Hypermagnetic knots and gravitational radiation at intermediate frequencies

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Abstract

The maximally gyrotropic configurations of the hypermagnetic field at the electroweak epoch can induce a stochastic background of relic gravitational waves with comoving frequencies ranging from the $\mu$Hz to the kHz. Using two complementary approaches we construct a physical template family for the emission of the gravitational radiation produced by the hypermagnetic knots. The current constraints and the presumed sensitivities of the advanced wide-band interferometers (both terrestrial and space-borne) are combined to infer that the lack of observations at intermediate frequencies may invalidate the premise of baryogenesis models based (directly or indirectly) on the presence of gyrotropic configurations of the hypermagnetic field at the electroweak epoch. Over the intermediate frequency range the spectral energy density of the gravitational waves emitted by the hypermagnetic knots at the electroweak epoch can exceed the inflationary signal even by nine orders of magnitude without affecting the standard bounds applicable on the stochastic backgrounds of gravitational radiation. The signal of hypermagnetic knots can be disambiguated, at least in principle, since the produced gravitational waves are polarized.

Keywords: cosmology, stochastic backgrounds of relic gravitons, hypermagnetic knots, anomalous magnetohydrodynamics, wide-band interferometers

(Some figures may appear in colour only in the online journal)

1. Introduction

The equations of the tensor modes of the geometry are not invariant under a Weyl rescaling of the four-dimensional metric [1], as suggested by Grishchuk well before the early formulations of conventional inflationary models. This means that, during inflation, the evolution of the space-time curvature induces a stochastic background of relic gravitons [1] with a spectral energy
density extending today from frequencies $\mathcal{O}(a\text{Hz})$ (i.e. $1\text{ aHz} = 10^{-18} \text{ Hz}$) up to frequencies $\mathcal{O}(\text{GHz})$ (i.e. $1\text{ GHz} = 10^2 \text{ Hz}$). In what follows, the cosmic background of relic gravitons will be described in terms of the spectral energy density $\Omega_{gw}(\nu, \tau_0)$, i.e. the energy density of the relic gravitational waves (measured in units of the critical energy density) per logarithmic interval of frequency and at the present (conformal) time $\tau_0$. The description of the energetic content of a stochastic background of gravitational radiation in terms of $\Omega_{gw}(\nu, \tau_0)$ goes back to the seminal contributions of [1] and has been consistently used since then to characterize the cosmic graviton background. The astrophysical sources of gravitational radiation, in contrast, are often described in terms of the spectral amplitude (measured in units of $1 \text{ Hz}^{-1}$) or even in terms of the dimensionless strain. The mutual relation between these three quantities can be found, for instance, in [2] as well as in various more recent references\(^1\). To clarify the results obtained here, it seems both useful and relevant to discuss, in some depth, the relation between the spectral energy density (i.e. $\Omega_{gw}(\nu, \tau_0)$), the spectral amplitude (i.e. $S_0(\nu, \tau_0)$ in what follows) and the dimensionless strain amplitude (i.e. $h_c(\nu, \tau_0)$ in what follows).

Given the Fourier amplitudes of the tensor modes of the geometry (see, e.g. equations (3.1) and (3.2)–(3.3)) the power spectrum $P_T(k, \tau)$ appears in the correlation function of the Fourier amplitudes

$$\langle h_y(\vec{k}, \tau)h_{imn}(\vec{p}, \tau) \rangle = \frac{2\pi^2}{k^2} S_{ijmn} P_T(k, \tau) \delta^{(3)}(\vec{k} + \vec{p}), \quad S_{ijmn} = \frac{1}{4} \sum_\lambda e^{(\lambda)}_y e^{(\lambda)}_{imn}, \quad (1.1)$$

where the result of the sum over the polarizations is given in equation (3.3); in equation (1.1) $k$ denotes the comoving wavenumber related to the comoving frequency in natural units as $\nu = k/(2\pi)$. At the present time $\tau_0$ the tensor power spectrum, the spectral amplitude $S_0(\nu, \tau_0)$ (measured in units of $\text{Hz}^{-1}$) and the dimensionless strain amplitude $h_c(\nu, \tau_0)$ are related in the following manner:

$$P_T(\nu, \tau_0) = 4\nu S_0(\nu, \tau_0), \quad \Omega_{gw}(k, \tau_0) = \frac{k^2}{12H^2} P_T(k, \tau_0). \quad (1.2)$$

In equation (1.2), as we shall reinstate in the forthcoming sections, $\mathcal{H} = aH$ where $a$ is the scale factor and $H$ is the Hubble rate. As a consequence of equations (1.1) and (1.2) the following chain of equalities can be easily derived:

$$S_0(\nu, \tau_0) = \frac{3a_0^2 H_0^2}{4\pi^2\nu^3} \Omega_{gw}(\nu, \tau_0) = 7.981 \times 10^{-43} \left(\frac{100 \text{ Hz}}{\nu}\right)^3 h_0^2 \Omega_{gw}(\nu, \tau_0) \text{ Hz}^{-1}, \quad (1.3)$$

where $H_0 = 100h_0 (\text{km s}^{-1}\text{Mpc}^{-1})$ is the present value of the Hubble rate and $h_0 = \mathcal{O}(0.7)$ its indetermination which appears as one of the for parameters of the concordance lore [3]. The dimensionless strain amplitude obeys $h_c^2(\nu, \tau_0) = 2\nu S_0(\nu, \tau_0)$ so that $h_c(\nu, \tau_0)$ becomes explicitly:

$$h_c(\nu, \tau_0) = 1.263 \times 10^{-20} \left(\frac{100 \text{ Hz}}{\nu}\right) \sqrt{h_0^2 \Omega_{gw}(\nu, \tau_0)}. \quad (1.4)$$

Since $\Omega_{gw}(\nu, \tau_0)$ is the quotient between the energy density of the gravitational waves and the critical energy density (i.e. $\rho_{\text{crit}} = 3H_0^2/(8\pi G)$ where, again, $H_0$ is the present Hubble rate) it is clear that in $h_c^2(\nu, \tau_0)$ the dependence on $h_0^2$ simplifies between the numerator and the denominator.

\(^1\)For the sake of conciseness the dependence of $\Omega_{gw}$ upon $\tau_0$ will be omitted when not strictly necessary. In this matter we follow the notations and the approach of the last paper of [3].
denominator. It is therefore practical to describe the stochastic backgrounds of gravitational radiation in terms of \( h_0^2 \Omega_{gw}(\nu, \tau_0) \) which does not depend explicitly upon \( h_0 \).

The normalization of the spectral energy density in critical units for frequencies of the order of the aHz is fixed by the tensor to scalar ratio (conventionally denoted by \( r_T \)). Between the aHz and \( 100 \) aHz a characteristic spectral signal is produced by those modes that exited the Hubble radius during inflation and re-entered after the beginning of the matter epoch. Since the analysis of the temperature and polarization anisotropies of the cosmic microwave background [3] demands that \( r_T < 0.1 \), the spectral energy density computed in conventional inflationary models is bounded from above at all frequencies and, in particular, at intermediate frequencies \( \muHz < \nu < 10 \kHz \) where \( h_0^2 \Omega_{gw}(\nu, \tau_0) < 1.6 \times 10^{-17} \) (see, in particular, the last paper of [3]). To draw a comparison, the recent detection of gravitational waves reported by the LIGO/Virgo collaboration corresponds to a dimensionless strain amplitude \( h_c = \mathcal{O}(10^{-21}) \) which is between eight and nine orders of magnitude larger than the stochastic background produced by the conventional inflationary models predicting, from equation (1.4), \( h_c = \mathcal{O}(10^{-29}) \).

The purpose of the present analysis is to derive a template family for the emission of gravitational waves from maximally gyrotropic configurations of the hypermagnetic field between the \( \muHz \) and the kHz. The spectral energy density of the gravitational waves obtained in this discussion exceeds the purported inflationary signal and will also be compared with the foreseen sensitivities of wide-band interferometers (both terrestrial and space-borne). In particular, as it will be clear from the phenomenological discussion, the spectral energy density of these peculiar sources can even be \( h_0^2 \Omega_{gw} = \mathcal{O}(10^{-8}) \) for the intermediate frequency range \( \muHz < \nu < \kHz \). Assuming, for the sake of concreteness, that \( \nu = \mathcal{O}(0.1) \kHz \) we have that \( h_c \) could even be as large as \( 10^{-24} \), that is to say five orders of magnitude larger than the inflationary signal in the same frequency range.

The layout of this topical paper is the following. In section 2 we shall briefly discuss the connections between the hypermagnetic knots and the electroweak physics. The gravitational waves emitted either by a single configuration or by a stochastic collection of hypermagnetic knots will be computed in section 3. Section 4 contains the phenomenological considerations. The concluding remarks are collected in section 5.

2. Hypermagnetic knots and electroweak physics

When the mass of the Higgs boson is \( \mathcal{O}(125 \GeV) \) [4] the phase diagram of the electroweak theory can only be scrutinized with non-perturbative methods. Lattice simulations [5] suggest the presence of a cross-over regime which is conceptually similar to the behaviour experienced by ordinary chemical compounds above their triple point. In this regime the collision of bubbles of the new phase cannot happen but gravitational waves can still be emitted provided the electroweak plasma hosts maximally gyrotropic configurations of the hypermagnetic field (dubbed hypermagnetic knots in [6]) for typical temperatures \( T_{ew} = \mathcal{O}(100 \GeV) \). Inside the electroweak particle horizon, hypermagnetic knots (HK in what follows) can be pictured as a collection of flux tubes (closed because of transversality) but characterized by a non-vanishing gyrotropy (i.e. \( \vec{B} \cdot \nabla \times \vec{B} \) where \( \vec{B} \) will denote throughout the comoving hypermagnetic field). The dynamical production of HK and Chern-Simons waves suggested in the past a viable mechanism for the generation of the baryon asymmetry of the Universe [6] (see also [7]). While this interesting possibility will be swiftly reviewed at the end of this section, as previously pointed out [6] the minimal comoving frequency of the gravitational waves potentially emitted by the HK is:
\[ \nu_{ew} = 25.03 \left( \frac{N_{\text{eff}}}{106.75} \right)^{1/4} \left( \frac{T_{ew}}{10^4 \text{GeV}} \right) \mu \text{Hz}, \tag{2.1} \]

where \( N_{\text{eff}} \) the effective number of relativistic degrees of freedom at \( T_{ew} \). While the smallest frequency of the emission is given by equation (2.1), the equations of anomalous magnetohydrodynamics [6, 8] govern the dynamics of the HK and determine, ultimately, the largest frequency of the spectrum.

The non-screened vector modes of the electroweak plasma correspond to the hypercharge field which has a chiral coupling to fermions. The axial currents may arise either as a finite density effect (implying a non-trivial evolution of the chemical potential), or they can be associated with the presence of an axion field. Anomalous magnetohydrodynamics (AMHD) aims at describing the dynamical evolution of the gauge fields in a plasma containing both vector and axial-vector currents: while the axial currents are not conserved because of the triangle anomaly, the vector currents are responsible for the Ohmic dissipation\(^2\). Even if AMHD admits a generally covariant formulation (see, in this respect, the last paper of [8]), for typical wavelengths smaller than the particle horizon at the electroweak epoch\(^3\), the relevant subset of the AMHD equations for the (comoving) hypermagnetic field and for the vorticity \( \vec{\omega} \) can be written as [8]:

\[ \partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{H}) + \frac{\nabla^2 \vec{B}}{4\pi \sigma} + \frac{\nabla \times (g_a \vec{\omega})}{4\pi \sigma} - \frac{\vec{B}}{4\pi \sigma}, \tag{2.2} \]

\[ \partial_t \vec{\omega} = \nabla \times (\vec{v} \times \vec{\omega}) + \frac{\nabla \times (\vec{J} \times \vec{B})}{a^2(\rho + p)} + \frac{\eta}{a^2(\rho + p)} \nabla^2 \vec{\omega}, \tag{2.3} \]

where \( \eta \) is the shear viscosity and \( \sigma \) is the comoving conductivity of the electroweak plasma\(^4\). Equations (2.2) and (2.3) hold when the two-fluid effects can be neglected in the slow branch of the AMHD spectrum (see, in particular, the first two papers of [8]). In equation (2.2) \( g_B \) and \( g_a \) denote, in a concise notation, the coefficients of the magnetic and the vortical currents [6, 8]. There are specific situations where the chemical potential and the axion field can be simultaneously present (see, respectively, the second paper of [6] and the last paper of [8]) both contributing to \( g_B \) (and possibly to \( g_a \)). However, in spite of the potential richness of the spectrum of AMHD (which has been explored elsewhere) for the present ends the most relevant observation is that the perfectly conducting limit suppresses the anomalous contributions.

According to equation (2.2) when the conductivity is very large the magnetic and the vortical currents are suppressed in comparison with the remaining term (which is the electroweak analog of the standard dynamo contribution). In a shorthand notation we then have that \( \partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{H}) + \mathcal{O}(g_B/\sigma) \). Defining the vector potential in the Coulomb gauge, equation (2.2) becomes, up to small corrections due to the conductivity, \( \partial_t A = \vec{v} \times (\nabla \times \vec{A}) \).

In the highly conducting limit, thanks to classic analyses\(^5\) the magnetic energy density will

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\(^2\)The AMHD equations differ from the ones where only chiral currents are present [9] at finite fermionic density. They are sometimes presented as the theoretical rationale for the chiral magnetic effect [10] originally discussed at the electroweak epoch [11] and today studied in the collisions of heavy ions. The name AMHD is technically preferable and goes back to [11] where the chiral magnetic effect was originally discussed by generalizing the analysis of finite density effects [9] to the case of finite conductivity.

\(^3\)This means that \( k/\mathcal{H}_{ew} = k_{ew} > 1 \). Recall that \( \mathcal{H}_{ew} = a_\text{ew}H_{ew} \); \( H_{ew} \) is the Hubble rate at the electroweak epoch and \( a_\text{ew} \) is the scale factor.

\(^4\)Equations (2.2) and (2.3) hold in the case of a conformally flat geometry of Friedmann-Robertson-Walker (FRW) type. Obviously in the radiation case \( a^4(\rho + p) = 4a^4/3 = \text{constant} \).

\(^5\)We remind here the results due to Fermi and Chandrasekhar (in connection with the stability of galactic arms) and, a little later, the analyses of Chandrasekhar Kendall and Woltjer [18].
then be minimized in a fiducial volume $V$ under the assumption of constant magnetic helicity by introducing the Lagrange multiplier $\zeta$. The configurations minimizing the functional

$$\mathcal{G} = \int_V d^3x \{ |\nabla \times \vec{A}|^2 - \zeta (\vec{A} \cdot (\nabla \times \vec{A})) \},$$

(2.4)

are the Beltrami fields satisfying $\nabla \times \vec{B} = \zeta \vec{B}$ where $1/\zeta = \lambda_B$ has dimensions of a length characterizing the spatial extension of the solution. The constant-$\zeta$ solutions represent the lowest state of hypermagnetic energy which a closed system may attain also in the case where anomalous currents are present, provided the ambient plasma is perfectly conducting [8]. The configurations minimizing the energy density with the constraint that the magnetic helicity be conserved coincide then with the ones obtainable in ideal magnetohydrodynamics (i.e. without anomalous currents). This observation explains why it is possible to derive hypermagnetic knot solutions in a hot plasma from their magnetic counterpart [8] (see, in this respect, [6, 10, 11]).

The gyrotropic configurations minimizing equation (2.4) exclude the backreaction on the flow. Indeed, because of the smallness of the hyperelectric fields the Ohmic current is given by $\vec{J} = \nabla \times \vec{B} / (4\pi)$; at the same time, $\nabla \times \vec{B} = \zeta \vec{B}$. Putting together the two previous observations the second term at the right hand side of equation (2.3) cancels exactly. Consequently HK have the unique property of allowing resistive decay of the field without introducing stresses on the evolution of the bulk velocity of the plasma. Thus, the maximal frequency of the spectrum is only be determined by the conductivity and it is given by [6, 8]:

$$\nu_\sigma = 58.28 \left( \frac{T_{ew}}{10^2 \text{GeV}} \right)^{1/2} \left( \frac{g'}{0.3} \right)^{-1} \left( \frac{N_{\text{eff}}}{106.75} \right)^{1/4} \text{kHz},$$

(2.5)

where $g'$ is the hypercharge coupling constant. Equation (2.5) is obtained by computing the comoving wavenumber $k_\sigma$ corresponding to the hypermagnetic diffusivity. In general terms we have that $k_\sigma^2 = \int d\tau' / (4\pi \sigma)$ however when the comoving conductivity is constant the previous expression simplifies as $k_\sigma \simeq \sqrt{\sigma H_{ew}}$. The role played by HK in the problem of baryogenesis is mainly related to the possible production of hypermagnetic gyrotropy [6, 7] and it might be useful to elucidate a bit the guiding logic of this proposal. Since the seminal work of Sakharov [12] various attempts have been made to account for the baryon asymmetry of the Universe (BAU in what follows). The standard lore of baryogenesis (see the first article of [13]) stipulates that during a strongly first-order electroweak phase transition the expanding bubbles are nucleated while the baryon number is violated by sphaleron processes. Given the current value of the Higgs mass, to produce a sufficiently strong (first-order) phase transition and to get enough $CP$ violation at the bubble wall, the standard electroweak theory must be appropriately extended. The second complementary lore for the generation of the BAU is leptogenesis (see the second article of [13]) which can be conventionally realized thanks to heavy Majorana neutrinos decaying out of equilibrium and producing an excess of lepton number ($L$ in what follows). The excess in $L$ can lead to the observed baryon number thanks to sphaleron interactions violating $(B + L)$. An admittedly less conventional perspective stipulates that the BAU could be the result of the decay of maximally helical configurations of the hypercharge field [6]. Indeed, while the $SU_L(2)$ anomaly is typically responsible for $B$ and $L$ non-conservation via instantons and sphalerons, the $U_A(1)$ anomaly might lead to the transformation of the infra-red modes of the hypercharge field into fermions [6, 9, 11]. As suggested in [6] the production of the BAU demands, in this context, the dynamical generation of the hypermagnetic gyrotropy $\mathcal{K}^{(B)}(\vec{x}, \tau) = \vec{B} \cdot \nabla \times \vec{B}$. 

The magnetic and kinetic gyrotropies play a crucial role in the mean-field dynamo [14] and the same notion occurs, with the due differences, in the present context. The production of a non-vanishing $K^{(b)}$ can be obtained, for instance, by studying the effective action describing the interaction of a dynamical pseudoscalar with hypercharge fields as proposed in [6] (see also [15])

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - V(\psi) - \frac{1}{4} Y_{\alpha\beta} Y^{\alpha\beta} + c \frac{\psi}{4M} Y_{\alpha\beta} \tilde{Y}^{\alpha\beta} \right],$$  

(2.6)

where $Y_{\alpha\beta}$ and $\tilde{Y}^{\alpha\beta}$ are, respectively, the hypercharge field strength and its dual. Note that $c = c_{\psi} \alpha'/\overline{2\pi}$ where $\alpha' = g^2/4\pi$ and $c_{\psi}$ is a numerical factor of order 1; in case $\psi$ would coincide with a conventional axion, $c_{\psi}$ could be computed from the Peccei-Quinn charges of all fermions present in the model. Even more recently this action has been generalized as:

$$S = \int d^4x \sqrt{-g} \left[ \mathcal{M}_\alpha^\rho(\varphi, \psi) Y_{\rho\alpha} Y^{\rho\alpha} - N_\rho^\alpha(\varphi, \psi) \tilde{Y}_{\rho\alpha} \tilde{Y}^{\rho\alpha} + Q_\rho^\sigma(\varphi, \psi) Y_{\rho\sigma} \tilde{Y}^{\rho\sigma} \right].$$  

(2.7)

The symmetric tensors $\mathcal{M}_\alpha^\rho(\varphi, \psi), N_\rho^\alpha(\varphi, \psi)$ and $Q_\rho^\sigma(\varphi, \psi)$ contain the couplings of the hypercharge either to the inflaton field itself (be it for instance $\varphi$) or to some other spectator field (be it for instance $\psi$). The typical derivative coupling arising in the relativistic theory of Casimir-Polder and Van der Waals interactions [16] is implicitly contained in equation (2.7) when $Q_\rho^\sigma(\varphi, \psi) = 0$: in this case equation (2.7) offers a viable framework for inflationary magnetogenesis [17] characterized by unequal electric and magnetic susceptibilities. To be even more specific, following the notations of the fourth paper of [7] the three symmetric tensors appearing in equation (2.7) can be parametrized as follows:

$$\mathcal{M}_\alpha^\rho = -\frac{\lambda_1}{4} \delta_\rho^\alpha - \frac{\lambda_2}{4} \delta_\rho^\alpha \psi^{\alpha}, \quad N_\rho^\alpha = -\frac{\lambda_3}{4} \rho^{\alpha} \pi^{\rho},$$

$$Q_\rho^\alpha = \frac{1}{4} \left[ \lambda_1(\varphi, \psi) \delta_\rho^\alpha + \lambda_2(\varphi, \psi) \pi^{\rho} \right].$$  

(2.8)

where $u_\rho = \partial_\rho \varphi / \sqrt{g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi}$ and $\pi_\rho = \partial_\rho \psi / \sqrt{g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi}$ are the normalized gradients of the scalar fields. When $u_\rho \to 0$ and $\pi_\rho \to 0$ equations (2.7) reduces to (2.6).

In spite of the specific mechanism for the production of $K^{(b)}$ the contribution of the hypermagnetic gyrotropy determines the comoving baryon to entropy ratio $\eta_b = n_b/\varsigma$ [6]:

$$\eta_b(\chi, \tau) = \frac{3\alpha' n_f}{8\pi H} \left( \frac{T}{\sigma} \right) K^{(b)}(\chi, \tau) \frac{1}{a^3 \rho_{\text{crit}}},$$  

(2.9)

where $\varsigma = 2\pi^2 T^3 N_{\text{eff}}/45$ is the entropy density of the plasma and $n_f$ is the number of fermionic generations; $N_{\text{eff}} = 106.75$ in the standard electroweak theory. Equation (2.9) holds when the rate of the slowest reactions in the plasma (associated with the right-electrons) is larger than the dilution rate caused by the hypermagnetic field itself [6].

The basic ideas discussed in the present section have been developed by different authors within slightly different approaches. An incomplete list of references can be found in [7]. The common aspect of the ideas presented in [7] is that gyrotropic configurations of the hypermagnetic field can seed, in some way, the baryon or lepton asymmetry. This idea of hypermagnetic baryogenesis or leptogenesis is exactly the one suggested in [6] even if the peculiar production mechanism of the gauge fields can be different.
3. Gravitational waves from hypermagnetic knots

The intermediate frequency range defined by $\nu_{ew}$ and $\nu_{sc}$ encompasses the operating window of space-borne interferometers (such as the (e)Lisa or the Bbo/Decigo projects [24]) and the frequencies where terrestrial wide-band interferometers (such as the Ligo/Virgo experiment [25]) are operating today (i.e. between few Hz and 10 kHz). Space-borne interferometers [such as (e)Lisa (laser interferometer space antenna), Bbo (Big Bang Observer), and Decigo (Deci-Hertz interferometer gravitational wave observatory)] might operate between few mHz and the Hz hopefully after 2032. Even in the most optimistic case the sensitivities of these instruments will not be immediately relevant for the stochastic backgrounds of cosmological origin. In particular the spectral energy density of the inflationary signal at intermediate frequency seems to be out of reach. The same might not be true for the signal coming from maximally gyrotropic configurations of the hypermagnetic field. To determine the template family for the emission of gravitational radiation we shall first analyze the case of a single knot configuration and then move to the case of stochastic collection of knots.

In a conformally flat geometry of Friedmann-Robertson-Walker type the background metric is given by $\bar{g}_{\mu\nu} = a^2(\tau)h_{\mu\nu}$ where $a$ is the scale factor, $\tau$ the conformal time coordinate and $h_{\mu\nu}$ is the Minkowski metric with signature mostly minus, i.e. $(+, -, -, -)$. In this background the amplitude of the tensor fluctuations of the geometry is defined as

$$\delta g_{\mu\nu} = -a^2 h_{\mu\nu}, \quad \delta h^\mu_{\nu} = h^\mu_{\nu} = 0,$$  \hfill (3.1)

i.e. $h_{\mu\nu}$ is, respectively, divergenceless and traceless. Moreover, given a triplet of mutually orthogonal unit vectors (e.g. $\hat{m}$, $\hat{n}$ and $\hat{q}$) the two tensor polarizations are defined as:

$$e^{(\ell)}_{ij}(\hat{q}) = (\hat{m}\hat{m}_j - \hat{n}\hat{n}_j), \quad e^{(\ell)}_{ij}(\hat{q}) = (\hat{m}\hat{n}_j + \hat{n}\hat{m}_j).$$ \hfill (3.2)

For the present ends the sum over the polarizations can be written as

$$\sum_\ell e^{(\ell)}_{ij}(\hat{q}) = p_ip_j + p_ip_j - p_ip_j,$$ \hfill (3.3)

where $p_{ij} = (\delta_{ij} - \hat{q}_i\hat{q}_j)$ is the transverse projector. By perturbing the Einstein-Hilbert action coupled to the sources to second order in the amplitude of the tensor modes of the geometry in a conformally flat FRW background we obtain [20]:

$$S_{gw} = \int d^3x d\tau \sqrt{-h} \left[ \frac{1}{8\ell_p^2}\bar{g}^{\alpha\beta}\partial_\alpha h_{ij}\partial_\beta h_{ij} - \frac{1}{2}\Pi_{ij}h_{ij} \right], \quad \ell_p = \sqrt{8\pi G},$$ \hfill (3.4)

where $\Pi_{ij}$ is the anisotropic stress produced by the HK; note that $\bar{g}$ is the determinant of the background metric $\bar{g}_{\mu\nu}$. After introducing the tensor normal mode $\mu_{ij} = ah_{ij}[1]$, the evolution equations derived from equation (3.4) can be expressed as:

$$\mu''_{ij} - \nabla^2\mu_{ij} - \frac{a''}{a}\mu_{ij} = -2\ell_p^2\bar{g}^{\lambda\beta}\partial_\lambda h_{ij}\partial_\beta h_{ij}, \quad \mu_{ij} = ah_{ij}. \hfill (3.5)$$

As previously remarked, the backreaction on the flow vanishes because the Ohmic current and the hypermagnetic field are parallel. Thus the HK the solution can be factorized as

$$\vec{B}(\vec{x}, \tau) = \vec{b}(\vec{x})f(\zeta, \tau), \quad f(\zeta, \tau) = \exp\left[-\zeta^2\tau/(4\pi\sigma)\right]$$ \hfill (3.6)

where $\vec{b}(\vec{x})$ minimizes equation (2.4) and $f(\zeta, \tau)$ accounts for the contribution of the Ohmic dissipation in the case of a single configuration with constant $\zeta$. According to the well known Chandrasekhar-Kendall representation [18], $\vec{b}(\vec{x})$ can always be expressed as:
\[ \vec{b}(\vec{x}) = \lambda H \vec{\nabla} \times [\vec{\nabla} \times (\vec{u} \Psi)] + \vec{\nabla} \times (\vec{u} \Psi), \]  

(3.7)

where \( \vec{u} \) is a unit vector denoting the direction of the knot and \( \Psi \) obeys \( \nabla^2 \Psi + \zeta^2 \Psi = 0 \). If \( \zeta \) is constant, the general representation of the hypermagnetic knot configuration can be achieved through the Chandrasekhar-Kendall representation [18]. Equation (3.7) leads to a gyrotropy that does not decrease at large distance scales but more realistic configurations where the gyrotropy does decrease at large distance scales can be found [6, 19].

The preferred direction of the knot introduces a difference in the evolution of the two polarizations of the gravitational wave. Indeed, the anisotropic stress associated with the HK is given by \( \Pi_\text{ij} = (B_i B_j - B^2 \delta_{ij}/(4\pi a^4(\tau))) \) and it can always be projected along the two tensor polarizations as \( \Pi_\text{ij} = (\Pi_\text{e}^\text{ij} + \Pi_\text{m}^\text{ij})/2 \) where \( \Pi_\text{e} \) and \( \Pi_\text{m} \) are, respectively,

\[ \Pi_\text{e}(\vec{x}, \tau) = \frac{1}{4\pi a^4(\tau)} [\langle \vec{B} \cdot \hat{\vec{n}} \rangle^2 - \langle \vec{B} \cdot \hat{\vec{n}} \rangle^2], \quad \Pi_\text{m}(\vec{x}, \tau) = \frac{1}{2\pi a^4(\tau)} \langle \vec{B} \cdot \hat{\vec{m}} \rangle (\vec{B} \cdot \hat{\vec{n}}). \]

(3.8)

Since the direction of the knot does not need to coincide with the direction of propagation of the gravitational wave, equation (3.8) implies that the emission is polarized. When the electroweak represents a portion of the radiation-dominated evolution (as assumed throughout) the scale factor evolves linearly in conformal time; thus \( a'' = 0 \) in equation (3.5) and the corresponding solution is:

\[ h_\text{ij}(\vec{x}, \tau) = -\frac{2f^2}{a(\tau)} \int d^3x' \int_{\tau_0}^\tau d\xi \mathcal{G}(\vec{x}, \vec{x'}; \tau, \xi) a^3(\xi) \Pi_\text{ij}(\vec{x}', \xi), \]

\[ \mathcal{G}(\vec{x}, \vec{x'}; \tau, \xi) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{k} e^{-ik(\vec{x}-\vec{x}')/y} \sin[k(\xi - \tau)]. \]

(3.9)

Thanks to equation (3.7) also the whole anisotropic stress of the HK can be factorized as \( \Pi_\text{ij}(\vec{x}, \tau) = \Pi_\text{ij}(\vec{x}) f(\sqrt{2\zeta}, \tau)/a^4(\tau) \). Using this observation and recalling that \( aH = a_{\text{ew}} H_{\text{ew}} \) is constant during radiation we have:

\[ h_\text{ij}(\vec{k}, \tau) = -\frac{2f^2 a_{\text{ew}}}{a(\tau)} \int d^3k \Pi_\text{ij}(\vec{k}) \mathcal{F}(k\tau)e^{-ik\vec{x}}, \]

\[ \mathcal{F}(k\tau) = \int_1^{k_{\text{ew}}} dy \frac{\sin(y - k\tau)}{y} \left[ 1 - \mathcal{O}\left(\frac{\zeta^2 y}{\sigma k}\right) \right], \quad y = k_{\text{ew}}. \]

(3.10)

Since we must integrate over all the modes inside the particle horizon at \( \tau_{\text{ew}} \) the lowest extremum of integration in equation (3.10) coincides with 1. To simplify the integrand even further we are just considering those scales which are not effected by the hypermagnetic diffusivity. Thus \( f(\sqrt{2\zeta}, \tau) \) can be expanded by neglecting all higher-order corrections and by positing, as it is in practice, that the signal is absent for scales smaller than the diffusivity scale. The approximate expression of \( \mathcal{F}(k\tau) \) appearing in equation (3.10) can then be written as:

\[ \mathcal{F}(k\tau) = \sin k\tau[\alpha - \text{Ci}(k\tau)] + \cos k\tau[\text{Si}(k\tau) - \beta], \]

(3.11)

where \( \alpha = \text{Ci}(1) = 0.33 \) and \( \beta = \text{Si}(1) = 0.94 \). Note that, in equation (3.11), the usual trigonometric integrals are defined as

\[ \text{Ci}(z) = -\int_z^\infty dt \frac{\cos t}{t}, \quad \text{Si}(z) = \int_z^\infty dt \frac{\sin t}{t}. \]

(3.12)

In the limit \( k\tau \gg 1 \) we have that \( \mathcal{F}(k\tau) \to (\pi/2 - \beta) \cos k\tau + \alpha \sin k\tau; \) this expression also implies \( |\mathcal{F}(k\tau)| < \epsilon \) where \( \epsilon = 0.7 \).
Having determined the emitted amplitude in the case of a single HK, we can now compute the energy density and, for this purpose, the energy-momentum tensor of the gravitational waves in a FRW background can be formally derived by varying the free part of the action (3.4) with respect to $g^\alpha\beta$ and the result is:

$$T^\nu_\mu = \frac{1}{4\ell_P^2} \left[ \partial_\mu h_{ij} \partial^\nu h_{ij} - \frac{1}{2} \delta^\nu_\mu \delta^{ij} \delta_{\mu\nu} \partial_\alpha h_{ij} \partial^\beta h_{ij} \right].$$

(3.13)

This expression due to Ford and Parker [20] relies on the physical observation that the two polarizations of the gravitational waves in a conformally flat background behave as a pair of minimally coupled scalar fields. However, since the energy and momentum of the gravitational field itself cannot be localized there is no unique expression for $T^\mu_\nu$, we must instead deal with pseudotensors whose definitions can be mathematically slightly different but are physically equivalent [21]. Indeed it has been argued in the past that different computational schemes lead exactly to the same spectral energy density for wavelengths shorter than the Hubble radius at each corresponding epoch. Note that the energy-momentum pseudo-tensor defined from the second-order variation of the Einstein tensor and the expression of equation (3.13) are formally slightly different and their peruse in the context of stochastic backgrounds of relic gravitons has been carefully discussed in [21] (see, in particular, the last paper).

With these necessary precisions the energy density defined from equation (3.13) is given by:

$$\rho_{gw} = \frac{1}{8\ell_P^2 a^2} \left[ \partial_\tau h_{ij} \partial_\tau h_{ij} + \partial_k h_{ij} \partial_k h_{ij} \right].$$

(3.14)

Since the first term at the right hand side of equation (3.14) is always subleading when the relevant modes are shorter than the particle horizon the energy density in critical units becomes:

$$\rho_{gw}/\rho_{cr} = \frac{N}{(1 + z_{eq})(1 + z_\Lambda)^3} \left( \frac{b^4}{3H^2_{eq} M_P^2} \right)^2, \quad N = \frac{e^2}{16\pi^2},$$

(3.15)

where $M_P = \ell_P^{-1}$ and $\Pi_{ij}(x) = b^4/(24\pi^2)$. Note that $1 + z_{eq} = 3228.9(h_0^2\Omega_M/0.134)$ is the redshift to equality while $1 + z_\Lambda = 0.703(\Omega_M/0.258)^{1/3}(\Omega_\Lambda/0.742)^{-1/3}$ is the redshift of $\Lambda$ dominance (recall that in the concordance model the dark energy component is simply parametrized in terms of a cosmological constant). Note that, after equality, the energy density of the gravitational waves still redshifts like radiation while the background is dominated by matter; after the moment of $\Lambda$-dominance the background energy density is instead constant. The overall result of these effects leads to the final form of equation (3.15).

Equation (3.15) gives the energy density emitted by a single HK in real space. A stochastic collection of HK is characterised, in Fourier space, by the two-point function:

$$\langle B_i(k, \tau) B_j(k', \tau') \rangle = \frac{2\pi^2}{k^2} \epsilon_{ijk} k^2 P_h(k, \tau, \tau') \delta^{(3)}(k + \vec{p}),$$

(3.16)

where the power spectrum $P_h(k, \tau)$ has the dimensions of an energy density. As in the case of a single knot the resistive decay of the field does not introduces stresses on the evolution of the bulk velocity. Equations (3.10) and (3.14) can still be used together with equation (3.16) for a direct computation of the spectral energy density in critical units:

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In the present paper $\ln$ denotes the natural logarithm while $\log$ denotes the common logarithm.
\[ \Omega_{gw}(k, \tau) = \frac{1}{\rho_{\text{crit}}} \frac{d\rho_{gw}}{d \ln k} = \frac{\rho_{gw}^2}{6a^3H^2} \int_{\tau_m}^\tau d\xi \int_{\tau_m}^\tau d\xi' a^3(\xi)a^3(\xi') \times \sin[\xi(\xi' - \tau)]\sin[\xi' - \tau)] \Omega(Q(\xi, \xi')) \left[ 1 + O\left( \frac{H^2}{k^2} \right) \right], \]  

(3.17)

where the subleading piece (second term in the squared bracket) comes from the time derivative of \( h_{ij} \) appearing in equation (3.14) which is negligible inside the Hubble radius. The power spectrum of the anisotropic stress appearing in equation (3.17) is quadratic in the power spectra of the HK and its explicit form is given by:

\[ P_{\Pi}(q, \xi, \xi') = \frac{5q^4}{96\pi^3a^3(\xi)a^3(\xi')} \int d^3k \int d^3p \int d^3k' \int d^3p' \delta^{(3)}(q - k) \delta^{(3)}(p - k') \left[ B_i(\kappa, \xi)B_j(\kappa, \xi')B_i(\kappa', \xi')B_j(\kappa', \xi') \right] \]

\[ \times \delta^{(3)}(\hat{\xi} - \hat{k} - \hat{p})\delta^{(3)}(\hat{\xi} - \hat{k}' - \hat{p}') \left[ B_i(\hat{\kappa}, \xi)B_j(\hat{\kappa}, \xi')B_i(\hat{\kappa}', \xi')B_j(\hat{\kappa}', \xi') \right]. \]  

(3.18)

The spectral energy density of equation (3.17) in critical units is dimensionless but scales with the square of the amplitude of the knot power spectrum. Equation (3.18) has therefore the same physical content of equation (3.15) where \( \rho_{gw}/\rho_{\text{crit}} \) scales with the square of the energy density of the HK. Notice, incidentally, that to derive equation (3.18) we need to evaluate the correlator of the anisotropic stresses. In particular after some algebra it is relatively straightforward to arrive at the following expression:

\[ \langle \Pi(\hat{q}, \xi) \Pi(\hat{q}', \xi') \rangle = \frac{1}{128\pi^3a^3(\xi)a^3(\xi')} \int d^3k \int d^3p \int d^3k' \int d^3p' \delta^{(3)}(\hat{q} - \hat{k} - \hat{p})\delta^{(3)}(\hat{q}' - \hat{k}' - \hat{p}') \left[ B_i(\kappa, \xi)B_j(\kappa, \xi')B_i(\kappa', \xi')B_j(\kappa', \xi') \right] \]

\[ \times \left( -\frac{1}{3} B_i(\hat{\kappa}, \xi)B_j(\hat{\kappa}, \xi')B_i(\hat{\kappa}', \xi')B_j(\hat{\kappa}', \xi') \right) \]  

(3.19)

The different correlation functions appearing in equation (3.19) can be computed explicitly in terms of the two-point functions of equation (3.16). After some algebra the result of equation (3.18) can be swiftly obtained by recalling that, according to the present notations, the power spectrum of the anisotropic stresses is defined as \( \langle \Pi(\hat{q}, \xi) \Pi(\hat{q}', \xi') \rangle = (2\pi^2/k^2)\delta^{(3)}(\hat{q} + \hat{q}') P_{\Pi}(q, \xi, \xi') \).

As in the case of equation (3.7) the evolution can be factored and also the power spectrum can be written as \( P_{\Pi}(k, \tau) = A_{hk}(k/k_{gw})^\beta \epsilon^{-2(\tau + \gamma)/\sigma} \). The power-law is the simplest parametrization for the power spectrum and it appears in the context of different models (see e.g. [6–8, 11]). Thus from equation (3.17) the spectral energy density can be expressed as:

\[ \Omega_{gw}(q, \tau_0) = \frac{5\pi^2a^3_{hk}}{288(1 + z_{eq})(1 + z_{L})^3H^2_{gw}} \mathcal{P}(q_{gw}) \left( \frac{q}{q_{gw}} \right)^{2\beta} \epsilon^{-2q_{gw}/q_{ew}} Q(q, q_{\sigma}, q_{ew}), \]

\[ Q(q, q_{\sigma}, q_{ew}) = \left\{ \begin{array}{l} 2 \left[ 1 - \left( \frac{q_{\sigma}}{q} \right)^\beta \right] + 2(\beta - 3) \left[ \frac{q_{\sigma}}{q} \right]^{\beta + 1} \left[ 1 - \left( \frac{q_{\sigma}}{q} \right)^{\beta + 1} \right] \\ - \frac{2}{3} \left[ \left( \frac{q}{q_{\sigma}} \right)^{3 - 2\beta} - 1 \right] + \frac{(\beta - 3)}{3(2 - \beta)} \left[ \left( \frac{q}{q_{\sigma}} \right)^{4 - 2\beta} - 1 \right] \end{array} \right\} \]  

(3.20)
Equation (3.20) holds, strictly speaking, for $0 < \beta < 1.5$; when $\beta \to 0$ and $\beta \to 1.5$ the explicit expression of $Q$ will contain some extra logarithms. In the range$^2$ of frequencies $\nu_{ew} < \nu < \nu_{\sigma}$ we have that $5\pi^{1/2}Q(\nu, \nu_{\sigma}, \nu_{ew})/288$ varies between 0.01 and 0.8. The same approximate values hold also in the limits $\beta \to 0$ and $\beta \to 1.5$. As long as the overall amplitude is not divergent of frequencies (as we just showed), the specific value of the numerical factors is not strictly essential for the determination of the template: the numerical factors can always be reshuffled in an overall amplitude which will be eventually constrained by phenomenology and (even more optimistically) by direct measurements. With this strategy in mind, from equation (3.20) the template for the emission of gravitational waves from a background of HK is:

$$\Omega_{gw}(\nu, \tau_0) = \frac{\Omega^2_B}{(1 + z_{eq})(1 + z_{\Lambda})^3} \left(\frac{\nu}{\nu_{ew}}\right)^4 e^{-2(\nu/\nu_{ew})^2}, \quad \nu \geq \nu_{ew},$$

(3.21)

where $\alpha = 2\beta$. All the numerical factors have then been reabsorbed into $\Omega^2_B$ which is chiefly determined by the dimensionless ratio $A_{ha}^2/(H_{ew}^2 M_p^2)$.

The template given by equation (3.21) can be constrained by using all the specific bounds applicable to the stochastic backgrounds of gravitational radiation of cosmological and astrophysical origin. As we shall see already by imposing these constraints a relevant portion of the parameter space can be excluded. The remaining regions may lead to a signal which could be observed, in principle, both by space-borne detectors and by terrestrial interferometers. The results of this analysis are reported in the following section.

4. Phenomenological considerations and sensitivities

To extract specific informations on $\Omega_B$ and $\alpha$, equation (3.21) can be compared with the current bounds applying to the stochastic backgrounds of relic gravitons. The large-scale bounds (constraining $r_T$ in the aHz region) are immaterial since, according to equation (3.21), $\nu_{ew}$ is the minimal frequency of the spectrum. For the same reason the pulsar timing limits [22] demand $\Omega(\nu_{pulsar}, \tau_0) < 1.9 \times 10^{-8}$ for a typical frequency $\nu_{pulsar} \simeq 10$ nHz. Since $\nu_{pulsar}$ roughly corresponds to the inverse of the observation time during which the pulsars timing has been monitored this potential constraint is always satisfied by equation (3.21) since $\nu_{ew} \ll \nu_{pulsar}$. A qualitatively different constraint stems from big-bang nucleosynthesis [23]. If there are some additional relativistic degrees of freedom (either bosonic or fermionic) the effect on the expansion rate will be the same as that of having some (perhaps a fractional number of) additional neutrino species. Before electron-positron annihilation we have $\rho_X = (7/8)\Delta N_\nu \rho_\gamma$ and after electron-positron annihilation we have $\rho_X = (7/8)(4/11)^{4/3} \Delta N_\nu \rho_\gamma \simeq 0.227 \Delta N_\nu \rho_\gamma$. The critical fraction of CMB photons can be directly computed from the value of the CMB temperature and it is given by $h_0^2\Omega_{\gamma} \equiv \rho_\gamma / \rho_{crit} = 2.47 \times 10^{-5}$. If the extra energy density component has stayed radiation-like until today, its ratio to the critical density is given by:

$$h_0^2 \frac{\rho_X}{\rho_c} = 5.61 \times 10^{-6} \Delta N_\nu \left(\frac{\Omega^2_B \Omega_{0,10}}{2.47 \times 10^{-5}}\right), \quad \rho_X = \frac{\rho_X}{3},$$

(4.1)

In case the additional species are gravitons, then equation (4.1) implies [23]:

$$h_0^2 \int_{\nu_{ew}}^{\nu} \Omega_{gw}(\nu, \tau_0)d\ln \nu = 5.61 \times 10^{-6} \Delta N_\nu \left(\frac{\Omega^2_B \Omega_{0,10}}{2.47 \times 10^{-5}}\right).$$

(4.2)

$^7$Needless to say, we can easily pass from wavenumbers to frequencies since $2\pi k = k$, as already mentioned in section 1.
The bounds on $\Delta N_{\nu}$ range from $\Delta N_{\nu} \leq 0.2$ to $\Delta N_{\nu} \leq 1$; the integrated spectral density is thus between $10^{-6}$ and $10^{-5}$. In general terms the lower extremum of integration should coincide with the present frequency corresponding to the Hubble rate at the time of big-bang nucleosynthesis; this quantity, conventionally denoted $\nu_{\text{bbn}}$, is of the order of 0.01 nHz for a putative nucleosynthesis temperature $\mathcal{O}(1)$ MeV and for an effective number of relativistic degrees of freedom corresponding to 10.75. In the range $\nu_{\text{bbn}} \leq \nu < \nu_{\text{ew}}$ the signal emitted from the HK is absent; therefore the lower extremum of integration in equation (4.2) coincides with the lowest frequency of the spectrum. Using equation (3.21) the integral of equation (4.2) can be performed explicitly. The common logarithm of the left hand side of equation (4.2) is illustrated in the left plot of figure 1. The thick line corresponds to the common logarithm of the right-hand side of equation (4.2) in the case $\Delta N_{\nu} = 0.2$. In the plane $(\Omega_B, \alpha)$ the region allowed by equation (4.2) should always be below the thick line. In the plot at the right of figure 1 the shaded area illustrates the portion of the parameter space where the bound of equation (4.2) is enforced and, simultaneously, $h_0^2 \Omega_{gw} > 10^{-16.5}$ for a typical frequency of the order of the mHz characterizing space-borne interferometers such as (e)Lisa and Bbo/Decigo. In the (e)Lisa case we shall assume a nominal sensitivity of $10^{-11}$ (for the spectral energy density) in a frequency range centered around the mHz. In the Bbo/Decigo case the sensitivity could even be $\mathcal{O}(10^{-15})$ and for a typical frequency range centered around the Hz. All in all, from the right plot of figure 2 we can just infer that for frequencies around the mHz the signal of HK can be larger than the inflationary signal, compatible with the nucleosynthesis constraint of equation (4.2) and even potentially detectable.

As we move to higher frequencies the portion of the parameter space compatible with the current bounds and reachable by the hoped sensitivities of terrestrial and space-borne detectors becomes smaller. This aspect is specifically illustrated in the fourfold plot of figure 2. In the two plots at the left we required $h_0^2 \Omega_{gw}$ to be larger than $10^{-12}$ while in the two plots at the right the shaded area illustrates the region where $h_0^2 \Omega_{gw} > 10^{-10}$. In all the plots of figure 2...
the current constraints discussed above have been consistently enforced. The Ligo/Virgo sensitivity to a stochastic background depends on $\alpha$. Unfortunately in its current configuration the experiment can only access regions where $\Omega_{gw} = O(10^{-4})$ [25]. These regions have been already excluded thanks to the current phenomenological constraints. Advanced Ligo/Virgo detectors might be far more sensitive and probe the shaded area of figure 2. By looking simultaneously at figures 1 and 2 we can conclude that at intermediate frequencies $\mu Hz \leq \nu \leq kHz$ the gravitational wave background induced by HK is always larger than the inflationary signal even if the allowed region of the parameter space is severely restricted by the current phenomenological bounds. Already the terrestrial interferometers (in their most advanced and sensitive configurations) will be able to exclude the region of white or blue spectra indices (i.e. $\alpha \geq -1$) and small amplitudes (i.e. $\Omega_B < 10^{-4}$). Conversely the space-borne detectors will preferentially probe the region of large amplitudes (i.e. $\Omega_B = O(10^{-3})$) and red spectral indices (i.e. $\alpha < -2$). The regions of red spectral indices can be already excluded on a theoretical ground since they would correspond to gyrotropic configurations of the hypermagnetic field.

Figure 2. The shaded areas in the two plots at the left illustrate the regions accessible to wide-band interferometers with putative sensitivity $O(10^{-12})$. Similarly the two plots at the right the sensitivity has been taken $O(10^{-10})$. The two plots at the top correspond to frequencies of the mHz which is the characteristic window of space-borne interferometers [24]. The two plots at the bottom concern a typical frequency of 0.1 kHz which is characteristic of terrestrial interferometers [25].
increasing (rather than decreasing) at large-distance scales. We are therefore left with the two tiny filled regions in the two plots at the bottom of figure 2.

5. Concluding remarks

The intermediate frequency range of the spectrum of relic gravitational radiation goes from few $\mu$Hz to 10kHz. This intermediate range encompasses the operating windows of space-borne interferometers (hopefully available twenty years from now) and of terrestrial detectors (already available but still insensitive to stochastic backgrounds of relic gravitons of cosmological origin). This statement can be understood by comparing the quoted sensitivities of the Ligo/Virgo experiments with the constraints imposed by the big-bang nucleosynthesis bound. Moreover, between few $\mu$Hz and 10kHz, the conventional inflationary models lead to relic gravitons whose spectral energy density can be (at most) of the order of $10^{-17}$.

There are however configurations of the hypermagnetic field carrying both magnetic helicity and magnetic gyrotropy, dubbed hypermagnetic knots, producing gravitational radiation with a spectrum ranging between $\nu_{ew} = \mathcal{O}(20) \mu$Hz and $\nu_{\sigma} = \mathcal{O}(50)$ kHz. In this paper we showed how to construct a physical template family for the emission of the gravitational radiation produced by the hypermagnetic knots. While between $\nu_{ew}$ and $\nu_{\sigma}$, the inflationary contribution implies a spectral energy density $h_0^2 \Omega_{gw}^{(s)} = \mathcal{O}(10^{-17})$, the signal due to hypermagnetic knots can be as large as $h_0^2 \Omega_{gw}^{(k)} = \mathcal{O}(10^{-8})$ without conflicting with current bounds applicable to stochastic backgrounds of gravitational radiation.

The lack of observation of gravitational waves between few $\mu$Hz and 10kHz will potentially exclude the presence of hypermagnetic knots configurations at the electroweak scale. Conversely the observation of a signal in the range that encompasses the operating windows of space-borne and terrestrial wide-band detectors will not necessarily confirm the nature of the source. Further scrutiny will be needed but the signal of the hypermagnetic knots can be disambiguated since the stochastic background of gravitational waves produced by the hypermagnetic knots is polarized. Last but not least a gravitational signal coming from maximally gyrotropic configurations of the hypercharge may offer an indirect test of the equations of anomalous magnetohydrodynamics whose spectrum includes hypermagnetic knots and Chern-Simons waves as low-frequency excitations.

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