On the Structure of the Nonlinear Vacuum Solutions in Extended Electrodynamics

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Abstract

In this paper, in the frame of Extended Electrodynamics (EED), we study some of the consequences that can be obtained from the introduced and used by Maxwell equations complex structure \( \mathcal{J} \) in the space of 2-forms on \( \mathbb{R}^4 \), and also used in EED. First we give the vacuum EED equations with some comments. Then we recall some facts about the invariance group \( G \) (with Lie algebra \( \mathfrak{g} \)) of the standard complex structure \( J \) in \( \mathbb{R}^2 \). After defining and briefly studying a representation of \( G \) in the space of 2-forms on \( \mathbb{R}^4 \) and the joint action of \( G \) in the space of the \( \mathcal{G} \)-valued 2-forms on \( \mathbb{R}^4 \) we consider its connection with the vacuum solutions of EED. Finally, we consider the case with point dependent group parameters and show that the set of the nonlinear vacuum EED-solutions is a disjoint union of orbits of the \( G \)-action, noting some similarities with the quantum mechanical eigen picture and with the QFT creation and annihilation operators.

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1 Introduction

Extended Electrodynamics (EED) [1] was brought to life in searching for appropriate ways to describe spatially-finite and time-stable electromagnetic field configurations propagating along a given direction in the 3-space. In view of the adequate enough to reality energy-momentum quantities and relations that follow from Maxwell theory, a basic requirement to the modification of Maxwell equations was to keep the same local energy-momentum quantities (i.e. the energy-momentum tensor) and relations. The 4-dimensional final form of the vacuum EED-equations was assumed to be

\[
dF \wedge *F = 0, \quad d *F \wedge F = 0, \quad F \wedge *d *F + *F \wedge *dF = 0, \tag{1}
\]

or in component form

\[
F^{\mu\nu}(dF)_{\mu\nu} = 0, \quad (dF)^{\mu\nu}(d *F)_{\mu\nu} = 0, \quad (dF)^{\mu\nu}(dF)_{\mu\nu} + F^{\mu\nu}(d *F)_{\mu\nu} = 0, \quad \mu < \nu,
\]

where \(*\) is the Hodge operator defined by the Minkowski metric \( \eta \) with signature \((-,-,-,+\)) on \( \mathbb{R}^4 \), and \( d \) is the exterior derivative.

We recall that Maxwell equations \( dF = 0, \quad d *F = 0 \) describe differentially the balance between the flows of the electric \( E \) and magnetic \( B \) vectors through finite closed and not-closed

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2-surfaces. This approach to description of the propagation of electromagnetic (EM) fields has, probably, been suggested by analogy with hydrodynamics. The substantial difference, however, is in, that in hydrodynamics the propagating vector quantities have a direct physical sense of mass-energy and momentum densities, which are considered in physics as universal conserved quantities, and this property allows to write down corresponding differential (local) energy-momentum balance relations. As we noted, in electrodynamics the flows of $E$ and $B$ have not a direct physical meaning of energy-momentum flows. This feature of the Maxwell approach to describe the field dynamics resulted in establishing the linear character of the equations for $F_{\mu\nu}$, and the D’Alembert linear evolution equation (i.e. $\Box U = 0$) for each of the field components appeared as a necessary condition. As a consequence, this lead to the impossibility to describe appropriately by exact solutions to the vacuum Maxwell equations finite portions of vacuum EM-radiation with soliton-like properties.

The 20th century physics proved, however, that the EM-radiation consists of (almost) non-interacting photons, which we consider as finite time-stable objects with soliton-like behavior, so, 3d soliton-like portions of EM-radiation are possible to be created experimentally. The soliton-like evolution of these vacuum electromagnetic 3d soliton-like configurations should have a consistent description even at non-quantum level. In an attempt to achieve this EED extended Maxwell equations to the nonlinear equations (1). In contrast to the Maxwell approach, equations (1) have a direct physical sense of local energy-momentum balance relations. Moreover, EED succeeded to keep the same basic energy-momentum expressions and relations from Maxwell theory, and at the same time it extends seriously the class of solutions. So that, all vacuum solutions to Maxwell equations are solutions to (1), but our equations (1) have more solutions satisfying the relations $dF \neq 0$, $d*F \neq 0$. These new solutions to (1) will be called further nonlinear, and will be the subject of study from a definite point of view. We begin with recalling the following result of EED specifying some of their properties.

For every nonlinear solution of (1) there exists a canonical coordinate system $(x, y, z, \xi = ct)$ in the Minkowski space-time in which the solution is fully represented by two functions: $\phi(x, y, \xi + \varepsilon z)$, $\varepsilon = \pm 1$, called amplitude function, and $\varphi(x, y, z, \xi), |\varphi| \leq 1$, called phase function, in the following way:

$$F = \varepsilon \phi \varphi dx \wedge dz + \varepsilon \phi \sqrt{1 - \varphi^2} dy \wedge dz + \phi \varphi dx \wedge d\xi + \phi \sqrt{1 - \varphi^2} dy \wedge d\xi. \quad (2)$$

Moreover, every nonlinear solution satisfies also:

$$(\delta F)^2 = (\delta * F)^2 < 0, \quad (\delta F)_{\sigma} (\delta * F)^{\sigma} = 0, \quad F_{\mu\nu} F^{\mu\nu} = F_{\mu\nu} (\ast F)^{\mu\nu} = 0.$$

where $\delta = \ast d \ast$ is the coderivative.

We recall that in this case the energy tensor

$$Q^\mu_{\nu} = -\frac{1}{4\pi} [(F_{\mu\sigma} F^{\nu\sigma} + (\ast F)_{\mu\sigma}(\ast F)^{\nu\sigma}]$$

has just one isotropic eigen direction [2], which is eigen direction for $F$ and $\ast F$ too, and in this coordinate system it is defined by the vector field $\zeta = -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}$.

In order to come to the nonlinear equations (1) EED considers the fields $(F, \ast F)$ as two components of a differential 2-form $\Omega = F \otimes e_1 + \ast F \otimes e_2$ on $\mathbb{R}^4$ with values in a 2-dimensional real vector space $V$, which as a vector space can be identified with $\mathbb{R}^2$. But a further study of the duality symmetry of the equations suggested this vector space $V$ to be identified with the Lie algebra of the corresponding Lie group of matrices of the kind

$$G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right|, \quad a, b \in \mathbb{R} \right\}, \quad (3)$$
which represents the dual symmetry of Maxwell vacuum equations. And further we consider what consequences can be made from this suggestion.

2 Some facts concerning the group $G$

The group $G$ (3) is a commutative 2-dimensional Lie group with respect to the usual product of matrices. In fact

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} m & -n \\ n & m \end{bmatrix} = \begin{bmatrix} am - bn & -(an + bm) \\ an + bm & am - bn \end{bmatrix}.$$

In addition, $G$ is a real 2-dimensional vector space with respect to the usual addition of matrices. A natural basis of the vector space $G$ is given by the two matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

Since the Lie group $G$ is commutative, its Lie algebra $\mathcal{G}$ is trivial, and as a vector space it coincides with the vector space $G$. Hence, every element $\alpha$ of $G$ and $\mathcal{G}$ can be represented as $\alpha = aI + bJ$, $a, b \in \mathbb{R}$. We have $J.J = -I$, so, $J$ defines a complex structure in $\mathcal{G}$. If $\alpha \in G$ is given by $\alpha = aI + bJ$, then

$$\alpha.J = -bI + aJ, \quad \alpha^{-1} = \frac{1}{a^2 + b^2}(aI - bJ), \quad \alpha^{-1}.J = \frac{1}{a^2 + b^2}(bI + aJ), \quad \det(\alpha) = a^2 + b^2.$$

The product of two matrices $\alpha = aI + bJ$ and $\beta = mI + nJ$ looks like $\alpha,\beta = (am - bn)I + (an + bm)J$. The commutativity of $G$ means symmetry, in particular, every $\alpha \in G$ is a symmetry of $J$:

$$\alpha.J = J.\alpha.$$

One sees that $G$ is the well known real matrix representation of the multiplicative group of the complex numbers, the complex conjugation is given by $\overline{\alpha} = aI - bJ$, and further we shall write $\alpha = aI + \varepsilon bJ$, $\varepsilon = \pm 1$.

3 The action of $G$ in the space of 2-forms on $\mathbb{R}^4$

We consider now the space $\mathbb{R}^4$ as a manifold, and denote by $(x^1, x^2, x^3, x^4 = x, y, z, \xi)$, where $\xi = ct$, the canonical coordinates of $\mathbb{R}^4$, and by $\Lambda^2(\mathbb{R}^4)$ the space of 2-forms on $\mathbb{R}^4$. As in [3] we introduce the following basis of $\Lambda^2(\mathbb{R}^4)$:

$$dx \wedge dy, \quad dx \wedge dz, \quad dy \wedge dz, \quad dx \wedge d\xi, \quad dy \wedge d\xi, \quad dz \wedge d\xi.$$

We recall from [3] that Maxwell equations $dF = 0$, $dJF = 0$ introduce just the complex structure $J$ in the space $\Lambda^2(\mathbb{R}^4)$, and $J$ acts on the above basis as follows:

$$J(dx \wedge dy) = -dz \wedge d\xi \quad J(dx \wedge dz) = dy \wedge d\xi \quad J(dy \wedge dz) = -dx \wedge d\xi \quad J(dx \wedge d\xi) = dy \wedge dz \quad J(dy \wedge d\xi) = -dx \wedge dz \quad J(dz \wedge d\xi) = dx \wedge dy.$$

Hence, in this basis the matrix of $J$ is off-diagonal with alternatively ordered $(-1, 1, -1, 1, -1, 1)$.

Let now $I$ is the identity map in $\Lambda^2(\mathbb{R}^4)$. We define a representation $\rho$ of $G$ in $\Lambda^2(\mathbb{R}^4)$ as follows:

$$\rho(\alpha) = \rho(aI + \varepsilon bJ) = aI + \varepsilon bJ.$$

So, the map $\rho$ is a linear map, it sends the elements of the linear space $G$ to the space of linear maps $L_{\Lambda^2(\mathbb{R}^4)}$ of $\Lambda^2(\mathbb{R}^4)$, so that every $\rho(\alpha)$ is a linear isomorphism, in fact, its determinant
\[ \det|\rho(\alpha)| \text{ is equal to } (a^2 + b^2)^3. \] The identity \( I \) of \( G \) is sent to the identity \( \mathcal{I} \) of \( \Lambda^2(\mathbb{R}^4) \), and the complex structure \( J \) of the vector space \( G \) is sent to the complex structure \( \mathcal{J} \) of \( \Lambda^2(\mathbb{R}^4) \). This map is surely a representation, because \( \rho(\alpha, \beta) = \rho(\alpha) \cdot \rho(\beta) \). In fact,

\[ \rho(\alpha, \beta) = \rho[(aI + \varepsilon bJ). (mI + \varepsilon nJ)] = \]

\[ \rho[(am - bn)I + \varepsilon(an + bm)J] = (am - bn)I + \varepsilon(an + bm)\mathcal{J}. \]

On the other hand

\[ \rho(\alpha) \cdot \rho(\beta) = \rho(aI + \varepsilon bJ). \rho(mI + \varepsilon nJ) \]

\[ = (aI + \varepsilon b\mathcal{J}). (mI + \varepsilon n\mathcal{J}) = (am - bn)I + \varepsilon(an + bm)\mathcal{J}. \]

We consider now the space \( \Lambda^2(\mathbb{R}^4, G) \) of \( G \)-valued 2-forms on \( \mathbb{R}^4 \). Every such 2-form \( \Omega \) can be represented as \( \Omega = F_1 \otimes I + F_2 \otimes J \), where \( F_1 \) and \( F_2 \) are 2-forms. We have the joint action of \( G \) in \( \Lambda^2(\mathbb{R}^4, G) \) as follows:

\[ [\rho(\alpha) \otimes \alpha], \Omega = \rho(\alpha). F_1 \otimes \alpha . I + \rho(\alpha). F_2 \otimes \alpha . J. \]

We obtain

\[ [\rho(\alpha) \otimes \alpha], \Omega = \]

\[ [(a^2 I + \varepsilon ab\mathcal{J})F_1 - (b^2 J + \varepsilon abI)F_2] \otimes I + [(b^2 J + \varepsilon abI)F_1 + (a^2 I + \varepsilon ab\mathcal{J})F_2] \otimes J. \]

In the special case \( \Omega = F \otimes I + J.F \otimes J \) it readily follows that

\[ [\rho(\alpha) \otimes \alpha], \Omega = (a^2 + b^2)\Omega. \] (5)

In this sense the forms \( \Omega = F \otimes I + J.F \otimes J \) may be called conformally equivariant with respect to the action of the group \( G \), and equivariant with respect to the subgroup \( SO(2) \subset G \), i.e. when \( a^2 + b^2 = 1 \).

**Remark.** The two partial actions will be denoted further by \( \rho(\alpha). \Omega \) and \( \alpha. \Omega \). So, the above conformal equivariance property may be written as

\[ \rho(\alpha). \Omega = \alpha^{-1}. (a^2 + b^2)\Omega = (aI - \varepsilon bJ). \Omega = (aF + \varepsilon bJF) \otimes I + J(aF + \varepsilon bJF) \otimes J \]

### 4 The action of \( G \) and the nonlinear solutions

We go back now to equations (1). We shall show that if \( \Omega = F \otimes I + J.F \otimes J \) is a solution, then \( \rho(\alpha). \Omega \) is also a solution. In order to do this in a coordinate free manner we shall represent equations (1) in a different form independent on the Minkowski metric. First we recall the general rule for multiplying vector valued differential forms. Let \( \Phi = \alpha^i \otimes e_i, i = 1, 2, \ldots, m \) and \( \Psi = \beta^j \otimes k_j, j = 1, 2, \ldots, n \) be two \( p \) and \( q \) differential forms on a manifold \( M \) with values in the vector spaces \( V^m \) and \( V^n \) with bases \( \{e_i\} \) and \( \{k_j\} \) respectively. Let \( f_1 : (\Lambda^p(M), \Lambda^q(M)) \rightarrow \Lambda^r(M) \) be a bilinear map, and \( f_2 : (V^m, V^n) \rightarrow V^s \) be a bilinear map. Now we can form the expression \( f_1(\alpha^i, \beta^j) \otimes f_2(e_i, k_j) \in \Lambda^r(M, V^s) \) which obviously is a \( V^s \) valued \( r \)-form on \( M \), and this last form we call a \( (f_1, f_2) \)-product of \( \Phi \) and \( \Psi \).
Let now $M \equiv \mathbb{R}^4$. We recall now the Poincaré isomorphism $\mathcal{P}$ between the 2-forms and the 2-vectors built by making use of the canonical volume forms $\omega = \partial_x \wedge \partial_y \wedge \partial_z \wedge \partial_\xi$ and $\omega^* = dx \wedge dy \wedge dz \wedge d\xi$:

$$\mathcal{P}(\partial_{x_i} \wedge \partial_{x_j}) = i(\partial_{x_i}) \circ i(\partial_{x_j}) \omega^*,$$

where $i(X)$ is the substitution operator induced by the vector field $X$. We make the composition $\mathcal{D} = -\mathcal{P} \circ \mathcal{J}$, which is also an isomorphism. So, if $\alpha$ and $\beta$ are $p$ and $q$ forms with $p \leq q$ we can form the $(q-p)$-form $i(\mathcal{D}\alpha)\beta$. Further, recalling our $\mathcal{G}$-valued 2-form $\Omega$, we choose the symmetrized tensor product $\vee$ for a bilinear map: $\vee : \mathcal{G} \times \mathcal{G} \to (\mathcal{G} \vee \mathcal{G})$. We are ready now to compute $(\vee, i)(\Omega, d\Omega)$; we obtain

$$(\vee, i)(\Omega, d\Omega) = [i(\mathcal{D}F)dF] \otimes I \vee I + [i(\mathcal{D}(JF))dJF] \otimes J \vee J + [i(\mathcal{D}F)dJF + i(\mathcal{D}(JF))dF] \otimes I \vee J.$$

Clearly, the equation $(\vee, i)(\Omega, d\Omega) = 0$ gives equations (1), where the Hodge * is replaced with the complex structure $J$.

Let now $\alpha = (aI + \varepsilon bJ) \in G$. We have $\rho(\alpha) \Omega = \rho(\alpha)F \otimes I + \rho(\alpha)JF \otimes J$. We shall show that if $\Omega$ is a solution then $\rho(\alpha)\Omega$ is also a solution. In fact,

$$i(\mathcal{D}(\rho(\alpha)F))d\rho(\alpha)F = a^2 i(\mathcal{D}F)dF + b^2 i(\mathcal{D}JF)dJF + \varepsilon ab [i(\mathcal{D}F)dJF + i(\mathcal{D}JF)dF].$$

It is seen that if $F$ satisfies (1) then the right side of the above relation is equal to zero since it is a polynomial of the numbers $(a, b)$ with coefficients equal to the three expressions staying in front of the basis vectors $I \vee I, J \vee J$ and $I \vee J$ of $(\vee, i)(\Omega, d\Omega)$. Similar expressions are obtained for $i(\mathcal{D}(\rho(\alpha)JF))d\rho(\alpha)JF$ and for $i(\mathcal{D}(\rho(\alpha)F))d\rho(\alpha)JF + i(\mathcal{D}(\rho(\alpha)JF))d\rho(\alpha)F$. Thus, $\rho(\alpha)\Omega$ is also a solution. Moreover, since $a$ and $b$ are constants, $(\rho(\alpha) \otimes \alpha)\Omega = (a^2 + b^2)\Omega$ is also a solution. Hence, the above defined action of $G$ defines a symmetry in the set of solutions to (1).

## 5 Point dependent group parameters

We are going now to see what happens if the group parameters become functions of the coordinates: $a = a(x, y, z, \xi), \ b = (x, y, z, \xi)$. For convenience, we shall write $\phi, \varphi \equiv u$ and $\phi, \sqrt{1-\varphi^2} \equiv p$. Then (2) becomes

$$F = \varepsilon udx \wedge dz + \varepsilon pdy \wedge dz + udx \wedge d\xi + pdy \wedge d\xi. \quad (6)$$

Now, it can be readily checked [4] that $F$ will satisfy (1) iff in the corresponding coordinate system $u$ and $p$ will satisfy the equation

$$L_\zeta \left[ det||\alpha(u, p)|| \right] = (u^2 + p^2)\xi - \varepsilon(u^2 + p^2)_z = 0 \quad (7)$$

where $L_\zeta$ is the Lie derivative with respect to the intrinsically defined vector field $\zeta$.

Consider now the 2-form

$$F_0 = \varepsilon dx \wedge dz + dx \wedge d\xi,$$

which, obviously, defines a (linear constant) solution, and define a map $f : \mathbb{R}^4 \to G$ as follows:

$$\alpha(x, y, z, \xi) = f(x, y, z, \xi) = u(x, y, z, \xi)I + \varepsilon p(x, y, z, \xi)J,$$
where the two functions \( u \) and \( p \) satisfy (7). Consider now the action of
\[
\rho(\alpha(x, y, z, \xi)) = u(x, y, z, \xi)I + \varepsilon p(x, y, z, \xi)J
\]
on \( F_o \). We obtain exactly \( F \) as given by (6), i.e. the (linear) solution \( F_o \) is transformed to a nonlinear solution. This suggests to check if this is true in general, i.e. if we have a solution (6) of (1) defined by the two functions \( u \) and \( p \), and we consider a map \( \alpha : \mathbb{R}^4 \to G \), such that the components \( a(x, y, z, \xi) \) and \( b(x, y, z, \xi) \) of \( \alpha = a(x, y, z, \xi)I + \varepsilon b(x, y, z, \xi)J \) satisfy (7), then whether the 2-form \( \tilde{F} = \alpha(x, y, z, \xi).F = [\alpha(x, y, z, \xi)I + \varepsilon b(x, y, z, \xi)J].F \) will satisfy (1)?

For \( \tilde{F} \) we obtain
\[
\tilde{F} = \rho(\alpha).F = (aI + \varepsilon bJ).F =
\]
\[
\varepsilon(au - bp)dx \wedge dz + \varepsilon(ap + bu)dy \wedge dz + (au - bp)dx \wedge d\xi + (ap + bu)dy \wedge d\xi.
\]
Now, \( \tilde{F} \) will satisfy (1) iff
\[
[(au - bp)^2 + (ap + bu)^2]z - \varepsilon[(au - bp)^2 + (ap + bu)^2]z = 0.
\]
This relation is equivalent to
\[
[(a^2 + b^2)z - \varepsilon(a^2 + b^2)z](u^2 + p^2) + [(u^2 + p^2)z - \varepsilon(u^2 + p^2)z](a^2 + b^2) = 0.
\]
This shows that if \( F(u, p) \) is a solution to (1), then \( \tilde{F}(u, p; a, b) = \rho(\alpha(a, b)).F(u, p) \) will be a solution to (1) iff the two functions \( (a, b) \) satisfy (7), i.e. iff \( \rho(\alpha(a, b)).F_o \) is a solution to (1). In other words: every nonlinear solution of (1), \( F(a, b) = [aI + \varepsilon bJ]F_o \), defines a map \( \Phi(a, b) : F(u, p) \to (\Phi F)(u, p; a, b) \), such that if \( F(u, p) \) is a solution to (1) then \( (\Phi F)(u, p; a, b) \) is also a solution to (1).

### 6 Interpretations

The above considerations allow a structural interpretation of the set of nonlinear solutions to (1). The whole set of nonlinear solutions to (1) divides to subclasses \( S(\tilde{r}) \) of the kind (6) (every such subclass is determined by the spatial direction \( \tilde{r} \) along which the solution propagates [4], it is the coordinate \( z \) in our consideration). Every solution \( F \) of a given subclass is obtained by means of the action of a solution of equation (7) on the corresponding \( F_o \), \( F(u, p) = \rho(\alpha(u, p)).F_o \). If \( \alpha(u, p) \neq 0 \) at some point of \( \mathbb{R}^4 \) then we have \( F^{-1} = \rho(\alpha^{-1}).F_o \) given by
\[
(F)^{-1} = \left(\frac{u}{u^2 + p^2}I - \varepsilon \frac{p}{u^2 + p^2}J\right).F_o.
\]
In view of these remarks equation (5) acquires a natural interpretation of eigen equation for the operator \( \rho(\alpha) \otimes \alpha \): every \( \Omega \) in a given subclass is an eigen state for any \( \rho(\alpha) \otimes \alpha \) since all states \( \Omega \) of this subclass describe propagation along the same spatial direction, or along the 4-dimensional direction defined by the only isotropic eigen vector \( \zeta \) of the energy-momentum tensor, and the eigen value \( (a^2 + b^2) \) is just the energy density in the point dependent case.

Further, the relation
\[
\rho(\alpha).\Omega_o = \rho(\alpha).((F_o \otimes I + J).F_o \otimes J) = F \otimes I + J.F \otimes J = \Omega
\]
suggests to consider \( \Omega_o \), as a "vacuum state", and the action of \( \rho(\alpha) \) on \( \Omega_o \) as a "creation operator", provided \( \alpha(x, y, z, \xi) \neq 0, (x, y, z, \xi) \in D \subset \mathbb{R}^4 \), satisfies (7) in \( D \). Since \( \rho(\alpha).\rho(\alpha^{-1}).\Omega_o = \)}
we see that every such "creation operator" in $D$ may be interpreted also as an "annihilation operator" in $D$.

Clearly, every solution $\Omega = \rho(\alpha)\Omega_o$ may be represented in various ways in terms of other solutions to (7) in the same domain $D$, i.e. we have an example of a nonlinear "superposition". Also, making use of the trigonometric representation of $\alpha$, we see that every solution in a natural way acquires the characteristics amplitude $\phi = |\alpha| = \sqrt{a^2 + b^2}$ and phase $\psi = \arccos(\varphi)$ (see relation (2)). Moreover, every $F = \rho(\alpha)F_o$ generates $(F)^n = \rho(\alpha^n)F_o$ and $\sqrt{F} = \rho(\sqrt{\alpha})F_o$.

If we consider the trivial bundle $\mathbb{R}^4 \times G \to \mathbb{R}^4$ then every such subclass of nonlinear solutions is generated by a subclass of sections $\alpha(u,p)$ where $(u,p)$ satisfy equation (7). Explicitly, the solution is obtained in the form of $\alpha(x,y,z,\xi)F_o$. Hence, instead of a linear structure, every such subclass of nonlinear solutions appears rather as an "orbit" of an action upon $F_o$ of those sections satisfying equation (7).

Another look at the situation is to identify $G$ with the field of complex numbers $\mathbb{C}$ and to consider the trivial bundle $\mathbb{R}^4 \times \mathbb{C}$. Then, considering those nonsingular sections $Z = (a,b)$ of this bundle the squared modules $|Z|^2 = (a^2 + b^2)$ of which satisfy (7), we see that the point-wise multiplication of these sections sends two solution-sections $Z_1 = (a,b)$ and $Z_2 = (u,p)$ to the solution-section $Z_1Z_2 = (au - bp, ap + bu)$.

## 7 Conclusion

This paper clearly shows that, the EED vacuum equations, as well as Maxwell vacuum equations, do not necessarily use pseudoeuclidean metric structures, the basic object used in both systems of equations is the complex structure $J$. The whole set of nonlinear solutions to equations (1) divides to subclasses, and each such subclass is determined by the direction of propagation in the 3-space, or by the corresponding eigen vector $\zeta$. The elements of each subclass are obtained by means of the action on a specially chosen 2-form $F_o$ of those (point-dependent) elements of $G$ which satisfy equation (7), i.e. which have invariant with respect to $\zeta$ determinant. So, every two nonlinear solutions determine another nonlinear solution provided their domains have non-empty section, i.e. we have a map inside a given subclass. Every nonlinear solution $F(u,p)$ acts as a symmetry transformation in its subclass, and we have $F_1(a,b).F_2(u,p) = F_2(u,p).F_1(a,b)$. Clearly, this structure of the subclass may be considered as generated by the usual multiplication of the complex numbers.

If we consider spatially finite elements $\bar{\alpha}(x,y,z,\xi)$, satisfying (7), we obtain $(3+1)$ soliton-like "creation operators" $\rho(\bar{\alpha})$, which produce $(3+1)$ soliton-like solutions $\rho(\bar{\alpha}).\Omega_o$ through acting on the "vacuum" state $\Omega_o$. So, the two solitons $\rho(\bar{\alpha})F_o$ and $\rho(\bar{\alpha}^{-1})F_o$ kill one another. All of these solutions have, of course, well defined amplitudes and phases. If we have a subclass of such soliton-like solutions with mutually non-overlapping 3d domains we obtain a flow of photon-like objects.

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