RELATIVE COMPLETIONS AND $K_2$ OF CURVES

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Abstract. We compute the completion of the special linear group over the coordinate ring of a curve over a number field $k$ relative to its representation in $SL_n(k)$, and relate this to the study of $K_2$ of the curve.

Introduction

The algebraic $K$-groups of curves defined over number fields are mysterious objects and have received a great deal of scrutiny. There is a plethora of conjectures about the groups $K_1$ and $K_2$ and a few results supporting these guesses. For example, if $A$ is the coordinate ring of a smooth affine curve over a number field $k$, then a conjecture of Vaserstein asserts that $SK_1(A)$ is torsion. This is supported by calculations of Raskind [11], who showed that $SK_1(A) \otimes \mathbb{Q}/\mathbb{Z} = 0$, and by more recent work of Østvaer and Rosenschon [10].

The group $K_2$ is even more troublesome, but the work of several authors ([2], [3], [4], [8], [12]) has shed some light on the structure of $K_2(A)$. For example, the existence of regulator maps shows that nontrivial elements exist; the rank of $K_2(A)$ is conjectured to be related to the number of infinite places of $k$ when $C$ is an elliptic curve; the second level of the rank filtration, $r_2K_2(A)_{\mathbb{Q}}$, vanishes when $C$ is an elliptic curve. Still, it is not known for any curve whether or not $K_2(A)$ has finite rank.

In this note, we use Deligne’s notion of relative completion to study the group $K_2(A)$. A full summary of this construction is given in Section 1, but the basic idea is the following. Let $S$ be a reductive algebraic group over a field $F$ and let $\rho : \Gamma \to S$ be a map of a discrete group $\Gamma$ into $S$ with Zariski dense image. The completion of $\Gamma$ relative to $\rho$ is a proalgebraic group $\mathcal{G}$ over $F$ together with a lift $\tilde{\rho} : \Gamma \to \mathcal{G}$ such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\tilde{\rho}} & \mathcal{G} \\
\downarrow^{\rho} & & \downarrow \\
S & & 
\end{array}
\]

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commutes. Using a suitably defined notion of the continuous cohomology of \( G, H^*_{cts}(G; F) \), there is an injective map
\[
H^2_{cts}(G; F) \longrightarrow H^2(\Gamma; F).
\]
This allows one to obtain a lower bound on the rank of \( H^2(\Gamma; F) \).

Here, we consider the group \( \Gamma = SL_n(A), n \geq 3, \) and \( \rho : SL_n(A) \rightarrow SL_n(k) \) given by reduction modulo the maximal ideal of a \( k \)-rational point on the curve (we assume one exists, for simplicity). The main result of this paper is the following calculation of the completion of \( SL_n(A) \) relative to \( \rho \).

**Theorem 3.1** The completion of \( SL_n(A) \) relative to \( \rho \) is isomorphic to the group \( SL_n(k[[T]]) \).

Unfortunately, this provides no information about the rank of \( H^2(SL_n(A)) \) (which is related to \( K_2(A) \)) in light of the following result.

**Proposition 4.1** For \( n \geq 3 \), \( H^2_{cts}(SL_n(k[[T]]); k) = 0 \).

Thus, to obtain a complete understanding of \( K_2(A) \), one needs to compute the group \( H_2(SL_n(A); \mathbb{Z}), n \geq 3 \). Such a calculation remains elusive, however.

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**Notation.** Throughout this paper, \( k \) denotes a number field and \( C \) denotes a smooth affine curve over \( k \). We denote the coordinate ring \( k[C] \) by \( A \). We assume, for simplicity, that \( C \) has a \( k \)-rational point \( x \) and we denote the associated maximal ideal of \( A \) by \( \mathfrak{m}_x \). We also assume that \( SK_1(A) \otimes \mathbb{Q} = 0 \). For a group \( G \), \( \Gamma^i G \) denotes the \( i \)-th term of the lower central series.

**Remark.** The results of this paper are also valid for finite fields, but the interest in that case is subsumed by Quillen’s calculations \([6]\).

1. **Relative Completions**

Relative completion is a generalization of the classical Malcev (or unipotent) completion. Proofs of the results in this section may be found in \([7]\) or \([9]\).

Let \( \Gamma \) be a discrete group and let \( S \) be a reductive linear algebraic group over a field \( F \). Suppose that \( \rho : \Gamma \rightarrow S \) is a homomorphism with Zariski dense image. The completion of \( \Gamma \) relative to \( \rho \) is a proalgebraic group \( \mathcal{G} \), defined over \( F \), which is an extension of \( S \) by a prounipotent group \( \mathcal{U} \),
\[
1 \rightarrow \mathcal{U} \rightarrow \mathcal{G} \rightarrow S \rightarrow 1,
\]
together with a lift \( \tilde{\rho} : \Gamma \rightarrow \mathcal{G} \) of \( \rho \). The group \( \mathcal{G} \) is required to satisfy the obvious universal mapping property. If \( S \) is the trivial group then this reduces to the usual unipotent completion (first defined by Malcev in the case \( F = \mathbb{Q} \)).
Examples of this construction may be found in [9]. We recall some relevant facts that will be used below. Suppose \( S \) is trivial so that we are considering the unipotent \( F \)-completion.

**Proposition 1.1.** Let \( G \) be a group with a filtration
\[
G = G^1 \supseteq G^2 \supseteq G^3 \supseteq \cdots
\]
satisfying the following conditions.

1. The graded quotients \( G^i/G^{i+1} \) are finite-dimensional \( F \)-vector spaces.
2. For all \( i \), \( (G^i/\Gamma^i G) \otimes F = 0 \).

Then \( \mathcal{U} = \varprojlim G/G^i \) is the unipotent \( F \)-completion of \( G \).

**Proof.** This is a modification of Proposition 3.5 of [9]. The unipotent \( F \)-completion of \( G \) is obtained as the inverse limit of the \( F \)-completions of each \( G/\Gamma^i G \). Since \( (G^i/\Gamma^i G) \otimes F = 0 \) for all \( i \), and since \( G^i/G^{i+1} \) is an \( F \)-vector space, the completion of \( G/\Gamma^i G \) over \( F \) is the group \( G/G^i \). Thus, \( \mathcal{U} = \varprojlim G/G^i \) is the unipotent \( F \)-completion of \( G \). \( \square \)

The extension
\[
1 \longrightarrow \mathcal{U} \longrightarrow G \longrightarrow S \longrightarrow 1
\]
is split ([9], p. 195). An obvious question to consider is the relationship between the group \( \mathcal{U} \), which is prounipotent, and the unipotent completion of the kernel of \( \rho : \Gamma \rightarrow S \). Denote this kernel by \( T \) and let \( \mathcal{T} \) be its unipotent \( F \)-completion. Then the map \( T \rightarrow \mathcal{U} \) induces a map \( \Phi : \mathcal{T} \rightarrow \mathcal{U} \). Let \( L \) be the image of \( \rho \).

**Proposition 1.2.** Suppose that \( H_1(T; F) \) is finite-dimensional. If the action of \( L \) on \( H_1(T; F) \) extends to a rational representation of \( S \) (for example, if \( L = S \)), then the kernel of \( \Phi \) is central in \( \mathcal{T} \).

**Proof.** See [9], Proposition 4.2. \( \square \)

**Proposition 1.3.** Suppose that \( H_1(T; F) \) is finite-dimensional. If \( \rho : \Gamma \rightarrow S \) is surjective, then \( \Phi : \mathcal{T} \rightarrow \mathcal{U} \) is surjective.

**Proof.** See [9], Proposition 4.3. \( \square \)

### 2. Continuous Cohomology

Suppose that \( \pi \) is a projective limit of groups,
\[
\pi = \varprojlim \pi_\alpha,
\]
and let \( F \) be a field. We define the continuous cohomology of \( \pi \) to be
\[
H^i_{cts}(\pi; F) = \varinjlim H^i(\pi_\alpha; F).
\]
For example, if \( \pi \) is the Galois group of a field extension \( L/K \), then \( H^i_{cts}(\pi; F) \) is simply the usual Galois cohomology.

This construction is relevant here in light of the following result.
Proposition 2.1. Let $\rho : \Gamma \to S$ be a split surjective representation and let $G$ be the completion relative to $\rho$. Assume that $H_1(T;F)$ is finite-dimensional ($T = \ker \rho$) and that $\Phi : T \to U$ is an isomorphism. Then the restriction map

$$H^2_{\text{cts}}(G;F) \to H^2(\Gamma;F)$$

is injective.

Proof. See [9], Corollary 5.5. \qed

3. The Completion of $SL_n(A)$

Recall that $k$ denotes a number field and $C$ is a smooth affine curve over $k$. The coordinate ring of $C$ is denoted by $A$. Assume that $x$ is a $k$-rational point of $C$ and let $m_x$ be the associated maximal ideal of $A$. We also assume that $SK_1(A) \otimes \mathbb{Q} = 0$. With these assumptions, we now proceed to compute the completion of $SL_n(A)$, $n \geq 3$.

Let $\rho : SL_n(A) \to SL_n(k)$ be the map induced by reducing modulo $m_x$. Note that $\rho$ is split surjective. Denote by $\hat{A}$ the $m_x$-adic completion of $A$; this is a complete regular local ring of dimension 1 and is thus isomorphic to the power series ring $k[[T]]$.

Theorem 3.1. The completion of $SL_n(A)$ relative to $\rho$ is the group $SL_n(\hat{A})$.

Proof. Let $K$ be the kernel of $\rho$ and define a central series $K^\bullet$ by

$$K^i = \{X \in SL_n(A) : X \equiv I \mod m_x^i \}.$$ 

Then for $n \geq 3$, we have isomorphisms [11]

$$K^i/E_n(A,m_x^i) \cong SK_1(A,m_x^i)$$

for all $i$. Here, $E_n$ denotes the subgroup generated by elementary matrices. By considering the long exact $K$-theory sequence associated to $(A,m_x^i)$, our hypothesis that $SK_1(A) \otimes \mathbb{Q} = 0$ yields a surjective map

$$K_2(A/m_x^i) \otimes \mathbb{Q} \to SK_1(A,m_x^i) \otimes \mathbb{Q}.$$ 

By choosing a uniformizing parameter $t$ at $x$, we obtain an isomorphism of rings $A/m_x^i \cong k[t]/t^i$. Note that for all $i$,

$$K_2(k[t]/t^i) \cong K_2(k[t]/t^i, (t)) \oplus K_2(k) \cong (\Omega_{k[|z|]}^1)^{i-1} \oplus K_2(k)$$

(see [5]) and hence $K_2(k[t]/t^i) \otimes \mathbb{Q} = 0$. Thus, $SK_1(A,m_x^i) \otimes \mathbb{Q} = 0$.

Now, since we have a sequence of inclusions

$$E_n(A,m_x^i) \subseteq \Gamma^i K \subseteq K^i,$$

and since $K^i/E_n(A,m_x^i)$ is torsion, we see that $K^i/\Gamma^i K$ is torsion. Note that the graded quotients $K^i/K^{i+1}$ are finite-dimensional $k$-vector spaces:

$$K^i/K^{i+1} \cong sl_n(m_x^i/m_x^{i+1}) \cong sl_n(k).$$
By Proposition 1.1, the unipotent $k$-completion of $K$ is the group

$$K(\hat{A}) = \varprojlim K/K^i.$$ 

This group fits into the split extension

$$(1) \quad 1 \rightarrow K(\hat{A}) \rightarrow SL_n(\hat{A}) \rightarrow SL_n(k) \rightarrow 1.$$ 

Now, if the completion of $SL_n(A)$ relative to $\rho$ is the extension

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{G} \rightarrow SL_n(k) \rightarrow 1,$$

we have a homomorphism $\Phi : K(\hat{A}) \rightarrow \mathcal{U}$. Note that $H_1(K; k) \cong K/K^2 \oplus (K/\Gamma^2 K \otimes k) \cong K/K^2$

is a finite-dimensional $k$-vector space. By Proposition 1.3 $\Phi$ is surjective and since the center of $K(\hat{A})$ is trivial, Proposition 1.2 implies that $\Phi$ is injective. Since the extension (1) is split, we have $\mathcal{G} \cong SL_n(\hat{A})$. \hfill \square

4. Application to $K$-theory

For $n \geq 3$, we have the following chain of isomorphisms:

$$H^2(SL_n(A); k) \cong H_2(SL_n(A); k)$$
$$\cong (H_2(SL_n(A); \mathbb{Z}) \otimes k) \oplus \text{Tor}^1(H_1(SL_n(A); \mathbb{Z}), k)$$
$$\cong (K_2(A) \otimes k) \oplus \text{Tor}^1(SK_1(A), k)$$
$$\cong K_2(A) \otimes k.$$

This is the primary motivation for calculating the completion of $SL_n(A)$. By Proposition 2.1 we have an injection

$$H^2_{cts}(SL_n(\hat{A}); k) \rightarrow H^2(SL_n(A); k),$$

and therefore we obtain a lower bound on $K_2(A) \otimes k$. Unfortunately, the lower bound is not useful.

**Proposition 4.1.** For $n \geq 3$, $H^2_{cts}(SL_n(\hat{A}); k) = 0$.

**Proof.** Note that $\hat{A} \cong k[[T]]$ and so we may as well consider the curve $C = \mathbb{A}^1$, $A = k[t]$. Then we have the following:

$$H^2_{cts}(SL_n(k[[T]]); k) \leftarrow H^2(SL_n(k[t]); k)$$
$$\cong H_2(SL_n(k[t]); k)$$
$$\cong H_2(SL_n(k); k)$$
$$\cong K_2(k) \otimes k$$
$$= 0.$$ 

\hfill \square
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