Strengthened Reeh-Schlieder Property and Scattering in Quantum Field Theories without Mass Gaps

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Abstract

We develop Haag-Ruelle scattering theory for Wigner particles in local relativistic Quantum Field Theory without assuming mass gaps or any other restrictions on the spectrum of the mass operator near the particle masses. Our approach is based on the Reeh-Schlieder property of the vacuum state. It is shown that a strengthened variant of this property, concerning the relative approximation error for single-particle states, implies the existence of scattering states.

1 Introduction

The infrared problem in Quantum Electrodynamics (QED) has attracted a lot of attention in the mathematical physics literature of the last decade. Consistent scattering theory has been developed for various physical processes involving charged particles (‘electrons’), neutral massive particles (‘atoms’) and massless particles (‘photons’). Some of these results were obtained in non-relativistic models of QED [CFP10; DyP13; MS14], others in the general setting of algebraic QFT [BR14; AD15; Dy05; Hrd13; DH14]. In spite of all these efforts, even the seemingly simple case of scattering of several atoms is still not fully under control.

This may be explained by the fact that atoms in QED constitute a prototypical example of an embedded particle. In other words, single-atom states correspond to eigenvalues of the mass operator which are not isolated, but embedded in a continuous mass spectrum, arising e.g. from states consisting of multiple lighter particles (photons). For the construction of scattering states, such background particles need to be separated from the desired single-atom states. In the framework of Haag-Ruelle theory, this separation could so far only be achieved with the help of technical assumptions¹ on the spectral measure of the mass operator near the particle masses. Such spectral conditions were first proposed by Herbst [Hrb71] and we might consider them to be a remnant of the original Haag-Ruelle mass-gap assumption [Ha58; Ru62; Hep65].

As the physical meaning of these assumptions has remained obscure, the existence of scattering states of atoms still lacks a conceptually clear explanation. Aiming at such an explanation, we develop Haag-Ruelle scattering theory for atoms, relying on certain non-local correlations of the vacuum state. The required condition is only slightly stronger than the well-established Reeh-Schlieder property and it permits to the best
of our knowledge the first proof of existence of scattering states of massive embedded Wigner particles without a priori requiring a spectral condition of Herbst type.

The Reeh-Schlieder property states that the vacuum $\Omega$ is cyclic for any algebra $\mathfrak{A}(\mathcal{O})$ of observables\(^2\) localized in a bounded space-time region $\mathcal{O}$. That is, given any vector $\Psi \in \mathcal{H}'$ (for example describing an atom essentially localized far from the region $\mathcal{O}$), there exists a family of observables $(A_\beta)_{\beta > 0}$ from $\mathfrak{A}(\mathcal{O})$ such that

$$\lim_{\beta \to 0} \| A_\beta \Omega - \Psi \| = 0.$$  \hspace{1cm} (1)

While $\| A_\beta \Omega \|$ clearly remains bounded, we note that the operator norms $\| A_\beta \|$ may tend to infinity as $\beta \to 0$. As it will be important for our investigation to quantify this growth, we will say that $\Psi$ is a vector of finite Reeh-Schlieder degree $\gamma$ if there exists a family of operators $(A_\beta)_{\beta > 0}$ localized in some fixed bounded space-time region $\mathcal{O}$, such that for some $\gamma > 0$ we have

$$\| A_\beta \| \leq \beta^{-\gamma} \quad \text{and} \quad \| A_\beta \Omega - \Psi \| \leq \beta.$$  \hspace{1cm} (2)

In this paper we will construct scattering states of configurations of atoms whose single-particle states are generated by such families with finite Reeh-Schlieder degree $\gamma$. Condition (2) is readily verified for free scalar fields\(^3\), but it seems that not much progress has been made in understanding such relations since the seminal work of Haag and Swieca [HS65]. In theories where Herbst’s spectral condition is satisfied, one can construct an operator family $(A_\beta)_{\beta > 0}$ satisfying a weakened variant (RS\(^9\)) of (2) (see concluding discussion), but the status of (2) in interacting theories is currently not clear and constitutes a difficult technical problem outside the scope of this work.

Let us now describe in non-technical terms the relevance of (2) for Haag-Ruelle scattering theory. Take a single-atom state $\Psi$ of finite Reeh-Schlieder degree and let $(A_\beta)_{\beta > 0}$ be a corresponding Reeh-Schlieder family from formula (2). Since $(A_\beta)_{\beta > 0}$ play a role of creation operators, it is technically convenient to smear them with the Fourier transform of a function $\hat{\chi} \in C_c^\infty(\mathbb{R}^4 \setminus \tilde{V}^-)$ yielding a family of almost-local operators

$$B_\beta := \int d^4x \chi(x) A_\beta(x), \quad (\beta > 0),$$  \hspace{1cm} (3)

where $A_\beta(x)$ denotes the translate of $A_\beta$ in space-time by $x$. Following the standard prescription we pick a regular positive-energy solution $f$ of the Klein-Gordon equation with the mass of the atom and set

$$B_\tau := \int d^4x f(\tau, x) B_{\beta(\tau)}(\tau, x), \quad \text{with} \quad \beta(\tau) := \tau^{-\mu}, \quad \mu > 0 \text{ fixed.}$$  \hspace{1cm} (4)

We will call $B_\tau$ an (approximating) creation operator of $\Psi$ since it has the property

$$\lim_{\tau \to \infty} B_\tau \Omega = (2\pi)^2 \hat{\chi}(H, P) \hat{f}(P) \left( \lim_{\tau \to \infty} A_{\beta(\tau)} \Omega \right) = (2\pi)^2 \hat{\chi}(H, P) \hat{f}(P) \Psi$$  \hspace{1cm} (5)

(see Proposition 3). That is, it asymptotically creates $\Psi$ from the vacuum up to an inessential function of the energy-momentum operators $(H, P)$ (which can be arranged to be equal to one if $\Psi$ has bounded energy). Since we inserted a Reeh-Schlieder family

\(^2\)In the case of QED these algebras should be generated by bounded functions of suitably smeared electromagnetic fields and the electric current, cf. [Bu86].

\(^3\)A free scalar field $\phi(f)$ is self-adjoint for real-valued $f$ and $\phi(f)\Omega$, $f \in C_c^\infty(\mathbb{R}^4)$, yield a dense subset of single-particle states. If $\text{supp } f \subset \mathcal{O}$ we can simply set $A_\beta := \phi(f) \exp(-\beta |\phi(f)|^2) \in \mathfrak{A}(\mathcal{O})$ to obtain Reeh-Schlieder families of arbitrarily small degrees $\gamma > 0$. For further examples see Appendix C.
in (4), we obtain convergence in (5) without the ergodic averaging used in earlier works [Dy06; Bu77]. We also note that (5) holds even for \( \Psi \) of infinite Reeh-Schlieder degree. The need to assume finiteness of the Reeh-Schlieder degree of \( \Psi \) arises only at the level of \( n \)-atom scattering states, \( n \geq 2 \) — the case to which we now proceed.

Let \( \Psi_1, \Psi_2 \) be two single-atom states with disjoint velocity supports and finite Reeh-Schlieder degree. Let \( B_{1\tau}, B_{2\tau} \) be the corresponding creation operators constructed as above. The scattering state describing these two atoms is given by the limit as \( \tau \to \infty \) of the family

\[
\Psi_\tau := B_{1\tau}B_{2\tau}\Omega.
\]

The conventional Cook-argument to establish convergence does not apply here due to the additional \( \tau \)-dependence via the Reeh-Schlieder family in (4). Therefore, we base our proof on a discretized analog of Cook’s argument involving summability of the telescopic expansion

\[
\|\Psi_{\tau_N} - \Psi_{\tau_0}\| \leq \sum_{k=0}^{N-1} \|\Psi_{\tau_{k+1}} - \Psi_{\tau_k}\| \tag{6}
\]

in the limit \( N \to \infty \) (here \( \tau_k := (1 + \rho)^k\tau_0 \), \( \tau_0 > 0 \), and \( \rho > 0 \) is sufficiently small). The first term in this sum has the form

\[
\Psi_{\tau_1} - \Psi_{\tau_0} = B_{1\tau_1}(B_{2\tau_1} - B_{2\tau_0})\Omega + (B_{1\tau_1} - B_{1\tau_0})B_{2\tau_0}\Omega. \tag{7}
\]

Exploiting locality and the fact that \( |\tau_1 - \tau_0| \) is small, we obtain that \( [(B_{1\tau_1} - B_{1\tau_0}), B_{2\tau_0}] \) is rapidly decreasing with \( \tau_0 \) and thus it suffices to study the expressions

\[
B_{1\tau_1}(B_{2\tau_1} - B_{2\tau_0})\Omega, \quad B_{2\tau_0}(B_{1\tau_1} - B_{1\tau_0})\Omega. \tag{8}
\]

Let us concentrate on the first term above: Thanks to the smearing operation (3) which restricts the energy-momentum transfers of the creation operators, we can write

\[
\|B_{1\tau_1}(B_{2\tau_1} - B_{2\tau_0})\Omega\| \leq \|B_{1\tau_1}E(\Delta)\| \|(B_{2\tau_1} - B_{2\tau_0})\Omega\|, \tag{9}
\]

where \( E(\Delta) \) is a projection onto a compact subset \( \Delta \) of the energy-momentum spectrum. Now exploiting formula (5) and results from [Bu90a], which give \( \|B_{1\tau_1}E(\Delta)\| \leq C\|A_{1\beta(\tau_1)}\| \), we can estimate (9) by

\[
\|A_{1\beta(\tau_1)}\|\|A_{2\beta(\tau_1)}\Omega - A_{2\beta(\tau_0)}\Omega\| \leq \|A_{1\beta(\tau_1)}\|\||A_{2\beta(\tau_1)}\Omega - \Psi_2\| + \|A_{2\beta(\tau_0)}\Omega - \Psi_2\| \tag{10}
\]

up to an overall constant, and the analysis of the second term in (8) gives an analogous bound. By substituting such estimates into (6), it is easy to obtain convergence of \( \Psi_\tau \), provided \( \Psi_1, \Psi_2 \) are of Reeh-Schlieder degree \( \gamma < 1 \) (cf. relations (2), (4)). A similar discussion of \( n \)-atom scattering states could suggest that single-atom states of arbitrarily small Reeh-Schlieder degree are needed. It turns out that this is not the case: by careful geometrical analysis and application of corresponding novel multi-operator clustering estimates (cf. Lemmas 8 and 16, respectively) we develop complete Haag-Ruelle scattering theory for single-atom states of arbitrarily large Reeh-Schlieder degree. Although atoms are our prime example, the construction works equally well for photons\(^5\), which demonstrates the robustness of our approach. We hope that this investigation will pave the way to a definite unifying solution of the problem of scattering of Wigner particles in algebraic QFT.

\(^4\)For clarity reasons we consider here only outgoing states. The incoming case \( \tau \to -\infty \) is analogous.

\(^5\)In contrast to atoms, scattering theory of photons is well understood since [Bu77].
This paper is structured as follows: In Section 2 we state the basic assumptions underlying this work and introduce the Reeh-Schlieder degree of Hilbert-space vectors. Section 3 gives an exposition of our variant of Haag-Ruelle creation operators and establishes some of their basic properties. Section 4 provides the fundamental technical tool of the discretized Cook's method: we derive rapid norm decay of non-equal time commutators of creation operators. In Sections 5 and 6 we establish clustering estimates and study their consequences relevant for refined handling of the norm growth of the creation operator approximants. All these results are then combined in Section 7 to prove convergence of scattering states and to establish their Fock structure in Section 8.

Acknowledgements
I am indebted to Klaus Fredenhagen for the suggestion to accelerate the convergence in the single-particle problem via the Reeh-Schlieder property. Similarly I would like to thank Wojciech Dybalski for encouragement and numerous insightful advice extended during the course of this work. Further I profited from helpful discussions with Sabina Alazzawi, Detlev Buchholz, Maximilian Butz, Daniela Cadamuro, and Yoh Tanimoto. Financial support from the Emmy Noether Programme of the DFG (grant DY107/2-1) is gratefully acknowledged.

2 Framework and assumptions

As the basis for our considerations we take a Haag-Kastler theory in the vacuum representation, i.e. a net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ of von Neumann algebras associated to bounded open regions $\mathcal{O} \subset \mathbb{R}^4$ in Minkowski space-time. Space-time translations by vectors $x = (t, \mathbf{x}) \in \mathbb{R}^4$ are represented on the Hilbert space $\mathcal{H}$ by a strongly-continuous group of unitary operators $U(t, \mathbf{x}) = e^{itH - i\mathbf{x} \cdot \mathbf{P}}$, generated by the strongly-commuting family of the self-adjoint energy-momentum operators $(H, \mathbf{P})$. Their joint spectral measure is denoted by $E(\Delta) := E_{(H, \mathbf{P})}(\Delta)$ for any Borel set $\Delta \subset \mathbb{R}^4$. The vacuum is a normalized translation-invariant vector $\Omega \in \mathcal{H}$. Finally, translations of operators $A \in \mathcal{B}(\mathcal{H})$ are induced by $U$ according to $A(x) := \alpha_x(A) := U(x)AU(x)^*$. We will use the following version of the Haag-Kastler postulates,

- **Isotony** $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$ (HK1)
- **Locality** $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'$ for $\mathcal{O}_1 \subset \mathcal{O}_2'$ (HK2)
- **Covariance** $\alpha_x(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O} + x)$ (HK3)
- **Uniqueness of $\Omega$** $E(\{0\})\mathcal{H} = \mathbb{C}\Omega$ (HK4)
- **Spectrum Condition** $\text{supp} E_{(H, \mathbf{P})} \subset \mathcal{V}^+$ (HK5)
- **Reeh-Schlieder Property** $\mathfrak{A}(\mathcal{O})\Omega = \mathcal{H}$ (HK6)

for any non-empty open bounded regions $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^4$ and any $x \in \mathbb{R}^4$. Here, $\mathfrak{A}(\mathcal{O})'$ is the commutant of $\mathfrak{A}(\mathcal{O})$ in $\mathcal{B}(\mathcal{H})$ and $\mathcal{O}' := \{y \in \mathbb{R}^4 : (y - x)^2 < 0 \forall x \in \mathcal{O}\}$ defines the causal complement of $\mathcal{O}$. Further, $\mathcal{V}^\pm := \{x \in \mathbb{R}^4 : x^2 \geq 0, \pm x^0 \geq 0\}$ is the future or past light cone, respectively. For future reference we denote by $\mathfrak{A}$ the $C^*$-inductive limit of the local net and by $H_m := \{p \in \mathbb{R}^4 : p^0 = \sqrt{p^2 + m^2}\}$ the mass hyperboloid of a particle of mass $m \geq 0$.

Next, we define the Reeh-Schlieder degree $\gamma_{\mathcal{RS}} \geq 0$ of a vector $\Psi \in \mathcal{H}$ as the infimum over all $\gamma \geq 0$ for which there exists an open bounded region $\mathcal{O}$ and a family

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\*We take the space-time metric with signature $(+,-,-,-)$.
of observables \((A_\beta)_{\beta > 0}\) from \(\mathfrak{A}(\mathcal{O})\) such that for all sufficiently small \(\beta > 0\) we have
\[
\|A_\beta \Omega - \Psi\| \leq \beta, \quad \|A_\beta\| \leq \beta^{-\gamma}.
\] (11)
We will call \((A_\beta)_{\beta > 0}\) a Reeh-Schlieder family (of degree \(\gamma\)). If no such family exists, we will say that \(\Psi\) is a vector of infinite Reeh-Schlieder degree. But we note that, by the standard Reeh-Schlieder property (HK6), at least the first inequality of (11) can always be satisfied for non-empty regions \(\mathcal{O} \subset \mathbb{R}^4\).

We amend the Haag-Kastler postulates by the following more specific assumptions, which can be seen in combination as a sharpened Wigner concept of a particle:

(HK5') In addition to (HK5), the relativistic mass operator
\[
M := \sqrt{H^2 - P^2}
\]
has an eigenvalue \(m \geq 0\). In other words \(E_m := E(H_m) \neq 0\).

(HK6') The single-particle subspace \(\mathcal{H}_m := E_m \mathcal{H}\) contains a dense subset of vectors of finite Reeh-Schlieder degree.

Under the above assumptions, (HK1) – (HK4), (HK5'), and (HK6'), our results from Sections 7 and 8 below allow to construct wave-operators and the S-matrix in the usual manner (see e.g. [Dy09] App. A).

3 Creation operators and their basic properties

Given a single-atom state \(\Psi_1 \in E(H_m) \mathcal{H}\) of mass \(m \geq 0\) we now want to find a corresponding family of creation operators \(B_\tau\), which is suitable for the construction of scattering states. By the Reeh-Schlieder property (HK6) we can always fix some non-empty bounded open region \(\mathcal{O} \subset \mathbb{R}^4\) and pick a corresponding family of local operators \((A_\beta)_{\beta > 0} \subset \mathfrak{A}(\mathcal{O})\) as in formula (1).

The Klein-Gordon equation will provide a free reference dynamics for comparison to the large-\(\tau\) asymptotics of the translated operator family \(A_\beta(\tau, x) := U(\tau, x)A_\beta U(\tau, x)^*\), \(x = (\tau, x) \in \mathbb{R}^4\), \(\beta > 0\), when taking the simultaneous limit \(\beta \to 0\). We will say that \(f : \mathbb{R}^4 \to \mathbb{C}\) is a regular positive-energy Klein-Gordon solution (of mass \(m \geq 0\)) if it can be written as
\[
f(t, x) = \int \frac{d^3k}{(2\pi)^3} e^{i(k \cdot x - i\omega_m(k)t)} \tilde{f}(k), \quad \omega_m(k) := \sqrt{k^2 + m^2},
\] (12)
where the wave-packet \(\tilde{f}\) has to be smooth and compactly supported. For the case \(m = 0\) we will also add the standard requirement \(0 \notin \text{supp} \tilde{f}\), as it leads to improved decay in the interior of the light cone which will be technically convenient in Section 4.

Taking a Reeh-Schlieder family \(A_\beta\) for a given single-particle state \(\Psi \in E(H_m) \mathcal{H}\) of mass \(m \geq 0\) and a regular positive-energy Klein-Gordon solution \(f\) of the same mass, we may modify the standard prescription for creation-operator approximants by admitting the following additional time-dependence of the smeared operators,
\[
A_\tau := \int d^3x f(\tau, x)A_{\beta(\tau)}(\tau, x).
\] (13)

If the Haag-Kastler net under consideration is obtained from a suitable Wightman theory (e.g. satisfying certain energy bounds [Bu90b]), property (HK6) holds as a consequence of the original results of Reeh and Schlieder [RS61]. Alternatively, (HK6) follows from assuming additivity of the Haag-Kastler net, see e.g. [A], Thm. 4.14.
For now it will suffice to demand that the scaling function $\beta$ satisfies $\beta(\tau) \to 0$ for $\tau \to \pm \infty$. The operator family $\mathcal{A}_\tau$ then already satisfies some properties which are characteristic for creation operators, as might be expected from the close similarity to standard Haag-Ruelle theory\(^9\). Before proceeding we would like to perform some further standard modifications needed for the multi-particle case, which will lead to improved differentiability and impose restrictions on energy-momentum transfers (see Proposition 3 (iii)).

Remark 1 (uniform differentiability of $\mathcal{A}_\beta$). By a standard smearing argument, restricting $\mathcal{A}_\beta$ (for fixed $\beta$) to the $*$-algebra of smooth operators $\mathfrak{A}_0(\mathcal{O})$, for which $(t,x) \mapsto \mathcal{A}_\beta(t,x)$ is arbitrarily often differentiable in norm, results in no loss of generality. It is important for our purposes that this smearing argument directly generalizes to yield uniformly differentiable families, i.e.

$$\|\partial_\alpha \mathcal{A}_\beta\| \leq C_\alpha \|\mathcal{A}_\beta\| \quad (14)$$

for all multi-indices $\alpha \in \mathbb{N}_0^4$ and some $\beta$-independent constants $C_\alpha$. In the following we will therefore assume that all appearing Reeh-Schlieder families $\mathcal{A}_\beta$ are smooth and uniformly differentiable.

Further it will be convenient to have at hand a related operator family with common compact energy-momentum transfers disjoint from a neighbourhood of the origin. To achieve this we have to give up strict localization and smear the family $\mathcal{A}_\beta$ with the Fourier transform of a function $\hat{\chi} \in C^\infty_c(\mathbb{R}^4 \setminus \bar{V}^-)$. We will denote the resulting family of almost-local\(^10\) operators by

$$B_\beta := \mathcal{A}_\beta(\chi).$$

With these preparations we can introduce our family of creation operator approximants.

Definition 2 (creation operator approximant). Let $\mathcal{A}_\beta \in \mathfrak{A}(\mathcal{O})$ be a uniformly differentiable Reeh-Schlieder family for $\Psi_1 \in E(H_m)\mathcal{H}$, $m \geq 0$. Fixing $\hat{\chi} \in C^\infty_c(\mathbb{R}^4 \setminus \bar{V}^-)$ we set $B_\beta := \mathcal{A}_\beta(\chi)$ and for $\tau \in \mathbb{R}$ and a regular positive-energy Klein-Gordon solution $f$ of the same mass $m$ we define creation-operator approximants as

$$B_\tau := \int d^3x \ f(\tau,x)B_\beta(\tau)(\tau,x). \quad (15)$$

We will often make use of the fact that $B_\tau$ are related to the simpler operator family $\mathcal{A}_\tau$ by convolution algebra. Let us collect the most important properties of these families of operators.

Proposition 3 (Basic properties of creation operators). For an arbitrary operator family $\mathcal{A}_\beta \in \mathfrak{B}(\mathcal{H})$ define $B_\beta$, $\mathcal{A}_\tau$ and $B_\tau$ as before. Then

(i) $B_\tau = \mathcal{A}_\tau(\chi)$.

(ii) $\|B_\tau\| \leq C \|\mathcal{A}_\tau\| \leq C'(1 + |\tau|^N) \|A_\beta(\tau)\|$ with suitable constants $C,C', N > 0$.

(iii) For any closed $\Delta \subset \mathbb{R}^4$, we have the energy-momentum transfer relations

$$B_\beta E(\Delta)\mathcal{H} \subset E(\Delta + \text{supp } \hat{\chi})\mathcal{H},$$

$$B_\tau^* E(\Delta)\mathcal{H} \subset E(\Delta - \text{supp } \hat{\chi})\mathcal{H}.$$
(iv) There exists a neighbourhood of zero \( \mathcal{U} \subset \mathbb{R}^4 \) such that \( B_\beta^*E(\mathcal{U}) = 0 \).

(v) \( B_\beta^*\Omega = 0 \).

(vi) If \( A_\beta \Omega \rightarrow \Psi_1 \in E(H_m)\mathcal{H} \) where \( m \geq 0 \) denotes the mass of \( f \), then

\[
\lim_{\tau \to \pm \infty} A_\tau \Omega = \tilde{f}(P) \Psi_1, \text{ and similarly } \lim_{\tau \to \pm \infty} B_\tau \Omega = \tilde{f}(P) \Psi'_1, \tag{16}
\]

with \( \Psi'_1 := \lim_{\beta \to 0} B_\beta \Omega = (2\pi)^2 \tilde{\chi}(H, P) \Psi_1 \).

Properties (iii)-(v) also hold with \( B_\tau \) in place of \( B_\beta \) without further modifications.

Proof. (i) is equivalent to \((\alpha_\tau(A_{\beta(\tau)}(\chi)))(f_\tau) = ((\alpha_\tau(A_{\beta(\tau)}))(f_\tau))(\chi)\), where \( f_\tau(x) := f(\tau, x) \), and this follows from convolution algebra. Property (ii) is a consequence of Hölder’s inequality \( \|A(f)\| \leq \|A\| \cdot \|f\|_1 \) and the standard polynomial bounds for spatial \( L^1 \)-norms of Klein-Gordon solutions [RS3, Appendix 1 to XI.3]. For the proof of relation (iii) we refer to the literature of Arveson spectral theory — e.g. [Arv80]. To establish (iv), we note that by assumption \( \text{supp} \tilde{\chi} \) is compact and disjoint from the closed set \( V^+ \), so that for a sufficiently small neighbourhood \( \mathcal{U} \) of the origin there holds \( (\mathcal{U} - \text{supp} \tilde{\chi}) \cap V^+ = \emptyset \). By (iii) and the spectrum condition (HK5) it follows that \( B_\beta^*E(\mathcal{U})\mathcal{H} \subset E(\mathcal{U} - \text{supp} \tilde{\chi})\mathcal{H} = \{0\} \). Identity (v) is a direct consequence of (iv), as \( \Omega \in E(\mathcal{U})\mathcal{H} \) for any neighbourhood of zero \( \mathcal{U} \). The relations for \( B_\tau \) follow by similar argument after using identity (i).

It remains to verify that \( A_\tau \) and \( B_\tau \) provide solutions for the single-particle problem (vi). By spectral calculus we obtain

\[
A_\tau \Omega = \tilde{f}(P)U(\tau)A_{\beta(\tau)} \Omega = \tilde{f}(P)e^{i(H-\omega_m(P))\tau}A_{\beta(\tau)} \Omega.
\]

As \( \Psi_1 \) is invariant under the unitaries \( V(\tau) := e^{i(H-\omega_m(P))\tau} \) we may directly estimate

\[
\|A_\tau \Omega - \tilde{f}(P)\Psi_1\| = \|A_\tau \Omega - \tilde{f}(P)V(\tau)\Psi_1\| \leq \|\tilde{f}\|_{\infty} \|A_{\beta(\tau)} \Omega - \Psi_1\|.
\]

The convergence of \( B_\tau \Omega \) follows then from (i) by writing \( B_\tau \Omega = (2\pi)^2 \tilde{\chi}(H, P)A_\tau \Omega \). \( \square \)

An important consequence of the energy-momentum transfer relation (iii) is the following energy bound. The key point is that the estimate can be made uniform in \( \tau \) relative to the norm of the underlying Reeh-Schlieder families, as long as we consider the restriction of creation operators to a subspace of bounded energy. Our analysis was somewhat inspired by Herdegen’s work [Hrd13], but we rely on different aspects of Buchholz’ results [Bu90a] given in Lemma 4.

Lemma 4 ([Bu90a], Lemma 2.2). Let \( K \subset \mathbb{R}^3 \) compact, \( B \in B(\mathcal{H}) \) and denote by \( P_n \) the orthogonal projection onto the intersection of the kernels of the \( n \)-fold products of translated operators \( B(x_1) \ldots B(x_n) \) for any configuration of \( x_1, \ldots, x_n \in \mathbb{R}^3 \). Then

\[
\left\| P_n \int_K d^3x (B^*B)(x)P_n \right\| \leq (n-1) \int_{\Delta K} d^3x \|B^*, B(x)\|, \tag{17}
\]

where integration on the right is over all element-wise differences \( \Delta K := K - K \).
**Proposition 5** (Energy bounds). Without further restrictions on the families of operators $A_\beta, A_{k\beta} \in \mathfrak{A}(O)$, we have for any compact $\Delta \subset \mathbb{R}^4$,

$$\|B_\tau E(\Delta)\| \leq C \|A_{\beta(\tau)}\|, \quad (17)$$

$$\|B_{1\tau} \ldots B_{n\tau} E(\Delta)\| \leq C \prod_{k=1}^{n} \|A_{k\beta(\tau_k)}\|, \quad (18)$$

where the constant $C$ depends on $\Delta$, $O$, $\text{supp } \hat{\chi}$, the number of operators $n$, and the corresponding wave packets $\hat{f}, \hat{f}_k$, but it is independent of $\tau$.

*Proof.* To establish (17), let $\Delta \subset \mathbb{R}^4$ be a given compact set. By a partition argument, we can assume that $\text{supp } \hat{\chi}$ is contained in a compact, convex set disjoint from $\tilde{V}^\perp$. The compact common energy-momentum transfer (cf. Proposition 3 (iii)) of $B_\tau$ then allows us to write

$$\|B_\tau E(\Delta)\| = \|E(\Delta + \text{supp } \hat{\chi})B_\tau E(\Delta)\| \leq \|E(\Delta')B_\tau\| = \|B_\tau^* E(\Delta')\|,$$

where $\Delta' := \Delta + \text{supp } \hat{\chi}$ is compact as well.

To make the connection with Lemma 4, we note that by iterated application of Proposition 3 (iii) and translation-invariance of finite-energy subspaces, we obtain

$$B_\beta^*(x_1) \ldots B_\beta^*(x_n) E(\Delta) \mathcal{H} \subset E(\Delta - \Sigma_n \text{supp } \hat{\chi}) \mathcal{H},$$

where $\Sigma_n \text{supp } \hat{\chi} := \{y_1 + \ldots + y_n : y_k \in \text{supp } \hat{\chi}\} = n \text{supp } \hat{\chi}$ due to convexity. By the Hyperplane Separation Theorem, we obtain $(\Delta' - \Sigma_n \text{supp } \hat{\chi}) \cap V^+ = \emptyset$ for sufficiently large $n \in \mathbb{N}$. This implies via the spectrum condition (HK5) that for such $n$, the projections $P_n$ appearing in Lemma 4 may be estimated from below by $E(\Delta') \mathcal{H} \subset P_n \mathcal{H}$. With these preparations we can estimate

$$\|B_\tau^* E(\Delta')\| \leq \|B_\tau^* P_n\| \leq \sup_{\Psi \in \mathcal{H}, \|\Psi\| = 1} \int d^3x |f(\tau, x)| \|B_\beta^*(\tau, x) P_n \Psi\|$$

$$\leq \left( \int d^3x |f(\tau, x)|^2 \right)^{1/2} \left( \sup_{\Psi \in \mathcal{H}, \|\Psi\| = 1} \int d^3x \|B_\beta^*(\tau, x) P_n \Psi\|^2 \right)^{1/2}.$$

The first factor is constant by the Plancherel identity (cf. Prop. 12 (iv)). For estimating the second factor we choose an arbitrarily large compact region $K \subset \mathbb{R}^3$ and obtain from Lemma 4 that

$$\sup_{\Psi \in \mathcal{H}, \|\Psi\| = 1} \int_K d^3x \|B_\beta^*(\tau, x) P_n \Psi\|^2 = \sup_{\Psi \in \mathcal{H}, \|\Psi\| = 1} \left\langle \Psi, P_n \int_K d^3x (B_\beta^*(\tau) B_\beta^*(\tau))(\tau, x) P_n \Psi \right\rangle$$

$$= \left\| P_n \int_K d^3x (B_\beta^*(\tau) B_\beta^*(\tau))(\tau, x) P_n \right\|$$

$$\leq (n - 1) \int_{\Delta \mathcal{K}} d^3x \left\| [B_\beta^*(\tau), B_\beta^*(\tau)] \right\|.$$

The family $B_\beta$ and its adjoint are uniformly almost-local (as defined in Appendix B), so that the remaining integral can be estimated by $2C_n \|A_\beta(\tau)\|^2 \cdot d^3$, where $d$ depends
only on the size of the localization region of \( A_\beta \). This yields a bound which is uniform in \( \Delta K \) and by taking \( K \not\supset \mathbb{R}^3 \) we obtain the energy bound for a single operator.

Then the bound (18) on multiple creation operators follows directly by induction: the compact common energy-momentum transfer of the family \( B_{\kappa t} \) yields

\[
\| B_{\tau_1} \ldots B_{\tau_n} E(\Delta) \| = \| B_{\tau_1} \ldots B_{\tau_{n-1}} E(\Delta + \text{supp} \tilde{\chi}) B_{\tau_n} E(\Delta) \| \\
\leq \| B_{\tau_1} \ldots B_{\tau_{n-1}} E(\Delta + \text{supp} \tilde{\chi}) \| \cdot \| B_{\tau_n} E(\Delta) \| \\
\leq C^{(n-1)}_{\Delta + \text{supp} \tilde{\chi}} \left( \prod_{k=1}^{n-1} \| A_k \| \right) \cdot C_\Delta \| A_{\kappa} \|.
\]

\[ \square \]

4 Geometry of non-equal time commutators

The goal of this section is to study the decay behaviour of commutators \([ B_{\tau_1}, B_{\tau_2} ]\) for distinct asymptotic parameters \( \tau_1 \neq \tau_2 \). The strongest known decay estimates for equal times \( \tau_1 = \tau_2 \) have been established for the case, where the defining Klein-Gordon solutions \( f_1, f_2 \) have disjoint support in momentum space [Hep65]. This corresponds to the physically reasonable assumption that the two particles will separate at large times. We will restrict our analysis to this setting and begin by reviewing required results on regular Klein-Gordon solutions \( f : \mathbb{R}^4 \rightarrow \mathbb{C} \) with mass \( m \geq 0 \), as defined in (12).

The geometry of the asymptotic behaviour of \( f \) can be intuitively understood in terms of the set of velocities corresponding to the momenta \( \mathbf{k} \in \text{supp} \tilde{f} \). Accordingly we define the velocity support of \( f \) by \( \Gamma_f := \{ \mathbf{k}/\omega_m(\mathbf{k}) : \mathbf{k} \in \text{supp} \tilde{f} \} \). Let us recall how this definition allows for a compact formulation of the classical result of Ruelle [Ru62] on the decay of Klein-Gordon solutions outside the velocity-support cone. We provide a unified treatment of the massive and massless case.

**Lemma 6** (velocity-support estimate). Let \( f \) be a regular solution of the Klein-Gordon equation with mass \( m \geq 0 \). The following estimate holds for any \( N \in \mathbb{N} \) with suitable constants \( C_N > 0 \) and any \( (t, \mathbf{x}) \in \mathbb{R}^4 \) satisfying \( \mathbf{x}/t \notin \Gamma_f \),

\[
|f(t, \mathbf{x})| \leq \frac{C_N}{\delta N |t|^N},
\]

where \( \delta \) denotes the distance of \( \mathbf{x}/t \) from the set \( \Gamma_f \).

For regular massive Klein-Gordon solutions, geometrical propagation properties such as the above can be found in various textbooks, e.g. [A] Thm. 5.3. We will skip the standard proof, which makes use of the non-stationary phase method (see e.g. [RS3], Appendix 1 to XI.3). Lemma 6 is applicable in particular in the case of \( \mathbf{x}/t \) approaching the velocity support \( \Gamma_f \). This will be needed later in Proposition 12 to establish certain norm estimates in the massless case.

For the purpose of rapid decay of commutators, it is actually sufficient to make use of Lemma 6 in some fixed neighbourhood \( U \supset \Gamma_f \). One obtains the following simple rapid-decay estimate with respect to time and space outside a corresponding enlarged neighbourhood of the cone generated by the velocity support.

**Corollary 7.** Let \( f \) be a regular solution of the Klein-Gordon equation with mass \( m \geq 0 \) and let \( U \supset \Gamma_f \) be any (slightly larger) neighbourhood of the velocity support. Then the restriction of \( \tilde{f} \) to the complement of the cone

\[
\Upsilon_U := \{(t, tv) \in \mathbb{R}^4, v \in U, t \in \mathbb{R} \}
\]
is rapidly decreasing, i.e. for any $N \in \mathbb{N}$ we have
\[ |f(t, x)| \leq C_N (1 + |t| + |x|)^{-N} \quad \forall (t, x) \in \mathbb{R}^4 \setminus \Upsilon_U, \]
with suitable $C_N > 0$ depending on $N$, $\tilde{f}$, and the distance between $\mathbb{R}^3 \setminus \Upsilon$ and $\Gamma_f$.

While our construction of collision states will make use of the creation operators $B_{k\tau}$, it is clear that additional technical difficulties arise due to the loss of strict locality when passing from localized Reeh-Schlieder families $A_{k\beta} \in \mathfrak{A}(\mathcal{O})$ (with $\mathcal{O}$ independent of $\beta$) to the almost-local operators $B_{k\beta} := A_{k\beta}(\chi)$. We recall that the thus obtained compact energy-momentum transfers of $B_{k\beta}$ were essential for establishing energy bounds in Proposition 5.

One strategy to resolve these complications, which makes arguments based on locality particularly transparent, is to first establish corresponding results for the operators $A_{k\tau}$, as these have better localization properties. Statements which are sufficiently stable under smearing can then be carried over to $B_{k\tau} = A_{k\tau}(\chi)$ (see Proposition 3 (i)). For this reason we want to additionally allow space-time translates $\alpha_x(A_{k\tau})$ with $x \in \mathbb{R}^4$ restricted to suitable bounded regions in space-time. We note for clarification that $\alpha_x(A_{k\tau+t})$ differs from $\alpha_x(A_{k\tau})(\chi)$ due to the time evolution of the Klein-Gordon solution and the underlying time-dependent Reeh-Schlieder family.

The geometrical content of Lemma 8 is illustrated in Figure 1. Regarding the depicted situation it is clear that in order to obtain rapid decay the allowed translation vectors $x = (x^0, x) \in \mathbb{R}^4$ will have to be subjected to a similar restriction as the time differences $|\tau_2 - \tau_1|$. In the context of causal distance estimates, it will be convenient to specify this restriction by introducing the norm $|x|_c := |x^0| + |x|$, where $|x| := \sqrt{x^2}$ denotes the Euclidean length of $x \in \mathbb{R}^3$. The centered open balls generated by this norm are the familiar double cones $C_R = \{ x \in \mathbb{R}^4 : |x|_c < R \}$ with radius $R > 0$.

**Lemma 8.** There exists a constant $C > 0$, such that for any $f_1, f_2$ with velocity supports separated by a positive distance $d > 0$, the following estimate holds for any $N \in \mathbb{N}$,
\(x \in \mathbb{R}^4\) and \(\tau_1, \tau_2 \in \mathbb{R}\) satisfying \(|x|_c + |\tau_2 - \tau_1| \leq Cd^2 \cdot \tau_{\min},\)

\[||[A_{1\tau_1}, \alpha_x(A_{2\tau_2})]|| \leq CN \cdot ||A_{1\beta(\tau_1)}|| \cdot ||A_{2\beta(\tau_2)}|| \cdot (1 + \tau_{\min})^{-N}. \tag{19}\]

Here, \(\tau_{\min} := \min(|\tau_1|, |\tau_2|)\) and the constants \(C_N\) depend only on \(N, f_k\) and the size of the localization regions of \(A_{k\beta} \).

**Proof.** We can assume without restriction that \(\tau_{\min} = |\tau_1|\). Further it is enough to establish (19) for \(|\tau_1|\) sufficiently large\(^{11}\), and for this case we will make use of a suitable common asymptotic decomposition of the Klein-Gordon solutions \(f_k\). By definition, the corresponding velocity supports \(\Gamma_{f_k}^0\) and \(\Gamma_{f_k}^1\) are closed subsets of the closed unit ball. Aiming at the application of Corollary 7, it is clear that we can find neighbourhoods \(U_1, U_2\) of the velocity supports \(\Gamma_{f_k}^0\) and \(\Gamma_{f_k}^1\), which are separated by a distance of at least \(d/2\) and which are contained in some fixed larger ball. For concreteness we may assume without loss of generality that \(v \in U_{1/2}\) always satisfy\(^{12}\) \(|v| \leq 2\).

Denoting by \(\mathcal{I}_{\mathcal{Y}U_k}\) the characteristic function of the cone \(\mathcal{Y}U_k\) (as defined in Corollary 7) we introduce the following decompositions into asymptotically dominant and negligible parts,

\[f_k = f_k^\dagger + f_k^\ddagger, \quad f_k^\dagger(x) := f_k(x) \cdot \mathcal{I}_{\mathcal{Y}U_k}(x),\]

and similarly \(A_{k\tau} = A_{k\tau}^\dagger + A_{k\tau}^\ddagger, (k = 1, 2)\), denote the induced decompositions of creation operators. By Corollary 7, we obtain

\[||A_{k\tau_k} - A_{k\tau_k}^\dagger|| = ||A_{k\tau_k}^\ddagger|| \leq C_N ||A_{1\beta(\tau_1)}|| \cdot (1 + |\tau_k|)^{-N}.\]

This implies that it is sufficient to analyse the commutator of the dominant parts as can be seen from the following estimate, which holds uniformly in \(x \in \mathbb{R}^4\),

\[||[A_{1\tau_1}, A_{2\tau_2}(x)]|| \leq ||[A_{1\tau_1}^\dagger, A_{2\tau_2}^\dagger(x)]|| + C_N \cdot ||A_{1\beta(\tau_1)}|| \cdot ||A_{2\beta(\tau_2)}|| \cdot (1 + \tau_{\min})^{-N}.\]

We will now verify that the commutator of the dominant parts vanishes for sufficiently large \(\tau_1\) in the claimed region of \(x\) and \(\tau_k\). As a standard consequence of the Haag-Kastler axioms we obtain

\[A_{k\tau_k}^\dagger \in \mathfrak{A}(O_{k\tau_k}), \quad \text{with} \quad O_{k\tau_k} := \mathcal{C}_R + \tau_k \cdot \{1\} \times U_k,\]

where we picked a sufficiently large radius \(R > 0\) such that the double cone \(\mathcal{C}_R\) provides a common bounded localization region of the families \(A_{k\beta} \). Then we have by covariance \(A_{2\tau_2}^\dagger(x) \in \mathfrak{A}(O_{2\tau_2} + x)\). To estimate the causal distance of any two points \(y_1 \in O_{1\tau_1}\) and \(y_2 \in O_{2\tau_2} + x\) from the respective support regions, we write them as \(y_1 = o_1 + \tau_1 \cdot (1, v_1),\)

\(y_2 = o_2 + \tau_2 \cdot (1, v_2) + x,\) with \(o_1, o_2 \in \mathcal{C}_R\) and \(v_k \in U_k\). We can then see that

\[y_2 - y_1 = [(\tau_2, \tau_2 v_2) - (\tau_1, \tau_1 v_1)] + o_2 + x - o_1.\]

In the end we will impose a suitable restriction on \(u := o_2 + x - o_1\) and therefore the space-like separation of \(y_1\) and \(y_2\) needs to be derived from the difference term inside the brackets, which we denote by \(w := (\tau_2, \tau_2 v_2) - (\tau_1, \tau_1 v_1)\). We compute

\[w^2 = (\tau_2 - \tau_1)^2 - (\tau_2 v_2 - \tau_1 v_1)^2,\]

\[|\tau_2 v_2 - \tau_1 v_1| = |\tau_2 v_2 - \tau_1 v_2 + \tau_1 (v_2 - v_1)| \geq |\tau_1 - \tau_2|, \quad |v_2 - v_1|, \tag{20}\]

\(^{11}\)On any bounded interval \(|\tau_k| \leq \tau_{\max}\) \((\tau_{\max} \) fixed), we may use Proposition 3 \((ii)\) to obtain \(||[A_{1\tau_1}, \alpha_x(A_{2\tau_2})]|| \leq C_{\tau_{\max}} \cdot ||A_{1\beta(\tau_1)}|| \cdot ||A_{2\beta(\tau_2)}||\), which is compatible with (19) for sufficiently large \(C_N\).

\(^{12}\)Such a bound will be important later in the proof. The concrete choice of the constant has no physical significance, but it will influence the magnitude of the proportionality constant \(C\) controlling time-differences in the statement of the lemma.
and thus
\[ w^2 \leq -\tau_1^2 (v_2 - v_1)^2 + 2 |\tau_1 - \tau_2| |\tau_1| |v_2 - v_1| |v_2| + (\tau_1 - \tau_2)^2. \]

We note that by the non-vanishing negative coefficient of the quadratic term, \( w \) will become space-like for large enough \(|\tau_1|\) if sufficient restrictions are placed on \(|\tau_2 - \tau_1|\).

By a similar argument also the perturbation of adding \( u \) can be controlled, as can be seen from
\[ (y_2 - y_1)^2 = w^2 + 2 w \cdot u + u^2 \leq w^2 + 2 |w|_c |u|_c + |u|_c^2, \tag{21} \]
where we used the Cauchy-Schwarz inequality. Now assume that \(|\tau_2 - \tau_1| + |x|_c \leq \bar{\rho} |\tau_1|\) for some constant \( \bar{\rho} > 0 \) (to be determined). Using that our choice of \( U_k \) implies \(|v_k| \leq 2, 0 < d \leq |v_2 - v_1| \leq 4 \), we can then further estimate
\[
\begin{align*}
  w^2 &\leq -d^2 \tau_1^2 + (16\bar{\rho} + \bar{\rho}^2)\tau_1^2, \\
  |w|_c &\leq 3 |\tau_2 - \tau_1| + 4 |\tau_1| \leq (4 + 3\bar{\rho}) |\tau_1|, \\
  |u|_c &\leq |o_1|_c + |o_2|_c + |x|_c \leq 2R + \bar{\rho} |\tau_1|. 
\end{align*}
\]

To simplify the resulting bound on \((y_2 - y_1)^2\), let us choose first \( \bar{\rho} \leq 1 \) and subsequently \(|\tau_1| \geq 2R/\bar{\rho} \). This allows us to eliminate unimportant scales by writing \(|w|_c \leq 7 |\tau_1|, \ |u|_c \leq 2 |\tau_1| \) and \( \bar{\rho}^2 \leq \bar{\rho} \). Then we obtain from (21) that with a suitable numerical constant \( C > 0 \),
\[
(y_2 - y_1)^2 \leq -d^2 \tau_1^2 + C^{-1} \bar{\rho} \tau_1^2.
\]

This proves that any choice \( 0 < \bar{\rho} < C d^2 \) \((\bar{\rho} \leq 1)\) leads to space-like localization regions of the dominant parts, and so by locality \([A_{1\tau_1}^k, A_{2\tau_2}^k(x)] = 0\) for \(|\tau_1| > 2R/\bar{\rho}\) under the assumed restriction on \(\tau_2\) and \(x\). \(\Box\)

With this technical preparation we can now establish asymptotic commutation of the creation operators \( B_{k\tau}^k \) with disjoint velocity supports at non-equal times. We can also appreciate now how the power-law scaling \( \beta(\tau) = |\tau|^{-\mu} \) (for large enough \(|\tau_1|\)), \( \mu > 0 \), plays a distinguished role: for this choice the norm terms \( \|A_{k\beta}^k(\tau)\| \) can be absorbed due to the rapid decay in Lemma 8. While these commutator estimates may still be improved in a suitably adapted setting, already the results of the next section will impose sharp restrictions on the scaling parameter \( \mu \).

**Theorem 9** (non-equal-time commutator estimate). Let \( A_{k\beta}^k, \ (k = 1, 2) \), be Reeh-Schlieder families of finite degree\(^{13}\), take regular Klein-Gordon solutions \( f_k \) with disjoint velocity supports and assume a fixed scaling \( \beta(\tau) = |\tau|^{-\mu} \) (for large enough \(|\tau_1|\)), \( \mu > 0 \). Setting \( \rho := C d^2 / 2 \in (0, 1) \) with \( C, d \) as in Lemma 8, there exists for any \( N \in \mathbb{N} \) a constant \( C_N > 0 \), such that for arbitrary \( \tau \in \mathbb{R} \) and all \( \tau_1, \tau_2 \) from the corresponding interval spanned by \( \tau \) and \( \tau + \rho \tau \),
\[
\| [B_{1\tau_1}^k, B_{2\tau_2}^k] \| \leq C_N (1 + |\tau|)^{-N}.
\]

**Proof.** We have \( B_{k\tau} = A_{k\tau}(\chi) \), with \( \chi \in \mathcal{S}(\mathbb{R}^4) \) and so we obtain
\[
\| [B_{1\tau_1}^k, B_{2\tau_2}^k] \| \leq \int d^4x d^4y \ |\chi(x)| \ |\chi(y)| \ \| [A_{1\tau_1}, A_{2\tau_2}(y - x)] \|. \tag{22}
\]
We decompose the integral into the region \(|x|_c \leq \rho |\tau| / 2\) and its complement, and similarly for the \(y\)-integration. As a consequence of our assumptions we have a polynomial

---

\(^{13}\)For Theorem 9, it is sufficient if the operator families \( A_{k\beta} \) are uniformly localized \((A_{k\beta} \in \mathfrak{A}(\mathcal{O}))\), with bounded \( \mathcal{O} \) independent of \( \beta \) and have at most polynomial norm growth \( \|A_{k\beta}\| \leq \beta^{-\gamma} \), \( (\gamma \geq 0) \).
bound $\|A_{k\beta(\tau)}\| \leq |\tau|^\mu$ and restricting to $\tau_1, \tau_2$ from the claimed interval we obtain for fixed $x \in \mathbb{R}^4$ that

$$\int d^4y \ |\chi(y)| \|[A_{1\tau_1}, A_{2\tau_2}(x-y)]\| \leq 2 \|\chi\|_1 \|f_{1\tau_1}\|_1 \|f_{2\tau_2}\|_1 \|A_{1\beta(\tau_1)}\| \|A_{2\beta(\tau_2)}\| \leq C |\tau|^M,$$

for some large enough $M > 0$ and the estimate holds uniformly in $x$. This now implies that the integral of (22) restricted to the outside region $|x|_c \geq \rho |\tau|/2$ is rapidly decreasing: we can estimate it by a product of the above polynomially bounded function with the rapidly decreasing function obtained by integrating $|\chi(x)|$ over the retracting regions given by $|x|_c \geq \rho |\tau|/2$. By a similar argument we can assume that also $|y|_c \leq \rho |\tau|/2$ and so we can write with suitable constants $C_N'$,

$$\|\left[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}\right]\| \leq \frac{C'_N}{1 + |\tau|^\mu} + \int d^4x \ d^4y \ |\chi(x)| \ |\chi(y)| \|[A_{1\tau_1}, A_{2\tau_2}(x-y)]\|.$$}

Assuming the given restriction $|\tau_1 - \tau_2| \leq \rho |\tau| \leq \rho \tau_{\min}$ we obtain $|\tau_2 - \tau_1| + |x-y|_c \leq 2\rho |\tau| \leq Cd^2 \tau_{\min}$. Therefore Lemma 8 is applicable, which yields

$$\int d^4x \ d^4y \ |\chi(x)| \ |\chi(y)| \|[A_{1\tau_1}, A_{2\tau_2}(x-y)]\| \leq \frac{C'_N}{(1 + \tau_{\min})^N} \|[A_{1\beta(\tau_1)}]\| \|[A_{2\beta(\tau_2)}]\|.$$}

As $\tau_{\min} \geq |\tau|$ we can proceed similarly as before and choose $N'$ large enough (depending on the desired decay order $N$, the scaling $\mu$, and $\|[A_{k\beta(\tau_2)}]\|$) to compensate for the polynomial growth of $\|[A_{k\beta(\tau_2)}]\|$.

It is clear that the same reasoning applies, if we replace one or more creation operator approximants by their adjoints. For later use in Section 8, we also mention the following equal-time result regarding double commutators with one additional creation operator which may have arbitrary velocity support. This follows from Theorem 9 by a well-known decomposition argument.

**Corollary 10** (decay of double commutators). In the setting of Theorem 9 let $\mathcal{B}_\tau$ be an additional creation operator approximant defined in terms of a regular Klein-Gordon solution $f$ (without restrictions on its velocity support) and an additional Reeh-Schlieder family $A_\beta$ of finite degree. Then,

$$\|[\mathcal{B}_\tau, \mathcal{B}_{1\tau}], \mathcal{B}_{2\tau}\| \leq C_N(1 + |\tau|)^{-N}.$$}

The same estimate holds if we replace one or more operators by their adjoints.

**Proof.** By a smooth decomposition of the wave packet $\tilde{f} = \tilde{f}_{1\tau} + \tilde{f}_{2\tau}$, such that the resulting commutators $[\mathcal{B}_{1\tau}, \mathcal{B}_{k\tau}]$ are both rapidly decreasing in norm, the result follows directly from Theorem 9 and the Jacobi identity.

The results of this section seem to be somewhat similar in spirit to Theorem 2 (ii) of [Hrd13], although their role in our verification of convergence of scattering states by discretized time sequences is quite different. A similar result can be found in [Du13].
5 Large space-like translations and clustering

In this section we prove the following clustering property for the operator families $A_{k\tau}$,

$$\lim_{\tau \to \infty} E_{\Omega}^+ [A_{1\tau}, A_{2\tau}]\Omega = 0,$$

with $E_{\Omega} := |\Omega\rangle\langle\Omega|$, $E_{\Omega}^+ := 1 - E_{\Omega}$, and where in contrast to Section 4 no restrictions are imposed on velocity supports. We will require that the scaling $\mu > 0$ has been chosen sufficiently small (depending on the Reeh-Schlieder degrees). Combined with the single-particle convergence established in Proposition 3 (vi), relation (23) implies that also the limit of $[A_{1\tau}, A_{2\tau}]\Omega$ exists and is proportional to the vacuum. Similarly as in Section 4 we will admit some relative translations of the two operators in (23), so that the results can be carried over to the corresponding expressions involving the operators $B_{k\tau}$ in Section 6. These estimates will play a key role for our proof of convergence of scattering states.

Our treatment is chiefly inspired by Section 3 of [Dy05] and corresponding earlier results of Buchholz [Bu77]. We rely similarly on space-like decay of matrix elements of local operators, as established by the well-known Araki-Hepp-Ruelle Theorem. For smooth operators $B \in \mathfrak{A}_0(\mathcal{O})$ a variant of this decay estimate may be conveniently expressed in terms of the norm $\|B\|_{\text{AHR}} := \|B\| + \|\partial_0 B\|$.

**Theorem 11** (Araki-Hepp-Ruelle [AHR62]). Let $A_k \in \mathfrak{A}_0(\mathcal{O}_{R_k})$, $k = 1, 2$. Then for any $|x| \geq 2(R_1 + R_2)$, we have

$$\left| \left\langle \Omega, A_1 U(x) E_{\Omega}^+ A_2 \Omega \right\rangle \right| \leq \frac{C_{\text{AHR}}(R_1 + R_2)^3}{|x|^2} \|A_1\|_{\text{AHR}} \|A_2\|_{\text{AHR}}. \quad (24)$$

The constant $C_{\text{AHR}}$ is universal, but we note that estimate (24) with its quadratic decay is specific to theories on physical Minkowski space-time $\mathbb{R}^4$.

To establish the clustering estimate (23) we will have to assume that $A_\beta \in \mathfrak{A}_0(\mathcal{O})$ for small enough $\beta > 0$ and that $\|A_\beta\|_{\text{AHR}}$ is not growing too fast. Both assumptions follow from the uniform differentiability property discussed in Remark 1. Further we will make use of the velocity support estimate of Lemma 6 supplemented by well-known globally valid norm estimates for Klein-Gordon solutions, which we collect in Proposition 12.

**Proposition 12.** Let $f$ be a regular solution of the Klein-Gordon equation with mass $m \geq 0$ and set $f_\tau(x) := f(\tau, x)$. Then for any $p \geq 1$ and $0 < \epsilon < 1$ the following estimates hold.

(i) $|f(t, x)| \leq C_N \epsilon^{-N}(1 + |t| + |x|)^{-N}$ for $|x| \geq (1 + \epsilon)|t|$ and any $N \in \mathbb{N}$.

(ii) If $m = 0$, then (i) holds also for any $|x| \leq (1 - \epsilon)|t|$.

(iii) For $m > 0$, we have $\|f_\tau\|_\infty \leq C(1 + |\tau|)^{-3/2}$ everywhere.

(iv) $\|f_\tau\|_\infty \leq C(1 + |\tau|)^{-1}$ everywhere.

(v) For $m > 0$, $\|f_\tau\|_p \leq C_p(1 + |\tau|^\frac{3}{2} - p^\frac{1}{2} \cdot (2-p^\frac{1}{2})^\frac{1}{2})$.

(vi) If $m = 0$, then $\|f_\tau\|_p \leq C_\epsilon \epsilon^\frac{1}{2} (1 + |\tau|^{2 - p^\frac{1}{2}})$ for any $\epsilon > 0$.

All appearing constants depend on the wave packet of $f$ and norms are taken in $L^p(\mathbb{R}^3)$.\]
Proof. (i) and (i₀) can be established as consequences of the velocity support estimate of Corollary 7. Note that for m = 0 we assumed \( \mathbf{0} \notin \text{supp} \hat{f} \). The global estimates (ii) and (ii₀) are proven e.g. in [RS3], Theorems XI.17 and XI.18. (iii) and (iii₀) with \( \epsilon = 1 \) follow by decomposing the integration according to the regions of validity of the respective versions of (i), (ii), i.e. for \( m = 0 \) we may take \( I_e := \{ ||x| - |\tau|| \leq d(|\tau|) \} \) and its complement. The present result for (iii₀) with \( 0 < \epsilon < 1 \) follows by setting \( d = d(\tau) = |\tau|^{-\nu} \) for any \( 0 < \nu < 1 \) with \( \nu := 1 - \epsilon \) and by making use of Lemma 6. Finally, (iv) is a consequence of the Plancherel identity. 

Lemma 13. Let the creation-operator approximants \( A_{k_\tau} \) be defined in terms of operator families \( A_{1_{\beta}} \) and \( A_{2_{\beta}} \) which are localized in the standard double cone \( \mathcal{C}_R \) (\( R > 0 \)). For any \( x_1, x_2 \in \mathbb{R}^4 \), we have

\[
\left\| E_\Omega^\perp [A_{1_{\tau}}(x_1^*), A_{2_{\tau}}(x_2)]\Omega \right\|^2 \leq \frac{C(R + |x_2 - x_1|_\nu)}{|\tau|^9} \cdot \left\| A_{1_{\beta}(\tau)} \right\|_{\text{AHR}} \left\| A_{2_{\beta}(\tau)} \right\|_{\text{AHR}},
\]

(25)

where \( |x|_\nu := |x^0| + |x| \). Here \( \kappa = 3/2 \) in the case of \( m > 0 \) and for \( m = 0 \) we can choose \( \kappa = 1 - \epsilon \) for any \( \epsilon > 0 \) with \( C \) depending on \( \epsilon \) and the wave packets \( f_k \).

Proof. By translation invariance, it is sufficient to establish the estimate for the relative translation by \( x := x_2 - x_1 \). We may express the norm square as a vacuum expectation value by writing

\[
\left\| E_\Omega^\perp [A_{1_{\tau}}(x_1^*), A_{2_{\tau}}(x_2)]\Omega \right\|^2 = \left\langle \Omega, [A_{2_{\tau}}(x)^*, A_{1_{\tau}}] E_\Omega^\perp [A_{1_{\tau}}^*, A_{2_{\tau}}(x)]\Omega \right\rangle
\]

\[
= \int d^4x_1 \ldots d^4x_4 \ f_{2_{\tau}}(x_1) f_{1_{\tau}}(x_2) f_{1_{\tau}}(x_3) f_{2_{\tau}}(x_4) K(\tau, x, x_1, \ldots, x_4)
\]

\[
\leq \int d^4x_1 \ldots d^4x_4 \ |f_{2_{\tau}}(x_1)| |f_{1_{\tau}}(x_2)| |f_{1_{\tau}}(x_3)| |f_{2_{\tau}}(x_4)| |K(\tau, x, x_1, \ldots, x_4)|,
\]

(26)

where due to time-translation invariance the matrix element \( K \) can be written as

\[
K := \left\langle \Omega, [A_{2_{\tau,x}}(x_1^*), A_{1_{\tau}}(x_2)] E_\Omega^\perp [A_{1_{\tau}}(x_3^*), A_{2_{\tau,x}}(x_4)]\Omega \right\rangle.
\]

For compact notation, we introduced the abbreviation \( A_{2_{\beta,x}} := \alpha_x(A_{2_{\beta}}) \) and we suppressed the \( \tau \)-dependence of \( \beta = \beta(\tau) \).

Now we can estimate \( K \) by combining its support properties resulting from locality (HK2) with the space-like decay estimates from Theorem 11 in a manner which seems to be originally due to Buchholz [Bu77]. More precisely, by covariance, isotony and the geometry of double cones, the standard double cone \( \mathcal{C}_{R^2 + |x|_\nu} \) provides a localization region for the translated operator family \( A_{2_{\beta,x}} \). Therefore \( K \) can only be non-zero if

\[
|x_1 - x_2| \leq R_1 + R_2 + |x|_c
\]

\[
|x_3 - x_4| \leq R_1 + R_2 + |x|_c
\]

(27)

are both satisfied. This (for fixed \( x \)) finite restriction on the relative differences \( x_1 - x_2 \) and \( x_3 - x_4 \) now allows for successfully estimating the integrand of (26) for large enough relative distance \( x_2 - x_4 \) “across” \( E_\Omega^\perp \) by means of Theorem 11. Restricting the integral (26) to the region subject to the constraints (27), which we shall denote by \( M \subset (\mathbb{R}^3)^4 \), we find that for points \( (x_1, \ldots, x_4) \in M \), the two appearing commutators can be localized in suitably translated double cones, whose radii can be simultaneously bounded from above by \( R' := 2(R_1 + R_2) + |x|_c \), i.e.

\[
C_1 := [A_{2_{\beta,x}}(x_1^*), A_{1_{\beta}}(x_2)] \in \mathfrak{A}(O_{x_2}), \quad O_{x_2} := \mathcal{C}_{R'} + (0, x_2),
\]

\[
C_2 := [A_{1_{\beta}}(x_3^*), A_{2_{\beta,x}}(x_4)] \in \mathfrak{A}(O_{x_3}), \quad O_{x_3} := \mathcal{C}_{R'} + (0, x_3).
\]
Note that $C_1$ and $C_2$ are both differentiable by the product rule, as a consequence of the assumed differentiability of the families $A_{k\beta}$. To apply Theorem 11 we subdivide $M$ into the region $M_1 := \{(x_1, \ldots, x_4) \in M : |x_2 - x_3| > 2R'\}$ and its complement $M_2 := M \setminus M_1$ and write

$$\left\| E_{21}^{1}[A_{1\tau}, A_{2\tau}(x)] \Omega \right\|^2 \leq I_{M_1} + I_{M_2},$$

where $I_{M_k}$ denotes the integration part of (26) over the subregion $M_k$. On $M_1$ we have by Theorem 11,

$$|K| \leq \frac{CAHR(2R')^3}{|x_2 - x_3|^2} C_A, \quad C_A := \|C_1\|_{AHR} \|C_2\|_{AHR} \leq 4 \|A_{1\beta}\|^2_{AHR} \|A_{2\beta}\|^2_{AHR}.$$

Also note that trivially $|K| \leq C_A$ holds everywhere. Here we made use of

$$\|C_2\|_{AHR} \leq \|A_1^* \cdot A_2\| + \|\partial_0 A_1^* \cdot A_2\| + \|A_1^* \cdot \partial_0 A_2\| \leq 2 \|A_1^*\| \|A_2\| + \|\partial_0 A_1^*\| \|A_2\| + \|A_1^*\| \|\partial_0 A_2\|,$$

and similarly for $\|C_1\|_{AHR}$, where we suppressed dependencies on $\beta$, $x$ and $x_k$. This allows us to estimate

$$I_{M_1} = \int_{M_1} d^3x_1 \cdots d^3x_4 |f_{2\tau}(x_1)| |f_{1\tau}(x_2)| |f_{1\tau}(x_3)| |f_{2\tau}(x_4)| \cdot |K(\tau, x, x_1, \ldots, x_4)|$$

$$\leq \int_{M_1} d^3x_1 \cdots d^3x_4 |f_{2\tau}(x_1)| |f_{1\tau}(x_2)| |f_{1\tau}(x_3)| |f_{2\tau}(x_4)| \cdot \frac{CAHR(2R')^3}{|x_2 - x_3|^2} C_A$$

$$= C_{AHR} C_A (2R')^3 \int_{|x_2 - x_3| > 2R'} d^3x_2 d^3x_3 \frac{|f_{1\tau}(x_2)| |f_{1\tau}(x_3)|}{|x_2 - x_3|^2} \int_{|x_1 - x_2| < R'} d^3x_1 \int_{|x_3 - x_4| < R'} d^3x_4 |f_{2\tau}(x_4)|$$

$$\leq (2R')^9 \|f_{2\tau}\|_\infty^2 \cdot C_{AHR} C_A \cdot \int_{|x_2 - x_3| > 2R'} d^3x_2 d^3x_3 \frac{|f_{1\tau}(x_2)| |f_{1\tau}(x_3)|}{|x_2 - x_3|^2} \frac{1}{|x_2 - x_3|^2}. \quad (28)$$

Here and below, all appearing $p$-norms ($1 \leq p \leq \infty$) are on $L^p(\mathbb{R}^3)$-spaces associated to spatial smearing functions. We proceed by first estimating the $d^3x_3$ subintegral for fixed $x_2$ using Cauchy-Schwarz (all integrals below over $\{x_3 \in \mathbb{R}^3 : |x_2 - x_3| > 2R'\}$)

$$\int d^3x_3 \frac{|f_{1\tau}(x_3)|}{|x_2 - x_3|^2} \leq \|f_{1\tau}\|_2 \cdot \left( \int \frac{d^3x_3}{|x_2 - x_3|^4} \right)^{1/2} \leq C_{R^{-1}} \|f_{1\tau}\|_2.$$

Here both terms are uniformly bounded in $\tau$, by the Plancherel identity or explicit computation\textsuperscript{14}, respectively.

Plugging this into the remaining $d^3x_2$-integration in (28), we have now shown that

$$I_{M_1} \leq C_{AHR} C_A C_{R^{-1}} (2R')^9 \|f_{2\tau}\|_\infty^2 \|f_{1\tau}\|_2 \|f_{1\tau}\|_1.$$

On $M_2$ we estimate similarly using $|K| \leq C_A$,

$$I_{M_2} \leq \int_{M_2} d^3x_1 \cdots d^3x_4 |f_{2\tau}(x_1)| |f_{1\tau}(x_2)| |f_{1\tau}(x_3)| |f_{2\tau}(x_4)| C_A$$

$$\leq C_A (2R')^9 \|f_{2\tau}\|_\infty^2 \|f_{1\tau}\|_1.$$

The result now follows from Proposition 12.\hfill \square

\textsuperscript{14}Performing the second integral in spherical coordinates around $x_2$ leads to the radial integration beginning at $2R' > R > 0$, which can be estimated uniformly in $|x|_c$ by a finite constant $C_{R^{-1}}$.\hfill \square
6 Consequences of the clustering estimate

With the clustering estimate for the operators $A_{k\tau}$ from Lemma 13 at hand, it is straightforward to prove clustering of the creation operators $B_{k\tau}$ for Reeh-Schlieder families of finite degree.

**Proposition 14.** For uniformly differentiable Reeh-Schlieder families $A_{1\beta}$, $A_{2\beta}$, and regular Klein-Gordon solutions $f_1$, $f_2$ of mass $m \geq 0$, we have

$$\left\| E_{\Omega}^\dagger B_{1\tau}^* B_{2\tau} \Omega \right\| \leq \frac{C}{|\tau|^{n/2}} \left\| A_{1\beta(\tau)} \right\|_{\text{AHR}} \left\| A_{2\beta(\tau)} \right\|_{\text{AHR}}.$$  

Here, $C$ depends on $\chi$, localization regions of $A_{k\beta}$, wave packets, and $\kappa$ (see Lemma 13).

**Proof.** As $B_{1\tau}^* \Omega = 0$, we can replace the product $B_{1\tau}^* B_{2\tau}$ acting on the vacuum by the commutator $[B_{1\tau}^*, B_{2\tau}]$. Making use of $B_{k\tau} = A_{k\tau}(\chi)$, $\chi \in \mathcal{F}(\mathbb{R}^4)$, and Lemma 13, we obtain

$$\begin{align*}
\left\| E_{\Omega}^\dagger [B_{1\tau}^* B_{2\tau}] \Omega \right\| &\leq \int d^4x_1 d^4x_2 \left| \chi(x_1) \right| \left| \chi(x_2) \right| \left\| E_{\Omega}^\dagger [A_{1\tau}(x_1), A_{2\tau}(x_2)] \Omega \right\| \\
&\leq \int d^4x_1 d^4x_2 \left| \chi(x_1) \right| \left| \chi(x_2) \right| \frac{C'(R + |x_1 - x_2|)^{3/2}}{|\tau|^{n/2}} \left\| A_{1\beta(\tau)} \right\|_{\text{AHR}} \left\| A_{2\beta(\tau)} \right\|_{\text{AHR}} \\
&= \frac{C}{|\tau|^{n/2}} \left\| A_{1\beta(\tau)} \right\|_{\text{AHR}} \left\| A_{2\beta(\tau)} \right\|_{\text{AHR}}.
\end{align*}$$  

For Reeh-Schlieder families of finite degree Proposition 14 simplifies further, yielding a constraint for admissible choices of scaling. In the following $\gamma$ always denotes the (finite) largest appearing degree, i.e. $\left\| A_{k\beta} \right\|_{\text{AHR}} \leq \beta^{-\gamma}$ for small enough $\beta > 0$ and all $k = 1, \ldots, n$. From now on we will also adopt the canonical scaling $\beta(\tau) := |\tau|^{-\mu}$, $\mu > 0$.

**Corollary 15.** Let the Reeh-Schlieder families $A_{1\beta}$, $A_{2\beta}$ have finite degrees. Under the assumptions of Proposition 14, there exists a $C > 0$ such that for large enough $\tau$ we have

$$\left\| E_{\Omega}^\dagger B_{1\tau}^* B_{2\tau} \Omega \right\| \leq C |\tau|^{2\gamma \mu - \kappa/2},$$  

Consequently for any $0 < \mu < \frac{\kappa}{2\gamma}$ we obtain

$$\lim_{\tau \to \infty} E_{\Omega}^\dagger B_{1\tau}^* B_{2\tau} \Omega = 0.$$  

**Proof.** Follows immediately from inserting $\left\| A_{k\beta(\tau)} \right\|_{\text{AHR}} \leq C' |\tau|^{-\gamma} = C' |\tau|^{-\mu}$ into the estimate of Proposition 14.

While Corollary 15 will be sufficient to establish the Fock structure of scattering states in Section 8, our proof of convergence relies on an extension of this result, which is concerned with the case of multiple creation operators. The resulting Lemma 16 combines energy bounds and clustering estimates in a novel way. It may be considered our main technical result.

**Lemma 16** (multi-operator clustering). For $\tau_1, \ldots, \tau_n \in \mathbb{R}$ denote by $|\tau_{\text{min}}| > 0$ and $|\tau_{\text{max}}|$ the minimum and maximum of absolute values $|\tau_k|$, $(1 \leq k \leq n)$, respectively. Then for large enough $\tau_{\text{min}}$

$$\left\| E_{\Omega}^\dagger \left( \prod_{k=1}^n B_{k\tau_k}^* B_{k\tau_k} \right) \Omega \right\| \leq C |\tau_{\text{max}}|^{2n\gamma \mu} \cdot |\tau_{\text{min}}|^{-\kappa/2}. \quad (29)$$  

The constant $C$ is independent of the $\tau_k$, but depends on the number of pairs $n$, wave packets, Reeh-Schlieder families, and the smearing function $\chi$.  

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Proof. We will show by induction that 

$$
\left\| E^\perp_\Omega \left( \prod_{k=1}^n B_{k\tau_k}^* B_{k\tau_k} \right) \Omega \right\| \leq C \sum_{j=1}^n \left( \prod_{k=1}^{j-1} \left\| E(\Delta)B_{k\tau_k}^* B_{k\tau_k} E(\Delta) \right\| \right) \left\| E^\perp_\Omega B_{j\tau_j}^* B_{j\tau_j} \Omega \right\| ,
$$

(30)

where $\Delta \subset \mathbb{R}^4$ is a large enough compact set depending on $\text{supp} \chi$ and the number of pairs $n \in \mathbb{N}$. From this we obtain by applying 2-operator clustering (Corollary 15), the energy bound of Proposition 5, and the finite-degree Reeh-Schlieder estimates that for large enough $|\tau_{\text{min}}|$, (29) holds as claimed. For $n = 1$, statement (30) has been established in Corollary 15. Assuming that (30) holds for $n - 1$ pairs, we write 

$$
\left\| E^\perp_\Omega \left( \prod_{k=1}^n B_{k\tau_k}^* B_{k\tau_k} \right) \Omega \right\| = \left\| E^\perp_\Omega \left( \prod_{k=1}^{n-1} B_{k\tau_k}^* B_{k\tau_k} \right) (E_\Omega + E^\perp_\Omega) B_{n\tau_n}^* B_{n\tau_n} \Omega \right\|
$$

$$
\leq \left\| E^\perp_\Omega \left( \prod_{k=1}^{n-1} B_{k\tau_k}^* B_{k\tau_k} \right) E^\perp_\Omega B_{n\tau_n}^* B_{n\tau_n} \Omega \right\|
$$

$$
+ \left\| E^\perp_\Omega \left( \prod_{k=1}^{n-1} B_{k\tau_k}^* B_{k\tau_k} \right) E^\perp_\Omega B_{n\tau_n}^* B_{n\tau_n} \Omega \right\| .
$$

(31)

Now we would like to estimate the second term by 2-operator clustering (Corollary 15). Regarding the applicability of energy bounds from Proposition 5, we note that by Proposition 3 (iii), $B_{n\tau_n}^* B_{n\tau_n}$ has compact energy-momentum transfer $\Delta' := \text{supp} \chi - \text{supp} \chi$. Therefore we can insert an energy-momentum projection onto $\Delta'$ (which commutes with $E^\perp_\Omega$) and estimate 

$$
\left\| E^\perp_\Omega \left( \prod_{k=1}^{n-1} B_{k\tau_k}^* B_{k\tau_k} \right) E(\Delta') E^\perp_\Omega B_{n\tau_n}^* B_{n\tau_n} \Omega \right\|
$$

$$
\leq \left\| E^\perp_\Omega \left( \prod_{k=1}^{n-1} B_{k\tau_k}^* B_{k\tau_k} \right) E(\Delta') \right\| \cdot \left\| E^\perp_\Omega B_{n\tau_n}^* B_{n\tau_n} \Omega \right\|
$$

$$
\leq \left( \prod_{k=1}^{n-1} \left\| E(\Delta)B_{k\tau_k}^* B_{k\tau_k} E(\Delta) \right\| \right) \cdot \left\| E^\perp_\Omega B_{n\tau_n}^* B_{n\tau_n} \Omega \right\| ,
$$

where we have chosen the compact set $\Delta \subset \mathbb{R}^4$ large enough (depending on $n$) to contain the sum of the energy-momentum transfers differences $\Delta'$ of all creation-annihilation operator pairs. Similarly we estimate the first term in (31) by making use of the one-dimensional nature of the projection $E_\Omega$, and the induction assumption,

$$
\left\| E^\perp_\Omega \left( \prod_{k=1}^{n-1} B_{k\tau_k}^* B_{k\tau_k} \right) E_\Omega B_{n\tau_n}^* B_{n\tau_n} \Omega \right\| = \left\| B_{n\tau_n} \Omega \right\|^2 \cdot \left\| E^\perp_\Omega \left( \prod_{k=1}^{n-1} B_{k\tau_k}^* B_{k\tau_k} \right) \Omega \right\|
$$

$$
\leq C \cdot \sum_{j=1}^n \left\| E^\perp_\Omega B_{j\tau_j}^* B_{j\tau_j} \Omega \right\| \cdot \left( \prod_{k=1}^{j-1} \left\| E(\Delta)B_{k\tau_k}^* B_{k\tau_k} E(\Delta) \right\| \right) .
$$

Here we also made use of the fact that $\left\| B_{n\tau_n} \Omega \right\| \leq C$ is uniformly bounded in $\tau_n$ by convergence to the corresponding single-particle state (see Proposition 3 (vi)). Taken together, these two estimates complete the induction step. \qed

A useful consequence of multi-operator clustering, which will be important for us later, is the boundedness of scattering-state approximants, i.e. vectors resulting from
iterated application of creation-operators to the vacuum. In fact, a similar result was used by Buchholz for the collision theory of massless bosons [Bu77]. While the proofs of Buchholz’ results can be simplified⁴ using methods from harmonic analysis [Bu90a], our construction is based on operator families $A_\beta$ with diverging norms in the limit $\beta \to 0$. This norm growth will be inherited by energy bounds for creation operators, if they are derived by means of Proposition 5. In the vacuum sector of a local theory however, we can establish uniform estimates on scattering-state approximants by relying on the good behaviour of $A_\beta \Omega$ via the previously established clustering properties, similarly as in [Bu77].

**Corollary 17.** Assume disjoint velocity supports. For any scaling $0 < \mu < \frac{\kappa}{\tau(n-1)}$, there exists a $C > 0$, such that for all sufficiently large $\tau \in \mathbb{R}$ and all $\tau_k$ from the corresponding interval spanned by $\tau$ and $\tau + \rho \tau$,

$$\|B_{\tau_1} \ldots B_{\tau n} \Omega\| \leq C,$$

with $\rho$ as in Theorem 9 (for $n = 1$, any $\mu \in (0, \infty)$ is admissible).

**Proof.** The proof is by induction on the number of particles $n$. For $n = 1$, the claim follows by convergence to the corresponding single-particle state as proven in Proposition 3 (ii). For the general case it will be sufficient to establish the claim for large enough $|\tau|$, as can be seen from the simple polynomial estimate of Proposition 3. Let us now assume that the statement holds for $n$ particles. For simplicity we set $B_k := B_{k \tau_k}$ and write

$$\|B_1 \ldots B_{n+1} \Omega\|^2 = \langle \Omega, B_{n+1}^* B_1 \ldots B_{n+1} \Omega \rangle = \langle \Omega, B_{n+1}^* B_{n+1} B_{n+1}^* B_1 \ldots B_n \Omega \rangle + \langle \Omega, B_{n+1}^* [B_{n}^* \ldots B_1^* B_1 \ldots B_n, B_{n+1}] \Omega \rangle,$$

where the absolute value of the second term is bounded, as it vanishes for $|\tau| \to \infty$ for any choice of scaling by the rapid decay of commutators (Theorem 9). This decay can compensate the norm growth of the creation-operator approximants, which is at most polynomial — even when using the naive estimate of Proposition 3.

Therefore it is sufficient to establish boundedness of the matrix element

$$\langle \Omega, B_{n+1}^* B_1 \ldots B_{n+1}^* B_{n}^* B_1 \ldots B_n \Omega \rangle = \langle \Omega, B_{n+1}^* (E_{\Omega} + E_{\Omega}^{\perp}) B_{n+1}^* B_{n+1} B_{n+1}^* B_1 \ldots B_n \Omega \rangle = \|B_{n+1} \Omega\|^2 \|B_1 \ldots B_n \Omega\|^2 + \langle \Omega, B_{n+1}^* B_{n+1} E_{\Omega} B_{n}^* B_{n} B_{n}^* B_1 \ldots B_n \Omega \rangle \rangle.$$

The first term of (32) provides the dominant contribution in the limit $|\tau| \to \infty$: its two factors are bounded by the induction assumption and the one-particle case. The second term can be written as the sum of $\langle \Omega, B_{n+1}^* B_{n+1} E_{\Omega} B_{n}^* B_{n} \ldots B_1 \Omega \rangle$ and further matrix elements involving at least one commutator of operators involving disjoint velocity supports. As before, the latter are rapidly decreasing by Theorem 9. We can conclude the proof by applying the Cauchy-Schwarz inequality to the remaining term

$$\left| \langle \Omega, B_{n+1}^* B_{n+1} E_{\Omega} B_{n}^* B_{n} \ldots B_1 \Omega \rangle \right| \leq \left\| E_{\Omega} B_{n+1}^* B_{n+1} \Omega \right\| \cdot \left\| E_{\Omega} B_{n}^* B_{n} \ldots B_1 \Omega \right\|,$$

where both factors vanish in the limit $|\tau| \to \infty$ for any sufficiently small choice of scaling $\mu$ by Lemma 16.

⁴See e.g. [AD15].
7 Convergence of scattering state approximants

For this section we adopt the standing assumptions that \( A_{1\beta}, \ldots, A_{n\beta} \) are Reeh-Schlieder families of finite degree and we take \( f_1, \ldots, f_n \) to be regular positive-energy Klein-Gordon solutions of the corresponding mass with pairwise disjoint velocity supports.

Theorem 18. Let the Reeh-Schlieder families \( A_{1\beta}, \ldots, A_{n\beta} \) have degrees less than some finite value \( \gamma > 0 \) and take a scaling exponent \( \mu \in (0, \frac{\kappa}{4(n-1)\gamma}) \) (\( \kappa \) as in Lemma 13).

(i) The family \( \Psi_\tau := B_{1\tau} \ldots B_{n\tau}, \Omega \) is convergent in norm as \( \tau \to \pm \infty \).

(ii) The limit is independent of the choice of \( \mu, A_{k\beta} \) and \( f_k \) within the specified restrictions, as long as the associated operators \( B_{k\tau} \) create on the vacuum the same family of single-particle states \( \Psi_k^{(1)} = \lim_{\tau \to \pm \infty} B_{k\tau} \Omega \).

Avoiding differentiability assumptions on \( A_{k\beta} \) with respect to the parameter \( \beta \), we will proceed by a discrete variant of Cook’s method, thereby reducing the convergence control over the convergence of the single-particle problem by Proposition 3 (vi).

The restrictions on the time differences to obtain rapid decay of commutators in Theorem 9 suggests to consider the restrictions of \( \Psi_\tau \) to sequences

\[
\tau_k = (1 + \rho)^k \tau_0, \quad \tau_0 \neq 0 \text{ arbitrary,}
\]

and \( \rho > 0 \) depending on the separation of velocity supports as explained in Theorem 9.

As preparation for proving Theorem 18 we will first show that we can relate the norm of differences \( \Psi_{\tau_2} - \Psi_{\tau_1} \) to corresponding single-particle expressions \( \|B_{k\tau_2} \Omega - B_{k\tau_1} \Omega\| \) at least “locally”, i.e. if we place sufficient restrictions on the differences \( |\tau_2 - \tau_1| \). We will give a unified account for proving both parts of Theorem 18 by comparing the scattering state approximants associated to two possibly distinct families of creation operators with comparable velocity supports. Thereto let \( A_{k\beta}, A'_{k\beta} \in A(\Omega) \) be uniformly differentiable Reeh-Schlieder families of finite degree, and choose regular positive-energy Klein-Gordon solutions \( f_1, \ldots, f_n \) and \( f_1', \ldots, f_n' \) of mass \( m \geq 0 \) such that all pairs with \( j \neq k \) (including mixed pairs \( f_j, f_k' \)) have disjoint velocity supports. We denote the corresponding creation operators by \( B_{k\tau}, B'_{k\tau} \) and set

\[
\Psi_\tau := B_{1\tau} \ldots B_{n\tau} \Omega, \quad \Psi'_\tau := B'_{1\tau} \ldots B'_{n\tau} \Omega.
\]

Remark 19 (change of scaling). Anticipating also the proof of Theorem 18 (ii), we may also allow the creation operator families \( B_{k\tau} \) and \( B'_{k\tau} \) to be defined using distinct choices of scaling \( \beta_k(\tau) := |\tau|^{-\mu_k}, \beta'_k(\tau) := |\tau|^{-\mu'_k} \). On the first reading, this detail can safely be ignored, but it is easily seen that the statement and proof of Lemma 20 can even be kept invariant under this generalization if we simply denote the smallest appearing scaling exponent by \( \mu := \min\{\mu_k, \mu'_k \ (1 \leq k \leq n)\} > 0 \). The required extensions of Theorem 9, Lemma 16, and Corollary 17 follow directly by similar considerations.

Lemma 20. Take \( \rho > 0 \) as given in Theorem 9 (using the smallest value suitable for all disjoint pairs of velocity supports), and choose sufficiently small scaling \( \mu > 0 \) (cf. Corollary 17). Then there exist constants \( C_1, C_2 > 0 \), such that for sufficiently large \( |\tau| \) \( > 0 \) and any subsequent choice of \( \tau_1, \tau_2 \) from the interval spanned by \( \tau \) and \( (1 + \rho)\tau \), we have

\[
\|\Psi_{\tau_2} - \Psi'_{\tau_2}\| \leq C_1 \sum_{k=1}^{n} \|B_{k\tau_2} \Omega - B'_{k\tau_1} \Omega\| + C_2 |\tau|^{\mu - \kappa/4}.
\]
Proof. For \( n = 1 \) the statement is trivial. For \( n \geq 2 \) we can estimate telescopically
\[
\| \Psi_{t_2} - \Psi'_{t_1} \| \leq \sum_{k=1}^{n} \| B_{t_2} \ldots B_{k-1} \cdot (B_{k \tau_2} - B'_{k \tau_1}) B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \|.
\]
The claim is obtained if the following estimate can be established for each \( 1 \leq k \leq n \),
\[
\| B_{t_2} \ldots B_{k-1} \cdot (B_{k \tau_2} - B'_{k \tau_1}) B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \|^2 \leq C_1 \| B_{k \tau_2} \Omega - B'_{k \tau_1} \Omega \|^2 + C_2 \| \tau \|^{2\gamma \mu - \kappa / 2}.
\] (34)

We will prove this inequality by making use of the rapid decay of restricted non-equal time commutators together with the energy bound and clustering. Introducing the abbreviation \( \Delta_r B_k := B_{k \tau_2} - B'_{k \tau_1} \), we can write
\[
\| B_{t_2} \ldots B_{k-1} \cdot (\Delta_r B_k) B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \|^2 = \langle \Omega, B'_{n \tau_1} \ldots B'_{k+1 \tau_1} (\Delta_r B_k)^* B'_{k-1 \tau_2} \ldots B'_{t_2} B_{t_2} \ldots B_{k-1} \cdot (\Delta_r B_k) B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \rangle
\leq \langle \Omega, B'_{n \tau_1} B_{t_2} \ldots B'_{k+1 \tau_1} B_{k-1} \cdot (\Delta_r B_k)^* B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \rangle + C_M \| \tau \|^{-M},
\] (35)
and the rapidly decreasing error can be subsumed into the \( C_2 \)-term of (34). To obtain eq. (35), we made multiple use of the non-equal-time commutator estimate\(^{10}\) of Theorem 9, which is sufficiently strong for overcompensating to any desired inverse polynomial order the asymptotic growth of the elementary estimate \( \| B_r \| \leq C_k (1 + |\tau|)^{N + \mu} \) and similar estimates for adjoints and primed operators (see Proposition 3).

The remaining term in (35) still contains the asymptotically dominant contribution, which we will now extract using the clustering estimate. Inserting an identity operator \((E_\Omega + E_{\Omega}')\) after \((\Delta_r B_k^*)(\Delta_r B_k)\) and making use of subadditivity and decay of commutators yields
\[
\| \langle \Omega, B'_{n \tau_1} B_{t_2} \ldots B'_{k+1 \tau_1} B_{k-1} \cdot (\Delta_r B_k)^* B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \rangle \| \leq \| E_\Omega B'_{n \tau_1} B_{t_2} \ldots B'_{k+1 \tau_1} B_{k-1} \cdot (\Delta_r B_k)^* B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \| \cdot \| (\Delta_r B_k^*)(\Delta_r B_k) \Omega \| + \| B_{t_2} \ldots B_{k-1 \tau_2} \cdot B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \|^2 \cdot \| (\Delta_r B_k) \Omega \|^2 + C_M \| \tau \|^{-M}.
\]
Both terms depend on the convergence speed of the single-particle problem, although — anticipating the results of Section 8 — we expect the second summand to be dominant for large \( \tau \): By boundedness of scattering state approximants (Corollary 17)
\[
\| B_{t_2} \ldots B_{k-1 \tau_2} \cdot B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \|^2 \leq C_1
\]
for suitable \( C_1 > 0 \). It remains to be shown that the first summand has the same asymptotics as the \( C_2 \)-term of (34). By the clustering result with multiple pairs of creation- and annihilation-operator approximants of Lemma 16, we obtain that
\[
\| E_\Omega B'_{n \tau_1} B_{t_2} \ldots B'_{k+1 \tau_1} B_{k-1} \cdot (\Delta_r B_k)^* B'_{k+1 \tau_1} \ldots B'_{n \tau_1} \Omega \| \leq C_2 \| \tau \|^{2(n-1)\gamma \mu - \kappa / 2},
\]
which also made use of the time restriction yielding \( |\tau| \leq |\tau_k| \leq (1 + \rho) |\tau| \). The second factor is estimated making use of the energy bound,
\[
\| (\Delta_r B_k^*)(\Delta_r B_k) \Omega \| = \| (\Delta_r B_k^*) E(\Delta) (\Delta_r B_k) \Omega \|
\leq \| (\Delta_r B_k^*) E(\Delta) \| \cdot \| (\Delta_r B_k) \Omega \|
\leq C_3 \| \tau \|^{\gamma \mu} \leq C_3 (1 + \rho) \gamma \mu \| \tau \|^{\gamma \mu} =: C_3' \| \tau \|^{\gamma \mu},
\]
where the energy-momentum projection onto the compact set \( \Delta := \text{supp} \hat\chi \) can be inserted due to \( \Delta_r B_k \Omega \in E(\Delta) \mathcal{H} \). Altogether we obtain (34), completing the proof. \( \square \)

\(^{10}\) For the status of Theorem 9 in the context of non-equal scaling, cf. Remark 19 and Footnote 13.
The convergence of scattering state approximants $\Psi_\tau$ is now easily established by iterated application of Lemma 20.

**Proof of Theorem 18. Ad (i).** We estimate by writing a telescopic sum and making use of subadditivity of the norm,

$$\|\Psi_{\tau_L} - \Psi_{\tau_0}\| \leq \sum_{k=1}^{L} \|\Psi_{\tau_k} - \Psi_{\tau_{k-1}}\|.$$  

We have by construction that $\tau_k, \tau_{k-1}$ are contained in the interval spanned by $\tau_{k-1}$ and $(1 + \rho)\tau_{k-1}$. Thus Lemma 20 is applicable with $B_{k\tau} = B'_{k\tau}$. Fixing the scaling parameter $\mu > 0$ such that $\delta := \kappa/4 - n\gamma \mu > 0$, all assumptions of Lemma 20 are satisfied and we obtain

$$\|\Psi_{\tau_L} - \Psi_{\tau_0}\| \leq \sum_{k=1}^{L} \left( C_1 \sum_{j=1}^{n} \|B_{j\tau_k} \Omega - B_{j\tau_{k-1}} \Omega\| + C_2 |\tau_{k-1}|^{-\delta} \right).$$  

(36)

Now, the single-particle convergence property of the Reeh-Schlieder families implies

$$\|B_{j\tau_k} \Omega - B_{j\tau_{k-1}} \Omega\| \leq \|B_{j\tau_k} \Omega - \Psi_j^{(1)}\| + \|\Psi_j^{(1)} - B_{j\tau_{k-1}} \Omega\| \leq C |\tau_{k-1}|^{-\mu},$$

where $\Psi_j^{(1)} = \lim_{\tau \to \pm \infty} B_j \tau \Omega$. Applying this estimate and inserting $\tau_k = (1 + \rho)k \tau_0$, we can take care of both terms in (36) by writing

$$\|\Psi_{\tau_L} - \Psi_{\tau_0}\| \leq C' \sum_{k=1}^{L} |\tau_{k-1}|^{-\mu'} = C' \tau_0^{-\mu'} \sum_{k=1}^{L} (1 + \rho)^{-\mu'(k-1)},$$  

(37)

with $\mu' := \min(\mu, \delta)$. Clearly, the geometric series is convergent for $L \to \infty$. Independence of the limit from the choice of the sequence $\tau_k$, i.e. convergence of $\Psi_\tau$ as a function of the continuous parameter $\tau$, may be inferred from a second invocation of Lemma 20 or directly from (37).

Ad (ii). This is another direct consequence of Lemma 20, which implies for equal times but distinct creation operators, with possibly distinct choices of scaling in the allowed region, that

$$\|\Psi_{\tau_L} - \Psi'_{\tau_0}\| \leq \sum_{j=1}^{n} \|B_{j\tau} \Omega - B'_{j\tau} \Omega\| + C_2 |\tau|^{-\delta},$$

where as before $\delta := \kappa/4 - n\gamma \mu > 0$. If $\lim_{\tau} B_{j\tau} \Omega = \lim_{\tau} B'_{j\tau} \Omega$, we obtain that the limits of $\Psi_\tau$ and $\Psi'_{\tau}$ coincide and that they are invariant under changes of scaling as claimed. \hfill $\Box$

### 8 Fock structure of scattering states

Finally, we want to establish the Fock structure of scattering states, which provides a simple formula for computing scalar products of any two scattering states in terms of their single-particle components. An important consequence is the non-vanishing of the limits defining the scattering states and it is the essential ingredient to establish the extension of wave operators to the full asymptotic Fock spaces (cf. [Dy09] App. A). With the clustering relation of creation-operators of Corollary 15 at hand, the arguments leading to the Fock structure of scattering states are well-known and we can not refrain from
rephrasing them, e.g. from [Dy05]. We will use the abbreviation \([n] := \{1, 2, \ldots, n\} \subset \mathbb{N}\) for finite subsets of natural numbers and \(\mathfrak{S}_n\) denotes the symmetric group of degree \(n\) in its defining representation, i.e. acting on \([n]\).

We now consider two scattering state approximants \((n, n' \in \mathbb{N}_0)\)

\[
\Psi_\tau := B_{1\tau} \ldots B_{n\tau} \Omega, \quad \Psi'_\tau := B'_{1\tau} \ldots B'_{n'\tau} \Omega,
\]
such that \(B_{k\tau}\) and \(B'_{k\tau}\) have disjoint velocity supports within each family. Assuming finite Reeh-Schlieder degrees, the \textit{outgoing} and \textit{incoming} scattering states \(\Psi^\pm := \lim_{\tau \to \pm \infty} \Psi_{\tau}\), respectively, are well-defined by Theorem 18 for sufficiently small choices of scaling \(\beta(\tau) = |\tau|^{-\mu}\), \(\mu > 0\), and similarly for \(\Psi'^\pm := \lim_{\tau \to \pm \infty} \Psi'_{\tau}\). We denote the corresponding single-particle states by \(\Psi^{(1)}_\tau := \lim_{\tau \to \infty} B_{k\tau} \Omega\), \((1 \leq k \leq n)\) and \(\Psi'^{(1)}_{k'} := \lim_{\tau \to \infty} B'_{k'\tau} \Omega\), \((1 \leq k' \leq n')\).

**Theorem 21** (Fock structure). The scalar products of any two outgoing scattering states of the above form are given by\(^{17}\)

\[
\langle \Psi^+, \Psi'^+ \rangle = \delta_{nn'} \sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^n \left\langle \Psi^{(1)}_k, \Psi'^{(1)}_{\pi(k)} \right\rangle, \quad (38)
\]

and similarly for incoming states.

**Proof.** For simplicity we treat only the outgoing case \(\tau \to +\infty\). By continuity of the scalar product, the left-hand side of (38) can be written as the limit \(\tau \to \infty\) of

\[
\langle \Psi_\tau, \Psi'_\tau \rangle = \langle B_{1\tau} \ldots B_{n\tau} \Omega, B'_{1\tau} \ldots B'_{n'\tau} \Omega \rangle, \quad (39)
\]

where we can assume identical scaling \(\mu > 0\) for both sides by Theorem 18 (ii). Now we perform induction with respect to the number of particles \(n'\) (assuming without restriction that \(n' \geq n\)). For each \(n'\) and \(n = 0\), statement (38) is equivalent to \(\|\Omega\| = 1\) for \(n = 0\) and \(\langle \Psi'^+, \Omega \rangle = 0\) for \(n' > 0\). The latter follows from eq. (39) and the spectral support argument of Proposition 3 (v).

Assuming now that (38) holds for \(n - 1\) particles, one can show by means of Corollary 10 and Corollary 15 that, up to terms vanishing for \(|\tau| \to \infty\), (39) equals

\[
\sum_{k=1}^{n'} \langle \Omega, B'_{n'\tau} \ldots B'_{2\tau} B'_{1\tau} \ldots B'_{k-1\tau} B'_{k+1\tau} \ldots B'_{n'\tau} \Omega | B_{1\tau} \ldots B_{n\tau} \Omega \rangle \to \infty
\]

\[
\sum_{k=1}^{n'} \left(\delta_{n-1,n'-1} \sum_{\pi \in \mathfrak{S}_{n-1}(1,k)} \prod_{l=2}^n \left\langle \Psi^{(1)}_{l}, \Psi'^{(1)}_{\pi(l)} \right\rangle \right),
\]

where \(\mathfrak{S}_{n-1}(1,k)\) denotes the set of bijective maps \(\pi\) between the two sets of numbers \([n] \setminus \{1\}\) and \([n] \setminus \{k\}\) and convergence is inferred from the induction assumption.

Note that while \(\mathfrak{S}_{n-1}(1,k)\) is by itself not a group (its elements are maps between different sets and thus cannot be composed), it can nevertheless be identified with the subset of \(\pi \in \mathfrak{S}_n\) for which \(\pi(1) = k\). This implies that

\[
\lim_{\tau \to \infty} \langle \Psi_\tau, \Psi'_\tau \rangle = \delta_{nn'} \sum_{k=1}^n \sum_{\pi \in \mathfrak{S}_n} \prod_{l=1}^n \left\langle \Psi^{(1)}_l, \Psi'^{(1)}_{\pi(l)} \right\rangle = \delta_{nn'} \sum_{\pi \in \mathfrak{S}_n} \prod_{l=1}^n \left\langle \Psi^{(1)}_l, \Psi'^{(1)}_{\pi(l)} \right\rangle. \quad \Box
\]

\(^{17}\)As usual, the right-hand side of (38) is consistently interpreted for \(n > n'\), yielding vanishing scalar products also in this case (as a consequence of the vanishing Kronecker delta \(\delta_{nn'}\)).

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9 Conclusions and outlook

We have established the existence and Fock structure of scattering states corresponding to single-particle states $\Psi_1 \in E(H_m)$. This assumption requires the existence of a family of local operators $(A_\beta)_{\beta > 0} \subset \mathfrak{A}(O)$ such that

$$\|A_\beta \Omega - \Psi_1\| \leq \beta, \quad \|A_\beta\| \leq \beta^{-\gamma}. \quad \text{RS}$$

Beyond (RS) our method has no further dependence on the concrete mechanism (e.g. additional ergodic averaging as in [Dy05]) yielding a limit of $A_\beta \Omega$ in the single-particle space. We have seen that the Haag-Ruelle construction can be adapted, so that any finite degree $\gamma$ is feasible. Thus an arbitrarily strong polynomial growth of $\|A_\beta\|$ relative to the convergence of $A_\beta \Omega$ to the single-particle vector $\Psi_1$ can be handled.

As mentioned in the introduction, Assumption (RS) is readily verified in free field theory (cf. also Appendix C). Its status in concrete interacting models or within the general axiomatic framework is beyond the scope of the present work and poses an interesting problem for future research. We will briefly summarize our current understanding regarding the validity of conditions of strengthened Reeh-Schlieder type and also give some additional supporting arguments for our approach to the construction of scattering states. We shall refrain from going into technical details, as we intend to provide them elsewhere.

(a) Quantitative improvements in the construction of scattering states regarding the strength of condition (RS) are possible. Most notably in theories with lower mass gap one can show that already $(A_\beta)_{\beta > 0} \subset \mathfrak{A}(O)$,

$$\|E(\Delta)(A_\beta \Omega - \Psi_1)\| \leq C_\Delta \beta, \quad \ln \|A_\beta\| \leq \beta^{-\gamma}, \quad \text{RS}^\flat$$

is sufficient for establishing scattering theory. Here $\Delta \subset \mathbb{R}^4$ is an arbitrary compact set, and $C_\Delta > 0$ does not depend on $\beta$. Intuitively, the stronger norm increase in (RS$^\flat$) may be compensated by the exponential space-like clustering in these models.

(b) It was already pointed out that previous constructions of scattering states of embedded (massive) particles commonly need to assume additional regularity of the spectral measure near the particle masses. Here we briefly comment on the relation of such regularity assumptions to conditions of Reeh-Schlieder type. For spectral regularity according to Herbst, one requires there exist local operators $A \in \mathfrak{A}(O)$ such that in addition to a nonvanishing single-particle component $E_m A \Omega$, one has for a suitable $\epsilon > 0$ and all small enough $\delta > 0$, [Hrb71; Dy05]18

$$\left\| E(H_m^\delta \setminus H_m)A \Omega \right\| \leq C\delta^\epsilon, \quad \text{where} \quad H_m^\delta := \bigcup_{|\mu - m| < \delta} H_\mu,$$  \quad (H)

and that the set of single particle vectors obtained from such operators is dense in the single particle space $E_m$. Starting from an operator $A \in \mathfrak{A}(O)$ as in (H), one can show by a very crude but general construction using differential operators that there exists a dense set of single particle states $\Psi_1 \in E_m$, which are generated by operators satisfying (RS$^\flat$), with $\gamma > 0$ inversely proportional to the Herbst constant $\epsilon$ from (H). Here we do not even need to invoke the Reeh-Schlieder property — one may make use of the non-local nature of the energy-projection $E(\Delta)$ in condition (RS$^\flat$) to generate single-particle states.

18Weakened variants of (H) have also been discussed recently, see e.g. [Hrd13; DH14].
states (even if $\Delta$ is larger than a subset of the mass hyperboloid). Improving upon this result appears to require a more detailed quantitative understanding of the non-local correlations implied by the Reeh-Schlieder theorem, which may be model-dependent — cf. also Appendix C.

(c) We restricted our analysis to uniformly localized Reeh-Schlieder families solely for technical convenience. The present method may be refined to admit families $A_\beta \in \mathfrak{A}(C_{R_\beta})$ similarly as in (RS), but localized in double cones $C_{R_\beta}$ of polynomially growing radii $R_\beta := \beta^{-N}$ (for some $N > 0$).

A similar delocalization commonly enters in previous approaches via ergodic averaging prescriptions [Hrb71; Dy05; Hrd13; DH14]. Due to the geometrical limitations discussed in Section 4, this delocalization appears to necessitate Herbst-type spectral conditions [Hrb71] in these works. Allowing a weakened localization $A_\beta \in \mathfrak{A}(C_{R_\beta})$ might help to understand the relation of such spectral conditions to the Reeh-Schlieder condition (RS).

A more concrete investigation of (RS) can be carried out using the concept of polarization-free generators [BBS01]. In this setting, we can derive a wedge-local variant of the Reeh-Schlieder condition from the domain condition $\Omega \in D(T^{1+\epsilon})$ for some $\epsilon > 0$, where $T \geq 0$ denotes the self-adjoint part of the polar decomposition of a suitable polarization-free generator $G = UT$. With this input we can proceed as in free field theory and set $A_\beta := UTe^{-\beta T^*}$ to obtain wedge-local Reeh-Schlieder families of degree $\gamma = \epsilon^{-1}$. If a corresponding variant of Theorem 11 holds for oppositely localized pairs of such wedge-local operators, as it is the case in purely massive theories [Fre85], our results may be extended to yield a construction of two-particle scattering states for embedded Wigner particles.

In this setting, it is problematic to imitate the Haag-Ruelle construction by directly smearing polarization-free generators $G$ due to the complicated structure of the domains $D(G)$. It has been shown that even ostensibly weak temperateness assumptions with respect to the action of space-time translations on $D(G)$ imply triviality of scattering in massive theories on Minkowski space with spatial dimension $s > 1$ [BBS01]. Therefore it is a subtle question whether the above domain condition is compatible with non-trivial scattering.\textsuperscript{19}

A Notation and Conventions

For the Minkowski space-time metric we use the convention $k \cdot x := k^0 x^0 - \mathbf{k} \cdot \mathbf{x}$ for $k, x \in \mathbb{R}^4$. Accordingly, the Fourier transform of a Schwartz functions $f \in \mathcal{S}(\mathbb{R}^4)$ is defined by

$$\hat{f}(k) := \frac{1}{(2\pi)^2} \int \mathcal{d}^4 x \, e^{ik \cdot x} f(x). \quad (40)$$

The wave-packet $\tilde{f}$ of a regular Klein-Gordon solution $f$ (as defined in Section 3), is related to a corresponding partial (spatial) inverse transform of $f_t(x) := f(t, x)$ at $t = 0$ by a factor $(2\pi)^{3/2}$.

\textsuperscript{19}Preliminary computations suggest that $\Omega \in D(T^{1+\epsilon})$ could be fulfilled in certain 1+1-dimensional integrable models with non-temperate polarization free generators $G$ [CT15] [Yoh Tanimoto, private communications]. A definite assessment requires the construction of a Borchers triple for these models, which has not yet been completed at the time of writing of this work.
The Fourier transform on the extended space $x = (x, s)$ and space-time $z = (x^0, x, s)$ (see Appendix C) is defined for $f \in \mathcal{S}((\mathbb{R}^5)$ and $f \in \mathcal{S}((\mathbb{R}^4)$ by

$$\hat{f}(\omega, k, \mu) := \frac{1}{(2\pi)^{3/2}} \int d^3x \ e^{i\omega x^0 - ik \cdot x - i\mu s} f(x^0, x, s),$$

$$\hat{f}(k, \mu) := \frac{1}{(2\pi)^2} \int d^4x \ e^{-ik \cdot x - i\mu s} f(x, s).$$

For $x = (t, x) \in \mathbb{R}^4$ we write $A(x) := \alpha_x(A) := U(x)AU(x)$ and similarly for $\alpha_t(A)$ and $\alpha_{x}(A)$. By weak integration, these automorphisms of the global algebra induce for given $A \in \mathfrak{A}$ (regular) operator-valued distributions

$$A(f) := \int d^dx \ f(x) \alpha_x(A), \quad f \in \mathcal{S}((\mathbb{R}^4)$$

and similar distributions $A(g)$ are obtained for spatial smearing with $g \in \mathcal{S}((\mathbb{R}^3)$.

**B Uniformly almost-local operator families**

An operator $A \in \mathfrak{A}$ is *almost-local* if there exists for any $r > 0$ a double-cone localized operator $A_r \in \mathfrak{A}(\mathcal{C}_r)$, such that for each $N \in \mathbb{N}$ with a suitable constant $C_N$ we have

$$\|A - A_r\| \leq \frac{C_N}{(1 + r)^N}. \tag{41}$$

For certain families $(A_\beta) \subset \mathfrak{A}$ of almost-local operators, the behaviour of corresponding constants $C_{N,\beta}$ in (41) with respect to the parameter $\beta > 0$ can be quantified in a simple manner.

**Proposition 22.** Let $A_\beta \in \mathfrak{A}(\mathcal{O})$ ($\beta > 0$) be an operator family localized in a fixed bounded region $\mathcal{O} \subset \mathbb{R}^3$ and let $\chi \in \mathcal{S}((\mathbb{R}^4)$. Then the family of almost-local operators $B_\beta := A_\beta(\chi)$ is uniformly almost-local relative to $\|A_\beta\|$ in the following sense: for each $\beta > 0$ there are $B_{\beta,r} \in \mathfrak{A}(\mathcal{C}_r)$ ($r > 0$), such that for all $N \in \mathbb{N}$

$$\exists C_N > 0 \forall \beta > 0 : \|B_\beta - B_{\beta,r}\| \leq \frac{C_N \|A_\beta\|}{1 + r^N}. \tag{42}$$

Notably, the constants $C_N$ are uniform in $\beta$. This also implies

$$\int d^3x \ \| [B_\beta, B_{\beta}(x)] \| \leq C_{\chi,\mathcal{O}} \|A_\beta\|^2. \tag{43}$$

**Proof.** Let us assume for concreteness that $A_\beta \in \mathfrak{A}(\mathcal{C}_R)$ with the double-cone radius $R > 0$ fixed. As $\chi \in \mathcal{S}((\mathbb{R}^4)$, we obtain natural candidates for approximating local operators

$$B_{\beta,r} := \int d^4x \ \chi(x) A_\beta(x) \in \mathfrak{A}(\mathcal{C}_r)$$

(for $r \leq R$ we simply set $B_{\beta,r} = 0$). By the rapid decay of $\chi$, we get for $r > 2R$

$$\|B_\beta - B_{\beta,r}\| \leq \|A_\beta\| \cdot \int d^4x \ |\chi(x)| \leq \frac{C_N \|A_\beta\|}{1 + (r - R)^N} \leq \frac{C_{\chi,\mathcal{O}} \|A_\beta\|}{1 + r^N}.$$

Together with the trivial estimate $\|B_\beta\| \leq \|A_\beta\| \|\chi\|_1$ for $r \leq 2R$, this implies (42).
To obtain (43) we use an $|x|$-dependent local approximation $B_{\beta,r}$ under the integral: choosing $r = r(x) := |x|/2$ the commutator $[B_{\beta,r(x)}, B_{\beta,r(x)}^*(x)]$ will vanish by locality and thereby we have reduced the integrand to terms proportional to the approximation error. More explicitly we rewrite the left-hand side as

$$\int d^3x \left\| (B_{\beta} - B_{\beta,r(x)}^*) + B_{\beta,r(x)}^* (B_{\beta}(x) - B_{\beta,r(x)}^*(x)) + B_{\beta,r(x)}^2 \right\| .$$

After expanding the commutator (preserving the two differences in brackets) and utilizing subadditivity, $\| [B_{\beta,r(x)}, B_{\beta,r(x)}^*(x)] \|$ vanishes for all $x$ by construction (due to locality). All remaining terms will contain at least one difference $B_{\beta} - B_{\beta,r(x)}$ or its translate. Using (42) we can now directly estimate the integral,

$$\left\| [B_{\beta} - B_{\beta,r(x)}, B_{\beta,r(x)}^*(x)] \right\| \leq 2 \| B_{\beta} - B_{\beta,r(x)} \| \| B_{\beta,r(x)}^*(x) \| \leq \frac{2CN\|A_{\beta}\|^2}{1 + r^N}.$$

Taking $N$ sufficiently large we obtain convergence of the integral and (43).

\[\Box\]

C Reeh-Schlieder Families in Generalized Free Models

Let us briefly discuss the status of condition (RS) for noninteracting theories with embedded mass shell. Generalized free theories have proven useful to study Herbst-type spectral conditions (H) ([Dy05], Sec. 4, see also [Hor90, Ch. 3.3, esp. p. 264 ff.] for a general review), and we think that the following considerations might also give some hints concerning strengthened Reeh-Schlieder properties in interacting theories\textsuperscript{20}. The generalized free field $\phi(f)$, $f \in \mathcal{F}(\mathbb{R}^4)$, may be interpreted as a certain superposition of ordinary free fields $\phi_{\mu}(f)$ of mass $\mu \geq 0$ with weight measure $d\rho(\mu)$ describing the mass spectrum of the theory. For our purposes, $\rho$ should consist of a delta measure at the desired particle mass $m \geq 0$ and some continuous background spectrum. We will take

$$\rho := \delta_m + \rho_{\text{cont}}, \quad \rho_{\text{cont}}(\Delta) := \int_{\Delta \cap [0,m+1]} d\mu \frac{1}{|\mu - m|^{1-\epsilon}} + \alpha \lambda(\Delta), \quad (44)$$

for Borel sets $\Delta \subset [0, \infty)$, where $\lambda$ denotes Lebesgue measure. The parameter $\epsilon > 0$ controls the regularity in the vicinity of the particle mass, i.e. regarding the Herbst condition (H). Additionally, the support properties of $\rho$, governed by $\alpha \in \{0,1\}$, are of (perhaps unexpected) relevance for the Reeh-Schlieder problem.

On the bosonic Fock space $\mathcal{F}_\rho := \Gamma(\mathcal{H}_{1,\rho})$ over the single-particle space $\mathcal{H}_{1,\rho} := L^2(\mathbb{R}^3) \otimes L^2([0,\infty),d\rho)$ we obtain a Wightman field in terms of the Segal operators $\Phi_S(\psi) := (a^*(\psi) + a(\psi))/\sqrt{2}$, $\psi \in \mathcal{H}_{1,\rho}$, for real-valued test functions $f \in \mathcal{F}(\mathbb{R}^4)$ by

$$\phi(f) = \Phi_S(\omega^{-1/2} \tilde{f}_+), \quad (45)$$

where the argument contains the restriction $\tilde{f}_+(\mu) := \int \omega(\mu,\omega,\mu,\omega)^2 + \mu^2$, and $\omega$ denotes the corresponding (unbounded) multiplication operator on $\mathcal{H}_{1,\rho}$. The representation of translation group is generated by the second quantization of the multiplication operators $\omega(\mu,\omega)$, and setting $W(f) := e^{i\phi(f)}$, we obtain a corresponding Haag-Kastler net for bounded open regions $\mathcal{O} \subset \mathbb{R}^4$ by

$$\mathcal{A}(\mathcal{O}) := \{ W(f) : f \in \mathcal{F}(\mathbb{R}^4), \supp f \subset \mathcal{O} \}''.$$

\textsuperscript{20}Due to vacuum polarization $\phi(f)\Omega$ cannot have sharp mass for interacting theories, i.e. there is some spectral background $E_{\omega,\alpha}^{\omega,\phi(f)}\Omega \neq 0$. Generalized free fields simulate this in a simplistic way via (44).
It will be convenient to adopt Landau’s formulation [Lan74], as it gives a simple reinterpretation of $\mathfrak{A}(O)$ in terms of time-zero fields. For Schwartz test functions $f(x)$, $x = (x^0, x, s)$, from here on assumed to be symmetric in $s$, where $s$ may be interpreted as new auxiliary space-like\(^{21}\) variable conjugate to the mass $\rho$, set $\phi(f) := \Phi_S(\omega^{-1/2} \hat{f})$, $\hat{f}_\mu(p, \mu) := \hat{f}(\omega_\mu(p, \mu))$. Analogously to (46), we obtain an extended net $\mathfrak{A}(O)$ on $\mathbb{R}^5$.

It is easily seen that extended field $\phi(f)$ and its time derivative $\beta(f) := -\hat{\phi}(\partial_t f)$ admit well-defined restrictions to time-zero fields

$$
\phi_0(f) = \Phi_S(\omega^{-1/2} \hat{f}), \quad \beta_0(f) = \Phi_S(\omega^{1/2} \hat{f})
$$

(47)

for test functions $f \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$ defined on the extended $(x, s)$-space. In terms of corresponding extended double cones $\mathcal{C}_R := \{(t, x, s) \in \mathbb{R}^{3+2} : |t| + \sqrt{x^2 + s^2} < R\}$, $(R > 0)$, Landau gave the following characterization of the net (46).

**Theorem 23.** [Lan74]. $\mathfrak{A}(\mathcal{C}_R) = \mathfrak{A}(\mathcal{C}_R^R)$. Furthermore, these algebras are generated by bounded functions of the time-zero fields (47) with test functions $f \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$ supported in the ball $B_R = \mathcal{C}_R|_{t=0}$.

**Proposition 24.** [Lan74]. If the defining measure $\rho$ of the generalized free field is exponentially decreasing, then $\mathfrak{A}(\mathcal{C}_R) = \mathfrak{A}(\mathcal{C}_R \times \mathbb{R})$.

For choosing $\alpha = 0$ in (44), we may conclude that the strengthened Reeh-Schlieder property holds for the net $\mathfrak{A}$: take a family of test functions $f_\beta \in C^\infty_c(\mathbb{R}^4)$, such that $\{0\} \times \text{supp} f_\beta \subset O \times \mathbb{R}$ and with Fourier transforms converging sufficiently rapidly to a smooth limit supported on the sharp-mass subset $\mathbb{R}^3 \times \{m\}$. By Proposition 24 we can make such a choice which is compatible with bounded functions of $\phi_\beta := \phi_0(f_\beta)$, such as $A_\beta := \phi_\beta e^{-\beta |\phi_\beta|^N}$, being contained in the local algebra $\mathfrak{A}(O)$, thus confirming the validity of (RS). Regarding (RS) we may summarize:

**Proposition 25.** For generalized free field models defined by (44) with $\alpha = 0$, there exists a dense set of sharp-mass single-particle states generated by Reeh-Schlieder families of arbitrarily small degree $\gamma > 0$ independently of the choice of $\epsilon$ in (44).

A fortiori, a continuity argument then shows that the sharp-mass free field net $\mathfrak{A}_{\text{sm}}(O)$ is a subset of $\mathfrak{A}(O)$. To obtain a non-trivial example we should thus choose $\alpha = 1$. We conclude with a short consideration of this difficult case, for which the assumptions of Proposition 24 are violated.

Given a bounded double-cone region $\mathcal{C}_R$ and a single-particle vector $\Psi_1 \in \mathcal{H}_{\mathfrak{A}, R}$ (say $\Psi_1 = \phi_0(f)\Omega$, with $f \in \mathcal{S}(\mathbb{R}^4)$ supported in a very large region) we would like to find a family of smeared field operators $\phi_\beta$ localized in $\mathcal{C}_R$, such that $\|\phi_\beta \Omega - \Psi_1\| \leq \beta$. For this purpose it will be convenient to introduce the following closed single-particle subspaces ($f \in \mathcal{S}(\mathbb{R}^4)$) in the setting of Theorem 23,

$$
\mathcal{H}_{\phi_0, B_R} := \{\phi_0(f)\Omega, \text{ supp} f \subset B_R\}, \quad \mathcal{H}_{\pi_0, B_R} := \{\pi_0(f)\Omega, \text{ supp} f \subset B_R\}.
$$

(48)

The orthogonal projections $P_\phi, P_\pi$ corresponding to (48) may be used to iteratively define approximations of $\Psi_1$ by vectors from (48) or equivalently, generated by $\mathcal{C}_R$-localized operators. Underlining error terms after each half-step we begin with

$$
\Psi_1 = P_\phi \Psi_1 + (1 - P_\phi) \Psi_1 = P_\phi \Psi_1 + P_\pi P_\phi \Psi_1 + P_\pi P_\phi^2 \Psi_1 + \ldots
$$

\(^{21}\)However the field $\phi(f)$ should not be expected to be local in the direction of $s$. 

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Similarly, after $N$ iterations the remaining error is given by $\| (P_\pi P_\varphi)^N \Psi_1 \|$. By the von Neumann alternating projection theorem [vN50, Thm. 13.7], $(P_\pi P_\varphi)^N$ in fact converges strongly to the orthogonal projection onto the intersection $\mathcal{H}_{\varphi,B_R} \cap \mathcal{H}_{\varphi,B_R} = (\mathcal{H}_{\varphi,B_R} + \mathcal{H}_{\varphi,B_R})^\perp$. The latter is trivial by the Reeh-Schlieder theorem, implying convergence of our iterative procedure. An upper bound on the degree of sharp-mass Reeh-Schlieder families along the lines of (RS) or (RS') may be inferred from the speed of convergence $\| (P_\pi P_\varphi)^N \Psi_1 \| \to 0$, $\Psi_1 \in E_m, \mathcal{H}_{\varphi,\rho}$ or equivalent geometrical information regarding the situation of $\Psi_1$ in relation to the spaces $(48)$. This is presently still under investigation.

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