Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor *

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Introduction

In the present work we prove a structure theorem for operators of the 0-th term of the $\mathcal{V}^Y_-$-filtration relative to a free divisor $Y$ of a complex analytic variety $X$. As an application, we give a formula for the logarithmic de Rham complex in terms of $\mathcal{V}^Y_0$-modules, which generalizes the classical formula for the usual de Rham complex in terms of $\mathcal{D}X$-modules, and the formula of Esnault-Viehweg in the case that $Y$ is a normal crossing divisor. Using this, we give a sufficient condition for perversity of the logarithmic de Rham complex. Now we comment on the contents of each part of the paper:

In the first section, we recall the concepts of logarithmic derivation and logarithmic form, as well as free divisor, all of them due to Kyogi Saito \[14\], and the definition of the ring $\mathcal{V}^Y_0(D_X)$ of logarithmic differential operators along $Y$.

In the second part, we study the logarithmic operators in the case that $Y$ is free. We give a structure theorem in which we prove that the ring of logarithmic differential operators is the polynomial algebra generated by the logarithmic derivations over the sheaf $\mathcal{O}X$ of holomorphic functions. As a consequence, $\mathcal{V}^Y_0(D_X)$ is a coherent sheaf. Thanks to this theorem, we can prove the equivalence between $\mathcal{V}^Y_0(D_X)$-modules and $\mathcal{O}X$-modules with logarithmic connections. Therefore, an $\mathcal{V}^Y_0(D_X)$-module (or logarithmic $\mathcal{D}X$-module) $\mathcal{M}$ defines a logarithmic de Rham complex $\Omega^X_*(\log Y)(\mathcal{M})$.

In the third part, we prove that the logarithmic de Rham complex is canonically isomorphic to the complex $R\mathcal{H}om\mathcal{V}^Y_0(D_X)(\mathcal{O}_X, \mathcal{M})$. To show this, we first construct a resolution of $\mathcal{O}_X$ as $\mathcal{V}^Y_0(D_X)$-module, which we call the logarithmic Spencer complex and denote by $Sp^*(\log Y)$.

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Finally, we give a sufficient condition for perversity of the logarithmic de Rham complex, which is a perverse sheaf if the symbols of a minimal generating set of logarithmic derivations form a regular sequence in the graded ring associated to the filtration by the order on $D_X$. This condition always holds in dimension 2.

Some results of this paper have been announced in [4]. We give here the complete proofs of all of the results announced in that note and other new results.

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1 Notations and Preliminaries

Let $X$ be a complex analytic variety of dimension $n$, and $Y$ a hypersurface of $X$ defined by the ideal $I$. We will denote by $D_X$ the sheaf of linear differential operators over $X$, $\text{Der}_\mathbb{C}(\mathcal{O}_X)$ the sheaf of derivations of $\mathcal{O}_X$, and $D_X[Y]$ the sheaf of meromorphic differential operators with poles along $Y$. Given a point $x$ of $Y$, we will denote by $I_x = (f)$, $\mathcal{O}_x$, $\text{Der}_\mathbb{C}(\mathcal{O})$ and $D$ the respective stalks at $x$. We will denote by $F^\bullet$ the filtration of $D_X$ by the order of the operators and $\Omega^\bullet_X[y]$ the meromorphic de Rham complex with poles along $Y$.

1.1 Logarithmic forms and logarithmic derivations.

Free divisors

We are going to recall some notions of [14] that we will use repeatedly:

A section $\delta$ of $\text{Der}_\mathbb{C}(\mathcal{O}_X)$, defined over an open set $U$ of $X$, is called a logarithmic derivation (or vector field) if for each point $x$ in $Y \cap U$, $\delta_x(I_x)$ is contained in the ideal $I_x$ (if $I = I_x = (f)$, it is sufficient that $\delta_x(f)$ belongs to $(f)\mathcal{O}$). The sheaf of logarithmic derivations is denoted by $\text{Der}(\log Y)$, and is a coherent $\mathcal{O}_X$-submodule of $\text{Der}_\mathbb{C}(\mathcal{O}_X)$ and a Lie subalgebra. We denote by $\text{Der}(\log f)$, or $\text{Der}(\log I)$, the stalks at $x$ of $\text{Der}(\log Y)$:

$$\text{Der}(\log f) = \{\delta \in \text{Der}_\mathbb{C}(\mathcal{O}) / \delta(f) \in (f)\}.$$ 

We say that a meromorphic $q$-form $\omega$ with poles along $Y$, defined in an open set $U$, is a logarithmic $q$-form along $Y$ or, simply, a logarithmic $q$-form, if for every point $x$ in $U$, $f \omega$ and $df \wedge \omega$ are holomorphic at $x$. The sheaf of logarithmic $q$-forms along $Y$ in $U$ is denoted by $\Omega^q_X(\log Y)(U)$. This definition gives rise to a coherent $\mathcal{O}_X$-module $\Omega^q_X(\log Y)$, whose stalks are:

$$\Omega^q(\log f) = \Omega^q_X(\log Y)_x = \{\omega \in \Omega^q_X[y]_x / f \omega \in \Omega^q, df \wedge \omega \in \Omega^{q+1}\}.$$
The logarithmic $q$-forms along $Y$ define a subcomplex of the meromorphic de Rham complex along $Y$, that we call the logarithmic de Rham complex and denote by $\Omega^*_X(\log Y)$.

Contraction of forms by vector fields defines a perfect duality between the $\mathcal{O}_X$-modules $\Omega^1_X(\log Y)$ and $\mathcal{D}\text{er}(\log Y)$, that we denote by $\langle \ , \ \rangle$. Thus, both of them are reflexive. In particular, when $n = \dim \mathbb{C}X = 2$, $\Omega^1_X(\log Y)$ and $\mathcal{D}\text{er}(\log Y)$ are locally free $\mathcal{O}_X$-modules of rank 2.

We say that $Y$ is free at $x$, or $I$ is a free ideal of $\mathcal{O}$, if $\text{Der}(\log I)$ is free as $\mathcal{O}$-module (of rank $n$). If $f \in \mathcal{O}$, we say that $f$ is free if the ideal $I = (f)$ is free. We say that $Y$ is free if it is at every point $x$. In this case, $\mathcal{D}\text{er}(\log Y)$ is a locally free $\mathcal{O}_X$-module of rank $n$. We can use the following criterion to determine when an hypersurface $Y$ is free at $x$:

**Saito’s Criterion:** The $\mathcal{O}$-module $\text{Der}(\log f)$ is free if and only if there exist $n$ elements $\delta_1, \delta_2, \ldots, \delta_n$ in $\text{Der}(\log f)$, with $\delta_i = \sum_{j=1}^n a_{ij}(z) \frac{\partial}{\partial z_j}$ ($i = 1, \ldots, n$), where $z = (z_1, z_2, \ldots, z_n)$ is a system of coordinates of $X$ centered in $x$, such that the determinant $\det(a_{ij})$ is equal to $af$, with $a \in \mathcal{O}$ a unit. Moreover, in this case, $\{\delta_1, \delta_2, \ldots, \delta_n\}$ is a basis of $\text{Der}(\log f)$.

When $Y$ is free, we have the equality: $\Omega^p_X(\log Y) \overset{\gamma^p}{=} \Omega^1_X(\log Y)$. Using the fact that $\Omega^1_X(\log Y) \cong \text{Hom}_{\mathcal{O}_X}(\text{Der}(\log Y), \mathcal{O}_X)$, we can construct a natural isomorphism:

$$\Omega^p_X(\log Y) \cong \text{Hom}_{\mathcal{O}_X}(\text{Der}(\log Y), \mathcal{O}_X),$$

defined locally by $\gamma^p(\omega_1 \wedge \cdots \wedge \omega_p)(\delta_1 \wedge \cdots \wedge \delta_p) = \det(\langle \omega_i, \delta_j \rangle)_{1 \leq i, j \leq p}$.

### 1.2 $\mathcal{V}$-filtration

We define the $\mathcal{V}$-filtration relative to $Y$ on $\mathcal{D}_X$ as in the smooth case ([10], [9]):

$$\mathcal{V}^\mathcal{Y}_k(\mathcal{D}_X) = \{ P \in \mathcal{D}_X / P(I^j) \subset I^{j-k}, \forall j \in \mathbb{Z} \}, \quad k \in \mathbb{Z},$$

where $I^p = \mathcal{O}_X$ when $p$ is negative. Similarly, $\mathcal{V}^\mathcal{Y}_k(\mathcal{D}) = \{ P \in \mathcal{D} / P(I^j) \subset I^{j-k}, \forall j \in \mathbb{Z} \}$, with $k$ an integer, and $I^p = \mathcal{O}$ when $p \geq 0$. In the case of $I = (f)$, we note $\mathcal{V}^\mathcal{Y}_k(\mathcal{D}) = \mathcal{V}^\mathcal{Y}_k(f)$.  

**Definition 1.2.1.** A logarithmic differential operator (or, simplify, a logarithmic operator) is a differential operator of degree 0 with respect to the $\mathcal{V}$-filtration.

We see that:

$$\text{Der}(\log Y) = \text{Der}_\mathbb{C}(\mathcal{O}_X) \cap \mathcal{V}^\mathcal{Y}_0(\mathcal{D}_X) = \mathcal{G}^1_1(\mathcal{V}^\mathcal{Y}_0(\mathcal{D}_X)),$$

$$F^1(\mathcal{V}^\mathcal{Y}_0(\mathcal{D}_X)) = \mathcal{O}_X \oplus \text{Der}(\log Y),$$

where the last expression is consequence of $F^1(\mathcal{D}_X) = \mathcal{O}_X \oplus \text{Der}_\mathbb{C}(\mathcal{O}_X)$. 

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Remark 1.2.2.– The inclusion $\text{Der}(\log Y) \subset \text{Gr}_{\mathcal{F}^\bullet} \left( \mathcal{V}_0^Y(D_X) \right)$ gives rise to a canonical graded morphism of graded algebras:

$$\kappa : \text{Sym}_{O_X}(\text{Der}(\log Y)) \longrightarrow \text{Gr}_{\mathcal{F}^\bullet} \left( \mathcal{V}_0^Y(D_X) \right).$$

Similarly, we have a canonical graded morphism of graded $O$-algebras:

$$\kappa_x : \text{Sym}_{O}(\text{Der}(\log I)) \longrightarrow \text{Gr}_{\mathcal{F}^\bullet} \left( \mathcal{V}_I^0(D) \right),$$

which is the stalk of $\kappa$ at $x$.

2 Logarithmic operators relative to a free divisor

2.1 The Structure Theorem

We denote by $\{ , \}$ the Poisson bracket defined in the graded ring $\text{Gr}_{\mathcal{F}^\bullet}(D)$ (cf. [12], [8]). Given two polynomials $F,G$ in $\text{Gr}_{\mathcal{F}^\bullet}(D) = O[\xi_1, \ldots, \xi_n]$:

$$\{F, G\} = \sum_{i=1}^n \frac{\partial F}{\partial \xi_i} \frac{\partial G}{\partial x_i} - \sum_{i=1}^n \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial \xi_i}.$$

Proposition 2.1.1.– Let $f$ be free. Consider a minimal system of generators $\{\delta_1, \delta_2, \ldots, \delta_n\}$ of Der(log $f$). Let $R_0$ be a polynomial in $\text{Gr}_{\mathcal{F}^\bullet}(D)$, homogeneous of order $d$, and such that there exist other polynomials $R_k$ in $\text{Gr}_{\mathcal{F}^\bullet}(D)$, with $k = 1, \ldots, d$, homogeneous of order $d - k$ such that:

$$\{R_k, f\} = fR_{k+1}, \quad (0 \leq k < d)$$

(we will say that $R_0$ verifies the property (II) for $R_1, R_2, \ldots, R_d$). Then there exist polynomials $H_j^k$ in $\text{Gr}_{\mathcal{F}^\bullet}(D)$, homogeneous of order $d - k - 1$, with $j = 1, \ldots, n$ and $k = 1, \ldots, d - 1$, such that:

a) $R_k = \sum_{j=1}^n H_j^k \sigma(\delta_j)$, where $\sigma(\delta_j)$ denotes the principal symbol of $\delta_j$.

b) $\{H_j^k, f\} = fH_j^{k+1} \ (1 \leq j \leq n, \ 0 \leq k < d - 1)$. This is the same as saying: $H_j^k$ verifies the property (II) for $H_j^{k+1}, \ldots, H_j^{d-1}$.

Proof: Let $A = (\alpha^j_i)$ be the square matrix whose rows are the coefficients of the basis $\{\delta_1, \delta_2, \ldots, \delta_n\}$ of Der(log $f$) with respect to the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$ of $\text{Der}_{\mathbb{C}}(O_X)$:

$$\delta_j = \sum_{i=1}^n \alpha^j_i \frac{\partial}{\partial x_i} = \alpha^j \bullet \partial^j,$$

with $j = 1, \ldots, n$, where we write $\partial$ instead of $\left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$. We consider the ring $O_{2n} = \mathbb{C}\{x_1, \ldots, x_2, \xi_1, \ldots, \xi_n\}$. Thanks to the Saito’s Criterion, we know that the set

$$\{\delta_1, \ldots, \delta_n; \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}\}$$

is regular.
is a basis of the $O_{2n}$-module $\text{Der}_{O_{2n}}(\log f)$. So, as we have, for $k = 1, \ldots, d$,

$$(f) \ni \{R_k, f\} = \sum_{i=1}^{n} (R_k)_{\xi_i} f_{x_i},$$

where $f_{x_i}$ represents $\frac{\partial f}{\partial x_i}$ and $(R_k)_{\xi_i}$ represents $\frac{\partial R_k}{\partial \xi_i}$, then there exist homogeneous polynomials $G^k_j$ in $\text{Gr}_F^\bullet(D)$, of degree $d - k - 1$, or null, with $j = 1, \ldots, n$ and $k = 1, \ldots, d - 1$, such that

$$((R_k)_{\xi_1}, (R_k)_{\xi_2}, \ldots, (R_k)_{\xi_n}) = \sum_{j=1}^{n} G^k_j \alpha^j.$$

Using the Euler relation $R_k = \frac{1}{d} \sum_{i=1}^{n} (R_k)_{\xi_i} \xi_i$, and as $\sigma(\delta_i) = \alpha^i \bullet \xi^t$, we obtain

$$R_k = \frac{1}{d} \sum_{i=1}^{n} \sum_{j=1}^{n} G^k_j \alpha_i^j \xi_i = \frac{1}{d} \sum_{j=1}^{n} G^k_j \sigma(\delta_j).$$

By Saito’s Criterion, the determinant of the matrix $A$ is equal to $uf$, with $u \in O$ invertible. Let $B = (b_{ij}) = \text{Adj}(A)^t$. We have:

$$((R_k)_{\xi_1}, (R_k)_{\xi_2}, \ldots, (R_k)_{\xi_n}) = \left(G^k_1, G^k_2, \ldots, G^k_n\right) A,$$

so

$$((R_k)_{\xi_1}, (R_k)_{\xi_2}, \ldots, (R_k)_{\xi_n}) B = g \left(G^k_1, G^k_2, \ldots, G^k_n\right).$$

Now:

$$g\{G^k_j, f\} = \{gG^k_j, f\} = \sum_{i=1}^{n} f_{x_i} \frac{\partial (gG^k_j)}{\partial \xi_i} = \sum_{i=1}^{n} f_{x_i} \sum_{i=1}^{n} \frac{\partial (R_k)_{\xi_i}}{\partial \xi_i} b_{ij} =$$

$$\sum_{i=1}^{n} b_{ij} \sum_{i=1}^{n} \frac{\partial^2 R_k}{\partial \xi_i \partial \xi_i} f_{x_i} = \sum_{i=1}^{n} b_{ij} \frac{\partial ((R_k, f))}{\partial \xi_i} = f \sum_{i=1}^{n} b_{ij} \frac{\partial R_{k+1}}{\partial \xi_i} = f \sum_{i=1}^{n} b_{ij} (R_{k+1})_{\xi_i} =$$

$$f \sum_{i=1}^{n} b_{ij} \sum_{p=1}^{n} G^k_{p+1} \alpha^p_i = f \sum_{p=1}^{n} G^k_{p+1} \sum_{i=1}^{n} b_{ij} \alpha^p_i = f gG^{k+1}.\]

Therefore,

$$\{G^k_j, f\} = f G^{k+1},$$

with $k = 0, \ldots, d - 2$ and $j = 0, \ldots, n$. We conclude by setting $H^k_j = \frac{1}{d} G^k_j$, for $j = 1, \ldots, n$ and $k = 0, \ldots, d - 1$. □

**Proposition 2.1.2.**— Let be $\{\delta_1, \delta_2, \ldots, \delta_n\}$ a basis of $\text{Der}(\log f)$. If a polynomial $R_0$ of $\text{Gr}_F^\bullet(D)$ is homogeneous and verifies the property (5) of the last proposition, we can find a differential operator $Q$ in $O[\delta_1, \delta_2, \ldots, \delta_n]$ such that $R_0$ is the symbol of $Q$.  

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Proof: We will do the proof by induction on the order of $R_0$. If $R_0 \in \mathcal{O}$, it is obvious. We suppose that the result holds if the order of $R_0$ is less than $d$. Now let $R_0$ of order $d$ verifying (III). By the last proposition there exist $n$ homogeneous polynomials $H_j^0$ of order $d - 1$ such that:

$$R_0 = \sum_{j=1}^{n} H_j^0 \sigma(\delta_j), \ H_j^0 \text{ verifies (III)} (j = 1, \ldots, n).$$

By induction hypothesis, there exist $Q_j \in \mathcal{O}[\delta_1, \delta_2, \ldots, \delta_n]$ such that $H_j^0 = \sigma(Q_j)$.

So
$$R_0 = \sum_{i=1}^{n} \sigma(Q_i)\sigma(\delta_i) = \sum_{i=1}^{n} \sigma(Q_i\delta_i) = \sigma(\sum_{i=1}^{n} Q_i\delta_i) = \sigma(Q)$$

and $Q = \sum_{i=1}^{n} Q_i\delta_i \in \mathcal{O}[\delta_1, \delta_2, \ldots, \delta_n]$. \qed

Remark 2.1.3.– Really, the previous argument proves that if $R_0$ verifies (III), then $R_0$ is a polynomial in $\mathcal{O}[\sigma(\delta_1), \ldots, \sigma(\delta_n)]$.

Theorem 2.1.4.– If $f$ is free and $\{\delta_1, \delta_2, \ldots, \delta_n\}$ is a basis of the $\mathcal{O}$-module $	ext{Der}(\log f)$, each logarithmic operator $P$ can be written in a unique way as a polynomial

$$P = \sum \beta_{i_1\ldots i_n} \delta_1^{i_1} \delta_2^{i_2} \cdots \delta_n^{i_n}, \quad \beta_{i_1\ldots i_n} \in \mathcal{O}.$$ 

In other words, the ring of logarithmic operators is the $\mathcal{O}$-subalgebra of $\mathcal{D}$ generated by logarithmic derivations:

$$\mathcal{V}_0^f(\mathcal{D}) = \mathcal{O}[\delta_1, \delta_2, \ldots, \delta_n] = \mathcal{O}[\text{Der}(\log f)].$$

Proof: The inclusion $\mathcal{O}[\delta_1, \delta_2, \ldots, \delta_n] \subseteq \mathcal{V}_0^f(\mathcal{D})$ is clear. We will prove the other inclusion by induction on the order of $P_0 \in \mathcal{V}_0^f(\mathcal{D})$. If the order of $P_0$ is zero, then it is a holomorphic function and the result is obvious. We suppose the result is true for every logarithmic operator $Q$ whose order is strictly less than $d$. Let $P_0$ be a logarithmic operator of order $d$. We know that:

$$[P_0, f] = f P_1,$$

with $P_1 \in \mathcal{V}_0^f(\mathcal{D})$. So, there exist several $P_k$, with $k = 0, \ldots, d$, such that $[P_k, f] = f P_{k+1}$. If we set $R_k = \sigma(P_k)$, in the case that $P_k$ has order $d - k$, and $R_k = 0$ otherwise, we obtain:

$$\{R_k, f\} = \{\sigma_{d-k}(P_k), f\} = \sigma_{d-k-1}([P_k, f]) = f \sigma_{d-k-1}(P_{k+1}) = f R_{k+1}.$$ 

By the previous proposition, there exists $Q$ in $\mathcal{O}[\delta_1, \delta_2, \ldots, \delta_n]$ of order $d$ and such that $\sigma(P_0) = \sigma(Q)$. As the order of $P_0 - Q \in \mathcal{V}_0^f(\mathcal{D})$ is strictly less than $d$, we apply the induction hypothesis to $P_0 - Q$ and obtain

$$P_0 = P_0 - Q + Q \in \mathcal{O}[\delta_1, \delta_2, \ldots, \delta_n],$$

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as we wanted.
On the other hand, using the structure of Lie algebra it is clear that we can write
a logarithmic operator as a $\mathcal{O}$-linear combination of the monomials $\{\delta_1, \ldots, \delta_n\}$. The uniqueness of this expression follows from the fact that these monomials are linearly independent over $\mathcal{O}$.

\[ \text{Remark 2.1.5.--} \] As a immediate consequence of the theorem (see the previous remark), we obtain an isomorphism:

\[ \text{Gr}_{F^*} \left( V^1_0(\mathcal{D}) \right) \cong \mathcal{O}[\sigma(\delta_1), \ldots, \sigma(\delta_n)]. \]

\[ \text{Corollary 2.1.6.--} \] If $Y$ is free at $x$, the morphism $\kappa_x$ from the symmetric algebra $\text{Sym}_{\mathcal{O}X}(\text{Der}(\log f))$ to $\text{Gr}_{F^*} \left( V^0_0(\mathcal{D}) \right)$ (see remark 2.1.2) is an isomorphism of graded $\mathcal{O}$-algebras. As a consequence, if $Y$ is a free divisor, the canonical morphism

\[ \kappa : \text{Sym}_{\mathcal{O}X}(\text{Der}(\log Y)) \to \text{Gr}_{F^*} \left( V^0_0(\mathcal{D}_X) \right) \]

is an isomorphism.

**Proof:** Let $x$ be in $X$ and $f \in \mathcal{O}$ a local reduced equation of $Y$ at a neighbourhood of $x$. Let $\{\delta_1, \ldots, \delta_n\}$ be a basis of $\text{Der}(\log f)$.

\[ \text{Der}(\log f) = \oplus_{i=1}^n \mathcal{O}\delta_i \cong \oplus_{i=1}^n \mathcal{O}\sigma(\delta_i). \]

The symmetric algebra of the $\mathcal{O}$-module $\text{Der}(\log f)$ is isomorphic to a polynomial ring:

\[ \text{Sym}_{\mathcal{O}}(\text{Der}(\log f)) \cong \mathcal{O}[\sigma(\delta_1), \ldots, \sigma(\delta_n)]. \]

We also have the inclusion:

\[ \oplus_{i=1}^n \mathcal{O}\sigma(\delta_i) = \text{Gr}_{F^*}^1 \left( V^0_0(\mathcal{D}) \right) \subset \text{Gr}_{F^*} \left( V^0_0(\mathcal{D}) \right), \]

where $\sigma(\delta_i)$ is the image of $\delta_i$ by the morphism $\kappa_x$. Therefore we conclude that the morphism $\kappa_x = \alpha^{-1}\beta$ is an isomorphism (see remark 2.1.3). On the other hand, the inclusion

\[ \text{Der}(\log Y) = \text{Gr}_{F^*}^1 \left( V^0_0(\mathcal{D}_X) \right) \subset \text{Gr}_{F^*} \left( V^0_0(\mathcal{D}_X) \right) \]

gives rise to a canonical graded morphism of graded $\mathcal{O}_X$-algebras (see remark 2.1.2): $\kappa : \text{Sym}_{\mathcal{O}_X}(\text{Der}(\log Y)) \to \text{Gr}_{F^*} \left( V^0_0(\mathcal{D}_X) \right)$, whose stalk at each point $x$ of $Y$ is the canonical graded isomorphism $\kappa_x$. So, $\kappa$ is also an isomorphism. \[ \square \]

**Corollary 2.1.7.--** $V^0_0(\mathcal{D}_X)$ is a coherent sheaf of rings.

**Proof:** By theorem 9.16 of [1] (p. 83), we have only to prove that $\text{Gr}_{F^*} \left( V^0_0(\mathcal{D}_X) \right)$ is coherent, but this sheaf is locally isomorphic to the polynomial ring $\mathcal{O}_X[T_1, \ldots, T_n]$, which is coherent (\[3, \text{lemma 3.2, VI, pg. 205}\]). \[ \square \]
2.2 Equivalence between $\mathcal{O}_X$-modules with a logarithmic connection and left $\mathcal{V}_Y^+(\mathcal{D}_X)$-modules.

**Definition 2.2.1.** (cf. [3]) Let $\mathcal{M}$ be a $\mathcal{O}_X$-module. A connection on $\mathcal{M}$, with logarithmic poles along $Y$, (or logarithmic connection on $\mathcal{M}$), is a $\mathbb{C}$-homomorphism $\nabla$,

$$\nabla : \mathcal{M} \rightarrow \Omega^1_X(\log Y) \otimes \mathcal{M},$$

that verifies Leibniz’s identity: $\nabla(hm) = dh \cdot m + h \cdot \nabla(m)$, where $d$ is the exterior derivative over $\mathcal{O}_X$. We will note $\Omega^q_X(\log Y)(\mathcal{M}) = \Omega^q_X(\log Y) \otimes \mathcal{M}$.

If $\delta$ is a logarithmic derivation along $Y$, it defines a $\mathbb{C}$-morphism:

$$\text{Der}(\log Y) \rightarrow \text{End}_{\mathbb{C}}(\mathcal{M}), \quad \delta \mapsto \nabla_\delta$$

where $\nabla_\delta(m) = \langle \delta, \nabla(m) \rangle$

**Remark 2.2.2.** A logarithmic connection $\nabla$ on $\mathcal{M}$ gives rise to a morphism of $\mathcal{O}_X$-modules

$$\nabla' : \text{Der}(\log Y) \rightarrow \text{Hom}_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$$

which verifies Leibniz’s condition: $\nabla'_\delta(fm) = \delta(f) \cdot m + f \cdot \nabla'_\delta(m)$.

Conversely, given $\nabla'$ verifying this condition, we define

$$\nabla : \mathcal{M} \rightarrow \Omega^1_X(\log Y)(\mathcal{M}),$$

with $\nabla(m)$ the element of $\Omega^1_X(\log Y)(\mathcal{M}) = \text{Hom}_{\mathcal{O}_X}(\text{Der}(\log Y), \mathcal{M})$ such that:

$$\nabla(m)(\delta) = \nabla'_\delta(m).$$

**Definition 2.2.3.** A logarithmic connection $\nabla$ is integrable if, for each pair $\delta$ and $\delta'$ of logarithmic derivations, it verifies:

$$\nabla_{[\delta, \delta']} = [\nabla_\delta, \nabla_{\delta'}],$$

where $[\ , \ ]$ represents the Lie bracket in $\text{Der}(\log Y)$ and the commutator in $\text{Hom}_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$.

Given a logarithmic connection $\nabla$ and the exterior derivative $d$, we can construct a morphism:

$$\nabla^q : \Omega^q_X(\log Y)(\mathcal{M}) \rightarrow \Omega^{q+1}_X(\log Y)(\mathcal{M}),$$

for each $q = 1, \cdots, n$. If $\omega$ and $m$ are sections of the sheaves $\Omega^q_X(\log Y)$ and $\mathcal{M}$:

$$\nabla^q(\omega \otimes m) = d\omega \otimes m + (-1)^q \omega \wedge \nabla(m).$$
The integrability condition is equivalent to $\nabla^q \circ \nabla^{q-1} = 0$, for every $q$ (cf. [3]).

**Definition 2.2.4.–** Let $\mathcal{M}$ be a $\mathcal{O}_X$-module, and $\nabla$ an integrable logarithmic connection along $Y$ on $\mathcal{M}$. With the above notation, we call the logarithmic de Rham complex of $\mathcal{M}$, and we denote by $\Omega^*_X(\log Y)(\mathcal{M})$, the complex (of sheaves of $\mathbb{C}$-vector spaces):

$$0 \to \mathcal{M} \xrightarrow{\nabla} \Omega^1_X(\log Y)(\mathcal{M}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{q}_X(\log Y)(\mathcal{M}) \xrightarrow{\nabla} \Omega^{q+1}_X(\log Y)(\mathcal{M}) \to 0.$$  

In the particular case where the $\mathcal{O}_X$-module $\mathcal{M}$ is equal to $\mathcal{O}_X$ and the logarithmic connection $\nabla$ is equal to the exterior derivative $d : \mathcal{O}_X \to \Omega^1_X(\log Y)$, the morphisms 

$$\nabla^q : \Omega^q_X(\log Y) \to \Omega^{q+1}_X(\log Y),$$

define the logarithmic de Rham complex of Saito.

We consider the rings $R_0 = \mathcal{O}_X \subset R_1$ and $R = \mathcal{O}^\vee_0(\mathcal{D}_X) = \bigcup_{k \geq 0} R_k$ ($1 \in R_0 \subset R$), with $R_k = F^k(\mathcal{O}^\vee_0(\mathcal{D}_X))$. The ring $\mathcal{G}r(R)$ is commutative and verifies

1) The canonical morphism $\alpha : \text{Sym}_{R_0}(\mathcal{G}r^1(R)) \to \mathcal{G}r(R)$, defined by $\alpha(s_1 \otimes \cdots \otimes s_t) = s_1 \cdots s_t$, is an isomorphism (see Corollary 2.1.6). With these conditions, $R_1$ is an $(R_0, R_0)$-bimodule, and a Lie algebra $[[x, y]] = xy - yx \in R_1$, because $\mathcal{G}r(R)$ is commutative. Moreover, $R_0$ is a sub-$\text{(R}_0, R_0\text{)}$-bimodule of $R_1$ such that the two induced structures of $R_0$-module over the quotient $R_1/R_0$ are the same.

Let $\mathbf{T}_{R_0}(R_1) = R_0 \oplus R_1 \oplus (R_1 \otimes R_0 R_1) \oplus \cdots$ be the tensor algebra of the $(R_0, R_0)$-bimodule $R_1$, and let $\psi : \mathbf{T}_{R_0}(R_1) \to R$ be the canonical morphism defined by the inclusion $R_1 \subset R$. We prove a reciprocal theorem of one Poincaré-Birkhoff-Witt theorem [3, theorem 3.1, p.198].

**Proposition 2.2.5.–** The morphism $\psi$ induces an isomorphism:

$$\phi : \mathbf{S} = \frac{\mathbf{T}_{R_0}(R_1)}{J} \cong R, \quad \phi((i(x_1) \otimes \cdots \otimes i(x_t)) + J) = x_1 x_2 \cdots x_t,$$

where $i$ the inclusion of $R_1$ in the tensor algebra, and $J$ is the two sided ideal generated by the elements:

a) $a - i(a), \ a \in R_0 \subset R_1$,  
b) $i(x) \otimes i(y) - i(y) \otimes i(x) - i([x, y]), \ x, y \in R_1$.

**Proof:** First, we check that the morphism $\phi : \mathbf{S} \to R$ is well defined:

$$\psi(a - i(a)) = a - a = 0, \ a \in R_0,$$

$$\psi(i(x) \otimes i(y) - i(y) \otimes i(x) - i([x, y])) = xy - yx - [x, y] = 0, \ x, y \in R_1.$$
The algebra $T_{R_0}(R_1)$ is graded, so it is filtered, and induces a filtration on the quotient. The induced morphism $\phi : S \to R$ is filtered:

$$\psi(a) = a \in R_0, \psi(i(x_1) \otimes \cdots \otimes i(x_t)) = x_1x_2 \cdots x_t \in R_t.$$ 

So, we can define a graded morphism of $R_0$-rings.

$$\pi : Gr(S) \to Gr(R),$$

$$\pi(\sigma_t(i(x_1) \otimes \cdots \otimes i(x_t) + J)) = \sigma'_t(x_1 \cdots x_t) = \overline{x_1} \cdots \overline{x_t},$$

where $x_i \in R_1$, $\overline{x_i} = \sigma'_t(x_1)$ is the class of $x_i$ in $R_1/R_0$, $\sigma_t(P)$ is the class of $P \in S$ in $Gr^t(S)$, and $\sigma'_t(Q)$ the class of $Q \in R_t$ in $Gr^t(R)$. Note that $Gr(S)$ is commutative: it is generated by the elements $\sigma_0(a + J), \sigma_1(i(x) + J)$, with $a \in R_0$, $x \in R_1$, and

$$[i(x) + J, i(y) + J] = i([x, y]) + J,$$

$$[a + J, i(x) + J] = i(ax - xa) + J = b + J, \quad b = ax - xa \in R_0.$$

On the other hand, the image of $R_0 \subset R_1$ in $S$ is exactly the part of degree zero of $S$, and then we obtain a morphism of $R_0$-modules from $Gr^1(R) = R_1/R_0$ to $Gr^1(S)$ which induces a morphism of $R_0$-algebras:

$$\rho : Sym_{R_0} \left( \frac{R_1}{R_0} \right) \to Gr(S),$$

$$\rho(\overline{x_1} \otimes \cdots \otimes \overline{x_t}) = \sigma_t(i(x_1) \otimes \cdots \otimes i(x_t) + J),$$

which is obviously surjective. The composition $\pi \rho$ is equal to $\alpha$, and, by property (1) of $R$, we deduce that $\rho$ is injective. As $\rho$ and $\pi \rho$ are isomorphisms, $\pi$ is as well, as we wanted to prove.

\textbf{Corollary 2.2.6.} Let $Y$ be a free divisor. Let $M$ be a $O_X$-module. An integrable logarithmic connection on $M$ gives rise to a left $V_0^Y(D_X)$-structure on $M$, and vice versa.

\textbf{Proof:} A $O_X$-module $M$ with an integrable logarithmic connection $\nabla$ has a natural structure of left $V_0^Y(D_X)$-module defined by its structure as $O_X$-module. Let $\mu$ be the morphism of $(O_X, O_X)$-bimodules:

$$\mu : R_1 = O_X \oplus Der(\log Y) \to End_C(M), \quad \mu(a)(m) = am, \quad \mu(\delta)(m) = \nabla_\delta(m).$$

$\mu$ induces a morphism $\nu : T_{R_0}(R_1) \to End_C(M)$, and, as $\nu(J) = 0$, we have a morphism

$$V_0^Y(D_X) \simeq \frac{T_{R_0}(R_1)}{J} \to End_C(M),$$

which defines an structure of $V_0^Y(D_X)$-module on $M$. 

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On the other hand, a left \( V_0^Y(D_X) \)-module structure on \( M \) defines an integrable logarithmic connection \( \nabla \) on the \( O_X \)-module \( M \):

\[
\nabla : \text{Der}(\log Y) \to \mathcal{E}\text{nd}_C(M), \quad \nabla_{\delta}(m) = \delta \cdot m.
\]

\[\square\]

**Remark 2.2.7.** A left \( V_0^Y(D_X) \)-module structure on \( M \) defines a logarithmic de Rham complex. In local coordinates \((U; x_1, \cdots, x_n)\), with \( \{\delta_1, \cdots, \delta_n\} \) a local basis of \( \text{Der}(\log Y) \) and \( \{\omega_1, \cdots, \omega_n\} \) its dual basis, the differential of the complex is defined by:

\[
\nabla^p(U)(\omega \otimes m) = d\omega \otimes m + \sum_{i=1}^{n} ((\omega_i \wedge \omega) \otimes \delta_i \cdot m),
\]

for any sections \( \omega \in \Omega^1_X(\log Y) \) and \( m \in M \). In the particular case of the left \( V_0^Y(D_X) \)-module \( O_X \), defined as \( V_0^Y(D_X) \)-module in a natural way \((P \cdot g = P(g), \) with \( g \) a holomorphic function and \( P \) a logarithmic operator\), this canonical structure of \( O_X \) as left \( V_0^Y(D_X) \)-module is obviously equivalent to the integrable logarithmic connection over \( O_X \) defined naturally by the exterior derivative \( (\nabla = d)\):

\[
\nabla_{\delta}(g) = \langle \delta, dg \rangle = \delta(g).
\]

## 3 The Logarithmic de Rham Complex

In this section, \( Y \) will be a free divisor.

### 3.1 The Logarithmic Spencer Complex

**Definition 3.1.1.** We call the logarithmic Spencer complex, and denote by \( \mathcal{S}p^\bullet(\log Y) \), the complex:

\[
\begin{align*}
0 \to V_0^Y(D_X) \otimes_{O_X} & \overset{n}{\wedge} \text{Der}(\log Y) \overset{\varepsilon_{-p}}{\to} \cdots \\
& 
\overset{\varepsilon_{-2}}{\to} V_0^Y(D_X) \otimes_{O_X} \overset{1}{\wedge} \text{Der}(\log Y) \overset{\varepsilon_{-1}}{\to} V_0^Y(D_X),
\end{align*}
\]

where

\[
\varepsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \delta_p) +
\]

\[
\sum_{1 \leq i < j \leq p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \hat{\delta_i} \wedge \cdots \wedge \hat{\delta_j} \wedge \cdots \wedge \delta_p), \quad (2 \leq p \leq n).
\]
We can augment this complex of left $\mathcal{V}_0^Y(D_X)$-modules by another morphism:

$$\varepsilon_0 : \mathcal{V}_0^Y(D_X) \to \mathcal{O}_X, \quad \varepsilon_0(P) = P(1).$$

We call the new complex $\tilde{S}_p^\bullet(\log Y)$.

This definition is essentially the same as the definition of the usual Spencer complex $S_p^\bullet$ of $\mathcal{O}_X$ (cf. [11, 2.1]) and generalizes the definition given by Esnault and Viehweg [7, App. A] in the case of a normal crossing divisor. We denote by $S_p^\bullet[*Y] = D_X[*Y] \otimes_{D_X} S_p^\bullet$ the meromorphic Spencer complex of $\mathcal{O}_X[*Y]$.

**Theorem 3.1.2.** The complex $S_p^\bullet(\log Y)$ is a locally free resolution of $\mathcal{O}_X$ as left $\mathcal{V}_0^Y(D_X)$-module.

**Proof:** To see the exactness of $\tilde{S}_p^\bullet(\log Y)$ we define a discrete filtration $G^\bullet$ such that it induces an exact graded complex (cf. [1, lemma 3.16]):

$$G^k \left( \mathcal{V}_0^Y(D_X) \otimes \wedge^p \text{Der}(\log Y) \right) = F^{k-p} \left( \mathcal{V}_0^Y(D_X) \otimes \wedge^p \text{Der}(\log Y) \right),$$

$$G^k(\mathcal{O}_X) = \mathcal{O}_X.$$

We have

$$G_{r,G^\bullet} \left( \mathcal{V}_0^Y(D_X) \otimes \wedge^p \text{Der}(\log Y) \right) = G_{r,F^\bullet} \left( \mathcal{V}_0^Y(D_X) \right) [-p] \otimes \wedge^p \text{Der}(\log Y),$$

$$G_{r,G^\bullet}(\mathcal{O}_X) = \mathcal{O}_X.$$

As the above filtrations are compatible with the differential of the complex $\tilde{S}_p^\bullet(\log Y)$, we can consider the complex $G_{r,G^\bullet} \left( \tilde{S}_p^\bullet(\log Y) \right)$:

$$0 \to G_{r,F^\bullet} \left( \mathcal{V}_0^Y(D_X) \right) [-n] \otimes_{\mathcal{O}_X} \wedge \text{Der}(\log Y) \xrightarrow{\psi} \cdots$$

$$\xrightarrow{\psi^{-1}} G_{r,F^\bullet} \left( \mathcal{V}_0^Y(D_X) \right) [-1] \otimes_{\mathcal{O}_X} \wedge \text{Der}(\log Y) \xrightarrow{\psi^{-1}} G_{r,F^\bullet} \left( \mathcal{V}_0^Y(D_X) \right) \xrightarrow{\psi} \mathcal{O}_X \to 0,$$

where the local expression of the differential is defined by:

$$\psi_p(G \otimes \delta_{j_1} \wedge \ldots \wedge \delta_{j_p}) = \sum_{i=1}^{p} (-1)^{i-1} G\sigma(\delta_{j_i}) \otimes \delta_{j_1} \wedge \ldots \wedge \widehat{\delta_{j_i}} \wedge \ldots \wedge \delta_{j_p}, \quad (2 \leq p \leq n).$$

$$\psi_{-1}(G \otimes \delta_i) = G\sigma(\delta_i), \quad \psi_0(G) = G_0,$$

with $\{\delta_1, \ldots, \delta_n\}$ a (local) basis of $\text{Der}(\log Y)$. This complex is the Koszul complex of the ring

$$G_{r,F^\bullet} \left( \mathcal{V}_0^Y(D_X) \right) \cong \text{Sym}_{\mathcal{O}_X}(\text{Der}(\log Y)).$$
with respect to the \( G_{T^*} \left( \mathcal{V}_0^\vee (\mathcal{D}_X) \right) \)-regular sequence \( \sigma(\delta_1), \ldots, \sigma(\delta_n) \) in the ring \( G_{T^*} \left( \mathcal{V}_0^\vee (\mathcal{D}_X) \right) \). Consequently, it is exact. \( \square \)

**Lemma 3.1.3.**— For every logarithmic operator \( P \in \mathcal{V}_0^\vee (\mathcal{D}) \), there exist, for each integer \( p \), a logarithmic operator \( Q \in \mathcal{V}_0^\vee (\mathcal{D}) \) and an integer \( k \) such that \( f^{-p}P = Qf^{-k} \).

**Proof:** We will prove the lemma by induction on the order of the logarithmic operator. If \( P \) has order 0, it is in \( \mathcal{O} \), and it is clear that \( f^{-p}P = Pf^{-p} \). Let \( P \) be of order \( d \), and consider the logarithmic operator \( [P, f^p] \), of order \( d - 1 \). By induction hypothesis, there exists an integer \( m \) such that:

\[
[P, f^{-p}]f^m \in \mathcal{V}_0^\vee (\mathcal{D}).
\]

Let \( k \) be the greatest of the integers \( m \) and \( p \). It is clear that:

\[
f^{-p}Pf^k = Pf^{k-p} - [P, f^{-p}]f^k \in \mathcal{V}_0^\vee (\mathcal{D}).
\]

This proves the result: \( Q = Pf^{k-p} - [P, f^{-p}]f^k. \) \( \square \)

**Remark 3.1.4.**— For every operator \( Q \) in \( \mathcal{D}_X [\star Y]_x \), we can always find a strictly positive integer \( m \) such that \( f^mQ \in \mathcal{V}_0^\vee (\mathcal{D}) \). Equivalently, for each meromorphic differential operator \( Q \), there exists a positive integer \( p \) and a logarithmic operator \( Q' \) such that we can write:

\[
Q = f^{-p}Q'.
\]

Now we introduce several morphisms that we will use later.

**Lemma 3.1.5.**— We have the following isomorphisms:

1. \( \mathcal{O}_X [\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^\vee (\mathcal{D}_X) \xrightarrow{\sim} \mathcal{D}_X [\star Y] \xrightarrow{\sim} \mathcal{V}_0^\vee (\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X [\star Y] \).
2. \( \alpha : \mathcal{D}_X [\star Y] \otimes_{\mathcal{V}_0^\vee (\mathcal{D})} \mathcal{O}_X \cong \mathcal{O}_X [\star Y], \quad \alpha (P \otimes g) = P(g). \)
3. \( \rho : \mathcal{D}_X [\star Y] \otimes_{\mathcal{V}_0^\vee (\mathcal{D})} \mathcal{D}_X [\star Y] \cong \mathcal{D}_X [\star Y], \quad \rho (P \otimes Q) = PQ. \)

**Proof:**

1. The inclusions \( \mathcal{V}_0^\vee (\mathcal{D}_X), \mathcal{O}_X [\star Y] \subset \mathcal{D}_X [\star Y] \) give rise to the previous isomorphisms of \( (\mathcal{V}_0^\vee (\mathcal{D}_X), \mathcal{O}_X [\star Y]) \)-modules. Locally:

\[
a f^{-k} \otimes P = a f^{-k} P = a Q \otimes f^{-p},
\]

with \( P \) and \( Q \) logarithmic operators such that \( f^{-k}P = Qf^{-p} \). We have seen how to obtain \( Q \) from \( P \) (lemma 3.1.3), and we can obtain \( P \) from \( Q \) in the same way. On the other hand, we saw in the previous remark how to express a meromorphic
defined locally in each degree by: $P$ isomorphism of complexes: $S$ of lemma 3.1.5, this complex is the same as

Composing both of them, we obtain a new inclusion:

\[
\mathcal{O}_X[\ast Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{O}_X \cong \mathcal{O}_X[\ast Y] \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X[\ast Y].
\]

3. We obtain this isomorphism of $\mathcal{D}_X[\ast Y]$-bimodules from the composition of the following isomorphisms:

\[
\mathcal{O}_X[\ast Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{D}_X[\ast Y] \cong \mathcal{D}_X[\ast Y] \\
\mathcal{O}_X[\ast Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \cong \mathcal{O}_X[\ast Y] \otimes_{\mathcal{O}_X} \mathcal{D}_X[\ast Y],
\]

where the isomorphism $\mathcal{O}_X[\ast Y] \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X[\ast Y]$ sends (locally) the tensor product $g_1 \otimes g_2$ to the meromorphic function $g_1, g_2$.

\[ \square \]

**Proposition 3.1.6.**— We have the following isomorphisms of complexes of $\mathcal{D}_X[\ast Y]$-modules:

(a) $\mathcal{D}_X[\ast Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^* \cong \mathcal{S}p^*[\ast Y]$.

(b) $\mathcal{D}_X[\ast Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^*(\log Y) \cong \mathcal{S}p^*[\ast Y]$.

**Proof:** (a) As $\mathcal{S}p^*$ is a subcomplex of $\mathcal{D}_X$-modules of $\mathcal{S}p^*[\ast Y]$, and $\mathcal{D}_X[\ast Y]$ is flat over $\mathcal{O}_X \otimes \mathcal{V}_0^Y(\mathcal{D}_X)$, the complex $\mathcal{D}_X[\ast Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^*$ is a subcomplex of $\mathcal{D}_X[\ast Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^*[\ast Y]$, (see lemma 3.1.5, 1.). But, by the third isomorphism of lemma 3.1.5, this complex is the same as $\mathcal{S}p^*[\ast Y]$. Hence, we have an injective morphism of complexes:

\[
\mathcal{D}_X[\ast Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^* \rightarrow \mathcal{S}p^*[\ast Y],
\]

defined locally in each degree by: $P \otimes Q \otimes \delta_1 \wedge \cdots \wedge \delta_p \mapsto P Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p)$. This morphism is clearly surjective and, consequently, an isomorphism.

(b) We consider $\mathcal{V}_0^Y(\mathcal{D}_X)$ as a subsheaf of $\mathcal{O}_X$-modules of $\mathcal{D}_X$. Using the fact that $\mathcal{O}_X \otimes \mathcal{O}_X \mathcal{D}er(\log Y)$ is $\mathcal{O}_X$-free, we have an inclusion

\[
\mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X \mathcal{D}er(\log Y) \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}er(\log Y).
\]

On the other hand, as $Y$ is free, we have a natural injective morphism from $\mathcal{O}_X \mathcal{D}er(\log Y)$ to $\mathcal{O}_X \mathcal{D}er_\mathbb{C}(\mathcal{O}_X)$ (cf. [2, AIII 88, Cor.]). As $\mathcal{D}_X$ is flat over $\mathcal{O}_X$, we have other inclusion:

\[
\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}er(\log Y) \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}er_\mathbb{C}(\mathcal{O}_X) \ (p \geq 0).
\]

Composing both of them, we obtain a new inclusion:

\[
\mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{D}er(\log Y) \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}er_\mathbb{C}(\mathcal{O}_X),
\]

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for \( p = 0, \ldots, n \). These inclusions give rise to an injective morphism of complexes of \( \mathcal{V}^Y_0(\mathcal{D}_X) \)-modules

\[
\mathcal{S}^p(\log Y) \hookrightarrow \mathcal{S}^p.*
\]

As \( \mathcal{D}_X[\ast Y] \) is flat over \( \mathcal{V}^Y_0(\mathcal{D}_X) \) (see lemma \[3.1.3\], 1.) we have an injective morphism of complexes of \( \mathcal{D}_X[\ast Y] \)-modules:

\[
\theta : \mathcal{D}_X[\ast Y] \otimes \mathcal{V}^Y_0(\mathcal{D}_X) \mathcal{S}^p(\log Y) \hookrightarrow \mathcal{D}_X[\ast Y] \otimes \mathcal{V}^Y_0(\mathcal{D}_X) \mathcal{S}^p. *
\]

defined by: \( \theta(P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p) \). This morphism is surjective, given \( P \) local section of \( \mathcal{D}_X[\ast Y] \), \( Q \) in \( \mathcal{D} \) and \( \delta_1, \cdots, \delta_n \) in \( \text{Der}_C(\mathcal{O}) \), we have:

\[
P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p) = \theta(P f^{-k}) \otimes Q' \otimes (f \delta_1 \wedge \cdots \wedge f \delta_p),
\]

with \( k > 0 \) and \( Q' \) a local section of \( \mathcal{V}^Y_0(\mathcal{D}_X) \) verifying \( f^k Q = Q' f^p \) (see lemma \[3.1.3\]). Composing \( \theta' \) with the isomorphism of (a), we obtain the isomorphism:

\[
\theta : \mathcal{D}_X[\ast Y] \otimes \mathcal{V}^Y_0(\mathcal{D}_X) \mathcal{S}^p(\log Y) \xrightarrow{\sim} \mathcal{S}^p(\ast Y),
\]

with local expression: \( \theta(P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = PQ \otimes (\delta_1 \wedge \cdots \wedge \delta_p) \). \( \square \)

### 3.2 The Logarithmic de Rham Complex

For each divisor \( Y \), we have a standard canonical isomorphism:

\[
\mathcal{H}om_{\mathcal{O}_X} \left( \overset{p}{\wedge} \text{Der}(\log Y), \mathcal{O}_X \right) \cong \mathcal{H}om_{\mathcal{V}^Y_0(\mathcal{D}_X)} \left( \mathcal{V}^Y_0(\mathcal{D}_X) \otimes \mathcal{O}_X, \overset{p}{\wedge} \text{Der}(\log Y), \mathcal{O}_X \right),
\]

defined by: \( \lambda^p(\alpha)(P \otimes \delta_1 \wedge \cdots \wedge \delta_p) = P (\alpha(\delta_1 \wedge \cdots \wedge \delta_p)) \).

Composing this isomorphism with the isomorphism \( \gamma^p \) defined in section \[1.1\], we can construct a natural morphism \( \psi^p = \lambda^p \circ \gamma^p : \)

\[
\Omega^p_X(\log Y) \overset{\psi^p}{\cong} \mathcal{H}om_{\mathcal{V}^Y_0(\mathcal{D}_X)} \left( \mathcal{V}^Y_0(\mathcal{D}_X) \otimes \overset{p}{\wedge} \text{Der}(\log Y), \mathcal{O}_X \right),
\]

for \( p = 0, \cdots, n \). Locally:

\[
\psi^p(\omega_1 \wedge \cdots \wedge \omega_p)(P \otimes \delta_1 \wedge \cdots \wedge \delta_p) = P (\det(\langle \omega_i, \delta_j \rangle)_{1 \leq i,j \leq p}).
\]

with \( \omega_i \ (i = 1, \cdots, n) \) local sections of \( \Omega^1_X(\log Y) \) and \( P \) a logarithmic operator.

Similarly, if \( \mathcal{M} \) is a left \( \mathcal{V}^Y_0(\mathcal{D}_X) \)-module, given an integer \( p \in \{1, \cdots, n\} \), there exist the following canonical isomorphisms:

\[
\gamma^p_{\mathcal{M}} : \Omega^p_X(\log Y) \otimes_{\mathcal{O}_X} \mathcal{M} \overset{\sim}{\rightarrow} \mathcal{H}om_{\mathcal{O}_X} \left( \overset{p}{\wedge} \text{Der}(\log Y), \mathcal{M}_X \right),
\]
\[ \lambda^p_M : \mathcal{H}om_{\mathcal{O}_X} \left( \hat{\kappa} \, \text{Der}(\log Y), \mathcal{M} \right) \twoheadrightarrow \mathcal{H}om_{\mathcal{V}_0^Y} \left( \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \hat{\kappa} \, \text{Der}(\log Y), \mathcal{M} \right), \]

\[ \psi^p_M = \lambda^p_M \circ \gamma^p_M : \Omega^p_X(\log Y)(\mathcal{M}) \twoheadrightarrow \mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)} \left( \mathcal{V}_0^Y(\mathcal{D}_X) \otimes \hat{\kappa} \, \text{Der}(\log Y), \mathcal{M} \right). \]

Locally:

\[ \psi^p_M(\omega_1 \wedge \cdots \wedge \omega_p \otimes m)(P \otimes \delta_1 \wedge \cdots \wedge \delta_p) = P \cdot \det(\langle \omega_i, \delta_j \rangle)_{1 \leq i, j \leq p} \cdot m. \]

**Theorem 3.2.1.** If \( \mathcal{M} \) is a left \( \mathcal{V}_0^Y(\mathcal{D}_X) \)-module (or, equivalently, is a \( \mathcal{O}_X \)-module with an integrable logarithmic connection), the complexes of sheaves of \( \mathbb{C} \)-vector spaces \( \Omega^p_X(\log Y)(\mathcal{M}) \) and \( \mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}^p(\log Y), \mathcal{M}) \) are canonically isomorphic.

**Proof:** The general case is solved if we prove the case \( \mathcal{M} = \mathcal{V}_0^Y(\mathcal{D}_X) \), using the isomorphisms:

\[ \Omega^p_X(\log Y)(\mathcal{M}) \cong \Omega^p_X(\log Y)(\mathcal{V}_0^Y(\mathcal{D}_X)) \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{M}, \]

\[ \mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}^p(\log Y), \mathcal{M}) \cong \mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}^p(\log Y), \mathcal{V}_0^Y(\mathcal{D}_X)) \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{M}. \]

For \( \mathcal{M} = \mathcal{V}_0^Y(\mathcal{D}_X) \), we obtain the right \( \mathcal{V}_0^Y(\mathcal{D}_X) \)-isomorphisms

\[ \phi^p = \psi^p_{\mathcal{V}_0^Y(\mathcal{D}_X)} : \Omega^p_X(\log Y)(\mathcal{V}_0^Y(\mathcal{D}_X)) \rightarrow \mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}^p(\log Y), \mathcal{V}_0^Y(\mathcal{D}_X)), \]

whose local expression are:

\[ \phi^p(\omega_1 \wedge \cdots \wedge \omega_p \otimes Q)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = P \cdot \det(\langle \omega_i, \delta_j \rangle) \cdot Q. \]

To prove that these isomorphisms produce a isomorphism of complexes we have to check that they commute with the differential of the complex. Thanks to the isomorphism (b) of the proposition [3.1.0],

\[ \mathcal{D}_X[\ast Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}^p(\log Y) \simeq \mathcal{S}^p(\log Y) \simeq \mathcal{V}_0^Y(\mathcal{D}_X[\ast Y]), \]

we obtain a natural morphism of complexes of sheaves of right \( \mathcal{V}_0^Y(\mathcal{D}_X) \)-modules:

\[ \tau^\bullet : \mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}^p(\log Y), \mathcal{V}_0^Y(\mathcal{D}_X)) \rightarrow \mathcal{H}om_{\mathcal{D}_X[\ast Y]}(\mathcal{S}^p(\log Y), \mathcal{D}_X[\ast Y]), \]

locally defined by:

\[ \tau^p(\alpha)(R \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = f^{-k} \alpha(R \otimes (f \delta_1 \wedge \cdots \wedge f \delta_p)), \]

(for any local sections \( \alpha \) of \( \mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}^p(\log Y), \mathcal{V}_0^Y(\mathcal{D}_X)) \), \( R \) of \( \mathcal{D}_X[\ast Y] \) and \( \delta_1, \ldots, \delta_p \) of \( \text{Der}_\mathbb{C}(\mathcal{O}_X) \)), where \( P \) is a local section of \( \mathcal{V}_0^Y(\mathcal{D}_X) \) such that \( Rf^{-p} = f^{-k}P \) (see lemma [3.1.3]). The morphisms \( \tau^i \) are injective, because:

\[ \alpha(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \tau^i(\alpha)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)). \]
Let us see the following diagram commutes:

$$\Omega^p_X(\log Y)(V_0^\vee(D_X)) \xrightarrow{j^p} \Omega^p_X[*Y](D_X[*Y])$$

$$\downarrow \phi^p \quad \# \quad \downarrow \Phi^p$$

$$\mathcal{H}om_{V_0^\vee(D_X)}(S_p^p(\log Y), V_0^\vee(D_X)) \xrightarrow{\tau^p} \mathcal{H}om_{D_X[*Y]}(S_p^p[*Y], D_X[*Y])$$

for each $p \geq 0$, where the $\Phi^p$ are the isomorphisms:

$$\Phi^p : \Omega^p_X[*Y](D_X[*Y]) \rightarrow \mathcal{H}om_{D_X[*Y]}\left(D_X[*Y] \otimes \mathcal{P}er_C(\mathcal{O}_X), D_X[*Y]\right),$$

$$\Phi^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q) (P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = P \cdot \det((\omega_i \cdot \delta_j)_{1 \leq i,j \leq p}) \cdot Q.$$

Given $\omega_1, \ldots, \omega_p$ local sections of $\Omega^1_X(\log Y)$, $Q$ and $R$ local sections of $D_X[*Y]$ and $\delta_1, \ldots, \delta_p$ local sections of $\mathcal{P}er_C(\mathcal{O}_X)$, we have

$$(\tau^p \circ \phi^p)((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)[R \otimes (\delta_1 \cdots \delta_p)] =$$

$$f^{-k} \phi^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)[P \otimes (f \delta_1 \wedge \cdots \wedge f \delta_p)] =$$

$$f^{-k} P \cdot \det(\langle \omega_i f \delta_j \rangle) \cdot Q = R \cdot f^{-p} \cdot \det(\langle \omega_i f \delta_j \rangle) \cdot Q = R \cdot \det(\langle \omega_i \delta_j \rangle) \cdot Q =$$

$$\Phi^p \circ j^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)[R \otimes (\delta_1 \cdots \delta_p)],$$

with $P$ a local section of $\mathcal{V}^\vee_0(D_X)$ such that $Rf^{-p} = f^{-k} P$.

But $\Phi^\bullet$, $j^\bullet$ and $\tau^\bullet$ are morphisms of complexes, and $\tau^\bullet$ is injective, hence we deduce that the $\phi^p$ commute with the differential and so define a isomorphism of complexes:

$$\phi^\bullet : \Omega^\bullet_X(\log Y)\left(V_0^\vee(D_X)\right) \rightarrow \mathcal{H}om_{V_0^\vee(D_X)}\left(S_p^\bullet(\log Y), V_0^\vee(D_X)\right),$$

as we wanted to prove.

\[\square\]

**Corollary 3.2.2.**— There exists a canonical isomorphism in the derived category:

$$\Omega^\bullet_X(\log Y)(\mathcal{M}) \cong R\mathcal{H}om_{V_0^\vee(D_X)}(\mathcal{O}_X, \mathcal{M}).$$

**Proof:** By theorem 3.1.2, the complex $S_p^\bullet(\log Y)$ is a locally free resolution of $\mathcal{O}_X$ as left $V_0^\vee(D_X)$-module. So, we have only to apply the theorem 3.2.1. \[\square\]

**Remark 3.2.3.**— In the specific case that $\mathcal{M} = \mathcal{O}_X$, we have that the complexes $\Omega^\bullet_X(\log Y)$ and $\mathcal{H}om_{V_0^\vee(D_X)}(S_p^\bullet(\log Y), \mathcal{O}_X)$ are canonically isomorphic and so, there exists a canonical isomorphism:

$$\Omega^\bullet_X(\log Y) \cong R\mathcal{H}om_{V_0^\vee(D_X)}(\mathcal{O}_X, \mathcal{O}_X).$$
Remark 3.2.4.– A classical problem is the comparison between the logarithmic de Rham complex and the meromorphic de Rham complex relative to a divisor $Y$,

$$\Omega^*_X[Y] \cong \mathcal{R}\text{Hom}_{\mathcal{D}X}(\mathcal{O}_X, \mathcal{O}_X[Y]) \cong \mathcal{R}\text{Hom}_{\mathcal{O}_Y^*(\mathcal{D}X)}(\mathcal{O}_X, \mathcal{O}_X[Y]).$$

If $Y$ is a normal crossing divisor, an easy calculation shows that they are quasi-isomorphic (cf. [6]). The same result is true if $Y$ is a strongly weighted homogeneous free divisor [5]. As a consequence of theorem 2.1.4, if $Y$ is an arbitrary free divisor, the meromorphic de Rham complex and the logarithmic de Rham complex are quasi-isomorphic if and only if:

$$0 = \mathcal{R}\text{Hom}_{\mathcal{D}X}(\mathcal{D}X \otimes \mathcal{V}_Y(\mathcal{D}X) \otimes \mathcal{O}_X, \mathcal{O}_X[Y]),$$

4 Perversity of the logarithmic complex

Now we consider the complex $\mathcal{D}X \otimes \mathcal{V}_Y(\mathcal{D}X) \mathcal{S}^{p,*}(\log Y)$:

$$0 \to \mathcal{D}X \otimes \mathcal{O}_X \xrightarrow{\varepsilon_1} \cdots \xrightarrow{\varepsilon_{n-1}} \mathcal{D}X \otimes \mathcal{S}^{1} \xrightarrow{\varepsilon_{n}} \mathcal{D}X,$$

where the local expressions of the morphisms are defined by:

$$\varepsilon_p(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \wedge \hat{\delta}_i \wedge \cdots \wedge \delta_p) + \sum_{1 \leq i < j \leq p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \hat{\delta}_i \wedge \cdots \wedge \hat{\delta}_j \wedge \cdots \wedge \delta_p), \quad (2 \leq p \leq n).$$

$$\varepsilon_{-1}(P \otimes \delta) = P\delta.$$

In the case that $Y$ is a free divisor, we can work at each point $x$ of $Y$ with a basis $\{\delta_1, \cdots, \delta_n\}$ of $\text{Der}(\log f)$, with $f$ a local reduced equation of $Y$ at $x$. 

**Proposition 4.0.5.–** If $\{\delta_1, \cdots, \delta_n\}$ is a basis of $\text{Der}(\log f)$, and the sequence $\{\sigma(\delta_1), \cdots, \sigma(\delta_n)\}$ is $\text{Gr}_{F^*}(\mathcal{D})$-regular, it verifies

$$\sigma(\mathcal{D}(\delta_1, \cdots, \delta_n)) = \text{Gr}_{F^*}(\mathcal{D})(\sigma(\delta_1), \cdots, \sigma(\delta_n)).$$

**Proof:** The inclusion $\text{Gr}_{F^*}(\mathcal{D})(\sigma(\delta_1), \cdots, \sigma(\delta_n)) \subset \sigma(\mathcal{D}(\delta_1, \cdots, \delta_n))$ is clair.

Let $F$ be the symbol of an operator $P$ of order $d$, with

$$P = \sum_{i=1}^{n} P_i \delta_i \in \mathcal{D}(\delta_1, \cdots, \delta_n).$$
We will prove by induction that $F = \sigma(P)$ belongs to $\text{Gr}_{F^*}(\mathcal{D})(\sigma_1, \cdots, \sigma_n)$, with $\sigma_i = \sigma(\delta_i)$. We will do the induction on the maximum order of the $P_i$ ($i = 1, \cdots, n$), order that we will denote by $k_0$. As $P$ has order $d$, $k_0$ is greater or equal to $d - 1$. If $k_0 = d - 1$, we have:

$$\sigma(P) = \sum_{i \in K} \sigma(P_i)\sigma_i,$$

with $K$ the set of subindexes $j$ such that $P_j$ has order $k_0$ in $\mathcal{D}$. We suppose that the result holds when $d - 1 \leq k_0 < m$. Let $F = \sigma(P)$, with $P = \sum_{i=1}^{n} P_i \delta_i$ and $k_0 = m$. There are two possibilities:

1. $F = \sigma(P) = \sum_{i \in K} \sigma(P_i)\sigma_i \in \text{Gr}_{F^*}(\mathcal{D})(\sigma_1, \cdots, \sigma_n)$, as we wanted to prove.

2. $\sum_{i \in K} \sigma(P_i)\sigma_i = 0$.

In this last case, as $\{\sigma_1, \cdots, \sigma_n\}$ is a $\text{Gr}_{F^*}(\mathcal{D})$-regular sequence, if we call $F_i$ the symbol $\sigma(P_i)$ in the case that $i \in K$ and 0 otherwise, we have:

$$(F_1, \cdots, F_n) = \sum_{i < j} F_{ij}(0, \cdots, 0, \sigma_j, 0, \cdots, 0, -\sigma_i, 0, \cdots, 0),$$

with $F_{ij} \in \text{Gr}_{F^*}(\mathcal{D})$ homogeneous polynomials of order $m - 1$. We choose, for $1 \leq i < j \leq n$, operators $Q_{ij}$, of order $m - 1$ in $\mathcal{D}$, such that $\sigma(Q_{ij}) = F_{ij}$, and define:

$$(Q_1, \cdots, Q_n) = (P_1, \cdots, P_n) - \sum_{i < j} Q_{ij} \left( (0, \cdots, 0, \sigma_j, 0, \cdots, 0, -\sigma_i, 0, \cdots, 0) - \alpha_{ij} \right),$$

where $\alpha_{ij}$ are the vectors with $n$ coordinates in $\mathcal{O}$ defined by the relations:

$$[\delta_i, \delta_j] = \sum_{k=1}^{n} a_{ij}^k \delta_k = \alpha_{ij} \cdot \delta,$$

with $\delta = (\delta_1, \cdots, \delta_n)$. These $Q_i$, of order $m$ in $\mathcal{D}$, verify

$$(\sigma_m(Q_1), \cdots, \sigma_m(Q_n)) =
(F_1, \cdots, F_n) - \sum_{i < j} F_{ij}(0, \cdots, 0, \sigma_j, 0, \cdots, 0, -\sigma_i, 0, \cdots, 0) = 0.$$

So, $Q_i$ has order $m - 1$ in $\mathcal{D}$. Moreover,

$$\sum_{i=1}^{n} Q_i \delta_i = \sum_{i=1}^{n} P_i \delta_i - \sum_{i < j} Q_{ij} (\delta_i \delta_j - \delta_j \delta_i - [\delta_i, \delta_j]) = \sum_{i=1}^{n} P_i \delta_i = P.$$
We apply the induction hypothesis to \( F = \sigma(P) \), with 
\[
P = \sum_{i=1}^{n} Q_i \delta_i,
\]
and obtain:
\[
\sigma(P) \in \text{Gr}_F^*(D)(\sigma_1, \cdots, \sigma_n).
\]
\[\blacksquare\]

**Proposition 4.0.6.**— Let \( \{\delta_1, \cdots, \delta_n\} \) be a basis of Der(log f). If the sequence \( \sigma(\delta_1), \cdots, \sigma(\delta_n) \) is a Gr\(_F^*\)(D)-regular sequence in Gr\(_F^*\)(D), the complex \( D \otimes_{\mathcal{O}}^\delta \mathcal{S}p^*(\log f) \) is a resolution of the quotient module \( D(D_{(\delta_1, \cdots, \delta_n)}) \).

**Proof:** We consider the complex \( D \otimes_{\mathcal{O}}^\delta \mathcal{S}p^*(\log f) \). We can augment this complex of \( D \)-modules by another morphism:
\[
\varepsilon_0 : D \to \frac{D}{D(\delta_1, \cdots, \delta_n)}, \quad \varepsilon_0(P) = P + D(\delta_1, \cdots, \delta_n).
\]
We denote by \( D \otimes_{\mathcal{O}}^\delta \mathcal{S}p^*(\log f) \) the new complex. To prove that this new complex is exact, we define a discrete filtration \( G^* \) such that the graded complex be exact (cf. [1, lemma 3.16]):
\[
G^k \left( D \otimes_{\mathcal{O}}^\delta \text{Der}(\log f) \right) = F^{k-p} \left( D \otimes_{\mathcal{O}}^\delta \text{Der}(\log f) \right),
\]
\[
G^k \left( \frac{D}{D(\delta_1, \cdots, \delta_n)} \right) = \frac{F^k(D) + D \cdot (\delta_1, \cdots, \delta_n)}{D(\delta_1, \cdots, \delta_n)}.
\]
Clairly the filtration is compatible with the differential of the complex. Moreover:
\[
\text{Gr}_{G^*} \left( D \otimes_{\mathcal{O}}^\delta \text{Der}(\log f) \right) = \text{Gr}_{F^*}^\circ(D)[-p] \otimes_{\mathcal{O}}^\delta \text{Der}(\log f),
\]
and, by the previous proposition,
\[
\text{Gr}_{G^*} \left( \frac{D}{D(\delta_1, \cdots, \delta_n)} \right) = \frac{\text{Gr}_{F^*}(D)}{\sigma(D \cdot (\delta_1, \cdots, \delta_n))} = \frac{\text{Gr}_{F^*}(D)}{\text{Gr}_{F^*}(D) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n))}.
\]
We consider the complex \( \text{Gr}_{G^*} \left( D \otimes_{\mathcal{O}}^\delta \mathcal{S}p^*(\log f) \right) \):
\[
0 \to \text{Gr}_{F^*}(D)[-n] \otimes_{\mathcal{O}}^\delta \text{Der}(\log f) \xrightarrow{\psi} \cdots \xrightarrow{\psi^2} \text{Gr}_{F^*}(D)[-1] \otimes_{\mathcal{O}}^\delta \text{Der}(\log f) \xrightarrow{\psi^1} \text{Gr}_{F^*}(D) \xrightarrow{\psi_0} \frac{\text{Gr}_{F^*}(D)}{\text{Gr}_{F^*}(D) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n))} \to 0,
\]
where the local expression of the differential is defined by:
\[
\psi_p(G \otimes_{\mathcal{O}}^\delta \delta_{j_1} \wedge \cdots \wedge \delta_{j_p}) = \sum_{i=1}^{p} (-1)^{i-1} G\sigma(\delta_{j_i}) \otimes_{\mathcal{O}}^\delta \delta_{j_1} \wedge \cdots \wedge \delta_{j_i} \wedge \cdots \wedge \delta_{j_p}, \quad (2 \leq p \leq n),
\]
\[ \psi_{-1}(G \otimes \delta_i) = G\sigma(\delta_i), \]
\[ \psi_0(G) = G + Gr(F) \cdot (\sigma(\delta_1), \ldots, \sigma(\delta_n)). \]

This complex is the Koszul complex of the ring \( Gr(F) \) with respect to the sequence \( \sigma(\delta_1), \ldots, \sigma(\delta_n) \). So we deduce that, if the sequence \( \sigma(\delta_1), \ldots, \sigma(\delta_n) \) is \( Gr(F) \)-regular in \( Gr(F) \), the complex \( Gr(G) \left( D \otimes V_f \theta \left( D \right) S p \left( \log f \right) \right) \) is exact. So, the complex \( D \otimes V_f \theta \left( D \right) S p \left( \log f \right) \) is exact too, and \( D \otimes V_f \theta \left( D \right) S p \left( \log f \right) \) is a resolution of \( \mathcal{D} \).

**Corollary 4.0.7.**— Let \( Y \) be a free divisor. With the conditions of the previous proposition (for each point \( x \) of \( Y \), there exists a basis \( \left\{ \delta_1, \ldots, \delta_n \right\} \) of \( \text{Der}(\log f) \) such that the sequence \( \sigma(\delta_1), \ldots, \sigma(\delta_n) \) is a \( Gr(F) \)-regular sequence), the sheaf \( \Omega_X \left( \log Y \right) \) is a perverse sheaf.

**Proof:** With the same conditions of the previous proposition, the homology of the complex \( D \otimes Y_f \theta \left( D \right) S p \left( \log Y \right) \) is concentrated in degree 0. All its homology groups are zero except the group in degree 0, which verifies:

\[ h^0 \left( D \otimes Y_f \theta \left( D \right) S p \left( \log Y \right) \right) = \frac{D_X}{D_X \cdot \text{Der}(\log Y)} = \frac{D_X}{D_X \cdot (\delta_1, \ldots, \delta_n)} = \mathcal{E}, \]

where \( \left\{ \delta_1, \ldots, \delta_n \right\} \) is a local basis of \( \text{Der}(\log Y) \). But \( \mathcal{E} \) is a holonomic \( D_X \)-module because:

\[ Gr(F) \left( \mathcal{E} \right) = \frac{Gr(F) \left( D_X \right)}{(\sigma(\delta_1), \ldots, \sigma(\delta_n))} \]

has dimension \( n \) (using the fact that \( \sigma(\delta_1), \ldots, \sigma(\delta_n) \) is a \( Gr(F) \)-regular sequence). So (using remark [3.2.3] for the first equality and theorem [3.1.2] for the last equality):

\[ \Omega_X \left( \log Y \right) = R \text{Hom}_{D_X} \left( D_X \otimes Y_f \left( D_X \right) \mathcal{O}_X, \mathcal{O}_X \right) = R \text{Hom}_{D_X} \left( D_X \otimes Y_f \left( D_X \right) S p \left( \log Y \right), \mathcal{O}_X \right) = R \text{Hom}_{D_X} \left( \frac{D_X}{D_X \left( \delta_1, \ldots, \delta_n \right)}, \mathcal{O}_X \right) \]

is a perverse sheaf (as solution of a holonomic \( D_X \)-module, cf. [11]).

**Corollary 4.0.8.**— Let \( Y \) be any divisor in \( X \), with \( \text{dim}_C X = 2 \). Then \( \Omega_X \left( \log Y \right) \) is a perverse sheaf.

**Proof:** We know that, if \( \text{dim}_C X = 2 \), any divisor \( Y \) in \( X \) is free [14]. So, we have only to check that the other hypothesis of the previous corollary
holds. We consider the symbols \( \{\sigma_1, \sigma_2\} \) of a basis \( \{\delta_1, \delta_2\} \) of \( \text{Der}(\log f) \), where \( f \) is a reduced equation of \( Y \). We have to see that they form a \( \text{Gr}_{F^*}(D) \)-regular sequence. If they do not, they have a common factor \( g \in \mathcal{O} \), because they are symbols of operators of order 1. If \( g \) is a unit, we divide one of them by \( g \) and eliminate the common factor. If \( g \) is not a unit, it would be in contradiction with Saito’s Criterion, because the determinant of the coefficients of the basis \( \{\delta_1, \delta_2\} \) would have as factor \( g^2 \), with \( g \) not invertible, and this determinant has to be equal to \( f \) multiplied by a unit.

\[ \square \]

**Remark 4.0.9.**— The regularity of the sequence of the symbols of a basis of \( \text{Der}(\log f) \) in \( \text{Gr}_{F^*}(D) \) is not necessary for the perversity of the logarithmic de Rham complex. For example, if \( X = \mathbb{C}^3 \) and \( Y \equiv \{ f = 0 \} \), with \( f = xy(x + y)(y + tx) \), \( f \) is a free divisor such that the graded complex

\[ \mathcal{G}_{\mathcal{T}^*}(D_X \otimes_{Y_0^*}(D_X) S^p*(\log Y)) = K(\sigma(\delta_1), \sigma(\delta_1), \sigma(\delta_3); \mathcal{G}_{F^*}(D_X)) \]

is not concentrated in degree 0, but the complex

\[ D_X \otimes_{Y_0^*}(D_X) S^p*(\log Y) \]

is. Moreover, in this case the dimension of \( S_{D_X(\delta_1, \delta_2, \delta_3)}^{D_X} \) is 3 and so, \( \Omega^*_{X}(\log Y) \) is a perverse sheaf.

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