Image-processing the topological charge density in the $CP^{N-1}$ model

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We study the topological charge density distribution using the two-dimensional $CP^{N-1}$ model. We numerically compute not only the topological susceptibility, which is a global quantity to probe topological properties of the whole system, but also the topological charge correlator with finite momentum. We perform Fourier power spectrum analysis for the topological charge density for various values of the coupling constant $\beta$. We propose to utilize the Fourier entropy as a measure to characterize spatial distribution patterns and demonstrate that the Fourier entropy exhibits nontrivial $\beta$ dependence. We also consider the snapshot entropy defined with the singular value decomposition, which also turns out to behave nonmonotonically with $\beta$.

I. INTRODUCTION

Gauge topology is a fundamental aspect of quantum field theory. Quantum chromodynamics (QCD) has a rich vacuum structure known as the QCD vacuum which realizes color confinement and chiral symmetry breaking. For theoretical considerations of these nonperturbative phenomena, it is crucial to take account of topological excitations in QCD, among which the most well understood is the QCD instanton. Indeed, the chiral condensate that spontaneously breaks chiral symmetry is induced by the interaction between fermionic zero modes associated with the instanton [1] (see Ref. [2] for a comprehensive review). Also, recently, there are significant progresses in theoretical studies of color confinement with the instanton with nontrivial holonomy (see, for example, proceedings articles in Ref. [3] and references therein, and also Ref. [4] for a recent review).

It is still a challenging problem how to access topologically nontrivial sectors in theory. For this purpose to study topological contents, the topological susceptibility, $\chi_t$, is the most common observable to quantify fluctuations with respect to the topological charge. Here, we note that the full topological susceptibility in QCD is sensitive to the chiral sector only [5] and it is clearly distinguished from the pure topological susceptibility, that is the topological susceptibility measured in the pure gluonic theory. Roughly speaking, if $\chi_t$ is large, the system should accommodate more instantons which enhance nonperturbative effects. In fact, the Witten-Veneziano formula relates the $\eta'$ mass and $\chi_t$ in the limit of large color number $[6,7]$; the axial anomaly induced mass is proportional to $\chi_t$. There are a countless number of precedent works on physics implications of $\chi_t$, and recently, dedicated studies of $\chi_t$ are also motivated from the axion cosmology $[8,9]$ (see Ref. [10] for a review). For more phenomenological applications of the topological susceptibility and related observables, see a lecture note in Ref. [11].

Since topological excitations are inherently nonperturbative, the information available from analytical considerations is limited, and the numerical Monte-Carlo simulation of QCD on the lattice is the most powerful approach [12] (for complementary attempts using chiral effective models, see Refs. [13–15]). Because the axial current is not conserved, the lattice measurement of $\chi_t$ suffers renormalization problems [16], and the precise determination of $\chi_t$ is a demanding task. There are some theoretical advances using a new idea like the gradient flow, which is already applied to measure the full topological susceptibility [17]. For more details about the lattice-QCD calculations, see a review in Ref. [18].

It is often theorists’ favorite approach to utilize simpler QCD-like theories that share important QCD features of interest. The two-dimensional $CP^{N-1}$ model is one of such idealized laboratories for QCD physics. The $CP^{N-1}$ model has the asymptotic freedom, the linear confining potential, and instantons [19–22]. Lately, it is a hot and growing area to investigate “resurgence” using the $CP^{N-1}$ model, that is, perturbative expandability and cancellation of ambiguity in each topological sector are intensely studied [23–25]. Also, we note that the $CP^{N-1}$ model has plenty of connections to condensed matter physics systems. There are several equivalent formulations of the $CP^{N-1}$ model such as SU($N$) Heisenberg ferromagnets, the tensor network with three-dimensional loop model which is generalized to the form of the tensor renormalization group, and so on [26–32]. In particular, in Heisenberg spin systems, the $CP^{N-1}$ model is relevant for a critical phase between the valence-bond solid and the antiferromagnetic phase [33, 34]. Furthermore, we point out that recent developments of cold atom experiments has enabled us to emulate the $CP^{N-1}$ model and its variants on the optical lattice [35]. In this way, although the $CP^{N-1}$ is a relatively simple and old model, many interesting studies are ongoing to the present date.

Thanks to simpler setup, the $CP^{N-1}$ model is far easier to analyze as compared to QCD or the pure gluonic theory, and nevertheless it provides us with fertile research opportunities. The dynamical mass generation
and the linear confining potential have been derived in the large \( N \) expansion [36, 37] as well as in the strong coupling expansion [38, 39]. Apart from those analytical methods, there are many works for the lattice Monte-Carlo simulation in the \( CP^{N-1} \) model [21, 22, 40, 41]. Here, let us make a remark that the \( CP^{N-1} \) action has a special analytic structure such that the sign problem is tamed even with nonzero chemical potential [42, 43] or \( \theta \) term [30, 44–47], which is a great advantage over QCD.

From the point of view of gauge topology, which is of our present interest, the \( CP^{N-1} \) model has a prominent feature that the topological charge can be defined geometrically [48] and it rigorously takes integer values even on the lattice. There are several caveats, however. Even with rigorous quantization, the physical interpretation may become subtle in some parameter regions. When the model coupling constant is far away from the region corresponding to the continuum limit, the physical lattice spacing would be too coarse to hold the instantons on the lattice. For further discussions on lattice artifacts such as finite size effects and the critical slowing down of the partition function, see Refs. [49–51] for example. Also, in the region with small \( N \), quantum fluctuations would blur instantons as quasi-particles, in other words, melt instantons. Therefore, we can perform the instanton gas analysis at large \( N \), while melting instantons causes the precocious scaling in the small-\( N \) region [52–55].

In this paper we revisit the topological properties in the \( CP^{N-1} \) model from a different perspective. The two-dimensional field configurations can be regarded as “image” data, and thus we will “image-process” the distribution of the topological charge density. As a first step, we perform the Fourier power spectrum analysis of the topological charge density. This power spectrum amounts to a momentum dependent generalization of the topological susceptibility, which has been discussed and partially measured also in QCD in Refs. [14, 56, 57]. The entire structures of the correlator in momentum space are quite informative, as we will reveal in this work, but such calculations are, if applied to QCD, too much resource consuming. Therefore, it would be much more convenient if there is any single measure extracting essential features of the topological charge density distribution. We will propose to use an entropy defined with the Fourier spectrum. Also, another candidate is to make use of the singular value decomposition (SVD) of image data. As a standard image-processing tool, the SVD is commonly employed; see Refs. [58–60] for physics applications of the SVD to image-process the spin configurations. In the same way as in the Fourier power spectrum analysis, interestingly, an entropy is constructed with the SVD eigenvalues, which will be referred to as the snapshot entropy.

This paper is organized as follows. In Sec. II we will review the \( CP^{N-1} \) model and the simulation algorithm. We will introduce physical observables including the Fourier and the snapshot entropies in Secs. II B and II C. Section III is devoted to the numerical setups and the consistency checks with the precedent lattice simulation results. In Sec. IV we will report our main numerical results, namely, the Fourier power spectrum analysis of the topological charge correlator in Sec. IV A, the Fourier and the snapshot entropies in Secs. IV B and IV C, and correlation between them in Sec. IV D. Finally, in Sec. V, we will make conclusions and some outlook.

### II. FORMULATION

We will make a brief overview of the numerical lattice simulation in the \( CP^{N-1} \) model. We adopt the method formulated in Ref. [40] to compute physical observables. We will also give an explanation of unconventional observables called the Fourier entropy and the snapshot entropy.

#### A. \( CP^{N-1} \) Model and the Monte-Carlo Method

The \( CP^{N-1} \) model is defined by the following partition function,

\[
Z = \int \mathcal{D}z \mathcal{D}\lambda \prod_x \delta(|z(x)|^2 - 1) e^{-\beta H} \tag{1}
\]

with the Hamiltonian,

\[
H = -N \sum_{x,\mu} [\bar{z}(x + \hat{\mu})z(x)\lambda_\mu(x) + (c.c.) - 2], \tag{2}
\]

where \( z(x) \) represents complex scalar fields with \( N \) components which are constrained as \( |z(x)|^2 = 1 \), and (c.c.) is the complex conjugate of the first term. The symbol \( \hat{\mu} \) denotes the unit lattice vector. Since there is no kinetic term, \( \lambda_\mu(x) \) is an auxiliary U(1) link variable. We will refer to \( \beta \) as “coupling constant” in analogy to quantum field theory, though it is literally the inverse temperature in the statistical mechanical sense. In some literature \( g \equiv 1/(N\beta) \) is often introduced, but we will consistently use \( \beta \) throughout this paper.

The Monte-Carlo simulation consists of two procedures called “update” and “step” which are explained respectively below. For one update we randomly choose a point \( x \), and calculate new \( z(x) \) and \( \lambda_\mu(x) \) at this point \( x \) according to the probability distributions, which will be explicitly given in the subsequent paragraphs. One step has \( L^2 \) updates, where \( L \) is the lattice size.

To make this paper self-contained, let us elucidate how to update \( z(x) \) at a chosen \( x \). We can update \( \lambda_\mu(x) \) similarly. According to a method called the “over-heat bath method” in Ref. [40] we successively update the configurations. It is important that we can write the \( z(x) \)-dependent piece of the Hamiltonian as

\[
H = -N(z(x), F_z(x)) + \cdots, \tag{3}
\]

where the ellipsis represents terms not involving \( z(x) \). In the above we introduced the inner product of the \( N \)
component scalar fields defined by \( \langle a, b \rangle \equiv \text{Re} \sum_i a_i b_i \). We can easily infer
\[
F_z(x) = \sum_{\mu} \left[ z(x - \hat{\mu}) \lambda_{\mu}(x - \hat{\mu}) + z(x + \hat{\mu}) \lambda_{\mu}(x) \right] \tag{4}
\]
from the Hamiltonian. The inner product of the \( N \) component complex scalars can be regarded as that of the \( 2N \) component real scalars, i.e., \( \langle a, b \rangle = \sum_i (\text{Re} a_i \text{Re} b_i + \text{Im} a_i \text{Im} b_i) \). Then, for this real inner product, we can define the relative angle \( \theta \) as
\[
\langle z(x), F_z(x) \rangle = |F_z(x)| \cos \theta(x). \tag{5}
\]
Here we used \( |z(x)| = 1 \). Because the above form depends only on the relative angle \( \theta(x) \), the choice of \( z(x) \) is not yet unique for the same \( \langle z(x), F_z(x) \rangle \). The over–heat bath method fixes new \( z(x) \) uniquely in such a way to disturb the system maximally.

For the update we first sample a new angle \( \theta \) (that is denoted as \( \theta^{\text{new}} \) below) according to the probability\(^1\),
\[
dp N(\theta) \propto d\theta (\sin \theta)^{2N-2} e^{\beta N |F_z(x)| \cos \theta}, \tag{6}
\]
where \( (\sin \theta)^{2N-2} \) appears from the measure given by the surface area of sphere in \( 2N \)-dimensional space.

We next specify new \( z(x) \) (that is denoted as \( z^{\text{new}}(x) \) below) with chosen \( \theta^{\text{new}} \). In the over–heat bath method, we take \( z^{\text{new}}(x) \) to minimize an overlap with original \( z(x) \), that is, \( \langle z^{\text{new}}(x), z(x) \rangle \) is minimized. This can be achieved with
\[
z^{\text{new}} = \cos \theta^{\text{new}} \frac{F_z}{|F_z|} - \left( 1 - \cos \theta \frac{F_z}{|F_z|} \right) \frac{\sin \theta^{\text{new}}}{\sin \theta}. \tag{7}
\]
After doing this for \( z(x) \), for the same \( x \), we perform the update, \( \lambda_{\mu}(x) \rightarrow \lambda_{\mu}^{\text{new}}(x) \), in the same way.

**B. Physical Observables**

In this work we measure physical observables in units of the lattice spacing \( a \). This means that \( \beta \)-dependence may enter through the running coupling \( \beta(a) \). If necessary, we can convert observables in the physical unit with a typical length scale, i.e., correlation length.

The energy density is one of the most elementary physical observables, that is given by
\[
E = \frac{1}{2NV} \langle H \rangle \tag{8}
\]
with the dimensionless volume \( V \equiv L^3 \). It is useful to keep track of \( E \) to monitor if the numerical simulation converges properly.

We shall define the correlation length. The U(1) invariance implies that the basic building block of local physical observables should be the following local operator (that is, a counterpart of a mesonic state in lattice QCD language),
\[
P_{ij}(x) = \bar{z}_i(x) z_j(x). \tag{9}
\]
The two-point correlation of \( P_{ij}(x) \) is
\[
G_P(x, y) = \langle \text{tr} P(x) P(y) \rangle - \frac{1}{N}, \tag{10}
\]
where the last term \( 1/N \) subtracts the disconnected part. In the perturbative regime near the scaling region, the correlation function in momentum space is expected to scale as
\[
\hat{G}_P(k) \sim \frac{Z_P}{\xi^2} \sum_\mu \frac{4 \sin^2(k_\mu/2)}{\xi_\mu^2}, \tag{11}
\]
where \( k_\mu \) is discretized as \( k_\mu = 2 \pi n_\mu / L \) with \( n_\mu = 0, 1, 2, \ldots, L - 1 \) for the lattice size \( L \) (where we use a notation slightly different from Ref. \([40]\)). We note that the above form assumes the periodic boundary condition at the spatial edges. Using the smallest nonzero momentum \( k(1, 0) \equiv (2\pi/L, 0) \), we can solve the correlation length as
\[
\xi_0^2 = \frac{\beta^2 \hat{G}_P(0)}{Z_P} \left[ \frac{\hat{G}_P(0)}{\hat{G}_P(k(1, 0))} - 1 \right]. \tag{12}
\]

as given in Ref. \([40]\). As we mentioned in the beginning of this section, dimensionless observables may scale with \( \xi_0 \). For example, in the scaling region in the large-\( N \) expansion, the analytical behavior is known as \([37]\)
\[
\frac{\beta^2 \hat{G}_P(0)}{\xi_0^2} = \frac{3}{2\pi} + O(N^{-1}). \tag{13}
\]

Our central interest in this work lies in the topological properties of the theory. Here, we adopt the following geometrical definition of the topological charge density,
\[
\rho(x) \equiv \frac{1}{2\pi} \left\{ \text{arg} \left[ \text{tr} P(x + \hat{1} + \hat{2}) P(x + \hat{1}) P(x) \right] + \text{arg} \left[ \text{tr} P(x + \hat{2}) P(x + \hat{1} + \hat{2}) P(x) \right] \right\}, \tag{14}
\]
where \( \text{arg} \) denotes the principal value of the complex argument within the interval \( (-\pi, \pi] \). Then the quantized topological charge is
\[
Q = \sum_x \rho(x). \tag{15}
\]
It is mathematically proven that this \( Q \) rigorously takes integer values even on discretized lattice, which is a big advantage to use the \( \mathbb{C}P^{N-1} \) model.

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\(^1\) The efficient procedure to deal with a sharply peaked function of \( \theta \) is the rejection sampling with Lorenzian fitting; see Ref. \([40]\) for details.
Using $Q$ as constructed above, we can define the topological susceptibility as
\[ \chi_t \equiv \sum_x (\rho(x) \rho(0)) = \frac{1}{V} (\langle Q^2 \rangle - \langle Q \rangle^2) . \] (16)

In the large-$N$ expansion the analytical behavior of the topological susceptibility is known as well. That is, in the leading order of the large-$N$ expansion, the topological susceptibility obtains as
\[ \chi_t = \frac{3m_0^2}{\pi N} + O(N^{-2}) , \] (17)
where $m_0$ is the vacuum expectation value of an auxiliary field which gives a dynamical mass for $z(x)$. In terms of the momentum cutoff $M_{\text{cut}}$, it can be expressed as
\[ \rho^2 = M_{\text{cut}}^2 e^{-4\pi \beta} , \] (18)
which leads to the following expression [19],
\[ \chi_t(\beta) = \frac{3M_{\text{cut}}^2}{\pi N} e^{-4\pi \beta} + O(N^{-2}) \] (19)
as a function of $\beta$ at the one-loop level. We can also express this using $\beta$-dependent $\xi_G$ whose leading form is [37]
\[ \xi_G^2 = \frac{1}{6m_0^2} + O(N^{-2}) = \frac{1}{6M_{\text{cut}}^2} e^{4\pi \beta} + O(N^{-2}) . \] (20)

The leading behavior of $\chi_t$ is thus characterized as
\[ \chi_t \xi_G^2 = \frac{1}{2\pi N} + O(N^{-2}) . \] (21)

C. Fourier and Snapshot Entropies

In this paper we will pay our special attention to the Fourier entropy and the snapshot entropy defined by spatial distribution of $\rho(x)$, which will be useful for our image-processing purpose.

We shall introduce the Fourier entropy as the Shannon entropy using the Fourier transformed topological charge density, $\tilde{\rho}(k)$. The normalization convention of discrete Fourier transform (DFT) is chosen as
\[ \tilde{\rho}(k) = \frac{1}{\sqrt{V}} \sum_{x^a} e^{-ik^a x} \rho(x) , \] (22)
where $x^a$ runs over $0, \ldots, L-1$ in units of $a = 1$. We first define the normalized Fourier spectrum as
\[ f_{\text{DFT}}(k) \equiv \frac{\tilde{\rho}(k)^2}{\sum_{k'} \tilde{\rho}(k')^2} , \] (23)
and then the Fourier entropy reads,
\[ S_{\text{DFT}} \equiv -\sum_k f_{\text{DFT}}(k) \ln f_{\text{DFT}}(k) . \] (24)
The entropy quantifies how the topological charge density distributes over space. For $k$-independent constant $\tilde{\rho}$, $S_{\text{DFT}}$ is saturated at $2\ln L$.

Another quantity which reflects the spatial pattern of $\rho(x)$ is the snapshot entropy defined with the singular values. Although our numerical simulation in the present work uses the square lattice, the procedure is applicable for general rectangular lattices. On the lattice the image of the topological charge density is regarded as an $L \times L$ real-valued matrix, and its SVD for $x \equiv (x_1, x_2)$ is written as
\[ \rho(x_1, x_2) = \sum_{n=1}^L \lambda^{(n)} U^{(n)}_{x_1} V^{(n)}_{x_2} . \] (25)

Here, $U^{(n)}$ and $V^{(n)}$ are two sets of orthonormal bases in $L$-dimensional vector space and $\lambda^{(n)}$ represents singular values sorted in descending order. The two-dimensional image of $U^{(n)}_{x_1} V^{(n)}_{x_2}$ is referred to as the $n$-th SVD layer, and the weight for each SVD layer is $\lambda^{(n)}$. Because all $\lambda^{(n)}$’s are nonnegative by construction, we can define the snapshot entropy as the Shannon entropy using $\lambda^{(n)}$. For this purpose we first normalize the singular values as
\[ f_{\text{SVD}}(n) \equiv \frac{\lambda^{(n)}}{\sum_{n'} \lambda^{(n')}} , \] (26)
and then the snapshot entropy reads,
\[ S_{\text{SVD}} \equiv -\sum_{n=1}^L f_{\text{SVD}}(n) \ln f_{\text{SVD}}(n) . \] (27)
The maximum value of $S_{\text{SVD}}$ is $\ln L$.

III. SIMULATION SETUPS AND CONSISTENCY CHECKS

We detail our numerical simulation processes. We have performed the calculations for $N = 2, 3, 10, 21$ with the $L = 32$ lattice size to see the $N$ dependence, and for $N = 10$ with $L = 32, 64$, and $128$ lattice sizes to see the $L$ dependence.

For each combination of $(N, L)$, we have initialized the configuration with the random start (i.e., hot start), and we have confirmed that thermalization is achieved by 2000 Monte-Carlo steps; we have checked this by comparing results in the hot and the cold starts. After thermalization we take 1000 sampling points with an interval by 100 steps to measure physical observables. For the error estimate of physical observables we use the standard Jack-knife method with (bin width) = 10. We have chosen the bin width and the interval to suppress the autocorrelation which is checked by the error estimate of the energy density at several $(N, L)$. Our simulation starts with $\beta = 0.1$ and we increase $\beta$ by 0.1 until $\beta = 1.5$, and for each $\beta$ we repeat the above procedures.
We have quantitatively checked the full consistency of our results and previous results in Ref. [40] for the energy density $E$, the correlation length $\xi_G$, and the topological susceptibility $\chi_t$ at several similar $(N, L)$, as listed in Table I.

Now, let us proceed to more detailed numerical results for respective physical observables. Figure 1 shows the energy density $E$ as a function of $\beta$ for $N = 10$ and $L = 32, 64, 128$. We note that this $E$ is a bare one measured in units of the lattice spacing $a$, and thus a part of the $\beta$-dependence appears from $\beta(a)$ as we pointed out. We make this plot just to see the volume dependence, and it is clear from Fig. 1 that the volume dependence is negligibly small within the error bars. Actually, $L = 32$ is already sufficiently close to the thermodynamic limit. We can understand this from the correlation length shown in Fig. 2. As long as $\beta$ is not too large (as is the case in the present work), the correlation length is significantly smaller than the lattice size $L = 32$ for any $N$, so that the finite size artifact is expected to be small already for $L = 32$.

In Fig. 3 we show $\chi_t$ as a function of $\beta$ for $L = 32$ and $N = 2, 3, 10, 21$. We see from Fig. 3 that $\chi_t$ is suppressed for larger $\beta$, which qualitatively agrees with exponential suppression in Eq. (19). We would point out that numerically obtained $N\chi_t$ has quite nontrivial $N$ dependence, while the convergence of $\xi_G^2$ at large $N$ is just monotonic. In fact, as noticed in Fig. 2, $\xi_G^2$ at $N = 10$ is very close to that at $N = 21$, from which one may want to conclude that $N = 10$ could be already a good approximation for the large-$N$ limit in which analytical formulas are known. In Fig. 3, however, $N\chi_t$ at $N = 10$ is not very close to the $N = 21$ results, and furthermore, around $\beta = 0.5$, the $N$ dependence is found to be nonmonotonic. Such prominent differences between $\xi_G^2$ and $N\chi_t$ clearly indicate that numerically obtained $N\chi_t$ must have more structures than simple scaling with $\xi_G^2$. The next section will be devoted to detailed analysis on this question.

| $(N, L)$     | $\beta$ | $E$         | $\xi_G^2$ | $\chi_t$ |
|--------------|---------|-------------|-----------|---------|
| $(2, 32)$    | 1.1     | 0.55680(43) | 18.7(61)  | 0.00798(35) |
| $(10, 32)$   | 0.7     | 0.78412(23) | 6.5(11)   | 0.00503(25) |
| $(10, 32)$   | 0.8     | 0.66613(23) | 25.4(16)  | 0.000832(44) |
| $(10, 60)$   | 0.8     | 0.66714(13) | 19.5(46)  | 0.000963(44) |
| $(21, 32)$   | 0.7     | 0.73888(14) | 15.7(8)   | 0.000465(32) |

| Previous Results |
|------------------|
| $(N, L)$     | $\beta$ | $E$         | $\xi_G^2$ | $\chi_t$ |
| $(2, 36)$     | 1.1     | 0.55593(14) | 12.11(48) | —       |
| $(10, 42)$    | 0.7     | 0.78402(13) | 5.52(14)  | 0.00505(11) |
| $(10, 30)$    | 0.8     | 0.66591(17) | 25.7(8)   | 0.000465(32) |
| $(10, 32)$    | 0.8     | 0.66801(8)  | 21.80(47) | 0.00101(4) |
| $(21, 36)$    | 0.7     | 0.73888(15) | 14.66(22) | —       |

TABLE I. Comparison of our results and previous results in Ref. [40] for various $(N, L)$ and $\beta$.

FIG. 1. Energy density $E$ as a function of $\beta$ for various $L$'s. The volume dependence is smaller than the dots.

FIG. 2. Correlation length squared, $\xi_G^2$, as a function of $\beta$ for $L = 32$ and various $N$'s.

IV. IMAGE PROCESSING OF THE TOPOLOGICAL CHARGE CORRELATOR

So far, we have discussed that our simulation results are fully consistent with the previous results. Since the validity has been confirmed, we are now going into more microscopic views of the spatial distribution of the topological charge density. For this image-processing purpose we perform the Fourier analysis and then we demonstrate that the Fourier entropy is a useful measure. We also discuss a relation to the SVD analysis which is known as a standard image-processing procedure.
we already observed decreasing $\chi$ in momentum space even when $\chi$ finite momenta, $\chi$. In other words, $\langle | \langle \rho(x) \rangle \rangle$ is nothing but the finite momentum extension of the topological susceptibility, i.e., $\chi_t(k^2)$ as defined in Ref. [56]. Hereafter, we shall use a simple notation of $\chi_t(k^2)$ to mean the Fourier power spectrum. We summarize our results for $\chi_t(k^2)$ computations for $\beta = 0.3, 0.7, 1.0$ in Fig. 4.

We chose these values of $\beta$ according to qualitative changes in the Fourier entropy as we will see later. For $(N, L) = (10, 32)$ we will find that $\beta \approx 0.7$ is a “threshold” for suppression of topological excitations, which we denote by $\beta_{th}$. As long as $\beta \lesssim \beta_{th}$, $\tilde{\rho}(k)$ spreads uniformly over momentum space (see the left panel in Fig. 4). We may say that the topological charge density is “white” then. In contrast to this, small momentum components are significantly suppressed for $\beta \gtrsim \beta_{th}$ and we interpret this behavior as suppression of topological excitations. In fact, it is natural that small momentum components of $\chi_t(k^2)$ are more diminished with increasing $\beta$; the spectral intensity at $k = 0$ is nothing but $\chi_t$, i.e., $\chi_t(0) = \langle | \tilde{\rho}(0) \rangle^2 \rangle = (Q^2/V) = \chi_t$ (where $\langle Q \rangle = 0$), and we already observed decreasing $\chi_t$ with increasing $\beta$ in Fig. 3.

It is intriguing that the topological charge correlator at finite momenta, $\chi_t(k^2) \neq 0$, has such a nontrivial shape in momentum space even when $\chi_t$ is (nearly) vanishing.

We can give a qualitative explanation for this structure. Although the topological charge itself is robust against perturbative fluctuations, the correlation function has a nonzero contribution from topologically trivial sector. Therefore, we can perform one-loop calculation to find nonzero $\chi_t(k^2)$ for $k$ large enough to justify perturbative treatments as

$$\chi_t(k^2) = \frac{3m_c^2}{\pi N} + \frac{3k^2}{10\pi N} - \frac{k^2}{2(2\pi)^2 \beta N},$$

where we approximately adopted a continuum theory and we note that the last term with negative sign would be suppressed for large $\beta$ which is assumed in the continuum limit. This quadratic rise of $\chi_t(k^2)$ accounts for our numerical results at $\beta > \beta_{th}$.

B. Fourier Entropy

It would be convenient if there is any observable whole value characterizes changes as shown in Fig. 4. We propose to use the Fourier entropy defined in Eq. (24).

Figure 5 shows the Fourier entropy as a function of $\beta$ for $L = 32$ and $N = 2, 3, 10, 21$. We see that the Fourier entropy stays constant for $\beta \lesssim \beta_{th}$, where $\beta_{th}$ turns out to depend on $N$. Then, the Fourier entropy starts dropping at $\beta_{th}$. As compared to Fig. 4, the threshold behavior is clearly manifested in Fig. 5.

We also checked the $L$ dependence of the Fourier entropy. We note that the saturated value of $S_{DFT}$ is $2 \ln L$ and thus it contains logarithmic $L$ dependence. Interestingly, we found that the subtracted Fourier entropy, $S_{DFT} - 2 \ln L$, seems to have a well-defined thermodynamic limit. That is, we show the subtracted Fourier entropy as a function of $\beta$ in Fig. 6 for $N = 10$ and $L = 32, 64, 128$. The subtracted Fourier entropy barely has $L$ dependence as confirmed in Fig. 6. This result is consistent with our previous discussion that $L = 32$ is already close to the thermodynamic limit.

C. Snapshot Entropy

The Fourier power spectrum is a useful device for the image processing, and an alternative is the SVD analysis which is suitable for course-graining the image. Let us compute the snapshot entropy using the SVD and check whether anything nontrivial appears near $\beta_{th}$ or not.

In Fig. 7 we plot $S_{SVD}$ as a function of $\beta$ for $L = 32$ and $N = 2, 3, 10, 21$, which is a SVD counterpart of the previous plot in Fig. 5.

Instead of clear threshold behavior in Fig. 5, we found that $S_{SVD}$ exhibits a dip around $\beta_{th}$ as seen in Fig. 7. Similarly to the Fourier entropy, in the case of $S_{SVD}$, the depth and the location of the dip depend on $N$. Interestingly, the $L$ dependence of $S_{SVD}$ is quite different from that of $S_{DFT}$. Figure 8 shows the subtracted snapshot entropy, $S_{SVD} - \ln L$, for $N = 10$ and $L = 32, 64, 128$. We
FIG. 4. Fourier power spectrum or $\chi_t(k^2)$ for $(N, L) = (10, 32)$ and $\beta = 0.3, 0.7, 1.0$. The momenta $k_1$ and $k_2$ run from $-\pi$ to $\pi$ in the lattice unit.

FIG. 5. Fourier entropy as a function of $\beta$ for $L = 32$ and various $N$.

FIG. 6. Subtracted Fourier entropy, $S_{\text{DFT}} - 2\ln L$, as a function of $\beta$ for $N = 10$ and various $L$.

FIG. 7. Snapshot entropy, $S_{\text{SVD}}$, as a function of $\beta$ for $L = 32$ and various $N$.

FIG. 8. Subtracted snapshot entropy, $S_{\text{SVD}} - \ln L$, as a function of $\beta$ for $N = 10$ and various $L$. 
see that some sizable $L$ dependence remains even after the subtraction, which makes a sharp contrast to Fig. 6. Such $L$ dependent results are highly nontrivial. We would point out an example of analytically calculable $S_{\text{SVD}}$: in a real random matrix theory, $S_{\text{SVD}}^{(\text{RM})} = \ln L - \pi/4$ is known \cite{58}. Thus, $L$ dependent $S_{\text{SVD}}$ may already indicate that the theory under consideration has some interesting features.

In order to locate $\beta_{\text{th}}$ for numerical simulations in a finite size box, $S_{\text{SVD}}$ is as useful as $S_{\text{DFT}}$. However, it is evident from Fig. 8 that the dip depth becomes shallower for larger $L$, implying that the dip may eventually disappear in the thermodynamic limit unless we know the proper $L$ scaling. Therefore, for a practical usage, $S_{\text{DFT}}$ would be a more tractable choice.

D. Correlation between Fourier and Snapshot Entropies

It would be instructive to clarify a possible connection of the Fourier spectrum and the SVD spectrum of the topological charge density. To this end we have calculated the Fourier spectrum of each SVD layer of the topological charge density. We note that each SVD layer is a direct product of two vectors, $U_{xz}^{(n)} V_{xz}^{(n)}$, so that the Fourier transform with respect to two spatial directions is trivially factorized, and we can separately discuss $\tilde{U}^{(n)}(k_1)$ [that is a Fourier transform of $U_{xz}^{(n)}$] and $\tilde{V}^{(n)}(k_2)$ [that is a Fourier transform of $V_{xz}^{(n)}$]. From symmetry between 1, 2 directions, clearly, it is sufficient to consider only $\langle |U^{(n)}(k_1)|^2 \rangle$ without loss of generality, and Fig. 9 shows our results. We recall that in our convention a smaller SVD layer index corresponds to a larger SVD eigenvalue. We see that the Fourier spectrum is “white” at $\beta \lesssim \beta_{\text{th}}$ for all the SVD layers as is the case in the left panel of Fig. 9. Some characteristic patterns start emerging around $\beta_{\text{th}} \approx 0.7$ as observed in the middle and the right panels of Fig. 9, namely, images with smaller layer index (i.e., larger SVD eigenvalue) are more dominated by large momentum modes, while images with larger layer index contain smaller momentum modes. Interestingly, this present situation is quite unusual: if the SVD is used for the image processing of ordinary snapshot photographs, usually, smaller SVD index layers would typically correspond to a partial image with smaller momenta or larger spatial domains. A general trend is that such an ordinary correspondence in the image processing holds for classical systems, and for quantum systems the correspondence could be reversed due to quantum fluctuations. It is a subtle question whether $\rho(x)$ belongs to classical or quantum class. Our numerical results suggest a quantum nature, so that if we want to course-grain $\rho(x)$, we should remove SVD layers from the smallest index.

V. CONCLUSIONS

We conducted the numerical Monte-Carlo simulation using the $\mathbb{C}P^{N-1}$ model. We first checked the simulation validity by comparing our results with the precedent studies for $N = 2, 3, 10,$ and $21$. We then scrutinized the spatial distribution of the topological charge density for various coupling constant $\beta$ by means of the Fourier analysis and the singular value decomposition. There, we found that the Fourier power spectrum of the topological charge density is rather structureless in momentum space for small $\beta$, while small momentum components become diminished for $\beta$ above a certain threshold $\beta_{\text{th}}$. At the same time, nontrivial structures (i.e., a drop in the Fourier entropy and a dip in the snapshot entropy) appear also around $\beta_{\text{th}}$. We clarified a correlation between the Fourier and the snapshot entropies. In contrast to the ordinary image-processing of picture images, SVD layers with larger SVD eigenvalues turn out to be dominated with higher momentum components. Thus, a cooling method could be implemented by removing the SVD layers from lower index (with larger SVD eigenvalue).

A striking finding in this work is that the Fourier power spectrum of the topological charge density or the finite momentum extended topological susceptibility, $\chi_t(k^2)$, has nontrivial momentum dependence even at large $\beta$ when the topological susceptibility itself is nearly vanishing. This indicates that the topological contents of the theory are not necessarily empty even when the topological susceptibility approaches zero if one explores finite momentum regions. One might think that nonzero topological fluctuations at finite momenta arise merely from perturbative loops and may not involve topological windings. This is true, and nevertheless, this situation is still interesting. We would recall that such property of $\chi_t(k^2)$ is reminiscent of the sphaleron rate which is a real-time quantity analytically continued from the topological susceptibility. In a seminal work in Ref. [61] it has been shown that the sphaleron rate has finite contributions from zero winding sector, though the analytical continuation makes such terms disappear in the topological susceptibility. In other words, the sphaleron rate is a finite frequency extension from $\chi_t$, and we are discussing a finite (spatial) momentum extension from $\chi_t$, and both are nontrivial.

In principle, it should be possible to calculate $\chi_t(k^2)$ in the pure gluonic theory (as well as in QCD), and then it would be interesting to ask whether the pure gluonic theory and QCD may lead to $\chi_t(k^2)$ with $k^2$ increasing behavior or not. Alternatively, the measurement of the Fourier entropy using the topological charge density should be feasible in any other theory in which the topological charge is defined. It is, however, tricky how to carry out the SVD for three or higher dimensional data, and so the applicability of the snapshot entropy is limited to two dimensional field theories.

An interesting future problem would be inclusion of farther interaction terms in the $\mathbb{C}P^{N-1}$ model, with
which nontrivial phase structures are realized. Another challenging problem in the $\mathbb{C}P^{N-1}$ model is how to identify the instantons and the bions numerically in the $\mathbb{C}P^{N-1}$ model simulations. In principle, all information on such special configurations should be encoded on the Fourier power spectrum of the topological charge density. Such questions as well as direct applications to QCD physics should deserve judicious investigations in the future.

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FIG. 9. SVD layer dependence of the Fourier spectrum for $(N,L) = (10,32)$ and $\beta = 0.3, 0.7, 1.0$. 

![Graph showing the Fourier spectrum for different values of $\beta$ and layer index.](image-url)
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