THE ERDŐS-ULAM PROBLEM, LANG’S CONJECTURE, AND UNIFORMITY

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Abstract. A rational distance set is a subset of the plane such that the distance between any two points is a rational number. We show, assuming Lang’s Conjecture, that the cardinalities of rational distance sets in general position are uniformly bounded, generalizing results of Solymosi-de Zeeuw, Makhul-Shaffaf, Shaffaf, and Tao. In the process, we give a criterion for certain varieties with non-canonical singularities to be of general type.

1. Introduction

A rational distance set is a subset $S$ of $\mathbb{R}^2$ such that the distance between any two points of $S$ is a rational number. In 1946, Ulam posed the following, based on a result of Anning-Erdős [AE45]:

Question 1. Is there a rational distance set that is dense for the Euclidean topology of $\mathbb{R}^2$?

While this problem is still open, Shaffaf [Sha18] and Tao [Tao14] independently showed that Lang’s Conjecture (Conjecture 2.2 below) implies that the answer to the Erdős-Ulam question is ‘no’. In fact, they showed that if Lang’s Conjecture holds, a rational distance set cannot even be dense for the Zariski topology of $\mathbb{R}^2$, i.e. must be contained in a union of real algebraic curves. Solymosi and de Zeeuw [SdZ10] proved (unconditionally, using Faltings’ proof of Mordell’s conjecture) that a rational distance contained in a real algebraic curve must be finite, unless the curve has a component which is either a line or a circle. Furthermore, any line (resp. circle) containing infinitely many points of a rational distance set must contain all but at most four (resp. three) points of the set.

One can rephrase the result of [SdZ10] as almost all points of an infinite rational distance set contained in a union of curves tend to concentrate on a line or circle. It is therefore natural to consider the ‘generic situation’, and so motivated by [SdZ10], we say that a subset $S \subseteq \mathbb{R}^2$ is in general position if no line contains all but at most four points of $S$, and no circle contains all but at most three points of $S$. For example, a set of seven points in $\mathbb{R}^2$ is in general position if and only if no line passes through $7 - 4 = 3$ of the points and no circle passes through $7 - 3 = 4$ of the points.

Before stating our main result, we note that assuming Lang’s conjecture, rational distance sets in general position are necessarily finite, by the above mentioned results [SdZ10, Sha18, Tao14].

Theorem 1.1. Assume Lang’s Conjecture. There exists a uniform bound on the the cardinality of a rational distance set in general position.

Erdős famously asked whether there exists a rational distance set with seven elements in general position. To stress the parallel between Lang’s Conjecture and Erdős’ question, we emphasize the following consequence of our main result.

Corollary 1.2. If there exist rational distance sets in general position of cardinality larger than any fixed constant, then Lang’s Conjecture does not hold.

Erdős’ question was answered in the affirmative by Kreisel and Kurz [KK08]. We are unaware of any examples of rational distance sets in general position of cardinality larger than seven. We
do note, however, that for sets of cardinality greater than seven, our notion of general position is strictly weaker than the one appearing previously e.g. in work of [KK08]. For them, general position means no three points contained on a line, and no four on any circle.

1.1. Method of proof. To prove Theorem 1.1, we note that points of rational distance sets lift to rational points of curves and surfaces of general type, (see [StZ10, Tao14]). We then show that Lang’s Conjecture implies uniform bounds for these sets of rational points, using results of Caporaso-Harris-Mazur [CHM97] and Hassett [Has96]. The idea of using uniformity to study rational distance sets first appeared in the paper of Makhul and Shaffaf [MS12], who used uniformity for curves [CHM97]. To obtain our Theorem 1.1 (which generalizes [MS12]), we use both uniformity for curves [CHM97] as well as uniformity for surfaces [Has96].

Our proof of Theorem 1.1 uses a result that certain (singular) complete intersections of four quadrics in \( \mathbb{P}^6 \) are of general type. This result was first shown by Tao [Tao14] (see also [Kov18]). We include in this paper a more general statement (Proposition 4.4 below), which implies Tao’s result and we think may be of interest in its own right.

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2. Background

Faltings proved in [Fal83] that a curve of genus at least two can have only finitely many points with rational coordinates (or more generally with coordinates in a number field). Varieties of general type provide the natural generalization of curves of genus at least two to higher dimensional varieties.

Definition 2.1. A projective variety \( X \) is of general type if for any desingularization \( \tilde{X} \to X \) the canonical divisor \( \omega_{\tilde{X}} \) is big.

In Lemma 4.2 and Proposition 4.4 we will see criteria for a variety to be of general type. Using this notion, Lang (and Bombieri for surfaces) proposed a generalization of Faltings’ theorem for varieties of higher dimension.

Conjecture 2.2 (Lang’s Conjecture). Let \( X \) be a projective variety of general type defined over a number field \( K \). Then the set of rational points \( X(K) \) is not Zariski dense in \( X \).

Conjecture 2.2 is known in full generality only for curves, where it reduces to Faltings’ Theorem [Fal83] and subvarieties of abelian varieties [Fal91]. The conjecture of Lang has a number of striking implications. One of the more spectacular ones is the following theorem of Caporaso-Harris-Mazur.

Theorem 2.3 ([CHM97, Theorem 1.1]). Assume Conjecture 2.2. For each number field \( K \) and positive integer \( g \geq 2 \), there exists an integer \( B(K, g) \) such that no smooth curve of genus \( g \) defined over \( K \) has more than \( B(K, g) \) rational points.

Following Caporaso-Harris-Mazur, Hassett proved the following implication of the Lang conjecture for surfaces that will be used in this paper [Has96].

Theorem 2.4 ([Has96, Theorem 6.2]). Assume Conjecture 2.2. Let \( X \to B \) be a flat family of projective surfaces defined over a number field \( K \), such that the general fiber is an integral surface of general type. For any \( b \in B(K) \) for which \( X_b \) is of general type, let \( N(b) \) be the sum of the degrees of the components of \( X_b(K) \). Then \( N(b) \) is bounded by a constant independent of \( b \).
Analogous statements were proven for (stably) integral points in [Abr97] and [ADT18].

3. Uniform bounds for rational distance sets

The goal of this section is to give a proof of Theorem 1.1. We begin with a well known lemma (see [Sha18, Lemma 2] for a proof) that says rational distance sets may be put in a standard form.

**Lemma 3.1.** Let $S \subset \mathbb{R}^2$ be a rational distance set. There exists a squarefree integer $k > 0$ and a sequence of translations, rotations and rational dilations of the plane that maps $S$ into the lattice \{(a, b\sqrt{k}) \mid a, b \in \mathbb{Q}\}. Moreover, the image of $S$ in this lattice is again a rational distance set.

Throughout this section, we let $x, y, z$ be the coordinates on $\mathbb{P}_C^2$ and identify $\mathbb{R}^2$ with the set of real points of affine open subset of $\mathbb{P}_C^2$ where $z = 1$. The following lemma summarizes some of the work of Solymosi-de Zeeuw [SDZ10, 3, 4]; we additionally give lower bounds for $\#(C \cap S)$ and upper bounds for the genus of the curve $D$. In Lemma 3.2 we do not consider curves of degree two, or degree three missing the two points at infinity, because the image of such curves under a suitable inversion of the plane is covered by the lemma (see the proof of Proposition 3.3).

**Lemma 3.2** (see also [SDZ10]). Let $S \subset \mathbb{R}^2$ be a rational distance set, let $C \subset \mathbb{P}_C^2$ be an irreducible projective curve of degree $d$ defined over $\mathbb{R}$, and suppose that $C \cap S$ is contained in the smooth locus of $C$. Suppose further that one of the following holds:

1. $d = 1$ and $\#(S \cap C) \geq 2$ and $\#(S \setminus C) \geq 5$.
2. $d = 3$ and $\#(S \cap C) \geq 9$ and $(\pm i, 1, 0) \in C$.
3. $d \geq 4$ and $\#(S \cap C) \geq 2d(d - 1)$.

There exists a smooth projective curve $D$ defined over $\mathbb{Q}$ whose genus is bounded below by 2 and bounded above by a function of $d$, and a morphism $\pi : D \to C$ defined over $\mathbb{R}$, such that $\pi(D(\mathbb{Q}))$ contains $S \cap C$.

**Proof.** By Lemma 3.1, we may assume that $S \subseteq \{(a, b\sqrt{k}) \in \mathbb{R}^2 \mid a, b \in \mathbb{Q}\}$ for some integer $k > 0$. Given $P = (a, b) \in \mathbb{Q}^2$, let $q_P := (x - az)^2 + k(y - bz)^2 \in \mathbb{Q}[x, y, z]$. The zero locus of $q_P$ in $\mathbb{P}_C^2$ is a union of two complex lines that contains a single real point (namely, $P$), where the two lines meet.

Let $\phi : \mathbb{P}_C^2 \to \mathbb{P}_C^2$ be the linear automorphism that sends $(x, y, z) \mapsto (x, y/\sqrt{k}, z)$. Let $C' := \phi(C)$ and $S' := \phi(S)$. Then $C'$ is defined over $\mathbb{Q}$, since it contains at least $\frac{1}{2}d(d + 3)$ points with rational coordinates in $\mathbb{P}_C^2$, namely the points of $S' \cap C'$. Let $f \in \mathbb{Q}[x, y, z]$ be a homogeneous equation for $C'$ of minimal degree. Let $g$ be the geometric genus of $C$ and $C'$. Note that $g \leq \frac{1}{2}(d - 1)(d - 2)$.

To construct $D$ and $\pi$, we distinguish between the three cases in the statement of the lemma.

1. Suppose $d = 1$ and $\#(S \setminus C) \geq 5$. Then there exist points $P_1, P_2, P_3 \in S' \setminus C'$ such that the union of the three pairs of complex lines defined by $q_{P_j} = 0$, where $j = 1, 2, 3$, meets the line $C'$ transversely at six points. Let $x, y, z, w$ be the coordinates of the weighted projective space $\mathbb{P}(1, 1, 1, 3)$, and let $D$ be the curve in $\mathbb{P}(1, 1, 1, 3)$ defined by $w^2 - q_{P_1}q_{P_2}q_{P_3} = f = 0$. Let $\pi' : D \to C'$ the projection onto the first three coordinates, and let $\pi := \phi^{-1} \circ \pi'$. Then $\pi(D(\mathbb{Q}))$ contains $C \cap S$. Furthermore, $\pi$ is a double cover with simple ramification over six points of $C$. It follows from Riemann-Hurwitz that $D$ has genus 2.

2. Suppose $d = 3$ and $\#(S \cap C) \geq 9$ and $(\pm i, 1, 0) \in C$. Then there exist points $P_1, P_2, P_3 \in S' \setminus C'$ such that each pair of complex lines defined by $q_{P_j} = 0$, where $j = 1, 2, 3$, meets $C'$ transversely at the two points $(\pm i, 1/\sqrt{k}, 0)$. Let $x, y, z, w$ be the coordinates of the weighted projective space $\mathbb{P}(1, 1, 1, 3)$, and let $D$ be the normalization of the curve in $\mathbb{P}(1, 1, 1, 3)$ defined by $w^2 - q_{P_1}q_{P_2}q_{P_3} = f = 0$. Let $\pi' : D \to C'$ the projection onto the first three
coordinates, and let \( \pi := \phi^{-1} \circ \pi' \). Then \( \pi(D(\mathbb{Q})) \) contains \( C \cap S \). Furthermore, \( \pi \) induces a double cover \( D \to \tilde{C} \) of the normalization \( \tilde{C} \) of \( C \) that has simple ramification over eight points. It follows from Riemann-Hurwitz that \( \mathcal{D} \) has genus \( 2g + 3 \).

(3) Suppose \( d \geq 4 \) and \( \#(S \cap C) \geq 2d(d - 1) \). Then there exists a point \( P \in S' \cap C' \) such that the pair of complex lines defined by \( q_P = 0 \) meet \( C' \) transversely away from \( P \). Let \( x, y, z, w \) be the coordinates of \( \mathbb{P}^3 \). Let \( \mathcal{D} \) be the normalization of the curve in \( \mathbb{P}^3 \) defined by \( w^2 - q_P = f = 0 \). Let \( \pi' : \mathcal{D} \to C' \) to be the projection onto the first three coordinates, and let \( \pi := \phi^{-1} \circ \pi' \). Then \( \pi(D(\mathbb{Q})) \) contains \( C \cap S \). Furthermore, \( \pi \) induces a double cover \( \mathcal{D} \to \tilde{C} \) of the normalization \( \tilde{C} \) of \( C \) that has simple ramification over \( 2(d - 1) \) points. It follows from Riemann-Hurwitz that \( \mathcal{D} \) has genus \( 2g - 2 + d \). \( \square \)

In the following proposition, we show that Lang’s Conjecture implies a stronger version of the results of [SdZ10] mentioned in the introduction. We say that an algebraic curve \( C \subset \mathbb{P}^2 \) is a circle if it has degree two and the intersection \( C \cap \mathbb{R}^2 \) is a circle in the usual sense.

**Proposition 3.3.** Assume Lang’s Conjecture. Let \( S \subset \mathbb{R}^2 \) be a rational distance set and let \( C \subset \mathbb{P}^2 \) be an irreducible projective curve of degree \( d \). Assume

1. \( C \) is neither a line nor a circle; or
2. \( C \) is a line and \( S \) contains at least five points not in \( C \); or
3. \( C \) is a circle and \( S \) contains at least four points not in \( C \);

then the cardinality of \( C \cap S \) is bounded by a constant that depends only on \( d \).

**Proof.** Since the number of singular points of \( C \) is bounded by its arithmetic genus \( \frac{1}{2}(d - 1)(d - 2) \), the number of points of \( S \) which are singular points of \( C \) is bounded uniformly in terms of \( d \). We may therefore assume that \( C \cap S \) is contained in the smooth locus of \( C \). We may further assume that \( \#(C \cap S) \geq \max\{4d(2d - 1) + 1, 10\} \), since otherwise there is nothing to prove. Then the curve \( C \) contains at least \( \frac{1}{2}d(d + 3) \) real points (namely, the points of \( C \cap \mathbb{R}^2 \)), so is defined over \( \mathbb{R} \). If \( C \) and \( S \) already satisfy the hypotheses of Lemma 3.2, then the proposition follows by applying the theorem of Caporaso-Harris-Mazur (Theorem 2.3) to the curve \( \mathcal{D} \) obtained from Lemma 3.2. Suppose that they do not.

Let \( P \in C \cap S \) be a point and let \( \phi : \mathbb{R}^2 \setminus \{P\} \to \mathbb{R}^2 \) be the inversion with respect to the unit circle centered at \( P \). Let \( S' := \phi(S \setminus \{P\}) \cup \{(0, 0)\} \); then \( S' \) is a rational distance set. Let \( \Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \) be the unique birational involution that agrees with \( \phi \) on the Zariski-dense subset \( \mathbb{R}^2 \setminus \{P\} \subset \mathbb{P}^2 \). For example, if \( P = (0, 0) \), then \( \Phi \) is given by \( \Phi(x, y, z) = (xz, yz, x^2 + y^2) \). If \( C' \subset \mathbb{P}^2 \) denotes the image of \( C \) under \( \Phi \), then the pair \((C', S')\) satisfies the hypotheses of Lemma 3.2. Indeed:

- The degree \( d' \) of the curve \( C' \) is at most \( 2d \), so
  \[
  \#(C' \cap S') \geq \#(C \cap S) - 1 \geq \max\{2d'(d' - 1), 9\}.
  \]
- If \( d = 2 \) and \((\pm i, 1, 0) \in C \) (so that \( C \) is a circle), then \( C' \) is a line and \( S' \) contains at least five points not in \( C' \).
- If \((\pm i, 1, 0) \not\in C \), then \( C' \) is of larger degree than \( C \) and contains the points \((\pm i, 1, 0)\).

Applying Lemma 3.2 to \((C', S')\) and then Theorem 2.3 to the curve \( \mathcal{D} \) obtained from Lemma 3.2 gives desired upper bound for \( \#(C \cap S) \) in terms of \( d \). \( \square \)

The following result strengthens the theorem of Shaffaf [Sha18] and Tao [Tao14] that rational distance sets are contained in real algebraic curves.
Proposition 3.4. Assume Lang’s Conjecture. There exists a positive integer $d$ such that every rational distance set is contained in a Zariski-closed proper subset $Z$ of $\mathbb{P}_C^2$. Moreover, the sum of the degrees of the components of $Z$ is at most $d$.

Proof. Let $S \subset \mathbb{R}^2$ be a rational distance set. By Lemma 3.1, we may assume that $S \subseteq \{(a, b \sqrt{k}) \in \mathbb{R}^2 \mid a, b \in \mathbb{Q}\}$ for some positive integer $k$, and we may also assume that $\# S \geq 4$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear automorphism that sends $(x, y) \mapsto (x, y/\sqrt{k})$. Let $S' := \phi(S)$. Let $(a_1, b_1), \ldots, (a_4, b_4) \in S'$ be four distinct points, let $x, y, z, r_1, r_2, r_3, r_4$ be the coordinates of $\mathbb{P}_C^6$, and let $V \subseteq \mathbb{P}^6$ be the intersection of the four quadrics with equations $r_j^2 = (x - a_j z)^2 + k(y - b_j z)^2$, where $j = 1, 2, 3, 4$.

If $\pi : V \to \mathbb{P}_C^2$ is the projection onto the first three coordinates, then $\pi(V(\mathbb{Q}))$ contains $S'$. Indeed, if $(u, v) \in S'$ then, for each $j = 1, 2, 3, 4$, the two points $(u, v \sqrt{k})$ and $(a_j, b_j \sqrt{k})$ are both in $S$, so the distance $s_j := \sqrt{(u - a_j)^2 + k(v - b_j)^2}$ is a rational number. Therefore, we conclude that $(u, v, 1, s_1, s_2, s_3, s_4) \in V(\mathbb{Q})$.

We will see in Proposition 5.1 below that the surface $V$ is of general type, and so by Lang’s Conjecture the Zariski-closure $\overline{V(\mathbb{Q})}$ of the set of rational points is a proper subset of $V$. By Hassett’s result (Theorem 2.4) applied to the family of all complete intersections of four quadrics in $\mathbb{P}^6$, there exists a constant $d$, independent of $S$ and the choice of the four points $(a_i, b_i)$, such that the sum of the degrees of the irreducible components of $\overline{V(\mathbb{Q})}$ is at most $d$.

The morphism $\pi : V \to \mathbb{P}_C^2$ is finite, so $\pi(\overline{V(\mathbb{Q})})$ is a Zariski-closed proper subset $Z \subset \mathbb{P}_C^2$, the sum of the degrees of the irreducible components of $Z$ is at most $d$, and $Z$ contains $S'$.

We now combine Propositions 3.3 and 3.4 to establish the main result of this paper.

Proof of Theorem 1.1. Let $S$ be a rational distance set. By Proposition 3.4, there exists a integer $d$, independent of $S$, and a Zariski-closed proper subset $Z \subset \mathbb{P}_C^2$ that contains $S$ and such that the sum of the degrees of the irreducible components of $Z$ is at most $d$. Thus $Z$ has at most $d$ irreducible components, each of which has degree at most $d$. The components of dimension zero each contain at most one point of $S$. By Proposition 3.3 and the assumption that the set $S$ is in general position, there exists a constant depending only on $d$, and hence independent of $S$, that bounds the number of points of $S$ contained in each of the one-dimensional irreducible components of $Z$. \hfill \Box

4. A CRITERION FOR GENERAL TYPE

The main result of this section is Proposition 4.4, which gives a criterion for varieties with non-canonical singularities to be of general type. We begin with a definition.

Definition 4.1. Let $X$ be a normal variety whose canonical divisor $K_X$ is $\mathbb{Q}$-Cartier. Let $f : \tilde{X} \to X$ be a resolution of singularities with irreducible exceptional divisors $E_1, \ldots, E_r$, and let $a_1, \ldots, a_r \in \mathbb{Q}$ be the corresponding discrepancies (see [KM98, Definition 2.22]), so that $K_{\tilde{X}} = f^*K_X + \sum_j a_j E_j$ in $\text{Pic}(\tilde{X}) \otimes \mathbb{Q}$. We say that $X$ has canonical singularities if $a_j \geq 0$ for all $j = 1, \ldots, r$.

We note that the above definition is independent of the choice of resolution. One can check whether a variety with canonical singularities is of general type without passing to a resolution. This observation is formalized by the following lemma.

Lemma 4.2. Let $X$ be a normal projective variety over $\mathbb{C}$. Suppose that the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier. If $X$ is of general type, then the canonical divisor $K_X$ is big. The converse holds if $X$ has canonical singularities.
Proof. Let \( \omega_X := \mathcal{O}_X(K_X) \) be the canonical sheaf of \( X \). Thus a section of \( \omega_X \) is a section of the top exterior power of the cotangent bundle of the smooth locus of \( X \). Let \( \rho : \tilde{X} \to X \) be a resolution of singularities. For each integer \( l > 0 \), there is a natural inclusion \( \tau : \mathcal{O} \omega_X^\otimes l \to \omega_X^{\otimes l} := (\omega_X^\otimes)^{\vee \vee} \) given by restriction of pluricanonical forms to the complement of the exceptional divisor of \( \rho \). Furthermore, if \( X \) has canonical singularities, then \( \tau \) is an isomorphism for each \( l \) such that \( \omega_X^{\otimes l} \) is locally free. Indeed, by definition of canonical singularities, pullbacks of sections of \( \omega_X^{\otimes l} \) do not have poles when regarded as rational sections of \( \omega_X^{\otimes l} \); pullback of sections defines the inverse of \( \tau \).

Proposition 4.4 partially generalizes the converse in the above lemma ("partially" because in this proposition we assume \( K_X \) is ample). Before stating it, we recall the definitions of Hilbert-Samuel multiplicity and ordinary multiple point.

**Definition 4.3.** Let \( X \) be an algebraic scheme over \( \mathbb{C} \). Let \( x \in X \) be a closed point.

1. There exists a unique polynomial \( H \in \mathbb{Q}[l] \) such that \( H(p) = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/m_x^p \) for all sufficiently large \( p \). The degree of \( H \) is equal to the Krull dimension \( d \) of \( \mathcal{O}_{X,x} \). The Hilbert-Samuel multiplicity of the local ring \( \mathcal{O}_{X,x} \) is equal to \( d! \) times the leading coefficient of \( H \). It is a positive integer that we denote by \( e(\mathcal{O}_{X,x}) \).

2. Let \( \pi : B \to X \) be the blowup of \( X \) at the ideal sheaf of \( X \). Let \( E \subseteq B \) be the exceptional divisor of \( \pi \). If \( E \) is smooth, we say that \( x \in X \) is an ordinary multiple point.

In terms of the canonical isomorphism \( E = \text{Proj}(\oplus_{l \geq 0} \mathcal{O}_E^l / \mathcal{O}_E^{l+1}) \), we have \( \mathcal{O}_E(1) = \mathcal{O}_B(-E)|_E \). By asymptotic Riemann-Roch, the Hilbert-Samuel multiplicity \( e(\mathcal{O}_{X,x}) \) is equal to the degree of \( E \) with respect to this ample invertible sheaf. If \( X \) has an ordinary multiple point at \( x \), then the blowup \( B \) is smooth in a neighborhood of \( E \), and so ordinary multiple points are isolated singularities.

The following criterion extends Lemma 4.2 when the set of non-canonical singularities of the variety consists of ordinary multiple points.

**Proposition 4.4.** Let \( X \) be a normal projective variety of dimension \( d \) over \( \mathbb{C} \) with \( \mathbb{Q} \)-Cartier and ample canonical divisor \( K_X \). Suppose that the set \( \Sigma \) of non-canonical singularities of \( X \) consists of ordinary multiple points. For each \( x \in \Sigma \), let \( E_x \) denote the exceptional divisor of the blowup of \( X \) at \( x \), and let \( a(E_x, X) \) denote the discrepancy of \( E_x \) with respect to \( X \). Suppose that

\[
\sum_{x \in \Sigma} |a(E_x, X)|^d \cdot e(\mathcal{O}_{X,x}) < K_X^d.
\]

Then \( X \) is of general type.

**Proof.** Let \( \pi : B \to X \) be the blowup of \( X \) at the points of \( \Sigma \). By definition of an ordinary multiple point, for each \( x \in \Sigma \), the exceptional divisor \( E_x \) is smooth, and hence \( B \) is smooth in a neighborhood of \( E_x \). Thus \( B \) has canonical singularities. By Lemma 4.2 it suffices to show that \( \omega_B \) is big.

For each non-canonical singularity \( x \in \Sigma \), write \( a_x := a(E_x, X) \in \mathbb{Q}_{<0} \). Thus \( K_B = \pi^* K_X + \sum_{x \in \Sigma} a_x E_x \in \text{Pic}(X) \otimes \mathbb{Q} \). In fact, let \( l_0 \) be the smallest positive integer \( l \) such that \( \omega_X^{\otimes l} := (\omega_X^\otimes)^{\vee \vee} \) is locally free. Then \( l_0 a_x \) is an integer for each \( x \in \Sigma \), and \( \omega_B^{\otimes l}(-l \sum_{x \in \Sigma} a_x E_x) = \pi^* \omega_X^{\otimes l} \) as subsheaves of \( \omega_B^{\otimes l} \otimes k(B) \) for all positive integers \( l \) divisible by \( l_0 \).

For each \( x \in \Sigma \), let \( m_x \subseteq \mathcal{O}_X \) denote the ideal sheaf of \( x \) in \( X \). Let \( n := \prod_{x \in \Sigma} m_x^{|a_x|} \). Then the image of the natural map \( \pi_* \omega_B^{\otimes l} \to \omega_X^{\otimes l} \) contains \( n^l \omega_X^{\otimes l} \) for all positive integers \( l \) divisible by \( l_0 \). Thus
\[ \Gamma(B, \omega_B^{\otimes l}) \] contains the kernel of the natural map

\[ \Gamma(X, \omega_X^{[l]}) \to \Gamma(X, \omega_X^{[l]} \otimes \mathcal{O}_X/n^l) \cong \mathcal{O}_X/n^l = \prod_{x \in \Sigma} \mathcal{O}_{X,x}/m_{x}^{[a_x]} \]

for all \( l \) divisible by \( l_0 \). The dimensions of the source and the target of \( \phi \) both grow as polynomials of degree \( d = \dim X \) in \( l \). By the asymptotic Riemann-Roch theorem, the leading coefficient of the source polynomial is \( \frac{1}{d!} K_X^d \). By the definition of the Hilbert-Samuel multiplicity, the target polynomial has leading coefficient \( \frac{1}{d!} \sum_{x \in S} |a_x|^d \cdot e(\mathcal{O}_{X,x}) \). The result follows. \( \square \)

5. Singular surfaces of general type

In this section we apply Proposition 4.4 to prove Proposition 5.1 below, which states that certain surfaces obtained as complete intersections of quadrics are of general type. Proposition 5.1 was used in the proof of the main result of this paper (Theorem 1.1). It generalizes a result proved by Tao [Tao14] (see also [Kov18]), who considered the case of four quadrics. Proposition 5.1 proves that the set of points in the plane with rational distances to any given rational distance set with (at least) four elements, lifts to the set of rational points of a variety of general type; see the proof of Proposition 3.4.

We fix the following setup. Let \( (a_1, b_1), \ldots, (a_m, b_m) \) be distinct points of \( \mathbb{R}^2 \). Let \( x, y, z, r_1, \ldots, r_m \) be the coordinates of \( \mathbb{P}^{2+m}_\mathbb{C} \). Let \( V \) be the intersection in \( \mathbb{P}^{2+m}_\mathbb{C} \) of the quadrics defined by \( r_j^2 = (x - a_jz)^2 + (y - b_jz)^2 \) with \( j = 1, \ldots, m \). Finally, let \( \pi : V \to \mathbb{P}^{2}_\mathbb{C} \) be the projection onto the first three coordinates.

Being a complete intersection of positive dimension, the surface \( V \) is connected, Cohen-Macaulay and Gorenstein, that is, has invertible canonical sheaf. Moreover, \( V \) has isolated singular points, by Lemma 5.3 below, hence is normal and irreducible.

Proposition 5.1. The surface \( V \) is of general type if \( m \geq 4 \).

To study the singularities of \( V \), it is convenient to work in the following setting. Let \( X \) be a smooth surface over \( \mathbb{C} \). Let \( f_1, \ldots, f_m \in \Gamma(X, \mathcal{O}_X) \) be nonzero sections. For each \( j = 1, \ldots, m \), let \( C_j \subseteq X \) be curve defined by \( f_j = 0 \). Let \( y_1, \ldots, y_m \) be the coordinates of the affine space \( \mathbb{A}^m_\mathbb{C} \). Let \( Z \subseteq X \times \mathbb{A}^m_\mathbb{C} \) be the subscheme defined by the equations \( y_j^2 = f_j \) with \( j = 1, \ldots, m \), and let \( \rho : Z \to X \) be the first projection. Note that \( \rho \) is finite and flat with fibers of length \( 2^m \), and is étale away from \( \rho^{-1}(C_1 \cup \cdots \cup C_m) \). Thus \( Z \) is smooth away from \( \rho^{-1}(C_1 \cup \cdots \cup C_m) \).

The following lemma relates the geometry of the curves \( C_1, \ldots, C_m \subseteq X \) with the singularities of the covering surface \( Z \).

Lemma 5.2. Let \( q \in Z \) be a closed point. Let \( p = \rho(q) \). Suppose that \( p \in C_1 \cap \cdots \cap C_r \), but \( p \not\in C_{r+1} \cup \cdots \cup C_m \).

1. Suppose that the curves \( C_1, \ldots, C_r \) are smooth at \( p \) and meet pairwise transversely at \( p \).
   (a) The surface \( Z \) is smooth at \( q \) if and only if \( r \leq 2 \).
   (b) If \( r \geq 3 \), then \( Z \) has an ordinary multiple point at \( q \). The exceptional divisor of the blowup of \( Z \) at \( q \) is a smooth complete intersection of \( r - 2 \) quadrics in \( \mathbb{P}^{2+r-1}_\mathbb{C} \).

2. Suppose that each of the curves \( C_1, \ldots, C_r \subseteq X \) has a node (i.e., an ordinary double point) at \( p \) and no two share a branch at \( p \). Then \( Z \) has an ordinary multiple point at \( q \). The exceptional divisor of the blowup of \( Z \) at \( q \) is a smooth complete intersection of \( r \) quadrics in \( \mathbb{P}^{1+r}_\mathbb{C} \).
Proof. Replacing $X$ with its étale cover $X' \subseteq X \times \mathbb{A}^{m-r}_C$ defined by the equations $y^2_j = f_j$ with $j = r + 1, \ldots, m$, we may assume that $m = r$. Then $Z$ is smooth at $w$ if and only if the differential form on $X \times \mathbb{A}^r_C$

$$(-2y_1dy_1 + df_1) \wedge \cdots \wedge (-2y_r dy_r + df_r)$$

is nonzero at $q$. Noting that $y_1 = \cdots = y_r = 0$ at $q$, part (1a) follows.

To prove (2), suppose that each of the curves $C_1, \ldots, C_r \subseteq X$ has a node at $p$ and that no two share a branch at $p$. After shrinking $X$ and replacing it by an étale cover (or, alternatively, after replacing $O_{X,p}$ by its completion), we may assume that there exists a regular system of parameters $u, v \in O_{X,p}$ such that

$$f_j = (u + s_j v)(u + t_j v) + \epsilon_j$$

for all $j = 1, \ldots, r$, where $s_1, t_1, \ldots, s_r, t_r$ are distinct complex numbers and $\epsilon_1, \ldots, \epsilon_r \in \mathbb{C}$. It is then straightforward to compute that the exceptional divisor of the blowup of $Z$ at $q$ is the intersection of the quadrics $y^2_j = (u + s_j v)(u + t_j v)$, where $j = 1, \ldots, r$, in the projectivized tangent space $\mathbb{P}^{1+r}_C$ of $X \times \mathbb{A}^r_C$ at $q$. Viewing this complete intersection as cover of the projective line $\mathbb{P}^1_C$ with coordinates $u, v$, it is easy to show that it is smooth. Thus (2) holds.

Finally, to prove (1b), suppose that the curves $C_1, \ldots, C_r$ are smooth and meet pairwise transversely at $p$, and $m \geq 3$. We may factor the map $\rho : Z \rightarrow X$ via the subscheme $X' \subseteq X \times \mathbb{A}^r_C$ defined by the equations $y^2_1 = f_1$ and $y^2_2 = f_2$. Let $\tilde{\rho} : X' \rightarrow X$ be the first projection, and let $p' \in X'$ be the unique point lying over $p \in X$. It is easy to show that the inverse images $\tilde{\rho}^{-1}C_3, \ldots, \tilde{\rho}^{-1}C_r \subseteq X'$ have nodes at $p'$. No two of them share a branch at $p'$, since both branches of $\tilde{\rho}^{-1}C_j$ map to the unique branch of $C_j$, for each $j = 3, \ldots, r$. Thus (1b) follows from (2).

We use the preceding lemma to describe the singularities of the surface $V$.

Lemma 5.3. The singularities of $V$ consist of $m2^{m-1} + 2$ ordinary multiple points. They are the $m2^{m-1}$ points of $V$ lying over the points $(a_1, b_1, 1), \ldots, (a_m, b_m, 1) \in \mathbb{P}^{2}_C$, and the two points of $V$ lying over $(1, \pm i, 0) \in \mathbb{P}^{2}_C$. The former singularities are ordinary double points. The exceptional divisors of the blowup of $V$ at the two singularities at infinity are smooth complete intersections of $m - 2$ quadrics in $\mathbb{P}^{2}_C$.

Proof. For each $j = 1, \ldots, m$, let $C_j \subset \mathbb{P}^{2}_C$ be the conic defined by the vanishing of $f_j := (x - a_j z)^2 + (y - b_j z)^2$. Writing

$$f_j = ((x - a_j z) + i(y - b_j z))(x - a_j z) - i(y - b_j z),$$

we see that $C_j$ is a pair of lines in meeting transversely at $(a_j, b_j, 1) \in \mathbb{P}^{2}_C$ for each $j = 1, \ldots, j$, and that $C_2, \ldots, C_m$ are obtained from $C_1$ by translations in the affine plane $\{z = 1\} \subset \mathbb{P}^{2}_C$. It follows that any two of the degenerate conics $C_j$ meet transversely at two points in the affine plane $\{z = 1\}$ and at the two points $(1, \pm i, 0)$. Furthermore, no three of them meet at a point in the affine plane $\{z = 1\}$. Hence the result follows from Lemma 5.2 with $\rho : Z \rightarrow X$ ranging over the restrictions of the projection $\pi : V \rightarrow \mathbb{P}^{2}_C$ to the open sets of the standard affine open cover of $\mathbb{P}^{2}_C$.

Proof of Proposition 5.1. The surface $V$ is a complete intersection of $m$ quadrics in $\mathbb{P}^{2m}_C$. By the adjunction formula, its canonical sheaf is

$$\omega_V = \omega_{\mathbb{P}^{2m}_C}(2m)|_V = O_V(m - 3),$$

which is ample if and only if $m \geq 4$. Thus

$$K^2_V = (m - 3)^2 \deg V = (m - 3)^2 2^m.$$
Let $\phi : B \to V$ be the blowup of $V$ at its singular points. Then $B$ is smooth by Lemma 5.3. Let $p \in V$ be a singular point. Let $E$ be the exceptional divisor of $\pi$ that lies over $p$. Let $a := a(E,V)$ be the discrepancy of $K_V$ at $E$. Then $K_B = \phi^* K_X + aE + F$ in $\text{Pic}(B) \otimes \mathbb{Q}$, where $F$ is a divisor supported on $B \setminus E$. Combining this equality with the adjunction formula $K_E = (K_B + E)|_E$, we obtain that $K_E = (a + 1)E$.

In the case where $\pi(p) \in \mathbb{P}^2_C$ is one of the points $(a_1, b_1, 1), \ldots, (a_m, b_m, 1)$, the singularity of $V$ at $p$ is an ordinary double point and hence canonical. Explicitly, the exceptional divisor $E$ is a smooth conic in $\mathbb{P}^2_C$. Hence $\omega_E = \mathcal{O}_E(-1) = \mathcal{O}_B(E)|_E$, which shows that $a = 0$.

In the case where $\pi(p) = (1, \pm i, 0)$, the exceptional divisor $E$ is a smooth complete intersection of $m - 2$ quadrics in $\mathbb{P}^m_C$. Its degree $2^{m-2}$ is equal to the Hilbert-Samuel multiplicity $e(\mathcal{O}_V(p))$. The canonical sheaf of $E$ is

$$\omega_E = \mathcal{O}_E(-m + 2(2 - m) - 2) = \mathcal{O}_B((4 - m)E)|_E.$$ 

Thus $a = 3 - m < 0$, so that $p$ is a non-canonical singularity in this case.

The result now follows from Proposition 4.4, which applies as

$$2|3 - m|^2 \cdot 2^{m-2} < (m - 3)^2 2^m$$

for all $m \geq 4$. 

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