AUTOMATON SEMIGROUP FREE PRODUCTS REVISITED

TARA BROUGH

Abstract. An improvement on earlier results on free products of automaton semigroups; showing that a free product of two automaton semigroups is again an automaton semigroup providing there exists a homomorphism from one of the base semigroups to the other. The result is extended by induction to give a condition for a free product of finitely many automaton semigroups to be an automaton semigroup.

1. Introduction

The free product of two semigroups $S = \text{sgp}(X_1 \mid R_1)$ and $T = \text{sgp}(X_2 \mid R_2)$, denoted $S \star T$, is the semigroup with presentation $\text{sgp}(X_1 \cup X_2 \mid R_1 \cup R_2)$.

The free product of two automaton semigroups $S$ and $T$ is always at least very close to being an automaton semigroup: adjoining an identity to $S \star T$ gives an automaton semigroup [1, Theorem 3]. In [1, Conjecture 5], the author and Cain conjectured that there exist finite semigroups $S$ and $T$ such that $S \star T$ is not an automaton semigroup. In [2, Theorems 2 to 4], the same authors showed that not only is every free product of finite semigroups an automaton semigroup, but whenever $S$ and $T$ are automaton semigroups either both containing an idempotent or both homogeneous, then $S \star T$ is an automaton semigroup. The construction in [2, Theorem 4] was modified by Welker [6] to considerably relax the hypothesis on the base semigroups: If $S$ and $T$ are automaton semigroups such that there exist automata for $S$ and $T$ with state sets $P$ and $Q$ respectively and maps $\phi : P \to Q$ and $\psi : Q \to P$ that extend to homomorphisms from $S$ to $T$ and from $T$ to $S$ respectively, then $S \star T$ is an automaton semigroup [6, Theorem 3.0.1]. This is a generalisation because the maps $\phi$ and $\psi$ can always be found if $S$ and $T$ either both contain idempotents (map all elements to a fixed idempotent in the other semigroup) or are both homogeneous (map all elements an arbitrary fixed element of the other semigroup); and Welker gives an example [6, Example 3.3.3] of a free product satisfying the hypothesis of her theorem but not of [2, Theorem 4].

In this paper we show that the hypothesis on the base semigroups can be relaxed still further: If $S$ and $T$ are two automaton semigroups such that there exists a homomorphism from one to the other, then their free product $S \star T$ is an automaton semigroup. We extend this result by induction to give a condition on a finite set of automaton semigroups that guarantees their free product is an automaton semigroup.

We assume an understanding of the basic theory and notation for automaton semigroups. For an introduction see [3], and for further background see [5].
The main theorem

Theorem 1. Let $S$ and $T$ be automaton semigroups such that there exists a homomorphism from $S$ to $T$. Then $S \ast T$ is an automaton semigroup.

Proof. Let $\phi : S \to T$ be a homomorphism. Let $A_1 = (Q_1, A, \delta_1)$ and $A_2 = (Q_2, B, \delta_2)$ be automata for $S$ and $T$ respectively. We may assume that $s\phi \in Q_2$ for all $s \in Q_1$. We construct an automaton $A = (Q, \delta)$ with $\Sigma(A) = S \ast T$. Let $Q = Q_1 \cup Q_2$ and

$$C = \{ \hat{a}b, \hat{a}b, \hat{a}b, \hat{a}b, \hat{a}b, \hat{a}b, \hat{a}b, \hat{a}b : a \in A, b \in B \}.$$ 

The symbols in $X := \{ \hat{a}b : a \in A, b \in B \}$ are called dominoes, and for $x \in X$ we call $x$ unmarked, $x^S$ S-marked, $x^T$ T-marked, and $x^\circ$ circled. The symbols $\$, $\$, $\$ are called gates and we refer to them respectively as open, half-open and closed gates. The transformation $\delta$ is defined as follows. For $s \in Q_1$, $t \in Q_2$, $a \in A$, $b \in B$ suppose that $(s, a)\delta_1 = (s_0, a_0)$ and $(t, b)\delta_2 = (t_0, b_0)$. Then the action of $Q$ on $Q \times C$ is given by

| $s$ | $t$ |
|-----|-----|
| $a$ | $(s_0, a_0)$ | $(t_0, a)$ |
| $b$ | $(s_0, b_0)$ | $(t_0, b)$ |
| $ab$ | $(s, a)$ | $(t, a)$ |
| $ba$ | $(s, b)$ | $(t, b)$ |
| $\$ | $(s\phi, \$)$ | $(t, \$)$ |
| $\$ | $(s\phi, \$)$ | $(t, \$)$ |
| $\$ | $(s, \$)$ | $(t, \$)$ |

This construction is heavily inspired by the construction in [2, Theorem 4], with the modification to that construction introduced by Welker [6]. The major difference is that the earlier construction used entirely different symbols to distinguish alternating products in $S \ast T$ depending on whether they started with an element of $S$ or with an element of $T$. In the current construction, we are able to distinguish all elements by their actions on $(X^T)^*$, where $X^T$ denotes the set of T-marked dominoes, or on $\overline{X}^*$. We begin by showing that words in $Q^+$ representing distinct elements of $S \ast T$ have distinct actions on $C^*$. For $w \in Q^+$, write $w = v_0u_1v_1 \ldots u_kv_n$ with $u_i \in Q_1^+, v_i \in Q_2^+$ and $v_i$ non-empty except for possibly $v_0, v_n$. Write strings in $X^*$ as $\alpha\beta$, where $\alpha \in A^*, \beta \in B^*$ are the strings obtained by reading off the first and second components respectively. Let $\gamma = \overline{\alpha_1, \beta_1, \$\alpha_2, \beta_2, \ldots, \$\alpha_n, \beta_n}$. Since $Q_2^+$ has no effect on unmarked dominoes or closed gates, $v_0$ leaves $\gamma$ unchanged; $u_1$ acts on $\alpha_1$ as in $A_1$, $S$ marks the first string of dominoes and half-opens the first gate, leaving the remainder of the string unchanged (acting by $u_1\phi \in Q_2^+$); then $v_1$ acts on $\beta_1$ as in $A_2$, $T$ marks the first string of dominoes, opens the first gate and leaves the remainder of the string unchanged. Thus
$\gamma \cdot w = \alpha_1 \cdot u_1 \beta_1 \cdot v_1 \delta_1 \ldots \delta_n \cdot u_n \beta_n \cdot v_n$. By induction $\gamma \cdot w = \alpha_1 \cdot u_1 \beta_1 \cdot v_1 \delta_1 \ldots \delta_n \cdot u_n \beta_n \cdot v_n$. Now let $w' \in Q^+$ such that $w, w'$ represent distinct elements of $S \ast T$. Write $w' = v'_0 u'_1 \ldots v'_m u'_m$ as we did for $w$. We may assume $m \leq n$, and define $u_i', v_i' = \epsilon$ for $m < i \leq n$. Let $k$ be minimal such that $u_k \not\in S \ u'_k$ or $v_k \not \in T \ v'_k$. If $k = 0$, then $w$ and $w'$ can already be distinguished by their actions on a string $[\alpha \beta]^T$, where $\beta \cdot v_0 \neq \beta \cdot v'_0$, since $[\alpha \beta]^T \cdot w = [\alpha \beta \cdot v_0]^T \neq [\alpha \beta \cdot v'_0]^T = [\alpha \beta]^T \cdot w'$. (Or the circled dominoes are $T$-marked if $n = 0$ or $m = 0$). If $k > 0$, choose $\alpha_k \in A^*, \beta_k \in B^*$ such that $\alpha_k \cdot u_k \neq \alpha_k \cdot u'_k$ or $\beta_k \cdot v_k \neq \beta_k \cdot v'_k$, and let $\gamma = \delta^{k-1}[\alpha \beta]$. Then $\gamma \cdot w = \delta^{k-1}[\alpha \cdot u_k \beta \cdot v_k] \neq \delta^{k-1}[\alpha \cdot u'_k \beta \cdot v'_k] = \gamma \cdot w'$. (Again the dominoes may be $T$-marked or even $S$-marked rather than circled.) Thus we can distinguish elements of $S \ast T$ by their actions on either $(X^T)^* \ast \bar{S}^*$. It remains to show that $A$ defines an action of $S \ast T$ on $C^*$. For this, it suffices to show that the action of $Q^+_2$ is an action of $S$ and the action of $Q^+_2$ is an action of $T$. The action of states in $Q_2$ is relatively straightforward, so we begin by showing that $\delta$ defines an action of $T$ on $C^*$. For clarity, we explain the this action by a series of observations.

(i) States in $Q_2$ recurse only to other states in $Q_2$.

(ii) The gates are irrelevant to the action of $Q^+_2$, since the effect will always be to open all the half-open gates and leave all the other gates unchanged; and no gate causes a change of state (from states in $Q_2$). Similarly, unmarked and circled dominoes are irrelevant, as they are unaffected by the action and never cause a change of state.

(iii) For $t \in Q^+_2, a \in A$ and $b \in B$ we have $(t, [a \ b]^T) \delta = (t, [a \ b]^T)$. Moreover, $S$-marked dominoes and do not occur in strings in $C^* \ast Q_2$. Thus the action of $Q^+_2$ on $C^*$ is unchanged if each $S$-marked domino is replaced by the corresponding $T$-marked domino.

(iv) Combining (ii) and (iii), we see that it suffices to consider the action of $Q^+_2$ on $(X^T)^*$. But this is determined entirely by the second component and is equivalent to the action of $Q^+_2$ on $B^*$ in the automaton $A_2$, hence $\delta$ restricted to $Q^+_2$ defines an action of $T$ on $C^*$.

The action of states in $Q_1$ is a little less straightforward because they can recurse to the state $e \in Q_2$. Again we proceed by a series of observations. Let $\gamma \in C^*$.

(i) Open gates and circled dominoes in the initial string $\gamma$ are irrelevant to the action of $Q^+$, so we may assume that $\gamma$ does not contain any such symbols.

(ii) States in $Q_1$ recurse only to other states in $Q_1$, except on closed and half-open gates. Thus let $\gamma = \gamma_1 y_{\gamma_2}$, where $y \in \{\bar{S}, \hat{S}\}$ and $\gamma_1$ contains no gates. If $w_1, w_2$ are two words in $Q^+_1$ representing the same element of $S$,

$$\gamma \cdot w_1 = (\gamma_1 \cdot w_1)\hat{S}(\gamma_2 \cdot w_2).$$
Since $\phi$ is a homomorphism and we have already shown that the action of $Q_2^+$ is an action of $T$, we have $\gamma_2 \cdot w_1 = \gamma_2 \cdot w_2$ and so it suffices to show that $\gamma_1 \cdot w_1 = \gamma_1 \cdot w_2$.

(iii) Since the automaton stays in states in $Q_1$ when acting on $\gamma_2$ by $w_i \in Q_1^+$, the action of $w_i$ on $\gamma_2$ will be identical if all $T$-marked dominoes are replaced by circled dominoes, and all unmarked dominoes replaced by $S$-marked dominoes. But the circled dominoes are irrelevant to the action, and so we may suppose $\gamma_2 \in (X^S)^*$. But the action of $Q_1^+$ on $S$-marked dominoes is determined entirely by the first component and is equivalent to the action of $Q_1^+$ on $A^*$ in the automaton $A_1$ for $S$, hence $\delta$ restricted to $Q_1^+$ defines an action of $S$ on $C^*$.

Thus $A$ defines an action of $S \ast T$ on $C^*$, and we have already observed that this action is faithful. Hence $\Sigma(A) \cong S \ast T$ and so $S \ast T$ is an automaton semigroup. □

It is easy to find examples of pairs of semigroups satisfying the hypothesis of Theorem 1 but not of [6, Theorem 3.0.1]. For example let $S$ be any free semigroup and $T$ any finite semigroup.

Theorem 1 generalises by induction to free products of finitely many semigroups, in the following way.

Corollary 2. Let $S_1, \ldots, S_n$ be a sequence of automaton semigroups such that for $1 \leq i \leq n-1$ there is a homomorphism $\phi_i : S_i \to S_j$ for some $j > i$. Then the free product $S_1 \ast S_2 \ast \ldots \ast S_n$ is an automaton semigroup.

Proof. The base case $n = 2$ is Theorem 1. Suppose the statement is true for some $n \geq 2$. Then $S := S_2 \ast S_2 \ast \ldots \ast S_n$ is an automaton semigroup, and the homomorphism $\phi_j : S_j \to S_j$ for some $j > 1$ extends to a homomorphism $\phi : S_1 \to S$, since $S_j$ is a subsemigroup of $S$. Thus $S_1 \ast S = S_1 \ast S_2 \ast \ldots \ast S_n$ is an automaton semigroup by Theorem 1. Hence, by induction, the statement is true for all $n$. □

In particular, the existence of a single idempotent guarantees that a free product of arbitrarily many automaton semigroups is an automaton semigroup.

Corollary 3. Let $S$ be a free product of finitely many automaton semigroups, one of which contains an idempotent. Then $S$ is an automaton semigroup.

Proof. Write $S = S_1 \ast S_2 \ast \ldots \ast S_n$ with each $S_i$ an automaton semigroup. Since the free product operation is commutative, we may assume $S_n$ contains an idempotent $e$. Then mapping all elements of $S_i$ to $e$ for $i < n$ gives the homomorphisms $\phi_i$ in the hypothesis of Corollary 2. □

3. Future work

The existence of a homomorphism from one base semigroup to another appears fundamental to the construction idea used in all results about free products of automaton semigroups to date. If the hypothesis of Theorem 1 is not in fact
necessary, it is likely that either the new examples will be in some sense ‘artificial’ or that a novel construction idea will be required.

The fact that the finiteness problem for automaton semigroups is undecidable \[4\], and that free products of finite automaton semigroups are automaton semigroups, leads us to conjecture that the ‘automaton-semigroupness’ problem for free products of automaton semigroups is undecidable, unless it turns out to be trivial.

**Conjecture 4.** One of the following holds:

- Every free product of two (and hence of finitely many) automaton semigroups is itself an automaton semigroup; or
- It is undecidable, given two automata \( A \) and \( B \), whether the free product \( \Sigma(A) \star \Sigma(B) \) is an automaton semigroup.

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