KRULL DIMENSION AND DEVIATION
IN CERTAIN PARAFREE GROUPS

SAL LIRIANO

SAL21458@yahoo.com

In memory of Marcelo Llarull (1958-2005)

Abstract. Hanna Neumann asked whether it was possible for two non-isomorphic residually nilpotent finitely generated (fg) groups, one of them free, to share the lower central sequence. G. Baumslag answered the question in the affirmative and thus gave rise to parafree groups. A group $G$ is termed parafree of rank $n$ if it is residually nilpotent and shares the same lower central sequence with a free group of rank $n$. The deviation of a $fg$ parafree group $G$ of rank $n$ is the difference $\mu(G) - n$, where $\mu(G)$ is the minimum possible number of generators of $G$.

Let $G$ be $fg$; then $\text{Hom}(G, \text{SL}_2(\mathbb{C}))$ inherits the structure of an algebraic variety, denoted by $R(G)$, which is an invariant of $fg$ presentations of $G$. If $G$ is an $n$ generated parafree group, then the deviation of $G$ is 0 iff $\text{Dim}(R(G)) = 3n$. It is known that for $n \geq 2$ there exist infinitely many parafree groups of rank $n$ and deviation 1 with non-isomorphic representation varieties of dimension $3n$. In this paper it is shown that given integers $n \geq 2$, and $k \geq 1$, there exists infinitely many parafree groups of rank $n$ and deviation $k$ with non-isomorphic representation varieties of dimension different from $3n$; in particular, there exist infinitely many parafree groups $G$ of rank $n$ with $\text{Dim}(R(G)) > q$, where $q \geq 3n$ is an arbitrary integer.

Structure of paper. New results in this paper are Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5. This paper is broken up into 3 parts: Introduction, Section one, Section two. In the introduction an outline of the mentality that guided this investigation is given, along with the proof of some preliminary results including the proof of Theorem 2, and Theorem 3. In Section One material involving sequences of primes, and groups associated with such sequences is developed. The section ends with a proof of Theorem 4. Section Two begins with a proof of Theorem 1, and ends with a proof of Theorem 5.

Introduction

Let $G$ be a finitely generated group. Then the set of homomorphisms from $G$ into $\text{SL}_2(\mathbb{C})$ inherits the structure of an algebraic variety denoted by $R(G)$. This algebraic variety, also known as the representation variety of $G$, is an invariant of finitely generated presentations of $G$. It exports into the theory of finitely generated groups the numerous invariants of Commutative Algebra and Algebraic Geometry. The object of this work is to continue the exploration with this invariant of a rather remarkable class of groups invented by G. Baumslag in the 1960’s, parafree groups. For the convenience of the reader, the definition of a parafree group follows.
Let $γ_n F$ stand for the $n$-th term of the Lower Central Series of the free group $F$. Then, a group $G$ is termed parafree if:

1) $G$ is residually nilpotent.
2) There exists a free group $F$ with the property that $G/γ_n G ≃ F/γ_n F$ for all $n ≥ 1$.

Now denote by $µ(G)$ the minimal number of generators of a group $G$. Define the rank of $G$, denoted here by $rk(G)$, to be $rk(G) = µ(G/γ_2 G)$. Further, define the deviation of the group $G$, here denoted by $δ(G)$, to be $δ(G) = µ(G) − rk(G)$. A parafree group $G$ is of rank $r$ if the free group in (2) above is also of rank $r$.

Perhaps two of the most fundamental invariants of an algebraic variety are its Krull dimension and reducibility status. As it is often the case, with these invariants can be associated other invariants. For example, let $c$ be a positive integer. Now denote by $N_c(V)$ the number of maximal irreducible components (mirc) of $V$ of dimension $c$. Then $N_c(V)$ is an invariant of the algebraic variety $V$. This invariant was introduced by the author in [L3] where it was shown that $N_4(R(𝔽)) = g$, where $g$ is the genus of the torus knot corresponding to the group $𝔽$. Subsequently, in [L4] the invariant $N_4(V)$ was employed in studying a class of parafree groups to show, amongst other surprising results, that given any parafree group $G$ of rank $n$, deviation 1, and with $Dim(R(G)) = 3n$, that there exist a set $P_n$ of parafree groups of rank $n$ and deviation 1 having the property that no two groups in $P_n$ have isomorphic representation varieties, and such that for any $G_j ∈ P_n$, it is the case that $N_3n(R(G_j)) < N_3n(R(G_j))$. In fact, the set, $\{N_3n(R(G_i)) | G_i ∈ P_n\}$ is infinite. This is significant given the bewildering likeness between a $fg$ free group and a $fg$ parafree group of the same rank with deviation other than 0. Indeed, a non-free $fg$ parafree group of rank $n$ agrees with a free group of the same rank on an infinite number of $fg$ torsion-free nilpotent quotients. The stage is now set to introduce the main results of this paper.

A direct result of W. Magnus' investigations is that a $fg$ parafree group of rank $n$ is free iff it has deviation zero; see [WM2], [WM]. In [L4] it was shown that an $n$-generated group $G$ is free, iff $Dim(R(G)) = 3n$. So it follows then that:

**Proposition 0.1.** If $G$ is an $n$ generated parafree group then the deviation of $G$ is zero iff $Dim(R(G)) = 3n$.

In other words, up to isomorphism a parafree group $G$ of rank $n$ and deviation zero is determined by an invariant of $R(G)$, namely its dimension. Admittedly, it was Proposition 0.1 that motivated the author to pursue more closely the possible connections between rank, deviation of a parafree group $G$, and the krull dimension of the algebraic variety $R(G)$. Indeed, all example of finitely generated parafree groups of rank $n$ considered in [L4] turned out to have dimension $R(G) = 3n$, that of the free group $F_n$, with which they shared the lower central sequence.\(^2\) In the face of it, this seems anything but surprising given that parafree groups share a host of properties with free groups, (see [L4], and [B1], [B2] for an account). But, since $fg$ parafree groups of finite rank can have arbitrarily large deviations, it did not seem at all unusual to inquire whether indeed it is possible that all finitely generated

\(^1\)A free group of rank $n$ is a parafree group of deviation 0 and rank $n$.

\(^2\)The lower central sequence of a group $G$ is the sequence given by $G/γ_nG$, where $γ_nG$ is the $n$-term of the lower central series.
parafree groups $G$ of rank $n$ have the property that $\text{Dim}(R(G)) = 3n$? The next proposition provides a bound for the dimension of the representation variety of a parafree group of rank $n$ and deviation $k$.

**Proposition 0.2.** Let $G$ be a parafree group of rank $n$ and deviation $k \geq 1$, then $\text{Dim}(R(G)) < 3(n + k)$.

*Proof.* If $G$ is a parafree group of rank $n$ and deviation $k$, then it can be easily seen that $G$ is generated by $n + k$ elements. To see this, recall the formula for the deviation $\delta(G)$ of a group introduced above: $\delta(G) = \mu(G) - rk(G)$, where $\mu(G)$ is the smallest possible number of generators of $G$. Inserting $k$ and $n$ as in the statement above yields $k + n = \mu(G)$. Thus $G$, since it is not free is a quotient group of a free group of rank $k + n$. Now, since a free group of rank $n + k$ has a faithful representation in $\text{SL}(2, \mathbb{C})$ by Sanov, (see [SN]), and since $R(F_{n+k})$ is an irreducible variety of dimension exactly $3(n + k)$, it follows that $\text{Dim}(R(G)) < \text{Dim}(R(F_{n+k})) = 3(n + k)$.

Next a theorem is unveiled that gives an answer of no to the question asking whether $\text{Dim}(R(G)) = 3n$ for any parafree group $G$ of rank $n$. In fact, it guarantees that for each integer $k \geq 1$ there exist an infinite set of rank 2 parafree groups of deviation $k$ having the property that their corresponding representation varieties are $4 + 2k$-dimensional, reducible, and pairwise not isomorphic, a fact that implies that the groups are not isomorphic. This is an amazing illustration of the sensitivity of some algebra-geometric invariants when unleashed in the study of fg groups.

**Theorem 1.** Given an integer $k \geq 1$ there exists an infinite set $S_k$ consisting of parafree groups of rank 2 and deviation $k$ having the property that for $G_i$, $G_j$ in $S_k$ with $i \neq j$ the following holds:

1. $\text{Dim}(R(G_i)) = 4 + 2k$.
2. If $i \neq j$, and $c = 4 + 2k$, then $N_c(R(G_i)) \neq N_c(R(G_j))$.
3. $R(G_i) \not\cong R(G_j)$.
4. Given any integer $m$ there exists an infinite subset $Q$ in $S_k$ such that for any $G_i \in Q$ and $c$ as immediately above $N_c(R(G_i)) \geq m$.

The next theorem is a consequence of Theorem 1. Because its proof is so dependent on Theorem 1, it is given immediately.

**Theorem 2.** Given any integer $r \geq 2$ and any integer $k \geq 1$ there exists an infinite set $S_{r,k}$ consisting of parafree groups of rank $r$ and deviation $k$ with the property that for each $G_i$ and $G_j$ with $i \neq j$ in $S_{r,k}$ it is the case that:

1. $\text{Dim}(R(G_i)) = 3r + 2k - 2$.
2. If $c = 3r + 2k - 2$, then $N_c(R(G_i)) \neq N_c(R(G_j))$.
3. $R(G_i) \not\cong R(G_j)$.
4. Given any integer $m$, there exists an infinite subset $Q$ in $S_{r,k}$ such that for any $G_i \in Q$ and $c$ as directly above $N_c(R(G_i)) \geq m$.

*Proof.* Let $k \geq 1$ be some fixed arbitrary integer. If $r = 2$, then replace $S_{r,k}$ by $S_k$, as in Theorem 1. Now, suppose that $r > 2$. Then let $S_{r,k} = \{F_{r-2} * G_i | G_i \in S_k\}$.

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3 In particular, $m$ can be $N_c(R(G))$ for any $G$ in $S_k$.
4 Adjust with $S_{r,k}$, and $c$ the statement of the footnote above.
The sets $S_{r,k}$ will satisfy all the statements of the theorem and consist of parafree groups of rank $r$ since the free product of two parafree groups is of rank the sum of the rank of their factors [B2]. The proof of part one is a trivial consequence of the fact that the dimension of a product of varieties is the sum of the dimensions of the factors, and that $\dim(R(F_{r-2})) = 3(r - 2)$. Part two follows from the fact that the representation variety of a free group in an irreducible linear algebraic group is irreducible. Three is a consequence of the fact that the number of $mirc$ is an invariant of the isomorphism class of an algebraic variety. Part four is a direct consequence of the fact that the sets $S_k$ in Theorem 1 are all infinite and well ordered by the $mirc$ counting function in dimension $3r + 2k - 2$. The proof is complete.

Undoubtedly, it is a well documented fact that cyclically pinched one-relator groups share a host of properties with free groups, a fact that given the likeness between non-free parafree groups and free groups, endows one relator parafree groups with cyclically pinched presentations with numerous additional free-like properties; the reader is encouraged to consult [B4], [B5], [FR], [L4] to learn of some of them.

For $k \geq 2$ the reader will see that the groups used in the proof of Theorem 1 turn out not to be one relator groups. Indeed, using Proposition 0.2 it is possible to deduce that an $n + 1$ generated parafree group of rank $n$ and deviation one always has $\dim(R(G)) \leq 3n + 2$, and using the main theorem of [L1] the next result follows immediately.

**Proposition 0.3.** Let $G$ be a a group on $n + 1$ generators and having a cyclically pinched presentation $< x_1, \cdots, x_n, y; W = y^p >$; then $\dim(R(G)) \leq 3n + 1$.

An immediate consequence of the above proposition is that any parafree group obtained by adjoining a root to a non-trivial element $W$ of the free group of rank $n \geq 2$ thus: $< x_1, \cdots, x_n, y; W = y^p >$, which is the method described by G. Baumslag in [B1], can only have a representation variety of dimension at most $3n + 1$. Indeed, until now all one relator parafree groups obtained in the described manner have resulted in groups having representation varieties of dimension precisely three times the rank of the free group to which the root was adjoined. But, is this always the case? The answer is provided by the next theorem.

**Theorem 3.** Given integer $k \geq 1$ there exist infinitely many parafree groups of rank 2 and deviation 1 with pairwise non-isomorphic representation varieties of dimension precisely 7 and such that $N_7(R(G)) \geq k$.

**Proof.** The set of groups in one to one correspondence with the set $S$ in Theorem 5 part (3) can be well ordered using the $mirc$ counting function $N_7(V)$ in dimension 7. Since the set $S$ is infinite, it follows that given any integer $k$ there exist only a finite number of integers smaller that $k$ and which correspond to the ordering

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5A group is termed cyclically pinched if it can be given a presentation of the type $G = \langle X \cup Y; W = V \rangle$, where $X$ is a finite set of generators, $W$ is a non-trivial word in the free group on $X$, and $Y$ is a finite set of generators, and the word $V$ a non-trivial word in the free group on $Y$.

6In particular, $k$ can be taken equal to $N_7(R(G))$ for any fixed $f$ parafree group $G$ with $\dim(R(G)) = 7$. 
imposed by the \textit{mir} counting function \(N_7(V)\) on the set of groups corresponding to \(S\). The result follows, since each of the groups that correspond to the set \(S\), besides being parafree of rank 1, has representation variety of dimension 7.

**Section One**

In this section material necessary for the proof of Theorem 1 and Theorem 5 is introduced. Because of the significant role it will play, it is prudent to begin by recalling a result developed and proved by the author in [L4], which in the sequel shall be referred to as the Coarse Sieve Theorem\(^7\), or the CS Theorem, for short.

**Coarse Sieve Theorem.**

Let \(S_{i=1}^\infty(G_i, \mathfrak{N}_i)\) be an infinite sequence of pairs \((G_1, \mathfrak{N}_1), (G_2, \mathfrak{N}_2), \ldots\) consisting of fg groups \(G_i\) and a corresponding normal subgroup \(\mathfrak{N}_i\) of \(G_i\). Let \(N_c(V)\) be the \textit{mir} counting function in dimension \(c\). Suppose that \(\text{Dim}(R(G_i)) = \text{Dim}(R(G_i/\mathfrak{N}_i)) = c\), and that the set \(S = \{N_c(R(G_j/\mathfrak{N}_j)) \mid (G_j, \mathfrak{N}_j) \in S_{i=1}^\infty(G_i, \mathfrak{N}_i)\}\) contains an infinite set of integer points. Then:

i) \(S_{i=1}^\infty(G_i, \mathfrak{N}_i)\) has an infinite subsequence \(S^2\) with the property that given two different pairs \((G_j, \mathfrak{N}_j), (G_k, \mathfrak{N}_k)\) in \(S^2\) then \(N_c(R(G_j)) \neq N_c(R(G_k))\).

ii) For different pairs \((G_j, \mathfrak{N}_j), (G_k, \mathfrak{N}_k)\) in \(S^2\) then \(R(G_j) \ncong R(G_k)\).

iii) The set of groups \(S_2 = \{G \mid G\text{ occurs in some term of } S^2\}\) can be well-ordered using the function \(N_c(V)\).

Next some preparation for the eventual deployment of the Coarse Sieve Theorem in the proof of Theorem 5, and Theorem 1 is introduced.

**k-Prime Sequences.** The Sieve of Eratosthenes can be used to find the terms of the ascending sequence of primes starting from the prime 3 and contained in the interval \([3, m]\), for any\(^8\) arbitrary positive integer \(m\). Inductively then one may list the ascending sequence of primes starting from 3; call the resulting sequence of primes: \(1P_{i=1}^\infty\). Now, given integer \(k \geq 1\) use \(1P_{i=1}^\infty\) to obtain the sequences \(2P_{i=1}^\infty, \ldots, kP_{i=1}^\infty\) specified in the following list\(^9\):

a) \(1P_{i=1}^\infty : P_1, P_2, P_3, \ldots\)

b) \(2P_{i=1}^\infty : (P_1, P_2), (P_3, P_4), \ldots\)

\[\vdots\]

t) \(kP_{i=1}^\infty : (P_1, P_2, \ldots, P_k), (P_{k+1}, \ldots, P_{2k}), \ldots\)

where \(P_1 = 3, P_2 = 5, P_3 = 7, \ldots\) is the infinite sequence of primes starting with 3, and (a) gives the sequence when \(k = 1\), and (b) gives the sequence when \(k = 2\), and so on.

\(^7\)The reason for the name is because under certain algebro-geometric conditions on the representation varieties of groups pertaining to the terms of an infinite sequence \(S_{i=1}^\infty(G_i, \mathfrak{N}_i)\), the theorem guarantees that the sequence \(S_{i=1}^\infty(G_i, \mathfrak{N}_i)\) has an infinite subsequence whose corresponding groups \(G_i\) have pairwise non-isomorphic representation varieties.

\(^8\)For sufficiently large \(m\) one of the many estimates of the prime counting function \(\pi(m)\) may be used to approximate the number of terms of the sequence in the interval \([3, m]\).

\(^9\)Note that \(1P_{i=1}^\infty\) may sometimes be written as \(P_{i=1}^\infty\) in the sequel.
Next, a procedure for associating a very particular free product of groups with each term of any of the above \(k\)-prime sequences is revealed. But first, it is important to mention that due to the presentation of certain parafree groups of importance in the sequel, the group
\[
(1-1) \quad G = \langle x_1, x_2; x_1^2x_2^2 = 1 \rangle
\]
shall play a fundamental role. Fortunately, some relevant aspects of the algebraic geometry of \(R(G)\) are well understood.

**Procedure 1.1.**
Let \(G = \langle x_1, x_2; x_1^2x_2^2 = 1 \rangle\), and let \(k \geq 1\) be any integer. Associate with any term of the sequence \(kP_i^\infty\) above, say the \(t\)-term \(kP_t\), where \(t\) is any integer \(\geq 1\), a unique group \(G_{kP_t}\) given by:
\[
(1-2) \quad G_{kP_t} = G \ast \mathbb{Z}_{P_{t-(t-1)k+1}} \ast \mathbb{Z}_{P_{t-(t-1)k+2}} \ast \cdots \ast \mathbb{Z}_{P_{tk}}
\]
So for example, the group associated with the third term \(P_3\) of the sequence \(P_i^\infty\) would be \(GP_3 = G \ast \mathbb{Z}_{P_3}\), which is nothing but the free product \(G \ast \mathbb{Z}_7\). The group associated with the second term \(2P_3\) belonging to the sequence \(2P_i^\infty\) is given by \(G2P_2\), which is nothing but \(G \ast \mathbb{Z}_{P_3} \ast \mathbb{Z}_{P_5}\), which really stands for: \(G \ast \mathbb{Z}_7 \ast \mathbb{Z}_{11}\).

**Lemma 1.1.** Let \(G = \langle x_1, x_2; x_1^2x_2^2 = 1 \rangle\). Then \(\text{Dim}(R(G)) = 4\), and \(R(G)\) is reducible with \(N_4(R(G)) = 1\).

**Proof.** This follows from the main theorem in [L3].

**Lemma 1.2.** Let \(G_{kP_j}\) be the group associated with the \(j\)-th term of sequence \(kP_i^\infty\), where \(j\) is any integer \(\geq 1\). Let \(c = 4 + 2k\). Then:
(1) \(\text{Dim}(R(G_{kP_j})) = c\);
(2) if \(k \geq 2\) then \(N_c(R(G_{kP_j})) = \frac{(P_{(t-1)k+1})^{(t-1)k+1-1} \cdots (P_k-1)}{2^c}\);
(3) if \(k = 1\) then \(N_c(R(G_{kP_j})) = \frac{(P-1)}{2}\).

**Proof.** \(G_{kP_j}\) is the free product of the group \(G\) in Lemma 1.1 and a free product of cyclics involving cyclic factors that are all of finite prime order \(t \geq 3\). The representation variety of a cyclic group of finite order \(t \geq 3\) can by elementary methods be shown to be 2 dimensional, and when \(t \geq 3\) is prime, to have \(\frac{t^2}{2}\) *mir* of dimension two. See [L1], or [L3]. This with the help of the next lemma, whose proof is left as an exercise, will yield the result sought.

**Lemma 1.3.** Let \(V\) and \(W\) be two algebraic varieties with \(\text{Dim}(V) = d\) with \(N_d(V) = y\), and with \(\text{Dim}(W) = r\) with \(N_r(W) = m\). Then \(\text{Dim}(V \times W) = d + r\) and \(N_{d+r}(V \times W) = ym\).

Now by Lemma 1.1, and (10), and by Lemma 1.3 it follows that \(\text{Dim}(R(G_{kP_j})) = 4 + 2k\) and that \(N_c(R(G_{kP_j})) = \frac{(P_{(t-1)k+1})^{(t-1)k+1-1} \cdots (P_k-1)}{2^c}\), if \(k \geq 2\). If \(k = 1\) the result is an immediate consequence of the fore stated. This completes the proof of Lemma 1.2.

As an immediate consequence of Lemma 1.2, is the following.
Lemma 1.4. If \( k \geq 1 \) is any integer, then \( \lim_{j \to \infty} N_{4+2k}(R(G_k P_j)) = \infty \).

Proof. See the formula for the number of 4 + 2k-dimensional maximal irreducible components in Lemma 1.2.

Yet one more procedure is necessary; it will for each integer \( k \geq 1 \), and each term in the sequence \( kP_{i=1}^\infty \) associate a group as follows:

Procedure 1.2. For each integer \( k \geq 1 \) associate with the \( n \)-th term, \( kP_n = (P_{k(n-1)+1}, P_{k(n-1)+2}, \ldots, P_{nk}) \), of the sequence \( kP_{i=1}^\infty \), a corresponding group \( pG_k P_n \) on the free generators \( \{x_1, x_2, y_1, y_2, \ldots, y_k\} \) given by the presentation:

1. \[< x_1, x_2, y_1, \ldots, y_k; x_1^{2^1} x_2^{2^1} = y_1^{P_{k(n-1)+1}}, x_1^{2^2} x_2^{2^2} = y_2^{P_{k(n-1)+2}}, \ldots, x_1^{2^k} x_2^{2^k} = y_k^{P_{nk}}>\]

Example a: if \( k = 1 \) and \( n = 2 \) the group associated with the \( P_2 \) term of the sequence \( 1P_{i=1}^\infty \) would be \( < x_1, x_2, y_1; x_1^{2^1} x_2^{2^1} = y_1^{P_2}> \) which is after evaluation the group \( < x_1, x_2, y_1; x_1^2 x_2^2 = y_1^2> \).

Example b: if \( k = 2 \) and \( n = 3 \) the group associated with the \( 2P_3 \) term of the sequence \( 2P_{i=1}^\infty \) would be \( < x_1, x_2, y_1, y_2; x_1^{2^1} x_2^{2^1} = y_1^{P_{2(n-1)+1}}, x_1^{2^2} x_2^{2^2} = y_2^{P_{2(n-1)+2}}> \) which is after evaluation the group \( < x_1, x_2, y_1, y_2; x_1^2 x_2^2 = y_1^2, x_1^2 x_2^2 = y_2^2> \).

Lemma 1.5. Given any integer \( k \geq 1 \) and any integer \( e \geq 1 \) then

1. \[\text{Dim}(R(pG_k P_e)) = \text{Dim}(R(G_k P_e)),\]

where \( pG_k P_e \) is the group associated with the \( e \)-th term of the sequence \( kP_{i=1}^\infty \) using Procedure 1.2, and \( G_k P_e \) is the group associated with the \( e \)-th term of the sequence \( kP_{i=1}^\infty \) using Procedure 1.1.

Proof. That \( \text{Dim}(R(pG_k P_e)) = 4 + 2k \) follows directly from Theorem 4. That \( \text{Dim}(R(G_k P_e)) = 4 + 2k \) is a direct consequence of Lemma 1.2.

Lemma 1.5.1. Given integer \( k \geq 1 \) and the group \( pG_k P_j \) associated with the \( j \)-th term of the sequence \( kP_{i=1}^\infty \) then:

1. \[\frac{pG_k P_j}{N(x_1^2 x_2^2)} \cong G_k P_j\]

where \( N(x_1^2 x_2^2) \) stands for the normal closure of the word \( x_1^2 x_2^2 \) in the group \( pG_k P_j \).

Proof. Applying Tietze transformations yields that:

\[\frac{pG_k P_j}{N(x_1^2 x_2^2)} = < x_1, x_2; x_1^2 x_2^2 = 1 > \ast \mathbb{Z} P_{(j-1)k+1} \ast \mathbb{Z} P_{(j-1)k+2} \ast \cdots \ast \mathbb{Z} P_{jk} = G_k P_j,\]

the desired result.

At this juncture, a series of additional lemma and observations to be employed in the proof of Theorem 4, will be introduced.
Lemma 1.7. Let $V$ be an algebraic variety and $W$ a subvariety. Then $\text{Dim}(V) = \text{Maximum} \{ \text{Dim}(V - W), \text{Dim}(W) \}$.

Proof. Obvious.

Given $a \in SL_2\mathbb{C}$ and $p \in \mathbb{Z}$, denote the set: $\{m|m^p = a, m \in SL_2\mathbb{C}\}$ by $\Omega(p,a)$. Further, given a positive integer $t$ define $\Omega^t(p,a) = \Omega(p,a) \times \cdots \times \Omega(p,a)$, taken $t$ times.

Lemma 1.71. All matrices of $SL_2\mathbb{C}$ having fixed trace $\alpha$ where $\alpha \neq \pm 2$ form an irreducible two dimensional variety.

Proof. Any two matrices in $SL_2\mathbb{C}$ with the same trace $\alpha \neq \pm 2$ are similar and also diagonalizable. Now define a regular map $\Gamma: SL_2\mathbb{C} \times n \rightarrow V$ given by $(m,n) \rightarrow m^{-1}nm$, where $n$ is a $SL_2\mathbb{C}$ matrix of trace $\alpha$ having zero in all entries that are not in its main diagonal. The map $\Gamma$ is a regular map onto $V$, where $V$ is the subvariety of $SL_2\mathbb{C}$ consisting of all matrices of trace $\alpha$. Suppose that $V$ is reducible; then $SL_2\mathbb{C} \times n$ is also reducible, a contradiction. That $V$ is two dimensional follows from the fact that $V$ is the zero locus in $SL_2\mathbb{C}$ of a non-constant polynomial, and an application of the Krull Principal Ideal Theorem.

Corollary 1.71. $\Omega(2,-I)$ is an irreducible two dimensional variety.

Proof. Using the Cayley-Hamilton Theorem and the characteristic polynomial for matrices in $SL_2\mathbb{C}$ one can deduce that $m \in \Omega(2,-I)$ iff $m$ has trace zero. Using Lemma 1.71 above the result follows.

The set of possible groups in the next theorem includes in it the groups $pGkP_j$, for any integer $k \geq 1$ and any integer $j \geq 1$. However, their description does not make use of the prime sequences introduced earlier; for that reason they have been named differently.

Theorem 4. Let $G_k = \langle x_1, x_2, y_1, \ldots, y_k; x_1^{y_1} x_2^{y_2} = y_1^{p_1}, \ldots, x_1^{y_k} x_2^{y_k} = y_k^{p_k} \rangle$, where the $p_1, \ldots, p_k$ are distinct primes all $\geq 3$, and $k$ is an integer $\geq 1$. Then

1. The groups $G_k$ are parafree of rank 2 deviation $k$;
2. $\text{Dim}(R(G_k)) = 4 + 2k$;
3. $R(G_k)$ is reducible.

Proof.

Proof 1: That the groups $G_k$ are parafree of rank 2 follows directly from [B1]. That they are of deviation $k$ is a consequence of the Grushko-Neumann Theorem since they map onto a free product of cyclics involving $k+2$ non-trivial factors.

Proof 2 and 3: Let $G_k = \langle x_1, x_2, y_1, \ldots, y_k; x_1^{y_1} x_2^{y_2} = y_1^{p_1}, \ldots, x_1^{y_k} x_2^{y_k} = y_k^{p_k} \rangle$, where the $p_1, \ldots, p_k$ are distinct primes all $> 2$, and $k$ is an integer $\geq 1$. It will be shown that $\text{Dim}(R(G_k)) = 4 + 2k$.

Let

\[\rho: R(G_k) \rightarrow (SL_2\mathbb{C})^2\]

be the projection map given by $\rho(m_1, m_2, \ldots, m_{k+1}, m_{k+2}) = (m_1, m_2)$. Since the $p_i$ are odd primes this map is onto by Lemma 1.8 in [L1].
Now for integers $i$ such that $1 \leq i \leq k$ define $S_i \subset R(G_k)$ as follows: $S_i = \{(m_1, m_2, \ldots, m_{k+1}, m_{k+2})|m_1^2 m_2^2 = \pm I\}$, and define $S^+_i$ as members of $S_i$ with $m_1^2 m_2^2 = I$; also define $S^-_i$ as those members of $S_i$ with $m_1^2 m_2^2 = -I$ in an analogous manner. Now let

(1-7) \[ S = \bigcup_{i=1}^{k} S_i, \]

and define also $S^+$ and $S^-$ in an analogous fashion as before. Now define

$$\rho(S_i) = \{(m_1, m_2)|m_1^2I_1 m_2^2I_2 = \pm I, m_1, m_2 \in SL_2C\}. $$

Also define $\rho(S^+_i)$ and $\rho(S^-_i)$ in an analogous manner as was done before. Define $\rho(S) = \bigcup_{i=1}^{k} \rho(S_i)$.

By Lemma 1.8 [L1] the regular map $\rho$ in (1-6) has the property that $\rho^{-1}$ restricted to the set

(1-8) \[ (SL_2C)^2 - \rho(S) \]

has finite fiber with coordinates over each point in (1-8) sitting in the set

(1-9) \[ R(G_k)^o = R(G_k) - S. \]

Thus by Proposition 2.5 in [L1] it follows that

(1-10) \[ \overline{Dim(R(G_k)) - S} = 6, \]

where the over line indicates the Zariski closure.

Thus to compute $Dim(R(G_k))$ all that needs to be known is $Dim(S)$, by Lemma 1.7. Note that $\Omega^2(2, -I)$ is a subset of $\rho(S)$, by Lemma 1.9 (below), and by Lemma 1.7.1 this is a 4 dimensional variety. So $Dim(\rho(S)) \geq 4$. But Theorem 0.2 of [L1], together with Lemma 3.6 of [L4] guarantees that $Dim(\rho(S)) = 4$, the desired result.

The subvariety $\rho(S_i)$ of $\rho(S)$ will play a central role as made evident by the next two lemmas, the first of which is obvious.

**Lemma 1.9.** $\Omega^2(2, -I) \subseteq \rho(S^+_i)$ for all $i \in \{1, 2, \ldots, k\}$.

**Lemma 1.91.** The fiber of $\rho$ over $\Omega^2(2, -I)$ is of constant dimension $2k$ at each point of $\Omega^2(2, -I)$.

**Proof.** Let $(m_1, m_2) \in \Omega^2(2, -I)$ then $\rho^{-1}(m_1, m_2) = (m_1, m_2) \times \Omega(p_1, I) \times \cdots \times \Omega(p_k, I)$ by Lemma 1.9. Now by Lemma 1.6 of [L1], $Dim((m_1, m_2) \times \Omega(p_1, I) \times \cdots \times \Omega(p_k, I)) = 0 + 2k$. Since the dimension of a product is the sum of the dimensions of the factors.

An immediate consequence of Lemma 1.9 and Lemma 1.91 is that

(1-11) \[ Dim(S) \geq 4 + 2k \]

since $S$ has a subvariety that maps via a regular map onto the 4 dimensional irreducible variety $\Omega^2(2, -I)$ and has fiber of constant dimension $2k$ over each point $(m_1, m_2) \in \Omega^2(2, -I)$. 


At this juncture it is only necessary to show that \( \dim(S) \leq 4 + 2k \). To achieve this the 4 dimensional variety \( \rho(S) \) will be decomposed into its maximal irreducible components \( C_{\alpha_1}, \ldots, C_{\alpha_s} \). Clearly, since \( \rho(S) \) is 4-dimensional, all components are of dimension at most 4. Further, for each \( i \) it is the case that \( \rho^{-1}(C_{\alpha_i}) \) is a closed subvariety of \( R(G_k) \). Now, without loss of generality let \((m_1, m_2)\) be any point in say \( C_{\alpha_i} \), then the fiber over \((m_1, m_2)\) has the form

\[(1-12) \quad (m_1, m_2) \times \Omega(p_{11}, m_1^{21} m_2^{31}) \times \cdots \times \Omega(p_{k1}, m_1^{2k} m_2^{3k}),\]

but this fiber is always of dimension at most \( 2k \) by Lemma 1.6 and Lemma 1.8 of [L1]. It follows then that \( \dim(S) \leq 4 + 2k \) since the dimension of \( \rho(S) \), the base, is 4. Now, by Lemma 1.7 it follows that \( \dim(R(G_k)) = \max\{6, 4 + 2k\} = 4 + 2k \), since \( k \geq 1 \). The reducibility of \( R(G_k) \) follows since there is always an irreducible component of dimension 6 and at least one irreducible component in \( S \) of dimension \( 4 + 2k \). This completes the proof of the theorem. The existence of the six dimensional component is a consequence of the following:

**Proposition 1.1.** Let \( V \) be an algebraic variety with \( \dim(V) = n \), and containing an open set \( V^o \) of strictly positive dimension \( m \leq n \). If \( V^0 \) does not intersect a closed \( n \)-dimensional set \( W \) of \( V \) made up exclusively of all the points of \( V \) that are not in \( V^o \), then \( V \) is reducible and contains at least one irreducible component of dimension \( m \).

**Proof.** Without loss of generality, one can assume that the \( m \)-dimensional variety \( \overline{V^o} \) is irreducible.\(^{10}\) Obviously \( \overline{V^o} \not\subseteq W \) since no point of an open set in \( \overline{V^o} \) lies in the closed set \( W \). It must be so then that \( V \) contains as a maximal irreducible component the \( m \)-dimensional variety \( \overline{V^o} \).

**Section Two**

All the necessary preliminary material is now in place for a proof of Theorem 1, and Theorem 5.

**Theorem 1.** Given an integer \( k \geq 1 \) there exists an infinite set \( S_k \) consisting of parafree groups or rank 2 and deviation \( k \) having the property that for \( G_i, G_j \) in \( S_k \) the following holds:

1. \( \dim(R(G_i)) = 4 + 2k \).
2. If \( i \neq j \) and \( c = 4 + 2k \), then \( N_c(R(G_i)) \neq N_c(R(G_j)) \).
3. \( R(G_i) \not\cong R(G_j) \).
4. Given any integer \( m \) there exists an infinite subset \( Q \) in \( S_k \) such that for any \( G_i \in Q \) and \( c \) as directly above \( N_c(R(G_i)) \geq m \).

**Proof.** Without loss of generality \( k \geq 1 \) can be fixed. Now, for each term, say the \( j \)-th term, of the sequence \( k \mathbb{P}_{i=1}^\infty \) the corresponding group \( pGkP_j \) is a parafree group or rank two by [B1], and since it maps onto a non-trivial free product of cyclics involving \( k + 2 \) factors it is of deviation \( k \).

Now, refer to the Coarse Sieve Theorem of Section One, and set \( G_i = pGkP_i \), where on the right side is the group corresponding to the \( i \)-term of the sequence

\(^{10}\)Overline stands for the Zariski closure.
Theorem 5.

Let \(G_{pq} = \langle a, b, c \mid b^p[a, b^q] = c^3 \rangle\), where \(p, q\) are different odd prime integers and where \([a, b]\) is defined to be \(a^{-1}b^{-1}ab\). Then the following assertions hold.

1. \(\text{Dim}(R(G_{pq})) = 7\).
2. \(R(G_{pq})\) is reducible.
3. There exists an infinite set \(S\) of distinct prime integer pairs \(S = \{(p, q) \mid p, q \in \mathbb{Z}_+\}\) having the property that for any two different pairs \((p, q)\) and \((p', q')\) in \(S\) the corresponding groups \(G_{pq}\) and \(G_{p'q'}\) have the property that \(R(G_{pq})\) is not isomorphic to \(R(G_{p'q'})\). Also, the set \(S\) can be well ordered using the \(micr\) counting function \(N_c(V)\) on representation varieties \(V\) corresponding to groups \(G_{pq}\) associated to \(S\).
4. The groups \(G_{pq}\) are residually torsion-free nilpotent.
5. The groups \(G_{pq}\) are parafree of rank 2 and deviation one.

Proof.

Proof 1: Let \(< x >\) stand for the infinite cyclic group and \(\mathbb{Z}_t\) for the cyclic group of order \(t\). Then the group \(< x > * \mathbb{Z}_p * \mathbb{Z}_q\) is a homomorphic image of the group \(G_{pq}\). Now \(\text{Dim}(< x > * \mathbb{Z}_p * \mathbb{Z}_q) = 7\); see [L1]. But, given any finitely generated group \(G\) and a normal subgroup \(N\) of \(G\) it is easy to see that \(R(G/N) \subseteq R(G)\). So \(\text{Dim}(R(G/N)) \leq \text{Dim}(R(G))\). Consequently \(\text{Dim}(G_{pq}) \geq 7\). That
$\text{Dim}(G_{pq}) \leq 7$ follows from the main theorem of [L1] which guarantees that any group with presentation $G = \langle x_1, \ldots, x_n, y; w = y^n \rangle$, $n$ an integer, and $w$ a word not involving the generator $y$, has the property that $\text{Dim}(R(G)) \leq 3n + 1$. Thus one is forced to conclude that $\text{Dim}(R(G_{pq})) = 7$.

Proof 2: The reducibility of $R(G_{pq})$ stems from the fact that if $H$ is a homomorphic image of $G$ of the same dimension as $G$ then $N_{\text{Dim}(R(G))}(R(G)) \geq N_{\text{Dim}(R(G))}(R(H))$. See [L4]. Also important is that the product of algebraic varieties is reducible when at least one of the factors is a reducible variety. But any finite non-trivial group has a reducible representation variety [L1].

Proof 3: The proof of part 3 employs the Coarse Sieve Theorem, and the sequence $2P_{i=1}^\infty$ defined in the beginning of Section One. Associate with each term of $2P_{i=1}^\infty$ a pair consisting of a group and a normal subgroup. For example with the $k$-th term $2P_k$ of the sequence associate the pair

$$(2-2) \quad (G_{2P_k}, N_{2P_k})$$

consisting of the group $G_{2P_k} = G_{\{P_{2k-1}, P_{2k}\}} = \langle a, b, c; b^{P_{2k-1}} = c^{P_{2k}} \rangle$, and the normal subgroup $N_{2P_k}$ of $G_{2P_k}$ generated by the word $\{b^{P_{2k-1}}\}$. Recall that $P_{2k-1}, P_{2k}$ are successive primes in the sequence of primes $P_{i=1}^\infty : 3, 5, 7, 11, 17, \ldots$

It is straightforward to see that associated with any term of the sequence $2P_{i=1}^\infty$, say the $k$-th term $2P_k$, it is so that for the corresponding pair $(G_{2P_k}, N_{2P_k})$ in (2-2) the following holds:

$$G_{2P_k}/N_{2P_k} \cong \langle a, b, c; b^{P_{2k-1}} = 1, c^{P_{2k}} = 1 \rangle$$

Thus, since the representation variety of a free product of a finite number of $fg$ groups is the product of the corresponding representation varieties of the factors, it follows that $\text{Dim}(R(G_{2P_k}/N_{2P_k})) = 7$. See [L1]. So $\text{Dim}(R(G_{2P_k})) = \text{Dim}(R(G_{2P_k}/N_{2P_k})) = 7$. Let $N_7(V)$ be the micr counting function in dimension 7, for any algebraic variety $V$.

Now, using elementary facts introduced in [L1], also in [L3], it follows immediately that $N_7(R(G_{2P_k}/N_{2P_k})) = \frac{(P_{2k-1})(P_{2k-1})}{2}$. Clearly, as $k$ tends to infinity $N_7(R(G_{2P_k}/N_{2P_k})$ also tends to infinity. So the set $\{N_7(R(G_{2P_k}/N_{2P_k})) | k \in 1, 2, 3, 4, \ldots \}$ has an infinite number of integer points. The conditions of the CS Theorem are thus satisfied, and consequently by wording the forth going in its language, one is guaranteed an infinite set $S$ of distinct odd primes integers pairs $S = \{(p, q)|p, q \in \mathbb{Z}_+ \}$ having the property that for different pair $(p, q)$ and $(p', q')$ in $S$ the corresponding groups $G_{pq}$ and $G_{p'q'}$ have the property that $R(G_{p'q'})$ is not isomorphic to $R(G_{p'q'})$.

Proof 4: In [6], G. Baumslag showed that if $G = \langle x_1, \ldots, x_n, y; w(x_1, \ldots, x_n) = y^t \rangle$ is such that $w(x_1, \ldots, x_n) \neq 1$ generates its own centralizer in $\langle x_1, \ldots, x_n \rangle$ and $t$ is a positive integer, that then $G$ is residually torsion free nilpotent. Under the definition of $[a, b]$ stipulated in the statement of the theorem the word $b^p[a, b^p]$, generates its own centralizer in the free group on $\{a, b\}$, and consequently the groups $G_{pq}$ are residually torsion free nilpotent.
Proof 5: G. Baumslag in [B1] introduced a result quite handy in building non-isomorphic parafree groups of the same rank as a previously given parafree group:

Let \( r \) and \( n \) be positive integers and \( H \) parafree of rank \( r \), and let \( (x) \) be the infinite cyclic group on \( (x) \). Further, let \( W \in H \). Suppose \( W \) is the \( k \) power of \( W' \) modulo \( \gamma_2 H \), where \( W' \) is itself not a power modulo \( \gamma_2 H \); also, assume that \( k \) and \( n \) are coprime and that \( G \) is residually nilpotent. Then the generalized free product \( G = \{ H * (x); W = x^n \} \) is parafree of rank \( r \).

Using Baumslag’s result and Part 4, one can see that the groups \( G_{pq} \) are parafree of rank 2. That they have deviation one stems from the fact that \( \dim(R(G_{pq})) = 7 \) and thus the number of generators necessary to generate a \( G_{pq} \) is at least 3. So the deviation for each of the \( G_{pq} \) is one.

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