Charged Nariai Black Holes With a Dilatonic Potential

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Abstract

The Reissner-Nordström-de Sitter black holes of standard Einstein-Maxwell theory with a cosmological constant have no analogue in dilatonic theories with a Liouville potential. The only exception are the solutions of maximal mass, the Charged Nariai solutions. We show that the structure of the solution space of the Dilatonic Charged Nariai black holes is quite different from the non-dilatonic case. Its dimensionality depends on the exponential coupling constants of the dilaton. We discuss the possibility of pair creating such black holes on a suitable background. We find conditions for the existence of Charged Nariai solutions in theories with general dilaton potentials, and consider specifically a massive dilaton.

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\section{Introduction}

This paper is motivated by two questions which have received much attention in recent years: the pair creation of black holes on a cosmological background via instantons \[1, 2, 3, 4, 5, 6, 7\], and the generalisation of cosmological black hole solutions to dilatonic theories \[8, 9, 10, 11, 12, 13, 14\].

Charged Nariai (CN) spacetimes are metrics which can be written as the direct topological product of 1 + 1-dimensional de Sitter space with a round two-sphere. They represent a pair of black holes immersed in de Sitter space. They admit a Euclidean section, which is given by the topological product of two two-spheres (not necessarily of the same radius). Thus they mediate a pair creation process on the background of de Sitter space, if such a background is available in a given theory. In this paper we investigate CN spacetimes occurring in dilatonic theories.

If nature is described by string theory, a dilaton field must be included in the low-energy gravitational action. The solutions and solution spaces of such theories can differ significantly from those of standard Einstein gravity. Dilatonic black holes have proven useful for the study of black hole entropy, thermodynamics and pair creation\[15, 16, 17, 18, 19\]. It is therefore of interest to look for dilatonic analogues to the standard black hole solutions.

Here we start from Einstein-Maxwell theory with a cosmological constant,

\[ L = (-g)^{1/2} \left( R - 2\Lambda - F^2 \right), \]

where

\[ F^2 = F_{\mu\nu}F^{\mu\nu}, \quad g = \det(g_{\mu\nu}). \]

This Lagrangian admits a three-parameter family of static charged black hole solutions:

\[ ds^2 = -U(r) \, dt^2 + U(r)^{-1}dr^2 + r^2 d\Omega_2^2, \]

where

\[ U(r) = 1 - \frac{2\mu}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2 \]

and \( d\Omega_2^2 \) is the metric on the unit two-sphere. The black holes can be either magnetically charged,

\[ F = Q \sin \theta \, d\theta \wedge d\phi, \]

or electrically charged,

\[ F = \frac{Q}{r^2} \, dt \wedge dr. \]
They are called Reissner-Nordström-de Sitter (RNdS) solutions.

The inclusion of a dilaton field will modify the Lagrangian, Eq. (1.1). We shall work in the so-called Einstein frame, where the dilaton field $\phi$ couples to the Maxwell field and to the cosmological term. The Lagrangian then reads:

$$L = \left(-g\right)^{1/2} \left(R - 2(\nabla \phi)^2 - 2\Lambda e^{2b\phi} - e^{-2a\phi} F^2 \right).$$

(1.7)

Because it arises naturally from a cosmological constant term in the string frame, we have chosen a Liouville potential for the dilaton. For definiteness, we shall stick to this potential through most of this paper. However, one can easily extend our methods to more general potentials; this is discussed in the final section.

The Lagrangian, Eq. (1.7), is invariant under the transformation

$$a \to -a, \quad b \to -b, \quad \phi \to -\phi.$$  

(1.8)

We fix a gauge by choosing $a \geq 0$. The variation of the action with respect to the metric, Maxwell, and dilaton fields yields the following equations of motion:

$$R_{\mu\nu} = 2\nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \Lambda e^{2b\phi} + 2e^{-2a\phi} \left(F_{\mu\lambda} F^\lambda_{\nu} - \frac{1}{4} g_{\mu\nu} F^2 \right),$$

(1.9)

$$0 = \nabla_\mu \left(e^{-2a\phi} F^{\mu\nu} \right),$$

(1.10)

$$2\nabla^2 \phi = -ae^{-2a\phi} F^2 + 2\Lambda be^{2b\phi}.$$  

(1.11)

One might hope that these equations give rise to a three-parameter family of dilaton black holes, which would be analogous to the RNdS black holes. Poletti and Wiltshire have shown, however, that such solutions do not exist for a Liouville potential with a positive cosmological constant [11]. This no-go theorem is related to the simple fact there exists no static de Sitter-type solution that could act as a background. Correspondingly, the theorem only bans dilatonic RNdS solutions which approach de Sitter space asymptotically.

The paper is outlined as follows. We begin by reviewing some properties of the RNdS solutions, in Sec. 2. We point out that they are all asymptotically de Sitter except for a two-dimensional subspace, the CN solutions, which are special in two ways. Firstly, unlike most other RNdS solutions, they admit smooth Euclidean sections. Therefore they can be pair created, as we discuss in Sec. 2.3. Secondly, because of their unusual asymptotics, the CN solutions are the only kind of RNdS solutions that avoid the Poletti-Wiltshire no-go theorem. Thus they can be generalised to dilatonic theories with a Liouville potential.
The Dilatonic Charged Nariai (DCN) solutions are easily found, since the dilaton will be constant. Nevertheless, the solution space that they form turns out to be quite interesting. In Sec. 3 we show that the range of some parameters differs significantly from the non-dilatonic case. We find that both the type of black hole charge, and the dimensionality of the solution space depends on the exponential coupling constants $a$ and $b$. Like the CN solutions, the DCN solutions admit a smooth Euclidean section. In Sec. 4 we calculate the Euclidean action of this instanton, and discuss whether it can mediate black hole pair creation. Finally, we show in Sec. 5 how to find DCN black holes for a general dilaton potential. As an example, we consider the case of a massive dilaton.

2 Standard Reissner-Nordström-de Sitter

2.1 Solution Space

In the RNdS solutions, Eq. (1.4) contains three parameters, $Q$, $\mu$ and $\Lambda$. $Q$ is the charge of the black hole, and $\mu$ is related to the mass. Strictly speaking, the cosmological constant $\Lambda$ is not a parameter of the solution space, but rather of the space of theories. Since the cosmological constant is nearly zero in our universe, however, a cosmological term in the action will realistically be due to the vacuum energy of a scalar field. In inflationary cosmology, this field starts out large and changes slowly over time. Its potential will act like an effective cosmological constant $\Lambda_{\text{eff}} > 0$. For many purposes it can be approximated by a fixed $\Lambda$, as we do here. $\Lambda_{\text{eff}}$, however, will depend on the value of the inflaton field. Thus it is a degree of freedom in the solutions, and is not specified by the theory. It is with a view to applications in cosmology, therefore, that we include $\Lambda$ among the solution space parameters.

What are the ranges of the three parameters? We shall take $\Lambda$ to be positive. $\mu$ must be non-negative to avoid naked singularities. For the same reason, the black hole may not be larger than the cosmological horizon. This limits its mass:

$$\mu^2 \Lambda \leq \frac{1}{18} \left[ 1 + 12Q^2 \Lambda + (1 - 4Q^2 \Lambda)^{3/2} \right].$$

In particular, $\mu^2 \Lambda$ can never exceed $2/9$. The charge is limited by the Bogomol’nyi bound, which we give here only approximately (see [3] for more details):

$$Q^2 \leq \mu^2 \left[ 1 + \frac{1}{3}(\mu^2 \Lambda) + O(\mu^4 \Lambda^2) \right].$$

4
Fig. 1 shows a two-dimensional projection of the parameter space for a fixed value of $\Lambda$. As special cases, the RNdS solutions include de Sitter space ($Q = \mu = 0$) and the Schwarzschild-de Sitter solutions ($Q = 0, \mu > 0$).

Figure 1: The Reissner-Nordström-de Sitter solutions are the points on or within the thick line. The plot is of the dimensionless quantities $\mu^2 \Lambda$ vs. $Q^2 \Lambda$.

2.2 Asymptotic Structure and Charged Nariai

The asymptotic structure of the RNdS solutions plays an important and problematic role when one tries to include a dilaton; therefore we shall dwell on it for a while. In a generic region of the parameter space, $U$ has three positive roots, which are denoted, in ascending order, by $r_i$, $r_o$ and $r_c$. They are the radii of the inner and outer black hole horizons and the cosmological horizon. The causal structure of the spacetime can be seen from its Carter-Penrose diagram, Fig. 2. An observer who ventures outside $r = r_c$ will find herself in region IV, where $r$ is a time coordinate and increases without bound. By Eq. (1.4), the spacetime looks more and more like de Sitter space as $r \to \infty$:

$$U(r) \to 1 - \frac{1}{3} \Lambda r^2.$$  \hfill (2.3)
Therefore the RNdS solutions represent a pair of black holes immersed in an asymptotically de Sitter universe. (The fact that it is a pair can be seen from the Carter-Penrose diagram.)

The causal structure will be different for solutions on the border of the parameter space, such as the de Sitter and neutral Schwarzschild-de Sitter universes, and the extreme ("cold") black holes (which have maximal charge at a given mass). Still, all these solutions have the same asymptotic structure, that is, they all look like de Sitter space at large distances from the black hole. Their Carter-Penrose diagrams look different from Fig. 2, but they all contain a region of type IV.

The only exception to this rule are the so-called Charged Nariai (CN) solutions, the two parameter family of solutions for which the mass is maximal at a given charge, as indicated in Fig. 1. In these solutions \( r_o = r_c \); therefore \( r \) is not a suitable coordinate for the region between the outer and cosmological horizons. An appropriate coordinate transformation was first given for the neutral case in Ref. [1]. It shows that as \( r_o \to r_c \), the region between \( r_o \) and \( r_c \) does not vanish. A refinement and a more detailed discussion can be found in the Appendix of Ref. [7]. The transformation can be readily generalised to include charged solutions [5]. We
set \( r_o = \rho - \epsilon, \ r_c = \rho + \epsilon \). With the coordinate transformation

\[
r = \rho + \epsilon \cos \chi, \ t = \frac{1}{U(\rho)} \epsilon \psi_I,
\]

(2.4)

the RNdS metric, Eq. (1.3), becomes

\[
ds^2 = \frac{1}{A} \left( - \sin^2 \chi d\psi_I^2 + d\chi^2 \right) + \frac{1}{B} d\Omega_2^2
\]

(2.5)
in the limit \( \epsilon \to 0 \), where

\[
A = \lim_{\epsilon \to 0} \frac{U(\rho)}{\epsilon^2}, \ B = \frac{1}{\rho^2},
\]

(2.6)

and \( d\Omega_2^2 \) is the metric on a round two-sphere of unit radius. The Maxwell field is given by

\[
F = Q \sin \theta \ d\theta \wedge d\phi
\]

(2.7)

(and therefore \( F^2 = 2B^2Q^2 \)) in the magnetic case, and by

\[
F = Q \frac{B}{A} \sin \chi \ d\chi \wedge d\psi_I
\]

(2.8)

(and therefore \( F^2 = -2B^2Q^2 \)) in the electric case.

It will be useful to define

\[
g = \frac{F^2}{2\Lambda}.
\]

(2.9)

\( g \) is positive (negative) for magnetic (electric) black holes. One can conveniently express the parameters \( A \) and \( B \) as

\[
A = \Lambda (1 - |g|), \ B = \Lambda (1 + |g|).
\]

(2.10)

Therefore we have \( A < B \) except in the neutral case, when \( A = B \). A metric with \( A > B \) does not admit a real Maxwell field. Since \( A \) must be positive, solutions exist only for \( |g| < 1 \). They can be parametrised by \( (\Lambda, g) \) or \( (\Lambda, F^2) \), for instance. Specification of the metric, i.e. of \( (A, B) \), determines the Maxwell field up to the type of charge. The same holds for the parameters \( (\Lambda, Q) \).

The CN metric, Eq. (2.5), is just the topological product of \( 1 + 1 \) dimensional de Sitter space with a round two-sphere. It is a homogeneous space-time with the same causal structure as \( 1 + 1 \) dimensional de Sitter space (see Fig. 3). In this degenerate solution, an observer sees a cosmological horizon on either side of her
(that is, in both the positive and negative $\chi$ direction). There is no black hole she could fall into. Like in de Sitter space, if she crosses the cosmological horizon (of a different observer), the space-time will look exactly the same to her as before.

This means, in particular, that the CN solutions do not possess a de Sitter-like asymptotic region. The radius of the two-spheres is constant: $r = B^{-1/2}$ everywhere. Precisely for this reason, the CN solutions avoid the Poletti-Wiltshire no-go theorem. We shall show in Sec. 3 that they possess dilatonic analogues.

We should stress that the CN solutions are quantum mechanically unstable. Because of quantum fluctuations, the radius $B$ of the two-spheres will vary slightly along the spatial variable $\chi$. There will be a minimum and a maximum two-sphere, which correspond to a black hole and a cosmological horizon [1, 7]. Thus the CN solutions necessarily become ordinary, slightly non-degenerate RNdS space-times. While this requires only a minute fluctuation of the horizon radii $r_o$ and $r_c$, the causal structure immediately reverts to that of Fig. 3.

### 2.3 Euclidean Solutions

Euclidean solutions of the Einstein equations (instantons) can be used for the semi-classical description of non-perturbative gravitational processes, such as the spontaneous quantum mechanical creation of a pair of black holes on some background.
Here we are interested in the pair creation rate of RNdS black holes. The appropriate background is de Sitter space, which the black hole solutions approach asymptotically. (de Sitter space is also the appropriate background for the CN solutions, since the quantum fluctuations of the geometry cannot be neglected in the semi-classical treatment.) One must find an instanton which can be analytically continued to match a spacelike surface $\Sigma$ of the Lorentzian background spacetime, and similarly one must find an instanton for the spacetime with the black holes. Then one calculates the Euclidean actions of these solutions, $I_{bg}$ and $I_{bh}$. Neglecting a prefactor, the pair creation rate $\Gamma$ is given by

$$\Gamma = \exp \left[ -2 \left( I_{bh} - I_{bg} \right) \right].$$

(2.11)

Only those Lorentzian black hole solutions which possess regular Euclidean sections can be pair created. In particular, not all RNdS solutions can be obtained as analytic continuations of smooth Euclidean solutions [3, 4, 5]. Of the entire three-parameter set, there are only three (intersecting) two-parameter subsets of RNdS solutions that can be pair created: the cold, the “lukewarm” ($Q = \mu$), and the CN black holes. We shall not bother ourselves with the cold and lukewarm cases, since they cannot be retained when we introduce a dilaton later. The CN black holes, however, will have dilatonic analogues, and luckily, they have a regular Euclidean section:

$$ds^2 = \frac{1}{A} \left( \sin^2 \chi d\psi_R^2 + d\chi^2 \right) + \frac{1}{B} d\Omega^2.$$  

(2.12)

This is the topological product of two round two-spheres of radii $A^{-1/2}$ and $B^{-1/2}$. The Maxwell field is given by

$$F = Q \sin \theta \, d\theta \wedge d\phi$$  

in the magnetic case, and by

$$F = -iQ \frac{B}{A} \sin \chi \, d\chi \wedge d\psi_R$$  

(2.14)

in the electric case. The Lorentzian solutions can be recovered by setting $\psi_R = i\psi_I$.

The Euclidean action is given by

$$I = -\frac{1}{16\pi} \int d^4x \, g^{1/2} \left( R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} \right) - \frac{1}{8\pi} \int_{\Sigma} d^3x \, h^{1/2}K,$$  

(2.15)

where $K$ is the trace of the second fundamental form and $h$ is the determinant of the three-metric on $\Sigma$. Since the sign of $F^2$ depends on the type of charge, it would
seem that electric and magnetic black holes would have different pair creation rates. Even worse, highly charged electric black holes \( g < -1/3 \) would have a lower action than de Sitter space and their pair creation would thus be enhanced relative to the background. However, this problem only appears because the above action is inappropriate for electrically charged black holes. Variation of this action will give the correct equations of motion only if the gauge potential \( A_\nu \) is held fixed on the boundary \( \Sigma \). This is appropriate for magnetic solutions, since the gauge potential determines the magnetic charge. Generally, the charge must be held fixed on variation of the action. In the electric case, however, this corresponds to fixing \( F^{\mu\nu} n_\mu \), rather than the gauge potential, on \( \Sigma \). In order to retain a stationary action, one must compensate this by introducing an extra boundary term \cite{4}:

\[
I_{(\text{electric})} = I - \frac{1}{4\pi} \int_{\Sigma} d^3x \ h^{1/2} F^{\mu\nu} n_\mu A_\nu, \quad (2.16)
\]

where \( n_\mu \) is the normal to \( \Sigma \).

For the background de Sitter instanton, the Euclidean action is

\[
I_{\text{dS}} = -\frac{3\pi}{2\Lambda}. \quad (2.17)
\]

For the CN instanton, we obtain an action of \(-\pi/\Lambda(1 + g)\) in the magnetic case, and \(-\pi/\Lambda(1 - g)\) in the electric case. Thus, for both cases,

\[
I_{\text{CN}} = -\frac{\pi}{\Lambda(1 + |g|)}. \quad (2.18)
\]

As one would hope, electric and magnetic black holes of the same charge thus have the same action, and the same pair creation rate,

\[
\Gamma_{\text{CN}} = \exp \left[ -2 (I_{\text{CN}} - I_{\text{dS}}) \right] = \exp \left[ -\frac{\pi}{\Lambda} \frac{1 + 3|g|}{1 + |g|} \right]. \quad (2.19)
\]

3 Dilatonic Charged Nariai Solutions

3.1 Solutions

We shall now present the dilatonic equivalent to the CN black hole solutions of the previous section. In a solution that is static and spherically symmetric, the dilaton
can only depend on the radius $r$, which is fixed in the CN case. Thus the dilaton will be constant. By comparing the Lagrangians, Eqs. (1.1) and (1.7), we see that the introduction of a constant dilaton leads only to a rescaling of the cosmological constant and the Maxwell field. For convenience we define

$$
\tilde{F}^{\mu \nu} = e^{-a\phi} F^{\mu \nu}, \quad (3.1)
$$

$$
\tilde{\Lambda} = e^{2b\phi} \Lambda, \quad (3.2)
$$

$$
\tilde{g} = \frac{\tilde{F}^2}{2\Lambda}, \quad (3.3)
$$

With these definitions the solutions take the same form as in the previous section, with $(\Lambda, g)$ replaced by $(\tilde{\Lambda}, \tilde{g})$:

$$
ds^2 = \frac{1}{\tilde{A}} \left( -\sin^2 \chi d\psi_1^2 + d\chi^2 \right) + \frac{1}{\tilde{B}} d\Omega_2^2, \quad (3.4)
$$

where

$$
\tilde{A} = \tilde{\Lambda} (1 - |\tilde{g}|), \quad \tilde{B} = \tilde{\Lambda} (1 + |\tilde{g}|). \quad (3.5)
$$

As before, the Maxwell field is given by

$$
F = Q \sin \theta \, d\theta \wedge d\phi \quad (3.6)
$$

in the magnetic case ($F^2 = 2Q^2 \tilde{B}^2$), and by

$$
F = Q \frac{B}{A} \sin \chi \, d\chi \wedge d\psi_1 \quad (3.7)
$$

in the electric case ($F^2 = -2Q^2 \tilde{B}^2$). The value of $\phi$ can be obtained from the dilaton equation of motion. Since the dilaton must be constant, the left hand side of Eq. (1.11) vanishes, and we get

$$
\phi = \frac{1}{2(a + b)} \ln \left( \frac{a}{b} \frac{F^2}{2\Lambda} \right) = \frac{1}{2(a - b)} \ln \left[ \pm \frac{a}{b} Q^2 \Lambda (1 + |\tilde{g}|)^2 \right], \quad (3.8)
$$

where the upper (lower) sign is for magnetic (electric) black holes.

Like the CN solutions, these solutions are quantum mechanically unstable. Quantum perturbations will cause the separation of the cosmological and black hole horizons. Since no non-degenerate static dilatonic RNdS solutions exist, the dilaton will be time dependent beyond the cosmological horizon in these perturbed solutions.

We shall see below that the space of DCN solutions is quite different from its non-dilatonic counterpart, in spite of the fact that they can be written in a deceptively similar form.
3.2 Solution Space

The CN and DCN solutions take the same form when written in terms of \((\Lambda, g)\) and \((\tilde{\Lambda}, \tilde{g})\). The variables \(g\) and \(\tilde{g}\), however, are of a fundamentally different nature. \(g\), as we have seen, parametrises the CN solutions along with \(\Lambda\). \(\tilde{g}\), on the other hand, is not a parameter of the solution space, since by Eqs. (1.11) and (3.3) \(\tilde{g}\) simplifies to

\[
\tilde{g} = \frac{b}{a}.
\]  

(3.9)

This simple result is central to the differences between the dilatonic and non-dilatonic CN solution spaces. It means that \(\tilde{g}\) is completely fixed by the couplings of the theory and does not constitute a degree of freedom. While most results for the DCN solutions can be obtained from analogous results for the CN solutions by replacing \((\Lambda, g)\) with \((\tilde{\Lambda}, \tilde{g})\) throughout, it is important to keep in mind that \(\tilde{g}\) is no longer a free parameter; it can be replaced by \(b/a\). In particular, the constraints on \(g\) (\(|g| < 1\)) will translate into constraints on the dilatonic theories that allow DCN solutions; this will be discussed in Sec. 3.3 below. Before that, it will be necessary to determine the degrees of freedom of the DCN solution space, and their range.

For this purpose we shall repeatedly exploit Eq. (3.9). Firstly, by Eq. (3.9), the type of the black hole charge is determined by the theory:

\[
\text{sgn}(F^2) = \text{sgn}(\tilde{g}) = \text{sgn}(b).
\]  

(3.10)

If \(b\) is positive, the charges are magnetic; if it is negative, they are electric. Electric-magnetic duality, in this case, is not a duality between pairs of solutions to the same theory, but between solutions in two different theories, \(b\) and \(-b\). We can now simplify Eq. (3.8) to

\[
\phi = \frac{1}{2a(1 - \tilde{g})} \ln \left[ \frac{(1 + |\tilde{g}|)^2}{|\tilde{g}|} Q^2 \Lambda \right].
\]  

(3.11)

Secondly, by Eq. (3.9), the DCN metric, Eq. (3.4), contains only one degree of freedom, \(\tilde{\Lambda}\), which determines its overall scale. The ratio \(\tilde{A}/\tilde{B}\), which is the more interesting geometrical information, is fixed from the start by the absolute value of \(\tilde{g}\):

\[
\frac{\tilde{A}}{\tilde{B}} = \frac{1 - |\tilde{g}|}{1 + |\tilde{g}|} = \frac{|a - b|}{|a + b|}.
\]  

(3.12)

Where did the other degree of freedom go? The CN metrics of standard Einstein-Maxwell theory carry two degrees of freedom. A variation of the field necessarily
changes the metric. This is no longer true in the dilatonic case. Holding \( \tilde{\Lambda} \) fixed, one can still vary \( Q \), or equivalently, \( \phi \). Let us therefore choose \( Q \) as a second parameter of the DCN solution space. But if the DCN metric does not contain information about the charge, maybe \( Q \) is just a gauge degree of freedom? The question is whether and how \( Q \) can be measured.

An observer in a CN universe can determine \( Q \) by observing the motion of a single test particle of unit charge. Sadly, this method fails for the DCN solutions. The particle will feel a force of \( |e^{-2a\phi}F^2|^{1/2} \), which is equal to \( |\tilde{F}^2|^{1/2} \) and therefore independent of \( Q \) for fixed \( \tilde{\Lambda} \). It cannot distinguish between different values of \( Q \). The observer is thus forced to resort to two test particles, each of unit charge. From their interaction strength she can infer the value of the dilaton. This in turn determines \( Q \). Therefore \( Q \) is a proper degree of freedom, in spite of the fact that it is not reflected in the metric.

What are the ranges of the parameters of the solution space, \( \tilde{\Lambda} \) and \( Q \)? Since \( \Lambda > 0 \), \( \tilde{\Lambda} \) can take on any positive value by Eq. (3.2). Freed from the geometrical constraints that would limit its range in the CN case, the charge can take on any real value except zero. (The case \( Q = 0 \), which implies \( b = 0 \), will be discussed below.)

3.3 Theory Space

The exponential coupling constants \( a \) and \( b \) form a two-parameter space of theories: \( a, b \in \mathbb{R}; a \geq 0 \). Let \( S(a, b) \) denote the space of DCN solutions admitted by the theory \((a, b)\). We shall show that \( S(a, b) \) is either one-dimensional, or two-dimensional, or empty, depending on \( a \) and \( b \) (see Fig. 4).

First consider the case \( b = 0 \). By the following chain of equivalences:

\[
\begin{align*}
b = 0 \quad &\quad \tilde{g} = 0 \quad \tilde{F}^2 = 0 \quad F^2 = 0 \quad Q = 0,
\end{align*}
\]

\( b = 0 \) if and only if \( Q = 0 \). There is no Maxwell field in this case, and so the dilaton effectively vanishes from the scene. The corresponding solution is the non-dilatonic neutral Nariai solution. \( \Lambda = A = B \) is the only degree of freedom. This is the only overlap of solution spaces of dilatonic and non-dilatonic theories.

Next consider the solutions described in Sec. 3.1 above, with two degrees of freedom, \( \tilde{\Lambda} \) and \( Q \). For \( b > 0 \) they are magnetically charged, and for \( b < 0 \) they are electrically charged. Since \( \tilde{A} \) must be positive, we obtain \( |b/a| < 1 \) by Eq. (3.6). Together with the condition that \( b \neq 0 \) and our gauge choice \( a \geq 0 \), this yields the
The dilatonic theories we consider can be labelled by the exponential coupling constants $a$ and $b$ that appear in the action. For $0 < |b| < a$, they admit the two-dimensional DCN solution space discussed above. For $b = 0$, only a one-dimensional solution space is allowed, the non-dilatonic neutral Nariai solutions. For other theories, no DCN-type solutions can be found.

\begin{equation}
0 < \frac{|b|}{a} < 1,
\end{equation}

for which a two-dimensional space of properly dilatonic CN solutions is admitted. It is interesting to note that these geometric considerations, which fail to limit the charge $Q$ of dilatonic black holes, constrain instead the range of dilatonic theories which admit DCN solutions at all.

The remaining theories have $|b| \geq a$, and $(a, b) \neq (0, 0)$. By Eq. (3.12) this would imply that $\tilde{A}$ were zero or negative. Therefore no DCN solutions are admitted in this case.

4 Euclidean Action and Pair Creation

Just like the non-dilatonic CN solutions of Sec. 3, the DCN solutions are analytic continuations of regular Euclidean solutions, given by the topological product of two
round two-spheres:

\[ ds^2 = \frac{1}{\tilde{A}} \left( \sin^2 \chi d\psi_R^2 + d\chi^2 \right) + \frac{1}{\tilde{B}} d\Omega_2^2, \]

(4.1)

with \( \tilde{A} \) and \( \tilde{B} \) given by Eq. (3.5). The Euclidean dilatonic action is

\[
I = -\frac{1}{16\pi} \int d^4x \, g^{1/2} \left( R - 2(\nabla \phi)^2 - 2\Lambda e^{2b\phi} - e^{-2a\phi} F^2 \right)
- \frac{1}{8\pi} \int_\Sigma d^3x \, h^{1/2} K.
\]

(4.2)

The action for the DCN instantons can be obtained from the CN action, Eq. (2.18), by replacing \((\Lambda, g)\) with \((\tilde{\Lambda}, \tilde{g})\):

\[
I_{\text{DCN}} = -\frac{\pi}{\tilde{\Lambda}(1 + |\tilde{g}|)}.
\]

(4.3)

These instantons do not, however, readily correspond to a pair creation process. For that purpose one would need a suitable background, and we have seen earlier that static de Sitter-like solutions do not exist in dilatonic theories. Even the abandonment of staticity alone will not help, since the dilaton would be pushed towards negative infinity by the cosmological constant. However, if we replace the fixed cosmological constant by a slowly decreasing effective cosmological constant \( \Lambda_{\text{eff}} \), the dilaton will only decrease as long as \( \Lambda_{\text{eff}} \) is large. For \(|b| \ll 1\) the dilaton will change slowly over time. Such solutions are well known in inflationary cosmology. They have been used by a number of authors \([20, 21, 22]\) for the description of extended chaotic inflation. Because of its similarity to de Sitter space, an inflating universe is a suitable background for the pair creation of black holes \([6, 7]\). With the modification \( \Lambda \rightarrow \Lambda_{\text{eff}} \), the DCN instantons will describe the nucleation of a dilatonic black hole pair immersed in an inflating universe. A detailed discussion of this effect is beyond the scope of this paper and will be presented separately. We shall only outline some of the expected results below.

In the absence of any rapidly evolving variables, one can approximate the background action by the de Sitter value \([3]\):

\[
I_{\text{DdS}} = -\frac{3\pi}{2\Lambda_{\text{eff}}}.
\]

(4.4)

Thus we obtain for the pair creation rate during inflation:

\[
\Gamma_{\text{DCN}} = \exp \left[ -2 (I_{\text{DCN}} - I_{\text{DdS}}) \right] = \exp \left[ -\frac{\pi}{\Lambda_{\text{eff}}} \frac{1 + 3|\tilde{g}|}{1 + |\tilde{g}|} \right].
\]

(4.5)
Again the form is similar to the result without a dilaton, Eq. (2.19). But the fact that \( \tilde{g} \) is not a degree of freedom will lead to an interesting qualitative difference. In inflation without a dilaton, the value of the inflaton field fixes the background effective cosmological constant, but \( g \) remains a degree of freedom. Therefore black holes can be produced at any time during inflation, as long as \( \Lambda_{\text{eff}} \) is small enough to allow at least one unit of charge \([7]\). In dilatonic inflation, however, both degrees of freedom, \( \tilde{\Lambda}_{\text{eff}} \) and \( Q \), are fixed by the values of the inflaton and dilaton fields. Since we would still expect the charge to be quantised, this means that black holes cannot be produced continuously throughout inflation, but only when the inflaton and dilaton fields have values that correspond to integer \( Q \).

5 Charged Nariai Solutions With a General Dilaton Potential

In Sec. 3 we considered a dilatonic potential of the Liouville type, for which the DCN solutions are particularly easy to analyse. Now we shall find the conditions for more general potentials to admit DCN solutions. As an example we will analyse the case of a massive dilaton.

Consider now a Lagrangian with a general dilaton potential \( V(\phi) \),

\[
L = (-g)^{1/2} \left( R - 2(\nabla \phi)^2 - 2V(\phi) - e^{-2a\phi} F^2 \right). \tag{5.1}
\]

(In Secs. 3 and 4 we chose \( V(\phi) = \Lambda e^{2b\phi} \).) The following definitions generalise the quantities \( \tilde{\Lambda} \), \( b \) and \( \tilde{g} \) that appeared in the previous sections:

\[
\tilde{\Lambda}(\phi) = V(\phi), \tag{5.2}
\]
\[
b(\phi) = \frac{V'}{2V}, \tag{5.3}
\]
\[
\tilde{g}(\phi) = \frac{b(\phi)}{a}, \tag{5.4}
\]

where a prime denotes differentiation with respect to \( \phi \). With these definitions, DCN solutions (if they exist) will again be given by Eqs. (3.4), (3.5), (3.6) and (3.7). For constant \( \phi \), the dilaton equation of motion becomes

\[
a e^{-2a\phi} F^2 = V', \tag{5.6}
\]
or equivalently,
\[ \tilde{g} e^{2a\phi} = (1 + |\tilde{g}|)^2 Q^2 \tilde{\Lambda}. \] (5.7)
Note that both \( \tilde{\Lambda} \) and \( \tilde{g} \) depend on \( \phi \). Generalising from Sec. 3, we note that DCN black holes exist if and only if this equation has at least one solution \( \phi_0 \) with \( V(\phi_0) > 0 \) and \( |\tilde{g}(\phi_0)| < 1 \). The first condition ensures the positivity of the cosmological constant, while the second ensures the positivity of the metric coefficient \( \tilde{A} \). Then it follows from Eq. (5.7) that the black holes are magnetic (electric) for positive (negative) \( \tilde{g} \). For \( \tilde{g} = 0 \) one obtains the non-dilatonic neutral Nariai solutions.

Let us consider a massive dilaton potential as an example:
\[ V_{\text{massive}} = m^2 \phi^2. \] (5.8)
Then the Lagrangian is invariant under the transformation
\[ a \to -a, \ \phi \to -\phi. \] (5.9)
Again we fix a gauge by choosing \( a \geq 0 \). For this potential, Eq. (5.7) becomes
\[ e^{-2a\phi} \phi (a\phi + 1)^2 = \frac{a}{Q^2 m^2}. \] (5.10)
The solution \( \phi = 0, a = 0 \) is excluded because of the condition \( V(\phi_0) > 0 \). \( \phi \) must be of the same sign as \( a \); thus, with our gauge choice, \( \phi > 0 \). We cannot solve Eq. (5.10) in closed form, but the constant value of \( \phi \) is of little interest anyway. Note, however, that the left hand side of Eq. (5.10) is zero for \( \phi = 0 \), then increases to reach a maximum at \( \phi = 1 \), and then decreases, approaching zero as \( \phi \to \infty \). The value of the maximum is \( 4/e^2 \), where \( e \) is Euler’s number. Thus the charge \( Q \) has a lower bound,
\[ |Q| \geq \frac{ea}{2m}, \] (5.11)
but no upper bound.

6 Summary
Charged Nariai solutions are important because they have a regular Euclidean section and can therefore mediate pair creation processes. Unlike all other RNdS black holes, the Charged Nariai solutions of Einstein-Maxwell theory possess an analogue
in dilatonic theories with a Liouville potential. The metric and Maxwell field of the DCN solutions can be written in a form very similar to the CN solutions. However, there are parameter pairs which parametrise the CN solutions, but whose dilatonic analogues do not parametrise the DCN solutions. Consequently, the solution spaces differ significantly. We found that the shape of the geometry is entirely fixed by the ratio of the dilaton coupling constants, \( \tilde{g} = b/a \). The black hole charge only affects the overall scale of the metric. The charge can thus be arbitrarily large; by Eq. (3.11), however, a large charge will lead to a large value of the dilaton, and thus to a very small Maxwell coupling. As a consequence, the force on a charged test particle turns out to be independent of the black hole charge. The electromagnetic coupling strength can be measured using two test particles; this in turn determines the black hole charge. The DCN solution space is two-dimensional only for \( 0 < |\tilde{g}| < 1 \). If \( \tilde{g} = 0 \), only the non-dilatonic neutral Nariai solution can be found. For other values of \( \tilde{g} \), no DCN solutions are admitted. The sign of \( b \) determines whether the black hole is magnetic, neutral, or electric.

The Euclidean DCN solutions themselves do not describe a pair creation process, since there is no dilatonic de Sitter solution which could act as a background. If the fixed cosmological constant is replaced by a slowly decreasing effective cosmological constant, however, one would expect the DCN instanton to mediate black hole pair creation during extended chaotic inflation.

We found conditions for the existence of Charged Nariai black holes in theories with a general dilaton potential. For the case of a massive dilaton, we showed that such solutions exist if the black hole is sufficiently charged.

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