CORRECTION TO THE PAPER “A FLEXIBLE CONSTRUCTION OF EQUIVARIANT FLOER HOMOLOGY AND APPLICATIONS”

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Abstract. We correct a mistake regarding almost complex structures on Hilbert schemes of points in surfaces in [HLS16]. The error does not affect the main results of the paper, and only affects the proofs of invariance of equivariant symplectic Khovanov homology and reduced symplectic Khovanov homology. We give an alternate proof of the invariance of equivariant symplectic Khovanov homology.

1. The mistake

At several points in Section 7 of our paper [HLS16] we assert that an almost complex structure $j$ on the algebraic surface $S$ used to define symplectic Khovanov homology induces an almost complex structure $\text{Hilb}^n(j)$ on the Hilbert scheme (or Douady space, following [Dou66]) $\text{Hilb}^n(S)$ of length $n$ subschemes of $S$. If $j$ is a complex structure this is true, but there is no known extension of the Hilbert scheme of points in a complex manifold to the almost-complex case. (Indeed, even the definition of the Hilbert scheme as a set depends on the complex structure.) See also Voisin’s paper [Voi00] for some interesting steps in this direction and further discussion.

This (false) principle is used in a “cylindrical” formulation of symplectic Khovanov homology in [HLS16, Lemma 7.10], which is then used in the proof of stabilization invariance for symplectic Khovanov homology in [HLS16, Section 7.4.1], equivariant symplectic Khovanov homology in [HLS16, Theorem 1.26], and reduced symplectic Khovanov homology in [HLS16, Theorem 7.25]. (See also Abouzaid-Smith’s paper [AS16] for a more careful cylindrical reformulation of the curves in symplectic Khovanov homology in certain cases, and Mak-Smith’s recent paper [MS] for a more general cylindrical reformulation.)

Below, we give a corrected, weaker version of the offending Lemma 7.10, and an independent proof of equivariant stabilization invariance (i.e., [HLS16, Theorem 1.26]). We have not been able to correct the proof of stabilization invariance of reduced symplectic Khovanov homology (i.e. [HLS16, Theorem 7.25]). So, reduced symplectic Khovanov homology is invariant under isotopies and handleslides, but is only conjectured to be invariant under stabilization.

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2. The corrected lemma

Recall that the curves we consider in the cylindrical formulation of symplectic Khovanov homology are maps
\[
\psi: (X, \partial X) \to (\mathbb{R} \times [0, 1] \times S, (\mathbb{R} \times \{0\} \times (\Sigma_{A_1} \cup \cdots \cup \Sigma_{A_n})) \cup (\mathbb{R} \times \{1\} \times (\Sigma_{B_1} \cup \cdots \cup \Sigma_{B_n})))
\]
where \(X\) is a Riemann surface with boundary and \(2n\) boundary punctures, \(\psi\) is asymptotic to \(\{-\infty\} \times [0, 1] \times x\) and \(\{+\infty\} \times [0, 1] \times y\), and \(\pi_{\mathbb{R} \times [0,1]} \circ \psi\) is an \(n\)-fold branched covering.

The incorrect Lemma 7.10 introduced the following condition for these maps:

\[(YC)\] The map \((\mathbb{I} \times \mathbb{I} \times i) \circ \psi: (X \setminus B(\psi)) \to \mathbb{R} \times [0, 1] \times \mathbb{C}\) is an embedding.

The following is a corrected version of Lemma 7.10:

**Lemma 7.10’** The set of holomorphic disks in the complex manifold \(\mathcal{Y}_n\) connecting \(x\) to \(y\) and which are transverse to the big diagonal \(\Delta\) is in bijection with the set of holomorphic maps as in Formula (7.9) satisfying condition \((YC)\).

**Remark 2.1.** To emphasize, the corrected Lemma 7.10’ is about a complex structure on \(S\) and the induced complex structure on \(\mathcal{Y}_n\), not an almost complex structure on \(S\) or \(\mathcal{Y}_n\). In particular, there is no assertion that the moduli spaces in Lemma 7.10’ are transversely cut out. The corrected Lemma 7.10’ is not used in the rest of this note.

3. Corrected proof of equivariant stabilization invariance

3.1. Background.

3.1.1. Skein triangles. The corrected proof of stabilization invariance is similar to our proof of stabilization invariance for the equivariant Heegaard Floer homology of branched double covers from [HLS16, Theorem 1.24]. Here are the analogues for symplectic Khovanov homology of the results about Heegaard Floer homology that proof used:

**Theorem 3.1.** [SS06] [Wal09] Symplectic Khovanov homology is a link invariant.

**Theorem 3.2.** [AS19] Let \(B, B’,\) and \(B''\) be collections of bridges which differ as shown in Figure 1 in part of the diagram and are small isotopic copies of each other in the rest of the diagram. For any collection of bridges \(A\) there is an exact triangle

\[
\begin{align*}
\text{HF}(\mathcal{K}_A, \mathcal{K}_B) & \quad \longrightarrow \quad \text{HF}(\mathcal{K}_A, \mathcal{K}_{B''}) \\
\text{HF}(\mathcal{K}_A, \mathcal{K}_{B'}) & \quad \longrightarrow \quad \text{HF}(\mathcal{K}_A, \mathcal{K}_B) \\
\end{align*}
\]
Further, there are elements $\alpha \in CF(K_B, K_{B'})$, $\beta \in CF(K_{B'}, K_{B''})$, and $\gamma \in CF(K_{B''}, K_B)$ so that the maps in the exact triangle come from counting holomorphic triangles on $(K_A, K_B, K_{B'})$, $(K_A, K_{B'}, K_{B''})$, and $(K_A, K_{B''}, K_B)$ with one corner at $\alpha$, $\beta$, and $\gamma$, respectively.

Proof. Using ideas of Abouzaid-Ganatra, Abouzaid-Smith show that there is an exact triangle of bimodules over the Fukaya category of $\mathcal{Y}_n$ relating the identity bimodule $I$, the composition of a cup and a cap, and the half-twist $\tau$; see [AS19, Proposition 7.4]. Evaluating these three bimodules on $K_B$ as the first object gives three one-sided modules---$K_B$, a module equivalent to $K_{B'}$, and a module equivalent to $K_{B''}$. Since equivalences preserve exact triangles, this implies that there is an exact triangle relating $K_B$, $K_{B'}$, and $K_{B''}$. In particular, the maps $K_B \to K_{B'}$, $K_{B'} \to K_{B''}$, and $K_{B''} \to K_B$ come from elements $\alpha \in CF(K_B, K_{B'})$, $\beta \in CF(K_{B'}, K_{B''})$, and $\gamma \in CF(K_{B''}, K_B)$, so that for any other Lagrangian $L$, counting holomorphic triangles on $(L, K_B, K_{B'})$ with a corner at $\alpha$ (respectively on $(L, K_{B'}, K_{B''})$ with a corner at $\beta$, on $(L, K_{B''}, K_B)$ with a corner at $\gamma$) gives an exact triangle relating $HF(L, K_B)$, $HF(L, K_{B'})$, and $HF(L, K_{B''})$. Taking $L = K_A$ gives the result.

Remak 3.3. For appropriate choices of diagrams, all of the generators of $CF(K_B, K_{B'})$, $CF(K_{B'}, K_{B''})$, and $CF(K_{B''}, K_B)$ are fixed by the $O(2)$-action.

3.1.2. Projected domains. While the arguments below take place in the Hilbert scheme, and avoid a cylindrical formulation of symplectic Khovanov homology, we will make use of the concept and some properties of projected domains from [HLS16, Section 7.1.3]. Fix a bridge diagram $(A = \{A_i\}, B = \{B_i\})$ for a link $L$. Choose a point $z_i$ in each connected component of $\mathbb{C} \setminus (A \cup B)$. Each $z_i$ gives a subvariety $\pi^{-1}(z_i) \times \text{Hilb}^{n-1}(S)$ of $\text{Hilb}^n(S)$ consisting of those length-$n$ subschemes where at least one point lies over $z_i$. Fix a neighborhood $U_i$ of $\pi^{-1}(z_i) \times \text{Hilb}^{n-1}(S)$, small enough that the closure of $U_i$ is disjoint from $K_A \cup K_B$.

Let $R$ denote the non-compact region of $\mathbb{C} \setminus (A \cup B)$. Then $\pi^{-1}(R) \times \text{Hilb}^{n-1}(S)$, the set of length-$n$ subschemes where at least one point lies over $R$, is an open subset of $\text{Hilb}^n(S)$. Order the $z_i$ above so that $z_0 \in R$.

We will only consider almost complex structures $J$ on $\mathcal{Y}_n$, compatible with the symplectic form described in Section 3.1.3 below, with the following three additional properties:

(J-1) The almost complex structure $J$ agrees with the standard complex structure $\text{Hilb}^n(j)$ on $U_i$. In particular, each $(\pi^{-1}(z_i) \times \text{Hilb}^{n-1}(S)) \cap \mathcal{Y}_n$ is a $J$-holomorphic submanifold.

(J-2) The almost complex structure $J$ agrees with the standard complex structure $\text{Hilb}^n(j)$ on $\pi^{-1}(R) \times \text{Hilb}^{n-1}(S)$.

(J-3) The almost complex structure $J$ agrees with the standard complex structure $\text{Hilb}^n(j)$ outside a compact subset. (This is not implied by the previous restriction because the fibers of $\pi$ are themselves non-compact.)

(Compare [OSz04, Definition 3.1].)

Since $(\pi^{-1}(z_i) \times \text{Hilb}^{n-1}(S)) \cap \mathcal{Y}_n$ is proper and disjoint from $K_A \cup K_B$, each $(\pi^{-1}(z_i) \times \text{Hilb}^{n-1}(S)) \cap \mathcal{Y}_n$ is dual to a relative cohomology class $PD[\pi^{-1}(z_i) \times \text{Hilb}^{n-1}(S)] \in H^2(\mathcal{Y}_n, K_A \cup K_B)$. Given a Whitney disk $u$ for $(K_A, K_B)$, let $n_{z_i}(u)$ be the result of evaluating the cohomology class $PD[\pi^{-1}(z_i) \times \text{Hilb}^{n-1}(S)]$ on $[u]$. The tuple $(n_{z_i}(u))$ is the projected domain of $u$. Sometimes, we think of the projected domain as an element of $H_2(\mathbb{C} \cup \{\infty\}, A \cup B)$, where $n_{z_i}(u)$ is the coefficient of the region containing $z_i$.

If $u$ is $J$-holomorphic then the projected domain of $u$ has the following properties:

(D-1) For each $i$, $n_{z_i}(u) \geq 0$. (Compare [OSz04, Lemma 3.2].)
The value \( n_\omega(u) = 0 \), and in fact \( u(D^2) \cap (R \times \text{Hilb}^{n-1}(S)) = \emptyset \).

(D-3) If \( n_\omega(u) = 0 \) for each \( i \) then \( u \) is constant.

The first two statements follow from positivity of intersections of complex submanifolds. The third follows from the fact that any such Whitney disk is homotopic to a constant disk, hence has zero area, and hence is itself constant.

Finally, no non-constant, \( J \)-holomorphic Whitney disk is entirely contained in the subspace

\[
(\pi^{-1}(R) \times \text{Hilb}^{n-1}(S)) \cup \bigcup_i U_i
\]

where we have imposed constraints on \( J \). So, the usual transversality arguments for \( J \)-holomorphic curves (see, e.g., [MS04, Chapter 3]) imply generic transversality for (one parameter families of) almost complex structures \( J \) in this class.

3.1.3. Symplectic forms and convexity at infinity. In [HLS16], we worked in the setting of symplectic manifolds which are convex at infinity, in the sense of [EG91]. An alternative notion of convexity comes from [Gro85, Section 0.4]: a symplectic manifold \( M \) is \textit{I-convex at infinity} (or simply \textit{I-convex}) if there is an exhaustion of \( M \) by open sets \( V_1 \subset V_2 \subset \cdots \) with \( \overline{V_i} \) compact and such that if \( u : D^2 \to M \) is \( I \)-holomorphic with \( u(\partial D^2) \subset V_i \) then \( u(D^2) \subset V_{i+1} \). For example, the maximum modulus theorem implies that any affine variety is \( I \)-convex. Observe also that the notion of a symplectic manifold being \( I \)-convex depends only on \( I \) near infinity (i.e., outside a compact set). As noted in [HLS], the arguments from [HLS16] work for symplectic manifolds which are \( I \)-convex at infinity for some \( G \)-invariant almost complex structure \( I \) defined outside a compact set.

In particular, the \( I \)-convex setting is convenient for symplectic Khovanov homology. Let \( I \) be the complex structure on \( \mathcal{Y}_n \) inherited from \( \text{Hilb}^n(S) \). The complex structure \( I \) is \( O(2) \)-invariant and, since \( \mathcal{Y}_n \) is an affine variety (see also [Man06, Theorem 1.2]), \( \mathcal{Y}_n \) is \( I \)-convex. Inspired by Perutz’s construction in [Per08b], in [AS16, Lemma 5.5], Abouzaid-Smith construct a Kähler form \( \omega' \) on \( \text{Hilb}^n(S) \) (with respect to \( I \)) whose restriction to \( \mathcal{Y}_n \) is exact and agrees with the product form outside a neighborhood of the diagonal. (In the notation of [AS16, Lemma 5.5], we choose \( \omega \) to be an exact Kähler form on \( S \).) We saw in [HLS, Lemma 4.24] that averaging \( \omega' \) gives an \( O(2) \)-invariant Kähler form on \( \mathcal{Y}_n \) (still with respect to \( I \)) which is still exact and still agrees with the product symplectic form outside a neighborhood of the diagonal.

If \( V \) is an open subset of \( \mathbb{C} \), then \( \mathcal{Y}_n \cap \text{Hilb}^n(\pi^{-1}(V)) \) is also \( I \)-convex, for the same reason that point (D-2), above, holds. In particular, complex structures satisfying condition (J-3) satisfy the convexity-at-infinity condition required for the constructions in [HLS16].

3.2. General Künneth theorem for the freed Floer complex. Another ingredient in our proof of Proposition 3.12 is a Künneth theorem for equivariant symplectic Khovanov homology. We give a general Künneth theorem for the equivariant Floer complex in this section and specialize to the case of symplectic Khovanov homology in the next section.

Remark 3.4. Throughout this section, the convexity assumption from [HLS16, Hypothesis 3.2] can be replaced with \( I \)-convexity for some \( G \)-invariant almost complex structure \( I \) defined outside a compact set. See Section 3.1.3 for further discussion and references.

Theorem 3.5. Suppose that \( H \) acts on \((M, L_0, L_1)\) and \( H' \) acts on \((M', L'_0, L'_1)\), both satisfying [HLS16, Hypothesis 3.2]. Then the action of \( H \times H' \) on \((M \times M', L_0 \times L'_0, L_1 \times L'_1)\)
satisfies \cite[Hypothesis 3.2]{HLS16} and there is a quasi-isomorphism
\[ \widehat{CF}^{H \times H'}(L_0 \times L_0', L_1 \times L_1') \simeq \widehat{CF}^{H}(L_0, L_1) \otimes_{\mathbb{F}_2} \widehat{CF}^{H'}(L_0', L_1') \]
of chain complexes over \( \mathbb{F}_2[H \times H'] \).

**Proof.** To keep notation simple, we will prove the result in the case that \( L_0 \cap L_1 \) and \( L_0' \cap L_1' \); the extension to non-transverse intersections is the same as \cite[Section 3.6]{HLS16}.

Observe that if \( \ihat{\mathcal{J}} \) (respectively \( \ihat{\mathcal{J}}' \)) is an eventually cylindrical almost complex structure on \( M \) (respectively \( M' \)) so that the moduli spaces of \( \ihat{\mathcal{J}} \)-holomorphic Whitney disks in \( M \) (respectively \( \ihat{\mathcal{J}}' \)-holomorphic Whitney disks in \( M' \)) are transversely cut out then the moduli space of \( (\ihat{\mathcal{J}} \times \ihat{\mathcal{J}}') \)-holomorphic Whitney disks in \( M \times M' \) are transversely cut out. More generally, given \( k \)-parameter families \( \ihat{\mathcal{J}}(t_1, \ldots, t_k) \) and \( \ihat{\mathcal{J}}'(t_1, \ldots, t_k) \) of eventually cylindrical almost complex structures on \( M \) and \( M' \), \( t_i \in [0,1] \), the moduli space of holomorphic Whitney disks with respect to the \( k \)-parameter family \( (\ihat{\mathcal{J}} \times \ihat{\mathcal{J}}') \) is transversally cut out if the moduli spaces with respect to \( \ihat{\mathcal{J}} \) and \( \ihat{\mathcal{J}}' \) and transversally cut out and intersect transversally in \( [0,1]^k \), in which case the moduli space with respect to \( (\ihat{\mathcal{J}} \times \ihat{\mathcal{J}}') \) is the fiber product, over \([0,1]^k\), of the moduli spaces with respect to \( \ihat{\mathcal{J}} \) and \( \ihat{\mathcal{J}}' \).

Next, observe that \( \mathcal{E}(H \times H') = (\mathcal{E} H) \times (\mathcal{E} H') \).

Now, fix sufficiently generic homotopy coherent diagrams \( F: \mathcal{E} H \to \mathcal{J}_M \) and \( F': \mathcal{E} H' \to \mathcal{J}_{M'} \). Construct a homotopy coherent diagram \( F F': \mathcal{E}(H \times H') \to \mathcal{J}_{M \times M'} \) as follows. On objects, define \( FF'(h, h') = F(h) \times F'(h') \). More generally, define
\[ FF'((f_n, f'_n), \ldots, (f_1, f'_1))(t_1, \ldots, t_{n-1}) = [F(f_n, \ldots, f_1)(t_1, \ldots, t_{n-1})] \times [F'(f'_n, \ldots, f'_1)(t_1, \ldots, t_{n-1})]. \]
Perturbing \( F \) and \( F' \) slightly, we may assume that the moduli spaces with respect to the family of almost complex structures \( F'(f'_n, \ldots, f'_1) \) are transverse to the moduli spaces with respect to \( F(f_n, \ldots, f_1) \), so \( FF' \) is sufficiently generic. Since \( F \) and \( F' \) were already generic, this perturbation does not change the functors \( G: \mathcal{E} H \to \Kom, G': \mathcal{E} H' \to \Kom \).

Given a sequence of morphisms \( h_0 \xrightarrow{f_1} h_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} h_k \) in \( \mathcal{E} H \) and a sequence of morphisms \( h'_0 \xrightarrow{f'_1} h'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_k} h'_k \) in \( \mathcal{E} H' \), a shuffle of these sequences is a sequence of morphisms \( (h_0, h'_0) \xrightarrow{g_0} \cdots \xrightarrow{g_{k+1}} (h_k, h'_k) \), where each \( g_i \) either has the form \( (f_j, \id) \) or \( (\id, f'_j) \), and the morphisms \( f_1, \ldots, f_k \) appear in order, once each, in this sequence, as do \( f'_1, \ldots, f'_k \).

For example, the three shuffles of \( h_0 \xrightarrow{f_1} h_1 \xrightarrow{f_2} h_2 \) and \( h'_0 \xrightarrow{f'_1} h'_1 \) are
\[
\begin{align*}
(h_0, h'_0) \xrightarrow{f_1 \times \id} (h_1, h'_0) \xrightarrow{f_2 \times \id} (h_2, h'_0) \xrightarrow{\id \times f'_1} (h_2, h'_1) \\
(h_0, h'_0) \xrightarrow{f_1 \times \id} (h_1, h'_0) \xrightarrow{1 \times f'_1} (h_1, h'_1) \xrightarrow{f_2 \times \id} (h_2, h'_1) \\
(h_0, h'_0) \xrightarrow{1 \times f'_1} (h_0, h'_1) \xrightarrow{f_1 \times \id} (h_1, h'_1) \xrightarrow{f_2 \times \id} (h_2, h'_1).
\end{align*}
\]
The shuffles correspond to permutations \( \sigma \in S_{k+\ell} \) so that \( \sigma|_{\{1, \ldots, k\}} \) and \( \sigma|_{\{k+1, \ldots, k+\ell\}} \) are increasing.

Notice that if \( (g_1, \ldots, g_{k+\ell}) \) is a shuffle then the moduli spaces of index \(-k - \ell + 1\) with respect to \( FF'(g_{k+\ell}, \ldots, g_1) \) are empty unless either \( k = 0 \) or \( \ell = 0 \). Indeed, the family of almost complex structures \( FF'(g_{k+\ell}, \ldots, g_1) \) factors through a map to \([0,1]^{k+\ell-2}\), so Maslov index \( 1 - k - \ell \) moduli spaces are empty. The exception is if \( k = 0 \) (respectively \( \ell = 0 \), in
which case the moduli space is identified with the moduli space of $F'(f_1', \ldots, f_i')$-holomorphic disks (respectively $F(f_k, \ldots, f_1)$-holomorphic disks), multiplied by constant disks in the other factor.

Let $G: \mathcal{E}H \to \text{Kom}$, $G': \mathcal{E}H' \to \text{Kom}$, and $GG': \mathcal{E}(H \times H') \to \text{Kom}$ be the homotopy coherent diagrams corresponding to $F$, $F'$, and $FF'$, respectively. With notation as in [HLS16, Definition 3.11], define a map

$$\eta: (\text{hocolim } G) \otimes (\text{hocolim } G') \to \text{hocolim } GG'$$

by

$$\eta((f_k, \ldots, f_1; \{0, 1\}^{\otimes k}; x) \otimes (f_{\ell}', \ldots, f_{i}'; \{0, 1\}^{\otimes \ell}; y)) = \sum_{\text{shuffles } g_1, \ldots, g_{k+\ell}} (g_{k+\ell}, \ldots, g_1; \{0, 1\}^{\otimes k+\ell}; x \otimes y).$$

We verify that $\eta$ is a chain map. The terms arising from taking the differential of $x$ or $y$ before or after applying $\eta$ clearly cancel in pairs, so we will ignore them from here on.

The remaining terms in $\partial \circ \eta$ are:

$$\sum_{i=1}^{k} \eta((f_k, \ldots, f_{i+1}; \{0, 1\}^{k-i} \otimes 0 \otimes \{0, 1\}^{k-i}; x) \otimes (f_{\ell}', \ldots, f_{i}'; \{0, 1\}^{\otimes \ell}; y))$$

$$+ \eta((f_k, \ldots, f_1; \{0, 1\}^{\otimes 1} \otimes 1 \otimes \{0, 1\}^{\otimes k}; x) \otimes (f_{\ell}', \ldots, f_{i}'; \{0, 1\}^{\otimes \ell}; y))$$

$$+ \sum_{i=1}^{k-1} \eta((f_k, \ldots, f_{i+1}; \{0, 1\}^{k-i}; G(f_i, \ldots, f_1)(\{0, 1\}^{i-1} \otimes x) \otimes (f_{\ell}', \ldots, f_{i}'; \{0, 1\}^{\otimes \ell}; y))$$

$$+ \eta((f_k, \ldots, f_{i+1} \circ f_i, \ldots, f_1; \{0, 1\}^{\otimes k-1}; x) \otimes (f_{\ell}', \ldots, f_{i}'; \{0, 1\}^{\otimes \ell}; y))$$

$$+ \eta((f_k, \ldots, f_{i+1}; \{0, 1\}^{\otimes k}; x) \otimes (f_{\ell}', \ldots, f_{i+1}'; \{0, 1\}^{\otimes \ell-i}; y))$$

$$+ \eta((f_k, \ldots, f_{i+1} \circ f_i, \ldots, f_{i+1}'; \{0, 1\}^{\otimes k-1}; x) \otimes (f_{\ell}', \ldots, f_{i+1}'; \{0, 1\}^{\otimes \ell-1}; y))$$

$$+ \eta((f_k, \ldots, f_{i+1}; \{0, 1\}^{\otimes k}; x) \otimes (f_{\ell-1}', \ldots, f_{i+1}'; \{0, 1\}^{\otimes \ell-1}; y)).$$

Call the terms on the six lines type A1–A6.

The remaining terms in $\partial \circ \eta$ are

$$\sum_{\text{shuffles } g_1, \ldots, g_{k+\ell}} \sum_{i=1}^{k+\ell-1} (g_{k+\ell}, \ldots, g_{i+1}; \{0, 1\}^{k+\ell-i}; GG'(g_i, \ldots, g_1)(\{0, 1\}^{i-1} \otimes (x \otimes y)))$$

$$+ (g_{k+\ell}, \ldots, g_{i+1} \circ g_i, \ldots, g_1; \{0, 1\}^{\otimes k+\ell-1}; (x \otimes y))$$

$$+ (g_{k+\ell-1}, \ldots, g_1; \{0, 1\}^{\otimes k+\ell-1}; (x \otimes y)).$$

Call the terms on the three lines type B1–B3.

- For type B1 terms, if some $g_j$, $1 \leq j \leq i$, is of the form $f \otimes \mathbb{I}$ while another $g_{j'}$, $1 \leq j' \leq i$, is of the form $\mathbb{I} \otimes f'$ then $GG'(g_i, \ldots, g_1) = 0$, because the corresponding
Corollary 3.6. Suppose that $H$ acts on both $(M, L_0, L_1)$ and $(M', L_0', L_1')$, both satisfying [HLS16, Hypothesis 3.2]. Endowing $(M \times M', L_0 \times L_0', L_1 \times L_1')$ with the diagonal action of $H$, there is a quasi-isomorphism

$$\widetilde{CF}^H(L_0 \times L_0', L_1 \times L_1') \simeq \widetilde{CF}^H(L_0, L_1) \otimes_{\mathbb{F}_2} \widetilde{CF}^H(L_0', L_1')$$

as chain complexes over $\mathbb{F}_2[H]$ with $H$ acting by the diagonal action on the right hand side.

Proof. With notation as in the proof of Theorem 3.5, the diagonal map $H \to H \times H$ induces an inclusion map $\Delta: \mathcal{E}H \hookrightarrow \mathcal{E}H \times \mathcal{E}H$. Composing with the functor $FF'$ gives a homotopy coherent $\mathcal{E}H$-diagram of almost complex structures $(FF') \circ \Delta$. The corresponding homotopy coherent diagram of chain complexes is $(GG') \circ \Delta$. There is an induced map

$$(3.7) \quad \text{hocolim}_{\mathcal{E}H}[(GG') \circ \Delta] \to \text{hocolim}_{\mathcal{E}H \times \mathcal{E}H}(GG').$$

This map clearly respects the $\mathbb{F}_2[H]$-module structure, and using Theorem 3.5, the two terms are quasi-isomorphic to $\widetilde{CF}^H(L_0 \times L_0', L_1 \times L_1')$ and $\widetilde{CF}^H(L_0, L_1) \otimes_{\mathbb{F}_2} \widetilde{CF}^H(L_0', L_1')$ over $\mathbb{F}_2[H]$.

Since for any object $h$ of $\mathcal{E}H$ (i.e., element $h \in H$) the inclusion of $G(h) \otimes G(h)$ into both $\text{hocolim}_{\mathcal{E}H \times \mathcal{E}H}(GG')$ and $\text{hocolim}_{\mathcal{E}H}[(GG') \circ \Delta]$ are quasi-isomorphisms, the map (3.7) is also a quasi-isomorphism. This proves the result. \qed

Corollary 3.8. Suppose that $H$ acts (symplectically) on symplectic manifolds $M, M', N$ and suppose there are $H$-invariant open subsets $V \subset M$, $V' \subset M'$ and $U \subset N$ containing $H$-invariant closed Lagrangians $L_0, L_1 \subset V$, $L_0', L_1' \subset V'$, and $K_0, K_1 \subset U$ such that the actions of $H$ on $(M, L_0, L_1)$, $(M', L_0', L_1')$, $(N, K_0, K_1)$ all satisfy [HLS16, Hypothesis 3.2], and $(U, K_0, K_1)$ is identified $H$-symplectically with the product $(V \times V', L_0 \times L_0', L_1 \times L_1')$. As in the proofs of Theorem 3.5 and Corollary 3.6, suppose that there exist systems of eventually cylindrical almost complex structures $F$ and $F'$ for $M$ and $M'$ and extensions $(FF') \circ \Delta$ of $(FF') \circ \Delta$ from $U$ to all of $N$, so that $F$, $F'$, and $(FF') \circ \Delta$ are regular for the freed complexes.
\[ CF^H(L_0, L_1), CF^H(L'_0, L'_1), \text{and } CF^H(K_0, K_1) \text{ and for which the defining holomorphic strips all lie inside } V, V' \text{ and } U. \] Then there is a quasi-isomorphism
\[ CF^H(K_0, K_1) \simeq CF^H(L_0, L_1) \otimes F_2 CF^H(L'_0, L'_1) \]
as chain complexes over \( F_2[H] \) with \( H \) acting by the diagonal action on the right hand term.

**Proof.** This follows immediately from Corollary 3.6. \( \square \)

3.3. The Künneth theorem for equivariant symplectic Khovanov homology. In this section we prove an equivariant version of Waldron’s Künneth theorem for symplectic Khovanov homology in [Wal09, Theorem 1.2].

We will use the following elementary lemma:

**Lemma 3.9.** Let \( p(z) \) be a complex polynomial which has simple roots and no zeros in the open unit disk \( D \). Then there is a smooth 1-parameter family of complex polynomials \( p_t(z) \), each with no zeros in \( D \), and only simple roots anywhere, interpolating between \( p(z) \) and the constant function 1.

**Proof.** The proof is by induction on the degree \( n \) of \( p(z) = a_0 + \cdots + a_n z^n \). By multiplying by a path in \( \mathbb{C}^n \) from 1 to \( 1/a_n \), we may assume \( p(z) \) is monic. A monic polynomial is uniquely determined by its roots. So, there is a path from \( p(z) \) to the polynomial \((z-2)(z-3)\cdots(z-n)\) simply by moving all the roots outside the unit disk. Next, let \( q(z) = (z-2)(z-3)\cdots(z-n) \) and consider the path of polynomials \( p_t(z) = [(1-t)z - n - 1]q(z) \).

The roots of \( p_t(z) \) are \( 2, 3, \ldots, n \) and \((n+1)/(1-t)\), all of which lie outside the unit circle and are distinct. The polynomial \( p_t(z) \) has degree \((n-1)\), and all roots outside the unit circle. By induction, this completes the proof. (Note that when concatenating the paths in the different steps of the proof, one needs to reparameterize the paths so the concatenation is smooth.) \( \square \)

**Proposition 3.10.** Given bridge diagrams \( L \) and \( L' \), there is a quasi-isomorphism of chain complexes over \( F_2[D_{2m}] \),
\[ C_{Kh}^{\text{symp, free}} (L \amalg L') \simeq C_{Kh}^{\text{symp, free}} (L) \otimes F_2 C_{Kh}^{\text{symp, free}} (L'), \]
where the right-hand side has the diagonal action of \( D_{2m} \).

**Proof.** For symplectic Khovanov homology itself, this is [Wal09, Theorem 1.2].

For the freed Floer complex, we will deduce this result from Corollary 3.8 after choosing suitable open sets \( U, V, \) and \( V' \) and deforming the symplectic forms on the open set \( U \) to be a product (without losing control of the holomorphic curves).

Let \( b_1, \ldots, b_n \) be the endpoints of the bridges in \( L \) (so \( n \) be the number of bridges in \( L \)) and \( b_{n+1}, \ldots, b_{n+2n'} \) the endpoints of the bridges in \( L' \). After an isotopy, we may assume there are disjoint open disks \( U_L, U_{L'} \subset \mathbb{C} \), containing \( L \) and \( L' \). Let
\[
S_L = \{ (u, v, z) \in \mathbb{C}^3 | u^2 + v^2 + (z - b_1) \cdots (z - b_n) = 0 \}
\]
\[
S_{L'} = \{ (u, v, z) \in \mathbb{C}^3 | u^2 + v^2 + (z - b_{n+1}) \cdots (z - b_{n+2n'}) = 0 \}
\]
\[
S_{L \amalg L'} = \{ (u, v, z) \in \mathbb{C}^3 | u^2 + v^2 + (z - b_1) \cdots (z - b_{n+2n'}) = 0 \}. 
\]

Let \( \tilde{U}_L \) (respectively \( \tilde{U}_{L'} \)) be the preimage of \( U_L \) (respectively \( U'_{L'} \)) in \( S_{L \amalg L'} \). Let \( V_L \) (respectively \( V_{L'} \)) be the preimage of \( U_L \) in \( S_L \) (respectively the preimage of \( U_{L'} \) in \( S_{L'} \)).
By Point (D-2) in Section 3.1, any holomorphic Whitney disk in $\mathcal{Y}_{n+n'}$ lies in the subspace $U := \text{Hilb}^n(\tilde{U}_L) \times \text{Hilb}^{n'}(\tilde{U}_{L'}) \cap \mathcal{Y}_{n+n'}$. If we let $\nabla$ denote the subspace of $\text{Hilb}^n(\tilde{U}_L)$ (respectively $\text{Hilb}^{n'}(\tilde{U}_{L'})$) where the projection to $\mathbb{C}$ has length less than $n$ (respectively $n'$) then
\[
[\text{Hilb}^n(\tilde{U}_L) \times \text{Hilb}^{n'}(\tilde{U}_{L'})] \cap \mathcal{Y}_{n+n'} = [\text{Hilb}^n(\tilde{U}_L) \setminus \nabla] \times [\text{Hilb}^{n'}(\tilde{U}_{L'}) \setminus \nabla].
\]

With respect to the restrictions of the averaged symplectic forms on $\text{Hilb}^{n+n'}(S_{L_{H\ddot{u}}})$ from [HLS, Lemma 4.24] (see also Section 3.1.3), this identification is a symplectomorphism, and it is also biholomorphic with respect to the standard complex structures.

Let
\[
V = \text{Hilb}^n(V_L) \setminus \nabla \subset \mathcal{Y}_n
\]
\[
V' = \text{Hilb}^{n'}(V_{L'}) \setminus \nabla \subset \mathcal{Y}_{n'}.
\]

By the general Künneth theorem for the freed Floer complex (Corollary 3.8) it suffices to show that the freed Floer complex of $(\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}, \Sigma_{B_1} \times \cdots \times \Sigma_{B_n})$ inside $\text{Hilb}^n(\tilde{U}_L) \setminus \nabla$ is quasi-isomorphic to their freed Floer complex inside $V$ (and similarly for $V'$).

By Lemma 3.9, the polynomials $p_0(z) = (z-b_1) \cdots (z-b_{2n+2\nu})$ and $p_1(z) = (z-b_1) \cdots (z-b_{2n})$ can be connected by a smooth family of polynomials $p_t(z)$, $t \in [0,1]$ whose roots in $U$ are exactly $b_1, \ldots, b_{2n}$, and so that all the roots of $p_t(z)$ are simple. Let $S_t = \{(u,v,z) \in \mathbb{C}^3 \mid u^2 + v^2 + p_t(z) = 0\}$ and let $V_t$ be the preimage of $U$ in $S_t$ ($t \in [0,1]$). The subspaces $V_t$ form a smooth family of open complex surfaces. (In particular, each $S_t$ is smooth, since $p_t$ has only simple roots.)

Each $V_t$ contains Lagrangian spheres $\Sigma_{A_i}$ and $\Sigma_{B_i}$ for $i = 1, \ldots, n$. Taking their Hilbert schemes gives a smooth family of complex manifolds $\text{Hilb}^n(V_t) \setminus \nabla$. Abouzaid-Smith’s construction of their Kähler form $\omega^t$ in [AS16, Lemma 5.5] gives a smooth 1-parameter family of Kähler forms on $\text{Hilb}^n(V_t)$, which restrict to $\text{Hilb}^n(V_t) \setminus \nabla$ as exact forms and which agree with the product form outside a neighborhood of the diagonal, so $\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}$ and $\Sigma_{B_1} \times \cdots \times \Sigma_{B_n}$ are Lagrangian. The averaging construction from [HLS, Lemma 4.24] then gives a smooth family of $O(2)$-invariant Kähler forms for which $\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}$ and $\Sigma_{B_1} \times \cdots \times \Sigma_{B_n}$ are still Lagrangian.

Let $V_t = V_0$ if $t < 0$ and $V_t = V_1$ if $t > 1$. The continuation map from the freed Floer complex of $(\text{Hilb}^n(V_0) \setminus \nabla, \Sigma_{A_1} \times \cdots \times \Sigma_{A_n}, \Sigma_{B_1} \times \cdots \times \Sigma_{B_n})$ to the freed Floer complex of $(\text{Hilb}^n(V_1) \setminus \nabla, \Sigma_{A_1} \times \cdots \times \Sigma_{A_n}, \Sigma_{B_1} \times \cdots \times \Sigma_{B_n})$ is defined by counting $\tilde{J}$-holomorphic sections of a bundle $E$ over $\mathbb{R} \times [0,1]$ whose fiber over $(t,s)$ is $V_t$, with boundary in the subbundle $F \subset E$ specified by $\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}$ and $\Sigma_{B_1} \times \cdots \times \Sigma_{B_n}$ over $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$, respectively. Here, the $\tilde{J}$ are suitable families of fiberwise almost complex structures $\tilde{J}$ satisfying analogues of (J-1)–(J-3).

We claim that the manifolds $\text{Hilb}^n(V_t) \setminus \nabla$ are uniformly $I$-convex in the following sense, which implies the continuation maps are well-defined. Let $j_t$ be the complex structure on $V_t$ and let $I_t = \text{Hilb}^n(j_t)$ be the complex structure on $\text{Hilb}^n(V_t)$ inherited from $V_t$. Let $J_t$ be a family of almost complex structures satisfying conditions (J-1)–(J-3) with respect to $j_t$ and $I_t$. In particular, assume that $J_t$ agrees with $I_t$ outside a compact set $K_t$, so that $\bigcup_t K_t$ is also compact. The almost complex structures $J_t$ give a fiberwise almost complex structure on $E$. By uniform $I$-convexity we mean there is a compact set $K'$, depending only on the $K_t$, so that if $u$ is a $J_t$-holomorphic section of $(E,F)$ then the image of $u$ is contained in $K'$.
Specifically, fix a family of $I_t$-holomorphic embeddings of the manifolds $\text{Hilb}^n(S_t) \setminus \nabla$ in $\mathbb{C}^N$ for some large $N$. Then $K'$ is the intersection of $\bigcup_t \text{Hilb}^n(V_t) \setminus \nabla$ with:

- $\text{Hilb}^n(V'_t)$ where $V'_t$ is the preimage of a slightly smaller open set $U'$ with $\overline{U'} \subset U$, and
- the polydisk $\{(z_1, \ldots, z_n) \in \mathbb{C}^N \mid |z_i| \leq R\}$ where $R$ is large enough that this set contains the Lagrangians and the compact sets $K_t$.

The fact that the image of a holomorphic curve $u$ lies in $K'$ follows from positivity of intersections (for the first term in the intersection, as in (D-2) above) and the maximum modulus theorem (for the second term in the intersection).

As noted above, this convexity is enough to ensure that the proof of invariance of the freed Floer complex [HLS16, Proposition 3.28] applies. Hence, the freed Floer complexes in $\text{Hilb}^n(V_0) \setminus \nabla$ and $\text{Hilb}^n(V_t) \setminus \nabla$ agree up to quasi-isomorphism, completing the proof. □

3.4. The basepoint action. The last ingredient in the proof of equivariant stabilization invariance is a module structure on symplectic Khovanov homology, analogous to one on Khovanov homology. Fix a bridge diagram $L$ and a basepoint $p \in L$ on one the $A$-arcs, say $A_1$. There is an action of $\mathbb{F}_2[X]$ on the symplectic Khovanov complex of $L$ defined as follows. Choose a preimage $p_i \in \Sigma_{A_i}$ lying over $p \in A_i$. For any $q \in \Sigma_{A_i}$, let $O_q \subset \mathcal{K}_A$ denote the codimension-2 subspace where one of the coordinates is $q$. Then the action of $X$ counts rigid holomorphic strips $u: \mathbb{R} \times [0,1] \to \mathcal{Y}_n$ with $u(0,0) \in O_{p_e}$. (The fact that such moduli spaces with a point constraint are transversely cut out is a straightforward adaptation of [MS04, Theorem 3.4.1] to the relative case.)

**Theorem 3.11.** The above action of $\mathbb{F}_2[X]$ on the symplectic Khovanov complex of $L$ satisfies the following properties:

- **(BP-1)** Multiplication by $X^2$ is homotopic to 0, so symplectic Khovanov homology inherits an action of $\mathbb{F}_2[X]/(X^2)$.
- **(BP-2)** If $p, p' \in L$ are different points then multiplication by $X$ at $p$ and at $p'$ commute up to homotopy.
- **(BP-3)** The symplectic Khovanov homology of the 1-bridge unknot is isomorphic to $\mathbb{F}_2[X]/(X^2)$.
- **(BP-4)** Up to homotopy, the chain maps associated to changes of almost complex structures on $\mathcal{Y}_n$, isotopies and handleslides of the $A$- and $B$-arcs which do not move the point $p$, and diffeomorphisms (which may move $p$), commute with multiplication by $X$.
- **(BP-5)** The maps in the skein exact triangle from Theorem 3.2 respect the action by $\mathbb{F}_2[X]/(X^2)$ (where corresponding points $p$ are used for the three diagrams).
- **(BP-6)** If $A_i, B_i, A_{i+1}$ are adjacent arcs in a bridge diagram so that the interior of $B_i$ does not intersect any $A$-arc and the interior of $A_i$ does not intersect any $B$-arc—that is, if the configuration looks like Figure 4(b)—and if $p \in A_i$ and $p' \in A_{i+1}$, then the actions by $p$ and $p'$ are homotopic.
- **(BP-7)** If $L$ is a disjoint union of two bridge diagrams $L_1 \sqcup L_2$ and $p$ is a basepoint on $L_1$ then the Künneth theorem $\mathcal{C}_{\text{Kh,symp}}(L) \simeq \mathcal{C}_{\text{Kh,symp}}(L_1) \otimes_{\mathbb{F}_2} \mathcal{C}_{\text{Kh,symp}}(L_2)$ (from [Wal09, Theorem 1.2] or the non-equivariant version of Proposition 3.10) intertwines the action of $X$ on $\mathcal{C}_{\text{Kh,symp}}(L)$ and the action of $X$ on $\mathcal{C}_{\text{Kh,symp}}(L_1) \otimes_{\mathbb{F}_2} \mathcal{C}_{\text{Kh,symp}}(L_2)$ induced from the action of $X$ on $\mathcal{C}_{\text{Kh,symp}}(L_1)$. That is, for appropriate choices of almost complex structures, $(m \otimes n) \ast_p X = (m \ast_p X) \otimes n$.

**Proof.** Some of these properties (namely (BP-1), (BP-2), (BP-5), and a part of (BP-4)) are fairly standard (see, e.g., [Sei08, Section 8] or [Per08a, Section 3.9], and references therein)
and hold for Lagrangian Floer homology more generally in the absence of disk bubbles, so we will only sketch the proofs of those properties and concentrate on the properties that are specific to symplectic Khovanov homology.

For Property (BP-1), let \( p'_e \) be a point in the fiber over \( p \) close to \( p_e \). For a generic choice of almost complex structure, if we choose \( p'_e \) close enough to \( p_e \) then the actions induced by \( p_e \) and \( p'_e \) agree. Observe that \( O_{p_e} \cap O_{p'_e} = \emptyset \). Counting holomorphic bigons with \( u(0,0) \in O_{p_e} \) and \( u(t,0) \in O_{p'_e} \) for some \( t > 0 \) gives a homotopy from multiplication by \( X^2 \) (corresponding to \( t \to \infty \)) to 0 (corresponding to \( t = 0 \)).

For Property (BP-2), let \( *_p X \) and \( *_{p'} X \) be the actions at \( p \) and \( p' \), respectively. Counting holomorphic disks with \( u(0,0) \in O_{p_e} \) and \( u(0,t) \in O_{p'_e} \), for any \( t \in \mathbb{R} \), gives a homotopy between \( *_p X *_{p'} X \) (for \( t \to -\infty \)) and \( *_{p'} X *_p X \) (for \( t \to +\infty \)).

For Property (BP-3), observe that, with respect to the standard complex structure on \( S \), there is an \( S^1 \)-family of holomorphic disks connecting the two generators of \( C_{Kh, symp}(U) \), one through each preimage of \( p \) on \( \Sigma_A = \mathcal{K}_A \). Indeed, the subset \( S_0 = \{(u,v,z) \in S \mid v = 0\} \) is bi-holomorphic to the cylinder \( \mathbb{R} \times S^1 \), and \( \Sigma_A \cap S_0 \) and \( \Sigma_B \cap S_0 \) are circles inside \( S_0 \) intersecting in two points. There are two holomorphic disks in \( S_0 \), transversally cut out, whose images under \( S^1 \) form a single family of holomorphic disks passing through each preimage of \( p \) on \( \Sigma_A \). It follows from a doubling argument and automatic transversality as in [HLS97] or, equivalently, a doubling argument and [MS04, Lemma 3.3.1], that these holomorphic disks are transversally cut out in \( S \). Next, consider the involution \( \tau: S \to S \), \( \tau(u,v,z) = (u, -v, z) \). Since the holomorphic disks inside \( \text{Fix}(\tau) = S_0 \) are transversally cut out, we can perturb the complex structure slightly to a \( \tau \)-invariant almost complex structure in which all holomorphic bigons are transversally cut out (see, e.g., [KS02, Section 5c]). Choose the preimage \( p_e \) of \( p \) to lie in \( S_0 \). Then any holomorphic bigons not contained in \( S_0 \) passing through \( p_e \) come in pairs exchanged by \( \tau \), and hence contribute 0 (mod 2) to the disk count.

Property (BP-4) is proved by considering holomorphic disks with cylindrical-at-infinity complex structures (for changes of complex structures or isotopies of the \( A \)- or \( B \)-arcs) or holomorphic triangles (for handleslides) with a similar point constraint. This in particular includes isotopies of \( B \)-arcs that pass over \( p \). For the case of a handleslide between \( A \)-arcs, there is an additional complication, so we spell out that case. Let \( A' \) be a collection of bridges obtained from the \( A \) bridges by a handleslide, arranged in the plane so that the \( A \)- and \( A' \)-bridges intersect only at their endpoints. Fix points \( p \in A_i \) and \( p' \in A'_i \), where \( A'_i \) is the \( A' \)-arc corresponding to \( A_i \), which is either a small translate of \( A_i \) or, if \( A_i \) is being handleslid, the result of the handleslide. Let \( p_e \in \Sigma_{A_i} \) over \( p \) and \( p'_e \in \Sigma_{A'_i} \) over \( p' \). We will focus on the case that \( p \) is on the arc being handleslid; the other cases are similar but easier.

Consider the 1-dimensional moduli spaces of holomorphic triangles with boundary on \( (\mathcal{K}_A, \mathcal{K}_{A'}, \mathcal{K}_B) \) with either a point along the \( A \)-edge mapped to \( O_{p_e} \) or a point along the \( A' \)-edge mapped to \( O_{p'_e} \), and the corner at \( (\mathcal{K}_A, \mathcal{K}_{A'}) \) mapped to the generator 1. This moduli space has six kinds of ends:

(TM-1) Ends corresponding to a bigon for \( (\mathcal{K}_A, \mathcal{K}_B) \) with a point mapping to \( O_{p_e} \) and a holomorphic triangle. These correspond to following the basepoint action for \( (\mathcal{K}_A, \mathcal{K}_B) \) by the isomorphism \( F: HF(\mathcal{K}_A, \mathcal{K}_B) \to HF(\mathcal{K}_{A'}, \mathcal{K}_B) \).

(TM-2) Ends corresponding to a bigon for \( (\mathcal{K}_{A'}, \mathcal{K}_B) \) with a point mapping to \( O_{p'_e} \) and a holomorphic triangle. These correspond to following \( F \) by the basepoint action for \( (\mathcal{K}_{A'}, \mathcal{K}_B) \).
(TM-3) Ends corresponding to a bigon for \((K_A, K_B)\) and a holomorphic triangle with a point constraint. These correspond to \(H \circ \partial\), where \(H\) is a homotopy defined by counting rigid triangles with a point constraint.

(TM-4) Ends corresponding to a bigon for \((K_{A'}, K_B)\) and a holomorphic triangle with a point constraint. These correspond to \(\partial \circ H\), where \(H\) is a homotopy defined by counting rigid triangles with a point constraint.

(TM-5) Ends corresponding to a bigon for \((K_A, K_{A'})\) with a point mapping to \(O_{p_1}\) and a holomorphic triangle.

(TM-6) Ends corresponding to a bigon for \((K_A, K_{A'})\) with a point mapping to \(O_{p_1}\) and a holomorphic triangle.

We need to show that the last two cases cancel. The triangles in the last two cases are the same, so we need to know that the number of bigons in the two cases agree (modulo 2). This count of bigons is exactly the basepoint action for the bridge diagram \((A, A')\), with either a basepoint on \(A\) or \(A'\). Since the differential on \(CF(K_A, K_{A'})\) is trivial for grading reasons, it suffices to verify that the two basepoint actions are homotopic.

By the proof of Proposition 3.10 (in the slightly simpler, non-equivariant case), it suffices to consider the case that \(A\) consists of only the two arcs involved in the handleslide. Label the endpoints of these arcs \(a_1, \ldots, a_4\), so that \(a_1\) and \(a_4\) are on the outer circle of \(A \cup A'\). (See Figure 2.) The generators of \(CF(K_A, K_{A'})\) are \(\{a_1, a_2\}, \{a_1, a_3\}, \{a_4, a_2\}, \) and \(\{a_4, a_3\}\). Label the arcs so that \(\{a_1, a_2\}\) is the top-graded generator, denoted 1 above. Let \(A_1\) be the arc with \(\partial A_1 = \{a_1, a_4\}\) and \(A_2\) the arc with \(\partial A_2 = \{a_2, a_3\}\).

Let \(\tau: \mathbb{C} \to \mathbb{C}\) be rotation by \(\pi\). Arrange that \(\tau\) exchanges \(A\) and \(A'\), and in particular exchanges \(A_1\) and \(A'_1\). There is an induced map \(\tau: \mathcal{Y}_n \to \mathcal{Y}_n\) which exchanges \(K_A\) and \(K_{A'}\).

The basepoint \(p\) lies on \(A_1\). There is a corresponding basepoint \(\tau(p)\) on \(A'_1\). Choose also a basepoint \(q\) on \(A_2\). We claim that:

1. \(\{a_1, a_2\} \ast_q X = \{a_1, a_3\}\) and \(\{a_4, a_2\} \ast_q X = \{a_4, a_3\}\).
2. The coefficient of \(\{a_1, a_3\}\) in \(\{a_1, a_2\} \ast_p X\) is 0, as is the coefficient of \(\{a_1, a_3\}\) in \(\{a_1, a_2\} \ast_{\tau(p)} X\).
3. The coefficient of \(\{a_4, a_2\}\) in \(\{a_1, a_2\} \ast_{\tau(p)} X\) is the same as the coefficient of \(\{a_4, a_3\}\) in \(\{a_1, a_3\} \ast_p X\).

Together, these three claims imply the result. Indeed, our goal is to show that \(\{a_1, a_2\} \ast_p X = \{a_1, a_2\} \ast_{\tau(p)} X\). By the second claim, \(\{a_1, a_2\} \ast_p X = \epsilon \{a_4, a_2\}\) and \(\{a_1, a_2\} \ast_{\tau(p)} X = \delta \{a_4, a_2\}\) for some \(\epsilon, \delta \in \mathbb{F}_2\); our goal is to show that \(\epsilon = \delta\). By Property (BP-2),

\[
\{a_1, a_2\} \ast_q X \ast_p X = \{a_1, a_2\} \ast_p X \ast_q X.
\]
For the left hand side, by the first claim, \( \{a_1, a_2\} \ast_q X = \{a_1, a_3\} \), and by the third claim, \( \{a_1, a_3\} \ast_p X = \delta \{a_4, a_3\} \). For the right hand side, \( \{a_1, a_2\} \ast_p X = \epsilon \{a_4, a_2\} \), and by the first claim, \( \epsilon \{a_4, a_2\} \ast_q X = \epsilon \{a_4, a_3\} \). Therefore, \( \epsilon = \delta \).

It remains to verify the three properties. Recall from Section 3.1 that a Whitney disk in \( \mathcal{Y}_n \) has a projected domain. For the first claim, observe that the projected domain of a holomorphic bigon counted for \( \{a_1, a_2\} \ast_q X \) is contained entirely in the bounded region of \( \mathbb{C} \setminus (A_2 \cup A'_2) \). Thus, the proof of the Künneth theorem shows that this computation is the same as in the 1-bridge unknot case, so this follows from Property (BP-3). Similarly, the second claim follows from the fact that there is no projected domain compatible with this action.

For the last claim we use the involution \( \tau \). Given a \( J \)-holomorphic Whitney disk \( u: \mathbb{R} \times [0, 1] \to \mathcal{Y}_n \) for \( (\mathcal{K}_A, \mathcal{K}_A') \) connecting \( \{a_1, a_2\} \) to \( \{a_4, a_2\} \) (so \( u(\mathbb{R} \times \{0\}) \subset \mathcal{K}_A \) and \( \lim_{s \to -\infty} u(s, t) = \{a_1, a_2\} \)), there is a corresponding \( (\tau \ast J) \)-holomorphic disk \( \tau \circ u: \mathbb{R} \times [0, 1] \to \mathcal{Y}_n \) for \( (\mathcal{K}_A', \mathcal{K}_A) \) connecting \( \{a_4, a_3\} \) to \( \{a_4, a_3\} \) (that is, \( \tau \circ u(\mathbb{R} \times \{0\}) \subset \mathcal{K}_A' \) and \( \lim_{s \to -\infty} (\tau \circ u)(s, t) = \{a_4, a_3\} \)). There is a holomorphic map \( \sigma: \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1] \) given by \( \sigma(s, t) = (1 - s, -t) \) (rotation around the middle of the strip). Then \( \tau \circ u \circ \sigma \) is a \( (\tau \ast J) \)-holomorphic Whitney disk for \( (\mathcal{K}_A, \mathcal{K}_A') \) connecting \( \{a_1, a_3\} \) to \( \{a_4, a_3\} \). Further, \( u \) passes through \( \mathcal{O}_{\tau(p_e)} \) if and only if \( \tau \circ u \circ \sigma \) passes through \( \mathcal{O}_{p_e} \). So, if we take \( p'_e = \tau(p_e) \) then the basepoint action using \( p'_e \) and the almost complex structure \( J \) agrees with the basepoint action using \( p_e \) and the almost complex structure \( \tau_*(J) \). Since the basepoint action on homology is independent of the choice of almost complex structure and the differential on the Floer complex is trivial, it follows that the coefficient of \( \{a_4, a_2\} \) in \( \{a_4, a_3\} \ast_p X \) is the same as the coefficient of \( \{a_4, a_3\} \) in \( \{a_1, a_3\} \ast_p X \), as desired. This concludes the proof of Property (BP-4).

Property (BP-5) follows from the same argument as the case of \( B \)-handleslides in Property (BP-4), by considering moduli spaces of holomorphic triangles with a point constraint. (Because only one edge of the triangle has a point constraint for the skein maps or a \( B \)-handleslide, there are no degenerations analogous to types (TM-5) and (TM-6), making these cases easier than the case of \( A \)-handleslides.)

For Property (BP-6), we use the well-known trick of moving the basepoint using handleslides and diffeomorphisms. Call a pair of adjacent \( A \) and \( B \) arcs a small pair if their interiors are disjoint from all bridges, for example as in Figure 4(b). Figure 3 shows how a small pair can be moved across an adjacent arc using handleslides and a diffeomorphism.
3.5. Proof of equivariant stabilization invariance.

**Proposition 3.12.** Let \( (\{A_i\}, \{B_i\}) \) and \( (\{A'_i\}, \{B'_i\}) \) be bridge diagrams for a link \( K \) which differ by a single stabilization, as in Figure 4(a,b). Then there is a quasi-isomorphism \( C_{Kh}^{symp, free}(\{A_i\}, \{B_i\}) \cong C_{Kh}^{symp, free}(\{A'_i\}, \{B'_i\}) \) over \( \mathbb{F}_2[D_{2m}] \).

**Proof.** Let \( L \) be a link diagram as in Figure 4(a), \( L_1 \) as in Figure 4(b), \( L_0 \) as in Figure 4(c), and \( L' \) as in Figure 4(d). Our goal is to show that there is a quasi-isomorphism

\[
C_{Kh}^{symp, free}(L) \cong C_{Kh}^{symp, free}(L_1)
\]

over \( \mathbb{F}_2[D_{2m}] \).

Let \( p \) be a point in \( L \) on the \( A \)-arc in the figure and \( q \) a point on \( A_{n+1} \). By Proposition 3.10, there is a quasi-isomorphism

\[
C_{Kh}^{symp, free}(L_0) \cong C_{Kh}^{symp, free}(L) \otimes C_{Kh}^{symp, free}(U)
\]

over \( \mathbb{F}_2[D_{2m}] \). On homology, this gives an isomorphism

\[
Kh_{symp}(L_0) \cong Kh_{symp}(L) \otimes Kh_{symp}(U).
\]

Theorem 3.11 gives two actions of \( \mathbb{F}_2[X]/(X^2) \), one coming from the point \( p \) and one from the point \( q \), and this isomorphism respects the actions. Write the action at \( p \) (respectively \( q \)) as \( \ast_p \) (respectively \( \ast_q \)). We claim that

\[
(3.13) \quad Kh_{symp}(L) \cong \{ m \in Kh_{symp}(L) \otimes Kh_{symp}(U) \mid m \ast_p X = m \ast_q X \}.
\]

Indeed, from Properties (BP-3) and (BP-7), \( a \otimes 1 + b \otimes X \) is an element of the right-hand side if and only if \( a \ast_p X = 0 \) and \( a = b \ast_p X \), but the second equation implies the first. So, this set is exactly \( \{(b \ast_p X \otimes 1) + (b \otimes X)\} \), which is isomorphic (as an \( \mathbb{F}_2 \)-vector space) to \( Kh_{symp}(L) \).

Now, consider the skein exact triangle

\[
\cdots \rightarrow Kh_{symp}(L') \rightarrow Kh_{symp}(L_0) \rightarrow Kh_{symp}(L_1) \rightarrow \cdots.
\]
By Theorem 3.1 and (the slightly simpler, non-equivariant case of) Proposition 3.10, this triangle is, in fact, a short exact sequence

$$0 \to \text{Kh}_{\text{symp}}(L') \to \text{Kh}_{\text{symp}}(L_0) \to \text{Kh}_{\text{symp}}(L_1) \to 0.$$ 

Let $g_* : \text{Kh}_{\text{symp}}(L_0) \to \text{Kh}_{\text{symp}}(L_1)$ be the map from this sequence. Since Property (BP-6) implies the actions at $p$ and $q$ are the same on $\text{Kh}_{\text{symp}}(L_1)$, by Property (BP-5) the map $g_*$ sends the image of $(*_p X - *_q X)$ to 0. Hence, by exactness and comparing dimensions, $g_*$ sends

$$\text{Kh}_{\text{symp}}(L_0)/\{m \in \text{Kh}_{\text{symp}}(L) \otimes \text{Kh}_{\text{symp}}(U) \mid m *_p X = m *_q X\} \cong \text{Kh}_{\text{symp}}(L)$$

isomorphically to $\text{Kh}_{\text{symp}}(L_1)$. There is a chain map over $\mathbb{F}_2[D_{2m}]$

$$f : C_{\text{Kh}}^{\text{symp,free}}(L) \to C_{\text{Kh}}^{\text{symp,free}}(L) \otimes_{\mathbb{F}_2} C_{\text{Kh}}^{\text{symp,free}}(U)$$

induced by $f(y) = y \otimes 1$. The induced map on homology is an isomorphism from $\text{Kh}_{\text{symp}}(L)$ to $\text{Kh}_{\text{symp}}(L)/\{m \in \text{Kh}_{\text{symp}}(L) \otimes \text{Kh}_{\text{symp}}(U) \mid m *_p X = m *_q X\}$.

Since the map $g_*$ is induced by counting holomorphic triangles with one corner at a $D_{2m}$-invariant intersection point, the map $g_*$ is induced by a map $g : C_{\text{Kh}}^{\text{symp,free}}(L_0) \to C_{\text{Kh}}^{\text{symp,free}}(L_1)$ (see, e.g., [HLS16, Proof of Proposition 3.25]). The composition $g \circ f : C_{\text{Kh}}^{\text{symp,free}}(L) \to C_{\text{Kh}}^{\text{symp,free}}(L_1)$ is the desired quasi-isomorphism. 

\begin{proof}[Proof of Theorem 1.26] This is immediate from isotopy and handleslide invariance, verified in our original proof of the theorem, and Proposition 3.12. \end{proof}

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