Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction

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Abstract

The generalized Drinfel’d-Sokolov hierarchies studied by de Groot-Hollowood-Miramontes are extended from the viewpoint of Sato-Wilson dressing method. In the $A_1^{(1)}$ case, we obtain the hierarchy that include the derivative nonlinear Schrödinger equation. We give two types of affine Weyl group symmetry of the hierarchy based on the Gauss decomposition of the $A_1^{(1)}$ affine Lie group. The fourth Painlevé equation and their Weyl group symmetry are obtained as a similarity reduction. We also clarify the connection between these systems and monodromy preserving deformations.

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1 Introduction

There are many brilliant works on the relation between Lie algebras and soliton equations. Among those works, the approach due to Drinfel’d and Sokolov [DS] is a milestone, and gives a method for classifying many soliton equations. Although extended version of their work has been proposed [dGHM, HM, BtK2], there still exist several soliton equations that are not treated along the line of Drinfel’d-Sokolov’s works. One example of those equations is a derivative nonlinear Schrödinger (dNLS) equation:

\[ iq_T = \frac{1}{2} q_{xx} + 2i q^2 \bar{q}_x + 4|q|^4 q, \tag{1.1} \]

which has been studied by several authors [ARS, GI, OS, T]. This integrable equation is a modification of the nonlinear Schrödinger (NLS) equation:

\[ iq_T = \frac{1}{2} q_{xx} + 4|q|^2 q. \tag{1.2} \]

Hereafter we will forget the complex structure of (1.1), (1.2) and consider nonlinear coupled equations,

\[
\begin{align*}
q_t &= \frac{1}{2} q_{xx} - 2q^2 r_x - 4q^3 r^2, \\
r_t &= -\frac{1}{2} r_{xx} - 2r^2 q_x + 4r^3 q^2,
\end{align*}
\tag{1.3}
\]

and

\[
\begin{align*}
q_t &= \frac{1}{2} q_{xx} + 4q^2 r, \\
r_t &= -\frac{1}{2} r_{xx} - 4qr^2.
\end{align*}
\tag{1.4}
\]

We note that (1.3), (1.4) is reduced to (1.1), (1.2), respectively, under the condition \( r = \bar{q}, \) \( X = ix, \) \( T = it. \) It is well-known that the hierarchy of soliton equations including NLS (1.4) is obtained as a Drinfel’d-Sokolov hierarchy of \( A_1^{(1)} \) homogeneous type.

The aim of the present article is threefold:

**Extension of the Drinfel’d-Sokolov formulation**

We extend the generalized Drinfel’d-Sokolov hierarchy [dGHM] from the viewpoint of Sato-Wilson dressing method. The extended version includes the \( \partial \text{NLS} \) equation (1.3) as an \( A_1^{(1)} \) case.
Description of affine Weyl group symmetry
There exist transformations of the ∂NLS equation, called Bäcklund transformations that relate two solutions of the ∂NLS equation. We construct two types of Bäcklund transformations that satisfy the relation of the $A_1^{(1)}$ affine Weyl group. We remark that our construction of the affine Weyl group symmetry is an extension of the work by Noumi and Yamada [NY1, NY2].

Algebraic description of similarity reduction
An interesting feature of the ∂NLS equation (1.1) is its connection to the fourth Painlevé equation (P$_{IV}$):
\[
y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \nu_1)y + \frac{\nu_2}{y},
\]
where $\nu_1, \nu_2 \in \mathbb{C}$ are parameters. In [ARS], Ablowitz, Ramani and Segur have shown that the self-similar solutions of ∂NLS satisfy $P_{IV}$ with a special case of the parameters. We give a systematic framework of similarity reductions of the ∂NLS hierarchy that gives $P_{IV}$ with full-parameters. We also give a relation to monodromy preserving deformation studied by Jimbo, Miwa and Ueno [JMU, JMI, JM2].

As for an application to discrete integrable systems, we consider a discrete equation,
\[
X_{n-1} + X_n + X_{n+1} = x + \frac{\kappa_1 n + \kappa_2 + \kappa_3 (-1)^n}{X_n}.
\]
This equation called the asymmetric discrete Painlevé I (dP$_1$) because a continuous limit of (1.6) with $\kappa_3 = 0$ is the first Painlevé equation. Grammaticos and Ramani [GR] obtained the equation (1.6) from the Schlesinger transformations, which are special type of Bäcklund transformations. We construct a Schlesinger transformations of the ∂NLS equation as an extension of affine Weyl group symmetry and obtain dP$_1$.

The equations, NLS (1.4), $P_{IV}$ (1.5), and dP$_1$ (1.6), share a class of rational solutions expressed by the Hermite polynomials [IY, NY2, OKS]. We clarify the algebraic structure of this class of solutions by using the fermionic representation of $\hat{sl}_2$.

2 Construction of the ∂NLS hierarchy

2.1 General framework
In this subsection, we outline our formulation of soliton equations based on the approach of Drinfel’d and Sokolov [BtK2, HGM, DS, W].

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra, and $(\ , \ )$ be the normalized invariant scalar product of $\mathfrak{g}$. The affine Lie algebra $\hat{\mathfrak{g}}$ associated to $(\mathfrak{g}, \left(\ , \right))$ can be realized as
\[
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,
\]
with the relations,
\[
[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}(X, Y)K, \\
[K, \hat{\mathfrak{g}}] = 0, \quad [d, X \otimes z^n] = nX \otimes z^n,
\]
for $X, Y \in \hat{\mathfrak{g}}$, $m, n \in \mathbb{Z}$.

To construct integrable hierarchies, Heisenberg subalgebras of $\hat{\mathfrak{g}}$ play a crucial role. It is known that non-equivalent Heisenberg subalgebras are classified by conjugacy classes of the Weyl group of $\mathfrak{g}$ [KP] [GHM]. We denote by $\mathcal{H}^{[w]}$ the Heisenberg subalgebra associated with the conjugacy class $[w]$: \[ \mathcal{H}^{[w]} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \Lambda_n^{[w]} \oplus \mathbb{C} K. \]

Once we fix a basis of Heisenberg subalgebra $\{ \Lambda_n^{[w]} \}_{n \in \mathbb{Z}}$, there is an associated gradation $d_w$ that is natural on $\{ \Lambda_n^{[w]} \}_{n \in \mathbb{Z}}$: \[ [d_w, \Lambda_n^{[w]}] = n \Lambda_n^{[w]} \]

The gradation $d_w$ induces a $\mathbb{Z}$-grading on $\hat{\mathfrak{g}}$: \[ \hat{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathfrak{g}}_j, \quad \hat{\mathfrak{g}}_j = \{ x \in \hat{\mathfrak{g}} : [d_w, x] = jx \}. \]

For an integer $k$, we use the notation \[ \hat{\mathfrak{g}}^\geq_k = \bigoplus_{j \geq k} \hat{\mathfrak{g}}_j, \quad \hat{\mathfrak{g}}^\leq_k = \bigoplus_{j < k} \hat{\mathfrak{g}}_j. \]

We now consider a Kac-Moody group $\hat{G}$ formed by exponentiating the action of $\hat{\mathfrak{g}}$ on an integrable module. Throughout this paper, we assume that the exponentiated action of an element of the positive degree subalgebra of $\mathcal{H}^{[w]}$ is well-defined. We remark that all of the representations used in what follows belong to this category. We denote by $\hat{G}^{[w]}_{\geq 0}$ and $\hat{G}^{[w]}_{\leq 0}$ the subgroups correspond to the subalgebras $\mathfrak{g}^{[w]}_{\geq 0}$ and $\mathfrak{g}^{[w]}_{\leq 0}$, respectively.

Starting from $g(0) \in \hat{G}$, we define time-evolutions with time variable $t = (t_1, t_2, \ldots)$ using the Heisenberg subalgebra $\{ \Lambda_n^{[w']} \}_{n \in \mathbb{Z}}$ associated with a conjugacy class $[w']$: \[ g(t) \overset{\text{def}}{=} \exp \left( \sum_{n > 0} t_n \Lambda_n^{[w']} \right) g(0), \quad (2.1) \]

which satisfies the following differential equation, \[ \frac{\partial g(t)}{\partial t_n} = \Lambda_n^{[w']} g(t), \quad n = 1, 2, \ldots \quad (2.2) \]

In what follows, we shall assume the existence and the uniqueness of the Gauss decomposition with respect to the gradation $d_w$: \[ g(t) = \{ g_{\leq 0}^{[w]}(t) \}^{-1} g_{\geq 0}^{[w]}(t), \quad g_{\leq 0}^{[w]}(t) \in \hat{G}_{\leq 0}^{[w]}, \quad g_{\geq 0}^{[w]}(t) \in \hat{G}_{\geq 0}^{[w]} \quad (2.3) \]

A detailed discussion about this assumption is in [BuK] [W] for instance. Note that the conjugacy classes of Weyl group $[w]$ of (2.3) and $[w']$ of (2.1) is not necessary equal.
From (2.2) and (2.3), we have
\[
\frac{\partial g_{<0}^{[w]}}{\partial t_n} = B_n g_{<0}^{[w]} - g_{<0}^{[w]} \Lambda_n^{[w']} ,
\]
(2.4)
\[
\frac{\partial g_{\geq0}^{[w]}}{\partial t_n} = B_n g_{\geq0}^{[w]} ,
\]
(2.5)
where \( B_n = B_n(t) \) is defined by
\[
B_n(t) \overset{\text{def}}{=} \left( g_{<0}^{[w]}(t) \Lambda_n^{[w']} g_{<0}^{[w]}(t)^{-1} \right)_{\geq0}^{[w]} \in \mathfrak{g}_{\geq0}^{[w]} .
\]
(2.6)

We call (2.4) and (2.5) the Sato-Wilson equations. The compatibility conditions for (2.4) or (2.5) give rise to the zero-curvature (or Zakharov-Shabat) equations,
\[
\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \quad m, n = 1, 2, \ldots ,
\]
(2.7)
which gives a hierarchy of soliton equations.

Note that de Groot et al. imposed the condition \( w' = w \) and in the definition (2.6) of \( B_n \), they used a projection with respect to \( d_w \) or a less gradation than \( d_w \) for an order of gradations [GHM]. However the formulas (2.4) and (2.5) is valid without the relation for \( w \) and \( w' \). In this sense, our formulation can be regarded as an extension of the generalized Drinfel’d-Sokolov hierarchy.

2.2 Hierarchy of the derivative NLS equation

Hereafter we consider only the \( \hat{\mathfrak{sl}}_2 \)-case to treat the \( \partial \)NLS hierarchy. The generators of \( \mathfrak{sl}_2 \) is denoted by \( E, F \) and \( H \) as usual:
\[
[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.
\]
We will use the abbreviation \( X_n = X \otimes z^n \) for \( X = E, F, H \).

In the case of \( \mathfrak{sl}_2 \), corresponding Weyl group is the symmetric group \( \mathfrak{S}_2 \) of order 2, generated by the simple transposition \( \sigma \). The gradation corresponds to \( \text{Id} \in \mathfrak{S}_2 \) is given by the element \( d \), and called “homogeneous”. The gradation corresponds to \( \sigma \in \mathfrak{S}_2 \) is called “principal”, given by \( d_p = 2d + \frac{1}{2}H_0 \).

We choose the Heisenberg subalgebra of homogeneous-type,
\[
\Lambda_n^{[h]} \overset{\text{def}}{=} H_n ,
\]
(2.8)
and the triangular decomposition of principal-type,
\[
\hat{\mathfrak{sl}}_2 = \left( \hat{\mathfrak{sl}}_2 \right)_{<0}^{[p]} \oplus \left( \hat{\mathfrak{sl}}_2 \right)_{\geq0}^{[p]} .
\]
In other words, we have chosen \( w' = \text{Id} \) in (2.1) and \( w = \sigma \) in (2.3). We stress that this choice does not fit to the condition \( w' \geq w \) in the sense of the Bruhat order, and
thus does not fall into the category treated in [dGHM]. Note that the homogeneous
Heisenberg subalgebra \( \{d, \Lambda^{[h]}\} = 2n \Lambda^{[h]} \).

We consider a formal series expansion of \( \log g^{[p]}_{<0}(t) \in \hat{\mathfrak{sl}}_2 \) as follows:

\[
\log g^{[p]}_{<0}(t) = \{q(t)E_{-1} + r(t)F_0\} + u(t)H_{-1} + \{v_1(t)E_{-2} + v_2(t)F_{-1}\} + w(t)H_{-2} + \cdots .
\]

(2.9)

By straightforward calculations, we can obtain the expression for \( B_1 \):

\[
B_1 = H_1 + (-2qE_0 + 2rF_1) + \{2qrH_0 - (qr + 2u)K\} .
\]

(2.10)

Lemma 1.

\[
\frac{\partial q}{\partial t} = 2v_1 + 2qu + \frac{4}{3}q^2 r, \quad \frac{\partial r}{\partial t} = -2v_2 + 2ru - \frac{4}{3}qr^2
\]

(2.11)

Proof. In the present case, the Sato-Wilson equation (2.4) with \( n = 1 \) is equivalent to

the following equation in \( \hat{\mathfrak{sl}}_2 \):

\[
\frac{\partial g^{[p]}_{<0}}{\partial t} (g^{[p]}_{<0})^{-1} = B_1 - g^{[p]}_{<0} H_1 (g^{[p]}_{<0})^{-1} .
\]

(2.12)

Comparing the \((\cdot)^{[p]}_{-1}\)-part of the both side of (2.12), we can derive the desirable result. \( \Box \)

A level-0 realization of \( \hat{\mathfrak{sl}}_2 \) is given by

\[
E_n \mapsto \begin{pmatrix} 0 & z^n \\ 0 & 0 \end{pmatrix}, \quad F_n \mapsto \begin{pmatrix} 0 & 0 \\ z^n & 0 \end{pmatrix}, \quad H_n \mapsto \begin{pmatrix} z^n & 0 \\ 0 & -z^n \end{pmatrix},
\]

\( K \mapsto 0, \quad d \mapsto z \frac{d}{dz} .
\]

(2.14)

Using this realization, we can express \( B_1 \) and \( B_2 \) as \( 2 \times 2 \) matrices:

\[
B_1 = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} + \begin{pmatrix} 0 & -2q \\ 2zr & 0 \end{pmatrix} + \begin{pmatrix} 2qr & 0 \\ 0 & -2qr \end{pmatrix},
\]

\[
B_2 = \begin{pmatrix} z^2 & 0 \\ 0 & -z^2 \end{pmatrix} + \begin{pmatrix} 0 & -2zq \\ 2z^2r & 0 \end{pmatrix} + \begin{pmatrix} 2zqr & 0 \\ 0 & -2zqr \end{pmatrix}
\]

\[
+ \begin{pmatrix} 0 & -q' \\ -zr' & 0 \end{pmatrix} + \begin{pmatrix} q'r - qr' - 2q^2 r^2 & 0 \\ 0 & qr' - q'r + 2q^2 r^2 \end{pmatrix} .
\]

(2.13)
These matrices give a Lax pair for the \(\partial\)NLS equation but are different from the conventional one (cf. [WS]). To reproduce the conventional Lax pair, we use the other level-0 realization of \(\hat{\mathfrak{sl}}_2\) given by

\[
E_n \mapsto \begin{pmatrix} 0 & \lambda^{2n+1} \\ 0 & 0 \end{pmatrix}, \quad F_n \mapsto \begin{pmatrix} 0 & 0 \\ \lambda^{2n-1} & 0 \end{pmatrix}, \quad H_n \mapsto \begin{pmatrix} \lambda^{2n} & 0 \\ 0 & -\lambda^{2n} \end{pmatrix},
\]

\[
K \mapsto 0, \quad d \mapsto \frac{1}{2} \left\{ z \frac{d}{dz} + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.
\]

(2.15)

From this realization, we obtain

\[
B_1 = \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & -2q \\ 2r & 0 \end{pmatrix} + \begin{pmatrix} 2qr & 0 \\ 0 & -2qr \end{pmatrix},
\]

\[
B_2 = \lambda^4 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda^3 \begin{pmatrix} 0 & -2q \\ 2r & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 2qr & 0 \\ 0 & -2qr \end{pmatrix}
+ \lambda \begin{pmatrix} 0 & -q' \\ -r' & 0 \end{pmatrix} + \begin{pmatrix} q'r - qr' - 2q^2r^2 & 0 \\ 0 & qr' - q'r + 2q^2r^2 \end{pmatrix}.
\]

For the latter use, we decompose \(g^{[p]}_{\geq 0}(t)\) into grade 0 and \(>0\) part:

\[
g^{[p]}_{\geq 0}(t) = g^{[p]}_0(t)g^{[p]}_{>0}(t).
\]

(2.16)

Substituting (2.16) into (2.5), we obtain

\[
\frac{\partial g^{[p]}_0}{\partial t_n} = \left( (g^{[p]}_0)^{-1} B_n g^{[p]}_0 \right)_0 g^{[p]}_0, \quad \frac{\partial g^{[p]}_{>0}}{\partial t_n} = \left\{ (g^{[p]}_0)^{-1} B_n g^{[p]}_0 - \left( (g^{[p]}_0)^{-1} B_n g^{[p]}_0 \right)_0 \right\} g^{[p]}_{>0}.
\]

From these differential equations together with (2.10), (2.13) and formal expansions,

\[
\log g^{[p]}_0(t) = \phi(t)H_0 + \psi(t)K, \quad \log g^{[p]}_{>0}(t) = a(t)E_0 + b(t)F_1 + c(t)H_1 + \cdots,
\]

it follows that the functions \(\phi(t), a(t), b(t)\) satisfy the equations,

\[
\frac{\partial \phi}{\partial t_1} = 2qr, \quad \frac{\partial \phi}{\partial t_2} = q'r - qr' - 2q^2r^2, \quad \frac{\partial a}{\partial t_1} = -2q e^{-2\phi}, \quad \frac{\partial a}{\partial t_2} = -q' e^{-2\phi}, \quad \frac{\partial b}{\partial t_1} = 2r e^{2\phi}, \quad \frac{\partial b}{\partial t_2} = -r' e^{2\phi}.
\]

(2.19)

(2.20)

(2.21)
2.3 Miura-type transformation to the NLS equation

The homogeneous hierarchy that includes the NLS equation is obtained by taking \( w' = \text{Id} \) in the time-evolution (2.1), same as \( \partial \text{NLS} \) hierarchy, and \( w = \text{Id} \) in the Gauss decomposition (2.3). We denote the result of the decomposition by

\[
g(t) = \{g^{[n]}_{<0}(t)\}^{-1}g^{[n]}_{\geq 0}(t), \quad g^{[n]}_{<0}(t) \in \widehat{G}_{<0}^{[\text{Id}]}, \quad g^{[n]}_{\geq 0}(t) \in \widehat{G}_{\geq 0}^{[\text{Id}]}.
\]

The relation between this system and the \( \partial \text{NLS} \) hierarchy is established by the Miura-type transformation, which is an analog of the Miura transformation in the case of the KdV and the mKdV equations. For \( g^{[n]}_{\geq 0}(t) \) of (2.24), we put

\[
G \overset{\text{def}}{=} \exp(-r(t)F_0)
\]

and consider the decomposition,

\[
g(t) = \{Gg^{[n]}_{<0}(t)\}^{-1}Gg^{[n]}_{\geq 0}, \quad Gg^{[n]}_{<0}(t) \in \widehat{G}_{<0}^{[\text{Id}]}, \quad Gg^{[n]}_{\geq 0}(t) \in \widehat{G}_{\geq 0}^{[\text{Id}]}.
\]

The assumption of the uniqueness of the Gauss decomposition causes

\[
g^{[n]}_{<0}(t) = Gg^{[n]}_{<0}(t) = \exp(gE_{-1} + uH_{-1} + \cdots) \in \widehat{G}_{<0}^{[\text{Id}]}, \quad (2.22)
\]

\[
g^{[n]}_{\geq 0}(t) = Gg^{[n]}_{\geq 0}(t) \in \widehat{G}_{\geq 0}^{[\text{Id}]}.
\]

These relations can be considered as Miura-type transformation in affine Lie group. By the equation (2.4), we can write

\[
\frac{\partial}{\partial t_n} - B_n = g^{[n]}_{<0} \left( \frac{\partial}{\partial t_n} - H_n \right) (g^{[n]}_{\geq 0})^{-1}.
\]

(2.24)

Then we can describe the transformation in terms of \( B_n \) by translating \( g^{[n]}_{<0}(t) \) to \( g^{[n]}_{\geq 0}(t) \) in (2.24) and we obtain

\[
\frac{\partial}{\partial t_n} - \ddot{B}_n \overset{\text{def}}{=} g^{[n]}_{<0} \left( \frac{\partial}{\partial t_n} - H_n \right) (g^{[n]}_{\geq 0})^{-1} = G \left( \frac{\partial}{\partial t_n} - B_n \right) G^{-1}.
\]

(2.25)

Note that this transformation preserves the zero-curvature equations (2.7). The relation between \( \ddot{B}_n \) and \( B_n \) can be described as follows:

\[
\ddot{B}_n = GB_nG^{-1} + \frac{\partial G}{\partial t_n}G^{-1}.
\]

(2.25)

For \( n = 1, 2 \), we obtain

\[
\ddot{B}_1 = H_1 + (-2qE_0 - (r' + 2qr^2)F_0) - (qr + 2u)K,
\]

\[
\ddot{B}_2 = H_2 + (-2qE_1 - (r' + 2qr^2)F_1) - q(r' + 2qr^2)H_0 - q'E_0 + \left( \frac{r'}{2} + qr^2 \right)'F_0 + \left( -4w - rv_1 - \frac{2}{3}qr^u - \frac{1}{3}q^2r^2 \right)K.
\]

(2.26)

Here we have used the \( \partial \text{NLS} \) equation (1.3) to eliminate \( r_t \) in \( \ddot{B}_2 \).

If we put

\[
\dot{r} = -\frac{r'}{2} - qr^2,
\]

(2.28)

then the zero-curvature equation for \( \ddot{B}_1 \) and \( \ddot{B}_2 \) gives the NLS equation (1.4).
2.4 Gauge transformation to a generalized $\partial$NLS equation

There are several different kind of derivative NLS equations [ARS, CLL, GI, KSS, KN, K]. By extending the approach of [WS], Kundu obtained the generalized $\partial$NLS equation [K],

$$
\begin{align*}
Q_t &= \frac{1}{2} Q'' + 2cQRQ' + 2(c-1)Q^2 R' - 2(c-1)(c-2)Q^3 R^2, \\
R_t &= -\frac{1}{2} R'' + 2cQR'R' + 2(c-1)R^2 Q' + 2(c-1)(c-2)Q R^3.
\end{align*}
$$

(2.29)

Here $c$ is a complex parameter. The equation (2.29) include the Kaup-Newell equation \((c = 1) [KN]\), the Chen-Lee-Liu equation \((c = 2) [CLL]\) and also (1.3) \((c = 0)\) as special cases. We can obtain the equation (2.29) by the gauge transformation of type (2.25) with respect to $g_0[p] t^{-c/2} = \exp(-c\phi/2)H_0$:

$$
\frac{\partial}{\partial t_n} - B_n \mapsto g_0[p] t^{-c/2} \left( \frac{\partial}{\partial t_n} - B_n \right) g_0[p] t^{c/2} = \frac{\partial}{\partial t_n} - g_0[p] t^{-c/2} B_n g_0[p] t^{c/2} + g_0[p] t^{-c/2} \frac{\partial g_0[p] t^{c/2}}{\partial t_n}
$$

and put

$$C_n \overset{\text{def}}{=} g_0[p] t^{-c/2} B_n g_0[p] t^{c/2} - g_0[p] t^{-c/2} \frac{\partial g_0[p] t^{c/2}}{\partial t_n}.
$$

Then for $n = 1, 2$, we have

$$
\begin{align*}
C_1 &= H_1 + (-2q e^{-c\phi} E_0 + 2r e^{c\phi} F_1) - (c-2)qr H_0 - (qr + 2u) K \\
C_2 &= H_2 + (-2q e^{-c\phi} E_1 + 2r e^{c\phi} F_2) + 2qr H_1 \\
&\quad + (-q e^{-c\phi} E_0 - r' e^{c\phi} F_1) + (1-c/2)(q' r - qr' - 2q^2 r^2) H_0 \\
&\quad + \left(-4w - r v_1 - \frac{2}{3} qr u - \frac{1}{3} q^2 r^2\right) K.
\end{align*}
$$

Here we have used the relation (2.19). We introduce the new variables

$$Q(t) \overset{\text{def}}{=} q e^{-c\phi}, \quad R(t) \overset{\text{def}}{=} r e^{c\phi}.
$$

By (2.19), the derivatives of these functions are written as

$$Q' = q' e^{-c\phi} - 2cQ^2 R, \quad R' = r' e^{c\phi} + 2cR^2 Q.
$$

Then the zero-curvature equation for $C_1$ and $C_2$ result in the equations (2.29). Especially, the Lax operators $C_1, C_2$ for the Kaup-Newell equation and the Chen-Lee-Liu equation realized as matrix form (2.15) are identified with that of [WS].

3 Actions of affine Weyl group to the $\partial$NLS hierarchy

In this section, we discuss symmetries of the $\partial$NLS hierarchy in terms of the affine Weyl group $W(A_1^{(1)}) = \langle s_0, s_1 \rangle$ with the relations $s_0^2 = s_1^2 = \text{Id.}$
Let $V$ be an integrable module of $\hat{\mathfrak{sl}}_2$. The affine Weyl group $W(A_1^{(1)})$ acts on $V$ as follows \cite{K}:

$$s_j = \exp(f_j) \exp(-e_j) \exp(f_j) \quad (j = 0, 1), \quad (3.1)$$

where $e_j, f_j$ are the Chevalley generators of $\hat{\mathfrak{sl}}_2$ given by

$$e_0 = F_1, \quad f_0 = E_{-1}, \quad h_0 = K - H_0,$$

$$e_1 = E_0, \quad f_1 = F_0, \quad h_1 = H_0.$$ 

Note that $e_j, h_j, f_j \ (j = 0, 1)$ have principal grades 1, 0, −1, respectively. Under the level-0 realization (2.14), we can describe them as follows:

$$s_0 \mapsto \left( \begin{array}{cc} 0 & z^{-1} \\ -z & 0 \end{array} \right), \quad s_1 \mapsto \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right). \quad (3.2)$$

The generators $s_0, s_1$ act naturally on $g(0)$ of (2.1) in two different ways.

### 3.1 Left-action

We consider the left-action of $s_j \ (j = 0, 1)$ of the form $s_j^{-1}g(0)$. Applying the principal Gauss decomposition (2.3) to $\exp[\sum_n t_n H_n] s_j^{-1}g(0)$, we define $s_j^L(g_{<0}(t))$ and $s_j^L(g_{\geq0}(t))$ as

$$\{s_j^L(g_{<0}(t))\}^{-1} s_j^L(g_{\geq0}(t)) = \exp \left[ \sum_{n>0} t_n H_n \right] s_j^{-1}g(0). \quad (3.3)$$

This decomposition induces an action of $s_j$ on the variables $q(t), r(t)$.

**Theorem 1.** Assume that the Gauss decomposition (3.3) exists uniquely. Then one can write down the action of $s_j \ (j = 0, 1)$ explicitly:

$$s_j^L : q(t) \mapsto -\frac{1}{q(-t)}, \quad r(t) \mapsto q(-t)r(-t) - \frac{1}{2} q'(-t), \quad (3.4)$$

$$s_j^L : q(t) \mapsto q(-t)r(-t)^2 + \frac{1}{2} r'(-t), \quad r(t) \mapsto -\frac{1}{r(-t)}. \quad (3.5)$$

Here $-t = (-t_1, -t_2, \ldots)$.

**Proof.** Using the relation $s_j H_n s_j^{-1} = -H_n \ (j = 0, 1, n = 1, 2, \ldots)$, one can rewrite (3.3) as

$$\{s_j^L(g_{<0}(t))\}^{-1} s_j^L(g_{\geq0}(t)) = s_j^{-1}g(-t) = \left\{g_{<0}^{[p]}(-t)s_j\right\}^{-1} g_{\geq0}^{[p]}(-t).$$

Next we consider the Gauss decomposition of $\left\{g_{<0}^{[p]}(-t)s_j\right\}^{-1}$:

$$\left\{g_{<0}^{[p]}(-t)s_j\right\}^{-1} = \left\{\hat{g}_{<0}^{(j)}(t)\right\}^{-1} \hat{g}_{\geq0}^{(j)}(t). \quad (3.6)$$

Assuming the uniqueness of the Gauss decomposition (3.3), one finds that

$$s_j^L(g_{<0}(t)) = \hat{g}_{<0}^{(j)}(t).$$
The decomposition (3.6) is equivalent to the condition,

\[ g^{[p]}_{<0}(-t)s_j \left\{ \tilde{g}^{(j)}_{<0}(t) \right\}^{-1} \in \tilde{G}^{[p]}_{\geq 0}. \]  

(3.7)

We introduce a formal series expansion of \( \log \tilde{g}^{(j)}_{<0}(t) \) as

\[
\log \tilde{g}^{(j)}_{<0}(t) = \left\{ \tilde{q}_j(t)E_{-1} + \tilde{r}_j(t)F_0 \right\} + \tilde{u}_j(t)H_{-1} \\
+ \left\{ \tilde{v}_{1,j}(t)E_{-2} + \tilde{v}_{2,j}(t)F_{-1} \right\} + \tilde{w}_j(t)H_{-2} + \cdots. \]  

(3.8)

Substituting \( g^{[p]}_{<0}(-t) \) (2.9), \( s_j \) (3.1) and (3.8) into \( g^{[p]}_{<0}(-t)s_j \left\{ \tilde{g}^{(j)}_{<0}(t) \right\}^{-1} \) and using the realization (2.14), we can rewrite the condition (3.7) as relations between the coefficients of \( g^{[p]}_{<0}(-t) \) (2.9) and \( \tilde{g}^{(j)}_{<0}(t) \) (3.8). For example, in the case of \( s_1 \), we have

\[
1 + r(-t)\tilde{r}_1(t) = 0, \\
u(-t) + \tilde{u}_1(t) + \frac{q(-t)r(-t) + \tilde{q}_1(t)\tilde{r}_1(t)}{2} = 0, \\
v (-t) + \frac{q(-t)r(-t)^2}{6} + \tilde{q}_1(t) + \frac{r(-t)\tilde{q}_1(t)\tilde{r}_1(t)}{2} + r(-t)\tilde{u}_1(t) = 0. 
\]

From these relations together with (2.11), we obtain (3.5). The \( s_0 \)-action (3.4) can be obtained in a similar way.

We remark that \( -s_1^r(q(-t)) \) coincides with \( \tilde{r} \) of (2.28). Thus \( q(t) \) and \( \tilde{r}(t) = -s_1^r(q(-t)) \) solve the NLS equation (1.4).

### 3.2 Right-action

Next we consider the right-action of \( s_j \) \( (j = 0, 1) \) of the form \( g(0)s_j \), which induces another action of \( s_j \) on the variables \( q(t), r(t) \) through the decomposition,

\[
\left\{ s_j^R(g^{[p]}_{<0}(t)) \right\}^{-1} s_j^R(g^{[p]}_{<0}(t)) = g(t) s_j. 
\]

(3.9)

**Theorem 2.** Assume that the Gauss decomposition (3.3) exists uniquely. Then one can write down the action of \( s_j \) \( (j = 0, 1) \) explicitly:

\[
s_0^R : q(t) \mapsto q(t) - \frac{1}{\tilde{\psi}_0(t)}, \quad r(t) \mapsto r(t), \]  

(3.10)

\[
s_1^R : q(t) \mapsto q(t), \quad r(t) \mapsto r(t) - \frac{1}{\tilde{\psi}_1(t)}. \]  

(3.11)

Here \( \tilde{\psi}_0(t) \) and \( \tilde{\psi}_1(t) \) satisfy the following differential equations,

\[
\frac{\partial \tilde{\psi}_0}{\partial t_2} = 2r - 4qr \tilde{\psi}_0, \quad \frac{\partial \tilde{\psi}_0}{\partial t_1} = -r' - 2(q' r - qr' - 2q^2 r^2) \tilde{\psi}_0, \]  

(3.12)

\[
\frac{\partial \tilde{\psi}_1}{\partial t_2} = -2q + 4qr \tilde{\psi}_1, \quad \frac{\partial \tilde{\psi}_1}{\partial t_1} = -q' + 2(q' r - qr' - 2q^2 r^2) \tilde{\psi}_1. \]  

(3.13)
Proof. We consider the Gauss decomposition of \( g_{\geq 0}(t)_j \):

\[
g_{\geq 0}(t)_j = \left\{ g_{<0}^{(j)}(t) \right\}^{-1} g_{\geq 0}(t). \tag{3.14}
\]

Assuming the uniqueness of the Gauss decomposition (3.9), one finds that

\[
s_R^{g_{<0}(t)}(g_{\geq 0}(t)) = g_{<0}(t)g_{\geq 0}(t). \tag{3.15}
\]

The decomposition (3.14) is equivalent to

\[
g_{\geq 0}(t)_j \left\{ g_{\geq 0}^{(j)}(t) \right\}^{-1} = \left\{ g_{<0}^{(j)}(t) \right\}^{-1} \in \, \hat{G}_{<0}. \tag{3.16}
\]

In the realization (2.14), substituting formal expansions \( g_{\geq 0}(t) = g_0^{[p]}(t)g_{>0}(t) \) (2.17), (2.18), \( \tilde{g}^{(j)}_{>0}(t) = \tilde{g}_0^{(j)}(t)\tilde{g}_{>0}^{(j)}(t) \)

\[
\log g^{(j)}_0(t) = \tilde{\phi}_j(t)H_0, \quad \log g^{(j)}_{>0}(t) = \tilde{a}_j(t)E_0 + \tilde{b}_j(t)F_1 + \tilde{c}_j(t)H_1 + \cdots,
\]

and \( s_j (3.1) \) to (3.16), we have

\[
be^{\tilde{\phi}_0 - \phi} = 1, \quad \tilde{b}\tilde{b}_0 = -1, \quad \ldots \nonumber
\]

\[
ae^{\phi - \tilde{\phi}_1} = 1, \quad a\tilde{a}_1 = -1, \quad \ldots \nonumber
\]

and

\[
\tilde{g}_0^{(0)}(t) = \exp[-e^{\phi + \tilde{\phi}_0}E_{-1}], \quad \tilde{g}_0^{(1)}(t) = \exp[-e^{-\phi - \tilde{\phi}_1}F_0].
\]

Therefore, we obtain \( s_j^{g_{<0}(t)}(g_{\geq 0}(t)) \) \((j = 0, 1) \) from (3.15). If we define

\[
\tilde{\psi}_0 = e^{-\phi - \tilde{\phi}_0} = e^{-2\phi}b, \quad \tilde{\psi}_1 = e^{\phi - \tilde{\phi}_1} = e^{2\phi}a \tag{3.17}
\]

we have the formulas (3.10), (3.11). The differential equations (3.12), (3.13) follow from (2.19), (2.20) and (2.21). \[\square\]

We remark that the right action can be described by the gauge transformation of the differential operators:

\[
\frac{\partial}{\partial t_n} - s_j(B_n) = \tilde{g}_{<0}^{(j)}(\frac{\partial}{\partial t_n} - B_n)(\tilde{g}_{<0}^{(j)})^{-1} \quad (j = 0, 1).
\]

This construction of the Weyl group action is essentially the same as that of Noumi and Yamada \([NY1, N]\). We will discuss this point in what follows (See Section 4.4).

### 3.3 Extended affine Weyl group

We denote by \( \pi \) the Dynkin diagram automorphism of \( A_1^{(1)} \)-type defined by

\[
\pi x_i = x_{i+1} \pi \quad (i = 0, 1, x = e, f, h),
\]

\[
\pi x_i = x_{i+1} x_i x_{i+1} = x_{i+1} \pi \quad (i = 0, 1, x = e, f, h).
\]
where the subscripts are understood as elements of \(\mathbb{Z}/2\mathbb{Z}\). We extend the affine Weyl group \(\tilde{W}(A_1^{(1)})\) by adding the element \(\pi\) that satisfies algebraic relations
\[
s_0^2 = s_1^2 = \pi^2 = 1, \quad \pi s_0 = s_1 \pi, \quad \pi s_1 = s_0 \pi.
\]
We denote the extended Weyl group by \(\tilde{W}(A_1^{(1)})\). In the level-0 realization of the Chevalley generators (2.14), the automorphism \(\pi\) are realized by the adjoint action of the matrix
\[
\begin{pmatrix}
0 & z^{-1/2} \\
-z^{1/2} & 0
\end{pmatrix}.
\]
As in the case of \(s_j\), the action of \(\pi\) on \(g(0)\) induces a transformation on solutions of the \(\partial\)NLS equation through the Gauss decomposition,
\[
\exp\left[\sum_n t_n H_n\right] \pi^{-1} g(0) \pi = \pi^{-1} g(-t) \pi = \begin{cases} 
\pi^{-1} g_{\leq 0}(-t) \pi^{-1} g_{\geq 0}(-t) \pi.
\end{cases}
\]
It follows that
\[
\pi : \begin{cases} 
q(t) \mapsto -r(-t), \quad r(t) \mapsto -q(-t), \\
\phi(t) \mapsto -\phi(-t), \quad a(t) \mapsto -b(-t), \quad b(t) \mapsto -a(-t).
\end{cases}
\]
The sets of transformations \(\langle s_0^L, s_1^L, \pi \rangle\) and \(\langle s_0^R, s_1^R, \pi \rangle\) satisfy the relation of the extended affine Weyl group (3.18) and thus we have obtained two different realizations of \(\tilde{W}(A_1^{(1)})\).

4 Similarity reduction and monodromy problem

In this section, we formulate a similarity condition of soliton equations in algebraic framework and consider the relation to monodromy problem of a linear ordinary differential system.

4.1 Similarity condition for soliton equation

First, we impose a constraint for the initial data \(g(0) = g(z; 0)\):
\[
[d, g(z; 0)] = \alpha H_0 g(z; 0) + \beta g(z; 0) H_0 + \gamma g(z; 0) K.
\]
Here \(\alpha, \beta, \gamma\) are complex parameters. This relation leads to the following constraint for \(g(t) = g(z; t)\) of (2.1):
\[
[d, g(z; t)] = \left(\alpha H_0 + \sum_{n>0} n t_n \frac{\partial}{\partial t_n}\right) g(z; t) + \beta g(z; t) H_0 + \gamma g(z; t) K,
\]
because the generators of the homogeneous Heisenberg subalgebra \(H_n\) satisfy the condition,
\[
\exp \left(\sum_{n>0} t_n H_n\right) \cdot d \cdot \exp \left(\sum_{n>0} t_n H_n\right) = d - \sum_{n>0} n t_n H_n.
\]
Note that $d = z \partial_z$ is the derivation for the homogeneous gradation. These conditions correspond to the similarity conditions for $g(z; 0)$ and $g(z; t)$:

$$g(\lambda z; 0) = \lambda^{\alpha H_0} g(z; 0) \lambda^{\beta H_0},$$

$$g(\lambda z; t) = \lambda^{\alpha H_0} g(z; t) \lambda^{\beta H_0},$$

by taking the exponential with respect to $\lambda$ of the operators of both hand side of (4.1), (4.2) respectively. By applying the Gauss decomposition to $g(z; t)$ with respect to the principal gradation, we obtain a constraint for $g_{\leq 0}^{[p]}(z; t)$ and $g_{\geq 0}^{[p]}(z; t)$ such as

$$[d, g_{<0}^{[p]}(z; t)] = [\alpha H_0, g_{<0}^{[p]}(z; t)] + \sum_{n>0} nt_n \frac{\partial g_{<0}^{[p]}(z; t)}{\partial t_n},$$

$$[d, g_{\geq 0}^{[p]}(z; t)] = (\alpha H_0 + \sum_{n>0} nt_n B_n) g_{\geq 0}^{[p]}(z; t) + \beta g_{>0}^{[p]}(z; t) H_0 + \gamma g_{>0}^{[p]}(z; t) K. \tag{4.5}$$

These conditions correspond to the similarity conditions:

$$g_{<0}^{[p]}(\lambda z; t) = \lambda^{\alpha H_0} g_{<0}^{[p]}(z; t) \lambda^{-\alpha H_0}, \quad g_{\geq 0}^{[p]}(\lambda z; t) = \lambda^{\alpha H_0} g_{\geq 0}^{[p]}(z; t) \lambda^{\beta H_0} \tag{4.6}$$

Especially, the first few components of $g_{<0}^{[p]}(z; t)$ (2.9) and $g_{\geq 0}^{[p]}(z; t)$ (2.17) satisfy the following conditions:

$$q(\tilde{t}) = \lambda^{-2\alpha-1} q(t), \quad r(\tilde{t}) = \lambda^{2\alpha} r(t), \quad \phi(\tilde{t}) = (\log \lambda^{-(\alpha + \beta)}) \phi(t), \tag{4.7}$$

$$a(\tilde{t}) = \lambda^{2\beta} a(t), \quad b(\tilde{t}) = \lambda^{-2\beta + 1} b(t). \tag{4.8}$$

**Proposition 1.** If we set

$$M = \alpha H_0 + \sum_{n>0} nt_n B_n, \tag{4.9}$$

then $M$ and $B_n$ $(n = 1, 2, \ldots)$ satisfy the zero-curvature equations:

$$\left[ z \frac{d}{dz} - M, \frac{\partial}{\partial t_n} - B_n \right] = 0. \tag{4.10}$$

**Proof.** By the definition (4.9) of $M$ and relations (2.21), (2.25), (4.4), (4.5), we can describe

$$z \frac{d}{dz} - M = g_{<0}^{[p]} \left( z \frac{d}{dz} - \alpha H_0 - \sum_{n>0} nt_n H_n \right) \left( g_{<0}^{[p]} \right)^{-1}$$

$$= g_{\geq 0}^{[p]} \left( z \frac{d}{dz} + \beta H_0 \right) \left( g_{\geq 0}^{[p]} \right)^{-1}.$$ 

Therefore, by multiplying $(g_{<0}^{[p]})^{-1}$ from the left and $g_{<0}^{[p]}$ from the right to the formula

$$\left[ z \frac{d}{dz} - \alpha H_0 - \sum_{n>0} nt_n H_n, \frac{\partial}{\partial t_m} - H_m \right] = 0 \quad (m = 1, 2, \ldots)$$

or

$$\left[ z \frac{d}{dz} + \beta H_0, \frac{\partial}{\partial t_m} - H_m \right] = 0 \quad (m = 1, 2, \ldots),$$

we have the equation (4.10). \qed
4.2 Monodromy problem and Painlevé IV

We now fix a positive integer \( l > 0 \) and restrict the operator for the time evolution to \( \exp[\sum_{n=0}^l t_n H_n] \), or we put \( t_{l+1} = t_{l+2} = \cdots = 0 \) in (4.3). Then \( M \) of (4.9) becomes a element of affine Lie algebra. Under the realization (2.14), we get a system of linear differential equations for a \( 2 \times 2 \) matrix \( Y = Y(z; t_1, \ldots, t_l) \):

\[
\begin{align*}
z \frac{\partial}{\partial z} Y &= MY, \\
\frac{\partial}{\partial t_n} Y &= B_n Y \quad (n = 1, \ldots, l).
\end{align*}
\]

(4.11)

This linear problem defines a monodromy preserving deformation of linear ordinary differential system, with regular singularity at \( z = 0 \) and irregular singularity of rank \( l \) at \( z = \infty \). We regard \( t_1, t_2, \ldots, t_l \) as a deformation parameter at \( \infty \), and \( \alpha, \beta \) as monodromy data at \( \infty, 0 \) respectively.

Hereafter, we set \( l = 2 \) and put \( t = t_2 = 1/2 \). Then \( M \) of (4.9) for \( B_1 \) (2.10) and \( B_2 \) (2.13) can be written as

\[
M = H_2 + (2qE_1 + 2rF_2) + (x + 2qr)H_1 \\
+ (-2xq + q'E_0 + (2xr - r')F_1) + (\alpha + k)H_0,
\]

(4.12)

where

\[
k = 2xqr + q'r - qr' - 2q^2r^2 = \left( x \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) \phi(t).
\]

(4.13)

The second equality is given by (2.19). We set

\[
\varphi = 2qr, \quad \psi_1 = -2x - \frac{q'}{q}, \quad \psi_0 = 2x - \frac{r'}{r}.
\]

(4.14)

The compatibility condition (4.10) for the linear system (4.11) for the restricted \( M \) of (4.12) and \( B_1 \) of (2.10) present the following system of differential equations:

\[
\begin{align*}
\varphi' &= -\varphi(\psi_1 + \psi_0), \\
\psi_1' &= \psi_1(2\varphi + \psi_1 + 2x) - 4\beta, \\
\psi_0' &= \psi_0(-2\varphi + \psi_0 - 2x) - 2(2\beta - 1).
\end{align*}
\]

(4.15-4.17)

In addition, the similarity condition for \( \phi \) of (4.7) fix the value of \( k \) (4.13):

\[
k = -\alpha - \beta.
\]

(4.18)

So we have the relation

\[
\psi_0 - \psi_1 - \varphi + \frac{2(\alpha + \beta)}{\varphi} = 2x.
\]

(4.19)

**Proposition 2.** Each of the quantities \( \varphi, \psi_1 \) and \( -\psi_0 \) solve the fourth Painlevé equation (1.5) with the following parameters:

\[
\begin{array}{c|cc}
\varphi & \nu_1 & \nu_2 \\
\psi_1 & \alpha - 3\beta + 1 & -2(\alpha + \beta)^2 \\
-\psi_0 & 2\alpha & -2(2\beta - 1)^2 \\
\end{array}
\]

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Proof. Differentiating (4.15), we obtain
\[ \varphi'' = -\varphi'\psi_0 - \varphi'(2(\varphi + x)(\psi_1 - \psi_0) + \psi_1^2 + \psi_0^2 - 8\beta + 2). \]

Then, by using the relations (4.15) and (4.19), we have PIV (1.3) for \(\varphi\). For \(\psi_1\), the relations (4.15), (4.16) and (4.19) give the equations,
\[ \psi_1' = 2\varphi\psi_1 + \psi_1^2 + 2x\psi_1 - 4\beta, \quad (4.20) \]
\[ (\varphi\psi_1)' = \frac{(\varphi\psi_1)^2}{\psi_1} - \varphi\psi_1 \left(\frac{4\beta}{\psi_1} + \psi_1\right) + 2(\alpha + \beta)\psi_1. \quad (4.21) \]

Then, differentiating the first equation (4.20) and eliminating \(\varphi\psi_1\) by using (4.20) and (4.21), we have the fourth Painlevé equation for \(\psi_1\). The other case \(-\psi_0\) can be treated in the similar way.

Remark 1. Ablowitz et al. presented the fourth Painlevé equation as a similarity reduction of \(\partial NLS\) [ARS]. In our notation, their results correspond to the equation for \(\varphi\). However, their result has only one parameter \(\beta\), and corresponds to the special case \(\alpha = -1/4\). We give the fourth Painlevé equation with full parameters.

Remark 2. Our system of linear equations (4.11) is not a special case of the generalized Painlevé systems given by Noumi and Yamada [NY1, N]. Their system is based on the similarity reduction of the principal hierarchy.

Jimbo and Miwa [JM2] showed that PIV is obtained as a similarity reduction of the NLS equation. Their results correspond to the Gauss decomposition of homogeneous-type in our setting. Since \(g_{<0}^h(z; t)\) (2.22) and \(g_{\geq0}^h(z; t)\) (2.23) satisfy the same similarity conditions as (4.6):
\[ g_{<0}^h(\lambda z; \tilde{t}) = \lambda^{\alpha H_0} g_{<0}^h(z; \tilde{t}) \lambda^{-\alpha H_0}, \quad g_{\geq0}^h(\lambda z; \tilde{t}) = \lambda^{\alpha H_0} g_{\geq0}^h(z; \tilde{t}) \lambda^{\beta H_0}, \]
so the solutions of the NLS equation (1.3) satisfy the conditions
\[ q(\tilde{t}) = \lambda^{-2\alpha-1} q(t), \quad \hat{r}(\tilde{t}) = \lambda^{2\alpha-1} \hat{r}(t). \]

By the same discussion as above, in the level-0 realization (2.14) of \(\tilde{B}_1\) (2.26) and \(\tilde{B}_2\) (2.27), we have the linear problem,
\[ \frac{\partial}{\partial z} Y = A(z)Y, \quad \frac{\partial}{\partial x} Y = B(z)Y, \]
with
\[ A(z) = z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} x & -2q \\ 2\hat{r} & -x \end{pmatrix} + z^{-1} \begin{pmatrix} \alpha + 2q\hat{r} & -2xq - q' \\ 2x\hat{r} - \hat{r}' & -(\alpha + 2q\hat{r}) \end{pmatrix}, \]
\[ B(z) = z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -2q \\ 2\hat{r} & 0 \end{pmatrix}. \]

These can be identified with the result of [JM2].
Furthermore, we use (4.14), (4.19) and (2.28) to show that
\[ \varphi \psi_1 = 4q \hat{r} + 2(\alpha + \beta). \] (4.22)

Applying the relation (4.22) to the compatibility condition
\[ \left[ \frac{\partial}{\partial z} - A(z), \frac{\partial}{\partial x} - B(z) \right] = 0, \]
we have (4.20), (4.21) and thus obtain the fourth Painlevé equation for \( \psi_1 \).

### 4.3 Relations to Hamiltonian system

In [O], Okamoto showed that the fourth Painlevé equation (1.5) is equivalent to the Hamilton system,
\[ y' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial y}, \]
with the polynomial Hamiltonian,
\[ H = yp^2 - y^2 p - 2xpy - 2\theta_0p + 2\theta_\infty y. \]

This is represented as the following system of equations for \( y \) and \( p \):
\[ y' = y(2p - y - 2x) - 2\theta_0, \]
\[ p' = p(2q - p + 2x) - 2\theta_\infty. \] (4.23)

Our system (4.15)–(4.17) can be identified with (4.23) in two different ways. Firstly, if we eliminate \( \psi_0 \) from (4.15) by using (4.19), we have
\[ \varphi' = \varphi(-2\psi_1 - \varphi - 2x) + 2(\alpha + \beta), \]
\[ \psi_1' = \psi_1(2\varphi + \psi_1 + 2x) - 4\beta, \]
which are equivalent to (4.23) with
\[ (y, p) = (\varphi, -\psi_1), \quad (\theta_0, \theta_\infty) = (-\alpha - \beta, 2\beta). \]

Secondly, if we eliminate \( \psi_1 \), we have (4.23) with
\[ (y, p) = (-\psi_0, -\varphi), \quad (\theta_0, \theta_\infty) = (\alpha + \beta, 2\beta - 1). \]

### 4.4 Weyl group symmetry for the fourth Painlevé equation

To construct a Weyl group symmetry for the similarity solution of the \( \partial NLS \) hierarchy, we examine a similarity conditions for \( s_j^{-1}g(0) \) and \( g(0)s_j \) \((j = 0, 1)\). We have
\[ [d, s_i^{-1}g(0)] = \left\{ (-\alpha - \frac{1}{2}) H_0 + \frac{h_i}{2} \right\} s_i^{-1}g(0) + \beta s_i^{-1}g(0)H_0 + \gamma s_i^{-1}g(0)K, \]
\[ [d, g(0)s_i] = \alpha H_0 g(0)s_i + g(0)s_i \left\{ (-\beta + \frac{1}{2}) H_0 - \frac{h_i}{2} \right\} + \gamma g(0)K \]
by using the relations (4.1) and
\[ [d, s_i] = \frac{1}{2} h_i s_i - \frac{1}{4} [H_0, s_i] \quad (i = 0, 1). \]

Therefore, we have two types of Weyl group actions for the parameters \( \alpha, \beta \)
\[
\begin{align*}
    s^L_0 : & \alpha \mapsto -\alpha - 1, \quad \beta \mapsto \beta, \quad \gamma \mapsto \gamma + \frac{1}{2}, \\
    s^L_1 : & \alpha \mapsto -\alpha, \quad \beta \mapsto \beta, \quad \gamma \mapsto \gamma,
\end{align*}
\]
\[
\begin{align*}
    s^R_0 : & \alpha \mapsto \alpha, \quad \beta \mapsto -\beta + 1, \quad \gamma \mapsto \gamma - \frac{1}{2}, \\
    s^R_1 : & \alpha \mapsto \alpha, \quad \beta \mapsto -\beta, \quad \gamma \mapsto \gamma.
\end{align*}
\]

Next we consider the right-action of the affine Weyl group under the similarity condition (4.1). Applying the relation (4.18) to (3.12) and (3.13), we have
\[
\begin{align*}
    (x \frac{\partial}{\partial x} + \frac{\partial}{\partial t}) \tilde{\psi}_0 &= 2xr - r' + 2(\alpha + \beta)\tilde{\psi}_0, \\
    (x \frac{\partial}{\partial x} + \frac{\partial}{\partial t}) \tilde{\psi}_1 &= -2xq - q' - 2(\alpha + \beta)\tilde{\psi}_1.
\end{align*}
\]

On the other hand, left-hand-side of these equations can be written in
\[
\begin{align*}
    (x \frac{\partial}{\partial x} + \frac{\partial}{\partial t}) \tilde{\psi}_0 &= (2\alpha + 1)\tilde{\psi}_0, \\
    (x \frac{\partial}{\partial x} + \frac{\partial}{\partial t}) \tilde{\psi}_1 &= -2\alpha\tilde{\psi}_1
\end{align*}
\]
by using (3.17), (4.7) and (4.8). Then, under the similarity condition, \( \tilde{\psi}_0 \) and \( \tilde{\psi}_1 \) can be expressed as
\[
\begin{align*}
    \tilde{\psi}_0 &= \frac{2xr - r'}{1 - 2\beta} = \frac{r\psi_0}{1 - 2\beta}, \\
    \tilde{\psi}_1 &= \frac{-2xq - q'}{2\beta} = \frac{q\psi_1}{2\beta},
\end{align*}
\]
(4.24)

We remark that in the realization (2.14), the right-action of the affine Weyl group is represented as a compatibility of the gauge transformation
\[
    s_0 Y = \left( 1 - \frac{1 - 2\beta}{r\psi_0}E_{-1} \right) Y, \quad s_1 Y = \left( 1 - \frac{2\beta}{q\psi_1}F_0 \right) Y
\]
for the linear system (4.11). This transformation is the same as the Weyl group symmetry of Painlevé type equation given by Noumi and Yamada [N].

In the level-0 realization (2.14), the action of the extended affine Weyl group can be obtained in the same manner. The matrix (3.19) satisfies the condition
\[
    [d, \pi] = -\frac{1}{4}[H_0, \pi],
\]
and then the relation
\[
    [d, \pi^{-1}g(0)\pi] = \left\{ \left( -\alpha - \frac{1}{2} \right) H_0 \right\} \pi^{-1}g(0)\pi + \pi^{-1}g(0)\pi \left\{ \left( -\beta + \frac{1}{2} \right) H_0 \right\}
\]
holds. Therefore the action of \( \pi \) for the parameters is given by
\[
    \pi : \alpha \mapsto -\alpha - \frac{1}{2}, \quad \beta \mapsto -\beta + \frac{1}{2}, \quad \gamma \mapsto \gamma.
\]

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4.5 Schlesinger transformations and discrete Painlevé equations

In the level-0 realization (2.14), we consider the local solutions of the linear system (4.11) at \( z = \infty \) and \( z = 0 \). They are obtained by the following formal serieses:

\[
Y^{(\infty)}(z; t) = g_{< 0}^{[p]}(z; t) \exp \left( (-\alpha \log z^{-1})H_0 + \sum_{n=1}^l t_n H_n \right),
\]

\[
Y^{(0)}(z; t) = g_{\geq 0}^{[p]}(z; t) \exp \left( (-\beta \log z)H_0 \right).
\]

Here \( g_{< 0}^{[p]}(z; t) \) and \( g_{\geq 0}^{[p]}(z; t) \) are the solutions of \( \partial NLS \) hierarchy with the similarity conditions (4.4) (4.5) and the set of parameters \((-\alpha, -\beta)\) corresponds to the monodromy exponents. The Schlesinger transformation relates the two solutions \( Y \) and \( Y' \) of the isomonodromy problem for the equation at hand corresponding to different sets of parameters. The change in parameters \((-\alpha, -\beta)\) are integers or half-integers.

In the case of Painlevé IV, the Schlesinger transformation can be understood in terms of the extended affine Weyl group. If we consider the transformation \( s R_1 \pi s L_1 \), the parameters \((-\alpha, -\beta)\) are transformed as

\[
(-\alpha, -\beta) \mapsto (-\alpha + \frac{1}{2}, -\beta + \frac{1}{2}) \quad (4.25)
\]

which corresponds to the Schlesinger transformation. Applying the realization (3.2), (3.19) of the extended affine Weyl group, we can describe this transformation as the compatibility condition of (4.11) with

\[
M = \begin{pmatrix} z^2 + (x + \varphi)z - \beta & -2qz + q\psi_1 \\ 2rz^2 + r\psi_0 z & -z^2 - (x + \varphi)z + \beta \end{pmatrix}
\]

and

\[
\overline{Y} = RY, \quad R = \begin{pmatrix} 0 & 0 \\ -r & 1 \end{pmatrix} z^{1/2} + \begin{pmatrix} 0 & 1/r \psi_0 \\ 0 & r/\psi_0 \end{pmatrix} z^{-1/2}.
\]

Note that \( \tilde{\psi}_0 \) is defined in (3.17). So the transformation of \( M \) is given by

\[
\overline{M} = RMR^{-1} + z \frac{\partial R}{\partial z} R^{-1}.
\]

Note that by the composition of left-action and \( \pi \), the sign of the variable \( x \) does not change. Using the relations (4.13) and (4.24), we obtain the image of \((\varphi, \psi_1)\) in terms of \((\varphi, \tilde{\psi}_1)\):

\[
\varphi = -2x - \varphi + \psi_0 + \frac{2(1 - \beta)}{\psi_0},
\]

\[
\tilde{\psi}_1 = -\psi_0.
\]
We remark that $\psi_0$ can be written in $(\varphi, \psi_1)$ by (4.19). Putting $\varphi = -2\chi_n$, $\psi_1 = -2\omega_n$ (and $\overline{\varphi} = -2\chi_{n+1}$, $\overline{\psi}_1 = -2\omega_{n+1}$) we find

$$\chi_n + \chi_{n-1} = x - \omega_n + \frac{\beta}{\omega_n};$$

$$\omega_n + \omega_{n+1} = x - \chi_n + \frac{\alpha + \beta}{2\chi_n}. $$

These equations are reduced to the discrete Painlevé I equation (1.6) by putting $X_{2n} = \omega_n$, $X_{2n-1} = \chi_n$ for $n \in \mathbb{N}$.

Note that the Schlesinger transformation to another direction represented by $s_R s_L$:

$$s_R s_L : (-\alpha, -\beta) \mapsto (-\alpha - \frac{1}{2}, -\beta + \frac{1}{2})$$

also gives the discrete Painlevé I equation for $\psi_1$ and $\varphi - 4\beta/\psi_1$.

5 Tau-functions and special solutions

In this section, we consider the basic representations of $\widehat{\mathfrak{sl}}_2$ [FK] to introduce “τ-functions”. Let $|\varpi_j\rangle$ be a highest weight vector associated with the highest weight $\varpi_j$ ($j = 0, 1$), i.e.,

$$e_i |\varpi_j\rangle = 0, \quad h_i |\varpi_j\rangle = \delta_{ij} |\varpi_j\rangle \quad (i, j = 0, 1),$$

$$f_0 |\varpi_1\rangle = f_1 |\varpi_0\rangle = 0.$$  

We denote by $L(\varpi_j)$ the basic representations with the highest weight $\varpi_j$, and by $L(\varpi_j)^\ast$ its dual space.

First we construct a realization of $L(\varpi_0) \oplus L(\varpi_1)$ on the space

$$V = C[x_1, x_2, \ldots] \otimes \left( \oplus_{n \in \mathbb{Z}} C e^{n\alpha/2} \right),$$

where $\alpha \in (\mathfrak{ch}_0 \oplus \mathfrak{ch}_1)^\ast$ satisfies $\alpha(h_0) = -2$, $\alpha(h_1) = 2$. The representation $(\rho, V)$ is given as follows:

$$\rho(H_j) (P(x) \otimes e^{n\alpha}) = \begin{cases} 
2 \frac{\partial P(x)}{\partial x_j} \otimes e^{n\alpha} & (j \geq 1), \\
2nP(x) \otimes e^{n\alpha} & (j = 0), \\
-jt_{-j} P(x) \otimes e^{n\alpha} & (j \leq -1),
\end{cases}$$

$$\rho(K) (P(x) \otimes e^{n\alpha}) = P(x) \otimes e^{n\alpha},$$

$$\rho(d) (P(x) \otimes e^{n\alpha}) = - \sum_{m=1}^{\infty} m x_m \frac{\partial P(x)}{\partial x_m} \otimes e^{n\alpha}.$$
Then the action of $E(z)$ and $F(z)$ is given by the following operators ("vertex operators"):

$$
\rho(E(z)) = e^{\eta(z,z)}e^{-2\eta(\partial_z,z^{-1})} \otimes e^{\delta z \cdot H_0},
\rho(F(z)) = e^{-\eta(z,z)}e^{2\eta(\partial_z,z^{-1})} \otimes e^{-\delta z \cdot H_0},
$$

with

$$
\eta(x,z) = \sum_{n=1}^{\infty} x^n z^n, \quad \eta(\partial_x, z^{-1}) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_x^n.
$$

The representation $(\rho, V)$ is decomposed as follows:

$$
V = V_0 \oplus V_1, \quad V_j = \mathbb{C}[x_1, x_2, \ldots] \otimes \left( \bigoplus_{n \in \mathbb{Z}} \mathbb{C} e^{(n+j)/2} \right) \quad (j = 0, 1).
$$

It is shown that each of the representation $V_j$ ($j = 0, 1$) is isomorphic to $L(\varpi_j)$ [FK], where the highest weight vector is given by $|\varpi_j\rangle = 1 \otimes e^{i\alpha/2}$.

Next we prepare several results on symmetric polynomials [M]. We denote by $p_j(z)$ the $j$-th elementary power-sum symmetric polynomial with respect to variables $z_1, \ldots, z_n$:

$$
p_j(z) = z_1^j + z_2^j + \cdots + z_n^j.
$$

The Schur polynomial $S_\lambda(x)$, labeled by the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is expressed by

$$
S_\lambda(x) = \det [s_{\lambda_i - i + j}(x)]_{1 \leq i, j \leq n},
$$

where $s_n(x)$ is the $n$-th elementary Schur polynomial defined by

$$
\exp[\eta(x, \lambda)] = \sum_{j=0}^{\infty} s_j(x) \lambda^j,
$$

and $s_n(x) = 0$ if $n < 0$.

We then introduce a scalar product in $\mathbb{C}[x_1, \ldots, x_n]$:

$$
\langle P(x), Q(x) \rangle = \frac{1}{n} \text{C.T.} \left[ P(x_j - \frac{p_j(z)}{n})Q(x_j - \frac{p_j(z)}{n}) \Delta(z) \Delta(z^{-1}) \right], \quad (5.1)
$$

where C.T.$[f(z)]$ denotes the constant term of $f(z) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ and $\Delta(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$, $\Delta(z^{-1}) = \prod_{1 \leq i < j \leq n} (z_i^{-1} - z_j^{-1})$. It is well-known that the Schur polynomials $\{S_\lambda\}$, associated with partitions $\{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \in \mathbb{Z}_{\geq 0}\}$, are pairwise orthogonal with respect to the scalar product $(5.1)$. The scalar product $(5.1)$ induces a scalar product on $V = V_0 \oplus V_1$:

$$
\langle P(x) \otimes e^{\alpha a}, Q(x) \otimes e^{\beta a} \rangle = \delta_{ab} \langle P(x), Q(x) \rangle,
$$

where $P(x), Q(x) \in \mathbb{C}[x]$ and $a, b \in \mathbb{Z}/2$. Since the Schur polynomials forms an orthogonal basis of $\mathbb{C}[x]$, an orthogonal basis of $V_0 \oplus V_1$ is given by $\{S_\lambda(x) \otimes e^{\alpha a}\}_{\lambda, a}$.

Following [BtKHM], we define $\tau$-functions associated with $g(t)$ of $(2.1)$ as

$$
\tau_n^{(j)}(t) = \langle 1 \otimes e^{(n+j)/2}, g(t)(1 \otimes e^{i\alpha/2}) \rangle
= \langle 1 \otimes e^{(n+j)/2}, \{g^{-1}_{<0}(t)\}^{-1}g^{[p]}(t)(1 \otimes e^{i\alpha/2}) \rangle
$$

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We can express $q(t), r(t)$ of (2.9) in terms of the $\tau$-functions:

$$q(t) = -\frac{\tau_1^{(0)}(t)}{\tau_0^{(0)}(t)}, \quad r(t) = -\frac{\tau_{-1}^{(1)}(t)}{\tau_0^{(1)}(t)}.$$  

As an example of concrete solutions, we construct polynomial-type $\tau$-functions, which are written in terms of the Schur polynomials. To this aim, we prepare the two lemmas:

**Lemma 2.** Let $n$ be an integer. We have

$$\rho(s_0 s_1)^n (1 \otimes e^{j\alpha/2}) = \epsilon_n (1 \otimes e^{(n+j/2)\alpha}),$$

where $j = 0, 1$ and $\epsilon_n = 1$ or $-1$ depending on the value of $n$.

*Proof.* This is a direct consequence of Lemma 12.6 of [K].

**Lemma 3.** (cf. [LY]) Let $k$ be a non-negative half-integer. We have the following expression of weight vectors:

$$\rho(e^{F_m})(1 \otimes e^{k\alpha}) = \sum_{n=0}^{2k-m} (-1)^{n/2} S_{\Box(n,2k-m-n)}(t) \otimes e^{(k-n)\alpha},$$

$$\rho(e^{E_m})(1 \otimes e^{-k\alpha}) = \sum_{n=0}^{2k-m} (-1)^{n/2} S_{\Box(2k-m-n,n)}(t) \otimes e^{(n-k)\alpha},$$

where the rectangular Young diagram $(k^n)$ is denoted by $\Box(n,k)$.

*Proof.* A proof can be given by same way as Theorem 1 of [LY]. We omit the detail.

Now we define $g_l(0)$ as

$$g_l(0) = \begin{cases} e^{f_l(s_0 s_1)^l} & (l \geq 0), \\ e^{f_0(s_0 s_1)^l} & (l < 0), \end{cases} \quad (5.2)$$

for an integer $l$. Using Lemmas 2, 3 we can calculate the corresponding $\tau$-functions explicitly:

$$\tau_n^{(0)} = \begin{cases} \epsilon_l(-1) \frac{(l-n)(l+n+1)}{2} S_{\Box(l-n,l+n)}(t) & (l \geq 0), \\ \epsilon_l(-1) \frac{(l-n)(l+n-1)}{2} S_{\Box(l-n,l+n-1)}(t) & (l < 0), \end{cases}$$

$$\tau_n^{(1)} = \begin{cases} \epsilon_l(-1) \frac{(l-n)(l+n+1)}{2} S_{\Box(l-n,l+n+1)}(t) & (l \geq 0), \\ \epsilon_l(-1) \frac{(l-n)(l+n-1)}{2} S_{\Box(l-n,l+n-1)}(t) & (l < 0). \end{cases}$$

These $\tau$-functions give rational solutions of the $\partial$NLS equation (1.3).

Furthermore, straightforward calculations show that $g_l(0)$ of (5.2) satisfies the reduction condition (4.1) with the following parameters:

$$l \geq 0 : \alpha = 0, \quad \beta = -l,$$

$$l < 0 : \alpha = -\frac{1}{2}, \quad \beta = \frac{1}{2} - l.$$
Hence we can perform the similarity reduction to the rational solutions given above and obtain rational solutions for the Painlevé IV. In this case, the Schur polynomials $p_n(t)$ are degenerated to the Hermite polynomials $H_n(t)$:

$$
\exp(zt_1 + z^2 t_2 + \cdots) \bigg|_{t_1=x, t_2=1/2, t_3=t_4=\ldots=0} = \exp(xz + z^2/2) = \sum_{n \in \mathbb{Z}} H_n(t) z^n.
$$

If we introduce discrete time evolution as (4.25),

$$
g_l(0; n) = s_1^R \pi s_1^L (g_l(0)) = s_0^{-1} \pi^{-1} (g_l(0)) \pi s_1,
$$

the corresponding rational solutions solve the discrete Painlevé equation (1.6) as discussed in the section 4.5. We remark that the rational solutions for the discrete Painlevé I (1.6) constructed in [OKS] are essentially the same as the above.

6 Concluding remarks

We have formulated the hierarchy of the $\partial$NLS equation and introduced a systematic method for similarity reductions to Painlevé-type equations. We used the fermionic representation of $\hat{sl}_2$ to construct rational solutions. We remark that the rational solutions can be expressed as ratio of Wronski-type determinants, which is discussed in [KK].

As pointed out by Okamoto [O], the fourth Painlevé equation has the Weyl group symmetry of $\tilde{W}(A_2^{(1)})$-type. Adler [A] and Noumi and Yamada [NY2] propose a new representation of $P_{IV}$, in which the $\tilde{W}(A_2^{(1)})$ symmetries become clearly visible. The Weyl group symmetry introduced in this article is isomorphic to $\tilde{W}(A_1^{(1)})$, which does not seems to be a subgroup of the $\tilde{W}(A_2^{(1)})$-symmetry discussed in [NY2, O]. To understand the relationship of our $\tilde{W}(A_1^{(1)})$-symmetry to whole symmetry of $P_{IV}$, it seems that we need to consider a larger group that contain both $\tilde{W}(A_1^{(1)})$ and $\tilde{W}(A_2^{(1)})$ as individual subgroups.

Though we limited ourselves to the $A_1^{(1)}$-case in this paper, our method may be applied to other type of affine Lie groups. For instance, in the case of $A_2^{(1)}$ non-standard hierarchy [KIK], we can obtain the fifth Painlevé equation with full parameters as a similarity reduction of the modified Yajima-Oikawa equation. We will discuss this subject elsewhere.

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