Completely geometric theory II: basics of quantum mechanics

S V Siparov
State University of Civil Aviation, St. Petersburg, 196210, RF
E-mail: sergey@siparov.ru

Abstract. The geometrical approach suggested earlier, makes it possible to investigate the regions both in mega- and micro worlds that cannot be accessed by the direct observation. This makes it possible not only to interpret the basic experiments of quantum mechanics in a new way but also to escape the paradoxes stemming from the wave function introduction. It also gives perspectives to adjust the quantum mechanics and the relativity theory.

In the previous paper [1], a completely geometric theory was constructed, the main result of which was the geodesic equation

\[
\frac{dy^i}{ds} + \left( \Gamma^i_{jk} + \frac{1}{2} g^{ik} \frac{\partial^2 \epsilon_{jk}}{\partial x^l} y^l \right) y^j y^l = 0
\]

for a weakly curved and weakly anisotropic space with metric \( g_{ik}(x,y) = \eta_{ik} + \epsilon_{ik}(x,y); \eta_{ik} = \text{diag}(-1,1,1) \), where \( \epsilon_{ik} = \sigma \xi_{ik}; \sigma << 1 \), and the metric field equation

\[
\Delta D^i - \frac{\partial^2 D^i}{\partial x^k} = -I^i
\]

for the kinematical tensor \( F_{ik} = \frac{\partial y_i}{\partial x^j} - \frac{\partial y_j}{\partial x^i}; y^i = \frac{\partial x^i}{\partial s} \).

The possibility and meaning of such a construction follows from Clifford’s idea of the experimental indistinguishability of physical and geometric fields. This idea was partially reflected in Einstein’s general relativity, in which the principle of equivalence was adopted, concerning the experimental indistinguishability of inertial forces and gravitational forces. It turned out [1] that the geometric approach allows one to present the main results of electrodynamics and hydrodynamics in a new language. Moreover, with its help it is possible [1] to cope with the difficulties encountered by general relativity on the galactic scale\(^1\), as well as to give an interpretation to a number of observations that are absent in modern theory.

In this regard, it seems promising to further develop the approach, which is based on a statement analogous to the principle of least action: there exists a geometry suitable for describing phenomena in experimental space such that all the motions of test bodies occur along the corresponding geodesics. At the same time, within the framework of the geometric approach, it is natural to assume that the stationary

---

\(^1\) Flat rotation curves of spiral galaxies and the anomalous refraction of light in gravitational lenses led to the introduction of the notion of dark matter that is self-contradictory. The decades’ search for the particles of dark matter also failed.
state is determined not by the principle of minimum potential energy, but by the principle of minimum occupied volume.

1. Principles of quantum mechanics and its basic experiments
Quantum mechanics arose as a result of the impossibility of interpreting all experiments either only from the wave or only from the corpuscular point of view. Let's list the main of these experiments:

1. Wave interpretation of a heated body radiation spectrum led to the "ultraviolet catastrophe". Planck put forward the hypothesis of discrete quanta of radiation, which made it possible to successfully describe the experiment.
2. The regularities of spectral terms and the result of Rutherford's experiment on the scattering of \( \alpha \)-particles allowed Bohr to formulate postulates about the existence of stationary states and about transitions between them.
3. The results of experiments on the scattering of electrons by crystals led de Broglie to the hypothesis that the electron has wave properties.

A self-consistent interpretation of these circumstances was proposed by Schrödinger, who refused to describe the microsystem and its behavior using classical concepts. Instead, he introduced the concept of a "wave function" satisfying a certain differential equation. The properties of this equation made its solutions satisfactorily describe the experimental results.

We see that at each stage of the quantum mechanics formation, formal assumptions of a paradoxical nature were made. Wave-corpuscle dualism, the absence of a visual meaning of the wave function and the impossibility of reconciling its instantaneous reduction with the relativity theory, as well as communicating physical meaning to the concepts of probability theory (probability waves, probability amplitude) constitute a conceptual problem of quantum mechanics.

2. Geometric approach
Let's apply a geometric approach to interpret the above experiments. After simplifications and transformations, the three-dimensional section of equation (1) will take the form

\[
\frac{d\vec{v}}{dt} = -\frac{c^2}{2} \nabla \varepsilon_{00} + \nabla(\vec{v}, \frac{c^2}{2} \frac{\partial \varepsilon_{00}}{\partial \vec{v}}) + [\vec{v}, \vec{\Omega}],
\]

where \( \vec{\Omega} = \frac{c^2}{2} rot \frac{\partial \varepsilon_{00}}{\partial \vec{v}} \). Note that the gradients of scalars on the right-hand side of the equation are polar vectors, while the cross product is an axial vector. However, one of the factors of the vector product is also the axial vector (curl), so equation (3) retains its meaning.

If the observed acceleration is absent, then \( \frac{d\vec{v}}{dt} = 0 \), and the trivial solution is \( \varepsilon_{00}(x, y) = const \), which reduces to the Minkowski metric for the flat space and corresponds to uniform rectilinear motion. However, equation (3) can have another solution. We will search for \( \vec{v} \) in the form

\[
\vec{v} = \vec{V} \exp i(\vec{k}, \vec{r}) - \omega t,
\]

and the correction \( \varepsilon_{00} \) to the metric tensor component in the form

\[
\varepsilon_{00} = \frac{1}{2} \frac{\nabla^2}{c^2} \frac{V^2}{2} \exp 2i(\vec{k}, \vec{r}) - \omega t,
\]

in both cases, referring to the real parts. Then

\[
\nabla \varepsilon_{00} = 2ik \varepsilon_{00}
\]

\[
\nabla(\vec{v}, \frac{c^2}{2} \frac{\partial \varepsilon_{00}}{\partial \vec{v}}) = c^2 2ik \varepsilon_{00},
\]
\[ \tilde{\Omega} = \frac{i}{2} [\tilde{V}, \tilde{k}] \exp\{i((\tilde{k}, \tilde{r}) - \omega t) \}, \]
and equation (3) gives
\[ \frac{d\tilde{\nu}}{dt} = i e^2 \left\{ \tilde{\nu} + [\tilde{\nu}, [\tilde{\nu}, \tilde{V}]] \right\} \phi, \tag{7} \]
where \( \tilde{\nu} = \frac{\tilde{V}}{\nu} \), and the dispersion relation has the form
\[ \omega \tilde{\nu} = -\frac{1}{2} V \left[ \tilde{k} + [\tilde{\nu}, [\tilde{\nu}, \tilde{k}]] \right] \exp\{i((\tilde{k}, \tilde{r}) - \omega t) \}. \tag{8} \]

Thus, when a weakly curved, weakly anisotropic space is used to simulate the experimental space in regions inaccessible for direct observation, the trajectory of free motion of a particle is a generalized helix. This circumstance makes it possible to naturally use the concept of phase to describe the motion of a particle. The radius \( R \) and the pitch \( \lambda \) of such a helical line are
\[ R = \frac{v \sin(\nu, \tilde{\Omega})}{|\tilde{\nu}|}, \lambda = 2\pi \frac{v \cos(\nu, \tilde{\Omega})}{|\tilde{\Omega}|}. \tag{9} \]

In general case, the axis of the helix may not be straight. If the axis of such a helical line is closed, then a selected finite area of motion appears. In the simplest case, such a closed line is a circle and the trajectory will lie on the surface of the torus. Note that the surface of a torus embedded in a 4-dimensional space has zero curvature, which has an additional important meaning.

3. Diffraction of electrons
Figure 1a. depicts Young's experiment for light waves. The position \( z_{\text{max}} \) of the interference maximum numbered \( n \) is determined by the expression \( z_{\text{max}} = \frac{nL}{d} \), i.e.
\[ d \cdot \tan \varphi = n\lambda. \tag{10} \]

| Figure 1. (a) Young’s experiment; (b) Electron’s trajectory in an anisotropic space. |

Sequential transmission of single electrons through the crystal diffraction grating also leads to the illumination of the screen only in certain directions. The experimenter controls the device of the electron gun, the parameters of the crystal and the relative position of the gun, obstacle and screen, and also measures the location of the traces of electrons entering the screen. Everything that happens in the experimental space is one or another interpretation that explains the results. If we assume that the electron has wave properties, then the wave corresponding to each individual electron passes through
the lattice, interacts with itself and, finally, is localized on the screen, forming a diffraction pattern. What is this wave?

Suppose that the electron has no wave properties, but the experimental space is slightly curved and slightly anisotropic. Then the electron moves along a helical line. Figure 1b shows an example of such a trajectory when an electron moves through a pair of slits, as in Young's experiment. Obviously, the electron can move in only such a direction towards the screen, the sine of which is equal to \( \sin \varphi = \frac{n \lambda}{d} \), i.e.

\[
d \cdot \sin \varphi = n \lambda,
\]

where \( \lambda \) is the pitch of the helix and \( d \) is the distance between the slits. The particle will not be able to move in all other directions, because the phase of its passage through the second slit will differ from the phase of the passage of the first slit; the particle will not enter the second slit of the obstacle and, thus, will not reach the screen. At small angles typical for experimental verification, the sine differs little from the tangent, therefore, the interpretation of the experiment using formula (11) instead of formula (10) will remain satisfactory, and the geometric approach will adequately describe the results of the experiment. Replacing \( d = \frac{D}{\cos \varphi} \), where \( D = 2R \) is the helix diameter, we obtain the expression for the position of the maxima

\[
D \cdot \tan \varphi = n \lambda,
\]

allowing to evaluate the parameters of the electron’s trajectory.

4. Model of the atom

Bohr’s semiclassical theory, which arose as a result of the interpretation of the results of Rutherford’s experiments and spectroscopic observations of hydrogen emission, provided for the existence of stationary states and transitions between them. Let us combine Bohr’s second postulate with the generalized Balmer formula and write down the expression for the energy of the emitted quantum

\[
|E_n - E_m| = C^{(*)} \left| \frac{n^2 - m^2}{m^2} \right|; n, m = 1, 2, 3; n \neq m,
\]

where \( C^{(*)} \) is proportional to the Rydberg constant.

Let the metric be such that the axis of the helix is a closed circle with radius \( R_A \) which is greater than the radius \( R \) of the helix. Then the particle will move along the surface of a torus. We will consider trajectories stationary if they are closed. Then

\[
n_1 2\pi R_A = n_2 \lambda,
\]

where \( n_1 \) and \( n_2 \) are integers, and in this case

\[
\tan (\vec{v}, \vec{\Omega}) = \frac{n_1}{n_2}
\]

is a rational number. This means that the characteristic linear size of the region, on the surface of which the trajectory of the particle is located during its finite motion, cannot be less than \( 2R \). Numerical estimates [2], consistent with independent experiments, give a reasonable estimate for the size of such an atom. Then the discrete states will correspond to the volumes of the tori with closed trajectories on the surface. The volume of the torus is

\[
V = 2\pi^2 R_A^2 R^2.
\]

Substitute here the expression for \( R \) from equation (9) and use formulas (4-6). Then, taking into account equation (15), we obtain

\[
V = 2\pi^2 R_A \frac{v^2 \sin^2 (\vec{v}, \vec{\Omega})}{|\vec{\Omega}|^2} = 2\pi^2 R_A \frac{v^2}{|\vec{\Omega}|^2} \frac{n_1^2}{n_1^2 + n_2^2}.
\]
For simplicity, let us assume that the number of large revolutions for different states is the same and
equal to 1, and the state is determined only by the number of small revolutions, i.e. $n_1 = 1; n_2 \equiv n$, and we get

$$V_n = 2\pi^2 R_A \frac{v^2}{|\Omega|^2} \frac{1}{1 + n^2}.$$  \hspace{1cm} (18)

Then

$$|V_n - V_m| = C^{(+)} \left| \frac{n^2 - m^2}{(1 + n^2)(1 + m^2)} \right|, n, m = 0, 1, 2, \ldots; n \neq m.$$ \hspace{1cm} (19)

Comparing the obtained expression with formula (13), we see that the regularity of the arrangement of the observed spectral lines remained unchanged. Thus, as in the previous section, this group of experiments also admits an adequate geometric interpretation.

The constant present in the last formula has the form

$$C^{(+)} = 2\pi^2 R_A \frac{v^2}{|\Omega|^2} = 2\pi^2 R_A \frac{4}{|[\omega, k]|^2} = 2R_A \lambda^2$$ \hspace{1cm} (20)

and is related both to the expression for the correction to the metric and to the size of the atom.

Further analysis can take into account the different number of revolutions along the great circumference of the torus; deformation of the torus and the formation of various nodes; two different directions of movement for each stationary state. All this provides opportunities for introducing additional quantum numbers if necessary.

5. The Planck formula

The Planck formula follows from the Boltzmann law on the distribution of states by energy, according to which the probability of a state with number $n$ is equal to $P_n = \frac{\exp(-E_n/kT)}{\sum_n \exp(-E_n/kT)}$. Then the average energy will be equal to $\langle E \rangle = \sum_n P_n E_n$.

Within the framework of the geometric approach, the state corresponds to the occupied volume, and the change in state is accompanied by the appearance or disappearance of one or several elementary toroidal vortices of the metric having an elementary volume

$$V_0 = 2\pi^2 R_A \frac{v^2}{|\Omega|^2}.$$ \hspace{1cm} (21)

The thermodynamic quantity $kT$, in accordance with its kinematic meaning, is determined by the square of the velocity

$$kT = b v^2.$$ \hspace{1cm} (22)

Then, denoting $x = \frac{2\pi^2 R_A}{b |\Omega|^2}$, we get

$$\langle V \rangle = V_0 \frac{\sum_n n \exp(-nx)}{\sum_n \exp(-nx)} = \frac{V_0}{\exp(x) - 1},$$ \hspace{1cm} (23)

as in the usual Planck formula. Since, according to equation (18), in the geometric approach, the volume $V_n$ is inversely proportional to the squares of integers, the formula will not change its usual form.
6. Equation of state

Thus, the geometric theory satisfactorily describes the basic experiments of quantum mechanics without invoking the ideas of wave-particle duality. As is known, Schrödinger did not derive his equation, but postulated it on the basis of an analogy between action in Jacobi theory and the phase of a wave in wave theory. In this case, the wave function used to describe the physical situation had no visual meaning, and the measured quantities served as parameters of the equation.

A typical example of the use of the Schrödinger equation is the well-known problem of a harmonic oscillator and its energy levels [3]. The one-dimensional stationary Schrödinger equation for a linear oscillator illustrates the well-known approach to describing the state of an atom. It has the form

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - \frac{m\omega^2 x^2}{2})\psi = 0$$  \hspace{1cm} (24)

and the solution

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\exp(-\frac{m\omega}{2\hbar} x^2)H_n(x\sqrt{\frac{m\omega}{\hbar}}),$$  \hspace{1cm} (25)

where

$$H_n(\xi) = (-1)^n \exp(\frac{\xi^2}{2}) \frac{d^n \exp(-\frac{\xi^2}{2})}{d\xi^n}$$  \hspace{1cm} (26)

are Hermite polynomials. The eigenvalues of equation (24), (in physical applications identified with energy levels), have the form

$$E_n = (n + \frac{1}{2})\hbar \omega; n = 0,1,2,...$$  \hspace{1cm} (27)

In the geometric approach, the state of an atom is characterized by the volume of the region available for the finite motion of the particle, which, in turn, is described by the geodesic equation. Using the kinematical tensor according to [1], instead of Maxwell's equations, one can obtain the geometric equation

$$i \frac{\partial \Psi_{(a)}}{\partial t} = c \cdot \text{rot}\Psi_{(a)},$$  \hspace{1cm} (28)

where \( \Psi_{(a)} = E + iH\), \( \vec{E}_{(a)} = -\nabla \varphi \) and \( \vec{H}_{(a)} = \text{rot}\vec{A} \), and the potentials have a metric meaning

$$\frac{c^2}{2} \varepsilon_{00} \equiv \varphi; \quad \frac{c^2}{2} \frac{\partial \varepsilon_{00}}{\partial \vec{v}} \equiv \frac{1}{c} \vec{A}. $$ \hspace{1cm} (29)

Besides, \( \vec{E}_{(a)} = ik\vec{A} \) and \( \vec{H}_{(a)} = i[\vec{k}, \vec{A}] \), where \( \vec{A} = \text{Re}\{A_0 e^{i(k\vec{r}-\omega t)}\} \), and the wave vector is \( \vec{k} = \frac{\omega}{c} \vec{n} \).

Then, to describe the stationary state, one can use the stationary equation

$$\text{rot}\psi_{(a)} = \vec{k}\psi_{(a)},$$  \hspace{1cm} (30)

whose eigenvalues are proportional to the zeros of the Bessel function of half-integer order [4]. They are also located at an equal distance from each other (see Figure 2).
Figure 2. Location of the zeros of the Bessel function of half-integer order.

For the components of the eigenfunctions of equation (30), written in a spherical coordinate system, the following expressions take place [4]

\[
(\psi_{r})_{\kappa} = c_{\kappa}(\lambda_{n,m} r)^{-1} \zeta_{n}(\lambda_{n,m} r) Y^{k}_{n}(\theta, \varphi) \\
(\psi_{\varphi} + i \psi_{\theta})_{\kappa} = c_{\kappa}(\lambda_{n,m} r)^{-1} \zeta_{n}(\lambda_{n,m} r) \Phi_{n}(\lambda_{n,m} r) H Y^{k}_{n}(\theta, \varphi),
\]
where

\[
\zeta_{n}(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z) \\
\Phi_{n}(\lambda_{n,m} r) = \int_{0}^{r} \exp\{i \lambda_{n,m} (r-t)\} \zeta_{n}(\lambda_{n,m} t) t^{-\frac{1}{2}} dt \\
HY^{k}_{n}(\theta, \varphi) = (\sin^{-1}(\partial_{\varphi} + i \partial_{\theta}) Y^{k}_{n}(\theta, \varphi)
\]

$Y^{k}_{n}(\theta, \varphi)$ - real spherical functions, $c_{\kappa}$ - rational numbers. These formulas are suitable for describing the motion of a curled fluid flow in a ball, there the flow velocity is $\psi_{\kappa}(x)$ at $n = 1, 2, 3, ..., and its vorticity $\text{rot} \psi_{\kappa}$, equal to $\lambda_{n,m} \psi_{\kappa}$, is nonzero at each point of the ball. According to Arnold's theorem [5], almost all flow streamlines of an ideal fluid that satisfy equation (30) are wound either on cylinders or on tori. The rest of the flows can have streamlines with a very complex topology. Both these statements are consistent with the assumptions mentioned in Section 4.

Thus, the use of the geometric approach to describe both experimental results and fundamental properties concerning the microworld does not contradict the known results and has a number of advantages.

References
[1] Siparov S V 2021 Proc. Conf. PIRT-2021 (Moscow: BMSTU)
[2] Siparov S V 2015 Proc. Conf. PIRT-2015 (Moscow: BMSTU) pp 483-501
[3] Landau L D and Lifshits E M 1963 Quantum mechanics (Moscow: Gosudarstvennoe Izdatel’stvo Fiziko-matematichskoi literatury)
[4] Sax R S 2013 Ufa mathematical journal 5 63
[5] Arnold V I 1965 C. R. Acad. Sci. Paris. 261 17