Non-Markovian dynamics of a two-level system in a bosonic bath and a Gaussian fluctuating environment with finite correlation time

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(Dated: May 27, 2020)

The character of evolution of an open quantum system is often encoded in the correlation function of the environment or equivalently in the spectral density function of the interaction. When the environment is heterogeneous, e.g. consists of several independent environments, with different spectral functions, the character of evolution could have some distinctive features allowing a control by adjusting properties of one of the sub-environments. We investigate non-Markovian evolution of a two-level system (qubit) under influence of three independent decoherence channels, two of them has classical nature and originate from interaction with a stochastic field, and one is a quantum channel formed by interaction with a bosonic bath. By modifying spectral densities of the channels, we study their impact on steady states of the two-level system, evolution of its density matrix and the equilibrium emission spectrums, noting inaccuracy of the rotation-wave approximation of the bath channel in comparison with the full-interaction one.

I. INTRODUCTION

In most realistic scenarios, a quantum system is coupled to its surroundings with inevitable processes of decoherence and dissipation. If the system-environment interaction satisfies certain conditions, e.g. weak coupling, uncorrelated initial states, and short environmental correlation times, evolution of an open quantum system (OQS) can be described by a Lindblad master equation with a constant Lindblad operator and positive decay rates [1, 2]. The correspondent dynamical map possesses the semigroup property and is called Markovian. In many optical problems the Markovian description works adequately, but the number of quantum systems where memory effects associated with the environment should be taken into consideration grows permanently [3] and includes such well-known systems as quantum dots [4, 5], micromechanical resonators [6] and superconducting qubits [7]. Many of the emerging quantum technologies, such as the single-photon sources for quantum communications [8], are founded on effects induced by interaction with non-Markovian environments. Non-Markovian effects are essential for problems involving the strong interplay of vibrational and electronic states, such as electron transport in natural photosynthetic systems [9, 10], light emission in complex organic molecules or solar cells [11], or semiconductor quantum dots [12–15], and also for non-equilibrium quantum processes, such as energy transport in molecular systems [16] or non-adiabatic processes in physical chemistry [17]. Non-Markovian effects are often considered as a resource for quantum information [18–20].

The main challenge in working with non-Markovian systems is the size of the environment state space. The goal of any OQS theory is to avoid integration of the full system. There are many approaches developed for this problem [3, 21]. In rare cases, exact solutions are known [22], in some others - effective weak coupling theories or perturbative expansions based on the projection operator techniques are possible [12, 23]. There are different diagrammatic and path-integral methods for which efficient Monte-Carlo schemes are known, for example, the Inchworm algorithm for the real-time diagrammatic Monte Carlo [24, 25]. Non-perturbative approaches include enlarging the state space of the system [26–28], capturing system's history by augmented density tensors (ADTs) [29–31], by a thermodynamic low-dimensional approximations [32], by mapping on an effective 1-D fermionic or bosonic chains [33], and many others. A common property of the non-perturbative methods is the strong dependence of overall computational complexity on the environment correlation time. Introduction of restrictive assumptions about the environment could lead to a better complexity in some cases [28, 34]. One of the most widely used non-perturbative approaches is the hierarchical equations of motions (HEOM) method [34–36], that represents the system-environment interaction memory kernel by an infinite system of recurrent differential equations for auxiliary reduced density matrices. This approach is capable of treating a great variety of spectral distribution functions [37–40] and works well in the high-temperature region, while the low-temperature region is computationally hard.

When an open system is coupled to several environments, either classical or quantum, its measurable properties might be affected by the induced correlation between the environments [41, 42]. If one of the environments is taken to be a stochastic environment, it becomes possible to control the system purely by noise [43]. The stochastic field alters the effective noise statistics of the joint environment, similar in this sense to dynamical decoupling creating an effective filter function [44, 45] by means of ordered artificial pulses. Simultaneous action of noises of the same decoherence mechanism has been studied in [46, 47], of different decoherence mechanisms, both dephasing and relaxation ones, in [48]. A discussion of interplay of a bath relaxation channel and a dephasing classical noise for a two-level and a three-level quantum systems can be found in [49–51].

Non-Markovian evolution has its own distinctive fea-
tures that are already visible in statistical quantities obtained from the reduced density matrix [52–55]. Many approaches to measure non-Markovianity degree of a quantum process are based on the reduced density matrix [3, 56]. Recently, several attempts have been made to employ two-time correlation functions to measure non-Markovian corrections to predictions of the quantum regression theorem (QRT). Besides, non-Markovianity is proved to be important for the equilibrium state of the reduced system [27].

In current work, we investigate non-Markovian evolution of a two-level system (TLS) that interacts with a composite environment consisting of an external stochastic field of arbitrary nature and a bosonic bath. The stochastic field and the bath act on the TLS independently and form two classical decoherence channels, one pure dephasing channel and one relaxation channel, and one quantum decoherence channel of the relaxation type. We capture non-Markovian evolution of the system exactly by utilizing the hierarchy equations of motion (HEOM) technique [35], which is numerically exact and doesn’t rely on any assumption, such as the magnitude of the TLS-environment coupling strength. We study evolution of a TLS in the composite environment numerically for the particular choice of the stochastic field and the bath sub-environments, a stochastic field described by a set of Ornstein-Uhlenbeck random processes and a bath with the high-temperature Drude spectral density. Adjusting the frequency cut-off and the coupling strength of the sub-environments, we analyze steady states of a TLS, evolution of the reduced density matrix, and the equilibrium emission spectrums.

The Hamiltonian of the system is often simplified by the rotating-wave approximation (RWA) that neglects processes that do not conserve energy, i.e. the ones involving simultaneous creation or annihilation of energy quanta in both the TLS subspace and the environment. It is known that the RWA is able to significantly alter the entire TLS dynamics for strong TLS-bath couplings [3, 59]. Wrongly used RWA, for example, may lead to incorrect values for the environmentally induced shifts to system frequencies [60] and affects non-Markovian properties of the evolution, as it does for interactions with bosonic environments at low temperatures [61], where rapidly oscillating terms, neglected by RWA, mostly determine non-Markovianity. To estimate the impact of the RWA, we developed HEOM capable of handling both the RWA and the non-RWA TLS-bath couplings equally accurate and provide comparison of the results.

The paper is organized as follows. In Sec. II we introduce the model, in Sec. III we derive the hierarchical equations of motion, in Sec. IV we present the Markovian approximation. Then we study the model numerically. In Sec. V we study steady states of a TLS, in Sec. VI - evolution of the reduced density matrix, and in Sec. VII - emission spectrums. Finally, we draw conclusions in Sec. VIII.

II. MODEL HAMILTONIAN AND INITIAL CONDITIONS

A. Model Hamiltonian

The full Hamiltonian for the system can be written as

$$\hat{H}_{AFB} = \hbar \omega_0 \hat{J}_0 + \sum_{s=1}^{\infty} \hbar \omega_s \hat{b}_s^+ \hat{b}_s + \hat{H}_{1B} + \hat{H}_{IF},$$

(1)

where $\omega_0$ is the frequency of a TLS, operators $\hat{J}_0, \hat{J}_+,$ and $\hat{J}_-$ are generators of an irreducible representation of $SU(2)$, $\hat{b}_s^+$ and $\hat{b}_s$ form a set of creation and annihiliation operators describing modes of a bosonic heat bath; $\hat{H}_{1B}$ is the Hamiltonian of interaction with a stochastic field and is defined as

$$\hat{H}_{1B} = \hbar \Omega(t) \hat{J}_0 + \hbar \left( \xi(t) \hat{J}_+ + \tilde{\xi}(t) \hat{J}_- \right).$$

(2)

In (2), $\Omega(t), \xi(t),$ and $\tilde{\xi}(t)$ are random functions, describing interaction with an external stochastic field. The term with $\Omega(t)$ forms a dephasing channel, the terms with $\xi(t)$ and $\tilde{\xi}(t)$ - a relaxation channel. $\hat{H}_{IF}$ is the Hamiltonian of interaction between the TLS and the bath.

The form of $\hat{H}_{1B}$ depends on whether the RWA is used or not. The exact form of the electric-dipole interaction (non-RWA) is expressed by the following interaction Hamiltonian

$$\hat{H}_{1B}^{\text{full}} = \sum_{s=1}^{\infty} g_s \left( \hat{J}_+ + \hat{J}_- \right) \left( \hat{b}_s^+ + \hat{b}_s \right),$$

(3)

If the RWA is used, it takes a more simple form

$$\hat{H}_{1B}^{\text{(RWA)}} = \sum_{s=1}^{\infty} \left( g_s \hat{J}_- \hat{b}_s^+ + \bar{g}_s \hat{J}_+ \hat{b}_s \right),$$

(4)

In (3) and (4), $g_s$ are the TLS-bath coupling constants. We place the TLS at the origin of coordinates, so the coupling constants are real.

Let us introduce in (3) and (4) a new auxiliary TLS operator $\hat{a}$ to represent the interaction Hamiltonians by one formula

$$\hat{H}_{1B} = \sum_{s=1}^{\infty} \left( g_s \hat{a} \hat{b}_s^+ + \bar{g}_s \hat{a}^+ \hat{b}_s \right).$$

(5)

The non-RWA interaction Hamiltonian (3) can be obtained from (5) by taking $\hat{a} = \hat{a}^+= \hat{J}_+ + \hat{J}_-$, and the RWA Hamiltonian (4) - by taking $\hat{a} = \hat{J}_- \text{ and } \hat{a}^+ = \hat{J}_+.$

Usually the interaction with a bosonic bath is defined via a spectral density function

$$J(\omega) = \sum_{s=1}^{\infty} \bar{g}_s g_s \delta (\omega - \omega_s),$$

(6)
which assumes positive frequencies. Often the domain of \( J(\omega) \) is extended to negative frequencies to obtain more symmetrical limits of integration by \( \omega \).

Let us assume that the bath characterizes by a Drude spectral density, which is an Ohmic spectral density with the algebraic cut-off \cite{3,36}. In the high-temperature limit it can be written as \cite{34,35}

\[
J_D(\omega) = c_J D \frac{\beta \hbar \omega}{\gamma_B + \omega^2},
\]

where \( c_J D = \gamma_B \hbar^2 \Delta_B^2 / \pi \), the parameter \( \Delta_B \) defines the coupling strength and represents the magnitude of damping. The parameter \( \gamma_B \) represents the width of the spectral distribution of the collective bath modes and is often called a cut-off frequency. Also \( \gamma_B \) is the reciprocal of the correlation time of the noise induced by the bath.

**B. Stochastic field**

The stochastic field is represented by the random functions \( \Omega(t) \) and \( \xi(t) \) in the TLS-field interaction Hamiltonian \eqref{eq:2}. By definition, \( \Omega(t) \) is a real random process and \( \xi(t) \) is a complex random process. Let us assume that \( \Omega(t) \) and \( \xi(t) \) are Markov processes of Ornstein-Uhlenbeck (OU) type \cite{35,62,63} and impose further restrictions on the form of \( \xi(t) \) process, considering it a sum of two real random processes \( \xi_1(t) \) and \( \xi_2(t) \):

\[
\begin{align*}
\xi(t) &= \xi_1(t) + i \xi_2(t), \\
\bar{\xi}(t) &= \xi_1(t) - i \xi_2(t).
\end{align*}
\]

An OU process \( \nu(t) \) is a Gaussian process with the exponential two-time correlation function

\[
\langle \nu(t) \nu(t') \rangle = \frac{\Delta^2}{\gamma_{\nu}} e^{-\gamma_{\nu}|t-t'|}
\]

and the Gaussian distribution function at equilibrium

\[
P_{eq}(\nu_0) = \frac{1}{\sqrt{2\pi \Delta^2_{\nu}}} \exp \left[ - \frac{\nu_0^2}{2 \Delta^2_{\nu}} \right].
\]

The transition probability of \( \nu(t) \) satisfies the Fokker-Planck equation

\[
\frac{\partial}{\partial t} P(\nu, t|\nu_0, t_0) = \gamma_{\nu} \frac{\partial}{\partial \nu} \left( \nu + \Delta^2_{\nu} \frac{\partial}{\partial \nu} \right) P(\nu, t|\nu_0, t_0),
\]

and can be expressed as

\[
P(\nu, t|\nu_0, t_0) = \left( \frac{\gamma_{\nu}}{2\pi \gamma_\nu \Delta^2_{\nu} (1 - \exp(-2\gamma_{\nu}(t - t_0)))} \right)^{1/2} \times \exp \left\{ -\frac{\gamma_{\nu} (\nu - \nu_0 \exp(-\gamma_{\nu}(t - t_0)))^2}{(2\gamma_{\nu}\Delta_{\nu}^2) (1 - \exp(-2\gamma_{\nu}(t - t_0)))} \right\}.
\]

The probability of an arbitrary realization of \( \nu(\tau) \) on the time segment \( \tau \in [t, t_0] \) with the boundary conditions \( \nu(t) = \nu \) and \( \nu(t_0) = \nu_0 \) can be expressed via the transition probabilities \( \langle 13 \rangle \) in the following way

\[
P[\nu(\tau)] = \lim_{N \to \infty} \prod_{j=1}^{N-1} \int_{X_\nu} d\nu_j \left( \prod_{j=0}^{N-1} P(\nu_{j+1}, t_{j+1}|\nu_j, t_j) \right)
\]

The parameters \( \Delta_{\nu} \) and \( \gamma_{\nu} \) of the stochastic field have similar physical meanings to the parameters \( \Delta_B \) and \( \gamma_B \) of the bath \eqref{eq:7}, respectively.

**C. Initial conditions**

Let us suppose that before the initial moment of time \( t = t_0 \) a TLS doesn’t interact with the bath and the stochastic field, and both of the environments are at equilibrium. Then the total density matrix at \( t = t_0 \) has the factorized form

\[
\hat{\rho}_{tot}(\Omega, \xi, \bar{\xi}, t_0) = \hat{\rho}^{(A)}(\Omega, \xi, \bar{\xi}) \otimes \hat{\rho}_{eq}^{(B)}(t_0),
\]

where the bath is taken in thermal equilibrium at the inverse temperature \( \beta = 1/(k_B T) \),

\[
\hat{\rho}_{eq}^{(B)}(t_0) = \int_{X_B} d\mu(\phi^{(B)}) \int_{X_B} d\mu(\phi'^{(B)}) \int_{X_B} d\mu(\phi''^{(B)}) \times \prod_{s=1}^{\infty} (1 - \rho(\omega_s)) \exp(\rho(\omega_s) \phi^{(B)|\phi'^{(B)|\phi''^{(B)}}}),
\]

and \( k_B \) is the Boltzmann constant, \( \rho(\omega_s) = \exp[-\beta \hbar \omega_s] \). The stochastic part of the initial conditions is contained in \( \hat{\rho}^{(A)}(\Omega, \xi, \bar{\xi}, t_0) \).

If we assume that there are no correlations between the TLS and the stochastic field at \( t = t_0 \), we get further factorization

\[
\hat{\rho}^{(A)}(\Omega, \xi, \bar{\xi}, t_0) = \hat{\rho}^{(A)}(t_0) P_{eq}(\Omega) P_{eq}(\xi_1) P_{eq}(\xi_2).
\]

**III. HIERARCHICAL EQUATIONS OF MOTION**

Let us introduce a coherent states basis \( |\phi\rangle \) in the bath subspace. The basis is a cross-product of the coherent states bases in each of the bath modes subspaces \( |\phi_s\rangle = \prod_s |\phi_s\rangle \), where \( s \) is the index of a bath mode. In the TLS subspace we introduce the basis of generalized coherent states \( |z\rangle \) (also known as the spin coherent states) \cite{64}. If we denote a vector of the full space as \( |z, \phi\rangle \), we can
write the total density matrix in the following way

\[ \rho_{tot}(t) = \int d\mu(\Omega) \int d\mu(\Omega') \times \int \chi_s d\mu(\tilde{\xi}, z) \int \chi_s d\mu(\tilde{z}', z') \times \int \chi_B d\mu(\tilde{B}^S), \phi(B)^S) \int \chi_B d\mu(\tilde{B}^S), \phi(B)^S) \times \rho_{tot}(\Omega, \xi, z, \phi(B)) \Omega(\bar{\xi}, z, \phi(B))(\Omega, \xi, z, \phi(B))(t) \]

where \( \chi_\Omega \) and \( \chi_\xi \) denote the sets of possible values of random processes \( \Omega(t) \) and \( \xi(t) \) at time \( t \), respectively; \( \chi_S \) denote the TLS subspace, and the integration over \( \chi_B \) is an abbreviation for integration by each of the bath modes subspaces

\[ \int \chi_B d\mu(\tilde{B}^S), \phi(B)^S) = \prod_{s=1}^{\infty} \int \chi_B \phi(B)^S) \]  

(19)

the infinitesimal measure on the atom subspace \( \chi_S \) has the next form

\[ d\mu(\tilde{z}, z) = \frac{2}{1 + z \tilde{z}^2} d[\tilde{z}, z], \]  

(20)

on the \( s \)-th bath mode subspace \( \chi_B^{(s)} \) - the next form

\[ d\mu(\tilde{B}^S), \phi(B)^S) = e^{-\phi(B)^S} d\mu(\tilde{B}^S), \phi(B)^S) \]

(21)

the stochastic field subspace measures are \( d\mu(\Omega) = P(\Omega)d\Omega \) and \( d\mu(\tilde{\xi}, \xi) = P(\xi_1)P(\xi_2)d\xi_1d\xi_2 \). In (20) and (21) the differential is \( d[\tilde{z}, z] = (1/\pi)dRe z dIm z \).

Let's take \( t = t_0 \) the initial moment of time and divide the time interval \([t_0, t]\) on \( N \) segments. At each time segment, evolution of the total density matrix is governed by a respective infinitesimal evolution operator. The total evolution operator \( U(t_0, t) \) is a product of infinitesimal operators at each of the segments. By inserting \( N - 1 \) identity operators at respective \( N - 1 \) moments of time and taking the limit \( N \to \infty \), we obtain matrix elements of the total evolution operator in the next form

\[ U(\Omega, \xi, z, \phi(B), t; \Omega_0, \xi_0, z_0, \phi_0(B), t_0) \]

\[ = \int \mathcal{D}[\Omega(\tau)] \int \mathcal{D}[\xi(\tau)] \int \mathcal{D}[\xi(\tau)] \int \mathcal{D}[\xi(\tau)] P(\Omega(\tau)|\phi_0(B), t_0) \]

(22)

where \( \mathcal{D}[\xi(\tau)] \) denotes functional integration over the set of trajectories starting at \( z(t_0) = z_0 \) and ending at \( z(t) = z_N = z \),

\[ \int \mathcal{D}[\xi(\tau)] = \lim_{N \to \infty} \int \mathcal{D}[\xi(\tau)] \int \mathcal{D}[\xi(\tau)] \int \mathcal{D}[\xi(\tau)] \int \mathcal{D}[\xi(\tau)] \int \mathcal{D}[\xi(\tau)] \]

(23)

and \( \mathcal{D}[\phi(B)(\tau)] \) denotes path integrals over the set of trajectories of all the bath modes,

\[ \int \mathcal{D}[\phi(B)(\tau)] \int \mathcal{D}[\phi(B)(\tau)] \int \mathcal{D}[\phi(B)(\tau)] \int \mathcal{D}[\phi(B)(\tau)] \int \mathcal{D}[\phi(B)(\tau)] \]

(24)

the functional \( S_A \) for the free TLS and has the next form

\[ S_A \left[ z(\tau); t, t_0 \right] \]

\[ = -i\hbar \lim_{N \to \infty} \sum_{j=0}^{N-1} \left( \ln \frac{2}{1 + z_j z_{j+1}} + \ln(z_{j+1}z_j) \right) \]

(25)

the functional \( S_B \) for the bath Hamiltonian

\[ S_B \left[ \phi(B); t, t_0 \right] \]

\[ = -i\hbar \lim_{N \to \infty} \sum_{j=0}^{N-1} \left( \phi_j(B)^* \phi_j(B) - \phi_{j+1}(B)^* \phi_j(B) \right) - \frac{i}{\hbar} H_B \left( \phi_{j+1}(B)^* \phi_j(B) \right) \]

(26)
where $\epsilon = (t - t_0) / N$ is the length of a time segment, $H_A, H_B, H_{IF}$, and $H_{IB}$ are symbols of $\hat{H}_A, \hat{H}_B, \hat{H}_{IF}$, and $\hat{H}_{IB}$ operators, respectively. All variables having a $j$ index correspond to the $j$-th time slice or the $j$-th identity operator. The semicolons in the operator symbols arguments separate variables taken at different slices. Those to the left correspond to the one-step-forward slice, with index $j + 1$, and to the right from a semicolon - to the $j$-the slice.

Tracing out the degrees of freedom of the bath and of the stochastic field, we obtain the TLS subsystem evolution, described by the reduced density matrix with the following matrix elements

$$
\rho^{(A)}(z, z', t) = \int_{C[z_0, t_0; z_0', t_0]} D[\bar{z}(\tau), z(\tau)]
$$

$$
\times \int_{C[z_0, t_0; z', t_0]} D[z'(\tau), z'(\tau)]
$$

$$
\times e^{\frac{i}{\hbar} S_A[z(\tau); t, t_0]} \rho^{(A)}(z_0, z'_0, t_0)
$$

$$
\times F_F [z(\tau), z'(\tau); t, t_0]
$$

$$
\times F_B [z(\tau), z'(\tau); t, t_0]
$$

$$
\times e^{-\frac{i}{\hbar} S_A[z'(\tau); t, t_0]},
$$

(27)

where we have utilized the form of the evolution operator (22) and the factorized from of the initial conditions (15, 17): $F_B [z(\tau), z'(\tau); t, t_0]$ denotes the bath influence functional

$$
F_B [z(\tau), z'(\tau); t, t_0] = \int_{C[z_0, t_0; z_0', t_0]} D[\bar{z}(\tau), \bar{z}(\tau)]
$$

$$
\times \int_{C[z_0, t_0; z', t_0]} D[\bar{z}'(\tau), \bar{z}'(\tau)]
$$

$$
\times e^{\frac{i}{\hbar} S_B[\bar{z}(\tau); t, t_0]} \rho^{(B)}(z_0, z'_0, t_0)
$$

$$
\times e^{-\frac{i}{\hbar} S_B[\bar{z}'(\tau); t, t_0]}
$$

(28)

and $F_F [z(\tau), z'(\tau); t, t_0]$ is the stochastic field influence functional, which is a product of influence functionals for each of the random processes

$$
F_F [z(\tau), z'(\tau); t, t_0] = \Pi_{\nu} F_{\nu} [z(\tau), z'(\tau); t, t_0],
$$

(29)

where $\nu \in \{\Omega, \xi_1, \xi_2\}$ and $F_{\nu} [z(\tau), z'(\tau); t, t_0]$ is represented by a functional integral of the next form

$$
F_{\nu} [z(\tau), z'(\tau); t, t_0] = \int_{C[z_0, t_0; z_0', t_0]} D[\nu(\tau), \nu(\tau)]
$$

$$
\times e^{-\frac{i}{\hbar} \int_{t_0}^{t} d\tau H_{\nu}(z(\tau); \nu(\tau))}
$$

$$
\times P[\nu(\tau)] P_{eq}(\nu_0)
$$

$$
\times e^{\frac{i}{\hbar} \int_{t_0}^{t} d\tau H_{\nu}(z'(\tau); \nu(\tau)),}
$$

(30)

The simple product form of the field influence functional (29) is a consequence of absence of TLS-field initial correlations (17) and independence of $\xi_1(t), \xi_2(t)$, and $\Omega(t)$. In (30) we have introduced symbols of interaction Hamiltonians with each of the random processes

$$
H_{I\nu}(z_{j+1}; z_j, \nu_j) = \nu_j V_{F\nu} (z_{j+1}; z_j),
$$

(31)

where $V_{\nu}$ are symbols of the operators, defined on the TLS subspace

$$
\dot{V}_{F\Omega} = \hbar \dot{\bar{J}}_0,
$$

(32)

$$
\dot{V}_{F\xi_1} = \hbar \left( J_+ + J_- \right),
$$

(33)

$$
\dot{V}_{F\xi_2} = i \hbar \left( J_+ - J_- \right).
$$

(34)

The hierarchical equations of motion (HEOM) are obtained from the reduced density matrix (27) by repetitive differentiation of the memory kernel, related to the influence functional. As a result, an infinite system of recurrent ODEs appears, involving time derivatives of auxiliary density matrices [34, 35].

At first, we consider a simplified problem, with the stochastic field turned off. From the two interaction Hamiltonians in Eq.(1) we keep only $H_{IB}$, so the expression (27) for the averaged reduced density matrix has only one influence functional $F_B [z(\tau), z'(\tau); t, t_0]$. Increasing $t$ by a small value $\epsilon$, we get

$$
\rho^{(A)}(z, z', t + \epsilon) = \int_{C[z_0, t_0; z_0', t_0]} D[\bar{z}(\tau), z(\tau)]
$$

$$
\times \int_{C[z_0, t_0; z', t_0]} D[z'(\tau), z'(\tau)]
$$

$$
\times e^{\frac{i}{\hbar} S_A[z(\tau); t, t_0]} \rho^{(A)}(z_0, z'_0, t_0)
$$

$$
\times F_F [z(\tau), z'(\tau); t + \epsilon, t_0]
$$

$$
\times F_B [z(\tau), z'(\tau); t + \epsilon, t_0]
$$

$$
\times e^{-\frac{i}{\hbar} S_A[z'(\tau); t + \epsilon, t_0]}.
$$

(35)

Taking values at $t + \epsilon$ involves one diagonal segment on the time axis, lying on the right from $t$ and of length $\epsilon$. Thus, the increment of the free-TLS action can be written as

$$
S_A [z(\tau); t + \epsilon, t_0] - S_A [z(\tau); t, t_0] = i \hbar \ln \left( 2 \left( 1 + z_N \bar{z}_N \right)^2 \right)
$$

(36)

$$
- i \hbar \ln (z_N) - \epsilon H_A (z, z_N).
$$

The influence functional $F_B [z(\tau), z'(\tau); t, t_0]$ can be found from the discret form of the path integral (28) by performing the bosonic Gaussian integration. For the relaxation-type interaction Hamiltonian (5), the following expression can be obtained

$$
F_B [z(\tau), z'(\tau); t, t_0] = \exp \left\{ \left( -\frac{i}{\hbar} \right)^2 \sum_{k=1}^{2} \int_{t_0}^{t} dt' \Phi_k [z(\tau), z'(\tau); t'] V_{k}^\nu (t') \right\},
$$

(37)
where we have introduced the influence phase functionals
\[ \Phi_k[z(\tau), z'(\tau); t] = \int_{t_0}^{t} dt' \int_{0}^{\infty} d\omega J(\omega) f_k(t, t', \omega), \] (38)
and
\[ f_1(t, t', \omega) = e^{i\omega(t-t')} \left( n_B(\omega) C^+_2(t') - (n_B(\omega) + 1) C^-_2(t') \right), \]
(39)
\[ f_2(t, t', \omega) = e^{-i\omega(t-t')} \left( (n_B(\omega) + 1) C^+_2(t') - n_B(\omega) C^-_2(t') \right). \]
(40)

Here, \( n_B(\omega) = \rho(\omega) / (1 - \rho(\omega)) \) and \( \rho(\omega) \) is defined in (16), \( C^+_2(t) \) and \( C^-_2(t) \) are continuous representation of symbols of the auxiliary operators \( \hat{a} \) and \( \hat{a}^+ \) respectively, the plus and minus signs originate from the forward and backward branches of the path integral; \( V^2_{f}(\tau) = C^+_1(t) - C^-_1(t), V^2_{z}(\tau) = C^+_2(t) - C^-_2(t), \) and \( J(\omega) \) is the bath spectral density (6).

In case of the non-RWA interaction with the bath (3), the equation (37) takes the form of the well-known influence functional of Feynman and Vernon [65]. Incrementing the time argument of \( F_B[z(\tau), z'(\tau); t, t_0] \) we get
\[ F_B[z(\tau), z'(\tau); t + \epsilon, t_0] \]
\[ = \left( 1 + \epsilon \sum_{k=1}^{2} \Phi^{(0)}_{B,k}(t + \epsilon) \Phi_k[z(\tau), z'(\tau); t] \right) \times F_B[z(\tau), z'(\tau); t, t_0]. \] (41)

where the functions of the boundary time are
\[ \Phi^{(0)}_{B,k}(t + \epsilon) = (-i/\hbar) V^z_k(t + \epsilon), \] (42)
that actually also depend on the previous time slice variables \( z_N \) and \( z'_N \).

By substituting (36) and (41) into (35), we obtain
\[ \rho^{(A)}(z, z', t + \epsilon) - \rho^{(A)}(z, z', t) \]
\[ = \epsilon \int_{\chi_s} d\mu(z_N, z_N) \int_{\chi_s} d\mu(z'_N, z'_N) \rho^{(A)}(z_N, z'_N, t) \]
\[ + \Phi^{(0)}_{B,k}(t + \epsilon) \sum_{k=1}^{2} \rho^{(A)}_{k}(z_N, z'_N, t), \] (43)

where we have introduced the auxiliary density matrix
\[ \rho^{(A)}_{k}(z_N, z'_N, t) \]
\[ = \int_{\mathcal{C}(x_s, t_0, z_N, t)} D[\tilde{\zeta}(\tau), z(\tau)] \]
\[ \times \int_{\mathcal{C}(x_s, t_0, z'_N, t)} D[\tilde{\zeta}'(\tau), z'(\tau)] \]
\[ \times \Phi_k[z(\tau), z'(\tau); t] F_B[z(\tau), z'(\tau); t, t_0] \]
\[ \times e^{iS_A[z(\tau); t, t_0]} \rho^{(A)}(z_0, z'_0, t_0) e^{-iS_A[z'(\tau); t, t_0]}, \] (44)

It can be seen from (44), that because \( \Phi_1[z(\tau), z'(\tau); t] \) is not equal to \( \Phi_2[z(\tau), z'(\tau); t] \), there are two branches in the recursion relation for the bath. The time-incremented form of (44) involves \( \Phi_1[z(\tau), z'(\tau); t + \epsilon] \) and \( \Phi_2[z(\tau), z'(\tau), t + \epsilon] \), which can be obtained from (38) by incrementing the time argument
\[ \Phi_k[z(\tau), z'(\tau); t + \epsilon] = \Phi_k[z(\tau), z'(\tau); t] \]
\[ + \epsilon \Phi^{(1)}_{B,k}(t + \epsilon) \]
(45)
and
\[ \Psi^{(1)}_{B,k}[z(\tau), z'(\tau); t] = \frac{1}{2} \int_{0}^{t} dt' \int_{-\infty}^{+\infty} d\omega J(\omega) f_k(t + \epsilon, t + \epsilon, \omega), \] (46)

and
\[ \Psi^{(1)}_{B,k}[z(\tau), z'(\tau); t] = \frac{1}{2} \int_{0}^{t} dt' \int_{-\infty}^{+\infty} d\omega J(\omega) \]
\[ \times \frac{\partial}{\partial t} f_k(t, t', \omega). \] (47)

In (46) and (47) the lower limits of integration by \( \omega \) have been extended to the negative infinity. The limits can be extended, if \( \hat{a} \) is a self-conjugated operator and the spectral density \( J(\omega) \) is an odd function of \( \omega \). It is completely fulfilled for the non-RWA interaction, but in the RWA case, \( \hat{a} \) is not a self-conjugated operator, and the extension imposes rather strong restrictions on both the coupling strength and the maximum time of the dynamics.

In (45) \( \Psi^{(1)}_{B,k}(t + \epsilon) \) depends only on the boundary time and yields an operator in the operator form of HEOM, while \( \Psi^{(1)}_{B,k}[z(\tau), z'(\tau); t] \) still depends on the whole path and contains the memory of the interaction. For the environments considered in current work, \( \Psi^{(1)}_{B,k}[z(\tau), z'(\tau); t] \) satisfies the following relation
\[ \Psi^{(1)}_{B,k}[z(\tau), z'(\tau); t] = \alpha^{(B)}_k \Phi_k[z(\tau), z'(\tau); t], \] (48)
where \( \alpha_k \) is a constant.

The relation (48) yields a system of ODEs for the auxiliary density matrices that can be translated to the operator form. The system of operator ODEs is of the HEOM
interaction with the stochastic field. We could perform the
operator $\hat{\Phi}^{(0)}_{B,1}\hat{\rho}_{m+1}(t)$ and $\hat{\Phi}^{(0)}_{B,2}\hat{\rho}_{mn+1}(t)$
(49)
\[
+ m\hat{\Phi}^{(1)}_{B,1}\hat{\rho}_{m-1}(t) + n\hat{\Phi}^{(1)}_{B,2}\hat{\rho}_{mn-1}(t),
\]
where we split the index of $44$ on two, one for each of the branches; by $\hat{H}_A^x$ we denote the commutator superoperator $\hat{H}_A^x\hat{\rho} = \hat{H}_A^x\hat{\rho} - \hat{\rho}\hat{H}_A^x$.

In (49) only the element with $m = 0$ and $n = 0$ has a physical meaning, others are just a representation for the bath memory kernel. By redefining the auxiliary memory functions (44) we can obtain an equivalent HEOM with adjusted coefficients, e.g. by making the substitutions
\[
\hat{\rho}^{(A)}_{m+1}(t) \rightarrow \hat{\rho}^{(A)}_{m+1}(t) \quad \text{and} \quad \hat{\rho}^{(A)}_{m-1}(t) \rightarrow (1/a)\hat{\rho}^{(A)}_{m-1}(t).
\]

In a similar way, we can investigate the case of interaction with the stochastic field. We could perform the Gaussian integration in a Wiener path integral, or just employ the results of [35], where interaction with a stochastic field is studied in a slightly different setting resembling the non-RWA TLS-bath interaction case (3). While in the present work the TLS-stochastic field interaction (2) is more like the RWA TLS-bath interaction (4), because of commutativity of classical random processes these two cases are quite similar. Following [35], where only one random process is present, we get a HEOM very similar to eq. (49), but with one index. For the full stochastic field, where the influence functional is a product of the form (29), the HEOM is indexed by three numbers.

In the general case of simultaneous interaction with both the bath and the stochastic field (27), the joint influence functional becomes a product of the influence functionals of the bath and the stochastic field. As a result, the HEOM contains five indexes. If we introduce the vector notation for indexes of the auxiliary density matrices
\[
m = (m_1, m_2, \ldots),
\]
\[
m_k + 1 = (m_1, m_2, \ldots, m_k + 1, \ldots),
\]
we could write the HEOM in the following form
\[
\frac{\partial}{\partial t}\hat{\rho}^{(A)}_m(t) = -\frac{i}{\hbar}\hat{H}_A^x\hat{\rho}^{(A)}_m(t)
+ \sum_k \left( m_k\alpha_k^{(B)}\hat{\rho}^{(A)}_m(t) + \hat{\Phi}^{(0)}_{B,k}\hat{\rho}^{(A)}_{m+1}(t) \right)
+ m_k\hat{\Phi}^{(1)}_{B,k}\hat{\rho}^{(A)}_{m+1}(t)
+ \sum_k \left( m_k\alpha_k^{(B)}\hat{\rho}^{(A)}_m(t) + \hat{\Phi}^{(0)}_{B,k}\hat{\rho}^{(A)}_{m+1}(t) \right)
+ m_k\hat{\Phi}^{(1)}_{B,k}\hat{\rho}^{(A)}_{m+1}(t),
\]
(50)
where $\hat{\rho}^{(A)}_m(t) = \hat{\rho}_0(t)$ and we make the summations over the field and the bath indexes explicit.

Eq. (50) describes evolution of any bath and stochastic field for which the recursion generation condition (41) is satisfied. In particular, the condition takes place for the Ornstein-Uhlenbeck stochastic field, for which the unknown constant $\alpha_k^{(F)}$ and the unknown operators $\hat{\Phi}^{(0)}_{F,k}$ and $\hat{\Phi}^{(1)}_{F,k}$ are
\[
\alpha_k^{(B)} = -\gamma_B,
\]
\[
\hat{\Phi}^{(0)}_{B,k} = a_B \left( -\frac{i}{\hbar} \right) \hat{c}_k^x,
\]
\[
\hat{\Phi}^{(1)}_{B,1} = \frac{c_J \pi}{a_B} \frac{\gamma_B}{2} \left( \hat{c}_k^2 - i\hbar\gamma_B \hat{c}_k^L \right),
\]
\[
\hat{\Phi}^{(1)}_{B,2} = \frac{c_J \pi}{a_B} \frac{\gamma_B}{2} \left( \hat{c}_k^2 - i\hbar\gamma_B \hat{c}_k^R \right),
\]
where $\nu_k$ corresponds to the k-th element of $\{\Omega, \xi_1, \xi_2\}$, $\hat{V}_{\xi_k} = \hat{V}_{\Omega_k}$ and $\hat{V}_{\Omega_k}$ are defined in (32-34).

The relation (48) is also satisfied for the high-temperature Drude bath (7). In the high-temperature limit, $n_B(\omega)$ in Eqs. (39, 40) contains a small parameter $\beta\hbar\omega$. Because $\omega$ is bounded by the effective cutoff frequency, which is defined by $\gamma_B$, the validity of the approximation is restricted by $\beta\hbar\Delta_B < 1$. In the region of validity, the HEOM coefficients take the next form
\[
\alpha_k^{(B)} = -\gamma_B,
\]
\[
\hat{\Phi}^{(0)}_{B,k} = a_B \left( -\frac{i}{\hbar} \right) \hat{c}_k^x,
\]
\[
\hat{\Phi}^{(1)}_{B,1} = \frac{c_J \pi}{a_B} \frac{\gamma_B}{2} \left( \hat{c}_k^2 - i\hbar\gamma_B \hat{c}_k^L \right),
\]
\[
\hat{\Phi}^{(1)}_{B,2} = \frac{c_J \pi}{a_B} \frac{\gamma_B}{2} \left( \hat{c}_k^2 - i\hbar\gamma_B \hat{c}_k^R \right),
\]
where $\hat{c}_1 = \hat{a}$, $\hat{c}_2 = \hat{a}^+$, and $\hat{c}_R$ is the super-operator acting from the right: $\hat{c}_R^x\hat{\rho} = \hat{\rho}\hat{c}_R^x$, whereas $\hat{c}_L$ is the one acting from the left. It is also convenient to use the renormalization constant $a_B = \hbar^2\Delta_B$ for better HEOM coefficients.

For the non-RWA interaction with a high-temperature Drude bath, it is possible to obtain a one-index HEOM [34]. This HEOM resembles the one for the stochastic field when there is only one random process is present, and the closeness increases with the temperature [35]. For the RWA interaction the minimum number of HEOM indexes is two because of the ambiguity in the transition from the path integral form to the operator form, also known as the quantization problem.

IV. MARKOVIAN APPROXIMATION

In Markovian approximation, evolution of the reduced density matrix of the TLS is described by a Markovian master equation of the Lindblad type. It’s general form
can be written as [21]
\[
\frac{d}{dt} \rho^{(A)}_{\text{int}}(t) = -\frac{i}{\hbar^2} \left[ \hat{H}_{LS}, \rho^{(A)}_{\text{int}}(t) \right] + \mathcal{D}_B \left( \rho^{(A)}_{\text{int}}(t) \right) + \mathcal{D}_F \left( \rho^{(A)}_{\text{int}}(t) \right),
\]
(58)
where \( \rho^{(A)}_{\text{int}}(t) \) stands for the density matrix in the interaction picture, \( \hat{H}_{LS} \) is the so-called Lamb shift Hamiltonian
\[
\hat{H}_{LS} = S_{LS} (\omega_0) \hat{J}_+ \hat{J}_- + S_{LS} (-\omega_0) \hat{J}_- \hat{J}_+,
\]
(59)
the symbol \( \mathcal{D}_B \left( \rho^{(A)}_{\text{int}}(t) \right) \) denotes the bath-related dissipator
\[
\mathcal{D}_B \left( \rho^{(A)}_{\text{int}}(t) \right) = \frac{2\pi}{\hbar^2} J (\omega_0) (n_B (\omega_0) + 1) \times \left( \hat{J}_- \rho^{(A)}_{\text{int}}(t) \hat{J}_- - \frac{1}{2} \left\{ \hat{J}_+ \hat{J}_-, \rho^{(A)}_{\text{int}}(t) \right\} \right),
\]
(60)
and \( \mathcal{D}_F \left( \rho^{(A)}_{\text{int}}(t) \right) \) is the stochastic field dissipator [66]
\[
\mathcal{D}_F \left( \rho^{(A)}_{\text{int}}(t) \right) = 2K_\alpha(t) \left( \hat{J}_0 \rho^{(A)}_{\text{int}}(t) \hat{J}_0 - \frac{1}{2} \left\{ \hat{J}_0^2, \rho^{(A)}_{\text{int}}(t) \right\} \right) + 2K_\beta(t) \left( \hat{J}_- \rho^{(A)}_{\text{int}}(t) \hat{J}_- - \frac{1}{2} \left\{ \hat{J}_- \hat{J}_-, \rho^{(A)}_{\text{int}}(t) \right\} \right),
\]
(61)
where we have used previously introduced designations, and \( K_\alpha(t) \) and \( K_\beta(t) \) are the integrals of correlation functions of corresponding random processes \( K_\nu(t) = \int_{t_0}^t \langle \nu(t) \bar{\nu} (t_1) \rangle dt_1 \).

In (59), \( S_{LS} (\omega) \) depends on the spectral density type. For the high-temperature limit of the Drude spectral density it has the form
\[
S_{LS}^{(HT)} (\omega) = c_{JD} \frac{2\pi \omega + \hbar \beta \gamma_B (2 \omega \ln |\gamma_B|/|\omega| - \pi \gamma_B)}{2 \gamma_B (\gamma_B^2 + \omega^2)},
\]
(62)

V. EQUILIBRIUM STATES OF A TWO-LEVEL SYSTEM

In contrast to the imaginary-time HEOM [36], which allow obtaining of equilibrium reduced density matrices by integrating to the specified temperature, real-time HEOM find steady states by propagating initial states to large times, where the reduced density matrix stops changing. We have propagated the reduced density matrix to its steady state by means of (50) for different parameter regimes and different types of the environment: an HT-Drude bath environment (the stochastic field is absent), a stochastic field environment (the HT-Drude bath is absent), and a composite bath-stochastic field environment. We investigate dependence of steady states on parameters of the bath spectral density and on parameters of the stochastic field, assuming that initially the full system was in the factorized state (15) and (17), both the bath and the field are taken at equilibrium, and the TLS - in the excited state. In that case the TLS density matrix is diagonal at any time moment, so it can be fully characterized by one parameter, that was chosen to be the population of the excited state of the TLS. Fig. 1 shows typical dependence of steady states of a TLS on parameters of the environment.

For a wide range of frequency cut-offs and coupling strengths tested, the stochastic field acting alone brings a TLS to a steady state, where both the excited and the ground states are equally possible. In case of the HT-Drude bath environment (the stochastic field is off), the steady states show no dependence on neither the bath frequency cut-off nor the bath coupling strength, which resembles the case of interaction with the stochastic field environment, with additional dependence on the bath temperature (see Fig. 1(c)). Because in our model the temperature dependence is included perturbatively, we can cover only a small range of temperatures near \( \beta = 0 \). In this range the excited state population tends to decrease with the inverse temperature \( \beta \), and the degree of the decrease is greater for the RWA interaction Hamiltonian (4). When \( \beta \) approaches zero, steady states of a TLS in the TLS-bath system and in the TLS-stochastic field system become equal. In that sense, a stochastic field is equal to an infinite-temperature bath. The Markovian approximation (58) gives similar results. In Fig. 1(c) it coincides with the non-RWA line.

When a TLS interacts with the composite environment (Figs. 1(a), 1(b)), the difference between stationary states of a TLS in the bath and in the stochastic field comes into play. Actually, the difference arises from the dependence of the stationary states on the bath temperature and the observation that a TLS in the stochastic field environment (the bath is absent) has the same stationary state as in the HT-Drude bath environment (the stochastic field is absent) of the infinite temperature. Let us take frequency cut-offs of all the random processes constituting the stochastic field equal and change them simultaneously (\( \gamma_\nu = \gamma_F \) for all \( \nu \)). Then, at the beginning, when the stochastic field frequency cut-off is zero \( \gamma_F = 0 \), a growth of the frequency cut-off lead to an increase of the field contribution and an increase of the excited state probability of the stationary state, because the infinite-temperature steady state lies higher, then we reach the maximum, and after it the effect of a growth
of the frequency cut-off is opposite (Fig. 1(a)). In case of the RWA coupling with the bath the overall impact of the stochastic field is greater and makes the stationary states lie above the ones of the non-RWA case. In comparison, when the stochastic field is off, the RWA steady states lie below (Fig. 1(c)). The Markovian approximation (58) tends to wrongly overestimate the field contribution for small frequency cut-offs and to give more rapid decrease for bigger cut-offs, with no maximum in between, because the line starts from the infinite-temperature steady state. If we change the roles of the stochastic field and the bath and start manipulating the frequency cut-off of the bath instead, we obtain the inverted picture, with the infinite-temperature steady state at \( \gamma_B = 0 \), then a minimum, and a region of constant growth after it.

The observed dependence on the environment frequency cut-off can be explained by the form of the environment spectral density (for a stochastic field it is the spectrum of the random process). Both the OU random process and the HT-Drude bath have spectrums with one peak. Because the spectrums have cut-offs in the algebraic form, the location of the peak depends on the frequency cut-off, and the peak shifts to the right with increasing of the frequency cut-off, from \( \omega = 0 \) to the TLS frequency \( \omega = \omega_0 \) and then further away. As a result, the impact of the environment being manipulated has the maximum when the peak is located at resonance \( \omega = \omega_0 \) and decreases in both directions from it.

If we fix frequency cut-offs of random processes of the stochastic field and vary the coupling strengths in a similar way, taking them all equal \( \Delta_\nu = \Delta_F \), the dependence of the steady states on the coupling strength \( \Delta_F \) has similar traits (Fig. 1(b)), but there are no maxima, because the spectrums of the random processes rise with the coupling strength monotonically. At \( \Delta_F = 0 \) the stochastic field is decoupled and we get the steady states of the TLS-bath system. When the TLS-stochastic field coupling is big, the stochastic field dominates the bath and the steady state approaches the steady state in the stochastic field environment.

If we look at the stochastic field in more detail, allowing arbitrary frequency cut-offs and coupling strengths for the underlying random processes, we find that each of the random processes acts differently. The processes \( \xi_\xi(t) \) and \( \Omega(t) \) don’t affect stationary states in the Markovian approximation and also the stationary probability of the excited state is always bigger for the non-RWA interaction in comparison with the RWA. As for \( \xi_\xi(t) \), it does affect stationary states in the Markovian approximation and the excited state stationary probability is higher for the non-RWA interaction than for the RWA interaction for small frequency cut-offs and coupling strengths, but is lower - for large frequency cut-offs and coupling strengths. When the stochastic field is presented by the \( \Omega(t) \) random process and constitutes a purely dephasing channel, it still contributes into the relaxation channel, but only if the TLS is also interacts with the bath.

VI. EVOLUTION OF THE REDUCED DENSITY MATRIX

Let us consider evolution of the reduced density matrix from the same initial state as we took for studying steady states of a TLS. In Fig. 2 we show evolution of the excited state population in the stochastic field environment and in Fig. 3 - in the HT-Drude bath environment, for different values of the frequency cut-off and the coupling strength of the environment. In Fig. 4 the impact of the stochastic field on the evolution of the TLS in the HT-Drude bath is shown.

One specific feature of the evolution is the presence of rapidly vanishing oscillations, which are more evident for the RWA interaction (4) with the bath environment (the field is off) and for interaction with the stochastic field environment (the bath is absent), which coupling type...
FIG. 2. Evolution of the excited state population of a TLS in the stochastic field (the bath is isolated) in dependence on (a) the field frequency cut-off $\gamma_F$ and (b) the field coupling strength $\Delta_F$. Orange denote non-Markovian curves, purple - Markovian ones. In (a) $\gamma_\nu/\omega_0 = \gamma_F/\omega_0 = \{0.2, 0.4, 0.8\}$ and $\Delta_\nu^2/\omega_0 = \Delta_F^2/\omega_0 = 1.6$, for (dotted, dashed, solid) curves, respectively, and in (b) $\gamma_\nu/\omega_0 = \gamma_F/\omega_0 = 0.4$, $\Delta_\nu^2/\omega_0 = \Delta_F^2/\omega_0 = \{0.4, 0.8, 1.6\}$.

FIG. 3. Evolution of the excited state population of a TLS in the HT-Drude bath (the stochastic field is off) in dependence on (a) the bath frequency cut-off $\gamma_B$ and (b) the bath coupling strength $\Delta_B$. Orange and green denote the RWA and non-RWA couplings respectively, purple - the Markovian approximation. In (a) $\gamma_B/\omega_0 = \gamma_F/\omega_0 = \{0.2, 0.4, 0.8\}$, $\Delta_B^2/\omega_0 = 1.6$, $\beta\hbar\omega_0 = 0.1$, for (dotted, dashed, solid) curves, respectively, and in (b) $\gamma_B/\omega_0 = 0.4\omega_0$, $\Delta_B^2/\omega_0 = \{0.4, 0.8, 1.6\}$, $\beta\hbar\omega_0 = 0.1$.

resembles the RWA-type coupling with the bath. These oscillations could also be found for the non-RWA coupling with the bath (3), but they are absent if the Markovian approximation (58) is applied. The oscillations are the distinctive feature of non-Markovian evolution and have a clear relation to the Rabi oscillations. Their amplitude should depend on the distance between the peak of the environment spectral density and the TLS resonance frequency. For high frequency cut-offs there are no significant oscillations, because the peak is located too far from the resonance frequency. It is also applicable to the Markovian approximation, because it implicitly implies interaction with a continuum of modes, which corresponds to large frequency cut-offs.

In evolution of the excited state population we could see a similar behavior that we observe for the steady states in Sec. V, relating to the location of the peak of the environment spectral function and its response to changes of the environment frequency cut-off. Increasing of the coupling strength always speeds up the evolution (Figs. 2(b) and 3(b)), the steady state is reached faster. For the frequency cut-off the situation is more complex: for small cut-offs it takes more time to reach equilibrium and it is also true for large cut-offs, while for moderate cut-offs it takes less time (Figs. 2(a) and 3(a)), which clearly relates to magnitude of the distance between the peak of the environment spectral density and the TLS resonance frequency.

For a TLS interacting with the HT-Drude bath environment, only one minimum and one maximum of the excited state population could be seen well, for a wide range of frequency cut-offs $\gamma_B$ (Fig. 3(a)) and coupling strengths $\Delta_B$ (Fig. 3(b)). For small $\gamma_B$ the difference between the minimum and the maximum is the biggest, and with the growth of $\gamma_B$ it gradually vanishes: the minimum slowly rises, the maximum lowers, but much faster, until they disappear; after this, the curve evenly
rises, falling down to the steady value of the excited state population. The main difference between the RWA and non-RWA evolutions lies in the value of the first minimum. For the RWA dynamics and strong couplings it is deeper and drops lower than the stationary value, while for the non-RWA dynamics it gradually vanishes when the coupling strength increases, lying strictly above the stationary value. Because the maximum lowers too, the difference between the maximum and the minimum has a clear extremum. In overall, the non-RWA dynamics seems like a smoothed version of the RWA dynamics. The bath temperature influences the dynamics via the stationary states (Fig. 1(c)).

Dynamics of a TLS in the stochastic field environment (Fig. 2) is similar to dynamics in an infinite-temperature HT-Drude bath with the RWA interaction Hamiltonian. We have the same dependence on frequency cut-offs and coupling strengths of the random processes \( \xi_1(t) \) and \( \xi_2(t) \), but dependence on \( \Omega(t) \) is different. As we have mentioned earlier, the interaction Hamiltonian of \( \Omega(t) \) commutes with the density matrix at all times, because the reduced density matrix is diagonal for the selected initial conditions, so it impacts the dynamics only in case of presence of another decoherence channel, e.g. one of the random processes \( \xi_1(t) \) and \( \xi_2(t) \) or the bath. The frequency cut-off of \( \Omega(t) \) influences the dynamics only via the change of the stationary value, an increase in the coupling strength slightly rises the minimum, practically not affecting the maximum.

In case of a TLS interacting with both the bath and the stochastic field (Fig. 4), presence of two environments speed up the evolution, i.e. equilibrium is reached earlier in comparison with interaction with only one of the environments. All curves are shifted by the stochastic field to the left as a direct consequence of the increase in the overall decoherence rate after appearance of additional decoherence channels. Evolution itself resembles the one for interaction with the bath (or the field), but with a larger coupling strength. It can be explained by the similarities between the two environments, as it has been said, the stochastic field environment is equivalent to an infinite-temperature HT-Drude bath, if the random process \( \Omega(t) \) is absent. Another effect of a stochastic field is the rising of the stationary value. The individual components of the stochastic field influence the dynamics differently: it seems not very sensible to changes in frequency cut-offs of any random process, but an increase in \( \Delta \omega \) rises the minimum, while an increase in \( \Delta \xi_1 \) lowers the minimum, and an increase in \( \Delta \xi_1 \) lowers the maximum.

VII. EMISSION SPECTRUM

We obtain emission spectrums of a TLS by applying the Fourier transform to the two-time correlation function \(< J_+ (t_2) J_- (t_1) >\). If Hamiltonian of the OQS don’t depend on time explicitly and depend only on the time increment, spectrum of the two-time correlation function, obtained by \( t_2 \) while assuming \( t_1 \) constant, is always real and may be considered as non-stationary emission spectrum. Often the limit \( t_1 \to \infty \) is taken, implying that an OQS reaches its equilibrium state before the first TLS operator is applied, so the emission spectrum doesn’t depend on \( t_1 \) and may be considered stationary. Both stationary and non-stationary emission spectrums can be found by means of the HEOMs in three steps. First, the initial state of the system is propagated to the time moment \( t_1 \) and the operator \( J_- \) is applied to each \( \hat{R}_m (t_1) \), modifying not only the reduced density matrix \( \hat{R}_q (t_1) \), but also the memory kernel, because it lies in the TLS subspace. Then the state \( \hat{J}_- \hat{\rho}_{tot} (t_1) \) is propagated to the time \( (t_2 - t_1) \) by the same HEOM, assuming \( t_1 \) as the initial moment of time, and then \( \hat{J}_+ \) is applied. Finally, we take the trace and perform the Fourier transform.

In Fig. 5 we show emission spectrums of an initially excited TLS for interaction with the stochastic field environment (the bath is absent) and in Fig. 6 - with the HT-Drude bath environment (the field is off) for several values of the frequency cut-off and the coupling strength of the environment. In Fig. 7 influence of the stochastic field on spectrums of a TLS in the HT-Drude bath is shown.

Emission spectrums for interaction with the HT-Drude bath environment (Fig. 6) are expectedly quite similar to the ones for interaction with a stochastic field environment (Fig. 5). For large frequency cut-offs the peak is almost centered at \( \omega_0 \), when the frequency cut-off falls, the peak shifts to the right and becomes less symmetrical, its left side rises faster than the right side, the peak widens, while its intensity falls. Then at some cut-off, which value depends on the coupling strength, a peak at \( \omega = 0 \) ap-
FIG. 5. Normalized by maximum value emission spectrums of a TLS in the stochastic field in dependence on (a) the field frequency cut-off $\gamma_F$ and (b) the field coupling strength. Orange denote non-Markovian curves, purple - Markovian ones. In (a) $\gamma_F/\omega_0 = (0.1, 0.2, 0.4)$, $\Delta^2_F/\omega_0 = 0.6$, for (dotted, dashed, solid) curves, respectively, and in (b) $\gamma_F/\omega_0 = 0.2$, $\Delta^2_F/\omega_0 = (0.4, 0.6, 0.8)$.

pears, becomes more distinct, rises up, and, eventually, becomes the dominant peak, while the first one disappears. For some values of the coupling strength and the frequency cut-off, the side peak at $\omega = 0$ is shifted to the right. It is more evident when intensity of the side peak, when it appears for the first time, is close to the main peak intensity. This behavior is more typical for the non-RWA coupling, though it is also can be found for the RWA coupling. When the side-peak appears with equal to the main peak intensity, the shift is the biggest. An increase of the coupling strength shifts the main peak to the right, but widens it much stronger. Because of the widening, the side peak can’t be seen clearly, but for large couplings it splits from the main and then starts to dominate the spectrum, shifting the peak to the left and gradually approaching $\omega = 0$. An increase of temperature makes the contour more asymmetrical, its left slope becomes less steep and the right -more steep. In the Markovian approximation the side peak never appears and the left slope of the main peak is much steeper than the right. Every non-RWA emission spectrum has a zero-point firmly located at $\omega = -\omega_0$ that never disappears for the parameter sets studied.

Impact of the stochastic field on the emission spectrums of a TLS in the bath environment is shown in Fig. 7. For the values of parameters selected, the Markovian approximation is clearly inaccurate either in absence nor in presence of the stochastic field. In absence of the stochastic field it gives a much narrower contour and without the side peak at left, in presence - the spectrum curve is too wide. The stochastic field smooths the negative frequency part of the spectrum and removes the zero-point for the non-RWA TLS-bath interaction. The stochastic field as if increases the TLS-bath coupling strength, making the side peak near $\omega = 0$ higher, shifting to the right and widening the main peak.

VIII. CONCLUSIONS

We have studied non-Markovian dynamics of a TLS interacting with two types of environments, a bosonic
βℏ
derived the hierarchical equations of motion [35] allowing us to study interaction of a TLS with each of the environments independently as well as their joint action and covering both the RWA and non-RWA types of environment couplings. The HEOM derived assumes factorized initial state of the total system, so any initial correlations between a TLS and the environment are not allowed.

We considered the high-temperature Drude spectral density of the bath and two widely used types of interaction between a TLS and a bath, the RWA interaction and the exact electric-dipole interaction. We took the environment at equilibrium: the bath was taken in the thermal state, and the stochastic field was assumed stationary.

It was shown, that the minimum number of HEOM indices for interaction with a stochastic field is one and for interaction with a bosonic bath is two. In case of the HT-Drude spectral density and the exact form of the TLS-bath interaction (non-RWA), the minimum number of HEOM indices is also one, and its coefficients resemble corresponding coefficients of the TLS-stochastic field HEOM, the coincidence is perfect for the infinite temperature bath and a stochastic field with only one random process ξ(t). This observation expresses similarities between the two cases that could be seen both in the reduced density matrix dynamics and the emission spectrums.

We performed numerical analysis of evolution of the reduced density matrix, the steady states, and emission spectrums of a TLS for a wide range of the bath and field frequency cut-offs, coupling strengths and bath temperatures. The evolution of the reduced density matrix exhibits rapidly vanishing oscillatory behavior and generally has either one minimum and one maximum visible or none of them. The amplitudes and locations depend on the frequency cut-off and the coupling strength of the environment. The emission spectrums in the Markovian approximation always have one peak, which widens and moves as a response to a change in the frequency cut-off and the coupling strength. In case of interaction with one of the environments, both for the stochastic field environment and the bath environment, if the coupling is rather strong or the cut-off is rather short, there are two peaks.

We also investigated the role of the rotating-wave approximation in non-Markovian regimes of evolution and the interplay between different parts of the composite environment, when a TLS interacts with both a stochastic field and a bath.

\[ \Delta_1 = \frac{\gamma_1}{\omega_0} = 0.2, \quad \Delta_2 = \frac{\gamma_2}{\omega_0} = 0.4, \quad \text{and} \quad \gamma_B = 0.2, \quad \Delta_B = 0.6, \quad \beta\hbar\omega_0 = 0.1. \]

FIG. 7. Normalized by maximum value emission spectrums of a TLS for simultaneous interaction with the stochastic field and the bath (solid curves) in comparison with the case of interaction with the bath only (dashed curves). Orange and green denote the RWA and non-RWA couplings respectively, purple - the Markovian approximation, γ_1/ω_0 = γ_2/ω_0 = 0.2, Δ_1/ω_0 = Δ_2/ω_0 = 0.4, and γ_B/ω_0 = 0.2, Δ_B/ω_0 = 0.6, βhω_0 = 0.1.
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