Master Thesis
On the almost everywhere and norm convergences of Nörlund means with respect to Vilenkin systems

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Preface

Unlike the classical theory of Fourier series which deals with decomposition of a function into sinusoidal waves the Vilenkin (Walsh) functions are rectangular waves. The development of the theory of Vilenkin-Fourier series has been strongly influenced by the classical theory of trigonometric series but there are a lot of differences also. The aim of my master thesis is to discuss, develop and apply the newest developments of this fascinating theory connected to modern harmonic analysis. In particular, we investigate Nörlund means but only in the case when their coefficients are monotone and prove convergence in Lebesgue and Vilenkin-Lebesgue points. Since almost everywhere points are Lebesgue and Vilenkin-Lebesgue points for any integrable functions we obtain almost everywhere convergence of such summability methods.
Key words

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Key words
Chapter 1

Introduction

1.1 Vilenkin Groups and Functions

Denote by \(N_+\) the set of the positive integers, \(N := N_+ \cup \{0\}\), \(Z\) the set of the integers, \(\mathbb{R}\) the real numbers, \(\mathbb{R}_+\) the positive real numbers, \(\mathbb{C}\) the complex numbers. Let \(m := (m_0, m_1, \ldots)\) be a sequence of positive integers not less than 2. Denote by

\[ Z_{m_k} := \{0, 1, \ldots, m_k - 1\} \]

the additive group of integers modulo \(m_k\).

Define the group \(G_m\) as the complete direct product of the groups \(Z_{m_k}\) with the product of the discrete topologies of \(Z_{m_k}\).

The direct product \(\mu\) of the measures

\[ \mu_k (j) := 1/m_k \quad (j \in Z_{m_k}) \]

is the Haar measure on \(G_m\) with \(\mu (G_m) = 1\).

If \(\sup_{n \in \mathbb{N}} m_n < \infty\), then we call \(G_m\) a bounded Vilenkin group. If the generating sequence \(m\) is not bounded, then \(G_m\) is said to be an unbounded Vilenkin group.

In this book we discuss only bounded Vilenkin groups, i.e. the case when \(\sup_{n \in \mathbb{N}} m_n < \infty\).

The elements of \(G_m\) are represented by sequences

\[ x := (x_0, x_1, \ldots, x_j, \ldots) \quad (x_j \in Z_{m_j}) \]

It is easy to give a base for the neighborhoods of \(G_m\):

\[ I_0 (x) := G_m, \]
\[ I_n (x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, \ n \in \mathbb{N}). \]
We call subsets \( I_n(x) \subset G_m \) Vilenkin intervals. Let 
\[
e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m \quad (n \in \mathbb{N}).
\]
If we define \( I_n := I_n(0), \) for \( n \in \mathbb{N} \) and \( \overline{T}_n := G_m \setminus I_n, \) then
\[
\overline{T}_N = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1} = \left( \bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_{N,k,l} \right) \bigcup \left( \bigcup_{k=1}^{N-1} I_{N,k,N} \right), \quad (1.1.1)
\]
where
\[
I_{N,k,l} := \begin{cases} 
I_N(0, \ldots, 0, x_k \neq 0, 0, \ldots, 0, x_l \neq 0, x_{l+1}, \ldots, x_{N-1}, \ldots), & \text{for} \quad k < l < N, \\
I_N(0, \ldots, 0, x_k \neq 0, x_{k+1} = 0, \ldots, x_{N-1} = 0, x_N, \ldots), & \text{for} \quad l = N.
\end{cases}
\]
If we define the so-called generalized number system based on \( m \) in the following way:
\[
M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),
\]
then every \( n \in \mathbb{N} \) can be uniquely expressed as
\[
n = \sum_{j=0}^{\infty} n_j M_j,
\]
where \( n_j \in \mathbb{Z}_{m_j} \quad (j \in \mathbb{N}_+) \) and only a finite number of \( n_j \)'s differ from zero.
The Vilenkin group can be metrizable with the following metric:
\[
\rho(x, y) := |x - y| := \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{M_{k+1}}, \quad (x, y \in G_m).
\]
For the natural numbers \( n = \sum_{j=1}^{\infty} n_j M_j \) and \( k = \sum_{j=1}^{\infty} k_j M_j \) we define
\[
n \hat{+} k := \sum_{i=0}^{\infty} (n_i \oplus k_i) M_{i+1}
\]
and
\[
n \hat{-} k := \sum_{i=0}^{\infty} (n_i \ominus k_i) M_{i+1},
\]
where
\[ a_i \oplus b_i := (a_i + b_i) \mod m_i, \quad a_i, b_i \in \mathbb{Z}_{m_i} \]
and \( \ominus \) is the inverse operation for \( \oplus \).

Next, we introduce on \( G_m \) an orthonormal system, which is called Vilenkin system (see [91, 92, 93]).
At first, we define the complex-valued function \( r_k (x) : G_m \to \mathbb{C} \), the generalized Rademacher functions, by
\[
    r_k(x) := \exp \left( 2\pi i x/m_k \right), \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}) .
\]  

(1.1.2)

Now, define the Vilenkin systems \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) by:
\[
    \psi_n(x) := \prod_{k=0}^{\infty} r_k^n(x), \quad (n \in \mathbb{N}).
\]  

(1.1.3)

Specifically, we call this system the Walsh-Paley system when \( m \equiv 2 \).

**Proposition 1.1.1** (see [91]) Let \( n \in \mathbb{N} \). Then
\[
    |\psi_n(x)| = 1,
    \psi_n(x - y) = \psi_n(x) \overline{\psi_n(y)}.
\]

The direct product \( \mu \) of the measures
\[
    \mu_k(\{j\}) := 1/m_k \quad (j \in \mathbb{Z}_{m_k})
\]
is the Haar measure on \( G_m \) with \( \mu (G_m) = 1 \). Translation of a subset \( I_n(x) \subset G_m \) by \( y \) is defined by \( \tau_y(I_n(x)) = \{I_n(x) + y\} \). Since \( \mu \) is a product measure we get that \( \mu (I_n(x)) = 1/M_n \) and
\[
    \mu (\tau_y(I_n(x))) = \mu (I_n(x) + y) = 1/M_n
\]
for all \( y \in G_m \). Hence,
\[
    \mu (\tau_y(I_n(x))) = \mu (I_n(x)).
\]

**Proposition 1.1.2** Let \( n, k \in \mathbb{N} \). Then
\[
    \int_{G_m} \psi_n d\mu = \begin{cases} 
    1 & n = 0, \\
    0 & n \neq 0.
\end{cases}
\]

Moreover, the Vilenkin systems are orthonormal, that is,
\[
    \int_{G_m} \psi_n \overline{\psi_k} d\mu = \begin{cases} 
    1 & n = k, \\
    0 & n \neq k.
\end{cases}
\]
1.2 \( L_p \) and weak-\( L_p \) Spaces

By a Vilenkin polynomial we mean a finite linear combination of Vilenkin functions. We denote the collection of Vilenkin polynomials by \( \mathcal{P} \).

Let \( L^0(G_m) \) represent the collection of functions which are almost everywhere limits with respect to a measure \( \mu \) of sequences in \( \mathcal{P} \).

For \( 0 < p < \infty \) let \( L^p(G_m) \) represent the collection of \( f \in L^0(G_m) \) such that
\[
\|f\|_p := \left( \int_{G_m} |f|^p \, d\mu \right)^{1/p}
\]
is finite.

Denote by \( L^\infty(G_m) \) the space of all \( f \in L^0(G_m) \) for which
\[
\|f\|_\infty := \inf \{ C > 0 : \mu \{ x \in G_m : |f| > C \} = 0 \} < +\infty.
\]
The space \( C(G_m) \) consist all continuous function for which
\[
\|f\|_C := \sup_{x \in G_m} |f(x)| < c < \infty.
\]

**Proposition 1.2.1** (see [88]) Well-known Minkowski’s integral inequality is given by
\[
\left\| \int_{G_m} f(\cdot, t) \, dt \right\|_p \leq \int_{G_m} \|f(\cdot, t)\|_p \, dt, \quad \text{for all } p \geq 1.
\]

The convolution of two functions \( f, g \in L^1(G_m) \) is defined by
\[
(f * g)(x) := \int_{G_m} f(x - t) g(t) \, dt \quad (x \in G_m).
\]

It is easy to see that
\[
(f * g)(x) = \int_{G_m} f(t) g(x - t) \, dt \quad (x \in G_m).
\]

**Proposition 1.2.2** Let \( f \in L^r(G_m) \), \( g \in L^1(G_m) \) and \( 1 \leq r < \infty \). Then \( f \ast g \in L^r(G_m) \) and
\[
\|f \ast g\|_r \leq \|f\|_r \|g\|_1.
\]

First we present the following very important proposition:
Proposition 1.2.3 Since the Vilenkin function $\psi_m$ is constant on $I_n(x)$ for every $x \in G_m$ and $0 \leq m < M_n$, it is clear that each Vilenkin function is a complex-valued step function, that is, it is a finite linear combination of the characteristic functions. On the other hand, notice that, by Lemma 1.3.3 (Paley’s Lemma), it yields that

$$
\chi_{I_m(t)}(x) = \frac{1}{M_n} \sum_{j=0}^{M_n-1} \psi_j(x-t), \quad x \in I_m(t),
$$

for each $x, t \in G_m$ and $n \in \mathbb{N}$. Thus each step function is a Vilenkin polynomial. Consequently, we obtain that the collection of step functions coincides with a collection of Vilenkin polynomials $\mathcal{P}$.

Since the Lebesgue measure is regular it follows that given $f \in L^1$ there exist Vilenkin polynomials $P_1, P_2, \ldots$, such that $P_n \to f$ a.e., as $n \to \infty$.

Moreover, any $f \in L^p(G_m)$ can be written in the form $f = g - h$ where the functions $g$, $h$ are almost everywhere limits of increasing sequences of non-negative Vilenkin polynomials. In particular, $\mathcal{P}$ is dense in the space $L^p$, for all $p \geq 1$.

The space weak $- L^p(G_m)$ consists of all measurable functions $f$, for which

$$
\|f\|_{\text{weak} - L^p} := \sup_{y > 0} \left\{ \mu \{ f > y \} \right\}^{1/p} < +\infty.
$$

Proposition 1.2.4 (see [88]) If $0 < p \leq \infty$, then $L^p(G_m) \subset \text{weak} - L^p(G_m)$ and

$$
\|f\|_{\text{weak} - L^p} \leq \|f\|_p.
$$

Proof. It is easy to see that

$$
\int_{G_m} |f(x)|^p \, dx \geq \int_{\{x : |f(x)| > y\}} |f(x)|^p \, dx \geq y^p \mu (|f| > y),
$$

which proves the proposition. 

An operator $T$ which maps a linear space of measurable functions on $G_m$ in the collection of measurable functions on $G_m$ is called sublinear if

$$
|T(f + g)| \leq |T(f)| + |T(g)| \quad \text{a.e. on } G_m \quad \text{and} \quad |T(\alpha f)| = |\alpha| |T(f)|
$$

for all scalars $\alpha$ and all $f$ in the domain of $T$. 
1.3 Dirichlet and Vilenkin-Fejér Kernels

If $f \in L^1(G_m)$ we can define the Fourier coefficients, the partial sums of Vilenkin-Fourier series, the Dirichlet kernels with respect to Vilenkin systems in the usual manner:

$$\hat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu, \quad (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{N}+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}+),$$

respectively. It is easy to see that

$$S_n f(x) = \int_{G_m} f(t) \sum_{k=0}^{n-1} \psi_k(x-t) d\mu(t) = \int_{G_m} f(t) D_n(x-t) d\mu(t) = (f \ast D_n)(x).$$

The next well-known identities with respect to Dirichlet kernels (see Lemmas 1.3.1 and 1.3.2, Lemma 1.3.3) will be used many times in the proofs of our main results:

**Lemma 1.3.1** (see [1]) Let $n \in \mathbb{N}$. Then

$$D_{j+Mn} = D_{Mn} + \psi_{Mn} D_j = D_{Mn} + r_n D_j, \quad j \leq (m_n - 1) M_n \quad (1.3.1)$$

and

$$D_{Mn-j}(x) = D_{Mn}(x) - \overline{\psi}_{Mn-1}(-x) D_j(-x) = D_{Mn}(x) - \psi_{Mn-1}(x) D_j(x), \quad j < M_n. \quad (1.3.2)$$

**Lemma 1.3.2** (see [1]) Let $n \in \mathbb{N}$ and $1 \leq s_n \leq m_n - 1$. Then

$$D_{s_n M_n} = D_{Mn} \sum_{k=0}^{s_n-1} \psi_{k M_n} = D_{Mn} \sum_{k=0}^{s_n-1} r_n^k \quad (1.3.3)$$

and

$$D_n = \psi_n \left( \sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j-n_j}^{m_j-1} r_j^k \right), \quad (1.3.4)$$

for $n = \sum_{i=0}^{\infty} n_i M_i$. 
Lemma 1.3.3 (see [9] and [31]) (Paley’s Lemma) Let $n \in \mathbb{N}$. Then

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases}$$

We also need the following estimate:

Lemma 1.3.4 (see [9]) Let $n \in \mathbb{N}$. Then

$$\|D_{M_n}\|_1 = 1.$$

Lemma 1.3.5 (see [9] and [31]) Let $x \in I_s \setminus I_{s+1}, \ s = 0, ..., N - 1$. Then

$$|D_n(x)| \leq cM_s$$

and

$$\int_{I_N} |D_n(x - t)| \, d\mu(t) \leq \frac{cM_s}{M_N},$$

where $c$ is an absolute constant.

It is obvious that

$$\sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} (D_k * f)(x)$$

$$= (f * K_n)(x) = \int_{G_m} f(t) K_n(x - t) \, d\mu(t),$$

where $K_n$ are the so called Fejér kernels:

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k.$$

Using Abel transformation we get another representation of Fejér means

$$\sigma_n f(x) = \sum_{k=0}^{n-2} \left(1 - \frac{k}{n}\right) \hat{f}(k) \psi_k(x)$$

We frequently use the following well-known result:

Lemma 1.3.6 (see [22]) Let $n > t, t, n \in \mathbb{N}$. Then

$$K_{M_n}(x) = \begin{cases} \frac{M_t}{1 - r_t(x)}, & x \in I_t \setminus I_{t+1}, \\ \frac{M_{t+1}}{2}, & x \in I_t, \\ 0, & \text{otherwise.} \end{cases}$$
The proof of the next lemma can easily be done by using Lemma 1.3.6.

**Lemma 1.3.7** Let \( n \in \mathbb{N} \) and \( x \in I_N^{k,l} \), where \( k < l \). Then

\[
K_{M_n}(x) = 0, \quad \text{if} \quad n > l, \\
|K_{M_n}(x)| \leq cM_k,
\]

and

\[
|K_{M_n}(x)| \leq c \sum_{s=0}^{n} M_s \sum_{r=1}^{m_s-1} \chi_{I_n(x-r,e_s)}
\]

Moreover,

\[
\int_{G_n} |K_{M_n}| \, d\mu \leq c < \infty,
\]

where \( c \) is an absolute constant.

We also need the following useful result:

**Lemma 1.3.8** (see [1] and [31]) Let \( t, s_n, \quad n \in \mathbb{N} \), and \( 1 \leq s_n \leq m_n - 1 \). Then

\[
s_nM_nK_{s_nM_n} = \sum_{l=0}^{s_n-1} \left( \prod_{i=0}^{l-1} r_n^i \right) M_nD_{M_n} + \left( \sum_{l=0}^{s_n-1} r_n^l \right) M_nK_{M_n}
\]

The next equality for Fejér kernels is very important for our further investigations:

**Lemma 1.3.9** (see [1] and [31]) Let \( n = \sum_{i=1}^{r} s_{n_i}M_{n_i} \), where \( n_1 > n_2 > \cdots > n_r \geq 0 \) and \( 1 \leq s_{n_i} < m_{n_i} \) for all \( 1 \leq i \leq r \) as well as \( n^{(k)} = n - \sum_{i=1}^{k} s_{n_i}M_{n_i} \), where \( 0 < k \leq r \). Then

\[
nK_n = \sum_{k=1}^{r} \left( \prod_{j=1}^{k-1} r_{n_j} \right) s_{n_k}M_{n_k}K_{s_{n_k}M_{n_k}} + \sum_{k=1}^{r-1} \left( \prod_{j=1}^{k-1} r_{n_j} \right) n^{(k)}D_{s_{n_k}M_{n_k}}.
\]
We will also frequently use the next estimation of the Fejér kernels:

**Corollary 1.3.10** Let \( n \in \mathbb{N} \). Then

\[
n |K_n| \leq c \sum_{l=(n)}^{[n]} M_l |K_{M_l}| \leq c \sum_{l=0}^{[n]} M_l |K_{M_l}|
\]  

(1.3.11)

where \( c \) is an absolute constant.

**Lemma 1.3.11** (see [1] and [31]) Let \( n \in \mathbb{N} \). Then, for any \( n,N \in \mathbb{N}_+ \), we have that

\[
\int_{G_m} K_n(x) d\mu(x) = 1, \quad (1.3.12)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |K_n(x)| d\mu(x) \leq c < \infty, \quad (1.3.13)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |K_n(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad (1.3.14)
\]

where \( c \) is an absolute constant.

**Lemma 1.3.12** (see [4, 9, 10, 54]) Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers, satisfying the condition (2.1.6). Then for any \( n,N \in \mathbb{N}_+ \),

\[
\int_{G_m} F_n^{-1}(x) d\mu(x) = 1, \quad (1.3.15)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_n^{-1}(x)| d\mu(x) \leq c < \infty, \quad (1.3.16)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_n^{-1}(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad (1.3.17)
\]

where \( c \) is an absolute constant.
Chapter 2

Nörlund Means of Vilenkin-Fourier series in Lebesgue Spaces

2.1 Introduction

In the literature, a point \( x \in G_m \) is called a Lebesgue point of \( f \in L^1(G_m) \), if

\[
\lim_{n \to \infty} M_n \int_{I_n(x)} f(t) \, dt = f(x) \quad \text{a.e. } x \in G_m.
\]

It is well-known (for details see [88] and [61]) that if \( f \in L^1(G_m) \) then almost every point is a Lebesgue point and the following important result holds true:

**Proposition 2.1.1** Let \( f \in L^1(G_m) \). Then, for all Lebesgue points \( x \),

\[
\lim_{n \to \infty} S_{M_n} f(x) = f(x).
\]

In the literature there are notations of \( n \)-th Nörlund \( L_n \) and Riesz \( R_n \) logarithmic means defined by

\[
L_n f := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k f}{n-k},
\]

\[
R_n f := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f}{k},
\]

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respectively, where 

\[ l_n := \sum_{k=1}^{n} \frac{1}{k}. \]

It is known that the Nörlund logarithmic mean has better approximation properties than the partial sums and that the Riesz logarithmic means is better than Fejér means in the same sense, but they have much more similar properties with them. In [23] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space \( L^1 \). Moreover, Gát and Goginava [24] proved that for each measurable function satisfying 

\[ \phi(u) = o\left(u \log^{1/2} u\right), \quad \text{as} \quad u \to \infty, \]

there exists an integrable function \( f \) such that 

\[ \int_{G_m} \phi(|f(x)|) \, d\mu(x) < \infty \]

and that there exists a set with positive measure such that the Nörlund logarithmic means of the function diverges on this set. It follows that weak-(1,1) type inequality does not hold for the maximal operator of Nörlund logarithmic means:

\[ L^* f := \sup_{n \in \mathbb{N}} |L_n f| \]

but there exists an absolute constant \( C_p \) such that 

\[ \|L^* f\|_p \leq C_p \|f\|_p, \quad \text{when} \quad f \in L^p, \quad p > 1. \]

Moreover (for details see [3]), if we consider the following restricted maximal operator \( \tilde{L}^*_\# f \), defined by

\[ \tilde{L}^*_\# f := \sup_{n \in \mathbb{N}} |L_{M_n} f|, \quad (M_k := m_0 \ldots m_{k-1}, \quad k = 0, 1, \ldots), \]

then

\[ y \mu\{\tilde{L}^*_\# f > y\} \leq c \|f\|_1, \quad f \in L^1(G_m), \quad y > 0. \]

Hence, if \( f \in L^1(G_m) \), then

\[ L_{M_n} f \to f, \quad \text{a.e. on} \quad G_m. \]
Fejér’s theorem shows that (see e.g books [15] and [16]) if one replaces ordinary summation by Cesàro summation $\sigma_n$ defined by

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f,$$

then the Féjér means of Fourier series of any integrable function converges a.e on $G_m$ to the function.

Goginava and Gogoladze [27] introduced the operator $W_A$ defined by

$$W_A f(x) := \sum_{s=0}^{A-1} M_s \sum_{r_s=1}^{m_s-1} \int_{I_A(x-r_s \epsilon_s)} |f(t) - f(x)| \, d\mu(t).$$

and define a Vilenkin-Lebesgue point of function $f \in L^1(G_m)$, as a point for which

$$\lim_{A \to \infty} W_A f(x) = 0$$

Moreover, they also proved that the following result is true:

**Proposition 2.1.2** Let $f \in L^1(G_m)$. Then

$$\lim_{n \to \infty} \sigma_n f(x) = f(x)$$

for all Vilenkin-Lebesgue points of $f$.

Moreover, the following is true:

**Proposition 2.1.3** Let $x \in G_m$ and $f \in L^1(G_m)$ is continuous at the point $x$. Then

$$\lim_{m \to \infty} \sigma_m f(x) = f(x).$$

If we consider the maximal operator of Féjér means $\sigma^*$ defined by:

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|$$

then

$$y\mu \{\sigma^* f > y\} \leq c \|f\|_1, \quad f \in L^1(G_m), \quad y > 0.$$
This result can be found in Zygmund [102] for trigonometric series, in Schipp [58] for Walsh series and in Pál, Simon [48] for bounded Vilenkin series.

The boundedness does not hold from Lebesgue space $L^1(G_m)$ to the space $L^1(G_m)$. On the other hand, if we consider restricted maximal operator $\tilde{\sigma}^*_\#$ of Féjer means defined by

$$\tilde{\sigma}^*_\# f := \sup_{n \in \mathbb{N}} |\sigma_{M_n} f|$$

then there exists a function $f \in L^1(G_m)$ such that

$$\|\tilde{\sigma}^*_\# f\|_1 = \infty.$$

In the one-dimensional case Yano [97] proved that

$$\|\sigma_n f - f\|_p \to 0, \quad \text{as} \quad n \to \infty, \quad (f \in L^p(G_m), \quad 1 \leq p \leq \infty).$$

However (see [32, 61]) the rate of convergence can not be better then $O(n^{-1})$ ($n \to \infty$) for non-constant functions, i.e., if $f \in L^p$, $1 \leq p \leq \infty$ and

$$\|\sigma_{M_n} f - f\|_p = o\left(\frac{1}{M_n}\right), \quad \text{as} \quad n \to \infty,$$

then $f$ is a constant function.

It is also known that (see e.g. the books [1] and [61]) for any $1 \leq p \leq \infty$ and $n \in \mathbb{N}$ we have the following estimate

$$\|\sigma_n f - f\|_p \leq c_p \omega_p\left(\frac{1}{M_n}, f\right) + c_p \sum_{s=0}^{N-1} \frac{M_s}{M_N} \omega_p\left(\frac{1}{M_s}, f\right).$$

where $\omega_p\left(\frac{1}{M_n}, f\right)$ is the modulus of continuity of function $f \in L^p$:

$$\omega_p\left(\frac{1}{M_n}, f\right) := \sup_{h \in I_n} \|f(\cdot - h) - f(\cdot)\|_p.$$

By applying this estimate, we immediately obtain that if $f \in \text{lip}(\alpha, p)$, i.e.,

$$\omega_p\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{M_n^\alpha}\right), \quad n \to \infty,$$

then
\[ \|\sigma_n f - f\|_p = \begin{cases} 
O \left( \frac{1}{MN} \right), & \text{if } \alpha > 1, \\
O \left( \frac{N}{MN} \right), & \text{if } \alpha = 1, \\
O \left( \frac{1}{MN} \right), & \text{if } \alpha < 1. 
\end{cases} \]

Another well-known summability method is the so-called \((C, \alpha)\)-means (Cesàro means) \(\sigma_n^\alpha\), which are defined by

\[ \sigma_n^\alpha f := \frac{1}{A_\alpha^n} \sum_{k=1}^{n} A_\alpha^{n-k} S_k f, \]

where

\[ A_\alpha^0 := 0, \quad A_\alpha^n := \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, \ldots \]

It is well-known that for \(\alpha = 1\) this summability method coincides with the Fejér summation and for \(\alpha = 0\) we just have the partial sums of the Vilenkin-Fourier series. Moreover, if we consider the maximal operator of the Cesàro means \(\sigma^{\alpha, *}\), defined by:

\[ \sigma^{\alpha, *} f := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f| \]

for \(0 < \alpha \leq 1\), then

\[ y \mu \{ \sigma^{\alpha, *} f > y \} \leq c \|f\|_1, \quad f \in L^1(G_m), \quad y > 0. \]

The boundedness of the maximal operator of the Riesz logarithmic means does not hold from \(L^1(G_m)\) to the space \(L^1(G_m)\). However,

\[ \|\sigma_n^\alpha f - f\|_p \to 0, \quad \text{when } n \to \infty, \quad (f \in L^p(G_m), \quad 1 \leq p \leq \infty). \]

Convergence and approximation in various norms of Vilenkin-Fejér means, \((C, \alpha)\)-means and Nörlund logarithmic means can be found in Blahota, Gát and Goginava \[5, 6, 7, 8, 9, 10, 11, 12, 13, 14\], Fine \[17, 18\], Fridli \[19\], Fujii \[20\], Goginava \[25, 26\], Gogolashvili Nagy and Tephnadze \[28, 29, 30\], Persson and Tephnadze \[50, 51\] (see also \[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\], Pál and Simon \[48\], Schipp \[58, 59, 60\], Simon \[62, 63\], Tephnadze \[86, 87, 88, 89\], Tephnadze and Zhizhiashvili \[98, 99, 100\]. Similar problems for the two-dimensional case can be found in Nagy \[39, 40, 41, 42\], Nagy and Tephnadze \[44, 45, 46, 47\].
The properties established in Lemma 1.3.11 ensure that kernel of the Fejér means $\{K_N\}_{N=1}^{\infty}$ form what is called an approximation identity.

**Definition 2.1.4** The family $\{\Phi_n\}_{n=1}^{\infty} \subset L^\infty(G_m)$ forms an approximate identity provided that

(A1) $\int_{G_m} \Phi_n(x) \, d(x) = 1$

(A2) $\sup_{n \in \mathbb{N}} \int_{G_m} |\Phi_n(x)| \, d\mu(x) < \infty$

(A3) $\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |\Phi_n(x)| \, d\mu(x) \to 0$, as $n \to \infty$, for any $N \in \mathbb{N}_+$.

The term “approximate identity” is used because of the fact that $\Phi_n * f \to f$ as $n \to \infty$ in any reasonable sense. In particular, the following results holds true (for details see the books [21] and [38]):

**Proposition 2.1.5** Let $f \in L^p(G_m)$, where $1 \leq p \leq \infty$ and the family $\{\Phi_n\}_{n=1}^{\infty} \subset L^\infty(G_m)$ forms an approximate identity. Then

$$\|\Phi_n * f - f\|_p \to 0 \quad \text{as} \quad n \to \infty.$$  

It is well-known that the $n$-th Nörlund mean $t_n$ and $T$ means $T_n$ for the Fourier series of $f$ are, respectively, defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k f \quad (2.1.1)$$

and

$$T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f, \quad (2.1.2)$$

where $\{q_k : k \in \mathbb{N}\}$ is a sequence of nonnegative numbers and

$$Q_n := \sum_{k=0}^{n-1} q_k.$$
Let \( \{q_k : k \geq 0\} \) be a sequence of nonnegative numbers where \( q_0 > 0 \). Then the summability method \((2.1.2)\) generated by \( \{q_k : k \geq 0\} \) is regular if and only if (see [36])

\[
\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = \infty.
\]

The representations

\[
t_n f(x) = \int_{G_m} f(t) F_n(x-t) \, d\mu(t)
\]

and

\[
T_n f(x) = \int_{G_m} f(t) F_n^{-1}(x-t) \, d\mu(t)
\]

play central roles in the sequel, where

\[
F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k
\]  \hspace{1cm} (2.1.3)

and

\[
F_n^{-1} := \frac{1}{Q_n} \sum_{k=1}^{n} q_k D_k
\]  \hspace{1cm} (2.1.4)

are called the kernels of Nörlund and \( T \) means, respectively.

Nörlund are generalizations of Fejér, \((C, \alpha)\) and Nörlund logarithmic means. According to all these facts it is of prior interest to study the behavior of operators related to Nörlund means of Fourier series with respect to orthonormal systems. The Nörlund summation are general summability methods, which satisfy the conditions (A1)-(A3). This means that all Nörlund and \( T \) means are approximation identity. According to all these facts it is of prior interest to study the behavior of operators related to Nörlund means of Fourier series with respect to orthonormal systems. Móricz and Siddiqi [37] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of \( L^p \) functions in norm. In particular, they proved that if \( f \in L^p(G_m) \), \( 1 \leq p \leq \infty \), \( n = M_j + k \), \( 1 \leq k \leq M_j \) \( (n \in \mathbb{N}_+) \) and \( \{q_k : k \in \mathbb{N}_+\} \) is sequence of non-negative numbers, such that

\[
\frac{n^\alpha-1}{Q_n^\alpha} \sum_{k=0}^{n-1} q_k^\alpha = O(1), \hspace{1cm} \text{for some} \hspace{0.5cm} 1 < \alpha \leq 2,
\]
then
\[ \|t_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{i=0}^{n-1} M_i q_{n-M_i} \omega_p \left( \frac{1}{M_i}, f \right) + C_p \omega_p \left( \frac{1}{M_j}, f \right), \]
when \( \{q_k : k \in \mathbb{N}\} \) is non-decreasing, while
\[ \|t_n f - f\|_p \leq \frac{C_p}{Q_n} \sum_{i=0}^{n-1} \left( Q_{n-M_j+1} - Q_{n-M_{j+1}} \right) \omega_p \left( \frac{1}{M_i}, f \right) + C_p \omega_p \left( \frac{1}{M_j}, f \right), \]
when \( \{q_k : k \in \mathbb{N}\} \) is non-increasing.

Let us define maximal operator of Nörlund means by
\[ t^* f := \sup_{n \in \mathbb{N}} |t_n f|. \]

If \( \{q_k : k \in \mathbb{N}\} \) is non-increasing and satisfying the condition
\[ \frac{1}{Q_n} = O \left( \frac{1}{n} \right), \text{ as } n \to \infty, \tag{2.1.5} \]
or if \( \{q_k : k \in \mathbb{N}\} \) is non-decreasing, satisfying the condition
\[ \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \text{ as } n \to \infty, \tag{2.1.6} \]
then
\[ y \mu \{t^* f > y\} \leq c \|f\|_1, \quad f \in L^1(G_m), \quad y > 0. \]

The boundedness of the such maximal operator of Nörlund means does not hold from \( L^1(G_m) \) to the space \( L^1(G_m) \). However
\[ \|t_n f - f\|_p \to 0, \quad \text{as} \quad n \to \infty, \quad (f \in L^p(G_m), \ 1 \leq p \leq \infty). \]
2.2. WELL-KNOWN AND NEW EXAMPLES OF NÖRLUND MEANS

2.2 Well-known and New examples of Nörlund Means

We define $B_n$ means as the class of Nörlund means, with monotone and bounded sequence $\{q_k : k \in \mathbb{N}\}$, such that

$$0 < q < \infty \quad \text{where} \quad q_\infty := \lim_{n \to \infty} q_n.$$

If the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, then we have that

$$nq_0 \leq Q_n \leq nq_\infty.$$

In the case when the sequence $\{q_k : k \in \mathbb{N}\}$ is non-increasing, then

$$nq_\infty \leq Q_n \leq nq_0.$$

In both cases we can conclude that conditions (2.1.6) and (2.1.5) are fulfilled.

Well-known examples of Nörlund means with monotone and bounded sequence $\{q_k : k \in \mathbb{N}\}$ is Fejér means

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f$$

It is evident that in this case conditions (2.1.6) and (2.1.5) are fulfilled.

The Cesàro means $\sigma_n^\alpha$ (sometimes also denoted $(C, \alpha)$) are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f$$

where

$$A_0^\alpha := 0, \quad A_n^\alpha := \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, ...$$

It is well-known that

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \quad (2.2.1)$$

$$A_n^\alpha - A_{n-1}^\alpha = A_{n-1}^{\alpha-1} \quad \text{and} \quad A_n^\alpha \sim n^\alpha. \quad (2.2.2)$$

It is obvious that

$$\frac{|q_n - q_{n+1}|}{n^{\alpha-2}} = O(1), \quad \text{as} \quad n \to \infty, \quad (2.2.3)$$
Nörlund Means of Vilenkin-Fourier series in Lebesgue Spaces

\[ \frac{q_0}{Q_n} = O \left( \frac{1}{n^\alpha} \right), \quad \text{as} \quad n \to \infty, \]  

(2.2.4)

and

\[ \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty. \]  

(2.2.5)

Let \( V_\alpha^n \) denote the Nörlund mean, where

\[ \{ q_k = (k + 1)^{\alpha - 1} : \ k \in \mathbb{N}, \ 0 < \alpha < 1 \}, \]

that is

\[ V_\alpha^n f := \frac{1}{Q_n} \sum_{k=1}^{n} (n - k - 1)^{\alpha - 1} S_k f. \]

It is obvious that

\[ \left| \frac{q_n - q_{n+1}}{n^{\alpha-2}} \right| = O \left( 1 \right), \quad \text{as} \quad n \to \infty, \]  

(2.2.6)

\[ \frac{q_0}{Q_n} = O \left( \frac{1}{n^\alpha} \right), \quad \text{as} \quad n \to \infty, \]  

(2.2.7)

and

\[ \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty. \]  

(2.2.8)

We just remind again that \( n \)-th Nörlund \( L_n \) and Riesz \( R_n \) logarithmic means are defined by the sequence \( \{ q_k = 1/k, \ k \in \mathbb{N}_+ \} \):

\[ L_n f := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k f}{n - k}, \quad R_n f := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f}{k}, \]

respectively, where

\[ l_n := \sum_{k=1}^{n} \frac{1}{k}. \]

It is evident that

\[ \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty \]  

(2.2.9)

and
Well-known and new examples of Nörlund Means

\[ \frac{q_0}{Q_n} = O \left( \frac{1}{\ln n} \right), \quad \text{as} \quad n \to \infty. \]  

Let \( U_n^\alpha \) denote the Nörlund mean, where

\[ \left\{ q_k = \frac{1}{(k+3)\ln^\alpha(k+3)} : k \in \mathbb{N}, \quad 0 < \alpha \leq 1 \right\}, \]

that is

\[ U_n^\alpha f := \frac{1}{Q_n} \sum_{k=1}^{n} \frac{S_k f}{(n-k-3)\ln^\alpha(n-k-3)}. \]

It is obvious that

\[ \frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty \]

and

\[ \frac{q_0}{Q_n} = \left\{ \begin{array}{ll} O \left( \frac{1}{\ln(\ln n)} \right), & \text{if} \quad \alpha = 1, \\ O \left( \frac{1}{\ln^{1-\alpha} n} \right), & \text{if} \quad 0 < \alpha < 1. \end{array} \right. \]

Let \( \alpha \in \mathbb{R}_+ \). If we define the sequence \( \{q_k : k \in \mathbb{N}\} \) by

\[ \{q_k = \log^\alpha(k+1) : k \in \mathbb{N} : \alpha > 0\}, \]

then we get the class of Nörlund means with non-decreasing coefficients:

\[ \beta_n^\alpha f := \frac{1}{Q_n} \sum_{k=1}^{n} \log^\alpha(n-k-1) S_k f. \]

It is obvious that

\[ \frac{n}{2} \log^\alpha(n/2) \leq Q_n \leq n \log^\alpha n. \]

It follows that

\[ \frac{1}{Q_n} \leq \frac{c}{n \log^\alpha n} = O \left( \frac{1}{n} \right) \to 0, \quad \text{as} \quad n \to \infty. \]

and

\[ \frac{q_{n-1}}{Q_n} \leq \frac{c \log^\alpha(n-1)}{n \log^\alpha n} = O \left( \frac{1}{n} \right) \to 0, \quad \text{as} \quad n \to \infty. \]
2.3 Kernels of Nörlund Means

Now we study kernels of Nörlund means with respect to Vilenkin systems. If we invoke Abel transformations for $a_j = A_j - A_{j-1}$, $j = 1, ..., n$,

\[
\sum_{j=1}^{n} a_j b_{n-j} = A_n b_0 + \sum_{j=1}^{n-1} A_j (b_j - b_{j+1}), \quad (2.3.1)
\]

\[
\sum_{j=M_N}^{n} a_j b_{n-j} = A_n b_0 - A_{M_N-1} b_{n-M_N} + \sum_{j=M_N}^{n-1} A_j (b_j - b_{j+1}), \quad (2.3.2)
\]

when $b_j = q_j$, $a_j = 1$ and $A_j = j$ for $j = 0, 1, ..., n$, then (2.3.1) and (2.3.2) give the following identities:

\[
Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^{n} q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n, \quad (2.3.3)
\]

\[
= \sum_{j=M_N}^{n-1} q_{n-j} = \sum_{j=M_N}^{n-1} q_{n-j} \cdot 1, \quad (2.3.4)
\]

\[
= \sum_{j=M_N}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n - (M_N - 1) q_{n-M_N}, \quad (2.3.5)
\]

Moreover, if we instead use the Abel transformations (2.3.1) and (2.3.2) for $b_j = q_{n-j}$, $a_j = D_j$ and $A_j = j K_j$ for any $j = 0, 1, ..., n - 1$ we get the identities:

\[
F_n = \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n \right), \quad (2.3.5)
\]

\[
= \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j, \quad (2.3.6)
\]

\[
= \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n - q_{n-M_N} (M_N - 1) K_{M_N-1} \right),
\]
Kernels of Nörlund Means

Analogously, if we use the Abel transformations (2.3.1) and (2.3.2) for \( b_j = q_j \), \( a_j = S_j \) and \( A_j = j\sigma_j \) for any \( j = 0, 1, ..., n - 1 \) we get the identities:

\[
t_nf = \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j\sigma_j f + q_0 n\sigma_n f \right) \tag{2.3.7}
\]

\[
\frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} S_j f \tag{2.3.8}
\]

\[
= \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-1} (q_{n-j} - q_{n-j-1}) j\sigma_j f + q_0 nK_n - q_{n-M_N}(M_N - 1)\sigma_{M_N-1} f \right).
\]

First we consider Nörlund kernels with respect to Vilenkin systems, which are generated by non-decreasing sequences:

**Lemma 2.3.1** Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers, satisfying the condition (2.1.6). Then

\[
|F_n| \leq \frac{c}{n} \sum_{j=0}^{n} M_j |K_{M_j}|,
\]

where \( c \) is an absolute constant.

**Proof.** Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing. Then, by using (2.1.6), we get that

\[
\frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) + q_0 \right) \leq \frac{q_{n-1} - q_n}{Q_n} \leq \frac{c}{n}.
\]

Hence, in view of (2.1.6) if we apply (1.3.11) in Corollary 1.3.10 and use the equality (2.3.5) we obtain that

\[
|F_n| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \right) \sum_{i=0}^{n} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{n} M_i |K_{M_i}|.
\]

The proof is complete. \( \square \)
We also state analogical estimate, but now without any restriction like (2.1.6):

**Lemma 2.3.2** Let \( n \geq M_N \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then

\[
\left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j \right| \leq \frac{c}{M_N} \left\{ \sum_{j=0}^{\lfloor n \rfloor} M_j |K_{M_j}| \right\},
\]

where \( c \) is an absolute constant.

**Proof.** Let \( M_N - 1 \leq j \leq n \). In the view of (1.3.11) in Corollary 1.3.10 we find that

\[
|K_j| \leq \frac{1}{j} \sum_{l=0}^{\lfloor j \rfloor} M_l |K_{M_l}| \leq \frac{1}{M_N - 1} \sum_{l=0}^{\lfloor n \rfloor} M_l |K_{M_l}|
\]

\[
\leq \frac{c}{M_N} \sum_{l=0}^{\lfloor n \rfloor} M_l |K_{M_l}|
\]

Since the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-decreasing we get that

\[
q_{n-M_N}(M_N - 1) = \sum_{j=0}^{M_N-1} q_{n-M_N} = \sum_{j=0}^{M_N-1} q_{n-M_N+j}
\]

\[
= q_{n-M_N} + q_{n-M_N+2} + \ldots + q_{n-M_N+(M_N-1)} \leq Q_n
\]

and

\[
\sum_{j=M_N}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n + q_{n-M_N}(M_N - 1)
\]

\[
\leq \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n + q_{n-M_N}(M_N - 1)
\]

\[
= \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n + q_{n-M_N}(M_N - 1)
\]

\[
= Q_n + q_{n-M_N}(M_N - 1) \leq 2Q_n.
\]
By using the Abel transformation (2.3.6) we can conclude that

\[
\frac{1}{Q_n} \sum_{j=0}^{n-1} q_n - q_{n-j} D_j \leq \frac{1}{Q_n} \left( \sum_{j=M_N^+}^{n-1} (q_n - q_{n-j-1}) j K_j + q_0 n K_{M_N} + q_n - M_N (M_N - 1) K_{M_N-1} \right) \leq \frac{c}{M_N} \sum_{i=0}^{[n]} M_i K_{M_i}
\]

The proof is complete.

Now we prove a lemma, which is very important for our further investigation to prove norm convergence in Lebesgue spaces of Nörlund means generated by non-decreasing sequences \( \{q_k : k \in \mathbb{N}\} \) in this chapter.

**Lemma 2.3.3** Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then, for any \( n, N \in \mathbb{N}_+ \),

\[
\int_{G_m} F_n(x) d\mu(x) = 1 \quad \text{(2.3.9)}
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_n(x)| d\mu(x) \leq c < \infty \quad \text{(2.3.10)}
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_n(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty \quad \text{(2.3.11)}
\]

where \( c \) is an absolute constant.

**Proof.** According to Lemma 1.3.4 we readily obtain (2.3.9). By using (1.3.13) in Corollary 1.3.11 combined with (2.3.3) and (2.3.5) we get that

\[
\int_{G_m} |F_n(x)| d\mu(x) \leq \frac{1}{Q_n} \sum_{j=1}^{n-1} (q_n - q_{n-j-1}) j \int_{G_m} |K_j| d\mu + \frac{q_0 n}{Q_n} \int_{G_m} |K_n| d\mu
\]

\[
\leq \frac{c}{Q_n} \sum_{j=1}^{n-1} (q_n - q_{n-j-1}) j + \frac{c q_0 n}{Q_n} < c < \infty
\]

so also (2.3.10) is proved.
According to (1.3.14) in Corollary 1.3.11 and also (2.3.3) and (2.3.5) we find that
\[
\int_{G_m \setminus I_N} |F_n| \, d\mu \leq \frac{1}{Q_n} \sum_{j=0}^{n-1} (q_{n-j} - q_{n-j-1}) j \int_{G_m \setminus I_N} |K_j| \, d\mu \\
+ \frac{q_0 n}{Q_n} \int_{G_m \setminus I_N} |K_n| \\
\leq \frac{1}{Q_n} \sum_{j=0}^{n-1} (q_{n-j} - q_{n-j-1}) j \alpha_j + \frac{q_0 n \alpha_n}{Q_n},
\]
where \( \alpha_n \to 0 \), as \( n \to \infty \). Since the sequence is non-decreasing, we can conclude that
\[
II = \frac{q_0 n \alpha_n}{Q_n} \leq \alpha_n \to 0, \quad \text{as} \quad n \to \infty.
\]
On the other hand, for any \( \varepsilon > 0 \) there exists \( N_0 \in \mathbb{N} \), such that
\[
\alpha_n < \varepsilon, \quad \text{when} \quad n > N_0.
\]
Moreover,
\[
I = \frac{1}{Q_n} \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j \alpha_j
\]
\[
= \frac{1}{Q_n} \sum_{j=1}^{N_0} (q_{n-j} - q_{n-j-1}) j \alpha_j + \frac{1}{Q_n} \sum_{j=N_0+1}^{n-1} (q_{n-j} - q_{n-j-1}) j \alpha_j = I_1 + I_2.
\]
Since sequence is non-decreasing, we can conclude that
\[
|q_{n-j} - q_{n-j-1}| < 2q_{n-1}
\]
\[
I_1 = \frac{1}{Q_n} \sum_{j=0}^{N_0} (q_{n-j} - q_{n-j-1}) j \alpha_j \leq \frac{2q_{n-1}N_0}{Q_n} \to 0, \quad \text{as} \quad n \to \infty
\]
and, furthermore,
\[
I_2 = \frac{1}{Q_n} \sum_{j=N_0+1}^{n-1} (q_{n-j} - q_{n-j-1}) j \alpha_j \leq \frac{\varepsilon}{Q_n} \sum_{j=N_0+1}^{n-1} (q_{n-j} - q_{n-j-1}) j
\]
\[
\leq \frac{\varepsilon}{Q_n} \sum_{j=0}^{n-1} (q_{n-j} - q_{n-j-1}) j < \varepsilon,
\]
and it follows that also \( I \to 0 \) so also (2.3.11) is proved. The proof is complete. ■
We also consider the kernel of Nörlund means with respect to the Vilenkin systems which are generated by non-increasing sequences \( \{ q_k : k \in \mathbb{N} \} \), but now with some new restrictions on the indexes:

**Lemma 2.3.4** Let \( \{ q_k : k \in \mathbb{N} \} \) be a sequence of non-increasing numbers satisfying the condition (2.1.5). Then

\[
|F_n| \leq \frac{c}{n} \left\{ \sum_{j=0}^{\lfloor n \rfloor} M_j |K_{M_j}| \right\},
\]

where \( c \) is an absolute constant.

**Proof.** Let the sequence \( \{ q_k : k \in \mathbb{N} \} \) be non-increasing and satisfying condition (2.1.5). Then

\[
\frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} - (q_{n-j} - q_{n-j-1}) + q_0 \right) \leq \frac{2q_0 - q_{n-1}}{Q_n} \leq \frac{2q_0}{Q_n} \leq \frac{c}{n}.
\]

Hence, if we apply (1.3.11) in Corollary 1.3.10 and invoke equalities (2.3.3) and (2.3.5), then we get that

\[
|F_n| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \right) \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}| \leq \frac{2q_0 - q_{n-1}}{Q_n} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}| \leq \frac{2q_0}{Q_n} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{\lfloor n \rfloor} M_i |K_{M_i}|.
\]

The proof is complete. \( \blacksquare \)
The next result is very important for our further investigation in this Chapter to prove norm convergence in Lebesgue spaces of Nörlund means generated by a non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \):

**Corollary 2.3.5** Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers satisfying the condition (2.1.5). Then, for any \( n, N \in \mathbb{N}_+ \),

\[
\int_{G_m} F_n(x) d\mu(x) = 1, \quad (2.3.12)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_n(x)| d\mu(x) \leq c < \infty, \quad (2.3.13)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_n(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad (2.3.14)
\]

where \( c \) is an absolute constant.

**Proof.** If we compare the estimation of \( K_n \) in Lemma 1.3.10 or \( F_n \) in Lemma 2.3.1 with the estimation of \( F_n \) in Lemma 2.3.4 we find that they are quite the same. Hence, the proof is analogous to those of Corollary 1.3.11 and Lemma 2.3.3, so, we leave out the details.

Finally we study some special subsequences of kernels of Nörlund and \( T \) means:

**Lemma 2.3.6** Let \( n \in \mathbb{N} \). Then

\[
F_{M_n}(x) = D_{M_n}(x) - \psi_{M_n-1}(x) \overline{F^{-1}_{M_n}}(x) \quad (2.3.15)
\]

and

\[
F^{-1}_{M_n}(x) = D_{M_n}(x) - \psi_{M_n-1}(x) \overline{F}_{M_n}(x). \quad (2.3.16)
\]

**Proof.** By using (1.3.2) in Lemma 1.3.1 we get that

\[
F_{M_n}(x) = \frac{1}{Q_{M_n}} \sum_{k=1}^{M_n} q_{M_n-k} D_k(x) = \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k D_{M_n-k}(x)
\]

\[
= \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k \left( D_{M_n}(x) - \psi_{M_n-1}(x) D_j(x) \right)
\]

\[
= D_{M_n}(x) - \psi_{M_n-1}(x) \overline{F^{-1}_{M_n}}(x)
\]
Hence, (2.3.15) is proved. Identity (2.3.16) is proved analogously so the proof is complete.

Next four lemmas will be used to prove norm convergence and almost everywhere convergence of subsequences of Nörlund means:

**Corollary 2.3.7** Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then, for any \( n, N \in \mathbb{N}_+ \),

\[
\int_{G_m} F_{M_n}^{-1}(x) d\mu(x) = 1, \quad (2.3.17)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_{M_n}^{-1}(x)| d\mu(x) \leq c < \infty, \quad (2.3.18)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_{M_n}^{-1}(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad (2.3.19)
\]

**Proof.** According to (2.3.16) the proof is a direct consequence of Lemmas 1.3.3 and 2.3.3 and Lemma 1.3.4. The proof is complete.

**Corollary 2.3.8** Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then, for any \( N \in \mathbb{N}_+ \),

\[
\int_{G_m} F_{M_n}(x) d\mu(x) = 1, \quad (2.3.20)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_{M_n}(x)| d\mu(x) \leq c < \infty, \quad (2.3.21)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_{M_n}(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad (2.3.22)
\]

**Proof.** According to (2.3.15) the proof is a direct consequence of Lemmas 1.3.3, 1.3.4 and 1.3.12. The proof is complete.
2.4 Norm Convergence of Nörlund Means in Lebesgue Spaces

First we consider norm convergence of Nörlund means with respect to Vilenkin systems:

**Theorem 2.4.1** Let \( f \in L^p(G_m) \) for \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then

\[
\|t_n f - f\|_p \to 0 \text{ as } n \to \infty.
\]

**Proof.** According to Lemma 2.3.3 we conclude that the conditions (A1), (A2) and (A3) in Theorem 2.1.5 are fulfilled, which implies the stated norm convergence.

The proof is complete. ■

**Theorem 2.4.2** Let \( f \in L^p(G_m) \) for \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers satisfying the condition (2.1.5). Then

\[
\|t_n f - f\|_p \to 0 \text{ as } n \to \infty.
\]

**Proof.** According to Corollary 2.3.5 we conclude that the conditions (A1), (A2) and (A3) in Theorem 2.1.5 are fulfilled and the stated norm convergence follows.

The proof is complete. ■

According to Theorems 2.4.1 and 2.4.2 we get the following result for Nörlund means:

**Corollary 2.4.3** Let \( f \in L^p(G_m) \) for \( p \geq 1 \). Then

\[
\|\sigma_n f - f\|_p \to 0 \text{ as } n \to \infty, \\
\|B_n f - f\|_p \to 0 \text{ as } n \to \infty, \\
\|\beta_n^\alpha f - f\|_p \to 0 \text{ as } n \to \infty,
\]

**Proof.** Since \( \sigma_n f \) and \( \beta_n^\alpha f \) are Nörlund means generated by non-decreasing sequences \( \{q_k : k \in \mathbb{N}\} \), the corresponding norm convergences are direct consequences of Theorem 2.4.1. In the case of \( B_n f \) means with non-decreasing sequence \( \{q_k : k \in \mathbb{N}\} \), this result is also a consequence of Theorem 2.4.1.

On the other hand, in the case of \( B_n f \) means with non-increasing sequence \( \{q_k : k \in \mathbb{N}\} \), this result is a consequence of Theorem 2.4.2 and (2.1.5).

The proof is complete. ■
2.5. CONVERGENCE OF NÖRLUND MEANS IN VILENKIN-LEBESGUE POINTS

Now, we consider subsequences of Nörlund means, generated by non-increasing sequences, but without any restrictions on the sequence \( \{q_k : k \in \mathbb{N}\} \):

**Theorem 2.4.4** Let \( f \in L^p(G_m) \) for \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then

\[
\|t_{M_n} f - f\|_p \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.** According to Corollary 2.3.8 we conclude that conditions (A1), (A2) and (A3) in Theorem 2.1.5 are fulfilled and the claimed norm convergence is proved. The proof is complete. 

2.5 Convergence of Nörlund Means in Vilenkin-Lebesgue points

Our first main result concerning convergence of Nörlund means reads:

**Theorem 2.5.1** a) Let \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers.

If the function \( f \in L^1(G_m) \) is continuous at a point \( x \), then

\[
t_n f(x) \to f(x), \quad \text{as} \quad n \to \infty.
\]

Furthermore,

\[
\lim_{n \to \infty} t_n f(x) = f(x)
\]

for all Vilenkin-Lebesgue points of \( f \in L^p(G_m) \).

b) Let \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers satisfying the condition (2.1.5).

If the function \( f \in L^1(G_m) \) is continuous at a point \( x \), then

\[
t_n f(x) \to f(x), \quad \text{as} \quad n \to \infty.
\]

Moreover,

\[
\lim_{n \to \infty} t_n f(x) = f(x)
\]

for all Vilenkin-Lebesgue points of \( f \in L^p(G_m) \).

**Proof.** Let \( \{q_k : k \in \mathbb{N}\} \) be a non-decreasing sequence. Suppose that \( x \) is either a point of continuity of a function \( f \in L^p(G_m) \) or Vilenkin-Lebesgue point of the function \( f \in L^p(G_m) \). According to Proposition 2.1.3 and Proposition 2.1.2 we can conclude that

\[
\lim_{n \to \infty} |\sigma_n f(x) - f(x)| = 0.
\]
Hence, by combining (2.3.3) and (2.3.7) we can conclude that
\[
|t_n f(x) - f(x)| \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_{n-j} - q_{n-j-1}) j |\sigma_j f(x) - f(x)| + q_0 n |\sigma_n f(x) - f(x)| \right) \\
\leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_{n-j} - q_{n-j-1}) j \alpha_j + \frac{q_0 n \alpha_n}{Q_n} := I + II,
\]
where \( \alpha_n \to 0, \) as \( n \to \infty. \)

Since the sequence \( \{q_k : k \in \mathbb{N}\} \) is non-decreasing, we can conclude that
\[
II \leq \alpha_n \to 0, \quad \text{as } n \to \infty.
\]

On the other hand, since \( \alpha_n \) converges to 0, we get that there exists an absolute constant \( A, \) such that \( \alpha_n \leq A \) for any \( n \in \mathbb{N} \) and for any \( \varepsilon > 0 \) there exists \( N_0 \in \mathbb{N} \), such that
\[
\alpha_n < \varepsilon \quad \text{when } \quad n > N_0.
\]

Hence,
\[
I = \frac{1}{Q_n} \sum_{j=1}^{N_0} (q_{n-j} - q_{n-j-1}) j \alpha_j + \frac{1}{Q_n} \sum_{j=N_0+1}^{n-1} (q_{n-j} - q_{n-j-1}) j \alpha_j \\
:= I_1 + I_2.
\]

Since
\[
|q_{n-j} - q_{n-j-1}| < 2q_{n-1} \quad \text{and} \quad \alpha_n < A,
\]
we obtain that
\[
I_1 = \frac{1}{Q_n} \sum_{j=1}^{N_0} (q_{n-j} - q_{n-j-1}) j \alpha_j \leq \frac{2A N_0 q_{n-1}}{Q_n} \to 0, \quad \text{as } n \to \infty
\]
and
\[ I_2 = \frac{1}{Q_n} \sum_{j=N_0+1}^{n-1} (q_{n-j} - q_{n-j-1}) j \alpha_j \]
\[ \leq \frac{\varepsilon}{Q_n} \sum_{j=0}^{n-1} (q_{n-j} - q_{n-j-1}) j \]
\[ \leq \frac{\varepsilon}{Q_n} \sum_{j=0}^{n-1} (q_{n-j} - q_{n-j-1}) j < \varepsilon. \]

We conclude that also \( I_2 \rightarrow 0 \) so a) is proved.

Assume now that the sequence is non-increasing and satisfying condition (2.1.5). To prove convergence in Vilenkin-Lebesgue points we use the estimations (2.3.3) and (2.3.7) to obtain that

\[ |t_n f - f(x)| \leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_{n-j-1} - q_{n-j}) j \alpha_j + \frac{q_0 \alpha_n}{Q_n} \]
\[ := III + IV, \]

where \( \alpha_n \rightarrow 0, \) as \( n \rightarrow \infty. \)

It is evident that

\[ IV \leq \frac{q_0 \alpha_n}{Q_n} \leq C \alpha_n \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \]

Moreover, for any \( \varepsilon > 0 \) there exists \( N_0 \in \mathbb{N}, \) such that \( \alpha_n < \varepsilon \) when \( n > N_0. \)

It follows that

\[ \frac{1}{Q_n} \sum_{j=1}^{n-2} (q_{n-j-1} - q_{n-j}) j \alpha_j \]
\[ = \frac{1}{Q_n} \sum_{j=1}^{N_0} (q_{n-j-1} - q_{n-j}) j \alpha_j + \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{n-j-1} - q_{n-j}) j \alpha_j \]
\[ := III_1 + III_2. \]

Since sequence is non-increasing, we can conclude that

\[ |q_{n-j} - q_{n-j-1}| < 2q_0. \]

Hence,
\[ III_1 \leq \frac{2q_0N_0}{Q_n} \to 0, \quad \text{as} \quad n \to \infty \]

and

\[
III_2 \leq \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{n-j-1} - q_{n-j}) j \alpha_j \\
\leq \frac{\varepsilon(n-1)}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{n-j} - q_{n-j-1}) \\
\leq \frac{\varepsilon(n-1)}{Q_n} (q_0 - q_{n-N_0}) \\
\leq \frac{2q_0\varepsilon(n-1)}{Q_n} < C\varepsilon.
\]

Hence, also \( III \to 0 \) so the proof of part b) is also complete.

**Corollary 2.5.2** Let \( f \in L^p(G_m) \), where \( p \geq 1 \). Then, for all Lebesgue points of \( f \in L^p(G_m) \),

\[
\sigma_n f \to f, \quad \text{as} \quad n \to \infty,
\]

\[
B_n f \to f, \quad \text{as} \quad n \to \infty,
\]

\[
\beta_{n}^\alpha f \to f, \quad \text{as} \quad n \to \infty.
\]

**Theorem 2.5.3** Let \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then

\[
\lim_{n \to \infty} t_{M_n} f(x) = f(x)
\]

for all Lebesgue points of \( f \in L^p(G_m) \).

**Proof.** By using Lemma 2.3.6 we get that

\[
t_{M_n} f(x) = \int_{G_m} f(t) F_n(x - t) d\mu(t)
\]

\[
= \int_{G_m} f(t) D_{M_n}(x - t) d\mu(t) - \int_{G_m} f(t) \psi_{M_n-1}(x - t) F^{-1}_{M_n}(x - t)
\]

\[
:= I - II.
\]
By applying Proposition 2.1.1 we get that $I = S_{M_n}f(x) \to f(x)$ for all Lebesgue points of $f \in L^p(G_m)$, where $p \geq 1$.

Moreover, according to Proposition 1.1.1 we find that

$$\psi_{M_n-1}(x-t) = \psi_{M_n-1}(x)\overline{\psi_{M_n-1}(t)}$$

and

$$II = \psi_{M_n-1}(x) \int_{G_m} f(t) \overline{F^{-1}_{M_n}(x-t)\psi_{M_n-1}(t)} d(t).$$

By combining Theorem 1.2.2 and Corollary 1.3.12 we find that the function

$$f(t) \overline{F^{-1}_{M_n}(x-t)} \in L^p(G_m) \quad \text{where} \quad p \geq 1 \quad \text{for any} \quad x \in G_m,$$

and $II$ are Fourier coefficients of an integrable function. Hence, according to the Riemann-Lebesgue Lemma we get that

$$II \to 0, \quad \text{as} \quad n \to \infty, \quad \text{for any} \quad x \in G_m.$$

The proof is complete.  

**Corollary 2.5.4** Let $f \in L^p(G_m)$, where $p \geq 1$. Then, for all Lebesgue points of $f \in L^p(G_m)$,

$$\sigma_{M_n}^\alpha f \to f, \quad \text{as} \quad n \to \infty,$$

$$V_{M_n}^\alpha f \to f, \quad \text{as} \quad n \to \infty,$$

$$U_{M_n}^\alpha f \to f, \quad \text{as} \quad n \to \infty.$$
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