Perturbative Metric of Charged Black Holes in Quadratic Gravity

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Abstract

We consider perturbative solutions to the classical field equations coming from a quadratic gravitational lagrangian in four dimensions. We study the charged, spherically symmetric black hole and explicitly give corrections up to third order (in the coupling constant $\beta$ multiplying the $R^{\mu\nu}R_{\mu\nu}$ term) to the Reissner–Nordström hole metric. We consider the thermodynamics of such black holes, in particular, we compute explicitly its temperature and entropy–area relation which deviates from the usual $S = A/4$ expression.

04.50.+h, 04.70.-s, 04.70.Dy, 04.70.Bw
I. INTRODUCTION

Quadratic or fourth-order gravity appears to be a good candidate to represent the low energy limit of the, yet unknown, quantum theory of gravity. In fact, it arises as the necessary counterterms one encounters when trying to renormalize semiclassical theory at one-loop level [1], and as the low-energy limit of string theory (ignoring the dilaton and antisymmetric fields) [2]. Consequently, it is interesting to study the classical solutions of such effective theory. In Ref. [3] we have developed a method to solve the field equations of the quadratic gravitational theory in four dimensions coupled to matter (see also Refs. [4] where a closely related approach was independently derived from a formal perturbation theory). The quadratic terms are written as a function of the matter stress tensor and its derivatives in such a way to have, order by order, Einsteinian field equations with an effective $T_{\mu \nu}$ as source. By successive perturbations around a solution to Einstein Gravity, which for us represent the zeroth order, one can build up approximate solutions. In Ref. [3], we applied this perturbative procedure to find first order solutions in the coupling constant $\beta$ for the charged black hole (where in this particular case $\alpha$ contributions vanish).

We consider the following Lagrangian formulation of quadratic theories

$$I = I_G + I_m = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -2\Lambda + R + \alpha R^2 + \beta R_{\mu \nu} R^{\mu \nu} + 16\pi L_m \right\}, \quad (1)$$

where we have set $G = 1$ and $c = 1$ for simplicity.

For the perturbative approach to properly work we consider only small curvatures, such that

$$\alpha |R| \ll 1, \quad |\beta R_{\mu \nu}| \ll 1. \quad (2)$$

The coupling constants $\alpha$ and $\beta$ are expected to be of the order of the Planck length squared. They already have upper bounds of the atomic size order from observations in the solar system, binary pulsars and cosmology.

It is worth to stress that the coupling constants $\alpha$ and $\beta$ must fulfill the, so called, no-tachyon constraints

$$3\alpha + \beta \geq 0, \quad \beta \leq 0, \quad (3)$$

that can be deduced upon linearization and asking for a real mass for both the scalar field $\phi$ related to $R$ and the spin-two field $\psi_{\mu \nu}$ related to $R_{\mu \nu}$ (see Ref. [4] for further details).

The field equations derived by extremizing the action $I$ can be rewritten to the $n$-th order approximation as [3]

$$R_{\mu \nu}^{(n)} - \frac{1}{2} R^{(n)} g_{\mu \nu}^{(n)} + \Lambda g_{\mu \nu}^{(n)} = 8\pi T_{\mu \nu}^{\text{eff} (n)} = 8\pi T_{\mu \nu} (g_{\mu \nu}^{(n)})$$

$$-\alpha H_{\mu \nu} (g_{\mu \nu}^{(n-1)}, T_{\mu \nu}^{(n-1)}) - \beta I_{\mu \nu} (g_{\mu \nu}^{(n-1)}, T_{\mu \nu}^{(n-1)}), \quad (4)$$

where the zeroth-order corresponds to the ordinary Einstein equations and
\[ H_{\mu\nu} = -2R_{\mu\nu} + 2g_{\mu\nu} \Box R - \frac{1}{2} g_{\mu\nu} R^2 + 2RR_{\mu\nu} , \tag{5} \]

and

\[
I_{\mu\nu} = -2R^\alpha_{\mu \nu} :_{\nu\alpha} + \Box R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Box R \\
+ 2R^\alpha_{\mu} R_{\alpha\nu} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} . \tag{6} \]

In Section II we explicitly compute the effective stress tensor, metric coefficients and horizon radius to third order in the coupling constant \( \beta \). In Section III we use these results to analyze the thermodynamic properties of charged black holes in the quadratic theory of gravity. We then briefly discuss the validity of the laws of thermodynamics for black holes in theories of gravitation different from Einstein’s general relativity. Finally, we discuss the relevance of the higher order \( \beta \)–corrections in the computations of the extreme black hole case, and in the “stabilization” of the quadratic solution around the general relativistic metric.

**II. CHARGED BLACK HOLE METRIC**

We shall now study spherically symmetric solutions to the quadratic field equations representing charged black holes. Its metric can be written (in the Schwarzschild gauge) as

\[ ds^2 = g_{tt}(r) dt^2 + g_{rr}(r) dr^2 + r^2 d\Omega^2 , \tag{7} \]

where \( d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \). The starting point is the General Relativistic solution, i.e. the Reissner–Nordström metric, \(-g_{tt}(r) = g_{rr}(r)^{-1} = (1 - 2M/r + Q^2/r^2)\).

The non-vanishing components of the energy-momentum tensor are \( \emph{6} \)

\[ T^t_t = T^r_r = -T^{\vartheta}_\vartheta = -T^\varphi_\varphi = -\frac{Q^2}{8\pi r^4} . \tag{8} \]

The exact metric, solution of the Einstein equations with the effective source, Eq. \( \emph{4} \), can be written as \( \emph{5} \) [in the Schwarzschild gauge, Eq. \( \emph{6} \)]

\[ g^{-1}_{rr}(r) = 1 - \frac{2M}{r} + \frac{8\pi}{r} \int_r^\infty \tilde{r}^2 T_{t\tilde{r}}^{\emph{eff}} d\tilde{r} , \tag{9} \]

\[ g_{tt}(r) = -g_{rr}(r)^{-1} \exp \left\{ 8\pi \int_r^\infty \left( T^r_r - T^t_t \right)^{\emph{eff}} \tilde{r} g_{rr}(\tilde{r}) d\tilde{r} \right\} . \tag{10} \]

We use Eq. \( \emph{8} \) as the zeroth order effective \( T_{\mu\nu} \) to compute the higher order \( T_{\mu\nu}^{\emph{eff}} \) (Eq. \( \emph{4} \)) we use a program of analytic manipulation within “Mathematica” \( \emph{3} \) to find up to third order corrections to the energy-momentum tensor.
\[ T_{r}^r = -\frac{Q^2}{8\pi r^4} + \frac{4\beta Q^2}{8\pi r^6} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) - \frac{\beta^2 Q^2}{5\pi r^8} \left(45 - 220\frac{M}{r} + 280\frac{M^2}{r^2} + 135\frac{Q^2}{r^2} - 345\frac{MQ^2}{r^3} + 106\frac{Q^4}{r^4}\right) - \frac{\beta^3 Q^2}{210\pi r^{10}} \left(75600 - 635040\frac{M}{r} + 1774080\frac{M^2}{r^2} + 430080\frac{Q^2}{r^2} - 2432640\frac{MQ^2}{r^3} - 1646400\frac{M^3}{r^3} + 847908\frac{Q^4}{r^4} + 3385200\frac{M^2 Q^2}{r^4} - 2323965\frac{MQ^4}{r^5} + 525476\frac{Q^6}{r^6}\right) + \mathcal{O}(\beta^4) , \]

(11)

\[ T_{t}^t = -\frac{Q^2}{8\pi r^4} + \frac{3\beta Q^2}{2\pi r^6} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) - \frac{\beta^2 Q^2}{5\pi r^8} \left(225 - 1140\frac{M}{r} + 1400\frac{M^2}{r^2} + 745\frac{Q^2}{r^2} - 1755\frac{MQ^2}{r^3} + 528\frac{Q^4}{r^4}\right) - \frac{\beta^3 Q^2}{710\pi r^{10}} \left(176400 - 1542240\frac{M}{r} + 4294080\frac{M^2}{r^2} + 1123920\frac{Q^2}{r^2} - 5993680\frac{MQ^2}{r^3} - 3841600\frac{M^3}{r^3} + 2005316\frac{Q^4}{r^4} + 7772240\frac{M^2 Q^2}{r^4} - 5069085\frac{MQ^4}{r^5} + 1074388\frac{Q^6}{r^6}\right) + \mathcal{O}(\beta^4) . \]

(12)

Since the effective energy-momentum tensor keeps its property of being traceless, the other two non-vanishing components can be deduced from the two above, i.e.

\[ T_{t}^\theta = T_{t}^\varphi = \frac{-1}{2} (T_{t}^t + T_{r}^r) , \]

(13)

Replacing this into Eqs. (9)–(11) we obtain the metric components

\[ g_{rr}^{-1}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{2\beta Q^2}{5r^4} \left(10 - \frac{15M}{r} + 6\frac{Q^2}{r^2}\right) + \frac{\beta^2 Q^2}{105r^6} \left(7560 - 31920\frac{M}{r} + 33600\frac{M^2}{r^2} + 17880\frac{Q^2}{r^2} - 36855\frac{MQ^2}{r^3} + 9856\frac{Q^4}{r^4}\right) + \frac{\beta^3 Q^2}{5005r^8} \left(14414400 - 110270160\frac{M}{r} + 272912640\frac{M^2}{r^2} + 71431360\frac{Q^2}{r^2} - 342838496\frac{MQ^2}{r^3} - 219739520\frac{M^3}{r^3} + 104276432\frac{Q^4}{r^4} + 404156480\frac{M^2 Q^2}{r^4} - 241626385\frac{MQ^4}{r^5} + 47273072\frac{Q^6}{r^6}\right) + \mathcal{O}(\beta^4) , \]

(14)

\[ g_{tt}(r) = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) + \frac{2\beta Q^2}{5r^4} \left(5 - \frac{5M}{r} + \frac{Q^2}{r^2}\right) - \frac{\beta^2 Q^2}{105r^6} \left(2520 - 8400\frac{M}{r} + 6720\frac{M^2}{r^2} + 3600\frac{Q^2}{r^2} - 5355\frac{MQ^2}{r^3} + 952\frac{Q^4}{r^4}\right) + \frac{\beta^3 Q^2}{r^8} \left(-720 + 4656\frac{M}{r} - 9600\frac{M^2}{r^2} - \frac{13504Q^2}{5r^2}\right) . \]
\[
\begin{align*}
&+ \frac{592864 \, MQ^2}{r_3} + 6272 \frac{M^3}{r^3} + \frac{3403504 \, Q^4}{r^4} + \frac{112064 \, M^2 Q^2}{r^4} \\
&+ \frac{2295703 \, MQ^4}{r^5} - \frac{1235296 \, Q^6}{r^6} + 64 M^2 Q^4 + \frac{11}{91} + \mathcal{O}(1/r^7) + \mathcal{O}(\beta^4). \tag{15}
\end{align*}
\]

Since the complete expression for \( g_{tt} \) is already rather complicated we do not pursue the computation of higher order corrections.

The validity of this metric will be assured if the condition Eq. (2) holds. In our case, this takes the form \(-\beta Q^2/r^4 \ll 1\). Thus, Eqs. (15)–(14) will be a good approximation to the (even extremely) charged black hole solutions in quadratic theories if

\[ r_H \gg \sqrt{-\beta}, \tag{16} \]

where \( r_H \) is the radial coordinate of the event horizon (in Schwarzschild’s gauge).

The radial coordinate of the horizon can be computed from making \( g_{tt} \) vanish given by Eq. (15). To first order in \( \beta \) we have (for non-extreme black holes)

\[
\begin{align*}
&\quad r_H = r_+ + \frac{\beta Q^2 \left(5r_+^2 - 3Q^2\right)}{5r_+^2 (r_+^2 - Q^2)} - \frac{\beta^2 Q^4 \left(2925r_+^6 - 6515Q^2 r_+^4 + 5095Q^4 r_+^2 - 1337Q^6\right)}{1050r_+^4 (r_+^2 - Q^2)^3} \\
&\quad - \frac{\beta^3 Q^2}{750750r_+^{11} (r_+^2 - Q^2)^5} \left(606000r_+^{14} - 4357050Q^2 r_+^{12} + 75284175Q^4 r_+^{10} - 79088475Q^6 r_+^8 \\
&\quad + 36916630Q^8 r_+^6 - 435320Q^{10} r_+^4 - 5714693Q^{12} r_+^2 + 1442637Q^{14}\right) + \mathcal{O}(\beta^4), \tag{17}
\end{align*}
\]

where \( r_+ = M + \sqrt{M^2 - Q^2} \).

We observe that the quadratic black hole shrinks with respect to the corresponding general relativistic one when we consider up to \( \beta^2 \) terms, but the next order already tends to stabilize the solution near the general relativistic value. In fact, we can expect \( r_H > 0 \) for a wide range of \( \beta \), thus ensuring the existence of charged black holes in quadratic theories of gravitation.

The extreme black hole will be now reached with a maximal charge different from the general relativistic one

\[
Q_{\text{extr}}^2 = M^2 + \frac{2}{5} \beta + \frac{4}{525} \frac{\beta^2}{M^2} + \mathcal{O}(\beta^3), \tag{18}
\]

thus \( r_H \) given by Eq. (17), will remain always bounded. In fact,

\[
r_{\text{extr}}^H = M + \frac{\beta}{5M} + \mathcal{O}(\beta^2). \tag{19}
\]

[Note that since \( g_{tt} \sim (r - r_{\text{extr}}^H)^2 \) for an extreme black hole to obtain \( r_{\text{extr}} \) to order \( \beta^n \) we need to know \( g_{tt} \) to order \( \beta^{2n} \)].

The horizon area, \( A_H \), is given by

\[
A_H = 4\pi r_H^2 = 4\pi r_+^2 + \frac{8\pi \beta Q^2}{5r_+^2 (Q^2 - r_+^2)} \left(3Q^2 - 5r_+^2\right)
\]
III. THERMODYNAMIC PROPERTIES OF THE PERTURBATIVE SOLUTION

In the theory of black holes in classical general relativity a mathematical analogy was discovered between certain laws of black hole mechanics and the ordinary laws of thermodynamics. With the discovery of quantum particle creation near black holes, this analogy appeared to be something more than a simple mathematical analogy.

Very recently, some aspects of this significative relation between black hole mechanics and thermodynamics have been studied in a more general context than in Einstein’s general relativity. In particular, here, we want to know for our approximate solution what is the explicit form of the laws of black hole mechanics.

As it is known the zeroth law of black hole mechanics asserts that the surface gravity $\kappa$ (where $\kappa$ is defined by $\xi^b \nabla_b \xi^a = \kappa \xi^a$ with $\xi^a$ is the null generator of horizon) is constant all over the event horizon of a stationary black hole. The proof of this law makes direct use of the specific form of Einstein field equations and matter dominant conditions [10], and it was not extended to others theories of gravity. Anyway, it can be easily shown that the constancy of the surface gravity trivially holds in the spherically symmetric case and can be computed using Eq. (21) below.

In fact, we can safely compute the Bekenstein-Hawking temperature from the surface gravity, $\kappa$, since this equation was found euclideanizing the black hole metric without making any assumption on the field equations that it fulfilled [4],

$$T_H = \frac{\kappa}{2\pi} = - \frac{1}{4\pi} \frac{g'_{tt}}{\sqrt{-g_{tt}}}, \quad \text{at } r=r_H,$$

where $g'_{tt} = \partial g_{tt}/\partial r$.

Thus, in our approximation, for non-extreme black holes we have,

$$T_H = \frac{1}{4\pi r_+^2} \left( r_+^2 - Q^2 \right) + \beta \frac{Q^4}{525r_+^8(Q^2 - r_+^2)^3} \left( 168Q^8 - 676Q^6r_+^2 + 791Q^4r_+^4 - 100Q^2r_+^6 - 225r_+^8 \right)$$

$$+ \beta^2 \frac{Q^4}{3003000\pi r_+^{11}(Q^2 - r_+^2)^5} \left( -1848000Q^{18} - 46018351687Q^{16}r_+^2 ight)$$

$$+ 85649730010Q^{14}r_+^4 + 215820442547Q^{12}r_+^6 - 752950763928Q^{10}r_+^8$$

$$+ 751221815775Q^8r_+^{10} - 212346839950Q^6r_+^{12} - 88152761275Q^4r_+^{14}$$

$$+ 46904894700Q^2r_+^{16} - 126126000r_+^{18} + \mathcal{O}(1/r_+) \right) + \mathcal{O}(\beta^4), \quad (22)$$
Note that, since $\beta < 0$, the effect of the quadratic gravitational theory corrections at first order in $\beta$ will be that of decreasing the black hole radiation temperature with respect to the general relativistic value (with the same $M$ and $Q$), leaving thus out open the possibility of switching off black hole evaporation and leaving behind a charged remnant with a mass of the order of the Planck mass. Higher order corrections, however, alternate sign. To further discuss this possibility one should have an exact solution and establish the values of $\beta$ that allow switching off of the black hole temperature. From Eq. (18) and Eq. (19) we can check that $T_H(Q_{\text{max}}) = 0$ up to order $\beta$.

The question of the first law of black hole mechanics in a general theory of gravity was largely discussed in Refs. [11,12] and clearly holds in our case, since the quadratic theory is derived from the diffeomorphism invariant lagrangian (1). Inverting and integrating the fundamental relation

$$dM = \frac{\partial M}{\partial A} \bigg|_Q dA + \frac{\partial M}{\partial Q} \bigg|_A dQ = T_H dS + \Phi_H dQ,$$

in Ref. [3], we found, to first order in $\beta$, an approximate expression for the entropy of the charged black hole, in which clearly appears not simply one quarter of the area of the horizon, but in general a more complicated function of the area, $A$ and charge, $Q$.

Recently, in Ref. [12], Wald has developed a mathematical rigorous method to derive the exact formal expression of the entropy of a stationary black hole in any general covariant theory of gravity. In fact, the entropy was found to be a local geometrical quantity expressed as an integral evaluated on an arbitrary cross-section of a killing horizon over the associated Noether charge.

In Ref. [13] Wald’s Noether charge techniques have been explicitly applied to a lagrangian theory such as (1), which is clearly generally covariant (invariant under general diffeomorphism transformations). Their general expression for the entropy can be written as

$$S = \frac{1}{4} \int_{\Sigma} d^2x \sqrt{h} \left[ 1 + 2\alpha R + \beta g^{\mu\nu} R_{\mu\nu} \right],$$

where $\Sigma$ is any arbitrary cross-section of the horizon, $d^2x \sqrt{h}$ is the intrinsic volume element and where $g^{\mu\nu}_\perp = g^{\mu\nu} - h^{\mu\nu} = (\xi^\mu \chi^\nu + \chi^\mu \xi^\nu)$ is the metric in the subspace normal to the horizon, with $\xi^\mu$ the stationary killing field and $\chi^\mu$ a vector field orthogonal $\Sigma$ satisfying $\chi^\rho \xi_\rho = 1$ on the horizon.

In fact, making use of expression (24), the entropy for our approximate solution takes the following form

$$S = \frac{1}{4} \int_{r=r_H} d\vartheta d\varphi \sqrt{g_{\vartheta\vartheta} g_{\varphi\varphi}} [1 + 2\beta g^{tt} R_{tt}] = \frac{1}{4} \left[ 1 + 16\pi \beta T^{\text{eff}}_t \right]_{r=r_H} \times \int_{r=r_H} d\vartheta d\varphi \sqrt{g_{\vartheta\vartheta} g_{\varphi\varphi}} \times$$

$$\left[ A^4 - 8\pi^2 \beta Q^2 \frac{A^2}{A^3} - 1024\pi^4 \beta^3 \frac{Q^2}{A^5} (A - 4\pi Q^2)^2 + 16384\pi^5 \beta^4 \frac{Q^2}{5A^7} (A - 4\pi Q^2)^{-1} \times \right.$$

$$\left. (25A^4 - 470\pi Q^2 A^3 + 3506\pi^2 Q^4 A^2 - 11120\pi^3 Q^6 A + 12448\pi^4 Q^8) + O(\beta^5) \right],$$

where we have taken the bifurcation surface as the cross-section of the horizon $\Sigma$ and where, in the charged spherically symmetric case, $R = 0$ and $g^{tt}_\perp = 2/g_{tt}$ is the only non vanishing component of the metric in the subspace normal to the horizon. In addition, Eq. (12) was
used for computing the term $g^{tt}R_{tt} = 8\pi T_{tt}^{\text{eff}}$, and the area of the event horizon is given by Eq. (20).

Note that the knowledge of the metric to $O(\beta^3)$ allowed us to compute the entropy up to $O(\beta^4)$. That means that from the expression of the Reissner–Nordström metric we can compute $S$ to $O(\beta)$, and thus recover the results of Refs. \cite{7,14} without having to solve any further field equation for $g_{\mu\nu}$. The vanishing of the term proportional to $\beta^2$ is a casual fact due to the particular dependence of $T_{tt}^{\text{eff}}$ to $O(\beta)$ (that order vanishes when evaluated on the horizon).

In Ref. \cite{13} it was noted that, since the action Eq. (1) “is linear in $\alpha$ and in $\beta$, the modifications to the entropy from higher curvature terms in the original action would only be linear in $\alpha$ and in $\beta$”. The precise meaning of this statement could not appear clear to the reader at first sight. When we write explicitly the entropy as a function of the horizon area and charge, we find corrections coming from quadratic curvature terms in the general formula Eq. (24) that are not linear, but of $(n+1)$-th order in $\beta$, if the metric contains $n$-th order terms in this coupling constant.

To continue our previous discussion on the validity of the laws of black hole mechanics in the case of our approximate solution, we consider the the second law of thermodynamics, which states that the entropy must be an always increasing function. The theorem that establishes that the area of the black hole horizon must be an always increasing quantity, was proved only for general relativity \cite{15}.

Although any proof of the second law of black hole mechanics in a general theory of gravity has not been given yet, it seems to exist a relation between its validity and the positive energy condition of the theory. In the case of lagrangian theory examined in this paper (Eq. (1)), the positive energy condition does not holds in general, as we can see from the linearized field equations given in Ref. [7], where the presence of the spin two massive tensorial field $\psi^{ab}$ allows for negative values of the energy conditions on the energy-momentum tensor.

As in general relativity, here, the third law of thermodynamics should hold in the form due Nerst: Loosely speaking, “it is very hard to reach $T_H = 0$”. The version due to Planck does not hold, since from Eqs. \cite{18,19} we have

$$ S(T_H = 0) = \pi M^2 - \frac{8\pi}{5} \beta + O(\beta^2/M^2). \quad (26) $$

This ground state is clearly degenerate due to the general relativistic term. The contribution of quadratic theories is (to first order) only to contribute with an universal constant (function of $\beta$).

With regards to the fourth law or the scaling laws fulfilled by the black hole critical exponents \cite{16,17}, we expect them to hold also for our approximate solution. In particular, the black hole critical exponents should be the same as in general relativity, since the Reissner–Nordström solution gives the leading behavior and since the validity of the universality hypothesis \cite{17}.

IV. DISCUSSION

The aim of this work was two-fold: First to see if, in practice, our perturbative approach allowed to be carried out to orders higher than the first. This has been shown to be possible
explicitly up to third order. The second objective of this work was to see if from the higher orders one could guess some recursive formulae that would lead us to an exact solution. So far, we have not fulfilled this expectation, but may be the reader can do it. We can, however, try to say something about the $\beta$–gravity (or $\beta$–sector of the quadratic theories): Although the first order corrections seem to indicate that $\beta$–gravity is weaker than General Relativity, we can see (for example in $Q_{\text{max}}$, $r_H$ and $T_H$) that to higher order we have alternateness in the sign of the corrections (we needed to go up to 3rd. order in $\beta$ to check that). Using the Noether charge method we have been able to express the black hole entropy in terms of the horizon area and charge up to 4th. order in $\beta$, and (in spite of the casual vanishing of the $O(\beta^2)$ term) we have shown that the dependence of $S$ on $\beta$ is not simply linear. We have also studied the extremely charged black hole. There, due to the particularity of this solution, to obtain $r_H^{\text{extr}}$ (and the other quantities evaluated on the extreme horizon such as $T_H$ and $S$) to the first order in $\beta$ we had to compute $g_{\mu\nu}$ to order $\beta^2$.

From the fact that our perturbative method can not go to the high curvature regime, unfortunately, we cannot say anything about the most interesting issue of the singularity at $r = 0$.

We are presently dealing with the rotation effects, i.e. the $\beta$ first–order corrections to the Kerr–Newman solution and the results will be published elsewhere [18].

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