Generalized cut and metric polytopes of graphs and simplicial complexes

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Received: 8 June 2017 / Accepted: 9 November 2018 / Published online: 13 November 2018
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Abstract
The metric polytope METP(Kn) of the complete graph on n nodes is defined by the triangle inequalities x(i, j) ≤ x(i, k) + x(k, j) and x(i, j) + x(j, k) + x(k, i) ≤ 2 for all triples i, j, k of {1, . . . , n}. The cut polytope CUTP(Kn) is the convex hull of the {0, 1} vectors of METP(Kn). For a graph G on n vertices the metric polytope METP(G) and cut polytope CUTP(G) are the projections of METP(Kn) and CUTP(Kn) on the edge set of G. The facets of the cut polytopes are of special importance in optimization and are studied here in some detail for many simple graphs. Then we define variants QMETP(G) for quasi-metrics, i.e. not necessarily symmetric distances and we give an explicit description by inequalities. Finally we generalize distances to m-dimensional area between m + 1 points and this defines an hemimetric. In that setting the generalization of the notion of graph is the notion of m-dimensional simplicial complex K for which we define a cone of hemimetric HMET(K).

Keywords Max-cut problem · Cut polytope · Metrics · Graphs · Cycles · Quasi-metrics · Hemimetrics

1 Introduction
The cut polytope is a natural polytope arising in the study of the maximum cut problem [6–8]. The cut polytope of the complete graph K_n has been widely investigated (see [22] and references therein). But the cut polytope of an arbitrary graph G arises naturally in the context of the max-cut problem over G where we restrict to edge weights on the edges of G. The cut polytope of a graph was studied in relation to minor
theory and polytope theory [26–28]. In [25] the diameter of the cut polytope of a graph is studied; in particular the diameter is at most $|V| - 1$. The dual description of the cut polytope, that is finding the set of its facet defining inequalities, is of special interest because of its tight link with the max cut problem. In this respect the cut polytopes of graphs have been used in Quantum Information Theory [1–3] because some cut polytopes are isomorphic to correlation polytopes (see the isomorphism in [22, Section 5.2]) and the facets give Bell inequalities. A remarkable result is the characterization of the facets of the cut polytopes of graphs without $K_5$ minor by Barahona [6] and Seymour [29]; see Theorem 2 below. In [9] lifting techniques for generating facets of cut polytope of graphs are described and in [20] the dual description of several small graphs is computed. This was used in [24] to compute Grothendieck constants of some graphs. In this work we continue this study of the facets and consider extension to other settings.

Given a graph $G = (V, E)$, for a vertex subset $S \subseteq V = \{1, \ldots, n\}$, the cut semimetric $\delta^G_S$ is a function on $E$ defined for $(x, y) \in E$ as

$$\delta^G_S(x, y) = \begin{cases} 1 & \text{if } |S \cap \{x, y\}| = 1, \\ 0 & \text{otherwise}. \end{cases} \quad (1)$$

The cut polytope $\text{CUTP}(G)$, respectively the cut cone $\text{CUT}(G)$, are defined as the convex hull, respectively the conic hull, of all such semimetrics.

The metric cone $\text{MET}(K_n)$ is the set of all semimetrics on $n$ points, i.e., the functions $d : \{1, \ldots, n\}^2 \to \mathbb{R}_{\geq 0}$ satisfying $d(i, i) = 0$, $d(i, j) = d(j, i)$ for $1 \leq i, j \leq n$ and the triangle inequalities

$$d(i, j) \leq d(i, k) + d(j, k) \quad \text{for } 1 \leq i, j, k \leq n. \quad (2)$$

The metric polytope $\text{METP}(K_n)$ is defined as the elements of $\text{MET}(K_n)$ satisfying the perimeter inequalities

$$d(i, j) + d(j, k) + d(k, i) \leq 2 \quad \text{for } 1 \leq i, j, k \leq n. \quad (3)$$

For a graph $G = (V, E)$ of order $|V| = n$, let $\text{MET}(G)$ and $\text{METP}(G)$ denote the projections of $\text{MET}(K_n)$ and $\text{METP}(K_n)$, respectively, on the subspace $\mathbb{R}^E$ indexed by the edge set of $G$. Clearly, $\text{CUT}(G)$ and $\text{CUTP}(G)$ are the projections of, respectively, $\text{CUT}(K_n)$ and $\text{CUTP}(K_n)$ on $\mathbb{R}^E$. We have the relaxation property

$$\text{CUT}(G) \subseteq \text{MET}(G) \quad \text{and} \quad \text{CUTP}(G) \subseteq \text{METP}(G).$$

In Sect. 2 we consider the structure of those polytopes and give the description of their facets for many graphs (see Tables 1 and 2). The data file of the groups and orbits of facets of considered polytopes is available from [30].

The construction of cuts and metrics is generalized in [4,17,19] to the non-symmetric setting. The cone $\text{QMET}(K_n)$ is defined as the quasi-metrics on $K_n$, i.e. functions $d : \{1, \ldots, n\}^2 \mapsto \mathbb{R}$ satisfying $d(i, i) = 0$, $d(i, j) \geq 0$ and
Table 1  Information on the cut polytope $\text{CUTP}(G)$ of some $K_5$-minor-free graphs $G$; the second column gives the number of vertices and edges; the third column gives the restricted automorphism group $A(G)$ (in case $A(G) > \text{Aut}(G)$); the fourth column gives the number of facets with the number of orbits in parenthesis.

| $G = (V, E)$ | $|V|, |E|$ | $A(G)$ | Number of facets (orbits) |
|-------------|-----------|---------|--------------------------|
| $M_8$       | 8, 2      | 16      | 184(4)                   |
| $M_6 = K_{3,3}$ | 6, 9      | 2(3!)²  | 90(2)                    |
| $K_{1,1,m}, m > 1$ | $m + 3, 3m + 3$ | 3!$m$! | 4 + 12$m$(2) |
| $K_{1,2,m}, m > 1$ | $m + 3, 3m + 2$ | $|\text{Aut}(K_{1,2,m})|$ | 8$m + 8\binom{m}{2}$(2) |
| $K_{3,m}, m \geq 3$ | $m + 3, 3m$ | $|\text{Aut}(K_{3,m})|$ | 6$m + 24\binom{m}{2}$(2) |
| $K_{2,m}, m > 2$ | $m + 2, 2m$ | $2^{m-1}m!|\text{Aut}(K_{2,m})|$* | 4$m^2$(1) |
| $K_{2,2}$ | 4, 4 | 6$|\text{Aut}(K_{2,2})|$* | 16(1) |
| $K_{1,1,m}, m > 1$ | $m + 2, 2m + 1$ | $2^{m-1}m!|\text{Aut}(K_{1,1,m})|$* | 4$m$(1) |
| $K_{m-1,m}, m > 1$ | $m + 1, m$ | $m!$ | 2$m$(1) |
| $A_{Prism_{6}}$ | 12, 24 | 24 | 2032(5) |
| $A_{Prism_{5}}$ | 10, 20 | 20 | 552(4) |
| $A_{Prism_{4}}$ | 8, 16 | 16 | 176(3) |
| $Prism_{7}$ | 14, 21 | 28 | 7394(6) |
| $Prism_{6}$ | 12, 18 | 24 | 2452(6) |
| $Prism_{5}$ | 10, 15 | 20 | 742(5) |
| $Prism_{3}$ | 6, 9 | 12 | 38(3) |
| Tr. Cube | 24, 36 | 48 | 230,200(6) |
| Sn. Cube | 24, 60 | 60 | 1,041,072(68) |
| Rh. Cuboctahedron | 24, 48 | 48 | 1,160,672(32) |
| Rh. Dodecahedron | 14, 24 | 48 | 8080(8) |
| Tr. Cuboctahedron | 48, 72 | 48 | 2,070,725,215,472(364) |
| Tr. Dodecahedron | 60, 90 | 120 | 22,822,824,384,652(26) |
| Icosidodecahedron | 30, 60 | 120 | 52,538,384(25) |
| Tr. Tetrahedron | 12, 18 | 24 | 540(4) |
| Cuboctahedron | 12, 24 | 48 | 1360(5) |
| Dodecahedron | 20, 30 | 120 | 167,164(8) |
| Icosahedron | 12, 30 | 120 | 1552(4) |
| Cube $K_{2}^{2}$ | 8, 12 | 48 | 200(3) |
| Octahedron $K_{2,2,2}$ | 6, 12 | 48 | 56(2) |
| Tetrahedron $K_{4}$ | 4, 6 | 6$|\text{Aut}(K_{4})|$* | 16(1) |

Moreover $Q\text{METP}(K_n)$ is defined by adding the following inequalities to the description of $Q\text{MET}(K_n)$:

$$d(i, j) \leq d(i, k) + d(k, j) \quad \text{for} \quad 1 \leq i, j, k \leq n.$$
Table 2  Same as Table 1 for some graphs $G$ with $K_5$-minor. Last column gives the $s$ of the orbits of facets which are $s$-cycle inequalities

| $G = (V, E)$ | $|V|, |E|$ | $A(G)$ | Number of facets (orbits) | Orbit’s $s$ |
|-------------|-------------|--------|--------------------------|-------------|
| Heawood graph | 14, 21 | 336 | 5,361,194 (9) | 2, 6, 8 |
| Petersen graph | 10, 15 | 120 | 3614 (4) | 2, 5, 6 |
| $M_{10}$ | 10, 15 | 20 | 1414 (5) | 2, 2, 4, 6 |
| $M_{12}$ | 12, 18 | 24 | 26,452(6) | 2, 2, 4, 7, 9 |
| $M_{14}$ | 14, 21 | 28 | 369,506(9) | 2, 2, 4, 8, 10 |
| $K_{5,5}$ | 10, 25 | $(2(5!))^2$ | 16,482,678,610 (1282) | 2, 4 |
| $K_{4,7}$ | 11, 28 | $4!^2$ | 271,596,584 (15) | 2, 4 |
| $K_{4,6}$ | 10, 24 | $4!^2$ | 23,179,008 (12) | 2, 4 |
| $K_{4,5}$ | 9, 20 | $4!^2$ | 983,560 (8) | 2, 4 |
| $K_{4,4}$ | 8, 16 | $2(4!)^2$ | 27,968 (4) | 2, 4 |
| $K_{3,3,3}$ | 9, 27 | $(3!)^4$ | 624,406,788 (2015) | 3, 4 |
| $K_{1,4,4}$ | 9, 24 | $2(4!)^2$ | 36,391,264 (175) | 3, 4 |
| $K_{1,3,5}$ | 9, 23 | $3!^2$ | 71,340 (7) | 3, 4 |
| $K_{1,3,4}$ | 8, 19 | $3!^2$ | 12,480 (6) | 3, 4 |
| $K_{1,3,3}$ | 7, 15 | $2(3!)^2$ | 684 (3) | 3, 4 |
| $K_{1,1,3,3}$ | 8, 21 | $(3!)^2$ | 432,552 (50) | 3, 3, 4 |
| $K_{1,2,2,2}$ | 7, 14 | $(3(2!))^3$ | 5864 (9) | 3, 3, 4 |
| $K_{1,1,2,2}$ | 6, 13 | $(4(2!)^2$ | 184 (4) | 3, 3, 4 |
| $K_{1,1,2,m, m > 2}$ | $m + 4, 4m + 5$ | $4m!$ | $8 + 20m + 8 \binom{m}{2}(16m - 15)(7)$ | 3, 3, 3, 4 |
| $K_{1,1,1,1,1,1, m > 1}$ | $m + 4, 4m + 6$ | $4m!$ | $8(8m^2 - 3m + 2)(4)$ | 3, 3 |
| $K_{1,1,1,1,1,1} = K_8 - K_3$ | 8, 25 | 360 | 2, 685, 152(82) | 3, 3 |
| $K_{1,1,1,1,1,2} = K_7 - K_2$ | 7, 20 | 240 | 31, 400(17) | 3, 3 |
| $K_7 - C_3$ | 7, 18 | 144 | 520(4) | 3, 3 |
| $K_7 - C_4$ | 7, 17 | 48 | 108(4) | 3, 3, 3 |
| $K_7 - C_5 = Pyr^2(C_5)$ | 7, 16 | 20 | 780(6) | 3, 3, 5 |
| $K_7 - C_6 = Pyr(Prym_3)$ | 7, 15 | 12 | 452(5) | 3, 3, 3, 4 |
| $K_7 - C_7$ | 7, 14 | 14 | 148(3) | 3, 4 |
| $Pyr(Prism_4)$ | 9, 20 | 48 | 10, 464(6) | 3, 4, 6 |
| $Pyr(Prism_5)$ | 11, 25 | 20 | 208, 133(22) | 3, 3, 4, 5, 7 |
| $Pyr(A-Prism_4)$ | 9, 24 | 16 | 389, 104(17) | 3, 3, 3, 4, 5 |
| $Pyr^2(C_6)$ | 8, 19 | 24 | 3, 432(7) | 3, 3, 6 |
| $Pyr^2(C_7)$ | 9, 22 | 28 | 14, 740(11) | 3, 3, 7 |
| Tr.Octahedron on $\mathbb{R}^2$ | 12, 18 | 48 | 62, 140(7) | 2, 2, 4, 6, 6 |

\[d(i, j) + d(j, k) + d(k, i) \leq 2 \text{ and } d(i, j) \leq 1 \text{ for } 1 \leq i, j, k \leq n.\]

For a graph $G$ the quasi-metric cone $\text{QMET}(G)$ and polytope $\text{QMETP}(G)$ are defined as projections of the above two cones and polytopes. In Theorems 3 and 4 we give
an inequality description of those projections. We also consider in Sect. 3 weightable quasi-metrics and we shortly consider the oriented cuts in this setting. As a consequence of this the optimization over the quasi-metric polytopes can be done in polynomial time even if their number of facets can be very large.

The notion of distance between 2 points can be generalized to the $m$-area $d(S)$ of a set $S$ of $m + 1$ points with $m > 1$. The triangle inequality for distances is generalized to the inequality

$$d(S) \leq \sum_{T \in \mathcal{K}, T \neq S} d(T)$$

for $\mathcal{K}$ a closed $m$-dimensional simplicial complex (see Sect. 4 for details). The functions $d$ satisfying above inequalities are called $m$-hemimetrics and are a natural generalization of metrics. In [14,15,18,21] we used another definition where the inequalities used were just the ones defined by $m + 1$ dimensional simplices. However, we pointed out in [12] that this definition is inadequate since it does not carry to simplicial complexes.

In Sect. 2 we study in more details the metric and cut polytopes of graphs, their symmetries, facets and hypermetric inequalities. In Sect. 3 we consider the quasi-metric case, the construction of the quasi-metric polytope and shortly the oriented cut polytope. In Sect. 4 we expose our construction of hemimetric of simplicial complexes.

### 2 Structure of cut polytopes of graphs

In this section, unless otherwise precised, $G$ is an undirected graph on $n$ vertices. The cut metric $\delta^G_S$ defined at Eq. (1) satisfies the relation $\delta^G_{[1, \ldots, n] - S} = \delta^G_S$.

For a set $S \subseteq \{1, \ldots, n\}$ and a function $d \in \mathbb{R}^E$ its switching (see [22] for details) $F_S(d) \in \mathbb{R}^E$ is a function on $E(G)$ defined for $(i, j) \in E$ by

$$F_S(d)(i, j) = \begin{cases} 1 - d(i, j) & \text{if } |S \cap \{i, j\}| = 1, \\ d(i, j) & \text{otherwise.} \end{cases}$$

The switching $F_S$ acts on the cuts in the following way $F_S(\delta^G_T) = \delta^G_{S \Delta T}$ with $\Delta$ denoting the symmetric difference. It is easy to see that if $G$ is connected then $\text{CUTP}(G)$ has exactly $2^{n-1}$ vertices. The polytope $\text{CUT}(G)$ can be interpreted as the projection of $\text{CUTP}(K_n)$ and $\text{CUT}(K_n)$ on $\mathbb{R}^E$.

The cone $\text{CUT}(K_n)$ is the set of all $n$-vertex semimetrics, which embed isometrically into some metric space $l_1$, and rational-valued elements of $\text{CUT}(K_n)$ correspond exactly to the $n$-vertex semimetrics, which embed isometrically, up to a scale $\lambda \in \mathbb{N}$, into the path metric of some $m$-cube $K^m_2$ [22]. The cone $\text{CUT}(K_n)$ is quite important in Analysis and Combinatorics and we refer to [22] for an in-depth overview. The enumeration of orbits of facets of $\text{CUT}(K_n)$ and $\text{CUTP}(K_n)$ for $n \leq 8$ was done in [5,20,23,32].

We not introduce a few graph theoretic notions that we need. In a graph $G$, a chordless cycle is any cycle, which is an induced subgraph. Clearly triangles and shortest cycle are chordless.
The complete graph $K_n$ is the graph on $n$ vertices will all pairs $(i, j)$ being edges. The complete multipartite graph $K_{n_1, \ldots, n_r}$ is the graph whose vertex sets is partitioned into $k$ sets $S_i$ of size $n_i$ such that two vertices are adjacent if and only if they belong to different parts. The cycle graph $C_n$ is the graph of $n$ vertices with the edges being $(i, i + 1)$ for $1 \leq i \leq n$. For a graph $G$ the pyramid $Pyr(G)$ is the graph formed by adding a new vertex $v$ adjacent to all the vertices of $G$. For an integer $m$ and $1 \leq i \leq m$, define $\phi_m(i) = i + 1$ for $i < m$ and $\phi_m(m) = 1$. By $Prism_m$ we denote the $2m$ vertex graph with edge set $(i, \phi_m(i)), (i, m + i)$ and $(m + i, m + \phi_m(i))$ for $1 \leq i \leq m$. By $APrism_m$ we denote the $2m$ vertex graph with edge set $(i, \phi_m(i)), (i, m + i)$, $(i, m + \phi_m(i))$ and $(m + i, m + \phi_m(i))$ for $1 \leq i \leq m$. By $M2m$ we denote the Möbius ladder on $2m$ vertices defined in Sect. 2.4. Denote by $P_m$ the path on $m$ vertices, with edges $(i, i + 1)$ for $1 \leq i \leq m - 1$.

An embedding of a graph in a manifold $M$ is an assignment of vertices to points in $M$ and of edges to paths in $M$. A graph is planar (resp., toroidal) if it can be embedded without crossing in the plane (resp., in the torus). It is called 1-planar if it can be embedded in the plane with each edge having at most one crossing. A graph $H$ is called a minor of a graph $G$ if it can be obtained from $G$ by deleting edges and vertices and by contracting edges. By Wagner’s theorem [33], a finite graph is planar if and only if it has no minors $K_5$ and $K_{3,3}$.

In this section we first consider the automorphism group of the cut polytopes in Sect. 2.1, then the edge, cycle inequalities, metric polytopes and hypermetric inequalities in Sect. 2.2. Then we consider the Platonic and Semiregular polyhedra in Sect. 2.3, the Möbius graphs in Sect. 2.4 and finally the complete-like graphs in Sect. 2.5.

The computations of dual descriptions were done with the software polyhedral [31] or by using the combinatorial description of Theorems 1, 2 when applicable.

### 2.1 Automorphism group of cut polytopes

For a graph $G$ the symmetry group $\text{Aut}(G)$ is formed by all symmetries of $G$, i.e. bijective mapping of the vertex set that maps edges to edges. The linear symmetry group $\text{Aut}(\text{CUTP}(G))$ is the set of affine symmetries of $\mathbb{R}^E$ preserving the polytope $\text{CUTP}(G)$ (see [10] for a computation technique). Any symmetry of $G$ induces a linear symmetry of $\mathbb{R}^E$ that preserves the cuts $\delta^G_S$ and so the polytope $\text{CUTP}(G)$. For any $U \subseteq \{1, \ldots, n\}$, the map $\delta^G_S \mapsto \delta^G_{U \Delta S}$ also defines a symmetry of $\text{CUTP}(G)$. Together, those form the restricted symmetry group $\text{ARes}(\text{CUTP}(G))$ of order $2|V| - 1|\text{Aut}(G)|$. The full symmetry group $\text{Aut}(\text{CUTP}(G))$ may be larger. In Tables 1 and 2, such cases are marked by *. Denote $21 - |V| |\text{Aut}(\text{CUTP}(G))| |A(G)|$.

For example, $|\text{Aut}(\text{CUTP}(K_n))|$ is $2^n - 1$ if $n \neq 4$ and $6 \times 2^3 4$ if $n = 4$ ([13]).

**Remark 1**

(i) If $G = (V, E)$ is $Prism_m$ $(m \neq 4)$, $APrism_m$ $(m > 3)$, $M2m$ and $Pyr^2(C_m)$ $(m > 3)$, then $\text{Aut}(G) = 4m$.

(ii) If $G$ is a complete multipartite graph with $t_1$ parts of size $a_1, \ldots, t_r$ parts of size $a_r$, with $a_1 < a_2 < \cdots < a_r$ and all $t_i \geq 1$, then $|\text{Aut}(G)| = \prod_{i=1}^r t_i!(a_i)!^{t_i}$.

**Conjecture 1**

(i) Among the complete multipartite graphs, all occurrences of $A(G) > |\text{Aut}(G)|$ are: $A(G) = m!2^{m-1}|\text{Aut}(G)|$ for $G = K_{2, m > 2}, K_{1,1,m > 1}$ and

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\( A(G) = 6|\text{Aut}(G)|, \) i.e., \( 2m! = 48, 6m! \) for \( G = K_{2,2} \) and \( K_{1,1,1,1} \), respectively.

(ii) For \( m \geq 3 \) we have \(|\text{Aut}(P_m)| = 2\) while \( A(P_m) = m! = (|V| - 1)! \).

(iii) For \( m \geq 4 \) we have \(|\text{Aut}(C_m)| = 2m\), while \( A(G) = \begin{cases} 2m! \text{ for } m = 4 \\ m! \text{ for } m \geq 5 \end{cases} \)

We verified that Conjecture 1 holds for \( m \leq 9 \).

2.2 Edge faces, cycle faces, metric polytope and hypermetric inequalities

Definition 1 Let \( G = (V, E) \) be a graph.

(i) Given an edge \( e \in E \), the edge inequality is

\[ x(e) \geq 0. \]

(ii) For a cycle \( C \), and an odd size set \( F \subseteq C \) the cycle inequality is

\[ x(F) - x(C \setminus F) \leq |F| - 1 \]

where \( x(U) = \sum_{u \in U} x(u) \).

We call a cycle of length \( s \) an \( s \)-cycle and the corresponding inequality an \( s \)-cycle inequality. The edge inequality is also called a 2-cycle inequality. The edge and cycle inequalities are valid on \( \text{CUTP}(G) \), since they are, clearly, valid on each cut: a cut intersects a cycle in a set of even cardinality. So, they define faces, but not necessarily facets. In fact, it holds:

Theorem 1 (i) The inequality \( x(e) \geq 0 \) is facet defining in \( \text{CUTP}(G) \) (also, in \( \text{CUT}(G) \)) if and only if \( e \) is not contained into a 3-cycle of \( G \).

(ii) A cycle inequality is facet defining in \( \text{CUTP}(G) \) (also, in \( \text{CUT}(G) \)) if and only if the corresponding cycle is chordless.

(iii) \( \text{METP}(G) \) is defined by all edge, bounding inequality \( x(e) \leq 1 \) and \( s \)-cycle inequalities, while \( \text{MET}(G) \) is defined by all edge inequalities and \( s \)-cycle inequalities with \( |F| = 1 \).

(i) and (ii) above were proved in [8], (iii) was proved in [7]; see also [22, Section 27.3]. The following Theorem, proved in [29] for cones and in [6] for polytopes, characterizes when the metric and cut polytope coincides:

Theorem 2 \( \text{CUT}(G) = \text{MET}(G) \) or, equivalently, \( \text{CUTP}(G) = \text{METP}(G) \) if and only if \( G \) does not have any \( K_5 \)-minor.

As a corollary of Theorem 2, we have that the facets of \( \text{CUTP}(G) \) (also, of \( \text{CUT}(G) \)) are determined by edge inequalities and cycle inequalities if and only if \( G \) does not have any \( K_5 \)-minor.

A 3-cycle inequality corresponds to the usual triangle inequality. Among all cycle inequalities, the triangle inequalities are the only inequalities defining facets of
2.3 Skeletons of Platonic and semiregular polyhedra

An exposition of the theory in the case of complete graph is in [22]. Denote by \( \text{CUTP}(G) \) the cut polytope of a graph. For example, the complete graph case it is possible that a negative type inequality defines a facet of \( \text{CUTP}(G) \). In contrast to \( \text{CUTP}(G) \), \( \text{CUTP}(K_n) \). Each edge \( e \) corresponds to two faces, \( x(e) \geq 0 \) and \( x(e) \leq 1 \). The \( 2|E| \) edge faces decompose into orbits, one for each orbit of edges of \( G \) under \( \text{Aut}(G) \). Let \( c_s \) denote the number of all chordless \( s \)-cycles in \( G \). There are \( 2^{s-1}c_s \) \( s \)-cycle faces, which decompose into orbits, one for each orbit of chordless \( s \)-cycles of \( G \) under \( \text{Aut}(G) \). For \( s \geq 2 \), \( s \)-cycle inequalities contain exactly \( 2^{|V|-s} \) cuts.

Given a sequence \( b_1, \ldots, b_n \) of integers, we define the inequality

\[
\text{scal}(b) = \sum_{(i,j) \in E(G)} x(i, j)b_ib_j \leq 0.
\]

If \( \sum_i b_i = 1 \) then the inequality is called hypermetric inequality, while for \( \sum_i b_i = 0 \) it is called negative type inequality. Both are valid on \( \text{CUTP}(G) \) and \( \text{CUTP}(K_n) \). An exposition of the theory in the case of complete graph is in [22]. Denote by \( \text{Tr}(p, q; r) \) the triangle inequality obtained as \( \text{scal}(b) \) with all non-zero \( b_i \) being \( b_p = b_q = 1 = -b_r \), and by \( \text{Pent}(p, q, r; u, v) \) the pentagonal inequality obtained as \( \text{scal}(b) \) with all non-zero \( b_i \) being \( b_p = b_q = b_r = 1 = -b_u = -b_v \). In contrast to the complete graph case it is possible that a negative type inequality defines a facet of the cut polytope of a graph. For example, \( \text{Pyr} \left( \text{APrism}_4 \right) \) has one facet determined by \( b = (1^3, (-1)^3, -2) \), see the data in [30] where the classification into edge, \( s \)-cycle, hypermetric and negative type inequality is done when possible.

2.3 Skeletons of Platonic and semiregular polyhedra

Let \( G \) be embedded without crossing in some oriented surface. Call face-bounding any cycle of \( G \), bounding a face in an embedding of \( G \).

Given a \( \text{Prism}_m \) (\( m \neq 4 \)) or an \( \text{APrism}_m \) (\( m \neq 3 \)), we call rung-edges the edges connecting two \( m \)-gons, and ring-edges other \( 2m \) edges.

Let \( P \) be an ordered partition \( X_1 \cup \cdots \cup X_{2t} = \{1, \ldots, m\} \) into ordered sets \( X_i \) of \( |X_i| \geq 3 \) consecutive integers. Call \( P \)-cycle of \( \text{Prism}_m \) the chordless \((m + 2t)\)-cycle obtained by taking the path \( X_1 \) on the, say, 1-st \( m \)-gon, then rung edge (in the same direction, then path \( X_2 \) on the 2-nd \( m \)-gon, etc. till returning to the path \( X_1 \). Any vertex of \( \text{Prism}_m \) can be taken as the 1-st element of \( X_1 \), in order to fix a \( P \)-cycle. So, a \( P \)-cycle defines an orbit of \( 2^{m+2t-1}2m \) \((m + 2t)\)-cycle facets of \( \text{CUTP}(\text{Prism}_m) \), except the case \((|X_1|, \ldots, |X_{2t}|) = (|X_2|, \ldots, |X_{2t}|, |X_1|)\) when the orbit is twice smaller.

A \( P \)-cycle of \( \text{APrism}_m \) is defined similarly, but we ask only \( |X_i| \geq 2 \) and rung edges, needed to change \( m \)-gon, should be selected, in the cases \( |X_i| = 2, 3 \) so that they not lead to a ring edge, i.e., a chord on \( P \). Clearly, \( P \)-cycles are all possible chordless \( t \)-cycles with \( t \neq 4, m \) for \( \text{Prism}_m \) and with \( t \neq 2, m \) for \( \text{APrism}_m \).

**Proposition 1** (i) If \( G \) is \( \text{Prism}_m \) (\( m \geq 5 \)), then all facets of \( \text{CUTP}(G) \) are:

1. orbit of \( 2m \) edge facets (from all \( m \) rung-edges)
2. orbit of \( 4m \) edge facets (from all \( 2m \) ring-edges);
3. orbit of \( 2^3p_4 = 8m \) 4-cycle facets (from all \( m \) 4-face-bounding 4-cycles);
4. orbit of \( 2^{m-1}p_m \) of \( m \)-cycle facets (from both \( m \)-face-bounding \( m \)-cycles);

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(5) orbits of cycle facets for all possible $P$-cycles.

(ii) If $G$ is $A_{\text{Prism}}(m \geq 4)$, then all facets of $\text{CUTP}(G))$ are:

1. orbit of $2^2p_3 = 8m$ 3-cycle facets (from all $2m$ 3-face-bounding 3-cycles);
2. orbit of $2^{m-1}p_m$ of $m$-cycle facets (from both $m$-face-bounding $m$-cycles);
3. orbits of cycle facets for all possible $P$-cycles.

**Proof** This follows from the enumeration of chordless cycles, Theorem 1 and 2. \qed

### 2.4 Möbius ladders

The Möbius ladder $M_{2m}$ has $2m$ vertices with edges $(2i - 1, 2i + 1)$ for $1 \leq i \leq m$, $(1, 2m), (2, 2m - 1)$ and $(2i, 2i + 2), (2i - 1, 2i + 1)$ for $1 \leq i \leq m - 1$. All graphs $M_{2m}$ are toroidal. The graph $M_{2m}$ has a $K_5$ minor if and only if $m \geq 5$.

**Conjecture 2** (i) For $m = 2k + 1 \geq 5$ a facet of $\text{CUTP}(M_{2m})$ is determined by the inequality

$$x(2m - 1, 2m) + x(2, 2m - 1) - 2x(1, 2m) + \sum_{i=1}^{k} 2x(4i - 1, 4i)$$

$$+ 2x(4i - 1, 4i + 1) + x(4i - 3, 4i - 2) - x(4i - 1, 4i)$$

$$+ x(4i - 3, 4i - 1) + x(4i, 4i + 2) \geq 0$$

(ii) For $m = 2k + 2 \geq 6$ a facet of $\text{CUTP}(M_{2m})$ is determined by the inequality

$$x(2m - 2, 2m) + 2x(2m - 3, 2m - 1) + x(2m - 1, 2m) + x(2, 2m - 1)$$

$$- 2x(1, 2m) + \sum_{i=1}^{k} x(4i - 2, 4i) + 2x(4i, 4i + 2) + 2x(4i - 3, 4i - 1)$$

$$+ x(4i - 1, 4i) + x(4i - 1, 4i + 1) - x(4i + 1, 4i + 2) \geq 0$$

For $m = 5, 6$ the orbit of those facets are all the facets disjoints from the ones of $\text{METP}(M_{2m})$.

### 2.5 Complete-like graphs

$K_n$ is toroidal only for $n = 5, 6, 7$, while it is 1-planar only for $n = 5, 6$. The planar complete multipartite graphs are: $K_{2,m}; K_{1,1,m}; K_{1,2,2}; K_{1,1,1,1} = K_4$ and their subgraphs. The complete multipartite graphs that are 1-planar but not planar are: $K_6; K_{1,1,1,6}; K_{1,1,2,3}; K_{2,2,2,2}; K_{1,1,1,2,2}$ and their subgraphs ([1][11]).

Given positive integers $a_1, \ldots, a_t$ with $t \geq 2$ and $1 \leq a_1 \leq \cdots \leq a_t$, let $G$ be the complete multipartite graph $K_{a_1, \ldots, a_t}$. All possible chordless cycles in $G$ are $c_3 = \sum_{1 \leq i < j < k < l} a_ia_ja_ka_l$ triangles and $c_4 = \sum_{1 \leq i < j < l} \binom{a_i}{2}\binom{a_j}{2}$ quadrangles. Hence, $c_3 > 0$ if and only if $t > 2$ and $c_4 > 0$ if and only if $t \geq 2$ and for at least two indexes $i$.
we have $a_t \geq 2$. So, among edge and cycle facets of CUTP($G$), only three such types are possible: $2|E|$ edge facets if $t = 2$, $4c_3$ 3-cycle facets and $8c_4$ 4-cycle facets.

All cases, when there are no other facets, i.e., when $G$ has no $K_5$-minor, are given in Table 1; note that the facets are simplices for $G = K_{2,2}$ and $K_{1,1,1,1,1}$. In particular, $G = K_{m+i} - K_m$, $m > 1$, has no $K_5$-minor only for $i = 1, 2, 3$. The facets of CUTP($G$) are the orbit of $2m$ edge facets for $i = 1$, the orbit of $2m$ 3-cycle facets for $i = 2$ and two orbits (of sizes $12m$ and 4) of 3-cycle facets for $i = 3$.

Some of remaining cases are presented in Table 2. For $G = K_{m+4} - K_m = K_{1,1,1,1,m+1}$ and $K_{1,1,2,m+2}$, the number of orbits stays constant for any $m$: 4 and 7, respectively (Check for $m \leq 5$).

If $G = K_{1,1,2,m}$ with $m \geq 3$, then CUTP($G$) has $8 + 20m + 8\binom{m}{2}(16m - 15)$ facets in 7 orbits: 3 orbits of 8, 4, $16m$ 3-cycle facets, one orbit of $8\binom{m}{2}$ 4-cycle facets and 3 orbits of $64\binom{m}{2}$, $64\binom{m}{2}$, $384\binom{m}{3}$ $\{0, \pm 1\}$-valued non cycle facets, having 4 values $-1$ and 11, 11, 12 values of 1. The partition is $\{1\}, \{2\}, \{3, 4\}, \{5, \ldots, m + 4\}$.

CUTP($K_{1,1,2,2}$) has 184 facets in 4 orbits: 2 orbits of $8 + 8$, 32 3-cycle facets, one orbit of 8 4-cycle facets and one orbits of 27 facets, represented by

$$\text{scal}(1, 1, 1, -1, -1, 0) + \text{scal}(0, 0, 1, 1, 0, -1) \leq 0.$$ 

The graph $G = K_{m+t} - K_m = K_{1, \ldots, 1, m}$ has a $K_5$-minor only if $t \geq 4$. The graph $K_{t+1}$ is an induced subgraph of $G$. If $m \geq 3$, then CUTP($G$) has 2 orbits of $4m\binom{t}{2}$ and $4\binom{1}{3}$ 3-cycle facets and, for $t < 4$ only, no other facets. The partition is $\{1\}, \ldots, \{t\}, \{t + 1, \ldots, t + m\}$.

If $G = K_{m+4} - K_m$, then CUTP($G$) has $8(8m^2 - 3m + 2)$ facets in 4 orbits: 2 orbits of $24m$, 16 3-cycle facets and 2 orbits of sizes $16m$, $128\binom{m}{2}$, represented by $P_{\text{ent}}(1, 2, 5; 3, 4)$ and

$$\text{scal}(1, 1, -1, 0, 1, -1, 0, \ldots, 0) + \text{scal}(0, 0, 0, -1, 1, 1, 0, \ldots, 0) \leq 0.$$ 

If $G = K_{m+5} - K_m$, then among many orbits of facets of CUTP($G$), 4 hypermetric are induced by the induced subgraph $K_6$ and its hypermetric facets. We check that for $m \leq 10$ one more orbit of facet is induced by the inequality

$$\text{scal}(1, -1, -1, 0, 0, 1, 1, 0, \ldots, 0) + \text{scal}(0, 0, 0, 1, 0, -1, 0, \ldots, 0) \leq 0.$$ 

Let $G = P_{\text{yr}}^2(C_m)$. Clearly, it is $K_4$, $K_5$ if $m = 2, 3$, respectively. For $m \geq 4$, it hold $A(G) = 4m$ and all chordless cycles $3m$ triangles and unique $m$-cycle. Any of $3m + 1$ edges belongs to a triangle. So, among orbits of facets of CUTP($G$), there are two (of size $8m$ and $4m$) orbits of 3-cycle facets and orbit of $2^{m-1}m$-cycle facets. All other facets for $m \leq 7$ are $\{0, \pm 1\}$-valued.

For odd $m = 2k + 1$ we checked that for $m \leq 13$ there is an orbit of $2^{m+1}$ facets of CUTP($P_{\text{yr}}^2(C_m)$) determined by the hypermetric inequality $\text{scal}((-1)^k, 1^{k+1}) \leq 0$.

Note that $K_7 - C_5 = P_{\text{yr}}^2(C_5)$. Now, $G = K_7 - C_4$ has $c_3 = 19$; CUTP($G$) has four orbits of facets: three (of sizes 48, 24, 4) of 3-cycle facets and one orbit of size 32, represented by $P_{\text{ent}}(4, 5, 6; 2, 7)$. 

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The motivation of those computations was to look for possible generalization of
Seymour’s result on graphs without $K_5$-minor. For many graphs with a $K_5$-minor there
is just one additional orbit of facet that is not a edge or cycle inequality and maybe
one could find a neat description of this orbit. However, a graph such as $Pyr^2(C_m)$ is
a $Prism^m$ to which one edge has been added and $Pyr^2(C_7)$ has already 11 orbits of
facets.

3 Quasi-metric polytopes over graphs

In this section we consider a generalization of the metric polytope of a graph to the
case of quasi-metrics, obtained by omitting the symmetry condition $d(i, j) = d(j, i)$
for all $i, j$ in the definition of semi-metric.

We first consider the construction in the complete graph case:

Definition 2 Given a fixed $n \geq 3$ we define:

(i) $Q_n$ to be the set of functions $d : \{1, \ldots, n\}^2 \rightarrow \mathbb{R}$ satisfying $d(i, i) = 0$ for
$1 \leq i \leq n$.

(ii) For $d \in Q_n$ and $1 \leq i, j, k \leq n$ the oriented triangle inequality is

$$d(i, j) \leq d(i, k) + d(k, j).$$

(iii) For $d \in Q_n$ and $1 \leq i, j \leq n$ the non-negativity inequality is $d(i, j) \geq 0$.
(iv) The quasi-metric cone $QMET(K_n)$ is the set of $d \in Q_n$ satisfying (ii) and (iii).
(v) The quasi-metric polytope $QMETP(K_n)$ is the set of $d \in QMET(K_n)$ satisfying
for all $1 \leq i, j, k \leq n$ the additional inequalities

$$d(j, i) + d(i, k) + d(k, j) \leq 2 \text{ and } d(i, j) \leq 1.$$

The cones $QMET(K_n)$ were introduced in [17,19], which considered questions
about adjacencies of facets, number of zeros of extreme rays and extension opera-
tions. There the notions of oriented cut and multicut quasi-metrics on $K_n$ were also
introduced and investigated.

Given a subset $S \subseteq \{1, \ldots, n\}$ and $d \in Q_n$, the oriented switching $F_S(d)$ is the
quasi-metric defined for $1 \leq i, j \leq n$ by

$$F_S(d)(i, j) = \begin{cases} 1 - d(j, i) & \text{if } |S \cap \{i, j\}| = 1, \\ d(i, j) & \text{otherwise.} \end{cases}$$

The symmetric group $\text{Sym}(n)$ and reversal operation $d(x, y) \mapsto d(y, x)$ acts on
$QMET(K_n)$ and define a group of size $2n!$. The oriented switchings, reversal and
$\text{Sym}(n)$ act on $QMETP(K_n)$ and determine a group of size $2^n n!$.

The cone $MET(K_n)$ and polytope $METP(K_n)$ are contained into $QMET(K_n)$ and
$QMETP(K_n)$ but we have another interesting subset (see [16] for more details):
Definition 3  Given $n \geq 3$ and a function $d \in Q_n$, $d$ is called weightable if there exist a vector $w \in \mathbb{R}^n$ such that for all $1 \leq i, j \leq n$

$$d(i, j) + w_i = d(j, i) + w_j.$$  

The set of weightable functions $d \in Q_n$ is called $WQ_n$.

We thus define the cone $WQMET(K_n) = QMET(K_n) \cap WQ_n$ and polytope $WQMETP(K_n) = QMETP(K_n) \cap WQ_n$. For $d \in WQ_n$ of weight $w$ and a set $S \subseteq \{1, \ldots, n\}$, one can easily see that $F_S(d) \in WQ_n$ for the same weight $w$. Thus oriented switchings preserve $WQMETP(K_n)$.

We are now in position to define the corresponding cones and polytopes on graphs:

Definition 4  Let $G$ be an undirected graph; we define $E(G)$ the set of edges. For an integer $n$ and a set $S$ of 2 element subsets of $\{1, \ldots, n\}$ we define $Dir(S)$ to be the set of directed $(x, y), (y, x)$ for $(x, y) \in S$. So, for each edge $e = (i, j)$ in $G$ we associate two directed edges $(i, j)$ and $(j, i)$ in $Dir(E(G))$.

(i) The space $Q(G)$ and $WQ(G)$ are the projections of the space $Q_n$ and $WQ_n$ on $\mathbb{R}^{Dir(E(G))}$.

(ii) The cones $QMET(G)$ and $WQMET(G)$ are the projections of the cones $QMET(K_n)$ and $WQMET(K_n)$ on $\mathbb{R}^{Dir(E(G))}$.

(iii) The polytopes $QMETP(G)$ and $WQMETP(G)$ are the projections of the polytopes $QMETP(K_n)$ and $WQMETP(K_n)$ on $\mathbb{R}^{Dir(E(G))}$.

We can now give a description by inequalities of $QMET(G)$:

Theorem 3  For a given graph $G$ the polyhedral cone $QMET(G)$ is the set of functions $d \in \mathbb{R}^{Dir(E(G))}$ such that

(i) For any directed edge $e = (i, j)$ of $G$ we have the inequality $d(i, j) \geq 0$.

(ii) For any cycle $(v_1, v_2, \ldots, v_m)$ of $G$ we have

$$d(v_1, v_m) \leq d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{m-1}, v_m). \quad (6)$$

The same results holds for $WQMET(G)$ by adding the extra condition that there exist a function $w$ such that $d(i, j) - d(j, i) = w_i - w_j$ for $1 \leq i, j \leq n$.

Proof  Our proof is adapted from the proof of [22, Theorem 27.3.3]. It is clear that the cycle inequalities (i) and (ii) are valid for $d \in QMET(K_n)$ and that edges of $G$ do not occur in their expression. Therefore, the inequalities are also valid for the projection.

The proof of sufficiency is done by induction on the number of edges and is more complicated. Suppose that the result is proved for $G + e$, i.e. $G$ to which an edge $e = (i, j)$ has been added. Suppose we have an element $x$ of $\mathbb{R}^{Dir(E(G))}$ satisfying all oriented cycle inequalities.

We need to find an extension of $x$, i.e. a function $y \in \mathbb{R}^{Dir(E(G)) \cup \{e\}}$. That is we need to find $y(i, j)$ and $y(j, i)$.
We let $P_{i,j}$ denote the set of directed paths from $i$ to $j$ in $G$. Assume first that $P_{i,j} \neq \emptyset$. For a directed path $P$ we write $x(P) = \sum_{f \in P} x(f)$. We define

$$u_{i,j} = \min_{P \in P_{i,j}} x(P).$$

Since $x$ is non-negative, we have $u_{i,j} \geq 0$. We then write

$$l_{i,j} = \max_{P \in P_{i,j}, f \in P} x(f^{-1}) - x(P \setminus \{f\})$$

with $f^{-1}$ the reversal of the directed edge $f$. If $P_{i,j} = \emptyset$, i.e., if the edge $e$ is connecting two connected components of $G$ then we set $l_{i,j} = u_{i,j} = 0$.

We have $l_{i,j} \leq u_{i,j}$ since otherwise we could take a path $p_u$ realizing the minimum $u_{i,j}$, a path $p_l$ and directed edge $f$ realizing the maximum $l_{i,j}$ put it together and get a counterexample to the oriented cycle inequality (ii).

So, we can find a value $y(i, j)$ such that

$$l_{i,j} \leq y(i, j) \leq u_{i,j}$$

and since $u_{i,j} \geq 0$ we can choose $y(i, j) \geq 0$. The same holds for $y(j, i)$. Therefore we found an extension of $x$ in $\mathbb{R}^{\text{Dir}(E(G) \cup \{e\})}$ and this proves the result for QMET($G$).

For WQMET($G$) we need to adjust the above construction in the induction proof. If $P_{i,j} = \emptyset$ then $i$ and $j$ belong to different connected components and so we can adjust the weights so that $w_i = w_j$. This is possible since the weights are determined up to a constant term in a connected component.

On the other hand if $P_{i,j}$ is not empty then the weight is already given and we should get in the end $y(i, j) - y(j, i) = w_i - w_j$. Actually this is not a problem since it can easily be shown that $u_{i,j} - u_{j,i} = w_i - w_j$ and $l_{i,j} - l_{j,i} = w_i - w_j$ and so the inductive construction works for WQMET($G$) as well.

Now we turn to the construction for the polytope case:

**Theorem 4** For a given graph $G$ the polytope QMETP($G$) is defined as the set of functions $\mathbb{R}^{\text{Dir}(E(G))}$ such that

(i) For any directed edge $e = (i, j)$ of $G$ the inequalities $0 \leq d(i, j) \leq 1$ hold.

(ii) For any oriented cycle $C = (v_1, v_2, \ldots, v_m)$ of $G$ and subset $F$ of odd size

$$\sum_{f=(v,v') \in F} d(v', v) - \sum_{f=(v,v') \in C \setminus F} d(v, v') \leq |F| - 1. \quad (7)$$

The same result holds for WQMETP($G$) with the extra condition that there exist a function $w$ such that $d(i, j) - d(j, i) = w_i - w_j$ for $1 \leq i, j \leq n$.

**Proof** The proof follows by remarking that the inequalities (i) and (ii) are the oriented switchings of the non-negative inequality and oriented cycle inequality (6). Thus the proof follows from Theorem 3 and the same proof strategy as [22, Theorem 27.3.3].
We conclude by mentioning a possible extension of cuts to the oriented case. Given an $n$-vertex graph $G$, and a set $S \subseteq \{1, \ldots, n\}$, an oriented cut is the function on $\text{Dir}(E(G))$ defined for $(x, y) \in \text{Dir}(E(G))$ by
\[
\delta'^G_S (x, y) = \begin{cases} 
1 & \text{if } x \in S, y \notin S \\
0 & \text{otherwise}.
\end{cases}
\]
The oriented cut $\delta'^G_S$ is a weightable quasi-metric for the weight $w$ defined by $w_i = -1$ for $i \in S$ and $w_i = 0$ otherwise. The conic hull of the oriented cuts is the oriented cut cone $\text{OCUT}(G)$. We have $F_S(\delta'^G_S) = \delta'^G_S$. We define the oriented cut polytope $\text{OCUTP}(G)$ to be the convex hull of the oriented switchings of the oriented cuts $F_S(\delta'^T_G)$ for $S \subseteq \{1, \ldots, n-1\}$, $T \subseteq \{1, \ldots, n\}$. The polytope $\text{OCUTP}(G)$ has at most $2^{2n-2}$ vertices and it has exactly that number if $G = K_n$.

Dual description computations show that $\text{OCUTP}(K_n) = \text{WQMETP}(K_n)$ for $n \leq 4$ while we have $\text{OCUTP}(K_5) \neq \text{WQMETP}(K_5)$. Based on that and analogy with Theorem 2 a natural conjecture would be that $\text{OCUTP}(G) = \text{WQMETP}(G)$ if $G$ has no $K_5$ minor. But this is not true since, for example, $\text{Prism}_3$ and $K_5 - K_2$ are counterexamples.

The quasi-metric polytopes introduced in this section deserve further study similar to the one paid to metric polytopes of graphs. However, computations done so far indicate that the dual description of the oriented cut polytopes is much more complicated than for cut polytopes of graphs.

4 Hemi-metric polytopes over simplicial complexes

We can also generate metrics to a measure of distance of more than two objects. Our approach differs from [14,15,18,21] and has the advantage of allowing to define it on simplicial complexes.

Throughout this section we fix $m \geq 1$ and $n \geq m + 1$. We consider by $\text{Set}_{n,m}$ the set of subsets of $m + 1$ points of $\{1, \ldots, n\}$.

**Definition 5**

(i) An $m$-dimensional simplicial complex $\mathcal{K}$ is formed by a subset of $\text{Set}_{n,m}$.

(ii) A $k$-simplex of $\mathcal{K}$ is a subset of $k + 1$ points contained in one subset $S \in \mathcal{K}$.

(iii) An $m$-dimensional simplicial complex $\mathcal{K}$ is a $m$-closed manifold if for every $(m - 1)$-simplex the number of $m$-simplices of $S$ containing it is even.

For the case $m = 1$ a closed manifold is a union of edge-disjoint cycles. We now proceed to defining the corresponding cycle inequalities:

**Definition 6** Given an $m$-dimensional simplicial complex $\mathcal{K}$ on $\{1, \ldots, n\}$, the hemi-metric cone $\text{HMET}(\mathcal{K})$ is formed by the functions $d$ on $\mathcal{K}$ satisfying:

(i) the non-negative inequalities
\[
d(\Delta) \geq 0 \text{ for all } \Delta \in \mathcal{K},
\]
(ii) and the simplicial inequalities

$$d(\Delta_i) \leq \sum_{1 \leq j \leq r, i \neq j} d(\Delta_j)$$

(8)

for all closed manifolds \(\{\Delta_1, \ldots, \Delta_r\} \subseteq K\) and \(1 \leq i \leq r\).

For \(m = 1\) the definition corresponds to the one of \(\text{MET}(G)\) in Theorem 1. If one takes the closed manifold to be a \(m + 1\) dimensional simplex \(\{\Delta_1, \ldots, \Delta_{m+2}\}\) then inequality (8) gives

$$d(\Delta_i) \leq \sum_{1 \leq j \leq m+2, j \neq i} d(\Delta_j) \text{ for } 1 \leq i \leq m+2$$

(9)

which was the formulation used in [14,15,18,21].

**Theorem 5** Let us take \(K\) an \(m\)-dimensional simplicial complex on \(n\) points. The cone \(\text{HMET}(K)\) is the projection of \(\text{HMET}(\text{Set}_{n,m})\) on the \(m\)-simplices included in \(K\).

**Proof** Like Theorem 3 our proof is adapted from [22, Theorem 27.3.3]. The inequalities for \(\text{HMET}(K)\) are clearly valid on \(\text{HMET}(\text{Set}_{n,m})\) which proves one inclusion.

We want to prove by induction on the cardinality of \(K\) the other inclusion. Suppose that we have a hemimetric \(d \in \text{HMET}(K)\) and a \(m\)-simplex \(\Delta \notin K\). We want to find a hemimetric \(d'\) on \(\text{HMET}(K \cup \{\Delta\})\), i.e., find a value of \(d(\Delta)\). For a subset \(S \subseteq \text{Set}_{n,m}\) we define

$$d(S) = \sum_{\Delta' \in S} d(\Delta').$$

Let us consider the set

$$W_{K,\Delta} = \{U \subseteq K : U \cup \{\Delta\} \text{ is a closed manifold}\}.$$

We now define the upper bound

$$u_{K,\Delta} = \min_{U \in W_{K,\Delta}} d(U).$$

We have \(u_{K,\Delta} \geq 0\) since \(d \in \text{HMET}(K)\) implies that \(d\) is non-negative.

The lower bound is formed by

$$l_{K,\Delta} = \max_{P \in W_{K,\Delta}, \Delta' \in P} d(\Delta') - d(P \setminus \{\Delta'\}).$$

Suppose that \(l_{K,\Delta} > u_{K,\Delta}\). We have \(u_{K,\Delta}\) realized by \(U_0\) and \(l_{K,\Delta}\) is realized by \(L_0\) and a simplex \(\Delta'_0 \in L_0\). The union \(L_0 \cup U_0\) is not necessarily a closed manifold since \(L_0 \cup U_0\) may share \(m\)-simplices. If that is so then we remove them and consider instead \(W_0 = L_0 \cup U_0 - L_0 \cap U_0\).
The inequality $l_{K,\Delta} > u_{K,\Delta}$ implies then
\[ d(\Delta'_0) > d(L_0 \setminus \{\Delta'_0\}) + d(U_0) = d(W_0 \setminus \{\Delta'_0\}) + 2d(L_0 \cap U_0) \geq d(W_0 \setminus \{\Delta'_0\}) \]
which violates the fact that $d \in \text{HMET}(K)$. Thus we can find a value $\alpha$ with
\[ l_{K,\Delta} \leq \alpha \leq u_{K,\Delta} \text{ and } \alpha \geq 0. \]

By setting $d(\Delta) = \alpha$ we extend $d$ to $\text{HMET}(K \cup \{\Delta\})$ and thus prove the theorem. □

The inequality set defining $\text{HMET}(K)$ is highly redundant but is still finite so, the cone $\text{HMET}(K)$ is actually polyhedral.

On the other hand, using only the simplex inequalities (9) gives a larger cone than $\text{HMET}(K)$. For example the Octahedron has 6 vertices and 8 triangular faces $\{\Delta_1, \ldots, \Delta_8\}$ and is a closed manifold in $\text{Set}_{6,2}$. Thus it determines in $\text{HMET}(\text{Set}_{6,2})$ the following inequality
\[ d(\Delta_1) \leq d(\Delta_2) + \cdots + d(\Delta_8) \]
which is not implied by the inequalities on the $m$-simplices (9) as can be proved using linear programming.

Acknowledgements The second author gratefully acknowledges support from the Alexander von Humboldt foundation and thanks the anonymous referees.

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