A NOTE ON THE LAPLACE TRANSFORM OF THE SQUARE IN THE CIRCLE PROBLEM

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ABSTRACT. If $P(x)$ is the error term in the circle problem, then it is proved that

$$
\int_0^\infty P^2(x)e^{-x/T} \, dx = \frac{1}{4} \left( \frac{T}{\pi} \right)^{3/2} \sum_{n=1}^\infty r^2(n)n^{-3/2} - T + O_\varepsilon (T^3\varepsilon),
$$

improving the earlier result with exponent $\frac{5}{6}$ in the error term. The new bound is obtained by using results of F. Chamizo on the correlated sum $\sum_{n \leq x} r(n)r(n+h)$, where $r(n)$ is the number of representations of $n$ as a sum of two integer squares.

1. Introduction

Let $r(n) = \sum_{n=a^2+b^2} 1$ denote the number of representations of $n \in \mathbb{N}$ as a sum of two integer squares. Thus $\frac{1}{4} r(n)$ is multiplicative and

$$
(1.1) \quad r(n) = 4 \sum_{d|n} \chi(d),
$$

where $\chi(n)$ is the non-principal character mod 4. A classical problem, with a rich history, is the circle problem. It consists of the estimation of the function

$$
(1.2) \quad P(x) = \sum_{n \leq x}^\prime r(n) - \pi x + 1,
$$

1991 Mathematics Subject Classification. Primary 11N37; Secondary 44A10.

Key words and phrases. Circle problem, Laplace transform, additive problems.
where, as usual, \( \sum' \) means that the last term in the sum is to be halved if \( x \) is an integer. One can estimate \( P(x) \) pointwise and in the mean square sense. M.N. Huxley [3] proved that

\[
P(x) = O(x^{23/73} \log^c x) \quad (c > 0, \frac{23}{73} = 0, 3150684 \ldots),
\]

which is the last in a series of improvements by the estimation of intricate exponential sums. The mean square formula for \( P(x) \) is written in the form

\[
\int_0^X P^2(x) \, dx = \left( \frac{1}{3 \pi^2} \sum_{n=1}^{\infty} r^2(n) n^{-3/2} \right) X^{3/2} + Q(X),
\]

where \( Q(X) \) is considered as the error term. The best known bound is

\[
Q(X) = O(X \log^2 X),
\]

proved long ago by I. Kátai [9]. From (1.4) and (1.5) one deduces that

\[
P(X) = \Omega(X^{1/4}),
\]

where as usual \( f = \Omega(g) \) means that \( \lim_{x \to \infty} f(x)/g(x) \neq 0 \). The omega-result (1.6) favours the long standing conjecture that

\[
P(X) = O_\varepsilon(X^{4 + \varepsilon}),
\]

where \( \varepsilon \) denotes arbitrarily small positive numbers, not necessarily the same ones at each occurrence. A comparison of (1.3) and (1.7) shows that there is a big gap between the known and conjectured pointwise estimates for \( P(x) \).

A useful representation of \( P(x) \) is the classical formula

\[
P(x) = x^{1/2} \sum_{n=1}^{\infty} r(n)n^{-1/2} J_1(2\pi \sqrt{x/n}),
\]

due to G.H. Hardy [2], where \( J_1 \) is the customary Bessel function. The series in (1.8) is boundedly, but not absolutely convergent. This causes problems in practice, and one can use the truncated form

\[
P(x) = -\frac{x^{1/4}}{\pi} \sum_{n \leq N} r(n)n^{-3/4} \cos(2\pi \sqrt{x/n} + \frac{\pi}{4}) + O_\varepsilon(x^{\varepsilon} + x^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{2}}),
\]

which is valid for \( x \geq 2, 2 \leq N \leq x^A \), and \( A > 0 \) is any constant. Trivial estimation of the sum in (1.9) (with \( N = x^{1/3} \)) yields at once the bound \( P(x) \ll_\varepsilon x^{\frac{4}{5} + \varepsilon} \).

**Acknowledgement.** I wish to thank F. Chamizo and T. Meurman for valuable remarks.
2. THE LAPLACE TRANSFORM OF $P^2(x)$

The difficulties encountered in evaluating mean square integrals like the one in (1.4) are less pronounced when the integrand is multiplied by an appropriate smooth function. In [5] the Laplace transforms of $P^2(x)$ and $\Delta^2(x)$ were evaluated, when $s = 1/T \to 0+$, and

\begin{equation}
\Delta(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4}, \quad d(n) = \sum_{\delta | n} 1
\end{equation}

is the error term in the classical Dirichlet divisor problem ($\gamma$ is Euler’s constant). It was proved that

\begin{equation}
\int_0^\infty P^2(x)e^{-x/T} \, dx = \frac{1}{4} \left( \frac{T}{\pi} \right)^{3/2} \sum_{n=1}^\infty r^2(n)n^{-3/2} - T + O_\varepsilon(T^{\alpha+\varepsilon})
\end{equation}

and

\begin{equation}
\int_0^\infty \Delta^2(x)e^{-x/T} \, dx = \frac{1}{8} \left( \frac{T}{2\pi} \right)^{3/2} \sum_{n=1}^\infty d^2(n)n^{-3/2} + T(A_1 \log^2 T + A_2 \log T + A_3) + O_\varepsilon(T^{\beta+\varepsilon}).
\end{equation}

The $A_j$’s are suitable constants ($A_1 = -1/(4\pi^2)$), and the constants $\frac{1}{2} \leq \alpha < 1$ and $\frac{1}{2} \leq \beta < 1$ are defined by the asymptotic formula

\begin{equation}
\sum_{n \leq x} r(n)r(n+h) = \frac{(-1)^h8x}{h} \sum_{d|h}(-1)^dE(x,h), E(x,h) \ll_\varepsilon x^{\alpha+\varepsilon},
\end{equation}

\begin{equation}
\sum_{n \leq x} d(n)d(n+h) = x \sum_{i=0}^2 (\log x)^i \sum_{j=0}^2 c_{ij} \sum_{d|h} \left( \frac{\log d}{d} \right)^j + D(x,h), D(x,h) \ll_\varepsilon x^{\beta+\varepsilon}.
\end{equation}

The $c_{ij}$’s are certain absolute constants, and the $\ll_\varepsilon$-bounds both in (2.4) and in (2.5) should hold uniformly in $h$ for $1 \leq h \leq x^{1/2}$. With the values $\alpha = 5/6$ of D. Ismoilov [6] and $\beta = 2/3$ of Y. Motohashi [11] it followed then that (2.2) and (2.3) hold with $\alpha = 5/6$ and $\beta = 2/3$. Motohashi’s fundamental paper (op. cit.) used the powerful methods of spectral theory of the non-Euclidean Laplacian. A variant of this approach was used recently by T. Meurman [10] to sharpen Motohashi’s
bound for $D(x, h)$ for ‘large’ $h$, specifically for $x^{7/6} \leq h \leq x^{2-\varepsilon}$, but the limit of both methods is $\beta = 2/3$ in (2.5).

Although one expects, by general analogies between the circle and divisor problems (see e.g., [4, Chapter 13]), that $\alpha = \beta$ holds (and that in fact $\alpha = \beta = 1/2$), proving this is difficult. If one wants to generalize the method of Motohashi or Meurman to $E(x, h)$, one encounters several difficulties. One stems from the fact that $r(n)$ is given by (1.1), while $d(n) = \sum_{\delta | n}^{}$ contains no characters. This is reflected in the following. Namely Meurman uses a Voronoi–type formula for sums of $d(n)F(n)$ ($F(x) \in C^1[a, b]$) when $n$ lies in a given residue class. Such a formula is easily derived from the summation formula (see M. Jutila [7])

$$\sum_{a \leq n \leq b}^{} d(n)e\left(\frac{nh}{k}\right)F(n) = \frac{1}{k} \int_{a}^{b} (\log x + 2\gamma - 2\log k)F(x)dx$$

$$+ \frac{1}{k} \sum_{n=1}^{\infty} d(n) \int_{a}^{b} (-2\pi e\left(-\frac{n\overline{h}}{k}\right)Y_0\left(\frac{4\pi}{k}\sqrt{nx}\right) + 4e\left(\frac{n\overline{h}}{k}\right)K_0\left(\frac{4\pi}{k}\sqrt{nx}\right))F(x)dx,$$

which is valid for $0 < a < b$, $F(x) \in C^1[a, b]$ and $(h, k) = 1$. However, the analogue of this formula for sums of $r(n)e\left(\frac{nh}{k}\right)F(n)$ is not so simple arithmetically. Namely M. Jutila analyzed this problem in his paper [8]. His equations (27) and (28) give

$$\sum_{a \leq n \leq b}^{} r(n)e\left(\frac{nh}{k}\right)F(n) = \pi k^{-2}G_Q(k, h) \int_{a}^{b} F(x)dx$$

$$+ 2\pi (2k)^{-1} \sum_{k=1}^{\infty} \tilde{r}(n)e\left(-\frac{n\overline{h}}{k}\right) \int_{a}^{b} J_0\left(\frac{2\pi}{k}\sqrt{xn}\right)F(x)dx,$$

where $\overline{h}$ is the multiplicative inverse of $h$ mod $k$ and

$$G_Q(k, h) = \left(\sum_{x=1}^{k} e\left(h\frac{x^2}{k}\right)\right)^2$$

is the square of the Gauss sum, so it is zero for $k = 4m + 2$ and $\chi(k)k$ for $k = 4m + 1$. When $k = 1$ we do get the ‘ordinary’ Voronoi formula for $r(n)$ (in which case $k^{-2}G_Q(k, h) = 1$), but for general $k$ the function $\tilde{r}(n)$ (it is small, being \leq 2r(n) \ll n^{\varepsilon}$) depends also on $k$. The outcome of this summation formula will be that we shall not get the ‘nice’ Kloosterman sum as happened in the case of $d(n)$, but some ‘twisted’ sums. In the case of $d(n)$ one used Kuznetsov’s trace formula for sums of Kloosterman sums, but in the case of $r(n)$ the analogue of this step is hard.
Nevertheless we can avoid these difficulties and appeal to results of F. Chamizo [1] to show that $\alpha = 2/3$ is indeed possible in (2.2), which is the limit of present methods coming from the use of spectral theory. Thus we have the following

**THEOREM.** We have

\begin{equation}
\int_0^\infty P^2(x)e^{-x/T} \, dx = \frac{1}{4} \left( \frac{T}{\pi} \right)^{3/2} \sum_{n=1}^\infty r^2(n)n^{-3/2} - T + O_\varepsilon(T^{4/3+\varepsilon}).
\end{equation}

3. Proof of the Theorem

We shall follow the method of [5] and use Theorem 4.3 of F. Chamizo [1]. This says that, uniformly for arbitrary $\alpha_m \in \mathbb{C}$ and $M > 1$, $N > 1$,

\begin{equation}
\sum_{M < m \leq 2M} \alpha_mE(N, m) \ll_\varepsilon ||\alpha||_2 (N^{2/3+\varepsilon}M^{1/2} + N^{1/3}M^{5/6+\varepsilon}),
\end{equation}

where $||\alpha||_2 = \left( \sum_{M < m \leq 2M} |\alpha_m|^2 \right)^{1/2}$ is the norm of the sequence $\{\alpha_m\}$. We also have by [1, Theorem 5.2] the pointwise estimate

\begin{equation}
E(N, m) \ll_\varepsilon N^{4+\varepsilon} m^{\frac{4}{3}} \quad (m \leq N).
\end{equation}

Actually Chamizo defines (see (2.4))

\[ E(N, h) = \sum_{n \leq N} r(n)r(n + h) - 8 \left( 2^{k+1} - 3 \right) \sigma \left( \frac{h}{2^k} \right) \frac{N}{h}, \]

where $2^k$ is the highest power of 2 dividing $h$. However it is not hard to see that

\begin{equation}
g(h) := \frac{(-1)^h8}{h} \sum_{d|h}(-1)^d d = \frac{8}{h} \left( 2^{k+1} - 3 \right) \sigma \left( \frac{h}{2^k} \right).\]

Namely if $k = 0$ then $h$ is odd and both expressions in (3.3) reduce to $8\sigma(h)/h$. If $k \geq 1$, then setting $H = h/2^k$ the identity becomes

\[ \sum_{d|2^k H,(2,H)=1} (-1)^d d = (2^{k+1} - 3)\sigma(H). \]
But the left-hand side equals
\[
\sum_{d | H} \left( (-1)^d d + (-1)^{2d} 2d + \ldots (-1)^{2^k} 2^k d \right)
\]
\[= -\sigma(H) + (2 + 2^2 + \ldots + 2^k)\sigma(H) \]
\[= (-1 + 2^{k+1} - 2)\sigma(H) = (2^{k+1} - 3)\sigma(H). \]

We start from (3.6) of [5], writing
\[
\sum_{n \leq t} r(n)r(n+h) = g(h)t + E(t, h) \quad (h^2 \leq t \leq T^{10}),
\]
where \(g(h)\) is given by (3.3). We recall the definition made in [5], namely
\[
f(t, h) := \left\{ -(\sqrt{t+h} - \sqrt{t})^2 + \frac{3(2t + h) + 2\sqrt{t(t+h)}}{16\pi^2 \sqrt{t(t+h)T}} \right\} t^{-3/4}(t+h)^{-3/4}
\]
and note that, for \(h^2 \leq t \leq T^{10},\)
\[
f(t, h) \ll h^2 t^{-5/2} + T^{-1} t^{-3/2}, \quad \frac{df(t, h)}{dt} \ll h^2 t^{-7/2} + T^{-1} t^{-5/2}.
\]

Then, as in [5], we can write
\[
\sum(T) = \sum_1(T) + \sum_2(T),
\]
where
\[
\sum_1(T) := \sqrt{\pi} T^{5/2} \sum_{h \leq T^5} g(h) \int_{h^2}^{T_{10}} e^{-\pi^2 T (\sqrt{t+h} - \sqrt{t})^2} f(t, h) \, dt,
\]
\[
\sum_2(T) := \sqrt{\pi} T^{5/2} \sum_{h \leq T^5} \int_{h^2}^{T_{10}} e^{-\pi^2 T (\sqrt{t+h} - \sqrt{t})^2} f(t, h) \, dE(t, h).
\]

We can evaluate \(\sum_1(T)\) (which provides the main terms in (2.6) plus an error term which is certainly \(\ll \sqrt{T}\)), as in [5]. The main task consists of the estimation of \(\sum_2(T),\) which contributes to the error term in (2.6). We effect this by an integration by parts. The integrated terms will be small, and we are left with the estimation of
\[
T^{5/2} \sum_{h \leq T^5} \int_{h^2}^{T_{10}} E(t, h) u(t, h) \, dt,
\]
where \(u(t, h)\) is a suitable function.
where \((h^2 \leq t \leq T^{10})\)

\[
\begin{align*}
u(t, h) &= \frac{d}{dt} \left( e^{-\pi^2 T(\sqrt{t+h}-\sqrt{T})^2} f(t, h) \right) \\
&\ll e^{-\frac{\pi h^2}{2}} \left( h^2 t^{-7/2} + T^{-1} t^{-5/2} + T h^4 t^{-9/2} \right).
\end{align*}
\]

Now write the above sum as

\[
T^{5/2} \int_1^{T^{10}} \sum_{h \leq t^{1/2}} E(t, h) u(t, h) \, dt,
\]

and divide the intervals of integration and summation into \(O(\log^2 T)\) subintervals of the form \([K, 2K]\) and \([H, 2H]\), respectively.

Note that (3.1) can be used with

\[
u(t, m) \ll \alpha_m = e^{-\frac{\pi m^2}{K}} \left( m^2 K^{-7/2} + T^{-1} K^{-5/2} + T m^4 K^{-9/2} \right),
\]

since the dependence of \(u(t, m)\) on \(t\) when \(t \in [K, 2K]\) is harmless. Thus we obtain a contribution which is

\[
\ll T^{5/2} \log^2 T \max_{K \ll T^{10}, H \ll \sqrt{K}} \int_K^{2K} \left| \sum_{H < h \leq 2H} E(t, h) u(t, h) \right| \, dt
\]

\[
\ll \varepsilon T^{5/2+\varepsilon} \max_{K \ll T^{10}, H \ll \sqrt{K}} e^{-TH^2/K} K H^{1/2} (K^{2/3} H^{1/2} + K^{1/3} H^{5/6}) \times
\]

\[
(H^2 K^{-7/2} + T^{-1} K^{-5/2} + T H^4 K^{-9/2})
\]

\[
\ll \varepsilon T^{5/2+\varepsilon} \max_{K \ll T^{10}, H \ll \sqrt{K}} e^{-\frac{\pi m^2}{2K}} (H^3 K^{-11/6} + HT^{-1} K^{-5/6} + T H^5 K^{-17/6}),
\]

since \(K^{1/3} H^{5/6} \leq K^{2/3} H^{1/2}\). Now using

\[
e^{-x} x^\alpha \leq e^{-\alpha} \alpha^\alpha \ll 1 \quad (x \geq 0, \alpha > 0 \text{ fixed})
\]

we obtain

\[
e^{-\frac{\pi m^2}{K}} H^3 K^{-11/6} \leq e^{-\frac{\pi m^2}{K}} \left( \frac{TH^2}{K} \right)^{\frac{11}{6}} T^{-\frac{11}{6}} \ll T^{-\frac{11}{6}},
\]

and likewise

\[
e^{-\frac{\pi m^2}{2K}} HT^{-1} K^{-5/6} \ll T^{-\frac{11}{6}}, \quad e^{-\frac{\pi m^2}{K}} TH^5 K^{-17/6} \ll T^{-\frac{11}{6}}.
\]
Since \( \frac{5}{2} - \frac{11}{6} = \frac{2}{3} \), we obtain
\[
\sum_2 (T) \ll \varepsilon T^{\frac{2}{3} + \varepsilon},
\]
which gives then (2.6).

Alternatively we may use (3.2) (although this is not uniform in \( m \), it is crucial that the exponent of \( m \) is small), namely
\[
E(N, m) \ll N^{\frac{2}{3} + \varepsilon} m^{\beta} \quad \text{with} \quad \beta = \frac{5}{42}.
\]
Since \( \exp(-TH^2/K) \leq T^{-50} \) for \( H \geq (50K \log T/T)^{1/2} \) and \( H \geq 1 \) has to hold, it follows that in the first bound in (3.5) it suffices to take the maximum over \( 1 \leq H \leq (50K \log T/T)^{1/2} \) and \( T/\log T \ll K \ll T^{50} \). Trivial estimation, (3.2) and (3.4) yield then a contribution which is
\[
\ll T^{\frac{2}{3} + \varepsilon} \max_{T/\log T \ll K \ll T^{10}} \left\{ \left( \frac{K}{T} \right)^{\frac{3}{2} + \beta} K^{\frac{2}{3} - \frac{\beta}{2}} + \left( \frac{K}{T} \right)^{\frac{1}{2} + \beta} T^{-1} K^{\frac{2}{3} - \frac{\beta}{2}} + T \left( \frac{K}{T} \right)^{\frac{5}{2} + \beta} K^{\frac{2}{3} - \frac{\beta}{2}} \right\}
\]
provided that \( 0 \leq \beta \leq 2/3 \), which in our case is amply satisfied since we can take \( \beta = 5/42 \). This furnishes another proof of the Theorem.

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