Complementarity and Classical Limit of Quantum Mechanics:
Energy Measurement aspects

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Abstract

In the present contribution we discuss the role of experimental limitations in the classical limit problem. We studied some simple models and found that Quantum Mechanics does not reproduce classical mechanical predictions, unless we consider the experimental limitations ruled by uncertainty principle. We have shown that the discrete nature of energy levels of integrable systems can be accessed by classical measurements. We have defined a precise limit for this procedure. It may be used as a tool to define the classical limit as far as the discrete spectra of integrable systems are concerned. If a diffusive environment is considered, we conclude that the ”lifetime” of discreteness is approximately 1/κ (κ is the diffusion constant), thus it was possible to relate the classical limit of a spectra with the action of an environment and experimental resolution.

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I. INTRODUCTION

Since the early twentieth century, the predictions of quantum mechanics were strange to most people and almost always seemed to contradict our common sense. There were no conclusive explanation for the disappearance of quantum effects in the macroscopic regime. Which of the quantum features we don’t see in our day life experience? One of the most disconcerting one is the wave-particle duality. This is in fact the first quantum feature that has a strong argument explaining its disappearance in the macroscopic world. The solution is in de Broglie expression for the wavelength. The length of the wave interference fringes becomes insignificant as we increase the classical action of the system. There are many other quantum features that are unobservable in the macroscopic regime, such as entanglement and discreteness of energy spectrum. Other important open question is about the classical regime. Does quantum mechanics reproduce the observed results of macroscopic experiments? The first try to answer this question is attributed to Bohr, the correspondence principle. Bohr’s correspondence principle [1]:

"Indeed, as adequate as the quantum postulates are in the phenomenological description of the atomic reactions, as indispensable are the basic concepts of mechanics and electrodynamics for the specification of atomic structures and for the definition of fundamental properties of the agencies with which they react. Far from being a temporary compromise in this dilemma, the recourse to essentially statistical considerations is our only conceivable means of arriving at a generalization of the customary way of description sufficiently wide to account for the features of individuality expressed by the quantum postulates and reducing to classical theory in the limiting case where all actions involved in the analysis of the phenomena are large compared with a single quantum. In the search for the formulation of such a generalization, our only guide has just been the so called correspondence argument, which gives expression for the exigency of upholding the use of classical concepts to the largest possible extent compatible with the quantum postulates."

Bohr also stated the postulate known as selection rule:

"This peculiar relation suggests a general law for the occurrence of transitions between stationary states. Thus we shall assume that even when the quantum numbers are small the possibility of transition between two stationary states is connected with the presence of a certain harmonic component in the motion of the system [2]. "

[1] Bohr, N. (1928). The correspondence principle. Nature, 121(3069), 580-589.
[2] Bohr, N. (1928). The correspondence principle and the quantum postulates. Nature, 121(3069), 589-590.
Bohr’s correspondence principle and his selection rule were later interpreted as:

1. “Quantum mechanical predictions must correspond to classical ones in the limit of large quantum numbers.”
2. “A selection rule must be valid for all possible quantum numbers. Thus, a selection rule must be valid for classical and quantum regimes.”

The Correspondence Principle of Quantum Mechanics is a widely used tool to study quantum to classical transition problem since its conception by Bohr. Nevertheless, Quantum and Classical Mechanics are incommensurable theories because their basic ontologies are different. The basic entities of Classical Mechanics (CM) are coordinates and velocities in configuration spaces. Quantum Mechanics (QM) deals with state vectors and operators in Hilbert space. Since both theories give acceptable descriptions of different experiments in limiting situations there should be a way of translating their predictions from one to the other, in going from QM to CM this is the so called classical limit problem. There are also attempts to explain experimental results usually in quantum domain within the framework of CM.

In the core of the Correspondence Principle there are two “beliefs” that has influenced the study of the classical limit problem. The first one is that classical mechanics is able to reproduce macroscopic experimental observations within an arbitrary precision in a such way that its predictions can be used for comparison with the quantum description of the experiment. The second one is that “typical quantum objects” and “typical classical objects” are subjected to the same conditions. Although the confirmation of these “beliefs” may be desirable, this is not necessary nor sufficient. We know that an experimental result, usually in classical domain, can be considered as a quantum one. This problem can be extended to a more general problem of comparing two theories. Suppose we have two theories A and B, and that it is common sense that theory A has a validity domain that includes the domain of B, i.e., A ⊃ B. Thus, it is expected that theory A reproduces all the results for B and it should also not be in contradiction with theory B in its domain. A typical example is special relativity, where we adopt slow velocity limit to yield Newtonian regime. Strictly speaking about classical mechanics, some points are relevant to our discussion:

1. Classical mechanics is a very successful theory, but it does not mean that there is
no uncertainty within its predictions. Inaccuracies are unavoidable, but in classical
domain Newtonian dynamics is in agreement with the observations.

2. It would be very interesting to obtain classical mechanics from quantum mechanics in
terms of a limit, such as high energy or mass. Although that, the parameter to be
used should not be a universal constant such as $\hbar$. For practical reasons it should be
a variable that can be manipulated in a laboratory.

3. There is not a known experimental criteria of general validity which determines if a
quantum state can be considered classical in all aspects or all possible measurements.

The above comments take us to philosophical questions, such as what is the objective of
a physical theory? What do we need to build a useful theory? Which criteria one should use
to compare two theories? It is common sense that a good physical theory must reproduce
the experimental results within the error bar. It would be more acceptable if it is able to
predict unknown phenomena. A physical theory must include mathematical entities which
the physicist should associate with the ones observed in laboratory. With the help of such
mathematical entities, one should be able to predict the time evolution of the observed physical entities. The last question is a subject of a stronger debate [5, 12–23]. If it was always
possible to measure the quantum state, it would be a most precise separability criteria. But
in fact we know how to measure Wigner functions only in some specific situations [23–29].

Let us go back to the general problem of theories $A$ and $B$. Suppose that an experimen-
talist makes an experiment in $B$ domain, and the result is compatible, within experimental
error bar, with theory $B$. If theory $A$ also gives a good description of the experiment and
its prediction is in accordance with the results, we can thus say that for this specific ex-
periment both theories are satisfactory, meaning that they are equivalent for this case. If
$Im(A) \cap Im(B) \neq \emptyset$, it is possible to have such an experimental result compatible with the-
tories $A$ and $B$: in this sense, the theories are not divergent. Due to the experimental errors
[21, 31], or inaccuracies [52], that are unavoidable and inherent to quantum theory [53], a
point in domain should be represented as a volume in its image. If all possible realizable
experiments in $B$ domain have their image in the intersection of theories image we can thus
say that theory $A$ successfully replaces $B$.

The investigation of the correspondence principle employing a quartic oscillator coupled
to a diffusive phase reservoir and taking into account the limited experimental resolution is
found in Ref. [21, 22]. They studied the system dynamics in perspective of an observable, the position or momentum.

In ref. [21] they show that the problem of the quantum-to-classical transition is close related to decoherence and inaccuracies of measurements on the correspondence of the two descriptions. They claimed that isolated systems and measurements with unlimited accuracy are mere idealizations.

In this contribution, the main goal is to investigate the role of experimental errors and of environment in the quantum-classical transition problem. We focus on the spectra discreteness. We propose a heuristic procedure (not deduced mathematically within the theory) to use spectroscopic information from model Hamiltonians and time energy Heisenberg relations in order to decide whether a quantum system can be described by CM. The quantum behavior is characterized by the discreteness of energy spectra. We are considering a gedankenexperiment, where the experimentalist does not know Quantum Mechanics but tries obtain the energy spectrum. Will he be able to conclude that it is discrete?

II. THE LARGE QUANTUM NUMBERS LIMIT

One controversial subject in quantum mechanics is the time-energy uncertainty relation \((\Delta t \Delta E \geq \hbar)\) and it is not our aim to bring this discussion to the present work. Nevertheless it is well known that time and frequency are conjugated variables in a pair of Fourier transforms in classical physics, the duration of a signal and the respective frequency are subjected to an unsharpness relation \((\Delta t \Delta \omega \geq \frac{1}{2})\) that is classical indeed. Frequency, in quantum theory, is another way of speaking about energy. An uncertainty relation between time and energy must be seriously considered, studied and interpreted, although there are objections due to the fact that time is not associated to a dynamical operator canonically conjugated to the hamiltonian. An exhaustive examination of this matter is made by Peres in his book [32]. On the other hand we have to assume that QM is not an objective description of physical reality. It only predicts the probability of occurrence of stochastic macroscopic events, following specified preparation procedures. These predictions are based on conceptions that are given to know indirectly - like all the microscopic elements as electrons, photons, etc. They have no substance at all, they are just perceived or understood by their interactions.
with measurement devices. We want to avoid these interpretational intricacies of QM by stating the problem in another way: how to describe the aim of measurement or what has actually to be done in a measurement process?

The main stream of our proposal is not face the discussion of the time-energy uncertainty relation, we face the problem of to decide under what conditions a system may be considered as Classical or Quantum. It is in this sense that we tackle the problem of dealing with the physical reality: how can one measure and what is in fact measured. Thus, we use the product of energy differences between neighbour levels by the corresponding classical period differences to compare with the time-energy uncertainty relation to classify a system as classical or quantum. If this product fulfills the time-energy uncertainty relation it is quantum otherwise it is classical.

To do this we add an element connected to this subject that concerns integrable systems: the decision wether a system is classical or quantum depends on the experimental apparatus which are essentially classical. One undoubtedly quantum feature is the discreteness of at least part of the energy spectrum. The essential idea here is that if one tries to measure the energy of the system in question using classical canonical pairs \(q, p\), the information about the quantum nature of the particle will be lost. In spite of this fact, as we show in what follows, the function

\[
y(n) = |\Delta E_n \Delta \tau_n|
\]

where \(\Delta E_n = (E_n - E_{n-1})/2\) is merely the energy difference between two neighbour levels, (it is the maximum uncertainty in energy for the state \(a|E_n\rangle + b|E_{n-1}\rangle\) ) and \(\Delta \tau_n = (\tau_n - \tau_{n-1})/2\), with \(\tau_n\) being the classical period associated to the energy \(E_n\). If the experiment has an accuracy \(\delta t\) it limits the period measurement precision, in real systems it is desirable that \(\delta t \ll \Delta \tau\), we assume that they are of same order, \(\Delta \tau \approx \delta t\).

The \(y(n)\) function can be heuristically justified if we consider the Bohr-Sommerfeld quantization rule for periodic systems, that states

\[
I = \oint pdq = 2\pi \hbar n
\]

thus we have

\[
\langle K \rangle = \frac{\pi \hbar n}{\tau},
\]

where \(\langle K \rangle\) is the the mean kinetic energy, \(\tau\) being the classical period associated with \(K\)
and $n$ is a quantum number. Observing that

$$\delta I = \oint \delta H dt$$

(4)

Consider two neighboring periodic motions of the same system. Then $H$ is a constant, and we have

$$\delta I = \tau \delta H.$$  

(5)

Note that (5) determines that the energy of the system depends only on $I$, then Bohr’s quantization rule determines the energy of the system, for more detail see [33].

The Bohr-Sommerfeld quantization rule makes a direct connection between kinetic energy and classical period, it was also demonstrated [34] that, for integrable systems, it is possible to reconstruct semiclassically the quantum state using classical dynamics. Since you know the Wigner function of the system it is possible to infer the period related to the state [34, 35]. Alternatively, we can use the semiclassical quantization rule [36, 37] for integrable systems, we have a similar result [37–39].

This function (4) is stated for a more clear definition of a classical limit. As we will show, the fact that one will only be able to see a continuum of energies does not mean that the classical limit has been reached. It only reveals that we have become “myope” to see the nature of the spectrum.

In what follows, we will consider a simple one dimensional systems with discrete spectra. Let us assume that a state has been prepared in the following way

$$|\Psi(0)\rangle = a |E_n\rangle + b |E_{n-1}\rangle$$

(6)

where $\{|E_n\rangle\}$ are eigenstates of the Hamiltonian $H$

$$\hat{H} |E_n\rangle = E_n |E_n\rangle$$

(7)

The variance in energy difference, for any time, of this state is given by

$$\Delta E = |a| |b| |E_n - E_{n-1}|$$

(8)

Using the normalization condition $|a|^2 + |b|^2 = 1$, it is easy to check that the maximum for this energy difference is achieved for $|a| = |b| = 1/\sqrt{2}$, and is given by
\[ \Delta E(n) = \frac{1}{2} |E_n - E_{n-1}|. \]  

(9)

How do we define the classical limit in this simple case? Let us assume the following

\[ \lim_{n \to \infty} \frac{\Delta E_n}{E_n} = 0. \]

The above expression means that as the energy increases \((E_n)\), the uncertainty in energy of the state \(|\Psi(t)\rangle\) becomes negligible as compared to \(E_n\). Let us discuss a few examples.

**Harmonic Oscillator**

For the harmonic oscillator we have

\[ \Delta E = \frac{1}{2} \hbar \omega \Rightarrow \frac{\Delta E}{E_n} = \frac{1/2}{(n + 1/2)} \]  

and we clearly have \( \lim_{n \to \infty} \frac{\Delta E_n}{E_n} = 0. \)

**Particle in a box**

In this case we have

\[ E_n = \hbar^2 \frac{n^2 \pi^2}{2ma^2}, \]  

where \(n = 1, 2, 3, \ dots, m\) is the mass of the particle and \(a\) is the box width. Then we have

\[ \Delta E_n = \hbar^2 / 4 \left| \frac{(2n - 1)\pi^2}{ma^2} \right|. \]  

(12)

Once again we have \( \lim_{n \to \infty} \frac{\Delta E_n}{E_n} \to 0. \)

The fact that this limit is zero is the argument usually found in textbooks in order to define the classical limit\([3, 40, 41]\). Of course it is true that when one considers the high energy limit, it becomes increasingly difficult to obtain good experimental resolution. However, as we show below, this limit does not necessarily imply that it is impossible to obtain the needed resolution. As we know, quantization has been observed in some macroscopic systems like superconducting Josephson junctions\([42]\). Besides, one can use something analogous to the Heisenberg microscope\([43]\) in order to obtain the velocity of the particle. In the case of a particle in a box, discussed above, once we know the velocity, the energy is easily obtained. The measurement of the velocity implies a perturbation of the position which depends on the characteristics of the apparatus and in principle, is independent of the quantum number \(n\). Thus, if one uses the adequate measurement the discrete character of a spectrum can be verified.
III. CLASSICAL MEASUREMENT OF THE ENERGY

When one deals with realistic systems, say, atoms, the spectrum is obtained from the emitted or absorbed electromagnetic radiation. A completely different approach is used for macroscopic systems. Within CM, for closed systems, the energy is a function of position and momentum \( E = f(p, q) \). So, from now on, we will call Classical Measurement of energy every process which makes use of the relation \( E = f(p, q) \) in order to obtain the energy of a given system, classical or quantum. As an example, we may look at the particle in a box again. In this case, the energy is a direct function of the velocity, so that once we determine the velocity, the energy will be defined. In practice, one may measure the time it takes for 2s inversions in the momentum and so determine the period of the motion. Classically, the period is given by

\[
\tau = \frac{2a}{v} = a\sqrt{\frac{2m}{E}}. \tag{13}
\]

In the above expression, we are considering the correspondent classical period for a specific energy eigenvalue. Since the energy levels are determined by QM, regardless of the energy scale one is talking about, the allowed values of the period are

\[
\tau_n = \frac{2a^2m}{\hbar n\pi}. \tag{14}
\]

So, the difference in period for two quantum neighboring levels is

\[
\Delta \tau = \frac{\tau_n - \tau_{n-1}}{2} = -\frac{a^2m}{\hbar\pi} \frac{1}{(n - 1)n}. \tag{15}
\]

From the above expression one can see that it becomes increasingly difficult to distinguish two higher adjacent energy levels by this method. Let us take a look at the product \( |\Delta E_n\Delta \tau_n| \) where \( \Delta E_n = (E_n - E_{n-1})/2 \) and \( \Delta \tau_n = (\tau_n - \tau_{n-1})/2 \). \( \tau_n \) is the classical period associated with the energy \( E_n \). It is interesting to observe the behavior of the function

\[
y(n) = |\Delta E_n\Delta \tau_n|. \tag{16}
\]

For the case in question,

\[
y(n) = \frac{\hbar\pi}{4} \frac{(2n - 1)}{(n - 1)n}. \tag{17}
\]

If we take \( n = 4 \), we find \( y(4) < \hbar/2 \), which means that we won’t have enough precision to verify whether the spectrum is discrete or continuous, since Quantum Mechanics forbids
such precision[54]. Here, we are just using the fact that the time-energy uncertainty relation has to be respected since we are considering that Quantum Mechanics must prevent Classical Mechanics. Also it is easy to see that \( \lim_{n \to \infty} y(n) = 0 \).

In the more realistic case of a hydrogenoid atom we have

\[
E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}
\]  

(18)

where \( Z e \) is the total charge interacting with the electron of charge \( e \), and \( \mu \) is the reduced mass of the system. As opposed to the previous case, the energy levels become closer as \( n \) grows. Thus, the question is: from which \( n \) can we say that the spectrum is continuous from the point of view of Classical Mechanics? In this case

\[
\Delta E = \frac{\mu Z^2 e^4}{4\hbar^2} \left| \frac{2n - 1}{n^2(n - 1)^2} \right|
\]  

(19)

\[
\Delta T = \frac{\pi \hbar^3}{Z^2 e^4 \mu} \left[ 3n^2 - 3n + 1 \right]
\]  

(20)

and

\[
y(n) = \frac{\pi \hbar (2n - 1) \left[ 3n^2 - 3n + 1 \right]}{4n^2(n - 1)^2}
\]  

(21)

Now, for the \( 1/r \) potential, we have \( y(n) < \hbar/2 \) for \( n = 9 \), so one won’t be able to observe the quantization for \( n \) bigger than 9 while using classical measurement of the energy.

Other interesting case is the Morse Potential, frequently used to describe the spectra of molecules[35]. The Morse potential is defined as

\[
U(x) = D \left( e^{-2\alpha x} - 2e^{-\alpha x} \right)
\]  

(22)

where \( D \) and \( \alpha \) are constants experimentally determined. For \( s \) waves, i.e., the orbital angular momentum is zero, we have

\[
E(n) = -D + \hbar \omega \left[ (n + 1/2) - \frac{1}{\zeta} (n + 1/2)^2 \right]
\]  

(23)

\( \zeta \) is also experimentally determined. The classical period corresponding to each \( E(n) \) is given by

\[
\tau(n) = 2\pi \sqrt{\frac{MRo^2}{2 |E(n)| \alpha^2}}
\]  

(24)

where \( Ro \) is a function of \( \alpha \) and \( \zeta \). It is easy to verify that, for the hydrogen molecule parameters, \( y(n) < \hbar/2 \) for all possible \( n \). Since the Morse potential is quasi-harmonic for
low energies we observe that it is not possible to distinguish neighboring discrete states with a classical measurement. From this example, we may conclude that any potential that is approximately harmonic have no assessable discrete spectrum through a classical measurement of the energy.

A. Harmonic Oscillator

For a harmonic oscillator, \( E = \frac{p^2}{2m} + \frac{1}{2}kq^2 \), its period is \( \tau = \frac{2\pi}{\omega} \) where \( \omega = \sqrt{\frac{k}{m}} \) which is energy independent, thus we conclude that it can not be used to characterize the spectrum. Another way of classically determining the energy can be obtained by measuring \( q \) and \( p \) at same time, therefore classical energy uncertain is given by

\[
\delta E = \frac{p}{m}\delta p + kq\delta q
\]  

(25)

Without lost of generality, we choose \( \delta p = \sqrt{\frac{m\omega}{2}}a \) and \( \delta q = \sqrt{\frac{h}{2m\omega}}a \) then we obtain the relation

\[
\delta p\delta q = a^2\hbar/2.
\]  

(26)

In case of \( a=1 \) we have the minimum uncertain defined under Hisenberg relation, in general we have

\[
\delta E = \left[ |p|\sqrt{\frac{\hbar\omega}{2m}} + k|q|\sqrt{\frac{\hbar}{2m\omega}} \right] a
\]  

(27)

then

\[
\delta E^2 = \frac{2}{\hbar} \left[ p\sqrt{\frac{\hbar\omega}{2m}} + kq\sqrt{\frac{\hbar}{2m\omega}} \right]^2 \delta p\delta q
\]  

(28)

From (10) we have

\[
\delta E = \frac{1}{2}\hbar\omega
\]  

(29)

thus we find

\[
\delta p\delta q = \frac{\hbar^3\omega^2}{4 \left[ |p|\sqrt{\frac{\hbar\omega}{2m}} + k|q|\sqrt{\frac{\hbar}{2m\omega}} \right]^2}.
\]  

(30)

Note that

\[
\left[ |p|\sqrt{\frac{\hbar\omega}{2m}} + k|q|\sqrt{\frac{\hbar}{2m\omega}} \right]^2 = \left( \frac{p^2}{2m} + \frac{1}{2}kq^2 + |pq|\omega \right) \hbar\omega,
\]  

(31)
if we consider the energy as \( E = \hbar \omega (n + 1/2) \), we find
\[
\delta p \delta q = \frac{\hbar^2}{4 (\hbar(n + 1/2) + |pq|)} \tag{32}
\]

From (32), we see that we need an experimental resolution that is avoided by Quantum Mechanics. Thus we can say that his spectrum cannot be resolved by Classical Energy Measurement.

**IV. NEIGHBORS DISTINGUISHABILITY**

Let us consider a quartic oscillator model. Its Hamiltonian is
\[
\hat{H}_0 = \hbar \omega \hat{a}^\dagger \hat{a} + \lambda \hbar^2 (\hat{a}^\dagger \hat{a})^2 \tag{33}
\]
where \( \hat{a} \) and \( \hat{a}^\dagger \) are destruction and creation operators of the harmonic oscillator, \( \omega \) is the natural frequency of the main oscillator, \( \lambda \) is the non-linear strength constant. The system has as energy levels
\[
E(n) = \hbar \omega n + \lambda \hbar^2 n^2. \tag{34}
\]

Its classical counterpart, with an energy given by (34) has a period given by
\[
\tau = \frac{2 \pi}{\omega + 2 \lambda n}, \tag{35}
\]
then we have
\[
y(n) = \frac{\pi \lambda n^2 [\omega + \lambda h(2n - 1)]}{[\omega + 2 \lambda h(n - 1)] [\omega + 2 \lambda h n]}. \tag{36}
\]

The case \( \lambda \hbar \ll \omega \) we obtain harmonic potential, as expected, and for \( \lambda \hbar \gg \omega \) we obtain the infinity well result:
\[
\lim_{\lambda \hbar/\omega \to 0} y(n) \approx \frac{\pi \lambda \hbar^2}{\omega}, \tag{37}
\]
\[
\lim_{\lambda \hbar/\omega \to \infty} y(n) \approx \frac{\pi \hbar (2n - 1)}{4(n - 1)n}. \tag{38}
\]

Considering \( \lambda \hbar \gg \omega \), in principle we can access the discreet spectra for \( n < 4 \).

Up to now, we didn’t consider the environment action. Real systems are never isolated, thus in a real experiment we can not ignore its action. Due to the impossibility of modeling the system and its surrounding from first principles, we are forced to treat the environment as a collection of harmonic oscillators, as usual. We use the model of the quartic oscillator
coupled to a bath of harmonic oscillators. This model is described by the
Hamiltonian
\[ \hat{H}_q = \hat{H}_0 + \hat{H}_{\text{int}} + \hat{H}_A \]  

where
\[ \hat{H}_0 = \hbar \omega \hat{a}^\dagger \hat{a} + \lambda \hbar^2 (\hat{a}^\dagger \hat{a})^2 \]  
is the Hamiltonian that governs the free evolution of the system of interest,
\[ \hat{H}_A = \sum_k \hbar \omega_k \hat{b}_k^\dagger \hat{b}_k \]  
is the Hamiltonian that governs the free evolution of the set of environmental harmonic
oscillators, and
\[ \hat{H}_{\text{int}} = \sum_k \hbar g_k (\hat{b}_k \hat{a}^\dagger + \hat{b}_k^\dagger \hat{a}) \]  
is the interaction Hamiltonian between the system of interest and the environment. In the
above equations, \( \omega_k \) is the natural frequency of the \( k \)-th oscillator of the bath, and \( g_k \) is the
coupling constant between the \( k \)-th bath oscillator and the system. For an initial Fock state
\( |b\rangle \), the density matrix evolution is given by
\[ \rho(t, b) = \sum_{l=0}^{\infty} \sum_{p=0}^{b} \frac{b! \cdot (p+l)! \cdot (l+p)! \cdot (b-p)!}{(p!)^2 \cdot (b+p)!} \]  

\[ \times \gamma(b+l-p, 0, t) \zeta(2p+1, 0, t) |p+l\rangle \langle p+l| . \]  

Here \( |n\rangle \) are Fock states and
\[ \gamma(n, t) = \frac{2\kappa \sinh(\Delta t)}{\Delta \cosh(\Delta t) + (i\lambda n + 2\kappa) \sinh(\Delta t)} \]  
\[ \zeta(n, t) = \frac{\Delta}{\Delta \cosh(\Delta t) + (i\lambda n + 2\kappa) \sinh(\Delta t)} \]

where \( \Delta = \sqrt{(i\lambda n + 2\kappa)^2 - 4\kappa^2} \)

and \( \kappa \) is the diffusion constant, see \[13, 14\].

This equation (43) indicates that an initial Fock state turns to a statistical mix state in a
short time ruled by \( \kappa \). Thus, if we prepare the system in an initial Fock state \( |b\rangle \), after a time
t, we can measure the system in a different Fock state. If we wish to distinguish neighboring
states we may use the fidelity. The time evolution of fidelity between $|b\rangle$ and $|b-1\rangle$, is given by

$$F(b, t) = Tr \left[ \rho(t, b) \rho(t, b-1) \right].$$  \hfill (46)

After some straightforward algebraic manipulations we obtain

$$F(b, t) = \sum_{l=0}^{\infty} \sum_{p=0}^{b} \sum_{p'=0}^{\min(b-1,p+l)} \frac{b ((b-1)!)((p+l)!)^2}{(p'!)^2 (p!)^2!!(b-p)!} \frac{1}{(p+l-p')!(b-p'-1)!} \times \gamma^{2b+2l-2p-1}(0,t)\zeta^{2(p+p')}(0,t).$$  \hfill (47)

In figure (1) we see $F(b, t)$ for $b=1,2,3,4$. Even for short time, $F(b, t)$ is different from zero, thus the gedankenexperiment of classical energy measurement has to be performed in a short time, what in practice turns it almost impossible since one needs a long time experiment for an accurate energy measurement.

In figure (2) we have the fidelity of the state $|b><b|_t$ and $|b>$ for $b=1, 5, 10$ and $15$. This is the probability of observing the state $|b>$ as function of time supposing we have prepared this state. As we can observe in this figure, the higher energy gets, the faster fidelity approaches zero (b). This means that we have a short time to measure the state before it becomes a complete statistical mixture.

If we consider a mixed state as (43) the $y(n)$ function has to be adapted. For the quartic oscillator que can assume that

$$\langle \tau \rangle \approx \frac{2\pi \hbar \langle \hat{N} \rangle}{\langle \hat{H}_0 \rangle},$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$. Thus, for an initial fock state $|b\rangle$, we have

$$\langle \hat{N}_b(b) \rangle = \sum_{l=0}^{\infty} \sum_{p=0}^{b} \frac{b! (p+l)!(p+l)}{(p!)^2!!(b-p)!} \times \gamma^{b+l-p}(0,t)\zeta^{2p+1}(0,t)$$  \hfill (48)

and

$$\langle \hat{H}_0(b) \rangle = \sum_{l=0}^{\infty} \sum_{p=0}^{b} \frac{b! (p+l)! \hbar \omega (p+l) + \lambda \hbar^2 (p+l)^2}{(p!)^2!!(b-p)!} \times \gamma^{b+l-p}(0,t)\zeta^{2p+1}(0,t).$$  \hfill (49)
Then we have \( \langle \Delta E_n \rangle = \left( \langle \hat{H}_0(b) \rangle - \langle \hat{H}_0(b - 1) \rangle \right) / 2 \) and \( \langle \Delta \tau_b \rangle = \left( \langle \tau_b \rangle - \langle \tau_{b-1} \rangle \right) / 2 \), and finally, for an initial state \( |b \rangle \) we obtain

\[
\langle y(b) \rangle = \langle \Delta E_b \rangle \langle \Delta \tau_b \rangle .
\]  

(50)

In figure (3) we have \( \langle y(b) \rangle \) of the state \( | b > < b \rangle(t) \) for \( b = 2, 5, 10 \) and \( 15 \) and \( \omega / \hbar \lambda = 0.10 \). In figure (4) we have \( \langle y(b) \rangle \) of the state \( | b > < b \rangle(t) \) for \( b = 2, 5, 10 \) and \( 15 \) and \( \omega / \hbar \lambda = 10 \). As we can see, the action of the environment reduces the chance of observing the spectra discreteness even for high non-linearity. The "lifetime" of discreteness is approximately \( 1/\kappa \), thus we connected the classical limit of a spectra with the action of an environment and experimental resolution.

V. CONCLUSION

In this work we have shown that the discrete nature of the energy levels can be accessed by classical measurements in some cases. We also defined a precise limit for this procedure using the function \( y(n) = |\Delta E_n \Delta \tau_n| \) and comparing it with the time-energy uncertainty principle. This maneuver gives us a complementarity principle and a well defined mathematical limit dictated by the experiment. Of course, the fact that we are not able to recognize the discrete nature of a spectrum does not necessarily mean it is not discrete. It only means how "myope" we are, suggesting that Classical Mechanics can be viewed as a blurring of essential aspects of Quantum Mechanics and also explains why it took so long to find quantum effects. Also, as observed by many works [5, 13, 14, 18, 22, 47–49] a quantum system is never isolated, thus we are forced to include environment action that, in the present case, turns impractical any classical energy measurement as defined in section III, what means that otherwise we show how to get the classical limit in terms of discrete nature of the spectra.

VI. REFERENCES

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Actually, it is largely accepted that decoherence, which is an inherently quantum process and does not exist in classical mechanics, works as a selection rule for the transition from quantum to classical.

In this context error means experimental resolution.

We should remark that there is not yet a quantum measurement error theory, although there is some effort in this direction.

The time-energy uncertainty relation $\Delta E \Delta t \geq \hbar/2$ is a first principle limitation and has nothing to do with experimental errors. A deeper discussion on the subject can be found in Refs. [31, 43, 50].
FIG. 1: Full thin line: we see the fidelity decay, $F(t)$, for $b = 1$, dotted line: $b = 5$, dashed line: $b = 10$. 
FIG. 2: Full thin line we see the fidelity of the state $|\psi(t)\rangle$ and $|\Psi(t)\rangle$ for $b = 1$, dotted line.
FIG. 3: Full thin line we see \langle y(b) \rangle of state \lvert b \rangle \langle b \lvert(t) for \ l = 2, dotted line \ l = 5, dashed line.
FIG. 4: Full thick line curve $\langle y(t) \rangle$ of state $|b\rangle\langle b| (t)$ for $b = 2$, dotted line $b = 5$, dashed line $b = 10$. 