Dark solitons of the Gross-Neveu model.

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We present N-soliton solutions for the classical (1+1)-dimensional Gross–Neveu model which satisfy non-zero boundary conditions. These solutions are obtained by direct method using some properties of the soliton matrices that appear in the framework of the Cauchy matrix approach.

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1. Introduction.

In this paper we consider the (1+1)-dimensional Gross–Neveu model with $N_f$ flavors of Dirac fermions [1],

$$L = i \sum_{n=1}^{N_f} \bar{\psi}_n \partial \psi_n + g \left( \sum_{n=1}^{N_f} \bar{\psi}_n \psi_n \right)^2$$  \hspace{1cm} (1.1)

(we will explain all the designations in what follows). This two-dimensional massless fermion asymptotically free field model has been introduced in 1974 in connection with the search for symmetry breaking, and since then has attracted a lot of interest in semiclassical field theory. The solutions of classical equations corresponding to the Lagrangian $L$ may be considered as candidates, or classical approximations, for the particles of the corresponding quantum theory. That is why the analytic solution of the classical model remains an actual problem (see, for example, [2–14]).

At the classical level, the Gross-Neveu model is closely related to the theory of integrable systems. In papers [15, 16] the authors found a class of the Gross-Neveu-like models which are completely integrable, and one can find there the inverse scattering transform which gives the possibility of deriving various solutions, in particular the soliton ones. Looking at more recent works [5–14] one can notice that, although the authors do not use the results of, say, [16] directly, they use various approaches developed in the theory of integrable systems: the Zakharov-Shabat scattering problem, Gelfand-Levitan-Marchenko equations (in [7–9]), or the theory of reflectionless potentials [17–19], Hirota ansatz and inverse scattering transform for the sinh-Gordon equation (in [5, 6, 10–14]).

The main object of this work is the $N_s$-soliton solutions for the classical Gross-Neveu model (1.1) or, in matrix form,

$$L = \text{Tr} \left( \bar{\Psi} \partial \Psi + g \left( \text{Tr} \bar{\Psi} \Psi \right)^2 \right)$$  \hspace{1cm} (1.2)
where $\Psi = \left(\psi_1, ..., \psi_N\right)$, which were derived in the paper [6] by Fitzner and Thies. Thus, the essential part of this paper may be viewed as an alternative derivation and representation of the results of [6]. The method used in what follows is a variant of the Cauchy matrix approach [20–25]. We start with an ansatz based on some class of matrices, the so-called ‘almost-intertwining’ matrices [26] that satisfy the ‘rank one condition’ [27–29], which is a particular case of the Sylvester equation [30–33]. An analysis of the properties of these matrices leads us to solutions of the ‘two-field’ model

$$\mathcal{L} = i \text{Tr} \Phi \partial \Psi + g (\text{Tr} \Phi \Psi)^2$$

(1.3)

(see section 2). The important point is that we do not need to solve the ‘consistency’ equations separately. We just show in sections 2.2 and 3 that the proposed ansatz possesses some reduction, that automatically resolves the consistency condition and leads from (1.3) to (1.2). This gives the possibility of obtaining the Gross-Neveu solitons, which we discuss in section 4.

2. Auxiliary system.

2.1. Almost-intertwining matrices.

The solutions we present here are built of the matrices defined by the equation

$$\mathbf{LA} - \mathbf{AR} = |k\rangle \langle b|$$

(2.1)

Here, $\mathbf{L}$ and $\mathbf{R}$ are constant diagonal $N_s \times N_s$ matrices, $\langle b| = (b_1, ..., b_{N_s})$-component row and $|k\rangle = (k_1, ..., k_{N_s})^T$.

The dependence of $\langle b|$ and $|k\rangle$ on the coordinates describing the model, $\langle b| = \langle b(\xi, \eta)|$ and $|k\rangle = |k(\xi, \eta)|$, is defined by

$$\partial_\xi \langle b| = \langle b| R^{-1}, \quad \partial_\eta \langle b| = \langle b| R$$

(2.3)

and

$$\partial_\xi |k\rangle = -L^{-1}|k\rangle, \quad \partial_\eta |k\rangle = -L|k\rangle$$

(2.4)

which leads to

$$\partial_\xi \mathbf{A} = |k_o\rangle \langle b_o|, \quad \partial_\eta \mathbf{A} = -|k\rangle \langle b|$$

(2.5)

where

$$\langle b_o| = \langle b| R^{-1}, \quad |k_o\rangle = L^{-1}|k\rangle.$$ (2.6)

In what follows, we use another set of rows and columns, this time of the length $N_f$, defined by

$$\langle \dot{1}|_Z = \langle 1|_Z - \langle b|GK_Z,$$

(2.7)

$$\langle \dot{2}|_Z = \langle 1|_Z |D^{-1}_Z - \langle b_o|GK_Z$$

and

$$| \dot{1}|_Z = |1\rangle + B_Z G|k\rangle,$$

(2.8)

$$| \dot{2}|_Z = |1\rangle + B_Z G|k_o\rangle.$$
Here,
\[ G = (1 + A)^{-1}, \]  
\[ Z = \{ z_1, \ldots, z_{N_f} \}, \]  
\[ \langle 1_Z \mid \text{ and } | 1_Z \rangle \]  
are \( N_f \)-row and \( N_f \)-column with all components equal to 1, and \( D_Z \) is the diagonal \( N_f \times N_f \)-matrix,
\[ D_Z = \text{diag} (z_1, \ldots, z_{N_f}) \]  
(2.10)
while \( B_Z \) and \( K_Z \) are rectangular matrices given by
\[ B_Z = \left( \frac{b_n}{R_n - z_m} \right)_{n=1,\ldots,N_f} \quad \text{and} \quad K_Z = \left( \frac{k_m}{L_m - z_n} \right)_{m=1,\ldots,N_s} \]  
(2.11)
where \( b_n \) and \( k_m \) are components of \( \langle b \mid \text{ and } | k \rangle \) (see (2.2)).

Applying the rules (2.3) and (2.4) to the definitions (2.7) and (2.8) one can obtain, by simple algebra, the following equations governing the ‘evolution’ of \( \langle 1_u Z \mid \) and \( | 2_v Z \rangle \):
\[ \partial_\xi \langle 1_u Z \mid = -\langle 1_u Z \mid D_Z^{-1} + u_o \langle 1_u Z \rangle, \]  
\[ \partial_\eta \langle 1_u Z \rangle = -\langle 1_u Z \rangle D_Z + v_o \langle 1_v Z \rangle \]  
(2.12)
and
\[ \partial_\xi | 1_v Z \rangle = D_Z^{-1} | 1_v Z \rangle - v_o | 1_v Z \rangle, \]  
\[ \partial_\eta | 1_v Z \rangle = D_Z | 1_v Z \rangle - u_o | 1_v Z \rangle \]  
(2.13)
where the functions \( u_o \) and \( v_o \) are defined by
\[ u_o = 1 - \langle b \mid | k_o \rangle, \]  
\[ v_o = 1 + \langle b_o \mid | k \rangle. \]  
(2.14)

2.2. Constraints.
The linear (with respect to \( \langle 1_u Z \mid \) and \( | 2_v Z \rangle \)) equations presented in the previous subsection are an important part of the approach of this paper. However, now we face a more difficult problem: we have to close system (2.12)–(2.14) (note that there is no obvious relationships between \( \langle 1_u Z \mid \text{ and } | 2_v Z \rangle \), \( u_o \) and \( v_o \)). Contrary to the derivation of (2.12) and (2.13), which is a straightforward procedure similar to one used repeatedly by various authors, the ‘closure’ problem is less trivial. In the framework of the theory of integrable systems, it is related to the so-called Bargmann constraints or the nonlinearization procedure. We do not discuss here the ‘theoretical’ aspects of this problem. Instead, we demonstrate that it is possible to relate the rows \( \langle 1_u Z \mid \) and the columns \( | 2_v Z \rangle \) to the functions \( u_o \) and \( v_o \) by careful analysis of the structure of the \( A \)-matrices or, in other words, of consequences of equation (2.1).

It can be shown (see Appendix A) that equation (2.1), together with the definitions (2.7), (2.8) and (2.14), imply
\[ \langle 1_u Z \mid 2_v Z \rangle = h u_o - \langle b \mid F_Z | k_o \rangle, \]  
\[ \langle 1_u Z \rangle | 2_v Z \rangle = h v_o - \langle b_o \mid F_Z | k \rangle \]  
(2.15)
where
\[ h = \sum_{n=1}^{N_f} z_n^{-1}, \]  
(2.16)
and
\[ F_Z = \hat{G} F_Z \hat{G} \]  
(2.17)
with
\[ \hat{F}_Z = \text{diag} \left( \sum_{n=1}^{N_f} \left[ \frac{1}{L_s - z_n} - \frac{1}{R_s - z_n} \right] \right) \quad (2.18) \]

(here, \( L_s \) and \( R_s \) are the elements of the diagonal matrices \( L \) and \( R \)).

As one can see from equations (2.15), the variables \( \langle \hat{u}_Z | \hat{v}_Z \rangle \), \( u_o \) and \( v_o \) which are involved in equations (2.12) and (2.13) are not enough, in the general case, to obtain a closed system. However, there exists a reduction of (2.1) that eliminates these difficulties.

The key point in our calculations is the fact (demonstrated in Appendix B) that the restriction \( L + R = 0 \) \((2.19)\)

leads to the following result:
\[ u_o = v_o \quad (2.20) \]

and
\[ 0 = \langle b|\hat{F}_Z|k_o \rangle + \langle b_o|\hat{F}_Z|k \rangle, \quad (2.21) \]

which together with (2.15) implies
\[ \langle \hat{u}_Z | \hat{\nu}_Z \rangle + \langle \hat{\nu}_Z | \hat{\nu}_Z \rangle = 2hu_o. \quad (2.22) \]

Thus, the restriction (2.19) converts equations (2.12) and (2.13) into a closed system for \( \langle \hat{u}_Z | \hat{v}_Z \rangle \).

2.3. **Matrices \( \Phi \) and \( \Psi \).**

Now, we rewrite equations (2.12), (2.13) and (2.22) in a matrix form. Consider the \( N_f \times 2 \) matrix
\[
\Phi = E^{-1} \left( \begin{array}{c} \langle \hat{v}_Z | \hat{\nu}_Z \rangle \\ \langle \hat{\nu}_Z | \hat{\nu}_Z \rangle \end{array} \right),
\]

and the \( 2 \times N_f \) matrix
\[ \Psi = \left( \begin{array}{c} \langle \hat{u}_Z | \\ \langle \hat{v}_Z | \end{array} \right) E, \quad (2.24) \]

where the \( N_f \times N_f \) diagonal matrix \( E \), which satisfies
\[ \partial_{\xi} E = D_{\hat{Z}}^{-1} E, \quad \partial_{\eta} E = D_{\hat{Z}} E, \quad (2.25) \]

is introduced to take into account the terms proportional to \( D_{\hat{Z}}^{\pm 1} \) in (2.12) and (2.13).

In terms of \( \Phi \) and \( \Psi \), equations (2.12) and (2.13) can be written as
\[
X_{\xi} \partial_{\xi} \Psi + X_{\eta} \partial_{\eta} \Psi = u_o \sigma_1 \Psi, \quad (2.26)
\]
\[
-(\partial_{\xi} \Phi) X_{\xi} - (\partial_{\eta} \Phi) X_{\eta} = v_o \Phi \sigma_1, \quad (2.27)
\]

where
\[
X_{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_{\eta} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.28)
\]

and \( \sigma_1 \) is the Pauli matrix, while equation (2.22) takes the form
\[ 2hu_o = \text{Tr} \, \Phi \sigma_1 \Psi. \quad (2.29) \]
To summarize, matrices $\Phi$ and $\Psi$ satisfy equations corresponding to the Lagrangian

$$
\mathcal{L} = 4h \text{Tr} \Phi \partial \Psi - (\text{Tr} \Phi \sigma_1 \Psi)^2
$$

(2.30)

with

$$
\partial = \begin{pmatrix}
\partial \xi & 0 \\
0 & \partial \eta
\end{pmatrix}.
$$

(2.31)

It is easy to see that (2.30) resembles the Gross-Neveu Lagrangian (1.2). The main difference is that the Lagrangian (2.30) is built of two matrices, $\Phi$ and $\Psi$. In the following section we discuss questions related to complex/Hermitian conjugation and establish that there is a natural reduction which links $\Phi$ and $\Psi$.

3. Involution.

It turns out that, if one works in the framework of the soliton ansatz used in this paper, the behavior of solutions under the complex/Hermitian conjugation is determined by whether the matrices $L$ are real or imaginary. Indeed, it is not difficult to show that the condition

$$
L^\dagger = -\epsilon L, \quad \epsilon = \pm 1
$$

(3.1)

implies

$$
A^\dagger = A, \quad G^\dagger = G.
$$

(3.2)

The requirement

$$
E^\dagger = E^{-1}
$$

(3.3)

leads to the restrictions

$$
z_n^* = \epsilon z_n \quad (n = 1, \ldots, N_f)
$$

(3.4)

and

$$
\xi^* = -\epsilon \xi, \quad \eta^* = -\epsilon \eta
$$

(3.5)

where $*$ stands for the complex conjugation.

This results in

$$
h^* = \epsilon h, \quad \Phi = \psi^\dagger S
$$

(3.6)

(3.7)

where

$$
S = \begin{pmatrix}
1 & 0 \\
0 & \epsilon
\end{pmatrix}
$$

(3.8)

and hence

$$
\mathcal{L} = 4h \text{Tr} \psi^\dagger S \partial \psi - \left(\text{Tr} \psi^\dagger S \sigma_1 \psi\right)^2.
$$

(3.9)

To conclude our analysis, we consider separately the cases $\epsilon = \pm 1$, and rewrite the Lagrangian (3.9) in terms of the Dirac matrices,

$$
\gamma^0 = \sigma_1 \quad \gamma^1 = i\sigma_2 \quad \gamma^5 = -\sigma_3
$$

(3.10)

and the adjoint matrix $\bar{\Psi}$ defined by

$$
\bar{\psi} = \psi^\dagger \gamma^0
$$

(3.11)
Gross-Neveu case (\(\epsilon = 1\)). To take into account the fact that in this case, as follows from (3.5), both \(\xi\) and \(\eta\) are pure imaginary we introduce two real variables,

\[
\xi = i(t - x), \quad \eta = i(t + x). \tag{3.12}
\]

Noting that \(S\) is the unit matrix and that \(h\) is real we can introduce the real coupling constant

\[
g = \frac{1}{2h} \tag{3.13}
\]

and rewrite the Lagrangian (3.9) (omitting an insignificant constant) as

\[
L = i \text{Tr} \bar{\Psi} \partial \Psi + g (\text{Tr} \bar{\Psi} \Psi)^2 \tag{3.14}
\]

with

\[
\bar{\phi} = \gamma^0 \partial_t + \gamma^1 \partial_x, \tag{3.15}
\]

which coincides with (1.2).

Case \(\epsilon = -1\). In this case, both \(\xi\) and \(\eta\) are real, \(S = \sigma_3\), the parameters \(z_n\) and hence the constant \(h\) are pure imaginary.

Introducing \(t\) and \(x\) by

\[
\xi = -t + x, \quad \eta = t + x, \tag{3.16}
\]

and \(g\) by

\[
g = \frac{i}{2h} \tag{3.17}
\]

(all of which are real), one can rewrite the Lagrangian (3.9) (again, omitting an insignificant constant) as

\[
L = i \text{Tr} \bar{\Psi} \bar{\phi} \Psi + g \left(\text{Tr} \bar{\Psi} \gamma^5 \Psi\right)^2, \tag{3.18}
\]

with \(\bar{\phi}\) defined in (3.15).

4. Solitons of the Gross-Neveu model.

Here, we would like to collect the results related to the Gross-Neveu model.

As follows from (3.1) with \(\epsilon = 1\), the matrix \(L\) is pure imaginary. Thus, we write it as

\[
L = i M = i \text{diag} (\mu_1, \ldots, \mu_{N_s}) \tag{4.1}
\]

with real \(\mu_m\). Equation (2.1) leads to

\[
A(t, x) = \left( C_{lm} \exp [\omega_l(t, x) + \omega_m(t, x)] \right)_{l, m = 1, \ldots, N_s} \tag{4.2}
\]

where

\[
\omega_m(t, x) = \left( \mu_m - \mu_m^{-1} \right) t + \left( \mu_m + \mu_m^{-1} \right) x \tag{4.3}
\]

or

\[
\omega_m(t, x) = \frac{2}{1 - v_m^2} (x - v_m t), \quad v_m = \frac{1 - \mu_m^2}{1 + \mu_m^2} \tag{4.4}
\]

and \(C_{lm}\) are constants given by

\[
C_{lm} = -i \frac{k_l^{(0)} b_m^{(0)}}{(\mu_l + \mu_m)} \tag{4.5}
\]

with \(b_m^{(0)} = b_m(0, 0)\) and \(k_l^{(0)} = k_l(0, 0)\) playing the role of parameters of the presented solutions.
The columns of the matrix $\Psi$ can be presented as

$$
\psi_n = e^{i \Theta_n} \begin{pmatrix}
1 - \langle b | G(L - z_n)^{-1} | k \rangle \\
z_n^{-1} + \langle b | L^{-1} G(L - z_n)^{-1} | k \rangle
\end{pmatrix}
$$

(4.6)

where the phases $\Theta_n = \Theta_n(t, x)$ are given by

$$
\Theta_n(t, x) = (z_n + z_n^{-1}) t + (z_n - z_n^{-1}) x
$$

(4.7)

or

$$
\psi_n(t, x) = e^{i \Theta_n(t, x)} \begin{pmatrix}
\det | z_n - i \text{MY}(t, x) | \\
z_n^{-1} \det | z_n \text{Y}(t, x) - i \text{M} |
\end{pmatrix}
$$

(4.8)

where

$$
Y = (1 + A)^{-1} (1 - A).
$$

(4.9)

To summarize, formulae (4.8) together with (4.1)–(4.3), (4.7) and (4.9) provide the $N_s$-soliton solutions for the Gross-Neveu model.

The function $u_o$, which in this case can be presented as

$$
uo = \frac{\det | 1 - A |}{\det | 1 + A |},
$$

(4.10)

satisfies the sinh-Gordon equation

$$
\frac{1}{4} \Box u_o = u_o^{-2} - u_o^2
$$

(4.11)

where $\Box = \partial_{tt} - \partial_{xx}$ (we prove these facts in Appendix C).

It is not difficult to obtain from (4.8) the behavior of $\psi_n$ in the asymptotic regions. For simplicity, we carry out this analysis under the following assumption:

$$
\mu_s > 0, \quad s = 1, \ldots, N_s.
$$

(4.12)

When $x \to -\infty$ with $t = \text{constant}$, all $\omega_m(t, x) \to -\infty$ which yields $\text{Y}(t, x) \to 1$ and

$$
\lim_{x \to -\infty} e^{-i \Theta_n(t, x)} \psi_n(t, x) = \phi_n^-
$$

(4.13)

with

$$
\phi_n^- = \begin{pmatrix} 1 \\ z_n^{-1} \end{pmatrix}.
$$

(4.14)

In a similar way one arrives at

$$
\lim_{x \to +\infty} e^{-i \Theta_n(t, x)} \psi_n(t, x) = \phi_n^+
$$

(4.15)

with

$$
\phi_n^+ = e^{i \delta_n} \begin{pmatrix} 1 \\ (-)^{N_s} z_n^{-1} \end{pmatrix}
$$

(4.16)

and

$$
\delta_n = 2 \sum_{s=1}^{N_s} \arg (z_n + i \mu_s).
$$

(4.17)
The limiting values of the condensate function $S$,

$$S = \text{Tr} \bar{\Psi} \Psi = \sum_{n=1}^{N_f} \bar{\psi}_n \psi_n,$$  \hspace{1cm} (4.18)

are given by

$$S(t, x) \to \begin{cases} 
1/g & \text{as } x \to -\infty \\
(-)^N/g & \text{as } x \to +\infty
\end{cases} \hspace{1cm} (4.19)$$

while the fermion density $Q$,

$$Q = \text{Tr} \Psi^\dagger \Psi = \sum_{n=1}^{N_f} \psi_n^\dagger \psi_n,$$  \hspace{1cm} (4.20)

satisfies

$$\lim_{x \to \pm \infty} Q(t, x) = Q^\infty = \text{constant} \hspace{1cm} (4.21)$$

where

$$Q^\infty = N_f + \sum_{n=1}^{N_f} z_n^{-2} \hspace{1cm} (4.22)$$

For the one-soliton solution $N_s = 1$, the matrix $L$ is scalar, $L = i\mu$ (we drop the subscript 1), and the one-soliton solution is characterized, except for the real set $\{z_1, \ldots, z_{N_f}\}$, by one velocity $v$, $v = (1 - \mu^2)/(1 + \mu^2)$, and one constant $C_{11}$, which without loss of generality can be set equal to unity, $C_{11} = 1$. The matrix $Y$ becomes

$$Y = -\tanh \omega, \hspace{1cm} (4.23)$$

where

$$\omega(t, x) = \frac{2}{1 - v^2} (x - vt). \hspace{1cm} (4.24)$$

Equation (4.8) can be rewritten as

$$\psi_n(t, x) = \frac{1}{2} e^{i\Theta_n(t, x)} \left[ \phi_n^+ + \phi_n^- + (\phi_n^+ - \phi_n^-) \tanh \omega \right] \hspace{1cm} (4.25)$$

or

$$\psi_n(t, x) = \frac{e^{i\Theta_n(t, x)}}{2 \cosh \omega} \left[ e^{-\omega} \phi_n^- + e^{\omega} \phi_n^+ \right], \hspace{1cm} (4.26)$$

where $\phi_n^\pm$ are the limits of $e^{-i\Theta_n} \psi_n$ defined earlier and given by

$$\phi_n^+ = \begin{pmatrix} 1 \\ z_n^{-1} \end{pmatrix}, \hspace{1cm} \phi_n^- = e^{i\delta_n} \begin{pmatrix} 1 \\ -z_n^{-1} \end{pmatrix}, \hspace{1cm} (4.27)$$

with

$$\delta_n = 2 \arg \left( z_n + i \frac{1 - v}{1 + v} \right). \hspace{1cm} (4.28)$$

The distribution of the condensate $S$, which is defined in (4.18), is given by

$$S = -\frac{1}{g} \tanh \omega, \hspace{1cm} (4.29)$$
Fig. 1 2-soliton solutions: fermion density $Q(t, x)$ (left pane) and condensate $S(t, x)$ (right pane). The parameters of the solution are given by $L = i \operatorname{diag}(0.2, 1.5)$, $\langle b(0, 0) \rangle = e^{i\pi/6}(1, 1)$, $|k(0, 0)| = e^{i\pi/3}(1, 1)^T$ and $Z = \{1\}$. The coordinate ranges are given by $t \in (-3.5, 3.5)$ and $x \in (-5.0, 5.0)$.

while the fermion density $Q$, defined in (4.20), can be presented as

$$Q = Q^\infty - \frac{A}{\cosh^2 \omega},$$

with $Q^\infty$ being defined in (4.22) and

$$A = (1 + \mu^2) \sum_{n=1}^{N_f} \frac{1}{z_n^2 + \mu^2}.$$  

Considering the more complex solutions, we present examples of two- and three-soliton solutions in figures 1 and 2.

5. Discussion.

To derive the solitons of the Gross-Neveu model we used rather standard technique from the theory of integrable systems. The Cauchy matrix approach, which appeared in 1980s as an alternative to the inverse scattering transform, was subsequently modified to become one of the easiest way to derive explicit solutions for integrable nonlinear equations. In this paper, and in many others, it is used just like an ansatz, which, if compared, for example, with the inverse scattering transform, is more straightforward, not restricted by imposing some boundary conditions beforehand, and rather flexible (see, e.g., [34]). Even in the framework of this paper one can note that our ansatz, with slight modifications, leads to solutions for both the Gross-Neveu model and its $\gamma^5$ variant (3.18). Clearly, it has its limitations. The soliton ansatz of this paper, that in context of other integrable models leads to ‘general’ N-soliton solutions, in the case of the Gross-Neveu equations provides less than one might anticipate. The solutions presented above belong to the so-called type I (the simplest) class,
Fig. 2 3-soliton solutions: fermion density $Q(t, x)$ (left pane) and condensate $S(t, x)$ (right pane). The parameters of the solution are given by $L = i \text{diag}(0.2, 1.5, 6.0)$, $|b(0, 0)| = e^{i\pi/6}(1, 1, 1)$, $|k(0, 0)| = e^{i\pi/3}(1, 1, 1)^T$ and $Z = \{1\}$. The coordinate ranges are given by $t \in (-3.5, 3.5)$ and $x \in (-5.0, 5.0)$.

according to the classification of [5]. Indeed, if we take notice of the $z_n$-dependence, the condensate function $S$ can be presented as

$$S(t, x, Z) = 2h(Z)u_v(t, x) \quad (5.1)$$

which leads to

$$\bar{\psi}_n \psi_n = \lambda_n S \quad (5.2)$$

where $\lambda_n$ is a constant, given by $\lambda_n = z_n^{-1} / \sum_{m=1}^{N_f} z_m^{-1}$. Thus, in our attempt to derive the $N_f$-flavor solitons we have actually obtained some kind of direct sum of $N_f = 1$ solitons (up to the linear global mixing $\Psi \rightarrow \Psi U$ where $U$ is a constant unitary matrix). This means that to find less trivial $N_f$-flavor solutions (even if we restrict ourselves to the classical Gross-Neveu model with finite $N_f$, i.e. without continuous constituent) we have to go beyond the soliton ansatz used above. However, this very important question is outside of the scope of this paper.

The last question we would like to mention is the question of terms. In the theory of integrable systems solutions like ones presented in this paper are usually called ‘solitons’ or, more precisely, ‘dark solitons’ where the word ‘dark’ indicates that they are solitons which satisfy constant non-zero boundary conditions. In the field theory, the more widely used term is ‘kink’. If one looks at the one-soliton solution (4.25) (or (4.29)), then there is no discordance: the tanh-function is what is usually associated with a kink. However, in the situation with two (or any even $N_s$) solitons the asymptotic behavior of both the condensate $S$ and the fermion density $Q$ differs from the kink-, or $N_s$-kink-like, one (see, for example, figure 1).
A. Derivation of (2.15).
From the definitions of $B_Z$ and $K_Z$, one can easily derive
\[
\langle 1_Z | B_Z = \langle b | \tilde{B}_Z, \quad K_Z | 1_Z \rangle = \hat{K}_Z | k \rangle
\]
and
\[
\begin{align*}
\langle 1_Z | D_Z^{-1} B_Z &= \langle b_0 | (h + \tilde{B}_Z), \\
K_Z D_Z^{-1} | 1_Z \rangle &= (h + \hat{K}_Z) | k_0 \rangle
\end{align*}
\]
where $h$ is defined in (2.16) and
\[
\tilde{B}_Z = \sum_{z \in Z} R^{-1}_z, \quad \hat{K}_Z = \sum_{z \in Z} L^{-1}_z.
\]
These equations, together with the definitions (2.7) and (2.8), lead to
\[
\begin{align*}
\langle 1_U | \hat{V}_z \rangle &= hu_o - \langle b | F_Z | k_0 \rangle \\
\langle 2_U | \hat{V}_z \rangle &= hv_o - \langle b_0 | F_Z | k \rangle
\end{align*}
\]
with
\[
F_Z = GK_Z B_Z G - \hat{B}_Z G + G \hat{K}_Z.
\]
To simplify this expression, one should note that
\[
K_Z B_Z = \sum_{n=1}^{N_f} | k_z_n \rangle \langle b_z_n |
\]
where
\[
\langle b_z | = \langle b | (R - z)^{-1}, \quad | k_z \rangle = (L - z)^{-1} | k \rangle,
\]
which, together with (2.1), leads to
\[
K_Z B_Z = A \hat{B}_Z - \hat{K}_Z A
\]
and, finally, to
\[
F_Z = G (\hat{K}_Z - \hat{B}_Z) G.
\]
Clearly, equations (A5) and (A10) coincide with (2.15) and (2.17) with $\hat{F}_Z = \hat{K}_Z - \hat{B}_Z$.

B. Proof of (2.20) and (2.21).
First, one has to note that the matrix $T$ that links $| k \rangle$ and $\langle b |$,
\[
| k \rangle = T | b^T \rangle
\]
where $| b^T \rangle = (\langle b |)^T$, and which is given by
\[
T = \text{diag} (\ldots, k_n / b_n, \ldots),
\]
as follows from equations (2.3) and (2.4) together with the restriction (2.19), does not depend on $\xi$ or $\eta$. 

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From (2.1) one can easily obtain $A^T = T^{-1}AT$, which holds for all $\xi$ and $\eta$ and which implies $G^T = T^{-1}GT$. Noting also that $|k_o\rangle = -T|b_o^T\rangle$, where $|b_o^T\rangle = (\langle b_o|)^T$, one can consequently obtain from (2.14)

$$u_o = 1 - \langle b|G|k_o\rangle$$
\[= 1 + \langle b|GT|b_o^T\rangle\]
\[= 1 + \langle b_o|TG^T|b\rangle\]
\[= 1 + \langle b_o|G|k\rangle\]
\[= v_o\]  \hspace{1cm} (B3)

which proves (2.20).

In a similar way, noting that

$$\langle k_o\rangle = -T\langle b^\gamma\rangle = T^{-1}\langle k\rangle\langle b_o|T,$$ \hspace{1cm} (B5)

and that the matrix $\hat{F}_Z$ in the definition (2.17) is diagonal (and hence commutes with $T$) one can obtain

$$F^T_Z = T^{-1}F_ZT$$ \hspace{1cm} (B6)

and then

$$\langle b|F_Z|k_o\rangle = \langle b\rangle = \text{Tr} F_Z|k_o\rangle\langle b|$$
\[= \text{Tr} (\langle k_o\rangle|b\rangle)^T F^T_Z\]
\[= -\text{Tr} T^{-1}\langle k\rangle\langle b_o|T \cdot T^{-1}F_ZT\]
\[= -\text{Tr} |k\rangle\langle b_o|F_Z\]
\[= -\langle b_o|F_Z|k\rangle\]  \hspace{1cm} (B7)

which proves (2.21).

C. Tau-functions.

The $\tau$-functions of the Gross-Neveu model can be defined as

$$\tau = \det |1 + A|.$$  \hspace{1cm} (C1)

From equation (2.1) with $R = -L$ and the definition (2.6) of $|k_o\rangle$ one can easily obtain

$$1 + A - |k_o\rangle\langle b| = 1 - L^{-1}AL$$ \hspace{1cm} (C2)

and then

$$1 - G|k_o\rangle\langle b| = GL^{-1}(1 - A)L$$ \hspace{1cm} (C3)

Taking the determinant of the last equation, using the identity $\det (1 + |u\rangle\langle v|) = 1 + \langle v||u\rangle$ and noting that $\det G = 1/\det |1 + A|$ one arrives at

$$u_o = \tau_-/\tau_+$$ \hspace{1cm} (C4)

which is (4.10).
The derivatives of \( \tau_{\pm} \) are given by

\[
\partial_\xi \ln \tau_{\pm} = \pm \langle b_0 | G_{\pm} | k_0 \rangle, \quad \partial_\eta \ln \tau_{\pm} = \mp \langle b | G_{\pm} | k \rangle, \quad (C5)
\]

where

\[
G_{\pm} = (1 \pm A)^{-1} \quad (C6)
\]

from which one can derive

\[
\partial_\xi \eta \ln \tau_{\pm} = 1 - (\tau_{\mp}/\tau_{\pm})^2 \quad (C7)
\]

and, taking into account (C4),

\[
\partial_\xi \eta \ln u_0 = u_o^2 - u_o^{-2} \quad (C8)
\]

which is nothing but (4.11).

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