Quantization of a Friedmann-Robertson-Walker Model with Gauge Fields in N=1 Supergravity *

P.V. Moniz†

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street, Cambridge, CB3 9EW
United Kingdom
DAMTP R95/36

Abstract

The purpose of this paper is to investigate a specific FRW model derived from the theory of N=1 supergravity with gauged supermatter. The supermatter content is restricted to a vector supermultiplet. This objective is particularly worthwhile. In fact, it was pointed in ref. [22] (Class. Quantum Grav. 12 (1995) 1343) that \( \Psi = 0 \) was the only allowed quantum state for N=1 supergravity with generic gauged supermatter subject to suitable FRW ansätze.

The ansätze employed here for the physical variables was presented in ref. [22]. The corresponding Lorentz and supersymmetry quantum constraints are then derived. Non-trivial solutions are subsequently found. A no-boundary solution is identified while another state may be interpreted as a wormhole solution. In addition, the usefulness and limitations of the ansätze are addressed. The implications of the ansätze with respect to the allowed quantum states are also discussed with a view on ref. [22].

1 Introduction

Research in supersymmetric quantum cosmology using canonical methods started about 20 years or so [1]-[3]. Since then, many other papers have appeared in the literature [4]-[28]. A review on the subject of canonical quantum N=1,2 supergravity can be found in ref. [29].

An important feature of N=1 supergravity is that it constitutes a “square-root” of gravity [4]. In fact, the Lorentz and supersymmetry constraints of the theory induce a set of coupled first-order differential equations which the quantum states ought to satisfy. Such physical states will subsequently satisfy the Hamiltonian constraints as a consequence of the supersymmetry constraints.

---

*PACS numbers: 04.60.-m, 04.65.+e, 98.80. H
†e-mail: prlm10@amtp.cam.ac.uk; PMONIZ@Delphi.com
algebra [1]-[3]. Supersymmetry may also play an important role when dealing with (ultraviolet) divergences in quantum cosmology and gravity [30] and removing Planckian masses induced by wormholes [4, 31]. Hence, canonical quantization methods may provide new and interesting insights as far as quantum supergravity theories are concerned.

Important results for Bianchi class-A models were recently achieved within pure N=1 supergravity. On the one hand, the Hartle-Hawking (no-boundary) [32] and wormhole (Hawking-Page) [33] states were found in the same spectrum of solutions [12, 13]. This result improved previous attempts [5]-[11] where only one of these states could be found, depending on the homogeneity conditions imposed on the gravitino [7]. The reason was an overly restricted ansatz for the wave function of the universe. More precisely, gravitational degrees of freedom have not been properly taken into account in the expansion of $\Psi$ in Lorentz invariant fermionic sectors. Another undesirable consequence of the ansatz used in [5]-[11] occurred from the inclusion of a cosmological constant [14]-[16]: Bianchi class-A models were found to have no physical states but the trivial one, $\Psi = 0$. However, an extension of the method in [12, 13] did led in ref. [17] to solutions of the form of exponentials of the Chern-Simons functional.

The introduction of supermatter [18, 19] in supersymmetric minisuperspaces provided challenging results. A scalar supermultiplet, constituted by complex scalar fields, $\phi, \bar{\phi}$ and their spin-$\frac{1}{2}$ partners, $\chi_A, \bar{\chi}_A'$ was considered in ref. [4, 9, 19, 20, 23]-[26]. A vector supermultiplet, formed by a gauge vector field $A_\mu^{(a)}$ and its supersymmetric partner, was added in ref. [21, 22]. A FRW model was considered in ref. [4, 9, 19, 20, 23, 24] while a Bianchi IX model was dealt with in ref. [25, 26]. Bianchi class-A models with Maxwell fields within N=2 supergravity were considered in ref. [27, 28].

Finding and identifying physical states in minisuperspaces obtained from supergravity theories with supermatter constitutes an important assignment:

- A wormhole state was obtained in ref. [4], but not in ref. [20]. We emphasize that the more general theory of N=1 supergravity with gauged supermatter [14] was employed in ref. [20]. The reason for the discrepancy between [4, 20] was analysed in ref. [23, 24] and related to the type of Lagrange multipliers and fermionic derivative ordering that were used.

- As far as a Hartle-Hawking state is concerned, some solutions present in the literature bear some of the properties corresponding to the no-boundary proposal [12]. Unfortunately, the supersymmetry constraints were not sufficient in determining the dependence of $\Psi$ with respect to the scalar field (cf. ref. [23, 24] for more details).

- The results found within the more general matter content used in ref. [21, 22] were disappointing: the only allowed physical state was $\Psi = 0$.

It is of interest to address some of these problems. On the one hand, the apparent absence of wormhole solutions [23, 24] and the difficulty to obtain an adequate Hartle-Hawking solution when scalar supermultiplets are considered. On the other hand, why non-trivial physical states are not permitted when all possible matter fields are present [21, 22].

The main purpose of this paper is precisely to investigate a particular FRW model obtained from N=1 supergravity with supermatter restricted to a vector supermultiplet. As a consequence, we hope to shed some light on the paradoxical situation found in ref. [21, 22], in spite of all the degrees of freedom thereby present.
In section 2 we will address the ansätze for the field variables. We summarize in subsection 2.1 the basic properties of the ansätze employed in the dimensional reduction of pure N=1 supergravity to a FRW model (see ref. [8, 9]). In subsection 2.2 we will consider our FRW case model. We put all scalar fields and corresponding supersymmetric partners equal to zero. The ansätze employed here (and in ref. [21, 22]) for the spin-1 field and corresponding fermionic partners are then described in some detail (see ref. [6]-[11] as well). The ansätze introduced in ref. [8, 9] for the case of N=1 pure supergravity are preserved under a combination of supersymmetry, local coordinates and Lorentz transformations. The preservation of our ansätze under combined gauge, supersymmetry, local coordinates and Lorentz transformations can be achieved only after further restrictions are imposed. It should be emphasized that a discussion on how gravitational, gravitino and supermatter fields can be accommodated within a \( k=1 \) FRW geometry and supersymmetry was not performed in ref. [21, 22]. Thus, the content of subsection 2.2 and following sections will surely contribute to our knowledge on supersymmetric minisuperspaces and associated features.

In section 3 we will derive the corresponding Lorentz and supersymmetry constraints. The results subsequently found will prove to be physically interesting. In contrast with ref. [21, 22], non-trivial solutions in different fermionic sectors are obtained. We identify a component of the Hartle-Hawking (no-boundary) solution [12, 33]. Another solution could be interpreted as a quantum wormhole state [33] (see ref. [37]). We stress that the Hartle-Hawking solution found here is part of the set of solutions shown in ref. [15], where a non-supersymmetric FRW minisuperspace with Yang-Mills fields was considered. Being N=1 supergravity a square-root of gravity, this result is particularly supportive. It further points that our approach and ansätze (see ref. [21, 22]) may be of some utility. Finally, our discussions and conclusions close this paper in section 4.

2 Ansätze for the field variables

The action for our model is obtained from the more general theory of N=1 supergravity with gauged supermatter [31]. We put all scalar fields and corresponding supersymmetric partners equal to zero (it should be noticed that Yang-Mills fields coupled to N=1 supergravity can be found in ref. [12]). The corresponding field variables will be constituted by a tetrad, \( e_A^A\nu \) (in two spinor component notation - cf., e.g., ref. [3]), gravitino fields, \( \psi_A^A, \bar{\psi}_A' \), (where a bar denotes Hermitian conjugation), a gauge spin-1 field, \( A_{\mu}^{(a)} \), (where \( (a) \) is a gauge group index) and the the corresponding spin-\( \frac{1}{2} \) partners, \( \lambda_A^{(a)}, \bar{\lambda}_A' \).

The restriction of this theory to a closed FRW model requires the introduction of specific ansätze for the fields mentioned above. This is discussed in the following subsections. We will start by reviewing how the tetrad and the gravitino fields can be chosen in pure N=1 supergravity [8, 9].

2.1 FRW model from pure N=1 supergravity

We choose the geometry to be that of a \( k=+1 \) Friedmann model with \( S^3 \) spatial sections, which are the spatial orbits of \( G=SO(4) \) – the (isometry) group of homogeneity and isotropy. The ansatz for the tetrad can then be written as

\[
e_{a\mu} = \begin{pmatrix} N(\tau) & 0 \\ 0 & a(\tau)E_{\bar{a}i} \end{pmatrix}, \quad e^{a\mu} = \begin{pmatrix} N(\tau)^{-1} & 0 \\ 0 & a(\tau)^{-1}E^{\bar{a}i} \end{pmatrix}.
\] (1)
where \( \hat{a} \) and \( i \) run from 1 to 3. \( E_{\hat{a}i} \) is a basis of left-invariant 1-forms on the unit \( S^3 \) with volume \( \sigma^2 = 2\pi^2 \). The spatial tetrad \( e^{AA'}_i \) satisfies the relation \( \partial_i e^{AA'}_j - \partial_j e^{AA'}_i = 2a^2e_{ijk}e^{AA'k} \) as a consequence of the group structure of \( SO(3) \), the isotropy (sub)group.

This ansatz reduces the number of degrees of freedom provided by \( e_{AA'\mu} \). If supersymmetry invariance is to be retained, then we need an Ansatz for \( \psi^A_\mu \) and \( \bar{\psi}^{A'}_{\bar{\mu}} \) which reduces the number of fermionic degrees of freedom as well \([8, 9]\). The Lagrange multipliers \( \psi^A_0 \) and \( \bar{\psi}^{A'}_{\bar{\mu}} \) are taken to be functions of time only. The Ansatz for the gravitino field further includes

\[
\psi^A_i = e^{AA'}_{i\bar{\mu}} \bar{\psi}^{A'}_{\bar{\mu}} , \quad \bar{\psi}^{A'}_{i\bar{\mu}} = e^{AA'}_{i\mu} \psi^A_{\mu},
\]

where we introduce the new spinors \( \psi^A_{\mu} \) and \( \bar{\psi}^{A'}_{\bar{\mu}} \) which are functions of time only. This means we truncate the general decomposition \( \psi^A_{BB'} = e_{BB'}^i \psi^A_i \)

\[
\psi^A_{BB'} = -2n^C_{B'} \gamma_{ABC} + \frac{2}{3} (\beta_A n_{BB'} + \beta_B n_{AB'}) - 2\varepsilon_{AB} n^C_{B'} \beta_C ,
\]

where \( \gamma_{ABC} = \gamma_{(ABC)} \), at spin \(-\frac{1}{2}\) mode level. I.e., \( \beta^A = \frac{2}{3} n^{AA'} \bar{\psi}^A_{A'} \sim \bar{\psi}^A \). However, it is important to stress that auxilairy fields were further required in \([8, 9]\) to balance the number of degrees of freedom in ansätze (1), (2).

Ansätze (1), (2) are preserved by a combination of local coordinate - \( \delta_{(lc)} \) - , Lorentz - \( \delta_{(L)} \) - and supersymmetry - \( \delta_{(s)} \) - transformations. In the 4-dimensional theory, these transformations are given by

\[
\begin{align*}
\delta_{(lc)} e^{AA'}_{\mu} &= \xi^\nu \partial_\nu e^{AA'}_{\mu} + e^{AA'}_{\nu} \partial_\mu \xi^\nu, \\
\delta_{(L)} e^{AA'}_{\mu} &= N^A_B e^{BA'}_{\mu} + N^{A'}_{B'} e^{AB'}_{\mu}, \\
\delta_{(s)} e^{AA'}_{\mu} &= -i \left( e^{A}_{\bar{\nu}} \bar{\psi}^{A'}_{\mu} + e^{A'}_{\nu} \psi^A_{\mu} \right), \\
\delta_{(lc)} \psi^A_{\mu} &= \xi^\nu \partial_\nu \psi^A_{\mu} + \psi^A_{\nu} \partial_\mu \xi^\nu, \\
\delta_{(L)} \psi^A_{\mu} &= N^A_B \psi^B_{\mu}, \\
\delta_{(s)} \psi^A_{\mu} &= 2D_\mu e^A = 2\partial_\mu e^A + \omega^{A}_{B\mu} e_B,
\end{align*}
\]

with the Hermitian conjugates of (4), (8), (9) and where \( \xi^\mu, N^{AB}, e^A \) are time-dependent vectors or spinors parametrizing, respectively, the above mentioned transformations.

In order for the ansatz (1) to be preserved under a combination of transformations (4), (8), (9) it ought to satisfy a relation as

\[
\delta e_i^{AA'} = P_1 \left[ e_{\mu}^{AA'} ; \psi^A_{\mu} \right] e_i^{AA'},
\]

where \( P_1 \) is an expression (spatially independent and possibly complex) where all spatial and spinorial indices have been contracted. Using expressions (4), (8), (9) with ansatz (8), defining

\[
\xi^i = -\xi^{AA'} \epsilon_{A'A}^i, \quad \xi^{AA'} = i\xi^{AB} n^A_B, \quad \xi^{A'B'} = 2\xi^{AB'} n^A_A n^B_B, \quad (11)
\]

and putting \( \xi^0 = 0 \) \( [8] \), we get

\[
\begin{align*}
\delta e_i^{AA'} &= -N^{AB} e_{iB}^{A'} - \bar{N}^{A'B'} e_{iB'}^{B'}, \\
&+ a^{-1} \xi^{AC} e_{iC}^{A'} + a^{-1} \xi^{A'C'} e_{iC'}^{A'}, \\
&+ i \left( e^A \psi^B_{iB'} + e^{A'} \bar{\psi}^{B'}_{iB'} e^A_{B'} \right).
\end{align*}
\]

(12)
Using now the spinorial relation
\[ \psi^{[A} \chi^{B]} = \frac{1}{2} \epsilon^{AB} \psi_C \chi^C, \quad (13) \]
we derive that
\[ \epsilon^A \psi^B e_i^A' = \epsilon^{(A} (\psi^B) e_i^{A'} + \frac{1}{2} \epsilon_C \psi^C e_i^{AA'}, \quad (14) \]
from which it follows that
\[
\begin{align*}
\delta \epsilon^{AA'}_{i} &= \left( -N^{AB} + a^{-1} \xi^{AB} + i \epsilon^{(A} (\psi^B) \right) e_B^{A'}_{i} \\
&+ \left( -\bar{N}^{A'B'} + a^{-1} \bar{\xi}^{A'B'} + i \epsilon^{(A'} (\bar{\psi}^{B')} \right) e_A^{B'i} \\
&+ \frac{i}{2} \left( \epsilon_C \psi^C + \bar{\epsilon}_C \bar{\psi}^C \right) e^{AA'}_{i} . \quad (15)
\end{align*}
\]
The relation (14) holds provided that the relations
\[ N^{AB} - a^{-1} \xi^{AB} - i \epsilon^{(A} (\psi^B) = 0, \quad \bar{N}^{A'B'} - a^{-1} \bar{\xi}^{A'B'} - i \epsilon^{(A'} (\bar{\psi}^{B')} = 0, \quad (16) \]
between the generators of Lorentz, coordinate and supersymmetry transformations are satisfied. Hence, we will achieve \( \delta \epsilon^{AA'}_{i} = C(t) \delta \epsilon^{AA'}_{i} \) with \( C(t) = \frac{1}{2} \left( \epsilon_C \psi^C + \bar{\epsilon}_C \bar{\psi}^C \right) \) and the ansatz (1) remains unchanged (see ref. [43]). Notice that any Grassman-algebra-valued field can be decomposed into a “body” or component along unity (which takes values in the field of real or complex numbers) and a “soul” which is nilpotent (see ref. [43]). The variation (15) with conditions (16) imply that \( \delta \epsilon^{AA'}_{i} \) exists entirely in the nilpotent (“soul”) part.

Let us now address ansatz (2). From eq. (1), (2) we get (cf. [9])
\[ 2 D_i \epsilon^A = \left[ 2 \left( \frac{\dot{a}}{a} + \frac{i}{a} \right) - i \frac{2}{N} \left( \psi_F \psi^F_0 + \bar{\psi}_{F'0} \bar{\psi}^{F'} \right) \right] n_{BA'} \epsilon^{AA'}_{i} \epsilon^B + \frac{i}{2} \epsilon^{(A'}_{i} \epsilon^B \psi_E \bar{\psi}^{E'} \epsilon^B . \quad (17) \]
Using then eq. (11), (13), (14), (15) and (17) it follows that
\[ \delta \psi^A_i = \frac{3i}{4} \epsilon^A \psi^B \bar{\psi}^{B'} e_{BB'} + a^{-1} \xi^{A'B'} \epsilon^A_{B'i} \bar{\psi}^{A'} + \left[ 2 \left( \frac{\dot{a}}{a} + \frac{i}{a} \right) - i \frac{2}{N} \left( \psi_F \psi^F_0 + \bar{\psi}_{F'0} \bar{\psi}^{F'} \right) \right] n_{BA'} \epsilon^{AA'}_{i} \epsilon^B . \quad (18) \]
Notice the factor \( \frac{3}{4} \) as different from [43]. Hence, to recover for eq. (18) a relation like
\[ \delta \psi^A_i = P_2 \left[ \epsilon^{AA'}_{i}, \psi^A \right] \epsilon^B \bar{\psi}^{A'} \psi^{A'}, \quad (19) \]
and its Hermitian conjugate (i.e., in order for ansatz (3) to be unchanged) the following conditions ought to be used.

First, we require \( \xi^{A'B'} = \xi^{AB} = 0 \) (see ref. [43]). In addition, we require that
\[ \left[ 2 \left( \frac{\dot{a}}{a} + \frac{i}{a} \right) - i \frac{2}{N} \left( \psi_F \psi^F_0 + \bar{\psi}_{F'0} \bar{\psi}^{F'} \right) \right] n_{BA'} \epsilon^B \sim P_2 \left[ \epsilon^{AA'}_{i}, \psi^A \right] \bar{\psi}^{A'}. \quad (20) \]
This means that the variation \( \delta \psi^A_i = D(t) \psi^A_i \) with \( D(t) = \left[ 2 \left( \frac{\dot{a}}{a} + \frac{i}{a} \right) - i \frac{2}{N} \left( \psi_F \psi^F_0 + \bar{\psi}_{F'0} \bar{\psi}^{F'} \right) \right] \) will have a component along unity (“body” of Grassman algebra) and another which is nilpotent.
(the “soul”: $\psi_F \psi_{F,0} + \bar{\psi}_{F,0} \bar{\psi}^{F'}$). Finally, the additional constraint $\psi^B \bar{\psi}^{B'} e_{BB'} = 0$ follows. This amounts to $\psi^B \bar{\psi}^{B'} \propto n^{B'B'}$, and so we can write

$$J_{AB} = \psi(A) \bar{\psi}^{B'} n_{B'B'} = 0.$$  \hfill (21)

The constraint $J_{AB} = 0$ has a natural interpretation as the reduced form of the Lorentz rotation constraint arising in the full theory. By requiring that the constraint $J_{AB} = 0$ be preserved under the same combination of transformations as used above, one finds equations which are satisfied provided the supersymmetry constraints $S_A = 0$, $\bar{S}_{A'} = 0$ hold. By further requiring that the supersymmetry constraints be preserved, one finds additionally that the Hamiltonian constraint $\mathcal{H} = 0$ should hold. The invariance of $\psi_i^A$ under transformations (21), (22), (23) leads to the Hermitian conjugates of the above expressions.

In the next section we will discuss in detail our ansätze (see ref. [21, 22]) for the physical variables of a $k = 1$ FRW model within N=1 supergravity with a vector supermultiplet. Namely, our objective is to access how these ansätze can be accommodated with both the FRW geometry and supersymmetry. In doing so we aim to provide relevant information that is absent in ref. [21, 22] and thus to improve our understanding of supersymmetric minisuperspaces and their current problems.

### 2.2 FRW model with gauge fields from N=1 supergravity with supermatter

Within the more general theory of N=1 supergravity with gauged supermatter, our field variables (together with scalar fields $\phi^I$, $\phi^{I*}$ and fermionic partners $\chi^i_A$, $\bar{\chi}^{i*}_{A'}$) are transformed as follows [34]. Under supersymmetry transformations - $\delta(\xi)$ - we have

$$\delta(\xi) e^{AA'} = -i \left( e^A \bar{\chi}^{A'}, \bar{\chi}^{A'} e^A \right),$$  \hfill (22)

$$\delta(\xi) \chi^i_A = 2 \hat{D}_i e^A - i \frac{1}{2} C^{\mu \nu} e_{BB'} \chi^i_{A'} C^{C'} \bar{\chi}^{A'} C^{C'},$$

$$\delta(\xi) \bar{\chi}^{i*}_{A'} = \frac{i}{2} \left( g_{\mu \nu} e^{AB} + C_{\mu \nu} e_{BB'} \right) e_{B'B'} \bar{\chi}^i_A C^{C'} \chi^i_A C^{C'},$$

$$\delta(\xi) \lambda^{(a)} = \frac{1}{4} \left( \frac{\partial K}{\partial \bar{\phi}^{J'}} \delta(\xi) \phi^J - \frac{\partial K}{\partial \phi^{J*}} \delta(\xi) \bar{\phi}^{J*} \right) \chi^{(a)}_A - i D^{(a)} e^A + \tilde{\bar{F}}^{(a)} e_{B'B'} \bar{\chi}^{A'} C^{C'} \chi^i_A C^{C'},$$

$$\delta(\xi) \phi^I = \sqrt{2} \epsilon^{AA'} \bar{\chi}^{A'} \chi^i_A,$$  \hfill (23)

$$\delta(\xi) \chi^i_A = i \sqrt{2} \epsilon^{AA'} \bar{\chi}^{A'} \left( \hat{D}_i e^A - \frac{i}{2} \sqrt{2} \bar{\psi}_{\mu C} \chi^i_{A'} C^{C'} \right) - \Gamma^{J}_{JK} \delta(\xi) \phi^J \chi^i_A,$$  \hfill (24)

$$\delta(\xi) \chi^i_A = i \sqrt{2} e_{AB} \bar{\chi}^{A'} \chi^i_A + \frac{1}{4} \left( \frac{\partial K}{\partial \bar{\phi}^{J'}} \delta(\xi) \phi^J - \frac{\partial K}{\partial \phi^{J*}} \delta(\xi) \bar{\phi}^{J*} \right) \chi^i_A - \sqrt{2} e_{A'B'} \bar{\chi}^{A'} \chi^i_A + \frac{1}{4} \left( \frac{\partial K}{\partial \bar{\phi}^{J'}} \delta(\xi) \phi^J - \frac{\partial K}{\partial \phi^{J*}} \delta(\xi) \bar{\phi}^{J*} \right) \chi^i_A - \sqrt{2} e_{A'B'} \bar{\chi}^{A'} \chi^i_A,$$  \hfill (25)

and their Hermitian conjugates. Here $\epsilon^{AB}$ is the alternating spinor, $\epsilon^A$ and $\bar{\epsilon}^A$ are odd (anticommuting) fields, $C^{\mu \nu} e_{BB'} = \frac{1}{4} \left( e^A e_{AA'} \bar{\chi}^{A'} + e^A e_{BB'} \chi^i_A e^A e_{BB'} \right)$, $\tilde{F}^{(a)} = F^{(a)} - i \left( \psi_{\mu A A'} \bar{\chi}^{A'} - \bar{\psi}_{\mu A A'} \chi^i_A \right)$. $\Gamma^{J}_{JK}$ is a Christoffel symbol derived from Kähler metric $g_{IJ} = \frac{\partial^2 K}{\partial \varphi^I \partial \varphi^J}$, $K$ is the Kähler potential, $P$ is a complex scalar-field dependent potential energy term, $D^{(a)}$ is a Killing potential.
and \( X^{I(a)} \) is a Killing vector on the Kähler manifold. In addition, \( D_I = \frac{\partial}{\partial \phi^I} + \frac{\partial K}{\partial \phi^I}, \hat{D}_\mu = \partial_\mu + \omega_\mu + \frac{1}{4} \left( \frac{\partial K}{\partial \phi^J} \hat{D}_\mu \phi^J - \frac{\partial K}{\partial \phi^K} \hat{D}_\mu \phi^K \right) + i \frac{2}{3} A^{(a)}_\mu Im F^{(a)}, \hat{D}_\mu = \partial_\mu - A^{(a)}_\mu X^{I(a)}, F^{(a)} = X^{I(a)} (\frac{\partial K}{\partial \phi^K} + i D^{(a)}) \) and \( I \) denotes Kähler indices.

For Lorentz transformations - \( \delta (L) \) - it follows that
\[
\delta (L)e^{\lambda \mu} = N^A_B e^{BA}_\mu + \tilde{N}^{A'}_{B'} e^{AB'}_\mu , \tag{28}
\]
\[
\delta (L)\psi^A_\mu = N^A_B \psi^B_\mu , \tag{29}
\]
\[
\delta (L)\phi^I = 0, \quad \delta (L)\chi^I_A = N_{AB} \chi^{IB} , \tag{30}
\]
\[
\delta (L)A^{(a)}_\mu = N^{a}_{\nu} A^{(a)}_\nu , \tag{31}
\]
\[
\chi^I_A = \xi^I_0 \phi^I_A , \tag{32}
\]
\[
\chi^I_A = \xi^I_0 \phi^I_A , \tag{33}
\]
\[
\delta (L)\chi^I_A = \xi^I_0 \partial \phi^I_A , \tag{34}
\]
\[
\delta (L)\chi^I_A = \xi^I_0 \partial \phi^I_A , \tag{35}
\]
\[
\delta (L)\phi^I = \xi^I_0 \partial \phi^I_A , \tag{36}
\]
considering the Hermitian conjugates as well.

Finally, for gauge transformations - \( \delta (g) \) - we get
\[
\delta (g)\psi^A_\mu = -\frac{i}{2} \zeta^{(a)} Im F^{(a)} \psi^A_\mu , \tag{37}
\]
\[
\delta (g)\chi^I_A = \zeta^{(a)} \frac{\partial X^{I(a)}}{\partial \phi^I} \chi^I_A + \frac{i}{2} \zeta^{(a)} Im F^{(a)} \chi^I_A , \tag{38}
\]
\[
\delta (g)A^{(a)}_\mu = \partial_\mu \zeta^{(a)} + k^{abc} \zeta^{(b)} A^{(c)}_\mu , \tag{39}
\]
\[
\delta (g)\lambda^{(a)}_A = k^{abc} \zeta^{(b)} \lambda^{(c)}_A - \frac{i}{2} \zeta^{(b)} Im F^{(b)} \lambda^{(b)}_A , \tag{40}
\]
and Hermitian conjugates.

Let us then consider the case\footnote{For the case of a gauge group SO(3) \( \sim SU(2) \) we stress that invariance under homogeneity and isotropy as well as gauge transformations requires that all components of \( \phi \) are zero \cite{3}. Only for SO(N), N>3 we may have \( \phi = (0,0,0, \phi_1, \ldots, \phi_{N-3}) \).} where \( \phi^I = 0 \), \( \chi^I_A = \tilde{\chi}^I_{A'} = 0 \) with a gauge group \( \hat{G} = SU(2) \).

An important consequence of not having scalar fields and their fermionic partners is that the Killing potentials \( D^{(a)} \), and related quantities are now absent. In fact, if we had complex scalar fields, a Kähler manifold could be considered with metric \( g_{IJ} \) on the space of \( (\phi^I, \phi^J) \). Analytic isometries that preserve the analytic structure of the manifold are associated with Killing vectors through \cite{34}
\[
X^{(b)} = X^{I(b)}(\phi^I) \frac{\partial}{\partial \phi^I} , \tag{41}
\]
\[
X^{*^{(b)}} = X^{I^{*}(b)}(\phi^{' I}) \frac{\partial}{\partial \phi^{' I}} .
\]
Killing’s equation integrability condition is equivalent to the statement that there exist scalar functions $D^{(a)}$ such that

$$ g_{I,J^*} X^{J^*(a)} = i \frac{\partial}{\partial \phi^I} D^{(a)}, $$
$$ g_{I,J^*} X^I(a) = -i \frac{\partial}{\partial \Phi^{J^*}} D^{(a)}. $$

\(42\)

The Killing potentials $D^{(a)}$ are defined up to constants $c^{(a)}$. They further satisfy

$$ \left[ X^I(a) \frac{\partial}{\partial \phi^I} + X^{I^*} \frac{\partial}{\partial \phi^{I^*}} \right] D^{(b)} = -f^{abc} D^{(c)}. $$

\(43\)

This fixes the constants $c^{(a)}$ for non-Abelian gauge groups. For $\hat{G} = SU(2)$ we take (see ref. [24]) $K = \ln(1 + \phi \bar{\phi})$ and the functions

$$ D^{(1)} = \frac{1}{2} \left( \frac{\phi + \bar{\phi}}{1 + \phi \bar{\phi}} \right), \quad D^{(2)} = -\frac{i}{2} \left( \frac{\phi - \bar{\phi}}{1 + \phi \bar{\phi}} \right), \quad D^{(3)} = -\frac{1}{2} \left( \frac{1 - \phi \bar{\phi}}{1 + \phi \bar{\phi}} \right), $$

\(44\)

follow. For $\phi = \bar{\phi} = 0$ this implies $D^{(1)} = D^{(2)} = 0$ and $D^{(3)} = -\frac{1}{2}$. However, being the $D^{(a)}$ fixed up to constants which are now arbitrarily, we can choose $D^{(3)} = 0$ consistently. Basically, $\phi = \bar{\phi} = 0$ also implies the inexistence of Killing vectors. Without complex scalar fields and fermionic partners it is meaningless to discuss Kähler manifolds, their analytical isometries and integrability conditions associated with $D^{(a)}$.

As we will explain in the following, we will use again the ansätze [1], [2] for the tetrad and gravitinos. Concerning the matter fields, the simplest choice would be to take $A^{(a)}_\mu$, $\lambda^{(a)}_A$ and Hermitian conjugates as time-dependent only. However, this is not sufficient in ordinary quantum cosmology with Yang-Mills fields. Special ansätze are required\(^2\) for $A^{(a)}_\mu$ (and $\phi$ as well) [35]-[39], which depend on the gauge group considered.

We will consider here the ansatz described in [33]-[39] for the vector field $A^{(a)}_\mu$ and also employed in [24]. This ansatz is the simplest one that allows vector fields to be present in a FRW geometry. In fact, only non-Abelian spin-1 fields can exist consistently within a FRW background [33]-[39]. More specifically, since the physical observables are to be $SO(4)$-invariant, the fields with gauge degrees of freedom may transform under $SO(4)$ if these transformations can be compensated by a gauge transformation. This is so since the physical observables are gauge invariant quantities. Fortunately there is a large class of fields satisfying the above conditions. These are the so-called $SO(4)$-symmetric fields, i.e., fields which are invariant up to a gauge transformation. Assuming a gauge group $\hat{G} = SO(3) \sim SU(2)$, the $SO(4)$-symmetric spin-1 field is taken to be

$$ A^{(a)}_\mu(t) \omega^\mu = \left( \frac{f(t)}{4} \varepsilon^{(a)(b)} T^{(a)(b)} \right) \omega^i. $$

\(45\)

Here $\{\omega^\mu\}$ represents the moving coframe $\{\omega^\mu\} = \{dt, \omega^i\}$, $\omega^i = \hat{E}^i_\epsilon dx^\epsilon$ ($i, \epsilon = 1, 2, 3$) of one-forms, invariant under the left action of $SU(2)$ and $T^{(a)(b)}$ are the generators of the $SU(2)$
gauge group, with \( \tau_{(a)} = -\frac{1}{2} \epsilon_{(a)(b)(c)} T_{(b)(c)} \) being the usual \( SU(2) \) matrices. The idea behind the ansatz (13) is to define a homomorphism of the isotropy group \( SO(3) \) to the gauge group. This homomorphism defines the gauge transformation which, for the symmetric fields, compensates the action of a given \( SO(3) \) rotation. Hence, the above form for the gauge field, where the \( A_0 \) component is taken to be identically zero. None of the gauge symmetries will survive: all the available gauge transformations are required to cancel out the action of a given \( SO(3) \) rotation. Thus, we will not have in our FRW case a gauge constraint\(^3\) \( Q^{(a)} = 0 \).

In addition, the ansatz (13) implies \( A_\mu^{(a)} \) to be parametrized by a single scalar function \( f(t) \). FRW cosmologies with this ansatz are totally equivalent to a FRW minisuperspace with an effective conformally coupled scalar field, but with a quartic potential instead of a quadratic one. Such minisuperspace sector simplifies considerably any analysis of the Hamiltonian constraints [35, 40] and this constituted another compelling argument to use ansatz (13).

We could now take as fermionic partner for \( A_\mu^{(a)} \) a simple spin-\( \frac{1}{2} \) field, like \( \chi_A \). This seems reasonable and in similarity with the case where only scalar fields are present [8, 9]. However, we will see that this choice would lead to some difficulties. Hence we will use the more general choice

$$\lambda_A^{(a)} = \lambda_A^{(a)}(t).$$  \tag{46}

Let us now address the consequences of the ansätze (1), (2), (15), (16) as far as the transformations (23), (24), (25) are concerned.

The tetrad (1) is not affected by any gauge transformation. Moreover, the Lorentz, local coordinate and supersymmetry transformations of the tetrad are precisely the same as in the case of pure N=1 supergravity. Hence, we see no reason to change the ansatz (1) and the comments present in subsection 2.1 will hold. Namely, that the ansatz (1) is preserved under a combination of Lorentz, local coordinate, (obviously gauge) and supersymmetry transformations. We should also stress that only the Rarita-Schwinger field is present in the transformations (22), (28), (32). Thus, to get a result similar to eq. (19) we have to employ the ansatz (2) without any change.

Concerning then the ansatz (2), we may neglect the transformation (37). In fact, from \( \tilde{\phi} = \tilde{\phi} = 0 \) it follows that \( F^{(a)} \) can be subsequently set equal to zero. The transformations (23) and (33) will contribute with \( 2D_\mu \epsilon_A^A \) to an expression equal to eq. (20). Let us address the remaining of transformation (23), namely the term

$$i \frac{1}{2} \left( g_{\mu\nu} \epsilon^{AB} + C_{\mu \nu} A^{AB} \right) \epsilon_B \lambda^{(a)c} \epsilon_{CC'} \bar{\lambda}^{(a)c'}. $$

This simplifies to

$$- \frac{3i}{8} \epsilon^A \lambda^{(a)c} \epsilon_{CC'} \bar{\lambda}^{(a)c'} - \frac{3i}{4} \epsilon_{C} \lambda^{(a)c} \epsilon_{A} \bar{\lambda}^{(a)c'}. $$ \tag{47}

In order to achieve an expression similar to (19) further conditions ought to be imposed. Equating the first term in (17) to zero gives essentially the contribution of the spin-\( \frac{1}{2} \) fields to the Lorentz constraint. This is an interesting way of analysing the Lorentz constraints when supermatter is present and has not been stressed previously in the current literature. We further need to consider the term \( \epsilon_{C} \lambda^{(a)c} \epsilon_{C} \bar{\lambda}^{(a)c'} \) as representing a field variable with indices \( A \) and \( i \), for each value of \( (a) \). Notice that to preserve the ansatz (2) in subsection 2.1 (see also ref. [13]) we had to require \( n_{AB} \epsilon^B \sim \bar{\psi}_B' \) which is not quite \( \tilde{\psi} \). Here we have to deal with the \( \lambda, \bar{\lambda} \)-fields and a similar step is necessary.

\(^3\)However, in the case of larger gauge group some of the gauge symmetries will survive. These will give rise, in the one-dimensional model, to local internal symmetries with a reduced gauge group. Therefore, a gauge constraint can be expected to play an important role in such a case and a study of such a model would be interesting.
Moving now to the homogeneous spin-1 field, the components of the ansatz (45) in the basis \( (E^i_c dx^i, \tau(a)) \) can be expressed as
\[
A^{(a)}_i = \frac{f}{2} \delta^{(a)}_i.
\] (48)

The local coordinate and Lorentz transformations will correspond to isometries and local rotations and these have been compensated by gauge transformations (cf. ref. [35]-[39] for more details). Hence, we need to impose the following condition

\[
\delta^{(a)} A^{(a)}_i = i a E^b_i \sigma_{bA'} \left( \epsilon^A A'(a) - \lambda^A(a) \epsilon A' \right).
\] (49)

Hence, we need to impose the following condition\(^4\)

\[
\begin{cases}
\sigma_{bA'} \left( \epsilon^A A'(a) - \lambda^A(a) \epsilon A' \right) = E(t) , & \text{if } (a) = b + 1 \\
\sigma_{bA'} \left( \epsilon^A A'(a) - \lambda^A(a) \epsilon A' \right) = 0 , & \text{if } (a) = b - 1 \\
\sigma_{bA'} \left( \epsilon^A A'(a) - \lambda^A(a) \epsilon A' \right) = 0 , & \text{if } (a) \neq b
\end{cases}
\] (50)

where \( E(t) \) is spatially independent and possibly complex, in order to obtain \( \delta A^{(a)} = P_3 \left[ e_{AA'}, \epsilon^A \right] \). From eq. (50) it follows that the preservation of the ansatz (45) will require \( \delta A^{(a)} \) to include a nilpotent (“soul”) component. This consequence is similar to (15), (16), (20) for the tetrad and gravitinos, and in accordance with the method introduced in ref. [9].

As far as the \( \lambda \)-fields are concerned, we obtain the following result using the ansätze (1), (2), (45), (46) and transformations (25), (31), (35), (40):

\[
\delta \lambda^{(a)}_A = - \frac{1}{2} F^{(a)}_{0i} \epsilon_A A^i n^B E_B + \frac{i}{2} \frac{F_{ij}}{E_B} \epsilon^{ijkl} h^2 n_{AA'} \epsilon_{A'B} \epsilon_B - \frac{i}{4} \tilde{\psi}_0^A \lambda^{(a)} A' n^B_{A'} \epsilon_B - \frac{i}{8} \tilde{\psi}_0^C n_{CC'} \lambda^{(a)} A' \epsilon_A + \kappa_{abc} \epsilon^b \lambda^{(a)}_A
\] (51)

where

\[
F^{(a)}_{0i} = \frac{f}{2} \delta^{(a)}_i ,
\]
\[
F^{(a)}_{ij} = \frac{1}{4} (2 f - f^2) \epsilon_{ij}(a) .
\] (52)

\(^4\)If we had chosen \( \lambda^{(a)}_A = \lambda_A \) for any value of \( (a) \) then we would not be able to obtain a consistent relation similar to (50). Namely, such that \( \delta A^{(1)}_1 \sim A^{(1)}_1 \) and \( \delta A^{(2)}_1 \sim A^{(2)}_1 = 0. \)
\[ \epsilon_{ijk} e_{AA'}^i e_{BB'}^j e_{CC'}^k = - i \hbar \frac{1}{2} \epsilon_{AB} n_{EA'} \epsilon_{D'B'} \left( - \epsilon_{C'}^F \epsilon_{A'}^D + n_{CC'} n_{DE'} \right) + i \hbar \frac{1}{2} \epsilon_{A'B'} n_{BB'} \epsilon_{DA'} \left( \epsilon_{C'}^D \epsilon_{C'}^E + n_{CC'} n_{DE'} \right). \] (53)

Using the bi-spinorial relation (13) and eq. (50), the last sixth terms in eq. (51) may be put in a more suitable form in order to get \( \delta \lambda^A_{(a)} = P_1 \left[ e_{AA'}, \psi_{(a)}, A_{(a)}, \lambda_{(a)} \right] \lambda_{(a)}^A \). This would require that the remaining terms (including the first to the 12th term in (51)) to satisfy a further condition equated to zero.

The above results concerning the transformations (22)-(40) imply that the ansätze for the physical variables are consistent with a FRW geometry and Lorentz, gauge and supersymmetry transformations. However, some restrictions (see eq. (16), (19)-(21), (47), (50), (51)) had to be imposed. It is reasonable to consider that such limitations would affect the possible spectrum of quantum solutions. In fact, it does and the solutions present in section 3 and ref. [21, 22] constitute just the expected consequence of these restrictions.

The purpose of section 3 is precisely to show both the usefulness and limitations of the ansätze for \( e^a_{\mu}, \psi^A_{\mu}, A^{(a)}, \lambda^{(a)} \) in deriving interesting results. We will reproduce previous results present in the literature regarding non-supersymmetric quantum cosmologies with Yang-Mills fields [35]. Being N=1 supergravity a square-root of gravity, our results are thus particularly supportive. Hence, our model could be regarded as valued approach in persuing a locally supersymmetric FRW with the most general gauge supermatter [21, 22]. Nevertheless, the solutions obtained here correspond to just a subset of the ones in ref. [35]. It seems then that our ansätze are not general enough in spite of their simplicity and adequacy. Hence, the spectrum of solutions is incomplete or drastically limited (\( \Psi = 0 \) in ref. [21, 22]).

### 3 Quantum constraints and solutions

Let us then solve explicitly the corresponding quantum Lorentz and supersymmetry constraints\(^5\). First we need to redefine the fermionic fields, \( \psi^A_{\mu} \) and \( \lambda^{(a)} \) in order to simplify the Dirac brackets [2, 9] following the steps described in [24].

For the \( \psi^A_\mu \)-field we introduce,

\[ \hat{\psi}^A = \frac{\sqrt{3}}{2i} \sigma^{a\frac{3}{2}} \psi^A, \quad \hat{\bar{\psi}}^A = \frac{\sqrt{3}}{2i} \sigma^{a\frac{3}{2}} \bar{\psi}^A, \] (54)

where the conjugate momenta are

\[ \pi_{\psi^A} = in_{AA'} \psi^{A'}, \quad \pi_{\bar{\psi}^A} = in_{AA'} \bar{\psi}^A. \] (55)

The Dirac brackets then become

\[ [\hat{\psi}^A, \hat{\psi}^{A'}]_D = in_{AA'}. \] (56)

Similarly for the \( \lambda^{(a)} \) field

---

\(^5\) The use of these constraints signifies that supersymmetry and Lorentz invariance is a feature of the reduced FRW model but subject to the (restrictive) conditions (16), (19)-21, (47), (50), (51).
\[
\hat{\lambda}^{(a)} = \frac{\sigma a^2}{2\pi} \lambda^{(a)}_A, \quad \hat{\lambda}'^{(a)}_A = \frac{\sigma a^2}{2\pi} \lambda^{(a)}_A',
\]
giving
\[
\pi_{\hat{\lambda}}^{(a)} = -in_{AA'} \hat{\lambda}'^{(a)}_A, \quad \pi_{\hat{\lambda}'}^{(a)} = -in_{AA'} \hat{\lambda}^{(a)}_A,
\]
with
\[
[\hat{\lambda}^{(a)}_A, \hat{\lambda}'^{(a)}_{A'}]_D = -i\delta^{ab} n_{AA'}.
\]
Furthermore,
\[
[a, \pi_a]_D = 1, \quad [f, \pi_f]_D = 1,
\]
and the rest of the brackets are zero.

It is simpler to describe the theory using only (say) unprimed spinors, and, to this end, we define
\[
\bar{\psi}_A = 2n^B_A \bar{\psi}^{B'}, \quad \bar{\lambda}^{(a)}_A = 2n^B_A \bar{\lambda}^{(a)}_{B'},
\]
with which the new Dirac brackets are
\[
[\bar{\psi}_A, \bar{\psi}_B]_D = i\epsilon_{AB}, \quad [\lambda^{(a)}_A, \bar{\lambda}^{(a)}_A]_D = -i\delta^{ab} \epsilon_{AB}.
\]
The rest of the brackets remain unchanged. Quantum mechanically, one replaces the Dirac brackets by anti-commutators if both arguments are odd (O) or commutators if otherwise (E):

\[
[E_1, E_2] = i[E_1, E_2]_D, \quad [O, E] = i(O, E)_D, \quad [O_1, O_2] = i(O_1, O_2)_D.
\]

Here, we take units with \(\hbar = 1\). The only non-zero (anti-)commutator relations are:

\[
\{\lambda^{(a)}_A, \lambda^{(b)}_B\} = \delta^{ab} \epsilon_{AB}, \quad \{\psi_A, \bar{\psi}_B\} = -\epsilon_{AB}, \quad [a, \pi_a] = [f, \pi_f] = i.
\]

We chose \((\bar{\lambda}^{(a)}_A, \psi_A, a, f)\) to be the coordinates of the configuration space, and \((\lambda^{(a)}_A, \bar{\psi}_A, \pi_a, \pi_f)\) to be the momentum operators in this representation. Hence

\[
\lambda_a \rightarrow -\frac{\partial}{\partial \lambda^{(a)}_A}, \quad \bar{\psi}_A \rightarrow \frac{\partial}{\partial \psi^B}, \quad \pi_a \rightarrow \frac{\partial}{\partial a}, \quad \pi_f \rightarrow -i \frac{\partial}{\partial f}.
\]

Following the ordering used in ref. [4], we put all the fermionic derivatives in \(S_A\) on the right. In \(\hat{S}_A\), all the fermionic derivatives are on the left. Implementing all these redefinitions, the supersymmetry constraints have the differential operator form

\[
S_A = -\frac{1}{2\sqrt{6}} a \psi_A \frac{\partial}{\partial a} - \sqrt{\frac{3}{2}} \sigma^2 a^2 \psi_A \\
- \frac{1}{8\sqrt{6}} \psi_B \psi^B \frac{\partial}{\partial \psi^B} - \frac{1}{4\sqrt{6}} \psi C \bar{\lambda}^{(a)}_C \frac{\partial}{\partial \lambda^{(a)}_A} \\
+ \frac{1}{3\sqrt{6}} \sigma^a_{AB'} \sigma^{bCC'} n^D_A n_C^{B'} \bar{\lambda}^{(a)}_D \psi_C \frac{\partial}{\partial \lambda^{(b)}_B} \\
+ \frac{1}{6\sqrt{6}} \sigma^a_{AB'} \sigma^{bAA'} n^D_A n_C^{B'} \bar{\lambda}^{(a)}_D \lambda^{(b)}_B \frac{\partial}{\partial \psi^E} \\
- \frac{1}{2\sqrt{6}} \psi_A \bar{\lambda}^{(a)}_C \frac{\partial}{\partial \lambda^{(a)}_C} + \frac{3}{8\sqrt{6}} \bar{\lambda}^{(a)}_A \lambda^{(a)}_C \frac{\partial}{\partial \psi^C} \\
+ \sigma^a_{AA'} n^{BA'} \bar{\lambda}^{(a)}_B \left(-\sqrt{\frac{2}{3}} \frac{\partial}{\partial f} + \frac{1}{8\sqrt{2}} (1 - (f - 1)^2) \sigma^2 \right)
\]
and

$$\bar{S}_A = \frac{1}{2\sqrt{6}} \frac{\partial}{\partial a} \frac{\partial}{\partial \psi^A} - \sqrt{\frac{3}{2}} \sigma^2 a^2 \frac{\partial}{\partial \psi^A}$$

$$- \frac{1}{8\sqrt{6}} \varepsilon_{BC} \frac{\partial}{\partial \psi^B} \frac{\partial}{\partial \psi^C} \psi_A + \frac{1}{4\sqrt{6}} \varepsilon_{BC} \frac{\partial}{\partial \psi^B} \frac{\partial}{\partial \lambda^{(a)}} \bar{A}^{(a)}$$

$$+ \frac{1}{3\sqrt{6}} \sigma^{AB} \sigma^{BC'} n_A \sigma_{C'} \frac{\partial}{\partial \psi^D} \frac{\partial}{\partial \lambda^{(a)B}} \bar{A}^{(b)}$$

$$+ \frac{1}{6\sqrt{6}} \sigma^{AB} \sigma^{BC'} n_A \sigma_{C'} \frac{\partial}{\partial \lambda^{(a)B}} \frac{\partial}{\partial \lambda^{(b)c}} \psi^D$$

$$+ \frac{1}{2\sqrt{6}} \frac{\partial}{\partial \psi^A} \frac{\partial}{\partial \lambda^{(a)B}} \bar{A}^{(a)B} + \frac{3}{8\sqrt{6}} \frac{\partial}{\partial \lambda^{(a)B}} \frac{\partial}{\partial \lambda^{(b)}A} \psi^B$$

$$+ n_A \sigma^{AB} \sigma^{BC} \left( \frac{2\sqrt{2}}{3} \frac{\partial}{\partial f} + \frac{1}{4\sqrt{2}} (1 - (f - 1)^2) \sigma^2 \right) \frac{\partial}{\partial \lambda^{(a)B}}. \quad (67)$$

When matter fields are taken into account the generalisation of the $J_{AB}$ constraint is:

$$J_{AB} = \psi (A \bar{A}^{B'} n_B n_B') - \lambda^{(a)B} \lambda^{(a)B'} n_B n_B' = 0. \quad (68)$$

The Lorentz constraint $J_{AB}$ implies that a physical wave function should be a Lorentz scalar. We can easily see that the most general form of the wave function

$$\Psi = A + B\psi^C \psi_C + d_a \lambda^{(a)C} \psi_C + c_{abc} \bar{A}^{(a)C} \lambda^{(b)} C \psi^D$$

$$+ c_{abc} \bar{A}^{(a)C} \lambda^{(b)} C \lambda^{(c)} D \psi^D + c_{abcd} \lambda^{(a)C} \lambda^{(b)} C \lambda^{(c)} D \lambda^{(d)} E \psi^E + d_{abcd} \bar{A}^{(a)C} \lambda^{(b)} C \lambda^{(c)} D \lambda^{(d)} E \psi^E$$

$$+ \mu_1 \lambda^{(2)C} \lambda^{(3)D} \lambda^{(1)E} \psi^E \psi^E$$

$$+ \mu_2 \lambda^{(1)C} \lambda^{(1)D} \lambda^{(2)E} \psi^E + \mu_3 \lambda^{(1)C} \lambda^{(1)D} \lambda^{(2)E} \lambda^{(3)E} \psi^E$$

$$+ F \lambda^{(1)C} \lambda^{(2)D} \lambda^{(3)E} \lambda^{(3)E} \lambda^{(3)E} \psi^F \psi^F. \quad (69)$$

where $A, B, ..., G$ are functions of $a, f$ only. This Ansatz contains all allowed combinations of the fermionic fields and is the most general Lorentz invariant function we can write down.

The next step is to solve the supersymmetry constraints $S_A \Psi = 0$ and $\bar{S}_A \Psi = 0$. Since the wave function \(\bar{A}\) is of even order in fermionic variables and stops at order $8$, the equations $S_A \Psi = 0$ and $\bar{S}_A \Psi = 0$ will be of odd order in fermionic variables and stop at order 7. Hence we will get ten equations from $S_A \Psi = 0$ and another ten equations from $\bar{S}_A \Psi = 0$. From $S_A \Psi = 0$ we obtain

$$- \frac{a}{2\sqrt{6}} \frac{\partial A}{\partial a} - \sqrt{\frac{3}{2}} \sigma^2 a^2 A = 0, \quad (70)$$

$$- \frac{\sqrt{2}}{3} \frac{\partial A}{\partial f} + \frac{1}{8\sqrt{2}} \left[ 1 - (f - 1)^2 \right] \sigma^2 A = 0. \quad (71)$$

These equations correspond, respectively, to terms linear in $\psi_A, \bar{A}^{(a)}$. Eq. \((70)\) and \((71)\) give the dependence of $A$ on $a$ and $f$, respectively. Solving these equations leads to $A = \bar{A}(a) \tilde{A}(a)$ as

$$A = \tilde{A}(f) e^{-3\sigma^2 a^2}, \quad (72)$$
\[ A = \hat{A}(a)e^{\frac{ia^2}{2}}(\frac{a^2}{4}+f^2), \]  
(73)

A similar relation exists for the \( \bar{S}_A \Psi = 0 \) equations, which from the \( \psi^A \lambda^{(1)}_E \lambda^{(1)}_E \lambda^{(2)}_E \lambda^{(2)}_E \lambda^{(3)}_E \lambda^{(3)}_E \) term in \( \Psi \) give for \( G = \hat{G}(a)\tilde{G}(f) \)

\[ G = \hat{G}(f)e^{3a^2a^2}, \]  
(74)

\[ G = \hat{G}(a)e^{\frac{3a^2}{4}(\frac{a^2}{4}-f^2)}. \]  
(75)

We notice that in our case study, differently to the case of ref. [4, 9, 20]-[26], we are indeed allowed to completely determine the dependence of \( A \) and \( G \) with respect to \( a \) and \( f \).

The solution (74), (75) is included in the Hartle-Hawking (no-boundary) solutions of ref. [35]. In fact, we basically recover solution (3.8a) in ref. [35] if we replace \( f \rightarrow f + 1 \). As it can be checked, this procedure constitutes the rightful choice according to the definitions employed in [38] for \( A^{(a)}_\mu \). Solution (74), (75) is also associated with an anti-self-dual solution of the Euclidianized equations of motion (cf. ref. [35, 37]). However, it is relevant to emphasize that not all the solutions present in [35] can be recovered here. In particular, the Gaussian wave function (74), (75), peaked around \( f = 1 \) (after implementing the above transformation), represents only one of the components of the wave function in ref. [33]. The wave function in ref. [35] is peaked around the two minima of the corresponding quartic potential. In our model, the potential terms correspond to a “square-root” of the potential present in [35].

Solution (72), (73) has the features of a (Hawking-Page) wormhole solution for Yang-Mills fields [33, 37], which nevertheless has not yet been found in ordinary quantum cosmology. However, in spite of (72), (73) being regular for \( a \rightarrow 0 \) and damped for \( a \rightarrow \infty \), it may not be well behaved when \( f \rightarrow -\infty \).

The equations obtained from the cubic and 5-order fermionic terms in \( S_A \Psi = 0 \) and \( \bar{S}_A \Psi = 0 \) can be dealt with by multiplying them by \( n_{EE'} \) and using the relation \( n_{EE'}n^{EA'} = \frac{1}{2}\epsilon_{EE'}^{AA'} \). Notice that the \( \sigma_a \) matrices are linear independent and are orthogonal to the \( n \) matrix. We would see that such equations provide the \( a, f \)-dependence of the remaining terms in \( \Psi \). It is important to point out that the dependence of the coefficients in \( \Psi \) corresponding to cubic fermionic terms on \( a \) and \( \phi, \bar{\phi} \) is mixed throughout several equations [4, 9]. However, in the present FRW minisuperspace with vector fields, the analogous dependence in \( a, f \) occurs in separate equations. The equations for cubic and 5-order fermionic terms further imply that any possible solutions are neither the Hartle-Hawking or a wormhole state. In fact, we would get \( \tilde{d}_{(a)} \sim a^5\hat{d}_{(a)}(a)\tilde{d}_{(a)}(f) \) and similar expressions for the other coefficients in \( \Psi \), with a prefactor \( a^n, n \neq 0 \). This behaviour has also been found in [4]. Hence, from their \( a \)-dependence equations these cannot be either a Hartle-Hawking or wormhole state. They correspond to other type of solutions which could be obtained from the corresponding Wheeler-DeWitt equation but with completely different boundary conditions.

Finally, it is worthy to notice that the Dirac bracket of the supersymmetry constraints (66), (67) induces an expression whose bosonic sector corresponds to the (decoupled) gravitational and vector field components of the Hamiltonian constraint in ref. [35]. This fact supports a relation between the ansätze (45), (46) and solutions (74), (75), within the context of \( N=1 \) supergravity being a square-root of gravity [4].

4 Discussions and Conclusions

Summarizing our work, we considered the canonical formulation of the more general theory of \( N = 1 \) supergravity with supermatter [28, 34] subject to a \( k = +1 \) FRW geometry. Ansätze
for the gravitational and gravitino fields, the gauge vector field $A^a_{\mu}$ and corresponding fermionic partners were then introduced. We set the scalar fields and their supersymmetric partners equal to zero. Our purpose was to initiate a discussion on the main result of ref. [21, 22]: the only allowed solution was $\Psi = 0$.

Concerning the ansätze employed here (and also in ref. [21, 22]) for the field variables, the following points are in order. It was clear from transformations (22)-(40) that the ansatz for the $k = +1$ FRW tetrad would have to be identical, either in subsection 2.1 (pure N=1 supergravity) or subsection 2.2 (N=1 supergravity with gauged supermatter). A consistent ansatz was also required for the gravitino fields in order for the tetrad ansatz (1) to be preserved under supersymmetry, Lorentz and local coordinate transformations. From the expression of $\delta e_{AA'}^i$ in the presence of supermatter it was straightforward to conclude that the ansatz for $\psi^A_i, \bar{\psi}^{A'}_{i}$ ought to be the same as in the pure N=1 case (see ref. [9]). In addition, both $\delta e_{AA'}^i, \delta \psi^A_i, \delta \bar{\psi}^{A'}_{i}$ (either in the pure or supermatter case) will include nilpotent ("soul") components [43] (see subsections 2.1, 2.2 and ref. [9]).

In addressing the variation of $\delta \psi^A_i$ using eq. (2) we had to combine supersymmetry, Lorentz and local coordinate transformations. Spin-$\frac{1}{2}$ fields $\left(\lambda^{(a)}_A, \bar{\lambda}^{(a)}_{A'}\right)$ were now present in $\delta(s)\psi^A_i$. For $\delta \psi^A_i \sim \psi^A_i$ to hold, further conditions were imposed on the gravitino and $\left(\lambda^{(a)}_A, \bar{\lambda}^{(a)}_{A'}\right)$ fields. One consequence was to clarify the contribution of the $\left(\lambda^{(a)}_A, \bar{\lambda}^{(a)}_{A'}\right)$ fields to the Lorentz constraints.

With respect to the vector field $A^a_{\mu}$, we chose a specific ansatz to simplify our calculations and already used in ordinary quantum cosmology with some success [35]-[39]. Ansatz (45) fulfilled such conditions. In fact, it induces FRW minisuperspaces with an effective conformally coupled scalar field with a quartic potential. Eq. (45) corresponds to a symmetric vector field, i.e., such that the action of isometries and local rotations are compensated by a gauge transformation. Furthermore, it has the simplest possible form for non-trivial non-Abelian vector fields and thus be accommodated in a FRW geometry. Our results showed that supersymmetry invariance could be achieved if condition (50) was imposed. This further implied that $\delta A^a_{\mu}$ involved a “soul” component, similarly to what occurred for the tetrad and gravitinos (see ref. [9]).

When addressing $\delta \lambda^{(a)}_A$, we were able to obtain $\delta \lambda^{(a)}_A \sim \lambda^{(a)}_A$ if further (restrictive) conditions were imposed. Overall, accommodating supersymmetry with homogeneity and isotropy requires specific conditions to be imposed on the physical fields. Expressions (16)-(21) are just some of them. When a vector supermultiplet is included, further and more severe restrictions are required. The inclusion of more (and different) fields and further restrictions will affect drastically the possible solutions. In particular, few or no non-trivial solutions would be derived in spite of the ansätze being physically adequate but not general enough.

The purpose of section 3 was precisely to show that interesting but limited physical features could be derived from our ansätze. A wave function constructed in the way mentioned in the previous section accommodates naturally the expectation that the early universe — earlier than an inflationary stage — might be dominated by radiation and associated fermionic fields. Our results constitute an approach towards such a supersymmetric scenario.

After a dimensional reduction, we derived the constraints for our one-dimensional model and solved the Lorentz and supersymmetry constraints. We then obtained non-trivial solutions. We found expressions that can be interpreted as corresponding to a wormhole (Hawking-Page) and (Hartle-Hawking) no-boundary solutions, respectively.

These results were quite supportive. Namely, the Hartle-Hawking solution found here corresponded to a component of the set of solutions obtained from a Wheeler-DeWitt equation in non-supersymmetric quantum cosmology [35]. That is consistent with our expectations, since
N=1 supergravity is a square root of gravity. Moreover, the Dirac bracket of the supersymmetry constraints (66), (67) induces an expression whose bosonic sector is the \((\text{decoupled})\) gravitational and vector field components of the Hamiltonian constraint in ref. \[35\].

As far as the problem of the null result in ref. \[21, 22\] is concerned, we hope our results may provide a new perspective on this issue. In the least, we know from the present paper that physical states in FRW models with gauged fields obtained from N=1 supergravity with supermatter indeed exist. Physical states also exit when solely scalar multiplets are concerned \[29\]. Thus, we could expect to merge both situations and hopefully obtain non-trivial states. It should be noticed however that so far \(\text{no}\) analytical solution has been found in non-supersymmetric FRW quantum cosmologies with vector and scalar fields.

We could speculate that \(\Psi = 0\) in ref. \[21, 22\] would possibly be a consequence of the ansätze for the physical variables. More precisely, that in spite of their simplicity the employed ansätze were not the \(\text{more general}\) ones. Hence, only a very particular solution \((\Psi = 0)\) was possible. Do notice as well that the solutions \((74), (75)\) in section 3 constitute only \(\text{part}\) of the set present in \[35\]. Hence, it would be tempting to relate the incompleteness of the solutions \((74), (75)\) (with respect to ref. \[35\]) to the ansätze \((15), (16)\) being also \(\text{incomplete}\). A more general ansätze (where supersymmetry, gauge, Lorentz and local coordinate transformations are accommodated \(\text{less restrictively — see eq. (16), (19)–(21), (47), (50), (51)}\)) could allow for a larger spectrum of solutions.

However, we must stress that the absence of \(\phi, \bar{\phi}\) and their fermionic partners played an important role as enhancing \(D^{(a)}\) to be set to zero. The presence of the Killing potentials in the supersymmetry constraints of \[21, 22\] constitute a relevant element in deriving \(\Psi = 0\). We hope to address all these issues in a future investigation.

Supersymmetric quantum cosmology certainly constitutes an active and challenging subject for further research (see ref. \[29\] for a review). Some problems that remain to be confronted have been raised throughout this paper. Other interesting issues which remain open and we are also aiming to address are the following:

a) Obtain conserved currents from \(\Psi\), as consequence of the Dirac-like structure of the supersymmetry constraints \[17\];

b) Test the validity of minisuperspace approximation in supersymmetric quantum cosmology;

c) Perform the canonical quantization of minisuperspaces and black-holes in N=2,3 supergravities.

ACKNOWLEDGEMENTS

The author is grateful to A.D.Y. Cheng and S.W. Hawking for pleasant conversations and for sharing their points of view and to O. Bertolami for useful comments and suggestions. Early motivation from discussions with O. Obregon and R. Graham are also acknowledged. This work was supported by JNICT/PRAXIS XXI Fellowship BPD/6095/95.

References

[1] C. Teitelboim, Phys. Rev. Lett. \textbf{38}, 1106 (1977).
[2] M. Pilati, Nucl. Phys. B 132, 138 (1978).
[3] P.D. D’Eath, Phys. Rev. D 29, 2199 (1984).
[4] L.J. Alty, P.D. D’Eath and H.F. Dowker, Phys. Rev. D 46, 4402 (1992).
[5] P.D. D’Eath, S.W. Hawking and O. Obregón, Phys. Lett. 300B, 44 (1993).
[6] P.D. D’Eath, Phys. Rev. D 48, 713 (1993).
[7] R. Graham and H. Luckock, Phys. Rev. D 49, R4981 (1994).
[8] P.D. D’Eath and D.I. Hughes, Phys. Lett. 214B, 498 (1988).
[9] P.D. D’Eath and D.I. Hughes, Nucl. Phys. B 378, 381 (1992).
[10] M. Asano, M. Tanimoto and N. Yoshino, Phys. Lett. 314B, 303 (1993).
[11] H. Luckock and C. Oliwa, Phys. Rev. D51 (1995) 5883.
[12] R. Graham and A. Csordás, Nontrivial fermion states in supersymmetric minisuperspace, in: Proceedings of the First Mexican School in Gravitation and Mathematical Physics, Guanajuato, Mexico, December 12-16, 1994 (gr-qc/9503054);
[13] R. Graham and A. Csordás, Phys. Rev. Lett. 74 (1995) 4926.
[14] P.D. D’Eath, Phys. Lett. B320, 20 (1994).
[15] A.D.Y. Cheng, P.D. D’Eath and P.R.L.V. Moniz, Phys. Rev. D49 (1994) 5246.
[16] A.D.Y. Cheng, P.D. D’Eath and P.R.L.V. Moniz, Gravitation and Cosmology 1 (1995) 12
[17] R. Graham and A. Csordás, Phys. Rev. D52 (1995) 5653
[18] A.D.Y. Cheng, P.D. D’Eath and P.R.L.V. Moniz, DAMTP-Report February R94/13,
[19] A.D.Y. Cheng, P.D. D’Eath and P.R.L.V. Moniz, Gravitation and Cosmology 1 (1995) 1
[20] A.D.Y. Cheng and P.R.L.V. Moniz, Int. J. Mod. Phys. D4 (1995) 189
[21] A.D.Y. Cheng, P.D. D’Eath and P.R.L.V. Moniz, Quantization of a Friedmann-Robertson-Walker model in N=1 Supergravity with Gauged Supermatter, in: Proceedings of the 1st Mexican School in Gravitation, Guanajuato, Mexico December 12-16 1994, gr-qc/9503009
[22] A.D.Y. Cheng, P.D. D’Eath and P.R.L.V. Moniz, Class. Quantum Grav. 12 (1995) 1343-1353
[23] P. Moniz, The Case of the Missing Wormhole State, in: Proceedings of the VI Moskow International Quantum Gravity Seminar, Moskow, Russia, 12-19 June 1995, to be published by World Scientific, DAMTP report R95/19, gr-qc/9506012.
[24] P. Moniz, Gen. Rel. Grav. 28 (1996) 97
[25] P. Moniz, Back to Basics? or How can supersymmetry be used in a simple quantum cosmological model, in: Proceedings of the 1st Mexican School in Gravitation, Guanajuato, Mexico December 12-16 1994, DAMTP report R95/20, gr-qc/9505002
[26] P. Moniz, *Quantization of the Bianchi type-IX model in N=1 Supergravity in the presence of Supermatter*, DAMTP Report R95/21, gr-qc/9505048, to be published in International Journal of Modern Physics A Vol. 11 No.6 (1996)

[27] A.D.Y. Cheng and P. Moniz, *Quantum Bianchi Models in N=2 Supergravity with Global O(2) Internal Symmetry* in: Proceedings of the VI Moskow International Quantum Gravity Seminar, Moskow, Russia, 12-19 June 1995, to be published by World Scientific, DAMTP report;

[28] A.D.Y. Cheng and P. Moniz, *Canonical Quantization of Bianchi Class A Models in N=2 Supergravity*, – accepted for publication in Modern Phys. Lett. A (1996).

[29] P.V. Moniz, *Supersymmetric Quantum Cosmology — Shaken not Stirred*, Int. J. Mod. Physics A (invited review), DAMTP report; gr-qc/9604025

[30] G. Esposito, *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries*, Springer Verlag (Berlin, 1993) and references therein.

[31] S.W. Hawking, Phys. Rev. D37 904 (1988).

[32] J.B. Hartle and S.W. Hawking, Phys. Rev. D 28, 2960 (1983).

[33] S.W. Hawking and D.N. Page, Phys. Rev. D 42, 2655 (1990).

[34] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd. ed. (Princeton University Press, 1992).

[35] O. Bertolami and J.M. Mourão, Class. Quantum Grav. 8 (1991) 1271;

[36] O. Bertolami and P.V. Moniz, Nuc. Phys. B439 (1995) 259

[37] O. Bertolami, J. Mourão, R. Picken and I. Volobujev, Int. J. Mod. Phys. A6 (1991) 4149.

[38] P.V. Moniz and J. Mourão, Class. Quantum Grav. 8, (1991) 1815;

[39] J.M. Mourão, P.V. Moniz and P.M. Sá, Class. Quantum Grav. 10 (1993) 517;

G. Gibbons and A. Steif, Phys. Lett. B320 245 (1994);

M.C. Bento, O. Bertolami, J.M. Mourão, P.V. Moniz and P.M. Sá, Class. Quantum Grav. 10 (1993) 285.

[40] O. Bertolami, Preprint Lisbon IFM-14/90, talk presented at the XIII International Colloquium on Group Theoretical Methods in Physics, Moscow, USSR June 1990, (Springer Verlag).

[41] O. Bertolami, J.M. Mourão, R.F. Picken and I.P. Volobujev, unpublished; S. Shabanov, talk presented at the First Iberian Meeting on Gravity, Évora, Portugal September 1992, edited by M.C. Bento, O. Bertolami, J.M. Mourão and R.F. Picken (World Scientific Press, 1993); see also N. Manton, Ann. Phys. 167 (1986) 328; N. Manton, Nuc. Phys. B193 (1981) 502.

[42] D.Z. Freedman and J. Schwarz, Phys. Rev. D15 (1977) 1007;

S. Ferrara, F. Gliozzi, J. Scherk and P.v. Nieuwenhuizen, Nuc. Phys. 117 (1976) 333.
[43] See e.g., P.C. Aichelburg and R. Güven, Phys. Rev. Lett. 51 (1983) 1613;
P. G.O. Freund, Introduction to Supersymmetry, (Cambridge U.P. – 1986);
B.S DeWitt, Supermanifolds, (Cambridge U.P.– 1984);
M. Henneaux and C. Teitelboim, Quantization of Gauge Systems, (Princeton U.P. – 1992).

[44] C. Isham and J. Nelson, Phys. Rev. D10 (1974) 3226.

[45] T. Christodoulakis and J. Zanelli, Phys. Lett. 102A (1984) 227;
T. Christodoulakis and J. Zanelli, Phys. Rev. D29 (1984) 2738;
T. Christodoulakis and C. Papadopoulos, Phys. Rev. D38 (1988) 1063

[46] P. D’Eath and J.J. Halliwell, Phys. Rev. D 35 (1987) 1100.

[47] P. Moniz, Rerum Universitas Sententia ex Susy, essay-DAMTP R96/13;
Conserved currents in supersymmetric quantum cosmology?, DAMTP R96/14.