On \(E\)-functions of semisimple Lie groups

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Abstract

We develop and describe continuous and discrete transforms of class functions on a compact semisimple, but not simple, Lie group \(G\) as their expansions into series of special functions that are invariant under the action of the even subgroup of the Weyl group of \(G\). We distinguish two cases of even Weyl groups—one is the direct product of even Weyl groups of simple components of \(G\) and the second is the full even Weyl group of \(G\). The problem is rather simple in two dimensions. It is much richer in dimensions greater than two—we describe in detail \(E\)-transforms of semisimple Lie groups of rank 3.

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1. Introduction

The paper can be considered as a completion of [5] and also [6] by the most complicated and interesting and potentially most useful version of the problem of the \(E\)-transform and their special functions (\(E\)-functions) of any semisimple, but not simple, compact Lie group \(G\). The rank \(n\) of \(G\) is the number of real variables of the \(E\)-functions.

Assuming that \(G = G_1 \times G_2\), there are two possibilities to introduce, what is then called, the even subgroup \(W^e\) of the Weyl group \(W\) of \(G\). Both options are mentioned in [5, 6] but only the simpler one of the two is considered there. Both options have the most valuable property: they admit development of multi-dimensional Fourier transforms, although rather different in each case. It is natural to expect that of the two versions each will find its field of optimal applications. Therefore, our aim here is to consider the option that was passed over in [5, 6].

Suppose a compact semisimple Lie group \(G\) is a product of two simple invariant subgroups, \(G = G_1 \times G_2\). We are interested in the even subgroup \(W^e\) of the Weyl group of \(G = G_1 \times G_2\).
The simpler of the two options is to take as the product $W^e(G_1) \times W^e(G_2)$, while the second option, investigated here, is to consider the full even subgroup $W^e(G_1 \times G_2)$. Since

$$W^e(G_1 \times G_2) \supset W^e(G_1) \times W^e(G_2),$$

the second option is clearly richer. There is an important implication of this fact for the class of functions that can be expanded in each option.

For example, in the simplest case when $G_1$ and $G_2$ are rank 1 simple Lie groups, $W^e(G_1) \times W^e(G_2)$ is of order 1, while $W^e(G_1 \times G_2)$ is of order 2.

The rank 3 groups are all considered in detail in view of their likely applicability. The $E$-functions of these cases were not studied before, with the exception $G = A_1 \times A_1 \times A_1$ which is a straightforward concatenation of three one-dimensional cases oriented in mutually orthogonal directions.

The pertinent standard properties of simple Lie groups and their $E$-functions are collected in section 2. Two types of even Weyl groups for semisimple Lie groups are introduced in section 3. The continuous and discrete orthogonality of corresponding $E$-functions is also included there. Explicit formulas of $E$-transforms for semisimple Lie groups of rank less than or equal to 3 are contained in section 4. Comments and follow-up questions are listed in section 5.

2. $E$-functions and their properties

In this section, we collect some facts from the basic theory of simple Lie groups, the Weyl groups and their orbit functions. More details can be found in [3] (for Lie groups), [4], [6] and [7] (for Weyl groups and orbit functions) as well as in the first papers of the authors [2, 5]. The facts we outline here are mostly to establish the notations and also for the sake of completeness of this paper.

Let $G$ be a compact simply connected simple Lie group of rank $n$. The Weyl group $W$ corresponding to $G$ is generated by $n$ reflections and is specified by its Coxeter–Dynkin diagram. We choose a non-orthogonal basis $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}^n$ of the simple roots of $G$. Then, $W_{\Pi}$ is the corresponding root system while

$$Q = \left\{ \sum_{i=1}^{n} a_i \alpha_i \mid a_i \in \mathbb{Z} \right\} = \mathbb{Z} \alpha_1 + \cdots + \mathbb{Z} \alpha_n$$

is the root lattice. Let $\langle , \rangle$ be a scalar product on $\mathbb{R}^n$. Put $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$ for any $\alpha \in \Pi$, namely $\Pi^\vee = \{\alpha^\vee_1, \ldots, \alpha^\vee_n\}$ the co-root basis and all linear combinations $\sum_{i=1}^{n} a_i \alpha^\vee_i$, $a_i \in \mathbb{Z}$ the co-root lattice $Q^\vee$. Relative length and angles between simple roots in $\Pi$ are given by the elements of the Cartan matrix $C = (c_{ij})_{i,j=1}^{n}$, namely $c_{ij} = \langle \alpha_i, \alpha^\vee_j \rangle$ for $i, j = 1, \ldots, n$. In addition to $\Pi$ and $\Pi^\vee$, we also introduce the two other bases. The basis of fundamental weights $\omega_1, \ldots, \omega_n$ is a dual basis of $\Pi^\vee$, i.e.

$$\langle \omega_i, \alpha^\vee_j \rangle = \delta_{ij}, \quad i, j = 1, \ldots, n,$$

while the basis of fundamental coweights $\omega^\vee_1, \ldots, \omega^\vee_n$ is dual to $\Pi$. The $\omega$-basis and $\alpha$-basis are also related by means of the Cartan matrix. In the matrix form,

$$\alpha = C \omega, \quad \alpha^\vee = C^T \omega^\vee.$$

Rank $n$ lattices $P = \mathbb{Z} \omega_1 + \cdots + \mathbb{Z} \omega_n$ and $P^\vee = \mathbb{Z} \omega^\vee_1 + \cdots + \mathbb{Z} \omega^\vee_n$ are called correspondingly the weight and coweight lattices. The weight lattice $P$ is dual to the co-root lattice $Q^\vee$, while $P^\vee$ is dual to $Q$. 

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We also define the set of dominant weights \( P^+ \) and the set of strictly dominant weights \( P^{++} \) as
\[
P^+ = \mathbb{Z}^{\geq 0} \omega_1 + \mathbb{Z}^{\geq 0} \omega_2 + \cdots + \mathbb{Z}^{\geq 0} \omega_n \supset P^{++} = \mathbb{Z}^{> 0} \omega_1 + \mathbb{Z}^{> 0} \omega_2 + \cdots + \mathbb{Z}^{> 0} \omega_n.
\]

To any simple root \( \alpha \in \Pi \), we associate the reflection \( r_\alpha \) acting on \( x \in \mathbb{R}^n \) as
\[
r_\alpha x = x - \langle \alpha, x \rangle \alpha^\vee.
\]
Then, the finite Weyl group is generated by \( r_i \equiv r_{\alpha_i}, i = 1, \ldots, n \).

Combining \( W \) with the translation group defined by \( Q^\vee \), we obtain the infinite affine Weyl group \( \hat{W} = Q \rtimes W \). A fundamental region \( F \subset \mathbb{R}^n \) for \( \hat{W} \) is the simplex which is specified by \( n + 1 \) vertices \( \{0, \frac{m_1}{m_1}, \ldots, \frac{m_n}{m_n}\} \), where \( m_1, \ldots, m_n \) are the coefficients of the highest root of \( G \) relative to \( \Pi \)—see [1].

For any \( \lambda \in \mathbb{R}^n \), denote by \( W(\lambda) \) its orbit with respect to the action of \( W \).

The even Weyl subgroup of \( W \) is the set of elements of even length, i.e.
\[
W_e = \langle r_{i_1} \cdots r_{i_k} | k \text{ is even}, i_j \in \{1, \ldots, n\} \rangle = \{ w \in W | \det w = 1 \}.
\]

It is a normal subgroup of \( W \) of index 2. For any \( \lambda \in \mathbb{R}^n \), its \( W_e \)-orbit is denoted by \( W_e(\lambda) \).

Similarly, we define a dual fundamental domain \( F^e \) of \( \hat{W} \) as
\[
F^e = F^\vee \cup r \text{ int}(F).
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2.1. \( E \)-functions

The \( E \)-functions are orbit functions of the symmetry group \( W^e \):
\[
E_\lambda(x) = \sum_{\mu \in W(\lambda)} e^{2\pi i \langle \mu, x \rangle}
\]
for \( x \in \mathbb{R}^n, \lambda \in P \). The \( E \)-functions are invariant under the action of both their corresponding even Weyl and even affine Weyl groups.

For any \( \lambda \in \mathbb{R}^n \), we denote the order of the stabilizer
\[
d_e^e \equiv |\text{Stab}_{W^e}(\lambda)|, \quad \text{Stab}_{W^e}(\lambda) = \{ w \in W^e | w\lambda = \lambda \}
\]
and introduce a different normalization of \( E \)-functions, namely
\[
\Xi_\lambda(x) = d_e^e E_\lambda(x) = \sum_{w \in W^e} e^{2\pi i \langle w\lambda, x \rangle}.
\]
2.2. Continuous orthogonality and continuous E-transforms

For any two weights from the set $P_{e} = P^{+} \cup rP^{+}$, corresponding E-functions are orthogonal on $F^{e}$:

$$\int_{F^{e}} \mathcal{E}_{\lambda}(x) \overline{\mathcal{E}_{\lambda}(x)} dx = |F^{e}| |W^{e}| \delta_{\lambda,\lambda'}.$$  \hfill (5)

Here, the overline denotes complex conjugation and $|F^{e}|$ the volume of the domain $F^{e}$. For the proof see [7].

The E-functions determine symmetrized Fourier series expansions

$$f(x) = \sum_{\lambda \in \mathcal{P}_{e}} c_{\lambda} \mathcal{E}_{\lambda}(x), \quad \text{where} \quad c_{\lambda} = \frac{1}{|F^{e}| |W^{e}| d_{\lambda}^{e}} \int_{F^{e}} f(x) \overline{\mathcal{E}_{\lambda}(x)} dx.$$

2.3. Discrete orthogonality and discrete E-transforms

For an arbitrary natural number $M$, the discrete calculus of $E$-functions is performed over the intersection $F_{M}^{e}$ of the finite group $\frac{1}{M}P^{e}/Q^{e}$ with the fundamental domain $F^{e}$, $F_{M}^{e} \equiv \frac{1}{M}P^{e}/Q^{e} \cap F^{e}$. The finite set of dominant weights $\Lambda_{M}^{e}$ labeling orthogonal E-functions can be chosen as the intersection of the quotient group $P/MQ$ with the augmented dual fundamental domain $MF_{e}^{e}$, $\Lambda_{M}^{e} \equiv MF_{e}^{e} \cap P/MQ$. For $x \in \frac{1}{M}P^{e}/Q^{e}$, we denote the orbit and its order by

$$W^{e} x = \{ wx \in \mathbb{R}^{n}/Q^{e} | w \in W^{e} \}, \quad \varepsilon^{e}(x) \equiv |W^{e} x|.$$  \hfill (6)

For $\lambda \in P/MQ$, we denote the order of the stabilizer

$$h_{\lambda}^{e} \equiv |\text{Stab}_{\lambda}^{e}(\lambda)|, \quad \text{Stab}_{\lambda}^{e}(\lambda) = \{ w \in W^{e} | w \lambda = \lambda \}.$$  \hfill (7)

For $\lambda, \lambda' \in \Lambda_{M}^{e}$, the discrete orthogonality relations hold [2]:

$$\sum_{x \in F_{M}^{e}} \varepsilon^{e}(x) \mathcal{E}_{\lambda}(x) \overline{\mathcal{E}_{\lambda'}(x)} = \text{det } C |W^{e}| M^{e} h_{\lambda}^{e} \delta_{\lambda,\lambda'},$$  \hfill (8)

and the discrete symmetrized E-functions expansion is given by

$$f(x) = \sum_{\lambda \in \Lambda_{M}^{e}} c_{\lambda} \mathcal{E}_{\lambda}(x), \quad \text{where} \quad c_{\lambda} = \frac{1}{\text{det } C |W^{e}| M^{e} h_{\lambda}^{e}} \sum_{x \in F_{M}^{e}} \varepsilon^{e}(x) f(x) \overline{\mathcal{E}_{\lambda}(x)}.$$  \hfill (9)

Example 2.1. In the case of the rank 1 group $A_{1}$, the Weyl group $W(A_{1})$ has two elements $\{id, r\}$, where $r$ is the reflection in the origin. Then $W^{e}(A_{1}) = \{id\}$. The Cartan matrix of $A_{1}$ is (2). Thus, we have $\alpha^{\vee} = \alpha$, $\omega = \omega^{\vee}$ and $\alpha = 2\omega$ with $\langle \omega, \omega \rangle = 1/2$. The weight lattice $P = \mathbb{Z}\omega$ and the dominant weights are $P^{+} = \mathbb{Z}^{\geq 0}\omega$ and $P_{e} = \mathbb{Z}\omega$. For any $a\omega \in \mathbb{R}$ it holds $W(a) = \{a, -a\}$, while $W^{e}(a) = \{a\}$. Since $F(A_{1}) = \{x\omega | 0 \leq x \leq 1\}$, we have from (2)

$$F^{e}(A_{1}) = \{x\omega | -1 < x \leq 1\}.$$  

Therefore, the corresponding E-function (4) for $a\omega \in P_{e}, a \in \mathbb{Z}$ and $x\omega \in F^{e}$ is

$$\mathcal{E}_{\lambda}(x) = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{i\pi (a-x)a} \, dx = e^{i\pi a x}.$$  

One can easily check property (5); in particular, for any $\lambda = a\omega, \lambda' = a'\omega \in \mathbb{Z}\omega$:

$$\int_{F^{e}(A_{1})} \mathcal{E}_{\lambda}(z) \overline{\mathcal{E}_{\lambda'}(z)} \, dz = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{i\pi (a-a')x} \, dx = e^{i\pi a x}.$$  


The discrete sets $F_M^c$ and $A_M^c$ have the form
\[
F_M^c(A_1) = \left\{ \frac{s_1}{M} \omega \mid M < s_1 \leq M \right\}
\]
\[
A_M^c(A_1) = \left\{ t_1 \omega \mid M < t_1 \leq M \right\}.
\]
The discrete orthogonality (8) has for $\lambda, \lambda' \in A_M^c(A_1)$ the form
\[
\sum_{z \in F_M^c(A_1)} \mathbb{E}_\lambda(z) \overline{\mathbb{E}_{\lambda'}(z)} = \sum_{n_1 = -M+1}^M e^{\frac{\pi i n_1 (\lambda, \lambda')}{4M}} = 2M \delta_{\lambda, \lambda'}.
\]

3. Even Weyl group of semisimple Lie groups

In this section, we will define the even Weyl group for the semisimple Lie groups. Let $G_1$, $G_2$ be two compact simply connected simple Lie groups of ranks $n_1$ and $n_2$, correspondingly. For $G_i$, $i = 1, 2$, we denote by $W_i$ the corresponding Weyl group, by $C_i$ its Cartan matrix, by $P_i$ the weight lattice, by $Q_i^\vee$ its co-root lattice and by $F_i$ the corresponding fundamental region. Then, for the semisimple Lie group $G = G_1 \times G_2$ of rank $n_1 + n_2$ we obtain that the corresponding Weyl group is $W = W_1 \times W_2$, its Cartan matrix is $C = \left( \begin{smallmatrix} C_1 & 0 \\ 0 & C_2 \end{smallmatrix} \right)$, the weight lattice $P = P_1 \times P_2$ both as lattices and as groups, analogously the co-root lattice $Q^\vee = Q_1^\vee \times Q_2^\vee$, its affine Weyl group $W_{aff} = W_1^aff \times W_2^aff = W \times Q^\vee$ and finally the fundamental region $F$ is the Cartesian product of $F_1$ and $F_2$.

There are two natural ways of defining the even Weyl subgroup of $W = W_1 \times W_2$ (even more in the case when $G$ has more then two factors). For the first, one continues the sequence of the notions for semisimple group in the last paragraph: the even Weyl subgroup of $W$ is a direct product of the corresponding even Weyl groups $W_1^e$ and $W_2^e$; we will denote it by
\[
W^{ee} = W_1^e \times W_2^e.
\]
Its affine even group is $W_{eaff}^{ee} = (W_1)^{eaff} \times (W_2)^{eaff} = W_1^e \times W_2^e \times Q^\vee = W^{ee} \times Q^\vee$ and its fundamental region $F^{ee} = F_1^e \times F_2^e$. The second possibility arises from (1); namely, we define the even subgroup $W$ as the set of the elements of even length of $W$. In this case, its affine group $W_{eaff} = W \times Q^\vee$ and its fundamental region $F^e = F_1 \times F_2 \cup \text{int}(r_1 F_1 \times F_2)$. In spite of the fact that the fundamental regions, where the expansion takes place as well as expansion functions, are different, for both cases we will have both continuous and discrete orthogonality.

3.1. Cartesian product $W^e(G_1) \times W^e(G_2)$

We consider the Cartesian product of two even Weyl groups $W^e(G_1) \equiv W_1^e$ and $W^e(G_2) \equiv W_2^e$, i.e.
\[
W^{ee} = W_1^e \times W_2^e.
\]
Since $|W_1^e| = \frac{1}{2} |W_1|$, we obtain that $|W^{ee}| = \frac{1}{2} |W_1||W_2|$. Let $\lambda, \mu \in \mathbb{R}^{n_1}$ and $\lambda, \mu \in \mathbb{R}^{n_2}$; then, for $v = (\lambda, \mu) \in \mathbb{R}^{n_1+n_2}$, we obtain for $W^{ee}$-orbits and stabilizers
\[
W^{ee}(v) = W_1^e(\lambda) \times W_2^e(\mu)
\]
\[
\text{Stab}_{W^{ee}}(v) = \text{Stab}_{W_1^e}(\lambda) \times \text{Stab}_{W_2^e}(\mu)
\]
and consequently
\[
d_v^{ee} = |	ext{Stab}_{W^{ee}}(v)| = d^{ee}_\lambda d^{ee}_\mu.
\]
Denote \( P_{ee} = (P_1)_e \times (P_2)_e \) and the normalized \( E \)-function corresponding to \( W_{ee} \) by \( \Xi_{ee} \). Then, for any \( \nu = (\lambda, \mu) \in P_{ee} \) and \( x = (x_1, x_2) \in \mathbb{R}^{n_1 \times n_2} \), the corresponding \( \Xi_{ee} \)-function is of the form
\[
\Xi_{ee}^\nu(x) = \sum_{w \in W_{ee}} e^{2\pi i (w_1 \cdot \lambda + w_2 \cdot \mu)} = \sum_{w_1 \in W_1} e^{2\pi i (w_1 \cdot \lambda)} \sum_{w_2 \in W_2} e^{2\pi i (w_2 \cdot \mu)} = \Xi_\lambda(x_1) \Xi_\mu(x_2). \tag{11}
\]
The functions \( \Xi_\lambda \) and \( \Xi_\mu \) are orbit functions defined by \( W_1^c \) and \( W_2^c \).

### 3.1.1. Continuous orthogonality and \( E \)-transforms.
Combining (11) with orthogonality (5) for \( E \)-functions for \( W_1^c \) and \( W_2^c \), we obtain the orthogonality for \( \Xi_{ee} \)-functions defined by \( W_{ee} \).

For any \( \nu = (\lambda, \mu) \in P_{ee} \),
\[
\int_{F_{ee}} \Xi_{ee}^\nu(x) \overline{\Xi_{ee}^\nu(x)} \, dx = \int_{F_{ee}} \Xi_\lambda(x_1) \overline{\Xi_\lambda(x_1)} \, dx_1 \int_{F_{ee}} \Xi_\mu(x_2) \overline{\Xi_\mu(x_2)} \, dx_2 = \left| F^\nu_{ee} \right| \left| W_{ee} \right| d\nu_{ee} \delta_{\nu \nu'}, \tag{12}
\]
where \( d\nu_{ee} \) is the measure on \( P_{ee} \).

Note that one can easily generalize this definition for the case \( G = G_1 \times \cdots \times G_k \). The even group is given by \( W_{ee} = W_1^c \times \cdots \times W_k^c \). For any \( \nu = (\nu_1, \ldots, \nu_k) \in P_{ee} \), and \( x = (x_1, \ldots, x_k) \in \mathbb{R}^{n_1 \times \cdots \times n_k} \),
\[
\Xi_{ee}^\nu(x) = \Xi_{\nu_1}(x_1) \cdots \Xi_{\nu_k}(x_k). \tag{13}
\]
The fundamental region is \( F_{ee} = F_1^c \times \cdots \times F_k^c \) and the orthogonal relations for any \( \nu, \nu' \in P_{ee} \) are
\[
\int_{F_{ee}} \Xi_{ee}^\nu(x) \overline{\Xi_{ee}^{\nu'}}(x) \, dx = \left| F_{ee} \right| \left| W_{ee} \right| d\nu_{ee} \delta_{\nu \nu'}. \tag{14}
\]

Let \( f \) be a function defined on \( F_{ee} \); then, we may expand \( f \) as a sum of \( \Xi_{ee} \)-functions:
\[
f(x) = \sum_{\nu \in P_{ee}} c_{\nu} \Xi_{ee}^\nu(x), \quad \text{where} \quad c_{\nu} = \frac{1}{\left| F_{ee} \right| \left| W_{ee} \right| d\nu_{ee}} \int_{F_{ee}} f(x) \overline{\Xi_{ee}^\nu(x)} \, dx. \tag{15}
\]

### 3.1.2. Discrete orthogonality and \( E \)-transforms.
For arbitrary natural numbers \( M_1, M_2 \), the discrete calculus of \( E \)-functions is performed over the intersection \( F_{M_1 M_2}^\nu = F_1^c \times F_2^c \) of the finite group \( \frac{1}{M_1} P_1^\nu / Q_1^\nu \times \frac{1}{M_2} P_2^\nu / Q_2^\nu \) with the fundamental domain \( F_{ee} = F_1^c \times F_2^c \),
\[
F_{M_1 M_2}^\nu = \left( \frac{1}{M_1} P_1^\nu / Q_1^\nu \times \frac{1}{M_2} P_2^\nu / Q_2^\nu \right) \cap F_{ee}. \tag{16}
\]
The finite set of dominant weights \( \Lambda_{M_1 M_2}^\nu \) labeling orthogonal \( E \)-functions can be chosen as the intersection of the quotient group \( P_1 / M_1 Q_1^\nu \times P_2 / M_2 Q_2^\nu \) with the augmented dual fundamental domain \( M_1 F_{M_1}^\nu \times M_2 F_{M_2}^\nu \),
\[
\Lambda_{M_1 M_2}^\nu = (P_1 / M_1 Q_1^\nu \times P_2 / M_2 Q_2^\nu) \cap (M_1 F_{M_1}^\nu \times M_2 F_{M_2}^\nu). \tag{16}
\]

For \( x = (x_1, x_2) \in \frac{1}{M_1} P_1^\nu / Q_1^\nu \times \frac{1}{M_2} P_2^\nu / Q_2^\nu \), we have for the orbit and its order
\[
\varepsilon^\nu(x) \equiv \left| W_{ee} x \right| = \varepsilon^\nu(x_1) \varepsilon^\nu(x_2). \tag{16}
\]
For \( \lambda = (\mu, \nu) \in P_1 / M_1 Q_1^\nu \times P_2 / M_2 Q_2^\nu \), we have for the order of the stabilizer
\[
\varepsilon_{\lambda}^\nu \equiv \left| \text{Stab}_{ee}^\nu(\lambda) \right| = \varepsilon_{\mu}^\nu \varepsilon_{\nu}. \tag{16}
\]
3.2. The full even group $W(G_1 \times G_2)$

Let $\Pi_i$ be a basis of simple roots of $G_i$. Denote $R = \{r_\alpha | \alpha \in \Pi_1 \cup \Pi_2\}$. Then, define the even group $W^e$ as a set of the elements of even length of $W = W_1 \times W_2$:

$$W^e(G_1 \times G_2) = \{r_1, \ldots, r_k \mid k \text{ is even } r_j \in R\} = \{w \in W_1 \times W_2 \mid |w| = 1\}.$$  \hfill (19)

In this case, $W^e(G)$ is a normal subgroup of $W$ of index 2, i.e. $|W^e| = \frac{1}{2}|W| = \frac{1}{2}|W_1||W_2|$. Take any generating reflection $r_i$ of $W_i$, $i = 1, 2$; then,

$$W^e(G_1 \times G_2) = W^e_1 \times W^e_2 \cup (r_1 W^e_1 \times r_2 W^e_2)$$

and for any $\lambda = (\mu, \nu) \in \mathbb{R}^{n_1+n_2}$:

$$W^e(G_1 \times G_2)(\lambda) = \{ (\eta, \theta), (r_1 \eta, r_2 \theta) | \eta \in W^e_1(\mu), \theta \in W^e_2(\nu)\}.$$  

The corresponding affine even group is $W^a(G_1 \times G_2) = W^e \rtimes Q^e$ and its fundamental domain is

$$F^e = F_1 \times F_2 \cup \text{int}(r_1 F_1 \times F_2) \subset \mathbb{R}^{n_1+n_2}. \hfill (20)$$

The choice of $r_1$ in (20) is arbitrary; we can choose any two adjacent copies of $F$. Finally, put $P_e = (P^e_1 \times P^e_2) \cup r_1 P^e_1 \times P^e_2$.

For any $\lambda = (\mu, \nu) \in P_e$ and $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$, the corresponding normalized $E^e$-function

$$\mathcal{E}_\lambda^e(x) = \sum_{w \in W^e} e^{2\pi i \langle w x, x \rangle} = \mathcal{E}_\mu(x_1) \mathcal{E}_\nu(x_2) + \mathcal{E}_{r_1 \nu}(x_1) \mathcal{E}_{r_2 \mu}(x_2). \hfill (21)$$

Note that for any $\lambda \in P_e$ function $\mathcal{E}_\lambda$ is $W_e^a$ invariant.

3.2.1. Continuous orthogonality and $E^e$-transforms.

The orthogonality of $E^e$-functions for any two weights from the set $P_e$ is given by

$$\int_{P_e} \mathcal{E}_\lambda^e(x) \overline{\mathcal{E}_\mu^e(x)} \, dx = |F^e||W^e|d^e_\lambda \delta_\lambda \mu, \hfill (22)$$

where $F^e$ is defined by (20). The size of the stabilizer $d^e_\lambda$ can be defined by (3), with $W^e$ of the form (19). For the proof see [7].

The $E^e$-functions determine symmetrized Fourier series expansions

$$f(x) = \sum_{\lambda \in P_e} c_\lambda \mathcal{E}_\lambda^e(x), \quad \text{where} \quad c_\lambda = \frac{1}{|F^e||W^e|d^e_\lambda} \int_{P_e} f(x) \overline{\mathcal{E}_\lambda^e(x)} \, dx. \hfill (23)$$
3.2.2. Discrete orthogonality and discrete $E$-transforms. Taking an arbitrary natural number $M$, the discrete calculus of $E$-functions is performed over the intersection $F^e_M$ of the finite group $P^\vee/Q^\vee$ with the fundamental domain $P^e$ of the form (20):

$$F^e_M = \frac{1}{M} (P_1^\vee \times P_2^\vee)/(Q_1^\vee \times Q_2^\vee) \cap F^e.$$

The finite set of dominant weights $\Lambda^e_M$ labeling orthogonal $E$-functions can be chosen using the dual fundamental domain $F^e_\vee = F_\vee^1 \times F_\vee^2 \cup \text{int}(r_1 F_\vee^1 \times F_\vee^2)$.

Then, the intersection of the quotient group $P/MQ$ with the augmented domain $MF^e_\vee$ forms the set

$$\Lambda^e_M = MF^e_\vee \cap (P_1 \times P_2)/[M(Q_1 \times Q_2)].$$

For $x \in \gamma (P^\vee/Q^\vee)$, we define the orbit and its size $\varepsilon(x)$ by equation (6) and similarly the stabilizer of $\lambda \in P/MQ$ and its size $h^\vee_\lambda$ by (7).

Then for $\lambda, \lambda' \in \Lambda^e_M$, the discrete orthogonality relations hold:

$$\sum_{x \in F^e_M} \varepsilon(x) \Xi^e_\lambda(x) \Xi^e_{\lambda'}(x) = \det C_{|W^e|} |M\gamma| h^\vee_\lambda h^\vee_{\lambda'} \delta_{\lambda \lambda'}.$$  

and the discrete symmetrized $E$-functions expansion is given by

$$f(x) = \sum_{\lambda \in \Lambda^e_M} c_{\lambda} \Xi^e_\lambda(x), \quad \text{where } c_{\lambda} = \frac{1}{\det C_{|W^e|} |M\gamma| h^\vee_\lambda} \sum_{x \in F^e_M} \varepsilon(x) f(x) \Xi^e_\lambda(x).$$

4. The $E$-transforms of semisimple Lie groups of rank $\leq 3$

In this section, we deal with expansions of continuous and discrete functions on both $P^e$ and $P^e$ into series of continuous and discrete $E$-functions and their inversion. There are four compact semisimple Lie groups of rank 3, namely $SU(2) \times SU(2) \times SU(2)$, $SU(2) \times SU(3)$, $SU(2) \times O(5)$ and $SU(2) \times G(2)$. We are using the following notation often used to denote the corresponding Lie algebras:

$$A_1 \leftrightarrow SU(2), \quad A_2 \leftrightarrow SU(3), \quad C_2 \leftrightarrow O(5), \quad G_2 \leftrightarrow G(2).$$

For a particular Lie group $G$ in order to apply the transformations (15), (18) and the transformations (23), (26), we need the following information:

1. the fundamental regions $F^e$ and $F^e$ and their volumes,
2. the infinite sets of weights $P^e$ and $P^e$,
3. the coefficients $d^e_\lambda$ and $d^e_\lambda$,
4. the explicit form of $E^e$- and $E^e$-functions,
5. the discrete sets $F^e_{M_{1,2}}$ and $F^e_M$,
6. the finite sets of weights $\Lambda^e_{M_{1,2}}$ and $\Lambda^e_M$ and
7. the coefficients $\varepsilon^e(x), \varepsilon^e(x)$ and $h^\vee_\lambda, h^\vee_\lambda, \det C$. 

8
4.1. The E-transforms of \( A_1 \times A_1 \)

Relative length and angles between the simple roots and fundamental weights of \( A_1 \times A_1 \) are given in terms of the Cartan matrix

\[
C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \det C = 4.
\]

(1) \( W^{ee}(A_1 \times A_1) \). The \( W^{ee} \)-orbit for the generic point \( a\omega_1 + b\omega_2 \in \mathbb{R}^2 \) is given by

\[
W^{ee}(a, b) = \{(a, b)\}
\]

and \( |W^{ee}| = 1 \). The fundamental region \( F^{ee} \) is given by

\[
F^{ee}(A_1 \times A_1) = \{ \omega_1^\vee + \omega_2^\vee | -1 < x, y \leq 1 \}
\]

and its volume is \( |F^{ee}| = 2 \). We have the weight lattice \( \Lambda^{ee}_{ee} \) of the form

\[
\Lambda^{ee}_{ee}(A_1 \times A_1) = \{ a\omega_1 + b\omega_2 | a, b \in \mathbb{Z} \}.
\]

Then, for any \( a\omega_1 + b\omega_2 \in \Lambda^{ee}_{ee} \), the corresponding \( E^{ee} \)-function at point \( x\omega_1^\vee + y\omega_2^\vee \) is

\[
\Xi^{ee}_{(a,b)}(x, y) = e^{i\pi(ax + by)}.
\]

The contour plots of some lowest \( \Xi^{ee} \) functions are plotted in figure 1.

The coefficients \( d^{ee}_{(a,b)} \) in continuous orthogonality relations (12) are all equal to 1.

The discrete grid \( F^{ee}_{M_1, M_2} \) has the explicit form

\[
F^{ee}_{M_1, M_2} = \left\{ \frac{s_1}{M_1} \omega_1^\vee + \frac{s_2}{M_2} \omega_2^\vee | s_i \in \mathbb{Z}, -M_i < s_i \leq M_i, i = 1, 2 \right\}
\]

and the corresponding grid \( \Lambda^{ee}_{M_1, M_2} \) of weights has the following form:

\[
\Lambda^{ee}_{M_1, M_2} = \{ t_1\omega_1 + t_2\omega_2 | t_i \in \mathbb{Z}, -M_i < t_i \leq M_i, i = 1, 2 \}.
\]

The discrete orthogonality relations of the functions \( \Xi^{ee} \) are of the form (17) with the resulting normalization coefficient equal to \( 4M_1 M_2 h_k^{ee} \). The coefficients \( e^{ee}(x) \) and \( h_k^{ee} \) in (17) are all equal to 1.
Figure 2. The contour plots of $\Xi^e$-functions of $A_1 \times A_1$ plotted over the fundamental domain $F^e$.

Table 1. The coefficients $d^e_{(a,b)}$, $\varepsilon^e(x)$, and $h^e_\lambda$ of continuous and discrete orthogonality relations of $A_1 \times A_1$. Assuming $a, b \neq 0, s_0, s_1, s_0', s_2 \neq 0$ and $t_0, t_1, t_0', t_2 \neq 0$.

| $\lambda \in \mathcal{P}_e$ | $d^e_{(a,b)}$ | $x \in F^e_{\mathcal{M}}$ | $\varepsilon^e(x)$ | $\lambda \in \mathcal{M}_e$ | $h^e_\lambda$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $(a, b)$ | 1 | $[s_0, s_1, s_0', s_2]$ | 2 | $[t_0, t_1, t_0', t_2]$ | 1 |
| $(0, b)$ | 1 | $[s_0, s_1, s_0', 0]$ | 2 | $[t_0, t_1, 0, t_2]$ | 1 |
| $(a, 0)$ | 1 | $[s_0, s_1, 0, s_2]$ | 2 | $[t_0, 0, t_0', t_2]$ | 1 |
| $(0, 0)$ | 2 | $[s_0, 0, s_0', s_2]$ | 2 | $[t_0, 0, 0, t_2]$ | 1 |
| | | $[s_0, 0, s_0', 0]$ | 1 | $[t_0, 0, 0, 0]$ | 2 |
| | | $[s_0, 0, 0, s_2]$ | 1 | $[t_0, 0, 0, t_2]$ | 2 |
| | | $[0, s_1, s_0', s_2]$ | 2 | $[t_0, t_1, t_0', t_2]$ | 1 |
| | | $[0, s_1, s_0', 0]$ | 1 | $[t_0, 0, 0, 0]$ | 2 |
| | | $[0, s_1, 0, s_2]$ | 1 | $[0, t_1, 0, t_2]$ | 2 |

(2) $W^e(A_1 \times A_1)$. The $W^e$-orbit for the generic point $a\omega_1 + b\omega_2 \in \mathbb{R}^2$ is given by

$$W^e(a, b) = \{(a, b), (-a, -b)\}$$

and $|W^e| = 2$. The fundamental region $F^e$ is given by

$$F^e(A_1 \times A_1) = \{x\omega_1^e + y\omega_2^e \mid 0 \leq x, y \leq 1\}$$

$$\cup \{-x\omega_1^e + y\omega_2^e \mid 0 < x, y < 1\}$$

and its volume is $|F^e| = 1$. We have the weight lattice $P_e$ of the form

$$P_e(A_1 \times A_1) = \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{Z}_{>0}\}$$

$$\cup \{-a\omega_1 + b\omega_2 \mid a, b \in \mathbb{N}\}.$$

Then, for any $a\omega_1 + b\omega_2 \in P_e$, the corresponding $E^e$-function at point $x\omega_1^e + y\omega_2^e$ is

$$\Xi^e_{(a,b)}(x, y) = 2 \cos(\pi(ax + by)).$$

The contour plots of some lowest $\Xi^e$ functions are plotted in figure 2.

The values of coefficients $d^e_{(a,b)}$, $\varepsilon^e(x)$, and $h^e_\lambda$ of continuous and discrete orthogonality relations (22) have been shown in table 1.
The discrete grid $F'_M$ has the explicit form
\[
F'_M = \left\{ \frac{s_1}{M} \omega_1^\vee + \frac{s_2}{M} \omega_2^\vee \mid s_1, s_2 \in \mathbb{Z}_{\leq 0}, s_1, s_2 \leq M \right\}
\cup \left\{ -\frac{s_1}{M} \omega_1^\vee + \frac{s_2}{M} \omega_2^\vee \mid s_1, s_2 \in \mathbb{N}, s_1, s_2 < M \right\}
\]
and the corresponding grid $\Lambda'_M$ of weights has the following form:
\[
\Lambda'_M = \left\{ t_1 \omega_1 + t_2 \omega_2 \mid t_1, t_2 \in \mathbb{Z}_{\geq 0}, t_1, t_2 \leq M \right\}
\cup \left\{ -t_1 \omega_1 + t_2 \omega_2 \mid t_1, t_2 \in \mathbb{N}, t_1, t_2 < M \right\}.
\]

The discrete orthogonality relations of the functions $\Xi^e$ are of the form (25) with the resulting normalization coefficient equal to $8M^2 h^e_\lambda$. We label each point $x = \frac{t_0}{M} \omega_1 + \frac{t_2}{M} \omega_2 \in F'_M$ by coordinates $[s_0, s_1, s'_0, s_2]$ such that $s_0 + s_1 = M, s'_0 + s_2 = M$. Similarly, we label each point from $\lambda = t_1 \omega_1 + t_2 \omega_2 \in \Lambda'_M$ by coordinates $[t_0, t_1, t'_0, t_2]$ such that $t_0 + t_1 = M, t'_0 + t_2 = M$. The coefficients $s^e(x)$ and $h^e_{\lambda}$ in (25) are listed in table 1.

4.2. The $E$-transforms of $A_1 \times A_2$

Relate length and angles between the simple roots and fundamental weights of $A_1 \times A_2$ are given in terms of the Cartan matrix $C = \left( \begin{smallmatrix} 0 & 2 \\ 2 & -1 \end{smallmatrix} \right)$, det $C = 6$.

1) $W^{ee}(A_1 \times A_2)$. The $W^{ee}$-orbit for the generic point $a \omega_1 + b \omega_2 + c \omega_3 \in \mathbb{R}^3$ is given by
\[
W^{ee}(a, b, c) = \{(a, b, c), (a, c, -b - c), (a, -b + c, b)\}
\]
and $|W^{ee}| = 3$. The fundamental region $F^{ee}$ is given by
\[
F^{ee}(A_1 \times A_2) = \{ x \omega_1^\vee + y \omega_2^\vee + z \omega_3^\vee \mid -1 < x \leq 1, y, z \geq 0, y + z \leq 1 \}
\cup \{ x \omega_1^\vee - y \omega_2^\vee + (y + z) \omega_3^\vee \mid -1 < x \leq 1, y, z > 0, y + z < 1 \}
\]
and its volume is $|F^{ee}| = 2/\sqrt{6}$. We have the weight lattice $P^{ee}$ of the form
\[
P^{ee}(A_1 \times A_2) = \{ a \omega_1 + b \omega_2 + c \omega_3 \mid a \in \mathbb{Z}, b, c \in \mathbb{Z}_{\geq 0} \}
\cup \{ a \omega_1 - b \omega_2 + (b + c) \omega_3 \mid a \in \mathbb{Z}, b, c \in \mathbb{N} \}.
\]

Then, for any $a \omega_1 + b \omega_2 + c \omega_3 \in P^{ee}$, the corresponding $E^{ee}$-function at point $x \omega_1^\vee + y \omega_2^\vee + z \omega_3^\vee$ is
\[
\Xi^{ee}_{(a, b, c)}(x, y, z) = e^{\pi i a x (e^{\frac{2\pi i}{3}}((2b+c)x+(b+2c)y) + e^{-\frac{2\pi i}{3}}((y+2z)b+(z-y)c) + e^{\frac{2\pi i}{3}}((y-c)b+(2z+y)c))}
\]
and the coefficients $d^{ee}_{(a, b, c)}$ in continuous orthogonality relations (12) have values in table 2.

The discrete grid $F^{ee}_{M_1, M_2}$ has the explicit form
\[
F^{ee}_{M_1, M_2} = \left\{ \frac{s_1}{M_1} \omega_1^\vee + \frac{s_2}{M_2} \omega_2^\vee + \frac{s_3}{M_2} \omega_3^\vee \mid s_1, s_2, s_3 \in \mathbb{Z}_{\geq 0}, s_1 < M_1, s_2, s_3 \leq M_2 \right\}
\cup \left\{ -\frac{s_1}{M_1} \omega_1^\vee + \frac{s_2}{M_2} \omega_2^\vee + \frac{s_3}{M_2} \omega_3^\vee \mid s_1 \in \mathbb{Z}, s_2, s_3 \in \mathbb{N}, s_1 < M_1, s_2 + s_3 < M_2 \right\}.
\]
Table 2. The coefficients $d_{\lambda}^{\sigma \epsilon}$ of continuous orthogonality of semisimple Lie groups of rank 3. Assuming $a, b, c \neq 0$.

| $\lambda \in P_\sigma$ | $A_1 \times A_2$ | $A_1 \times C_2$ | $A_1 \times G_2$ | $A_1 \times A_2 \times A_3$ |
|------------------------|------------------|------------------|------------------|------------------|
| $(a, b, c)$            | 1                | 1                | 1                | 1                |
| $(0, b, c)$            | 1                | 1                | 1                | 1                |
| $(a, 0, c)$            | 1                | 1                | 1                | 1                |
| $(a, b, 0)$            | 1                | 1                | 1                | 1                |
| $(0, 0, c)$            | 3                | 4                | 6                | 1                |
| $(0, 0, 0)$            | 3                | 4                | 6                | 1                |

and the corresponding grid $\Lambda_{M, M_1}^{\sigma \epsilon}$ of weights has the following form:

$$\Lambda_{M, M_1}^{\sigma \epsilon} = \{t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 | t_1 \in \mathbb{Z}, t_2, t_3 \in \mathbb{Z}_0^+, -M_1 < t_1 \leq M_1, t_2 + t_3 \leq M_2 \}
\cup \{t_1 \omega_1 - t_2 \omega_2 + (t_2 + t_3) \omega_3 | t_1 \in \mathbb{Z}, t_2, t_3 \in \mathbb{N}, -M_1 < t_1 \leq M_1, t_2 + t_3 < M_2 \}.$$  

The discrete orthogonality relations of the functions $\Xi^{\sigma \epsilon}$ are of the form (17). We label each point $x = \frac{x_1}{M} \omega_1 + \frac{x_2}{M} \omega_2 + \frac{x_3}{M} \omega_3 \in F_{M, M_1}^{\sigma \epsilon}$ by coordinates $[s_0, s_1, s_2, s_3, s_1]$ such that $s_0 + s_1 = M_1$ and $s_0 + s_2 + s_3 = M_2$. Similarly, we label each point from $\lambda = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \in \Lambda_{M, M_1}^{\sigma \epsilon}$ by coordinates $[t_0, t_1, t_2, t_3, t_1]$ such that $t_0 + t_1 = M_1$ and $t_0 + t_2 + t_3 = M_2$. The coefficients $\xi^{\sigma \epsilon}(x)$ and $\kappa^{\sigma \epsilon}(x)$ in (17) are listed in table 4.

(2) $W^\sigma(A_1 \times A_2)$. The $W$-orbit for the generic point $a \omega_1 + b \omega_2 + c \omega_3 \in \mathbb{R}^3$ is given by $W^\sigma(a, b, c) = \{(a, b, c), (a, c, -b - c), (a, -b - c, b), (-a, -b, b + c), (-a, b + c, -c), (-a, -c, -b)\}$ and $|W^\sigma| = 6$. The fundamental region $F^\sigma$ is given by

$$F^\sigma(A_1 \times A_2) = \{x \omega_1^\sigma + y \omega_2^\sigma + z \omega_3^\sigma | 0 \leq x \leq 1, y, z \geq 0, y + z \leq 1\}
\cup \{x \omega_1^\sigma - y \omega_2^\sigma + (y + z) \omega_3^\sigma | 0 < x < 1, y, z > 0, y + z < 1\}$$
and its volume is $|F^\sigma| = 1/\sqrt{6}$. We have the weight lattice $P_\sigma$ of the form

$$P_\sigma(A_1 \times A_2) = \{a \omega_1 + b \omega_2 + c \omega_3 | a, b, c \in \mathbb{Z}_0^+\}
\cup \{a \omega_1 - b \omega_2 + (b + c) \omega_3 | a, b, c \in \mathbb{N}\}.$$  

Then, for any $a \omega_1 + b \omega_2 + c \omega_3 \in P_\sigma$, the corresponding $E^\sigma$-function at point $x \omega_1^\sigma + y \omega_2^\sigma + z \omega_3^\sigma$ is

$$\Xi_{(a, b, c)}(x, y, z) = e^{i \pi ax} (e^{\frac{2i \pi}{M}((2b+c)y+(b+2c)z)} + e^{-\frac{2i \pi}{M}((2b+c)y+(b+2c)z)}) + e^{i \pi ay} (e^{\frac{2i \pi}{M}((c+a)y+(c+b)z)} + e^{-\frac{2i \pi}{M}((c+a)y+(c+b)z)}) + e^{i \pi az} (e^{\frac{2i \pi}{M}((b+c)y+(b+c)z)} + e^{-\frac{2i \pi}{M}((b+c)y+(b+c)z)})$$
and the coefficients $d_{\sigma (a,b,c)}^{\epsilon}$ in continuous orthogonality relations (22) have values as shown in table 3.

The discrete grid $F_{M}^{\sigma \epsilon}$ has the explicit form

$$F_{M}^{\sigma \epsilon} = \left\{ \frac{s_1}{M} \omega_1^\sigma + \frac{s_2}{M} \omega_2^\sigma + \frac{s_3}{M} \omega_3^\sigma | s_1, s_2, s_3 \in \mathbb{Z}_0^+, s_1 \leq M, s_2 + s_3 \leq M \right\}
\cup \left\{ \frac{s_1}{M} \omega_1^\sigma - \frac{s_2}{M} \omega_2^\sigma + \frac{s_3}{M} \omega_3^\sigma | s_1, s_2, s_3 \in \mathbb{N}, s_1 < M, s_2 + s_3 < M \right\}$$

and the corresponding grid $\Lambda_{M}^{\sigma \epsilon}$ of weights has the following form:

$$\Lambda_{M}^{\sigma \epsilon} = \{t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 | t_1, t_2, t_3 \in \mathbb{Z}_0^+, t_1 \leq M, t_2 + t_3 \leq M \}
\cup \{t_1 \omega_1 - t_2 \omega_2 + (t_2 + t_3) \omega_3 | t_1, t_2, t_3 \in \mathbb{N}, t_1 < M, t_2 + t_3 < M \}.$$
4.3. The E-transforms of $A_1 \times C_2$

Relative length and angles between the simple roots and fundamental weights of $A_1 \times C_2$ are given in terms of the Cartan matrix $\mathfrak{C}$ and $|= 4$. Similarly, we label each point from $\lambda = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \in \Lambda_0$ by coordinates \((t_1, t_2, t_3)\) such that $t_0 + t_1 = M$ and $t_0 + t_2 + t_3 = M$. The coefficients $\varepsilon(x)$ and $h(x)$ in (25) are listed in Table 5.


The discrete orthogonality relations of the functions $\Xi$ are of the form (25). We label each point $x = \frac{1}{2} \omega_1 + \frac{1}{2} \omega_2 + \frac{1}{2} \omega_3 \in F_{ee}$ by coordinates \([s_0, s_1, s_2, s_3]\) such that $s_0 + s_1 = M$ and $s_1 + s_2 + s_3 = M$. Similarly, we label each point from $\lambda = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \in \Lambda_0$ by coordinates \([t_0, t_1, t_2, t_3]\) such that $t_0 + t_1 = M$ and $t_0 + t_2 + t_3 = M$. The coefficients $\varepsilon(x)$ and $h(x)$ in (25) are listed in Table 5.

4.3. The E-transforms of $A_1 \times C_2$

Relative length and angles between the simple roots and fundamental weights of $A_1 \times C_2$ are given in terms of the Cartan matrix $\mathfrak{C}$ and $|= 4$. Similarly, we label each point from $\lambda = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \in \Lambda_0$ by coordinates \((t_1, t_2, t_3)\) such that $t_0 + t_1 = M$ and $t_0 + t_2 + t_3 = M$. The coefficients $\varepsilon(x)$ and $h(x)$ in (25) are listed in Table 5.

The discrete orthogonality relations of the functions $\Xi$ are of the form (25). We label each point $x = \frac{1}{2} \omega_1 + \frac{1}{2} \omega_2 + \frac{1}{2} \omega_3 \in F_{ee}$ by coordinates \([s_0, s_1, s_2, s_3]\) such that $s_0 + s_1 = M$ and $s_1 + s_2 + s_3 = M$. Similarly, we label each point from $\lambda = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \in \Lambda_0$ by coordinates \([t_0, t_1, t_2, t_3]\) such that $t_0 + t_1 = M$ and $t_0 + t_2 + t_3 = M$. The coefficients $\varepsilon(x)$ and $h(x)$ in (25) are listed in Table 5.
The coefficients $\epsilon^{xy}$ and $h^{\text{cr}}_{xy}$ of discrete orthogonality relations of semisimple Lie groups of rank 3. Assuming $s_0, s_1, s_0', s_2, s_3 \neq 0$ and $t_0, t_1, t_0', t_2, t_1 \neq 0$.

| $x \in \mathcal{F}_{M_1, M_2}^{\text{cr}}$ | $\epsilon^{xy}$ | $\lambda \in \Lambda_{M_1, M_2}^{\text{cr}}$ | $h^{\text{cr}}_{xy}$ |
|-----------------|----------------|------------------|------------------|
| $[0, s_0, s_0, s_2, s_1]$ | 3 4 6 | $[t_0, t_1, t_0', t_2, t_1]$ | 1 1 1 |
| $[s_0, s_1, s_0, s_2, 0]$ | 3 4 6 | $[t_0, t_1, t_0', 0, t_3]$ | 1 1 1 |
| $[s_0, s_1, s_0', s_2, s_3]$ | 3 4 6 | $[t_0, t_1, 0, t_2, t_3]$ | 1 1 1 |
| $[s_0, s_1, s_0', 0, s_3]$ | 1 1 1 | $[t_0, t_1, t_0', 0, 0]$ | 3 4 6 |
| $[s_0, s_1, s_0, s_2, 0]$ | 1 2 2 | $[t_0, t_1, 0, t_2, 0]$ | 3 4 3 |
| $[s_0, s_1, s_0, s_2, 0]$ | 1 1 1 | $[t_0, t_1, 0, 0, t_1]$ | 3 2 2 |
| $[s_0, 0, s_0, s_2, s_3]$ | 3 4 6 | $[t_0, t_1, t_0', 0, t_3]$ | 1 1 1 |
| $[s_0, 0, s_0, s_2, 0]$ | 3 4 6 | $[t_0, t_1, t_0', 0, t_3]$ | 1 1 1 |
| $[s_0, 0, s_0', s_2, s_3]$ | 3 4 6 | $[t_0, t_1, 0, t_2, t_3]$ | 1 1 1 |
| $[s_0, 0, s_0', 0, s_3]$ | 1 1 1 | $[t_0, t_1, t_0', 0, 0]$ | 3 4 6 |
| $[s_0, 0, s_0, s_2, 0]$ | 1 2 2 | $[t_0, t_1, 0, t_2, 0]$ | 3 4 3 |
| $[s_0, 0, 0, 0, s_1]$ | 1 1 3 | $[t_0, t_1, 0, 0, 0]$ | 3 2 2 |
| $[0, s_1, s_0, s_2, s_1]$ | 3 4 6 | $[0, t_1, t_0', t_2, t_3]$ | 1 1 1 |
| $[0, s_1, s_0', s_2, s_3]$ | 3 4 6 | $[0, t_1, t_0', t_2, 0]$ | 1 1 1 |
| $[0, s_1, s_0', 0, s_3]$ | 1 1 1 | $[0, t_1, t_0', 0, 0]$ | 3 4 6 |
| $[0, s_1, s_0, s_2, 0]$ | 1 2 2 | $[0, t_1, 0, t_2, 0]$ | 3 4 3 |
| $[s_0, 0, 0, 0, s_1]$ | 1 1 3 | $[0, t_1, 0, 0, t_1]$ | 3 2 2 |

and the corresponding grid $\Lambda_{M_1, M_2}^{\text{cr}}$ of weights has the following form:

\[ \Lambda_{M_1, M_2}^{\text{cr}} = \{ t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 | t_1, t_2, t_3 \in \mathbb{Z}_{\geq 0}, -M_1 < t_1 \leq M_1, t_2 + 2t_3 \leq M_2 \} \]

The discrete orthogonality relations of the functions $\Xi^{xy}$ are of the form (17). We label each point $x = \frac{s_0}{M_1} \omega_1^x + \frac{s_1}{M_2} \omega_2^x + \frac{s_3}{M_2} \omega_3^x \in \mathcal{F}_{M_1, M_2}^{\text{cr}}$ by coordinates $[s_0, s_1, s_0', s_2, s_3]$ such that $s_0 + s_1 = M_1$ and $s_0' + 2s_2 + s_3 = M_2$. Similarly, we label each point from $\lambda = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \in \Lambda_{M_1, M_2}^{\text{cr}}$ by coordinates $[t_0, t_1, t_0', t_2, t_1]$ such that $t_0 + t_1 = M_1$ and $t_0' + t_2 + 2t_3 = M_2$. The coefficients $\epsilon^{xy}$ and $h^{\text{cr}}_{xy}$ in (17) are listed in Table 4.

(2) $W^*(A_1 \times C_2)$. The $W^*$-orbit for the generic point $a \omega_1 + b \omega_2 + c \omega_3 \in \mathbb{R}^3$ is given by

\[ W^*(a, b, c) = \{(a, \pm b, \pm c), (a, \pm (2c + b), \mp (b + c)), (-a, \pm b, \mp (b + c)), (-a, \pm (2c + b), \mp (b + c))\} \]

and $|W^*| = 8$. The fundamental region $F^*$ is given by

\[ F^*(A_1 \times C_2) = \{ x \omega_1^x + y \omega_2^y + z \omega_3^z | 0 \leq x \leq 1, y, z \geq 0, 2y + z \leq 1 \} \]

\[ \cup \{ x \omega_1^x - y \omega_2^y + (z + 2y) \omega_3^z | 0 < x < 1, y, z > 0, 2y + z < 1 \} \]
4.4. The E-transforms of $A_1 \times G_2$

Relative length and angles between the simple roots and fundamental weights of $A_1 \times C_2$ are given in terms of the Cartan matrix $C = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $\det C = 2$.

(1) $W^{\text{re}}(A_1 \times G_2)$. The $W^{\text{re}}$-orbit for the generic point $a \omega_1 + b \omega_2 + c \omega_3 \in \mathbb{R}^3$ is given by

$W^{\text{re}}(a, b, c) = \{ (a, \pm b, \pm c), (a, \pm (2b + c), \mp (3b + c)), (a, \mp (b + c), \pm (3b + 2c)) \}$

and $|W^{\text{re}}| = 6$. The fundamental region $F^{\text{re}}$ is given by

$F^{\text{re}}(A_1 \times G_2) = \{ x \omega_1^\vee + y \omega_2^\vee + z \omega_3^\vee \mid -1 < x \leq 1, y, z \geq 0, 2y + 3z \leq 1 \}$

$\cup \{ x \omega_1^\vee - y \omega_2^\vee + (y + z) \omega_3^\vee \mid -1 < x \leq 1, y, z > 0, 2y + 3z < 1 \}$

and its volume is $|F^{\text{re}}| = \sqrt{6}/6$. We have the weight lattice $P^{\text{re}}$ of the form

$P^{\text{re}}(A_1 \times C_2) = \{ a \omega_1 + b \omega_2 + c \omega_3 \mid a, b, c \in \mathbb{Z} \}$

$\cup \{ a \omega_1 - b \omega_2 + (c + 3b) \omega_3 \mid a \in \mathbb{Z}, b, c \in \mathbb{N} \}$.

Then, for any $a \omega_1 + b \omega_2 + c \omega_3 \in P^{\text{re}}$, the corresponding $E^{\text{re}}$-function at point $x \omega_1^\vee + y \omega_2^\vee + z \omega_3^\vee$ is

$\Sigma^{\text{re}}_{(a, b, c)}(x, y, z) = 2e^{\text{str} \pi}(\cos(2\pi((2b + c)y + (3b + 2c)z)) + 2 \cos(2\pi((b + c)y + cz)))$

and the coefficients $d^{\text{re}}_{(a, b, c)}$ in continuous orthogonality relations (12) have values as shown in Table 2.
The discrete grid \( F^e_{M,M_e} \) has the explicit form

\[
F^e_{M,M_e} = \left\{ \frac{s_1}{M_1} \omega_1^e + \frac{s_2}{M_2} \omega_2^e + \frac{s_3}{M_2} \omega_3^e \mid s_1 \in \mathbb{Z}, s_2, s_3 \in \mathbb{Z}^\geq 0, 
-M_1 < s_1 \leq M_1, 2s_2 + 3s_3 \leq M_2 \right\}
\]

and the corresponding grid \( \Lambda^e_{M,M_e} \) of weights has the following form:

\[
\Lambda^e_{M,M_e} = \left\{ t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \mid t_1 \in \mathbb{Z}, t_2, t_3 \in \mathbb{Z}^\geq 0, -M_1 < t_1 \leq M_1, 3t_2 + 2t_3 \leq M_2 \right\}
\]

\[
\cup \left\{ t_1 \omega_1 + t_2 \omega_2 + (t_3 + 3t_2) \omega_3 \mid t_1 \in \mathbb{Z}, t_2, t_3 \in \mathbb{N}, -M_1 < t_1 \leq M_1, 3t_2 + 2t_3 < M_2 \right\}
\]

The discrete orthogonality relations of the functions \( \Xi^e \) are of the form (17). We label each point \( x = \frac{s_1}{M_1} \omega_1^e + \frac{s_2}{M_2} \omega_2^e + \frac{s_3}{M_2} \omega_3^e \) in \( F^e_{M,M_e} \) by coordinates \([s_0, s_1, s_0', s_2, s_3] \) such that \( s_0 + s_1 = M_1 \) and \( s_0' + 2s_2 + 3s_3 = M_2 \). Similarly, we label each point from \( \lambda = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 \) in \( \Lambda^e_{M,M_e} \) by coordinates \([t_0, t_1, t_0', t_2, t_3] \) such that \( t_0 + t_1 = M_1 \) and \( t_0' + 3t_2 + 2t_3 = M_2 \). The coefficients \( e^e(x) \) and \( h^e_{\lambda} \) in (17) are listed in Table 4.
4.5. The E-transforms of $A_1 \times A_1 \times A_1$

Relative length and angles between the simple roots and fundamental weights of $A_1 \times A_1 \times A_1$ are given in terms of the Cartan matrix $C = \begin{pmatrix} \frac{2}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$, det $C = 8$.

(1) $W^{ee}(A_1 \times A_1 \times A_1)$. The $W^{ee}$-orbit for the generic point $a\omega_1 + b\omega_2 + c\omega_3 \in \mathbb{R}^3$ is given by

$$W^{ee}(a, b, c) = \{(a, \pm b, \pm c), (a, \pm(2b + c), \mp(3b + c)), (a, \mp(b + c), \pm(3b + 2c)) \}$$

and $|W^{ee}| = 12$. The fundamental region $F^{ee}$ is given by

$$F^{ee}(A_1 \times A_1) = \{x\omega_1^\vee + y\omega_2^\vee + z\omega_3^\vee \mid -1 < x, y, z \leq 1\}$$

and its volume is $|F^{ee}| = 2\sqrt{2}$. We have the weight lattice $P_{ee}$ of the form

$$P_{ee}(A_1 \times A_1) = \{a\omega_1 + b\omega_2 + c\omega_3 \mid a, b, c \in \mathbb{Z}\}$$
Then, for any $a\omega_1 + b\omega_2 + c\omega_3 \in P_e$, the corresponding $F^e$-function at point $x\omega_1^y + y\omega_2^y + z\omega_3^y$ is

$$
\Xi_{(a,b,c)}^e(x, y, z) = e^{i(a x + by + cz)}
$$

and the coefficients $d_{(a,b,c)}^e$ in continuous orthogonality relations (12) have values as shown in table 2.

The discrete grid $F_{M_1,M_2,M_3}^e$ has the explicit form

$$
F_{M_1,M_2,M_3}^e = \left\{ \frac{s_1}{M_1} \omega_1^y + \frac{s_2}{M_2} \omega_2^y + \frac{s_3}{M_3} \omega_3^y \mid s_i \in \mathbb{Z}, -M_i < s_i \leq M_i, i = 1, 2, 3 \right\}
$$

and the corresponding grid $\Lambda_{M_1,M_2,M_3}^e$ of weights has the following form:

$$
\Lambda_{M_1,M_2,M_3}^e = \{ t_1\omega_1 + t_2\omega_2 + t_3\omega_3 \mid t_i \in \mathbb{Z}, -M_i < t_i \leq M_i, i = 1, 2, 3 \}.
$$

The discrete orthogonality relations of the functions $\Xi^e$ are of the form (17) with the resulting normalization coefficient equal to $8M_1M_2M_3h_3^{e,\nu}$. The coefficients $e^e(x)$ and $h_3^{e,\nu}$ in (17) are all equal to 1.

(2) $W^e(\Lambda_1 \times \Lambda_1 \times \Lambda_1)$. The $W^e$-orbit for the generic point $a\omega_1 + b\omega_2 + c\omega_3 \in \mathbb{R}^3$ is given by

$$
W^e(a, b, c) = \{(a, \pm b, \pm c), (-a, \pm b, \mp c)\}
$$

and $|W^e| = 4$. The fundamental region $F^e$ is given by

$$
F^e(\Lambda_1 \times \Lambda_1 \times \Lambda_1) = \left\{ x\omega_1^y + y\omega_2^y + z\omega_3^y \mid 0 \leq x, y, z \leq 1 \right\}
$$

$$
\cup \left\{ -x\omega_1^y + y\omega_2^y + z\omega_3^y \mid 0 < x, y, z < 1 \right\}
$$

and its volume is $|F^e| = 1/\sqrt{2}$. We have the weight lattice $P_e$ of the form

$$
P_e(\Lambda_1 \times \Lambda_1 \times \Lambda_1) = \{ a\omega_1 + b\omega_2 + c\omega_3 \mid a, b, c \in \mathbb{Z}^3 \}
$$

$$
\cup \{ -a\omega_1 + b\omega_2 + c\omega_3 \mid a, b, c \in \mathbb{N} \}.
$$

Then, for any $a\omega_1 + b\omega_2 + c\omega_3 \in P_e$, the corresponding $F^e$-function at point $x\omega_1^y + y\omega_2^y + z\omega_3^y$ is

$$
\Xi_{(a,b,c)}^e(x, y, z) = 2e^{i\pi a x} \cos \pi(by + cz) + 2e^{-i\pi a x} \cos \pi(by - cz)
$$

and the coefficients $d_{(a,b,c)}^e$ in continuous orthogonality relations (22) have values as shown in table 3.

The discrete grid $F_M^e$ has the explicit form

$$
F_M^e = \left\{ \frac{s_1}{M} \omega_1^y + \frac{s_2}{M} \omega_2^y + \frac{s_3}{M} \omega_3^y \mid s_1, s_2, s_3 \in \mathbb{Z}^3, s_1, s_2, s_3 \leq M \right\}
$$

$$
\cup \left\{ -\frac{s_1}{M} \omega_1^y + \frac{s_2}{M} \omega_2^y + \frac{s_3}{M} \omega_3^y \mid s_1, s_2, s_3 \in \mathbb{N}, s_1, s_2, s_3 < M \right\}
$$

and the corresponding grid $\Lambda_M^e$ of weights has the following form:

$$
\Lambda_M^e = \{ t_1\omega_1 + t_2\omega_2 + t_3\omega_3 \mid t_1, t_2, t_3 \in \mathbb{Z}^3, t_1, t_2, t_3 \leq M \}
$$

$$
\cup \{ -t_1\omega_1 + t_2\omega_2 + t_3\omega_3 \mid t_1, t_2, t_3 \in \mathbb{N}, t_1, t_2, t_3 < M \}.
$$

The discrete orthogonality relations of the functions $\Xi^e$ are of the form (25) with the resulting normalization coefficient equal to $32M^3h_3^{e,\nu}$. We label each point $x = \frac{t_0}{M} \omega_1^y + \frac{t_1}{M} \omega_2^y + \frac{t_2}{M} \omega_3^y \in F_M^e$ by coordinates $[s_0, s_1, s_2, s_3, s_4, s_5, s_6]$ such that $s_0 + s_1 = M$, $s_0' + s_2 = M$ and $s_3' + s_3 = M$. Similarly, we label each point from $\Lambda = t_1\omega_1 + t_2\omega_2 + t_3\omega_3 \in \Lambda_M^e$ by coordinates $[t_1, t_2, t_3', t_3'' , t_3''']$ such that $t_0 + t_1 = M$, $t_0' + t_2 = M$ and $t_0'' + t_3 = M$. The coefficients $e^e(x)$ and $h_3^{e,\nu}$ in (25) are listed in table 6.
5. Concluding remarks

- Given a function \( f : F^r \to \mathbb{C} \) or \( f : F^r \to \mathbb{C} \), we may define interpolating functions in the usual way:

\[
\Xi_{M,M}^{ce}(x) = \sum_{\lambda \in \Lambda_M} \tilde{c}_\lambda \Xi_{\lambda}^{ce}(x), \quad x \in \mathbb{R}^{n_1 + n_2}
\]

\[
\Xi_{M}^e(x) = \sum_{\lambda \in \Lambda_M^e} c_\lambda \Xi_{\lambda}^{e}(x), \quad x \in \mathbb{R}^{n_1 + n_2}
\]

where \( \tilde{c}_\lambda \), \( c_\lambda \) are given by formulas (18) and (26), respectively. These interpolating functions then coincide with \( f \) on the grids \( F^r_{M,M} \), \( F^r_M \). Interpolation properties of \( \Xi_{M,M}^{ce} \), \( \Xi_{M}^{e} \) as well as convergence of these functional series deserve further study.

- Product-to-sum decompositions have for both \( E^r \)- and \( E^c \)-functions the following straightforward form:

\[
\Xi_{\lambda}^{ce}(x) \Xi_{\lambda'}^{ce}(x) = \sum_{\omega \in W} \Xi_{\omega + \omega', \lambda}^{ce}(x), \quad x \in \mathbb{R}^{n_1 + n_2}, \lambda, \lambda' \in \mathbb{Z}.
\]
\[ \Xi^\varepsilon_{\lambda}(x) \Xi^{\varepsilon}_{\lambda'}(x) = \sum_{w \in W^v} \Xi^\varepsilon_{\lambda+w\lambda'}(x), \quad x \in \mathbb{R}^{n_1+n_2}, \lambda, \lambda' \in P. \]

- For semisimple \( G = G_1 \times G_2 \) with \( W = W_1 \times W_2 \), there are another possible classes of special functions. One may consider \( C \)- or \( S \)-functions related to component \( W_1 \) and \( E \)-functions related to component \( W_2 \) or vice versa. Such ‘mixed’ functions then inherit properties from the corresponding simple components, such as being eigenfunctions of the Laplace operator with different boundary conditions.
- Similarly as the common exponential function can be seen as the sum of the cosine and sine functions, the \( E \)-functions are realized in any dimension as the sum of the appropriate \( C \)- and \( S \)-function. Such functions are considered here for the cases where the underlying Lie group is semisimple but not simple. Recent discovery of additional families of \( W \)-invariant (skew-invariant) functions \([8]\) opens several new possibilities of studying \( E \)-like functions in our problems.

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