Classical and quantum cosmology of multigravity

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Abstract
Recently, a multigraviton theory on a simple closed circuit graph corresponding to the discretization of $S^1$ compactification of the Kaluza–Klein (KK) theory has been considered. In the present paper, we extend this theory to that on a general graph and study what modes of particles are included. Furthermore, we generalize it in a possible nonlinear theory based on the vierbein formalism and study classical and quantum cosmological solutions in the theory. We found that scale factors in a solution for this theory repeat acceleration and deceleration.

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1. Introduction

Both astronomical and cosmological data seem to require the presence of yet directly undetected dark matter and dark energy in the Universe. The necessity for these mysterious components occurs at distances where the gravitational interaction is not understood sufficiently. This suspicious coincidence inspires a search for modifications of the general relativity at large distances. It is important to study massive and multigraviton theory for understanding cosmology and unification. In the linear-field theory, gravitons have the Fierz–Pauli (FP) type masses \cite{1}. But there is an ambiguity in its nonlinear generalization. Thus far we studied the linear multigraviton theory on a circle corresponding to $S^1$ compactification of the KK theory with dimensional deconstruction \cite{2}. This model is an extended version of Hamamoto's model \cite{3} for a massive graviton.

In this paper, we construct the FP Lagrangian for multigravitons associated with a general graph and investigate what modes of particles are included. Furthermore, we extend it to nonlinear theory based on the vierbein formalism \cite{4, 5}. The nonlinear extensions of

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multigraviton theory have been studied by many authors [6]. In the present paper we focus on the semiclassical sector of the theory which governs the evolution of the Universe; in other words, we will not consider nonlocal contributions and terms with higher derivatives in the possible complete theory here.

The features of our model are as follows. (i) Gravitons as the fluctuation from Minkowski space time have the FP type masses [1]. (ii) This model is based on a generalized dimensional deconstruction method. So, the mass spectrum in the model can be tuned more easily than in the KK theory. (iii) The mass term has a reflection symmetry assigned at each vertex and an exchange symmetry assigned at each edge of a graph.

In this paper, beginning with the theoretical description of graph, we introduce the dimensional deconstruction [7, 8] and description of the linear theory of multigravity as the basis of our model in section 2. A nonlinear extension of the model is proposed in section 3. In section 4, we consider the vacuum cosmological solutions of the case associated with the four-site star graph and the four-site path graph. The study on the quantum cosmological model is exhibited in section 5. Finally, we summarize our work and give remarks about the outlook in section 6.

2. Multigraviton theory on a general graph

2.1. FP on a graph

We consider the matrix representation of the graph theory3. A graph $G$ is a pair of $V$ and $E$, where $V$ is a set of vertices (sites) while $E$ is a set of edges (links). An edge connects two vertices; two vertices located at the ends of an edge $e$ are denoted as $o(e)$ and $t(e)$. Then, we introduce two matrices, an incidence matrix and a graph Laplacian, associated with a specific graph. The incidence matrix $E$ represents the condition of connection or structure of a graph, and the graph Laplacian $\Delta$ can be obtained by $E^T E$, where $E^T$ is the transposed matrix of $E$. By use of these matrices, a quadratic form of vectors $a^T \Delta a (\equiv a^T E^T E a)$ can be written as a sum of $(a_{i(e)} - a_{o(e)})^2$. If all $a_i (i = 1, 2, \ldots, \#V)$, the components of $a$, take the same value, $E^T a = 0$ and then $\Delta a = 0$.

So, we consider the Lagrangian for massive gravitons $h_{\mu
u}^v$ on each vertex with the St"{u}ckelberg vector fields $A^\mu_e$ on each edge and a scalar field $\phi^v$ on each vertex:

$$L_m = L_0 - \frac{m^2}{2} \sum_{v \in V} \left[ h_{\mu\nu}^v (E^T E h_{\mu\nu})^v - h^v (E^T E h)^v \right]$$

$$- 2 \sum_{v \in V} \left[ m (E A^\mu)^v + \partial_\mu \phi^v \right] \left( \partial_\nu h_{\mu\nu}^v - \partial^\mu h^v \right) - \frac{1}{2} \sum_{e \in E} \left( \partial_\mu A^\nu_e - \partial_\nu A^\mu_e \right)^2,$$

where $L_0$ is the linearized Einstein–Hilbert Lagrangian:

$$L_0 = \sum_{v \in V} \left[ -\frac{1}{2} \partial_\mu h_{\mu\nu}^v \partial^\nu h_{\mu\nu}^v + \partial_\mu h_{\mu\nu}^v \partial_\nu h_{\nu\nu}^v - \partial_\mu h_{\mu\nu}^v \partial_\nu h^v + \frac{1}{2} \partial_\mu h_{\mu\nu}^v \partial^\nu h^v \right],$$

and $h^v \equiv h_{\mu\nu}^v h_{\mu\nu}^v$.

This action is invariant under the following transformations:

$$h_{\mu\nu}^v \rightarrow h_{\mu\nu}^v + \partial_\mu \xi^v - \partial_\nu \xi^v, \quad A^\mu_e \rightarrow A^\mu_e + m (E^T E \xi^\mu_e - \partial_\mu \xi^e), \quad \phi^v \rightarrow \phi^v + m (E \xi)^v,$$

(2.3)

3 Please see [9] for a brief review of the application of graph theory to field theory, and textbooks [10, 11] for algebraic graph theory.
where $\xi^v$ and $\zeta^e$ are the parameters on each vertex and each edge respectively. The massive modes of vector and scalar fields are absorbed by the massive modes of graviton fields due to the symmetry in the manner of St"uckelberg.

Now we examine the gauge fixing of the Lagrangian. Suppose the following gauge fixing terms:

$$L_{gf} = - \sum_{v \in V} \left[ \partial_\nu h^{\nu \mu v} - \frac{1}{2} \partial^\mu h^v - m(E A^\mu)^v - \partial^\mu \phi^v \right]^2 - \sum_{e \in E} \left[ \partial_\mu A^\mu e - \frac{m}{2} (E^T h)^e + m(E^T \phi)^e \right]^2; \quad (2.4)$$

then the gauge-fixed Lagrangian becomes

$$L_m + L_{gf} = \frac{1}{2} H^{\mu \nu} (\partial^2 - m^2 E E^T) \left( H_{\mu \nu} - \frac{1}{2} H \eta_{\mu \nu} \right) + A^{\mu} (\partial^2 - m^2 E E^T) A_{\mu} + 3 \phi (\partial^2 - m^2 E E^T) \phi, \quad (2.5)$$

where $H_{\mu \nu} = h_{\mu \nu} + \phi \eta_{\mu \nu}$. Here the indices $\nu$ and $e$, and the notion of the sum over them are omitted.

In the following subsections, we will see that the mass spectra of fields in the Lagrangian for specific graphs with large number of vertices are similar to those of a five-dimensional model with a compactified extra space.

2.2. Dimensional deconstruction

It is assumed that we put fields on vertices or edges. An idea that there are four dimensional fields on the sites (vertices) and links (edges), dubbed as dimensional deconstruction, is introduced by Arkani-Hamed et al [7, 8]. In this scheme, the square of mass matrix is proportional to the Laplacian of the associated graph.

In the case of a cycle graph (a ‘closed circuit’) with $N$ sites (denoted as $C_N$, and $C_{60}$ is shown in figure 1 for example), when $N$ becomes large, the model on the graph coincides with the five-dimensional theory with $S^1$ (circle) compactification. In other words, the mass scale of the model $f$ over $N$ corresponds to the inverse of the circumference of the circle:

$$M^2_\ell = 4 f^2 \left( \sin \pi \ell / N \right)^2 \quad \rightarrow \quad M^2_\ell = \left( 2 \pi \ell / L \right)^2, \quad (f / N \rightarrow 1 / L). \quad (2.6)$$
The mass spectrum is given by the eigenvalues of the graph Laplacian of $C_N$, which can be expressed as

$$\Delta = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{pmatrix}. \quad (2.7)$$

For a cycle graph, the linear graviton model presented in the previous subsection coincides with the model proposed in [2]. The model is a most general linear multigraviton theory on a generic graph.

2.3. Particle content in the multigraviton theory on a graph

For this model, we investigate what modes of particles are included. Although any graph is available for the model, here we consider two types, a cycle graph $C_N$ and a path graph $P_N$. The path graph has a simple structure like a chain, and has two ends ($i = 1$ and $N$) and the $i$th vertex is adjacent to the $(i - 1)$th and $(i + 1)$th vertices ($1 < i < N$). For example, we show $C_4$ and $P_4$ in figure 2.

The incidence matrix for $C_4$ is defined as

$$E(C_4) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \quad (2.8)$$

and then

$$E(C_4)E(C_4)^T = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} = E(C_4)^T E(C_4). \quad (2.9)$$

The eigenvalues of $EE^T$ are $\{0, 2, 2, 4\}$ for $C_4$. 

On the other hand, the incidence matrix for \( P_4 \) is given by

\[
E(P_4) = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1 \\
0 & 0 & -1
\end{pmatrix}.
\] (2.10)

Thus

\[
E(P_4)E(P_4)^T = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix}
\] (2.11)

and

\[
E(P_4)^T E(P_4) = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}
\] (2.12)

are different in their sizes. The eigenvalues of \( EE^T \) are \( \{0, 2 - \sqrt{2}, 2 + \sqrt{2}\} \) and those of \( E^TE \) are \( \{2 - \sqrt{2}, 2, 2 + \sqrt{2}\} \) for \( P_4 \). For \( P_N \), it is known that the Laplacian eigenvalues are \( 4 \sin^2 \frac{k\pi}{2N} \) \( (k = 0, 1, \ldots, N - 1) \). If we introduce a mass scale \( f \) and consider the large \( N \) limit as in (2.6), we find \( 4f^2 \sin^2 \frac{k\pi}{2N} \to \left( \frac{\pi k}{L} \right)^2 \) where \( f/N = 1/L \). This spectrum corresponds to that of the compactification on \( S^1/Z_2 \), where the circumference of \( S^1 \) is \( 2L \).

In the multigraviton theory associated with the cycle graph \( C_N \) \( (#V = N, #E = N) \), \( N - 1 \) massive spin two’s, a massless spin two, \( N - 1 \) massive vectors, a massless vector, \( N - 1 \) massive scalars and a massless scalar seem to be included, as seen from the gauge-fixed Lagrangian (2.5). The mass spectra of different spin fields are the same, except for zero modes. This is due to the fact that the eigenvalues of \( EE^T \) and ones of \( E^TE \) are the same except for zero eigenvalues.

However, \( N - 1 \) massive spin two, a massless spin two, a massless vector and a massless scalar are left physically, because massive vectors and massive scalars are absorbed by massive spin-two fields to form massive gravitons with five degrees of freedom each.

Similarly, in the model associated with the path graph \( P_N \) \( (#V = N, #E = N - 1) \), \( N - 1 \) massive spin two’s, a massless spin two, and a massless scalar are left physically, and the massless vector mode is absent.

The limits of \( N \) to infinity in the cases of \( C_N \) and \( P_N \) realize the KK theory with \( S^1 \) and \( S^1/Z_2 \) compactification, respectively.

3. Nonlinear extension of a multigraviton theory on a tree graph

Now we will consider a nonlinear extension of the linear theory. Following Nibbelink et al [4, 5], we introduce a useful ‘tool’:

\[
\langle ABCD \rangle \equiv -\epsilon_{abcd}e^{\mu\nu\rho\sigma} A_{\mu}^{a} B_{\rho}^{b} C_{\nu}^{c} D_{\sigma}^{d},
\] (3.1)

where \( \epsilon \) is the totally antisymmetric tensor. Using this expression, we have the Einstein–Hilbert term replacing \( A \) and \( B \) by vierbeins and \( C \) and \( D \) by the curvature 2-form. In addition, because the fourth power of vierbein in the angle bracket is equal to the determinant of vierbeins \( \langle eee e \rangle = \langle e^4 \rangle = 24|e| \), this expression means that the Einstein–Hilbert term and the cosmological term have the similar structure.
Figure 3. The star graph $K_{1,3}$.

We now assume that the following term is assigned for each edge of a graph:

$$\langle (e_1 e_1 - e_2 e_2)^2 \rangle,$$

where $e_1$ and $e_2$ are the vierbeins at two ends of one edge. Note that this term has a reflection symmetry $e \leftrightarrow -e$ at each vertex and an exchange symmetry $e_1 \leftrightarrow e_2$ at each edge.

In the weak field limit, i.e., $e_1 = \eta + f_1$, $e_2 = \eta + f_2$,

$$\langle (e_1 e_1 - e_2 e_2)^2 \rangle = 8((f_1) - (f_2)^2) + O(f^3),$$

where $\eta$ is the Minkowski metric, and $[f] = \text{tr} f$ for notational simplicity. This quadratic term corresponds to the FP mass term.

On the other hand, the Einstein–Hilbert term $\frac{1}{2} |e|^2 R$ contains the kinetic terms of a graviton in the lowest order up to the total derivative:

$$\frac{1}{2} |e|^2 R = -\frac{1}{2} \partial_\mu f_\mu \partial^\nu f_\nu + \partial_\mu f^\mu f^\nu - \partial_\mu f_\nu \partial_\nu f_\mu - \frac{1}{2} \partial_\mu f^\mu f + O(f^3),$$

and $\frac{1}{2} R$ contains the following terms in the first order:

$$\frac{1}{2} R = -\partial^\mu \partial_\mu f + \partial_\mu \partial_\nu f + O(f^2).$$

In the case of a tree graph (a graph with no closed circuit—the path graph $P_N$ is a tree graph, for example), we have the nonlinear Lagrangian of multigraviton theory without a higher derivative and nonlocal terms,

$$L_m = \frac{1}{2} \exp \Phi \sum_{e \in V} |e_\nu| R_\nu + \frac{M^2}{24} \sum_{e \in E} \langle (e_\nu e_\nu - e_\nu e_\nu)^2 \rangle,$$

where $R_\nu$ is the scalar curvature associated with $e_\nu$ and $M^2 \equiv 3m^2/2$. The scalar zero-mode field $\Phi$ can be identified as $\phi_1 = \phi_2 = \cdots = \Phi$.

4. Classical cosmology of the multigraviton theory

Now we consider two vacuum cosmological models, associated with a four-site star graph and a four-site path graph respectively. Both the star graph and the line graph are tree graphs. The star graph consists of one central vertex and the other vertices adjacent to the central one. The star graph $K_{1,3}$ is shown in figure 3.

It is known that the asymmetric part of $f$ can be omitted [12].
The incidence matrix for $K_{1,3}$ is

$$E(K_{1,3}) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.1)$$

Thus

$$E(K_{1,3})E(K_{1,3})^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \quad (4.2)$$

and

$$E(K_{1,3})^T E(K_{1,3}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \quad (4.3)$$

One can see that the eigenvalues of $EE^T$ are $\{0, 1, 1, 4\}$ and those of $E^TE$ are $\{1, 1, 4\}$ for the star graph $K_{1,3}$. For $K_{1,N-1}$, $N$ eigenvalues of the Laplacian are $\{0, 1, \ldots, 1, N\}$. The degeneracy of $N-2$ eigenvalues ($=1$) is apparently due to the symmetry of the star graph.

In the case of the star graph, the associated Lagrangian for multigravitons is as follows:

$$L_{\text{star}} = \frac{1}{2} \exp \Phi \sum_{i=1}^{4} |e_i| R_i + \frac{M^2}{24} \sum_{i=2}^{4} (e_i e_i - e_i e_j)^2, \quad (4.4)$$

where $e_1$ is on the center of the graph. On the other hand, the Lagrangian in the case of the path graph is

$$L_{\text{path}} = \frac{1}{2} \exp \Phi \sum_{i=1}^{4} |e_i| R_i + \frac{M^2}{24} \sum_{i=1}^{3} (e_i e_j - e_i e_{j+1})^2, \quad (4.5)$$

where $e_1$ and $e_4$ are on each end of the graph.

Now let us introduce the setting for cosmology. We assume the homogeneous Universe with a spatially constant scalar field $\Phi(t)$ and the following metric:

$$g_{\mu\nu} dx^\mu dx^\nu = -e^{-\Phi(t)} dt^2 + e^{-\Phi(t)} A_i^2(t) (dr^2 + r^2 d\Omega_i^2), \quad (4.6)$$

where $A_i(t)$ ($i = 1, \ldots, 4$) are the scale factors. Then,

$$\langle (e_i e_i - e_i e_j)^2 \rangle = e^{-2\Phi(t)} (e^{\Phi(t)} - e^{\Phi(t)})(e^{2\Phi(t)} - e^{2\Phi(t)}), \quad (4.7)$$

where $a_i(t) \equiv \ln A_i(t)$.

We show the results of numerical calculations for the two models on the same appropriate initial conditions in figures 4 and 5. In both cases the scalar field $\Phi$ behaves similarly and in each case scale factors repeat the increase and the decrease. The oscillation of the scale factors in the path graph case include more different modes than that of the scale factors in the case of the star graph where the degeneracy of eigenvalues exists.

The star graph model has more symmetries than the path graph model. Therefore a lot of modes in the star graph are degenerate, while there is no degeneracy in the spectrum of the line graph. In the path graph case, increase of the number of sites gives the more complicated behaviors of the scale factors. On the other hand, in the star graph case, the symmetries are preserved even if the number of sites increases. Therefore, the behaviors of scale factors are very similar to those in the four-site model, essentially.
5. Quantum cosmology of the multigraviton theory

5.1. The Wheeler–DeWitt equation

In the previous section, we have seen the oscillatory behavior in the evolution of scale factors. As a qualitative analysis, we only show the characteristic solutions. In fact, oscillations must be dependent on the initial conditions. What are the natural conditions? To study the initial state, we have to consider quantum behavior of cosmology. Note that the quantum cosmology of multigraviton theory has never been studied yet as far as we know.

In this section we consider a minimal model based on a graph $P_2$, which is shown in figure 6.

This model has two gravitons\(^5\) or two scale factors. The Lagrangian density is given by

\[
\mathcal{L} = \frac{1}{2\kappa^2} \exp \Phi [|e_g| R_g + |e_f| R_f] + \frac{M^2}{24} ((e_g e_g - e_f e_f)^2) + \text{(surface terms)},
\]

where two the graviton fields are labeled by $g$ and $f$. This model in this case is very similar to $f\cdot g$ gravity [17] or bigravity [18], but our model also contains a massless scalar field.

We take the metric ansatz as follows:

\[
d s_f^2 = e^{-\Phi} \left( -N^2 \, dt^2 + A^2 \, d\Omega_1^2 \right),
\]

\[
d s_g^2 = e^{-\Phi} \left( -N^2 \, dt^2 + B^2 \, d\Omega_2^2 \right).
\]

\(^5\) In this case, the eigenvalues of mass are 0 and $2M/\sqrt{3}$. 

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**Figure 4.** Numerical solutions of $a$’s and $\Phi$ in the case of the four-site star graph.
These choices are equivalent to the field redefinitions \( g_{\mu\nu} = e^{-\Phi} g_{\mu\nu}^{(E)} \) so that

\[
L = \frac{1}{2\kappa^2} \left\{ e^{\alpha(E)} R_x^{(E)} + e^{\beta(E)} R_f^{(E)} \right\} + \ldots
\]

and often quoted as the choice of the Einstein frame. Here we assume that \( A, B \) and \( \Phi \) depend only on \( t \), and \( d\Omega_3^2 = dx^2 + dy^2 + dz^2 \). The lapse function will be set as \( N = 1 \) after the calculation (by the redefinition of \( t \)). Each metric is homogeneous, isotropic and flat in the Einstein frame, in the present analysis. Then the action reads

\[
\int L dt = \int dt \left[ \frac{3}{2N\kappa^2} \left\{ e^{3\alpha} (-4\dot{\alpha}^2 + \Phi^2) + e^{3\beta} (-4\dot{\beta}^2 + \Phi^2) \right\} + NM^2 e^{-2\Phi} (e^\alpha + e^\beta) (e^\alpha - e^\beta)^2 \right], \tag{5.4}
\]

where \( \alpha = \ln A \) and \( \beta = \ln B \). The dot indicates the derivative with respect to \( t \). The conjugate variables are

\[
\pi_\alpha = -\frac{12e^{3\alpha}\dot{\alpha}}{N\kappa^2}, \quad \pi_\beta = -\frac{12e^{3\beta}\dot{\beta}}{N\kappa^2}, \quad \pi_\Phi = \frac{3(e^{3\beta} + e^{3\beta})\Phi}{N\kappa^2}; \tag{5.5}
\]

thus, we obtain the Hamiltonian of the Universe as

\[
H = N \left[ -\frac{\kappa^2}{24} \left\{ e^{-3\alpha}\pi_\alpha^2 + e^{-3\beta}\pi_\beta^2 \right\} + \frac{\kappa^2}{6(e^{3\alpha} + e^{3\beta})}\pi_\Phi^2 - M^2 e^{-2\Phi} (e^\alpha + e^\beta) (e^\alpha - e^\beta)^2 \right]. \tag{5.6}
\]
From the Hamiltonian, we obtain the Wheeler–DeWitt (WDW) equation for the wavefunction of the Universe $\Psi(\alpha, \beta, \Phi)$ [13]. Although there are ambiguities in the ordering, we adopt the simple replacement of conjugate variables by the derivatives with respect to the corresponding dynamical variables$^6$. The WDW equation for the present model is

$$
-\frac{\kappa^2}{24} \left( e^{-3\alpha} \frac{\partial^2}{\partial \alpha^2} + e^{-3\beta} \frac{\partial^2}{\partial \beta^2} \right) + \frac{\kappa^2}{6(3e^{3\alpha} + 3e^{3\beta})} \frac{\partial^2}{\partial \Phi^2} + M^2 e^{-2\Phi} (e^\alpha + e^\beta)(e^{\alpha} - e^{\beta})^2 \right] \Psi(\alpha, \beta, \Phi) = 0. 
$$

(5.7)

Now we introduce new variables $x$ and $y$. They are defined as

$$
x = \frac{\alpha + \beta}{2}, \quad y = \frac{\alpha - \beta}{2}. 
$$

(5.8)

Since

$$
\frac{\partial}{\partial \alpha} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \beta} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right),
$$

(5.9)

the WDW equation (5.7) is rewritten as

$$
-\frac{\kappa^2}{6} \left( \cosh 3y \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2 \sinh 3y \frac{\partial^2}{\partial x \partial y} \right) + \frac{\kappa^2}{6 \cosh 3y} \frac{\partial^2}{\partial \Phi^2} + 64M^2 e^{-4\Phi} e^{6x} \cosh y \sinh^2 y \Psi(x, y, \phi) = 0,
$$

(5.10)

where we have also introduced $\phi = \Phi/2$ for simplicity.

5.2. Wave-packet solutions

To analyze the WDW equation (5.10), we assume the wave packet ansatz. The wave packet in quantum cosmology was originally introduced in [14, 15], and is utilized recently for various models such as in [16]. The use of the wave packet is crucial for the case with no special ‘initial’ state served as in the case with the positively curved homogeneous space.

The general form of the wavefunction is written as

$$
\Psi(x, y, \phi) = \sum_n C_n(x, \phi) \psi_n(x, y, \phi),
$$

(5.11)

where

$$
-\frac{\kappa^2}{6} \frac{\partial^2}{\partial y^2} + 64M^2 e^{-4\phi} e^{6x} \cosh y \sinh^2 y \psi_n(x, y, \phi) = E_n(x, \phi) \psi_n(x, y, \phi).
$$

(5.12)

We assume that $x$ and $\phi$ are the slowly evolving variables, while $y$ is a rapidly changing variable$^7$. In other words, we assume $\partial_x \ln C_n \ll 1$ and $\partial_\phi \ln C_n \ll 1$.

Further we approximate the equation if $y$ has a small amplitude. Then

$$
-\frac{\kappa^2}{6} \frac{\partial^2}{\partial y^2} + 64M^2 e^{6x} e^{-4\phi} y^2 \psi_n(x, y, \phi) = E_n(x, \phi) \psi_n(x, y, \phi).
$$

(5.13)

If $x$ and $\phi$ are slowly developing variables, this is no other than the equation for a harmonic oscillator. The differential equation

$$
\psi''(y) - ky^2 \psi(y) + c \psi(y) = 0
$$

(5.14)

Another plausible choice is the adoption of the Laplacian in the minisuperspace. The qualitative behavior is not changed by the choice of the operator orderings.

This assumption leads to a Universe with the increasing mean size, which looks like our present Universe. Some violent evolutions can occur in the very early Universe, but we do not consider the possibility here.

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$^7$ This assumption leads to a Universe with the increasing mean size, which looks like our present Universe. Some violent evolutions can occur in the very early Universe, but we do not consider the possibility here.
has the solution
\[ \psi_n(y) = \exp \left( -\frac{1}{2} \sqrt{b} y^2 \right) H_n (|b|^{1/4} y), \quad (5.15) \]
where \( H_n \) is the Hermite polynomial in the definition of Mathematica and \( \psi \) is normalizable if
\[ c = c_n = 2 \sqrt{b} \left( n + \frac{1}{2} \right), \quad n = \text{integer}. \quad (5.16) \]

Therefore the approximation gives the solution of (5.13) which leads to
\[ E_n(x, \phi) = \frac{k^2}{6} c_n, \quad (5.17) \]
where \( c_n \) is given by (5.16) with
\[ b = \frac{6}{k^2} 64M^2 e^{3x-4\phi}. \quad (5.18) \]

Now the differential equation for \( C_n \) becomes
\[ \left[ -\frac{k^2}{6} \frac{\partial^2}{\partial x^2} + \frac{k^2}{6} \frac{\partial^2}{\partial \phi^2} + E_n(x, \phi) \right] C_n(x, \phi) = 0, \quad (5.19) \]
and can be approximated as
\[ \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \phi^2} - 16M \sqrt{\frac{6}{k^2}} \left( n + \frac{1}{2} \right) e^{3x-2\phi} \right] C_n(x, \phi) = 0. \quad (5.20) \]

Further rewriting variables as
\[ X = \frac{3x - 2\phi}{\sqrt{5}}, \quad Z = \frac{3\phi - 2x}{\sqrt{5}}, \quad (5.21) \]
leads to
\[ \left[ \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Z^2} - 16M \sqrt{\frac{6}{k^2}} \left( n + \frac{1}{2} \right) e^{3X/2} \right] C_n(X, Z) = 0. \quad (5.22) \]

Finally, separating variables as \( C_n(X, Z) = f_k(Z)\varphi_{kn}(X) \) according to
\[ \left[ \frac{\partial^2}{\partial Z^2} + k^2 \right] f_k(Z) = 0, \quad (5.23) \]
gives the solution
\[ f_k(Z) = e^{-ikZ}, \quad \varphi_{kn}(X) = K_{\frac{3k}{\sqrt{5}}} \left[ \frac{2\sqrt{6}e^{3X/2}}{\sqrt{5}} \right], \quad (5.24) \]
where \( K_{\nu} \) is the modified Bessel function of the second kind with
\[ a_n = 16M \sqrt{\frac{6}{k^2}} \left( n + \frac{1}{2} \right). \quad (5.25) \]
The wave packet can be written in the form
\[ \Psi = \sum_{n=0}^{\infty} \psi_n(y) \int_{-\infty}^{\infty} dk A_n(k) K_{\frac{3k}{\sqrt{5}}} \left[ \frac{2\sqrt{6}e^{3X/2}}{\sqrt{5}} \right] e^{-ikZ}. \quad (5.26) \]
The wavefunction behaves oscillatory in the region \( X < 0 \) and exponentially damps in the region \( X > 0 \). This is because the exponential potential ‘wall’ in (5.22). The amplitude with respect to \( X \) has a maximum peak at \( X \sim 0 \) independently to \( k \). Therefore the general wave packet, in which \( A(k) \) is taken to be a Gaussian, has a peak at \( X \sim 0 \) because other peaks are destructively superposed.

The Universe with \( X \sim 0 \) is preferred in general. Even in classical solution, oscillatory \( y \) leads to \( x \sim 2/3\phi \) can be confirmed.
5.3. Comparison to the case with no oscillation

If we assume ‘classically’ $y \sim 0$, i.e. $\alpha = \beta$, the WDW equation reads

$$\left[ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \phi^2} \right] \psi = 0 \quad (5.27)$$

or

$$\left[ \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Z^2} \right] \psi = 0. \quad (5.28)$$

The solution of this differential equation is

$$\psi = f_1(x - \phi) + f_2(x + \phi) = g_1(X - Z) + g_2(X + Z). \quad (5.29)$$

This shows very different behaviors from the ‘correct’ solution of the WDW equation. No typical peak can be expected. This is rather trivial, but this comparison reminds us the fact that there is at least zero-point oscillation in any oscillatory quantum system.

6. Conclusion and outlook

We have studied the simple and Lorentz-invariant theory of multigraviton and have shown typical cosmological solutions. We focused our attention on the models associated with the four-site star graph and the path graph and found that vacuum cosmological solutions with the scale factors show the repeated accelerating and decelerating expansions. The differences between these two models were discussed from a viewpoint about symmetries. By using a simplest model, we also qualitatively showed that the oscillatory behavior is considered as necessary in quantum universe. We should investigate more plausible and applicable solutions for classical as well as quantum cosmology, including usual matter.

To this end, we should study how the gravitons and the scalar field couple to various matter fields. To consider various coupled fields, incorporation of supersymmetry or supergravity is also of much interest. Permitting higher derivative terms and nonlocal terms in the action will bring more possibilities to the completion of nonlinearity and be worth studying still.

As for future works, from the mathematical point of view, it is interesting to construct models with the use of generic graphs, such as weighted graphs, fractals and so on.

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