Asymptotic transition from Fourier series to integrals in LGT

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Abstract

It is shown that in asymptotic transition from Fourier series to integrals an error and ambiguity may arise. Ambiguity reduces to a possibility of addition of some distribution to the result. Properties of such distributions are studied and conditions are established under which ambiguity doesn’t arise. Method for correction computation is suggested and conditions for correction turning to zero are specified.

1 Introduction

In a continuum transition in a gauge theory on a lattice with length \( N_\mu \) in \( \mu = 0, 1, 2, 3 \) directions and periodic border conditions, Fourier series in a discrete variable \( \varphi_\mu = 2\pi n_\mu/N_\mu \) \( (n_\mu = 0, ..., N_\mu - 1) \) must transform into series in a continual variable \( 0 \leq \varphi_\mu < 2\pi [1, 2] \). In a continuum limit \( N_\mu \to \infty \) lattice spacings \( a_\mu \) in \( \mu \) directions turn into zero and physical lengths \( a_\mu N_\mu = L_\mu \) remain finite. In a thermodynamical limit \( L_1 L_2 L_3 \to \infty \), Fourier series in a continual variable must transform into Fourier integrals. Formal change series for integrals is applicable only for “smooth enough” functions. Although the lattice action include only regular functions, in a limit \( a_\mu \to 0 \) they become, in a general case, distributions. This may appear not only in gluodynamics, but in presence mater fields as well, as it is shown by fermion determinant computation on extremely anisotropic lattice [3]. In this paper we consider transitions from Fourier series to integrals for tempered distributions \( F(x) \in S' \) (which satisfy some additional restrictions specified below) that are defined as functional \( \langle F(x), \phi(x) \rangle \) on fast decreasing test functions \( \phi(x) \in S \) and that may be represented as finite order derivatives of \( F(x) = G^{(n)}(x) \), \( n < \infty \) of some tempered functions \( |G(x)| < |x|^{\sigma}, x \to \pm\infty, \sigma < \infty \) [4, 5, 6].

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A transition from series to integrals requires an analytical continuation function of a discrete variable to continual one, that is unique only for functions which satisfy set of conditions of a Carlson theorem version [7, 8]. However, in a case of lattice gauge theories (LGT) such conditions may appear to be too restrictive. Therefore we shall determine what class function may introduce an ambiguity in a general case and conditions under which no ambiguity appears will be specified.

2 Transition from Fourier series to Fourier integral

The transition Fourier series into integral with the infinite increasing of period is used in [9] for the definition of the Fourier integral. One starts from locally summable function \( F(x) \) and constructs a truncated function \( \overline{F}(x) \) as

\[
\overline{F}(x) = \begin{cases} F(x) & -\pi \tau < x < \pi \tau \\ 0 & \text{otherwise} \end{cases}
\]

(1)

after that function \( \overline{F}(x) \) is extended periodically (with the period \( 2\pi \tau \)) along the whole real axis \( x \) by the change \( \overline{F}(x) \rightarrow \overline{F}(x/\tau) \) where \( \overline{F}(\varphi) \) is the periodic function\(^1\) with the period \( 2\pi \) and

\[
\overline{F}(x/\tau) = F(x) = \overline{F}(x);
\]

(2)

for \(-\pi \tau < x < \pi \tau\). Periodic function \( \overline{F}(x/\tau) \) may be represented as the Fourier series

\[
\overline{F}(x/\tau) = \sum_{n=-\infty}^{\infty} F_n \exp \{inx/\tau\}
\]

(3)

where Fourier coefficients \( F_n \) are given by the standard expression

\[
F_n = \frac{1}{2\pi \tau} \int_{-\pi \tau}^{\pi \tau} \overline{F}(x/\tau) \exp \{-inx/\tau\} \, dx
\]

(4)

After [9] we split the region of summation \(-\infty < n < \infty\) into intervals (‘bursts’) \( \tau k - \tau \delta k/2 < n < \tau k + \tau \delta k/2 \) and \textit{approximately} change\(^2\) the sum in each interval for its value in the midpoint

\[
\frac{1}{\tau \delta k} \sum_{n=\tau k - \tau \delta k/2}^{\tau k + \tau \delta k/2} \exp \{inx/\tau\} F_n \simeq [\exp \{inx/\tau\} F_n]_{n=\tau k} = \exp \{ikx\} F_{k \tau}
\]

(5)

\(^1\)Further a tilde marks periodic functions with the period \( 2\pi \).

\(^2\)Although term ‘approximately’ is not specified in [9], one may interpret it so that the approximate equality is assumed to become an exact one with simultaneous decreasing \( \delta k \) and \( 1/\tau \).
After formal change $n \to k\tau$ we get from (4)

$$
\frac{1}{2\pi} \int_{-\pi\tau}^{\pi\tau} \tilde{F}(x/\tau) \exp\{-ikx\} \, dx = f_k \delta k
$$

(6)

where it is defined

$$
f_k = \tau F_{k\tau}
$$

(7)

It is assumed that after proceeding to limit $\tau \to \infty$, the expression (6) is converted into the inverse Fourier transform

$$
f_k = \frac{1}{2\pi} \int_{-\pi\tau}^{\pi\tau} \tilde{F}(x/\tau) \exp\{-ikx\} \, dx
$$

(8)

Although initially the small value $\delta k$ is treated as an arbitrary one, if we claim that (5) and (6) become exact equations in a limit $\tau \to \infty$, then from (6) we conclude, that it may happen only if we impose the condition

$$
\tau \delta k = 1
$$

(9a)

so that each 'burst' includes only one element between $\tau k - 1/2$ and $\tau k + 1/2$ and this 'burst' contains the sole term $\exp\{ikx\} F_{k\tau}$. In other words, the transition from series to integral is done, in fact, by the formal substitution

$$
n \to k\tau; \sum_{n=-\infty}^{\infty} \to \tau \int_{-\infty}^{\infty} dk; \quad F_n \to F_{k\tau} \equiv \frac{1}{\tau} f_k.
$$

(10)

As it will be shown, the formal transition

$$
f_k^{\text{formal}} = \lim_{\tau \to \infty} \tau F_{k\tau}
$$

(11)

is admissible only under some conditions and the Fourier transform of $f_k^{\text{formal}}$ may differ from $F(x)$, so we introduce for it a specific notation

$$
F^{\text{formal}}(x) = \int_{-\infty}^{\infty} f_k^{\text{formal}} \exp\{ikx\} \, dk.
$$

(12)

However, if $F(x)$ is computed as

$$
F(x) = \lim_{\tau \to \infty} \tilde{F}(x/\tau),
$$

(13)

the result must be true without any additional conditions on $\tilde{F}(x/\tau)$ (and consequently on $F_{k\tau}$) therefore this result we treat as an exact one.

### 3 Ambiguity introduced by the formal transition

The formal transition is fully justified, when it used in [9] for definition of Fourier integral. When such definition is done, even for some restricted class
of functions, it may be extended even on arbitrary distributions and it needs no reference to the original Fourier series. One-to-one correspondence is needed only for the function and its Fourier transform. In this paper, however, just the transition from Fourier series to Fourier integral and their interrelations are of particular interest. Hence the degree of ambiguity in such transition deserves a more detailed consideration.

For a formal transition it is necessary to extend discrete function $F_n$ to continuous one $F_t$. We claim that $F_n$ may be reconstructed uniquely from $F_t$, i.e., $F_t$ must be continuous at integer values of argument $t = n$. Since distributions do not obligatory have a definite value at the point, we confine ourself to such $F_t \in S'$, which are locally continuous at integer values of argument $t = n$ and denote a family of such distributions as $S'_Z$.

Recall that the distribution is locally continuous at $t = t_0$, if there exists a function $\psi_0 \left[ t_0 \right]$ continuous in the arbitrary small but finite vicinity of $t_0$ such that

$$\langle \Psi_t, \phi (t) \rangle = \langle \psi_0 \left[ t_0 \right], \phi (t) \rangle$$

for all $\phi (t) \in S$ with a support located in this vicinity [4, 5].

Since any linear combination of distributions from $S'_Z$ is locally continuous at $t = n$, $S'_Z$ forms a linear space as well and $S'_Z \subset S'$. It is clear, that $\mathcal{F}[F_t]$ for $F_t \in S'_Z$ also forms a linear space due to linearity of Fourier transform and we denote it as $S'_F$.

It evident, that the analytical extension $F_n \rightarrow F_t$ is not unique in a general case, since one may add to $F_t$ any distribution $\omega_t \in S'_Z$ that locally turns into zero at all integer values of $t$. The family of such distributions $\{\omega_t\}$ forms linear space $\Omega$. Indeed, $\Omega$ includes zero element and $\alpha_1 \omega_{1,t} + \alpha_2 \omega_{2,t} \in \Omega$ for any $\omega_{k,t} \in \Omega$ and arbitrary complex numbers $\alpha_k$. The family of Fourier transforms

$$\omega (\varphi) = \int_{-\infty}^{\infty} \exp \{i\varphi t\} \omega_t dt, \quad \omega_t \in \Omega$$

forms linear space $\Omega^F = \{\omega (\varphi)\}$.

If there are some restrictions on family of functions to which $F_t$ must belong, it may lead to restrictions on $\omega_t$ and partially or completely reduce ambiguity. In particular, if it is demanded that $F_t$ must satisfy set of conditions of a Carlson theorem version [7, 8] an extension $F_n \rightarrow F_t$ will be unique. However, if one demands $F_n$ to be represented in a form [26] and takes this expression as an analytical extension $F_n \rightarrow F_t$, such extension do not satisfy Carlson theorem conditions, because in

$$\lim_{r \rightarrow \infty} \frac{1}{r} \ln \frac{|F_{c+ir}|}{|F_{c-ir}|} \leq 2\pi, \quad c = const,$$

the equality sign cannot be excluded.

Nonetheless, this extension is unique. Indeed, let us assume that one may add to $F_t$ some $\omega_t \in \Omega$, which may be represented as

$$\omega_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\omega} (\varphi) \exp \{-i\varphi t\} d\varphi.$$

4
However, if $\omega_t$ turns into zero for all integer $t = n$ it leads to $\tilde{\omega}(\varphi) \equiv 0$ and, consequently, to $\omega_t \equiv 0$, not only for ordinary functions, but for distributions. In other words, there is only one distribution in $\Omega$ that may be represented in a form (17), and namely $\omega_{0,t} \equiv 0$.

4 Properties of $\Omega$ and $\Omega^F$ spaces

For a more detailed clarification of $\omega_t \in \Omega$ properties we need the following procedure. Let us put in correspondence to any $F(\varphi) \in S'$ some distribution $\tilde{F}(\varphi) \equiv \hat{\Sigma} F(\varphi) \equiv \sum_{n=-\infty}^{\infty} F(\varphi + 2\pi n)$. (18)

Procedure (18), is used not infrequently (see e.g. [9]), but as a rule it is applied to functions $F(\varphi)$ that decrease quite rapidly with $\varphi \to \pm \infty$. We apply it to distributions.

As it is known (see e.g. [5, 6]) distribution $F(\varphi)$ may be represented as $F(\varphi) = F_+ (\varphi + i\varepsilon) - F_- (\varphi - i\varepsilon)$, where $\varepsilon$ is a routine positive parameter, which is regarded as arbitrary small but finite and $F_+/F_-$ are some functions regular in the upper/lower half-plane $\varphi$. If $F(\varphi) \in S'$ one can choose

$$F_+ (\varphi) = \int_0^\infty F_t e^{it\varphi} dt, \quad F_- (\varphi) = - \int_0^\infty F_t e^{it\varphi} dt$$

for Fourier integrals and

$$F_+ (\varphi) = \sum_{n=0}^{\infty} F_n e^{i\varphi n}, \quad F_- (\varphi) = - \sum_{n=-\infty}^{-1} F_n e^{i\varphi n},$$

for series, because in this case $F_t \in S'$ and $F_n$ is discrete tempered function, so $F_+/F_-$ are regular in the upper/lower half-plane $\varphi$.

Such representation is equal to Abel-Poisson regularization of Fourier integrals $F(\varphi) = \mathcal{F}[F_t, \varphi] = \int_{-\infty}^{\infty} F_t e^{it\varphi} dt \rightarrow \int_{-\infty}^{\infty} F_t e^{it\varphi - \varepsilon|t|} dt$ (21) and series $\tilde{F}(\varphi) \equiv \sum_{n=-\infty}^{\infty} F_n e^{i\varphi n} \rightarrow \sum_{n=-\infty}^{\infty} F_n e^{i\varphi n - \varepsilon|n|}$, (22) that provides a uniform convergence which is necessary for changing the order of integration and summation. If the opposite is not specified, it is assumed that such regularization is done.

Writing (18) as

$$\tilde{F}(\varphi) = \hat{\Sigma} F(\varphi) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\varphi t} \delta (t - n) F_t dt,$$  (23)
we see that the integrand contains a product of distributions, that is undefined in a general case. However, since we confine ourself to \( F_t \in S'_Z \) and such \( F_t \) are locally continuous at \( t = n \), then product \( \delta (t - n) F_t \) is always defined. Therefore, procedure (18) holds to \( F(\varphi) \in S'_Z \) and from (23) we obtain

\[
\tilde{F}(\varphi) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\varphi t} \delta (t - n) F_t dt = \sum_{n=-\infty}^{\infty} e^{i\varphi n} F_n, \quad (24)
\]

Hence \( \tilde{F}(\varphi) \) is a periodic function and coefficients of its Fourier series are defined by a standard expression

\[
F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(\varphi) \exp \{-i\varphi n\} d\varphi. \quad (25)
\]

It is easy to see that \( \omega(\varphi) \in \Omega^F \) if and only if

\[
\hat{\Sigma} \omega(\varphi) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{i\varphi t\} \delta (t - m) \omega_t dt \equiv \bar{\omega}(\varphi) = 0. \quad (26)
\]

Since \( \hat{\Sigma} \) turns into zero any \( \omega(\varphi) \in \Omega^F \), then \( \Omega^F \) belongs to kernel of the operator \( \hat{\Sigma} \).

Any periodic function

\[
\tilde{\Phi}(\varphi/\tau) = \sum_{n=-\infty}^{\infty} \phi_n \exp \{in\varphi/\tau\} \quad (27)
\]

with the period \( 2\pi \tau \), where \( \tau \) is not a rational number, belongs to kernel of the operator \( \hat{\Sigma} \) as well, since

\[
\hat{\Sigma} \tilde{\Phi}(\varphi/\tau) = \sum_{n=-\infty}^{\infty} \phi_n \exp \{in\varphi/\tau\} \sum_{k=-\infty}^{\infty} \delta \left(\frac{n}{\tau} - k\right) \equiv 0 \quad (28)
\]

If \( \tau \) is equal to rational number, then \( \hat{\Sigma} \tilde{\Phi}(\varphi/\tau) \sim \delta (0) \), hence functions with such period do not belong to kernel of the operator \( \hat{\Sigma} \).

Let us now introduce an operator

\[
\hat{\Delta} = \sum_{n=-\infty}^{\infty} \hat{\Delta}_{n,\epsilon} \quad (29)
\]

where any of operator \( \hat{\Delta}_{n,\epsilon} \) preserves only small vicinity \( \Delta_{n,\epsilon} = (n - \epsilon_n, n + \epsilon_n) \) of the integer \( n \) in the support of distribution \( \Psi_t \in S' \)

\[
\hat{\Delta}_{n,\epsilon} \Psi_t \equiv \begin{cases} 
\Psi_t & t \in \Delta_{n,\epsilon} \\
0 & t \notin \Delta_{n,\epsilon}
\end{cases}, \quad (30)
\]

where \( 0 < \epsilon_n \leq \epsilon \) and \( \epsilon \) is an arbitrary small but finite number.
For $F(\varphi) \in S'_2$ we define

$$\hat{\Delta} F(\varphi) \equiv \int_{-\infty}^{\infty} \hat{\Delta} F_t \exp \{i\varphi t\} \, dt = \sum_{m=-\infty}^{\infty} \int_{m+\epsilon}^{m-\epsilon} F_t \exp \{i\varphi t\} \, dt.$$ (31)

Since $F_t$ are locally continuous at $t = m$, we can rewrite (31) as

$$\hat{\Delta} F(\varphi) = 2 \frac{\sin (\varphi \epsilon)}{\varphi} \sum_{m=-\infty}^{\infty} F_m e^{i\varphi m} \frac{2 \sin (\varphi \epsilon)}{\varphi} \tilde{F}(\varphi).$$ (32)

Therefore $\hat{\Delta} F(\varphi) = O(\epsilon)$ for small enough $\epsilon$. At the same time

$$\hat{\Delta} \hat{\Sigma} F(\varphi) = \hat{\Delta} \sum_{n=-\infty}^{\infty} F(\varphi + 2\pi n) = \hat{F}(\varphi) \sum_{n=-\infty}^{\infty} 2 \sin [(\varphi + 2\pi n) \epsilon] \frac{\varphi + 2\pi n}{(\varphi + 2\pi n)}.$$ (33)

Applying Poisson formula for series summation, from (33) we obtain

$$\sum_{n=-\infty}^{\infty} \frac{2 \sin (\varphi + 2\pi n) \epsilon}{(\varphi + 2\pi n)} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin x \epsilon}{\pi x} e^{i(x-\varphi)m} \, dx = 1, \quad 0 < \epsilon < 1,$$ (34)

which means

$$\hat{\Delta} \hat{\Delta} F(\varphi) = \hat{\Delta} \hat{\Sigma} F(\varphi) = \hat{F}(\varphi).$$ (35)

It should be noted that it is difficult to study behavior of $\hat{\Delta}$ for $\epsilon \to 0$ in a framework of the distribution theory. Indeed, in this limit $\left(\hat{\Delta}, \varphi\right) \to 0$ for any test function $\varphi$. In distribution theory it means $\hat{\Delta} \to 0$. At the same time product $\hat{\Delta} \delta(\varphi - n)$ becomes undefined in the same limit. Therefore, $\epsilon$ will be treated always as an arbitrary small but finite parameter.

### 5 General form of distributions in $\Omega$ and $\Omega^F$

Let us show that any $\omega(\varphi) \in \Omega^F$ may be represented as

$$\omega(\varphi) = \rho(\varphi + \pi) - \rho(\varphi - \pi), \quad \rho(\varphi) \in S',$$ (36)

with some restrictions on $\rho(\varphi)$.

Indeed, if $\omega(\varphi) \in \Omega^F$, then $\rho(\varphi)$ may be defined as

$$\rho(\varphi) = \sum_{k=0}^{\infty} \omega(\varphi + \pi (2k + 1)).$$ (37)

It is evident that (36) define $\rho(\varphi)$ only up to some periodic function with a period $2\pi$. On the other hand, if $\omega(\varphi)$ is given by (36) then

$$\omega_\ell = \left(e^{i\pi \ell} - e^{-i\pi \ell}\right) \rho_\ell,$$ (38)
where
\[ \rho_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\varphi) \exp \{-it\varphi\} d\varphi. \]  
(39)

So if singularity of \( \rho_t \) at integer \( t = n \) is weak enough and do not compensate zeros of \( (e^{it} - e^{-it}) \) and \( (e^{it} - e^{-it}) \rho_t \in S' \), then \( \rho(\varphi + \pi) - \rho(\varphi - \pi) \in \Omega^F \).

It follows from (26) that
\[ \hat{\Sigma} \omega (\varphi) = \hat{\Sigma} \rho (\varphi + \pi) - \hat{\Sigma} \rho (\varphi - \pi) = \lim_{N \to \infty} s_N, \]  
(40)

where
\[ s_N = \sum_{n=-N}^{N-1} [\rho (\varphi + \pi + 2\pi n) - \rho (\varphi + 2\pi n)], \]
\[ = \rho (\varphi + \pi - 2\pi N) - \rho (\varphi + \pi + 2\pi N), \]  
(41)

and one may write
\[ \hat{\Sigma} \omega (\varphi) = \lim_{N \to \infty} \int_{-\infty}^{\infty} \rho_t \left( e^{-2it\pi N} - e^{2it\pi N} \right) e^{i(\varphi + \pi)t} dt, \]  
(42)

Therefore, \( \hat{\Sigma} \omega (\varphi) = 0 \), if
\[ \lim_{N \to \infty} \rho_t \exp \{-it2\pi N\} - \lim_{N \to \infty} \rho_t \exp \{it2\pi N\} = 0. \]  
(43)

To compute \( \lim_{N \to \infty} \rho_t \exp \{\pm it2\pi N\} \) one may use an asymptotic expansion \[10\]. In particular, if
\[ \rho_t = a_+ (t + i0)^{-\lambda_+} + a_- (t - i0)^{-\lambda_-}, \]  
(44)

where \( \lambda_\pm \) and \( a_\pm \) are some constants, then
\[ \hat{\Sigma} \omega (\varphi) = \left( \frac{a_+ e^{i\pi \lambda_+} \lim_{N \to \infty} N^{\lambda_+ - 1}}{(2\pi)^{\lambda_+} \Gamma (\lambda_+)} - \frac{a_- e^{-i\pi \lambda_-} \lim_{N \to \infty} N^{\lambda_- - 1}}{(2\pi)^{\lambda_-} \Gamma (\lambda_-)} \right) (1 + O (1/N)) \]  
(45)

and we see that \( \lambda_\pm < 1 \) is enough for \( \hat{\Sigma} \omega (\varphi) = 0 \).

Let us assume that some \( q_t \in S'_Z \) does not turn into zero for some integer \( t = n \), so \( q_t \notin \Omega \). It is clear that \( (e^{it} - e^{-it}) \frac{1}{m} q_t \in \Omega \) for any finite \( m > 0 \). Since
\[ \lim_{m \to \infty} \left\langle \phi (t), (e^{it} - e^{-it}) \frac{1}{m} q_t \right\rangle = \left\langle \phi (t), q_t \right\rangle, \]  
(46)

for all \( \phi \in \mathcal{S} \), then space \( \Omega \) is incomplete.

Let us consider as an example Fourier series with coefficients
\[ F_n = \frac{1}{n - i\sigma}, \; \Im \sigma = 0. \]  
(47)
Direct extension, that is simple change \( n \to t \), gives \( F_n = \frac{1}{n-i\sigma} \to \frac{1}{t-i\sigma} = F_t \). If, however, one takes into account

\[
\tilde{F}(\varphi) = \sum_{n=-\infty}^{\infty} \frac{e^{i n \varphi}}{n-i\sigma} = 2i \pi e^{-\sigma \varphi_{\text{mod} \; 2\pi}} \left( \theta(\varphi_{\text{mod} \; 2\pi}) + \frac{1}{e^{2\pi \sigma} - 1} \right), \tag{48}
\]

where \( \theta(t) \) is the Heaviside function and use \( \Re \) for analytical extension, one gets

\[
F_t = \frac{1}{t-i\sigma} + \frac{1}{t-i\sigma \sin \pi \sigma}, \tag{49}
\]

so

\[
\omega_t = \frac{1}{t-i\sigma \sin \pi \sigma}. \tag{50}
\]

Therefore, direct extension gives for \( \tilde{F}(\varphi) \) expression which differs from (48) in a function \( \omega(\varphi) = 2\pi i e^{-\sigma \varphi} \left( \theta(\varphi+\pi) - \theta(\varphi-\pi) \right) \) for \(-\pi \leq \varphi < \pi\). For a finite \( x \) it leads to

\[
F(x) - F^{\text{formal}}(x) = \lim_{\tau \to \infty} \omega \left( \frac{x}{\tau} \right) = 2\pi i \lim_{\tau \to \infty} e^{-\sigma x/\tau} \left( \frac{\theta(x/\tau + \pi)}{e^{2\pi \sigma} - 1} - \frac{\theta(x/\tau - \pi)}{1 - e^{-2\pi \sigma}} \right) = \frac{2\pi i}{e^{2\pi \sigma} - 1}, \tag{51}
\]

where

\[
F^{\text{formal}}(x) = \int_{-\infty}^{\infty} \frac{1}{k - i\theta} \exp \{ikx\} \, dk = 2\pi i \theta(x). \tag{52}
\]

In conclusion we wish to point that since \( \hat{\Delta} \hat{\Delta} = \hat{\Delta} \), then \( \hat{\Delta} \) as well as \( 1 - \hat{\Delta} \) are projectors. So, if for some \( \xi_t \in S' \) product \( \left( 1 - \hat{\Delta} \right) \xi_t \) exists and belongs to \( S' \) at least for small enough \( \epsilon \), then \( \left( 1 - \hat{\Delta} \right) \xi_t \in \Omega \). Indeed, let us write \( \hat{\Delta} \) in a form

\[
\hat{\Delta} = \sum_{n=\infty}^{\infty} \left( \theta(t+n+\epsilon) - \theta(t+n-\epsilon) \right) = \sum_{m=\infty}^{\infty} \frac{\sin(2\pi mc)}{\pi m} \exp \{i2\pi mt\}. \tag{53}
\]

For \( R_t \in S' \) product \( \left( 1 - \hat{\Delta} \right) R_t \in \Omega \) if and only if convolution for Fourier transforms of cofactors \( \mathcal{F} \left( 1 - \hat{\Delta} \right) \ast \mathcal{F} (R_t|\varphi) = R_\Omega (\varphi) \) belong to \( \Omega^\mathcal{F} \). In accordance with (31) Fourier transform of \( \hat{\Delta} \) may computed as

\[
\hat{\Delta} \delta (\varphi) = \int_{-\infty}^{\infty} \hat{\Delta} \exp \{-i\varphi t\} \frac{dt}{2\pi} = \mathcal{F} \left( \hat{\Delta}|\varphi \right) = \sum_{m=\infty}^{\infty} \frac{\sin(2\pi mc)}{\pi m} \delta (\varphi - 2\pi m), \tag{54}
\]
which leads to
\[
R_\Omega (\varphi) = R (\phi) - \sum_{m=-\infty}^{\infty} \frac{\sin (2\pi m \epsilon)}{\pi m} R (\phi - 2\pi m).
\] (55)

Taking into account
\[
\sum_{m=-\infty}^{\infty} \frac{\sin (2\pi m \epsilon)}{\pi m} = 1, \ 0 < \epsilon < 1,
\] (56)
one can get \( \hat{\Sigma} R_\Omega (\varphi) = 0 \) that leads to \( R_\Omega (\varphi) \in \Omega^F \). In agreement with (37) we can write \( R_\Omega (\varphi) \) as
\[
R_\Omega (\varphi) = r (\varphi + \pi) - r (\varphi - \pi), \ r (\varphi) = \sum_{k=0}^{\infty} R_\Omega (\varphi + \pi (2k + 1)).
\] (57)

It means, that \( (1 - \hat{\Delta}) R_t \in \Omega \) for any \( R_t \in S' \).

6 Error introduced by the formal transition

Applying the Poisson formula to (3) we get
\[
\tilde{F} (x/\tau) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} F_n \exp \left\{ in \frac{x}{\tau} - 2\pi i mn \right\} dn = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \tau F_{k\tau} \exp \{ ikx - 2\pi i mk\tau \} dk.
\] (58)

Therefore, to remove an error introduced by a formal transition one must introduce a correction
\[
\Xi (x) \equiv F (x) - F^{formal} (x) = \int_{-\infty}^{\infty} \lim_{\tau \to \infty} \tau F_{k\tau} \exp \{ ikx - 2\pi i mk\tau \} dk.
\] (59)

If \( k = 0 \) is regular point of \( \lim_{\tau \to \infty} \tau F_{k\tau} = f_k^{formal} \), then \( \lim_{\tau \to \infty} \tau F_{k\tau} \exp \{ \pm 2\pi i mk\tau \} = 0 \) for all \( m \neq 0 \), and we get
\[
\lim_{\tau \to \infty} \tilde{F} (x/\tau) = \int_{-\infty}^{\infty} f_k^{formal} \exp \{ ikx \} dk = F (x).
\] (60)

Let us consider the case when an extension \( F_n \to F_t \) is chosen in such a way that \( f_k \) appears to be singular at \( k = 0 \). For instance, if \( F_n = n^{-\lambda} \) for \( n > 0 \) and \( F_n = 0 \) for \( n \leq 0 \), one may choose \( F_t = t_+^{-\lambda} \). If regularization parameter doesn’t depend on \( \tau \), it may be replaced for \(+0\) and one may write
\( f_k(\tau) = f_{k+i0}^{(+)} - f_{k-i0}^{(-)} \), where \( f_k^{(\pm)} \) are regular in the upper/lower half-plane of a complex variable \( k \). In this case the correction may be written as

\[
\Xi(x) = \lim_{\tau \to \infty} \Xi(x, \tau) = \lim_{\tau \to \infty} \sum_{m \neq 0} F_{k\pm0}^{formal}(x - 2\pi m \tau).
\] (61)

Physical value must disappear at infinity so for them \( \Xi(x, \tau) \to 0 \) with \( \tau \to \infty \). For gauge fields this condition is not mandatory and \( \Xi(x) \) may differ from zero at finite \( x \).

With \( \tau \to \infty \) correction \( \Delta(x) \) must disappear for any physical value \( F(x) \). It is not true, however, for gauge fields. For instance, if \( F_{k\pm0}^{formal}(x) = \exp\{ -|x/\tau - 2\pi| \} \) for \( |x| < \tau \) we see that \( \Xi(x) = \sinh \frac{x}{\tau} \) doesn’t turn into zero at finite \( x \) and \( \tau \to \infty \).

Let us finally consider the case when regularization parameter depends on \( \tau \), namely, when \( f_k(\tau) \) may be represented as

\[
f_k(\tau) = f_{k+i0}^{(+)} - f_{k-i0}^{(-)},
\] (62)

where \( \sigma > 0 \). For \( \tau \to \infty \) we get \( f_k(\tau) \to f_k^{formal} = f_{k+i0}^{(+)} - f_{k-i0}^{(-)} \), hence generally \( f_k^{formal} \) must be treated as the distribution.

For \( \Xi(x) = 0 \) it is enough that \( e^{2i\pi m \tau} f_{k\pm0}^{(\pm)} \to 0 \) with \( \tau \to \infty \) for all \( m \neq 0 \). A general method for studying an asymptotic behavior of \( e^{ikx} f_{k\pm0}^{(\pm)} \) for \( x \to \pm \infty \) was developed in [11]. An extension of such method to a case when regularization parameter is finite and turns into zero only with \( \tau \to \infty \) was suggested in [12]. If \( f_k(\tau) \) may be represented in a form (62), some operation can be suggested, after which for studying an asymptotic behavior of \( e^{2i\pi m \tau} f_{k\pm0}^{(\pm)} \) the method of [11] can be used. Indeed, let us integrate \( e^{2i\pi m \tau} f_{k\pm0}^{(\pm)} \) over real \( k \). Then, taking into account that \( f_k^{(\pm)} \) are regular in the upper/lower half-plane \( k \), we may shift the integration path \( (-\infty, \infty) \to (-\infty \mp i(\sigma/\tau - 0), \infty \mp i(\sigma/\tau - 0)) \) and after changing variable \( k \to k \mp i(\sigma/\tau - 0) \) we obtain

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{ ik2\pi m \tau \} f_{k\pm0}^{(\pm)} dk = e^{\pm 2i\pi m \sigma} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{ ik2\pi m \tau \} f_{k+i0}^{(\pm)} dk,
\] (63)

that allows to write symbolically

\[
\exp\{ ik2\pi m \tau \} f_{k\pm0}^{(\pm)} = \exp\{ ik2\pi m \tau \} e^{\pm 2i\pi m \sigma} f_{k\pm0}^{(\pm)}
\] (64)

for studying an asymptotic behavior of \( f_k^{(\pm)} \) one may apply the method developed in [11]. For \( \tau \to \infty \) we find

\[
e^{2i\pi m \tau} f_{k\pm0}^{(\pm)} = 0, \quad \pm m > 0
\] (65)

and

\[
e^{2i\pi m \tau} f_{k\pm0}^{(\pm)} = e^{-2\pi|m|\sigma} e^{2i\pi m \tau} f_{k\pm0}^{(\pm)} = e^{-2\pi|m|\sigma} \sum_{n=0}^{\infty} b_n^{(\pm)} \delta^{(n)}(k), \quad \mp m > 0,
\] (66)
where coefficients $b_n^{(+)}$ are defined by specific form of $f_k^{(+)}$ and depend on $m\tau$. For power type distributions $b_n^{(\pm)}$ are computed in [10]. A general method for studying an asymptotic behavior of multidimensional distributions was developed in [13]. Application of this method to studying of some specific issues of transition to continuum and thermodynamic limit in lattice gauge series we consider in subsequent papers.

In the case considered above an expression for the correction takes the form

$$
\Xi(x) = \lim_{\tau \to \infty} \sum_{m \neq 0} e^{-2\pi|m|\sigma} F_{formal}(x - 2\pi m \tau).
$$

(67)

Referring again to the example (48) we see that

$$
f_{k}^{(-)} = \lim_{\tau \to \infty} \tau F_{k\tau} = \lim_{\tau \to \infty} \frac{1}{k - i\sigma/\tau} \to \frac{1}{k - i\sigma}.
$$

At the same time, from (64) we get symbolic relation

$$
f_k(\tau) = f_k^{(-)} = \frac{1}{k - i\sigma/\tau} = e^{-2\pi|m|\sigma} \frac{1}{k - i0}.
$$

(68)

Hence, for $\tau \to \infty$ and $m \neq 0$ we get

$$
\frac{1}{k - i0} \exp\{2\pi im\tau k\} = 2\pi i\delta(k)
$$

(69)

and consequently

$$
\lim_{\tau \to \infty} \sum_{m=-\infty}^{\infty} \exp\{2\pi im\tau k\} \frac{1}{k - i\sigma/\tau} = \frac{1}{k - i0} + \frac{2\pi i}{\exp\{2\sigma\}} \delta(k),
$$

(70)

that coincides with Fourier transform of (48).

7 Conclusions

Studying system behavior in a finite volume with subsequent transition to thermodynamic limit provides beneficial regularization and is used in many domains of physics (see e.g. [14, 15]). In the present paper we show that if in the course of regularization removing, Fourier series in LGT are formally changed for integrals, it may introduce an ambiguity. Such ambiguity arises due to lack of uniqueness in the analytical extension of function of discrete argument (Fourier coefficients) to continuous one. Moreover, formal change introduces an error, which may grow in the course of regularization removal. All that may lead to noncoincidence in the results obtained by different methods. Furthermore, unreasonable discrepancy may arise among gauge theories with different expressions for actions.

It is shown that mentioned ambiguity reduced to a possibility of addition of some distribution $\omega(x/\tau) \in \Omega^F$ to the result. Properties of spaces $\Omega$ and $\Omega^F$ were studied. There are established conditions under which analytical extension is unique even in a case when conditions of Carlson theorem are not
satisfied. A method is also suggested for the computation of correction to remove the error, which may appear after formal transition. Conditions under which such correction turns to zero are specified.

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References

[1] Creutz M, 1983 Quarks, gluons and lattices, (Cambridge MMP)
[2] Montvay I, Munster G 1997 Quantum fields on a lattice (Cambridge Univ. Press)
[3] Petrov V K 1999 Theor.Math.Phys.121 1628
[4] Gelfand I M and Shilov G E 1964 Generalized functions. Properties and Operations (Academic Press, New York).
[5] Vladimirov V S 1979 Generalized functions in mathematical physics (MIR)
[6] Bremermann H 1965 Distributions, complex variables and Fourier transforms (Addison-Wesley)
[7] Bieberbach L 1955 Analytische fortsetzung (Springer Verlag)
[8] Hille E 1962 Analytic Function Theory (Ginn and Company, New York) volume II
[9] Schwartz L 1966 Mathematics for the physical sciences (Addison-Wesley)
[10] Brychkov Yu A and Prudnikov A P, Integralnye preobrazovania obobscheniyh funktsij, Moscow, Nauka, 1977
[11] Brychkov Yu A and Shirokov YuM 1970 Teor.Mat.Fiz. 4 301, Brychkov Yu A 1970, Teor.Mat.Fiz. 5 98, 1973 Teor.Mat.Fiz. 15 375, 1975 Teor.Mat.Fiz. 23 191
[12] Petrov V K 2004 Asymptotic series for distributions (Preprint hep-lat/0411031)
[13] Vladimirov VS, Drozhzhinov Yu N and Zav’yalov BI 1986 Multidimensional Tauberian theorems for generalized functions (Nauka, Moscow)
[14] Aoki S et al. 2004 Phys. Rev. D70 034503.
[15] Orth B, Lippert T and Schilling K 2005 Phys.Rev.D72 014503