Integrable 3D lattice model in M-theory

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ABSTRACT: It is argued that the supersymmetric index of a certain system of branes in M-theory is equal to the partition function of an integrable three-dimensional lattice model. The local Boltzmann weights of the lattice model satisfy a generalization of Zamolodchikov’s tetrahedron equation. In a special case the model is described by a solution of the tetrahedron equation discovered by Kapranov and Voevodsky and by Bazhanov and Sergeev.

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1 Introduction

Over the past decade a family of integrable two-dimensional (2D) lattice models have been embedded into string theory [1–8]. Using open strings ending on a stack of D5-branes, one can construct the eight-vertex model and its trigonometric and rational limits [7]. Combined with string dualities, the brane construction provides a unified understanding [7] of the appearances of these models in supersymmetric quantum field theories (QFTs) in diverse spacetime dimensions [4, 5, 9–19].

In this paper I propose an analogous construction in one dimension higher: the supersymmetric index of a certain brane system in M-theory computes the partition function of an integrable 3D lattice model. In a special case the local Boltzmann weights of the model are given by a solution of Zamolodchikov’s tetrahedron equation [20], a 3D analog of the Yang-Baxter equation. More generally, the model is built from eight types of local
Boltzmann weights. They satisfy 16 quartic relations, which have appeared in [21, 22] and will be collectively referred to here as the supertetrahedron equation.

The brane system in question consists of M5-branes that intersect a 3-torus $T^3$ along 2-tori. The 3D lattice model emerges on the periodic cubic lattice formed by these 2-tori. The excited states of the system contributing to the supersymmetric index contains M2-branes suspended between M5-branes along the edges of the lattice. These M2-branes play the roles of spin variables for the lattice model.

The masses of the M2-branes are determined by the positions of M5-branes in the directions transverse to $T^3$. When the positions are adjusted in such a way that all M2-branes become massless, the supersymmetric index has a description in terms of a 3D topological quantum field theory (TQFT) defined on $T^3$. The M5-branes are represented by surface defects in the TQFT. This description enables the supersymmetric index to be decomposed into contributions localized around the vertices of the lattice. These local contributions are the local Boltzmann weights of the lattice model.

For each M5-brane, there are two discrete choices for the directions of its worldvolume. The two choices lead to two kinds of surface defect in the TQFT, which we distinguish by the signs $+$ and $-$. An edge of the lattice is where two surface defects intersect. The Hilbert space for the spin variable living there depends on the pair of signs $(\sigma_1, \sigma_2)$ representing the types of the two defects: the bosonic Fock space if $\sigma_1 = \sigma_2$ and the fermionic Fock space if $\sigma_1 \neq \sigma_2$. A local Boltzmann weight arises at a vertex of the lattice where three surface defects intersect, and is therefore labeled by a triplet of signs $(\sigma_1, \sigma_2, \sigma_3)$.

The topological invariance, together with the fact that the TQFT has hidden extra dimensions in the 11D spacetime, implies that the correlation function is invariant under movement of surface defects [2, 3, 23, 24]. A local version of this statement is an equivalence between two different arrangements of four surface defects forming tetrahedra. In the language of the lattice model the equivalence translates to an equation satisfied by local Boltzmann weights, and there is one equation for each quadruple of signs $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ which specifies the signs of the four defects. The set of equations thus obtained is the supertetrahedron equation. For $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (+++)$ and $(-,-,-,-)$, the equations take the form of the ordinary tetrahedron equation.

When the M5-branes are freed from the adjusted configuration and the M2-branes become massive, the TQFT description is deformed by continuous parameters corresponding to the M5-brane positions. These parameters control the twisting of the periodic boundary conditions of the lattice model. The supertetrahedron equation and the existence of the twisting parameters together imply the integrability of the model.

The tetrahedron equation is a highly overdetermined system of equations. If a 3D lattice model (of the vertex type) has a single type of spin variable that takes $N$ different values, then the local Boltzmann weights are encapsulated in an $N^3 \times N^3$ matrix, called the R-matrix of the model. The tetrahedron equation, on the other hand, has $N^6 \times N^6$ components.

For the lattice model constructed from branes in this paper, constraints imposed on the local Boltzmann weights are even more stringent. The local Boltzmann weights are defined by eight R-matrices $R^{\sigma_1\sigma_2\sigma_3}$, $\sigma_1, \sigma_2, \sigma_3 = \pm$, satisfying intricate relations gener-
alizing the tetrahedron equation. Although there are symmetries that reduce the number of independent R-matrices to three, the supertetrahedron equation may still appear too constraining to admit a solution.

Remarkably, a solution of the supertetrahedron equation is known [21, 22]. For this solution, \( R^{+++} \) is the R-matrix discovered independently by Kapranov and Voevodsky [25] and by Bazhanov and Sergeev [26], whereas \( R^{++-} \) was obtained in [26] where it was called an L-operator. The remaining R-matrices were found by Yoneyama [22]. Furthermore, these R-matrices possess all the properties that we expect for the R-matrices of our model to have, such as a charge conservation rule and involutivity. Therefore, I propose that the lattice model constructed from branes is described by this solution of the supertetrahedron equation.

Probably the most convincing evidence for this proposal is that the behavior of the brane system under string dualities correctly reproduces the behavior of the 3D lattice model defined by this solution under reduction along one of the directions of the lattice. By reduction to 10D and T-duality, the brane system under consideration is mapped to a brane configuration studied in [7, 8], which constructs a 2D lattice model defined by a trigonometric solution of the Yang-Baxter equation whose symmetry algebra is a general linear Lie superalgebra \( gl(L_+|L_-) \). This is consistent with the results of [21, 26, 27].

Turning the logic around, the brane construction of this paper explains characteristic features of the above solution of the supertetrahedron equation. For example, the R-matrices are constant and the spectral parameters, which are crucial for the integrability of the model, enter the partition function as twisting parameters for the periodic boundary conditions. This is in contrast with familiar solutions of the Yang-Baxter equation describing integrable 2D lattice models, for which spectral parameters directly appear in the R-matrices. As we will see, this feature finds a natural explanation in the brane construction.

The rest of the paper is organized as follows. In section 2 we introduce the brane system whose supersymmetric index is argued to define an integrable 3D lattice model. Section 3 discusses this system from the point of view of 3D TQFT. We explain how the structure of a 3D TQFT arises from the special case of the brane configuration, how the 3D TQFT gives rise to a 3D lattice model in the presence of surface defects, and how the existence of extra dimensions implies the tetrahedron equation. In section 4 we deduce key properties of the lattice model based on the brane construction. Finally, in section 5 we identify a solution of the supertetrahedron equation that has the properties deduced in the previous section and which I propose describes this lattice model. An analysis of the supersymmetry preserved by the brane system is carried out in appendix A.

2 3D lattice model constructed from branes

We will construct an integrable 3D lattice model in M-theory formulated in the 11D space-time

\[
\mathbb{R}_0 \times \mathbb{T}^3_{123} \times \mathbb{R}^3_{156} \times \mathbb{R}^2_{78} \times \mathbb{R}^2_{9},
\]

(2.1)
endowed with the flat metric
\[ \eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + \cdots + (dx^9)^2 + (dx^{\natural})^2. \] (2.2)

where \( \mathbb{R}^n_{\mu_1\mu_2...\mu_n} \) is an \( n \)-dimensional Euclidean space with coordinates \( (x^{\mu_1}, x^{\mu_2}, \ldots, x^{\mu_n}) \) and \( T^3_{123} \) is a 3-torus with periodic coordinates \( (x^1, x^2, x^3) \), respectively. We use the notation \( \natural = 10 \). We will write \( S_{\mu} \) to denote a circle with periodic coordinate \( x^{\mu} \).

To this spacetime we introduce six types of M5-branes, which we call type \( i\sigma \), \( i = 1, 2, 3 \), \( \sigma = +, - \). M5-branes of different types have worldvolumes extending in different directions, as listed in table 1. We will say that an M5-brane of type \( i\sigma \) has sign \( \sigma \) and refer to M5-branes of either type \( i^+ \) or type \( i^- \) simply as M5-branes of type \( i \). The orientations of the M5-branes are chosen in such a way that the setup is symmetric under the cyclic permutation \( (1, 4) \rightarrow (2, 5) \rightarrow (3, 6) \rightarrow (1, 4) \) and the exchange \( (7, 8) \leftrightarrow (9, \natural) \) of directions.

This configuration of M5-branes is invariant under 1/8 of the supersymmetry of M-theory. The unbroken supersymmetry is generated by two supercharges \( Q^+, Q^- \), defined and analyzed in appendix A.

These supercharges have a couple of properties that are important to us. One is that \( Q^2_+ + Q^2_- = \frac{1}{2} (H - Z^{(2)}_{14} - Z^{(2)}_{25} - Z^{(2)}_{36} - Y) \), (2.3)

where \( H = P^0 \) is the Hamiltonian, \( Z^{(2)}_{14}, Z^{(2)}_{25}, Z^{(2)}_{36} \) are components of a 2-form charge \( Z^{(2)} \), and \( Y \) is a linear combination of components of a 5-form charge \( Z^{(5)} \). (The momentum \( P \) and the charges \( Z^{(2)}, Z^{(5)} \) commute with the supercharges and with each other.) Since \( Q^\pm \) are hermitian, the right-hand side is nonnegative.

Another property of \( Q^\pm \) is the invariance under the action of the antidiagonal subgroup of the rotation groups \( \text{SO}(2)_{78} \) of \( \mathbb{R}^2_{78} \) and \( \text{SO}(2)_{9\natural} \) of \( \mathbb{R}^2_{9\natural} \). In other words, \( Q^\pm \) commute with the difference \( J_{78} - J_{9\natural} \) of the generators \( J_{78} \) of \( \text{SO}(2)_{78} \) and \( J_{9\natural} \) of \( \text{SO}(2)_{9\natural} \):

\[ [J_{78} - J_{9\natural}, Q^\pm] = 0. \] (2.4)
To make use of this property, we place the M5-branes at either the origin of \( \mathbb{R}_7 \) or the origin of \( \mathbb{R}_9 \). Then, the M5-brane system is invariant under the action of the antidiagonal subgroup of \( \text{SO}(2)_7 \times \text{SO}(2)_9 \), and its Hilbert space \( \mathcal{H} \) (the subspace of the Hilbert space of M-theory consisting of states that contain this configuration of M5-branes) is acted on by \( J_7 - J_9 \) in addition to \( Q_\pm \).

Now, let us define the supersymmetric index of the M5-brane system by

\[
Z = \text{tr}_\mathcal{H}(\langle (-1)^F e^{i\theta(J_7 - J_9)} e^{-\beta(H - E_0)} \rangle),
\]

where \( E_0 \) is the ground state energy. The index \( Z \) can be computed by the Euclidean path integral in which the imaginary time \( i x^0 \) is periodic with period \( \beta \). In this language, we have chosen a periodic boundary condition for fermions, which is responsible for the appearance of the fermion parity operator \( (-1)^F \). The insertion of \( e^{i\theta(J_7 - J_9)} \) means that the periodic boundary condition is twisted: if the time direction is viewed as the interval \([0, \beta] \) with the end points identified, the Hilbert space at \( i x^0 = \beta \) is glued back to the Hilbert space at \( i x^0 = 0 \) after \( \mathbb{R}_7 \) and \( \mathbb{R}_9 \) are rotated by angles \( +\theta \) and \( -\theta \), respectively.

The Hilbert space \( \mathcal{H} \) of the M5-brane system is graded by \( J_7 \) \(- J_9 \), which we normalize to take integer values: \( \mathcal{H} \) has the decomposition

\[
\mathcal{H} = \bigoplus_{j=-\infty}^{\infty} \mathcal{H}^j,
\]

where \( \mathcal{H}^j \) is the eigenspace of \( J_7 - J_9 \) with eigenvalue \( j \). Hence, the partition function is a formal power series

\[
Z = \sum_{j=-\infty}^{\infty} Z_j q^j, \quad q = e^{i\theta},
\]

with the coefficients given by

\[
Z_j = \text{tr}_{\mathcal{H}^j}(\langle (-1)^F e^{-\beta(H - E_0)} \rangle).
\]

Eventually, we will identify \( Z \) with the partition function of an integrable 3D lattice model. To see how such a model can arise from \( Z \), let us look at the configuration of M5-branes a little more closely. The M5-branes generically do not intersect with each other in the 11D spacetime because any two of them can be separated in \( \mathbb{R}_6 \). Inside \( T^3 \), however, they represent intersecting 2-tori that form a periodic cubic lattice\(^1\). Given a lattice made up by M5-branes, the operation of taking its supersymmetric index produces a complex number. In this sense, the brane system defines a physical model associated with a 3D lattice.

Although this is certainly what we want, it is a rather weak conclusion. Why should the model thus defined by the M5-branes be a lattice model in the usual sense of the term in statistical mechanics, that is, a model describing interactions of “spin variables” located at lattice points?

\(^1\)Analogous cubic lattice configurations in M-theory were considered in [28] where the authors were motivated by an observed structure of galaxy superclusters.
The answer is because only special states contribute to $Z$. We can write $Z$ as the sum of a trace over $\ker Q_+ = \ker Q_+^2$ and a trace over the orthogonal complement $(\ker Q_+)^\perp$ of $\ker Q_+$. In the latter subspace $\ker Q_+$ is invertible. (In the eigenspace of $Q_+^2$ with eigenvalue $q_+^2 > 0$, we have $Q_+^{-1} = Q_+ / q_+^2$.) The action of $Q_+$ gives a one-to-one correspondence between the bosonic states and the fermionic states in $(\ker Q_+)^\perp$, and their contributions to $Z$ cancel. As a result, only the states in $\ker Q_+$ contribute to $Z$. The same argument applies to $Q_-$ (which leaves $\ker Q_+$ invariant). Hence, $Z$ receives contributions only from $\ker Q_+ \cap \ker Q_-$, which is the space of states whose energy satisfies the Bogomol'nyi-Prasad-Sommerfield (BPS) condition:

$$H = Z_{14}^{(2)} + Z_{25}^{(2)} + Z_{36}^{(2)} + Y.$$  

(2.9)

The relevant BPS states are represented by configurations of M2-branes and M5-branes added to the system on top of those M5-branes that are already present. BPS M2-branes contribute to $Z_{14}^{(2)}$, $Z_{25}^{(2)}$, $Z_{36}^{(2)}$, whereas BPS M5-branes contribute to $Y$.

We can introduce, for example, an M5-brane along $\mathbb{R}^{5,1}_{0145678}$. However, this M5-brane has an infinite spatial volume, so adding it to the system increases the energy by an infinite amount and BPS states containing it do not contribute to $Z$. By the same token, any BPS states with additional M5-branes make no contributions to $Z$ because those M5-branes necessarily have infinite spatial volumes. 2 For this reason, on the states relevant to $Z$ the value of $Y$ is fixed to the value for the initial brane configuration, which we have called $E_0$. In the definition (2.5) of $Z$, we subtracted $E_0$ from $H$ to offset the ground state energy so that $Z$ can be nonvanishing.

In contrast, BPS M2-branes can have finite spatial area. There are three types of BPS M2-branes, which we call type $i$, $i = 1, 2, 3$. An M2-brane of type $i$ extends in the directions of $\mathbb{R}_0 \times S_i \times \mathbb{R}_{i+3}$. For example, the spatial area of an M2-brane of type 1 is finite if it stretches between two M5-branes, one of type 2 and one of type 3, both of which are localized at points in $\mathbb{R}_4$. Therefore, the contributions to $Z$ come from those BPS states that only have finite-area M2-branes excited. Writing $\mathcal{H}_{\text{BPS}}$ for the space of such BPS states, we have:

$$Z_j = \tr_{\mathcal{H}_{\text{BPS}}^j}((-1)^F e^{-\beta(Z_{14}^{(2)} + Z_{25}^{(2)} + Z_{36}^{(2)})}),$$

(2.10)

where

$$\mathcal{H}_{\text{BPS}}^j = \mathcal{H}_{\text{BPS}} \cap \mathcal{H}^j.$$  

(2.11)

Let us focus on a single vertex of the lattice and ask what sort of configurations of M2-branes are possible around it. Six edges meet at the vertex, two from the intersection of M5-branes of type 1 and type 2, two from type 2 and type 3, and two from type 3 and type 1; see figure 1. Take the pair of edges where the M5-branes of type 2 and type 3 intersect.

2From the expression (A.27) of $Y$ we see that there may also be BPS states that have nonzero $Z_{01456}^{(5)}$, $Z_{0146}^{(5)}$, $Z_{0256}^{(5)}$, or $Z_{0789}^{(5)}$. Such a BPS state corresponds to a spacetime in which $\mathbb{T}^{12}_1 \times \mathbb{R}^{25}_2$, $\mathbb{T}^{13}_2 \times \mathbb{R}^{56}_3$, $\mathbb{T}^{14}_3 \times \mathbb{R}^{67}_4$, or $\mathbb{R}^{123456789}$ is replaced by a Taub-NUT space. Since a Taub-NUT geometry becomes an infinite D6-brane in type IIA string theory, it does not contribute to $Z$. In section 3 we will modify the system so that it has a Taub-NUT space in place of $\mathbb{R}^{123456789}$.
Along these edges we can suspend M2-branes of type 1. The number of suspended M2-branes may be different for the two edges. Similarly, we can suspend some numbers of M2-branes of type 2 between the M5-branes of type 3 and type 1 along the pair of edges coming from these M5-branes, and some numbers of M2-branes of type 3 between the M5-branes of type 1 and type 2 along the remaining pair of edges.

Thus we can assign an integer to each edge of the lattice, namely the number of M2-branes suspended between M5-branes along that edge. We can interpret this integer as a spin variable of the lattice model. Near a vertex the M2-branes along the six edges combine in some manner, giving rise to interactions among the spin variables.

The picture that emerges from the above consideration is that the supersymmetric index of the brane system defines what is known as a vertex model in statistical mechanics: spin variables live on the edges and interact at the vertices of a lattice.

We have understood, more or less, how a lattice model arises from the brane system. An important question remains to be answered, however: why should this model be integrable?

3 TQFT origin of the model

Before trying to answer the question of integrability, we should first understand the origin of the lattice model better. To this end let us consider the special case of the above brane system in which the positions of the M5-branes are adjusted so that all of them intersect the origin of $\mathbb{R}^3_{456}$. This is not so severe a restriction because, as it turns out, the general case merely differs by twisting of the periodic boundary conditions of the lattice model.

We will see that the supersymmetric index of the brane system in this special case has the structure of a 3D TQFT, and this structure is what underlies the emergence of the lattice model. The line of reasoning in this section in large part follows similar arguments given in [2, 3], which adapt ideas of Costello [23, 24] to brane constructions of gauge theories.

3.1 From branes to 3D TQFT

While we restrict ourselves to the special case of the brane system as just mentioned, we also make the following two generalizations to the setup.

First, we replace $T^3_{123} \times \mathbb{R}^3_{456}$ with the cotangent bundle $T^*M$ of a closed oriented Riemannian 3-manifold $M$.\footnote{More precisely, we isometrically embed $M$ into a Calabi-Yau threefold $X$ as a special Lagrangian} The total space of $T^*M$ is the phase space of a particle...
moving in $M$, locally parametrized by coordinates $(x^1, x^2, x^3)$ on $M$ and their conjugate momenta $(p_1, p_2, p_3)$. The zero section of $T^*M$, where $p_1 = p_2 = p_3 = 0$, may be identified with $M$. In the original setup, $T^*_3\mathbb{R}^3 \times \mathbb{R}^4_{156}$ can be regarded as $T^*M$ by the identification $p_i = x^{i+3}$, $i = 1, 2, 3$. There, the M5-branes wrap submanifolds parametrized by the coordinates $(x^2, x^3, p_1), (x^3, x^1, p_2)$ or $(x^1, x^2, p_3)$, and we have specialized the positions of the M5-branes in $\mathbb{R}^4_{156}$ so that these submanifolds intersect within the zero section. In the generalized setup, we require each M5-brane to wrap the conormal bundle $N^*\Sigma$ of some oriented surface $\Sigma \subset M$. This is a subbundle of $T^*M|_{\Sigma}$ consisting of all cotangent vectors that annihilate the tangent vectors of $\Sigma$. For two surfaces $\Sigma_1, \Sigma_2 \subset M$ intersecting along a curve, their conormal bundles $N^*\Sigma_1, N^*\Sigma_2$ intersect solely in the zero section $M \cap T^*M$ along $\Sigma_1 \cap \Sigma_2$; if $(x, p) \in N^*\Sigma_1 \cap N^*\Sigma_2$, then $p(v_1) = p(v_2) = 0$ for all $v_1 \in T_x\Sigma_1$ and $v_2 \in T_x\Sigma_2$, which implies $p = 0$ since $T_x\Sigma_1 + T_x\Sigma_2 = T_xM$.

Second, we replace $\mathbb{R}^4_{789}$ with a Taub-NUT space $TN$. This space is a hyperkähler manifold and can be presented as a circle fibration over $\mathbb{R}^3$. The circle fiber shrinks to a point at the origin of $\mathbb{R}^3$. Along a half-line emanating from the origin, the radius of the fiber increases and approaches an asymptotic value $R$. The fibers over the half-line thus make a cigar shape homeomorphic to $\mathbb{R}^2$. We choose a line through the origin of the base $\mathbb{R}^3$ and view it as the union of two half-lines $\mathbb{R}^+$ and $\mathbb{R}^-$ meeting at the origin. The fibers over $\mathbb{R}^+$ and $\mathbb{R}^-$ form two cigars, $D^+$ and $D^-$, touching each other at the tips. We wrap each M5-brane on either $D^+$ or $D^-$. In summary, in the modified brane system the spacetime is

$$\mathbb{R}_0 \times T^*M \times TN, \quad (3.1)$$

and each M5-brane is supported on a submanifold of the form

$$\mathbb{R}_0 \times N^*\Sigma \times D^{\sigma[\Sigma]}, \quad (3.2)$$

The worldvolume of an M5-brane is specified by a signed surface $(\Sigma, \sigma[\Sigma])$, a pair of an oriented surface $\Sigma \subset M$ and a sign $\sigma[\Sigma] \in \{+, -\}$. If we choose $M = T^3$ and take the limit $R \to \infty$ (in which $TN$ reduces to $\mathbb{R}^4$), the modified spacetime becomes the one considered in section 2; the M5-branes are allowed to wrap more general surfaces in $T^3$ but required to intersect the origin of $\mathbb{R}^4_{156}$. As in the original brane system, the modified system preserves two supercharges.

Let us discuss implications of the modifications we have made to the brane system one by one.

By forcing the M5-branes to intersect $\mathbb{R}^4_{156}$ at the origin, we have reduced the spatial areas of BPS M2-branes suspended between M5-branes to zero. Thus we have made all M2-branes massless. Then, the states in $\mathcal{H}_{\text{BPS}}$ all have the same energy $H = E_0$, and the index $Z$ reduces to

$$\hat{Z} = \sum_{j=-\infty}^{\infty} \text{tr}_{\mathcal{H}_{\text{BPS}}} (-1)^F q^j. \quad (3.3)$$

submanifold, which is always possible [29]. A neighborhood of $M \subset X$ can be identified with a neighborhood of the zero section of $T^*M$. For the following discussion it is sufficient to consider this neighborhood.
Table 2. The D4-D6 brane configuration defining a 3D TQFT. The symbols $\circ-$ and $-\circ$ mean that the brane extends along the positive and negative half-lines, respectively.

|     | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| D6  | - | - | - | - | - | - | - | o | o | o |
| D4_{1+} | - | * | - | - | - | - | - | o | o | o |
| D4_{2+} | - | - | * | - | - | - | - | o | o | o |
| D4_{3+} | - | - | - | * | - | - | - | o | o | o |
| D4_{1-} | - | * | - | - | - | - | - | o | o | o |
| D4_{2-} | - | - | * | - | - | - | - | o | o | o |
| D4_{3-} | - | - | - | * | - | - | - | o | o | o |

The integer coefficient of $q^j$ in $\hat{Z}$ is known as the Witten index of $H^j$. What happens to the M2-branes when we replace $T_{123}^3 \times R_{456}^3$ with $T^*M$? Suppose that two M5-branes wrap surfaces $\Sigma_1, \Sigma_2 \subset M$ that intersect along a curve. The cotangent bundle $T^*(\Sigma_1 \cap \Sigma_2)$ of this curve may be considered as a submanifold of $T^*M$ via the sequence of maps $T^*(\Sigma_1 \cap \Sigma_2) \cong T(\Sigma_1 \cap \Sigma_2) \hookrightarrow TM \cong T^*M$, where the isomorphisms between the tangent and cotangent bundles are induced by the canonical symplectic forms. (The canonical symplectic form of $T^*M$ is $\sum_{i=1}^3 dx^i \wedge dp_i$.) An M2-brane suspended between these two M5-branes is BPS if its spatial extent lies in $T^*(\Sigma_1 \cap \Sigma_2)$. Since both M5-branes intersect $T^*(\Sigma_1 \cap \Sigma_2)$ only along the curve $\Sigma_1 \cap \Sigma_2 \subset M \subset T^*M$, such an M2-brane has a zero spatial area. Hence, the index is still a series of the form (3.3).

The coefficients being Witten indices, $\hat{Z}$ is invariant under continuous deformations of parameters of the system: although some states may enter or leave $H^j_{BPS}$ under deformations, due to supersymmetry such changes occur in boson-fermion pairs and their contributions to the Witten index of $H^j$ cancel. In particular, $\hat{Z}$ is protected against deformations of the metric of $M$ and the shapes of surfaces in $M$ on which the M5-branes are supported. Since the M5-brane worldvolumes are completely specified by these surfaces and the choice of their signs, we may think of $\hat{Z}$ as a topological invariant of the configuration of signed surfaces in $M$.

Then we replaced $R_{123}^4 \times R_{456}^3$ with the Taub-NUT space $TN$. The spacetime now contains a spatial circle around which all M5-branes wrap, the circle fiber of $TN$. If we reduce M-theory to type IIA string theory on this circle, the spacetime becomes $R_0 \times T^*M \times R^3$ and a D6-brane with worldvolume $R_0 \times T^*M \times \{0\}$ appears. An M5-brane with worldvolume $R_0 \times N^*\Sigma \times D^3[\Sigma]$ is converted to a D4-brane with worldvolume $R_0 \times N^*\Sigma \times R^3[\Sigma]$, ending on the D6-brane. The resulting D4-D6 brane system is summarized in table 2, where $D^\pm$ are chosen to be the cigars along $R^3_7$ in the base $R^3_{123}$ of TN.

The low-energy physics of a D6-brane is described by a 7D QFT, and the D4-branes create 4D defects in this theory. This QFT description is sufficient for us as we are interested in $\hat{Z}$, which receives contributions only from zero-energy states. To compute $\hat{Z}$, we perform a Wick rotation to Euclidean signature, place the theory on $S_0 \times T^*M$, and twist the periodic boundary condition in the time direction by the $U(1)$ symmetry originating from the rotation symmetry of the circle fibers of $TN$. In the presence of the codimension-two
defects created by the D4-branes, the path integral yields $\hat{Z}$.

The 7D theory on $S_0 \times T^*M$ may be regarded as a 3D theory on $M$, with fields taking values in infinite-dimensional spaces. For example, the theory has three real scalar fields parametrizing the positions of the D6-brane in $\mathbb{R}^3_{\mathcal{S}_0}$, and they can be thought of as sections of a fiber bundle over $M$ whose fiber at $x \in M$ is the space of maps from $S_0 \times T^*_x M$ to $\mathbb{R}$. From this point of view, $\hat{Z}$ is captured by a 3D QFT which has two types of surface defects distinguished by signs.

Combining what we have deduced, we reach the following conclusion: the reduced index $\hat{Z}$ defines a 3D TQFT that assigns to a configuration of signed surfaces in a 3-manifold $M$ a formal power series in $q$ and $q^{-1}$ with integer coefficients.

### 3.2 From 3D TQFT to 3D lattice model

With the understanding of $\hat{Z}$ as a 3D TQFT with surface defects, we can say more about its structures using properties of TQFT. We argue that for $M = \mathbb{T}^3$ and surface defects making a cubic lattice, $\hat{Z}$ is the partition function of a 3D lattice model.

Suppose that a 3D TQFT is placed on a 3-manifold $M$. For simplicity let us assume that the theory has only one type of surface defects. Given a configuration of surface defects, we wish to compute their correlation function.

Our strategy is to use the cutting-and-gluing property of QFT. We dice $M$ up into hexahedra $H_\alpha$, $\alpha = 1, \ldots, N$, in such a way that each hexahedron contains an intersection of three surface defects. This is possible if the configuration of surfaces is generic, which we assume is the case. Since we are considering a TQFT, detailed shapes of the hexahedra are irrelevant. An example of a hexahedron is illustrated in figure 2.

For each hexahedron, we choose boundary conditions on its faces, edges and vertices. To a hexahedron $H_\alpha$ equipped with a choice $b_\alpha$ of boundary conditions, the TQFT assigns a value $W_\alpha(b_\alpha)$ which is an element of a commutative ring $\Lambda$. Once boundary conditions are chosen for all hexahedra, we have a set of values $\{W_1(b_1), \ldots, W_N(b_N)\}$. As we vary the boundary conditions, these values also vary, giving a function $W = (W_1, \ldots, W_N)$ from the set of boundary conditions to $\Lambda^N$. According to the gluing property of QFT, the value assigned to the entire configuration of surface defects in $M$ is reconstructed from this function as

$$
\sum_{(b_1, \ldots, b_N) \in B} \prod_{\alpha} W_\alpha(b_\alpha),
$$

---

**Figure 2.** Left: a hexahedron containing three intersecting surfaces. Right: the boundary conditions on the faces, edges and corners represented by balls; they are spin variables of a lattice model.
Figure 3. Reformulation of the spin model arising from the hexahedral decomposition as a vertex model. After the spin variables on the edges are doubled and those on the corners are tripled, the nine spin variables on and around each face are combined into one spin variable placed on the face.

where the sum is taken over the set \( B \) of consistent boundary conditions. The consistency means that glued faces, edges or vertices have matching boundary conditions.

This procedure of reconstructing the correlation function of surface defects from hexahedra parallels how the partition function of a classical spin model is defined. Spin variables reside on the faces, edges and vertices of the hexahedra and take values in appropriate sets of boundary conditions. To a hexahedron \( H_\alpha \) with a specified configuration \( b_\alpha \) of spin variables, the model assigns a value \( W_\alpha(b_\alpha) \), called the local Boltzmann weight for the spin configuration. The partition function of the model is defined to be the product of the local Boltzmann weights for all hexahedra, summed over the set \( B \) of all spin configurations.

We can reformulate this spin model as a model that has spin variables only on the faces of the hexahedra. Imagine making a hexahedron from six quadrilaterals. On each quadrilateral there are nine spin variables, one on the face, four on the edges and four on the vertices. We can think of these nine spin variables as collectively specifying a single “big” spin variable, which we can conveniently place on the face. A hexahedron in the reformulated model therefore has six spin variables that are worth 54 original spin variables, but for most configurations of these 54 spin variables we set the local Boltzmann weight to zero. The only nonvanishing configurations are those for which the spin variables match on the glued edges and vertices. For these configuration, the local Boltzmann weight is simply set to the corresponding value in the original model. See figure 3 for illustration.

In the reformulated model the spin variables live on the faces of the hexahedra, or equivalently, on the edges of the lattice formed by the surface defects. The local Boltzmann weights are assigned to the vertices of the lattice, located at the centers of the hexahedra. Rephrased in this way, we see that the correlation function of surface defects in the TQFT coincides with the partition function of a vertex model on the lattice formed by the surface defects.

Applying this argument to the TQFT \( \hat{\mathcal{Z}} \), we deduce that \( \hat{\mathcal{Z}} \) computes the partition function of a vertex model, with local Boltzmann weights valued in \( \Lambda = \mathbb{Z}[[q, q^{-1}]] \).

3.3 Commuting transfer matrices

Because of its M-theory origin, the lattice model arising from the 3D TQFT \( \hat{\mathcal{Z}} \) has a special property which lattice models produced by other 3D TQFTs do not possess in general: its
transfer matrices commute with each other. As we will explain, this property is closely related to the integrability of the model.

To facilitate discussions let us introduce various notations. There are \( L_i \) M5-branes of type \( i \), which we call \( \text{M5}_i[\ell_i] \), \( \ell_i = 1, \ldots, L_i \). The M5-brane \( \text{M5}_i[\ell_i] \) is located at \( x^i(\text{M5}_i[\ell_i]) \in S_i \) and \( x^\mu(\text{M5}_i[\ell_i]) \in \mathbb{R}_\mu \) for \( \mu \in \{4,5,6\} \setminus \{i+3\} \). We choose the cyclic ordering of the M5-branes such that

\[
x^i(\text{M5}_i[1]) \leq x^i(\text{M5}_i[2]) \leq \cdots \leq x^i(\text{M5}_i[L_i]) \leq x^i(\text{M5}_i[1]) .
\]  

The index \( i \) is understood modulo 3 when appearing as a label of a brane. The variable \( \ell_i \) is defined modulo \( L_i \), reflecting the periodic boundary condition. The worldvolume of \( \text{M5}_i[\ell_i] \) makes a surface \( \Sigma_i[\ell_i] \) inside \( T_{123}^3 \), which is a 2-torus. The three surfaces \( \Sigma_1[\ell_1] \), \( \Sigma_2[\ell_2] \), \( \Sigma_3[\ell_3] \) intersect at a vertex \( v[\ell_1, \ell_2, \ell_3] \) of the \( L_1 \times L_2 \times L_3 \) cubic lattice made by the 2-tori. The adjacent vertices \( v[\ell_1 - \delta_{11}, \ell_2 - \delta_{12}, \ell_3 - \delta_{13}] \) and \( v[\ell_1, \ell_2, \ell_3] \) are connected by an edge \( e_i[\ell_1, \ell_2, \ell_3] \). See figure 4.

Given a configuration \( \mathbf{n} \) of the spin variables, the lattice model assigns a local Boltzmann weight to every vertex. As in the discussion in section 3.2, let us assume that all M5-branes have the same sign, say +. Then we have only one type of surface defect and one type of local Boltzmann weight. Writing \( n_i[\ell_1, \ell_2, \ell_3] \) for the value of the spin variable on \( e_i[\ell_1, \ell_2, \ell_3] \), let

\[
\mathcal{R}[\ell_1, \ell_2, \ell_3] n_1[\ell_1, \ell_2, \ell_3] n_2[\ell_1, \ell_2, \ell_3] n_3[\ell_1, \ell_2, \ell_3] \]

be the local Boltzmann weight at \( v[\ell_1, \ell_2, \ell_3] \), determined by the spin variables on the six edges connected to \( v[\ell_1, \ell_2, \ell_3] \). The partition function of the lattice model is defined by

\[
\hat{Z} = \sum_{\mathbf{n} \in \mathbb{N}} \prod_{\ell_1, \ell_2, \ell_3} \mathcal{R}[\ell_1, \ell_2, \ell_3] n_1[\ell_1, \ell_2, \ell_3] n_2[\ell_1, \ell_2, \ell_3] n_3[\ell_1, \ell_2, \ell_3] ,
\]  

where \( \mathbb{N} \) is the set of all spin configurations.
The local Boltzmann weight (3.6) can be regarded as a matrix element with six indices. Let $V_i[\ell_1, \ell_2, \ell_3]$ be a complex vector space spanned by the set of all values the spin variable on $e_i[\ell_1, \ell_2, \ell_3]$ can take. The collection of all possible values of the local Boltzmann weight at $v[\ell_1, \ell_2, \ell_3]$ define a linear operator

$$\mathcal{R}[\ell_1, \ell_2, \ell_3]: V_1[\ell_1, \ell_2, \ell_3] \otimes V_2[\ell_1, \ell_2, \ell_3] \otimes V_3[\ell_1, \ell_2, \ell_3] \rightarrow V_1[\ell_1 + 1, \ell_2, \ell_3] \otimes V_2[\ell_1, \ell_2 + 1, \ell_3] \otimes V_3[\ell_1, \ell_2, \ell_3 + 1], \quad (3.8)$$

called the $R$-matrix at $v[\ell_1, \ell_2, \ell_3]$.

With the help of the $R$-matrix we can map our 3D classical spin model to a 2D quantum spin model. The idea is to pick one of the periodic directions of the lattice, say the “vertical” direction $S_3$, and think of it as a time direction with a discrete time coordinate $\ell_3$. A time slice at time $\ell_3$ is a “horizontal” 2-torus intersecting the vertical edges $e_3[\ell_1, \ell_2, \ell_3]$, $\ell_1 = 1, \ldots, L_1$, $\ell_2 = 1, \ldots, L_2$. The vector space $V_3[\ell_1, \ell_2, \ell_3]$ is interpreted as the Hilbert space for a quantum mechanical degree of freedom attached to $e_3[\ell_1, \ell_2, \ell_3]$. The total Hilbert space on the time slice is the tensor product

$$V_3[\ell_3] = \bigotimes_{\ell_1, \ell_2} V_3[\ell_1, \ell_2, \ell_3]. \quad (3.9)$$

Across the horizontal torus $\Sigma_3[\ell_3]$, a state in $V_3[\ell_3]$ changes to another state in $V_3[\ell_3 + 1]$. An initial state at $\ell_3 = 1$ evolves in $L_3$ steps to a final state at $\ell_3 = L_3 + 1$, which is required by the periodic boundary condition to be the same as the initial state.

Discrete time evolution in the $x^3$-direction is generated by linear operators called transfer matrices. The layer-to-layer transfer matrix

$$\mathcal{T}_3[\ell_3]: V_3[\ell_3] \rightarrow V_3[\ell_3 + 1] \quad (3.10)$$

at time $\ell_3$ is the composition of all $R$-matrices assigned to the vertices lying on $\Sigma_3[\ell_3]$, with an appropriate trace taken according to the periodicity of the torus:

$$\mathcal{T}_3[\ell_3] = \text{tr}_{\ell_2} \bigotimes_{\ell_2} V_1[\ell_1, \ell_2, \ell_3] \otimes \bigotimes_{\ell_1} V_2[\ell_1, \ell_2, \ell_3] \left( \prod_{\ell_1, \ell_2} \mathcal{R}[\ell_1, \ell_2, \ell_3] \right). \quad (3.11)$$

In this formula the operator ordering is chosen in accordance with the orientations of the relevant edges. Using the transfer matrix we can write the partition function as

$$\hat{Z} = \text{tr}_{V_3[1]} (\mathcal{T}_3[L_3] \cdots \mathcal{T}_3[2] \mathcal{T}_3[1]). \quad (3.12)$$

From the point of view of the TQFT, the vector space $V_i[\ell_1, \ell_2, \ell_3]$ is, roughly speaking, the Hilbert space on the faces of hexahedra that intersect $e_i[\ell_1, \ell_2, \ell_3]$. It is the Hilbert space the theory assigns to a quadrilateral intersected by the surface defects $\Sigma_{i+1}[\ell_{i+1}]$ and $\Sigma_{i+2}[\ell_{i+2}]$. (We use the same symbol $\Sigma_i[\ell_i]$ to refer to the surface defect placed on the surface $\Sigma_i[\ell_i]$.) With this description it is clear that the spaces $V_i[\ell_1, \ell_2, \ell_3]$ for different values of $\ell_i$ are all isomorphic since they are determined solely by the types of the intersecting surface defects, which we have been assuming to be all +. The space $V_3[\ell_3]$ is
likewise independent of $\ell_3$; it is the Hilbert space of the theory on a 2-torus intersected by the surface defects $\Sigma_1[\ell_1]$, $\ell_1 = 1, \ldots, L_1$, and $\Sigma_2[\ell_2]$, $\ell_2 = 1, \ldots, L_2$. Since the theory is topological, a state evolves trivially except when it encounters some objects. In the present case, those objects are the horizontal surface defects, and the transfer matrix $\mathcal{T}_3[\ell_3]$ implements the action of $\Sigma_3[\ell_3]$ on the Hilbert space.

Now, take a horizontal surface defect $\Sigma_3[\ell_3]$ and slide it vertically upward. The topological invariance of the theory implies that continuous deformations of the geometry of the surface defects do not affect $\hat{Z}$ as long as the topology of the configuration remains the same. Therefore, $\hat{Z}$ is invariant as $\Sigma_3[\ell_3]$ slides upward until the point when it hits the next surface defect $\Sigma_3[\ell_3 + 1]$. If we further move $\Sigma_3[\ell_3]$ past $\Sigma_3[\ell_3 + 1]$ and interchange their vertical positions, $\hat{Z}$ might change since the topology is altered. At least the structure of a TQFT does not prohibit such a change.

This is, however, not what happens because our theory is more than just a 3D TQFT: it has extra eight dimensions in the 11D spacetime of M-theory. Recall that the surface defects are M5-branes, which intersect the origin of $\mathbb{R}^{3,56}$. Imagine that when we try to move $\Sigma_3[\ell_3]$ past $\Sigma_3[\ell_3 + 1]$, we displace $M5_3[\ell_3]$ in $\mathbb{R}^{3,5}_2$ to avoid collision with $M5_3[\ell_3 + 1]$. During this “detour” the Witten index of $\mathcal{H}'$, calculating the integer coefficient of $q^3$ in $\hat{Z}$, remains unchanged since nothing is discontinuous about this process. It follows that when $M5_3[\ell_3]$ comes back to the origin of $\mathbb{R}^{3,5}_2$ and is placed above $M5_3[\ell_3 + 1]$ in $S_3$, the value of $\hat{Z}$ is the same as the beginning of the process.

Translated to the quantum mechanical language, what we have just found means that the transfer matrices $\hat{\mathcal{T}}_3[\ell_3]$ and $\hat{\mathcal{T}}_3[\ell_3 + 1]$ commute:

$$\hat{\mathcal{T}}_3[\ell_3 + 1]\hat{\mathcal{T}}_3[\ell_3] = \hat{\mathcal{T}}_3[\ell_3]\hat{\mathcal{T}}_3[\ell_3 + 1].$$

(3.13)

Under the current assumption that the signs of the M5-branes are all $+$, this equation is vacuous since $\hat{\mathcal{T}}_3[\ell_3]$ and $\hat{\mathcal{T}}_3[\ell_3 + 1]$ are actually the same operator. As soon as this assumption is dropped the equation becomes nontrivial.

In section 4.6 we will see that the transfer matrices of the lattice model defined by the full index $Z$ also commute with each other. In this case the commutativity is meaningful even for transfer matrices coming from surface defects of the same sign because they depend on additional continuous parameters. In fact, the commutativity and the existence of continuous parameters imply the integrability of the model.

### 3.4 Tetrahedron equation

We have seen that $\hat{Z}$ is invariant under vertical movement of horizontal surface defects. The partition function also has an invariance under local deformations of surface defects. This local invariance leads to a highly overdetermined system of equations satisfied by the R-matrix, known as Zamolodchikov’s tetrahedron equation [20].

In the 3D TQFT $\hat{Z}$, consider a configuration of four surface defects that form a tetrahedron. Introducing an ordering among the four surfaces, we call them $\Sigma_1$, $\Sigma_2$, $\Sigma_3$, $\Sigma_4$ from the smallest to the largest:

$$\Sigma_1 < \Sigma_2 < \Sigma_3 < \Sigma_4.$$  

(3.14)
For \( a < b < c \), let
\[
(\mathbf{ab}) = \Sigma_a \cap \Sigma_b, \quad (\mathbf{abc}) = \Sigma_a \cap \Sigma_b \cap \Sigma_c.
\] (3.15)

The ordering of the surfaces induces a lexicographic order of the vertices of the tetrahedron:
\((\mathbf{abc}) < (\mathbf{def})\) if \(abc < def\), read as 3-digit numbers. We orient the edges so that they point from a smaller vertex to a larger vertex. See the left-hand side of figure 5.

The R-matrix at the vertex \((\mathbf{abc})\)
\[
R_{(abc)} : V_{(bc)} \otimes V_{(ac)} \otimes V_{(ab)} \rightarrow V_{(bc)} \otimes V_{(ac)} \otimes V_{(ab)}
\] (3.16)
is an endomorphism of the tensor product of the vector spaces \(V_{(bc)}\), \(V_{(ac)}\), \(V_{(ab)}\) assigned to the three lines \((bc)\), \((ac)\), \((ab)\). The topological invariance and the existence of extra dimensions \(\mathbb{R}^3_{456}\) imply that \(\hat{Z}\) remains invariant under a local deformation of the configuration that transforms the tetrahedron into another tetrahedron. The equality of the Boltzmann weights associated with the two tetrahedra leads to the tetrahedron equation:
\[
R_{(234)}R_{(134)}R_{(124)}R_{(123)} = R_{(123)}R_{(124)}R_{(134)}R_{(234)}.
\] (3.17)

The commutativity of transfer matrices (3.13) follows from the tetrahedron equation and the invertibility of the \(R\)-matrix, as explained in section 4.6.

4 Properties of the model

We have explained how a 3D lattice model originates from the brane system in the special case in which the positions of the M5-branes are adjusted so that all M2-branes become massless. Now we consider the case in which all M2-branes are massive: for distinct \(i, j, k \in \{1, 2, 3\}\), we assume
\[
x^{i+3}(\text{M}5_j[\ell_j]) \neq x^{i+3}(\text{M}5_k[\ell_k])
\] (4.1)
for all \(\ell_j = 1, \ldots, L_j\) and \(\ell_k = 1, \ldots, L_k\). The massless case is recovered as a limit of the massive case. In this section we discuss various properties that the model in this general case is expected to possess. In particular, we will explain why the model is integrable.
4.1 Spin variables

The spin variables of the model are nonnegative integers that count the numbers of M2-branes placed along the edges of the lattice. More precisely, \( n_i[\ell_1, \ell_2, \ell_3] \) is the number of M2-branes suspended between the M5-branes \( \text{M5}_{i+1}[\ell_1, \ell_2] \) and \( \text{M5}_{i+2}[\ell_1, \ell_2] \) along \( e_i[\ell_1, \ell_2, \ell_3] \). (Recall that the index \( i \) labeling a brane is understood modulo \( 3 \).)

The range in which the spin variable \( n_i[\ell_1, \ell_2, \ell_3] \) takes values depends on the number of directions shared by the worldvolumes of the two M5-branes. Let \( \sigma(\text{M5}_i[\ell_i]) \in \{+,-\} \) be the sign of \( \text{M5}_i[\ell_i] \). Then,

\[
  n_i[\ell_1, \ell_2, \ell_3] \in \begin{cases}
    \{0, 1, 2, \ldots \} & (\sigma(\text{M5}_{i+1}[\ell_{i+1}]) = \sigma(\text{M5}_{i+2}[\ell_{i+2}])); \\
    \{0, 1\} & (\sigma(\text{M5}_{i+1}[\ell_{i+1}]) \neq \sigma(\text{M5}_{i+2}[\ell_{i+2}])).
  \end{cases}
\]

(4.2)

The reason the number of M2-branes is restricted if \( \sigma(\text{M5}_{i+1}[\ell_{i+1}]) \neq \sigma(\text{M5}_{i+2}[\ell_{i+2}]) \) is that in this case the ground state of an M2-brane stretched between the two M5-branes is fermionic [30], hence no more than one such M2-brane is allowed to exist without becoming non-BPS. This fact is related to the “s-rule” for the D3-D5-NS5 brane system of Hanany and Witten [31] by reduction on \( S_{i+1} \) to type IIA string theory and subsequent T-duality on \( S_{i+2} \).

4.2 Charge conservation

There are also constraints on the numbers of M2-branes that can be placed on a set of six edges meeting at a vertex. To see why such constraints exist, consider reduction on \( S_0 \) to type IIA string theory. Then, the M5-brane \( \text{M5}_i[\ell_i] \) becomes a D4-brane \( \text{D4}_i[\ell_i] \) whose low-energy physics is described by 5D abelian super Yang-Mills theory, and M2-branes ending on \( \text{D4}_i[\ell_i] \) become fundamental strings whose ends behave as electrically charged particles in this theory. The conservation of electric charge leads to relations among the numbers of M2-branes.

Consider the fundamental strings coming from the M2-branes living on the four edges \( e_1[\ell_1, \ell_2, \ell_3], e_2[\ell_1, \ell_2, \ell_3], e_1[\ell_1 + 1, \ell_2, \ell_3] \) and \( e_2[\ell_1, \ell_2 + 1, \ell_3] \). They create charged particles in the theory on \( \text{D4}_3[\ell_3] \). We can view the edges \( e_1[\ell_1, \ell_2, \ell_3] \) and \( e_2[\ell_1, \ell_2, \ell_3] \) as the worldlines of two incoming charged particles. They scatter at the vertex \( v[\ell_1, \ell_2, \ell_3] \) and fly away as two outgoing charged particles whose worldlines are \( e_1[\ell_1 + 1, \ell_2, \ell_3] \) and \( e_2[\ell_1, \ell_2 + 1, \ell_3] \). Each string creates a charge of unit magnitude, but the sign of the charge depends on the position of the string relative to \( \text{D4}_3[\ell_3] \). For strings along \( e_1[\ell_1, \ell_2, \ell_3] \) and \( e_1[\ell_1 + 1, \ell_2, \ell_3] \), the charges created by them are positive if \( x^4(\text{M5}_2[\ell_2]) < x^4(\text{M5}_3[\ell_3]) \) and negative if \( x^4(\text{M5}_2[\ell_2]) > x^4(\text{M5}_3[\ell_3]) \). Similarly, for strings along \( e_2[\ell_1, \ell_2, \ell_3] \) and \( e_2[\ell_1, \ell_2 + 1, \ell_3] \), the charges are positive if \( x^5(\text{M5}_1[\ell_1]) < x^5(\text{M5}_3[\ell_3]) \) and negative if \( x^5(\text{M5}_1[\ell_1]) > x^5(\text{M5}_3[\ell_3]) \).

Equating the charges before and after the scattering, we obtain the relation

\[
  n_1[\ell_1, \ell_2, \ell_3] - n_2[\ell_1, \ell_2, \ell_3] = n_1[\ell_1 + 1, \ell_2, \ell_3] - n_2[\ell_1, \ell_2 + 1, \ell_3].
\]

(4.3)

\footnote{The charge assignments are symmetric under the cyclic permutation \((1, 4) \rightarrow (2, 5) \rightarrow (3, 6) \rightarrow (1, 4)\): in \( \text{D4}_1[\ell_1] \), the other ends of these strings are charged positively if \( x^5(\text{M5}_3[\ell_3]) < x^5(\text{M5}_1[\ell_1]) \) and negatively if \( x^5(\text{M5}_3[\ell_3]) > x^5(\text{M5}_1[\ell_1]) \).}
where we have defined

\[
  n_i[\ell_1, \ell_2, \ell_3] = \begin{cases} 
  +n_i[\ell_1, \ell_2, \ell_3] & (x^{i+3}(M5_{i+1}[\ell_{i+1}]) < x^{i+3}(M5_{i+2}[\ell_{i+2}])) \\
  -n_i[\ell_1, \ell_2, \ell_3] & (x^{i+3}(M5_{i+1}[\ell_{i+1}]) > x^{i+3}(M5_{i+2}[\ell_{i+2}])) \end{cases},
\]

(4.4)

More generally, the charge conservation implies

\[
  n_i[\ell_1, \ell_2, \ell_3] - n_j[\ell_1, \ell_2, \ell_3] = n_i[\ell_1 + \delta_{1i}, \ell_2 + \delta_{2i}, \ell_3 + \delta_{3i}] - n_j[\ell_1 + \delta_{1j}, \ell_2 + \delta_{2j}, \ell_3 + \delta_{3j}].
\]

(4.5)

Summing this equation over \( \ell_j \), we find that for distinct \( i, j, k \in \{1, 2, 3\} \), the quantity

\[
  n_i(\Sigma_k[\ell_k]) = \sum_{\ell_j} n_i[\ell_1, \ell_2, \ell_3]
\]

(4.6)

is independent of \( \ell_i \). In other words, the charge flowing in the \( x^i \)-direction on \( \Sigma_k[\ell_k] \) is conserved. Further summing over \( \ell_k \), we obtain a constant

\[
  n_i = \sum_{\ell_k} n_i(\Sigma_k[\ell_k]) = \sum_{\ell_j, \ell_k} n_i[\ell_1, \ell_2, \ell_3],
\]

(4.7)

which is the total charge flowing in the \( x^i \)-direction in the system.

### 4.3 Partition function

For a spin configuration \( \mathbf{n} \), let \( \mathcal{H}_{\text{BPS}}^\mathbf{n} \) be the subspace of \( \mathcal{H}_{\text{BPS}} \) that contain \( n_i[\ell_1, \ell_2, \ell_3] \) M2-branes on \( e_i[\ell_1, \ell_2, \ell_3] \). The partition function of the model is given by the sum

\[
  Z = \sum_{\mathbf{n} \in \mathbb{N}} Z_\mathbf{n}
\]

(4.8)

over all spin configurations of the index \( Z_\mathbf{n} \) of \( \mathcal{H}_{\text{BPS}}^\mathbf{n} \):

\[
  Z_\mathbf{n} = \text{tr}_{\mathcal{H}_{\text{BPS}}^\mathbf{n}}((-1)^F q^{J_8 - J_{36}} e^{-\beta(H - E_0)}).
\]

(4.9)

For each \( \mathbf{n} \), the factor \( e^{-\beta(H - E_0)} \) in the trace is constant and can be calculated as follows.

Each BPS M2-brane has a mass equal to the product of its area and the unit M2-brane charge \( Q_{\text{M2}} \). The mass of the \( n_i[\ell_1, \ell_2, \ell_3] \) M2-branes placed along the edge \( e_i[\ell_1, \ell_2, \ell_3] \) is

\[
  E_i[\ell_1, \ell_2, \ell_3] = Q_{\text{M2}} n_i[\ell_1, \ell_2, \ell_3] \left( x^i(M5_i[\ell_i]) - x^i(M5_i[\ell_i - 1]) \right) \times \left( x^{i+3}(M5_{i+2}[\ell_{i+2}]) - x^{i+3}(M5_{i+1}[\ell_{i+1}]) \right).
\]

(4.10)

For a state in \( \mathcal{H}_{\text{BPS}}^\mathbf{n} \), the total mass of the M2-branes on the edges is

\[
  \sum_{\ell_1, \ell_2, \ell_3} 3 \sum_{i=1}^3 E_i[\ell_1, \ell_2, \ell_3] = 3 \sum_{i=1}^3 Q_{\text{M2}} e_i \left( \sum_{\ell_{i+2}} n_i(\Sigma_{i+2}[\ell_{i+2}]) x^{i+3}(M5_{i+2}[\ell_{i+2}]) - \sum_{\ell_{i+1}} n_i(\Sigma_{i+1}[\ell_{i+1}]) x^{i+3}(M5_{i+1}[\ell_{i+1}]) \right),
\]

(4.11)
where
\[ c_i = \sum_{\ell_i} (x^i(M5_\ell[i]) - x^i(M5_\ell[i] - 1)) \] (4.12)
is the circumference of \( S_i \). The configuration of M2-branes close to the vertices may be complicated, but this part, being localized, does not contribute to the energy.

Hence, \( Z_n \) takes the form
\[ Z_n = \hat{Z}_n \prod_{\ell_1} \lambda_2(\Sigma_1[\ell_1])^{n_2(\Sigma_1[\ell_1])} \prod_{\ell_2} \lambda_3(\Sigma_2[\ell_2])^{n_3(\Sigma_2[\ell_2])} \prod_{\ell_3} \lambda_4(\Sigma_3[\ell_3])^{n_4(\Sigma_3[\ell_3])}, \] (4.13)
with
\[ \lambda_j(\Sigma_i[\ell_i]) = \exp(-\beta Q_{M2} x^{j+3}(M5_i[\ell_i])) \] (4.14)
and
\[ \hat{Z}_n = \text{tr}_{\text{BPS}}((-1)^F q^{c_1 - J_0}). \] (4.15)
The factor \( \hat{Z}_n \) is the summand for \( n \in \mathbb{N} \) in the definition (3.7) of \( \hat{Z} \) (when the signs of all surfaces are +).

The factor \( \lambda_j(\Sigma_i[\ell_i])^{n_j(\Sigma_i[\ell_i])} \) is independent of the \( x^j \)-coordinate of the point on \( \Sigma_i[\ell_i] \) at which it is evaluated. If it is evaluated between, say \( x^j(\Sigma_j[L_j]) \) and \( x^j(\Sigma_j[1]) \), then it can be interpreted as the contribution from twisting of the periodic boundary condition for \( \Sigma_i[\ell_i] \) in the \( x^j \)-direction. The TQFT interpretation of this factor is a Wilson loop of a background gauge field on \( \Sigma_i[\ell_i] \) wrapped around \( S_j \); the \( x^j \)-component of the gauge field is \(-\epsilon_{ijk} \beta Q_{M2} x^{j+3}(M5_i[\ell_i])\), where \( k \in \{1, 2, 3\} \setminus \{i, j\} \) and \( \epsilon_{ijk} \) is the component of the completely antisymmetric tensor with \( \epsilon_{123} = 1 \). Since the Wilson loop is gauge invariant, it can be evaluated in any gauge. In particular, we can choose a gauge such that the gauge field is zero everywhere except between \( x^j(\Sigma_j[L_j]) \) and \( x^j(\Sigma_j[1]) \), thereby connecting to the interpretation as a twisting parameter for the periodic boundary condition.

The Wilson loop interpretation is not just an analogy. Consider the type IIA setup described in section 4.2, which is obtained by reduction on \( S_0 \). Each D4-brane in that setup supports a \( U(1) \) gauge field as well as five scalar fields parametrizing its transverse positions. Take \( D4_3[\ell_3] \) for definiteness. The components \( A_1, A_2, A_6 \) of the gauge field \( A \) on \( D4_3[\ell_3] \) combines with the scalar fields \( \phi_4, \phi_5, \phi_3 \) parametrizing the positions in \( \mathbb{R}_4, \mathbb{R}_5, S_3 \) to form a partial complex \( GL(1) \) gauge field
\[ A = (A_1 + i\phi_4)dx^1 + (A_2 + i\phi_5)dx^2 + (A_6 + i\phi_3)dx^6. \] (4.16)
(In fact, the BPS sector the worldvolume theory on \( D4_3[\ell_3] \) is described by Chern-Simons theory on \( T_{12} \times \mathbb{R}_6 \) with gauge field \( A \) and gauge group \( GL(1) \) [32–34].) At low energies \( A \) can be considered as nondynamical. The value of \( A \) can be taken to be zero here because we did not turn on background fields on the M5-branes, whereas the value of \( \phi^j \) sets the \( x^j \)-coordinate of \( D4_3[\ell_3] \) (and hence of \( M5_3[\ell_3] \)). The coupling of \( A \) to open strings ending on \( D4_3[\ell_3] \) produces the Wilson loop.
4.4 R-matrices

We have thus reduced the computation of $Z_n$ to that of $\hat{Z}_n$. As explained in section 3.2, the latter is given by the product of the local Boltzmann weights for the configuration $n$. The local Boltzmann weights are described by the R-matrices assigned to the vertices, defined via the TQFT that emerges in the limit in which all M2-branes become massless. Let us deduce some properties these R-matrices should have.

First of all, we need to address one important subtlety which we did not discuss in section 3: the massless limit is realized by going to a special point in the configuration space of the M5-branes, but the brane system may not behave continuously at this point. For example, M2-branes may be created or annihilated when two M5-branes pass through each other. In order to describe the lattice model using the R-matrices, we need to specify the relative positions of the M5-branes from which we start to take the massless limit. We will not try to present a completely satisfactory treatment of this point. Instead, in what follows we will content ourselves with the following particular choice of the relative positions.

To the surface defects in the TQFT forming the cubic lattice, we introduce the ordering

$$\Sigma_1[1] < \Sigma_1[2] < \cdots < \Sigma_1[L_1] < \Sigma_2[1] < \cdots < \Sigma_2[L_2] < \Sigma_3[1] < \cdots < \Sigma_3[L_3].$$  \hspace{1cm} (4.17)

For any two surfaces $\Sigma_a$, $\Sigma_b$ such that $\Sigma_a < \Sigma_b$ and $\Sigma_a \cap \Sigma_b \neq \emptyset$, we require that the corresponding M5-branes $M_5a$ and $M_5b$, which intersect along $\Sigma_a \cap \Sigma_b \subset M \subset T^* M$, are infinitesimally separated in the fiber direction so that $M_5a$ is “closer” to the base than $M_5b$. In other words, we take the massless limit from a configuration such that

$$x^{k+3}(M_5_i[\ell_i]) < x^{k+3}(M_5_j[\ell_j]), \quad i < j, \quad k \in \{1, 2, 3\} \setminus \{i, j\}. \hspace{1cm} (4.18)$$

With this convention, all factors of R-matrices involved in the computation of $\hat{Z}$ are specified unambiguously.

Let $\Sigma_a$, $\Sigma_b$, $\Sigma_c$ be three surfaces such that $\Sigma_a < \Sigma_b < \Sigma_c$ and $\Sigma_a \cap \Sigma_b \cap \Sigma_c \neq \emptyset$. The space $V_{(ab)}$ assigned to $\Sigma_a \cap \Sigma_b$ is isomorphic to a vector space $V^{\sigma_a \sigma_b}$ determined by the signs $\sigma_a$ of $\Sigma_a$ and $\sigma_b$ of $\Sigma_b$. It is either the bosonic Fock space $F^{(0)}$ or the fermionic Fock space $F^{(1)}$:

$$V^{\sigma_a \sigma_b} = \begin{cases} F^{(0)} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}|n\rangle & (\sigma_a = \sigma_b); \\ F^{(1)} = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle & (\sigma_a \neq \sigma_b). \end{cases} \hspace{1cm} (4.19)$$

The state $|1\rangle \in F^{(1)}$ is fermionic, whereas $|0\rangle \in F^{(1)}$ and all states in $F^{(0)}$ are bosonic. The R-matrix $R_{(abc)}$ assigned to the intersection $\Sigma_a \cap \Sigma_b \cap \Sigma_c$ is a linear operator

$$R^{\sigma_a \sigma_b \sigma_c} : V^{\sigma_a \sigma_c} \otimes V^{\sigma_a \sigma_c} \otimes V^{\sigma_a \sigma_b} \to V^{\sigma_a \sigma_c} \otimes V^{\sigma_a \sigma_c} \otimes V^{\sigma_a \sigma_b}. \hspace{1cm} (4.20)$$

The matrix elements of $R^{\sigma_a \sigma_b \sigma_c}$ are defined by

$$R^{\sigma_a \sigma_b \sigma_c}(|l\rangle \otimes |m\rangle \otimes |n\rangle) = \sum_{l', m', n'} |l'\rangle \otimes |m'\rangle \otimes |n'\rangle (R^{\sigma_a \sigma_b \sigma_c})_{lmn}^{l'm'n'}. \hspace{1cm} (4.21)$$
Some of them are easily determined. The matrix element \( (R_{\sigma_\alpha \sigma_\beta \sigma_\gamma})^{000}_{000} \) corresponds to the unique situation in which there is no M2-brane, so we have
\[
(R_{\sigma_\alpha \sigma_\beta \sigma_\gamma})^{000}_{000} = 1. \tag{4.22}
\]
With the convention (4.18), \( |l\rangle \otimes |m\rangle \otimes |n\rangle \in V_{\sigma_\beta \sigma_\gamma} \otimes V_{\sigma_\alpha \sigma_\gamma} \otimes V_{\sigma_\alpha \sigma_\beta} \) has the charges \( l = l', m = -m, n = n \). Charge conservation then implies that \( (R_{\sigma_\alpha \sigma_\beta \sigma_\gamma})^{l'n'm'}_{lmn} = 0 \) unless \( l + m = l' + m', \quad m + n = m' + n'. \tag{4.23} \)

Note that \( l + n \equiv l' + n' \mod 2 \). Since either none of \( V_{(ab)}, V_{(ac)}, V_{(bc)} \) or exactly two of them are the fermionic Fock space, \( R_{\sigma_\alpha \sigma_\beta \sigma_\gamma} \) is a bosonic operator.

The R-matrix also has properties that reflect symmetries of the brane system. Consider a rotation that exchanges \( R_{78} \) and \( R_{9} \). This is a symmetry of the brane system if the signs of all M5-branes are reversed at the same time. The definition of \( Z \), however, involves the operator \( q^{J_{78} - J_9} \) whose exponent changes sign under the rotation. It follows that
\[
(R_{\sigma_\alpha \sigma_\beta \sigma_\gamma})^{-} = R_{\sigma_\alpha \sigma_\beta \sigma_\gamma}, \tag{4.24}
\]
where \( R_{\sigma_\alpha \sigma_\beta \sigma_\gamma} \) is \( R_{\sigma_\alpha \sigma_\beta \sigma_\gamma} \) with \( q \) replaced by \( q^{-1} \). The reflection \( x^\mu \rightarrow -x^\mu, \mu = 1, 2, \ldots, 6 \), leaving the symplectic structure of \( T^* M \) invariant, is likewise a symmetry. This reverses the ordering of surfaces (4.17), so we find
\[
(R_{\sigma_\alpha \sigma_\beta \sigma_\gamma})^{l'm'n'}_{lmn} = (R_{\sigma_\alpha \sigma_\beta \sigma_\gamma})^{n'm'l'}_{nml}. \tag{4.25}
\]

(Imagine the parity reversal of the brane picture in figure 4.)

Lastly, there are properties that follow from the protected nature of \( \hat{Z} \) in the same way as how the commutativity of transfer matrices was deduced in section 4.6. One such property is that the R-matrix is an involution:
\[
R_{\sigma_\alpha \sigma_\beta \sigma_\gamma} = (R_{\sigma_\alpha \sigma_\beta \sigma_\gamma})^{-1}. \tag{4.26}
\]
This property expresses the equality between the two configurations of surfaces shown in figure 6. Another property of this kind is the supertetrahedron equation, to which we now turn.

### 4.5 Supertetrahedron equation

The most remarkable property of the R-matrices is that they solve the supertetrahedron equation. We have introduced this equation in section 3.4 in the case in which all surfaces...
have the same sign and hence all states are bosonic. In this case the supertetrahedron equation reduces to the ordinary tetrahedron equation \((3.17)\). In general, the surfaces can have either sign and states can be bosonic or fermionic. These extra degrees of freedom make the equation more complex.

The supertetrahedron equation can be written most concisely in the language of Fock spaces. Consider four surfaces \(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\) as in figure 5. On the intersections of these surfaces live the six spaces \(V_{(12)}, V_{(13)}, V_{(14)}, V_{(23)}, V_{(24)}, V_{(34)}\). The space \(V_{(ab)}\) is a Fock space generated from a vacuum state \(|0\rangle_{(ab)}\) by the action of a creation operator \(c_{(ab)}\). We introduce the Fock space \(V_{1234}\) generated from a vacuum state \(|0\rangle_{1234}\) by the action of the creation operators \(c_{(12)}, c_{(13)}, c_{(14)}, c_{(23)}, c_{(24)}, c_{(34)}\); it is spanned by the states of the form

\[
|n_{(12)} n_{(13)} n_{(14)} n_{(23)} n_{(24)} n_{(34)}\rangle |0\rangle_{1234},
\]

where \(n_{(ab)} \in \mathbb{Z}_{\geq 0}\) or \(\{0, 1\}\) depending on whether \(c_{(ab)}\) is bosonic or fermionic. For distinct \(a, b, c, d \in \{1, 2, 3, 4\}\), we have an isomorphism

\[
V_{1234} \cong V_{(ab)} \otimes V_{(ac)} \otimes V_{(ad)} \otimes V_{(bc)} \otimes V_{(bd)} \otimes V_{(cd)}
\]

as vector spaces. The notation

\[
|l\rangle_{(ab)} |m\rangle_{(ac)} |n\rangle_{(ad)} |p\rangle_{(bc)} |q\rangle_{(bd)} |r\rangle_{(cd)} = c_{(ab)}^l c_{(ac)}^m c_{(ad)}^n c_{(bc)}^p c_{(bd)}^q c_{(cd)}^r |0\rangle_{1234}
\]

makes the isomorphism manifest.

In this notation the R-matrix at \((abc)\) is represented in \(V_{1234}\) by the operator

\[
R_{(abc)} = \sum_{l,m,n,l',m',n'} (R_{\sigma^a \sigma^b \sigma^c})^{l'm'n'}_{lmn} |l'\rangle_{(bc)} |m'\rangle_{(ac)} |n'\rangle_{(ad)} |n\rangle_{(ab)} |m\rangle_{(ac)} \langle m|_{(ac)} \langle n|_{(bc)} \langle l|.
\]

The action of \(R_{(abc)}\) is defined by

\[
R_{(abc)} |l\rangle_{(bc)} |m\rangle_{(ac)} |n\rangle_{(ad)} |p\rangle_{(bc)} |q\rangle_{(bd)} |r\rangle_{(cd)} = \sum_{l',m',n'} (R_{\sigma^a \sigma^b \sigma^c})^{l'm'n'}_{lmn} |l'\rangle_{(bc)} |m'\rangle_{(ac)} |n'\rangle_{(ad)} |p\rangle_{(bc)} |q\rangle_{(bd)} |r\rangle_{(cd)}.
\]

Translating the two brane configurations in figure 5 to the operator language we obtain the \textit{supertetrahedron equation}

\[
R_{(234)} R_{(134)} R_{(124)} R_{(123)} = R_{(123)} R_{(124)} R_{(134)} R_{(234)}.
\]

The tetrahedron equation is usually expressed as an equality between operators acting on the right-hand side of the isomorphism \((4.28)\). Let us recast the supertetrahedron equation \((4.32)\) in this form. Doing so introduces nontrivial sign factors.

Let us define the fermion number operator \(F\) by

\[
F |n\rangle = \begin{cases} 
0 & (|n\rangle \in \mathcal{F}^{(0)}) \\
n |n\rangle & (|n\rangle \in \mathcal{F}^{(1)}) 
\end{cases},
\]
and let $P$ be the operator that swaps factors in the tensor product of two spaces with an appropriate sign:

$$P(|m⟩ \otimes |n⟩) = (-1)^{F \otimes F} |n⟩ \otimes |m⟩).$$  \hfill (4.34)

We define the operator

$$\tilde{R}^{\sigma_a \sigma_b \sigma_c} = R^{\sigma_a \sigma_b \sigma_c} (P \otimes 1)(1 \otimes P)(P \otimes 1).$$  \hfill (4.35)

The matrix elements of $\tilde{R}^{\sigma_a \sigma_b \sigma_c} : \mathcal{V}^{\sigma_a \sigma_b} \otimes \mathcal{V}^{\sigma_b \sigma_c} \otimes \mathcal{V}^{\sigma_b \sigma_c} \rightarrow \mathcal{V}^{\sigma_a \sigma_c} \otimes \mathcal{V}^{\sigma_a \sigma_c} \otimes \mathcal{V}^{\sigma_a \sigma_b}$ are given by

$$(\tilde{R}^{\sigma_a \sigma_b \sigma_c})_{lmn}^{l'm'n'} = (-1)_{l}^{mn} \delta_{\sigma_a \sigma_b} \delta_{\sigma_c \sigma_c} + mn \delta_{\sigma_a \sigma_b} \delta_{\sigma_b \sigma_c} + nl \delta_{\sigma_a \sigma_b} \delta_{\sigma_b \sigma_c} (R^{\sigma_a \sigma_b \sigma_c})_{l'm'n'},$$ \hfill (4.36)

where $\delta_{\sigma_a \sigma_b} = 1 - \delta_{\sigma_a \sigma_b}$. Finally, we rename the vector spaces involved in the supertetrahedron equation as

$$\mathcal{V}_1 = \mathcal{V}^{\sigma_1 \sigma_4}, \quad \mathcal{V}_2 = \mathcal{V}^{\sigma_2 \sigma_4}, \quad \mathcal{V}_3 = \mathcal{V}^{\sigma_2 \sigma_3}, \quad \mathcal{V}_4 = \mathcal{V}^{\sigma_1 \sigma_4}, \quad \mathcal{V}_5 = \mathcal{V}^{\sigma_1 \sigma_3}, \quad \mathcal{V}_6 = \mathcal{V}^{\sigma_1 \sigma_2}.$$ \hfill (4.37)

In terms of the operators and spaces introduced above, the supertetrahedron equation can be expressed as the following equality between homomorphisms from $\mathcal{V}_3 \otimes \mathcal{V}_3 \otimes \mathcal{V}_4 \otimes \mathcal{V}_5 \otimes \mathcal{V}_2 \otimes \mathcal{V}_1$ to $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \mathcal{V}_4 \otimes \mathcal{V}_5 \otimes \mathcal{V}_6$:

\begin{equation}
\begin{split}
\tilde{R}^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \tilde{R}^{\sigma_1 \sigma_3 \sigma_4} P_{23} P_{36} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} P_{34} = P_{34} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} P_{34} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} P_{12} P_{45} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4}.
\end{split}
\end{equation}

(4.38)

The subscripts on an operator indicate the positions in the relevant tensor product space at which the operator acts. In components,

\begin{equation}
\begin{split}
\sum_{n_1', n_2', n_3', n_4', n_5', n_6'} (-1)^{n_1 n_6} \delta_{n_3 n_4} \delta_{n_1 n_2} + n_2 n_5 \delta_{n_2 n_3} + n_4 n_5 \delta_{n_3 n_4} \delta_{n_1 n_2} \\
\times \left( \tilde{R}^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \right)_{n_1 n_2 n_3 n_4}^{n_1' n_2' n_3'} \left( \tilde{R}^{\sigma_1 \sigma_3 \sigma_4} \right)_{n_4 n_5 n_6}^{n_4' n_5' n_6'} \left( \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \right)_{n_1 n_2 n_3}^{n_1' n_2' n_3'} = \\
\sum_{n_1', n_2', n_3', n_4', n_5', n_6'} (-1)^{n_1 n_6} \delta_{n_3 n_4} \delta_{n_1 n_2} + n_2 n_5 \delta_{n_2 n_3} + n_4 n_5 \delta_{n_3 n_4} \delta_{n_1 n_2} \\
\times \left( \tilde{R}^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \right)_{n_1 n_2 n_3}^{n_1' n_2' n_3'} \left( \tilde{R}^{\sigma_1 \sigma_3 \sigma_4} \right)_{n_4 n_5 n_6}^{n_4' n_5' n_6'} \left( \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \right)_{n_1 n_2 n_3}^{n_1' n_2' n_3'}.
\end{split}
\end{equation}

(4.39)

Defining $\tilde{R}^{\sigma_a \sigma_b \sigma_c}_{\alpha,\beta,\gamma} \in \text{End}(\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \mathcal{V}_4 \otimes \mathcal{V}_5 \otimes \mathcal{V}_6)$ by

\begin{equation}
\begin{split}
\tilde{R}^{\sigma_a \sigma_b \sigma_c}_{\alpha,\beta,\gamma} (|n_\alpha⟩ \otimes \cdots \otimes |n_\beta⟩ \otimes \cdots \otimes |n_\gamma⟩ \otimes \cdots) = \\
\sum \left( \tilde{R}^{\sigma_a \sigma_b \sigma_c} \right)_{n_\alpha n_\beta n_\gamma}^{n_\alpha' n_\beta' n_\gamma'} (|n_\alpha'⟩ \otimes \cdots \otimes |n_\beta'⟩ \otimes \cdots \otimes |n_\gamma'⟩ \otimes \cdots),
\end{split}
\end{equation}

(4.40)

we can also write the supertetrahedron equation as

\begin{equation}
\begin{split}
\tilde{R}^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} (\mathcal{F}_1 \mathcal{F}_6 + \mathcal{F}_2 \mathcal{F}_5) \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \\
= (-1)^{\mathcal{F}_1 \mathcal{F}_6 + \mathcal{F}_2 \mathcal{F}_5} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \\
= (-1)^{\mathcal{F}_1 \mathcal{F}_6 + \mathcal{F}_2 \mathcal{F}_5} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \tilde{R}^{\sigma_1 \sigma_2 \sigma_3} \tilde{R}^{\sigma_1 \sigma_2 \sigma_4} \
\end{split}
\end{equation}

(4.41)

The sign factors are transparent in this expression.
The exchange of indices

The remaining eight are obtained from the above relations by the replacement

\[ R = R^{+++} \in \text{End}(\mathcal{F}^{(0)} \otimes \mathcal{F}^{(0)} \otimes \mathcal{F}^{(0)}), \]

\[ L = R^{++-} \in \text{End}(\mathcal{F}^{(1)} \otimes \mathcal{F}^{(1)} \otimes \mathcal{F}^{(0)}), \]

\[ M = R^{+-+} \in \text{End}(\mathcal{F}^{(0)} \otimes \mathcal{F}^{(1)} \otimes \mathcal{F}^{(1)}), \]

\[ N = R^{-++} \in \text{End}(\mathcal{F}^{(1)} \otimes \mathcal{F}^{(0)} \otimes \mathcal{F}^{(1)}). \]

In view of the properties of R-matrices deduced in section 4.4, these operators satisfy

\[ R_{lmn}^{m'n'} = R_{mnl}^{n'm'}, \]

\[ L_{lmn}^{m'n'} = M_{mnl}^{n'm'}, \]

\[ N_{lmn}^{m'n'} = N_{mnl}^{n'm'}. \]

and

\[ R = R^{---} \in \text{End}(\mathcal{F}^{(0)} \otimes \mathcal{F}^{(0)} \otimes \mathcal{F}^{(0)}), \]

\[ L = R^{--+} \in \text{End}(\mathcal{F}^{(1)} \otimes \mathcal{F}^{(1)} \otimes \mathcal{F}^{(0)}), \]

\[ M = R^{+-+} \in \text{End}(\mathcal{F}^{(0)} \otimes \mathcal{F}^{(1)} \otimes \mathcal{F}^{(1)}), \]

\[ N = R^{++-} \in \text{End}(\mathcal{F}^{(1)} \otimes \mathcal{F}^{(0)} \otimes \mathcal{F}^{(1)}). \]

Eight of the 16 relations are

\[
\begin{align*}
(++++) & \quad \hat{R}_{1,2,3} \hat{R}_{1,4,5} \hat{R}_{2,4,6} \hat{R}_{3,5,6} = \hat{R}_{3,5,6} \hat{R}_{2,4,6} \hat{R}_{1,4,5} \hat{R}_{1,2,3}, \\
(++++) & \quad \hat{L}_{1,2,3} \hat{L}_{1,4,5} \hat{L}_{2,4,6} \hat{L}_{3,5,6} = \hat{R}_{3,5,6} \hat{L}_{2,4,6} \hat{L}_{1,4,5} \hat{L}_{1,2,3}, \\
(++++) & \quad \hat{N}_{1,2,3} \hat{N}_{1,4,5} \hat{R}_{2,4,6} \hat{L}_{3,5,6} = \hat{L}_{3,5,6} \hat{R}_{2,4,6} \hat{N}_{1,4,5} \hat{N}_{1,2,3}, \\
(++++) & \quad \hat{M}_{1,2,3} \hat{M}_{1,4,5} \hat{R}_{2,4,6} \hat{L}_{3,5,6} = \hat{N}_{3,5,6} \hat{N}_{2,4,6} \hat{R}_{1,4,5} \hat{N}_{1,2,3}, \\
(----) & \quad \hat{N}_{1,2,3} \hat{L}_{1,4,5} \hat{N}_{2,4,6} \hat{L}_{3,5,6} = \hat{N}_{3,5,6} \hat{N}_{2,4,6} \hat{L}_{1,4,5} \hat{N}_{1,2,3}, \\
(----) & \quad \hat{N}_{1,2,3} \hat{L}_{1,4,5} \hat{N}_{2,4,6} \hat{L}_{3,5,6} = \hat{N}_{3,5,6} \hat{N}_{2,4,6} \hat{L}_{1,4,5} \hat{N}_{1,2,3}, \\
(----) & \quad \hat{N}_{1,2,3} \hat{L}_{1,4,5} \hat{N}_{2,4,6} \hat{L}_{3,5,6} = \hat{N}_{3,5,6} \hat{N}_{2,4,6} \hat{L}_{1,4,5} \hat{N}_{1,2,3}, \\
(----) & \quad \hat{N}_{1,2,3} \hat{L}_{1,4,5} \hat{N}_{2,4,6} \hat{L}_{3,5,6} = \hat{N}_{3,5,6} \hat{N}_{2,4,6} \hat{L}_{1,4,5} \hat{N}_{1,2,3},
\end{align*}
\]

where \( \hat{n} \) is the number operator defined by

\[ \hat{n}|n\rangle = n|n\rangle. \]

The remaining eight are obtained from the above relations by the replacement \( q \rightarrow q^{-1} \).

The exchange of indices \((1,2) \leftrightarrow (6,5)\) interchanges relations (4.55) and (4.56) and relations (4.54) and (4.60) (without the underlines).


## 4.6 Integrability

In section 3.3 we deduced that the transfer matrices commute in the special case in which all M2-branes are massless. A surface defect $\Sigma$ in the emergent TQFT comes with a single discrete parameter, the sign $\sigma(\Sigma)$ specifying which of the planes $\mathbb{R}^{78}_{T}$ and $\mathbb{R}^{2}_{Z}$ the corresponding M5-brane covers. Accordingly, the transfer matrix $T_3[\ell_3]$ for $\hat{Z}$ has a discrete parameter $\sigma(M5[\ell_3])$. In our discussion in section 3.3, we chose this sign to be $+$ and wrote $\hat{T}_3[\ell_3]$ as $\hat{\mathcal{T}}_3[\ell_3]$.

The transfer matrix $T_3[\ell_3]$ of the model defined by the index $Z$, as opposed to its massless limit $\hat{Z}$, has additional continuous parameters, namely the coordinates of M5$_3[\ell_3]$ in $\mathbb{R}^2$. Expanding $T_3[\ell_3]$ in these parameters, we obtain a series of operators acting on the Hilbert space of the equivalent 2D quantum spin model, and we can declare that one of them is the Hamiltonian. If the commutativity of transfer matrices continues to hold, then all of these operators commute with each other. The commutativity therefore implies that the model is quantum integrable: there is a series of commuting conserved charges generated by the transfer matrix.

The commutativity of transfer matrices can be argued as follows. Fix the size of the lattice and the positions of the M5-branes in $\mathbb{R}^3_3$. Then, the energy of a BPS state is determined solely by the charges flowing on the surface defects according to formula (4.11). Let $E$ be the spectrum of $H - E_0$ in $\mathcal{H}_{\text{BPS}}$, which is discrete, and let $\mathcal{H}_E^j$ be the subspace of $\mathcal{H}$ in which $H - E_0 = E$ and $J_{78} - J_{32} = j$. We can write the partition function as

$$Z = \sum_{E \in E} \sum_{j = -\infty}^{\infty} \text{tr}_{\mathcal{H}_{\text{BPS}}^E} (-1)^F e^{-\beta_E q^j},$$

(4.62)

where $\mathcal{H}_{\text{BPS}}^E = \mathcal{H}_E^j \cap \mathcal{H}_{\text{BPS}}$. The coefficient of $e^{-\beta_E q^j}$ is the Witten index of $\mathcal{H}_E^j$ and invariant under continuous movement of the M5-branes in $T_{123}$. In particular, we can switch the positions of M5$_3[\ell_3]$ and M5$_3[\ell_3 + 1]$ in $S_3$ continuously. Throughout the process the two M5-branes do not intersect in the 11D spacetime, so $Z$ remains the same. Hence, $T_3[\ell_3]$ and $T_3[\ell_3 + 1]$ commute.

Since the above argument is a little abstract, we provide below a more direct derivation of the commutativity of transfer matrices using the supertetrahedron equation and charge conservation.

In figure 5, replace the surface $\Sigma_1$ with a stack of surfaces $\Sigma_1[\ell_1]$, $\ell_1 = 1, \ldots, L_1$, that are all parallel to $\Sigma_1$. Similarly, replace $\Sigma_2$ with a stack of parallel surfaces $\Sigma_2[\ell_2]$, $\ell_2 = 1, \ldots, L_2$. These surfaces should be thought of as (parts of) the 2-tori with the same names in the lattice of the 3D lattice model. The surfaces $\Sigma_3$ and $\Sigma_4$ will be identified with $\Sigma_3[\ell_3]$ and $\Sigma_4[\ell_3 + 1]$. We place these surfaces in such a way that their intersections look like the corresponding horizontal layers of the 3D lattice, and introduce the following ordering of the surfaces:

$$\Sigma_1[1] < \Sigma_1[2] < \cdots < \Sigma_1[L_1] < \Sigma_2[1] < \Sigma_2[2] < \cdots < \Sigma_2[L_2] < \Sigma_3 < \Sigma_4.$$  

(4.63)
Let $\mathcal{V}_{(1[\ell_1]2[\ell_2])}$ and $\mathcal{V}_{(a[\ell_a]b)}$, $a = 1, 2, b = 3, 4$, be the Fock spaces assigned to $\Sigma_1[\ell_1] \cap \Sigma_2[\ell_2]$ and $\Sigma_a[\ell_a] \cap \Sigma_b$. We use $\mathcal{V}_{(12)}$ and $\mathcal{V}_{(ab)}$ to denote the Fock spaces such that

$$\mathcal{V}_{(12)} \cong \bigotimes_{\ell_1, \ell_2} \mathcal{V}_{(1[\ell_1]2[\ell_2])}, \quad \mathcal{V}_{(ab)} \cong \bigotimes_{\ell_a} \mathcal{V}_{(a[\ell_a]b)}.$$  \hspace{1cm} (4.64)

Also, let $\mathcal{R}_{(1[\ell_1]2[\ell_2])}$ and $\mathcal{R}_{(a[\ell_a]b)}$, $a = 1, 2$, denote the R-matrices at $\Sigma_1[\ell_1] \cap \Sigma_2[\ell_2] \cap \Sigma_b$ and $\Sigma_a[\ell_a] \cap \Sigma_3 \cap \Sigma_4$, and define

$$\mathcal{R}_{(134)} = \prod_{\ell_1} \mathcal{R}_{(1[\ell_1]34)}, \quad \mathcal{R}_{(234)} = \prod_{\ell_2} \mathcal{R}_{(2[\ell_2]34)}, \quad \mathcal{R}_{(12b)} = \prod_{\ell_1, \ell_2} \mathcal{R}_{(1[\ell_1]2[\ell_2]b)}.$$  \hspace{1cm} (4.65)

The symbols $\prod$ and $\prod$ mean that the operator product is arranged in the lexicographic order of the vertices from left to right and from right to left, respectively. By repeated application of the supertetrahedron equation (4.32) we obtain the equality

$$\mathcal{R}_{(234)}\mathcal{R}_{(134)}\mathcal{R}_{(124)}\mathcal{R}_{(123)} = \mathcal{R}_{(123)}\mathcal{R}_{(124)}\mathcal{R}_{(134)}\mathcal{R}_{(234)}$$  \hspace{1cm} (4.66)

between endomorphisms of $\mathcal{V}_{134} \cong \mathcal{V}_{(12)} \otimes \mathcal{V}_{(13)} \otimes \mathcal{V}_{(14)} \otimes \mathcal{V}_{(23)} \otimes \mathcal{V}_{(24)} \otimes \mathcal{V}_{(34)}$.

By charge conservation, $\hat{n}_{(1[\ell_1]3)} + \hat{n}_{(1[\ell_1]4)}$, $\sum_{\ell_2} \hat{n}_{(1[\ell_2]3)} - \hat{n}_{(1[\ell_2]4)}$ and $\sum_{\ell_2} \hat{n}_{(2[\ell_2]4)} + \hat{n}_{(34)}$ commute with $\mathcal{R}_{(234)}$. Multiplying both sides of (4.66) by

$$\left( \frac{\sum_{\ell_2} \hat{n}_{(2[\ell_2]3)} - \hat{n}_{(1[\ell_2]4)} - \sum_{\ell_2} \hat{n}_{(2[\ell_2]4)} + \hat{n}_{(34)}}{\prod_{\ell_2} \mu_1[\ell_2]^{n_{(2[\ell_2]3)} + n_{(2[\ell_2]4)}}} \right) \mathcal{R}_{(134)}^{-1}$$  \hspace{1cm} (4.67)

and taking the supertrace over $\mathcal{V}_{(23)} \otimes \mathcal{V}_{(24)} \otimes \mathcal{V}_{(34)}$, \footnote{By taking the supertrace over $\mathcal{V}_{(34)} \otimes \mathcal{V}_{(24)} \otimes \mathcal{V}_{(23)}$, we mean that we multiply by $(-1)^{F_{(34)} + \sum_{\ell_2} F_{(2[\ell_2]4)} + \sum_{\ell_2} F_{(3[\ell_2]2)}}$, sandwich by $|l\rangle_{(34)} \prod_{\ell_2} |m[\ell_2]\rangle_{(2[\ell_2]4)} \prod_{\ell_2} |n[\ell_2]\rangle_{(2[\ell_2]3)}$ and its dual, and sum over $l, m[\ell_2], n[\ell_2], \ell_2 = 1, \ldots, L_2$.} we obtain

$$S_{\bullet 23}S_{\bullet 24}S_{\bullet 23} = S_{\bullet 23}S_{\bullet 24}S_{\bullet 34}.$$  \hspace{1cm} (4.68)

where

$$S_{\bullet 2b} = \text{str}_{\mathcal{V}(2b)} \left( \prod_{\ell_2} \left( \frac{\rho_b}{\mu_1[\ell_2]} \right) \mathcal{R}_{(12b)} \right), \quad S_{\bullet 34} = \text{str}_{\mathcal{V}(34)} \left( (\nu_3\nu_4)^{-1} \mathcal{R}_{(134)} \right).$$  \hspace{1cm} (4.69)

Charge conservation implies that $\hat{n}_{(1[\ell_1]3)} + \hat{n}_{(1[\ell_1]4)}$, $\sum_{\ell_1} \hat{n}_{(1[\ell_1]3)}$ and $\sum_{\ell_1} \hat{n}_{(1[\ell_1]4)}$ commute with $S_{\bullet 34}$. Multiplying both sides of (4.68) by

$$\left( \prod_{\ell_1} \frac{\hat{n}_{(1[\ell_1]3)} - \hat{n}_{(1[\ell_1]4)}}{\rho_3 \mu_2[\ell_1]^{n_{(1[\ell_1]3)} + n_{(1[\ell_1]4)}}} \right) S_{\bullet 34}^{-1}$$  \hspace{1cm} (4.70)

and taking the supertrace over $\mathcal{V}_{(13)} \otimes \mathcal{V}_{(14)}$, we find $T_b T_3 = T_2 T_4$ with

$$T_b = \text{str}_{\mathcal{V}(b)} \left( \prod_{\ell_1} \left( \frac{\rho_b}{\mu_2[\ell_1]} \right) \mathcal{R}_{(12b)} \right).$$  \hspace{1cm} (4.71)
Finally, take $\Sigma_3 = \Sigma_3[\ell_3]$, $\Sigma_4 = \Sigma_3[\ell_3 + 1]$ and set
\[
\frac{\nu_b}{\mu_1[\ell_2]} = \frac{\lambda_1(\Sigma_3[\ell_3 + \delta_b])}{\lambda_1(\Sigma_2[\ell_2])}, \quad \frac{\rho_b}{\mu_2[\ell_1]} = \frac{\lambda_2(\Sigma_3[\ell_3 + \delta_b])}{\lambda_2(\Sigma_1[\ell_1])}.
\]

Then we obtain
\[
T_3[\ell_3 + 1]T_3[\ell_3] = T_3[\ell_3]T_3[\ell_3 + 1],
\]
where $T_3[\ell_3]$ can be written, in the notations of section 3.3, as
\[
T_3[\ell_3] = \text{str}_{\ell_2} \otimes \nu_1[1, \ell_2, \ell_3] \otimes \nu_2[1, 1, \ell_3] \left( \prod_{\ell_1, \ell_2} R[\ell_1, \ell_2, \ell_3] \right. \times \frac{\lambda_1(\Sigma_3[\ell_3]) \hat{n}_1(\Sigma_1[\ell_3])}{\prod_{\ell_2} \lambda_1(\Sigma_2[\ell_2])} \frac{\lambda_2(\Sigma_3[\ell_3]) \hat{n}_2(\Sigma_1[\ell_3])}{\prod_{\ell_1} \lambda_2(\Sigma_1[\ell_1])} \left. \right) .
\]

(Here we have suppressed complicated sign factors by employing the Fock space notation for the R-matrices.) The operator $T_3[\ell_3]$ is the transfer matrix with twisted periodic boundary conditions, with the twisting in the $x^4$-direction controlled by the parameter $\lambda_i(\Sigma_3[\ell_3])$: the partition function can be expressed as
\[
Z = \text{str}_{\nu_3[1]} \left( T_3[L_3] \cdots T_3[2] T_3[1] \prod_{\ell_2} \lambda_3(\Sigma_2[\ell_2]) \hat{n}_3(\Sigma_1[\ell_1]) \right) .
\]

Thus we have demonstrated the integrability of the model.

### 4.7 Reduction to 2D lattice models

Given any 3D lattice model on a periodic cubic lattice, one can always view it as a 2D lattice model by treating one of the directions of the lattice as internal degrees of freedom. If the 3D lattice model is integrable, then the resulting 2D lattice model is also integrable, in the sense that the equivalent 1D quantum spin model has commuting transfer matrices with continuous parameters.

For the 3D lattice model constructed from branes, we can make a stronger statement since its R-matrices satisfy the supertetrahedron equation. Replace $\Sigma_4$ in figure 5 with a stack of surfaces $\Sigma_4[\ell]$, $\ell = 1, \ldots, L$. The product R-matrix
\[
R_{(ab4)} = R_{(ab4[L])} R_{(ab4[L−1])} \cdots R_{(ab4[1])}
\]
satisfies the equation
\[
R_{(234)}R_{(134)}R_{(124)}R_{(123)} = R_{(123)}R_{(124)}R_{(134)}R_{(234)} .
\]

Multiplying both sides by
\[
R_{(123)}^{-1} \left( \begin{array}{c} \frac{z_1}{z_2} \\ \frac{z_1}{z_3} \end{array} \right) \hat{n}_{(12)} - \hat{n}_{(23)} \left( \begin{array}{c} \frac{z_1}{z_2} \\ \frac{z_1}{z_3} \end{array} \right) \hat{n}_{(13)} + \hat{n}_{(23)} \right) R_{(123)}^{-1}
\]

\[\boxed{\text{(4.78)}}\]
and taking the supertrace over $V_{(23)} \otimes V_{(13)} \otimes V_{(12)}$, we obtain the Yang-Baxter equation

$$S_{23} \left( \frac{z_2}{z_3} \right) S_{13} \left( \frac{z_1}{z_3} \right) S_{12} \left( \frac{z_1}{z_2} \right) = S_{12} \left( \frac{z_2}{z_3} \right) S_{13} \left( \frac{z_1}{z_2} \right) S_{23} \left( \frac{z_1}{z_2} \right),$$  

(4.79)

where

$$S_{ab}(z) = \text{str}_{V_{ab}} \left( z^{\hat{n}_{ab}} R_{(ab|L)} \right) \cdots R_{(ab|1)}).$$  

(4.80)

See figure 7 for a graphical representation of the Yang-Baxter equation. The operator $S_{ab}(z)$ is the R-matrix describing the local Boltzmann weights of a 2D lattice model, and in this context the parameter $z$ is called the spectral parameter. The integrability of the 2D lattice model is a consequence of the Yang-Baxter equation with spectral parameter.

For comparison with known results in the literature, let us take

$$\sigma_1 = \sigma_2 = \sigma_3 = +.$$  

(4.81)

Then, the sign factors in the supertetrahedron equation (4.41) drop out. Since the sign of $\Sigma_4[\ell]$ can be either $+$ or $-$, we obtain a set of $2^L$ Yang-Baxter equations. Choose an $L$-tuple

$$\epsilon = (\epsilon_1, \ldots, \epsilon_L) \in \{0, 1\}^L$$  

(4.82)

and take

$$\sigma(\Sigma_4[\ell]) = (-1)^{\epsilon_\ell}.$$  

(4.83)

The corresponding solution of the Yang-Baxter equation is

$$S^{(\epsilon)}(z) = \text{tr}_{F^{(0)}} \left( z^{\hat{n}_{\epsilon L}} \cdots \hat{S}^{(\epsilon_1)} \right) \in \text{End}(F^{(\epsilon)} \otimes F^{(\epsilon)}),$$  

(4.84)

where we have defined

$$S^{(0)} = R, \quad S^{(1)} = L$$  

(4.85)

and

$$F^{(\epsilon)} = F^{(\epsilon_1)} \otimes \cdots \otimes F^{(\epsilon_1)}.$$  

(4.86)

The vector space $F^{(\epsilon)}$ is the Fock space with $L_+$ bosonic creation operators and $L_-$ fermionic creation operators, where $L_+$ and $L_-$ are the numbers of 0 and 1 in $\epsilon$, respectively. Let $F^{(\epsilon)}_n$ be the level-$n$ subspace of $F^{(\epsilon)}$, spanned by the vectors of the form

$$|n_{L_1} \otimes \cdots \otimes n_{L_1} \rangle \in F^{(\epsilon_L)} \otimes \cdots \otimes F^{(\epsilon_1)}, \quad n_1 + \cdots + n_{L_1} = n.$$  

(4.87)

By charge conservation, $S^{(\epsilon)}$ leaves $F^{(\epsilon)}_l \otimes F^{(\epsilon)}_m$ invariant. Thus, $S^{(\epsilon)}$ decomposes as

$$S^{(\epsilon)} = \bigoplus_{l,m=0}^{\infty} S^{(\epsilon)}_{l,m}, \quad S^{(\epsilon)}_{l,m} \in \text{End}(F^{(\epsilon)}_l \otimes F^{(\epsilon)}_m),$$  

(4.88)

and each summand satisfies the Yang-Baxter equation. The Lie supergroup $\text{GL}(L_+|L_-)$ acts on $F^{(\epsilon)}_n$ by identifying the creation operators with the standard basis vectors of
Figure 7. A graphical representation of the Yang-Baxter equation that follows from the tetrahedron equation. The vertical direction is periodic.

\[
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \hat{z} \\
\hline
\text{D}3_{1+} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{D}3_{2+} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{D}5_{3+} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{D}3_{1-} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{D}3_{2-} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{D}5_{3-} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{F}1_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{F}1_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{F}1_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Table 3. A configuration of D5-branes, D3-branes and fundamental strings.

\[
\mathbb{C}^{L_+|L_-} \quad \text{As a GL}(L_+|L_-)-module,} \quad \mathcal{F}_n^{(\epsilon)} = \bigoplus_{l,m \in \mathbb{Z}_{\geq 0}} S^l(C^{L_+}) \otimes \Lambda^m(C^{L_-}). \tag{4.89}
\]

Here $S^n(V)$ and $\Lambda^n(V)$ denote the $n$th symmetric and antisymmetric powers of $V$.

From the point of view of the brane system, the reduction of the 3D lattice model to a 2D lattice model amounts to reducing M-theory on $S_3$ to type IIA string theory. Let us further apply T-duality along the periodic Euclidean time direction $S_0$ and S-duality. Equivalently, we can reduce the M-theory setup on $S_0$ and then take T-duality on $S_3$. We will adopt the second presentation and let $\hat{S}_3$ denote the circle dual to $S_3$. If we do so, the M5-branes become D3-branes and D5-branes, and the M2-branes become fundamental strings. The resulting configuration is summarized in table 3. The situation considered above is when the signs of the D3-branes are all $+$ and the sign of the $\ell$th D5-brane is $(-1)^\ell$.

This D3-D5 brane system was studied in [8]. (In fact, a more general setup with D3-branes of both signs was considered there.) Let us summarize the results of [8] pertinent to the present discussion.

The $L_+$ D5-branes of sign $+$ produce 6D $\mathcal{N} = 2$ super Yang-Mills theories with gauge group $U(L_+)$. The twisting of the periodic boundary condition in the $x^0$-direction induces a Ramond-Ramond 2-form background field deforming this theory. The deformation has the
effect of reducing the relevant BPS sector of the 6D theory to 4D Chern-Simons theory \cite{23, 24, 35} with gauge group GL($L_\pm$) and coupling $h \propto \theta$ \cite{7, 32, 36}. The $L_-$ D5-branes of sign $-$ likewise produce 4D Chern-Simons theory with gauge group GL($L_-$) and coupling $-\hbar$. Finally, open strings connecting D5-branes of opposite signs produce 4D Chern-Simons theory with gauge group GL($L_-$) and coupling $\propto \theta$ \cite{7, 32, 36}. Together, these ingredients combine into 4D Chern-Simons theory on $T^2_{12} \times \hat{\mathbb{C}} \times \mathbb{R}_6$ with gauge group GL($L_+ | L_-$) and coupling constant $\hbar$.

The appearance of 4D Chern-Simons theory naturally explains the origin of the integrable 2D lattice model \cite{23, 24}. The cylinder $\hat{\mathbb{C}} \times \mathbb{R}_6$, equipped with the standard complex structure, is conformally equivalent to a punctured plane $\hat{\mathbb{C}}^\times_{36}$. 4D Chern-Simons theory formulated on $T^2_{12} \times \hat{\mathbb{C}}^\times_{36}$ is topological on $T^2_{12}$ and holomorphic on $\hat{\mathbb{C}}^\times_{36}$. It has Wilson line operators lying in $T^2_{12}$ and supported at points in $\hat{\mathbb{C}}^\times_{36}$. We can wrap these Wilson lines around $T^2_{12}$ to form a lattice. The correlation function of such a configuration of Wilson lines computes the partition function of a 2D lattice model, defined by a trigonometric solution of the Yang-Baxter equation associated with the Lie algebra $\mathfrak{g}$ of the gauge group of the theory. Crossing of two Wilson lines yields a factor of the R-matrix, evaluated in the representations of the Wilson lines and with the spectral parameter given by the ratio of the coordinates of the two lines in $\hat{\mathbb{C}}^\times_{36}$.

The D3-branes create line defects in the 4D Chern-Simons theory, extending in the $x^1$-direction and the $x^2$-direction. In the setup studied in \cite{8}, where the holomorphic directions were taken to be $\mathbb{C}$, it was shown that a D3-brane intersecting the D5-branes realizes a quantum mechanical system with global symmetry GL($L_+ | L_-$), coupled to the 4D Chern-Simons theory via gauging. Open strings localized at the intersection of the D3-brane and a D5-brane give rise to an oscillator algebra, which is bosonic if the two branes have the same sign and fermionic otherwise. If we fix the number of open strings attached to the D3-brane to $n$, the Hilbert space of the quantum mechanical system is projected to $\mathcal{F}^n_\epsilon$. Integrating out this quantum mechanical system leaves a Wilson line in the representation $\mathcal{F}^\epsilon_n$. When two such Wilson lines intersect in $T^2_{12}$, they exchange gluons. These gluons are open strings stretched between two D3-branes. The interaction produces a rational $gl(L_+ | L_-)$ R-matrix acting on the tensor products of the representations of the Wilson lines.

The setup is slightly different in the case at hand in that the holomorphic directions are $\hat{\mathbb{C}}^\times$ here. Correspondingly, we expect that the R-matrix $S^{(\epsilon)}$ is a trigonometric version of this rational $gl(L_+ | L_-)$ R-matrix.

5 Identification of the model

A solution of the supertetrahedron equation (4.38) that has all of the properties deduced in section 4 is known \cite{22}. I propose that this solution describes the 3D lattice model constructed from branes.
The R-matrices for the solution in question have the following nonzero matrix elements:

\[
\tilde{R}_{nml}^{m'n'} = \delta_l^{l+m'} \delta_{m+n'} \sum_{\lambda, \mu \in \mathbb{Z}_{\geq 0}} (-1)^\lambda q^{(n'-m)+(n+1)\lambda+\mu(\mu-n)} \left( \frac{q^2}{q^{2\mu}} \right) \left( \frac{q^2}{q^{2\lambda}} \right) \left( m \right) \left( \lambda \right) q^2 , \tag{5.1}
\]

\[
\tilde{\mathcal{L}}_{n00}^{01} = \tilde{\mathcal{L}}_{n11}^{11} = \delta_n^{n'} , \quad \tilde{\mathcal{L}}_{n10}^{01} = -\delta_n^{n'} q^{n+1} , \quad \tilde{\mathcal{L}}_{n01}^{10} = \delta_n^{n'} q^n ,
\]

and \( \tilde{M}_{lmn}^{m'n'} = \tilde{L}_{nml}^{m'n'} \). Here

\[
(q)_n = \prod_{k=1}^{n} (1 - q^k) , \quad \binom{m}{n}_q = \frac{(q)_m}{(q)_{m-n}(q)_n} , \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{5.4}
\]

are the q-factorial, the q-binomial and the q-number, and the q-binomial is understood to be zero unless \( 0 \leq n \leq m \).

The operator \( \mathcal{R} \) is a solution of the RRRRR relation (4.53) that was constructed by Kapranov and Voevodsky [25] from the intertwiner of two irreducible representations of the quantum coordinate ring \( A_q(\mathfrak{sl}(3)) \). (The relevant representations are reducible if \( q \) is a root of unity. When \( q \) is a primitive root of unity of odd order, they contain finite-dimensional irreducible quotient representations and the corresponding R-matrix was obtained in [37].) Later, this R-matrix was rediscovered by Bazhanov and Sergeev [26] in a different but equivalent [38] form. The operator \( \mathcal{L} \) which solves the RLLL relation (4.54) was discovered in [26]. Finally, Yoneyama [22], building on earlier works by Sergeev [39] and Kuniba, Okado and Yamada [40], provided a uniform characterization of \( \mathcal{R} \), \( \mathcal{L} \), \( \mathcal{M} \) and \( \mathcal{N} \) as transition matrices between the Poincaré-Birkhoff-Witt bases for nilpotent subalgebras of \( U_q(\mathfrak{sl}(L_+ + L_-)) \) with \( L_+ + L_- = 3 \).

The matrix elements of the above R-matrices are polynomials in \( q \) and \( q^{-1} \) with integer coefficients, in agreement with the integrality of the reduced index \( \tilde{Z} \). The R-matrices satisfy the normalization condition (4.22), the charge conservation rule (4.23) and the symmetry (4.25). Moreover, they have the involutivity (4.26). The relation \( \mathcal{R}^{-1} = \mathcal{R} \) is proved in [38]. In [22] it is shown that

\[
\sum_{l', \mu', \nu'} \tilde{L}^{l'n'n'}_{\nu'l'm'l'} \tilde{L}^{l'm'n'}_{l'nml} = \sum_{l', \mu', \nu'} (-1)^{\mu'} \tilde{N}^{l'n'n'}_{\nu'l'm'l'} (-1)^{\mu} \tilde{N}^{l'm'n'}_{nml} = \delta_l^{l'm'} \delta_m^{m'} \delta_n^{n'} . \tag{5.5}
\]

Since \( (-1)^{l+m+l'} \mathcal{L}^{l'm'n'} = \mathcal{L}^{l'm'n'} \) and \( (-1)^{l+n+l'+m+m'} \mathcal{N}^{l'm'n'} = \mathcal{N}^{l'm'n'} \), we have \( \mathcal{L}^{-1} = \mathcal{L} \) and \( \mathcal{N}^{-1} = \mathcal{N} \).

\[\text{In the notation of [22], } \tilde{R}_{nml}^{m'n'} , \tilde{\mathcal{L}}_{nml}^{m'n'} , \tilde{\mathcal{N}}_{nml}^{m'n'} , \tilde{\mathcal{M}}_{lmn}^{m'n'} , \text{are written as } \mathcal{R}_{lmn}^{m'n'} , \mathcal{L}_{lmn}^{m'n'} , \mathcal{N}_{lmn}^{m'n'} , \mathcal{M}_{lmn}^{m'n'} , \text{respectively.} \]
This solution also has the expected relation to solutions of the Yang-Baxter equation. In [26] it was found that the trace of the product of \( L \) copies of \( R \) and the trace of the product of \( L \) copies of \( L \) produce trigonometric sl\((L)\) R-matrices valued in the direct sum of all symmetric tensor representations and in the direct sum of all antisymmetric tensor representations, respectively. This result was extended in [21, 27] to more general combinations of R-matrices. In [27], the trace of the product of \( L_+ \) copies \( R \) and \( L_- \) copies \( L \) was studied. It was shown that \( S^{(\epsilon)}_{lm} \) is an R-matrix acting on the tensor product of two irreducible modules \( W^{(\epsilon)}_{l} \), \( W^{(\epsilon)}_{m} \) of a Hopf algebra \( U(\epsilon) \) called the generalized quantum group of type \( A \), which is an affine analog of the quantized enveloping algebra \( U_q(\mathfrak{gl}(L_+|L_-)) \) of \( \mathfrak{gl}(L_+|L_-) \). We identify \( W^{(\epsilon)}_{n} \) with \( F^{(\epsilon)}_{n} \).

One of the interesting features of this solution is that the R-matrices are constant, and the spectral parameters of the solutions of the Yang-Baxter equations and transfer matrices originate from twist parameters for the periodic boundary conditions. As we saw in section 4.3, this feature is nicely explained in the brane picture: the only continuous parameters of the model, the positions of the M5-branes in \( \mathbb{R}^3_{456} \), can be identified with Wilson loops of complex gauge fields in a dual frame and, as such, show up in the partition function as twist parameters.

As a last piece of evidence, we point out that there is another way of reducing the 3D lattice model to a 2D lattice model. Let us consider the circle \( S_3 \) as the interval \([-c_3/2, +c_3/2]\) with the two ends identified. Instead of simply reducing the lattice along \( S_3 \), we can first take the orbifold \( S_3/\mathbb{Z}_2 \) by the \( \mathbb{Z}_2 \)-action \( x^3 \mapsto -x^3 \) and then perform reduction. For the orbifolding to be possible, an M5-brane located at \( x^3 \in S_3 \) must have its image at \(-x^3\), unless it is located at one of the fixed points at \( x^3 = 0 \) and \( x^3 = \pm c_3/2 \). The orbifold \( S_3/\mathbb{Z}_2 \) can also be regarded as the interval \([0, c_3/2]\) sandwiched by a pair of M9-branes [41]. Each M9-brane provides a boundary state of the lattice model, and we expect that a product of R-matrices evaluated between the two boundary states is a solution of the Yang-Baxter equation. There is indeed such a construction in the lattice model [43, 44]. In fact, two different boundary states were introduced in [44]. The difference between the two is probably whether a “half” M5-brane is stuck on the fixed point or not.

It may be possible to reproduce the partition function and the R-matrices of the 3D lattice model as quantities calculated in QFTs describing the brane system. An interesting problem in this regard is to identify the quantum mechanical system supported on the intersection of three M5-branes, one of type \( 1\sigma_1 \), one of type \( 2\sigma_2 \) and one of type \( 3\sigma_3 \), and some numbers of M2-branes of any type. The supersymmetric index of this system is equal to a matrix element of the R-matrix \( R^{\sigma_1\sigma_2\sigma_3} \). The system can be thought of as a junction of six theories, each living on a stack of M2-branes suspended between a pair of M5-branes, and this viewpoint may be helpful in approaching the problem. Another interesting problem is to concretely describe the 3D TQFT discussed in section 3.1, using the 7D theory on the D6-brane or its dimensional reduction on \( S_0 \). The R-matrices are obtained from correlation functions of intersecting surface defects in this theory, which we can try to compute in perturbation theory in \( q \) or \( q^{-1} \). A quantitative verification of
the proposal of this paper, along these lines or with any other methods, is left for future research.

A Analysis of supersymmetry

In this appendix we analyze the supersymmetry preserved by the M5-brane configuration introduced in section 2. We will mostly follow the notations of [45].

A.1 Spinors in ten and eleven dimensions

Let $\Gamma_\mu$, $\mu = 0, 1, \ldots, 9, \natural$, be the 11D gamma matrices. They generate the Clifford algebra, defined by the anticommutation relation

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}, \quad (A.1)$$

where $\eta = \text{diag}(-1, +1, +1, \ldots, +1)$ is the 11D Minkowski metric. We write $\Gamma_{\mu_1\mu_2\ldots\mu_k}$ for the antisymmetrized product of $\Gamma_{\mu_1}, \Gamma_{\mu_2}, \ldots, \Gamma_{\mu_k}$; if the indices are all distinct, $\Gamma_{\mu_1\mu_2\ldots\mu_k} = \Gamma_{\mu_1} \Gamma_{\mu_2} \cdots \Gamma_{\mu_k}$.

There are two inequivalent irreducible representations of the Clifford algebra with dimension greater than one. We pick one by requiring

$$\Gamma_{0123456789\natural} = +1. \quad (A.2)$$

(The other irreducible representation has $\Gamma_{0123456789\natural} = -1$.) This representation is 32-dimensional. The gamma matrices are represented by matrices acting on a complex 32-dimensional vector space, an element of which is called a Dirac spinor.

The minus of the transpose of the gamma matrices $\{-\Gamma^T_\mu\}$ also satisfy the defining relations of this irreducible representation, so they are related to $\{\Gamma_\mu\}$ by a change of basis. It follows that there exists a charge conjugation matrix $C$ such that

$$C\Gamma_\mu C^{-1} = -\Gamma^T_\mu. \quad (A.3)$$

Since $C^{-1}C^T$ commutes with the gamma matrices, it is proportional to the identity matrix.

The quadratic elements $\{i\Gamma_{\mu\nu}\}$ generate the action of the Lorentz group $SO(10, 1)$ on the Clifford algebra and Dirac spinors. Since $C\Gamma_{\mu\nu}C^{-1} = -\Gamma^T_{\mu\nu}$, Dirac spinors $\zeta$ and $C\zeta$ transform in the dual representations of $SO(10, 1)$. If $\zeta, \chi$ are two Dirac spinors, then $\bar{\chi}\Gamma_{\mu_1\mu_2\ldots\mu_k}\chi$ transforms in the same way as a component of an antisymmetric tensor of type $(0,k)$, where the Majorana conjugate $\bar{\zeta}$ of $\zeta$ is defined by

$$\bar{\zeta} = \zeta^T C. \quad (A.4)$$

There is a basis, called a Majorana basis, such that all gamma matrices are real $32 \times 32$ matrices, $\Gamma_0$ is antisymmetric and the others are symmetric. In such a basis we can take $C = \Gamma^0$. A Dirac spinor $\epsilon$ that is real in a Majorana basis is called a Majorana spinor.
A.2 Supersymmetry preserved by the M5-brane system

11D supersymmetry is generated by 32 hermitian fermionic conserved charges $Q_\alpha$, $\alpha = 1, \ldots, 32$, satisfying the anticommutation relations

$$\{Q_\alpha, Q_\beta\} = - (\Gamma^\mu C^{-1})_{\alpha\beta} P_\mu - \frac{1}{2} (\Gamma^{\mu\nu} C^{-1})_{\alpha\beta} Z^{(2)}_{\mu\nu} - \frac{1}{5!} (\Gamma^{\mu\nu\rho\tau\sigma} C^{-1})_{\alpha\beta} Z^{(5)}_{\mu\nu\rho\tau\sigma} , \quad (A.5)$$

where $P$ is the momentum, $Z^{(2)}$ is a 2-form charge and $Z^{(5)}$ is a 5-form charge, all commuting with $Q_\alpha$ and with each other. Under the Lorentz transformations the supercharges transform as the components of a spinor $Q$. In what follows we choose a Majorana basis so that the gamma matrices and the parameters of supersymmetry transformations are all real.

The M5-branes of type $i+$, $i = 1, 2, 3$, are invariant under supersymmetry transformations generated by $\bar{\epsilon}Q$ if the constant Majorana spinor $\epsilon$ satisfies

$$\epsilon = \Gamma i^{i+3} 1^{012378} \epsilon. \quad (A.6)$$

Similarly, the M5-branes of type $i−$ are invariant for $\epsilon$ satisfying

$$\epsilon = \Gamma i^{i+3} 1^{012398} \epsilon. \quad (A.7)$$

The signs of the right-hand sides of the above equations are determined by the orientations of the M5-branes, which we have chosen in such a way that the brane system is invariant under the cyclic permutation $(1, 4) \rightarrow (2, 5) \rightarrow (3, 6) \rightarrow (1, 4)$ and the exchange $(7, 8) \leftrightarrow (9, 9)$ of directions.

The presence of the M5-branes leads to six constraint equations for $\epsilon$, but not all of them are independent. Imposing these six constraints is equivalent to requiring that $\epsilon$ satisfies one of them, say

$$\epsilon = \Gamma^{14} 1^{012378} \epsilon = - \Gamma^{023478} \epsilon, \quad (A.8)$$

as well as the conditions

$$\Gamma^{ij+3} \epsilon = \Gamma^{jj+3} \epsilon, \quad i, j = 1, 2, 3, \quad (A.9)$$

and

$$\Gamma^{78} \epsilon = \Gamma^{99} \epsilon. \quad (A.10)$$

Equivalently, $\epsilon$ must satisfy the following four independent equations:

$$\Gamma^{023478} \epsilon = - \epsilon, \quad (A.11)$$

$$\Gamma^{1245} \epsilon = + \epsilon, \quad (A.12)$$

$$\Gamma^{2356} \epsilon = + \epsilon, \quad (A.13)$$

$$\Gamma^{789} \epsilon = - \epsilon. \quad (A.14)$$

The products of gamma matrices in these four equations are symmetric matrices commuting with each other, hence simultaneously diagonalizable. Moreover, they square to the identity matrix and are traceless (by the cyclicity of trace and the anticommutation relations of the
gamma matrices), so their eigenvalues are $+1$ and $-1$ and the corresponding eigenspaces have the same dimension. Therefore, imposing each of the above condition reduces the dimension of the space of possible choices of $\epsilon$ by half. In total, there is a two-dimensional space of solutions to the constraint equations.

The complex matrices $i\Gamma^{78}$ and $i\Gamma^{98}$ are hermitian, traceless, square to the identity matrix and commute with each other and with those products of gamma matrices that appear in the above conditions. Let $\zeta$ be a simultaneous eigenvector with eigenvalue $+1$:

$$i\Gamma^{78}\zeta = i\Gamma^{98}\zeta = \zeta.$$  \hfill (A.15)

We normalize $\zeta$ so that

$$\zeta^\dagger \zeta = 1$$  \hfill (A.16)

and define the complex supercharge

$$Q_\zeta = \bar{\zeta} Q$$  \hfill (A.17)

corresponding to $\zeta$. Using two hermitian supercharges $Q_+, Q_-$ we can write

$$Q_\zeta = Q_+ + iQ_-.$$  \hfill (A.18)

The complex conjugate $\zeta^*$ of $\zeta$ satisfies $i\Gamma^{78}\zeta^* = i\Gamma^{98}\zeta^* = -\zeta^*$ and $Q_\zeta^\dagger = \bar{\zeta}^\dagger Q$.

Let us calculate the anticommutation relation

$$\{Q_\zeta^\dagger, Q_\zeta\} = (\zeta^\dagger \Gamma^0 \Gamma^\mu \zeta) P_\mu + \frac{1}{2}(\zeta^\dagger \Gamma^0 \Gamma^{\mu\nu} \zeta) Z^{(2)}_{\mu\nu} + \frac{1}{5!}(\zeta^\dagger \Gamma^0 \Gamma^{\mu\nu\rho\sigma\tau} \zeta) Z^{(5)}_{\mu\nu\rho\sigma\tau}.$$  \hfill (A.19)

The choice of $\zeta$ is unique up to an overall phase factor but the anticommutator is independent of this factor, so the right-hand side is completely determined by the conditions that fix $\zeta$ up to a phase. In other words, the coefficients in front of the charges on the right-hand side can be reduced to multiples of $\zeta^\dagger \zeta$ by the constraint equations for $\epsilon$ written above. We need not use (A.15) since $\{Q_\zeta^\dagger, Q_\zeta\}$ is symmetric with respect to $\zeta$ and $\zeta^*$. We can reduce $\zeta^\dagger \Gamma^0 \Gamma^\mu \zeta$ to a multiple of $\zeta^\dagger \zeta$ only if $\mu = 0$:

$$\zeta^\dagger \Gamma^0 \Gamma^0 \zeta = -\zeta^\dagger \zeta = -1.$$  \hfill (A.20)

and reduce $\zeta^\dagger \Gamma^0 \Gamma^{\mu\nu} \zeta$ only if $\{\mu, \nu\} = \{i, i + 3\}$ with $i = 1, 2, 3$:

$$\zeta^\dagger \Gamma^0 \Gamma^{i+3} \zeta = -\zeta^\dagger \Gamma^0 \Gamma^{i+3} \Gamma^{023478} \Gamma^{2356} \Gamma^{0123456789} \zeta = -\zeta^\dagger \Gamma^{i+3} \Gamma^{01239} \zeta = -1.$$  \hfill (A.21)

As for $\zeta^\dagger \Gamma^0 \Gamma^{\mu\nu\rho\sigma\tau} \zeta$, we can reduce it if $\{\mu, \nu, \rho, \sigma, \tau\} \in \{1, 2, 3, i, i + 3, 7, 8\} \setminus \{i\}$ for $i = 1, 2, 3$ using (A.6):

$$\zeta^\dagger \Gamma^0 \Gamma^{i+3} \Gamma^{12378} \zeta = 1,$$  \hfill (A.22)

if $\{\mu, \nu, \rho, \sigma, \tau\} \in \{4, 5, 6, 7, 8\}$ using (A.11) and (A.13):

$$\zeta^\dagger \Gamma^0 \Gamma^{45678} \zeta = -\zeta^\dagger \Gamma^0 \Gamma^{023478} \Gamma^{2356} \zeta = 1.$$  \hfill (A.23)

if $\{\mu, \nu, \rho, \sigma, \tau\} \in \{0, i, j, i + 3, j + 3\}$ for distinct $i, j \in \{1, 2, 3\}$ using (A.9):

$$\zeta^\dagger \Gamma^0 \Gamma^{0ij+3+j+3} \zeta = \zeta^\dagger \Gamma^{i+3} \Gamma^{j+3} \zeta = -1.$$  \hfill (A.24)
and if \( \{\mu, \nu, \rho, \sigma, \tau\} \in \{0, 7, 8, 9, \natural\} \) using (A.14):
\[
\zeta^\dagger \Gamma^0 \Gamma^{0789\natural} \zeta = -\zeta^\dagger \Gamma^{789\natural} \zeta = 1.
\] (A.25)

Also, \( \zeta^\dagger \Gamma^{\mu\nu\rho\sigma\tau} \zeta \) can be reduced to a multiple of \( \zeta^\dagger \zeta \) in those cases that are related to the above cases by the exchange \((7, 8) \leftrightarrow (9, \natural)\). In the other cases the coefficients vanish.

Thus we find
\[
\{Q^1, Q_\zeta\} = -P_0 - Z_{14}^{(2)} - Z_{25}^{(2)} - Z_{96}^{(2)} - Y,
\] (A.26)

where

\[
Y = Z_{23478}^{(5)} - Z_{13578}^{(5)} + Z_{12678}^{(5)} + Z_{2349}^{(5)} - Z_{1359}^{(5)} + Z_{1269}^{(5)} + Z_{01245}^{(5)} + Z_{01346}^{(5)} + Z_{02356}^{(5)} - Z_{0789}^{(5)}.
\] (A.27)

Finally, we mention two properties of \( Q_\zeta \) which are important in connection with the brane setup in section 3. The conditions (A.9) can be rewritten as
\[
(\Gamma^{ij} + \Gamma^{i+3j+3}) \epsilon = 0, \quad i, j = 1, 2, 3.
\] (A.28)

These equations say that \( \epsilon \) is invariant under the action of the diagonal subgroup of \( \text{SO}(3)_{123} \times \text{SO}(3)_{456} \), hence so is \( Q_\zeta \). By the same token, condition (A.10) shows that \( Q_\zeta \) is invariant under the action of the antidiagonal subgroup of \( \text{SO}(2)_{78} \times \text{SO}(2)_{9\natural} \). From these properties we immediately see that the only component of \( P \) that can appear in \( \{Q^1, Q_\zeta\} \) is \( P_0 \) since the other components are not invariant under either the diagonal subgroup of \( \text{SO}(3)_{123} \times \text{SO}(3)_{456} \) or the antidiagonal subgroup of \( \text{SO}(2)_{78} \times \text{SO}(2)_{9\natural} \).

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