LIE GROUPS LOCALLY ISOMORPHIC TO GENERALIZED HEISENBERG GROUPS

HIROSHI TAMARU AND HISASHI YOSHIDA

ABSTRACT. We classify connected Lie groups which are locally isomorphic to generalized Heisenberg groups. For a given generalized Heisenberg group $N$, there is a one-to-one correspondence between the set of isomorphism classes of connected Lie groups which are locally isomorphic to $N$ and a union of certain quotients of noncompact Riemannian symmetric spaces.

1. Introduction

Generalized Heisenberg groups were introduced by Kaplan ([5]), and have been studied in many fields in mathematics. Especially, generalized Heisenberg groups, endowed with the left-invariant metric, provide beautiful examples in geometry. They provide many examples of symmetric-like Riemannian manifolds, that is, Riemannian manifolds which share some properties with Riemannian symmetric spaces (see [1] and the references). Compact quotients of some generalized Heisenberg groups provide isospectral, but nonisometric manifolds (see Gordon [3] and the references). Generalized Heisenberg groups also provide remarkable examples of solvmanifolds, by taking certain solvable extension. The constructed solvmanifolds are now called Damek-Ricci spaces, after Damek-Ricci ([2]) showed that they are harmonic spaces (see also [1]).

The aim of this paper is to classify connected Lie groups which are locally isomorphic to generalized Heisenberg groups. They provide interesting examples of nilmanifolds, that is, Riemannian manifolds on which nilpotent Lie groups act isometrically and transitively. They might be a good prototype of the study of non simply-connected and noncompact nilmanifolds. This is one of our motivation.

Other motivation comes from the interesting structure of the moduli space, the set of isomorphism classes of connected Lie groups locally isomorphic to a generalized Heisenberg group. If one thought about abelian Lie groups or classical Lie groups, the moduli space seems to be a finite set. But, in the case of a generalized Heisenberg group $N$, there is a continuous family of connected Lie groups which are locally isomorphic to given $N$ (except for the classical Heisenberg groups). Our main theorem states that the moduli space is bijective to

$$\prod_{r=0}^{m} (\text{SL}_r(\mathbb{Z})\backslash \text{SL}_r(\mathbb{R})/\text{SO}_r(\mathbb{R})).$$

2000 Mathematics Subject Classification. Primary 53C30; Secondary 22E25.

Key words and phrases. Generalized Heisenberg groups, automorphism groups, local isomorphisms of Lie groups.

The first author was supported in part by Grant-in-Aid for Young Scientists (B) 17740039, The Ministry of Education, Culture, Sports, Science and Technology, Japan.
where $m$ is the dimension of the center $Z(N)$. It seems to be surprising that the moduli spaces depend only on $\dim Z(N)$.

This paper is organized as follows. In Section 2 we study the structure of the automorphism group of a metric Lie algebra and of a two-step nilpotent metric Lie algebra. We recall the definition of a generalized Heisenberg group, and describe the action of the automorphism group on the center, in Section 3. In particular, this action is equivalent to the action of the conformal group $CO_m(\mathbb{R})$. Our main theorem is proved in Section 4. For the proof, we have to classify lattices in $\mathbb{R}^m$ up to $CO_m(\mathbb{R})$-conjugation. In case of $\dim Z(N) = 2$, the component $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R})$ is the quotient of the real hyperbolic plane $\mathbb{R}H^2$ by the modular group $SL_2(\mathbb{Z})$, which have been well studied. Using the structure theorem of $SL_2(\mathbb{Z}) \backslash \mathbb{R}H^2$, we give explicit descriptions of connected Lie groups which are locally isomorphic to $N$ in this case, in Section 5.

The authors would like to thank Professor Makoto Matsumoto, Professor Nobuo Tsuzuki, Dr. Akira Ishii, Dr. Nobuyoshi Takahashi, Dr. Masao Tsuzuki and Dr. Takuya Yamaguchi for their useful advices and kind encouragements. The authors are also grateful to Mr. Tadashi Kashiwa and Mr. Hironao Kato for useful discussions.

2. Preliminaries on automorphism groups

For the classification of connected Lie groups locally isomorphic to a given simply-connected Lie group $G$, we need to know the action of the automorphism group on the center of $G$. In this section we study the structure of the automorphism group of a two-step nilpotent metric Lie algebra.

We start with an arbitrary metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$, a Lie algebra with an inner product. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{g}$. Since the automorphism group $Aut(\mathfrak{g})$ preserves $\mathfrak{z}$, we first study the structure of

$$N_{GL(\mathfrak{g})}(\mathfrak{z}) := \{ \sigma \in GL(\mathfrak{g}) : \sigma(\mathfrak{z}) \subset \mathfrak{z} \}.$$ 

Let us define an additive group

$$Hom(\mathfrak{v}, \mathfrak{z}) := \{ \beta : \mathfrak{v} \rightarrow \mathfrak{z} : \text{linear} \}.$$

Each element $\beta \in Hom(\mathfrak{v}, \mathfrak{z})$ acts on $\mathfrak{g}$ by $T_\beta$ in the natural way, that is,

$$T_\beta(z) := z, \quad T_\beta(v) := \beta(v) + v, \quad \text{for } z \in \mathfrak{z} \text{ and } v \in \mathfrak{v}.$$ 

Lemma 2.1. $N_{GL(\mathfrak{g})}(\mathfrak{z}) = Hom(\mathfrak{v}, \mathfrak{z}) \times (GL(\mathfrak{z}) \times GL(\mathfrak{v}))$.

Proof. Let $(h, g) \in GL(\mathfrak{z}) \times GL(\mathfrak{v})$ and $\beta \in Hom(\mathfrak{v}, \mathfrak{z})$. It is easy to see that $(h, g)^{-1} \circ T_\beta \circ (h, g) = T_{h^{-1} \circ \beta \circ \circ g}$. Thus $GL(\mathfrak{z}) \times GL(\mathfrak{v})$ normalizes $Hom(\mathfrak{v}, \mathfrak{z})$ and the inclusion "⊂" is clear. To show the converse, let $T \in N_{GL(\mathfrak{g})}(\mathfrak{z})$. Since $T$ preserves $\mathfrak{z}$, there exist $h \in GL(\mathfrak{z})$, $g \in GL(\mathfrak{v})$ and $\beta \in Hom(\mathfrak{v}, \mathfrak{z})$ such that

$$T(z) = h(z), \quad T(v) = \beta(v) + g(v).$$

It concludes that $T = T_{\beta \circ h^{-1} \circ \circ g} \in Hom(\mathfrak{v}, \mathfrak{z}) \times (GL(\mathfrak{z}) \times GL(\mathfrak{v}))$. \hfill \Box

Let us now consider two-step nilpotent metric Lie algebras $(\mathfrak{n}, \langle , \rangle)$. In this case $Aut(\mathfrak{n})$ can be decomposed into two groups, according to Lemma 2.1. We need

$$Aut_n(\mathfrak{n}) := Aut(\mathfrak{n}) \cap (GL(\mathfrak{z}) \times GL(\mathfrak{v})).$$

Lemma 2.2. For a two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle , \rangle)$, we have
(1) $\text{Hom}(v, \mathfrak{z}) \subset \text{Aut}(n)$, and therefore $\text{Aut}(n) = \text{Hom}(v, \mathfrak{z}) \times \text{Aut}_v(n)$, and

(2) $\text{Aut}_v(n) = \{(h, g) \in \text{GL}(\mathfrak{z}) \times \text{GL}(v) : [gu, gv] = h[u, v] \ (\forall u, v \in v)\}$.

**Proof.** Since $n$ is of two-step nilpotent, one can immediately see that $[v, u] \subset \mathfrak{z}$. This leads easily that $\text{Hom}(v, \mathfrak{z}) \subset \text{Aut}(n)$. Lemma 2.1 states that $\text{Aut}(n) \subset \text{Hom}(v, \mathfrak{z}) \times (\text{GL}(\mathfrak{z}) \times \text{GL}(v))$. We can conclude (1) by the following group-theoretic property: Let $K \subset H_1 \times H_2$ and $H_3 \subset K$, then $K = H_1 \times (H_2 \cap K)$ holds. The claim (2) is an easy consequence of the assumption that $n$ is of two-step nilpotent. \hfill \Box

We note that Lemma 2.2 is obtained by Saal ([7, Proposition 2.3]) for generalized Heisenberg algebras.

### 3. Automorphism groups of $H$-type groups

In this section, we recall the definition of a generalized Heisenberg algebra $(n, \langle , \rangle)$ and describe the action of the automorphism group $\text{Aut}(n)$ on the center of $n$.

**Definition 3.1.** A two-step nilpotent metric Lie algebra $(n, \langle , \rangle)$ is called a generalized Heisenberg algebra or an $H$-type algebra if

$$J_z^2 = -\langle z, z \rangle \text{id}_n \ (\forall z \in \mathfrak{z}),$$

where the operator $J : \mathfrak{z} \rightarrow \text{End}(v)$ is defined by

$$\langle J_z u, v \rangle = \langle z, [v, u] \rangle \quad \text{for } z \in \mathfrak{z}, u, v \in v.$$

The connected and simply-connected Lie group $N$ with Lie algebra $n$, endowed with the induced left-invariant metric, is called a generalized Heisenberg group or an $H$-type group.

We refer [5] and [1] for $H$-type groups and algebras.

We would like to know the action of the automorphism group $\text{Aut}(N)$ on the center $Z(N)$ of an $H$-type group $N$. Since $N$ is simply-connected and the exponential map $\exp : n \rightarrow N$ is a diffeomorphism, it is sufficient to investigate the action of the automorphism group $\text{Aut}(n)$ on the center $\mathfrak{z}$ of $n$. Note that the following proposition can be obtained as a corollary of the description of the automorphism group $\text{Aut}(n)$ by Saal ([7]). But, we will give a proof here, since we determine the action of $\text{Aut}(n)$ on $\mathfrak{z}$ directly and hence the arguments become slightly simpler. Denote the conformal group by

$$\text{CO}(\mathfrak{z}) := \{dg \in \text{GL}(\mathfrak{z}) \mid d \in \mathbb{R}^\times, \ g \in \text{O}(\mathfrak{z})\}.$$

**Proposition 3.2.** For an $H$-type algebra $(n, \langle , \rangle)$, the action of $\text{Aut}(n)$ on the center $\mathfrak{z}$ is equivalent to the action of the conformal group $\text{CO}(\mathfrak{z})$ on $\mathfrak{z}$.

**Proof.** First of all, Lemma 2.2 states that $\text{Aut}(n) = \text{Hom}(v, \mathfrak{z}) \times \text{Aut}_v(n)$. By definition, $\text{Hom}(v, \mathfrak{z})$ acts trivially on $\mathfrak{z}$, therefore the actions of $\text{Aut}(n)$ and $\text{Aut}_v(n)$ coincide on $\mathfrak{z}$. Hence, the group we would like to know is nothing but

$$D := \{h \in \text{GL}(\mathfrak{z}) : \exists g \in \text{GL}(v) : (h, g) \in \text{Aut}_v(n)\}.$$

For the proof of $D = \text{CO}(\mathfrak{z})$, we need the following groups:

- Let $\text{Cliff}(n)$ denote the subgroup of $\text{Aut}(n)$ generated by $\{(-|z|^2 \rho_z, J_z) : z \in \mathfrak{z} - \{0\}\}$, where $\rho_z$ is the reflection in $\mathfrak{z}$ with respect to $(\mathbb{R}z)^\perp$.
- Let $\text{Pin}(n)$ be the subgroup of $\text{Cliff}(n)$ generated by $\{(-\rho_z, J_z) : |z| = 1\}$. 

We refer [5] and [1] for $H$-type groups and algebras.

We would like to know the action of the automorphism group $\text{Aut}(N)$ on the center $Z(N)$ of an $H$-type group $N$. Since $N$ is simply-connected and the exponential map $\exp : n \rightarrow N$ is a diffeomorphism, it is sufficient to investigate the action of the automorphism group $\text{Aut}(n)$ on the center $\mathfrak{z}$ of $n$. Note that the following proposition can be obtained as a corollary of the description of the automorphism group $\text{Aut}(n)$ by Saal ([7]). But, we will give a proof here, since we determine the action of $\text{Aut}(n)$ on $\mathfrak{z}$ directly and hence the arguments become slightly simpler. Denote the conformal group by

$$\text{CO}(\mathfrak{z}) := \{dg \in \text{GL}(\mathfrak{z}) \mid d \in \mathbb{R}^\times, \ g \in \text{O}(\mathfrak{z})\}.$$
Note that \((-|z|^2 \rho_z, J_z) \in \text{GL}(\mathfrak{z}) \times \text{GL}(\mathfrak{v})\) defines an automorphism of \(\mathfrak{n}\). Let \(l := \dim \mathfrak{v}, m := \dim \mathfrak{z}\), and \(\{z_1, \ldots, z_m\}\) be an orthonormal basis of \(\mathfrak{z}\).

Claim 1: the action of \(\text{Cliff}(\mathfrak{n})\) on \(\mathfrak{z}\) coincides with that of \(\text{CO}(\mathfrak{z})\). In case of \(m\) is even, the action of \(\text{Pin}(\mathfrak{n})\) on \(\mathfrak{z}\) coincides with that of \(\text{O}(\mathfrak{z})\), since \(\det(-\rho_z) = -1\). Therefore \(\text{Cliff}(\mathfrak{n})\) acts on \(\mathfrak{z}\) as \(\text{CO}(\mathfrak{z})\). In case of \(m\) is odd, the action of \(\text{Pin}(\mathfrak{n})\) on \(\mathfrak{z}\) coincides with that of \(\text{SO}(\mathfrak{z})\), since \(\det(-\rho_z) = 1\). Thus \(\text{Cliff}(\mathfrak{n})\) acts on \(\mathfrak{z}\) as \(\mathbb{R}^\times \cdot \text{SO}(\mathfrak{z})\), but \(m\) is odd, which implies \(\mathbb{R}^\times \cdot \text{SO}(\mathfrak{z}) = \mathbb{R}^\times \cdot \text{O}(\mathfrak{z}) = \text{CO}(\mathfrak{z})\).

Claim 2: \(D \supset \text{CO}(\mathfrak{z})\). This is a direct consequence of Claim 1.

Claim 3: \(D \subset \text{CO}(\mathfrak{z})\). Let us take \(h \in D\). By definition, there exists \(g \in \text{GL}(\mathfrak{v})\) such that \((h, g) \in \text{Aut}_\mathbb{v}(\mathfrak{n})\). Elementary linear algebra leads to

\[
\exists k_1, k_2 \in \text{O}(\mathfrak{z}) \text{ such that } k_1 k_2 = \text{diag}(d_1, \ldots, d_m) \text{ and } d_i > 0.
\]

Put \(h_0 := \text{diag}(d_1, \ldots, d_m)\), which is the diagonal matrices (note that we consider matrices representation with respect to the basis \(\{z_1, \ldots, z_m\}\)). Since Claim 2 implies that \(\text{Cliff}(\mathfrak{n})\) has a subgroup which acts on \(\mathfrak{z}\) as \(\text{O}(\mathfrak{z})\), there exists \(g_1, g_2 \in \text{GL}(\mathfrak{v})\) such that \((k_1, g_1), (k_2, g_2) \in \text{Cliff}(\mathfrak{n}) \subset \text{Aut}_\mathbb{v}(\mathfrak{n})\). Let \(g_0 := g_1 g_2\). One has

\[
(h_0, g_0) = (k_1, g_1)(h, g)(k_2, g_2) \in \text{Aut}_\mathbb{v}(\mathfrak{n}).
\]

The condition for \((h_0, g_0)\) to be an automorphism is

\[
[g_0 u, g_0 v] = h_0[u, v] \quad (\forall u, v \in \mathfrak{v}).
\]

By taking the inner product with \(z_i\) \((i = 1, \ldots, m)\), we have

\[
(z_i, [g_0 u, g_0 v]) = (z_i, h_0[u, v]) \quad (\forall u, v \in \mathfrak{v}).
\]

This leads that \(g_0 J_z z_0 = d_i J_z z_i\). By taking the determinants of each sides, one has \((\det g_0)^2 = d_i^2\). This means that \(d_i\) is independent of \(i\) (recall that \(d_i > 0\)). Therefore \(h_0 = d \cdot \text{id}_\mathfrak{z}\), where \(d := d_1 = d_2 = \cdots = d_m\). We conclude that \(h = dk_1^{-1} k_2^{1} \in \text{CO}(\mathfrak{z})\), which completes the proof.

\[
\square
\]

4. Main Theorem

In this section, we classify connected Lie groups which are locally isomorphic to \(H\)-type groups \(N\). We refer [4], [8] for the theory of Lie groups and homogeneous spaces.

Let \(G\) be a connected and simply-connected Lie group, and denote by

\[
LI(G) := \{G'\}; \ G' \text{ is a connected Lie group and locally isomorphic to } G)/\text{isom}.
\]

An arbitrary connected Lie group can be represented by \(G/\Gamma\), where \(G\) is a connected and simply-connected Lie group and \(\Gamma\) is a discrete subgroup contained in the center \(Z(G)\) of \(G\). Furthermore, \(G/\Gamma_1\) and \(G/\Gamma_2\) are isomorphic to each other if and only if \(\Gamma_1\) and \(\Gamma_2\) are conjugate by \(\text{Aut}(\Gamma)\). Therefore, one has

\[
LI(G) \cong \{\Gamma \subset Z(G); \ \Gamma \text{ is a discrete subgroup}\}/\text{Aut}(\Gamma).
\]

We now study \(H\)-type groups \(N\). Denote the Lie algebra of \(N\) by \(\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}\), where \(\mathfrak{z}\) is the Lie algebra of \(Z(N)\). Via a Lie group isomorphism \(\exp : \mathfrak{z} \rightarrow Z(N)\), discrete subgroups in \(Z(N)\) corresponds to lattices in \(\mathfrak{z} \cong \mathbb{R}^m\), where \(m = \dim \mathfrak{z}\). Note that \(\text{Aut}(\mathfrak{z}) \cong \text{Aut}(\mathfrak{n})\), since \(N\) is simply-connected. Therefore we have that

\[
LI(N) \cong \{\Lambda \subset \mathfrak{z}; \ \Lambda \text{ is a lattice}\}/\text{CO}(\mathfrak{z}).
\]

Hence, Proposition \([3, 2]\) leads that

\[
LI(N) \cong \{\Lambda \subset \mathfrak{z}; \ \Lambda \text{ is a lattice}\}/\text{CO}(\mathfrak{z}).
\]
Theorem 4.1. Let $N$ be an $H$-type group with $m$-dimensional center. The moduli space $LI(N)$ of isomorphism classes of connected Lie groups which are locally isomorphic to $N$ satisfies

$$LI(N) \cong \prod_{r=0}^{m} (SL_r(\mathbb{Z}) \backslash SL_r(\mathbb{R}) / SO_r(\mathbb{R})),$$

where we define $SL_0(\mathbb{R}) \backslash SL_0(\mathbb{Z}) / SO_0(\mathbb{R}) = \{0\}$.

Proof. We fix an orthonormal basis $\{z_1, \ldots, z_m\}$ of $\mathfrak{z}$. Then one can identify $\mathfrak{z} = \mathbb{R}^m$, and thus

$$LI(N) = \{ \Lambda \subset \mathbb{R}^m; \Lambda \text{ is a lattice}\} / CO_m(\mathbb{R}).$$

Here we define

$$LI_r(N) := \{ \Lambda \subset \mathbb{R}^m; \Lambda \text{ is a lattice of rank } r\} / CO_m(\mathbb{R}).$$

On the other hand, the natural inclusion map $SL_r(\mathbb{R}) \to GL_r(\mathbb{R})$ induces

$$SL_r(\mathbb{Z}) \backslash SL_r(\mathbb{R}) / SO_r(\mathbb{R}) \cong GL_r(\mathbb{Z}) \backslash GL_r(\mathbb{R}) / CO_r(\mathbb{R}),$$

since $CO_r(\mathbb{R}) \cap SL_r(\mathbb{R}) = \{0\}$ and $GL_r(\mathbb{Z}) \cap SL_r(\mathbb{R}) = SL_r(\mathbb{Z})$. We will show the theorem by constructing a bijective map from $GL_r(\mathbb{Z}) \backslash GL_r(\mathbb{R}) / CO_r(\mathbb{R})$ to $LI_r(N)$. First of all, we define

$$\varphi : GL_r(\mathbb{R}) \to \{ \Lambda \subset \mathbb{R}^m; \text{rank } \Lambda = r\} \colon \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \mapsto \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r,$$

where $u_i \in M_{1,r}(\mathbb{R})$, $\tilde{u}_i := (u_i, 0, \ldots, 0) \in M_{1,m}(\mathbb{R})$. Denote by $\tilde{\varphi}$ the induced map,

$$\tilde{\varphi} : GL_r(\mathbb{Z}) \backslash GL_r(\mathbb{R}) / CO_r(\mathbb{R}) \to LI_r(N) : u \mapsto [\varphi(u)],$$

where $[\varphi(u)]$ is the $CO_m(\mathbb{R})$-conjugate class of the lattice $\varphi(u)$.

Claim 1: $\tilde{\varphi}$ is well-defined. Let $u \in GL_r(\mathbb{R})$, $C \in GL_r(\mathbb{Z})$ and $g \in CO_r(\mathbb{R})$. One has $[\varphi(Cug)] = [\varphi(Cu)]$, since $CO_r(\mathbb{R})$ is a subgroup of $CO_m(\mathbb{R})$ naturally. It can also be easily checked that $\varphi(u) = \varphi(v)$ (that is, they are the same lattice) if and only if $u = Cv$ for some $C \in GL_r(\mathbb{Z})$. Hence one has $\varphi(Cu) = \varphi(u)$. We conclude that $[\varphi(Cug)] = [\varphi(u)]$.

Claim 2: $\tilde{\varphi}$ is surjective. Let $\Lambda = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r$ be a lattice so that $[\Lambda] \in LI_r(N)$. Then there exists $g \in CO_m(\mathbb{R})$ such that $u_ig = (u_i, 0, \ldots, 0)$ (for every $i = 1, \ldots, r$).

Since rank $\Lambda = r$, one has

$$u := \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \in GL_r(\mathbb{R})$$

and $\varphi(u)g^{-1} = \Lambda$. We conclude that $[\varphi(u)] = [\Lambda]$.

Claim 3: $\tilde{\varphi}$ is injective. Let $u, v \in GL_r(\mathbb{R})$, and assume that $[\varphi(u)] = [\varphi(v)]$. Then there exists $g \in CO_m(\mathbb{R})$ such that $\varphi(u) = \varphi(v)g$. Since all the basis of these lattices are of the form $(u_i, 0, \ldots, 0)$, one can see that there exists $g' \in CO_r(\mathbb{R})$ such that $\varphi(u) = \varphi(v)g' = \varphi(vg')$. They are the same lattice, which implies that $u = Cvg'$ for some $C \in GL_r(\mathbb{Z})$. We conclude that $u$ and $v$ are in the same equivalent class. □
5. The case of two dimensional center

In this section we explicitly describe the classification of Lie groups which are locally isomorphic to $H$-type groups with two dimensional center.

It is known that an $H$-type group $N$ with two dimensional center is the complex Heisenberg group, that is,

$$N = \left\{ \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_k & \gamma \\ & 1 & \ddots & \vdots & \beta_k \\ & & \ddots & 1 & \vdots \\ & & & 1 & \beta_1 \\ & & & & 1 \end{bmatrix} : \alpha_i, \beta_i, \gamma \in \mathbb{C} \right\}. $$

We denote the element in the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$, for $\gamma \in \mathbb{C}$, by

$$u(\gamma) := \begin{bmatrix} 0 & \cdots & 0 & \gamma \\ & \ddots & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix}. $$

Obviously $\{u(1), u(i)\}$ is an orthonormal basis of $\mathfrak{z}$. Define

$$D_1 := \{ \rho \in \mathbb{C} ; \ \text{Im} \ \rho > 0, \ |\rho| \leq 1, \ |\rho - 1| \geq 1, \ |\rho + 1| \geq 1 \}. $$

**Theorem 5.1.** Let $N$ be a complex Heisenberg group. A connected (real) Lie group which is locally isomorphic to $N$ is isomorphic to either

1. $N$,
2. $N/\mathbb{Z} \exp u(1)$, or
3. $N/\mathbb{Z} \exp u(1) \times \mathbb{Z} \exp u(\rho)$ with $\rho \in D_1$.

For $\rho, \rho' \in D_1$, two Lie groups $N/\mathbb{Z} \exp u(1) \times \mathbb{Z} \exp u(\rho)$ and $N/\mathbb{Z} \exp u(\rho')$ are isomorphic if and only if

(i) $\{\rho, \rho'\} = \{ie^{i\theta}, ie^{-i\theta}\}$ for $0 \leq \theta \leq \pi/6$, or
(ii) $\{\rho, \rho'\} = \{e^{i\theta} - 1, -e^{-i\theta} + 1\}$ for $0 < \theta \leq \pi/3$.

**Proof.** Theorem 5.1 states that the moduli space of a connected Lie group which is locally isomorphic to $N$ is bijective to

$$\prod_{r=0}^{2} (\text{SL}_r(\mathbb{Z})\backslash \text{SL}_r(\mathbb{R})/\text{SO}_r(\mathbb{R})). $$

Note that $\text{SL}_0(\mathbb{Z})\backslash \text{SL}_0(\mathbb{R})/\text{SO}_0(\mathbb{R}) = \{\text{pt}\} = \{N\}$, which corresponds to simply-connected one. Furthermore, $\text{SL}_1(\mathbb{Z})\backslash \text{SL}_1(\mathbb{R})/\text{SO}_1(\mathbb{R}) = \{\text{pt}\}$, which means that $N/\mathbb{Z} u(\rho)$ are isomorphic to each other for all $\rho \in \mathbb{C}$. Therefore, in this case, it is isomorphic to $N/\mathbb{Z} u(1)$.

Now we have to study $\text{SL}_2(\mathbb{Z})\backslash \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$. It is known that $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}) = \mathbb{R}^+ \cdot \text{SL}_2(\mathbb{R})/\text{CO}_2^+(\mathbb{R})$ is bijective to the real hyperbolic plane (or the upper half complex plane)

$$\mathbb{R}H^2 := \{z \in \mathbb{C} ; \ \text{Im} \ z > 0 \}. $$

Note that the bijective correspondence is induced from the following linear fractional transformation:

$$p : \mathbb{R}^+ \cdot \text{SL}_2(\mathbb{R}) \rightarrow \mathbb{R}H^2 : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ai + b)/(ci + d). $$
Furthermore, note that the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{R}^+ \cdot \text{SL}_2(\mathbb{R})/\text{CO}_2^+(\mathbb{R})$ induced from the left action on $\mathbb{R}^+ \cdot \text{SL}_2(\mathbb{R})$ is equivalent to the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{R}H^2$ via the linear fractional transformation.

It is well known (see [6], [9], for example) that representative of the orbit under the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{R}H^2$ can be taken as

$$D_2 := \{ z \in \mathbb{R}H^2; |z| \geq 1, -1/2 \leq \text{Re } z \leq 1/2 \}.$$

Note that $z, z' \in D_2$ are in the same $\text{SL}_2(\mathbb{Z})$-orbit if and only if

(i)' $\{z, z'\} = \{ie^{i\theta}, ie^{-i\theta}\}$ for $0 \leq \theta \leq \pi/6$, or

(ii)' $\{z, z'\} = \{1/2 + iy, -1/2 + iy\}$ for $y \geq \sqrt{3}/2$.

We translate these results into matrices, via the linear fractional transformation $p$. Direct calculations show that

$$p \left( \begin{array}{cc} 1 & 0 \\ r \cos \theta & r \sin \theta \end{array} \right) = (1/r)e^{i\theta}.$$ 

Therefore the $\text{SL}_2(\mathbb{Z})$-orbit in $\mathbb{R}^+ \cdot \text{SL}_2(\mathbb{R})/\text{CO}_2^+(\mathbb{R})$ have to meet $p(D_3)$, where $D_3 \subset \mathbb{R}^+ \cdot \text{SL}_2(\mathbb{R})$ is defined by

$$D_3 := \left\{ \left[ \begin{array}{cc} 1 & 0 \\ r \cos \theta & r \sin \theta \end{array} \right]; 0 < \theta < \pi, 0 < r \leq 1, -1/2 \leq (1/r) \cos \theta \leq 1/2 \right\}.$$ 

The above matrix represents the lattice $\mathbb{Z}u(1)+\mathbb{Z}(re^{i\theta})$ in $\mathfrak{g}$, and hence, represents the discrete subgroup $\mathbb{Z}\exp u(1) \times \mathbb{Z}\exp u(re^{i\theta})$ in $\mathbb{Z}(N)$. The domain $D_1$ can be obtained by the expression of $D_3$, and the equivalence conditions (i) and (ii) come from the conditions (i)' and (ii)', respectively. □

References

1. J. Berndt, F. Tricerri, L. Vanhecke, Generalized Heisenberg groups and Damek-Ricci harmonic spaces. Lecture Notes in Mathematics, 1598. Springer-Verlag, Berlin, 1995.
2. E. Damek, F. Ricci, Harmonic analysis on solvable extensions of $H$-type groups. J. Geom. Anal. 2 (1992), no. 3, 213–248.
3. C. Gordon, Isospectral closed Riemannian manifolds which are not locally isometric. J. Differential Geom. 37 (1993), no. 3, 639–649.
4. S. Helgason, Differential geometry, Lie groups, and symmetric spaces. Graduate Studies in Mathematics, 34. Amer. Math. Soc., Providence, RI, 2001.
5. A. Kaplan, Riemannian nilmanifolds attached to Clifford modules. Geom. Dedicata 11 (1981), no. 2, 127–136.
6. D. Mumford, Tata lectures on theta. I. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman. Progress in Mathematics, 28. Birkhauser Boston, Inc., Boston, MA, 1983.
7. L. Saal, The automorphism group of a Lie algebra of Heisenberg type. Rend. Sem. Mat. Univ. Politec. Torino 54 (1996), no. 2, 101–113.
8. M. Sugiura, Theory of Lie Groups. (in Japanese) Kyoritsu Shuppan Co., Ltd, Tokyo, 2000.
9. H. Umemura, Theory of Elliptic Functions: Analysis of Elliptic Curves. (in Japanese) University of Tokyo Press, 2000.

E-mail address: tamaru@math.sci.hiroshima-u.ac.jp

E-mail address: r060141m@mbox.nagoya-u.ac.jp