Breaking CPT by mixed non-commutativity

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The mixed component of the non-commutative parameter $\theta_{\mu M}$, where $\mu = 0, 1, 2, 3$ and $M$ is an extra dimensional index may violate four-dimensional CPT invariance. We calculate one and two-loop induced couplings of $\theta_{\mu 5}$ with the four-dimensional axial vector current and with the CPT odd dim=6 operators starting from five-dimensional Yukawa and $\mathcal{U}(1)$ theories. The resulting bounds from clock comparison experiments place a stringent constraint on $\theta_{\mu 5}$, $|\theta_{\mu 5}|^{-3/2} \gtrsim 5 \times 10^{13}$ GeV. Orbifold projection and/or localization of fermions on a 3-brane lead to CPT-conserving physics, in which case the constraints on $\theta_{\mu 5}$ are softened.

I. INTRODUCTION

Non-commutative field theories and their realizations in string theory have been a subject of intensive theoretical research over the past few years (See, e.g. \cite{1,2}). Much of this excitement has gone totally unnoticed by particle phenomenology for the following simple reason. The presence of the antisymmetric tensor $\theta_{\mu \nu}$ as a constant background violates Lorentz invariance, a possibility excluded to an impressive accuracy by various low-energy precision measurements (See, e.g. \cite{3}). In our previous paper, Ref. \cite{4}, we have shown that in the low-energy effective interaction linearized in the non-commutative parameter, $\theta_{\mu \nu}$ couples to the nucleon spin, $\bar{N} \sigma_{\mu \nu} N$ with the strength proportional to the cube of the characteristic hadronic scale, $\Lambda_{\text{hadr}} \sim 1$ GeV. This analysis has been done at the tree level in order to avoid potential problems with the issues of renormalizability of the non-commutative theories, the necessity to introduce a cutoff, etc. This coupling generates an additional, magnetic-field-independent, contribution to the nucleon Larmor frequency. Therefore, this interaction has the signature of a constant magnetic field of a fixed direction and can be searched through a precise monitoring of sidereal variation of the magnetic field. Ref. \cite{4} places the limit on the possible size of such an interaction at the level of $\nu=100$ nHz. Comparing it with the result of the theoretical calculation in \cite{5}, one arrives at an incredibly strong constraint, $\Lambda_{\text{NC}} = 1/\sqrt{\theta} \gtrsim 5 \times 10^{14}$ GeV. If non-commutativity is realized somehow only in the leptonic sector, $\theta_{\mu \nu} \bar{N} \sigma_{\mu \nu} N$ interaction is still generated, although with $(\alpha/\pi)^2$ suppression compared to the non-commutative QCD case. This relaxes the limit down to $10^{12}$ GeV level. Later it was argued in Refs. \cite{6,7} that $\sigma_{\mu \nu} \theta_{\mu \nu}$ operator can be generated at the loop level with quadratically divergent integral which may bring even tighter bounds which scale with the cutoff.

Subsequent analyses have addressed the possibility to observe $\theta_{\mu \nu}$ in future collider experiments \cite{8}, and in the neutral $K$ and $B$ meson systems \cite{9}. If the former cannot do much better than $\Lambda_{\text{NC}} \sim 1$ TeV, the neutral kaons could in principle be quite sensitive to $\Lambda_{\text{NC}}$. However, even with the most favorable assumptions that the non-commutativity generates somehow an effective $\Delta F = 2$ transition $\theta(d\bar{Q}s)(d\bar{Q}s)$ of unsuppressed strength, one can get only to the level $\Lambda_{\text{NC}} \sim 10^5$ GeV. This is in the range already excluded by the clock comparison experiments! Generically, any attempt to construct a fully non-commutative Standard Model (e.g. Ref. \cite{10}) will have to comply with the bound obtained in \cite{4} which would make $\theta_{\mu \nu}$ totally unobservable for conventional particle physics experiments. (A possible exception could be measurements of the refraction index over cosmological distances \cite{11} or the search for the imprints of non-commutative inflation \cite{12}, when the characteristic energy scales could be quite high and compensate for the extreme smallness of $\Lambda_{\text{NC}}^{-2}$.) Other relevant works on the observational consequences of non-commutativity include \cite{13}.

In this paper we study phenomenological consequences of mixed non-commutativity $\theta_{\mu M}$, where $\mu$ is a normal 4-d index and $M$ is along extra dimensions. The presence of such a component is perceived by a four-dimensional observer as a constant 4-vector, which obviously breaks Lorentz invariance. The purpose of this work is to show that in certain classes of models this background may also break 4-dimensional CPT invariance. Indeed, the transformation of $\theta_{\mu M}$ under the CPT reflection is similar to the behavior of charge times the $\mathcal{U}(1)$ field strength, $eF_{\mu M}$, for which we know that $\text{CPT}(eF_{\mu M}) = \text{CPT}(e\theta/\partial x^M A_\mu) = -1$. Note that parity is defined here in a 4-dimensional sense and thus $P(\theta/\partial x^M) = 1$.

Unlike the previous case with the breaking of Lorentz invariance by $\theta_{\mu \nu}$ for which no plausible low-energy physics motivation exists, one can think of baryogenesis-motivated reasons to study the CPT non-invariant interactions. Namely, we refer to an interesting idea \cite{14} that the breaking of CPT effectively comprises two out of three Sakharov’s conditions needed for baryogenesis, requiring only the breaking of the baryon number
as an extra ingredient. If baryogenesis happens at the early cosmological epoch with characteristic energy scales $E^4 \sim 1/(\theta_{\mu\phi})^2$, a dramatic power-like suppression by $\Lambda_{NC}$ may be lifted.

We study the possibility of CPT violation by $\theta_{\mu\phi}$ and find that it is possible in the 5-dimensional model with Dirac fermions and scalar particles and/or gauge bosons living in all 5 dimensions. After the compactification of the fifth dimension, the zero level Kaluza-Klein mode of the fermion acquires the coupling with $\theta_{\mu\phi}$ through the axial-vector current or the CPT non-invariant $\text{dim}=6$ operators. Specifying this to the case of QED, we effectively get $\theta_{\mu\phi} \partial_\mu F_{\alpha\beta} N \phi^\alpha \gamma_5 N$ which results in the strong bounds on $\theta_{\mu\phi}$. On the other hand, we show that in the models with intrinsically 4-d fermions, e.g. localized on a 3+1-dimensional domain wall, the CPT may be conserved and even only powers of $\theta_{\mu\phi}$ may appear in the low-energy effective lagrangian.

II. 5-D MODEL WITH LOW-ENERGY 4-D CPT VIOLATION

We begin by reminding that the nonvanishing commutation relations

$$[\hat{x}^a, \hat{\phi}^b] = i\theta^{ab} \quad [\hat{\phi}^a, \hat{x}^b] = 0 \quad (1)$$

in the coordinate space $(a, b, c = 0, 1, 2, 3, M, ..N)$ lead to the modification of the interaction terms in the field theory through the Moyal product, given by

$$\phi_1 \ast \phi_2(x) = e^{i\theta^{\mu\nu} \partial_\mu \partial_\nu} \phi_1(x + \xi)\phi_2(x + \zeta)|_{\xi = \zeta = 0}. \quad (2)$$

Working in the framework of an effective field theory, and assuming smallness of $E^2\theta$, where $E$ is the characteristic energy scale in the problem, we linearize the $\ast$-product to get a combination

$$\phi_1 \ast \phi_2(x) = \phi_1 \phi_2(x) + i\frac{1}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \phi_1 \frac{\partial}{\partial x^\nu} \phi_2 + ... \quad (3)$$

Specializing the second term in this expansion to the case of $\theta_{\mu\phi}$, we immediately observe the presence of $\partial/\partial y$ derivative along the fifth dimension. We take this dimension to be compact with the radius smaller than $10^{-17}$ cm so that the Kaluza Klein modes of fermions, gauge bosons and scalars are heavy enough to avoid particle physics constraints. Our goal is to integrate out these heavy states and obtain the low-energy effective action for the zero-level KK modes. Lowest energy modes do not have any $y$-dependence. Therefore, $\theta_{\mu\phi}$ may only appear in the loop-induced amplitudes with excited KK modes inside. Moreover, since the term we are interested in contains $i\theta/\partial y = p_5$ in one of the vertices, we have to find an amplitude that would contain an odd function of $p_5$ in the propagators because otherwise the loop amplitude will vanish upon summation of positive and negative $p_5$ modes. An odd function of $p_5$ may come only from the fermion propagators. Therefore, in order to get a nonvanishing result linear in $\theta_{\mu\phi}$ we have to allow fermions to propagate in a full five-dimensional space.

As a warm-up exercise we consider a two-loop-generated $\bar{\psi}N\phi^M \psi$ amplitude in the non-commutative scalar-fermion theory with Yukawa interaction $\lambda \bar{\psi}\psi \phi + i\theta_{MN} \bar{\psi} \partial_N \psi \partial_M \phi$. This calculation is similar to the 4-dimensional calculation performed in Ref. [6]. The $\mu 5$ component of $\theta$ couples to $\bar{\psi}N\gamma_5 \psi = -\bar{\psi}N\gamma_5 \psi$, which is a CPT-odd operator in four dimensions. Taking into account the contribution of the first excited KK level, we arrive at the following two-loop induced amplitude,

$$\theta_{\mu5} (\bar{\psi} \gamma_\mu \gamma_5 \psi) \times \frac{\lambda^4 m}{32\pi^4} M^2 \ln(\Lambda_{UV}^2/M^2). \quad (4)$$

In this formula $m$ and $M$ are the masses of the zeroth and first KK levels and $\Lambda_{UV}$ is an ultraviolet cutoff. The summation over the Kaluza Klein tower diverges as $N^4$. Thus we would have to cut it at some value $N_{\text{max}}$ which exactly corresponds to an initial $\Lambda_{UV}^2$ divergence of the 5-dimensional two-loop integral.

We would like to comment at this point that in a supersymmetric four-dimensional theory there can be a natural scale for the ultraviolet cutoff related to the soft breaking masses, $\Lambda_{UV} \sim m_{soft}$. We base this argument on the similarity of $\theta_{MN} \bar{\psi}N\phi^M \psi$ operator with the fermion anomalous magnetic moment which is not supersymmetrizable and thus must vanish in the exact SUSY limit [17]. We believe, however, that the explicit calculation of $\theta_{MN} \bar{\psi}N\phi^M \psi$ interaction in softly broken supersymmetric theory is needed to clarify this matter. To stay on the conservative side, we shall assume that the cutoff is not higher than few hundred GeV. Having the answer [17] at hand, one can try to determine the level of sensitivity to $\theta_{\mu\phi}$ of different CPT and Lorentz violation searching experiments [18] by identifying $\phi$ with the Higgs boson and $\lambda$ with the light fermion Yukawa coupling. Clearly, one would have a serious suppression because all the relevant Yukawa couplings are quite small.

Thus, we are bound to explore gauge theories and we limit our discussion to the case of the five-dimensional non-commutative $U(1)$. One would naively guess that with the use of the non-commutative QCD in the bulk, the effective interaction is going to be at least two orders more suppressed than $\theta_{\mu\phi}$ configuration. Indeed, multiple KK states accelerate the renormalization group running and already after few thresholds known that multiple KK states accelerate the renormalization group running and already after few thresholds are taken into account, $\alpha_s \sim \alpha_s [20]$.

First, we note that $n\theta_{MN} \bar{\psi}N\phi_{\mu\phi} \psi$ is not generated at one-loop level. We also disagree with the claim of the Ref. [6] that $\theta_{MN} \bar{\psi}N\phi^M \psi$ interaction can be generated at one-loop level in a non-commutative $SU(N)$ theory simply because the amplitude, calculated in [6], is proportional to $\bar{\psi} - m$ and therefore vanishes on-shell. In what follows we restrict our calculation to the one-loop level and compute the effective dimension 6 operators.
Clearly, the use of higher dimensional operator will generically be suppressed by, say, \( \Lambda_{QCD}^2 / \Lambda_{UV}^4 \) compared to the leading dimension operator, \( m \psi \sigma_{MN} \bar{\psi} \). In our case, this suppression is not going to be dramatic since we choose a low value for the cutoff.

\[
\kappa_p \simeq 0.11; \quad \kappa_n \simeq 0.08.
\]

The use of the constituent quark model would produce 50% larger results.

One can treat \( \delta_p F_{ab} \) in eq. (3) in two different ways. The most straightforward approach would be to try to estimate the photon loop nucleon self-energy diagram with one of the vertices given by eq. (3). One can easily check that the Lorentz structure of such a diagram will be proportional to \( \theta_{if} \gamma_{5} N \). As to the numerical result of ultraviolet divergent integration, we cannot estimate it better than \( O(\varepsilon \times \text{loop factor} \times \Lambda^3) \) where \( \Lambda \) can be anywhere between \( m_\pi \) and \( m_p \). Such a result could be considered as an order-of-magnitude estimate at best.

We prefer to use a different method and directly calculate the nuclear matrix element of interaction (8). We exploit the fact that the gradient of the electric field inside a large nucleus is approximately constant,

\[
\delta_i F_{ij} \simeq \delta_{ij} \frac{Z e}{R^3} \simeq \frac{Z e}{A} \text{fm}^{-3}.
\]

For a non-relativistic nucleus, inside the nucleus, \( V \) reduces to the product of the nucleon spin operator and \( \theta_{i5} \). The wave function of an external nucleon is concentrated mainly inside the nuclear radius \( R \). Therefore, the nuclear matrix element reduces to a trivial angular part. Assuming one valence nucleon with orbital momentum \( L \) above closed shells, we arrive at the final form of the interaction between the nuclear spin \( I \) and the external vector \( \theta_{i5} \)

\[
V = \frac{20}{3} \frac{Z}{A} N_{\text{max}} \alpha^2 \kappa_{p(n)} a_{LI} (\vec{\theta}) \text{fm}^{-3}.
\]

Here \( a_{LI} = \langle \vec{S} \cdot \vec{I} \rangle_{LI} \) is a trivial combination of \( I(I+1) \) and \( L(L+1) \). \( a = 1 \) for \( L = 0 \).

The two most sensitive experiments, Ref. [3] and Ref. [4], use \(^{199} \text{Hg} \) and \(^{129} \text{Xe} \) whose spins are carried by external neutrons. Choosing \( N_{\text{max}} = 1 \) and comparing the size of \( V \) with the experimental accuracy of Ref. [4], \(|V| < 2\pi \times 100 \text{ nHz} \), we deduce the level of sensitivity to the presence of mixed non-commutativity

\[
|\theta_{i5}| \lesssim (5 \cdot 10^{11} \text{GeV})^{-2}
\]

Interestingly enough, we did not gain anything in terms of \( \Lambda_{NC} \) compared to our previous limit on \( \theta_{ij} \), even though in addition to Lorentz symmetry the effective interactions (3) and (4) violate CPT. This is because in the case of mixed non-commutativity, we had to resort to a loop level and used low energy (small) value of \( \alpha \). This limit can be improved by one or two orders of magnitude if one computes the two-loop level NCQCD induced \( \theta_{MN} \psi \sigma_{MN} \bar{\psi} \) interaction.

Do the results (3) and (4) mean that any five-dimensional model with \( \theta_{i5} \neq 0 \) would violate CPT in the low-energy regime? The answer is no, as we can easily construct a counterexample. Indeed, let us consider a

\[
\begin{align*}
\text{FIG. 1. One-loop diagrams in the 5-d non-commutative } \\
U(1) \text{ theory, which generate CPT non-invariant dim=6 operators in four dimensions. Heavy dot represents the interaction term with } \theta_{ij} \text{ which can be placed at any vertex in (a) non-commutative box diagram + photon permutation, (b) non-commutative “bosonic” penguin. This diagram does not exist in a normal, commutative } U(1) \text{ theory. (c) non-commutative “fermionic” penguin.}
\end{align*}
\]

The relevant set of diagrams, Fig. 1, contains boxes, “bosonic” and “fermionic” penguins with all possible insertions of the \( \theta \)-dependent vertices. We follow the linearized version of the Feynman rules for the non-commutative \( U(1) \) theory given, for example, in [21]. As in the previous example, loops contain heavy KK modes. We simplify our calculation by working in zeroth order in \( 1/N_{\text{max}} \).

The interaction (3) obviously breaks CPT. In addition to the EDM-like interaction, \( F_{ab} \delta_i \psi_i \) (eq. 3), there is an extra derivative \( \partial_\mu \psi \) which changes sign under either \( P \) or \( T \) transformation. In order to obtain phenomenological consequences of this interaction, we specialize the result (3) to the case of the light quarks. For the matrix elements \( \Delta \) operator of \( \bar{\psi} \sigma_{\mu\nu} \gamma_5 \psi \) operators over a nucleon, we use the results of lattice [22] and QCD sum rule [23] calculations: \( \Delta u = \Delta d = 0.8 \) and \( \Delta s = \Delta \delta_p \simeq -0.2 \) Thus we get

\[
V = \frac{5e^3 N_{\text{max}}}{48\pi^2} \partial_\mu F_{ab} \theta_{i5} (\kappa_p \bar{\psi} \sigma_{\alpha\beta} \gamma_5 p + \kappa_n \bar{n} \sigma_{\alpha\beta} \gamma_5 n)
\]

where

\[
\begin{align*}
\end{align*}
\]
model where all fermions stay confined to a 3+1 domain wall and only gauge bosons live in the bulk. Then the parity along the extra dimension, $y \rightarrow -y$ will forbid any odd powers of coupling of $\theta_{\mu 5}$ to the four-dimensional fermions. Another obvious trick which helps to get rid of CPT violation at low energies is the orbifold projection in 5 dimensions. The same argument of exact parity in $y$ coordinate will prohibit it. As the result of orbifolding and/or fermion localization, only quadratic couplings in $\theta_{\mu 5}$ could be important. Let us estimate a possible level of sensitivity to $\theta_{\mu 5}$ in this case.

There are many four-fermion operators that may appear and violate Lorentz invariance. It is easy to see, for example, that $c \theta_{\mu 5} \theta_{\nu 5} (\bar{q} \gamma_\mu \gamma_5 q) (\bar{q} \gamma_\nu q)$ may arise due to the interaction with $SU(2)$ KK gauge bosons. The component $\theta_5 \theta_5$ will couple to the spin of the external neutron in the mercury nucleus with the strength proportional to the nuclear density, $0.17 \text{fm}^{-3}$. Assuming again that $c \sim 100 \text{GeV}^2$ times the loop factor, we arrive at the level of sensitivity $\Lambda_{\text{NC}} \sim 10^7 \text{GeV}$. It is significantly milder constraint, although severe enough to exclude the influence of $\theta_{\mu 5} \theta_{\nu 5}$ on any plausible terrestrial particle physics experiments.

### III. CONCLUSIONS

The phenomenological consequences of the CPT violation are well understood, but the explicit models which break CPT are hard to find. In this paper we have demonstrated that the violation of the four-dimensional CPT invariance is possible in the presence of mixed non-commutativity $\theta_{\mu 5}$. In particular, this happens when fermions are allowed to propagate in the five-dimensional bulk. We have shown that the Yukawa or gauge interaction of these fermions generate an effective four-dimensional $\bar{\psi} \gamma_\mu \gamma_5 \gamma_\nu \psi$ as well as higher dimensional CPT-violating operators. Of course, an important ingredient in this picture is an indefinite parity for a fermion propagating in the five dimensional space.

Curiously enough, the tightest constraints in the case of $\theta_{\mu 5}$ do not arise from truly CPT-violation oriented experiments. We exploit the Lorentz non-invariance of this background and estimate the level of sensitivity of the clock comparison experiments [5] and [18] to the non-commutative scale as $(\theta_{\mu 5})^{-1/2} \sim 5 \cdot 10^{-1} \text{GeV}$.

An interesting feature of our result is the non-decoupling of heavy KK modes due to the ultraviolet enhancement brought by $\theta_{MN}$. This behavior may change in the non-commutative SUSY theories which phenomenological consequences obviously deserve more careful analysis.

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