Guaranteed cost estimation and control for a class of nonlinear systems subject to actuator saturation

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A B S T R A C T

The problems of guaranteed cost estimation (GCE) and guaranteed cost control (GCC) concern designing a state observer or a controller, respectively, such that some performance is maintained below an upper bound. This paper provides a matrix inequality-based observer/controller design procedure to perform GCE and GCC in a class of nonlinear systems affected by actuator saturation. In particular, this class of systems corresponds to those for which the origin of the state space is an equilibrium point when null inputs are considered, and the nonlinearity is differentiable with respect to the state and linear with respect to the saturated input. Simulation results obtained using a numerical example and a rotational single-arm inverted pendulum are used to illustrate the effectiveness of the proposed design procedure.

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1. Introduction

The problem of optimal control consists in finding a control law for a given system so that some performance criterion is minimized. For a nonlinear generic system, this problem can be solved using Pontryagin’s maximum principle or solving the Hamilton-Jacobi-Bellman (HJB) equation [8,13,49]. However, in most of the cases, obtaining the analytical solution is a hard task, which motivated the search for alternative approaches to perform the above-mentioned minimization. A quite successful approach is the so-called guaranteed cost control (GCC), which was proposed first by Chang and Peng [3] as a way to guarantee the performance for uncertain systems by requiring it to be below some upper bound. Early results on GCC were obtained in the 1990s by Jan R. Petersen and his colleagues, who studied the design of robust state-feedback controllers that minimize an upper bound on a quadratic cost function [25,26]. A computationally efficient framework for finding the optimal guaranteed cost controller was provided by linear matrix inequalities (LMIs: see [31] for a tutorial), such that several approaches were developed, e.g. [6,43]. LMIs were also used in [19] to achieve GCC in bilateral teleoperation systems which included time-varying delays and model uncertainty. Although the initial attention of the research community was devoted to linear time invariant (LTI) systems, soon it was driven towards other classes of systems which could take into account variability in time, for example linear systems with varying parameters (LPV) [28,30] or fuzzy Takagi-Sugeno (TS) systems [9,41,42,46]. GCC is still a very active field of research, with several works appearing every year in the literature, mainly dealing with nonlinear systems, e.g. [11,15,33,39,45].

Strictly related to GCC is the problem of guaranteed cost estimation (GCE) in which instead of designing a controller, one wishes to design a state observer that minimizes some upper bound on a performance criterion, which is a function of the estimation error and the measurement noise. [25,26] showed that GCE extended the much celebrated Kalman filter to uncertain systems. A GCE-based approach was later developed by Petersen [24] using a class of state estimators that include copies of the globally Lipschitz system nonlinearities within them. More recently, Ishihara et al. [12] developed an approach based on regularization and penalty functions to solve the optimal filtering problem for discrete-time systems with norm-bounded parametric uncertainties.

Another topic that has attracted much attention by the control theory research community is how to deal with saturation nonlinearities. Saturation can be found everywhere in physical applications, since real-world actuators are constrained in the number of deliverable control actions. The control techniques that ignore these actuator limits can be affected by degraded performance or...
instability of the closed-loop system. Hence, the analysis and synthesis of control systems with saturating actuators has been investigated by several works, see e.g. \cite{4,20,27,37,38}. The developed approaches can be divided into two main categories: the two-step paradigm (also referred to as anti-windup compensation \cite{14,50}) ignores the saturation at the controller design step, and handles it by adding a compensator; on the other hand, the one-step paradigm (also called direct control design \cite{7,36}) takes into account the saturation during the controller design phase. Among the most recent results concerning this topic, one may mention \cite{48}, where the problem of input saturation was solved by introducing an auxiliary design system, Shahri et al. \cite{32}, which employed the Lyapunov direct method for the stability analysis of fractional order linear systems subject to input saturation, and \cite{29}, which proposed a virtual actuator-based fault tolerant control strategy to deal with actuator saturations in unstable linear systems.

This work is motivated by the big importance held by the optimal design of observer and controller gains in automatic control systems. The literature review has shown that, although there exist a few results on GCC for systems with input saturation, e.g. \cite{17,44,47}, the problem of GCC for these systems has not been considered yet. Hence, this paper aims at developing a design procedure that addresses both the GCC and the GCC for a class of nonlinear saturating systems, while at the same time analysing the case in which the controller uses the estimate produced by the observer in order to update the control law.

More specifically, this paper proposes a matrix inequality-based guaranteed cost estimation and control design procedure for a class of discrete-time nonlinear systems subject to actuator saturation. This class of systems corresponds to those for which the origin of the state space is an equilibrium point when null inputs are considered, and the nonlinearity is differentiable with respect to the state and linear with respect to the saturated input. It is worth highlighting that these nonlinearities, which have been considered previously in the context of fault estimation by Zhu \cite{2,40}, encompass cases which cannot be dealt using the traditional bounding box method \cite{35}. Hence, an alternative approach based on the application of the mean value theorem, as described by Lewis et al. \cite{1,23}, must be obtained.

The contributions of the paper can be resumed as follows:

1. a polytopic approach based on the application of the mean value theorem is described for the characterization of a class of discrete-time nonlinear systems subject to actuator saturation;
2. sufficient conditions for the synthesis of a state observer that achieves GCC and a state-feedback controller that achieves GCC for the above-mentioned class of systems are provided in the form of an LMI-based feasibility or optimization problem;
3. it is shown that for the above-mentioned class of systems, the celebrated separation principle holds only one-way in the sense that the observer can be designed independently from the controller, but the converse is not true. Hence, sufficient conditions for the design of an estimate-feedback guaranteed cost controller are obtained in the form of bilinear matrix inequalities (BMIs).

The paper is structured as follows. Section 2 describes the notation and some lemmas which are used in the proofs of the theoretical results. In Section 3, the class of considered nonlinear systems is defined, and the different design problems considered in this paper are formulated. Section 4 provides LMI-based sufficient conditions for the design of the state observer. Section 5 is devoted to providing LMI-based sufficient conditions for the design of the state-feedback controller. In Section 6, the state-feedback control law is replaced by an estimate-feedback control, and BMI-based conditions for the design of the controller gain are obtained. Section 7 summarizes the final procedure for designing and implementing the components of the control system that provide GCC and GCE. The theoretical results are illustrated by means of an illustrative example in Section 8, whereas an application to a nonlinear rotational single-arm inverted pendulum is given in Section 9. Finally, the main conclusions are outlined in Section 10.

2. Notation and preliminaries

For a real symmetric matrix $A \in \mathbb{R}^{n \times n}$, the notation $A > 0$ ($A < 0$) stands for a positive (negative) definite matrix and indicates that all the eigenvalues of $A$ are positive (negative). Given a matrix $A \in \mathbb{R}^{n \times n}$ with $A > 0$, the symbol $\mathcal{E}_A$ denotes the ellipsoid:

$$\mathcal{E}_A = \{ x \in \mathbb{R}^n : x^T A x \leq 1 \}.$$  

The symbol $\text{Co}(A, i = 1, \ldots, N)$ denotes the convex hull of a finite number of $N$ vertex matrices:

$$\text{Co}(A, i = 1, \ldots, N) = \left\{ \sum_{i=1}^{N} \mu_i A_i \mid \sum_{i=1}^{N} \mu_i = 1, \mu_i \geq 0 \forall i = 1, \ldots, N \right\}.$$  

Given a vector $u \in \mathbb{R}^m$, the symbol $\sigma(\cdot)$ denotes the standard saturation function, such that $\sigma(u) = [\sigma(u_1), \sigma(u_2), \ldots, \sigma(u_m)]^T$, where $\sigma(u_i) = \text{sgn}(u_i) \min\{1, |u_i|\}$. Given a matrix $A \in \mathbb{R}^{m \times n}$, the symbol $\mathcal{L}_A(A)$ denotes:

$$\mathcal{L}_A(A) = \{ x \in \mathbb{R}^n : \sigma(Ax) = Ax \}.$$  

The following lemmas are used throughout the paper.

**Lemma 1.** Given two matrices $K, H \in \mathbb{R}^{m \times n}$ and a vector $x \in \mathbb{R}^n$, suppose that $|H_i x| \leq 1$ for $i = 1, 2, \ldots, m$, where $H_j$ denotes the $j$th row of $H$. Then:

$$\sigma(Kx) \in \text{Co}\left\{D_i K x + D_i^T H x \mid i = 1, \ldots, 2^m\right\},$$

where the matrices $D_i$ are all the possible $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0, and $D_i^T = I - D_i$.

**Proof.** See \cite{10}. \hfill $\Box$

**Lemma 2.** Given the matrix $H \in \mathbb{R}^{m \times n}$, if the following inequality holds for $j = 1, \ldots, m$:

$$\begin{bmatrix} P & PH_j^T \\ H_j P & 1 \end{bmatrix} > 0,$$

then $|H_j x| \leq 1 \forall x \in \mathcal{E}_{P^{-1}}$.

**Proof.** See \cite{21}. \hfill $\Box$

3. Problem formulation

Consider the following discrete-time nonlinear system:

$$x_{k+1} = A x_k + g(x_k, \sigma(u_k)),$$

$$y_k = C x_k,$$  

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^p$ is the output, and $A$ and $C$ are constant matrices of appropriate dimensions, and the nonlinear function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to satisfy the following assumptions:

1. $g(x, \sigma(u))$ is affine in $\sigma(u)$, so that it can be rewritten as:

$$g(x, \sigma(u)) = f(x) + F(x)\sigma(u),$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ appropriate functions such that $f(0) = 0$ and $F(0) \neq 0$. Note that a consequence of this fact is that $g(0, 0) = 0$.
2. \( g(x, \sigma(u)) \) is differentiable with respect to \( x \) with bounded partial derivatives:
\[
\frac{\partial^2 g}{\partial x^2}_{ij} \leq \bar{a}_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n. \tag{6}
\]

By applying the mean value theorem \([5]\), the following relationship holds:
\[
g(a, \sigma(u)) - g(b, \sigma(u)) = M(a, b, \sigma(u))(a - b), \tag{7}
\]
for some matrix \( M(\cdot) \) obtained as follows:
\[
M(a, b, \sigma(u)) = \begin{bmatrix}
\frac{\partial g}{\partial x}(c_1, \sigma(u)) \\
\vdots \\
\frac{\partial g}{\partial x}(c_n, \sigma(u))
\end{bmatrix}, \tag{8}
\]
with:
\[
c_1, \ldots, c_n \in \{c \in \mathbb{R}^n : c = \alpha a + \beta b, \alpha + \beta = 1, \alpha \beta \geq 0 \}. \tag{9}
\]

Taking into account lower and upper bounds \( \underline{a}_{ij}, \bar{a}_{ij} \) and each possible permutation of these bounds, matrices \( M_i \in \mathbb{R}^{m \times n} \), \( i = 1, \ldots, N \), can be obtained such that:
\[
M(a, b, \sigma(u)) \in \mathbb{M} = \text{Co}(M_i, i = 1, \ldots, N). \tag{10}
\]
so that:
\[
g(a, \sigma(u)) - g(b, \sigma(u)) \in \mathbb{M}(a - b). \tag{11}
\]
Moreover, taking into account \((5)\), the following holds:
\[
g(x, \sigma(a)) - g(x, \sigma(b)) = F(x)(\sigma(a) - \sigma(b)). \tag{12}
\]
Hereafter, the problems of observer and controller design are formulated.

3.1. State observer design

Let us consider a nonlinear discrete-time observer of the form:
\[
\hat{x}_{k+1} = A\hat{x}_k + g(\hat{x}_k, \sigma(u_k)) + K_0(y_k - C\hat{x}_k), \tag{13}
\]
where \( \hat{x}_k \) denotes the estimate of the state \( x_k \) and \( K_0 \) denotes the observer gain to be designed. Then, the dynamics of the estimation error \( e_k = x_k - \hat{x}_k \) is given by:
\[
e_{k+1} = x_{k+1} - \hat{x}_{k+1} = \hat{A}e_k + s_k, \tag{14}
\]
where \( \hat{A} = A - K_0C \) and \( s_k = g(x_k, \sigma(u_k)) - g(\hat{x}_k, \sigma(u_k)) \). Apart from the asymptotical convergence to zero of the estimation error, a bound on the following cost function:
\[
J_0 = \sum_{k=0}^{\infty} e_k^T Q e_k, \tag{15}
\]
with given \( Q > 0 \), is considered as objective for the design of the observer. Hence, the GCC design problem can be formulated as follows.

**Problem 1.** (State observer design problem) Given \( Q > 0 \), \( Q > 0 \), \( Q > 0 \) and a scalar \( \gamma > 1 \), design \( K_0 \) such that the dynamics of the estimation error \((14)\) is asymptotically stable with:
\[
J_0 < \gamma e_0^T Q e_0. \tag{16}
\]

3.2. State-feedback controller design

To design a robust controller, we consider the following control law:
\[
u_k = K_c x_k, \tag{17}
\]
where \( K_c \) is the controller gain to be designed. Substituting the above equation into \((3)\) gives the following closed-loop system:
\[
x_{k+1} = Ax_k + g(x_k, \sigma(K_c x_k)). \tag{18}
\]
Let us note that, since \( g(0, 0) = 0 \), then \( g(x_k, \sigma(K_c x_k)) \) can be rewritten as:
\[
g(x_k, \sigma(K_c x_k)) = g(x_k, \sigma(K_c x_k)) - g(0, \sigma(K_c x_k)) + g(0, \sigma(K_c x_k)) - g(0, 0). \tag{19}
\]
Taking into account \((11)\) and \((12)\), the following is obtained:
\[
g(x_k, \sigma(K_c x_k)) - g(0, \sigma(K_c x_k)) \in \mathbb{M}x_k, \tag{20}
\]
so that:
\[
g(x_k, \sigma(K_c x_k)) \in \mathbb{M}x_k + F(0)\sigma(K_c x_k). \tag{21}
\]
The following objectives are taken into account for the design of the controller: (i) asymptotical convergence to zero of \( x_k \) when \( x_0 \) belongs to the ellipsoid \( \mathcal{E}_Q \) defined by a given matrix \( Q > 0 \); and (ii) bound on the following cost function:
\[
J_c = \sum_{k=0}^{\infty} (x_k^T Q x_k + \sigma(u_k)^T Q \sigma(u_k)), \tag{23}
\]
with given \( Q > 0 \) and \( Q > 0 \). Hence, the GCC design problem can be formulated as follows.

**Problem 2.** (State-feedback controller design problem) Given matrices \( Q > 0 \), \( Q > 0 \), \( Q > 0 \) and the scalar \( \gamma > 1 \), design \( K_c \) such that \((18)\) is asymptotically stable and:
\[
J_c < \gamma e_0^T Q e_0, \tag{24}
\]
when \( e_0 \in \mathcal{E}_Q \).

3.3. Estimate-feedback controller design

The controller design problem previously formulated (Problem 2) assumes that the real state \( x_k \) is available for feedback. However, a more realistic situation is the one in which the estimated state should be used instead, i.e. \((17)\) changes into:
\[
u_k = K_c \hat{x}_k, \tag{25}
\]
where \( \hat{x}_k \) is the estimated state given by the observer \((13)\). In this case, the question about whether it is possible or not to design the observer and the controller separately arises.

Let us consider the interaction of the system \((3)\) and \((4)\), the observer \((13)\) and the control law \((25)\) such that the overall system obeys (see Fig. 1 for a block diagram depicting their structures and interconnections):
\[
e_{k+1} = \hat{A}e_k + g(x_k, \sigma(K_c \hat{x}_k)) - g(\hat{x}_k, \sigma(K_c \hat{x}_k)). \tag{26}
\]
\[
x_{k+1} = Ax_k + g(x_k, \sigma(K_c \hat{x}_k)). \tag{27}
\]
Let us note that:
\[
g(x_k, \sigma(K_c \hat{x}_k)) = g(x_k, \sigma(K_c \hat{x}_k)) - g(0, \sigma(K_c \hat{x}_k)) + g(0, \sigma(K_c \hat{x}_k)) - g(0, 0). \tag{28}
\]
Taking into account \((11)\):
\[ g(x_k, \sigma(K_x\hat{x}_k)) - g(0, \sigma(K_x\hat{x}_k)) \in Mx_k, \]  \hspace{1cm} (29)  
\[ g(x_k, \sigma(K_x\hat{x}_k)) - g(\hat{x}_k, \sigma(K_x\hat{x}_k)) \in M(x_k - \hat{x}_k) = Me_k. \]  \hspace{1cm} (30)  
Moreover, due to (12):
\[ g(0, \sigma(K_x\hat{x}_k)) - g(0, 0) = F(0)\sigma(K_x\hat{x}_k). \]  \hspace{1cm} (31)  
Hence:
\[ g(x_k, \sigma(K_x\hat{x}_k)) \in Mx_k + F(0)\sigma(K_x\hat{x}_k), \]  \hspace{1cm} (32)  
which means that the overall system can be put in the equivalent form:
\[ x_{k+1} = (A + M)x_k + F(0)\sigma(K_x\hat{x}_k - K_e e_k). \]  \hspace{1cm} (33)  
\[ e_{k+1} = (\tilde{A} + M)e_k, \]  \hspace{1cm} (34)  
where \( \hat{x}_k = x_k - e_k \) has been used.

From (33) to (34) it can be seen that, due to the nonlinear term \( \sigma(K_x\hat{x}_k - K_e e_k) \), the separation principle holds only one-way, in the sense that, while the observer can be designed independently from the controller, this is not true for the controller, whose design procedure should be modified to take into account the effect of the evolution of \( e_k \), driven by the specific choice of the observer gain \( K_e \) on the nonlinearity \( \sigma(\cdot) \). In order to deal with this situation, the requirements of Problem 2 are changed by requiring them to hold for \( [x_0^T, e_0^T]^T \in E_{s}^{Q,S} \), where the ellipsoid \( E_{s}^{Q,S} \) is defined by a given matrix:

\[ \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0, \]  \hspace{1cm} (35)  
and that \( J_e < \gamma_e(x_0^T Q x_0 + e_0^T Q e_0) \). Note that with this choice, in case \( x_0 = 0 \), then \( J_e < \gamma_e x_0^T Q x_0 \) is obtained, whereas if \( e_0 = 0 \), then the case \( J_e < \gamma_e x_0^T Q x_0 \) is recovered. Hence, the GCC design problem can be modified as follows:

**Problem 3.** (Estimate-feedback controller design problem) Given matrices \( Q > 0, S > 0, Q_x > 0, Q_e > 0 \), the observer gain \( K_e \) and the scalar \( \gamma_e > 1 \), design the controller gain \( K_c \) such that (33) and (34) are asymptotically stable and:
\[ J_e < \gamma_e(x_0^T Q x_0 + e_0^T Q e_0). \]  \hspace{1cm} (36)  
when \( [x_0^T, e_0^T]^T \in E_{s}^{Q,S} \).

**4. Design of the state observer**

The objective of this section is to solve Problem 1 by obtaining sufficient conditions for the synthesis of the state observer (13), which are given by the following theorem.

**Theorem 1.** Let \( P > 0, \gamma_o > 1 \) and \( U \) be such that the following holds:
\[ P - \gamma_o Q_e < 0. \]  \hspace{1cm} (37)  
\[ \begin{bmatrix} Q_e - P & 0 \\ P(A + M_i) - UC & - P \end{bmatrix} < 0, \]  \hspace{1cm} (38)  
where the nonlinear discrete–time observer given by (13), with gain calculated as \( K_o = P^{-1}U \), is such that (14) is asymptotically stable and \( J_o < \gamma_o e_0^T Q_o e_0 \).

**Proof.** Let us consider the following inequality:
\[ \Delta V_k + e_k^T Q e_k < 0 \quad \text{for} \quad k = 0, \ldots, \infty. \]  \hspace{1cm} (39)  
where \( \Delta V_k = V_{k+1} - V_k \), with \( V_k = e_k^T P e_k \). Since \( e_k^T Q e_k > 0 \forall e_k \), if inequality (39) holds then \( \Delta V_k < 0 \) which corresponds to the Lyapunov condition for asymptotic stability of (14). Then, by summing (39) from 0 to \( \infty \), the following is obtained:
\[ \sum_{k=0}^{\infty} e_k^T Q e_k = J_o < V_0 = e_0^T P e_0. \]  \hspace{1cm} (40)
which, due to (37) ensuring that $V_0 < \gamma_0 e_0^T Q_0 e_0$ for any initial condition $e_0$, proves that $J_k < \gamma_0 e_0^T Q_0 e_0$ [22].

The remaining of the proof shows that inequality (39) follows from (38). In fact, taking into account (13), (39) can be rewritten as:

$$\Delta V_k + e_k^T Q_0 e_k = e_k^T (\bar{A}^T P \bar{A} - P + Q_0) e_k + e_k^T \bar{A}^T P \bar{s}_k$$

$$+ s_k^T P \bar{a}_k + s_k^T P \bar{s}_k < 0. \quad (41)$$

From (11), it follows that:

$$s_k = (x_k, \sigma(u_k)) - (\bar{x}_k, \sigma(\bar{u}_k)) \in M(x_k - \bar{x}_k) = M e_k,$$

so that (41) is satisfied if:

$$\bar{A}^T P \bar{A} - P + Q_0 + \bar{A}^T P M^T \bar{P} A + M^T P M^T P M \preceq 0, \quad \forall M \in \mathbb{M}. \quad (43)$$

which, using Schur complements, leads to:

$$\begin{bmatrix} Q_0 - P & (\bar{A}^T + M^T) \bar{P} \\ P(\bar{A} + M) & \bar{P} \end{bmatrix} \preceq 0, \quad \forall M \in \mathbb{M}. \quad (44)$$

Replacing:

$$P \bar{A} = \bar{P} A - P \bar{K}_\sigma C = P \bar{A} - UC,$$

into (44), where $U = PK_\sigma$, leads to:

$$\begin{bmatrix} Q_0 - P & \ast \\ P(\bar{A} + M) & -\bar{P} \end{bmatrix} \preceq 0, \quad \forall M \in \mathbb{M}. \quad (46)$$

which is satisfied if (38) holds, thus completing the proof. \qed

**Remark 1.** The problem of determining the observer gain matrix described by Theorem 1 can be treated as an optimization problem in which the cost performance index $\gamma_0$ is minimized.

### 5. Design of the state-feedback controller

The objective of this section is to solve Problem 2 by obtaining sufficient conditions for the synthesis of the controller (17) for the system (3)-(4), which are given by the following theorem.

**Theorem 2.** Let $P > 0$, $\gamma > 1$ and $I, Z$ be such that the following holds:

$$\begin{bmatrix} Q_0 & I \\ I & P \end{bmatrix} \succeq 0. \quad (47)$$

$$\begin{bmatrix} P & Z^T_j & 1 \\ Z_j & I \end{bmatrix} \succeq 0, \quad j = 1, \ldots, m. \quad (48)$$

$$\begin{bmatrix} (A + M)^T P + F(0) \phi & \ast & \ast & \ast \\ P & \ast & \ast & \ast \\ Q_k^{1/2} & 0 & \ast & \ast \end{bmatrix} \succeq 0, \quad i = 1, \ldots, 2^m, \quad (50)$$

with $\phi_i = D_i^T \Gamma + D_i^T Z$, where $Z_j$ denotes the $j$th row of the matrix $Z$, the matrices $D_i$ are all the possible $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0, and $D_i^0 = I - D_i$. Then the state-feedback control law (17), with gain calculated as $K_c = \Gamma P^{-1}$, is such that (18) is asymptotically stable and $J_k < \gamma_0 e_0^T Q_0 e_0$ when $x_0 \in \mathcal{E}_0$.

**Proof.** In order to ensure that Problem 2 is solved, let us define the function $V_k = \xi_k^T P^{-1} \xi_k$, $P > 0$, and let us require that the ellipsoid $\mathcal{E}_0$ is contained in $\mathcal{E}_{p^{-1}}$, which is equivalent to $Q - P^{-1} \succeq 0$ and, by Schur complements, to (47). Then, to ensure asymptotic stability for $x_0 \in \mathcal{E}_0$, it is sufficient to ensure it for $x_0 \in \mathcal{E}_{p^{-1}}$, which together with the constraint on $J_k$, leads to the following constraints on $V_k$:

$$\Delta V_k + \xi_k^T Q \xi_k + \sigma(K_c \xi)^T Q_k \sigma(K_c \xi) < 0, \quad \forall M \in \mathbb{M}. \quad (51)$$

$$P^{-1} - \gamma \xi_k < 0, \quad (52)$$

where $\Delta V_k = V_{k+1} - V_k$.

By defining:

$$\bar{x}_k = [\xi_k^T, \sigma(K_c \xi)^T]^T, \quad (53)$$

and taking into account (22), the inequality (51) is satisfied if:

$$\begin{bmatrix} \Lambda_{11}^\sigma & \Lambda_{12}^\sigma \\ \Lambda_{21}^\sigma & \Lambda_{22}^\sigma \end{bmatrix} \bar{x}_k = (\bar{x}_k^T \Lambda^\sigma \bar{x}_k < 0, \quad \forall M \in \mathbb{M}. \quad (54)$$

where:

$$\Lambda_{11}^\sigma = (A + M)^T P^{-1} (A + M) - P^{-1} + Q_k,$$

$$\Lambda_{12}^\sigma = (A + M)^T P^{-1} F(0),$$

$$\Lambda_{22}^\sigma = F(0)^T P^{-1} F(0) + Q_k.$$

According to Lemmas 1–2, by introducing an auxiliary feedback matrix $K_c$ and the constraint (48), which enforces $\mathcal{E}_{p^{-1}} \subseteq \mathcal{E}_c (H_c)$ and that is obtained from (2) through the change of variable $Z = H_c P$, then $\sigma(K_c \xi)$ can be placed into the convex hull of a group of linear feedbacks:

$$\sigma(K_c \xi) \in \text{Co}[D_i K_c \xi + D_i^T H_c \xi, i = 1, \ldots, 2^m]. \quad (55)$$

Hence, from (55) and by convexity of the function $V_k$, it follows:

$$\bar{x}_k^T \Lambda^\sigma \bar{x}_k \leq \max_{i=1,\ldots,2^m} \bar{x}_k^T \Lambda_i^\sigma \bar{x}_k. \quad (56)$$

where:

$$\Lambda_i^\sigma = \Sigma_i^T P^{-1} \Sigma_i - P^{-1} + Q_k + \psi_i^T P \psi_i, \quad (57)$$

with:

$$\Sigma_i = A + M + F(0) \psi_i. \quad (58)$$

By requiring that $\Lambda_i^\sigma < 0$ for $i = 1, \ldots, 2^m$, and applying Schur complements, with an appropriate congruence transformation, the following is obtained:

$$\begin{bmatrix} -P & P \Sigma_i^T & P Q_k^{1/2} & P \psi_i^T \\ \Sigma_i P & -\bar{P} & 0 & 0 \\ Q_k^{1/2} P & 0 & -I & 0 \\ \psi_i P & 0 & 0 & -Q_k^{-1} \end{bmatrix} \preceq 0, \quad \forall M \in \mathbb{M}. \quad (59)$$

which, by replacing:

$$V_k = (A + M)^T P \Sigma_i - P^{-1} + Q_k + \psi_i^T P \psi_i,$$

$$=egin{bmatrix} (A + M)^T P + F(0) D_i K_c P + F(0) D_i^T H_k P \\ (A + M)^T P + F(0) \Gamma + F(0) D_i^T Z \end{bmatrix}, \quad (60)$$

where $\Gamma = K_c P$, leads to:

$$\begin{bmatrix} -P & (A + M)^T P + F(0) \phi_i & \ast & \ast & \ast \\ (A + M)^T P + F(0) \phi_i & -\bar{P} & \ast & \ast & \ast \\ Q_k^{1/2} P & 0 & -I & \ast & \ast \\ \phi_i & 0 & 0 & -Q_k^{-1} \end{bmatrix} \preceq 0, \quad i = 1, \ldots, 2^m, \quad \forall M \in \mathbb{M}. \quad (61)$$

which is satisfied if (49) holds. By applying Schur complements to (52), (50) is obtained, which completes the proof. \qed

**Remark 2.** Also in this case, the problem of determining the controller gain matrix described by Theorem 2 can be treated as an optimization problem in which the cost performance index $\gamma_0$ is minimized.
6. Design of the estimate-feedback controller

The objective of this section is to solve Problem 3 by obtaining sufficient conditions for the synthesis of the controller (25) for the system (3) and (4) with state observer (13), which are given by the following theorem.

**Theorem 3.** Given the observer gain \( K_o \) (hence, the matrix \( \hat{A} \)), let \( P > 0 \), \( \gamma_c > 1 \) and the matrices \( \mathcal{K}_c, \mathcal{K}_r \) be such that:

\[
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix} - P \succeq 0,
\]

\[
P - \frac{1}{\gamma_c} \begin{bmatrix}
Q_0 & 0 \\
0 & Q_j
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
P & 
\begin{bmatrix}
H_{c,j}^T \\
-H_{c,j}
\end{bmatrix}
\end{bmatrix} \succeq 0, \quad j = 1, \ldots, m
\]

(62)

(63)

(64)

(65)

(66)

(67)

(68)

(69)

(70)

(71)

(72)

(73)

which, by means of Schur complements, and an appropriate congruence transformation, leads to:

\[
\begin{bmatrix}
P & \begin{bmatrix}
A + M_1 + F(0)\phi_i & -F(0)\phi_i \\
0 & \hat{A} + M_2
\end{bmatrix} \\
0 & -\phi_i^T Q_0 \phi_i \
\end{bmatrix} > 0, \quad i = 1, \ldots, 2^m \quad \forall M_1, M_2 \in \mathbb{M}.
\]

(73)

which is satisfied if (65) holds, thus completing the proof. \( \Box \)

Note that when applying Theorem 6, the matrix \( \hat{A} \) is considered to be known, since the observer gain \( K_o \) is assumed to be designed beforehand using Theorem 1. Taking the above considerations into account, similarly to previous cases, an optimization problem concerning minimization of the cost performance index \( \gamma_c \) can be defined. It is worth highlighting that the conditions provided by Theorem 6 are bilinear matrix inequalities (BMIs) due to the product between the unknown variables \( P \) and \( \phi_i \), hence their resolution suffers from being a non-convex problem.

7. Design and implementation procedure for the guaranteed cost estimation and control

The problem of determining the state observer and controller gain matrices is solved using the results given by Theorems 1–3. It is done by an optimization problem subject to minimization of the cost performance indexes \( \gamma_o \) and \( \gamma_c \). The design and implementation procedure can be summarized as follows: *Off-line computation:*

1. Obtain a representation of the system of interest as in (3) and (4);
2. Calculate the Jacobians of the function \( g(\cdot) \) with respect to state and input;
3. Compute lower and upper bounds for the elements of the Jacobians and use them to obtain (10) using the bounding box approach;
4. (Observer design) Obtain the observer gain \( K_o \) by solving the optimization problem:

\[
\min \gamma_o
\]

subject to \( (37)-(38) \);

5. (Controller design) Obtain the controller gain \( K_c \) by solving the optimization problem:

\[
\min \gamma_c
\]

subject to \( (47)-(50) \);

*On-line computation:*

1. Compute the state estimate using (13);
2. Compute the control action using (25).

**Remark 3.** The above procedure summarizes the necessary steps for the design of the state observer or the state-feedback controller. It could be applied to the case of the estimate-feedback controller, albeit some minor changes.

8. Numerical example

Let us consider the following system:

\[
x_1(k + 1) = a_{11} x_1(k) - 0.5 x_2(k) + 0.1 x_3(k) + \frac{\cos(x_1(k)) \sigma_1(u_1(k))}{3 x_1^2(k) + 2},
\]

\[
x_2(k + 1) = -0.2 x_1(k) + a_{22} x_2(k) + 0.1 x_3(k) + \frac{\sigma_2(u_2(k))}{2 + x_2^2(k)}.
\]

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\(x_3(k+1) = 0.1x_1(k) - 0.1x_2(k) + a_{32}x_3(k),\)

where the state variable \(x_1(k)\) is assumed to be measured, which can be reshaped in the form (3) and (4) by considering:

\[
A = \begin{bmatrix}
a_{11} & -0.5 & 0.1 \\
-0.2 & a_{22} & 0.1 \\
0.1 & -0.1 & a_{33}
\end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.
\]

The nonlinear function \(g(\cdot)\) is differentiable with respect to \(x\) and \(\sigma(u)\):

\[
\frac{\partial g(\cdot)}{\partial x} = \begin{bmatrix} -\frac{3\sin(x_1)x_1^2 + 6\cos(x_1)x_1 + 2\sin(x_1)}{(3x_1^2 + 2)^2} \sigma_1(u_1) \\
0 \\
0 \\
0 \end{bmatrix},
\]

\[
\frac{\partial g(\cdot)}{\partial \sigma(u)} = \begin{bmatrix} \cos(x_1) \\
\frac{2}{3x_1^2} \\
0 \\
0 \\
0 \\
0 \end{bmatrix} - F(x).
\]

Note that the following holds:

\[-0.52 \leq -\frac{3\sin(x_1)x_1^2 + 6\cos(x_1)x_1 + 2\sin(x_1)}{(3x_1^2 + 2)^2} \sigma_1(u_1) \leq 0.52,\]

\[-0.23 \leq -\frac{2x_2}{(2 + x_1^2)^2} \sigma_2(u_2) \leq 0.23.\]

Also, \(g(0,0) = 0\) and:

\[
F(0) = \begin{bmatrix} 0.5 & 0 \\
0 & 0.5 \\
0 & 0 \end{bmatrix}.
\]

Taking into account the above computed bounds, it is possible to obtain the set defined in (10) as the convex combination of the following four matrices:

\[
M_1 = \begin{bmatrix} -0.52 & 0 & 0 \\
0 & -0.23 & 0 \\
0 & 0 & 0 \end{bmatrix},
\]

\[
M_2 = \begin{bmatrix} -0.52 & 0 & 0 \\
0 & 0.23 & 0 \\
0 & 0 & 0 \end{bmatrix},
\]

\[
M_3 = \begin{bmatrix} 0.52 & 0 & 0 \\
0 & -0.23 & 0 \\
0 & 0 & 0 \end{bmatrix},
\]

\[
M_4 = \begin{bmatrix} 0.52 & 0 & 0 \\
0 & 0.23 & 0 \\
0 & 0 & 0 \end{bmatrix}.
\]

\[
8.1. \text{Open-loop stable equilibrium}
\]

In this subsection, we will assume that \(a_{11} = a_{22} = a_{33} = 0.6\), so that the origin of the state-space is an open-loop stable equilibrium point. Let us consider \(Q_s = I\), and three different observer gains \(K^o_2\), \(K^o_3\) and \(K^o_c\), where \(K^o_2\) has been obtained through the minimization of \(\gamma_o\) using Theorem 1 [16,34], whereas \(K^o_3\) and \(K^o_c\) are observer gains designed by requiring only the stabilization of the estimation error dynamics. The observer gains are as follows:

\[
K^o_2 = \begin{bmatrix} 0.82 \\
-0.47 \\
0.21 \\
\end{bmatrix},
\]

\[
K^o_3 = \begin{bmatrix} 0.63 \\
-0.88 \\
-0.10 \\
\end{bmatrix},
\]

\[
K^o_c = \begin{bmatrix} 1.03 \\
-1.24 \\
-0.18 \\
\end{bmatrix}.
\]

which deliver minimized values of \(\gamma_o\) as follows: \(\gamma^o_2 = 3.70\), \(\gamma^o_3 = 70.78\), \(\gamma^o_c = 442.73\). In order to validate the proposed design technique, different simulations starting from initial conditions \(x_0\) on the unit sphere \(S\), with \(x_0 = 0\), have been performed. Then, Fig. 2 shows the evolution of the following signals:

\[
\tilde{f}(k|K^o_c) = \max_{k=1}^K \sum_{i=0}^k e_i^T Q_e e_i |K_0 = K^o_c,\]

\[
(74)
\]

yielding the improved upper bounds \(\gamma^o_{cb} = 19.8\).

\[
8.2. \text{Open-loop unstable equilibrium}
\]

In this subsection, we will assume that \(a_{11} = 1.2, a_{22} = 1.2\) and \(a_{33} = 0.7\), so that the origin of the state-space is an open-loop unstable equilibrium point. Let us consider \(Q_s = I\), and three different observer gains \(K^o_2\), \(K^o_3\) and \(K^o_c\), where \(K^o_2\) has been obtained solving

\[
\gamma_o^c = \left[ \begin{array}{ccc}
50 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right],
\]

\[
K^c_2 = \left[ \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 50
\end{array} \right].
\]

three different controller gains have been designed, as follows:

\[
K^c_2 = \begin{bmatrix} -1.22 & 1.00 & -0.20 \\
0.31 & -0.76 & -0.07 \end{bmatrix},
\]

\[
K^c_3 = \begin{bmatrix} -1.04 & 1.39 & -0.26 \\
0.41 & -1.20 & -0.18 \end{bmatrix},
\]

\[
K^c_c = \begin{bmatrix} -1.15 & 1.11 & -0.82 \\
0.61 & -1.12 & 0.75 \end{bmatrix}.
\]
the minimization of $\gamma_0$ using Theorem 1, whereas $K_o^d$ and $K_o^e$ are observer gains designed by requiring only the stabilization of the estimation error dynamics. The observer gains are as follows:

$$K_o^d = \begin{bmatrix} 2.12 \\ -2.33 \\ 0.43 \end{bmatrix}, \quad K_o^e = \begin{bmatrix} 2.11 \\ -2.37 \\ 0.41 \end{bmatrix}, \quad K_o^f = \begin{bmatrix} 2.14 \\ -2.41 \\ 0.43 \end{bmatrix}. $$

which deliver minimized values of $\gamma_0$ as follows: $\gamma_0^d = 199.09$, $\gamma_0^e = 5528$, $\gamma_0^f = 1686.8$. Let us note that in this case, if compared to the open-loop stable one, very small changes in the elements of the observer gains cause big variations in the values of the computed upper bounds. Moreover, Fig. 6, which shows the evolution of the signal (74), $i \in \{d, e, f\}$, illustrates the increase in conservativeness of the proposed methodology when open-loop unstable plants are considered. For the sake of completeness in the presentation of the results, the upper and lower envelopes of the estimation error trajectories are shown in Fig. 7.

Subsequently, selecting $Q = 100I$, $Q_o^d = Q_o^e$, $Q_o^f = Q_o^g$, $Q_o^I = Q_o^e$ and $Q_u = I$, the following controller gains have been designed:

$$K_c^d = \begin{bmatrix} -2.38 & 1.02 & -0.20 \\ 0.29 & -1.85 & -0.15 \end{bmatrix},$$

$$K_c^e = \begin{bmatrix} -2.40 & 1.05 & -0.31 \\ 0.43 & -2.34 & -0.18 \end{bmatrix}.$$
Fig. 4. Evolution of $\hat{f}(k|K_i)$ and upper bounds $\gamma_i$, $i \in \{a, b, c\}$ (open-loop stable, state-feedback).

Fig. 5. Envelopes of $x(k)$ with controller gains $K_a^c$, $K_b^c$, $K_c^c$ (open-loop stable, state-feedback).

$$K_i^f = \begin{bmatrix} -2.34 & 1.34 & -0.76 \\ 0.74 & -2.10 & 0.65 \end{bmatrix}$$

each one solving the minimization problem described in Section 5, obtaining $\gamma_d^c = 7.37$, $\gamma_e^c = 9.81$ and $\gamma_f^c = 16.40$, respectively. Fig. 8 shows the signal calculated using (75), $i \in \{d, e, f\}$, which demonstrates that $f(k|K_i^c) < \gamma_i^c$ is satisfied in all simulations. As in the previous case, the controller gain that provides a faster convergence to zero of the state variable $x_1(k)$ is $K_d^c$, whereas $K_e^c$ and $K_f^c$ provide a faster convergence of $x_2(k)$ and $x_3(k)$, respectively. For the sake of completeness, Fig. 9 shows the upper and lower envelopes of the state trajectories for initial conditions on the frontier of $S$.

This can be seen also from Fig. 3, where the upper and lower envelopes of the estimation error trajectories are plotted, showing that $K_d^o$ provides a faster convergence to zero of the estimation error.

9. Application to a rotational single-arm inverted pendulum

Let us consider the following nonlinear system describing the dynamics of a rotational single-arm inverted pendulum [18]:

$$x_1(k + 1) = x_1(k) + T x_2(k).$$
\[ x_2(k+1) = T_s \frac{g}{l} \sin(x_1(k)) + (1 - T_s \frac{b}{ml^2})x_2(k) + \frac{T_s}{ml^2} \sigma(u_1(k)), \]

where \( T_s = 0.01[s] \) is the sampling time, \( m = 0.2[kg] \) is the mass of the pendulum, \( l = 0.15[m] \) is the length of the pendulum, whereas \( b = 0.0067[kgm^2s^{-1}] \) and \( g = 9.81[m/s^2] \) are the friction coefficient and the gravitational acceleration, respectively. Assuming that the state variable \( x_1(k) \) (angle of the pendulum) is measured, the pendulum model can be reshaped as:

\[
A = \begin{bmatrix} 1 & T_s \frac{b}{ml^2} \\ 0 & 1 - \frac{T_s}{ml^2} \end{bmatrix}, \quad g(\cdot) = \begin{bmatrix} T_s \frac{g}{l} \sin(x_1(k)) \\ T_s \frac{g}{l} \cos(x_1(k)) \end{bmatrix} + \frac{T_s}{ml^2} \sigma(u_1(k))
\]

\[
C = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

The nonlinear function \( g(\cdot) \) is differentiable with respect to \( x \) and \( \sigma(u) \):

\[
\frac{\partial g(\cdot)}{\partial x} = \begin{bmatrix} T_s \frac{g}{l} \cos(x_1(k)) \\ T_s \frac{g}{l} \sin(x_1(k)) \end{bmatrix}
\]

\[
\frac{\partial g(\cdot)}{\partial \sigma(u)} = \begin{bmatrix} 0 \\ T_s \frac{g}{l} \cos(x_1(k)) \end{bmatrix} = F(x).
\]
Fig. 8. Evolution of the signals $\tilde{J}_i(k|K^c_i)$ and upper bounds $\gamma^c_i$, $i \in \{d, e, f\}$ (open-loop unstable, state-feedback).

Fig. 9. Envelopes of the state trajectories $x(k)$ with different controller gains $K^c_d$, $K^c_e$, $K^c_f$.

Note that the following holds:

$$-0.6540 \leq T_s g \cos(x_1(k)) \leq 0.6540.$$  

Also, $g(0, 0) = 0$ and:

$$F(0) = \begin{bmatrix} 0 \\ 2.2222 \end{bmatrix}.$$  

Taking into account the above computed bound, it is possible to obtain (10) as the convex combination of the following matrices:

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0.6540 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ -0.6540 & 0 \end{bmatrix}$$

Taking into consideration $Q_e = I$, three different observer gain $K^o_a$, $K^b_o$ and $K^c_o$ have been obtained:

$$K^o_a = \begin{bmatrix} 1.0378 \\ 3.7197 \end{bmatrix}, \quad K^b_o = \begin{bmatrix} 0.4750 \\ 4.9875 \end{bmatrix}, \quad K^c_o = \begin{bmatrix} 0.5750 \\ 7.8375 \end{bmatrix},$$

where $K^o_a$ has been obtained through the minimization of $\gamma_o$ using Theorem 1 and whereas $K^b_o$ and $K^c_o$ provide only the stabilization of the estimation error dynamics. Using initial conditions $x_0$, Fig. 10, shows the evolution of (74) for $i \in \{a, b, c\}$. Also for this case, confirms that observer gain matrices $K^o_a$ provide the best performance (see blue line). Moreover, Fig. 11 shows the upper and lower envelopes of the estimation error trajectories, confirms, that $K^o_a$ pro-
Fig. 10. Evolution of $f(k|K_i^o)$, $i \in \{a, b, c\}$, and upper bound $\gamma^2_{a,b,c}Q_e e_o$.

Fig. 11. Envelopes of the error $e(k)$ with observer gains $K_a^o, K_b^o, K_c^o$.

Fig. 12. Evolution of $f^*(k|K_i^c)$ and upper bounds $\gamma^*_i$, $i \in \{a, b, c\}$ (state-feedback).
vides the smallest estimation error with respect to time and initial condition.

Subsequently, selecting $Q_d = \text{diag}(10, 10000)$, $Q_0^\bullet = \text{diag}(0.0001, 0.0001)$, $Q_d^\bullet = \text{diag}(0.0001, 0.0001)$ and $Q_d^a = Q_0^a = 0.01$, the following controller gains have been designed:

$$K_0^c = \begin{bmatrix} -1.1967 \\ -0.2703 \end{bmatrix},$$

$$K_0^\bullet = \begin{bmatrix} -1.1578 \\ -0.2968 \end{bmatrix},$$

$$K_0^a = \begin{bmatrix} -1.0410 \\ -0.3750 \end{bmatrix},$$

with $\gamma^c = 15279$, $\gamma^\bullet = 19996$ and $\gamma^a = 501.74$, respectively. Fig. 12 shows the signal calculated using (75), $i \in \{a, b, c\}$, which demonstrates that $f_i(k|K_i^c) < \gamma_i$ is satisfied in all simulations. For this example, the controller gain that provides a faster convergence to zero of the state variable $x_1(k)$ and $x_2(k)$ is $K_0^c$. Finally, Fig. 13 shows the upper and lower envelopes of the state trajectories for initial conditions on the frontier of $\mathcal{S}$.

10. Conclusions

This paper has discussed the design of a state observer and a state-feedback controller that provide guaranteed cost estimation and guaranteed cost control, respectively, for a class of nonlinear systems affected by actuator saturations. The considered systems correspond to those for which the origin of the state space is an equilibrium point when null inputs are considered, and the nonlinearity is differentiable with respect to the state and linear with respect to the saturated input.

It has been shown that when both designs are considered separately, the procedure consists in solving LMIs, which is efficient to do using available solvers. The simulation results have shown the main characteristics of the proposed guaranteed cost design method, and the fact that less conservative solutions are found when the origin is an open-loop stable equilibrium.

On the other hand, it has been shown that in the more realistic situation in which a state estimate-feedback should be used, e.g., due to the lack of availability of some state variables for measurement, it is not possible to design the controller without taking into account the observer. In this case, the design procedure relies on bilinear matrix inequalities (BMIs). Some experiments using a BMI solver have shown that, although the proposed design procedure is viable in some cases, it suffers in returning a solution due to non-convexity issues.

In spite of the advantages of the proposed approach, the performance of the closed-loop system is affected by the conservativeness brought by the use of a quadratic Lyapunov function with constant Lyapunov matrix and constant observer/controller matrices. Future work will explore other types of Lyapunov functions which can decrease the conservativeness of the design procedure and the use of gain-scheduled (state-dependent) observer/controller gains. Moreover, other important directions for further research are the conversion of the BMIs obtained for computing the estimate-feedback controller gain into more computationally convenient LMIs, and the development of a procedure for the joint design of the observer and controller gain for estimate-feedback guaranteed cost estimation and control.

Declaration of Competing Interest

None.

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