On subspaces of Kloosterman zeros and permutations of the form $L_1(x^{-1}) + L_2(x)$

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Abstract. Permutations of the form $F = L_1(x^{-1}) + L_2(x)$ with linear functions $L_1, L_2$ are closely related to several interesting questions regarding CCZ-equivalence and EA-equivalence of the inverse function. In this paper, we show that $F$ cannot be a permutation if the kernel of $L_1$ or $L_2$ is too large. A key step of the proof is a new result on the maximal size of a subspace of $\mathbb{F}_2^n$ that contains only Kloosterman zeros, i.e. a subspace $V$ such that $K_n(v) = 0$ for all $v \in V$ where $K_n(v)$ denotes the Kloosterman sum of $v$.

Keywords: Inverse function · permutation polynomials · Kloosterman sums · EA-equivalence · CCZ-equivalence.

1 Introduction

Vectorial Boolean functions play a big role in the design of symmetric cryptosystems as design choices for Sboxes. The linear and differential properties of vectorial Boolean functions are a measure of resistance against linear $[20]$ and differential $[4]$ attacks.

Definition 1. A function $F: \mathbb{F}_2^n \to \mathbb{F}_2^n$ has differential uniformity $d$, if

$$d = \max_{a \neq 0, b} \{x: F(x) + F(x + a) = b\}.$$  

A function with differential uniformity 2 is called almost perfect nonlinear (APN) on $\mathbb{F}_2^n$.

To resist differential attacks, a vectorial Boolean function should have low differential uniformity. As the differential uniformity is always even, the APN functions yield the best resistance against differential attacks.

Definition 2. The Walsh transform $W_F$ of a function $F: \mathbb{F}_2^n \to \mathbb{F}_2^n$ is defined as follows:

$$W_F(a,b) = \sum_{x \in \mathbb{F}_2^n} (-1)^{Tr(aF(x)+bx)}.$$
The nonlinearity of $F$ is defined as

$$nl(F) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^n}} |W_F(a, b)|.$$  \hspace{1cm} (1)

The higher the nonlinearity of a vectorial Boolean function, the better is its resistance to linear attacks.

We recall some concepts of equivalence of vectorial Boolean functions. We use an approach using graphs of functions (see e.g. [10]) which we will denote by $G_F = \{(x, F(x)) : x \in \mathbb{F}_{2^n}\}$.

**Definition 3.** Two functions $F_1, F_2 : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are CCZ-equivalent if there are linear mappings $\alpha, \beta, \gamma, \delta : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ and $a, b \in \mathbb{F}_{2^n}$ such that $L(G_{F_1}) + (a,b) = G_{F_2}$ where $L : \mathbb{F}_{2^n}^2 \to \mathbb{F}_{2^n}^2$ is a bijective mapping defined by

$$L(x, y) = (\alpha(x) + \beta(y), \gamma(x) + \delta(y))$$

for $x, y \in \mathbb{F}_{2^n}$.

We call $F_1$ and $F_2$ extended affine equivalent (EA-equivalent) if a mapping $L$ defined as above can be found with $\beta = 0$ and affine equivalent if a mapping $L$ can be found with $\beta = \gamma = 0$.

Clearly, affine equivalence implies EA-equivalence, which in turn implies CCZ-equivalence. It is also obvious that the size of the image set is invariant under affine equivalence. Nonlinearity and differential uniformity are invariant under CCZ-equivalence [3]. Thus, investigating functions that are CCZ- or EA-equivalent to functions with low differential uniformity/high nonlinearity is interesting.

**Outline.** In this paper, we focus on EA- and CCZ-equivalence to the inverse function. This is a particularly interesting case because of the good cryptographic properties of the inverse function. In the second section, we show that some questions about CCZ- and EA-equivalence to a function $F$ are related to the existence of permutation polynomials of the form $L_1(F(x)) + L_2(x)$. Accordingly, we investigate the existence of permutation polynomials of the form $L_1(x^{-1}) + L_2(x)$. We show that this problem is related to Kloosterman zeros, i.e. elements whose Kloosterman sum is zero. In Section 3 we give an upper bound on the maximal size of a subspace of $\mathbb{F}_{2^n}$ that contains only Kloosterman zeros. We believe that this result is interesting on its own. Using this result, we show in Section 4 that there are no permutations of the form $L_1(x^{-1}) + L_2(x)$ if $\ker(L_1)$ or $\ker(L_2)$ is large.

### 2 EA- and CCZ-equivalence via permutation polynomials

The results in this section give the motivation for our investigations.
Proposition 1 ([3, Proposition 3]). Let $F, F': \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$. The function $F'$ is EA-equivalent to $F$ or $F^{-1}$ (if it exists) if and only if there exists a linear permutation $L = (L_1, L_2)$ on $\mathbb{F}_{2^n}^2$ such that $L(G_F) = G_{F'}$ and $L_1$ depends only on one variable. More precisely, if $L_1(x, y)$ depends only on $x$ then $F'$ is EA-equivalent to $F$ and if it depends only on $y$ it is EA-equivalent to $F^{-1}$.

Note that similar results to the following are used in [4, 2, 3].

Proposition 2. Let $F: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$.

- If no permutation of the form $F(x) + L(x)$ exists where $L(x) \neq 0$ is a linearized polynomial, then every permutation that is EA-equivalent to $F$ is already affine equivalent to $F$. In particular, if additionally $F$ is not a permutation, then there are no permutations EA-equivalent to $F$.

- If no permutation of the form $L_1(F(x)) + L_2(x)$ exists where $L_1, L_2$ are linearized polynomials with $L_1 \neq 0, L_2 \neq 0$, then every function that is CCZ-equivalent to $F$ is EA-equivalent to either $F$ or $F^{-1}$. Moreover, all permutations that are CCZ-equivalent to $F$ are affine equivalent to $F$ or $F^{-1}$.

Proof. Assume that no permutation of the form $F(x) + L(x)$ exists with $L \neq 0$. Further, let $F_2$ be a permutation EA-equivalent to $F$. By the definition of EA-equivalence, there exists a bijective mapping $L: \mathbb{F}_{2^n}^2 \to \mathbb{F}_{2^n}^2$ defined by $L = (\alpha(x), \gamma(x) + \delta(y))$ with linear mappings $\alpha, \gamma, \delta: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ such that

$$L(x, F(x)) + (a, b) = (\alpha(x), \gamma(x) + \delta(F(x))) = (\pi(x), \pi(F_2(x)))$$

for some permutation $\pi: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$. Since $L$ is bijective, $\delta$ has to be bijective. Moreover, since $F_2$ is a permutation, $\pi(F_2(\mathbb{F}_{2^n})) = \mathbb{F}_{2^n}$, so $\gamma(x) + \delta(F(x))$ is a permutation, or, equivalently, $\delta^{-1}(\gamma(x)) + F(x)$ is a permutation. Observe that $\delta^{-1}(\gamma(x))$ is linear. This implies that $\gamma = 0$ and $F_2$ is affine equivalent to $F$.

Let now $F_2$ be a function CCZ-equivalent to $F$ and assume that no permutation of the form $L_1(F(x)) + L_2(x)$ with $L_1, L_2 \neq 0$ exists. By the definition of CCZ-equivalence, there is a bijective mapping $L = (\alpha(x) + \beta(y), \gamma(x) + \delta(y))$ with linear mappings $\alpha, \beta, \gamma, \delta: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ such that

$$L(x, F(x)) + (a, b) = (\alpha(x) + \beta(F(x)), \gamma(x) + \delta(F(x))) = (\pi(x), \pi(F_2(x)))$$

for some permutation $\pi: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$. In particular, $\alpha(x) + \beta(F(x))$ must be a permutation. By our assumption, then $\alpha = 0$ or $\beta = 0$. Using Proposition 1 we infer that $F_2$ is EA-equivalent to $F$ or $F^{-1}$.

Now assume that $F_2$ is additionally a permutation. Then $\gamma(x) + \delta(F(x))$ is a permutation as well, which again implies $\gamma = 0$ or $\delta = 0$. Since $L$ is a permutation, we have $\delta = 0$ if $\alpha = 0$ and $\gamma = 0$ if $\beta = 0$. In the first case, $F_2$ is affine equivalent to $F^{-1}$ and in the second case affine equivalent to $F$. □
Proposition 2 shows that very strong conclusions can be drawn when no permutations of the form \(L_1(F(x)) + L_2(x)\) with \(L_1 \neq 0\) and \(L_2 \neq 0\) exist. The following proposition gives a criterion when such a function is a permutation. For a linear mapping \(L\), we denote by \(L^*\) its adjoint mapping with respect to the bilinear form \((x, y) = \text{Tr}(xy)\)

\[ (x, y) = \text{Tr}(xy) \]

where \(\text{Tr}\) is the absolute trace mapping, i.e. we have

\[ \text{Tr}(L(x)y) = \text{Tr}(xL^*(y)) \]

for all \(x, y \in \mathbb{F}_{2^n}\). Further, we define for a subset \(A \subseteq \mathbb{F}_{2^n}\)

\[ A^\perp = \{ x \in \mathbb{F}_{2^n} : \text{Tr}(ax) = 0 \text{ for all } a \in A \} \]

Proposition 3. Let \(F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}\) and \(L_1, L_2\) be linear mappings. The function \(L_1(F(x)) + L_2(x)\) is a permutation if and only if

\[ W_F(L_1^*(b), L_2^*(b)) = 0 \]

for all \(b \in \mathbb{F}_{2^n}^*\).

Proof. A function is a permutation if and only if all of its component functions are balanced, i.e.

\[ 0 = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}(b(L_1(F(x)) + L_2(x)))} \]

\[ = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}(L_1^*(b)F(x) + L_2^*(b)x)} = W_F(L_1^*(b), L_2^*(b)) \]

for all \(b \in \mathbb{F}_{2^n}^*\). \(\square\)

It was shown in [8] that no permutation of the form \(x^d + L(x)\) exists when there is an \(a \in \mathbb{F}_{2^n}\) such that \(\text{Tr}(ax^d)\) is bent. In [16] a characterization of all permutations of the form \(x^{2^i+1} + L(x)\) over \(\mathbb{F}_{2^n}\) with \(\gcd(i, n) = 1\) was given, as well as some results for the more general case \(x^d + L(x)\). Permutations of the form \(x^{2^i+1} + L(x)\) over \(\mathbb{F}_{2^n}\) with \(\gcd(i, n) > 1\) were recently considered in [21].

A particularly interesting case are functions of the type \(L_1(x^{-1}) + L_2(x)\) because of the good cryptographic properties (nonlinearity/differential uniformity) of the inverse function. Here we use as usual the convention \(0^{-1} = 0\). Li and Wang proved the following:

**Theorem 1 ([17]).** Let \(F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}\) defined by \(F = x^{-1} + L(x)\) with some linear mapping \(L(x) \neq 0\). If \(n \geq 5\) then \(F\) is not a permutation.

**Remark 1.** Functions of the type \(L_1(x^{-1}) + L_2(x)\) were also considered in odd characteristic in [12]. It was shown that they are never permutations in characteristic \(\geq 5\) (except for the trivial cases \(L_1 = 0\) or \(L_2 = 0\)). In characteristic 3, no permutations of the type \(x^{-1} + L(x)\) with \(L \neq 0\) exist except for sporadic cases in the small fields \(\mathbb{F}_3\) and \(\mathbb{F}_9\).
In this paper, we consider the more general case, i.e. functions of the type \( L_1(x^{-1}) + L_2(x) \) where \( L_1, L_2 \) are linear polynomials over \( \mathbb{F}_{2^n} \). In the case of the inverse function, the Walsh transform is closely connected to Kloosterman sums.

**Definition 4.** For \( a \in \mathbb{F}_{2^n} \), the Kloosterman sum of \( a \) over \( \mathbb{F}_{2^n} \) is defined as

\[
K_n(a) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{Tr(x^{-1} + ax)}.
\]

Clearly, \( K_n(a) = W_F(1, a) \) for \( F(x) = x^{-1} \). More precisely, we have

\[
W_F(a, b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{Tr(ax^{-1} + bx)} = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{Tr(x^{-1} + abx)} = K_n(ab)
\]

if \( a \neq 0 \) using the substitution \( x \mapsto ax \). If \( a = 0 \) and \( b \neq 0 \), we have \( K_n(ab) = W_F(a, b) = 0 \).

Proposition 3 can thus be stated using Kloosterman sums:

**Corollary 1.** Let \( F(x) = x^{-1} \) be the inverse mapping on \( \mathbb{F}_{2^n} \) and \( L_1, L_2 \) be linear mappings over \( \mathbb{F}_{2^n} \). Then \( L_1(x^{-1}) + L_2(x) \) is a permutation if and only if

\[
K_n(L_1^*(b)L_2^*(b)) = 0
\]

for all \( b \in \mathbb{F}_{2^n} \) and \( \ker(L_1^*) \cap \ker(L_2^*) = \{0\} \).

**Proof.** By Proposition 3, \( L_1(x^{-1}) + L_2(x) \) is a permutation if and only if \( W_F(L_1^*(b), L_2^*(b)) = 0 \) for all \( b \neq 0 \). If \( b \in \ker(L_1^*) \cap \ker(L_2^*) \), then \( W_F(L_1^*(b), L_2^*(b)) = 2^n \neq 0 \). In the other cases \( W_F(L_1^*(b), L_2^*(b)) = K_n(L_1^*(b)L_2^*(b)) \) by the considerations above. \( \Box \)

## 3 Vector spaces of Kloosterman zeros

Corollary 1 motivates us to investigate Kloosterman zeros, i.e. elements \( a \in \mathbb{F}_{2^n} \) with \( K_n(a) = 0 \). Kloosterman zeros have attracted interest before for the construction of bent and hyperbent functions (see for example [9], [6]). Generally, not much is known about the number and distribution of Kloosterman zeros. It was shown that for all \( n \), Kloosterman zeros exist [13] (note that this is not true in characteristic \( \geq 5 \) [12]). Moreover, it was shown that for \( n > 4 \), Kloosterman zeros are never contained in proper subfields of \( \mathbb{F}_{2^n} \) [19]. In this section, we give an upper bound for the size of vector spaces that contain exclusively Kloosterman zeros.

Let \( Q : \mathbb{F}_{2^n} \to \mathbb{F}_2 \) be the quadratic form defined by

\[
Q(x) = \sum_{0 \leq i < j < n} x^{i+2j}
\]

for all \( x \in \mathbb{F}_{2^n} \). Note that if \( m_n \) is the minimal polynomial of \( a \in \mathbb{F}_{2^n} \) over \( \mathbb{F}_2 \) of degree \( d \), then \( Q(a) \) is the third coefficient of \( m_n^{n/d} \). This in particular shows that \( Q(a) \in \mathbb{F}_2 \). Let \( B(x, y) = Q(x) + Q(y) + Q(x+y) \) be the associated bilinear form.

Our main tool in this section is the following theorem that gives a necessary condition for Kloosterman zeros.
Theorem 2 (13). Let \( n \geq 4 \). Then \( K_n(a) \equiv 0 \pmod{16} \) if and only if \( \text{Tr}(a) = 0 \) and \( Q(a) = 0 \).

We denote by \( \text{rad}(B) = \{ y \in \mathbb{F}_{2^n} : B(x,y) = 0 \text{ for all } x \in \mathbb{F}_{2^n} \} \) the radical of \( B \) and by \( \text{rad}(Q) = \text{rad}(B) \cap Q^{-1}(\{0\}) \) the radical of \( Q \). A quadratic form is called non-degenerate if \( \text{rad}(Q) = \{0\} \). We first compute the radical of \( Q \).

Lemma 1. We have

\[
\text{rad}(Q) = \begin{cases} 
\{0,1\}, & n \equiv 1 \pmod{4} \\
\{0\}, & \text{else}.
\end{cases}
\]

Proof. We have

\[
B(x,y) = \sum_{0 \leq i < j < n} x^{2^i+2^j} + \sum_{0 \leq i < j < n} y^{2^i+2^j} + \sum_{0 \leq i < j < n} (x+y)^{2^i+2^j}
\]

\[
= \sum_{i \neq j} x^{2^i} y^{2^j} + \sum_{i=0}^{n-1} x^{2^i} \sum_{j \neq i} y^{2^j}
\]

\[
= \sum_{i=0}^{n-1} x^{2^i} (\text{Tr}(y) + y^{2^i}) = \sum_{i=0}^{n-1} (xy)^{2^i} + \text{Tr}(y) \sum_{i=0}^{n-1} x^{2^i}
\]

\[
= \text{Tr}(xy) + \text{Tr}(x) \text{Tr}(y) = \text{Tr}((y + \text{Tr}(y))x).
\]

Assume \( y \in \text{rad}(B) \), i.e. \( B(x,y) = 0 \) for all \( x \in \mathbb{F}_{2^n} \). Then \( y + \text{Tr}(y) = 0 \), so \( \text{rad}(B) = \{0\} \) if \( n \) is even and \( \text{rad}(B) = \{0,1\} \) if \( n \) is odd (since in this case \( \text{Tr}(1) = 1 \)). One can easily verify that

\[
Q(1) = \frac{n(n-1)}{2} = \begin{cases} 
0, & n \equiv 0,1 \pmod{4} \\
1, & n \equiv 2,3 \pmod{4}
\end{cases}
\]

and the result follows. \( \square \)

We denote by \( N(Q(x) = a) \) the number of solutions of \( Q(x) = a \) over \( \mathbb{F}_{2^n} \). Because of the connection of \( Q \) to the minimal polynomial, the value \( N(Q(x) = a) \) was investigated in [11],[22],[5] in relation to irreducible polynomials with prescribed coefficients. In particular, the value \( N(Q(x) = 0) \) was determined. We summarize some of their results in the following theorem.

Theorem 3. Let \( N(Q(x) = 0) \) be the number of solutions of \( Q(x) = 0 \) over \( \mathbb{F}_{2^n} \). Then \( N(Q(x) = 0) = 2^{n-1} + e \) where

\[
e = \begin{cases} 
-2^{\frac{n-2}{2}}, & n \equiv 0,2 \pmod{8} \\
2^{\frac{n-1}{2}}, & n \equiv 1 \pmod{8} \\
0, & n \equiv 3,7 \pmod{8} \\
2^{\frac{n-2}{2}}, & n \equiv 4,6 \pmod{8} \\
-2^{\frac{n-1}{2}}, & n \equiv 5 \pmod{8}.
\end{cases}
\]

The following theorem is well known (see e.g. [18]).
Theorem 4 (Classification of quadratic forms). Let \( f : V \to \mathbb{F}_2 \) with \( \dim(V) = n \) be a quadratic form with \( \dim(\text{rad}(f)) = w \). \( f \) is equivalent to one of three forms:

\[
\begin{align*}
\quad & f \simeq \sum_{i=1}^{v} x_i y_i & \text{(hyperbolic case)} \\
\quad & f \simeq z + \sum_{i=1}^{v} x_i y_i & \text{(parabolic case)} \\
\quad & f \simeq x_1^2 + y_1^2 + \sum_{i=1}^{v} x_i y_i & \text{(elliptic case)}
\end{align*}
\]

where \( v = (n - w)/2 \). The value of \( N(f(x) = 0) \) depends only on \( n, w \) and the type of the quadratic form, in particular

\[
N(f(x) = 0) = 2^{n-1} + \Lambda(f)2^{\frac{n+w-2}{2}},
\]

with

\[
\Lambda(f) = \begin{cases} 
1, & \text{if } f \text{ is hyperbolic} \\
0, & \text{if } f \text{ is parabolic} \\
-1, & \text{if } f \text{ is elliptic}.
\end{cases}
\]

Remark 2. Just using the classification of quadratic forms in Theorem 4 and the determination of the radical in Lemma 1 we can give a simple alternative proof of the cases \( n \equiv 3, 7 \pmod{8} \) in Theorem 3. Indeed, in these cases \( Q \) is necessarily parabolic which immediately gives the value for \( N(Q(x) = 0) \).

With Theorems 3 and 4 we can immediately identify the type of \( Q \) depending on the value of \( n \). The following proposition gives us an upper bound for the size of subspaces on that a quadratic form vanishes.

Proposition 4. Let \( f : V \to \mathbb{F}_2 \) be a non-degenerate quadratic form on a vector space \( V \) over \( \mathbb{F}_2 \) with \( \dim(V) = n \). Further, let \( W \) be a subspace of \( V \) such that \( Q(w) = 0 \) for all \( w \in W \). Then

\[
\dim(W) \leq \begin{cases} 
\frac{n}{2}, & \text{if } f \text{ is hyperbolic} \\
\frac{n-1}{2}, & \text{if } f \text{ is parabolic} \\
\frac{n-2}{2}, & \text{if } f \text{ is elliptic}.
\end{cases}
\]

The bounds are sharp.

Proof. The result follows from the classification of quadratic forms. In the hyperbolic case we have

\[
f \simeq \sum_{i=1}^{n/2} x_i y_i
\]

so \( \dim(W) \leq \frac{n}{2} \) since every pair \( x_i, y_i \) contributes at most one dimension to \( W \). Similarly, in the parabolic case \( z = 0 \) is forced, so \( \dim(W) \leq \frac{n-1}{2} \) and in the elliptic case \( x_1 = y_1 = 0 \) is forced so \( \dim(W) \leq \frac{n-2}{2} \). \( \Box \)
We are now able to give an upper bound on the size of vector spaces that contain only Kloosterman zeros.

**Theorem 5.** Let $V$ be a $\mathbb{F}_2$-vector space over $\mathbb{F}_2^n$ with $K_n(v) = 0$ for all $v \in V$ and $n \geq 4$. Then $\dim V \leq d$ where

$$d = \begin{cases} \frac{n-2}{2}, & n \equiv 0, 2 \pmod{8} \\ \frac{n-1}{2}, & n \equiv 1, 3, 7 \pmod{8} \\ \frac{n}{2}, & n \equiv 4, 6 \pmod{8} \\ \frac{n-3}{2}, & n \equiv 5 \pmod{8}. \end{cases}$$

**Proof.** From the Theorems 3 and 4 we deduce that $Q$ is elliptic if $n \equiv 0, 2, 5 \pmod{8}$, hyperbolic if $n \equiv 1, 4, 6 \pmod{8}$ and parabolic of $n \equiv 3, 7 \pmod{8}$. In the cases $n \not\equiv 1, 5 \pmod{8}$ the quadratic form $Q$ is non-degenerate by Lemma 1 and the result follows from Proposition 4.

If $n \equiv 1, 5 \pmod{8}$ we have $\text{rad}(Q) = \{0, 1\}$ by Lemma 1. We then apply Proposition 4 to the non-degenerate form $Q|_{\{0, 1\}^\perp}$ of dimension $n - 1$. We set $H = \{0, 1\}^\perp = \{x : \text{Tr}(x) = 0\}$. It follows that $
abla(V \cap H) \leq \begin{cases} \frac{n-1}{2}, & n \equiv 1 \pmod{8} \\ \frac{n}{2}, & n \equiv 5 \pmod{8}. \end{cases}$

Observe that $V$ has to be contained in $H$ by Theorem 2, so $\dim V = \dim(V \cap H)$, yielding our result. $\square$

We want to give a brief remark that shows that our approach yields a slightly better bound than an approach using the Weil bound for Kloosterman sums. The following identity for sums of Kloosterman sums over a vector space was given in [7, Proposition 3]: For any subspace $V$ of $\mathbb{F}_2^n$ with $\dim(V) = k$ we have

$$\sum_{a \in V} (K_n^2(a) - K_n(a)) = 2^{n+k} - 2^{n+1} + 2^k \sum_{u \in V^\perp} K_n(\frac{1}{u}).$$

If $V$ contains exclusively Kloosterman zeros, we get

$$0 = 2^{n+k} - 2^{n+1} + 2^k \sum_{u \in V^\perp} K_n(\frac{1}{u}).$$

Bounding the Kloosterman sum in the right hand side of the equation using the Weil bound $|K_n(a)| \leq 2^{\frac{n}{2}+1}$, we get

$$0 \geq 2^{n+k} - 2^{n+1} - 2^k 2^{n-k} 2^{\frac{n}{2}+1} = 2^{n+k} - 2^{n+1} - 2^{\frac{3n}{2}+1}.$$  

This shows that $k = \dim(V) \leq \frac{n}{2} + 1$. If $n = 2m$ is even we also get a bound on the size of vector spaces of Kloosterman zeros using the existence of a large subfield. As shown in [19], Kloosterman sums are never contained in proper subfields. Hence, every vector space $V$ with $\dim(V) > m$ cannot contain exclusively Kloosterman zeros because $V$ has nontrivial intersection with the subfield $\mathbb{F}_{2^m}$. We conclude that for $n$ even the size of a vector space of Kloosterman zeros is bounded by $\frac{n}{2}$.

If $n \not\equiv 4, 6 \pmod{8}$, these bounds are slightly worse than the ones we gave in Theorem 5.
4 Permutations of the form $L_1(x^{-1}) + L_2(x)$

We now apply the results from the previous section to the functions of the form $F = L_1(x^{-1}) + L_2(x)$. First, we need a well-known lemma on the adjoint mapping. We include a simple proof for the convenience of the reader.

**Lemma 2.** Let $L : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be a linear mapping and $L^*$ its adjoint mapping. Then $\dim(\text{im}(L^*)) = \dim(\text{im}(L))$ and $\dim(\ker(L^*)) = \dim(\ker(L))$.

*Proof.* Let $v \in \text{im}(L^*)$ and $w \in \ker(L)$. We can write $v = L^*(x)$ for some $x \in \mathbb{F}_{2^n}$. Then $(v, w) = (L^*(x), w) = (x, L(w)) = (x, 0) = 0$, so $\text{im}(L^*) \subseteq \ker(L) \cap \ker(L^*)$, in particular $\dim(\text{im}(L^*)) \leq \dim(\ker(L))$. The other inequality holds with $L^* = L$.

The statement on the kernel follows from $\dim(\text{im}(L)) + \dim(\ker(L)) = n$. \qed

**Theorem 6.** Let $F = L_1(x^{-1}) + L_2(x)$ where $L_1 \neq 0$ and $L_2 \neq 0$ are linearized polynomials over $\mathbb{F}_{2^n}$ with $n > 4$. If $\max(\dim(\ker(L_1)), \dim(\ker(L_2))) > d$, where $d$ is defined as in Theorem 3 then $F$ does not permute $\mathbb{F}_{2^n}$.

*Proof.* Assume that $F$ is a permutation. Observe that $F(x)$ is a permutation if and only if $F(x^{-1}) = L_1(x) + L_2(x^{-1})$ is a permutation, so we can assume without loss of generality that $\dim(\ker(L_1)) \geq \dim(\ker(L_2))$.

We can further assume that $\ker(L_1)$ and $\ker(L_2)$ are both nontrivial. Indeed, if $L_2$ is a permutation then $L_2^{-1}(F(x^{-1})) = x^{-1} + L_2^{-1}(L_1(x))$ is also a permutation, contradicting Theorem 3.

By Corollary 3 we have $\ker(L_1) \cap \ker(L_2) = \{0\}$ and $K_n(L_1(b)L_2(b)) = 0$ for all $b \in \mathbb{F}_{2^n}$. Set $e = \dim \ker(L_1) = \dim \ker L_1$. Choose $0 \neq c \in \ker(L_2)$. The set

$$V = L_1(c + \ker(L_1)) \cdot L_2(c + \ker(L_1)) = L_1(c) \cdot L_2(c + \ker(L_1))$$

is a vector space that is contained in the image set of $L_1(b)L_2(b)$, in particular $K_n(v) = 0$ for all $v \in V$. Since $\ker(L_1) \cap \ker(L_2) = \{0\}$ we have $\dim(V) = e$. Theorem 3 then implies that $e \leq d$. \qed

We actually conjecture the following.

**Conjecture 1.** Let $F = L_1(x^{-1}) + L_2(x)$ where $L_1 \neq 0$ and $L_2 \neq 0$ are linearized polynomials over $\mathbb{F}_{2^n}$ with $n > 4$. Then $F$ does not permute $\mathbb{F}_{2^n}$.

With Proposition 2 Conjecture 1 implies the following (recall that the inverse mapping is an involution):

**Conjecture 2.** Let $n > 4$. Every function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ that is CCZ equivalent to the inverse function is already EA equivalent to it. Moreover, if $F$ is additionally a permutation then $F$ is affine equivalent to the inverse function.
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