A COMPARISON BETWEEN THE BISMUT-LOTT TORSION
AND THE IGUSA-KLEIN TORSION

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ABSTRACT. We consider a fibration with compact fiber together with a unitarily flat complex vector bundle over the total space. Under the assumption that the fiberwise cohomology admits a filtration with unitary factors, we construct Bismut-Lott analytic torsion classes. The analytic torsion classes obtained satisfy Igusa’s and Ohrt’s axiomatization of higher torsion invariants. As a consequence, we obtain a higher version of the Cheeger-Müller/Bismut-Zhang theorem: for trivial flat line bundles, the Bismut-Lott analytic torsion classes coincide with the Igusa-Klein higher topological torsions up to a normalization.

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0. INTRODUCTION

We consider a unitarily flat complex vector bundle $(F, \nabla^F)$ over a closed manifold $X$ whose cohomology with coefficients in $F$ vanishes, i.e., $H^\bullet(X, F) = 0$. Franz [15], Reidemeister [38] and de Rham [13] constructed a topological invariant associated with $(F, \nabla^F)$, known as the Reidemeister-Franz topological torsion (RF-torsion). RF-torsion is the first algebraic-topological invariant which distinguishes certain homotopy-equivalent topological spaces [15, 38]. RF-torsion could be extended to the case $H^\bullet(X, F) \neq 0$ [13, 29, 42]. The construction of RF-torsion is based on the complex of simplicial chains in $X$ with values in $F$. 

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By replacing the complex of simplicial chains by the de Rham complex, Ray and Singer [37] obtained an analytic version of RF-torsion, known as the Ray-Singer analytic torsion (RS-torsion). In the same paper, Ray and Singer conjectured that RF-torsion and RS-torsion are equivalent.

Ray-Singer conjecture was proved independently by Cheeger [12] and Müller [31]. Their result is now known as the Cheeger-Müller theorem. Bismut, Zhang and Müller simultaneously considered its extension. Müller [32] extended the Cheeger-Müller theorem to the unimodular case, i.e., the induced metric on the determinant line bundle det $F$ is flat. Bismut and Zhang [8] extended the Cheeger-Müller theorem to arbitrary flat complex vector bundle. There are also various extensions to equivariant cases [9, 25, 26].

Wagoner [41] conjectured that RF-torsion and RS-torsion can be extended to invariants of a fiber bundle, i.e., a fibration $M \to S$ together with a flat complex vector bundle $(F, \nabla^F)$ over $M$.

Bismut and Lott [7] confirmed the analytic side of Wagoner’s conjecture. They extended RS-torsion to analytic torsion forms (BL-torsion). Indeed, Bismut and Lott proved a Riemann-Roch-Grothendieck type formula (RRG formula) for flat vector bundles. The analytic torsion forms come as a refinement of their RRG formula at the level of differential forms. Their work makes the analytic torsion into a consequence of the local index theory.

Inspired by the work of Bismut and Lott, Igusa [21] confirmed the topological side of Wagoner’s conjecture by constructing higher topological torsions, known as the Igusa-Klein higher topological torsion (IK-torsion). Goette, Igusa and Williams [20, 19] used IK-torsion to detect the exotic smooth structure of fiber bundles. Dwyer, Weiss and Williams [14] constructed another version of higher topological torsion (DWW-torsion).

The relation among these higher torsions is a natural and important research topic. We expect a higher version of the Cheeger-Müller/Bismut-Zhang theorem in full generality.

By extending the proof of the Bismut-Zhang theorem [8], Bismut and Goette [5] established a higher Cheeger-Müller/Bismut-Zhang theorem under the assumption that there exist a fiberwise Morse function $f : M \to \mathbb{R}$ and a fiberwise Riemannian metric such that the fiberwise gradient of $f$ is Morse-Smale [39]. Goette [17, 18] extended the results in [5] to arbitrary fiberwise Morse functions. Bismut and Goette [5] also extended BL-torsion to the equivariant case. And there are related works [6, 11]. We refer to the survey by Goette [16] for an overview on higher torsion invariants. Goette also proposed a program extending the argument in [17, 18] to functions with both non-degenerate critical points and birth-death critical points.

An alternative approach to the higher Cheeger-Müller/Bismut-Zhang theorem is based on Igusa’s work [22]. Igusa axiomatized higher torsion invariants of fibrations (equipped with trivial flat complex line bundles). He considered a fibration $M \to S$ with closed oriented fiber $Z$ such that $H^\ast(Z)$ is unipotent. He stated two axioms, called the additivity axiom and the transfer axiom, and showed that an invariant of $M/S$ satisfies the axioms if and only if it is a linear combination of IK-torsion and the higher Miller-Morita-Mumford class [28, 30, 33] of $M/S$. Badzioch, Dorabiala, Klein and Williams [1] showed that DWW-torsion satisfies Igusa’s axioms.
One of the key results in this paper states that BL-torsion satisfies Igusa’s axioms. As a consequence, in Theorem 0.1, we establish a higher Cheeger-Müller/Bismut-Zhang theorem for fibrations equipped with trivial flat complex line bundles. The proof is based on the results of Ma [27] and ours [36]. Ma’s result [27], which describes the behavior of BL-torsion under the composition of submersions, shows that BL-torsion satisfies the transfer axiom. Our result [36], which describes the behavior of BL-torsion under gluing, shows that BL-torsion satisfies the additivity axiom. Our result [36] is a higher version of the gluing formula obtained by Brüning and Ma [10]. And there are related results [26, 35, 40, 43, 44]. In particular, in [35], we gave a purely analytic proof of [10]. The technique applied in [35] is closely related to [36].

Ohrt [34] axiomatized higher torsion invariants of certain fibrations equipped with unitarily flat complex vector bundles. He considered a smooth manifold $S$ whose fundamental group $\pi_1(S)$ is finite, a fibration $M \to S$ with simple closed oriented fiber $Z$, and a unitarily flat complex vector bundle $F$ over $M$ with finite holonomy group such that $H^\bullet(Z, F)$ is unipotent. He showed that an invariant of $(M/S, F)$ satisfies his axioms if and only if it is a linear combination of IK-torsion and the higher Miller-Morita-Mumford class of $(M/S, F)$.

In this paper, we also show that BL-torsion satisfies Ohrt’s axioms. Moreover, under the same assumptions as in [34], we establish a higher Cheeger-Müller/Bismut-Zhang theorem.

Let us now give more details about the matter of this paper.

**Odd characteristic form and torsion form.** Let $M$ be a smooth manifold. Let $(F, \nabla^F)$ be a flat complex vector bundle over $M$ with flat connection $\nabla^F$. Let $g^F$ be a Hermitian metric on $F$. We will view $g^F$ as a map from $F$ to $F^*$. Following [8, (4.1)] and [7, (1.31)], we define

$$(0.1) \quad \omega(F, g^F) = (g^F)^{-1}\nabla^F g^F \in \Omega^1(M, \text{End}(F)).$$

We fix a square root of $i$, denoted by $i^{1/2}$. In what follows, the choice of square root will be irrelevant. Let $\varphi : \Omega^\bullet(M) \to \Omega^\bullet(M)$ be such that

$$(0.2) \quad \varphi \omega = (2\pi i)^{-k/2} \omega \quad \text{for} \ \omega \in \Omega^k(M).$$

Let $f$ be an odd polynomial, i.e., $f(-z) = -f(z)$. Following [7, (1.34)], we define

$$(0.3) \quad f(\nabla^F, g^F) = (2\pi i)^{1/2} \varphi \left[ f\left(\frac{\omega(F, g^F)}{2}\right)\right] \in \Omega^{\text{odd}}(M).$$

Bismut and Lott [7, §I] showed that $f(\nabla^F, g^F)$ is a closed real form and its cohomology class

$$(0.4) \quad f(\nabla^F) := [f(\nabla^F, g^F)] \in H^{\text{odd}}(M).$$
is independent of \( g^F \). For a graded flat complex vector bundle \((F^\bullet = \bigoplus_k F^k, \nabla F^\bullet = \bigoplus_k \nabla F^k)\) and a Hermitian metric \( g^{F^\bullet} = \bigoplus_k g^{F^k} \) on \( F^\bullet \), we denote

\[
\begin{align*}
 f(\nabla F^\bullet, g^{F^\bullet}) &= \sum_k (-1)^k f(\nabla F^k, g^{F^k}) \in \Omega^{\text{odd}}(M), \\
 f(\nabla F^k) &= \sum_k (-1)^k f(\nabla F^k) \in H^{\text{odd}}(M).
\end{align*}
\]

(0.5)

If \( f \) is an odd formal power series, the constructions above still make sense. In this paper, we always take

\[
f(z) = z e^{z^2}.
\]

(0.6)

Now let

\[
(E^\bullet, \nabla E^\bullet, \partial) : 0 \to E^0 \to \cdots \to E^n \to 0
\]

be an exact sequence of flat complex vector bundles over \( M \). More precisely, \((E^\bullet, \partial)\) is an exact sequence of complex vector bundles over \( M \), and \( \partial \) commutes with the flat connection \( \nabla E^\bullet \). By [7, Thm. 2.19], we have

\[
f(\nabla E^\bullet) = 0 \in H^{\text{odd}}(M).
\]

(0.8)

Let \( g^{E^\bullet} = \bigoplus_k g^{E^k} \) be a Hermitian metric on \( E^\bullet \). The torsion form [7, Def. 2.20] is a real even differential form \( \mathcal{T}(\nabla E^\bullet, \partial, g^{E^\bullet}) \) on \( M \) satisfying

\[
d\mathcal{T}(\nabla E^\bullet, \partial, g^{E^\bullet}) = f(\nabla E^\bullet, g^{E^\bullet}).
\]

(0.9)

We remark that (0.8) and (0.9) can be extended to the case where (0.7) is not necessarily exact [7, §II].

**R.R.G. formula and analytic torsion form.** Let

\[
\pi : M \to S
\]

be a fibration with closed fiber \( Z \). Let \( o(TZ) \) be the orientation line of the fiberwise tangent bundle \( TZ \). Let

\[
e(TZ) \in H^{\dim Z}(M, o(TZ))
\]

(0.11)

be the Euler class of \( TZ \) (see [8, (3.17)]).

Let

\[
(F, \nabla F)
\]

(0.12)

be a flat complex vector bundle over \( M \). Let \( H^\bullet(Z, F) \) be the fiberwise cohomology of \( Z \) with coefficients in \( F \). Then \( H^\bullet(Z, F) \) is a graded complex vector bundle over \( S \) equipped with a canonical flat connection \( \nabla H^\bullet(Z, F) \) (see [7, Def. 2.4]). Bismut and Lott [7, Thm. 3.17] established the following Riemann-Roch-Grothendieck type formula

\[
f(\nabla H^\bullet(Z, F)) = \int_Z e(TZ) f(\nabla F) \in H^{\text{odd}}(S).
\]

(0.13)

Now let \( T^HM \subseteq TM \) be a complement of \( TZ \), let \( g^{TZ} \) be a Riemannian metric on \( TZ \), let \( g^F \) be a Hermitian metric on \( F \). Let \( \nabla^{TZ} \) be the Bismut connection [3, Def.
this paper, we introduce a closed form (see (0.17)). Let $g^{H^*(Z,F)}$ be the $L^2$-metric on $H^*(Z,F)$ associated with $g^{TZ}$. Bismut and Lott [7, Def. 3.22] constructed a real even differential form $\mathcal{T}(T^H M, g^{TZ}, g^F)$ on $S$ such that

\begin{equation}
(0.15)\quad d \mathcal{T}(T^H M, g^{TZ}, g^F) = \int_Z e(TZ, \nabla^{TZ}) f(\nabla^F, g^F) - f(\nabla^{H^*(Z,F)}, g^{H^*(Z,F)}) \, .
\end{equation}

We call $\mathcal{T}(T^H M, g^{TZ}, g^F)$ the Bismut-Lott analytic torsion form. Zhu [43, §2] extended the R.R.G. formula (0.13) as well as the analytic torsion form to the case where $Z$ is a compact manifold with boundaries and $T^H M, g^{TZ}, g^F$ are product on a tubular neighborhood of the boundary (see [43, (2.30)-(2.35)]).

**Torsion classes.** For a smooth manifold $S$, there is a canonical bijection

\begin{equation}
(0.16)\quad \{\text{flat complex vector bundles over } S\}/\text{isomorphism} \cong \{\text{linear representations of }\pi_1(S)\}/\text{conjugation} \, .
\end{equation}

Let $(E, \nabla^E)$ be flat complex vector bundle over $S$. If $(E, \nabla^E)$ corresponds to a unitary representation of $\pi_1(S)$, we call $(E, \nabla^E)$ a unitarily flat complex vector bundle, and we sometimes simply say that $(E, \nabla^E)$ is unitary. A flat complex vector bundle is unitary if and only if it admits a flat Hermitian metric. For a unitarily flat complex vector bundle $(E, \nabla^E)$ and two flat Hermitian metrics $g^E, g^{E'}$ on $E$, we can find an automorphism of flat complex vector bundle $\phi : E \to E$ such that $g^{E'} = \phi^* g^E$.

Let $(E, \nabla^E)$ be a flat complex vector bundle. If there is a filtration by flat subbundles

\begin{equation}
(0.17)\quad E = E_r \supseteq E_{r-1} \supseteq \cdots \supseteq E_0 = 0
\end{equation}

such that $E_j/E_{j-1}$, equipped with the flat connection induced by $\nabla^E$, is unitary for each $j$, we say that $(E, \nabla^E)$ is filtered by flat subbundles with unitary factors. Moreover, if $E_j/E_{j-1}$ is a trivial flat complex line bundle for each $j$, we say that $(E, \nabla^E)$ is unipotent.

**From now on, we always assume that $(F, \nabla^F)$ is unitary, and $H^*(Z,F)$ is filtered by flat subbundles with unitary factors.** Let $g^F$ be a flat Hermitian metric on $F$. In this paper, we introduce a closed form (see (2.5) and (2.6))

\begin{equation}
(0.18)\quad \mathcal{T}_\zeta(T^H M, g^{TZ}, g^F) = \mathcal{T}^{>0}(T^H M, g^{TZ}, g^F) + \text{correction terms} \in \Omega^{\text{even} \geq 2}(S) \, ,
\end{equation}

where $\mathcal{T}^{>0}(T^H M, g^{TZ}, g^F)$ consists of the components of $\mathcal{T}(T^H M, g^{TZ}, g^F)$ of positive degree. The correction terms are torsion forms of certain exact sequences induced by a filtration of $H^*(Z,F)$ with unitary factors, which were introduced by Ma [27, Def. 3.1]. The Bismut-Lott analytic torsion class (see Definition 2.1) is defined as

\begin{equation}
(0.19)\quad \tau^{\text{BL}}(M/S, F) := [\mathcal{T}_\zeta(T^H M, g^{TZ}, g^F)] \in H^{\text{even} \geq 2}(S) \, .
\end{equation}

We will see that $\tau^{\text{BL}}(M/S, F)$ is uniquely determined by the fibration $\pi : M \to S$ and the flat complex vector bundle $(F, \nabla^F)$. One of the many reasons to drop the
component of degree zero in (0.18) is to make $\tau_{BL}(M/S, F)$ independent of $g^F$. Goette [16, Def. 2.8] constructed $\tau_{BL}(M/S, F)$ for $H^\bullet(Z, F)$ unitary.

We further assume that $H^\bullet(Z, F)$ is unipotent, and the fiber of $\pi : M \to S$ is oriented, i.e., the fiberwise tangent bundle $TZ$ is oriented. Igusa [21] constructed a higher topological torsion

\[(0.20) \quad \tau_{IK}(M/S, F) \in H^{\text{even}}(S),\]

which we call the Igusa-Klein higher topological torsion.

**Higher Cheeger-Müller/Bismut-Zhang theorem for trivial flat complex line bundles.** Let $\pi : M \to S$ be a smooth fibration with closed oriented fiber. Let $\mathbb{1}$ be the trivial flat complex line bundle over $M$.

We assume that $H^\bullet(Z, F)$ is unipotent, and the fiber of $\pi : M \to S$ is oriented, i.e., the fiberwise tangent bundle $TZ$ is oriented.

Igusa [21] constructed a higher topological torsion

\[(0.20) \quad \tau_{IK}(M/S, F) \in H^{\text{even}}(S),\]

which we call the Igusa-Klein higher topological torsion.

**Theorem 0.1.** For any positive integer $k$ and any smooth fibration $\pi : M \to S$ with closed oriented fiber $Z$ such that $H^\bullet(Z)$ is unipotent, we have

\[(0.22) \quad \frac{2^{4k} ((2k)!)^2}{(4k+1)!} \tau_{BL}^{2k}(M/S, \mathbb{1}) = -\frac{(2k)!}{(2\pi)^{2k}} \tau_{IK}^{2k}(M/S, \mathbb{1}) + \frac{\zeta'(-2k)}{2} \left[ \int_Z e(TZ) \text{ch}(TZ) \right]^{4k} \in H^{4k}(S).\]

Here the coefficient on the left hand side of (0.22) is exactly the Chern normalization introduced by Bismut and Goette [5, Def. 2.37]. And the second term on the right hand side of (0.22) is a special case of the characteristic class $0J(\cdot, \cdot)$ constructed by Bismut and Goette [5, Def. 7.3] (see also [16, (3.20)])]. In [22, §3.3], this term is called the higher Miller-Morita-Mumford class [28, 30, 33].

We remark that

\[(0.23) \quad \tau_{BL}^{2k+1}(M/S, \mathbb{1}) = 0 \in H^{4k+2}(S).\]

This is due to Bismut and Lott [7] (see also Theorem 2.5).

By [22, Cor. 4.8], if the fiber of $M/S$ is even-dimensional, then

\[(0.24) \quad \tau_{IK}^{2k}(M/S, \mathbb{1}) = \frac{(-1)^k \zeta(2k+1)}{4} \left[ \int_Z e(TZ) \text{ch}(TZ) \right]^{4k}.\]

By (0.24) and the fact that

\[(0.25) \quad \zeta(2k+1) = \frac{(-1)^k 2^{2k+1} \pi^{2k}}{(2k)!} \zeta'(-2k),\]

Theorem 0.1 is equivalent to the following theorem.
**Theorem 0.1'.** Under the same assumptions as in Theorem 0.1, if $Z$ is odd-dimensional, then
\[
\frac{2^k((2k)!)^2}{(4k+1)!}\tau_{2k}^{BL}(M/S, 1) = -\frac{(2k)!}{(2\pi)^{2k}}\tau_{2k}^{IK}(M/S, 1) \in H^{4k}(S),
\]
if $Z$ is even-dimensional, then
\[
\tau_{2k}^{BL}(M/S, 1) = 0 \in H^{4k}(S).
\]

Now we briefly explain the proof of Theorem 0.1'.

One of the key ingredients in the proof is Igusa's axiomatization of higher torsion invariants [22]. We consider an invariant $\tau$ assigning a cohomology class
\[
\tau(M/S) \in H^{\text{even}}(S)
\]
to any smooth fibration $\pi : M \to S$ with closed oriented fiber $Z$ such that $H^*(Z)$ is unipotent. Similarly to (0.21), we denote by $\tau_k(M/S) \in H^{2k}(S)$ the components of $\tau(M/S)$ of degree $2k$. By [22, Cor. 4.5], if $\tau$ satisfies Axioms 1-3 in §1.4 with $F = 1$, then there exist $a_{2k}, b_{2k} \in \mathbb{R}$ such that for any $M/S$ under consideration, we have
\[
\tau_{2k}(M/S) = a_{2k}\tau_{2k}^{IK}(M/S, 1) \quad \text{if } Z \text{ is odd-dimensional},
\]
\[
\tau_{2k}(M/S) = b_{2k} \left[ \int_Z e(TZ)\mathrm{ch}(TZ) \right]^{[4k]} \quad \text{if } Z \text{ is even-dimensional}.
\]

Axiom 1 trivially holds for $\tau^{BL}$. We will show that Axiom 2 holds for $\tau^{BL}$ using our recent result [36]. And we will show that Axiom 3 holds for $\tau^{BL}$ using the result of Ma [27]. Here the calculation is quite straightforward.

Now we know that for certain $a_{2k}, b_{2k} \in \mathbb{R}$, the identities in (0.29) hold with $\tau(M/S)$ replaced by $\tau_{2k}^{BL}(M/S, 1)$. To find $a_{2k}, b_{2k}$, it is sufficient to consider $S^n$-bundles with $n = 1, 2$.

**Partial result for unitarily flat complex vector bundles.** A topological space $X$ is called simple, if $X$ is connected, the fundamental group $\pi_1(X)$ is abelian, and $\pi_1(X)$ acts trivially on the higher homotopy groups $\pi_{>2}(X)$.

Let $S$ be a smooth manifold with $\pi_1(S)$ finite. Let $\pi : M \to S$ be a smooth fibration with simple closed oriented fiber. Let $(F, \nabla^F)$ be a unitarily flat complex vector bundle over $M$ with finite holonomy group.

We assume that $H^*(Z, F)$ is unipotent.

Similarly to (0.21), we denote by $\tau_{k}^{BL}(M/S, F), \tau_{k}^{IK}(M/S, F) \in H^{2k}(S)$ the components of $\tau^{BL}(M/S, F), \tau^{IK}(M/S, F)$ of degree $2k$.

The following theorem partially confirms the higher Cheeger-Müller/Bismut-Zhang theorem proposed by Goette [16, Thm. 5.5].

**Theorem 0.2.** Under the assumptions above, for any positive integer $k$, we have
\[
\frac{2^{4k}((2k)!)^2}{(4k+1)!}\tau_{2k}^{BL}(M/S, F) = -\frac{(2k)!}{(2\pi)^{2k}}\tau_{2k}^{IK}(M/S, F) + \frac{\zeta'(-2k)\tau_{2k}^{IK}(M/S, F)}{2} \left[ \int_Z e(TZ)\mathrm{ch}(TZ) \right]^{[4k]} \in H^{4k}(S),
\]
for any nonnegative integer \( k \), we have
\[
(0.31) \quad 2^{4k+2} \frac{(2k+1)!}{(4k+3)!} \tau_{2k+1}^{BL} (M/S, F) = -\frac{(2k+1)!}{(2\pi)^{2k+1}} \tau_{2k+1}^{IK} (M/S, F) \in H^{4k+2}(S) .
\]

One of the key ingredients in the proof of Theorem 0.2 is Ohrt’s axiomatization of higher torsion invariants [34]. We will show that \( \tau_{BL} \) satisfies Ohrt’s axioms (see Axioms 1-7 in §1.4). Then, by [34, Theorem 0.1], there exist \( a_{2k}, b_{2k} \in \mathbb{R} \) such that for any \((M/S, F)\) under consideration, we have
\[
(0.32) \quad 2^{4k} \frac{(2k)!}{(4k+1)!} \tau_{2k}^{BL} (M/S, F) = a_{2k} \tau_{2k}^{IK} (M/S, F) + b_{2k} \operatorname{rk} F \left[ \int Z e(TZ) \operatorname{ch}(TZ) \right]^{[4k]} .
\]

We can find \( a_{2k}, b_{2k} \) by considering \( \mathbb{S}^n \)-bundles equipped with trivial flat complex line bundles. And (0.31) follows from a similar argument.

**Notations.** We summarize some frequently used notations and conventions.

For a smooth manifold \( S \), we denote by \( \Omega^k(S) \) the vector space of differential forms on \( S \) of degree \( k \). In particular, we have \( \Omega^0(S) = \mathcal{C}^\infty(S) \). We denote
\[
(0.33) \quad \Omega^\bullet(S) = \bigoplus_k \Omega^k(S) , \quad \Omega^{even/odd}(S) = \bigoplus_{k \ even/odd} \Omega^k(S) .
\]

For \( \omega = \sum_k \omega_k \in \Omega^\bullet(S) \) with \( \omega_k \in \Omega^k(S) \), we denote
\[
(0.34) \quad \omega^{[k]} = \omega_k , \quad \omega^{[>k]} = \sum_{j>k} \omega_j .
\]

Following [7, Def. 1.10], we denote by
\[
(0.35) \quad Q^S \subseteq \Omega^{even}(S)
\]
the vector subspace of real even forms on \( S \), and denote by
\[
(0.36) \quad Q^{S,0} \subseteq Q^S
\]
the vector subspace of exact real even forms on \( S \).

For a vector bundle \( E \) over \( S \), we denote by \( \Omega^k(S, E) \) the vector space of differential forms on \( S \) of degree \( k \) with values in \( E \). In particular, we have \( \Omega^0(S, E) = \mathcal{C}^\infty(S, E) \). We use the notation \( \Omega^\bullet(S, E) \) in the same way as in (0.33).

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1. Preliminaries

1.1. Torsion form. This subsection follows [7, §II].

Let \( S \) be a smooth manifold. Let
\[
(1.1) \quad (E^\bullet, \nabla^{E^\bullet}, \partial) : 0 \to E^0 \to \cdots \to E^n \to 0
\]
be an exact sequence of flat complex vector bundles over $S$. We extend the action

\[ \partial : \mathcal{C}^\infty(S, E^i) \to \mathcal{C}^\infty(S, E^{i+1}) \]

to $\partial : \Omega^k(S, E^i) \to \Omega^k(S, E^{i+1})$ such that

\[ \partial(\tau \otimes w) = (-1)^k \tau \otimes \partial w \quad \text{for } \tau \in \Omega^k(S) \text{ and } w \in \mathcal{C}^\infty(S, E^i) . \]

We extend the action $\nabla^{E^*} : \mathcal{C}^\infty(S, E^*) \to \Omega^1(S, E^*)$ to $\nabla^{E^*} : \Omega^k(S, E^*) \to \Omega^{k+1}(S, E^*)$ such that

\[ \nabla^{E^*}(\tau \otimes w) = dt \otimes w + (-1)^k \tau \wedge \nabla^{E^*} w \quad \text{for } \tau \in \Omega^k(S) \text{ and } w \in \mathcal{C}^\infty(S, E^*) . \]

Under the assumptions and the conventions above, we have

\[ \Delta^2 = 0 , \quad (\nabla^{E^*})^2 = 0 , \quad \partial \nabla^{E^*} + \nabla^{E^*} \partial = 0 . \]

Set

\[ A'' = \partial + \nabla^{E^*} . \]

By (1.4) and (1.5), we have

\[ (A'')^2 = 0 . \]

Thus $A''$ is a flat superconnection in the sense of [7, §I].

Let $g^{E^*} = \bigoplus_{k=0}^n g^{E^k}$ be a Hermitian metric on $E^*$. Similarly to (0.1), we define

\[ \omega^{E^*} = (g^{E^*})^{-1} \nabla^{E^*} g^{E^*} \in \Omega^1(S, \text{End}(E^*)) . \]

Let $\partial^*$ be the adjoint of $\partial$. Let $A'$ be the adjoint superconnection of $A''$ in the sense of [7, §I]. By [7, §II(b)], we have

\[ A' = \partial^* + \nabla^{E^*} + \omega^{E^*} . \]

Set

\[ X = \frac{1}{2}(A' - A'') = \frac{1}{2}(\partial^* - \partial) + \frac{1}{2} \omega^{E^*} \in \Omega^*(S, \text{End}(E^*)) . \]

Let $N^{E^*}$ be the number operator on $E^*$, i.e., $N^{E^*} |_{E^k} = k \text{ Id}$. For $t > 0$, let $X_t$ be the operator $X$ with the metric $g^{E^*}$ replaced by $t^{N^{E^*}} g^{E^*}$. We have

\[ X_t = \frac{1}{2} (t \partial^* - \partial) + \frac{1}{2} \omega^{E^*} . \]

We denote

\[ \chi'(E^*) = \sum_{k=0}^n (-1)^k k \text{ rk} (E^k) . \]

Recall that $f(z) = z e^{z^2}$ and $f'(z) = (1 + 2z^2) e^{z^2}$. Recall that $Q^S$ was defined in (0.35). The following definition is due to Bismut and Lott [7, Def. 2.20].

**Definition 1.1.** The torsion form associated with $(\nabla^{E^*}, \partial, g^{E^*})$ is defined as

\[ \mathcal{F} (\nabla^{E^*}, \partial, g^{E^*}) = - \int_0^{+\infty} \left\{ \varphi \text{ Tr} \left[ (-1)^{N^{E^*}} \frac{N^{E^*}}{2} f'(X_t) \right] - \frac{1}{2} \chi'(E^*) f' \left( \frac{1+i t}{2} \right) \right\} \frac{dt}{t} \in Q^S . \]

By [7, Thm. 2.13, Prop. 2.18], the integrand in (1.12) is integrable.
Let \( f(\nabla^{E^*}, g^{E^*}) \in \Omega^{\text{odd}}(S) \) be as in (0.5). By [7, Thm. 2.22], we have
\[
(1.13) \quad d\mathcal{T}(\nabla^{E^*}, \partial, g^{E^*}) = f(\nabla^{E^*}, g^{E^*}).
\]

The following theorem is due to Bismut and Lott [7, Thm. 1.8(iv)].

We will use the notation in (0.34).

**Theorem 1.2.** If the quadruple \((E^*, \nabla^{E^*}, \partial, g^{E^*})\) is the complexification of a real quadruple \((E^*_R, \nabla^{E^*_R}, \partial_R, g^{E^*_R})\), then
\[
(1.14) \quad \mathcal{T}^{[k]}(\nabla^{E^*}, \partial, g^{E^*}) = 0 \quad \text{for } k \equiv 2 \pmod{4}.
\]

**Proof.** Note that \(N^{E^*}\) is self-adjoint, the proof of [7, Thm. 1.8(iv)] yields
\[
(1.15) \quad \text{Tr} \left[ (-1)^{N^{E^*}} \frac{N^{E^*}}{2} f'(X_i) \right] = (-1)^{k(k-1)/2} \text{Tr} \left[ (-1)^{N^{E^*}} \frac{N^{E^*}}{2} f'(X_i) \right]^{[k]}.
\]

From Definition 1.1 and the fact that \(\mathcal{T}(\nabla^{E^*}, \partial, g^{E^*})\) is real, we obtain (1.14). This completes the proof. \(\Box\)

By the end of this subsection, we introduce several notations.

For a flat complex vector bundle \((E, \nabla^{E})\) over \(S\) and Hermitian metrics \(g^E, g^{E'}\) on \(E\), we denote by \(\mathcal{T}(g^E, g^{E'})\) the torsion form of the exact sequence
\[
(1.16) \quad 0 \rightarrow E \xrightarrow{1} E 
\]
where the first \(E\) is equipped with the metric \(g^E\) and the second \(E\) is equipped with the metric \(g^{E'}\).

For a flat complex vector bundle \((E, \nabla^{E})\) over \(S\), a flat subbundle \(F \subseteq E\) and Hermitian metrics \(g^F, g^E, g^{E/F}\) on \(F, E, E/F\), we denote by \(\mathcal{T}(g^F, g^E, g^{E/F})\) the torsion form of the exact sequence
\[
(1.17) \quad 0 \rightarrow F \rightarrow E \rightarrow E/F 
\]
equipped with metrics \(g^F, g^E, g^{E/F}\).

1.2. Torsion form of filtration with unitary factors. Let \((E, \nabla^{E})\) be a flat complex vector bundle over \(S\). We assume that \(E\) is filtered by flat subbundles with unitary factors. More precisely, there is a filtration of \(E\) by flat subbundles
\[
(1.18) \quad E^*_r : E = E_r \supseteq E_{r-1} \supseteq \cdots \supseteq E_1 \supseteq E_0 = 0,
\]
such that \(F_j := E_j/E_{j-1}\) admits a flat Hermitian metric for each \(j\).

Let \(g^E\) be a Hermitian metric on \(E\). Let \(g^{E_j}\) be the restricted metric on \(E_j\). Let \(g^{F_j}\) be a flat Hermitian metric on \(F_j\). Recall that \(\mathcal{T}(\cdot, \cdot, \cdot)\) was defined in the paragraph containing (1.17). Following [27, Def. 3.1], we define
\[
(1.19) \quad \mathcal{T}(E^*_r, g^{F^*_r}, g^E) = - \sum_{j=1}^{r} \mathcal{T}(g^{E_{j-1}}, g^{E_j}, g^{F_j}).
\]

By (0.5), (1.13) and (1.19), we have
\[
(1.20) \quad d\mathcal{T}(E^*_r, g^{F^*_r}, g^E) = f(\nabla^E, g^E). \]

Recall that \(Q^{S, 0}\) was defined in (0.36). We will use the notation in (0.34).
Theorem 1.3. For

\( E^j_\bullet : E = E^j_\bullet \supseteq E^j_{s-1} \supseteq \cdots \supseteq E^j_1 \supseteq E^j_0 = 0 \)

another filtration with unitary factors and \( g^{F_j} \) flat Hermitian metric on \( F^j_j := E^j_j/E^j_{j-1} \), we have

\[
\mathcal{T}^{>0}(E^j_\bullet, g^{F^j_\bullet}, g^E) - \mathcal{T}^{>0}(E^\bullet_\bullet, g^{F^\bullet_\bullet}, g^E) \in Q^{S,0}.
\]

Proof. The proof consists of several steps.

**Step 1.** We prove (1.22) under the assumption that \( r = s \) and \( E^j_j = E_j \) for each \( j \).

From (1.19) and our assumptions, we get

\[
\mathcal{T}(E^j_\bullet, g^{F^j_\bullet}, g^E) - \mathcal{T}(E^\bullet_\bullet, g^{F^\bullet_\bullet}, g^E)
\]

\[
= - \sum_{j=1}^r \left( \mathcal{T}(g^{E^j_{j-1}}, g^{E_j}, g^{F^j_j}) - \mathcal{T}(g^{E^j_{j-1}}, g^{E^j_j}, g^{F^j_j}) \right).
\]

We consider the following commutative diagram of flat complex vector bundles over \( S \) with exact rows and columns,

\[
\begin{array}{c}
0 \rightarrow E_{j-1} \rightarrow E_j \rightarrow F_j \rightarrow 0 \\
0 \rightarrow E_{j-1} \rightarrow E_j \rightarrow F_j \rightarrow 0,
\end{array}
\]

where all the maps are canonical embeddings and projections. We equip \( E_{j-1}, E_j \) in (1.24) with the metrics \( g^{E_{j-1}}, g^{E_j} \), equip the lower \( F_j \) in (1.24) with the metric \( g^{F_j} \), and equip the upper \( F_j \) in (1.24) with the metric \( g^{F^j_j} \). Applying [7, Theorem A1.4] to (1.24), we obtain

\[
\mathcal{T}(g^{F_j}, g^{F^j_j}) = \mathcal{T}(g^{E^j_{j-1}}, g^{E_j}, g^{F^j_j}) - \mathcal{T}(g^{E^j_{j-1}}, g^{E^j_j}, g^{F^j_j}) \text{ modulo } Q^{S,0}.
\]

On the other hand, since \( g^{F_j}, g^{F^j_j} \) are flat Hermitian metrics, \( \mathcal{T}(g^{F_j}, g^{F^j_j}) \) is a locally constant function on \( S \). As a consequence, we have

\[
\mathcal{T}^{>0}(g^{F_j}, g^{F^j_j}) = 0.
\]

From (1.23), (1.25) and (1.26), we obtain (1.22).

**Step 2.** We prove (1.22) under the assumption that the filtration \( E^j_\bullet \) is a refinement of \( E_\bullet \), i.e., there is a map \( \sigma : \{1, \cdots, r\} \rightarrow \{1, \cdots, s\} \) such that \( E_j = E^\sigma_{\sigma(j)} \) for each \( j \).

We may and we will assume that \( s = r + 1 \) and there exists \( a \in \mathbb{N} \) such that

\[
E^j_j = E_j \text{ for } j = 1, \cdots, a - 1, \quad E^j_{j-1} = E_{a+1} \text{ for } j = a + 1, \cdots, s.
\]

As a consequence, we have

\[
F^j_j = F_j \text{ for } j = 1, \cdots, a - 1, \quad F^j_{j-1} = F_{a+1} \text{ for } j = a + 2, \cdots, s.
\]

By Step 1, we may further assume that

\[
g^{F^j_j} = g^{F_j} \text{ for } j = 1, \cdots, a - 1, \quad g^{F^j_{j-1}} = g^{F_{j-1}} \text{ for } j = a + 2, \cdots, s.
\]
By (1.19) and (1.27)-(1.29), we have
\begin{equation}
\mathcal{T}(E'_*, g^{F'_a}, g^F) - \mathcal{T}(E'_*, g^{F'_a}, g^F) = \mathcal{T}(g^{E_{a-1}}, g^{E_a}, g^{F_{a+1}}) - \mathcal{T}(g^{E_{a-1}}, g^{E_a}, g^{F_{a+1}}).
\end{equation}

We consider the following commutative diagram of flat complex vector bundles over $S$ with exact rows and columns,
\begin{equation}
\begin{array}{c}
0 \\ \downarrow \\
F'_{a+1} \rightarrow E_{a-1} \rightarrow E_a \rightarrow F_a \rightarrow 0 \\
\downarrow \downarrow \downarrow \\
\text{Id} \\
0 \\
\end{array}
\begin{array}{c}
0 \\ \downarrow \\
F'_{a+1} \rightarrow E'_{a-1} \rightarrow E'_a \rightarrow F'_a \rightarrow 0 \\
\downarrow \downarrow \downarrow \\
\text{Id} \\
0 \\
\end{array}
\end{equation}

where all the maps are canonical embeddings and projections. Applying [7, Theorem A1.4] to (1.31), we obtain
\begin{equation}
\mathcal{T}(g^{F'_a}, g^{F_a}, g^{F_{a+1}}) - \mathcal{T}(g^{F'_a}, g^{F_a}, g^{F_{a+1}}) = \mathcal{T}(g^{E_{a-1}}, g^{E_a}, g^{F_{a+1}}) - \mathcal{T}(g^{E_{a-1}}, g^{E_a}, g^{F_{a+1}}) \pmod{Q^{S,0}}.
\end{equation}

On the other hand, since $g^{F'_a}, g^{F_a}, g^{F_{a+1}}$ are flat Hermitian metrics, $\mathcal{T}(g^{F'_a}, g^{F_a}, g^{F_{a+1}})$ is a locally constant function on $S$. As a consequence, we have
\begin{equation}
\mathcal{T}^{>0}(g^{F'_a}, g^{F_a}, g^{F_{a+1}}) = 0.
\end{equation}

From (1.30), (1.32) and (1.33), we obtain (1.22).

**Step 3.** We prove (1.22) under the assumption that $r = s$ and there exists $a \in \mathbb{N}$ such that $E_j = E'_j$ for $j \neq a$.

By Step 2, we may replace the filtration $E_*$ by
\begin{equation}
E_r \supset \cdots \supset E_{a+1} \supset E_a \supset E'_a \supset E_a \cap E'_a \supset E_{a-1} \supset \cdots \supset E_0 = 0,
\end{equation}
and replace the filtration $E'_*$ by
\begin{equation}
E_r \supset \cdots \supset E_{a+1} \supset E_a \supset E'_a \supset E_a \cap E'_a \supset E_{a-1} \supset \cdots \supset E_0 = 0.
\end{equation}

Equivalently, we may and we will assume that
\begin{equation}
E_{a+1} = E'_{a+1} = E_a \supset E'_a, \quad E_{a-1} = E'_{a-1} = E_a \cap E'_a.
\end{equation}

As a consequence, we have
\begin{equation}
E_{a+1}/E_{a-1} = F_a \oplus F'_a.
\end{equation}

Hence
\begin{equation}
E_r \supset \cdots \supset E_{a+1} \supset E_{a-1} \supset \cdots \supset E_0 = 0.
\end{equation}
is a filtration with unitary factors. Note that both $E_\ast$ and $E_\ast'$ are refinements of the filtration (1.38), from Step 2, we obtain (1.22).

**Step 4.** We prove (1.22).

Similarly to the proof of Jordan-Hölder theorem, we can find $E_0^0, \ldots, E_m^0$ filtrations of $E$ by flat subbundles with unitary factors such that

- for each $i, j$, we have $E_j^i \in \{ E_k \cap E_l^j : k = 0, \ldots, r, l = 0, \ldots, s \};$
- for $i = 0, \ldots, m - 1$, there exists $a_i \in \mathbb{N}$ such that $E_j^i = E_{j+1}^i$ for $j \neq a_i$;
- $E_0^0$ is a refinement of $E_\ast$, and $E_m^m$ is a refinement of $E_\ast'$.

For each $i, j$, let $F_j^i = E_j^i / E_{j-1}^i$, let $g^{F_j^i}$ be a flat Hermitian metric on $F_j^i$. By Step 2, we have

$$\mathcal{T}^{[>0]}(E_\ast, g^{F_\ast}, g_E) - \mathcal{T}^{[>0]}(E_\ast, g^{F_\ast'}, g_E) \subset Q^{S,0},$$

On the other hand, by Step 3, we have

$$\mathcal{T}^{[>0]}(E_\ast, g^{F_\ast}, g_E) - \mathcal{T}^{[>0]}(E_{i+1}^i, g^{F_{i+1}^i}, g_E) \subset Q^{S,0} \quad \text{for } i = 0, \ldots, m - 1.$$

From (1.39) and (1.39), we obtain (1.22). This completes the proof. □

**Definition 1.4.** We define

$$\mathcal{T}(E, g^E) = \mathcal{T}^{[>0]}(E_\ast, g^{F_\ast}, g_E) \subset Q^S / Q^{S,0}.$$  

By Theorem 1.3, $\mathcal{T}(E, g^E)$ is well-defined.

By Definition 1.4 and (1.20), we have

$$d\mathcal{T}(E, g^E) = f(\nabla^E, g_E)^{[>1]} .$$

Similarly to Theorem 1.2, we have the following result.

**Theorem 1.5.** If the triple $(E, \nabla^E, g^E)$ is the complexification of a real triple $(E_\mathbb{R}, \nabla^{E_\mathbb{R}}, g^{E_\mathbb{R}})$, then

$$\mathcal{T}^{[k]}(E, g^E) = 0 \quad \text{for } k \equiv 2 \pmod{4} .$$

**Proof.** Let

$$E = \overline{E}, \overline{E}_{r-1} \supset \cdots \supset \overline{E}_1 \supset \overline{E}_0 = 0$$

be the conjugated filtration. Since $E_\ast$ is with unitary factors, so is $\overline{E}_\ast$. We denote $\overline{F}_j = \overline{E}_j / \overline{E}_{j-1}$. Let $g^{\overline{F}_j}$ be the Hermitian metric on $\overline{F}_j$ defined by $\overline{g}_j^{\overline{F}_j}(v, \overline{v}) = g_{\overline{F}_j}(\overline{v}, \overline{v})$. Since $g^{\overline{F}_j}$ is a flat Hermitian metric, so is $\overline{F}_j$. Let $g_{\overline{E}_j}$ be the restricted metric of $g^E$ on $\overline{E}_j$. Similarly to Theorem 1.2, the proof of [7, Thm. 1.8(iv)] yields

$$\mathcal{T}^{[k]}(g^{\overline{E}_{j-1}}, g^{\overline{E}_j}, g^{F_j}) = \mathcal{T}^{[k]}(g^{E_{j-1}}, g^{E_j}, g^{F_j}) = (-1)^{(k-1)/2} \mathcal{T}^{[k]}(g^{E_{j-1}}, g^{E_j}, g^{F_j}) .$$
By Definition 1.4, (1.19) and (1.45), for \( k > 0 \), we have
\[
\mathcal{T}^{[k]}(E, g^E) = -\sum_{j=1}^{r} \mathcal{T}^{[k]}(g_{E_{j-1}}, g_{E_j}, g_{F_j})
\]
\[
= -(-1)^{(k-1)/2} \sum_{j=1}^{r} \mathcal{T}^{[k]}(g_{E_{j-1}}, g_{E_j}, g_{F_j}) = (-1)^{(k-1)/2} \mathcal{T}^{[k]}(E, g^E).
\]

From (1.46), we obtain (1.43). This completes the proof. \( \square \)

Now we consider an exact sequence of flat complex vector bundles over \( S \),
\[
(E^\bullet, \partial) : 0 \to E^0 \to E^1 \to \cdots \to E^n \to 0.
\]
We assume that each \( E^k \) is filtered by flat subbundles with unitary factors.

Let \( g^{E^\bullet} = \bigoplus_k g^{E^k} \) be a Hermitian metric on \( E^\bullet \). We define
\[
\mathcal{T}(E^\bullet, g^{E^\bullet}) = \sum_{k=0}^{n} (-1)^k \mathcal{T}(E^k, g^{E^k}) \in Q^S/Q^{S,0}.
\]

By (0.5), (1.42) and (1.48), we have
\[
d\mathcal{T}(E^\bullet, g^{E^\bullet}) = f(\nabla^{E^\bullet}, g^{E^\bullet})^{>1}.
\]
Let \( \mathcal{T}(E^\bullet, \partial, g^{E^\bullet}) \) be the torsion form of \((E^\bullet, \partial)\) equipped with the metric \( g^{E^\bullet} \). We have
\[
d\mathcal{T}(E^\bullet, \partial, g^{E^\bullet}) = f(\nabla^{E^\bullet}, g^{E^\bullet}).
\]
From (1.49) and (1.50), we see that \( \mathcal{T}(E^\bullet, g^{E^\bullet}) - \mathcal{T}^{>0}(E^\bullet, \partial, g^{E^\bullet}) \) is closed.

**Theorem 1.6.** The following identity holds in \( Q^S/Q^{S,0} \),
\[
\mathcal{T}(E^\bullet, g^{E^\bullet}) = \mathcal{T}^{>0}(E^\bullet, \partial, g^{E^\bullet}).
\]

**Proof.** The proof consists of two steps.

**Step 1.** We prove (1.51) under the assumption that \( n = 2 \), i.e., (1.47) is given by
\[
0 \to E^0 \xrightarrow{\alpha} E^1 \xrightarrow{\beta} E^2 \to 0.
\]
We can find filtrations with unitary factors
\[
E^0 = E^0_r \supseteq \cdots \supseteq E^0_0 = 0, \quad E^1 = E^1_{r+s} \supseteq \cdots \supseteq E^1_0 = 0,
\]
\[
E^2 = E^2_{r+s} \supseteq \cdots \supseteq E^2_r = 0
\]
such that
\[
E^1_j = \alpha(E^0_j) \text{ for } j = 1, \ldots, r, \quad E^2_j = \beta(E^1_j) \text{ for } j = r+1, \ldots, r+s.
\]
For each \( i, j \), we denote \( F^i_j = E^i_j/E^i_{j-1} \). Then we have
\[
F^0_j = E^1_j \text{ for } j = 1, \ldots, r, \quad F^2_j = F^1_j \text{ for } j = r+1, \ldots, r+s.
\]
For ease of notations, we denote \( F_j = F_j^1 \). For each \( j \), let \( g^{F_j} \) be a flat Hermitian metric on \( F_j \). For each \( i, j \), we denote by \( g^{E^i_j} \) the restricted metric of \( g^{E_j} \) on \( E^i_j \). By Definition 1.4, (1.19) and (1.48), we have

\[
\mathcal{T}(E^*, g^{E*}) = \sum_{j=1}^{r} \left( \mathcal{T}^{[>0]}(g^{E^1_{j-1}}, g^{E^1_j}, g^{F_j}) - \mathcal{T}^{[>0]}(g^{E^0_{j-1}}, g^{E^0_j}, g^{F_j}) \right) \\
+ \sum_{j=r+1}^{r+s} \left( \mathcal{T}^{[>0]}(g^{E^1_{j-1}}, g^{E^1_j}, g^{F_j}) - \mathcal{T}^{[>0]}(g^{E^2_{j-1}}, g^{E^2_j}, g^{F_j}) \right).
\]

(1.56)

For \( j = 1, \ldots, r \), applying [7, Theorem A1.4] to the commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \rightarrow & E^1_{j-1} & \rightarrow & E^1_j & \rightarrow & F_j & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow \text{Id} & & \\
0 & \rightarrow & E^0_{j-1} & \rightarrow & E^0_j & \rightarrow & F_j & \rightarrow & 0,
\end{array}
\]

(1.57)

we obtain

\[
\mathcal{T}(g^{E^1_{j-1}}, g^{E^1_j}, g^{F_j}) - \mathcal{T}(g^{E^0_{j-1}}, g^{E^0_j}, g^{F_j}) = \mathcal{T}(g^{E^0_j}, g^{E^1_j}) - \mathcal{T}(g^{E^0_{j-1}}, g^{E^1_{j-1}}) \mod Q^{S,0}.
\]

(1.58)

As a consequence, we have

\[
\sum_{j=1}^{r} \left( \mathcal{T}(g^{E^1_{j-1}}, g^{E^1_j}, g^{F_j}) - \mathcal{T}(g^{E^0_{j-1}}, g^{E^0_j}, g^{F_j}) \right) = \mathcal{T}(g^{E^0_j}, g^{E^1_j}) \mod Q^{S,0}.
\]

(1.59)

For \( j = r + 1, \ldots, r + s \), applying [7, Theorem A1.4] to the commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \rightarrow & E^2_{j-1} & \rightarrow & E^2_j & \rightarrow & F_j & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow \text{Id} & & \\
0 & \rightarrow & E^1_{j-1} & \rightarrow & E^1_j & \rightarrow & F_j & \rightarrow & 0 \\
& & \uparrow \text{Id} & & \uparrow & & \uparrow & & \\
E^0 & \rightarrow & E^0 & \rightarrow & 0 & & 0 & & ,
\end{array}
\]

(1.60)

we obtain

\[
\mathcal{T}(g^{E^1_{j-1}}, g^{E^1_j}, g^{F_j}) - \mathcal{T}(g^{E^2_{j-1}}, g^{E^2_j}, g^{F_j}) = \mathcal{T}(g^{E^0_j}, g^{E^1_j}) - \mathcal{T}(g^{E^0_{j-1}}, g^{E^1_{j-1}}) \mod Q^{S,0}.
\]

(1.61)
As a consequence, we have

\[
\sum_{j=r+1}^{r+s} \left( \mathcal{J}(g^{E_j}_{r-1}, g^{E_j}_{r}, g^{F_j}) - \mathcal{J}(g^{E_j}_{r-1}, g^{E_j}_{r}, g^{F_j}) \right)
= \mathcal{J}(g^{E_0}, g^{E_1}) - \mathcal{J}(g^{E_0}, g^{E_1}) \mod Q^{S,0}.
\]

From (1.56), (1.59), (1.62), we obtain

\[
\mathcal{J}(E^*, g^{E^*}) = \mathcal{J}[>0](g^{E_0}, g^{E_1}, g^{E^2}) \mod Q^{S,0},
\]

which is equivalent to (1.51) with \( n = 2 \).

Step 2. We prove (1.51) by induction on \( n \).

The cases \( n = 1, 2 \) are proved in Step 1. In the sequel, we assume that \( n \geq 3 \). Let \( K^{n-1} \subseteq E^{n-1} \) be the kernel of \( E^{n-1} \to E^n \), which is also the image of \( E^{n-2} \to E^{n-1} \). We have an exact sequence

\[
(1.64)

(\tilde{E}^*, \partial) : 0 \to E^0 \to \cdots \to E^{n-2} \to K^{n-1} \to 0.
\]

Let \( g^{K^{n-1}} \) be the restricted metric of \( g^{E^{n-1}} \) on \( K^{n-1} \). Let \( g^{\tilde{E}^*} \) be the metric on \( \tilde{E}^* \) defined by \( (g^{E^i})_{i=0, \ldots, n-2} \) and \( g^{K^{n-1}} \). Let \( \mathcal{J}(\tilde{E}^*, \partial, g^{\tilde{E}^*}) \) be the torsion form of \( (\tilde{E}^*, \partial) \) equipped with the metric \( g^{\tilde{E}^*} \).

Applying [7, Theorem A1.4] to the commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
\uparrow & & & & & & & & \downarrow \\
0 & \to & E^0 & \to & \cdots & \to & E^{n-2} & \to & E^{n-1} & \to & E^n & \to & 0 \\
& & & & & & & & \downarrow \text{Id} & & \downarrow \text{Id} & & \downarrow \text{Id} & & \downarrow \\
& & & & & & & & 0 & \to & E^0 & \to & \cdots & \to & E^{n-2} & \to & K^{n-1} & \to & 0 & \to & 0
\end{array}
\]

we obtain

\[
\mathcal{J}(E^*, \partial, g^{E^*}) - \mathcal{J}(\tilde{E}^*, \partial, g^{\tilde{E}^*})
= (-1)^n \mathcal{J}(g^{K^{n-1}}, g^{E^{n-1}}, g^{E^n}) \mod Q^{S,0}.
\]

On the other hand, by Step 1 and (1.48), we have

\[
\mathcal{J}(E^*, g^{E^*}) - \mathcal{J}(\tilde{E}^*, g^{\tilde{E}^*})
= (-1)^n \left( \mathcal{J}(K^{n-1}, g^{K^{n-1}}) - \mathcal{J}(E^{n-1}, g^{E^{n-1}}) + \mathcal{J}(E^n, g^{E^n}) \right)
= (-1)^n \mathcal{J}[>0](g^{K^{n-1}}, g^{E^{n-1}}, g^{E^n}) \mod Q^{S,0}.
\]
From (1.66) and (1.67), we obtain
\[
\mathcal{T}^{[>0]}(E^\bullet, \partial, g^{E^\bullet}) - \mathcal{T}^{[>0]}(\widetilde{E}^\bullet, \partial, g^{\widetilde{E}^\bullet}) = \mathcal{T}(E^\bullet, g^{E^\bullet}) - \mathcal{T}(\widetilde{E}^\bullet, g^{\widetilde{E}^\bullet}) \mod Q^{S,0}.
\]
This completes the proof by induction. \[\square\]

### 1.3. Analytic torsion form.

This subsection follows [7, §III].

Let \(\pi : M \to S\) be a smooth fibration with compact fiber \(Z\). Here \(Z\) may have boundaries. Let \(T^H M \subseteq TM\) be a complement of \(TZ\), which induces the following identification,
\[
\Lambda^\bullet(T^* M) = \Lambda^\bullet(T^H M) \otimes \Lambda^\bullet(T^* Z) \simeq \pi^*(\Lambda^*(T^* S)) \otimes \Lambda^*(T^* Z).
\]

Let \((1.71) (F, \nabla^F)\) be a flat complex vector bundle over \(M\). Set \(F^\bullet = \Omega^\bullet(Z, F)\), which is a graded complex vector bundle of infinite dimension over \(S\). From (1.70), we get a formal identity
\[
\Omega^\bullet(M, F) = \Omega^\bullet(S, F^\bullet).
\]

For \(U \in TS\), let \(U^H \in T^H M\) be such that \(\pi_* U^H = U\). For \(U \in C^\infty(S, TS)\), let \(L^- U\) be the Lie differentiation operator acting on \(\Omega^\bullet(M, F)\). For \(U \in C^\infty(S, TS)\) and \(s \in \Omega^\bullet(S, F^\bullet) = \Omega^\bullet(M, F)\), we define
\[
\nabla_U^F s = L^- U s.
\]
Then \(\nabla^F\) is a connection on \(F^\bullet\) preserving the grading.

Let \(P_T Z : TM \to TZ\) be the projection with respect to the decomposition \(TM = T^H M \oplus TZ\). For \(U, V \in C^\infty(S, TS)\), set
\[
\nabla(U, V) = P_T Z[U^H, V^H] \in C^\infty(M, TZ).
\]
Then
\[
\nabla \in C^\infty(M, \pi^*(\Lambda^2(T^* S)) \otimes TZ).
\]

Let \(i_T \in C^\infty(M, \pi^*(\Lambda^2(T^* S)) \otimes \text{End}(\Lambda^*(T^* Z)))\) be the interior multiplication by \(\nabla\).

The flat connection \(\nabla^F\) (resp. \(\nabla^F|_Z\)) naturally extends to an exterior differentiation operator on \(\Omega^*(M, F)\) (resp. \(\Omega^*(Z, F) = \mathcal{F}^\bullet\)), which we denote by \(d^M\) (resp. \(d^Z\)). In the sense of [7, §II(a)], the operator \(d^M\) is a superconnection of total degree 1 on \(F^\bullet\).

By [7, Prop. 3.4], we have
\[
d^M = d^Z + \nabla^F + i_T.
\]

Let \(g^{TZ}\) be a Riemannian metric on \(TZ\). Let \(g^F\) be a Hermitian metric on \(F\).

We denote \(Y = \partial Z\). If \(Y \neq \emptyset\), we assume that \(T^H M\), \(g^{TZ}\) and \(g^F\) are product on a tubular neighborhood of \(Y\) (see [43, (2.30)-(2.35)]).
Let \(g^{\mathcal{F}}\) be the \(L^2\)-metric on \(\mathcal{F}\) associated with \(g^{TZ}, g^F\). Let \(d^{M,*}, d^{Z,*}, \nabla^{\mathcal{F}}\) be the formal adjoints of \(d^M, d^Z, \nabla^{\mathcal{F}}\) with respect to \(g^{\mathcal{F}}\) in the sense of \([7, \text{Def. 1.6}]\). By \([7, \text{Prop. 3.7}]\), we have
\[
\tag{1.77}
d^{M,*} = d^{Z,*} + \nabla^{\mathcal{F}},
\]
where \(\mathcal{T}^* \in \mathcal{C}^\infty(M, \pi^*\big(\Lambda^2(T^*Z) \ominus T^*Z\big))\) is the dual of \(\mathcal{T}\) with respect to \(g^{TZ}\).

Let \(N^{TZ}\) be the number operator on \(\Lambda^*(T^*Z)\), i.e., \(N^{TZ}|_{\Lambda^k(T^*Z)} = k \text{Id}\). Then \(N^{TZ}\) acts on \(\mathcal{F}\) in the obvious way. For \(t > 0\), let \(d^{M,*}_t\) be the formal adjoints of \(d^M\) with respect to the metric \(t^{N^{TZ}} g^{\mathcal{F}}\). We have
\[
\tag{1.78}
d^{M,*}_t = td^{Z,*} + \nabla^{\mathcal{F}}, - \frac{1}{t} \mathcal{T}^*.
\]
Set
\[
\tag{1.79}
D_t = \frac{1}{2}(d^{M,*}_t - d^M).
\]
By (1.76), (1.78) and (1.79), we have
\[
\tag{1.80}
D_t^2 = -\frac{t}{4} (d^{Z,*}d^Z + d^Zd^{Z,*}) + \text{positive degree terms}.
\]
Here \(d^{Z,*}d^Z + d^Zd^{Z,*}\) is the fiberwise Hodge Laplacian. If \(Y = \partial Z \neq \emptyset\), we put absolute boundary condition (see \([43, (2.52)]\)) on \(Y\), then \(d^{Z,*}d^Z + d^Zd^{Z,*}\) is self-adjoint.

Let \(\text{End}_{tr}(\mathcal{F}) \subseteq \text{End}(\mathcal{F})\) be the subbundle of trace class operators. Recall that \(f'(z) = (1 + 2z^2)e^{z^2}\). By (1.80), we have
\[
\tag{1.81}
f'(D_t) \in \Omega^*(S, \text{End}_{tr}(\mathcal{F})).
\]
Let \(\text{Tr} : \text{End}_{tr}(\mathcal{F}) \to \mathbb{C}\) be the trace map, which extends to
\[
\tag{1.82}
\text{Tr} : \Omega^*(S, \text{End}_{tr}(\mathcal{F})) \to \Omega^*(S).
\]

Let \(H^*(Z, F)\) be the fiberwise cohomology of \(Z\) with coefficients in \(F\). Then \(H^*(Z, F)\) is a graded complex vector bundle over \(S\). We denote
\[
\tag{1.83}
\chi'(Z, F) = \sum_{k=0}^{\dim Z} (-1)^k \text{rk}(H^k(Z, F))\).
\]

Recall that \(\varphi\) was defined in (0.2). Recall that \(Q^S\) was defined in (0.35).

\text{Bismut and Lott \([7, \text{Def. 3.22}]\) gave the following definition for \(Z\) closed. Zhu \([43, \text{Def. 2.18}]\) extended the definition to the case \(Y = \partial Z \neq \emptyset\).}

\textbf{Definition 1.7.} The analytic torsion form associated with \((T^HM, g^{TZ}, g^F)\) is defined as
\[
\mathcal{T}(T^HM, g^{TZ}, g^F) = -\int_0^{+\infty} \left\{ \varphi \text{Tr} \left[ (-1)^{N^{TZ}} \frac{N^{TZ}}{2} f'(D_t) \right] - \frac{\chi'(Z, F)}{2} \right. \\
\left. - \left( \frac{\dim Z \text{rk}(F)\chi(Z)}{4} - \frac{\chi'(Z, F)}{2} \right) f'(\frac{i\sqrt{t}}{2}) \right\} \frac{dt}{t} \in Q^S.
\]
By \([7, \text{Thm 3.21}]\) \([43, \text{Thm 2.17}]\), the integrand in (1.84) is integrable.
Let $\nabla^{TZ}$ be the Bismut connection [3, Def. 1.6] on $TZ$ associated with $T^HM$ and $g^{TZ}$. Let $\nabla^{TY}$ be the Bismut connection on $TY$ associated with $T^HM$ and $g^{TZ}|_{TY}$. Let $\nabla^{H_*(Z,F)}$ be the canonical flat connection on $H_*(Z,F)$ (see [7, Def. 2.4]). Let $g^{H_*(Z,F)}$ be the $L^2$-metric on $H_*(Z,F)$ associated with $g^{TZ}, g^F$. By [7, Thm 3.23] [43, Thm 2.19], we have

$$d\mathcal{F}(T^HM, g^{TZ}, g^F) = \int_Z e(TZ, \nabla^{TZ}) f(\nabla^F, g^F) + \frac{1}{2} \int_Y e(TY, \nabla^{TY}) f(\nabla^F, g^F) - f(\nabla^{H_*(Z,F)}, g^{H_*(Z,F)}).$$

(1.85)

The following theorem is an infinite dimensional analogue of Theorem 1.2.

We will use the notation in (0.34).

**Theorem 1.8.** If the triple $(F, \nabla^F, g^F)$ is the complexification of a triple $(F_{\mathbb{R}}, \nabla^{F_{\mathbb{R}}}, g^{F_{\mathbb{R}}})$, then

$$\mathcal{F}^{[k]}(T^HM, g^{TZ}, g^F) = 0 \text{ for } k \equiv 2 \text{ (mod 4)}. \quad (1.86)$$

**Proof.** Similarly to Theorem 1.2, we apply the argument in [7, Thm. 1.8(iv)] with $E^*$ replaced by $\mathcal{F}^*$ and $X_t$ replaced by $D_t$. Though $\mathcal{F}^*$ is infinite dimensional, the argument still works. \qed

1.4. **Igusa’s and Ohrt’s axioms.** This subsection follows [22, §3] and [34, §2.2].

We consider an invariant $\tau$ assigning a cohomology class

$$\tau(M/S, F) \in H^{even}(S) \quad (1.87)$$

to any triple $(\pi : M \to S, F, \nabla^F)$ satisfying

- $\pi : M \to S$ is a smooth fibration with closed fiber;
- $(F, \nabla^F)$ is a unitarily flat complex vector bundle over $M$;
- $H^*(Z,F)$ is filtered by flat subbundles with unitary factors.

We will state several axioms. If $\tau$ satisfies all the axioms, we call $\tau$ a higher torsion invariant.

For ease of notations, a triple under consideration will be denoted by $F \to M \to S$.

Let $F \to M \to S$ be a triple under consideration. Let $S'$ be a smooth manifold. Let $\varphi : S' \to S$ be a smooth map. Let $F' \to M' \to S'$ be the pull-back of $F \to M \to S$.

**Axiom 1** (naturality) We have

$$\tau(M'/S', F') = \varphi^* \tau(M/S, F) \in H^{even}(S'). \quad (1.88)$$

Let $F_j \to M_j \to S$ with $j = 1, 2$ be two triples with boundaries. More precisely, $M_1, M_2$ are allowed to have boundaries, and the other assumptions still hold. We assume that there is a diffeomorphism $\varphi : \partial M_1 \to \partial M_2$. We further assume that there is an isomorphism between flat vector bundles

$$\phi : F|_{\partial M_1} \to \varphi^*(F|_{\partial M_2}). \quad (1.89)$$

We can glue $F_1$ and $F_2$ together to a unitarily flat complex vector bundle $F := F_1 \cup_\varphi F_2$ over $M := M_1 \cup_\varphi M_2$. Similarly, for $j = 1, 2$, we construct

$$DM_j = M_j \cup_{Id} M_j, \quad DF_j = F_j \cup_{Id} F_j. \quad (1.90)$$
Then $DF_j$ is a unitarily flat complex vector bundle over $DM_j$.

**Axiom 2** (additivity) We have

\begin{equation}
\tau(M/S, F) = \frac{1}{2} \tau(DM_1/S, DF_1) + \frac{1}{2} \tau(DM_2/S, DF_2).
\end{equation}

Let $F \to M \to S$ be a triple under consideration. Let $\xi$ be a real vector bundle of rank $n + 1$ over $M$. Let $q : S^n(\xi) \to M$ be the $S^n$-bundle associated. More precisely, for any norm $|\cdot|$ on $\xi$, the manifold $S^n(\xi)$ is diffeomorphic to $\{v \in \xi : |v| = 1\}$.

**Axiom 3** (transfer) We have

\begin{equation}
\tau(S^n(\xi)/S, q^*F) = \chi(S^n) \tau(M/S, F) + \int_Z e(TZ) \tau(S^n(\xi)/M, q^*F),
\end{equation}

where $\chi(S^n)$ is the Euler characteristic of $S^n$, $e(TZ)$ is the Euler class of $TZ$, and $q^*F$ is the pull-back of $F$.

Let $\mathbb{1} \to M \to S$ be a triple under consideration, where $\mathbb{1}$ is the trivial flat line bundle over $M$. We will use the notation in (0.34).

**Axiom 4** (triviality) For $k \in \mathbb{N}$, we have

\begin{equation}
\tau^{[4k+2]}(M/S, \mathbb{1}) = 0 \in H^{4k+2}(S).
\end{equation}

Let $F_j \to M \to S$ with $j = 1, \ldots, m$ be a family of triples under consideration.

**Axiom 5** (additivity of coefficients) We have

\begin{equation}
\tau\left(M/S, \bigoplus_{j=1}^m F_j\right) = \sum_{j=1}^m \tau(M/S, F_j).
\end{equation}

Let $F \to M \overset{\pi}{\to} S$ be a triple under consideration. Assume that there is a finite covering $p : M \to M_\infty$ and a fibration $\pi_\infty : M_\infty \to S$ such that $\pi = \pi_\infty \circ p$.

**Axiom 6** (induction) We have

\begin{equation}
\tau(M/S, F) = \tau(M_\infty/S, p_*F),
\end{equation}

where $p_*F$ is the direct image of $F$.

We consider the universal complex line bundle $L$ over $\mathbb{C}P^\infty$. Let $\alpha \in \mathbb{C}$ be a root of unity. Let $n$ be a positive integer such that $\alpha^n = 1$. Let $L^\otimes n$ be the $n$-th tensor product of $L$. Let $S^1(L^\otimes n)$ be the $S^1$-bundle associated with $L^\otimes n$. Let $F_\alpha$ be a flat complex line bundle over $S^1(L^\otimes n)$ such that

- the pull-back of $F_\alpha$ to $S^1(L)$ is a trivial flat complex line bundle;
- the holonomy of $F_\alpha$ along the fiber of $S^1(L^\otimes n) \to \mathbb{C}P^\infty$ equals $\alpha$.

Let $\tau_k(S^1(L^\otimes n)/\mathbb{C}P^\infty, F_\alpha)$ be the component of $\tau(S^1(L^\otimes n)/\mathbb{C}P^\infty, F_\alpha)$ of degree $2k$. We identify $\mathbb{Q}/\mathbb{Z}$ with the roots of unity in $\mathbb{C}$.

**Axiom 7** (continuity) For $k \in \mathbb{N}$, the map

\begin{equation}
\mathbb{Q}/\mathbb{Z} \to H^{2k}(\mathbb{C}P^\infty) = \mathbb{R}
\end{equation}

\begin{equation}
\alpha \mapsto \frac{1}{n^k} \tau_k(S^1(L^\otimes n)/\mathbb{C}P^\infty, F_\alpha)
\end{equation}
is continuous. Here, to show that the map (1.96) is well-defined, we need Axiom 1 and the following fact: let $\varphi : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be such that $\varphi^* L \simeq L^\otimes n$, then the pull-back map $\varphi^* : H^{2k}(\mathbb{C}P^\infty) \to H^{2k}(\mathbb{C}P^\infty)$ equals $n^k \text{Id}$. 

2. Analytic torsion class

2.1. Construction. Let

\[ \pi : M \to S \]  

be a smooth fibration with compact fiber. For $s \in S$, we denote $Z_s = \pi^{-1}(s)$. We will omit the index $s$ when we refer to the generic fiber. Here $Z$ may have boundaries. Let

\[ (F, \nabla_F) \]

be a unitarily flat complex vector bundle over $M$. Let $H^\bullet(Z, F)$ be the fiberwise cohomology with coefficients in $F$. Let $\nabla^{H^\bullet(Z, F)}$ be the canonical flat connection on $H^\bullet(Z, F)$. We assume that each $H^k(Z, F)$ is filtered by flat subbundles with unitary factors.

Let $T^H M \subseteq TM$ be a complement of $TZ$. Let $g^{TZ}$ be a Riemannian metric on $TZ$. Let $g^F$ be a flat Hermitian metric on $F$. Let $\mathcal{T}(T^H M, g^{TZ}, g^F)$ be the Bismut-Lott analytic torsion form, which we view as an element in $Q^S/Q^S,0$. Let $g^{H^\bullet(Z, F)}$ be the $L^2$-metric on $H^\bullet(Z, F)$ associated with $g^{TZ}, g^F$. From (1.85) and the assumptions above, we get

\[ d\mathcal{T}(T^H M, g^{TZ}, g^F) = -f(\nabla^{H^\bullet(Z, F)}, g^{H^\bullet(Z, F)}) . \]

Let $\mathcal{F}(\nabla^{H^\bullet(Z, F)}, g^{H^\bullet(Z, F)})$ be as in (1.48). By (1.49), we have

\[ d\mathcal{F}(\nabla^{H^\bullet(Z, F)}, g^{H^\bullet(Z, F)}) = f(\nabla^{H^\bullet(Z, F)}, g^{H^\bullet(Z, F)})^{[>1]} . \]

We define

\[ \mathcal{T}_{\text{cl}}(T^H M, g^{TZ}, g^F) = \mathcal{T}^{[>0]}(T^H M, g^{TZ}, g^F) + \mathcal{F}(\nabla^{H^\bullet(Z, F)}, g^{H^\bullet(Z, F)}) . \]

From (2.3)-(2.5), we get

\[ d\mathcal{T}_{\text{cl}}(T^H M, g^{TZ}, g^F) = 0 . \]

Definition 2.1. The analytic torsion class of $(\pi : M \to S, F)$ is defined as

\[ \tau_{\text{BL}}(M/S, F) = \left[ \mathcal{T}_{\text{cl}}(T^H M, g^{TZ}, g^F) \right] \in H^{\text{even}>2}(S) . \]

A standard argument using the functoriality and the closedness of $\mathcal{T}_{\text{cl}}(T^H M, g^{TZ}, g^F)$ shows that $\tau_{\text{BL}}(M/S, F)$ is independent of $T^H M, g^{TZ}, g^F$.

If $H^\bullet(Z, F)$ is unitary, Definition 2.1 is equivalent to [16, Def. 2.8].
2.2. Additivity. In this subsection, we use the notations in §2.1. And we assume that \( Z \) is closed.

Let \( N \subseteq M \) be a hypersurface cutting \( M \) into two pieces, which we denote by \( M'_1, M'_2 \). Assume that \( \pi|_N : N \to S \) is surjective and \( N \) is transversal to \( Z_s \) for any \( s \in S \). Then \( \pi|_N : N \to S \) is a fibration. Let \( N \subseteq U \subseteq M \) be a tubular neighborhood such that \( \pi|_U : U \to S \) is isomorphic to the fibration \( \pi|_N \circ \text{pr}_2 : (−1, 1) \times N \to S \). Set
\[
M_1 = M'_1 \cup U, \quad M_2 = M'_2 \cup U, \quad M_3 = U.
\]

For \( j = 1, 2, 3 \), we have a fibration
\[
\pi_j := \pi|_{M_j} : M_j \to S.
\]

For \( s \in S \), we denote \( Z_{j,s} = \pi_j^{-1}(s) \). For convenience, we will use the notations \( M_0 = M, Z_{0,s} = Z_s, \pi_0 = \pi \) etc.

\[
\begin{array}{c}
\text{Figure 1. fibrations } \pi_j : M_j \to S \text{ with } j = 0, 1, 2, 3 \\
\end{array}
\]

For \( j = 0, 1, 2, 3 \), the fiberwise cohomology \( H^\bullet(Z_j, F) \) is a graded flat complex vector bundle over \( S \). We assume that each \( H^k(Z_j, F) \) is filtered by flat subbundles with unitary factors. Then
\[
\tau_{\text{BL}}(M_j/S, F) \in H^{\text{even} \geq 2}(S) \text{ with } j = 0, 1, 2, 3
\]
are well-defined.

**Theorem 2.2.** The following identity holds,
\[
\tau_{\text{BL}}(M/S, F) = \tau_{\text{BL}}(M_1/S, F) + \tau_{\text{BL}}(M_2/S, F) - \tau_{\text{BL}}(M_3/S, F).
\]

**Proof.** For \( j = 0, 1, 2, 3 \), let \( T^HM_j \subseteq TM_j \) be the restriction of \( T^HM \) to \( M_j \), let \( g^{TZ_j} \) be the restricted metric of \( g^{TZ} \) on \( TZ_j \). Let
\[
\mathcal{S}(T^HM_j, g^{TZ_j}, g^F) \in Q^S
\]
be the Bismut-Lott analytic torsion form of \( (\pi_j : M_j \to S, F) \).
We consider the Mayer-Vietoris exact sequence
\[(2.13)\quad \cdots \to H^k(Z, F) \to H^k(Z_1, F) \oplus H^k(Z_2, F) \to H^k(Z_3, F) \to \cdots ,\]
which is an exact sequence of flat complex vector bundles over \(S\). For \(j = 0, 1, 2, 3\), let \(g^{H^k(Z, F)}\) be the \(L^2\)-metric on \(H^k(Z, F)\) associated with \(g^{TZ}, g^F\). Let
\[(2.14)\quad \mathcal{T}_F \in Q^S\]
be the torsion form of the exact sequence (2.13) equipped with metrics \((g^{H^k(Z, F)})_{j=0,1,2,3}\).

By [36, Thm. 0.1], we have
\[(2.15)\quad \mathcal{T}_F + \sum_{j=0}^3 (-1)^{j(j-3)/2} \mathcal{T}( T^H M_j, g^{T Z_j}, g^F) = 0 \text{ modulo } Q^S.\]

On the other hand, by Theorem 1.6 and (1.48), we have
\[(2.16)\quad \mathcal{T}^{[>0]}_F = \sum_{j=0}^3 (-1)^{j(j-3)/2} \mathcal{T}( H_j^*, g^{H_j^*} ) \text{ modulo } Q^S.\]

By Definition 2.1 and (2.5), we have
\[(2.17)\quad \tau_{BL}(M_j/S, F) = \left[ \mathcal{T}^{[>0]}( T^H M_j, g^{T Z_j}, g^F) + \mathcal{T}( H_j^*(Z_j, F), g^{H_j^*(Z, F)} ) \right].\]

From (2.15)-(2.17), we obtain (2.11). This completes the proof. \(\square\)

2.3. **Transfer and induction.** In this subsection, we use the notations in §2.1. And we assume that \(Z\) is closed.

Let \(\xi\) be a real vector bundle of rank \(n + 1\) over \(M\). Let
\[(2.18)\quad q : S^n(\xi) \to M\]
be the \(S^n\)-bundle associated. We denote by \(X\) the fiber of \(\pi \circ q : S^n(\xi) \to S\). We denote by \(Y\) the fiber of \(q : S^n(\xi) \to M\). Recall that \(Z\) is the fiber of \(\pi : M \to S\). The notations are summarized in the following commutative diagram,
\[(2.19)\quad q^* F \longrightarrow S^n(\xi) \quad \xymatrix{ X \ar[d] \ar[dr] \ar[r] & Z \ar[d] \\ F \ar[r] & M \ar[r] & S .} \]

The fiberwise cohomology \(H^k(Y, q^* F)\) is a graded flat complex vector bundle over \(M\). Since \(Y \simeq S^n\), we have
\[(2.20)\quad H^0(Y, q^* F) = H^n(Y, q^* F) = F , \quad H^k(Y, q^* F) = 0 \text{ for } k \neq 0, n .\]

The fiberwise cohomologies \(H^k(Z, H^k(Y, q^* F))\) and \(H^k(X, q^* F)\) are graded flat complex vector bundles over \(S\). Moreover, we have
\[(2.21)\quad H^k(X, q^* F) = H^k(Z, H^0(Y, q^* F)) \oplus H^{k-n}(Z, H^n(Y, q^* F)) = H^k(Z, F) \oplus H^{k-n}(Z, F) .\]
From (2.20) and (2.21), we see that both $H^\bullet(Y, q^* F)$ and $H^\bullet(X, q^* F)$ are filtered by flat subbundles with unitary factors. Hence

$$(2.22) \quad \tau^{\text{BL}}\left(S^n(\xi)/M, q^* F\right) \in H^{\text{even} \geq 2}(M), \quad \tau^{\text{BL}}\left(S^n(\xi)/S, q^* F\right) \in H^{\text{even} \geq 2}(S)$$

are well-defined.

The following theorem is a direct consequence of [27, Thm. 0.1].

**Theorem 2.3.** The following identity holds,

$$(2.23) \quad \tau^{\text{BL}}\left(S^n(\xi)/S, q^* F\right) = \chi(S^n)\tau^{\text{BL}}(M/S, F) + \int_Z e(TZ)\tau^{\text{BL}}\left(S^n(\xi)/M, q^* F\right).$$

**Proof.** Let $g^{TX}, g^{TY}$ be Riemannian metrics on $TX, TY$. Recall that $g^{TZ}$ is a Riemannian metric on $T Z$. Recall that $g^F$ is a flat Hermitian metric on $F$.

- Let $g^{H^\bullet(X, q^* F)}$ be the $L^2$-metric on $H^\bullet(X, q^* F)$ associated with $g^{TX}, q^* g^F$.
- Let $g^{H^\bullet(Y, q^* F)}$ be the $L^2$-metric on $H^\bullet(Y, q^* F)$ associated with $g^{TY}, q^* g^F$.
- Let $g^{H^\bullet(Z, H^\bullet(Y, q^* F))}$ be the $L^2$-metric on $H^\bullet(Z, H^\bullet(Y, q^* F))$ associated with $g^{TZ}, g^{H^\bullet(Y, q^* F)}$.
- Recall that $g^{H^\bullet(Z, F)}$ is the $L^2$-metric on $H^\bullet(Z, F)$ associated with $g^{TZ}, g^F$.

We assume that the volume of any fiber $Y$ with respect to $g^{TY}$ equals 1. Then, under the isomorphism (2.20), we have

$$g^{H^\bullet(Y, q^* F)} = g^{H^\bullet(Y, q^* F)} = g^F$$

and

$$g^{H^\bullet(Z, H^\bullet(Y, q^* F))} = g^{H^\bullet(Z, H^\bullet(Y, q^* F))} = g^{H^\bullet(Z, F)}.$$

Under the isomorphism (2.21), we may view $g^{H^\bullet(Z, H^\bullet(Y, q^* F))} \oplus g^{H^\bullet-n(Z, H^\bullet(Y, q^* F))}$ as a metric on $H^\bullet(X, q^* F)$. Recall that $\mathcal{T}(\cdot, \cdot)$ was defined in the paragraph containing (1.16). For a graded flat complex vector bundle $(E^\bullet, \nabla^{E^\bullet})$ over $S$ and Hermitian metrics $g^{E^\bullet*} = \bigoplus_k g^{E^k}, g^{E^\bullet*} = \bigoplus_k g^{E^k}$ on $E^\bullet$, we denote

$$\mathcal{T}(g^{E^\bullet*}, g^{E^\bullet*}) = \sum_k (-1)^k \mathcal{T}(g^{E^k}, g^{E^k}).$$

By Theorem 1.6, (1.48), (2.25) and (2.26), we have

$$\mathcal{T}[>0]\left(g^{H^\bullet(Z, H^\bullet(Y, q^* F))} \oplus g^{H^\bullet-n(Z, H^\bullet(Y, q^* F))}, g^{H^\bullet(X, q^* F)}\right)$$

$$(2.27) = (1 + (-1)^n) \mathcal{T}\left(H^\bullet(Z, F), g^{H^\bullet(Z, F)}\right) - \mathcal{T}\left(H^\bullet(X, q^* F), g^{H^\bullet(X, q^* F)}\right)$$

$$= \chi(S^n) \mathcal{T}\left(H^\bullet(Z, F), g^{H^\bullet(Z, F)}\right) - \mathcal{T}\left(H^\bullet(X, q^* F), g^{H^\bullet(X, q^* F)}\right) \text{ modulo } Q^{S,0}.$$
be the Bismut-Lott analytic torsion form of \((q : S^n(\xi) \to M, q^*F)\). Let \(T^H_{\text{tot}}S^n(\xi) \subseteq T\Sigma^n(\xi)\) be a complement of \(TX\). Let

\[
\mathcal{T}(T^H_{\text{tot}}S^n(\xi), g^TX, q^*g^F) \in Q^S/Q^{S,0}
\]

be the Bismut-Lott analytic torsion form of \((\pi \circ q : S^n(\xi) \to S, q^*F)\). By [27, Thm. 0.1], we have

\[
\mathcal{T}(T^H_{\text{tot}}S^n(\xi), g^TX, q^*g^F) = \mathcal{T}(T^HM, g^{TZ}, g^{H(Y,q^*F)}) + \int_Z e(TZ) \mathcal{T}(T^HS^n(\xi), g^{TY}, q^*g^F) + \mathcal{T}(g^{H^*(Z,H^n(\xi,q^*F))} \oplus g^{H^*\cdot n(Z,H^n(\xi,q^*F))}, g^{H^*H^n(X,q^*F)}) \pmod{Q^{S,0}}.
\]

Here we remark that \(\mathcal{T}(T^HM, g^{TY}, q^*g^F)\) is closed. This follows from (2.3) with \((\pi : M \to S, F)\) replaced by \((q : S^n(\xi) \to M, q^*F)\).

By (2.24) and the fact that \(g^F\) is a flat Hermitian metric, we have

\[
\mathcal{T}(H^*(Y,q^*F), g^{H^*H^n(Y,q^*F)}) = 0.
\]

By Definition 2.1, (2.5) and (2.32), we have

\[
\tau^{BL}(S^n(\xi)/M, q^*F) = \left[ \mathcal{T}^{>0}(T^HS^n(\xi), g^{TY}, q^*g^F) \right].
\]

By Definition 2.1 and (2.5), we have

\[
\tau^{BL}(S^n(\xi)/S, q^*F) = \left[ \mathcal{T}^{>0}(T^HS^n(\xi), g^TX, q^*g^F) + \mathcal{T}(H^*(X,q^*F), g^{H^*H^n(X,q^*F)}) \right].
\]

On the other hand, by (2.20) and (2.24), we have

\[
\mathcal{T}(T^HM, g^{TZ}, g^{H^*(Y,q^*F)}) = (1 + (-1)^n) \mathcal{T}(T^HM, g^{TZ}, g^F) = e(S^n) \mathcal{T}(T^HM, g^{TZ}, g^F).
\]

From Definition 2.1, (2.5), (2.27) and (2.31)-(2.35), we obtain (2.23). This completes the proof.

Now we assume that there is a finite covering

\[
p : M \to M_\sim
\]

and a fibration

\[
\pi_\sim : M_\sim \to S
\]

such that \(\pi = \pi_\sim \circ p\). Let \(Z_\sim\) be the fiber of \(\pi_\sim : M_\sim \to S\). Let \(\pi_*F\) be the direct image of \(F\), which is flat complex vector bundle over \(M_\sim\). The notations are summarized in the following commutative diagram,
Let \( g^{p,F} \) be the Hermitian metric on \( g^{p,F} \) induced by \( g^F \). Since \( g^F \) is a flat Hermitian metric, so is \( g^{p,F} \). The fiberwise cohomology \( H^\bullet(Z_\sim, p_\ast F) \) is a graded flat complex vector bundle over \( S \). Moreover, we have
\[
H^\bullet(Z, F) = H^\bullet(Z_\sim, p_\ast F).
\]
Since \( H^k(Z, F) \) is filtered by flat subbundles with unitary factors, so is \( H^k(Z_\sim, p_\ast F) \). Hence
\[
\tau^{\mathrm{BL}}(M_\sim/S, p_\ast F) \in H^{\text{even} \geq 2}(S)
\]
is well-defined.

The following theorem is a direct consequence of [27, Thm. 0.1].

**Theorem 2.4.** The following identity holds,
\[
\tau^{\mathrm{BL}}(M/S, F) = \tau^{\mathrm{BL}}(M_\sim/S, p_\ast F).
\]

**Proof.** Let \( g^{TZ_\sim} \) be a Riemannian metric on \( TZ_\sim \). Let \( g^{H^\bullet(Z_\sim, p_\ast F)} \) be the \( L^2 \)-metric on \( H^\bullet(Z_\sim, p_\ast F) \) associated with \( g^{TZ_\sim}, g^{p,F} \). Under the isomorphism (2.39), we may view \( g^{H^\bullet(Z_\sim, p_\ast F)} \) as a metric on \( g^{H^\bullet(Z,F)} \). By Theorem 1.6, (1.48) and (2.26), we have
\[
\mathcal{T}^{[>0]}(g^{H^\bullet(Z_\sim, p_\ast F)}, g^{H^\bullet(Z,F)})
\]
(2.42)
\[
= \mathcal{T}(H^\bullet(Z_\sim, p_\ast F), g^{H^\bullet(Z_\sim, p_\ast F)}) - \mathcal{T}(H^\bullet(Z, F), g^{H^\bullet(Z,F)}) \mod Q^{S,0}.
\]

Let \( T^H M_\sim \subseteq TM_\sim \) be a complement of \( TZ_\sim \). Let
\[
\mathcal{T}^{[>0]}(T^H M_\sim, g^{TZ_\sim}, g^{p,F}) \in Q^S/Q^{S,0}
\]
be the Bismut-Lott analytic torsion form of \( (\pi_\sim : M_\sim \to S, p_\ast F) \). Since \( g^F \) is a flat Hermitian metric and \( p : M \to M_\sim \) is a finite covering, the Bismut-Lott analytic torsion form of \( (p : M \to M_\sim, F) \) vanishes. Then, by [27, Thm. 0.1], we have
\[
\mathcal{T}(T^H M, g^{T^Z}, g^F)
\]
(2.44)
\[
= \mathcal{T}(T^H M_\sim, g^{TZ_\sim}, g^{p,F}) + \mathcal{T}(H^\bullet(Z_\sim, p_\ast F), g^{H^\bullet(Z,F)}) \mod Q^{S,0}.
\]

By Definition 2.1 and (2.5), we have
\[
\tau^{\mathrm{BL}}(M_\sim/S, p_\ast F)
\]
(2.45)
\[
= \left[ \mathcal{T}^{[>0]}(T^H M_\sim, g^{TZ_\sim}, g^{p,F}) + \mathcal{T}(H^\bullet(Z_\sim, p_\ast F), g^{H^\bullet(Z,F)}) \right].
\]
From Definition 2.1, (2.5), (2.42), (2.44) and (2.45), we obtain (2.41). This completes the proof. \( \square \)

### 2.4. Triviality

In this subsection, we use the notations in §2.1.

For a positive integer \( k \), let
\[
\tau_k^{\mathrm{BL}}(M/S, F) \in H^{2k}(S)
\]
be the component of \( \tau^{\mathrm{BL}}(M/S, F) \) of degree \( 2k \).

The following theorem is due to Bismut and Lott [7].

**Theorem 2.5.** If \( (F, \nabla^F) \) is a trivial flat complex line bundle over \( M \), then
\[
\tau_{2k+1}^{\mathrm{BL}}(M/S, F) = 0 \text{ for } k \in \mathbb{N}.
\]

**Proof.** This is a direct consequence of Theorems 1.5, 1.8, Definition 2.1 and (2.5). \( \square \)
2.5. **Circle bundle.** Let \( n \) be a positive integer. Let \( L \) be a complex line bundle over \( S \). Let \( L^\otimes n \) be \( n \)-th tensor power of \( L \). Let

\[
\pi_n : S^1(L^\otimes n) \to S
\]

be the \( S^1 \)-bundle associated with \( L^\otimes n \). Let

\[
q_n : S^1(L) \to S^1(L^\otimes n)
\]

be the \( n \)-covering induced by \( L \to L^\otimes n, s \mapsto s^\otimes n \). Let \( \alpha \in \mathbb{C}^* \) such that \( \alpha^n = 1 \). Let \( (F_\alpha, \nabla^{F_\alpha}) \) be the unique flat complex line bundle over \( S^1(L^\otimes n) \) such that

- The pull-back \( q_n^*(F_\alpha, \nabla^{F_\alpha}) \) is a trivial flat complex line bundle over \( S^1(L) \).
- The holonomy of \( (F_\alpha, \nabla^{F_\alpha}) \) along the fiber of \( \pi_n : S^1(L^\otimes n) \to S \) equals \( \alpha \).

For \( k \) an integer greater than 1, let \( \text{Li}_k \) be the polylogarithm function

\[
\text{Li}_k : \{ z \in \mathbb{C} : |z| \leq 1 \} \to \mathbb{C}
\]

\[
z \mapsto \sum_{m=1}^{\infty} m^{-k} z^m.
\]

Let \( \omega \in H^2(S) \) be the first Chern class of \( L^\otimes n \).

The following theorem is due to Bismut and Lott [7].

**Theorem 2.6.** The following identity holds,

\[
\tau^{BL}(S^1(L^\otimes n)/S, F_\alpha) = \sum_{k \geq 0 \text{ even}} (-1)^{k/2} \frac{(2k + 1)!}{2^{2k}(k!)^2(2\pi)^k} \text{Re}(\text{Li}_{k+1}(\alpha)) \omega^k
\]

\[
+ \sum_{k > 0 \text{ odd}} (-1)^{(k-1)/2} \frac{(2k + 1)!}{2^{2k}(k!)^2(2\pi)^k} \text{Im}(\text{Li}_{k+1}(\alpha)) \omega^k.
\]

**Proof.** For \( \alpha \neq 1 \), the identity (2.51) is a direct consequence of [7, Cor. 4.14]. Since (2.51) concerns the analytic torsion forms of positive degree, the proof of [7, Cor. 4.14] equally implies (2.51) for \( \alpha = 1 \). \( \square \)

The following theorem is due to Igusa and Klein [24] (see also [23, Thm. 3.1]).

**Theorem 2.7.** The following identity holds,

\[
\tau^{IK}(S^1(L^\otimes n)/S, F_\alpha) = \sum_{k \text{ even}} (-1)^{(k+2)/2} \frac{1}{k!} \text{Re}(\text{Li}_{k+1}(\alpha)) \omega^k
\]

\[
+ \sum_{k \text{ odd}} (-1)^{(k+1)/2} \frac{1}{k!} \text{Im}(\text{Li}_{k+1}(\alpha)) \omega^k.
\]

2.6. **Proofs of Theorems 0.1', 0.2.**

**Proof of Theorem 0.1'.** Recall that Axioms 1-7 were stated in §1.4. Axiom 1 trivially holds for \( \tau^{BL} \). By Theorem 2.2, Axiom 2 holds for \( \tau^{BL} \). By Theorem 2.3, Axiom 3 holds for \( \tau^{BL} \). Hence \( \tau^{BL} \) satisfies Igusa’s axiomatization [22, §3].

By [22, Cor. 4.5], there exists \( a_{2k} \in \mathbb{R} \) such that for any \( M/S \) with odd-dimensional fiber, we have

\[
\sum_{k \geq 0} \frac{2^{4k}}{(4k + 1)!} (2k)! \tau^{BL}_{2k}(M/S, \underline{1}) = a_{2k} \tau^{IK}_{2k}(M/S, \underline{1}).
\]
Now let $M/S$ be a circle bundle. From Theorems 2.6, 2.7, we get

\[(2.54) \quad a_{2k} = -\frac{(2k)!}{(2\pi)^{2k}}.\]

By [22, Cor. 4.5], there exists $b_{2k} \in \mathbb{R}$ such that for any $M/S$ with even-dimensional fiber $Z$, we have

\[(2.55) \quad \tau_{2k}^{\text{BL}}(M/S, \mathbb{1}) = b_{2k} \left[ \int_Z e(TZ)\text{ch}(TZ) \right]^{[4k]} \in H^{4k}(S).\]

Now let $M/S$ be a $S^2$-bundle. Let $TZ$ be a Riemannian metric on $TZ$ such that the volume of any fiber of $M/S$ equals 1. Let $g^1$ be the canonical metric on $\mathbb{1}$. Then the $L^2$-metric on $H^\bullet(Z)$ is flat. By Definition 2.1, we have

\[(2.56) \quad \tau_{2k}^{\text{BL}}(M/S, \mathbb{1}) = \left[ \mathcal{F}(T^H M, g^{TZ}, g^1) \right]^{[4k]} \in H^{4k}(S).\]

On the other hand, by [7, Thm. 3.26], we have

\[(2.57) \quad \mathcal{F}(T^H M, g^{TZ}, g^1) = 0.\]

Hence $\tau_{2k}^{\text{BL}}(M/S, \mathbb{1}) = 0$ and $b_{2k} = 0$. This completes the proof. \(\square\)

**Proof of Theorem 0.2.** In the proof of Theorem 0.1, we showed that Axioms 1-3 hold for $\tau^{\text{BL}}$. By Theorem 2.5, Axiom 4 holds for $\tau^{\text{BL}}$. Axiom 5 trivially holds for $\tau^{\text{BL}}$. By Theorem 2.4, Axiom 6 holds for $\tau^{\text{BL}}$. By Theorem 2.6, Axiom 7 holds for $\tau^{\text{BL}}$. Hence $\tau^{\text{BL}}$ satisfies Ohrt's axiomatization [34, §2.2].

By [34, Theorem 0.1], there exist $a_{2k}, b_{2k} \in \mathbb{R}$ such that for any $(M/S, F)$ under consideration, we have

\[(2.58) \quad \frac{2^{4k}((2k)!)^2}{(4k + 1)!} \tau_{2k}^{\text{BL}}(M/S, F) = a_{2k} \tau_{2k}^{\text{IK}}(M/S, F) + b_{2k} \text{rk} F \left[ \int_Z e(TZ)\text{ch}(TZ) \right]^{[4k]}.
\]

We view (2.58) as a system of equations of $a_{2k}, b_{2k}$. Taking $F = \mathbb{1}$, we get

\[(2.59) \quad \frac{2^{4k}((2k)!)^2}{(4k + 1)!} \tau_{2k}^{\text{BL}}(M/S, \mathbb{1}) = a_{2k} \tau_{2k}^{\text{IK}}(M/S, \mathbb{1}) + b_{2k} \left[ \int_Z e(TZ)\text{ch}(TZ) \right]^{[4k]}.
\]

Comparing Theorem 0.1 with (2.59), we know that

\[(2.60) \quad a_{2k} = -\frac{(2k)!}{(2\pi)^{2k}}, \quad b_{2k} = \frac{c'(2k)}{2}\]

is a solution of (2.59). On the other hand, by [22, Prop. 4.6, 4.7], the solution of (2.59) is unique. Hence (2.60) is the unique solution of (2.58).

By [34, Theorem 0.1], there exist $c_{2k+1} \in \mathbb{R}$ such that for any $(M/S, F)$ under consideration, we have

\[(2.61) \quad \frac{2^{4k+2}((2k + 1)!)^2}{(4k + 3)!} \tau_{2k+1}^{\text{BL}}(M/S, F) = c_{2k+1} \tau_{2k+1}^{\text{IK}}(M/S, F).\]

Now let $M/S$ be a circle bundle. From Theorems 2.6, 2.7, we get

\[(2.62) \quad c_{2k+1} = -\frac{(2k + 1)!}{(2\pi)^{2k+1}}.\]
This completes the proof.

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