COHOMOLOGY, EXTENSIONS AND AUTOMORPHISMS OF BRACES

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Abstract. We formalize the definition of low dimensional cohomology of a left brace with coefficients in a trivial brace with actions, study its connections with extensions of left braces and construct Wells’ like exact sequence relating second cohomology group with inducible automorphisms of extensions of left braces.

1. Introduction

Classification of set-theoretic solutions of the Yang-Baxter equation, proposed by Drinfeld [11], is a wide open problem. In 2007, Rump [21] introduced the notion of braces in connection with non-degenerate set theoretic solutions of the Yang-Baxter equation. The definition of a brace, which we use here and is frequently used in the contemporary literature, is slightly different but equivalent to the one given by Rump, and is due to Cedo, Jespers and Onkinski [9]. An algebraic structure \((E, +, \circ)\) is said to be a left brace if \((E, +)\) is an abelian group, \((E, \circ)\) is a group and, for all \(a, b, c \in E\), the following compatibility condition holds:

\[ a \circ (b + c) + a = a \circ b + a \circ c. \]

A right brace can be defined analogously. In the present article, we shall consider only left braces.

As is well known now (see [9]), a left brace gives rise to a non-degenerate involutive set theoretic solution of the Yang-Baxter equation (and vice-versa). Not only this, braces have deep connections with many other algebraic structures; linear cycle sets, radical rings, Hopf-Galois extensions, bijective 1-cocycles, to name some. So each new construction of a left brace, contributes to the solutions of the Yang-Baxter equation and enriches the class of other related structures. Constructing new algebraic structure from the existing ones is always an important and interesting problem. So is the case for braces. We mention some such constructions. Notion of semidirect product of braces was introduced in [22], which was generalised to asymmetric product of braces in [19]. Asymmetric product of braces was further studied in [3], where the authors also studied the wreath product of braces. Matched product of braces was investigated in [1]. Iterated matched product of braces was taken up in [2]. Our interest in this note is the cohomology and extension theory of braces.

Vendramin [15] studied dynamical extensions of finite cycle sets and its relationship with dynamical co-cycles. The topic was further studied by Castelli, Catino, Miccoli and Pinto [8] for quasi-linear cycle sets. Dynamical extensions give rise to non-degenerate involutive multipermutation solutions of the Yang-Baxter equation. Extensions of bijective 1-cocycles were investigated by Ben David and Ginosar [5]. Homology and cohomology theories for solution sets of the Yang-Baxter equations were developed by Carter, Elhambadi and Saito [7]. Different homology theories for various structures related to solutions of the Yang-Baxter equations were extensively investigated by Lebed and Vendramin [17]. They also defined second cohomology group for a cycle set \(X\) with coefficients in an abelian group \(A\) and established a bijective correspondence between the second cohomology group and equivalence classes of extensions of \(X\) by \(A\). Further, a general homology and cohomology theory was developed for linear cycle sets by Lebed and Vendramin [16] with trivial actions, where the authors extensively explored close connections between second cohomology of linear cycle sets and extension theory. Analogous results for left braces

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are also obtained in most of the cases by the authors. J.A. Guccione and J.J. Guccione [12] further investigated the ideas of [10] and used ‘perturbation lemma’ for computing first and second cohomology groups and extensions of cyclic linear cycles set acting trivially on an abelian group. Bachiller [1] considered extensions of left braces with actions and characterised equivalent extension in the form of certain cocycles and relations among them.

In the present article, we formalize the ideas developed in [1] in cohomological settings showing that the extension theory in the present case encompasses the extension theory investigated in [16], and present an exact sequence connecting automorphism groups of braces with second cohomology. Let $H := (H, +, \circ)$ be a left brace and $I := (I, +)$ an abelian group viewed as a trivial left brace. Suppose that $(H, \circ)$ acts on $(I, +)$ from left by an action $\nu$ and from right by an action $\sigma$. The images of $h \in H$ in $\text{Aut}(I, +)$, the automorphism group of the group $(I, +)$, under $\nu$ and $\sigma$ are denoted by $\nu_h$ and $\sigma_h$ respectively. Of particular interest are the pair of actions $(\nu, \sigma)$ which satisfy

$$\nu_{h_1 + h_2}(\sigma_{h_1 + h_2}(y)) + y = \nu_{h_1}(\sigma_{h_1}(y)) + \nu_{h_2}(\sigma_{h_2}(y))$$

for all $h_1, h_2 \in H, y \in I$. Such pairs will be called good pair of actions. For a given good pair of actions, we define $H^2_N(H, I)$, $i = 1, 2$, the $i$th cohomology group of $H$ with coefficients in $I$ (see Section 3). Let $\text{Ext}(H, I)$ denote the set of equivalence classes of extensions of $H$ by $I$ (see Section 2 for the definition). It turns out that

$$\text{Ext}(H, I) = \bigsqcup_{(\nu, \sigma)} \text{Ext}_{\nu, \sigma}(H, I),$$

where the pairs $(\nu, \sigma)$ run over all good pairs of actions of $H$ on $I$ and $\text{Ext}_{\nu, \sigma}(H, I)$ is the set of all equivalence classes of extensions of $H$ by $I$ whose corresponding good pair of actions is $(\nu, \sigma)$ (see Corollary 3.3). We can now state our first result

**Theorem A.** Let $H$ be a left brace, which acts on a trivial brace $I$ by a good pair of actions $(\nu, \sigma)$. Then there exists a bijection between $H^2_N(H, I)$ and $\text{Ext}_{\nu, \sigma}(H, I)$.

Let $Z^1_N(H, I)$ denote the group of all derivations from $H$ to $I$ as defined in Section 3. For an extension $\mathcal{E} : 0 \to I \to E \overset{\pi}{\to} H$ of $H$ by $I$, where $I$ is viewed as an ideal of $E$, we denote by $\text{Autb}_I(E)$ the set of all brace automorphisms of $E$ which normalise $I$. It follows that $\text{Autb}_I(E)$ is a subgroup of $\text{Aut}(E)$, the group of all brace automorphisms of $E$. For the extension $\mathcal{E}$, as explained in Section 3, one can associate a unique good pair of actions $(\nu, \sigma)$. Let

$$C_{(\nu, \sigma)} := \{ (\phi, \theta) \in \text{Autb}(H) \times \text{Autb}(I) \mid \nu_h = \theta^{-1} \nu_{\phi(h)} \theta \text{ and } \sigma_h = \theta^{-1} \sigma_{\phi(h)} \theta \}.$$  

Our next result is an analog for braces of an exact sequence, called the fundamental exact sequence of Wells [23], relating derivations, automorphisms and cohomology group of groups. For unexplained symbols in the following result see Section 5.

**Theorem B.** Let $\mathcal{E} : 0 \to I \to E \overset{\pi}{\to} H$ be a extension of a left brace $H$ by a trivial left brace $I$ such that $[\mathcal{E}] \in \text{Ext}_{\nu, \sigma}(H, I)$. Then we have the following exact sequence of groups

$$0 \to Z^1_N(H, I) \to \text{Autb}_I(E) \overset{\rho(\mathcal{E})}{\longrightarrow} C_{\nu, \sigma} \overset{\omega(\mathcal{E})}{\longrightarrow} H^2_N(H, I),$$

where $\omega(\mathcal{E})$ is, in general, only a derivation.

We remark that the fundamental exact sequence in group theoretical setting has been revisited, reformulated and applied by many authors in the past, for example, see [13, 14, 18, 20]. The exposition which we present here is analogous to the one in [14]. A unified treatment of the fundamental exact sequence of Wells with various applications is carried out in all fine details in [19] Chapter 2. A similar exact sequence for cohomology, extensions and automorphisms of quandles was constructed in [1].

We close this section with a quick layout of the article. Section 2 contains basic definitions and observation on left braces. In Section 3 we formalise the ideas on extension theory developed in [1] in
2. Preliminaries

An algebraic structure $(E, +, \circ)$ is said to be a left brace if $(E, +)$ is an abelian group, $(E, \circ)$ is a group and, for all $a, b, c \in E$, the following compatibility condition holds:

$$a \circ (b + c) + a = a \circ b + a \circ c. \quad (1)$$

Notice that the identity element $0$ of $(E, +)$ coincides with the identity element $1$ of $(E, \circ)$.

For a left brace $E$ and $a \in E$, define a map $\lambda_a : E \to E$ by

$$\lambda_a(b) = a \circ b - a$$

for all $b \in E$. The automorphism group of a group $G$ is denoted by $\text{Aut}(G)$. We have the following result which was proved by Rump [21] in linear cycle set settings.

**Lemma 2.1.** For each $a \in E$, the map $\lambda_a$ is an automorphism of $(E, +)$ and the map $\lambda : (E, \circ) \to \text{Aut}(E, +)$ given by $\lambda(a) = \lambda_a$ is a group homomorphism.

A subbrace $I$ of a left brace $E$ is said to be a left ideal of $E$ if $\lambda_a(y) \in I$ for all $a \in E$ and $y \in I$. A left ideal of $E$ is said to be an ideal if $(I, \circ)$ is a normal subgroup of $(E, \circ)$. An ideal $I$ of $E$ is said to be central if $y \circ a = a \circ y = a + y$ for all $a \in A$ and $y \in I$.

The following is an easy but important observation, which will be used several times in what follows.

**Lemma 2.2.** Let $E$ be a left brace. Then for all $a, b \in E$, the following hold:

(i) $a + b = a \circ \lambda_a^{-1}(b)$.

(ii) $a \circ b = a + \lambda_a(b)$.

Let $E_1$ and $E_2$ be two left braces. A map $f : E_1 \to E_2$ is said to be a brace homomorphism if $f(a + b) = f(a) + f(b)$ and $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in E_1$. A one-to-one and onto brace homomorphism from $E_1$ to itself is called an automorphism of $E_1$. The kernel of a homomorphism $f : E_1 \to E_2$ is defined to be the subset $\{a \in E_1 \mid f(a) = 0\}$ of $E_1$. It turns out that $\text{Ker}(f)$, the kernel of $f$, is an ideal of $E_1$. The set of all brace automorphisms of a left brace $E$, denoted by $\text{Autb}(E)$, is a group.

Let $H$ and $I$ be two left braces. By an extension of $H$ by $I$, we mean a left brace $E$ with an exact sequence

$$\mathcal{E} := 0 \to I \xrightarrow{i} E \xrightarrow{\pi} H \to 0,$$

where $i$ and $\pi$, respectively, are injective and surjective brace homomorphisms. An extension $\mathcal{E}$ is said to a central extension if the image of $I$ under $i$ is a central ideal of $E$. A set map $s : H \to E$ is called a set-theoretic section of $\pi$ if $\pi(s(h)) = h$ for all $h \in H$ and $s(0) = 0$. The abbreviation ‘st-section’ will be used for ‘set-theoretic section’ throughout.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two extension of $H$ by $I$, that is,

$$\mathcal{E} : 0 \to I \xrightarrow{i} E \xrightarrow{\pi} H \to 0$$

and

$$\mathcal{E}' : 0 \to I \xrightarrow{i'} E' \xrightarrow{\pi'} H \to 0$$
are exact sequences of braces. The extensions $E$ and $E'$ are said to be equivalent if there exists a brace homomorphism $\phi : E \to E'$ such that the following diagram commutes:

$$
\begin{array}{cccccc}
0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & H & \longrightarrow & 0 \\
\downarrow \text{Id} & & \downarrow \phi & & \downarrow \text{Id} & & \downarrow \text{Id} & & \\
0 & \longrightarrow & I & \longrightarrow & E' & \longrightarrow & H & \longrightarrow & 0.
\end{array}
$$

The set of all equivalence classes of extensions of $H$ by $I$ is denoted by $\text{Ext}(H, I)$.

Let $H$ be a left brace. An abelian brace $I$, that is, $(I, \circ)$ is also abelian, is said to be an $H$-bi-module if there exists a group homomorphism $\nu : (H, \circ) \to \text{Aut}(I)$ and a group anti-homomorphism $\sigma : (H, \circ) \to \text{Aut}(I)$ as defined in Section 3 by \ref{5} and \ref{6} respectively. Thus $I$ becomes an $H$-bi-module. All $H$-bi-modules $I$, in this paper, will be taken trivial braces. More distinctly, $I$ will always denote a trivial brace throughout.

Let $H$ and $H'$ be left braces, and $I$ and $I'$ be $H$ and $H'$ bi-modules with $(\nu, \sigma)$ and $(\nu', \sigma')$ respectively, as actions. Let $\alpha : H' \to H$ and $\zeta : I \to I'$ be homomorphisms of braces. The pair $(\alpha, \zeta)$ is said to be compatible with the pairs of actions $(\nu, \sigma)$ and $(\nu', \sigma')$ if the following diagram commutes for both left and right actions:

$$
\begin{array}{ccc}
H \times I & \xrightarrow{(\nu, \sigma)} & I \\
\uparrow \alpha & \downarrow \zeta & \\
H' \times I' & \xrightarrow{(\nu', \sigma')} & I'.
\end{array}
$$

More precisely, $(\alpha, \zeta)$ is compatible with the actions, if

$$
\zeta(\nu_{\alpha(h')}(y)) = \nu'_{\alpha(h')}(\zeta(y))
$$

and

$$
\zeta(\sigma_{\alpha(h')}(y)) = \sigma'_{\alpha(h')}(\zeta(y)).
$$

Let $I$ and $I'$ be two $H$-bi-modules with actions $(\nu, \sigma)$ and $(\nu', \sigma')$ respectively. A map $\zeta : I \to I'$ is called $H$-bi-module homomorphism if $(\text{Id}, \zeta)$ is compatible with the pairs of actions $(\nu, \sigma)$ and $(\nu', \sigma')$, where $\text{Id} : H \to H$ is the identity homomorphism.

3. Second Cohomology and Extensions of Braces

Let $H := (H, +, \circ)$ be a brace and $I := (I, +)$ an abelian group viewed as a trivial brace. Let $\nu : (H, \circ) \to \text{Aut}(I, +)$ and $\sigma : (H, \circ) \to \text{Aut}(I, +)$, respectively, be left and right group actions of $(H, \circ)$ on $(I, +)$. We denote the image of $h \in H$ under $\nu$ and $\sigma$ by $\nu_h$ and $\sigma_h$, respectively. We call a pair $(\nu, \sigma)$ of such actions a good pair if, for all $h_1, h_2 \in H$, $y \in I$, the following relation holds:

$$
\nu_{h_1 + h_2}(\sigma_{h_1 + h_2}(y)) + y = \nu_{h_1}(\sigma_{h_1}(y)) + \nu_{h_2}(\sigma_{h_2}(y)).
$$

Let $\text{Sh}_{r,j-r}$ be the set of all permutations $p$ on the set $\{1, \ldots, j\}$ such that $p(1) \leq \cdots \leq p(r)$, $p(r + 1) \leq \cdots \leq p(j)$, where $1 \leq r \leq j - 1$. For given $i \geq 0$, $j \geq 1$, $r$ in the range $1 \leq r \leq j - 1$ and $(h_1, \ldots, h_i, h_{i+1}, \ldots, h_{i+j}) \in H^{i+j}$, an expression of the form

$$
\sum_{p \in \text{Sh}_{r,j-r}} (-1)^p(h_1, \ldots, h_i, h_{i+p-1}(1), \ldots, h_{i+p-1}(j))
$$
is called a partial shuffle.

For $i \geq 0$, $j \geq 1$, let $\text{Fun}(H^{i+j}, I)$ denote the abelian group of all functions from $H^{i+j}$ to $I$ and $C^i_{H^j} := C^i_{H^j}(H, I)$ denote the subgroup of $\text{Fun}(H^{i+j}, I)$ consisting of all functions whose linearisation vanishes on all partial shuffles defined above and whose values on all degenerate tuples is zero. A tuple $(h_1, \ldots, h_n) \in H^n$ is said to be degenerate if $h_i = 0$ for at least one $i$, $1 \leq i \leq n$. Set $C^0_N := I$, $C^1_N := \bigoplus_{i+j=n} C^{i+j}_{N^i}$. We are mainly interested in small values of $n$, that is, $n \leq 3$. To be more precise, we take $C^0_N = C^0_{\nu^1}, C^1_N = C^0_{h^2} \oplus C^1_{\nu^2}$ and $C^3_N = C^0_{\nu^3} \oplus C^1_{\nu^2, h} \oplus C^1_{\nu^1}^2$. We wish to constitute the following zero-sequence:

$$C^0_N \xrightarrow{\partial^0} C^1_N \xrightarrow{\partial^1} C^2_N \xrightarrow{\partial^2} C^3_N.$$

Define $\partial^0 : C^0_N \to C^1_N$ by $\partial^0(y) = f_y$, where

$$f_y(h) = \nu_h(\sigma_h(y)) - y$$

and $\partial^1 : C^1_N \to C^2_N$ by $\partial^1(\theta) = (g, f)$, where $\theta \in C^1_N$ and for $h_1, h_2 \in H$,

$$g(h_1, h_2) = \theta(h_2) - \theta(h_1 + h_2) + \theta(h_1),$$

$$f(h_1, h_2) = \nu_{h_1}(\theta(h_2)) - \theta(h_1 \circ h_2) + \nu_{h_1 \circ h_2}(\sigma_{h_2} \nu_{h_1}^{-1}(\theta(h_1))).$$

Next define $\partial^2 : C^2_N \to C^3_N$ by

$$\partial^2(f, g) = (\partial^0_{\nu^2}(g), \partial^0_{h^2}(g) - \partial^1_{\nu^1}(f), \partial^1_{h^1}(f)),$$

where $(g, f) \in C^2_N$ and, for $h_1, h_2, h_3 \in H$, $\partial^0_{\nu^2} : C^2_N \to C^0_N$, $\partial^0_{h^2} : C^2_N \to C^0_N$, $\partial^1_{\nu^1} : C^1_N \to C^{1,2}_N$ and $\partial^1_{h^1} : C^{1,1}_N \to C^{2,1}_N$ are defined by

$$\partial^0_{\nu^2}(g)(h_1, h_2, h_3) = g(h_2, h_3) - g(h_2, h_3) + g(h_1 + h_2, h_3) - g(h_1, h_2),$$

$$\partial^0_{h^2}(g)(h_1, h_2, h_3) = \nu_{h_1}(g(h_2, h_3)) - g(h_1 \circ h_2, h_1 \circ h_3) + g(h_1, h_1 \circ (h_2 + h_3)), $$

$$\partial^1_{\nu^1}(f)(h_1, h_2, h_3) = f(h_2, h_2) - f(h_1, h_2 + h_3) + f(h_1, h_3),$$

$$\partial^1_{h^1}(f)(h_1, h_2, h_3) = \nu_{h_1}(f(h_2, h_3)) - f(h_1 \circ h_2, h_3) + f(h_1, h_2 \circ h_3) - \nu_{h_1 \circ h_2}(\sigma_{h_2} \nu_{h_1}^{-1}(f(h_1, h_2))).$$

We can now prove

**Lemma 3.1.** Let $(\nu, \sigma)$ be a good pair of actions of $H$ on $I$. Then the sequence $C^0_N \xrightarrow{\partial^0} C^1_N \xrightarrow{\partial^1} C^2_N \xrightarrow{\partial^2} C^3_N$ is indeed a zero-sequence.

**Proof.** Since $(\nu, \sigma)$ is a good pair of actions, it follows that $f_y(h_1 + h_2) = f_y(h_1) + f_y(h_2)$. So, for $y \in C^0_N$, take $\partial^0(y) = f_y$ and $\partial^1(f_y) = (g, f)$. Then for all $h_1, h_2 \in H$, we get

$$g(h_1, h_2) = f_y(h_2) - f_y(h_1 + h_2) + f_y(h_1) = 0.$$
Lemma 3.2. Let \( \nu, \sigma \in N \) such that \( \nu, \sigma \neq 0 \). Then we have 
\[
\partial_v^{1,1}(f)(h_1, h_2, h_3) = 0 \quad \text{for all } h_1, h_2, h_3 \in H.
\]
We compute each component separately.
\[
\partial_h^{0,2}(g)(h_1, h_2, h_3) = \nu_{h_1}(g(h_2, h_3)) - g(h_1 \circ h_2, h_1 \circ h_3) + g(h_1, h_1 \circ (h_2 + h_3))
\]
\[
+ \nu_{h_1}(\theta(h_1)) - \nu_{h_1}(\theta(h_2 + h_3)) + \nu_{h_1}(\theta(h_2))
\]
\[
- \theta(h_1 \circ h_3) + \theta(h_1 + h_1 \circ (h_2 + h_3)) - \theta(h_1 \circ h_2)
\]
\[
+ \theta(h_1) - \theta(h_1 + h_1 \circ (h_2 + h_3)) + \theta(h_1 \circ (h_2 + h_3))
\]
\[
= \nu_{h_1}(\theta(h_3)) - \nu_{h_1}(\theta(h_2 + h_3)) + \nu_{h_1}(\theta(h_2)) - \theta(h_1 \circ h_3)
\]
\[
- \theta(h_1 \circ h_2) + \theta(h_1) + \theta(h_1 \circ (h_2 + h_3))
\]

and
\[
\partial_v^{1,1}(f)(h_1, h_2, h_3) = f(h_1, h_2) - f(h_1, h_2 + h_3) + f(h_1, h_3)
\]
\[
= \nu_{h_1}(\theta(h_2)) - \theta(h_1 \circ h_2) + \nu_{h_1}(\sigma_{h_2}(\nu_{h_1}^{-1}(\theta(h_1))))
\]
\[
- \nu_{h_1}(\theta(h_2 + h_3)) + \theta(h_1 \circ (h_2 + h_3)) - \nu_{h_1}(\theta(h_2 + h_3)) - \theta(h_1 \circ h_3)
\]
\[
+ \nu_{h_1}(\theta(h_3)) - \theta(h_1 \circ h_3) + \nu_{h_1}(\sigma_{h_3}(\nu_{h_1}^{-1}(\theta(h_1)))).
\]

Subtracting the second expression from the first and solving on the right hand side, we get
\[
\partial_h^{0,2}(g)(h_1, h_2, h_3) - \partial_v^{1,1}(f)(h_1, h_2, h_3) = \theta(h_1) + \nu_{h_1}(\nu_{h_2 + h_3}(\nu_{h_1}^{-1}(\theta(h_1))))
\]
\[
- \nu_{h_2}(\sigma_{h_2}(\nu_{h_1}^{-1}(\theta(h_1)))) - \nu_{h_3}(\sigma_{h_3}(\nu_{h_1}^{-1}(\theta(h_1)))).
\]

The pair \((\nu, \sigma)\) being good pair, we finally get
\[
\partial_v^{0,2}(g)(h_1, h_2, h_3) - \partial_v^{1,1}(f)(h_1, h_2, h_3) = \theta(h_1) - \nu_{h_1}(\nu_{h_1}^{-1}(\theta(h_1))) = 0.
\]

The proof is now complete. \(\square\)

Let \( Z^i_N(H, I) := \text{Ker}(\partial^i) \) and \( B^i_N(H, I) := \text{Im}(\partial^{i-1}) \) for \( i = 1, 2 \). Define
\[ H^i_N(H, I) := Z^i_N(H, I)/B^i_N(H, I), \]
the \( i \)-th cohomology group of \( H \) with coefficients in \( I \). Elements of \( Z^i_N(H, I) \) and \( B^i_N(H, I) \) are called \( i \)-cocycles and \( i \)-coboundaries respectively. Two 2-cocycles \((\beta_1, \tau_1)\) and \((\beta_2, \tau_2)\) are said to be cohomologous if \((\beta_1, \tau_1) - (\beta_2, \tau_2) \in B^2_N(H, I)\); more precisely, if there exists a \( \theta \in C^2_N \) such that
\[
(\beta_1 - \beta_2)(h_1, h_2) = \theta(h_1) - \theta(h_1 + h_2) + \theta(h_1)
\]
and
\[
(\tau_1 - \tau_2)(h_1, h_2) = \nu_{h_1}(\theta(h_2)) - \theta(h_1 \circ h_2) + \nu_{h_1}(\sigma_{h_2}(\nu_{h_1}^{-1}(\theta(h_1))))
\]
for all \( h_1, h_2 \in H \). The elements of \( Z^1_N(H, I) \) are also called derivations.

The next result follows from the definitions.

Lemma 3.2. Let \((g, f) \in C^2_N\). Then, for all \( h_1, h_2 \in H \), the following hold:

(i) \( g(h_1, h_2) = g(h_2, h_1) \).

(ii) \( g(h_1, 0) = g(0, h_2) = 0 \).

(iii) \( f(h_1, 0) = f(0, h_2) = 0 \).

Let \((\beta, \tau) \in C^2_N\). Define on \( H \times I \), the following operations:
\[
(h_1, y_1) + (h_2, y_2) = (h_1 + h_2, y_1 + y_2 + \beta(h_1, h_2)).
\]
\[
(h_1, y_1) \circ (h_2, y_2) = (h_1 \circ h_2, \nu_{h_1 \circ h_2}(\sigma_{h_2}(\nu_{h_1}^{-1}(y_1))) + \nu_{h_1}(y_2) + \tau(h_1, h_2)).
\]

Then we have
**Theorem 3.3.** Let \((\nu, \sigma)\) be a good pair of actions of \(H\) on \(I\). Then \(H \times I\) takes the structure of a left brace under the operations defined in the preceding para if and only if \((\beta, \tau) \in Z_N^2(H, I)\).

**Proof.** If \((\beta, \tau) \in Z_N^2(H, I)\), then, on the lines of the proof of [1, Theorem 3.3], using Lemma 3.2 it follows that the operations under hypothesis turn \(H \times I\) into a left brace. Let \((E, +, \circ)\) denote the brace defined on \(H \times I\) with the operations under hypothesis. Then, that \((\beta, \tau) \in Z_N^2(H, I)\) follows from the associativity of \('+\)' and \('\circ'\) and the compatibility condition [1] of the brace \(E\). \(\square\)

The left brace structure on \(H \times I\) (as given in Theorem 3.3) is an extension of \(H\) by \(I\), which we denote by the 6-tuple data \((H, I, \nu, \sigma, \beta, \tau)\) with the exact sequence

\[
0 \to I \to (H, I, \nu, \sigma, \beta, \tau) \to H \to 0,
\]

where \(i(y) = (0, y)\) and \(\pi(h, y) = h\). Throughout the paper, for such extensions, we'll always take a fixed st-section \(s : H \to (H, I, \nu, \sigma, \beta, \tau)\) of \(\pi\) given by \(s(h) = (h, 0)\).

Let \(0 \to I \to E \to H \to 0\) be an extension of \(H\) by \(I\). For the ease of notation, we shall view \(I\) as an ideal of \(E\) through the given embedding. Let \(s : H \to E\) be any st-section of \(\pi\). Then, for all \(h \in H\) and \(y \in I\), we define \(\nu, \sigma : (H, \circ) \to Aut(I, +)\) by

\[
\nu_h(y) := s(h) \circ y - s(h) = \lambda_{s(h)}(y) \quad (5)
\]

and

\[
\sigma_h(y) := s(h)^{-1} \circ y \circ s(h). \quad (6)
\]

It is not difficult to see that, for a given st-section \(s\), \(\nu\) is a homomorphism and \(\sigma\) is an anti-homomorphism from \((H, \circ)\) to \(Aut(I, +)\). Thus \(\nu\) and \(\sigma\) are, respectively, left and right actions of \((H, \circ)\) on \((I, +)\).

With this setting, we have

**Proposition 3.4.** Let \(0 \to I \to E \to H \to 1\) be an extension of a left brace \(H\) by a trivial brace \(I\).

1. Then the actions \(\nu\) and \(\sigma\) are independent of the choice of an st-section. Moreover, the pair \((\nu, \sigma)\) is a good pair of actions of \(H\) on \(I\).

2. Equivalent extensions have the same pair of actions.

**Proof.** Let \(s_1\) and \(s_2\) be two different st-sections of \(\pi\). Let \((\sigma, \nu)\) and \((\sigma', \nu')\) be actions corresponding to \(s_1\) and \(s_2\) respectively. We know that for each \(h \in H\), there exists \(y_h \in I\) such that \(s_2(h) = y_h \circ s_1(h)\). Thus, using the triviality of the brace \(I\), for all \(h \in H\) and \(y \in I\), we get

\[
\sigma'_h(y) = s_2(h)^{-1} \circ y \circ s_2(h)
\]

\[
= (s_1(h))^{-1} \circ (y_h)^{-1} \circ y \circ y_h \circ s_1(h)
\]

\[
= s_1(h)^{-1} \circ y \circ s_1(h)
\]

\[
= \sigma_h(y)
\]

and

\[
\nu'_h(y) = s_2(h) \circ y - s_2(h)
\]

\[
= y_h \circ s_1(h) \circ y - y_h \circ s_1(h)
\]

\[
= y_h \circ (s_1(h) \circ y - s_1(h)) - y_h
\]

\[
= y_h + (s_1(h) \circ y - s_1(h)) - y_h
\]

\[
= \nu_h(y).
\]

This shows the independence of the actions on the choice of an st-section. That \((\nu, \sigma)\) is a good pair is proved by Bachiller [1, page 1675]. This proves assertion (1).
For assertion (2), let $E$ and $E'$ be two equivalent extensions of $H$ by $I$. Then there exists a homomorphism $\phi : E \to E'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
0 & \longrightarrow & I \\
\downarrow \text{id} & & \downarrow \phi \\
E & \longrightarrow & H \\
\downarrow \phi & & \downarrow \text{id} \\
0 & \longrightarrow & I \\
E' & \longrightarrow & H \\
\end{array}
$$

Let $s$ be an st-section of $\pi$. By the commutativity of the diagram, it follows that $s' : H \to E'$, given by $s'(h) := \phi(s(h))$ for all $h \in H$, is an st-section of $\pi'$, and $\phi(y) = y$ for all $y \in I$. Let $(\sigma, \nu)$ and $(\sigma', \nu')$ be actions corresponding to $s$ and $s'$ respectively. Then, using the fact that $\phi$ is a homomorphism, we have

$$
\sigma'_h(y) = s'(h)^{-1} \circ y \circ s(h) = \phi(s(h))^{-1} \circ y \circ \phi(s(h)) = \phi(s(h)^{-1} \circ y \circ s(h)) = \phi(\sigma_h(y)) = \sigma_h(y)
$$

for all $h \in H$ and $y \in I$. Thus, $\sigma$ and $\sigma'$ are the same actions. Similarly, it also follows that the actions $\nu$ and $\nu'$ are the same, which completes the proof. \qed

Recall that $\text{Ext}(H, I)$ denotes the set of equivalence classes of all extensions of $H$ by $I$. Equivalence class of an extension $E : 0 \to I \to E \to H \to 0$ is denoted by $[E]$. As a consequence of the preceding proposition, it follows that each equivalence class of extension of $H$ by $I$ admits a unique good pair of actions $(\nu, \sigma)$ of $H$ on $I$, which we call the corresponding pair of actions. Let $\text{Ext}_{\nu, \sigma}(H, I)$ denote the equivalence classes of those extensions of $H$ by $I$ whose corresponding pair of actions is $(\nu, \sigma)$. We can easily establish

**Corollary 3.5.** $\text{Ext}(H, I) = \bigsqcup_{(\nu, \sigma)} \text{Ext}_{\nu, \sigma}(H, I)$.

Next, for the extension $0 \to I \to E \xrightarrow{\beta} H \to 0$, define maps $\beta, \tau : H \times H \to I$ by

$$
\beta(h_1, h_2) := s(h_1) + s(h_2) - s(h_1 + h_2)
$$

and

$$
\tau(h_1, h_2) := \nu_{h_1 \circ h_2}(s(h_1) \circ h_2) - s(h_1 \circ h_2) = s(h_1) \circ s(h_2) - s(h_1 \circ h_2),
$$

where $\nu$ is defined in (6).

**Lemma 3.6.** Let $0 \to I \to E \xrightarrow{\nu \circ} H \to 0$ be an extension of $H$ by $I$ and $\nu, \sigma$ are, respectively, the left and right actions of $H$ on $I$ defined by (5) and (6). Then $(\beta, \tau) \in Z^2_{\nu}(H, I)$, where the maps $\beta$ and $\tau$ are defined in (7) and (8). Moreover, the cohomology class of $(\beta, \tau)$ is independent of the choice of an st-section of $\pi$.

**Proof.** Let $s$ be an st-section of $\pi$. That $\partial^{0,2}_{\nu}(\beta) = 0$ and $\partial^{1,1}_{\nu}(\tau) = 0$ follows from the associativity of ‘$+$’ and ‘$\circ$’ in $E$ respectively; more precisely, one achieves this by using the identities: (i) $s(h_1) + (s(h_2) + s(h_3)) = s(h_1) + s(h_2) + s(h_3)$ and (ii) $s(h_1) \circ (s(h_2) \circ s(h_3)) = s(h_1) \circ s(h_2) \circ s(h_3)$. We demonstrate computations in one case. Using the definition of $\tau$, for all $h_1, h_2, h_3 \in H$, we have

$$
\begin{align*}
\tau(h_1, h_2) &= (s(h_1) \circ \tau(h_2, h_3) + s(h_2 \circ h_3)) \\
&= s(h_1) \circ \tau(h_2, h_3) + s(h_1) \circ s(h_2 \circ h_3) \\
&= \nu_{h_1}(\tau(h_2, h_3)) + \tau(h_1, h_2 \circ h_3) + s(h_1 \circ h_2 \circ h_3).
\end{align*}
$$
On the other hand

\[ (s(h_1) \circ s(h_2)) \circ s(h_3) = s(h_1 \circ h_2) \circ \nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2)) \circ s(h_3) \]
\[ = s(h_1 \circ h_2) \circ s(h_3) \circ \sigma_{h_3}(\nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2))) \]
\[ = s(h_1 \circ h_2 \circ h_3) \circ \nu_{h_1,ob_2}^{-1}(\tau(h_1 \circ h_2, h_3)) \circ \sigma_{h_3}(\nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2))) \]
\[ = s(h_1 \circ h_2 \circ h_3) \circ \nu_{h_1,ob_2}^{-1}(\tau(h_1 \circ h_2, h_3)) + s(h_1 \circ h_2 \circ h_3) \circ \sigma_{h_3}(\nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2)))) \]
\[ = s(h_1 \circ h_2 \circ h_3) \circ \nu_{h_1,ob_2}^{-1}(\tau(h_1 \circ h_2, h_3)) \]
\[ + s(h_1 \circ h_2 \circ h_3) \circ \sigma_{h_3}(\nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2))) \]
\[ = s(h_1 \circ h_2 \circ h_3) \circ \nu_{h_1,ob_2}^{-1}(\tau(h_1 \circ h_2, h_3)) \]
\[ + s(h_1 \circ h_2 \circ h_3) \circ \sigma_{h_3}(\nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2)))) \]
\[ = \tau(h_1 \circ h_2, h_3) + s(h_1 \circ h_2 \circ h_3) + \nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2))). \]

Comparing the preceding two expressions, we get

\[ \nu_{h_1}(\tau(h_2, h_3)) - \tau(h_1 \circ h_2, h_3) + \tau(h_1, h_2 \circ h_3) - \nu_{h_1,ob_2}^{-1}(\tau(h_1, h_2))) = 0. \]

That \( \partial H^2(\beta) - \partial H^1(\tau) = 0 \) follows from the brace condition, that is, \( s(h_1) \circ (s(h_2) + s(h_3)) + s(h_1) = s(h_1) \circ s(h_2) + s(h_1) \circ s(h_3) \), by similar computations. We refer the reader to [1, pages 1674-75] for other computations. This shows that \( \beta, \tau \in Z_N^2(H, I) \).

For the second assertion, let \( s \) and \( s' \) be two st-sections of \( \pi \). Let \( (\beta, \tau) \) and \( (\beta', \tau') \) be 2-cocycles corresponding to \( s \) and \( s' \) respectively. Notice that \( s(h) - s'(h) \in I \). Define a map \( \theta : H \to I \) by \( \theta(h) = s(h) - s'(h) \) for all \( h \in H \). A straightforward computation then shows that the 2-cocycles \( (\beta, \tau) \) and \( (\beta', \tau') \) differ by \( \partial^1(\theta) \). This completes the proof. \( \square \)

We are now ready to prove the main result of this section (Theorem A).

**Theorem 3.7.** Let \( H := (H, +, \circ) \) be a left brace and \( I := (I, +) \) an abelian group viewed as a trivial brace. Let \( (\nu, \sigma) \) be a good pair of actions of \( H \) on \( I \). Then there is a bijection between \( H^2_N(H, I) \) and \( \text{Ext}_{\nu, \sigma}(H, I) \).

**Proof.** Let \( (\tau, \beta) \in Z_N^2(H, I) \). Then it follows from Theorem 3.3 that the pair \( (\tau, \beta) \) gives rise to an extension \( (H, I, \nu, \sigma, \beta, \tau) \). Let \( (\beta_1, \tau_1), (\beta_2, \tau_2) \in Z_N^2(H, I) \) be cohomologous. Then there exists \( \theta \in C_N^1 \) such that

\[ (\beta_1 - \beta_2)(h_1, h_2) = \theta(h_2) - \theta(h_1 + h_2) + \theta(h_1) \]

and

\[ (\tau_1 - \tau_2)(h_1, h_2) = \nu_{h_1}(\theta(h_2)) - \theta(h_1 + h_2) + \nu_{h_1,ob_2}(\sigma_{h_2}(\nu_{h_1}^{-1}(\theta(h_1)))) \]

for all \( h_1, h_2 \in H \). Let \( (H, I, \nu, \sigma, \beta_1, \tau_1) \) and \( (H, I, \nu, \sigma, \beta_2, \tau_2) \) be the corresponding extensions for \((\beta_1, \tau_1)\) and \((\beta_2, \tau_2)\) respectively. Define a map \( \phi : (H, I, \nu, \sigma, \beta_1, \tau_1) \to (H, I, \nu, \sigma, \beta_2, \tau_2) \) by

\[ \phi(h, y) := (h, y + \theta(h)). \]

Consider the following diagram:

\[
\begin{array}{ccc}
0 & \to & I \xrightarrow{i} (H, I, \nu, \sigma, \beta_1, \tau_1) \xrightarrow{\pi} H \to 0 \\
\text{Id} & & \phi \downarrow \quad \text{Id} \\
0 & \to & I \xrightarrow{i'} (H, I, \nu, \sigma, \beta_2, \tau_2) \xrightarrow{\pi'} H \to 0.
\end{array}
\]
Notice that the linearity of $\phi$ in $'+'$ and $'o'$ follows from the equations $[11]$ and $[10]$ respectively. It is now obvious that the preceding diagram is commutative. Thus the extensions $(H, I, \nu, \sigma, \beta_1, \tau_1)$ and $(H, I, \nu, \sigma, \beta_2, \tau_2)$ are equivalent, and therefore the map $\psi : H^2_N(H, I) \to \text{Ext}_{\nu, \sigma}(H, I)$ given by

$\psi([(\beta, \tau)]) = [(H, I, \nu, \sigma, \beta, \tau)]$

is well defined.

We now show that $\psi$ is surjective. Let $\mathcal{E} : 0 \to I \to E \to H \to 0$ be any extension whose corresponding pair of actions is $(\nu, \sigma)$. Let $s : H \to E$ be an st-section of $\pi$. Then, by Lemma 3.6 we get $(\beta, \tau) \in Z^2_N(H, I)$ as defined in $[9]$ and $[8]$. Applying $\psi$, we have $\psi([(\beta, \tau)]) = [(H, I, \nu, \sigma, \beta, \tau)]$. Now the surjectivity of $\psi$ is equivalent to showing that $\mathcal{E}$ and $(H, I, \nu, \sigma, \beta, \tau)$ are equivalent. More precisely, the cohomology class $[(\beta, \tau)]$ of $(\beta, \tau)$ will be a pre-image of the extension $\mathcal{E}$. We remark that this mechanism is independent of the choice of an st-section $s$ of $\pi$, because whatever $s$ we start with, we finally get an extension equivalent to $\mathcal{E}$. So, for the surjectivity of $\psi$, we only need to establish the commutativity of the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & I & \overset{i}{\longrightarrow} & E & \overset{\pi}{\longrightarrow} & H & \longrightarrow & 0 \\
\id & & \downarrow{\phi} & & \phantom{\pi} & & \phantom{\id} & & \phantom{\pi} \\
0 & \longrightarrow & I & \overset{i'}{\longrightarrow} & (H, I, \sigma, \nu, \beta, \tau) & \overset{\pi'}{\longrightarrow} & H & \longrightarrow & 0,
\end{array}
\]

where $\phi$ is a brace homomorphism. As mentioned above, $\pi'(h, y) = h$ and $s'$, given by $s'(h) := (h, 0)$, is an st-section of $\pi'$, which we’ll use here. Notice that every $g \in E$ can be uniquely written as $g = s(h) + y$ for some $h \in H$ and $y \in I$. Define $\phi : E \to (H, I, \nu, \sigma, \beta, \tau)$ by

$\phi(s(h) + y) = (h, y)$.

It is easy to see that $\phi$ is linear under $'+'$ and $'o'$. Notice that $s(h) \circ y = s(h) + \nu_h(y)$ and $s(h) + y = s(h) \circ \nu_h^{-1}(y)$ for all $h \in H$ and $y \in I$. Let $s(h_1) + y_1$ and $s(h_2) + y_2$ be two elements of $E$. Then

$\phi((s(h_1) + y_1) \circ (s(h_2) + y_2)) = \phi(s(h_1) \circ \nu_h^{-1}(y_1) \circ s(h_2) \circ \nu_h^{-1}(y_2))$

$= \phi(s(h_1) \circ s(h_2) \circ \sigma_{h_2}(\nu_h^{-1}(y_1)) \circ \nu_h^{-1}(y_2))$

$= \phi(s(h_1 \circ h_2) \circ \nu_h^{-1}(\sigma_{h_2}(\nu_h^{-1}(y_1))) \circ \nu_h^{-1}(y_2))$

$= \phi(s(h_1 \circ h_2) + \nu_{h_1 \circ h_2}(\sigma_{h_2}(\nu_h^{-1}(y_1))) + \nu_h^{-1}(y_2))$

$= \phi((h_1 \circ h_2, y_1, y_2))$

$= \phi(s(h_1) + y_1) \circ \phi(s(h_2) + y_2)$.

Hence $\phi$ is a brace homomorphism. Also $\phi(0, y) = (0, y) = i'(y)$ and

$\pi(s(h) + y) = h = \pi'(h, y) = \pi'(\phi(s(h) + y))$,

which establishes the commutativity of the preceding diagram.

Finally, we proceed to show the injectivity of $\psi$. Let $(\beta_1, \tau_1), (\beta_2, \tau_2) \in Z^2_N(H, I)$ such that $\psi([(\beta_1, \tau_1)]) = \psi([(\beta_2, \tau_2)])$. More precisely, we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & I & \overset{i}{\longrightarrow} & (H, I, \nu, \sigma, \beta_1, \tau_1) & \overset{\pi}{\longrightarrow} & H & \longrightarrow & 0 \\
\id & & \downarrow{\phi} & & \phantom{\pi} & & \phantom{\id} & & \phantom{\pi} \\
0 & \longrightarrow & I & \overset{i'}{\longrightarrow} & (H, I, \nu, \sigma, \beta_2, \tau_2) & \overset{\pi'}{\longrightarrow} & H & \longrightarrow & 0,
\end{array}
\]

where $\phi$ is a brace homomorphism. Let $s$ and $s'$ be the st-sections of $\pi$ and $\pi'$ given by $s(h) = (h, 0)$ and $s'(h) = (h, 0)$, respectively, for all $h \in H$. It now follows from the commutativity of the diagram that, for each $h \in H$, $\phi(h, 0) = (h, y_h)$ for some $y_h \in I$. Notice that for a given $h \in H$, $y_h$ is unique. Define
\( \theta : H \rightarrow I \) by \( \theta(h) = y_h \). Since \( s(0) = (0, \ 0) \), it follows that \( \theta \in C_N^1 \). By a regular computation, one can show that

\[
(\beta_1 - \beta_2)(h_1, h_2) = \theta(h_2) - \theta(h_1 + h_2) + \theta(h_1)
\]

and

\[
(\tau_1 - \tau_2)(h_1, h_2) = \nu_{h_1}(\theta(h_2)) - \theta(h_1 \circ h_2) + \nu_{h_1 \circ h_2}(\sigma_{h_2}(\nu_{h_1}^{-1}(\theta(h_1))))
\]

for all \( h_1, h_2 \in H \). This simply means that \((\beta_1, \tau_1)\) and \((\beta_2, \tau_2)\) are cohomologous, and the proof is complete.

Let \( H \) be a left brace and \( I \) a trivial brace such that \( H \) acts trivially on \( I \), that is, both \( \nu_h \) and \( \sigma_h \) are trivial automorphism of \((I, +)\) for all \( h \in H \). It is immediate, in this case, that \((\nu, \sigma)\) is a good pair of actions. Let \([[\beta, \tau]] \in H^2_N(H, I)\) and \( E := (H, I, \sigma, \nu, \tau, \beta) \) be the corresponding extension of \( H \) by \( I \). It turns out that the image of \( I \) in \( E \) is a central ideal of \( E \). On the other hand, if

\[
E : 0 \rightarrow I \rightarrow E \xrightarrow{\pi} H \rightarrow 0
\]

is a central extension, then \( E \in \text{Ext}_{\nu, \sigma}(H, I) \), where \((\nu, \sigma)\) is a trivial action of \( H \) on \( I \). Let the set of all equivalence classes of central extension of \( H \) by \( I \) be denoted by \( \text{CExt}(H, I) \). Then, as a special case of Theorem 3.7 we get the following brace theoretic analog of [16, Theorem 5.8].

**Corollary 3.8.** Let \( H := (H, +, \circ) \) be a left brace which acts trivially on a trivial brace \( I := (I, +) \). Then there is a bijection between \( H^2_N(H, I) \) and \( \text{CExt}(H, I) \).

We conclude this section with explicit computation of \( H^2_N(H, I) \), where \( H = \mathbb{Z}/2\mathbb{Z} \) and \( I = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) with two good pairs of actions; one non-trivial and other trivial. It is easy to see that the pair \((\nu, \sigma)\) with \( \nu_h(a, b) = (a + b + h, b) \), \( \sigma_h(a, b) = (a, b + a + h) \) is a good pair of actions, where \( h \in H \) and \( a, b \in I \). This pair of actions was considered by Bachiller [H]. Let \((\beta, \tau)\) be a 2-cocycle. Notice that both \( \beta \) as well as \( \tau \) are determined by their values at \((1, 1) \in H \times H \). An easy computation then reveals that the following are the only choices for \((\beta(1, 1), \tau(1, 1))\):

\[
\{(0, 0), (0, 0), (0, 0), (1, 0), (0, 0)\text{, and } (1, 0), (0, 0)\}.
\]

For clarity, let us take these cocycles as \((\beta_1, \tau_1), (\beta_2, \tau_2), (\beta_3, \tau_1), (\beta_3, \tau_2)\) respectively. If we define \( \theta(0) = (0, 0) \) and \( \theta(1) = (0, 1) \), then \( \theta \in C_N^1 \); further \( \beta_1, \tau_1 = \partial^1(\theta) \), that is, \((\beta_1, \tau_1)\) is a 2-coboundary. We claim that none of \((\beta_2, \tau_1)\) and \((\beta_2, \tau_2)\) is a 2-coboundary, but these are cohomologous. Contrarily, assume that \((\beta_2, \tau_2)\) is a 2-coboundary. Then there exists \( \theta' \in C_N^1 \) such that \( \partial^1(\theta') = (\beta_2, \tau_2) \). So \( \theta(0) = (0, 0) \) and let \( \theta(1) = (y_1, y_2) \). Now

\[
\beta_2(1, 1) = \theta(1) - \theta(0) + \theta(1) = (y_1, y_2) + (y_1, y_2) = (0, 0),
\]

which is not possible, since \( \beta_2(1, 1) = (1, 0) \). Hence \((\beta_2, \tau_2)\) is not a 2-coboundary. The same computation also shows that \((\beta_2, \tau_1)\) too is not a 2-coboundary. Finally, notice that \((\beta_2, \tau_2) - (\beta_2, \tau_1) = (\beta_1, \tau_2)\), which we have already shown to be a 2-coboundary. Hence \((\beta_2, \tau_2)\) and \((\beta_2, \tau_1)\) are cohomologous, and therefore

\[
H^2_N(H, I) = \{[(\beta_1, \tau_1)], [(\beta_2, \tau_2)]\} \cong \mathbb{Z}/2\mathbb{Z}.
\]

Now we consider the trivial action, that is, \( \nu_h = \sigma_h = \text{Id} \) for all \( h \in H \). As above, for computing 2-cocycles \((\beta, \tau)\) of \( H \), it is enough to know their values on \((1, 1) \). A detailed computation shows that \((\beta_i, \tau_j)\), \( 1 \leq i, j \leq 4 \), are all possible 2-cocycles of \( H \), where \( \beta_1 = (0, 0) = \tau_1 \), \( \beta_2 = (0, 1) = \tau_2 \), \( \beta_3 = (1, 0) = \tau_3 \) and \( \beta_4 = (1, 1) = \tau_4 \). Let \( \theta \in C_N^1 \). Notice that \( \theta(0) = 0 \). Then \( \partial^1(\theta) = (g, f) \), where

\[
g(h_1, h_2) = f(h_1, h_2) = \theta(h_2) - \theta(h_1 + h_2) + \theta(h_1) = 0
\]

for all \( h_1, h_2 \in H \), since \( H \) is a trivial brace and the action is trivial. Hence \( B^2_N(H, I) \) is the trivial group, and therefore \( H^2_N(H, I) \) is isomorphic to the elementary abelian group of order 16.
4. COHOMOLOGY OF BRACES : A SPECIAL CASE

In this section we generalize the cohomological results of [10] Sections 2, 3 for braces. Let $(H, +, \circ)$ be a left brace and $(I, +)$ an abelian group. Let $\nu : (H, \circ) \to \text{Aut}(I, +)$ and $\sigma : (H, \circ) \to \text{Aut}(I, +)$, respectively, again be left and right group actions of $(H, \circ)$ on $(I, +)$.

Set $C^n(H; I) := \text{Fun}(H^n, I)$ for $n \geq 1$, where $H^n$ denotes the cartesian product of $n$ copies of $H$. Notice that $C^n(H; I)$ is an abelian group for each $n$. For $n \geq 0$, define the maps $\partial^n : C^n \to C^{n+1}$ by

\[
(\partial^n f)(h_1, \ldots, h_{n+1}) = \nu_{h_i}(f(h_2, \ldots, h_{n+1})) + \sum_{i=1}^{n} (-1)^i f(h_1, \ldots, h_i \circ h_{i+1}, \ldots, h_{n+1}) + (-1)^{n+1} \nu_{h_1 \circ h_2 \circ \cdots \circ h_n}(\sigma_{h_{n+1}}(\nu_{h_1 \circ h_2 \circ \cdots \circ h_n}(f(h_1, \ldots, h_n)))�.
\]

Define

$RC^n(H; I) = \{ f \in \text{Fun}(H^n, I) \mid f \text{ is linear in the } \nu \text{th co-ordinate} \}$.

Then the restriction of $\partial^n$, which we still denote by $\partial^n$, gives the map

$\partial^n : RC^n(H; I) \to RC^{n+1}(H; I)$

for each $n \geq 1$. Further, let $RC^*_n(H; I)$ denote the set of all $f \in RC^n(H; I)$ which vanish on all degenerate tuples. This becomes a subgroup of $RC^n(H; I)$ and the restriction of $\partial^n$ gives a map from $RC^*_n(H; I) \to RC^{n+1}_*(H; I)$.

**Theorem 4.1.** For $n \geq 1$, $(C^n(H; I), \partial^n)$, $(RC^n(H; I), \partial^n)$ and $(RC^*_n(H; I), \partial^n)$ are cochain complexes.

**Proof.** We shall only indicate a proof of the fact that $(C^n(H; I), \partial^n)$ is a cochain complex. Other assertion can be easily verified from this. We are required to show that $\partial^{n+1} \partial^n = 0$. We can write

$\partial^n = \sum_{i=0}^{n+1} (-1)^i \partial_{n,i}$

where

\[
(\partial_{n,0} f)(h_1, \ldots, h_{n+1}) = \nu_{h_1}(f(h_2, \ldots, h_{n+1})),
\]

\[
(\partial_{n,i} f)(h_1, \ldots, h_{n+1}) = f(h_1, \ldots, h_i \circ h_{i+1}, \ldots, h_{n+1}), \quad 1 \leq i \leq n,
\]

\[
(\partial_{n,n+1} f)(h_1, \ldots, h_{n+1}) = \nu_{h_1 \circ h_2 \cdots \circ h_n}(\sigma_{h_{n+1}}(\nu_{h_1 \circ h_2 \cdots \circ h_n}(f(h_1, \ldots, h_n)))�.
\]

By a direct computation one can easily show that $\partial_{n,i} \partial_{n-1,j} = \partial_{n,j+1} \partial_{n-1,i}$ for $0 \leq i, j \leq n$, which implies $\partial^n \partial^{n-1} = 0$. To demonstrate, we compute in one case. Let $i = 0$ and $j \geq 1$ be a fixed integer in the appropriate range. Then

$\partial_{n,0}(\partial_{n-1,j} f)(h_1 \ldots h_{n+1}) = \nu_{h_1}(\partial_{n-1,j} f)(h_2, \ldots, h_{n+1}) = \nu_{h_1}(f(h_2, \ldots, h_{j+1} \circ h_{j+2}, \ldots, h_{n+1})).$

On the other hand

$\partial_{n,j+1}(\partial_{n-1,0} f)(h_1, \ldots, h_{n+1}) = (\partial_{n-1,0} f)(h_1, \ldots, h_{j+1} \circ h_{j+2}, \ldots, h_{n+1}) = \nu_{h_1}(f(h_2, \ldots, h_{j+1} \circ h_{j+2}, \ldots, h_{n+1})).$

Hence $\partial_{n,0} \partial_{n-1,j} = \partial_{n,j+1} \partial_{n-1,0}$. \[\square\]
Let $H^n(H, I)$, $RH^n(H, I)$ and $RH^2_N(H, I)$, respectively, denote the $n$th cohomology groups of the complexes $(C^n(H, I), \partial^n)$, $(RC^n(H, I), \partial^n)$ and $(RC^2_N(H, I), \partial^n)$.

Let $\text{STExt}_{\nu, \sigma}(H, I)$ be the set of equivalence classes of all extensions of $H$ by $I$ with fixed good pair of actions $(\nu, \sigma)$, and which split as extensions of abelian groups. Let $(\beta, \tau) \in H^2_N(H, I)$. If we take $\beta = 0$, then it follows that $\tau \in RH^2_N(H, I)$. More precisely, $RH^2_N(H, I) = \{ \tau : H \times H \to I \mid (0, \tau) \in H^2_N(H, I) \}$. The next result now follows from Theorem 3.7 by taking $\beta = 0$.

**Theorem 4.2.** There exists a one-to-one correspondence between $RH^2_N(H, I)$ and $\text{STExt}_{\nu, \sigma}(H, I)$.

Let $(\nu, \sigma)$ be a good pair of actions of $H$ on $I$. Let $H^2_N((H, \circ), I)$ denote the second cohomology group of $(H, \circ)$ with coefficients in the abelian group $I$, where $(H, \circ)$ acts on $I$ through $\sigma$. For a 2-cocycle $f$ from $RC^2_N(H; I)$ we can define a map $f' : (H, \circ) \times (H, \circ) \to I$ by setting

$$f'(h_1, h_2) = \nu^{-1}_{h_1, h_2}(f(h_1, h_2))$$

for all $h_1, h_2 \in (H, \circ)$. It turns out that $f'$ is a 2-cocycle of $(H, \circ)$ with coefficients in $I$. This gives rise to the following

**Proposition 4.3.** The cohomology group $RH^2_N(H, I)$ of braces embeds in the cohomology group $H^2_N((H, \circ), I)$ of groups under the association $\{ f' \}$. 

Let $H$ and $H'$ be left braces, and $I$ and $I'$ be $H$ and $H'$ bi-modules with $(\nu, \sigma)$ and $(\nu', \sigma')$, respectively, as actions. Let $\alpha : H' \to H$ and $\zeta : I \to I'$ be homomorphisms of braces. The pair $(\alpha, \zeta)$ is said to be compatible with the pairs of actions $(\nu, \sigma)$ and $(\nu', \sigma')$ if the diagram commutes for both left and right actions. Recall that, equivalently saying, $(\alpha, \zeta)$ is compatible with the actions, if

$$\zeta(\nu_{\alpha(h')} (y)) = \nu'_{\alpha(h')} (\zeta(y))$$

and

$$\zeta(\sigma_{\alpha(h')} (y)) = \sigma'_{\alpha(h')} (\zeta(y)).$$

Let $(\alpha, \zeta)$ be compatible with the actions $(\nu, \sigma)$ and $(\nu', \sigma')$. Let us fix an integer $n \geq 1$. For $f \in C^n(H, I)$, define $f' : H'^n \to I'$ by

$$f'(h_1', \ldots, h_n') = \zeta(f(\alpha(h_1), \ldots, \alpha(h_n)))$$

for $h_i' \in H'$. Since $(\alpha, \zeta)$ is compatible with the actions, it easily follows that $f' \in C^n(H', I')$. Define $(\alpha, \zeta)^n : C^n(H, I) \to C^n(H', I')$ by

$$(\alpha, \zeta)^n(f) = f'$$

for all $f \in C^n(H, I)$. It is not difficult to see that $(\alpha, \zeta)^n$ is a homomorphism and the following diagram commutes

$$C^n(H, I) \xrightarrow{\partial^n} C^n(H', I') \xrightarrow{(\alpha, \zeta)^n} C^n(H', I'') \xrightarrow{\partial'^n}$$

Further straightforward computations now give the following

**Theorem 4.4.** For each $n \geq 1$, the homomorphism $(\alpha, \zeta)^n : C^n(H, I) \to C^n(H', I')$, defined in the preceding para, induces a homomorphism from $H^n(H, I)$ to $H^n(H', I')$, which further induces homomorphisms from $RH^n(H, I)$ to $RH^n(H', I')$ and $RH^2_N(H, I)$ to $RH^2_N(H', I')$.

Let $K$ be a left ideal of $H$ and $\alpha : K \to H$ be the embedding. For a pair of actions $(\nu, \sigma)$ of $H$ on $I$, $(\nu', \sigma')$ is a pair of actions of $K$ on $I$, where $\nu' = \nu|_K$ and $\sigma' = \sigma|_K$. Let $\text{Id} : I \to I$ be the identity homomorphism. Obviously $(\alpha, \text{Id})$ is compatible with these actions. Hence, as a consequence of the preceding theorem, we have

**Corollary 4.5.** There exists a homomorphism from $RH^2_N(H, I)$ to $RH^2_N(K, I)$. 


Such a homomorphism is called the restriction homomorphism, denoted by res$^H_I$.

5. Wells’ type exact sequence for braces

In this section we establish various group actions on the set Ext($H, I$) and construct an exact sequence connecting certain automorphism groups of $E$ with $H^2_N(H, I)$ for a given extension

$$E : 0 \to I \xrightarrow{i} E \xrightarrow{\tau} H \to 0$$

of left braces. We start with an action of $\text{Autb}(H) \times \text{Autb}(I)$ on $\text{Ext}(H, I)$.

Let $H$ be any left brace and $I$ a trivial brace. For a pair $(\phi, \theta) \in \text{Autb}(H) \times \text{Autb}(I)$ of brace automorphisms and an extension

$$E : 0 \to I \xrightarrow{i} E \xrightarrow{\tau} H \to 0$$

of $H$ by $I$, we can define a new extension

$$E^{(\phi, \theta)} : 0 \to I \xrightarrow{i^\theta} E \xrightarrow{\phi^{-1}\tau^\tau} H \to 0$$

of $H$ by $I$. Let

$$E_1 : 0 \to I \xrightarrow{i} E_1 \xrightarrow{\tau} H \to 0$$

and

$$E_2 : 0 \to I \xrightarrow{i'} E_2 \xrightarrow{\tau'} H \to 0$$

be two equivalent extensions of $H$ by $I$. Then it is not difficult to show that the extensions $E_1^{(\phi, \theta)}$ and $E_2^{(\phi, \theta)}$ are also equivalent for any $(\phi, \theta) \in \text{Autb}(H) \times \text{Autb}(I)$. Thus, for a given $(\phi, \theta) \in \text{Autb}(H) \times \text{Autb}(I)$, we can define a map from $\text{Ext}(H, I)$ to itself given by

$$[E] \mapsto [E^{(\phi, \theta)}]. \quad (12)$$

If $\phi$ and $\theta$ are identity automorphisms, than obviously $E^{(\phi, \theta)} = E$. It is also easy to see that

$$[E]^{(\phi_1, \theta_1)(\phi_2, \theta_2)} = ([E]^{(\phi_1, \theta_1)})^{(\phi_2, \theta_2)}.$$  

We conclude that the association $(12)$ gives an action of the group $\text{Autb}(H) \times \text{Autb}(I)$ on the set $\text{Ext}(H, I)$.

As we know that $\text{Ext}(H, I) = \bigcup_{(\nu, \sigma)} \text{Ext}_{(\nu, \sigma)}(H, I)$. Let $(\nu, \sigma)$ be an arbitrary but fixed good pair of actions of $H$ on $I$. Let $C_{(\nu, \sigma)}$ denote the stabiliser of $\text{Ext}_{(\nu, \sigma)}(H, I)$ in $\text{Autb}(H) \times \text{Autb}(I)$; more explicitly

$$C_{(\nu, \sigma)} = \{(\phi, \theta) \in \text{Autb}(H) \times \text{Autb}(I) \mid \nu_h = \theta^{-1}\nu(h)\theta \text{ and } \sigma_h = \theta^{-1}\sigma(h)\theta\}.$$  

Notice that $C_{(\nu, \sigma)}$ is a subgroup of $\text{Autb}(H) \times \text{Autb}(I)$, and it acts on $\text{Ext}_{(\nu, \sigma)}(H, I)$ by the same rule as given in $(12)$.

Next we consider an action of $C_{(\nu, \sigma)}$ on $H^2_N(H, I)$. Let $(\phi, \theta) \in \text{Aut}(H) \times \text{Aut}(I)$ and $f \in \text{Fun}(H^n, I)$, where $n \geq 1$ be an integer. Define $f^{(\phi, \theta)} : H^n \to I$ by setting

$$f^{(\phi, \theta)}(h_1, h_2, \ldots, h_n) := \theta^{-1}(f(\phi(h_1), \phi(h_2), \ldots, \phi(h_n))).$$

It is not difficult to see that the group $\text{Autb}(H) \times \text{Autb}(I)$ acts on the group $\text{Fun}(H^n, I)$ as well as on the group $C^N_N(H, I)$, by automorphisms, given by the association

$$f \mapsto f^{(\phi, \theta)}.$$

It is also obvious that $C_{(\nu, \sigma)}$ acts on both of these sets. We are interested in the action of $C_{(\nu, \sigma)}$ on $H^2_N(H, I)$. The association $(13)$ induces an action of $C_{(\nu, \sigma)}$ on $C^N_N = C^N_{0, 2}(H, I) \oplus C^N_{1, 1}(H, I)$ by setting

$$((\beta, \tau)) \mapsto ((\beta(\phi, \theta), \tau(\phi, \theta))). \quad (14)$$
Lemma 5.1. For $(\phi, \theta) \in C_{(\nu, \sigma)}$, the following hold:

(i) If $((\beta, \tau)) \in Z_{N}^{2}(H, I)$, then $((\beta^{(\phi, \theta)}, \tau^{(\phi, \theta)})) \in Z_{N}^{2}(H, I)$.

(ii) If $((\beta, \tau)) \in B_{N}^{2}(H, I)$, then $((\beta^{(\phi, \theta)}, \tau^{(\phi, \theta)})) \in B_{N}^{2}(H, I)$.

Hence the association \[14\] gives an action of $C_{(\nu, \sigma)}$ on $H_{N}^{2}(H, I)$ by automorphisms if we define

$$[(\beta, \tau)]^{(\phi, \theta)} = [((\beta^{(\phi, \theta)}, \tau^{(\phi, \theta)}))].$$

Proof. That $\partial^{0,2}_{\nu}(\beta^{(\phi, \theta)})$ is a zero function in $C_{N}^{3}$ holds trivially. Since $(\phi, \theta) \in C_{(\nu, \sigma)}$, we have

$$\nu_{h_{2}}((\theta^{-1}(\tau(\phi(h_{2}), \phi(h_{3})))) = \theta^{-1}(\nu_{\phi(h_{1})}(\tau(\phi(h_{2}), \phi(h_{3})))).$$

and

$$\nu_{h_{1}h_{2}}(\sigma_{h_{2}}(\nu^{-1}_{h_{1}}(\phi(h_{1}), \phi(h_{2})))) = \theta^{-1}(\nu_{\phi(h_{1})}(\nu^{-1}_{h_{2}}(\sigma_{h_{2}}(\nu^{-1}_{h_{1}}(\tau(\phi(h_{1}), \phi(h_{2}))))).$$

Using these identities, for all $h_{1}, h_{2}, h_{3} \in H$, it follows that

$$\partial^{0,1}_{B_{0}^{1}}(\tau^{(\phi, \theta)})(h_{1}, h_{2}, h_{3}) = \theta^{-1}(\partial^{0,1}_{B_{0}^{1}}(\tau(\phi(h_{1}), \phi(h_{2}), \phi(h_{3})))) = 0.$$

This proves assertion (i).

Now assume that $((\beta, \tau)) \in B_{N}^{2}(H, I)$. Then there exists a $\lambda \in C_{N}^{3}$ such that $(\beta, \tau) = \partial^{1}(\lambda)$. It is now not difficult to see that

$$((\beta^{(\phi, \theta)}, \tau^{(\phi, \theta)})) = \partial^{1}(\lambda^{(\phi, \theta)}),$$

which establishes assertion (ii). That the association under consideration gives an action of $C_{(\nu, \sigma)}$ on $H_{N}^{2}(H, I)$ by automorphisms, is now straightforward, which completes the proof. \[\square\]

Remark 15. The action of $C_{(\nu, \sigma)}$ on $H_{N}^{2}(H, I)$, defined in the preceding lemma, can be transferred on $\text{Ext}_{(\nu, \sigma)}(H, I)$ through the bijection given in Theorem 3.7. Notice that the resulting action of $C_{(\nu, \sigma)}$ on $\text{Ext}_{(\nu, \sigma)}(H, I)$ agrees with the action defined by \[12\].

We now consider the action of $H_{N}^{2}(H, I)$ onto itself by right translation, which is faithful and transitive. Again using Theorem 3.7 we can transfer this action on $\text{Ext}_{(\nu, \sigma)}(H, I) = \{[(H, I, \nu, \sigma, \beta, \tau)] \mid [(\beta, \tau)] \in H_{N}^{2}(H, I)\}$. More precisely, for $[(\beta, \tau)] \in H_{N}^{2}(H, I)$, the action is given by

$$[(H, I, \nu, \sigma, \beta, \tau)]^{[(\beta, \tau)]} = [(H, I, \nu, \sigma, \beta + \beta, \tau + \tau)]$$

for all $[(H, I, \nu, \sigma, \beta, \tau)] \in \text{Ext}_{(\nu, \sigma)}(H, I)$. Notice that this action is also faithful and transitive.

Consider the semidirect product $\Gamma := C_{(\nu, \sigma)} \rtimes H_{N}^{2}(H, I)$ under the action defined in Lemma 5.1. We wish to define an action of $\Gamma$ on $\text{Ext}_{(\nu, \sigma)}(H, I)$. For $(c, h) \in \Gamma$ and $[\mathcal{E}] \in \text{Ext}_{(\nu, \sigma)}(H, I)$, define

$$[\mathcal{E}]^{(c, h)} = ([\mathcal{E}]^{c})^{h}.$$ \[16\]

Lemma 5.2. The rule in \[16\] gives an action of $\Gamma$ on $\text{Ext}_{(\nu, \sigma)}(H, I)$.

Proof. Notice that for $(c_{1}, h_{1}), (c_{2}, h_{2}) \in \Gamma$, $(c_{1}, h_{1}) \cdot (c_{2}, h_{2}) = (c_{1}c_{2}, h_{1}^{c_{2}}h_{2})$. So, it is enough to show that $([\mathcal{E}]^{h})^{c} = ([\mathcal{E}]^{c})^{h}$ for each $c \in C_{(\nu, \sigma)}$, $h \in H_{N}^{2}(H, I)$ and $[\mathcal{E}] \in \text{Ext}_{(\nu, \sigma)}(H, I)$. We know that $[\mathcal{E}] = [(H, I, \nu, \sigma, \beta, \tau)]$ for some $[(\beta, \tau)] \in H_{N}^{2}(H, I)$. Then, for $h = [(\beta_{h}, \tau_{h})] \in H_{N}^{2}(H, I)$, we have

$$([\mathcal{E}]^{h})^{c} = [(H, I, \nu, \sigma, (\beta + \beta_{h})^{c}, (\tau + \tau_{h})^{c})]$$

$$= [(H, I, \nu, \sigma, \beta^{c} + \beta_{h}^{c}, \tau^{c} + \tau_{h}^{c})]$$

$$= ([(H, I, \nu, \sigma, \beta^{c}, \tau^{c})]^{h})^{c}$$

$$= ([\mathcal{E}]^{c})^{h}.$$
The proof is now complete. □

Let \([\mathcal{E}] \in \text{Ext}_{\nu,\sigma}(H, I)\) be a fixed extension. Since the action of \(H^2_N(H, I)\) on \(\text{Ext}_{\nu,\sigma}(H, I)\) is transitive and faithful, for each \(c \in C_{(\nu,\sigma)}\), there exists a unique element (say) \(h_c\) in \(H^2_N(H, I)\) such that \([\mathcal{E}]^c = [\mathcal{E}]^{h_c}\).

We thus have a well defined map \(\omega(\mathcal{E}) : C_{(\nu,\sigma)} \to H^2_N(H, I)\) given by

\[
\omega(\mathcal{E})(c) = h_c \tag{17}
\]

for \(c \in C_{(\nu,\sigma)}\).

**Lemma 5.3.** The map \(\omega(\mathcal{E}) : C_{(\nu,\sigma)} \to H^2_N(H, I)\) given in (17) is a derivation with respect to the action of \(C_{(\nu,\sigma)}\) on \(H^2_N(H, I)\) given in (14).

**Proof.** Let \(c_1, c_2 \in C_{(\nu,\sigma)}\) and \(\omega(\mathcal{E})(c_1 c_2) = h_{c_1 c_2}\). Thus, by the definition of \(\omega(\mathcal{E})\), \([\mathcal{E}]^{c_1 c_2} = [\mathcal{E}]^{h_{c_1} h_{c_2}}\). Using the fact that \([\mathcal{E}]^h = ([\mathcal{E}]^c)^h\) for each \(c \in C_{(\nu,\sigma)}\), \(h \in H^2_N(H, I)\), we have

\[
[\mathcal{E}]^{h_{c_1} h_{c_2}} = ([\mathcal{E}]^{c_1})^{h_{c_2}} = ([\mathcal{E}]^{h_{c_1}})^{c_2} = ([\mathcal{E}]^{c_1})((h_{c_1})^{c_2} = ([\mathcal{E}]^{c_2})((h_{c_2})^{c_2} = ([\mathcal{E}]^{h_{c_2}})((h_{c_2})^{c_1} = [\mathcal{E}]((h_{c_2} + (h_{c_1})^{c_1}).
\]

Since the action of \(H^2_N(H, I)\) on \(\text{Ext}_{\nu,\sigma}(H, I)\) is faithful, it follows that \(h_{c_1 c_2} = (h_{c_1})^{c_2} + h_{c_2}\). This implies that \(\omega(\mathcal{E})(c_1 c_2) = (\omega(\mathcal{E})(c_1))^{c_2} + \omega(\mathcal{E})(c_2)\); hence \(\omega(\mathcal{E})\) is a derivation. □

Let

\[
\mathcal{E} : 0 \to I \to E \xrightarrow{\pi} H
\]

be an extension of a left brace \(H\) by a trivial brace \(I\) such that \([\mathcal{E}] \in \text{Ext}_{\nu,\sigma}(H, I)\). Let \(\text{Autb}_I(E)\) denote the subgroup of \(\text{Autb}(E)\) consisting of all automorphisms of \(E\) which normalize \(I\), that is,

\[
\text{Autb}_I(E) := \{ \gamma \in \text{Autb}(E) \mid \gamma(y) \in I \text{ for all } y \in I \}.
\]

For \(\gamma \in \text{Autb}_I(E)\), set \(\gamma_I := \gamma|_I\), the restriction of \(\gamma\) to \(I\), and \(\gamma_H\) to be the automorphism of \(H\) induced by \(\gamma\). More precisely, \(\gamma_H(h) = \pi(\gamma(s(h)))\) for all \(h \in H\), where \(s\) is an st-section of \(\pi\). Notice that the definition of \(\gamma_H\) is independent of the choice of an st-section. Define a map \(\rho(\mathcal{E}) : \text{Autb}_I(E) \to \text{Autb}(H) \times \text{Autb}(I)\) by

\[
\rho(\mathcal{E})(\gamma) = (\gamma_H, \gamma_I).
\]

Although \(\omega(\mathcal{E})\) is not a homomorphism, but we can still talk about its set theoretic kernel, that is,

\[
\text{Ker}(\omega(\mathcal{E})) = \{ c \in C_{(\nu,\sigma)} \mid [\mathcal{E}]^c = [\mathcal{E}] \}.
\]

With this setting we have

**Proposition 5.4.** For the extension \(\mathcal{E}\), \(\text{Im}(\rho(\mathcal{E})) \subseteq C_{(\nu,\sigma)}\) and \(\text{Im}(\rho(\mathcal{E})) = \text{Ker}(\omega(\mathcal{E}))\).
Proof. For the first assertion, we are required to show that \( \nu_h = \gamma^{-1}_I \nu_{\gamma_I(h)} \gamma_I \) and \( \sigma_h = \gamma^{-1}_I \sigma_{\gamma_I(h)} \gamma_I \) for all \( h \in H \). Let \( s \) be an st-section of \( \pi \) and \( x \in E \). Notice that \( \gamma^{-1}_I \) is the restriction of \( \gamma^{-1} \) on \( I \). Also notice that for a given \( x \in E \), \( (\pi(x)) = x \circ y_h \) for some \( y_h \in I \). Now for \( h \in H \) and \( y \in I \), we have

\[
\gamma^{-1}_I \nu_{\gamma_I(h)} \gamma_I(y) = \gamma^{-1}_I (\nu_{\pi(\gamma(s(h))))(\gamma(y))}
= \gamma^{-1}_I (s(\pi(\gamma(s(h)))) \circ \gamma(y) - s(\gamma(s(h))))
= \gamma^{-1}_I (\gamma(s(h)) \circ y_{\gamma(s(h)))} \circ \gamma(y) - \gamma(s(h)) \circ y_{\gamma(s(h))})
= s(h) \circ \gamma^{-1}_I (y_{\gamma(s(h)))} \circ y - s(h) \circ \gamma^{-1}_I (y_{\gamma(s(h))})
= s(h) \circ (\gamma^{-1}_I (y_{\gamma(s(h)))} + y) - s(h) \circ \gamma^{-1}_I (y_{\gamma(s(h))})
= s(h) \circ y - s(h) \quad \text{(using 1)}
= \nu_h(y).
\]

Hence \( \nu_h = \gamma^{-1}_I \nu_{\gamma_I(h)} \gamma_I \). One can similarly show that \( \sigma_h = \gamma^{-1}_I \sigma_{\gamma_I(h)} \gamma_I \).

Now we prove the second assertion. Let \( \rho(\mathcal{E})(\gamma) = (\gamma_H, \gamma_I) \) for \( \gamma \in \text{Aut}_I(E) \). We know that \( s(\pi(x)) = x + y_\pi \) for some \( y_\pi \in I \). Thus we have

\[
\gamma^{-1}_H (s(\pi(\gamma(s(h)))) = \pi(\gamma^{-1}_I (s(\pi(\gamma(s(h))))))
= \pi(\gamma^{-1}_I (\gamma(s(h)) + y_{\gamma(s(h))}))
= \gamma(h),
\]

which implies that the diagram

\[
\begin{array}{c}
0 \longrightarrow I \longrightarrow E \stackrel{\pi}{\longrightarrow} H \longrightarrow 0 \\
\text{Id} \downarrow \quad \gamma \downarrow \quad \text{Id} \downarrow \\
0 \longrightarrow I \longrightarrow \gamma_I \longrightarrow E \stackrel{\gamma^I_H}{\longrightarrow} H \longrightarrow 0
\end{array}
\]

commutes. Hence \( [\mathcal{E}]^{(\gamma_H \cdot \gamma_I)} = [\mathcal{E}] \), which shows that \( \text{Im}(\rho(\mathcal{E})) \subseteq \text{Ker}(\omega(\mathcal{E})) \).

Conversely, if \( (\phi, \theta) \in \text{Ker}(\omega(\mathcal{E})) \), then there exists a brace homomorphism \( \gamma : E \rightarrow E \) such that the diagram

\[
\begin{array}{c}
0 \longrightarrow I \longrightarrow E \stackrel{\pi}{\longrightarrow} H \longrightarrow 0 \\
\text{Id} \downarrow \quad \gamma \downarrow \quad \text{Id} \downarrow \\
0 \longrightarrow I \longrightarrow \theta \longrightarrow E \stackrel{\phi^{-1}_I}{\longrightarrow} H \longrightarrow 0
\end{array}
\]

commutes. It is now obvious that \( \gamma \in \text{Aut}_I(E) \), \( \theta = \gamma_I \) and \( \phi = \gamma_H \). Hence \( \rho(\mathcal{E})(\gamma) = (\phi, \theta) \), which completes the proof. \( \square \)

Continuing with the above setting, set \( \text{Aut}^{H,I}(E) := \{ \gamma \in \text{Aut}_I(E) \mid \gamma_I = \text{Id}, \gamma_H = \text{Id} \} \). Notice that \( \text{Aut}^{H,I}(E) \) is precisely the kernel of \( \rho(\mathcal{E}) \). Hence, using Proposition 5.4, we get

**Theorem 5.5.** Let \( \mathcal{E} : 0 \rightarrow I \rightarrow E \stackrel{\pi}{\rightarrow} H \) be an extension of a left brace \( H \) by a trivial brace \( I \) such that \( [\mathcal{E}] \in \text{Ext}_{\pi,J}(H,I) \). Then we have the following exact sequence of groups

\[
0 \rightarrow \text{Aut}^{H,I}(E) \rightarrow \text{Aut}_I(E) \stackrel{\rho(\mathcal{E})}{\longrightarrow} C_{(\nu,J)} \stackrel{\omega(\mathcal{E})}{\longrightarrow} H^2_N(H,I),
\]

where \( \omega(\mathcal{E}) \) is, in general, only a derivation.

We further prove

**Proposition 5.6.** Let \( \mathcal{E} : 0 \rightarrow I \rightarrow E \stackrel{\pi}{\rightarrow} H \) be an extension of \( H \) by \( I \) such that \( [\mathcal{E}] \in \text{Ext}_{\nu,J}(H,I) \). Then \( \text{Aut}^{H,I}(E) \cong Z_N^1(H,I) \).
Proof. We know that every element \( x \in E \) has a unique expression of the form \( x = s(h) + y = s(h) \circ \nu_h^{-1}(y) \) for some \( h \in H \) and \( y \in I \). Let us define a map \( \eta : Z^2_N(H, I) \to \text{Aut}^{H,I}(E) \) by
\[
\eta(\lambda)((s(h) + y) = s(h) + \lambda(h) + y,
\]
where \( \lambda \in Z^2_N(H, I) \). Notice that the image of \( \eta(\lambda) \) is independent of the choice of an st-section. We claim that \( \eta(\lambda) \in \text{Aut}^{H,I}(E) \). For \( h_1, h_2 \in H \) and \( y_1, y_2 \in I \), we know that
\[
\tau(h_1, h_2) = \nu_{h_1 \circ h_2}^{-1}(s(h_1 \circ h_2)^{-1} \circ s(h_1) \circ s(h_2))
\]
as defined in [\( \star \)]. Set \( \tau_1 = \nu_{h_1 \circ h_2}^{-1} \tau \). Also notice that \( s(h) \circ y = s(h) + \nu_h(y) \). We then have
\[
\eta(\lambda)((s(h_1) + y_1) \circ (s(h_2) + y_2)) = \eta(\lambda)((s(h_1) \circ \nu_h^{-1}(y_1)) \circ (s(h_2) \circ \nu_h^{-1}(y_2)))
\]
and
\[
\eta(\lambda)((s(h_1) + y_1) \circ (s(h_2) + y_2)) = \eta(\lambda)((s(h_1) \circ \nu_h^{-1}(y_1)) \circ (s(h_2) \circ \nu_h^{-1}(y_2)))
\]
That \( \eta \) is injective, follows from the injectivity of \( s \) and the fact that \( \lambda(0) = 0 \). Surjectivity of \( \eta(\lambda) \) is immediate. This shows that \( \eta(\lambda) \) is an automorphism of \( E \). We also have \( \eta(\lambda)(s(h)) = s(h) + \lambda(h) \) and \( \eta(\lambda)(y) = y \) for all \( h \in H \) and \( y \in I \). Hence \( \eta(\lambda) \in \text{Aut}^{H,I}(E) \).

We’ll now define a map \( \zeta : \text{Aut}^{H,I}(E) \to Z^2_N(H, I) \). For \( \gamma \in \text{Aut}^{H,I}(E) \) and \( h \in H \), there exists a unique element (say) \( y_h^\gamma \in I \) such that \( \gamma(s(h)) = s(h) + y_h^\gamma \) for some unique \( y_h^\gamma \in I \). Thus, for \( h \in H \), define \( \zeta \) by
\[
\zeta(\gamma)(h) = y_h^\gamma.
\]
Notice that this is independent of the choice of an st-section. Since \( \sigma_h^{-1} = \sigma_{h^{-1}} \), it follows that
\[
\gamma(s(h)^{-1}) = s(h)^{-1} \circ \sigma_h^{-1}((\nu_h^{-1}((y_h^\gamma)^{-1}))
\]
for all \( h \in H \). Further, since \( (s(h_1) \circ h_2)^{-1} \circ s(h_1) \circ s(h_2) \in I \) for all \( h_1, h_2 \in H \), we have
\[
s(h_1 \circ h_2)^{-1} \circ s(h_1) \circ s(h_2) = \gamma(s(h_1 \circ h_2)^{-1} \circ s(h_1) \circ s(h_2))
\]
\[
= \gamma(s(h_1 \circ h_2)^{-1}) \circ \gamma(s(h_1)) \circ s(h_2)
\]
\[
= s(h_1 \circ h_2)^{-1} \circ \sigma_{h_1 \circ h_2}^{-1}(\nu_{h_1 \circ h_2}^{-1}(y_{h_1 \circ h_2}^\gamma)^{-1}) \circ s(h_1) \circ \nu_h^{-1}(y_h^\gamma)
\]
\[
= s(h_2) \circ \nu_h^{-1}(y_h^\gamma)
\]
\[
= s(h_1 \circ h_2)^{-1} \circ \sigma_{h_1 \circ h_2}^{-1}(\nu_{h_1 \circ h_2}^{-1}(y_{h_1 \circ h_2}^\gamma)^{-1}) \circ s(h_1) \circ \nu_h^{-1}(y_h^\gamma)
\]
which, using \( (y_{h_1 \circ h_2}^\gamma)^{-1} = -y_{h_1 \circ h_2}^\gamma \), implies that
\[
\sigma_{h_1 \circ h_2}^{-1}(\nu_{h_1 \circ h_2}^{-1}(y_{h_1 \circ h_2}^\gamma)^{-1}) = \sigma_{h_1^{-1}}(\nu_h^{-1}(y_h^\gamma)) + \sigma_{h_1 \circ h_2}^{-1}(\nu_{h_2}^{-1}(y_{h_2}^\gamma)).
\]
This, on further simplification, finally gives
\[
y_{h_1 \circ h_2}^\gamma = \nu_{h_1 \circ h_2} \sigma_h(\nu_h^{-1}(y_h^\gamma)) + \nu_{h_1}(y_{h_2}^\gamma).
\]
For \( h_1, h_2 \in H \), it easily follows that
\[
y_{h_1+h_2} = y_{h_1} + y_{h_2}.
\]
We have now proved that \( \zeta(\gamma) \) is a derivation. That both \( \eta \) and \( \zeta \) are homomorphisms, and \( \eta \zeta \) and \( \zeta \eta \) are, respectively, the identity elements of \( \text{Aut}^H(E) \) and \( Z_N(H, I) \) is obvious. Hence \( \text{Aut}^H(E) \cong Z_N(H, I) \), and the proof is complete.

We finally get the following Wells’ like exact sequence for braces (Theorem B).

**Theorem 5.7.** Let \( E : 0 \rightarrow I \rightarrow E \xrightarrow{\nu} H \) be a extension of a left brace \( H \) by a trivial brace \( I \) such that \([E] \in \text{Ext}_{\nu,*}(H, I)\). Then we have the following exact sequence of groups
\[
0 \rightarrow Z_N^1(H, I) \rightarrow \text{Aut}_I(E) \xrightarrow{\rho(E)} C_{\nu,*} \xrightarrow{\omega(E)} H_N^2(H, I),
\]
where \( \omega(E) \) is, in general, only a derivation.

A pair of automorphisms \((\phi, \theta) \in C_{\nu,*}\) is said to be inducible if there exists an automorphism \( \gamma \in \text{Aut}(E) \) such that \((\gamma_H, \gamma_I) = (\phi, \theta)\). Theorem 5.7 tells that a pair \((\phi, \theta) \in C_{\nu,*}\) is inducible if and only if \((\phi, \theta) \in \text{Ker}(\omega(E))\). When \( H_N^2(H, I) = 0 \), as a consequence of Theorem 5.7, we get the following

**Corollary 5.8.** Let \( E : 0 \rightarrow I \rightarrow E \xrightarrow{\nu} H \) be a extension of a left brace \( H \) by a trivial brace \( I \) such that \([E] \in \text{Ext}_{\nu,*}(H, I)\) and \( H_N^2(H, I) = 0 \). Then we have the following exact sequence of groups
\[
0 \rightarrow Z_N^1(H, I) \rightarrow \text{Aut}_I(E) \xrightarrow{\rho(E)} C_{\nu,*} \xrightarrow{\omega(E)} 0.
\]
Hence every pair \((\phi, \theta) \in C_{\nu,*}\) is inducible.

We remark that when \((\nu, \sigma)\) is a trivial action, then \( C_{\nu,*} = \text{Aut}(H) \times \text{Aut}(I) \). So, if \( H_N^2(H, I) = 0 \), then every pair \((\phi, \theta) \in \text{Aut}(H) \times \text{Aut}(I)\) is inducible.

We now present a module theoretic interpretation of inducible pairs. Let \( I \) and \( I' \) be two \( H \)-bi-modules with good pairs of actions \((\nu, \sigma)\) and \((\nu', \sigma')\) respectively. Recall that a map \( \zeta : I \rightarrow I' \) is called \( H \)-bi-module homomorphism if \((\text{Id}, \zeta)\) is compatible (as defined in Section 2) with the pairs of actions \((\nu, \sigma)\) and \((\nu', \sigma')\).

Let \( E : 0 \rightarrow I \rightarrow E \rightarrow H \rightarrow 0 \) be an extension of a left brace \( H \) by a trivial brace \( I \). Then \( I \) can be viewed as an \( H \)-bi-module through the corresponding good pair of actions \((\nu, \sigma)\) as defined in (4) and (5). An automorphism \( \phi \) of the brace \( H \) defines a new \( H \)-bi-module structure on \( I \) given by \((\nu \phi, \sigma \phi)\), which we denote by \( I_\phi \), where \( \nu \phi(h) = \nu(h) \) and \( \sigma \phi(h)(y) = s(\phi(h))^{-1} y s(\phi(h)) \) for \( h \in H \) and \( y \in I \). It is not difficult to show that the automorphism \( \phi \) induces an isomorphism \( \phi^* \) of cohomology groups \( \phi^* : H^2_N(H, I) \rightarrow H^2_N(H, I_\phi) \) defined by
\[
\phi^*([(\beta, \tau)]) = [(\beta(\phi, \text{Id}), \tau(\phi, \text{Id}))].
\]
Further, any \( H \)-bi-module isomorphism \( \theta : I \rightarrow I_\phi \) induces an isomorphism \( \theta^* \) of cohomology groups \( \theta^* : H^2_N(H, I) \rightarrow H^2_N(H, I_\phi) \) given by
\[
\theta^*([(\beta, \tau)]) = [(\beta(\text{Id}, \theta^{-1}), \tau(\text{Id}, \theta^{-1}))].
\]

With this set-up, we have

**Theorem 5.9.** A pair of automorphisms \((\phi, \theta) \in \text{Aut}(H) \times \text{Aut}(I)\) is inducible if and only if the following conditions hold:

(i) \( \theta : I \rightarrow I_\phi \) is an isomorphism of \( H \)-bi-modules.

(ii) \( \theta^*([(\beta, \tau)]) = \phi^*([(\beta, \tau)]) \).
Proof. Let \((\phi, \theta)\) be inducible. Then there exists an automorphism \(\gamma \in \text{Aut}b_1(E)\) such that \((\phi, \theta) = (\gamma_H, \gamma_I).\) For \(h \in H\) and \(y \in I,\) we know that \(\gamma((s(h)) = s(\gamma_H(h)) \circ y_h\) for some \(y_h \in I.\) We have

\[
\gamma_I(\nu_h(y)) = \gamma(s(h) \circ y - s(h)) = \gamma(s(h)) \circ \gamma(y) - \gamma(s(y)) = I_H(h) \circ y_h \circ \gamma(y) - s(\gamma_H(h)) \circ y_h = s(\gamma_H(h)) \circ \gamma_I(y) - s(\gamma_H(h))
\]

Similarly it also follows that

\[
\gamma_I(\sigma_h(y)) = \sigma_{\gamma_H(h)}(\gamma_I(y)).
\]

This shows that \((\text{Id}, \gamma_I)\) is compatible with the pairs of actions \((\nu, \sigma)\) and \((\nu \gamma_H, \sigma \gamma_I);\) hence condition (i) holds.

For each \(h \in H,\) there exists a unique element (say) \(\lambda(h)\) in \(I\) such that \(\gamma(s(h)) = s(\gamma_H(h)) + \lambda(h).\) It turns out that \(\lambda : H \to I\) given by \(h \mapsto \lambda(h)\) lies in \(C^*_N.\) Given elements \(x_1, x_2 \in E\) have unique expressions of the form \(x_1 = s(h_1) + y_1\) and \(x_2 = s(h_2) + y_2\) for some \(h_1, h_2 \in H\) and \(y_1\) and \(y_2 \in I.\) Now

\[
\gamma(x_1 + x_2) = \gamma(s(h_1) + s(h_2) + \beta(h_1, h_2) + y_1 + y_2)
\]

On the other hand, we have

\[
\gamma(x_1) + \gamma(x_2) = s(\gamma_H(h_1)) + s(\gamma_H(h_2)) + \lambda(h_1) + \lambda(h_2) + \gamma_I(y_1 + y_2).
\]

Since \(\gamma(x_1 + x_2) = \gamma(x_1) + \gamma(x_2),\) preceding two equations give

\[
\gamma_I(\beta(h_1, h_2)) - \beta(\gamma_H(h_1), \gamma_H(h_1)) = \lambda(h_1) - \lambda(h_1 + h_2) + \lambda(h_2).
\]

Notice that \(s(h) + \lambda(h) = s(h) \circ \nu_h^{-1}(\lambda(h))\) and \(s(h) \circ \lambda(h) = s(h) + \nu_h(\lambda(h)).\) By this and using \((18)\) and \((19),\) we have

\[
\gamma(x_1 \circ x_2) = \gamma((s(h_1) + y_1) \circ ((s(h_2) + y_2)) = \gamma(s(h_1 + h_2) \circ (\nu^{-1}_{h_1, h_2}(\sigma_{\nu^{-1}_{h_1}}(y_1))) + \nu^{-1}_{h_2, h_2}(y_2)) = s(\gamma_H(h_1) + \gamma_H(h_2)) + \lambda(h_1 + h_2) + \gamma_I(\nu_{h_1}(y_2) + \gamma_I(\nu_{h_2}(y_2)) + \gamma_I(h_1, h_2)) = s(\gamma_H(h_1) + \gamma_H(h_2)) + \lambda(h_1 + h_2) + \nu_{\gamma:h_1, h_2}((\nu^{-1}_{\gamma:h_1}(y_1))) + \nu^{-1}_{\gamma:h_2}(\nu_{\gamma:h_2}(y_2)))
\]

By similar computations, on the other hand, we get

\[
\gamma(x_1) \circ \gamma(x_2) = s(\gamma_H(h_1) \circ h_2) + \nu_{\gamma:h_1, h_2}(\sigma_{\nu_{\gamma:h_1}}(\nu^{-1}_{\gamma:h_1}(y_1))) + \nu^{-1}_{\gamma:h_1}(y_1)\]

Preceding two equations give

\[
\gamma_I(\tau(h_1, h_2)) - \tau(\gamma_H(h_1), \gamma_H(h_2)) = \nu_{\gamma:h_1, h_2}(\sigma_{\nu_{\gamma:h_1}}(\nu^{-1}_{\gamma:h_1}(\lambda(h_1)))) - \lambda(h_1 \circ h_2) + \nu_{\gamma:h_1}(\lambda(h_2)),
\]

which, alongwith \((20)\) proves condition (ii), that is, \(\gamma^*_I([\beta, \tau]) = \gamma^*_I([\beta, \tau])\).

Conversely, let \((\phi, \theta) \in \text{Aut}b(H) \times \text{Aut}b(I)\) satisfy conditions (i) and (ii). Condition (ii) guarantees the existence of a map \(\lambda : H \to I\) in \(C^*_N\) such that

\[
(\beta^{(\phi, \text{Id})}, \tau^{(\phi, \text{Id})}) - (\beta^{(\text{Id}, \theta^{-1})}, \tau^{(\text{Id}, \theta^{-1})}) = \partial^1(\lambda).
\]
Define the map \( \gamma : E \to E \), for all \( h \in H \) and \( y \in I \), by

\[
\gamma(s(h) + y) = s(\phi(h)) + \lambda(h) + \theta(y).
\]

A routine check then shows that \( \gamma \in \text{Aut}_b(I) \), \( \gamma_H = \phi \) and \( \gamma_I = \theta \). The proof is now complete. \( \square \)

We conclude with a reduction argument on extension and lifting problem for automorphisms of braces. We’ll only deal with brace extensions from \( \text{STExt}_{\nu, \sigma}(H, I) \), with \( H \) of finite order, for a given good pair of actions \( (\nu, \sigma) \) of \( H \) on \( I \). Let us start with the following fixed extension of a finite left brace \( H \) by a trivial brace \( I \) lying in \( \text{STExt}_{\nu, \sigma}(H, I) \):

\[
\mathcal{E} : 0 \to I \to E \to H \to 0.
\]

For this extension \( \mathcal{E} \), by Theorem 5.6, we get the exact sequence of groups

\[
0 \to \text{Aut}^H(I) \to \text{Aut}_b(I) \xrightarrow{\rho(\mathcal{E})} C_{(\nu, \sigma)} \xrightarrow{\omega(\mathcal{E})} \text{RH}^2_N(H, I),
\]

where \( \omega(\mathcal{E}) \) is a derivation.

Let \( P_i \) denote a Sylow \( p_i \)-subgroup of \((H, \phi)\), where \( p_i \) is a prime divisor of the order of \( H \). Notice that \( P_i \) is also the Sylow \( p_i \)-subgroup of \((H, \circ)\); hence \( P_i \) becomes a left ideal of \( H \). Let \( R_i \) denote the \( P_i \)-pre-image of \( \nu \) in \( E \). Notice that \( (\nu^i, \sigma^i) \) is a good pair of actions of \( P_i \) on \( I \), where \( \nu^i := \nu|_{P_i} \) and \( \sigma^i := \sigma|_{P_i} \), and the extension

\[
\mathcal{E}_i : 0 \to I \to R_i \to E \to H \to 0
\]

lies in \( \text{STExt}_{\nu^i, \sigma^i}(H, I) \). Set

\[
C_{(\nu^i, \sigma^i)} := \{(\phi, \theta) \in \text{Aut}(P_i) \times \text{Aut}(I) \mid \nu^i_h = \theta^{-1}\nu^i_{\phi(h)}\theta \text{ and } \sigma^i_h = \theta^{-1}\sigma^i_{\phi(h)}\theta\}.
\]

The extension \( \mathcal{E}_i \) now gives the exact sequence of groups

\[
0 \to \text{Aut}^{P_i}\mathcal{I}(R_i) \to \text{Aut}_b(R_i) \xrightarrow{\rho(\mathcal{E}_i)} C_{(\nu^i, \sigma^i)} \xrightarrow{\omega(\mathcal{E}_i)} \text{RH}^2_N(P_i, I).
\]

On the other hand, denote by \( \text{Aut}_{b, R_i}(E, I) \) the subgroup of \( \text{Aut}_b(H, I) \) consisting of all automorphism of \( E \) normalising \( I \) as well as \( R_i \), and set \( C_i^{(\nu, \sigma)} := \{(\phi, \theta) \in C_{(\nu, \sigma)} \mid \phi(P_i) = P_i\} \). Then for the extension

\[
\mathcal{E} : 0 \to I \to E \to H \to 0,
\]

we get the following exact sequence of groups from (21):

\[
0 \to \text{Aut}^{H, I}(E) \to \text{Aut}_{b, R_i}(E) \xrightarrow{\rho(\mathcal{E}_i)} C_i^{(\nu, \sigma)} \xrightarrow{\omega(\mathcal{E}_i)} \text{RH}^2_N(P_i, I).
\]

Let \( \text{res}^H_{P_i} : \text{RH}^2_N(H, I) \to \text{RH}^2_N(P_i, I) \) be the restriction homomorphisms as defined in Corollary 1.5. Define \( r^H_{P_i} : C_i^{(\nu, \sigma)} \to C_i^{(\nu^i, \sigma^i)} \) by

\[
r^H_{P_i}(\phi, \theta) = (\phi|_{P_i}, \theta).
\]

Using the definition of \( \omega(\mathcal{E}_i) \), we now get the following commutative diagram:

\[
\begin{array}{ccc}
C_i^{(\nu, \sigma)} & \xrightarrow{\omega(\mathcal{E}_i)} & \text{RH}^2_N(H, I) \\
\downarrow r^H_{P_i} & & \downarrow \text{res}^H_{P_i} \\
C_i^{(\nu^i, \sigma^i)} & \xrightarrow{\omega(\mathcal{E}_i)} & \text{RH}^2_N(P_i, I).
\end{array}
\]

Recall that \( \text{RH}^2_N((H, \circ), I) \) denotes the second cohomology group of the group \((H, \circ)\) with coefficients in \( I \), where the right action of \( H \) on \( I \) is through \( \sigma \). Similarly \( \text{RH}^2_N((P_i, \circ), I) \) denotes the second cohomology group of \((P_i, \circ)\) with coefficients in \( I \), where the right action of \( P_i \) on \( I \) is through \( \sigma^i = \sigma|_{P_i} \). By Proposition
there exist embeddings \( \iota : RH^2_N(H, I) \to H^2_N((H, \circ), I) \) and \( \iota_1 : RH^2_N(P_i, I) \to H^2_N((P_i, \circ), I) \). We now get the following commutative diagram:

\[
\begin{array}{ccl}
RH^2_N(H, I) & \xrightarrow{\iota} & H^2_N((H, \circ), I) \\
\downarrow{\text{res}_P} & & \downarrow{\text{res}_P} \\
RH^2_N(P_i, I) & \xrightarrow{\iota_1} & H^2_N((P_i, \circ), I).
\end{array}
\]

(23)

Let \( \pi(H) := \{p_1, \ldots, p_r\} \) be the set of all distinct prime divisors of the order of \( H \). Please note the ad-hoc use of \( \pi \). With this set-up, we finally have

**Theorem 5.10.** Let \( (\phi, \theta) \in C_{(\nu, \sigma)} \) be such that \( (\phi|_{P_i}, \theta) \in C_{(\nu^i, \sigma^i)} \) is inducible for some Sylow \( p_i \)-subgroup of \( H \) for each \( p_i \in \pi(H) \). Then \( (\phi, \theta) \) is inducible.

Conversely, if \( (\phi, \theta) \in C_{(\nu, \sigma)} \) is inducible, then \( (\phi|_{P_i}, \theta) \in C_{(\nu^i, \sigma^i)} \) is inducible whenever \( \phi(P_i) = P_i \).

**Proof.** Let \( (\phi, \theta) \in C_{(\nu, \sigma)} \) be as in the statement and \( \omega(\mathcal{E})(\phi, \theta) = [f] \). Then by the hypothesis and the commutativity of diagram (22), it follows that \( \text{res}_P^H([f]) \) is the identity element of \( RH^2_N(P_i, I) \). Further, by the commutativity of diagram (23), we have \( \text{res}_P^H([f']) \) is the identity element of \( H^2_N((P_i, \circ), I) \). Being in the cohomology of groups, we can now use corestriction-restriction homomorphism result [6] (9.5) Proposition. (ii) to deduce that \( k[f'] \) is the zero cohomology class of \( H^2_N((H, \circ), I) \), where \( k \) is the index of \( P_i \) in \( H \). Since this is true for at least one Sylow \( p_i \)-subgroup for each \( p_i \in \pi(H) \), it follows that \( [f] \) is the identity element of \( H^2_N((H, \circ), I) \), which, \( \iota \) being an embedding, implies that \( [f] \) is the zero cohomology class of \( H^2_N(H, I) \). Hence \( (\phi, \theta) \) is inducible.

The converse part is left as an exercise for the reader. \( \square \)

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