CONFORMAL METRICS WITH PRESCRIBED FRACTIONAL SCALAR CURVATURE ON CONFORMAL INFINITIES WITH POSITIVE FRACTIONAL YAMABE CONSTANTS

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Abstract. Let \((X, g^+)\) be an asymptotically hyperbolic manifold and \((M, \hat{h})\) its conformal infinity. Our primary aim in this paper is to introduce the prescribed fractional scalar curvature problem on \(M\) and provide solutions under various geometric conditions on \(X\) and \(M\). We also obtain the existence results for the fractional Yamabe problem in the end-point case, e.g., \(n = 3\), \(\gamma = 1/2\) and \(M\) is non-umbilic, etc. Every solution we find turns out to be smooth on \(M\).

1. Introduction

The main objective of this paper is to introduce and examine the prescribed fractional scalar curvature problem, the nonlocal version of the prescribed scalar curvature problem which has been served as one of the central problems in conformal geometry.

Suppose that \(X^{n+1}\) is an \((n + 1)\)-dimensional smooth manifold with boundary \(M^n\) and \(\rho\) is a defining function for \(M\), namely, a function in \(X\) satisfying \(\rho > 0\) in \(X\), \(\rho = 0\) on \(M\) and \(d\rho \neq 0\) on \(M\). We say that a metric \(g^+\) in \(X\) is conformally compact if there is a defining function \(\rho\) such that \(\bar{g} := \rho^2 g^+\) is a compact metric on the closure \(\bar{X}\) of \(X\). If \([\hat{h}]\) denotes the conformal class of the metric \(\hat{h} = \bar{g}|_M\) on \(M\), then \((M, [\hat{h}])\) is called the conformal infinity of the manifold \(X\). An asymptotically hyperbolic manifold is a conformally compact manifold such that all sectional curvatures tend to \(-1\) as each point approaches to \(M\).

Assume that \((X, g^+)\) is conformally compact and Einstein. It is called Poincaré-Einstein and a special example of asymptotically hyperbolic manifolds. In [24], Graham and Zworski introduced the fractional conformal Laplacian \(P^\gamma [g^+, \hat{h}]\) on the conformal infinity \((M, [\hat{h}])\) with \(n > 2\gamma\), a pseudo-differential operator which satisfies the conformal covariance property

\[
P^\gamma [g^+, w^{n-2\gamma} \hat{h}] u = w^{-\frac{n-2\gamma}{n-2\gamma}} P^\gamma [g^+, \hat{h}](wu)
\]

for all \(u, w \in C^\infty(M)\) such that \(w > 0\) on \(M\), and

\[
\sigma(P^\gamma [g^+, \hat{h}]) = \sigma((-\Delta_{\hat{h}})^\gamma)
\]

where \(\sigma\) denotes the principal symbol and \((-\Delta_{\hat{h}})^\gamma\) is the fractional power of the Laplace-Beltrami operator on \(M\) (see also [38, 31, 11, 21]). If \(\gamma = 1\) or 2, it agrees with the classical conformal Laplacian and the Paneitz operator, respectively. More generally, it is the same as the GJMS operator [23] for each \(\gamma \in \mathbb{N}\), constructed via the ambient metric.

Set the fractional scalar curvature (or \(\gamma\)-scalar curvature) by \(Q^\gamma [g^+, \hat{h}] := P^\gamma [g^+, \hat{h}](1)\). For Poincaré-Einstein manifolds, if \(\gamma = 1\) or 2, it is nothing but the scalar curvature and Branson’s \(Q\)-curvature up to a constant multiple, respectively. In this paper, we study the prescribed fractional scalar curvature problem or the prescribed \(\gamma\)-scalar curvature problem for \(\gamma \in (0, 1)\), which addresses:

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Given a smooth function \( f \) on the boundary \( M \), can one find a metric \( \hat{h} \in [\hat{h}] \) on \( M \) whose \( \gamma \)-scalar curvature \( Q^\gamma[g^+, \hat{h}_0] \) is \( f? \)

By virtue of (1.1), it is equivalent to look for a solution to the equation

\[
P^\gamma[g^+, \hat{h}]u = f u^{n+2\gamma \over n-2\gamma} \quad \text{and} \quad u > 0 \quad \text{on} \ (M^n, \hat{h}) \tag{1.3}
\]

provided that \( n > 2\gamma \).

This type of the problems dates back to at least the work of Kazdan and Warner in 1970s. In 32, they proved that for a compact manifold \( M^n \) with \( n \geq 3 \), any smooth function \( f \) that is somewhere negative on \( M \) can be a scalar curvature, and every smooth function \( f \) is a scalar curvature if and only if \( M \) admits a metric whose scalar curvature is positive. While they attempted to make use of (1.3) with \( \gamma = 1 \), there exists a function \( f \) such that the equation does not have a solution. To resolve this obstacle, they introduced an auxiliary diffeomorphism \( \varphi \) on \( M \) and changed the right-hand side of (1.3) with \((f \circ \varphi) u^{(n+2\gamma)/(n-2\gamma)}\), which gives additional flexibility to guarantee the existence of \( u \).

Later, by studying the associated constrained minimization problem to (1.3) with \( \gamma = 1 \), Escobar and Schoen 18 proved that (1.3) with \( \gamma = 1 \) has a solution if \((M, \hat{h})\) is a locally conformally flat compact manifold whose scalar curvature \( R[\hat{h}] \) is positive and fundamental group is non-trivial, and the function \( f \) is positive somewhere on \( M \) and vanishes up to order \( n-2 \) at the maximum point. (It is notable that the vanishing condition on \( f \) is satisfied automatically if \( n = 3 \)) They also considered when a locally conformally flat manifold \( M \) has the vanishing scalar curvature. Using a similar idea, Aubin and Hebey 5 and Hebey and Vaugon 26 studied when the Weyl tensor is not entirely zero on \( M \).

For a smooth compact Riemannian manifold \((X, \bar{g})\) with boundary \((M, \hat{h})\), Escobar 17 introduced the prescribed mean curvature problem, which is deeply related to (1.3) with \( \gamma = 1/2 \), and solved it under various geometric settings. For instance, if \( X \) has the positive Sobolev quotient, one can attain a solution provided that either \( f \) is somewhere positive on \( M \) and its Laplacian is sufficiently small compared to the norm of the umbilic tensor at the maximum point, or \( X \) is locally conformally flat, not conformally diffeomorphic to the Euclidean ball, \( M \) is umbilic, \( f \) is positive and \( \nabla^k f = 0 \) for \( 1 \leq k \leq n-2 \) at the maximum point.

As we will see, our theorems provide extensions of the above mentioned results, which cover all \( \gamma \in (0, 1) \) and most of situations such that the fractional Yamabe constant \( \Lambda^\gamma(M, [\hat{h}]) \) (see 2.10 for its definition) is positive.

If \((X, g^+)\) is the Poincaré ball and its conformal infinity \((M^n, \hat{h})\) is the standard sphere, Eq. (1.3) is said to be the fractional Nirenberg problem. By means of the stereographic projection, it is reduced to the problem

\[
(-\Delta)^\gamma u = f u^{2\gamma \over n-2\gamma} \quad \text{and} \quad u > 0 \quad \text{on} \ \mathbb{R}^n. \tag{1.4}
\]

For \( \gamma \in (0, 1) \), by employing the Caffarelli-Silvestre extension 7, Jin et al. 27 showed that the solution set of (1.4) is compact if \( f \) is positive somewhere on \( \mathbb{R}^n \) and vanishes up to order \( n-2\gamma \) at each critical point. Also, in 28, they deduced the existence result from the compactness theorem of 27 and the degree counting argument. (Compare this with our Theorem 1.6 especially, paying attention to the vanishing condition on \( f \).) On the other hand, the authors in 11 13 obtained some existence criterions regarding a topological condition on the level set of \( f \), by establishing Euler-Hopf type index formulae. Furthermore, Abdelhedi et al. 2 dealt with the case when the vanishing order of \( f \) is in \((1, n - 2\gamma)\). For \( \gamma \in (0, n/2) \), Jin et al. 29 recently extended the results of 27 28 by analyzing (1.4) through its integral representations instead of appealing the extension theorem.

If \( f \) is a constant on \( M \), we call (1.3) the fractional Yamabe problem or the \( \gamma \)-Yamabe problem. As in the classical case (i.e., \( \gamma = 1 \)), if \( M \) is the standard sphere, the solution set of (1.3) or (1.4) with \( f = 1 \) consists of the bubbles \( w_{\lambda, \sigma} \) defined in (2.15) (refer to 27).
Theorem 1.8] for its proof, which induce the loss of compactness in the fractional Sobolev space \( H^{s}(\mathbb{R}^n) \) due to the scaling invariance. For general manifolds and \( \gamma \in (0,1) \), \([13]\) has been investigated by several researchers \([21, 22, 33, 37]\) and most of cases are covered up to now. Also, Qing and Raske \([31]\) studied it assuming that \( \gamma \in [1,n/2) \) and \( M \) is locally conformally flat manifold with positive Yamabe constant and Poincaré exponent less than \((n-2\gamma)/2\).

The next theorem describes the result of Kim et al. \([33]\), which generalizes the pioneering works of González and Qing \([21]\) and González and Wang \([22]\).

**Theorem A** (Kim, Musso, Wei \([33]\)). Assume that the first \( L^2(X) \)-eigenvalue \( \lambda_1(-\Delta_{g+}) \) of the Laplace-Beltrami operator \(-\Delta_{g+}\) satisfies

\[
\lambda_1(-\Delta_{g+}) > \frac{n^2}{4} - \gamma^2. \tag{1.5}
\]

Suppose also that \( \rho \) is the geodesic defining function in \((X, g^+)\) associated to \((M, \hat{h})\), which is a unique defining function splitting the metric \( \bar{g} = \rho^2 g^+ \) as \( d\rho^2 + h_\rho \) near \( M \) with a family of metrics \( \{h_\rho\}_\rho \) on \( M \) such that \( h_0 = \hat{h} \). If one of the conditions

(a) \( n \geq 2, \gamma \in (0,1/2) \) and the boundary \((M, \hat{h})\) of \((\bar{X}, \bar{g})\) has a point where the mean curvature \( H \) is negative;

(b) \( n \geq 4, \gamma \in (0,1) \), \((M, \hat{h})\) is the non-umbilic boundary of \((\bar{X}, \bar{g})\) and

\[
R[\gamma^+] + n(n+1) = o(\rho^2) \quad \text{as } \rho \to 0 \text{ uniformly on } M \tag{1.6}
\]

where \( R[\gamma^+] \) denotes the scalar curvature of \((X, g^+)\);

(c) \( n > 3+2\gamma, \gamma \in (0,1) \), \((M, \hat{h})\) is the umbilic boundary of \((\bar{X}, \bar{g})\) such that a covariant derivative \( R_{ppp}[\bar{g}] \) of the component \( R_{pp}[\bar{g}] \) of the Ricci tensor on \((\bar{X}, \bar{g})\) is negative somewhere on \( M \), and \((1.6)\) holds;

(d) \( n > 4+2\gamma, \gamma \in (0,1) \), \((M, \hat{h})\) is the umbilic non-locally conformally flat boundary of \((\bar{X}, \bar{g})\) and

\[
R[\gamma^+] + n(n+1) = o(\rho^4) \quad \text{as } \rho \to 0 \text{ uniformly on } M, \text{ where } \bar{x} \text{ is a local coordinate on } M \tag{1.7}
\]

is true, then the \( \gamma \)-Yamabe problem is solvable.

**Remark 1.1.** For the precise meaning of the solvability of the \( \gamma \)-Yamabe problem, see Theorem 1.8 below. We have two additional remarks on the previous theorem.

1. The sign of \( H \) or \( R_{ppp}[\bar{g}] \) at a given point on \( M \) is intrinsic, namely, independent of the choice of the representative of the conformal class \([\hat{h}]\) on \( M \). The proof of this fact is given in \([21\) Lemma 2.3] and \([22\) Lemma 2.3], respectively. On the other hand, if \((1.6)\) is valid, then \( H = 0 \) on \( M \). Refer to our Lemma 2.6.

2. Notice that here we replaced the condition on (d) (refer to (1.19) of \([33]\)) with a slightly simpler one \((1.7)\). This is possible because \([33\) Lemma 3.2] can be derived under the sole condition \((1.7)\). See Lemma 2.7.

The first main theorem of this paper deals with the existence of positive solutions to \((1.3)\) when one of the geometric assumptions (a), (b), (c) and (d) in Theorem A is valid and the function \( f \) has a suitable behavior.

**Theorem 1.2.** Assume that \((1.5)\) holds, \( \Lambda^2(M, [\hat{h}]) > 0 \) and \( f \) is a smooth function positive somewhere on \( M \). If one of the following hypotheses

(A) condition (a) holds;

(B) condition (b) holds, \( f \) achieves a global maximum point at a nonumbilic point \( y \) of \( M \) and

\[
\frac{-\Delta_h f(y)}{f(y)} < c^1_{n,\gamma} \|\Pi\|^2_{h}(y) \tag{1.8}
\]

for some constant \( c^1_{n,\gamma} > 0 \) depending only on \( n \) and \( \gamma \);
(C) condition (c) holds, \( f \) achieves a global maximum point at \( y \in M \), \(- \Delta_{\hat{h}} f(y) = 0\) and \( R_{ppq}[\hat{g}](y) < 0;\)

(D) condition (d) holds, \( f \) achieves a global maximum point at a non-locally conformally flat point \( y \) of \( M \),

\[-\Delta_{\hat{h}} f(y) = 0 \quad \text{and} \quad \frac{-(-\Delta_{\hat{h}})^2 f(y)}{f(y)} < c_{n,\gamma}^2 \|W\|_{h}^2(y) \quad (1.9)\]

for some constant \( c_{n,\gamma}^2 > 0 \) depending only on \( n \) and \( \gamma \)

is met, then the prescribed \( \gamma \)-scalar curvature problem \((1.3)\) is solvable. Here

- \( \Lambda^\gamma(M, [\hat{h}]) > 0 \) is the fractional Yamabe constant whose definition is introduced in \((2.10)\);
- \( \Delta_{\hat{h}} \) is the Laplace-Beltrami operator on \((M, \hat{h})\), a negative semi-definite operator;
- \( \Pi \) is the second fundamental form on \((M, \hat{h}) \subset (\bar{X}, \bar{g})\) and \( \|\Pi\|_{h} \) is its 2-tensor norm;
- \( W \) is the \((0, 4)\) Weyl tensor on \((M, \hat{h})\) and \( \|W\|_{h} \) is its 4-tensor norm.

Remark 1.3. For the precise meaning of the solvability of the \( \gamma \)-scalar curvature problem, see Theorem 1.8. We have two additional remarks on the previous theorem.

(1) We always have \(-\Delta_{\hat{h}} f(y) \geq 0\) since \( y \in M \) is assumed to be a maximum point.

(2) In \((1.4)\) and \((1.9)\), the exact values of the positive constants \( c_{n,\gamma}^1 \) and \( c_{n,\gamma}^2 \) are presented, which are optimal in view of the energy expansion.

In fact, we can extend the above theorems to the end-point case. First, inspired by Marques \cite{36} and Almaraz \cite{33} for the boundary Yamabe problem, we can prove the following result on the 1/2-Yamabe problem. It validates the expectation in Remarks 1.2 (4) and 1.4 (3) of \cite{33}.

**Theorem B.** Suppose that \( \gamma = 1/2 \) and \((1.5)\) has the validity. If one of the conditions

- (b') \( n = 3 \), \((M, \hat{h})\) is the non-umbilic boundary of \((\bar{X}, \bar{g})\) and \((1.6)\) holds;
- (c') \( n = 4 \), \((M, \hat{h})\) is the umbilic boundary of \((\bar{X}, \bar{g})\), \( R_{ppq}[\bar{g}] \) is negative at some point on \( M \) and \((1.6)\) holds;
- (d') \( n = 5 \), \((M, h)\) is the umbilic non-locally conformally flat boundary of \((\bar{X}, \bar{g})\) and \((1.7)\) holds

is true, then the \( \gamma \)-Yamabe problem is solvable.

Although the 1/2-Yamabe problem and the boundary Yamabe problem match in analytic sense in that their energy functionals are identical modulo ignorable higher order terms, conformal changes of the metric are permitted only on the conformal infinity \( M \) in the 1/2-Yamabe problem unlike the boundary Yamabe problem. Theorem B confirms that the decay assumptions on the scalar curvature \( R[g^\gamma] \) given as \((1.6)\) and \((1.7)\) take away the difference of these problems.

Based on the previous theorem, we are able to deduce the following result.

**Theorem 1.4.** Assume that \( \gamma = 1/2 \), \((1.5)\) holds, \( \Lambda^\gamma(M, [\hat{h}]) > 0 \) and \( f \) is a smooth function positive somewhere on \( M \). If one of the following hypotheses

- (B') condition (b') holds and \( f \) achieves a global maximum point at a nonumbilic point \( y \) of \( M \);
- (C') condition (c') holds, \( f \) achieves a global maximum point at \( y \in M \), \(- \Delta_{\hat{h}} f(y) = 0\) and \( R_{ppq}[\hat{g}](y) < 0;\)
- (D') condition (d') holds, \( f \) achieves a global maximum point at a non-locally conformally flat point \( y \) of \( M \) and \(- \Delta_{\hat{h}} f(y) = 0 \)

is satisfied, then the prescribed \( \gamma \)-scalar curvature problem \((1.3)\) is solvable.

**Remark 1.5.** Our proof for the above theorem produces analogous results to (B), (B'), (D) and (D') for the prescribed mean curvature problem, which extend the work of Escobar \cite{17}.
In [17, Theorem 3.3], the result corresponding to (B) was obtained for the prescribed mean curvature problem under the additional assumption that $n \geq 6$. It is notable that $\max_M f = 1$ is implicitly assumed in condition (3.20) of [17].

For the classical prescribed scalar curvature problem, corresponding to $\gamma = 1$, an analogous result to (D) is obtained by Aubin and Hebey [5]. Our statement also quantifies how the bi-Laplacian of $f$ can be large by providing the explicit value of $\gamma_{n,\gamma}^2$, which can be applied in the case of [5] as well. See also Hebey and Vaugon [24] that further investigated in this direction.

Suppose now that $(X, g^+)$ is Poincaré-Einstein, and either $(M, \bar{h})$ is locally conformally flat or $n = 2$. Let $\rho$ be the geodesic defining function in $X$ associated to $(M, \bar{h})$. In [33, Proposition 1.5], it was shown that for each $y \in M$, there exists Green’s function $G(\cdot, y)$ on $\bar{X} \setminus \{y\}$ which solves

$$
\begin{cases}
-\text{div}_g(\rho^{1-2\gamma}\nabla G(\cdot, y)) + E(\rho) G(\cdot, y) = 0 & \text{in } (X, \bar{g}), \\
\partial_\nu G(\cdot, y) := -\kappa_\gamma \left( \lim_{\rho \to 0^+} \rho^{1-2\gamma} \partial G(\cdot, y) \right) = \delta_y & \text{on } M
\end{cases}
$$

in the distribution sense where $\kappa_\gamma := 2^{2\gamma-1} \Gamma(\gamma)/\Gamma(1-\gamma) > 0$ and $\delta_y$ is the Dirac measure at $y$. Consult Proposition [A] below for the motivation of the equation. In [33, Conjecture 1.6], it was conjectured that $G$ has the form

$$
G(x, y) = g_{n,\gamma} d_g(x, y)^{-(n-2\gamma)} + A + \Psi(d_g(x, y))
$$

for any $x \in \bar{X}$ near $y \in M$, where $g_{n,\gamma} > 0$ is a suitable normalizing constant, $A \in \mathbb{R}$ and $\Psi$ is a function in $\mathbb{R}$ satisfying

$$
|\Psi(t)| \leq C|t|^{\min\{1,2\gamma\}} \quad \text{and} \quad |\Psi'(t)| \leq C|t|^{\min\{0,2\gamma-1\}}
$$

for $t$ small. Recently, its verification is proposed in [37, Corollary 6.1]. Under the validity of (1.10) with a technical condition $A > 0$, we can deduce the following theorem.

**Theorem 1.6.** Suppose that $\gamma \in (0,1)$, (1.5) holds, $(X, g^+)$ is a Poincaré-Einstein manifold, $\Lambda^+(M, [\bar{h}]) > 0$ and $f$ is a smooth function positive somewhere on $M$. Assume also that the expansion (1.10) of Green’s function is valid and $A > 0$. If one of the following hypotheses

(E) $n > 2\gamma$, $(M, \bar{h})$ is locally conformally flat, $f$ achieves a global maximum $y \in M$ and $\nabla^m f(0) = 0$ for $m = 1, \ldots, [n-2\gamma]$; or

(E') $n = 2$, $f$ achieves a global maximum $y \in M$ and $\nabla^m f(0) = 0$ for $m = 1, [2-2\gamma]$ is true, then the prescribed $\gamma$-scalar curvature problem (1.3) is solvable. Here $[x]$ denotes the smallest integer greater than or equal to $x$, which implies that (E') is automatically satisfied if $n = 2$ and $\gamma \in [1/2,1)$.

**Remark 1.7.** (1) Suppose that $\gamma = 1$ and either $n \leq 7$ or $M$ is locally conformally flat. In this situation, the positive mass theorem of Schoen and Yau [40, 41] implies that $A \geq 0$, and the condition $A > 0$ holds if and only if $M$ is not conformally equivalent to the standard sphere $\mathbb{S}^n$. Currently, to formulate and prove an analogue of the positive mass theorem for $\gamma \in (0,1)$ is left as a challenging open problem.

(2) The vanishing or flatness condition on $f$ in (E) and (E') corresponds to the results of Jin, Li and Xiong [27, Theorems 1.2, 1.3] for the fractional Nirenberg problem on the standard $n$-dimensional unit sphere.

The proof of Theorems [12, 13, 16] and [B] are based on the constrained minimization technique, employing the Chang-González extension theorem stated in Proposition [A]. In particular, the statement in the above theorems that ‘the $\gamma$-Yamabe problem or the prescribed $\gamma$-scalar curvature problem is solvable’ actually means the existence of a weak solution in the
weighted Sobolev space $H^{1,2}(X; \rho^{1-2\gamma})$ (see the next paragraph to Proposition A for its definition) to
\[
\begin{align*}
-u\text{div}_{\bar{g}}(\rho^{1-2\gamma}\nabla U) + E(\rho)U &= 0 & \text{in } (X, \bar{g}), \\
U > 0 & \text{in } \bar{X}, \quad U = u & \text{on } M,
\end{align*}
\]
\[
\tag{1.11}
\]
The next regularity theorem assures that it indeed gives a classical solution to (1.3) under the general condition $H = 0$ on $M$.

**Theorem 1.8.** Suppose that $\gamma \in (0, 1)$, \textup{(}5.5\textup{)} holds, $\Lambda^\gamma(M, [\hat{h}]) > 0$, $f \in C^\infty(M)$ and the mean curvature $H$ on $(M, \hat{h}) \subset (\bar{X}, \bar{g})$ vanishes. Then the trace $u$ on $M$ of each weak solution $U \in H^{1,2}(X; \rho^{1-2\gamma})$ to (1.11) belongs to the space $C^\infty(M)$ and solves (1.3) in a classical sense.

This result has been regarded to be true since the study on the fractional Yamabe problem was initiated. See [21, Proposition 3.2] for example. However, looking at the problem more carefully, one realizes that there is a technical issue arising from the fact that the metric $\bar{g}$ does not have to be the Euclidean one $\delta_\cdot$. More precisely, unlike the settings of \textup{[6, 27, 19, 8]}, deviation of $\bar{g}_{(n+1)(n+1)}$ from $\delta_{(n+1)(n+1)} = 1$ makes the partial derivative $\partial_\rho U$ of a solution $U$ to (1.11) appear as an inhomogeneous term of the equation satisfied by each difference quotient of $U$ in a tangent direction to $M$ (see (5.2)). Unfortunately, it seems not easy to treat $\partial_\rho U$ directly. To bypass this issue, we shall pick a representative of the conformal class $[\hat{h}]$ on $M$ such that the exponential map at some point is a local volume preserving map, whose existence is guaranteed by Cao [9] and Günther [25]. Then it will suffice to control $\rho \partial_\rho U$ instead of $\partial_\rho U$, which is a rather simple task owing to the scaling property of (1.11).

The paper is organized as follows. In Section 2 we introduce background materials such as the Chang-González extension theorem, the definition of the weighted Sobolev space $H^{1,2}(X; \rho^{1-2\gamma})$, the fractional Yamabe constant $\Lambda^\gamma(M, [\hat{h}])$ and so forth. In Sections 3 and 4, we derive a sufficient condition to assure the existence of a solution to (1.3) and describe several situations when the condition holds, proving Theorems 1.2 and 1.4 and 1.6. Section 5 is devoted to present regularity property of solutions to (1.3), and especially, the proof of Theorem 1.8. Finally, in Appendix A we prove Theorem B by employing various arguments from the papers [21, 33, 36, 3] on the fractional and the boundary Yamabe problem.

**Notations.**
- We use $2^\ast := (n + 2\gamma)/(n - 2\gamma)$ and Einstein summation convention throughout the paper.
- We always assume that $1 \leq i, j, k, l \leq n$ and $1 \leq a, b \leq N := n + 1$.
- $C > 0$ is a generic constant which can vary from line to line.
- For any $x \in \bar{\mathbb{R}}^N_+$ and $r > 0$, we set by $B^N(x, r)$ the $N$-dimensional open ball of radius $r$ and center $x$, and $B^N_+(x, r) = B^N(x, r) \cap \mathbb{R}_+^N$.
- Let $C^\alpha(\Omega) = C^{[\alpha], \alpha^{-[\alpha]}}(\Omega)$ for a number $0 < \alpha \notin \mathbb{N}$ and a subset $\Omega$ of $\mathbb{R}^n$.

2. Preliminaries

2.1. Extension results, functional spaces and the fractional Yamabe constant. Let $n > 2\gamma$, $\gamma \in (0, 1)$, $(X^{n+1}, g^\ast)$ an asymptotically hyperbolic manifold, $(M^n, [\hat{h}])$ its conformal infinity and $\rho$ a geodesic defining function of $(M, \hat{h})$. Also we write $P^\gamma[g^\ast, \hat{h}]$ to denote the fractional conformal Laplacian on $(M, \hat{h})$. Since the metric $g^\ast$ is always fixed, we will use the notation $P^\gamma_h = P^\gamma[g^\ast, \hat{h}]$ for simplicity.

We begin this section by recalling the local interpretation of the operator $P^\gamma_h$. Set
\[
\mathcal{D} := \{U = u + v\rho^{2\gamma} + o(\rho^{2\gamma}) \in C^\infty(X) \cap C^0(\bar{X}) : u, v \in C^\infty(M)\}.
\]
Proposition A (Chang, González [11]). Let \( \bar{g} = \rho^2 g^+ \) be the smooth metric on \( \bar{X} \) and \( H \) the mean curvature on \( (M, \bar{h}) \subset (\bar{X}, \bar{g}) \). Define

\[
E(\rho) = \rho^{-1-\gamma}(-\Delta_{\bar{g}}^+ - s(n-s))\rho^{n-s} \quad \text{in} \ X \quad \text{where} \ s := \frac{n}{2} + \gamma,
\]

which can be shown to be

\[
E(\rho) = -\left( \frac{n-2\gamma}{2} \right) \left( \frac{\partial \sqrt{g}}{\sqrt{|g|}} \right) \rho^{-2\gamma} \quad \text{in} \ M \times (0, r_0)
\]

(2.2) for some small \( r_0 > 0 \) (refer to [15] Remark 2.2 for its derivation). Suppose that (1.3) holds, \( H = 0 \) on \( M \) if \( \gamma \in (1/2, 1) \) and \( u \in C^\infty(M) \) is given.

(1) Let \( U \in D \) be a solution to

\[
\begin{aligned}
-\text{div}_{\bar{g}} (\rho^{1-2\gamma} \nabla U) + E(\rho) U &= 0 \quad \text{in} \ (X, \bar{g}), \\
U &= u \quad \text{on} \ M,
\end{aligned}
\]

(2.3) whose unique existence is guaranteed by [31] [24] [11]. Then it holds that

\[
\partial^\gamma_j U = -\kappa \gamma \left( \lim_{\rho \to 0^+} \rho^{1-2\gamma} \frac{\partial U}{\partial \rho} \right) = \begin{cases} 
\frac{P^*_h u}{\rho^*_h} & \text{for} \ \gamma \in (0, 1) \ \{1/2\}, \\
\frac{P^*_h u - \left( \frac{n-1}{2} \right) H u}{\rho^*_h} & \text{for} \ \gamma = 1/2.
\end{cases}
\]

(2.4)

(2) There is a defining function \( \rho^* \) such that

\[
E(\rho^*) = 0 \quad \text{on} \ X \quad \text{and} \quad \rho^*(\rho) = \rho(1 + O(\rho^{2\gamma})) \quad \text{near} \ M,
\]

(2.5) which is called adaptive. In addition, if we denote \( \tilde{U} = (\rho/\rho^*)^{(n-2\gamma)/2} U \in D \), then it solves

\[
\begin{aligned}
-\text{div}_{\bar{g}^*} \left( (\rho^*)^{1-2\gamma} \nabla \tilde{U} \right) &= 0 \quad \text{in} \ (X, \bar{g}^*), \\
\partial^\gamma_j \tilde{U} &= -\kappa \gamma \left( \lim_{\rho \to 0^+} (\rho^*)^{1-2\gamma} \frac{\partial \tilde{U}}{\partial \rho} \right) = \frac{P^*_h u - Q^*_h u}{\rho^*_h} \quad \text{on} \ M,
\end{aligned}
\]

(2.6)

where \( \bar{g}^* := (\rho^*)^2 g^+ \) and \( Q^*_h \) is the fractional scalar curvature.

Let \( H^{1,2}(X; \rho^{1-2\gamma}) \) be the weighted Sobolev space with weight \( \rho^{1-2\gamma} \) as the completion of the space \( D \) in \( (2.1) \) with respect to the norm

\[
||U||_{H^{1,2}(X; \rho^{1-2\gamma})} := \left( \int_X \rho^{1-2\gamma} (|\nabla U|_{\bar{g}}^2 + U^2) dv_{\bar{g}} \right)^{1/2} < \infty.
\]

(2.7)

By (2.5), \( H^{1,2}(X; (\rho^*)^{1-2\gamma}) \) is a Hilbert space equivalent to \( H^{1,2}(X; \rho^{1-2\gamma}) \).

In addition, for any element \( U \in H^{1,2}(X; \rho^{1-2\gamma}) \), we set the energy \( I^\gamma(U) \) of \( U \) associated with (2.3) and (2.4) as

\[
I^\gamma(U) = \begin{cases} 
\kappa \gamma \left( \int_X (\rho^{1-2\gamma} |\nabla U|_{\bar{g}}^2 + E(\rho) U^2) dv_{\bar{g}} \right) & \text{if} \ \gamma \in (0, 1) \ \{1/2\}, \\
\kappa \gamma \left( \int_X (\rho^{1-2\gamma} |\nabla U|_{\bar{g}}^2 + E(\rho) U^2) dv_{\bar{g}} + \left( \frac{n-1}{2} \right) \int_M H U^2 dv_h \right) & \text{if} \ \gamma \in \{1/2\},
\end{cases}
\]

(2.8)

and the energy \( J^\gamma(U) \) of \( U \) associated with (2.6) as

\[
J^\gamma(U) = \kappa \gamma \int_X (\rho^*)^{1-2\gamma} |\nabla U|_{\bar{g}}^2 dv_{\bar{g}} + \int_M Q^*_h U^2 dv_h.
\]

(2.9)

Let \( H^\gamma(M) \) be the fractional Sobolev space realized as the space of traces of functions in \( H^{1,2}(X; \rho^{1-2\gamma}) \). Then it holds that \( H^\gamma(M) \hookrightarrow L^{2^\gamma+1}(M) \) and we can define the fractional Yamabe constant as

\[
\Lambda^\gamma(M, [\hat{h}]) = \inf_{u \in H^\gamma(M) \setminus \{0\}} \frac{\int_M u P^*_h u dv_h}{(\int_M |u|^{2^\gamma+1} dv_h)^{\frac{n-2\gamma}{n}}},
\]

(2.10)
Let us set also
\[
\Lambda^\gamma(X, [\hat{h}]) = \inf_{U \in H^{1,2}(X; \rho^{1-2\gamma}) \setminus \{0\}} \frac{\Gamma^\gamma(U)}{\int_M |U|^{2+1} dv_{\hat{h}}} \tag{2.11}
\]
and \(\tilde{\Lambda}^\gamma(X, [\hat{h}])\) by replacing \(\Gamma^\gamma(U)\) in (2.11) with \(J^\gamma(U)\). The result of Case \[10\] shows that
\[
\Lambda^\gamma(M, [\hat{h}]) = \tilde{\Lambda}^\gamma(X, [\hat{h}]) = \Lambda^\gamma(X, [\hat{h}]) > -\infty \tag{2.12}
\]
under the validity of (1.5). We remark that the Poincaré-Einstein manifolds were treated in \[10\], but the arguments in it can be generalized to any asymptotically hyperbolic manifold provided \(H = 0\) for \(\gamma \in (1/2, 1)\).

Fix any \(\gamma \in (0, 1) \setminus \{1/2\}\) and assume that \((M, [\hat{h}])\) is a conformal infinity with positive fractional Yamabe constant. In the remaining part of this subsection, we briefly explain how to extend the operator \(P^\gamma_h : C^\infty(M) \to C^\infty(M)\) to \(P^\gamma_h : H^\gamma(M) \to H^{-\gamma}(M) := (H^\gamma(M))^*\).

**Lemma 2.1.** If \(\gamma \in (0, 1) \setminus \{1/2\}\), \((1.5)\) has the validity, \(\Lambda^\gamma(M, [\hat{h}]) > 0\), and \(H = 0\) on \(M\) for \(\gamma \in (1/2, 1)\), then \(\Gamma^\gamma(U) \geq 0\) for all \(U \in H^{1,2}(X; \rho^{1-2\gamma})\). Furthermore, \(\|U\|_\gamma := \sqrt{\Gamma^\gamma(U)}\) serves as an equivalent norm to the standard \(H^{1,2}(X; \rho^{1-2\gamma})\)-norm introduced in (2.7).

**Proof.** The first claim trivially comes from (2.11) and (2.12). The proof of the second claim can be found in \[12\], Lemma 3.4. \(\Box\)

**Lemma 2.2.** Given \(\gamma \in (0, 1) \setminus \{1/2\}\), the operator \(P^\gamma_h : H^\gamma(M) \to H^{-\gamma}(M)\) is well-defined and bounded. Moreover, if (1.5) is true, \(\Lambda^\gamma(M, [\hat{h}]) > 0\), and \(H = 0\) on \(M\) for \(\gamma \in (1/2, 1)\), then for any \(u \in H^\gamma(M)\), there exists a unique function \(U \in H^{1,2}(X; \rho^{1-2\gamma})\) such that (2.3) and (2.4) hold in weak sense.

**Proof.** Given a fixed element \(u_0 \in H^\gamma(M)\), let \(\{u_m\}_{m \in \mathbb{N}} \subset C^\infty(M)\) such that \(u_m \to u_0\) in \(H^\gamma(M)\). For each \(v \in H^\gamma(M)\), we define
\[
\int_M (P^\gamma_h u_0)v \, dv_{\hat{h}} := \lim_{m \to \infty} \int_M (P^\gamma_h u_m)v \, dv_{\hat{h}}.
\]
Then (1.2) tells us that it is well-defined and in fact independent of a choice of \(\{u_m\}_{m \in \mathbb{N}}\). Also, the boundedness of \(P^\gamma_h : H^\gamma(M) \to H^{-\gamma}(M)\) clearly holds.

For each \(u_m \in C^\infty(M)\), let \(U_m \in \mathcal{D}\) be a unique solution to (2.3). We get from Lemma 2.1 that
\[
\sup_{m \in \mathbb{N}} \|u_m\|^2_{H^{1,2}(X; \rho^{1-2\gamma})} \leq \Gamma^\gamma(U_m) = \int_M (P^\gamma_h u_m)v \, dv_{\hat{h}} \leq C \sup_{m \in \mathbb{N}} \|u_m\|^2_{H^\gamma(M)} < \infty.
\]
Therefore, \(u_m \to U_0 \in H^{1,2}(X; \rho^{1-2\gamma})\) for some \(U_0 \in H^{1,2}(X; \rho^{1-2\gamma})\). It is plain to verify that \(U_0\) solves (2.3) and (2.4) with \(u = u_0\) weakly. The uniqueness of \(U_0\) again results from Lemma 2.1 \(\Box\)

It is notable that the conformal covariance property (1.1) still holds for \(u \in H^\gamma(M)\).

**2.2. Bubbles.** Let \(N = n+1\), \(\mathbb{R}^N_+ = \{(\bar{x}, x_N) : \bar{x} \in \mathbb{R}^n, x_N > 0\}\) and \(\dot{H}^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})\) be the homogeneous weighted Sobolev space with weight \(x_N^{1-2\gamma}\). As can be seen in \[21\] Proposition 1.3, there exists the optimal constant \(S_{n,\gamma} > 0\) depending only on \(n \in \mathbb{N}\) and \(\gamma \in (0, 1)\) such that
\[
\|U(\cdot, 0)\|^2_{L^{2\gamma+1}(\mathbb{R}^n)} \leq S_{n,\gamma} \|U\|^2_{\dot{H}^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})} := S_{n,\gamma} \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} |\nabla U(\bar{x}, x_N)|^2 \, dx \tag{2.13}
\]
for any function \(U \in \dot{H}^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})\). Besides, if \(W_{\lambda,\sigma}\) is a bubble, that is, the unique solution of
\[
\begin{cases}
-\text{div}(x_N^{1-2\gamma} \nabla W_{\lambda,\sigma}) = 0 & \text{in } \mathbb{R}^N_+,

\partial_\nu W_{\lambda,\sigma} := -\kappa \gamma \lim_{|x_N| \to 0+} x_N^{1-2\gamma} \frac{\partial U}{\partial x_N} = (-\Delta)^\gamma w_{\lambda,\sigma} = w_{\lambda,\sigma}^{2*} & \text{on } \mathbb{R}^n
\end{cases}
\tag{2.14}
\]
where
\[ w_{\lambda,\sigma}(x) := \alpha_{n,\gamma} \left( \frac{\lambda}{x^2 + |x - \sigma|^2} \right)^{n-1/2} \] for \( x \in \mathbb{R}^n \) \hfill (2.15)
and \( \alpha_{n,\gamma} > 0 \) is a certain constant relying only on \( n \) and \( \gamma \), then the equality of (2.13) is attained by \( U = cW_{\lambda,\sigma} \) for any \( c > 0 \), \( \lambda > 0 \) and \( \sigma \in \partial \mathbb{R}^N \). If \( \gamma = 1/2 \), the function \( W_{\lambda,\sigma} \) can be explicitly written as
\[ W_{\lambda,\sigma}(x) = \alpha_{n,1/2} \left( \frac{\lambda}{(x + \sigma)^2 + |x - \sigma|^2} \right)^{n-1/2} \] for \( x = (\bar{x}, x_N) \in \mathbb{R}^n_+ \). \hfill (2.16)
We also note that \( W_{\lambda,0}(\cdot, x_N) \) is radially symmetric for each \( x_N > 0 \). An immediate consequence of (2.12) and (2.13) is that
\[ \nabla_h \rho \leq \frac{C}{2} \rho^{2\lambda} \] for any \( \lambda, \sigma \in \mathbb{R} \) for any \( \rho \in \mathbb{R}_+ \) and so on, if \( \rho > 0 \).

2.3. Results on the metric. Let us recall the expansion of the metric \( \bar{g} \) on \( \mathbb{X} \) near the boundary \( M \). Its proof can be found in Escobar [16] Lemma 3.1.

**Lemma 2.3.** Given any point \( y \in M \), let \( x = (\bar{x}, x_N) \in \mathbb{R}^n_+ \) be the Fermi coordinate on \( \mathbb{X} \) around \( y \). Then it holds that
\[ \bar{g}^{ij}(x) = \delta_{ij} + 2\Pi_{ij}x_N + \frac{1}{2}R_{ikjl}[\bar{h}]x_kx_l + \frac{1}{3}R_{ikjl}R_{kl} - \frac{1}{6}R_{ij}[\bar{h}]x_i x_j \] \[ \sqrt{\bar{g}}(x) = 1 - nHx_N + \frac{n}{2}(n^2H^2 - ||\Pi||^2 - R_{NN}x_N + O(|x|)) \] for \( x \in B^N_+(0, r_1) \) with \( r_1 > 0 \) small. Here \( 1 \leq i,j,k,l \leq n \),
- \( \Pi \) is the second fundamental form on \( (\mathbb{X}, \bar{h}) \subset (\mathbb{X}, \bar{g}) \) and \( ||\Pi||^2 = \bar{g}^{ij}\bar{g}^{kl}\Pi_{ij}\Pi_{kl} \);
- \( \bar{h} \) is the mean curvature on \( M \), that is, the average of the diagonal component of \( \Pi \);
- \( R_{ikjl}[\bar{h}] \) and \( R_{NN}[\bar{g}] \) are components of the full Riemannian curvature tensors on \( (\mathbb{M}, \bar{h}) \) and \( (\mathbb{X}, \bar{g}) \), respectively;
- \( R_{ij}[\bar{h}] \) and \( R_{NN}[\bar{g}] \) denote components of the Ricci tensors on \( (\mathbb{X}, \bar{h}) \) and \( (\mathbb{X}, \bar{g}) \), respectively.

Commas mean partial derivatives and all the coefficients are computed at \( y \).

**Remark 2.4.** In general, it holds that \( |E(\rho)| \leq C\rho^{-2\gamma} \) in \( X \) for some \( C > 0 \), so the energy functionals \( \mathcal{E}(U) \) or \( \mathcal{E}^*(U) \) in (2.8) and (2.9) are well-defined for all \( U \in H^{1,2}(X; \rho^{1-2\gamma}) \) and \( \gamma \in (0, 1/2) \) (see the proof of [15] Lemma 3.1). Because the coefficient of \( x_i x_N \) contains \( H_{ij} \), one of \( x_i x_j x_N \) contains \( H_{ijk} \) and so on, if \( H = 0 \) on \( M \), then we have that \( |\partial_{\rho}\sqrt{\bar{g}}(x)| \leq C\rho^{-1-2\gamma} \) in \( X \). In light of (2.22), this in turn implies that \( |E(\rho)| \leq C\rho^{1-2\gamma} \) in \( X \), and \( \mathcal{E}^*(U) \) or \( \mathcal{E}(U) \) are well-defined for all \( U \in H^{1,2}(X; \rho^{1-2\gamma}) \) and \( \gamma \in (0, 1) \).

We also have the following higher order expansion of the metric \( \bar{g} \) due to Marques [36].

**Lemma 2.5.** Given \( y \in M \), let \( x = (\bar{x}, x_N) \in \mathbb{R}^N_+ \) be the Fermi coordinate on \( \mathbb{X} \) around \( y \). If \( \Pi = \Pi_i = \Pi_{ij} = 0 \) at \( y \), it holds that
\[ \sqrt{\bar{g}}(x) = 1 - \frac{1}{12}R_{ijk}[\bar{h}]x_kx_j + \frac{1}{2}R_{NN}[\bar{g}]x_N^2x_i - \frac{1}{6}R_{NN}[\bar{g}]x_N^2 \] \[ - \frac{1}{20}R_{ijk}[\bar{h}]x_kx_jx_i + \frac{1}{9}R_{mijkl}[\bar{h}]x_kx_lx_i \] \[ - \frac{1}{4}R_{NN}[\bar{g}]x_N^2x_i \] \[ - \frac{1}{24}R_{NN}[\bar{g}]x_N^2x_i + 1 \] \[ + 2(R_{NN}[\bar{g}])^2 \] \[ x_N^N + O(|x|^3) \]
and
\[
\hat{g}^{ij}(x) = \delta_{ij} + \frac{1}{3} R_{ikjl} \hat{h}[i]x_kx_l + R_{iNjN}[\hat{g}]x_N^2 + \frac{1}{6} R_{ikjl;m} \hat{h}[i]x_kx_lx_m + R_{iNjN;k}[\hat{g}]x_N^2x_k \\
+ \frac{1}{3} R_{iNjN,N}[\hat{g}]x_N^3 + \left( \frac{1}{20} R_{ikjl;m6}[\hat{h}] + \frac{1}{15} R_{iksl}[\hat{h}]R_{ijkl}[\hat{h}] \right) x_kx_lx_mx_n \\
+ \left( \frac{1}{2} R_{iNjN,kl}[\hat{g}] + \frac{1}{3} \text{Sym}_{ij} \left( R_{iksl}[\hat{h}]R_{NjN}[\hat{g}] \right) \right) x_N^2x_k + \frac{1}{3} R_{iNjN;kN}[\hat{g}]x_N^3x_k \\
+ \frac{1}{12} (R_{iNjN,NN}[\hat{g}] + 8R_{iNjN}[\hat{g}]R_{NjN}[\hat{g}])x_N^4 + O(|x|^5)
\]
for \( x \in B_+^N(0,r_1) \). Here \( 1 \leq i,j,k,l,m,q \leq n \), semicolons mean covariant derivatives and every tensor is computed at \( y \).

The next lemmas explain how to choose a good conformal metric \( \hat{h} \) on \( M \) and control extrinsic quantities such as the mean curvature \( H \) or the second fundamental form \( \Pi \) on \( M \) by utilizing conditions (1.6) or (1.7).

**Lemma 2.6.** Suppose that \((X^N, g^+)\) is an asymptotically hyperbolic manifold with conformal infinity \((M, [\hat{h}] )\). If (1.6) holds, then there exist a representative \( \bar{h}_0 \in [\hat{h}] \), the geodesic boundary defining function \( \rho_0 \) associated to \( \bar{h}_0 \) and \( \bar{g}_0 = \rho_0^2 g^+ \) such that
\[
H = 0 \text{ on } M, \quad R_{ij}[\bar{h}_0](y) = 0 \quad \text{and} \quad R_{i\rho \rho_k}[\bar{g}_0](y) = \frac{1 - 2n}{2(n-1)}||\Pi_0(y)||^2 \quad (2.18)
\]
for a fixed point \( y \in M \), where \( \Pi_0 \) is the second fundamental form on \((M, \bar{h}_0) \subset (X, \bar{g}_0)\).

**Proof.** The proof can be found in [33, Lemma 2.4]. \( \square \)

The next lemma is a slight generalization of [33, Lemma 3.2] in that the decay condition of \( R[g^+] \) is relieved compared to that of [33].

**Lemma 2.7.** For \( n \geq 3 \), let \((X^N, g^+)\) be an asymptotically hyperbolic manifold such that the conformal infinity \((M^n, [\hat{h}] )\) is umbilic. If (1.7) holds, then there exist a representative \( \bar{h}_0 \in [\hat{h}] \), the geodesic boundary defining function \( \rho_0 \) \((= x_N \text{ near } M)\) associated to \( \bar{h}_0 \) and the metric \( \bar{g}_0 = \rho_0^2 g^+ \) such that (2.18) is true,

1. \( \text{Sym}_{ijk} R_{ijkl}[\bar{h}_0](y) = 0 \), \( \text{Sym}_{ijkl} \left( R_{ijkl}[\bar{h}_0] + \frac{2}{9} R_{miqj} \bar{h}_0 R_{mkql} \bar{h}_0 \right)(y) = 0 \),
2. \( \Pi = 0 \text{ on } M \), \( R_{NN;N}[\bar{g}_0](y) = R_{N\lambda N}[\bar{g}_0](y) = 0 \),
3. \( R_{\alpha\beta}[\bar{g}_0](y) = -\frac{n||W_0(y)||^2}{6(n-1)} \), \( R_{NN;\alpha\beta}[\bar{g}_0](y) = -\frac{||W_0(y)||^2}{12(n-1)} \),
4. \( R_{iNjN}[\bar{g}_0](y) = R_{ij}[\bar{g}_0](y), \quad R_{NN;NN}[\bar{g}_0](y) = \frac{3}{2n} R_{NN}[\bar{g}_0](y) - 2(R_{ij}[\bar{g}_0](y))^2 \),
5. \( R_{iNjN;ij}[\bar{g}_0](y) = \left( 3 - \frac{n}{2n} \right) R_{NN}[\bar{g}_0](y) - (R_{ij}[\bar{g}_0](y))^2 - \frac{||W_0(y)||^2}{12(n-1)} \)

for a fixed point \( y \in M \), where ||\( W_0 ||\) is the norm of the Weyl tensor \( W_0 := W[\hat{h}_0] \) of \((M, \bar{h}_0)\).

**Proof.** Fix \( y \in M \). In view of the previous lemma and the existence of a conformal normal coordinate [34, Theorem 5.2] (see also [9, 25]), we may assume that the metric \( \bar{h}_0 \) on \( M \) satisfies (2.18) and (1) in the statement. Then we see from the umbilicity of \( M \) and Lemma 2.5 that \( \Pi(y) = 0 \) and that (1.7) is equivalent to
\[
2n \left[ -R_{NN}[\bar{g}_0] - R_{NN;N}[\bar{g}_0]x_N - \frac{1}{2} R_{NN;N}[\bar{g}_0]x_N - \frac{1}{2} R_{NN;ij}[\bar{g}_0]x_i x_j \right. \\
\left. - \frac{1}{2} R_{NN;N}[\bar{g}_0]x_N x_i + \left( \frac{1}{2} (R_{NN}[\bar{g}_0])^2 - \frac{1}{6} R_{NN;NN}[\bar{g}_0] - \frac{1}{3} (R_{iNjN}[\bar{g}_0])^2 \right) x_N^2 \right]
\]
where the last equation should be modified adequately if \( f \) is nonempty. If we set \( \beta, f \in (1, 2^* ) \) for (1.3) is to examine the existence of positive solutions to a more general class of problems and \( \beta \) for \( \gamma = 1/2 \). 

3.1. Constraint set and accompanying quantities. Throughout this section, we always assume that \( f \) is a smooth function on \( M \), somewhere positive. One way to search solutions for (1.3) is to examine the existence of positive solutions to a more general class of problems

\[
P_h^\beta u = f|u|^\beta-1 u \quad \text{on} \quad (M^n, \hat{g})
\]

for \( \beta \in (1, 2^* ] \). Thanks to Proposition \( \Box \) and Lemma 2.2, it can be interpreted as

\[
\begin{aligned}
&-\text{div}_g(\rho^{1-2\gamma} \nabla U) + E(\rho)U = 0 \quad \text{in} \quad (X, \bar{g}), \\
&U = u \quad \text{on} \quad M, \\
&\partial_\nu U = f|u|^\beta-1 u \quad \text{on} \quad M,
\end{aligned}
\]

where the last equation should be modified adequately if \( \gamma = 1/2 \). 

Since \( f \) is positive at some point on \( M \), the constraint set \( C_{\beta, f} \) given as

\[
C_{\beta, f} = \left\{ U \in H^{1,2}(X; \rho^{1-2\gamma}) : U = u \quad \text{on} \quad M, \int_M f|u|^\beta+1 dv_h = 1 \right\}
\]

is nonempty. If we set

\[
\Theta^\gamma(\beta, f) = \inf_{U \in C_{\beta, f}(M)} I^\gamma(U), \quad \tilde{\Theta}^\gamma(\beta, f) = \inf_{U \in \tilde{C}_{\beta, f}(M)} J^\gamma(U)
\]

is nonempty. If we set

\[
\Theta^\gamma(\beta, f) = \inf_{U \in C_{\beta, f}(M)} I^\gamma(U), \quad \tilde{\Theta}^\gamma(\beta, f) = \inf_{U \in \tilde{C}_{\beta, f}(M)} J^\gamma(U)
\]
Lemma 3.1. Suppose that (1.5) is valid, \( H = 0 \) for \( \gamma \in (1/2, 1) \), \( \Lambda^\gamma(M,[\hat{h}]) > 0 \) and \( \beta \in (1,2^*) \). Then it holds that
\[
\Theta^\gamma(\beta,f) = \Theta^\gamma(\beta,f) = \bar{\Theta}^\gamma(\beta,f) \geq 0.
\]
Proof. By the condition \( \Lambda^\gamma(M,[\hat{h}]) > 0 \), [10] Theorem 1.1] and the density argument, we have
\[
0 \leq \Theta^\gamma(\beta,f) \leq \min \{ \Theta^\gamma(\beta,f), \bar{\Theta}^\gamma(\beta,f) \}.
\]
Suppose that \( \{u_m\}_{m \in \mathbb{N}} \subset C^\infty(M) \) is a minimizing sequence of \( \Theta^\gamma(\beta,f) \). Owing to (1.5), for each \( u_m \), the eigenvalue problem
\[
-\Delta_g v - s(n-s)v = 0 \quad \text{in } X, \quad s := \frac{n}{2} + \gamma
\]
has a solution of the form
\[
v_m = F\rho^{n-s} + G\rho^s, \quad F,G \in C^\infty(X), \quad F|_{\rho=0} = u_m
\]
(see for instance [38, 31, 24]) where \( \rho \) is the geodesic defining function associated to \((M,\hat{h})\).

In [11], the authors proved that \( U_m := \rho^{-(n-s)}v_m \in H^{1,2}(X;\rho^{1-2\gamma}) \) is a solution of (2.3) with \( u = u_m \) and \( \partial^\gamma U_m = P_h^\gamma u_m \) on \( M \). Thus, by putting \( U_m \) in (2.3), we observe
\[
\Theta^\gamma(\beta,f) = \Gamma(U_m) = \int_M u_mP_h^\gamma u_m dv_h \to \Theta^\gamma(\beta,f).
\]
Similarly, we have \( \bar{\Theta}^\gamma(\beta,f) \leq \Theta^\gamma(\beta,f) \). This finishes the proof. \( \square \)

If \( H = 0 \) on \( M \) and \( U \in C_{\beta,f} \) achieves the infimum \( \Theta^\gamma(\beta,f) \), then it solves
\[
\begin{aligned}
&\text{div}_\gamma(\rho^{1-2\gamma}\nabla U) + E(\rho)U = 0 \quad \text{in } (X,\bar{g}), \\
&\partial^\gamma U = \Theta^\gamma(\beta,f)|U|^{\beta-1}U \quad \text{on } M
\end{aligned}
\]
(3.7)
in the weak sense. Since \( |U| \) also attains \( \Theta^\gamma(\beta,f) \), we may assume that \( 0 \leq U \neq 0 \) in \( X \). Then Remark [5,2] below implies that \( U \) is in fact positive in \( X \) and \( \bar{\Theta}^\gamma(\beta,f) > 0 \) (cf. [21] Theorem 3.5, Corollary 3.6)). Therefore a constant multiple of \( U \) gives a positive solution of (3.2). In the next subsection, we shall provide a criterion which guarantees a minimizer \( U \in C_{\beta,f} \).

3.2. Subcritical approximation. The main goal of this subsection is to show

Proposition 3.2. Suppose that (1.5) holds, \( H = 0 \) for \( \gamma \in (1/2,1) \), \( \Lambda^\gamma(M,[\hat{h}]) > 0 \) and \( f \) is a smooth function positive somewhere on \( M \). For any \( U \in C_{2^*,f} \), it is valid that
\[
\left( \max_{x \in M} f(x) \right)^{\frac{n-2n}{n}} \Theta^\gamma(2^*,f) \leq \bar{\Theta}^\gamma(H^N,[\hat{h}_c]) \]
(3.8)
where \( \Theta^\gamma(2^*,f) \) and \( \bar{\Theta}^\gamma(H^N,[\hat{h}_c]) \) are quantities defined in (3.5) and (2.17), respectively. Moreover, if the strict inequality in (3.8) holds, there exists a function \( U_{2^*} \in C_{2^*,f} \) attaining \( \bar{\Theta}^\gamma(2^*,f) \). It can be chosen to be positive on \( X \), and so it gives a weak solution to (1.11).

Remark 3.3. (1) Escobar and Schoen [18], and Escobar [17] gave the proof of the above result for \( \gamma = 1 \) and 1/2, which is rather sketchy. Aubin [4] also proved it provided that \( \gamma = 1 \) and \( f \) is positive everywhere on \( M \).
(2) Setting \( f = 1 \) recovers the solvability criterion for the fractional Yamabe problem, which appeared firstly in González and Qing [21, Theorem 1.4]. Our argument is a bit more complicated than that of [21], because we allow the situation that \( f \) attains a negative value. See the proof of Lemmas 3.6 and 3.7.

(3) Suppose that

\[
-\infty < \overline{\mathcal{X}}(X, \lceil \hat{h} \rceil) < \overline{\mathcal{X}}(\mathbb{H}^N, [\hat{h}_c])
\]

(3.9) has the validity, which is the case when one of the conditions (a)-(d) in Theorem A is true. Then there exists a small number \( \epsilon > 0 \) which depends only on the underlying manifold \((X^{n+1}, g^+)\), its boundary \((M^n, [\hat{h}])\) and \( \gamma \in (0, 1) \) such that if \( f \) is a positive function which satisfies \( \sup_M f \leq (1+\epsilon) \inf_M f \), then the strict inequality in (3.8) holds. To verify it, one may estimate \( \widehat{\Theta}^*(2^*, f) \) with the constant function \( (\int_M f dv_h)^{-1/(2^*+1)} \in C_{2^*, f} \). This argument was given by Aubin [4] for \( \gamma = 1 \).

The former part of Proposition 3.2 can be deduced immediately. Assume that there exists a number \( r_2 > 0 \) small enough so that for each fixed point \( y \in M \), the Fermi coordinate around \( y \) is well-defined in its \((2r_2)\)-geodesic neighborhood in \( X \). Let also \( \chi \) be a smooth radial cut-off function in \( \mathbb{R}^n_+ \) which satisfies

\[
0 \leq \chi \leq 1 \quad \text{on} \quad \mathbb{R}^n_+ \quad \text{and} \quad \chi = \begin{cases} 1 & \text{in} \quad B^+_n(0,1) \\ 0 & \text{outside} \quad B^+_n(0,2). \end{cases}
\]

(3.10)

Proof of Proposition 3.2. Derivation of (3.8). We will adapt the proof of [21, Theorem 1.4]. Choose \( y \in M \) such that \( f(y) = \max_{x \in M} f(x) > 0 \) and consider its Fermi coordinate (identifying \( y \) with the origin in \( \mathbb{R}^n \)). If \( \chi_{r_2} := \chi(r_2 \cdot) \in C^\infty_c(\mathbb{R}^n_+) \), then

\[
0 < \mu_0 := \int_{\mathbb{R}^n(0,2r_2)} f(x_{r_2}w_{1,0})^{2^*+1}\sqrt{|\hat{h}|}d\bar{x} = f(y)\int_{\mathbb{R}^n} w_{1,0}^{2^*+1}d\bar{x} + o(1)
\]

for small \( \epsilon > 0 \), where \( o(1) \to 0 \) as \( \epsilon \to 0 \). Hence the function \( U_\epsilon := \mu_0 (2^*+1) \chi_{r_2} W_{\epsilon,0} \in C_{2^*, f} \), and by (2.17),

\[
\left( \max_{x \in M} f(x) \right)^{\frac{n-2\gamma}{n}} \widehat{\Theta}^*(2^*, f) \leq f(y)^{\frac{n-2\gamma}{n}} \mu_0^{\frac{n-2\gamma}{n}} \kappa_7 \int_{B^n_+(0,2r_2)} \frac{x_N^{1-2\gamma}}{\sqrt{|\nabla(\chi_{r_2} W_{\epsilon,0})|}} + o(1) = \overline{\mathcal{X}}^*(\mathbb{H}^N, [\hat{h}_c]) + o(1).
\]

Taking \( \epsilon \to 0 \), we obtain (3.8) in light of Lemma 3.3. \( \square \)

On the other hand, we need several preliminary lemmas to prove the latter part of Proposition 3.2.

The next result follows from the standard variational argument with the compactness of the trace operator \( H^{1,2}(X; \rho^{1-2\gamma}) \hookrightarrow L^\beta(M) \) for \( \beta \in (2, 2^*+1) \) and the strong maximum principle given in Remark 5.2 so we omit the proof.

**Lemma 3.4.** Suppose that (1.5) holds, \( H = 0 \) for \( \gamma \in (1/2, 1) \), \( \Lambda^\gamma(M, [\hat{h}]) > 0 \) and \( f \) is a smooth function positive somewhere on \( M \). For all \( \beta \in (1, 2^*) \), there exists a positive minimizer \( \overline{U}_\beta \in C_{\beta, f}(M) \) of \( \overline{\Theta}^*(\beta, f) \), which solves

\[
\begin{cases}
-\text{div} g^\gamma (\rho^*)^{1-2\gamma} \nabla U = 0 & \text{in} \ (X, \tilde{g}^*), \\
\partial^\gamma U = \overline{\Theta}^*(\beta, f) \lvert U \rvert^{\beta-1} U - Q_h^\gamma U & \text{on} \ M
\end{cases}
\]

(3.11)

(refer to (3.3) and (3.21)).

The following lemma shows that \( \overline{\Theta}^*(\beta, f) \) is upper semi-continuous from the left at \( \beta = 2^* \). Note that finiteness of \( \overline{\Theta}^*(2^*, f) \) is assured by (3.8).
Lemma 3.5. It holds that
\[ \limsup_{\beta \to 2^{-}} \tilde{\Theta}^\gamma(\beta, f) \leq \tilde{\Theta}^\gamma(2^*, f). \] (3.12)

Proof. Observe that for each \( U \in \mathcal{D} \) (see (2.11)) such that \( \int_M f |U|^{\beta+1} dv_\beta > 0 \), we have
\[ \limsup_{\beta \to 2^{-}} \tilde{\Theta}^\gamma(\beta, f) \leq \limsup_{\beta \to 2^{-}} \frac{J^\gamma(U)}{\left( \int_M f |U|^{\beta+1} dv_\beta \right)^{\frac{\beta+1}{\beta+2}}} \leq \frac{J^\gamma(U)}{\left( \int_M f |U|^{2^*+1} dv_\beta \right)^{\frac{2^*+1}{2^*}}} . \]
Taking the infimum over \( U \in \mathcal{D} \), we deduce the desired inequality (3.12). \( \square \)

We also need a uniform regularity result on the family of functions \( \tilde{U}_\beta \).

Lemma 3.6. Suppose that the assumptions in Lemma 3.4 hold. For each \( \beta \in (1, 2^*) \), let \( \tilde{U}_\beta \in C_{\beta, f}(M) \) be a positive minimizer of \( \tilde{\Theta}^\gamma(\beta, f) \). If \( \Lambda^\gamma(M, [\tilde{h}]) > 0 \), then there exists a constant \( C > 0 \) and small \( \varepsilon > 0 \) such that
\[ \sup_{\beta \in [2^*-\varepsilon, 2^*)} ||\tilde{U}_\beta||_{H^1,2(X, \rho^{1-2\gamma})} \leq C \quad \text{and} \quad \sup_{\beta \in [2^*-\varepsilon, 2^*)} ||\tilde{U}_\beta||_{C^\alpha(K)} \leq C \] (3.13)
for some \( \alpha \in (0, 1) \) and \( K := \{ y \in M : f(y) \leq 0 \} \).

Proof. By (2.12) and (3.12), we see
\[ \Lambda^\gamma(M, [\tilde{h}]) \left( \int_M \tilde{U}_\beta^{2^*+1} dv_\beta \right)^{\frac{n-2\gamma}{n}} \leq J^\gamma(\tilde{U}_\beta) = \tilde{\Theta}^\gamma(\beta, f) \leq 2\tilde{\Theta}^\gamma(2^*, f) \] (3.14)
for \( \beta \in [2^*-\varepsilon, 2^*) \) for a sufficiently small number \( \varepsilon > 0 \). Thus
\[ \int_X (\rho^*)^{1-2\gamma} |\nabla \tilde{U}_\beta|_{\gamma}^2 dv_{\gamma} \leq 2\tilde{\Theta}^\gamma(2^*, f) + ||Q^\gamma_k||_{L^\infty(M)} \int_M \tilde{U}_\beta^2 dv_\beta, \]
and the right-hand side is uniformly bounded in \( \beta \in [2^*-\varepsilon, 2^*) \). On the other hand, [15, Lemma 3.1] implies that the norm
\[ \left( \int_X (\rho^*)^{1-2\gamma} |\nabla U|_{\gamma}^2 dv_{\gamma} + \int_M U^2 dv_\beta \right)^{1/2} \]
for \( U \in H^{1,2}(X; \rho^{1-2\gamma}) \)
is equivalent to the standard \( H^{1,2}(X; \rho^{1-2\gamma}) \)-norm. As a result, we establish the first inequality in (3.13).

To deduce the second inequality in (3.13), it suffices to verify that there exists an \( \eta \)-neighborhood \( B_\eta(K) \subset M \) of \( K \) such that
\[ \sup_{\beta \in [2^*-\varepsilon, 2^*)} ||\tilde{U}_\beta||_{L^\infty(B_\eta(K))} \leq C \] (3.15)
Then together the De Giorgi-Nash-Moser estimate stated in Lemma 5.1 (2), one can get the \( C^\alpha(K) \)-uniform estimate for \( \{\tilde{U}_\beta\}_{\beta \in [2^*-\varepsilon, 2^*)} \) for some \( \alpha \in (0, 1) \).

We will apply the blow-up argument close to the proof of [21 Theorem 1.4]. Suppose that there exist sequences \( \beta_m \to 2^*, \tilde{U}_m := \tilde{U}_{\beta_m} \) and \( y_m \in B_2(K) \) such that
\[ M_m := \tilde{U}_m(y_m) = ||\tilde{U}_m||_{L^\infty(B_\eta(K))} \to \infty, \quad \tilde{\Theta}^\gamma(\beta_m, f) \to \tilde{\Theta}_0 \leq \tilde{\Theta}^\gamma(2^*, f) \]
and \( y_m \to y_0 \in B_\eta(K) \) as \( m \to \infty \). Take a Fermi coordinate system around \( y_0 \) (identified with \( 0 \in \mathbb{R}_+^N \)) and define
\[ \tilde{V}_m(x) := M_m^{-1} \tilde{U}_m(\delta_m x + y_m, \delta_m x_N) \quad \text{for} \quad x = (\bar{x}, x_N) \in B_2^N(0, 2r_2/\delta_m) \]
where \( \delta_m := M_m^{-1/2(2^*)} \) and \( r_2 \) is a small number. Let also \( \chi_{r_2/\delta_m} = \chi(\delta_m \cdot r_2) \) where \( \chi \in C^\infty_c(\mathbb{R}_+^N) \) is an arbitrary function such that (3.10) holds. Then \( V_m := \chi_{r_2/\delta_m} \tilde{V}_m \) solves
\[ \begin{aligned}
-\text{div}_{\gamma_m}((\rho_m)^{-1-2\gamma} \nabla V_m) &= 0 & & \text{in} & & B_2^N(0, r_2/\delta_m) \\
\partial_{n_\gamma}^\gamma V_m &= \tilde{\Theta}^\gamma(\beta_m, f) f_m V_m^{2^*-1} - (Q_k^\gamma)_{m,\delta_m} V_m & & \text{on} & & B^n(0, r_2/\delta_m) 
\end{aligned} \]
where
\[
g^m_m(x, x_N) := \tilde{g}^*(\delta_m \bar{x} + y_m, \delta_m x_N), \quad \rho_m(x, x_N) := \rho^*(\delta_m \bar{x} + y_m, \delta_m x_N),
\]
and \(f_m, (Q^*_h)_m\) are similarly defined. Also,
\[
\int_{B^N_{2}(0, 2r_2/\delta_m)} x_n^{1-2\gamma}|\nabla V_m|^2 dx \leq C \int_{B^N_{2}(0, 2r_2/\delta_m)} x_n^{1-2\gamma} \left( |\nabla \tilde{V}_m|^2 + |\nabla \chi_{r_2/\delta_m}|^2 \tilde{V}_m^2 \right) dx
\]
for some constant \(C_0 > 0\) where the exponent of \(M_m\) in the leftmost side is always negative since \(\beta_m < 2^*\). Therefore \(V_m \to V_0\) strongly in \(C^\alpha(\mathbb{R}^N_+)\) and weakly \(H^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})\) for some nonzero bounded function \(V_0\) and \(\alpha' \in (0, 1)\). It is easy to check that \(V_0\) is a solution of
\[
\begin{cases}
-\text{div}(x_n^{1-2\gamma} \nabla V_0) = 0 & \text{in } \mathbb{R}^N_+, \\
\partial_n V_0 = \tilde{\Theta}_0 f(y_0) V_0 & \text{on } \mathbb{R}^n.
\end{cases}
\]
If \(\tilde{\Theta}_0 = 0\) or \(f(y_0) \leq 0\), a contradiction immediately arises since it should hold that \(V_0 = 0\) in \(\mathbb{R}^N_+\). Suppose that \(\tilde{\Theta}_0 > 0\) and \(f(y_0) > 0\). For any \(\delta > 0\), one can select small \(\eta > 0\) so that \(f(y) \leq \delta\) for any \(y \in B_{\eta}(K)\). By the classification theorem \([27\text{ Theorem 1.8}]\) of Eq. \((3.17)\), we know that \(V_1 := (\tilde{\Theta}_0 f(y_0))^{(n-2\gamma)/(4\gamma)} V_0\) is the bubble \(W_{\lambda, 0}\) for some \(\lambda > 0\). Consequently,
\[
C_0 \geq \|V_0\|_{H^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})} = \left(\tilde{\Theta}_0 f(y_0)\right)^{-\frac{n-2\gamma}{n}} \|W_{\lambda, 0}\|_{H^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})} \geq (\tilde{\Theta}^\gamma(2^*, f) \delta)^{-\frac{n-2\gamma}{n}} \|W_{1, 0}\|_{H^{1,2}(\mathbb{R}^N_+; x_N^{1-2\gamma})},
\]
which is a contradiction to \((3.16)\) provided \(\delta > 0\) small enough. Hence \((3.15)\) is true, thereby completing the proof. \(\square\)

If \(\Lambda^\gamma(M, [\tilde{h}]) > 0\), we are able to improve Lemma \(3.3\) by showing the \(\tilde{\Theta}^\gamma(\beta, f)\) is continuous from the left at \(\beta = 2^*\). Unlike the fractional Yamabe problem, it is less clear here due to the negative part of \(f\).

**Lemma 3.7.** Under the assumption in Lemma \(3.4\), we have
\[
\lim_{\beta \to 2^*} \tilde{\Theta}^\gamma(\beta, f) = \tilde{\Theta}^\gamma(2^*, f).
\]

**Proof.** Let \(\tilde{U}_\beta \in C_\beta, f\) be the positive minimizer of \(\tilde{\Theta}^\gamma(\beta, f)\). Then we infer from Hölder’s inequality and Lemma \(3.6\) that for any given \(\delta > 0\), there exists \(\varepsilon > 0\) such that
\[
\int_M f \tilde{U}_\beta^{2^*+1} dv_h \geq \int_M f \tilde{U}_\beta^{\beta+1} dv_h - \delta = 1 - \delta
\]
for all \(\beta \in [2^* - \varepsilon, 2^*)\). Therefore we have
\[
\tilde{\Theta}^\gamma(2^*, f) \leq \frac{J^\gamma(U)}{(\int_M f \tilde{U}_\beta^{2^*+1} dv_h)^{\frac{n-2\gamma}{n}}} \leq (1 - \delta)^{-\frac{n-2\gamma}{n}} \tilde{\Theta}^\gamma(\beta, f)
\]
for \(\beta \in [2^* - \varepsilon, 2^*)\). Taking \(\delta \to 0\), we prove the assertion. \(\square\)

We are now ready to conclude the proof of Proposition \(3.2\).

**Completion of the proof of Proposition 3.2.** By \((3.13)\), there exists a nonnegative function \(\tilde{U}_2^* \in H^{1,2}(X; \rho^{1-2\gamma})\) such that \(\tilde{U}_\beta \to \tilde{U}_2^*\) weakly in \(H^{1,2}(X; \rho^{1-2\gamma})\). Thanks to Lemma \(3.1\) \((2)\), it is not hard to see that \(\tilde{U}_2^*\) is Hölder continuous on \(\overline{X}\) and solves \((3.11)\) with \(\beta = 2^*\). Besides, by the strong maximum principle in Remark \(5.2\) we have that \(\tilde{U}_2^* > 0\) on \(\overline{X}\) unless it is trivial.
We claim that $\tilde{U}_{2*}$ is nonzero provided that the strict inequality in (3.18) holds. As in [30, Proposition 2.5], one can prove that for any $\epsilon > 0$, there exists $A(\epsilon) > 0$ such that

$$\left( \int_M |U|^{2*+\epsilon}dv_{\hat{h}} \right)^{\frac{2}{2*+\epsilon}} \leq (1 + \epsilon)S_{n,\gamma} \int_X (\rho^*)^{1-2\gamma}|\nabla U|_g^2dv_y + A(\epsilon) \int_M U^2dv_{\hat{h}}$$

for any $U \in H^{1,2}(X; \rho^{1-2\gamma})$. Since $J^\gamma(\tilde{U}_\beta) = \tilde{\Theta}^\gamma(\beta, f)$, it follows that

$$\left( \int_M \tilde{U}_{\beta}^{2*+\epsilon}dv_{\hat{h}} \right)^{\frac{2}{2*+\epsilon}} \leq (1 + \epsilon)S_{n,\gamma}\kappa^{-1}_\gamma \tilde{\Theta}^\gamma(\beta, f) + A'(\epsilon) \int_M \tilde{U}_\beta^2dv_{\hat{h}}$$

(3.18)

where $A'(\epsilon) := (S_{n,\gamma} + \epsilon)\kappa^{-1}_\gamma \|Q^\gamma_h\|_{L^\infty(M)} + A(\epsilon)$. Meanwhile, we get from $\tilde{U}_\beta \in C_{\beta,f}$ and Hölder’s inequality that

$$1 \leq \left( \max_{x \in M} f(x) \right) |M|^{\frac{2* - \beta}{2*+\epsilon}} \left( \int_M \tilde{U}_{\beta}^{2*+\epsilon}dv_{\hat{h}} \right)^{\frac{\beta+1}{2*+\epsilon}}.$$  

Thus, putting (3.18), (3.19), (2.17) and Lemma 3.7 together, we obtain

$$1 - (1 + 2\epsilon) \left( \max_{x \in M} f(x) \right)^{\frac{n-2\gamma}{n}} \tilde{\Theta}^\gamma(2*, f) \left( \mathcal{N}'(\mathbb{H}^N, [\hat{h}_c]) \right)^{-1} \leq C \int_M \tilde{U}_\beta^2dv_{\hat{h}}$$

for $\beta$ close to $2^*$. Now the left-hand side is positive if $\epsilon > 0$ is chosen small enough. Hence the $L^2(M)$-norm of $\tilde{U}_{2*}$, or $\tilde{U}_2$, itself, is nonzero.

Thanks to the assumption $\Lambda^\gamma(M, [\hat{h}]) > 0$ and (3.14), we obtain

$$\tilde{\Theta}^\gamma(2*, f) \int_M f\tilde{U}_{2*}^{2*+\epsilon}dv_{\hat{h}} = J^\gamma(\tilde{U}_{2*}) \geq 0$$

(3.20)

by testing $\tilde{U}_{2*}$ in (3.11) with $\beta = 2^*$. Hence Lemma 3.4 yields that $\tilde{\Theta}^\gamma(2*, f) > 0$. Notice that we cannot have that $\tilde{\Theta}^\gamma(2*, f) = 0$, since we would get $U_{2*} = 0$ on $M$ if it is so. By reasoning in the same way, we obtain $\mu_1 := \int_M f\tilde{U}_{2*}^{2*+\epsilon}dv_{\hat{h}} > 0$ as well. Using the lower semi-continuity of $J^\gamma$ and Lemma 3.7 we see

$$\tilde{\Theta}^\gamma(2*, f) \mu_1 = J^\gamma(\tilde{U}_{2*}) \leq \liminf_{\beta \to 2^*-} J^\gamma(\tilde{U}_\beta) = \lim_{\beta \to 2^*-} \tilde{\Theta}^\gamma(\beta, f) = \tilde{\Theta}^\gamma(2^*, f),$$

so $\mu_1 \in (0, 1]$. Now if we set $V = \mu_1^{-1/(2^*+\epsilon)}\tilde{U}_{2*} \in C_{2*,f}$, we deduce from (3.20) that

$$\tilde{\Theta}^\gamma(2*, f) \leq J^\gamma(V) = \frac{1-n-2\gamma}{n-2\gamma} J^\gamma(\tilde{U}_{2*}) = \frac{2}{\mu_1^2} \tilde{\Theta}^\gamma(2*, f).$$

Thus $\mu_1 = 1$ and $\tilde{U}_{2*} \in C_{2*,f}$ is a minimizer of $\tilde{\Theta}^\gamma(2*, f)$. Let $U_{2*} = (\rho^*/\rho)^{(n-2\gamma)/2}\tilde{U}_{2*} > 0$ on $\overline{X}$. Then it is an element in $C_{2*,f}$ which attains $\tilde{\Theta}^\gamma(2*, f)$ and solves (3.7). In view of the discussion at the end of Subsection 3.1 the proof is completed.

4. Existence results

4.1. Proof of Theorems 1.2 and 1.4. In light of Proposition 3.2, we just need to verify that the strict inequality in (3.8) holds in each situation. As in the previous section, we assume that for each fixed $y \in M$, the Fermi coordinate around $y$ is well-defined in its $(2r_2)$-geodesic neighborhood on $\overline{X}$.

Condition (A). Let $\chi \in C^\infty_c(\mathbb{R}^N_+)$ be a cut-off function satisfying (3.10) and $\chi_{r_2} = \chi(\cdot/r_2)$. In [33, Proposition 2.5], it is proved that for any small $\epsilon > 0$

$$I^\gamma(\chi_{r_2}W_{\epsilon,0}) \leq \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} |\nabla W_{1,0}|^2dx$$

$$+ \epsilon \kappa_\gamma H(y) \left[ \frac{2n^2 - 2n + 1 - 4\gamma^2}{2(1-2\gamma)} \right] \int_{\mathbb{R}^N_+} x_N^{2-2\gamma} |\nabla W_{1,0}|^2dx + o(\epsilon)$$
where $I^{\gamma}$ is the functional given in (2.8) and $W_{c,0}$ is the bubble defined in terms of Eq. (2.11). On the other hand, if $y \in M$ is the maximum point of $f$, then
\[
\frac{f(y)}{\int_{M} f(x) (x^2 + 1) dw_{h}} = \left( \int_{\mathbb{R}^n} \frac{w^{2+1}_{1,0} d\tilde{\varepsilon}}{w^{2+1}_{1,0} d\tilde{\varepsilon}} \right)^{-1} + o(\varepsilon^2).
\]

Combining these two estimates and utilizing Lemma 3.1, we conclude that
\[
\left( \max_{x \in M} f(x) \right)^{\frac{n-2}{n}} \Theta^{\gamma}(2^*, f) = \left( \frac{f(y)}{\int_{M} f(x) (x^2 + 1) dw_{h}} \right)^{\frac{n-2}{n}} I^{\gamma}((x_2)_{W_{c,0}}) - \Theta^{\gamma}(2^*, f) + o(\varepsilon^2)
\]
\[
< \tilde{\lambda}((\mathbb{H}^N, [\hat{h}]), [\mathbb{H}^N, [\hat{h}]])
\]

From Proposition 3.2, we obtain a weak solution to (1.11).

CONDITION (B). If we pick $\hat{h}_0 \in [\hat{h}]$ satisfying (2.18) and define
\[
\Psi_1(x) := C_1 \Pi_{ij}(y) x_i x_j x_{N+1} \partial_i W_{c,0}(x) \quad \text{for} \quad x \in \mathbb{R}^n
\]
where $1 \leq i, j \leq n$, $r := |x|$ and $C_1 \in \mathbb{R}$, then the computation in the proof of [33, Proposition 2.8] shows
\[
I^{\gamma}((x_2)_{W_{c,0}} + \Psi_1)) \leq C_2 \int_{\mathbb{R}^n} x^2 |\nabla W_{1,0}|^2 dx
\]
\[
- \varepsilon^2 \kappa \Pi(y)^2 \gamma M_1(n, \gamma) \int_{\mathbb{R}^n} x^2 |\nabla W_{1,0}|^2 dx + o(\varepsilon^2)
\]
for the optimal number $C_1 \in \mathbb{R}$. Here
\[
M_1(n, \gamma) := \left( 1 + \gamma \right) \left[ \frac{3n^2 + n(16\gamma^2 - 22) + 20(1 - \gamma^2)}{n(n-1)(1-\gamma^2)} \right] + \frac{16(n-1)(1-\gamma^2)}{n(3n^2 + n(2 - 8\gamma^2) + 4\gamma^2 - 4)} > 0 \quad \text{for} \quad n \geq 4, \gamma \in (0, 1).
\]

Since $y \in M$ is the maximum point of the function $f$, we have
\[
\int_{M} f(x) (x^2 + 1) dw_{\hat{h}_0} = f(y) \int_{\mathbb{R}^n} w^{2+1}_{1,0} d\tilde{\varepsilon}
\]
\[
+ \varepsilon^2 \left( \frac{\Delta f(y)}{2n} \right) \int_{\mathbb{R}^n} |\tilde{\varepsilon}| w^{2+1}_{1,0} d\tilde{\varepsilon} + o(\varepsilon^3)
\]
where the number 0 in the integrand of the left-hand side stands for the value of $\Psi_1$ on $M$. As remarked in the proof of [17, Theorem 3.3], we still have (1.8) with the same value of $c_{n,\gamma}^1$ even if $\hat{h}$ is replaced with $\hat{h}_0$. Therefore, choosing
\[
c_{n,\gamma}^1 = \left[ \frac{2n(n-2)}{n-2} \right] \left[ \frac{\int_{\mathbb{R}^n} x^2 |\nabla W_{1,0}|^2 dx}{\int_{\mathbb{R}^n} x^2 |\nabla W_{1,0}|^2 dx} \right] M_1(n, \gamma)
\]
\[
= \left[ \frac{16n(n-1)(1-\gamma^2)}{(n-2)(n-2+2\gamma)(n-2-2\gamma)} \right] M_1(n, \gamma),
\]
where the second equality can be computed as in [21, 33], we obtain
\[
\left( \max_{x \in M} f(x) \right)^{\frac{n-2}{n}} \Theta^{\gamma}(2^*, f) = \left( \frac{\hat{\lambda}}{\tilde{\lambda}} \right)(\mathbb{H}^N, [\hat{h}]) + \varepsilon^2 c_{n,\gamma}^{11} \left( -\frac{\Delta f(y)}{f(y)} - \frac{\pi_{n,\gamma}}{\Pi(y)} \right) + o(\varepsilon^2)
\]
\[
< \tilde{\lambda}((\mathbb{H}^N, [\hat{h}]), [\mathbb{H}^N, [\hat{h}])
\]
for some $c_{n,\gamma}^{11} > 0$. Proposition 4.2 implies the existence of a solution to (1.11).
CONDITION (B'). Putting (A.2) and the estimate
\[ \int_M f(\chi_{r^2(w_{r,0} + C_1^2 H_{ij} x_j x_j \partial_{ij} w_{r,0}))}^3 \, dv_{h_0} = f(y) \int_{\mathbb{R}^n} w_1^{2} \, d\bar{x} + O(\epsilon^2) \]
together, we find
\[ \left( \max_{x \in M} f(x) \right)^{\frac{2}{3}} \Theta^2(2, f) \leq \Xi^2(\mathbb{H}^4, [\hat{h}_c]) - C \epsilon^2 |\log \epsilon| + O(\epsilon^2) < \Xi^2(\mathbb{H}^4, [\hat{h}_c]) \]
for some $C > 0$. Therefore there exists a solution to (1.11).

CONDITIONS (C) and (C'). As before, we pick $\hat{h}_0 \in [\hat{h}]$ satisfying (2.18). Then, under condition (c), we have
\[ I^\gamma(\chi_{r^2 \psi_{\epsilon,0}}) \leq \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} |\nabla W_{1,0}|^2 \, dx \]
\[ + \epsilon^3 \kappa_\gamma R_{NN,N}(y) \left[ \frac{4n^2 - 12n + 9 - 4\gamma^2}{24n(3-2\gamma)} \right] \int_{\mathbb{R}^N_+} x_N^{4-2\gamma} |\nabla W_1|^2 \, dx + o(\epsilon^3) \]
as computed in [33, Proposition 3.4]. Since $-\Delta_{\hat{h}} f(y) = 0$ and $w_{\epsilon,0}$ is radially symmetric, (4.3) gives
\[ \int_M f(\chi_{r^2 w_{\epsilon,0}})^{2\gamma+1} \, dv_{h_0} = f(y) \int_{\mathbb{R}^n} w_1^{2\gamma+1} \, d\bar{x} + O(\epsilon^4). \]
Hence (4.1) holds provided that $R_{NN,N}(y) < 0$ and (1.11) has a solution. A similar calculation can be conducted when (c') holds.

CONDITION (D). Select $\hat{h}_0 \in [\hat{h}]$ satisfying all the conditions imposed in Lemma 2.7 and write $\hat{h}_0 = uh$ for some positive function $u$ on $M$. Set also a function
\[ \Psi_{2\epsilon} = C_2 R_{ijN} [\hat{g}](y) x_i x_j x_N r^{-1} \partial_{ij} W_{\epsilon,0} \text{ in } \mathbb{R}^N_+ \] (4.5)
where $1 \leq i, j \leq n$, $r = |\bar{x}|$ and $C_2 \in \mathbb{R}$. From the proof of [33, Proposition 3.7], we see
\[ I^\gamma(\chi_{r^2(W_{\epsilon,0} + \Psi_{2\epsilon})}) \leq \kappa_\gamma \int_{\mathbb{R}^N_+} x_N^{1-2\gamma} |\nabla W_{1,0}|^2 \, dx \]
\[ - \epsilon^4 \kappa_\gamma \left[ ||W_0||^2 \mathcal{M}_{21}(n, \gamma) + (R_{ij} [\hat{g}])^2 \mathcal{M}_{22}(n, \gamma) \right] \int_{\mathbb{R}^N_+} x_N^{5-2\gamma} |\nabla W_{1,0}|^2 \, dx + o(\epsilon^4) \]
for an optimally chosen number $C_2 \in \mathbb{R}$. Here, the metric $\hat{g}$ is replaced with $\hat{g}_0$, $W_0 = W[\hat{h}_0]$, \[ \mathcal{M}_{21}(n, \gamma) := \frac{15n^4 - 120n^3 + 2n^2(17 - 2\gamma^2) - 80n(5 - 2\gamma^2) + 48(4 - 5\gamma^2 + \gamma^4)}{7680n(n-1)(n-3)(1+\gamma)(1-\gamma)(2-\gamma)} \]
and $\mathcal{M}_{22}(n, \gamma)$ is a number depending only on $n$ and $\gamma$. The constants $\mathcal{M}_{21}(n, \gamma)$ and $\mathcal{M}_{22}(n, \gamma)$ are positive for any $n > 4 + 2\gamma$ and $\gamma \in (0, 1)$. Because $y \in M$ is the maximum point of the function $f$ at which $\Delta_{\hat{h}} f = 0$, we have
\[ \Delta_{\hat{h}_0} f(y) = u^{-1} \Delta_{\hat{h}} f(y) = 0 \quad \text{and} \quad \partial_{ij} f(y) = 0 \quad \text{for each } 1 \leq i, j \leq n, \]
where the latter assertion can be checked by mathematical induction on $n$. Besides, one can obtain using Lemma 2.7 (1) that
\[ \int_M f(\chi_{r^2 w_{\epsilon,0}})^{2\gamma+1} \, dv_{h_0} = f(y) \int_{\mathbb{R}^n} w_1^{2\gamma+1} \, d\bar{x} \]
\[ + \epsilon^4 \left[ \frac{(\Delta)^2 f(y)}{8n(n+2)} \right] \int_{\mathbb{R}^n} |\bar{x}|^4 w_1^{2\gamma+1} \, d\bar{x} + O(\epsilon^5). \] (4.6)
Meanwhile, the function $u$ can be assumed to satisfy $u(y) = 1$ and $u_i(y) = 0$ (see Lee-Parker 34 Section 5)). It follows that
\[ (-\Delta_{\hat{h}_0})^2 f(y) = u(y)^{-2} \left[ (-\Delta_{\hat{h}})^2 f + (n-2) \partial_{ij} u \partial_{ij} f \right](y) = (-\Delta_{\hat{h}})^2 f(y). \] (4.7)
Moreover, it is a well-known fact that the (1, 3)-Weyl tensor is invariant under the conformal transformation. Hence
\[ ||W_0||^2 = \hat{h}_0^{ij} \hat{h}_0^{kl} \hat{h}_0^{pq} \hat{h}_0^{rs} (W_0)_{klpq} (W_0)_{ijrs} = \hat{h}_0^{ij} \hat{h}_0^{kl} (W_0)_p^q \hat{h}_0^{ij} \hat{h}_0^{kl} (W_0)_q^p \]
\[ = u^{-2} \hat{h}_0^{ij} \hat{h}_0^{kl} \hat{W}^{ij}_{klpq} \hat{W}^{pq}_{ijkl} = ||W||^2 \]
(4.8)
at y (here, the indices \( i, j, k, l, p, q, r \) and \( s \) range from 1 to \( n \)). By (4.7) and (4.8), assumption (1.9) should be still valid with the same value of \( c_{n, \gamma}^2 > 0 \), even after we substitute \( \hat{h}_0 \) for \( \hat{h} \). As a consequence, our selection
\[ c_{n, \gamma}^2 = \frac{8n(n - 2)(n - 4)}{n - 2} \left[ \frac{\int_{\mathbb{R}^N} x^N_2 |\nabla W_{1,0}|^2 dx}{\int_{\mathbb{R}^N} x^N_2 |\nabla W_{1,0}|^2 dx} \right] a_{21}(n, \gamma) \]
\[ = \left[ \frac{1024(n - 3)(n - 1)(2 - \gamma)(1 - \gamma)\gamma}{3(n - 2)\gamma(n - 2 - \gamma)(1 - 2\gamma)(n - 4 + \gamma)(n - 4 + 2\gamma)} \right] a_{21}(n, \gamma), \]
(4.9)
where the second equality comes from the arguments in [21, 33], allows us to deduce
\[ \left( \max_{x \in M} f(x) \right)^{\frac{n - 2\gamma}{n}} \Theta^\gamma(2^*, f) \]
\[ \leq \nabla \left( \frac{(-\Delta \hat{h})^2 f(y)}{f(y)} + c_{n, \gamma}^2 ||W||^2(y) + c_{n, \gamma}^{22} (R_{ij} \hat{h}_{ij})^2 \right) + o(\epsilon^4) \]
for some \( c_{n, \gamma}^{21}, c_{n, \gamma}^{22} > 0 \).

**Condition (D').** The desired inequality (4.1) follows from [A, 3] and (L.6).

4.2. **Proof of Theorem 1.6.** We shall give a brief sketch of the proof of Theorem 1.6 under (E). Let \( \Phi_{\epsilon, r_2} \) be a Schoen-type test function constructed in (4.9) (with \( \theta_0 = r_2 \)) of Kim et al. [33], which is equal to the bubble \( W_{\epsilon, 0} \) in \( X \cap B^N(0, r_2) \) and a constant multiple of Green’s function \( G(x, 0) \) in \( X \setminus B^N(0, 2r_2) \). It is nonnegative in \( X \) and satisfies
\[ \Gamma^\gamma(\Phi_{\epsilon, r_2}) \leq \kappa \gamma \int_{\mathbb{R}^n} x^N_2 |\nabla W_{1,0}|^2 dx - c_{n, \gamma}^3 A e^{-2\gamma} + o(e^{-2\gamma}) \]
for some \( c_{n, \gamma}^3 > 0 \) (see the proof of [33 Proposition 4.5]). The vanishing condition on \( f \) implies
\[ \int_{M} f \phi_{\epsilon, r_2}^{2+} d\nu_{\epsilon, r_2} \geq \int_{M \cap B^N(0, r_2)} f w_{\epsilon, 0}^{2+} d\nu_{\epsilon, r_2} \]
\[ = f(y) \int_{\mathbb{R}^n} w_{\epsilon, 0}^{2+} d\nu_{\epsilon, r_2} + o(e^{-2\gamma}). \]

Hence a strict inequality in (4.8) holds. A similar argument works for the case (E').

5. **Regularity of solutions to the prescribed fractional scalar curvature problems**

5.1. **Local regularity results.** Suppose that \( n > 2\gamma \) and \( \gamma \in (0, 1) \setminus \{1/2\} \). In this subsection, we present several regularity results for degenerate elliptic equations having the form
\[ \begin{cases} -\text{div}(x^N_2 F) + x^N_2 BU = x^N_2 G + \text{div}(x^N_2 F) & \text{in } B_r, \\ \partial_{\nu} U = aU + b & \text{on } \partial B_r \end{cases} \]
(5.1)
where \( B_r := B^N_+(0, r) \) and \( \partial B_r := B^N(0, r) \) for a fixed radius \( r > 0 \). We introduce conditions:
\[(R1) \quad A = (A_{iN})_{i=1}^N \in C^2(\mathbb{R}^N, \mathbb{R}^{N \times N}) \text{ satisfies that } A_{iN} = 0 \text{ in } B_r \text{ for } i = 1, \cdots, n, \text{ and the uniform ellipticity condition } \Lambda_1 |\xi|^2 \leq A(x)|\xi| \leq \Lambda_2 |\xi|^2 \text{ holds for all points } x \in \mathbb{R}^N, \]
\[ x \in B_r \text{ and some constants } 0 < \Lambda_1 \leq \Lambda_2. \]
Suppose that functions $A, B, F, G$ and $b$ satisfy (R1), (R2) and (R3), and $U ∈ H^{1,2}(B_r; x_N^{1−2γ})$ is a weak solution of (5.1).

(1) There is a small number $δ_0 > 0$ relying only on $n, γ$ and $r$ such that if $∥a∥_{L^n/(2γ)(∂B_r)} < δ_0$, then

$$∥U∥_{H^{1,2}(B_r/2; x_N^{1−2γ})} + ∥U∥_{L^n/(2γ)(∂B_{r/2})} \leq C \left( ∥U∥_{L^2(B_r; x_N^{1−2γ})} + ∥F∥_{L^q(B_r; x_N^{1−2γ})} + ∥G∥_{L^q_2(B_r; x_N^{1−2γ})} + ∥b∥_{L^p(∂B_r)} \right)$$

where $C > 0$ depends only on $n, γ, r, A_1, A_2$ and $∥B∥_{L^{q_2}(B_r; x_N^{1−2γ})}$.

(2) Assume that $a ∈ L^1(∂B_r)$ for $p_1 > n/(2γ)$. Then $U ∈ C^{α}(∂B_{r/2})$ for some $α ∈ (0, 1)$ and

$$∥U∥_{C^{α}(∂B_{r/2})} \leq C \left( ∥U∥_{L^2(B_r; x_N^{1−2γ})} + ∥F∥_{L^q(B_r; x_N^{1−2γ})} + ∥G∥_{L^q_2(B_r; x_N^{1−2γ})} + ∥b∥_{L^p(∂B_r)} \right)$$

where $C > 0$ depends only on $n, γ, r, A_1, A_2$, $∥B∥_{L^{q_2}(B_r; x_N^{1−2γ})}$ and $∥a∥_{L^p(∂B_r)}$.

Proof. One can justify (1) by following the argument in [27] Proposition 2.3, Lemma 2.8 but utilizing the optimal Sobolev inequality [13] Lemma 3.1.

The assertion (2) follows from the standard argument involving the Moser iteration technique. Refer to [27] Proposition 2.6 and [19] Proposition 3 which cover the case that $A$ is the identity and $F = 0$ in $B_r$.

Remark 5.2. On the course of the proof of Lemma 5.1 (2), one gets the weak Harnack inequality. Suppose that the hypotheses of Lemma 5.1 (2) hold and $U ≥ 0$ in $B_r$. Then there is a number $q_0 > 0$ such that

$$\inf_{B_{r/4}} U + ∥F∥_{L^{q_1}(B_r; x_N^{1−2γ})} + ∥G∥_{L^{q_2}(B_r; x_N^{1−2γ})} + ∥b∥_{L^p(∂B_r)} ≥ C∥U∥_{L^{q_0}(B_r/2; x_N^{1−2γ})}$$

where $C > 0$ depends only on $n, γ, r, A_1, A_2$, $∥B∥_{L^{q_2}(B_r; x_N^{1−2γ})}$ and $∥a∥_{L^p(∂B_r)}$. For its derivation, see [27] Proposition 2.6 and [19] Proposition 2 which treat the situation that $A$ is the identity and $F = 0$ in $B_r$.

From this result, one obtains the strong maximum principle: If $F, G = 0$ in $B_r$, $b = 0$ on $∂B_r$ and $U ≥ 0$ attains $0$ at some point in $B_r$, then $U = 0$ in $B_r$.

Secondly, we establish regularity of the derivatives of solutions to (5.1) in the $x$-variables. Let us introduce a terminology: For any function $V ∈ R_N$ and $h ∈ R^n$ with $∥h∥$ small, define

$$D_hV(\bar{x}, x_N) = V(\bar{x} + h, x_N) − V(\bar{x}, x_N)/∥h∥$$

for $(\bar{x}, x_N) ∈ R_N$.

We call $D_hV$ a difference quotient of $V$.

Lemma 5.3. Suppose that $A$ satisfies (R1) and $U ∈ H^{1,2}(B_r; x_N^{1−2γ})$ is a weak solution of (5.1). Additionally, assume that

(R11) there exists $A_{NN} ∈ C^2(\mathbb{R}_r)$ such that $A_{NN}(x) = 1 + A'_{NN}(x)x_N$ on $\mathbb{R}_r$, which implies that $|D_hA_{NN}| + |D_hA_{NN}(x)| + ⋯ + |D_hA_{NN}(x)| ≤ Cx_N$ on $\mathbb{R}_r$;

(R21) $F_a(\partial_jF_a)^n_{i=1} ∈ C_{\alpha'}(\mathbb{R}_r)$ for $α' ∈ (0, 1)$ and $B, \nabla_{\bar{x}}B, G, \nabla_{\bar{x}}G ∈ L^{\infty}(B_r)$;

(R22) $\nabla_{\bar{x}}F_a = (\partial_jF_a)^n_{j=1} ∈ L^{q_1}(B_r; x_N^{1−2γ})$ and $\nabla_{\bar{x}}^2B, \nabla_{\bar{x}}^2G ∈ L^{q_2}(B_r; x_N^{1−2γ})$ for $q_1 > n − 2γ + 2$ and $q_2 > (n − 2γ + 2)/2$;
(R31) $a, \nabla_a a, \nabla_a^2 a \in L^{p_1}(\partial'B_r)$ and $b, \nabla_b b, \nabla_b^2 b \in L^{p_2}(\partial'B_r)$ for $p_1, p_2 > p/(2\gamma)$. Then there exists $0 < r' < r$ such that $\nabla_{\bar{x}}U, \nabla_{\bar{x}}^2 U \in C^\alpha(\overline{B_{r/2}})$ for some $\alpha \in (0, 1)$.

Proof. We shall show that $\nabla_{\bar{x}}U$ is Hölder continuous on $\overline{B_{r/2}}$. For each $h \in \mathbb{R}^n$ with $|h|$ small, the corresponding difference quotient $D_hU$ of $U$ solves

$$
\begin{cases}
-\text{div}(x_N^{1-2\gamma}A\nabla (D_hU)) + x_N^{1-2\gamma}B(D_hU) = x_N^{1-2\gamma}\tilde{G} + \text{div}(x_N^{1-2\gamma}\tilde{F}) & \text{in } B_{r/2}, \\
\partial_{\bar{x}}(D_hU) = a(D_hU) + [(D_ha)U_h + D_hb] & \text{on } \partial'B_{r/2}
\end{cases}
$$

(5.2)

where $U_h(x, x_N) := U(x + h, x_N)$,

$$\tilde{F} := (D_hF_1 + (D_hA_{ij})(\partial_jU_h), D_hF_N + (D_hA_{iN})(\partial_NU_h)),
\tilde{G} := D_hG - (D_hB)U_h.$$

By (R11) and Lemma 5.1 (2), we have

$$
\|D_hU\|_{C^{\alpha}(\overline{B_{r/12}})} \leq C \left(\|\nabla U\|_{L^2(B_{r/4}; x_N^{1-2\gamma})} + \|\nabla_{\bar{x}}U\|_{L^2(B_{r/4}; x_N^{1-2\gamma})} + \|x_N\partial_NU\|_{L^2(B_{r/4}; x_N^{1-2\gamma})}
+ \|\nabla_{\bar{x}}F\|_{L^2(B_{r/4}; x_N^{1-2\gamma})} + \|\nabla_{\bar{x}}G\|_{L^2(B_{r/4}; x_N^{1-2\gamma})} + \|\nabla_{\bar{x}}B\|_{L^2(B_{r/4}; x_N^{1-2\gamma})}
+ \|\nabla_{\bar{x}}a\|_{L^p(\partial'B_r)} + \|\nabla_{\bar{x}}b\|_{L^p(\partial'B_r)}\right)
$$

where $C > 0$ depends only on $n, \gamma, r, A, B, \|a\|_{L^p(\partial'B_r)}$ and $\|U\|_{L^\infty(B_r)}$. In view of (R21) and (R31), it is sufficient to check that $x_N\partial_NU \in L^\infty(B_{r/4})$ and $\nabla_{\bar{x}}U \in L^q(B_{r/6}; x_N^{1-2\gamma})$ for any $q > 1$.

We apply the rescaling argument to prove the first claim. For a fixed point $x_0 = (\bar{x}_0, x_{N0}) \in B_{r/4}$ and any element $x \in B^N((0,1), 1/2)$, we set $x' = (x', x_N') = (\bar{x}_0 + x_{N0}x, x_{N0}x_N) \in B^N(x_0, x_{N0}/4)$, $\tilde{U}(x) = U(x')$, $\tilde{A}(x) = A(x')$, etc. Then $\tilde{U}$ is a solution to the equation

$$
-\text{div}\left(x_N^{1-2\gamma}\tilde{A}\nabla \tilde{U}\right) + x_{N0}^2x_N^{1-2\gamma}\tilde{B}\tilde{U} = x_{N0}^2x_N^{1-2\gamma}\tilde{G} + x_{N0}\text{div}\left(x_N^{1-2\gamma}\tilde{F}\right)
$$

in $B^N((0,1), 1/2)$. It is uniformly elliptic, so an application of [20 Theorem 8.32] and (R21) give

$$
\|x_N\partial_NU(x_0)\| \leq \|\partial_N\tilde{U}\|_{C^{\alpha'}(\overline{B^N(0,1/4)})}
\leq C \left(\|\tilde{U}\|_{L^\infty(B_{r/2})} + \|\tilde{F}\|_{C^{\alpha'}(\overline{B_r})} + \|\tilde{G}\|_{L^\infty(B_r)}\right)
$$

(5.3)

where $C > 0$ depends only on $n, r, A$ and $\|B\|_{L^\infty(B_r)}$.

To examine the second claim, let us select

$$
k = \begin{cases}
\|\nabla_{\bar{x}}F\|_{L^2(B_{r/4}; x_N^{1-2\gamma})} + \|\nabla_{\bar{x}}G\|_{L^2(B_{r/4}; x_N^{1-2\gamma})} + \|\nabla_{\bar{x}}B\|_{L^2(B_{r/4}; x_N^{1-2\gamma})}
+ \|\nabla_{\bar{x}}a\|_{L^p(\partial'B_r)} + \|\nabla_{\bar{x}}b\|_{L^p(\partial'B_r)} & \text{if it is nonzero}, \\
\text{any positive number} & \text{otherwise}.
\end{cases}
$$

In the latter case, we send $k \to 0$ at the last step. For a fixed $M > 0$ and $m \geq 0$, we define

$$
V_h = (D_hU)_+ + k = \max\{D_hU, 0\} + k, \quad V_{h,M} = \min\{V_h, M\}, \quad Z_{h,M,m} = V_{h,M}^{m/2}V_h.
$$

Let $\chi \in C^\infty(\overline{B_r})$ be a cut-off function satisfying (5.10) and $|\nabla \chi| \leq C\chi$ in $\mathbb{R}^N_+$. Setting $\chi_{r/6} = \chi(6r/r)$, we test (5.2) with $\Xi_{h,M,m} := \chi_{r/6}V_{h,M} - k^{m+1}$. Then a bit of calculation exploiting Hölder’s inequality, Young’s inequality, the Sobolev inequality, the Sobolev trace.
inequality and (R11) shows
\[
\int_{B_r} x_N^{1-2\gamma} |\nabla (x_r / 6 Z_{h,M,m})|^2 dx \\
\leq C \left[ \int_{B_r} x_N^{1-2\gamma} \chi_{r/6}^2 \left( V_{h,M}^{m-1} V_h + V_{h,M}^m |\nabla V_h|^2 \right) dx + \int_{B_r} x_N^{1-2\gamma} (\chi_{r/6} Z_{h,M,m})^2 dx \right] \\
\leq C \left[ \int_{B_r} x_N^{1-2\gamma} (|\nabla U_h| + |x_N \partial N U_h|) |\nabla \Xi_{h,M,m}| dx + \int_{B_r} x_N^{1-2\gamma} (\chi_{r/6} Z_{h,M,m})^2 dx \right] 
\]  \tag{5.4}
for $C > 0$ depending only on $n, \gamma, r, A, B, \|a\|_{L^p(\partial B_r)}, \|U\|_{L^\infty(B_r)}$ and $m$. (The boundary integrals appearing here can be controlled as in the proof of \[27\, Proposition 2.6.\]) Because of the first claim, the first integral in the rightmost side of (5.4) is bounded by
\[
\int_{B_r} x_N^{1-2\gamma} \left( |\nabla U_h| + \|x_N \partial N U\|_{L^\infty(B_r/2)} \right) |\nabla \Xi_{h,M,m}| dx \leq \varepsilon \int_{B_r} x_N^{1-2\gamma} \chi_{r/6}^2 \left( V_{h,M}^m |\nabla V_h|^2 \right) dx \\
+ C \left[ \int_{B_r} x_N^{1-2\gamma} \chi_{r/6}^2 \left( |\nabla U_h|^2 + \|x_N \partial N U\|^2_{L^\infty(B_r/2)} \right) dx + \int_{B_r} x_N^{1-2\gamma} (\chi_{r/6} Z_{h,M,m})^2 dx \right]
\]
for a small $\varepsilon > 0$. As a consequence, by taking $M \to \infty$ and applying the Sobolev inequality, we reach at
\[
\|V_h\|_{L^{m+2}(B_{r/6}; x_N^{1-\gamma})}^m + \int_{B_{r/3}} x_N^{1-2\gamma} V_h^m |\nabla U_h|^2 dx + \|x_N \partial N U\|_{L^\infty(B_r/2)}^2 
\]
whenever the right-hand side is finite.

Since $\nabla U \in L^2(B_r; x_N^{1-\gamma})$, the $L^2(B_r/3; x_N^{1-\gamma})$-norm of $V_h$ is uniformly bounded in $h \in \mathbb{R}^n$ with small $|h| > 0$. By taking $m = 0$ in the above estimate, we deduce that the $L^2(n-2\gamma+2)/(n-2\gamma)(B_{r/6}; x_N^{1-\gamma})$-norm of $V_h$ is uniformly bounded in $h$, and so $(\nabla \bar{U})_+ \in L^2(n-2\gamma+2)/(n-2\gamma)(B_{r/6}; x_N^{1-\gamma})$. The same argument applied to $(D_h U)_- + k$ where $(D_h U)_- := \max \{-D_h U, 0\}$ gives us that $(\nabla \bar{U})_- \in L^2(n-2\gamma+2)/(n-2\gamma)(B_{r/6}; x_N^{1-\gamma})$ as well. Repeating this procedure many times (and adjusting $r > 0$), we discover
\[
\|\nabla \bar{U}\|_{L^q(B_{r/6}; x_N^{1-\gamma})} \leq C \left( \|\nabla \bar{U}\|_{L^1(B_{r/2}; x_N^{1-\gamma})} + \|x_N \partial N U\|_{L^\infty(B_r/2)} + k \right)
\]
for any $q > 1$ where $C > 0$ depends only on $n, \gamma, r, A, B, \|a\|_{L^p(\partial B_r)}, \|U\|_{L^\infty(B_r)}$ and $q$. This justifies the second claim. The fact that $\nabla \bar{U} \in H^{1,2}(B_{r/6}; x_N^{1-\gamma})$ follows from [5.4] as a by-product.

In a similar way, one can prove that $\nabla \bar{U}$ is Hölder continuous on $B_{r/14}$. Because of (R22), the only nontrivial part is to check that $\|x_N \partial N_i U\|_{L^\infty(B_{r/24})} < \infty$ for $i = 1, \ldots, n$. By elliptic regularity, there holds that
\[
|(x_N \partial N_i U)(x_0)| \leq \left\| \partial N_i \bar{U} \right\|_{C^{\min(n, \alpha')}(B_N((0,1), 1/4))} \\
\leq C \left( \|\partial_i U\|_{C^{\alpha'}(B_{r/12})} + \|x_N \partial N U\|_{C^{\alpha'}(B_N((0,1), 1/2))} \right) \\
+ \|\nabla F\|_{C^{\alpha'}(B_{r/5})} + \|\nabla G\|_{L^\infty(B_r)} + \|\nabla B\|_{L^\infty(B_r)} 
\]
denote $x' = (\bar{x}_0 + x_{N0}\bar{x}, x_{N0}x_N)$ and $y' = (\bar{x}_0 + x_{N0}\bar{y}, x_{N0}y_N)$, then $x', y' \in B^N(x_0, x_{0}/2)$ and
\[
|x_N\partial_N U(x) - x_N\partial_N U(y)| = \left| (x_{N0}x_N)(\partial_N U)(x') - (x_{N0}y_N)(\partial_N U)(y') \right|
\leq x_{N0} \left| (x_N - y_N)(\partial_N U)(x') + |y_N|(\partial_N U)(x') - (\partial_N U)(y') \right|
\leq 2 \|x_N\partial_N U\|_{L^\infty(B_{r/4})} |x_N - y_N| + \frac{3}{2} \left\| \partial_N U \right\|_{C^{\alpha'}(B_{r/2})} |x - y|^\alpha'.
\]
Thus we see from (5.3) that
\[
\left\| x_N\partial_N U \right\|_{C^{\alpha'}(B^N((0, 1)/2))} \leq C \left( \|U\|_{L^\infty(B_{r/2})} + \|F\|_{C^{\alpha'}(B_{r/2})} + \|G\|_{L^\infty(B_{r/2})} \right).
\]
This proves the assertion and concludes the proof. \qed

Modifying the above argument slightly, one gets the following result.

**Corollary 5.4.** Assume that $A$ satisfies (R1) and
(R12) $A \in C^{\alpha}(B_r, \mathbb{R}^{N \times N})$ and $A_{NN}(x) = 1 + A'_{NN}(x)x_N$ in $B_r$ for some $A'_{NN} \in C^\infty(B_r)$;
(R23) $\nabla^m_x F_a \in C^\alpha(B_r)$ and $\nabla^m_x B, \nabla^m_x G \in L^\infty(B_r)$ for some $\alpha' \in (0, 1)$ and all $m \in \mathbb{N}$.
Moreover, suppose that $U \in H^{1,2}(B_r; x_N^{1-2\gamma}) \cap L^\infty(B_r)$ is a positive function in $B_r$ and weakly solves
\[
-\text{div}(x_N^{1-2\gamma} A \nabla U) + x_N^{1-2\gamma} BU = x_N^{1-2\gamma} G + \text{div}(x_N^{1-2\gamma} F) \quad \text{in } B_r,
\]
\[
\partial_N U = f \quad \text{on } \partial' B_r,
\]
for some $f \in C^\infty(B_r)$ and $\beta \in (1, 2^\gamma]$. Then, for any $m \in \mathbb{N}$, there exist $\alpha \in (0, 1)$ and $0 < r' < r$ such that $\nabla^m_x U \in C^\alpha(B_{r'})$.

**Proof.** We can argue as in the proof of the previous lemma to show that $\nabla^m_x U \in C^\alpha(B_{r/12})$. The only difference is the way to deal with the boundary integral in deducing (5.4). We easily observe that $\partial_N D_h U = (D_h f) U_h^\beta + f (D_h U)^\beta$ on $\partial B_{r/2}$. In addition, if we redefine
\[
k = \left\{ \begin{array}{ll}
\|\nabla^m_x F\|_{L^1(B_r; x_N^{1-2\gamma})} + \|\nabla^m_x G\|_{L^2(B_r; x_N^{1-2\gamma})} + \|\nabla^m_x B\|_{L^2(B_r; x_N^{1-2\gamma})} & \text{if it is nonzero},
\
\text{any positive number} & \text{otherwise},
\end{array} \right.
\]
and set $\Xi_{h, m, m}$ as before, then we get $|D_h U|^\beta \leq C \|U\|_{L^\infty(\partial B_r)}^{\beta - 1} |D_h U|$ in $\partial B_{r/2}$ so that
\[
\int_{\partial' B_r} (|D_h f| U_h^\beta + |f| |D_h U|^\beta) \Xi_{h, m, m} d\bar{x}
\leq C \|U\|_{L^\infty(\partial B_r)}^{\beta - 1} \int_{\partial' B_r} \left( \|U\|_{L^\infty(\partial B_r)} \frac{|D_h f|}{k} + \|f\|_{L^\infty(\partial B_r)} \left( x_{r/6} \Omega_{h, m, m} \right)^2 d\bar{x} \right)
\]
\[
\leq C \|U\|_{L^\infty(\partial B_r)}^{\beta - 1} \int_{\partial' B_r} \left( \|U\|_{L^\infty(\partial B_r)} \frac{|D_h f|}{k} + \|f\|_{L^\infty(\partial B_r)} \left( x_{r/6} \Omega_{h, m, m} \right)^2 d\bar{x} \right)
\]
where $C > 0$ depends only on $\beta$. Now, one can derive (5.4) having this estimate in hand and following the proof of [27] Proposition 2.6).

Hölder continuity of higher order derivatives $\nabla^m_x U$ for $m \in \mathbb{N}$ can be achieved by iteration of this argument. The positivity of $U$ guarantees that $U^{\beta - m}$ is bounded away from 0 for any $\beta \in (1, 2^\gamma]$, so $|D_h U^{\beta - m}| \leq C |D_h U|$ in $B_r$ for some $C > 0$. \qed

We next proceed to prove Hölder regularity of the weighted normal derivative $x_N^{1-2\gamma} \partial_N U$ of a solution $U$ to (5.1) from that of the solution and its tangential derivatives $\nabla^m_x U, \nabla^m_x U$ as well as suitable regularity on $A, B, F, G, a$ and $b$.

**Lemma 5.5.** Suppose that $U \in H^{1,2}(B_r; x_N^{1-2\gamma})$ is a weak solution of (5.1) where the matrix $A$ satisfies (R1). Furthermore, assume that $U, \nabla^m_x U, \nabla^m_x U \in C^\alpha(B_r)$,
In addition, we suppose that the mean curvature \( \gamma \) vanishes. If \( f \) satisfies a certain condition, then the standard elliptic estimate works if \( X \) is a geodesic defining function in \( \Omega \). Then \( x_N^{-1-2\gamma} \partial N U \in C^{\min(\alpha,2-2\gamma)}(B_{r/2}) \).

**Proof.** The first equation in (5.1) can be rewritten as

\[-\partial_N(x_N^{-1-2\gamma}A_N \partial N U) = x_N^{-1-2\gamma} [\partial_i(A_{ij} \partial_j U) - BU] + G + \partial_a F_a =: x_N^{-1-2\gamma} Q \text{ in } B_r.\]

Hence it holds that

\[-A_{NN}(x_0) \cdot x_N^{-1-2\gamma} \partial N U(x_0) = \kappa^{-1} A_{NN}(\bar{x}_0,0)(aU + b)(\bar{x}_0,0) + \int_0^{x_N(0)} x_N^{-1-2\gamma} Q(\bar{x}_0,x_N) dx_N =: Q(x_0)\]

for any point \( x_0 = (\bar{x}_0,0) \in \overline{B}_r \). From this relation and the bound that \( A_1 \leq A_{NN}(x_0) \leq A_2 \), coming from the assumption on \( A \), we immediately observe that

\[\|x_N^{-1-2\gamma} \partial N U\|_{L^\infty(B_r)} \leq C \left( \|U\|_{L^\infty(B_r)} + \|b\|_{L^\infty(\partial B_r)} + \sum_{i=1}^2 \|\nabla_i U\|_{L^\infty(B_r)} + \sum_{\alpha=1}^N \|\partial_\alpha F_\alpha\|_{L^\infty(B_r)} + \|G\|_{L^\infty(B_r)} \right)\]

where \( C > 0 \) depends only on \( n, \gamma, r, A_1, A_2 \), \( \|a\|_{L^\infty(\partial B_r)} \) and \( \|B\|_{L^\infty(\partial B_r)} \). In addition, a simple computation reveals that there exists \( C > 0 \) counting only on \( \gamma \) and \( r \) such that

\[\|Q\|_{C^{\min(\alpha,2-2\gamma)}(B_{r/2})} \leq C \sup_{x_N \in (0,r/2)} \|Q(\cdot,x_N)\|_{C^{\alpha}(\overline{B}_{r/2})}.\]

Therefore

\[\|x_N^{-1-2\gamma} \partial N U\|_{C^{\min(\alpha,2-2\gamma)}(B_{r/2})} \leq C \left( \|U\|_{C^{\alpha}(\overline{B}_r)} + \|b\|_{C^{\alpha}(\overline{B}_r)} + \sup_{x_N \in (0,r/2)} \|Q(\cdot,x_N)\|_{C^{\alpha}(\overline{B}_{r/2})} \right)\]

\[\leq C \left( \sum_{\ell=0}^2 \|\nabla_\ell U\|_{C^{\alpha}(\overline{B}_r)} + \|b\|_{C^{\alpha}(\overline{B}_r)} \right)\]

\[+ C \sup_{x_N \in (0,\bar{r})} \left( \sum_{\alpha=1}^N \|\partial_\alpha F_\alpha(\cdot,x_N)\|_{C^{\alpha}(\overline{B}_r)} + \|G(\cdot,x_N)\|_{C^{\alpha}(\overline{B}_r)} \right)\]

where the constant \( C > 0 \) depends only on \( n, \gamma, r, A_1 \), \( \|a\|_{C^{\alpha}(\overline{B}_r)} \) and \( \|B\|_{C^{\alpha}(\overline{B}_r)} \). \( \square \)

### 5.2. Proof of Theorem **1.8**

Finally, collecting all the results obtained in the previous subsection together, we deduce a regularity result on our main equation (1.3) or its extension (1.11). It particularly validates Theorem **1.8**

**Proposition 5.6.** Assume that \( (X^N, g^+) \) is an asymptotically hyperbolic manifold, \( (M^n, [\hat{h}]) \) is its conformal infinity, \( \rho \) is the geodesic boundary defining function of \( (M, \hat{h}) \) and \( \tilde{g} = \rho^2 g^+ \). In addition, we suppose that the mean curvature \( H \) on \( (M, \hat{h}) \) as a submanifold of \( (X, \tilde{g}) \) vanishes. If \( f \in C^\infty(M) \) and \( U \in H^{1,2}(X; \rho^{1-2\gamma}) \) weakly solves (1.11), then the trace \( u \in H^\gamma(M) \) of \( U \) on \( M \) is in fact of class \( C^\infty(M) \) and a classical solution to (1.11). Moreover, \( \nabla_\nu^m U \) and \( \rho^{1-2\gamma} \partial N U \) are Hölder continuous on \( \overline{X} \) for every \( m \in \mathbb{N} \).

**Proof.** The standard elliptic estimate works if \( \gamma = 1/2 \) as confirmed by Cherrier [14], so we assume that \( \gamma \in (0,1) \setminus \{1/2\} \).

Fix any \( y \in M \) and choose a smooth metric \( \hat{h}_y \in [\hat{h}] \) on \( M \) such that \( \|\hat{h}_y\|_1 \approx 1 \) around \( y \), whose existence was deduced by Cao [9] and Günther [23]. Let \( \rho_y \) be the associated geodesic defining function in \( X \), \( \tilde{g}_y = \rho^2_y \hat{h}_y \) and \( w_y \in C^\infty(M) \) a positive function satisfying
\[ \hat{h}_y = w_y^{4/(n-2\gamma)} u \] on \( M \). According to Lemma 2.2, it holds that \( \tilde{g}_y = 1 + O(\rho_y) \) in \( X \) near \( y \). As a matter of fact, we have that \( \tilde{g}_y = 1 + O(\rho_y^2) \) by the condition \( H = 0 \) on \( M \).

In view of Lemma 2.2, we know that \( P_\gamma h = fu^{2\gamma} \) in \( H^{-\gamma}(M) \). By the conformal covariance property (1.2), the function \( u_y := w_y^{-1} u \in H^2(M) \) weakly solves \( P_\gamma u_y = f u^{2\gamma} \) on \( M \). Besides, there is a solution \( U_y \in H^{1,2}(X; \rho^{1-2\gamma}) \) to (1.11) (where the subscript \( y \) is attached suitably) such that \( U_y = u_y \) on \( M \).

Denote by \( x \) the Fermi coordinate on \((\bar{X}, \bar{g}_y)\) around \( y \). We assume that it is defined in the geodesic half-ball \( B_{\bar{g}}(y, r) := \{ \bar{y} \in X : \text{dist}_{\bar{g}}(\bar{y}, \bar{y}) < r \} \) with \( B_{\bar{g}} = B_{\bar{g}}^*(0, r) \subset \mathbb{R}^N \). Then, owing to Remark 2.4 and the assumption \( H = 0 \) on \( M \), the term \( E(\rho(y)) = E(x_N) \) can be expressed as \( x_N^{1-2\gamma} B \) for some function \( B \in C^\infty(\overline{B_r}) \). Consequently, if we set \( A_{ab} = \sqrt{|\bar{g}_y|} g_y^{ab} \) and \( F_a = G = 0 \) on \( B_r \), then the equation (1.11) of \( U_y \) can be described as (5.1) with \( \alpha = u_y^{n-1}, b = 0 \) on \( \partial^* B_r \) or (5.5) with \( \beta = 2^*, \) where conditions (R1), (R12), (R2), (R23), (R4), (R3) and (R32) are all fulfilled.

We first regard (1.11) as (5.1). By employing the Sobolev trace inequality, we see that \( a \in L^{n/(2\gamma)}(\partial B_r) \). Thus we can apply Lemma 5.1 (1) to derive that \( a \in L^{p_1}(\partial B_r/2) \) for some \( p_1 \geq n/(2\gamma) \). Because of Lemma 5.1 (2), this implies that \( U_y \in C^\infty(\overline{B_r/4}) \) for some \( \alpha \in (0, 1) \). Furthermore, regarding (1.11) as (5.1), we realize from Corollary 5.4 that \( \nabla^m_{x} u_y(0) \) exists for all \( m \in \mathbb{N} \). This means that \( u \) is infinitely differentiable at \( y \), and for \( y \) is arbitrary, it leads that \( u \in C^\infty(M) \).

The last assertion in the statement can be proved via (3.10) and Lemma 5.2. This concludes the proof. \[ \square \]

**Appendix A. End-point Case of the Fractional Yamabe Problem**

This section is devoted to proving Theorem 1.3. By Proposition 3.2, it is enough to verify (3.9). Three mutually exclusive cases (B'), (C') and (D') will be handled in Propositions A.1, A.2 and A.3, respectively. Our proof will be rather sketchy, so we ask the reader to consult the papers [3, 16, 17, 35, 36] on the boundary Yamabe problem for more detailed explanation under analogous settings.

**Proposition A.1.** Suppose that \( n = 3, \gamma = 1/2, (M, \hat{h}) \) is the non-umbilic boundary of \((\bar{X}, \bar{g})\) and (1.4) is valid. Then (3.9) holds.

**Proof.** We fix a non-umbilic point \( y \in M \) and set \( \Psi_{1\epsilon} \) as in (4.2). It suffices to prove that

\[ \Gamma(\chi_{r_2}(W_{\epsilon,0} + \Psi_{1\epsilon})) \leq \bar{X}(\mathbb{H}^{4}, [\hat{h}_c]) + O(\epsilon^2) \]  
(A.1)

for some \( C_1 \in \mathbb{R} \) where \( \chi_{r_2} \) is the cut-off function defined after (3.10). Then the argument of Marques [36, pp. 400–403] will give

\[ \Gamma(\chi_{r_2}(W_{\epsilon,0} + \Psi_{1\epsilon} + C'_{1} II_{ij}(y)x_i x_j \partial_{ij} W_{\epsilon,0})) \leq \bar{X}(\mathbb{H}^{4}, [\hat{h}_c]) - C'_{1} \log \epsilon + O(\epsilon^2) \]  
(A.2)

for some \( C > 0 \) and small \( C'_{1} \in \mathbb{R} \), which in particular tells us that (3.9) holds.

We may suppose that \( \bar{g} \) on \( \bar{X} \) and \( \hat{h} \) on \( M \) satisfy (2.18). As in [33, Lemma 2.6], using Lemma 2.3, (2.16) and the identity \( \partial_{ij} W_{\epsilon,0}(x) = x_i r_2^{-1} \partial_{ij} W_{\epsilon,0}(x) \) which holds for all \( x \in \mathbb{R}^N_+ \), we calculate

\[ \int_{X} |\nabla(\chi_{r_2} W_{\epsilon,0})|^2 d\bar{v}_{\bar{g}} = \int_{\mathbb{R}^N_+} |\nabla W_{\epsilon,0}|^2 dx \]

\[ + \left( 3 II_{ik}(y) II_{kj}(y) + R_{iNjN}[\bar{g}](y) \right) \int_{B_{\mathbb{R}^N_+}(0, r_2)} x_N^2 \partial_i W_{\epsilon,0} \partial_j W_{\epsilon,0} dx \]

\[ - \frac{1}{2} \left( \|II(y)\|^2 + R_{NN}[\bar{g}](y) \right) \int_{B_{\mathbb{R}^N_+}(0, r_2)} x_N^2 |\nabla W_{\epsilon,0}|^2 dx + O(\epsilon^2) \]
Suppose that the metrics $\bar{\gamma} = (\partial_x W_{1,0}, \cdots, \partial_n W_{1,0})$ and $|S^2| = 4\pi$ is the surface measure of the unit 2-sphere. Owing to (2.2) and Lemma 2.3, we also have

$$E(x_N) = (\|\Pi(y)\|^2 + R_{NN}\bar{g}(y))(1 + O(|x|)) = -\frac{1}{4}\|\Pi(y)\|^2(1 + O(|x|))$$

in $B^1_+(0,2r_2)$. Thus

$$\int_{X} E(\rho)(\chi_{r_2} W_{\epsilon,0})^2 dv_y = -\frac{1}{4}\|\Pi(y)\|^2 c^2 \int_{B^1_+(0,2r_2)} W^2_{y,0} dx + O(\epsilon^2)$$

$$= -\frac{\pi}{16} \alpha_{3,1/2}^2 |S^2|\|\Pi(y)\|^2 c^2 \log \left(\frac{r_2}{\epsilon}\right) + O(\epsilon^2).$$

Consequently, we derive from the definition (2.8) of $I^\gamma$ that

$$I^\gamma(\chi_{r_2} W_{\epsilon,0}) = \int_{\mathbb{R}^4_+} |\nabla W_{1,0}|^2 dx + \frac{\pi}{24} \alpha_{3,1/2}^2 |S^2|\|\Pi(y)\|^2 c^2 \log \left(\frac{r_2}{\epsilon}\right) + O(\epsilon^2).$$

Moreover, a direct computation shows

$$I^\gamma(\chi_{r_2} W_{\epsilon,0} + \Psi_{1\epsilon})$$

$$= I^\gamma(\chi_{r_2} W_{\epsilon,0}) + 4H_{ij}(y) \int_{B^1_+(0,r_2)} \partial_i W_{\epsilon} \partial_j \Psi_{1\epsilon} dx + \int_{B^1_+(0,r_2)} |\nabla \Psi_{1\epsilon}|^2 dx + O(\epsilon^2)$$

$$= \int_{\mathbb{R}^4_+} |\nabla W_{1,0}|^2 dx + (1 + 4C_1 + 4C_1^2) \frac{\pi}{24} \alpha_{3,1/2}^2 |S^2|\|\Pi(y)\|^2 c^2 \log \left(\frac{r_2}{\epsilon}\right) + O(\epsilon^2)$$

(cf. [33 Proposition 2.8]). Hence, choosing $C_1 = -1/2$, we observe the validity of (A.1). This completes the proof.

Taking the above proposition into consideration, one can guess that the local geometry on $M$ still allows us to derive (3.9) when $n = 3$, $\gamma \in (0,1/2)$, $(M, \hat{h})$ is the non-umbilic boundary of $(\bar{X}, \bar{g})$ and (1.6) holds. However, there seems a computational difficulty in employing test functions whose forms are similar to that of the function in the previous proof.

**Proposition A.2.** Suppose that $n = 4$, $\gamma = 1/2$, $(M, \hat{h})$ is the umbilic boundary of $(\bar{X}, \bar{g})$, $R_{NN}\bar{g}(y) < 0$ for some $y \in M$ and (1.6) is valid. Then (3.9) holds.

**Proof.** Suppose that the metrics $\bar{g}$ and $\hat{h}$ satisfy (2.18). Since $\Pi = 0$ on $M$ by the assumption, we know that $R_{NN}\bar{g}(y) = 0$. By (2.2) and Lemma 2.3,

$$E(x_N) = \frac{3}{2} \left( R_{NN}\bar{g}|x_i + \frac{1}{2} R_{NN,N}\bar{g}|x_N + O(|x|^2) \right)$$

in $B^1_+(0,2r_2)$. It follows that

$$I^\gamma(\chi_{r_2} W_{\epsilon,0}) = \int_{\mathbb{R}^4_+} |\nabla W_{1,0}|^2 dx + \frac{3}{4} \frac{R_{NN}\bar{g}(y)}{e^3} \int_{B^4_+(0,r_2^{-1})} x_N W^2_{1,0} dx$$

$$+ R_{NN}\bar{g}(y) e^3 \int_{B^4_+(0,r_2^{-1})} x_N^3 \left( \frac{1}{12} |\nabla_x W_{1,0}|^2 - \frac{1}{6} |\nabla W_{1,0}|^3 \right) dx + O(\epsilon^3)$$

$$= \int_{\mathbb{R}^4_+} |\nabla W_{1,0}|^2 dx + \frac{3}{32} \alpha_{4,1/2}^2 |S^3| R_{NN}\bar{g}(y) e^3 \log \left(\frac{r_2}{\epsilon}\right) + O(\epsilon^3).$$
Proposition A.3. Assume that \( n = 5, \gamma = 1/2, (M, \hat{h}) \) is the umbilic non-locally conformally flat boundary of \((\overline{X}, \hat{g})\) and (1.7) is valid. Then (3.9) holds.

Proof. We assume that \( \hat{h} \) is the representative of its conformal class satisfying all the properties listed in Lemma 2.7 and the Weyl tensor at \( y \in M \) is nontrivial. Then a tedious but straightforward computation gives

\[
\int_X |\nabla (\chi_{r_2} W_{e,0})|^2 d\hat{v}_\hat{g} = \int_{\mathbb{R}^6_+} |\nabla W_{e,0}|^2 dx + \frac{1}{2} R_{iNjN,k}\hat{l}[\hat{g}] \int_{B^6_+(0,r_2)} x^2_N x_k x_l \partial_i W_{e,0} \partial_j W_{e,0} dx \\
+ \frac{1}{12} (R_{iNjN;k}\hat{l}[\hat{g}] + 8 R_{iNkN}\hat{l}[\hat{g}] R_{kNjN}\hat{l}[\hat{g}]) \int_{B^6_+(0,r_2)} x^4_N \partial_i W_{e,0} \partial_j W_{e,0} dx \\
- \frac{1}{4} R_{NN;ij}\hat{l}[\hat{g}] \int_{B^6_+(0,r_2)} x^2_N x_i x_j |\nabla W_{e,0}|^2 dx \\
- \frac{1}{24} (R_{NN;NN}\hat{l}[\hat{g}] + 2(R_{ij}\hat{l}[\hat{g}])^2) \int_{B^6_+(0,r_2)} x^4_N |\nabla W_{e,0}|^2 dx + O(\epsilon^4) \\
= \int_{\mathbb{R}^6_+} |\nabla W_{1,0}|^2 dx + \frac{\pi}{640} \alpha^2_{5,1/2} |\overline{\gamma}|^4 \left[ -\frac{1}{8} \|W\|^2 - \frac{1}{2} R_{NN}\hat{l}[\hat{g}] + 2(R_{ij}\hat{l}[\hat{g}])^2 \right] \epsilon^4 \log \left( \frac{T_0}{\epsilon} \right) \\
+ \frac{\pi}{640} \alpha^2_{5,1/2} |\overline{\gamma}|^4 \left[ \frac{5}{24} \|W\|^2 - \frac{3}{2} R_{NN}\hat{l}[\hat{g}] \right] \epsilon^4 \log \left( \frac{T_0}{\epsilon} \right) + O(\epsilon^4)
\]

where all tensors are evaluated at \( y \in M \). Furthermore, we obtain from (2.2) and Lemma 2.5 that

\[
E(x_N) = 2R_{NN;i}\hat{l}[\hat{g}] x_i + R_{NN;ij}\hat{l}[\hat{g}] x_i x_j + R_{NN;Nl}\hat{l}[\hat{g}] x_N x_l \\
+ \frac{1}{3} (R_{NN;NN}\hat{l}[\hat{g}] + 2(R_{ij}\hat{l}[\hat{g}])^2) x^2_N + O(|x|^3)
\]

in \( B^6_+(0,2r_2) \) and so

\[
\int_X E(\rho)(\chi_{r_2} W_{e,0})^2 d\hat{v}_\hat{g} = \frac{\pi}{640} \alpha^2_{5,1/2} |\overline{\gamma}|^4 \left[ -\frac{5}{12} \|W\|^2 + 2 R_{NN}\hat{l}[\hat{g}] \right] \epsilon^4 \log \left( \frac{T_0}{\epsilon} \right) + O(\epsilon^4)
\]

in light of Lemma 2.7 (3) and (4). As a result,

\[
\Gamma(\chi_{r_2} W_{e,0}) = \int_{\mathbb{R}^6_+} |\nabla W_{1,0}|^2 dx + \frac{\pi}{640} \alpha^2_{5,1/2} |\overline{\gamma}|^4 \left[ -\frac{1}{3} \|W\|^2 + 2(R_{ij}\hat{l}[\hat{g}])^2 \right] \epsilon^4 \log \left( \frac{T_0}{\epsilon} \right) + O(\epsilon^4).
\]

We now recall the function \( \Psi_{2\epsilon} \) defined in (1.3). With this, one can compute

\[
\Gamma(\chi_{r_2} (W_{e,0} + \Psi_{2\epsilon})) = \int_{\mathbb{R}^6_+} |\nabla W_{1,0}|^2 dx + \frac{\pi}{640} \alpha^2_{5,1/2} |\overline{\gamma}|^4 \left[ -\frac{1}{3} \|W\|^2 + (2 + 24C_2 + 56C_2^2)(R_{ij}\hat{l}[\hat{g}])^2 \right] \epsilon^4 \log \left( \frac{T_0}{\epsilon} \right) + O(\epsilon^4).
\]

If we take \( C_2 = -3/14 \), then \( 2 + 24C_2 + 56C_2^2 = -4/7 < 0 \), yielding the validity of (3.9). \( \square \)
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