Analytical Solutions to the Navier-Stokes-Poisson Equations with Density-dependent Viscosity and with Pressure

Ling Hei Yeung*

Manwai Yuen†

Department of Applied Mathematics, The Hong Kong Polytechnic University,
Hung Hom, Kowloon, Hong Kong

Revised 05-May-2010

Abstract

We study some particular solutions to the Navier-Stokes-Poisson equations with density-dependent viscosity and with pressure, in radial symmetry. With extension of the previous known blowup solutions for the Euler-Poisson equations / pressureless Navier-Stokes-Poisson with density-dependent viscosity, we constructed the corresponding analytical blowup solutions for the Navier-Stokes-Poisson Equations with density-dependent viscosity and with pressure.

Key Words: Fluids, Navier-Stokes-Poisson Equations, Density-dependent Viscosity, With Pressure, Blowup, Free Boundary, Self-similar Solutions, Global Solutions, Gaseous stars, Semi-conductor models

---

*E-mail address: lightisgood2005@yahoo.com.hk
†E-mail address: nevetsayuen@hotmail.com
1 Introduction

The evolution of a self-gravitating fluid can be formulated by the isentropic Euler-Poisson equations of the following form:

\[
\begin{cases}
\rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\
\rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \nabla P = -\delta \rho \nabla \Phi + \text{vis}(\rho, \vec{u}), \\
\Delta \Phi(t, x) = \alpha(N) \rho - \Lambda,
\end{cases}
\]

where \( \alpha(N) \) is a constant related to the unit ball in \( \mathbb{R}^N \): \( \alpha(1) = 2; \alpha(2) = 2\pi \) and for \( N \geq 3 \),

\[
\alpha(N) = N(N-2)Vol(N) = N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2 + 1)},
\]

where \( Vol(N) \) is the volume of the unit ball in \( \mathbb{R}^N \) and \( \Gamma \) is a Gamma function. And as usual, \( \rho = \rho(t, \vec{x}) \) and \( \vec{u} = \vec{u}(t, \vec{x}) \in \mathbb{R}^N \) are the density and the velocity respectively. \( P = P(\rho) \) is the pressure. And \( \Lambda \) is the background constant.

When \( \delta = 1 \), the system can model fluids that are self-gravitating, such as gaseous stars. In addition, the evolution of the simple cosmology can be modelled by the dust distribution without pressure term. This describes the stellar systems of collisionless and gravitational \( n \)-body systems. And the pressureless Euler-Poisson equations can be derived from the Vlasov-Poisson-Boltzmann model with the zero mean free path. For \( N = 3 \), the equations are the classical (non-relativistic) descriptions of a galaxy in astrophysics. See [2] and [3], for details about the systems.

When \( \delta = -1 \), the system is the compressible Euler-Poisson equations with repulsive forces. The equation \( (1)_{3} \) is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons. In this case, the system can be viewed as a semiconductor model. See [5] and [6] for detailed analysis of the system.

When \( \delta = 0 \), the potential forces are ignored. The system is called Euler / Navier-Stokes equations. See [4], [16] and [17] for detailed analysis of the system.

In the above system, the self-gravitational potential field \( \Phi = \Phi(t, x) \) is determined by the density \( \rho \) through the Poisson equation.
And $vis(\rho, \vec{u})$ is the viscosity function:

$$vis(\rho, \vec{u}) := \nabla(\mu(\rho) \nabla \cdot \vec{u}).$$

(3)

In this article, we seek the radial solutions

$$\rho(t, \vec{x}) = \rho(t, r) \text{ and } \vec{u} = \frac{\vec{x}}{r} V(t, r) =: \frac{\vec{x}}{r} V,$$

(4)

with $r = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$. By the standard computation, the Euler-Poisson equations in radial symmetry can be written in the following form:

$$\begin{cases}
\rho_t + V \rho_r + \rho V_r + \frac{N-1}{r} \rho V = 0, \\
\rho (V_t + V V_r) + P_r(\rho) + \delta \rho \Phi_r(\rho) = \mu_r(\rho) \left(\frac{N-1}{r} V + V_r\right) + \mu(\rho) \left[-\left(\frac{N-1}{r}\right) V + \frac{N-1}{r} V_r + V_{rr}\right].
\end{cases}$$

(5)

Under a common assumption for, the viscosity function can be defined:

$$\mu(\rho) := \kappa \rho^\theta,$$

(6)

where $\kappa$ and $\theta \geq 0$ are the constants. For the study of the above system, the readers may refer [13], [16], [18] and [21]. In particular, when $\theta = 0$, that returns the expression for the only $V$-dependent viscosity function:

$$\begin{cases}
\rho_t + V \rho_r + \rho V_r + \frac{N-1}{r} \rho V = 0, \\
\rho (V_t + V V_r) + P_r(\rho) + \delta \rho \Phi_r(\rho) = \kappa \left[V_{rr} + \frac{N-1}{r} V_r - \left(\frac{N-1}{r}\right) V\right],
\end{cases}$$

(7)

which are the usual form of the Navier-Stokes-Poisson equations. The equations (1) and (2) ($vis(\rho, u) \neq 0$) are the compressible Navier-Stokes equations with forcing term. The equation (1) is the Poisson equation through which the gravitational potential is determined by the density distribution of the density itself. Thus, the system (1) is called the Navier-Stokes-Poisson equations.

Here, if the $vis(\rho, u) = 0$, the system is called the Euler-Poisson equations. In this case, the equations can be viewed as a prefect gas model. For $N = 3$, (1) is a classical (nonrelativistic) description of a galaxy, in astrophysics. See [3], [15] for details about the system.

$P = P(\rho)$ is the pressure. The $\gamma$-law can be applied on the pressure $P(\rho)$, i.e.

$$P(\rho) = K \rho^\gamma := \frac{\rho^\gamma}{\gamma},$$

(8)
which is a common hypothesis. The constant $\gamma = c_P/c_v \geq 1$, where $c_P$, $c_v$ are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in (5). With $K = 0$, we call that the system is pressureless.

For the local existence of the Euler-Poisson equations, the interested reader may refer the results of Makino [14], Gambin [9] and Bezard [1].

In the following, we always seek solutions in radial symmetry. Thus, the Poisson equation (1) is transformed to

$$r^{N-1} \Phi_{rr}(t, x) + (N - 1) r^{N-2} \Phi_r = \alpha(N) r^{N-1},$$

$$\Phi_r = \frac{\alpha(N)}{r^{N-1}} \int_0^r (\rho(t, s) - \Lambda) s^{N-1} ds. \tag{10}$$

In this paper, we concern about blowup solutions for the $N$-dimensional pressureless Navier-Stokes-Poisson equations with the density-dependent viscosity. And our aim is to construct a family of blowup solutions.

Historically in astrophysics, Goldreich and Weber [11] constructed the analytical blowup (collapsing) solutions of the 3-dimensional Euler-Poisson equations for $\gamma = 4/3$ and for the non-rotating gas spheres. After that, Makino [15] obtained the rigorously mathematical proof of the existence of such kind of blowup solutions. And in [7], Deng, Xiang and Yang extended the above blowup solutions in $\mathbb{R}^N$ $(N \geq 3)$. Recently, Yuen obtained the blowup solutions in $\mathbb{R}^2$ with $\gamma = 1$ in [21]. The family of the analytical solutions are rewritten as:

for $N \geq 3$ and $\gamma = (2N - 2)/N$, in [7]:

$$\rho(t, r) = \begin{cases}
\frac{1}{a(t)^N} \dot{y}(\frac{r}{a(t)})^{N/(N-2)}, & \text{for } r < a(t)Z_\mu; \\
0, & \text{for } a(t)Z_\mu \leq r.
\end{cases}$$

$$\dot{a}(t) = -\lambda a(t)^{N-1}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1,$$

$$\dot{y}(z) + \frac{N-1}{z} \ddot{y}(z) + \frac{\alpha(N)}{(2N-2)K} y(z)^{N/(N-2)} = \mu, \quad y(0) = \alpha > 0, \quad \dot{y}(0) = 0,$$ \tag{11}

where $\mu = [N(N-2)\lambda]/(2N-2)K$ and the finite $Z_\mu$ is the first zero of $y(z)$;
for \( N = 2 \) and \( \gamma = 1 \), in [21]

\[
\begin{align*}
\rho(t, r) &= \frac{1}{a(t)^2} e^{y(r/a(t))}, \quad V(t, r) = \frac{\dot{a}(t)}{a(t)} r, \\
\dot{a}(t) &= \frac{\lambda}{a(t)} a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \\
\ddot{y}(z) + \frac{1}{z} \dot{y}(z) + \frac{2\pi}{K} e^{-y(z)/\lambda} &= \mu, \quad y(0) = \alpha, \quad \dot{y}(0) = 0,
end{align*}
\]

where \( K > 0, \mu = \frac{2\lambda}{K} \) with a sufficiently small \( \lambda \) and \( \alpha \) are constants.

For the construction of analytical solutions to the Euler, Navier-Stokes and pressureless Navier-Stokes-Poisson equations, readers may refer to the recent results in [20], [19], [23] and [24].

In this article, it is natural to extend the more general results to cover the full system (5) — Navier-Stokes-Poisson equations with density-dependent viscosity and with pressure. In short, we successfully deduce the system (5) into the much simpler ordinary differential equations.

**Theorem 1** For the Navier-Stokes-Poisson equations in \( \mathbb{R}^N (N \geq 2) \) with \( \gamma = \frac{2N-2}{N} \), (7) there exists a family of solutions, those are:

\[
\begin{align*}
\rho(t, r) &= \begin{cases}
\frac{f(z)}{a(t)^{2N}}, & \text{for } r < a(t) Z_{\mu}, \\
0, & \text{for } a(t) Z_{\mu} \leq r,
\end{cases} \\
V(t, r) &= \frac{\dot{a}(t)}{a(t)} r,
\end{align*}
\]

where \( f(z) \) and \( a(t) \) are the following functions and the finite \( Z_{\mu} \) is the first zero of \( f(z) \):

(1) with \( \lambda = 0 \), and

(1a) \( \theta = \frac{2N-3}{N} \),

\[
\begin{align*}
a(t) &= mt + n, \\
\gamma K f(z)^{\gamma-2} \dot{f}(z) - m \kappa \theta N f(z)^{\theta-2} \dot{f}(z) + \frac{\delta a(N)}{2N} \int_0^z f(s) s^{N-1} ds, \quad f(0) = \alpha > 0,
\end{align*}
\]

where \( m, n > 0 \) and \( \alpha \) are constant;

(1b) \( \theta = \frac{3N-4}{2N} \),

\[
\begin{align*}
a(t) &= (mt + n)^{2/N}, \\
\gamma K f(z)^{\gamma-2} \dot{f}(z) - 2m \kappa \theta f(z)^{\theta-2} \dot{f}(z) + \frac{\delta a(N)}{2N} \int_0^z f(s) s^{N-1} ds = \frac{2(N-2)m^2}{N^2} z, \quad f(0) = \alpha > 0.
\end{align*}
\]

In particular, for \( m < 0 \), the solutions (14) and (15) blow up at the finite time \( T = -m/n \).
(2) with \( \delta \Lambda > 0 \) and \( \theta = \frac{2N-3}{N} \), we have

\[
\begin{align*}
a(t) &= e^{\sqrt{\frac{2\alpha(N)}{N}} t}, \\
(K \gamma - N \delta \alpha(N) \Lambda) f(z)^{\theta-2} f(z) + \frac{\delta \alpha(N)}{N} f(z)^{\theta} f(s)s^{N-1} ds, & f(0) = \alpha, \ f(0) = \alpha > 0.
\end{align*}
\]

(16)

2 Separable Solutions

Before we present the proof of Theorem 1, Lemmas 3 and 12 of [22] are quoted here.

Lemma 2 (Lemma 3 of [22]) For the equation of conservation of mass in radial symmetry:

\[
\rho_t + V \rho_r + \rho V_r + \frac{N-1}{r} \rho V = 0,
\]

(17)

there exist solutions,

\[
\rho(t, r) = \frac{f(r/a(t))}{a(t)^N}, \ V(t, r) = \frac{\dot{a}(t)}{a(t)} r,
\]

(18)

with the form \( f \geq 0 \in C^1 \) and \( a(t) > 0 \in C^1 \).

It is clear to check our solutions to satisfy the Navier-Stokes-Poisson equations:

Proof. As we use the functional structure of the above lemma, our solutions (13) fit well to the mass equation (5). 1.

(1a) For the momentum equation (5), we have:

\[
\begin{align*}
\rho(V_t + V \cdot V_r) + K(\rho^\gamma)_r - \kappa(\rho^\theta)_r & \left( \frac{N-1}{r} V + V_r \right) - \kappa \rho_r (V_r + \frac{N-1}{r} V_r - \frac{N-1}{r^2} V) \\
+ \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \rho s^{N-1} ds & = \rho \frac{\dot{a}(t)}{a(t)} r + K \left( \frac{f(r/a(t))}{a(t)^N} \right)^\gamma_r - \kappa \left( \frac{f(r/a(t))}{a(t)^N} \right)^\theta_r \frac{N \dot{a}(t)}{a(t)} + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f(r/a(t))}{a(t)^N} s^{N-1} ds \\
= \gamma K \left[ \frac{f(r/a(t))}{a(t)^N} \right]^\gamma \frac{\partial}{\partial r} \left[ \frac{f(r/a(t))}{a(t)^N} \right] - \kappa \theta \left[ \frac{f(r/a(t))}{a(t)^N} \right]^\theta \frac{\partial}{\partial r} \left[ \frac{f(r/a(t))}{a(t)^N} \right] \frac{N \dot{a}(t)}{a(t)} \\
+ \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f(r/a(t))}{a(t)^N} s^{N-1} ds,
\end{align*}
\]

(19) (20) (21) (22) (23)
with $a(t) = mt + n$.

Then, we have:

$$
= \gamma K \left[ \frac{f(r/a(t))}{a(t)^N} \right]^{-1} \frac{\dot{f}(r/a(t))}{a(t)^N} - \kappa \theta \left[ \frac{f(r/a(t))}{a(t)^N} \right]^{\theta-1} \frac{\dot{f}(r/a(t))}{a(t)^N+2} N \dot{a}(t) + \rho \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f(r/a(t))}{a(t)^N} s^{N-1} ds \\
= \rho \left\{ \frac{\gamma K f(\frac{r-a(t)}{a(t)^N})^{-2} j(\frac{r-a(t)}{a(t)^N})}{a(t)^{N(\gamma-2)+N+1}} - \kappa \theta N f(\frac{r-a(t)}{a(t)^N})^{\theta-2} j(\frac{r-a(t)}{a(t)^N}) \dot{a}(t) + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f(r/a(t))}{a(t)^N} s^{N-1} ds \right\} \\
= \rho \frac{\gamma K f(\frac{r-a(t)}{a(t)^N})^{-2} j(\frac{r-a(t)}{a(t)^N})}{a(t)^{N(\gamma-2)+2}} - \kappa \theta N f(\frac{r-a(t)}{a(t)^N})^{\theta-2} j(\frac{r-a(t)}{a(t)^N}) \dot{a}(t) + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^{r/a(t)} f(s) s^{N-1} ds \right\} .
$$

(24)

And we plug with the condition for $\gamma = \frac{2N-2}{N}$ and $\theta = \frac{2N-3}{N}$ and with the new variable $z := r/a(t)$:

$$
= \rho \frac{\gamma K f(z)^{-2} j(z) - m \kappa \theta N f(z)^{\theta-2} j(z) + \frac{\delta \alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds}{a(t)^{N-1}} .
$$

(25)

In the theorem, we require the ordinary differential equation for $y(z)$ is

$$
\begin{cases}
\gamma K f(z)^{-2} j(z) - m \kappa \theta N f(z)^{\theta-2} j(z) + \frac{\delta \alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds = 0, \\
y(0) = \alpha > 0.
\end{cases}
$$

(27)

Therefore, our solutions satisfy the momentum equation.

(1a) Similarly, for $\Lambda = 0$, $\gamma = \frac{2N-2}{N}$ and $\theta = \frac{3N-4}{2N}$, we have:

$$
\rho(V_t + V \cdot V_r) + K (\rho \gamma)_r - \kappa (\rho \theta)_r \left( \frac{N-1}{r} V + V_r \right) - \kappa \rho_r (V_{rr} + \frac{N-1}{r} V_r - \frac{N-1}{r^2} V) \\
+ \rho \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \rho s^{N-1} ds
$$

(29)

$$
= \dot{\theta}(t) + K \left[ \left( \frac{f(r/a(t))}{a(t)^N} \right)^\gamma \right]_r - \kappa \left[ \left( \frac{f(r/a(t))}{a(t)^N} \right)^\theta \right]_r N \dot{a}(t) + \rho \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f(r/a(t))}{a(t)^N} s^{N-1} ds.
$$

(30)

$$
= \rho \frac{\varepsilon^2}{(mt+n)^2} V + \gamma K \left[ \frac{f(r/a(t))}{a(t)^N} \right]^{-1} \frac{\partial}{\partial r} \left[ \frac{f(r/a(t))}{a(t)^N} \right]
$$

(31)

$$
- \kappa \theta \left[ \frac{f(r/a(t))}{a(t)^N} \right]^{\theta-1} \frac{\partial}{\partial r} \left[ \frac{f(r/a(t))}{a(t)^N} \right] 2m(mt+n)^{-1} + \rho \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f(r/a(t))}{a(t)^N} s^{N-1} ds.
$$

(32)

(33)
with \( a(t) = (mt + n)^{2/N} \).

Then, we have:

\[
\begin{align*}
\rho &= \frac{2(2 - N)m^2r}{N^2(mt + n)^2} + \gamma K \left[ \frac{f \left( \frac{r}{a(t)} \right)}{a(t)^N} \right]^{\gamma - 1} \frac{\dot{f} \left( \frac{r}{a(t)} \right)}{a(t)^{N+1}} \\
-2m\kappa \theta &\left[ \frac{f \left( \frac{r}{a(t)} \right)}{a(t)^N} \right]^{\theta - 1} \frac{\dot{f} \left( \frac{r}{a(t)} \right)}{a(t)^{N+2}} (mt + n)^{(2 - N)/N} + \rho \frac{\delta \alpha(N)}{rN - 1} \int_0^r \frac{f \left( \frac{s}{a(t)} \right)}{a(t)^N} s^{N-1} ds \\
\rho &= \frac{2(2 - N)m^2r}{N^2(mt + n)^2} + K\gamma f \left( \frac{r}{a(t)} \right)^{\gamma - 2} \frac{\dot{f} \left( \frac{r}{a(t)} \right)}{a(t)^N} \\
-2m\kappa \theta &\left[ \frac{f \left( \frac{r}{a(t)} \right)}{a(t)^N} \right]^{\theta - 2} \frac{\dot{f} \left( \frac{r}{a(t)} \right)}{a(t)^{N+2}} (mt + n)^{(2 - N)/N} + \rho \frac{\delta \alpha(N)}{rN - 1} \int_0^r \frac{f \left( \frac{s}{a(t)} \right)}{a(t)^N} s^{N-1} ds \\
\rho &= \frac{2(2 - N)m^2r}{N^2(mt + n)^2} + \gamma K f \left( \frac{r}{a(t)} \right)^{\gamma - 2} \frac{\dot{f} \left( \frac{r}{a(t)} \right)}{a(t)^N} \\
-2m\kappa \theta f \left( \frac{r}{a(t)} \right)^{\theta - 2} \frac{\dot{f} \left( \frac{r}{a(t)} \right)}{a(t)^{N+2}} (mt + n)^{(2 - N)/N} + \rho \frac{\delta \alpha(N)}{rN - 1} \int_0^r \frac{f \left( \frac{s}{a(t)} \right)}{a(t)^N} s^{N-1} ds \\
\end{align*}
\]

with \( \gamma = \frac{2N - 2}{N} \) and \( \theta = \frac{3N - 4}{2N} \) for \( N \geq 2 \).

and with the new variable \( z := r/a(t) \):

\[
\begin{align*}
\rho &= \frac{2(2 - N)m^2}{N^2} z + \gamma K f(z)^{\gamma - 2} \frac{\dot{f}(z)}{z^{N-1}} - 2m\kappa \theta f(z)^{\theta - 2} \frac{\dot{f}(z)}{z^{N-1}} + \frac{\delta \alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds .
\end{align*}
\]

In the theorem, we require the ordinary differential equation for \( f(z) \):

\[
\begin{align*}
\gamma K f(z)^{\gamma - 2} \frac{\dot{f}(z)}{z^{N-1}} - 2m\kappa \theta f(z)^{\theta - 2} \frac{\dot{f}(z)}{z^{N-1}} + \frac{\delta \alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds &= \frac{2(N-2)m^2}{N^2} z , \quad (41)
\end{align*}
\]

Therefore, our solutions satisfy the momentum equation.
(2) for $\delta \Lambda > 0$ and $\gamma = \theta = \frac{2N-2}{N}$, we have:

$$
\rho (V_t + V \cdot V_r) + K (\rho^\gamma)_r - \kappa (\rho^\theta)_r \left( \frac{N-1}{r} V + V_r \right) - \kappa \rho_r (V_{rr} + \frac{N-1}{r} V_r - \frac{N-1}{r^2} V) \tag{42}
$$

$$
+ \frac{\delta \alpha(N) \rho}{r^{N-1}} \int_0^r (\rho s^{N-1} - \Lambda) ds \tag{43}
$$

$$
= \frac{\rho \delta \alpha(N) \Lambda}{N} r + K \left( \left[ \frac{f'({\pi/\tau(t)})}{a(t)^N} \right] \right) r - \left( \kappa \left[ \frac{f'({\pi/\tau(t)})}{a(t)^N} \right] \right) N \dot{\alpha}(t) \tag{44}
$$

$$
+ \frac{\rho \delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f'({\pi/\tau(t)})}{a(t)^N} s^{N-1} ds - \rho \frac{\delta \alpha(N) \Lambda}{r^{N-1}} \int_0^r s^{N-1} ds. \tag{45}
$$

With $a(t) = e^{\sqrt{\frac{\delta \alpha(N) \Lambda}{N}} t}$, then we get:

$$
= K \gamma \left[ \frac{f'({\pi/\tau(t)})}{a(t)^N} \right] \gamma^{-1} \dot{f} \left( \frac{\pi}{\tau(t)} \right) - \kappa \theta \left[ \frac{f'({\pi/\tau(t)})}{a(t)^N} \right] \gamma^{-2} \dot{f} \left( \frac{\pi}{\tau(t)} \right) \sqrt{N \delta \alpha(N) \Lambda} + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f'({\pi/\tau(t)})}{a(t)^N} s^{N-1} ds \tag{46}
$$

$$
= \rho \left\{ \gamma K \left[ \frac{f'({\pi/\tau(t)})}{a(t)^N} \right] \gamma^{-2} \dot{f} \left( \frac{\pi}{\tau(t)} \right) - \kappa \theta \sqrt{N \delta \alpha(N) \Lambda} \left[ \frac{f'({\pi/\tau(t)})}{a(t)^N} \right] \gamma^{-2} \dot{f} \left( \frac{\pi}{\tau(t)} \right) + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f'({\pi/\tau(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{47}
$$

$$
= \rho \left\{ \gamma K f'({\pi/\tau(t)}) \gamma^{-2} \dot{f} \left( \frac{\pi}{\tau(t)} \right) - \kappa \theta \sqrt{N \delta \alpha(N) \Lambda} f \left( \frac{\pi}{\tau(t)} \right) \gamma^{-2} f' \left( \frac{\pi}{\tau(t)} \right) + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f'({\pi/\tau(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{48}
$$

$$
= \frac{\rho}{a(t)^{N-1}} \left\{ \gamma K f'({\pi/\tau(t)}) \gamma^{-2} \dot{f} \left( \frac{\pi}{\tau(t)} \right) - \kappa \theta \sqrt{N \delta \alpha(N) \Lambda} f \left( \frac{\pi}{\tau(t)} \right) \gamma^{-2} f' \left( \frac{\pi}{\tau(t)} \right) + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f'({\pi/\tau(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{49}
$$

$$
= \frac{\rho}{a(t)^{N-1}} \left\{ \gamma K f'({\pi/\tau(t)}) \gamma^{-2} \dot{f} \left( \frac{\pi}{\tau(t)} \right) - \kappa \theta \sqrt{N \delta \alpha(N) \Lambda} f \left( \frac{\pi}{\tau(t)} \right) \gamma^{-2} f' \left( \frac{\pi}{\tau(t)} \right) + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f'({\pi/\tau(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{50}
$$

$$
= \frac{\rho}{a(t)^{N-1}} \left\{ \gamma K f'({\pi/\tau(t)}) \gamma^{-2} \dot{f} \left( \frac{\pi}{\tau(t)} \right) - \kappa \theta \sqrt{N \delta \alpha(N) \Lambda} f \left( \frac{\pi}{\tau(t)} \right) \gamma^{-2} f' \left( \frac{\pi}{\tau(t)} \right) + \frac{\delta \alpha(N)}{r^{N-1}} \int_0^r \frac{f'({\pi/\tau(t)})}{a(t)^N} s^{N-1} ds \right\} \tag{51}
$$
with $\gamma = \theta = \frac{2N-2}{N}$.

We require the corresponding ordinary differential equations $f(z)$ with $z := r/a(t)$:

$$\begin{cases}
(\gamma K - \kappa \theta \sqrt{N} \delta \alpha(N) \Lambda) f(z)^{-2} \dot{f}(z) + \frac{\delta \alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds = 0 \\
y(0) = \alpha > 0.
\end{cases}
$$

The proof is completed.

**Remark 3** If the complex number solutions $(\rho, u) \in \mathbb{C}^{N+1}$ is considered, we have the corresponding solutions for $\delta \Lambda < 0$:

$$\begin{cases}
\rho(t, r) = \frac{f(r/a(t))}{a(t)\sqrt{r}},
V(t, r) = i \sqrt{-\frac{\delta \Lambda}{N}} r, \\
a(t) = e^{i \sqrt{-\frac{\delta \Lambda}{N}} t},
\end{cases}
$$

$$\begin{cases}
(\gamma K - i \kappa \theta \sqrt{-N} \delta \alpha(N) \Lambda) f(z)^{-2} \dot{f}(z) + \frac{\delta \alpha(N)}{z^{N-1}} \int_0^z f(s) s^{N-1} ds = 0, \\
f(0) = \alpha,
\end{cases}
$$

where $i$ is the complex constant.

**Remark 4** Our method can be easily extended to the corresponding systems with linear damping:

$$\begin{cases}
\rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\
\rho \vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} + \beta \rho \vec{u} + \nabla P = -\delta \rho \nabla \Phi + vis(\rho, \vec{u}), \\
\Delta \Phi(t, x) = \alpha(N) \rho - \Lambda,
\end{cases}
$$

where $\beta > 0$ is a constant.

**References**

[1] M. Bezard, *Existence locale de solutions pour les equations d'Euler-Poisson. (French) [Local Existence of Solutions for Euler-Poisson Equations]* Japan J. Indust. Appl. Math. **10** (1993), no. 3, 431–450.

[2] J. Binney and S. Tremaine, *Galactic Dynamics*, Princeton Univ. Press, 1994.

[3] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Univ. of Chicago Press, 1939.
[4] G.Q. Chen and D.H. Wang, *The Cauchy Problem for the Euler Equations for Compressible Fluids*, Handbook of Mathematical Fluid Dynamics, Vol. I, 421-543, North-Holland, Amsterdam, 2002.

[5] C. F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, Plenum, New York (1984).

[6] P.L. Lions, *Mathematical Topics in Fluid Mechanics*. Vols. 1, 2, 1998, Oxford: Clarendon Press, 1998.

[7] Y.B. Deng, J.L. Xiang and T. Yang, *Blowup Phenomena of Solutions to Euler-Poisson Equations*, J. Math. Anal. Appl. 286 (1)(2003), 295-306.

[8] H. H. Fliche and R. Triay, *Euler-Poisson-Newton Approach in Cosmology*, Cosmology and Gravitation, 346–360, AIP Conf. Proc., 910, Amer. Inst. Phys., Melville, NY, 2007.

[9] P. Gamblin, *Solution reguliere a temps petit pour l’equation d’Euler-Poisson. (French) [Small-time Regular Solution for the Euler-Poisson Equation]* Comm. Partial Differential Equations 18 (1993), no. 5-6, 731–745.

[10] R. T. Glassey, *The Cauchy Problem in Kinetic Theory*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[11] P. Goldreich and S. Weber, *Homologously Collapsing Stellar Cores*, Astrophys. J. 238, 991 (1980).

[12] R. Kippenhahn, A. Weigert, *Stellar Structure and Evolution*, Springer-Verlag, 1990.

[13] L.D. Landau and E.M. Lifshitz, E. M. (1959). *Fluid Mechanics*. Translated from the Russian by J. B. Sykes and W. H. Reid. Course of Theoretical Physics. Vol. 6. London: Pergamon Press.

[14] T. Makino, *On a Local Existence Theorem for the Evolution Equation of Gaseous Stars*, Patterns and waves, 459–479, Stud. Math. Appl., 18, North-Holland, Amsterdam, 1986.

[15] T. Makino, *Blowing up Solutions of the Euler-Poisson Equation for the Evolution of the Gaseous Stars*, Transport Theory and Statistical Physics 21 (1992), 615-624.
[16] T. Nishida, *Equations of Fluid Dynamics-free Surface Problems*, Comm. Pure Appl. Math., XXXIX (1986), 221–238.

[17] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. x+500 pp.

[18] T. Yang, Z.A. Yao and C. J. Zhu, *Compressible Navier-Stokes Equations with Density-dependent Viscosity and Vacuum*, Comm. Partial Differential Equations 26 (2001), no. 5-6, 965–981.

[19] L.H. Yeung and M.W. Yuen, *Analytical Solutions to the Navier-Stokes Equations with Density-dependent Viscosity and with Pressure*, J. Math. Phys., 50 (2009), No. 8, 083101, 6pp.

[20] M.W. Yuen, *Blowup Solutions for a Class of Fluid Dynamical Equations in $R^N$*, J. Math. Anal. Appl. 329 (2007), 1064-1079.

[21] M.W. Yuen, *Analytical Blowup Solutions to the 2-dimensional Isothermal Euler-Poisson Equations of Gaseous Stars*, J. Math. Anal. Appl. 341 (2008), 445-456.

[22] M.W. Yuen, *Analytical Solutions to the Navier-Stokes Equations*, J. Math. Phys., 49 (2008), 113102, 10pp.

[23] M.W. Yuen, *Analytical Blowup Solutions to the Pressureless Navier-Stokes-Poisson Equations with Density-dependent Viscosity in $R^N$*, Nonlinearity 22 (2009), 2261-2268.

[24] M.W. Yuen, *Analytically Periodic Solutions to the Three-dimensional Euler-Poisson Equations of Gaseous Stars with a Negative Cosmological Constant*, Class. Quantum Grav. 26 (2009), 235011 (8pp).