Phi \) is a composition of two functions \( f \) and \( g \), where \( f \) is increasing. Moreover, the function \( g \) could be characterized by the implicit function theorem, which yields for \( z \in [z_1, z_2] \) that

\[
\frac{\partial g(z)}{\partial z} = -\frac{\frac{\partial}{\partial z} \tilde{F}(g(z), z)}{\frac{\partial}{\partial x} \tilde{F}(g(z), z)}.
\]

Since \( g(z) \) is a maximum of the function \( \gamma(y,z) - K(x_0) \) it satisfies that \( \frac{\partial g(z)}{\partial z} < 0 \). Thus, \( \Phi \) is strictly decreasing and has exactly one fixed point.

However, if Assumption 2.6 holds, the condition of Theorem 4.2 cannot be satisfied. Indeed, we obtain under this assumption the reverse inequality. Thus, it seems that there were no practical uniqueness criteria for the problem at hand. We present an example illustrating this in Section 7.2.

5. The Mean Field Control Problem

We now consider the situation that the market participants cooperate in the sense that they agree to choose a common strategy. The mean field control problem therefore consists of maximizing

\[
J_x(R) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n: \tau_n \leq T} \left( \gamma(X^{\tau_n}_{\tau_n-}, \mathbb{E}[X^R]) - K \right) \right]
\]

over all invariant admissible strategies \( R \). In the class of threshold strategies \( R(y), y \geq y_0 \), the optimization problem is easily solved. In this case, we explicitly know

\[
\mathbb{E}[\hat{X}^R_{\infty}] = \int_{-\infty}^{y} x \pi_{y_0,y}(x)dx.
\]

Hence, we just have to maximize the explicitly given real function

\[
v(y) := J_x(R(y), R(y)) = \frac{\gamma \left( y, \mathbb{E}[\hat{X}^R(y)] \right) - K}{\xi(y)}.
\]

However, the problem (7) is a non-standard stochastic control problem due to the expectation-term. Therefore, it is by no means clear that threshold strategies are indeed optimal in the class of all invariant admissible strategies. In the following theorem, we prove this fact by splitting up the problem as follows

\[
\sup_{R} J_x(R, R) = \sup_{z \in [z_1, z_2]} \sup_{R \text{ with } \mathbb{E}[X^R_{\infty}] = z} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n: \tau_n \leq T} \left( \gamma(X^{\tau_n}_{\tau_n-}, z) - K \right) \right]
\]

and then utilizing a Lagrange-type approach to reduce the restricted problem to a standard problem. Here, \( z_1, z_2 \) are given according to Lemma 2.3.

Before stating the result, we first need a stronger version of Assumption 2.4 to deal with certain auxiliary control problems with running costs. A similar discussion as in the proof of Lemma 2.7 could be carried out, but we avoid this here.

Assumption 5.1. 1. For every \( z \in [z_1, z_2] \) and every \( \lambda > 0 \) there is a unique critical point \( y = y_z \in (y_0, \infty) \) of

\[
\frac{\gamma(y, z) - K - \lambda \mathbb{E}_y \left[ \int_0^{\tau_n} X_s ds \right]}{\xi(y)}
\]

and this is a global maximum.
2. For all pairs \((y, z)\) describing a critical point as given before, it holds
\[
\frac{\partial^2 \gamma(y, z)}{\partial y^2} - K - \lambda \mathbb{E}_{w_0} \left[ \int_0^\infty X_e ds \right] < 0
\]

Theorem 5.2. Under Assumptions 2.1, 2.4, and 5.1, the value of the problem (7) is
\[
\sup_y H(y), \quad H(y) = \gamma \left( y, \mathbb{E} \left[ X^{R(y)}_{\infty} \right] \right) - K
\]
and if \(y^*\) is a maximizer of \(H\), then the threshold strategy \(R = R(y^*)\) is optimal in the class of all invariant admissible strategies.

Proof. By the previous discussion, it is enough to consider, for fixed \(z \in [z_1, z_2]\), the restricted problem
\[
\sup_{R \text{ with } \mathbb{E}[X^R_{\infty}] = z} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n, \tau_n \leq T} \left( \gamma(X^{R}_{\tau_n -}, z) - K \right) \right]
\]
and prove that a threshold strategy is optimal. Following a standard Lagrange approach, we consider, for fixed \(\lambda \geq 0\), the associated unconstrained problem
\[
\sup_{R \text{ with } \mathbb{E}[X^R_{\infty}] = z} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n, \tau_n \leq T} \left( \gamma(X^{R}_{\tau_n -}, z) - K \right) - \lambda \left( \mathbb{E}[X^{R}_{\infty}] - z \right) \right]
\]
(9)
By standard calculus, it holds that
\[
\mathbb{E} \left[ \dot{X}^R_{\infty} \right] = \lim_{T \to \infty} \mathbb{E} \left[ \dot{X}^R_T \right] = \lim_{T \to \infty} \mathbb{E}_x \left[ X^{R}_{T} \right] = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T X^R_t dt \right],
\]
and hence problem (10) may be rewritten as
\[
\sup_{R \text{ with } \mathbb{E}[X^R_{\infty}] = z} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n, \tau_n \leq T} \left( \gamma(X^{R}_{\tau_n -}, z) - K \right) - \int_0^T \lambda X^R_t dt \right] + \lambda z.
\]
As \(\lambda z\) is just a constant, this problem is a standard ergodic impulse control problem and by Proposition 3.6 and Assumption 5.1, there exists a unique threshold \(y = y(\lambda, z)\) such that \(R(y)\) is optimal. Denote by \(y(z)\) the unique threshold value satisfying \(z = \mathbb{E}[X^{R(y)}_{\infty}]\).
Then we obtain that all maximizers \(z\) in (8) fulfill \(y(z) \leq y(0, z)\). Indeed, assume that \(y(z) > y(0, z)\). Since \(y(z)\) is decreasing in the second component, we obtain
\[
J_x(\lambda y(z), y(z)) \leq J_x(\lambda y(0, z), y(z)) \leq J_x(\lambda y(0, z), y(z)),
\]
which is a contradiction. Moreover, for all \(\lambda > 0\),
\[
\lim_{\lambda \to \infty} y(\lambda, z) = y_0 \leq y(z)
\]
and, again by the implicit function theorem and Assumption 5.1, the function \(\lambda \mapsto y(\lambda, z)\) is continuous. Due to Lemma 2.3,
\[
\lim_{\lambda \to \infty} \int w \Pi^{R(y(\lambda, z))}(dw) \leq z \leq \lim_{\lambda \to 0} \int w \Pi^{R(y(\lambda, z))}(dw).
\]
Therefore, there is a \(\lambda_z\) such that
\[
z = \int w \Pi^{R(y(\lambda_z, z))}(dw).
\]
As \( R(y(\lambda, z)) \) is an (unconstrained) maximizer for
\[
\sup_{R} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \sum_{n: \tau_n \leq T} \gamma(X^R_{\tau_n}, z) \right] - \lambda z \left( \mathbb{E}[X^R_{\infty}] - z \right)
\]
and fulfills
\[
z = \mathbb{E}[\hat{X}^R_{\infty}(y(\lambda, z))],
\]
it is a maximizer for \( 9 \) as well, proving the result.

6. Computation and Comparison of Solutions

We have shown that, both for the mean field game and for the mean field equilibrium, there is a solution being a threshold strategy. In both cases the threshold can be obtained by maximizing the function
\[
G : (y_0, \infty) \times \mathbb{R}_+; (x_1, x_2) \mapsto \gamma(x_1, \mathbb{E}[\hat{X}^R_{\infty}(x_2)]) - K \xi(x_1)
\]
in a certain way. We remark that \( G \) is differentiable, since \( x_2 \mapsto \mathbb{E}[\hat{X}^R_{\infty}(x_2)] \) is differentiable due to \( 3 \).

The optimal thresholds \( y^p \) for the cooperative mean field problem is given as the solutions of
\[
0 = \frac{\partial}{\partial y} G(y, y) = \frac{\partial}{\partial x_1} G(y, y) + \frac{\partial}{\partial x_2} G(y, y)
\]
and the threshold(s) \( y^g \) constituting an equilibrium for the competitive mean field game are given as the solution(s) of
\[
0 = \frac{\partial}{\partial x_1} G(y, y).
\]

Our assumptions, in particular the assumption that \( \gamma(x, z) \) is strictly decreasing in \( z \) for all \( x \), yield the following comparison result stating that the threshold under competition is larger than in the cooperative regime. Therefore, in our (oversimplified) model, competition leads to a higher average volume of wood in the forest stands.

**Theorem 6.1.** Let \( \gamma(x, z) \) be strictly decreasing in \( z \) for all \( x \). Then for all threshold values \( y^p \) being solutions of the mean field problem and all threshold values \( y^g \) being solutions of the mean field game we obtain \( y^p \leq y^g \).

**Proof.** Since \( \gamma(x, z) \) is strictly decreasing in \( z \) and \( x \mapsto \mathbb{E}[\hat{X}^R_{\infty}(x)] \) is increasing (see Lemma 3.3), we obtain that \( G(x, z) \) is also decreasing in \( z \). Assume that there is an equilibrium threshold \( y^g \) and a threshold for the mean field control problem \( y^p \) such that \( y^g < y^p \). Since \( y^g \) is an equilibrium we obtain that
\[
G(y^p, y^g) \leq G(y^g, y^g).
\]
Since \( y^p \) is a solution of the mean field control problem we have
\[
G(y^g, y^p) \leq G(y^p, y^p).
\]
Together with the fact that \( G \) is strictly decreasing in the second argument we obtain
\[
G(y^g, y^g) \leq G(y^p, y^p) < G(y^p, y^g) \leq G(y^g, y^g),
\]
which is a contradiction.
7. Example  To illustrate our results, first, we consider the case of a classical logistic stochastic growth model. This is, the controlled process follows the dynamics

\[ dX_t = X_t(a - bX_t)dt + \beta X_t dW_t, \]

where \( a, b, \beta \) are positive constants. This diffusion is well-studied. We refer to [22] for the results we use here and further references. The results there also yield that our assumptions are fulfilled.

Using the notation \( q := 1/2 - a\beta^{-2} \), it is well-known that \( X \) converges towards a unique stationary distribution if \( q < 0 \) and converges to 0 a.s. for \( q > 0 \). We assume \( q < 0 \) in the following. Speed measure and scale function are, resp., given by the densities

\[
\begin{align*}
  s(x) &= x^{2q-1} \exp \left( \frac{2}{\beta^2} b(x-1) \right), \\
  m(x) &= \frac{2}{\beta^2} x^{-2q-1} \exp \left( -\frac{2}{\beta^2} b(x-1) \right),
\end{align*}
\]

so that for each \( y \) the expectation

\[
E[X_R(y)] = \int_{-\infty}^{y} x \pi_{y,0}(x) dx
\]

can be calculated according to (3). The function \( \xi \) is known (semi-) explicitly:

\[
\xi(y) = \frac{1}{\beta^2 |q|} \left( \log \left( \frac{y}{y_0} \right) + \sum_{n=1}^{\infty} \frac{1}{(1-2q)n} \frac{(\rho y)^n}{n} - \sum_{n=1}^{\infty} \frac{1}{(1-2q)n} \left( \frac{\rho y_0}{n} \right)^n \right),
\]

where \( (u)_n = u(u+1) \cdots (u+n-1) \) denotes the Pochhammer symbol and \( \rho := 2b\beta^{-2} \).

(The series may be represented using hypergeometric functions.) For the mean field game all we have to find are thresholds \( y^g \) such that

\[
y^g = \arg \max_y \gamma \left( y, E[X_R(y)] \right) - K \xi(y).
\]

For the mean field problem, the function

\[
y \mapsto \arg \max_y \gamma \left( y, E[X_R(y)] \right) - K \xi(y)
\]

has to be optimized. As all expressions are known explicitly, this task can be carried out straightforwardly. Here we use the following set of parameters

\[
q := -1, \ b := 1/2, \ \beta := 1, \ y_0 := 1, \ K := 1.
\]

7.1. A reward function yielding unique solutions  Let us first choose \( \gamma(y, z) = (y - y_0)/z \). Then we obtain for the game a unique equilibrium threshold \( y^g \) with corresponding value 0.2256. For the control problem the optimizer is \( y^p \approx 4.23 \) with value 0.2446. As discussed in Section 6, the threshold \( y^g \) is lower that \( y^p \). This corresponds to expected volumes of wood \( E \left[ X_{\infty}^{R(y^p)} \right] \approx 1.624 \) and \( E \left[ X_{\infty}^{R(y^g)} \right] \approx 1.78 \). The value in the problem is of course higher then the value in the game.

7.2. A reward function yielding multiple equilibria  Let us now consider the reward function

\[
\gamma : [y_0, \infty) \times (0, \infty) \to \mathbb{R}_+; \ (y, z) \mapsto \frac{y - y_0}{1 + \exp(10(z - 1.9))}.
\]
Having this logistic dependence on $z$ indeed yields three equilibria which are approximately at the points $y^1_g \approx 4.55$, $y^2_g \approx 6.8$ and $y^3_g \approx 55.5$. While the first and the last one of them are stable in the sense that when starting with a value $y_1$ in an interval around the equilibrium point the iteration used to numerically determine the equilibrium points defined by

$$y_{n+1} = \arg \max_{\tilde{y}} \frac{\gamma \left( \tilde{y}, \mathbb{E}^{X_{\mathbb{R}}^{R(y_n)}} \right) - K}{\xi(\tilde{y})}$$

for all $n \in \mathbb{N}$ will converge to $y^1_g$ and $y^3_g$ respectively, this is not the case for $y^2_g$.

A. Solving the Auxiliary Control Problem(s) In this section we prove Proposition 3.6. For the ease of presentation we will write $\gamma(x)$ for $\gamma(x, z)$ as $z$ is assumed to be fixed in this section. First, we derive a verification result. Thereafter, we present a candidate for the value function and the optimal threshold by relying on an associated stopping problem, for which we then prove that it indeed satisfies the conditions of the verification result.

Lemma A.1 (Verification result). (i) Let $g$ be a measurable function on $\mathbb{R}_+$, let $u$ be defined by

$$u(x, y) = \gamma(x) - \gamma(y) - K - g(x) + g(y)$$

for all $x, y \in \mathbb{R}_+, y \leq x$ and assume

(a) $M = (g(X_t) - \int_0^t (h(X_s) + \rho) ds)_{t \geq 0}$ is a supermartingale under $\mathbb{P}_x$ for all $x \in \mathbb{R}_+$,

(b) $\limsup_{T \to \infty} \frac{\mathbb{E}_x g(X^Q_T)}{T} \geq 0$ for all admissible $Q, x \in \mathbb{R}_+$,

(c) $u(x, y_0) \leq 0$ for all $x \in \mathbb{R}_+$,

Then

$$\nu^b(x) \leq \rho \text{ for all } x \in \mathbb{R}_+.$$

(ii) If furthermore $Q^* = (\tau^*_n)_{n \in \mathbb{N}}$ is admissible and such that

(a) Using the notation $M^Q_t := g(X^Q_t) - \int_0^t (h(X^Q_s) + \rho) ds$ for all $Q$, we have

$$\mathbb{E}_x \left( M^Q_{\tau^*_n} - M^Q_{\tau^*_{n-1}} \right) = 0 \text{ for all } n \in \mathbb{N}, \ x \geq 0,$$

(b) $\lim_{T \to \infty} \frac{\mathbb{E}_x g(X^Q_T)}{T} = 0 \text{ for all } x \in \mathbb{R}_+$,

(c) $u(X^Q_{\tau^*_n}, y_0) = 0 \ \mathbb{P}_x$-a.s. for all $x \in \mathbb{R}_+$,

then

$$\nu^b(x) = \rho, \text{ for all } x \in \mathbb{R}_+$$

and $Q^*$ is optimal.
Proof. We first fix an admissible \( Q = (\tau_n)_{n \in \mathbb{N}} \) and \( T > 0 \). Since the process \( X^Q \) runs uncontrolled on each stochastic interval \([\tau_{k-1}, \tau_k)\), the optional sampling theorem yields that \( \mathbb{E}_x [M^Q_{\tau_k \land T} - M^Q_{\tau_{k-1} \land T}] \leq 0 \) for each \( k \in \mathbb{N} \), \( x \in \mathbb{R}_+ \). Hence

\[
\mathbb{E}_x \left[ \sum_{n \in \mathbb{N} : \tau_n \leq T} (\gamma(X^Q_{\tau_n^-}) - K) - \int_0^T h(X^Q_s) \, ds \right]
\]

\[
\leq \mathbb{E}_x \left[ \sum_{n \in \mathbb{N} : \tau_n \leq T} (\gamma(X^Q_{\tau_n^-}) - K) - \sum_{k=1}^{\infty} (M^Q_{\tau_k \land T} - M^Q_{\tau_{k-1} \land T}) - \int_0^T h(X^Q_s) \, ds \right]
\]

\[
= \mathbb{E}_x \left[ \sum_{n \in \mathbb{N} : \tau_n \leq T} (\gamma(X^Q_{\tau_n^-}) - K) - \sum_{k=1}^{\infty} (g(X^Q_{\tau_{k-1} \land T}) - g(X^Q_{\tau_k \land T})) - \int_{\tau_{k-1} \land T}^{\tau_k \land T} (h(X^Q_s) + \rho) \, ds \right] - \int_0^T h(X^Q_s) \, ds
\]

\[
= \mathbb{E}_x \left[ \sum_{1 \leq n : \tau_n \leq T} (\gamma(X^Q_{\tau_n^-}) - K - g(X^Q_{\tau_n^-}) + g(y_0)) \right] - \mathbb{E}_x \left[ g(X^Q_T) \right] + g(x) + \rho T
\]

\[
\leq -\mathbb{E}_x g(X^Q_T) + g(x) + \rho T,
\]

where we use \(\text{i.(c)}\) and that \(\gamma(y_0) = 0\) by Assumption 2.4.

Dividing by \(T\) and taking the limit \(T \to \infty\), we obtain the first assertion using \(\text{i.(b)}\).

The additional assumptions in \(\text{ii.}\) guarantee that we have equality in each step for the strategy \(Q^*\).

We now provide a candidate and verify that this candidate satisfies the assumptions in Lemma A.1. The intuition for our candidate below is as follows: We first find one value \(y\) from which shifting the process back to \(y_0\) yields the maximal reward per time unit. Due to the continuity of \(X\), we will always hit this point \(y\) and therefore going on like this should yield an optimal strategy. More precisely, define

\[
y^* := \arg \max_{y \in [y_0, \bar{y}]} \frac{\gamma(y) - K - \mathbb{E}_{y_0} \left( \int_0^{\tau_y} h(X_s) \, ds \right)}{\zeta(y)},
\]

\[
\rho^* := \max_{y \in [y_0, \bar{y}]} \frac{\gamma(y) - K - \mathbb{E}_{y_0} \left( \int_0^{\tau_y} h(X_s) \, ds \right)}{\zeta(y)},
\]

and \(Q^* = R(y^*)\). Moreover, set for all \(x \in \mathbb{R}_+\)

\[
g(x) := \sup_{\tau \in T_{y_0}} \mathbb{E}_x \left[ \gamma(X_{\tau}) - K - \int_0^{\tau} (h(X_s) + \rho^*) \, ds \right],
\]

where

\[
T_{y_0} := \{\tau \text{ stopping time } |X_{\tau} \geq y_0 \text{ a.s.}\}.
\]

Note that \(g(x)\) is the value function of a classical stopping problem.

**Lemma A.2.** The function \(g\), the constant \(\rho^*\) and the strategy \(R(y^*)\) fulfil the requirements in Lemma A.7.
Proof. \( M = (g(X_t) - \int_0^t (h(X_s) + \rho^*) ds)_{t \geq 0} \) is a supermartingale: This is well-known by the standard theory of optimal stopping, but we give a direct proof here using the reverse optional sampling theorem. Let \( \tau \) be a bounded stopping time. Then, using the time shift operator \( \theta \),

\[
\mathbb{E}_x [M_{\tau}] = \mathbb{E}_x \left[ g(X_{\tau}) - \int_0^{\tau} (h(X_s) + \rho^*) ds \right] \\
= \mathbb{E}_x \left[ \sup_{\sigma \in \mathcal{T}_{\mathcal{F}_\tau}} \mathbb{E}_x \left[ g(X_{\sigma}) - K - \int_0^{\sigma \wedge \tau} (h(X_s) + \rho^*) ds \right] \right] \\
= \mathbb{E}_x \left[ \sup_{\sigma \in \mathcal{T}_{\mathcal{F}_\tau}} \mathbb{E}_x \left[ g(X_{\sigma \wedge \tau}) - K - \int_0^{\sigma \wedge \tau} (h(X_s) + \rho^*) ds \mid \mathcal{F}_\tau \right] \right] \\
\leq \mathbb{E}_x \left[ \sup_{\sigma \in \mathcal{T}_{\mathcal{F}_\tau}} \mathbb{E}_x \left[ \gamma(X_{\sigma}) - K - \int_0^{\sigma \wedge \tau} \rho^* ds \mid \mathcal{F}_\tau \right] \right] \\
\leq \sup_{\sigma \in \mathcal{T}_{\mathcal{F}_\tau}} \mathbb{E}_x \left[ \gamma(X_{\sigma}) - K - \int_0^{\sigma \wedge \tau} (h(X_s) + \rho^*) ds \mid \mathcal{F}_\tau \right] \\
= g(x) \\
= M_0
\]

The inequalities (i).c and (ii).c hold: For this we investigate the stopping problem associated to \( g \). As a first step we note that by the general theory of optimal stopping, first entrance times into non-empty closed sets \( S_\epsilon \) are \( \epsilon \)-optimal. (Note that problems with linear running costs are time homogenous, see, e.g., [29]). Since \( X \) has continuous sample paths and due to the one-sided nature of the attainable stopping times, we have that \( \mathbb{P}_{x_0} \)-a.s. the first entrance into \( S_\epsilon \) is identical to the first hitting times \( \tau_{y_\epsilon} \), where \( y_\epsilon = \min S_\epsilon \). Thus, we obtain

\[
g(y_0) = \sup_{y \geq y_0} \mathbb{E}_{y_0} \left[ \gamma(X_{\tau_{y_0}}) - K - \int_0^{\tau_{y_0}} (h(X_s) + \rho^*) ds \right] \\
= \sup_{y \geq y_0} \left[ \gamma(y) - K - \mathbb{E}_{y_0} \left[ \int_0^{\tau_{y_0}} h(X_s) ds \right] \right] - \max_{z \geq y_0} \frac{\gamma(z) - K - \mathbb{E}_{y_0} \left[ \int_0^{\tau_{y_0}} h(X_s) ds \right]}{\xi(z)} \xi(y) \\
= 0.
\]

Furthermore, we see that by construction the first hitting time of \( y^* \) is optimal. In particular, this yields \( g(y^*) = \gamma(y^*) - K \).

Since immediate stopping is possible, we have \( g \geq \gamma - K \). This implies that

\[
u(x, y_0) = \gamma(x) - \gamma(y_0) - K - g(x) + g(y_0) = \gamma(x) - K - g(x) \leq 0,
\]

which is (i).c. Analogously, we obtain, since \( y^* \) is in the stopping region that

\[
u(\tau_{y^* -}, y_0) = u(y^*, y_0) = 0.
\]

(ii).a holds: The observation that \( g(y^*) = \gamma(y^*) - K \) directly yields that

\[
\mathbb{E}_x [M_{\tau_{y^* -}} - M_{\tau_{y^* -}}] = \mathbb{E}_{y_0} \left[ g(X_{\tau_{y^* -}}) - \int_0^{\tau_{y^*}} (h(X_s) + \rho^*) ds \right] - \mathbb{E}_{y_0} \left[ g(y^*) - \int_0^{\tau_{y^*}} h(X_s) ds \right] \\
= \mathbb{E}_{y_0} \left[ g(y^*) - \int_0^{\tau_{y^*}} h(X_s) ds - g(y_0) \right] - \rho^* \mathbb{E}_{y_0} [\tau_{y^*}] \\
= g(y^*) - (\gamma(y^*) - K) = 0.
\]

The transversality conditions (i).b and (ii).b hold: As pointed out in Remark 3.2, for any threshold strategy \( R(y) \), \( y \geq y_0 \), the controlled process \( X^{R(y)} \) has an invariant
distribution. Denote the invariant measure for $Q^* = R(y^*)$ by $\Pi^Q$. Thus, for all $y \in \mathbb{R}_+$ we obtain

$$\lim_{T \to \infty} E_y [g(X^Q_T)] = \int_0^{y^*} g(x) \pi^Q(dx).$$

Because $h$ is bounded on $[0, y^*]$ and $\tau_{y^*}$ is an optimal stopping time for the problem with value function $g$, we obtain for some $d > 0$

$$g(x) = \gamma(y^*) - K - E_x \left[ \int_0^{\tau_{y^*}} (h(X^*_s) + \rho^*) \, ds \right] \\
\geq \gamma(y^*) - K - d E_x \tau_{y^*} \\
= \gamma(y^*) - K - d \left( \int_x^{y^*} (S(y^*) - S(y)) m(y) \, dy + (S(y^*) - S(x)) M[0, x] \right) \\
\geq \gamma(y^*) - K - d \int_x^{y^*} (S(y^*) - S(y)) m(y) \, dy.$$

As $0$ was assumed to be an entrance-boundary, the last integral is finite and hence $g$ is bounded from below. Since $\gamma$ is continuous, the function $g$ is furthermore bounded from above on compacts. Therefore, we obtain that

$$\lim_{T \to \infty} E_x \left[ g(X^Q_T) \right] = 0$$

for all $x \in \mathbb{R}_+$.

Now take an arbitrary strategy $Q$ and denote with $X^r$ the process $X$ reflected at $y_0$. By the same coupling argument as in the proof of Lemma 2.3, we can assume $X^r \leq X^Q$. It is well known that in our setting $X^r$ has an invariant distribution $\Pi^r$. Further, as mentioned above, $g$ is bounded on $(0, y^*_0)$. Utilizing that $\gamma$ is increasing and $h$ is a positive function a simple calculation yields that $g$ is increasing, which implies that $g(X^Q) \geq g(X^r)$. In total we obtain for all $y \in \mathbb{R}_+$

$$\liminf_{T \to \infty} E_y [g(X^Q_T)] \geq \liminf_{T \to \infty} E_y [g(X^r_T)] = \int g(x) \Pi^r(dx) \in \mathbb{R}$$

and therefore for all $y \in \mathbb{R}_+$

$$\limsup_{T \to \infty} E_y [g(X^Q_T)] / T \geq \liminf_{T \to \infty} E_y [g(X^r_T)] / T = 0.$$

\[ \square \]

**Remark A.3.** The renewal reward theorem directly yields that for each $y > y_0$ the value we get by using $R(y)$ equals

$$\gamma(y) - K - E_{y_0} \left[ \int_0^{\tau_{y^*}} h(X^*_s) \, ds \right] / \xi(y).$$

Hence, whenever $y^*$ is the unique maximizer of

$$y \mapsto \gamma(y) - K - E_{y_0} \left[ \int_0^{\tau_{y^*}} h(X^*_s) \, ds \right] / \xi(y),$$

then $R(y^*)$ is the unique optimal strategy in the set of threshold strategies.
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