A translational flavor symmetry in the mass terms of
Dirac and Majorana fermions

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Requiring the effective mass term for a category of fundamental Dirac or Majorana
fermions of the same electric charge to be invariant under the translational trans-
formations \(\psi_{\alpha L(R)} \rightarrow \psi_{\alpha L(R)} + n_\alpha z_{\psi L(R)}\) in the flavor space, where \(n_\alpha\) and \(z_{\psi L(R)}\)
stand respectively for the flavor-dependent complex numbers and a constant spinor
field anticommuting with the fermion fields, we show that \(n_\alpha\) can be identified as
the elements \(U_{\alpha i}\) in the \(i\)-th column of the unitary matrix \(U\) used to diagonalize the
 corresponding Hermitian or symmetric fermion mass matrix \(M_\psi\), and \(m_i = 0\) holds
accordingly. We find that the reverse is also true. Now that the mass spectra of
charged leptons, up- and down-type quarks are all strongly hierarchical and current
experimental data allow the lightest neutrino to be massless, we argue that the zero
mass limit for the first-family fermions and the translational flavor symmetry behind
it should be a natural starting point for building viable fermion mass models.

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I. INTRODUCTION

The standard model (SM) of particle physics has proved to be a huge success, but its flavor sector remains unsatisfactory in the sense that all the flavor parameters are theoretically undetermined. Going beyond the SM, one has to introduce some more free parameters in connection with the massive neutrinos. One way out of this situation is to determine or constrain the flavor structures of leptons and quarks with the help of certain proper family symmetries, such that some testable predictions for fermion masses and flavor mixing parameters (or their correlations) can be achieved. So far a lot of efforts have been made along this line of thought toward building phenomenologically viable models [1, 2], but the true flavor dynamics is still unclear. At the present stage a logical candidate for the underlying flavor symmetry should at least help interpret some salient features of the observed fermion mass spectra and flavor mixing patterns.

In 2006 Friedberg and Lee put forward a novel idea to constrain the flavor texture of three Dirac neutrinos [3]. In the basis of a diagonal charged-lepton mass matrix, they required the Dirac neutrino mass term $L_\nu$ to keep unchanged under the transformation $\nu_\alpha \rightarrow \nu_\alpha + z_\nu$ (for $\alpha = e, \mu, \tau$), where $z_\nu$ is a constant spinor field anticommutating with the neutrino fields $\nu_\alpha$ [4]. The constraint of this translational symmetry of $L_\nu$ on the neutrino mass matrix $M_\nu$ is so strong that $\det (M_\nu) = 0$ holds, implying that one of the neutrino masses $m_i$ (for $i = 1, 2, 3$) must vanish (i.e., one of the neutrinos is a Goldstone-like fermion [4, 5]). A very instructive neutrino mixing pattern $U = U_{TBM}O_{13}$ in correspondence to $m_2 = 0$ can then be obtained [3, 6–9], where $U_{TBM}$ stands for the well-known “tribimaximal” flavor mixing pattern [10–12] and $O_{13}$ is a unitary rotation matrix in the complex $(1, 3)$ plane. In this scenario, however, a proper symmetry breaking term has to be introduced to assure $m_2 > m_1$ as indicated by current neutrino oscillation data [13].

In fact, as early as in 1979, ’t Hooft had imposed a quite similar translational displacement $\phi(x) \rightarrow \phi(x) + \Lambda$ on the Lagrangian $L_\phi$ for a renormalizable scalar field theory, where $\Lambda$ is a constant scalar field commutating with $\phi$ [14]. It turns out that both the mass and the self-coupling parameter of $\phi$ have to vanish in order to guarantee the invariance of $L_\phi$, which can be referred to as a Goldstone-type symmetry [15, 16], under the above transformation. So one generally expects that the translational symmetry of an effective Lagrangian may provide a simple and natural way to understand why the mass of a fermion or boson in this
physical system is vanishing or vanishingly small \(^1\), although its deep meaning remains a puzzle at present.

In this paper we point out that the translational transformation \(\nu_i \to \nu_i + z_\nu\) for a massless neutrino field \(\nu_i\) (i.e., \(m_i = 0\) for either \(i = 1\) or \(i = 3\)) is equivalent to the translational transformation \(\nu_\alpha \to \nu_\alpha + U_{ai} z_\nu\) in the flavor space, where \(U_{ai}\) (for \(\alpha = e, \mu, \tau\)) denote the corresponding neutrino flavor mixing matrix elements and \(z_\nu\) represents a constant spinor field anticommuting with the neutrino fields. Then we are going to show that the effective mass terms of Dirac or Majorana fermions may all have a kind of translational symmetry under the discrete shifts \(\psi_{\alpha L(R)} \to \psi_{\alpha L(R)} + n_\alpha z_\psi L(R)\) for a constant spinor field \(z_\psi L(R)\) in the flavor space, if and only if \(m_i = 0\) holds and \(n_\alpha = U_{ai}\) are the elements in the \(i\)-column of the unitary matrix \(U\) used to diagonalize the corresponding Hermitian or symmetric fermion mass matrix \(M_\psi\). We find that the reverse is also true. Given the very facts that all the fundamental fermions of the same nonzero electric charge have a rather strong mass hierarchy and current neutrino oscillation data allow the lightest neutrino to be (almost) massless, the zero mass limit for the first-family fermions and the translational symmetry behind it can serve as a very natural starting point for building viable fermion mass models toward understanding the observed patterns of both the Cabibbo-Kobayashi-Maskawa (CKM) quark flavor mixing matrix \([17, 18]\) and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton flavor mixing matrix \([19–21]\).

II. TRANSLATION OF A MASSLESS NEUTRINO FIELD

It is well known that the massless photon travels at the speed of light in free space and its electromagnetic field obeys the equation of motion \(\Box A = 0\). This equation is invariant under the translational transformation \(A \to A + A_0\), where \(A_0\) denotes a constant vector field commuting with \(A\). Similarly, a massless neutrino also travels at the speed of light in free space and its field \(\nu_i\) (for \(i = 1\) or \(3\)) satisfies the Dirac equation \(i\gamma^\mu \partial_\mu \nu_i = 0\), which is invariant under the translational transformation

\[
\nu_i \to \nu_i + z_\nu ,
\]

\(^1\) If there were no flavor mixing in the lepton or quark sector, the flavor and mass eigenstates of a fermion would be identical with each other. In this case the kinetic energy and mass terms of a fermion field, which does not involve any self-interactions, would be closely analogous to those of a scalar field.
where \( z_\nu \) is a constant spinor field anticommuting with the neutrino fields. This observation means that such a translation of the massless neutrino field is consistent with Einstein’s principle of constancy of light velocity for all the inertial reference systems in vacuum.

Although current neutrino oscillation data do allow the existence of a massless neutrino species, the other two neutrino species must be massive. In this case the massless neutrino field and its two massive counterparts can form three quantum superposition states — the neutrino flavor eigenstates, which directly participate in the standard weak interactions. This kind of mismatch between the flavor and mass eigenstates of three neutrinos is described by a \( 3 \times 3 \) unitary matrix \( U \) as follows:

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}_L =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}_L.
\]

(2)

As a consequence, the translational transformation made for the massless neutrino field \( \nu_i \) with \( m_i = 0 \) in Eq. (1) requires that the three neutrino flavor states \( \nu_\alpha \) (for \( \alpha = e, \mu, \tau \)) transform in the following way:

\[
\nu_\alpha_L \rightarrow \nu_\alpha_L + U_{\alpha i} z_\nu.
\]

(3)

So the aforementioned Friedberg-Lee transformation \([3]\) is equivalent to taking \( U_{e2} = U_{\mu2} = U_{\tau2} = 1/\sqrt{3} \) in correspondence to \( m_2 = 0 \).

Now that the translational transformation described by Eq. (3) in the flavor space is equivalent to the translation of a massless neutrino field described by Eq. (1), one should be able to show that \( m_i = 0 \) must hold if the effective Majorana (or Dirac) neutrino mass term keeps invariant under the transformation in Eq. (3). In the subsequent sections we are going to demonstrate that this is really the case.

III. MAJORANA NEUTRINOS

First of all, let us consider the effective mass term of the three known neutrinos (i.e., \( \nu_e \), \( \nu_\mu \) and \( \nu_\tau \)) by assuming that they have the Majorana nature \(^2\)

\[
-\mathcal{L}_M = \frac{1}{2} \sum_\alpha \sum_\beta [\bar{\nu}_L \langle m \rangle_{\alpha \beta} (\nu_L)^c] + \text{h.c.}.
\]

(4)

\(^2\) Of course, an effective Majorana mass term for the right-handed neutrino fields in the canonical seesaw mechanism \([22-26]\) can be discussed in an analogous way.
where \( \alpha \) and \( \beta \) run over the flavor indices \( e, \mu \) and \( \tau \), “\( c \)” denotes the charge conjugation, and \( \langle m \rangle_{\alpha\beta} = \langle m \rangle_{\beta\alpha} \) are the elements of the 3 \( \times \) 3 Majorana neutrino mass matrix \( M_\nu \). Since \( M_\nu \) can be diagonalized via the unitary transformation \( U^\dagger M_\nu U^* = \text{diag}\{m_1, m_2, m_3\} \), we have the expressions

\[
\langle m \rangle_{\alpha\beta} = \sum_i (m_i U_{\alpha i} U_{\beta i}) ,
\]
for \( i = 1, 2 \) and 3. The unitarity of \( U \) allows us to prove

\[
\sum_\alpha [U^*_{\alpha j} \langle m \rangle_{\alpha\beta}] = m_j U_{\beta j} ,
\]
\[
\sum_\beta [\langle m \rangle_{\alpha\beta} U^*_{\beta j}] = m_j U_{\alpha j} ,
\]
\[
\sum_\alpha \sum_\beta [U^*_{\alpha j} \langle m \rangle_{\alpha\beta} U^*_{\beta j}] = m_j ,
\]
which are essentially equivalent to one another and all proportional to the neutrino mass \( m_j \) (for \( j = 1, 2 \) or 3). Following Eq. (3), now we make the same translational transformation for the left-handed neutrino fields in the flavor space \(^3\)

\[
\nu_{\alpha L} \to \nu_{\alpha L} + \nu_{\alpha L} z_\nu ,
\]
Then \( L_M \) becomes

\[
-L_M' = -L_M + \frac{1}{2} m_j \left[ \overline{\nu_\nu} z_\nu^c + \sum_\alpha [U_{\alpha j} \overline{\nu_{\alpha L}}] z_\nu^c + \overline{\nu_\nu} \sum_\beta \left( U_{\beta j} (\nu_{\beta L}^c) \right) \right] .
\]
It becomes clear that \( L'_M = L_M \) will hold under the above transformation if and only if \( m_j = 0 \) holds. Namely, one of the three neutrinos must be massless if the effective Majorana neutrino mass term \( L_M \) keeps invariant under the discrete shifts of \( \nu_{\alpha L} \) made in Eq. (7), which helps provide a novel link between the two sides of one coin (i.e., the mass and flavor mixing issues of the Majorana neutrinos). The point is that the three flavor-dependent coefficients \( U_{\alpha j} \) of \( z_\nu \) constitute the \( j \)-th column of the unitary matrix \( U \) used to diagonalize \( M_\nu \), which corresponds to \( m_j = 0 \). In comparison, most of the popular global discrete flavor

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\(^3\) Different from Ref. [3] and some other references, where only the Dirac neutrinos and the simplest flavor-independent transformation \( \nu_\alpha \to \nu_\alpha + z \) are taken into account, here we have considered a more generic and flavor-dependent transformation for the Majorana neutrino fields. This nontrivial treatment will allow us to establish a direct and thus more transparent connection between the vanishing neutrino mass and the corresponding neutrino mixing matrix elements.
symmetries can help predict very specific neutrino mixing patterns but leave the neutrino mass spectrum unconstrained \[27–29\].

Given the fact of \(m_2 > m_1\), one is left with either \(m_1 = 0\) (normal ordering) or \(m_3 = 0\) (inverted ordering) for the neutrino mass spectrum. In either case the other two neutrino masses can be determined by inputting the experimental values of two independent neutrino mass-squared differences, and only a single nontrivial Majorana CP phase of \(U\) survives \[30\].

In the basis of a diagonal charged-lepton mass matrix, the unitary matrix \(U\) appearing in Eqs. (5)—(8) is just the PMNS matrix \(U_{\text{PMNS}}\) which link the neutrino mass eigenstates \(\nu_i\) (for \(i = 1, 2, 3\)) to the neutrino flavor eigenstates \(\nu_\alpha\) (for \(\alpha = e, \mu, \tau\)). Focusing on the possibility of \(m_1 = 0\), one may adopt the original Kobayashi-Maskawa (KM) parametrization \[18\] for \(U_{\text{PMNS}}\) so as to make the expressions of \(U_{\nu_1}\) in the first column of \(U_{\text{PMNS}}\) as simple as possible. Explicitly,

\[
U_{\text{PMNS}} = \begin{pmatrix}
    c_1 & s_1 c_3 & s_1 \hat{s}_3^* \\
    -s_1 c_2 & c_1 c_2 c_3 + s_2 \hat{s}_3 & c_1 c_2 \hat{s}_3^* - s_2 c_3 \\
    -s_1 s_2 & c_1 s_2 c_3 - c_2 \hat{s}_3 & c_1 s_2 \hat{s}_3^* + c_2 c_3
\end{pmatrix} P_\nu
\]

(9)

with the definitions \(c_i \equiv \cos \theta_i\), \(s_i \equiv \sin \theta_i\), \(\hat{s}_3 \equiv s_3 e^{i \phi}\) and \(P_\nu \equiv \text{diag}\{1, e^{i \sigma}, 1\}\), where \(\phi\) and \(\sigma\) denote the Dirac and Majorana CP-violating phases, respectively. Provided \(U_{e1} = 2/\sqrt{6}\) and \(U_{\mu_1} = U_{\tau_1} = -1/\sqrt{6}\) are assumed (i.e., \(\theta_1 = \arcsin(1/\sqrt{3}) \simeq 35.3^\circ\) and \(\theta_2 = 45^\circ\) are taken \[31\]), for example, one will obtain

\[
U_{\text{PMNS}} = \frac{1}{\sqrt{6}} \begin{pmatrix}
    2 & \sqrt{2} & 0 \\
    -1 & \sqrt{2} & -\sqrt{3} \\
    -1 & \sqrt{2} & \sqrt{3}
\end{pmatrix}
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & c_3 & \hat{s}_3^* \\
    0 & -\hat{s}_3 & c_3
\end{pmatrix} P_\nu ;
\]

(10)

namely, \(U_{\text{PMNS}} = U_{\text{TBM} O_{23}} P_\nu\) with \(O_{23}\) being a unitary rotation matrix in the complex \((2, 3)\) plane. This simple flavor mixing pattern, which was first proposed in 2006 \[32, 33\], remains favored in today’s neutrino phenomenology. A combination of Eqs. (5) and (10) allows us to reconstruct the Majorana neutrino mass matrix in the chosen basis:

\[
M_\nu = \begin{pmatrix}
    a & a & a \\
    a & a & a \\
    a & a & a
\end{pmatrix} + \begin{pmatrix}
    0 & c & -c \\
    c & b + 2c & -b \\
    -c & -b & b - 2c
\end{pmatrix},
\]

(11)

in which \(a \equiv (m_2 c_3^2 + m_3 \hat{s}_3^2) / 3\), \(b \equiv (m_2 \hat{s}_3^2 + m_3 c_3^2) / 2\) and \(c \equiv (m_2 \hat{s}_3^2 - m_3 \hat{s}_3^2) c_3 / \sqrt{6}\) with \(\overline{m}_2 \equiv m_2 e^{2i \sigma}\) are defined. The simple structure of \(M_\nu\) depends on the simple choice of \(U_{\nu_1}\).
and is suggestive of certain simple flavor symmetries which can be used for the explicit model building [33]. Note that in this example the PMNS matrix $U_{\text{PMNS}}$ only possesses a partial $\mu$-$\tau$ permutation symmetry characterized by $U_{\mu 1} = U_{\tau 1}$ instead of $U_{\mu i} = U_{\tau i}$ (for $i = 1, 2, 3$) [34], and the possibility of a flavor democracy for the first column of $U_{\text{PMNS}}$ (i.e., $U_{e1} = U_{\mu 1} = U_{\tau 1}$) has been discarded from the beginning since it is definitely incompatible with current neutrino oscillation data [13].

At this point it is worth emphasizing that the translational symmetry of $\mathcal{L}_M$ is useful in the following two aspects: on the one hand, it forces the smallest neutrino mass to be zero; on the other hand, it helps to partly constrain the texture of $M_\nu$ and the pattern of $U_{\text{PMNS}}$. Nevertheless, such constraints cannot be explicitly achieved unless the coefficients of $z_\nu$ in the translational transformation of $\nu_\alpha L$ in Eq. (7) are specified, as we have already seen from the example taken in Eqs. (10) and (11). Note that the model building exercises based on most of the discrete flavor symmetries actually face a similar problem [1, 2]: one usually has to follow a somewhat contrived way to assign the irreducible representations of a flavor symmetry group to the relevant fermion and scalar fields in a concrete model, and the primary guideline in this connection is fully empirical or phenomenological — just to fit the available experimental data as well as possible. The same is true of the charged fermion sector, unfortunately, as one will see later on. Before a convincing breakthrough is made in flavor dynamics, we find it useful to explore all the possibilities from the bottom up.

Provided $M_\psi$ is neither diagonal nor Hermitian, one can diagonalize $M_\psi M_\psi^\dagger$ with the help of a unitary transformation $U_\psi^\dagger M_\psi M_\psi^\dagger U_\psi = \text{diag}\{m_{e^2}, m_{\mu^2}, m_{\tau^2}\}$. In this case the PMNS lepton flavor mixing matrix is expressed as $U_{\text{PMNS}} = U_\psi^\dagger U_\nu$, where $U_\nu$ is equivalent to the unitary matrix $U$ used to diagonalize $M_\nu$ in the above discussions. Of course, the mass term of the charged leptons or Dirac neutrinos may also possess a possible translational flavor symmetry of this kind, so may the mass term of the up- or down-type quarks.

**IV. DIRAC FERMIONS**

We proceed to discuss the Dirac fermion mass terms in an opposite way. If the massive neutrinos are of the Dirac nature, it is possible to treat them on the same footing as the charged leptons and quarks. Without loss of any generality, a Dirac fermion mass matrix $M_\psi$ can always be taken to be Hermitian after a proper choice of the flavor basis in the
SM or its extensions which have no flavor-changing right-handed currents [35]. In this case one may diagonalize $M_\psi$ via the unitary transformation $V^\dagger M_\psi V = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_i$ being the eigenvalues of $M_\psi$ (i.e., $m_i = |\lambda_i|$ are physical masses of the charged or neutral fermions under consideration, either for leptons or for quarks). The effective mass term for a given category of the fundamental fermions with the same electric charge can be written in the chosen basis as follows:

$$-\mathcal{L}_D = \sum_\alpha \sum_\beta \overline{\psi}_{\alpha L} \langle m \rangle_{\alpha \beta} \psi_{\beta R} + \text{h.c.},$$

in which the subscripts $\alpha$ and $\beta$ run over the flavor indices of the Dirac neutrinos, charged leptons, up-type quarks or down-type quarks, and

$$\langle m \rangle_{\alpha \beta} = \langle m \rangle^*_{\beta \alpha} = \sum_i (\lambda_i V_{\alpha i} V^*_{\beta i})$$

holds thanks to the Hermiticity of $M_\psi$. Now let us require that $\mathcal{L}_D$ keep unchanged under a translational transformation of the left- and right-handed fermion fields in the flavor space,

$$\psi_{\alpha L(R)} \rightarrow \psi_{\alpha L(R)} + n_\alpha z_{\psi L(R)},$$

where $n_\alpha$ and $z_{\psi L(R)}$ are the flavor-dependent complex numbers and a constant spinor field anticommuting with the fermion fields 4, respectively. We find that $\mathcal{L}_D$ will be invariant with respect to the transformation made in Eq. (14) if and only if the conditions

$$\sum_\alpha [n^{\ast}_\alpha \langle m \rangle_{\alpha \beta}] = 0,$$

$$\sum_\beta [\langle m \rangle_{\alpha \beta} n_\beta] = 0,$$

$$\sum_\alpha \sum_\beta [n^{\ast}_\alpha \langle m \rangle_{\alpha \beta} n_\beta] = 0$$

are satisfied. Given these constraints, it is easy to show that the determinant of $M_\psi$ vanishes, indicating that one of the three fermion masses $m_i$ (for $i = 1, 2, 3$) must be exactly vanishing.

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4 If $z_{\psi L(R)}$ is spacetime-dependent, one should also take into account the relevant kinetic energy term. That would make things more complicated. But one example of this kind for the right-handed neutrinos, which are the gauge SU(2)$_L$ singlets, has been discussed in Ref. [36].
Substituting Eq. (13) into Eq. (15), we immediately obtain

\[
\sum_i \left[ \lambda_i \sum_{\alpha} (n^*_\alpha V_{\alpha i}) V^*_{\beta i} \right] = 0,
\]

\[
\sum_i \left[ \lambda_i V_{\alpha i} \sum_{\beta} (V^*_{\beta i} n_{\beta}) \right] = 0,
\]

\[
\sum_i \left[ \lambda_i \sum_{\alpha} (n^*_\alpha V_{\alpha i}) \sum_{\beta} (V^*_{\beta i} n_{\beta}) \right] = 0 .
\] (16)

If the sum of \( n^*_\alpha V_{\alpha i} \) over \( \alpha \) (or equivalently, the sum of \( V^*_{\beta i} n_{\beta} \) over \( \beta \)) in Eq. (16) were nonzero, one would be left with an interesting but phenomenologically-disfavored relationship \( V_{\beta i}/V_{\beta j} = constant \) for \( \beta \) taking different flavors in connection with \( m_k = 0 \), where \( i, j \) and \( k \) run cyclically over 1, 2 and 3. So this nontrivial solution has to be discarded.

It is therefore straightforward to obtain the other nontrivial solution to Eq. (16):

\[
\sum_{\alpha} (n^*_\alpha V_{\alpha i}) = \sum_{\beta} (V^*_{\beta i} n_{\beta}) = 0 ,
\] (17)

in correspondence to \( \lambda_i \neq 0 \). One can see that \( n_\alpha \propto V_{\alpha j} \) and \( n_\beta \propto V_{\beta j} \) with \( j \neq i \) satisfy Eq. (17). Imposing the normalization condition on \( n_\alpha \), we simply take \( n_\alpha = V_{\alpha j} \) and \( n_\beta = V_{\beta j} \) associated with the \( j \)-th mass eigenvalue \( \lambda_j \), and substitute them into Eq. (16). We are then left with the consistent results

\[
\lambda_j V^*_{\beta j} = 0 , \quad \lambda_j V^*_{\alpha j} = 0 , \quad \lambda_j = 0 .
\] (18)

In other words, the invariance of a Dirac fermion mass term \( \mathcal{L}_D \) under the translational transformation made in Eq. (14) implies that the three flavor-dependent complex numbers \( n_\alpha \) can be fully determined by the elements of \( V \) in its \( j \)-th column corresponding to \( \lambda_j = 0 \).

The reasoning made above for the Dirac fermions can also be extended to the Majorana neutrinos, or vice versa. While the possibility of \( m_1 = 0 \) (or \( m_3 = 0 \)) is still allowed by current experimental data, none of \( m_\epsilon = 0 \), \( m_u = 0 \) and \( m_d = 0 \) are true in nature. But the observed striking hierarchies \( m_\epsilon \ll m_\mu \ll m_\tau \), \( m_\mu \ll m_\tau \ll m_t \) and \( m_d \ll m_s \ll m_b \) [13] indicate that the zero mass limit for the first-family charged fermions is actually a reasonable starting point for model building, and the nonzero but small values of \( m_\epsilon, m_u \) and \( m_d \) can be naturally attributed to either the tree-level perturbations [37–39] or the loop-level corrections [40, 41].
Note that $m_u = 0$ used to be the most economical solution to the strong CP problem in quantum chromodynamics (QCD) [42, 43], but it has been discarded today. In any case $m_u = 0$ can be regarded as a straightforward consequence of the translational symmetry of $L_D$ for the up-type quarks as discussed above, and it is well in tune with the naturalness principle advocated by 't Hooft [14]. The finite values of $m_u, m_d, m_e$ and $m_1$ (or $m_3$) can therefore be generated from some slight breaking of such an unconventional flavor symmetry. To keep the flavor mixing pattern obtained in the zero mass limit unspoiled, the simplest phenomenological way to break the translational symmetry of $L_D$ (or $L_M$) is just to add a diagonal and flavor-universal mass term [3, 7, 8]. Namely, $L_D$ in Eq. (12) can now be written as follows:

$$\mathcal{L}_D = \sum_\alpha \sum_\beta \left[ \bar{\psi}_{\alpha L} (m)_{\alpha \beta} \psi_{\beta R} \right] + m_0 \sum_\alpha \left[ \bar{\psi}_{\alpha L} \psi_{\alpha R} \right] + \text{h.c.}, \quad (19)$$

where $m_0$ measures the explicit symmetry breaking effect and its magnitude is expected to be comparable with the mass of the electron, the up quark or the down quark. In this case the smallness of $m_0$ implies that $L_D$ may still have an approximate translational symmetry.

To illustrate, let us consider the up- and down-quark sectors in the $m_u = m_d = 0$ limits by simply taking

$$V_q = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & -\sqrt{3} \\ -1 & \sqrt{2} & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_q & s_q \\ 0 & -s_q & c_q \end{pmatrix}, \quad (20)$$

where $c_q \equiv \cos \vartheta_q$ and $s_q \equiv \sin \vartheta_q$ (for $q = u$ or $d$). In this case the CKM flavor mixing matrix turns out to be

$$V_{\text{CKM}} = V_u^T V_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta \\ 0 & -\sin \vartheta & \cos \vartheta \end{pmatrix}, \quad (21)$$

where $\vartheta = \vartheta_d - \vartheta_u$ is a nontrivial but small quark mixing angle. Note that the structural parallelism between $V_u$ and $V_d$ (or equivalently, between $M_u$ and $M_d$) assures that the resulting $V_{\text{CKM}}$ should not deviate far from the identity matrix [6]. Combining Eqs. (13) and 5.

5 A similar term of the form $\frac{1}{2} m_0 \sum_\alpha [\bar{\nu}_{\alpha L} (\nu_{\alpha L})^c]$ can be added to $L_M$ in Eq. (4), where $m_0$ characterizes a slight breaking of the translational symmetry of $L_M$ and its magnitude is comparable with the smallest neutrino mass $m_1$ (or $m_3$).

6 In comparison, the charged-lepton and neutrino mass matrices must be quite different in their textures so as to give rise to significant effects of lepton flavor mixing.
(21), we find that the reconstructed up-type quark mass matrix $M_u$ has the same form as $M_\nu$ in Eq. (11) with the replacements $a \equiv (m_c c_u^2 + m_t s_u^2)/3$, $b \equiv (m_c s_u^2 + m_t c_u^2)/2$ and $c \equiv (m_c - m_t) c_u s_u/\sqrt{6}$, so does the down-type quark mass matrix $M_d$ with the corresponding replacements. To generate the masses for the first-family quarks together with the other two flavor mixing angles and CP violation, one needs to properly break the translational symmetry in the up- and down-quark sectors.

If the translational symmetry of a Dirac or Majorana mass term is obtained at a superhigh energy scale, it may also be broken at the electroweak scale due to the quantum corrections described by the renormalization-group equations (RGEs) [44]. But it has been shown that a nonzero value of $m_1$ (or $m_3$) of $\mathcal{O}(10^{-13})$ eV cannot be generated from $m_1 = 0$ (or $m_3 = 0$) unless the two-loop RGE running effects are taken into account for the Majorana neutrinos [45–47], and such a tiny mass and the associated Majorana CP phase do not play any seeable role in neutrino physics. The masses of the first-family Dirac fermions are in general insensitive to the two-loop RGE-induced quantum corrections either [48–50].

V. SUMMARY

It has been a common belief in particle physics that behind different families of the fundamental fermions is some kind of flavor symmetry which can help understand the salient features of fermion mass spectra and flavor mixing patterns. Although the origin of the massive Majorana neutrinos may be quite different from that of the charged leptons and quarks, their flavor structures are likely to share the same symmetry. In this connection many flavor symmetries (either Abelian or non-Abelian, either continuous or discrete, either local or global) have been tried in the past decades, but new ideas are always called for in order to pin down the true flavor dynamics. That is why we highlight a possible translational flavor symmetry associated with the mass terms of fundamental Dirac or Majorana fermions in this work. In particular, we point out that the translational transformation $\nu_i \rightarrow \nu_i + z_\nu$ for a massless neutrino field $\nu_i$ is equivalent to the translational transformation $\nu_\alpha \rightarrow \nu_\alpha + U_{\alpha \nu} z_\nu$ in the flavor space with $U_{\alpha \nu}$ being the corresponding neutrino flavor mixing matrix elements and $z_\nu$ denoting a constant spinor field anticommuting with the neutrino fields.

If the effective mass term for a category of Dirac or Majorana fermions of the same electric charge keeps unchanged under the transformation $\psi_{\alpha L(R)} \rightarrow \psi_{\alpha L(R)} + n_\alpha z_\psi \psi_{L(R)}$ in
the flavor space, where $n_\alpha$ and $z_{\psi R}$ stand respectively for the flavor-dependent complex numbers and a constant spinor field anticommuting with the fermion fields, we have shown that $n_\alpha$ can be simply identified as the elements $U_{\alpha i}$ in the $i$-th column of the unitary matrix $U$ used to diagonalize the corresponding Hermitian or symmetric fermion mass matrix $M_\psi$, and $m_i = 0$ holds accordingly. The reverse is also true, and thus this translational flavor symmetry can constrain both the fermion mass spectrum and the flavor mixing pattern.

Given the very facts that all the fundamental fermions of the same nonzero electric charge have a strong mass hierarchy and current experimental data allow the lightest neutrino to be (almost) massless, we expect that the zero mass limit for the first-family fermions and the flavor symmetry behind it should be a natural starting point for model building and may even help shed light on the secret of flavor dynamics.

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