An unexpected topological censor

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Morris-Thorne wormholes with a cosmological constant $\Lambda$ have been studied extensively, even allowing $\Lambda$ to be replaced by a space variable scalar. These wormholes cannot exist, however, if $\Lambda$ is both space and time dependent. Such a $\Lambda$ will therefore act as a topological censor.

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I. INTRODUCTION

Wormholes are handles or tunnels in the spacetime topology connecting two separate and distinct regions of spacetime. These regions may be part of our Universe or of different universes. The pioneer work of Morris and Thorne [1] has shown that macroscopic wormholes may be actual physical objects, provided that certain energy conditions are violated. Several wormhole studies have added the cosmological constant $\Lambda$ ([2], [3], [4]). For a detailed discussion with an extensive list of references, see Alcaniz [16]. For variable-\(\Lambda\) models having a "big bounce," Ref. [18] discusses the big bang, as well as the cosmic strings with $\Lambda = \Lambda(\tau)$ ([5], [6]).

When Einstein first introduced the cosmological constant into his field equations in 1917, he was still striving for consistency with Mach’s principle. From the standpoint of cosmology, however, $\Lambda$ served to create a kind of repulsive pressure to yield a stationary Universe. Eventually Zel’dovich identified $\Lambda$ with the vacuum energy density due to quantum fluctuations [5].

It has been proposed from time to time that the "constant" is actually a variable parameter. For example, in discussing a family of asymptotically flat globally regular solutions to the Einstein field equations, Dymnikova [6] notes that the source term corresponds to an $r$-dependent $\Lambda$. Assuming that $\Lambda$ does indeed have the form $\Lambda = \Lambda(r)$, Rahaman, et. al., [7] obtained a class of wormhole solutions, while Ray, et. al., [8] studied various models that are applied to the classical electron of the Lorentz type. Cosmic strings with $\Lambda = \Lambda(r)$ are discussed in Ref. [9]. In Ref. [10] the variable $\Lambda$ is derived from a higher spatial dimension and manifests itself as an energy-density for the vacuum.

Another widely discussed possibility is a space- and time-dependent $\Lambda$, i.e., $\Lambda = \Lambda(r,t)$, suggested by recent observations of high redshift Type Ia supernovae [11], [12], [13], [14], [15]. For a detailed discussion with an extensive list of references, see Alcaniz [16]. For various $\Lambda$-decay scenarios from the original high value during inflation to the present, see Ref. [17] and references therein. Ref. [18] discusses the big bang, as well as the "big bounce," referring to variable-$\Lambda$ models having a non-singular origin.

The purpose of this paper is to show that if $\Lambda$ is both space and time dependent, so that $\Lambda = \Lambda(r,t)$, then a wormhole of the Morris-Thorne type will have a curvature singularity at the center.

II. BACKGROUND

Using units in which $c = G = 1$, our starting point is the Einstein-de Sitter metric

\[
ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}} + r^2(d\theta^2 + \sin^2\theta
d\phi^2), \tag{1}
\]

which is the unique solution of the vacuum Einstein field equations for a spherically symmetric spacetime with a positive cosmological constant. The line element reduces to the Schwarzschild line element if $\Lambda = 0$. The wormhole metric in Ref. [1],

\[
ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta
d\phi^2), \tag{2}
\]

provides a motivation for the following metric, proposed by Delgaty and Mann [2]:

\[
ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{M(r)}{r} - \frac{\Lambda(r,t)r^2}{3}} + r^2(d\theta^2 + \sin^2\theta
d\phi^2). \tag{3}
\]

(4)

(In Ref. [2], $\Lambda$ is fixed, while the constant 1 is incorporated in the function $M$.) Here $\Phi(r)$ is called the redshift function. If $\Lambda = 0$, then $M(r) = b(r)$. So $M(r)$ will be called the shape function; thus $M(r_0) = r_0$. (Recall that in Eq. (2), the sphere of radius $r = r_0$ is the throat of the wormhole.) Qualitatively, $M(r)$ has the form shown in Fig. 1. Observe that $\Lambda$ is a positive function of both $r$ and $t$.

According to Ref. [19], since the wormhole described by the metric in Eq. (3) is dynamic, there are actually two throats on opposite sides of the center $r = r_1$. This center is determined implicitly (for any fixed $t$) from the equation

\[
1 - \frac{M(r)}{r} - \frac{\Lambda(r,t)r^2}{3} = 0. \tag{4}
\]
So for any fixed \( t \), a solution to Eq. (5) is a fixed point \( F(r_1) = r_1 \). (See Fig. 1.) Since \( M \), \( \Lambda \), and \( r \) are all positive, \( M(r_1) < r_1 \). So \( r_1 > r_0 \). Since the sphere \( r = r_1 \) is the center of the wormhole, \( r = r_0 \) is not in the manifold, while each throat is a sphere with time-dependent radius \( r_2 > r_1 \).

### III. THE FAILED SOLUTION

To study the presumptive wormhole solution, it is necessary to compute the components of the Riemann curvature and Einstein tensors using the following orthonormal basis:

\[
\begin{align*}
\theta^0 &= e^{\phi(r)} dt, \\
\theta^1 &= \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r,t)r^2}{3} \right)^{-1/2} dr, \\
\theta^2 &= r d\theta, \\
\theta^3 &= r \sin \theta d\phi.
\end{align*}
\]

Some of the components of the Einstein tensor are listed next:

\[
G_{\hat{r}\hat{r}} = \frac{M'(r)}{r^2} + \Lambda(r,t) + \frac{1}{3} \frac{\partial}{\partial r} \Lambda(r,t),
\]

\[
G_{\hat{r}\hat{t}} = \frac{2}{r} \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r,t)r^2}{3} \right) \Phi'(r)
\]

\[
- \frac{M(r)}{r^3} - \frac{\Lambda(r,t)}{3},
\]

\[
G_{\hat{t}\hat{r}} = \frac{r}{3} e^{-\phi(r)} \frac{\partial}{\partial t} \Lambda(r,t) \times \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r,t)r^2}{3} \right)^{-1/2}.
\]

From the Einstein field equations with cosmological constant,

\[
G_{\hat{a}\hat{b}} + \Lambda g_{\hat{a}\hat{b}} = 8\pi T_{\hat{a}\hat{b}},
\]

we obtain

\[
T_{\hat{a}\hat{b}} = \frac{1}{8\pi} (G_{\hat{a}\hat{b}} + \Lambda g_{\hat{a}\hat{b}}).
\]

So

\[
T_{\hat{t}\hat{t}} = \rho(r,t) = \frac{1}{8\pi} \left( \frac{M'(r)}{r^2} + \frac{1}{3} \frac{\partial}{\partial r} \Lambda(r,t) \right),
\]

\[
T_{\hat{r}\hat{r}} = p(r,t) = \frac{1}{8\pi} \left[ 2 \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r,t)r^2}{3} \right) \Phi'(r)
\right.
\]

\[
- \frac{M(r)}{r^3} + \frac{2}{3} \Lambda(r,t) \right],
\]

\[
T_{\hat{t}\hat{r}} = T_{\hat{r}\hat{t}} = - f(r,t) = \frac{1}{8\pi} \frac{r}{3} e^{-\phi(r)} \frac{\partial}{\partial t} \Lambda(r,t) \times
\]

\[
\left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r,t)r^2}{3} \right)^{-1/2},
\]

where \( f(r,t) \) is usually interpreted as the energy flux in the outward radial direction.

Now let us assume that at the throat \( (r = r_2) \) the usual flare-out conditions have been met and that for every \( t \) the weak energy condition (WEC) has been violated. (The WEC states that given the stress-energy tensor \( T_{\hat{a}\hat{b}} \), the inequality \( T_{\hat{a}\hat{b}} \mu^a \mu^b \geq 0 \) holds for all time-like vectors and, by continuity, all null vectors.) So for the radial outgoing null vector \((1,1,0,0)\) we therefore have

\[
T_{\hat{a}\hat{b}} \mu^a \mu^b = \rho + p \pm 2f < 0.
\]

In this manner all the conditions for the existence of a wormhole appear to have been met. However, the real problem does not depend on any violation of the WEC: in view of Eq. (14), we have for any given \( t \)

\[
1 - \frac{M(r_1)}{r_1} - \frac{\Lambda(r_1,r_1)r_1^2}{3} = 0
\]

at the center \( r = r_1 \). Hence \( f(r,t) \) cannot be a finite quantity as long as \( \partial \Lambda(r_1,t) / \partial t \neq 0 \). Similarly, the components \( G_{\hat{t}\hat{b}} \) and \( G_{\hat{b}\hat{t}} \), which are proportional to the lateral pressure \( p_l \), cannot be finite as long as \( \Lambda \) is time-independent radius \( r_2 > r_1 \).
dependent:

\[
G_{\delta\theta} = G_{\phi\phi} = -e^{-2\Phi(r)} \times \\
\left[ \frac{r^2}{6} \frac{\partial^2}{\partial t^2} \Lambda(r, t) \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right)^{-1} + \right.
\]

\[
+ \left. \frac{r^4}{12} \left( \frac{\partial}{\partial t} \Lambda(r, t) \right)^2 \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right)^{-2} \right] + \text{other terms.} \tag{15}
\]

Finally, it is shown in Ref. [1] that for a wormhole to be traversable by humanoid travelers, the radial tidal constraint must be met: \(|R_{t\tau}^{\tau}\)| \(\leq (10^8 \text{ m})^{-2}\), where \(R_{t\tau}^{\tau}\) is a component of the Riemann curvature tensor. This component is given by

\[
R_{t\tau}^{\tau} = -e^{-2\Phi(r)} \times \\
\left[ \frac{r^2}{6} \frac{\partial^2}{\partial t^2} \Lambda(r, t) \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right)^{-1} + \right.
\]

\[
+ \left. \frac{r^4}{12} \left( \frac{\partial}{\partial t} \Lambda(r, t) \right)^2 \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right)^{-2} \right] + \left. \frac{1}{2} \Phi'(r) \left( \frac{M'(r)}{r} - \frac{M(r)}{r^2} \right) \right)
\]

\[
+ 2 \frac{r^2}{3} \Lambda(r, t) + \frac{1}{3} \frac{r^2}{\partial r} \Lambda(r, t) \right). \tag{16}
\]

Because of Eq. (14), we see that, once again, the right-hand side of Eq. (16) cannot be finite at the center as long as \(\Lambda\) is time dependent. The same problem arises with the lateral tidal constraints. So even if the earlier problems did not occur, the wormhole would not be traversable.

## IV. A DIVERGENT SCALAR QUANTITY

The singularities encountered so far could conceivably be removed by a suitable coordinate transformation. To show that the spacetime is singular, we need a scalar quantity that becomes infinite. To this end we list the components of the Ricci tensor. First we define the function

\[
H(r) = \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right) \left( -\Phi''(r) - [\Phi'(r)]^2 \right) + \frac{1}{2} \Phi'(r) \left( \frac{M'(r)}{r} - \frac{M(r)}{r^2} \right)
\]

\[
+ \frac{2}{3} \Lambda(r, t) + \left. \frac{1}{3} \frac{r^2}{\partial r} \Lambda(r, t) \right). \nonumber
\]

Then

\[
R_{ii} = -H(r) + \frac{2}{r} \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right) \Phi'(r),
\]

\[
R_{\tau\tau} = H(r) + \frac{1}{r} \left( \frac{rM'(r) - M(r)}{r^2} \right) + \frac{2}{3} \Lambda(r, t) + \frac{1}{3} \frac{r^2}{\partial r} \Lambda(r, t),
\]

\[
R_{\phi\phi} = - \frac{1}{r} \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right) \Phi'(r) + \frac{1}{2r} \left( \frac{rM'(r) - M(r)}{r^2} \right) + \frac{2}{3} \Lambda(r, t) + \frac{1}{3} \frac{r^2}{\partial r} \Lambda(r, t),
\]

\[
R_{\phi\phi} = R_{t\tau}^{\tau} = - \frac{1}{r} \left( 1 - \frac{M(r)}{r} - \frac{\Lambda(r, t)r^2}{3} \right) \Phi'(r) + \frac{1}{2r} \left( \frac{rM'(r) - M(r)}{r^2} \right) + \frac{2}{3} \Lambda(r, t) + \frac{1}{3} \frac{r^2}{\partial r} \Lambda(r, t).
\]

Now consider the square of the curvature scalar

\[
R_{\alpha\beta}R^{\alpha\beta} = R_{ii}R_{\tau\tau} + 2R_{i\tau}R_{\tau i} + R_{ii}R_{\tau\tau} + R_{\phi\phi}R_{\phi\phi} + R_{\phi\phi}R_{\phi\phi} + R_{t\tau}^{\tau}R_{t\tau}^{\tau}.
\]

For any fixed \(t\) (that is, for any fixed time-slice), the term \(R_{ii}R_{\tau\tau}\) is divergent for some \(r = r_1\) whenever

\[
\frac{\partial}{\partial t} \Lambda(r, t) \neq 0.
\]

Being a scalar quantity, it diverges at the center in all coordinate systems.

## V. DISCUSSION

As we have seen, because of Eqs. (13), (15), and (16), the energy flux, lateral pressure, and curvature cannot be finite at the center of the wormhole as long as

\[
\frac{\partial}{\partial t} \Lambda(r, t) \neq 0.
\]

Since the scalar quantity \(R_{\alpha\beta}R^{\alpha\beta}\) also diverges, there is a curvature singularity at the center. It follows that for a wormhole of the Morris-Thorne type to exist, \(\Lambda\) must not be time dependent. In the language of the topological censorship principle [21,22], causal curves originating from and ending in a simply connected asymptotic region do not see any non-trivial topology and can therefore be deformed to a curve contained entirely within the asymptotic region. In the present situation, an ingoing radial null geodesic continues to move inward and so cannot pass through the wormhole and probe the topology. A time-dependent \(\Lambda\) will therefore act as a topological censor for wormholes of the Morris-Thorne type.
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