Approval of the method of modeling of regenerating systems by the example of modeling the process of functioning of the information system taking into account service of various types of applications

M V Zamoryonov, V Y Kopp and D V Zamoryonova
Sevastopol State University, 33, Universitetskaya str., Sevastopol, 299053, Russia
E-mail: zamoryonoff@gmail.com

Abstract. The article verifies the method of modeling regenerative systems with a common phase state space using the example of phase enlargement of the model of the information system functioning process taking into account the servicing of various types of applications. The distribution functions of random variables obtained using the method proposed in this work, based on the theorem on the distribution functions of the residence time in a subset of continuous states, and obtained using the classical phase enlargement algorithm, are compared, and the transition probabilities of the enlarged system and the stationary distribution of the embedded circuit are determined and compared Markov. It shows the coincidence of the results of phase enlargement carried out by the method proposed in the work and the classical algorithm of phase enlargement. This work clearly demonstrates the simplicity and convenience of the proposed method of phase enlargement due to the fact that there is no need to determine the stationary distribution of the embedded Markov chain for systems with a common phase state space, which is a rather complicated task that requires solving systems of integral equations containing sums and differences of variables. Currently, a general solution to this problem is unknown, although there are solutions for individual cases.

1. Introduction
Much attention is paid to the problem of modeling semi-Markov systems with a common phase state space in the literature [1–8]. It is considered in more detail in the writings of Academician V.S. Korolyuk [9–12].

The modeling process itself includes a number of stages, among which one of the most important and rather complicated is the stage of determining the stationary distribution of the embedded Markov chain. The complexity of this stage is explained by the fact that in order to determine the required parameters it is often necessary to solve systems of integral equations with kernels containing functions of the sum and difference of variables. There are no general methods for solving such problems; only certain particular solutions are known.

If it is possible to determine the stationary distribution of the embedded Markov chain, then the process of further modeling can be greatly simplified using the phase enlargement algorithm proposed in [10].
The authors developed a method of phase enlargement of regenerating systems [13], which does not require determining the stationary distribution of the embedded Markov chain for a system with a common phase state space.

2. Problem formulation

The purpose of this article is to test the method of modeling regenerative systems using an example of modeling, taking into account the servicing of various types of applications.

The proposed method is based on the proved theorem [13] on the distribution functions (DF) of residence time in a subset of continuous states.

The DF of the difference between random variable (RV) $\alpha$ and RV $\beta_\kappa$ is the recovery time in the stream of the same name generated by the DF $F_\kappa (t) \alpha$ RV $\beta$, provided that it has the $\alpha > \beta_\kappa$, form:

$$F_{\alpha - \beta_\kappa} (t) = P\{[\alpha - \beta_\kappa]^+ \leq t\} = \frac{\int_0^\infty [F_\kappa (t + y) - F_\kappa (y)]h_\kappa (y)dy}{\int_0^\infty [1 - F_\kappa (x)]h_\kappa (dx)},$$

(1)

where $h_\kappa (y) = \sum_{k=1}^\infty f_\kappa^{(*)k} (y) = \sum_{k=1}^\infty f_{2,k} (y) -$ density recovery function [6]; $*$ - a sign of a convolution operation.

We note that in order to verify the method, it is necessary to obtain the characteristics of an enlarged system and compare them with the corresponding characteristics obtained using algorithm of phase enlargement (APE) [12, 15].

There are information systems that continuously monitor various constantly changing parameters of an object. In the case when a quick response of the control system according to the monitoring results is required, the demand arriving at the analysis unit cannot wait and with the busy analysis unit is lost.

In this case, the studied system can be considered as a single-channel system of mass service with losses and one serving device [16]. The time between claims arrivals is a random variable $\beta$ with an absolutely continuous distribution function $G(t) = P\{\beta < t\}$ and density $g(t)$. If the system is busy servicing the requirements, then the requirements arriving in the system are denied.

Suppose that the stream of requirements coming into the system contains type requirements. At the time the service begins, requirements are instantly determined by its type. The probability that the requirement is of the $k$-th type is equal to $p_k$, $\sum_{k=1}^n p_k = 1$. The duration of service requirements depends on its type. The service requirement of the $k$-th type is a random variable $\alpha_k$ with an absolutely continuous distribution function $F_k (t) = P\{\alpha_k < t\}$ and density $f_k (t)$, $k = 1, n$.

In other words, the requirement service time has a phase type distribution with a distribution function:

$$F(t) = \sum_{k=1}^n p_k F_k (t).$$

The expression for the distribution function $F(t)$ can be interpreted as the ability of an information system to service requirements in $n$ various phases. At the time the service starts, the demand is sent with a probability to $p_k$ the $k$-th phase, where a random time is served $\alpha_k$, after which the service is considered completed. If you schematically depict the process of servicing requirements, then the phases of which only one passes are parallel to each other (Figure 1). The arrows entering the service phases indicate the probabilities of receipt of a requirement for this phase. Since the information system can contain no more than one application, then no more than one phase can be occupied at a time.
We assume that the random variables are independent $\alpha_k$, $\beta$ have finite mathematical expectations $E\alpha_k$, $E\beta$ and variances $D\alpha_k$, $D\beta$. The service system considered in this section is a system $GI/G/1/0$ in the generally accepted classification of D. Kendall.

We describe the functioning of the system by a semi-Markov process $S(t)$ [10, 11, 16–18] with phase space:

$$E = \{0x, 1kx, 1k, k = 1, n\}.$$

We decipher the status codes:
- $1k$ - the requirement of the type received in the system began to be serviced;
- $0x$ - the requirement service has ended, there is time left before the next requirement arrives in the system;
- $1kx$ - the requirement received in the system is lost, the device is busy servicing the requirements of the $i$th type, until the end of which there is time $x$.

The transition graph is shown in Figure 2.

The residence times of the system in states are determined by the formulas:
\[ \theta_{0x} = x, \theta_{ik} = \alpha_k \Lambda \beta, \theta_{1x} = x \Lambda \beta , \]

where \( \Lambda \) – the minimum sign, \( k = 1, n \).

3. Problem Solution

When using APE [10] in [15] we defined:

- transition probabilities of the enlarged SM system:

\[ \hat{P}_{0x}^{ik} = p_k, \hat{P}_{ik}^{ks} = P(\alpha_k > \beta) = \int_0^\infty f_k(t)G(t)dt; \hat{P}_{ik}^{dx} = P(\beta > \alpha_k) = \int_0^\infty f_k(t)\bar{G}(t)dt; \]

\[ \hat{P}_{1x}^{id} = \int_0^\infty \int_0^\infty h_i(t) f_j(t+x)G(x)dxdt / \int_0^\infty h_j(t)\bar{G}(t)dt, \]

(3)

where \( h_i(t) = \sum_{n=1}^\infty g^{(n)}(t) \) – density recovery function;

\[ \hat{P}_{1x}^{dy} = \int_0^\infty \int_0^\infty h_i(t) f_j(t+x)\bar{G}(x)dxdt / \int_0^\infty h_j(t)\bar{G}(t)dt; \]

(4)

- stationary distribution of the embedded Markov chain:

\[ \hat{\rho}_{1x} = \rho \int_0^\infty h_i(t)\bar{F}_x(t)dt; \]

(5)

\[ \hat{\rho}_{(0x)} = \sum_{k=1}^n \rho_{ik}; \]

(6)

\[ \rho_{ik} = \frac{\rho p_k}{p_n}, k = 1, n. \]

- DF of the time the system was in these states:

\[ \hat{F}_{1x}(t) = 1 - \bar{G}(t); \int_0^\infty h_i(y)\bar{F}_x(t+y)dy / \int_0^\infty h_i(y)\bar{F}_x(y)dy, \]

(7)

\[ \hat{F}_{0x}(t) = \frac{\sum_{k=1}^n \rho_{ik} \int_0^\infty f_j(y)V_x(y,t)dy}{\sum_{k=1}^n \rho_{ik}}; \]

(8)

where \( V_x(t,x) \) - DF direct residual time \( \beta_i \) for the recovery process generated by a RV \( \beta \).

We determine the transition probabilities of the enlarged system using the method proposed by the authors.

\[ \bar{P}_{0x}^{ik} = p_k, \bar{P}_{ik}^{ks} = P(\alpha_k > \beta) = \int_0^\infty f_k(t)G(t)dt; \bar{P}_{ik}^{dx} = P(\beta > \alpha_k) = \int_0^\infty f_k(t)\bar{G}(t)dt; \]

\[ \bar{P}_{1x}^{id} = \int_0^\infty \int_0^\infty f_k(t) f_j(t+x)G(x)dxdt / \int_0^\infty f_j(t)\bar{G}(t)dt, \]

\[ \bar{P}_{1x}^{dy} = \int_0^\infty \int_0^\infty f_k(t) f_j(t+x)\bar{G}(x)dxdt / \int_0^\infty f_j(t)\bar{G}(t)dt, \]

\[ \bar{P}_{1x}^{kk} = P(x > \beta) = \int_0^\infty f_s^{1k}(t)G(t)dt; \bar{P}_{1x}^{dx} = P(\beta > x) = \int_0^\infty f_s^{1k}(t)\bar{G}(t)dt, \]

where \( f_s^{1k}(t) \) is the density of the DF \( F_s^{1k}(t) \), a \( F_s^{1k}(t) \) is determined by the theorem (1):
\[ F_{x}^{1k}(t) = P\{[\alpha - \beta_{x}] \leq t\} = \frac{\int_{0}^{\infty} [F_{x}(t+x) - F_{x}(x)]h_{x}(x)dx}{\int_{0}^{\infty} h_{x}(x)dx} = 1 - \frac{\int_{0}^{\infty} F_{x}(t+x)h_{x}(x)dx}{\int_{0}^{\infty} F_{x}(x)h_{x}(x)dx}. \] (9)

From here:

\[ \bar{P}_{1k}^{1k} = \frac{\int_{0}^{\infty} \int_{0}^{\infty} h_{x}(t)F_{x}(t+x)G(x)dxdt}{\int_{0}^{\infty} h_{x}(t)F_{x}(t)dt}; \] (10)

\[ \bar{P}_{10}^{1k} = \frac{\int_{0}^{\infty} \int_{0}^{\infty} h_{x}(t)F_{x}(t+x)\bar{G}(x)dxdt}{\int_{0}^{\infty} h_{x}(t)\bar{F}_{x}(t)dt}. \] (11)

As can be seen, the obtained expressions (10) and (11) completely coincide with the corresponding expressions (3) and (4).

Let us determine the DF of the time the system was in states based on the method of modeling regenerating systems proposed by the authors.

From the expression (2) it is clear that it is necessary to determine the DF \( \bar{F}_{1k}^{1k}(t) \) and random variables - the residence times of the system in enlarged states:

\[ \bar{F}_{1k}^{1k}(t) = 1 - \bar{G}_{1}(t) \cdot \bar{F}_{x}^{1k}(t) \] (12)

\[ \bar{F}_{0x}^{1k}(t) = \sum_{k} \frac{\rho_{1k} F_{x}^{2k}(t)}{\sum_{k} \rho_{1k}}. \] (13)

In accordance with the method of modeling regenerating systems developed by the authors, the required DF \( F_{x}^{2k}(t) \) has the form:

\[ F_{x}^{2k}(t) = P_{1k}^{0x} \cdot \frac{\int_{0}^{\infty} [G(t+x) - G(x)]f_{x}(x)dx}{\int_{0}^{\infty} G(t)f_{x}(t)dt} + P_{1k}^{1k} \cdot \frac{\int_{0}^{\infty} [G(t+y) - G(y)]f_{x}^{1k}(y)dy}{\int_{0}^{\infty} G(t)f_{x}^{1k}(t)dt}, \] (14)
\[
\tilde{F}_{0x}(t) = \frac{rol_{11}}{rol_{11} + rol_{12}} \cdot F_{x}^{11}(t) + \frac{rol_{12}}{rol_{11} + rol_{12}} \cdot F_{x}^{12}(t) = \\
= \frac{rol_{11}}{rol_{11} + rol_{12}} \left[ p_{01}^{11} \cdot \int_{0}^{t} [G(t + x) - G(x)] f_{1}(x) \, dx + p_{11}^{11} \cdot \int_{0}^{t} [G(t + y) - G(y)] f_{1}^{11}(y) \, dy \right] + \\
+ \frac{rol_{12}}{rol_{11} + rol_{12}} \left[ p_{01}^{12} \cdot \int_{0}^{t} [G(t + x) - G(x)] f_{2}(x) \, dx + p_{12}^{12} \cdot \int_{0}^{t} [G(t + y) - G(y)] f_{2}^{12}(y) \, dy \right]
\]

\[
\tilde{F}_{1x}(t) = 1 - G(t)^{0.5} \cdot \frac{\int_{0}^{t} h_{x}(y) F_{1}(t + y) \, dy}{\int_{0}^{t} h_{x}(y) \, dy}.
\]

As an example, we consider a system with two types of claims.

The initial data for modeling are:
- DF \( G(t) \) - time between requirements;
- DF \( F_{1}(t) \) - service time requirements of the 1st type;
- DF \( F_{2}(t) \) - service time requirements of the 2nd type;
- \( P_{1} \) and \( P_{2} \) - the probabilities of receipt of claims of the 1st and 2nd types, respectively.

These RFs are distributed according to the generalized Erlang law of the second order with parameters \( \alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2}, \gamma_{1}, \gamma_{2} \), respectively; moreover:

\[
g(t) = \frac{\alpha_{1} \alpha_{2}(e^{-\alpha_{1}t} - e^{-\alpha_{2}t})}{\alpha_{2} - \alpha_{1}}, \quad f_{1}(t) = \frac{\beta_{1} \beta_{2}(e^{-\beta_{1}t} - e^{-\beta_{2}t})}{\beta_{2} - \beta_{1}}, \quad f_{2}(t) = \frac{\gamma_{1} \gamma_{2}(e^{-\gamma_{1}t} - e^{-\gamma_{2}t})}{\gamma_{2} - \gamma_{1}}, \quad P_{1} = P_{2} = 0.5
\]

where \( \alpha_{1} = 0.33 \, \text{h}^{-1}, \, \alpha_{2} = 1.0 \, \text{h}^{-1}, \, \beta_{1} = 0.083 \, \text{h}^{-1}, \, \beta_{2} = 0.25 \, \text{h}^{-1}, \, \gamma_{1} = 0.667 \, \text{h}^{-1}, \, \gamma_{2} = 2.0 \, \text{h}^{-1} \).

The resulting DF simulation \( \tilde{F}_{0x}(t) \) and \( \tilde{F}_{0x}(t) \) downtime of the information system are presented in Figure 3, and their distribution density (DD) \( \tilde{F}_{0x}(t) \) and \( \tilde{F}_{0x}(t) \) – in Figure 4.
We compare the values of the mathematical expectation of the function; we have obtained and the mathematical expectation determined using expression (8).

The mathematical expectation of the distribution function we obtained is:

$$3.328482968 \text{ h},$$

whereas we defined using expression (8) for:

$$3.328482968 \text{ h}.$$

It is easy to state that the mathematical expectations, like the graphs in Figures 3 and 4, coincide.

4. Conclusion

The studies showed a complete coincidence of the results, which proves the correctness of the developed method. The proposed method, together with the path method [19] and the trajectory method [20] developed by the authors, allows one, with further research, to begin the task of automated construction of semi-Markov models.

5. Acknowledgments

The studies were supported by a grant from the Russian Foundation for Basic Research, No. 19-01-00704.

References

[1] Grabski F 2015 Semi-Markov Processes: Applications in System Reliability and Maintenance (Elsevier)
[2] Janssen J and Raimondo M 2006 Applied Semi-Markov Processes (Springer Science + Business Media)
[3] Korolyuk V S and Limnios N 2005 Stochastic Systems in Merging Phase Space (World Scientific, Imperial Coledge Press)
[4] Limnios N and Oprisan G 2001 Semi-Markov Processes and Reliability (Springer Science+Business Media)
[5] Jansen J and Limnios N, eds 1999 Semi-Markov Models and Applications (Kluwer Academic Publishers)
[6] Silvestrov D and Silvestrov S 2017 Nonlinearly Perturbed Semi-Markov Processes (Springer)
[7] Kashтанов V A and Medvedev A 12002 Reliability theory of complex systems (theory and practice) (Moscow: European Center for Quality)
[8] Shurenkov V M 1989 Ergodic processes of Markov (Moscow: Nauka)
[9] Korolyuk V S and Turbin A F 1982 *Markov recovery processes in systems reliability problems* (Kiev: Naukova Dumka)

[10] Korolyuk V S 1989 *Stochastic models of systems* (Kiev: Naukova Dumka)

[11] Korolyuk V S 1981 Superposition of Markov recovery processes *Cybernetics* **4** 121–124

[12] Korolyuk V S 1976 *Semi-Markov processes and their applications* (Kiev: Naukova Dumka)

[13] Zamoryonov M V, Kopp V Ya and Zamoryonova D V 2018 A method for modeling regenerative systems *Mathematical modeling* [in Russian – Matematicheskoe modelirovanie] **30(6)** 134–144

[14] Beichelt F and Franken P 1988 *Reliability and maintenance. The mathematical approach* (Moscow: Radio and communications)

[15] Zamoryonov M V, Kopp V Ya and Zamoryonova D V 2017 Defining information system characteristics with account of different applications service types *Int. Conf. on Industrial Engineering, Applications and Manufacturing (ICIEAM)* DOI: 10.1109/ICIEAM.2017.8076397

[16] Peschansky A I 2013 *Semi-Markov Models of One-Server Loss Queues with Recurrent Input* (Germany: LAP LAMPERT Academic Publishing)

[17] Reinshke K and Ushakov I A 1988 *Assessment of system reliability using graphs* (Moscow: Radio and communications)

[18] Obzherin Y E and Boyko E G 2015 *Semi-Markov Models: Control of Restorable Systems with Latent Failures* (Elsevier, Academic Press)

[19] Zamorenov M V, Kopp V Ya, Zamoryonova D V and Skidan A A 2016 Modeling the process of functioning of a service device with non-depreciating failures by the path method *Bulletin of Tula State University. Technical science* **7** Part 1 (Tula: Publishing house of TulSU) 71–82

[20] Kopp V Ya, Zamorenov M V, Obzherin Yu E and Linar M Yu 2016 Using the trajectory method to build a semi-Markov model of the structure “technological cell - drive” *Scientific and technical bulletin of Peter the Great St. Petersburg Polytechnic University. Computer science. Telecommunications. Management* **3(247)** 23–34