Ribbon graphs and the Temperley-Lieb Algebra

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Abstract

Let $n$ be a nonnegative integer, we use ribbon $n$–graph diagrams and the Yamada polynomial skein relations to construct an algebra $\mathcal{Y}_n$ which is shown to be closely related to the Temperley-Lieb Algebra. We prove that the algebra $\mathcal{Y}_2$ is isomorphic to some quotient of a three variables polynomial algebra. Then, we give a family of generators for the algebra $\mathcal{Y}_3$.

Key Words. Ribbon graphs, Temperley-Lieb Algebra, Yamada polynomial.

MSC. 57M25, 05C10.

1 Introduction

Throughout this paper, a graph is the geometric realization of a finite CW-complex of dimension 1. Furthermore, we assume that all vertices (0-cells) have valency greater than 2. A spatial graph is an embedding of a graph into the three-dimensional Euclidean space $\mathbb{R}^3$. The theory of spatial graphs is considered as a natural extension of knot theory. Therefore, many of the techniques and problems of knot theory have their counterparts in spatial graph theory. A natural question that arose after the discovery of the Jones polynomial and other quantum invariants of links, was to define invariants of Jones type for spatial graphs. In that direction, Yamada \cite{Yamada} introduced a topological invariant of spatial graphs, hereafter referred to as the Yamada polynomial. It is a one variable Laurent polynomial $Y(A)$ which can be defined recursively on
planar diagrams of spatial graphs.

The Jones and the Kauffman bracket polynomials are closely related to the Temperley-Lieb algebra $\tau_n$. Actually, one can construct these polynomials through representations of the Artin braid group into $\tau_n$. The original motivation of the present paper is to explore the possibility of a similar algebraic interpretation of the Yamada polynomial.

Ribbon graphs are geometrical objects that appeared as a natural generalization of framed links by Reshetikhin and Turaev in [9]. Let $n$ be a nonnegative integer. A ribbon $n$–graph is a compact oriented surface embedded into $\mathbb{R}^2 \times I$ which meets the boundary of $\mathbb{R}^2 \times I$ orthogonally exactly along the $2n$ segments $\{[i-1/10, i+1/10] \times \{0\} \times \{0, 1\}, i = 1, \ldots, n\}$. It is worth mentioning that the precise definition requires some other technical arrangements the discussion of which is postponed to Section 3. Ribbon graphs are represented by planar graph diagrams generalizing link diagrams. Let $S_n$ be the set of all ribbon $n$–graph diagrams and $R = \mathbb{Z}[A^{\pm 1}, d^{-1}]$, where $d = -A^2 - A^{-2}$. Let $Y_n$ be the free $R$-module generated by all elements of $S_n$. We define $\mathcal{Y}_n$ to be the quotient module of $Y_n$ by the Yamada relations in Section 3. This module admits a natural algebra structure. The product of two ribbon $n$–graphs $G$ and $G'$ is defined as illustrated by Figure 1.

Here are the main results in this paper.

**Theorem 1.1.** The algebra $\mathcal{Y}_2$ is the free additive $R$-algebra with multiplicative elements $1_2, V$ and $X$ pictured below.
Corollary 1.1. The algebra $\mathcal{Y}_2$ is isomorphic to the quotient of the commutative algebra $R[1, X, V]$ by the ideal generated by $V^2 - (d^2 - 1)V, VX - (d - d^{-1})V$ and $X^2 = (d - 2d^{-1})X + d^{-2}V$.

Theorem 1.2. The algebra $\mathcal{Y}_3$ is generated by the 6 elements $A, B, C, F, G$ and $N$ in Figure 3.

This paper is outlined as follows. In Section 2, we briefly review some properties of the Temperley-Lieb algebra needed in the sequel. In Section 3, we define ribbon graphs and we introduce the algebra $\mathcal{Y}_n$. The proofs of the main results are given in Section 4. Finally, Section 5 explores the connection between $\mathcal{Y}_n$ and the Temperley-Lieb algebra.
2 The Temperley-Lieb algebra

Temperley-Lieb algebras appeared first in the context of statistical physics. With the discovery of the Jones polynomial, these algebras offered a new approach for the study of the quantum invariants of links and three-manifolds. This section is a brief introduction to the theory of Temperley-Lieb algebras from the knot theory viewpoint.

Let $n$ be a nonnegative integer, an $n$-tangle $T$ is a one-dimensional sub-manifold of $\mathbb{R}^2 \times I$, such that the boundary of $T$ is made up of $2n$ points $\{(i, 0, 0), (i, 0, 1); 1 \leq i \leq n\}$. As usual, tangles are considered up to isotopies of $\mathbb{R}^2 \times I$ fixing the boundary pointwise. It is well known that the study of tangles up to isotopy is equivalent to the study of their planar diagrams in $\mathbb{R} \times I$ up to Reidemeister moves keeping the boundary fixed pointwise.

Let $\mathcal{T}_n$ be the free $\mathcal{R}$-module generated by the set of all $n$-tangles. We define $\tau_n$ to be the quotient of $\mathcal{T}_n$ by the smallest submodule containing all elements of the form:

$$\bigcirc \cup L - dL$$
$$L - A L_0 - A^{-1} L_\infty,$$

where $\bigcirc$ is the trivial circle, $d = -A^2 - A^{-2}$ and $L, L_0$ and $L_\infty$ are three tangle diagrams which are identical everywhere except in a small disc where they look as pictured below.

\begin{center}
\begin{tabular}{ccc}
\hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
$L$ & $L_0$ & $L_\infty$
\end{tabular}
\end{center}

If we equip the module $\tau_n$ with the standard product of tangles, then $\tau_n$ turns out to be an algebra which is isomorphic to the Temperley-Lieb algebra. A set of generators $(U_i)_{0 \leq i \leq n-1}$ of $\tau_n$ is illustrated below.
Let \((f_i)_{0 \leq i \leq n-1}\) denote the family of Jones-Wenzl projectors in \(\tau_n\). This family is defined by the following recursive formulas:

\[
\begin{align*}
  f_0 &= U_0, \\
  f_{k+1} &= f_k - \mu_{k+1} f_k U_{k-1} f_k,
\end{align*}
\]

where \(\mu_1 = d^{-1}\) and \(\mu_{k+1} = (d - \mu_k)^{-1}\).

In particular, we have \(f_1 = 1 - d^{-1}U_1\). The elements \(f_k\) enjoy the following properties: \(f_k^2 = f_k\) and \(f_i U_j = U_j f_i = 0\) for \(j \leq i\). See [5] for more details.

### 3 Graph Algebra

Ribbon graphs have been introduced by Reshetikhin and Turaev in [9]. They appeared as a natural generalization of framed links. We begin this section by a brief review of the definition of ribbon graphs. More details about these objects can be found in [10]. Then, we will define the graph algebra \(\mathcal{Y}_n\), which will appear as an extension of the theory of Temperley-Lieb algebra.

Let \(n \geq 1\) be an integer. A ribbon \(n\)-graph \(G\) is a compact oriented surface embedded into \(\mathbb{R}^2 \times I\) which can be decomposed into a finite collection of annuli, coupons (small rectangles) and ribbons (long bands), such that:

(i) annuli do not meet each other and do not meet ribbons or coupons.

(ii) ribbons never meet each other, but they may meet coupons at their bases.

(iii) \(G\) meets \(\mathbb{R}^2 \times \{0, 1\}\) orthogonally exactly in bases of certain ribbons. The intersection is a collection of segments \(\{[i - 1/10, i + 1/10] \times \{0\} \times \{0, 1\} ; i = 1, \ldots, n\}\).

Ribbon \(n\)-graphs are considered up to isotopies of \(\mathbb{R}^2 \times I\) fixing the boundary pointwise and preserving the decomposition into annuli, coupons and ribbons. According to [10], ribbon
\( n \)-graphs can be represented by planar diagrams, where coupons are represented by vertices, annuli are represented by circles and long bands by either an ordinary graph edge, a half edge (arc connecting a vertex and a boundary point) or an arc connecting two boundary points.

\[ \text{Figure 4} \]

The study of ribbon graphs up to isotopy is equivalent to the study of their planar diagrams modulo planar isotopies and the extended Reidemeister moves in Figure 5.

\[ \text{Figure 5} \]

Let \( S_n \) be the set of all ribbon \( n \)-graph diagrams and \( \mathcal{R} = \mathbb{Z}[A^{\pm 1}, d^{-1}] \), where \( d = -A^2 - A^{-2} \). Let \( Y_n \) be the free \( \mathcal{R} \)-module generated by all elements of \( S_n \). We define \( Y_n \) to be the quotient module of \( Y_n \) by the smallest submodule containing all expressions of the form:
These relations are referred to as the Yamada skein relations. In each relation, the pictures represent planar diagrams of ribbon $n$–graphs which are identical except in small disk where they look as pictured. In the same way as for tangles, a multiplicative structure can be defined on $\mathcal{Y}_n$. The identity relative to this product is the ribbon graph made up of $n$–parallel ribbons, this element is denoted hereafter by $1_n$. The product of two elements $G$ and $G'$ is the ribbon $n$–graph $GG'$ obtained by putting $G$ over $G'$ as pictured below:

4 Proofs

In this section we give the proofs of Theorem 1.1, Corollary 1.1 and Theorem 1.2. We begin by describing a family of generators of the $\mathcal{R}$–module $\mathcal{Y}_n$.

Lemma 4.1. The $\mathcal{R}$–module $\mathcal{Y}_n$ is generated by all ribbon $n$–graph diagrams with no crossings, no cycles and no ordinary edges.
Proof. Let $G$ be a ribbon $n$-graph diagram. One can apply the first Yamada relation to smooth all the crossings of the diagram. Therefore, $G$ is expressed as a linear combination, with coefficients in $\mathcal{R}$ of ribbon $n$-graph diagrams each of which has no crossings. In the next step, we use the Yamada deletion-contraction relation to delete all graph edges. Hence, our graph is written as a linear combination of diagrams which have no ordinary edges. Now, we can remove all cycles using Yamada relations (3) and (4). Finally, our graph $G$ is expressed as a linear combination of diagrams each of which has no crossings, no edges and no cycles.

Proof of Theorem 1.1. According to Lemma 4.1, the module $\mathcal{Y}_2$ is generated by the three elements $1_2$, $V$ and $X$ pictured in Figure 2. So is the algebra $\mathcal{Y}_2$.

Proof of Corollary 1.1. The proof is straightforward by applying Yamada relations as illustrated below:

\[
V^2 = (d^2 - 1)V
\]

\[
VX = XV = (d - d^{-1})V
\]

\[
X^2 = (d - 2d^{-1})X + d^{-2}V
\]
Proof of Theorem 1.2. According to Lemma 4.1, the module $\mathcal{Y}_3$ is generated by the following 15 elements:

\begin{align*}
A & = 1 \\
B & = \text{Diagram}
\end{align*}

Now, we use the multiplication structure to reduce these 15 generators of the module to the six generators of the algebra $\mathcal{Y}_3$. This reduction is briefly illustrated by these 4 kind of operations

1) It can be easily seen that: $D = BC$ and $E = CB$.

2) We have $J = BF$ as depicted below

\begin{align*}
\text{Diagram} & \quad = \quad \text{Diagram}
\end{align*}

Similarly we get $M = FB$, $K = CG$ and $L = GC$.

3) According to the picture below we have: $FG = N - d^{-1}H$ which implies that $H = N - dFG$.

\begin{align*}
\text{Diagram} & \quad = \quad \text{Diagram} \quad - \quad \text{Diagram}
\end{align*}
Similarly, $GF = N - d^{-1}I$ which implies that $I = N - dFG$.

4) Finally, using the deletion contraction formula as in the picture below, we show that $MK = N - d^{-1}P$. This implies that: $P = d(N - MK) = d(N - FBCG)$.

![Deletion Contraction Formula Diagram]

5 Relationship between $\mathcal{Y}_n$ and $\tau_n$

The purpose of this section is to discuss the relationship between the two algebras $\mathcal{Y}_n$ and $\tau_n$. In the case of skein modules of three-manifolds, we defined a homomorphism from the graph skein module to the Kauffman bracket skein module $[1, 2]$. An analogous of this homomorphism is defined here. Let $\varphi_n : Y_n \mapsto \tau_{2n}$ be the linear map that associates to each ribbon $n$-graph diagram $G$ the linear combination of diagrams obtained from $G$ by replacing each edge, half edge and arc of $G$ by two planar strands with a projector $f_1 = \parallel -d^{-1} \bigcup \bigcap$ in the cable, and by replacing each vertex of $G$ by a diagram as follows (the figure illustrates the case of a four-valent vertex)

![Diagram Illustrating Graph Replacement]

In this picture, writing an integer 2 beneath an edge $e$ means that this edge has to be replaced by 2 parallel ones. Notice that $\varphi_n$ is defined on the generators of the free $\mathcal{R}$-module, then extended by linearity to all elements of $Y_n$. Using the same graphic calculations as in $[1]$ (Lemma 3.4), one can check easily that $\varphi_n$ defines a map $\Phi_n$ from $\mathcal{Y}_n$ to $\tau_{2n}$. Obviously, $\Phi_n$ is a homomorphism of algebras. The following picture illustrates how to compute $\Phi_2(1_2)$. 
Theorem 5.1. The homomorphism $\Phi_n : \mathcal{Y}_n \mapsto \tau_{2n}$ is injective.

Proof. Remind first that the $\mathcal{R}$-module $\tau_n$ has a standard base consisting of all diagrams of $n-$arcs with no crossings joining the $2n$ boundary points pairwise. The dimension of $\tau_n$ is the Catalan number $\frac{C_{2n}}{n+1}$. Now, let $t$ be an element of the standard base of $\tau_{2n}$. If we take the union of $t$ with the $n$ segments $[i, i+1] \times \{0\} \times \{0, 1\}$ for $i$ odd, then we get a 1-dimensional manifold which bounds a surface $\Sigma(t)$ in $\mathbb{R} \times [0, 1]$. The surface $\Sigma(t)$ retracts by deformation (in $\mathbb{R} \times [0, 1]$) on an graph diagram $g(t)$. The picture below illustrates this construction in the case of an element $t \in \tau_6$.

![Diagram](image_url)

**Figure 7**

Now, we shall prove that the kernel of $\Phi_n$ is trivial. Let $g_1, \ldots, g_s$ be distinct generators of $\mathcal{Y}_n$ as described in Lemma 4.1. Let $r_1, \ldots, r_s$ be elements of $\mathcal{R}$ such that $\Phi_n(r_1g_1 + \ldots + r_s g_s) = 0$. We know that $\Phi_n(g_i)$ is expressed as a linear combination of the standard generators of $\tau_{2n}$. Among the elements which appear in this combination, let $g_i^1$ be the generator whose surface $\Sigma(g_i^1)$ has the minimum connected components. Obviously, this surface retracts by deformation on $g_i$. Moreover, it is easy to see that $\Phi_n(r_1g_1 + \ldots + r_s g_s) = 0$ implies that $r_1g_1^1 + \ldots + r_s g_s^1 = 0$ which leads to $r_i = 0$ for all $1 \leq i \leq s$. Hence, $\Phi_n$ is injective.
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