VANISHING OF GROMOV-WITTEN INVARIANTS OF
PRODUCT OF $\mathbb{P}^1$

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ABSTRACT. We prove some vanishing conditions on the Gromov-
Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$.

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0. Introduction

0.1. Gromov-Witten invariants. Denote by $\mathbb{P}[n]$ the product of $n$
projective lines:

$$\mathbb{P}[n] := \mathbb{P}^1 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1.$$ 

Denote by

$$\overline{M}_{g,n}(\mathbb{P}[n], d)$$

the moduli space of stable maps to $\mathbb{P}[n]$ with degree $d = (d_1, d_2, \ldots, d_n)$. It has canonical virtual fundamental class $[\overline{M}_{g,n}(\mathbb{P}[n], d)]^{vir}$ of dimension $(1 - g)(n - 3) + m + 2 \sum_i d_i$. Let $\pi$ be the morphism to the moduli space of stable curves determined by the domain,

$$\pi : \overline{M}_{g,n}(\mathbb{P}[n], d) \to \overline{M}_{g,n}.$$ 

For $c_k = (c_{1,k}, c_{2,k}, \ldots, c_{n,k}) \in \mathbb{Z}_2^n$, Gromov-Witten invariants are defined by
GW_{g,m,d}^{\mathbb{P}[n]}(\tau_{k_1}c_1, \ldots, \tau_{k_m}c_m) := \int_{[\mathcal{M}_{g,m}(\mathbb{P}[n],d)]^\text{vir}} \prod_{i=1}^m \pi^*(\psi_i)^{k_i} \text{ev}_i^*(\otimes_{j=1}^n H_j^{c_{j,i}}).

Gromov-Witten invariants have been studied more than 20 years, see \cite{6,7} for an introduction to the subject. In this paper, we prove the following vanishing conditions on Gromov-Witten invariants of \(\mathbb{P}[n]\).

**Theorem 1.** We have

\[ GW_{g,m,d}^{\mathbb{P}[n]}(\tau_{k_1}c_1, \ldots, \tau_{k_m}c_m) = 0, \]

if the following conditions hold.

(i) \((g - 1 + \sum_{k=1}^m c_{ik})\) is odd for \(1 \leq i \leq n\),

(ii) \(3g - 3 + m - \sum_{i=1}^m k_i < n\).

**0.2. Quasimap invariants and wall-crossing formula.** Let \(G := (\mathbb{C}^*)^n\) act on \((\mathbb{C}^2)^n\) by the standard diagonal action componentwisely on each component \(\mathbb{C}^2\) so that its associated GIT quotient

\( (\mathbb{C}^2)^n // G = \mathbb{P}[n]. \)

With this set-up, Ciocan-Fontanine and Kim defined the quasimap moduli space

\[ Q_{g,m}(\mathbb{P}[n], d) \]

with the canonical virtual fundamental class \([Q_{g,k}(\mathbb{P}[n], d)]^\text{vir}\). See \cite{1} \cite{5} \cite{15} for an introduction. For \(c_k = (c_{1,k}, c_{2,k}, \ldots, c_{n,k}) \in \mathbb{Z}_2^n\), we define quasimap invariants of \(\mathbb{P}[n]\) by

\[ Q_{g,m,d}^{\mathbb{P}[n]}(\tau_{k_1}c_1, \ldots, \tau_{k_m}c_m) := \int_{[Q_{g,m}(\mathbb{P}[n],d)]^\text{vir}} \prod_{i=1}^m \pi^*(\psi_i)^{k_i} \text{ev}_i^*(\otimes_{j=1}^n H_j^{c_{j,i}}). \]

In \cite{3}, the authors studied the relationship between quasimap invariants and Gromov-Witten invariants. By applying the general theorem to \(\mathbb{P}[n]\), we have the following proposition.

**Proposition 2.** (\cite{3}) For \(c_k = (c_{1,k}, c_{2,k}, \ldots, c_{n,k}) \in \mathbb{Z}_2^n\), we have

\[ GW_{g,m,d}^{\mathbb{P}[n]}(\tau_{k_1}c_1, \ldots, \tau_{k_m}c_m) = Q_{g,m,d}^{\mathbb{P}[n]}(\tau_{k_1}c_1, \ldots, \tau_{k_m}c_m) \]

i.e. quasimap invariants and Gromov-Witten invariants are same for \(\mathbb{P}[n]\). In section 3, we prove the following theorem which is equivalent to Theorem \cite{1} by Proposition \cite{2}.
Theorem 3. We have
\[ Q_{g,n,d}[\tau_1 c_1, \ldots, \tau_m c_m] = 0, \]
if the following conditions hold.
(i) \((g - 1 + \sum_{k=1}^n c_{ik})\) is odd for \(1 \leq i \leq n\),
(ii) \(3g - 3 + m - \sum_{i=1}^m k_i < n\).

To prove Theorem 3, we apply the localization strategy introduced by Givental [9, 10, 13] for Gromov-Witten theory to the quasimap theory of \(\mathbb{P}[n]\).

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1. Preliminaries

1.1. \(T\)-equivariant theory. Let \(T := (\mathbb{C}^*)^2\) act on \(\mathbb{P}[n]\) standardly on each component. Denote by
\[ \lambda = (\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \ldots, \lambda_n, \bar{\lambda}_n) \]
\(T\)-equivariant parameters. We will use following specializations throughout the paper:
\[ \lambda_k + \bar{\lambda}_k = 0, \text{ for } 1 \leq k \leq n. \]

First we set the notation for the cohomology basis and its dual basis. Let \(\{p_i\}\) be the set of \(T\)-fixed points of \(\mathbb{P}[n]\) and let \(\phi_i\) be the basis of \(H_T^*(\mathbb{P}[n])\) defined by satisfying followings:
\[ \phi_i|_{p_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \]

Let \(\phi^i\) be the dual basis with respect to the \(T\)-equivariant Poincaré pairing, i.e.,
\[ \int_Y \phi_i \phi^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \]
For each \(T\)-fixed point \(p_i \in \mathbb{P}[n]\), Let
\[ e_i = e(T_{p_i}(\mathbb{P}[n])) \]
be the equivariant Euler class of the tangent space of \(\mathbb{P}[n]\) at \(p_i\).

The action of \(T\) on \(\mathbb{P}[n]\) lifts to \(Q_{g,0}(\mathbb{P}[n], d)\). The localization formula of [11] applied to the virtual fundamental class \(\left[Q_{g,0}(\mathbb{P}[n], d)\right]^{vir}\) will play a fundamental role in our paper. The \(T\)-fixed loci are represented in
terms of dual graphs, and the contributions of the $T$-fixed loci are given by tautological classes. The formulas are standard, see [12, 15].

1.2. genus 0 invariants. In this section, we review the genus zero theory. Integrating along the virtual fundamental class

$$[Q_{0,k}(\mathbb{P}[n],d)]^{\text{vir}}$$

we define correlators $\langle \ldots \rangle_{0,k,d}^{0+}$ as follows. For $\gamma_i \in H^*_T(\mathbb{P}[n]) \otimes \mathbb{Q}(\lambda)$,

$$\langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_1^{a_k} \rangle_{0,k,d}^{0+} = \int_{[Q_{0,k}(\mathbb{P}[n],d)]^{\text{vir}}} \prod_i ev_i^*(\gamma_i) \psi_1^{a_i},$$

where $\psi_i$ is the psi-class associated to the $i$-th marking and $ev_i$ is the $i$-th evaluation map.

Let $Q_{g,k}(\mathbb{P}[n],d)^{T,p_i}$ be the fixed loci of $Q_{g,k}(\mathbb{P}[n],d)$ whose elements have domain components only over $p_i$. Integrating along the localized cycle class

$$[Q_{0,k}(d)^{T,p_i}]^{\text{vir}}$$

we also define local correlators $\langle \ldots \rangle_{0,k,\beta}^{0+}$ and $\langle\langle \ldots \rangle\rangle_{0,k,\beta}^{0+}$ as follows:

$$\langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_1^{a_k} \rangle_{0,k,d}^{0+} = \int_{[Q_{0,k}(\mathbb{P}[n],d)^{T,p_i}]^{\text{vir}}} \prod_i ev_i^*(\gamma_i) \psi_1^{a_i};$$

$$\langle\langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_1^{a_k} \rangle\rangle_{0,k,d}^{0+} = \sum q \prod_{i=1}^d \langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_1^{a_k}, t, \ldots, t \rangle_{0,k+m,\beta}^{0+},$$

where $q = (q_1, \ldots, q_n)$ are formal Novikov variables and $t = \sum_i t_i H_i \in H^*_T(\mathbb{P}[n])$ where $H_i$ is the pull back of hyperplane class in $i$-th $\mathbb{P}^1$.

1.3. insertions of 0+ weighted marking. For the explicit calculations of various genus 0 invariants, we need to introduce the notion of 0+ weighted marking. We briefly recall the definitions from [4].

Denote by

$$Q_{g,k|m}(\mathbb{P}[n],d)$$
the moduli space of genus \(g\) (resp. genus zero), degree \(d\) stable quasimaps to \(\mathbb{P}[n]\) with ordinary \(k\) pointed markings and infinitesimally weighted \(m\) pointed markings, see [4] for more explanations.

Denote by

\[Q^{0+,0+}_{g,k|m}(\mathbb{P}[n], d)^T_{p_i}\]

the \(T\)-fixed part of \(Q^{0+,0+}_{g,k|m}(\mathbb{P}[n], d)\), whose domain components of universal curves are only over \(p_i\).

Let \(\tilde{H}_i \in H^*([((\mathbb{C}^2)^n/G)]\) be a lift of \(H_i \in H^*(\mathbb{P}[n])\), i.e. \(\tilde{H}_i|_{\mathbb{P}[n]} = H_i\).

For \(\gamma_i \in H^*_T(\mathbb{P}[n]) \otimes \mathbb{Q}(\lambda), t = \sum_i t_i \tilde{H}_i, \delta_j \in H^*_T([((\mathbb{C}^2)^n/G), \mathbb{Q}),\) denote

\[
\langle \gamma_1 \psi^{a_1}, ..., \gamma_k \psi^{a_k}; \delta_1, ..., \delta_m \rangle^{0+,0+}_{0,k|m,d}
= \int_{[Q^{0+,0+}_{0,k|m}(\mathbb{P}[n], d)^T_{p_i}]^{vir}} \prod_i \hat{ev}_i^* (\gamma_i) \psi^{a_i}_i \prod_j \hat{ev}_j^* (\delta_j) ;
\]

\[
\langle \gamma_1 \psi^{a_1}, ..., \gamma_k \psi^{a_k} \rangle^{0+,0+}_{0,k}
= \sum_{m,d} \frac{q^d}{m!} \langle \gamma_1 \psi^{a_1}, ..., \gamma_k \psi^{a_k}; t, ..., t \rangle^{0+,0+}_{0,k|m,d} ;
\]

\[
\langle \gamma_1 \psi^{a_1}, ..., \gamma_k \psi^{a_k}; \delta_1, ..., \delta_m \rangle^{0+,0+,p_i}_{0,k|m,d}
= \int_{[Q^{0+,0+}_{0,k|m}(\mathbb{P}[n], d)^T_{p_i}]^{vir}} \prod_i \hat{ev}_i^* (\gamma_i) \psi^{a_i}_i \prod_j \hat{ev}_j^* (\delta_j) ;
\]

\[
\langle \gamma_1 \psi^{a_1}, ..., \gamma_k \psi^{a_k} \rangle^{0+,0+,p_i}_{0,k}
= \sum_{m,d} \frac{q^d}{m!} \langle \gamma_1 \psi^{a_1}, ..., \gamma_k \psi^{a_k}; t, ..., t \rangle^{0+,0+,p_i}_{0,k|m,d} ,
\]

where \(\hat{ev}_j\) is the evaluation map to \([((\mathbb{C}^2)^n/G)]\) at the \(j\)-th infinitesimally weighted marking.

Let \(\{\phi_i\}\) be the equivariant basis of \(H^*_T(\mathbb{P}[n])\) satisfying [1]. Let us define \(U_i, S, V_{ij}\) by
\[ U_{p_i}^{[n]} := \langle \langle 1, 1 \rangle \rangle_{0,2}^{0+} \in \mathbb{Q}[q, t](\lambda) ; \]
\[ S^{P[n]}(\gamma) := \sum_{i,\beta} \phi^i q^d \langle \langle \phi_i, \psi \rangle \rangle_{0,2}^{0+} \in H^*_T(\mathbb{P}[n])[\lbrack z^{-1}, q, t \rbrack] \otimes \mathbb{Q}(\lambda) ; \]
\[ V_{ij}^{P[n]}(x, y) := \sum_d q^d \langle \langle \phi_i, \phi_j \rangle \rangle_{0,2}^{0+} \in \mathbb{Q}[\langle x^{-1}, y^{-1}, q, t \rangle](\lambda) ; \]

We recall the following WDVV equation from [3],

\[ e_i V_{ij}(x, y) e_j = \frac{\sum_k S(\phi_k)|_{p_i, z = x} S(\phi_k)|_{p_j, z = y}}{x + y} \]

(2) 

To study the properties of \( S \) and \( V_{ij} \), we need to recall infinitesimal \( I \)-function defined in [4]. Denote by \( I \) the infinitesimal \( I \)-function \( J^{0+, 0^+} \) in [4]. Note that \( S \) coincide with the definition of infinitesimal \( S \)-operator \( S^{0+, 0^+} \) in [4]. From (5.1.3) in [4], we know explicit form of \( I \) of \( P[n] \)

\[ (3) \quad I^{P[n]} = \prod_{i=1}^{n} e^{t_i(H_i + d_i z) / z} \frac{q_i^{d_i}}{\prod_{k=1}^{d_i} (H_i - \lambda_i + k z)(H_i - \overline{\lambda}_i + k z)} \in \mathbb{R}[[z, z^{-1}, q, t]](\lambda). \]

To find the explicit forms of \( S \)-operators, we recall following proposition from [12].

**Proposition 4.** (Proposition 2.4 in [12]) There are unique coefficients \( a_i(z, q) \in \mathbb{Q}(\lambda)[z][\lbrack q \rbrack] \) making

\[ \sum_i a_i(z, q) \partial_{\phi_i} \mathbb{G}^{P[n]} = \gamma + O(1/z). \]

Furthermore LHS coincides with \( S^{P[n]}(\gamma) \).

1.4. **Ordered graphs.** Let the genus \( g \) and the number of markings \( n \) for the moduli space be in the stable range

\[ 2g - 2 + n > 0. \]

We can organize the \( T \)-fixed loci of \( Q_{g,n}(\mathbb{P}[n], d) \) according to decorated graphs. A **decorated graph** \( \Gamma \) consists of the data \( (V, E, N, g) \) where

(i) \( V \) is the vertex set,
(ii) $E$ is the edge set (including possible self-edge),
(iii) $N : \{1, 2, \ldots, n\} \to V$ is the marking assignment,
(iv) $g : V \to \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$g = \sum_{v \in V} g(v) + h^1(\Gamma).$$

The markings $L = \{1, \ldots, n\}$ are often called legs.

Let $G_{g,n}$ be the set of decorated graph as defined as above. The flags of $\Gamma$ are the half-edges\footnote{In our convention, markings are not flags.}. Let $F$ be the set of flags. For each $\Gamma \in G_{g,n}$, we choose an ordering

$$(4) \quad \nu_T : V \to \{1, 2, \ldots, |V|\}.$$ 

Let $G_{g,n}^{\text{ord}}$ be the set of decorated graph with fixed choice of orderings on vertices. We will sometimes identify $V$ with $\{1, 2, \ldots, |V|\}$ by \footnote{In our convention, markings are not flags.}

1.5. Universal ring. Denote by

$$R$$ 

the ring generated by

$$\lambda_{1,j}^{\pm 1}, \lambda_{2,j}^{\pm 1}, \ldots, \lambda_{n,j}^{\pm 1}, \ j \in \mathbb{N}$$

with relations

$$(5) \quad \lambda_{k,j}^2 = 1, \text{ for } 1 \leq k \leq n, \ j \in \mathbb{N}.$$ \footnote{In our convention, markings are not flags.}

- A monomial in $R$ is called canonical if degree of $\lambda_{i,j}$ are 1 or 0 for all $1 \leq i \leq n, \ j \in \mathbb{N}$.
- The length of canonical monomial is the number of $\lambda_{i,j}$ whose degree is 1.
- For a monomial $m \in R$, the length of $m$ is the length of unique canonical monomial equal to $m$ in $R$.

The ring $R$ will play a fundamental role in the proof of the main theorem.

**Definition 5.** For monomial $m \in R$, we say $m$ has type $(a; b) = (a_1, a_2, \ldots, a_n; b) \in \mathbb{N}^n \times \mathbb{N}$ if $m$ has following form;

$$m = \prod_j \prod_i \lambda_{i,j}^{a_{ij}} f$$

where $\sum_j a_{i,j} = a_i$ and $f$ is a monomial whose length is less than or equal to $b$. 
Definition 6. For a polynomial $f \in \mathbb{R}$, we say $f$ has type $(a; b)$ if $f$ is a sum of monomials of type $(a; b)$.

The following two lemmas are easy to check.

Lemma 7. If $m_1$ has type $(a_1; b_1)$ and $m_2$ has type $(a_2; b_2)$, then $m_1 m_2$ has type $(a_1 + a_2; b_1 + b_2)$.

Lemma 8. If $f \in \mathbb{R}$ has type $(a; b)$ with odd $a_i$ for $0 \leq i \leq n$ and $b < n$,

$$\sum_{\lambda_{i,j}=\pm 1} f = 0.$$ 

The notation $\sum_{\lambda_{i,j}=\pm 1} f$ in Lemma 8 denotes the sum of evaluations of $f \in \mathbb{R}$ with all possible choice of $\lambda_{i,j} = 1$ or $-1$.

1.6. Universal polynomial. We review here the definition of universal polynomial $P$ from [14]. The universal polynomial play the essential role in calculating higher genus invariants from genus 0 invariants, see [9, 13, 14] for more explanations.

Let $t_0, t_1, t_2, \ldots$ be formal variables. The series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \ldots$$

in the additional variable $c$ plays a basic role. The variable $c$ will later be replaced by the first Chern class $\psi_i$ of a cotangent line over $\overline{M}_{g,n}$,

$$T(\psi_i) = t_0 + t_1 \psi_i + t_2 \psi_i^2 + \ldots,$$

with the index $i$ depending on the position of the series $T$ in the correlator.

Let $2g - 2 + n > 0$. For $a_i \in \mathbb{Z}_{\geq 0}$ and $\gamma \in H^*(\overline{M}_{g,n})$, define the correlator

$$\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle\rangle_{g,n} = \sum_{k \geq 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{i=1}^k T(\psi_{n+i}).$$

We consider $\mathbb{C}(t_1)[t_2, t_3, \ldots]$ as $\mathbb{Z}$-graded ring over $\mathbb{C}(t_1)$ with

$$\deg(t_i) = i - 1 \quad \text{for} \quad i \geq 2.$$

Define a subring of homogeneous elements by

$$\mathbb{C}\left[\frac{1}{1 - t_1}\right][t_2, t_3, \ldots]_{\text{Hom}} \subset \mathbb{C}(t_1)[t_2, t_3, \ldots].$$
We easily see
\[
\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle\rangle_{g,n} |_{t_0=0} \in \mathbb{C} \left[ \frac{1}{1-t_1} \right] [t_2, t_3, \ldots]_{\text{Hom}}.
\]

Using the leading terms (of lowest degree in \(\frac{1}{1-t_1}\)), we obtain the following result.

**Lemma 9.** The set of genus 0 correlators
\[
\left\{ \langle\langle 1, \ldots, 1 \rangle\rangle_{0,n} |_{t_0=0} \right\}_{n \geq 4}
\]
freely generate the ring \(\mathbb{C}(t_1)[t_2, t_3, \ldots]\) over \(\mathbb{C}(t_1)\).

By Lemma 9, we can find a unique representation of \(\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \rangle\rangle_{g,n} |_{t_0=0}\) in the variables
\[
\left\{ \langle\langle 1, \ldots, 1 \rangle\rangle_{0,n} |_{t_0=0} \right\}_{n \geq 3}.
\]
The \(n = 3\) correlator is included in the set \(\langle\rangle\) to capture the variable \(t_1\). For example, in \(g = 1\),
\[
\langle\langle 1, 1 \rangle\rangle_{1,2} |_{t_0=0} = \frac{1}{24} \left( \frac{\langle\langle 1, 1, 1, 1 \rangle\rangle_{0,5} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3} |_{t_0=0}} - \frac{\langle\langle 1, 1, 1 \rangle\rangle_{0,4} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3} |_{t_0=0}} \right),
\]
\[
\langle\langle 1 \rangle\rangle_{1,1} |_{t_0=0} = \frac{1}{24} \frac{\langle\langle 1, 1, 1 \rangle\rangle_{0,4} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3} |_{t_0=0}}.
\]

A more complicated example in \(g = 2\) is
\[
\langle\rangle_{2,0} |_{t_0=0} = \frac{1}{1152} \frac{\langle\langle 1, 1, 1, 1, 1 \rangle\rangle_{0,6} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^{2} |_{t_0=0}} - \frac{7}{1920} \frac{\langle\langle 1, 1, 1, 1 \rangle\rangle_{0,5} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^{3} |_{t_0=0}} + \frac{1}{360} \frac{\langle\langle 1, 1, 1 \rangle\rangle_{0,4}^{3} |_{t_0=0}}{\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^{4} |_{t_0=0}}.
\]

**Definition 10.** For \(\gamma \in H^*(\overline{M}_{g,k})\), let
\[
P_{g,n}^{a_1, \ldots, a_n, \gamma}(s_0, s_1, s_2, \ldots) \in \mathbb{Q}(s_0, s_1, \ldots)
\]
be the unique rational function satisfying the condition
\[
\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} | \gamma \rangle\rangle_{g,n} |_{t_0=0} = P_{g,n}^{a_1, a_2, \ldots, a_n, \gamma} |_{s_i = \langle\langle 1, \ldots, 1 \rangle\rangle_{0, i+3} |_{t_0=0}}.
\]

We will use following notation.
\[
P_{h, n}^{\psi_{1}^{k_1}, \ldots, \psi_{n}^{k_n}} [H_{h}^{p_i}]_{0^+} : =
\]
\[
P_{h,1}^{k_1, \ldots, k_n, H_{h}^{p_i}} (\langle\langle 1, 1, 1 \rangle\rangle_{0,3}^{0^+}, \langle\langle 1, 1, 1 \rangle\rangle_{0,4}^{0^+}, \ldots).
\]
2. Higher genus series on $\mathbb{P}[n]$

We review here the now standard method used by Givental to express genus $g$ descendent correlators in terms of genus 0 data, see [10, 12, 13, 14] for more explanations.

**Definition 11.** Let us define equivariant genus $g$ generating function $F_{g,m}^T(q) \in \mathbb{C}[[q]](\lambda)$ of $\mathbb{P}[n]$ as following;

$$F_{g,m}^T[\tau_1 c_1, \ldots, \tau_m c_m](q) := \sum_{d_1, d_2, \ldots, d_n \geq 0} q^{d_1} q^{d_2} \ldots q^{d_n} \int_{[Q_{g,0}(\mathbb{P}[n], d)]} \pi^* H^{c_1,j}_j \prod_{i=1}^m \pi^* (\psi_i)^k_i \text{ev}^*_i (\otimes_{j=1}^n H^{c_{j,i}}_j) \in \mathbb{C}[[q]](\lambda).$$

We apply the localization strategy introduced first by Givental for Gromov-Witten theory to the quasimap invariants of $\mathbb{P}[n]$. We write the localization formula as

$$F_{g,m}^T[\tau_1 c_1, \ldots, \tau_m c_m]|_{\lambda_i = 1, \lambda_i = -1} = \sum_{\lambda_i, j = \pm 1} \sum_{\Gamma \in G_{g,0}^{\text{ord}}} \text{Cont}_\Gamma$$

where Cont$_\Gamma$ is the contribution to $F^g$ of the $T$-fixed loci associated $\Gamma$. We have the following formula for Cont$_\Gamma$.

$$\text{Cont}_\Gamma = \frac{1}{\text{Aut}(\Gamma)} \sum_{A, B \in \mathbb{Z}^{F(\Gamma)}_{\geq 1} \times \mathbb{Z}^{L(\Gamma)}_{\geq 1}} \prod_v \prod_{e \in E(\Gamma)} \prod_{l \in L(\Gamma)} \text{Cont}_\Gamma^{A,B}(v) \text{Cont}_\Gamma^A(e) \text{Cont}_\Gamma^B(l).$$

We need explanations for the Cont$_\Gamma^{A,B}(v)$, Cont$_\Gamma^A(e)$ and Cont$_\Gamma^B(l)$. Let $v \in V(\Gamma)$ be a vertex of genus $g(v)$.

$$\text{Cont}_\Gamma^{A,B}(v) = P[\psi^{a_1-1}, \psi^{a_2-1}, \ldots, \psi^{a_{\alpha}-1}, \psi^{b_1-1+k_{j_1}}, \ldots, \psi^{b_{\beta}+k_{j_{\beta}}}/H_{g(v)}^{\nu(v)}]^{0+,v}_{g(v), \alpha+\beta}$$

where,

- $H_{g(v)}^{\nu(v)} = \prod_{k=1}^n \frac{\prod_{j=1}^l (\lambda_{k,v} - (-\lambda_{k,v}) - c_j)}{\lambda_{k,v} - (-\lambda_{k,v})}$
- $(a_1, a_2, \ldots, a_\beta)$ are components of $A \in \mathbb{Z}^{F(\Gamma)}_{\geq 1}$ associated to flags of $v$ and $(b_1, b_2, \ldots, b_\beta)$ are components of $B \in \mathbb{Z}^{L(\Gamma)}_{\geq 1}$ associated to legs attached to $v$.
- $\lambda_{k,v}$ is the weight of $k$-th $\mathbb{C}^*$ at $\nu(v)$.
- $\{c_j|1 \leq j \leq g(v)\}$ are chern roots of Hodge bundle on $M_{g(v),k}$. 
Let $e \in E(\Gamma)$ be an edge between $v_i$ and $v_j$.

$$\text{Cont}_\Gamma^A(e) = \left[ e^{-\frac{\tau^P[n]}{x} - \frac{\tau^F[n]}{y}} e_i \nabla_{ij}^P[n](x, y) e_j \right]_{x^{a_1-1}y^{a_2-1}}$$

where $(a_i, a_j)$ are component of $A \in \mathbb{Z}^{F(\Gamma)}$ associated to flags $(v_i, e)$ and $(v_j, e)$. The notation $[\ldots]_{x^{a_1-1}y^{a_2-1}}$ above denotes the coefficient of $x^{a_1-1}y^{a_2-1}$ in the series expansion of the argument. The bar above $U_i$ and $V_{ij}$ means evaluation $t = 0$.

Let $l \in L(\Gamma)$ be the $i$-th leg with insertion $\tau_i c_i$.

$$\text{Cont}_\Gamma^B(l) = \left[ e^{-\frac{\tau^P[n]}{x} - \frac{\Sigma^F[n]}{y} (\otimes_j H^c_{ij})} \right]_{x^a},$$

where $a$ is component of $B$ associated to leg $l$.

**Remark 12.** By (4), $\text{Cont}_\Gamma^A(v), \text{Cont}_\Gamma^A(e)$ and $\text{Cont}_\Gamma^B(l)$ can be considered as elements in $\mathbb{R}[q, x^{-1}, y^{-1}]$.

Finally we obtain the following result.

**Proposition 13.** For $\mathbb{P}[n]$, We have

$$F^T_{g,m}[\tau_{i_1} c_1, \ldots, \tau_{i_m} c_m]|_{\lambda_i = 1, \lambda_i = -1} = \sum_{\lambda_i, j = \pm 1} \sum_{\Gamma \in G^{\text{ord}}_{g,0}} \frac{1}{\text{Aut}(\Gamma)} \sum_{A, B \in \mathbb{Z}^{F(\Gamma)}_{\geq 1} \times \mathbb{Z}^{L(\Gamma)}_{\geq 1}} \prod_v \prod_e \prod_l \text{Cont}_\Gamma^A(v) \text{Cont}_\Gamma^A(e) \text{Cont}_\Gamma^B(l),$$

where $\text{Cont}_\Gamma^A(v), \text{Cont}_\Gamma^A(e)$ and $\text{Cont}_\Gamma^B(l)$ are defined as above.

### 3. Proof of main theorem

**3.1. Overview.** Using (3) and Proposition 4 we can explicitly calculate the S-operator of $\mathbb{P}[n]$.

**Proposition 14.** For $\{i_1, i_2, \ldots, i_n\} = \{0, 1\}$,

$$S^P[n](H_1^{i_1}, H_2^{i_2}, \ldots, H_n^{i_n}) = \left( z \frac{d}{dt_1} \right)^{i_1} \left( z \frac{d}{dt_2} \right)^{i_2} \ldots \left( z \frac{d}{dt_n} \right)^{i_n} \mathbb{P}[n].$$

If we restrict $S$-operator to the fixed points, it admits Birkhoff factorizations. See [2] for more explanations.
Definition 15. For fixed point $p \in (\mathbb{P}[n])^T$, let us define $R^i_{p,k} \ldots i_n$ by following:

\[
S^{\mathbb{P}[n]}(H_1^{i_1}H_2^{i_2} \ldots H_n^{i_n})|_p = e^{\sum_{p,k}^{\mathbb{P}[n]} \lambda_{1,p}^{\delta_{11}} \lambda_{2,p}^{\delta_{12}} \ldots \lambda_{n,p}^{\delta_{1n}} (\sum_k R^i_{p,k} \ldots i_n z^k) \in \mathbb{R}[z, z^{-1}, q, t]([\lambda]).}
\]

Here $\lambda_{k,p}$ is the weight of $k$-th $C^*$ at $p$. We consider $\lambda_{k,p}$ as an element of $\mathbb{R}$ by identifying $\lambda_{k,p}$ with $\lambda_{k,1}$.

Proposition 16. $R^i_{p,k} \ldots i_n$ has type $(0, k)$.

Proof. Applying Prop[14] to (3), the proof follows from following lemma.

Lemma 17. We have following assymptotic expansion of local $S$-operators of $\mathbb{P}^1$ for fixed point $p \in (\mathbb{P}^1)^T$.

\[
S^{\mathbb{P}^1}(1)|_p = e^{\sum_k^{\mathbb{P}^1} \lambda_p^k \delta(k) z^k},
\]

\[
S^{\mathbb{P}^1}(H)|_p = e^{\sum_k^{\mathbb{P}^1} \lambda_p^k \delta(k) z^k}
\]

where $\delta(k) = 0$ if $k$ is even, $\delta(k) = 1$ if $k$ is odd. Here $Q_{ik}$ are constant with respect to $\lambda_p$.

Remark 18. We can calculate $Q_{ik}$ in closed form using Picard-Fuchs equation of $I$-function. Since this is not needed in our paper, we leave the details to the readers.

Proposition 19. Let $\Gamma \in G_{g,0}^{\text{ord}}$ be a decorated graph.

(1) $\text{Cont}_\Gamma^A(v)$ has type $(g(v) - 1; 3g(v) - 3 + N_v(A) + N_v(B))$, where $N_v(A) = \sum_{i=1}^k (2 - a_i)$ and $N_v(B) = \sum^\beta_{i=1} (2 - b_i - k_i)$. Here $(a_1, \ldots, a_n)$ are components of $A$ associated to $v$ and $(b_1, \ldots, b_\beta)$ are components of $B$ associated to $v$. The notation $\pi$ for $a \in N$ means $(a, \ldots, a) \in N^n$.

(2) $\text{Cont}_\Gamma$ has type $(g - 1 + \sum_i c_{i,1}, \ldots, g - 1 + \sum_i c_{i,n}; 3g - 3 + m - \sum_{i=1}^m k_i)$.

Proof. For the first parts, if we expand

\[
\text{Cont}_\Gamma^A(v) = \sum_{l_1, l_2, \ldots, l_{g(v)} \geq 0} b_{l_1, l_2, \ldots, l_{g(v)}} [\psi_{a_1 - 1}, \psi_{a_2 - 1}, \ldots, \psi_{a_n - 1}, \psi_{b_1 + k_1}, \ldots, \psi_{b_\beta + k_\beta} [H_{g(v)}]_v^{0+, v}]^{0+, v}
\]

\[
\text{Cont}_\Gamma = \sum_{l_1, l_2, \ldots, l_{g(v)} \geq 0} c_{l_1, l_2, \ldots, l_{g(v)}} [\psi_{a_1 - 1}, \psi_{a_2 - 1}, \ldots, \psi_{a_n - 1}, \psi_{b_1 + k_1}, \ldots, \psi_{b_\beta + k_\beta} [H_{g(v)}]_v^{0+, v}]^{0+, v}
\]


where \((a_1, a_2, \ldots, a_n)\) are components of \(A\) associated to \(v\) and \((b_1, \ldots, b_j)\) are components of \(B\) associated to \(v\). We can check the followings.

- \([\psi]^{a_1-1}, [\psi]^{a_2-1}, \ldots, [\psi]^{a_n-1}, [\psi]^{b_1-1+k_{j_1}}, \ldots, [\psi]^{b_\beta+k_{j_\beta}}|c_1, c_2, \ldots, c_g|e_v\) has type \((0; 3g(v) - 3 - \sum_{i=1}^{g(v)} i + \sum_{i=1}^{a} (2 - a_i) + \sum_{i=1}^{\beta} (2 - b_i - k_{j_i}))\).

The proof of first part follows from Lemma \(7\).

For the second parts, let \(e\) be an edge connecting \(v_i\) and \(v_j\). Then, by Prop \(16\) and \(2\), we can check that

\[
\text{Cont}_A^A(e) = \left(-1\right)^{k+l} e^{-\frac{\nu_{[x]}^0}{x} - \frac{\nu_{[y]}^0}{y}} \sum_{i=1}^{l} \left(1 + (k - 1) + (l - 1)\right) \left(-1\right)^{k+l+i+1} e^{-\frac{\nu_{[x]}^0}{x} - \frac{\nu_{[y]}^0}{y}} \sum_{k} \left[S_{x,y}^0(\phi_k)|_{v_i,z=x} S_{v_j}^0(\phi^k)|_{v_j,z=y}\right] \left[S_{x,y}^0(\phi_k)|_{v_i,z=x} S_{v_j}^0(\phi^k)|_{v_j,z=y}\right]
\]

has type \((0; 1 + (k - 1) + (l - 1))\). Here \((k, l)\) are components of \(A\) associated to flags \((v_i, e)\) and \((v_j, e)\). Let \(l\) be the \(i\)-th leg with insertion \(\tau_k, c_i\). Then similarly,

\[
\text{Cont}_A^A(l) = e^{-\frac{\nu_{[x]}^0}{x} - \frac{\nu_{[y]}^0}{y}} \sum_{j} \left(\otimes_{y}^0 H_j^{c_{j,i}}\right)
\]

has type \((c_{1,i}, c_{2,i}, \ldots, c_{n,i}; a - 1)\), where \(a\) is the component of \(B\) associated to \(l\). By multiplying all contributions of \(\Gamma\), the proof of second part follows from Lemma \(7\). 

\[\square\]

### 3.2. Proof of Theorem \(3\)

Consider the following decomposition of \(F_{g,m}^T[\tau_{k_1}, c_1, \ldots, \tau_{k_m} c_m]\).

\[
F_{g,m}^T[\tau_{k_1} c_1, \ldots, \tau_{k_m} c_m]|_{\lambda_i = 1, \bar{\lambda}_i = -1} = \sum_{\lambda_i, \bar{\lambda}_i = \pm 1} \sum_{\Gamma \in G_{g,0}} \text{Cont}_\Gamma.
\]

Each \(\text{Cont}_\Gamma\) has type \((g - 1 + \sum_i c_{i,1}, \ldots, g - 1 + \sum_i c_{i,n}; 3g - 3 + m - \sum_{i=1}^{m} k_i)\) by Proposition \(19\). Using Lemma \(8\), we obtain the following result for \(n > 3g - 3 + m - \sum_{i=1}^{m} k_i\) and odd \(g - 1 + \sum_i c_{i,j}\).

\[
\sum_{\lambda_i, \bar{\lambda}_i = \pm 1} \text{Cont}_\Gamma = 0.
\]
Therefore we conclude
\[
F^T_{g,m}[\tau_{k_1}c_1, \ldots, \tau_{k_m}c_m]|_{\lambda_i=1, \bar{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1} \sum_{\Gamma \in \mathcal{G}^{ord}_{g,0}} \text{Cont}_\Gamma = 0,
\]
for \( n > 3g - 3 + m - \sum_{i=1}^m k_i \) and odd \( g - 1 + \sum_i c_{i,j} \). The \( q \)-coefficients in \( F^T_{g,m}[\tau_{k_1}c_1, \ldots, \tau_{k_m}c_m] \) of degree \( d \) with virtual dimension 0 are independent of \( \lambda \). Theorem 3 is an immediate consequence.

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