Martingale and Pathwise Solutions to the Stochastic Zakharov-Kuznetsov Equation with Multiplicative Noise

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Abstract

We study in this article the stochastic Zakharov-Kuznetsov equation driven by a multiplicative noise. We establish, in space dimensions two and three the global existence of martingale solutions, and in space dimension two the global pathwise uniqueness and the existence of pathwise solutions. New methods are employed in the passage to the limit on a special type of boundary conditions and in the verification of the pathwise uniqueness of martingale solutions with a lack of regularity, where both difficulties arise due to the partly hyperbolic feature of the model.

Keywords: Zakharov-Kuznetsov equation, Korteweg-de Vries equation, stochastic fluid dynamics.

1 Introduction

We consider the stochastic Zakharov-Kuznetsov equation subject to multiplicative random noise

\[ du + (\Delta u_x + cu_x + uu_x) \, dt = f \, dt + \sigma(u) \, dW(t), \]  

(1.1)

where \( u = u(x, x^\perp, t), \) \( x^\perp = y \) or \( x^\perp = (y, z) \) in a limited domain \( \{(x, x^\perp), 0 < x < 1, x^\perp \in (-\pi/2, \pi/2)^d, d = 1, 2\}. \)

The deterministic ZK equation describes the propagation of nonlinear ionic-sonic waves in a plasma submitted to a magnetic field directed along the \( x \)-axis. It has been derived formally in a long wave, weakly nonlinear regime from the Euler-Poisson system in [ZK74], [LS82] and [LLS13] (see also [BPS81] and [BPS83] for more general physical backgrounds). When \( u \) depends only on \( x \) and \( t \), the ZK equation reduces to the classical Korteweg-de Vries (KdV) equation. Recently the ZK equation has caught considerable attention (see e.g. ([Fam06]), [Fam08], [LT13], [DL13], [BF13], [ST10] and [STW12]), not only because it is closely related with the physical phenomena but also because it is the start to explore more general problems that are partly hyperbolic (such as the inviscid primitive equations).

To capture the phenomena similar to those of more realistic fluid systems, random waves have attracted interests nowadays. For example, the stochastic KdV equation has been studied extensively (see [dBD09], [GB06] and [DP01]), where the main focus are on Wick-type SPDEs ([ZZ10] and [Liu07]) and exact solutions of the stochastic KdV equation under additive noise ([HR09]). However, to the best of our knowledge, there has been no result so far for the stochastic ZK equation driven by multiplicative noise. In particular, although (1.1) bears some similarity to the model studied in e.g. [DGHT11], the situation is in fact totally different and we can not use the results derived there; specifically, the operator \( Au := \Delta u_x + cu_x \) is neither symmetric nor coercive in our case (see Remark 1.1 for details).

In the present article, we extend the results of global existence (in space dimensions two and three) and uniqueness (in space dimension two) of weak solutions in [STW12] to the stochastic case. This initial program of well posed-ness will serve as the foundation for the investigation of the qualitative and quantitative properties of solutions in both the deterministic and stochastic cases and facilitate the comparison of these two models. Note that here we have different notions of solutions, namely, the martingale solutions and the pathwise solutions. In the former notion,
the stochastic basis is not specified in the beginning and is viewed as part of the unknown, while in the latter case, the stochastic basis is fixed in advance as part of the assumptions. To pass from the martingale to the pathwise solutions, we apply the extension of the Yamada-Watanabe theorem ([YW71]) to the infinite dimension (see [GK96]), that is, the pathwise solutions exist whenever there exists a pathwise unique martingale solution.

One of the main novelties of this paper is the treatment to the boundary conditions, which are more complicated than the usual Dirichlet or periodic ones. Firstly, it is not clear whether all the boundary conditions are still preserved after the application of the Skorokhod embedding theorem (Theorem 2.4 in [DPZ92]) since the underlying stochastic basis has been changed. To solve this problem, a measurability result concerning Hilbert spaces is developed (Lemma 5.2). Secondly, we have extended the trace results in [STW12] to the stochastic setting by establishing the trace properties of the linearized ZK equation depending on the probabilistic parameters (Lemma 5.3 and 5.5). This method can be used to deal with the non-conventional boundary conditions in other circumstances in future.

A further novelty is contained in the proof of the pathwise uniqueness (Section 4.2). Difficulties arise with the derivation of the energy inequality for the difference of the solutions due to the lack of regularity. Moreover, the method in the deterministic case (see [STW12]) can not be adapted to the stochastic case if we just use the stochastic version of the Gronwall lemma established in [GHZ09] (see also [MR04]), as issues would arise in passage to the limit on the terms involving stopping times. We overcome this difficulty by establishing a variant of the stochastic Gronwall lemma (Lemma 5.11), where we find that in certain situations we can weaken the hypotheses so as to avoid the stopping times. This idea of making use of Lemma 5.11 to deal with the insufficient regularity of the solutions to verify the pathwise uniqueness, we believe, is not only suited for this model but can also be applied to other classes of equations.

The present article is organized as follows. In Section 2 we make precise the assumptions on the problem (1.1) and the stochastic framework. In Section 3, we study the parabolic regularization of equation (1.1) in dimensions two and three. We show by Galerkin truncation the existence and uniqueness of the global pathwise solution \( u^\epsilon \) which is sufficiently regular for the subsequent calculations and more importantly, we establish the uniform estimates independent of \( \epsilon \), which we use in Section 4.1 to develop the compactness argument when \( \epsilon > 0 \) varies. Then with application of the Skorokhod embedding theorem, which leads to strong convergence of some subsequence, we obtain the global existence of martingale solutions to (1.1) in dimensions two and three. In Section 4.2 we prove in dimension two the pathwise uniqueness of martingale solutions and by the Gyöngy-Krylov Theorem deduce the global existence of pathwise solution.

Finally in the Appendix, we present the generalized trace results, the measurability result and the adapted stochastic Gronwall lemma, among the other existing results used in the article.

2 Stochastic ZK equation

We consider the stochastic ZK equation with multiplicative noise

\[
du + (\Delta u_x + cu_x + uu_x) dt = f dt + \sigma(u) dW(t),
\]

(2.1)
evolving in a rectangular or parallelepiped domain, namely, in \( \mathcal{M} = (0,1)_x \times (-\pi/2, \pi/2)^d \), with \( d = 1 \) or \( 2 \). In the sequel, we will use the notations \( I_x = (0,1)_x \), \( I_y = (-\pi/2, \pi/2)_y \) and
\( I_z = (-\pi/2, \pi/2) \). Here \( \Delta u = u_{xx} + \Delta^\perp u \), \( \Delta^\perp u = u_{yy} \) or \( u_{yy} + u_{zz} \). We assume that \( f \) is a deterministic function, and the white noise driven stochastic term \( \sigma(u) \, dW(t) \) is in general state dependent. As in [STW12], we assume the boundary conditions on \( x = 0, 1 \) to be

\[
|u|_{x=0} = |u|_{x=1} = u_{x}|_{x=1} = 0. \tag{2.2}
\]

For the boundary conditions in the \( y \) and \( z \) directions, we can choose either the Dirichlet boundary conditions

\[
u = 0 \text{ at } y = \pm \frac{\pi}{2} \quad \text{and} \quad z = \pm \frac{\pi}{2}, \tag{2.3}
\]

or the periodic boundary conditions

\[
|u|_{y = \frac{\pi}{2}} = |u|_{y = -\frac{\pi}{2}} = 0 \quad \text{and} \quad u|_{z = \frac{\pi}{2}} = u|_{z = -\frac{\pi}{2}} = 0. \tag{2.4}
\]

The initial condition reads:

\[
u(0) = u_0. \tag{2.5}
\]

For the simplicity of the presentation, we will mostly study the Dirichlet case (2.1)-(2.3) and (2.5). We will just make some remarks concerning the closely related space periodic case when (2.3) is replaced by (2.4).

### 2.1 Stochastic framework

In order to define the term \( \sigma(u) \, dW(t) \) in (2.1), we recall some basic notions and notations of stochastic analysis from [DGHT11]. For further details and background, see e.g. [PR07], [FG95], [Fla08], [Ben95] and [DPZ92].

To begin with we fix a stochastic basis

\[
\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W_k\}_{k \geq 1}), \tag{2.6}
\]

that is a filtered probability space with \( \{W_k\}_{k \geq 1} \) a sequence of independent standard one-dimensional Brownian motions relative to \( \{\mathcal{F}_t\}_{t \geq 0} \). In order to avoid unnecessary complications below we may assume that \( \mathcal{F}_t \) is complete and right continuous (see [DPZ92]).

We fix a separable Hilbert space \( \mathcal{U} \) with an associated orthonormal basis \( \{e_k\}_{k \geq 1} \). We may formally define \( W \) by taking \( W = \sum_{k=1}^\infty W_k e_k \). As such \( W \) is said to be a ‘cylindrical Brownian motion’ evolving over \( \mathcal{U} \).

We next recall some basic definitions and properties of spaces of Hilbert-Schmidt operators. For this purpose we suppose that \( X \) is any separable Hilbert space with the associated norm and inner product written as \( \| \cdot \|_X, \langle \cdot, \cdot \rangle_X \). We denote by

\[
L_2(\mathcal{U}, X) = \left\{ R \in L(\mathcal{U}, X) : \sum_k |R e_k|_X^2 < \infty \right\},
\]

the space of Hilbert-Schmidt operators from \( \mathcal{U} \) to \( X \). We know that the definition of \( L_2(\mathcal{U}, X) \) is independent of the choice of the orthonormal basis \( \{e_k\}_{k \geq 1} \) in \( X \). By endowing this space with the inner product \( \langle R, T \rangle_{L_2(\mathcal{U}, X)} = \sum_k \langle R e_k, T e_k \rangle_X \), we may consider \( L_2(\mathcal{U}, X) \) as itself being a
Hilbert space. Again this scalar product can be shown to be independent of the orthonormal basis \( \{e_k\}_{k \geq 1} \).

We also define the auxiliary space \( \mathcal{U}_0 \supset \mathcal{U} \) via

\[
\mathcal{U}_0 := \left\{ v = \sum_{k \geq 0} a_k e_k : \sum_k \frac{a_k^2}{k^2} < \infty \right\},
\]

equipped with the norm \( |v|_{\mathcal{U}_0}^2 = \sum_k \frac{a_k^2}{k^2}, v = \sum_k a_k e_k \). Note that the embedding of \( \mathcal{U} \subset \mathcal{U}_0 \) is Hilbert-Schmidt. Moreover, using standard martingale arguments combined with the fact that each \( W_k \) is almost surely continuous (see [DPZ92]) we obtain that, for almost every \( \omega \in \Omega \),

\( W(\omega) \in C([0, T], \mathcal{U}_0) \).

Given an \( X \)-valued predictable process \( \Psi \in L^2(\Omega; L^2((0, T), L_2(\mathcal{U}, X))) \), one may define the Itô stochastic integral

\[
M_t := \int_0^t \Psi \, dW = \sum_k \int_0^t \Psi_k \, dW_k,
\]
as an element in \( \mathcal{M}_X^2 \), that is the space of all \( X \)-valued square integrable martingales. In the sequel we will use the Burkholder-Davis-Gundy inequality which takes the form

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} \left| \int_0^s \Psi \, dW(t) \right|^r \right) \leq c_1 \mathbb{E} \left( \left( \int_0^T \| \Psi \|_{L_2(\mathcal{U}, X)}^2 \, dt \right)^{r/2} \right),
\]  \hspace{1cm} \text{(2.7)}

valid for any \( r \geq 1 \). Here \( c_1 \) is an absolute constant depending only on \( r \).

### 2.2 Conditions imposed on \( \sigma, f \) and \( u_0 \).

Given any pair of Banach spaces \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \), we denote by \( Bnd_u(\mathcal{X}_1, \mathcal{X}_2) \), the collections of all continuous mappings

\[
\Psi : \mathcal{X}_1 \to \mathcal{X}_2,
\]
such that

\[
\| \Psi(u) \|_{\mathcal{X}_2} \leq c_B (1 + \| u \|_{\mathcal{X}_1}), \quad u \in \mathcal{X}_1,
\]  \hspace{1cm} \text{(2.9)}

for some constant \( c_B \). In addition, if

\[
\| \Psi(u) - \Psi(v) \|_{\mathcal{X}_2} \leq c_U \| u - v \|_{\mathcal{X}_1}, \quad \forall u, v \in \mathcal{X}_1,
\]  \hspace{1cm} \text{(2.10)}

for some constant \( c_U \), we say that \( \Psi \in \text{Lip}_u(\mathcal{X}_1, \mathcal{X}_2) \). In the sequel we will consider time dependent families of such mappings \( \Psi = \Psi(t) \) and require that (2.9) and (2.10) hold for a.e. \( t \) with the same constants \( c_B, c_U \) for all \( t \)'s under consideration.

We shall assume throughout the work that

\[
\sigma : [0, \infty) \times L^2(\mathcal{M}) \to L_2(\mathcal{U}, L^2(\mathcal{M})).
\]  \hspace{1cm} \text{(2.11)}

Here \( \mathcal{U} \) and \( L_2(\mathcal{U}, L^2(\mathcal{M})) \) are as introduced above. Moreover we assume that for a.e. \( t \),

\[
\sigma(t) \in Bnd_u(L^2(\mathcal{M}), L_2(\mathcal{U}, L^2(\mathcal{M}))) \cap Bnd_u(\mathcal{E}_1, L_2(\mathcal{U}, \mathcal{E}_1)),
\]  \hspace{1cm} \text{(2.12)}
and
\[ \sigma(t) \in Lip_u(L^2(M), L_2(U, L^2(M))), \tag{2.13} \]
where
\[ \Xi_1 := \{ u \in H^2(M) \cap H^1_0(M), \ u_x|_{x=1} = 0 \}. \tag{2.14} \]
When proving pathwise uniqueness of martingale solutions and the existence of pathwise solutions in Section 4.2, we will additionally suppose that for a.e. \( t \),
\[ \sigma(t) \in Lip_u(L^2(M), L_2(U, \Xi_1)). \tag{2.15} \]
Furthermore in the sequel \( \sigma \) is a measurable function of \( t \) and all the corresponding norms of \( \sigma(t) \) are essentially (a.e.) bounded in time.

Finally we state the assumptions for the initial condition \( u_0 \) and for \( f \). On the one hand, in Section 4.1, where we consider only the case of martingale solutions, since the stochastic basis is an unknown of the problem, we will only be able to specify \( u_0 \) as an initial probability measure \( \mu_{u_0} \) on the space \( L^2(M) \) such that
\[ \int_{L^2(M)} |u|^6_{L^2(M)} \, d \mu_{u_0}(u) < \infty, \tag{2.16} \]
and we assume that \( f \) is deterministic,
\[ f = f(x, x^\perp, t) \in L^6(0, T; L^2(M)). \tag{2.17} \]
On the other hand, for pathwise uniqueness and the existence of pathwise solutions in Section 4.2, where the stochastic basis \( S \) is fixed in advance we assume that, relative to this basis, \( u_0 \) is an \( L^2(M) \)-valued random variable such that
\[ u_0 \in L^7(\Omega; L^2(M)) \text{ and } u_0 \text{ is } \mathcal{F}_0 \text{ measurable,} \tag{2.18} \]
and \( f \) is deterministic,
\[ f = f(x, x^\perp, t) \in L^7(0, T; L^2(M)). \tag{2.19} \]

3 Regularized stochastic ZK equation

As indicated above we consider the Dirichlet case, i.e. (2.1)-(2.3) and (2.5). The domain is \( \mathcal{M} = I_x \times (-\pi/2, \pi/2)^d, \) in \( \mathbb{R}^{d+1} \) with \( d = 1 \) or 2. In order to study this system, we will use a parabolic regularization of equation (2.1), as in [ST10]. That is, for \( \epsilon > 0 \) “small”, we consider the stochastic parabolic equation of the 4-th order in space:
\[ \begin{cases} \, du^\epsilon + \left[ \Delta u^\epsilon + cu^\epsilon + u^\epsilon u^\epsilon + \epsilon \left( \frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4} \right) \right] \, dt = f^\epsilon \, dt + \sigma(u^\epsilon) \, dW(t), \, \\
\, u^\epsilon(0) = u_0^\epsilon, \end{cases} \tag{3.1} \]
supplemented with the boundary conditions (2.2), (2.3) and the additional boundary conditions
\[ u^\epsilon_{yy}|_{y=\pm \pi/2} = u^\epsilon_{zz}|_{z=\pm \pi/2} = 0, \tag{3.2} \]
\[ u'_{xx}|_{x=0} = 0. \] (3.3)

In the case of martingale solutions in Section 4.1, observing that the space \( L^2(\Omega; \Xi_1) \cap L^{22/3}(\Omega; L^2(\mathcal{M})) \) is dense in \( L^6(\Omega; L^2(\mathcal{M})) \), we can use e.g. the Fourier series to construct an approximate family \( \{u'_0\}_{\epsilon>0} \) which is \( \mathcal{F}_0 \) measurable, such that, as \( \epsilon \to 0 \):

\[
\begin{align*}
\quad u'_0 &\in L^2(\Omega; \Xi_1) \cap L^{22/3}(\Omega; L^2(\mathcal{M})), \\
\quad u'_0 &\to u_0 \text{ in } L^6(\Omega; L^2(\mathcal{M})).
\end{align*}
\] (3.4)

Similarly there exists a family of deterministic functions \( \{f^\epsilon\}_{\epsilon>0} \) such that as \( \epsilon \to 0 \):

\[
\begin{align*}
\quad f^\epsilon &\in L^{22/3}(0,T; L^2(\mathcal{M})), \\
\quad f^\epsilon &\to f \text{ in } L^6(0,T; L^2(\mathcal{M})).
\end{align*}
\] (3.6)

In the case of pathwise solutions in Section 4.2, in the same way we can deduce the existence of the approximate families \( \{u'_0\}_{\epsilon>0} \) and \( \{f^\epsilon\}_{\epsilon>0} \) satisfying (3.4) and (3.6) respectively, and such that as \( \epsilon \to 0 \):

\[
\begin{align*}
\quad u'_0 &\to u_0 \text{ in } L^7(\Omega; L^2(\mathcal{M})), \\
\quad f^\epsilon &\to f \text{ in } L^7(0,T; L^2(\mathcal{M})).
\end{align*}
\] (3.8)

For notational convenience, as in [STW12], we recast (3.1) in the form

\[
\begin{align*}
\quad d\tilde{u}^\epsilon &= (-Au^\epsilon - B(u^\epsilon) - \epsilon Lu^\epsilon + f^\epsilon) dt + \sigma(u^\epsilon) dW(t), \\
\quad u^\epsilon(0) &= u'_0,
\end{align*}
\] (3.10)

where

\[
\begin{align*}
\quad Au &= \Delta u_x + cu_x, \quad \forall \ u \in D(A), \\
\quad B(u,v) &= uv_x \in H^{-1}(\mathcal{M}), \quad \forall \ u \in L^2(\mathcal{M}), \ v \in H^1(\mathcal{M}), \\
\quad Lu &= u_{xxxx} + u_{yyyy} + u_{zzzz}, \quad \forall \ u \in H^4(\mathcal{M}).
\end{align*}
\] (3.11)

with \( D(A) = \{ u \in L^2(\mathcal{M}) : Au \in L^2(\mathcal{M}), \ u = 0 \text{ on } \partial\mathcal{M}, \ u_x = 0 \text{ at } x = 1 \} \). Note that a trace theorem proven in [STW12] shows that if \( u \in L^2(\mathcal{M}) \) and \( Au \in L^2(\mathcal{M}) \) then the traces of \( u \) on \( \partial\mathcal{M} \) and of \( u_x \) at \( x = 1 \) make sense.

**Remark 3.1.** As mentioned in the Introduction, although we can rewrite (2.1) as

\[
\quad du + (Au + B(u)) dt = f dt + \sigma(u) dW(t),
\] (3.12)

which is similar to the equation studied in [DGHT11], the models are actually different. Indeed, the operator \( A \) does not satisfy the assumptions in [DGHT11]; for example, \( A \) is not symmetric. In fact, for the adjoint \( A^* \) and its domain \( D(A^*) \), we have

\[
\begin{align*}
\quad D(A^*) &= \{ \bar{u} \in L^2(\mathcal{M}) : \bar{A}\bar{u} \in L^2(\mathcal{M}), \bar{u} = 0 \text{ on } \partial\mathcal{M}, \bar{u}_x = 0 \text{ at } x = 0 \}, \\
\quad A^*\bar{u} &= -((\Delta\bar{u}_x + c\bar{u}_x), \ \bar{u} \in D(A^*). \quad (3.13)
\end{align*}
\]

For more details see Section 2.3.2 in [STW12].
3.1 Definition of solutions

We first introduce the necessary operators and functional spaces. We will denote by \((\cdot, \cdot)\) and \(|\cdot|\) the inner product and the norm of \(L^2(M)\). The space \(\Xi_1\) defined in (2.14) is endowed with the scalar product and norm \([\cdot, \cdot]_2\), \(|\cdot|_2\):

\[
[u, v]_2 = (u_{xx}, v_{xx}) + (u_{yy}, v_{yy}) + (u_{zz}, v_{zz}),
\]

\[
[u]_2^2 = |u_{xx}|^2 + |u_{yy}|^2 + |u_{zz}|^2,
\]

which make it a Hilbert space. Note that since \(|\Delta u| + |u|\) is a norm on \(H^1_0 \cap H^2\) equivalent to the \(H^2\)-norm, \([\cdot]_2\) is a norm on \(\Xi_1\) equivalent to the \(H^2\)-norm. Thanks to the Riesz theorem, we can associate to the scalar product \([\cdot, \cdot]_2\) the isomorphism \(L\) from \(\Xi_1\) onto \(\Xi_1'\), where \(L\) denotes the abstract operator corresponding to the differential operator \(L\). Then considering the Gelfand triple \(\Xi_1 \subset H := L^2(M) \subset \Xi_1'\), we introduce \(L^{-1}(H)\) the domain of \(L\) in \(H\), which is the space

\[
\Xi_2 = \left\{ u \in \Xi_1 \cap H^4(M), u_{yy} |_{y=\pm \frac{\pi}{2}} = u_{zz} |_{z=\pm \frac{\pi}{2}} = u_{xx} |_{x=0} = 0 \right\}.
\]

The operator \(L^{-1}\) is self-adjoint and compact in \(H\). It possesses an orthonormal set of eigenvectors which is complete in \(H\), and which we denote by \(\{\phi_i\}_{i \geq 1}\). Note that all the \(\phi_i\) belong to \(\Xi_2\) which is the domain of \(L\) in \(H\). Hence we have

\[
(Lu, v) = [u, v]_2, \quad u \in \Xi_2, \quad v \in \Xi_1.
\]

We now introduce the following definitions.

**Definition 3.1.** (Global martingale solutions for the regularized ZK equation) Fix an \(\epsilon > 0\). For the case of martingale solutions, we only specify the measure \(\mu_{u_0}\) to be the probability measure of \(u_0\) on \(\Xi_1\) which satisfies

\[
\int_{L^2(M)} |u|^{22/3} d\mu_{u_0}(u) < \infty,
\]

(3.16)

\[
\int_{\Xi_1} |u|^2 d\mu_{u_0}(u) < \infty,
\]

(3.17)

and \(f^\epsilon\) and \(\sigma\) satisfy (3.6), (2.12) and (2.13).

A pair \((\tilde{S}, \tilde{u}^\epsilon)\) is a global martingale solution to the regularized stochastic ZK equation (3.1)-(3.3), (2.2) and (2.3) (in the Dirichlet case), if \(\tilde{S} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}}, \{\tilde{W}^k\}_{k \geq 1})\) is a stochastic basis, and \(\tilde{u}^\epsilon(\cdot) : \tilde{\Omega} \times [0, \infty) \to \Xi_1\) is an \{\mathcal{F}_t\} adapted process such that:

\[
\tilde{u}^\epsilon \in L^{22/3}(\tilde{\Omega}; L^\infty(0, T; L^2(M))) \cap L^2(\tilde{\Omega}; L^\infty([0, T]; \Xi_1) \cap L^2(0, T; \Xi_2)),
\]

(3.18)

and

\[
\tilde{u}^\epsilon(\cdot, \omega) \in C([0, T]; L^2_w(M))) \tilde{\mathbb{P}} - \text{a.s.},
\]

(3.19)

where \(L^2_w(M)\) is \(L^2(M)\) equipped with the weak topology, and the law of \(\tilde{u}^\epsilon(0)\) is \(\mu_{u_0}\), defined as above, i.e. \(\mu_{u_0}(E) = \tilde{\mathbb{P}}(\tilde{u}^\epsilon(0) \in E)\), for all Borel subsets \(E\) of \(\Xi_1\), and finally \(\tilde{u}^\epsilon\) almost surely satisfies

\[
\tilde{u}^\epsilon(t) + \int_0^t (A\tilde{u}^\epsilon + B(\tilde{u}^\epsilon) + \epsilon L\tilde{u}^\epsilon - f^\epsilon) \, ds = \tilde{u}^\epsilon(0) + \int_0^t \sigma(\tilde{u}^\epsilon) \, d\tilde{W},
\]

(3.20)

as an equation in \(L^2(M)\) for every \(0 \leq t \leq T\).
Definition 3.2. (Global pathwise solutions for the regularized ZK equation; Uniqueness)

Let \( S := (\Omega, F, \{ F_t \}_{t \geq 0}, \mathbb{P}, \{ W^k \}_{k \geq 1}) \) be a fixed stochastic basis and assume that \( u_0, \sigma \) and \( f^\varepsilon \) satisfy (3.4), (2.12), (2.13) and (3.6).

(i) For any fixed \( \varepsilon > 0 \), a random process \( u^\varepsilon \) is a global pathwise solution to (3.1)-(3.3), (2.2) and (2.3) if \( u^\varepsilon \) is an \( F_t \) adapted process in \( L^2(M) \) so that (relative to the fixed-given-basis \( S \)) (3.18)-(3.20) hold.

(ii) Global pathwise solutions of (3.1)-(3.3), (2.2) and (2.3) are said to be global (pathwise) unique if given any pair of pathwise solutions \( u^\varepsilon, v^\varepsilon \) which coincide at \( t = 0 \) on a subset \( \Omega_0 \) of \( \Omega \), \( \Omega_0 = \{ u^\varepsilon(0) = v^\varepsilon(0) \} \), then

\[
\mathbb{P}\{ 1_{\Omega_0}(u^\varepsilon(t) = v^\varepsilon(t)) \} = 1, \ 0 \leq t \leq T. \tag{3.21}
\]

In the sequel, we will prove that there exists a unique global pathwise solution \( u^\varepsilon \) to (3.1)-(3.3), (2.2) and (2.3), which is sufficiently regular for the calculations in Section 4.1 to be fully legitimate without any need of further regularization. The existence of such a solution is basically classical (see e.g. [DPZ92], [Fla08], [FG95] and [DGHT11]) for a parabolic problem like this, but we will make partly explicit the construction of \( u^\varepsilon \) because we need to see how the estimates depend or not on \( \varepsilon \).

3.2 Pathwise solutions in dimensions 2 and 3

With the above definitions, we can state the main result of section 3:

Theorem 3.1. When \( d = 1, 2 \), suppose that, relative to a fixed given stochastic basis \( S \), \( u_0 \) satisfies (3.4), and that \( f^\varepsilon \) and \( \sigma \) satisfy (3.6), (2.12) and (2.13), with \( \varepsilon > 0 \) fixed arbitrary. Then there exists a unique global pathwise solution \( u^\varepsilon \) which satisfies (3.1) and the boundary conditions (2.2), (2.3), (3.2) and (3.3).

To prove this theorem, we first use a Galerkin scheme to derive the estimates indicating a compactness argument based on fractional Sobolev spaces and tightness properties of the truncated sequence. Then by the Skorokhod embedding theorem (see Theorem 2.4 in [DPZ92], also [Bil86] and [Jak97]) we can pass to the limit in the Galerkin truncation and hence obtain the global existence of martingale solutions. Finally we deduce the existence of global pathwise solutions using pathwise uniqueness of martingale solutions and the Gyöngy-Krylov Theorem (Theorem 5.1 of the Appendix). Here we will only present in details the derivation of the estimates, which will be utilized in the subsequent investigations of the stochastic ZK equation in Section 4.1.1.

We start the proof of Theorem 3.1 by introducing the Galerkin system. We define \( P^n \) as the orthogonal projector from \( L^2(M) \) onto \( H^n \), the space spanned by the first \( n \) eigenfunctions of \( \mathcal{L}, \phi_1, ..., \phi_n \). We consider the Galerkin system as follows

\[
\begin{cases}
  du^n + (A^n u^n + B^n(u^n)) \ dt + \varepsilon L u^n \ dt = f^n \ dt + \sigma^n(u^n) \ dW(t), \\
  u^n(0) = P_n u_0,
\end{cases}
\tag{3.22}
\]

where \( u^n \) maps \( \Omega \times [0, T] \) into \( H^n \), \( A^n u^n := P^n A u^n, B^n(u^n) := P^n B(u^n) \), and \( \sigma^n(u^n) := P^n(\sigma(u^n)) \). In (3.22), \( \varepsilon \) being fixed, we write for simplicity \( u^n \) for \( u^{\varepsilon,n} \) and \( f^n \) for \( f^{\varepsilon,n} \).
3.2.1 Estimates independent of $\epsilon$ and $n$

We first derive the following estimates on $u^{\epsilon,n}$ independent of $\epsilon$ and $n$.

**Lemma 3.1.** With the same assumptions as in Theorem 3.1, if $u_0$ and $f^\epsilon$ satisfy (3.5) and (3.7) respectively, then the following estimates hold for $u^n = u^{\epsilon,n}$ independently of $\epsilon$ and $n$:

\[
\begin{align*}
\left|u^{\epsilon,n}\right|_{x=0} & \text{ remains bounded in } L^2(\Omega; L^2(0, T; L^2(I_{x\perp}))), \\
\sqrt{\epsilon}u^{\epsilon,n} & \text{ remains bounded in } L^2(\Omega; L^2(0, T; \Xi_1)), \\
u^{\epsilon,n} & \text{ remains bounded in } L^6(\Omega; L^\infty(0, T; L^2(\mathcal{M}))).
\end{align*}
\]

If we further assume that (3.8) and (3.9) hold, then

\[
u^{\epsilon,n} \text{ remains bounded in } L^7(\Omega; L^\infty(0, T; L^2(\mathcal{M}))),
\]

with the bounds in (3.24)-(3.27) independent of both $\epsilon$ and $n$.

**Proof.** We start by applying the Itô formula to (3.22). This yields

\[
d\left|u^n\right|^2 = 2 \left(u^n, \mathcal{N}^n(u^n)\right) dt + 2 \left(u^n, \sigma^n(u^n) dW(t)\right) + \left|\sigma^n(u^n)\right|^2_{L^2(0, T; L^2(\mathcal{M}))} dt,
\]

where $\mathcal{N}^n(u^n) := -A^n u^n - B^n(u^n) - \epsilon Lu^n + f^n$, and

\[
\begin{align*}
\left(u^n, \mathcal{N}^n(u^n)\right) &= -\left(u^n, A^n u^n\right) - \left(u^n, B^n(u^n)\right) - \epsilon\left(u^n, Lu^n\right) + \left(u^n, f^n\right) \\
&= -\left(u^n, A^n u^n\right) - \left(u^n, B(u^n)\right) - \epsilon\left(u^n, Lu^n\right) + \left(u^n, f^n\right).
\end{align*}
\]

To compute the right-hand side of (3.29) we remember that by (3.23), for every $t$, $u^n(t) \in H^n = \text{span } (\phi_1, ..., \phi_n) \in \Xi_2$ a.s., since all the $\phi_i$ belong to $\Xi_2$. Hence in particular $u^n(t)$ satisfies the boundary conditions in (3.15). We drop the super index $n$ for the moment and perform the following calculations a.s. exactly as in [STW12]:

- $(Au, u) = \int_M \Delta u_x u \, dM + c \int_M u_x u \, dM$
- $(Bu, u) = 0,$
- $\varepsilon(Lu, u) = \varepsilon \int_M \left(|u_{xx}|^2 + |u_{yy}|^2 + |u_{zz}|^2\right) \, dM.$
Hence by (3.28) we find
\[ d|u^n|^2 + \left( |u^n_x|_{L^2(I_{x_0})}^2 + 2\epsilon |u^n|^2 \right) dt = 2(f^n, u^n) dt + \left| \sigma^n(u^n) \right|_{L^2(\Omega, L^2(\mathcal{M}))}^2 dt + 2(u^n, \sigma^n(u^n) dW(t)). \] (3.30)

Integrating both sides from 0 to s with 0 \leq s \leq r \leq T, taking the supremum over [0, r], we have
\[
\sup_{0 \leq s \leq r} |u^n(s)|^2 + \int_0^r \left( |u^n_x|^2_{L^2(I_{x_0})} + 2\epsilon |u^n|^2 \right) dt \\
\leq |u_0^n|^2 + 2 \int_0^r |(f^n, u^n)| dt + \int_0^r \left| \sigma^n(u^n) \right|_{L^2(\Omega, L^2(\mathcal{M}))}^2 dt \\
+ 2 \sup_{0 \leq s \leq r} \left| \int_0^s (u^n, \sigma^n(u^n) dW(t)) \right|. 
\] (3.31)

Raising both sides to the power \( p/2 \) for \( p \geq 2 \), then taking expectations, we obtain with the Minkowski inequality and Fubini’s Theorem
\[
\mathbb{E} \sup_{0 \leq s \leq r} |u^n(s)|^p \leq \mathbb{E}|u_0^n|^p + 2 \mathbb{E} \int_0^r |(f^n, u^n)|^{p/2} dt \\
+ \mathbb{E} \int_0^r \left| \sigma^n(u^n) \right|_{L^2(\Omega, L^2(\mathcal{M}))}^p dt \\
+ 2 \mathbb{E} \left( \sup_{0 \leq s \leq r} \left| \int_0^s (u^n, \sigma^n(u^n) dW(t)) \right| \right)^{p/2}, 
\] (3.32)

where \( \leq \) means \( \leq \) up to an absolute multiplicative constant. Here and below \( c' \) indicates an absolute constant, whereas \( \eta, \kappa, \) and the \( \kappa_i \) indicate constants depending on the data \( u_0, f, \) etc. These constants may be different at each occurrence. We estimate the terms on the right-hand side of (3.32) a.s. and for a.e. \( t \):
\[
|(f^n, u^n)|^{p/2} \leq |f^n|^{p/2} |u^n|^{p/2} \leq |u^n|^p + |f^n|^p, \\
\left| \sigma^n(u^n) \right|_{L^2(\Omega, L^2(\mathcal{M}))}^p \leq \text{by (2.12)} \leq c_B^p (|u^n| + c')^p \leq |u^n|^p + c';
\]

for the stochastic term, we use the Burkholder-Davis-Gundy inequality (see (2.7)) and (2.12)
\[
\mathbb{E} \left( \sup_{0 \leq s \leq r} \left| \int_0^s (u^n, \sigma^n(u^n) dW(t)) \right| \right)^{p/2} \\
\leq \mathbb{E} \sup_{0 \leq s \leq r} \left| \int_0^s (u^n, \sigma^n(u^n) dW(t)) \right|^{p/2} \\
\leq c_1 \mathbb{E} \left[ \left( \int_0^r |u^n|^2 \left| \sigma^n(u^n) \right|_{L^2(\Omega, L^2(\mathcal{M}))}^2 dt \right)^{p/4} \right] \\
\leq \mathbb{E} \left[ \left( \sup_{0 \leq s \leq r} |u^n|^2 \right) \left( \int_0^r 1 + |u^n|^2 dt \right)^{p/4} \right] \\
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq r} |u^n|^p + c' \mathbb{E} \int_0^r |u^n|^p dt + c'.
\]
Applying the above estimates to (3.32), we obtain
\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq r} |u^n(s)|^p \leq \mathbb{E} |u_0^n|^p + c' \mathbb{E} \int_0^r |u^n(t)|^p \, dt + \mathbb{E} \int_0^r |f^n(t)|^p \, dt + c'. \tag{3.33}
\]
Since \( \mathbb{E} \int_0^r |u^n(t)|^p \, dt \leq \int_0^T \mathbb{E} \sup_{0 \leq t \leq r} |u^n(t)|^p \, dt \), setting \( \mathbb{E} \sup_{0 \leq s \leq r} |u^n(s)|^p =: U(r) \), with (3.33) we deduce
\[
U(r) \leq U(0) + c' \int_0^r U(t) \, dt + \int_0^r \mathbb{E} |f^n(t)|^p \, dt + c',
\]
for every \( 0 \leq r \leq T \). Hence applying the (deterministic) Gronwall lemma, we obtain for \( p \geq 2 \),
\[
\mathbb{E} \sup_{0 \leq r \leq T} |u^n(r)|^p \leq \mathbb{E} |u_0^n|^p + \mathbb{E} \int_0^T |f^n(t)|^p \, dt + c'. \tag{3.34}
\]
Letting \( p = 6 \), thanks to (3.5) and (3.7), we deduce that
\[
\mathbb{E} \sup_{0 \leq r \leq T} |u^n(r)|^6 \leq \kappa_1, \tag{3.35}
\]
for a constant \( \kappa_1 \) depending only on \( u_0, f, T \) and \( \sigma \), and independent of \( \epsilon \) and \( n \); this implies (3.26). Similarly, setting \( p = 7 \) in (3.34), we infer (3.27) from (3.8) and (3.9). Finally, setting \( p = 2 \) in (3.33), along with (3.31) we obtain (3.24) and (3.25).

### 3.2.2 Estimates dependent on \( \epsilon \)

We now derive estimates independent of \( n \) only, that is, valid for fixed \( \epsilon \).

**Lemma 3.2.** With the same assumptions as in Theorem 3.1, the following estimates hold for \( u^n = u^{\epsilon,n} \), as \( n \to \infty \) and \( \epsilon > 0 \) remains fixed:
\[
u^{\epsilon,n} \text{ remains bounded in } L^{22/3}(\Omega; L^\infty(0, T; L^2(M))),
\tag{3.36}
\]
\[
u^{\epsilon,n} \text{ remains bounded in } L^2(\Omega; L^\infty(0, T; \Xi_1)),
\tag{3.37}
\]
\[
u^{\epsilon,n} \text{ remains bounded in } L^2(\Omega; L^2(0, T; \Xi_2)).
\tag{3.38}
\]

**Proof.** Setting \( p = 22/3 \) in (3.34), we infer (3.36) from (3.4) and (3.6).

Returning to (3.22), we apply the Itô formula to (3.22) and obtain an evolution equation for the \( \Xi_1 \) norm:
\[
d[|u^n|^2] = 2(Lu^n, N(u^n)) \, dt + 2(Lu^n, \sigma^n(u^n) \, dW(t)) + ||\sigma^n(u^n)||_{L^2(\mu, \Xi_1)}^2 \, dt,
\tag{3.39}
\]
where \( N^n(u^n) \) has been defined before. Similar to (3.29) we have a.s. and for a.e. \( t \):
\[
(Lu^n, N^n(u^n)) = -(Lu^n, Au^n) - (Lu^n, B(u^n)) - \epsilon |Lu^n|^2 + (Lu^n, f^n).
\tag{3.40}
\]

\(^1\)Note that here \( f^{\epsilon,n} \) is actually independent of \( \omega \in \Omega \) and the symbol \( \mathbb{E} \) in front of the corresponding term is not needed. However in Section 4.2.1 we will use another version of this calculation in which \( f^{\epsilon,n} \) is replaced by \( g' \) which depends on \( \omega \); hence we leave \( \mathbb{E} \) in front of the term involving \( f^{\epsilon,n} \) in view of the calculations in Section 4.2.1.
By (3.39) and (3.40) we deduce

\[
d[u_n]^2 + \epsilon |Lu_n|^2 dt = -2(Lu_n, Au_n) dt - 2(Lu_n, B(u^n)) dt + 2(Lu_n, f^n) dt + 2 \left( Lu_n, \sigma^n(u^n) dW(t) \right) + \|\sigma^n(u^n)\|^2_{L^2(U, \Omega_1)} dt.
\]

Integrating both sides from 0 to s with 0 ≤ s ≤ T, taking the supremum over [0, T], then taking expectations, we arrive at

\[
\mathbb{E} \sup_{0 \leq s \leq T} [u_n]^2 + \mathbb{E} \int_0^T |Lu_n|^2 dt \leq \mathbb{E} |u_0|^2 + 2 \mathbb{E} \int_0^T |(Lu_n, Au_n)| dt + 2 \mathbb{E} \int_0^T |(Lu_n, B(u^n))| dt + 2 \mathbb{E} \int_0^T |(Lu_n, f^n)| dt + 2 \mathbb{E} \int_0^T \|\sigma^n(u^n)\|^2_{L^2(U, \Omega_1)} dt.
\]

(3.42)

We estimate a.s. each term on the right-hand side of (3.42); we emphasize that the estimates depend on \( \epsilon \) but not on \( n \) or on \( \omega \in \Omega \):

\[
|(Lu_n, Au_n)| \leq |Lu_n||u_n|_{H^3(M)} \leq |Lu_n||u_n|_{H^2}^{1/4}|u_n|^{3/4} \leq |Lu_n|^{7/4}|u_n|_{L^2}^{1/4} \leq \frac{\epsilon}{4}|Lu_n|^2 + \eta(\epsilon)|u_n|^2,
\]

where \( \eta(\epsilon) \) depends on \( \epsilon \). For the term \(|(Lu_n, B(u^n))|\), we first estimate a.s. \(|B(u^n)|\) in dimension three:

\[
|u_n u_n| \leq |u_n|^{3/2}_{H^3(M)} |u_n|^{1/2}_{H^2(M)} \leq (\text{by interpolation in dimension three, } |u_n|_{H^3(M)} \lesssim |u_n|^{3/4}_{H^2(M)} |u_n|^{1/4}_{H^4(M)}),
\]

and

\[
|u_n|_{H^2(M)} \lesssim |u_n|^{1/2}_{H^3(M)} |u_n|^{1/2}_{H^2(M)} \lesssim |u_n|^{9/8}_{H^2(M)} |u_n|^{3/8}_{H^4(M)} |u_n|^{1/4}_{H^2(M)} \lesssim |u_n|^{11/8} |Lu_n|^{5/8}.
\]

(3.43)

Hence

\[
|(Lu_n, B(u^n))| \leq |B(u^n)| |Lu_n| \leq (\text{by (3.43)}) \lesssim |u_n|^{11/8} |Lu_n|^{13/8} \lesssim \frac{\epsilon}{4}|Lu_n|^2 + \eta(\epsilon)|u_n|^{22/3},
\]

where \( \eta(\epsilon) \) depends on \( \epsilon \). For the stochastic term, we have

\[
\mathbb{E} \sup_{0 \leq s \leq T} \left| \int_0^s (Lu_n, \sigma^n(u^n) dW(t)) \right| \leq (\text{by the Burkholder-Davis-Gundy inequality (2.7)})
\]

\[
\leq c_1 \mathbb{E} \left[ \left( \int_0^T |Lu_n|^2 \|\sigma^n(u^n)\|^2_{L^2(U, L^2(M))} dt \right)^{1/2} \right]
\]

\[
\leq c_1 c_B^2 \mathbb{E} \left[ \left( \int_0^T |Lu_n|^2 (1 + |u_n|^2) dt \right)^{1/2} \right]
\]

\[
\leq \eta(\epsilon) \mathbb{E} \sup_{0 \leq s \leq T} |u_n|^2 + \frac{\epsilon}{4} \mathbb{E} \int_0^T |Lu_n|^2 dt + \eta(\epsilon),
\]

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where $\eta(\epsilon)$ depends on $\epsilon$.

For the term $\mathbb{E} \int_0^T ||\sigma^n(u^n)||_{L^2(\mathcal{M})}^2 \, dt$, we infer from (2.12) that

$$
\mathbb{E} \int_0^T ||\sigma^n(u^n)||_{L^2(\mathcal{M})}^2 \, dt \lesssim \mathbb{E} \int_0^T (1 + [u^n]^2) \, dt.
$$

Collecting all the above estimates, along with (3.42) we deduce

$$
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq T} [u^n(s)]^2 + \frac{\epsilon}{4} \mathbb{E} \int_0^T |Lu^n|^2 \, dt \\
\leq \mathbb{E} [u^n|^2/2 + \eta(\epsilon) \mathbb{E} \int_0^T |u^n|^2 \, dt + \eta(\epsilon) \mathbb{E} \int_0^T |u^n|^2/2 \, dt + \eta(\epsilon).
$$

Hence we can apply (3.36) and (3.25) to (3.44), and we obtain (3.37) and (3.38). Thus we have completed the proof of Lemma 3.2.

3.2.3 Estimates in fractional Sobolev spaces.

We will apply the compactness result based on fractional Sobolev spaces in Lemma 5.2 (of the Appendix) with

$$
\mathcal{Y} := L^2(0, T; H^\alpha_0(\mathcal{M})) \cap W^{\alpha,2}(0, T; \Xi_2'), \quad 0 < \alpha < \frac{1}{2},
$$

where $\Xi_2'$ is the dual of $\Xi_2$ relative to $L^2(\mathcal{M})$. For that purpose we will need the following estimates on fractional derivatives of $u^{\epsilon,n}$.

**Lemma 3.3.** With the same assumptions as in Theorem 3.1, we have

$$
\mathbb{E}|u^{\epsilon,n}|_{\mathcal{Y}} \leq \kappa_2(\epsilon),
$$

$$
\mathbb{E} \left| u^{\epsilon,n}(t) - \int_0^t \sigma(u^{\epsilon,n}) \, dW(s) \right|^2_{H^1(0,T;\Xi_2')} \leq \kappa_3,
$$

$$
\mathbb{E} \left| \int_0^t \sigma(u^{\epsilon,n}) \, dW(s) \right|^2_{W^{\alpha,6}(0,T;L^2(\mathcal{M}))} \leq \kappa_4, \quad \forall \alpha < \frac{1}{2},
$$

where $\kappa_2(\epsilon)$ is independent of $n$ (but may depend on $\epsilon$ and other data), while $\kappa_3$ and $\kappa_4$ depend only on $u_0$, $f$, $T$ and $\sigma$, and are independent of $\epsilon$ and $n$.

**Proof.** (3.22) can be written as

$$
u^n(t) = u^n_0 - \int_0^t A^n u^n \, ds - \int_0^t B^n(u^n) \, ds \\
- \epsilon \int_0^t L u^n \, ds + \int_0^t f^n \, ds + \int_0^t \sigma^n(u^n) \, dW(s) \\
:= J_1^n + J_2^n + J_3^n + J_4^n + J_5^n + J_6^n.
$$
For $J_2^n$, fixing $u^+ \in D(A^\ast)$ we have a.s. and for a.e. $t$
\[
\left| (A^n u^n, u^+) \right| = \left| (u^n, A^* P^n u^+) \right| \leq |u^n| \left| P^n u^+ \right|_{D(A^\ast)} \leq (\text{since } \Xi_2 \subseteq D(A^\ast)) \leq |u^n||u^+|_{\Xi_2}.
\]
Hence
\[
|A^n u^n|_{\Xi_2} \lesssim |u^n|.
\]
(3.50)

With (3.50) and (3.26) we obtain
\[
E|J_2^n|_{W^{1,6}(0,T;\Xi_2)} is bounded independently of $n$ and $\epsilon$.
\]
(3.51)

For $J_3^n$, firstly we observe that $\forall u^+ \in \Xi_2$ (dropping the super index $n$ for the moment),
\[
\left| (B(u), u^+) \right| = \left| \int_{M} \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) u^+ \, d\mathcal{M} \right| = \frac{1}{2} \left| \int_{M} u^2 u^+ \, d\mathcal{M} \right|
\]
\[
\leq \frac{1}{2} |u|^2 |u^+_x|_{L^\infty(M)}
\]
\[
\leq (\text{with } H^3(M) \subseteq L^\infty(M) \text{ in dimension 3})
\]
\[
\lesssim |u|^2 |u^+_x|_{H^3(M)}
\]
\[
\lesssim |u|^2 |u^+|_{\Xi_2},
\]
(3.52)

hence
\[
\left| (B^n(u^n), u^+) \right| = \left| (B(u^n), P^n u^+) \right| \lesssim |u^n|^2 |P^n u^+|_{\Xi_2} \leq |u^n|^2 |u^+|_{\Xi_2},
\]
(3.53)

which implies that $|B^n(u^n)|_{\Xi_2} \lesssim |u^n|^2$. This along with (3.26) implies that
\[
E|B^n(u^n)|_{L^2(0,T;\Xi_2)} is bounded independently of $n$ and $\epsilon$,
\]
(3.54)

and hence
\[
E|J_3^n|_{H^1(0,T;\Xi_2)} is bounded independently of $n$ and $\epsilon$.
\]
(3.55)

For $J_4^n$, we have, $\forall u^+ \in \Xi_2$, $\left| (Lu^n, u^+) \right| = \left| (u^n, Lu^+) \right| \leq |u^n||Lu^+|$. Hence $|Lu^n|_{\Xi_2} \lesssim |u^n|.$

Thus
\[
E \int_0^T |Lu^n|_{\Xi_2}^2 \, dt \leq 2 E \int_0^T |u^n|^2 \, dt.
\]

Multiplying both sides by $\epsilon^2$, we obtain with (3.26)
\[
E|J_4^n|_{H^1(0,T;\Xi_2)} is bounded independently of $n$ and $\epsilon$.
\]
(3.56)

For $J_5^n$, Lemma 5.7 implies that, $\forall \alpha < \frac{1}{2}$,
\[
E \left| \int_0^t \sigma^n(u^n(s)) \, dW(s) \right|_{W^{0,6}(0,T;L^2(M))}^6 \lesssim E \int_0^t |\sigma^n(u^n(s))|_{L^2(M)}^6 \, ds
\]
\[
\lesssim E \int_0^t |\sigma(u^n(s))|_{L^2(M)}^6 \, ds
\]
\[
\leq (\text{by (2.12)})
\]
\[
\leq cB E \int_0^t (1 + |u^n|)^6 \, ds.
\]

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This together with (3.26) implies that
\[ E \left| J_0^n \right|^2_{W^{a,\alpha}(0,T; L^2(\mathcal{M}))} \] is bounded independently of \( n \) and \( \epsilon \), \( \forall \alpha < \frac{1}{2} \). (3.57)

Hence we obtain (3.48). Collecting the estimates (3.51) and (3.55)-(3.57), we obtain
\[ E|u^n|_{W^{a,2}(0,T; \Xi_2)} \] is bounded independently of \( n \) and \( \epsilon \), \( \alpha < \frac{1}{2} \). (3.58)

By (3.37) we deduce
\[ E|u^n|_{L^2(0,T; H^1_0(M))] } \] is bounded independently of \( n \), (3.59)
but the bounds may depend on \( \epsilon \). From (3.58) and (3.59) we obtain (3.46).

Observing from (3.49) that \( u^n(t) - \int_0^t \sigma^n(u^n) \, dW(s) = J_1^n + J_2^n + J_3^n + J_4^n + J_5^n \), and applying (3.51), (3.55) and (3.56), we obtain (3.47) as desired. \( \square \)

Remark 3.2. See Lemma 4.3 below for a variant of the proof of Lemma 3.3 leading to the analogue of bounds in (3.46)-(3.48) but independent of \( \epsilon \). Note however that the proof in Lemma 4.3 for \( u^\epsilon \) can not be applied here to \( u^{n,\epsilon} \), because multiplication by \( \sqrt{1+x} \) does not commute with \( P^n \), which prevents us from deducing for now the estimates derived from (4.10) below.

Proof of Theorem 3.1. The rest of the proof of Theorem 3.1 is classical (see e.g. \[ FG95 \] and \[ DGHT11 \]). Applying Lemma 5.2 (of the Appendix) and Chebychev’s inequality to the estimates (3.46)-(3.48), we can use the same technic as that for the proof of Lemma 4.1 in \[ DGHT11 \] to derive the compactness and tightness properties of the sequences \( (u^{\epsilon,n}(t), W(t)) \) in \( n \) for fixed \( \epsilon \). Then we apply the Skorokhod embedding theorem to construct some subsequence \( \{ (u^{\epsilon,nk}(t), W(t)) \} \) that converges strongly as \( n_k \to \infty \), upon shifting the underlying probability basis. Then we pass to the limit on the Galerkin truncation (3.22) as \( n_k \to \infty \) (\( \epsilon \) fixed). Note that we do not need to worry about passing to the limit on the boundary conditions, because they are all well-defined (and conserved) thanks to (3.38). Thus, we have established the existence of martingale solutions to the regularized stochastic ZK equation (3.1)-(3.3), (2.2) and (2.3) in the sense of Definition 3.1.

As for the pathwise solutions, we first prove the pathwise uniqueness of martingale solutions, and then by the Gyöngy-Krylov Theorem we obtain the global existence of pathwise solutions in the sense of Definition 3.2. To conclude, we have completed the proof of Theorem 3.1. \( \square \)

We will develop these steps below in more details in the more complicated case when \( \epsilon \to 0 \).

4 Passage to the limit as \( \epsilon \to 0 \) to study the stochastic ZK equation

We now aim to study the stochastic solutions to the ZK equation basically by passing to the limit as \( \epsilon \to 0 \) in (3.1) and the boundary conditions (2.2), (2.3), (2.2) and (2.3).

Definition of solutions of the ZK equation. The definition of the martingale and pathwise solutions for the ZK equation are essentially the same as that for the regularized equation, with the necessary changes in the assumptions, equations and the function spaces.
Definition 4.1. (Global Martingale Solutions) Let $\mu_{u_0}$ be the probability measure of $u_0$ given as in (2.16) on $L^2(\mathcal{M})$ and assume that (2.12), (2.13) and (2.17) hold. A global martingale solution to the stochastic ZK equation (2.1)-(2.3) and (2.5) (in the Dirichlet case) is defined as in Definition 3.1 as a pair $(\hat{S}, \hat{u})$, such that
\[
\hat{u} \in L^0(\tilde{\Omega}; L^\infty(0, T; L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega}; L^2(0, T; H^1_0(\mathcal{M}))),
\]
and $\hat{u}$ satisfying almost surely
\[
\hat{u}(t) + \int_0^t (\Delta \hat{u}_x + c\hat{u}_x + \hat{u}\hat{u}_x) ds = \hat{u}(0) + \int_0^t f ds + \int_0^t \sigma(\hat{u}) d\hat{W}(s);
\]
the equality in (4.3) is understood in the sense of distributions on $D(\mathcal{M})$ for every $0 \leq t \leq T$. Moreover $\hat{u}$ vanishes on $\partial \mathcal{M}$ (since $\hat{u} \in L^2(\tilde{\Omega}; L^2(0, T; H^1_0(\mathcal{M})))$) and $\hat{u}_x \mid_{x=1} = 0$. For the latter, we observe that according to Lemma 4.5 below, $\tilde{u}_x \mid_{x=1} = 0$ makes sense in a suitable space for any $\tilde{u}$ satisfying (4.1) and (4.3).

Definition 4.2. (Global Pathwise Solutions; Uniqueness)
Let $\mathcal{S} := (\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1})$ be a fixed stochastic basis and suppose that $u_0$ is an $L^2(\mathcal{M})$-valued random variable (relative to $\mathcal{S}$) satisfying (2.18). We suppose that $\sigma$ and $f$ satisfy (2.12), (2.13), (2.15) and (2.19).

(i) A global pathwise solution $u$ of (2.1)-(2.3) and (2.5) is defined as in Definition 3.2 with (3.18)-(3.20) replaced by (4.1)-(4.3). Also note that $u$ vanishes on $\partial \mathcal{M}$ (because $u \in L^2(\Omega; L^2((0, T); H^1_0(\mathcal{M})))$) and $u_x \mid_{x=1} = 0$ which makes sense for the same reasons as for the martingale solution.

(ii) Global pathwise uniqueness is defined in the same way as in Definition 3.2.

The strategy is the same as that in the case of the regularized stochastic ZK equation in Section 3.2: we first derive the global existence of martingale solutions, then prove the pathwise uniqueness of martingale solutions and hence deduce the existence of global pathwise solutions.

4.1 Martingale solutions in dimensions 2 and 3
All the subsequent proofs are valid for $d = 1$ or 2, except for (4.7) and (4.9), and for the uniqueness in Section 4.2.2, which are only valid for $d = 1$ (space dimension two).

Theorem 4.1. When $d = 1$ or 2, suppose that $\mu_0$ satisfies (2.16), that $\sigma$ and $f$ maintain (2.12), (2.13) and (2.17). Then there exists a global martingale solution $(\hat{S}, \hat{u})$ of (2.1)-(2.3) and (2.5) in the sense of Definition 4.1.

Furthermore, when $d = 1$, and if additionally $f$ and $\sigma$ satisfy (2.19) and (2.15), then the martingale solution is pathwise unique (see Proposition 4.3 below).

To prove Theorem 4.1, similar to the case of the regularized stochastic ZK equation, we first derive the estimates leading to weak convergence, then using the Skorokhod embedding theorem we upgrade the weak convergence into the strong convergence, with the probability basis shifted. Special measures will be taken to pass to the limit in the boundary conditions.
4.1.1 Estimates and developments independent of $\epsilon$.

We begin the proof of Theorem 4.1 by deriving the estimates on $u^\epsilon$ valid as $\epsilon \to 0$. We observe that we can prove the estimates in (3.24)-(3.27) under the new assumptions in Theorem 4.1.

**Lemma 4.1.** With the assumptions of Theorem 4.1, when $d = 1, 2$, we have the following estimates valid as $\epsilon \to 0$:

\[
\begin{align*}
|u^\epsilon|^2 \bigg|_{x=0} & \text{ remains bounded in } L^2(\Omega; L^2(0, T; L^2(I_{x^\perp}))), \\
\sqrt{\epsilon}u^\epsilon & \text{ remains bounded in } L^2(\Omega; L^2(0, T; \Xi_1)), \\
u^\epsilon & \text{ remains bounded in } L^6(\Omega; L^\infty(0, T; L^2(\mathcal{M}))),
\end{align*}
\]

(4.4)

If we additionally assume that $u_0$ and $f$ satisfy (2.18) and (2.19), then we have

\[
u^\epsilon \text{ remains bounded in } L^7(\Omega; L^\infty(0, T; L^2(\mathcal{M}))).
\]

(4.7)

**Proof.** The estimates follow from (3.24)-(3.26) (or (3.27)) by passing to the lower limit first in $n$ and then in $\epsilon$ using the lower semicontinuity of the norms; indeed e.g. to show (4.6), with (3.26) we obtain \[|\epsilon u^\epsilon|^2 \bigg|_{x=0} \leq \liminf_n |\epsilon u^\epsilon,n|^2 \bigg|_{x=0} \leq \kappa'_1, \]
for a constant $\kappa'_1$ independent of $\epsilon$.

**Lemma 4.2.** The assumptions are those of Theorem 4.1 with $d = 1$ or $2$. We have the following estimates valid as $\epsilon \to 0$:

\[
u^\epsilon \text{ remains bounded in } L^2(\Omega; L^2(0, T; H^1_0(\mathcal{M}))).
\]

(4.8)

If furthermore we suppose that $u_0$ and $f$ satisfy (2.18) and (2.19), and $d = 1$, then we have

\[
u^\epsilon \text{ remains bounded in } L^{7/2}(\Omega; L^2(0, T; H^1_0(\mathcal{M}))).
\]

(4.9)

**Remark 4.1.** We will use (4.7) and (4.9) only when dealing with the pathwise uniqueness (see the calculations leading to (4.82) below).

**Proof of Lemma 4.2.** The proof does not follow promptly from the estimates on the $u^\epsilon,n$ as that of (4.4)-(4.7), but they are derived directly from the solutions $u^\epsilon$ of the regularized equations; this is in fact the reason for which we introduced this regularization. Note that the solutions $u^\epsilon$ are sufficiently regular for the following calculations to be valid.

We start by multiplying (3.10) with $\sqrt{1+x}$, to find

\[
d(\sqrt{1+x} u^\epsilon) = \sqrt{1+x} \mathcal{N}(u^\epsilon) \, dt + \sqrt{1+x} \sigma(u^\epsilon) \, dW(t),
\]

(4.10)

where again $\mathcal{N}(u^\epsilon) := -Au^\epsilon - B(u^\epsilon) - \epsilon Lu^\epsilon + f^\epsilon$. Applying the Itô formula to (4.10), we obtain

\[
d(\sqrt{1+x} u^\epsilon)^2 = 2 \left( \sqrt{1+x} u^\epsilon, \sqrt{1+x} \mathcal{N}(u^\epsilon) \right) \, dt
+ 2 \left( \sqrt{1+x} u^\epsilon, \sqrt{1+x} \sigma(u^\epsilon) \, dW(t) \right)
+ \|\sqrt{1+x} \sigma(u^\epsilon)\|_{L^2(\Omega)}^2 \, dt.
\]

(4.11)
We drop the super index \( \epsilon \) for the moment and with exactly the same calculations as in the deterministic case (see the proof of Theorem 3.1 in [STW12]), performed a.s. and for a.e. \( t \), we have:

\[
2 \left( \sqrt{1 + x} \, u, \sqrt{1 + x} \, \mathcal{N}(u) \right) = -|\nabla u|^2 - 2|u_x|^2 - (1 - 2\epsilon) \left| u_x \right|_{L^2(I_{\epsilon^2})}^2 - 2\epsilon \left( |\sqrt{1 + x} \, u_{xx}|^2 + |\sqrt{1 + x} \, u_{yy}|^2 + |\sqrt{1 + x} \, u_{zz}|^2 \right) + 2(f, (1 + x) \, u) + \frac{2}{3} \int_{\mathcal{M}} u^3 \, d\mathcal{M} + c|u|^2. \tag{4.12}
\]

Integrating both sides of (4.11) in \( t \) from 0 to \( s \), \( 0 \leq s \leq T \), we find with (4.12) that when say \( \epsilon \leq 1/4 \),

\[
\int_0^s |\nabla u|^2 \, dt \leq |\sqrt{1 + x} \, u_0|^2 + 2 \int_0^s (f^\epsilon, (1 + x)u^\epsilon) \, dt + \frac{2}{3} \int_0^s |u^\epsilon|_{L^3(\mathcal{M})}^3 \, dt + c \int_0^s |u^\epsilon|^2 \, dt + \int_0^s \left| \sqrt{1 + x} \, \sigma(u^\epsilon) \right|_{L^2(\mathcal{M})}^2 \, dt + 2 \int_0^s ((1 + x)u^\epsilon, \sigma(u^\epsilon) \, dW(t)). \tag{4.13}
\]

For the first term on the right-hand side, using \( H^{1/2}(\mathcal{M}) \subset L^3(\mathcal{M}) \) in dimension three, we have

\[
|u^\epsilon|_{L^3(\mathcal{M})}^3 \leq c' |u^\epsilon|^{3/2}|\nabla u^\epsilon|^{3/2} \leq \frac{1}{4} |\nabla u|^2 + c'|u^\epsilon|^6;
\]

hence taking expectations on both sides of (4.13) and using Hölder’s inequality, we obtain

\[
\frac{1}{2} \mathbb{E} \int_0^s |\nabla u|^2 \, dt \leq 2 \mathbb{E} |u_0|^2 + \mathbb{E} \int_0^s |f^\epsilon|^2 \, dt + c' \mathbb{E} \int_0^s |u^\epsilon|^6 \, dt + c' \int_0^s \left| \sqrt{1 + x} \, \sigma(u^\epsilon) \right|_{L^2(\mathcal{M})}^2 \, dt. \tag{4.14}
\]

Here the stochastic term vanishes. We find with (4.6) and (4.14)

\[
\mathbb{E} \int_0^T |\nabla u|^2 \, dt \leq \kappa_5, \tag{4.15}
\]

for a constant \( \kappa_5 \) depending only on \( u_0, f, T \) and \( \sigma \), and independent of \( \epsilon \); this implies (4.8).

Returning to (4.13), when \( d = 1 \), we have

\[
\int_0^s |u^\epsilon|_{L^3(\mathcal{M})}^3 \, dt \leq (H^{1/3}(\mathcal{M}) \subset L^3(\mathcal{M}) \text{ in dimension } 2) \leq \int_0^s |u^\epsilon|^2 |\nabla u^\epsilon| \, dt d\mathcal{M} \tag{4.16}
\]

\[
\leq c' \sup_{0 \leq t \leq s} |u^\epsilon(t)|^4 + \frac{1}{3} \left( \int_0^s |\nabla u^\epsilon| \, dt \right)^2.
\]

Hence (4.13) implies

\[
\frac{1}{2} \int_0^s |\nabla u|^2 \, dt \leq 2|u_0|^2 + \int_0^s |f^\epsilon|^2 \, dt + c' \int_0^s |u^\epsilon|^2 \, dt + c' \sup_{0 \leq t \leq s} |u^\epsilon(t)|^4 + c' \int_0^s ((1 + x)u^\epsilon, \sigma(u^\epsilon) \, dW(t)). \tag{4.17}
\]

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Taking the supremum over \([0, T]\), raising both sides to the power \(7/4\), then taking expectations, we obtain with Minkowski’s inequality and Fubini’s Theorem:

\[
\frac{1}{2} \mathbb{E} \left( \int_0^T |\nabla u^\epsilon|^2 \, dt \right)^{7/4} \lesssim \mathbb{E}|u_0^\epsilon|^{7/2} + \mathbb{E} \int_0^T |f^\epsilon|^{7/2} \, dt + \mathbb{E} \int_0^T |u^\epsilon|^7 \, dt + c'
\]

(4.18)

For the stochastic term, we have

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \int_0^s \left| \left( 1 + x \right) u^\epsilon, \sigma(u^\epsilon) \, dW(t) \right| \right]^{7/4}
\]

\[
\leq \mathbb{E} \sup_{0 \leq s \leq T} \left| \int_0^s \left( 1 + x \right) u^\epsilon, \sigma(u^\epsilon) \, dW(t) \right|^{7/4}
\]

\[
\leq (\text{by the Burkholder-Davis-Gundy inequality (2.7)})
\]

\[
\leq c_1 \mathbb{E} \left[ \left( \int_0^T |u^\epsilon|^2 \, dt \right)^{7/8} \right]
\]

\[
\lesssim \mathbb{E} \left[ \left( \int_0^T |u^\epsilon|^4 \, dt \right)^{7/8} \right] + c'.
\]

This together with (4.18) implies

\[
\frac{1}{2} \mathbb{E} \left( \int_0^T |\nabla u^\epsilon|^2 \, dt \right)^{7/4} \lesssim \mathbb{E}|u_0^\epsilon|^{7/2} + \mathbb{E} \int_0^T |f^\epsilon|^{7/2} \, dt + \mathbb{E} \int_0^T |u^\epsilon|^7 \, dt + 2 \mathbb{E} \left[ \left( \sup_{0 \leq s \leq T} \int_0^s \left| \left( 1 + x \right) u^\epsilon, \sigma(u^\epsilon) \, dW(t) \right| \right]^{7/4} + c'.
\]

(4.19)

Hence (4.19) and (4.7) imply

\[
\mathbb{E} \left( \int_0^T |\nabla u^\epsilon|^2 \, dt \right)^{7/4} \leq \kappa_6,
\]

(4.20)

for a constant \(\kappa_6\) depending only on \(u_0, f, T\) and \(\sigma\), and independent of \(\epsilon\); this implies (4.9).

The proof of Lemma 4.2 is complete. \(\Box\)

**Estimates in fractional Sobolev spaces.**

**Lemma 4.3.** With the same assumptions as in Theorem 4.1 and \(d = 1, 2\), we have

\[
\mathbb{E}|u^\epsilon|_3^2 \leq \kappa_7,
\]

(4.21)

\[
\mathbb{E} \left| u^\epsilon(t) - \int_0^t \sigma(u^\epsilon) \, dW(s) \right|_{H^1(0, T; \Xi_2)}^2 \leq \kappa_8,
\]

(4.22)

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\[
E \left| \int_0^t \sigma(u^\epsilon) \, dW(s) \right|^2 \leq \kappa_9, \quad \forall \alpha < \frac{1}{2},
\]
where \( \mathcal{Y} \) is defined as in (3.45), and \( \kappa_7, \kappa_8 \) and \( \kappa_9 \) are independent of \( \epsilon \).

**Proof.** By repeating the proof of Lemma 3.3 (see Remark 3.2), we see that we can obtain for \( u^\epsilon \) the estimates analog to (3.46)-(3.48) independent of \( \epsilon \). The only point is to derive the estimate of \( E|u^\epsilon|^2_{L^2(0,T; H^1_0(\mathcal{M}))} \) being bounded independently of \( \epsilon \) (see (3.59) correspondingly). For that we just need the estimate (4.8). Hence Lemma 4.3 is proven. \( \square \)

### 4.1.2 Compactness arguments for \( \{(u^\epsilon, W)\}_{\epsilon > 0} \)

With these estimates independent of \( \epsilon \) in hand, we can establish the compactness of the family \((u^\epsilon(t), W(t))\). For this purpose we consider the following phase spaces:

\[
\mathcal{X}_u = L^2(0,T; L^2(\mathcal{M})) \cap C(0,T; H^{-5}(\mathcal{M})), \quad \mathcal{X}_W = C(0,T; \Omega_0), \quad \mathcal{X} = \mathcal{X}_u \times \mathcal{X}_W.
\]

We then define the probability laws of \( u^\epsilon(t) \) and \( W(t) \) respectively in the corresponding phase spaces:

\[
\mu^\epsilon(\cdot) = \mathbb{P}(u^\epsilon \in \cdot),
\]

and

\[
\mu_W(\cdot) = \mu_W^\epsilon(\cdot) = \mathbb{P}(W \in \cdot).
\]

This defines a family of probability measures \( \mu^\epsilon := \mu_u^\epsilon \times \mu_W^\epsilon \) on the phase space \( \mathcal{X} \). We now show that this family is tight in \( \epsilon \). More precisely:

**Lemma 4.4.** We suppose that \( d = 1,2 \), and the hypotheses of Theorem 4.1 hold. Consider the measures \( \mu^\epsilon \) on \( \mathcal{X} \) defined according to (4.25) and (4.26). Then the family \( \{\mu^\epsilon\}_{\epsilon > 0} \) is tight and therefore weakly compact over the phase space \( \mathcal{X} \).

**Proof.** We can use the same technic as in the proof of Lemma 4.1 in [DGHT11]. The main idea is to apply Lemma 5.2 (of the Appendix) and Chebychev’s inequality to (4.21)-(4.23). \( \square \)

**Strong convergence as \( \epsilon \to 0 \).** Since the family of measures \( \{\mu^\epsilon\} \) associated with the family \((u^\epsilon(t), W(t))\) is weakly compact on \( \mathcal{X} \), we deduce that \( \mu^\epsilon \) converges weakly to a probability measure \( \mu \) on \( \mathcal{X} \) up to a subsequence. We can apply the Skorokhod embedding theorem (see Theorem 2.4 in [DPZ92], also [Bil86] and [Jak97]²) to deduce the strong convergence of a further subsequence, that is:

**Proposition 4.1.** Suppose that \( \mu_0 \) is a probability measure on \( L^2(\mathcal{M}) \) that satisfies (2.16). Then there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), and a subsequence \( \epsilon_k \) of random vectors \((\tilde{u}^{\epsilon_k}, W^{\epsilon_k})\) with values in \( \mathcal{X} \) (defined in (4.24)) such that

(i) \((\tilde{u}^{\epsilon_k}, W^{\epsilon_k})\) have the same probability distributions as \((u^{\epsilon_k}, W^{\epsilon_k})\).

(ii) \((\tilde{u}^{\epsilon_k}, W^{\epsilon_k})\) converges almost surely as \( \epsilon_k \to 0 \), in the topology of \( \mathcal{X} \), to an element \((\tilde{u}, \tilde{W}) \in \mathcal{X}\), i.e.

\[
\tilde{u}^{\epsilon_k} \to \tilde{u} \text{ strongly in } L^2(0,T; H^1(\mathcal{M})) \cap C([0,T]; H^{-5}(\mathcal{M})) \ a.s.,
\]

²particularly in [Jak97], the theorem applies to \( \mathcal{X} \) as a Polish space, that is, a separable completely metrizable topological space.
\[ \tilde{W}^{\epsilon_k} \rightarrow \tilde{W} \text{ strongly in } C([0,T]; \mathcal{U}_0) \text{ a.s.,} \]  

(4.28)

where \((\tilde{u}, \tilde{W})\) has the probability distribution \(\mu\).

(iii) \(\tilde{W}^{\epsilon_k}\) is a cylindrical Wiener process, relative to the filtration \(\tilde{\mathcal{F}}_t^{\epsilon_k}\), given by the completion of the \(\sigma\)-algebra generated by \{\((\tilde{u}^{\epsilon_k}(s), \tilde{W}^{\epsilon_k}(s)); s \leq t\}\).

(iv) For each fixed \(\epsilon_k\), \(\tilde{u}^{\epsilon_k} \in L^2(\tilde{\Omega}; L^2(0,T; \Xi_2))\). Moreover, all the statistical estimates on \(u^{\epsilon_k}\) are valid for \(\tilde{u}^{\epsilon_k}\), in particular, (4.6) and (4.8) hold.

(v) Each pair \((\tilde{u}^{\epsilon_k}, \tilde{W}^{\epsilon_k})\) satisfies (3.1) as an equation in \(L^2(\mathcal{M})\) a.s., and satisfies the boundary conditions (2.2), (2.3), (3.2) and (3.3) thanks to (iv), that is, \(\tilde{u}^{\epsilon_k}(t)\) is adapted to \(\tilde{\mathcal{F}}_t^{\epsilon_k}\), and

\[
\begin{align*}
\frac{d\tilde{u}^{\epsilon_k}}{dt} &= (-A\tilde{u}^{\epsilon_k} - B(\tilde{u}^{\epsilon_k}) - \epsilon_k L\tilde{u}^{\epsilon_k} + f^{\epsilon_k}) dt + \sigma(\tilde{u}^{\epsilon_k}) d\tilde{W}^{\epsilon_k}(t), \\
\tilde{u}^{\epsilon_k} &= 0 \text{ on } \partial \mathcal{M}, \quad \tilde{u}^{\epsilon_k}_{x}(x=1) = 0, \\
\tilde{u}^{\epsilon_k}_{xx}(x=0) &= \tilde{u}^{\epsilon_k}_{yy}(y=\pm \frac{x}{2}) = \tilde{u}^{\epsilon_k}_{zz}(z=\pm \frac{x}{2}) = 0, \\
\tilde{u}^{\epsilon_k}(0) &= \tilde{u}^{\epsilon_k}_{0}.
\end{align*}
\]

(4.29)

**Proof.** (i) and (ii) follow directly from the Skorokhod embedding theorem.

To prove (iv), we first observe that thanks to Lemma 5.9 (of the Appendix), the space \(L^2(0,T; \Xi_2)\) is a Borel set in the space \(\mathcal{X}_u\), and hence the integration \(\int_{L^2(0,T; \Xi_2)} |u|^2 d\mu_u^{\epsilon_k}(u)\) makes sense, and by (i) we have for each \(\epsilon_k\),

\[
\mathbb{E}|u^{\epsilon_k}|^2_{L^2(0,T; \Xi_2)} = \int_{L^2(0,T; \Xi_2)} |u|^2 d\mu_u^{\epsilon_k}(u) = \tilde{\mathbb{E}}|\tilde{u}^{\epsilon_k}|^2_{L^2(0,T; \Xi_2)} < (\text{by (3.18)}) < \infty.
\]

In the same way we would prove that all estimates on \(u^{\epsilon}\) are valid for \(\tilde{u}^{\epsilon_k}\), particularly (4.6) and (4.8).

To prove (v), we define

\[
\tilde{M}^{\epsilon_k} := \int_0^T \left| \tilde{u}^{\epsilon_k}(t) + \int_0^t A\tilde{u}^{\epsilon_k} + B(\tilde{u}^{\epsilon_k}) + \epsilon_k L\tilde{u}^{\epsilon_k} - f^{\epsilon_k} ds - \tilde{u}^{\epsilon_k}(0) - \int_0^t \sigma(\tilde{u}^{\epsilon_k}) d\tilde{W}^{\epsilon_k}(s) \right|^2 dt;
\]

then we can use the exact same technique in [Ben95] to prove \(\mathbb{E} \frac{\tilde{M}^{\epsilon_k}}{1 + \tilde{M}^{\epsilon_k}} = 0\). Hence we obtain (4.29).

\[\Box\]

### 4.1.3 Passage to the limit

Now equipped with the strong convergences in (4.27), we can consider passing to the limit on the regularized equation (4.29) as \(\epsilon_k \rightarrow 0\). Note that (4.29) is the version of (3.1) provided by the Skorokhod embedding theorem.

Thanks to (4.6) and (4.8), we deduce the existence of an element

\[
\tilde{u} \in L^6(\tilde{\Omega}; L^\infty(0,T; L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega}; L^2(0,T; H^1_0(\mathcal{M}))),
\]

(4.30)

and a subsequence still denoted as \(\epsilon_k\) such that

\[
\tilde{u}^{\epsilon_k} \rightharpoonup \tilde{u} \text{ weak-star in } L^6(\tilde{\Omega}; L^\infty(0,T; L^2(\mathcal{M}))),
\]

(4.31)
\[ \tilde{u}^{\epsilon_k} \to \tilde{u} \text{ weakly in } L^2(\tilde{\Omega}; L^2(0, T; H^1_0(M))). \] (4.32)

Fixing \( u^\natural \in \Xi_2 \), by (4.32) and (4.31) we can pass to the limit in the linear terms.

For the nonlinear term, for every \( u^\natural \in \Xi_2 \), we write a.s. and for a.e. \( t \):

\[
\left| \int_0^t \left( B(\tilde{u}^{\epsilon_k}) - B(\tilde{u}), u^\natural \right) \, ds \right|
= \frac{1}{2} \left| \int_0^t \left( (\tilde{u}^{\epsilon_k} - \tilde{u})(\tilde{u}^{\epsilon_k} + \tilde{u}), u^\natural_x \right) \, ds \right|
\leq \frac{1}{2} \int_0^t |\tilde{u}^{\epsilon_k} - \tilde{u}| |\tilde{u}^{\epsilon_k} + \tilde{u}| |u^\natural_x|_{L^\infty(M)} \, ds
\leq \frac{1}{2} \int_0^t |\tilde{u}^{\epsilon_k} - \tilde{u}|^2 + |\tilde{u}^{\epsilon_k} + \tilde{u}|^2 \, ds.
\] (4.33)

Thus with (4.27) and (4.6), we deduce that

\[
\int_0^t \left( B(\tilde{u}^{\epsilon_k}), u^\natural \right) \, ds \to \int_0^t \left( B(\tilde{u}), u^\natural \right) \, ds \text{ for a.e. } (\tilde{\omega}, t) \in \tilde{\Omega} \times (0, T). \] (4.34)

We next establish the convergence for the nonlinear term in the space \( L^1(\tilde{\Omega} \times (0, T)) \). We calculate as in (3.53),

\[
E \int_0^T \left| \int_0^t \left( B(\tilde{u}^{\epsilon_k}), u^\natural \right) \, ds \right|^2 \, dt \lesssim E \int_0^T |\tilde{u}^{\epsilon_k}|^4 |u^\natural|^2_{\Xi_2} \, ds \lesssim |u^\natural|^2_{\Xi_2} E \int_0^T |\tilde{u}^{\epsilon_k}|^4 \, ds.
\]

Thus by (4.6), we have

\[
\left\{ \int_0^t \left( B(\tilde{u}^{\epsilon_k}), u^\natural \right) \, ds \right\}_{\epsilon_k > 0} \text{ is uniformly integrable for all } \epsilon_k \in L^1(\tilde{\Omega} \times (0, T)).
\]

Hence thanks to the Vitali convergence theorem, we conclude that

\[
\int_0^t \left( B(\tilde{u}^{\epsilon_k}), u^\natural \right) \, ds \to \int_0^t \left( B(\tilde{u}), u^\natural \right) \, ds \text{ in } L^1(\tilde{\Omega}) \times (0, T). \] (4.35)

For the stochastic term, by (4.27) we obtain

\[
|\tilde{u}^{\epsilon_k} - \tilde{u}|^2 \to 0, \text{ for a.e. } (\tilde{\omega}, t) \in \tilde{\Omega} \times (0, T).
\] (4.36)

Thus, along with (2.13) we deduce

\[
||\sigma(\tilde{u}^{\epsilon_k}) - \sigma(\tilde{u})||_{L^2(M, H)} \to 0, \text{ for a.e. } (\tilde{\omega}, t) \in \tilde{\Omega} \times (0, T).
\]
On the other hand, we observe that

$$
\sup_{\epsilon_k} \mathbb{E} \left( \int_0^T |\sigma(\tilde{u}^{\epsilon_k})|_{L^2(\mathcal{M}, H)}^6 \, ds \right) \lesssim \sup_{\epsilon_k} \mathbb{E} \left( \int_0^T (1 + |\tilde{u}^{\epsilon_k}|^6) \, ds \right),
$$

where we made use of (2.12). We therefore infer from (3.26) that $|\sigma(\tilde{u}^{\epsilon_k})|_{L^2(\mathcal{M}, H)}$ is uniformly integrable for $\epsilon_k$ in $L^q(\tilde{\Omega} \times (0, T))$ for any $q \in [1, 6]$. With the Vitali convergence theorem we deduce that, for all such $q \in [1, 6)$,

$$
\sigma(\tilde{u}^{\epsilon_k}) \to \sigma(\tilde{u}) \text{ in } L^q(\tilde{\Omega}; L^q((0, T), L_2(\mathcal{M}, H))).
$$

Particularly (4.37) implies the convergence in probability of $\sigma(\tilde{u}^{\epsilon_k})$ in $L^2((0, T), L_2(\mathcal{M}, H))$. Thus, along with the assumption (4.28), we apply Lemma 5.6 (of the Appendix) and deduce that

$$
\int_0^t \sigma(\tilde{u}^{\epsilon_k}) \, dW^{\epsilon_k} \to \int_0^t \sigma(\tilde{u}) \, dW, \text{ in probability in } L^2((0, T); L^2(\mathcal{M})).
$$

By the Vitali convergence theorem using the estimates involving (2.7) and (4.37), from (4.38) we infer a stronger convergence result:

$$
\int_0^t \sigma(\tilde{u}^{\epsilon_k}) \, dW^{\epsilon_k} \to \int_0^t \sigma(\tilde{u}) \, dW, \text{ in } L^2(\tilde{\Omega}; L^2((0, T); L^2(\mathcal{M}))).
$$

Hence we can pass to the limit in (3.1), and obtain (4.3) as an equation in $\Xi'_2$.

For the initial condition, since (4.27) and (4.30) imply that $\tilde{u}^{\epsilon} \in L^\infty(0, T; L^2(\mathcal{M})) \cap C([0, T]; H^{5}(\mathcal{M}))$ a.s., hence $\tilde{u}^{\epsilon}$ is weakly continuous with values in $L^2(\mathcal{M})$ a.s.; then (4.2) follows.

Having shown that the limit $\tilde{u}$ almost surely satisfies (4.3) in the sense of distributions on $\mathcal{D}(\mathcal{M})$, we want now to address the question of the boundary conditions. We need to be more careful because of the lack of regularity (see Lemma 4.5 below).

**Passage to the limit on the boundary conditions.** Since $\tilde{u} \in L^2(0, T; H^0_0(\mathcal{M}))$ a.s. (see (4.32)), we deduce that $\tilde{u}$ satisfies the Dirichlet boundary conditions. Hence there remains to show that the boundary condition

$$
\tilde{u}_x |_{x=1} = 0,
$$

is satisfied almost surely. This boundary condition is the object of Lemma 4.5 below where we show that $\tilde{u}_x |_{x=1}$ is well defined when $\tilde{u} \in L^6(\tilde{\Omega}; L^\infty(0, T; L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega}; L^2(0, T; H^1(\mathcal{M})))$, and satisfies an equation like (4.3).

**Lemma 4.5.** We assume that $\tilde{u} \in L^6(\tilde{\Omega}; L^\infty(0, T; L^2(\mathcal{M}))) \cap L^2(\tilde{\Omega}; L^2(0, T; H^1(\mathcal{M})))$ satisfies (4.3) almost surely in the sense of distributions on $\mathcal{D}(\mathcal{M})$, for every $0 \leq t \leq T$. Then

$$
\tilde{u}_x, \tilde{u}_{xx} \in C_x(I_x; \mathcal{B}), \quad \text{where } \mathcal{B} = L^{5/4}(\tilde{\Omega}; H^{-3}((0, T) \times I_{x \perp})),
$$

and, in particular,

$$
\tilde{u}_x |_{x=0,1} \text{ and } \tilde{u}_{xx} |_{x=0,1},
$$

are well defined in $\mathcal{B}$. 25
Proof. If $\tilde{u}$ almost surely satisfies (4.3), then $\tilde{U} := \int_0^t \tilde{u} \, ds$ satisfies

$$\frac{\partial \tilde{U}}{\partial t} + \Delta \frac{\partial \tilde{U}}{\partial x} + c \frac{\partial \tilde{U}}{\partial x} = F \text{ a.s.,}$$

where $F := \tilde{u}_0 - \int_0^t B(\tilde{u}) \, ds + \int_0^t f \, ds + \int_0^t \sigma(\tilde{u}) \, d\tilde{W}(s)$.

For the term $\int_0^t B(\tilde{u}) \, ds$, we note that by (4.10) in [ST10],

$$|\tilde{u}\tilde{u}_x|_{L^{9/8}(M)} \leq |\tilde{u}|^{2/3} |\nabla \tilde{u}|^{4/3}, \text{ for a.e. } t \text{ and a.s.}$$

Hence we have a.s.

$$\left| \int_0^t B(\tilde{u}) \, ds \right|_{L^{5/4}(0,T;L^{9/8}(M))}^{5/4} = \int_0^T |\tilde{u}\tilde{u}_x|_{L^{9/8}(M)}^{5/4} dt \lesssim \int_0^T \left( |\tilde{u}|^{5/6} + |\nabla \tilde{u}|^{5/3} \right)^{6/5} dt \lesssim \int_0^T |\tilde{u}|^5 + |\nabla \tilde{u}|^2 dt. \quad (4.44)$$

Since $\tilde{u} \in L^6(\tilde{\Omega}; L^\infty(0,T; L^2(M))) \cap L^2(\tilde{\Omega}; L^2(0,T; H^1(M)))$, taking expectations on both sides of (4.44) we have

$$\tilde{E} \left| \int_0^t B(\tilde{u}) \, ds \right|_{L^{5/4}(0,T;L^{9/8}(M))}^{5/4} < \infty,$$

that is

$$\int_0^t B(\tilde{u}) \, ds \text{ belongs to } L^{5/4}(\tilde{\Omega}; L^{5/4}(0,T; L^{9/8}(M))),$$

and hence belongs to $L^{5/4}(I_x; L^{5/4}(\tilde{\Omega} \times (0,T) \times I_x)).$ \hspace{1cm} (4.45)

For the term $\int_0^t \sigma(\tilde{u}) \, d\tilde{W}(s)$, from (4.39) we deduce that $\int_0^t \sigma(\tilde{u}) \, d\tilde{W}(s) \in L^2(\tilde{\Omega}; L^2(0,T; L^2(M))$. Applying the above estimates, we obtain that

$F$ belongs to $L^{5/4}(\tilde{\Omega}; L^{5/4}(0,T; L^{9/8}(M)))$, and hence belongs to $L^{5/4}(I_x; L^{5/4}(\tilde{\Omega} \times (0,T) \times I_x)).$ \hspace{1cm} (4.46)

Hence Lemma 5.3 (of the Appendix) applies with $p = 5/4$ and $E = L^{5/4}(\tilde{\Omega} \times (0,T) \times I_x)$, and from (5.7) we have

$$\tilde{U}_x \text{ and } \tilde{U}_{xx} \text{ belong to } C_x(I_x; L^{5/4}(\tilde{\Omega}; H^{-2}((0,T) \times I_x))). \quad (4.47)$$

Since $\tilde{U}_x(t) = \int_0^t \tilde{u}_x \, ds$, we have $\frac{d\tilde{U}_x(t)}{dt} = \tilde{u}_x(t)$; differentiation in time maps continuously $H^{-2}(0,T)$ into $H^{-3}(0,T)$ and from (4.47) we thus infer (4.41) and (4.42). \hfill \square

26
We now need to show that the boundary condition $\tilde{u}^{\epsilon_k}_{x} |_{x=1} = 0$, “passes to the limit” to imply (4.40). The idea is to apply Lemma 5.5 (of the Appendix) to $\tilde{U}^{\epsilon_k}(t) := \int_{0}^{t} \tilde{u}^{\epsilon_k} ds$. Rewriting (4.29) in an integral form and rearranging, we obtain a.s.

$$\tilde{u}^{\epsilon_k}(t) + \int_{0}^{t} \Delta \tilde{u}^{\epsilon_k}_{x} ds + c \int_{0}^{t} \tilde{u}^{\epsilon_k}_{x} ds + \epsilon_k \int_{0}^{t} L \tilde{u}^{\epsilon_k} ds$$

$$= \tilde{u}^{\epsilon_k}_0 - \int_{0}^{t} B(\tilde{u}^{\epsilon_k}) ds + \int_{0}^{t} f^{\epsilon_k} ds + \int_{0}^{t} \sigma(\tilde{u}^{\epsilon_k}) d\tilde{W}^{\epsilon_k}(s).$$

Hence for almost every $\tilde{\omega}$, $\tilde{U}^{\epsilon_k}$ satisfies the linearized parabolic regularized equation:

$$\begin{cases}
  \frac{\partial \tilde{U}^{\epsilon_k}}{\partial t} + \Delta \tilde{U}^{\epsilon_k} + c \frac{\partial \tilde{U}^{\epsilon_k}}{\partial x} + \epsilon_k L \tilde{U}^{\epsilon_k} = F^{\epsilon_k}, \\
  \tilde{U}^{\epsilon_k}|_{x=0} = \tilde{U}^{\epsilon_k}|_{x=1} = \tilde{U}^{\epsilon_k}_{x=0} = 0,
\end{cases}$$

where $F^{\epsilon_k} := \tilde{u}^{\epsilon_k}_0 - \int_{0}^{t} B(\tilde{u}^{\epsilon_k}) ds + \int_{0}^{t} f^{\epsilon_k} ds + \int_{0}^{t} \sigma(\tilde{u}^{\epsilon_k}) d\tilde{W}^{\epsilon_k}(s)$.

For the term $\int_{0}^{t} B(\tilde{u}^{\epsilon_k}) ds$, by the same calculations as those leading to (4.44), we infer from (4.6) and (4.8) that

$$\mathbb{E} \left| \int_{0}^{t} B(\tilde{u}^{\epsilon_k}) ds \right|^{5/4} \text{ is bounded independently of } \epsilon_k.$$  (50)

By (4.39) we deduce that $\int_{0}^{t} \sigma(\tilde{u}^{\epsilon_k}) d\tilde{W}^{\epsilon_k}(s)$ remains bounded in $L^{2}(\tilde{\Omega}; L^{2}((0,T); L^{2}(\mathcal{M}))$. Collecting all the previous estimates we conclude that $\mathbb{E}|F^{\epsilon_k}|^{5/4}_{L^{5/4}(0,T; L^{5/4}(\mathcal{M}))}$ is bounded independently of $\epsilon_k$, and hence

$$F^{\epsilon_k} \text{ is bounded independently of } \epsilon_k \text{ in } L^{5/4}(I_{x}^{\perp}; L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x}^{\perp})).$$

Applying Lemma 5.5 (of the Appendix) with $p = 5/4$, $\tilde{\mathcal{E}} = L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x}^{\perp})$ and $\tilde{\mathcal{B}} = L^{2}(\tilde{\Omega}; H_{1}^{-1}(0,T; L^{2}(I_{x}^{\perp}))) + L^{2}(\tilde{\Omega}; L_{2}^{2}(0,T; H^{-4}(I_{x}^{\perp}))) + L^{5/4}(\tilde{\Omega} \times (0,T) \times (I_{x}^{\perp})))$, we deduce that $\tilde{U}^{\epsilon_k}_{x=1}$ converges to $\tilde{U}_{x=1}$ weakly in $\tilde{\mathcal{B}}$. Hence

$$\tilde{U}_{x=1}(t) = 0,$$  (51)

a.s. and for a.e. $t \in (0,T)$. Since $\tilde{U}_{x=1}(t) = \int_{0}^{t} \tilde{u}_{x=1} ds$, thanks to the Lebesgue differentiation theorem, we infer from (4.51) that $\tilde{u}_{x=1}(t) = 0$ a.s. and for a.e. $t \in (0,T)$. Thus we have finished the proof of Theorem 4.1.

4.2 Pathwise solutions in dimension 2 ($d = 1$)

We aim to establish the existence of pathwise solutions when $d = 1$, that is:

**Theorem 4.2.** When $d = 1$, assume that, relative to a fixed stochastic basis $S$, $u_{0}$ satisfies (2.18), and that $\sigma$ and $f$ satisfy (2.12), (2.13), (2.15) and (2.19) respectively. Then there exists a unique global pathwise solution $u$ which satisfies (2.1)-(2.3) and (2.5) in the sense of Definition 4.2.
To prove this theorem, we first establish the pathwise uniqueness of martingale solutions and then apply the Gyöngy-Krylov Theorem (Theorem 5.1 of the Appendix). The difficulty lies in deducing the pathwise uniqueness due to a lack of regularity of the martingale solutions (see (4.1) and (4.2)). Adapting the idea from the deterministic case (see [STW12]), we introduce a preliminary result concerning the existence and uniqueness of global pathwise solutions to the linearized stochastic ZK equation with additive noise. More importantly, we establish an energy inequality, which leads to a suitable estimate of the difference of the solutions for the application of the version of the stochastic Gronwall lemma given in Lemma 5.11 below.

4.2.1 Linearized stochastic ZK equation with additive noise \((d = 1)\)

**Proposition 4.2.** When \(d = 1\), let \(S\) be a fixed stochastic basis, that is 
\[
S := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1}).
\]
We consider the linearized stochastic ZK equation \((c = 0)\),
\[
\begin{aligned}
d \mathcal{R} + \Delta \mathcal{R}_x \, dt &= g \, dt + h \, dW(t), \\
\mathcal{R}(0) &= \mathcal{R}_0,
\end{aligned}
\]  
(4.52)
with the boundary conditions (2.2) and (2.3) for \(\mathcal{R}\). We assume that 
\[
\mathcal{R}_0 \in L^2(\Omega; L^2(\mathcal{M})),
\]  
(4.53)
and \(h\) and \(g\) are given predictable processes relative to the stochastic basis \(S\), such that 
\[
g \in L^2(\Omega; L^{4/3}(0, T; L^{4/3}(\mathcal{M}))) \cap L^2(\Omega; L^2(0, T; \Xi^2)),
\]  
(4.54)
and 
\[
h \in L^2(\Omega; L^2(0, T; L_2(\mathcal{M}))) \cap L^2(\Omega; L^2(0, T; L_2(\mathcal{M}))),
\]  
(4.55)
Then there exists a unique global pathwise solution \(\mathcal{R}\) to (4.52) which satisfies (2.2) and (2.3), and such that 
\[
\mathcal{R} \in L^2(\Omega; L^\infty(0, T; L^2(\mathcal{M}))) \cap L^2(\Omega; L^2(0, T; H^1_0(\mathcal{M}))),
\]  
(4.56)
and 
\[
\mathcal{R}(\cdot, \omega) \in C([0, T]; L_w^2(\mathcal{M})) \text{ a.s..}
\]  
(4.57)
Furthermore \(\mathcal{R}\) satisfies the following energy inequality for any stopping time \(\tau_b\) with \(0 \leq \tau_b \leq T\),
\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq \tau_b} |\mathcal{R}(s)|^2 + \mathbb{E} \int_0^{\tau_b} |\nabla \mathcal{R}|^2 \, dt \\
\leq \mathbb{E} |\mathcal{R}(0)|^2 + 2 \mathbb{E} \int_0^{\tau_b} |(g, (1 + x)\mathcal{R})| \, dt + c' \mathbb{E} \int_0^{\tau_b} ||h||_{L^2(\mathcal{M})}^2 \, dt.
\]  
(4.58)
**Proof.** We will first show the existence of the solutions, which is similar to that of the nonlinear case, but only easier because the use of a compactness argument and the derivation of strong convergence are not necessary for the linearized model. Then we will verify the uniqueness of the solutions, which is direct since the noise is additive. More precisely, the difference of two
solutions satisfies a deterministic equation depending on the parameters $\omega \in \Omega$. Finally, we will deduce the energy inequality (4.58) utilizing the duality between the spaces to which $g$ and $R$ each belongs.

We start by proving the existence of pathwise solutions with application of the parabolic regularization:

$$
\begin{aligned}
\left\{
\begin{array}{l}
dR^\epsilon + (\Delta R^\epsilon + \epsilon LR^\epsilon) \, dt = g^\epsilon \, dt + h \, dW(t), \\
R^\epsilon(0) = R_0^\epsilon,
\end{array}
\right.
\end{aligned}
$$

supplemented with the boundary conditions (2.2), (2.3) and the additional boundary conditions (3.2), (3.3). As in Section 3, there exist $\{R_0^\epsilon\}_{\epsilon > 0}$, a family of elements in the space $L^2(\Omega; \Xi_1) \cap L^{22/3}(\Omega; L^2(\mathcal{M}))$ which are $\mathcal{F}_0$ measurable, and such that, as $\epsilon \to 0$,

$$
R_0^\epsilon \to R_0 \text{ in } L^2(\Omega; L^2(\mathcal{M})) \text{ strongly};
$$

and there exist $\{g^\epsilon\}_{\epsilon > 0}$, a family of predictable processes relative to the stochastic basis $\mathcal{S}$, so that

$$
g^\epsilon \in L^\infty(\Omega; L^{22/3}(0, T; L^2(\mathcal{M}))),
$$

$$
g^\epsilon \to g \text{ in } L^2(\Omega; L^{4/3}(0, T; L^{4/3}(\mathcal{M}))) \text{ strongly as } \epsilon \to 0.
$$

Since (4.55) corresponds to (2.12) and (2.13), we can use a proof similar to that of Theorem 3.1 to deduce the existence and uniqueness of the global pathwise solution $R^\epsilon$ for each fixed $\epsilon$. Note that for the proof of existence, although $g^\epsilon$ depends on $\omega$, it will not be a problem for us; this is essentially because we can prove the existence of a pathwise solution without referring to any compactness argument.

In the sequel, we will derive the estimates independent of $\epsilon$, then pass to the limit on the parabolic regularization, where again we need to pay special attention to the boundary conditions.

(i) Preliminary estimates independent of $\epsilon$. We will prove the following bounds on $R^\epsilon$ as $\epsilon \to 0$:

$$
R^\epsilon \text{ remains bounded in } L^2(\Omega; L^\infty(0, T; L^2(\mathcal{M}))),
$$

$$
R^\epsilon \text{ remains bounded in } L^2(\Omega; L^2(0, T; H^1_0(\mathcal{M}))).
$$

We start by multiplying both sides of (4.59) by $\sqrt{1 + x}$ and applying the Itô formula, we find

$$
\begin{aligned}
d|\sqrt{1 + x} R^\epsilon|^2 = 2(\sqrt{1 + x} R^\epsilon, \sqrt{1 + x} Q(R^\epsilon)) \, dt \\
+ 2(\sqrt{1 + x} R^\epsilon, \sqrt{1 + x} h \, dW(t)) + ||\sqrt{1 + x} h||^2_{L^2(\mathcal{U}, L^2(\mathcal{M}))} \, dt,
\end{aligned}
$$

where $Q(R^\epsilon) := -\Delta R^\epsilon - \epsilon LR^\epsilon + g^\epsilon$. Let some stopping times $\tau_a$, $\tau_b$ be given so that $0 \leq \tau_a \leq \tau_b \leq T$; we integrate (4.65) from $\tau_a$ to $s$ and take the supremum over $[\tau_a, \tau_b]$. After taking expected values, and by the same calculations as those leading to (4.13), we obtain that when
\[ \epsilon \leq \frac{1}{4}, \]

\[ \mathbb{E} \sup_{\tau_a \leq s \leq \tau_b} |R^\epsilon(s)|^2 + \mathbb{E} \int_{\tau_a}^{\tau_b} |\nabla R^\epsilon|^2 dt \leq 2 \mathbb{E} |R^\epsilon(\tau_a)|^2 + 2 \mathbb{E} \int_{\tau_a}^{\tau_b} |(g^\epsilon, (1 + x)R^\epsilon)| dt \]

\[ + 2 \mathbb{E} \sup_{\tau_a \leq s \leq \tau_b} \int_s^{\tau_b} ((1 + x)R^\epsilon, h dW(t)) \]

\[ + 2 \mathbb{E} \int_{\tau_a}^{\tau_b} \|h\|^2_{L^2(\mathcal{U},L^2(\mathcal{M}))} dt. \]  

For the stochastic term, we have

\[ \mathbb{E} \sup_{\tau_a \leq s \leq \tau_b} \left| \int_s^{\tau_b} ((1 + x)R^\epsilon, h dW(t)) \right| \lesssim (\text{by the Burkholder-Davis-Gundy inequality}) \]

\[ \lesssim \mathbb{E} \left[ \left( \int_{\tau_a}^{\tau_b} |R^\epsilon|^2 \|h\|^2_{L^2(\mathcal{U},L^2(\mathcal{M}))} dt \right)^{1/2} \right] \]

\[ \lesssim \frac{1}{4} \mathbb{E} \sup_{\tau_a \leq s \leq \tau_b} |R^\epsilon|^2 + c' \mathbb{E} \int_{\tau_a}^{\tau_b} \|h\|^2_{L^2(\mathcal{U},L^2(\mathcal{M}))} dt. \]

Hence (4.66) implies

\[ \frac{1}{2} \mathbb{E} \sup_{\tau_a \leq s \leq \tau_b} |R^\epsilon(s)|^2 + \mathbb{E} \int_{\tau_a}^{\tau_b} |\nabla R^\epsilon|^2 dt \leq 2 \mathbb{E} |R^\epsilon(\tau_a)|^2 \]

\[ + 2 \mathbb{E} \int_{\tau_a}^{\tau_b} |(g^\epsilon, (1 + x)R^\epsilon)| dt \quad (4.67) \]

\[ + c' \mathbb{E} \int_{\tau_a}^{\tau_b} \|h\|^2_{L^2(\mathcal{U},L^2(\mathcal{M}))} dt. \]

To estimate the term \( \mathbb{E} \int_{\tau_a}^{\tau_b} |(g^\epsilon, (1 + x)R^\epsilon)| dt \), we observe that a.s.

\[ |(g^\epsilon, (1 + x)R^\epsilon)| \leq |g^\epsilon|_{L^{4/3}(\mathcal{M})} |(1 + x)R^\epsilon|_{L^4(\mathcal{M})} \]

\[ \leq (\text{by Sobolev embedding in dimension 2}) \]

\[ \leq |g^\epsilon|_{L^{4/3}(\mathcal{M})} |\nabla R^\epsilon|^{1/2} |R^\epsilon|^{1/2} \]

\[ \leq c' |g^\epsilon|_{L^{4/3}(\mathcal{M})} |R^\epsilon|^{2/3} + \frac{1}{4} |\nabla R^\epsilon|^2 \]

\[ \leq c' |g^\epsilon|_{L^{4/3}(\mathcal{M})} (|R^\epsilon|^2 + 1) + \frac{1}{4} |\nabla R^\epsilon|^2. \]

Applying the above estimates to (4.67) we obtain

\[ \frac{1}{2} \mathbb{E} \sup_{\tau_a \leq s \leq \tau_b} |R^\epsilon(s)|^2 + \frac{1}{2} \mathbb{E} \int_{\tau_a}^{\tau_b} |\nabla R^\epsilon|^2 dt \]

\[ \leq 2 \mathbb{E} |R^\epsilon(\tau_a)|^2 + 2c' \mathbb{E} \int_{\tau_a}^{\tau_b} |g^\epsilon|_{L^{4/3}(\mathcal{M})}^{4/3} |R^\epsilon|^2 dt \]

\[ + \mathbb{E} \int_{\tau_a}^{\tau_b} 2c' |g^\epsilon|_{L^{4/3}(\mathcal{M})}^{4/3} + c'' \|h\|^2_{L^2(\mathcal{U},L^2(\mathcal{M}))} dt. \]
Thanks to (4.61) and (4.55), we can apply the stochastic Gronwall lemma (Lemma 5.10 below) to (4.68) to find

$$\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq T} |\mathcal{R}(s)|^2 + \frac{1}{2} \mathbb{E} \int_0^T |\nabla \mathcal{R}|^2 dt \lesssim \mathbb{E} |\mathcal{R}_0|^2 + \mathbb{E} \int_0^T |g^{1/3}_{L^4(M)}(\omega, t) + ||h||_{L^2(\mathcal{U}, L^2(M))} dt. \quad (4.69)$$

Thanks to (4.62), we have

$$|g^{1/3}_{L^4(M)}(\omega, t)| \lesssim \mathbb{E} \left[ \int_0^T |g^{1/3}_{L^4(M)} dt \right]^{3/2} + c' \lesssim |g^{1/3}_{L^4(M)}(\omega, t)| + c';$$

hence (4.69) implies that

$$\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq T} |\mathcal{R}(s)|^2 + \frac{1}{2} \mathbb{E} \int_0^T |\nabla \mathcal{R}|^2 dt \lesssim \mathbb{E} |\mathcal{R}_0|^2 + |g^{1/3}_{L^4(M)}(\omega, t)| + c' + ||h||_{L^2(\mathcal{U}, L^2(M))} dt. \quad (4.69)$$

Hence along with (4.54) and (4.55), we obtain (4.63) and (4.64).

(ii) Estimates in fractional Sobolev spaces. By the same proof as for Lemma 4.3, with \( \mathcal{Y} \) defined as in (3.45), we derive the following estimates independent of \( \epsilon \),

$$\mathbb{E} |\mathcal{R}|^2 \lesssim \kappa_8, \quad (4.70)$$

with \( \kappa_8 \) independent of \( \epsilon \). This estimate will be useful to prove the continuity in time in (4.57).

(iii) Passage to the limit as \( \epsilon_k \to 0 \). With (4.63), (4.64) and (4.70), we deduce the following weak convergences, for a subsequence \( \epsilon_k \to 0 \):

$$\mathcal{R}^{\epsilon_k} \rightharpoonup \mathcal{R} \text{ weakly in } L^2(\Omega; L^2(0, T; H_0^1(M))) \cap L^2(\Omega; W^{\alpha, 2}(0, T; \mathbb{E})), \quad (4.71)$$

$$\mathcal{R}^{\epsilon_k} \rightharpoonup \mathcal{R} \text{ weak star in } L^2(\Omega; L^\infty(0, T; L^2(M))). \quad (4.72)$$

We can thus pass to the weak limit in (4.59) and obtain

$$\left\langle \mathcal{R}(t), \mathcal{R}_\delta \right\rangle + \int_0^t \left\langle \Delta \mathcal{R}_x - g, \mathcal{R}_\delta \right\rangle ds = \left\langle \mathcal{R}_0, \mathcal{R}_\delta \right\rangle + \int_0^t \left\langle h, \mathcal{R}_\delta \right\rangle dW, \quad (4.73)$$

for almost every \((\omega, t) \in \Omega \times (0, T)\) and every \( \mathcal{R}_\delta \in \mathbb{E} \).

To pass to the limit on the boundary conditions (2.2) and (2.3), we use the same idea as in Section 4.1.3. Firstly, we can prove an analogue of Lemma 4.5; that is, \( \mathcal{R}_x \big|_{x=1} \) is well defined if \( \mathcal{R} \in L^2(\Omega; L^\infty(0, T; L^2(M))) \cap L^2(\Omega; L^2(0, T; H^1(M))) \), and satisfies an equation like (4.73). To show this, we just need to observe that thanks to (4.54), Lemma 5.3 applies with \( p = 4/3 \) and \( \mathcal{E} = L^{4/3}(\Omega \times (0, T) \times I_{x+}) \). Secondly, we can pass to the limit on the boundary conditions applying Lemma 5.5 (of the Appendix) with \( p = 4/3 \) and \( \mathcal{E} = L^{4/3}(\Omega \times (0, T) \times I_{x+}) \) and \( \tilde{\mathcal{B}} = L^2(\Omega; H^{-1}_t(0, T; L^2(I_{x+}))) + L^2(\Omega; L^2_0(0, T; H^{-3}(I_{x+}))) + L^{4/3}(\Omega \times (0, T) \times (I_{x+})). \)

To prove (4.57), we infer from (4.71) that \( \mathcal{R} \in W^{\alpha, 2}(0, T; \mathbb{E} \mathcal{Y}_2) \cap L^\infty(0, T; L^2(M)) \) a.s., and hence \( \mathcal{R} \in C(0, T; H^{-3}(\mathcal{M})) \cap L^\infty(0, T; L^2(\mathcal{M})) \) a.s. Thus \( \mathcal{R} \) is weakly continuous with values in \( L^2(\mathcal{M}) \) almost surely, which implies (4.57).
To conclude, we have proven the existence of a global pathwise solution $R$ which satisfies (4.52), (2.2) and (2.3).

(iv) Global pathwise uniqueness. We assume that $R_1, R_2$ are two solutions of (4.52); setting $R = R_1 - R_2$, we substract the equation (4.52) for $R_1$ from that for $R_2$; we obtain that almost surely

$$
\begin{cases}
\frac{\partial R}{\partial t} + \Delta R_x = 0, \\
R_0 = 0.
\end{cases}
$$

(4.74)

With (4.56), we have $R \in L^\infty(0, T; L^2(M)) \cap L^2(0, T; H^1_0(M))$ a.s.. Hence we can apply Lemma 3.2 in [STW12] and deduce that $\frac{d}{dt} |R|^2 \leq 0$ for a.e. $\omega \in \Omega$ and $t \geq 0$; thus $R(\omega) = 0$ follows whenever $R_0(\omega) = 0$.

(v) Passage to the limit to obtain energy inequality (4.58). From (4.67), we obtain when $\tau_n = 0$,

$$
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq \tau_n} |R^\epsilon(s)|^2 + \mathbb{E} \int_0^T |\nabla R^\epsilon|^2 dt \leq 2 \mathbb{E} |R_0|^2 + 2 \mathbb{E} \int_0^\tau (g^\epsilon, (1 + x)R^\epsilon) dt
$$

$$
+ c' \mathbb{E} \int_0^\tau ||h||_{L^2(\Omega; L^2(M))} dt.
$$

(4.75)

We infer from (4.71) and (4.72) that for any $\tau_b$ with $0 \leq \tau_b \leq T$,

$$
R^\epsilon t \leq \tau_b \rightarrow R_{1t \leq \tau_b} \text{ weakly in } L^2(\Omega; L^2(0, T; H^1_0(M))),
$$

$$
R^\epsilon t \leq \tau_b \rightarrow R_{1t \leq \tau_b} \text{ weak star in } L^2(\Omega; L^\infty(0, T; L^2(M))),
$$

and hence we can pass to the lower limit on the left-hand-side of (4.75). To pass to the limit on the term $\mathbb{E} |R_0|^2$, we use (4.60).

For the term $\mathbb{E} \int_0^\tau (g^\epsilon, (1 + x)R^\epsilon) dt$, we first note that in dimension 2,

$$
\left( \int_0^T |R^\epsilon|_{L^4(M)}^4 ds \right)^{1/4} \leq \left( \int_0^T |R^\epsilon|^2 |\nabla R^\epsilon|^2 ds \right)^{1/4}
$$

$$
\leq \sup_{0 \leq s \leq T} |R^\epsilon(s)|^{1/2} \left( \int_0^T |\nabla R^\epsilon|^2 ds \right)^{1/4}
$$

$$
\leq 2 \sup_{0 \leq s \leq T} |R^\epsilon(s)| + 2 \left( \int_0^T |\nabla R^\epsilon|^2 ds \right)^{1/2}.
$$

Squaring both sides and taking the expectations we can use (4.63) and (4.64) to obtain that, as $\epsilon \rightarrow 0$,

$$
R^\epsilon \text{ remains bounded in } L^2(\Omega; L^4(0, T; L^4(M))),
$$

(4.76)

and hence a subsequence of $R^\epsilon$ converges weakly in the space $L^2(\Omega; L^4(0, T; L^4(M)))$, which is the dual of $L^2(\Omega; L^{4/3}(0, T; L^{4/3}(M)))$. Since

$$
\mathbb{E} \int_0^\tau (g^\epsilon, (1 + x)R^\epsilon) dt = \mathbb{E} \int_0^T |1_{t \leq \tau_b}(g^\epsilon, (1 + x)R^\epsilon)| dt = \mathbb{E} \int_0^T |(g^\epsilon 1_{t \leq \tau_b}, (1 + x)R^\epsilon)| dt,
$$

32
we see that with (4.62), \( g'1_{t \leq \tau_b} \to g1_{t \leq \tau_b} \) strongly in \( L^2(\Omega; L^{4/3}(0, T; L^{4/3}(\mathcal{M}))) \), and hence the convergence of \( \mathbb{E} \int_0^T |(g', (1 + x)R')| \, dt \) follows.

Thus we can pass to the lower limit on the left-hand side of (4.75) and to the limit on the right-hand side of (4.75), and thus deduce (4.58). Hence we have completed the proof of Proposition 4.2. \( \square \)

**4.2.2 Global pathwise uniqueness for the full stochastic ZK equation (\( d = 1 \))**

The following result establishes the pathwise uniqueness of martingale solutions to (2.1)-(2.3) and (2.5).

**Proposition 4.3.** When \( d = 1 \), suppose that \((\tilde{S}, \tilde{u})\) and \((\tilde{S}, \tilde{v})\) are two global martingale solutions of (2.1)-(2.3) and (2.5), relative to the same stochastic basis. We assume that the conditions imposed in Definition 4.2 hold. We define

\[
\Omega_0 = \{ \tilde{u}(0) = \tilde{v}(0) \}.
\]

Then \( \tilde{u}, \tilde{v} \) are indistinguishable on \( \Omega_0 \) in the sense that

\[
\mathbb{P}(\mathbf{1}_{\Omega_0}(\tilde{u}(t) = \tilde{v}(t))) = 1, \text{ } \forall \text{ } 0 \leq t \leq T. \tag{4.77}
\]

**Proof.** We will mainly use (4.58) from Proposition 4.2 and the version of the stochastic Gronwall lemma given in Lemma 5.11 below. We define \( \mathcal{R} = \tilde{u} - \tilde{v} \). Due to the bilinear term \( B(\tilde{u}) \), when attempting to estimate \( \mathcal{R} \), the terms that involve only \( \tilde{u} \) or \( \tilde{v} \) will arise. To deal with this issue we define the stopping times

\[
\tau^{(m)} = \inf_{t \geq 0} \left\{ \sup_{s \in [0, t]} |\tilde{u}|^2 + \int_0^t |\nabla \tilde{u}|^2 \, ds + \sup_{s \in [0, t]} |\tilde{v}|^2 + \int_0^t |\nabla \tilde{v}|^2 \, ds \geq m \right\},
\]

\[
= \sup_{t \geq 0} \left\{ \sup_{s \in [0, t]} |\tilde{u}|^2 + \int_0^t |\nabla \tilde{u}|^2 \, ds + \sup_{s \in [0, t]} |\tilde{v}|^2 + \int_0^t |\nabla \tilde{v}|^2 \, ds \leq m \right\}. \tag{4.78}
\]

We deduce from (4.30) that \( \lim_{m \to \infty} \tau^{(m)} = \infty \). Define \( \mathcal{R} = \mathbf{1}_{\Omega_0} \mathcal{R} \), and the result will follow once we show that for any \( m \),

\[
\mathbb{E} \left( \sup_{[0, \tau^{(m)} \wedge T]} |\mathcal{R}|^2 \right) = 0. \tag{4.79}
\]

Subtracting the equation (2.1) for \( \tilde{v} \) from that for \( \tilde{u} \), multiplying both sides by \( \mathbf{1}_{\Omega_0} \), we arrive at the following equation for \( \mathcal{R} \),

\[
\begin{align*}
\begin{cases}
\quad d\mathcal{R} + \Delta\mathcal{R}_x \, dt &= \left( -c\mathcal{R}_x + \mathbf{1}_{\Omega_0}(B(\tilde{v}) - B(\tilde{u})) \right) \, dt + \mathbf{1}_{\Omega_0}(\sigma(\tilde{u}) - \sigma(\tilde{v})) \, d\tilde{W}(t), \\
\quad \mathcal{R}(0) &= 0.
\end{cases} \tag{4.80}
\end{align*}
\]

Hence together with the stochastic basis \( \tilde{S} \), we can regard \( \mathcal{R} \) as a global pathwise solution to (4.80) written as (4.52) with the boundary conditions (2.2) and (2.3), where \( g = -c\mathcal{R}_x + \mathbf{1}_{\Omega_0}(B(\tilde{v}) - B(\tilde{u})) \) and \( h = \mathbf{1}_{\Omega_0}(\sigma(\tilde{u}) - \sigma(\tilde{v})) \). To apply Proposition 4.2, now we only need to show that (4.54) and (4.55) are satisfied. We infer from (3.54) that \( g \in L^2(\Omega; L^2(0, T; \Xi^2)) \).
To show that $g$ also belongs to the space $L^2(\tilde{\Omega}; L^{4/3}(0, T; L^{4/3}(M)))$, we first note that $\mathcal{R}_x \in L^2(\tilde{\Omega}; L^2((0, T); L^2(M)))$. Next we estimate $B(\bar{u})$. By the Sobolev embedding theorem in dimension 2, we deduce that $|B(\bar{u})|_{L^{4/3}(M)} \leq c' |\bar{u}|_{L^4(M)} |\bar{u}_x| \leq c' |\bar{u}|^{1/2} |\nabla \bar{u}|^{1/2} |\bar{u}_x|$, and hence almost surely

\[
\left( \int_0^T |B(\bar{u})|^{4/3}_{L^{4/3}(M)} \, dt \right)^{3/2} \leq c_2 \left( \int_0^T |\bar{u}|^{2/3} |\nabla \bar{u}|^2 \, dt \right)^{3/2} \leq c' \left( \sup_{t \in [0, T]} |\bar{u}|^{2/3} \int_0^T |\nabla \bar{u}|^2 \, dt \right)^{3/2} \leq c' \sup_{t \in [0, T]} |\bar{u}| \left( \int_0^T |\nabla \bar{u}|^2 \, dt \right)^{3/2} \leq c' \sup_{t \in [0, T]} |\bar{u}|^7 + \left( \int_0^T |\nabla \bar{u}|^2 \, dt \right)^{7/4} .
\]  

(4.81)

Since (4.7) and (4.9) imply that $\bar{u}$ and $\tilde{v}$ both belong to the space $L^2((\tilde{\Omega}; L^{\infty}((0, T); H^1_0(M))) \cap L^{7/2}(\tilde{\Omega}; L^2((0, T); H^1_0(M)))$, taking expectations on both sides of (4.81) we obtain

\[B(\bar{u}) \text{ belong to } L^2(\tilde{\Omega}; L^{4/3}(0, T; L^{4/3}(M))).\]  

(4.82)

To conclude we infer that $g \in L^2(\tilde{\Omega}; L^{4/3}(0, T; L^{4/3}(M)))$. We infer from (2.15) that

\[||h||_{L_2(\mathfrak{g}^0; \Xi_1)} = ||\sigma(\bar{u}) - \sigma(\tilde{v})||_{L_2(\mathfrak{g}^0; \Xi_1)} \leq c_U ||\mathcal{R}||,\]

which implies that $h \in L^2(\tilde{\Omega}; L^2(0, T; L_2(\mathfrak{g}^0, \Xi_1)))$. Similarly, By (2.13) we can deduce that $h \in L^2(\tilde{\Omega}; L^2(0, T; L_2(\mathfrak{g}^0, L^2(M))))$. Thus we have proven that $h$ satisfies (4.55).

Now Proposition 4.2 applies, and we obtain (4.58) for any $\tau_b$ with $0 \leq \tau_b \leq \tau^m \wedge T$, $\tau^m$ defined as in (4.78) (for notation simplicity, we will write $\tau^m := \tau^m \wedge T$ from now on). We then estimate the right-hand side of (4.58). Thanks to (4.78), we see that

\[\mathcal{R}(\cdot \wedge \tau^m) \in L^{\infty}(\tilde{\Omega}; L^{\infty}((0, T); L^2(M))) \cap L^{\infty}(\tilde{\Omega}; L^2((0, T); H^1_0(M))),\]  

(4.83)

and hence the following calculations are all legitimate for $t \in (0, \tau^m)$. We observe that a.s. and for a.e. $t$:

\[
|\langle g, (1 + x)\bar{R} \rangle| = | -c(\bar{R}_x, (1 + x)\bar{R}) + (B(\tilde{v}) - B(\bar{u})) \rangle (1 + x)\bar{R})| \leq (\text{by (5.1) of the Appendix}) \leq \frac{c}{2} |\mathcal{R}|^2 + \left( |\mathcal{R}_x, (1 + x)\bar{u}_x - \frac{1}{2}(\tilde{v} + (1 + x)\bar{v}_x) \right) | \leq (\text{with } \gamma(t) = |\bar{u}_x(t)| + |\tilde{v}(t)| + |\bar{v}_x(t)|) \leq \frac{c}{2} |\mathcal{R}|^2 + c' \gamma(t) |\mathcal{R}|_{L^4(M)} \leq \frac{c}{2} |\mathcal{R}|^2 + c' \gamma(t) |\mathcal{R}| \leq \frac{c}{2} |\nabla \mathcal{R}|^2 + c' \gamma(t) |\mathcal{R}| \leq \frac{c}{2} |\nabla \mathcal{R}|^2 + c' \gamma(t) |\mathcal{R}| \leq \frac{c}{2} |\nabla \mathcal{R}|^2 + c' \gamma(t) |\mathcal{R}|^2 .
\]  

(4.84)
Applying (4.84) to (4.58), with \( \overline{R}(0) = 0 \) we obtain
\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq \tau_b} |\overline{R}(s)|^2 + \frac{1}{2} \mathbb{E} \int_0^{\tau_b} |\nabla \overline{R}|^2 \, dt \leq c' \mathbb{E} \int_0^{\tau_b} \gamma(t)^2 |\overline{R}|^2 \, dt,  \tag{4.85}
\]
for any stopping time \( \tau_b \) with \( 0 \leq \tau_b \leq \tau^{(m)} \). Along with (4.83), the version of the stochastic Gronwall Lemma given in Lemma 5.11 below applies. Hence we obtain (4.79). This completes the proof of Proposition 4.3.

Thanks to the pathwise uniqueness of martingale solutions, we can apply the Gyöngy-Krylov Theorem to prove the existence of the pathwise solutions (for more details, see [DGHT11]).

**Proof of Theorem 4.2.** We consider the families \((u^\epsilon, u^\epsilon', W)\), where \(u^\epsilon\) and \(u^\epsilon'\) are pathwise solutions to the parabolic regularization \((3.1)-(3.3), (2.2)\) and \((2.3)\). Then by (4.25) and (4.26), we can define the joint distributions of \((u^\epsilon, u^\epsilon', W)\) as \(\nu^{\epsilon, \epsilon'} = \mu^\epsilon_u \times \mu^\epsilon_u' \times \mu_W \) on the phase space \( \mathcal{X}_u \times \mathcal{X}_u \times \mathcal{X}_W \) (\( \mathcal{X}_u \) and \( \mathcal{X}_W \) defined in (4.24)). With the same argument as for Lemma 4.4, we can show that the family \(\{\nu^{\epsilon, \epsilon'}\}\) is tight in \(\epsilon\) and \(\epsilon'\). By the Skorokhod embedding theorem, we deduce the existence of a family \((\tilde{u}^\epsilon, \tilde{u}^\epsilon', \tilde{W})\) defined on a different probability space which converges almost surely to an element \((\tilde{u}, \tilde{u}, \tilde{W})\). By the same proof as for Proposition of 4.1, we can show that \((\tilde{u}^\epsilon, \tilde{W})\) and \((\tilde{u}^\epsilon', \tilde{W})\) both satisfy (i)-(v). In particular, \((\tilde{u}^\epsilon, \tilde{u}^\epsilon')\) have the same probability distributions, \(\mu^\epsilon_u \times \mu^\epsilon_u'\), as \((u^\epsilon, u^\epsilon')\), and the family \(\{\mu^\epsilon_u \times \mu^\epsilon_u'\}_{\epsilon, \epsilon' > 0}\) is tight and hence converges weakly to a probability measure \(\mu_1\), defined by \(\mu_1(\cdot) = \mathbb{P}(\tilde{u} = \tilde{u} \in \cdot)\). It is clear that \(\tilde{u}\) and \(\tilde{\tilde{u}}\) are both martingale solutions to \((2.1)-(2.3)\) and \((2.5)\), hence by the pathwise uniqueness (Proposition 4.3), \(\tilde{u} = \tilde{\tilde{u}}\) a.s.

Thus
\[
\mu_1(\{(u, v) \in \mathcal{X}_u \times \mathcal{X}_u : u = v\}) = \mathbb{P}(\tilde{u} = \tilde{\tilde{u}} \in \mathcal{X}_u) = 1.
\]

We can apply the Gyöngy-Krylov Theorem (Theorem 5.1 of the Appendix) and deduce that the original family \(u^\epsilon\) defined on the initial probability space converges in probability, and hence converges almost surely up to a subsequence, to an element \(u\) in the topology of \(\mathcal{X}_u\). Thus we can pass to the limit on the regularized equation as explained in details in Section 4.1.3. To conclude we have established the existence of a pathwise solutions to \((2.1)-(2.3)\) and \((2.5)\), and we have completed the proof of Theorem 4.2.

**Remark 4.2.** For the space periodic case, that is, \((2.1)\) and the boundary and initial conditions \((2.2), (2.5)\) and \((2.4)\), the results will be the same with the Dirichlet case as discussed above. The reasoning will be similar as in [STW12].

## 5 Appendix

In Section 5.1 and 5.2, we recall some results of deterministic nature. In Section 5.3 to 5.7 we present some results of probabilistic nature.

### 5.1 Properties of \(B(u)\)

In the article, we use the following properties of \(B(u)\) defined in (3.11).
Lemma 5.1. Suppose \( u, v \in H^1_0 \), then with \( R = u - v \), we have

\[
((B(v) - B(u), (1 + x)R) = (R^2, (1 + x)u_x - \frac{1}{2}(v + (1 + x)v_x)).
\]

(5.1)

Proof.

\[
((B(v) - B(u), (1 + x)R) = (R_{ux}, (1 + x)R) + (vR_x, (1 + x)R) \\
= (R^2, (1 + x)u_x) + \frac{1}{2}(R^2, v + (1 + x)v_x) \\
= (R^2, (1 + x)u_x - \frac{1}{2}(v + (1 + x)v_x)).
\]

\[\square\]

5.2 Compact embedding theorems

We recall the theorems from [FG95] and [Fla08] (see also [Tem95] for Lemma 5.2).

Definition 5.1. (The Fractional Derivative Space) We assume that \( H \) is a separable Hilbert space. Given \( \tilde{p} \geq 2 \), \( \alpha \in (0, 1) \), \( W^{\alpha, \tilde{p}}(0, T; H) \) denotes the Sobolev space of all \( h \in L^{\tilde{p}}(0, T; H) \) such that

\[
\int_0^T \int_0^T \frac{|h(t) - h(s)|_{H}^\tilde{p}}{|t - s|^{1+\alpha p}} dt ds < \infty,
\]

which is endowed with the Banach norm

\[
|h|_{W^{\alpha, \tilde{p}}(0, T; H)} = \left( \int_0^T \int_0^T \frac{|h(t) - h(s)|_{H}^\tilde{p}}{|t - s|^{1+\alpha p}} dt ds \right)^{1/\tilde{p}} < \infty.
\]

(5.2)

(5.3)

Lemma 5.2. (i) Let \( \mathcal{E}_0 \subset \mathcal{E} \subset \mathcal{E}_1 \) be Banach spaces, \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) reflexive, with continuous injections and a compact embedding of \( \mathcal{E}_0 \) in \( \mathcal{E} \). Let \( 1 < p < \infty \) and \( \alpha \in (0, 1) \) be given. Let \( \mathcal{Y} \) be the space

\[
\mathcal{Y} := L^p(0, T; \mathcal{E}_0) \cap W^{\alpha, \tilde{p}}(0, T; \mathcal{E}_1),
\]

(5.4)

endowed with the natural norm. Then the embedding of \( \mathcal{Y} \) in \( L^p(0, T; \mathcal{E}) \) is compact.

(ii) If \( \mathcal{E} \subset \bar{\mathcal{E}} \) are two Banach spaces with \( \mathcal{E} \) compactly embedded in \( \bar{\mathcal{E}} \), \( 1 < p < \infty \) and \( \alpha \in (0, 1) \) satisfy

\[
\alpha p > 1,
\]

then the space \( W^{\alpha, \tilde{p}}(0, T; \mathcal{E}) \) is compactly embedded into \( \mathcal{C}([0, T]; \bar{\mathcal{E}}) \).

5.3 Some generalized trace results

The following trace result is an extension of the linear case of Lemma 3.1 of [STW12].
Lemma 5.3. Let $u$ be a random process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $F$ is a given function such that
\[ F \in L^p(I_x; \mathcal{E}), \quad \text{where } \mathcal{E} = L^p(\Omega \times (0,T) \times I_{x^\perp}), \quad p > 1. \] (5.5)
We assume that $u \in L^2(\Omega; L^2(0,T; H^1(\mathcal{M})))$ satisfies almost surely the following linear equation
\[ u_t + \Delta u_x + cu_x = F, \] (5.6)
that is, almost surely we have
\[ u(t) + \int_0^t (\Delta u_x + cu_x)ds = u(0) + \int_0^t F ds, \]
in the sense of distributions on $D(\mathcal{M})$ for every $0 \leq t \leq T$. Then
\[ u_x, u_{xx} \in C^1(I_x; \mathcal{B}), \quad \mathcal{B} = L^2(\Omega; H^{-2}((0,T) \times I_{x^\perp})) \cap \mathcal{E}. \] (5.7)
and, in particular,
\[ u_x|_{x=0,1} \text{ and } u_{xx}|_{x=0,1}, \] (5.8)
are well defined in $\mathcal{B}$.

Proof. We write equation (5.6) in the form
\[ u_{xxx} = F - cu_x - \Delta u_x - u_t. \]
Then clearly we have
\[ u_{xxx} \in L^{p\wedge 2}(I_x; L^2(\Omega; H^{-2}((0,T) \times I_{x^\perp})) \cap \mathcal{E}), \quad p \geq 1, \] (5.9)
which implies that (5.7) holds.

We use Lemma 5.3 in the proof of Lemma 4.5 with $p = 5/4$ and $\mathcal{E} = L^{5/4}(\hat{\Omega} \times (0,T) \times I_{x^\perp})$, and in the proof of Proposition 4.2 with $p = 4/3$ and $\mathcal{E} = L^{4/3}(\Omega \times (0,T) \times I_{x^\perp})$.

Next we recall another trace result from [STW12].

Lemma 5.4. Let $\mathcal{B}$ be a reflexive Banach space and let $p \geq 1$. We assume that two families of functions $u^\epsilon$ and $g^\epsilon \in L^p_x(I_x; \mathcal{B})$ satisfy
\[
\begin{align*}
    u^\epsilon_{xxx} + cu^\epsilon_{xxxx} &= g^\epsilon, \\
    u^\epsilon(0) &= u^\epsilon(1) = u^\epsilon_x(1) = u^\epsilon_{xx}(0) = 0,
\end{align*}
\]
and that $g^\epsilon$ is bounded in $L^p_x(I_x; \mathcal{B})$ as $\epsilon \to 0$. Then $u^\epsilon_{xx}$ and $u^\epsilon_x$ is bounded in $L^\infty_x(I_x; \mathcal{B})$ as $\epsilon \to 0$. Furthermore, for any subsequences $u^\epsilon \to u$ converging (strongly or weakly) in $L^q_x(I_x; \mathcal{B})$, $1 < q$, $u^\epsilon_x(1)$ converges to $u_x(1)$ in $\mathcal{B}$ (weakly at least), and hence $u_x(1) = 0$.

We are now ready to prove the following trace result generalized from the argument in [STW12].
Lemma 5.5. Let \( \{u^\epsilon\}_{\epsilon>0} \) be a family of random processes, all defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We consider the following linearized regularized equation

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\quad u^\epsilon_t + \Delta u^\epsilon_x + cu^\epsilon_x - \epsilon Lu^\epsilon = F^\epsilon, \\
\quad u^\epsilon|_{x=0} = u^\epsilon|_{x=1} = u^\epsilon|_{x=+} = 0,
\end{array}
\right.
\end{aligned}
\tag{5.10}
\]

where \( F^\epsilon \) remains bounded in the reflexive Banach space \( L^p(\Omega \times (0,T)), \) with \( \tilde{\mathcal{E}} := L^p(\Omega \times (0,T) \times I_{x+}), \)
\( p > 1. \) We suppose that \( u^\epsilon \) almost surely satisfies (5.10), that is, almost surely we have

\[
u^\epsilon(t) + \int_0^t (\Delta u^\epsilon_x + cu^\epsilon_x - \epsilon Lu^\epsilon - F^\epsilon) \, ds = u^\epsilon(0),
\]

in the sense of distributions on \( \mathcal{D}(\mathcal{M}) \) for every \( 0 \leq t \leq T. \) We assume that \( u^\epsilon \) converges weakly to some \( u \) in \( L^2(\Omega; L^2(0,T; H^1_0(\mathcal{M}))) \) as \( \epsilon \to 0, \) then \( u^\epsilon_x(1) \) converges to \( u_x(1) \) in \( \tilde{\mathcal{B}} \) specified below, and hence \( u^\epsilon_x(1) = 0. \)

Proof. By (5.10) we have

\[
u^\epsilon_{xxx} + \epsilon u^\epsilon_{xxxx} = F^\epsilon - u^\epsilon_x - cu^\epsilon_x - \Delta^\epsilon u^\epsilon_{xx} - \epsilon u^\epsilon_{yyyy} - \epsilon u^\epsilon_{zzzzz}.
\]

We call the right hand side \( g^\epsilon. \) It is easy to observe that, since \( u^\epsilon \) remains bounded in \( L^2(I_x; L^2(0,T) \times I_{x+}) \) as \( \epsilon \to 0, \) then \( g^\epsilon \) remains bounded in the reflexive Banach space \( L^p_x(I_x; \tilde{\mathcal{B}}), \ p > 1, \) where

\[
\tilde{\mathcal{B}} = L^2(\Omega; H_t^{-1}(0,T; L^2(I_{x+}))) + L^2(\Omega; L^2_t(0,T; H^{-4}(I_{x+}))) + \tilde{\mathcal{E}}.
\]

Thus we can apply Lemma 5.4 with this space \( \tilde{\mathcal{B}} \) and obtain the convergence of the boundary term \( u^\epsilon_x(1). \) \( \square \)

Lemma 5.5 is applied in Section 4.1.3 with \( p = 5/4 \) and \( \tilde{\mathcal{E}} = L^{5/4}(\tilde{\Omega} \times (0,T) \times I_{x+}) \) and in the proof of Proposition 4.2, with \( p = 4/3, \ \tilde{\mathcal{E}} = L^{4/3}(\Omega \times (0,T) \times I_{x+}) \) and \( \tilde{\mathcal{B}} = L^2(\Omega; H_t^{-1}(0,T; L^2(I_{x+}))) + L^2(\Omega; L^2_t(0,T; H^{-4}(I_{x+}))) + L^{4/3}(\Omega \times (0,T) \times (I_{x+})). \)

5.4 Convergence theorem for the noise term

The following convergence theorem for the stochastic integrals is used to facilitate the passage to the limit in the parabolic regularization approximation. The statements and proofs can be found in [Ben95], [GK96] and [DGHT11].

Lemma 5.6. Let \( \{\Omega, \mathcal{F}, \mathbb{P}\} \) be a fixed probability space, and \( \mathcal{X} \) a separable Hilbert space. Consider a sequence of stochastic bases \( S_n := (\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_{t \geq 0}, \mathbb{P}, W^n) \), such that each \( W^n \) is a cylindrical Brownian motion (over \( \Omega \)) with respect to \( \{\mathcal{F}_t^n\}_{t \geq 0}. \) We suppose that the \( \{G^n\}_{n \geq 1} \) are a sequence of \( \mathcal{X} \)-valued \( \mathcal{F}_t^n \) predictable processes so that \( G^n \in L^2((0,T); L_2(\Omega, \mathcal{X})) \) a.s.. Finally consider \( S := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W) \) and a function \( G \in L^2((0,T); L_2(\Omega, \mathcal{X})), \) which is \( \mathcal{F}_t \) predictable. If

\[
W^n \to W \text{ in probability in } L^2([0,T]; \mathcal{M}_0),
\]

\[
G^n \to G \text{ in probability in } L^2((0,T); L_2(\Omega, \mathcal{X})),
\]

then

\[
\int_0^t G^n \, dW^n \to \int_0^t G \, dW \text{ in probability in } L^2((0,T); \mathcal{X}).
\]
Then we have the following lemma based on the Burkholder-Davis-Gundy inequality and the notion of fractional derivatives in Definition 5.1 (whose proof can be found in [FG95]).

**Lemma 5.7.** Let $q \geq 2$, $\alpha > \frac{1}{2}$ be given so that $q\alpha > 1$. Then for any predictable process $h \in L^q(\Omega \times (0, T); L_2(\mathcal{U}, H))$, we have

$$\int_0^t h(s) \, dW(s) \in L^q(\Omega; W^{\alpha,q}(0, T; H)),$$

and there exists a constant $c' = c'(q, \alpha) \geq 0$ independent of $h$ such that

$$E \left| \int_0^t h(s) \, dW(s) \right|^q_{W^{\alpha,q}(0, T; H)} \leq c'(q, \alpha) E \int_0^t |h(s)|^q_{L_2(\mathcal{U}, H)} \, ds. \quad (5.12)$$

### 5.5 Some probability tools

We recall the Gyöngy-Krylov Theorem from [GK96], which is used in proving the existence of pathwise solutions.

**Theorem 5.1.** Let $\mathcal{X}$ be a Polish space. Suppose that $\{Y_m\}$ is a sequence of $\mathcal{X}$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mu_{k,m}\}_{k,m \geq 1}$ be the sequence of joint laws of $\{Y_m\}_{m \geq 1}$, that is

$$\mu_{k,m}(E) := ((Y_k, Y_m) \in E), \quad E \in \mathcal{B}(\mathcal{X} \times \mathcal{X}).$$

Then $\{Y_m\}$ converges in probability if and only if for every subsequence of joint probability measures, $\{\mu_{k,m}\}_{l \geq 0}$, there exists a further subsequence which converges weakly to a probability measure $\mu$ such that

$$\mu(\{(u, v) \in \mathcal{X} \times \mathcal{X} : u = v\}) = 1. \quad (5.13)$$

### 5.6 The Jakubowski-Skorokhod representation theorem

We recall the following result from [Ond10].

**Lemma 5.8.** Let $A_1$ be a topological space such that there exists a sequence $\{f_m\}$ of continuous functions $f_m : A_1 \to \mathcal{R}$ that separate points of $A_1$. Let $A_2$ be a Polish space, that is, a separable completely metrizable topological space, and let $I : A_2 \to A_1$ be a continuous injection. Then $I(B)$ is a Borel set in $A_1$ whenever $B$ is Borel in $A_2$.

The following result is a special case of Lemma 5.8.

**Lemma 5.9.** Let $A_1$ be a separable Hilbert space. Assume that $A_2$ is a separable Hilbert space continuously injected into $A_1$. Then $A_2$ is a Borel set of $A_1$.

**Proof.** Firstly, it is clear that any separable Hilbert space $A_1$ satisfies the hypotheses of Lemma 5.8. Since $A_2$ is a separable Hilbert space, hence it is a Polish space. Now in Lemma 5.8, let $B$ be $A_2$, which of course is a Borel set of $A_2$. Then $I(B) = I(A_2) = A_2$ is a Borel set in $A_1$ thanks to Lemma 5.8.

We use Lemma 5.9 in the proof of Proposition 4.1.
5.7 The adapted stochastic Gronwall lemma

We first recall the stochastic Gronwall lemma from [GHZ09] (see also [MR04]), then we present a variant result which is used in the proof of Proposition 4.3.

**Lemma 5.10.** Fix $T > 0$. We assume that 

$$X, Y, Z, M : [0, T) \times \Omega \to \mathbb{R},$$

are real valued, non-negative stochastic processes. Let $0 \leq \tau < T$ be a stopping time so that

$$\mathbb{E} \int_0^\tau (MX + Z) \, ds < \infty. \quad (5.14)$$

We suppose, moreover that for some fixed constant $\kappa$,

$$\int_0^\tau M \, ds < \kappa, \text{ a.s.} \quad (5.15)$$

Suppose that for all stopping times $\tau_a, \tau_b$ with $0 \leq \tau_a \leq \tau_b \leq \tau$ we have

$$\mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Y \, ds \right) \leq C_0 \mathbb{E} \left( X(\tau_a) + \int_{\tau_a}^{\tau_b} (MX + Z) \, ds \right), \quad (5.16)$$

where $C_0$ is a constant independent of the choice of $\tau_a, \tau_b$. Then

$$\mathbb{E} \left( \sup_{t \in [\tau_0, \tau]} X + \int_{\tau_0}^\tau Y \, ds \right) \leq C \mathbb{E} \left( X(0) + \int_{\tau_0}^\tau Z \, ds \right), \quad (5.17)$$

where $C = C(C_0, T, \kappa)$.

When $X(0) = 0$ and $Z = 0$ we can weaken the hypotheses by requiring that (5.16) only holds for $\tau_a = 0$ and all $\tau_b, 0 \leq \tau_b \leq \tau$. We then obtain

**Lemma 5.11.** We assume that $X(0) = 0$ and $Z = 0$ in Lemma 5.10 and that (5.16) holds only for $\tau_a = 0$ and all $\tau_b, 0 \leq \tau_b \leq \tau$, that is:

$$\mathbb{E} \left( \sup_{t \in [0, \tau_b]} X + \int_0^{\tau_b} Y \, ds \right) \leq C_0 \mathbb{E} \left( X(0) + \int_0^{\tau_b} MX \, ds \right), \quad (5.18)$$

where $C_0$ is a constant independent of the choice of $\tau_b$. Then the calculation (5.17) holds true and reduces to

$$\mathbb{E} \sup_{t \in [0, \tau]} X = \mathbb{E} \int_0^\tau Y \, ds = 0. \quad (5.19)$$

**Proof.** Step 1. We first show how to construct a finite sequence of stopping times

$$0 < \tau_1 < \ldots < \tau_N < \tau_{N+1} = \tau \text{ a.s.,}$$

so that

$$\int_{\tau_j}^{\tau_{j+1}} M \, ds < \frac{1}{2C_0} \text{ a.s., } \forall j = 1, \ldots, N. \quad (5.20)$$
We construct the sequence inductively. We start with time 0. We assume that \( \tau_{j-1} \) is found. Then define
\[
\tau_j := \inf_{t \geq 0} \left\{ \int_{\tau_{j-1}}^t M \, ds < \frac{1}{2C_0} \right\} \land \tau,
\]
and \( \tau_j > 0 \) is well-defined since \( M > 0 \) and it satisfies (5.15). Hence we have
\[
\int_{\tau_{j-1}}^{\tau_j} M \, ds \geq \frac{1}{2C_0}, \quad \forall j \geq 1 \text{ such that } \tau_j < \tau. \quad (5.21)
\]
Now we claim that there exists a finite integer \( N \) such that \( \tau_N = \tau \), and
\[
N \leq 2C_0\kappa + 1, \quad \text{a.s..} \quad (5.22)
\]
We show this by contradiction; suppose that (5.22) is not true, then \( N - 1 > 2C_0\kappa \), and hence
\[
\int_0^{\tau_{N+1}} M \, ds = \sum_{j=1}^{N-1} \int_{\tau_j}^{\tau_{j+1}} M \, ds + \int_{\tau_N}^\tau M \, ds \geq \text{ with (5.21) } \geq (N-1)\frac{1}{2C_0} > \kappa.
\]
But this contradicts with (5.15). Hence (5.22) is proven, and we can choose the integer \( N = \lceil 2C_0\kappa + 1 \rceil \).

Step 2. Letting \( \tau_b = \tau_1 \) in (5.18), we have
\[
\mathbb{E} \left( \sup_{t \in [0, \tau_1]} X + \int_0^{\tau_1} Y \, ds \right) \leq C_0 \mathbb{E} \int_0^{\tau_1} M X \, ds; \quad (5.23)
\]
from (5.23), (5.14) and (5.20) we infer
\[
\mathbb{E} \left( \frac{1}{2} \sup_{t \in [0, \tau_1]} X + \int_0^{\tau_1} Y \, ds \right) = 0. \quad (5.24)
\]
Thanks to (5.24), for every \( \tau_b \geq \tau_1 \) a.s., we find that
\[
\mathbb{E} \sup_{t \in [0, \tau_b]} X = \mathbb{E} \sup_{t \in [\tau_1, \tau_b]} X,
\]
\[
\mathbb{E} \int_0^{\tau_b} Y \, ds = \mathbb{E} \int_{\tau_1}^{\tau_b} Y \, ds,
\]
\[
\mathbb{E} \int_0^{\tau_b} M X \, ds = \mathbb{E} \int_{\tau_1}^{\tau_b} M X \, ds. \quad (5.25)
\]
Thus from (5.25) and (5.18), we infer that for every \( \tau_b \geq \tau_1 \) a.s.,
\[
\mathbb{E} \left( \sup_{t \in [\tau_1, \tau_b]} X + \int_{\tau_1}^{\tau_b} Y \, ds \right) \leq C_0 \mathbb{E} \left( \int_{\tau_1}^{\tau_b} M X \, ds \right). \quad (5.26)
\]
Setting \( \tau_b = \tau_2 \) in (5.26), we have
\[
\mathbb{E} \left( \sup_{t \in [\tau_1, \tau_2]} X + \int_{\tau_1}^{\tau_2} Y \, ds \right) \leq C_0 \mathbb{E} \left( \int_{\tau_1}^{\tau_2} M X \, ds \right); \quad (5.27)
\]
with (5.27), (5.14) and (5.20) we deduce

\[ E \left( \frac{1}{2} \sup_{t \in [\tau_1, \tau_2]} X + \int_{\tau_1}^{\tau_2} Y \, ds \right) = 0. \]  

(5.28)

Hence by finite induction up to \( N \) we obtain (5.19).

\[ \square \]

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