PODLEŠ' QUANTUM SPHERE: DUAL COALGEBRA AND CLASSIFICATION OF COVARIANT FIRST ORDER DIFFERENTIAL CALCULUS

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ABSTRACT. The dual coalgebra of Podleš' quantum sphere \( O_q(S^2_c) \) is determined explicitly. This result is used to classify all finite dimensional covariant first order differential calculi over \( O_q(S^2_c) \) for all but exceptional values of the parameter \( c \).

1. Introduction

Podleš’ quantum sphere \( O_q(S^2_c) \) \cite{Pod87} is one of the best investigated examples of a quantum space, i.e. of a comodule algebra over the \( q \)-deformed coordinate ring of some affine algebraic group. Nevertheless, classification of covariant first order differential calculus (FODC) over \( O_q(S^2_c) \) in the sense of Woronowicz \cite{Wor89} has so far been achieved only under additional assumptions and in low dimensions. In \cite{Pod92} certain 2-dimensional covariant FODC over \( O_q(S^2_c) \) which in many respects behave similarly as their classical counterparts have been classified. It turned out that only in the so called quantum subgroup case \( c = 0 \) such a calculus exists and is then uniquely determined. All covariant FODC which as right modules are freely generated by the differentials of the generators \( e_i \), \( i = -1, 0, 1 \) of \( O_q(S^2_c) \) have been determined in \cite{AS94}. It was shown by computer calculations that for all but exceptional values of \( c \) exactly one such calculus exists. Finally in \cite{Her02} a general notion of dimension of covariant FODC was introduced and all 2-dimensional covariant FODC over \( O_q(S^2_c) \) have been classified.

In the present paper all finite dimensional covariant FODC over \( O_q(S^2_c) \) for all but exceptional values of \( c \) are classified. It turns out, that for generic \( c \) there exists precisely one irreducible covariant FODC for any irreducible \( O_q(S^2_c) \)-subcomodule of \( O_q(S^2_c) \). The submodule \( \mathbb{C} \cdot 1 \) corresponds to the trivial calculus while in general the irreducible differential calculus has the same dimension as the corresponding \( O_q(SL(2)) \)-submodule. For generic \( c \) any covariant FODC over \( O_q(S^2_c) \) can be uniquely written as a direct sum of irreducible FODC. The exceptional cases include the quantum subgroup case \( c = 0 \).

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The main tool on this way is the notion of quantum tangent space introduced for quantum groups in [Vor89] and generalized to a large class of quantum spaces in [HK01]. Podleś' quantum sphere can be obtained as right \( K_c \)-invariant elements in \( \mathcal{O}_q(\text{SL}(2)) \) where \( K_c \) denotes a left coideal subalgebra of \( U_q(\mathfrak{sl}_2) \) generated by one twisted primitive element \( X_c \). The notion of quantum tangent space allows one to identify finite dimensional covariant FODC over \( \mathcal{O}_q(\mathfrak{sl}_2) \) with finite dimensional left subcomodules \( T_\varepsilon \subset \mathcal{O}_q(\mathfrak{sl}_2)^\circ \) of the dual coalgebra which are right \( K_c \)-invariant and contain the counit \( \varepsilon \). Thus as a first step towards classification the dual coalgebra \( \mathcal{O}_q(\mathfrak{sl}_2)^\circ \) is determined explicitly in Theorem 3.1. It turns out that for all but exceptional values of \( c \) the restriction \( \mathcal{O}_q(\text{SL}(2))^{\circ} \to \mathcal{O}_q(\mathfrak{sl}_2)^\circ \) is onto.

Next, the subspace \( F(\mathcal{O}_q(\mathfrak{sl}_2)^\circ, K_c) \) of elements of \( \mathcal{O}_q(\mathfrak{sl}_2)^\circ \) with finite right \( K_c \)-action is determined. The action of the generator \( X_c \) induces a \( U_q(\mathfrak{sl}_2) \)-action on \( F(\mathcal{O}_q(\mathfrak{sl}_2)^\circ, K_c) \) such that the decomposition into irreducible \( U_q(\mathfrak{sl}_2) \)-modules corresponds to the decomposition into right \( K_c \)-invariant left \( \mathcal{O}_q(\mathfrak{sl}_2)^\circ \)-comodules. To calculate \( F(\mathcal{O}_q(\mathfrak{sl}_2)^\circ, K_c) \) explicit results of [MS99] are employed.

The quantum tangent spaces of the covariant FODC constructed in [Her98] are calculated. It turns out that for generic \( c \) the resulting tangent spaces cover all covariant FODC over \( \mathcal{O}_q(\mathfrak{sl}_2) \) can be constructed by this method. Moreover it is shown in Proposition 5.2 that these FODC are free left and right \( \mathcal{O}_q(\mathfrak{sl}_2) \)-modules and inner calculi.

The ordering of this paper is as follows. In Section 2 the definition and some properties of \( \mathcal{O}_q(\mathfrak{sl}_2) \) are recalled. Section 3 serves to give a complete description of the dual coalgebra \( \mathcal{O}_q(\mathfrak{sl}_2)^\circ \). The main idea on this way is to show that all representations of \( \mathcal{O}_q(\mathfrak{sl}_2) \) can be written as direct sums of representations of certain localizations of \( \mathcal{O}_q(\mathfrak{sl}_2) \). These localizations are seen to be isomorphic to \( U_q(\mathfrak{b}_-)^{\text{op}} \) and the dual coalgebra of \( U_q(\mathfrak{b}_-)^{\text{op}} \) is known [Jos93]. In Section 4 the subspace \( F(\mathcal{O}_q(\mathfrak{sl}_2)^\circ, K_c) \) is determined and decomposed into \( U_q(\mathfrak{sl}_2) \)-modules. The notion of covariant FODC and quantum tangent space are recalled in the last section. Combination of the above steps lead to the classification result in Theorem 5.1.

If not stated otherwise all notations and conventions coincide with those introduced in [KS97]. All through this paper \( q \in \mathbb{C} \setminus \{0\} \) will be assumed not to be a root of unity. For any element \( a \) of a coalgebra \( \mathcal{A} \) with counit \( \varepsilon \) define \( a^+ := a - \varepsilon(a) \) and for any subset \( \mathcal{B} \subset \mathcal{A} \) set \( \mathcal{B}^+ := \{b^+ \mid b \in \mathcal{B}\} \).

2. Podleś' Quantum Sphere

Let \( u^j_i, i, j = 1, 2 \), denote the matrix coefficients of the vector representation of \( U_q(\mathfrak{sl}_2) \), i.e. the generators of the quantum group \( \mathcal{O}_q(\text{SL}(2)) \). In the notation of [Pod87] the matrix coefficients of the three dimensional representation of \( U_q(\mathfrak{sl}_2) \)
are given by
\[
(\pi^i_j)_{i,j=-1,0,1} = \begin{pmatrix}
  u_2^2 u_2^2 & -(q^2 + 1) u_2^1 u_1^2 & -q u_1^2 u_1^1 \\
  -q^{-1} u_1^2 u_2^2 & 1 + (q + q^{-1}) u_2^1 u_1^2 & u_1^1 u_1^2 \\
  -q^{-1} u_1^1 u_2^1 & (q + q^{-1}) u_2^1 u_1^1 & u_1^1 u_1^1
\end{pmatrix}.
\]

For \( c = (\varepsilon(e_{-1}) \varepsilon(e_1) : \varepsilon(e_0)^2) \in \mathbb{C} P^1 \) Podleś’ quantum sphere \( \mathcal{B} = \mathcal{O}_q(S^2_{\varepsilon}) \) \cite{Pod87} is isomorphic to the subalgebra of \( \mathcal{O}_q(\text{SL}(2)) \) generated by \( e_i = \sum_{j=-1,0,1} \varepsilon(e_j) \pi^i_j, \) \( i = -1, 0, 1. \) The algebra \( \mathcal{O}_q(S^2_{\varepsilon}) \) obtains the structure of a right \( \mathcal{O}_q(\text{SL}(2)) \)-comodule algebra by \( \Delta(e_i) = e_j \otimes \pi^i_j. \) A complete set of defining relations of \( \mathcal{O}_q(S^2_{\varepsilon}) \) is given by

\[
(1 + q^2)(e_{-1} e_1 + q^{-2} e_1 e_{-1}) + e_0^2 = \rho
\]

\[
-q^2 e_{-1} e_0 + e_0 e_{-1} = \lambda e_{-1}
\]

\[
(1 + q^2)(e_{-1} e_1 - e_1 e_{-1}) + (1 - q^2) e_0^2 = \lambda e_0
\]

\[
e_1 e_0 - q^2 e_0 e_1 = \lambda e_1
\]

where \( \rho = q^{-2}(q^2 + 1)^2 \varepsilon(e_{-1}) \varepsilon(e_1) + \varepsilon(e_0)^2 \) and \( \lambda = (1 - q^2) \varepsilon(e_0). \) For \( c \neq \infty \) one can choose \( \varepsilon(e_0) = 1, \varepsilon(e_{-1}) = \varepsilon(e_1). \) Then \( \lambda = 1 - q^2 \) and \( \rho = (q + q^{-1})^2 c + 1. \)

Defining \( A = (1 + q^2)^{-1}(1 - e_0) \) the above relations can be rewritten as

\[
(2.1) \quad e_{-1} e_1 = A - A^2 + c
\]

\[
(2.2) \quad e_1 e_{-1} = q^2 A - q^4 A^2 + c
\]

\[
(2.3) \quad e_1 A = q^2 A e_1
\]

\[
(2.4) \quad e_{-1} A = q^{-2} A e_{-1}.
\]

Similarly for \( c = \infty \) choose \( \varepsilon(e_0) = 0 \) and \( \varepsilon(e_{-1}) = \varepsilon(e_1) = 1, \) i.e. \( \lambda = 0 \) and \( \rho = (q + q^{-1})^2. \) Defining \( A = -(1 + q^2)^{-1} e_0 \) the above relations are equivalent to

\[
(2.5) \quad e_{-1} e_1 = -A^2 + 1
\]

\[
(2.6) \quad e_1 e_{-1} = -q^4 A^2 + 1
\]

\[
(2.7) \quad e_1 A = q^2 A e_1
\]

\[
(2.8) \quad e_{-1} A = q^{-2} A e_{-1}.
\]

Define linear functionals \( f_\lambda, \lambda \in \mathbb{C} \setminus \{0\} \) and \( g \) in the dual Hopf algebra \( \mathcal{O}_q(\text{SL}(2))^\circ \) of \( \mathcal{O}_q(\text{SL}(2)) \) by

\[
(2.9) \quad f_\lambda((u_j^i)) = \begin{pmatrix}
  \lambda & 0 \\
  0 & \lambda^{-1}
\end{pmatrix}, \quad g((u_j^i)) = \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}
\]

and

\[
(2.10) \quad \Delta f_\lambda = f_\lambda \otimes f_\lambda, \quad \Delta g = g \otimes \varepsilon + \varepsilon \otimes g.
\]
Recall that the dual pairing \([KS97]\) between \(U_q(\mathfrak{sl}_2)\) and \(\mathcal{O}_q(\text{SL}(2))\) induces linear functionals \(E\) and \(F\) in \(\mathcal{O}_q(\text{SL}(2))^\circ\) satisfying
\[
(2.11) \quad E((u_j^i)) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad F((u_j^i)) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
and
\[
(2.12) \quad \Delta E = E \otimes K + \varepsilon \otimes E, \quad \Delta F = F \otimes \varepsilon + K^{-1} \otimes F
\]
where \(K = f_q^{-1}\). Let \(\mathcal{U} \subset \mathcal{O}_q(\text{SL}(2))^\circ\) denote the algebra generated by the functionals \(f_\lambda, \lambda \in \mathbb{C} \setminus \{0\}\), \(E\), \(F\) and \(g\). For transcendental \(q\) the Hopf algebra \(\mathcal{U}\) is isomorphic to \(\mathcal{O}_q(\text{SL}(2))^\circ\) \([Jos95, 9.4.9]\). The above functionals satisfy the relations
\[
f_\lambda f_\mu = f_{\lambda \mu}, \quad f_\lambda E = \lambda^{-2} E f_\lambda, \quad f_\lambda F = \lambda^2 F f_\lambda, \quad f_\lambda g = g f_\lambda, \quad E g = (g + 2) E, \quad F g = (g - 2) F,
\]
\[
EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.
\]

Note that the subalgebra of \(\mathcal{O}_q(\text{SL}(2))^\circ\) generated by \(E, F, K\) and \(K^{-1}\) is isomorphic to \(U_q(\mathfrak{sl}_2)\) \([KS97, 4.4.1]\). Evaluating the functionals \(f_\lambda, g, E\) and \(F\) on the matrix coefficients \(\pi_j^i\) one obtains
\[
f_\lambda((\pi_j^i)) = \lambda^{-2} E_{-1}^{-1} + E_0^0 + \lambda^2 E_1^0
\]
\[
g((\pi_j^i)) = -2E_{-1}^{-1} + 2E_1^1
\]
\[
E((\pi_j^i)) = -(q^2 + 1) E_0^{-1} + E_1^0
\]
\[
F((\pi_j^i)) = -q^{-1} E_{-1}^{-1} + (q + q^{-1}) E_0^0
\]
where \(E_j^i, i, j = -1, 0, 1\) denotes the \(3 \times 3\)-matrix with entry 1 at position \((i, j)\) and zero elsewhere. The right comodule structure of \(\mathcal{O}_q(S^2_\mathbb{C})\) induces a left action of \(\mathcal{U}\) on \(\mathcal{O}_q(S^3_\mathbb{C})\) which is given by
\[
(2.15) \quad f_\lambda \triangleright e_i = \lambda^{2i} e_i
\]
\[
(2.16) \quad g \triangleright e_i = 2ie_i
\]
\[
(2.17) \quad E \triangleright e_i = -(q^2 + 1) \delta_{i,0} e_{-1} + \delta_{i,1} e_0
\]
\[
(2.18) \quad F \triangleright e_i = -q^{-1} \delta_{i,-1} e_0 + (q + q^{-1}) \delta_{i,0} e_1.
\]

For \(n \in \mathbb{N}_0/2\) set \(c(n) = -1/(q^n + q^{-n})^2\). Since \(q\) is not a root of unity \(c(n) \neq c(m)\) for all \(n, m \in \mathbb{N}_0/2, n \neq m\). Define subsets of \(\mathbb{C}P^1\) by
\[
J_1 := \{ c \in \mathbb{C}P^1 \mid c \neq c(n) \quad \forall n \in \mathbb{N}/2 \setminus \mathbb{N} \}
\]
\[
J_2 := \{ c \in \mathbb{C}P^1 \mid c \neq c(n) \quad \forall n \in \mathbb{N}_0/2 \}
\]

It is known \([MS95]\) Rem. 4.5.3] that the following statements are equivalent:
1. $c \in J_1$
2. $\mathcal{O}_q(S^2) \cong \{ b \in \mathcal{O}_q(SL(2)) \mid X(b_{(1)})b_{(2)} = 0 \}$ for a twisted primitive element $X = \alpha(K^{-1} - 1) + \beta K^{-1}E + \gamma F \in \mathcal{U}$

and

$$c = \begin{cases} \frac{\beta \gamma q^{-1}}{\alpha^2(q - q^{-1})^2} & \text{if } \alpha \neq 0, \\ \infty & \text{if } \alpha = 0 \text{ and } \beta \gamma \neq 0. \end{cases}$$

Calculating the pairing between $X$ and the explicit generators $e_i \in \mathcal{O}_q(SL(2))$ chosen above one obtains $-\varepsilon(e_1)(q - q^{-1})\alpha = \gamma$ and $\beta = q\gamma$ in the case $c \neq \infty$. Similarly for $c = \infty$ one obtains $\alpha = 0$ and $\beta = q\gamma$. Thus the embeddings from above are realized by

$$X_c = \begin{cases} qK^{-1}E + F & \text{if } c = \infty, \\ K & \text{if } c = 0, \\ -(c^{1/2}(q - q^{-1}))^{-1}(K^{-1} - 1) + qK^{-1}E + F & \text{else} \end{cases}$$

for any square root $\varepsilon(e_1) = c^{1/2}$ of $c$. Define $K_c = \mathbb{C}[X_c] \subset U_q(\mathfrak{so}_2)$. If $c \in J_2$ then any finite dimensional $U_q(\mathfrak{so}_2)$-module is a direct sum of irreducible $K_c$-modules and therefore $\mathcal{O}_q(SL(2))$ is a faithfully flat left (and right) $\mathcal{O}_q(S^2)$-module [MS99, Thm. 5.2].

3. The Dual Coalgebra $\mathcal{B}^\circ = \mathcal{O}_q(S^2)^\circ$

To understand the dual coalgebra [Swe69, Sect. 6.0] $\mathcal{O}_q(S^2)^\circ$ of Podleš’ quantum sphere it is useful to consider first the dual Hopf algebra $(U_q(\mathfrak{b}_-)^{\text{op}})^\circ$ where $U_q(\mathfrak{b}_-) \subset U_q(\mathfrak{so}_2)$ denotes the subalgebra generated by $F, K$ and $K^{-1}$. Further let $U_0, U_q(\mathfrak{n}_+), U_q(\mathfrak{n}_-)$ and $U_q(\mathfrak{b}_+)$ denote the subalgebra of $U_q(\mathfrak{so}_2)$ generated by $\{K, K^{-1}\}, E, F$ and $\{E, K, K^{-1}\}$, respectively. By [Jos95, Thm. 2.1.8] the dual Hopf algebra $(U_0)^\circ$ is isomorphic to the commutative Hopf algebra

$$\mathbb{C}[\gamma, \chi_\lambda \mid \lambda \in \mathbb{C} \setminus \{0\}]/(\chi_\lambda\chi_\mu = \chi_{\lambda\mu}, \chi_1 = 1)$$

where $\gamma(K) = 1$, $\chi_\lambda(K) = \lambda$ and the coalgebra structure is given by

$$\Delta\gamma = \gamma \otimes 1 + 1 \otimes \gamma$$

$$\Delta\chi_\lambda = \chi_\lambda \otimes \chi_\lambda.$$ 

The subalgebra $U_q(\mathfrak{n}_+) \subset U_q(\mathfrak{so}_2)$ is a right $U_0$-comodule with coaction

$$\delta_R(E^i) = E^i \otimes K^{-i}$$

and therefore has a left $(U_0)^\circ$-module structure. The corresponding left crossed product algebra $U_q(\mathfrak{n}_+) \rtimes (U_0)^\circ$ is a Hopf algebra with $\Delta E = 1 \otimes E + E \otimes \chi_{q^{-2}}$ containing $U_q(\mathfrak{b}_+)$ where $K \in U_q(\mathfrak{b}_+)$ corresponds to $\chi_{q^{-2}}$. The dual pairing of Hopf algebras (in the conventions of [KS97, 6.3.1])

$$\langle \cdot, \cdot \rangle : U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)^{\text{op}} \to \mathbb{C}$$
given by \( \langle K, K \rangle = q^{-2}, \langle K, F \rangle = \langle E, K \rangle = 0 \) and \( \langle E, F \rangle = 1/(q^{-1} - q) \) extends to a pairing of Hopf algebras
\[
\langle \cdot, \cdot \rangle : (U_q(n_+) \rtimes (U_0)^\circ) \otimes U_q(b_-)^{\text{op}} \to \mathbb{C}
\]
such that
\[
\langle \gamma, K \rangle = 1, \quad \langle \chi_\lambda, K \rangle = \lambda, \quad \langle \gamma, F \rangle = \langle \chi_\lambda, F \rangle = 0.
\]

**Lemma 3.1.** For \( a \in U_q(n_+), u \in U_0, b \in U_q(n_-) \) and \( f \in (U_0)^\circ \) one has
\[
\langle af, bu \rangle = f(u) \langle a, b \rangle.
\]
In particular the pairing (3.3) is non-degenerate.

*Proof.* Note first that \( \langle f, b \rangle = \varepsilon(b) f(1) \) and therefore
\[
\langle f, bu \rangle = \langle f(1), u \rangle \langle f(2), b \rangle = \varepsilon(b) \langle f, u \rangle.
\]
Using this relation one calculates
\[
\langle af, bu \rangle = \langle a, b_{(1)} u_{(1)} \rangle \langle f, b_{(2)} u_{(2)} \rangle = \langle a, bu_{(1)} \rangle \langle f, u_{(2)} \rangle = \langle a, b \rangle \langle f, u \rangle
\]
where in the last equation the property \( \langle a, bu \rangle = \varepsilon(u) \langle a, b \rangle \) of (3.2) is used. The non-degeneracy of (3.3) now follows from the non-degeneracy of (3.2). \( \Box \)

By the above lemma the map of Hopf algebras
\[
\Phi : (U_q(n_+) \rtimes (U_0)^\circ) \to (U_q(b_-)^{\text{op}})^\circ
\]
induced by (3.3) is injective. The following result is proven in [Jos95, 9.4.8] for transcendental \( q \). Yet it also holds for \( q \in \mathbb{C} \setminus \{0\} \) not a root of unity and is reproduced here in our setting for the convenience of the reader.

**Proposition 3.1.** The map \( \Phi \) is an isomorphism.

*Proof.* Recall that there is a canonical isomorphism \( U_q(b_-)^{\text{op}} \cong U_q(n_-) \otimes U_0 \) of vector spaces. Let \( J \subset U_q(b_-)^{\text{op}} \) denote any two sided ideal of finite codimension. Then \( J \) contains some ideal \( I \subset U_0 \) of finite codimension and \( (U_q(n_-)^+)^n \) for some \( n \in \mathbb{N} \). Therefore \( J \) contains the left ideal
\[
(U_q(n_-)^+)^n \otimes U_0 + U_q(n_-) \otimes I \subset U_q(n_-) \otimes U_0
\]
of finite codimension. Thus
\[
(U_q(b_-)^{\text{op}}/J)^\circ \subset ((U_q(n_-)/(U_q(n_-)^+)^n) \otimes (U_0/I))^* = (U_q(n_-)/(U_q(n_-)^+)^n)^* \otimes (U_0/I)^* \subset U_q(n_+) \otimes (U_0)^{\text{op}}
\]
where in the last inclusion one uses that \( U_q(n_+) \) is the graded dual of \( U_q(n_-) \) via the pairing (3.2). By Lemma 3.1 one obtains \( U_q(n_+) \otimes (U_0)^{\text{op}} \subset \text{Im} \Phi \) and therefore \( \Phi \) is onto. \( \Box \)
Lemma 3.2. Any finite dimensional representation \( \mu : O_q(S^2_c) \to \text{End}(V) \) is a direct sum \( \mu = \mu_0 \oplus \mu_{\neq 0} \) where \( \mu_0(A) \) is nilpotent and \( \mu_{\neq 0}(A) \) is invertible. In particular the coalgebra \( O_q(S^2_c) \) is a direct sum \( C_0 \oplus C_{\neq 0} \) where \( C_0 \) and \( C_{\neq 0} \) denote the coalgebras of matrix coefficients of finite dimensional representations of \( O_q(S^2_c) \) with nilpotent and invertible \( A \) action, respectively. In addition

1. if \( c \neq c(n) \) for all \( n \in \mathbb{N} \) then \( \mu_{\neq 0} = 0 \).
2. if \( c \neq 0 \) then \( \mu_0(e_{\pm 1}) \) are isomorphisms.
3. if \( c = c(n) \) for some \( n \in \mathbb{N} \) then there exists exactly one indecomposable representation \( \mu_n : O_q(S^2_c) \to \text{End}(V) \) such that \( \mu_n(A) \) is invertible. This representation is \( n \)-dimensional.
4. if \( c = 0 \) then \( C_0 = C_{0+} \oplus C_{00} \oplus C_{0-} \) where \( C_{0\pm} \) (resp. \( C_{00} \)) denotes the coalgebra of matrix coefficients of finite dimensional representations with invertible action of \( e_{\pm 1} \) (with nilpotent action of \( e_1 \) and \( e_{-1} \)).

Proof. Relation (2.3) and (2.4) imply that \( e_1 \) and \( e_{-1} \) transform the generalized eigenspace \( V_{\lambda} \) of \( A \) with corresponding eigenvalue \( \lambda \) to the generalized eigenspace \( V_{q^{2}\lambda} \) and \( V_{q^{2}\lambda} \), respectively. Set \( V_{\neq 0} := \oplus_{\lambda \neq 0} V_{\lambda} \). Then \( V = V_0 \oplus V_{\neq 0} \) is a direct sum of representations of \( O_q(S^2_c) \).

Since \( q \) is not a root of unity \( e_1 \) and \( e_{-1} \) act nilpotently on \( V_{\neq 0} \). Assume that \( v \in V_{\neq 0} \) is an eigenvector of \( A \) with eigenvalue \( \lambda \) such that \( e_{-1} v = 0, e_1^2 v = 0 \) and \( w := e_1^{n-1} v \neq 0 \). Then relations (2.2), (2.4) and (2.1), (2.5) applied to \( v \) and \( w \) respectively imply

\[
\begin{align*}
0 &= q^2 \lambda - q^4 \lambda^2 + c & \quad & \text{for } c \neq \infty \\
0 &= q^{-(2(n-1))} \lambda - q^{-4(n-1)} \lambda^2 + c \\
0 &= -q^4 \lambda^2 + 1 \\
0 &= -q^{-4(n-1)} \lambda^2 + 1 & \quad & \text{for } c = \infty.
\end{align*}
\]

The second set of equations cannot be fulfilled as \( q \) is not a root of unity. The first set of equations implies \( c = c(n) \) and therefore proves 1).

Since \( \mu_0(A) \) is nilpotent the second statement follows from (2.1) and (2.3).

To prove the third statement assume first that there exists \( u \in V_{\neq 0} \) such that \( (A-\nu)^2 u = 0 \) but \( (A-\nu) u \neq 0 \) for some \( \nu \in \mathbb{C} \setminus \{0\} \). Applying \( e_{-1} \) several times we may assume using the notations from above that \( \nu = \lambda \) and \( (A-\nu) u = v \). Then (3.3) implies \( \lambda = q^{n-2}/(q^n + q^{-n}) \). The relation \( e_{-1} v = 0 \) implies that \( e_{-1} u \) is an eigenvector of \( A \) with corresponding eigenvalue \( q^2 \lambda \) or \( e_{-1} u = 0 \). Suppose that \( e_{-1} u = 0 \) for some \( k \geq 1 \) and \( e_{-1}^k u \neq 0 \). Then on the one hand the eigenvalue of \( A \) corresponding to \( e_{-1}^k u \) coincides with \( q^{2k-2} \lambda \) while on the other hand by (3.5) it is equal to \( \lambda \). Therefore \( k = 1 \) and \( e_{-1} u = 0 \). By equation (2.2)
and \((A - \lambda)^2 u = 0\) one now obtains

\[(q^2 - 2q^4\lambda)Au + (c + q^4\lambda^2)u = -q^2\frac{q^n - q^{-n}}{q^n + q^{-n}}Au + (c + q^4\lambda^2)u = 0.\]

As \(n \geq 1\) and \(q^{2n} \neq 1\) this is a contradiction to the assumption that \(u\) is not an eigenvector of \(A\). Thus \(A\) is diagonalisable. The relations \(\text{(3.3)}\) imply that all eigenvalues of \(A\) lie in the set \(\{ q^{n-2k}/(q^n + q^{-n}) \mid k = 1, 2, \ldots, n \}\). In view of \(\text{(2.1)}\) and \(\text{(2.2)}\) the eigenspaces for different eigenvalues are isomorphic and \(V_{\neq 0}\) is the direct sum of the \(O_q(S_c^2)\)-orbits of the elements of an arbitrary basis of \(V_\lambda\). These orbits have dimension \(n\).

To validate the last statement note first that any finite dimensional representation \(\mu : O_q(S_0^2) \to \text{End}(V)\) is a direct sum \(\mu = \mu_+ \oplus \mu_-\), \(V = V_+ \oplus V'\) where \(\mu_+(e_1)\) is invertible and \(\mu_-'(e_1)\) is nilpotent. Indeed, \(\text{(2.3)}\) implies that \(AV_+ \subset V_+\) and \(AV' \subset V'\). On the other hand \(\text{(2.1)}\) leads to

\[e_{-1}V_+ = e_{-1}e_1V_+ = (A - A^2)V_+ \subset V_+\]

and \(e_1^kV' = 0\) yields

\[e_1^{k+1}e_{-1}V' = e_1^k(q^2A - q^4A^2)V' \subset e_1^kV' = 0.\]

Note then that \(\text{(2.1)}, \text{(2.3)}\) and the nilpotency of \(\mu(A)\) imply that \(\mu_+(e_{-1})\) is nilpotent. Similarly \(\mu_- = \mu_0 \oplus \mu_-\) where \(\mu_-'(e_{-1})\) is nilpotent and \(\mu_- (e_{-1})\) is invertible.

The inclusion \(O_q(S_c^2) \subset O_q(SL(2))\) of right \(O_q(SL(2))\)-comodule algebras induces a map of right \(O_q(SL(2))^\circ\)-module coalgebras \(O_q(SL(2)) \to O_q(S_0^2)^\circ\). For \(m, l \in \mathbb{N}_0\) and \(\lambda \in \mathbb{C} \setminus \{0\}\) let \(\psi_{\lambda, l}^m\) denote the image of \(f_\lambda g^mE\) under this projection. It follows from \(\text{(2.13)}\) that \(f_\lambda = f_{-\lambda}\) on \(O_q(S_c^2)\) and therefore the definition of \(\psi_{\lambda, m}^l\) does not depend on the choice of a root of \(\mu\).

**Theorem 3.1.** The following sets form a vector space basis of \(O_q(S_c^2)^\circ\).

1. If \(c \notin \{ 0, c(n) \mid n \in \mathbb{N} \} \): \(\{ \psi_{\lambda, l}^m \mid \lambda \in \mathbb{C} \setminus \{0\}, m, l \in \mathbb{N}_0 \}\).
2. If \(c = c(n)\), \(n \in \mathbb{N}\): \(\{ \psi_{\lambda, l}^m \mid \lambda \in \mathbb{C} \setminus \{0\}, m, l \in \mathbb{N}_0 \} \cup \mathcal{B}_n\) where \(\mathcal{B}_n\) denotes any basis of the \(n^2\)-dimensional subspace \(C_{\neq 0}\) of \(O_q(S_c^2)^\circ\).
3. If \(c = 0\): \(\{ E^kF^l \mid k, l \in \mathbb{N}_0 \} \cup \{ \chi_{\lambda}^+g^mF^l, \chi_{\lambda}^-g^mE^l \mid \lambda \in \mathbb{C} \setminus \{0\}, l, m \in \mathbb{N}_0 \}\) where \(\chi_{\lambda}^±\) is the character on \(O_q(S_c^2)\) defined by \(\chi_{\lambda}^±(e_i) = \delta_{i0} + \delta_{i±1}\lambda^±1\).

**Proof.** Consider the Hopf subalgebra \(\mathcal{O}_{q^2}(SO(3)) \subset O_q(SL(2))\) generated by the matrix coefficients \(\{ \pi_j^i \mid i, j = -1, 0, 1 \}\) and let \(J\) denote the intersection of the two-sided ideal \((u_2^1)\) with \(\mathcal{O}_{q^2}(SO(3))\). There is an isomorphism of Hopf algebras \(\mathcal{O}_{q^2}(SO(3))/J \to U_q(b_-)^\circ\)

\[u_2^1u_2^1 \mapsto K^{-1}, \quad u_2^1u_2^2 \mapsto (1 - q^2)F, \quad u_1^1u_1^1 \mapsto K\]

such that the functionals \(E, f_\lambda, g \in \mathcal{O}_{q^2}(SO(3))^\circ\) given by \(\text{(2.9)}\) and \(\text{(2.11)}\) correspond to \(E, \chi_{\lambda}^2, 2\gamma \in U_q(n_-) \times (U_0)^\circ = (U_q(b_-)^\circ)^\circ\).
For \( c \neq 0 \) the sequence
\[
\mathcal{O}_q(S^2_c) \hookrightarrow \mathcal{O}_q^\circ(SO(3)) \rightarrow \mathcal{O}_q^\circ(SO(3))/J \rightarrow U_q(b_-)^{\text{op}}
\]
induces an isomorphism \( \mathcal{O}_q(S^2_c)(e_{-1}) \rightarrow U_q(b_-)^{\text{op}} \)
\[
\begin{align*}
e_{-1} & \mapsto \varepsilon(e_{-1})K^{-1}, \quad e_0 \mapsto \varepsilon(e_{-1})(q^3 - q^{-1})F + \varepsilon(e_0), \\
e_1 & \mapsto -\varepsilon(e_{-1})(q - q^{-1})^2KF^2 - \varepsilon(e_0)(q - q^{-1})KF + \varepsilon(e_1)K
\end{align*}
\]
where \( \mathcal{O}_q(S^2_c)(e_{-1}) \) denotes the localization of \( \mathcal{O}_q(S^2_c) \) with respect to the left and right Ore set \( \{e^*_{-1} | n \in \mathbb{N}_0\} \). Thus by Lemma 3.2(2) and Proposition 3.1 one obtains
\[
C_0 \cong \mathcal{O}_q(S^2_c)(e_{-1})^\circ \cong (U_q(b_-)^{\text{op}})^\circ \cong U_q(n_+) \times (U_0)^\circ
\]
and the basis element \( \chi \gamma^m E^l \in U_q(n_+) \times (U_0)^{\circ} \) corresponds to \((1/2)^m \chi^m \gamma^m \). This proves (1) and one obtains (2) taking into account that for \( c = c(n) \) the representation \( \mu \neq 0 \) is irreducible.

In the case \( c = 0 \) consider the embedding
\[
\mathcal{O}_q(S^2_0) \hookrightarrow \mathcal{O}_q^\circ(SO(3)), \quad e_i \mapsto \pi_i^{-1} + \pi^0_i.
\]
Similarly to the case \( c \neq 0 \) this induces an isomorphism \( \mathcal{O}_q(S^2_0)(e_{-1}) \rightarrow U_q(b_-)^{\text{op}} \)
given by (3.7) with \( \varepsilon(e_i) = \delta_0 + \delta_{i,-1} \). Thus by Lemma 3.2
\[
C_0 = (U_q(b_-)^{\text{op}})^\circ \cong U_q(n_+) \times (U_0)^{\circ}
\]
and the basis element \( \chi \gamma^m E^l \in U_q(n_+) \times (U_0)^{\circ} \) corresponds to \((1/2)^m \chi \gamma^m \). The subcoalgebra \( C_{0+} \) is dealt with analogously replacing the two-sided ideal \((u_1^2)\) by \((u_1^2)\) and \( U_q(b_-)^{\text{op}} \) by \( U_q(b_+)^{\text{op}} \). The component \( C_{00} \) has been shown to coincide with \( U_q(sl_2)/(K-1)U_q(sl_2) \) in [HK01], Lem. 5.2, Cor. 3.8]. The elements \( \{E^kF^l | k,l \in \mathbb{N}_0\} \) form a basis of the coalgebra \( U_q(sl_2)/(K-1)U_q(sl_2) \).

\section*{4. Local Finiteness for the \( K_- \)-Action on \( \mathcal{O}_q(S^2_c)^\circ \)}

From now on and for the rest of this paper assume that \( 0 \neq c \in J_2 \). For \( \mathcal{B} = \mathcal{O}_q(S^2_c) \) recall that \( \mathcal{B}^c \) is a right \( \mathcal{U} \)-module. Define
\[
F(\mathcal{B}^c, K_c) = \{f \in \mathcal{B}^c | \dim(f K_c) < \infty\}.
\]
If \( f \in \mathcal{B}^c \) is the restriction of an element \( f' \in \mathcal{U} \) to \( \mathcal{B} \) and \( k \in K_c \) then
\[
fk = k(0)S^{-1}(k_{-1})f'k_{-2}|_B = S^{-1}(k(0))f'k_{-1}|_B
\]
as \( K_c \) is a left \( \mathcal{U} \)-comodule and \( k|_B = k(1)|_E \). Thus \( F(\mathcal{U})|_B \subset F(\mathcal{B}^c, K_c) \) where for any Hopf algebra \( A \)
\[
F(A) = \{a \in A | \dim(\text{ad}A)a < \infty\}, \quad (\text{ad}b)a = b_{(1)}aS(b_{(2)}).
\]

\begin{lemma}
The vector space \( F(\mathcal{B}^c, K_c) \) is a right \( F(\mathcal{U}) \)-module left \( \mathcal{B}^c \)-comodule. Any element of \( F(\mathcal{B}^c, K_c) \) is contained in a finite dimensional right \( K_c \)-submodule left \( \mathcal{B}^c \)-subcomodule.
\end{lemma}
Proof. For \( f \in F(\mathcal{U}) \), \( u \in F(B^0, K_c) \) consider \( V = uK_c \) and \( W = (\text{ad}\mathcal{U})f \). Then for any \( k \in K_c \)
\[
(u \cdot f)k = uk_{(0)} \cdot S^{-1}(k_{(-1)})fk_{(-2)} \in V \cdot W.
\]
Therefore \( F(B^0, K_c) \) is a right \( F(\mathcal{U}) \)-module.

Let \( \bar{V} \) denote the left \( B^0 \)-comodule generated by \( V \). The vector space \( \bar{V} \) is finite dimensional. Applying the coaction to the second factor of \( k_{(-1)} \otimes uk_{(0)} \in \mathcal{U} \otimes V \) one obtains
\[
k_{(-2)} \otimes u_{(1)}k_{(-1)} \otimes u_{(2)}k_{(0)} \in U \otimes B^0 \otimes V
\]
and therefore
\[
u_{(1)} \otimes u_{(2)}k = u_{(1)}k_{(-1)}S^{-1}(k_{(-2)}) \otimes u_{(2)}k_{(0)} \in B^0 \otimes \bar{V}.
\]
Thus \( F(B^0, K_c) \) is a left \( B^0 \)-comodule and \( u \in \bar{V} \subset \bar{V}K_c \).

**Lemma 4.2.** Any left \( B^0 \) subcomodule \( W \subset F(B^0, K_c) \) is a \( (\mathbb{C}\setminus\{0\}) \)-graded vector space where
\[
\text{deg}(g^m f_\mu E^l) = \mu.
\]

**Proof.** Consider an arbitrary element \( u \in W \subset F(B^0, K_c) \). By Theorem 3.1.1 one can assume that \( u = \sum_{\mu} f_\mu a^\mu \) for some \( a^\mu \) which are linear combinations of basis vectors \( g^m E^l \), \( m, l \in \mathbb{N}_0 \). By the explicit form (2.11), (2.12) of the coproduct of \( g \) and \( E \) and by Theorem 3.1.1 one can write
\[
\Delta u = \sum_{\mu} f_\mu \otimes f_\mu a^\mu + \sum_i u_i^1 \otimes u_i^2
\]
where \( \{u_i^1, f_\mu\} \) is a set of linear independent elements in \( B^0 \). As \( W \) is a left \( B^0 \)-comodule \( f_\mu a^\mu \in W \) for all \( \mu \).

Let \( F_\mu(B^0, K_c) \) denote the subspace of elements of degree \( \mu \) in \( F(B^0, K_c) \).

**Lemma 4.3.** \( F(B^0, K_c) \subset \bar{F} := \text{Lin}_\mathbb{C}\{\psi^l_\lambda | l \in \mathbb{N}_0, \lambda \in \mathbb{C}\setminus\{0\}\} \).

**Proof.** Consider an arbitrary element \( u \in F_\mu(B^0, K_c) \). By Theorem 3.1.1 one can assume that \( u = \sum_{i=0}^m g^i a_i \) for some \( a_i \in F \) such that \( \text{deg}(a_i) = \mu \) and \( a_m \neq 0 \). Suppose that \( m \geq 1 \). Applying the coaction to \( u \) one obtains
\[
\Delta u = f_\mu g^{m-1} \otimes (ma_m g + a_{m-1}) + \sum_i u_i^1 \otimes u_i^2
\]
where \( \{f_\mu g^{m-1}, u_i^1\} \) is a linearly independent set of elements of \( B^0 \). Thus, as \( F(B^0, K_c) \) is a left \( B^0 \)-comodule, we can assume \( m = 1 \). By similar arguments one can assume that \( u = g f_\mu + a_0 \) and also \( f_\mu \in F(B^0, K_c) \).

One checks by direct computation that \( (\text{ad}\mathcal{U})E \) is a three dimensional vector space and therefore \( E \in F(\mathcal{U}) \). By Lemma 4.1 this implies \( f_\mu E^m \in F(B^0, K_c) \) for all \( m \in \mathbb{N}_0 \). Thus \( g f_\mu \in F(B^0, K_c) \).
Direct calculation using (2.13) and (2.19) leads to

\[(4.1) \quad g f_\mu E^l X_c = q(q^{2l} - \mu^4)g f_{q\mu} E^{l+1} - 4q\mu^4 f_{q\mu} E^{l+1} + \sum_{i=0}^{l} a_i E^i \]

where \(a_i \in \text{Lin}_C \{g f_\nu, f_\nu \mid \nu \in \mathbb{C} \setminus \{0\}\}. Further

\[(4.2) \quad f_\mu E^l X_c = q(q^{2l} - \mu^4) f_{q\mu} E^{l+1} + \alpha(q^{2l} - \mu^2) f_{q\mu} E^l + \alpha(\mu^2 - 1) f_\mu E^l + \frac{[l]}{q - q^{-1}} f_\mu E^{l-1} \]

where as in (2.19) \(\alpha = 0\) if \(c = \infty\) and \(\alpha = -(c^{1/2}(q - q^{-1}))^{-1}\) else. By (4.1)

\[g f_\mu(X_c)^k = q^k \left( \prod_{i=0}^{k-1} (q^{2i} - (q^i \mu)^4) \right) g f_{q^k\mu} E^k \]

\[- 4 \sum_{j=0}^{k-1} (q^j \mu)^4 q^j \left( \prod_{i=0}^{k-1} (q^{2i} - (q^i \mu)^4) \right) f_{q^k\mu} E^j + \ldots \]

where \(\ldots\) denotes terms containing only smaller powers of \(E\). Therefore \(g f_\mu \in F(B^o)\) implies \(\mu^4 = q^{-2(k-1)}\) for some \(k \in \mathbb{N}\). Then for \(l \geq 0\)

\[g f_\mu(X_c)^{k+l} = -4(q^{k-1} \mu)^4 q^{k+l} \left( \prod_{i=0}^{k+l-1} (q^{2i} - (q^i \mu)^4) \right) f_{q^k\mu} E^{k+l} + \ldots \]

again up to expressions containing only smaller powers of \(E\). As

\[q^{2(k+l)} - (q^{k+l} \mu)^4 = q^{2(k+l)}(1 - q^{2(l+1)}) \neq 0 \quad \text{for all } l \geq 0 \]

the coefficient of \(f_{q^k\mu} E^{k+l}\) does not vanish. This is a contradiction to the assumption \(g f_\mu \in F(B^o, K_c)\).

To shorten notation let \(\psi_\lambda^l\) denote the basis element \(\psi_\lambda^{0l}\) of \(\tilde{F}\). Define three maps \(\phi, \varphi, \kappa : \tilde{F} \to \tilde{F}\) by

\[(4.3) \quad \phi(\psi_\lambda^l) = -\frac{q^{l}[l]}{q - q^{-1}} \psi_{q^2\lambda}^{l-1} + \alpha q(q^{2l} - \lambda) \psi_{q^2\lambda}^l + q^2(q^{2l} - \lambda^2) \psi_{q^2\lambda}^{l+1} \]

\[\varphi(\psi_\lambda^l) = \lambda^{-1} \frac{q^{-l}[l]}{q - q^{-1}} \psi_{q^{-2}\lambda}^{l-1} \]

\[\kappa(\psi_\lambda^l) = \lambda \psi_\lambda^l. \]

In view of (4.2) this means

\[(4.4) \quad \psi_\lambda^l X_c = q^{-1} \phi(\psi_\lambda^l) + \lambda \varphi(\psi_\lambda^l) + \alpha(1 - \lambda^{-1}) \kappa(\psi_\lambda^l). \]
Note that
\[ \phi \circ \varphi - \varphi \circ \phi = \frac{\kappa - \kappa^{-1}}{q - q^{-1}}, \quad \kappa \circ \phi = q^2 \phi \circ \kappa, \quad \kappa \circ \varphi = q^{-2} \varphi \circ \kappa, \]
i.e. the operators \( \phi, \varphi \) and \( \kappa \) yield a representation \( \rho : U_q(\mathfrak{sl}_2) \to \text{End}(\tilde{F}), \)
\[ \rho(E) = \phi, \quad \rho(F) = \varphi, \quad \rho(K) = \kappa. \]

**Lemma 4.4.** For any finite dimensional subspace \( V \subset \tilde{F} \) the following statements are equivalent.

1. \( \Delta V \subset B \otimes V \) and \( VK_c \subset V \)
2. \( V \) is a left \( U_q(\mathfrak{sl}_2) \)-module via \( \rho \).

**Proof.** (1)\( \Rightarrow \) (2) As in Lemma 4.2 one obtains that \( V \) is \( (\mathbb{C}\{0\}) \)-graded. Then the assertion follows from (4.4). To verify (2)\( \Rightarrow \) (1) note that
\[ \Delta \psi^l_{\lambda} = \sum_{r=0}^{l} \binom{l}{r} q^{-r(l-r)} \psi^r_{\lambda} \otimes \psi^{l-r}_{q^{-2}r \lambda} = \sum_{r=0}^{l} b_r \psi^r_{\lambda} \otimes \varphi^r(\psi^l_{\lambda}) \]
where \( b_r \in \mathbb{C} \) depend on \( r \) and \( \lambda \) but not on \( l \).
\[ \square \]

Lemma 4.3 implies that \( F(B^o, K_c) \) is a \( \rho \)-invariant subspace of \( \tilde{F} \). Recall that an element \( \psi \in V \setminus \{0\} \) is called a highest weight vector of a \( U_q(\mathfrak{sl}_2) \)-module \( V \) with highest weight \( \lambda \) if \( K^{-1} \psi = \lambda \psi \) and \( F \psi = 0 \).

**Proposition 4.1.** There exists a decomposition of \( U_q(\mathfrak{sl}_2) \)-modules
\[ F(B^o, K_c) = \bigoplus_{\lambda \in J^c} V_{\lambda} \]
such that
\[ J^c = \{ q^{-l} \mid l \in 2\mathbb{N}_0 \} \quad \text{for } c \notin \{ \infty, (q^r - q^{-r})^{-2} \mid r \in \mathbb{N}/2 \} \]
\[ J^\infty = \{ \pm q^{-l} \mid l \in 2\mathbb{N}_0 \} \]
\[ J^{(q^r - q^{-r})^{-2}} = \{ q^{-l}, -q^{-k} \mid l \in 2\mathbb{N}_0, k \in 2r + 2\mathbb{N}_0 \}, \quad r \in \mathbb{N}/2 \]
where the components \( V_{\pm q^l} \) are \((l+1)\)-dimensional irreducible \( U_q(\mathfrak{sl}_2) \)-modules of highest weight \( \pm q^l \).

**Proof.** Lemma 4.1 and Lemma 4.4, 1.\( \Rightarrow \) 2., imply that the \( U_q(\mathfrak{sl}_2) \)-module \( F(B^o, K_c) \) can be written as a direct sum
\[ F(B^o, K_c) = \bigoplus_{\lambda \in J^c} V_{\lambda} \]
of finite dimensional irreducible \( U_q(\mathfrak{sl}_2) \)-modules. Here \( J^c \subset \mathbb{C} \setminus \{0\} \) denotes the subset of nonzero complex numbers \( \lambda \) such that \( \phi \) operates nilpotently on \( \psi^0_{\lambda} = f_{\sqrt{\lambda}} \). Indeed, by (4.3) the set \( \{ \psi^0_{\lambda} \mid \lambda \in J^c \} \) is a basis of all highest weight vectors of \( F(B^o, K_c) \) with respect to the \( U_q(\mathfrak{sl}_2) \)-module structure. It remains to show that \( J^c \) is of the form given in the proposition.
Note that \( \phi^{l+1}(\psi_\lambda^0) = 0 \) and \( \phi^l(\psi_\lambda^0) \neq 0 \) imply \( \lambda = \pm q^{-l} \). In this case the mapping
\[
\phi : \text{Lin}_\mathbb{C}\{\psi_{q^k \lambda}^k | k = 0, \ldots, l\} \to \text{Lin}_\mathbb{C}\{\psi_{q^k \lambda}^{k+2} | k = 0, \ldots, l\}
\]
is given by the matrix
\[
\begin{pmatrix}
-(\pm q^l - 1)\alpha & -\hat{q}^{-1}q^0[1] & 0 & \cdots & 0 \\
q(1 - q^{2l}) & -(\pm q^l - q^2)\alpha & -\hat{q}^{-1}q^1[2] & \ddots & \vdots \\
0 & q(q^2 - q^{2l}) & -(\pm q^l - q^4)\alpha & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -\hat{q}^{-1}q^{l-1}[l] \\
0 & \cdots & 0 & q(q^{2(l-1)} - q^{2l}) & -(\pm q^l - q^{2l})\alpha
\end{pmatrix}
\]
with respect to the bases \( \psi^k_\mu^\lambda (\mu = q^2\lambda \text{ and } \mu = q^{2+2}\lambda, \text{ respectively}) \), \( \hat{q} = q - q^{-1} \).

Recall that \( \beta = q \) and \( \gamma = 1 \). Using \( q(1 - q^{2k}) = -\hat{q}q^{k+1}[k] \) the map \( \phi \) can be written with respect to the bases \( \psi^k_\mu^\lambda := (-\hat{q})^k q^{-(l-k)(l-k+1)/2} \psi^k_\mu^\lambda \) as
\[
(4.5) \quad q^{l+1}
\begin{pmatrix}
(q^{-l} \mp 1)\alpha & [1]\gamma & 0 & \cdots & 0 \\
-q^{-l}[l]\beta & (q^{2-l} \mp 1)\alpha & [2]\gamma & \ddots & \vdots \\
0 & q^{2-l}[l-1]\beta & (q^{4-l} \mp 1)\alpha & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & [l]\gamma \\
0 & \cdots & 0 & q^{l-2}[1]\beta & (q^l \mp 1)\alpha
\end{pmatrix}
\]

In the case of minus signs in the diagonals this matrix is up to the overall factor precisely the matrix \( M_l \) describing the transpose of the left action of \( X_c \) on the \((l+1)\)-dimensional irreducible \( U_q(\mathfrak{sl}_2) \)-module \( V \) \([\text{MS99}, \text{Sect. } 4]\). By \([\text{MS99}, \text{Prop. } 4.2]\) the matrix \( M_l \) is known to have \( l + 1 \) not necessarily distinct eigenvalues
\[
\rho_r = \frac{\alpha}{2}(q^r - q^{-r})^2 + \frac{1}{2}(q^{2r} - q^{-2r})R, \quad r \in I_l = \{-l/2, 1 - l/2, \ldots, l/2\}
\]
where \( R^2 = \alpha^2 + \frac{4\beta q^{-1}}{(q - q^{-1})^2} \). In particular \( M_l \) has eigenvalue 0 if and only if \( l \) is even or
\[
0 = \rho_r \rho_{-r} = -(q^r - q^{-r})^2 \left( \alpha^2 + \beta q^{-1} \left( \frac{q^r + q^{-r}}{q - q^{-1}} \right)^2 \right).
\]
The second case is equivalent to \( c = c(n) \) for some \( n \in I_l \). As this case is excluded by assumption \( q^{-l} \in J^c \) if and only if \( l \) is even.

Let \( \mathbb{C}w \) denote the one-dimensional representation of \( U_q(\mathfrak{sl}_2) \) uniquely determined by \( E \cdot w = 0, F \cdot w = 0, K \cdot w = -w \). By means of a base change the matrix \([1.3]\) corresponding to \(-q^{-l}\) can be transformed into the matrix of the transpose of the left \( X_c \)-action on the finite dimensional \( U_q(\mathfrak{sl}_3) \)-module \( \mathbb{C}w \otimes V \). The eigenvalues of this action can be computed by means of \([\text{MS99}, \text{Prop. } 4.6]\).
In particular the $X_c$-action has a nontrivial kernel if and only if

$$0 = (\rho_r + 2\alpha)(\rho_{-r} + 2\alpha) = \left( q^r + q^{-r} \right)^2 \left( \alpha^2 - \beta \gamma q^{-1} \left( \frac{q^r - q^{-r}}{q - q^{-1}} \right)^2 \right)$$

for some $r \in I$. This equation is equivalent to $c = \frac{1}{(q^r - q^{-r})^2}$, $r \neq 0$ or $c = \infty$, $r = 0$. Notice that $(q^r - q^{-r})^2 - c(n)^{-1} = (q^{r+n} + q^{-(r+n)})((q^{r-n} + q^{-(r-n)}) \neq 0$ for all $r, n \in \mathbb{N}_0 / 2$ and therefore these cases are not excluded.

5. Differential Calculus over $O_q(S^2_c)$

For the convenience of the reader the notion of differential calculus from [Wor89] is recalled. A first order differential calculus (FODC) over an algebra $B$ is a $B$-bimodule $\Gamma$ together with a $C$-linear map $d : B \to \Gamma$ such that $\Gamma = \text{Lin}_C\{a \cdot dbc \mid a, b, c \in B\}$ and $d$ satisfies the Leibniz rule

$$d(ab) = a \cdot db + da \cdot b.$$

Let in addition $A$ denote a Hopf algebra and $\Delta_B : B \to B \otimes A$ a right $A$-comodule algebra structure on $B$. If $\Gamma$ possesses the structure of a right $A$-comodule

$$\Delta_\Gamma : \Gamma \to \Gamma \otimes A$$

such that

$$\Delta_\Gamma(ab c) = (\Delta_B a)((d \otimes \text{id})\Delta_B b)(\Delta_B c)$$

then $\Gamma$ is called right covariant. A FODC $d : B \to \Gamma$ over $B$ is called inner if there exists an element $\omega \in \Gamma$ such that $dx = \omega x - x \omega$ for all $x \in B$. For further details on first order differential calculi consult [KS97].

Let $U$ denote a Hopf algebra with bijective antipode and $L \subset U$ a left coideal subalgebra, i.e. $\Delta_L : L \to U \otimes L$. Consider a tensor category $\mathcal{C}$ of finite dimensional left $U$-modules. Let $A := U_0^\mathcal{C}$ denote the dual Hopf algebra generated by the matrix coefficients of all $U$-modules in $\mathcal{C}$. Assume that $A$ separates the elements of $U$. Define a right coideal subalgebra $B \subset A$ by

$$(5.1) \quad B := \{ b \in A \mid \langle u, b(b) \rangle b(2) = 0 \text{ for all } u \in L^+ \},$$

where $L^+ = \{ u - \varepsilon(u) \mid u \in L \}$. Assume $L$ to be $\mathcal{C}$-semisimple, i.e. the restriction of any $U$-module in $\mathcal{C}$ to the subalgebra $L \subset U$ is isomorphic to the direct sum of irreducible $L$-modules. By [MS99] Theorem 2.2 (2) this implies that $A$ is a faithfully flat $B$ module.

In this situation right covariant first order differential calculi over $B$ can be classified via certain left ideals of $B^+$ [Her02]. More explicitly the subspace

$$(5.2) \quad \mathcal{L} = \left\{ \sum_i a_i^+ \varepsilon(b_i) \bigg| \sum_i da_i b_i = 0 \right\} \subset B^+$$
is a left ideal which determines the differential calculus uniquely. To this left ideal one associates the vector space
\[ T^e = \{ f \in B^e \mid f(x) = 0 \text{ for all } x \in L \} \]
and the so called quantum tangent space
\[ T = (T^e)^+ = \{ f \in T^e \mid f(1) = 0 \}. \]
The dimension of a first order differential calculus is defined by
\[ \dim \Gamma = \dim_C \Gamma / \Gamma B^+ = \dim_C B^+ / L. \]

**Proposition 5.1.** ([HK01] Cor. 1.2) There is a one to one correspondence between \( n \)-dimensional covariant FODC over \( B \) and \((n+1)\)-dimensional subspaces \( T^e \subset B^e \) such that \( \varepsilon \in T^e \), \( \Delta T^e \subset B^e \otimes T^e \), \( T^e L \subset T^e \).

A covariant FODC \( \Gamma \) over \( B \) is called irreducible if it does not possess any nontrivial quotient (by a right covariant \( B \)-bimodule). Note that this property is equivalent to the property that \( T^e_\Gamma \) does not possess any right \( L \)-invariant left \( B^e \)-subcomodule \( \tilde{T}^e \) such that \( C : \varepsilon \not\subset \tilde{T} \not\subset T^e_\Gamma \).

For a family of right covariant FODC \( (\Gamma_i, d_i)_{i=1,\ldots,k} \) define \( d = \oplus_i d_i : B \to \oplus_i \Gamma_i \). Then \( \Gamma = B d B \subset \oplus_i \Gamma_i \) is a covariant FODC with differential \( d \) which is called the sum of the calculi \( \Gamma_1, \ldots, \Gamma_k \) [HS98]. The left ideal corresponding to \( \Gamma \) is given by \( L_\Gamma = \cap_i L_{\Gamma_i} \) and therefore the relation \( T_{\Gamma} = T_{\Gamma_1} + \cdots + T_{\Gamma_k} \) of quantum tangent spaces holds. A sum of covariant differential calculi is called a direct sum if \( \Gamma = \bigoplus_i \Gamma_i \) is a direct sum of bimodules. This condition is equivalent to \( T_{\Gamma} = \bigoplus_i T_{\Gamma_i} \).

As an immediate consequence of Corollary 5.1, Lemma 4.4 and Proposition 4.1 one obtains the following classification result for differential calculi over \( O_q(S^2_c) \).

**Theorem 5.1.** Assume \( 0 \neq c \in J_2 \). For \( \lambda \in J^c \) let \( \Gamma_\lambda \) denote the uniquely determined covariant FODC over \( O_q(S^2_c) \) such that \( T^e_{\Gamma_\lambda} = V_\lambda + C \varepsilon \). Then \( \Gamma_\lambda \) is irreducible and any finite dimensional covariant FODC \( \Gamma \) over \( O_q(S^2_c) \) is isomorphic to a direct sum
\[ \Gamma = \bigoplus_{\lambda \in J} \Gamma_\lambda \]
for some finite subset \( J \subset J^c \).

For any covariant FODC \( \Gamma \) with corresponding left ideal \( L \) and quantum tangent space \( T \) consider the projection
\[ P_r : \Gamma \otimes_B A \to \Gamma \otimes_B A, \quad \gamma \otimes a \mapsto \gamma(1) \otimes S(\gamma(2)) \varepsilon(a) \]
on onto the subspace \( (\Gamma \otimes_B A)^{\text{inv}} \subset \Gamma \otimes_B A \) of right coinvariant elements. The relation \( db \otimes a = d(b(1)) \otimes S(b(2))b(3)a \) implies that the right \( A \)-module \( \Gamma \otimes_B A \) is
generated by the elements \( P_r(db \otimes 1), b \in B \). For any \( a = \sum_i a_i^+ \varepsilon(b_i) \in L \) where \( \sum_i da_i b_i = 0 \) one obtains
\[
P_r(da \otimes 1) = P_r \left( \sum_i da_i \otimes b_i \right) = 0.
\]
Therefore \( P_r \) induces a well defined surjection
\[
B^+ / L \to \left( \Gamma \otimes_B A \right)_{\text{inv}}, \quad b \mapsto P_r(db \otimes 1).
\]

(5.4)

**Lemma 5.1.** The pairing
\[
(\Gamma \otimes_B A)_{\text{inv}} \times T \to \mathbb{C}, \quad (da \otimes b, X) \mapsto X(a)\varepsilon(b)
\]
is non-degenerate. Further \( b \in L \) if and only if \( b \in B^+ \) and \( P_r(db \otimes 1) = 0 \).

**Proof.** To verify the first statement note that by construction the elements \( P_r(db \otimes 1), b \in B \) separate \( T \). On the other hand (5.4) implies \( \dim \mathbb{C}((\Gamma \otimes_B A)_{\text{inv}}) \leq \dim \mathbb{C} B^+ / L = \dim \mathbb{C} T \) and therefore \( T \) separates \( (\Gamma \otimes_B A)_{\text{inv}} \) and (5.4) is an isomorphism.

**Lemma 5.2.** Let \( W \subset B \) be a right \( A \)-subcomodule then \( dW \) generates \( \Gamma \) as a right \( B \)-module if and only if the elements of \( W \) separate the quantum tangent space \( T\Gamma \). If \( \dim W = \dim \Gamma \) and the elements of \( W \) separate \( T\Gamma \) then \( \Gamma \) is a free right \( B \)-module generated by the differentials of an arbitrary basis of \( W \).

**Proof.** Let \( \Gamma' \subset \Gamma \) denote the right \( B \)-module generated by \( dW \). Then as \( A \) is a faithfully flat left \( B \)-module
\[
\Gamma' = \Gamma \iff \Gamma' \otimes_B A = \Gamma \otimes_B A \iff (\Gamma' \otimes_B A)_{\text{inv}} = (\Gamma \otimes_B A)_{\text{inv}}.
\]
Now, if \( W \) separates \( T\Gamma \) then \( (\Gamma' \otimes_B A)_{\text{inv}} \) separates \( T\Gamma \) and therefore by Lemma 5.1 it coincides with \( (\Gamma \otimes_B A)_{\text{inv}} \). Conversely, if \( \Gamma' = \Gamma \) then \( (\Gamma' \otimes_B A)_{\text{inv}} \) separates \( T\Gamma \) and therefore the elements of \( W \) separate \( T\Gamma \). This proves the first statement.

To prove the second statement let \( \Gamma'' \) denote the free right \( B \)-module generated by the differentials of an arbitrary basis \( e_1, \ldots, e_k \) of \( W \). Then, as above,
\[
\Gamma'' = \Gamma \iff \Gamma'' \otimes_B A = \Gamma \otimes_B A \iff P_r(de_i \otimes 1), i = 1, \ldots, k \text{ form a basis of } (\Gamma \otimes_B A)_{\text{inv}}.
\]
In view of Lemma 5.1 this property is equivalent to the nondegeneracy of the pairing between \( W \) and \( T\Gamma \).

Combining the above Lemma with Theorem 5.1 one can now classify all covariant FODC over \( \mathcal{O}_q(S^2_c) \) generated as right \( \mathcal{O}_q(S^2_c) \)-modules by the differentials \( de_i, i = -1, 0, 1 \). The straightforward calculations of the pairing of the tangent spaces with the generators \( e_i, i = -1, 0, 1 \), are omitted.
Corollary 5.1. For $c \in J_2 \setminus \{0, \infty, (q^{1/2} - q^{-1/2})^{-2}\}$ there exists exactly one covariant FODC $\Gamma_q$ over $\mathcal{O}_q(S^2)$ which is generated by $\{d_i | i = -1, 0, 1\}$ as a right $\mathcal{O}_q(S^2)$-module. The elements $\{d_i | i = -1, 0, 1\}$ form a right $\mathcal{O}_q(S^2)$-module basis of this calculus.

For $c = \infty$ there exist exactly three covariant FODC over $\mathcal{O}_q(S^2)$ which are generated by $\{d_i | i = -1, 0, 1\}$ as a right $\mathcal{O}_q(S^2)$-module. One of them, $\Gamma_{-1}$, is one-dimensional, the elements $\{d_i | i = -1, 0, 1\}$ form a right $\mathcal{O}_q(S^2)$-module basis of each of the other two calculi $\Gamma_{\pm q^{-2}}$.

For $c = (q^{1/2} - q^{-1/2})^{-2}$ there exist exactly two covariant FODC over $\mathcal{O}_q(S^2)$ which are generated by $\{d_i | i = -1, 0, 1\}$ as a right $\mathcal{O}_q(S^2)$-module. One of them, $\Gamma_{-q^{-1}}$, is two-dimensional, the elements $\{d_i | i = -1, 0, 1\}$ form a right $\mathcal{O}_q(S^2)$-module basis of the other calculus $\Gamma_{q^{-2}}$.

For generic value of $c$ the above corollary reproduces the results obtained in [AS94] by means of computer calculations.

The odd dimensional covariant FODC $\Gamma_{q^{1/l}}$, $l \in 2\mathbb{N}$, for arbitrary $c$ and $\Gamma_{q^{-1/l}}$, $l \in 2\mathbb{N}$, for $c = \infty$ can be explicitly constructed by a method by U. Hermisson. To match the above conventions the relevant lemma from [Her98], [Her02] is cited in terms of right comodule algebras.

Let $\mathcal{A}$ denote a coquasitriangular Hopf algebra with universal $r$-form $r$ and $\mathcal{B}$ a right $\mathcal{A}$-comodule algebra. Let $\nu$ be a comodule algebra endomorphism of $\mathcal{B}$. Let further $W \subset \mathcal{B}$ denote a finite dimensional right $\mathcal{A}$ subcomodule and $W' = \text{Hom}(W, \mathbb{C})$ the dual comodule defined by $(\Delta f)(w) = (f \otimes S^{-1})\Delta w$ for $w \in W$, $f \in W'$. More explicitly, if $\{b_1, \ldots, b_N\}$ is a basis of $W$ and $\{\gamma^1, \ldots, \gamma^N\} \subset W'$ the dual basis then $\Delta b_i = b_j \otimes \psi_i^j$ implies $\Delta \gamma^i = \gamma^j \otimes S^{-1}(\psi_i^j)$.

Lemma 5.3. The free right $\mathcal{B}$-module $W' \otimes \mathcal{B}$ can be endowed with a right $\mathcal{A}$-covariant $\mathcal{B}$-bimodule structure by

$$afb := f(0)\nu(a(0))b r(a(1), f(1)), \quad a, b \in \mathcal{B}, f \in W'$$

and will be denoted by $\Gamma_{r, \nu, W}$. Moreover, if $\omega = \sum_{i=1}^N \gamma^i b_i \in \Gamma_{r, \nu, W}$ denotes the canonical invariant element then $d : \mathcal{B} \rightarrow \Gamma_{r, \nu, W}$, $db := \omega b - b\omega$, defines a covariant FODC $(dB \cdot \mathcal{B}, d)$ over $\mathcal{B}$.

Lemma 5.4. The quantum tangent space of the differential calculus $\Gamma$ described in Lemma 5.3 is the linear span of the functionals $\chi_i \in \mathcal{B}^\circ$, $i = 1, \ldots, N$, defined by

$$\chi_i(a) = r(\nu(a), S^{-1}(b_i)) - \varepsilon(b_i)\varepsilon(a).$$
The standard universal
the differential calculus $\Gamma$ from Lemma 5.3 satisfies
with highest weight vector $b_1 = e_1^n$. For $\nu = \text{id}$ the quantum tangent space $T$ of the
differential calculus $\Gamma$ from Lemma 5.1 satisfies
\[
\text{Therefore by Lemma 5.1 the identification (5.5) follows. Now (5.5) implies } \Gamma_{q^{-2n}} = \Gamma_{r, \text{Id}, V(n)}.
\]

Let $V(n), n \geq 1$, denote the $(2n+1)$-dimensional $U_q(\mathfrak{sl}_2)$-submodule of $O_q(S_\infty^2)$ with highest weight vector $b_1 = e_1^n$. For $\nu = \text{id}$ the quantum tangent space $T$ of the
differential calculus $\Gamma$ from Lemma 5.1 satisfies
\[
\text{The standard universal } r\text{-form of } O_q(\text{SL}(2)) \text{ is defined by}
\]
\[
r(u_i^j \otimes u_i^k) = q^{-1/2} \begin{cases} q & \text{if } i = j = k = l \\ 1 & \text{if } i = j \neq k = l \\ q^{-1} & \text{if } j < i = l \\ 0 & \text{else.} \end{cases}
\]

In particular $r(a \otimes u_2^n) = 0$ for all $a \in O_q(\text{SL}(2))$ and therefore $\tilde{\chi} : B \rightarrow \mathbb{C},$
\[
\tilde{\chi}(a) := \varepsilon(e_1)^{-n} r(a, S^{-1}(e_1^n)) = \varepsilon(e_1)^{-n} r(S(a), e_1^n) = r(S(a), (u_1^n)^{2n}) = r(a, (u_2^n)^{2n})
\]
is a character which satisfies
\[
\tilde{\chi}(e_i) = q^{-2ni} \varepsilon(e_i).
\]

Since $\tilde{\chi} = \psi_{q^{-2n}}^0 \in V_{q^{-2n}}$ and $\dim V_{q^{-2n}} \oplus \mathbb{C} \varepsilon = 2n + 2 \geq \dim T^\varepsilon$ one obtains
\[
T^\varepsilon = V_{q^{-2n}} \oplus \mathbb{C} \varepsilon. \quad \text{Hence the differential calculus } \Gamma \text{ coincides with } \Gamma_{q^{-2n}}. \quad \text{Similarly the differential calculus } \Gamma_{-q^{-l}}, \ l \in 2\mathbb{N}, \text{ over } O_q(S_\infty^2) \text{ can be realized using the comodule algebra endomorphism } \nu : e_i \mapsto -e_i.
\]

Note that
\[
(\Gamma_{q^{-2n}} \otimes_B \mathcal{A})_{inv} = (\Gamma_{r, \text{Id}, V(n)} \otimes_B \mathcal{A})_{inv}
\]
where as above $V(n)$ denotes the $(2n+1)$-dimensional representation of $U_q(\mathfrak{sl}_2)$. Indeed, by the above remarks $\Gamma = \Gamma_{q^{-2n}}$ can be considered as a right $B$-submodule of $\Gamma_{r, \text{Id}, V(n)}$ and as $\mathcal{A}$ is a flat $B$-module this implies $\Gamma_{q^{-2n}} \otimes_B \mathcal{A} \subset \Gamma_{r, \text{Id}, V(n)} \otimes_B \mathcal{A}$.

As $\dim(\Gamma_{r, \text{Id}, V(n)} \otimes_B \mathcal{A})_{inv} = 2n+1$ by construction and $\dim(\Gamma_{q^{-2n}} \otimes_B \mathcal{A})_{inv} = 2n+1$ by Lemma 5.1 the identification (5.4) follows. Now (5.3) implies $\Gamma_{q^{-2n}} \otimes_B \mathcal{A} = \Gamma_{r, \text{Id}, V(n)} \otimes_B \mathcal{A}$ and by faithful flatness of $\mathcal{A}$ this in turn gives $\Gamma_{q^{-2n}} = \Gamma_{r, \text{Id}, V(n)}$. Thus one has the following proposition.
Proposition 5.2. For any $0 \neq c \in J_2$ the FODC $\Gamma_{q^{-2n}}$, $n \in \mathbb{N}$, is isomorphic to $\Gamma_{r,Id,V(n)}$. For $c = \infty$ the FODC $\Gamma_{-q^{-2n}}$ is isomorphic to $\Gamma_{r,\nu,V(n)}$ where $\nu(e_i) = -e_i$. In particular $\Gamma_{\pm q^{-2n}}$ are free left and right $\mathcal{B}$ modules and inner first order differential calculi.

Remark. Covariant FODC over $O_q(S^2_0)$ are qualitatively different from those over $O_q(S^2_c)$, $0 \neq c \notin J_2$. Let $\tilde{\Gamma}_{kl}$ denote the $k + l$-dimensional FODC over $O_q(S^2_0)$ with quantum tangent space $\tilde{T}_{kl} = \text{Lin}_C \{ E^i F^j \mid 0 \leq i \leq k, 0 \leq j \leq l, (i,j) \neq (0,0) \}$. By Proposition 5.1 and [HK01, Lemma 5.3] any covariant FODC over $O_q(S^2_0)$ can be written as a (not necessarily direct) sum of calculi $\tilde{\Gamma}_{kl}$ for certain $k, l$. In particular the only irreducible calculi are $\tilde{\Gamma}_{10}$ and $\tilde{\Gamma}_{01}$ constructed in [Kol01].

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