A Connection Formula of the Hahn–Exton $q$-Bessel Function

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Abstract. We show a connection formula of the Hahn–Exton $q$-Bessel function around the origin and the infinity. We introduce the $q$-Borel transformation and the $q$-Laplace transformation following C. Zhang to obtain the connection formula. We consider the limit $p \to 1^-$ of the connection formula.

Key words: Hahn–Exton $q$-Bessel function; $q$-Borel transformation; connection problems

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1 Introduction

In this paper, we show a connection formula of the Hahn–Exton $q$-Bessel function $J^{(3)}_{\nu}(x; q)$. At first, we review the Bessel function and $q$-analogues of the Bessel function. The Bessel equation

$$\frac{d^2 u}{dz^2} + \frac{1}{z} \frac{du}{dz} + \left( 1 - \frac{\nu^2}{z^2} \right) u = 0$$

has a solution $u(z) = J_{\nu}(z), J_{-\nu}(z)$. Here, the Bessel function $J_{\nu}(z)$ is

$$J_{\nu}(z) = \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^{\nu} 0F_1 \left( -\nu + 1, -\frac{z^2}{4} \right).$$

The degenerated confluent hypergeometric function $0F_1(-, \alpha, z)$ is defined by

$$0F_1(-, \alpha, z) = \sum_{n \geq 0} \frac{1}{(\alpha)_n n!} z^n, \quad (\alpha)_n = \alpha \{ \alpha + 1 \} \cdots \{ \alpha + (n - 1) \}.$$

Both $J_{\nu}(z)$ and $J_{-\nu}(z)$ are linearly independent if $\nu \notin \mathbb{Z}$.

It is known that there exists three different $q$-analogues of the Bessel function.

$$J^{(1)}_{\nu}(x; q) := \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \left( \frac{x}{2} \right)^{\nu} 2F_1 \left( 0, 0; q^{\nu+1}; q, -\frac{x^2}{4} \right), \quad |x| < 2,$$

$$J^{(2)}_{\nu}(x; q) := \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \left( \frac{x}{2} \right)^{\nu} 0F_1 \left( -; q^{\nu+1}; q, -\frac{q^{\nu-1}x^2}{4} \right), \quad x \in \mathbb{C},$$

$$J^{(3)}_{\nu}(x; q) := \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} x^{\nu} 1F_1 \left( 0; q^{\nu+1}; q, qx^2 \right), \quad x \in \mathbb{C}.$$

Here,

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & n \geq 1, \end{cases}$$

\begin{align*}
\frac{d^2 u}{dz^2} + \frac{1}{z} \frac{du}{dz} + \left( 1 - \frac{\nu^2}{z^2} \right) u &= 0, \\
J_{\nu}(z) &= \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^{\nu} 0F_1 \left( -\nu + 1, -\frac{z^2}{4} \right), \\
0F_1(-, \alpha, z) &= \sum_{n \geq 0} \frac{1}{(\alpha)_n n!} z^n, \quad (\alpha)_n = \alpha \{ \alpha + 1 \} \cdots \{ \alpha + (n - 1) \}. \
\end{align*}
\[(a; q)_\infty = \lim_{n \to \infty} (a; q)_n\]

and

\[ (a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty. \]

Moreover, the basic hypergeometric series \(r\varphi_s\) is

\[ r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left( -1 \right)^n q^{\frac{u(n-1)}{2}} x^n. \]

The first and the second one are called Jackson’s first and second \(q\)-Bessel function and the third one is called the Hahn–Exton \(q\)-Bessel function. They satisfy the following \(q\)-difference equations:

\[ J^{(1)}_\nu : \quad u(xq) - \left(q^\nu/2 + q^{-\nu/2}\right) u(xq^{1/2}) + \left(1 + \frac{x^2}{4}\right) u(x) = 0, \]

\[ J^{(2)}_\nu : \quad \left(1 + \frac{qx^2}{4}\right) u(xq) - \left(q^\nu/2 + q^{-\nu/2}\right) u(xq^{1/2}) + u(x) = 0, \]

\[ J^{(3)}_\nu : \quad u(xq) - \left\{ \left(q^\nu/2 + q^{-\nu/2}\right) - q^{-\nu/2+1} x^2\right\} u(xq^{1/2}) + u(x) = 0. \] \hspace{1cm} (1)

The limits of these \(q\)-analogues of the Bessel function are the Bessel function when \(q \to 1^-\):

\[ \lim_{q \to 1^-} J^{(k)}_\nu ((1 - q)x; q) = J_\nu(x), \quad k = 1, 2 \]

and

\[ \lim_{q \to 1^-} J^{(3)}_\nu ((1 - q)x; q) = J_\nu(2x). \]

The relation between \(J^{(1)}_\nu(x; q)\) and \(J^{(2)}_\nu(x; q)\) was found by Hahn [3] as follows:

\[ J^{(2)}_\nu(x; q) = \left(-\frac{x^2}{4}; q\right)_\infty J^{(1)}_\nu(x; q). \] \hspace{1cm} (2)

Connection problems of the \(q\)-difference equation between the origin and the infinity are studied by G.D. Birkhoff [1]. We review connection formulae for several \(q\)-difference functions.

1. Watson’s formula. In 1910 [6], Watson showed the connection formula of the basic hypergeometric function \(2\varphi_1\) as follows:

\[ 2\varphi_1(a, b; c; q; x) = \frac{(b, c/a; q)_\infty (ax, q/ax; q)_\infty}{(c, b/a; q)_\infty (x, q/x; q)_\infty} 2\varphi_1(a, aq/c; aq/b; q; cq/abx) + \frac{(a, c/b; q)_\infty (bx, q/bx; q)_\infty}{(c, a/b; q)_\infty (x, q/x; q)_\infty} 2\varphi_1(b, bq/c; bq/a; q; cq/abx). \]

2. Connection formula of \(J^{(1)}_\nu(x; q)\). C. Zhang has given some connection formulae for the solutions of the \(q\)-difference equations of confluent type [7, 8] and [9]. In [8], Zhang has shown connection formulae for \(J^{(1)}_\nu(x; q)\) and \(J^{(2)}_\nu(x; q)\). The connection formula of \(J^{(1)}_\nu(x; q)\) is given by

\[ \left(\frac{\alpha}{\sqrt{\beta}}; \beta\right)_\infty 2\varphi_1 \left(p^{\nu+\frac{1}{2}}, -p^{\nu+\frac{1}{2}}; -p; p; \frac{\alpha}{\sqrt{\beta}}\right). \]
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Definition 1. For any $\nu \in \mathbb{C}^*$, given by the following series:

$$\theta_p \left( -\frac{z}{x} \right) = \frac{1}{\theta_p \left( -\frac{z}{x} \right)} \left\{ \frac{\theta_p \left( -\frac{z^2}{x} \right)}{(q, q^{-\nu}; q)_\infty} 2\varphi_1 \left( 0, 0; q^{\nu+1}; q, -\frac{x^2}{4} \right) \right\}$$

where $q = p^2$ and $\alpha^2 = -4q^{3/2}$.

The connection formula of $J^{(2)}_\nu(x; q)$ is obtained by (3) and (2). But it is not known the connection formula of the Hahn–Exton $q$-Bessel function.

The Hahn–Exton $q$-Bessel equations (1) has two analytic solutions $u(x) = J^{(3)}_\nu(x; q)$ and $J^{(3)}_{-\nu}(xp^{-\nu})$ around $x = 0$ and has one analytic solution $z(x) = \frac{1}{\theta_p(-p^{\nu+2}/x)} \sum_{n \geq 0} a_n x^{-n}$, $a_0 = 1$. We show a connection formula of $J^{(3)}_\nu(x; q)$ in Section 2 as follows:

**Theorem 1.** For any $x \in \mathbb{C}^* \setminus [p^{\nu+2}; p]$, we have

$$z \left( \frac{1}{x} \right) = \frac{1}{(p^{2\nu}, p; p)_\infty} \theta_p \left( -\frac{p^{2\nu+2}}{x} \right) \frac{\theta_p \left( -\frac{z^2}{x} \right)}{(q, q^{\nu}; q)_\infty} 1\varphi_1 \left( 0, p^{1+2\nu}; p, x \right)$$

$$+ \frac{1}{(p^{2\nu}, p; p)_\infty} \frac{\theta_p \left( -\frac{z^2}{x} \right)}{(q, q^{\nu}; q)_\infty} 1\varphi_1 \left( 0, p^{1-2\nu}; p, p^{-2\nu} x \right).$$

Here, $\theta_p(\cdot)$ is the theta function of Jacobi and $[\lambda; q]$ is the $q$-spiral (see Section 2). We use the $q$-Borel transformation and the $q$-Laplace transformation which is defined by C. Zhang in [8].

In Section 3, we consider the limit $p \to 1^-$ of the connection formula. If we take a suitable limit $p \to 1^-$ of (4), we obtain

$$H^{(2)}_{\nu} \left( \sqrt{x} \right) = \frac{-ie^{i\nu \pi i}}{\sin \nu \pi} \left\{ J_{\nu} \left( \sqrt{x} \right) - e^{-\nu \pi i} J_{-\nu} \left( \sqrt{x} \right) \right\}.$$  

Here, $H^{(2)}_{\nu}(z)$ is the Hankel function of the second kind. Thus we obtain a connection formula of the Bessel function as a limit $p \to 1^-$ of (4).

2 The connection formula

In this section, we give a connection formula of the Hahn–Exton $q$-Bessel function. We introduce the $p$-Borel transformation and the $p$-Laplace transformation to obtain the connection formula between the origin and the infinity. These transformations are useful to consider connection problems. We assume that $q \in \mathbb{C}^*$ satisfies $0 < |q| < 1$ and $q = p^2$. The $q$-difference operator $\sigma_q$ is given by $\sigma_q f(x) = f(qx)$.

2.1 The theta function of Jacobi

Before we study connection problems, we review the theta function of Jacobi. The theta function of Jacobi is given by the following series:

**Definition 1.** For any $x \in \mathbb{C}^*$,

$$\theta_q(x) = \theta(x) := \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n.$$
We denote by $\theta_q(x)$ or more shortly $\theta(x)$. The theta function satisfies Jacobi’s triple product identity:

$$\theta(x) = (q, -x, -\frac{q}{x}; q)_\infty.$$

The theta function satisfies the $q$-difference equation as follows

$$\theta(q^kx) = q^{-\frac{k(k-1)}{2}}x^{-k}\theta(x), \quad \forall x \in \mathbb{C}^*.$$

The theta function has the inversion formula $x\theta(1/x) = \theta(x)$. For all fixed $\lambda \in \mathbb{C}^*$, we define a $q$-spiral $[\lambda; q] := \lambda q^Z = \{\lambda q^k : k \in \mathbb{Z}\}$. We remark that $\theta(\lambda q^k/x) = 0$ if and only if $x \in [-\lambda; q]$.

### 2.2 The Hahn–Exton $q$-Bessel function

The Hahn–Exton $q$-Bessel function is defined by

$$J^{(3)}_\nu(x; q) := (q^{\nu+1}; q)_\infty x^\nu \sum_{n \geq 0} (-1)^n q^{\frac{n(n-1)}{2}} (q^{\nu+1}; q)_n (qx^2)^n.$$

The function $J^{(3)}_\nu(x; q)$ satisfies the $q$-difference equation

$$\left[\sigma_p^2 - \left\{(p^{\nu} + p^{-\nu}) - x^2p^{2-\nu}\right\} \sigma_p + 1\right] y(x) = 0. \tag{5}$$

If we replace $\nu$ by $-\nu$ and $x$ by $xp^{-\nu}$, we obtain $J^{(3)}_{-\nu}(xp^{-\nu}; q)$ which is another solution of (5) around the origin. This solution corresponds to the classical Neumann function $Y_\nu(x)$ [5]. We consider the behavior of equation (5) around the infinity. We set $1/t$, formally $t^2 \mapsto t$ and $z(t) = y(1/t)$. Then $z(t)$ satisfies

$$\left[\sigma_p^2 - \left\{(p^{\nu} + p^{-\nu}) - \frac{p^{2-\nu}}{t}\right\} \sigma_p + 1\right] z(t) = 0. \tag{6}$$

We set $E(t) = 1/\theta_p(-p^{\nu+2}t)$ and $f(t) = \sum_{n \geq 0} a_n t^n$, $a_0 = 1$. We assume that $z(t)$ can be described as

$$z(t) = E(t)f(t) = \frac{1}{\theta_p(-p^{\nu+2}t)} \left(\sum_{n \geq 0} a_n t^n\right).$$

Since $E(t)$ satisfies the following $q$-difference equation

$$\sigma_p E(t) = -p^{\nu+2}tE(t), \quad \sigma_p^2 E(t) = p^{2\nu+5}t^2E(t),$$

we can check out that the function $f(t)$ satisfies the equation

$$\left\{p^{2\nu+5}t^2\sigma_p^2 + p^{\nu+2}(p^{\nu} + p^{-\nu})t\sigma_p - \sigma_p + 1\right\} f(t) = 0. \tag{7}$$

### 2.3 The $p$-Borel transformation and the $p$-Laplace transformation

We define the $p$-Borel transformation and the $p$-Laplace transformation to solve the equation (7), following Zhang [8].
**Definition 2.** For \( f(t) = \sum_{n \geq 0} a_n t^n \), the \( p \)-Borel transformation is defined by
\[
g(\tau) = (B_p f)(\tau) := \sum_{n \geq 0} a_n \tau^{-\frac{n(n-1)}{2}} \tau^n,
\]
and the \( p \)-Laplace transformation is given by
\[
(L_p g)(t) := \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau) \frac{d\tau}{\tau}.
\]
Here, \( r_0 > 0 \) is enough small number.

The \( p \)-Borel transformation is considered as a formal inverse of the \( p \)-Laplace transformation.

**Lemma 1.** We assume that the function \( f \) can be \( p \)-Borel transformed to the analytic function \( g(\tau) \) around \( \tau = 0 \). Then,
\[
L_p \circ B_p f = f.
\]

**Proof.** We can prove this lemma calculating residues of the \( p \)-Laplace transformation around the origin. \hfill \Box

The \( p \)-Borel transformation has the following operational relation.

**Lemma 2.** For any \( l, m \in \mathbb{Z}_{\geq 0} \),
\[
B_p (\tau^m \sigma_p^l) = p^{-\frac{m(m-1)}{2}} \tau^m \sigma_p^l \cdot B_p.
\]

Applying the \( p \)-Borel transformation to the equation (7) and using Lemma 2, \( g(\tau) \) satisfies the first order difference equation
\[
g(p\tau) = (1 + p^{2\nu + 2} \tau) (1 + p^2 \tau) g(\tau).
\]
Since \( g(0) = 1 \), we get an infinite product of \( g(\tau) \):
\[
g(\tau) = \frac{1}{(p^{2\nu + 2} \tau; p)_\infty(p^2 \tau; p)_\infty}.
\]
Then \( g(\tau) \) has single poles at
\[
\left\{ -p^{-2\nu - 2 - k}, -p^{-2 - k}; k \in \mathbb{Z}_{\geq 0} \right\}.
\]

We set
\[
0 < r < r_0 := \min \left\{ \frac{1}{|p^{2\nu + 2}|}, \frac{1}{|p^2|} \right\}
\]
and choose the radius \( r > 0 \) such that \( 0 < r < r_0 \). By Cauchy’s residue theorem, the \( p \)-Laplace transform of \( g(\tau) \) is
\[
f(t) = \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau) \frac{d\tau}{\tau} \theta_p \left( \frac{t}{\tau} \right)
\]
\[
= -\sum_{k \geq 0} \text{Res} \left\{ g(\tau) \theta_p \left( \frac{t}{\tau} ; \tau = -p^{-2\nu - 2 - k} \right) \right\} - \sum_{k \geq 0} \text{Res} \left\{ g(\tau) \theta_p \left( \frac{t}{\tau} ; \tau = -p^{-2 - k} \right) \right\},
\]
where \( 0 < r < r_0 \). To calculate the residue, we use the following lemma.
Definition 3. For any \( k \in \mathbb{N}, \lambda \in \mathbb{C}^* \), we have

\[
1. \quad \text{Res} \left\{ \frac{1}{(\tau/\lambda; p; \infty)} \frac{1}{\tau} : \tau = \lambda p^{-k} \right\} = \frac{(-1)^{k+1} p^{k(k+1)}}{(p;p)_{k}(p;p)_{\infty}},
\]

\[
2. \quad \frac{1}{(\lambda p^{-k}; p; \infty)} = \frac{(-\lambda)^{-k} p^{k(k+1)}}{(\lambda; p)_{\infty}(p/\lambda; p)_{k}}, \quad \lambda \not\in p^{\mathbb{Z}}.
\]

Summing up all of the residues, we obtain the convergent series \( f(t) \) as follows

\[
f(t) = \theta_{p} \left( -p^{2\nu+2t} \right) \frac{1}{(p^{-2\nu}; p; \infty)} \varphi_{1}(0, p^{1+2\nu}; p, x) + \theta_{p} \left( -p^{2t} \right) \frac{1}{(p^{2\nu}; p; \infty)} \varphi_{1}(0, p^{1-2\nu}; p, p^{-2\nu} x),
\]

where \( xt = 1 \). Therefore, we acquire the connection formula for \( z(t) = E(t)f(t) \).

3 The limit of the connection formula

In this section, we show that the limit \( p \to 1^- \) of the connection formula gives a connection formula of the Bessel function. At first, we assume that \( 0 < p < 1 \) and \( 0 < \sqrt{p} < 1 \). For the Bessel function, we set the Hankel function of the first and the second kind \( H^{(1)}_{\nu}(z) \) and \( H^{(2)}_{\nu}(z) \).

Definition 3. The Hankel function of the first kind is given by

\[
H^{(1)}_{\nu}(z) := \frac{\Gamma \left( \frac{1}{2} - \nu \right)}{\pi i \sqrt{\pi}} \left( \frac{z}{2} \right)^{-\nu} \int_{1+i\infty}^{1+\infty} e^{izt} (t^2 - 1)^{-\nu} dt, \quad -\pi < \arg z < 2\pi.
\]

The Hankel function of the second kind is defined by

\[
H^{(2)}_{\nu}(z) := \frac{\Gamma \left( \frac{1}{2} - \nu \right)}{\pi i \sqrt{\pi}} \left( \frac{z}{2} \right)^{-\nu} \int_{1-\infty}^{-1+i\infty} e^{itz} (t^2 - 1)^{\nu} dt, \quad 2\pi < \arg z < 2\pi.
\]

The contour for \( H^{(1)}_{\nu}(z) \) is a path starting from \( t = +1 + \infty i \), rounding the circle around \( t = 1 \) counterclockwise, and going back to \( t = +1 + \infty i \). Moreover, the contour for \( H^{(2)}_{\nu}(z) \) is a path starting from \( t = -1 + \infty i \), rounding the circle around \( t = 1 \) clockwise, and going back to \( t = -1 + \infty i \).

The Hankel functions can be written by \( J_{\nu}(z) \):

\[
H^{(1)}_{\nu}(z) = \frac{ie^{-\nu \pi i}}{\sin \nu \pi} \left\{ J_{\nu}(z) - e^{\nu \pi i} J_{-\nu}(z) \right\}, \quad (8)
\]

\[
H^{(2)}_{\nu}(z) = -\frac{ie^{\nu \pi i}}{\sin \nu \pi} \left\{ J_{\nu}(z) - e^{-\nu \pi i} J_{-\nu}(z) \right\}. \quad (9)
\]

The Hankel functions have asymptotic expansions around \( z = 0 \) [4]:

\[
H^{(1)}_{\nu}(z) \sim \left( \frac{2}{\pi z} \right)^{1/2} e^{i\zeta} \sum_{s \geq 0} i^{s} \frac{A_{s}(\nu)}{z^{s}}, \quad -\pi + \delta \leq \arg z \leq 2\pi - \delta,
\]

\[
H^{(2)}_{\nu}(z) \sim \left( \frac{2}{\pi z} \right)^{1/2} e^{-i\zeta} \sum_{s \geq 0} (-i)^{s} \frac{A_{s}(\nu)}{z^{s}}, \quad -2\pi + \delta \leq \arg z \leq \pi - \delta,
\]

as \( z \to \infty \). Here, \( \delta \) is an any small constant,

\[
A_{s}(\nu) = \frac{(4\nu^{2} - 1^{2})(4\nu^{2} - 2^{2}) \cdots \{ 4\nu^{2} - (2s - 1)^{2} \}}{s! 8^{s}}
\]

and

\[
\zeta = z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi.
\]

In this sense, (8) and (9) considered as connection formula of the Bessel equation.
3.1 Limit of the connection formula

We rewrite the connection formula in Theorem 1 in order to take a limit \( p \to 1^- \). We set new functions \( h_\nu(t; p) \) and \( J^\nu_t(x; p) \). We set \( h_\nu(t; p) := (p^{1/2}, p^{1/2}; p)_\infty z(t) \). For any \( x \in \mathbb{C}^* \setminus [-\lambda; p) \) and \( \lambda \in \mathbb{C}^* \), \( J^+_{\nu,\lambda}(x; p) \) is

\[
J^+_{\nu,\lambda}(x; p) := \frac{(p^{\nu+1}; p)_\infty}{(p; p)_\infty} \frac{\theta_p\left(\frac{x^{\nu}}{p}\right)}{\theta_p\left(\frac{x}{p}\right)} \varphi_1\left(0; p^{1+2\nu}; p, x\right).
\]

Similarly, \( J^-_{\nu,\lambda}(x; p) \) is

\[
J^-_{\nu,\lambda}(x; p) := \frac{(p^{\nu+1}; p)_\infty}{(p; p)_\infty} \frac{\theta_p\left(\frac{x^{\nu}}{p}\right)}{\theta_p\left(\frac{x}{p}\right)} \varphi_1\left(0; p^{1+2\nu}; p, p^2x\right).
\]

We remark that the function \( \theta_p(\lambda x^{\nu}/x)/\theta_p(\lambda/x) \) satisfies the following \( q \)-difference equation

\[ u(px) = p^{\nu}u(x), \]

which is also satisfied by the function \( u(x) = x^{\nu} \). We remark that the pair \((J^+_{\nu,\lambda}(x; p), J^-_{\nu,\lambda}(x; p))\) gives a fundamental system of solutions of equation (6) if \( \nu \not\in \mathbb{Z} \). We set the function \( C^+_{\nu}(\lambda, t; p) \) and \( C^-_{\nu}(\lambda, t; p) \) as follows:

**Definition 4.** For any \( \lambda \in \mathbb{C}^* \), \( C^+_{\nu}(\lambda, t; p) \) is

\[
C^+_{\nu}(\lambda, t; p) := \frac{(p^{1/2}, p^{1/2}; p)_\infty}{(p^{\nu+1}, p^{2\nu}; p)_\infty} \frac{\theta_p(-p^{2\nu+2t})}{\theta_p(-p^{\nu+2t})} \frac{\theta_p(\lambda t)}{\theta_p(\lambda p^{\nu} t)}.
\]

Similarly, the function \( C^-_{\nu}(\lambda, t; p) \) is

\[
C^-_{\nu}(\lambda, t; p) := \frac{(p^{1/2}, p^{1/2}; p)_\infty}{(p^{-\nu+1}, p^{2\nu}; p)_\infty} \frac{\theta_p(-p^{-2t})}{\theta_p(-p^{-\nu+2t})} \frac{\theta_p(\lambda t)}{\theta_p(\lambda p^{-\nu} t)}.
\]

Then, \( C^+_{\nu}(\lambda, t; p) \) and \( C^-_{\nu}(\lambda, t; p) \) are single valued as a function of \( t \). The function \( C^+_{\nu}(\lambda, t; p) \) and \( C^-_{\nu}(\lambda, t; p) \) are the \( p \)-elliptic functions. By using these new functions, our connection formula is rewritten by

\[
h_\nu\left(\frac{1}{x}; p\right) = C^+_{\nu}\left(\lambda, \frac{1}{x}; p\right) J^+_{\nu}(x; p) + C^-_{\nu}\left(\lambda, \frac{1}{x}; p\right) J^-_{\nu,\lambda}(x; p).
\]

**Theorem 2.** For any \( x \in \mathbb{C}^* \setminus (-\infty, 0] \) where \( \arg x \in (-\pi, \pi) \), we have

\[
\lim_{p \to 1^-} h_\nu\left(\frac{1}{(1-p)^2x}; p\right) = -ie^{-\nu\pi i} H^{(2)}_{2\nu}(2\sqrt{x}).
\]

Here, \( H^{(2)}_{2\nu}(\cdot) \) is the Hankel function of the second kind.

The aim of this section is to give a proof of the theorem above.

By the definition, \( h_\nu\left(1/((1-p)^2x); p\right) \) can be described as follows

\[
h_\nu\left(\frac{1}{(1-p)^2x}; p\right) = \left\{ \begin{array}{ll}
(p^{1/2}, p^{1/2}; p)_\infty & (1-p)^{2\nu} \\
(p^{-2\nu}, p^{2\nu}; p)_\infty & \end{array} \right\} \left\{ \begin{array}{l}
\theta_p\left(-\frac{p^{2\nu+2}}{x(1-p)^2}\right) \\
\theta_p\left(-\frac{p^{2\nu+2}}{x(1-p)^2}\right)
\end{array} \right\} (1-p)^{-2\nu} \times \{1\varphi_1\left(0; p^{1+2\nu}; p, (1-p)^2x\right)\}.
\]
Lemma 4. For any \( \nu \in \mathbb{C}^+ \setminus \mathbb{Z} \), we have

\[
\lim_{p \to 1^-} \frac{(p^{\frac{1}{2}}, p^{\frac{1}{2}}; p)_{\infty}}{(p^{1-2\nu}, p; p)_{\infty}} (1-p)^{2\nu} = \frac{1}{\sin(2\nu \pi) \Gamma(2\nu + 1)}.
\]

Proof. We can check out as follows

\[
\frac{(p^{\frac{1}{2}}, p^{\frac{1}{2}}; p)_{\infty}}{(p^{1-2\nu}, p; p)_{\infty}} (1-p)^{2\nu} = \left\{ \frac{(p;p)_{\infty}}{(p^{2\nu}; p)_{\infty}} (1-p)^{2\nu} \right\} \frac{\Gamma(1\nu)}{\Gamma_1(2\nu)} = \frac{(p;p)_{\infty}}{(p^{2\nu}; p)_{\infty}} (1-p)^{2\nu} = \frac{1}{\sin(2\nu \pi) \Gamma(2\nu + 1)}.
\]

Here, \( \Gamma_q(\cdot) \) is Jackson’s \( q \)-gamma function which is defined by

\[
\Gamma_q(x) := \frac{(q;q)_{\infty}}{(q^x;q)_{\infty}} (1-q)^{1-x}, \quad 0 < q < 1.
\]

This function satisfies \( \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x) [2] \). Therefore,

\[
\lim_{p \to 1^-} \frac{(p^{\frac{1}{2}}, p^{\frac{1}{2}}; p)_{\infty}}{(p^{1-2\nu}, p; p)_{\infty}} (1-p)^{2\nu} = \frac{\Gamma(-2\nu)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)}.
\]

By Euler’s reflection formula of the gamma function, we get

\[
\frac{\Gamma(-2\nu)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)} = \frac{1}{\sin(2\nu \pi) \Gamma(2\nu + 1)}.
\]

Therefore, we get the conclusion.

If we replace \( \nu \) by \(-\nu\), we get the limit

\[
\lim_{p \to 1^-} \frac{(p^{\frac{1}{2}}, p^{\frac{1}{2}}; p)_{\infty}}{(p^{2\nu}, p; p)_{\infty}} (1-p)^{-2\nu} = \frac{1}{\sin(2\nu \pi) \Gamma(1-2\nu)}.
\]

In [8], the following proposition can be found:

Proposition 1. For any \( x \in \mathbb{C}^+ \) \((-\pi < \arg x < \pi)\), we have

\[
\lim_{p \to 1^-} \frac{\theta_p \left( \frac{p^{\nu_1}}{(1-p^2)x} \right)}{\theta_p \left( \frac{p^{\nu_2}}{(1-p^2)x} \right)} (1-p^2)^{\nu_2-\nu_1} = x^{\nu_2-\nu_1},
\]

and

\[
\lim_{p \to 1^-} \frac{\theta_p \left( \frac{-p^{\nu_1}}{(1-p^2)x} \right)}{\theta_p \left( \frac{-p^{\nu_2}}{(1-p^2)x} \right)} (1-p^2)^{\nu_2-\nu_1} = (-x)^{\nu_2-\nu_1}.
\]
Corollary 1. Therefore, we obtain the following relation.

We remark that (\(\sqrt{p}; \sqrt{p}\))\(\infty\) can be rewritten as follows [2]:

\[
\frac{(\sqrt{p}; \sqrt{p})_\infty}{(-\sqrt{p}; \sqrt{p})_\infty} = \sum_{n \in \mathbb{Z}} (-1)^n (\sqrt{p})^{n^2} = \theta_p(-\sqrt{p}).
\]

We obtain the conclusion. ■

Therefore, we obtain the following relation.

Corollary 1. For any \(x \in \mathbb{C}^* (-\pi < \arg x \leq \pi)\) and fixed constant \(K\), we have

\[
\theta_p(-\sqrt{p})\theta_p \left( \frac{-K}{x} \right) = \theta_{\sqrt{p}} \left( \sqrt{\frac{K}{x}} \right) \theta_{\sqrt{p}} \left( -\sqrt{\frac{K}{x}} \right).
\]

Proof. From Jacobi’s triple product identity and \((a^2; q^2)_n = (a, -a; q)_n\), we obtain

\[
\frac{(\sqrt{p}; \sqrt{p})_\infty}{(-\sqrt{p}; \sqrt{p})_\infty} \theta_p \left( \frac{-K}{x} \right) = \theta_{\sqrt{p}} \left( \sqrt{\frac{K}{x}} \right) \theta_{\sqrt{p}} \left( -\sqrt{\frac{K}{x}} \right).
\]

We remark that \((\sqrt{p}; \sqrt{p})_\infty/(\sqrt{p}; \sqrt{p})_\infty\) can be rewritten as follows [2]:

\[
\frac{(\sqrt{p}; \sqrt{p})_\infty}{(-\sqrt{p}; \sqrt{p})_\infty} = \sum_{n \in \mathbb{Z}} (-1)^n (\sqrt{p})^{n^2} = \theta_p(-\sqrt{p}).
\]

We obtain the conclusion. ■

Lemma 5. For any \(x \in \mathbb{C}^* (-\pi < \arg x \leq \pi)\) and fixed constant \(K\), we have

\[
\theta_{\sqrt{p}} \left( \frac{\nu + 1}{(1-p)^{\nu + 1}} \right) = \theta_{\sqrt{p}} \left( \sqrt{\frac{1}{1-p}} \right) \theta_{\sqrt{p}} \left( \sqrt{\frac{1}{1-p}} \right).
\]

Proof. From Jacobi’s triple product identity and \((a^2; q^2)_n = (a, -a; q)_n\), we obtain

\[
\frac{(\sqrt{p}; \sqrt{p})_\infty}{(-\sqrt{p}; \sqrt{p})_\infty} \theta_p \left( \frac{-K}{x} \right) = \theta_{\sqrt{p}} \left( \sqrt{\frac{K}{x}} \right) \theta_{\sqrt{p}} \left( -\sqrt{\frac{K}{x}} \right).
\]

We remark that \((\sqrt{p}; \sqrt{p})_\infty/(\sqrt{p}; \sqrt{p})_\infty\) can be rewritten as follows [2]:

\[
\frac{(\sqrt{p}; \sqrt{p})_\infty}{(-\sqrt{p}; \sqrt{p})_\infty} = \sum_{n \in \mathbb{Z}} (-1)^n (\sqrt{p})^{n^2} = \theta_p(-\sqrt{p}).
\]

We obtain the conclusion. ■

Lemma 6. For any \(x \in \mathbb{C}^* (-\pi, 0] (-\pi < \arg x \leq \pi)\), we have

1. \(\lim_{p \to 1^{-}} \theta_p \left( \frac{-\nu^{\nu + 2}}{x(1-p)^{\nu + 1}} \right) (1-p)^{-2\nu} = e^{\nu \pi i} x^2\) and

2. \(\lim_{p \to 1^{-}} \theta_p \left( \frac{-\nu^{\nu + 2}}{x(1-p)^{\nu + 1}} \right) (1-p)^{2\nu} = e^{-\nu \pi i} x^{-\nu}\).

Proof. Combining Proposition 1 and Corollary 1, we consider the limit \(\sqrt{p} \to 1^{-}\) as follows:

\[
\theta_p \left( \frac{\nu^{\nu + 2}}{x(1-p)^{\nu + 1}} \right) (1-p)^{-2\nu} = \theta_{\sqrt{p}} \left( \frac{\nu^{\nu + 1}}{(1-p)^{\nu + 1}} \right) \theta_{\sqrt{p}} \left( \frac{\nu^{\nu + 1}}{(1-p)^{\nu + 1}} \right) (1-p)^{-2\nu}
\]

\[
= \left\{ \theta_{\sqrt{p}} \left( (\sqrt{p})^{2\nu + 2} \frac{1}{(1-(\sqrt{p})^{2})^{\nu}} \right) \right\} \left\{ 1 - (\sqrt{p})^2 \right\}^{-\nu}
\]

\[
\times \left\{ \theta_{\sqrt{p}} \left( (\sqrt{p})^{2\nu + 2} \frac{1}{(1-(\sqrt{p})^{2})^{\nu}} \right) \right\} \left\{ 1 - (\sqrt{p})^2 \right\}^{-\nu}
\]

\[
\to (\sqrt{x})^{\nu} \cdot (-\sqrt{x})^{\nu} = (x)^{\nu} = e^{\nu \pi i} x^{\nu}, \quad \sqrt{p} \to 1^{-}.
\]

Similarly, we can prove the latter one. We obtain the conclusion. ■
We consider the last part.

**Lemma 7.** For any $x \in \mathbb{C}^*$, we have
\[
\lim_{p \to 1^-} \varphi_1(0; p^{1+2\nu}; p, (1-p)^2 x) = {}_0F_1(-, 1 + 2\nu; -x)
\]
and
\[
\lim_{p \to 1^-} \varphi_1(0; p^{1-2\nu}; p, p^{-2\nu} (1-p)^2 x) = {}_0F_1(-, -2\nu; -x).
\]

**Proof.** We check each of the term of
\[
\varphi_1(0; p^{1+2\nu}; p, (1-p)^2 x) = \sum_{n \geq 0} \frac{1}{(p^{1+2\nu}; p)_n} (-1)^n p^{\frac{n(n-1)}{2}} \left\{ (1-p)^2 x \right\}^n.
\]
For any $n \geq 0$,
\[
\frac{1}{(p^{1+2\nu}; p)_n} (-1)^n p^{\frac{n(n-1)}{2}} \left\{ (1-p)^2 x \right\}^n
= \frac{(1-p)^n (1-p)^n}{(p^{1+2\nu}; p)_n (p; p)_n} n^{\frac{n(n-1)}{2}} (-x)^n \to \frac{1}{(1+2\nu)_n \cdot n!} (-x)^n, \quad p \to 1^-.
\]
Summing up all terms, we get
\[
\sum_{n \geq 0} \frac{1}{(1+2\nu)_n \cdot n!} (-x)^n = {}_0F_1(-, 1 + 2\nu; -x).
\]
Therefore, we obtain the conclusion. Similarly, we can prove the latter.

We give the proof of Theorem 2.

**Proof.** Apply Lemma 4, Lemma 6 and Lemma 7 to (10), we obtain
\[
\nu \left( \frac{1}{(1-p)^2 x}; p \right) \to \left\{ \frac{1}{\sin(2\nu \pi)} \Gamma(1+2\nu) \right\} e^{i \nu \pi i} \nu F_1(-, 1+2\nu; -x)
\]
\[
+ \left\{ \frac{1}{\sin(2\nu \pi)} \Gamma(1-2\nu) \right\} e^{-i \nu \pi i} \nu F_1(-, -2\nu; -x)
\]
\[
= -e^{i \nu \pi i} J_{2\nu} (2\sqrt{x}) + e^{i \nu \pi i} J_{-2\nu} (2\sqrt{x}) \cdot \frac{\sin(2\nu \pi)}{i} H^{(2)}_{2\nu} (2\sqrt{x}), \quad p \to 1^-.
\]
Therefore, we acquire the conclusion.

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A Connection Formula of the Hahn–Exton \(q\)-Bessel Function

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