COSTRATIFICATION AND ACTIONS OF TENSOR-TRIANGULATED CATEGORIES

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Abstract. We develop the theory of costratification in the setting of relative tensor-triangular geometry, in the sense of Stevenson, providing a unified approach to classification results of Neeman and Benson–Iyengar–Krause, while laying the foundations for future applications. In addition, we introduce and study prime localizing submodules and prime colocalizing hom-submodules, in the first case, generalizing objectwise-prime localizing tensor-ideals. We apply our results to show that the derived category of quasi-coherent sheaves over a noetherian separated scheme is costratified.

Contents

1. Introduction 1
2. Preliminaries 3
3. Stratification–costratification 10
4. Prime submodules 16
5. Smashing submodules 20
6. Derived categories of noetherian rings and schemes 24
References 26

1. Introduction

The theory of cosupport and costratification in tensor-triangulated categories was initiated by Benson–Iyengar–Krause [BIK12], inspired by the classification of colocalizing subcategories of the derived category of a commutative noetherian ring by Neeman [Nee11]. Their main application was the classification of Hom-closed colocalizing subcategories of the stable module category of a finite group. Compared to the theory of support and stratification [BIK08, BIK11a, BIK11b, BHS23] (which is related to the classification of localizing subcategories) the theory of cosupport has not been explored as much. In particular, a theorem that unifies the aforementioned classifications has not yet been stated and the machinery of [BIK12] depends on the action of a commutative noetherian ring on the triangulated category involved. The aim of this paper is to provide such a theorem and develop the theory of cosupport in the context of relative tensor-triangular geometry [Ste13]. The advantage to this approach is that it allows our theory to apply in cases of triangulated categories that are not necessarily tensor-triangulated but are endowed with an action of a tensor-triangulated category. For instance, singularity categories of commutative noetherian rings [Kra05, Ste14b]. This will be the subject of future work. It should be noted that progress on the topic of costratification has also been made independently by Barthel–Castellana–Heard–Sandars, focusing on the Balmer–Favi support [BCHS23].

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Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category and let $\mathcal{K}$ be a compactly generated triangulated category endowed with an action of $\mathcal{T}$; see Section 2 for details. In the special case where $\mathcal{K} = \mathcal{T}$, the action is given by the tensor product of $\mathcal{T}$.

Section 2 consists of preliminary material — including a very brief account of the Balmer–Favi support [BF11] and the smashing spectrum and the small smashing support [BS23] — and establishes concepts and basic lemmas that we will be using throughout the paper.

In Section 3, we introduce the notion of a good support–cosupport pair on $\mathcal{T}$ taking values in a space $S$, which induces a support–cosupport pair on $\mathcal{K}$. For example, one could take the small smashing support–cosupport or the Balmer–Favi support–cosupport or the BIK support–cosupport. In our first main result, Theorem 3.22, we prove that $\mathcal{K}$ is costratified (meaning that there is a bijective correspondence between certain subsets of $S$ and the collection of colocalizing hom-submodules of $\mathcal{K}$) if and only if $\mathcal{K}$ satisfies two conditions: the colocal-to-global principle and cominimality. These two conditions are in a sense dual to the more well-established local-to-global principle and minimality, which were introduced in [BIK11] and studied further in [Ste13, Ste17, Ste18, BHS23]. Moreover, in Corollary 3.28, we prove that the local-to-global principle implies the colocal-to-global principle.

Section 4 is devoted to the study of prime localizing submodules and prime colocalizing hom-submodules of $\mathcal{K}$, the former specializing to the class of objectwise-prime localizing tensor-ideals when $\mathcal{K} = \mathcal{T}$; see [BS23, Ver23]. We prove that if $\mathcal{K}$ is costratified, then there is a bijective correspondence between points of a certain subspace of $S$ and prime colocalizing hom-submodules of $\mathcal{K}$, obtaining a complete description of the latter (Theorem 4.10). The correspondence is given by Theorem 3.22. In addition, we analyze the relation of prime localizing submodules and prime colocalizing hom-submodules with the Action Formula (which generalizes the Tensor Product Formula) and the Internal-Hom Formula.

In Section 5, we show that costratification of $\mathcal{K}$ can be reduced to costratification of certain smashing localizations of $\mathcal{K}$; see Theorem 5.4. For instance, if $\mathcal{K} = \mathcal{T}$ and the action of $\mathcal{T}$ on itself is the tensor product of $\mathcal{T}$ and the support–cosupport theory one fixes is given by the smashing spectrum $\text{Spc}^e(\mathcal{T})$ and the small smashing support (assuming that $\text{Spc}^e(\mathcal{T})$ is $T_D$) we have the following: Provided that $\mathcal{T}$ satisfies the colocal-to-global principle, $\mathcal{T}$ is costratified if and only if $\mathcal{T}/P$ is costratified, for all $P \in \text{Spc}^e(\mathcal{T})$; see Corollary 5.5. As a direct consequence of Corollary 5.5, we have: If $\text{Spc}^e(\mathcal{T}) = \bigcup_{j \in J} V_{S_j}$ is a cover of $\text{Spc}^e(\mathcal{T})$ by closed subsets such that each $\mathcal{T}/S_j$ satisfies cominimality, then (provided that $\mathcal{T}$ satisfies the colocal-to-global principle) $\mathcal{T}$ is costratified; see Corollary 5.6. In Corollary 5.7 and Corollary 5.8, we state the analogous results for the Balmer spectrum $\text{Spc}(\mathcal{T}^c)$ and the Balmer–Favi support (assuming that every point of $\text{Spc}(\mathcal{T}^c)$ is visible). In Theorem 5.9, we prove a generalization of Corollary 5.8 by replacing $\mathcal{T}$ with a $\mathcal{T}$-module $\mathcal{K}$.

Finally, in Section 6, we present our application: Using the general machinery developed throughout, we first give in Theorem 6.5 a more streamlined proof of Neeman’s classification of the colocalizing subcategories of the derived category of a commutative noetherian ring and then we combine Theorem 6.5 with Corollary 5.8 to show that the derived category of quasi-coherent sheaves over a noetherian separated scheme is costratified.
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2. Preliminaries

2.A. Actions and basic lemmas. Throughout, $\mathcal{T} = (\mathcal{T}, \otimes, 1)$ will be a rigidly-compactly generated tensor-triangulated category (big tt-category) as in [BF11], i.e., $\mathcal{T}$ is a tensor-triangulated category with arbitrary coproducts and it is generated by its compact objects as a localizing subcategory. Moreover, the (essentially small) subcategory $\mathcal{T}_c$ of compact objects of $\mathcal{T}$ is a tensor-triangulated subcategory and the rigid objects of $\mathcal{T}$ coincide with the compact objects. The internal-hom functor of $\mathcal{T}$ will be denoted by $[-, -]: \mathcal{T}^{\text{op}} \times \mathcal{T} \to \mathcal{T}$.

Let $\mathcal{K}$ be a compactly generated triangulated category and let $\ast: \mathcal{T} \times \mathcal{K} \to \mathcal{K}$ be an action of $\mathcal{T}$ on $\mathcal{K}$, in the sense of [Ste13]. In short, $\ast: \mathcal{T} \times \mathcal{K} \to \mathcal{K}$ is a coproduct-preserving triangulated functor in each variable such that there exist natural (in all variables) isomorphisms $\alpha_{X,Y,A}: X \ast (Y \ast A) \xrightarrow{\cong} (X \otimes Y) \ast A$ and $l_A: 1 \ast A \xrightarrow{\cong} A$, $\forall X, Y \in \mathcal{T}$, $\forall A \in \mathcal{K}$. The natural isomorphism $\alpha$ is called the associator and the natural isomorphism $l$ is called the unitor. There is also a host of coherence conditions that need to be satisfied; we refer the reader to the aforementioned source for details. We call $\mathcal{K} = (\mathcal{K}, \ast)$ a $\mathcal{T}$-module.

By definition, for every object $X \in \mathcal{T}$, the functor $X \ast -: \mathcal{K} \to \mathcal{K}$ is a coproduct-preserving triangulated functor. Hence, by Brown representability, $X \ast -$ admits a right adjoint $[X, -]: \mathcal{K} \to \mathcal{K}$. Assembling these right adjoints yields a functor $[-, -]: \mathcal{T}^{\text{op}} \times \mathcal{K} \to \mathcal{K}$ that we call the relative internal-hom. Since $[1, -]_\ast$ is the right adjoint of $1 \ast - \cong \text{Id}_{\mathcal{K}}$, it holds that $[1, -]_\ast \cong \text{Id}_\mathcal{K}$. Specifically, the composite $m := \text{Id}_\mathcal{K} \to [1, 1 \ast -]_\ast \xrightarrow{[1, l]} [1, -]_\ast$, where the first map is the unit of adjunction, is a natural isomorphism (which we call the hom-unitor).

Lemma 2.1. Let $X, Y \in \mathcal{T}$ and $A \in \mathcal{K}$. Then there exists a natural (in all variables) isomorphism $\beta_{X,Y,A}: [X \otimes Y, A]_\ast \xrightarrow{\cong} [X, [Y, A]_\ast]_\ast$, called the hom-associator.

Proof. Let $B \in \mathcal{K}$. By the adjunction between the action and the relative internal-hom and the relation $(X \otimes Y) \ast B \cong (Y \otimes X) \ast B \cong Y \ast (X \ast B)$, we have:

\[
\text{Hom}_\mathcal{K}(B, [X \otimes Y, A]_\ast) \cong \text{Hom}_\mathcal{K}((X \otimes Y) \ast B, A) \\
\cong \text{Hom}_\mathcal{K}((Y \otimes X) \ast B, A) \\
\cong \text{Hom}_\mathcal{K}(Y \ast (X \ast B), A) \\
\cong \text{Hom}_\mathcal{K}(X \ast B, [Y, A]_\ast) \\
\cong \text{Hom}_\mathcal{K}(B, [X, [Y, A]_\ast]_\ast).
\]

Consequently, for $B = [X \otimes Y, A]_\ast$, the image of the identity morphism on $[X \otimes Y, A]_\ast$ under the above series of isomorphisms gives a natural (in all variables) isomorphism $\beta_{X,Y,A}: [X \otimes Y, A]_\ast \xrightarrow{\cong} [X, [Y, A]_\ast]_\ast$. \qed

Notation 2.2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two $\mathcal{T}$-modules. For $i = 1, 2$, the relative internal-hom of $\mathcal{K}_i$ will be denoted by $[-, -]_i$. Let $X, Y \in \mathcal{T}$ and $A \in \mathcal{K}_i$. The associator and unitor natural isomorphisms will be denoted by $\alpha_{X,Y,A}^i$ and $l_A^i$, respectively. The hom-associator and hom-unitor natural isomorphisms will be denoted by $\beta_{X,Y,A}^i$ and $m_A^i$, respectively. The unit and the counit of the action-hom adjunction will
be denoted by \( u_{X,A}^i: A \to [X, X \ast_i A]_i \) and \( c_{X,A}^i: X \ast_i [X, A]_i \to A \), respectively.

We denote by \( \sigma_{X,Y}: X \otimes Y \cong Y \otimes X \) the symmetry natural isomorphism. We denote by \( c_{X,-}: X \otimes [X, -] \to \text{Id}_T \) the counit of the adjunction \( X \otimes - \dashv [X, -] \). Set \( X^\vee := [X, 1] \) and define the morphism \( ev_{X,A}^i: X \ast_i A \to [X^\vee, A]_i \) as the following composite:

\[
X \ast_i A \xrightarrow{u_{X,Y,X \ast_i A}^i} [X^\vee, X \ast_i (X \ast_i A)]_i \xrightarrow{[X^\vee, \sigma_{X,Y,X \ast_i A}^i]} [X^\vee, (X \otimes X) \ast_i A]_i \xrightarrow{[X^\vee, - \ast_i A]_i} [X^\vee, A]_i.
\]

**Definition 2.3.** A functor \( F: \mathcal{K}_1 \to \mathcal{K}_2 \), between \( \mathcal{T} \)-modules \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \), is called action-preserving if there is a natural isomorphism \( \phi: F(- \ast_1 -) \to - \ast_2 F(-) \) between the functors \( F(- \ast_1 -), - \ast_2 F(-): \mathcal{T} \times \mathcal{K}_1 \to \mathcal{K}_2 \) such that, for all \( X, Y \in \mathcal{T} \) and for all \( A \in \mathcal{K}_2 \), the following diagrams commute:

\[
\begin{align*}
F(X \ast_1 (Y \ast_1 A)) & \xrightarrow{\phi_{X,Y \ast_1 A}} X \ast_2 F(Y \ast_1 A) & \xrightarrow{X \ast_2 \phi_{Y,A}} X \ast_2 (Y \ast_2 FA) \\
F((X \otimes Y) \ast_1 A) & \xrightarrow{\phi_{X \otimes Y,A}} (X \otimes Y) \ast_2 FA,
\end{align*}
\]

**Definition 2.4.** A functor \( G: \mathcal{K}_2 \to \mathcal{K}_1 \), between \( \mathcal{T} \)-modules \( \mathcal{K}_2 \) and \( \mathcal{K}_1 \), is called hom-preserving if there is a natural isomorphism \( \psi: [-, G(-)]_1 \to G[-, -]_2 \) between the functors \([-, G(-)]_1, G[-, -]_2: \mathcal{T}^{\text{op}} \times \mathcal{K}_2 \to \mathcal{K}_1 \) such that, for all \( X, Y \in \mathcal{T} \) and for all \( B \in \mathcal{K}_2 \), the following diagrams commute:

\[
\begin{align*}
[X, [Y, GB]_1]_1 & \xrightarrow{[X, \psi_{Y,B}]_1} [X, G[Y, B]_2]_1 \xrightarrow{\psi_{X,[Y,B]_2}} G[X, [Y, B]_2]_2 \\
[X \otimes Y, GB]_1 & \xrightarrow{\psi_{X \otimes Y, B}} G[X \otimes Y, B]_2,
\end{align*}
\]

**Lemma 2.5.** Let \( \mathcal{T} \) and \( \mathcal{K} \) be triangulated categories with \( \mathcal{T} \) compactly generated and let \( F_1, F_2: \mathcal{T} \to \mathcal{K} \) be coproduct-preserving triangulated functors (or contravariant triangulated functors that send coproducts to products). If there is a natural transformation \( \theta: F_1 \to F_2 \) such that \( \theta_x \) is an isomorphism, for all \( x \in \mathcal{T}^c \), then \( \theta \) is a natural isomorphism.
Proof. The subcategory $\mathcal{X} = \{ X \in \mathcal{T} \mid \theta_X: F_1 X \to F_2 X \text{ is an isomorphism} \}$ is a localizing subcategory of $\mathcal{T}$ that contains $\mathcal{T}^c$. Consequently, $\mathcal{X} = \mathcal{T}$ and this proves the statement. □

Lemma 2.6. Let $F: \mathcal{K}_1 \to \mathcal{K}_2$ be a coproduct and action-preserving triangulated functor between $\mathcal{T}$-modules and let $G: \mathcal{K}_2 \to \mathcal{K}_1$ be the right adjoint to $F$. Then $G$ is hom-preserving. If $G$ is coproduct-preserving, then $G$ is action-preserving. If $F$ is product-preserving, then $F$ is hom-preserving.

Proof. We denote by $\eta: \text{Id}_{\mathcal{K}_1} \to GF$ and $\varepsilon: FG \to \text{Id}_{\mathcal{K}_2}$ the unit and the counit, respectively, of the adjunction $F \dashv G$. Let $A \in \mathcal{K}_1$, $B \in \mathcal{K}_2$ and $X \in \mathcal{T}$. Then

$$\text{Hom}_{\mathcal{K}_1}(A, [X, GB]_1) \cong \text{Hom}_{\mathcal{K}_1}(X \star A, GB)$$
$$\cong \text{Hom}_{\mathcal{K}_2}(F(X \star A), B)$$
$$\cong \text{Hom}_{\mathcal{K}_2}(X \star FA, B)$$
$$\cong \text{Hom}_{\mathcal{K}_2}(FA, [X, B]_2)$$
$$\cong \text{Hom}_{\mathcal{K}_1}(A, G[X, B]_2).$$

For $A = [X, GB]_1$, the image of the identity morphism on $[X, GB]_1$ under the above series of isomorphisms provides a natural (in both variables) isomorphism $\psi_{X,B}: [X, GB]_1 \to G[X, B]_2$ that satisfies the conditions of Definition 2.4, showing that $G$ is hom-preserving. More precisely, $\psi_{X,B}$ is the following composite:

$$[X, GB]_1 \xrightarrow{\eta_{[X, GB]_1}} GF[X, GB]_1 \xrightarrow{G\psi_{X,F[X, GB]_1}} G[X, X \star_2 F[X, GB]_1]_2$$

$$G[X, F(X \star_1 [X, GB]_1)]_2 \xrightarrow{G[X,F]_1} G[X, FGB]_2$$

$$G[X, \varepsilon B]_2 \xrightarrow{G}[X, B]_2.$$

Now suppose that $G$ preserves coproducts. We define a natural transformation $\xi_{X,B}: X \star_1 GB \to G(X \star_2 B)$ as the composite:

$$X \star_1 GB \xrightarrow{\eta_{X \star_1 GB}} GF(X \star_1 GB) \xrightarrow{G\phi_{X,GB}} G(X \star_2 FGB) \xrightarrow{G(X \star_2 \varepsilon B)} G(X \star_2 B).$$

We claim that the square

$$\begin{array}{ccc}
X \star_1 GB & \xrightarrow{\xi_{X,B}} & G(X \star_2 B) \\
\downarrow \text{ev}_{X,GB}^{\downarrow} & & \downarrow \text{Gev}_{X,B}^{\downarrow} \\
[X^\vee, GB]_1 & \xrightarrow{\psi_{X^\vee,B}} & G[X^\vee, B]_2
\end{array}$$

(2.7)
commutes. First, square (2.7) can be expanded as follows:

\[
\begin{align*}
X *_1 GB & \xrightarrow{\eta_{X*1,GB}} GF(X *_1 GB) \xrightarrow{Gev^1_{X,GB}} G(X *_2 FGB) \xrightarrow{G(X *_2 FB)} G(X *_2 B) \\
\xrightarrow{ev^1_{X,GB}} & (1) \xrightarrow{Gev^1_{X,GB}} G(X *_2 FGB) \xrightarrow{G(X *_2 FB)} G(X *_2 B) \\
[X^\vee,GB]_1 & \xrightarrow{\eta_{X*,GB}} GF[X^\vee,GB]_1 \xrightarrow{2} G[X^\vee,FGB]_2 \xrightarrow{G[X^\vee,FB]} G[X^\vee,B]_2 \\
& \xrightarrow{Gev^1_{X^\vee,GB}} G[X^\vee,F(X^\vee,GB)]_2 \xrightarrow{G[X^\vee,F(X^\vee,GB)]_2} G[X^\vee,F(X^\vee,GB)]_2 \xrightarrow{G[X^\vee,F(X^\vee,GB)]_2} G[X^\vee,F(X^\vee,GB)]_2 \\
& \xrightarrow{Gev^1_{X^\vee,GB}} G[X^\vee,F(X^\vee,GB)]_2 \xrightarrow{G[X^\vee,F(X^\vee,GB)]_2} G[X^\vee,F(X^\vee,GB)]_2 \\
& \xrightarrow{Gev^1_{X^\vee,GB}} G[X^\vee,F(X^\vee,GB)]_2 \xrightarrow{G[X^\vee,F(X^\vee,GB)]_2} G[X^\vee,F(X^\vee,GB)]_2 \\
& \xrightarrow{Gev^1_{X^\vee,GB}} G[X^\vee,F(X^\vee,GB)]_2 \xrightarrow{G[X^\vee,F(X^\vee,GB)]_2} G[X^\vee,F(X^\vee,GB)]_2
\end{align*}
\]

where square (1) commutes by naturality of \( \eta \) and square (3) commutes by naturality of \( ev^1_{X^\vee,GB} \). Therefore, in order to show that (2.7) commutes, it suffices to show that diagram (2) commutes. We will prove this slightly more generally. We claim that the following diagram commutes:

\[
\begin{align*}
F(X *_1 A) & \xrightarrow{\phi_{X^\vee,A}} X *_2 FA \\
F[X^\vee,A]_1 & \xrightarrow{ev^1_{X^\vee,A}} [X^\vee,FA]_2 \\
[X^\vee,F[X^\vee,A]]_1 & \xrightarrow{u^1_{X^\vee,F[X^\vee,A]}_1} [X^\vee,F(X^\vee,A)]_2 \\
[X^\vee,F[X^\vee,A]]_1 & \xrightarrow{[X^\vee,F(X^\vee,A)]_2} [X^\vee,F(X^\vee,A)]_2.
\end{align*}
\]

Set

\[
\begin{align*}
f_1 & = u^1_{X^\vee,F[X^\vee,X^\vee*1(X*1,A)_1]} \quad g_1 = [X^\vee,\phi_{X^\vee,X^\vee*1(X*1,A)_1}^{-1}]_1 \quad \text{[2]} \\
f_2 & = u^1_{X^\vee,F[X^\vee,(X^\vee*1,X^\vee*1)_1]} \quad g_2 = [X^\vee,\phi_{X^\vee,X^\vee*1(X^\vee*1,X^\vee*1)_1}^{-1}]_1 \quad \text{[2]} \\
f_3 & = u^1_{X^\vee,F[X^\vee,(X^\vee*1)_1]} \quad g_3 = [X^\vee,\phi_{X^\vee,X^\vee*1(X^\vee*1)_1}^{-1}]_1 \quad \text{[2]} \\
f_4 & = u^1_{X^\vee,F[X^\vee,1+1]} \quad g_4 = [X^\vee,\phi_{X^\vee,X^\vee*1(X^\vee*1)_1}^{-1}]_1 \quad \text{[2]} \\
f_5 & = u^1_{X^\vee,F[X^\vee,A]} \quad g_5 = [X^\vee,\phi_{X^\vee,X^\vee*1(X^\vee*1)_1}^{-1}]_1 \quad \text{[2]} \\
\end{align*}
\]

\[
\begin{align*}
h_1 & = [X^\vee,X^\vee*2 \phi_{X,A}]_2 \circ [X^\vee,\phi_{X^\vee,X^\vee*1(X*1,A)_1}]_2 \circ [X^\vee,Fc^1_{X^\vee*1(X*1,A)_1}]_2 \quad \text{[2]} \\
h_2 & = [X^\vee,\phi_{X^\vee,X^\vee*1(X*1,A)_1}]_2 \circ [X^\vee,Fc^1_{X^\vee*1(X^\vee*1,X^\vee*1)_1}]_2 \quad \text{[2]} \\
h_3 & = [X^\vee,\phi_{X^\vee*1(X^\vee*1)_1}]_2 \circ [X^\vee,Fc^1_{X^\vee*1(X^\vee*1)_1}]_2 \quad \text{[2]} \\
h_4 & = [X^\vee,\phi_{X,A}]_2 \circ [X^\vee,Fc^1_{X^\vee,1+1}]_2 \quad \text{[2]} \\
h_5 & = [X^\vee,Fc^1_{X^\vee,A}]]_2 \quad \text{[2]}
\end{align*}
\]
and expand diagram (2.8) as below:

\[
F(X_1, A) \xrightarrow{\phi_{A,A}} X_2, FA \\
\downarrow \quad \downarrow \\
F(X^i, X^s_1(X_1, A)) \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F(X^i, X^s_1(X_1, A)) \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F[X^i, (X^s_1(X_1, A)] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F[X^i, (X^s_1(X_1, A)] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F[X^i, (X^s_1(X_1, A)] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F[X^i, (X^s_1(X_1, A)] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F[X^i, (X^s_1(X_1, A)] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F[X^i, (X^s_1(X_1, A)] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])] \\
\downarrow \quad \downarrow \\
F[X^i, (X^s_1(X_1, A)] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2F[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, F(X^s_1[X^i, X^s_1(X_1, A)]] \xrightarrow{\phi_{X_1, A}} [X^i, X^s_2([X^i, FA])]
\]

It is clear that the squares (1), (2), ..., (12) commute. Thus, it remains to show that \(u_{X^i, X^s_1FA}^{2} \circ \phi_{X, A} = h_1 \circ g_1 \circ f_1 \circ F_{X^i, X^s_1 A}^{1} \). Indeed,

\[
h_1 \circ g_1 \circ f_1 \circ F_{X^i, X^s_1 A}^{1} = [X^i, X^s_2 \phi_{X, A}]_2 \circ [X^i, \phi_{X^i, X^s_1 A}]_2 \\
\circ [X^i, F_{X^i, X^s_1 A}^{1}]_2 \circ [X^i, \phi_{X^i, X^s_1 A}]_2 \\
\circ u_{X^i, A}^{2} \circ F_{X^i, X^s_1 A}^{1} \\
= [X^i, X^s_2 \phi_{X, A}]_2 \circ [X^i, \phi_{X^i, X^s_1 A}]_2 \\
\circ [X^i, F_{X^i, X^s_1 A}^{1}]_2 \circ [X^i, \phi_{X^i, X^s_1 A}]_2 \\
\circ [X^i, X^s_2 F_{X^i, X^s_1 A}^{1}]_2 \circ u_{X^i, A}^{2} \circ F_{X^i, X^s_1 A}^{1} \\
= [X^i, X^s_2 \phi_{X, A}]_2 \circ [X^i, \phi_{X^i, X^s_1 A}]_2 \\
\circ u_{X^i, A}^{2} \circ F_{X^i, X^s_1 A}^{1} \\
= u_{X^i, X^s_2 A}^{2} \circ \phi_{X, A}.
\]

We conclude that diagram (2.8) commutes and, as a result, square (2.7) commutes. If \( X \in \mathcal{T} \), then by [Stel13, Lemma 4.6], ev_{X, \_}^{1} is an isomorphism. Consequently, the restriction of \( \xi_{\_B} : - *_2 GB \to G(\_ *_2 B) \) to the compact objects of \( \mathcal{T} \) is a natural isomorphism. Since the triangulated functors \( - *_1 GB \) and \( G(\_ *_2 B) \) are coproduct-preserving, it follows by Lemma 2.5 that \( \xi_{\_B} \) is a natural isomorphism. It is easy to verify that the conditions of Definition 2.3 are satisfied. We conclude that \( G \) is action-preserving.
The proof that if $F$ preserves products, then $F$ is hom-preserving is similar and left to the interested reader. □

Let $\mathcal{K}$ be a $\mathcal{T}$-module. A subcategory $\mathcal{L} \subseteq \mathcal{K}$ is called a localizing submodule if $\mathcal{L}$ is a localizing subcategory such that $X \ast A \in \mathcal{L}$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{L}$. The collection of localizing submodules of $\mathcal{K}$ is denoted by $\text{Loc}^{\ast}(\mathcal{K})$. A subcategory $\mathcal{C} \subseteq \mathcal{K}$ is called a colocalizing hom-submodule if $\mathcal{C}$ is a colocalizing subcategory such that $[X, A]_{\ast} \in \mathcal{C}$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{C}$. The collection of colocalizing hom-submodules of $\mathcal{K}$ is denoted by $\text{Coloc}^{\ast}(\mathcal{K})$. Let $A$ be an object of $\mathcal{K}$. The localizing (resp. colocalizing) submodule of $\mathcal{K}$ generated (resp. cogenerated) by $A$, i.e., the smallest localizing (resp. colocalizing) submodule of $\mathcal{K}$ that contains $A$, is denoted by $\text{loc}^{\ast}(A)$ (resp. $\text{coloc}^{\ast}(A)$). Specializing to the case $\mathcal{K} = \mathcal{T}$ and $\ast = \otimes$, we obtain the notions of localizing tensor-ideal and colocalizing left hom-ideal.

The subcategory $\text{Ann}_{\mathcal{T}}(\mathcal{X}) := \{X \in \mathcal{T} \mid X \ast = 0\} = \bigcap_{A \in \mathcal{K}} \text{Ker}(\ast A) \subseteq \mathcal{T}$ is a localizing tensor-ideal of $\mathcal{T}$, which is called the annihilator of $\mathcal{X}$ in $\mathcal{T}$. If $\text{Ann}_{\mathcal{T}}(\mathcal{X}) = 0$ (for instance, when $\mathcal{K} = \mathcal{T}$ and $\ast = \otimes$) then $\mathcal{X}$ is called a conservative $\mathcal{T}$-module.

**Lemma 2.9.** Let $\mathcal{X}_{1}, \mathcal{X}_{2}$ be two $\mathcal{T}$-modules and let $A \subseteq \text{Ob}(\mathcal{X}_{1})$ and $B \subseteq \text{Ob}(\mathcal{X}_{2})$.

(a) If $F : \mathcal{X}_{1} \to \mathcal{X}_{2}$ is a coproduct and action-preserving triangulated functor, then $F(\text{loc}^{\ast}(A)) \subseteq \text{loc}^{\ast}(FA)$.

(b) If $F : \mathcal{X}_{1}^{\text{op}} \to \mathcal{X}_{2}$ is a triangulated functor that sends coproducts to products and $F(X \ast A) \cong [X, FA]_{\ast}$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{X}_{1}$, then $F(\text{loc}^{\ast}(A)) \subseteq \text{coloc}^{\ast}(FA)$.

(c) If $G : \mathcal{X}_{2} \to \mathcal{X}_{1}$ is a product and hom-preserving triangulated functor, then $G(\text{coloc}^{\ast}(B)) \subseteq \text{coloc}^{\ast}(GB)$.

**Proof.** We will prove (a). The subcategory $\mathcal{X} = \{A \in \mathcal{X}_{1} \mid FA \in \text{loc}^{\ast}(FA)\}$ is a localizing submodule of $\mathcal{X}_{1}$ that contains $A$. Therefore, $\mathcal{X}$ contains $\text{loc}^{\ast}(A)$, proving the statement. The proofs of (b) and (c) are similar. □

2.B. Balmer–Favi support. Let $\text{Spc}(\mathcal{T}^{c})$ be the Balmer spectrum; see [Bal05]. Then a point $p \in \text{Spc}(\mathcal{T}^{c})$ is called visible (weakly visible in [BHS23]) if there exist Thomason subsets $V, W$ of $\text{Spc}(\mathcal{T}^{c})$ such that $\{p\} = V \cap (\text{Spc}(\mathcal{T}^{c}) \setminus W)$ [BF11, Ste13]. Particularly, if $\text{Spc}(\mathcal{T}^{c})$ is noetherian, then every point of $\text{Spc}(\mathcal{T}^{c})$ is visible. The subsets $V$ and $W$ correspond to thick tensor-ideals $\mathcal{T}_{V}$, $\mathcal{T}_{W}$ of compact objects. Let $\mathcal{T}_{V} = \text{loc}^{\otimes}(\mathcal{T}_{V})$ and $\mathcal{T}_{W} = \text{loc}^{\otimes}(\mathcal{T}_{W})$. (It should be noted that the localizing subcategories generated by $\mathcal{T}_{V}$ and $\mathcal{T}_{W}$ are already tensor-ideals.) Since the ideals $\mathcal{T}_{V}$ and $\mathcal{T}_{W}$ are compactly generated, they are smashing ideals [Mil92]. Therefore, they have associated left and right idempotents $e_{V}, f_{V}$ and $e_{W}, f_{W}$, respectively. Let $g_{p} = e_{V} \otimes f_{W}$. Then the objects $\{g_{p} \mid p \in \text{Spc}(\mathcal{T}^{c}) \text{ and } p \text{ is visible}\}$ are pairwise-orthogonal tensor-idempotents. Let $X$ be an object of $\mathcal{T}$ and assume that all points of $\text{Spc}(\mathcal{T}^{c})$ are visible. Then the Balmer–Favi support of $X$ is $\text{Supp}(X) = \{p \in \text{Spc}(\mathcal{T}^{c}) \mid g_{p} \otimes X \neq 0\}$.

**Lemma 2.10** ([BF11, Proposition 7.17]). The map $\text{Supp}(\ast) : \text{Ob}(\mathcal{T}) \to \mathcal{P}(\text{Spc}(\mathcal{T}^{c}))$, $X \mapsto \text{Supp}(X)$ satisfies the following properties:

(a) $\text{Supp}(0) = \emptyset$ & $\text{Supp}(1) = \text{Spc}(\mathcal{T}^{c})$.

(b) $\text{Supp}(\bigcup_{i \in J} X_{i}) = \bigcup_{i \in J} \text{Supp}(X_{i})$.

(c) $\text{Supp}(\Sigma X) = \text{Supp}(X)$. 

Supp(Y) ⊆ Supp(X) ∪ Supp(Z), for any triangle X → Y → Z.
(e) Supp(X ⊗ Y) ⊆ Supp(X) ∩ Supp(Y).
(f) Supp(x ⊗ y) = Supp(x) ∩ Supp(y), ∀x, y ∈ Tc.

Let V ⊆ Spc(Tc) be a Thomason subset with complement U. The quotient T/TV is denoted by T(U). Since TV is smashing, T(U) is a big tt-category and moreover, Spc(T(U)c) ≃ U; see [BF07, Proposition 1.11].

2.C. Smashing support. Following [BS23], we briefly recall some facts concerning the smashing spectrum of a big tt-category T. We denote by S°(T) the lattice of smashing tensor-ideals of T. Then, under the hypothesis that S°(T) is a spatial frame, there is a space Spc°(T) associated with S°(T) via Stone duality, called the smashing spectrum of T that consists of the meet-prime smashing ideals of T, i.e., those smashing ideals P such that S₁ ∩ S₂ ⊆ P ⇒ S₁ ⊆ P or S₂ ⊆ P, ∀S₁, S₂ ∈ S°(T). The open subsets of Spc°(T) stand in bijection with the smashing tensor-ideals and are of the form Us = { P ∈ Spc°(T) | S ∉ P }, where S ∈ S°(T). The closed subsets of Spc°(T) are of the form Vs = { P ∈ Spc°(T) | S ⊆ P }. A point P ∈ Spc°(T) is called locally closed if { P } = Us ∩ Vs, for some smashing ideals S, R. If P is locally closed, then the ideal R can be replaced by P in the sense that { P } = Us ∩ Vp. Each smashing ideal S corresponds to a left Rickard idempotent eS and a right Rickard idempotent fS, which are the images of the tensor-unit of T under the associated acyclization and localization functors, respectively; see [BF11]. If P is locally closed and { P } = Us ∩ Vp, then the Rickard idempotent associated with P is ΓP = eS ⊗ fP. If every point of Spc°(T) is locally closed, then Spc°(T) is called TD.

Let X be an object of T. The big smashing support of X is the subset supp°(X) = { P ∈ Spc°(T) | X ∉ P }. If Spc°(T) is TD, then the small smashing support of X is Supp°(X) = { P ∈ Spc°(T) | ΓP ⊗ X ≠ 0 }. It holds that Supp°(X) ⊆ supp°(X), with the two being equal if X ∈ Tc. The analogous properties stated in the following lemma hold for the small smashing support as well.

Lemma 2.11 ([BS23, Lemma 3.2.8]). The map supp°(-) : Ob(T) → P(Spc°(T)), X → supp°(X) satisfies the following properties:
(a) supp°(0) = ∅ & supp°(1) = Spc°(T).
(b) supp°(⊔E⊂I Xᵢ) = ∪E⊂I supp°(Xᵢ).
(c) supp°(ΣX) = supp°(X).
(d) supp°(Y) ⊆ supp°(X) ∪ supp°(Z), for any triangle X → Y → Z.
(e) supp°(X ⊗ Y) ⊆ supp°(X) ∩ supp°(Y).
(f) supp°(x ⊗ y) = supp°(x) ∩ supp°(y), ∀x, y ∈ Tc.

There is a surjective continuous map ψ : Spc°(T) → Spc(Tc)c, where Spc(Tc)c denotes the Hochster dual of Spc(Tc), that sends P ∈ Spc°(T) to P ∩ Tc. According to [BS23, Corollary 5.1.5], the map ψ is a homeomorphism if and only if T satisfies the Telescope Conjecture (meaning that every smashing ideal is generated by compact objects). For further discussion on the smashing spectrum and related concerns, see also [Ver23].

Hypothesis 2.12. Throughout the paper, whenever we state results concerning the smashing spectrum Spc°(T), we will always assume that the frame S°(T) of smashing ideals of T is a spatial frame.
3. Stratification–costratification

Fix a big tt-category $\mathcal{T}$ and a $\mathcal{T}$-module $\mathcal{K}$. We always assume that $\mathcal{K}$ is compactly generated. Let us recall some well-known facts concerning Brown–Comenetz duals of compact objects. Let $x$ be a compact object of $\mathcal{T}$. Then $H_x := \text{Hom}_\mathcal{T}(\text{Hom}_\mathcal{T}(x,-), \mathbb{Q}/\mathbb{Z}) : \mathcal{T}^{\text{op}} \to \text{Ab}$ is a cohomological functor that sends coproducts to products. So, by Brown Representability, $H_x$ is representable. The representing object of $H_x$ is denoted by $Ix$ and is called the Brown–Comenetz dual of $x$. The functor $\text{Hom}_\mathcal{T}(-,Ix)_{|\mathcal{T}_x}$ is an injective object of $\text{Mod}(\mathcal{T}^e)$, the abelian category of additive functors $\{\mathcal{T}^e\}^{\text{op}} \to \text{Ab}$; see [Nee98]. Hence, by [Kra00, Theorem 1.8] (see also [Bel00, Theorem 8.6]) $Ix$ is pure-injective. Choosing a skeleton for the subcategory of compact objects, the product of the associated Brown–Comenetz duals is denoted by $I$. Being a product of pure-injective objects, $I$ is also pure-injective. Using the fact that $\mathcal{T}$ is compactly generated, one can easily check that $I$ is a cogenerator of $\mathcal{T}$ in the sense that $\text{Hom}_\mathcal{T}(X,\Sigma^nI) = 0$, $\forall n \in \mathbb{Z}$ implies that $X = 0$. It holds that $\mathcal{T} = \text{coloc}(I)$. This follows from the fact that the Brown–Comenetz duals of the compact objects form a perfect cogenerating set for $\mathcal{T}$; see [Kra02]. We use the symbol $I_\mathcal{T}$ if there is any possibility for confusion. Similarly, $\mathcal{K}$ has a pure-injective cogenerator $I_\mathcal{K}$.

One could choose any object $X \in \mathcal{T}$ (resp. $A \in \mathcal{K}$) such that $\mathcal{T} = \text{coloc}(X)$ (resp. $\mathcal{K} = \text{coloc}(A)$) and state everything that follows with respect to $X$ (resp. $A$) instead of $I_\mathcal{T}$ (resp. $I_\mathcal{K}$). The discussion above simply provides a concrete construction of such cogenerators.

3.A. Support–cosupport. Fix a topological space $S$.

Definition 3.1 (See also [BHS23, Definition 7.1]). A support data for $\mathcal{T}$ with values in $S$ is a map $s : \text{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ that satisfies the following properties:

(a) $s(0) = \emptyset$ & $s(1) = S$.
(b) $s(\prod X_i) = \bigcup s(X_i)$.
(c) $s(\Sigma X) = s(X)$.
(d) $s(Y) \subseteq s(X) \cup s(Z)$, for any triangle $X \to Y \to Z$.
(e) $s(X \otimes Y) \subseteq s(X) \cap s(Y)$.

Definition 3.2. A cosupport data for $\mathcal{T}$ with values in $S$ is a map $c : \text{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ that satisfies the following properties:

(a) $c(0) = \emptyset$ & $c(I) = S$.
(b) $c(\prod X_i) = \bigcup c(X_i)$.
(c) $c(\Sigma X) = c(X)$.
(d) $c(Y) \subseteq c(X) \cup c(Z)$, for any triangle $X \to Y \to Z$.
(e) $c([X,Y]) \subseteq c([X,I]) \cap c(Y)$.

Remark 3.3. Let $c : \text{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ be a cosupport data. If $[-,I]$ preserves triangles (see Hypothesis 3.24) it is a straightforward verification, using the properties of $[-,I]$, that setting $s(X) = c([X,I])$ gives rise to a support data $s : \text{Ob}(\mathcal{T}) \to \mathcal{P}(S)$.

Definition 3.4. Let $c : \text{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ be a cosupport data. The support data $s : \text{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ defined by $s(X) = c([X,I])$ is called the support data induced by $c$. We say that $(s,c)$ is a support–cosupport pair.
Lemma 3.5. Let $\Gamma : S \to \text{Ob}(\mathcal{T})$ be a map such that $\Gamma_s := \Gamma(s) \neq 0, \forall s \in S$. Then the maps
\[
\begin{align*}
s_{\Gamma} : \text{Ob}(\mathcal{T}) &\to \mathcal{P}(S), \quad s_{\Gamma}(X) = \{ s \in S \mid \Gamma_s \otimes X \neq 0 \} \\
c_{\Gamma} : \text{Ob}(\mathcal{T}) &\to \mathcal{P}(S), \quad c_{\Gamma}(X) = \{ s \in S \mid [\Gamma_s, X] \neq 0 \}
\end{align*}
\]
are a support and cosupport data, respectively. Moreover, $s_{\Gamma}$ is induced by $c_{\Gamma}$.

Proof. That $s_{\Gamma}$ and $c_{\Gamma}$ are a support and cosupport data, respectively, follows from the fact that $\Gamma_s \otimes -$ is a coproduct-preserving triangulated functor and $[\Gamma_s, -]$ is a product-preserving triangulated functor. Let $X \in \mathcal{T}$. The claim that $s_{\Gamma}$ is induced by $c_{\Gamma}$ follows from the isomorphism $[\Gamma_s, [X, I]] \cong [\Gamma_s \otimes X, I]$ and the fact that $I$ is a cogenerator of $\mathcal{T}$. □

Definition 3.6. A support–cosupport pair $(s_{\Gamma}, c_{\Gamma})$ is called good if it is induced by a map $\Gamma : S \to \text{Ob}(\mathcal{T})$ such that $\Gamma_s \otimes \Gamma_r = 0, \forall s \neq r$ and $\Gamma_s \otimes \Gamma_s \cong \Gamma_s \neq 0, \forall s \in S$.

Remark 3.7. An important feature of a good support–cosupport pair $(s_{\Gamma}, c_{\Gamma})$ is that $s_{\Gamma}(\Gamma_s) = c_{\Gamma}(\Gamma_s, I) = \{ s \}, \forall s \in S$.

Example 3.8. The following are good support–cosupport pairs on $\mathcal{T}$:
\( \text{(a)} \) Assuming that the frame of smashing ideals of $\mathcal{T}$ is a spatial frame, the big smashing support–cosupport pair $(\text{supp}^s, \text{cosupp}^s)$:
\[
\begin{align*}
\text{supp}^s(X) &= \{ P \in \text{Spc}^e(\mathcal{T}) \mid f_P \otimes X \neq 0 \}, \\
\text{cosupp}^s(X) &= \{ P \in \text{Spc}^e(\mathcal{T}) \mid [f_P, X] \neq 0 \}.
\end{align*}
\]
Since $\text{Ker}(f_P \otimes -) = P$, it holds that $P \in \text{supp}^s(X)$ if and only if $X \not\in P$. Using the equality $P^\perp = \text{Im}(f_P \otimes -)$, one can deduce that $\text{Ker}[f_P, -] = (P^\perp)^\perp$, so $P \in \text{cosupp}^s(X)$ if and only if $X \not\in (P^\perp)^\perp$.
\( \text{(b)} \) Assuming further that $\text{Spc}(\mathcal{T})$ is $T_D$, the small smashing support–cosupport pair $(\text{Supp}^s, \text{Cosupp}^s)$:
\[
\begin{align*}
\text{Supp}^s(X) &= \{ P \in \text{Spc}^e(\mathcal{T}) \mid \Gamma_P \otimes X \neq 0 \}, \\
\text{Cosupp}^s(X) &= \{ P \in \text{Spc}^e(\mathcal{T}) \mid [\Gamma_P, X] \neq 0 \}.
\end{align*}
\]
\( \text{(c)} \) Assuming that every point of $\text{Spc}(\mathcal{T})$ is visible, the Balmer–Favi support–cosupport pair $(\text{Supp}, \text{Cosupp})$:
\[
\begin{align*}
\text{Supp}(X) &= \{ p \in \text{Spec}(\mathcal{T}) \mid g_p \otimes X \neq 0 \}, \\
\text{Cosupp}(X) &= \{ p \in \text{Spec}(\mathcal{T}) \mid [g_p, X] \neq 0 \}.
\end{align*}
\]
\( \text{(d)} \) If $R$ is a graded commutative noetherian ring and $\mathcal{T}$ is $R$-linear, the BIK support–cosupport pair $(\text{supp}_R, \text{cosupp}_R)$:
\[
\begin{align*}
\text{supp}_R(X) &= \{ p \in \text{Spec}(R) \mid \Gamma_p 1 \otimes X \neq 0 \}, \\
\text{cosupp}_R(X) &= \{ p \in \text{Spec}(R) \mid [\Gamma_p 1, X] \neq 0 \}.
\end{align*}
\]
See [BIK11a, BIK12].

Remark 3.9. Suppose that $\text{Spc}^e(\mathcal{T})$ is $T_D$ and let $P \in \text{Spc}^e(\mathcal{T})$ with associated idempotent $\Gamma_P \equiv e_S \otimes f_P$. If $X$ is an object of $\mathcal{T}$ such that $P \in \text{Cosupp}^s(X)$, then $0 \neq [\Gamma_P, X] = [e_S \otimes f_P, X] \cong [e_S, [f_P, X]]$. Hence, $[f_P, X] \neq 0$. In other words, $P \in \text{cosupp}^s(X)$. This shows that $\text{Cosupp}^s(X) \subseteq \text{cosupp}^s(X), \forall X \in \mathcal{T}$.
Lemma 3.10. Assuming that $\text{Spec}^c(\mathcal{T})$ is $T_D$, the small and big smashing cosupports coincide, i.e., $\text{Cosupp}^c(X) = \text{cosupp}^c(X)$, $\forall X \in \mathcal{T}$, if and only if every point of $\text{Spec}^c(\mathcal{T})$ is closed, i.e., $\text{Spec}^c(\mathcal{T})$ is $T_1$.

Proof. Since $\text{Supp}^c(\cdot) = \text{Cosupp}^c([\cdot, I])$ and $\text{supp}^c(\cdot) = \text{cosupp}^c([\cdot, I])$, if the small and big smashing cosupports coincide, then so do the small and big smashing supports. By [Ver23, Lemma 4.28], it follows that $\text{Spec}^c(\mathcal{T})$ is $T_1$. Conversely, if $\text{Spec}^c(\mathcal{T})$ is $T_1$ and $P \in \text{Spec}^c(\mathcal{T})$, then $V_P = \{P\}$. This implies that $\Gamma_P = f_P$. So, $[\Gamma_P, X] = 0$ if and only if $[f_P, X] = 0$, for all $X \in \mathcal{T}$. Hence, $\text{Cosupp} = \text{cosupp}^c$. □

Remark 3.11. Assume that $\text{Spec}^c(\mathcal{T})$ is $T_D$ and consider the small smashing support–cosupport. Then $\text{Cosupp}^c(1) = \{P \in \text{Spec}^c(\mathcal{T}) \mid [\Gamma_P, 1] \neq 0\}$. There are many cases where $\text{Cosupp}^c(1) \neq \text{Spec}^c(\mathcal{T})$. For instance, $\text{Cosupp}^c(Z_p) = \{(p)\} \neq \text{Spec}^c(D(Z_p)) \cong \text{Spec}(Z_p) = \{(0), (p)\}$. For more examples and results concerning the cosupport in derived categories of commutative noetherian rings, see [Tho18].

Let $(s_{\Gamma}, c_{\Gamma})$ be a support–cosupport pair on $\mathcal{T}$ induced by a map $\Gamma: S \to \text{Ob}(\mathcal{T})$ and define the maps

$$s_{\Gamma}^*: \text{Ob}(\mathcal{X}) \to \mathcal{P}(S), s_{\Gamma}^*(A) = \{s \in S \mid \Gamma_s \ast A \neq 0\},$$

$$c_{\Gamma}^*: \text{Ob}(\mathcal{X}) \to \mathcal{P}(S), c_{\Gamma}^*(A) = \{s \in S \mid [\Gamma_s, A] \neq 0\}.$$

Lemma 3.12. The maps $s_{\Gamma}^*$ and $c_{\Gamma}^*$ satisfy the following properties:

(a) $s_{\Gamma}^*(0) = \emptyset$.
(b) $s_{\Gamma}^*(\amalg_i A_i) = \bigcup s_{\Gamma}^*(A_i)$.
(c) $s_{\Gamma}^*(\Sigma A) = s_{\Gamma}^*(A)$.
(d) $s_{\Gamma}^*(B) \subseteq s_{\Gamma}^*(A) \cup s_{\Gamma}^*(C)$, for any triangle $A \to B \to C$ of $\mathcal{X}$.
(e) $s_{\Gamma}^*(C \ast A) \subseteq s_{\Gamma}(X) \cap s_{\Gamma}^*(A)$.
(f) $c_{\Gamma}^*(0) = \emptyset$.
(g) $c_{\Gamma}^*(\amalg_i A_i) = \bigcup c_{\Gamma}^*(A_i)$.
(h) $c_{\Gamma}^*(\Sigma A) = c_{\Gamma}^*(A)$.
(i) $c_{\Gamma}^*(B) \subseteq c_{\Gamma}^*(A) \cup c_{\Gamma}^*(C)$, for any triangle $A \to B \to C$ of $\mathcal{X}$.
(j) $c_{\Gamma}^*([X, A]_\ast) \subseteq c_{\Gamma}^*([X, I_{\mathcal{T}}]) \cap c_{\Gamma}^*(A)$.

Proof. The argument is essentially the same as the one given in Lemma 3.5. The property $c_{\Gamma}^*([X, A]_\ast) \subseteq c_{\Gamma}^*([X, I_{\mathcal{T}}]) \cap c_{\Gamma}^*(A)$ follows from Lemma 2.1. □

3.B. The (co)local-to-global principle and (co)minimality. We define two pairs of inclusion-preserving maps

$$\mathcal{P}(S) \xrightarrow{\tau_{s_{\Gamma}^*}} \text{Loc}^c(\mathcal{X}) \quad \& \quad \mathcal{P}(S) \xrightarrow{\tau_{c_{\Gamma}^*}} \text{Coloc}^{\text{hom}}(\mathcal{X})$$

by the formulas

$$\tau_{s_{\Gamma}^*}(W) = \{A \in \mathcal{X} \mid s_{\Gamma}^*(A) \subseteq W\} \quad \& \quad \sigma_{s_{\Gamma}^*}(\mathcal{L}) = \bigcup_{A \in \mathcal{L}} s_{\Gamma}^*(A),$$

$$\tau_{c_{\Gamma}^*}(W) = \{A \in \mathcal{X} \mid c_{\Gamma}^*(A) \subseteq W\} \quad \& \quad \sigma_{c_{\Gamma}^*}(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} c_{\Gamma}^*(A).$$

It is clear from the properties of $s_{\Gamma}^*$ and $c_{\Gamma}^*$ that the maps $\tau_{s_{\Gamma}^*}, \sigma_{s_{\Gamma}^*}, \tau_{c_{\Gamma}^*}, \sigma_{c_{\Gamma}^*}$ are well-defined. Moreover, $\text{Im} \sigma_{s_{\Gamma}^*} \subseteq \mathcal{P}(\sigma_{s_{\Gamma}^*}(\mathcal{X}))$ and $\text{Im} \sigma_{c_{\Gamma}^*} \subseteq \mathcal{P}(\sigma_{c_{\Gamma}^*}(\mathcal{X}))$. In fact, $\sigma_{s_{\Gamma}^*}(\mathcal{X}) = \sigma_{c_{\Gamma}^*}(\mathcal{X}) = c_{\Gamma}^*(I_{\mathcal{X}})$. The first equality follows from the adjunction $\Gamma_s \ast - \dashv$
Remark 3.14. Since we will always work with a fixed support–cosupport pair induced by a map $\Gamma: S \to \text{Ob}(\mathcal{T})$, we will omit the reference to $\Gamma$ in Definition 3.13 and say “$\mathcal{K}$ is stratified” and “$\mathcal{K}$ is costratified”, respectively. We will mention explicit support–cosupport pairs where appropriate.

Example 3.15 ([Kra05, Ste14b]). Let $R$ be a commutative noetherian ring. Then the singularity category $S(R) = \text{K}_{ac}(\text{Inj} R)$, which is the homotopy category of acyclic complexes of injective $R$-modules, is a compactly generated triangulated category and there is an action $*: D(R) \times S(R) \to S(R)$: If $X \in D(R)$ and $A \in S(R)$, then $X * A = \tilde{X} \otimes_R A$, where $\tilde{X}$ is a K-flat resolution of $X$. In this instance, $\sigma_{\Gamma*}(S(R))$ is $\text{Sing}(R)$ the singular locus of $R$. If $R$ is locally a hypersurface, then $S(R)$ is stratified by the action of $D(R)$: There is a bijective correspondence between localizing subcategories of $S(R)$ and subsets of $\text{Sing}(R)$; see [Ste14b, Theorem 6.13].

Definition 3.16 (For (a), see [Ste13, Definition 6.1]).
(a) $\mathcal{K}$ satisfies the local-to-global principle if

$$\text{loc}^*(A) = \text{loc}^*(\Gamma_s * A \mid s \in S), \forall A \in \mathcal{K}.$$ 

(b) $\mathcal{K}$ satisfies minimality if, for all $s \in S$, $\text{loc}^*(\Gamma_s * A \mid A \in \mathcal{K})$ is minimal in $\text{Loc}^*(\mathcal{K})$ in the sense that it does not contain any non-zero proper localizing submodule of $\mathcal{K}$.

(i) $\mathcal{K}$ satisfies the colocal-to-global principle if

$$\text{coloc}^\text{hom}(A) = \text{coloc}^\text{hom}(\Gamma_s, A) \mid s \in S), \forall A \in \mathcal{K}.$$

(ii) $\mathcal{K}$ satisfies cominimality if, for all $s \in S$, $\text{coloc}^\text{hom}(\Gamma_s, I\mathcal{K})$ is minimal in $\text{Coloc}^\text{hom}(\mathcal{K})$ in the sense that it does not contain any non-zero proper colocalizing hom-submodule of $\mathcal{K}$.

Remark 3.17. Let $X \in \mathcal{J}$. Since $\mathcal{K} = \text{coloc}(I\mathcal{K}) = \text{coloc}^\text{hom}(I\mathcal{K})$, by Lemma 2.9 for the functor $[\mathcal{K}, -]_s$, we have $\text{coloc}^\text{hom}([\mathcal{K}, I\mathcal{K}]_s) = \text{coloc}^\text{hom}([X, A]_s \mid A \in \mathcal{K})$. In particular, $\text{coloc}^\text{hom}([\Gamma_s, I\mathcal{K}]_s) = \text{coloc}^\text{hom}([\Gamma_s, A]_s \mid A \in \mathcal{K})$, $\forall s \in S$.

Remark 3.18. It is clear from the definition of $s^*_\Gamma$ and $c^*_\Gamma$ that

$$\text{loc}^*(\Gamma_s * A \mid s \in S) = \text{loc}^*(\Gamma_s * A \mid s \in s^*_\Gamma(A)),$$

$$\text{coloc}^\text{hom}(\Gamma_s, A)_s \mid s \in S) = \text{coloc}^\text{hom}(\Gamma_s, A)_s \mid s \in c^*_\Gamma(A)).$$

In addition, if $\mathcal{K}$ satisfies the local-to-global (resp. colocal-to-global) principle, then $s^*_\Gamma$ (resp. $c^*_\Gamma$) detects vanishing, i.e., $s^*_\Gamma(A) = \emptyset \Rightarrow A = 0$ and similarly for $c^*_\Gamma$. For the case $\mathcal{K} = \mathcal{J}$ and $\ast = \otimes$, it holds that codetection implies detection, since $\emptyset = s_T(X) = c_T([X, I])$ implies $[X, I] = 0$, so $X = 0$. 

[Γ_s, −]_s. The second equality is a special case of Lemma 3.19 using the fact that $\mathcal{K} = \text{coloc}^\text{hom}(I\mathcal{K})$. If $\mathcal{K}$ is a conservative $\mathcal{J}$-module, then $\Gamma_s * − \neq 0$, $\forall s \in S$. Hence, in this case, $\sigma_{\Gamma_s}(\mathcal{K}) = S$. For $\mathcal{K} = \mathcal{J}$ and $\ast = \otimes$, we obtain the maps $\tau_{\Gamma_s}, \sigma_{\Gamma_s}, \tau_{\Gamma_s}, \sigma_{\Gamma_s}$. 

Definition 3.13.

(a) $\mathcal{K}$ is stratified by $\Gamma$ if $\tau_{\Gamma_s}$ and $\sigma_{\Gamma_s}$, between $\mathcal{P}(c^*_\Gamma(I\mathcal{K}))$ and $\text{Loc}^*(\mathcal{K})$, are mutually inverse bijections.

(b) $\mathcal{K}$ is costratified by $\Gamma$ if $\tau^*_{\Gamma_s}$ and $\sigma^*_{\Gamma_s}$, between $\mathcal{P}(c^*_\Gamma(I\mathcal{K}))$ and $\text{Coloc}^\text{hom}(\mathcal{K})$, are mutually inverse bijections.
In particular, if

It holds that

Proof. We will prove the case of \( \sigma_{c_T^*} \). Let \( s \) be an element of \( S \). Then

\[
\begin{align*}
\forall A \in \mathcal{A} \quad s \notin \bigcup_{A \in \mathcal{A}} c_T^*(A) & \iff A \subseteq \ker[\Gamma_s, -]_*, \\
\iff \coloc_{\text{hom}}(A) \subseteq \ker[\Gamma_s, -]_* & \iff s \notin \bigcup_{A \in \colon \text{coloc}_{\text{hom}}(A)} c_T^*(A) \equiv \sigma_{c_T^*}(\coloc_{\text{hom}}(A)).
\end{align*}
\]

Remark 3.20. If \( (s_T, c_T) \) is a good support-cosupport pair on \( T \), then it holds that \( c_T^*([\Gamma_s, A]_*) \subseteq \{s\} \). Hence, if \( [\Gamma_s, A]_* \neq 0 \) (i.e., \( s \in c_T^*([\Gamma_s, A]_*) \)) then \( c_T^*([\Gamma_s, A]_*) = \{s\} \). In particular, if \( s \in c_T^*([\Gamma_s, I_K]_*) \), then \( c_T^*([\Gamma_s, I_K]_*) = \{s\} \).

Lemma 3.21. It holds that \( \sigma_{c_T^*} \circ \tau_{c_T^*} = \text{Id} \) and \( \sigma_{c_T^*} \circ \tau_{c_T^*} = \text{Id} \), where both composites are restricted to \( \mathcal{P}(c_T^*(I_K)) \). In particular, the respective restrictions of \( \tau_{c_T^*} \) and \( \tau_{c_T^*} \) are injective, while \( \sigma_{c_T^*} \) and \( \sigma_{c_T^*} \) are surjective.

Proof. We will prove that \( \sigma_{c_T^*} \circ \tau_{c_T^*} = \text{Id} \) (restricted to \( \mathcal{P}(c_T^*(I_K)) \)). Let \( W \) be a subset of \( c_T^*(I_K) \). Clearly \( (\sigma_{c_T^*} \circ \tau_{c_T^*})(W) \subseteq W \), since \( (\sigma_{c_T^*} \circ \tau_{c_T^*})(W) = \bigcup_{c_T^*(A) \subseteq W} c_T^*(A) \). Let \( s \) be an element of \( W \). Then \( s \in c_T^*([\Gamma_s, I_K]_*) = \{s\} \subseteq W \). Therefore, \( s \in (\sigma_{c_T^*} \circ \tau_{c_T^*})(W) \), completing the proof.

Theorem 3.22. Let \( (s_T, c_T) \) be a good support-cosupport pair on \( T \).

(a) \( \mathcal{K} \) is stratified with respect to \( (s_T, c_T) \) if and only if \( \mathcal{K} \) satisfies the local-to-global principle and minimality.

(b) \( \mathcal{K} \) is costratified with respect to \( (s_T, c_T) \) if and only if \( \mathcal{K} \) satisfies the colocal-to-global principle and cominimality.

Proof. We will only prove (b), since (a) is proved analogously. Suppose that \( \mathcal{K} \) is costratified. Then \( \sigma_{c_T^*} \) is injective. Let \( A \) be an object of \( \mathcal{K} \). Then

\[
\begin{align*}
\sigma_{c_T^*}(\coloc_{\text{hom}}([\Gamma_s, A]_* | s \in c_T^*(A))) & = \bigcup_{s \in c_T^*(A)} c_T^*([\Gamma_s, A]_*) \\
& = \bigcup_{s \in c_T^*(A)} \{s\} \\
& = c_T^*(A) \\
& = \sigma_{c_T^*}(\coloc_{\text{hom}}(A)),
\end{align*}
\]

where the first and last equalities are due to Lemma 3.19. Since \( \sigma_{c_T^*} \) is injective, it follows that \( \coloc_{\text{hom}}([\Gamma_s, A]_* | s \in c_T^*(A)) = \coloc_{\text{hom}}(A) \). Thus, \( \mathcal{K} \) satisfies the colocal-to-global principle. In particular, \( c_T^* \) detects vanishing.

Let \( s \) be an element of \( c_T^*(I_K) \) and \( A \) a non-zero object in \( \coloc_{\text{hom}}([\Gamma_s, I_K]_*) \). Then \( \emptyset \neq c_T^*(A) \subseteq c_T^*([\Gamma_s, I_K]_*) = \{s\} \). Therefore, \( c_T^*(A) = c_T^*([\Gamma_s, I_K]_*) \). Since \( \sigma_{c_T^*} \) is injective, \( \coloc_{\text{hom}}(A) = \coloc_{\text{hom}}([\Gamma_s, I_K]_*) \). Hence, \( \coloc_{\text{hom}}([\Gamma_s, I_K]_*) \) is minimal.
Suppose that \( \mathcal{K} \) satisfies the colocal-to-global principle and cominimality. Let \( \mathfrak{C} \in \text{Coloc}^{\text{hom}}(\mathcal{K}) \). Clearly, \( \mathfrak{C} \subseteq (\tau_\mathfrak{C} \circ \sigma_\mathfrak{C})(\mathfrak{C}) \). Let \( A \in (\tau_\mathfrak{C} \circ \sigma_\mathfrak{C})(\mathfrak{C}) \), i.e., \( c_\mathfrak{C}^*(A) \subseteq \sigma_\mathfrak{C}^*(\mathfrak{C}) \). Then

\[
\text{coloc}^{\text{hom}}(A) = \text{coloc}^{\text{hom}}\left( [\Gamma_s, A]_s \mid s \in c_\mathfrak{C}^*(A) \right) \\
\subseteq \text{coloc}^{\text{hom}}\left( [\Gamma_s, I_{\mathcal{K}}]_s \mid s \in c_\mathfrak{C}^*(A) \right) \\
\subseteq \text{coloc}^{\text{hom}}\left( [\Gamma_s, I_{\mathcal{K}}]_s \mid s \in \sigma_\mathfrak{C}^*(\mathfrak{C}) \right) \\
\subseteq \mathfrak{C}.
\]

The first equality is due to the colocal-to-global principle. The first containment relation follows from Remark 3.17, while the second containment relation is clear. For the third containment, if \( s \in \sigma_\mathfrak{C}^*(\mathfrak{C}) \), then there exists an object \( B \in \mathcal{C} \) such that \( [\Gamma_s, B]_s \neq 0 \). Since \( [\Gamma_s, B]_s \in \text{coloc}^{\text{hom}}([\Gamma_s, I_{\mathcal{K}}]_s) \) and the latter is minimal, it follows that \( \text{coloc}^{\text{hom}}([\Gamma_s, I_{\mathcal{K}}]_s) = \text{coloc}^{\text{hom}}([\Gamma_s, B]_s) \subseteq \mathfrak{C} \). We infer that \( A \in \mathfrak{C} \), proving that \( (\tau_\mathfrak{C} \circ \sigma_\mathfrak{C})(\mathfrak{C}) = \mathfrak{C} \). So, \( \sigma_\mathfrak{C}^* \) is injective and thus, \( \sigma_\mathfrak{C}^* \) is bijective. This shows that \( \mathcal{K} \) is costratified. \( \Box \)

**Remark 3.23.** Theorem 3.22 (b) could be stated slightly more generally, replacing a good support–cosupport pair \((s_\Gamma, c_\Gamma)\), in the sense of Definition 3.6, with one that satisfies the property stated in Remark 3.20, i.e., if \( A \) is an object of \( \mathcal{K} \) such that \( [\Gamma_s, A]_s \neq 0 \), then \( c_\Gamma^*([\Gamma_s, A]_s) = \{s\} \). Similarly, the analogous property for Theorem 3.22 (a) is: if \( A \) is an object of \( \mathcal{K} \) such that \( \Gamma_s \ast A \neq 0 \), then \( s_\Gamma^*([\Gamma_s \ast A]_s) = \{s\} \). This observation will be useful in Section 6, where we consider the support–cosupport for objects of the derived category of a commutative noetherian ring defined by the residue fields.

3.C. **Local-to-global implies colocal-to-global.** Let \((s_\Gamma, c_\Gamma)\) be a (not necessarily good) support–cosupport pair on \( \mathcal{T} \). For our next result, we need an additional assumption.

**Hypothesis 3.24.** We further assume that the relative internal-hom of \( \mathcal{K} \) is a triangulated functor in the first variable, i.e., \([-,-]_s: \mathcal{T}^{\text{op}} \to \mathcal{K} \) preserves triangles, for all \( A \in \mathcal{K} \). This is true, e.g., if \( \mathcal{K} \) satisfies a formulation of May’s TC3 axiom ([May01]) replacing the tensor product of \( \mathcal{T} \) with the action of \( \mathcal{T} \) on \( \mathcal{K} \). The proof of [Mur07, Theorem C.1] goes through verbatim. Our assumption is satisfied by all known examples.

**Lemma 3.25.** Suppose that \( \mathcal{T} = \text{loc}^{\circ}(G) \). Then the following hold:

(a) \( \text{loc}^*(A) = \text{loc}^*(G \ast A), \forall A \in \mathcal{K} \).

(b) \( \text{coloc}^{\text{hom}}(A) = \text{coloc}^{\text{hom}}([G, A]_s), \forall A \in \mathcal{K} \) (under Hypothesis 3.24).

**Proof.** We will prove (b). The inclusion \( \text{coloc}^{\text{hom}}([G, A]_s) \subseteq \text{coloc}^{\text{hom}}(A) \) is clear. Since \( \mathcal{T} = \text{loc}^{\circ}(G) \), it holds that \( 1 \in \text{loc}^{\circ}(G) \). By Lemma 2.9 for the functor \([-,-]_s \), it follows that \( A \cong [1, A]_s \in \text{coloc}^{\text{hom}}([G, A]_s) \). Therefore, \( \text{coloc}^{\text{hom}}(A) \subseteq \text{coloc}^{\text{hom}}([G, A]_s) \). The proof of (a) is analogous. \( \Box \)

**Remark 3.26.** An easy generalization of Lemma 3.25 is the following: If \( \mathcal{T} = \text{loc}^{\circ}(\mathcal{G}) \), for a collection of objects \( \mathcal{G} \), then \( \forall A \in \mathcal{K}: \text{loc}^*(A) = \text{loc}^*\left( G \ast A \mid G \in \mathcal{G} \right) \) and \( \text{coloc}^{\text{hom}}(A) = \text{coloc}^{\text{hom}}\left( [G, A]_s \mid G \in \mathcal{G} \right) \).
Proposition 3.27 (See also [Ste13, Proposition 6.8]). If \( \mathcal{T} \) satisfies the local-to-global principle, then \( \mathcal{K} \) satisfies the local-to-global principle and (under Hypothesis 3.24) the colocal-to-global principle.

Proof. Since \( \mathcal{T} \) satisfies the local-to-global principle, we have \( \mathcal{T} = \text{loc}^\otimes (\Gamma_s \mid s \in S) \).

Hence, by Remark 3.26, \( \text{loc}^\ast (A) = \text{loc}^\ast (\Gamma_s \ast A \mid s \in S) \) and \( \text{coloc}^\text{hom}(A) = \text{coloc}^\text{hom}(\left[\Gamma_s, A\right]_s \mid s \in S) \), for all \( A \in \mathcal{K} \). This proves the statement. \( \square \)

Corollary 3.28. Under Hypothesis 3.24 for the case \( \mathcal{K} = \mathcal{T} \), if \( \mathcal{T} \) satisfies the local-to-global principle, then \( \mathcal{T} \) satisfies the colocal-to-global principle.

Example 3.29. Let \( R \) be a graded commutative noetherian ring such that \( \mathcal{T} \) is \( R \)-linear and consider the BIK support–cosupport \( (\text{supp}_R, \text{cosupp}_R) \), which takes values in \( \text{supp}_R(1) \subseteq \text{Spec}(R) \) — this may not be an equality. As explained in [BHS23, Corollary 7.11], if \( \mathcal{T} \) is stratified in the sense of BIK, then \( \text{supp}_R(1) \) is homeomorphic to \( \text{Spc}(\mathcal{T}^\circ) \) and the BIK support is identified with the Balmer–Favi support under this homeomorphism. It then follows that \( \mathcal{T} \) is stratified by the Balmer–Favi support. Now since the tensor-idempotents \( \Gamma_p(1) \) (defining the Balmer support) and the tensor-idempotents \( g_p \) (defining the Balmer–Favi support) have the same support (which is \{p\}) it follows that \( \text{loc}^\otimes (\Gamma_p(1)) = \text{loc}^\otimes (g_p) \). Applying Lemma 2.9 for the functor \( [-, I] \) (taking into account Hypothesis 3.24) it follows that \( \text{coloc}^\text{hom}(\left[\Gamma_p, I\right]) = \text{coloc}^\text{hom}(\left[g_p, I\right]) \). By Corollary 3.28, \( \mathcal{T} \) satisfies the colocal-to-global principle with respect to the Balmer–Favi support. Taking into account Theorem 3.22, we conclude that if \( \mathcal{T} \) is BIK-stratified, then: \( \mathcal{T} \) is Balmer–Favi-costratified if and only if \( \mathcal{T} \) is BIK-costratified if and only if \( \text{coloc}^\text{hom}(\left[\Gamma_p, I\right]) \) is minimal, for all \( p \in \text{supp}_R(1) \). If \( \mathcal{T} = \text{Mod}(kG) \) is the stable module category of the group algebra of a finite group \( G \), then \( \mathcal{T} \) is BIK-costratified by the canonical action of \( H^*(G, k) \); see [BIK12, Theorem 11.13]. We infer that \( \text{Mod}(kG) \) is Balmer–Favi-costratified.

4. PRIME SUBMODULES

In this section we introduce the classes of prime localizing submodules and hom-prime colocalizing submodules of a given \( \mathcal{T} \)-module \( \mathcal{X} \). The class of prime localizing submodules generalizes the class of objectwise-prime localizing tensor-ideals [BS23, Ver23] in the context of relative tensor-triangular geometry, while the class of hom-prime colocalizing submodules specializes to the class of hom-prime colocalizing left hom-ideals if \( \mathcal{X} = \mathcal{T} \).

4.A. Prime localizing and colocalizing submodules. As before, \((\text{sp}_p, \text{cp}_p)\) will be a good support-cosupport pair on \( \mathcal{T} \) with values in a space \( S \). Given \( \mathcal{L} \in \text{Loc}^\ast(\mathcal{X}) \) and \( \mathcal{C} \in \text{Coloc}^\text{hom}(\mathcal{X}) \), we define two subcategories of \( \mathcal{T} \) as follows:

\[
\mathcal{L}^\otimes = \{ X \in \mathcal{T} \mid X \ast \mathcal{X} \subseteq \mathcal{L} \},
\]

\[
\mathcal{C}^\otimes = \{ X \in \mathcal{T} \mid [X, \mathcal{X}]_* \subseteq \mathcal{C} \},
\]

where \( X \ast \mathcal{X} := \text{loc}^\ast (X \ast A \mid A \in \mathcal{K}) \) and \( [X, \mathcal{X}]_* := \text{coloc}^\text{hom}([X, A]_* \mid A \in \mathcal{K}) \), with the latter also being equal to \( \text{coloc}^\text{hom}([X, I\mathcal{X}]_*) \). Evidently, if \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \), then \( \mathcal{L}_1^\otimes \subseteq \mathcal{L}_2^\otimes \) and if \( \mathcal{C}_1 \subseteq \mathcal{C}_2 \), then \( \mathcal{C}_1^\otimes \subseteq \mathcal{C}_2^\otimes \).

Remark 4.1. Clearly, \( \mathcal{L}^\otimes \) is a localizing tensor-ideal of \( \mathcal{T} \) and \( \mathcal{C}^\otimes \) is closed under coproducts, suspensions and the tensor product. Under Hypothesis 3.24, \( \mathcal{C}^\otimes \) is also closed under cones and so, \( \mathcal{C}^\otimes \) is a localizing tensor-ideal of \( \mathcal{T} \).
Remark 4.2. If \( \mathcal{K} = \mathcal{T} \) and \( * = \otimes \), then \( \mathcal{L}^{\oplus L} = \mathcal{L} \). The inclusion \( \mathcal{L}^{\oplus L} \subseteq \mathcal{L} \) follows from the equality \( X \otimes \mathcal{T} = \text{loc}(X) \), while the inclusion \( \mathcal{L} \subseteq \mathcal{L}^{\oplus L} \) holds because \( \mathcal{L} \) is a tensor-ideal.

Definition 4.3.
(a) A proper localizing submodule \( \mathcal{L} \subseteq \mathcal{K} \) is called prime if \( X \ast A \in \mathcal{L} \) implies \( X \in \mathcal{L}^{\oplus L} \) or \( A \in \mathcal{L} \).
(b) A proper colocalizing hom-submodule \( \mathcal{C} \subseteq \mathcal{K} \) is called hom-prime if \( [X, A]_s \in \mathcal{C} \) implies \( X \in \mathcal{C}^{\oplus C} \) or \( A \in \mathcal{C} \).

Remark 4.4. If \( \mathcal{K} = \mathcal{T} \) and \( * = \otimes \), then the notion of prime localizing submodule recovers the notion of objectwise-prime localizing tensor-ideal; see Remark 4.2. The notion of hom-prime colocalizing hom-submodule provides the notion of hom-prime colocalizing left hom-ideal.

Lemma 4.5. Let \( \mathcal{L} \) be a prime localizing submodule of \( \mathcal{K} \) and let \( \mathcal{C} \) be a hom-prime colocalizing submodule of \( \mathcal{K} \). Then \( \mathcal{L}^{\oplus L} \) and \( \mathcal{C}^{\oplus C} \) are objectwise-prime, in the sense that if \( X \otimes Y \in \mathcal{L}^{\oplus L} \), then \( X \in \mathcal{L}^{\oplus L} \) or \( Y \in \mathcal{L}^{\oplus L} \) and similarly for \( \mathcal{C}^{\oplus C} \).

Proof. We will prove that \( \mathcal{C}^{\oplus C} \) is objectwise-prime. The proof for \( \mathcal{L}^{\oplus L} \) is analogous. Let \( X, Y \) be objects of \( \mathcal{T} \) such that \( X \otimes Y \in \mathcal{C}^{\oplus C} \). Then \( [X \otimes Y, A]_s \in \mathcal{C}, \forall A \in \mathcal{K} \).

By Lemma 2.1, \( [X, [Y, A]_s] \cong [X \otimes Y, A]_s \). Since \( \mathcal{C} \) is hom-prime, \( X \in \mathcal{C}^{\oplus C} \) or \( [Y, A]_s \in \mathcal{C} \). If \( X \notin \mathcal{C}^{\oplus C} \), then \( [Y, A]_s \in \mathcal{C}, \forall A \in \mathcal{K}, \) i.e., \( Y \in \mathcal{C}^{\oplus C} \). This proves that \( \mathcal{C}^{\oplus C} \) is objectwise-prime.

The main result of this section, i.e., Theorem 4.10, is a consequence of the following series of lemmas.

Lemma 4.6. The following statements hold:
(a) \( \text{Ker}(\Gamma_s \otimes -) \subseteq \text{Ker}(\Gamma_s \ast -)^{\oplus L} = \text{Ker}([\Gamma_s, A, -]_s)^{\oplus C}, \forall s \in S \).
(b) If \( \mathcal{K} \) is conservative, then \( \text{Ker}(\Gamma_s \otimes -) = \text{Ker}(\Gamma_s \ast -)^{\oplus L}, \forall s \in S \).
(c) If \( \text{Ker}(\Gamma_s \otimes -) = \text{Ker}(\Gamma_s \ast -)^{\oplus L}, \forall s \in S \) and \( \text{sr} \) detects vanishing, then \( \mathcal{K} \) is conservative.

Proof. Let \( X \) be an object of \( \mathcal{T} \). Then we have \( X \in \text{Ker}(\Gamma_s \ast -)^{\oplus L} \) if and only if \( \Gamma_s \ast X \ast A \cong (\Gamma_s \otimes X) \ast A = 0, \forall A \in \mathcal{K} \), which is equivalent to \( (\Gamma_s \otimes X) \ast - = 0 \). Similarly, using the isomorphism \( [\Gamma_s \otimes X, -]_s \cong [\Gamma_s, X, -]_s \), one deduces that \( X \in \text{Ker}([\Gamma_s, A, -]_s)^{\oplus C} \) if and only if \( [\Gamma_s \otimes X, -]_s = 0 \). Since \( (\Gamma_s \otimes X) \ast - \cong [\Gamma_s \otimes X, -]_s \), these two functors are either both the zero functor on \( \mathcal{K} \) or none of them is the zero functor. Therefore, \( \text{Ker}(\Gamma_s \ast -)^{\oplus L} = \text{Ker}([\Gamma_s, A, -]_s)^{\oplus C} \). Since \( \text{Ker}(\Gamma_s \ast -)^{\oplus L} = \{ X \in \mathcal{T} \mid (\Gamma_s \otimes X) \ast - = 0 \} \), it immediately follows that \( \text{Ker}(\Gamma_s \otimes -) \subseteq \text{Ker}(\Gamma_s \ast -)^{\oplus L} \). This proves (a).

If \( \mathcal{K} \) is conservative and \( (\Gamma_s \otimes X) \ast - = 0 \), then \( \Gamma_s \otimes X = 0 \). Hence, \( \text{Ker}(\Gamma_s \otimes -) = \text{Ker}(\Gamma_s \ast -)^{\oplus L} \). This proves (b).

Let \( X \in \mathcal{T} \) such that \( X \ast - = 0 \). Then \( (\Gamma_s \otimes X) \ast - \cong X \ast (\Gamma_s \ast -) = 0, \forall s \in S \).

Therefore, \( X \in \text{Ker}(\Gamma_s \ast -)^{\oplus L} \). This implies that \( X \in \text{Ker}(\Gamma_s \otimes -), \forall s \in S \), i.e., \( \Gamma_s \otimes X = 0, \forall s \in S \). Equivalently, \( \text{sr}(X) = \emptyset \). Since \( \text{sr} \) detects vanishing, it follows that \( X = 0 \). This proves (c).

Lemma 4.7.
(a) Let \( \mathcal{L} \) be a prime localizing submodule of \( \mathcal{K} \). There is at most one \( s \in c^*_+(I_x) \) such that \( \mathcal{L} \subseteq \text{Ker}(\Gamma_s \ast -) \).
(b) Let $C$ be a hom-prime colocalizing submodule of $K$. There is at most one $s \in c_I^s(I_K)$ such that $C \subseteq \text{Ker}[\Gamma_s, -]_s$.

**Proof.**

(a) Similar to (b).

(b) Let $s \in c_I^s(I_K)$ and suppose that $C \subseteq \text{Ker}[\Gamma_s, -]_s$. Let $r \in S$ such that $r \neq s$ and let $A \in K$. Then $[\Gamma_s, [\Gamma_r, A]]_s = 0 \in C$. Since $C$ is hom-prime, $\Gamma_s \in C^{\otimes C} \subseteq \text{Ker}[\Gamma_s, -]_s^{\otimes C} = \text{Ker}(\Gamma_s * -)^{\otimes L}$ or $[\Gamma_s, A]_s \in C \subseteq \text{Ker}[\Gamma_s, -]_s$. For the equality $\text{Ker}[\Gamma_s, -]_s^{\otimes C} = \text{Ker}(\Gamma_s * -)^{\otimes L}$, see Lemma 4.6. The former of the two does not hold since $[\Gamma_s, A]_s = 0$ if and only if $[\Gamma_s * - = 0$, but $s \in c_I^s(I_K)$ which means that $[\Gamma_s * - \neq 0$. It follows that $C$ contains all objects $[\Gamma_r, A]_s$, for $r \neq s$ and $A \in K$. So, if $C \subseteq \text{Ker}[\Gamma_r, -]_s$, for $r \neq s$ and $r \in c_I^s(I_K)$, then $[\Gamma_r, A]_s \in \text{Ker}[\Gamma_s, -]_s$, $\forall A \in K$. It follows that $[\Gamma_r, -]_s = 0$; thus, $\Gamma_r * - = 0$, which is false since $r \in c_I^s(I_K)$.

**Lemma 4.8.** If $K$ satisfies the colocal-to-global principle, then it holds $\forall s \in S$: $\text{Ker}[\Gamma_s, -]_s = \text{coloc}^{\text{hom}}([\Gamma_r, I_K]_s \mid r \neq s)$.

**Proof.** Let $r, s \in S$ such that $r \neq s$. Then $[\Gamma_r, I_K]_s \in \text{Ker}[\Gamma_s, -]_s$. Therefore, $\text{coloc}^{\text{hom}}([\Gamma_r, I_K]_s \mid r \neq s) \subseteq \text{Ker}[\Gamma_s, -]_s$. Let $A \in \text{Ker}[\Gamma_s, -]_s$. Then $s \notin c_I^s(A)$. Since $K$ satisfies the colocal-to-global principle,

$$\text{coloc}^{\text{hom}}(A) = \text{coloc}^{\text{hom}}([\Gamma_r, A]_s \mid r \in c_I^s(A))$$

$$= \text{coloc}^{\text{hom}}([\Gamma_r, A]_s \mid r \neq s)$$

$$\subseteq \text{coloc}^{\text{hom}}([\Gamma_r, I_K]_s \mid r \neq s).$$

See Remark 3.18 for the second equality and Remark 3.17 for the containment relation. Hence, $A \in \text{coloc}^{\text{hom}}([\Gamma_r, I_K]_s \mid r \neq s)$, completing the proof.

**Lemma 4.9.** Let $C \in \text{Coloc}^{\text{hom}}(K)$. Then $\tau(c_I^s(C)) = \bigcap_{s \in c_I^s(I_K)} \text{Ker}[\Gamma_s, -]_s$.

If $K$ is costratified, then $C = \bigcap_{s \in \text{Ker}[\Gamma_s, -]_s} \text{Ker}[\Gamma_s, -]_s$.

**Proof.** Let $A \in K$. Then $A \notin \bigcap_{s \in \text{Ker}[\Gamma_s, -]_s} \text{Ker}[\Gamma_s, -]_s$ if and only if there exists $s \in S$ such that $C \subseteq \text{Ker}[\Gamma_s, -]_s$ and $[\Gamma_s, A]_s \neq 0$. Equivalently, $s \notin \sigma(c_I^s(C))$ and $s \in c_I^s(A)$. In other words, $c_I^s(A) \subseteq \sigma(c_I^s(C))$. Since $\tau(c_I^s(C))$ consists precisely of those $A \in K$ such that $c_I^s(A) \subseteq \sigma(C)$, it follows that $\tau(c_I^s(C)) = \bigcap_{s \in \text{Ker}[\Gamma_s, -]_s} \text{Ker}[\Gamma_s, -]_s$. Finally, if $K$ is costratified, then $C = \tau(c_I^s(C))$, which proves the statement (the indexing set of the intersection involved in the claimed equalities can be verified to consist of points $s \in c_I^s(I_K)$ since if $s \notin c_I^s(I_K)$, then $[\Gamma_s, -]_s = 0$ and so $\text{Ker}[\Gamma_s, -]_s = K$ so the intersection is not affected).

**Theorem 4.10.** Let $K$ be a costratified $T$-module. Then there is a bijective correspondence between hom-prime colocalizing submodules of $K$ and points of $c_I^s(I_K)$. A point $s \in c_I^s(I_K)$ is associated with $\text{Ker}[\Gamma_s, -]_s = \text{coloc}^{\text{hom}}([\Gamma_r, I_K]_s \mid r \neq s)$.

**Proof.** Let $C \in \text{Coloc}^{\text{hom}}(K)$ be hom-prime. Then $C = \bigcap_{s \in c_I^s(I_K)} \text{Ker}[\Gamma_s, -]_s$, by Lemma 4.9. It follows by Lemma 4.7 that $C$ must be contained in $\text{Ker}[\Gamma_s, -]_s$, for a unique $s \in c_I^s(I_K)$. Conclusion: $C = \text{Ker}[\Gamma_s, -]_s$, for a unique $s \in c_I^s(I_K)$. The equality $\text{Ker}[\Gamma_s, -]_s = \text{coloc}^{\text{hom}}([\Gamma_r, I_K]_s \mid r \neq s)$ was proved in Lemma 4.8.
The following observation, which is of independent interest and will not play a role in the sequel, showcases a conceptual similarity between the theory of actions of tensor-triangulated categories and the theory of associated primes of modules over rings. To see this, recall the following result: If \( R \) is a ring and \( M \) is a non-zero \( R \)-module such that for every non-zero submodule \( N \subseteq M \), it holds that \( \text{Ann}_R(M) = \text{Ann}_R(N) \), then \( \text{Ann}_R(M) \) is a prime ideal of \( R \).

**Proposition 4.11.** Let \( \mathcal{L} \) be a non-zero localizing submodule of \( \mathcal{K} \) such that for every non-zero localizing submodule \( \mathcal{L}' \) of \( \mathcal{K} \) with \( \mathcal{L}' \subseteq \mathcal{L} \), it holds that \( \text{Ann}_\mathcal{T}(\mathcal{L}) = \text{Ann}_\mathcal{T}(\mathcal{L}') \). Then \( \text{Ann}_\mathcal{T}(\mathcal{L}) \) is an objectwise-prime localizing ideal of \( \mathcal{T} \).

**Proof.** Let \( X, Y \in \mathcal{T} \) such that \( X \otimes Y \in \text{Ann}_\mathcal{T}(\mathcal{L}) \). This means that \( (X \otimes Y) \ast \mathcal{L} = 0 \).

4.B. The action and internal-hom formulas.

**Definition 4.12.**

(a) \( \mathcal{K} \) satisfies the Action Formula (AF) if
\[
s_\ast^\mathcal{T}(X \ast A) = s_\mathcal{T}(X) \cap s_\mathcal{T}(A), \quad \forall X \in \mathcal{T}, \, \forall A \in \mathcal{K}.
\]

(b) \( \mathcal{K} \) satisfies the Internal-Hom Formula (IHF) if
\[
c_\ast^\mathcal{T}([X, A]) = c_\mathcal{T}([X, I_\mathcal{T}]) \cap c_\mathcal{T}(A), \quad \forall X \in \mathcal{T}, \, \forall A \in \mathcal{K}.
\]

(Recall that \( c_\mathcal{T}([X, I_\mathcal{T}]) = s_\mathcal{T}(X) \).)

**Lemma 4.13.**

(a) If \( \mathcal{K} \) satisfies the Action Formula, then \( \text{Ker}(\Gamma_s \ast -) \) is a prime localizing submodule, \( \forall s \in S \). If \( \mathcal{K} \) is a conservative \( \mathcal{T} \)-module, then the converse holds.

(b) If \( \mathcal{K} \) satisfies the Internal-Hom Formula, then \( \text{Ker}[\Gamma_s, -]_s \) is a hom-prime colocalizing hom-submodule, \( \forall s \in S \). If \( \mathcal{K} \) is a conservative \( \mathcal{T} \)-module, then the converse holds.

**Proof.**

(a) Similar to (b).

(b) The Internal-Hom Formula can be restated as follows: if \( [\Gamma_s \otimes X, A]_s = 0 \) then \( \Gamma_s \otimes X = 0 \) or \( [\Gamma_s, A]_s = 0 \) — the converse holds by the definition of cosupport. So, if \( [X, A]_s \in \text{Ker}[\Gamma_s, -]_s \), then \( X \in \text{Ker}(\Gamma_s \otimes -) \subseteq [\text{Ker}[\Gamma_s, -]]_s^C \) or \( A \in \text{Ker}[\Gamma_s, -]_s^\circ \); for the first alternative, see Lemma 4.6. This means that \( \text{Ker}[\Gamma_s, -]_s \) is hom-prime. Now if \( [\Gamma_s, -]_s \) is hom-prime and \( \mathcal{K} \) is a conservative \( \mathcal{T} \)-module, then \( \text{Ker}(\Gamma_s \otimes -) = \text{Ker}[\Gamma_s, -]_s^\circ \). Therefore, if \( [\Gamma_s \otimes X, A]_s = 0 \), then \( \Gamma_s \otimes X = 0 \) or \( [\Gamma_s, A]_s = 0 \), which is precisely the statement of the Internal-Hom Formula.

**Proposition 4.14.**

(a) If \( \mathcal{T} \) satisfies minimality, then \( \mathcal{K} \) satisfies the Action Formula and (under Hypothesis 3.24) the Internal-Hom Formula.

(b) If \( \mathcal{K} \) is a conservative \( \mathcal{T} \)-module and \( \mathcal{K} \) satisfies cominimality, then \( \mathcal{K} \) satisfies the Internal-Hom Formula.

(c) If \( \mathcal{T} \) satisfies the Internal-Hom Formula, then \( \mathcal{T} \) satisfies the Action Formula.

**Proof.** Let \( s \in S, \, X \in \mathcal{T}, \, A \in \mathcal{K} \).
(a) If \( s \in \sf(X) \cap \sf^*_s(A) \), then \( \Gamma_s \otimes X \neq 0 \) and \( \Gamma_s \ast A \neq 0 \). Since \( \text{loc}^\otimes(\Gamma_s) \) is minimal, it follows that \( \Gamma_s \in \text{loc}^\otimes(\Gamma_s \otimes X) \). Hence, \( \Gamma_s \ast A \in \text{loc}^\ast((\Gamma_s \otimes X) \ast A) \). Since \( \Gamma_s \ast A \neq 0 \), it holds that \( \Gamma_s \ast (X \ast A) \cong (\Gamma_s \otimes X) \ast A \neq 0 \). In other words, \( s \in \sf^*_s(X \ast A) \). Conclusion: \( \K \) satisfies AF.

Now suppose that \( s \in \sf([X, I_T]) \cap \sf^*_s(A) \). Then \( \Gamma_s \otimes X \neq 0 \) and \( \Gamma_s, A \neq 0 \) (recall that \( \sf([X, I_T]) = \sf(X) \)). Since \( \text{loc}^\otimes(\Gamma_s) \) is minimal, it follows that \( \Gamma_s \in \text{loc}^\otimes(\Gamma_s \otimes X) \). Hence, \( \Gamma_s, A \in \text{coloc}^\text{hom}((\Gamma_s \otimes X, A)_s) \). It follows that \( \Gamma_s \otimes X, A \ast \neq 0 \), i.e., \( s \in \sf^*_s([X, A]_s) \). Conclusion: \( \K \) satisfies IHF.

(b) Suppose that \( s \in \sf([X, I_T]) \cap \sf^*_s(A) \). Then \( \Gamma_s \otimes X \neq 0 \) and \( \Gamma_s, A \neq 0 \). Aiming for contradiction, assume that \( \Gamma_s \otimes X, A \ast = 0 \). Then \( A \in \text{Ker}[\Gamma_s \otimes X, -]_s \).

Thus, \( \Gamma_s \otimes X, [I]_s = \Gamma_s \otimes X, [I]_s \neq 0 \). So, \( \Gamma_s \otimes X, - \ast = 0 \). By the \( \text{coloc}^\text{hom}(\Gamma_s, A)_s = \text{coloc}^\text{hom}(\Gamma_s, A)_s \subset \text{Ker}(\text{loc}^\text{hom}(\Gamma_s, A)_s) \). Hence, \( \text{coloc}^\text{hom}(\Gamma_s, A)_s \neq 0 \), i.e., \( \Gamma_s \otimes X \in \text{Ann}_T(\K) \). Since \( \K \) is a conservative \( T \)-module, \( \Gamma_s \otimes X = 0 \), which is a contradiction. Conclusion: \( \K \) satisfies IHF.

(c) Let \( X, Y \in T \). Then \( \sf([X \otimes Y] = \sf([X, Y, I_T]) = \sf([X, Y, I_T]) \cap \sf([X, Y, I_T]) = \sf([X, Y, I_T]) \cap \sf([X, Y, I_T]) \). Conclusion: IHF implies AF. \( \square \)

Remark 4.15. If \( \K = T \), then the statement of the Action Formula is: \( \sf([X \otimes Y]) = \sf([X, Y, I_T]) \), \( \forall X, Y \in T \). This is known as the Tensor Product Formula (which does not hold in general); see [BF11, BHS23]. See also [Bal20] for a support theory that does satisfy the Tensor Product Formula. On the other hand, the Internal-Hom Formula states: \( \sf([X, Y]) = \sf(X) \cap \sf(Y) \), \( \forall X, Y \in T \). For the BIK support, this is equivalent to stratification of \( T \); see [BIK12, Theorem 9.5].

5. Smashing submodules

Let \( \K \) be a \( T \)-module. Recall our assumption that \( \K \) is compactly generated. A smashing submodule of \( \K \) is a smashing subcategory \( \M \subset \K \) that is also a submodule. Specifically, the quotient functor \( j_M : \K \to \K/M \) is a coproduct-preserving and essentially surjective triangulated functor that has a right adjoint \( k_M : \K/M \to \K \) (which is necessarily fully faithful) that preserves coproducts — and products since it is a right adjoint. By Brown representability, \( k_M \) has a right adjoint \( \ell_M : \K \to \K/M \) (which is necessarily essentially surjective) that preserves products. By the relations \( j_M k_M \cong \text{Id} \cong \ell_M k_M \), it follows that \( j_M \) and \( \ell_M \) take the same values on the image of \( k_M \), which is \( \M \perp \). The set of smashing submodules of \( \K \) is denoted by \( \S^\perp(\K) \).

Next we describe the action of \( T \) on \( \K/M \) induced by the action of \( T \) on \( \K \). The category \( T \times \K/M \) is a triangulated category that is the quotient of \( T \times \K \) over \( 0 \times \M \), with the quotient functor \( T \times \K \to T \times \K/M \) being \( \text{Id}_T \times j_M \). Since \( 0 \times \M \) is contained in the kernel of \( j_M \circ * \), it follows that \( j_M \circ * \) factors through \( T \times \K/M \) via a functor \( * : T \times \K/M \to \K/M \). It is straightforward to check that this functor is an action of \( T \) on \( \K/M \). If \( X \in T \) and \( A \in \K/M \), then \( X \ast A = j_M(X \ast B) \). The functor \( j_M : \K \to \K/M \) is action-preserving. We denote by \( [\ast, -]_s : T \times \K/M \to \K/M \) the relative internal-hom of \( \K/M \). By Lemma 2.6, \( k_M \) is action and hom-preserving and \( \ell_M \) is hom-preserving. Moreover, since \( I_\K \) (the product of the Brown–Comenetz duals of the compact objects of \( \K \)) is a pure-injective cogenerator of \( \K \) and \( \ell_M \) is an essentially surjective right adjoint, it follows that \( \ell_M(I_\K) \) is a pure-injective cogenerator of \( \K/M \). In particular, \( \K/M = \text{coloc}(\ell_M(I_\K)) \).
Now we describe the colocalizing hom-submodules of $\mathcal{K}/\mathcal{M}$. The functor $k_{\mathcal{M}}$ gives a bijective correspondence between the colocalizing subcategories of $\mathcal{K}/\mathcal{M}$ and the colocalizing subcategories of $\mathcal{K}$ contained in $\mathcal{M}^\perp$. Since $k_{\mathcal{M}}$ is hom-preserving, this bijection restricts to colocalizing hom-submodules, i.e., the maps

$$\text{Coloc}^{\text{hom}}(\mathcal{K}/\mathcal{M}) \xrightarrow{k_{\mathcal{M}}} \{ \mathcal{C} \in \text{Coloc}^{\text{hom}}(\mathcal{K}) \mid \mathcal{C} \subseteq \mathcal{M}^\perp \} \quad (5.1)$$

are mutually inverse inclusion-preserving bijections. An observation that will be useful in the sequel is that $k_{\mathcal{M}}\text{coloc}^{\text{hom}}(j_{\mathcal{M}}(A)) = \text{coloc}^{\text{hom}}(k_{\mathcal{M}}j_{\mathcal{M}}(A))$, $\forall A \in \mathcal{K}$.

Let $(s_\Gamma, c_\Gamma)$ be a good support–cosupport pair on $\mathcal{T}$. We denote the induced support–cosupport on $\mathcal{K}/\mathcal{M}$ by $(s_{\Gamma}^M, c_{\Gamma}^M)$. Specifically,

$$s_{\Gamma}^M(j_{\mathcal{M}}(A)) = \{ s \in S \mid j_{\mathcal{M}}(\Gamma_s * A) \neq 0 \},$$

$$c_{\Gamma}^M(j_{\mathcal{M}}(A)) = \{ s \in S \mid [\Gamma_s, j_{\mathcal{M}}(A)]_s \neq 0 \}.$$

Then $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle if

$$\text{coloc}^{\text{hom}}(j_{\mathcal{M}}(A)) = \text{coloc}^{\text{hom}}([\Gamma_s, j_{\mathcal{M}}(A)]_s \mid s \in S), \forall A \in \mathcal{K}$$

and $\mathcal{K}/\mathcal{M}$ satisfies cominimality if $\text{coloc}^{\text{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_s)$ is a minimal colocalizing hom-submodule of $\mathcal{K}/\mathcal{M}$ for all $s \in S$. Finally, let $S_{\mathcal{M}} = \{ s \in S \mid [\Gamma_s, I_{\mathcal{K}}]_s \in \mathcal{M}^\perp \}$.

**Proposition 5.2.** Let $\mathcal{M} \in S^*(\mathcal{K})$. The following are equivalent:

(a) $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle.

(b) $\text{coloc}^{\text{hom}}(B) = \text{coloc}^{\text{hom}}([\Gamma_s, B]_s \mid s \in S), \forall B \in \mathcal{M}^\perp$.

As a result, if $\mathcal{K}$ satisfies the colocal-to-global principle, then $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle.

**Proof.** Let $A$ be an object of $\mathcal{K}$ and set

$$\mathcal{C}_1 = \text{coloc}^{\text{hom}}(j_{\mathcal{M}}(A)),
\mathcal{C}_2 = \text{coloc}^{\text{hom}}([\Gamma_s, j_{\mathcal{M}}(A)]_s \mid s \in S),
\mathcal{D}_1 = \text{coloc}^{\text{hom}}(k_{\mathcal{M}}j_{\mathcal{M}}(A)),
\mathcal{D}_2 = \text{coloc}^{\text{hom}}([\Gamma_s, k_{\mathcal{M}}j_{\mathcal{M}}(A)]_s \mid s \in S).$$

Under the bijection (5.1), $\mathcal{C}_1$ corresponds to $\mathcal{D}_1$, while $\mathcal{C}_2$ corresponds to $\mathcal{D}_2$ (recall that $k_{\mathcal{M}}$ is hom-preserving). So, if $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle, then $\mathcal{C}_1 = \mathcal{C}_2$. Hence, $\mathcal{D}_1 = \mathcal{D}_2$. Since $\text{Im} k_{\mathcal{M}}j_{\mathcal{M}} = \mathcal{M}^\perp$, (b) follows. On the other hand, if (b) holds, then $\mathcal{D}_1 = \mathcal{D}_2$. As a result, $\mathcal{C}_1 = \mathcal{C}_2$, i.e., $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle. This proves (a).

If $\mathcal{K}$ satisfies the colocal-to-global principle, then

$$\text{coloc}^{\text{hom}}(A) = \text{coloc}^{\text{hom}}([\Gamma_s, A]_s \mid s \in S), \forall A \in \mathcal{K},$$

so the equality certainly holds for $A \in \mathcal{M}^\perp$. Therefore, $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle by the equivalence (a) $\Leftrightarrow$ (b).

**Proposition 5.3.** Suppose that $s \in S_{\mathcal{M}}$. Then $\text{coloc}^{\text{hom}}([\Gamma_s, I_{\mathcal{K}}]_s)$ is a minimal colocalizing hom-submodule of $\mathcal{K}$ if and only if $\text{coloc}^{\text{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_s)$ is a minimal colocalizing hom-submodule of $\mathcal{K}/\mathcal{M}$.\[\square\]
results for the Balmer spectrum and the Balmer–Favi support. What one needs

\[ \text{Corollary 5.6.} \text{ Suppose that } \mathcal{K} \text{ satisfies cominimality if and only if } \mathcal{K}/M_s \text{ satisfies cominimality, for all } s \in S. \]

\[ \text{Proof.} \text{ The result is a direct consequence of Theorem 5.4, taking into account the preceding discussion.} \]

\[ \text{Corollary 5.5.} \text{ Suppose that } \mathcal{T} \text{ satisfies cominimality if and only if } \mathcal{T}/P \text{ satisfies cominimality, for all } P \in \mathcal{T}. \]

\[ \text{Proof.} \text{ The result is a direct consequence of Theorem 5.4, taking into account the preceding discussion.} \]

\[ \text{Corollary 5.6.} \text{ Suppose that } \mathcal{T} \text{ satisfies cominimality if and only if } \mathcal{T}/P \text{ satisfies cominimality, for all } P \in \mathcal{T}. \]

\[ \text{Proof.} \text{ The result is a direct consequence of Theorem 5.4, taking into account the preceding discussion.} \]
Corollary 5.7. Suppose that every point of $\text{Spc}(\mathcal{T}^c)$ is visible. Then:

(a) $\mathcal{T}$ satisfies cominimality if and only if $\mathcal{T}/\text{loc}^0(\mathfrak{p})$ satisfies cominimality, for all $\mathfrak{p} \in \text{Spc}(\mathcal{T}^c)$.

(b) Suppose that $\mathcal{T}$ satisfies the colocal-to-global principle. Then $\mathcal{T}$ is costratified if and only if $\mathcal{T}/\text{loc}^0(\mathfrak{p})$ is costratified, for all $\mathfrak{p} \in \text{Spc}(\mathcal{T}^c)$.

Corollary 5.8. Suppose that every point of $\text{Spc}(\mathcal{T}^c)$ is visible and that $\text{Spc}(\mathcal{T}^c) = \bigcup_{j \in J} U_j$ is a cover of $\text{Spc}(\mathcal{T}^c)$ by complements of Thomason subsets. Let $V_j$ be the complement of $U_j$. If $\mathcal{T}(U_j)$ satisfies cominimality, for all $j \in J$, then $\mathcal{T}$ satisfies cominimality. If, moreover, $\mathcal{T}$ satisfies the colocal-to-global principle, then $\mathcal{T}$ is costratified.

In view of future applications involving singularity categories of schemes, we need a version of Corollary 5.8 for the more general case where $\mathcal{T}$ acts on $\mathcal{X}$. Let $\mathcal{S}$ be a compactly generated localizing tensor-ideal of $\mathcal{T}$ and set $\mathcal{M} = \mathcal{S} \ast \mathcal{X}$. Then $\mathcal{M}$ is a compactly generated localizing submodule of $\mathcal{X}$; see [Ste13, Section 4]. The action of $\mathcal{T}$ on $\mathcal{X}$ induces, as already discussed previously, an action of $\mathcal{T}$ on $\mathcal{X}/\mathcal{M}$. Because of the way $\mathcal{M}$ is defined, it follows that there is an induced action of $\mathcal{T}/\mathcal{S}$ on $\mathcal{X}/\mathcal{M}$ and a colocalizing subcategory of $\mathcal{X}/\mathcal{M}$ is a hom $\mathcal{T}$-submodule if and only if it is a hom $\mathcal{T}/\mathcal{S}$-submodule.

Assuming that every point of $\text{Spc}(\mathcal{T}^c)$ is visible, let $V$ be a Thomason subset of $\text{Spc}(\mathcal{T}^c)$ and let $U = \text{Spc}(\mathcal{T}^c) \setminus V$ and consider the localizing tensor-ideal $\mathcal{T}_V$ generated by those compact objects of $\mathcal{T}$ whose support is contained in $V$. By definition, $\mathcal{T}_V$ is compactly generated and hence smashing, so there are associated left and right (respectively) idempotents $e_V$ and $f_V$ such that $\mathcal{T}_V = \text{loc}^0(e_V) = \text{Ker}(f_V \otimes -) = \text{Im}(e_V \otimes -)$. We denote by $\mathcal{T}(U)$ the category $\mathcal{T}/\mathcal{T}_V$. It holds that $\text{Spc}(\mathcal{T}(U)^c) = U$ and we will treat this homeomorphism as an identification. Let $\mathcal{K}_V = \mathcal{T}_V \ast \mathcal{X}$ and let $\mathcal{K}(U) = \mathcal{X}/\mathcal{K}_V$. By the previous paragraph, $\mathcal{K}_V$ is a compactly generated localizing submodule of $\mathcal{X}$ and there is an induced action of $\mathcal{T}(U)$ on $\mathcal{K}(U)$ such that a colocalizing subcategory of $\mathcal{K}(U)$ is a hom $\mathcal{T}$-submodule if and only if it is a hom $\mathcal{T}(U)$-submodule. Further, $\mathcal{K}_V = \text{Im}(e_V \ast -) = \text{Ker}(f_V \ast -)$ and $\mathcal{K}_V^\perp = \text{Im}[e_V, -] = \text{Ker}[f_V, -]$. By this last observation, it follows that $S_{\mathcal{K}_V} := \{ \mathfrak{p} \in \text{Spc}(\mathcal{T}^c) \mid [g_{\mathfrak{p}}, I_{\mathcal{X}}], \in \mathcal{K}_V^\perp \} = U$.

The following result is the analogue of [Ste13, Theorem 8.11] for colocalizing hom-submodules.

Theorem 5.9. Suppose that every point of $\text{Spc}(\mathcal{T}^c)$ is visible and that $\text{Spc}(\mathcal{T}^c) = \bigcup_{j \in J} U_j$ is a cover of $\text{Spc}(\mathcal{T}^c)$ by complements of Thomason subsets. Let $V_j$ be the complement of $U_j$. If $\mathcal{K}(U_j)$ (as a $\mathcal{T}(U_j)$-module) satisfies cominimality, for all $j \in J$, then $\mathcal{K}$ satisfies cominimality. If, moreover, $\mathcal{K}$ satisfies the colocal-to-global principle, then $\mathcal{K}$ is costratified.

Proof. If $\mathfrak{p} \in \text{Spc}(\mathcal{T}^c)$, then there exists $j_{\mathfrak{p}} \in J$ such that $\mathfrak{p} \in U_{j_{\mathfrak{p}}}$. Fix such a $j_{\mathfrak{p}} \in J$, for each $\mathfrak{p} \in \text{Spc}(\mathcal{T}^c)$. Then we have a collection $\{ \mathcal{K}_{V_{j_{\mathfrak{p}}}} \}_{\mathfrak{p} \in \text{Spc}(\mathcal{T}^c)}$ of
smashing submodules of \( X \) such that \( p \in S_{X_{U_{j_k}}} \) since the latter is equal to \( U_{j_k} \).
The result now follows by an immediate application of Theorem 5.4. \( \square \)

### 6. Derived categories of noetherian rings and schemes

Throughout, \( R \) will denote a commutative noetherian ring. In the article [Nee11], Neeman proved that there is a bijective correspondence between colocalizing subcategories of \( D(R) \) and subsets of \( \text{Spec}(R) \). In this section, we give a more streamlined proof of Neeman’s theorem by using the general machinery we developed; specifically Theorem 3.22 and Corollary 3.28. As a direct consequence, we obtain a complete description of the \( \text{RHom-prime} \) colocalizing subcategories of \( D(R) \) in terms of the residue fields. Further, using Corollary 5.8, we prove that the derived category of quasi-coherent sheaves over a noetherian separated scheme is costratified.

**Remark 6.1.** Let \( X \) be a quasi-compact separated scheme. By [Mur07, Proposition C.13], \( D(X) \) the derived category of quasi-coherent sheaves over \( X \) satisfies Hypothesis 3.24 (with \( D(R) \) being the special case \( X = \text{Spec}(R) \)). In particular, this allows us to apply Corollary 3.28 later.

#### 6.A. Noetherian rings

We will use the cosupport taking values in \( \text{Spec}(R) \) defined by the residue fields \( k(p) \). More specifically, if \( X \in D(R) \), then \( \text{Cosupph}(X) = \{ p \in \text{Spec}(R) \mid \text{RHom}_R(k(p), X) \neq 0 \} \). We use the notation \( \text{Cosupph} \) to avoid conflict with the Balmer–Favi cosupport. Note that since \( D(R) \) is generated by its tensor-unit, every colocalizing subcategory of \( D(R) \) is a left \( \text{RHom-ideal} \). We denote by \( J_R \) the cogenerator of \( D(R) \) that is the product of the Brown-Comenetz duals of the compact objects.

**Lemma 6.2.** Let \( p \in \text{Spec}(R) \). Then \( \text{RHom}_R(k(p), X) \cong \bigoplus_{i \in \mathbb{Z}} \Sigma^i k(p)^{(J_i)} \cong \prod_{i \in \mathbb{Z}} \Sigma^i k(p)^{(J_i)} \), for some sets \( J_i \), for all \( X \in D(R) \). The same holds for the complex \( \text{RHom}_R(X, k(p)) \).

**Proof.** Let \( E \) be a \( K \)-injective resolution of \( X \). Then \( \text{RHom}_R(k(p), X) \) is the Hom-complex \( \text{Hom}_R(k(p), E) \). This is a complex of \( k(p) \)-vector spaces, therefore it must be quasi-isomorphic to its cohomology complex with zero differential (which also has \( k(p) \)-vector spaces as terms; thus coproducts of copies of \( k(p) \)). For \( \text{RHom}_R(X, k(p)) \), pick a \( K \)-projective resolution of \( X \) instead of a \( K \)-injective resolution and argue in an identical manner. \( \square \)

**Lemma 6.3.** Let \( X \) be an object of \( D(R) \) such that \( \text{RHom}_R(k(p), X) \neq 0 \). Then \( \text{coloc}(k(p)) = \text{coloc}(\text{RHom}_R(k(p), X)) \subseteq \text{coloc}(X) \).

**Proof.** It holds that \( \text{RHom}_R(k(p), X) \cong \prod_{i \in \mathbb{Z}} \Sigma^i k(p)^{(J_i)} \). Since \( k(p)^{(J_i)} \rightarrow k(p)^{J_i} \) is a map of \( k(p) \)-vector spaces, it must split. So, \( k(p)^{J_i} \) is a summand of \( k(p)^{J_i} \). This implies that \( k(p)^{(J_i)} \in \text{coloc}(k(p)) \) and consequently, \( \prod_{i \in \mathbb{Z}} \Sigma^i k(p)^{(J_i)} \in \text{coloc}(k(p)) \). Thus, \( \text{coloc}(\text{RHom}_R(k(p), X)) \subseteq \text{coloc}(k(p)) \). By the fact that \( k(p) \) is a summand of \( \text{RHom}_R(k(p), X) \), it follows that \( \text{coloc}(k(p)) \subseteq \text{coloc}(\text{RHom}_R(k(p), X)) \). Since \( D(R) \) is generated by its tensor-unit, every colocalizing subcategory of \( D(R) \) is a left \( \text{RHom-ideal} \). Hence, \( \text{RHom}_R(k(p), X) \in \text{coloc}(X) \). This completes the proof. \( \square \)

**Proposition 6.4.** The category \( D(R) \) satisfies the colocal-to-global principle (in particular, \( \text{Cosupph detects vanishing} \) and, for each \( p \in \text{Spec}(R) \), it holds that \( \text{Cosupph}(k(p)) = \{ p \} \).
Proof. Since $D(R)$ satisfies the local-to-global principle [Nee92], by Corollary 3.28, $D(R)$ satisfies the colocal-to-global principle and, by Remark 3.18, Cosupp detects vanishing. Hence, $D(R)$ satisfies the colocal-to-global principle, $D(R)$ satisfies the colocal-to-global principle and, by Remark 3.18, Cosupp detects vanishing and Cosupp of $D(R)$ is the Balmer–Favi support–cosupport. Therefore, by Theorem 3.22, $D(R)$ satisfies both the colocal-to-global principle and cominimality; so, Theorem 3.22 implies that $D(R)$ is costratified; see also Remark 3.23.

\[ \square \]

Theorem 6.6. The RHom-prime colocalizing subcategories of $D(R)$ correspond to points of $\text{Spec}(R)$. The correspondence is given by associating $p \in \text{Spec}(R)$ with $\text{Ker} \text{RHom}_R(k(p), X) = \text{coloc} \big( \text{RHom}_R(k(p), I_R) \big)$.

Proof. Since $D(R)$ is costratified, and clearly a conservative $D(R)$-module, Theorem 4.10 implies that the RHom-prime colocalizing subcategories of $D(R)$ are precisely of the form $\text{Ker} \text{RHom}_R(k(p), X) = \text{coloc} \big( \text{RHom}_R(k(p), I_R) \big)$, and the claimed equality is due to Lemma 6.3.

\[ \square \]

Remark 6.7. One could also choose to work with the Balmer–Favi support (or the smashing support since $D(R)$ satisfies the Telescope Conjecture [Nee92]; see also [Ver23, Lemma 7.2] and [BS23, Section 6]). There is a homeomorphism between $\text{Spc}(D(R))$ and $\text{Spec}(R)$ [Nee92]. Using this homeomorphism, we can express the Balmer–Favi support–cosupport via $\text{Spec}(R)$. For each $p \in \text{Spec}(R)$, the Balmer–Favi idempotents associated with $p$ is $g_p = K_p \otimes R_p$, where $K_p$ is the stable Koszul complex and $R_p$ is the localization of $R$ at $p$. The objects $g_p$ are orthogonal tensor-idempotents, so they define a support–cosupport pair: Let $X \in D(R)$. Then $\text{Supp}(X) = \{ p \in \text{Spec}(R) \mid g_p \otimes X \neq 0 \}$ and $\text{Cosupp}(X) = \{ p \in \text{Spec}(R) \mid \text{RHom}_R(g_p, X) \neq 0 \}$. It holds that $\text{loc}(g_p) = \text{loc}(k(p))$ [Ste18b, Lemma 3.22]. Therefore, $\text{coloc}(\text{RHom}_R(g_p, X)) = \text{coloc}(\text{RHom}_R(k(p), X)) = \text{coloc}(k(p))$, with the last equality by Lemma 6.3 (provided that $\text{RHom}_R(k(p), X) \neq 0$). Since $D(R)$ is stratified by the Balmer–Favi support [BHS23, Theorem 5.8], in particular it satisfies the local-to-global principle, $D(R)$ must also satisfy the colocal-to-global principle; see Corollary 3.28. The equality $\text{coloc}(\text{RHom}_R(g_p, X)) = \text{coloc}(k(p))$ shows that $D(R)$ satisfies cominimality with respect to the Balmer–Favi support–cosupport. Therefore, by Theorem 3.22, $D(R)$ is costratified with respect to the Balmer–Favi support–cosupport.

Example 6.8 ([Ste14a]). We include an example of a category that is not costratified. Let $R$ be an absolutely flat ring that is not semi-artinian. Then there exists a superdecomposable injective $R$-module $E$. Let $p \in \text{Spec}(R)$. Then $\text{RHom}_R(k(p), E) =$.
Hom_R(k(p), E). If there was a non-zero map k(p) → E, then (as k(p) is simple and injective since R is absolutely flat) k(p) would have to be a summand of E, which leads to a contradiction. This shows that RHom_R(k(p), E) = 0, for all p ∈ Spec(R), i.e., Cosupp(E) = ∅; showcasing the failure of the cosupport to detect vanishing and consequently, the failure of the colocal-to-global principle. As a result, the local-to-global principle cannot hold either, since it implies the colocal-to-global principle.

6.B. Noetherian schemes. Let X be a noetherian separated scheme and denote by D(X) the derived category of quasi-coherent sheaves over X. Then D(X) is a big tt-category whose subcategory of compact objects is D^perf(X) the subcategory of perfect complexes. The Balmer spectrum of D(X) is homeomorphic to the underlying space of X [Tho97]. The notion of support we consider is the Balmer–Favi support. However, since D(X) satisfies the Telescope Conjecture [Ste13], one can also choose either the usual homological support or the small smashing support, as they are all identified under the bijections Spc^*(D(X)) ∼ Spc(D^perf(X)) ∼ X. Theorem 6.9. Let X be a noetherian separated scheme. Then D(X) is costratified.

Proof. By [Ste13, Corollary 8.13], D(X) is stratified. In particular, D(X) satisfies the local-to-global principle. Hence, by Corollary 3.28, D(X) satisfies the colocal-to-global principle. So, it suffices to prove cominimality. Let \{U_i\}_{i∈I} be an open affine cover of X. As X is noetherian, any open subset of X is quasi-compact, so its complement is Thomason. The corresponding smashing localization D(X)(U_i) is equivalent to D(U_i). The latter is costratified (in particular it satisfies cominimality) by Theorem 6.5. The result follows by Corollary 5.8. □

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