Hyper-Kähler Calabi Metrics, $L^2$ Harmonic Forms, Resolved M2-branes, and AdS$_4$/CFT$_3$ Correspondence

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ABSTRACT

We obtain a simple explicit expression for the hyper-Kähler Calabi metric on the cotangent bundle of $\mathbb{CP}^{n+1}$, for all $n$, in which it is constructed as a metric of cohomogeneity one with $SU(n+2)/U(n)$ principal orbits. These results enable us to obtain explicit expressions for an $L^2$-normalisable harmonic 4-form in $D = 8$, and an $L^2$-normalisable harmonic 6-form in $D = 12$. We use the former in order to obtain an explicit resolved M2-brane solution, and we show that this solution is invariant under all three of the supersymmetries associated with the covariantly-constant spinors in the 8-dimensional Calabi metric. We give some discussion of the corresponding dual $\mathcal{N} = 3$ three-dimensional field theory. Various other topics are also addressed, including superpotentials for the Calabi metrics and the metrics of exceptional $G_2$ and Spin(7) holonomy in $D = 7$ and $D = 8$. We also present complex and quaternionic conifold constructions, associated with the cone metrics whose resolutions are provided by the Stenzel $T^*S^{n+1}$ and Calabi $T^*\mathbb{CP}^{n+1}$ metrics. In the latter case we relate the construction to the hyper-Kähler quotient. We then use the hyper-Kähler quotient to give a quaternionic rederivation of the Calabi metrics.
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1 Introduction

The conjectured AdS/CFT correspondence [1, 2, 3] asserts that a background of the form AdS$_d \times \mathcal{M}_m$ in string theory or M-theory will have an associated dual description as a conformal field theory on the $(d-1)$-dimensional boundary of AdS$_d$. Work in this area has focused predominantly on the case where the $m$-dimensional Einstein space $\mathcal{M}_m$ is a sphere, implying that the supersymmetry of the CFT will be maximised. However, if $\mathcal{M}_m$ is taken to be some other Einstein space that admits Killing spinors, then the associated dual CFT will again be supersymmetric, but with a lesser degree of supersymmetry. Any such background can be viewed as the decoupling limit of a $(d-1)$-brane solution, in which the usual flat space transverse to the brane is replaced by the Ricci-flat metric $d\hat{s}^2$ on the cone over $\mathcal{M}_m$,

$$d\hat{s}^2 = dr^2 + r^2 d\Sigma^2,$$

(1.1)

where $d\Sigma^2$ is the Einstein metric on $\mathcal{M}_m$ with Ricci-tensor $R_{ab} = (m-1) g_{ab}$. This was discussed in detail for type IIB backgrounds of the form AdS$_5 \times \mathcal{M}_5$, viewed as the decoupling limit of D3-branes, with $\mathcal{M}_5$ taken to be the $U(1)$ bundle over $S^2 \times S^2$ known as $T^{1,1}$, in [4].

Ricci-flat cone metrics are singular at the apex of the cone, where the $\mathcal{M}_m$ principal orbits degenerate to a point. In many cases a “resolution” of the singularity is possible, in which the degeneration near the origin is instead locally of the form

$$\mathcal{M}_m \longrightarrow S^p \times \tilde{\mathcal{M}}_q,$$

(1.2)

where $S^p$ denotes a round $p$-sphere whose radius tends to zero at the origin, and $\tilde{\mathcal{M}}_q$ is a manifold whose size remains non-vanishing at the origin. Provided that the rate of collapse of the $S^p$ factor is appropriate, the region near the origin smoothly approaches $\mathbb{R}^{p+1} \times \tilde{\mathcal{M}}_q$ locally [5]. At large distance the resolved metric asymptotically approaches the original cone metric. Since $\mathcal{M}_m$ is usually a homogeneous manifold in the cases of interest (where $\mathcal{M}_m$ admits Killing spinors), it follows that the resolutions of the cone metrics will then be described by manifolds of cohomogeneity one, with a compact symmetry group acting transitively on the principal orbits.

In the case of $\mathcal{M}_5 = T^{1,1}$, complete resolved Ricci-flat Kähler metrics are known for

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1There is, of course, always a “trivial” cone metric over the round sphere, $\mathcal{M}_m = S^m$, for which the “conifold” is just the $(m+1)$-dimensional Euclidean manifold, and there is no singularity at the apex. This corresponds to the special case $q = 0$. 

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which the degeneration (1.2) takes any of the forms

\[ T^{1,1} \rightarrow S^2 \times S^3, \quad S^3 \times S^2, \quad S^1 \times (S^2 \times S^2). \] (1.3)

A further possibility that now arises is to consider additional branes in the type IIB theory, that can wrap around the non-trivial cycle in the transverse space. This gives rise to so-called “fractional D3-branes” \([5, 7]\). In terms of the description within type IIB supergravity, the wrapping of additional branes corresponds to turning on additional fluxes for the R-R and NS-NS 3-forms, which are now set equal to the real and imaginary parts of a self-dual harmonic 3-form in the resolved transverse-space metric. Physically, one of the consequences of turning on the additional flux is to perturb the background away from AdS\(_5 \times T^{1,1}\) in the decoupling limit, meaning that the dual field theory on the boundary is perturbed away from a conformally-invariant one. This has the important consequence that scale invariance is broken, and a mass gap can now open up in the spectrum of the operator determining the glueball mass. Thus the non-conformal phase with the additional flux is a confining phase.

Another situation of considerable interest is M-theory backgrounds of the form AdS\(_4 \times \mathcal{M}_7\), which correspond to decoupling limits of M2-branes. A detailed discussion of the conifolds where \(\mathcal{M}_7\) is taken to be \(Q^{1,1,1}\) (a \(U(1)\) bundle over \(S^2 \times S^2 \times S^2\) with winding numbers 1 over each \(S^2\)) or \(M^{2,3}\) (a \(U(1)\) bundle over \(S^2 \times CP^2\) with winding numbers 2 and 3 over \(S^2\) and \(CP^2\)) was given in \([8]\). For these, and various other choices for the Einstein manifold \(\mathcal{M}_7\), one can again replace the singular cone metrics on the 8-dimensional transverse spaces by smoothed-out Ricci-flat resolutions. There are three basic kinds of such irreducible 8-manifolds that admit covariantly-constant spinors, namely those with exceptional Spin(7) holonomy, Ricci-flat Kähler holonomy \(SU(4)\), and hyper-Kähler holonomy \(Sp(2) \equiv \text{Spin}(5)\). These admit \(\mathcal{N} = 1, 2\) and 3 covariantly-constant spinors respectively.

Resolved M2-brane solutions using transverse 8-spaces that are resolutions with Spin(7) and \(SU(4)\) holonomy were constructed in \([8, 9, 10]\). In the Spin(7) example the metric is a resolution of a cone over \(S^7\). However, the metric is not just the trivial Euclidean one because the Einstein metric on the 7-sphere at the base of the cone is the non-standard “squashed” one, and not the usual round-sphere metric. Ricci-flat Kähler examples with \(SU(4)\) holonomy were constructed for cases where \(\mathcal{M}_7\) is \(Q^{1,1,1}\) or \(M^{2,3}\). In another example \(\mathcal{M}_7\) was again locally \(S^7\), but this time although the base of the cone has the round \(S^7\) metric, its global topology is actually \(S^7/\mathbb{Z}_4\) and so again the resolved 8-manifold is not merely the trivial Euclidean space.\(^2\) Another example of a similar kind was based on the

\(^2\)In fact this example is a higher-dimensional analogue of the 4-dimensional Eguchi-Hanson metric \([12]\), for which the principal orbits were shown in \([13]\) to be \(S^3/\mathbb{Z}_2\) rather than \(S^3\).
manifold $M_7 = SU(3)/U(1)$. In all these cases, there exist $L^2$-normalisable harmonic 4-forms in resolved transverse 8-manifolds, and these can be used in order to modify the usual 4-form field strength background of an M2-brane, leading to deformed M2-brane solutions. The notion of “wrapping” does not arise here, and correspondingly the additional field-strength contribution gives no flux integral at infinity. The physical interpretation of these deformed M2-brane solutions is therefore rather different from that for the fractional D3-branes.

The principal theme in the present paper is to study the third type of resolved M2-brane solution mentioned above, where the transverse 8-manifold is hype-Kähler, with Spin(5) holonomy. In fact it can be shown that there is only one such complete non-singular 8-metric of cohomogeneity one [14], namely the hyper-Kähler Calabi metric [15]. In order to give an explicit construction of the associated deformed M2-brane solution it is necessary to find an explicit expression for the harmonic 4-form. The usual presentations of the Calabi metric do not lend themselves to obtaining such explicit results in any convenient way. Accordingly, we begin the paper by finding a more explicit way to construct the Calabi metric. In fact we shall give a general construction for the hyper-Kähler Calabi metrics in all dimensions $D = 4n + 4$. The manifolds for these solutions are $T^*\mathbb{CP}^{n+1}$, the cotangent bundles of the complex projective spaces $\mathbb{CP}^{n+1}$. Our approach is along the lines of the construction in [14, 1]. In an earlier paper [10] we applied these methods in order to produce fully explicit expressions for the Stenzel metrics on $T^*S^{n+1}$ [18], for which the principal orbits were $SO(n+2)/SO(n)$. In this paper we apply a similar technique to the case of $T^*\mathbb{CP}^{n+1}$, for which the principal orbits are $SU(n+2)/U(n)$.

Our procedure for obtaining the hyper-Kähler solutions is by making an ansatz for metrics of cohomogeneity one whose principal orbits are $SU(n+2)/U(n)$, with undetermined functions of $r$ parameterising the various radii in the level sets. Following the procedure of [14], when obtain first-order equations for these functions from the integrability conditions for the existence of a hyper-Kähler structure. We find that the solution of these equations leads to the following simple expression for the hyper-Kähler Calabi metric in $D = 4n + 4$ dimensions:

$$ds^2 = \frac{dr^2}{1-r^{-4}} + \frac{1}{4} r^2 (1-r^{-4}) \lambda^2 + r^2 (\nu_1^2 + \nu_2^2) + \frac{1}{2} (r^2 - 1) (\sigma_1^2 + \sigma_2^2) + \frac{1}{2} (r^2 + 1) (\Sigma_1^2 + \Sigma_2^2) ,$$

where $(\lambda, \nu_1, \nu_2, \sigma_1, \sigma_2, \Sigma_1, \Sigma_2)$, with $\alpha$ running over $n$ values, are left-invariant 1-forms on the coset $SU(n+2)/U(n)$.

We also show how the Einstein equations for the metric ansatz, viewed as a Lagrangian...
system, can be cast into a form where they can be derived from a superpotential, whose associated first-order equations again admit the hyper-Kähler Calabi metrics as solutions. In fact by making restrictions in the original parameterisation of the metrics we can obtain two inequivalent superpotentials, one leading to the hyper-Kähler solutions, and the other to Ricci-flat Kähler solutions on the complex line bundles over $SU(n + 2)/(U(n) \times U(1))$.

Having obtained explicit expressions for the hyper-Kähler Calabi metrics in a fashion that is suitable for our purposes, we then look for middle-dimension harmonic forms. In particular, we obtain explicit results for an $L^2$-normalisable (anti) self-dual harmonic 4-form in the $D = 8$ metric, and an $L^2$-normalisable (anti) self-dual harmonic 6-form in the $D = 12$ metric. The $D = 8$ result allows us to build a resolved M2-brane solution. We also show that all three of the covariantly-constant spinors of the hyper-Kähler 8-metric remain Killing spinors of the resolved M2-brane solution. This implies that the dual 3-dimensional gauge theory on the boundary of AdS$_4$ will have $\mathcal{N} = 3$ supersymmetry.

In the remainder of the paper, we give some discussion of the physical significance of this, and previously-obtained resolved brane solutions. We analyse the symmetry groups of the dual field theories. In particular, we emphasise that one of the purposes of the brane resolution is to break the conformal symmetry in order to achieve confinement. This leads to very different mechanisms for confinement in $D = 4$ and $D = 3$ gauge theories. In $D = 4$, an additional fractional 5-brane flux is needed for the brane-resolution, whilst in $D = 3$ the breaking of the conformal phase is caused by a perturbation of relevant operators, associated with the pseudoscalar fields of the gauge theory in the Higgs branch.

After a concluding section the paper ends with a set of appendices. In appendix A we present a summary of complete non-compact Ricci-flat metrics of cohomogeneity one in dimensions $D = 4, 6, 7$ and $8$. These are the dimensions of greatest interest from the point of view of constructing brane resolutions in string theory and M-theory. We include derivations of superpotential formulations for the Einstein equations leading to metrics of $G_2$ holonomy in $D = 7$, and Spin(7) holonomy in $D = 8$. Appendix B contains a summary of some results for higher-dimensional generalisations of the Taub-NUT metric. Appendix C contains some further results for the $T^*\mathbb{CP}^{n+1}$ construction in the specific case of $n = 1$, where more general superpotentials can be found. Finally, in appendix D we present a construction of complex and quaternionic conifolds. In particular, we show how the Einstein metric on the base of the quaternionic cone, which is the asymptotic form of the principal orbits in the Calabi metrics, arises as the hyper-Kähler quotient of the flat metric on $\mathbb{H}^{n+2}$. We then use the hyper-Kähler quotient construction to give a quaternionic rederivation of the Calabi
metric in the form \((1.4)\).

2 Metrics on co-tangent bundle of \(\mathbb{C}P^{n+1}\)

In this section, we present an explicit construction of metrics of cohomogeneity one on the co-tangent bundle of \(\mathbb{C}P^n\). Following the approach of \cite{14}, we then show how the conditions for Ricci-flatness can be reduced to a system of first-order equations, by requiring the existence of a hyper-Kähler structure. We obtain solutions to these equations, and thereby obtain fully explicit expressions for the hyper-Kähler Calabi metrics that will be used in subsequent sections. (Later, in appendix D, we shall rederive the same metrics using the hyper-Kähler quotient construction.)

2.1 Metrics of cohomogeneity one on \(T^*\mathbb{C}P^{n+1}\)

Paralleling the construction of the Stenzel metrics on \(T^*(S^{n+1})\) in \cite{10}, here we start with the generators of \(SU(n+2)\), and their associated left-invariant 1-forms \(L_A^B\), where \(L_A^A = 0\) and \((L_A^B)^† = L_B^A\), which satisfy the exterior algebra

\[
dL_A^B = i L_A^C \wedge L_C^B. \tag{2.1}\]

We then split the \(SU(n+2)\) index \(A\) as \(A = (1, 2, \alpha)\). The idea will be to define generators in the coset \(SU(n+2)/U(n)\), and those that lie in the denominator group \(U(n)\). Clearly the latter will include the \(SU(n)\) generators

\[
\tilde{L}_\alpha^\beta \equiv L_\alpha^\beta + \frac{1}{n} Q \delta_\alpha^\beta, \tag{2.2}\]

where the trace subtraction is written in terms of the \(U(1)\) generator \(Q\), defined to be

\[
Q \equiv L_1^1 + L_2^2. \tag{2.3}\]

There is one other \(U(1)\) generator, namely

\[
\lambda \equiv L_1^1 - L_2^2. \tag{2.4}\]

The generators of the coset will then be the complement of \(\tilde{L}_\alpha^\beta\) and \(Q\), namely \(\lambda\) and

\[
\sigma^\alpha \equiv L_1^\alpha, \quad \Sigma^\alpha \equiv L_1^\alpha, \quad \nu \equiv L_1^2. \tag{2.5}\]

Note that \(\sigma^\alpha\), \(\Sigma^\alpha\) and \(\nu\) are all complex, while \(\lambda\) is real.
In general, we find on decomposing (2.1) that the exterior derivatives are given by

\[ d\sigma^\alpha = \frac{1}{2} \lambda \wedge \sigma^\alpha + i \nu \wedge \Sigma^\alpha + \frac{1}{2}(1 + 2/n) Q \wedge \sigma^\alpha + i \sigma^\beta \wedge \bar{L}_\beta^\alpha, \]

\[ d\Sigma^\alpha = -\frac{1}{2} \lambda \wedge \Sigma^\alpha + i \tilde{\nu} \wedge \sigma^\alpha + \frac{1}{2}(1 + 2/n) Q \wedge \Sigma^\alpha + i \Sigma^\beta \wedge \bar{L}_\beta^\alpha, \]

\[ d\nu = i \lambda \wedge \nu + i \sigma^\alpha \wedge \Sigma_\alpha, \]

\[ d\lambda = 2i \nu \wedge \tilde{\nu} + i \sigma^\alpha \wedge \bar{\sigma}_\alpha - i \Sigma^\alpha \wedge \bar{\Sigma}_\alpha, \]

\[ dQ = i \sigma^\alpha \wedge \bar{\sigma}_\alpha + i \Sigma^\alpha \wedge \Sigma_\alpha, \]

\[ d\bar{L}_\alpha^\beta = i \bar{\sigma}_\alpha \wedge \sigma^\beta + i \Sigma_\alpha \wedge \Sigma^\beta - \frac{1}{n} i (\bar{\sigma}_\gamma \wedge \sigma^\gamma + \Sigma_\gamma \wedge \Sigma^\gamma) \delta^\beta_\alpha + i \bar{L}_\alpha^\gamma \wedge \bar{L}_\gamma^\beta. \]

The \((4n + 4)\)-dimensional metric of cohomogeneity one on the co-tangent bundle of \(\mathbb{C}P^{n+1}\) will be written as

\[ ds^2 = dt^2 + a^2 |\sigma^\alpha|^2 + b^2 |\Sigma^\alpha|^2 + c^2 |\nu|^2 + f^2 \lambda^2, \]

where the metric functions \(a, b, c\) and \(f\) all depend only on \(t\).

### 2.2 Curvature calculations

Define real 1-forms as follows:

\[ \sigma_\alpha = \sigma_{1\alpha} + i \sigma_{2\alpha}, \quad \Sigma_\alpha = \Sigma_{1\alpha} + i \Sigma_{2\alpha}, \quad \nu = \nu_1 + i \nu_2, \]

which therefore satisfy

\[ d\sigma_{1\alpha} = -\frac{1}{2} \lambda \wedge \sigma_{2\alpha} - \nu_1 \wedge \Sigma_{2\alpha} - \nu_2 \wedge \Sigma_{1\alpha} - \hat{Q} \wedge \sigma_{2\alpha} - \sigma_{1\beta} \wedge L_{2\beta}^\alpha - \sigma_{2\beta} \wedge L_{1\beta}^\alpha, \]

\[ d\sigma_{2\alpha} = \frac{1}{2} \lambda \wedge \sigma_{1\alpha} + \nu_1 \wedge \Sigma_{1\alpha} - \nu_2 \wedge \Sigma_{2\alpha} + \hat{Q} \wedge \sigma_{2\alpha} + \sigma_{1\beta} \wedge L_{1\beta}^\alpha - \sigma_{2\beta} \wedge L_{2\beta}^\alpha, \]

\[ d\Sigma_{1\alpha} = \frac{1}{2} \lambda \wedge \Sigma_{2\alpha} - \nu_1 \wedge \sigma_{2\alpha} + \nu_2 \wedge \sigma_{1\alpha} - \hat{Q} \wedge \Sigma_{2\alpha} - \Sigma_{1\beta} \wedge L_{2\beta}^\alpha - \Sigma_{2\beta} \wedge L_{1\beta}^\alpha, \]

\[ d\Sigma_{2\alpha} = -\frac{1}{2} \lambda \wedge \Sigma_{1\alpha} + \nu_1 \wedge \sigma_{1\alpha} + \nu_2 \wedge \sigma_{2\alpha} + \hat{Q} \wedge \Sigma_{1\alpha} + \Sigma_{1\beta} \wedge L_{1\beta}^\alpha - \Sigma_{2\beta} \wedge L_{2\beta}^\alpha, \]

\[ d\nu_1 = -\lambda \wedge \nu_2 + 2 \sigma_{1\alpha} \wedge \Sigma_\alpha - 2 \sigma_\alpha \wedge \Sigma_{1\alpha}, \]

\[ d\nu_2 = \lambda \wedge \nu_1 + \sigma_{1\alpha} \wedge \Sigma_\alpha + 2 \sigma_\alpha \wedge \Sigma_{1\alpha}, \]

\[ d\lambda = 2 \sigma_{1\alpha} \wedge \sigma_\alpha - 2 \Sigma_{1\alpha} \wedge \Sigma_\alpha + 4 \nu_1 \wedge \nu_2, \]

where we have defined \(\hat{Q} = \frac{1}{2}(1 + 2/n) Q\), and \(\bar{L}_\alpha^\beta = L_{1\alpha}^\beta + i L_{2\alpha}^\beta\). Note that since \(\bar{L}_\alpha^\beta\) is hermitean, it follows that \(L_{1\alpha}^\beta\) is symmetric, and \(L_{2\alpha}^\beta\) is antisymmetric.

In addition, we need to know that for \(\hat{Q}, L_{1\alpha}^\beta\) and \(L_{2\alpha}^\beta\), which lie outside the coset, we have

\[ d\hat{Q} = (1 + \frac{2}{n}) (\sigma_{1\alpha} \wedge \sigma_\alpha + \Sigma_{1\alpha} \wedge \Sigma_\alpha), \]

\[ dL_{1\alpha}^\beta (a) = \frac{1}{2} \lambda \wedge \sigma^\alpha + \sigma_{1\beta} \wedge L_{2\beta}^\alpha, \]

\[ dL_{2\alpha}^\beta (a) = -\frac{1}{2} \lambda \wedge \Sigma^\alpha + \sigma_{2\beta} \wedge L_{1\beta}^\alpha, \]

\[ d\Sigma^\alpha (a) = \frac{1}{2} \lambda \wedge \Sigma^\alpha, \]

\[ d\lambda (a) = 2 \sigma_{1\alpha} \wedge \Sigma^\alpha - 2 \Sigma_{1\alpha} \wedge \Sigma^\alpha + 4 \nu_1 \wedge \nu_2. \]
In an obvious notation, these are

\[ R \alpha \beta \gamma = \sigma_1 \alpha \wedge \sigma_2 \beta - \sigma_1 \beta \wedge \sigma_2 \alpha - \Sigma_1 \alpha \wedge \Sigma_2 \beta - \Sigma_1 \beta \wedge \Sigma_2 \alpha + \frac{2}{n} (\sigma_1 \gamma \wedge \sigma_2 \gamma + \Sigma_1 \gamma \wedge \Sigma_2 \gamma) \delta_\alpha^\beta - L_{1\alpha} \gamma \wedge L_{2\alpha} \beta - L_{2\alpha} \gamma \wedge L_{1\gamma} \beta, \]  

\[ R_{2\alpha} \beta = \sigma_1 \alpha \wedge \sigma_2 \beta + \sigma_2 \alpha \wedge \sigma_1 \beta + \Sigma_1 \alpha \wedge \Sigma_1 \beta + \Sigma_2 \alpha \wedge \Sigma_2 \beta + L_{1\alpha} \gamma \wedge L_{1\gamma} \beta - L_{2\alpha} \gamma \wedge L_{2\gamma} \beta. \]  

In terms of the real 1-forms, the metric \((2.7)\) becomes

\[ ds^2 = dt^2 + a^2 (\sigma_1^2 + \sigma_2^2) + b^2 (\Sigma_1^2 + \Sigma_2^2) + c^2 (\nu_1^2 + \nu_2^2) + f^2 \lambda^2. \]  

In the obvious orthonormal basis,

\[ e^0 = dt, \quad e^0 = f \lambda, \quad e^i = c \nu_1, \quad e^2 = c \nu_2, \]

\[ e^1 = a \sigma_1, \quad e^2 = a \sigma_2, \quad e^1 = b \Sigma_1, \quad e^2 = b \Sigma_2, \]

we find after some calculation that the Ricci tensor is diagonal, and there are just five inequivalent eigenvalues, associated with the \(dt, \sigma_{i\alpha}, \Sigma_{i\alpha}, \nu_i\) and \(\lambda\) directions respectively.

In an obvious notation, these are

\[ R_0 = -2n \left( \frac{\bar{a}}{a} + \frac{\bar{b}}{b} \right) - \frac{2\bar{c}}{c} - \frac{\bar{f}}{f}, \]

\[ R_a = -\frac{\bar{a}}{a} \frac{(2n-1)a^2}{ab} - \frac{2\bar{a}}{ac} - \frac{\bar{a}}{af} + \frac{a^4 - b^4 - c^4}{a^2 b^2 c^2} + \frac{2(n+2)}{a^2} - \frac{2f^2}{a^4}, \]

\[ R_b = -\frac{\bar{b}}{b} \frac{(2n-1)b^2}{ab} - \frac{2\bar{b}}{bc} - \frac{\bar{b}}{bf} + \frac{b^4 - a^4 - c^4}{a^2 b^2 c^2} + \frac{2(n+2)}{b^2} - \frac{2f^2}{b^4}, \]

\[ R_c = -\frac{\bar{c}}{c} \frac{c^2}{c^2} - \frac{2\bar{c}}{ac} - \frac{2\bar{c}}{bc} - \frac{\bar{c}}{cf} + n \left( \frac{c^4 - a^4 - b^4}{a^2 b^2 c^2} \right) + \frac{2(n+2)}{c^2} - \frac{8f^2}{c^4}, \]

\[ R_f = -\frac{\bar{f}}{f} - \frac{2n}{a f} - \frac{2n \bar{a}}{bf} - \frac{2\bar{c}}{cf} + f^2 \left( \frac{2n}{a^2} + \frac{2n}{b^4} + \frac{8}{c^4} \right). \]  

### 2.3 Hyper-Kähler solutions of the Einstein equations

We are principally interested in obtaining the hyper-Kähler Calabi metrics, which should arise as particular solutions of the Ricci-flat Einstein equations obtained above. The easiest way to find these metrics is to follow the procedure of Dancer and Swann [14], of writing down the ansätze for the three hyper-Kähler forms, and then imposing the conditions of covariant-constancy.

We seek three simultaneous Kähler forms \(J_i\). It quickly becomes evident, after consulting \((2.9)\), that the following are three candidates for the Kähler forms:

\[ J_1 = f \ dt \wedge \lambda + c^2 \nu_1 \wedge \nu_2 + a^2 \sigma_{1\alpha} \wedge \sigma_{2\alpha} - b^2 \Sigma_{1\alpha} \wedge \Sigma_{2\alpha}, \]

\[ J_2 = c \ dt \wedge \nu_1 - c f \lambda \wedge \nu_2 + a b (\sigma_{1\alpha} \wedge \Sigma_{2\alpha} - \sigma_{2\alpha} \wedge \Sigma_{1\alpha}), \]

\[ J_3 = c \ dt \wedge \nu_2 + c f \lambda \wedge \nu_1 + a b (\sigma_{1\alpha} \wedge \Sigma_{1\alpha} + \sigma_{2\alpha} \wedge \Sigma_{2\alpha}). \]
Note that the $a$, $b$, $c$ and $f$ prefactors are determined completely, up to signs, by the fact that each term in each $J_i$ should be just a wedge product of vielbeins (see (2.12). The structure of the $dt$, $\lambda$, $\nu_1$ and $\nu_2$ terms in each case is determined by the requirement that these must supply the “remainder” of the Kähler form, which must span the total dimension of the manifold, and so structurally these terms follow once the terms involving $\sigma_{i\alpha}$ and $\Sigma_{i\alpha}$ are settled. The precise details can be fixed easily, by looking first at the terms involving $dt$, when one imposes $dJ_i = 0$. Note also that we have already incorporated some of the easily-determined results from these equations, in making the specific selections of $\pm$ signs presented in (2.14).

Now, imposing $dJ_i = 0$, we find that $dJ_1 = 0$ leads to the equations

$$
\frac{d(a^2)}{dt} = 2f, \quad \frac{d(b^2)}{dt} = 2f, \quad \frac{d(c^2)}{dt} = 4f, \quad a^2 + b^2 = c^2. \quad (2.15)
$$

Similarly, we find that imposing $dJ_2 = 0$ gives

$$
\frac{d(ab)}{dt} = c, \quad \frac{d(cf)}{dt} = c, \quad ab = cf. \quad (2.16)
$$

Finally, imposing $dJ_3 = 0$ just gives the same equations as from $dJ_2 = 0$. We see that (2.15) and (2.16) are precisely the first-order equations obtained by Dancer and Swann [14].

It is straightforward to solve these equations. In terms of the new radial variable $r$, related to $t$ by $dt = h\,dr$, we find

$$
a^2 = \frac{1}{2}(r^2 - 1), \quad b^2 = \frac{1}{2}(r^2 + 1), \quad c^2 = r^2, \quad f^2 = \frac{1}{4}r^2(1 - r^{-4}), \quad h^2 = (1 - r^{-4})^{-1}. \quad (2.17)
$$

The Calabi metric in $D = 4n + 4$ dimensions is then given by (2.11). One can easily verify that the Ricci tensor, given by (2.13), does indeed vanish. Note that, remarkably, the expressions for the metric functions are completely independent of the dimension $D = 4n+4$.

In fact by specialising to $n = 0$, in which case there are no 1-forms $\sigma^\alpha$ or $\Sigma^\alpha$ at all, we recover the well-known Eguchi-Hanson metric. The characteristic $(1 - r^{-4})$ functions of this metric thus continue to appear in all the higher-dimensional hyper-Kähler Calabi metrics.

### 2.4 Geometry of the Calabi metrics

It is evident from (2.11) and (2.17) that the radial coordinate runs from $r = 1$ to $r = \infty$. Asymptotically, at large $r$, the metric approaches the cone over the homogeneous tri-Sasaki Einstein manifold $SU(n + 2)/U(n)$:

$$
ds^2 \sim dr^2 + \frac{1}{4}r^2(\lambda^2 + 2|\sigma_\alpha|^2 + 2|\Sigma_\alpha|^2 + 4|\nu|^2). \quad (2.18)
$$
Near to \( r = 1 \), it is helpful to introduce a new radial coordinate \( \rho \), defined by \( \frac{1}{2}(r^2 - 1) = \rho^2 \). In terms of this, we find that near \( \rho = 0 \) the metric approaches

\[
ds^2 \sim d\rho^2 + \rho^2 \lambda^2 + \rho^2 |\sigma_\alpha|^2 + |\Sigma_\alpha|^2 + |\nu|^2,
\]

and so the principal \( SU(n + 2)/U(n) \) orbits collapse down to a \( \mathbb{C}P^{n+1} \) bolt. The geometry near \( \rho = 0 \) is locally \( \mathbb{R}^{2n+2} \times \mathbb{C}P^{n+1} \).

### 2.5 Complex structures in the Calabi metrics

The Calabi metrics are hyper-Kähler, and so there are three complex structure tensors, whose multiplication rules are those of the imaginary unit quaternions. After lowering the upper index in each case to give a 2-form, we see from (2.12) and (2.14) that the three associated Kähler forms are

\[
J_1 = e^0 \wedge e^\tilde{0} + e^1 \wedge e^\tilde{1} + e^{2\alpha} \wedge e^{2\tilde{\alpha}} - e^{1\alpha} \wedge e^{2\tilde{\alpha}}, \\
J_2 = e^0 \wedge e^1 - e^0 \wedge e^\tilde{2} + e^{1\alpha} \wedge e^{2\tilde{\alpha}} - e^{2\tilde{\alpha}} \wedge e^{1\tilde{\alpha}}, \\
J_3 = e^0 \wedge e^\tilde{2} + e^\tilde{0} \wedge e^\tilde{1} + e^{1\alpha} \wedge e^{1\tilde{\alpha}} + e^{2\alpha} \wedge e^{2\tilde{\alpha}}.
\]

(2.20)

If we choose \( J = J_1 \) to define a privileged complex structure, then we see from (2.20) that we can define a complex vielbein basis

\[
e^0 \equiv e^0 + i e^\tilde{0}, \quad e^\# \equiv e^1 + i e^2, \quad e^\alpha \equiv e^{1\alpha} + i e^{2\alpha}, \quad e^{\tilde{\alpha}} \equiv e^{1\tilde{\alpha}} - i e^{2\tilde{\alpha}},
\]

in terms of which

\[
J_1 = \frac{i}{2} (e^0 \wedge e^\tilde{0} + e^\# \wedge e^{\tilde{\#}} + e^\alpha \wedge e^{\tilde{\alpha}} + e^{\tilde{\alpha}} \wedge e^{\tilde{\alpha}}),
\]

(2.22)

which is, of course, of type \((1,1)\). The two complex combinations \( K_+ \equiv J_2 \pm i J_3 \) of the other two Kähler forms are then given by

\[
K_+ = e^0 \wedge e^\# + i e^\alpha \wedge e^{\tilde{\alpha}}, \quad K_- = e^0 \wedge e^{\tilde{\#}} - i e^{\tilde{\alpha}} \wedge e^\tilde{\alpha}.
\]

(2.23)

These are pure \((2,0)\) and \((0,2)\) respectively, as expected.

Note that the \( SU(n + 2) \) factor in the \( U(n + 2) \) isometry group acts triholomorphically; i.e. it leaves invariant all three complex structures in (2.20). The remaining \( U(1) \) factor in the isometry group leaves \( J_1 \) invariant, while rotating \( J_2 \) into \( J_3 \).

Using (2.9) and (2.10), we can write the Kähler forms \( J_i \) locally in terms of 1-form potentials, \( J_i = dA_i \), with

\[
A_1 = \frac{1}{4} r^2 \lambda - \frac{1}{4} Q, \quad A_2 = \frac{1}{2} (r^4 - 1)^{1/2} \nu_1, \quad A_3 = \frac{1}{2} (r^4 - 1)^{1/2} \nu_2.
\]

(2.24)
Note that unlike $A_2$ and $A_3$, we cannot give a general expression for $A_1$ using quantities intrinsic to the $SU(n + 2)/U(n)$ coset within the framework we are using here. Of course if we introduced coordinates for the coset, $Q$ would then be expressible in terms of them. In appendix D, we show how $A_1$ acquires an interpretation as the connection on the $U(1)$ Hopf fibres in a hyper-Kähler quotient construction of the Calabi metrics.

3 The eight-dimensional case; Calabi metric on $T^*\mathbb{CP}^2$

Our principle motivation for studying the hyper-Kähler Calabi metrics is in order to obtain explicit results for the 8-dimensional example, which will allow us to construct a new explicit resolved M2-brane solution. Accordingly, in this section we shall present some further details of the 8-dimensional example, including in particular complete expressions for its Riemann curvature.

3.1 Conventions and curvature

The 8-dimensional case corresponds to setting $n = 1$ in the general results of the previous section. The $\alpha$ indices now only takes a single value, namely $\alpha = 3$, and so the traceless “$SU(1)$ generators” $\tilde{L}_{\alpha \beta}$ vanish here. The coset is $SU(3)/U(1)$. Rather than following the general notation of section 2, it is more convenient here to adopt a different labelling of the real 1-forms, corresponding to the real and imaginary parts of $\sigma_\alpha, \Sigma_\alpha$ (with $\alpha = 3$ here in $D = 8$) and $\nu$ as follows:

$$\sigma^3 \equiv \sigma_1 + i \sigma_2, \quad \Sigma^3 \equiv \Sigma_1 + i \Sigma_2, \quad \nu \equiv \nu_1 + i \nu_2.$$  \hspace{1cm} (3.1)

Thus the 8-metric will be

$$ds_8^2 = dt^2 + a^2 (\sigma_1^2 + \sigma_2^2) + b^2 (\Sigma_1^2 + \Sigma_2^2) + c^2 (\nu_1^2 + \nu_2^2) + f^2 \lambda^2,$$ \hspace{1cm} (3.2)

with the metric functions given by (2.17). We define the obvious orthonormal basis

$$e^0 = dt, \quad e^1 = a \sigma_1 \quad e^2 = a \sigma_2 \quad e^3 = b \Sigma_1 \quad e^4 = b \Sigma_2 \quad e^5 = c \nu_1 \quad e^6 = c \nu_2 \quad e^7 = f \lambda.$$ \hspace{1cm} (3.3)

(Note that for convenience we are now replacing the previous indices $(\bar{0}, \bar{1}, \bar{2})$ of the generic case $(\bar{3}, \bar{3})$ by $(7, 5, 6)$ respectively.)

It is useful to note, from the general expressions in section 2.2, that the exterior derivatives of the left-invariant 1-forms in $D = 8$ are given by

$$d\sigma_1 = -\frac{1}{2} \lambda \wedge \sigma_2 - \nu_1 \wedge \Sigma_2 - \nu_2 \wedge \Sigma_1 - \frac{3}{2} Q \wedge \sigma_2,$$

\[ d\sigma_2 = \frac{1}{2} \lambda \wedge \sigma_1 + \nu_1 \wedge \Sigma_1 - \nu_2 \wedge \Sigma_2 + \frac{3}{2} Q \wedge \sigma_1, \]
\[ d\Sigma_1 = \frac{1}{2} \lambda \wedge \Sigma_2 - \nu_1 \wedge \sigma_2 + \nu_2 \wedge \sigma_1 - \frac{3}{2} Q \wedge \Sigma_2, \]
\[ d\Sigma_2 = -\frac{1}{2} \lambda \wedge \Sigma_1 + \nu_1 \wedge \sigma_1 + \nu_2 \wedge \sigma_2 + \frac{3}{2} Q \wedge \Sigma_1, \]
\[ d\nu_1 = -\lambda \wedge \nu_2 - \sigma_2 \wedge \Sigma_1 + \sigma_1 \wedge \Sigma_2, \]
\[ d\nu_2 = \lambda \wedge \nu_1 + \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2, \]
\[ d\lambda = 2\sigma_1 \wedge \sigma_2 - 2\Sigma_1 \wedge \Sigma_2 + 4\nu_1 \wedge \nu_2. \quad (3.4) \]

In addition, we need to know that for \( Q \), which lies outside the coset, we have
\[ dQ = 2\sigma_1 \wedge \sigma_2 + 2\Sigma_1 \wedge \Sigma_2. \quad (3.5) \]

Using the expressions \((2.14)\) for the three Kähler forms, we find that in the orthonormal basis \((3.3)\) they are given by
\[ J_1 = e^0 \wedge e^7 + e^1 \wedge e^2 - e^3 \wedge e^4 + e^5 \wedge e^6, \]
\[ J_2 = e^0 \wedge e^5 + e^6 \wedge e^7 + e^1 \wedge e^4 - e^2 \wedge e^3, \]
\[ J_3 = e^0 \wedge e^6 - e^5 \wedge e^7 + e^1 \wedge e^3 + e^2 \wedge e^4. \quad (3.6) \]

It is straightforward to evaluate the curvature 2-forms \( \Theta_{ab} \) for the 8-dimensional Calabi metric. After some algebra, we find that they are given by
\[ \Theta_{01} = -\frac{1}{r^4} (-e^0 \wedge e^1 + e^2 \wedge e^7 + e^3 \wedge e^6 + e^4 \wedge e^5), \]
\[ \Theta_{03} = -\frac{1}{r^4} (e^0 \wedge e^3 + e^1 \wedge e^6 - e^2 \wedge e^5 + e^4 \wedge e^7), \]
\[ \Theta_{05} = -\frac{2}{r^4} (e^0 \wedge e^5 - e^6 \wedge e^7), \]
\[ \Theta_{07} = \frac{2}{r^4} (e^1 \wedge e^2 + e^3 \wedge e^4) + \frac{4}{r^6} (e^0 \wedge e^7 - e^5 \wedge e^6), \]
\[ \Theta_{12} = \frac{4}{r^2} (e^1 \wedge e^2 + e^3 \wedge e^4) + \frac{2}{r^4} (e^0 \wedge e^7 - e^5 \wedge e^6), \]
\[ \Theta_{13} = -\frac{2}{r^2} (e^1 \wedge e^3 - e^2 \wedge e^4), \]
\[ \Theta_{15} = -\frac{1}{r^4} (e^0 \wedge e^4 + e^1 \wedge e^5 + e^2 \wedge e^6 - e^3 \wedge e^7), \]
\[ \Theta_{17} = \frac{1}{r^2} (e^0 \wedge e^2 + e^1 \wedge e^7 + e^3 \wedge e^5 - e^4 \wedge e^6), \]
\[ \Theta_{34} = \frac{4}{r^2} (e^1 \wedge e^2 + e^3 \wedge e^4) + \frac{2}{r^4} (e^0 \wedge e^7 - e^5 \wedge e^6), \]
\[ \Theta_{35} = \frac{1}{r^2} (e^0 \wedge e^2 + e^1 \wedge e^7 + e^3 \wedge e^5 - e^4 \wedge e^6), \]
\[ \Theta_{37} = \frac{1}{r^4} (e^0 \wedge e^4 + e^1 \wedge e^5 + e^2 \wedge e^6 - e^3 \wedge e^7), \]
\[
\Theta_{56} = -\frac{2}{r^4} (e^1 \wedge e^2 + e^3 \wedge e^4) - \frac{4}{r^0} (e^0 \wedge e^7 - e^5 \wedge e^6),
\]
\[
\Theta_{57} = -\frac{2}{r^6} (e^0 \wedge e^6 - e^5 \wedge e^7),
\]

where we are using the orthonormal basis (3.3). (Note that we have listed only those which are “inequivalent.” For example, \(\Theta_{02}\) will be just like \(\Theta_{01}\), with appropriate renumberings.)

It is interesting to note that here, and indeed for the hyper-Kähler Calabi metrics in all dimensions \(D = 4n + 4 \geq 8\), certain orthonormal components of the Riemann tensor fall off only as \(1/r^2\). These are components that are absent when \(n = 0\), and so the curvature of the Eguchi-Hanson metric falls of exceptionally rapidly, in comparison to the hyper-Kähler Calabi metrics in higher dimensions.

### 3.2 Covariantly-constant spinors

From the expressions (3.7) for the curvature 2-forms of the 8-dimensional Calabi metric, it is easy to study the integrability conditions \(\Theta_{ab} \tilde{\Gamma}_{ab} \eta = 0\) for covariantly-constant spinors, and hence to determine the number of independent ones. It is not hard to establish that in fact the content of these integrability conditions is completely implied by the following subset of the conditions, namely

\[
(\Gamma_{07} - \Gamma_{56}) \eta = (\Gamma_{12} + \Gamma_{34}) \eta = (\Gamma_{01} - \Gamma_{27} - \Gamma_{30} - \Gamma_{45}) \eta = 0.
\]

(3.8)

Furthermore, it follows from these that there are exactly 3 linearly-independent covariantly-constant spinors, as one would expect for a hyper-Kähler metric in eight dimensions. In fact, in the natural choice of spin frame following from (3.3), it follows that the conditions \(D \eta \equiv d \eta + \frac{1}{4} \omega_{ab} \Gamma_{ab} \eta = 0\) for covariant-constancy reduce, after using (3.8), to simply \(d \eta = 0\). Thus in this frame the three covariantly-constant spinors are just the three solutions of the algebraic conditions (3.8), with constant components.

### 4 Superpotentials and first-order systems

In this section, we shall make a more detailed investigation of the equations resulting from requiring Ricci-flatness for the general class of metrics of cohomogeneity one on \(T^* \mathbb{C}P^{n+1}\) that were introduced in section 2. In particular, we shall show how one can obtain the hyper-Kähler Calabi metrics as solutions of a first-order system that follows from a superpotential. This is achieved by first making certain algebraic restrictions on the metric functions \(a, b, c\) and \(f\) appearing in the ansatz (2.7). We shall also show how a different first-order system,
with an *inequivalent* superpotential, can be obtained by making a different set of algebraic restrictions on \(a, b, c\) and \(f\). In this latter case, the first-order system implies Ricci-flat Kähler geometry, with \(SU(2n + 2)\) holonomy, rather than the hyper-Kähler geometry, with \(Sp(n + 1)\) holonomy, of the previous first-order system. For general dimensions \(D = 4n + 4\) the unrestricted system, with all four of \(a, b, c\) and \(f\) remaining as independent dynamical variables, appears not to admit a description in terms of first-order equations and a superpotential. However, we find that such a description is possible for the special case of \(n = 1\), corresponding to \(D = 8\) dimensions. In fact the resulting first-order equations imply \(Spin(7)\) holonomy, which is an exceptional case in the Berger classification \[17\]. This presumably accounts for the non-existence of a superpotential for the general 4-variable system in dimensions \(D = 4n + 4 \geq 12\). In the following three subsections we derive the 4-variable superpotential for \(Spin(7)\) holonomy in \(D = 8\), and then in arbitrary dimensions \(D = 4n + 4\) the 2-variable superpotential for hyper-Kähler \(Sp(n + 1)\) holonomy and the inequivalent 2-variable superpotential for Ricci-flat Kähler \(SU(2n + 2)\) holonomy.

### 4.1 Superpotential for \(Spin(7)\) holonomy in \(D = 8\)

By considering the \(G_{00}\) component of the Einstein tensor constructed from \(R_{AB}\) in section 2.2, which defines a Hamiltonian \(H = T + V\), we can derive the equations for Ricci-flatness from the Lagrangian \(L = T - V\), together with the constraint \(H = 0\). Specialising to \(n = 1\), we find that after changing radial variable to \(\eta\) defined by \(dt = (a b c)^2 f d\eta\), \(T\) and \(V\) are given by

\[
\begin{align*}
T &= 2\alpha'^2 + 2\beta'^2 + 2\gamma'^2 + 8\alpha' \beta' + 8\alpha' \gamma' + 8\beta' \gamma' + 4\alpha' \sigma' + 4\beta' \sigma' + 4\gamma' \sigma', \\
V &= -12 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) + 2 \left( \frac{a^4 + b^4 + c^4}{a^2 b^2 c^2} \right) + 2f^2 \left( \frac{1}{a^4} + \frac{1}{b^4} \right) + \frac{8f^2}{c^4},
\end{align*}
\]

(4.1)

where \(a = e^\alpha, b = e^\beta, c = e^\gamma\) and \(f = e^\sigma\). The prime denotes a derivative with respect to the new radial variable \(\eta\).

Defining the “DeWitt metric” \(g_{ij}\) by \(T = \frac{1}{2}g_{ij} \alpha^i \alpha^j\), where \(\alpha^i = (\alpha, \beta, \gamma, \sigma)\), we find after some calculation that the potential \(V\) can be written in terms of a superpotential \(W\) as

\[
V = -\frac{1}{2}g^{ij} \frac{\partial W}{\partial \alpha^i} \frac{\partial W}{\partial \alpha^j},
\]

(4.2)

with \(W\) given by

\[
W = 4abc f (a^2 + b^2 + c^2) - 2f^2 (2a^2 b^2 - a^2 c^2 - b^2 c^2).
\]

(4.3)
The first-order equations following from this superpotential formulation of the equations, namely $\alpha'' = g^{ij} \partial W/\partial \alpha^j$, are

$$
\begin{align*}
\alpha' &= -e^{3\alpha+\beta+\gamma+\sigma} + e^{\alpha+3\beta+\gamma+\sigma} + e^{\alpha+\beta+3\gamma+\sigma} - e^{2\beta+2\gamma+2\sigma}, \\
\beta' &= e^{3\alpha+\beta+\gamma+\sigma} - e^{\alpha+3\beta+\gamma+\sigma} + e^{\alpha+\beta+3\gamma+\sigma} - e^{2\alpha+2\gamma+2\sigma}, \\
\gamma' &= e^{3\alpha+\beta+\gamma+\sigma} + e^{\alpha+3\beta+\gamma+\sigma} - e^{\alpha+\beta+3\gamma+\sigma} + 2e^{2\alpha+2\beta+2\sigma}, \\
\sigma' &= -2e^{2\alpha+2\beta+2\gamma+2\sigma} + e^{2\beta+2\gamma+2\sigma} + e^{2\alpha+2\gamma+2\sigma},
\end{align*}
$$

Note that another way of writing them is to go back to the original radial variable $t$. In terms of this, we have

$$
\begin{align*}
\dot{\alpha} &= \frac{b^2 + c^2 - a^2}{abc} - \frac{f}{a^2}, \\
\dot{\beta} &= \frac{a^2 + c^2 - b^2}{abc} - \frac{f}{b^2}, \\
\dot{\gamma} &= \frac{a^2 + b^2 - c^2}{abc} + \frac{2f}{c^2}, \\
\dot{\sigma} &= -\frac{2f}{c^2} + \frac{f}{a^2} + \frac{f}{b^2}.
\end{align*}
$$

If we substitute the above first-order equations into the expressions for the curvature 2-forms, and then calculate the integrability condition for covariantly-constant spinors, $\Theta_{ab} \Gamma_{ab} \eta = 0$, we find that there is exactly one solution. Thus these are the first-order equations for Spin(7) holonomy.

It is not clear how to solve these first-order equations in general. However, a special solution can be obtained by setting $b = a$ and $f = -c/2$, which, it can be verified, is consistent with (4.3). The equations then reduce to $\dot{\alpha} = 3c/(2a)$ and $\dot{c} = 1 - c^2/a^2$, and it is easily seen that in terms of a new radial variable $r$ defined by $dr = c dt$, these equations lead to the solution $a^2 = 9r^2/10$ and $c^2 = 1 - r^{-10/3}$. Thus we obtain the 8-dimensional Ricci-flat metric

$$
\begin{align*}
ds_8^2 &= \frac{dr^2}{1 - r^{-10/3}} + \frac{9r^2}{100} \left(1 - r^{-10/3}\right) \left(\lambda^2 + 4\nu_1^2 + 4\nu_2^2\right) + \frac{9r^2}{10} \left(\sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2\right).
\end{align*}
$$

This is structurally very similar to the 8-metric of Spin(7) holonomy constructed in [24, 25], which was defined on the complete manifold of an $\mathbb{R}^4$ bundle over $S^4$. The difference here is that the manifold for (4.6) is the analogous $\mathbb{R}^4$ bundle over $\mathbb{C}P^2$. Indeed, one can easily verify by looking at the integrability conditions $\Theta_{ab} \Gamma_{ab} \eta = 0$ for covariantly-constant spinors that there is exactly one solution, and so the metric (4.3) has Spin(7) holonomy. Unfortunately it suffers from a conical singularity at $r = 1$, since the topology of the 3-dimensional fibres over $\mathbb{C}P^2$, with their metric $(\lambda^2 + 4\nu_1^2 + 4\nu_2^2)$, is $RP^3$ rather than $S^3$. 

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4.2 Lagrangian formulation for the general $D = 4n + 4$ metrics

In dimensions $D = 4n + 4 \geq 12$ we can obtain the general 4-variable Ricci-flat equations from a Lagrangian $L = T - V$, together with the constraint $T + V = 0$, where

$$
T = 2n(2n-1)(\alpha'^2 + \beta'^2) + 2\gamma'^2 + 8n^2 \alpha' \beta' + 8n (\alpha' \gamma' + \beta' \gamma') + 4n (\alpha' \sigma' + \beta' \sigma') + 4\gamma' \sigma',
$$

$$
V = -4(n+2)\left(\frac{n}{a^2} + \frac{n}{b^2} + \frac{1}{c^2}\right) + 2n \left(\frac{a^4 + b^4 + c^4}{a^2 b^2 c^2}\right) + 2n f^2 \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{8}{c^4}\right).
$$

As usual we have written $a = e^\alpha$, $b = e^\beta$, $c = e^\gamma$, $f = e^\sigma$, and a prime denotes a derivative with respect to the new radial variable $\eta$ defined by $dt = (ab)^{2n} c^2 d\eta$.

From the kinetic energy in (4.7), which may be written as $T = \frac{1}{2} g_{ij} \dot{\alpha}^i \dot{\alpha}^j$, where $\alpha^i = (\alpha, \beta \gamma, \sigma)$, we can read off the metric $g_{ij}$:

$$
g_{ij} = \begin{pmatrix}
4n(2n-1) & 8n^2 & 8n & 4n \\
8n^2 & 4n(2n-1) & 8n & 4n \\
8n & 8n & 4 & 4 \\
4n & 4n & 4 & 0
\end{pmatrix}.
$$

Although $V$ can be written in terms of a superpotential, as in (4.2), (4.3), for the special case $D = 4n + 4 = 8$, this does not appear to be possible for $n \geq 2$. This is not unreasonable, since the occurrence of Spin(7) as a holonomy group in $D = 8$ is exceptional, with no higher-dimensional analogue.

4.3 Superpotentials for hyper-Kähler geometry

In this subsection, we shall show that by imposing an appropriate set of algebraic restrictions on the four metric functions $a$, $b$, $c$ and $f$, it is possible to reduce the general Lagrangian formulation of the previous subsection to one that does allow a reformulation in terms of a superpotential for all dimensions $D = 4n + 4$, for which the associated first-order equations imply the special holonomy $Sp(n + 1)$ of hyper-Kähler geometry. Specifically, we impose the two algebraic equations contained within (2.15) and (2.16), namely

$$
a^2 + b^2 = c^2, \quad ab = cf.
$$

It is easily verified that these conditions are compatible with the Einstein equations.

Using these conditions to eliminate $c$ and $f$, we find that for the reduced system where only $a$ and $b$ remain, the kinetic and potential energies in (4.7) become

$$
T = \frac{1}{2} g_{ij} \frac{\partial \alpha^i}{\partial \eta} \frac{\partial \alpha^j}{\partial \eta}.
$$
\[ V = -\frac{2(ab)^{4n}}{(a^2 + b^2)^2} \left[ n(2n + 1) a^8 + 2(4n^2 + 5n + 2) a^6 b^2 + 2(6n^2 + 9n + 2) a^4 b^4 + 2(4n^2 + 5n + 2) a^2 b^6 + n(2n + 1) b^8 \right]. \] (4.10)

where \( \alpha^i = (a, b) \), and the new radial variable \( \eta \) is now given by \( dt = (ab)^{2n} c^2 f d\eta = (ab)^{2n+1} (a^2 + b^2)^{1/2} d\eta \). The sigma-model metric components in \( T \) are given by

\[
\begin{align*}
g_{11} &= 4 \left( \frac{n+1)(2n+1) a^4 + 2(2n^2 + 2n + 1) a^2 b^2 + n(2n+1) b^4}{(a^2 + b^2)^2} \right), \\
g_{12} &= 4 \left( \frac{n+1)(2n+1) a^4 + (4n^2 + 6n+1) a^2 b^2 + (n+1)(2n+1) b^4}{(a^2 + b^2)^2} \right), \\
g_{22} &= 4 \left( \frac{n(2n+1) a^4 + 2(2n^2 + 2n+1) a^2 b^2 + (n+1)(2n+1) b^4}{(a^2 + b^2)^2} \right). 
\end{align*}
\] (4.11)

Note that because of the non-linear nature of the algebraic substitution in (4.9), the sigma-model is now a non-linear one.

We find that \( V \) can now be expressed in terms of a superpotential, \( V = -\frac{1}{2} g^{ij} \partial_i W \partial_j W \), where \( W \) is given by

\[ W = \frac{2(ab)^{2n}}{a^2 + b^2} \left[ (2n+1) (a^4 + b^4) + 4(n+1) a^2 b^2 \right]. \] (4.12)

The first order equations derived from the superpotential are rather simple, given (in terms of the original radial variable \( t \)) by

\[
\dot{a} = \frac{b}{\sqrt{a^2 + b^2}}, \quad \dot{b} = -\frac{a}{\sqrt{a^2 + b^2}}. \] (4.13)

Despite the rather complicated \( n \)-dependence of \( T \) and \( V \), these first-order equations are independent of \( n \). This should not be surprising, since we expect that after imposing the algebraic restrictions (4.9), the solutions should reproduce the Calabi hyper-Kähler metrics, for which the metric functions take the \( n \)-independent form (2.17). One can in fact verify that after substituting (4.3) and (1.13) into the general expressions for the Riemann curvature, the integrability conditions \( \Theta_{ab} \Gamma_{ab} \eta = 0 \) imply the existence of \( (n+2) \) covariantly-constant spinors, as one expects for hyper-Kähler metrics in \( D = 4n + 4 \) dimensions.

One can of course directly the first-order equations equations (4.13), and one then obtains the hyper-Kähler Calabi solutions. If we introduce \( \rho \) by defining \( dt = (a^2 + b^2)^{1/2} d\rho \) then (1.13) are easily solved for \( a \) and \( b \). Substituting back to get the remaining metric functions, we then obtain

\[
a = \sinh \rho, \quad b = \cosh \rho, \quad c^2 = \cosh 2\rho, \quad f^2 = \frac{\sinh^2 2\rho}{4\cosh 2\rho}, \quad h^2 = \cosh 2\rho. \] (4.14)
The further coordinate redefinition \( r^2 = \cosh 2\rho \) gives back the previous form of the solution,
\[
a^2 = \frac{1}{2}(r^2 - 1), \quad b^2 = \frac{1}{2}(r^2 + 1), \quad c^2 = r^2, \quad f^2 = \frac{1}{4}r^2(1 - r^{-4}), \quad h^2 = (1 - r^{-4})^{-1}.
\]
\[ (4.15) \]

### 4.4 Superpotentials for Ricci-flat Kähler geometry

There is a different specialisation of the original \((a, b, c, f)\) system that can be made, which allows the construction of a different superpotential whose first-order equations imply \(SU(2n + 2)\) holonomy, i.e. Ricci-flat Kähler geometry. To obtain this case, we need to set
\[
b = a, \quad c = (\sqrt{2})a,
\]
leaving just \(a\) and \(f\) as dynamical variables. These conditions are compatible with the Einstein equations. In this case the kinetic and potential energies in \((4.17)\) reduce to
\[
T = 2(2n + 1)(4n + 1)\alpha'^2 + 4(2n + 1)\alpha'\sigma',
\]
\[
V = -8(2n + 1)\alpha'^{8n} f^2 [2(n + 1)a^2 - f^2],
\]
where \(\sigma'\) denotes a derivative with respect to \(\eta\), now given by \(dt = 2a^{4n+2} f d\eta\). Thus we have
\[
g_{ij} = 4(2n + 1) \begin{pmatrix} 4n + 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]
\[ (4.18) \]

We find that this system can be described by a superpotential \(W\), such that \(V = -\frac{1}{2}g^{ij} \partial_i W \partial_j W\), which is given by
\[
W = 4a^{4n} [(n + 1) a^2 + (2n + 1) f^2].
\]
\[ (4.19) \]

The associated first-order equations are then
\[
\dot{a} = \frac{f}{a}, \quad \dot{f} = \frac{(n + 1) a^2 - (2n + 1) f^2}{a^2},
\]
when expressed in terms of the original radial variable \(t\). It should be emphasised that in the special case of \(n = 1\), where we were able to obtain the superpotential \((4.3)\) for Spin(7) holonomy, the superpotential in \((4.19)\) with \(n = 1\) is not simply the one in \((4.3)\) subject to the constraint \((4.16)\). The superpotential \((4.3)\) has an overall factor \(f\), but this is not the case for the superpotential \((4.19)\).

One can verify from the integrability conditions that these first-order equations imply the existence of two covariantly-constant spinors, and hence \(SU(2n + 2)\) holonomy. This corresponds to Ricci-flat Kähler geometry.
It is easy to solve the first-order equations (4.20). If we define a new radial variable $r$ by $dt = a/(\sqrt{2} f) dr$, then the first equation immediately implies $a = r/\sqrt{2}$. It is then straightforward to solve for $f$. From these, the rest of the metric functions follow. Thus we find

$$a^2 = b^2 = \frac{1}{2} r^2, \quad c^2 = r^2, \quad f^2 = \frac{1}{4} U r^2, \quad h^2 = U^{-1},$$

(4.21)

where

$$U = 1 - \left(\frac{r_0}{r}\right)^{4n+4}.$$  (4.22)

The resulting Ricci-flat Kähler metric in fact lies within the class constructed in [21, 22]. It is a complex line bundle over the homogeneous Einstein-Kähler manifold $SU(n+2)/(U(n) \times U(1))$.

5 Harmonic forms in the Calabi metrics

In this section, we construct $L^2$-normalisable middle-dimension harmonic forms in the Calabi metrics of dimension 8 and 12. Our principal interest will be in the case of the 8-dimensional hyper-Kähler Calabi metric, since this will allow us to obtain an explicit solution for a new resolved M2-brane. By way of an introduction, we shall begin by presenting the already-known $L^2$-normalisable harmonic 2-form in the 4-dimensional Calabi metric (i.e. the Eguchi-Hanson solution). In the subsequent two subsections following this, we present our new results for the $L^2$-normalisable harmonic 4-form and 6-form in the hyper-Kähler Calabi metrics of dimensions 8 and 12 respectively. A fourth subsection contains a discussion of a regular but non $L^2$-normalisable harmonic 2-form in the Calabi metrics of arbitrary dimension.

5.1 $L^2$-normalisable harmonic 2-form in $D = 4$

The 4-dimensional Calabi metric arises by taking $n = 0$ in the formulae in section 2, in which case (2.11) and (2.17) reduce to the well-known Eguchi-Hanson metric. In this specific case we adopt the notation that the orthonormal frame $(e^0, e^1, e^2, e^3)$ of the general formalism in section 2 will be written simply as $(e^0, e^1, e^2, e^3)$, and so

$$e^0 = h dr, \quad e^1 = c \nu_1, \quad e^2 = c \nu_2, \quad e^3 = f \lambda.$$  (5.1)

The three Kähler forms (2.20) become

$$J_1 = e^0 \wedge e^3 + e^1 \wedge e^2, \quad J_2 = e^0 \wedge e^1 + e^2 \wedge e^3, \quad J_3 = e^0 \wedge e^2 + e^3 \wedge e^1.$$  (5.2)
There is an $L^2$-normalisable harmonic 2-form, given by

$$G_{(2)} = \frac{1}{r^4} (e^0 \wedge e^3 - e^1 \wedge e^2).$$

(5.3)

Note that in conventions where the Kähler forms are self-dual, this harmonic form is anti-self-dual.

We may define an holomorphic complex vielbein $(\tilde{e}^0, \tilde{e}^1)$ with respect to any of the three complex structures. From (5.2), these can be chosen as follows:

$$J_1: \quad \tilde{e}^0 = e^0 + i e^3, \quad \tilde{e}^1 = e^1 + i e^2,$$

$$J_2: \quad \tilde{e}^0 = e^0 + i e^1, \quad \tilde{e}^1 = e^2 + i e^3,$$

$$J_3: \quad \tilde{e}^0 = e^0 + i e^2, \quad \tilde{e}^1 = e^3 + i e^1.$$ (5.4)

In terms of these three different holomorphic bases, we see that the harmonic 2-form (5.3) is given by

$$J_1: \quad G_{(2)} = \frac{i}{2r^4} (e^0 \wedge \tilde{e}^0 - \tilde{e}^1 \wedge \tilde{e}^1),$$

$$J_2: \quad G_{(2)} = \frac{i}{2r^4} (e^0 \wedge \tilde{e}^1 - \tilde{e}^0 \wedge \tilde{e}^1),$$

$$J_3: \quad G_{(2)} = \frac{1}{2r^4} (e^0 \wedge \tilde{e}^1 + \tilde{e}^0 \wedge \tilde{e}^1),$$ (5.5)

Note that $G_{(2)}$ is a type $(1, 1)$ form with respect to all three complex structures $J_i$, and it is orthogonal to all three of the $J_i$. This normalisable harmonic 2-form was used to construct resolved supersymmetric heterotic 5-brane and dyonic string in [9].

### 5.2 $L^2$-normalisable harmonic 4-form in $D = 8$

The structure of the middle-dimension harmonic forms in the higher-dimensional Calabi metrics is somewhat more complicated. In this section, we obtain an explicit expression for an $L^2$-normalisable harmonic 4-form in the 8-dimensional Calabi metric.

We begin by considering the 4-form structures that arise from calculating $\frac{1}{2} J_1 \wedge J_1$, $\frac{1}{2} J_2 \wedge J_2$ and $\frac{1}{2} J_3 \wedge J_3$. These are in fact all anti-self-dual, in our conventions. Then, we shall take these structures, replace the coefficients “1” of each anti-self-dual pair of terms by an arbitrary function of $r$, and adopt this as our ansatz for an anti-self-dual harmonic 4-form $G_{(4)}$. (We may also consider an ansatz for a self-dual harmonic 4-form where now we first change the previous anti-self-dual pairs, and then give these arbitrary functions of $r$ as coefficients. However, this does not give any $L^2$-normalisable harmonic 4-form.)
From (3.6), it follows that the ansatz for \( G_{(4)} \) will be
\[
G_{(4)} = f_1 (e^0 \wedge e^7 \wedge e^1 \wedge e^2 - e^3 \wedge e^4 \wedge e^5 \wedge e^6) \\
+ f_2 (e^0 \wedge e^7 \wedge e^3 \wedge e^4 - e^1 \wedge e^2 \wedge e^5 \wedge e^6) \\
+ f_3 (e^0 \wedge e^7 \wedge e^5 \wedge e^6 - e^1 \wedge e^2 \wedge e^3 \wedge e^4) \\
+ g_1 (e^0 \wedge e^1 \wedge e^4 \wedge e^5 - e^2 \wedge e^3 \wedge e^6 \wedge e^7) \\
+ g_2 (e^0 \wedge e^2 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 \wedge e^7) \\
+ h_1 (e^0 \wedge e^1 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5 \wedge e^7) \\
+ h_2 (e^0 \wedge e^2 \wedge e^4 \wedge e^6 - e^1 \wedge e^3 \wedge e^5 \wedge e^7). \tag{5.6}
\]

(Note that some structures from \( J_2 \wedge J_2 \) and \( J_3 \wedge J_3 \) just produce a repetition of ones already present from \( J_1 \wedge J_1 \), and so we have only 7 functions and not 9.)

The vielbeins are given by (3.3), the metric functions are given by (4.15), and the exterior derivatives of the various 1-forms in (3.3) are given by (3.4). The 1-form \( Q \), which lies outside the \( SU(3)/U(1) \) coset, should drop out of the expressions for \( dG_{(4)} \), and this implies the following further conditions
\[
g_2 = -g_1, \quad h_2 = h_1. \tag{5.7}
\]

Imposing \( dG_{(4)} = 0 \) now gives a system of five first-order equations for the remaining five functions. We find that there is a solution regular at \( r = 1 \), given by
\[
f_2 = \frac{1}{r^4(r^2 + 1)^2}, \quad h_1 = \frac{1}{r^2(r^2 + 1)^3}, \tag{5.8}
\]

\[
\text{together with}
\]
\[
f_1 = f_2 + 2h_1, \quad f_3 = -2h_1, \quad g_1 = h_1, \quad g_2 = -h_1, \quad h_2 = h_1. \tag{5.9}
\]

Thus the harmonic 4-form is given by
\[
G_{(4)} = f_2 (e^0 \wedge e^7 \wedge e^5 \wedge e^6) \wedge (e^1 \wedge e^2 + e^3 \wedge e^4) + 2h_1 (e^0 \wedge e^7 + e^3 \wedge e^4) \wedge (e^1 \wedge e^2 - e^5 \wedge e^6) \\
+ h_1 (e^0 \wedge e^6 - e^5 \wedge e^7) \wedge (e^1 \wedge e^3 + e^2 \wedge e^4) \\
+ h_1 (e^0 \wedge e^5 + e^6 \wedge e^7) \wedge (e^1 \wedge e^4 - e^2 \wedge e^3). \tag{5.10}
\]

The magnitude of \( G_{(4)} \) is given by
\[
|G_{(4)}|^2 = 96(f_2^2 + 2f_2 h_1 + 6h_1^2) = \frac{96(r^8 + 6r^6 + 16r^4 + 6r^2 + 1)}{r^8(r^2 + 1)^6}. \tag{5.11}
\]
It is evident that $G_{(4)}$ is $L^2$-normalisable, since it is regular at $r = 1$ and $|G_{(4)}|^2$ falls off as $1/r^{12}$ at large $r$. In section 6, we shall use this harmonic form in order to obtain a new explicit resolved M2-brane solution.

It is again instructive to re-express the harmonic 4-form in terms of a complex holomorphic vielbein basis. As in the previous subsection, we can use any of the three $J_i$ in order to define the complex structure. Let us first consider $J_1$. From (2.20), after changing to our $D = 8$ notation, we see that using $J_1$ as the complex structure we can define the holomorphic vielbeins

$$
\begin{align*}
\epsilon^0 &= e^0 + ie^7, \quad \epsilon^1 = e^1 + ie^2, \quad \epsilon^2 = e^3 + ie^3, \quad \epsilon^3 = e^5 + ie^6. 
\end{align*}
$$

(5.12)

It is useful also to define the following real 2-forms:

$$
\begin{align*}
x_0 &= i\epsilon^0 \wedge \bar{\epsilon}^0, \quad x_1 = i\epsilon^1 \wedge \bar{\epsilon}^1, \quad x_2 = i\epsilon^2 \wedge \bar{\epsilon}^2, \quad x_3 = i\epsilon^3 \wedge \bar{\epsilon}^3, 
\end{align*}
$$

(5.13)

In terms of these, we find that the harmonic 4-form (5.10) is given by

$$
G_{(4)} = f (x_0 - x_3) (x_1 - x_2) + 2g [ (x_0 - x_2) (x_1 - x_3) - i\epsilon^0 \wedge \bar{\epsilon}^3 \wedge \bar{\epsilon}^1 \wedge \epsilon^2 + i\epsilon^0 \wedge \bar{\epsilon}^3 \wedge \epsilon^1 \wedge \epsilon^2].
$$

(5.14)

where the functions $f$ and $g$ are given by

$$
\begin{align*}
f &= \frac{1}{r^4 (r^2 + 1)}, \quad g = \frac{1}{r^2 (r^2 + 1)^3}.
\end{align*}
$$

(5.15)

If instead we use a complex holomorphic vielbein basis defined by $J_2$, then we have

$$
\begin{align*}
\epsilon^0 &= e^0 + ie^5, \quad \epsilon^3 = e^6 + ie^7, \quad \epsilon^1 = e^1 + ie^4, \quad \epsilon^2 = e^2 - ie^3.
\end{align*}
$$

(5.16)

In terms of these, we now find that (5.10) is given by $G_{(4)} = fU_1 + gU_2$, with $f$ and $g$ given by (5.15) and

$$
\begin{align*}
U_1 &= i(\epsilon^0 \wedge \epsilon^3 - \epsilon^0 \wedge \epsilon^3) \wedge (\epsilon^1 \wedge \epsilon^2 + \epsilon^1 \wedge \epsilon^2), \\
U_2 &= U_1 + (x_0 + x_3)(x_1 + x_2) - 2x_1x_2 - 2x_0x_3 - 2i(\epsilon^0 \wedge \epsilon^3 \wedge \epsilon^1 \wedge \epsilon^2 - \epsilon^0 \wedge \epsilon^3 \wedge \epsilon^1 \wedge \epsilon^2).
\end{align*}
$$

(5.17)

Finally, we could instead use $J_3$ to define the complex structure. This gives an expression for $G_{(4)}$ that is very similar to (5.17). It is worth remarking that $G_{(4)}$ is of type $(2, 2)$ with respect to all three of the complex structures $J_i$. It is also orthogonal to all three $J_i$. 

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5.3 $L^2$-normalisable harmonic 6-form in $D=12$

The discussion becomes more complicated for the Calabi metric in the next higher dimension, namely for the harmonic 6-form in $D=12$. First, we consider a complex vielbein that is holomorphic with respect to the complex structure $J_1$:

$$e^0 = e^0 + i e^5, \quad e^1 = e^{1,1} + i e^{2,1}, \quad e^2 = e^{1,2} + i e^{2,2},$$
$$e^3 = e^{1,1} - i e^{2,1}, \quad e^4 = e^{1,2} - i e^{2,2}, \quad e^5 = e^1 + i e^2. \quad (5.18)$$

We also define the following (real) 2-forms:

$$x_0 = i e^0 \wedge e^0, \quad x_1 = i e^1 \wedge e^1, \quad x_2 = i e^2 \wedge e^2, \quad x_3 = i e^3 \wedge e^3, \quad x_4 = i e^4 \wedge e^4, \quad x_5 = i e^5 \wedge e^5. \quad (5.19)$$

We then proceeded by making a rather general ansatz for the 6-form, with as-yet undetermined functions of $r$ as coefficients for each structure in the ansatz. After calculations of some complexity, we find after imposing (anti) self-duality, and $dG_{(6)} = 0$, that there is an $L^2$-normalisable harmonic 6-form, given by

$$G_{(6)} = f (x_0 - x_5) [3x_1 x_2 + 3x_3 x_4 - x_1 x_3 - x_2 x_4 - 2x_1 x_4 - 2x_2 x_3$$
$$- e^1 \wedge e^2 \wedge e^3 \wedge e^4 - e^1 \wedge e^2 \wedge e^3 \wedge e^4]$$
$$+ 2g \left[ S - (x_0 - x_5) (e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^3 \wedge e^4)$$
$$- 3i (x_1 - x_3) (e^0 \wedge e^5 \wedge e^0 \wedge e^5 \wedge e^2 \wedge e^4)$$
$$- 3i (x_2 - x_4) (e^0 \wedge e^5 \wedge e^3 \wedge e^3 - e^0 \wedge e^5 \wedge e^3 \wedge e^3) \right], \quad (5.20)$$

where

$$S \equiv (x_0 - x_5) (3x_1 x_2 + 3x_3 x_4 - x_1 x_3 - x_2 x_4 - 2x_1 x_4 - 2x_2 x_3)$$
$$+ 3(x_0 + x_5)(x_1 x_2 - x_3 x_4) - 3x_0 x_5 (x_1 + x_2 - x_3 - x_4)$$
$$- 3x_1 x_2 (x_3 + x_4) + 3x_3 x_4 (x_1 + x_2). \quad (5.21)$$

The two functions $f$ and $g$ are given by

$$f = \frac{1}{r^4 (r^2 + 1)^2}, \quad g = \frac{1}{r^2 (r^2 + 1)^4}. \quad (5.22)$$

The harmonic 6-form is normalisable, and anti-self-dual. It is orthogonal to all three Kähler forms $J_i$, and it is manifest from (5.20) and (5.21) that it is of type $(3, 3)$. The magnitude of $G_{(6)}$ is given by

$$|G_{(6)}|^2 = 15 \times 6! (f^2 + 4f g + 16g^2). \quad (5.23)$$
We may instead express the harmonic 6-form in terms of a complex basis that is holomorphic with respect to the complex structure $J_2$. From (2.20), we may choose

$$
\epsilon^0 = e^0 + i e^1, \quad \epsilon^5 = e^5 - i e^6, \\
\epsilon^1 = e^{1,1} + i e^{2,1}, \quad \epsilon^2 = e^{1,2} + i e^{2,2}, \quad \epsilon^3 = e^{2,1} - i e^{1,1}, \quad \epsilon^4 = e^{2,2} - i e^{1,2}.
$$

(5.24)

In terms of this basis, we find $G_6 = f U_1 + g U_2$ with $f$ and $g$ given again by (5.22), and

$$
U_1 = e^0 \wedge e^5 \wedge [(x_1 + x_3)(x_2 + x_4) - 2x_1 x_3 - 2x_2 x_4 - 5P \\
- 2e^1 \wedge e^3 \wedge e^1 - 2\epsilon^1 \wedge \epsilon^3 \wedge \epsilon^4] + \text{c.c.,}
$$

$$
U_2 = U_1 - 6e^0 \wedge e^5 \wedge S - 6e^0 \wedge e^5 \wedge \bar{S} \\
+ 3(e^1 \wedge \epsilon^3 + e^1 \wedge \epsilon^3) \wedge [(x_0 + x_5)(x_2 + x_4) - 2x_2 x_4 - 2x_0 x_5] \\
+ 3(e^2 \wedge \epsilon^4 + e^2 \wedge \epsilon^4) \wedge [(x_0 + x_5)(x_1 + x_3) - 2x_1 x_3 - 2x_0 x_5],
$$

(5.25)

where c.c. indicates that the complex conjugate of the entire expression presented for $U_1$ is to be added, and here we have defined

$$
P \equiv e^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 + e^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 + e^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 + e^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4,
$$

$$
S \equiv e^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 + e^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 + e^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4.
$$

(5.26)

Note that $G_6$ is of type $(3,3)$ in this basis also.

### 5.4 A non-normalisable harmonic 2-form

Consider the $U(1)$ Killing vector dual to the direction of the 1-form $\lambda$. If we lower the index, we get a Killing 1-form

$$
A = f^2 \lambda.
$$

(5.27)

Because it is a Killing 1-form, and the metric is Ricci-flat, we have that $d^* dA = 0$, as well, of course, as $d dA = 0$, and so $dA$ is harmonic. We find that

$$
dA = \frac{2f'}{h} e^0 \wedge e^5 + \frac{2f^2}{a^2} e^{1\alpha} \wedge e^{2\alpha} - \frac{2f^2}{b^2} e^{1\bar{\alpha}} \wedge e^{2\bar{\alpha}} + \frac{4f^2}{c^2} \epsilon^1 \wedge \epsilon^2,
$$

\[ \equiv J_1 + G_{(2)}, \tag{5.28} \]

where $J_1$ is the first Kähler form in (2.20), and

$$
G_{(2)} = \frac{1}{r^4} (e^0 \wedge e^5 - \epsilon^1 \wedge \epsilon^2) + \frac{1}{r^2} \left( e^{1\alpha} \wedge e^{2\alpha} + e^{1\bar{\alpha}} \wedge e^{2\bar{\alpha}} \right).
$$

(5.29)
Since we know that \( J_1 \) itself, being a Kähler form, is harmonic, and we know that \( dA \) is harmonic, it follows that \( G_{(2)} \) is harmonic. It is not in general \( L^2 \) normalisable, since \( |G_{(2)}|^2 \) is proportional to \( r^{-4} \), but it is regular and square-integrable at \( r = 1 \). Note that in the special case of \( D = 4n + 4 = 4 \) dimensions, for which the Calabi metric is just the Eguchi-Hanson solution, the terms proportional to \( 1/r^2 \) in (5.29) are absent and so \( G_{(2)} \) reduces to precisely the \( L^2 \)-normalisable harmonic 2-form given in (5.3) in this case.

6 A resolved M2-brane in the Calabi 8-metric

6.1 The M2-brane solution, and its supersymmetry

In section 5.2 we constructed an \( L^2 \)-normalisable anti-self-dual harmonic 4-form (5.10) in the eight-dimensional hyper-Kähler Calabi metric. We may use this in order to obtain a new resolved M2-brane solution.

As a consequence of the Chern-Simons modification to the equation of the motion of the 3-form potential in \( D = 11 \) supergravity, namely

\[
d\hat{F}_{(4)} = \frac{1}{2} F_{(4)} \wedge F_{(4)}.
\]

(6.1)
it is possible to construct a deformed M2-brane, given by

\[
ds_{11}^2 = H^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + H^{1/3} ds_8^2,
\]

\[
F_{(4)} = d^3x \wedge dH^{-1} + m G_{(4)},
\]

(6.2)
where \( G_{(4)} \) is the harmonic anti-self-dual 4-form in the Ricci-flat transverse space \( ds_8^2 \), and the function \( H \) satisfies

\[
\Box H = -\frac{1}{16} m^2 G_{(4)}^2.
\]

(6.3)

Warped reductions of this type, were also discussed in [29, 30, 31].

Using the expression (3.2) for \( ds_8^2 \), and (5.11) for \(|G_{(4)}|\), it follows that the solution for the metric function \( H \) in the corresponding resolved M2-brane is

\[
H = c_0 + \frac{m^2}{48} \int_r^\infty dr' \left( \frac{k^2}{\sqrt{g}} \int_1^{r'} dr'' \sqrt{g} |G_{(4)}|^2 \right)
\]

\[
= c_0 + \frac{m^2 (5r^6 + 25r^4 + 48r^2 + 40)}{160r^2 (r^2 + 1)^5}.
\]

(6.4)

At small distance, i.e. \( r \to 1 \), the function \( H \) tends to a constant. At large \( r \), \( H \) has the asymptotic behaviour

\[
H \sim c_0 + \frac{m^2}{32r^6} - \frac{m^2}{80r^{10}} - \frac{m^2}{32r^{14}} + \cdots.
\]

(6.5)
In terms of proper distance $\rho$ defined by $d\rho = h \, dr$, we have

$$H \sim c_0 + \frac{m^2}{32 \rho^6} - \frac{7m^2}{160 \rho^{10}} + \frac{3m^2}{224 \rho^{14}} + \cdots .$$  \hspace{1cm} (6.6)

In the decoupling limit, where $c_0$ can be dropped, the resolved M2-brane approaches AdS$_4 \times \Sigma_7$ at large distance, where $\Sigma_7$ is the $SU(3)/U(1)$ Einstein space known as the $N^{1,1}$ manifold. The solution therefore interpolates between this geometry at large distance and a geometry that is locally $M_3 \times R^4 \times S^4$ at short distance, where $M_3$ denotes 3-dimensional Minkowski spacetime.

We now turn to a discussion of the supersymmetry of our new resolved M2-brane solution. The criterion for supersymmetry is

$$G_{abcd} \Gamma^{bcd} \eta = 0 ,$$  \hspace{1cm} (6.7)

where $\eta$ is a covariantly-constant spinor in the 8-dimensional Ricci-flat metric. Substituting (5.10) into this, and making use of the conditions (3.8) that define the three covariantly-constant spinors in the 8-dimensional Calabi metric, we find that (6.7) is satisfied by all three of the covariantly-constant spinors. To be precise, the constraint (6.7) is consistent with the linear relations amongst the coefficients of the harmonic 4-form given in (5.9). Thus our solution is a supergravity dual of an $\mathcal{N} = 3$ conformal field theory.

It is possible to perform a dimensional reduction of the M2-brane on the principal orbits $N^{1,1}$ of the transverse space, and thereby obtain a four-dimensional domain wall, given by

$$ds_4^2 = r^7 (1 - r^{-4})^{3/2} \left[ H^{1/2} (-dt^2 + dx_1^2 + dx_2^2) + H^{3/2} \frac{dr^2}{1 - r^{-4}} \right],$$  \hspace{1cm} (6.8)

which is asymptotically AdS$_4$. It is easy to verify that the minimally coupled scalar wave equation in this background has a discrete spectrum, indicating the confinement of the dual conformal gauge theory. We shall discuss the properties of the dual field theories in more detail in the following section.

### 6.2 General features of resolved brane solutions

It is worth pausing at this point to make some observations about certain general features of the resolved M2-brane solution, and their relation to corresponding properties of harmonic 4-form. In fact although we shall make these remarks in the specific context of the resolved M2-brane, they are quite general, and apply equally well to any resolved brane solution in which an additional field strength is taken to be proportional to an harmonic form.

There are two characteristics of the harmonic form that are relevant. Firstly, there is the question of regularity, i.e. whether or not the form is well-behaved at the “origin,”
which lies at $r = 1$ in our present example. Secondly, there is the issue of normalisability, and in particular, whether the harmonic form is square-integrable at large distance. If the form is in $L^2$, such as the 4-form that we have obtained in this paper, then it satisfies both the conditions of short-distance regularity and large-distance square-integrability. In other examples, such as the harmonic 3-form in the deformed fractional D3-brane solution of [7], it is regular at the origin but it is not square-integrable at large distance. In yet further examples, such as the fractional D3-brane using the “small resolution” of [26, 27], there is a harmonic form that is neither regular at short distance nor square-integrable at large distance.

The regularity of $G_{(4)}$ at short distance implies that the lower limit in the $r''$ integral in (6.4) can be taken to be the “origin” at $r = r_0 = 1$, which ensures that the solution (6.2) has no naked singularity at $r = 1$. The $L^2$-normalisability of the $G_{(4)}$ ensures that the electric M2-brane charge is then a (non-vanishing) finite constant, given by

$$Q = m^2 \int_{r_0}^{\infty} dr \, \sqrt{g} \, |G_{(4)}|^2.$$  \tag{6.9}

By contrast, in the deformed fractional D3-brane of [7] the analogous D3-brane “charge” grows logarithmically with radius, as a consequence of the lack of large-distance square-integrability. The solution is, however, still regular at short distance. On the other hand in the small resolution of the D3-brane [26, 27], as well as the logarithmic growth of the “charge” resulting from the lack of square-integrability at large distance, there is also a “repulson” singularity at short distance, resulting from the short-distance irregularity of the harmonic 3-form in that example.

7 Comments on dual gauge three-dimensional theories

7.1 M2-brane/CFT$_3$ correspondence

Having obtained the resolved M2-brane on the 8-dimensional hyper-Kähler Calabi metric, it is of interest to study this solution, and others, from the point of view of the AdS$_4$/CFT$_3$ correspondence. A large class of explicit M2-brane solutions on various supersymmetric 8-dimensional manifolds was obtained in [6, 10, 11]. The properties of the 8-dimensional manifolds are tabulated in appendix A. In this section we shall make comments on properties of the associated dual field theories as well.
7.1.1 Symmetry groups of the field theory

The global symmetries of the matter sector of the dual gauge theory is determined by the symmetry of the Ricci flat manifold, specifying the coordinates of the transverse space of the p-brane. Namely, the matter multiplets form representations of the global symmetry group of the SYM theory, which is isomorphic to the isometry group of the transverse space. These multiplets are expected to transform as bi-fundamentals of the SYM gauge factors, in accordance with the quiver theory that specifies the world-volume theory of the resolved brane. The global symmetry groups and the other properties of the various M2-branes constructed in this and the previous papers \([9, 10, 11]\) are summarised in the following table.

| ∏ N | K | Global Symmetry | Flow Para. | γ |
|-----|----|-----------------|------------|
| 1   | \(S^7\) | \(SO(5) \times SO(3)\) | \(4\)   |
| 0   | \(S^7\) | \(U(4)\) | \(8\) |
| 0   | \(N^{1,1}\) | \(U(3)\) | \(8\) |
| 2   | \(M^{2,3}\) | \(SU(3) \times U(2)\) | \(4\) |
| 2   | \(Q^{1,1,1}\) | \(SU(2)^3 \times U(1)\) | \(4\) |
| 2   | \(Q^{1,1,1}\) | \(SU(2)^3 \times U(1)\) | \(4\) |
| 3   | \(N^{1,1}\) | \(U(3)\) | \(4\) |

Table 1: Global symmetries of the transverse spaces, determining the global symmetries of the dual three-dimensional gauge theories for resolved M2-branes. The \(\mathcal{N} = 1\) solution uses the 8-manifold of Spin(7) holonomy. The \(\mathcal{N} = 0\) and \(\mathcal{N} = 2\) examples use Ricci-flat Kähler 8-manifolds. The example with \(K = Q^{1,1,1}\) is listed twice because there are two different resolutions, one with \(S^2 \times S^2\) degenerate orbits, and the other with \(S^2 \times S^2 \times S^2\). In the \(\mathcal{N} = 3\) example the transverse space is the 8-dimensional hyper-Kähler Calabi metric.

It is worth remarking that the two \(\mathcal{N} = 0\) cases in Table 1 arise because the associated harmonic 4-forms do not satisfy the supersymmetry criterion \((6.7)\), even though the 8-metrics themselves admit two covariantly-constant spinors. Thus in these two cases the procedure of breaking the conformal symmetry by resolving the cone metric simultaneously leads to a breaking of supersymmetry.
The “flow parameter” $\gamma$ in Table 1 measures the deviation from an asymptotically AdS$_4$ behaviour at large proper distance $\rho$. It is defined by looking at the behaviour of the metric function $H$ in (6.2) at large distance, in terms of the proper-distance coordinate $\rho$:

$$H = c_0 + \frac{Q}{\rho^6}(1 - \frac{c}{\rho^\gamma} + \cdots). \quad (7.1)$$

The actual determination of the local gauge symmetry and the spectrum of the matter super-multiplets, in the absence of reliable field theory calculations (such as in the case of $S^5/\mathbb{Z}_2$ orbifold blow-up, which leads, in the infra-red, to the SYM theory on the conifold (see [4] in four-dimensional field theory)), often relies on symmetry and an intuitive approach. In certain cases, this is enough. For example, the isometry group of the transverse space for AdS$_5 \times S^5$ is $SO(6) \times U(1)$, and the six scalars form an irreducible (vector) representation of $SO(6)$. It follows that all the six scalars must carry the same representation of the local gauge group, namely $SU(N)$. For more complicated cases such as our resolved branes, such a determination has so far only been achieved for D3-branes or M2-branes in which the transverse-space manifolds are of the toric variety. In these cases the geometric constraints of the conifold allow one to determine the so-called D-term constraints and fix the superpotential interactions for the massless matter supermultiplets. For the $T^{1,1}$ conifold, the local gauge group turns out to be $SU(N) \times SU(N)$, which can be enhanced to $SU(N) \times SU(N + M)$ when additional fractional D5-branes wrap around the 2-cycle of the $T^{1,1}$ space [72]. The D-terms for the $Q^{1,1,1}$ and $M^{2,3}$ spaces were determined in [33] and [34] respectively. From these D-term constraints, the gauge groups can be shown to be $SU(N) \times SU(N) \times SU(N)$ and $SU(N) \times SU(N)$ respectively [8].

### 7.1.2 Brane resolution and confinement

In the resolved M2-brane construction there are two stages to the resolution. The first stage consists of resolving the cone manifold (i.e. conifold) itself, and replacing it with a complete Ricci-flat manifold that asymptotically approaches the original conifold at large distance. Having done this, the usual construction of an M2-brane is typically singular at small distance. At the second stage the singularity is resolved by setting the $F^{(4)}$ field strength equal to a regular harmonic self-dual 4-form in the transverse 8-metric. If the harmonic 4-form is normalisable as well, then the AdS structure is preserved at large distance. In this case, in the ultraviolet regime (the gravity dual at large distance) the dual field theory approaches a three-dimensional conformal field theory with less than maximal supersymmetry.

In the case of the M2-brane, with an 8-dimensional transverse space that has the asym-
As a consequence, the dual field theory in the infrared (the gravity dual at small distance) is described as a perturbation of the CFT by relevant operators, identified with pseudo-scalar fields of the dual field theory [28].

One might wonder why it is necessary to consider the resolution of the solution, since although the original cone metric is singular at the origin, this singularity is smoothed by the presence of the M2-brane located on it, which implies that the geometry near the cone singularity becomes $\text{AdS}_4 \times \Sigma$, where $\Sigma$ is a smooth Einstein manifold. This solution might seem already to be a perfectly satisfactory supergravity dual for the three-dimensional conformal field theory with reduced supersymmetry, since the solution is regular everywhere. However, it is easy to verify that the minimally-coupled scalar wave equation for the $\text{AdS}_4$ horospherical metric,

$$\left.d^2 s_{\text{AdS}_4} = \frac{1}{z^2} \left( -dt^2 + dx_1^2 + dx_2^2 + dz^2 \right), \right.$$

has a continuous spectrum with no mass gap. To be specific, although the UV boundary provides an infinite wall to the Schrödinger potential, it vanishes in the IR region where $z \to \infty$. Thus there would be no indication of confinement in the dual gauge theory.

The picture is rather different for the resolved M2-branes. First of all, the geometry of the resolved conifold itself becomes $\mathbb{R}^n \times \Sigma_{8-n}$ at small distance, where $\rho$ is the proper distance from the origin. The usual construction of the M2-brane would imply that we have $H = 1/\rho^{n-2}$ at small distance, which would then result in a singularity. However, if the 8-manifold has a harmonic self-dual 4-form $G_{(4)}$ that is regular at short distance, and we set $F_{(4)} = m \ G_{(4)}$, then with the integration constants chosen as in (6.4), $H$ will approach a non-vanishing constant at small $\rho$, and hence the solution becomes regular there. The geometry in the infrared region then becomes a smooth manifold

$$M_3 \times R^n \times \Sigma_{8-n}, \quad (7.3)$$

where $M_3$ denotes 3-dimensional Minkowski spacetime. This implies that the transverse space at short distance tends to the finite values of the metric coefficients, rather than an AdS horizon. In turn, this means that the Schrödinger potential for the minimally-coupled
scalar wave equation is of the infinite-well type, and hence that the spectrum is discrete, indicating confinement of the corresponding gauge theory.

The decoupling limit of the resolved M2-brane interpolates between $M_3 \times R^n \times \Sigma_{8-n}$ in the IR region and $\text{AdS}_4 \times \Sigma_7$ at the UV boundary region. Owing to the fact that the harmonic 4-form $G^{(4)}$ does not carry a non-vanishing flux for the M5-brane charge, one expects that the flow from the conformal point of $\text{AdS}_4 \times \Sigma_7$ to a non-conformal $M_3 \times R^n \times \Sigma_{8-n}$ will be described by a perturbation by relevant operators, associated with the pseudoscalar fields of the gauge theory in the Higgs branch. Thus we see that these relevant operators play an important role in the context of confinement. This is very different from the D3-brane resolution, and hence the mechanism for confinement for the $\mathcal{N} = 2$, $D = 4$ gauge theory, where the additional fractional wrapped D5-brane flux is needed [5, 7].

### 7.2 D2-brane/QFT$_3$ correspondence

The D2-brane differs from the M2-brane in two important respects. First of all, the D2-brane does not have an asymptotic AdS structure, and hence the dual gauge theory is not a conformal one. Secondly, it is possible to wrap additional branes, which can be either D4-branes or NS-NS 5-branes, on non-trivial cycles in the 7-dimensional transverse space. The properties of some non-trivial complete Ricci-flat 7-manifold are summarised in appendix A. The D2-branes with wrapped fractional D4-branes or NS-NS 5-branes in these manifolds were constructed in [4, 11], and they were shown to preserve $\mathcal{N} = 1$ supersymmetry [11]. Here, we list the global symmetries of the transverse spaces, which in turn determine the global symmetries of the spectrum in the dual field theory.

| $K$ | Brane Charge | Global Symmetry |
|-----|--------------|-----------------|
| $\mathbb{C}P^3$ | $D2(N), D4(M)$ | $SO(5) \times SO(3)$ |
| Flag$_6$ | $D2(N), D4(M)$ | $SU(3) \times SU(2)$ |
| $S^3 \times S^3$ | $D2(N), NS5(M)$ | $SO(4) \times SO(3)$ |

Table 2: Global symmetries of dual three dimensional field theories of resolved D2-branes.

We should like at this point to venture into trying to address the spectrum of the dual field theory, in particular for the resolved D2-brane with the base $S^3 \times S^3$, for which the topology of the transverse space is $\mathbb{R}^4 \times S^3$. The topology can be encoded by embedding
this space into $R^8$, and imposing the constraint \[ (7.4) \]

\[
\sum_{i=1}^{4} x_i^2 - \sum_{j=1}^{4} y_j^2 = r > 0,
\]

where the $x_i$ coordinates parameterise the $S^3$. Reinterpreting this constraint in terms of quaternionic variables \[ (7.4) \]: \( \{x_i\} \rightarrow q_1 = \sum_i u_i \sigma_i \) for \( \{y_i\} \rightarrow q_2 = \sum_i v_i \sigma_i \), where $\sigma_i$ are the quaternionic generators, allows one to rewrite it as:

\[
|q_1|^2 - |q_2|^2 = r.
\] 

The global symmetry $[SU(2)_1 \times SU(2)_2]_L \times [SU(2)_{1+2}]_R$ of the manifold acts on the quaternionic fields $q_{1,2}$ by the left and right $[SU(2)_{1,2}]_L,R$ multiplications, respectively. One may attempt to identify these fields with the matter fields of the dual theory, and they transform under the above global symmetry as: $q_1 = (2,1,2)_{N,N+M}$, $q_2 = (1,2,2)_{N,N+M}$. Of course the conjectured form of the matter spectrum would have to be confirmed by independent calculation. Here we have also tacitly assumed that the gauge group of the theory is $SU(N) \times SU(N+M)$.

The dual field theory has $\mathcal{N} = 1$ supersymmetry, and as a consequence the assumed interaction terms for the proposed matter fields would be of the form

\[
W = \epsilon_{BD} \epsilon_{B'D'} \epsilon_{AA'} \epsilon_{CC'} [q_1]_{AB} [q_2]_{CD} [q_1]_{A'B'} [q_2]_{C'D'},
\] 

where an implicit summation over gauge indices is understood.

This fractional D2-brane is obtained if the transverse seven-dimensional manifold has a non-trivial 3-cycle, around whose dual 3-cycle an NS-NS 5-brane can wrap. In this case, we take $G_{(3)} = \omega_{(3)}$, where $\omega_{(3)}$ is the harmonic form associated with the 3-cycle. The function $H$ in the M2-brane metric (6.2) is given at large distance by \[ (7.7) \]

\[
H = c_0 + \frac{m^2}{4r^4} + \frac{Q}{r^3} + \cdots.
\]

There is now a term with the slower fall-off $1/r^4$ than the usual $1/r^5$ term for the standard M2-brane, owing to the fact that $\omega_3$ is not $L^2$ normalisable at large $r$. In particular, the deformed solution no longer has a well-defined ADM mass. In other words, the effective 4-form “electric charge” is proportional to $r^6 H'$ which is $\propto m^2 r$ thus confirming the indication in the dual field theory that the difference of the inverses of the two gauge-group factors, $g_1^{-2} - g_2^{-2}$, is proportional to $M \sim m^2$. As should be the case in three-dimensional field theories, this difference grows linearly with energy (which on the gravity side it proportional to the distance $c$).
On the other hand fractional D2-brane solutions associated with the \( \mathbb{CP}^3 \) or Flag_6 base manifolds arise from the existence of a non-trivial 4-cycle around whose dual 2-cycle a D4-brane can wrap [11]. If \( \omega_{(4)} \) denotes the associated harmonic 4-form in the transverse 7-space, we can set \( G_{(4)} = \omega_{(4)} \) in (6.3) in order to obtain a solution. One finds that the metric function \( H \) in the M2-brane solution (6.2) is given at large distance, for a suitable normalisation for \( \omega_{(4)} \), by [28]

\[
H = c_0 + \frac{Q}{r^5} - \frac{m^2}{4r^6} + \cdots.
\]

The fractional D2-brane carries an electric charge \( Q \) and a magnetic charge \( m \) for \( F_{(4)} \), while the 3-form, given by \( F_{(3)} = m r^{-4} dr \wedge \ast_6 \omega_{(4)} \), has vanishing flux integral. This result indicates that in contrast to the previous example, here in the dual three-dimensional theory the leading contribution to the running of the difference of gauge couplings is absent.

When the transverse space is flat, corresponding to the maximally-supersymmetry case, the D2-brane can be viewed as a periodic array of isotropic M2-branes, and hence the usual D2-brane is a limit of the M2-brane [38]. Such a connection between the M2-brane and the D2-brane becomes much less obvious in the non-trivial 7-geometries. The supersymmetry suggests that the fractional D2-brane might be related to an M2-brane resolved on an 8-manifold of Spin(7) holonomy. We leave this for future investigation.

## 8 Conclusions

In this paper, we have given an explicit construction of the hyper-Kähler Calabi metrics, for all dimensions \( D = 4n+4 \). This uses the fact that the metrics have cohomogeneity one, with principal orbits described by the coset \( SU(n+2)/U(n) \). We began by making an ansatz for such metrics, with undetermined functions of the radial coordinate as scaling factors for the various terms in the homogeneous level sets. Following the procedure of Dancer and Swann [14], we then obtained first-order equations for these functions as integrability conditions for the existence of a hyper-Kähler structure. Our solution of these equations gives very simple expressions for the hyper-Kähler Calabi metrics.

Having obtained the Calabi solutions, it is of interest also to see how they can arise as the solutions of a system of first-order equations derivable from a superpotential. In fact from a Lagrangian formulation of the Einstein equations for our original metric ansatz we were able in general to derive two inequivalent superpotential descriptions; one leading to the hyper-Kähler solutions, and the other to Ricci-flat Kähler solutions on the complex line bundles over \( SU(n+2)/(U(n) \times U(1)) \).
The Calabi metric in dimension $D = 4n + 4$ is defined on a complete non-compact manifold that is asymptotic to the cone over $SU(n+2)/U(n)$. Thus it can be viewed as a resolution of the conifold where the base of the cone is $SU(n+2)/U(n)$. In an appendix, we studied this conifold, using a quaternionic construction in which one starts from $\mathbb{H}^{n+2}$ with coordinates $q_A$ and imposes the quadratic constraint $q_A^T q_A = 0$, which imposes three real conditions, together with $q_A^T q_A = 1$ to set the radius of the cone to unity. We showed how the Einstein metric on the $SU(n+2)/U(n)$ base of the cone arises as the restriction of the flat metric on $\mathbb{H}^{n+2}$ under the quadratic constraints, together with a further $U(1)$ Hopf fibration. This is the hyper-Kähler quotient construction, which we then applied also in the non-singular case, thereby obtaining an alternative derivation of the Calabi metrics in the form (1.4). We also showed how this contrasts with the complex conifold construction, where the base of the cone is instead $SO(n+2)/SO(n)$. In this case the analogous constraints on the coordinates of $\mathbb{C}^{n+2}$ directly yield the manifold of the $SO(n+2)/SO(n)$ base, with no need for a further Hopf reduction. However, in this example the Einstein metric on $SO(n+2)/SO(n)$ is not simply the metric inherited from the flat metric on $\mathbb{C}^{n+2}$; instead, it is obtained by adding an additional invariant term which corresponds to a “squashing” of the Hopf fibres in $SO(n+2)/SO(n)$ viewed as a $U(1)$ bundle over the real Grassmannian $SO(n+2)/(SO(n) \times SO(2))$.

One of our motivations for obtaining the explicit construction of the hyper-Kähler Calabi metrics was in order to allow us to solve explicitly for $L^2$-normalisable middle-dimension harmonic forms. In particular, our principle interest was in the 8-dimensional Calabi metric, which can be used in order to obtain a resolved M2-brane solution of $D = 11$ supergravity. We therefore solved explicitly for the $L^2$-normalisable harmonic 4-form in the 8-dimensional Calabi metric. We also obtained the result for the normalisable harmonic 6-form in the 12-dimensional Calabi metric.

Using the harmonic 4-form in the 8-dimensional Calabi metric, we then constructed a deformed M2-brane solution. We showed that all three of the covariantly-constant spinors in the Calabi metric remain as supersymmetries of the deformed M2-brane. It follows that in the decoupling limit, the dual gauge theory on the 3-dimensional boundary of AdS$_4$ has $\mathcal{N} = 3$ supersymmetry.

We also discussed the physical significance of this, and previously-obtained resolved brane solutions, and analysed the symmetry groups of the dual field theories. Resolved brane solutions lead to a breaking on conformal invariance in the dual field theories, and this can provide a mechanism for confinement. In fact the mechanisms for the cases of dual
field theories in $D = 4$ and $D = 3$ are quite different. In $D = 4$, an additional fractional 5-brane flux is needed for the brane-resolution, whilst in $D = 3$ the breaking of the conformal phase is caused by a perturbation of relevant operators, associated with the pseudoscalar fields of the gauge theory in the Higgs branch.

A Ricci-flat manifolds in $D = 4$, 6, 7 and 8

In this appendix, we present a summary of known irreducible complete non-compact Ricci-flat manifolds of cohomogeneity one in the dimensions that are of relevance in string theory and M-theory. In particular, we shall discuss only those cases where the metric admits covariantly-constant spinors, so that the associated resolved $p$-brane solutions might be supersymmetric. This means that the spaces must be either Ricci-flat Kähler, or hyper-Kähler, or else with exceptional holonomy $G_2$ in $D = 7$ or Spin(7) in $D = 8$. In all the cases that we shall consider, the metrics are of cohomogeneity one. Most of the cases are asymptotically conical, of the form $\mathbb{R} \times K$, where $K = G/H$ is a homogeneous space whose metric approaches an Einstein metric of positive curvature as $r$ tends to infinity. The principal orbits $G/H$ typically degenerate at the “centre,” at the minimum value $r = r_0$ of the radial coordinate. The requirements of regularity of the manifold at $r = r_0$ imply that the degeneration must be of the form $S^m \times \tilde{K}$, where $S^m$ is a round sphere whose proper radius tends to zero at the appropriate rate so that the manifold is locally of the form $\mathbb{R}^{m+1} \times \tilde{K}$ near $r = r_0$.

A.1 Ricci-flat Kähler metrics in $D = 4$

First, we list the case of 4 dimensions. Here the only examples within the class described above are Kähler (and hence hyper-Kähler), with principal orbits that are locally $S^3$. The most relevant for our purposes, since it is asymptotically-conical, is the Eguchi-Hanson solution [12]. The global structure of this manifold was first described in [13], where it was shown that the principal orbits are $RP^3 \equiv S^3/\mathbb{Z}_2$ rather than $S^3$, and so it provided the first example of an ALE space.
Table 3: Complete non-compact Ricci-flat 4-manifolds with covariantly-constant spinors [12, 19, 20]. They are all Kähler, with $SU(2)$ holonomy, and hence they are also hyper-Kähler. The $S^3$ principal orbits in Taub-NUT degenerate to a point at the origin, just as in flat space. (This, and its higher-dimensional analogues, are discussed in appendix B.) The tilde in the expression $\tilde{T}^*\mathbb{RP}^3$ for the Atiyah-Hitchin manifold denotes that it is the universal covering space of the co-tangent bundle of $\mathbb{RP}^3$. Only the Eguchi-Hanson manifold is asymptotically conical.

A.2 Ricci-flat Kähler metrics in $D = 6$

Next, we turn to $D = 6$. Here, there are two possibilities for the principal orbits, namely $S^5$ or $T^{1,1}$. These correspond to $U(1)$ bundles over $\mathbb{CP}^2$ or $S^2 \times S^2$ respectively.

Table 4: Complete non-compact Ricci-flat 6-manifolds. These are all Kähler, and so they have $SU(3)$ holonomy and admit two covariantly-constant spinors. The notation $X \ltimes Y$ is used to signify a bundle over $Y$ with fibre $X$. In the final row, $T^*S^3$ denotes the co-tangent bundle of $S^3$. Metrics for the first two examples are obtained in [21, 22], and more general such metrics for the second example are obtained in [27]. The metric for the third example is in [4, 23], and for the fourth in [6, 18, 7].
A.3 Ricci-flat metrics of \( G_2 \) holonomy in \( D = 7 \)

In \( D = 7 \) the only irreducible manifolds within the class we are listing here are ones with exceptional \( G_2 \) holonomy. Three of these are known explicitly.

| \( K \) \( G/H \) | Isometry \( G \times H \) | Degenerate Orbit \( \tilde{K} \) | Harmonic \( 3 \)-Form? | Manifold |
|---|---|---|---|---|
| \( \mathbb{C}P^3 \) \( \frac{SO(5)}{SO(3) \times SO(2)} \) | \( SO(5) \times SO(3) \) | \( S^4 \) | \( L^2 \) | \( \mathbb{R}^3 \times S^4 \) |
| Flag\(_6\) \( \frac{SU(3)}{U(1) \times U(1)} \) | \( SU(3) \times SU(2) \) | \( \mathbb{C}P^2 \) | \( L^2 \) | \( \mathbb{R}^3 \times \mathbb{C}P^2 \) |
| \( S^3 \times S^3 \) \( \frac{SO(4) \times SO(3)}{SO(3)} \) | \( SO(4) \times SO(3) \) | \( S^3 \) | Reg, not \( L^2 \) | \( \mathbb{R}^4 \times S^3 \) |

Table 5: Complete non-compact Ricci-flat 7-manifolds, which all have exceptional \( G_2 \) holonomy. Flag\(_6\) denotes the 6-dimensional flag manifold \( SU(3)/(U(1) \times U(1)) \). The metrics are all asymptotically conical, with principal orbits that approach Einstein metrics on \( \mathbb{C}P^3 \), Flag\(_6\) and \( S^3 \times S^3 \). The \( \mathbb{C}P^3 \) and Flag\(_6\) Einstein metrics are “squashed,” rather than the standard Kähler ones. The Einstein metric on \( S^3 \times S^3 \) is again “squashed,” and is not the standard direct-product metric. The metrics for all three examples are obtained in [24, 25].

Topologically, the manifolds can be described as the bundle of self-dual 2-forms over \( S^4 \) or \( \mathbb{C}P^2 \), and the spin bundle over \( S^3 \), respectively.

For the sake of completeness, we shall present here the metrics, and the superpotentials from which the Ricci-flat solutions can be derived, for the three \( G_2 \) examples. We shall describe them in the order listed in Table 5. In fact the relevant equations are the same for both the \( \mathbb{R}^3 \) bundle over \( S^4 \) and the \( \mathbb{R}^3 \) bundle over \( \mathbb{C}P^2 \). The metrics can be written in the form

\[
\begin{align*}
    ds_7^2 &= dt^2 + c^2 (d\mu^i + \epsilon_{ijk} A^j \mu^k)^2 + a^2 ds_4^2, \\
     \mu^i \mu^i &= 1 \quad (i = 1, 2, 3) \quad \text{and} \quad A^i \text{ is the } SU(2) \text{ connection of the } 4\text{-dimensional quaternionic Kähler Einstein manifold with metric } ds_4^2.
\end{align*}
\]

This can be chosen to be \( S^4 \) or \( \mathbb{C}P^2 \). Defining a new radial coordinate \( \eta \) by \( dt = a^4 c^2 d\eta \), the Ricci-flat conditions can be derived from the Lagrangian \( L = T - V \), together with the constraint \( T + V = 0 \), where

\[
\begin{align*}
    T &= \frac{1}{2} g_{ij} \alpha^i \alpha^j = 2 \gamma^2 + 16 \alpha' \gamma' + 12 \alpha^2, \\
    V &= -2 e^{2 \gamma + 8 \alpha} - 12 e^{4 \gamma + 6 \alpha} + 2 e^{6 \gamma + 4 \alpha},
\end{align*}
\]
where \( a = e^\alpha \), \( c = e^\gamma \) and a prime denotes \( d/d\eta \). We find that the potential \( V \) can be expressed in terms of a superpotential \( W \), such that \( V = -\frac{3}{2}g^{ij}(\partial W/\partial \alpha^i)(\partial W/\partial \alpha^j) \), with

\[
W = 4e^{\gamma + 3\alpha} + 4e^{3\gamma + 2\alpha}.
\] (A.3)

The first-order equations \( d\alpha^i/d\eta = \frac{1}{2}g^{ij}\partial W/\partial \alpha^j \), re-expressed in terms of the original radial variable \( t \), give

\[
\dot{a} = \frac{c}{a}, \quad \dot{c} = 1 - \frac{c^2}{a^2}.
\] (A.4)

The solution, after a further change of radial variable, gives the complete Ricci-flat metric on the \( \mathbb{R}^3 \) bundle over \( S^4 \) or \( \mathbb{C}P^2 \) that was obtained in [24, 25]:

\[
ds_t^2 = \frac{dr^2}{1 - r^{-4}} + \frac{1}{7r^2}(1 - r^{-4})(d\mu^i + \epsilon_{ijk} A^j \mu^k)^2 + \frac{1}{7}r^2 ds_4^2.
\] (A.5)

For the \( \mathbb{R}^4 \) bundle over \( S^3 \), the form of the metric is

\[
ds_t^2 = dt^2 + c^2(\nu_1^2 + \nu_2^2 + \nu_3^2) + a^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2),
\] (A.6)

where \( \nu_i = \Sigma_i - \frac{1}{2}\sigma_i \), and the quantities \( \sigma_i \) and \( \Sigma_i \) are left-invariant 1-forms on two copies of \( SU(2) \). In terms of a new radial variable \( \eta \) defined by \( dt = a^3 c^3 d\eta \), the Ricci-flat conditions can be derived from the Lagrangian \( L = T - V \) with constraint \( T + V = 0 \), where

\[
T = 3\gamma^2 + 3\alpha^2 + 9\alpha' \gamma',
\]

\[
V = -\frac{3}{4}e^{4\gamma + 6\alpha} - \frac{3}{4}e^{6\gamma + 4\alpha} + \frac{3}{64}e^{8\gamma + 2\alpha}.
\] (A.7)

We find that \( V \) can be obtained from a superpotential, given by

\[
W = \frac{3}{2}e^{2\gamma + 3\alpha} + \frac{3}{8}e^{4\gamma + \alpha},
\] (A.8)

The first-order equations following from this superpotential, written in terms of \( t \), are therefore

\[
\dot{a} = \frac{c}{4a}, \quad \dot{c} = \frac{1}{2} - \frac{c^2}{8a^2}.
\] (A.9)

The solution of these equations, in terms of a new radial variable \( r \), gives the complete Ricci-flat metric on the \( \mathbb{R}^4 \) bundle over \( S^3 \) that was obtained in [24, 25]:

\[
ds_r^2 = \frac{dr^2}{1 - r^{-3}} + \frac{1}{9r^2}(1 - r^{-3})(\nu_1^2 + \nu_2^2 + \nu_3^2) + \frac{1}{12}r^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2).
\] (A.10)
A.4 Ricci-flat metrics in $D = 8$

In $D = 8$ there are three kinds of irreducible manifolds within the class we are listing here, with exceptional holonomy $\text{Spin}(7)$, Ricci-flat Kähler holonomy $\text{SU}(4)$, or hyper-Kähler holonomy $\text{Sp}(2) \equiv \text{Spin}(5)$. They admit 1, 2 or 3 covariantly-constant spinors respectively.

| $K$          | $G/H$                  | Isometry   | Degenerate Orbit $\tilde{K}$ | Harmonic 4-Form? | Manifold        | Holonomy |
|--------------|------------------------|------------|-------------------------------|------------------|-----------------|----------|
| $S^7$        | $\frac{\text{SO}(5)}{\text{SO}(3)}$ | $\text{SO}(5) \times \text{SO}(3)$ | $S^4$            | $L^2$            | $\mathbb{R}^4 \times S^4$ | Spin(7)  |
| $S^7/\mathbb{Z}_4$ | $\frac{\text{SU}(4)}{\text{SU}(3)}$ | $\text{U}(4)$ | $\mathbb{C}P^3$             | $L^2$            | $\mathbb{C} \times \mathbb{C}P^3$ | $\text{SU}(4)$ |
| $N^{1,1}$    | $\frac{\text{SU}(3)}{\text{U}(1)}$ | $\text{U}(3)$ | Flag $6$                     | $L^2$            | $\mathbb{C} \times \tilde{K}$ | $\text{SU}(4)$ |
| $M^{2,3}$    | $\frac{\text{SU}(3) \times \text{SU}(2)}{\text{SU}(2) \times \text{U}(1)}$ | $\text{SU}(3) \times \text{U}(2)$ | $S^2 \times \mathbb{C}P^2$ | $L^2$            | $\mathbb{C} \times \tilde{K}$ | $\text{SU}(4)$ |
| $Q^{1,1,1}$  | $\frac{\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)}{\text{U}(1) \times \text{U}(1) \times \text{U}(1)}$ | $\text{SU}(2)^3 \times \text{U}(1)$ | $(S^2)^3$        | $L^2$            | $S^2 \times \tilde{K}$     | $\text{SU}(4)$ |
| $Q^{1,1,1}$  | $\frac{\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)}{\text{U}(1) \times \text{U}(1) \times \text{U}(1)}$ | $\text{SU}(2)^3 \times \text{U}(1)$ | $(S^2)^2$        | $L^2$            | $S^2 \times \tilde{K}$     | $\text{SU}(4)$ |
| $V_{5,2}$    | $\frac{\text{SO}(5)}{\text{SO}(3)}$ | $\text{SO}(5) \times \text{U}(1)$ | $S^4$            | $L^2$            | $T^*S^4$        | $\text{SU}(4)$ |
| $N^{1,1}$    | $\frac{\text{SU}(3)}{\text{U}(1)}$ | $\text{U}(3)$ | $\mathbb{C}P^2$             | $L^2$            | $T^*\mathbb{C}P^2$ | Spin(5)  |

Table 6: Complete non-compact Ricci-flat 8-manifolds. The example with Spin(7) holonomy is obtained in [24, 25]. The first four Kähler examples are obtained in [21, 22], and more general metrics for the third of these are obtained in [10], and for the fourth in [10, 11]. The fifth Kähler example is obtained in [10, 11], and the final Kähler example is contained in the construction of [18], and obtained explicitly in [10]. Note that the embeddings of $\text{SO}(3)$ in the descriptions of $S^7$ and $V_{5,2}$ as $\text{SO}(5)/\text{SO}(3)$ are such that the 5 of $\text{SO}(5)$ decomposes to $1 + 2 + 2$ and $1 + 1 + 3$ respectively. The hyper-Kähler Calabi metric is obtained in [15], and a useful further discussion is to be found in [14]. The fully explicit metric, in a form that is most useful for our present purposes, is obtained in the present paper.

For completeness, we shall present the metric, superpotential and solution for the complete Ricci-flat metric of Spin(7) holonomy on the $\mathbb{R}^4$ bundle over $S^4$. The metric is of the form

$$ds_8^2 = dt^2 + c^2 (\nu_1^2 + \nu_2^2 + \nu_3^2) + a^2 d\Omega_4^2,$$

where $\nu_i = \sigma_i - A^i$, and $A^i$ denotes the potential for the $\text{SU}(2)$ Yang-Mills instanton over the unit 4-sphere with metric $d\Omega_4^2$. In terms of a new radial variable $\eta$, defined by $dt = a^4 c^3 d\eta$,
the Ricci-flat conditions can be derived from the Lagrangian $L = T - V$, with the constraint $T + V = 0$, where

$$
T = 6\gamma'^2 + 24\alpha' \gamma' + 12 \alpha'^2, \\
V = -\frac{3}{2}e^{4\gamma + 8\alpha} - 12e^{6\gamma + 6\alpha} + 3e^{8\gamma + 4\alpha},
$$

(A.12)

One finds that $V$ can be derived from a superpotential

$$
W = 3e^{2\gamma + 4\alpha} + 6e^{4\gamma + 2\alpha}.
$$

(A.13)

In terms of the original radial variable $t$, the first-order equations following from this superpotential are

$$
\dot{a} = \frac{3c}{2a}, \quad \dot{c} = \frac{1}{2} - \frac{c^2}{a^2}.
$$

(A.14)

The solution of these equations yields the complete Ricci-flat metric of Spin(7) holonomy obtained in [24, 25]:

$$
ds^2 = \frac{dr^2}{1 - r^{-10/3}} + \frac{9}{100}r^2 (1 - r^{-10/3}) (\sigma_i - A^i)^2 + \frac{9}{20}r^2 d\Omega^2_{4},
$$

(A.15)

Note that the first-order equations (A.14) are the same, after trivial rescalings, as those obtained for the 8-metrics of Spin(7) holonomy in section 4.1.

B Higher-dimensional generalisations of Taub-NUT

Although the Taub-NUT metric [19] and its higher-dimensional generalisations [41] are not asymptotically conical, it is useful to gather the results for these metrics here, since they provide further explicit examples of complete Ricci-flat metrics. The construction gives generalisations in all even dimensions, but only the 4-dimensional Taub-NUT metric itself is Kähler. (See also [42] for a recent discussion of some higher-dimensional metrics of the Taub-NUT type.)

Consider the general class of complex lines-bundle metrics

$$
ds^2 = h^2 dr^2 + c^2 \sigma^2 + a^2 d\Sigma^2_m
$$

(B.1)

where $d\Sigma^2_m$ is the Fubini-Study metric on the unit $\mathbb{C}P^m$ (with cosmological constant $\lambda = 2m + 2$, where $R_{ij} = \lambda \delta_{ij}$). The 1-form $\sigma$ is given by

$$
\sigma = dz + A,
$$

(B.2)

---

3This superpotential was obtained in [39]. A system of first-order equations that gives rise to the metric of [24, 25] was also obtained in [40].

41
where $dA = 2J$, and $J$ is the Kähler form on $\mathbb{CP}^m$.

In the vielbein basis $\hat{e}^0 = h\,dr$, $\hat{e}^0 = c\,\sigma$, $\hat{e}^a = a\,e^a$, the Ricci tensor of $d\hat{s}^2$ is given by

\begin{align*}
\hat{R}_{00} &= -\frac{n}{h_a} \left( \frac{d'}{h} \right)' - \frac{1}{h_c} \left( \frac{c'}{h} \right)'
\hat{R}_{0\bar{0}} &= -\frac{n\,a'\,c'}{h^2\,a\,c} - \frac{1}{h_c} \left( \frac{c'}{h} \right)'
\hat{R}_{ij} &= -\left[ \frac{1}{a\,h} \left( \frac{d'}{h} \right)' + \frac{a'\,c'}{h^2\,a\,c} + \frac{(n-1)\,a'^2}{h^2\,a^2} - \frac{\lambda}{a^2} + \frac{2c^2}{a^4} \right] \delta_{ij},
\end{align*}

(B.3)

where $n = 2m$ is the real dimension of the $\mathbb{CP}^m$ base space.

Making the coordinate gauge choice $h\, c = 1$, we find that the combination $\hat{R}_{00} - \hat{R}_{0\bar{0}} = 0$ of the Ricci-flat conditions implies that $a'' + a^{-3} = 0$, which gives

$$a^2 = r^2 - 1.$$  

(B.4)

The equation $\hat{R}_{00} = 0$ then implies that $c^2 = k\, f$, where $k$ is a constant and $f$ satisfies the equation

$$(r^2 - 1)^2 f'' + n\, r (r^2 - 1) f' - 2n\, f = 0.$$  

(B.5)

We require the solution that is non-singular at $r = 1$. This is given by

$$f = r (r^2 - 1)^{-m} \int_1^r s^{-2} (s^2 - 1)^m \, ds,$$  

(B.6)

(recalling that $n = 2m$), which, in terms of hypergeometric functions, is

$$f = (-1)^m \, r (r^2 - 1)^{-m} \left( \frac{2^{2m} (m!)^2}{(2m)!} - \frac{1}{r} \, _2F_1[-\frac{1}{2}, -m, \frac{1}{2}, r^2] \right).$$  

(B.7)

(For any integer $m$, $f$ reduces to a rational function of $r$. The first few cases are presented below.) It can then be verified that the remaining Einstein equation $\hat{R}_{ij} = 0$ is satisfied provided that the constant $k$ in $c^2 = k\, f$ is chosen to be $k = 2m + 2$. It is useful to note from (B.6) that $f$ satisfies the first-order equation

$$r (r^2 - 1) f' + [1 + (n - 1)\, r^2] f = r^2 - 1.$$  

(B.8)

Thus we arrive at the Ricci-flat metric

$$d\hat{s}^2 = \frac{dr^2}{(2m + 2)\, f} + (2m + 2)\, f \, \sigma^2 + (r^2 - 1)\, d\Sigma_m^2.$$  

(B.9)

For $r$ close to 1, if we write $r = 1 + \frac{1}{2}\, \rho^2$, it follows from (B.6) that

$$f \sim \frac{\rho^2}{2m + 2},$$  

(B.10)
and the metric approaches

\[ ds^2 = d\rho^2 + \rho^2 (\sigma^2 + d\Sigma_m^2). \quad (B.11) \]

Since \( dA = 2J \), it follows that \( \sigma^2 + d\Sigma_m^2 \) is precisely the metric on the unit \((2m+1)\)-sphere, and so near \( r = 1 \) the metric \((B.9)\) smoothly approaches the origin of \( \mathbb{R}^{2m+2} \) written in spherical polar coordinates. At large \( r \) we have

\[ f \sim \frac{1}{2m-1}, \quad (B.12) \]

and so the metric is asymptotically flat with the cylindrical geometry \( \mathbb{R}^{2m+1} \times S^1 \).

When \( m = 1 \), the resulting 4-dimensional metric is precisely Taub-NUT. For larger \( m \), we get higher-dimensional generalisations. The first few examples are

\[
\begin{align*}
    ds^2_4 &= \frac{r+1}{4(r-1)} dr^2 + \frac{4(r-1)}{r+1} \sigma^2 + (r^2-1) d\Sigma^2_1, \\
    ds^2_6 &= \frac{(r+1)^2}{2(r-1)(r+3)} dr^2 + \frac{2(r-1)(r+3)}{(r+1)^2} \sigma^2 + (r^2-1) d\Sigma^2_2, \\
    ds^2_8 &= \frac{5(r+1)^3}{8(8-1)(r^2+4r+5)} dr^2 + \frac{8(r-1)(r^2+4r+5)}{5(r+1)^3} \sigma^2 + (r^2-1) d\Sigma^2_3, \\
    ds^2_{10} &= \frac{7(r+1)^4}{2(2-1)(5r^3+25r^2+47r+35)} dr^2 + \frac{2(r-1)(5r^3+25r^2+47r+35)}{7(r+1)^4} \sigma^2 + (r^2-1) d\Sigma^2_4.
\end{align*}
\]

The generalised Taub-NUT metrics are not Kähler, except for the special case \( n = 2m = 2 \) of the original 4-dimensional example. One way to see that they are not in general Kähler is to note that if they were, then, being Ricci flat, they would admit two covariantly-constant spinors \( \eta \). These would have to satisfy the integrability conditions \( R_{abcd} \Gamma_{cd} \eta = 0 \). To show that these cannot be satisfied for \( m \geq 2 \) it suffices to look at the components \( \Theta_{0i} \) of the curvature 2-form:

\[ \Theta_{0i} = k Q_1 e^0 \wedge e^i + k Q_2 J_{ij} e^0 \wedge e^j, \quad (B.14) \]

where

\[ Q_1 = \frac{1 - r^2 + [3 + (n - 1) r^2] f}{2(r^2 - 1)^2}, \quad Q_2 = \frac{1 - r^2 + [1 + (n + 1) r^2] f}{2r (r^2 - 1)^2}. \quad (B.15) \]

It is evident that the associated integrability condition is

\[ Q_1 \Gamma_{0i} \eta + Q_2 J_{ij} \Gamma_{ij} \eta = 0, \quad (B.16) \]

and that this can only be satisfied by a non-vanishing \( \eta \) if \( Q_1 = \pm Q_2 \). It is straightforward to check from the expression \([B.7]\) that this can only occur if \( n = 2 \).
C A more general $T^*\mathbb{CP}^2$ construction in $D = 8$

In our construction of the Calabi metrics in section 2, we took the $U(1)$ generator within the coset to be $\lambda = L_1^1 - L_2^2$, while the $U(1)$ generator corresponding to the combination $Q = L_1^1 + L_2^2$ was taken to lie outside the coset. It is possible to take a more general embedding of the $U(1)$ generators, in which we have $\lambda$ within the coset and $Q$ outside the coset given by

$$
\lambda = (L_1^1 - L_2^2) \cos \delta + (L_1^1 + L_2^2) \sin \delta, \\
Q = -(L_1^1 - L_2^2) \sin \delta + (L_1^1 + L_2^2) \cos \delta,
$$

(C.1)

where $\delta$ is a fixed angle that specifies the embedding. It can take a discrete infinity of possible values, and the case discussed in section 2 that leads to the Calabi hyper-Kähler metrics corresponds to $\delta = 0$.

We shall just present results for the more general $\delta \neq 0$ embeddings in $D = 8$ here, since again we find that there exists a superpotential leading to first-order equations that imply $\text{Spin}(7)$ holonomy. Specifically, we find that the superpotential (4.3) now becomes

$$
W = 4a^3 b c f + 4a b^3 c f + 4a b c^3 f - 4a^2 b^2 f^2 \cos \delta \\
- 2\sqrt{2} a^2 c^2 f^2 \sin(\delta - \frac{1}{4} \pi) + 2\sqrt{2} b^2 c^2 f^2 \cos(\delta - \frac{1}{4} \pi). \tag{C.2}
$$

Correspondingly, the first-order equations (4.5) generalise to

$$
\dot{\alpha} = \frac{b^2 + c^2 - a^2}{a b c} - \frac{\sqrt{2} f \cos \tilde{\delta}}{a^2}, \\
\dot{\beta} = \frac{a^2 + c^2 - b^2}{a b c} + \frac{\sqrt{2} f \sin \tilde{\delta}}{b^2}, \\
\dot{\gamma} = \frac{a^2 + b^2 - c^2}{a b c} + \frac{\sqrt{2} f (\cos \tilde{\delta} - \sin \tilde{\delta})}{c^2}, \\
\dot{\sigma} = -\frac{\sqrt{2} f (\cos \tilde{\delta} - \sin \tilde{\delta})}{c^2} + \frac{\sqrt{2} f \cos \tilde{\delta}}{a^2} - \frac{\sqrt{2} f \sin \tilde{\delta}}{b^2}, \tag{C.3}
$$

where we have defined $\tilde{\delta} = \delta - \frac{1}{4} \pi$.

D Complex and quaternionic conifolds

D.1 Complex conifolds

Let us first briefly review the construction of complex conifolds, which describe the cones over $SO(n+2)/SO(n)$. These all admit smooth resolutions to give the Stenzel metric
of dimension $D = 2n + 2$ on $T^*S^{n+1}$, the co-tangent bundle of $S^{n+1}$. The case $n = 1$ corresponds to Eguchi-Hanson and $n = 2$ corresponds to the deformed conifold of $[3,4]$ that was used in the construction of the fractional D3-brane. The case $n = 3$ is an 8-dimensional metric, which was used in $[6,7]$ in order to construct a smooth deformed M2-brane. The conifold for $n = 3$ was discussed in $[4]$.

For arbitrary $n$, we generalise the discussion in $[3,4]$, and consider complex coordinates $z_A$ on $\mathbb{C}^{n+2}$, subject to the quadratic constraint
\[ \sum_A z_A^2 = 0. \]  
(D.1)

The rays under which $z_A$ is identified with $\lambda z_A$ for all $\lambda$ form a cone. Fixing the radius to unity, without loss of generality, we have also
\[ \sum_A |z_A|^2 = 1. \]  
(D.2)

The quadric $\sum z_A^2$ in (D.1) is invariant under $z_1 \rightarrow z'_1 = A_{AB} z_B$ for complex matrices $A_{AB}$ satisfying $A^T A = 1$, i.e. in $SO(n+2,\mathbb{C})$, while (D.2) is invariant for $A^\dagger A = 1$, i.e. under $U(n+2)$. Writing $A = P + i Q$ where $P$ and $Q$ are real, we see that from $(A^T - A^\dagger)A = 0$ we shall have $Q^T Q = 0$, and hence by taking the trace we get $Q_{AB} Q_{AB} = 0$, which implies $Q = 0$. The intersection of $SO(n+2,\mathbb{C})$ and $U(n+2)$ is therefore $SO(n+2)$, and this acts transitively on the base of the cone defined by (D.1) and (D.2). A fiducial point on this manifold, for example
\[ z_1 = \frac{1}{\sqrt{2}}, \quad z_2 = \frac{i}{\sqrt{2}}, \quad z_3 = z_4 = \cdots = z_{n+2} = 0, \]  
(D.3)
is stabilised by $SO(n)$, acting on the last $n$ of the $z_A$. Thus the base of the cone is the coset space $SO(n+2)/SO(n)$. There is also an additional $U(1)$ factor in the isometry group, giving $SO(n+2)(\times U(1))$ in total. This arises because the fact that $\sum_A z_A^2$ vanishes in (D.1) allows the additional symmetry transformation $z_A \rightarrow e^{i\gamma} z_A$, which is also an invariance of (D.2).

The Einstein metric on the coset $SO(n+2)/SO(n)$ that forms the base of the cone can be constructed in terms of the $z_A$ coordinates, as
\[ ds^2 = \frac{n}{n+1} \left[ \sum_A |dz_A|^2 - k^2 \left| \sum_A \bar{z}_A d z_A \right|^2 \right], \]  
(D.4)

where $k$ is a constant “squashing parameter” whose value we shall determine presently. To show this, let $Z$ be the column vector of $z_A$ coordinates, so $Z^T = (z_1, z_2, \ldots, z_{n+2})$, and let
Let $Z_0$ be the fiducial point defined above, i.e., $Z_0^T = (1, i, 0, \ldots, 0)/\sqrt{2}$. Then as we saw above, we may express the general point $Z$ as

$$Z = P Z_0,$$

where $P$ is a (real) orthogonal matrix in $SO(n+2)$. We shall therefore have

$$\sum_A |d z_A|^2 = d Z_0^\dagger d Z = Z_0^\dagger d P^T d P Z_0 = Z_0^\dagger K^T K Z_0,$$

$$\sum_A \bar{z}_A dz_A = Z_0^\dagger K Z_0,$$

where we have defined the 1-form $K$ in the Lie algebra of $SO(n+2)$ by $K \equiv P^T d P$. This may be expanded in terms of the $SO(n+2)$ generators $T_{AB}$ as

$$K = \frac{1}{2} L_{AB} T_{AB},$$

where $L_{AB}$ are left-invariant 1-forms on the group $SO(n+2)$.

If we define the basis $(n+2)$-vectors $X_A$ by $X_A^T = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where there is a zero in every component except for a 1 at the $A$'th position, then for $SO(n+2)$ generators $T_{AB}$ in the fundamental representation we shall have

$$T_{AB} X_C = \delta_{BC} X_A - \delta_{AC} X_B.$$

Writing the fiducial point as $Z_0 = (X_1 + i X_2)/\sqrt{2}$, it is now an elementary exercise to show that

$$Z_0^\dagger K^T K Z_0 = \frac{1}{2} \sigma_1^2 + \frac{1}{2} \bar{\sigma}_1^2 + \nu^2, \quad Z_0^\dagger K Z_0 = i \nu,$$

where we have defined

$$\nu \equiv L_{12}, \quad \sigma_i \equiv L_{1i}, \quad \bar{\sigma}_i \equiv L_{2i}, \quad 3 \leq i \leq n+2.$$

Note that $\nu$, $\sigma_i$ and $\bar{\sigma}_i$ are the left-invariant 1-forms on the coset $SO(n+2)/SO(n)$ that we introduced in [10] in our construction of the Stenzel metrics on $T^* S^{n+1}$. Thus we see that the proposed metric (D.4) on the $SO(n+2)/SO(n)$ coset that forms the base of the cone can be written as

$$ds^2 = \frac{n}{n+1} \left[ \frac{1}{2} \sigma_i^2 + \frac{1}{2} \bar{\sigma}_i^2 + (1 - k^2) \nu^2 \right].$$

This metric may be compared with the form of the Stenzel metric in $D = 2n + 2$ dimensions [18], in the form obtained in [11];

$$d \hat{s}^2 = c^2 d \nu^2 + c^2 \nu^2 + a^2 \sigma_i^2 + b^2 \bar{\sigma}_i^2,$$
where
\[ a^2 = R^{1/(n+1)} \coth r, \quad b^2 = R^{1/(n+1)} \tanh r, \quad c^2 = \frac{1}{n+1} R^{-n/(n+1)} (\sinh 2r)^n, \]
(D.13)
and
\[ R(r) \equiv \int_0^r (\sinh 2u)^n du. \]
(D.14)
From this, it is easy to see that at large \( r \) the Stenzel metric approaches
\[ ds^2 = d\rho^2 + \rho^2 ds_0^2, \]
(D.15)
where
\[ ds_0^2 = \frac{n}{n+1} \left[ \frac{1}{2} \sigma_i^2 + \frac{1}{2} \bar{\sigma}_i^2 + \frac{n}{n+1} \nu^2 \right]. \]
(D.16)
This metric, which is necessarily Einstein, is the required form for the metric on \( SO(n+2)/SO(n) \) on the base of the cone. Comparing with (D.11), we see that it agrees provided that we choose
\[ k^2 = \frac{1}{n+1}. \]
(D.17)
The metric (D.4) for the case \( n = 2 \) was obtained in [6]. This case is rather special, in that the transitively-acting group is \( SO(4) \) which factorises as \( SU(2) \times SU(2) \), and so the principal orbits can be described as the 5-manifold \( T^{1,1} \) which is a \( U(1) \) bundle over \( S^2 \times S^2 \), whose metric was given in [43].

To summarise, we have shown that the Stenzel metrics, which are regular and asymptotic to cones over \( SO(n+2)/SO(n) \), have the structure at large radius given by (D.14), with the metric on the principal orbits approaching (D.16). This Einstein metric on \( SO(n+2)/SO(n) \) is the one that is needed in the conifold construction of the cone over \( SO(n+2)/SO(n) \), and we have shown that it can be written in terms of the conifold coordinates \( z_A \) as (D.4), with the constant \( k \) given by (D.17). It should be emphasised, therefore, that the Einstein metric on \( SO(n+2)/SO(n) \) is not obtained simply by restricting the flat metric \( d\bar{z}_A dz_A \) on \( \mathbb{C}^{n+2} \) by the quadratic constraints (D.1) and (D.2). Rather, we must subtract the appropriate multiple of \( |\bar{z}_A dz_A|^2 \), as given by (D.4) and (D.17). This amounts to a homogeneous squashing of the fibres in \( SO(n+2)/SO(n) \), viewed as a \( U(1) \) bundle over the real Grassmannian \( SO(n+2)/(SO(n) \times SO(2)) \).

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4Since \( ds^2 \) in (D.12), and its asymptotic form (D.15), is Ricci flat, it follows that \( ds_0^2 \) in (D.15) must be Einstein, with \( R_{ab} = 2n g_{ab} \).
D.2 Quaternionic conifolds and the hyper-Kähler quotient

D.2.1 Quaternionic conifolds

We saw in the previous subsection how the complex conifold construction of [6] for \( D = 6 \), and [4] for \( D = 8 \), generalises to arbitrary complex dimensions, and that the Stenzel metrics on \( T^*S^{n+1} \) can be viewed as the resolutions of the singular metrics on the cones. Here, we carry out a similar analysis for the analogous quaternionic conifolds. The construction, known as the hyper-Kähler quotient, shows explicitly how the Calabi metrics on \( T^*CP^{n+1} \) can be viewed as the resolutions of the singular metrics on the quaternionic cones. Some care is needed in the construction, on account of the non-commutativity of the quaternions.

It is useful to write a quaternion \( q \) in terms of two complex quantities \( u \) and \( v \):

\[
q = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}.
\]  

We begin by considering \((n+2)\) quaternions \( q_A \) on \( \mathbb{H}^{n+2} \), subject to the quadratic constraint

\[
\sum_A q_A^T q_A = 0,
\]  

where \( q_A^T \) denotes transposition of the \( 2 \times 2 \) matrix introduced in (D.18). The rays where \( q_A \) is identified with \( q_A \lambda \), with \( \lambda \) any quaternion, define a cone,\(^5\) and fixing the radius to unity gives the intersection with the unit \((4n+7)\)-sphere

\[
\sum_i q_i^A q_A = 1.
\]  

Under transformations \( q_A \rightarrow q'_A = A_{AB} q_B \), with

\[
q_A \equiv \begin{pmatrix} u_A & -\bar{v}_A \\ v_A & \bar{u}_A \end{pmatrix}, \quad A_{AB} = \begin{pmatrix} \alpha_{AB} & \bar{\beta}_{AB} \\ \beta_{AB} & \bar{\alpha}_{AB} \end{pmatrix},
\]  

we see that \( q_A^T q_A \) is invariant for matrices \( A \) in \( SO(n+2, \mathbb{H}) = SO^*(2n+4) \), while \( q_A^\dagger q_A \) is invariant for matrices \( A \) in \( Sp(n+2) = USp(2n+4) \). In terms of the complex components

\(^5\)Note that unlike the complex conifold discussed previously, here one has to distinguish between left-multiplication and right-multiplication by quaternions. In fact (D.19) is slightly too weak a condition to specify the cone, since it is also invariant under the left-multiplication \( q_A \rightarrow \kappa q_A \), where \( \kappa \) is a quaternion satisfying \( \kappa^T \kappa = 1 \). In fact by writing (D.19) in terms of the complex components \( u_a \) and \( v_a \) as in (D.18), we see that it implies \( u_A^2 + \bar{v}_A^2 = 0 \) and \( \bar{u}_A v_A - \bar{v}_A u_A = 0 \), which is just three real conditions, and not four as one might have expected. As we shall see later, the factoring out of the residual \( U(1) \) implies that we should perform a Hopf reduction on the fibres corresponding to this \( U(1) \) coordinate.
\( \alpha_{AB} \) and \( \beta_{AB} \), these conditions are
\[
\begin{align*}
SO(n+2, \mathbb{H}) : & \quad \alpha^T \alpha + \beta^T \beta = 1, \quad \alpha^\dagger \beta - \beta^\dagger \alpha = 0, \\
Sp(n+2) : & \quad \alpha^\dagger \alpha + \beta^\dagger \beta = 1, \quad \alpha^T \beta - \beta^T \alpha = 0.
\end{align*}
\] (D.22)

The intersection of \( SO(n+2, \mathbb{H}) \) and \( Sp(n+2) \) can be seen by defining \( \alpha = \alpha_R + i \alpha_I, \beta = \beta_R + i \beta_I \), where the matrices \( \alpha_R, \alpha_I, \beta_R \) and \( \beta_I \) are all real. Taking differences of the two lines in (D.22), and then taking the imaginary parts, we learn that \( \alpha^\dagger \beta - \beta^\dagger \alpha = 0 \), and hence \( \alpha_I = 0, \beta_I = 0 \). Having learned that the matrices \( \alpha \) and \( \beta \) are real, it is easy to see that by defining the complex matrix \( S \equiv \alpha + i \beta \), we shall have
\[
S^\dagger S = (\alpha^T - i \beta^T)(\alpha + i \beta),
\]
and so the remaining conditions in (D.22) are satisfied if and only if \( S \) is unitary, \( S^\dagger S = 1 \).

The \( U(1) \) factor in \( U(n+2) = SU(n+2) \times U(1) \) is irrelevant here, and so this shows that the manifold defined by (D.19) and (D.20) is invariant under \( SU(n+2) \), which acts transitively on the base of the cone. As in the complex case there is in fact an additional \( U(1) \) isometry, coming from the fact that the right-hand side in (D.19) is zero, implying that the \( q_A \) can all undergo the phase-scaling transformation \( q_A \rightarrow e^{i \gamma} q_A \), which also leaves (D.20) invariant. Thus the total isometry group is \( U(n+2) \).

The coset structure can be seen by choosing a fiducial point on the rays satisfying (D.19) and (D.20), such as
\[
q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad q_3 = q_4 = \cdots = q_{n+2} = 0.
\] (D.24)

This has \( U(n) \) as the stability subgroup contained within the transitively-acting \( SU(n+2) \) obtained above, consisting of matrices \( S \) built as above, but with
\[
\alpha_{11} = \alpha_{22} = 1, \quad \alpha_{12} = \alpha_{21} = \beta_{12} = \beta_{21} = \beta_{11} = \beta_{22} = 0.
\] (D.25)

Thus the base of the cone is the homogeneous manifold \( SU(n+2)/U(n) \).

The coset \( SU(n+2)/U(n) \) is precisely the one that we used in section 2, for the principal orbits of the manifolds of cohomogeneity one that led to the hyper-Kähler Calabi metrics on \( T^* \mathbb{C}P^{n+1} \). Thus these smooth manifolds are resolutions of the quaternionic cones in \( \mathbb{H}^{n+2} \) defined by (D.19).

As in the previous discussion for the complex conifolds, here too we can give an explicit expression for the Einstein metric on the coset space \( SU(n+2)/U(n) \) that forms the base.

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of the cone, in terms of the quaternionic conifold coordinates. To do this, it is advantageous
to introduce further notation as follows. We define complex column vectors $U$ and $V$
by $U^T = (u_1, u_2, \ldots, u_{n+2})$ and $V^T = (v_1, v_2, \ldots, v_{n+2})$ respectively, and then define the
quaternionic column vector $Z$ by

$$Z \equiv U + j V,$$  \hspace{1cm} (D.26)

where $j$ is an imaginary quaternion unit, which anticommutes with $i$ used in the construction
of the complex quantities, with $ij = -ji = k$, etc. It is easy to see that the original conditions
(D.19) and (D.20) are nothing but

$$Z^T Z = 0, \quad Z^\dagger Z = 1.$$  \hspace{1cm} (D.27)

We also now define the special unitary matrix $P$ by

$$P \equiv \alpha + \beta j,$$  \hspace{1cm} (D.28)

where as above $\alpha$ and $\beta$ are real matrices satisfying (D.22). It is easy to see that the action
of $U(n+2)$ on the original quaternionic coordinates $q_A$, which are related to $u_A$ and $v_A$ as in (D.21),
is now expressible as

$$Z \rightarrow Z' = P Z = \tilde{P} e^{j \psi} Z,$$  \hspace{1cm} (D.29)

where we have extracted the phase from $P$ in $e^{j \psi}$, so that $\det \tilde{P} = 1$. The fiducial point
(D.24) is now given by

$$Z_0 = U_0 + j V_0 = \frac{1}{\sqrt{2}} X_1 - \frac{1}{\sqrt{2}} k X_2,$$  \hspace{1cm} (D.30)

where $X_A$ is the same set of basis vectors as in the complex case (zero except for a 1 at
$A$'th position).

Following analogous steps to those for the complex conifold, we now introduce the 1-form
$K \equiv P^\dagger dP$ in the Lie algebra of $U(n+2)$;

$$K \equiv P^\dagger dP = \tilde{P}^\dagger d\tilde{P} + j d\psi = j L_A^B T_B^A + j d\psi,$$  \hspace{1cm} (D.31)

where $T_B^A$ are the (hermitean) generators of $SU(n+2)$ (constructed using $j$ as the complex
unit), satisfying $(T_B^A)^\dagger = T_A^B$. In the fundamental representation we shall have

$$T_A^B X_C = \delta_C^B X_A - \frac{1}{n+2} \delta_A^B X_C.$$  \hspace{1cm} (D.32)

\textsuperscript{6}Qua representation of $U(n+2)$, $P$ is equivalent to the special unitary matrix $S$ introduced above equation
(D.23). Since we are now changing to a description of the quaternions in terms of the three imaginary units
(i,j,k) rather than in terms of complex $2 \times 2$ matrices, we need to use the $j$ unit in the construction of $P$.  

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(The second term, which is a trace subtraction, will not enter in the subsequent calculations since $L_A^A = 0$.) Noting that we have $L_A^B k = k L_B^A$, it is now a straightforward exercise to derive the following:

$$dZ^\dagger dZ = Z_0^\dagger K Z_0 = \frac{1}{2}(\sigma_{1\alpha}^2 + \sigma_{2\alpha}^2 + \Sigma_{1\alpha}^2 + \Sigma_{2\alpha}^2) + \nu_1^2 + \nu_2^2 + \frac{1}{4}\lambda^2 + \frac{1}{4}(d\psi + \frac{1}{2}Q)^2, \quad (D.33)$$

where the real 1-forms $(\sigma_{1\alpha}, \Sigma_{1\alpha}, \Sigma_{2\alpha}, \nu_1, \nu_2, \lambda, Q)$ appearing here are as defined in section 2 of the paper. Note in particular that $Q = L_1^1 + L_2^2$ is the $U(1)$ generator that lies outside the coset.

In our proof of the relation between the conditions (D.22) and $S^\dagger S = 1$ (or $P^\dagger P = 1$) in (D.23), we noted that the phase of $S$ (or $P$) is irrelevant, since it factors out in $S^\dagger S = 1$. Therefore in our construction of the metric on conifold, we should project orthogonally to the orbit of the $U(1)$ fibres $\psi$ that parameterise this phase (see (D.31)). Thus we see from (D.33) that the metric $dZ^\dagger dZ$ induced from the original flat metric on $\mathbb{H}^{n+2}$, after this projection perpendicular to the $U(1)$ Hopf $U(1)$ fibres, is

$$ds^2 = dZ^\dagger dZ \bigg|_{\text{perp}} = \frac{1}{2}(\sigma_{1\alpha}^2 + \sigma_{2\alpha}^2 + \Sigma_{1\alpha}^2 + \Sigma_{2\alpha}^2) + \nu_1^2 + \nu_2^2 + \frac{1}{4}\lambda^2. \quad (D.34)$$

We may now compare this with the asymptotic form of the metric on the $SU(n+2)/U(n)$ principal orbits in the hyper-Kähler Calabi metrics, which are given by (1.4). At large distance we have

$$ds^2 = dr^2 + r^2 ds_0^2, $$

$$= dr^2 + r^2 \left[\frac{1}{2}(\sigma_{1\alpha}^2 + \sigma_{2\alpha}^2 + \Sigma_{1\alpha}^2 + \Sigma_{2\alpha}^2) + \nu_1^2 + \nu_2^2 + \frac{1}{4}\lambda^2\right]. \quad (D.35)$$

Thus the metric on the $SU(n+2)/U(n)$ base of the asymptotically-conical large-distance limit of the Calabi metrics is precisely the same as the one we obtained in (D.34) from our quaternionic conifold construction above. An interesting feature is that in this hyper-Kähler case we are finding that the induced metric coming simply from $dZ^\dagger dZ$, followed by a Hopf reduction on the $U(1)$ fibres associated with the transformation $q_A \rightarrow \kappa q_Q$, yields the required Einstein metric on the $SU(n+2)/U(n)$ base of the cone. By contrast, in the complex conifold case that we discussed previously, we instead obtained the required Einstein metric on the $SO(n+2)/SO(n)$ base of the cone by subtracting an appropriate multiple of $|Z^\dagger dZ|^2$ from the metric directly induced from $|dZ^\dagger dZ|^2$. The difference between the two cases is a reflection of the more rigid structure of hyper-Kähler metrics, in comparison to Ricci-flat Kähler metrics.
D.2.2 Hyper-Kähler quotient

In this section we make contact with the hyper-Kähler quotient construction of the Calabi metrics, as described, for example, in [44]. We present a very simple rederivation of the Calabi metrics in the notation of this paper, using the hyper-Kähler quotient construction.

The construction proceeds as follows. Let us define a column vector of quaternions $W$ by

$$W \equiv rZ,$$

where, as in section (D.2.1), $Z$ is normalised so that $Z^\dagger Z = 1$, and so we have $W^\dagger W = r^2$. We now deform the original cone metric, and modify (D.19), which would imply $W^TW = 0$, so that instead

$$W^TW = -\ell^2,$$

where $\ell$ is a real constant. We can achieve this, which implies $Z^TZ = -\ell^2/r^2$, while still maintaining $Z^\dagger Z = 1$, by changing our fiducial point (D.30) so that now $Z_0$ is given by

$$Z_0 = \frac{1}{\sqrt{2}} \left[ \left(1 - \frac{\ell^2}{r^2}\right)^{1/2} X_1 - k \left(1 + \frac{\ell^2}{r^2}\right)^{1/2} X_2 \right].$$

It is now a simple exercise to calculate the metric

$$ds^2 \equiv dW^\dagger dW = dr^2 + r^2 dZ^\dagger dZ = dr^2 + r^2 Z_0^\dagger K^\dagger K Z_0 + r^2 dZ_0^\dagger dZ_0,$$

giving

$$d\hat{s}^2 = \left(1 - \frac{\ell^4}{r^4}\right) -1 dr^2 + \frac{1}{4} r^2 \left(1 - \frac{\ell^4}{r^4}\right) \lambda^2 + r^2 (\nu_1^2 + \nu_2^2)
+ \frac{1}{2} \left(r^2 - \ell^2\right) (\sigma_1^2 + \sigma_2^2) + \frac{1}{2} \left(r^2 + \ell^2\right) (\Sigma_1^2 + \Sigma_2^2)
+ r^2 \left(d\psi + \frac{1}{2} Q - \frac{\ell^2}{2r^2} \lambda\right)^2.$$

We see that after projecting orthogonally to the vector $\partial/\partial\psi$ along the $U(1)$ fibres, we end up with hyper-Kähler Calabi metric in precisely the form we derived in section 2, with scale parameter $\ell = 1$. Note that the final term in (D.40) is nothing but $r^2 (d\psi - 2A_1)^2$, where $A_1$ is the potential for the Kähler form $J_1$ obtained in (2.24).

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7 The quaternion $q$ in [44] is related to our $W$, and the complex components $u$ and $v$ in (D.18) are related to the $q$ in [44] by $q = r (u - vj)$ (the roles of left and right multiplication in [44] are interchanged in comparison to the conventions we have adopted here). The $U(1)$ action in our discussion is equivalent to transforming the $q$ of [44] according to $q \rightarrow q e^{i\ell}$. 

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