CAUSALITY IN 1+1 DIMENSIONAL YUKAWA MODEL-I

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Abstract

We study the 1+1 dimensional Yukawa theory, in a certain limit of its parameters $g, M, m$ (as suggested by the study of causality in presence of bound states in this model [1]). We study the bound state formation in the model. In the limit $g \to \infty, M \to \infty$, in a certain specific manner, we show that there are a large number of bound states of which at least the low lying states are described by the non-relativistic Schrodinger equation. We show that, in this limit, the excited bound states are unstable and deem to decay quickly (lifetime $\tau \to 0$) by emission of scalar (s) in this particular limit. The mass of the ground state is not significantly affected by higher order quantum corrections and by proper choice of parameters, involving only small changes, can be adjusted to be equal to the mass of the scalar. As a result of quantum effects, the state of the meson mixes with the lowest bound state and may be dominated by the latter. We show that in this detailed sense, a scalar meson in Yukawa model can be looked upon as a bound state of a fermion-anti fermion pair formed.

1 Preliminary

We consider the 1 + 1 dimensional Yukawa theory,

$$\mathcal{L} = \bar{\psi} [i \not{\partial} - M + g \phi] \psi + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^{2} \phi^{2}$$

and expand $\phi = \phi_c + \phi_q$, where $\phi_c$ is the classical field obeying the classical equation of motion:

$$[\partial^2 + m^2] \phi_c = g \langle \bar{\psi} \psi \rangle,$$

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leading to the Yukawa potential, and

\[ \mathcal{L}_I = g \bar{\psi} \psi \phi_q + \ldots \]  

(3)

is the quantum interaction Lagrangian. [This division is more or less like what we do when we discuss radiation from an atom (see e.g. [2])]. The fermion anti-fermion pair interacts in the first instance via \( \phi_c \), the classical Yukawa potential and form bound states. In absence of \( \mathcal{L}_I \), these are the exact stationary states of the system (like the H-atom has them if interaction of an electron with a photon were neglected). The excited states, however, are unstable against decay via emission of scalar(s) through the interaction Lagrangian \( \mathcal{L}_I \). We want to show that in a certain limit of parameters \( g, m, M \), (suggested by a study of causality in this model, [1]) there is only one stable bound state and the scalar particle can be identified with it. All other bound states are highly unstable (\( \tau \to 0 \)).

The motivation of the work derives from the study of causality in the Yukawa field theory model [1]. We have considered the limit \( g \to \infty, M \to \infty \) and \( \frac{g^2}{M} = \text{const.} \) of the field theory model and shown that the model reduces to a \( \phi^3 \) nonlocal field theory which does not preserve causality. For the interpretation to hold, it is crucial to know the bound state nature of 1 + 1 dimensional Yukawa model in this limit. We find that in this limit, only one bound state, Yukawa ground state, is stable and has nonzero lifetime. This result has been employed in [1], to interpret the result in terms of the scalar field representing the Yukawa bound state.

\section{Non-relativistic Bound States}

We shall consider the fermion anti-fermion system (mass \( M \) each) in a bound state via the attractive Yukawa potential [This is a solution of (2), under appropriate conditions.].

\[ V(x) = -g^2 e^{-m|x|} \frac{e^{-m|x|}}{2m} \]

We shall assume that the mass \( M \) is large, and that the bound state is non-relativistic. (This will be justified later). For the relative motion, the non-relativistic Schrodinger equation is satisfied (reduced mass is \( \frac{M}{2} \)):

\[ -\frac{\hbar^2}{M} \frac{d^2 \psi}{dx^2} - g^2 e^{-m|x|} \frac{e^{-m|x|}}{2m} \psi = E \psi \]
where \( E = -\varepsilon < 0 \) is the energy of a bound state. We introduce the change of the independent variable: \( y = e^{-m|x|/2} \). Under this change, the equation transforms to

\[
y^2 \frac{d^2 \psi}{dy^2} + y \frac{d \psi}{dy} + \frac{2M y^2}{m^3} y^2 \psi + \frac{4EM}{m^2} \psi = 0
\]

[The equation, now, has to be solved separately for \( x > 0 \) and \( x < 0 \) and the solutions have to be tested for continuity of \( \psi \) and \( \psi' \) at \( x = 0 \)]. A further rescaling:

\[
t = \xi y \equiv \sqrt{\frac{2Mg^2}{m^3}} y = \sqrt{\frac{2Mg^2}{m^3}} e^{-m|x|/2}
\]

will put the equation in the form of the Bessel equation \([3, 4]\):

\[
\psi'' + \frac{1}{t} \psi' + \left( 1 - \frac{\nu^2}{t^2} \right) \psi = 0.
\]

with \( \nu \equiv +\sqrt{\frac{4\varepsilon M}{m^2}} \). [We recall that as \( \varepsilon \) is always less than the depth of the well \( \frac{g}{2} \), the allowed values of \( \nu \) will turn out to be less than \( \xi \)]. We note that as \( x \to \pm\infty, t \to 0 \). As \( x \to 0^\pm, t \to \xi \). The boundary condition at \( x \to \pm\infty \) requires that \( \psi(0) \to 0 \). This requires that we take the solution which is regular at \( t = 0 \), viz. \( J_\nu(t) \) and drop the singular solution \( N_\nu(t) \). The discrete values of \( \varepsilon \) are determined from the conditions of continuity at \( x = 0 \):

1. \( \psi(x \to 0^-) = \psi(x \to 0^+) \Rightarrow \psi(t) \) is continuous as \( t \to \xi \) from either side.
2. \( \frac{d}{dx} \psi(x \to 0^-) = \frac{d}{dx} \psi(x \to 0^+) \Rightarrow \psi'(t \to \xi; x \to 0^-) = -\psi'(t \to \xi; x \to 0^+) \)

These boundary conditions can materialize in two distinct ways: As the potential is even, the non-degenerate states will have a definite parity[?]; so that,

1. the wave-function is even and \( \frac{d}{dx} \psi(x \to 0^-) = \frac{d}{dx} \psi(x \to 0^+) = 0; \) (and \( \psi(x = 0) \neq 0 \));
2. the wave-function is odd: \( \psi(x = 0) = 0, \) (and \( \frac{d}{dx} \psi(x \to 0^-) = \frac{d}{dx} \psi(x \to 0^+) \neq 0 \));

In the case 1 above, \( \psi(t) \) satisfies: \( \psi'(t) = 0 \) at \( t = \xi \). In the case 2 above, \( \psi(t) \) satisfies: \( \psi(t) = 0 \) at \( t = \xi \). Thus, the energy eigenvalues are determined by:

1. \( J_\nu(\xi) = 0, \) or
2. \( J'_\nu(\xi) = 0 \)
[where, from the properties of $J_{\nu}(\xi)$, both $J_{\nu}(\xi)$ and $J'_{\nu}(\xi)$ cannot simultaneously vanish for a $\xi > 0$ ]]. The energy dependence enters through the order $\nu \equiv + \sqrt{\frac{4\epsilon M}{m}}$ of $J_{\nu}$.

3 Properties of Bound States and Relevant Properties of $J_{\nu}(\xi)$

For the ground state, being an even wavefunction, $J'_{\nu}(\xi) = 0$. $\nu$ is so determined that $\xi$ is the first positive zero of $J'_{\nu}(\xi)$. The ground state wave-function is

$$\psi = C J_{\nu}(\xi \exp \left[-m|x|/2\right])$$

(4)

The relevant properties of the ground state wave-function are:

1. $\nu < \xi$ as expected and for a large $\xi$, $\frac{\nu}{\xi} \to 1$ from below. This implies that

   $$\frac{\xi}{g^2/2m} \to 1$$

   from below. The exact behavior is given as [8],

   $$\xi = \nu + 0.808618 \nu^{1/3} + O \left(\nu^{-1/3}\right) \equiv \nu + \alpha \nu^{1/3} + O \left(\nu^{-1/3}\right)$$

(5)

This can be numerically verified: See Table 1.

2. $\frac{\xi^2 - \nu^2}{\xi^2} \to 0$. The quantity $\xi^2$ is proportional to the maximum depth of the well $\frac{g^2}{2m}$; $\nu^2$ likewise is proportional to the magnitude of the ground state energy eigenvalue. So, $(\xi^2 - \nu^2)$ gives an upper bound on the kinetic energy of the ground state. We shall consider the depth of the potential approximately equal to $2M$. The relation $\frac{\xi^2 - \nu^2}{\xi^2} \to 0$ then implies: $\frac{\langle KE \rangle}{g^2/2m} = \frac{\langle KE \rangle}{2M} \to 0$; or that the ground state is non-relativistic. Evidently, it will also apply to low lying excited states.

3. $\frac{\xi^2 - \nu^2}{\xi^2} \to 0$. It behaves as $\frac{\xi^2 - \nu^2}{\xi^2} \sim \nu^{-2/3}$. This follows from (5), which implies,

   $$(\xi^2 - \nu^2) = (\xi - \nu)(\xi + \nu)$$

   $$= [0.808618 \nu^{1/3} + O \left(\nu^{-1/3}\right)]$$

   $$\times \left[2\nu + 0.808618 \nu^{1/3} + O \left(\nu^{-1/3}\right)\right]$$

(6)
\[ \frac{(\xi^2 - \nu^2)}{\xi^2} \nu^{2/3} = 1.617236 \nu^2 + O\left(\nu^{-2/3}\right) \]

\[ = 1.617236 + O\left(\nu^{-2/3}\right) \]

We first note that the (7) tells us that

\[ \frac{KE}{M} \sim \nu^{-2/3} << 1 \]

and offers a justification for treating the ground state non-relativistically. Further, the specific behavior implies that the root mean square momentum \( \sqrt{\langle p^2 \rangle} \lesssim M^{5/9} \). This behavior will be used in deciding the cut-off \( \Lambda \) on the quantum correction to propagator.

4. Let \( \nu' \) be the largest value satisfying \( J_{\nu'}(\xi) = 0 \) for a given \( \xi \). This state corresponds to the first exited state. Then [9],

\[ \xi \approx \nu' + 1.8588 \nu'^{1/3} \equiv \nu' + \beta \nu'^{1/3} \]

Equations (5) and (9) imply

\[ \frac{\nu^2 - \nu'^2}{\nu^2} = 2(\beta - \alpha) \nu^{-2/3} + O\left(\nu^{-4/3}\right) = 2.1 \nu^{-2/3} + O\left(\nu^{-4/3}\right) \]

Recalling that \( \nu^2 - \nu'^2 \propto \Delta \varepsilon \), the energy gap between the first excited state and the ground state. This implies that,

\[ \frac{\Delta \varepsilon}{2M} = 2.1 \left( \frac{m}{2\sqrt{2M}} \right)^{2/3} \]

\[ \frac{\Delta \varepsilon}{m} = 2.1 \left( \frac{2M}{m} \right) \left( \frac{m}{2\sqrt{2M}} \right)^{2/3} = 2.1 \times \left( \frac{M}{m} \right)^{1/3} > 1 \]

Thus, the energy gap \( \Delta \varepsilon >> m \). Kinematics does not forbid excited state be unstable against the decay:

\[ \text{Excited state} \rightarrow \text{ground state} + \text{a scalar} \]
This holds also for higher excited states.

5. The excited states can decay to the ground state or lower excited states by emission of a scalar(s). We shall present the calculation regarding this in section 5. The decay width is \( \sim g^2 \) and is large as \( g \) and \( M \to \infty \). The excited states are very unstable in this limit. The life-time of excited states will \( \to 0 \) as \( g \) increases.

These set of results above can be verified numerically [12] (See table 1): In table 1, we have numerically evaluated various quantities. We found it convenient to take a several simple values of \( \nu \) and found the smallest solution \( \xi \) for it. We see that consistently, the ratio \( \frac{\nu}{\xi} \) approaches 1 from below. We next study how \( \frac{\nu}{\xi} \) approaches 1 from below. We compute \( \frac{\xi^2 - \nu^2}{\xi^2} \nu^{2/3} \) and note that it approaches a constant. For each \( \xi \), we find the largest solution \( \nu' \) satisfying \( J_{\nu'}(\xi) = 0 \) corresponding to the first excited state. We have had to round the value to an integer (which explains the fluctuations in the last column.).

### Table 1

| \( \nu \) | \( \xi \) | \( \frac{\nu}{\xi} \) | \( \frac{\xi^2 - \nu^2}{\xi^2} \nu^{2/3} \) | \( \nu' \) | \( \nu^2 - \nu'^2 \) | \( \frac{\nu^2 - \nu'^2}{\nu + \nu'} \) |
|---|---|---|---|---|---|---|
| 100 | 103.76838 | 0.9637 | 0.0713118 | 1.535893693 | 95 | 975 | 2.1038008 |
| 150 | 154.30972 | 0.9721 | 0.055078 | 1.554393383 | 144 | 1764 | 2.21701654 |
| 199 | 203.73309 | 0.9768 | 0.0459239 | 1.564782456 | 193 | 2352 | 2.02799167 |
| 300 | 305.4238 | 0.9822 | 0.0352012 | 1.576909638 | 283 | 4151 | 2.07085695 |
| 500 | 506.42703 | 0.9873 | 0.0252208 | 1.588151557 | 492 | 7936 | 2.00389353 |

4 Connecting Mass with Net Bound State Energy

We have taken a fermion-anti-fermion pair of mass \( M \) each, bound together in a Yukawa potential of maximum depth \( \approx \frac{g^2}{2m} \). We have assumed that \( M \gg m \). We want to ultimately show that the bound state can actually be identified with the scalar mass \( m \) of the particle that gives rise to the Yukawa potential. In order that this can be done, first of all, there is a need for consistency in the mass of the bound state and of the scalar. We shall show that by making a small adjustment (much less than \( M \)) in the depth of the well \( \frac{g^2}{2m} \) we can make the net energy equal the mass of the scalar \( m \). We would like,

\[
\text{Total relativistic energy of a bound state} = 2M - \varepsilon = m
\]
\[
\text{i.e. } 2M - \frac{m^2 \nu^2}{4M} = m
\]

\[
m^2 \nu^2 = 1 - \frac{m}{2M} \approx 1 - \frac{\sqrt{2}}{\nu}
\]

This together with (5) leads to

\[
\frac{g^2}{2m} = \xi^2 \frac{m^2}{8M^2} = \left( \frac{\xi}{\nu} \right)^2 \left( 1 - \frac{m}{2M} \right) \approx \left( 1 + \alpha \nu^{-2/3} \right)^2 \left( 1 - \frac{\sqrt{2}}{\nu} \right) \approx 1 - \frac{\sqrt{2}}{\nu} + \frac{2\alpha}{\nu^{2/3}} + ...
\]

\[
\frac{g^2/2m}{2M} = 1 + \frac{2\alpha \nu^{1/3} - \sqrt{2}}{\nu} + O(\nu^{-4/3})
\]

Compared to \( \frac{g^2/2m}{2M} = 1 \), this is a slight relative adjustment in the depth, which moreover vanishes as \( M \to \infty \). We shall see the purpose of this minute adjustment more clearly in a future section.

## 5 Decay of Excited States

We would like to argue next that the exited states are highly unstable and decay to suitable lower excited state(s), including the ground state. The lifetime of the excited states tends to zero as \( g \to \infty \). [We cannot base the result simply on dimensional analysis because the result depends on an overlap integral involving \( J_\nu \). \( \nu \) depends upon the dimensionless ratio \( M/m \)]. To begin with, we shall assume that \( g \) is not too large, and we can employ time-dependent perturbation theory in the Schrodinger picture. Let the \( n \)-th bound state \([(n-1)\text{-th excited state}] have energy \( 2M - \varepsilon_n \) when at rest. Let the bound state under consideration be in the \( n \)'th state at \( t = 0 \). We write the Hamiltonian operator in the Schrodinger picture as,

\[
H = H_H (t = 0) = H_0 + gH_I (t = 0)
\]

\[
H_I (t = 0) = \int dx : \overline{\psi} (x, 0) \psi (x, 0) \phi_q (x, 0).
\]

Here, \( H_0 \) is the free Hamiltonian, with the classical Yukawa interaction, that in the non-relativistic approximation leads to bound states of a fermion anti-fermion.
pair and \( H_I \) is the interaction Hamiltonian (from (3)) that leads in particular, in first order perturbation theory, to a decay to a lower state with the emission of a scalar. Let the wave-function of the \( m \)th bound state in the momentum space be \( \beta_m (q) \), where \( 2q \) is the relative momentum between the quark anti-quark pair. We denote by \( \int dp \beta_m (p) |p, -p, 0 \rangle \otimes |0 \rangle_b \) the initial state, where \( p \) and \( -p \) are the momenta of fermion and anti-fermion relative to its rest-frame, \( 0 \) is the momentum of the CM and \( |0 \rangle_b \) is the vacuum state for bosons. The final state is \( \int dq \beta_m (q) |\frac{P}{2} + q, \frac{P}{2} - q, P \rangle \otimes |l \rangle \). We express the relevant terms in \( H_I \) in terms of creation and destruction operators:

\[
H_I (t = 0) = \int \frac{dl_1 dp_1 dp_2}{(2\pi)^3 \sqrt{8E_{l_1} E_{p_1} E_{p_2}}} \left\{ a_{l_1}^\dagger \left[ d_{p_1}^\dagger d_{p_2} \overline{\psi} (p_1) u (p_2) 2\pi \delta (l_1 + p_1 - p_2) \right] - b_{p_2}^\dagger b_{p_1} \overline{\psi} (p_1) v (p_2) 2\pi \delta (l_1 + p_2 - p_1) + ... \right\}
\]

\[
\equiv H_I^{(1)} + H_I^{(2)}
\]

where we have expanded\(^3\).

\[
\psi (x) = \int \frac{dp}{(2\pi) \sqrt{2E_p}} \left[ d_p u (p) e^{-ipx} + b_{p}^\dagger v (p) e^{ipx} \right]
\]

\[
\phi (x) = \int \frac{dp}{(2\pi) \sqrt{2E_p}} \left[ a_p e^{-ipx} + a_{p}^\dagger e^{ipx} \right]
\]

\[
\{d (p), d^\dagger (q)\} = 2\pi \delta (p - q)
\]

\[
|p\rangle = \sqrt{2E_p} d^\dagger (p) |0\rangle
\]

\[
\langle p | p' \rangle = 2\pi \delta (p - p') 2E_p \text{ etc.}
\]

\( H_I^{(1)} \) consists of a term that leaves anti-quarks unaffected but destroys and creates a quark and a scalar. \( H_I^{(2)} \) on the other hand consists of a term that leaves quarks unaffected but destroys and creates an anti-quark and a scalar. We note that

\(^3\)We shall use the conventions in [13].
\[ d_p | p' \rangle = 2\pi \sqrt{2E_p} \delta (p - q). \] We have to compute the matrix element,

\[ \widetilde{M} \equiv \int dp dq \beta_n (p) \beta_m^* (q) \langle l | \otimes \left( \frac{P}{2} + q, \frac{P}{2} - q, P \right) H_I (t = 0) | p, -p, 0 \rangle \otimes | 0 \rangle \]

\[ \equiv \widetilde{M}_1 + \widetilde{M}_2 \]

where,

\[ \widetilde{M}_1 = \int dp dq \left\{ \frac{dl_1 dp_1 dp_2}{(2\pi)^2 \sqrt{4E_p E_p}} \left[ \beta_n (p) \beta_m^* (q) \left[ d_{p_1}^\dagger d_{p_2}^\dagger \pi (p_1) u (p_2) \delta (l_1 + p_1 - p_2) \right] \right] \right\} \]

\[ \times \delta (l_1 - l) | p, -p, 0 \rangle \]

\[ = -\int dp dq dl_1 dp_1 dp_2 \left\{ \left[ \pi (p_1) u (p_2) \delta (l_1 + p_1 - p_2) \right] \right\} \]

\[ \times \delta (l_1 - l) \delta (p_2 - p) \delta \left( p_1 - q - \frac{P}{2} \right) 2\pi \delta \left( -p + q - \frac{P}{2} \right) 2E_p \beta_n (p) \beta_n^* (q) \]

\[ \equiv -\delta (l + P) M_1 \]

\[ \widetilde{M}_2 = -\int dq \overline{u} \left( -q - \frac{P}{2} \right) v \left( -q + \frac{P}{2} \right) \beta_n \left( q - \frac{P}{2} \right) \beta_m^* (q) \delta (l + P) 2\pi 2E_{q + \frac{P}{2}} \]

\[ \equiv -\delta (l + P) M_2 \]

Let

\[ H_{mn} = \langle m | gH_I | n \rangle = -g \delta (l + P) [M_1 + M_2] \equiv -g \delta (l + P) M \]

We shall be using formulas corresponding to discrete normalization rather than continuous. Hence, we make the replacements [10]:

\[ \frac{L \delta_{p,p'}}{2\pi} \leftrightarrow \delta (p - p') ; \quad d_p^\dagger = 1/\sqrt{(2L)} d_p^\dagger (p) ; \quad ||p \rangle = 1/\sqrt{2LE_p} | p \rangle ; \quad a_p^\dagger = 1/\sqrt{(2L)} a_p^\dagger (p) \]
[Here, $d^\dagger_p$ and $|p\rangle$ refers to the discrete formulation and $d^\dagger (p)$ and $|p\rangle$ to continuum formulation.]. We have, for the continuum matrix element, now called $\tilde{H}_{mn}$,

$$\tilde{H}_{mn} \leftrightarrow [2L]^{3/2} \sqrt{M_n M_m E_l} H_{mn} = -g \delta (l + P) \mathcal{M} \leftrightarrow -g \frac{L}{2\pi} \delta_{l,-p} \mathcal{M}$$

where, from now on, we shall denote by $H_{mn}$ the matrix element in discrete case, which then is,

$$H_{mn} = -\frac{cg}{\sqrt{L}} \mathcal{M} \delta_{l,-p}$$

where $c \propto 1/\sqrt{M_n M_m E_l}$. We now first suppose that the coupling $g$ is weak. Then, we can legitimately employ the first order time-dependent perturbation theory. Initial state of the system is the $n$th bound state. The amplitude that the system is found in a bound state $m +$ a scalar particle is given by [2],

$$C_m (t) = -i \int_0^t dt' e^{i\omega_m t'} H_{mn} = i \frac{cg}{\sqrt{L}} \mathcal{M} \delta_{l,-p} \int_0^t dt' e^{i\omega_m t'} = \frac{cg}{\sqrt{L}} \mathcal{M} \delta_{l,-p} \frac{(1 - e^{i\omega_m t})}{\omega_m}$$

So the transition probability is ($\delta_{l,-p}^2 = \delta_{l,-p}$),

$$P_m (t) = |C_m (t)|^2 = \frac{e^2 g^2}{L} |\mathcal{M}|^2 \frac{\sin^2 \left( \frac{\omega_m t}{2} \right)}{\left( \frac{\omega_m}{2} \right)^2} \delta_{l,-p} = g^2 |\mathcal{M}|^2 \frac{\pi t^2}{L} \delta \left( \frac{\omega_m}{2} \right) \delta_{l,-p}; \quad t \to \infty$$

where we have employed,

$$\lim_{t \to \infty} \frac{1}{\pi t} \frac{\sin^2 tx}{x^2} = \delta (x)$$

Transition rate, which is probability per unit time, is

$$R_{m\to n} = (\text{const})g^2 |\mathcal{M}|^2 \frac{1}{M_n M_m E_l L} \delta \left( \frac{\omega_m}{2} \right) \delta_{l,-p};$$
\[
\frac{2\pi}{L} \delta (\omega_{mn}) = \frac{2\pi}{L} \delta \left( -M_n + M_m + \sqrt{l^2 + m^2 + \frac{l^2}{2M_m}} \right) \\
= \frac{2\pi}{L} \delta (l - l_1) \frac{1}{\omega_{mn}} \frac{1}{|l|} \leftrightarrow \delta_{l,l_1} \frac{1}{\omega} \frac{1}{l} \\
\simeq \delta_{l,l_1} \text{ assuming } M_m >> l >> m
\]

Thus, the contribution to the width of \( n \) state from this decay \( n \rightarrow m \) is

\[
\Gamma_{n \rightarrow m} \propto g^2 |\mathcal{M}|^2 \left[ M_n M_mE_l \right]^{-1}
\]

The dimensions of \( \Gamma \) are that of \( M \). Hence, dimensions of \( |\mathcal{M}|^2 \) is \( M^2 \). Since the former is a function of \( \xi \) and \( \nu \) and \( M \), we have \( \Gamma_{n \rightarrow m} \propto Mf \left[ \frac{m}{M} \right] \). Let us apply this to the transition from 1st excited state to ground state. Then,

\[
M_n = m, \quad M_m = m + 2.1 \left( \frac{M}{m} \right)^{1/3} m, \quad E_l \simeq 2.1 \left( \frac{M}{m} \right)^{1/3} m,
\]

so that

\[
\left[ M_n M_mE_l \right]^{-1} \sim \frac{1}{m^3 \times (2.1)^2} \left( \frac{m}{M} \right)^{2/3}
\]

This calculation is completed in the Appendix A.

We can’t borrow the results directly from the standard treatment of the radiation problem in textbooks (or with minor modifications). Major differences from the standard treatment of the radiation problem and the present case are:

1. Former is a 3-dimensional problem. Phase space is different.
2. It involves photon wavefunction rather than the scalar wavefunction.
3. Latter involves overlap integral involving large order Bessel functions. If the overlap integral tends to zero fast enough in the limit we have considered despite a factor of \( g^2 \) in the amplitude and despite increasing phase space, the result could be spoiled. [The overlap integral, in fact, does tend to zero but not fast enough]. This is the reason for investigation the relevant section in detail.
6 Some Issues

Before we proceed, we have a number of issues to settle: This will be done largely in appendices. But, here we list them and state our conclusions and how we use them. Approximations we made:

- The constituents of the bound state are non-relativistic.
- The $O(g^2)$ interactions are sufficient to determine the bound state structure.
- Renormalization effects on mass are ignorable. Quantum corrections to the classical mass of the scalar and whether that can alter our conclusions.

The last issue will be discussed in appendix B. The first two will be discussed in appendix C.

7 Identifying Scalar Field with the Stable Bound State

Consider the full propagator for the scalar field for $x_0 > y_0$ in presence of bound states. We have,

$$\Delta_F (x - y) = \langle 0 | \phi (x) \phi (y) | 0 \rangle \equiv \langle \Phi (x) | \Phi (y) \rangle$$

$$= \langle \Phi (x) | I | \Phi (y) \rangle$$

$$= \langle \Phi (x) | \{ \Sigma_n | \Psi_n \rangle \langle \Psi_n | \} | \Phi (y) \rangle$$

where $| \Psi_n \rangle$ is a stable state with the same quantum numbers as $| \Phi (y) \rangle$. As shown in the earlier section, there is only one such bound state, the ground state, and there will be a set of scattering states. We shall assume a spectral representation for the exact propagator:

$$\Delta_F (p) \equiv F.T. \{ \Delta_F (x - y) \} \equiv \int_0^\infty \frac{\rho (\sigma^2)}{p^2 - \sigma^2 + i\varepsilon} d\sigma^2$$

with,

$$\rho (\sigma^2) = Z\delta (\sigma^2 - m_1^2) + B\delta (\sigma^2 - m_2^2) + \rho_1 (\sigma^2)$$

where, $m_1$ is the rest-mass of the ground bound state, $B > 0$, and we assume

$$\rho_1 (\sigma^2) \geq 0$$
\[ \rho_1 (\sigma^2) = 0, \quad \sigma^2 < 4M^2 \]

\[ \int_{4M^2}^{\infty} \rho_1 (\sigma^2) \, d\sigma^2 = 1 - Z - \mathcal{B} \]

\[ < 1 \]

Here, we have assumed a relation similar to one that is thought to hold in LSZ formulation:

\[ 1 = Z + \int_{4M^2}^{\infty} \rho(\sigma^2) d\sigma^2 \]

. Then,

\[ \Delta_F (p) = \frac{Z}{p^2 - m^2 + i\epsilon} + \frac{\mathcal{B}}{p^2 - m_1^2 + i\epsilon} + \int_{4M^2}^{\infty} \frac{d\sigma^2 \rho_1 (\sigma^2)}{p^2 - \sigma^2 + i\epsilon} \]

We note that usually in a QFT, \( Z \to 0 \) as more and more channels in propagator are taken into account. Hence, we should have, \( \mathcal{B} + \int_{4M^2}^{\infty} \rho_1 (\sigma^2) = 1 \). Now, we study the propagator for \( p^2 \) small,

\[ - \int_{4M^2}^{\infty} \frac{d\sigma^2 \rho_1 (\sigma^2)}{p^2 - \sigma^2 + i\epsilon} \approx \int_{4M^2}^{\infty} \frac{d\sigma^2 \rho_1 (\sigma^2)}{\sigma^2} < \frac{1}{4M^2} \int_{4M^2}^{\infty} d\sigma^2 \rho_1 (\sigma^2) \lesssim \frac{1}{4M^2} \to 0 \]

as \( M \to \infty \), where use has been made of \( \int_{4M^2}^{\infty} \rho_1 (\sigma^2) d\sigma^2 < 1 \).

[An alternate argument can also be given: The last term in the above depends on \( \rho_1 (\sigma^2) \) which in turn depends on probability of finding a scattering state of an invariant mass \( \sigma \), where \( \sigma > 2M \). We expect this to be rather less sensitive to \( m \). Now, \( C = C (g, m, M) = C \left[ \frac{g^2}{mM}, \frac{m}{M} \right] \). Now, \( C \) is dimensionless, hence \( C = C \left[ \frac{g^2}{mM}, \frac{m}{M} \right] \). To the lowest order, this quantity is \( O(g^2) \) and also insensitive to \( m \). Hence, \( C \sim \frac{g^2}{mM} \frac{m}{M} \sim \frac{g^2}{M^2} \). For, similar reasons, we expect that, in higher orders, \( C \) is a function of single dimensionless variable \( g^2/M^2 : C = C [g^2/M^2] \). In the limit under consideration, \( C \to C[0] = \lim_{g \to 0} C [g^2/M^2] = \) the \( O(g^2) \) result \( \sim \frac{g^2}{M^2} \).]
Hence, we have,

\[ \lim_{M \to \infty} \frac{C}{4M^2} = 0 \quad (10) \]

and we find that, the corrections to propagator is saturated by the ground bound state. Now, if \( Z \to 0; \mathcal{B} \to 1 \); the propagator itself will be represented fully by the propagation of bound state.

8 Uses of this Formulation

We summarize the uses of this formulation as follows:

- This formulation allows one to look the scalar field as representing a propagating bound state.
- In particular, the interaction Lagrangian of the scalar fields, obtained by integrating with respect to \( \psi \), is that due to a tight bound state of \( \psi, \bar{\psi} \). Non-locality in the interaction of scalar field can be conceivably understood as due to the bound state nature of a scalar \( \phi \).
- One cannot distinguish if it is a theory of an elementary field \( \phi \) or a composite bound state.
- This formulation has been useful in study of whether bound state formation can affect causality of the theory [1].

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Appendix A

In this appendix, we shall perform an explicit calculation of the decay width for the process 1st excited state \( \rightarrow \) ground state + a scalar. We have seen, in section 5 that this is given by

\[ \Gamma \propto g^2 \frac{1}{(2.1)^2 m^3} \left( \frac{m}{M} \right)^{2/3} |\mathcal{M}|^2 \]
Here, we need to estimate $|\mathcal{M}|^2$. To do this, consider first $\mathcal{M}_1$:

$$\mathcal{M}_1 \propto \int dq\tilde{u} \left( q + \frac{P}{2} \right) u \left( q - \frac{P}{2} \right) \beta_n \left( q + \frac{P}{2} \right) \beta^*_m (q) E_{q - \frac{P}{2}}$$

Now,

$$\beta_n (q) = \int_{-\infty}^{\infty} dx\, C J_\nu (\xi \exp (-m|x|)) \exp (-iqx)$$

$$= \frac{1}{m} \int_0^\infty dXC \, J_\nu (\xi \exp (-X)) \exp (-i(q/m)X) + ....$$

$$= \frac{1}{m} C F \left[ \frac{q}{m}, \xi \right] + ....$$

And

$$C = \left\{ \int_{-\infty}^{\infty} dxJ^2_\nu (\xi \exp (-m|x|)) \right\}^{-1/2}$$

$$= \left\{ -\frac{1}{m} \int_\xi^0 \frac{dt}{t} J^2_\nu (t) + ....... \right\}^{-1/2}$$

$$= m^{1/2} f (\xi)$$

Thus, we find,

$$\beta_n (q) = \frac{1}{\sqrt{m}} \tilde{F} \left[ \frac{q}{m}, \xi \right]$$

Setting $\frac{q}{m} = Q$ and letting $m \to 0$, we obtain,

$$\int dq\tilde{u} \left( q + \frac{P}{2} \right) u \left( q - \frac{P}{2} \right) \beta_n \left( q + \frac{P}{2} \right) \beta^*_m (q) E_{q - \frac{P}{2}}$$

$$= \int dQ\tilde{u} \left( mQ + \frac{P}{2} \right) u \left( mQ - \frac{P}{2} \right) \tilde{F}_n \left( Q + \frac{P}{2m}, \xi \right) F^*_m (Q, \xi) E_{mQ - \frac{P}{2}}$$
In the limit $m \to 0$ and $M \to \infty$, we can set

$$\tilde{u}\left(mQ + \frac{P}{2}\right) u \left(mQ - \frac{P}{2}\right) \approx \tilde{u}\left(\frac{P}{2}\right) u \left(-\frac{P}{2}\right) \sim M$$

and also,

$$E_{mQ - \frac{P}{2}} \to M$$

Then the expression for $\mathcal{M}_1$ is proportional to

$$M^2 \int dQ \tilde{F}_n \left(Q + \frac{P}{2m}, \xi\right) F^*_m (Q, \xi) |\xi \to \infty$$

We recognize the above manipulations equivalent to proving:

$$\mathcal{M}_1 \propto \int dq \tilde{u} \left(q + \frac{P}{2}\right) u \left(q - \frac{P}{2}\right) \beta_n \left(q + \frac{P}{2}\right) \beta^*_m \left(q\right) E_{q - \frac{P}{2}}$$

$$= M^2 \int dq \beta_n \left(q + \frac{P}{2}\right) \beta^*_m \left(q\right)$$

Now,

$$\int_{-\infty}^{\infty} dq \beta_n \left(q + \frac{P}{2}\right) \beta^*_m \left(q\right)$$

$$= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dx C J_\nu \left(\xi \exp (-m|x|)\right) \exp \left(-i \left[q + \frac{P}{2}\right] x\right)$$

$$\times \int_{-\infty}^{\infty} d'x C' J_\mu \left(\xi \exp (-m|x'|)\right) \exp \left(iq x'\right)$$

$$\propto \int_{-\infty}^{\infty} dx C C' J_\nu \left(\xi \exp (-m|x|)\right) J_\mu \left(\xi \exp (-m|x|)\right) \exp \left(-i \frac{P}{2} x\right)$$

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Let us employ $\ln \xi - m|\mathbf{x}| = \ln t; \quad dx = \frac{1}{\pm m} \frac{dt}{t}$ so that

$$\int_{-\infty}^{\infty} dq \beta_n \left( q + \frac{P}{2} \right) \beta_m^* (q)$$

$$\propto \frac{1}{m} \int_0^\xi \frac{dt}{t} CC' J_\nu (t) J_\mu (t) \left[ \exp \left( -i \frac{P}{2m} (\ln \xi - \ln t) \right) + c.c. \right]$$

$$= \frac{1}{m} \exp \left( -i \frac{P}{2m} \ln \xi \right) \int_0^\xi \frac{dt}{t} CC' J_\nu (t) J_\mu (t) \exp \left( i \frac{P}{2m} \ln t \right) + c.c.$$

Suppose,

$$I = \int_0^\xi \frac{dt}{t} J_\nu (t) J_\mu (t) \exp \left( i \frac{P \ln t}{2m} \right) \quad (11)$$

We shall now express

$$I = \int_0^{\alpha \sqrt{\nu}} \frac{dt}{t} J_\nu (t) J_\mu (t) \exp \left( i \frac{P \ln t}{2m} \right)$$

$$+ \int_\alpha^{\sqrt{\nu}} \frac{dt}{t} J_\nu (t) J_\mu (t) \exp \left( i \frac{P \ln t}{2m} \right)$$

$$\equiv I_1 + I_2$$

where $\alpha$ has been chosen so that the following approximation in the first integral $I_1$ can be made small with any desired degree of accuracy [7].

$$J_\nu (t) \simeq \frac{t^\nu}{\nu!} \quad ; 0 \leq t < \alpha \sqrt{\nu}$$

So that $I_1$ can be ignored. Moreover, in the second integral $I_2$, we note that $|J_\nu (t)| < 1$ for $t$ being real. Then

$$|I_2| < \int_\alpha^{\sqrt{\nu}} \frac{dt}{t} = \ln \frac{\xi}{\alpha \sqrt{\nu}} \sim \frac{1}{2} \ln \xi$$
is, in fact, not a power-law-behaved quantity. $I_2$, for the present transition, can be estimated as follows: We look at the graphs of $J_\nu$ and $J_\nu'$, we find that the product $J_\nu J_\nu'$ is sharply peaked around a value $0 << t_0 < \xi$, so that to a leading approximation

$$I_2 \approx \exp (i z_0) \int_0^\xi \frac{dt}{t} J_\nu (t) J_\mu (t)$$

where $z_0 = \frac{P \ln t_0}{2m}$. Putting this in the expression for $I$ in $\mathcal{M}_1$, we find, using $z = \frac{P \ln \xi}{2m}$

$$\mathcal{M}_1 \propto \frac{M^2}{m} \left[ \exp i(z - z_0)I_0 + c.c. \right]$$

$$= \frac{M^2}{m} \left[ 2 \cos(z - z_0)I_0 \right]$$

Noting that,

$$z - z_0 = \frac{P}{2m} (\ln \xi - \ln t_0)$$

Thus, the magnitude of a typical contribution to $\Gamma$ indeed blows in the limit $M \to \infty$. This implies that the first excited state is totally unstable in this limit. We can see a similar conclusion plausible for higher excited states.

**Appendix B: The Quantum Corrections to the Mass of the Scalar**

The mass parameter we have been using in the above is the classical mass. We need to see if the quantum corrections to the mass of the scalar can alter the conclusions. The quantum corrections, to $O(g^2)$, can be calculated by calculating the self-energy of the scalar, now in a model that admits bound states. As shown in the section 5, in the limit $g \to \infty$, $M \to \infty$, there is only stable bound state. As argued in section 7, we can find the self-energy by effectively saturating the propagator by the lowest bound state. We shall estimate the self-energy as follows: Lowest bound state has a wave-function given by $|1\rangle$. Let us first consider the scalar on mass-shell with momentum $p$ with $p^2$ on-shell. We can always go to the rest-frame of the scalar. When the scalar decomposes between a quark-antiquark pair, they carry momenta $\frac{q}{2}, -\frac{q}{2} + p$ with an amplitude $\Phi_0 (q)$. $\Phi_0 (q)$ is the Fourier transform of the normalized ground state wave-function $CJ_\nu (\xi \exp [-m|x|/2])$. We assume that the pair is near
mass-shell and hence have \( q^2 \approx 0 \). The contribution of this intermediate state to the self-energy of scalar is

\[
i \Sigma(p) = -g^2 \int d^2k \frac{1}{(2\pi)^2} \frac{Tr \left[ (k + M)(k + \not{p} + M) \right]}{(k^2 - M^2)((k + p)^2 - M^2)} \Phi_0(k) \Phi_0(-k)
\]

In \( \Phi_0(k) \), \( k \) is effectively a one-momentum. We can carry out a Wick rotation, so that

\[
i \Sigma(p) = -g^2 i \int d^2k \frac{1}{(2\pi)^2} \frac{Tr \left[ (k + M)(k + \not{p} + M) \right]}{(k^2 + M^2)((k + p)^2 + M^2)} \Phi_0(k) \Phi_0(-k)
\]

\( k \) now is a Euclidean momentum. We now recall that \( \Phi_0(k) \) falls off rapidly beyond \( (k)^2 > \Lambda^2 \). We can then estimate the integral by putting a cut-off \( \Lambda \) and find:

\[
\Sigma(p) = \frac{g^2}{4\pi} \int_0^1 d\alpha \left[ \ln \left( \frac{\Lambda^2 + M^2 - \alpha(1 - \alpha)p^2}{M^2 - \alpha(1 - \alpha)p^2} \right) - \frac{2\Lambda^2}{\Lambda^2 + M^2 - \alpha(1 - \alpha)p^2} \right]
\]

For \( p^2 = 0 \), we have\(^4\)

\[
\Sigma(0) = \frac{g^2}{4\pi} \left[ \ln \left( \frac{\Lambda^2 + M^2}{M^2} \right) - \frac{2\Lambda^2}{\Lambda^2 + M^2} \right]
\]

Taking into account the momentum behavior of \(^1\), we shall assume that \( \Lambda \) can be chosen as \( \sim M^{5/9} \). Then, we can assume \( \Lambda << M \) and, we have the quantum correction to \( m^2 \), viz.

\[
\delta m^2 = -\Sigma(0) = \frac{g^2}{4\pi} \left[ \frac{\Lambda^2}{M^2} \right]
\]

which goes to zero as \( M^{-2/9} \) as \( M \rightarrow \infty \). Thus, the mass of the scalar is stable against quantum corrections in this limit.

\(^4\)Mass correction is usually evaluated at \( p^2 = m^2 \); however the difference is small and doesn’t affect the conclusion.
Appendix C: The Bound States and Use of the Schrodinger Equation

In using non-relativistic equation, which is a second order in $g$ i.e. $O(g^2)$, to obtain the bound states, we have made several approximations:

- The constituents of the bound state are non-relativistic.
- The $O(g^2)$ interactions are sufficient to determine the bound state structure.
- Renormalization effects on mass are ignorable.

We have already addressed to the last question (see appendix B). Further, in our present case, we have shown that $\frac{\langle KE \rangle}{2M} \sim \nu^{-2/3}$ (Please see (8)). As long as $\nu$ is large, i.e. for the low lying states, kinetic energy is small compared to the mass of fermions, and the constituents are non-relativistic. As to the higher order corrections, we can employ Bethe-Salpeter approach [11]. We consider the next order correction to the non-relativistic momentum space wave-function. Let $\Phi(p)$ be the momentum space wave-function as calculated from the Schrodinger equation. The next order diagram (See Fig-1) is

![Figure 1: O(g^4) diagram.](image)

$$\tilde{\Phi}(p) = g^2 \int \frac{d^2q}{(2\pi)^2} \frac{i}{p + q - M + i\varepsilon} \Phi(p + q) \frac{i}{p + q - M + i\varepsilon} \frac{i}{q^2 - m^2 + i\varepsilon}$$

For the ground state, $\Phi(p + q)$ is damped out for $|p + q| \geq O(M^{5/9})$. Thus, the effective range of $|p + q|$ inside the integral $<< M$. It is not difficult to see that this integral is suppressed by a dimensionless factor of $O\left(\frac{g^2}{M^2}\right)$. [The singular depen-
dence on $m$ is at worst logarithmic]. Now,

$$\frac{g^2}{M^2} = \frac{g^2}{mM} \frac{m}{M} \sim \frac{m}{M} \ll 1$$

Thus, in this particular limit, the higher order quantum corrections are indeed negligible. We note in passing that Harindranath and Perry have dealt with a problem of bound states between two different species of fermions [4] in light-front field theory for the 1+1 dimension Yukawa problem. They have shown the connection between the $O(g^2)$ quantum correction term and the Schrodinger equation (Please see Appendix C of reference [4]). It holds under the conditions that (i) the constituents are non-relativistic and (ii) the $O(g^4)$ and higher order terms in the relevant equations can be ignored.

References

[1] A. Haque and S.D. Joglekar arXiv:1004.2344v3 [hep-th].

[2] See e.g. “Quantum Mechanics” by E. Merzbacher second edition, published by John Wiley and sons, Inc. New York 1970.

[3] O. W. Greenberg Phys. Rev. 147, 1077 (1966).

[4] A. Harindranath, Robert J. Perry and J. Shigemitsu Phys. Rev. D 46, 4580 - 4602 (1992) [See appendix];

[5] A. Harindranath and Robert J. Perry Phys. Rev. D 43, 4051 - 4602 (1991) [See appendix C].

[6] See e.g. “Quantum Mechanics”, L.Landau and I. Lifshitz , Pergamon Press, London, 1960.

[7] G.N. Watson, “The treatise on the theory of Bessel functions” Cambridge Mathematical Library,

[8] Please see reference [7], page 521,

[9] Please see reference [7], page 521.

[10] See e.g. Relativistic Quantum Fields, J.D. Bjorken and S.D. Drell, we employ 1-dimensional analogue.

[11] E. Salpeter and H. Bethe Phys. Rev. 84,1232,(1951).
[12] We have made use of the interactive web-site: cose.math.bas.bg/webMathematica/webComputing/BesselZeros.jsp

[13] M. E. Peskin, and D. V. Schroeder, An introduction to quantum field theory (Westview, Boulder, Colo., 2003).
A self-consistent bound state model for meson

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Abstract

We study the 1+1 dimensional Yukawa theory, in a certain limit of its parameters $g, M, m$ (as suggested by the study of causality in presence of bound states in this model \cite{1}). We study the bound state formation in the model. In the limit $g \rightarrow \infty, M \rightarrow \infty$, in a certain specific manner, we show that there are a large number of bound states of which at least the low lying states are described by the non-relativistic Schrodinger equation. We show that, in this limit, the excited bound states are unstable and deem to decay quickly (lifetime $\tau \rightarrow 0$) by emission of scalar (s) in this particular limit. The mass of the ground state is not significantly affected by higher order quantum corrections and by proper choice of parameters, involving only small changes, can be adjusted to be equal to the mass of the scalar. As a result of quantum effects, the state of the meson mixes with the lowest bound state and may be dominated by the latter. We show that in this detailed sense, a scalar meson in Yukawa model can be looked upon as a bound state of a fermion-anti fermion pair formed.

1 Preliminary

We start with the 1+1 dimensional Yukawa theory,

\[ \mathcal{L} = \bar{\psi} \left[ i \gamma \cdot \partial - M + g \phi \right] \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \] (1)

We expand $\phi = \phi_c + \phi_q$, where $\phi_c$ is the classical field obeying the classical equation of motion:

\[ \left[ \partial^2 + m^2 \right] \phi_c = g \langle \bar{\psi} \psi \rangle, \] (2)

leading to the Yukawa potential, and

\[ \mathcal{L}_T = g \bar{\psi} \psi \phi_q + .... \] (3)

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is the quantum interaction Lagrangian. [This division is more or less like what we do when we discuss radiation from an atom. (see e.g. [2])] The fermion anti-fermion pair interacts in the first instance via $\phi_c$, the classical Yukawa potential and form bound states. In absence of $\mathcal{L}_I$, these are the exact stationary states of the system (like the H-atom has them if interaction of an electron with a photon were neglected). The excited states, however, are unstable against decay via emission of scalar(s) through the interaction Lagrangian $\mathcal{L}_I$. We want to show that in a certain limit of parameters $g, m, M$, (suggested by a study of causality in this model, [1]) there is only one stable bound state and the scalar particle can be identified with it. All other bound states are highly unstable. ($\tau \to 0$).

2 Non-relativistic Bound States

We shall consider the fermion anti-fermion system (mass $M$ each) in a bound state via the attractive Yukawa potential [This is a solution of (2), under appropriate conditions].

$$V(x) = -g^2 e^{-m|x|}/2m$$

We shall assume that the mass $M$ is large, and that the bound state is non-relativistic. (This will be justified later.) For the relative motion, the non-relativistic Schrodinger equation is satisfied (reduced mass is $\frac{M}{2}$):

$$-\frac{\hbar^2}{M} \frac{d^2 \psi}{dx^2} - g^2 e^{-m|x|}/2m \psi = E \psi$$

where $E = -\varepsilon < 0$ is the energy of a bound state. We introduce the change of the independent variable: $y = e^{-m|x|/2}$. Under this change, the equation transforms to

$$y^2 \frac{d^2 \psi}{dy^2} + y \frac{d\psi}{dy} + \frac{2Mg^2}{m^3} y^2 \psi + \frac{4EM}{m^2} \psi = 0$$

[The equation, now, has to be solved separately for $x > 0$ and $x < 0$ and the solutions have to be tested for continuity of $\psi$ and $\psi'$ at $x = 0$. A further rescaling:

$$t = \xi y \equiv \sqrt{\frac{2Mg^2}{m^3}} y = \sqrt{\frac{2Mg^2}{m^3}} e^{-m|x|/2}$$

will put the equation in the form of the Bessel
equation \[3, 4\]:

\[
\psi'' + \frac{1}{t}\psi' + \left(1 - \frac{\nu^2}{t^2}\right)\psi = 0.
\]

with \(\nu \equiv +\sqrt{\frac{4\varepsilon M}{m^2}}\). [We recall that as \(\varepsilon\) is always less than the depth of the well \(\frac{\varepsilon^2}{2m}\), the allowed values of \(\nu\) will turn out to be less than \(\xi\)]. We note that as \(x \to \pm\infty\), \(t \to 0\). As \(x \to 0^\pm\), \(t \to \xi\). The boundary condition at \(x \to \pm\infty\) requires that \(\psi(0) \to 0\). This requires that we take the solution which is regular at \(t = 0\), viz. \(J_\nu(t)\) and drop the singular solution \(N_\nu(t)\). The discrete values of \(\varepsilon\) are determined from the conditions of continuity at \(x = 0\):

1. \(\psi(x \to 0^-) = \psi(x \to 0^+) \Rightarrow \psi(t)\) is continuous as \(t \to \xi\) from either side.
2. \(\frac{d}{dx}\psi(x \to 0^-) = \frac{d}{dx}\psi(x \to 0^+) \Rightarrow \psi'(t \to \xi; x \to 0^-) = -\psi'(t \to \xi; x \to 0^+)\)

These boundary conditions can materialize in two distinct ways: As the potential is even, the non-degenerate states will have a definite parity\[6\]; so that,

1. the wave-function is even and \(\frac{d}{dx}\psi(x \to 0^-) = \frac{d}{dx}\psi(x \to 0^+) = 0\); (and \(\psi(x = 0) \neq 0\) );
2. the wave-function is odd: \(\psi(x = 0) = 0\), (and \(\frac{d}{dx}\psi(x \to 0^-) = \frac{d}{dx}\psi(x \to 0^+) \neq 0\) );

In the case 1 above, \(\psi(t)\) satisfies: \(\psi'(t) = 0\) at \(t = \xi\). In the case 2 above, \(\psi(t)\) satisfies: \(\psi(t) = 0\) at \(t = \xi\). Thus, the energy eigenvalues are determined by:

1. \(J_\nu(\xi) = 0\), or
2. \(J'_\nu(\xi) = 0\)

[where, from the properties of \(J_\nu(\xi)\), both \(J_\nu(\xi)\) and \(J'_\nu(\xi)\) cannot simultaneously vanish for a \(\xi > 0\) \[7\]]. The energy dependence enters through the order \(\nu \equiv +\sqrt{\frac{4\varepsilon M}{m^2}}\) of \(J_\nu\).
3 Properties of Bound States and Relevant Properties of $J_\nu (\xi)$

For the ground state, being an even wavefunction, $J'_\nu (\xi) = 0$. $\nu$ is so determined that $\xi$ is the first positive zero of $J'_\nu (\xi)$. The ground state wave-function is

$$\psi = C J_\nu (\xi \exp [-m|x|/2])$$

(4)

The relevant properties of the ground state wave-function are:

1. $\nu < \xi$ as expected and for a large $\xi$, $\nu/\xi \to 1$ from below. This implies that

$$\frac{\varepsilon}{g^2/2m} \to 1 \text{ from below.}$$

The exact behavior is given as [8],

$$\xi = \nu + 0.808618 \nu^{1/3} + O (\nu^{-1/3}) \equiv \nu + \alpha \nu^{1/3} + O (\nu^{-1/3})$$

(5)

This can be numerically verified: See Table 1.

2. $\frac{\xi^2 - \nu^2}{\xi^2} \to 0$. The quantity $\xi^2$ is proportional to the maximum depth of the well $g^2/2m$, $\nu^2$ like-wise is proportional to the magnitude of the ground state energy eigenvalue. So, $(\xi^2 - \nu^2)$ gives an upper bound on the kinetic energy of the ground state. We shall consider the depth of the potential approximately equal to $2M$. The relation $\frac{\xi^2 - \nu^2}{\xi^2} \to 0$ then implies: $\frac{<KE>}{g^2/2m} = \frac{<KE>}{2M} \to 0$; or that the ground state is non-relativistic. Evidently, it will also apply to low lying excited states.

3. $\frac{\xi^2 - \nu^2}{\xi^2} \to 0$. It behaves as $\frac{\xi^2 - \nu^2}{\xi^2} \sim \nu^{-2/3}$. This follows from (5), which implies,

$$\left(\xi^2 - \nu^2\right) = (\xi - \nu) (\xi + \nu)$$

$$= \left[0.808618 \nu^{1/3} + O (\nu^{-1/3})\right]$$

$$\times \left[2\nu + 0.808618 \nu^{1/3} + O (\nu^{-1/3})\right]$$

$$= 1.617236 \nu^{4/3} + 0.654 \nu^{2/3} + O (1)$$

(6)

$$\frac{(\xi^2 - \nu^2)}{\xi^2} \nu^{2/3} = 1.617236 \frac{\nu^2}{\xi^2} + O (\nu^{-2/3})$$

(7)

$$= 1.617236 + O (\nu^{-2/3})$$
We first note that the (7) tells us that
\[ KE \sim \nu^{-2/3} \ll 1 \] (8)
and offers a justification for treating the ground state non-relativistically. Further, the specific behavior implies that the root mean square momentum \( \sqrt{\langle p^2 \rangle} \ll M^{5/9} \). This behavior will be used in deciding the cut-off \( \Lambda \) on the quantum correction to propagator.

4. Let \( \nu' \) be the largest value satisfying \( J_{\nu'}(\xi) = 0 \) for a given \( \xi \). This state corresponds to the first exited state. Then \[ \xi \approx \nu' + 1.8588 \nu'^{1/3} \equiv \nu' + \beta \nu'^{1/3} \] (9)

Equations (5) and (9) imply
\[ \frac{\nu^2 - \nu'^2}{\nu^2} = 2 (\beta - \alpha) \nu^{-2/3} + O \left( \nu^{-4/3} \right) = 2.1 \nu^{-2/3} + O \left( \nu^{-4/3} \right) \]

Recalling that \( \nu^2 - \nu'^2 \propto \Delta \varepsilon \), the energy gap between the first excited state and the ground state. This implies that,
\[ \frac{\Delta \varepsilon}{2M} = 2.1 \left( \frac{m}{2 \sqrt{2} M} \right)^{2/3} \]
\[ \frac{\Delta \varepsilon}{m} = 2.1 \left( \frac{2M}{m} \right) \left( \frac{m}{2 \sqrt{2} M} \right)^{2/3} = 2.1 \times \left( \frac{M}{m} \right)^{1/3} > 1 \]

Thus, the energy gap \( \Delta \varepsilon \gg m \). Kinematics does not forbid excited state be unstable against the decay:

\[ \text{Excited state} \rightarrow \text{ground state} + \text{a scalar} \]

This holds also for higher excited states.

5. The excited states can decay to the ground state or lower excited states by emission of a scalar(s). We shall present the calculation regarding this in section 5. The decay width is \( \sim g^2 \) and is large as \( g \) and \( M \to \infty \). The excited states are very unstable in this limit. The life-time of excited states will \( \rightarrow 0 \) as \( g \) increases.
Table 1

| $\nu$ | $\xi$ | $\nu \xi$ | $\frac{\xi^2 - \nu^2}{\xi^2}$ | $\frac{\xi^2 - \nu^2}{\xi^2} \nu^2$ | $\nu'$ | $\nu^2 - \nu'^2$ | $\frac{\nu^2 - \nu'^2}{\nu^{1/3}}$ |
|-------|-------|----------|----------------|-------------------------------|------|----------------|----------------|
| 100   | 103.76838 | 0.9637  | 0.0713118 | 1.535893693 | 95   | 975 | 2.1038008 |
| 150   | 154.30972 | 0.9721  | 0.055078  | 1.554393383 | 144 | 1764 | 2.21701654 |
| 199   | 203.73309 | 0.9768  | 0.0459239 | 1.564782456 | 193 | 2352 | 2.02799167 |
| 300   | 305.4238  | 0.9822  | 0.0352012 | 1.576909638 | 283 | 4151 | 2.07085695 |
| 500   | 506.42703 | 0.9873  | 0.0252208 | 1.588151557 | 492 | 7936 | 2.00389353 |

These set of results above can be verified numerically [12]. (See table 1): In table 1, we have numerically evaluated various quantities. We found it convenient to take a several simple values of $\nu$ and found the smallest solution $\xi$ for it. We see that consistently, the ratio $\frac{\nu}{\xi}$ approaches 1 from below. We next study how $\frac{\nu}{\xi}$ approaches 1 from below. We compute $\frac{\xi^2 - \nu^2}{\xi^2} \nu^2$ and note that it approaches a constant. For each $\xi$, we find the largest solution $\nu'$ satisfying $J_{\nu'}(\xi) = 0$ corresponding to the first excited state. We have had to round the value to an integer (which explain the fluctuations in the last column.)

4 Connecting Mass with Net Bound State Energy

We have taken a fermion-anti-fermion pair of mass $M$ each, bound together in a Yukawa potential of maximum depth $\approx \frac{g^2}{2m}$. We have assumed that $M >> m$. We want to ultimately show that the bound state can actually be identified with the scalar mass $m$ of the particle that gives rise to the Yukawa potential. In order that this can be done, first of all, there is a need for consistency in the mass of the bound state and of the scalar. We shall show that by making a small adjustment (much much less than M) in the depth of the well $\frac{g^2}{2m}$ we can make the net energy equal the mass of the scalar $m$. We would like,

Total relativistic energy of a bound state = $2M - \varepsilon = m$

i.e. $2M - \frac{m^2 \nu^2}{4M} = m$

$\frac{m^2 \nu^2}{8M^2} = 1 - \frac{m}{2M} \approx 1 - \frac{\sqrt{2}}{\nu}$
This together with (5) leads to

\[
\frac{g^2}{2m^2} = \frac{\xi^2}{8M^2} \approx \left(1 + \frac{\alpha \nu^{-2/3}}{\nu} \right)^2 \left(1 - \frac{2}{\nu} \right) \approx 1 - \frac{\sqrt{2}}{\nu} + \frac{2\alpha}{\nu^{2/3}} + \ldots
\]

\[
\frac{g^2/2m}{2M} = 1 + \frac{(2\alpha \nu^{1/3} - \sqrt{2})}{\nu} + O(\nu^{-4/3})
\]

Compared to \(\frac{g^2/2m}{2M} = 1\), this is a slight relative adjustment in the depth, which moreover vanishes as \(M \to \infty\). We shall see the purpose of this minute adjustment more clearly in a future section.

5 Decay of Excited States

We would like to argue next that the excited states are highly unstable and decay to suitable lower excited state(s), including the ground state. The lifetime of the excited states tends to zero as \(g \to \infty\). [We cannot base the result simply on dimensional analysis because the result depends on an overlap integral involving \(J_{\nu, \nu}\) depends upon the dimensionless ratio \(M/m\)].

To begin with, we shall assume that \(g\) is not too large, and we can employ time-dependent perturbation theory in the Schrödinger picture. Let the \(n\)-th bound state [(\(n - 1\)]-th excited state] have energy \(2M - \epsilon_n\) when at rest. Let the bound state under consideration be in the \(n\)th state at \(t = 0\). We write the Hamiltonian operator in the Schrödinger picture as,

\[
H = H_H (t = 0) = H_0 + gH_I (t = 0)
\]

\[
H_I (t = 0) = \int dx : \bar{\psi} (x, 0) \psi (x, 0) \phi_q (x, 0) :
\]

Here, \(H_0\) is the free Hamiltonian, with the classical Yukawa interaction, that in the non-relativistic approximation leads to bound states of a fermion anti-fermion pair and \(H_I\) is the interaction Hamiltonian (from (3)) that leads in particular, in first order perturbation theory, to a decay to a lower state with the emission of a scalar. Let the wave-function of the \(m\)th bound state in the momentum space be \(\beta_m (q)\), where \(2q\) is the relative momentum between the quark anti-quark pair. We denote by \(\int dp \beta_n (p) |p, -p, 0\rangle \otimes |0\rangle_b\) the initial state, where \(p\) and \(-p\) are the momenta of fermion and anti-fermion relative to its rest-frame, 0 is the momentum of the CM and \(|0\rangle_b\) is the vacuum state for bosons. The final state is
\[ \int dq \beta_m (q) \langle \frac{\mathbf{P}}{2} + q, \frac{\mathbf{P}}{2} - q, \mathbf{P} \rangle \otimes |l \rangle. \]

We express the relevant terms in \( H_I \) in terms of creation and destruction operators:

\[
H_I (t = 0) = \int \frac{dl_1 dp_1 dp_2}{(2\pi)^3} \sqrt{8E_{l_1} E_{p_1} E_{p_2}} \left\{ a_{l_1}^\dagger \left[ d_{p_1}^\dagger d_{p_2}^\dagger \mathbf{\Pi} (p_1) u (p_2) \right] 2\pi \delta (l_1 + p_1 - p_2) \right\} \\
- b_{p_2}^\dagger b_{p_1} \mathbf{\Pi} (p_1) v (p_2) 2\pi \delta (l_1 + p_2 - p_1) + \ldots \right\} \\
\equiv H_I^{(1)} + H_I^{(2)}
\]

where we have expanded \(^3\).

\[
\psi (x) = \int \frac{dp}{(2\pi)^3} \sqrt{2E_p} \left[ d_p u (p) e^{-ipx} + b_p^\dagger v (p) e^{ipx} \right]
\]

\[
\phi (x) = \int \frac{dp}{(2\pi)^3} \sqrt{2E_p} \left[ a_p e^{-ipx} + a_p^\dagger e^{ipx} \right]
\]

\[ \{ d (p), d^\dagger (q) \} = 2\pi \delta (p - q) \]

\[ |p\rangle = \sqrt{2E_p} d^\dagger (p) |0\rangle \]

\[ \langle p | p' \rangle = 2\pi \delta (p - p') 2E_p \]

\[ H_I^{(1)} \] consists of a term that leave anti-quarks unaffected but destroy and create a quark and a scalar. \( H_I^{(2)} \) on the other hand consists of a term that leave quarks unaffected but destroy and create a anti-quark and a scalar. We note that \( d_p |p'\rangle = 2\pi \sqrt{2E_p} \delta (p - q) \). We have to compute the matrix element,

\[
\widetilde{M} \equiv \int dp dq \beta_n (p) \beta_m^* (q) \langle l | \otimes \left\langle \frac{\mathbf{P}}{2} + q, \frac{\mathbf{P}}{2} - q, \mathbf{P} \left| H_I (t = 0) |p, -p, 0\rangle \otimes |0\rangle \right\rangle \\
\equiv \widetilde{M}_1 + \widetilde{M}_2
\]

\(^3\)We shall use the conventions in [13].

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where,

\[ \tilde{M}_1 = \int dpdq \left( \frac{P}{2} + q, \frac{P}{2} - q, P \right) \]

\[
\times \int \frac{dl_1 dp_1 dp_2}{(2\pi)^2 \sqrt{4E_{p_1} E_{p_2}}} \left\{ \beta_n(p) \beta_m^*(q) \left[ d_{p_1}^\dagger d_{p_2} \bar{u}(p_1) u(p_2) \delta (l_1 + p_1 - p_2) \right] \right\}
\]

\[
\times \delta (l_1 - l) |p, -p, 0\rangle
\]

\[ = -\int dpdqdl_1 dp_1 dp_2 \left\{ \left[ \bar{u}(p_1) u(p_2) \delta (l_1 + p_1 - p_2) \right] \right\}
\]

\[
\times \delta (l_1 - l) \delta (p_2 - p) \delta \left( p_1 - q - \frac{P}{2} \right) 2\pi \delta \left( -p + q - \frac{P}{2} \right) 2E_p \beta_n(p) \beta_m^*(q)
\]

\[ = -\int dq \bar{u} \left( q + \frac{P}{2} \right) u \left( q - \frac{P}{2} \right) \beta_n \left( q - \frac{P}{2} \right) \beta_m^*(q) \]

\[
2\pi 2E_{q - \frac{P}{2}} \delta (l + P)
\]

\[ \equiv -\delta (l + P) \mathcal{M}_1
\]

\[ \tilde{M}_2 = -\int dq \bar{u} \left( -q - \frac{P}{2} \right) v \left( -q + \frac{P}{2} \right) \beta_n \left( q + \frac{P}{2} \right) \beta_m^*(q) \delta (l + P) 2\pi 2E_{q + \frac{P}{2}}
\]

\[ \equiv -\delta (l + P) \mathcal{M}_2
\]

Let

\[ H_{mn} = \langle m | gH_I | n \rangle = -g\delta (l + P) [\mathcal{M}_1 + \mathcal{M}_2] \equiv -g\delta (l + P) \mathcal{M}
\]

We shall be using formulas corresponding to discrete normalization rather than continuous. Hence, we make the replacements [10]:

\[ \frac{L\delta_{p,p'}}{2\pi} \leftrightarrow \delta (p - p') ; \quad d_p^\dagger = 1/\sqrt{(2L)} d^\dagger (p) ; \quad |p\rangle = 1/\sqrt{2LE_p} |p\rangle ; \quad a_p^\dagger = 1/\sqrt{(2L)} a^\dagger (p)
\]

[Here, \( d_p^\dagger \) and \( |p\rangle \) refers to the discrete formulation and \( d^\dagger (p) \) and \( |p\rangle \) to continuum formulation.]. We have, for the continuum matrix element, now called \( \tilde{H}_{mn} \),

\[ \tilde{H}_{mn} \leftrightarrow [2L]^{3/2} \sqrt{M_n M_m E_l} H_{mn} = -g\delta (l + P) \mathcal{M} \leftrightarrow -g\frac{L}{2\pi} \delta_{l,-P} \mathcal{M} \]
where, from now on, we shall denote by $H_{mn}$ the matrix element in discrete case, which then is,

$$H_{mn} = -\frac{cg}{\sqrt{L}} \mathcal{M} \delta_{l'-P}$$

where $c \propto 1/\sqrt{M_n M_m E_l}$. We now first suppose that the coupling $g$ is weak. Then, we can legitimately employ the first order time-dependent perturbation theory. Initial state of the system is the $nth$ bound state. The amplitude that the system is found in a bound state $m +$ a scalar particle is given by [2],

$$C_m(t) = -i \int_0^t dt' e^{i\omega_{mn}t'} H_{mn} = i \frac{cg}{\sqrt{L}} \mathcal{M} \delta_{l'-P} \int_0^t dt' e^{i\omega_{mn}t'} = \frac{cg}{\sqrt{L}} \mathcal{M} \delta_{l'-P} \frac{(1 - e^{i\omega_{mn}t})}{\omega_{mn}}$$

where. So the transition probability is ($\delta_{l'-P} = \delta_{l-P}$),

$$P_m(t) = |C_m(t)|^2 = \frac{c^2 g^2}{L} |\mathcal{M}|^2 \sin^2 \left(\frac{\omega_{mn}t}{2}\right) \delta_{l'-P} = g^2 |\mathcal{M}|^2 \pi t \frac{c^2}{L} \delta \left(\frac{\omega_{mn}}{2}\right) \delta_{l'-P}; \quad t \to \infty$$

where we have employed,

$$\lim_{t \to \infty} \frac{1}{\pi t} \sin tx x^2 = \delta(x)$$

Transition rate, which is probability per unit time, is

$$R_{m \to n} = (\text{const}) g^2 |\mathcal{M}|^2 \pi \frac{1}{M_n M_m E_l L} \delta \left(\frac{\omega_{mn}}{2}\right) \delta_{l'-P};$$

$$\frac{2\pi}{L} \delta (\omega_{mn}) = \frac{2\pi}{L} \delta \left(-M_n + M_m + \sqrt{l^2 + m^2 + \frac{l^2}{2M_m}}\right)$$

$$= \frac{2\pi}{L} \delta (l - l_1) \frac{1}{\omega_{mn}} \bigg|_{l=l_1} \leftrightarrow \delta_{l' l_1} \frac{1}{\omega_l + \frac{l}{M_m}}$$
\[ \simeq \delta_{l,l_i} \quad assuming \quad M_m >> l >> m \]

Thus, the contribution to the width of \( n \)-th state from this decay \( n \to m \) is

\[ \Gamma_{n \to m} \propto g^2 |\mathcal{M}|^2 \left[ M_n M_mE_l \right]^{-1} \]

The dimensions of \( \Gamma \) are that of \( M \). Hence, dimensions of \( |\mathcal{M}|^2 \) is \( M^2 \). Since the former is a function of \( \xi \) and \( \nu \) and \( M \), we have \( \Gamma_{n \to m} \propto M f \left[ \frac{m}{M} \right] \). Let us apply this to the transition from 1st excited state to ground state: then,

\[ M_n = m, \quad M_m = m + 2.1 \left( \frac{M}{m} \right)^{1/3} m, \quad E_l \simeq 2.1 \left( \frac{M}{m} \right)^{1/3} m, \]

so that

\[ \left[ M_n M_mE_l \right]^{-1} \simeq \frac{1}{m^3 \times (2.1)^2} \left( \frac{m}{M} \right)^{2/3} \]

This calculation is completed in the Appendix A.

### 6 Some Issues

Before we proceed, we have a number of issues to settle: This will be done largely in appendices. But, here we list them and state our conclusions and how we use them. Approximations we made:

- The constituents of the bound state are non-relativistic.
- The \( O(g^2) \) interactions are sufficient to determine the bound state structure.
- Renormalization effects on mass are ignorable. Quantum corrections to the classical mass of the scalar and whether that can alter our conclusions.

The last issue will be discussed in appendix A. The first two will be discussed in appendix B.
7 Identifying Scalar Field with the Stable Bound State

Consider the full propagator for the scalar field for \( x_0 > y_0 \) in presence of bound states. We have,

\[
\Delta_F (x - y) = \langle 0| \phi (x) \phi (y) |0\rangle \equiv \langle \Phi (x) | \Phi (y) \rangle = \langle \Phi (x) | \mathcal{I} | \Phi (y) \rangle = \langle \Phi (x) | \{ \Sigma_n | \Psi_n \rangle \langle \Psi_n | \} | \Phi (y) \rangle
\]

where \( |\Psi_n\rangle \) is a stable state with the same quantum numbers as \( |\Phi (y)\rangle \). As shown in the earlier section, there is only one such bound state, the ground state, and there will be a set of scattering states. We shall assume a spectral representation for the exact propagator:

\[
\Delta_F (p) \equiv F.T. \{ \Delta_F (x - y) \} \equiv \int_0^\infty \frac{\rho (\sigma^2)}{p^2 - \sigma^2 + i\varepsilon} d\sigma^2
\]

with,

\[
\rho (\sigma^2) = Z\delta (\sigma^2 - m^2) + B\delta (\sigma^2 - m_1^2) + \rho_1 (\sigma^2)
\]

where, \( m_1 \) is the rest-mass of the ground bound state, \( B > 0 \), and we assume

\[
\rho_1 (\sigma^2) \geq 0
\]

\[
\rho_1 (\sigma^2) = 0, \quad \sigma^2 < 4M^2
\]

\[
\int_{4M^2}^{\infty} \rho_1 (\sigma^2) d\sigma^2 = 1 - Z - B
\]

\[
< 1
\]

Here, we have assumed a relation similar to one that is thought to hold in LSZ formulation:

\[
1 = Z + \int_{4M^2}^{\infty} \rho (\sigma^2) d\sigma^2
\]

. Then,

\[
\Delta_F (p) = \frac{Z}{p^2 - m^2 + i\varepsilon} + \frac{B}{p^2 - m_1^2 + i\varepsilon} + \int_{4M^2}^{\infty} \frac{d\sigma^2}{p^2 - \sigma^2 + i\varepsilon} \rho_1 (\sigma^2)
\]
We note that usually in a QFT, \( Z \to 0 \) as more and more channels in propagator are taken into account. Hence, we should have, \( B + \int_{4M^2}^{\infty} \rho_1 (\sigma^2) \cdot \frac{1}{4M^2} \int_{4M^2}^{\infty} d\sigma^2 \rho_1 (\sigma^2) \lesssim \frac{1}{4M^2} \to 0 \)

as \( M \to \infty \), where use has been made of \( \int_{4M^2}^{\infty} d\sigma^2 \rho_1 (\sigma^2) < 1 \).

[An alternate argument can also be given: The last term in the above depends on \( \rho_1 (\sigma^2) \) which in turn depends on probability of finding a scattering state of an invariant mass \( \sigma \), where \( \sigma > 2M \). We expect this to be rather less sensitive to \( m \). Now, \( C = C (g, m, M) = C \left[ \frac{g^2}{mM}, \frac{m}{M} \right] \). Now, \( C \) is dimensionless, hence

\[
\text{\( C = C \left[ \frac{g^2}{mM}, \frac{m}{M} \right] \). To the lowest order, this quantity is } O(g^2) \text{ and also insensitive to } m. \text{ Hence, } C \sim \frac{g^2}{mM} m \sim \frac{g^2}{M^2}. \text{ For, similar reasons, we expect that, in higher orders, } C \text{ is a function of single dimensionless variable } g^2/M^2 : \text{ } C = C [g^2/M^2]. \text{ In the limit under consideration, } C \to C[0] = \lim_{g \to 0} C [g^2/M^2] = \text{ the } O(g^2) \text{ result } \sim \frac{g^2}{M^2}. \text{ Hence, we have,}

\[
\lim_{M \to \infty} \frac{C}{4M^2} = 0
\]

and we find that, the corrections to propagator is saturated by the ground bound state. Now, if \( Z \to 0; B \to 1 \); the propagator itself will be represented fully by the propagation of bound state.

8 Uses of this Formulation

- This formulation allows one to look the scalar field as representing a propagating bound state.

- In particular, the interaction Lagrangian of the scalar fields, obtained by integrating with respect to \( \psi \), is that due to a tight bound state of \( \psi, \overline{\psi} \). Non-locality in the interaction of scalar field can be conceivably understood as due to the bound state nature of a scalar \( \phi \).

- One cannot distinguish if it is a theory of an elementary field \( \phi \) or a composite
bound state.

- This formulation has been useful in study of whether bound state formation can affect causality of the theory [1].

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Appendix A

In this appendix, we shall perform an explicit calculation of decay width for the process 1st excited state $\rightarrow$ ground state + a scalar. We have seen, in section 5, that this is given by

$$
\Gamma \propto g^2 \frac{1}{(2\pi)^2 m^3} \left( \frac{m}{M} \right)^{2/3} |\mathcal{M}|^2
$$

Here, we need estimate $|\mathcal{M}|^2$. To do this, consider first $\mathcal{M}_1$:

$$
\mathcal{M}_1 \propto \int dq \bar{u} \left( q + \frac{P}{2} \right) u \left( q - \frac{P}{2} \right) \beta_n \left( q + \frac{P}{2} \right) \beta^*_m (q) E_{q-P/2}
$$

Now, consider,

$$
\beta_n(q) = \int_{-\infty}^{\infty} dx \, C J_\nu (\xi \exp (-m|x|)) \exp (-iqx)
$$

$$
= \frac{1}{m} \int_{0}^{\infty} dX C J_\nu (\xi \exp (-X)) \exp (-i(q/m)X) + ....
$$

$$
= \frac{1}{m} C \, F \left[ \frac{q}{m}, \xi \right] + ....
$$

And

$$
C = \left\{ \int_{-\infty}^{\infty} dx J^2_\nu (\xi \exp (-m|x|)) \right\}^{-1/2}
$$
\[
\left\{ -\frac{1}{m} \int_{\xi}^{0} \frac{dt}{t} J_{\nu}^{2} (t) + \ldots \right\}^{-1/2}
\]
\[
= m^{1/2} f (\xi)
\]
Thus, we find,
\[
\beta_{n} (q) = \frac{1}{\sqrt{m}} \bar{F} \left[ \left[ \frac{q}{m}, \xi \right] \right]
\]
Setting \( \frac{q}{m} = Q \) and letting \( m \to 0 \), we obtain,
\[
= \int dq \bar{u} \left( q + \frac{P}{2} \right) u \left( q - \frac{P}{2} \right) \beta_{n} \left( q + \frac{P}{2} \right) \beta_{m}^{*} (q) E_{q - \frac{P}{2}}
\]
\[
= \int dQ \bar{u} \left( mQ + \frac{P}{2} \right) u \left( mQ - \frac{P}{2} \right) \bar{F}_{n} \left( Q + \frac{P}{2m}, \xi \right) F_{m}^{*} (Q, \xi) E_{mQ - \frac{P}{2}}
\]
As \( m \to 0 \) and hence \( M \to \infty \), we can set
\[
\bar{u} \left( mQ + \frac{P}{2} \right) u \left( mQ - \frac{P}{2} \right) \approx \bar{u} \left( \frac{P}{2} \right) u \left( -\frac{P}{2} \right) \sim M
\]
and also,
\[
E_{mQ - \frac{P}{2}} \to M
\]
Then the expression for \( M_{1} \) is proportional to
\[
M^{2} \int dQ \bar{F}_{n} \left( Q + \frac{P}{2m}, \xi \right) F_{m}^{*} (Q, \xi) \bigg|_{\xi \to \infty}
\]
We recognize the above manipulations equivalent to proving:
\[
\mathcal{M}_{1} \propto \int dq \bar{u} \left( q + \frac{P}{2} \right) u \left( q - \frac{P}{2} \right) \beta_{n} \left( q + \frac{P}{2} \right) \beta_{m}^{*} (q) E_{q - \frac{P}{2}}
\]
\[
= M^{2} \int dq \beta_{n} \left( q + \frac{P}{2} \right) \beta_{m}^{*} (q)
\]
Now,

\[
\begin{align*}
&= \int_{-\infty}^{\infty} dq \beta_n \left( q + \frac{P}{2} \right) \beta_m^* (q) \\
&= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dx C J_\nu (\xi \exp (-m|x|)) \exp \left( -i \left[ q + \frac{P}{2} \right] x \right) \\
&\times \int_{-\infty}^{\infty} dx' C' J_\mu (\xi \exp (-m|x'|)) \exp (iqx') \\
&\propto \int_{-\infty}^{\infty} dx C' C J_\nu (\xi \exp (-m|x|)) J_\mu (\xi \exp (-m|x|)) \exp \left( -i \frac{P}{2} x \right)
\end{align*}
\]

We employ \( \ln \xi - m|x| = \ln t \); \( dx = \frac{1}{\mp m} dt \)

\[
= \frac{1}{m} \int_0^\xi \frac{dt}{t} CC' J_\nu (t) J_\mu (t) \left[ \exp \left( -i \frac{P}{2m} (\ln \xi - \ln t) \right) + c.c. \right]
\]

\[
= \frac{1}{m} \exp \left( -i \frac{P}{2m} \ln \xi \right) \int_0^\infty \frac{dt}{t} CC' J_\nu (t) J_\mu (t) \exp \left( i \frac{P}{2m} \ln t \right) + c.c.
\]

We write,

\[
I = \int_0^\xi \frac{dt}{t} J_\nu (t) J_\mu (t) \exp \left( i \frac{P}{2m} \ln t \right) \tag{11}
\]

We shall write

\[
I = \int_0^{\alpha \sqrt{\nu}} \frac{dt}{t} J_\nu (t) J_\mu (t) \exp \left( i \frac{P}{2m} \ln t \right) + \int_{\alpha \sqrt{\nu}}^\xi \frac{dt}{t} J_\nu (t) J_\mu (t) \exp \left( i \frac{P}{2m} \ln t \right)
\]

\[
\equiv I_1 + I_2
\]

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where \( \alpha \) has been chosen so that the following approximation in the first integral \( I_1 \) can be made small with any desired degree of accuracy

\[
J_\nu(t) \simeq \frac{t^\nu}{\nu!} \quad ; 0 \leq t < \alpha \sqrt{\nu}
\]

So that \( I_1 \) can be ignored

In the second integral, we note: \( |J_\nu(t)| < 1; t \) real. Then

\[
|I_2| < \int_{\alpha \sqrt{\nu}}^\xi \frac{dt}{t} = \ln \frac{\xi}{\alpha \sqrt{\nu}} \sim 1/2 \ln \xi
\]

i.e., in fact, not a power-law-behaved quantity. In fact, we can estimate \( I_2 \), for the present transition, as follows: We look at the graphs of \( J_\nu \) and \( J_\nu' \), we find that the product \( J_\nu J_\nu' \) is sharply peaked around a value \( 0 << t_0 < \xi \), so that to a leading approximation

\[
I_2 \approx \exp(iz_0) \int_{\alpha \sqrt{\nu}}^\xi \frac{dt}{t} J_\nu(t) J_\nu(t)
\]

where \( z_0 = \frac{P \ln t_0}{2m} \) Putting this in the expression for \( I \) in \( \mathcal{M}_1 \), we find, using \( z = \frac{P \ln \xi}{2m} \)

\[
\mathcal{M}_1 \propto \frac{M^2}{m} [\exp i(z - z_0)I_0 + c.c.] = \frac{M^2}{m} [2 \cos(z - z_0)I_0]
\]

Noting that

\[
z - z_0 = \frac{P}{2m} (\ln \xi - \ln t_0)
\]

Thus, the magnitude of a typical contribution to \( \Gamma \) indeed blows in the limit \( M \to \infty \). This implies that the first excited state is totally unstable in this limit. We can see that a similar conclusion plausible for higher excited states.
Appendix B: The Quantum Corrections to the Mass of the Scalar

The mass parameter we have been using in the above is the classical mass. We need to see if the quantum corrections to the mass of the scalar can alter the conclusions. The quantum corrections, to $O(g^2)$, can be calculated by calculating the self-energy of the scalar, now in a model that admits bound states. As shown in the section 5, in the limit $g \to \infty$, $M \to \infty$, there is only stable bound state. As argued in section 7, we can find the self-energy by effectively saturating the propagator by the lowest bound state: i.e. we shall estimate the self-energy as follows: Lowest bound state has a wave-function given by (4). Let us first consider the scalar on mass-shell with momentum $p$ with $p^2$ on-shell. We can always go to the rest-frame of the scalar. When the scalar decomposes between a quark-antiquark pair, they carry momenta $\frac{q}{2}, -\frac{q}{2} + p$ with an amplitude $\Phi_0(q)$. $\Phi_0(q)$ is the Fourier transform of the normalized ground state wave-function $CJ_\nu(\xi \exp[-m|x|/2])$. We assume that the pair is near mass-shell and hence have $q^2 \approx 0$. The contribution of this intermediate state to the self-energy of scalar is

$$i \Sigma(p) = -g^2 \int d^2 k \frac{1}{(2\pi)^2} \frac{Tr[(k + M)(k + p + M)]}{(k^2 - M^2)((k + p)^2 - M^2)} \Phi_0(k) \Phi_0(-k)$$

In $\Phi_0(k)$, $k$ is effectively a one-momentum. We can carry out a Wick rotation, so that

$$i \Sigma(p) = -g^2 i \int d^2 k \frac{1}{(2\pi)^2} \frac{Tr[(k + M)(k + p + M)]}{(k^2 + M^2)((k + p)^2 + M^2)} \Phi_0(k) \Phi_0(-k)$$

$k$ now is a Euclidean momentum. We now recall that $\Phi_0(k)$ falls of rapidly beyond $(k)^2 > \Lambda^2$. We can then estimate the integral by putting a cut-off $\Lambda$ and find:

$$\Sigma(p) = \frac{g^2}{4\pi} \int_0^1 d\alpha \left[ \ln \left( \frac{\Lambda^2 + M^2 - \alpha(1 - \alpha)p^2}{M^2 - \alpha(1 - \alpha)p^2} \right) - \frac{2\Lambda^2}{\Lambda^2 + M^2 - \alpha(1 - \alpha)p^2} \right]$$
For $p^2 = 0$, we have:

$$
\Sigma(0) = \frac{g^2}{4\pi} \left[ \ln \left( \frac{\Lambda^2 + M^2}{M^2} \right) - \frac{2\Lambda^2}{\Lambda^2 + M^2} \right]
$$

Taking into account the momentum behavior of (11), we shall assume that $\Lambda$ can be chosen as $\sim M^{5/9}$. Then, we can assume $\Lambda \ll M$ and, we have the quantum correction to $m^2$, viz.

$$
\delta m^2 = -\Sigma(0) = \frac{g^2}{4\pi} \left[ \frac{\Lambda^2}{M^2} \right]
$$

which goes to zero as $M^{-2/9}$ as $M \rightarrow \infty$. Thus, the mass of the scalar is stable against quantum corrections in this limit.

**Appendix C: The Bound States and Use of the Schrodinger Equation**

In using non-relativistic equation, which is a second order in $g$ i.e. $O(g^2)$, to obtain the bound states, we have made several approximations:

- The constituents of the bound state are non-relativistic.
- The $O(g^2)$ interactions are sufficient to determine the bound state structure.
- Renormalization effects on mass are ignorable.

We have already addressed to the last question. Further, in our present case, we have shown that $\frac{\langle KE \rangle}{2\nu^2} \sim \nu^{-2/3}$ (Please see (8)). As long as $\nu$ is large, i.e. for the low lying states, kinetic energy is small compared to the mass of fermions, and the constituents are non-relativistic. As to the higher order corrections, we can employ Bethe-Salpeter approach [11]. We consider the next order correction to the non-relativistic momentum space wave-function. Let $\Phi(p)$ be the momentum space wave-function as calculated from the Schrodinger equation. The next order diagram (See Fig-1) is

---

4Mass correction is usually evaluated at $p^2 = m^2$; however the difference is small and doesn’t affect the conclusion.
For the ground state, \( \Phi(p + q) \) is damped out for \(|p + q| \geq O(M^{5/9})\). Thus, the effective range of \(|p+q|\) inside the integral \( \ll M \). It is not difficult to see that this integral is suppressed by a dimensionless factor of \( O\left(\frac{g^2}{M^2}\right) \). The singular dependence on \( m \) is at worst logarithmic. Now,

\[
\frac{g^2}{M^2} = \frac{g^2}{mM} \frac{m}{M} \sim \frac{m}{M} \ll 1
\]

Thus, in this particular limit, the higher order quantum corrections are indeed negligible. We note in passing that Harindranath and Perry have dealt with a problem of bound states between two different species of fermions in light-front field theory for the 1+1 dimension Yukawa problem. They have shown the connection between the \( O(g^2) \) quantum correction term and the Schrodinger equation (Please see Appendix C of reference [5]). It holds under the conditions that (i) the constituents are
non-relativistic and (ii) the $O(g^4)$ and higher order terms in the relevant equations can be ignored.

References

[1] A. Haque and S.D. Joglekar (in preparation)

[2] See e.g. “Quantum Mechanics” by E. Merzbacher second edition, published by “John Wiley and sons, Inc. New York 1970.

[3] O. W. Greenberg Phys. Rev. 147, 1077 (1966)

[4] A. Harindranath, Robert J. Perry and J. Shigemitsu Phys. Rev. D 46, 4580 - 4602 (1992) [See appendix];

[5] A. Harindranath and Robert J. Perry Phys. Rev. D 43, 4051 - 4062 (1991) [See appendix C]

[6] See e.g. “Quantum Mechanics”, L.Landau and I. Lifshitz , Pergamon Press, London, 1960)

[7] G.N. Watson, “The treatise on the theory of Bessel functions” Cambridge Mathematical Library

[8] Please see reference [7], page 521.

[9] Please see reference [7], page 521,

[10] see e.g. Relativistic Quantum Fields, J.D. Bjorken and S.D. Drell, we employ 1-dimensional analogue.

[11] E. Salpeter and H. Bethe Phys. Rev. 84,1232,(1951)

[12] We have made use of the interactive web-site: cose.math.bas.bg/webMathematica/webComputing/BesselZeros.jsp

[13] M. E. Peskin, and D. V. Schroeder, An introduction to quantum field theory (Westview, Boulder, Colo., 2003)