Form factors and correlation functions of an interacting spinless fermion model

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Abstract

Introducing the fermionic $R$-operator and solutions of the inverse scattering problem for local fermion operators, we derive a multiple integral representation for zero-temperature correlation functions of a one-dimensional interacting spinless fermion model. Correlation functions particularly considered are the one-particle Green's function and the density-density correlation function both for any interaction strength and for arbitrary particle densities. In particular for the free fermion model, our formulae reproduce the known exact results. Form factors of local fermion operators are also calculated for a finite system.

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1 Introduction

The evaluation of correlation functions has been one of the challenging problems in research on quantum many-body systems in one-dimension, since most of the intriguing phenomena induced by underlying strong quantum fluctuations are theoretically described through correlation functions. In these systems, it is widely known that the existence of models which are solvable by means of Bethe ansatz (see [1, 2] for example). Though the exact computation of correlation functions, of course, is still tremendously difficult even in such models, several analytical methods have been recently developed to derive manageable expressions for correlation functions, especially in the spin-1/2 Heisenberg XXZ chain.

In 1990s, correlation functions of the spin-1/2 XXZ chain at zero temperature and for zero magnetic field have been expressed as multiple integral forms derived by the $q$-vertex operator approach [3–5]. An alternative method combining the algebraic Bethe ansatz with

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solutions to the quantum inverse problem has been provided for the XXZ chain in arbitrary magnetic fields [6–8] (see also [9] for a recent review). This method can be generalized to the finite-temperature and/or the time-dependent correlation functions [10–14].

In general, one-dimensional (1D) quantum spin systems are mapped to 1D fermion systems through the Jordan-Wigner transformation. For the spin-1/2 XXZ chain, the corresponding system is a spinless fermion model with the nearest neighbor hopping and interaction. In the thermodynamic limit, the bulk quantities in the XXZ chain are exactly the same as those in the spinless fermion model. Namely, in the 1D quantum systems, the difference of the statistics between the spin and the fermion does not show up, as long as we concentrate on their bulk quantities.

The situation, however, is radically changed when the quantities accompanying a change of the number of particles are considered. For instance, let us consider the (equal-time) one-particle Green’s function \( \langle c_1 \, c_{m+1}^\dagger \rangle \) for the spinless fermion model and the transverse spin-spin correlation function \( \langle \sigma_1^z \, \sigma_{m+1} \rangle \) for the XXZ chain, which are intuitively the same. In fact, due to the difference of the statistics, or equivalently the nonlocality of the Jordan-Wigner transformation, both the correlation functions exhibit completely different behavior. For instance, one observes an oscillatory behavior for the one-particle Green’s function (referred to as the \( k_F \)-oscillation, where \( k_F \) is the Fermi momentum), which is peculiar to the fermion systems. In contrast, for the transverse spin-spin correlation function, such an oscillatory behavior does not appear \(^1\).

As mentioned above, exact expressions for the correlation functions of the XXZ chain have already been proposed in the form of multiple integral representations. Unfortunately, once the Jordan-Wigner transformation is performed and the XXZ chain is considered instead of the spinless fermion model, it is difficult to trace the difference of the statistics in the framework of multiple integral representations. Namely, to derive a manageable expression of correlation functions for the spinless fermion model, we must directly treat the fermion system from the very beginning.

In this paper, introducing the fermionic \( R \)-operator [15] which acts on the fermion Fock space, we directly treat the spinless fermion model without mapping to the XXZ chain. Combining the method provided in the XXZ chain [7] with solutions to the inverse scattering problem of local fermion operators [16], we derive a multiple integral representing the equal-time one-particle Green’s function and the density-density correlation function at zero temperature both for any interaction strengths and for arbitrary particle densities. Our formulae reproduce the known results for the free fermion model. In addition to the correlation functions, we also compute form factors for local fermion operators, which might be useful for systematic evaluations of the spectral functions for the spinless fermion model.

This paper is organized as follows. In the subsequent section, we introduce the fermionic \( R \)-operator and the transfer operator constructed by the \( R \)-operator. Then we briefly review the algebraic Bethe ansatz for the spinless fermion model. The scalar product of a Bethe state with an arbitrary state is presented in section 3. Combining solutions of the inverse scattering problem with the scalar product, we compute form factors of local fermion operators. Multiple integral representations for correlation functions are derived in section 4. In section 5, using the multiple integral representations, we explicitly evaluate correlation functions for the free

\(^1\) In fact, there could exist an oscillatory behavior in the transverse spin-spin correlation function, which, however, can be eliminated by a gauge transformation.
2 Spinless fermion model

The Hamiltonian of the spinless fermion model on a periodic lattice with $M$ sites is defined as

$$H = H_0 - \mu_c \sum_{j=1}^{M} \left( \frac{1}{2} - n_j \right),$$

$$H_0 = t \sum_{j=1}^{M} \left\{ c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + 2\Delta \left( \left( \frac{1}{2} - n_j \right) \left( \frac{1}{2} - n_{j+1} \right) - \frac{1}{4} \right) \right\},$$

(2.1)

where $c_j^\dagger$ and $c_j$ are the fermionic creation and annihilation operators at the $j$th site, respectively, satisfying the canonical anti-commutation relations

$$\{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0, \quad \{c_j^\dagger, c_k\} = \delta_{jk}. \quad (2.2)$$

Here $t$ and $\Delta$ are real constants characterizing the nature of the ground state, and $\mu_c$ denotes the chemical potential coupling to the density operator $n_j = c_j^\dagger c_j$. By use of the Jordan-Wigner transformation

$$c_j = \exp \left[ i\pi \sum_{k=1}^{j-1} n_k \right] \sigma_j^+, \quad c_j^\dagger = \exp \left[ -i\pi \sum_{k=1}^{j-1} n_k \right] \sigma_j^-, \quad n_j = \frac{1}{2} (1 - \sigma_j^z), \quad (2.3)$$

the spinless fermion model (2.1) can be mapped to the spin-1/2 XXZ chain with an external magnetic field $h$:

$$H_{XXZ} = J \sum_{j=1}^{M} \left\{ \sigma_j^+ \sigma_{j+1}^- + \sigma_{j+1}^+ \sigma_j^- + \frac{\Delta}{2} \left( \sigma_j^z \sigma_{j+1}^z - 1 \right) \right\} - \frac{h}{2} \sum_{j=1}^{M} \sigma_j^z, \quad (2.4)$$

where we have changed the variables $t$ and $\mu_c$ in (2.1) as $t \rightarrow J$ and $\mu_c \rightarrow h$.

The difference between the spinless fermion model and the corresponding XXZ chain lies only in the difference of boundary conditions. Therefore, in the thermodynamic limit, the quantities without accompanying a change of the number of particles such as the density-density correlation function $\langle n_1 n_{m+1} \rangle$ are exactly the same as those such as a longitudinal spin-spin correlation function $\langle (1 - \sigma_j^z)(1 - \sigma_{m+1}^z) \rangle/4$ for the XXZ chain. As mentioned in the preceding section, however, the quantities changing the number of particles such as the one-particle Green’s function $\langle c_1 c_{m+1}^\dagger \rangle$ exhibit completely different behavior from those such as the transverse spin-spin correlation function $\langle \sigma_1^+ \sigma_{m+1}^z \rangle$, due to the nonlocality of the Jordan-Wigner transformation (2.3). Though the exact expression of $\langle \sigma_1^+ \sigma_{m+1}^z \rangle$ for the XXZ chain has already been presented in [7], it is difficult to convert it to the expression for the spinless fermion model, as long as we concentrate on the XXZ chain (2.4). Namely we should treat the original spinless fermion model without mapping to the XXZ chain. In this section, introducing the fermionic $R$-operator [15], we directly consider the spinless fermion model (2.1).
The fermionic $R$-operator is defined by

$$R_{12}(\lambda) = 1 - n_1 - n_2 + \frac{\text{sh} \lambda}{\text{sh}(\lambda + \eta)}(n_1 + n_2 - 2n_1n_2) + \frac{\text{sh} \eta}{\text{sh}(\lambda + \eta)}(c_1^\dagger c_2 + c_2^\dagger c_1),$$

(2.5)

which acts on $V_1 \otimes_s V_2$. Here $V_j$ is a two-dimensional fermion Fock space whose normalized orthogonal basis is given by $|0\rangle_j$ and $|1\rangle_j := c_j^\dagger |0\rangle_j$, where $c_j|0\rangle_j = 0$ and $\otimes_s$ denotes the super tensor product. Note that this fermionic $R$-operator satisfies the Yang-Baxter equation [15]:

$$R_{12}(\lambda_1 - \lambda_2)R_{13}(\lambda_1)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda_1 - \lambda_2).$$

(2.6)

Identifying one of the two fermionic Fock spaces with the quantum space $\mathcal{H}_m$, we define the $L$-operator at the $m$th site by

$$L_m(\lambda) = R_{am}(\lambda - \xi_m),$$

(2.7)

where $\xi_m$ are inhomogeneous parameters assumed to be arbitrary complex numbers. The fermionic monodromy operator is then constructed as a product of the $L$-operators:

$$T(\lambda) = L_M(\lambda) \cdots L_1(\lambda) = A(\lambda)(1 - n_a) + B(\lambda)c_a + c_a^\dagger C(\lambda) + D(\lambda)n_a.$$

(2.8)

Thanks to the Yang-Baxter equation (2.6), the fermionic transfer operator defined by

$$T(\lambda) = \text{str}_a T(\lambda) = a\langle 0|T(\lambda)|0\rangle_a - a\langle 1|T(\lambda)|1\rangle_a = A(\lambda) - D(\lambda),$$

(2.9)

constitutes a commuting family: $[T(\lambda), T(\mu)] = 0$, where the dual fermion Fock space is spanned by $a|0\rangle$ and $a|1\rangle$ with $a|1\rangle := a|0\rangle c_a$ and $a|0\rangle c_a^\dagger = 0$.

The Hamiltonian of the spinless fermion model $H_0$ (2.1) can be expressed as the logarithmic derivative of the fermionic transfer operator $T(\lambda)$ in the homogeneous limit $\xi_m \to \eta/2$:

$$H_0 = t \text{sh}(\eta) \frac{\partial}{\partial \lambda} \ln T(\lambda) \bigg|_{\lambda = \xi_m = \eta/2}, \quad \Delta = \text{ch} \eta.$$

(2.10)

Due to the Yang-Baxter equation (2.6), one immediately sees that the following relation is valid for arbitrary spectral parameters $\lambda_1$ and $\lambda_2$:

$$R_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2).$$

(2.11)

This leads to the commutation relations among the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ constructing the monodromy operator (2.8). Let us explicitly write down some of these relations, which we will use later:

$$B(\mu)B(\lambda) = B(\lambda)B(\mu), \quad C(\mu)C(\lambda) = C(\lambda)C(\mu),$$

$$A(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)A(\mu) - g(\lambda, \mu)B(\mu)A(\lambda),$$

$$D(\mu)B(\lambda) = -f(\mu, \lambda)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda),$$

$$C(\mu)A(\lambda) = f(\mu, \lambda)A(\lambda)C(\mu) - g(\mu, \lambda)A(\mu)C(\lambda),$$

$$C(\mu)D(\lambda) = -f(\lambda, \mu)D(\lambda)C(\mu) + g(\lambda, \mu)D(\mu)C(\lambda),$$

$$C(\mu)B(\lambda) + B(\lambda)C(\mu) = g(\lambda, \mu)(D(\lambda)A(\mu) - D(\mu)A(\lambda)),$$

(2.12)
Utilizing commutation relations in (2.12), one finds that the state (2.15) becomes an eigen-energy satisfying the following linear integral equation:

\[
\text{of the Bethe roots:}
\]

the ground state reduces to the following linear integral equation for the distribution function by a function of the chemical potential \( \mu \):

Then the corresponding eigenvalue \( \tau (\mu) \) (i.e. \( \mathcal{T}(\mu)|\psi\rangle = \tau (\mu)|\psi\rangle \)) is given by

Note that the existence of an extra factor \((-1)^{N+1}\) in (2.16) and (2.17) reflects the fermionic structure of the system, which does not appear for the XXZ chain (2.4). The relation (2.10) leads to the explicit expression of the energy spectrum per site \( e_0 \) of the spinless fermion model:

In the thermodynamic limit, where \( M \to \infty, N \to \infty, \) and \( N/M \) is a constant determined by a function of the chemical potential \( \mu_c \), the Bethe ansatz equation (2.16) characterizing the ground state reduces to the following linear integral equation for the distribution function of the Bethe roots:

In the above, the integral kernel \( K(\lambda) \) and the function \( t(\lambda, \mu) \) are respectively defined by

The integration contour \( \mathcal{C} = [-\Lambda_c, \Lambda_c] \) is given by \( \epsilon(\pm \Lambda_c) = 0 \), where \( \epsilon(\lambda) \) is the dressed energy satisfying the following linear integral equation:

\[
-2 \pi \iota \rho_{\text{tot}}(\lambda) + \int_{\mathcal{C}} K(\lambda - \mu) \rho_{\text{tot}}(\mu) d\mu = t(\lambda, \frac{\eta}{2}).
\]
Namely, for $-1 < \Delta \leq 1$ ($\eta = -i\zeta$, $\zeta > 0$), the contour $\mathcal{C}$ is defined by an interval on the real axis. In particular, at $\mu_c \to 0$, $\Lambda_{\mu_c} \to \infty$. On the other hand, for $\Delta > 1$ ($\eta < 0$), the Bethe roots are distributed on the imaginary axis. In particular, at $\mu_c = 0$, $\Lambda_{\mu_c} = -\pi i/2$. Then the energy density (2.18) explicitly reads

$$e_0 = \tanh(\eta) \int_{\mathcal{C}} t \left( \lambda, \frac{\eta}{2} \right) \rho_{\text{tot}}(\lambda) d\lambda - \mu_c \left( \frac{1}{2} - \langle n_j \rangle \right) = \int_{\mathcal{C}} t \left( \lambda, \frac{\eta}{2} \right) \varepsilon(\lambda) d\lambda - \frac{\mu_c}{2},$$

(2.22)

where $\langle n_j \rangle$ is the particle density given by

$$\langle n_j \rangle = \int_{\mathcal{C}} \rho_{\text{tot}}(\lambda) d\lambda.$$

(2.23)

Note that the above expression of the ground state energy (2.22), the distribution function (2.19) and the dressed energy (2.21) are exactly the same as those for the XXZ chain [1, 2].

For later convenience, let us define the inhomogeneous distribution function $\rho(\lambda, \xi)$ [7] as the solution of the integral equation:

$$-2\pi i \rho(\lambda, \xi) + \int_{\mathcal{C}} K(\lambda - \mu) \rho(\mu, \xi) d\mu = t(\lambda, \xi).$$

(2.24)

Correspondingly, the inhomogeneous total distribution function $\rho_{\text{tot}}(\lambda, \{\xi\})$ is defined by

$$\rho_{\text{tot}}(\lambda, \{\xi\}) = \frac{1}{M} \sum_{m=1}^{M} \rho(\lambda, \xi_m),$$

(2.25)

which reduces to $\rho_{\text{tot}}(\lambda)$ (2.19), when one takes the homogeneous limit $\xi_m \to \eta/2$.

In particular, at zero chemical potential $\mu_c = 0$ (corresponding to the half filling case $\langle n_j \rangle = 1/2$), one has

$$\rho(\lambda, \xi) = \begin{cases} 
\frac{1}{2\pi} \frac{1}{\text{ch}(\lambda + \frac{\eta}{2} - \xi)} & \text{for } |\Delta| < 1, \zeta = i\eta \\
\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\varepsilon^{2n}(\lambda + \frac{\eta}{2} - \xi)}{\text{ch}(n\eta)} & \text{for } \Delta > 1, q = e^{\eta},
\end{cases}$$

(2.26)

where $\vartheta_n(\lambda, q)$ is the Jacobi theta function.

### 3 Scalar products and form factors

#### 3.1 Scalar Products

Our main aim is to evaluate the correlation function of the spinless fermion model (2.1):

$$\langle \mathcal{O}_1^\dagger \mathcal{O}_{m+1} \rangle = \frac{\langle \psi_g \mid \mathcal{O}_1^\dagger \mathcal{O}_{m+1} \mid \psi_g \rangle}{\langle \psi_g \mid \psi_g \rangle},$$

(3.1)

where $|\psi_g\rangle$ is the ground state constructed by substituting an appropriate solution $\{\lambda\}$ of the Bethe ansatz equation (2.16) into (2.15), and $\langle \psi_g \|$ is the dual state: $\langle \psi_g \| = \langle \Omega | \prod_{j=1}^{N} C(\lambda_j)$. In
this paper we concentrate on the one-particle Green’s function \( \langle O_j = c_j, c_j^\dagger \rangle \) and the density-density correlation function \( \langle O_j = n_j \rangle \). Here and in what follows, we consider the “inhomogeneous” model (2.8) for convenience. The correlation functions of the original Hamiltonian (2.1) or (2.10) can be obtained by taking the homogeneous limit \( \xi_m \rightarrow \eta/2 \).

To apply the algebraic Bethe ansatz to the computation of (3.1), we must express local fermionic operators in terms of elements of the monodromy operators (2.8). In fact, such an expression has already been obtained by solving the quantum inverse scattering problem for local fermion operators [16].

**Theorem 3.1** [16] For the inhomogeneous spinless fermion model (2.8) with arbitrary inhomogeneous parameters \( \xi_m \ (m = 1, \ldots M) \), the local fermionic operators acting on the \( j \)th space can be expressed in terms of the elements of the fermion monodromy operator as

\[
c_j^\dagger = \prod_{k=1}^{j-1} T(\xi_k) B(\xi_j) \prod_{k=1}^{j} T^{-1}(\xi_k), \quad c_j = \prod_{k=1}^{j-1} T(\xi_k) C(\xi_j) \prod_{k=1}^{j} T^{-1}(\xi_k),
\]

\[
n_j = -\prod_{k=1}^{j-1} T(\xi_k) D(\xi_j) \prod_{k=1}^{j} T^{-1}(\xi_k), \quad 1 = \prod_{k=1}^{M} T(\xi_k).
\]

To proceed the evaluation by use of the theorem 3.1, one needs to calculate the action of the operators \( A, B, C \) and \( D \) on an arbitrary state \( \langle \psi \rangle = \langle \Omega | \prod_{j=1}^{N} C(\lambda_j) \rangle \), where \( \{ \lambda \} \) is a set of arbitrary complex numbers (not necessary the Bethe roots). After simple but tedious computation utilizing (2.12), one obtains

\[
\langle \Omega | \prod_{j=1}^{N} C(\lambda_j) A(\lambda_{N+1}) = \sum_{b=1}^{N+1} a(\lambda_b) \frac{\prod_{j=1}^{N+1} \text{sh}(\lambda_b - \lambda + \eta)}{\prod_{j=1}^{N+1} \text{sh}(\lambda_j - \lambda_b)} \langle \Omega | \prod_{j=1}^{N+1} C(\lambda_j),
\]

\[
\langle \Omega | \prod_{j=1}^{N} C(\lambda_j) D(\lambda_{N+1}) = (-1)^N \sum_{a=1}^{N+1} d(\lambda_a) \frac{\prod_{j=1}^{N} \text{sh}(\lambda_a - \lambda_j + \eta)}{\prod_{j=1}^{N} \text{sh}(\lambda_j - \lambda_a)} \langle \Omega | \prod_{j=1}^{N+1} C(\lambda_j),
\]

\[
\langle \Omega | \prod_{j=1}^{N} C(\lambda_j) B(\lambda_{N+1}) = (-1)^{N-1} \sum_{a=1}^{N+1} d(\lambda_a) \frac{\prod_{k=1}^{N+1} \text{sh}(\lambda_a - \lambda_k + \eta)}{\prod_{k=1}^{N+1} \text{sh}(\lambda_k - \lambda_a)}
\]

\[
\times \sum_{a', a' \neq a} \frac{a(\lambda_{a'})}{\prod_{j=1}^{N+1} \text{sh}(\lambda_{N+1} - \lambda_{a'} + \eta)} \langle \Omega | \prod_{j=1}^{N+1} C(\lambda_j).
\]

Note that the extra sign factors \((-1)^N\) and \((-1)^{N+1}\) appearing in the last two relations reflect the fermionic nature of the system (cf. (3.7) and (3.8) in [7]). Finally the action of the operator \( C(\lambda) \) is free.

In the above, one distinguishes the terms containing \( a(\lambda_{N+1}) \) or \( d(\lambda_{N+1}) \) and the others, which are referred to as the direct and indirect terms, respectively. Due to the existence of indirect terms, the explicit form of the scalar product \( \langle \Omega | \prod_{j=1}^{N} C(\mu_j) \prod_{j=1}^{N} B(\lambda_j) \rangle | \Omega \rangle \) is required to compute the expectation value (3.1), where \( \{ \lambda_j \}_{j=1}^{N} \) are solutions of the Bethe
ansatz equation and \( \{\mu_j\}_{j=1}^{N} \) are arbitrary complex numbers. This can be explicitly evaluated in the same way as in \([1, 17]\). The difference between the present system and the XXZ chain stems from the difference of the commutation relations (2.12). Here we write down only the result.

**Proposition 3.1** The scalar product between a Bethe state and an arbitrary state

\[
S_N(\{\mu\}|\{\lambda\}) = \langle \Omega | \prod_{j=1}^{N} C(\mu_j) \prod_{j=1}^{N} B(\lambda_j) | \Omega \rangle
\]  

(3.4)

can be expressed as follows:

\[
S_N(\{\mu\}|\{\lambda\}) = S_N(\{\lambda\}|\{\mu\}) = (-1)^{N(N-1)/2} \frac{\prod_{a,b=1}^{N} \text{sh}(\lambda_a - \mu_b + \eta) \prod_{a<\beta} \text{sh}(\lambda_a - \lambda_b) \text{sh}(\mu_b - \mu_a)}{\prod_{a,b=1}^{N} \text{sh}(\lambda_a - \lambda_b) \text{sh}(\mu_b - \mu_a)} \det_N \Psi(\{\mu\}|\{\lambda\}),
\]

(3.5)

where \( \{\lambda_j\}_{j=1}^{N} \) are Bethe roots, \( \{\mu_j\}_{j=1}^{N} \) are arbitrary complex parameters. The \( N \times N \) matrix \( \Psi(\{\mu\}|\{\lambda\}) \) is defined by

\[
\Psi_{jk}(\{\mu\}|\{\lambda\}) = t(\lambda_j, \mu_k) - (-1)^{N+1} d(\mu_k) t(\mu_k, \lambda_j) \prod_{a=1}^{N} \frac{\text{sh}(\mu_k - \lambda_a + \eta)}{\text{sh}(\mu_k - \lambda_a - \eta)}.
\]

(3.6)

and \( \det_N \) denotes the determinant of an \( N \times N \) matrix.

Reflecting the fermion statistics, the extra factor \((-1)^{N+1}\) enters into the above expression. This factor, however, is eliminated by that in the Bethe ansatz equation (2.16), when one takes the limit \( \{\mu\} \to \{\lambda\} \), which gives the square of the norm of the Bethe vector:

\[
S_N(\{\lambda\}|\{\lambda\}) = (-1)^{N(N-1)/2} \prod_{a,b=1}^{N} \frac{\text{sh}(\lambda_a - \lambda_b + \eta)}{\text{sh}(\lambda_a - \lambda_b - \eta)} \det_N \Phi(\{\lambda\}),
\]

(3.7)

where

\[
\Phi_{jk}(\{\lambda\}) = \delta_{jk} \left[ d''(\lambda_j) \frac{d(\lambda_j)}{d(\lambda_j)} - \sum_{a=1}^{N} K(\lambda_j - \lambda_a) \right] + K(\lambda_j - \lambda_k).
\]

(3.8)

Set \( \{\mu\} = \{\xi_1, \ldots, \xi_n\} \cup \{\lambda_{n+1}, \ldots, \lambda_N\} \) in (3.5). Then the ratio of the determinants of \( \Psi \) and \( \Phi \) can be evaluated in the thermodynamic limit:

\[
\frac{\det_N \Psi}{\det_N \Phi} = \prod_{a=1}^{n} (M \rho_{\text{tot}}(\lambda_a, \{\xi\}))^{-1} \det_n \rho(\lambda_j, \xi_k),
\]

(3.9)

where \( \rho(\lambda, \xi) \) and \( \rho_{\text{tot}}(\lambda, \{\xi\}) \) are defined as (2.24) and (2.25), respectively.

### 3.2 Form factors

Utilizing the scalar product obtained in the preceding subsection, we derive determinant representations of the form factors for a finite system. These formulae might be useful in calculating the spectral functions of the present model.
The form factors of the local fermion operators are defined by

\[ F_N^{-}(m|\{\mu\}|\{\lambda\}) = \langle \Omega | \prod_{j=1}^{N+1} C(\mu_j) c_m^\dagger \prod_{k=1}^{N} B(\lambda_k)|\Omega \rangle, \]

\[ F_N^{+}(m|\{\lambda\}|\{\mu\}) = \langle \Omega | \prod_{k=1}^{N} C(\lambda_k)c_m \prod_{j=1}^{N+1} B(\mu_j)|\Omega \rangle, \]

\[ F_N^{x}(m|\{\mu\}|\{\lambda\}) = \langle \Omega | \prod_{j=1}^{N} C(\mu_j)(1-2m_j) \prod_{k=1}^{N} B(\lambda_k)|\Omega \rangle, \quad (3.10) \]

where \( \{\lambda_k\}_{k=1}^{N} \) and \( \{\mu_j\}_{j=1}^{N+1} \) (or \( \{\mu_j\}_{j=1}^{N} \)) are solutions of the Bethe ansatz equation (2.16). To evaluate them, first we substitute solutions of the quantum inverse scattering problem for local fermion operators (3.2) into the above expressions. For instance, \( F_N^{x}(m|\{\mu\}|\{\lambda\}) \) can be written as

\[ F_N^{x}(m|\{\mu\}|\{\lambda\}) = 2\langle \Omega | \prod_{j=1}^{N} C(\mu_j) \prod_{l=1}^{m-1} T(\xi_l) A(\xi_m) \prod_{l=m+1}^{M} T(\xi_l) \prod_{k=1}^{N} B(\lambda_k)|\Omega \rangle - S_N(|\{\mu\}|\{\lambda\}). \quad (3.11) \]

Then, noting that the states \( \prod_{k=1}^{N} B(\lambda_k)|\Omega \rangle \) and \( \langle \Omega | \prod_{j=1}^{N} C(\mu_j) \) are the Bethe states, we remove the products of the transfer matrices by (2.17). Finally applying the determinant representation of the scalar product (3.5), and using a simple identity \( \prod_{j=1}^{N} \prod_{k=1}^{M} f(\lambda_j, \xi_k) = 1 \) derived by the Bethe ansatz equation (2.16), we obtain the determinant representations of the form factors. They are summarized into the following proposition.

**Proposition 3.2** The form factors \( F_N^{-}(m|\{\mu\}|\{\lambda\}) \) and \( F_N^{+}(m|\{\lambda\}|\{\mu\}) \) for local fermion operators \( c_m^\dagger \) and \( c_m \) are respectively given by the following determinant representations:

\[ F_N^{-}(m|\{\mu\}|\{\lambda\}) = (-1)^{N(N+1)} \begin{array}{c} \phi_{m-1}^N(\{\mu\}) \prod_{j=1}^{N+1} \text{sh}(\mu_j - \xi_m + \eta) \\ \phi_{m}^N(\{\lambda\}) \prod_{k=1}^{N} \text{sh}(\lambda_k - \xi_m + \eta) \\ \text{det}_{N+1} H^{-}(m|\{\mu\}|\{\lambda\}) \end{array} \]

\[ \times \prod_{j<k}^{N+1} \text{sh}(\mu_j - \mu_j) \prod_{\alpha<\beta}^{N} \text{sh}(\lambda_\alpha - \lambda_\beta), \]

\[ F_N^{+}(m|\{\lambda\}|\{\mu\}) = \phi_{m}^N(\{\lambda\}) \phi_{m-1}^N(\{\mu\}) \frac{\phi_{m+1}^N(\{\mu\}) \phi_{m+1}^N(\{\lambda\})}{\phi_{m-1}^N(\{\mu\}) \phi_{m+1}^N(\{\lambda\})} F_N^{-}(m|\{\mu\}|\{\lambda\}), \quad (3.12) \]

where the coefficient \( \phi_{m}^N(\{\lambda\}) \) is

\[ \phi_{m}^N(\{\lambda\}) = \prod_{j=1}^{N} \prod_{k=1}^{m} f(\lambda_j, \xi_k), \quad (3.13) \]

and the \( (N+1) \times (N+1) \) matrix \( H^{-}(m|\{\mu\}|\{\lambda\}) \) is defined by

\[ H^{-}_{jk} = \frac{\text{sh} \eta}{\text{sh}(\mu_j - \lambda_k)} \left[ \prod_{a=1}^{N+1} \text{sh}(\mu_a - \lambda_k + \eta) - (-1)^N \text{d}(\lambda_k) \prod_{a=1}^{N+1} \text{sh}(\mu_a - \lambda_k - \eta) \right] \quad \text{for} \quad k \leq N, \]

\[ H^{-}_{jN+1} = t(\mu_j, \xi_m). \quad (3.14) \]
On the other hand, the form factor $F_N^\pm(m|\mu\rangle\langle\lambda|)$ for the operator $1 - 2n_m$ is given by

\[
F_N^\pm(m|\mu\rangle\langle\lambda|) = (-1)^{N(N-1)} \frac{\phi^{N-1}_{m-1}(\mu)}{\phi^{N-1}_{m-1}(\lambda)} \prod_{j=1}^{N} \frac{\text{sh}(\mu_j - \xi_m + \eta)}{\text{sh}(\lambda_j - \xi_m + \eta)} \times \frac{\det(H(\{(\mu)\}(\lambda)))) - 2P(m|\mu\rangle\langle\lambda|))}{\prod_{j<k}^{N} \text{sh}(\lambda_j - \lambda_k) \text{sh}(\mu_k - \mu_j)},
\]

where $H(\{(\mu)\}(\lambda))$ is an $N \times N$ matrix defined as

\[
H_{jk} = \frac{\text{sh} \eta}{\text{sh}(\mu_j - \lambda_k)} \left[ \prod_{a \neq j} \text{sh}(\mu_a - \lambda_k + \eta) - (-1)^{N+1} \text{sh}(\lambda_k) \prod_{a \neq j} \text{sh}(\mu_a - \lambda_k - \eta) \right],
\]

and $P(m|\mu\rangle\langle\lambda|$) is an $N \times N$ matrix of rank one,

\[
P_{jk}(m) = t(\mu_j,\xi_m) \prod_{a=1}^{N} \text{sh}(\lambda_a - \lambda_k + \eta).
\]

Here we have set $a(\lambda) = 1$. Note that, in the derivation of $F_N^\pm(m|\mu\rangle\langle\lambda|)$, we have used the orthogonality of the Bethe vectors, and the formula that the determinant of the sum of an arbitrary $N \times N$ matrix $A$ and an $N \times N$ matrix of rank one $B$ can be given by

\[
det(A + B) = detA + \sum_{l=1}^{N} \det A^{(l)} \quad A^{(l)}_{jk} = \begin{cases} A_{jk} & \text{for } k \neq l \\ B_{jk} & \text{for } k = l \end{cases}.
\]

Comparing the form factor for the local spin operator of the third component $\sigma^z_m$ in the XXZ chain [18], one finds that the extra factor $(-1)^{N+1}$ appears in (3.16). This factor, however, is canceled by that in the Bethe ansatz equation (2.16). Hence, in the thermodynamic limit where the distribution of the Bethe roots (2.26) characterizing the ground state is the same as that of the XXZ chain, $F_N^\pm(m|\mu\rangle\langle\lambda|)$ (3.15) coincides with the form factor of the $\sigma^z_m$ for the XXZ chain. In contrast to this, for the form factor $F_N^\pm(m|\mu\rangle\langle\lambda|)$ (3.12), the factor $(-1)^N$ in (3.14) can not be eliminated by the Bethe ansatz equation (2.16). Namely, compared with the form factors of the $\sigma^z_m$ for the XXZ chain, the extra minus sign remains in (3.14). This essential difference caused by the statistics between the fermion and the spin induces the different behavior of the correlation functions $\langle c_1 c_{m+1}^\dagger \rangle$ and $\langle \sigma^+_1 \sigma^-_{m+1} \rangle$.

4 Multiple integral representations for correlation functions

In this section, we derive multiple integrals representing the one-particle Green’s function and the density-density correlation function. Inserting the expression of the local fermion operators $c_j^\dagger$ and $c_j$ (3.2) into (3.1), one obtains

\[
\langle c_1 c_{m+1}^\dagger \rangle = \frac{\langle \psi_g|C(\xi_1) \prod_{a=2}^{m} (A - D)(\xi_a) B(\xi_{m+1}) \prod_{k=1}^{m+1} (A - D)^{-1}(\xi_k)|\psi_g\rangle}{\langle \psi_g|\psi_g \rangle}.
\]

\[\text{Note that the overall factor } (-1)^{N(N-1)/2} \text{ in (3.15) and (3.12) can be eliminated by the normalization of the Bethe vector (see (3.7))}.\]
Note that \(\langle c_1^\dagger c_{m+1}\rangle = -\langle c_1 c_{m+1}\rangle\).

Following [7], for the density-density correlation function, we conveniently introduce an operator \(Q_{1,m}\) as \(Q_{1,m} = \sum_{k=1}^{m} n_k\) and consider the expectation value of the generating function \(\exp(\beta Q_{1,m}) (\beta \in \mathbb{C})\):

\[
\langle \exp(\beta Q_{1,m}) \rangle = \frac{\langle \psi_g | \prod_{a=1}^{m} (A - e^{\beta}D)(\xi_a) \prod_{b=1}^{m} (A - D)^{-1}(\xi_b) | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle}.
\] (4.2)

Then the density-density correlation can be expressed as

\[
\langle n_1 n_{m+1} \rangle = \left. \frac{1}{2} \mathcal{D}_m^2 \frac{\partial^2}{\partial \beta^2} \langle \exp(\beta Q_{1,m}) \rangle \right|_{\beta = 0},
\] (4.3)

where \(\mathcal{D}_m\) denotes the lattice derivative defined by \(\mathcal{D}_m f(m) = f(m+1) - f(m)\) and \(\mathcal{D}_m^2 f(m) = f(m + 1) - 2f(m) + f(m - 1)\). In fact, there exist several manners to express the density-density correlation by the generating function [7]. For instance, \(\langle n_1 n_{m+1} \rangle\) is also written by using \(\langle \exp(\beta Q_{1,m}) n_{m+1} \rangle\) as

\[
\langle n_1 n_{m+1} \rangle = \left. \mathcal{D}_m \frac{\partial}{\partial \beta} \langle \exp(\beta Q_{1,m}) n_{m+1} \rangle \right|_{\beta = 0},
\]

\[
\langle \exp(\beta Q_{1,m}) n_{m+1} \rangle = -\frac{\langle \psi_g | \prod_{a=1}^{m} (A - e^{\beta}D)(\xi_a)D(\xi_{m+1}) \prod_{b=1}^{m+1} (A - D)^{-1}(\xi_b) | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle}.
\] (4.4)

To evaluate the correlation functions (4.1)–(4.4), we must calculate the multiple action of \((A - e^{\beta}D)(x)\) on an arbitrary state \(\langle \psi = |\Omega\rangle \prod_{j=1}^{N} C(\lambda_j)\), where \(\{\lambda\}\) are arbitrary complex numbers. Utilizing the commutation relations among \(A\), \(D\) and \(C\) (2.12), one finds that the action on a general state can be written as

\[
\langle \psi | \prod_{a=1}^{m} (A - e^{\beta}D)(x_a) \rangle = \sum_{n=0}^{p} \sum_{|\lambda^+| = |x^+| = n} R_n(\{x^+\}|\{x^-\}|\{\lambda^+\}|\{\lambda^-\}) \langle \Omega | \prod_{a=1}^{n} C(x_a^+) \prod_{b=1}^{N-n} C(\lambda_b^-) \rangle,
\] (4.5)

where \(|\lambda^+|, |x^+|\), etc denote the number of elements of \(\{\lambda^+\}, \{x^+\}\), etc; \(p = \min(m, N)\).

**Proposition 4.1** The coefficient \(R_n(\{x^+\}|\{x^-\}|\{\lambda^+\}|\{\lambda^-\})\) in (4.5) is given by

\[
R_n(\{x^+\}|\{x^-\}|\{\lambda^+\}|\{\lambda^-\}) = S_n(\{x^+\}|\{\lambda^+\}|\{\lambda^-\}) \prod_{a=1}^{m-n} \left[ a(x_a^-) \prod_{b=1}^{n} f(x_a^+, x_b^-) \prod_{b=1}^{N-n} f(\lambda_b^-, x_b^-) \right] - e^{\beta}d(x_a^-) \prod_{b=1}^{n} \{- f(x_a^+, x_b^-)\} \prod_{b=1}^{N-n} \{- f(x_a^-, \lambda_b^-)\},
\] (4.6)

where the highest coefficient is expressed as

\[
S_n(\{x^+\}|\{\lambda^+\}|\{\lambda^-\}) = \left. \frac{\prod_{a,b=1}^{m} \text{sh}(x_a^+ - \lambda_b^- + \eta)}{\prod_{a < b} \text{sh}(\lambda_a^+ - \lambda_b^+) \text{sh}(x_a^+ - x_b^+)} \right| \det_n M_{jk}
\] (4.7)
with
\[ M_{jk} = a(\lambda_j^+) t(x_k^+, \lambda_j^+) \prod_{a=1}^{N-n} f(\lambda_a^+, \lambda_j^+) \]
\[ + e^\beta d(\lambda_j^+) t(\lambda_j^+, x_k^+) \prod_{a=1}^{N-n} \{ -f(\lambda_j^+, \lambda_a^-) \} \prod_{b=1}^{n} \left\{ -\frac{\text{sh}(\lambda_j^+ - x_b^+ + \eta)}{\text{sh}(\lambda_j^+ - x_b^+ - \eta)} \right\}. \] (4.8)

The proof is completely parallel to that in the XXZ chain [7]. The difference of the expression stems from the difference of the commutation relation of \( C \) and \( D \), which reflects the fermionic nature of the present system.

If \( |\psi\rangle \) is a Bethe vector (or equivalently the eigenstate of the transfer operator (2.9)), one further has the following expression which is directly applicable to the computation of the generating function (4.2).

**Corollary 4.1** If \( \langle \psi | = \langle \Omega | \prod_{j=1}^{N} C(\lambda_j) \) is a Bethe vector, namely if \( \{ \lambda \} \) are the solutions to the Bethe ansatz equation (2.16), the coefficient \( R_n(\{\xi^+\}|\{\xi^-\} |\{\lambda^+\} |\{\lambda^-\}) \) of the action \( \prod_{a=1}^{m} (A - e^\beta D)(\xi_a) \) is given by
\[ R_n(\{\xi^+\}|\{\xi^-\} |\{\lambda^+\} |\{\lambda^-\}) = \tilde{S}_n(\{\xi^+\}|\{\lambda^+\}) \]
\[ \times \prod_{a=1}^{n} \prod_{b=1}^{m-n} f(\xi_a^+, \xi_b^-) \prod_{a=1}^{N-n} \prod_{b=1}^{m-n} f(\lambda_a^-, \xi_b^-) \prod_{a=1}^{N-n} \prod_{b=1}^{n} f(\lambda_a^-, \lambda_b^+), \] (4.9)

where
\[ \tilde{S}_n(\{\xi^+\}|\{\lambda^+\}) = \frac{\prod_{a,b=1}^{n} \text{sh}(\xi_a^+ - \lambda_b^+) \eta}{\prod_{a,b=1}^{n} \text{sh}(\lambda_b^+ - \lambda_a^+) \text{sh}(\xi_a^+ - \xi_b^+)} \text{det}_n \tilde{M}_{jk}. \] (4.10)

Here the \( n \times n \) matrix \( \tilde{M} \) is defined as
\[ \tilde{M}_{jk} = t(\xi_k^+, \lambda_j^+) + e^\beta t(\lambda_j^+, \xi_k^+) \prod_{a=1}^{n} \left[ \frac{\text{sh}(\lambda_a^+ - \lambda_j^+ + \eta) \text{sh}(\lambda_j^+ - \xi_a^+ + \eta)}{\text{sh}(\lambda_j^+ - \lambda_a^+ + \eta) \text{sh}(\xi_a^+ - \lambda_j^+ + \eta)} \right]. \] (4.11)

Note that here we have set \( a(\lambda) = 1 \) and \( d(\xi) = 0 \), and used the fact that the parameters \( \{ \lambda \} \) satisfy the Bethe ansatz equation (2.16).

Now we would like to compute the correlation functions. First let us consider the generating function (4.2) of the density-density correlation \( \langle n_1 n_{m+1} \rangle \). Noting that \( |\psi_\beta\rangle \) is the Bethe vector characterizing the ground state, one finds that the numerator in (4.2) can be expressed by \( \langle \psi_\beta | \prod_{a=1}^{n} (A - e^\beta D)(\xi_a) |\psi_\beta\rangle / \prod_{a=1}^{m} \tau(\xi_a) \), where \( \tau(\lambda) \) is the corresponding eigenvalue of the transfer matrix (see (2.17)). Then from Corollary 4.1, Proposition 3.1 and the norm of the Bethe vector (3.7), the generating function (4.2) is given by a determinant form. By \( d(\xi) = 0 \), it immediately follows that the determinant form is the same as that for the generating function of the longitudinal spin-spin correlation function \( \langle \sigma_i^+ \sigma_{m+1}^z \rangle \) [7]. Thus utilizing the method proposed in [7], we arrive at the following proposition.
Proposition 4.2 In the thermodynamic limit $M \to \infty$, the ground state expectation value of the generating function $\exp(\beta Q_{1,m})$ is expressed as

$$
\langle \exp(\beta Q_{1,m}) \rangle = \sum_{n=0}^{m} \frac{1}{(n!)^2} \int_{\mathcal{C}} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \prod_{a=1}^{n} \prod_{b=1}^{m} \frac{\text{sh}(z_a - \xi_b + \eta) \text{sh}(\lambda_a - \xi_b)}{\text{sh}(z_a - \xi_b) \text{sh}(\lambda_a - \xi_b + \eta)}
\times W_n(\{\lambda\}|\{z\}) \det_n(\tilde{M}_{jk}(\{\lambda\}|\{z\})) \det_n[\rho(\lambda_j, z_k)],
$$

where

$$
W_n(\{\lambda\}|\{z\}) = \prod_{a=1}^{n} \prod_{b=1}^{n} \frac{\text{sh}(\lambda_a - z_b + \eta) \text{sh}(z_b - \lambda_a + \eta)}{\text{sh}(\lambda_a - \lambda_b + \eta) \text{sh}(z_a - z_b + \eta)},
$$

and

$$
\tilde{M}_{jk}(\{\lambda\}|\{z\}) = t(z_k, \lambda_j) + e^{\beta} t(\lambda_j, z_k) \prod_{a=1}^{n} \frac{\text{sh}(\lambda_a - \lambda_j + \eta) \text{sh}(\lambda_j - z_a + \eta)}{\text{sh}(\lambda_j - \lambda_a + \eta) \text{sh}(z_a - \lambda_j + \eta)}. \tag{4.14}
$$

The contour $\mathcal{C}$ depends on the regime and the magnitude of the chemical potential, while $\Gamma$ encircles $\{\xi\}_{j=1}^{m}$ and does not contain any other singularities. In the above expression, the homogeneous limit $\xi_j = \eta/2$ can be taken trivially. In this case, $\Gamma$ surrounds the point $\eta/2$.

Note that here to replace the sums over the partition of the set $\{\xi\}$ with the set of contour integrals around $\{\xi\}$, we have used the following relation:

$$
\sum_{\{\xi\} = \{\xi^+\} \cup \{\xi^-\}} \prod_{a=1}^{n} \prod_{b=1}^{m-n} f(\xi^+_a; \xi^-_b) \mathcal{F}(\{\xi^+\})
\quad = \frac{1}{n!} \int_{\mathcal{C}} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \prod_{a=1}^{n} \prod_{b=1}^{m-n} f(z_a, \xi_b) \prod_{a=1}^{n} \prod_{b=1}^{m-n} \frac{\text{sh}(z_a - z_b)}{\text{sh}(z_a - z_b + \eta)} \mathcal{F}(\{z\}), \tag{4.15}
$$

where $\mathcal{F}(\{z\})$ is assumed to be a symmetric function of $n$ variables $\{z\}$, and to be analytic in the vicinities of $z_j = \xi_k$. As for the sum over the partition of the Bethe roots $\{\lambda\}$ (2.16) characterizing the ground state, we have used the following relation, which is valid in the thermodynamic limit:

$$
\lim_{M \to \infty} \frac{1}{M^n} \sum_{\{\lambda\} = \{\lambda^+\} \cup \{\lambda^-\}} \mathcal{F}(\{\lambda^+\}) = \frac{1}{n!} \int_{\mathcal{C}} d^n \lambda \prod_{j=1}^{n} \rho_{\text{tot}}(\lambda_j, \{\xi\}) \mathcal{F}(\{\lambda\}), \tag{4.16}
$$

where $\mathcal{F}(\{\lambda\})$ is assumed to be a symmetric function of $n$ variables $\{\lambda\}$, and to be zero when any two of its variables are the same.

The multiple integral (4.12) representing the generating function of the density-density correlation function is exactly the same as that for the longitudinal spin-spin correlation function [7]. The density-density correlation function $\langle n_1 n_{m+1} \rangle$ can be obtained by taking the derivative with respect to $\beta$ and $m$ as in (4.3).
As for the generating function \( \langle \exp(\beta Q_{1,m}) \rangle_{n_{m+1}} \) in (4.4), the numerator can be written as \( \langle \psi_\lambda | \prod_{a=1}^{m} (A - e^{\beta} D)(\xi_a) D(\xi_{m+1}) | \psi_\lambda \rangle / \prod_{a=1}^{m} \tau(\xi_a) \). Applying Corollary 4.1, and then using the second relation in (3.3), one can obtain a determinant form of the generating function. By the same method as in the derivation of (4.12), the following multiple integral representing \( \langle \exp(\beta Q_{1,m}) \rangle_{n_{m+1}} \) can be derived in the thermodynamic limit.

**Proposition 4.3** The ground state expectation value of the operator \( \langle \exp(\beta Q_{1,m}) \rangle_{n_{m+1}} \) in the thermodynamic limit is written as

\[
\langle \exp(\beta Q_{1,m}) \rangle_{n_{m+1}} = \sum_{n=0}^{m} \frac{1}{(n!)^2} \int_{\Gamma} \prod_{j=1}^{n} dz_j \int_{\mathcal{C}} \prod_{a=1}^{m} \prod_{b=1}^{m} \frac{d\lambda}{2\pi i} \frac{d\lambda}{d\lambda_{m+1}} \frac{sh(z_a - \xi_b + \eta) \cdot sh(\lambda_a - \xi_b)}{sh(z_a - \xi_b) \cdot sh(\lambda_a - \xi_b + \eta)}
\]

\[
\times \prod_{a=1}^{n} \frac{sh(\lambda_a - \xi_{m+1})}{sh(z_a - \xi_{m+1})} \prod_{a=1}^{n} \frac{sh(\lambda_{m+1} - \xi_a + \eta)}{sh(\lambda_{m+1} - \lambda_a + \eta)} W_n(\{\lambda\} | \{z\}) \times \det_n[\widetilde{M}_{jk}(\{\lambda\} | \{z\})] \det_{n+1}[\rho(\lambda_j, \xi_k), \ldots, \rho(\lambda_{j}, \xi_{m+1})].
\]

(4.17)

Here the integration contours \( \Gamma \) and \( \mathcal{C} \) are taken as in Proposition 4.2. In this representation, the homogeneous limit \( \xi_j = \eta/2 \) can be trivially taken.

Note that the above expression agrees with that for \( \langle \exp(\beta Q_{1,m}(1-\sigma^z_{m+1})/2) \rangle \) [7], as expected.

Finally we would like to consider the equal-time one particle Green’s function \( \langle c_1^\dagger c_{m+1} \rangle \) (4.1). Because the operators \( B \) and \( C \) enter into the numerator in (4.1), the computation is more complicated than that for the generating functions of the density-density correlation function. Nevertheless, applying Proposition 4.1 and then utilizing the third relation in (3.3), one finds the method used above is still applicable for the evaluation of \( \langle c_1^\dagger c_{m+1} \rangle \).

**Proposition 4.4** The equal-time one-particle Green’s function \( \langle c_1^\dagger c_{m+1} \rangle \) in the thermodynamic limit can be represented as a multiple integral as

\[
\langle c_1^\dagger c_{m+1} \rangle = -\sum_{n=0}^{m-1} \frac{1}{n!(n+1)!} \int_{\Gamma} \prod_{j=1}^{n} dz_j \int_{\mathcal{C}} \prod_{a=1}^{n+1} \prod_{b=1}^{n+1} \frac{d\lambda}{2\pi i} \frac{d\lambda}{d\lambda_{m+2}} \frac{sh(z_a - \xi_b + \eta)}{sh(z_a - \xi_b)}
\]

\[
\times \prod_{a=1}^{n+1} \prod_{b=2}^{n+1} \frac{sh(\lambda_a - \xi_b)}{sh(\lambda_a - \xi_b + \eta)} \prod_{a=1}^{n+1} \prod_{b=1}^{n+1} \frac{sh(\lambda_{m+2} - \lambda_a + \eta)}{sh(\lambda_{m+2} - \lambda_a - \eta)}
\]

\[
\times \frac{\widehat{W}_n(\{\lambda\} | \{z\})}{sh(\lambda_{m+1} - \lambda_{m+2} + \eta)} \det_{n+1} \widetilde{M}_{jk} \det_{n+2}[\rho(\lambda_j, z_1), \ldots, \rho(\lambda_{j}, \xi_{m+1}), \rho(\lambda_{j}, \xi_{m+1})],
\]

(4.18)

where the function \( \widehat{W}_n(\{\lambda\} | \{z\}) \) and the \( (n+1) \times (n+1) \) matrix are respectively defined as

\[
\widehat{W}_n(\{\lambda\} | \{z\}) = \frac{\prod_{a=1}^{n+1} \prod_{b=1}^{n+1} sh(\lambda_a - \xi_b + \eta) sh(z_b - \lambda_a + \eta)}{\prod_{a=1}^{n+1} \prod_{b=1}^{n+1} sh(\lambda_a - \lambda_b + \eta) \prod_{a=1}^{n+1} \prod_{b=1}^{n+1} sh(z_a - z_b + \eta)},
\]

(4.19)

and

\[
\widetilde{M}_{jk} = \begin{cases} 
  t(z_k, \lambda_j) + t(\lambda_j, z_k) \prod_{a=1}^{n} sh(\lambda_a - \lambda_j + \eta) \prod_{b=1}^{n+1} sh(\lambda_{j} - \xi_b + \eta) & \text{for } j \leq n \\
  t(z_k, \xi_1) & \text{for } j = n+1.
\end{cases}
\]

(4.20)
Corollary 4.2

The equal-time one-particle Green’s function for the homogeneous case at zero chemical potential can be expressed as

\[ \langle c_1 c_{m+1}^\dagger \rangle = \sum_{n=0}^{m-1} \frac{1}{n!(n+1)!} \int_{\Gamma} \prod_{j=1}^{n+1} \frac{dz_j}{2\pi i} \int_{C} d^{n+2} \lambda \prod_{a=1}^{n+1} \left( \frac{\text{sh}(z_a + \frac{\eta}{2})}{\text{sh}(z_a - \frac{\eta}{2})} \right)^m \]

\[ \times \prod_{a=1}^{n+1} \left( \frac{\text{sh}(\lambda_a - \frac{\eta}{2})}{\text{sh}(\lambda_a + \frac{\eta}{2})} \right)^m \prod_{a=1}^{n+1} \left[ \text{sh}(\lambda_{n+1} + \lambda_a + \eta) \text{sh}(\lambda_{n+2} - z_a) \right] \prod_{a=1}^{n+1} \left[ \text{sh}(\lambda_{n+1} - \lambda_a - \eta) \text{sh}(\lambda_{n+2} - \lambda_a) \right] \]

\[ \times \frac{\hat{W}_m(\{\lambda\}, \{z\})}{\text{sh}(\lambda_{n+1} - \lambda_{n+2})} \text{det}_{n+1} \hat{M}_{jk} \text{det}_{n+2} [\rho(\lambda_j, z_1), \ldots, \rho(\lambda_j, z_{n+1}), \rho(\lambda_j, \frac{\eta}{2})], \quad (4.21) \]

where \( \hat{M}_{jk} \) is defined as (4.20) for \( j \leq n \) and \( \hat{M}_{jk} = t(z_k, \frac{\eta}{2}) \) for \( j = n+1 \).

Compared with the transverse spin-spin correlation function \( \langle \sigma_1^+ \sigma_{m+1}^- \rangle \) in [7]^3, one finds the sign in front of the second term of \( \hat{M}_{jk} \) for \( j \leq n \) changes.

^3There exists a typo in equation (6.13) in [7] for the transverse spin-spin correlation function \( \langle \sigma_1^+ \sigma_{m+1}^- \rangle \). It should be corrected by multiplying by \(-1\).
5 Free fermion model

Using the multiple integral representations derived in the preceding section, we reproduce the exact expressions of correlation functions for the free fermion model.

Let us consider the density-density correlation function \( \langle n_1 n_{m+1} \rangle \) first. As mentioned before, the generating functions \( \exp(\beta Q_{1,m}) \) (4.12) for \( \langle n_1 n_{m+1} \rangle \) is the same as that for the spin-spin correlation functions \( \sigma_{1}^\pi \sigma_{m+1}^\pi \) for the XXZ chain. Hence the same method described in [19] can be applied to derive the exact expression of \( \langle n_1 n_{m+1} \rangle \). Here we derive it by using (4.17) and (4.4). Set \( \eta = -\pi i/2 \). Then \( M_{jk} \) (4.14) in (4.17) is factorized as \( \hat{M}_{jk} = 2(e^\beta - 1)/\sh(2(\lambda_j - z_k)) \), which significantly simplifies the integral representation.

After taking the derivative of (4.17) with respect to \( \beta \) and setting \( \beta = 0 \), one finds all the terms except for \( n = 1 \) vanish. Substituting \( \rho(\lambda, z) = i/(\pi \sh(2(\lambda - z)) \), and taking the lattice derivative, one obtains

\[
\langle n_1 n_{m+1} \rangle = \frac{i}{2\pi^3} \oint_{\Gamma} dz \int_{\mathcal{C}} d^2\lambda \frac{\sh(\lambda_1 - \lambda_2)\varphi^m(z)\varphi^{-m}(\lambda_1)}{\sh(\lambda_1 - z)\sh(\lambda_2 - z) \ch(2\lambda_1) \ch(2\lambda_2)}. \tag{5.1}
\]

where

\[
\varphi(\lambda) = \frac{\sh(\lambda - \frac{\pi i}{2})}{\sh(\lambda + \frac{\pi i}{2})}. \tag{5.2}
\]

The integral on \( \Gamma \) can be evaluated by considering the residues outside the contour \( \Gamma \) i.e. at the points \( z = \lambda_1 \) and \( \lambda_2 \). It immediately follows that

\[
\langle n_1 n_{m+1} \rangle = \frac{1}{\pi^2} \int_{\mathcal{C}} d^2\lambda \frac{1 - \varphi^m(\lambda)\varphi^{-m}(\lambda)}{\ch(2\lambda_1) \ch(2\lambda_2)}. \tag{5.3}
\]

Changing the variables \( \ch(2\lambda) = -1/\cos k \ (k \in [k_F, 2\pi - k_F]) \), where \( k_F \in [\pi/2, \pi] \) is the Fermi momentum determined by \( \cos k_F = -1/\ch\Lambda_{\mu_c} \) (see section 2), and identifying \( d\lambda = -dk/(2\cos k) \), one arrives at

\[
\langle n_1 n_{m+1} \rangle = \frac{1}{4\pi^2} \int_{k_F}^{2\pi-k_F} dk_1 \int_{k_F}^{2\pi-k_F} dk_2 (1 - e^{-im(k_1-k_2)}) = \langle n_j \rangle^2 - \frac{1 - \cos(2k_F m)}{2\pi^2 m^2} \quad k_F \in [\frac{\pi}{2}, \pi], \tag{5.4}
\]

where \( \langle n_j \rangle = 1 - k_F/\pi \) is the particle density given by (2.23). This expression agrees with the well-known result. At zero chemical potential \( \mu_c = 0 \), where the Fermi momentum corresponds to \( k_F = \pi/2 \), one finds

\[
\langle n_1 n_{m+1} \rangle = \frac{1}{4} - \frac{1 - (-1)^m}{2\pi^2 m^2}. \tag{5.5}
\]

Next let us discuss the one-particle Green’s function \( \langle c_1 c_{m+1}^\dagger \rangle \) (4.18). Since \( M_{jk} = 0 \) for \( j \leq n \), only the term corresponding to \( n = 0 \) survives in (4.18). Namely

\[
\langle c_1 c_{m+1}^\dagger \rangle = \frac{1}{2\pi^3} \oint_{\Gamma} dz \int_{\mathcal{C}} d\lambda_1 \int_{\mathcal{C}} d\lambda_2 \frac{\ch(\lambda_1 - z) \ch(\lambda_2 - z) \varphi^m(z)}{\sh(z + \frac{\pi i}{4}) \sh(z - \frac{\pi i}{4}) \ch(\lambda_1 - \lambda_2)} \times \left[ \frac{1}{\sh(2(\lambda_1 - z)) \sh(2\lambda_2 - \frac{\pi i}{4})} - \frac{1}{\sh(2(\lambda_2 - z)) \sh(2\lambda_1 + \frac{\pi i}{4})} \right]. \tag{5.6}
\]
To evaluate the above integral explicitly, we shift the \( \lambda_2 \)-contour \( \tilde{C} \) to \( C \). Since the integrand is antisymmetric with respect to the exchange of \( \lambda_1 \) and \( \lambda_2 \), the resulting integral vanishes: we only have to take into account the poles surrounded by the \( \lambda_2 \)-contour \( -C \cup \tilde{C} \) i.e. \( \lambda_2 = -\pi i/4, z \). After evaluating the integral with respect to \( z \) by the method used in the calculation for the density-density correlation function, one has

\[
\langle c^\dagger_1 c_{m+1}^\dagger \rangle = \frac{-1}{2\pi} \int_C d\lambda \frac{\varphi^m(\lambda)}{\text{sh}(\lambda + \pi i/4) \text{sh}(\lambda - \pi i/4)}.
\]  

(5.7)

Changing the variables as explained before, one finally obtains

\[
\langle c^\dagger_1 c_{m+1}^\dagger \rangle = \frac{-1}{2\pi} \int_{k_F}^{2\pi - k_F} dk e^{imk} = \frac{\sin(k_F m)}{\pi m} \quad k_F \in \left[\frac{\pi}{2}, \pi\right].
\]  

(5.8)

This also coincides with the well-known result. In particular at zero chemical potential \( \mu_c = 0 \) \((k_F = \pi/2)\), one sees

\[
\langle c^\dagger_1 c_{m+1}^\dagger \rangle = -(-1)^m \frac{\sin(\pi m)}{\pi m}.
\]  

(5.9)

6 Summary and discussion

In this paper, correlation functions for the spinless fermion model have been discussed by using the fermionic \( R \)-operator and the algebraic Bethe ansatz. Applying solutions of the inverse scattering problem, we derived determinant representations for form factors of local fermion operators. In addition, multiple integrals representing the density-density correlation function and the equal-time one-particle Green’s function were obtained both for arbitrary interaction strengths and particle densities. In particular for the free fermion model, these formulae reduce to the known exact results.

Before closing this paper, we would like to remark possible generalizations of our formulae. A generalization to the finite temperature case is of importance. In fact, the quantum transfer matrix, which is a powerful tool for study of finite-temperature properties for strongly correlated systems in one-dimension, has already been provided for the spinless fermion model [20]. Because the algebraic relations among the elements of monodromy operators in the quantum transfer matrix approach are completely the same as (2.12), the relations derived in this paper are applicable without essential changes.

It will also be very interesting to compute the spectral function for the spinless fermion model. Considering several excited states and inserting the corresponding Bethe roots into the determinant representations for the form factors (3.12) and the scalar product (3.5), one can explicitly compute the spectral function for a finite system. The method developed in the study of the dynamical structure factors for the XXZ chain (see [21–26] for example) will also be useful in computations of the spectral function of the spinless fermion model. Comparison with the arguments in [27] is also interesting.

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References

[1] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, Quantum Inverse Scattering Method and Correlation Functions (Cambridge University Press, Cambridge, 1993).

[2] M. Takahashi, Thermodynamics of One-Dimensional Solvable Models (Cambridge University Press, Cambridge, 1999).

[3] M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki, Phys. Lett. A 168 (1992) 256.

[4] M. Jimbo and T. Miwa, Algebraic Analysis of Solvable Lattice Models, (American Mathematical Society, Providence, RI, 1995).

[5] M. Jimbo and T. Miwa, J. Phys. A 29 (1996) 2923.

[6] N. Kitanine, J.M. Maillet and V. Terras, Nucl. Phys. B 567 (2000) 554.

[7] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, Nucl. Phys. B 641 (2002) 487.

[8] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, Nucl. Phys. B 712 (2005) 600.

[9] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, hep-th/0505006.

[10] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, Nucl. Phys. B 729 (2005) 558.

[11] F. Göhmann, A. Klümper and A. Seel, J. Phys. A 37 (2004) 7625.

[12] F. Göhmann, A. Klümper and A. Seel, J. Phys. A 38 (2005) 1833.

[13] F. Göhmann and N. P. Hasenclever and A. Seel, J. Stat. Mech. 0510 (2005) P10015.

[14] K. Sakai, J. Phys. A 40 (2007) 7523.

[15] Y. Umeno, M. Shiroishi and M. Wadati, J. Phys. Soc. Jpn. 67 (1998) 1930.

[16] F. Göhmann and V.E. Korepin, J. Phys. A 33 (2000) 1199.

[17] N.A. Slavnov, Theor. Math. Phis. 79 (1989) 1605.

[18] N. Kitanine, J.M. Maillet, V. Terras, Nucl. Phys. B 554 (1999) 647.

[19] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, Nucl. Phys. B 642 (2002) 433.

[20] K. Sakai, M. Shiroishi, J. Suzuki and Y. Umeno, Phys. Rev. B 60 (1999) 5186.

[21] D. Biegel, M. Karbach and G. Müller, Europhys. Lett. 59 (2002) 882.

[22] J. Sato and M. Shiroishi and M. Takahashi, J. Phys. Soc. Jpn. 73 (2004) 3008.

[23] J.-S. Caux, R. Hagemans, J.-M. Maillet, J. Stat. Mech. (2005) P09003.
[24] J.-S. Caux and J. M. Maillet, Phys. Rev. Lett. 95 (2005) 077201.

[25] R. G. Pereira, J. Sirker, J.-S. Caux, R. Hagemans, J. M. Maillet, S. R. White, I. Affleck, Phys. Rev. Lett. 96 (2006) 257202.

[26] J.-S. Caux and R. Hagemans, J. Stat. Mech. 0612 (2006) P12013.

[27] M. Khodas, M. Pustilnik, A. Kamenev, L.I. Glazman, Phys. Rev. B 76 (2007) 155402.