Fully Abstract and Robust Compilation
And how to reconcile the two, abstractly

Carmine Abate\textsuperscript{1}, Matteo Busi\textsuperscript{2}, and Stelios Tsampas\textsuperscript{3}

\begin{flushleft}
\textsuperscript{1} MPI-SP Bochum, Germany
carmine.abate@mpi-sp.org
\textsuperscript{2} Università di Pisa, Italy
matteo.busi@di.unipi.it
\textsuperscript{3} KU Leuven, Belgium
stelios.tsampas@cs.kuleuven.be
\end{flushleft}

\textbf{Abstract.} The most prominent formal criterion for secure compilation is \emph{full abstraction}, the preservation and reflection of contextual equivalence. Recent work introduced \emph{robust compilation}, defined as the preservation of \emph{robust satisfaction} of hyperproperties, i.e., their satisfaction against arbitrary attackers. In this paper, we initially set out to compare these two approaches to secure compilation. To that end, we provide an exact description of the hyperproperties that are robustly satisfied by programs compiled with a fully abstract compiler, and show that they can be meaningless or trivial. We then propose a novel criterion for secure compilation formulated in the framework of Mathematical Operational Semantics (MOS), guaranteeing both full abstraction and the preservation of robust satisfaction of hyperproperties in a more sensible manner.

\textbf{Keywords:} secure compilation, fully abstract compilation, robust hyperproperty preservation, language-based security, Mathematical Operational Semantics

\textbf{Remark.} To ease reading, we highlight the elements of source languages in \textcolor{blue}{blue, sans-serif}, the target elements in \textbf{red, bold} and the common ones in black \cite{37}.

\section{1 Introduction}

Due to the complexity of modern computing systems, engineers make substantial use of \emph{layered design}. Higher layers hide details of the lower ones and come with abstractions that ease reasoning about the system itself \cite{43}. A layered design of programming languages allows to benefit from modules, interfaces or dependent types of a \emph{source, high-level} language to write well-structured programs, and execute them efficiently in a \emph{target, low-level} language, after \emph{compilation}. Unfortunately, an attacker may exploit the lack of abstractions at the low-level to mount a so-called \emph{layer-below attack} \cite{43}, which is otherwise impossible at the high-level \cite{19, 20}.

\textit{Secure compilation} \cite{39} devises both principles and proof techniques to preserve the (security-relevant) abstractions of the source and prevent layer-below
attacks. Abadi \cite{abadi2001} hinted that secure compilers must respect *equivalences*, as some security properties can be expressed in terms of indistinguishability w.r.t. arbitrary attackers, or *contextual equivalence*. Fully abstract compilers preserve and reflect (to avoid trivial translations) contextual equivalence.

Two decades of successes \cite{abadi2001, abate2014, abate2013, abate2014b, abate2014c, abate2014d, abate2014e, abate2014f} made full abstraction the gold-standard for secure compilation. However, some ad-hoc examples from recent literature \cite{abate2014g, abate2014h} showed that fully abstract compilers may still introduce bugs that were not present in source programs, e.g.,

**Example 1 (See also Appendix E.5 of \cite{abate2014g}).** Consider source programs to be functions \( \mathbb{B} \to \mathbb{N} \) (from booleans to natural numbers) and target ones to be functions \( \mathbb{N} \to \mathbb{N} \). Define contextual equivalence to be equality of outputs on equal inputs. Next, identify \( \mathbb{B} \) with \( \{0, 1\} \subseteq \mathbb{N} \), and compile a program \( P \) to \( \llbracket P \rrbracket : \mathbb{N} \to \mathbb{N} \) that coincides with \( P : \mathbb{B} \to \mathbb{N} \) on \( \{0, 1\} \) and returns a default value – denoting a bug – otherwise,

\[
\llbracket P \rrbracket (n) = \begin{cases} 
P(n) & \text{for } n = 0, 1 \\
42 & \text{otherwise}
\end{cases}
\]

\( \llbracket \cdot \rrbracket \) is fully abstract, yet a source program that “never outputs 42”, will no longer enjoy this property.

This simple example underlines the fact that if a security property like “never output 42” is not captured by contextual equivalence, there is no guarantee it will be preserved by a fully abstract compiler. Abadi \cite{abadi2001} tellingly wrote

\[
[\ldots] \text{we still have only a limited understanding of how to specify and prove that a translation preserves particular security properties. [\ldots]}
\]

Abate et al. \cite{abate2014g} proposed to specify security in terms of *hyperproperties*, sets of sets of traces of observable events \cite{abate2014i}. In this setting, they consider a compiler *secure* only if it robustly preserves a class of hyperproperties, i.e., if it preserves their satisfaction against arbitrary attackers. For Example \cite{abadi2001} “never output 42” can be specified as a *safety* hyperproperty, where function inputs and outputs are the observable events. The above compiler \( \llbracket \cdot \rrbracket \) is *not* secure according to Abate et al. \cite{abate2014g}, as it does not robustly preserve the class of safety hyperproperties. More generally, each particular class of hyperproperties, e.g., the one for data integrity or the one for data confidentiality \cite{abate2014j}, determines a precise formal secure compilation criterion.

Despite the introduction of the robust criteria, full abstraction is still widely adopted \cite{abate2014g, abate2014k, abate2014l, abate2014m}, for at least two reasons. First, contextual equivalence can model security properties such as noninterference \cite{abate2014n}, isolation \cite{abate2014o}, well-bracketed control flow or local state encapsulation \cite{abate2014p} for programs that don’t expose events externally. Second, even though fully abstract compilers do not in general preserve data integrity or confidentiality, they often do so in practice.

Fully abstract and robust compilation both embody valuable notions of secure compilation and neither is stronger than the other nor are they orthogonal,
which makes us believe their relation deserves further investigation. Our goal is
to have criteria with well understood security guarantees for compiled programs,
so that both users and developers of compilers may decide which criterion better
fits their needs. For that, we assume an abstract trace semantics, collecting ob-
servables events and internal steps, is given for both source and target languages,
and start our quest not by asking if a given fully abstract compiler preserves all
hyperproperties, but which ones do and which ones do not preserve.

Contributions. First, we make explicit the guarantees given by full abstraction
w.r.t. arbitrary source hyperproperties. We achieve this by showing that for ev-
every fully abstract compiler \([\cdot]\), there exists a translation or interpretation of
source hyperproperties into target ones, \(\tilde{\tau}\), such that if \(P\) robustly satisfies a
source hyperproperty \(H\), \([P]\) robustly satisfies \(\tilde{\tau}(H)\) (Theorem 1). However, we
observe that a fully abstract compiler may fail to preserve the robust satisfac-
tion of some hyperproperty, as \(\tilde{\tau}\) may map interesting hyperproperties to trivial
ones (Example 2). We then provide a sufficient and necessary condition to pre-
serve the robust satisfaction of hyperproperties (Corollary 1), but argue that it
is unfeasible to be proven true for an arbitrary fully abstract compiler. To over-
come the above issues, we introduce a novel criterion, that we formulate in the
abstract framework of Mathematical Operational Semantics (MOS). We show
that our novel criterion implies full abstraction and the preservation of robust
satisfaction of arbitrary hyperproperties (Section 5). We illustrate effectiveness
and realizability of our criterion in Example 3.

2 Fully abstract and robust compilation

Let us briefly recall the intuition of fully abstract and robust compilation, and
provide their rigorous definitions. We refer the interested reader to [3, 4, 39] for
more details.

2.1 Fully abstract compilation

Abadi [1] proposed fully abstract compilation to preserve security properties
such as confidentiality and integrity when these are expressed in terms of indis-
tinguishability w.r.t. the observations of arbitrary attackers, the latter modeled
as execution contexts. For a concrete example, if no source context \(C_S\) can dis-
tinguish a program \(P_1\) that uses some confidential data \(k\) from a program \(P_2\)
that does not, we can deduce that \(k\) is kept confidential by \(P_1\). Thus, a compiler
\([\cdot]\) from a source language to a target one, that aims to preserve confidentiality,
must ensure that also \([P_1]\) and \([P_2]\) are equivalent w.r.t. the observations of
any target context \(C_T\). To avoid trivial translations, one typically asks for the
reflection of the equivalence as well.

Definition 1 (Fully abstract compilation [1]). A compiler \([\cdot]\) is fully ab-

\[ (\forall C_S. C_S[P_1] \approx C_S[P_2]) \leftrightarrow (\forall C_T. C_T[[P_1]] \approx C_T[[P_2]]) \]
where \( C_S, C_T \) denote source and target contexts resp., \( \approx, \approx \) denote the two contextual equivalences, i.e., equivalence relations on programs.

Notice that the security notions one can preserve and reflect with a fully abstract compiler are those captured by the contextual equivalence relation \( \approx \), that determines both the meaningfulness and the effectiveness of full abstraction. Indeed, if \( \approx \) is too coarse-grained, some interesting security properties may be ignored. Dually, if \( \approx \) is too fine-grained, equivalent source programs may not have counterparts that are equivalent in the target. In Section 3, we pick \( \approx \) to be equality of execution traces which, under mild assumptions \([22, 32]\), coincides with other common choices of \( \approx \) (see also Section 6).

### 2.2 Robust compilation

Abate et al. \([4]\) suggest a family of secure compilation criteria that depend on the security notion one is interested in preserving. The key idea in their criteria is the preservation of robust satisfaction, i.e., satisfaction of (classes of) security properties against arbitrary attackers, modeled as contexts. More concretely, Abate et al. \([3, 4]\) assume that every execution of a program exposes a trace of observable events \( t \in \text{Trace} \) for a fixed set \( \text{Trace} \) and model interesting security notions like data integrity, confidentiality or observational determinism as sets of sets of traces, i.e., hyperproperties denoted by \( H \in \wp(\wp(\text{Trace})) \) \([15]\).

**Definition 2 (Robust satisfaction \([3, 4]\)).** A program \( P \) robustly satisfies a hyperproperty \( H \) iff \( \forall C. C \strut_P \models H \), where \( C \strut_P \models H \equiv \text{beh}(C \strut_P) \in H \) and \( \text{beh}(C \strut_P) \) is the set of all traces that can be observed when executing \( C \strut_P \).

Secure compilation criteria can then be defined as the preservation of robust satisfaction of classes of hyperproperties such as safety or liveness \([4]\), in this paper we consider the class of all hyperproperties and robust hyperproperty preservation \((\text{RHP}^\tau \text{ from } [3])\). For that, consider a function \( \tau \) that takes a source-level hyperproperty and returns its interpretation (or translation) at the target level. Intuitively, a compiler \( \llbracket \cdot \rrbracket \) is \( \text{RHP}^\tau \) if, for any source hyperproperty \( H \) robustly satisfied by \( P \), its interpretation \( \tau(H) \) is robustly satisfied by \( \llbracket P \rrbracket \), formally:

**Definition 3 (Robust hyperproperty preservation).** A compiler \( \llbracket \cdot \rrbracket \) preserves the robust satisfaction of hyperproperties according to a translation \( \tau : \wp(\wp(\text{Trace})) \rightarrow \wp(\wp(\text{Trace})) \) iff the following \( \text{RHP}^\tau \) holds:

\[
\text{RHP}^\tau \equiv \forall P \forall H \in \wp(\wp(\text{Trace})). (\forall C_S. C_S \strut_P \models H) \Rightarrow \\
(\forall C_T. C_T \strut_T \models H) \Rightarrow \tau(H)
\]

when \( \tau \) is clear from the context we simply say that \( \llbracket \cdot \rrbracket \) is robust.

\( \text{RHP}^\tau \) can be formulated without quantification on hyperproperties \([3, 4]\).

**Lemma 1 (Property-free \( \text{RHP}^\tau \)).** For a compiler \( \llbracket \cdot \rrbracket \), \( \text{RHP}^\tau \) is equivalent to

\[\forall P \forall C_T \exists C_S. \text{beh}^T(C_T \strut_T) = \tau(\text{beh}^S(C_S \strut_P))\]

\( \tau(\text{beh}^S(C_S \strut_P)) \) is a shorthand for \( \tau(\{\text{beh}^S(C_S \strut_P)\}) \).
Notice that, while Definition 3 describes — through $\tau$ — the target guarantees for $\llbracket P \rrbracket$ against arbitrary target contexts, Lemma 1 enables proofs by back-translation. In fact, similarly to fully abstract compilation [39], one can prove that a compiler is $\text{RHP}^\tau$ by exhibiting a so-called back-translation map producing a source context $C_S$ whose interaction with $P$ exposes “the same” observables as $C_T$ does with $\llbracket P \rrbracket$.

Remark 1 (RHP$^\tau$ by back-translation). RHP$^\tau$ holds if there exists a back-translation function $bk$ such that for any $C_T$ and any $P$, $bk(C_T(\llbracket P \rrbracket)) = C_S$ is such that $\text{beh}_T(C_T(\llbracket P \rrbracket)) = \tau(\text{beh}_S(C_S[P]))$.

3 Comparing FAC and RHP$^\tau$

In the previous section we defined fully abstract compilation as the preservation and reflection of contextual equivalence, i.e., what the contexts can observe about programs. Instead, $\text{RHP}^\tau$ was defined as the preservation of (robust satisfaction of) hyperproperties of externally observable traces of events. To enable any comparison, we first provide an intuition on how to accommodate the mismatch in observations between full abstraction and $\text{RHP}^\tau$ (see Appendix A.2 for all the details). We assume the operational semantics of our languages exhaustively specify the execution of programs in contexts, including both internal steps and steps that expose externally observable events like inputs and outputs. Also, we say that a trace is abstract if it collects both internal steps and externally observable events. In a slight abuse of notation, we still denote with $\text{beh}(\llbracket P \rrbracket)$ the set of all the possible abstract traces allowed by the semantics when executing $P$ in $C$. Moreover, since hyperproperties just express predicates over events, we now write $\text{beh}(C[P]) \in H$ to mean that the traces of events for $C[P]$ satisfy the hyperproperty $H$. Finally, we elect to express contextual equivalence as the equality of the (sets of) abstract traces in an arbitrary context.

Definition 4 (Equality of $\text{beh}(\cdot)$). For programs $P_1, P_2$ and a context $C$,

$$C[P_1] \approx C[P_2] \iff \text{beh}(C[P_1]) = \text{beh}(C[P_2])$$

In Section 4 we discuss other common choices for $\approx$ such as equi-termination, and the hypotheses under which they are equivalent to ours. We now instantiate Definition 1 on the contextual equivalence from Definition 4 and make explicit the notion of fully abstract compilation we are going to use from now on. Note how we are only interested in the preservation of contextual equivalence, as reflection is often subsumed by compiler correctness (e.g., in absence of internal non-determinism) [1, 39].

Definition 5 (FAC). For a compiler $\llbracket \cdot \rrbracket$, FAC is the following predicate

$$\text{FAC} \equiv \forall P_1 P_2. \forall C_S. \text{beh}_S(C_S[P_1]) = \text{beh}_S(C_S[P_2]) \Rightarrow \forall C_T. \text{beh}_T(C_T(\llbracket P_1 \rrbracket)) = \text{beh}_T(C_T(\llbracket P_2 \rrbracket))$$
Abate et al. [4], Patrignani and Garg [41] have previously investigated the relation between FAC as in Definition 5 and RHP. In particular, Abate et al. [4] showed that FAC does not imply any of the robust criteria, with an example similar to the one we sketched in Section 1. In this section, we provide further evidence of this fact: a fully abstract compiler that does not preserve the robust satisfaction of a security-relevant hyperproperty, namely noninterference. More details on the example can be found in Appendix A.1.

Example 2. Let Source and Target to be two WHILE-like languages [32] with a mutable state. A state \( s \in S \triangleq (\text{Var} \rightarrow \mathbb{N}) \) assigns every variable \( v \in \text{Var} \) a natural number. We assume \( \text{Var} \) to be partitioned into a “high” (private) and a “low” (public) part. We write \( v \in \text{Var}_H \) (\( v \in \text{Var}_L \), resp.) to denote that the variable \( v \) is private (public, resp.). Partial programs are defined in the same way in both Source and Target, whereas whole programs, or terms, are obtained by filling the hole(s) of a context with a partial program (Figure 1).

The only context in Source is \([\cdot]\), called the identity context and such that for any \( P \), \( [P] = P \). Instead, contexts in Target additionally include \([\cdot]\) that is able to observe the internal event \( H \) (intuitively, a form of information leakage that is not observed by source contexts) and report it by emitting \( ! \).

\[
\langle P \rangle ::= \text{skip} \mid v := \langle \text{expr} \rangle \mid \langle P \rangle ; \langle P \rangle \mid \text{while} \langle \text{expr} \rangle \langle P \rangle
\]

\[
\langle C_S \rangle ::= [\cdot] \quad \langle C_T \rangle ::= [\cdot] \mid [\cdot]
\]

Fig. 1. \( \langle P \rangle \) defines the syntax of both Source and Target partial programs, where \( \langle \text{expr} \rangle \) denotes the usual arithmetic expressions over \( \mathbb{N} \). \( \langle C_S \rangle \) and \( \langle C_T \rangle \) define instead the contexts of Source and Target, respectively.

The semantics of Source and Target are partially given in Figure 2. Rule asnL is for assignments that do not involve high variables. asnH is for assignments of high variables, and – upon a change in their value – the internal trace \( H \) is emitted. The Target counterparts, asnL and asnH, work similarly. Finally, the most interesting rule is bang2, where we see how context \([\cdot]\) reports a \( ! \) upon encountering an \( H \).

\[
\begin{align*}
\text{asnL} & \quad v \in \text{Var}_L \\
& \quad s, v := e \rightarrow s[\{v \leftarrow e\}], \checkmark \\
\text{asnH} & \quad v \in \text{Var}_H \\
& \quad s, v := e \xrightarrow{H} s[\{v \leftarrow e\}], \checkmark \\
\text{bang2} & \quad s, p \xrightarrow{H} s', p' \\
& \quad s, [\cdot] \xrightarrow{!} s', p'
\end{align*}
\]

Fig. 2. Selected rules of Source and Target.

For example, consider a high variable \( v \in \text{Var}_H \) and the Source program \( P \triangleq v := 42 \). When \( P \) is plugged in the identity context \([\cdot]\), the resulting behavior is \( \text{beh}_S([P]) = \{ s \cdot H \cdot s' \cdot \checkmark \mid s \in S \land s' = s[\{v + 42\}] \} \). Intuitively, the traces in \( \text{beh}_S([P]) \) express that the execution starts in a state \( s \), then a high
variable is updated (\(\mathcal{H}\)) leading to state \(s'\) and then the program terminates (✓). For the same \(v \in \text{Var}_H\), target program \(P \triangleq v := 42\) in \([\cdot]\), we have that \(\text{beh}_T([P]) = \{ s \cdot ! \cdot s' \cdot ✓ \mid s \in S \land s' = s_{[v \leftarrow 42]} \}\). Notice the additional ! w.r.t. the source, due to the fact that the context observed a change in a high variable. Informally, we say that a program satisfies noninterference if, executing it in two low-equivalent initial states, it transitions to two low-equivalent states. More rigorously, noninterference can be defined for both Source and Target as the following hyperproperty \(NI \in \varphi(\varphi(\text{Trace}))\),

\[
NI = \{ \pi \in \varphi(\text{Trace}) \mid \forall t_1, t_2 \in \pi. \ t_1^0 =_L t_2^0 \Rightarrow t_1 =_L t_2 \}
\]

where \(t_i^0\) stands for the first observable of the trace \(t_i\) and \(=_L\) denotes the fact that two states are low-equivalent (i.e., they coincide on all \(x \in \text{Var}_L\)). Also, we lift the notation to traces and write \(t_1 =_L t_2\) to denote that \(t_1\) and \(t_2\) are pointwise low-equivalent. More precisely, \(=_L\) ignores all occurrences of \(\mathcal{H}\) (as it is internal) and compares traces observable-by-observable, relating ✓ and ! to themselves and comparing states with the above notion of low-equivalence.

The identity compiler preserves trace equality (see Lemma 3 for the proof), but does not preserve the robust satisfaction of noninterference as the Target context \(\lceil \cdot \rceil\) can detect changes in high variables and report a !.

On the one hand, \(\text{RHP}^\tau\) provides an explicit description of the target hyperproperty \(\tau(\mathcal{H})\) that is guaranteed to be robustly satisfied after compilation under the hypothesis that \(\mathcal{H}\) is robustly satisfied in the source. However, \(\text{RHP}^\tau\) does not imply the preservation of contextual equivalence (or trace equality) because hyperproperties cannot specify which traces are allowed for every single context. On the other hand, it is possible that \(\text{FAC}\) does not preserve (the robust satisfaction of) hyperproperties, because contextual equivalence may not capture some hyperproperty such as noninterference, as shown in Example 2. So, what kind of hyperproperties a FAC compiler is guaranteed to preserve? If \(P\) robustly satisfies \(\mathcal{H}\) (possibly not captured by \(\approx\)), what is the hyperproperty that is robustly satisfied by \([P]\) for \([\cdot]\) being FAC?

We answer this question by defining a map \(\tilde{\tau} : \varphi(\varphi(\text{Trace}_S)) \rightarrow \varphi(\varphi(\text{Trace}_T))\) so that \(\text{FAC}\) implies \(\text{RHP}^{\tilde{\tau}}\). The map \(\tilde{\tau}\) enjoys an optimality condition making it the best possible description of the target guarantee for programs compiled by a FAC compiler.

**Theorem 1.** If \([\cdot]\) is FAC, then there exists a map \(\tilde{\tau}\) such that \([\cdot]\) is \(\text{RHP}^{\tilde{\tau}}\). Moreover, \(\tilde{\tau}\) is the smallest (pointwise) with this property.

To avoid any misunderstanding, we stress the fact that, akin to [36, Theorem 1], neither the existence, nor the optimality of \(\tilde{\tau}\) can be used to argue that a FAC compiler \([\cdot]\) is reasonably robust. The robustness of \([\cdot]\) depends on the image of \(\tilde{\tau}\) on the hyperproperties of interest: it should not be trivial, e.g., \(\tilde{\tau}(\text{NI}_S) = \top\) like in Example 2 nor distort the intuitive meaning of the hyperproperty itself, e.g., \(\tilde{\tau}(\text{NI}_S) = \text{“never output 42”}\). In a setting in which observables are coarse enough to be common to source and target traces, i.e., \(\text{Trace}_S = \text{Trace}_T\), it is possible to establish whether \(\tilde{\tau}(\mathcal{H})\) has “the same meaning” as \(\mathcal{H}\):
Corollary 1. If $\llbracket \cdot \rrbracket$ is FAC, then for every hyperproperty $H$, $\llbracket \cdot \rrbracket$ preserves the robust satisfaction of $H$ iff $\tilde{\tau}(H) \subseteq H$, where $\tilde{\tau}$ is the map from Theorem 1.

The rigorous definition of $\tilde{\tau}$ and the proof of Theorem 1 and Corollary 1 can be found in Appendix A. Here, we only mention that the definition of $\tilde{\tau}$ requires information on the compiler itself, thus it can be unfeasible to compute and assess the meaningfulness of $\tilde{\tau}(H)$. Corollary 1 partially mitigates this problem by allowing to approximate $\tilde{\tau}(H)$ rather than computing it, e.g., by showing an intermediate $K$ such that $\tilde{\tau}(H) \subseteq K \subseteq H$. We leave as future work any approximation techniques for $\tilde{\tau}$ that would make substantial use of Corollary 1.

To overcome the issues highlighted above, we extend the categorical approach to secure compilation of Tsampas et al. [52] and propose an abstract criterion that implies both FAC and RHP$_\tau$ for a $\tau$ defined via co-induction and therefore independent of the compiler. In Section 4 we shall summarize the underlying theory before introducing our criterion in Section 5.

4 Secure compilation, categorically

The basis of our approach and that of Tsampas et al. [52] is the framework of Mathematical Operational Semantics (MOS) [54]. Here, we briefly explain how MOS gives a mathematical description of programming languages as well as (secure) compilers and show how our earlier Example 2 fits such a framework. We refer the interested reader to the seminal paper of Turi and Plotkin [54] and the excellent introductory material of Klin [27] for more details. Further examples and applications can be found in the literature [52, 53, 55].

4.1 Distributive laws and operational semantics

The main idea of MOS is that the semantics of programming languages, or systems in general, can be formally described through distributive laws (i.e., natural transformations of varying complexity) of a syntax functor $\Sigma$ over a behavior functor $B$ in a suitable category (in our case the category Set of sets and total functions [27]). The functor $\Sigma : \text{Set} \to \text{Set}$ represents the algebraic signature of the language and thus acts as an abstract description of its syntax. Instead, the functor $B : \text{Set} \to \text{Set}$ describes the behavior of the language in terms of its observable events (e.g., the behavior of a non-deterministic labeled transition system can be modeled by the functor $BX = \wp(X)^\Delta$, where $\Delta$ is a set of trace labels [56]).

Recall now the languages Source and Target of Example 2. The syntax functor for Source for a set of terms $X$ builds terms $\Sigma X$ according to (the sum of all) the constructors of the language:

$$\Sigma X \triangleq \top \uplus (N \times E) \uplus (X \times X) \uplus (E \times X),$$

where $E$ is the set of arithmetic expressions. The behavior functor for Source is a map that for an arbitrary set $X$, updates a store $s \in S$, and either terminates.
(✓) or returns another term in $X$, possibly recording that some high-variable
has been modified ($\mathcal{H}$):

$$B \times X \triangleq (S \times (\text{Maybe } \mathcal{H}) \times (X \uplus \checkmark))^S.$$  

In Target, the syntax functor is $\Sigma X = \Sigma X \uplus X$, where the extra occurrence of
$X$ corresponds to the target context $[\cdot]$, and $B \times X \triangleq (S \times (\text{Maybe } (\mathcal{H} \uplus !)) \times (X \uplus \checkmark))^S$. We explicitly notice that syntactic “holes” are represented by the identity
functor $\text{Id}_X = X$ and, to make this connection clearer, the syntax functor for Source can be equivalently written as $\Sigma \triangleq \top \uplus (N \times E) \uplus (\text{Id} \times \text{Id}) \uplus (E \times \text{Id})$.

Next, we can define the operational semantics, a distributive law of $\Sigma$ over $B$, in the format of a GSOS law [27, Section 6.3]. A GSOS law of $\Sigma$ over $B$ is
a natural transformation $\rho : \Sigma (\text{Id} \times B) \Rightarrow B \Sigma^*$, where $\Sigma^*$ is the free monad
over $\Sigma$.

For instance, the rules of sequential composition in Source (see seq1 and seq2 in Figure I) correspond to the following component of the GSOS law $\rho : \Sigma (\text{Id} \times B) \Rightarrow B \Sigma^*$:

$$\begin{aligned}
(p, f) : (q, g) \mapsto & \lambda s. \begin{cases} 
(s', \delta, p' ; q) & \text{if } f(s) = (s', \delta, p') \\
(s', \delta, q) & \text{if } f(s) = (s', \delta, \checkmark) 
\end{cases}
\end{aligned}$$

Here, $p, q \in X$ with $X$ a generic set of terms, i.e., $p$ and $q$ can be programs,
contexts or programs within a context, and $f, g \in B_X$. The image of $\rho$ is an
element of $B \Sigma^* X = (S \times (\text{Maybe } \mathcal{H}) \times (\Sigma^* X \uplus \checkmark))^S$, depending on whether $p$
transitions to a term $p'$ (thus involving seq2), or terminates with $\checkmark$ (seq1).

Lastly, we informally recall that when the formal semantics of a language is
given through a GSOS law $\rho : \Sigma (\text{Id} \times B) \Rightarrow B \Sigma^*$, for $\Sigma, B : \text{Set} \rightarrow \text{Set}$, the
set of programs is (isomorphic to) the initial algebra $A = \Sigma^* \emptyset$, while the final
coa lgebra $Z = B^\infty$ describes the set of all possible behaviors.

Remark 2. A distributive law $\rho$ induces a map $f : A \rightarrow Z$ that assigns to every
closed term or program its behaviors as specified by the law $\rho$ itself.

For Source and Target from Example 2 $f$ and $f$ are just another, equivalent
representation of $\text{beh}_S (\cdot)$ and $\text{beh}_T (\cdot)$, e.g., for $v$ private variable,

$$f([v := 42]) = \lambda s. \langle s|_{x := 42}, (\mathcal{H}, \checkmark) \rangle$$

$$f([v := 42]) = \lambda s. \langle s|_{x := 42}, (\checkmark) \rangle$$

In other words, map $f : A \rightarrow Z$ is the abstract counterpart of map $\text{beh} (\cdot)$ that
assigns to every program the set of all its possible execution traces.

4.2 Maps of distributive laws as fully abstract compilers

Watanabe [55] first introduced maps of distributive laws (MoDL) as well-behaved
translations between two GSOS languages. Tsampas et al. [52] showed how
MoDL can also be used as a formal, abstract criterion for secure compilation.
Let us recall the definition of MoDL for two GSOS laws in the same category.

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$\Sigma^*$ is the free monad over $\Sigma$ and $B^\infty$ is the co-free comonad over $B$ [27, Ch. 5].
Definition 6 (MoDL). A map of distributive law between \( \rho : \Sigma(\text{Id} \times B) \Rightarrow B\Sigma^* \) and \( \rho : \Sigma(\text{Id} \times B) \Rightarrow B\Sigma^* \) is a pair of natural transformations \( s : \Sigma \Rightarrow \Sigma^* \) and \( b : B \Rightarrow B \) such that the following diagram commutes,

\[
\begin{array}{c}
\Sigma(\text{Id} \times B) \\
\downarrow \quad \downarrow \rho \\
\Sigma(\text{Id} \times B) \\
\end{array}
\begin{array}{c}
\rightarrow B\Sigma^* \\
\downarrow b \circ B \circ s^* \\
\rightarrow B\Sigma^* \\
\end{array}
\]

where \( s^* : \Sigma^* \Rightarrow \Sigma^* \) extends \( s : \Sigma \Rightarrow \Sigma^* \) to a morphism of free monads, i.e., to terms of arbitrary depth via structural induction.

The diagram in Definition 6 expresses a form of compatibility of the source and the target semantics. Considering any source term, executing it w.r.t. the source semantics \( \rho \) and then translating the behavior (together with the resulting source term) is equivalent to first compiling the source term (and translating the behavior of its subterms) and then executing it w.r.t. the target semantics \( \rho \).

We recall that the set of source (resp. target) programs is \( A \triangleq \Sigma^* \) (resp.), and that \( \llbracket \cdot \rrbracket \triangleq s^*_\emptyset : A \rightarrow A \) is the compiler induced by \( s \). On the behaviors side, the natural transformation \( b : B \Rightarrow B \) induces a translation of behaviors \( d := b^\infty \circ Z \rightarrow Z \) where \( Z \triangleq B^\infty \uplus \top \). The compiler \( \llbracket \cdot \rrbracket = s^*_\emptyset \) preserves (and also reflects when all the components of \( b \) are injective) bisimilarity (see [52], Section 4.3). Whenever bisimilarity coincides with trace equality (see Definition 4), for example under the assumption of determinacy \( \sigma \), the following holds ([52]).

Corollary 2. In absence of internal non-determinism, MoDL implies FAC.

Similarly to FAC, the definition of MoDL does not ensure that \( \llbracket \cdot \rrbracket = s^*_\emptyset \) is robust. Indeed, the obvious embedding compiler from Example 2 is a MoDL (let \( s = i_1 \) and \( b = (S \times (\text{Maybe } i_1) \times (1 \uplus \top))^S \)). Intuitively, MoDL adequately captures the fact that compilation preserves the behavior of terms, but fails to capture the observations target contexts can make on compiled terms.

5 Reconciling fully abstract and robust compilation

To account for the shortcoming of MoDL, we introduce a new, complementary definition that allows reasoning explicitly on the semantic power of contexts in some target language relative to contexts in a source language. This definition acts (in conjunction with MoDL) as an abstract criterion of robust compilers.

For the new definition, we elect to qualify some constructors in \( \Sigma \) as contexts constructors so that \( \Sigma \triangleq \mathcal{C} \cup \mathcal{P} \) where \( \mathcal{C} \) defines the constructors for contexts and \( \mathcal{P} \) for all the rest. We also assume that the GSOS law \( \rho : \Sigma(\text{Id} \times B) \Rightarrow \Sigma^*B \) respects this “logical partition” of \( \Sigma \) in that \( \rho = [B \ i_1 \circ \rho_1, \ \rho_2] \) where \( \rho_1 : \mathcal{C}(\text{Id} \times B) \Rightarrow B\mathcal{C}^* \) and \( \rho_2 : \mathcal{P}(\text{Id} \times B) \Rightarrow B\Sigma^* \).

It is possible to eliminate the hypothesis of determinacy when \( B \) is an endofunctor over categories richer that \textbf{Set}, e.g., \textbf{Rel} the category of sets and relations.

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6 It is possible to eliminate the hypothesis of determinacy when \( B \) is an endofunctor over categories richer that \textbf{Set}, e.g., \textbf{Rel} the category of sets and relations.
**Definition 7 (MMoDL).** A many layers map of distributive laws (MMoDL) between \( \rho : \Sigma(\text{Id} \times B) \rightarrow B \times \Sigma^* \) and program \( \rho : \Sigma(\text{Id} \times B) \rightarrow B \Sigma^* \) is given by natural transformations \( b : B \Rightarrow B \) and \( t : \mathcal{E} \Rightarrow \mathcal{E}^* \) making the following commute:

\[
\begin{array}{c}
\mathcal{C}(\text{Id} \times B) \\
\downarrow^{\mathcal{C}(\Sigma_1, \rho)} \\
\mathcal{C}(\text{Id} \times B) \Sigma^* \\
\downarrow^{\mathcal{C}(\Sigma_1, \rho)} \\
\mathcal{C}(\text{Id} \times B) \Sigma^* \\
\mathcal{C}(\text{Id} \times B) \Sigma^* \\
\end{array}
\]

The top-left object, \( \mathcal{C}(\text{Id} \times B) \), represents a target context which is filled with some source term, whose subterms exhibit some source behavior. In both paths, the plugged source terms are initially evaluated w.r.t. the source semantics. On the upper path, we first back-translate \( \mathcal{C}(\text{Id} \times B) \) to the target context using \( t \), then we run the resulting program w.r.t. the source semantics \( (\rho_1) \), and finally we translate the resulting behavior back to the target via \( b \). Instead, in the lower path we first translate the resulting behavior through \( \mathcal{C}(\text{Id} \times b) \), then we let the target context observe \( (\rho_1) \), and finally we back-translate the behavior via \( B \times \).

To relate MMoDL with RHP\(^T\), we formulate the latter in the framework of MOS. Recall (see Remark [1]) that RHP\(^T\) holds if there exists a back-translation map \( bk \) that for every target context \( C_T \) and program \( P \), produces a source context \( bk(C_T, P) = C_s \) such that \( \text{beh}_T(C_T[\llbracket P \rrbracket]) = \tau(\text{beh}_S(C_s[\llbracket P \rrbracket])) \).

**Remark 3 ((Abstract) RHP\(^T\)).** For \( \tau : Z \rightarrow Z \), a compiler \( [\cdot] \) is RHP\(^T\) iff there exists \( bk \) such that

\[
\tau \circ f \circ \text{plug} \circ bk = f \circ \text{plug} \circ id \times [\cdot],
\]

where \( f : A \rightarrow Z \) associates to every program its behaviors as specified by \( \rho \) (see Remark [2]) and \( \text{plug} \) is the operation of plugging a term into a context.

We are now ready to state our second contribution, namely that the pairing of a MoDL \( (s, b) \) and a MMoDL \( (t, b) \) gives an (abstract) RHP\(^T\) compiler.

**Theorem 2 (MMoDL imply RHP\(^T\)).** Let \( s : \Sigma \Rightarrow \Sigma^* \), \( b : B \Rightarrow B \) and \( t : \mathcal{E} \Rightarrow \mathcal{E}^* \) such that \( (s, b) \) and \( (t, b) \) are respectively a MoDL and a MMoDL from \( \rho : \Sigma(\text{Id} \times B) \Rightarrow B \Sigma^* \) to \( \rho : \Sigma(\text{Id} \times B) \Rightarrow B \Sigma^* \). The compiler \( [\cdot] = s^* \) is (abstract) RHP\(^T\) for \( \tau = b^* \) coinductively induced by \( b \).

**Proof (Sketch).** The back-translation \( bk := t^* \times id \) satisfies the equation in Remark [3] (details in Appendix D.2). \( \square \)

Before fixing the compiler from Example [2] to make it satisfy both MoDL (Definition [1]) and MMoDL (Definition [7]), let us see why the back-translation mapping both target contexts to the identity source context \( [] \) is not a MMoDL. Let \( v \in \text{Var}_H \) be a private variable, on the upper path of Definition [7] we have:

\[
\begin{array}{c}
[v := 42] \\
\downarrow^{\mathcal{C}(\Sigma_1, \rho)} \\
[\checkmark], \lambda s.(s[v := 42], \mathcal{H}) \\
\downarrow^t \\
[\checkmark], \ldots \mathcal{H} \\
\downarrow^{\rho_1} \\
\checkmark, \ldots \mathcal{H} \\
\downarrow^b \\
\checkmark, \ldots \mathcal{H}
\end{array}
\]

Note how the identity context fails to report \( ! \). On the lower path, we have instead:
\[ v := 42 \quad \varepsilon^*(\Sigma_\pi, \rho) \xrightarrow{\text{\[\_\_\_\_\_\_]}} [\checkmark], \lambda_* (s_{[v := 42]}, H) \xrightarrow{\varepsilon (\text{Id} \times b)} [\checkmark], \ldots H \xrightarrow{\rho_1} \checkmark, \ldots ! \xrightarrow{\text{Bt}^*} \checkmark, \ldots ! \]

Here, it is evident that the context \[\_\_\_\_\_\_\_] \text{“picks up”} H \text{ and reports !, unlike } [\_\_\_\_\_\_] .

**Example 3 (Example 2, revisited).** We now show how to fix the compiler from Example 2 by making it \text{RHP}^\tau \text{ for a suitable } \tau . For that, we first need to slightly modify the language \text{Target}. The idea is that variable assignments in \text{Target} should now be \text{sandboxed, so that the interactions with the context } [\_ \_ ] \text{ do not expose sensitive information. Formally, we extend the algebraic signature of } \text{Target with a constructor for sandboxing assignments, i.e., } \Sigma \triangleright (E \times \text{Id}), \text{ so that } \text{Target terms are generated by grammar}

\[
\langle P \rangle := \text{skip} \mid v := \langle \text{expr} \rangle \mid \langle P \rangle \langle P \rangle \mid \text{while} \ (\text{expr}) \langle P \rangle \mid \mid v := \langle \text{expr} \rangle ]
\]

where the semantics of \[\_ \_\_\_\_\_\_\] is described in Figure 3. We can now define the new

\[
\begin{array}{c}
sb1 \quad s, p \xrightarrow{H} s', \checkmark \\
sb2 \quad s, [ p ] \rightarrow s', \checkmark
\end{array}
\]

(i.e., fixed) compiler \[\_\_\_\_\_\_\_] and the appropriate map \(\tau\), so that \[\_\_\_\_\_\_\_] is \text{RHP}^\tau . Both \[\_\_\_\_\_\_\_] and \(\tau\) are determined by the natural transformations \(s\), \(t\), and \(b\), such that \((s, b)\) is a MoDL and \((t, b)\) is a MMoDL. The natural transformation \(s : \Sigma \Rightarrow (\Sigma \triangleright (E \times \text{Id}))^*_1\), and therefore the inductively defined compiler \[\_\_\_\_\_\_\_\_\_] \triangleq s_0^*, \text{ wraps assignments in the sandbox } [\_ \_ \_ \_ ] \text{, i.e., } [v := e] = [v := e] \text{ and acts as the identity on other terms. The natural transformation } t : \mathcal{C} \Rightarrow \mathcal{C}^* \text{ maps every } \text{Target context to the identity context } [\_ \_ \_ \_ ] . \text{ Finally, the translation of behaviors } b : B \Rightarrow B \text{ erases the occurrences of } H, \text{ implying that the compiled terms are not expected to report changes in high variables.}

Recall that the diagram from Definition 4 failed to commute for Example 2 because \((s, b)\) being a MoDL imposed \(b\) to not erase any occurrences of \(H\). The same diagram for the new \text{Target} language and natural transformations \(s\), \(b\), and \(t\) now commutes. More specifically, in the upper path we have

\[
[ v := 42 ] \xrightarrow{\varepsilon^*(\Sigma_\pi, \rho)} [\checkmark], \lambda_* (s_{[v := 42]}, H) \xrightarrow{\varepsilon (\text{Id} \times b)} [\checkmark], \ldots H \xrightarrow{\rho_1} \checkmark, \ldots ! \xrightarrow{\text{Bt}^*} \checkmark, \ldots !
\]

while in the lower path we get

\[
[v := 42] \xrightarrow{\varepsilon^*(\Sigma_\pi, \rho)} [\checkmark], \lambda_* (s_{[v := 42]}, H) \xrightarrow{\varepsilon (\text{Id} \times b)} [\checkmark], \ldots \xrightarrow{\rho_1} \checkmark, \ldots \xrightarrow{\text{Bt}^*} \checkmark, \ldots !
\]

We point the reader interested to Appendix B for more details in showing that the above \((s, b)\) is a MoDL and that \((t, b)\) is a MMoDL.

Hereafter, we discuss one of the benefits of the abstract definitions presented so far, namely that we can easily compute \(\tau\), and immediately establish if programs that robustly satisfy \(\text{Nl}_\Sigma\) (noninterference in \text{Source}) are compiled to programs that robustly satisfy \(\text{Nl}_\tau\). In order to do so, we need to connect \(Z\) and \(Z\) to traces and hyperproperties of \text{Source} and \text{Target}. Elements of \(Z\) are functions that assign to every \(s \in S\) a new state \(s'\) and \text{maybe an extra symbol like...}
or $\mathcal{H}$, and a continuation, i.e., another function of the same type. Traces are instead sequences of stores possibly exhibiting the extra symbols $\mathcal{H}$ and $!$. It is easy to show (see Lemma 5) that every trace corresponds to an element of \( Z \) – the function that returns the head of the trace and continues as the tail of the same trace – and that every function in \( Z \) corresponds to a set of traces – one trace for every fixed \( s \in S \). Thus, we can prove that \( \tau \) maps (the set of functions in \( Z \) corresponding to) \( \text{NI}_5 \) to a subset of (the set of functions corresponding to) \( \text{NI}_T \), i.e., the compiler \( \llbracket \cdot \rrbracket \) preserves robust satisfaction. \( \blacksquare \)

6 Related work

In this section, we discuss related work regarding origins and applications of full abstraction, trace based criteria, MoDL and relevant proof techniques.

**Full abstraction** was introduced to relate the operational and the denotational semantics of programming languages [44]. A denotational semantics of a language is said to be fully abstract w.r.t. an operational one for the same language if the same denotation is given to contextually equivalent terms, i.e., those terms that result the *same* when evaluated according to the operational semantics. Common choices to establish when the result of the evaluation is the *same*, and hence to define contextual equivalence, are *equi-convergence* and *equi-divergence* (e.g., in [13, 14, 26, 32, 42]). Notice that there is no loss of generality with these choices, if (and only if!) contexts are powerful enough [32], e.g., when all inputs can be thought as part of the context, and the context itself may select one final value as the result of the execution or diverge.

Fully abstract translations as in Definition 1 have been adopted for comparing expressiveness of languages (see, e.g., the works by Mitchell [32] and Patrignani et al. [42]), but Gorla and Nestmann [23] showed that they may lead to false positive results. The interested reader can find out more in Appendix C, where we also sketch how to use \( \text{RHP}^? \) for expressiveness comparisons.

**Full abstraction and secure compilation** Abadi [1] originally proposed to use full abstraction to preserve security properties in translations from a source language \( L_1 \) to a target one \( L_2 \). A fully abstract translation or compiler preserves and reflects equivalences, and can therefore be a way to preserve security properties when these are expressed as equivalences. Remarkable examples from the literature are given by Bowman and Ahmed [13], Busi et al. [14] and Skorstengaard et al. [47]. In the first two works the authors model contexts so that contextual equivalence captures (forms of) noninterference and preserve it through a fully abstract translation. Skorstengaard et al. [47] consider a source language with well-bracketed control flow (WBCF) and local state encapsulation (LSE), then model target contexts so that these two properties are captured by contextual equivalence and, they exhibit a fully abstract translation so that both WBCF and LSE are guaranteed also in the target. We stress the fact that, all security properties that are not captured by contextual equivalence are not necessarily preserved by a fully abstract compiler, thus allowing for counterexamples similar to Example 1. Finally, it is worth noting that fully abstract compilation
does not prevent source programs to be insecure, nor suggests how to fix them, quoting Abadi [1]:

An expression of the source language $L_1$ may be written in a silly, incompetent, or even malicious way. For example, the expression may be a program that broadcasts some sensitive information—so this expression is insecure on its own, even before any translation to $L_2$. Thus, full abstraction is clearly not sufficient for security […]

Beyond full abstraction Several definitions of “well-behaved translations” exist, depending both on the scenario and on the properties one aims to preserve during the translation. For example, if the goal is to preserve functional correctness, then it is natural to require the compiled program to simulate the source one [3]. This can be expressed both as a relation between the operational semantics of the source and the target (see for example [31, 41, 51]), or extrinsically as a relation between the execution traces of programs before and after compilation [3, 12, 51].

Trace based criteria for compiler correctness The CompCert [12, 31] and CakeML [50] projects are milestones in the formal verification of compilers. Preservation of functional correctness can be expressed in both cases in terms of execution traces [3]. For the CompCert compiler, executing $\llbracket P \rrbracket$ w.r.t. the target semantics yields the same observable events as executing $P$ w.r.t. the source semantics, as long as $P$ does not encounter an undefined behavior. Similarly, CakeML ensures that executing $\llbracket P \rrbracket$ w.r.t. the target semantics yields the same observable events as executing $P$ w.r.t. the source semantics, as long as there is enough space in target memory. In both cases, correctness is proven by exhibiting a simulation between $\llbracket P \rrbracket$ and $P$.

Trace based criteria for secure compilation Similarly to what happens for functional correctness, relations between the execution traces of a program and of its compiled version, can be used to express preservation of noninterference through compilation [3, 10, 34]. The simulation-based techniques introduced in CompCert sometimes suffice also to show the preservation of noninterference, e.g., when the source and the target semantics are equipped with a notion of leakage [3, Sections 5.2-5.4]. However, in more general cases a stronger, cube-shaped simulation is needed (see [3, Section 5.5], and [10, 34]). Stewart et al. [48] propose a variant of CompCert that also gives some guarantees w.r.t. source contexts, and their compilation in the target. Still, this does not guarantee security against arbitrary target contexts, that can be strictly stronger than source ones. Abate et al. [3, 4] propose a family of criteria with the goal of preserving satisfaction of (classes of) security properties against arbitrary contexts. Also, they show that their criteria can be formulated in at least two equivalent ways. The first one explicitly describes the target guarantees ensured for compiled programs, for example which safety properties are guaranteed for programs written in unsafe languages and compiled according to the criterion proposed by Abate et al. [2] (see their Appendix A). The second way is instead more amenable to proofs, e.g., by enabling proofs by back-translation Abate et al. [2, Fig. 4].

Maps of Distributive Laws (MoDL) Mathematical Operational Semantics (MOS) and distributive laws ensure well-behavedness of the operational semantics of a
language while also providing a formal description for it. Such semantics have
been given for languages with algebraic effects [5] and for stochastic calculi [28].
In their biggest generality distributive laws are defined between monads and
comonads [27], but it is often convenient to consider the slightly less general
GSOS laws that correspond bijectively to GSOS rules [6, 27, 45].

Proof techniques for fully abstract compilation include both cross-language log-
ical relations between source and compiled programs [13, 39, 47] and back-
translation of target contexts into source ones [14, 18, 39]. The latter technique
sometimes exploits information from execution traces [18], and can be adapted
also to some of the robust criteria of Abate et al. [4]. Ongoing work is aiming
to formalize the back-translation technique needed to prove some of the robust
preservation of safety (hyper)properties in the Coq proof assistant [2, 51]. The
best results in mechanization of secure compilation criteria have been achieved
for the criteria that can be proven via simulations, especially when extending
the CompCert proof scripts, e.g., [9]. The complexity of many proofs is relatively
contained as they show a forward simulation — the source program simulates the
one in the target — and “flip” it into a backward one — the compiled program
simulates the source one — with a general argument. We are not aware of mech-
anized proofs for MoDL, but we believe it would be convenient to first express
maps between GSOS laws as relations between GSOS rules (see also Section 7).

7 Conclusions and future work

The scope of this work has been to clarify the guarantees provided by criteria
for secure compilation, make them explicit and immediately accessible to users
and developers of (provably) secure compilers. We investigated the relation be-
tween fully abstract and robust compilation, provided an explicit description of
the hyperproperties robustly preserved by a fully abstract compiler, and noticed
that these are not always meaningful, nor of practical utility. We have therefore
introduced a novel criterion that ensures both fully abstract and robust com-
ilation, and such that the meaningfulness of the hyperproperty guaranteed to
hold after compilation can be easily established. The proposed example shows
that our criterion is achievable.

Future work will focus on proof techniques for MoDL and MMOoDL that are
amenable to formalization in a proof assistant. For that we can either build on
existing formalizations of polynomial functors as containers, or exploit the cor-
respondence between GSOS laws and GSOS rules, and characterize MoDL and
MMoDL as relations between source and target rules. Another interesting line of
work consists in devising over (under) approximation for the map $\tilde{\tau}$ from The-
orem 1 and use our Corollary 1 to establish whether existing fully abstract
compilers preserve (violate) a given hyperproperty.

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A Supplement to Section 3

In this appendix Source and Target denote source and target languages and \([\cdot]\) a compiler from Source to Target. Let Trace denote the set of traces; \(\psi(\text{Trace})\) be the set of trace properties; and \(\varphi(\psi(\text{Trace}))\) of hyperproperties. Most of our results hold also when source and target traces differ, so that we use \(\text{Trace}_S\) and \(\text{Trace}_T\) to denote them respectively and similarly (hyper)properties. When \(\text{Trace}_S = \text{Trace}_T\) (like in Corollary 1) we omit colors for traces and (hyper)properties. Hereafter we consider program equivalence to be equality of traces in arbitrary context and \(\text{FAC}\) from Section 3 looks as follows:

\[
\text{FAC} \equiv \forall P_1 P_2. \ (\forall C_S. \ \text{beh}(C_S [P_1]) = \text{beh}(C_S [P_2])) \Rightarrow \\
(\forall C_T. \ \text{beh}(C_T [[P_1]]) = \text{beh}(C_T [[P_2]]))
\]

where \(\text{beh}(C [P]) = \{ t \mid C [P] \rightsquigarrow t \}\). When talking about reflection of contextual equivalence we refer to the following

Definition 8 (Reflection of contextual equivalence). \([\cdot]\) reflects contextual equivalence if and only if

\[
\forall P_1 P_2. \ (\forall C_T. \ \text{beh}(C_T [[P_1]]) = \text{beh}(C_T [[P_2]])) \Rightarrow \\
(\forall C_S. \ \text{beh}(C_S [P_1]) = \text{beh}(C_S [P_2]))
\]

Denote by \(\text{Ctx}_S\) and \(\text{Ctx}_T\) the classes of source and target contexts and define the relation

\[
\mathcal{R}_{[\cdot]} \subseteq (\text{Ctx}_S \rightarrow \varphi(\text{Trace}_S)) \times (\text{Ctx}_T \rightarrow \varphi(\text{Trace}_T))
\]

as

\[
f_S \mathcal{R}_{[\cdot]} g_T \iff \exists P. \ (f_S = \lambda C_S. \ \text{beh}(C_S [P])) \land \\
(g_T = \lambda C_T. \ \text{beh}(C_T [[P]]))
\]

In accordance with the result by Parrow [36], stating that for any \(\text{FAC}\) compiler there exists an injective mapping between the equivalence classes of \(\approx\) and those of \(\approx\), we prove the following lemma:

Lemma 2. If \([\cdot]\) is \(\text{FAC}\), then \(\mathcal{R}_{[\cdot]}\) is a partial function defined over all functions of the form \(\lambda C_S. \ \text{beh}(C_S [P])\) for some partial source program \(P\). The function is injective if \([\cdot]\) also reflects contextual equivalence.
Proof. We first show $\mathcal{R}_{\mathcal{I}}$ is a partial function i.e., that every element of $\text{Ctx}_S \to \phi(\text{Traces})$ is related at most with one element of $\text{Ctx}_T \to \phi(\text{Trace}_T)$. Assume $f_S \mathcal{R}_{\mathcal{I}} g^1_T$ and $f_S \mathcal{R}_{\mathcal{I}} g^2_T$. By definition,

$$\exists p^1. \ (f_S = \lambda C_S. \ beh(C_S [p^1])) \\
\quad \land (g^1_T = \lambda C_T. \ beh(C_T [\llbracket p^1\rrbracket]))$$

and

$$\exists p^2. \ (f_S = \lambda C_S. \ beh(C_S [p^2])) \\
\quad \land (g^2_T = \lambda C_T. \ beh(C_T [\llbracket p^2\rrbracket])).$$  

Notice now that the same $f_S$ appears in the two equations above, thus for an arbitrary $C_S$, $f_S = beh(C_S [p_1]) = beh(C_S [p_2])$. So that

$$\forall C_S. \ beh(C_S [p_1]) = beh(C_S [p_2])$$

and by (preservation direction of) FAC

$$\forall C_T. \ beh(C_T [\llbracket p_1\rrbracket]) = beh(C_T [\llbracket p_2\rrbracket])$$

meaning that $g^1_T$ and $g^2_T$ are point-wise equal.

We now assume that $\llbracket \cdot \rrbracket$ also reflects contextual equivalence (see Definition 8), and show injectivity of $\mathcal{R}_{\mathcal{I}}$. Let

$$f^1_S = \lambda C_S. \ beh(C_S [p_1]) \quad f^2_S = \lambda C_S. \ beh(C_S [p_2])$$

with $f^1_S \neq f^2_S$ then, there exists some $C_S$ such that $beh(C_S [p_1]) \neq beh(C_S [p_2])$. By the (contrapositive of) reflection in the definition of FAC we deduce there exists $C_T$ such that $beh(C_T [\llbracket p_1\rrbracket]) \neq beh(C_T [\llbracket p_2\rrbracket])$ which implies $\mathcal{R}_{\mathcal{I}} (f^1_S) \neq \mathcal{R}_{\mathcal{I}} (f^2_S)$ and $\mathcal{R}_{\mathcal{I}}$ is injective. 

Abate et al. [3] define $\tilde{\tau} : \phi(\text{Traces}_S) \to \phi(\text{Trace}_T)$ and then lift it to $\phi(\phi(\text{Traces}_S)) \to \phi(\phi(\text{Trace}_T))$. Instead, we define $\tilde{\tau} : \phi(\phi(\text{Traces}_S)) \to \phi(\phi(\text{Trace}_T))$ directly. Moreover, if one is willing to specialize it to trace properties this can be done by $\tilde{\tau}(\pi_S) = \tilde{\tau}(\{\pi_S\})$.

We can now prove the following

**Theorem 1.** If $\llbracket \cdot \rrbracket$ is FAC, then there exists a map $\tilde{\tau}$ such that $\llbracket \cdot \rrbracket$ is $\text{RHP}^\tilde{\tau}$. Moreover, $\tilde{\tau}$ is the smallest (pointwise) with this property.

**Proof.** By definition, if $C_S [P] \models H_S$ there exists $\pi_{C_S} \in H_S$ such that $beh(C_S [P]) = \pi_{C_S}$. By Lemma 2 $\mathcal{R}_{\mathcal{I}}$ is defined on the following function:

$$f^p_{H_S} : \text{Ctx}_S \to \phi(\text{Traces}_S)$$

$$f^p_{H_S} = \lambda C_S. \ \pi_{C_S}$$
and by definition of $\mathcal{R}_{\mathfrak{I}}\mathfrak{I}$, for every target context $C_T$

$$\text{beh}(C_T[H]) = ((\mathcal{R}_{\mathfrak{I}}\mathfrak{I}) \mathfrak{I}_h \mathfrak{I}) C_T).$$

Thus, $[P]$ satisfies the following target hyperproperty

$$\{(\mathcal{R}_{\mathfrak{I}}\mathfrak{I}) \mathfrak{I}_h \mathfrak{I}) C_T \mid C_T \in \text{Ctx}_T\}$$

in an arbitrary target context $C_T$.

We define

$$\tilde{\tau}(H_S) = \{(\mathcal{R}_{\mathfrak{I}}\mathfrak{I}) \mathfrak{I}_h \mathfrak{I}) C_T \mid C_T \in \text{Ctx}_T \land P \in \text{Prgs} \land (\forall C_S. C_S[P] \models H_S)\}$$

Notice that $[\cdot]$ is $\text{RHP}^\tau$ for such a $\tilde{\tau}$ by construction.

We now show minimality of $\tilde{\tau}$ i.e., that for every $\alpha$ such that $[\cdot]$ is $\text{RHP}^\alpha$, $\forall H_S \in \varphi(\varphi(\text{Trace}))$. $\tilde{\tau}(H_S) \subseteq \alpha(H_S)$. If $\tilde{\tau}(H_S) = \emptyset$ then $\emptyset \subseteq \alpha(H_S)$. If $\tau_T \in \tilde{\tau}(H_S)$, then by construction $\tau_T = \text{beh}(C_T'[P])$ for some $C_T'$ and some $P$ such that $\forall C_S. C_S[P] \models H_S$. By $\text{RHP}^\alpha$ deduce $\forall C_T. C_T'[P] \models \alpha(H_S)$ and hence in particular $C_T'[P] \models \alpha(H_S)$ that means $\tau_T = \text{beh}(C_T'[P]) \in \alpha(H_S)$ and concludes the proof.

The next corollary shows how to use Theorem 1 to establish if a compiler preserves or not the robust satisfaction of a single hyperproperty. Corollary 1 assumes that source and target traces are the same, and hence the set $\varphi(\varphi(\text{Trace}))$, that we don’t color any longer.

**Corollary 1.** If $[\cdot]$ is $\text{FAC}$, then for every hyperproperty $H$, $[\cdot]$ preserves the robust satisfaction of $H$ iff $\tilde{\tau}(H) \subseteq H$, where $\tilde{\tau}$ is the map from Theorem 1.

**Proof.** For the implication from left to right assume $[\cdot]$ preserves robust satisfaction of $H$, meaning that $\forall P. (\forall C. C[P] \models H) \Rightarrow (\forall C. C'[P] \models H)$. It follows that the compiler is $\text{RHP}^h$ where $h$ is defined as

$$h(H') = \begin{cases} H & \text{if } H = H' \\ \top & \text{o.w.} \end{cases}$$

and $\top = \{\pi \mid \pi \in \varphi(\text{Trace})\}$ is trivially robustly satisfied by any program. By minimality of $\tilde{\tau}$ it must be $\tilde{\tau}(H) \subseteq h(H) = H$.

For the implication from right to left we need to show that $\forall P. (\forall C. C[P] \models H) \Rightarrow (\forall C. C'[P] \models H)$.

Assume that $P$ robustly satisfies $H$ in the source, i.e., $(\forall C. C[P] \models H)$, since the compiler is $\text{RHP}^\tau$ we deduce $(\forall C. C'[P] \models \tilde{\tau}(H))$. Unfolding the definition of $\models$, we have $\forall C. \text{beh}(C'[P]) \in \tilde{\tau}(H)$ with $\tilde{\tau}(H) \subseteq H$, so that $\forall C. \text{beh}(C'[P]) \in H$. 

\qed
A.1 Details of Example 2

\[ (expr) ::= n \in \mathbb{N} \mid v \in \text{Var} \mid (expr) \ (bin) \ (expr) \mid (un) \ (expr) \]

\[ (P) ::= \text{skip} \mid v ::= (expr) \mid (P);\ (P) \mid \text{while} \ (expr) \ (P) \]

\[ (C_S) ::= [] \]

\[ (P) ::= \text{skip} \mid v ::= (expr) \mid (P);\ (P) \mid \text{while} \ (expr) \ (P) \]

\[ (C_T) ::= [\cdot] \mid [\cdot] \]

![Diagram](image)

Fig. 4. Operational semantics of Source.

For \( S \) being the set of stores and \( \mathcal{H}, !, \checkmark \not \in S \), we define the abstract traces to be

\[ \text{Trace}_S \overset{\Delta}{=} \{ t : \checkmark \mid t \in (S \cup \{ \mathcal{H} \})^* \cup (S \cup \{ \mathcal{H} \})^\omega \} \]

\[ \text{Trace}_T \overset{\Delta}{=} \{ t : \checkmark \mid t \in (S \cup \{ \mathcal{H}, ! \})^* \cup (S \cup \{ \mathcal{H}, ! \})^\omega \} \]

where for a set \( X \), \( X^* \) denotes the set of finite sequences of elements of \( X \) and \( X^\omega \) the infinite ones.

**Definition 9 (Observable and Internal events).** We call observable events states, termination or \( !, \) i.e., elements of \( S \cup \{ !, \checkmark \} \), and call \( \mathcal{H} \) internal event.

**Definition 10 (Low-equivalence).** We define low-equivalence as the following equivalence relation \( =_L \) over observable events.

- \( \forall s_1, s_2 \in S = \text{Var}_L \cup \text{Var}_H \rightarrow \mathbb{N}, \ s_1 =_L s_2 \ i.f.f \ \forall v \in \text{Var}_L. \ s_1(v) = s_2(v) \)
$\forall v \in Var_H, s(v) \neq [e]_s$

$\forall v \in Var_L, \ s(v) := e \to s_{\preceq \\{v\}} \rightarrow \checkmark$

$\forall v \in Var_L, \ s(v) := e \to s_{\preceq \\{v\}} \rightarrow \checkmark$

$s, \ p \rightarrow s', \ \checkmark$

$s, [p] \rightarrow s', \ p'$

$s, [p] \rightarrow s', \ p'$

\[ \text{Fig. 5. The semantics of Target includes the rules of the semantics of Source where} \]
\[ \text{the ones for assignments are changed as depicted here. The semantics also includes two} \]
\[ \text{new rules for the “stronger” context } [\cdot] \text{ that allow to observe !.} \]

$! = _L !, \ \checkmark = _L \checkmark$

Two traces $t_1, t_2 \in \text{Traces}_S$ (or $t_1, t_2 \in \text{Trace}_T$) are low-equivalent iff the sequences of
$s, !, \checkmark$ appearing on the two traces are pointwise low-equivalent, i.e.,

$t_1 = _L t_2 \iff \forall i. t_1^i = _L t_2^i,$

where $t_i^i$ is the $i$-th observable event appearing in $t$.

**Definition 11 (Source and target noninterference).** Source and target noninterference, $\text{NI}_S \in \wp(\wp(\text{Traces}_S))$ and $\text{NI}_T \in \wp(\wp(\text{Trace}_T))$ resp., are defined as following.

$$\text{NI}_S = \{ \pi \in \wp(\text{Traces}_S) \mid \forall t_1, t_2 \in \pi. t_1^0 = _L t_2^0 \Rightarrow t_1 = _L t_2 \}$$

$$\text{NI}_T = \{ \pi \in \wp(\text{Trace}_T) \mid \forall t_1, t_2 \in \pi. t_1^0 = _L t_2^0 \Rightarrow t_1 = _L t_2 \}$$

**Lemma 3.** The identity compiler between Source and Target preserves trace equality, but does not preserve robust satisfaction of noninterference.

**Proof (Sketch).** ($id$ preserves trace equality). Consider source programs $P_1, P_2$ such that $\text{beh}_S([P_1]) = \text{beh}_S([P_2])$, then in the target, for the identity context $[\cdot]$, $\text{beh}_T([P_1]) = \text{beh}_S([P_1])$ and the same holds for $P_2$, so that $\text{beh}_T([P_1]) = \text{beh}_T([P_2])$. The interaction with the context $[\cdot]$ may expose also the extra symbol $!$, this however – because of equivalence in the source – happens at the same point of the execution for both $[P_1]$ and $[P_2]$, so that $\text{beh}_T([P_1]) = \text{beh}_T([P_2])$.

($id$ does not preserve robust satisfaction of noninterference). Let $P := v := 42$ and $v \in Var_H$. $P$ trivially robustly satisfies $\text{NI}_S$ in the source, but $[P] := v := 42$ violates target noninterference $\text{NI}_T$ in the context $[\cdot]$. Indeed, let $s, \bar{s}$ be two states such that $s(v) = 1$ and $\bar{s}(v) = 42$. When executing $[P]$ in the initial state $s$, the trace observed is $s \cdot ! \cdot s_{\preceq 42} \cdot \checkmark$, while when executing in $\bar{s}$, the state is not be updated and $!$ never exposed.
A.2 An insertion for reconciling observations

This section is inspired to the hierarchy of semantics of transition systems by Cousot [16]. The high level intuition is to consider (for source and target languages) abstract traces $Trace^A$ collecting both the externally observable events and internal states to be finite or infinite sequences of states and concrete $Trace^E$ that instead collect only externally observable events, and that intuitively are obtained from the abstract ones by simply hiding or abstracting the internal states. Rigorously we assume a Galois insertion [17] between their powersets.

Remark 4 (Trace insertion). We assume a Galois insertion,

$$\alpha : (\wp(Trace^A), \subseteq) \Rightarrow (\wp(Trace^E), \subseteq) : \gamma$$

with $\alpha$ abstraction map and $\gamma$ concretization.

Remark 5 (Lifting of an insertion). Let $A, C \in \textbf{Set}$. Every $\alpha : (\wp(A), \subseteq) \Rightarrow (\wp(C), \subseteq) : \gamma$ Galois insertion lifts to a Galois insertion $\bar{\alpha} : (\wp(\wp(A)), \subseteq) \Rightarrow (\wp(\wp(C)), \subseteq) : \bar{\gamma}$, by $\bar{\alpha}(X) = \{\alpha(x) \mid x \in X\}$ and $\bar{\gamma}(Y) = \{\gamma(y) \mid y \in Y\}$.

Remark 6 (Hyperproperty insertion). For every $H^E \in \wp(\wp(Trace^E))$, we have $\bar{\alpha}(\bar{\gamma}(H^E)) = H^E$, where $\bar{\alpha}$, $\bar{\gamma}$ are given in Remark 4.

Thanks to Remark 6 we can define hyperproperties on the traces of events only, like we did with noninterference, but regard it as elements of the abstract $\wp(\wp(Trace^A))$, i.e., sets of sets of abstract traces.

B Detail on Example 3

Lemma 4. $(s, b)$ is a MoDL and $(b, t)$ is a MMoDL.

Proof (Sketch). We just show that $(s, b)$ and $(b, t)$ satisfy the definition of MoDL and MMoDL resp. for the case of assignments of a high variable $v$ with an expression $e$. The other cases are immediate.

– To prove that $(s, b)$ satisfies the definition of MoDL, it suffices to show how to build the following diagram:

$$
v := e, \lambda s. t(s) \rightarrow \checkmark, \lambda s. (s[v:=e], \mathcal{H})$$

$$\Downarrow$$

$$\Downarrow$$

$$|v := e|, \lambda s. t(s) \rightarrow \checkmark, \lambda s. (s[v:=e], \checkmark)$$

Right-down path: First execute according to $\rho$ and update the state with the new expression $e$ for the variable $v$. Here, $\mathcal{H}$ is exposed because the variable is private. Finally, apply $b$ that erases the $\mathcal{H}$ thus reaching $\checkmark, \lambda s. (s[v:=e], \checkmark)$. 
Down-right path: First “compile” to the sandboxed target assignment
and then execute in the target reaching again $\checkmark$, $\lambda s.s[v \leftarrow e], \checkmark$. (Here $H$ is not exposed thanks to the sandbox).

- Similarly to the above, we now prove that $(b, t)$ satisfy the definition of MMoDL by building the following diagram:

$$
\begin{array}{c}
\begin{array}{c}
[v := e], \lambda s. t(s) \rightarrow [\checkmark], \lambda s.s[v := e], \checkmark \\
\downarrow \\
[\checkmark], \lambda s.s[v := e], H \\
\downarrow \\
[\checkmark], \lambda s.s[v := e], \checkmark \\
\downarrow \\
\checkmark, \lambda s.s[v := e], \checkmark \rightarrow \checkmark, \lambda s.s[v := e], \checkmark
\end{array}
\end{array}
$$

Right-down path: The inner subterm reduces to $\checkmark$ (application of $\rho$).
Translate its behavior in the target by $b$, then execute the context $\rho_1$ and back-translate the (consumed) context (i.e., no changes).

Down-right path: The inner subterm reduces to $\checkmark$ (application of $\rho$).
Then apply $t$ that back-translates the target context to the identity source one and execute it. Finally, translate the behavior in the target.

Lemma 5 (From Traces to $Z$ and back). There exist injective maps $\varphi : \text{Trace}_S \rightarrow Z$ and $\Psi : Z \rightarrow \varphi(\text{Trace}_S)$, and similarly $\varphi : \text{Trace}_T \rightarrow Z$ and $\Psi : Z \rightarrow \varphi(\text{Trace}_T)$.

Proof. The maps $\varphi$ and $\varphi$ are similar, so we give only one definition, $\varphi : \text{Trace} \rightarrow Z$ that associates to every trace $t$ the function $f \in Z$ such that $f(s)$ is the pair made of the head of $t$ and the tail of $t$ as continuation,

$$
\varphi : \text{Trace} \rightarrow Z
$$

$$
t \mapsto \left\{ \begin{array}{ll}
\lambda s \in S.\langle \text{head}(t), \varphi(\text{tail}(t)) \rangle & \text{if } \text{head} \circ \text{tail}(t) \in S \cup \{ \checkmark \} \\
\lambda s \in S.\langle \text{head}(t), \langle \text{Some } x, \varphi(\text{tail}(t)) \rangle \rangle & \text{if } \text{head} \circ \text{tail}(t) = x \in \{!, H\}
\end{array} \right.
$$

$\varphi$ is clearly injective as if $t_1 \neq t_2$ then they differ at some finite point, or equivalently $\text{head} \circ \text{tail}^n(t_1) \neq \text{head} \circ \text{tail}^n(t_2)$ for some $n \in \mathbb{N}$, so that the two functions $\varphi(t_1)$ and $\varphi(t_2)$ are different.

Also, $\Psi$ and $\Psi$ are similar, so we define $\Psi$ as follows. To every element $f \in Z$ we associate a set $\Psi(f)$ consisting of all traces $t$ such that $t^{i+1}$ is the evaluation
of the $i^{th}$ continuation of $f$ on a fixed $s_0 \in S$,

$$\Psi : Z \to \wp(\text{Trace})$$

$$l \mapsto \{ t \mid t^0 = \pi_1^2(l(s_0)) \land \forall i, t^{i+1} = \pi_1(k(s_0, i+1, l(t^i))) \land s_0 \in S \}$$

where $k(s_0, i, l)$ is defined, for every $s_0 \in S$ by induction on $i$,

$$k(s_0, 1, l) = \pi_2(l(s_0))$$

$$k(s_0, i+1, l) = k(s_0, i, l(\pi_1^2(k(s_0, i, l(s_0))))$$

$\Psi$ is injective because if $f_1 \neq f_2$ then they will differ when evaluated on some state or on their continuation.

**Remark 7 (beh = $\Psi \circ f$).** For every $C, P$, for $f : A \to Z$ and for $\varphi$ and $\Psi$ as in Lemma 5 $\text{beh}(C[P]) = \Psi(f(C[P]))$. Vice versa due to injectivity of $\Psi$, $f(C[P]) = \Psi^{-1}(\text{beh}(C[P]))$.

**Lemma 6.** For every hyperproperty $H \in \wp(\wp(\text{Trace}_S))$ and source program $P$, if $P$ robustly satisfies $H$, then $[P]$ robustly satisfies $H := \Psi \circ \tau \circ \Psi^{-1}(H)$.

**Proof.** Let $P$ be a source program that robustly satisfies $H$ and let $C_T$ be an arbitrary target context. Notice that since $P$ robustly satisfies $H$, then for any $C_S$, $\text{beh}_S(C_S[P]) \in H$, and from Remark 7 $f(C_S[P]) \in \Psi^{-1}(H) = \{ \Psi^{-1}(\pi) \mid \pi \in H \}$. Recall that the compiler is (abstract) $\text{RHP}^\tau$, i.e.,

$$\tau \circ f \circ \text{plug} \circ \text{bk} = f \circ \text{plug} \circ \text{id} \times []$$

and let $C_S$ be the source context obtained by back-translation, from Equation (1) we have,

$$\tau(f(C_S[P])) = f(C_T[[P]])$$

so that $f(C_T[[P]]) \in \{ \tau(F) \mid F \in \Psi^{-1}(H) \}$, where $\tau(F) = \{ \tau(x) \mid x \in F \}$. From Remark 7 it follows that $\text{beh}_T(C_T[[P]]) = \Psi(f(C_T[[P]])) \in \Psi \circ \tau \circ \Psi^{-1}(H)$.

**Lemma 7.** For every $P$, if $P$ robustly satisfies $\text{NI}_S$, $[P]$ robustly satisfies $\text{NI}_T$.

**Proof.** From Lemma 6 we have to show that the set $\Psi \circ \tau \circ \Psi^{-1}(\text{NI}_S)$ is included in $\text{NI}_T$. This is immediate because $\tau$ that simply erases the occurrences of $\mathcal{H}$ (and source traces do not expose $!$), and $\Psi$, $\Psi^{-1}$ do not modify the events on the traces.
Mitchell [32] suggested to compare the expressiveness of languages by providing a fully abstract translation between two languages that also respects a certain homomorphism condition. The homomorphism condition (R1 from Section 3 of [32]) requires that the (denotational) semantics of $\llbracket C \rrbracket$ coincides with the one of $\llbracket C \rrbracket[P]$, and is naturally satisfied by canonical compilers like the one recently proposed by Patrignani et al. [42]. Ignoring the homomorphism condition of Mitchell [32] may lead to “false positive” results, as highlighted by Gorla and Nestmann [23]. For example, it is possible to define a fully abstract compiler, that does not respect the homomorphic condition, between the class of Turing machines and that of (deterministic) finite-state automata, despite the fact that they are clearly not equi-expressive [11, 23]. Hereafter we sketch how RHP $\tau$ rules out this false positive result, and therefore can be thought as a criterion to compare expressiveness of languages. To support this claim, we consider as source programs Turing machines (TM) and as target ones finite automata (DFA). Source and target contexts simply pass an arbitrary string on $\{0, 1\}$ to the TM or to the DFA which accept or reject. Traces are arbitrary strings over $\{0, 1\}$ and the traces that can be observed during execution of a TM or of a DFA in a context, are those that are accepted by it. Let $\llbracket \cdot \rrbracket : TM \to DFA$ any translation between the two languages, for example the fully abstract translation from [11, 22], $\llbracket \cdot \rrbracket$ cannot be RHP $\tau$ with $\tau$ the identity map. For a proof, assume $\llbracket \cdot \rrbracket$ is RHP $\tau$ and $M$ be the TM that accepts all and only the strings in $\{0^n1^n \mid n \in N\}$, the DFA $\llbracket M \rrbracket$ would accept exactly the same strings, a contradiction ([24] Exercise 4.1.1).

D MoDL and MMoDL

D.1 Background

Definition 12 (GSOS laws). A GSOS law of $\Sigma$ over $B$ is a natural transformation $\rho : \Sigma(\text{Id} \times B) \Longrightarrow B\Sigma^*$, where $\Sigma^*$ is the free monad over $\Sigma$.

Remark 8 ($\rho$ extends to $\lambda$). We will make use of a result from [29] stating that GSOS laws $\rho : \Sigma(\text{Id} \times B) \Longrightarrow B\Sigma^*$ are equivalent to natural transformations $\lambda : \Sigma^*(\text{Id} \times B) \Longrightarrow B\Sigma^*$ respecting the structure of $\Sigma^*$ as follows:

$$
\begin{array}{c}
\Sigma^*(\text{Id} \times B) \\
\downarrow_{\mu_{(\text{Id} \times B)}} \\
\Sigma^*(\text{Id} \times B) \\
\downarrow_{\lambda} \\
B\Sigma^*
\end{array}
$$

$$
\begin{array}{c}
\Sigma^*(\text{Id} \times B) \\
\downarrow_{\eta_{(\text{Id} \times B)}} \\
\Sigma^*(\text{Id} \times B) \\
\downarrow_{\lambda} \\
B\Sigma^*
\end{array}
$$

$$
\begin{array}{c}
\Sigma^*(\text{Id} \times B) \\
\downarrow_{\eta_{(\text{Id} \times B)}} \\
\Sigma^*(\text{Id} \times B) \\
\downarrow_{\lambda} \\
B\Sigma^*
\end{array}
$$

$$
\begin{array}{c}
\text{Id} \times B \\
\downarrow_{\eta_{(\text{Id} \times B)}} \\
\text{Id} \times B \\
\downarrow_{\lambda} \\
B\Sigma^*
\end{array}
$$

$$
\begin{array}{c}
\text{Id} \times B \\
\downarrow_{\eta_{(\text{Id} \times B)}} \\
\text{Id} \times B \\
\downarrow_{\lambda} \\
B\Sigma^*
\end{array}
$$
The respective λ can be constructed via structural induction on Σ*. The reader may assume that given a GSOS law ρ, λ is its extended version and vice versa.

Remark 9 (f as the bialgebra morphism induced by ρ [24]). Let a : ΣA → A (A = Σ*∅) the initial Σ-algebra and z : Z → BZ final B-coalgebra (Z = B^∞⊤). A GSOS law ρ : Σ(Id × B) ⇒ BΣ* induces:

- a Σ-algebra g : ΣZ → Z
- a B-coalgebra h : BA → A
- a unique morphism of bialgebras f : A → Z such that the following diagram

\[
\begin{array}{ccc}
\Sigma A & \xrightarrow{a} & A \\
\Sigma f & \downarrow & \downarrow f \\
\Sigma Z & \xrightarrow{g} & Z \\
\end{array}
\]

\[
\begin{array}{ccc}
& & h \\
Bf & \downarrow & \\
BZ & \xrightarrow{z} & \\
\end{array}
\]

Following the concrete definitions, we qualify certain constructors as being (execution) contexts and the rest as being terms. In the categorical setting of GSOS laws, this means that the syntax Σ : Set → Set is C ⊎ P, with C : Set → Set being the context constructors and P : Set → Set standing for terms. Furthermore, we acknowledge that a term may also consists of contexts (but not vice versa), as we want terms to be able to interact themselves with contexts.

Definition 13 (plug [52]). The plugging operation, plug, is defined via (strong) initiality. For a : ΣA → A and q : (⊤ ⊎ C) ⊎ C → C being the initial algebras of Σ and ⊤ ⊎ C respectively and where C = (⊤ + C)*∅, we have

\[
\begin{array}{ccc}
(T ⊎ C)A × A & \xrightarrow{q \times 1} & C × A \\
(T ⊎ C)C × A & \xrightarrow{(st, π_2)} & (T ⊎ C)(C × A) × A \\
(T ⊎ C)A × A & \xrightarrow{[1 \circ π_2, 1 \circ π_1]} & (T × A) ⊎ (CA × A) \\
\end{array}
\]

Intuitively, in T ⊎ C, the T part denotes an actual hole to be filled in. Thus, the initial algebra of T ⊎ C are all finite combinations of context constructors that have one or more holes in them. The plugging operation simply returns the term-to-be-plugged upon encountering a hole. We may write c[p] to denote plug(c, p).

Remark 10 (Assumption on ρ). For any language we introduce, we want the contexts to be closed under the semantics, therefore from now on we consider GSOS laws ρ : Σ(Id × B) ⇒ BΣ* consisting of ρ_1 : C(Id × B) ⇒ BC* and ρ_2 : Π(Id × B) ⇒ BΣ* with ρ = [Bt_1 ∘ ρ_1, ρ_2]. This is true for any GSOS law introduced later on.
We recall the definitions of MoDL ad MMoDL.

**Definition 6 (MoDL).** A map of distributive law between \( \rho : \Sigma (\text{Id} \times B) \Rightarrow B \Sigma^* \) and \( \rho : \Sigma (\text{Id} \times B) \Rightarrow B \Sigma^* \) is a pair of natural transformations \( s : \Sigma \Rightarrow \Sigma^* \) and \( b : B \Rightarrow B \) such that the following diagram commutes,

\[
\begin{array}{ccc}
\Sigma (\text{Id} \times B) & \xrightarrow{\rho} & B \Sigma^* \\
\downarrow s \circ \Sigma (\text{Id} \times b) & & \downarrow b \circ s \circ \Sigma (\text{Id} \times b) \\
\Sigma (\text{Id} \times B) & \xrightarrow{\rho} & B \Sigma^*
\end{array}
\]

where \( s^* : \Sigma^* \Rightarrow \Sigma^* \) extends \( s : \Sigma \Rightarrow \Sigma^* \) to a morphism of free monads, i.e., to terms of arbitrary depth via structural induction.

**Definition 7 (MMoDL).** A many layers map of distributive laws (MMoDL) between \( \rho : \Sigma (\text{Id} \times B) \Rightarrow B \Sigma^* \) and \( \rho : \Sigma (\text{Id} \times B) \Rightarrow B \Sigma^* \) is given by natural transformations \( b : B \Rightarrow B \) and \( t : C \Rightarrow C \) making the following commute:

\[
\begin{array}{cccc}
\mathcal{C} \Sigma (\text{Id} \times B) & \xrightarrow{\mathcal{C}^* (\Sigma \pi_1, \rho)} & \mathcal{C} (\text{Id} \times B) \Sigma^* & \xrightarrow{t} & \mathcal{C}^* (\text{Id} \times B) \Sigma^* \\
\downarrow \mathcal{C} (\Sigma (\text{Id} \times B)) & & \downarrow \mathcal{C} (\Sigma (\text{Id} \times B)) & & \downarrow \mathcal{C} (\Sigma (\text{Id} \times B)) \\
\mathcal{C} (\text{Id} \times B) \Sigma^* & \xrightarrow{\mathcal{C}^* (\text{Id} \times b)} & \mathcal{C} (\text{Id} \times B) \Sigma^* & \xrightarrow{\rho_1} & \mathcal{B} \mathcal{C}^* \Sigma^* \\
\mathcal{C} (\text{Id} \times B) \Sigma^* & \xrightarrow{\mathcal{C} (\Sigma \pi_2, \rho)} & \mathcal{C} (\Sigma (\text{Id} \times B)) \Sigma^* & \xrightarrow{\rho_2} & \mathcal{B} \mathcal{C}^* \Sigma^* \\
\mathcal{C} (\text{Id} \times B) \Sigma^* & \xrightarrow{\mathcal{C} (\Sigma \pi_1, \rho)} & \mathcal{C} (\Sigma (\text{Id} \times B)) \Sigma^* & \xrightarrow{\rho_3} & \mathcal{B} \mathcal{C}^* \Sigma^* \\
\mathcal{C} (\text{Id} \times B) \Sigma^* & \xrightarrow{\mathcal{C} (\Sigma \pi_2, \rho)} & \mathcal{C} (\Sigma (\text{Id} \times B)) \Sigma^* & \xrightarrow{\rho_4} & \mathcal{B} \mathcal{C}^* \Sigma^*
\end{array}
\]

**D.2 MMoDL and RHP\(^*\)**

In order to prove Theorem 2 we consider the result of plugging a **Source** term \( p : A \) in a **Target** context \( c : C \) as an element of \( C \Sigma^* A \), and obtain a **B**-coalgebra structure on \( C \Sigma^* A \).

**Definition 14 (cross-language plug).** We write \( m \) to denote the operation of plugging \( p \) in \( c \) and define it via (strong) initiality:

\[
\begin{array}{cccc}
(\top \uplus \mathcal{C}) C \times A & \xrightarrow{q \times 1} & C \times A \\
(\sigma_1, \pi_2) & & \Downarrow m \\
(\top \uplus \mathcal{C})(C \times A) \times A & \xrightarrow{(\top \times A) \uplus (\mathcal{C} \mathcal{C}^* A \times A)} & (\top \times A) \uplus (\mathcal{C} \mathcal{C}^* A \times A) & \xrightarrow{(i_1, i_2, i_3, i_4)} & A \uplus \mathcal{C} \mathcal{C}^* A & \xrightarrow{\cong} & \mathcal{C}^* A
\end{array}
\]

**Remark 11 (a useful B-coalgebra).** \( \rho_1 \) induces the following B-coalgebra structure on \( H^* A \),

\[
\mathcal{C}^* A \xrightarrow{\mathcal{C}^*(1, h)} \mathcal{C}^*(\text{Id} \times B) A \xrightarrow{\mathcal{C}^*(1 \times b)} \mathcal{C}^*(\text{Id} \times B) A \xrightarrow{\rho_1} \mathcal{B} \mathcal{C}^* A
\]

The crucial fact about this B-coalgebra is that it is in a (functional) bisimulation with \( b \circ h : A \to B A \). More specifically:
Lemma 8. Let GSOS laws $\rho : \Sigma(\text{Id} \times B) \Rightarrow B\Sigma^*$ and $\rho : \Sigma(\text{Id} \times B) \Rightarrow B\Sigma^*$, so that $\Sigma \models \mathcal{Q}$, $\Sigma \models \mathcal{P}$ and $\rho$ respect the assumption in Remark 14. If $(s, b) : \rho \Rightarrow \rho$ is a MoDL and $(t, b) : \rho \Rightarrow \rho$ is an MMoDL, then $a^* \circ t^* : \mathcal{C}^*A \Rightarrow \mathcal{C}^*A$ is a $B$-coalgebra homomorphism from $\rho_1 \circ \mathcal{C}(1 \times b) \circ \mathcal{C}^*(1, h) : \mathcal{C}^*A \Rightarrow B\mathcal{C}^*A$ to $b \circ h : A \Rightarrow BA$.

Proof. We first show that $t^* : \mathcal{C}^*A \Rightarrow \mathcal{C}^*A$ is a bisimulation by induction on the structure of $\mathcal{C}^*A \cong A \uplus \mathcal{C}^*A$. By naturality of $\rho$ we see that

Assuming that the theorem holds for the inner $\mathcal{C}^*A$ (the inductive hypothesis), we only need to show that $1 \uplus (t \circ t^*)$ is a $B$-coalgebra homomorphism from the above coalgebra structure on $A \uplus \mathcal{C}^*A$ to the one on $A \uplus \mathcal{C}^*A$. We will do so by proving that the graph of $1 \uplus (t \circ t^*)$ is a bisimulation. We have the following commutative diagram:

Where $r : \Sigma^*A \Rightarrow \Sigma\Sigma A$ is defined by induction on the free monad $\Sigma^*$ as the inductive extension of the algebra structure $[\alpha^{-1}, \Sigma, \alpha] : A \uplus \Sigma^*A \Rightarrow \Sigma A$, where $\alpha$ is the initial algebra of $\Sigma$. Intuitively, $r$ simply exposes the outer layer of $\Sigma^*A$ and “merges” the inner layers into $A$. The top rectangle commutes due to the inductive hypothesis and the bottom due to the fact that $(t, b)$ is a MMoDL. We note that $t \circ \mathcal{C}r \circ \mathcal{C}_1 \circ t^* \neq t \circ \mathcal{C}t^*$, however, since $\mathcal{C}r \circ \mathcal{C}_1$ only “re-packages” the syntax layers, we know that the image of $(t \circ \mathcal{C}r \circ \mathcal{C}_1 \circ t^*, t \circ \mathcal{C}t^*)$ is a bisimulation. This makes the graph of $t \circ \mathcal{C}t^*$ a bisimulation thus concluding (the base case is the trivial identity bisimulation) that $t^* : \mathcal{C}^*A \Rightarrow \mathcal{C}^*A$ is a bisimulation. Map $a^*_1$ is a $B$-coalgebra homomorphism and hence also a $B$-coalgebra homomorphism, which wraps up the proof.
We are now almost ready to present our main theorem. But first, we introduce the following two “structural” lemmata that we eventually make use of.

**Lemma 9 (Blue lemma).** Let syntax functors $\Sigma$ and $\Sigma'$ so that $\Sigma \triangleq C \uplus P$, $\Sigma' \triangleq C \uplus P'$. For any natural transformation $s : \Sigma \Rightarrow \Sigma'$, we have

$$
\begin{array}{ccc}
C \times A & \xrightarrow{\mathbb{1} \times [\ ]} & C \times A \\
\downarrow m & & \downarrow \text{plug} \\
C^* A & \xrightarrow{a_1^* \circ c^*} & A
\end{array}
$$

Where $[\ ] : A \to A$ is the compiler determined inductively by $s$, i.e., $[\ ] = s^*_0$.

**Lemma 10 (Purple lemma).** Let syntax functors $\Sigma$ and $\Sigma'$ so that $\Sigma \triangleq C \uplus P$, $\Sigma' \triangleq C \uplus P'$. For any back-translation $t : C \Rightarrow C'$, we have

$$
\begin{array}{ccc}
C \times A & \xrightarrow{t' \times 1} & C \times A \\
\downarrow m & & \downarrow \text{plug} \\
C^* A & \xrightarrow{a_1^* \circ t^*} & A
\end{array}
$$

Where $t' : C \to C$ is the translation determined inductively by $t$, i.e., $t' = t^*_0$.

**Theorem 2 (MMoDL imply RHP$_\tau$).** Let $s : \Sigma \Rightarrow \Sigma^*$, $b : B \Rightarrow B$ and $t : C \Rightarrow C'$ such that $(s, b)$ and $(t, b)$ are (respectively) a MoDL and a MMoDL from $\rho : \Sigma(\text{Id} \times B) \Rightarrow B\Sigma^*$ to $\rho : \Sigma(\text{Id} \times B) \Rightarrow B\Sigma^*$. The compiler $[\ ] = s^*_0$ is (abstract) RHP$_\tau$ for $\tau = b^\omega$ coinductively induced by $b$.

**Proof.** We have to show that the following diagram commutes:

Starting from top-middle node $C \times A$, which is the pairing of a Target context with a Source term, the left-outer path follows the back-translation of the Target context $(t' \times 1)$, the plug of the Source term to the back-translated context, then the semantics map $f$ mapping a Source term to its behavior in Source and finally the behavioral translation $\tau$. On the other side, the right-outer path consists of the evaluation $- f -$ of the plug of the Target context and the compiled Source term.

To prove Theorem [2] it suffices to show that the inner rectangles commute:
– The top-left (purple) rectangle commutes due to Lemma 10.
– The top-right (blue) rectangle commutes due to Lemma 9.
– The bottom (green) rectangle commutes due to \((t, b)\) being a MMoDL and
\((s, b)\) a MoDL with the following argument.

Notice that for the bottom green rectangle, it suffices to show that every arrow
is a \(B\)-coalgebra homomorphism, this because the bottom-right node \(Z\) is the
final coalgebra \(z : Z \to BZ\). We know that \(f\) is a \(B\)-coalgebra homomorphism
by construction while \(\tau \circ f\) is a \(B\)-coalgebra homomorphism due to the fact that
\((s, b)\) is a MoDL (and \(b \circ h : A \to BA\) is a \(B\)-coalgebra). It is for the same reason
that \(\llbracket \cdot \rrbracket : A \to A\) is a \(B\)-coalgebra homomorphism which makes it simple to show
that \(a_1^* \circ C^* \llbracket \cdot \rrbracket\) is also a \(B\)-coalgebra homomorphism (\(C^*A\) is a \(B\) – coalgebra for
Remark 11). Finally, \(a_1^* \circ t^*\) is a \(B\)-coalgebra homomorphism for Lemma 8.