The spectrum of random \( k \)-lifts of large graphs (with possibly large \( k \))

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Abstract

We study random \( k \)-lifts of large, but otherwise arbitrary graphs \( G \). We prove that, with high probability, all eigenvalues of the adjacency matrix of the lift that are not eigenvalues of \( G \) are of the order \( O \left( \sqrt{\Delta \ln(kn)} \right) \), where \( \Delta \) is the maximum degree of \( G \). Similarly, and also with high probability, the “new” eigenvalues of the Laplacian of the lift are all in an interval of length \( O \left( \sqrt{\ln(nk)/d} \right) \) around 1, where \( d \) is the minimum degree of \( G \).

We also prove that, from the point of view of Spectral Graph Theory, there is very little difference between a random \( k_1 k_2 \ldots k_r \)-lift of a graph and a random \( k_1 \)-lift of a random \( k_2 \)-lift of \ldots of a random \( k_r \)-lift of the same graph.

The main proof tool is a concentration inequality for sums of random matrices that was recently introduced by the author.

1 Introduction

Let \( G \) be a graph with vertex set \( V \) and edge set \( E \). A \( k \)-lift of \( G \) is a graph \( G^{(k)} \) with vertex set \( V \times [k] \) and edge set:

\[
E^{(k)} \equiv \cup_{vw \in E} M_{vw}
\]

where each \( M_{vw} \) is a matching of the sets \( \{(v, 1), (v, 2), \ldots, (v, k)\} \) and \( \{(w, 1), (w, 2), \ldots, (w, k)\} \).

In more intuitive terms: each vertex of \( G \) is replaced by \( k \) copies of itself and each edge \( vw \in E \) is replaced by a matching of the copies of \( v \) and \( w \).

There have been many recent results about random \( k \)-lifts of graphs where \( G \) is fixed and \( k \rightarrow +\infty \). Here “random” means that the matchings \( M_{vw} \) are chosen independently and each of them is uniformly distributed. A lot is now known about properties of \( G^{(k)} \) such as connectivity \(^3,\) chromatic number \(^4,\) spectral distribution \(^3,\) \(^6\) and the existence of perfect matchings \(^17\).

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A disjoint line of work has considered 2-lifts of arbitrary (possibly large) graphs $G$. The goal in this case was to provide an explicit construction of some 2-lift with good spectral properties, so that arbitrarily large expanders can be efficiently constructed via successive 2-lifts [3].

In this paper we study a scenario that is quite natural but, to the best of our knowledge, new: random $k$-lifts of large graphs $G$. We obtain non-trivial results only when the minimum degree of $G$ is $\gg \ln(|V|/k)$, but $G$ and $k$ are otherwise arbitrary. For concrete examples, one may think of random $n$-lifts of graphs on $n$ vertices and minimal degree $\ln(1+\epsilon)n$; or of $2^{\sqrt{n}}$-lifts of $(n/2)$-regular graphs on $n$ vertices.

Our focus will be on the spectra of the adjacency matrix and Laplacian of the random lift. These two matrices are the central objects of Spectral Graph Theory and their eigenvalues can be used to estimate many parameters of graphs, including the diameter, distances between distinct subsets, discrepancy-like properties, path congestion, cuts, chromatic number and the mixing time for random walk; see e.g. [4, 8, 9]. Our main theorem is a first indication of what the above parameters are for the random lifts we consider. In fact, our theorem works even for a relaxed definition of random lifts where the $M_{vw}$ need not be uniformly distributed.

We first need some preliminaries. Let $A$ and $A^{(k)}$ be the adjacency matrix of the graph $G$ and of its $k$-lift $G^{(k)}$ (resp.). We will see in Section 3.3 that the spectrum of $A^{(k)}$ always contains the spectrum of $A$ in the sense of multisets: any eigenvalue of $A$ with multiplicity $m$ is an eigenvalue of $A^{(k)}$ with multiplicity $\geq m$. The same holds for the spectra of the Laplacians $\mathcal{L}^{(k)}$ and $\mathcal{L}$ of $G^{(k)}$ and $G$ (respectively).

Let $\text{new}(A^{(k)})$ be the difference between the spectrum of $A^{(k)}$ and the spectrum of $A$ and define $\text{new}(\mathcal{L}^{(k)})$ similarly. $\text{new}(A^{(k)})$ is also a multiset: if $\lambda$ has multiplicity $m_1$ in the spectrum of $A$ and multiplicity $m_2$ in the spectrum of $A^{(k)}$, it occurs $m_2 - m_1$ times in $\text{new}(A^{(k)})$. Our main result is:

**Theorem 1.1** With the above notation, let $n = |V|$ be the number of vertices in $G$. Also let $d$ and $\Delta$ be the minimum and maximum degrees in $G$ (respectively). Assume that the matchings $\{M_{vw}\}_{vw \in E}$ are chosen independently and that for each $vw \in E$ and $\ell, r \in [k]$:

$$\mathbb{P}(\{(v, \ell), (w, r)\} \in M_{vw}) = \frac{1}{k}.$$ 

Then for all $\delta \in (0, 1)$,

$$\mathbb{P}\left(\sup_{\eta \in \text{new}(A^{(k)})} |\eta| \leq 16\sqrt{\Delta \ln(2nk/\delta)}\right) \geq 1 - \delta$$

and

$$\mathbb{P}\left(\sup_{\beta \in \text{new}(\mathcal{L}^{(k)})} |1 - \beta| \leq 16\sqrt{\frac{\ln(2nk/\delta)}{d}}\right) \geq 1 - \delta.$$
This is interesting even in the case \( k = 2 \). It is known \([5]\) that any \( d \)-regular graph has a two-lift whose new eigenvalues are all \( O \left( \sqrt{\ln d} \right) \). However, a typical random 2-lift of \( G_n \) might have at least one eigenvalue equal to \( d \). One example (also from \([5]\)) consists of \( n/(d+1) \) disconnected \((d+1)\)-cliques; the new eigenvalue \( d \) comes from there being a clique whose lift consists of two disconnected cliques. [It is possible to find connected examples with similar behavior.] Notice that the probability of there being such a clique is \( 1 - o(1) \) even when \( d = \lceil c \sqrt{\ln n} \rceil \) for some small constant \( c > 0 \). On the other hand, the Theorem shows that there exists some \( C > 0 \) such that for any \( \epsilon > 0 \), if \( d \geq C \ln n/\epsilon^2 \), then the largest new eigenvalue is \( \leq \epsilon d \) with probability \( \geq 1 - 1/n^2 \).

On the other hand, we note that the largest eigenvalue of \( A^{(k)} \) is always between \( d \) and \( \Delta \) and the eigenvalues of \( L^{(k)} \) are always between 0 and 2 \([7]\). Hence our result for the adjacency matrix is trivial if \( \Delta \leq \ln(nk/\delta) \) and the bound for the Laplacian is trivial when \( d \leq \ln(nk/\delta) \).

One corollary of Theorem 1.1 is the following result.

**Corollary 1.1** In the setting of Theorem 1.1, let \( k = k_1 \ldots k_s \) with \( k_1, \ldots, k_s \in \mathbb{N} \setminus \{0,1\} \) and consider two different random graphs:

- \( G^{(k)} \) is a maximally random \( k \)-lift of \( G \): that is to say, each random matching \( M_{vw} \) appearing in the construction of \( G^{(k)} \) is uniformly distributed over all matchings of \( \{(v,i)\}_{i=1}^k \) and \( \{(w,j)\}_{j=1}^k \), and the matchings are independent.
- \( \tilde{G}^{(k)} = G_s \) where \( G_0 = G \) and, for each \( 1 \leq i \leq s \), \( G_i \) is a maximally random \( k_i \)-lift of \( G_{i-1} \) (conditionally on \( G_0, G_1, \ldots, G_{i-1} \)).

Let \( A^{(k)} \) and \( L^{(k)} \) denote the adjacency matrix and Laplacian of \( G^{(k)} \) and define \( \tilde{A}^{(k)} \) and \( \tilde{L}^{(k)} \) similarly. Then (with an appropriate labelling of the vertices of the two graphs):

\[
P \left( \| A^{(k)} - \tilde{A}^{(k)} \| \leq 32 \sqrt{\Delta \ln(4nk/\delta)} \right) \geq 1 - \delta
\]

and

\[
P \left( \| L^{(k)} - \tilde{L}^{(k)} \| \leq 32 \sqrt{\frac{\ln(4nk/\delta)}{d}} \right) \geq 1 - \delta.
\]

This is interesting because the distributions of \( G^{(k)} \) and \( \tilde{G}^{(k)} \) can be very different. For instance, let \( k_1 = k_2 = \cdots = k_s = 2 \). If \( s \) is constant and the number of vertices is large enough, all \( 2^s! \) possible permutations will be seen in the matchings of \( \{v\} \times [k] \) with \( \{w\} \times [k] \) for \( vw \in E \). On the other hand, only \( 2^s \) possible permutations will be seen in \( \tilde{G}^{(k)} \).

Theorem 1.1 will be deduced from a recent concentration result for sums of independent random matrices. In what follows \( \mathbb{C}^{d \times d}_{\text{Herm}} \) is the space of \( d \times d \) Hermitian matrices with complex entries, the expectations of matrices are defined entrywise and \( \| \cdot \| \) is the operator norm. [See Section 2.2 and Section 2.4 for these and related definitions.]
Theorem 1.2 (Corollary 7.1 in [18]) Let $X_1, \ldots, X_m$ be mean-zero independent random matrices, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{C}^{d \times d}_{\text{Herm}}$ and such that there exists a $M > 0$ with $\|X_i\| \leq M$ almost surely for all $1 \leq i \leq m$. Define:

$$\sigma^2 \equiv \text{the largest eigenvalue of } \sum_{i=1}^{m} \mathbb{E}[X_i^2].$$

Then for all $t \geq 0$,

$$\mathbb{P}\left(\left\|\sum_{i=1}^{m} X_i\right\| \geq t\right) \leq 2d e^{-\frac{\sigma^2 t^2}{8d^2 + 4Mt}}.$$

Given this bound, Theorem 1.1 follows quite easily, while other proof techniques for bounding spectra of random matrices (such as the trace method [15, 13, 14, 16] and the discrepancy-based ideas of Feige and Ofek [11]) can be quite technical. In our setting, Theorem 1.2 is also an improvement over other general concentration bounds for random matrices, most notably the operator Chernoff bound of Ahlswede and Winter [1] and the matrix Hoeffding bound of Christofides and Markström [6]. A key advantage of Theorem 1.2 over related results is that its “variance” term can be much smaller, especially in the graph-theoretical setting; this is discussed in more detail in Remark 7.1 of [18].

The remainder of the paper is organized as follows. After the preliminary Section 2, we collect some basic facts about $k$-lifts in Section 3. We prove the Theorem and its Corollary in Section 4. The last Section presents some extensions and open questions.

2 Preliminaries

2.1 Basic notation

For a natural number $m \in \mathbb{N}\setminus\{0\}$, $[m]$ is the set of all integers $1 \leq i \leq m$.

We will frequently speak of multisets $S$. Given a ground set $\mathcal{S}$ (which will usually be $\mathbb{R}$), a multiset $S$ is defined by a function $m_S : \mathcal{S} \to \mathbb{N}$. Informally, we will let think of $S$ as a set where each $x \in A$ appears $m_S(x)$ times and we will refer to this quantity as the multiplicity of $x$. We say that $x$ belongs to $S$ ($x \in S$) if $m_S(x) > 0$.

For two multisets $S_1, S_2$ over the same ground set $\mathcal{S}$ and with corresponding functions $m_{S_1}, m_{S_2}$, we say that $S_1 \subset S_2$ if for all $x \in S$ $m_{S_1}(x) \leq m_{S_2}(x)$. The difference $S_2 \setminus S_1$ is the multiset where each $x \in S$ has multiplicity $\max\{m_{S_2}(x) - m_{S_1}(x), 0\}$. 
2.2 Linear algebra

For given \( d, d_c \in \mathbb{N}\setminus\{0\}, \mathbb{R}^{d_r \times d_c} \) (resp. \( \mathbb{C}^{d_r \times d_c} \)) is the space of \( d_r \times d_c \) matrices with entries in \( \mathbb{R} \) (resp. \( \mathbb{C} \)).

For \( A \in \mathbb{R}^{d_r \times d_c}, A^\dagger \in \mathbb{R}^{d_c \times d_r} \) is the transpose of \( A \); similarly, for \( B \in \mathbb{C}^{d_r \times d_c}, B^* \in \mathbb{C}^{d_c \times d_r} \) is the conjugate transpose of \( B \). We identify \( \mathbb{R}^d \) and \( \mathbb{C}^d \) with \( \mathbb{R}^{d \times 1} \) and \( \mathbb{C}^{d \times 1} \) (resp.), so that the standard inner product of \( x, y \in \mathbb{R}^d \) is \( x^\dagger y \).

\( \mathbb{C}^{d \times d}_{\text{Herm}} \) is the space of \( d \times d \) Hermitian matrices, which are the \( A \in \mathbb{C}^{d \times d} \) with \( A^* = A \). Similarly, \( \mathbb{R}^{d \times d}_{\text{Sym}} \) is the space of all \( d \times d \) real matrices that are symmetric in the sense that \( A = A^\dagger \).

For a vector \( v \in \mathbb{R}^d \) or \( \mathbb{C}^d \), \( \| v \| \) is its Euclidean norm. The operator norm of \( A \in \mathbb{R}^{d \times d} \) is:

\[
\| A \| \equiv \max_{v \in \mathbb{R}^d, \| v \|=1} \| Av \|.
\]

Finally, the canonical basis vectors for \( \mathbb{R}^d \) is denoted by \( e_1, e_2, \ldots, e_d \).

2.2.1 The spectral theorem

We recall the standard spectral theorem: for any \( A \in \mathbb{R}^{d \times d}_{\text{Sym}} \) there exists a set \( S \subset \mathbb{R} \) and orthogonal projections \( \{P_\alpha\}_{\alpha \in S} \) with orthogonal ranges such that:

\[
\sum_{\alpha \in S} \alpha P_\alpha = A \quad \text{and} \quad \sum_{\alpha \in S} P_\alpha = I_d,
\]

where \( I_d \) is the \( d \times d \) identity matrix. The numbers \( \alpha \in S \) are called the eigenvalues of \( A \) and the vectors \( v \) in the range of \( P_\alpha \) are eigenvectors corresponding to a given \( \alpha \). The spectrum of \( A \), denoted by \( \text{spec}(A) \), is the multiset where each \( \alpha \in S \) appears with multiplicity equal to the rank of \( P_\alpha \).

One useful consequence of the spectral decomposition is that \( \| A \| = \max_{\alpha \in \text{spec}(A)} |\alpha| \).

2.2.2 Tensor products

It will be convenient to represent the matrices of lifts via tensor products. The tensor product of \( \mathbb{R}^{d_1} \) and \( \mathbb{R}^{d_2} \), denoted by \( \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2} \), is the set of all formal linear combinations of vectors of the form \( e_{i_1} \otimes e_{i_2} \) with \( 1 \leq i_b \leq d_b \) for \( b = 1, 2 \). [We will abuse notation and assume that \( e_i \in \mathbb{R}^{d_1} \cap \mathbb{R}^{d_2} \) for \( i \leq \min\{d_1, d_2\} \).]

Similarly, if \( v_b = \sum_{j_b=1}^{d_b} v_{b,j_b} e_{j_b} \) (\( b = 1, 2 \)), the tensor product of \( v_1 \otimes v_2 \) is defined by the “distributive rule”:

\[
v_1 \otimes v_2 \equiv \sum_{j_1=1}^{d_1} \sum_{j_2=1}^{d_2} v_{1,j_1} v_{2,j_2} e_{j_1} \otimes e_{j_2}.
\]
There exists a unique inner product on $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$, denoted by $(\cdot, \cdot)$, such that for all $v_1, w_1 \in \mathbb{R}^{d_1}$ and $v_2, w_2 \in \mathbb{R}^{d_2}$,
\[ (v_1 \otimes v_2, w_1 \otimes w_2) = (v_1^\top v_2)(v_1^\top w_2). \]

Moreover, the tensor product of $A_1 \in \mathbb{R}^{d_1 \times d_1}$ and $A_2 \in \mathbb{R}^{d_2 \times d_2}$ is the unique linear operator $A_1 \otimes A_2$ from $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$ to itself that satisfies:
\[
\forall 1 \leq i_1 \leq d_1, \forall 1 \leq i_2 \leq d_2, \; (A_1 \otimes A_2)(e_{i_1} \otimes e_{i_2}) = (A_1 e_{i_1}) \otimes (A_2 e_{i_2}).
\]

One can check that if $A_1 \in \mathbb{R}^{d_1 \times d_1}_{\text{Sym}}$ and $A_2 \in \mathbb{R}^{d_2 \times d_2}_{\text{Sym}}$, then $A_1 \otimes A_2$ is self-adjoint in the sense that:
\[
\forall u, v \in \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}, \; (u, (A_1 \otimes A_2)v) = ((A_1 \otimes A_2)u, v).
\]

In general, one still has:
\[
\forall u, v \in \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}, \; (u, (A_1 \otimes A_2)v) = ((A_1^\dag \otimes A_2^\dag)u, v).
\]

i.e. $A_1^\dag \otimes A_2^\dag$ is the adjoint of $A_1 \otimes A_2$.

Notice that $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$ is isomorphic to $\mathbb{R}^{d_1 d_2}$, in the sense that any bijection $\psi : [d_1] \times [d_2] \to [d_1 d_2]$ can be “lifted” to an invertible, inner-product-preserving linear map:
\[
\Psi : \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2} \to \mathbb{R}^{d_1 d_2}
\]
defined by the rule $\Psi(e_i \otimes e_j) = e_{\psi(i,j)}$, $(i,j) \in [d_1] \times [d_2]$. Under this map, self-adjoint maps over $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$ correspond to symmetric matrices over $\mathbb{R}^{d_1 d_2}$ and vice versa. Therefore, one may also state a spectral theorem over $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$; we omit the details.

### 2.3 Concepts from Graph Theory

For our purposes a graph $G = (V, E)$ consists of a finite set $V$ of vertices and a set $E$ of edges, which are subsets of size 2 of $V$. Unless otherwise noted, we will assume that $V = [n]$ for some integer $n \geq 2$, where $[n] \equiv \{1, 2, \ldots, n\}$. We will write edges as unordered pairs $vw$ or $\{v, w\}$ and make no distinction between $vw$ and $wv$. The degree $d_G(v)$ of a vertex $v$ is the number of $w \in V \setminus \{v\}$ such that $vw \in E$.

Assume that $V = [n]$, or more generally, that the elements of $V$ are labelled $v_1, \ldots, v_n$. The adjacency matrix of $G$ is the $n \times n$ matrix $A \in \mathbb{R}^{n \times n}_{\text{Sym}}$ with zeros on the diagonal and such that, for all $1 \leq i < j \leq n$, the $(i,j)$-th entry of $A$ is 1 if $v_i v_j \in E$ and 0 otherwise. When $V = [n]$,
this reads:

\[ A \equiv \sum_{ij \in E} (e_i e_j^\dagger + e_j e_i^\dagger). \tag{2.2} \]

The Laplacian \( \mathcal{L} \) of \( G \) is the matrix:

\[ \mathcal{L} = I_n - T A T \]

where \( T \) is the \( n \times n \) diagonal matrix whose \((i, i)\)-th entry is \( d_G(i)^{-1/2} \) if \( d_G(i) \neq 0 \), or 0 if \( d_G(i) = 0 \). If all degrees are non-zero, one can write this as follows:

\[ \mathcal{L} = I_n - \sum_{ij \in E} \frac{(e_i e_j^\dagger + e_j e_i^\dagger)}{\sqrt{d_G(i)d_G(j)}}. \tag{2.3} \]

### 2.4 Probability with matrices

We will be dealing with random matrices (and random linear operators) throughout the paper.

Following common practice, we will always assume that we have a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) in the background where all random variables are defined.

Call a map \( X : \Omega \to \mathbb{C}^{d \times d}_{\text{Herm}} \) a random \( d \times d \) Hermitian matrix (or a \( \mathbb{C}^{d \times d}_{\text{Herm}} \)-valued random variable) if for each \( 1 \leq i, j \leq n \), the function \( X(i, j) : \Omega \to \mathbb{C}^{d \times d}_\text{Herm} \) corresponding to the \((i, j)\)-th entry of \( X \) is \( \mathcal{F} \)-measurable, or equivalently, if for each Borel subset \( S \subset \mathbb{C}^{d \times d}_\text{Herm} X^{-1}(S) \in \mathcal{F} \).

We say that \( X \) is integrable if all the entries of \( X \) are integrable, one defines \( \mathbb{E}[X] \) entrywise: the \((i, j)\)th entry of \( \mathbb{E}[X] \) is \( \mathbb{E}[X(i, j)] \). We will also use analogous definitions for \( X : \Omega \to \mathbb{R}^{d \times d}_{\text{Sym}} \).

[We will essentially ignore all measurability and integrability issues in the remainder of the paper. These can be dealt with in a rather straightforward manner.]

One can easily check that if \( X \) is a random integrable \( d \times d \) Hermitian matrix and \( A \in \mathbb{C}^{d \times d}_{\text{Herm}} \) is deterministic, \( \mathbb{E}[AX] = A \mathbb{E}[X] \). If the entries of \( X \) are also square integrable, one may define a “matrix variance” \( \mathbb{V}(X) \equiv \mathbb{E}[(X - \mathbb{E}[X])^2] \) and deduce that:

\[ \mathbb{V}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \tag{2.4} \]

### 3 Lifts of graphs

Our goal here is to review the construction of lifts of graphs outlined in the introduction and to prove some elementary facts that will be useful later on. Other perspectives on these objects can be found in [3].

Recall that a matching of two finite, disjoint, non-empty sets \( A, B \) is a set of pairs:

\[ \mathcal{M} = \{ \{a_i, b_i\} : i = 1, \ldots, m \} \]
where \((a_1, \ldots, a_m)\) is a permutation of the elements of \(A\) and \((b_1, \ldots, b_m)\) is a permutation of the elements of \(B\). Notice that the existence of a matching \(M\) as above implies that \(|A| = |B| = m\).

Now let \(G\) be a graph with vertex set \(V = [n]\) and edge set \(E\). Given \(k \in \mathbb{N} \setminus \{0, 1\}\), a \(k\)-lift \(G^{(k)}\) of \(G\) is determined by a choice of matchings:

\[
\{M_{ij} : ij \in E\},
\]

where for each \(ij \in E\) \(M_{ij}\) is a matching of \(\{i\} \times [k]\) and \(\{j\} \times [k]\). \(G^{(k)}\) is the graph with vertex set \([n] \times [k]\) and edge set \(E^{(k)} = \cup_{ij \in E} M_{ij}\).

### 3.1 Graph matrices and tensor products

It is convenient to represent the matrices corresponding to \(G^{(k)}\) in the tensor space \(\mathbb{R}^n \otimes \mathbb{R}^k\). That is to say, we will write down a linear operator \(A^{(k)}\) over \(\mathbb{R}^n \otimes \mathbb{R}^k\) such that for all \((i, \ell), (j, r) \in [n] \times [k]\),

\[
(e_i \otimes e_{\ell}, A^{(k)}(e_j \otimes e_r)) = \begin{cases} 
1 & \text{if } \{(i, \ell), (j, r)\} \in E^{(k)}; \\
0 & \text{otherwise}.
\end{cases}
\]

This is satisfied by:

\[
A^{(k)} = \sum_{ij \in E^{(k)}} (e_i e_j^\dagger) \otimes (e_{\ell} e_r^\dagger) + (e_j e_i^\dagger) \otimes (e_r e_{\ell}^\dagger).
\]

Another way of writing \(A^{(k)}\) will be more useful later on:

\[
A^{(k)} = \sum_{ij \in E} e_i e_j^\dagger \otimes V_{(i,j)} + e_j e_i^\dagger \otimes V_{(j,i)}, \quad \text{where } V_{(i,j)} \text{ is defined as:}
\]

\[
V_{(i,j)} \equiv \sum_{(\ell, r) \in [k]^2 : \{(i, \ell), (j, r)\} \in M_{ij}} e_{\ell} e_r^\dagger.
\] (3.1)

We emphasize that the definition of \(V_{(i,j)}\) is not symmetric with respect to \(i, j\): in fact, a simple computation shows that \(V_{(i,j)} = V_{(j,i)}^{-1}\).

The Laplacian \(\mathcal{L}^{(k)}\) of \(G^{(k)}\) can be similarly written as a linear operator over \(\mathbb{R}^n \otimes \mathbb{R}^k\). The key point to notice is that all copies of \(i \in [n]\) in \(G^{(k)}\) have the same degree, i.e.,

\[
\forall \ell \in [k], \ d_{G^{(k)}}((i, \ell)) = d_G(i).
\]

A simple calculation (omitted) shows that:

\[
\mathcal{L}^{(k)} = I_n \otimes I_k - \sum_{ij \in E} \frac{e_i e_j^\dagger \otimes V_{(i,j)} + e_j e_i^\dagger \otimes V_{(j,i)}}{\sqrt{d_G(i)d_G(j)}}.
\] (3.3)
3.2 Old and new eigenvalues

We now draw a connection between the spectrum and eigenvalues of $A$ and $A^{(k)}$. All arguments here also appear on previous papers on graph lifts (e.g. [3, 13]).

**Proposition 3.1** The spectrum of the adjacency matrix $A$ of $G$ is contained in the spectrum of $A^{(k)}$ (counting multiplicities). Moreover, if

$$\text{new}(A^{(k)}) \equiv \text{spec}(A^{(k)}) \setminus \text{spec}(A)$$

is the difference of the two spectra as multisets,

$$\max_{\eta \in \text{new}(A^{(k)})} |\eta| = \|A^{(k)} - A \otimes \Pi_k\|$$

where $\Pi_k$ is the $k \times k$ matrix with all entries equal to $1/k$.

Essentially the same argument shows a related result for the Laplacian $L^{(k)}$ of $G^{(k)}$ (proof omitted).

**Proposition 3.2** The spectrum of the Laplacian $L$ of $G$ is contained in the spectrum of $L^{(k)}$ (counting multiplicities). Moreover, if

$$\text{new}(L^{(k)}) \equiv \text{spec}(L^{(k)}) \setminus \text{spec}(L)$$

is the difference of the two spectra as multisets,

$$\max_{\eta \in \text{new}(L^{(k)})} |1 - \eta| = \|L^{(k)} - (I_n \otimes I_k - (I - L) \otimes \Pi_k)\|$$

where $\Pi_k$ is the $k \times k$ matrix with all entries equal to $1/k$.

**Proof:** [of Proposition 3.1] Let $1_k \in \mathbb{R}^k$ be the vector with all coordinates equal to 1. Notice that $V_{(i,j)}1_k = \Pi_k 1_k = 1_k$ for all $i, j$ with $ij \in E$. Therefore, for all vectors $v \in \mathbb{R}^n$,

$$A^{(k)}(v \otimes 1_k) = (Av) \otimes 1_k = (A \otimes \Pi_k)1_k.$$

In particular, if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, $v \otimes 1_k$ is an eigenvector of both $A^{(k)}$ and $A \otimes \Pi_k$, with the same eigenvalue $\lambda$ for both matrices. It follows that each eigenvalue $\lambda$ of $A$ with multiplicity $m$ is an eigenvalue of $A^{(k)}$ with multiplicity $\geq m$, which is the first assertion in the Proposition.

Any new eigenvalue $\eta \in \text{new}(A^{(k)})$ must correspond an eigenvector $w \in \mathbb{R}^n \otimes \mathbb{R}^k$ that is orthogonal to $v \otimes 1_k$ for all eigenvectors $v$ of $A$ corresponding to “old” eigenvalues. Since the
eigenvectors of $A$ span $\mathbb{R}^n$, any $w$ as above must be orthogonal to the subspace:

$$H \equiv \{ v \otimes 1_k : v \in \mathbb{R}^n \} \subset \mathbb{R}^n \otimes \mathbb{R}^k.$$ 

In particular,

$$\max_{\eta \in \text{new}(A^{(k)})} ||\eta|| = \max_{w \in H^\perp} \|A^{(k)}w\|.$$ 

To finish, we must show that the RHS equals $\|A^{(k)} - A \otimes \Pi_k\|$. We have already seen that the operators $A^{(k)}$ and $A \otimes \Pi_k$ have $H$ as an invariant subspace and that their restrictions to that subspace are equal. This implies that $H^\perp$ must also be invariant and moreover:

$$\|A^{(k)} - A \otimes \Pi_k\| = \max_{w \in H^\perp : \|w\|=1} \|A^{(k)}w - (A \otimes \Pi_k)w\|.$$ 

Now notice that:

$$H^\perp \equiv \text{span}\{x \otimes y : x \in \mathbb{R}^n, y \in \mathbb{R}^k, y \perp 1_k\}.$$ 

Moreover, for all $x \otimes y$ as above,

$$(A \otimes \Pi_k)(x \otimes y) = (Ax) \otimes (\Pi_k y) = 0$$

since $\Pi_k$ is the projection onto the line spanned by $1_k$. By linearity, this implies that $(A \otimes \Pi_k)w = 0$ for all $w \in H^\perp$, which results in the desired equality:

$$\|A^{(k)} - A \otimes \Pi_k\| = \max_{w \in H^\perp : \|w\|=1} \|A^{(k)}w\|.$$ 

$\square$

## 4 Main proofs

Propositions 3.1 and 3.2 show that in order to prove Theorem 1.1, one must bound the difference between certain matrices. We attack this problem from the perspective of concentration of measure. As it turns out, $A \otimes \Pi_k$ is the expected value of $A^{(k)}$ and $I_n \otimes I_k - (I_n - \mathcal{L}) \otimes \Pi_k$ is the expected value of $\mathcal{L}^{(k)}$. The concentration inequality in Theorem 1.2 will ensure that $A^{(k)}$ and $\mathcal{L}^{(k)}$ are likely to be close to their respective expected values. One this is achieved, Theorem 1.1 and its Corollary will easily follow.

### 4.1 Proof of the main theorem

In this section we prove Theorem 1.1.
Proof: [of Theorem 1.1] We start with the result for the adjacency matrix. Proposition 3.1 implies that it is necessary and sufficient to prove that:

\[
\text{Goal} \quad \mathbb{P} \left( \| A^{(k)} - A \otimes \Pi_k \| \leq 16 \sqrt{\Delta \ln(2nk/\delta)} \right) \geq 1 - \delta. \tag{4.1}
\]

We will restate this as a concentration bound for the sum of random matrices. Recall from Section 3.1 that:

\[
A^{(k)} = \sum_{ij \in E} Z_{ij} \text{ where } Z_{ij} = e_j e_i^\dagger \otimes V_{(i,j)} + e_i e_j^\dagger \otimes V_{(j,i)}. 
\]

We notice that all $Z_{ij}$ are self-adjoint, as attested by (2.1) and the fact that $V_{(i,j)}^\dagger = V_{(j,i)}$ (cf. Section 3.1).

The matrices $V_{(i,j)}$ and $V_{(j,i)}$ are determined by the random matching $M_{ij}$. Since these matchings are independent, the $\{Z_{ij}\}_{ij \in E}$ are also independent. Let us now compute $\mathbb{E}[Z_{ij}]$ for a fixed $ij \in E$. It is not hard to show that this is:

\[
\mathbb{E}[Z_{ij}] = (e_j e_i^\dagger + e_i e_j^\dagger) \otimes \Pi_k. \tag{4.2}
\]

The $(\ell, r)$-th entry of $V_{(i,j)}$ is an indicator random variable that is equal to 1 iff $(i, \ell)$ and $(j, r)$ are connected in the matching. By assumption, this happens with probability $1/k$, therefore each entry of $V_{(i,j)}$ has expected value $1/k$. This implies that $\mathbb{E}[V_{(i,j)}] = \Pi_k$. We deduce that:

\[
\mathbb{E}[Z_{ij}] = (e_j e_i^\dagger + e_i e_j^\dagger) \otimes \Pi_k. \tag{4.2}
\]

Now employ (2.2) to deduce that:

\[
\sum_{ij \in E} \mathbb{E}[Z_{ij}] = \left(\sum_{ij \in E} e_j e_i^\dagger + e_i e_j^\dagger \right) \otimes \Pi_k = A \otimes \Pi_k.
\]

In other words,

\[
A^{(k)} - A \otimes \Pi_k = \sum_{ij \in E} (Z_{ij} - \mathbb{E}[Z_{ij}]) \tag{4.3}
\]

is a sum of independent, self-adjoint random linear operators with mean 0. One may recall from Section 2.2.2 that self-adjoint linear operators over $\mathbb{R}^n \otimes \mathbb{R}^k$ correspond to symmetric matrices over $\mathbb{R}^{nk}$; therefore, we can apply Theorem 1.2 to the above sum once we compute the variance parameter $\sigma^2$ and the uniform bound $M$.

We start with $M$. $Z_{ij}$ is the adjacency matrix of a graph that has all degrees equal to 1.
matrices are defined. We obtain:

\[ d \text{ parameter in this case is linear operators } \{ \} \]

is the largest entry on the diagonal, which is \( \max_i \).

But the matrix on the RHS is diagonal with non-negative entries, hence its largest eigenvalue \( \lambda \).

Now recall from Section 3.1 that \( B \).

Given two symmetric matrices \( A, B \),

Summing up those terms, we arrive at:

\[ \sum_{ij} E \left( (Z_{ij} - E[Z_{ij}])^2 \right) = \left[ \sum_{ij} (e_i^t e_i + e_j^t e_j) \right] \otimes (I_k - \Pi_k) = \sum_{i=1}^n d_G(i) e_i e_i^t \otimes (I_k - \Pi_k). \]

Given two symmetric matrices \( B_1, B_2 \), the eigenvalues of \( B_1 \otimes B_2 \) are precisely the products of the form \( \lambda_1 \lambda_2 \) with \( \lambda_i \in \text{spec}(B_i) \), \( i = 1, 2 \). To apply this above, notice that \( \Pi_k \) is a rank-1 projection, hence the eigenvalues of \( I_k - \Pi_k \) are 0 and 1. It follows that:

\[ \left\| \sum_{i=1}^n d_G(i) e_i e_i^t \otimes (I_k - \Pi_k) \right\| = \left\| \sum_{i=1}^n d_G(i) e_i e_i^t \right\|. \]

But the matrix on the RHS is diagonal with non-negative entries, hence its largest eigenvalue is the largest entry on the diagonal, which is \( \max_i d_G(i) = \Delta \). We deduce that one may take \( \sigma^2 = \Delta \).

We now apply Theorem 1.2 with \( \sigma^2 = \Delta \) and \( M = 2 \) to the sum of the independent random linear operators \( \{ Z_{ij} - E[Z_{ij}] \}_{ij \in E} \), which is \( A(k) - A \otimes \Pi_k \) (cf. (1.3)). Moreover, the dimension parameter in this case is \( d = nk \) because that is the dimension of the space \( \mathbb{R}^n \otimes \mathbb{R}^k \) where the matrices are defined. We obtain:

\[ \mathbb{P} \left( \| A(k) - A \otimes \Pi_k \| \geq t \right) \leq 2nk e^{-\frac{t^2}{8\Delta + 1}}. \]
Taking $t = 16 \max \{ \sqrt{\Delta \ln(2nk/\delta)}, \ln(2nk/\delta) \}$ makes the RHS smaller than $\delta$. This implies the desired result if $\Delta \geq \ln(2nk/\delta)$. However, notice that $\|A^{(k)}\| \leq \Delta$, as $G^{(k)}$ is a graph of maximal degree $\Delta$; and similarly, $\|A \otimes \Pi_k\| \leq \Delta$. Therefore, we have $\|A^{(k)} - A \otimes \Pi_k\| \leq 2\Delta$ always and this implies that we still have the postulated bound if $\Delta \leq \ln(2nk/\delta)$, as in that case $16\sqrt{\Delta \ln(2nk/\delta)} \geq 16\Delta$. This proves (4.1), which (as seen above) is equivalent to the desired assertion via Proposition 3.1.

The proof for the Laplacian is quite similar and we will present it in less detail. We use Proposition 3.2 in order to restate the desired inequality as:

$$\text{[Goal]} \quad \mathbb{P} \left( \|L^{(k)} - (I_n \otimes I_k - (I - \mathcal{L}) \otimes \Pi_k)\| \leq 16\sqrt{\frac{\ln(2nk/\delta)}{d}} \right) \geq 1 - \delta. \quad (4.4)$$

Using equations (2.3) and (3.3), we see that:

$$I_n \otimes I_k - (I - \mathcal{L}) \otimes \Pi_k - L^{(k)} = \sum_{ij \in E} \frac{e_i e_i^\dagger \otimes (V_{i,j} - \Pi_k) + e_j e_j^\dagger \otimes (V_{j,i} - \Pi_k)}{\sqrt{d_G(i)d_G(j)}}$$

$$= \sum_{ij} \frac{Z_{ij} - \mathbb{E}[Z_{ij}]}{\sqrt{d_G(i)d_G(j)}} \otimes (I_k - \Pi_k) \quad (4.5)$$

with the same $Z_{ij}$ from the first part. The terms in the sum are again independent matrices with mean 0 and we will apply Theorem 1.2 to their sum. For this, we need to compute the corresponding $M$ and $\sigma^2$.

For the parameter $M$, we observe that, since $d$ is the minimum degree and $\|Z_{ij} - \mathbb{E}[Z_{ij}]\| \leq 2$ (as shown before),

$$\left\| \frac{Z_{ij} - \mathbb{E}[Z_{ij}]}{\sqrt{d_G(i)d_G(j)}} \right\| \leq \frac{2}{\sqrt{d_G(i)d_G(j)}} \leq \frac{2}{d},$$

hence we may take $M = 2/d$. Each term in the sum has variance:

$$\mathbb{E} \left[ \left( \frac{Z_{ij} - \mathbb{E}[Z_{ij}]}{\sqrt{d_G(i)d_G(j)}} \right)^2 \right] = \frac{1}{d_G(i)d_G(j)} \mathbb{E} \left[ (Z_{ij} - \mathbb{E}[Z_{ij}])^2 \right] = \frac{(e_i e_i^\dagger + e_j e_j^\dagger) \otimes (I_k - \Pi_k)}{d_G(i)d_G(j)}.$$

The sum of these terms is:

$$\sum_{ij \in E} \frac{(e_i e_i^\dagger + e_j e_j^\dagger) \otimes (I_k - \Pi_k)}{d_G(i)d_G(j)} = \sum_{i=1}^n \left( \sum_{j: i \in E} \frac{e_i e_i^\dagger}{d_G(i)d_G(j)} \right) \otimes (I_k - \Pi_k).$$

Again we have a tensor product of a diagonal matrix with another matrix whose eigenvalues are
either 0 or 1. We deduce as before that the operator norm is at most:

$$\max_i \sum_{j,i \in E} \frac{1}{d_G(i)d_G(j)} \leq \max_i \sum_{j \in E} \frac{1}{d_G(i)} = \frac{1}{d}.\$$

Therefore, we may take $\sigma^2 = 1/d$.

Apply now Theorem 1.2 and (4.5) to deduce that:

$$P\left(\|L^{(k)} - (I_n \otimes I_k - (I - \mathcal{L}) \otimes \Pi_k)\| \geq t\right) \leq 2nk e^{-\frac{t^2d}{8+8t}}.\$$

Taking:

$$t \equiv 16 \max \left\{ \sqrt{\frac{\ln(2nk/\delta)}{d}}, \frac{\ln(2nk/\delta)}{d} \right\}$$

makes the RHS $\leq \delta$ and implies the desired result when $\ln(2nk/\delta)/d \leq 1$. However, any graph Laplacian has spectrum contained in $[0, 2]$; this implies that $\|L^{(k)} - (I_n \otimes I_k - (I - \mathcal{L}) \otimes \Pi_k)\| \leq 4$ always. In particular, the bound claimed in (4.4) holds even if $\ln(2nk/\delta)/d > 1$. This finishes the proof of (4.4), which implies the Theorem (cf. Proposition 3.2).

**4.2 Proof of the corollary**

**Proof:** [of Corollary 1.1] We only present the proof of the adjacency matrix; the argument for the Laplacian is exactly the same.

The adjacency matrix $A^{(k)}$ of the graph $G^{(k)}$ satisfies:

$$P\left(\|A^{(k)} - A \otimes \Pi^{(k)}\| \leq 16\sqrt{\Delta \ln(4nk/\delta)}\right) \geq 1 - \frac{\delta}{2}.\quad (4.6)$$

This is precisely what we showed in the course of the proof of Theorem 1.1 and also follows from applying the Theorem in conjunction with Proposition 3.1.

We claim that the same bound holds for $\tilde{A}^{(k)}$, after a suitable relabelling of the vertices. The vertex set of this graph is $[n] \times K$ where

$$K = [k_1] \times [k_2] \times \cdots \times [k_s].$$

A simple induction argument shows that $\tilde{G}^{(k)}$ is also a lift of $G$, in the sense that its edge set $\tilde{E}^{(k)}$ is a union:

$$\tilde{E}^{(k)} = \bigcup_{ij \in E} \mathcal{M}_{ij},$$

where $\mathcal{M}_{ij}$ is a matching of $\{i\} \times K$ and $\{j\} \times K$.

It is easy to see that these matchings are independent, because they correspond to successive
matchings of the lifted images of distinct edges of \( G \). Moreover, two vertices \((i, \ell_1, \ldots, \ell_s) \in \{i\} \times K\) and \((j, r_1, \ldots, r_s) \in j \times \{j\} \times K\) are matched in \( \tilde{M}_{ij} \) if \((i, \ell_1)\) is matched to \((j, r_1)\) in \( G_1 \) and \((i, r_1, r_2)\) is matched to \((j, r_1, r_2)\) in \( G_2 \) and \( \ldots \) \((i, \ell_1, \ldots, \ell_s)\) is matched to \((j, r_1, \ldots, r_s)\) in \( G_s \). The recipe for constructing \( G_s \) implies that the probability of this event is:

\[
P \left( \{(i, \ell_1, \ldots, \ell_s), (j, r_1, \ldots, r_s)\} \in \tilde{M}_{ij} \right) = \frac{1}{k_1 k_2 \ldots k_s} = \frac{1}{k}.
\]

Thus if we label the elements of \( K \) with the numbers 1, 2, \ldots, \( k \), we see that \( \tilde{G}^{(k)} \) satisfies the assumptions of the Theorem. It follows that, just as in the case of \( G^{(k)} \),

\[
P \left( \left\| \tilde{A}^{(k)} - A \otimes \Pi^{(k)} \right\| \leq 16 \sqrt{\Delta \ln (4nk/\delta)} \right) \geq 1 - \frac{\delta}{2}.
\]

Putting this together with (4.6) finishes the proof. \( \square \)

5 Extensions and open questions

Lifts of Markov chains. The argument we showed can be applied to lifts of weighted graphs, or equivalently, of \textit{reversible Markov chains}. Let \( P \) be the transition matrix of an irreducible Markov chain on \([n]\) that is reversible with respect to a probability measure \( \pi \), meaning that \( \pi(i)P(i, j) = \pi(j)P(j, i) \) for all \( 1 \leq i, j \leq n \). [This implies that \( P \) has \( n \) real eigenvalues.]

Choose a matching \( M_{ij} \) for each pair \( 1 \leq i \leq j \leq n \) in the same way as in Theorem 1.1 and consider a Markov chain \( P^{(k)} \) on \([n] \times [k]\) with transition probabilities given by:

\[
P^{(k)}((i, r), (j, \ell)) = \begin{cases} P(i, j) & \{(i, r), (j, \ell)\} \in M_{ij}; \\ 0 & \text{if not.} \end{cases}
\]

One can show (proof omitted) that the spectrum of \( P^{(k)} \) contains that of \( P \) and that all new eigenvalues of \( P^{(k)} \) satisfy:

\[
P \left( \max_{\eta \in \text{new}(P^{(k)})} |\eta| \leq 16 \sqrt{c_P \ln (nk/\delta)} \right) \geq 1 - \delta,
\]

where

\[
c_P \equiv \max_{i \in [n]} \sum_{j=1}^{n} \frac{\pi(j)P(j, i)^2}{\pi(i)}.
\]

To prove this, one only needs to consider the symmetric matrix \( Q \) with entries equal to

\[
Q(i, j) \equiv \sqrt{\frac{\pi(i)}{\pi(j)}} P(i, j)
\]
(which has the same spectrum as $P$) and the corresponding matrix $Q^{(k)}$ for the lifted chain $P^{(k)}$, which is reversible with respect to the probability distribution:

$$\pi^{(k)}(i, \ell) = \pi(i)/k \quad ((i, \ell) \in [n] \times [k]).$$

Notice that the parameter $c_P$ always satisfies:

$$c_P \leq \max_{i \in [n]} \left\{ \left( \max_{r \in [n]} \frac{\pi(j)P(j, i)}{\pi(i)} \right) \sum_{j=1}^{n} \pi(j)P(j, i) \right\} = \max_{(i, r) \in [n]^2} P(r, i).$$

**Sharpness of the bound:** We do now know if the bound in Theorem 1.1 can be improved. For instance, could it be the case that all new eigenvalues of the adjacency matrix are $O\left(\sqrt{\Delta}\right)$ with high probability, at least when the minimum degree is $\Omega\left(\ln n\right)$? This would be similar to the Erdős-Rényi random graph [11] and also related to results on random regular graphs [14]. An analysis of the proof of Theorem 1.1 shows that the only obstacle to obtaining such a bound is the $d$ term in Theorem 1.2, but that term is known to be necessary in general [18]. However, it might be possible to obtain better concentration bounds in the graph-theoretic setting, at least for “well-behaved” base graphs $G$.

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