THE ACTION OF AFFINE DIFFEOMORPHISMS ON THE RELATIVE
COHOMOLOGY OF ABELIAN COVERS OF THE FLAT PILLOWCASE

CHENXI WU

Abstract. We calculate the action of the group of affine diffeomorphisms on the relative
cohomology of square-tiled surfaces that are normal abelian covers of the flat pillowcase,
and as an application, answer a question raised by Smillie and Weiss.

1. Introduction

In a paper in preparation by Smillie and Weiss on horocycle or bit closures they ask for
the existence of a square-tiled surface $M$ with cone point sets $\Sigma$ constructed as a normal
abelian branched cover of the flat pillowcase, such that there is a direct sum decomposition
$H^1(M, \Sigma; \mathbb{C}) = N \oplus H$ preserved by the action of the group of orientation preserving affine
diffeomorphisms, there is a positive or negative definite hermitian norm on $N$ invariant
under the affine diffeomorphism group action, and the affine diffeomorphism group action
on $N$ does not factor through a discrete group. We will construct such examples in Section
6.

In answering this question, we give a comprehensive treatment on relative cohomology
of branched abelian covers of the flat pillowcase, the affine diffeomorphism group action on
it, as well as the invariant subspaces and invariant Hermitian form under this action. This
problem is related to the monodromy of the hypergeometric functions which dates back to
Euler and is outlined in [DM86]. Wright [Wri12] described the Hodge form and invariant
direct sum decomposition on the absolute cohomology of such surfaces under the action
of a subgroup of the Veech group, and calculated the Lyapunov exponents of this action
by showing that the projectivization of the group action factors through a triangle group.
Forni-Matheus-Zorich [FMZ11], Bouw-Möller [BM10], Deligne-Mostow [DM86], McMullen
[McM13] and Eskin-Kontsevich-Zorich [EKZ10] have also done similar computations in dif-
ferent contexts. Matheus and Yoccoz [MY09] calculated the action of the full affine group
on relative cohomology for two specific abelian branched covers. By modifying some of the
ideas in their articles, as well as some arguments similar to [DM86] and [Thu98], we are
able to describe the action of the affine group on relative cohomology and establish the
existence of abelian covers as required by Smillie and Weiss’s paper. Hubert-Schmithüsen
[HSd] also gave a proof of the non-discreteness in some cases through Lyapunov exponents
and Galois conjugate. The author thanks his thesis advisor John Smillie for suggesting the
problem and many helpful conversations.
The existence of examples answering the question of Smillie and Weiss follows from a decomposition of cohomology into invariant components, which is done in Theorem 3.1, a signature calculation of the Hodge form on each component, and a discreteness criteria. Here we give an alternative, self-contained treatment. We will give a description of the affine diffeomorphism group of these surfaces in section 2. In section 3, we describe the action of affine diffeomorphism group on relative cohomology and show the existence of direct sum decomposition. In section 4, we calculated the signature of the Hodge Hermitian form. The reader is warned that our definition of Hodge Hermitian form is different from other definitions in the literature. In section 5 we described a useful subgroup of affine diffeomorphism group to work with. In section 6 construct examples that answer the question of Smillie and Weiss. If we only need to construct certain examples, it can also be done with discreteness criteria and signature calculation in [DM86].

We will now set up some notation to describe normal branched covers of the pillowcase. Let $P$ be the unit flat pillowcase with four cone points $z_1, z_2, z_3$ and $z_4$ of cone angle $\pi$ as follows:

![Figure 1.](image-url)

Let $G$ be a finite group and $g = (g_1, \ldots, g_4) \in G^4$ a 4-tuple of elements in $G$ such that $g_1g_2g_3g_4 = 1$. Let $M = M(G, g)$, be the connected normal branched cover of $P$ branching at $z_1, \ldots, z_4$, with deck transformation group $G$ acting on the left. The loop $l_j$ around $z_j$ in counter-clockwise direction on $P$ based in $B^1$ lifts to a path from the preimage of $B^1$ in the $g$-th sheet of the cover to the preimage of $B^1$ in the $gg_j$-th sheet. In other words, $g$ gives a group homomorphism from

$$\pi_1(P - \{z_1, z_2, z_3, z_4\}) = \langle l_1, l_2, l_3, l_4 | l_1l_2l_3l_4 = 1 \rangle$$

to $G$. Here the homomorphism defined by $g$ sends $l_j$ to $g_j \in G$. The connectedness of $M$ is equivalent to the condition that $\{g_1, \ldots, g_4\}$ generate $G$. Let $\Sigma$ denote the set of preimages of all points $z_j$, $j = 1, \ldots, 4$. The surface $M$ has a half translation structure induced by the half translation structure on $P$. Let $\text{Aff}(M, \Sigma)$ denote the group of orientation preserving
affine diffeomorphisms from $M$ to itself that sends $\Sigma$ to $\Sigma$. When the orders of $g_j$ are all even, all the holonomies are translations and $M$ is a translation surface. When the order of $g_j$ is 2, the corresponding vertex has cone angle $2\pi$. When none of the orders of $g_j$ is 2, $\Sigma$ consists of actual cone points of $M$, in which case $\text{Aff}$ is the affine diffeomorphism group.

The decomposition of $P$ into two squares in figure 1 induces a cell decomposition on $M(G, g)$, which can be described as $|G|$-copies of pair of squares labeled by elements in $G$ as $B^1_g$, $B^2_g$, that are glued together by identifying edges $e^j_g$ and $e^{j'}_g$ when $j = j'$ and $g = g'$, so that the directions indicated by the arrows match:

![Figure 2](image)

For example, in our notation the Wollmilchsau [For06][HSa] is $M(Z/4, (1,1,1,1))$, can be presented as the union of the following squares with indicated gluings:

![Figure 3](image)

As another example, let $G = Z/3$ and $g = (0,1,1,1)$. In this case $M = M(G, g)$ is a half translation surface and the gluing is as follows:
Now we describe the action of the deck group $G$ on $M(G,g)$. An element $h \in G$ sends $B^k_g$ to $B^k_{hg}$ and $e^k_g$ to $e^k_{hg}$. The deck group action induces a right $G$-action on $H^1(M, \Sigma; \mathbb{C})$ that makes it a right $G$-module.

2. Affine diffeomorphisms

From now on we assume that $G$ is abelian, though many of our arguments work for any finite group. At the end of this section we will point out the modification required in the non-Abelian case.

We calculate $\text{Aff} = \text{Aff}(M(G,g))$ in a way inspired by the coset graph description used in [Sch04]. One distinction is that we consider the whole affine diffeomorphism group while [Sch04] considers only the Veech group. Fixing $G$, let $V$ be the set of all 4-tuples $h = (h_1, h_2, h_3, h_4)$ such that $\{h_1, h_2, h_3, h_4\}$ generates $G$ and $h_1 h_2 h_3 h_4 = 1$, each is associated with a square-tiled surface $M(G,h)$ which is equipped with a cell decomposition labeled as in figure 2. By construction, an element $F$ in $\text{Aff}$ induces an automorphism of the deck group $G$ by $g \mapsto FgF^{-1}$, i.e. there is a group homomorphism $\text{Aff} \to \text{Aut}(G)$. We denote the kernel of this homomorphism as $\Gamma$. Because $\text{Aut}(G)$ is a finite group, $\Gamma$ is a subgroup of $\text{Aff}$ with finite index.

We will show that all orientation preserving affine diffeomorphisms between various $M(G,h)$ that preserves $\Sigma$ are compositions of a finite affine diffeomorphisms, which we call basic affine diffeomorphisms, which we will describe below. In our discussion we will be dealing with both translation surfaces and half translation surface surfaces. It will be convenient to view the derivative of an affine diffeomorphism as an element of $\text{PGL}(2,\mathbb{R}) = \text{GL}(2,\mathbb{R})/\{\pm I\}$. We will call an affine translation diffeomorphism a half translation equivalence when its derivative is 1 in $\text{PGL}(2,\mathbb{G})$.

Now we define four of the five classes of the basic affine diffeomorphisms:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Figure 4.}
\end{figure}
(i) Rotation: \( t_{(h_1, h_2, h_3, h_4)} \), \( h_j \in G \) is a map from \( M(G, (h_2, h_3, h_1, h_4)) \) to \( M(G, (h_1, h_2, h_3, h_4)) \) and sends \( B^1_e \) of \( M(G, (h_2, h_3, h_1, h_4)) \) to \( B^1_e \) of \( M(G, (h_1, h_2, h_3, h_4)) \) by rotating counterclockwise by \( \pi/2 \).

(ii) Deck transformation: \( r_{g, h} \), \( g \in G \), \( h \in G^4 \) is the deck transformation \( g \) in \( M(G, h) \).

(iii) Interchange of \( B^1 \) and \( B^2 \): \( f_{(h_1, h_2, h_3, h_4)} \), \( h_j \in G \) is a map from \( M(G, (h_2, h_1, h_1^{-1} h_4 h_1, h_2 h_3 h_2^{-1})) \) to \( M(G, (h_1, h_2, h_3, h_4)) \) which interchanges \( B^1_g \) and \( B^2_g \) by a rotation of \( \pi \).

(iv) Relabeling: \( m_\psi \), \( \psi \in Aut(G) \) is a map from \( M(G, h) \) to \( M(G, \psi(h)) \) and sends \( B^1_g \) to \( B^1_{\psi(g)} \). Its derivative is 1.

We claim that any half translation equivalence from \( M(G, h) \) to \( M(G, h') \) can be written as composition of basic affine diffeomorphisms \( t^2 \), \( r \), \( f \) and \( m \). Because by our assumption they preserve \( \Sigma \), they can be seen as a permutation of unit squares that tile \( M \) and \( M' \).

More precisely, any half translation equivalence is completely determined by the following data: i) the induced automorphism \( \psi \) of deck group, ii) a number \( j = 1 \) or \( 2 \), i.e. whether or not we interchange \( B^1 \) and \( B^2 \), an element \( g \in G \), such that \( F_0(B^1_g) = B^2_g \), and whether \( F_0^{-1}(e^1_g) \) is \( e^1_e \) or \( e^3_e \). Here \( e \) is the unit in \( G \). We can now use \( m \) to deal with \( \psi \), then use \( r \) and \( f \) to send \( B^1_e \) to \( B^2_g \), and if needed precompose with \( t^2 \).

For general orientation preserving affine diffeomorphism \( F \), \( DF \) will be in \( PSL(2, \mathbb{Z}) \).

We add another class of basic affine diffeomorphisms:

(v) Shearing: \( s_{(h_1, h_2, h_3, h_4)} \), \( h_j \in G \) is a map from \( M(G, (h_1 h_2 h_1^{-1}, h_1, h_3, h_4)) \) to \\
\( M(G, (h_1, h_2, h_1 h_3 h_4)) \) that sends \( e^3_1 \) to \( e^3_1 \) and has derivative \( \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \).

Because the derivative of \( s \) and \( t \) generate \( PSL(2, \mathbb{Z}) \), by successively composing with \( s \) and \( t \) we can reduce to the case when derivative is identity. Hence any affine diffeomorphisms between \( M(G, h) \), \( h \in V \) that sends \( \Sigma \) to \( \Sigma \), or more specifically, any element in \( \text{Aff} \), is a composition of the five classes of maps described above. Because \( m \) commutes with other 4 classes of diffeomorphisms, i.e.
\[
m_\psi t_h = t_\psi(h)m_\psi \\
m_\psi r_{g,h} = r_{\psi(g),\psi(h)}m_\psi \\
m_\psi f_h = f_{\psi(h)}m_\psi \\
m_\psi s_h = s_{\psi(h)}m_\psi
\]
any \( F \in \text{Aff} \) can be written as \( F = F_1m_\psi \) where \( F_1 \) is a composition of \( t \), \( s \), \( r \) and \( f \), while \( \psi \) is the automorphism of deck group induced by \( F \). Hence, elements in \( \Gamma \) can be written as successive compositions of \( t, r, f \) and \( s \).

As in [Sch04], consider the directed graph \( D \) with vertex set \( V \), each element \( h \in V \) corresponding to a surface \( M(G, h) \), the edges in the graph corresponding to basic affine
diffeomorphisms. Paths starting and ending at $M(G, g)$ correspond to elements in $\text{Aff}$. Now the fact that any affine diffeomorphism is a successive composition of $t$, $s$, $r$, $f$, and $m$ means that the map from the set of such paths to $\text{Aff}$ is surjective. Similarly, let $D_0$ be graph $D$ with those edges corresponding to $m$ removed, then the set of paths starting and ending at $g$ in $D_0$ maps surjectively to $\Gamma$.

Consider the example $G = \mathbb{Z}/6$, $g = (1, 1, 1, 3)$. This example is the Ornithorynque [FM08]. In the following figure we give the connected component of $D$ that contains $g$, with loops corresponding to deck transformation (i.e. all the $r$ arrows) omitted:

![Diagram](image)

**Figure 5.**

This method of calculating $\text{Aff}$ does not use the fact that $G$ is abelian in any important way. When $G$ is non-abelian, we can define $\Gamma$ in a slightly different way. In general, we let $\Gamma$ be the elements in $\text{Aff}(M)$ that induce an inner automorphism on $G$, then elements in $\Gamma$ are compositions of $t$, $s$, $r$, $f$ as well as $m_\psi$ where $\psi$ is an element of an inner automorphism of $G$.

3. Invariant decomposition of relative cohomology

We start by giving a direct sum decomposition of $H^1(M(G, g), \Sigma)$, and calculate the dimension of the summands as well as the action of the affine diffeomorphisms on them.
**Theorem 3.1.** Let $\Delta$ be the set of irreducible representations of a finite abelian group $G$, i.e. homomorphisms from $G$ to $\mathbb{C}^*$. Let $M = M(G, \mathfrak{g})$, $\Sigma \subset M$ be defined as in section 1, then

$$H^1(M, \Sigma; \mathbb{C}) = \bigoplus_{\rho \in \Delta} H^1(\rho)$$

where $H^1(\rho)$ are invariant subspaces under $\Gamma \subset \text{Aff}$ as defined in section 1, and are permuted by the action of $\text{Aff}$. The dimension of $H^1(\rho)$ is 3 if $\rho$ is the trivial representation, 2 if otherwise.

A compatible splitting for the absolute cohomology $H^1(M)$ was described in [Wri12]. In the case of absolute cohomology the summands can have dimensions 0, 1 or 2. $H^1(\rho)$ can also be described as cohomology with twisted coefficients as in [DM86], [Thu98].

**Proof.** Consider the relative cellular cochain complex

$$0 \to C^1(M, \Sigma; \mathbb{C}) \to C^2(M, \Sigma; \mathbb{C})$$

We can identify $C^1(M, \Sigma; \mathbb{C})$ with $(\mathbb{C}[G])^4$ as a right-$G$ module by writing $m \in C^1(M, \Sigma; \mathbb{C})$ as

$$(\sum_g m(e^1_g)g^{-1}, \sum_g m(e^2_g)g^{-1}, \sum_g m(e^3_g)g^{-1}, \sum_g m(e^4_g)g^{-1}) \in (\mathbb{C}[G])^4$$

identify $C^2(M, \Sigma; \mathbb{C})$ with $(\mathbb{C}[G])^2$ as a right-$G$ module by writing $n \in C^2(M, \Sigma; \mathbb{C})$ as

$$(\sum_g n(B^1_g)g^{-1}, \sum_g n(B^2_g)g^{-1}) \in (\mathbb{C}[G])^2$$

then the coboundary map from $C^1$ to $C^2$ is

$$(1) \quad d^1(a, b, c, d) = (a + b + c + d, a + g_2 b + g_2 g_3 c + g_2 g_3 g_4 d)$$

Hence:

$$(2) \quad H^1(M, \Sigma; \mathbb{C}) = \{(a, b, c, d) \in (\mathbb{C}[G])^4 : a + b + c + d = a + g_2 b + g_2 g_3 c + g_2 g_3 g_4 d = 0\}$$

Because $\mathbb{C}[G]$ is semisimple [Ser77], it splits into simple algebras $\mathbb{C}[G] = \bigoplus_{\rho \in \Delta} D_\rho$, where $D_\rho$ is the simple subalgebras of $\mathbb{C}[G]$ corresponding to irreducible representation $\rho$. The splitting of the algebra gives a splitting of the claim complex $0 \to C^1 \to C^2$, hence a splitting of the cohomology:

$$(3) \quad H^1(M, \Sigma; \mathbb{C}) = \bigoplus_{\rho \in \Delta} H^1(\rho), H^1(\rho) = \{(a, b, c, d) \in D_\rho^4 : d^1(a, b, c, d) = 0\}$$

The image of $d^1$ in $C^2(M, \Sigma; \mathbb{C})$ is

$$(1, 1)\mathbb{C}[G] \oplus (0, 1)M \subset (\mathbb{C}[G])^2 = C^2(M, \Sigma; \mathbb{C})$$

where $M$ is the right ideal generated by $\{g_2 - 1, g_2 g_3 - 1, g_2 g_3 g_4 - 1\}$. This is because

$$d^1(a, b, c, d) = (a + b + c + d, a + g_2 b + g_2 g_3 c + g_2 g_3 g_4 d)$$

$$= (a + b + c + d, a + b + c + d) + (0, (g_2 - 1)b) + (0, (g_2 g_3 - 1)c) + (0, (g_2 g_3 g_4 - 1)d)$$

$$= (a + b + c + d)(1, 1) + ((g_2 - 1)b + (g_2 g_3 - 1)c + (g_2 g_3 g_4 - 1)d)(0, 1)$$
Now we show that $\mathbb{C}[G] = \mathbb{C} \oplus M$. Because $(1 - a) + (1 - b)a = 1 - ba$, if $g$ is a product of elements in $\{g_2, g_2g_3, g_2g_3g_4\}$ then $1 - g \in M$. Also, because $M$ is connected, $\{g_2, g_2g_3, g_2g_3g_4\}$ generates $G$, hence $M$ is generated by all elements of the form $1 - g$ for any $g \in G$, therefore $\mathbb{C}[G] = \mathbb{C} \oplus M$, where $\mathbb{C}$ is the trivial sub-algebra generated by $\sum_{g \in G} g$. Because $\mathbb{C}[G]$ is semisimple,

$$H^1(M, \Sigma; \mathbb{C}) = \ker(d^1) \rightarrow C^1(M, \Sigma; \mathbb{C}) \rightarrow \text{im}(d^1)$$

splits, hence we have

(4) $$H^1(M, \Sigma; \mathbb{C}) = (\mathbb{C}[G])^4/(\mathbb{C}[G] \oplus M) = (\mathbb{C}[G])^2 \oplus \mathbb{C}$$

Therefore, as $G$-module $H^1(\rho) \cong \mathbb{C}^3$ when $\rho$ is the trivial representation, $H^1(\rho) \cong D^2_1$ if otherwise. Because $G$ is abelian, $\dim_{\mathbb{C}} D_\rho = 1$, so $\dim_{\mathbb{C}} H^1(\rho) = 3$, if $\rho$ is trivial, $\dim_{\mathbb{C}} H^1(\rho) = 2$, otherwise.

In the previous section we describe elements in $\text{Aff}$ as compositions of elementary affine diffeomorphisms $t_h, s_h, r_{g,h}, f_h$ and $m_\psi$, and elements in $\Gamma$ as compositions of elementary affine diffeomorphisms $t_h, s_h, r_{g,h}$ and $f_h$. We will show the invariance of $H^1(\rho)$ under $\Gamma$ by explicitly describing the action of elementary affine diffeomorphisms. The induced map of $t_h, s_h, r_{g,h}, f_h$ from $H^1(M(G, h), \Sigma; \mathbb{C})$ to some $H^1(M(G, h), \Sigma; \mathbb{C})$ are as follows:

(5) $$t^*_h([a, b, c, d]) = [b, c, d, a]$$

(6) $$s^*_h([a, b, c, d]) = [-h_1a, a + b, c, d + h_1a]$$

(7) $$r^*_{g,h}([a, b, c, d]) = [ag, bg, cg, dg]$$

(8) $$f^*_h([a, b, c, d]) = [-a, -h_2h_3h_4d, -h_2h_3c, -h_2b]$$

From equation (3) we know that they all preserve decomposition $H^1(\ast, \Sigma; \mathbb{C}) = \bigoplus_\rho H^1(\rho)$, hence all summands $H^1(\rho)$ are invariant under $\Gamma$.

Furthermore, $m_\psi$ is a diffeomorphism from $M(G, \psi^{-1} h)$ to $M(G, h)$, and the map it induced from $H^1(M(G, h), \Sigma; \mathbb{C})$ to $H^1(M(G, \psi(h)), \Sigma; \mathbb{C})$ is

$$m^*_\psi([\sum_{g \in G} a_g g, \sum_{g \in G} b_g g, \sum_{g \in G} c_g g, \sum_{g \in G} d_g g]) = [\sum_{g \in G} a_g \psi^{-1}(g), \sum_{g \in G} b_g \psi^{-1}(g), \sum_{g \in G} c_g \psi^{-1}(g), \sum_{g \in G} d_g \psi^{-1}(g)]$$

which, according to equation (3), would send $H^1(\rho)$ to $H^1(\psi^{-1} \rho)$. In other words, elements in $\text{Aff}$ permute $H^1(\rho)$.

$\square$
Remark 1. In certain situations $\Gamma = \text{Aff}$. This happens when the $g_j$ are all of different order, or when $G$ is $\mathbb{Z}/n$, $n \geq 4$ and $g = (1, 1, 1, n - 3)$. In these cases $H^1(\rho)$ are all invariant under $\text{Aff}$. Our argument here is similar to, but not completely the same as those used in [MY09].

4. The signature of the Hodge form

Now we define and calculate the signature of an invariant Hermitian form on $H^1(\rho)$ as in [Thu98] and [DM86].

The Hodge form $A_G$, or area form as in [Thu98], on $H^1(M, \Sigma; \mathbb{C})$ is defined as $\frac{1}{2\pi}$ of the cup product with coefficient pairing $\mathbb{C} \times \mathbb{C} \to \mathbb{C} : (z, z') \mapsto z\overline{z'}$ on $(M, \Sigma)$

$$H^1(M, \Sigma; \mathbb{C}) \times H^1(M, \Sigma; \mathbb{C}) \to H^2(M, \Sigma; \mathbb{C}) = \mathbb{C}$$

In other words,

$$A_G([a, b, c, d], [a', b', c', d']) = \frac{1}{4i}((b, a')_G - (a, b')_G + (d, c')_G - (c, d')_G$$

$$-(h_2b, a')_G + (a, h_2b')_G - (h_2h_3h_4d, h_2h_3c')_G + (h_2h_3c, h_2h_3h_4d')_G$$

where $(\cdot, \cdot)_G$ is the positive definite Hermitian norm on $\mathbb{C}[G]$ defined as

$$(\sum_g a_gg^{-1}, \sum_g b_gg^{-1})_G = \sum_g a_g\overline{b_g}$$

Alternatively, if elements in $H^1(M, \Sigma; \mathbb{C})$ are represented by closed differential forms, $A_G$ can be written as $A_G(\alpha, \beta) = \frac{1}{2\pi} \int_M \alpha \wedge \overline{\beta}$.

By definition $A_G$ is invariant under the $\Gamma$-action. Furthermore, from (9) and the fact that different $D_\rho$ are orthogonal under $(\cdot, \cdot)_G$, we know that $H^1(\rho)$ for different representation $\rho$ are orthogonal to each other under $A_G$. When $\rho$ is the trivial representation, $A_G = 0$ on $H^1(\rho)$.

Now we assume $\rho$ to be a non-trivial representation. Because we will deal with Hermitian forms that may be degenerate, we denote the signature of a Hermitian form as $(n_0, n_+, n_-)$, where $n_0, n_+, n_-$ are the number of 0, positive and negative eigenvalues respectively.

We will prove the following theorem:

**Theorem 4.1.** The signature of the area form $A_G$ on $H^1(\rho)$, $\rho \neq 1$ is $(n_0, \frac{\theta_1}{2\pi}, \frac{\theta_2}{2\pi} - 1, \frac{\theta_3}{2\pi}) = (4 - \frac{\theta_1 + \theta_2}{2\pi}, \frac{\theta_2}{2\pi} - 1, \frac{\theta_3}{2\pi} - 1)$, where $\theta_1 = \sum_{j=1}^{4} \text{arg}(\rho(g_j))$, $\theta_2 = \sum_{j=1}^{4} \text{arg}(\rho(g_j)^{-1})$. The number $n_0$ is also the number of indices $j$ such that $\rho(g_j) = 1$.

**Proof.** In the case when $\rho(g_1g_2) = \rho(g_2g_3) = 1, a = c, b = d$, and $A_G$ is $2|G|$ times the area of parallelogram with side $a$ and $b$, i.e. proportional to the cross product of two vectors on the complex plane, which has signature $(0, 1, 1) = (4 - \frac{\theta_1 + \theta_2}{2\pi}, \frac{\theta_2}{2\pi} - 1, \frac{\theta_3}{2\pi} - 1)$. 
Now we consider the case when \( \rho(g_1g_2) \neq 1 \) or \( \rho(g_2g_3) \neq 1 \). Without losing generality we assume \( \rho(g_1g_2) \neq 1 \). From (1), (3) we know that any \((a, b, c, d) \in H^1(\rho)\) satisfies
\[
a + b + c + d = 0
\]
\[
a + \rho(g_2)b + \rho(g_2g_3)c + \rho(g_2g_3g_4)d = 0
\]
Because \( \rho(g_3g_4) = \rho(g_1g_2)^{-1} \neq 1 \), we can solve \((b, d)\) from these two equations as linear functions of \((a, c)\), i.e. rewrite these equations can be written as \((b, d) = (a, c)A\) where \(A\) is a 2-by-2 matrix.

Consider subspaces \( H^1_a = \{(a, b, 0, d) \in H^1(\rho)\}, H^1_{a'} = \{(0, b, c, d) \in H^1(\rho)\}\), then by the previous arguments \( \dim(H^1_a) = \dim(H^1_{a'}) = 1 \) and \( H^1(\rho) = H^1_a \oplus H^1_{a'} \). We will show that they are orthogonal subspaces under \( \Gamma \). For any \((a, b, 0, d) \in H^1_a(\rho), (0, b', c', d') \in H^1_{a'}(\rho)\), because \((*, *)_G\) is \(G\)-invariant and \(d^1(a, b, 0, d) = d^1(0, b', c', d') = 0\), we have
\[
A_G([a, b, 0, d], [0, b', c', d']) = \frac{1}{2i}(-(a, b')_G + (d, c')_G + (a, h_2b')_G - (h_2h_3h_4d, h_2h_3c')_G)
\]
\[
= \frac{1}{2i}((b, b')_G + (d, b')_G - (d, b')_G - (d, d')_G - (h_2b, h_2b')_G)
\]
\[
- (h_2h_3h_4d, h_2h_3) + (h_2h_3h_4d, h_2h_3d')_G + (h_2h_3h_4d, h_2b')_G
\]
\[
= 0
\]
In other words, \(A_G\) is diagonalized under \(H^1(\rho) = H^1_a \oplus H^1_{a'}\).

Now we show that the signature of \(A_G\) on \(H^1_a\) is \((3 - \frac{\theta_1a+\theta_2a}{2\pi}, \frac{\theta_2a}{2\pi} - 1, \frac{\theta_1a}{2\pi} - 1)\), where \(\theta_1a = \arg(\rho(g_1)) + \arg(\rho(g_2)) + \arg(\rho(g_3g_4)), \theta_2a = \arg(\rho(g_1)^{-1}) + \arg(\rho(g_2)^{-1}) + \arg(\rho(g_3g_4)^{-1})\). From (10) and (11) we know that
\[
H^1_a = \{t(\rho(g_2) - \rho(g_1^{-1}), \rho(g_1^{-1}) - 1, 0, \rho(g_2) - 1) : t \in D_\rho\}
\]
If \( \rho(g_2) = 0 \), equation (12) becomes \(H^1_a = \{(t, -t, 0, 0) : t \in D_\rho\}\). If \( \rho(g_1) = 0 \), (12) becomes \(H^1_a = \{(t, 0, 0, -t) : t \in D_\rho\}\). In both cases the signature is \((1, 0, 0) = (3 - \frac{\theta_1a+\theta_2a}{2\pi}, \frac{\theta_2a}{2\pi} - 1, \frac{\theta_1a}{2\pi} - 1)\). If neither \( \rho(g_1) \) nor \( \rho(g_2) \) is 1, \( \theta_1a + \theta_2a = 6\pi \), and \( \theta_1a \) is either \(2\pi\) or \(4\pi\). From (9) and (12) we know that \(A_G\) is positive on \(H^1_a\) definite when \(\theta_1a = 2\pi\) and negative definite on \(H^1_a\) when \(\theta_1a = 4\pi\), i.e. the signature of \(A_G\) on \(H^1_a\) is 
\[
(3 - \frac{\theta_1a+\theta_2a}{2\pi}, \frac{\theta_2a}{2\pi} - 1, \frac{\theta_1a}{2\pi} - 1)
\]
We can calculate the signature of \(A_G\) on \(H^1_{a'}\) similarly. Because \(A_G\) is diagonalized under \(H^1(\rho) = H^1_a \oplus H^1_{a'}\), the \(n_0, n_+\) and \(n_-\) of \(A_G\) on \(H^1(\rho)\) can be obtained by adding the \(n_0, n_+\) and \(n_-\) of \(A_G\) on \(H^1_a\) and \(H^1_{a'}\).

5. A SUBGROUP OF \(\Gamma\)

In this section we introduce a subgroup \(\Gamma_1\) of \(\Gamma\) of finite index, which is easier to work with than \(\Gamma\). In section 6, we will give a criteria for non-discreteness of the action of \(\Gamma\) by
analyzing the action of this subgroup of finite index.

There is a homomorphism \( D : \text{Aff} \to SL(2, \mathbb{Z}) \) that sends an affine diffeomorphism to its derivative. Because elements of \( \text{Aff} \) preserves \( \Sigma \), \( \ker(D) \) is finite. Consider two elements in \( \Gamma \) which are liftings of the horizontal and vertical Dehn twists of the pillowcase

\[
\gamma_1 = s(g_1, g_2, g_3, g_4)^s(g_2, g_1, g_3, g_4)
\]
\[
\gamma_2 = t(g_1, g_2, g_3, g_4)^s(g_2, g_3, g_4, g_1)^s(g_3, g_2, g_4, g_1)^t(g_1, g_2, g_3, g_4)
\]

\( D\gamma_1 \) and \( D\gamma_2 \) generates the level 2 congruence subgroup of \( SL(2, \mathbb{Z}) \), hence the group generated by them is a subgroup of \( \text{Aff} \) of finite order. From (5), (6) we have:

(13) \[ \gamma_1^*(a, b, c, d) = (g_1g_2a, b + a - g_1a, c, d + g_1a - g_1g_2a) \]

(14) \[ \gamma_2^*(a, b, c, d) = (a + g_2b - g_2g_3b, g_2g_3b, c + b - g_2b, d) \]

Denote the group generated by \( \gamma_1 \) and \( \gamma_2 \) as \( \Gamma_1 \). When restricted to \( H^1(\rho) \),

(15) \[ \gamma_1^*(a, b, c, d) = (\rho(g_1g_2)a, b + a - \rho(g_1)a, c, d + \rho(g_1)a - \rho(g_1g_2)a) \]

(16) \[ \gamma_2^*(a, b, c, d) = (a + \rho(g_2)b - \rho(g_2g_3)b, \rho(g_2g_3)b, c + b - \rho(g_2)b, d) \]

When \( \rho = 1 \) is the trivial representation, the \( \Gamma_1 \) action on \( H^1(\rho) \) is trivial. When \( \rho \) is non-trivial, the \( \Gamma \) action on \( H^1(\rho) = \mathbb{C}^2 \) induces an action on \( \mathbb{C}P^1 \) under projectivization. The map on \( \mathbb{C}P^1 \) induced by \( \gamma_1 \) is parabolic if and only if \( \rho(g_1g_2) = 1 \). Similarly, \( \gamma_2 \) is parabolic if and only if \( \rho(g_2g_3) = 1 \). If they are not parabolic they are elliptic.

Furthermore, when \( \rho(g_2) = 1 \) and all other \( \rho(g_j) \neq 1 \), the \( \Gamma_1 \) action \( H^1(\rho) \) is not semisimple. We can see this as follows: by equations (15) and (16), \( (1, -1, 0, 0) \) is the only common eigenvector of \( \gamma_1^* \) and \( \gamma_2^* \) in \( H^1(\rho) \), hence \( H^1_\alpha(\rho) = \{(t, -t, 0, 0)\} \) is the only 1-dimensional subspace of \( H^1(\rho) \) invariant under \( \Gamma_1 \), i.e. in this case the \( \Gamma_1 \) action on \( H^1(\rho) \) is not semisimple. Furthermore, from the discussions in Section 6, 7 we know the \( \Gamma_1 \) action on \( H^1(\rho) \) is semisimple in other cases. In other words, we have:

**Proposition 5.1.** The \( \Gamma_1 \) action on \( H^1(\rho) \) is not semisimple if and only if exactly one of the four complex numbers \( \rho(g_1), \rho(g_2), \rho(g_3), \rho(g_4) \) is 1.

6. The spherical case and polyhedral groups

The Hodge norm \( A_G \) on \( H^1(\rho) \) induces a metric, hence a geometric structure on the projectivization \( \mathbb{P}(H^1(\rho)) = \mathbb{C}P^1 \) under the \( \Gamma \)-action. When \( A_G \) is positive definite or negative definite, it induces a spherical structure on \( \mathbb{C}P^1 \). When \( A_G \) has signature \((1, 0, 1)\), it induces a Euclidean structure on \( \mathbb{C}P^1 - \{0 : 1\} \). When \( A_G \) has signature \((1, 1, 0)\), it induces a Euclidean structure on \( \mathbb{C}P^1 - \{1 : 0\} \). Finally, when the signature of \( A_G \) is \((0, 1, 1)\), it induces a hyperbolic structure on a disc \( D \) in \( \mathbb{P}H^1(\rho) \), which consists of the image \( \{\alpha \in H^1(\rho) : A(\alpha, \alpha) > 0\} \). In this section we will describe the spherical case, and in the next section we will describe the remaining cases.
When $A_G$ is positive definite or negative definite, the generators of $\Gamma_1$, $\gamma_1$ and $\gamma_2$, act as finite order rotations with different fixed points, and their orders are the orders of $\rho(g_1g_2)$ and $\rho(g_2g_3)$ in $\mathbb{C}^*$ respectively, hence by the ADE classification [Dic59] of finite subgroups of $SO(3)$ we know that if both the orders of $\rho(g_1g_2)$ and $\rho(g_2g_3)$ are greater than 5 the action of $\Gamma$ on $H^1(\rho)$ can not factor through a discrete group.

**Example:** Let $G$ be the subgroup of $(\mathbb{Z}/120)^3$ spanned by $g_1 = (20, 0, 0), g_2 = (0, 15, 0), g_3 = (0, 0, 12), g_4 = (100, 105, 108), g = (g_1, g_2, g_3, g_4), M = M(G, g), \rho(g_1) = e^{\pi i/3}, \rho(g_2) = e^{\pi i/4}, \rho(g_3) = e^{\pi i/5}, \rho(g_4) = e^{73\pi i/60}$. Then by Theorem 3.1 and Remark 1 $H^1(\rho)$ is invariant under $\text{Aff}$ with an invariant complement, by section 4 the Hodge form is positive definite on $H^1(\rho)$, and by the argument above the $\text{Aff}$ action on $H^1(\rho)$ is not discrete. In other words, $M$ and the decomposition $H^1(M, \Sigma; \mathbb{C}) = H^1(\rho) \oplus (\bigoplus_{\rho' \neq \rho} H^1(\rho'))$ satisfy all the conditions mentioned in the beginning of section 1.

Furthermore, using [Cox73], we can list all possible 4-tuples $(\rho(g_1), \rho(g_2), \rho(g_3), \rho(g_4))$ such that the $\Gamma_1$ action on $\mathbb{P}(H^1(\rho)) = \mathbb{CP}^1$ factors through a finite group. Denote the arguments of $\rho(g_j)$ as $2t_j \pi, j = 1, \ldots, 4$, then the $\Gamma_1$ action is finite if and only if $t_1, t_2, t_3, t_4$ is a permutation of one of the 4-tuples in the table below:
Here $d$ and $n$ are positive integers, and the last column shows the discrete subgroup of $SO(3)$ it corresponds to.

Examples of $M$ and $\rho$ that satisfies the conditions in Section 1 can be built from any 4-tuple of positive rational numbers not on the above list that sum up to 1 or 3. For example, $(1/8, 1/8, 1/8, 5/8)$ is not on the list, so let $G = \mathbb{Z}/8$, $g_1 = g_2 = g_3 = 1$, $g_4 = 5$, $\rho(g_1) = \rho(g_2) = \rho(g_3) = e^{\pi i/4}$, $\rho(g_4) = e^{5\pi i/4}$ satisfies the conditions in Section 1. This is an abelian cover of flat pillowcase that satisfy the conditions in section 1 with the smallest number of squares.

| $t_1$   | $t_2$   | $t_3$   | $t_4$   | Group                      |
|---------|---------|---------|---------|----------------------------|
| $d/2n$  | $d/2n$  | $(n-d)/2n$ | $(n-d)/2n$ | Dihedral group             |
| 1/12    | 1/4     | 1/4     | 5/12    | Tetrahedral group          |
| 1/24    | 5/24    | 7/24    | 11/24   | Octahedral group           |
| 1/60    | 11/60   | 19/60   | 29/60   | Icosahedral group          |
| 1/6     | 1/6     | 1/6     | 1/2     | Tetrahedral group          |
| 1/12    | 1/6     | 1/6     | 7/12    | Octahedral group           |
| 1/30    | 19/30   | 1/6     | 1/6     | Icosahedral group          |
| 1/30    | 3/10    | 3/10    | 11/30   | Icosahedral group          |
| 1/20    | 3/20    | 7/20    | 9/20    | Icosahedral group          |
| 1/15    | 2/15    | 4/15    | 8/15    | Icosahedral group          |
| 1/10    | 3/10    | 3/10    | 3/10    | Icosahedral group          |
| 1/10    | 1/10    | 7/30    | 17/30   | Icosahedral group          |
| 1/10    | 1/10    | 1/10    | 7/10    | Icosahedral group          |
| 7/60    | 13/60   | 17/60   | 23/60   | Icosahedral group          |
| 1/6     | 1/6     | 7/30    | 13/30   | Icosahedral group          |
| $(2n-d)/2n$ | $(2n-d)/2n$ | $(n+d)/2n$ | $(n+d)/2n$ | Dihedral group             |
| 7/12    | 3/4     | 3/4     | 11/12   | Tetrahedral group          |
| 13/24   | 19/24   | 17/24   | 23/24   | Octahedral group           |
| 31/60   | 41/60   | 49/60   | 59/60   | Icosahedral group          |
| 1/2     | 5/6     | 5/6     | 5/6     | Tetrahedral group          |
| 5/12    | 5/6     | 5/6     | 11/12   | Octahedral group           |
| 5/6     | 5/6     | 11/30   | 29/30   | Icosahedral group          |
| 19/30   | 7/10    | 7/10    | 29/30   | Icosahedral group          |
| 11/20   | 13/20   | 17/20   | 19/20   | Icosahedral group          |
| 7/15    | 11/15   | 13/15   | 14/15   | Icosahedral group          |
| 7/10    | 7/10    | 7/10    | 9/10    | Icosahedral group          |
| 13/30   | 23/30   | 9/10    | 9/10    | Icosahedral group          |
| 3/10    | 9/10    | 9/10    | 9/10    | Icosahedral group          |
| 37/60   | 43/60   | 47/60   | 53/60   | Icosahedral group          |
| 17/30   | 23/30   | 5/6     | 5/6     | Icosahedral group          |
7. The hyperbolic and Euclidean cases and triangle groups

In this section we complete the description of $\Gamma_1$-action by describing the signature $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ cases as triangle groups. In the $(0, 1, 1)$ case, $\sum_j \arg(\rho(g_j)) = 4\pi$. Without losing generality we assume $\arg(\rho(g_1)) + \arg(\rho(g_2)) \geq 2\pi$, $\arg(\rho(g_2)) + \arg(\rho(g_3)) \geq 2\pi$.

By Remark 1 and Section 5, $\gamma_1$ acts on $D$ as a rotation by $\arg(\rho(g_1 g_2))$ when $\rho(g_1 g_2) \neq 1$ and as a parabolic transform when $\rho(g_1 g_2) = 1$, $\gamma_2$ acts on $D$ as a rotation by $\arg(\rho(g_2 g_3))$ when $\rho(g_2 g_3) \neq 1$ and as a parabolic transform when $\rho(g_2 g_3) = 1$. Furthermore, $\gamma_1$ and $\gamma_2$ generate an index 2 subgroup of a triangle group, and the angles of the triangle are $|\pi - \arg(\rho(g_1 g_2))/2|$, $|\pi - \arg(\rho(g_2 g_3))/2|$ and $|\pi - \arg(\rho(g_1 g_3))/2|$. \cite{Wri12} has calculated the Lyapunov exponents from the area of this triangle.

When the signature of $A_G$ is $(1, 1, 0)$ or $(1, 0, 1)$, only one $\rho(g_j)$ is equal to 1. Without losing generality assume $\rho(g_2) = 1$, then $(a, b, c, d) \mapsto b/a$ sends $H^1(\rho)$ to $\mathbb{C}$, and under this map $\Gamma_1$ acts on $\mathbb{C} = \mathbb{C} - \{\infty\}$ as an index-2 subgroup of a Euclidean triangle group. The angles of the triangle are $\arg(\rho(g_1))/2$, $\arg(\rho(g_3))/2$ and $\arg(\rho(g_4))/2$ when

$$\arg(\rho(g_1)) + \arg(\rho(g_3)) + \arg(\rho(g_4)) = 2\pi$$

i.e. when the signature of $A_G$ is $(1, 1, 0)$. The angles of the triangles are $\pi - \arg(\rho(g_1))/2$, $\pi - \arg(\rho(g_3))/2$ and $\pi - \arg(\rho(g_4))/2$ when

$$\arg(\rho(g_1)) + \arg(\rho(g_3)) + \arg(\rho(g_4)) = 4\pi$$

i.e. when the signature of $A_G$ is $(1, 0, 1)$.

When two of the four $\rho(g_j)$ are equal to 1, then the $\Gamma_1$ action on $H^1(\rho)$ factors through a finite abelian group.

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