Homological Algebra on Graded Posets

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Introduction

This work has its origin in a problem related with $p$-local finite groups \[10\]. These algebraic objects are triples $(S, F, L)$ where $S$ is a finite $p$-group and $F$ and $L$ are categories which encode fusion information among the subgroups of $S$. In fact, they contain the data needed to build topological spaces which behave as $p$-completions of classifying spaces of finite groups $(BG^p)$. While a finite group give rise to a $p$-local finite group in a natural way, there are $p$-local finite groups which do not arise from finite groups. These are called exotic $p$-local finite groups, and examples have already been found in \[10\], \[26\], \[36\] and \[14\].

Prof. A. Viruel wondered a few years ago if these exotic $p$-local finite group arise or not from an infinite group. The work of Aschbacher and Chermak shows that this is the case for some of the Solomon 2-local finite groups \[26\].

An approach to find a candidate to this infinite group is to use the normalizer decomposition for a $p$-local finite group $(S, F, L)$ \[27\]. It gives a description of the nerve $|L|$ as a homotopy colimit (in the sense of Bousfield and Kan \[8\]) of spaces which has the homotopy type of classifying spaces of finite groups:

$$\text{hocolim}_{\pi dC} \delta_C \simeq |L|.$$  

Here, $\pi dC$ is a partially ordered set (poset for short) built from the $p$-local finite group $(S, F, L)$ and $\delta_C : \pi dC \rightarrow \text{Top}$ is a functor which satisfies

$$\delta_C(\sigma) \simeq B\text{Aut}_L(\sigma)$$

for every $\sigma \in \pi dC$ (where $B : \text{Grp} \rightarrow \text{Top}$ is a functor which sends a discrete small group to a classifying space for it). If this homotopy colimit has the homotopy type of an Eilenberg-MacLane space

$$\text{hocolim}_{\pi dC} \delta_C \simeq B\pi,$$

then $\pi$ is a candidate for infinite group from which the original $p$-local finite group arises. Thus, the problem we have focused on is:

**Problem 1.** Given a diagram of groups $G : C \rightarrow \text{Grp}$ and a cone $\tau : G \Rightarrow \pi$, when is the natural map

$$\text{hocolim}_{C} BG \rightarrow B\pi$$

a homotopy equivalence? And, in case it is not, can we compute the fiber of this map?
A classical result of J.H.C. Whitehead states that if $\mathcal{C}$ is the pushout category

\[
\begin{array}{ccc}
  b & \rightarrow & c \\
  \uparrow & & \downarrow \\
  a & \rightarrow & c
\end{array}
\]

and $\pi$ is the amalgamated product $G(b) *_{G(a)} G(c)$, i.e., the direct limit $\lim_{\to} G$, then $\hocolim_{\mathcal{C}} BG \simeq B\pi$. Thus, in this case we have an affirmative answer to our problem. It is worthwhile noticing that, also in this particular case, we have that the category $\mathcal{C}$ is contractible and that the arrows $\tau_a$, $\tau_b$ and $\tau_c$ are monomorphisms:

\[
\begin{array}{ccc}
  G(b) & \xrightarrow{\tau_b} & c \\
  \downarrow & & \downarrow \\
  G(a) & \xrightarrow{\tau_a} & G(b) *_{G(a)} G(c) \\
  \downarrow & & \downarrow \\
  G(c) & \xrightarrow{\tau_c} & G(b) *_{G(a)} G(c)
\end{array}
\]

Studying further the fiber in the general case we find that:

**Theorem 2.** Let $G : \mathcal{C} \to \text{Grp}$ be a functor and let $\tau : G \Rightarrow \lim_{\to} G$ be the limit cone to the direct limit $\lim_{\to} G$. Assume $\mathcal{C}$ is contractible and $\tau$ is a monomorphism on each object. Then, if $F$ is the fiber of the map

\[\hocolim_{\mathcal{C}} BG \to B\lim_{\to} G,\]

we have that:

- $F$ is simply connected, and
- $H_j(F) = \lim_{\to} H_{j-1}$ for each $j \geq 2$, where $H : \mathcal{C} \to \text{Ab}$ is a functor.

Thus, when we assume the same hypothesis as in Whitehead’s Theorem, it turns out that the homology groups are given by the higher direct limits $\lim_{\to} H$ for a functor $H : \mathcal{C} \to \text{Ab}$ from $\mathcal{C}$ to abelian groups. Before continue put

**Definition 3.** Let $H : \mathcal{C} \to \text{Ab}$ be a functor. Then we say that it is \textit{lim-acyclic} if $\lim_{\to} H = 0$ for $j \geq 1$.

Then it is clear by G.W. Whitehead’s Theorems [43], that $F$ is homotopic to a point, i.e., $\hocolim_{\mathcal{C}} BG \simeq B\lim_{\to} G$, if and only if $H$ is lim-acyclic. Thus, we have reduced the Homotopy Theory Problem 1 to the following problem of Homological Algebra:

**Problem 4.** Given a functor $H : \mathcal{C} \to \text{Ab}$, when does it hold that $H$ is lim-acyclic? And, in case it does not hold, can we compute $\lim_{\to} H$?
This is a very general problem, and we shall assume one additional hypothesis to attack it. It consist in restricting the category $\mathcal{C}$: mainly because the category $\mathbf{sdC}$ considered above is a graded partially ordered set (a graded poset for short) we assume that $\mathcal{C}$ is a graded poset. These are special posets in which we can assign an integer to each object (called the degree of the object) in such a way that preceding elements are assigned integers which differs just in 1. Thus a graded poset can be divided into a set of “layers” (the objects of a fixed degree), and these layers are linearly ordered. This restriction is no so hard as simplicial complexes and subdivision categories are graded posets. Moreover, each $\mathcal{CW}$-complex is (strong) homotopy equivalent to a graded poset.

Coming back to Problem 4 consider the functor $H: \mathcal{C} \to \text{Ab}$ as an object of the abelian category $\text{Ab}^\mathcal{C}$ of functors from $\mathcal{C}$ to abelian groups. Because the higher limits $\lim_{\rightarrow j} H$ are the left derived functors of the direct limit functor $\lim_{\rightarrow} : \text{Ab}^\mathcal{C} \to \mathcal{C}$, then $H$ projective is enough for $H$ $\lim_{\rightarrow}$-acyclic (as left derived functors vanishes on projective objects). This is a partial answer to Problem 4 inasmuch that we know the projective objects of $\text{Ab}^\mathcal{C}$. In order to study these projective objects we introduce

**Definition 5.** Let $H: \mathcal{C} \to \text{Ab}$ be a functor over the graded poset $\mathcal{C}$, and let $i_0$ be an object of $\mathcal{C}$. Then define

$$\text{Coker}(i_0) = H(i_0)/\langle \{ H(\alpha) | \alpha : i \to i_0, \alpha \neq 1_{i_0} \} \rangle.$$  

That is, $\text{Coker}(i_0)$ is the quotient of the value of $H$ on $i_0$ by the images of the non-trivial morphisms arriving to $i_0$. For example, if $i_0$ is a minimal object of the graded poset $\mathcal{C}$, i.e., it has no non-trivial arriving arrows, then $\text{Coker}(i_0) = H(i_0)$. The next theorem relates the projectiveness of $H$ in $\text{Ab}^\mathcal{C}$ with that of $\text{Coker}(i_0)$ in $\text{Ab}$:

**Theorem 6.** Let $\mathcal{C}$ be a graded poset and suppose $H \in \text{Ab}^\mathcal{C}$. Then $H$ is projective in $\text{Ab}^\mathcal{C}$ if and only if

- $\text{Coker}(i_0)$ is projective in $\text{Ab}$ for each object $i_0$ of $\mathcal{C}$.
- $H$ is pseudo-projective.

This theorem characterizes the projective objects of $\text{Ab}^\mathcal{C}$. Although the second condition, pseudo-projectiveness, is technical, and thus I do not state it here for simplicity, both conditions in the theorem are easy to check for a given functor $H \in \text{Ab}^\mathcal{C}$. As we commented above, $H$ projective implies that $H$ is $\lim_{\rightarrow}$-acyclic, but this last condition is clearly weaker than $H$ being projective. Can we weak the condition of projectiveness and still have $\lim_{\rightarrow}$-acyclicity? Then answer is yes and next theorem provide us the weaker condition:

**Theorem 7.** Let $\mathcal{C}$ be a graded poset and suppose $H \in \text{Ab}^\mathcal{C}$. If $H$ is pseudo-projective then $H$ is $\lim_{\rightarrow}$-acyclic.

Thus pseudo-projectiveness gives also an answer to Problem 4. This last theorem is proven by constructing a spectral sequence for a given functor $H \in \text{Ab}^\mathcal{C}$ over a graded poset $\mathcal{C}$. It is the grading of $\mathcal{C}$ which allows us to define certain filtered differential graded $\mathbb{Z}$-module from which we build the spectral sequence:
**Proposition 8.** Let $\mathcal{C}$ be a graded poset and suppose $H \in \text{Ab}^\mathcal{C}$. Then there is a homological type spectral sequence $E^{*,*}_r$ with target $\lim_{\to} H$.

To answer the second question in Problem 4 we have made a “dimension shifting argument” in the sense it is made in group cohomology \[11\]. This is done by constructing a short exact sequence

$$0 \Rightarrow K_1 \Rightarrow H' \Rightarrow H \Rightarrow 0,$$

where the functor $H'$ is pseudo-projective. Iterating this process we obtain

**Lemma 9.** Let $\mathcal{C}$ be a graded poset and $H : \mathcal{C} \to \text{Ab}$ a functor. Then, for each $j \geq 1$,

$$\lim_{\to}^j H = \lim_{\to}^1 K_{j-1},$$

where $H = K_0, K_1, K_2, \ldots$ are functors in $\text{Ab}^\mathcal{C}$.

We have also given an interpretation of the higher limit of order 1, $\lim_{\to}^1 H$, as a flow problem in the directed graph associated to $\mathcal{C}$. All these tools can be applied to an example told to us by A. Libman. This example shows that the conditions $\mathcal{C}$ contractible and $\tau$ a monomorphism on each object are not enough to have $\text{hocolim}_{\mathcal{C}} BG \simeq B \lim_{\to} G$ in general. We compute the fiber of the map $\text{hocolim}_{\mathcal{C}} BG \to B \lim_{\to} G$ by the methods above:

**Example 10.** For any group $\pi$ consider the graded poset $\mathcal{C}$ and the functor $G : \mathcal{C} \to \text{Grp}$ with values

Then there is a fibration

$$\bigvee_{\alpha \in \pi \setminus \{1\}} (S^2)_\alpha \to \text{hocolim}_C BG \to B\pi.$$

All the results concerning higher direct limits are contained in Chapter 3, while the spectral sequence is introduced in Chapter 2. Results about the homotopy colimit, as Theorem 2 and Example 10 are presented in Chapter 7. This chapter also contains a proof of the theorem of J.H.C. Whitehead about the pushout, and a proof of that the classifying space of a locally finite group is the homotopy colimit of the classifying spaces of its subgroups. In order to wide our view of the developments till here we decided to dualize all the Homological Algebra results. This is made in Chapter 4 and the results are summarized below:
Definition 11. Let \( H : C \to \text{Ab} \) be a functor. Then we say that it is \( \text{lim-acyclic} \) if 
\[
\lim_{\leftarrow}^j H = 0 \quad \text{for} \quad j \geq 1.
\]

Definition 12. Let \( H : C \to \text{Ab} \) be a functor over the graded poset \( C \), and let \( i_0 \) be an object of \( C \). Then define
\[
\text{Ker}(i_0) = \bigcap_{\alpha : i_0 \to i, \alpha \neq 1} H(\alpha).
\]

Theorem 13. Let \( C \) be a graded poset and suppose \( H \in \text{Ab}^C \). Then \( H \) is injective in \( \text{Ab}^C \) if and only if

- \( \text{Ker}(i_0) \) is injective in \( \text{Ab} \) for each object \( i_0 \) of \( C \).
- \( H \) is pseudo-injective.

Theorem 14. Let \( C \) be a graded poset and suppose \( H \in \text{Ab}^C \). If \( H \) is pseudo-injective then \( H \) is \( \text{lim-acyclic} \).

Lemma 15. Let \( C \) be a graded poset and \( H : C \to \text{Ab} \) a functor. Then, for each \( j \geq 1 \),
\[
\lim_{\leftarrow}^j H = \lim_{\leftarrow}^1 C_{j-1},
\]
where \( H = C_0, C_1, C_2, \ldots \) are functors in \( \text{Ab}^C \).

If we take \( H = c_Z : C \to \text{Ab} \) in this lemma, i.e., \( H \) equals the functor of constant value \( Z \), we obtain an approach to compute the integer cohomology groups \( H^*(|C|; Z) = \lim_{\leftarrow}^* c_Z \) of the realization \( |C| \) of the graded poset \( C \). From now onwards we assume that \( H = c_Z \), and we explain how a sharpener version of Lemma 15 is obtained for \( H = c_Z \) (this is made in detail in Chapter 5).

First step is imposing some extra structure in the graded poset \( C \). In fact, this extra structure is not very restrictive as each \( CW \)-complex is still homotopy equivalent to a graded poset which carries this extra information. We denote this structure by \( J \) and we call it a covering family for \( C \). It consist of a collection of subsets of objects of the category \( C \). We explain this briefly.

This collection of subsets, \( J \), is of “local” nature in the sense that for each object \( i_0 \) of \( C \) we have several subsets of objects of \( C \). More precisely, for a fixed \( i_0 \), we must choose subsets of objects of the subcategory \((i_0 \downarrow C)\). This under category corresponds to the full subcategory of \( C \) with objects \( \{i | \exists i_0 \to i\} \), i.e., it is exactly composed by the objects greater or equal to \( i_0 \). What we need on each category \((i_0 \downarrow C)\) amounts to a subset \( J_p^{i_0} \) of objects from \((i_0 \downarrow C)\) for each appropriate degree \( p \) (recall that objects of \( C \) are graded). An example shall make this clearer:
Example 16. Suppose \((i_0 \downarrow \mathcal{C})\) has the following shape

\[
\begin{array}{c}
\text{Example 16.} \\
\text{Suppose (}i_0 \downarrow \mathcal{C} \text{) has the following shape}
\end{array}
\]

where subindexes point out the degree of each object. We can define \(J_{i_0}^0 = \{a\}\), \(J_{i_0}^1 = \{c, d\}\), and \(J_{i_0}^2 = \{g\}\).

The collection of subsets \(\mathcal{J}\) must fulfill a “covering” condition and a “inheritance” condition. The “covering” condition states that each object of \((i_0 \downarrow \mathcal{C})\) of a given degree \((p - 1)\) must be preceded by an object of \(J_p^{i_0}\). The “inheritance condition” states that if \(i \in J_p^{i_0}\) then \(J_i^p \subseteq J_p^{i_0}\). We come back to the earlier example to show what this means:

Example 17. The “covering” condition for the object \(e\) of degree 0 is fulfilled as it is preceded by the object \(c \in J_{i_0}^1\):

\[
\begin{array}{c}
\text{Example 17.} \\
\text{The “covering” condition for the object e of degree 0 if fulfilled as it is preceded by the object c \(c \in J_{i_0}^1\):}
\end{array}
\]

(elements in \(\mathcal{J}\) are boldfaced). It is also fulfilled for \(f\) and for \(g\) as \(d \in J_{i_0}^1\) precedes \(f\) and \(c\) and \(d\) precede \(g\). If we define \(J_{d}^1 = \{d\}\) and \(J_{d}^0 = \{g\}\):

\[
\begin{array}{c}
\text{Example 17.} \\
\text{The “covering” condition for the object e of degree 0 is fulfilled as it is preceded by the object c \(c \in J_{i_0}^1\):}
\end{array}
\]

then the object \(d \in J_{i_0}^1\) fulfills the “inheritance” condition because \(J_0^d = \{g\} \subseteq J_{i_0}^0 = \{g\}\).

The existence of this family \(\mathcal{J}\) of subsets for each object \(i_0\) of \(\mathcal{C}\), plus a numerical condition (adequateness) involving the number of elements in the subsets of \(\mathcal{J}\), gives:

Proposition 18. Let \(\mathcal{C}\) be a graded poset category and let \(\mathcal{J}\) be an adequate covering family for \(\mathcal{C}\). Then, for each \(j \geq 1\),

\[
\lim_{j \to \infty} c_Z = \lim_{i \to \infty} F_{j-1},
\]

where \(c_Z = F_0, F_1, F_2, \ldots\) are functors in \(\text{Ab}\) which take free abelian groups as values.
The sequence of functors $F_0, F_1, F_2, \ldots$ of Proposition 18 have some advantages over the sequence of functors $C_0, C_1, C_2, \ldots$ of Lemma 15 (applied to the functor $H = c_Z$):

- $F_j(i_0)$ is a free abelian group for each object $i_0$ of $C$,
- $F_j(i_0)$ has far less generators than $C_j(i_0)$ for each object $i_0$ of $C$, and
- $\lim \leftarrow F_j \simeq \lim \leftarrow F_j|C_{j+1},j$, where $C_{j+1,j}$ is the full subcategory of $C$ with objects of degrees $j + 1$ and $j$.

The third statement above is the main feature of $F_j$. It is a generalization of the fact that the connected components of a simplicial complex are computed by just looking to the vertices and to the edges, i.e., to the objects of degrees 0 and 1. The nice properties of $F_j$ causes our main theorem about integer cohomology:

**Theorem 19.** Let $C$ be a bounded graded poset for which there exist an adequate covering family $\mathcal{J}$ and an adequate global covering family $\mathcal{K}$. Then $|C|$ is acyclic if and only if $|K_0|$ equals the number of connected components of $C$. Moreover, in this case $H^0(|C|; \mathbb{Z}) = \mathbb{Z}^{|K_0|}$.

Here, we are using the concept of adequate *global* covering family, which plays the role of covering family for the whole category $C$ instead of for each subcategory $(i_0 \downarrow C)$. The family $\mathcal{K}$ is composed of subsets $K_p$, one for each appropriate degree $p$. This theorem reduces the $\lim \leftarrow$-acyclicity of the functor $c_Z \in \text{Ab} C$, i.e., the acyclicity of the space $|C|$, to the integral equation $|K_0| = |\pi_0(C)| = |\pi_0(|C|)|$.

Although the hypotheses in Theorem 19 look very hard, the existence of the family $\mathcal{J}$ is automatic for $C^{\text{op}}$ (of course $C^{\text{op}}$ and $C$ have the same homotopy type) when $C$ is a simplicial complex or a subdivision category. In fact, it applies to $C^{\text{op}}$ whenever the graded poset $C$ is locally as a simplicial complex, i.e., such that $(C \downarrow i_0)$ is isomorphic to the poset of all non-empty sets of a finite set (with inclusion as order relation). We call these posets *simplicial-like* posets. Thus, the difficult point resides in finding a global family $\mathcal{K}$.

In Chapter 6 we describe a particular case where this global family $\mathcal{K}$ exists, yielding a proof of part of a conjecture of Webb. The conjecture is related with the Brown’s complex [11] (denoted $S_p(G)$) of a finite group $G$ and a prime $p$. Webb conjectured that the orbit space $S_p(G)/G$ is contractible. This orbit space has as objects the $G$-conjugation classes of chains of subgroups of $S_p(G)$. The conjecture was first proven by Symonds in [40], generalized for blocks by Barker [4, 5] and extended to arbitrary (saturated) fusion system by Linckelmann [28].

The works of Symonds and Linckelmann prove the contractibility of the orbit space by showing that it is simply connected and acyclic, and invoking G.W. Whitehead’s theorem. Both proofs of acyclicity work on the subposet of normal chains (introduced by Knörr and Robinson [25] for groups). Symonds uses the results from Thevenaz and Webb [41] that the subposet of normal chains is $G$-equivalent to Brown’s complex. Linckelmann proves on his own that, also for fusion systems, the orbit space and the orbit space on the normal chains has the same cohomology [28, Theorem 4.7].

We apply Theorem 19 to prove that the orbit space on the normal chains is acyclic. The definition of the global covering family $\mathcal{K}$ is related with the pairing defined by
Linckelmann in \[28\], Definition 4.7. We describe \(\mathcal{K}\) briefly: let \((S, \mathcal{F})\) be a saturated fusion system where \(S\) is a \(p\)-group and consider its subdivision category \(\mathcal{S}(\mathcal{F})\) and the poset \([[S(\mathcal{F})]]\). An object of degree \(n\) in the orbit space of the normal chains \([\mathcal{S}_d(\mathcal{F})]\) is an \(\mathcal{F}\)-isomorphism class of chains 

\[
[Q_0 < Q_1 < \ldots < Q_n]
\]

where the \(Q_i\) are subgroups of \(S\) normal in \(Q_n\). The subcategory \([[\mathcal{S}_d(\mathcal{F})]] \downarrow [Q_0 < \ldots < Q_n]\) has objects \([Q_{i_0} < \ldots < Q_{i_m}]\) with \(0 \leq m \leq n\) and \(0 \leq i_1 < i_2 < \ldots < i_m \leq n\) (see \[28\], 2). For example, \([[\mathcal{S}_d(\mathcal{F})]] \downarrow [Q_0 < Q_1 < Q_2]\) is

\[
[Q_0] \rightarrow [Q_0 < Q_1] \rightarrow [Q_0 < Q_1 < Q_2].
\]

Then it is clear that \([[\mathcal{S}_d(\mathcal{F})]]\) is a simplex-like poset and thus Theorem 19 applies to \([[\mathcal{S}_d(\mathcal{F})]]^{op}\). The definition of the global family \(\mathcal{K}\) follows

\[
K_n = \{[Q_0 < \ldots < Q_n] | [Q_0 < \ldots < Q_n] = [Q'_0 < \ldots < Q'_n] \Rightarrow \cap_{i=0}^{n} N_S(Q'_i) = Q'_n\}.
\]

It is easy to see that \([[\mathcal{S}_d(\mathcal{F})]]^{op}\) has just one connected component. Thus, by Theorem 19 we can conclude that \([[\mathcal{S}_d(\mathcal{F})]]^{op}\) is acyclic if it is the case that \(|K_0| = 1\). By definition

\[
K_0 = \{[Q_0] | [Q_0] = [Q'_0] \Rightarrow N_S(Q'_0) = Q'_0\}.
\]

Because the unique subgroup of the \(p\)-group \(S\) whose normalizer equals \(S\) is \(S\) itself we have that \(|K_0| = 1\).

This finishes the exposition of the main results of the present work. By chapter, the contents are the following:

• Chapter 1: Notation and preliminaries. The notation used throughout the work is introduced, as well as the definitions of derived functors and the normalization theorems for simplicial abelian groups. Likewise, graded posets are defined, which are the categories over which the main results of the work are developed. This kind of category have been chosen because it is the prototype of the category \(\mathcal{s}dC\) used in the normalizer decomposition mentioned above \[27\]. Moreover, they include simplicial complexes (Section 1.4) and subdivision categories \[28\], and they contain all the homotopic information of \(CW\)-complexes (Section 1.5).

• Chapter 2: A spectral sequence. A spectral sequence is built from differential graded modules associated to a functor \(H : \mathcal{C} \rightarrow \text{Ab}\), where \(\mathcal{C}\) is a graded poset. The target of this spectral sequence is the higher limits \(\lim^* H\). In a similar way another spectral sequence is built whose target is the higher inverse limits \(\lim^* H\).
• Chapter 3: Higher direct limits. Projective objects in the abelian category $\text{Ab}^C$ are characterized, where $C$ is a graded poset. Moreover, due to the spectral sequence from Chapter 2, another family of objects of $\text{Ab}^C$ with vanishing higher direct limits is found, they are the pseudo-projective functors. Finally, a way of reducing the higher limit $\lim_j H$ of order $j$ to limits of lower order is given, and some applications of these results are worked out.

• Chapter 4: Higher inverse limits. This chapter is the dualization of Chapter 3, in which the injective objects of $\text{Ab}^C$ are described and another family of functors with vanishing higher inverse limits is found (the pseudo-injective functors). As application some tools for the computation of integer cohomology of categories are developed in Chapter 5.

• Chapter 5: Cohomology. Through some additional structure (called covering families) on the graded poset $C$ it is proven that such a category is acyclic if and only if some integral equation involving geometrical elements of $C$ is satisfied.

• Chapter 6: Application: Webb’s conjecture. Results from the earlier chapter are used to prove the cohomological part of Webb’s conjecture, which has been proven in its maximal generality in [28]. This conjecture states that the orbit space of the $p$-subgroups poset of a finite group $G$ is contractible.

• Chapter 7: Application: homotopy colimit. Although the study of homotopy colimits is the origin of this work, it appears here as an application of earlier chapters. It is proven Whitehead’s Theorem, which states that the pushout of Eilenberg-MacLane spaces with injective morphisms is an Eilenberg-MacLane space, and an example is given of explicit computation of the fiber $F$. Also it is proven the well known fact that the classifying space of a locally finite group is the homotopy colimit of the classifying spaces of its finite subgroups.

Finally, we comment other applications where we hope the theory can contribute with some results. First of them is to translate to the initial diagram of groups $G : C \rightarrow \text{Grp}$ the conditions of pseudo-projectiveness over $H$ (the functor $H$ comes from Theorem [2]). In this way it should be obtained a generalized Whitehead’s Theorem for diagrams larger than the pushout. Moreover, it would be interesting to study its relation with Mather’s cube Theorem.

The second application is take up again the original problem about $p$-local finite groups. We have already found conditions which imply that the functor $\delta_C : \mathcal{sd}C \rightarrow \text{Top}$ factorizes as

$$
\begin{array}{ccc}
\mathcal{sd}C & \xrightarrow{\delta_C} & \text{Top} \\
\downarrow B & & \downarrow \\
\text{Grp} & & \\
\end{array}
$$

This makes possible to apply the theory to a “honest” diagram of groups. Moreover, these conditions imply that $\mathcal{sd}C$ is contractible, fulfilling one of the hypotheses of
Thus, the next step is determine when is $|\mathcal{L}|$ an Eilenberg-MacLane space or compute the fiber $F$

$$F \to |\mathcal{L}| \to B\pi$$

for appropriate group $\pi$ in favorable cases. For example, it can be shown that all the $p$-rank 2 $p$-local finite groups for $p$ odd (which are described in [14]) have an Eilenberg-MacLane space as non-completed classifying space $|\mathcal{L}|$.

The third possible application is related to Quillen’s conjecture [34], which states that the poset of the non-trivial $p$-subgroups of a finite group $G$, $\mathcal{S}_p(G)$ (Brown’s complex [11]), is contractible if and only if $\mathcal{O}_p(G) \neq 1$ (where $\mathcal{O}_p(G)$ is the largest normal $p$-subgroup of $G$). A stronger formulation of this conjecture is that the integer reduced cohomology of this poset is trivial if and only if $\mathcal{O}_p(G) \neq 1$ (this formulation is used in [8]). Due to Theorem 19 this stronger formulation is equivalent to find an adequate covering family $\mathcal{K}(G)$ of $\mathcal{S}_p(G)$ for each finite group $G$ which satisfies the condition

$$|\mathcal{K}(G)| = 1 \Leftrightarrow \mathcal{O}_p(G) \neq 1.$$ 

By results of Bouc [7] and of Thévenaz and Webb [41] we can also work with the poset of non-trivial $p$-radical subgroups or with the poset of non-trivial elementary abelian $p$-subgroups.

Final note: For reference it follows a list of the main points of the introduction and their corresponding statements in the manuscript. For simplicity some statements over the boundedness of the graded poset $\mathcal{C}$ were omitted in the introduction. These conditions are automatically satisfied in the applications and for finite graded posets. We add them bracketed in the list (as well as other comments).

- Definition 3 = Definition 3.2.1
- Definition 5 = Definition 3.1.1
- Definition 11 = Definition 4.2.1
- Definition 12 = Definition 4.1.1
- Definition of pseudo-projectiveness = Definition 3.1.5
- Definition of pseudo-injectiveness = Definition 4.1.3
- Definition of covering family = Definition 5.2.1
- Definition of adequate covering family = Definition 5.2.7
- Definition of global covering family = Definition 5.3.3
- Definition of adequate global covering family = Definition 5.3.4
- Theorem 2 = Theorem 7.0.4 with $G_0 = \varinjlim G$
- Theorem 6 = Lemma 3.1.2 + Lemma 3.1.7 + Proposition 3.1.10 ($\mathcal{C}$ must be bounded below)
- Theorem 7 = Theorem 3.2.3 ($\mathcal{C}$ must be bounded below)
- Theorem 13 = Lemma 4.1.2 + Lemma 4.1.7 + Proposition 4.1.10 ($\mathcal{C}$ must be bounded above)
- Theorem 14 = Theorem 4.2.3 ($\mathcal{C}$ must be bounded above)
- Proposition 8 = Proposition 2.0.4 + Proposition 2.0.5 (also in Chapter 2 is built the dual spectral sequence with target $\varprojlim H$)
- Proposition 18 = Proposition 5.2.8 ($\mathcal{C}$ must be bounded above).
• Example 10 = Example 7.2.1
• Examples 16 and 17 = come from Section 5.4.
• Simplex-like posets = Definition 1.4.1 + Section 5.4 (relation with covering families).
• Graded posets = Section 1.3 + Section 1.5 (realize all the CW-homotopy types).
• Locally finite groups = Example 7.0.10.
• Properties of functors $F_0, F_1, F_2...$ = Proposition 5.2.8 + Section 5.2.2 + Lemma 5.3.1.
• Whitehead’s Theorem = Example 7.0.6.
Resumen en castellano

Introducción.

Este trabajo tiene su origen en un problema relacionado con grupos \( p \)-locales finitos \([10]\). Estos objetos algebraicos son triples \((S, \mathcal{F}, \mathcal{L})\) donde \(S\) es un \( p \)-grupo finito y \(\mathcal{F}\) y \(\mathcal{L}\) son categorías que codifican la información de fusión entre los subgrupos de \(S\). De hecho, contienen los datos necesarios para construir espacios topológicos que se comportan como \( p \)-completaciones de espacios clasificadores de grupos finitos \(BG^\wedge_p\). Mientras que un grupo finito da lugar a un grupo \( p \)-local finito de una forma natural, hay grupos \( p \)-locales finitos que no surgen de este modo. Son los llamados grupos \( p \)-locales finitos exóticos, de los cuales se han encontrado ya varias familias \([10], [26], [36]\) y \([14]\).

El profesor A. Viruel formuló hace varios años la cuestión de si estos grupos \( p \)-locales finitos exóticos provienen o no de un grupo infinito. El trabajo de Aschbacher y Chermak muestra que esto ocurre en algunos de los grupos finitos 2-locales de Solomon \([26]\).

Un enfoque para encontrar candidatos a este grupo infinito es usar la descomposición por normalizadores \([27]\) para un grupo \( p \)-local finito \((S, \mathcal{F}, \mathcal{L})\). Esta descomposición da una descripción del nervio \(|\mathcal{L}|\) como un colínite homotópico de espacios del tipo de homotopía de espacios clasificadores de grupos finitos:

\[
\text{hocolim}_{\bar{sdC}} \delta_C \simeq |\mathcal{L}|.
\]

Aquí, \(\bar{sdC}\) es cierto poset construido a partir del grupo \( p \)-local finito \((S, \mathcal{F}, \mathcal{L})\) y \(\delta_C : \bar{sdC} \to \text{Top}\) es un funtor que satisface

\[
\delta_C(\sigma) \simeq B \text{Aut}_\mathcal{L}(\sigma),
\]

para todo \(\sigma \in \bar{sdC}\) (donde \(B : \text{Grp} \to \text{Top}\) asocia funtorialmente a cada grupo discreto un espacio clasificador). Si este colínite homotópico tiene el tipo de homotopía de un espacio de Eilenberg-MacLane

\[
\text{hocolim}_{\bar{sdC}} \delta_C \simeq B\pi,
\]

entonces \(\pi\) es un candidato a grupo infinito del que proviene el grupo \( p \)-local finito original.

Así que el problema del que partió este trabajo es si dado un diagrama de grupos \(G : \mathcal{C} \to \text{Grp}\) y un cono \(\tau : G \Rightarrow \pi\) la aplicación natural

\[
\text{hocolim}_\mathcal{C} BG \to B\pi
\]
es o no una equivalencia homotópica. Con este propósito estudiamos la fibra $F$ de esta aplicación (Capítulo 7). Bajo hipótesis débiles $F$ es simplemente conexa y su homología está dada por los límites directos superiores $\lim_{\to}j H$, donde $H : C \to Ab$.

Hemos estudiado entonces condiciones para que estos límites derivados se anulen y herramientas para su cómputo (Capítulo 3), así como los resultados duales para límites inversos superiores $\lim_{\leftarrow}j H$ (Capítulo 4). Estos resultados están fundamentados en una sucesión espectral (Capítulo 2), y tienen aplicaciones al cálculo de cohomología entera de categorías (Capítulo 5), como el caso de la conjetura de Webb (Capítulo 6).

Una versión más detallada del contenido es la siguiente:

- **Capítulo 1: Notación y preliminares.** Se introducen las notaciones que se usarán así como las definiciones adecuadas de funtores derivados y los teoremas de normalización para grupos abelianos simpliciales. Así mismo, se introducen los posets graduados, que son las categorías sobre las que se desarrollan los resultados principales de este trabajo. Este tipo de categorías ha sido elegido ya que es el prototipo de la categoría $sdC$ usada en la descomposición antes mencionada [27]. Además incluye a los complejos simpliciales (Sección 1.4) y a las categorías de subdivisión [28], y contiene toda la información homotópica de los $CW$-complejos (Sección 1.5).

- **Capítulo 2: Un sucesión espectral.** Se construye a partir de módulos diferenciales graduados una sucesión espectral asociada a un funtor $H : P \to Ab$, donde $P$ es un poset graduado. El límite de esta sucesión espectral es los límites superiores $\lim_{\to}j H$. Análogamente se construye otra sucesión espectral que converge a los límites inversos superiores $\lim_{\leftarrow}j H$.

- **Capítulo 3: Límites directos superiores.** Se caracterizan los objetos proyectivos en la categoría abeliana $\text{Ab}^P$ donde $P$ es un poset graduado. Además, gracias a la sucesión espectral del Capítulo 2, se encuentra otra familia de objetos de $\text{Ab}^P$ cuyos límites directos superiores se anulan. Finalmente se muestran aplicaciones de estos resultados.

- **Capítulo 4: Límites inversos superiores.** Es la dualización del Capítulo 3, en el cual se tratan los objetos inyectivos de $\text{Ab}^P$ y otra familia de funtores cuyos límites inversos superiores se anulan. Como aplicación se desarrollan herramientas para el cálculo de cohomología de categorías en el Capítulo 5.

- **Capítulo 5: Cohomología.** A través de una estructura adicional sobre el poset graduado $P$ se establece que dicha categoría es acíclica si y sólo si se verifica cierta ecuación entera en la que están involucrados elementos geométricos de $P$.

- **Capítulo 6: Aplicación: la conjetura de Webb.** Se usan los resultados del capítulo anterior para probar la parte cohomológica de la conjetura de Webb, que ha sido demostrada en su máxima generalidad en [28]. Esta conjetura afirma que el espacio de órbitas del poset de los $p$-subgrupos de un grupo finito $G$ es contráctil.
• **Capítulo 7: Aplicación: colímite homotópico.** Aunque el estudio de colímites homotópicos es el origen de este trabajo, aparece aquí como aplicación de los capítulos anteriores. Se prueba el Teorema de Whitehead, el cual afirma que el pushout de espacios de Eilenberg-MacLane con aplicaciones inyectivas es un espacio de Eilenberg-MacLane, y se da un ejemplo de cálculo explícito de la fibra $F$. También se demuestra el hecho conocido de que el espacio clasificador en un grupo localmente finito es el colímite homotópico de los espacios clasificadores de sus subgrupos finitos.

Finalmente, comentamos otras aplicaciones donde esperamos que la teoría pueda contribuir con algunos resultados:

- Trasladar al diagrama inicial de grupos $G: C \to Grp$ las condiciones sobre $H$ que implican que $\lim_{\longrightarrow} H = 0$ para $j > 0$. De esta forma se obtendría una versión generalizada del Teorema de Whitehead para diagramas más grandes que el pushout. Estudiar la relación con el Teorema del cubo de Mather.

- Habiendo encontrado ya condiciones que implican que el funtor $\delta_C: \mathcal{sd}C \to Top$ factoriza como

$$
\begin{array}{ccc}
\mathcal{sd}C & \xrightarrow{\delta_C} & Top \\
\downarrow & & \downarrow \\
Grp & \xrightarrow{B} & Top
\end{array}
$$

determinar cuando $|\mathcal{L}|$ es un espacio de Eilenberg-MacLane o calcular la fibra $F$

$$
F \to |\mathcal{L}| \to B\pi
$$

para un grupo $\pi$ adecuado en casos favorables. Por ejemplo, se puede demostrar que el espacio clasificador sin completar $|\mathcal{L}|$ de todos los grupos $p$-locales finitos de $p$-rango 2 con $p$ impar (descritos en [14]) es un espacio de Eilenberg-MacLane.

- Un posible enfoque de la conjetura de Quillen [34], la cual afirma que el poset de los $p$-subgrupos no triviales de un grupo finito $G$ (complejo de Brown [11]) es contráctil si y sólo si $O_p(G) \neq 1$ (donde $O_p(G)$ es el mayor $p$-subgrupo normal de $G$). Una formulación más fuerte de esta conjetura es que la cohomología entera reducida de este poset es trivial si y sólo si $O_p(G) \neq 1$ (esta formulación es usada, por ejemplo, en [3]).

Gracias a los resultados del Capítulo 5 esta formulación más fuerte es consecuencia de encontrar para cada grupo finito $G$ un subconjunto adecuado $K(G)$ del complejo de Brown’s que satisface la ecuación entera

$$
|K(G)| = 1
$$

exactamente cuando $O_p(G) \neq 1$. Por resultados de Bouc [7] y de Thévenaz y Webb [41] también podemos trabajar con el poset de los subgrupos $p$-radicales no triviales o con el poset de los $p$-subgrupos elementales abelianos no triviales.
Capítulo 1. Notación y preliminares.

1.1. Categorías. En esta sección se enumeran las categorías que se usarán a lo largo del trabajo, y que son las siguientes: Set (conjuntos), Grp (grupos), Ab (grupos abelianos), Top (espacios topológicos), SSet (conjuntos simpliciales) y Cat (categoría cuyos objetos son las categorías pequeñas y cuyas flechas son los funtores entre ellas). Los objetos y flechas de una categoría $C$ se denotan por $\text{Ob}(C)$ y $\text{Hom}(C)$ respectivamente.

Después se introducen los conceptos de categoría conexa así como las sobre-categorías $(S \downarrow c_0)$ y bajo-categorías $(c_0 \downarrow S)$ dado un funtor $S : C \to D$ y un objeto $c$ de $C$. Particularizando a $S = 1_C : C \to C$ se definen las categorías de objetos sobre $(C \downarrow c_0)$ y bajo $(c_0 \downarrow C)$ un objeto dado $c$ de la categoría $C$.

La principal referencia para esta sección es [29].

1.2. Grupos y grupos abelianos. Tras introducir las notaciones multiplicativas y aditivas para Grp y Ab respectivamente, se fijan las notaciones para subgrupo generado, producto directo y suma directa (coproducto) en la categoría de grupos abelianos Ab. Así mismo se introducen ciertos grupos abelianos (infinito cíclico $\mathbb{Z}$, finito cíclico $\mathbb{Z}_n$, racionales $\mathbb{Q}$, $\mathbb{Z}[p^\infty]$ con $p$ primo) y algunas definiciones relativas a homomorfismos en Ab.

Seguidamente se introducen tanto los funtores $F : C \to \text{Grp}$ como las transformaciones naturales entre ellos que se comportan de manera inyectiva o sobreyectiva sobre cada flecha o sobre cada objeto respectivamente. De la referencia [22] se extrae algunos hechos básicos de Ab, incluyendo la descripción exacta de sus objetos proyectivos e inyectivos.

Finalmente se introducen las aplicaciones $f : A \to B$ de Ab donde $A$ y $B$ son libres, y tales que el conúcleo $B/f(A)$ es libre. Una aplicación inyectiva de este tipo es un isomorfismo cuando $A$ y $B$ tienen el mismo rango finito ($\text{rk} A = \text{rk} B$).

1.3. Posets graduados. En esta sección se introduce el tipo de categoría sobre el que se establecerán los principales resultados de este trabajo. No son sino ciertos conjuntos parcialmente ordenados (posets) para los que existe una función de grado $\text{deg} : \text{Ob}(P) \to \mathbb{Z}$ cuyos valores sobre objetos que se preceden se diferencian en una unidad.

Tras varios ejemplos y algunos hechos básicos se prueba que la función de grado se puede extender a los morfismos del poset graduado, y se definen los subconjuntos $\text{Ob}_S(C)$ y $\text{Hom}_S(C)$ como los objetos y flechas, respectivamente, cuyo grado pertenece a un subconjunto fijado $S \subseteq \mathbb{Z}$.

Acotación en posets graduados. En esta subsección se definen los posets graduados $\mathcal{P}$ cuya función de grado $\text{deg} : \text{Ob}(\mathcal{P}) \to \mathbb{Z}$ tiene imagen acotada superior o inferiormente o ambas. Varios ejemplos ilustran los diferentes casos, así como la no acotación.

Grafos y posets graduados. Se introduce el grafo (dirigido) asociado a un poset graduado $\mathcal{P}$ como aquél que tiene los mismos objetos y tiene por aristas las flechas de grado 1 de $\mathcal{P}$, es decir, las flechas que corresponden a dos elementos que se preceden.
en $P$. A partir de aquí se introducen los conceptos de árbol y árbol maximal para posets graduados.

1.4. Complejos simpliciales. Después de definir los objetos geométricos que constituyen los complejos simpliciales, se los caracteriza como aquellos posets que localmente son simplex-like, y tales que cada par de elementos que tienen una cota inferior poseen un ínfimo $[18]$. Además, se muestra que todo conjunto simplicial puede ser visto como un poset graduado.

1.5. Realización y tipo de homotopía. En este punto se ha visto que tenemos las inclusiones

$$
\text{Conjuntos simpliciales} \subseteq \text{Posets graduados} \subseteq \text{Categorías.}
$$

Usando que todo espacio topológico tiene el tipo de homotopía (débil) de un CW-complejo $[43]$, los cuales a su vez tienen el tipo de homotopía de un complejo simplicial $[23]$, y usando también la subdivisión baricéntrica $[38]$ se llega a

**Proposición.** Para cualquier espacio $X \in \text{Top}$ existe un poset graduado $P$ y una equivalencia de homotopía débil:

$$
X \simeq |NP|.
$$

Esto significa que la restricción a posets graduados no implica pérdida alguna desde el punto de vista homotópico.

1.6. Funtores derivados del límite directo e inverso. En esta sección se dan definiciones para los funtores derivados izquierdos $\lim_{\to}$ del funtor límite directo $\lim : \text{Ab}^C \to \mathcal{C}$ y para los funtores derivados derechos $\lim_{\leftarrow}$ del límite inverso $\lim_{\leftarrow} : \text{Ab}^C \to \mathcal{C}$, donde $\mathcal{C}$ es una categoría pequeña.

Sabemos $[30]$ IX, Proposition 3.1] que $\text{Ab}^C$ es una categoría abeliana, y que $\lim_{\to}$ y $\lim_{\leftarrow}$ son exactos por la derecha y por la izquierda respectivamente porque son los adjuntos por la izquierda y por la derecha del funtor $\text{Ab} \to \text{Ab}^C$ definido por $A \mapsto c_A$, donde $c_A : \mathcal{C} \to \text{Ab}$ es el funtor de valor constante $A$.

En vez de definir los funtores derivados a través de resoluciones proyectivas e inyectivas se definen como la homología (cohomología) de ciertos complejos de cadena (cocadena), basándose en $[17]$ Appendix II.3, $[8]$ XII.5.5] y $[8]$ XI.6.2].

1.7. Teorema de Normalización. En esta sección se enuncian los Teoremas de Normalización para grupos abelianos simpliciales y cosimpliciales, los cuales se pueden encontrar en $[31]$ 22.1, 22.3] y $[19]$ VIII.1] respectivamente. Estos teoremas establecen que la homología (cohomología) de un grupo abeliano simplicial (cosímplexial) se puede calcular omitiendo los simples (cosímplices) degenerados.
Capítulo 2. Una sucesión espectral.

En este capítulo se introducen un par de sucesiones espectrales que tienen por límites \( \lim_i F \) y \( \lim' F \), donde \( F : C \to \text{Ab} \) es un funtor y \( C \) es un poset graduado, y se dan condiciones de convergencia (débil). La construcción se hace a partir de módulos diferenciales graduados [32].

Estas sucesiones espectrales son el pilar fundamental en los desarrollos posteriores que se realizan en este trabajo. Las páginas iniciales se calculan como homología (cohomología) de subgrupos abelianos simpliciales (cosimpliciales) cuyos simplices (cosímplexes) son cadenas que empiezan y acaban con objetos de \( C \) en los cuales la función de grado toma valores fijos.

2.1. Ejemplos. En esta sección se describen numerosos ejemplos para ilustrar la construcción de las sucesiones espectrales en casos concretos. En particular se trabaja sobre las siguientes categorías:

- **pushout** \( a \xrightarrow{f} b \xleftarrow{g} c \),
- **pullback** \( a \xrightarrow{f} b \xleftarrow{g} c \),
- **telescopio** \( a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \xrightarrow{f_3} a_3 \xrightarrow{f_4} a_4 \ldots \)
- **ciclo** \( a \xrightarrow{f} b \xleftarrow{g} c \xrightarrow{h} d \).

Algunos de los comportamientos particulares de las sucesiones espectrales que se observan en estos ejemplos se generalizan en capítulos posteriores a teoremas de ámbito general.

Capítulo 3. Límites directos superiores.

En este capítulo se encuentran condiciones suficientes para que los límites derivados superiores de un funtor dado \( F : C \to \text{Ab} \) se anulen, es decir, para que \( \lim_i F = 0 \) si \( i \geq 1 \). A estos funtores se les llamará funtores \( \lim \)-áciclicos. Así mismo se desarrollan ciertas herramientas para el cálculo de estos límites superiores en el caso general, y se encuentran aplicaciones basándose en una analogía con el concepto de flujo en un grafo.

El desarrollo es como sigue: en la Sección 3.1 se caracterizan los objetos proyectivos de la categoría abeliana \( \text{Ab}^\mathcal{P} \), donde \( \mathcal{P} \) es un poset graduado. Recordemos que los funtores proyectivos son \( \lim \)-áciclicos. A partir de esta caracterización se deduce una condición más débil, \( \text{pseudoproyectividad} \), la cual también implica \( \lim \)-aciclicidad. Este hecho se prueba en la Sección 3.2 mediante el uso de las sucesiones espectrales del Capítulo 2.
En la Sección 3.3 se construye un functor pseudo-proyectivo asociado a un functor arbitrario $F : \mathcal{P} \to \text{Ab}$. Mediante su uso y un argumento clásico de “cambio de dimensiones” se demuestra como reducir el cálculo de un límite superior $\lim_i F$ al cálculo de un límite superior de orden 1 $\lim_{i \to 1} G$, donde $G$ es un functor que se construye a partir de $F$.

Por último, en la Sección 3.4, se aplican los resultados de la sección anterior al caso en el que cierta subcategoría, denotada core($\mathcal{P}$), de la categoría $\mathcal{P}$ es un árbol. Todo el capítulo está ilustrado con ejemplos en los se concretan las ideas teóricas desarrolladas.

3.1. Objetos proyectivos de $\text{Ab}^\mathcal{P}$. Dado un functor $F : \mathcal{P} \to \text{Ab}$, donde $\mathcal{P}$ es un poset graduado, podemos considerar, para cada objeto $i \in \text{Ob}(\mathcal{P})$, el cociente de $F(i)$ por las imágenes $F(\alpha)$ donde $\alpha \in \text{Hom}(\mathcal{P})$ es un morfismo no trivial que termina en $i$. A este cociente lo denotamos Coker$(i)$.

Pues bien, en esta sección se demuestra que si $F$ es proyectivo en $\text{Ab}^\mathcal{P}$ entonces:

- Coker$(i)$ es proyectivo en $\text{Ab}$ para todo $i \in \text{Ob}(\mathcal{P})$.
- $F$ verifica una condición técnica adicional que llamamos pseudo-proyectividad y la cual, en casos extremos, se reduce a la inyectividad de los morfismos $F(\alpha)$, $\alpha \in \text{Hom}(\mathcal{P})$.

Posteriormente se demuestra que si el poset graduado $\mathcal{P}$ es acotado inferiormente entonces estas dos condiciones mencionadas arriba implican la proyectividad de $F$, con lo cual se obtiene una caracterización completa de los proyectivos de $\text{Ab}^\mathcal{P}$.

3.2. Pseudo-proyectividad. Recuérdese que la proyectividad de un objeto implica que los funtores derivados izquierdos se anulan sobre él, en particular los objetos proyectivos de $\text{Ab}^\mathcal{P}$ son $\lim_{i \to}$-acíclicos. Teniendo en cuenta que la pseudo-proyectividad es más débil que la proyectividad en $\text{Ab}^\mathcal{P}$ (ver Sección 3.1), es natural preguntarse si esa condición implica por sí sola $\lim_{i \to}$-aciclicidad.

Esta sección se dedica exclusivamente a probar este hecho, que la pseudo-proyectividad implica $\lim_{i \to}$-aciclicidad. Para ello se hace un uso intensivo de las sucesiones espectrales del Capítulo 2. Se asume la hipótesis adicional de que $\mathcal{P}$ es acotado inferiormente, lo cual permite hallar una página de la sucesión spectral $E_r$, para $r$ suficientemente grande, en la cual probar que las contribuciones al límite son cero.

3.3. Calculando límites superiores. En esta sección se asocia a cada functor $F : \mathcal{P} \to \text{Ab}$ un functor pseudo-proyectivo $F'$ y una transformación natural $F' \Rightarrow F$. Este functor tiene varias propiedades interesantes, entre las que destaca ser pseudo-proyectivo y por tanto, gracias a la Sección 3.2, $\lim_{i \to}$-acíclico.

La sucesión exacta corta

$0 \Rightarrow K_F \Rightarrow F' \Rightarrow F \Rightarrow 0$

nos da $\lim_{i \to} F = \lim_{i \to} K_F$ para $i > 1$. La iteración de este argumento reduce el cálculo de todos los límites superiores de $F$ al cálculo de límites de orden 1 sobre sucesivos
funtores. Como aplicación se obtiene que un funtor (con morfismos inyectivos) sobre un árbol es \( \lim\rightarrow \)-acíclico.

3.4. \( \lim_{\rightarrow 1} \) como un problema de flujo. Los problemas de flujo son un tópico en la Teoría de Grafos \([6]\). Un flujo no es más que la asignación de valores a las aristas de un grafo de forma que en cada vértice el flujo entrante iguala al saliente. Mientras que clásicamente estos valores son números naturales, un sencillo argumento permite describir los 1-ciclos (del complejo de cadena cuya homología en grado 1 es \( \lim_{\rightarrow 1} F \)) como “flujos” cuyo valor en la arista \( \alpha : i_0 \to i_1 \) es un elemento de \( F(i_0) \).

 Esto da inmediatamente que \( \lim_{\rightarrow 1} F = 0 \), es decir, todo 1-ciclo es 1-borde, si y sólo si cada “flujo” se escribe como suma de ciertos flujos triviales minimales. Posteriormente, la relativa rigidez de la estructura de un poset graduado permite re-escribir cada “flujo” como un “flujo” sobre cierta subcategoría de \( \mathcal{P} \) denotada \( \text{core}(\mathcal{P}) \).

 Finalmente, se obtiene que la aciclicidad de \( \text{core}(\mathcal{P}) \), es decir, que esta subcategoría sea un árbol, junto con la inyectividad de \( F \) implican la \( \lim\rightarrow \)-aciclicidad de este último.

Capítulo 4. Límites inversos superiores.

Este capítulo es la dualización del Capítulo 3 y su desarrollo es como sigue:

- **4.1** Se caracterizan los objetos inyectivos de \( \text{Ab}^\mathcal{P} \).
- **4.2** Se prueba, mediante el uso de las sucesiones espectrales del Capítulo 2, que la pseudo-inyectividad implica \( \lim_{\to 1} \)-aciclicidad.
- **4.3** Mediante sucesiones exactas cortas se reducen el cálculo de \( \lim_{\leftrightarrow 1} F \) a límites \( \lim_{\to 1} G \), donde \( G \) es un funtor que se construye a partir de \( F \).

 Como aplicación se han desarrollado herramientas para el cálculo de cohomología de categorías y, debido a su extensión, esto constituye el Capítulo 5

Capítulo 5. Cohomología

En esta sección, y como aplicación de la teoría desarrollada en el capítulo anterior, de desarrollan herramientas para el cálculo de la cohomología \( H^*(\mathcal{P}; \mathbb{Z}) \) de un poset graduado \( \mathcal{P} \), bajo ciertas hipótesis estructurales adicionales sobre \( \mathcal{P} \).

Los conceptos introducidos a lo largo de las distintas secciones son de naturaleza local, en el sentido de que dependen de las subcategorías \( (i \downarrow \mathcal{P}) \) y la restricciones \( F_{(i\downarrow\mathcal{P})} \) donde \( i \) es un objeto arbitrario de \( \mathcal{P} \).

Inicialmente se asume que \( F \) es un funtor que toma por valores grupos abelianos libres, para especializarse posteriormente al caso en que \( F = c_\mathbb{Z} \) y de este modo calcular la cohomología de \( \mathcal{P} \). La última sección describe una familia de posets graduados, los \emph{simplex-like} posets, a la cual pertenecen los complejos simpliciales, y cuyos miembros verifican las hipótesis estructurales adicionales.

La idea que guía todo el capítulo es la siguiente: en el capítulo anterior se construyó una sucesión exacta corta

\[
0 \Rightarrow F \Rightarrow F' \Rightarrow C_F \Rightarrow 0,
\]
donde $F'$ es un funtor $\lim$-acíclico, con objeto de reducir el cálculo de los límites inversos superiores de $\overline{F}$ a límites de orden inferior sobre otro funtor. Los funtores $F'$ y $C_F$ puede ser “muy grandes” (en el sentido de cantidad de generadores), y con objeto de disminuir este tamaño se considera otro funtor Ker$_F$ “más pequeño”, y se buscan condiciones bajo las cuales podamos construir sucesiones exactas cortas del tipo

$$0 \Rightarrow F \Rightarrow \text{Ker}'_F \Rightarrow G \Rightarrow 0$$

iterativamente.

5.1. **Funciones $p$-condensados.** En esta sección se considera el funtor Ker$_F$ asociado al funtor $F : \mathcal{P} \rightarrow \text{Ab}$, y cuyo valor en $i \in \text{Ob}(\mathcal{P})$ es la intersección de los morfismos no triviales que salen de $i$. Fijado el entero $p$, diremos que $F$ es $p$-condensado si

- $F(i) = 0$ si $\text{deg}(i) < p$, y
- Ker$_F(i) = 0$ si $\text{deg}(i) > p$.

Como veremos, esta condición significa que la información del funtor $F$ está contenida de alguna manera en sus objetos de grado $p$. Técnicamente nos permite construir la sucesión exacta corta deseada:

$$0 \Rightarrow F \Rightarrow \text{Ker}'_F \Rightarrow G \Rightarrow 0.$$

La sección termina dando una caracterización de cuándo el funtor $G$ de la sucesión exacta corta de arriba es $(p + 1)$-condensado. Esto es importante ya que el objetivo final del capítulo es hacer esta construcción iterativamente. Además, esta caracterización se usa en la siguiente sección para reducir el problema de cuándo es $G$ $(p + 1)$-condensado a un conjunto de ecuaciones enteras con coeficientes que dependen de la estructura local de $\mathcal{P}$.

5.2. **Familias recubridoras.** Dado un poset graduado $\mathcal{P}$, una familia recubridora $\mathcal{J}$ del mismo es una familia de subconjuntos $J^i_p \subseteq (i \downarrow \mathcal{P})_p$, donde $i$ recorre los objetos de $\mathcal{P}$ y $p$ recorre los grados $p \leq \text{deg}(i)$, sometida a ciertas restricciones combinatorias. Como se ve la existencia o no de una familia recubridora sólo depende de la estructura local de $\mathcal{P}$. Esto permite (ver Sección 5.4) construir familias recubridoras para posets que tiene una estructura local homogénea y adecuada, como son los simplex-like posets.

Un funtor $F : \mathcal{P} \rightarrow \text{Ab}$ que toma por valores grupos abelianos libres, se dirá que es $\mathcal{J}$-determinado, donde $\mathcal{J}$ es una familia recubridora para $\mathcal{P}$, si la información de $F$ está contenida en los objetos de los subconjuntos de $\mathcal{J}$. Así se llega a la

**Proposición [5.2.3].** Sea $\mathcal{P}$ un poset graduado y $\mathcal{J}$ una familia recubridora para $\mathcal{P}$. Si el funtor $F$ es $p$-condensado y $\mathcal{J}$-determinado entonces el funtor $G$ de la sucesión exacta corta

$$0 \Rightarrow F \Rightarrow \text{Ker}'_F \Rightarrow G \Rightarrow 0$$

es $(p + 1)$-condensado y $\mathcal{J}$-determinado si se verifican un conjunto de ecuaciones enteras.
Este teorema se puede aplicar iterativamente sin más que comprobar si ciertas ecuaciones enteras se verifican.

**Familias recubridoras adecuadas.** Esta subsección se centra en el caso $F = c_Z$, con objeto de calcular la cohomología $H^*(\mathcal{P}; \mathbb{Z})$. Si $\mathcal{J}$ es una familia recubridora para el poset graduado $\mathcal{P}$, diremos que es adecuada si podemos aplicar el Teorema 5.2.3 iterativamente, es decir, si las ecuaciones enteras involucradas se verifican.

La condición de adecuación de $\mathcal{J}$ se define, de nuevo, mediante ciertas ecuaciones enteras que dependen de la estructura local de $\mathcal{P}$. En el caso de que $\mathcal{J}$ sea adecuada se obtiene una sucesión de fundores $F_0 = c_Z, F_1, F_2,...$ cada uno de los cuales encaja en una sucesión exacta corta

$$0 \Rightarrow F_p \Rightarrow \ker F_p \Rightarrow F_{p+1} \Rightarrow 0.$$

**Bases y morfismos locales.** En esta subsección se dan descripciones explícitas de bases para los grupos abelianos libres que toman como valores los fundores $F_0, F_1, F_2,...$ construidos en la subsección anterior. Así mismo se dan descripciones explícitas de los morfismos $F_p(\alpha)$, donde $\alpha \in \text{Hom}(\mathcal{P})$. Una vez más, estas descripciones dependen de la estructura local de $\mathcal{P}$ (y de la familia $\mathcal{J}$ considerada).

### 5.3. Comportamiento global

Centrándose aún en el caso $F = c_Z$, y considerando la sucesión de fundores $F_0, F_1, F_2,...$ descrita en la Subsección 5.2.1, se describen en esta sección propiedades de los límites superiores de $F$ que no dependen de la familia recubridora $\mathcal{J}$ elegida para construir la mencionada sucesión de fundores (en la subsección 5.2.2 se describieron propiedades que sí dependen de $\mathcal{J}$).

En particular, se prueba un hecho análogo a que la cohomología en grado $n$ de un CW-complejo sólo depende del $(n+1)$-esqueleto, y una fórmula para la característica de Euler, que se reduce a la conocida suma alternada del número de objetos de cada grado en caso de que $\mathcal{P}$ sea un simplex-like poset (ver Sección 5.4).

**Familias recubridoras globales.** En esta subsección se encuentra el teorema principal de este capítulo. Se comienza definiendo el concepto de familia recubridora global $\mathcal{K}$, que a diferencia de su versión local, es decir, de familia recubridora, es una familia de subconjuntos $K_p \subseteq \text{Ob}_p(\mathcal{P})$ donde $p$ recorre los grados posibles del poset graduado $\mathcal{P}$. Obtenemos entonces el

**Teorema 5.3.6.** Sea $\mathcal{P}$ un poset graduado para el que existe una familia recubridora adecuada $\mathcal{J}$ y una familia recubridora global adecuada $\mathcal{K}$. Entonces $\mathcal{P}$ es acíclico, es decir, $H^i(\mathcal{P}; \mathbb{Z}) = 0$ para $i > 0$, si y sólo si $|K_0|$ iguala el número de componentes conexas de $\mathcal{P}$.

Como se ve, se ha reducido un problema de Álgebra Homológica, la aciclicidad de $\mathcal{P}$, a una ecuación entera que involucra elementos geométricos de $\mathcal{P}$.

### 5.4. Simplex-like posets

Un simplex-like poset no es más que un poset graduado $\mathcal{P}$ cuyos subcategorías locales $(i \downarrow \mathcal{P}^{op})$ son isomorfas al poset de subdivisión del poset lineal $0 < 1 < ... < \text{deg}(i)$. Por ejemplo, para $\text{deg}(i) = 2$ la categoría
(i \downarrow \mathcal{P}^{op}) tendría el aspecto:

\[ \begin{array}{c}
\text{\scalebox{0.5}{\includegraphics{diagram.png}}}
\end{array} \]

Si \mathcal{P} es simplex-like entonces, como se prueba en esta sección, existe una familia recubridora adecuada para \mathcal{P}^{op}, lo cual se usará en el Capítulo 6. La sección acaba reenunciando los resultados de la Sección 5.3 en el caso de simplex-like posets.

Capítulo 6. Aplicación: la conjetura de Webb.

Denotemos por \( S_p(G) \) el complejo de Brown para el primo \( p \), que fue introducido en [11]. Webb conjeturó que el espacio de órbitas \( S_p(G)/G \) es contráctil (como espacio topológico), lo cual fue probado por Symonds en [40], extendido a bloques por Barker [4, 5] y extendido a sistemas de fusión (saturados) arbitrarios por Linckelmann [28].

Los trabajos de Symonds y Linckelmann prueban la contractibilidad del espacio de órbitas mostrando que es simplemente conexo y acíclico, e invocando el Teorema de Whitehead. Ambas pruebas de aciclicidad trabajan con el subposet de las cadenas normales. Symonds usa los resultados de Thévenaz y Webb [41] sobre que el subposet de las cadenas normales es \( G \)-equivalente al complejo de Quillen. Linckelmann prueba por su cuenta que el espacio de órbitas y el espacio de órbitas sobre las cadenas normales tienen la misma cohomología entera [28, Theorem 4.7].

En este capítulo se aplican los resultados del Capítulo 5 para dar una prueba alternativa de que el espacio de órbitas sobre las cadenas normales es acíclico. Para ello, se considera un sistema de fusión saturado \((S, \mathcal{F})\) y el espacio de órbitas sobre las cadenas normales correspondiente, denotado \([S_{\triangle}(\mathcal{F})]\). Fácilmente se comprueba que este poset es un simplex-like poset, por lo cual (ver Sección 5.4) existe una familia recubridora adecuada para \([S_{\triangle}(\mathcal{F})]^{op}\).

Para construir una familia global adecuada \( \mathcal{K} \) para \([S_{\triangle}(\mathcal{F})]^{op}\) se usa el mismo emparejamiento que usa Linckelmann [28, Definition 4.7]. Del hecho de que el único subgrupo del \( p \)-grupo \( S \) que iguala a su normalizador es el propio \( S \) se deduce que \( |K_0| = 1 \). Esto, junto con que el poset \([S_{\triangle}(\mathcal{F})]^{op}\) es conexo, nos da, gracias al Teorema 5.3.6, la aciclicidad de este poset.

La prueba de que la familia \( \mathcal{K} \) definida a través del emparejamiento es una familia recubridora global adecuada es bastante técnica, y se postpone a la Subsección 6.1.

6.1. \( \mathcal{K} \) es una familia recubridora global adecuada. El subconjunto \( K_n \) se define como aquellas clases de isomorfismo de cadenas normales \([Q_0 < \ldots < Q_n]\) que tienen como normalizador al propio \( Q_n \) para cualquier representante \( Q_0 < \ldots < Q_n \). Los detalles técnicos usan profusamente los resultados de [10, Appendix], junto al hecho de que el sistema de fusión bajo consideración \((S, \mathcal{F})\) es saturado.
Capítulo 7. Aplicación: colímite homotópico.

En este capítulo se estudia el problema de cuando el colímite homotópico de un diagrama de espacios clasificadores de grupos coincide con el espacio clasificador del colímite de los grupos. Para ello, dado un diagrama de grupos $G: \mathcal{P} \to \text{Grp}$ y un cono $\tau: G \Rightarrow G_0$ se estudia la fibra $F$ de la aplicación

$$\text{hocolim } BG \to BG_0.$$ 

La relación con los desarrollos previos consiste en que la homología de $F$ se calcula como los límites directos derivados para cierto funtor $H: \mathcal{P} \to \text{Ab}$. Este hecho es el teorema principal de este capítulo, y su prueba se postpone a la Sección 7.1.

El resto del capítulo se dedica a ejemplos de aplicación del teorema, como son el Teorema de Whitehead sobre el pushout y los siguientes:

- Si $G_0$ es un grupo localmente finito entonces se tiene que
  $$\text{hocolim}_{G \subseteq G_0, G \text{ finito}} BG \simeq BG_0.$$ 

- Para cualquier grupo $G_0$ tenemos
  $$\text{hocolim}_{G \subseteq G_0, G \text{ p-subgrupo finito normal}} BG \simeq B(\lim_{G \subseteq G_0, G \text{ p-subgrupo finito normal}} G).$$

### 7.1. Demostración del teorema

La demostración del teorema está basada en la descripción de la fibra $F$ como el colímite homotópico de las fibras sobre cada objeto [12]. También se usa la sucesión espectral de Van Kampen [39], la sucesión espectral en homología de Bousfield-Kan [8] y la sucesión larga de homotopía para fibraciones.

### 7.2. Otro ejemplo

En esta subsección final se aplican el teorema central de este capítulo junto con las herramientas desarrolladas para límites directos derivados a un ejemplo concreto propuesto por A. Libman. Se obtiene una fibración

$$\bigvee_{a \in G_0 \setminus \{1\}} (S^2)_a \to \text{hocolim } BG \to BG_0$$

donde el colímite homotópico se toma sobre cierto poset graduado de dimensión 2.
CHAPTER 1

Notation and Preliminaries

1.1. Categories

Throughout this work we use the following familiar categories:

• Set, the category of sets,
• Grp, the category of groups,
• Ab, the category of abelian groups,
• Top, the category of topological spaces with arrows continuous maps,
• SSet, the category of simplicial sets,
• Cat, the category with objects the small categories and with arrows the functors between them,

and the pointed versions SSet*, and Top*. Any other category is assumed to be a small category without explicit mention. The objects of a category \( C \) are denoted by \( \text{Ob}(C) \) and the arrows by \( \text{Hom}(C) \). If \( c, c' \in \text{Ob}(C) \) then \( \text{Hom}_C(c, c') \) denotes the arrows in \( C \) from \( c \) to \( c' \). Any functor \( F : C \to D \) is covariant if not stated otherwise.

If \( s, s' \in \text{Ob}(\text{Set}) \) and \( k \in s' \) then the constant function from \( s \) to \( s' \) of value \( k \) is denoted by \( c_k \). Now we recall some concepts in Category Theory (see [29]):

**Definition 1.1.1.** A category \( C \) is connected if for any two objects \( c, c' \in \text{Ob}(C) \) exists a chain of morphisms in \( C \) between \( c \) and \( c' \):

\[
    c \to c_1 \leftarrow c_2 \to \ldots c_{n-1} \to c_n \leftarrow c'
\]

**Definition 1.1.2.** Let \( S : D \to C \) be a functor and \( c_0 \in \text{Ob}(C) \). The category of objects \( S \)-under \( c_0 \), \( (c_0 \downarrow S) \), has objects all the pairs \( (f, d) \), where \( d \in \text{Ob}(D) \) and \( f \in \text{Hom}_C(c_0, S(d)) \), and arrows \( h : (f, d) \to (f', d') \) those arrows \( h : d \to d' \) in \( D \) for which \( S(h) \circ f = f' \).

**Definition 1.1.3.** Let \( S : D \to C \) be a category and \( c_0 \in \text{Ob}(C) \). The category of objects \( S \)-over \( c_0 \), \( (S \downarrow c_0) \), has objects all the pairs \( (f, d) \), where \( d \in \text{Ob}(D) \) and \( f \in \text{Hom}_C(S(d), c_0) \), and arrows \( h : (f, d) \to (f', d') \) those arrows \( h : d \to d' \) such that \( f' \circ S(h) = f \).

In particular, for the identity functor \( 1_C : C \to C \), we have the categories of objects under and over \( c_0 \in \text{Ob}(C) \) respectively:

- \( (c_0 \downarrow C) \overset{\text{def}}{=} (c_0 \downarrow 1_C) \) and
- \( (C \downarrow c_0) \overset{\text{def}}{=} (1_C \downarrow c_0) \).
1.2. Groups and abelian groups

For the product of two elements \( g, g' \in G \in \text{Grp} \) the multiplicative notation \( gg' \) is used, while for \( a, a' \in A \in \text{Ab} \) the additive notation \( a + a' \) is used. Another notations for the category \( \text{Ab} \) are:

- \( \sum_{i \in I} A_i \) is the subgroup of \( A \in \text{Ab} \) generated by the subgroups \( A_i \subseteq A \).
- \( \bigoplus_{i \in I} A_i \) denotes the direct sum (coproduct) of the abelian groups \( \{ A_i; \ i \in I \} \), and \( \prod_{i \in I} A_i \) denotes its direct product.
- \( \mathbb{Z} \): infinite cyclic group, \( \mathbb{Z}_n \): order \( n \) cyclic group, \( \mathbb{Q} \): rational numbers as additive group, \( \mathbb{Z}[\{ p \} \infty], p \text{ prime} \): the subgroup of \( \mathbb{Q}/\mathbb{Z} \) generated by the cosets \( 1/p^n + \mathbb{Z} \) for \( n \geq 0 \). Recall that \( \mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Z}[p \infty] \).
- \( \mathbb{Z} \times_n \mathbb{Z} \) denotes the homomorphism \( m \mapsto nm \) and \( \mathbb{Z} \rightarrow \mathbb{Z}/n \) is the canonical projection, where \( n \geq 0 \).
- If \( f : A \rightarrow B \) and \( g : C \rightarrow D \) are homomorphisms then \( f \oplus g : A \oplus C \rightarrow B \oplus D \) is given by \( a \oplus c \mapsto f(a) \oplus g(c) \) and, in case \( A = C \), \( f \times g : A \rightarrow B \oplus D = B \times D \) is given by \( a \mapsto f(a) \oplus g(a) = (f(a), g(a)) \). Finally, in case \( B = D \), \( f + g : A \oplus C \rightarrow B \) is defined by \( a \oplus c \mapsto (f + g)(a \oplus c) = f(a) + g(c) \).

Consider a functor \( F : \mathcal{C} \rightarrow \text{Grp} \) with the category of groups as target category. The functor \( F \) is called \( \text{monic} \) if \( F(f) \) is a monomorphism for each \( f \in \text{Hom}(\mathcal{C}) \). A natural transformation \( \tau : F \Rightarrow F' \) between functors \( F, F' : \mathcal{C} \rightarrow \text{Grp} \) is called \( \text{monic} \) if \( \tau_i \) is a monomorphism for each \( i \in \text{Ob}(\mathcal{C}) \).

Similarly we call a functor \( F : \mathcal{C} \rightarrow \text{Grp} \) \( \text{epic} \) if \( F(f) \) is an epimorphism for each \( f \in \text{Hom}(\mathcal{C}) \). A natural transformation \( \tau : F \Rightarrow F' \) between functors \( F, F' : \mathcal{C} \rightarrow \text{Grp} \) is called \( \text{epic} \) if \( \tau_i \) is an epimorphism for each \( i \in \text{Ob}(\mathcal{P}) \). Notice that, by the natural inclusion \( \text{Ab} \subseteq \text{Grp} \), we have defined the terms \( \text{monic} \) and \( \text{epic} \) also for functors \( \mathcal{C} \rightarrow \text{Ab} \) and natural transformations between them.

We collect a few basic facts about the category \( \text{Ab} \) of abelian groups:

1. The projective objects of \( \text{Ab} \) are the free abelian groups [22, III, Theorem 18].
2. The injective objects of \( \text{Ab} \) are direct sums of \( \mathbb{Q} \) and \( \mathbb{Z}[p \infty] \) for various primes \( p \) [22, III, Theorem 21].
3. Subgroups of free groups are free [22, II, Theorem 15].
4. Finitely generated torsion free groups are free [22, II, Theorem 16].

We are also interested in maps \( A \xrightarrow{f} B \) between free abelian groups which have free cokernel.

**Definition 1.2.1.** Let \( A \xrightarrow{f} B \) be a map between free abelian groups. We say that \( f \) is \( \text{pure} \) if \( \text{Coker}(f) \) is a free abelian group.
If \( A \cong \mathbb{Z}^n \) is a finitely generated free abelian group we call \( \text{rk}(A) \overset{\text{def}}{=} n \). We have the following property of pure maps, which will be used repeatedly in successive sections,

**Lemma 1.2.2.** Let \( f: A \to B \) be a map in \( \text{Ab} \) between free abelian groups of the same rank. If \( f \) is pure and injective then it is an isomorphism.

**Proof.** The short exact sequence of free abelian groups

\[
0 \to A \xrightarrow{f} B \to \text{Coker}(f) \to 0
\]

implies that

\[
\text{rk} A - \text{rk} B + \text{rk}(\text{Coker}(f)) = 0
\]

and thus,

\[
\text{rk}(\text{Coker}(f)) = 0
\]

and \( \text{Coker}(f) = 0 \). \( \Box \)

**Definition 1.2.3.** Let \( F: \mathcal{C} \to \text{Ab} \) be a functor. We say that \( F \) is free if \( F(i) \) is a free abelian group for each \( i \in \text{Ob}(\mathcal{C}) \).

Notice that this does not imply that \( F \) is a free object in \( \text{Ab}^\mathcal{C} \). For example, consider the category \( \mathcal{C} = \cdot \to \cdot \) and the functor \( F \in \text{Ab}^\mathcal{C} \) with values

\[
\mathbb{Z} \xrightarrow{0} \mathbb{Z}.
\]

The functor \( F \) is free (Definition 1.2.3). However, it is not projective by Corollary 3.1.12, and thus \( F \) is not a free object in \( \text{Ab}^\mathcal{C} \).

### 1.3. Graded posets

In this section we define a special kind of categories, graded posets, which are the main ingredient in most of the results of this work.

We begin defining posets:

**Definition 1.3.1.** A poset is a category \( \mathcal{P} \) in which, given objects \( p \) and \( p' \),

- there is at most one arrow \( p \to p' \), and
- if there are arrows \( p \to p' \) and \( p' \to p \) then \( p = p' \).

In any poset \( \mathcal{P} \) define a binary relation on its objects \( \text{Ob}(\mathcal{P}) \) with \( p \leq p' \) if and only if there is an arrow \( p \to p' \). Then \( \leq \) is reflexive, symmetric and transitive, i.e., \( (\text{Ob}(\mathcal{P}), \leq) \) is a partial order. Conversely any partial order determines a poset in which the arrows are exactly those ordered pairs \( (p, p') \) for which \( p \leq p' \).

It is worthwhile noticing that if \( \mathcal{P} \) is a poset and \( p_0 \in \mathcal{P} \) then the categories \( (\mathcal{P} \downarrow p_0) \) and \( (p_0 \downarrow \mathcal{P}) \) defined in Section 1.1 are exactly the full subcategories of \( \mathcal{P} \) with objects \( \{ p \mid \exists p_0 \to p \} \) and \( \{ p \mid \exists p \to p_0 \} \) respectively. We define also the categories \( (\mathcal{P} \downarrow p_0)_* \) and \( (p_0 \downarrow \mathcal{P})_* \), as the full subcategories of \( \mathcal{P} \) with objects \( \{ p \mid \exists p_0 \to p, p \neq p_0 \} \) and \( \{ p \mid \exists p \to p_0, p \neq p_0 \} \) respectively.
Definition 1.3.2. If $P$ is a poset and $p < p'$ then $p$ precedes $p'$ if $p \leq p'' \leq p'$ implies that $p = p''$ or $p' = p''$.

Most of the results of this work are about the following types of posets:

Definition 1.3.3. Let $P$ be a poset. $P$ is called graded if there is a function $\text{deg}: \text{Ob}(P) \to \mathbb{Z}$, called the degree function of $P$, which is order preserving and that satisfies that if $p$ precedes $p'$ then $\text{deg}(p') = \text{deg}(p) + 1$. If $p \in \text{Ob}(P)$ then $\text{deg}(p)$ is called the degree of $p$.

Notice that the degree function associated to a graded poset is not unique (consider the translations $\text{deg}' = \text{deg} + c_k$ for $k \in \mathbb{Z}$). According to the definition the degree function increases in the direction of the arrows: we say that this degree function is increasing. If the degree function is order reversing and satisfies the alternative condition that $p$ precedes $p'$ implies $\text{deg}(p') = \text{deg}(p) - 1$, i.e., $\text{deg}$ decreases in the direction of the arrows, then we say that $\text{deg}$ is a decreasing degree function. Clearly both definitions are equivalent (by taking $\text{deg}' = -\text{deg}$). A poset which is graded satisfies some structural conditions:

Lemma 1.3.4. If $P$ is a graded poset and $p < p'$ then there is an integer $n$ and a finite chain

$$p = p_0 < p_1 < p_2 < \ldots < p_{n-1} < p_n = p'$$

where $p_i$ precedes $p_{i+1}$ for $i = 0, \ldots, n - 1$. Moreover, if

$$p = q_0 < q_1 < q_2 < \ldots < q_{m-1} < q_m = p'$$

is another finite chain where $q_i$ precedes $q_{i+1}$ for $i = 0, \ldots, m - 1$ then $m = n$.

The proof of this lemma is straightforward.

Example 1.3.5. The “pushout category” $b \leftarrow a \rightarrow c$, the “telescope category” $a \rightarrow b \rightarrow c \rightarrow \ldots$, and the opposite “telescope category” $\ldots \rightarrow c \rightarrow b \rightarrow a$ are graded posets. The integers $\mathbb{Z}$ is a graded poset. The rationals $\mathbb{Q}$ with the usual order is a poset but it is not a graded poset by the first condition of Lemma 1.3.4.

As in the next example, when drawing a poset, we picture just the arrows $p \rightarrow p'$ where $p$ precedes $p'$.

Example 1.3.6. The poset

![Diagram of a poset with arrows between $p_1$, $p_2$, $p$, $q_1$, and $p'$]

is not graded by the second condition of Lemma 1.3.4.
If $\mathcal{P}$ is a graded poset we can “extend” the degree function $\text{deg}$ to the morphisms set $\text{Hom}(\mathcal{P})$ by $\text{deg}(p \to p') = |\text{deg}(p') - \text{deg}(p)|$. By the preceding lemma this number does not depend on the degree function. Whenever $\mathcal{P}$ is a graded poset we denote by $\text{Ob}_n(\mathcal{P})$ the objects of degree $n$ and by $\text{Hom}_n(\mathcal{P})$ the arrows of degree $n$ of the graded poset $\mathcal{P}$. More generally:

**Definition 1.3.7.** Let $S \subset \mathbb{Z}$ and let $\mathcal{P}$ be a graded poset with degree function $\text{deg}$. Then $\mathcal{P}_S$ is the full subcategory of $\mathcal{P}$ with objects $p$ such that $\text{deg}(p) \in S$, $\text{Ob}_S(\mathcal{P}) = \{p \in \text{Ob}(\mathcal{P})|\text{deg}(p) \in S\}$ and $\text{Hom}_S(\mathcal{P})$ is the set $\{f \in \text{Hom}(\mathcal{P})|\text{deg}(f) \in S\}$.

**1.3.1. Boundedness on graded posets.** For some results we restrict to:

**Definition 1.3.8.** A graded poset $\mathcal{P}$ with increasing degree function $\text{deg}$ is bounded below (bounded above) if the set $\text{deg}(\mathcal{P}) \subset \mathbb{Z}$ has a lower bound (an upper bound). If the degree function $\text{deg}$ of $\mathcal{P}$ is decreasing then $\mathcal{P}$ is bounded below (bounded above) if and only if $\text{deg}(\mathcal{P}) \subset \mathbb{Z}$ has an upper bound (a lower bound). Notice that upper boundedness means that the category $\mathcal{P}$ ends if you move in the direction of the arrows, and lower boundedness means that $\mathcal{P}$ finishes moving in the opposite direction.

If $\mathcal{P}$ is bounded below and over then $N \overset{\text{def}}{=} \max(\text{deg}(\mathcal{P})) - \min(\text{deg}(\mathcal{P}))$ exists and is finite. By Lemma [1.3.4] this number does not depend on the degree function. We call it the dimension of $\mathcal{P}$, and we say that $\mathcal{P}$ is $N$-dimensional.

**Example 1.3.9.** The “pushout category” $b \leftarrow a \to c$ is 1-dimensional, the “telescope category” $a \to b \to c \to \ldots$ is bounded below but it is not bounded over. The opposite “telescope category” $\ldots \to c \to b \to a$ is bounded over but it is not bounded below.

Notice that in a bounded above (below) graded poset there are maximal (minimal) elements, but that the existence of maximal (minimal) objects does not guarantee boundedness.

**Remark 1.3.10.** When dealing with cohomology some assumptions on a bounded above graded poset $\mathcal{P}$ shall be done. In particular, we shall assume that $(i \downarrow \mathcal{P})_n = (i \downarrow \mathcal{P})_{(n)}$ (Definition [1.3.7]) is finite for each $i \in \text{Ob}(\mathcal{P})$, and that all the maximal elements of $\mathcal{P}$ have the same degree.

**1.3.2. Graphs and graded posets.** Let $\mathcal{P}$ be a graded poset. The undirected graph associated to $\mathcal{P}$ has vertices the objects $\text{Ob}(\mathcal{P})$ and edges the arrows of degree 1, $\text{Hom}_1(\mathcal{P})$. The directed graph associated to $\mathcal{P}$ a has vertices the objects $\text{Ob}(\mathcal{P})$ and oriented edges the oriented arrows of degree 1, $\text{Hom}_1(\mathcal{P})$.

A graded poset $\mathcal{P}$ is a tree if it is associated undirected graph is a tree (it contains no cycle). A tree or maximal tree of $\mathcal{P}$ is a subcategory $\mathcal{P}'$ such that the undirected subgraph associated to the graded poset $\mathcal{P}'$ is a tree or maximal tree respectively of the undirected graph associated to $\mathcal{P}$.

**1.4. Simplicial complexes**

A simplicial complex \([15]\) is a pair $K = (V, S)$ where $V$ is a set and $S$ is a collection of finite subsets of $V$ satisfying
1. Notation and Preliminaries

(1) $\sigma \in S$, $\sigma' \subseteq \sigma \Rightarrow \sigma' \in S$.

(2) $v \in V \Rightarrow \{v\} \in S$.

Elements of $V$ are called *vertices* and elements of $S$ are called *simplices*. If $K = (V,S)$ is a simplicial complex we can associate to it the poset with objects $S$ and inclusion as order relation. This poset verify the following property:

**Definition 1.4.1.** Let $\mathcal{P}$ be a poset. Then $\mathcal{P}$ is *simplex-like* if for all $p \in \text{Ob}(\mathcal{P})$ the category $(\mathcal{P} \downarrow p)$ is isomorphic to the poset of all non-empty subsets of a finite set (with inclusion as order relation).

In fact, [18, 3.1], we have that a poset $\mathcal{P}$ arises from a simplicial complex as above if and only if it is a simplex-like poset and any two elements of $\mathcal{P}$ which have a lower bound have an infimum, i.e., a greatest lower bound. If the poset $\mathcal{P}$ arises from the simplicial complex $K = (V,S)$ then there is a map $\text{dim} : \mathcal{P} \to \mathbb{Z}$ which assigns to each simplex $s \in S$ its dimension $\text{dim}(s) \in \mathbb{Z}$. If $s$ is a simplex of $S$ then all the subsets of $s$ are in $S$ too. This implies that preceding simplices of $\mathcal{P}$ differ just in one dimension, and thus the function $\text{dim}$ is an increasing degree function and $\mathcal{P}$ is a graded poset.

1.5. Realization and homotopy type

According to Sections 1.3 and 1.4 we have inclusions

$\text{Simplicial complexes} \subseteq \text{Graded posets} \subseteq \text{Categories}$.

On the one hand, we can realize in Top a simplicial complex $K$ as a space of formal linear combinations appropriately topologized [15, 3]. On the other hand, for a category $\mathcal{C}$ we can consider its *nerve* $N\mathcal{C}$, which is a simplicial set $N\mathcal{C} \in \text{SSet}$, and the realization in Top of this simplicial set [15, 3]. We denote by $|K|$ and $|N\mathcal{C}|$ the realizations of simplicial complexes and categories respectively.

For a given simplicial complex $K$ consider the graded poset $\mathcal{P}$ associated to it (Section 1.4). The simplicial complex whose simplices consist of all of the totally ordered subsets of $\mathcal{P}$ is exactly the barycentric subdivision [38, 3.3] of $K$, $sdK$. By [15, 3,4] there are homeomorphisms $|K| \cong |sdK| \cong |N\mathcal{P}|$. Thus, as the realization of a simplicial complex $K$ we can consider either of $|K|$, $|sdK|$ or $|N\mathcal{P}|$.

1.5.1. Homotopy type. From the homotopy viewpoint restricting to graded posets means no loss:

**Proposition 1.5.1.** For any space $X \in \text{Top}$ there is a graded poset $\mathcal{P}$ and a weak homotopy equivalence:

$$X \simeq |N\mathcal{P}|.$$ 

**Proof.** It is a well known fact [43, Theorem V.3.2] that $X$ has the weak homotopy type of a $CW$-complex. In fact, by [23, Theorem 2C.5], $X$ also has the weak homotopy type of a simplicial complex $K$, $X \simeq |K|$. By the comments above we have $|K| \cong |sdK| \cong |N\mathcal{P}|$ for certain graded poset $\mathcal{P}$, and thus $X \simeq |N\mathcal{P}|$. \qed
Remark 1.5.2. By the proof of the Theorem if $X$ is a CW-complex then there is a (strong) homotopy equivalence $X \simeq |NP|$.

1.6. Derived functors of direct and inverse limit

In this section we give definitions for the left derived functors $\lim_i$ of the direct limit $\lim_i : \text{Ab} C \to C$ and for the right derived functors $\lim_i'$ of the inverse limit $\lim_i' : \text{Ab}^i C \to C$ for any small category $C$.

Notice [30] IX, Proposition 3.1] that $\text{Ab} C$ is an abelian category in which the short exact sequences are the object-wise ones, and that $\lim_i$ and $\lim_i'$ are right exact and left exact respectively because they are left adjoint and right adjoint respectively to the functor $\text{Ab} \to \text{Ab} C$ which maps $A \mapsto c_A$.

It is well known ([8], XI.6.1] and its dual) that in $\text{Ab} C$ there are enough projectives and injectives so we can define the derived functors of $\lim_i$ and $\lim_i'$. Instead of considering projective and injective resolutions for objects of $\text{Ab} C$, the definitions of $\lim_i$ and $\lim_i'$ we expound here have computational purposes and are based on [17] Appendix II.3. They also appear in [8] XII.5.5, [8] XI.6.2. In [19] p.409ff.] there is a summary.

Denote by $NC$ the nerve of the small category $C$ and by $\sigma \in NC_n$ an $n$-simplex of the nerve, that is, a chain of morphisms $\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \sigma_{n-1} \xrightarrow{\alpha_n} \sigma_n$. Given a covariant functor $F : C \to \text{Ab}$ consider the simplicial abelian group with simplices

$$C_n(C, F) = \bigoplus_{\sigma \in NC_n} F_\sigma,$$

where $F_\sigma = F(\sigma_0)$. The face map $d_i$ for $0 \leq i \leq n$ is the unique homomorphism which makes commute the diagram

$$
\begin{array}{ccc}
C_n(C, F) & \xrightarrow{d_i} & C_{n-1}(C, F) \\
\downarrow \pi_\sigma & & \downarrow \pi_{d_i(\sigma)} \\
F_\sigma & \xrightarrow{id^*} & F_{d_i(\sigma)}
\end{array}
$$

for every $\sigma \in NC_n$, where

$$id^* = \begin{cases}
F(\alpha) : F(\sigma_0) \to F(\sigma_1) & \text{for } i = 0 \\
Id_{F(\sigma_0)} : F(\sigma_0) \to F(\sigma_0) & \text{for } 0 < i \leq n.
\end{cases}$$

The degeneracy map $s_i$ for $0 \leq i \leq n$ is the unique homomorphism which makes commute the diagram

$$
\begin{array}{ccc}
C_n(C, F) & \xrightarrow{s_i} & C_{n+1}(C, F) \\
\downarrow \pi_\sigma & & \downarrow \pi_{s_i(\sigma)} \\
F_\sigma & \xrightarrow{id_F(\sigma_0)} & F_{s_i(\sigma)}
\end{array}
$$
for every $\sigma \in NC_n$.

This simplicial object gives rise to a chain complex, the Moore complex, $(C_s(C, F), d)$ with differential of degree $-1$, $d : C_n(C, F) \to C_{n-1}(C, F)$, $d = \sum_{i=0}^{n} (-1)^i d_i$.

**Definition 1.6.1.** Let $C$ be a small category and let $F$ be a covariant functor $F : C \to \text{Ab}$, then the $i$-left derived functor of $\lim : \text{Ab}^C \to C$ is

$$\lim_{\leftarrow i}(F) := H_i(C_s(C, F), d).$$

For the inverse limit $\lim : \text{Ab}^C \to C$ consider the cosimplicial abelian group with simplices:

$$C^n(C, F) = \prod_{\sigma \in NC_n} F^\sigma,$$

where $F^\sigma = F(\sigma_n)$. The coface map $d^i$ for $0 \leq i \leq n+1$ is the unique homomorphism which makes commute the diagram

$$C^n(C, F) \xrightarrow{d^i} C^{n+1}(C, F) \xrightarrow{\pi_{\sigma}} F^\sigma$$

for every $\sigma \in NC_{n+1}$, where

$$id^* = \begin{cases} F(\alpha_{n+1}) : F(\sigma_n) \to F(\sigma_{n+1}) & \text{for } i = n+1 \\ id_{F(\sigma_{n+1})} : F(\sigma_{n+1}) \to F(\sigma_{n+1}) & \text{for } 0 \leq i \leq n \end{cases}$$

The codegeneracy map $s^i$ for $0 \leq i \leq n$ is the unique homomorphism which makes commute the diagram

$$C^{n+1}(C, F) \xrightarrow{s^i} C^n(C, F) \xrightarrow{\pi_{\sigma}} F^\sigma$$

for every $\sigma \in NC_n$.

This cosimplicial object gives rise to a cochain complex $(C^*(C, F), d)$ with differential of degree $1$, $d : C^n(C, F) \to C^{n+1}(C, F)$, $d = \sum_{i=0}^{n+1} (-1)^i d_i$.

**Definition 1.6.2.** Let $C$ be a small category and let $F$ be a covariant functor $F : C \to \text{Ab}$, then the $i$-right derived functor of $\lim : \text{Ab}^C \to C$ is

$$\lim_{\rightarrow i}(F) := H^i(C^s(C, F), d).$$

For every short exact sequence of natural trasformations

$$0 \to F \to G \to H \to 0$$

in $\text{Ab}^C$ there exists a pair of long exact sequences of derived functors

$$\cdots \to \lim_{\rightarrow 1} F \to \lim_{\rightarrow 1} G \to \lim_{\rightarrow 1} H \to \lim\to F \to \lim\to G \to \lim\to H \to 0,$$

$$\cdots \to \lim_{\leftarrow 1} H \leftarrow \lim_{\leftarrow 1} G \leftarrow \lim_{\leftarrow 1} F \leftarrow \lim\leftarrow G \leftarrow \lim\leftarrow H \leftarrow \lim\leftarrow F \leftarrow 0.$$
We use the following notation for the obvious inclusions and projections:

\[ F_\sigma \xhookrightarrow{\iota_\sigma} C_n(C, F) \xrightarrow{\pi_\sigma} F_\sigma, \]

\[ F^\sigma \xhookrightarrow{\sigma} C^n(C, F) \xrightarrow{\pi^\sigma} F^\sigma. \]

1.7. Normalization Theorem

We shall use the Normalization Theorem for simplicial abelian groups in order to compute the homology of these. It states roughly that the homology of a simplicial abelian group can be computed removing the degenerate simplices. The theorem we state below is contained in [31, 22.1, 22.3]. It can be also found in [19, III.2.1, III.2.4].

Let \( S \) be a simplicial abelian group and let \((S, \sum (-1)^i d_i)\) be the Moore chain complex associated to \( S \). Consider the chain complex \( NS \) with \( n \)-chains

\[ NS_n = \bigcap_{i=0}^{n-1} \text{Ker}(d_i : S_n \to S_{n-1}) \]

and with differential \((-1)^n d_n : NS_n \to NS_{n-1}\). Define \( DS \) as the chain subcomplex \( DS \) of \((S, \sum (-1)^i d_i)\) which \( n \)-chains are generated by the degenerate elements of \( S \), that is, by the elements in the image of some \( s_i \).

**Theorem 1.7.1.** Let \( S \) be a simplicial abelian group, then:

\[ H_*(S) = H_*(NS) = H_*(S/DS). \]

The dual version for cosimplicial abelian groups appears in [19, VIII.1] and [8, X.7.1]. Let \( C \) be a cosimplicial abelian group and consider the cochain complex \((C, \sum (-1)^i d^i)\). There is a cochain complex \( NC \) with \( n \)-cochains

\[ NC^n = \bigcap_{i=0}^{n-1} \text{Ker}(s^i : C^n \to C^{n-1}) \]

and with differential \( \sum (-1)^i d^i \). Define \( DC \) as the subcomplex of \((C, d = \sum (-1)^i d^i)\) which \( n \)-cochains are generated by the elements in the image of some \( d^i \).

**Theorem 1.7.2.** Let \( C \) be a cosimplicial abelian group, then:

\[ H^*(C) = H^*(NC) = H^*(C/DC). \]

**Remark 1.7.3.** We shall apply these theorems to simplicial and cosimplicial abelian groups coming from a diagram as in Section [1.6]. In fact, we shall use the quotient chain complex \( S/DS \) when dealing with \( C_n(C, F) \) and the cochain subcomplex \( NC \) when dealing with \( C^n(C, F) \) without explicit mention.
CHAPTER 2

A spectral sequence

In this section we shall construct spectral sequences with targets \( \lim_{\to} F \) and \( \lim_{\leftarrow} F \) for \( F : \mathcal{C} \to \text{Ab} \) with \( \mathcal{C} \) a graded poset. Some conditions for (weak) convergence shall be given. We build the spectral sequences starting from filtered differential modules (see \[32\], where the notion of weak convergence we use is also given).

First consider the complex \((C^\ast(C, F), d)\) defined in Section 1.6 and choose a decreasing degree function \( \text{deg} \) over the objects of \( \mathcal{C} \). There is a decreasing filtration of this complex given by

\[
L^p C^n(C, F) = \bigoplus_{\sigma \in \mathcal{N}_n, \text{deg}(\sigma_n) \geq p} F_{\sigma}.
\]

It is straightforward that the triple \((C^\ast(C, F), d, L^\ast)\) is a filtered differential graded \( \mathbb{Z} \)-module, so it yields a spectral sequence \((E^\ast, \ast_r, d_r)\) of cohomological type whose differential \( d_r \) has bidegree \((r, 1 - r)\). The \( E^p, \ast_1 \) page is given by

\[
E^p_{1, \ast} \simeq H^{p+q}(L^0C/L^{p-1}C).
\]

The differential graded \( \mathbb{Z} \)-module \( L^pC/L^{p-1}C \) is in fact a simplicial abelian group because the face operators \( d_i \) and the degeneracy operators \( s_i \) respect the filtration \( L^\ast \). The \( n \)-simplices are

\[
(L^pC/L^{p-1}C)_n = \bigoplus_{\sigma \in \mathcal{N}_n, \text{deg}(\sigma_n) = p} F_{\sigma}.
\]

Moreover, for each \( p \), \( L^pC/L^{p-1}C \) can be filtered again by the condition \( \text{deg}(\sigma_0) \leq p' \) to obtain a homological type spectral sequence. Then arguing as above we obtain:

**Proposition 2.0.4.** For a (decreasing) graded poset and a functor \( F : \mathcal{C} \to \text{Ab} \):

- There exists a cohomological type spectral sequence \( E^p, \ast_r \) with target \( \lim_{\to} F \).
- There exists a homological type spectral sequence \( (E^p)^{p, \ast}_r \) with target the column \( E^p_{1, \ast} \) for each \( p \).

Notice that the column \( E^p_{1, \ast} \) is given by the cohomology of the simplicial abelian group formed by the simplices that end on objects of degree \( p \), the column \( (E^p)^{p, \ast}_r \) is given by the homology of the simplicial abelian group formed by the simplices that end on degree \( p \) and begin on degree \( p' \), and all the differentials in the spectral sequences above are induced by the completely described differential of \((C^\ast(C, F), d)\).

An advantage of handling simplicial abelian groups instead of chain complexes is the chance to use the Normalization Theorem [1.7.1].
As \( \bigcup_p L^p C_n = C_n \) and \( \bigcap_p L^p C_n = 0 \) for each \( n \) the spectral sequence \( E^{*\,*}_\bullet \) converges weakly to its target. In case the map \( \text{deg} \) has a bounded image, i.e., when \( C \) is \( N \) dimensional, the filtration \( L^\bullet \) is bounded below and over, and so \( E^{*\,*}_\bullet \) collapses after a finite number of pages. The same assertions on weak converge and boundedness hold for the spectral sequences \( (E^p)^{*,*}_\bullet \).

If we proceed in reverse order, i.e., filtrating first by the degree of the beginning object and later by the degree of the ending object, we obtain:

**Proposition 2.0.5.** For a (decreasing) graded poset and a functor \( F : C \to \text{Ab} \):

- There exists a homological type spectral sequence \( E^{*\,*}_\bullet \) with target \( \lim_{\rightarrow} F \).
- There exists a cohomological type spectral sequence \( (E^p)^{*,*}_\bullet \) with target the column \( E^1_{p,*} \) for each \( p \).

If the degree function we take is increasing then the appropriate conditions for the filtrations are \( \text{deg}(\sigma_n) \leq p \) and \( \text{deg}(\sigma_0) \geq p' \), and the spectral sequences obtained in Propositions 2.0.4 and 2.0.5 are of homological (cohomological) type instead of cohomological (homological) type.

For the case of the cochain complex \((C^\bullet(C, F), d)\) defined in Section 1.6 the choices for the filtrations are \( \text{deg}(\sigma_n) \leq p \) and \( \text{deg}(\sigma_0) \geq p' \) for a decreasing degree function and \( \text{deg}(\sigma_n) \geq p \) and \( \text{deg}(\sigma_0) \leq p' \) for an increasing one. Analogously we obtain spectral sequences with target \( \lim_{\leftarrow} F \) which columns in the first page are computed by another spectral sequence. In this case we can use the Normalization Theorem 1.7.2 to compute cohomology of the cosimplicial abelian groups appearing in the page 1 of these spectral sequences.

Table 1 shows a summary of the types of the spectral sequences for all the cases. The statements on weak convergence and boundedness apply to any of the spectral sequences of the table.

**Remark 2.0.6.** Recall from Remark 1.7.3 the chain (cochain) complex chosen to normalize a simplicial (cosimplicial) abelian group. It is straightforward that normalizing the simplicial (cosimplicial) abelian groups that computes the page 1 of the
spectral sequences above has the same effect as considering the spectral sequences of the normalizations of $C_*(\mathcal{C}, F)$ ($C^*(\mathcal{C}, F)$).

2.1. Examples

Next there are some examples that show how the spectral sequences just built work. In this section $F$ denotes a covariant functor $F : \mathcal{P} \to \text{Ab}$ where $\mathcal{P}$ is a graded poset.

These examples also serve as preamble to Chapters 3 and 4 where conditions are found such that $\varprojlim F = 0$ and $\varprojlim^i F = 0$ for $i > 0$ respectively. The behaviour of the spectral sequences in these examples resembles the general results in this thesis (Theorems 3.2.3 and 4.2.3).

In particular, in the pushout example $F$ monic implies $\varprojlim_1 F = 0$ (compare with Definition 3.1.5, Remark 3.1.6 and Theorem 3.2.3). For the pullback example we have that $F$ epic implies $\varprojlim^1 F = 0$ (compare with Definition 4.1.5, Remark 4.1.6 and Theorem 4.2.3). The telescope example shows the importance that in a graded poset every morphism factors as composition of morphisms of degree 1 (compare with Step 1 in the proof of Theorem 3.2.3). Finally, the “cycle” example shows that the existence of cycles in $\mathcal{P}$ may prevent $\varprojlim_1 F = 0$ (compare with Corollaries 3.3.12 and 3.4.10). In this example also appears a heuristic version of pseudo-projectiveness (Definition 3.1.5). This property is related (Theorem 3.2.3) to the vanishing of $\varprojlim F$.

Example 2.1.1. Pushout: Consider $\mathcal{P}$ the “pushout category”:

$$
\begin{array}{ccc}
a & f & b \\
& g & \\
& c
\end{array}
$$

This category is a graded poset with increasing degree function indicated by the subscripts:

$$
\begin{array}{ccc}
a_1 & f & b_0 \\
& g & \\
& c_1
\end{array}
$$

Although in this case it is trivial to compute the derived functors we apply the earlier propositions to have a taste of it. So we filter by the ending object $(\sigma_n)$ to obtain a homological type spectral sequence $E_{s,s}^*$ converging to $\varprojlim F$. We do not filter a second time since this case is too simple. The column $E_{p,s}^1$ is given by the homology of a simplicial abelian group. In fact, by Theorem 1.7.1, $E_{1,s}^1$ is the homology of the quotient chain complex of non-degenerated simplices ending in degree 1, that is, the homology of

$$
\begin{array}{ccc}
\ldots & 0 & F_f \oplus F_g \\
& F(f) \oplus F(g) & F_a \oplus F_c
\end{array}
$$

Analogously, $E_{0,s}^1$ is the homology of the quotient chain complex of non-degenerated simplices ending in degree 0, that is, the homology of

$$
\begin{array}{ccc}
\ldots & 0 & F_b
\end{array}
$$
So the page $E^1$ looks like

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & F_b & k & (F(f) @+ F(g)) & 0 \\
0 & 0 & cok(F(f) @+ F(g)) & 0 \\
0 & 0 & 0 & 0
\end{array}
$$

and the only nontrivial differential $d_1$ is the restriction of $d_F(b) + id_F(b) : F(b) @+ F(b) \rightarrow F(b)$. It is clear that the spectral sequence collapses at $E^2$. Clearly $\varprojlim \mathbb{F} = 0$ for $i \geq 2$. Notice that if $F(f)$ and $F(g)$ are monomorphisms then $E^1_{1,0} = 0$ and $\varprojlim \mathbb{F} = 0$.

The extension problem for $\varprojlim \mathbb{F} = \varprojlim \mathbb{F}$ gives the short exact sequence:

$$0 \rightarrow F(b) / \text{Ker}(F(f)) + \text{Ker}(F(g)) \rightarrow \varprojlim \mathbb{F} = cok(F(f) \times F(g)) \rightarrow cok(F(f) @+ F(g)) \rightarrow 0.$$

Using any of the spectral sequences for $\varprojlim \mathbb{F}$ it is straightforward that

$$\varprojlim \mathbb{F} = \begin{cases} 
F(b) & \text{for } i = 0 \\
0 & \text{if } i \geq 1.
\end{cases}$$

**Example 2.1.2. Pullback:** Consider $\mathcal{P}$ the “pullback category”:

$$
\begin{array}{ccc}
& c & \\
\downarrow & \downarrow & \\
a & g & b,
\end{array}
$$

This category is a graded poset with decreasing degree function indicated by the subscripts:

$$
\begin{array}{ccc}
c_1 & \\
\downarrow & \\
a_1 & b_0.
\end{array}
$$

We filter by the initial object $(\sigma_0)$ to obtain a cohomological type spectral sequence $E^*,*$ converging to $\varprojlim \mathbb{F}$. The column $E^1_{*,*}$ is given (Theorem 1.7.2) by the cohomology of the normalized cochain complex of simplices beginning in degree $p$. So $E^1_{1,*}$ is the cohomology of

$$
\begin{array}{ccc}
\ldots & 0 & F^f @+ F^g \\
\downarrow & \downarrow & \downarrow \\
F^f @+ F^g & \downarrow & F^a @+ F^c,
\end{array}
$$

and $E^0_{1,*}$ the cohomology of

$$
\begin{array}{ccc}
\ldots & 0 & F^b \\
\downarrow & \\
F^b.
\end{array}$$
So the page $E_1$ looks like

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & F(b) & \text{coker}(F(f) \oplus F(g)) & 0 \\
0 & 0 & \text{ker}(F(f) \oplus F(g)) & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

and the only nontrivial differential $d_1$ is induced by $id_{F(b)} \times id_{F(b)} : F(b) \rightarrow F(b) \oplus F(b)$. The spectral sequence collapses at $E_2$. Notice that if $F(f)$ and $F(g)$ are epimorphisms then $E_1^{1,0} = 0$ and $\lim^1 F = 0$.

The extension problem for $\lim^0 F = \lim F$ gives the short exact sequence:

$0 \rightarrow \text{ker}(F(f) \oplus F(g)) \rightarrow \lim F = \text{ker}(F(f) - F(g)) \rightarrow \text{Im}(F(f)) \cap \text{Im}(F(g)) \rightarrow 0.$

For $\lim_{\rightarrow i} F$ it holds that

\[
\lim_{\rightarrow i} F = \begin{cases}
F(b) & \text{for } i = 0 \\
0 & \text{if } i \geq 1.
\end{cases}
\]

**Example 2.1.3. Telescope:** Consider $\mathcal{P}$ the “telescope category”:

\[
a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \xrightarrow{f_3} a_3 \xrightarrow{f_4} \ldots
\]

where the subscript indicates the name of the object and the value of an increasing degree function which makes $\mathcal{P}$ a graded poset. The same homological type spectral sequence of the pushout example for $\lim_{\rightarrow i} F$ has as column $E_1^{p,\ast}$ the homology of the normalized chain complex of simplices ending in degree $p$. These chain complexes become more and more complicated as $p$ grows. To get more insight we filter a second time by the condition $\deg(\sigma_0) \geq p'$ to obtain for each $p$ a cohomological type spectral sequence $(E_p)^{p',\ast}$ converging to $E_1^{p,\ast}$. The column $(E_p)^{p',\ast}$ is given by the homology of the simplicial abelian group formed by the simplices that end on degree $p$ and begin on degree $p'$.

We want to have a look at $\lim_{\rightarrow p} F$. Here we write an informal discussion, for precise statements look at the proof of Theorem 3.2.3. The contributions to $\lim_{\rightarrow p} F$ from the second spectral sequence for $p' < p$ come from the homology at degree 1 of normalized chain complexes

\[
\ldots \xrightarrow{d} F_{\sigma} = F(a_{p'}) \xrightarrow{d} 0
\]

where $\sigma = a_{p'} \xrightarrow{f_{p'} \cdot f_{p'+1}} a_p$ with $p' < p$. If $p' < p - 1$, for $x \in F(p')$ take the 2-chain

\[
y = i_{\sigma'}(-x)
\]

where $\sigma' = a_{p'} \xrightarrow{f_{p'} \cdot f_{p'+1}} a_{p'+1} \xrightarrow{f_{p'} \cdot f_{p'+2}} a_p$. 

Then $d(y) = d_0(y) - d_1(y) + d_2(y) = 0 - (-x) + 0 = x$, and so if $p' < p - 1$ there is no contribution to $\lim_{\to 1} F$. Notice that the key fact in the argument above is that every morphism of degree greater than 1 can be written as composition of morphisms of degree 1. Notice that at this point the calculus of $\lim_{\to 1} F$ have been simplified a lot.

The computation of $\lim^{\to 1} F$ is simplified too. In this case the contributions to $\lim^{\to 1} F$ come from the cohomology of normalized cochain complexes

$$\cdots \xrightarrow{d} F^\sigma = F(a_{p'}) \xrightarrow{d} 0$$

where $\sigma = a_{p'} \xrightarrow{f_{p'}} a_{p'} a_p$ with $p' < p$. For $p' < p - 1$, if $x \in F(p)$ is a cohomological class then $d(x) = 0$, and so $\pi^{\sigma'}(d(x)) = x = 0$ where $\sigma' = a_{p'} \xrightarrow{f_{p' + 1}} a_{p' + 1} \xrightarrow{f_{p'} \cdots f_{p' + 2}} a_p$. Thus if $p' < p - 1$ there is no contribution to $\lim^{\to 1} F$.

**Example 2.1.4. Cycle:** Consider $\mathcal{P}$ the following category

\[
\begin{array}{cccc}
  a & f & c \\
  & g \\
  b & i & d. \\
\end{array}
\]

It is a graded poset with increasing degree function

\[
\begin{array}{cccc}
  a_1 & f & c_2 \\
  & g \\
  b_1 & i & d_2. \\
\end{array}
\]

The associated undirected graph has a cycle. A direct computation shows that $\lim_{\to 1} F$ consists of the tuples $(x_f, x_g, x_h, x_i) \in F_f \times F_g \times F_h \times F_i = F(a) \times F(a) \times F(b) \times F(b)$ such that $x_f + x_g = 0$, $F(f)(x_f) + F(h)(x_h) = 0$, $x_h + x_i = 0$ and $F(g)(x_g) + F(i)(x_i) = 0$. This system of equations may have non-trivial solutions. For example, if we consider the constant functor $F = c_\mathbb{Z}$ then $\lim_{\to 1} F = H_1(|N\mathcal{P}|) = H_1(S^1) = \mathbb{Z}$, and so the solution is cyclic infinite. What happens if we add an initial object $e$ to $\mathcal{P}$?
In case $F$ is the constant functor of value $\mathbb{Z}$ then $\lim_{\to 1} F$ is the first group of homology of the cone over $S^1$, which is a contractible space, and so $\lim_{\to 1} F = 0$. What happens for an arbitrary $F$? The image by the differential from $C_2(\mathcal{P}, F)$ to $C_1(\mathcal{P}, F)$ of the tuple

$$(x_{jf}, x_{jg}, x_{kh}, x_{ki}) \in F_{jf} \times F_{jg} \times F_{kh} \times F_{ki} = F(e) \times F(e) \times F(e) \times F(e)$$

is the tuple

$$(F(j)(x_{jf}), F(j)(x_{jg}), F(k)(x_{kh}), F(k)(x_{ki}), x_{jg} + x_{kh} + x_{ki}, -x_{jg} + x_{kh}, -x_{jg} + x_{ki})$$

of

$$F_f \times F_g \times F_h \times F_i \times F_j \times F_k \times F_{f\circ j} \times F_{i\circ k}$$

which equals

$$F(a) \times F(a) \times F(b) \times F(b) \times F(e) \times F(e) \times F(e) \times F(e).$$

Applying the arguments of the preceding examples we have that for a class

$$[(x_f, x_g, x_h, x_i, x_j, x_k, x_{f\circ j}, x_{i\circ k})]$$

in the kernel of the differential at $C_1(\mathcal{P}, f)$ we can take

- $x_{f\circ j} = x_{i\circ k} = 0$ because $f \circ j$ and $i \circ k$ can be factored by morphisms of lower degree.
- $x_j = x_k = 0$ if $F(j)$ and $F(k)$ are monomorphisms, as $j$ and $k$ are the only arrows arriving to their ending objects.

So supposing that $F(j)$ and $F(k)$ are monomorphisms we can take as representative of a class in the kernel a tuple $(x_f, x_g, x_h, x_i, 0, 0, 0, 0)$ such that $x_f + x_g = 0$, $F(f)(x_f) + F(h)(x_h) = 0$, $x_h + x_i = 0$ and $F(g)(x_g) + F(i)(x_i) = 0$, as before.

Now, suppose that for every $x_a \in F(a)$ and every $x_b \in F(b)$ such that $F(f)(x_a) = F(h)(x_b)$ there exists $x_e \in F(e)$ such that $F(j)(x_e) = x_a$ and $F(k)(x_e) = x_b$. This condition is natural as it is related with the projectiveness of $F$ in $\text{Ab}^\mathcal{P}$ (cf. Section 3.1).

For a tuple $(x_f, x_g, x_h, x_i, 0, 0, 0, 0)$ in the kernel, we have that $F(f)(x_f) = F(h)(-x_h)$ and so, by hypothesis, exists $x_e \in F(e)$ with $F(j)(x_e) = x_f$ and $F(k)(x_e) = -x_h$. Take now the 2-chain of $C_2(\mathcal{P}, F)$ $y = (x_e, -x_e, -x_e, x_e)$. Then the differential of $y$ is

$$(x_f, -x_f, x_h, -x_h, x_e - x_e, -x_e + x_e, -(x_e - x_e), -(x_e + x_e))$$

which equals $(x_f, x_g, x_h, x_i, 0, 0, 0, 0)$. Thus with these assumptions $\lim_{\to 1} F = 0$. 

2.1. EXAMPLES
CHAPTER 3

Higher direct limits

3.1. Projective objects in $\text{Ab}^P$.

Consider the abelian category $\text{Ab}^P$ for some graded poset $P$. In this section we shall determine the projective objects in $\text{Ab}^P$. One of the main features of projective objects is that derived functors vanish on them. Recall that in $\text{Ab}$ the projective objects are well known, and are exactly the free abelian groups. Along the rest of the section $P$ denotes a graded poset.

Suppose $F \in \text{Ab}^P$ is projective. How does $F$ look? Consider $i_0 \in \text{Ob}(P)$. We show that the quotient of $F(i_0)$ by the images of the non-identity morphisms arriving to $i_0$ is projective. To prove it, write

**Definition 3.1.1.** $\text{Im}(i_0) = \sum_{i : i \neq i_0} \text{Im} F(\alpha)$ (or $\text{Im}(i_0) = 0$ if the index set of the sum is empty) and $\text{Coker}(i_0) = F(i_0)/\text{Im}(i_0)$.

Let $F \in \text{Ab}^P$ be a projective functor. For any diagram in $\text{Ab}$ as the following

\[
\begin{array}{ccc}
C\text{oker}(i_0) & \rightarrow & 0 \\
\downarrow & & \\
A_0 & \xrightarrow{\pi_0} & B_0 \\
\uparrow & \nearrow \rho_0 & \\
& \sigma_0 & \\
\end{array}
\]

we want to find $\rho_0$ that makes it commutative. Consider the atomic functors $A, B : P \rightarrow \text{Ab}$ which take the values on objects

\[
A(i) = \begin{cases} A_0 & \text{for } i = i_0 \\ 0 & \text{for } i \neq i_0 \end{cases}
\]

\[
B(i) = \begin{cases} B_0 & \text{for } i = i_0 \\ 0 & \text{for } i \neq i_0 \end{cases}
\]

and on morphisms

\[
A(\alpha) = \begin{cases} 1_{A_0} & \text{for } \alpha = 1_{i_0} \\ 0 & \text{for } \alpha \neq 1_{i_0} \end{cases}
\]

\[
B(\alpha) = \begin{cases} 1_{B_0} & \text{for } \alpha = 1_{i_0} \\ 0 & \text{for } \alpha \neq 1_{i_0} \end{cases}
\]

and the natural transformations $\sigma : F \Rightarrow B$ and $\pi : A \Rightarrow B$ given by

\[
\sigma(i) = \begin{cases} \sigma_0 \circ p & \text{for } i = i_0 \\ 0 & \text{for } i \neq i_0 \end{cases}
\]
\[ \pi(i) = \begin{cases} \pi_0 & \text{for } i = i_0 \\ 0 & \text{for } i \neq i_0 \end{cases} \]

where \( p \) is the projection \( F(i_0) \to \text{Coker}(i_0) \). \( A \xrightarrow{\pi} B \to 0 \) is exact as \( A_0 \xrightarrow{\pi_0} B_0 \to 0 \) is. It is straightforward that \( \pi \) is a natural transformation. The key point in checking that \( \sigma \) is a natural transformation is that for \( \alpha : i_1 \to i_0, \alpha \neq 1_{i_0} \) the diagram

\[
\begin{array}{ccc}
F(i_1) & \xrightarrow{F(\alpha)} & F(i_0) \\
\downarrow{\sigma(i_1)} & & \downarrow{\sigma_0 \circ p} \\
B(i_1) & \xrightarrow{0} & B_0
\end{array}
\]

must commute. And it does because \( p \circ F(\alpha) = 0 \) for every \( \alpha \neq 1_{i_0} \).

So, as \( F \) is projective, this data gives a natural transformation \( \rho \) which makes commutative the diagram of natural transformations

\[
\begin{array}{ccc}
F & \xrightarrow{\pi} & A \\
\downarrow{\rho} & \xleftarrow{\sigma} & \downarrow{\pi_0} \\
\sigma & \xrightarrow{p} & A_0
\end{array}
\]

which restricts over \( i_0 \) to

\[
\begin{array}{ccc}
F(i_0) & \xrightarrow{\rho(i_0)} & \text{Coker}(i_0) \\
\downarrow{p} & & \downarrow{\sigma_0} \\
A_0 & \xrightarrow{\pi_0} & B_0
\end{array}
\]

Then \( \rho_0 \) exists if and only if \( \ker(p) = \text{Im}(i_0) \leq \ker \rho(i_0) \). To check that this condition holds take \( x = \sum_{j=1,..,k} F(\alpha_j)(x_j) \) in \( \text{Im}(i_0) \) for \( 1_{i_0} \neq \alpha_j : i_j \to i_0 \) \( j = 1,..,k \). Then

\[
\rho(i_0)(x) = \sum_{j=1,..,k} \rho(i_0)(F(\alpha_j)(x_j)) = \sum_{j=1,..,k} A(\alpha_j)(\rho(i_j)(x_j)) = \sum_{j=1,..,k} A(\alpha_j)(0) = 0.
\]

We have just proven

**Lemma 3.1.2.** Let \( F : \mathcal{P} \to \text{Ab} \) be a projective functor over a graded poset \( \mathcal{P} \). Then \( \text{Coker}(i_0) \) is projective for every object \( i_0 \in \text{Ob}(\mathcal{P}) \).

This means that we can write

\[ F(i_0) = \text{Im}(i_0) \oplus \text{Coker}(i_0) \]

with \( \text{Coker}(i_0) \) free for every \( i_0 \in \text{Ob}(\mathcal{P}) \), and also that

**Example 3.1.3.** For the category \( \mathcal{P} \) with shape

\[
. \to .
\]
the functor \( F : \mathcal{P} \rightarrow \text{Ab} \) with values
\[
\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}
\]
is not projective as Coker on the right object equals the non-free abelian group \( \mathbb{Z}/n \).

Now that we know a little about the values that a projective functor \( F : \mathcal{P} \rightarrow \text{Ab} \) takes on objects we can wonder about the values \( F(\alpha) \) for \( \alpha \in \text{Hom}(\mathcal{P}) \). Do they have any special property? Recall that a feature of graded posets is that there is at most one arrow between any two objects, and also that

**Remark 3.1.4.** If \( \mathcal{P} \) is graded then for any \( i_0 \in \text{Ob}(\mathcal{P}) \)
\[
\text{Im}(i_0) = \sum_{i \rightarrow i_0, \text{deg}(\alpha) = 1} \text{Im}(F(\alpha))
\]
because every morphism factors as composition of morphisms of degree 1.

We prove that the following property holds for \( F \):

**Definition 3.1.5.** Let \( F : \mathcal{P} \rightarrow \text{Ab} \) be a functor over a graded poset \( \mathcal{P} \) with degree function \( \text{deg} \). Given \( d \geq 0 \) we say that \( F \) is \( d \)-pseudo-projective if for any \( i_0 \in \text{Ob}(\mathcal{P}) \) and \( k \) different objects \( i_j \in \text{Ob}(\mathcal{P}) \), arrows \( \alpha_j : i_j \rightarrow i_0 \) with \( \text{deg}(\alpha_j) = d \), and \( x_j \in F(i_j) \) \( j = 1, ..., k \) such that
\[
\sum_{j=1,...,k} F(\alpha_j)(x_j) = 0
\]
we have that \( x_j \in \text{Im}(i_j) \) \( j = 1, ..., k \). If \( F \) is \( d \)-pseudo-projective for each \( d \geq 0 \) we call \( F \) pseudo-projective.

**Remark 3.1.6.** In case \( k = 1 \) and \( \text{Im}(i_1) = 0 \) the condition states that \( F(\alpha_1) \) is a monomorphism. Notice that any functor is 0-pseudo-projective as the identity is a monomorphism.

Before proving that projective functors \( F \) over a graded poset verify this property we define two functors Coker and Coker' and natural transformations \( \sigma \) and \( \pi \) that fit in the diagram
\[
\begin{array}{ccc}
\text{Coker} & \overset{\sigma}{\rightarrow} & \text{Coker} \\
\downarrow{\pi} & & \downarrow{0} \\
\text{Coker'} & & 
\end{array}
\]
for any functor \( F : \mathcal{P} \rightarrow \text{Ab} \) with \( \mathcal{P} \) a graded poset. We begin defining Coker. Because for every \( \alpha : i_1 \rightarrow i_0 \) holds that \( F(\alpha)(\text{Im}(i_1)) \leq \text{Im}(i_0) \) we can factor \( F(\alpha) \) as in the diagram
\[
\begin{array}{ccc}
\text{Coker}(i_1) & \overset{F(\alpha)}{\rightarrow} & \text{Coker}(i_0) \\
\downarrow & & \downarrow \\
\text{Coker}(i_1) & \overset{F(\alpha)}{\rightarrow} & \text{Coker}(i_0).
\end{array}
\]
In fact, if $\alpha \neq 1_{i_1}$, then $F(\alpha) \equiv 0$ by definition. Because the identity $1_{i_0}$ cannot be factorized (by non-identity morphisms) in a graded poset then we have a functor Coker with value Coker$(i)$ on the object $i$ of $\mathcal{P}$ and which maps the non-identity morphisms to zero. Coker is a kind of “discrete” functor. Also it is clear that there exists a natural transformation $\sigma : F \Rightarrow \text{Coker}$ with $\sigma(i)$ the projection $F(i) \twoheadrightarrow \text{Coker}(i)$.

Now we define $\text{Coker}'$ from Coker in a similar way as free diagrams are constructed. Let $\text{Coker}'$ be defined on objects by

$$\text{Coker}'(i_0) = \bigoplus_{\alpha : i \to i_0} \text{Coker}(i).$$

For $\beta \in \text{Hom}(\mathcal{P})$, $\beta : i_1 \to i_0$, $\text{Coker}'(\beta)$ is the only homomorphism which makes commute the diagram

$$\begin{array}{ccc}
\text{Coker}'(i_1) & \xrightarrow{\text{Coker}'(\beta)} & \text{Coker}'(i_0) \\
\downarrow & & \downarrow \\
\text{Coker}(i) & \xrightarrow{1} & \text{Coker}(i)
\end{array}$$

for each $\alpha : i \to i_1$. In the bottom row of the diagram, the direct summands Coker$(i)$ of Coker$(i_1)$ and Coker$(i_0)$ correspond to $\alpha : i \to i_1$ and to the composition $i \xrightarrow{\alpha} i_1 \xrightarrow{\beta} i_0$ respectively.

Then there exists a candidate to natural transformation $\pi : \text{Coker}' \Rightarrow \text{Coker}$ which value $\pi(i)$ is the projection $\pi(i) : \text{Coker}'(i) \twoheadrightarrow \text{Coker}(i)$ onto the direct summand corresponding to $1_i : i \to i$. Thus, $\pi$ is a natural transformation if for every $\beta : i_1 \to i_0$ with $i_1 \neq i_0$ the following diagram is commutative

$$\begin{array}{ccc}
\text{Coker}'(i_1) & \xrightarrow{\text{Coker}'(\beta)} & \text{Coker}'(i_0) \\
\downarrow{\pi(i_1)} & & \downarrow{\pi(i_0)} \\
\text{Coker}(i_1) & \xrightarrow{0} & \text{Coker}(i_0).
\end{array}$$

It is clear that this square commutes if the identity $1_{i_0}$ cannot be factorized (by non-identity morphisms), and this holds in a graded poset.

Now we have the commutative triangle

$$\begin{array}{ccc}
\text{Coker}' & \xrightarrow{\pi} & \text{Coker} \\
\downarrow{=} & & \downarrow{=} \\
\text{Coker} & \xrightarrow{\rho} & \text{Coker} \xrightarrow{=} 0
\end{array}$$

where the natural transformation $\rho$ exists because $F$ is projective. To prove that $F$ is $d$-pseudo-projective for some $d \geq 0$ take $i_0 \in \text{Ob}(\mathcal{P})$, $k$ objects $i_1, \ldots, i_k$, arrows $\alpha_j : i_j \to i_0$ with $\text{deg}(\alpha_j) = d$ and elements $x_j \in F(i_j)$ for $j = 1, \ldots, k$ such that

$$\sum_{j=1,\ldots,k} F(\alpha_j)(x_j) = 0.$$
To visualize what is going on consider the diagram above near $i_0$ for $k = 2$

where $\pi$ is not drawn completely for clarity. Recall that we are supposing that \{x_1, \ldots, x_k\} is such that $\sum_{j=1, \ldots, k} F(\alpha_j)(x_j)$ = 0. Then

$$0 = \rho(i_0)(0) = \sum_{j=1, \ldots, k} \rho(i_0)(F(\alpha_j)(x_j)) = \sum_{j=1, \ldots, k} Coker'(\alpha_j)(\rho(i_j)(x_j)).$$

Now consider the projection $p_{j_0}$ for $j_0 \in \{1, \ldots, k\}$ from $Coker'(i_0)$ onto the direct summand $Coker(i_{j_0}) \hookrightarrow Coker(i_0)$ which corresponds to $\alpha_{j_0} : i_{j_0} \rightarrow i_0$

$$Coker'(i_0) \twoheadrightarrow Coker(i_{j_0}).$$

Then

(1) $$0 = p_{j_0}(0) = p_{j_0}(\rho(i_0)(0)) = \sum_{j=1, \ldots, k} p_{j_0}(Coker'(\alpha_j)(\rho(i_j)(x_j))).$$

For any $y = \bigoplus_{\alpha : i \rightarrow i_j} y_\alpha \in Coker'(y_j)$

$$p_{j_0}(Coker'(\alpha_j)(y)) = \sum_{\alpha : i \rightarrow i_j, \alpha \rho = \alpha_{j_0}} y_\alpha.$$

So if $y_j = \rho(i_j)(x_j) = \bigoplus_{\alpha : i \rightarrow i_j} y_{j, \alpha} \in Coker'(i_j)$ then

$$p_{j_0}(Coker'(\alpha_j)(\rho(i_j)(x_j))) = \sum_{\alpha : i \rightarrow i_j, \alpha \rho = \alpha_{j_0}} y_{j, \alpha}.$$

This last sum runs over $\alpha : i_{j_0} \rightarrow i_j$ such that the following triangle commutes

Because we are in a graded poset and $deg(i_j) = d$ for each $j = 1, \ldots, k$ then the only chance is $i_j = i_{j_0}$ and $\alpha = 1_{i_{j_0}}$. Because the objects $i_1, \ldots, i_k$ are different this implies that $j = j_0$ too. Thus

$$p_{j_0}(Coker'(\alpha_j)(\rho(i_j)(x_j))) = \begin{cases} y_{j_0, 1_{i_{j_0}}} & \text{for } j = j_0 \\ 0 & \text{for } j \neq j_0 \end{cases}$$
and Equation (1) becomes
\[ 0 = p_{j_0}(0) = y_{j_0,1_{j_0}}. \]
Notice now that \( y_{j_0,1_{j_0}} \) is the evaluation of \( \pi(i_{j_0}) \) on \( y_{j_0} = \rho(i_{j_0})(x_{j_0}) \) and then
\[ 0 = y_{j_0,1_{j_0}} = \pi(i_{j_0})(\rho(i_{j_0})(x_{j_0})) = \sigma_{i_{j_0}}(x_{j_0}). \]
This last equation means that \( x_{j_0} \) goes to zero by the projection \( F(i_{j_0}) \colon \text{Coker}(i_{j_0}) = F(i_{j_0})/\text{Im}(i_{j_0}) \), and then
\[ x_{i_{j_0}} \in \text{Im}(i_{j_0}). \]
As \( j_0 \) was arbitrary this completes the proof of

**Lemma 3.1.7.** Let \( F : \mathcal{P} \to \text{Ab} \) be a projective functor over a graded poset \( \mathcal{P} \). Then \( F \) is pseudo-projective.

**Example 3.1.8.** For the category \( \mathcal{P} \) with shape
\[ \cdot \to \cdot \]
the functor \( F : \mathcal{P} \to \text{Ab} \) with values
\[ \mathbb{Z}^\text{red}_n \to \mathbb{Z}/n \]
is not projective as \( \text{red}_n \) is not injective, in spite of the Coker’s are \( \mathbb{Z} \) and 0, which are free abelian.

Now we define pre-projective objects

**Definition 3.1.9.** Let \( F : \mathcal{P} \to \text{Ab} \) be a functor over a graded poset \( \mathcal{P} \). We call \( F \) pre-projective if

1. for any \( i_0 \in \text{Ob}(\mathcal{P}) \) \( \text{Coker}(i_0) \) is projective.
2. \( F \) is pseudo-projective.

Till now we have obtained that projective functors \( \mathcal{P} \to \text{Ab} \) over graded posets are pre-projective. In fact, as the next proposition shows, the restriction we did to graded posets is worthwhile:

**Proposition 3.1.10.** Let \( F : \mathcal{P} \to \text{Ab} \) be a pre-projective functor over a graded poset \( \mathcal{P} \). If \( \mathcal{P} \) is bounded below then \( F \) is projective.

**Proof.** We can suppose that the degree function \( \text{deg} \) on \( \mathcal{P} \) is increasing and takes values \( \{0, 1, 2, 3, \ldots\} \), and that \( \text{Ob}_0(\mathcal{P}) \neq \emptyset \).

To see that \( F \) is projective in \( \text{Ab}^\mathcal{P} \), given a diagram of functors with exact row as shown, we must find a natural transformation \( \rho : F \Rightarrow A \) making the diagram commutative:

\[
\begin{array}{c}
\begin{array}{c}
F \\
\longrightarrow \\
\rho \\
\sigma \\
\end{array} \\
\begin{array}{c}
A \\
\longrightarrow \\
B \\
0 \\
\end{array}
\end{array}
\]

We define \( \rho \) inductively, beginning on objects of degree 0 and successively on object of degrees 1, 2, 3, ...
So take \( i_0 \in \text{Ob}_0(\mathcal{P}) \) of degree 0, and restrict to the diagram in \( \text{Ab} \) over \( i_0 \). By Definition 3.1.9, as \( \text{Im}(i_0) = 0 \), \( F(i_0) = \text{Coker}(i_0) \) is projective. So we can close the following triangle with a homomorphism \( \rho(i_0) \):

\[
\begin{array}{ccc}
F(i_0) & \xrightarrow{\sigma(i_0)} & 0 \\
\downarrow{\rho(i_0)} & & \\
A(i_0) & \xrightarrow{\pi(i_0)} & B(i_0)
\end{array}
\]

As there are no arrows between degree 0 objects we do not worry about \( \rho \) being a natural transformation. Now suppose that we have defined \( \rho \) on all objects of \( \mathcal{P} \) of degree less than \( n \) (\( n \geq 1 \)), and that the restriction of \( \rho \) to the full subcategory generated by these objects is a natural transformation and verifies \( \pi \circ \rho = \sigma \).

The next step is to define \( \rho \) on degree \( n \) objects. So take \( i_0 \in \text{Ob}_n(\mathcal{P}) \) and consider the splitting

\[ F(i_0) = \text{Im}(i_0) \oplus \text{Coker}(i_0) \]

where

\[ \text{Im}(i_0) = \sum_{i \xrightarrow{\alpha} i_0, \text{deg}(\alpha) = 1} \text{Im} F(\alpha). \]

To define \( \rho(i_0) \) such that it makes commutative the diagram

\[
\begin{array}{ccc}
\text{Im}(i_0) \oplus \text{Coker}(i_0) & \xrightarrow{\rho(i_0)} & 0 \\
\downarrow{\pi(i_0)} & \sigma(i_0) & \\
A(i_0) & \xrightarrow{\pi(i_0)} & B(i_0)
\end{array}
\]

we define it on \( \text{Im}(i_0) \) and \( \text{Coker}(i_0) \) separately. For \( \text{Coker}(i_0) \), as it is a projective abelian group, we define it by any homomorphism that makes commutative the diagram above when restricted to \( \text{Coker}(i_0) \). For \( \text{Im}(i_0) \) take \( x = \sum_{j=1,..,k} F(\alpha_j)(x_j) \) where \( \{i_1,..,i_k\} \) are \( k \) different objects, \( \alpha_j : i_j \to i_0, \text{deg}(\alpha_j) = 1 \) and \( x_j \in F(i_j) \) for \( j = 1,..,k \) (see Remark 3.1.4). Then define

\[ \rho(i_0)(x) = \sum_{j=1,..,k} (A(\alpha_j) \circ \rho(i_j))(x_j). \]

To check that \( \rho(i_0)(x) \) does not depend on the choice of the \( i_j \)'s, \( \alpha_j \)'s and \( x_j \)'s we have to prove that

\[ \sum_{j=1,..,k} F(\alpha_j)(x_j) = 0 \Rightarrow \sum_{j=1,..,k} (A(\alpha_j) \circ \rho(i_j))(x_j) = 0. \]

So suppose that

\[ \sum_{j=1,..,k} F(\alpha_j)(x_j) = 0. \]

(2)
Then using that $F$ is 1-pseudo-projective and Remark 3.1.4 we obtain objects $i_{j,j'}$, arrows $\alpha_{j,j'}$ of degree 1, and elements $x_{j,j'}$ for $j = 1, \ldots, k$, $j' = 1, \ldots, k_j$ such that

$$
(3) \sum_{j'=1,\ldots,k_j} F(\alpha_{j,j'})(x_{j,j'}) = x_j
$$

for every $j \in \{1, \ldots, k\}$. Notice that possibly not all the objects $i_{j,j'}$ are different.

Replacing Equation (3) in Equation (2) we obtain

$$
(4) \sum_{j = 1, \ldots, k, j' = 1, \ldots, k_j} F(\alpha_j \circ \alpha_{j,j'})(x_{j,j'}) = 0.
$$

Because in a graded poset there is at most one arrow between two objects, the condition $i_{j,j'} = i_{j',j''} = i$ implies $\alpha_j \circ \alpha_{j,j'} = \alpha_{j'} \circ \alpha_{j',j''} : i \to i_0$. So, considering objects $i \in \text{Ob}(\mathcal{P})$, we can rewrite (4) as

$$
(5) \sum_{i \in \text{Ob}(\mathcal{P})} F(\alpha_j \circ \alpha_{j,j'})(\sum_{j,j' | i_{j,j'} = i} x_{j,j'}) = 0.
$$

Call $\{i'_1, \ldots, i'_m\} = \{i_{j,j'} | j = 1, \ldots, k, j' = 1, \ldots, k_j\}$ where these sets have $m$ elements. Call $\beta_l = \alpha_j \circ \alpha_{j,j'}$ if $i'_l = i_{j,j'}$ and $y_l = \sum_{j,j' | i_{j,j'} = i'_l} x_{j,j'}$ for $l = 1, \ldots, m$. Notice that $\text{deg}(\beta_l) = 2$ for each $l$. Then Equation (5) becomes

$$
(6) \sum_{l = 1, \ldots, m} F(\beta_l)(y_l) = 0.
$$

Now we repeat the same argument: applying that $F$ is 2-pseudo-projective and the Remark 3.1.4 to Equation (6) we obtain objects $i''_{l,l'}$, arrows $\beta_{l,l'}$ of degree 1, and elements $y_{l,l'}$ for $l = 1, \ldots, m$, $l' = 1, \ldots, k'_l$ such that

$$
(7) \sum_{l' = 1, \ldots, k'_l} F(\beta_{l,l'})(y_{l,l'}) = y_l
$$

for every $l \in \{1, \ldots, m\}$. Substituting (7) in (6)

$$
\sum_{l = 1, \ldots, m, l' = 1, \ldots, k'_l} F(\beta_l \circ \beta_{l,l'})(y_{l,l'}) = 0.
$$

Now proceed as before regrouping the terms in this last equation.

In a finite number of steps, after a regrouping of terms as above, we find objects $i''_s$, arrows $\gamma_s$, and elements $z_s$ of degree 0 for $s = 1, \ldots, r$ which verify an equation

$$
(8) \sum_{s = 1, \ldots, r} F(\gamma_s)(z_s) = 0.
$$

Then pseudo-injectivity gives that $z_s \in \text{Im}(i''_s)$ for each $s$. As $\text{deg}(i''_s) = 0$ then $\text{Im}(i''_s) = 0$ and so $z_s = 0$ (notice that $z_s = 0$ for $s = 1, \ldots, r$ does not imply $x_j = 0$ for any $j$).

Recall that we want to prove that

$$
(9) \sum_{j = 1, \ldots, k} (A(\alpha_j) \circ \rho(i_j))(x_j) = 0.
$$
Substituting (3) in $\sum_{j=1,\ldots,k} (A(\alpha_j) \circ \rho(i_j))(x_j)$ we obtain
\[
\sum_{j=1,\ldots,k} (A(\alpha_j) \circ \rho(i_j))(x_j) = \sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j) \circ \rho(i_j) \circ F(\alpha_{j,j'}))(x_{j,j'})
\]
\[
= \sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j) \circ A(\alpha_{j,j'}) \circ \rho(i_{j,j'}))(x_{j,j'})
\]
\[
= \sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j \circ \alpha_{j,j'}) \circ \rho(i_{j,j'}))(x_{j,j'}),
\]
as $\rho$ is natural up to degree less than $n$. Then regrouping terms
\[
\sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j \circ \alpha_{j,j'}) \circ \rho(i_{j,j'}))(x_{j,j'}) = \sum_{i \in \text{Ob}(P)} (A(\alpha_j \circ \alpha_{j,j'} \circ \rho(i_{j,j'}))(x_{j,j'})) = \sum_{i \in \text{Ob}(P)} (A(\beta_i \circ \rho(i'_i))(y_i).
\]
Then, after a finite number of steps, we obtain
\[
\sum_{j=1,\ldots,k} (A(\alpha_j) \circ \rho(i_j))(x_j) = \sum_{s = 1,\ldots,r} (A(\gamma_s) \circ \rho(i''_s))(z_s) = 0
\]
as $z_s = 0$ for each $z = 1,\ldots,r$.

So we have checked that $\rho(i_0)(x)$ does not depend on the choice of $i_j, \alpha_j$ and $x_j$. It is straightforward that $\rho(i_0)$ on $\text{Im}(i_0)$ defined in this way is a homomorphism of abelian groups.

It remains to prove that $\pi(i_0) \circ \rho(i_0) = \sigma(i_0)$ when restricted to $\text{Im}(i_0)$. So take $x = \sum_{j=1,\ldots,k} F(\alpha_j)(x_j)$ in $\text{Im}(i_0)$. Then
\[
\pi(i_0)(\rho(i_0))(x) = \sum_{j=1,\ldots,k} (\pi(i_0) \circ A(\alpha_j) \circ \rho(i_j))(x_j)
\]
\[
= \sum_{j=1,\ldots,k} (B(\alpha_j) \circ \pi(i_j) \circ \rho(i_j))(x_j), \pi \text{ is a natural transformation}
\]
\[
= \sum_{j=1,\ldots,k} (B(\alpha_j) \circ \sigma(i_j))(x_j), \text{by the inductive hypothesis}
\]
\[
= \sum_{j=1,\ldots,k} (\sigma(i_0) \circ F(\alpha_j))(x_j), \sigma \text{ is a natural transformation}
\]
\[
= \sigma(i_0)(x)
\]
Defining $\rho(i_0)$ in this way for every $i_0 \in \text{Ob}_n(P)$ we have now $\rho$ defined on all objects of $P$ of degree less or equal than $n$. Finally, to complete the inductive step we have to prove that $\rho$ restricted to the full subcategory over these objects is a natural transformation. Take $\alpha : i \to i_0$ in this full subcategory. If the degree of $i_0$ is less
than $n$ then the commutativity of
\[
\begin{array}{c}
F(i) \xrightarrow{F(\alpha)} F(i_0) \\
\downarrow \rho(i) \downarrow \rho(i_0) \\
A(i) \xrightarrow{A(\alpha)} A(i_0)
\end{array}
\]
is granted by the inductive hypothesis. Suppose that the degree of $i_0$ is $n$. Take $x' \in F(i)$. Because $\mathcal{P}$ is graded there exists $\alpha_1 : i_1 \to i_0$ of degree 1 and $\alpha' : i \to i_1$ such that $\alpha = \alpha_1 \circ \alpha'$:
\[
\begin{array}{c}
i \\
\downarrow \alpha' \downarrow \alpha_1 \\
i_1
\end{array}
\]
Write $x = F(\alpha)(x') = F(\alpha_1)(x_1)$ where $x_1 = F(\alpha')(x')$. Then, by definition of $\rho(i_0)$ on $\text{Im}(i_0)$,
\[
\rho(i_0)(x) = (A(\alpha_1) \circ \rho(i_1))(x_1) = (A(\alpha_1) \circ \rho(i_1))(F(\alpha')(x')) = (A(\alpha_1) \circ \rho(i_1) \circ F(\alpha'))(x') = (A(\alpha_1 \circ \alpha') \circ \rho(i))(x'), \text{ $\rho$ is natural up to degree less than $n$}
\]
and so the diagram commutes. 

**Remark 3.1.11.** As the following example shows the condition of lower boundedness of $\mathcal{P}$ in Theorem 3.1.10 cannot be dropped:

Consider the inverse ‘telescope category’ $\mathcal{P}$ with shape
\[
\ldots \to \cdot \to \cdot \to \cdot
\]
It is a graded poset which is not bounded below. Consider the functor of constant value $\mathbb{Z}/p$, $c_{\mathbb{Z}/p} : \mathcal{P} \to \text{Ab}$:
\[
\ldots \to \mathbb{Z}/p \to \mathbb{Z}/p \to \mathbb{Z}/p
\]
It is straightforward that it is a pre-projective functor as all the cokernels are zero and all the arrows are injective. But it is not a projective object of $\text{Ab}^\mathcal{P}$ because, in that case, the adjoint pair $c : \text{Ab} \leftrightarrow \text{Ab}^\mathcal{P} : \lim$ would give that $\mathbb{Z}/p$ is projective in $\text{Ab}$ (right adjoints preserve projectives, see [13, 3.2, Ex7]).

Thus for the categories that are graded and bounded below we have the useful

**Corollary 3.1.12.** Let $\mathcal{P}$ be a bounded below graded poset and let $F : \mathcal{P} \to \text{Ab}$ be a functor. Then $F$ is projective if and only if it is pre-projective.

This corollary yields the following examples. The degree functions $\text{deg}$ for the bounded below graded posets appearing in the examples are indicated by subscripts $i_{\text{deg}(i)}$ on the objects $i \in \text{Ob}(\mathcal{P})$ and take values $\{0, 1, 2, 3, \ldots\}$. 

Example 3.1.13. For the ‘pushout category’ $\mathcal{P}$ with shape

$$
\begin{array}{ccc}
a_0 & \xrightarrow{f} & b_1 \\
\downarrow{g} & & \downarrow{} \\
c_1 & & \\
\end{array}
$$

a functor $F : \mathcal{P} \to \text{Ab}$ is projective if and only if

- $F(a), F(b)/\text{Im } F(f)$ and $F(c)/\text{Im } F(g)$ are free abelian.
- $F(f)$ and $F(g)$ are monomorphisms.

Example 3.1.14. For the ‘telescope category’ $\mathcal{P}$ with shape

$$
\begin{array}{cccccccc}
a_0 & \xrightarrow{f_1} & a_1 & \xrightarrow{f_2} & a_2 & \xrightarrow{f_3} & a_3 & \xrightarrow{f_4} & \cdots \\
\end{array}
$$

a functor $F : \mathcal{P} \to \text{Ab}$ is projective if and only if

- $F(a_0)$ is free abelian.
- $F(a_i)/\text{Im } F(f_i)$ is free abelian, $F(f_i \circ f_{i-1} \circ \cdots \circ f_0)$ is a monomorphism and $\ker F(f_i \circ f_{i-1} \circ \cdots \circ f_{i-d}) \subseteq \text{Im } F(f_{i-d-1})$ for $d = 0, 1, \ldots, i - 1$ for each $i = 1, 2, 3, 4, \ldots$.

3.2. Pseudo-projectivity

Consider a functor $F : \mathcal{P} \to \text{Ab}$ over a graded poset $\mathcal{P}$. In this section we look for, and find, conditions on $F$ such that $\lim_{\longrightarrow} F = 0$ for $i \geq 1$, i.e., we want conditions such that the left derived functors of the right exact functor $\lim_{\longrightarrow}$ vanish on $F$. Fix the following notation

**Definition 3.2.1.** Let $\mathcal{P}$ be a graded poset and $F : \mathcal{P} \to \text{Ab}$. We say $F$ is $\lim_{\longrightarrow}$-acyclic if $\lim_{\longrightarrow} F = 0$ for $i \geq 1$.

Recall that for projective objects it holds that any left derived functor vanishes. So, from Proposition 3.1.10, we obtain firstly that

**Proposition 3.2.2.** Let $F : \mathcal{P} \to \text{Ab}$ be a pre-projective functor over a bounded below graded poset $\mathcal{P}$. Then $F$ is $\lim_{\longrightarrow}$-acyclic.

Because being $\lim_{\longrightarrow}$-acyclic is clearly weaker that being projective we can wonder if is it possible to weaken the hypothesis on Proposition 3.2.2 keeping the thesis of $\lim_{\longrightarrow}$-acyclicity. The answer is yes and the following theorem states the appropriate conditions. Notice that we have removed the condition (I) of Definition 3.1.9.

**Theorem 3.2.3.** Let $F : \mathcal{P} \to \text{Ab}$ be a pseudo-projective functor over a bounded below graded poset $\mathcal{P}$. Then $F$ is $\lim_{\longrightarrow}$-acyclic.

*Proof.* We can suppose that the degree function $\text{deg}$ on $\mathcal{P}$ is increasing and takes values $\{0, 1, 2, 3, \ldots \}$, and that $\text{Ob}_0(\mathcal{P}) \neq \emptyset$. To compute $\lim_{\longrightarrow} F$ we use the (normalized, Remark 2.0.6) spectral sequences corresponding to the third row of Table 1 in Chapter 2. That is, we first filter by the degree of the end object of each
simplex to obtain a homological type spectral sequence $E_{p**}$. To compute the column
$E_{p**}^1$ we filter by the degree of the initial object of each object to obtain cohomological
type spectral sequences $(E_p)^{p**}$. Fix such an $p$. Notice that to prove that $\lim_p F = 0$ is enough to show that $E_{p-t}^1$ is
zero for every $p$. The contributions to $E_{p-t}^1$ come from $(E_p)^{p-p-t}$ for $p' \leq p - t$
(we are using normalized (Remark 2.0.6) spectral sequences). We prove that
$$(E_p)^{p-p-t} = 0$$
if $r$ is big enough for each $p$ and $p' \leq p - t$. This implies that $\lim_{p'} F = 0$.

Consider the increasing filtration $L_*$ of $C_*(P, F)$ that gives rise to the spectral
sequence $E_{p*}$. The $n$-simplices are
$$L_1 = L_1 C_n(P, F) = \bigoplus_{\sigma \in NP, n, deg(\sigma) \leq p} F_\sigma.$$ 

For each $p$ we have a decreasing filtration $M_p$ of the quotient $L_1/L_1^{p-1}$ that gives rise
to the spectral sequence $(E_p)^{p**}$ and which $n$-simplices are
$$(M_p)^{p'} = \bigoplus_{\sigma \in NP, n, deg(\sigma) \geq p', deg(\sigma) = p} F_\sigma.$$ 

For $p' \leq p - t$ the abelian group $(E_p)^{p',q'}$ at the $t = -(p' + q') + p$ simplices is given by
$$(E_p)^{p',q'} = (M_p)^{p'} \cap d^{-1}((M_p)^{p'+r})/((M_p)^{p'+1} \cap d^{-1}((M_p)^{p'+r}) + (M_p)^{p'} \cap d((M_p)^{p'-r+1}))$$
where $d$ is the differential of the quotient $L_1/L_1^{p-1}$ restricted to the subgroups of the
filtration $(M_p)^*$. For $r > p - p' + (t - 1)$ there are not $(t - 1)$-simplices beginning
in degree at least $p' + r > p - (t - 1)$ and ending in degree $p$, i.e., $(M_p)^{p'-r} = 0$.
Because $P$ is bounded below for $r$ big enough $(M_p)^{p'-r+1} = (M_p)^{0}_{t+1} = (L_1/L_1^{p-1})_{t+1}$,
i.e., $(M_p)^{p'-r+1}$ equals all the $(t+1)$-simplices that end on degree $p$. Thus there exists
$r$ such that
$$E_{p} E_{p, r}^{p',q'} = (M_p)^{p'} \cap d^{-1}(0)/(M_p)^{p'+1} \cap d^{-1}(0) + (M_p)^{p'} \cap d((M_p)^{0}_{t+1}).$$

Fix such an $r$ and take $[x] \in (E_{p}, E_{p, r})^{p',q'}$ where
$$x = \bigoplus_{\sigma \in NP, deg(\sigma) \geq p', deg(\sigma) = p} x_\sigma$$
and $d(x) = 0$. Notice that by definition there is just a finite number of summands
$x_\sigma \neq 0$ in the expression (11) for $x$. We prove that $[x] = 0$ in three steps:

**Step 1:** In this first step we find a representative $x'$ for $[x]$
$$x' = \bigoplus_{\sigma \in NP, deg(\sigma) \geq p', deg(\sigma) = p} x_\sigma$$
such that $deg(\alpha_1) = 1$ for every $\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t$ with
$x_\sigma' \neq 0$. 

Take \( \sigma \) such that \( x_\sigma \neq 0 \) and suppose that \( \text{deg}(\alpha_1) > 1 \), i.e., \( \text{deg}(\sigma_0) < \text{deg}(\sigma_1) - 1 \). Then, as in a graded poset every morphism factors as composition of degree 1 morphisms, there exists an object \( \sigma_* \) of degree \( \text{deg}(\sigma_0) < \text{deg}(\sigma_*) < \text{deg}(\sigma_1) \) and arrows \( \beta_1: \sigma_0 \to \sigma_* \) and \( \beta_2: \sigma_* \to \sigma_1 \) with \( \alpha_1 = \beta_2 \circ \beta_1 \).

\[
\begin{array}{c}
\sigma_0 \xrightarrow{\alpha_1} \sigma_* \xleftarrow{\beta_1} \sigma_1.
\end{array}
\]

Call \( \tilde{\sigma} \) to the \((t + 1)\)-simplex \( \sigma = \sigma_0 \xrightarrow{\beta_1} \sigma_* \xrightarrow{\beta_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t \) and consider the \((t + 1)\)-chain of \( (M_p)_t^0 : y = i_\sigma(-x_\sigma) \). Its differential in \( L^p/L^{p-1} \) equals

\[
d(y) = d_0(y) - d_1(y) + \sum_{i=2,\ldots,t} (-1)^i d_i(y) = d_0(y) + i_\sigma(x_\sigma) + \sum_{i=2,\ldots,t} (-1)^i d_i(y).
\]

Notice that the first morphisms appearing in the simplices \( d_0(\tilde{\sigma}) \) and \( d_i(\tilde{\sigma}) \) for \( i = 2, \ldots, t \) have degree \( \text{deg}(\beta_2) \) and \( \text{deg}(\beta_1) \) respectively, which are strictly less than \( \text{deg}(\alpha_1) \). Also notice that \( d(y) \in (M_p)_t^0 \cap d((M_p)_t^0) \) (which is zero in Equation \((10)\)).

Taking the (finite) sum of the chains \( y \) for each term \( x_\sigma \) we find that \( [x] = [x'] \) where

\[
x' = \bigoplus_{\sigma \in NP_t, \text{deg}(\sigma_0) \geq p', \text{deg}(\sigma_t) = p} x'_\sigma.
\]

and the maximum of the degrees of the morphisms \( \alpha_1 \) of the simplices

\[
\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t
\]

with \( x'_\sigma \neq 0 \) is smaller than this maximum computed for \( x \). So repeating this process a finite number of times we find a representative as wished. For simplicity we write also \( x \) for this representative.

**Step 2:** By Step 1 we can suppose that \( \text{deg}(\alpha_1) = 1 \) for every

\[
\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t
\]

with \( x_\sigma \neq 0 \). Now our objective is to find a representative \( x' \) for \([x]\)

\[
x' = \bigoplus_{\sigma \in NP_t, \text{deg}(\sigma_0) = p', \text{deg}(\sigma_t) = p} x'_\sigma,
\]

i.e., such that the expression for \( x' \) runs over simplices \( \sigma \) with begin in degree \( p' \). Begin writing \( x \) as

\[
x = \bigoplus_{i=p',\ldots,p-t} x_i
\]

where

\[
x_i = \bigoplus_{\sigma \in NP_t, \text{deg}(\sigma_0) = i, \text{deg}(\sigma_t) = p} x_\sigma.
\]
Notice that the index \( i \) just goes to \( p-t \) (and not to \( p \)) because we are using normalized (Remark \([2.0.6]\)) spectral sequences. Now we prove

**Claim 3.2.3.1.** For each \( i \) from \( i = p - t \) to \( i = p' \) there exists a representative \( x'_i \) for \([x]\)

\[
x'_i = \bigoplus_{\sigma \in NP_t, i \geq \deg(\sigma_0) \geq p', \deg(\sigma_1) = p} (x'_i)_\sigma
\]

such that

\[
(x'_i)_\sigma \neq 0 \quad \text{and} \quad \deg(\sigma_0) < i \implies \deg(\alpha_1) = 1.
\]

Notice that taking \( i = p' \) in the claim, the step 2 is finished. The case \( i = p - t \) in the claim is fulfilled taking \( x'_{p-t} = x \) (by step 1). Suppose the statement of the claim holds for \( i \). Then we prove it for \( i - 1 \). We have \( x'_i \) such that

\[
x'_i = \bigoplus_{\sigma \in NP_t, i \geq \deg(\sigma_0) \geq p', \deg(\sigma_1) = p} (x'_i)_\sigma,
\]

d\( (x'_i) = 0 \) and \([x] = [x'_i] \). The differential \( d \) on \( L^p/L^{p-1} \) restricts to

\[
d : (M_p)^{p'}_t \to (M_p)^{p'}_{t-1}
\]

and carries \( z \in F_\sigma \to \bigoplus_{\sigma \in NP_t, \deg(\sigma_0) \geq p', \deg(\sigma_1) = p} F_\sigma = (M_p)^{p'}_t \) to

\[
d(z) = \sum_{j=0, \ldots, t-1} (-1)^j d_j(z)
\]

with \( d_j(z) \in F_{d_j(\sigma)} \to (M_p)^{p'}_{t-1} \). Notice that the initial object of \( d_j(\sigma) \) is \( \sigma_1 \) for \( j = 0 \) and \( \sigma_0 \) for \( j = 1, \ldots, t-1 \). Also notice that the final object of \( d_j(\sigma) \) is \( \sigma_t \) for \( j = 0, \ldots, t-1 \).

By hypothesis \( d(x'_i) = 0 \). So for every \( \epsilon \in NP_{t-1} \) with \( \deg(\epsilon_0) \geq p' \) and \( \deg(\epsilon_{t-1}) = p \) we can apply the projection

\[
\pi_\epsilon : (M_p)^{p'}_{t-1} \to F_\epsilon
\]

and obtain \( \pi_\epsilon(d(x'_i)) = 0 \). If \( \deg(\epsilon_0) > i \) then the remarks on the differential above and condition \([12]\) imply that

\[
\pi_\epsilon(d(x'_i)) = \sum_{\sigma \in NP_t, \deg(\sigma_0) = i, d_0(\sigma) = \epsilon} F(\alpha_1)((x'_i)_\sigma)
\]

and thus

\[
(13) \quad 0 = \sum_{\sigma \in NP_t, \deg(\sigma_0) = i, d_0(\sigma) = \epsilon} F(\alpha_1)((x'_i)_\sigma)
\]

for each \( \epsilon \in NP_{t-1} \) with \( \deg(\epsilon_0) > i \) and \( \deg(\epsilon_{t-1}) = p \). Notice that each summand \( (x'_i)_\sigma \) with \( \sigma \in NP_t \), \( \deg(\sigma_0) = i \) and \( \deg(\sigma) = p \) appears in one and just one equation as \([13]\) (take \( \epsilon = d_0(\sigma) \)).

Fix an \( \epsilon \in NP_{t-1} \) with \( \deg(\epsilon_0) > i \) and \( \deg(\epsilon_{t-1}) = p \) and consider the associated Equation \([13]\). Then, as \( F \) is \((i - \deg(\epsilon_0))-\text{pseudo-projective}, (x'_i)_\sigma \in \text{Im}(\sigma_0) \) for every \( \sigma \in NP_t \) with \( \deg(\sigma_0) = i \) and \( d_0(\sigma) = \epsilon \). This means that for every such a \( \sigma \) there
exists \( k_{\sigma} \) objects of degree \((i - 1)\), namely \( i_{\sigma}^1, \ldots, i_{\sigma}^{k_{\sigma}} \), arrows \( \beta_{\sigma}^i : i_{\sigma}^i \to \sigma_0 \) and elements \( x_{\sigma}^j \in F(i_{\sigma}^j) \) for \( j = 1, \ldots, k_{\sigma} \) such that

\[
(x'_{i})_{\sigma} = \sum_{j=1}^{k_{\sigma}} F(\beta_{\sigma}^j)(x_{\sigma}^j).
\]

Consider the \((t + 1)\)-simplices for \( j = 1, \ldots, k_{\sigma} \)

\[
\sigma^j = i_{\sigma}^j \xrightarrow{\beta_{\sigma}^j} \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t
\]

and the \((t + 1)\)-chain of \((M_p)_{t+1}^{i-1}

\[
y_{\sigma} = \bigoplus_{j=1}^{k_{\sigma}} i_{\sigma}^j(x_{\sigma}^j).
\]

The differential of \( y_{\sigma} \) is

\[
d(y_{\sigma}) = d_0(y_{\sigma}) + \sum_{j=1}^{t} (-1)^j d_j(y_{\sigma})
\]

\[
= d_0(y_{\sigma}) + R_{\sigma}, \text{ where } R_{\sigma} = \sum_{j=1}^{t} (-1)^j d_j(y_{\sigma})
\]

\[
= \sum_{j=1}^{k_{\sigma}} i_{\sigma}^j(F(\beta_{\sigma}^j)(x_{\sigma}^j)) + R_{\sigma}
\]

\[
= \sum_{j=1}^{k_{\sigma}} i_{\sigma}(F(\beta_{\sigma}^j)(x_{\sigma}^j)) + R_{\sigma}
\]

\[
= i_{\sigma}(\sum_{j=1}^{k_{\sigma}} F(\beta_{\sigma}^j)(x_{\sigma}^j)) + R_{\sigma}
\]

\[
= i_{\sigma}((x'_{i})_{\sigma}) + R_{\sigma}
\]

where the last equality is due to (14). Notice that \( R_{\sigma} \) lives in the subgroup \( \bigoplus_{\sigma \in NP_t, deg(\sigma_0) = i-1} F_{\sigma} \subseteq (M_p)_{t}^{p'} \) of simplices beginning at degree \((i - 1)\). Repeating the same construction for each \( \sigma \in NP_t \) with \( deg(\sigma_0) = i \) and \( d_0(\sigma) = \epsilon \) we obtain \( y_{\epsilon} = \sum_{\sigma} y_{\sigma} \) such that

\[
d(y_{\epsilon}) = \bigoplus_{\sigma \in NP_{t}, deg(\sigma_0) = i, d_0(\sigma) = \epsilon} (x'_{i})_{\sigma} + R_{\epsilon}
\]

where \( R_{\epsilon} \) lives in the subgroup \( \bigoplus_{\sigma \in NP_{t-1}, deg(\sigma_0) = i-1} F_{\sigma} \subseteq (M_p)_{t-1}^{p'} \). Repeating the same argument for every \( \epsilon \in NP_{t-1} \) with \( deg(\epsilon_0) > i \) and \( deg(\epsilon_t-1) = p \) we obtain \( y = \sum_{\epsilon} y_{\epsilon} \) such that

\[
d(y) = \bigoplus_{\sigma \in NP_{t-1}, deg(\sigma_0) = i, d_0(\sigma_t) = p} (x'_{i})_{\sigma} + R
\]

where \( R \) lives in the subgroup \( \bigoplus_{\sigma \in NP_{t-1}, deg(\sigma_0) = i-1, d_0(\sigma_t) = p} F_{\sigma} \subseteq (M_p)_{t}^{p'} \). By construction \( y \in (M_p)_{t+1}^{i-1} \subseteq (M_p)_{t+1}^{0} \) and \( d(y) \in (M_p)_{t}^{i-1} \subseteq (M_p)_{t}^{p'} \). Thus \( d(y) \in (M_p)_{t}^{p'} \cap d((M_p)_{t+1}^{0}) \). Then, by (10), [14], \( [x'_{i}] = [x_{i} - d(y)] = [x'_{i-1}] \) where

\[
x'_{i-1} = \bigoplus_{\sigma \in NP_{t}, i > deg(\sigma_0) \geq p'} (x'_{i})_{\sigma} + R
\]
is a representative that lives in
\[ \bigoplus_{\sigma \in \mathcal{NP}_t, \deg(\sigma_0) \geq p'} F_\sigma \subseteq (M_p)_t^{p'} \]
as wished. That condition (12) holds is clear from the definition of \( x'_{i-1} \).

**Step 3:** By Step 2 we can suppose that
\[ x = \bigoplus_{\sigma \in \mathcal{NP}_t, \deg(\sigma_0) = p', \deg(\sigma_1) = p} x_\sigma. \]

Our objective now is to see that there exists \( y \in (M_p)_t^0 \) with \( d(y) = x \). This implies that \([x] = 0\) and finishes the proof of the theorem. We need the

**Claim 3.2.3.2.** There exist chains \( x_i \in (M_p)_t^0 \) for \( i = p', \ldots, 0 \) and \( y_i \in (M_p)_t^{i+1} \) for \( i = p', \ldots, 1 \) such that
\[ d(y_i) = x_i + x_{i-1} \]
for \( i = p', \ldots, 1 \) with \( x_p' = x \) and \( x_0 = 0 \) such that

1. \( x_i \) lives on \( \bigoplus_{\sigma \in \mathcal{NP}_t, \deg(\sigma_0) = i, \deg(\sigma_1) = p} F_\sigma \subseteq (M_p)_t^0 \) for \( i = p', \ldots, 0 \).
2. \( d(x_i) = 0 \) for \( i = p', \ldots, 0 \).

Notice that the claim finishes Step 3: as \( x_0 = 0 \) then \( x_1 = d(y_1), x_2 = d(y_2) - x_1 = d(y_2 - y_1), x_3 = d(y_3) - x_2 = d(y_3 - y_2 + y_1), \ldots, x = x_p' = d(y_p') - x_{p'-1} = d(y_p' - y_{p'-1} + \ldots + (-1)^{p'+1}y_1) \) where \( y_p' - y_{p'-1} + \ldots + (-1)^{p'+1}y_1 \in (M_p)_t^{p'} \).

Define \( x_{p'} = x \). Then condition (11) and (12) are satisfied for \( i = p' \). We construct \( y_i \) and \( x_{i-1} \) from \( x_i \) recursively beginning on \( i = p' \). The arguments are similar to those used in step 2.

The differential \( d \) on \( L^p/L^{p-1} \) restricts to
\[ d : (M_p)_t^0 \to (M_p)^0_{t-1}. \]
As \( d(x_{p'}) = d(x) = 0 \), for every \( \epsilon \in \mathcal{NP}_{t-1} \) with \( \deg(\epsilon_{t-1}) = p \) we can apply the projection
\[ \pi_\epsilon : (M_p)_{t-1}^0 \to F_\epsilon \]
and obtain \( \pi_\epsilon(d(x)) = 0 \). If \( \deg(\epsilon_0) > p' \) then
\[ \pi_\epsilon(d(x)) = \sum_{\sigma \in \mathcal{NP}_t, d_0(\sigma) = \epsilon} F(\alpha_1)(x_\sigma) \]
and thus
\[ 0 = \sum_{\sigma \in \mathcal{NP}_t, d_0(\sigma) = \epsilon} F(\alpha_1)(x_\sigma) \]
for each \( \epsilon \in \mathcal{NP}_{t-1} \) with \( \deg(\epsilon_0) > p' \) and \( \deg(\epsilon_{t-1}) = p \). Notice that each summand \( x_\sigma \) with \( \sigma \in \mathcal{NP}_t \), \( \deg(\sigma_0) = p' \) and \( \deg(\sigma) = p \) appears in one and just one equation as (16) (take \( \epsilon = d_0(\sigma) \)). Using now pseudo-injectivity we build as before \( y_\sigma, y_\epsilon = \sum_\sigma y_\sigma \).
3.3. Computing Higher Limits

and \( y = \sum_{\epsilon} y_{\epsilon} \), where \( \epsilon \) runs over \( \epsilon \in NP_{t-1} \) with \( \text{deg}(\epsilon_0) > p' \) and \( \text{deg}(\epsilon_{t-1}) = p \), such that

\[
d(y) = x + R
\]

with \( R \) living in \( \bigoplus_{\sigma \in NP, \text{deg}(\sigma_0) = p'_{-1}, \text{deg}(\sigma_{t-1}) = p} F_{\sigma} \subseteq (M_p)_t^0 \). Call \( y_{p'} \stackrel{\text{def}}{=} y \) and \( x_{p'_{-1}} = R \).

Then Equation (15) is satisfied. Condition (1) for \( i = p'_{-1} \) holds by the construction of \( y_{p'} \) and condition (2) for \( i = p'_{-1} \) holds because \( d(x_{p'_{-1}}) = d(R) = d(d(y) - x) = d^2(y) - d(x) = 0 - 0 = 0 \) as \( d \) is a differential and \( d(x) = 0 \) by hypothesis. The construction of \( y_i \) and \( x_{i-1} \) from \( x_i \) is totally analogous to the construction of \( y_{p'} \) and \( x_{p'_{-1}} \) that we have just made.

After we have built \( y_1 \) and \( x_0 \) if we try to build \( y = \sum_{\epsilon} y_{\epsilon} \) and \( R \) from \( x_0 \) we find that, because there are not objects of negative degree (thus if \( z \in \text{Im}(i') \) where \( \text{deg}(i') = 0 \) then \( z = 0 \)), \( x_0 = 0 \).

The following examples come from Example (3.1.13). They show the weaker conditions that are needed for \( \text{lim} \overrightarrow{\text{acyclic}} \) instead of projectiveness.

**Example 3.2.4.** For the “pushout category” \( \mathcal{P} \) with shape

\[
\begin{array}{ccc}
a_0 & \xrightarrow{f} & b_1 \\
\downarrow{g} & & \\
\downarrow{c_1} & & \\
\end{array}
\]

a functor \( F : \mathcal{P} \rightarrow \text{Ab} \) is \( \text{lim} \overrightarrow{\text{acyclic}} \) if \( F(f) \) and \( F(g) \) are monomorphisms.

For the “telescope category” \( \mathcal{P} \) with shape

\[
\begin{array}{cccccccc}
a_0 & \xrightarrow{f_1} & a_1 & \xrightarrow{f_2} & a_2 & \xrightarrow{f_3} & a_3 & \xrightarrow{f_4} & \ldots \\
\end{array}
\]

a functor \( F : \mathcal{P} \rightarrow \text{Ab} \) is \( \text{lim} \overrightarrow{\text{acyclic}} \) if \( F(f_1 \circ f_{i-1} \circ \ldots \circ f_1) \) is a monomorphism and \( \text{Ker} F(f_1 \circ f_{i-1} \circ \ldots \circ f_{i-d+1}) \subseteq \text{Im} F(f_{i-d}) \) for \( d = 1, 2, 3, \ldots, i-1 \) for each \( i = 2, 3, 4, \ldots \)

Notice that for this is enough that \( F(f_i) \) is a monomorphism for each \( i = 1, 2, 3, \ldots \)

**3.3. Computing higher limits**

Theorem 3.2.3 shows that over a bounded below graded poset pseudo-projectivity is enough for \( \text{lim} \overrightarrow{\text{acyclic}} \). But it turns out that pseudo-projectivity is not necessary for \( \text{lim} \overrightarrow{\text{acyclic}} \):

**Example 3.3.1.** For the “pullback category” \( \mathcal{P} \) with shape

\[
\begin{array}{ccc}
a_0 & \xrightarrow{f} & b_0 \\
\downarrow{g} & & \\
\downarrow{c_1} & & \\
\end{array}
\]

a functor \( F : \mathcal{P} \rightarrow \text{Ab} \) is pseudo-projective if

- \( F(f) \) and \( F(g) \) are monomorphisms.
- \( \text{Im} F(f) \cap \text{Im} F(g) = 0 \).
But a straightforward calculus shows $\lim_{\to i} F = 0$ for $i \geq 1$ for any $F$.

This shows that pseudo-projectivity is not necessary for $\lim$-acyclicity. However, we shall see how pseudo-projectivity allows us to obtain a better knowledge of the higher limits $\lim_{\to i} F$. We begin with

**Definition 3.3.2.** Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a graded poset $\mathcal{P}$. $F' : \mathcal{P} \to \text{Ab}$ is the functor which takes values on objects

$$F'(i_0) = \bigoplus_{\alpha : i \to i_0} F(i)$$

for $i_0 \in \text{Ob}(\mathcal{P})$. For $\beta \in \text{Hom}(\mathcal{P})$, $\beta : i_1 \to i_0$, $F'(\beta)$ is the only homomorphism which makes commute the diagram

$$\begin{tikzcd}
F'(i_1) \ar{r}{F'(\beta)} & F'(i_0) \\
F(i) \ar{u} \ar[swap]{r}{\alpha : i \to i_1} & F(i) \ar{u}
\end{tikzcd}$$

for each $\alpha : i \to i_1$.

$F'$ is built from $F$ as $\text{Coker}'$ was built from $\text{Coker}$ in Section 3.1. It mimics the construction of free objects in $\text{Ab}^\mathcal{P}$. Notice that $\text{Coker}_{F'}(i) = F(i)$ for each $i \in \text{Ob}(\mathcal{P})$. A nice property of $F'$ is

**Lemma 3.3.3.** Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a graded poset. Then for each $G \in \text{Ab}^\mathcal{P}$ there is a bijection

$$\text{Hom}_{\text{Ab}^\mathcal{P}}(F', G) \xrightarrow{\varphi} \prod_{i \in \text{Ob}(\mathcal{P})} \text{Hom}_{\text{Ab}}(F(i), G(i)).$$

**Proof.** $\varphi$ is given by

$$\varphi(\nu : F' \Rightarrow G)_i = (F(i) \xrightarrow{i_{i_0}} F'(i) \xrightarrow{\nu_i} G(i)).$$

For a family $\tau = \{\tau_i\}_{i \in \text{Ob}(\mathcal{P})} \in \prod_{i \in \text{Ob}(\mathcal{P})} \text{Hom}_{\text{Set}}(F(i), G(i))$ define the natural transformation $\psi(\tau) : F' \Rightarrow G$ on the object $i_0 \in \text{Ob}(\mathcal{P})$ as the only homomorphism which makes commute the diagram

$$\begin{tikzcd}
F'(i_0) \ar{r}{\psi(\tau)_{i_0}} & F(i_0) \\
F(i) \ar{u} \ar[swap]{r}{\tau_i} & G(i) \ar{u} \ar{u}{G(\alpha)}
\end{tikzcd}$$

for every $\alpha : i \to i_0$. Then both compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity. \qed

Another interesting property of $F'$ is the following
Lemma 3.3.4. Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a graded poset. Then
\[
\varprojlim F' \cong \bigoplus_{i \in \text{Ob}(\mathcal{P})} F(i).
\]

**Proof.** It is straightforward using the previous lemma. Notice that the cone \( \eta : F' \to \varprojlim F' \) is given by the homomorphisms \( \eta_i \) for each \( i_0 \in \text{Ob}(\mathcal{P}) \) which make commutative the diagrams
\[
\begin{array}{ccc}
F'(i_0) & \xrightarrow{\eta_{i_0}} & \bigoplus_{i \in \text{Ob}(\mathcal{P})} F(i) \\
\downarrow & & \downarrow \\
F(i)_{\alpha : i \to i_0} & \xrightarrow{1} & F(i)
\end{array}
\]
for each \( \alpha : i \to i_0 \). This description shall be useful later. \( \square \)

The main feature of $F'$ we shall use is

**Lemma 3.3.5.** Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a graded poset. Then $F'$ is pseudo-projective.

**Proof.** Take \( k \) objects \( i_j \in \text{Ob}(\mathcal{P}) \), arrows \( \alpha_j : i_j \to i_0 \) with \( \text{deg}(\alpha_j) = d \) and \( y_j \in F'(i_j) \) \( (j = 1, \ldots, k) \) such that
\[
\sum_{j=1}^k F'(\alpha_j)(y_j) = 0 \text{ (in } F'(i_0)).
\]
We want that \( y_j \in \text{Im}_{F'}(i_j) \) for \( j = 1, \ldots, k \). Write \( y_j = \bigoplus_{\alpha:i \to i_j} y_{j, \alpha} \).

Fix \( j_0 \in \{1, \ldots, k\} \) and consider the projection \( p_{j_0} \) from \( F'(i_0) \) onto the direct summand \( F'(i_{j_0}) \hookrightarrow F'(i_0) \) which corresponds to \( \alpha_{j_0} : i_{j_0} \to i_0 \)
\[
F'(i_0) \xrightarrow{p_{j_0}} F'(i_{j_0}).
\]
For any \( y = \bigoplus_{\alpha:i \to i_j} y_{\alpha} \in F'(y_j) \),
\[
p_{j_0}(F'(\alpha_j)(y)) = \sum_{\alpha : i \to i_j, \alpha \circ \alpha = \alpha_{j_0}} y_{\alpha}.
\]
So, for \( y_j = \bigoplus_{\alpha:i \to i_j} y_{j, \alpha} \in F'(i_j) \) we have
\[
p_{j_0}(F'(\alpha_j)(y_j)) = \sum_{\alpha : i \to i_j, \alpha \circ \alpha = \alpha_{j_0}} y_{j, \alpha}.
\]
This last sum runs over \( \alpha : i_{j_0} \to i_j \) such that the following triangle commutes
\[
\begin{array}{ccc}
i_{j_0} & \xrightarrow{\alpha_{j_0}} & i_0 \\
\downarrow & & \downarrow \\
i_j & \xrightarrow{\alpha} & i_0 \\
\end{array}
\]
Because we are in a graded poset and \( \deg(i_j) = d \) for each \( j = 1, \ldots, k \) then the only chance is \( i_j = i_{j_0} \) and \( \alpha = 1_{i_{j_0}} \). Because the objects \( i_1, \ldots, i_k \) are different this implies that \( j = j_0 \) too. Thus

\[
p_{j_0}(F'(\alpha_j)(y_j)) = \begin{cases} y_{j_0,1_{i_{j_0}}} & \text{for } j = j_0 \\ 0 & \text{for } j \neq j_0. \end{cases}
\]

Then

\[
0 = p_{j_0}(\sum_{j=1,\ldots,k} F'(\alpha_j)(y_j)) = p_{j_0}(\sum_{j=1,\ldots,k} F'(\alpha_j)(y_j)) = \sum_{j=1,\ldots,k} p_{j_0}(F'(\alpha_j)(y_j)) = y_{j_0,1_{i_{j_0}}}
\]

As \( j_0 \) was arbitrary this means that \( y_{j,1_{i_j}} = 0 \) for each \( j \in \{1, \ldots, k\} \). Now it is clear by the definition of \( F' \) that \( y_j \in \text{Im}_{F'(i_j)} \) for \( j = 1, \ldots, k \).

\[\square\]

**Remark 3.3.6.** The epic natural transformation \( G' \Rightarrow F \), where \( G = \mathbb{Z} \circ \mathcal{U} \circ F \) with \( \mathcal{U} : \text{Ab} \to \text{Set} \) the forgetful functor and \( \mathbb{Z} : \text{Set} \to \text{Ab} \) the free abelian group on a set, is the usual way to prove that \( \text{Ab}^\mathcal{C} \) has enough projectives for any small category \( \mathcal{C} \).

By Lemma 3.3.3 for the family of homomorphisms \( \{F(i) \overset{1_F(i)}{\to} F(i)\}_{i \in \text{Ob}(\mathcal{P})} \) we have a natural transformation \( \pi : F' \Rightarrow F \). It is clear that \( \pi : F' \Rightarrow F \Rightarrow 0 \) is exact in \( \text{Ab}^\mathcal{P} \). Thus we can consider the object-wise kernel of \( \pi : F' \Rightarrow F \) to obtain a short exact sequence of functors

\[
0 \Rightarrow K_F \Rightarrow F' \xrightarrow{\pi} F \Rightarrow 0.
\]

If \( \mathcal{P} \) is bounded below then the long exact sequence (see Section 1.6) associated to this short exact sequence gives

\[
\lim_{\to j} F = \begin{cases} \lim_{\to j} K_F & j > 1 \\ \text{Ker}\{\lim_{\to j} K_F \to \lim_{\to j} F'\} & j = 1 \end{cases}
\]

because \( F' \) is \( \lim_{\to} \)-acyclic (it is pseudo-projective by Lemma 3.3.5 and apply Theorem 3.2.3). So writing \( K_0 \overset{\text{def}}{=} F \) and \( K_1 \overset{\text{def}}{=} K_F \), we have

\[
\lim_{\to j} F = \text{Ker}\{\lim_{\to j} K_1 \to \lim_{\to j} K'_0\}
\]

where the map \( \lim_{\to j} K_1 \to \lim_{\to j} K'_0 \) comes from the long exact sequence of derived functors associated to a short exact sequence

\[
0 \Rightarrow K_1 \Rightarrow K'_0 \Rightarrow K_0 \Rightarrow 0.
\]
Also we obtained that \( \underline{\lim}_j F = \underline{\lim}_{j-1} K_1 \) for \( j \geq 2 \) and that \( \underline{\lim}_{0} K_0' = \bigoplus_{i \in \text{Ob}(\mathcal{P})} K_0(i) \). Thus applying the same machinery to the functor \( K_1 \) we have a short exact sequence

\[
0 \Rightarrow K_2 \Rightarrow K_1' \Rightarrow K_1 \Rightarrow 0
\]

and

\[
\underline{\lim}_j F = \underline{\lim}_{j-1} K_1 = \text{Ker}\{\underline{\lim}_{j-2} K_j \rightarrow \underline{\lim}_{j-1} K_{j-1}'\}
\]

with \( \underline{\lim}_j K_1' = \bigoplus_{i \in \text{Ob}(\mathcal{P})} K_1(i) \). Recursively we obtain short exact sequences

\[
0 \Rightarrow K_j \Rightarrow K_{j-1}' \Rightarrow K_{j-1} \Rightarrow 0
\]

and

\[
\underline{\lim}_j F = \underline{\lim}_{j-1} K_1 = \underline{\lim}_{j-2} K_2 = \ldots = \underline{\lim}_{j-1} K_{j-1} = \text{Ker}\{\underline{\lim}_{j-1} K_j \rightarrow \underline{\lim}_{j-2} K_{j-1}'\}
\]

for every \( j \geq 1 \), where \( \underline{\lim}_j K_{j-1}' = \bigoplus_{i \in \text{Ob}(\mathcal{P})} K_{j-1}(i) \).

**Lemma 3.3.7.** Let \( \mathcal{P} \) be a bounded below graded poset and \( F : \mathcal{P} \rightarrow \text{Ab} \) a functor. Then there are functors \( K_j : \mathcal{P} \rightarrow \text{Ab} \) for \( j = 0, 1, 2, \ldots \) with \( K_0 = F \) and \( K_1 = \text{Ker}(F' \Rightarrow F) \) such that

\[
\underline{\lim}_j F = \underline{\lim}_{j-1} K_1 = \underline{\lim}_{j-2} K_2 = \ldots = \underline{\lim}_{j-1} K_{j-1} = \text{Ker}\{\underline{\lim}_{j-1} K_j \rightarrow \underline{\lim}_{j-2} K_{j-1}'\}
\]

for each \( j = 0, 1, 2, \ldots \).

The values \( F'(i_0) \) and \( K_F(i_0) \) can be very big because they contain a copy of \( F(i) \) for each \( i \rightarrow i_0 \). This can be improved considering the functor \( \text{Coker} : \mathcal{P} \rightarrow \text{Ab} \) in Section 3.1. Suppose that for every \( i_0 \) there is a section \( s_{i_0} : \text{Coker}(i_0) \rightarrow F(i_0) \) to the projection \( F(i_0) \rightarrow \text{Coker}(i_0) \) (for example if \( \text{Coker}(i_0) \) is free for each \( i_0 \) or if \( F \) is an epic functor). Then by Lemma 3.3.3 there is a natural transformation \( \text{Coker}' \Rightarrow F \). If \( \mathcal{P} \) is bounded below then it is easy to see by induction on the degree of objects that this natural transformation is object-wise surjective. Notice that \( \text{Coker}'(i_0) \) is, in general, smaller than \( F'(i_0) \).

**Lemma 3.3.8.** Let \( \mathcal{P} \) be a bounded below graded poset and let \( F : \mathcal{P} \rightarrow \text{Ab} \) be an epic functor. Then there is a short exact sequence of functors

\[
0 \Rightarrow K \Rightarrow \text{Coker}' \Rightarrow F \Rightarrow 0
\]

where \( \text{Coker}' \) is \( \underline{\lim} \)-acyclic.

We finish this section with some examples of \( \underline{\lim} \)-acyclic functors:

**Example 3.3.9.** Let \( \mathcal{P} \) be a graded poset with initial object \( i_0 \). Then \( \mathcal{P} \) is contractible and thus \( H_i(\mathcal{P}; M) = 0 \) for \( i \geq 1 \) and any trivial coefficients \( M \in \text{Ab} \). We can prove this by taking the functor \( F_M : \mathcal{P} \rightarrow \text{Ab} \) which takes the value \( M \) on \( i_0 \) and 0 otherwise. Then \( (F_M)' : \mathcal{P} \rightarrow \text{Ab} \) is the functor of constant value \( M \) and thus \( H_i(\mathcal{P}; M) = \underline{\lim}_0 (F_M)' \) for \( i \geq 0 \). These higher limits vanish for \( i \geq 1 \) because \( (F_M)' \) is pseudo-projective (Lemma 3.3.5). Finally, \( H_0(\mathcal{P}; M) = \underline{\lim}_0 (F_M)' = M \) by Lemma 3.3.4.
Definition 3.3.10. Let \( \mathcal{P} \) be graded poset. We say \( \mathcal{P} \) is a rooted tree if \( \mathcal{P} \) is a tree and \( \mathcal{P} \) has a initial object.

Example 3.3.11. The category with shape
\[
\begin{array}{c}
\rightarrow & c & \rightarrow & e \\
\downarrow & & & \\
b & \rightarrow & d & \rightarrow & f & \rightarrow & g
\end{array}
\]
is a tree but not a rooted tree. The category with shape
\[
\begin{array}{c}
\rightarrow & d \\
\uparrow & & & \\
a & \rightarrow & b & \rightarrow & e \\
\downarrow & & & & \downarrow \\
& & c
\end{array}
\]
is a rooted tree.

Corollary 3.3.12. Let \( \mathcal{P} \) be a rooted tree. Then any monic functor \( F: \mathcal{P} \to \text{Ab} \) is \( \lim\rightarrow \)-acyclic.

Proof. Just check that a monic functor \( F: \mathcal{P} \to \text{Ab} \) over a rooted tree \( \mathcal{P} \) is pseudo-projective. \( \square \)

Example 3.3.13. Any monic functor over the push-out category or the “telescope category” is \( \lim\rightarrow \)-acyclic.

3.4. \( \lim_1 \) as a flow problem.

In this section we give an interpretation of the first derived limit \( \lim_1 F \) in terms of flow problems. While in classical flows on directed graphs (see [6, III.1]) a non-negative integer is associated to each edge \( i_0 \to i_1 \), we associate a value from the abelian group \( F(i_0) \). Here, \( F \) is a functor \( F: \mathcal{P} \to \text{Ab} \) and \( \mathcal{P} \) is a graded poset. Notice that, by Section 3.3, all the higher limits \( \lim_i F \) can be reduced to first derived functors \( \lim_1 K_{i-1} \) for suitable \( K_{i-1} \).

Recall Definition 1.6.1. It is straightforward that \( \lim_1 F \) equals a quotient \( M/N \) where \( N \subseteq M \) are abelian subgroups of \( \bigoplus_{i_0 \to i_1 \in \mathcal{P}_1} F(i_0) \).

Notice that \( N\mathcal{P}_1 = \text{Hom}(\mathcal{P}) \) and that an element \( x = \{x_{\alpha}\}_{\alpha \in \text{Hom}(\mathcal{P})} \) of \( \bigoplus_{i_0 \to i_1 \in \mathcal{P}_1} F(i_0) \) belongs to \( M \) just in case
\[
\sum_{i \sim i_0} F(\alpha)(x_{\alpha}) = \sum_{i_0 \to i} x_{\alpha}
\]
for every \( i_0 \in \text{Ob}(\mathcal{P}) \). The subgroup \( N \) is the image of the differential

\[
C_2(\mathcal{P}, F) \xrightarrow{d} C_1(\mathcal{P}, F)
\]

and it is generated by

\[
x_\alpha \oplus (-x)_{\beta \circ \alpha} \oplus (F(\alpha)(x))_{\beta}
\]

where \( i_0 \xrightarrow{\alpha} i_1 \) and \( i_1 \xrightarrow{\beta} i_2 \) are any two composable morphisms and \( x \in F(i_0) \).

**Definition 3.4.1.** Let \( \mathcal{P} \) be a graded poset and let \( F : \mathcal{P} \to \text{Ab} \) be a functor. A *generalized flow* is an element \( x = \{x_\alpha\}_{\alpha \in \text{Hom}(\mathcal{P})} \) with finitely many terms different from zero such that

\[
\sum_{i_0 \xrightarrow{\alpha} i \in \text{Hom}(\mathcal{P})} F(\alpha)(x_\alpha) = \sum_{i_0 \xrightarrow{\alpha} i \in \text{Hom}(\mathcal{P})} x_\alpha
\]

for every \( i_0 \in \text{Ob}(\mathcal{P}) \). A minimal trivial flow is a generalized flow

\[
x_\alpha \oplus (-x)_{\beta \circ \alpha} \oplus (F(\alpha)(x))_{\beta}
\]

where \( i_0 \xrightarrow{\alpha} i_1 \) and \( i_1 \xrightarrow{\beta} i_2 \) are any two composable morphisms and \( x \in F(i_0) \).

With these definitions it is clear that

**Lemma 3.4.2.** \( \lim_{\rightarrow 1} F = 0 \) if and only if any generalized flow can be written as sum of minimal trivial flows.

There is also an easy geometrical interpretation of \( M \) on the directed graph associated to \( \mathcal{P} \) (see 1.3.2): because in the graded poset \( \mathcal{P} \) any morphism factors as composition of morphisms of degree 1, then in the class \( \pi \in \lim_{\rightarrow 1} F = M/N \) of a generalized flow \( x = \{x_\alpha\}_{\alpha \in \text{Hom}(\mathcal{P})} \) there is a representative \( x' = \{x'_\alpha\}_{\alpha \in \text{Hom}_1(\mathcal{P})} \) which takes nonzero values just in degree 1 morphisms. To obtain \( x' \) is enough to use the cycles

\[
(-x_\gamma)_{\gamma_1} \oplus (x_\gamma)_{\gamma} \oplus (F(\alpha)(-x_\gamma))_{\gamma_2}
\]

where \( \gamma = \gamma_2 \circ \gamma_1 \) is a morphism of degree greater than 1, arguing by induction on \( \max\{\text{deg}(\alpha)|x_\alpha \neq 0\} \) (morphisms of degree 0 do not appear as we assume that we are using normalized chain complexes 1.7).

**Definition 3.4.3.** Let \( \mathcal{P} \) be a graded poset and let \( F : \mathcal{P} \to \text{Ab} \) be a functor. A *flow* is an element \( x = \{x_\alpha\}_{\alpha \in \text{Hom}_1(\mathcal{P})} \) with finitely many terms different from zero such that

\[
\sum_{i_0 \xrightarrow{\alpha} i, \text{deg}(\alpha) = 1} F(\alpha)(x_\alpha) = \sum_{i_0 \xrightarrow{\alpha} i, \text{deg}(\alpha) = 1} x_\alpha
\]

for every \( i_0 \in \text{Ob}(\mathcal{P}) \).

Again we have

**Lemma 3.4.4.** \( \lim_{\rightarrow 1} F = 0 \) if and only if any flow can be written as sum of minimal trivial flows.
Consider the directed graph associated to $\mathcal{P}$ (see 1.3.2). Then a flow corresponds to a choice of a value $x_\alpha \in F(i_0)$ for each edge $i_0 \xrightarrow{\alpha} i_1$ of this graph, i.e., for each morphism of degree 1 of $\mathcal{P}$:

$$i_0 \xrightarrow{x_\alpha} i_1.$$ 

In order for $x$ to be a flow, for each $i_0 \in \text{Ob}(\mathcal{P})$ we must have the equality

$$\sum_{i_0 \xrightarrow{\alpha, \deg(\alpha)=1} i_0} F(\alpha)(x_\alpha) = \sum_{i_0 \xrightarrow{\alpha, \deg(\alpha)=1} i_0} x_\alpha.$$

For example, for a vertex $i_0$ as

where $\text{Hom}_1(\mathcal{P}, i_0) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\text{Hom}_1(i_0, \mathcal{P}) = \{\alpha_4, \alpha_5, \alpha_6\}$, this condition reads

$$F(\alpha_1)(x_{\alpha_1}) + F(\alpha_1)(x_{\alpha_2}) + F(\alpha_3)(x_{\alpha_3}) = x_{\alpha_4} + x_{\alpha_5} + x_{\alpha_6}.$$

A minimal trivial flow can be represented as the generalized flow

(but notice that the arrows are not edges of the directed graph associated to $\mathcal{P}$ in general).

**Example 3.4.5.** Consider the graded poset $\mathcal{P}$ with shape (it was considered before on Example 2.1.4)

![Diagram](image)

This representation corresponds exactly with the directed graph associated to $\mathcal{P}$, because only the degree 1 morphisms are displayed. Consider the functor
3.4. $\mathrm{lim}_1$ AS A FLOW PROBLEM.

$F : \mathcal{P} \to \text{Ab}$ with constant value $\mathbb{Z}$. Then the following is a flow

![Diagram of a flow problem with labeled arrows and values]

This flow can be written as sum of the following minimal trivial flows:

- $a \to c$ with $3$.
- $e \to a$ with $3$.
- $e \to b$ with $-3$.
- $d \to e$ with $-3$.
- $e \to c$ with $3$.
- $b \to d$ with $-7$.
- $e \to b$ with $-3$.
- $c \to e$ with $3$.

This naive interpretation of $\mathrm{lim}_1 F$ has some consequences, which will be useful later. Recall (1.3) that $(p_0 \downarrow \mathcal{P})_*$ is the full subcategories of $\mathcal{P}$ with objects $\{p | \exists p_0 \to p, p \neq p_0\}$.

**Definition 3.4.6.** Let $\mathcal{P}$ be a graded poset. Write $\mathcal{P}'$ for the full subcategory of $\mathcal{P}$ with objects all but those objects $i_0$ such that there are no arriving arrows to $i_0$ and such that $(i_0 \downarrow \mathcal{P})_*$ is empty or connected. Write $\mathcal{P}^n = (\mathcal{P}^{n-1})'$ for $n \geq 1$ and $\mathcal{P}^0 = \mathcal{P}$. We have inclusions

$$\mathcal{P}^n \subseteq \mathcal{P}^{n-1} \subseteq \ldots \subseteq \mathcal{P}^1 = \mathcal{P}' \subseteq \mathcal{P}^0 = \mathcal{P}$$

Then define the core of $\mathcal{P}$, denoted $\text{core}(\mathcal{P})$, as the inverse limit $\lim_{\leftarrow n} \mathcal{P}^n$ in the complete category of small categories.
Notice that if \( \mathcal{P} \) is finite then we obtain \( \text{core}(\mathcal{P}) \) from \( \mathcal{P} \) after a finite number of times of applications of the operator \((\cdot)'\).

**Example 3.4.7.** Consider the graded poset \( \mathcal{P} \) with shape

\[
\begin{array}{ccc}
    a & \overset{f}{\rightarrow} & c \\
    \downarrow g & & \downarrow c \\
    e & \overset{f}{\rightarrow} & d \\
    \downarrow k & & \downarrow d \\
    b & \overset{i}{\rightarrow} & d \\
\end{array}
\]

Then \( \mathcal{P}^1 = \mathcal{P}' \) is the category

\[
\begin{array}{ccc}
    a & \overset{f}{\rightarrow} & c \\
    \downarrow g & & \downarrow c \\
    b & \overset{i}{\rightarrow} & d \\
\end{array}
\]

and \( \mathcal{P}^2 = \mathcal{P}'' = \text{core}(\mathcal{P}) \) is

\[
\begin{array}{ccc}
    a & \overset{f}{\rightarrow} & c \\
    \downarrow g & & \downarrow d \\
\end{array}
\]

**Lemma 3.4.8.** Let \( \mathcal{P} \) be a finite graded poset and let \( F : \mathcal{P} \rightarrow \text{Ab} \) be a functor. Then any class \( \varpi \in \lim_{\rightarrow} F \) has a representative \( x' = \{x'_\alpha\}_{\alpha \in \text{Hom}(\text{core}(\mathcal{P}))} \) which takes nonzero values just in morphisms of the core of \( \mathcal{P} \).

**Proof.** By the definition of the core of \( \mathcal{P} \) it is enough to prove that any generalized flow \( x \) is in the same class that a generalized flow \( x' = \{x'_\alpha\}_{\alpha \in \mathcal{P}'} \) which takes nonzero values just in morphisms of \( \mathcal{P}' \).

Take any generalized flow \( x \) over \( \mathcal{P} \) and its class \( \varpi \in \lim_{\rightarrow} F \). By the above remarks we can suppose that \( x \) is in fact a flow, i.e., takes nonzero values just in degree 1 morphisms.

Consider any object \( i_0 \) which is in \( \mathcal{P} \) but not in \( \mathcal{P}' \). Then there are no arriving arrows to \( i_0 \) and \( (i_0 \downarrow \mathcal{P})_* \) is empty or connected. If it is empty then there is nothing to do. Thus, assume that \( (i_0 \downarrow \mathcal{P})_* \) is non-empty and connected. The equation for \( x \) on \( i_0 \) becomes

\[
0 = \sum_{i_0 \overset{\alpha}{\rightarrow} i, \deg(\alpha) = 1} x_\alpha.
\]
3.4. \( \text{Lim}_1 \) AS A FLOW PROBLEM.

Fix \( i_0 \overset{\alpha}{\to} i_* \) with \( \deg(\alpha) = 1 \). Now, take any other arrow \((\alpha \neq \beta) \ i_0 \overset{\beta}{\to} j_*\) of degree 1 (if it does not exist such a \( \beta \) then we are done). Recall that \((i_0 \downarrow \mathcal{P})_*\) is connected. Then there exists a zigzag of morphisms connecting \( i_* \) and \( j_* \). Because \( \mathcal{P} \) is a graded poset we can assume that the zig-zag has the following shape:

\[
\begin{array}{c}
\alpha = \alpha_0 \\
\alpha = \alpha_1 \\
\alpha = \alpha_{n-1} \\
\beta = \alpha_n \\
\end{array}
\]

\[
\begin{array}{c}
i_* = j_0 \\
j_2 \\
j_{n-1} \\
j_* = j_n \\
k_n \\
k_1 \\
k_{i+1} \\
k_{i+1} \\
j_{i+1} \\
i_0 \\
j_i \\
j_{i+1} \\
\end{array}
\]

where \( \deg(\alpha_i) = 1 \) for \( i = 0, \ldots, n \). For each \( i = 0, \ldots, n - 1 \) consider the following diamond with commutative triangles

\[
\begin{array}{c}
\alpha_i \\
\alpha_{i+1} \\
\alpha_{i+1} \\
\beta = \alpha_n \\
\end{array}
\]

\[
\begin{array}{c}
i_0 \\
j_i \\
j_{i+1} \\
k_{i+1} \\
j_{i+1} \\
\end{array}
\]

and the two minimal trivial flows

\[
\begin{array}{c}
\alpha_i \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_{i+1} \\
\end{array}
\]

\[
\begin{array}{c}
i_0 \\
x_{\beta} \\
-x_{\beta} \\
k_{i+1} \\
j_{i+1} \\
\end{array}
\]

and

\[
\begin{array}{c}
i_0 \\
x_{\beta} \\
-x_{\beta} \\
k_{i+1} \\
\end{array}
\]

\[
\begin{array}{c}
j_{i+1} \\
\alpha_{i+1} \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]

\[
\begin{array}{c}
F(\alpha_i)(x_{\beta}) \\
-x_{\beta} \\
F(\alpha_{i+1})(-x_{\beta}) \\
\end{array}
\]
If we add up these two minimal trivial flows for all \( i = 0, \ldots, n - 1 \) we obtain a generalized flow \( y_{j*} = 0 \) and such that, for \( \gamma : i_0 \to i, \)

\[
y_{j*}\gamma = \begin{cases} 
x_\beta & \text{if } \gamma = \alpha \\
-x_\beta & \text{if } \gamma = \beta \\
0 & \text{otherwise.}
\end{cases}
\]

Thus, \( x' = x + y_{j*} \) is a representative for \( x \) which verifies, for \( \gamma : i_0 \to i, \)

\[
x'_\gamma = \begin{cases} 
x_\alpha + x_\beta & \text{if } \gamma = \alpha \\
0 & \text{if } \gamma = \beta \\
x_\gamma & \text{otherwise.}
\end{cases}
\]

Notice that, by construction, if \( x'_\gamma \neq 0 \) then either \( x_\gamma \neq 0 \) or \( \gamma \) has origin in \( i \) and \( i \in \text{Ob}(\mathcal{P}') \). Doing the same construction for each arrow \( i_0 \to j \) different from \( \alpha \) (and of degree 1) with \( x_{i_0 \to j} \neq 0 \) (recall that there is a finite number of these arrows) we obtain a representative \( x'' \) for \( x \) such that, for \( \gamma : i_0 \to i, \)

\[
x''_\gamma = \begin{cases} 
x_\alpha + \sum_{\beta : i_0 \to j, \deg(\beta) = 1, \beta \neq \alpha} x_\beta & \text{if } \gamma = \alpha \\
0 & \text{if } \gamma = \beta \\
x_\gamma & \text{otherwise.}
\end{cases}
\]

Then, by Equation (17), \( x'_\alpha = 0 \) and \( x'_\gamma = 0 \) for each arrow \( \gamma \) with origin in \( i_0 \). Again by construction if \( x''_\gamma \neq 0 \) then either \( x_\gamma \neq 0 \) either \( \gamma \) has origin in \( i \) and \( i \in \text{Ob}(\mathcal{P}') \). As it was described before the lemma, there is a representative \( x'' \) for \( x \) which takes non-zero values just in morphisms of degree 1. Moreover, if \( x''_\gamma \neq 0 \) then either \( x_\gamma \neq 0 \) either \( \gamma \) has origin in \( i \) and \( i \in \text{Ob}(\mathcal{P}') \).

Then, we repeat the process (but with \( x'' \) instead of \( x \)) for any object \( i_0 \) which is in \( \mathcal{P} \) but no in \( \mathcal{P}' \) and for which there is a morphism \( \alpha \) with origin in \( i_0 \) and with \( x_\alpha \neq 0 \). There is a finite number of such objects.

Thus, finally, we obtain a representative for \( \mathfrak{T} \) which takes values different from zero on morphism which do not begin on objects from \( \text{Ob}(\mathcal{P}) \setminus \text{Ob}(\mathcal{P}') \), i.e., on morphisms from \( \text{Hom}(\mathcal{P}') \).

Because \( \mathcal{P} \) is finite by hypothesis, then \( \text{core}(\mathcal{P}) \) is reached in a finite number of computations \( \mathcal{P}, \mathcal{P}', (\mathcal{P}')', \ldots, \text{core}(\mathcal{P}) \) and the lemma is proven.

Next, we present some conditions on vanishing of higher limits derived from the previous lemma:

**Corollary 3.4.9.** Let \( \mathcal{P} \) be a finite graded poset with \( \text{core}(\mathcal{P}) = \emptyset \). Then every functor \( F : \mathcal{P} \to \text{Ab} \) is \( \lim\rightarrow \)-acyclic.

**Proof.** That \( \lim\rightarrow F = 0 \) results directly from the lemma above. The limit \( \lim\rightarrow_i F \) with \( i \geq 2 \) is given by \( \lim\rightarrow K_{i-1} \) where \( K_{i-1} : \mathcal{P} \to \text{Ab} \) is a functor (see section 3.3).

**Corollary 3.4.10.** Let \( \mathcal{P} \) be a finite graded poset such that \( \text{core}(\mathcal{P}) \) is a tree. If \( F : \mathcal{P} \to \text{Ab} \) is a functor with \( F|_{\text{core}(\mathcal{P})} \) monic then \( F \) is \( \lim\rightarrow \)-acyclic.
3.4. \( \text{Lim}_1 \) AS A FLOW PROBLEM.

**Proof.** For \( \lim_{i \to 1} F \) notice that the class \( \bar{x} \in \lim_{i \to 1} F \) of any generalized flow \( x \) has as representative a flow \( x' \) living in \( \text{core}(\mathcal{P}) \). Then the equations for \( x' \) to be a flow imply that \( x' \equiv 0 \) as \( \text{core}(\mathcal{P}) \) is a tree and \( F \) restricted to \( \text{core}(\mathcal{P}) \) is monic.

The limit \( \lim_{i \to i} F \) with \( i \geq 2 \) is given by \( \lim_{i \to 1} K_{i-1} \) where \( K_{i-1} : \mathcal{P} \to \text{Ab} \) is a monic functor (see Section 3.3). Then use the same argument. \( \square \)

This implies, in particular, that (cf. Corollary 3.3.12):

**Corollary 3.4.11.** Let \( \mathcal{P} \) be a finite tree. If \( F : \mathcal{P} \to \text{Ab} \) is a monic functor then \( F \) is \( \text{lim}_i \)-acyclic.

**Proof.** If \( \mathcal{P} \) is a tree then \( \text{core}(\mathcal{P}) \) is a subcategory of \( \mathcal{P} \) and thus it is a tree too. Then use the previous corollary. \( \square \)
CHAPTER 4

Higher inverse limits

4.1. Injective objects in $\text{Ab}^P$.

Consider the abelian category $\text{Ab}^P$ for some graded poset $P$. In this section we shall determine the injective objects in $\text{Ab}^P$ following dual arguments to those of Chapter 3.1. Recall that in $\text{Ab}$ the injective objects are well known, and are exactly the direct sums of $\mathbb{Q}$ and $\mathbb{Z}[p^\infty]$ for various primes $p$. Along the rest of the section, $P$ denotes a graded poset.

Suppose $F \in \text{Ab}^P$ is injective. How does $F$ look? We show that the intersection of the kernels of the non-identity morphisms with source $i_0$ is an injective abelian group. To prove it, write

**Definition 4.1.1.** $\text{Ker}(i_0) = \bigcap_{i_0 \to i, \alpha \neq 1_{i_0}} \text{Ker}(F(\alpha))$ (or $\text{Ker}(i_0) = F(i_0)$ if the index set of the intersection is empty) and $\text{Coim}(i_0) = F(i_0)/\text{Ker}(i_0)$.

For any diagram in $\text{Ab}$ as the following

\[
\begin{array}{c}
A_0 \xleftarrow{\lambda_0} B_0 \xrightarrow{\rho_0} 0 \\
\end{array}
\]

we want to find $\rho_0$ that makes it commutative. Consider the atomic functors $A, B : P \to \text{Ab}$ which take the values on objects

\[
A(i) = \begin{cases} 
A_0 & \text{for } i = i_0 \\
0 & \text{for } i \neq i_0 
\end{cases}
\]

\[
B(i) = \begin{cases} 
B_0 & \text{for } i = i_0 \\
0 & \text{for } i \neq i_0 
\end{cases}
\]

and on morphisms

\[
A(\alpha) = \begin{cases} 
1_{A_0} & \text{for } \alpha = 1_{i_0} \\
0 & \text{for } \alpha \neq 1_{i_0} 
\end{cases}
\]

\[
B(\alpha) = \begin{cases} 
1_{B_0} & \text{for } \alpha = 1_{i_0} \\
0 & \text{for } \alpha \neq 1_{i_0} 
\end{cases}
\]

and the natural transformations $\sigma : B \Rightarrow F$ and $\lambda : A \Rightarrow B$ given by

\[
\sigma(i) = \begin{cases} 
j \circ \sigma_0 & \text{for } i = i_0 \\
0 & \text{for } i \neq i_0 
\end{cases}
\]

\[
\lambda(i) = \begin{cases} 
j_0 \circ \sigma_0 & \text{for } i = i_0 \\
0 & \text{for } i \neq i_0 
\end{cases}
\]
\[\lambda(i) = \begin{cases} 
\lambda_0 & \text{for } i = i_0 \\
0 & \text{for } i \neq i_0 
\end{cases}\]

where \(j\) is the inclusion \(\text{Ker}(i_0) \hookrightarrow F(i_0)\). Then \(0 \Rightarrow B \xrightarrow{\lambda_0} A\) is exact as \(0 \Rightarrow B_0 \xrightarrow{\lambda_0} A_0\) is so. It is straightforward that \(\lambda\) is a natural transformation. The key point in checking that \(\sigma\) is a natural transformation is that for \(\alpha : i_0 \to i_1, \alpha \neq 1_{i_0}\) the diagram

\[
\begin{array}{ccc}
F(i_0) & \xrightarrow{F(\alpha)} & F(i_1) \\
\downarrow{j \circ \sigma_0} & & \downarrow{\sigma(i_1)} \\
B_0 & \to & B(i_1)
\end{array}
\]

must commute. And it does because \(F(\alpha) \circ j = 0\) for every \(\alpha \neq 1_{i_0}\).

So, as \(F\) is injective, this data gives a natural transformation \(\rho\) which makes commutative the diagram of natural transformations

\[
\begin{array}{ccc}
F & \Rightarrow & \\
\uparrow{\rho} & \uparrow{\sigma} & \\
A & \xleftarrow{\lambda} & B \leftarrow 0
\end{array}
\]

which restricts over \(i_0\) to

\[
\begin{array}{ccc}
F(i_0) & \xrightarrow{\rho(i_0)} & \text{Ker}(i_0) \\
\downarrow{j} & & \\
A_0 & \xleftarrow{\rho_0} & B_0 \leftarrow 0
\end{array}
\]

Then \(\rho_0\) exists if and only if \(\text{Im}(\rho(i_0)) \subseteq \text{Ker}(i_0)\). To check that this condition holds take \(\alpha : i_0 \to i_1\) with \(\alpha \neq 1_{i_0}\) and \(a \in A_0\). Then

\[F(\alpha)(\rho(i_0)(a)) = \rho(i_1)(A(\alpha)(a)) = \rho(i_1)(0) = 0.\]

We have just proven

**Lemma 4.1.2.** Let \(F : \mathcal{P} \to \text{Ab}\) be an injective functor over a graded poset \(\mathcal{P}\). Then \(\text{Ker}(i_0)\) is injective for every object \(i_0 \in \text{Ob}(\mathcal{P})\).

This means that we can write

\[F(i_0) = \text{Ker}(i_0) \oplus \text{Coim}(i_0)\]

with \(\text{Ker}(i_0)\) injective for every \(i_0 \in \text{Ob}(\mathcal{P})\), and also that

**Example 4.1.3.** For the category \(\mathcal{P}\) with shape

\[
\cdot \to \cdot
\]
the functor \( F : \mathcal{P} \to \text{Ab} \) with values
\[
\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Z}[p^\infty],
\]
is not injective as \( \text{Ker} \) on the left object equals the non-injective abelian group \( \mathbb{Z} \).

Now that we know a little about the values that an injective functor \( F : \mathcal{P} \to \text{Ab} \) takes on objects we can wonder about the values \( F(\alpha) \) for \( \alpha \in \text{Hom}(\mathcal{P}) \). Do they have any special property? Recall that a feature of graded posets is that there is at most one arrow between any two objects, and also that

**Remark 4.1.4.** If \( \mathcal{P} \) is graded then for any \( i_0 \in \text{Ob}(\mathcal{P}) \)
\[
\text{Ker}(i_0) = \bigcap_{i_0 \to i, \deg(\alpha) = 1} \text{Ker} F(\alpha)
\]
because every morphism factors as composition of morphisms of degree 1.

We prove that the following property holds for \( F \):

**Definition 4.1.5.** Let \( F : \mathcal{P} \to \text{Ab} \) be a functor over a graded poset \( \mathcal{P} \) with degree function \( \deg \). Fix an integer \( d \geq 0 \). If for any \( i_0 \in \text{Ob}(\mathcal{P}) \), different objects \( \{i_j\}_{j \in J} \subseteq \text{Ob}(\mathcal{P}) \), arrows \( \alpha_j : i_0 \to i_j \) with \( \deg(\alpha_j) = d \) and elements \( x_j \in \text{Ker}(i_j) \) for each \( j \in J \), there is \( y \in F(i_0) \) with
\[
F(\alpha_j)(y) = x_j
\]
for each \( j \in J \), we call \( F \) \( d \)-pseudo-injective. If \( F \) is \( d \)-pseudo-injective for each \( d \geq 0 \) we call \( F \) pseudo-injective.

**Remark 4.1.6.** In case \( J = \{1\} \) and \( \text{Ker}(i_1) = F(i_1) \) we are claiming that the homomorphism \( F(\alpha_1) : F(i_0) \to F(i_1) \) is surjective. Notice that any functor is 0-pseudo-injective as the identity is an epimorphism.

Before proving that injective functors \( F \) over a graded poset verify this property we define two functors \( \text{Ker} \) and \( \text{Ker}' \) and natural transformations \( \sigma \) and \( \lambda \) that fit in the diagram

\[
\begin{array}{ccc}
F & \downarrow & \\
\text{Ker}' & \leftarrow & \text{Ker} \\
& \lambda \uparrow & \\
& 0 & \end{array}
\]
for any functor \( F : \mathcal{P} \to \text{Ab} \) with \( \mathcal{P} \) a graded poset. We begin defining \( \text{Ker} \). Because for every \( \alpha : i_1 \to i_0 \) holds that \( F(\alpha)(\text{Ker}(i_1)) \leq \text{Ker}(i_0) \) we can factor \( F(\alpha) \) as in the diagram

\[
\begin{array}{ccc}
F(i_1) & \xrightarrow{F(\alpha)} & F(i_0) \\
\uparrow & & \uparrow \\
\text{Ker}(i_1) & \xrightarrow{F(\alpha)\mid_{\text{Ker}(i_1)}} & \text{Ker}(i_0).
\end{array}
\]
In fact, if \( \alpha \neq 1_{i_1} \), then \( F(\alpha)|_{\text{Ker}(i_1)} \equiv 0 \) by definition. Because the identity \( 1_{i_0} \) cannot be factorized (by non-identity morphisms) in a graded poset then we have a functor Ker with value Ker\((i)\) on the object \( i \) of \( P \) and which maps the non-identity morphisms to zero. Ker is a kind of “discrete” functor. Also it is clear that exists a natural transformation \( \sigma : \text{Ker} \Rightarrow F \) with \( \sigma(i) \) the inclusion \( \text{Ker}(i) \hookrightarrow F(i) \).

Now we define Ker’ from Ker in a dual way as free diagrams are constructed. Let Ker’ be defined on objects by

\[
\text{Ker'}(i_0) = \bigoplus_{\alpha : i_0 \rightarrow i} \text{Ker}(i).
\]

For \( \beta \in \text{Hom}(P), \beta : i_1 \rightarrow i_0 \), Ker’\((\beta)\) is the only homomorphism which makes commute the diagram

\[
\begin{array}{ccc}
\text{Ker'}(i_1) & \xrightarrow{\text{Ker'}(\beta)} & \text{Ker'}(i_0) \\
\uparrow & & \uparrow \\
\text{Ker}(i)_{i_1 \rightarrow i} & \xrightarrow{1} & \text{Ker}(i)_{i_0 \rightarrow i}
\end{array}
\]

for each \( \alpha : i_1 \rightarrow i \) that factors through \( \beta \)

and the diagram

\[
\begin{array}{ccc}
\text{Ker'}(i_1) & \xrightarrow{\text{Ker'}(\beta)} & \text{Ker'}(i_0) \\
\downarrow & & \downarrow \\
\text{Ker}(i)_{i_1 \rightarrow i} & \xrightarrow{0} & \text{Ker}(i)_{i_0 \rightarrow i}
\end{array}
\]

for each \( \alpha : i_1 \rightarrow i \) that does not factor through \( \beta \).

Then there exists a candidate to natural transformation \( \lambda : \text{Ker} \Rightarrow \text{Ker'} \) whose value \( \lambda(i) \) is the inclusion of Ker\((i)\) into the direct summand Ker\((i)_{1_{i_1} \rightarrow i}\) corresponding to the identity on \( i \). \( \lambda \) is a natural transformation if for every \( \beta : i_1 \rightarrow i_0 \) with \( i_1 \neq i_0 \) the following diagram is commutative

\[
\begin{array}{ccc}
\text{Ker'}(i_1) & \xrightarrow{\text{Ker'}(\beta)} & \text{Ker'}(i_0) \\
\lambda(i_1) \downarrow & & \lambda(i_0) \downarrow \\
\text{Ker}(i_1) & \xrightarrow{0} & \text{Ker}(i_0).
\end{array}
\]

It is straightforward that this square commutes because \( 1_{i_1} : i_1 \rightarrow i_1 \) cannot be factored through \( \beta \).
Now we have the commutative triangle

\[
\begin{array}{c}
\text{Ker'} \\
\downarrow \quad \rho \\
\text{Ker}
\end{array}
\Rightarrow
\begin{array}{c}
\text{Ker'} \\
\downarrow \\
\text{Ker}
\end{array}
\Rightarrow
\begin{array}{c}
\text{0}
\end{array}
\]

where the natural transformation \( \rho \) exists because \( F \) is injective. To prove that \( F \) is pseudo-injective take \( i_0 \in \text{Ob}(P) \), different objects \( \{i_j\}_{j \in J} \subseteq \text{Ob}(P) \), arrows \( \alpha_j : i_0 \to i_j \) with \( \deg(\alpha_j) = d \) and elements \( x_j \in \text{Ker}(i_j) \) for each \( j \in J \). To visualize what is going on consider the diagram above near \( i_0 \) in case \( J = \{1, 2\} \)

Consider the element \( x \in \text{Ker'}(i_0) = \bigoplus_{\alpha : i_0 \to i} \text{Ker}(i) \) given in components by

\[
x_{\alpha} = \begin{cases} 
x_j & \text{if } \alpha = \alpha_j : i_0 \to i_j \text{ for some } j \in J \\
0 & \text{otherwise.}
\end{cases}
\]

Fix \( j_0 \in J \) and consider the map \( \text{Ker'}(\alpha_{j_0}) : \text{Ker'}(i_0) \to \text{Ker'}(i_{j_0}) \). Because all the maps \( \{\alpha_j\}_{j \in J} \) have degree \( d \) then we have a commutative triangle

\[
\begin{array}{c}
i_0 \\
\downarrow \quad \alpha_j \\
i_j
\end{array}
\Rightarrow
\begin{array}{c}
i_0 \\
\downarrow \quad \beta \\
i_j
\end{array}
\Rightarrow
\begin{array}{c}
i_{j_0}
\end{array}
\]

only in case \( j = j_0 \) and \( \beta = 1_{j_0} \). This means that \( \text{Ker'}(\alpha_{j_0}) \) maps \( x \in \text{Ker'}(i_0) \) to \( x_{j_0} \in \text{Ker}(i_{j_0}) \to \text{Ker'}(i_{j_0}) \).

Thus for each \( j \in J \),

\[
(18) \quad \text{Ker'}(\alpha_j)(x) = \lambda(i_j)(x_j).
\]

Set \( y \overset{\text{def}}{=} \rho(i_0)(x) \) and take \( j \in J \). Then

\[
F(\alpha_j)(y) = F(\alpha_j)(\rho(i_0)(x)) = \rho(i_0)(\text{Ker'}(\alpha_j)(x)) = \rho(i_0)(\lambda(i_j)(x_j)) = \sigma(i_0)(x_j) = x_j
\]

and we are done.

This completes the proof of

**Lemma 4.1.7.** Let \( F : P \to \text{Ab} \) be an injective functor over a graded poset \( P \). Then \( F \) is pseudo-injective.
Example 4.1.8. For the category $\mathcal{P}$ with shape
\[
\cdot \rightarrow \cdot
\]
the functor $F : \mathcal{P} \rightarrow \text{Ab}$ with values
\[
\mathbb{Z}[p^\infty] \hookrightarrow \mathbb{Q}/\mathbb{Z},
\]
where $p$ is prime, is not injective as the inclusion $\iota$ is not surjective, in spite of the Ker’s are $\mathbb{Q}/\mathbb{Z}$ and 0, which are injective objects of Ab.

Now we define pre-injective objects

Definition 4.1.9. Let $F : \mathcal{P} \rightarrow \text{Ab}$ be a functor over a graded poset $\mathcal{P}$. We call $F$ pre-injective if

1. for any $i_0 \in \text{Ob}(\mathcal{P})$ Ker$(i_0)$ is injective.
2. $F$ is pseudo-injective.

Till now we have obtained that injective functors $\mathcal{P} \rightarrow \text{Ab}$ over graded posets are pre-injective. In fact, as the next proposition shows, the restriction we did to graded posets is worthwhile:

Proposition 4.1.10. Let $\mathcal{P}$ be a bounded above graded poset and $F : \mathcal{P} \rightarrow \text{Ab}$ be a functor. Then $F$ is injective if and only if it is pre-injective.

Proof. If $F$ is injective then Lemmas 4.1.2 and 4.1.7 prove that $F$ is pre-injective. So assume that $F$ is an pre-injective functor.

We can suppose that the degree function deg on $\mathcal{P}$ is decreasing and takes values $\{..., 3, 2, 1, 0\}$, and that $\text{Ob}_0(\mathcal{P}) \neq \emptyset$.

To see that $F$ is injective in $\text{Ab}^\mathcal{P}$, given a diagram of functors with exact row as shown, we must find a natural transformation $\rho : A \Rightarrow F$ making the triangle commutative

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & B \\
\downarrow{\rho} & & \downarrow{\rho} \\
F & & \end{array}
\]

We define $\rho$ inductively, beginning on objects of degree 0 and successively on objects of degrees 1, 2, 3, ...

So take $i_0 \in \text{Ob}_0(\mathcal{P})$ of degree 0, and restrict to the diagram in $\text{Ab}$ over $i_0$. By Definition 4.1.9, as Ker$(i_0) = F(i_0)$, $F(i_0)$ is an injective abelian group. So we can close the following triangle with a homomorphism $\rho(i_0)$

\[
\begin{array}{ccc}
A(i_0) & \xrightarrow{\lambda(i_0)} & B(i_0) \\
\downarrow{\rho(i_0)} & & \downarrow{\rho(i_0)} \\
F(i_0) & & \end{array}
\]

As there are no arrows between degree 0 objects then $\rho$ restricted to the full subcategory with objects of degree 0 is trivially a natural transformation. Now suppose
that we have defined $\rho$ on all objects of $\mathcal{P}$ of degree less than $n$ ($n \geq 1$), and that the restriction of $\rho$ to the full subcategory generated by these objects is a natural transformation and verifies $\rho \circ \lambda = \sigma$.

The next step is to define $\rho$ on degree $n$ objects. So take $i_0 \in \text{Ob}_n(\mathcal{P})$ and consider the splitting

$$
F(i_0) = \text{Ker}(i_0) \oplus \text{Coim}(i_0)
$$

where

$$
\text{Ker}(i_0) = \bigcap_{i_0 \to i, i \neq i_0} \text{Ker} F(\alpha).
$$

To define $\rho(i_0)$ such that it makes commutative the diagram

$$
\begin{array}{c}
\text{Ker}(i_0) \oplus \text{Coim}(i_0) \\
\rho(i_0) \downarrow \searrow \sigma(i_0) \\
A(i_0) \overset{\lambda(i_0)}{\longrightarrow} B(i_0) \longrightarrow 0,
\end{array}
$$

we define by components $\rho(i_0) = \rho(i_0)_K \oplus \rho(i_0)_C$ with $\rho(i_0)_K : A(i_0) \to \text{Ker}(i_0)$ and $\rho(i_0)_C : A(i_0) \to \text{Coim}(i_0)$. Define $\rho(i_0)_K$ as a homomorphism which closes the diagram

$$
\begin{array}{c}
\text{Ker}(i_0) \\
\rho(i_0)_K \downarrow \searrow p_{i_0} \\
F(i_0) \\
\rho(i_0)_K \downarrow \searrow \sigma(i_0) \\
A(i_0) \overset{\lambda(i_0)}{\longrightarrow} B(i_0) \longrightarrow 0
\end{array}
$$

where $p_{i_0}$ is the projection. This homomorphism $\rho(i_0)_C$ does exist because $\text{Ker}(i_0)$ is an injective abelian group.

Defining $\rho(i_0)_C$ needs a little bit more of work:

**Claim 4.1.10.1.** Fix $a \in A(i_0)$. Then for each $l = 0, \ldots, n - 1$ there is $y_l \in F(i_0)$ such that

$$
F(\alpha)(y_l) = \rho(i)(A(\alpha)(a))
$$

for each $\alpha : i_0 \to i$ with $\text{deg}(i) \leq l$.

Notice that $\rho(i)$ in Equation (19) is defined by the induction hypothesis and because $\text{deg}(i) \leq l < n$.

We prove Claim 4.1.10.1 by induction. The base case reduces to find $y_0 \in F(i_0)$ such that

$$
F(\alpha)(y_0) = \rho(i)(A(\alpha)(a))
$$

for each $\alpha : i_0 \to i$ with $\text{deg}(i) = 0$. Notice that, as there are no objects of negative degree, then $\text{Ker}(i) = F(i)$ for each $i$ with $\text{deg}(i) = 0$. Then $\rho(i)(A(\alpha)(a)) \in F(i) =$
Ker(i) for each i with deg(i) = 0 and thus y₀ exists because F is n-pseudo-injective (taking J \( \defeq \{ i \mid \exists \alpha : i₀ \to i \text{ and } \deg(i) = 0 \} \).

Now we prove the inductive step of Claim 4.1.10. Suppose we have \( y_{l-1} \) verifying the hypothesis of the claim for \( l < n \). Then we construct \( y_l \). Consider any object \( i \) of degree \( l(< n) \) such that there is an arrow \( \alpha : i₀ \to i \). We have the elements \( \rho(i)(A(\alpha)(a)) \) and \( F(\alpha)(y_{l-1}) \) in the abelian group \( F(i) \). Moreover, for any \( \alpha' : i \to i' \) with \( \alpha' \neq 1_i \) it holds that

\[
F(\alpha')(F(\alpha)(y_{l-1})) = F(\alpha' \circ \alpha)(y_{l-1}) = \rho(i')A(\alpha' \circ \alpha)(a)
\]

by the induction hypothesis on \( y_{l-1} \) and because \( \deg(i') < \deg(i) = l \). Also we have that

\[
F(\alpha')(\rho(i)(A(\alpha)(a))) = \rho(i')A(\alpha' \circ \alpha)(a)
\]

because \( \rho \) is a natural transformation on objects of degree less than \( n \) and because \( \deg(i) = l < n \).

Equations (20) and (21) give that \( \rho(i)(A(\alpha)(a)) - F(\alpha)(y_{l-1}) \in \text{Ker}(i) \). Then considering \( J' \defeq \{ i \mid \exists \alpha : i₀ \to i \text{ and } \deg(i) = l \} \) and applying that \( F \) is \( (n-l) \)-pseudo-injective we obtain \( y' \in F(i₀) \) such that

\[
F(\alpha)(y') = \rho(i)(A(\alpha)(a)) = F(\alpha)(y_{l-1})
\]

for each \( \alpha : i₀ \to i \) with \( \deg(i) = l \). Define

\[
y_l \defeq y_{l-1} + y'.
\]

To check that \( y_l \) satisfies the statement of the claim take \( \alpha : i₀ \to i \) with \( \deg(i) \leq l \). If \( \deg(i) = l \) then

\[
F(\alpha)(y_l) = F(\alpha)(y_{l-1} + y') = F(\alpha)(y_{l-1}) + (\rho(i)(A(\alpha)(a)) - F(\alpha)(y_{l-1})) = \rho(i)(A(\alpha)(a)).
\]

If \( \deg(i) < l \) then

\[
F(\alpha)(y_l) = F(\alpha)(y_{l-1} + y') = \rho(i)(A(\alpha)(a)) + F(\alpha)(y').
\]

In fact, \( F(\alpha)(y') = 0 \): factor \( \alpha \) as

\[
\begin{array}{ccc}
  i₀ & \rightarrow & i \\
   & \alpha & \downarrow \\
   & \beta & \downarrow \\
   & i' & \rightarrow
\end{array}
\]

where \( i' \) has degree \( l \). Then

\[
F(\alpha)(y') = F(\beta)(F(\alpha')(y')) = F(\beta)(\rho(i')(A(\alpha')(a)) - F(\alpha')(y_{l-1})) = \rho(i)(A(\alpha)(a)) - F(\alpha)(y_{l-1}) = 0
\]

by using the induction hypothesis on \( y_{l-1} \) and on the naturality of \( \rho \). This finishes the proof of the claim.
4.1. INJECTIVE OBJECTS IN $\text{Ab}^2$.

Now we define $\rho(i_0)_C$. Take $a \in A(i_0)$ and define $\rho(i_0)_C(a) = \overline{y} \in \text{Coim}(i_0)$, where $y \in F(i_0)$ is such that

$$F(\alpha)(y) = \rho(i)(A(\alpha)(a))$$

for each $\alpha : i_0 \to i$ with $\alpha \neq 1_{i_0}$. The element $y$ is obtained by taking $y = y_l$ with $l = n - 1$ in Claim 4.1.10.1.

It is straightforward that $\rho(i_0)_C$ is well defined because if $y' \in F(i_0)$ also verifies that $F(\alpha)(y) = \rho(i)(A(\alpha)(a))$ for each $\alpha : i_0 \to i$ with $\alpha \neq 1_{i_0}$ then $y - y' \in \text{Ker}(i_0)$ and thus $\overline{y} = \overline{y'}$ in $\text{Coim}(i_0)$. Also it is clear that $\rho(i_0)_C$ is a homomorphism of abelian groups.

It remains to prove that $\rho(i_0) \circ \lambda(i_0) = \sigma(i_0)$. Take $b \in B(i_0)$ and write $\sigma(i_0)(b) = p_{i_0}(\sigma(i_0)(b)) \oplus \sigma(i_0)(b) \in F(i_0)$. We want to see that $\rho(i_0)_K(\lambda(i_0)(b)) = p_{i_0}(\sigma(i_0)(b))$ and that $\rho(i_0)_C(\lambda(i_0)(b)) = \sigma(i_0)(b)$. The equation

$$\rho(i_0)_K(\lambda(i_0)(b)) = p_{i_0}(\sigma(i_0)(b))$$

holds by definition of $\rho(i_0)_K$. Moreover, for each $\alpha : i_0 \to i$ with $\alpha \neq 1_{i_0}$, we have that

$$F(\alpha)(\sigma(i_0)(b)) = \sigma(i)(B(\alpha)(b))$$

as $\sigma$ is a natural transformation, and that

$$\rho(i)(A(\alpha)(\lambda(i_0)(b))) = \rho(i)(\lambda(i)(B(\alpha)(b))) = \sigma(i)(B(\alpha)(b))$$

as $\lambda$ is a natural transformation and as $\rho \circ \lambda = \sigma$ holds on objects of degree less than $n$ by the induction hypothesis. Then, by the definition of $\rho(i_0)_C$,

$$\rho(i_0)_C(\lambda(i_0)(b)) = \overline{\sigma(i_0)(b)}.$$

Defining $\rho(i_0)$ in this way for every $i_0 \in \text{Ob}_n(\mathcal{P})$ we have now $\rho$ defined on all objects of $\mathcal{P}$ of degree less or equal than $n$. The last thing to do in order to complete the inductive step is to prove that $\rho$ restricted to the full subcategory over these objects is a natural transformation. Take $\alpha : i_0 \to i$ of degree different from zero in this full subcategory. If the degree of $i_0$ is less than $n$ then the commutativity of

$$
\begin{array}{ccc}
F(i_0) & \xrightarrow{F(\alpha)} & F(i) \\
\rho(i_0) \downarrow & & \downarrow \rho(i) \\
A(i_0) & \xrightarrow{A(\alpha)} & A(i)
\end{array}
$$

is granted by the inductive hypothesis. Suppose that the degree of $i_0$ is $n$. Take $a \in A(i_0)$. Then $\rho(i_0)(a) = \rho(i_0)_K(a) \oplus \rho(i_0)_C(a) = \rho(i_0)_K(a) \oplus \overline{y}$ where $y \in F(i_0)$ is such that

$$F(\alpha')(y) = \rho(i')(A(\alpha')(a))$$

for each $\alpha' : i_0 \to i'$ with $\alpha' \neq 1_{i_0}$. Then

$$F(\alpha)(\rho(i_0)_K(a)) = 0$$

because $\rho(i_0)_K(a) \in \text{Ker}(i_0)$. Thus

$$F(\alpha)(\rho(i_0)(a)) = F(\alpha)(y)$$
and

\[ F(\alpha)(y) = \rho(i)(A(\alpha)(a)) \]

by the construction of \( y \).

This proposition yields the following examples. The degree functions \( deg \) for the bounded above graded posets appearing in the examples are indicated by subscripts \( i_{deg(i)} \) on the objects \( i \in \text{Ob}(\mathcal{P}) \) and take values \{..., 3, 2, 1, 0\}.

**Example 4.1.11.** For the “pullback category” \( \mathcal{P} \) with shape

\[
\begin{array}{ccc}
a_0 & \xrightarrow{f} & c_1 \\
\downarrow{g} & & \\
b_0 & \xrightarrow{f} & c_1
\end{array}
\]

a functor \( F : \mathcal{P} \to \text{Ab} \) is injective if and only if

- \( F(c) \), \( \text{Ker}(F(f)) \) and \( \text{Ker}(F(g)) \) are injective abelian groups.
- \( F(f) \) and \( F(g) \) are epimorphisms.

For the inverse “telescope category” \( \mathcal{P} \) with shape

\[
\begin{array}{cccccccc}
... & a_4 & \xrightarrow{f_4} & a_3 & \xrightarrow{f_3} & a_2 & \xrightarrow{f_2} & a_1 & \xrightarrow{f_1} & a_0 \\
\end{array}
\]

a functor \( F : \mathcal{P} \to \text{Ab} \) is injective if and only if

- \( F(a_0) \) is an injective abelian group.
- \( \text{Ker}(F(f_i)) \) is an injective abelian group and \( F(f_{i-d} \circ f_{i-d+1} \circ \ldots \circ f_i) \) is an epimorphism for \( d = 0, 1, \ldots, i - 1 \) for each \( i = 1, 2, 3, 4, \ldots \).

### 4.2. Pseudo-injectivity

Consider a functor \( F : \mathcal{P} \to \text{Ab} \) over a small category \( \mathcal{P} \). In this section we look for, and find, conditions on \( F \) such that \( \lim_i F = 0 \) for \( i \geq 1 \), i.e., we want conditions such that the right derived functors of the left exact functor \( \lim \) vanishes on \( F \). We restrict along this section to graded posets \( \mathcal{P} \). Fix the following notation

**Definition 4.2.1.** Let \( \mathcal{P} \) be a graded poset and \( F : \mathcal{P} \to \text{Ab} \). We say \( F \) is \textit{lim-acyclic} if \( \lim_i F = 0 \) for \( i \geq 1 \).

Recall that for injective objects it holds that any right derived functor vanishes. So, from Proposition 4.1.10 we obtain firstly that

**Proposition 4.2.2.** Let \( F : \mathcal{P} \to \text{Ab} \) be a pre-injective functor over a bounded above graded poset \( \mathcal{P} \). Then \( F \) is \textit{lim-acyclic}.

Because being \( \textit{lim}-\text{acyclic} \) is clearly weaker than being injective we can wonder if is it possible to weaken the hypothesis on Proposition 4.2.2 keeping the thesis of \( \textit{lim}-\text{acyclicity} \). The answer is yes and the following theorem states the appropriate conditions. Notice that we have removed the condition [11] of Definition 4.1.9.
Theorem 4.2.3. Let $F : \mathcal{P} \rightarrow \text{Ab}$ be a pseudo-injective functor over a bounded above graded poset $\mathcal{P}$. Then $F$ is $\varprojlim$-acyclic.

**Proof.** We can suppose that the degree function $\text{deg}$ on $\mathcal{P}$ is decreasing and takes values $\{..., 3, 2, 1, 0\}$, and that $\text{Ob}_0(\mathcal{P}) \neq \emptyset$. To compute $\varprojlim F$ we use the normalized (see (2.0.6)) spectral sequences corresponding to the sixth row of Table 1 (of Chapter 2).

Fix $t \geq 1$. To prove $\varprojlim F = 0$ it is enough to show that $E_{1,t}^{p,t-p}$ is zero for every $p$. The contributions to $E_{1,t}^{p,t-p}$ come from $(E_p)^{\infty}_{p',p-p'-t}$ for $p' \leq p - t$ (we are using normalized spectral sequences, Remark 2.0.6). We prove that

$$
(E_p)^{r}_{p',p-p'-t} = 0
$$

if $r$ is big enough for each $p$ and $p' \leq p - t$. This implies that $\varprojlim F = 0$.

Consider the decreasing filtration $L^*$ of $C^*(\mathcal{P}, F)$ that gives rise to the spectral sequence $E_{*,*}^r$. The $n$-simplices are

$$
L_n^p = L^pC^n(\mathcal{P}, F) = \prod_{\sigma \in \mathcal{P}_n, \text{deg}(\sigma) \geq p} F^\sigma.
$$

For each $p$ we have an increasing filtration $M_p^*$ of the quotient $L^p/L^{p+1}$ that gives rise to the spectral sequence $(E_p)^{*,*}_r$, and which $n$-simplices are

$$
(M_p)_n^{p'} = \prod_{\sigma \in \mathcal{P}_n, \text{deg}(\sigma) = p, \text{deg}(\sigma) \leq p'} F^\sigma.
$$

For $p' \leq p - t$ the abelian group $(E_p)^{r}_{p',q'}$ at the $t = -(p' + q') + p$ simplices is given by

$$(E_p)^{r}_{p',q'} = (M_p)_t^{p'} \cap d^{-1}((M_p)_{t+1}^{p'-r})/(M_p)_t^{p'-1} \cap d^{-1}((M_p)_{t+1}^{p'-r}) + (M_p)_t^{p'} \cap d((M_p)_{t-1}^{p'+r-1})$$

where $d$ is the differential of the quotient $L^p/L^{p+1}$ restricted to the subgroups of the filtration $(M_p)^*$. For $r > p' - p + t + 1$ there are not $(t+1)$-simplices beginning in degree $p$ and ending in degree less or equal to $p' - r < p - (t + 1)$, i.e., $(M_p)_t^{p'-r} = 0$. Also $r$ big enough implies $(M_p)_{t-1}^{p'+r-1} = (M_p)_{t-1}^{t-1} = (L^p/L^{p+1})_{t-1}$, i.e., $(M_p)_t^{p'+r-1}$ equals all the $(t-1)$-simplices that begin on degree $p$. Thus there exists $r$ such that

$$(E_p)^{r}_{p',q'} = (M_p)_t^{p'} \cap d^{-1}(0)/(M_p)_t^{p'-1} \cap d^{-1}(0) + (M_p)_t^{p'} \cap d((M_p)_{t-1}^p)$$

Fix such an $r$ and take $[x] \in (E_p)^{r}_{p',q'}$ where

$$
x \in (M_p)_t^{p'} = \prod_{\sigma \in \mathcal{P}_t, \text{deg}(\sigma) = p, \text{deg}(\sigma) \leq p'} F^\sigma
$$

and $d(x) = 0$. We prove that $[x] = 0$ in two steps:

**Step 1:** In this first step we find a representative $x' \in (M_p)_t^{p'} \cap d^{-1}(0)$ for $[x]$ such that $x'_{\sigma} = 0$ for every $\sigma = \sigma_0 \overset{\alpha_1}{\longrightarrow} \sigma_1 \overset{\alpha_2}{\longrightarrow} \cdots \overset{\alpha_{t-1}}{\longrightarrow} \sigma_{t-1} \overset{\alpha_t}{\longrightarrow} \sigma_t$ with $\text{deg}(\sigma_{t-1}) \leq p'$. 


Notice that because \( P \) is bounded above then
\[
(M_p)_p^{p'} = \prod_{\sigma \in NP_t, \deg(\sigma_0) = p, \deg(\sigma) \leq p'} F^\sigma = \prod_{\sigma \in NP_t, \deg(\sigma_0) = p, 0 \leq \deg(\sigma) \leq p'} F^\sigma.
\]

We need the following:

**Claim 4.2.3.1.** For each \( l = 0, \ldots, p' \) there exists a representative \( x'_l \in (M_p)_l^{p'} \cap d^{-1}(0) \) for \([x]\) such that \((x'_l)_\sigma = 0 \) for every
\[
\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{l-1}} \sigma_l
\]
with \( \deg(\sigma_{l-1}) \leq l \).

Notice that taking \( l = p' \) in the claim the step 1 is finished. The case \( l = 0 \) in the claim is fulfilled by taking \( x'_0 = x \) because there are no objects of negative degree.

Now we build \( x'_l \) from \( x'_{l-1} \) for \( 1 \leq l \leq p' \). For that we need the

**Claim 4.2.3.2.** For each \( m = 0, \ldots, l \) there exists a representative \( x'_{l,m} \in (M_p)_l^{p'} \cap d^{-1}(0) \) for \([x]\) such that \((x'_{l,m})_\sigma = 0 \) for every
\[
\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{l-1}} \sigma_l
\]
with \( \deg(\sigma_{l-1}) < l \) or \( \deg(\sigma_{l-1}) \leq l \) and \( \deg(\sigma_l) < m \).

Notice that taking \( m = l \) in Claim 4.2.3.2 we obtain \( x'_l \overset{\text{def}}{=} x'_{l,l} \) such that \((x'_l)_\sigma = 0 \) if \( \deg(\sigma_{l-1}) < l \), i.e., such that \( x'_l \) satisfies Claim 4.2.3.1. Define \( x'_{l,0} \overset{\text{def}}{=} x'_{l-1} \). This satisfies the claim for \( m = 0 \) as there is no object of negative degree. Now we build \( x'_{l,m} \) from \( x'_{l,m-1} \) for \( 1 \leq m \leq l \).

By hypothesis \( d(x'_{l,m-1}) = 0 \). The differential \( d \) is the restriction of the induced differential on \( L^p/L^{p+1} \) to
\[
d : (M_p)_l^{p'} \to (M_p)_{l+1}^{p'}.
\]

Recall that
\[
d = \sum_{j=1, \ldots, t+1} (-1)^j d^j
\]
and that, for \( \sigma \in NP_{t+1} \), the final object of \( d_j(\sigma) \) is \( \sigma_{t+1} \) for each \( j = 1, 2, \ldots, t \) and \( \sigma_t \) for \( j = t + 1 \).

Thus for every \( \epsilon \in NP_{t+1} \) with \( \deg(\epsilon_0) = p \) and \( \deg(\epsilon_{t+1}) \leq p' \), we can apply the projection
\[
\pi^\epsilon : (M_p)_l^{p'} \to F^\epsilon
\]
and obtain that \( \pi^\epsilon(d(x'_{l-1})) = 0 \).

Now fix \( \sigma \in NP_t \) with \( \deg(\sigma_0) = p \), \( \deg(\sigma_{t-1}) = l \) and \( \deg(\sigma_t) = m-1 \). Consider the object \( \sigma_t \in Ob_{m-1}(P) \) and any morphism \( \alpha : \sigma_t \to i \) with \( \alpha \neq 1_{\sigma_t} \). Write
\[
\epsilon \overset{\text{def}}{=} \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t \xrightarrow{\alpha} i.
\]
Notice that \( \deg(e) = p \) and \( \deg(e_{l+1}) = \deg(i) < \deg(\sigma_t) = m - 1 \leq l - 1 < p' \). By hypothesis \( \pi^t(d(x'_{i,m-1})) = 0 \). In fact, as \( x'_{i,m-1} \) fulfills Claim 4.2.3.2 then \( \pi^t(d(x'_{i,m-1})) = F(\alpha)((x'_{i,m-1})_\sigma) \) and so
\[
F(\alpha)((x'_{i,m-1})_\sigma) = 0.
\]

Because \( \alpha : \sigma_t \rightarrow i \) was arbitrary we have proven that
\[
(x'_{i,m-1})_\sigma \in \text{Ker}(\sigma_t)
\]
for each \( \sigma \in NP_t \) with \( \deg(\sigma_0) = p, \deg(\sigma_{t-1}) = l \) and \( \deg(\sigma_t) = m - 1 \).

Consider now any \( \tau \in NP_{t-1} \) with \( \deg(\tau_0) = p \) and \( \deg(\tau_{t-1}) = l \). For the object \( \tau_{t-1} \) of degree \( l \) consider all the non-trivial morphisms from \( \tau_{t-1} \) to objects of degree \( m - 1 \). Define \( \Delta_\tau = \{ \alpha \in \text{Hom}_F(\tau_{t-1}, i)|\deg(i) = m - 1 \} \).

Notice that for each \( \alpha \in \Delta_\tau \) and the \( t \)-simplex
\[
\tau_\alpha \defeq \tau_0 \approx_1 \tau_1 \ldots \approx t_{l-2} \approx_{t-1} \tau_{t-1} \approx i
\]
we have proven before that \( (x'_{i,m-1})_{\tau_\alpha} \in \text{Ker}(i) \). Then, as \( F \) is \( (m - 1 - l) \)-pseudo-injective, there is an element \( x_\tau \in F(\tau_{t-1}) \) such that
\[
F(\alpha)(x_\tau) = (x'_{i,m-1})_{\tau_\alpha}
\]
for every \( \alpha \in \Delta_\tau \).

Consider now the \((t - 1)\)-chain \( y \in (M_p)_{t-1}^p \) given by
\[
y_\tau = \begin{cases} x_\tau & \text{if } \deg(\tau_{t-1}) = l \\ 0 & \text{otherwise.} \end{cases}
\]

If fact, as \( y \) takes non zero values just on \( \tau \) with \( \deg(\tau_{t-1}) = l \leq p' \), then \( y \in (M_p)_t^{p'} \) and thus \( d(y) \in (M_p)^{p'} \). \((M_p)^* \) is a filtration of a differential complex. This means that \( d(y) \in (M_p)_t^{p'} \cap d((M_p)_t^{p'}) \).

Define \( x'_{i,m} = x'_{i,m-1} - d(y) \). Then \( x'_{i,m} \in (M_p)^p \) and \( d(x'_{i,m}) = 0 \). Thus \([x'_{i,m}] = [x_{i,m}] \). That \( x'_{i,m} \) fulfills Claim 4.2.3.2 is clear by construction. This finishes the proofs of Claims 4.2.3.2 and 4.2.3.1.

**Step 2:** By the step 1 we can suppose that
\[
x_\sigma = 0
\]
for \( \sigma \in NP_t \) with \( \deg(\sigma_0) = p \) and \( \deg(\sigma_{t-1}) \leq p \). Our objective now is to see that there exists \( y \in (M_p)_{t-1}^p \) with \( d(y) = x \). This implies that \([x] = 0 \) and finishes the proof of the theorem. We need the

**Claim 4.2.3.3.** There are chains \( x_i \in (M_p)_t^p \) for \( i = p', p' + 1, \ldots, p - t - 1, p - t, p - t + 1 \) and \( y_i \in (M_p)_{t-1}^p \) for \( i = p', p' + 1, \ldots, p - t - 1, p - t \) such that
\[
d(y_i) = x_i + x_{i+1}
\]
for \( i = p', p' + 1, \ldots, p - t - 1, p - t \) with \( x_{p'} = x \) and \( x_{p-t+1} = 0 \). Moreover

(I) \( (x_i)_\sigma = 0 \) if \( \deg(\sigma_{t-1}) \leq i \) for \( i = p', \ldots, p - t + 1 \)

(II) \( d(x_i) = 0 \) for \( i = p', \ldots, p - t + 1 \).
Notice that the claim finishes the step 2: as \( x_{p-t+1} = 0 \) then \( x_{p-t} = d(y_{p-t}) \),
\[
x_{p-t+1} = d(y_{p-t+1}) - x_{p-t} = d(y_{p-t+1} - y_{p-t}) \quad x_{p-t+2} = d(y_{p-t+2} - y_{p-t+1} + y_{p-t}) \quad x = x' = d(y_{p'}) - x_{p'+1} = d(y_{p'} - y_{p'+1} + \ldots + (-1)^{p'-p+1} y_{p-t})
\]
where \( y_{p'} - y_{p'+1} + \ldots + (-1)^{p'-p+1} y_{p-t} \in (M_p)^p \).

Define \( x_{p'} \overset{\text{def}}{=} x \). Then conditions (I) and (II) are satisfied for \( i = p' \). We construct \( y_{i+1} \) from \( x_i \) recursively beginning with \( i = p' \). The element \( y_i \) is built as in Claim 4.2.3.1, the only difference being that now \( y_i \) lies in \((M_p)^{p}_{t-1} \subseteq (M_p)^{p}_{t-1} \), but not in \((M_p)^{p'}_{t-1} \). Then write \( x_{i+1} \overset{\text{def}}{=} d(y_i) - x_i \). Notice that \( d(x_{i+1}) = d^2(y) - d(x_i) = 0 - 0 = 0 \) by the induction hypothesis on \( i \). Also notice that \( x_{i+1} \) satisfies (I) of the Claim 4.2.3.3 by the construction of \( y_i \).

When constructing \( y_{p-t} \) and \( x_{p-t+1} \) from \( x_{p-t} \) notice that, because \( \tau \in N P_{t-1} \) with \( \deg(\tau) = p \) and \( \deg(\tau_{t-1}) = p - (t - 1) \) imply that every morphism in \( \tau \) is of degree 1, then the condition \( d_j(\sigma) = \tau \) for \( \sigma \in N P_t \) only holds for \( j = t \). This implies that \( x_{p-t+1} = 0 \) by the construction of \( y_{p-t} \).

\[\square\]

**Example 4.2.4.** For the “pullback category” \( P \) with shape

\[
\begin{array}{c}
a_0 \\
\downarrow f \\
b_0 \\
\downarrow g \\
c_1
\end{array}
\]

a functor \( F : P \to \text{Ab} \) is \( \lim \)-acyclic if \( F(f) \) and \( F(g) \) are epimorphisms.

For the inverse ‘telescope category’ \( P \) with shape

\[
\ldots \xrightarrow{a_4} \xrightarrow{f_4} \xrightarrow{a_3} \xrightarrow{f_3} \xrightarrow{a_2} \xrightarrow{f_2} \xrightarrow{a_1} \xrightarrow{f_1} \xrightarrow{a_0}
\]

a functor \( F : P \to \text{Ab} \) is \( \lim \)-acyclic if \( F(f_i \circ f_{i-1} \circ \ldots \circ f_1) \) is an epimorphism for \( d = 0, 1, \ldots, i - 1 \) for each \( i = 1, 2, 3, 4, \ldots \).

### 4.3. Computing higher limits

Theorem 4.2.3 shows that over a bounded above graded poset pseudo-injectivity is enough for \( \lim \)-acyclicity. But it turns out that pseudo-injectivity is not necessary for \( \lim \)-acyclicity:

**Example 4.3.1.** For the ‘pushout category’ \( P \) with shape

\[
\begin{array}{c}
a_0 \\
\downarrow f \\
b_1 \\
\downarrow g \\
c_1
\end{array}
\]

a functor \( F : P \to \text{Ab} \) is pseudo-injective if and only if \( F(a) \overset{F(f) \oplus F(g)}{\longrightarrow} F(b) \oplus F(c) \) is an epimorphism. But a straightforward calculus shows that \( \lim F = 0 \) for \( i \geq 1 \) for any \( F \).
Anyway, we shall see how pseudo-injectivity allows us to obtain a better knowledge of the higher limits \( \lim_{\leftarrow}^i F \). We begin with

**Definition 4.3.2.** Let \( F : \mathcal{P} \to \text{Ab} \) be a functor over a graded poset \( \mathcal{P} \). Then \( F' : \mathcal{P} \to \text{Ab} \) is the functor which takes values on objects

\[
F'(i_0) = \bigoplus_{\alpha : i_0 \to i} F(i)
\]

for \( i_0 \in \text{Ob}(\mathcal{P}) \). For \( \beta \in \text{Hom}(\mathcal{P}), \beta : i_1 \to i_0, \) \( F'(\beta) \) is the only homomorphism which makes the diagram

\[
\begin{array}{ccc}
F'(i_1) & \xrightarrow{F'(\beta)} & F'(i_0) \\
\uparrow & & \uparrow \\
F(i_{i_1 \to i}) & = & F(i_{i_0 \to i})
\end{array}
\]

to commute for each \( \alpha : i_1 \to i \) that factors through \( \beta \)

\[
\begin{array}{ccc}
i_1 & \xrightarrow{\beta} & i_0 \\
\downarrow & & \downarrow \\
& \xrightarrow{i} & \\
\end{array}
\]

and the diagram

\[
\begin{array}{ccc}
F'(i_1) & \xrightarrow{F'(\beta)} & F'(i_0) \\
\uparrow & & \uparrow \\
F(i_{i_1 \to i}) & \xrightarrow{0} & \\
\end{array}
\]

for each \( \alpha : i_1 \to i \) that does not factor through \( \beta \).

Notice that \( F' \) is built from \( F \) as \( \text{Ker}' \) was built from \( \text{Ker} \) in Section 4.1 and that \( \text{Ker}_{F'}(i) = F(i) \) for each \( i \in \text{Ob}(\mathcal{P}) \). A nice property of \( F' \) is

**Lemma 4.3.3.** Let \( F : \mathcal{P} \to \text{Ab} \) be a functor over a graded poset. Then for each \( G \in \text{Ab}^{\mathcal{P}} \) there is a bijection

\[
\text{Hom}_{\text{Ab}^{\mathcal{P}}}(G, F') \xrightarrow{\phi} \prod_{i \in \text{Ob}(\mathcal{P})} \text{Hom}_{\text{Ab}}(G(i), F(i)).
\]

The proof is analogous to that of Lemma 3.3.3. Another interesting property of \( F' \) is the following

**Lemma 4.3.4.** Let \( F : \mathcal{P} \to \text{Ab} \) be a functor over a graded poset. Then

\[
\lim_{\leftarrow}^i F' \cong \prod_{i \in \text{Ob}(\mathcal{P})} F(i).
\]

**Proof.** It is straightforward using the previous lemma. \( \square \)

The main feature of \( F' \) we shall use is
Lemma 4.3.5. Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a graded poset. Then $F'$ is pseudo-injective.

Proof. It is straightforward. \qed

Remark 4.3.6. The monic natural transformation $F \Rightarrow G'$, where $G = \mathbb{Q}/\mathbb{Z}^{\text{HoF}}$ with $U : \text{Ab} \to \text{Set}$ the forgetful functor and $\mathbb{Q}/\mathbb{Z}^{-} : \text{Set} \to \text{Ab}$ the power on a set, is a way to prove that $\text{Ab}^C$ has enough injectives for any small category $C$ (see [13, 243ff.] on how to construct the maps $F(i) \to G(i)$ for $i \in \text{Ob}(C)$).

By Lemma 4.3.3 there is a unique natural transformation $\lambda : F \Rightarrow F'$ corresponding to the family of homomorphisms $\{F(i) \xrightarrow{1_{F(i)}} F(i')\}_{i \in \text{Ob}(\mathcal{P})}$. It is clear that $\lambda$ is a monic natural transformation. Thus we can consider the object-wise co-kernel of $\lambda$ to obtain a short exact sequence of functors

$$0 \Rightarrow F \xrightarrow{\lambda} F' \Rightarrow C \Rightarrow 0.$$  

If $\mathcal{P}$ is bounded above then the long exact sequence (see Section 1.6) associated to this short exact sequence gives

$$\lim_{\leftarrow} F = \begin{cases} \lim_{\leftarrow} C_F & j > 1 \\ \text{Coim}\{\lim_{\leftarrow} F' \to \lim_{\leftarrow} C_F\} & j = 1 \end{cases}$$

because $F'$ is $\lim$-acyclic (it is pseudo-injective by Lemma 4.3.5 and apply Theorem 4.2.3). Arguing as in the direct limit case we have

Lemma 4.3.7. Let $\mathcal{P}$ be a bounded above graded poset and let $F : \mathcal{P} \to \text{Ab}$ be a functor. Then there are functors $C_j : \mathcal{P} \to \text{Ab}$ for $j = 0, 1, 2, \ldots$ with $C_0 = F$ and $C_1 = \text{Coim}(F \Rightarrow F')$ such that

$$\lim_{\leftarrow} F = \lim_{\leftarrow} C_1 = \lim_{\leftarrow} C_2 = \ldots = \lim_{\leftarrow} C_j = \text{Coim}\{\lim_{\leftarrow} C'_{j-1} \to \lim_{\leftarrow} C_j\}$$

for each $j = 0, 1, 2, \ldots$.

The values $F'(i_0)$ and $C_F(i_0)$ can be very big. This can be improved considering the functor $\text{Ker} : \mathcal{P} \to \text{Ab}$ of Section 4.1. Suppose that $\mathcal{P}$ is bounded above and choose a family of homomorphisms $F(i) \to \text{Ker}(i)$ such that the choice for the objects $i_0$ which do not have any departing arrow is the identity $F(i_0) \to F(i_0) = \text{Ker}(i_0)$. Then the natural transformation $\lambda : F \Rightarrow \text{Ker}'$ given by Lemma 4.3.3 is monic in case $F$ is monic. We have

Lemma 4.3.8. Let $\mathcal{P}$ be a bounded above graded poset and let $F : \mathcal{P} \to \text{Ab}$ be a monic functor. Then there is a short exact sequence of functors

$$0 \Rightarrow F \Rightarrow \text{Ker}' \Rightarrow C \Rightarrow 0$$

where $\text{Ker}'$ is $\lim$-acyclic.
CHAPTER 5

Cohomology

In this chapter we develop further the tools of Chapter 4 in order to compute the integer cohomology of graded posets $P$ under some structural assumptions, i.e., the existence of a “covering family”. Throughout this chapter $P$ shall be a bounded above graded poset with decreasing degree function $\text{deg}$ which takes the values $\{\ldots, 3, 2, 1, 0\}$, and with $\text{Ob}_0(P) \neq \emptyset$. Conditions [1.3.10] are assumed for $P$ in the whole chapter.

The first two sections are devoted to the computation of higher limits $\lim_{\leftarrow}^i F$, where $F : P \to \text{Ab}$ takes free groups as values. After this, we specialize in compute $H^i(P; \mathbb{Z}) = \lim_{\leftarrow}^i c\mathbb{Z}$, whereas $c\mathbb{Z} : P \to \text{Ab}$ is the constant functor of value $\mathbb{Z}$. Finally, we apply it to simplex-like posets.

5.1. $p$-condensed functors

Recall that any family of homomorphisms $\{F(i) \to \text{Ker}(i)\}_{i \in \text{Ob}(P)}$ gives a functor $\lambda : F \Rightarrow \text{Ker}'_F$ by Lemma [1.3.3]. If $\lambda$ is monic then we have a short exact sequence of natural transformations

$$0 \to F \to \text{Ker}'_F \to G \to 0$$

where $\text{Ker}'_F$ is $\lim$-acyclic. This implies that $\lim^1 F = \text{Coim}(\lim \text{Ker}'_F \to \lim G)$ and $\lim^i F = \lim^{-1} G$ for $i \geq 2$. Notice that $\lim \text{Ker}'_F$ is known by Lemma [1.3.4].

Next we find conditions on $F$ such that we can build a monic natural transformation $\lambda$.

**Definition 5.1.1.** Let $F : P \to \text{Ab}$ be a functor. Suppose $P$ has decreasing degree function $\text{deg} : \text{Ob}(P) \to \{\ldots, 3, 2, 1, 0\}$ and let $0 \leq p \in \mathbb{Z}$. We say that $F$ is $p$-condensed if

(a) $F(i) = 0$ if $\text{deg}(i) < p$, and
(b) $\text{Ker}_F(i) = 0$ if $\text{deg}(i) > p$.

Notice that constant functors are 0-condensed. If the functor $F$ is $p$-condensed then we can consider the natural transformation $\lambda : F \Rightarrow \text{Ker}_F$ given by Lemma [1.3.3] for the maps $\tau_i : F(i) \to \text{Ker}_F(i)$

$$\tau_i = \begin{cases} 1_F(i) & \text{if } \text{deg}(i) = p \\ 0 & \text{otherwise.} \end{cases}$$
Notice that we have

\[
\text{Ker}_F(i) = \begin{cases} 
F(i) & \text{if } \deg(i) = p \\
0 & \text{otherwise}
\end{cases}
\]

by hypothesis (a) and (b) in Definition 5.1.1. In fact, the functor \( \text{Ker}'_F \) takes values on objects

(25) \[
\text{Ker}'_F(i_0) = \prod_{i \in (i_0 \downarrow \mathcal{P})_p} F(i)
\]

where \((i_0 \downarrow \mathcal{P})_p = \{ i \in \text{Ob}(\mathcal{P}) | \exists i_0 \to i, \deg(i) = p \}\) and \( \text{Ker}_F(\beta) \) for \( \beta : i_0 \to i_1 \) is induced by the projections determined by \((i_1 \downarrow \mathcal{P})_p \subset (i_0 \downarrow \mathcal{P})_p \). The homomorphism \( \lambda_i : F(i) \to \text{Ker}'_F(i) \) is given by

\[
\lambda_i = \prod_{i \in (i_0 \downarrow \mathcal{P})_p} F(\alpha_i)
\]

where \( \alpha_i : i_0 \to i \) is the unique arrow from \( i_0 \) to \( i \in (i_0 \downarrow \mathcal{P})_p \). So \( \lambda_i \) is a kind of "diagonal". An easy induction argument on \( \deg(i) \in \{ p, p+1, \ldots \} \) shows that \( \lambda \) is a monic natural transformation. We have obtained

**Lemma 5.1.2.** Let \( F : \mathcal{P} \to \text{Ab} \) be a \( p \)-condensed functor. Then there is a short exact sequence

\[
0 \longrightarrow F \xrightarrow{\lambda} \text{Ker}'_F \longrightarrow G \longrightarrow 0.
\]

\( G \) is obtained by taking the object-wise co-image of \( \lambda \). On the object \( i_0 \), \( G \) takes the value

(26) \[
G(i_0) = \prod_{i \in (i_0 \downarrow \mathcal{P})_p} F(i)/\lambda_{i_0}(F(i_0)).
\]

It is clear that \( G \) verifies condition (a) of Definition 5.1.1 for \( p+1 \), but in general condition (b) does not hold for \( G \) and \( p+1 \). More precisely, if \( \deg(i_0) > p+1 \) then \( \text{Ker}_G(i_0) = 0 \) is equivalent to the natural map

\[
F(i_0) \to \varprojlim_{(i_0 \downarrow \mathcal{P})_*} F
\]

being an isomorphism, where \((i_0 \downarrow \mathcal{P})_* \) is the full subcategory of \( \mathcal{P} \) with objects \( \{ i | \exists i_0 \to i, i \neq i_0 \} \). This natural map is a monomorphism by condition (b) of \( F \) being \( p \)-condensed. So, \( \text{Ker}_G(i_0) = 0 \) if and only if \( F(i_0) \to \varprojlim_{(i_0 \downarrow \mathcal{P})_*} F \) is surjective. This is a local property. We summarize these results in the following:

**Lemma 5.1.3.** Let \( F : \mathcal{P} \to \text{Ab} \) be a \( p \)-condensed functor. Then there is a short exact sequence

\[
0 \longrightarrow F \xrightarrow{\lambda} \text{Ker}'_F \longrightarrow G \longrightarrow 0.
\]

Moreover, \( G \) is \( (p+1) \)-condensed if and only if for each object \( i_0 \) of degree greater than \( p+1 \), we have \( F(i_0) \xrightarrow{\approx} \varprojlim_{(i_0 \downarrow \mathcal{P})_*} F \).
5.2. Covering families

In this section we study a bit further the condition given in Lemma 5.1.3 which implies that the $G$ is $(p + 1)$-condensed, where the functor $G$ is defined by the short exact sequence

$$0 \longrightarrow F \xrightarrow{\lambda} \text{Ker}' \xrightarrow{\gamma} G \longrightarrow 0,$$

and $F$ is a $p$-condensed functor. This condition states that $G$ is $(p + 1)$-condensed if and only if for each object $i_0$ of degree greater than $p + 1$ it holds that the map

$$(27) F(i_0) \rightarrow \lim_{\leftarrow (i_0 \downarrow \mathcal{P})_p} F$$

is an isomorphism. Recall that this map is a monomorphism if $F$ is $p$-condensed.

Fix $i_0$ of degree greater than $p + 1$ and consider the map given by restriction

$$\lim_{\leftarrow (i_0 \downarrow \mathcal{P})_p} F = \text{Hom}(i_0 \downarrow \mathcal{P})_p(c_Z, F) \rightarrow \prod_{i \in J} F(i)$$

over the subset $J \subseteq (i_0 \downarrow \mathcal{P})_p$. If this restriction map turns out to be injective (notice that it is injective for $J = (i_0 \downarrow \mathcal{P})_p$ because $F$ is $p$-condensed) then the composition

$$F(i_0) \rightarrow \lim_{\leftarrow (i_0 \downarrow \mathcal{P})_p} F \rightarrow \prod_{i \in J} F(i)$$

is also injective. If $F$ is a free functor (Definition 1.2.3) then both groups $F(i_0)$ and $\prod_{i \in J} F(i)$ are free abelian groups (because we are assuming Remark 1.3.10). If the map

$$F(i_0) \rightarrow \prod_{i \in J} F(i)$$

is pure then, by Lemma 1.2.2, the condition $\text{rk} F(i_0) = \sum_{i \in J} \text{rk} F(i)$ implies that this composition is an isomorphism and so $F(i_0) \cong \lim_{\leftarrow (i_0 \downarrow \mathcal{P})_p} F$. Thus we study the subsets $J \subseteq \text{Ob}(\mathcal{P})$ that make this restriction map a pure monomorphism:

**Definition 5.2.1.** Let $\mathcal{P}$ be a bounded above graded poset with decreasing degree function $\text{deg}$ which takes values $\{\ldots, 3, 2, 1, 0\}$. A family of subsets $J = \{ J^i_p \}_{i_0 \in \text{Ob}(\mathcal{P})}$, $0 \leq p \leq \text{deg}(i_0)$ with $J^i_p \subseteq (i_0 \downarrow \mathcal{P})_p$ is a covering family if

a) For each $i_0$ and $0 \leq p < \text{deg}(i_0)$ it holds that $\bigcup_{i \in J^i_p} (i \downarrow \mathcal{P})_p = (i_0 \downarrow \mathcal{P})_p$

b) For each $i_0$, $0 \leq p < \text{deg}(i_0)$ and $i \in J^i_p$ it holds that $J^i_p \subseteq J^i_{p+1}$

Notice that the definition above does not depend on a functor defined over the category $\mathcal{P}$. Also, we have $J^i_{\text{deg}(i_0)} = \{ i_0 \}$ by [ ]. The next definition states the relation we expect between a covering family and a $p$-condensed free functor

**Definition 5.2.2.** Let $\mathcal{P}$ be a bounded above graded poset, $J$ be a covering family and $F : \mathcal{P} \rightarrow \text{Ab}$ be a $p$-condensed free functor. We say that $F$ is $J$-determined if
for any object \(i_0\) of degree greater than \(p + 1\) the restriction map

\[
\lim_{(i_0 \downarrow \mathcal{P})^*} F \to \prod_{i \in J^0_p} F(i)
\]

is a monomorphism and the map

\[
F(i_0) \to \prod_{i \in J^0_p} F(i)
\]

is pure. If \(\deg(i_0) = p + 1\) then we require that the last map above is a pure monomorphism.

The main feature of covering families is that allow freeness plus \(\mathcal{J}\)-determinacy to pass from \(F\) to \(G\). For an object \(i_0\) with \(\deg(i_0) \geq p + 1\) notice that the map

\[
F(i_0) \to \prod_{i \in J^0_p} F(i)
\]

is a pure monomorphism as consequence of Definition 5.2.2. The condition in Definition 5.2.2 for \(\deg(i_0) = p + 1\) is added in order to obtain that \(G\) is a free functor. Notice that the following proposition restricts to functors which take free abelian groups as values.

**Proposition 5.2.3.** Let \(\mathcal{P}\) be a bounded above graded poset and \(\mathcal{J}\) a covering family. Assume that \(F: \mathcal{P} \to \text{Ab}\) is \(p\)-condensed, free and \(\mathcal{J}\)-determined and consider the functor \(G\) defined by

\[
0 \to F \xrightarrow{\lambda} \text{Ker} F' \xrightarrow{\pi} G \to 0.
\]

Then, if for each object \(i_0\) with \(\deg(i_0) \geq p + 1\) it holds that \(\text{rk } F(i_0) = \sum_{i \in J^0_p} \text{rk } F(i)\) then \(G\) is \((p + 1)\)-condensed, free and \(\mathcal{J}\)-determined.

**Proof.** Notice that the hypothesis implies that for any object \(i_0\) of degree \(\deg(i_0) > p + 1\) the two maps

\[
F(i_0) \to \lim_{(i_0 \downarrow \mathcal{P})^*} F \to \prod_{i \in J^0_p} F(i)
\]

are isomorphisms. In particular, \(F(i_0) \iso \lim_{(i_0 \downarrow \mathcal{P})^*} F\) and so \(G\) is \((p + 1)\)-condensed.

If \(\deg(i_0) = p + 1\) then the map

\[
F(i_0) \to \prod_{i \in J^0_p} F(i)
\]

is an isomorphism by hypothesis. Next we prove that \(G\) is a free functor. Consider any \(i \in \text{Ob} (\mathcal{P})\) with \(\deg(i) \geq p + 1\) (if \(\deg(i) < p + 1\) then \(G(i) = 0\)) and the short exact sequence of abelian groups

\[
0 \to F(i) \xrightarrow{\lambda} \text{Ker} F'(i) \xrightarrow{\pi_i} G(i) \to 0.
\]
Then it is straightforward that the map

\[ s_i : \text{Ker}_F(i) = \prod_{j \in (i_0 \downarrow P)_p} F(j) \to \prod_{j \in J_p} F(j) \xrightarrow{\sim} F(i) \]

is a section of \( \lambda \), i.e. \( s_i \circ \lambda_i = 1_{F(i)} \) (use that the restriction map \( F(i) \to \prod_{j \in J_p} F(j) \) is injective). This implies that the short exact sequence above splits and so \( G(i) \) is a subgroup of the free abelian group \( \text{Ker}_F(i) \), and thus it is free as well. Next we prove that \( G \) is \( J \)-determined.

Take \( i_0 \) of degree \( n = \text{deg}(i_0) \) greater than \( p+2 \). We first check that the restriction map

\[ \lim_{(i_0 \downarrow P)_+} G \to \prod_{i \in J_{p+1}^{i_0}} G(i) \]

is injective. Consider any element \( \psi \in \lim_{(i_0 \downarrow P)_+} G \) which is in the kernel of the restriction map above. Notice that, as \( \text{deg}(i_0) > p+2 \), we can consider the subset \( J_{p+2}^{i_0} \subseteq (i_0 \downarrow P)_+ \). If for any \( j \in J_{p+2}^{i_0} \) it holds that \( \psi_j(1) = 0 \) then \( \psi = 0 \) by Definition 5.2.11 and because \( G \) is \( (p+1) \)-condensed.

Thus take \( j \in J_{p+2}^{i_0} \). We want to see that \( x \overset{\text{def}}{=} \psi_j(1) = 0 \). Recall the short exact sequence of abelian groups

\[ 0 \to F(j) \xrightarrow{\lambda_j} \text{Ker}_F(j) \xrightarrow{\pi_j} G(j) \to 0 \]

and take \( y \in \text{Ker}_F(j) \) such that \( \pi_j(y) = x \). Recall that \( \text{Ker}_F(j) = \prod_{i \in (j \downarrow P)_p} F(i) \) and denote by \( \alpha_i : j \to i \) the unique arrow from \( j \) to \( i \) for \( i \in (j \downarrow P)_p \).

Now consider the restriction \( y|_{i} \in \prod_{i \in J_p} F(i) \). Because \( \text{deg}(j) = p+2 > p+1 \) the map \( F(j) \to \prod_{i \in J_p} F(i) \) is an isomorphism by hypothesis. Then there exists a unique \( z \in F(j) \) with \( F(\alpha_i)(z) = y_i \) for each \( i \in J_p \subseteq (j \downarrow P)_p \). If we prove that \( F(\alpha_i)(z) = y_i \) for each \( i \in (j \downarrow P)_p \) then \( \lambda_j(z) = y \). This implies that \( x = \pi_j(y) = \pi_j(\lambda_j(z)) = 0 \) and finishes the proof.

Thus take \( i \in (j \downarrow P)_p \). By Definition 5.2.11 there is \( \beta_i : i' \to i \) with \( i' \in J_{p+1}^j \). Write \( \beta_{i'} : j \to i' \) for the unique arrow from \( j \) to \( i' \). It holds that \( \alpha_i = \beta_{i'} \circ \beta_{i'} \).

By Definition 5.2.11 we have that \( J_{p+1}^j \subseteq J_{p+1}^j \). Thus \( G(\beta_{i'})(x) = G(\beta_{i'})(\psi_j(1)) = \psi_{i'}(1) = 0 \) as \( \psi \) is in the kernel of the restriction map. The short exact sequence

\[ 0 \to F(i') \xrightarrow{\lambda_{i'}} \text{Ker}_F(i') \xrightarrow{\pi_{i'}} G(i') \to 0 \]

implies that there exists \( t_{i'} \in F(i') \) such that \( \lambda_{i'}(t_{i'}) = \text{Ker}_F(i')(y) \). Consider \( z_{i'} = F(\beta_{i'})(z) \). We have that \( z_{i'} \) and \( t_{i'} \) have the same image by the restriction map

\[ \lim_{P_{i'}} F \to \prod_{i \in J_{i'}^p} F(i) \]

because \( J_{i'}^p \subseteq J_{i'}^p \). Because \( F \) is \( J \)-determined then this restriction map is a monomorphism and so \( z_{i'} = t_{i'} \). This implies that

\[ F(\alpha_i)(z) = F(\beta_{i'} \circ \beta_{i'})(z) = F(\beta_{i'})(z_{i'}) = F(\beta_{i'})(t_{i'}) = y_i \]
and the proof of the restriction map being injective is finished.

Now we check that the map
\[ \omega : G(i_0) \to \prod_{i \in J_{p+1}^0} G(i) \]
is pure. Take \( z \in \prod_{i \in J_{p+1}^0} G(i) \) such that there exists \( x \in G(i_0) \) with \( n \cdot z = \omega(x) \) for some \( n \neq 0 \). We have to check that there exists \( x' \in G(i_0) \) with \( z = \omega(x') \), or equivalently, that \( x = n \cdot x' \) for some \( x' \in G(i_0) \). Recall once more the short exact sequence of abelian groups
\[ 0 \to F(i_0) \xrightarrow{\lambda_{i_0}} \ker F(i_0) \xrightarrow{\pi_{i_0}} G(i_0) \to 0 \]
and take \( y \in \ker F(i_0) \) with \( \pi_{i_0}(y) = x \). We are going to build \( h \in F(i_0) \) such that
\( y - \lambda_{i_0}(h) = n \cdot y', \) i.e., such that for any \( i \in (i_0 \downarrow \mathcal{P})_p \) the element \( (y - \lambda_{i_0}(h))_i = y_i - F(i_0 \to i)(h)_i \in F(i) \) is divisible by \( n \). This implies that \( x = n \cdot x' \) with \( x' = \pi_{i_0}(y') \).

Notice that by hypothesis for each \( j \in J_{p+1}^0 \), \( G(i_0 \to j)(x) = n \cdot z_j \in G(j) \).

This implies that there exist \( h_j \in F(j) \) and \( y_j \in \ker F(j) \) such that \( \ker F(i_0 \to j)(y) - \lambda_j(h_j) = n \cdot y_j \), i.e., such that for each \( i \in (j \downarrow \mathcal{P})_p \subseteq (i_0 \downarrow \mathcal{P})_p \) we have that
\[ y_i - F(j \to i)(h_j)_i = n \cdot y_i \]
to take \( y_j \) with \( \pi_j(y_j) = z_j \).

To build \( h \) we use the map
\[ \tau : \prod_{i \in J_p^0} F(i) \xrightarrow{\cong} F(i_0) \]
given by hypothesis, which is the inverse of the map
\[ F(i_0) \to \prod_{i \in J_p^0} F(i). \]

For each \( i \in J_p^0 \subseteq (i_0 \downarrow \mathcal{P})_p \) choose, by Definition 5.2.1b, \( j(i) \in J_{p+1}^0 \) such that there is an arrow \( j(i) \to i \). Then set \( h_i = F(j(i) \to i)(h_{j(i)})_i \in F(i) \), where \( h_{j(i)} \) is built as before. Define \( h \overset{\text{def}}{=} \tau(\eta) \). By construction \( F(i_0 \to i)(h)_i = F(j(i) \to i)(h_{j(i)}) \) for each \( i \in J_p^0 \) (but not for an arbitrary \( i \in (i_0 \downarrow \mathcal{P})_p \).

With this definition for \( h \) we check now that \( y_i - F(i_0 \to i)(h)_i \) is divisible by \( n \) for each \( i \in (i_0 \downarrow \mathcal{P})_p \). This finishes the proof. Fix \( i \in (i_0 \downarrow \mathcal{P})_p \) and \( j_i \in J_{p+1}^0 \) such that there is an arrow \( j_i \to i \) (we are not assuming that \( j_i = j(i) \) if \( i \in J_p^0 \)). On the one hand we have by hypothesis that
\[ y_k - F(j_i \to k)(h_{j_i}) = n \cdot (y_{j_i})_k \]
for each \( k \in \mathcal{P}_p^i \). In particular,
\[ y_k - F(j_i \to k)(h_{j_i}) = n \cdot (y_{j_i})_k \]
for each \( k \in J_p^0 \). Set \( h' \overset{\text{def}}{=} F(i_0 \to j_i)(h) \). Because \( j_i \in J_{p+1}^0 \) then, by Definition 5.2.1b, \( J_p^i \subseteq J_p^0 \) and thus by construction for any \( k \in J_p^i \)
\[ y_k - F(i_0 \to k)(h) = y_k - F(j(k) \to k)(h_{j(k)}) = n \cdot (y_{j(k)})_k. \]
Notice that \( F(i_0 \to k)(h) = F(j_i \to k)(F(i_0 \to j_i)(h)) = F(j_i \to k)(h') \). On the other hand, we have obtained

\[
(29) \quad y_k - F(j_i \to k)(h') = n \cdot (y_{j(k)})_k
\]

for each \( k \in J_i^{j_i} \).

Now write \( \eta_k = (y_{j(k)})_k - (y_{j_i})_k \) for each \( k \in J_i^{j_i} \) and write \( h'' = \tau(\eta) \in F(j_i) \) where \( \tau \) is the inverse of the map \( F(j_i) \to \prod_{i \in J_i^{j_i}} F(i) \).

By Equations (28) and (29) it is straightforward that the elements \( h_{j_i} - n \cdot h'' \) and \( h' \) have the same image by this map. Then, as this map is injective by hypothesis, \( h' = h_{j_i} - n \cdot h'' \).

Thus \( y_i - F(i_0 \to i)(h) = y_i - F(j_i \to i)(h') = y_i - F(j_i \to i)(h_{j_i} - n \cdot h'') \), and this equals

\[
\eta_k - F(j_i \to k)(h') = n \cdot (y_{j_i})_k + n \cdot F(j_i \to k)(h'').
\]

If \( \text{deg}(i_0) = p + 2 \) we have to see that the map

\[
\omega: G(i_0) \to \prod_{i \in J_i^{j_i}} G(i)
\]

is a pure monomorphism. To prove that \( \omega \) is a monomorphism use the proof above starting where \( \psi_j \) is considered for an arbitrary object \( j \) of degree \( p + 2 \). The proof of \( \omega \) being pure is exactly the same as above.

\[\square\]

**Remark 5.2.4.** Notice that, in the conditions of the proposition, and assuming that \( F \) takes finitely generated free abelian groups as values, we have the following formula for the rank of the free abelian group \( G(i_0) \) for \( \text{deg}(i_0) \geq p + 1 \)

\[
\text{rk}(G(i_0)) = \sum_{i \in (i_0 \downarrow P)_p} \text{rk} F(i) - \text{rk} F(i_0)
\]

(recall that we are assuming \[3.10\]). This is so because of the short exact sequence of free abelian groups

\[
0 \to F(i_0) \xrightarrow{\lambda_{i_0}} \text{Ker}_{F}(i_0) \xrightarrow{\pi_{i_0}} G(i_0) \to 0.
\]

**Remark 5.2.5.** Consider again the map \( s_{i_0} : \text{Ker}_{F}(i_0) \to F(i_0) \) with \( s_{i_0} \circ \lambda_{i_0} = 1_{F(i_0)} \) built in the proof of the previous proposition for \( \text{deg}(i_0) \geq p + 1 \). To \( s_{i_0} \) corresponds the monomorphism

\[
G(i_0) \xrightarrow{\delta_{i_0}} \text{Ker}_{F}(i_0)
\]

given by

\[
\pi_{i_0}(x) \mapsto x - (\lambda_{i_0} \circ s_{i_0})(x),
\]
which satisfies $\pi_{i_0} \circ \delta_{i_0} = 1_{G(i_0)}$. It is straightforward that, by construction, $\text{Im} \delta_{i_0} = \prod_{i \in (i_0 \downarrow P) \setminus i_0^p} F(i)$, and thus

$$G(i_0) \cong \prod_{i \in (i_0 \downarrow P) \setminus i_0^p} F(i).$$

Moreover, $x = \delta_{i_0}(y)$ is the only preimage of $y$ by $\pi_{i_0}$ which verifies $x_i = 0$ for $i \in J_{i_0}^p$.

The main consequence of the previous proposition is that it reduces the problem of whether $G$ is $(p+1)$-condensed to some integral equations. Moreover, this procedure can be applied recursively because the resulting functor $G$ turns out to be $(p+1)$-condensed, free and $J$-determined, and so the proposition applies to $G$ too. Notice that the ranks of $G$ are given by Remark 5.2.4.

5.2.1. Adequate covering families. In this section we apply the work developed in Sections 5.1 and 5.2 to compute the cohomology with integer coefficients of the realization of a bounded above graded poset $P$ equipped with a covering family $J$.

To compute the abelian group $H^p(P; \mathbb{Z})$ for $p \geq 1$ we compute the higher limit $\lim_{\leftarrow} c_{\mathbb{Z}}$ where $c_{\mathbb{Z}} : P \to \text{Ab}$ is the functor of constant value $\mathbb{Z}$ which sends every morphism to the identity $1_{\mathbb{Z}}$. We begin studying the extra conditions that the covering family $J$ must satisfy to apply iteratively the Proposition 5.2.3 beginning on $c_{\mathbb{Z}}$.

First, notice that $c_{\mathbb{Z}}$ is 0-condensed (we are assuming 1.3.10) and free (Definition 1.2.3). By Definition 5.2.2, $c_{\mathbb{Z}}$ is $J$-determined as 0-condensed functor if and only if for each $i_0 \in \text{Ob}(P)$ with $\deg(i_0) \geq 2$ the set $J_{i_0}^0$ intersects each connected component of $(i_0 \downarrow P)_*$. The dimensional equation in Proposition 5.2.3 for $i_0 \in \text{Ob}(P)$ with $\deg(i_0) \geq 1$ becomes $\text{rk} c_{\mathbb{Z}}(i_0) = 1 = |J_{i_0}^0| = \sum_{i \in J_{i_0}^0} \text{rk} c_{\mathbb{Z}}(i)$. Thus, $c_{\mathbb{Z}}$ is $J$-determined as 0-condensed functor if and only if $(i_0 \downarrow P)_*$ is connected for $\deg(i_0) \geq 2$ and $|J_{i_0}^0| = 1$ for $\deg(i_0) \geq 1$.

The successive applications of Proposition 5.2.3 give, by the dimensional equation in the statement of the Proposition 5.2.3, the following structural conditions linking $P$ and $J$:

**Definition 5.2.6.** Let $P$ be a bounded above graded poset. Define, inductively on $p$, the number $R_{i_0}^p$ for each object $i_0$ with $\deg(i_0) \geq p$ by $R_{i_0}^0 = 1$ and by

$$R_{i_0}^p = \sum_{i \in (i_0 \downarrow P)_{p-1}} R_i^{p-1} - R_{i_0}^{p-1}$$

for $p \geq 1$.

**Definition 5.2.7.** Let $P$ be a bounded above graded poset and $J$ be a covering family for $P$. We say that $J$ is adequate if $(i_0 \downarrow P)_*$ is connected for $\deg(i_0) \geq 2$, and if we have the equality

$$R_{i_0}^p = \sum_{i \in J_{i_0}^p} R_i^p$$
for \( p \geq 0 \) and \( \deg(i_0) \geq p + 1 \).

**Proposition 5.2.8.** Let \( \mathcal{P} \) be a bounded above graded poset and let \( \mathcal{J} \) be an adequate covering family. Then there is a sequence of functors \( F_0, F_1, F_2, \ldots \) defined by \( F_0 \overset{def}{=} c_Z : \mathcal{P} \rightarrow \text{Ab} \) and by the short exact sequence

\[
0 \longrightarrow F_{p-1} \xrightarrow{\lambda_{p-1}} \ker'_{F_{p-1}} \xrightarrow{\pi_p} F_p \longrightarrow 0
\]

for \( p = 1, 2, 3, \ldots \). Moreover, \( F_p \) is \( p \)-condensed, free and \( \mathcal{J} \)-determined for \( p \geq 0 \). For \( \deg(i_0) \geq p \) we have \( \text{rk} F_p(i_0) = R_{i_0}^p \).

**Proof.** We prove by induction the following

**Claim 5.2.8.1.** For \( p = 0, 1, 2, \ldots, N \) there exist functors \( F_0, F_1, F_2, \ldots, F_N \) given by \( F_0 \overset{def}{=} c_Z : \mathcal{P} \rightarrow \text{Ab} \) and by a short exact sequence

\[
0 \longrightarrow F_{p-1} \xrightarrow{\lambda_{p-1}} \ker'_{F_{p-1}} \xrightarrow{\pi_p} F_p \longrightarrow 0
\]

for \( p = 1, 2, 3, \ldots, N \). Moreover, for any \( p = 0, 1, 2, \ldots, N \), \( F_p \) is \( p \)-condensed, free and \( \mathcal{J} \)-determined and for \( \deg(i_0) \geq p \) we have

\[
\text{rk} F_p(i_0) = R_{i_0}^p.
\]

The base case of the claim, \( N = 0 \), holds by the arguments before the proposition and because \( \text{rk} F_0(i_0) = \text{rk} c_Z(i_0) = 1 = R_{i_0}^0 \) for any \( i_0 \in \text{Ob}(\mathcal{P}) \).

Suppose that the claim holds for \( N \geq 0 \). Then we prove it for \( N + 1 \). We apply Proposition 5.2.3 to the \( N \)-condensed, free and \( \mathcal{J} \)-determined functor \( F_N \) given by the induction hypothesis. Thus we have to check that for every object \( i_0 \) of degree greater or equal to \( N + 1 \) the following equality holds

\[
\text{rk} F_N(i_0) = \sum_{i \in J_N^{i_0}} \text{rk} F_N(i).
\]

By the induction hypothesis this equations is exactly

\[
R_N^{i_0} = \sum_{j \in I_N^{i_0}} R_N^j,
\]

and this holds by the definition of adequate covering family. By Remark 5.2.4 the rank of \( F_{N+1}(i_0) \) for \( \deg(i_0) \geq N + 1 \) is given by

\[
\text{rk} F_{N+1}(i_0) = \sum_{i \in (i_0 \downarrow \mathcal{P})_N} \text{rk} F_N(i) - \text{rk} F_N(i_0)
\]

which, by the induction hypothesis again, equals

\[
\text{rk} F_{N+1}(i_0) = \sum_{i \in (i_0 \downarrow \mathcal{P})_N} R_N^i - R_N^{i_0}.
\]

Thus, \( \text{rk} F_{N+1}(i_0) = R_{N+1}^{i_0} \) by Definition 5.2.6. □
5.2.2. Local basis and morphisms. In the previous section we found a sequence of functors $F_0 = c_Z, F_1, F_2, \ldots$ to compute the cohomology of a graded poset $\mathcal{P}$ equipped with an adequate covering family $\mathcal{J}$. Moreover, we found the dimension of $F_p(i)$ for any $i \in \mathcal{P}$. In this section we shall build inductively an explicit basis for $F_p(i)$.

For $p = 0$ and $i_0 \in \text{Ob}(\mathcal{P})$ we choose the basis $B^0_{i_0} = \{1\}$ of $F_0(i_0) = \mathbb{Z}$. For $p + 1 \geq 1$ and $\text{deg}(i_0) \geq p + 1$ notice that we have an isomorphism (cf. 5.2.5)

$$F_{p+1}(i_0) \cong \prod_{i \in (i_0 \downarrow \mathcal{P})_p \setminus J^0_p} F_p(i)$$

which sends $y \in F_{p+1}(i_0)$ to the only element $x \in \text{Ker}_F^0(i_0)$ which projects on $y$ by $\pi_{i_0}$ and with $x_i = 0$ for each $i \in J^0_p$.

**Lemma 5.2.9.** Let $\mathcal{P}$ be a bounded above graded poset and let $\mathcal{J}$ be an adequate covering family. Let $c_Z, F_1, F_2, \ldots$ be the sequence of functors given by Proposition 5.2.8. Then there are basis $B^i_p$ of $F_p(i)$ for $p \geq 0$ and $\text{deg}(i) \geq p$ such that, for $p \geq 0$ and $\text{deg}(i_0) \geq p + 1$, the map $\pi_{i_0}$ of the short exact sequence

$$0 \to F_p(i_0) \to \prod_{i \in (i_0 \downarrow \mathcal{P})_p} F_p(i) \to F_{p+1}(i_0) \to 0$$

maps $\bigcup_{i \in (i_0 \downarrow \mathcal{P})_p \setminus J^0_p} B^i_p$ bijectively onto $B^0_{p+1}$.

**Remark 5.2.10.** Notice that for $y = \pi_{i_0}(x) \in F_{p+1}(i_0)$, the expression of $y$ in terms of the basis of the lemma above,

$$y = \sum_{i \in (i_0 \downarrow \mathcal{P})_p \setminus J^0_p} \sum_{l = 1, \ldots, R^i_p} y^i_l \cdot \pi_{i_0}(e^i_l),$$

where $B^i_p = \{e^i_{p,1}, \ldots, e^i_{p,R^i_p}\}$, corresponds to the expression of $\delta_{i_0}(y) = x - (\lambda_{i_0} \circ s_{i_0})(x) \in \prod_{i \in (i_0 \downarrow \mathcal{P})_p \setminus J^0_p} F_p(i)$,

$$\delta_{i_0}(y) = \sum_{i \in (i_0 \downarrow \mathcal{P})_p \setminus J^0_p} \sum_{l = 1, \ldots, R^i_p} y^i_l \cdot e^i_l$$

in terms of the basis $\bigcup_{i \in (i_0 \downarrow \mathcal{P})_p \setminus J^0_p} B^i_p$.

Next we are interested in the expression of $F_p(\alpha)$ with respect to the basis of Lemma 5.2.9. Fix the map $\alpha : i_1 \to i_2$ with $\text{deg}(i_1) \geq \text{deg}(i_2) \geq p \geq 1$. The basis $B^{i_1}_p$ is in 1-1 correspondence with the set

$$\bigcup_{i \in (i_1 \downarrow \mathcal{P})_p \setminus J^1_p} B^i_{p-1},$$

and the basis $B^{i_2}_p$ is in 1-1 correspondence with the set

$$\bigcup_{i \in (i_2 \downarrow \mathcal{P})_p \setminus J^2_p} B^i_{p-1}.$$
5.3. Global behaviour.

Take an element $\pi_{i_1}(e_{p-1,l})$ of $B_{p}^{i_1}$, where $i \in (i_1 \downarrow \mathcal{P})_p \setminus J_{p}^{i_1}$ and $l \in 1, \ldots, R_{p-1}^{i_1}$. Then

$$F_p(\alpha)(\pi_{i_1}(e_{p-1,l})) = \pi_{i_2}(\text{Ker}_{F_{p-1}}'(\alpha)(e_{p-1,l})).$$

We have

$$\text{Ker}_{F_{p-1}}'(\alpha)(e_{p-1,l}) = \begin{cases} e_{p-1,l}^i & \text{if } i \in (i_2 \downarrow \mathcal{P})_{p-1} \\ 0 & \text{otherwise}, \end{cases}$$

and thus

$$F_p(\alpha)(e_{p-1,l}) = \begin{cases} \pi_{i_2}(e_{p-1,l}) & \text{if } i \in (i_2 \downarrow \mathcal{P})_{p-1} \\ 0 & \text{otherwise}. \end{cases}$$

Suppose that $i \in (i_2 \downarrow \mathcal{P})_{p-1}$. If $i \notin J_{p-1}^{i_2}$ then $\pi_{i_2}(e_{p-1,l})$ is an element of $B_{p}^{i_2}$ and so $F_p(\alpha)$ sends the element $\pi_{i_1}(e_{p-1,l})$ to the element $\pi_{i_2}(e_{p-1,l})$. If $i \in J_{p-1}^{i_2}$ then we have to find the coefficients of $\pi_{i_2}(e_{p-1,l})$ with respect to the basis $B_{p}^{i_2}$. By the Remark 5.2.10 these coefficients equal the coefficients of $\delta_{i_2}(\pi_{i_2}(e_{p-1,l}))$ with respect to the basis $\bigcup_{i \in (i_2 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_2}} B_{p-1}^{i}$ of $\prod_{i \in (i_2 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_2}} F_{p-1}(i)$. This proves the following

**Lemma 5.2.11.** Let $\mathcal{P}$ be a bounded above graded poset and $\mathcal{J}$ an adequate covering family. Let $\alpha : i_1 \rightarrow i_2$ with $\deg(i_1) \geq \deg(i_2) \geq p \geq 1$ be a map in $\mathcal{P}$. Then $F_p(\alpha)$ is the identity when restricted to

$$\langle \bigcup_{i \in (i_1 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_1} \cap (i_2 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_2}} B_{p-1}^{i} \rangle$$

**Remark 5.2.12.** Notice that if $i_2 \in J_{p-1}^{i_2}$ then Definition 5.2.1 implies that $J_{p-1}^{i_2} \subseteq J_{p-1}^{i_1}$. Then, for $i \in (i_1 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_1}$, if $i \in (i_2 \downarrow \mathcal{P})_{p-1}$ then $i \notin J_{p-1}^{i_2}$. Thus, in this case, $(i_1 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_1} \cap (i_2 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_2} = (i_2 \downarrow \mathcal{P})_{p-1} \setminus J_{p-1}^{i_2}$ and the lemma applies to the whole $F_p(i_2) \subseteq F_p(i_1)$.

### 5.3. Global behaviour.

In the previous section we saw (Proposition 5.2.8) how the local properties of a graded poset $\mathcal{P}$ equipped with an adequate covering family $\mathcal{J}$ give rise to a sequence $F_0 = c_{\mathcal{Z}}, F_1, F_2, \ldots$ of functors. In this section we study some global properties of $\mathcal{J}$, as well as we define global covering families.

The first global point to notice is that the sequence of functors from Proposition 5.2.8 does not depend on the adequate covering family $\mathcal{J}$. Thus, two or more adequate covering families can be considered for the same graded poset and they still give rise to the same sequence of functors.

Next we focus on the short exact sequence

$$0 \longrightarrow F_{p-1} \xrightarrow{\lambda_{p-1}} \text{Ker}_{F_{p-1}}' \xrightarrow{\pi_p} F_p \longrightarrow 0$$
for \( p \geq 1 \) of Proposition 5.2.8. The beginning of the long exact sequence of this short exact sequence is

\[
0 \to \lim_{\leftarrow} F_{p-1} \to \lim_{\leftarrow} \ker_{F_{p-1}} \to \lim_{\leftarrow} F_p \to H^p(\mathcal{P}; \mathbb{Z}) \to 0
\]

by Lemma 4.3.5, where \( t_{p-1} = \lambda_{p-1} \) and \( \omega_p = \tilde{\pi}_p \).

Notice that the three inverse limits appearing above are free abelian groups as the corresponding functors take free abelian groups as values. In fact, for the middle term we have the exact description

\[
\lim_{\leftarrow} \ker_{F_{p-1}} \cong \prod_{i \in \text{Ob}(\mathcal{P})} F_{p-1}(i)
\]

given by Lemma 4.3.4. It turns out that also there is a simpler description for \( \lim_{\leftarrow} F_p \), which can be interpreted as the analogue to the fact that the cohomology on degree \( n \) depends only on the \( n+1 \) skeleton (recall that \( F_p(i) = 0 \) if \( \deg(i) < p \)):

**Lemma 5.3.1.** Let \( \mathcal{P} \) be a bounded above graded poset and let \( \mathcal{J} \) be an adequate covering family. Let \( c_\mathbb{Z}, F_1, F_2, \ldots \) be the sequence of functors given by Proposition 5.2.8 Then

\[
\lim_{\leftarrow} F_p \cong \lim_{\leftarrow} F_p|_{\mathcal{P}(p+1,p)}
\]

for each \( p \geq 0 \).

**Proof.** Consider the restriction map

\[
\lim_{\leftarrow} F_p \to \lim_{\leftarrow} F_p|_{\mathcal{P}(p+1,p)}.
\]

This map is clearly a monomorphism because \( F_p \) is a \( p \)-condensed functor. To see that it is surjective take \( \psi \in \lim_{\leftarrow} F_p|_{\mathcal{P}(p+1,p)} \). We want to extend \( \psi \) to each \( i \in \text{Ob}(\mathcal{P}) \) with \( \deg(i) > p + 1 \). We do it inductively on \( \deg(i) \).

Notice that (see the beginning of the proof of Proposition 5.2.3)

\[
F_p(i) \to \lim_{\leftarrow} F_p|_{(i \downarrow \mathcal{P})^*}
\]

is an isomorphism for \( \deg(i) > p + 1 \). For \( \deg(i) = p + 2 \) we have that \( j \in (i \downarrow \mathcal{P})^* \) implies that \( \deg(j) \leq p + 1 \). Then there is a unique way of extending \( \psi \) to \( \tilde{\psi}(i) \). Once we have extended \( \psi \) to \( \mathcal{P}_{(p+2,p+1,p)} \) we proceed with an induction argument. That the extension that we are building is actually a functor is again due to that \( F_p \) is \( p \)-condensed.

Also, from Equations (30) and (31), we have the following formula, analogue to that of the euler characteristic for CW-complexes:

**Lemma 5.3.2.** Let \( \mathcal{P} \) be a bounded graded poset for which exists an adequate covering family. Then

\[
\sum_i (-1)^i \text{rk } H^i(\mathcal{P}; \mathbb{Z}) = \sum_i (-1)^i \sum_{p \in \text{Ob}(\mathcal{P})} R^i_p.
\]
5.3.1. **Global covering families.** Recall that covering families were defined as subsets of the local categories \((i \downarrow \mathcal{P})\) for \(i \in \text{Ob}(\mathcal{P})\), where \(\mathcal{P}\) is a bounded above graded poset. In this section we define global covering families by subsets of the whole category \(\mathcal{P}\), imitating some of the local features of the (local) covering families.

**Definition 5.3.3.** Let \(\mathcal{P}\) be a bounded above graded poset for which there exists an adequate covering family, and consider the sequence of functors \(F_0 = c_Z, F_1, F_2, \ldots\) given by Proposition 5.2.8. A **global covering family** is a family of subsets \(K = \{K_p\}_{p \geq 0}\) with \(K_p \subseteq \text{Ob}_p(\mathcal{P})\) such that for \(p \geq 1\) the restriction map

\[
\lim_{\leftarrow} F_p \to \prod_{i \in K_p} F_p(i)
\]

is a monomorphism and the map

\[
\prod_{i \in \text{Ob}_{p-1}(\mathcal{P}) \setminus K_{p-1}} F_{p-1}(i) \to \prod_{i \in K_p} F_p(i)
\]

is pure. For \(p = 0\) we require that the first map above is a pure monomorphism.

Notice that the definition does not depend on the particular adequate covering family used to obtain the sequence of functors (as the sequence of functors does not depend on it). The map

\[
\prod_{i \in \text{Ob}_{p-1}(\mathcal{P}) \setminus K_{p-1}} F_{p-1}(i) \to \prod_{i \in \text{Ob}_{p-1}(\mathcal{P})} F_{p-1}(i)
\]

used in the definition is obtained from (30) as:

\[
\prod_{i \in \text{Ob}_{p-1}(\mathcal{P}) \setminus K_{p-1}} F_{p-1}(i) \cong \prod_{i \in \text{Ob}_{p-1}(\mathcal{P})} F_{p-1}(i)
\]

For the rest of the section suppose we have a graded poset \(\mathcal{P}\) with adequate covering family \(\mathcal{J}\) and global covering family \(\mathcal{K}\). Consider the free, condensed and \(\mathcal{J}\)-determined functors \(c_Z, F_1, F_2, \ldots\) given by Proposition 5.2.8.

For each \(p \geq 1\) we have, at the local level, a short exact sequence

\[
0 \to F_{p-1}(i_0) \xrightarrow{\lambda_{i_0}} \prod_{i \in (i_0 \downarrow \mathcal{P})_p} F_{p-1}(i) \xrightarrow{\pi_{i_0}} F_p(i_0) \to 0
\]

for \(\text{deg}(i_0) \geq p\), and at the global level we have the short exact sequence

\[
0 \longrightarrow \lim_{\leftarrow} F_{p-1} \xrightarrow{\lim_{\leftarrow} \iota_{p-1}} \lim_{\leftarrow} \ker F_{p-1} \xrightarrow{\omega_p} \lim_{\leftarrow} F_p \longrightarrow H^p(\mathcal{P}; \mathbb{Z}) \longrightarrow 0
\]

Recall that it was a set of integral equations involving \(\mathcal{J}\) (the adequateness property) that gave rise to the local short exact sequences above. Next we study if the condition \(H^p(\mathcal{P}; \mathbb{Z}) = 0\) can also be stated in terms of integral equations involving \(\mathcal{K}\).

We begin defining adequate global covering families:
Definition 5.3.4. Let $\mathcal{P}$ be a bounded above graded poset for which exists an adequate covering family and let $\mathcal{K}$ be a global covering family. We say that $\mathcal{K}$ is adequate if

$$\sum_{i \in \text{Ob}_{p-1}(\mathcal{P})} R^i_{p-1} = \sum_{i \in K_{p-1}} R^i_{p-1} + \sum_{i \in K_p} R^i_p$$

for $p \geq 1$ (the numbers $R^i_p$ depend on the local shape of $\mathcal{P}$ and were defined in Definition 5.2.6).

Consider the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \lim_{\leftarrow} F_{p-1} & \longrightarrow & \lim_{\leftarrow} \lim_{\leftarrow} F_{p-1} & \longrightarrow & H^p(\mathcal{P}; \mathbb{Z}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \prod_{i \in K_{p-1}} F_{p-1}(i) & & \prod_{i \in K_p} F_p(i) & & & & \\
\end{array}
$$

Recall that the six abelian groups involved in the diagram are free abelian groups but, possibly, $H^p(\mathcal{P}; \mathbb{Z})$.

Suppose first that $H^p(\mathcal{P}; \mathbb{Z}) = 0$ for a fixed $p \geq 1$. Then, as $\lim_{\leftarrow} \text{Ker}_{F_{p-1}} \cong \prod_{i \in \text{Ob}_{p-1}(\mathcal{P})} F_{p-1}(i)$ and $\text{rk} F_{p-1}(i) = R^i_{p-1}$ (Proposition 5.2.8), we have the equality

$${\text{rk}} \lim_{\leftarrow} F_{p-1} - \sum_{i \in \text{Ob}_{p-1}(\mathcal{P})} R^i_{p-1} + \text{rk} \lim_{\leftarrow} F_p = 0.$$ 

By definition of adequate covering family we also have the equation

$$\sum_{i \in K_{p-1}} R^i_{p-1} - \sum_{i \in \text{Ob}_{p-1}(\mathcal{P})} R^i_{p-1} + \sum_{i \in K_p} R^i_p = 0.$$ 

From these two equations it is clear that

$$\sum_{i \in K_{p-1}} R^i_{p-1} = \text{rk} \lim_{\leftarrow} F_{p-1} \Leftrightarrow \sum_{i \in K_p} R^i_i = \text{rk} \lim_{\leftarrow} F_p.$$ 

Thus, in case $H^p(\mathcal{P}; \mathbb{Z}) = 0$ for $p \geq 1$, we obtain that the condition

$$(32) \quad \sum_{i \in K_p} R^i_i = \text{rk} \lim_{\leftarrow} F_p$$

holds for each $p \geq 0$ if and only if it holds for some $p \geq 1$.

Now we work in the opposite way: does Equation (32) for each $p \geq 1$ imply that $H^p(\mathcal{P}; \mathbb{Z}) = 0$ for each $p \geq 1$? Recall that, by the definition of global covering family, the map

$$\lim_{\leftarrow} F_0 \rightarrow \prod_{i \in K_0} F_0(i)$$

is a pure monomorphism. Then Equation (32) for $p = 0$ implies that this map is in fact an isomorphism. Suppose now, inductively, that the map

$$\lim_{\leftarrow} F_{p-1} \rightarrow \prod_{i \in K_{p-1}} F_{p-1}(i)$$

is an isomorphism for some $p \geq 1$. Then, by the definition of adequate covering family, we have

$$\sum_{i \in K_{p-1}} R^i_{p-1} = \text{rk} \lim_{\leftarrow} F_{p-1} \Rightarrow \sum_{i \in K_p} R^i_i = \text{rk} \lim_{\leftarrow} F_p.$$ 

Thus, in case Equation (32) holds for each $p \geq 1$, we obtain that the condition

$$(32) \quad \sum_{i \in K_p} R^i_i = \text{rk} \lim_{\leftarrow} F_p$$

holds for each $p \geq 0$ if and only if it holds for some $p \geq 1$. 

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is an isomorphism and that Equation (32) holds for \( p \). Because \( \mathcal{K} \) is a global covering family the map
\[
\varprojlim F_p \to \prod_{i \in K_p} F_p(i)
\]
is a monomorphism. This implies that the quotient map
\[
\varprojlim \ker_{F_{p-1}} / \varprojlim F_{p-1} \to \prod_{i \in K_p} F_p(i)
\]
is injective too. By the induction hypothesis
\[
\varprojlim F_{p-1} \cong \prod_{i \in K_{p-1}} F_{p-1}(i)
\]
and thus
\[
\varprojlim \ker_{F_{p-1}} / \varprojlim F_{p-1} = \prod_{i \in \text{Ob}_{p-1}(\mathcal{P}) \setminus K_{p-1}} F_{p-1}(i).
\]
As \( \mathcal{K} \) is a global covering family the map
\[
\prod_{i \in \text{Ob}_{p-1}(\mathcal{P}) \setminus K_{p-1}} F_{p-1}(i) \to \prod_{i \in K_p} F(i)
\]
is pure with \( \text{rk} \varprojlim \ker_{F_{p-1}} / \varprojlim F_{p-1} = \sum_{i \in K_p} R^i_p = \text{rk} \prod_{i \in K_p} F(i) \) (\( \mathcal{K} \) is adequate). This implies that it is an isomorphism, that
\[
\varprojlim \ker_{F_{p-1}} / \varprojlim F_{p-1} \to \prod_{i \in K_p} F_p(i)
\]
is an isomorphism too, that
\[
\varprojlim \ker_{F_{p-1}} \to \prod_{i \in K_p} F_p(i)
\]
and
\[
\varprojlim \ker_{F_{p-1}} \to \varprojlim F_p
\]
are epimorphisms and that
\[
\varprojlim F_p \to \prod_{i \in K_p} F_p(i)
\]
is an isomorphism. Moreover, \( H^p(\mathcal{P}; \mathbb{Z}) = 0 \). We have proven

**Theorem 5.3.5.** Let \( \mathcal{P} \) be a bounded above graded poset for which there exists an adequate covering family and an adequate global covering family \( \mathcal{K} \). Suppose there exists \( p_0 \geq 1 \) such that
\[
\text{rk} \varprojlim F_{p_0} = \sum_{i \in K_{p_0}} R^i_{p_0}.
\]
Then
\[
H^p(\mathcal{P}; \mathbb{Z}) = 0 \text{ for } p \geq 1 \Leftrightarrow \text{rk} \varprojlim F_p = \sum_{i \in K_p} R^i_p \text{ for } p \geq 0.
\]
The beauty of this theorem is that it states the fact of \( \mathcal{P} \) being acyclic in terms of integral equations. At first glance it seems that the numbers \( \text{rk}\lim_{\leftarrow} F_p \) are totally unknown. However, recall that \( \text{rk}\lim_{\leftarrow} F_p = \text{rk}\lim_{\leftarrow} c_Z \) is the number of connected components of \( \mathcal{P} \) (see also Lemma 5.3.1) and, if \( \mathcal{P} \) is bounded below with \( p_0 \in \mathbb{Z} \) minimal such that \( \text{Ob}_{p_0}(\mathcal{P}) \neq \emptyset \), it turns out from the definition of global covering family that \( K_{p_0} = \text{Ob}_{p_0}(\mathcal{P}) \) and \( \text{rk}\lim_{\leftarrow} F_{p_0} = \sum_{i \in \text{Ob}_{p_0}(\mathcal{P})} R_p^i = \sum_{i \in K_{p_0}} R_{p_0}^i \). Thus, as \( R_{p_0}^i = 1 \) for each \( i \in \text{Ob}(\mathcal{P}) \), we obtain

**Theorem 5.3.6.** Let \( \mathcal{P} \) be a bounded graded poset for which there exist an adequate covering family and an adequate global covering family \( K \). Then \( \mathcal{P} \) is acyclic if and only if \( |K_0| \) equals the number of connected components of \( \mathcal{P} \). Moreover, in this case \( H^0(\mathcal{P}, \mathbb{Z}) = \mathbb{Z}^{|K_0|} \).

### 5.4. Simplex-like posets.

Recall (Section 1.4) that simplicial complexes can be viewed as a special kind of posets: simplex-like posets. Another examples of simplex-like posets are subdivision categories (see [28]) and the category \( \mathcal{d}C \) used by Libman in [27] for the normalizer decomposition for \( p \)-local finite groups. In this section we shall see that if \( \mathcal{P} \) is a simplex-like poset then it is a graded poset, and \( \mathcal{P}^{\text{op}} \) is a bounded above graded poset that can be equipped with an adequate covering family. We begin showing that \( \mathcal{P} \) and \( \mathcal{P}^{\text{op}} \) are graded:

**Lemma 5.4.1.** Let \( \mathcal{P} \) be a simplex-like poset. Then \( \mathcal{P} \) and \( \mathcal{P}^{\text{op}} \) are graded posets, \( \mathcal{P} \) is bounded below and \( \mathcal{P}^{\text{op}} \) is bounded above.

**Proof.** That \( \mathcal{P} \) and \( \mathcal{P}^{\text{op}} \) are posets is immediate. To see that \( \mathcal{P} \) is graded recall that, by definition of simplex-like poset, for any \( p \in \text{Ob}(\mathcal{P}) \) the subcategory \( (\mathcal{P} \downarrow p) \) is isomorphic to the poset of all non-empty subsets of a finite set \( T \) (with inclusion as order relation). Define \( \text{deg}(p) = |T| - 1 \). Then \( \text{deg} : \text{Ob}(\mathcal{P}) \to \mathbb{Z} \) is a decreasing degree function which assigns 0 to minimal elements and \( \mathcal{P} \) is graded. The same function \( \text{deg} \) is an increasing degree function for \( \mathcal{P}^{\text{op}} \) which assigns 0 to maximal elements. To see that \( \mathcal{P} \) and \( \mathcal{P}^{\text{op}} \) are bounded just apply the definition. \( \square \)

It is straightforward that the poset \( \mathcal{P} \) is simplex-like \( \mathcal{P} \) if and only if for each \( p \in \text{Ob}(\mathcal{P}) \) the subcategory \( (p \downarrow \mathcal{P}^{\text{op}}) = (\mathcal{P} \downarrow p)^{\text{op}} \) is isomorphic for some \( n \geq 0 \) to the poset \( \Delta_{\text{deg}(p)} \) defined below:

**Definition 5.4.2.** For any integer \( n \geq 0 \) define the poset \( \Delta_n \) as the subdivision category of the poset \( 0 \to 1 \to 2 \to \ldots \to n \), i.e., it has as objects the sequences of integers \( [n_0, n_1, \ldots, n_k] \) which are increasing \( n_0 < n_1 < \ldots < n_k \), and with \( 0 \leq n_j \leq n \). The arrows are generated by the morphisms

\[
d_l : [n_0, n_1, \ldots, n_l] \to [n_0, n_1, \ldots, \hat{n}_l, \ldots, n_k]
\]

for \( l = 0, \ldots, k \). It is a bounded above and below graded poset with degree function \( \text{deg}(n_0, n_1, \ldots, n_k) = k \).
Example 5.4.3. \( \Delta_2 \) looks like

\[
\begin{array}{ccc}
[0, 1] & \longrightarrow & [0] \\
\downarrow & & \downarrow \\
[0, 1, 2] & \longrightarrow & [0, 2] \quad [1] \\
\downarrow & & \downarrow \\
[1, 2] & \longrightarrow & [2].
\end{array}
\]

Next, we shall see that there is an adequate covering family for \( \Delta_n \). This is a first step to equip the whole \( \mathcal{P}^{op} \) with an adequate covering family.

**Lemma 5.4.4.** There is an adequate covering family \( J = \{ J^i_p \}_{i \in \text{Ob}(\Delta_n), 0 \leq p \leq \text{deg}(i)} \) for \( \Delta_n \). Moreover, for \( p \geq 0 \) and \( \text{deg}(i) = q \geq p \) we have

\[
R^i_p = \sum_{l=0}^{p} (-1)^{(p-l)} (q+1)
\]

**Proof.** For the object \( i_0 = [0, 1, 2, \ldots, n] \in \text{Ob}(\Delta_n) \) of degree \( n \) define the subsets \( J^i_p \subseteq (i_0 \downarrow \Delta_n)_p \) for \( 0 \leq p \leq n \) as the sequences of increasing integers \( [n_0, n_1, \ldots, n_{p-1}, n_p] \) which biggest value is \( n \), i.e., such that \( n_p = n \). This definition coincides with the following inductive one:

\[
J^i_0 = \{ [n] \}
\]

and

\[
J^i_p = \{ [s, n], s \in (i_0 \downarrow \Delta_n)_{p-1} \setminus J^i_{p-1} \}
\]

for \( p = 1, 2, \ldots, n \).

For any other object \( i \in \text{Ob}(\Delta_n) \) different from \( i_0 = [0, 1, 2, \ldots, n] \) of degree \( \text{deg}(i) = m < n \) we define the subsets \( J^i_p \subseteq (i \downarrow \Delta_n)_p \) for \( 0 \leq p \leq m \) using the definition above for \( \Delta_m \) instead of \( \Delta_n \) and taking into account the natural isomorphism \( \Delta_m \cong (\Delta_n)^i \) (see the remark below). Purely combinatorial arguments show that \( J \) is a covering family for \( \Delta_n \). \( \square \)

**Example 5.4.5.** For \( n = 2 \) have \( J^{[0,1,2]}_0 = \{ [2] \}, J^{[0,1,2]}_1 = \{ [0, 2], [1, 2] \}, J^{[0,1,2]}_2 = \{ [0, 1, 2] \} \):

\[
\begin{array}{ccc}
[0, 1] & \longrightarrow & [0] \\
\downarrow & & \downarrow \\
[0, 1, 2] & \longrightarrow & [0, 2] \quad [1] \\
\downarrow & & \downarrow \\
[1, 2] & \longrightarrow & [2].
\end{array}
\]

Also, \( J^{[0,1]}_0 = \{ [1] \}, J^{[0,2]}_0 = \{ [2] \} \) and \( J^{[1,2]}_0 = \{ [2] \} \)
Remark 5.4.6. For this covering family we have the following statement, which is stronger than Definition [5.2.11]: \( J_q^i = J_q^{i_0} \cap (i \downarrow \Delta_n)_q \) for each \( i_0, 0 \leq p \leq \text{deg}(i_0), \ i \in J_p^{i_0} \) and \( 0 \leq q \leq \text{deg}(i) \). Moreover, we have that \( R_p^i = 1 \) if \( \text{deg}(i) = p \) by using the binomial expansion of \( (1 - 1)^{p+1} = 0 \).

Remark 5.4.7. Notice that any isomorphism of categories \( \varphi : \Delta_m \to \mathcal{C} \) is determined by the values \( \varphi([a]) \) for \( 0 \leq a \leq m \) \( ([a] \in (\Delta_m)_0) \). If \( \mathcal{C} = (\Delta_n)^i \) the natural isomorphism \( \varphi : \Delta_m \to (\Delta_n)^i \) used in the lemma above is the only order preserving isomorphism, i.e., the only one such that \( a < b \) if and only if \( \varphi([a]) < \varphi([b]) \).

Now we reach the main result of this section:

Lemma 5.4.8. If \( \mathcal{P} \) is a simplex-like poset then there is an adequate covering family for the bounded above graded poset \( \mathcal{P}^{op} \).

Proof. Above we have defined isomorphisms of categories \( (p \downarrow \mathcal{P}^{op}) \simeq \Delta_n \) for each \( p \in \text{Ob}(\mathcal{P}) \), and we know that \( \Delta_n \) can be equipped with an adequate covering family. To build an adequate covering family \( \mathcal{K} = \{ K_p^i \}_{i \in \text{Ob}(\mathcal{P}^{op}), 0 \leq p \leq \text{deg}(i_0) \} \) we have just to choose appropriately the isomorphisms \( (p \downarrow \mathcal{P}^{op}) \simeq \Delta_n \).

Consider the degree function \( \text{deg} : \text{Ob}(\mathcal{P}^{op}) \to \mathbb{Z} \) defined in Lemma [5.4.4] and the set \( T \) of the maximal elements of \( \mathcal{P}^{op} \), i.e., \( T = \{ p \in \text{Ob}(\mathcal{P}^{op}) | \text{deg}(p) = 0 \} \). Choose a total order \( < \) for \( T \) (suppose \( T \) is finite or use the Axiom of Choice [24]). Then, given \( p \in \text{Ob}(\mathcal{P}^{op}) \), consider the subset \( (p \downarrow \mathcal{P}^{op})_0 \subseteq T \) and the restriction \( ((p \downarrow \mathcal{P}^{op})_0, <) \) of the total order from \( T \). There is a unique isomorphism \( \varphi_p : (p \downarrow \mathcal{P}^{op}) \simeq \Delta_{\text{deg}(p)} \) which induces an order preserving map \( ((p \downarrow \mathcal{P}^{op})_0, <) \simeq (\Delta_{\text{deg}(p)})_0 = \{ [0], [1], [2], ..., [\text{deg}(p)] \} \).

Denote by \( \mathcal{J} \) the covering family for \( \Delta_{\text{deg}(p)} \) of Lemma [5.4.4] and define, for \( 0 \leq n \leq \text{deg}(p) \),

\[
K_p^n = \varphi_p^{-1}(J_{\varphi_p(p)}^n).
\]

Then \( \mathcal{K} \) fulfills condition [a] of Definition [5.2.1] because for \( 0 \leq n < \text{deg}(p) \)

\[
(p \downarrow \mathcal{P}^{op})_n = \varphi_p^{-1}((\Delta_{\text{deg}(p)})_n) = \varphi_p^{-1}\left( \bigcup_{i \in J_{\varphi_p(p)}^n} (i \downarrow \Delta_{\text{deg}(p)})_n \right) = \bigcup_{i \in \varphi_p^{-1}(K_{\varphi_p(p)}^n)} \varphi_p^{-1}((i \downarrow \Delta_{\text{deg}(p)})_n) = \bigcup_{i \in K_{\varphi_p(p)}^n} (i \downarrow \mathcal{P}^{op})_n.
\]

To check condition [b] of Definition [5.2.1] take \( i \in K_{\varphi_p(p)}^n \) for some \( 0 \leq n < \text{deg}(p) \) and call \( \mathcal{J}' \) to the covering family for \( \Delta_{n+1} \) of Lemma [5.4.4]. We want to see that \( K_n^i \subseteq K_n^p \). Recalling the natural inclusion \( (i \downarrow \mathcal{P}^{op}) \subseteq (p \downarrow \mathcal{P}^{op}) \) this is equivalent to

\[
\varphi_i^{-1}(J_{\varphi_p(p)}^n) \subseteq \varphi_p^{-1}(J_{\varphi_p(p)}^n).
\]
and to
\[ \psi(J_{n}^{\varphi_{i}(i)}) \subseteq J_{n}^{\varphi_{p}(p)} \]
where \( \psi = \varphi_{p} \circ \varphi_{i}^{-1} \). By construction \( \psi \) is order preserving (see Remark 5.4.7) and thus this inclusion holds.

Next, we list more familiar re-statements of results about covering families applied to simplex-like posets:

**Lemma 5.4.9.** Let \( \mathcal{P} \) be a simplex-like poset and consider the bounded above graded poset \( \mathcal{P}^{op} \) (for which exists an adequate covering family by Lemma 5.4.8). Let \( \mathcal{K} \) be a global covering family for \( \mathcal{P}^{op} \). Then \( \mathcal{K} \) is adequate if and only if
\[ | \text{Ob}_{p-1}(\mathcal{P}) | = | K_{p-1} | + | K_{p} | \]
for \( p \geq 1 \).

**Proof.** Apply Remark 5.4.6 to Definition 5.2.7. \( \square \)

**Lemma 5.4.10.** Let \( \mathcal{P} \) a simplex-like poset. Then
\[ \sum_{i} (-1)^{i} \text{rk} H^{i}(\mathcal{P}; \mathbb{Z}) = \sum_{i} (-1)^{i} | \text{Ob}_{i}(\mathcal{P}) |. \]

**Proof.** Use Remark 5.4.6 and observe that \( \mathcal{P} \simeq \mathcal{P}^{op} \) and that \( \text{Ob}(\mathcal{P}) = \text{Ob}(\mathcal{P}^{op}) \). Then apply Lemma 5.3.2. \( \square \)
CHAPTER 6

Application: Webb’s conjecture

Denote by $S_p(G)$ the Brown’s complex of the finite group $G$ for the prime $p$, whose elements are the non-trivial $p$-subgroups of $G$, and that was introduced by Brown [11]. Webb conjectured that the orbit space $S_p(G)/G$ (as topological space) is contractible. This conjecture was first proven by Symonds in [40], generalized for blocks by Barker [4,5] and extended to arbitrary (saturated) fusion system by Linckelmann [28].

The works of Symonds and Linckelmann prove the contractibility of the orbit space by showing that it is simply connected and acyclic, and invoking Whitehead’s Theorem. Both proofs of acyclicity work on the subposet of normal chains (introduced by Knorr and Robinson [25] for groups). Symonds uses the results from Thévenaz and Webb [41] that the subposet of normal chains is $G$-equivalent to Brown’s complex. Linckelmann proves on his own that, also for fusion systems, the orbit space and the orbit space on the normal chains has the same cohomology [28, Theorem 4.7].

In this chapter we shall apply the results of Chapter 5 to prove in an alternative way that the orbit space on the normal chains is acyclic.

Let $(S,\mathcal{F})$ be a saturated fusion system where $S$ is a $p$-group. Consider its subdivision category $S(\mathcal{F})$ (see [28, 2]) and the poset $[S(\mathcal{F})]$. An object in $[S(\mathcal{F})]$ is an $\mathcal{F}$-isomorphism class of chains $[Q_0 < Q_1 < ... < Q_n]$ where the $Q_i$’s are subgroups of $S$. The subcategory $([S(\mathcal{F})] \downarrow [Q_0 < ... < Q_n])$ has objects $[Q_{i_0} < ... < Q_{i_m}]$ with $0 \leq m \leq n$ and $0 \leq i_1 < i_2 < ... < i_m \leq n$ (see [28, 2] again). For example, $([S(\mathcal{F})] \downarrow [Q_0 < Q_1 < Q_2])$ is

Then it is clear that $[S(\mathcal{F})]$ is a simplex-like poset. Following Linckelmann’s notation denote by $S_<(\mathcal{F})$ the full subcategory of $S(\mathcal{F})$ which objects $Q_0 < ... < Q_n$ with $Q_i \triangleleft Q_n$ for $i = 0, ..., n$. Also, denote by $[S_<(\mathcal{F})]$ the subdivision category of $S_<(\mathcal{F})$, which is a sub-poset of $[S(\mathcal{F})]$.

Our goal in this chapter is to prove that $H^n([S_<(\mathcal{F})];\mathbb{Z}) = 0$ for $n \geq 1$ and $H^0([S_<(\mathcal{F})];\mathbb{Z}) = \mathbb{Z}$. It is straightforward that $[S_<(\mathcal{F})]$ is a simplex-like poset and
thus, by Lemma 5.4.8, there exists an adequate covering family for the bounded above graded poset $[S_{\triangle}(F)]^{op}$. We shall build an adequate global covering family for $[S_{\triangle}(F)]^{op}$ in order to apply Theorem 5.3.6.

The definition of the global covering family is as follows, and it is related with the pairing defined by Linckelmann in [28] Definition 4.7. The notion of paired chains was used by Knorr and Robinson in several forms throughout [25].

**Definition 6.0.11.** For the graded poset $[S_{\triangle}(F)]^{op}$ define the subsets $\mathcal{K} = \{K_n\}_{n \geq 0}$ by

$$K_n = \{ [Q_0 < \ldots < Q_n] \mid [Q_0 < \ldots < Q_n] = [Q'_0 < \ldots < Q'_n] \Rightarrow \cap_{i=0}^n N_S(Q'_i) = Q'_n \}$$

**Lemma 6.0.12.** The family $\mathcal{K} = \{K_n\}_{n \geq 0}$ defined in 6.0.11 is a global covering family for $[S_{\triangle}(F)]^{op}$.

The proof is postponed to the next section. The argument uses the properties of the (local) covering family of the simplex-like category $[S_{\triangle}(F)]^{op}$ and that the chains $Q_0 < \ldots < Q_n$ can be ordered by $|Q_n|$. The fact that $\mathcal{K}$ is defined through a pairing provides (see next section) a bijection $\psi : \text{Ob}_n([S_{\triangle}(F)]^{op}) \setminus K_n \rightarrow K_{n+1}$, which gives

**Lemma 6.0.13.** The global covering family $\mathcal{K} = \{K_n\}_{n \geq 0}$ defined in 6.0.11 is adequate for $[S_{\triangle}(F)]^{op}$.

**Proof.** For any $n \geq 0$ we have

$$\text{Ob}_n([S_{\triangle}(F)]^{op}) = K_n \cup (\text{Ob}_n([S_{\triangle}(F)]^{op}) \setminus K_n).$$

Then the bijection (Lemma 6.1.2)

$$\text{Ob}_n([S_{\triangle}(F)]^{op}) \setminus K_n \rightarrow K_{n+1}$$

gives

$$|\text{Ob}_n([S_{\triangle}(F)]^{op})| = |K_n| + |K_{n+1}|$$

as wished (see Lemma 5.4.9). \qed

As $[S(F)]$ is connected then so is $[S_{\triangle}(F)]^{op}$. Thus, $\varprojlim c_Z = 1$ over $[S_{\triangle}(F)]^{op}.$ Also, by elementary properties of $p$-groups it is clear that $K_0 = \{[S]\}$, i.e., the only subgroup of $S$ which equals its normalizer in $S$ is $S$ itself, and $|K_0| = 1$. Then Theorem 5.3.6 gives

**Theorem 6.0.14.** Let $(S,F)$ be a saturated fusion system. Then $H^n([S_{\triangle}(F)]^{op}; Z) = 0$ for $n \geq 1$ and $H^0([S_{\triangle}(F)]^{op}; Z) = Z$.

### 6.1. $\mathcal{K}$ is an adequate global covering family.

In this section we prove that the family $\mathcal{K} = \{K_n\}_{n \geq 0}$ defined in 6.0.11 is an adequate global covering family for $[S_{\triangle}(F)]^{op}$. We use terminology and results from [10] Appendix.

For any chain $Q_0 < \ldots < Q_n$ in $S_{\triangle}(F)$ define the following subgroup of automorphisms of $Q_n$

$$A_{Q_0 < \ldots < Q_n} = \{ \alpha \in \text{Aut}(Q_n) \mid \alpha(Q_i) = Q_i, i = 0, \ldots, n \}.$$
Then,
\[ N_S^{A_{Q_0} \prec \ldots \prec Q_n} (Q_n) = \cap_{i=0}^n N_S(Q_i). \]

If \([Q_0 < \ldots < Q_n] = [Q'_0 < \ldots < Q'_n]\) then there is \(\varphi \in \text{Iso}_F(Q_n, Q'_n)\) with \(Q'_i = \varphi(Q_i)\) for \(i = 0, \ldots, n\) and
\[ \varphi A_{Q_0 \prec \ldots \prec Q_n} \varphi^{-1} = A_{Q'_0 \prec \ldots \prec Q'_n}. \]

By [10] A.2(a) \(Q_n\) is fully \(A_{Q_0 \prec \ldots \prec Q_n}\)-normalized if and only if \(|N_S^{A_{Q_0} \prec \ldots \prec Q_n} (Q_n)|\) is maximum among \(|N_S^{A_{Q'_0} \prec \ldots \prec Q'_n} (Q_n)|\) with \([Q'_0 < \ldots < Q'_n] = [Q_0 < \ldots < Q_n]\), i.e., if and only if \(\cap_{i=0}^n N_S(Q'_i)\) is maximum among \(\cap_{i=0}^n N_S(Q_i)\) with \([Q'_0 < \ldots < Q'_n] = [Q_0 < \ldots < Q_n]\). Notice that in the isomorphism class of chains \([Q_0 < \ldots < Q_n]\) always there is a representative \(Q'_0 < \ldots < Q'_n\) which is fully \(A_{Q'_0 \prec \ldots \prec Q'_n}\)-normalized, and that any two representaties \(Q'_0 < \ldots < Q'_n\) and \(Q''_0 < \ldots < Q''_n\) of \([Q_0 < \ldots < Q_n]\) which are fully \(A_{Q'_0 \prec \ldots \prec Q'_n}\)-normalized and fully \(A_{Q''_0 \prec \ldots \prec Q''_n}\)-normalized respectively verify
\[ |\cap_{i=0}^n N_S(Q'_i)| = |\cap_{i=0}^n N_S(Q''_i)|. \]

Thus, Definition 6.1.1 is equivalent to

**Definition 6.1.1.** For the graded poset \([S_c(\mathcal{F})]^{op}\) define the subsets \(\mathcal{K} = \{K_n\}_{n \geq 0}\) by
\[ K_n = \{[Q'_0 < \ldots < Q'_n] | Q'_n \text{ fully } A_{Q'_0 \prec \ldots \prec Q'_n}, \cap_{i=0}^n N_S(Q'_i) = Q'_n\}. \]

**Lemma 6.1.2.** For any \(n \geq 0\) there is a bijection
\[ \psi : \text{Ob}_n([S_c(\mathcal{F})]^{op}) \setminus K_n \rightarrow K_{n+1}. \]

**Proof.** Take \([Q_0 < \ldots < Q_n] \in \text{Ob}_n([S_c(\mathcal{F})]^{op}) \setminus K_n\) and a representantive \(Q'_0 < \ldots < Q'_n\) which is fully \(A_{Q'_0 \prec \ldots \prec Q'_n}\)-normalized. Then \(\cap_{i=0}^n N_S(Q'_i) > Q'_n\). Define \(\psi([Q_0 < \ldots < Q_n]) = [Q'_0 < \ldots < Q'_n < \cap_{i=0}^n N_S(Q'_i)]\). The proof is divided in four steps:

**a)** \(\psi\) is well defined. Take another representatvie \(Q''_0 < \ldots < Q''_n\) which is fully \(A_{Q''_0 \prec \ldots \prec Q''_n}\)-normalized. Then, by [10] A.2(c), there is a morphism
\[ \varphi \in \text{Hom}_F(N_S^{A_{Q''_0} \prec \ldots \prec Q''_n} (Q'_n), N_S^{A_{Q''_0} \prec \ldots \prec Q''_n} (Q''_n)) \]
with \(\varphi(Q'_i) = Q''_i\) for \(i = 0, \ldots, n\). As \(\cap_{i=0}^n N_S(Q'_i) = |\cap_{i=0}^n N_S(Q''_i)|\) then \(\varphi\) is an isomorphism onto \(N_S^{A_{Q''_0} \prec \ldots \prec Q''_n} (Q'_n)\) and thus
\[ [Q'_0 < \ldots < Q'_n < \cap_{i=0}^n N_S(Q'_i)] = [Q''_0 < \ldots < Q''_n < \cap_{i=0}^n N_S(Q''_i)]. \]

**b)** \(\psi([Q_0 < \ldots < Q_n])\) belongs to \(K_{n+1}\). We have \(\psi([Q_0 < \ldots < Q_n]) = [Q'_0 < \ldots < Q'_n < \cap_{i=0}^n N_S(Q'_i)]\) where \(Q'_0 < \ldots < Q'_n\) is fully \(A_{Q'_0 \prec \ldots \prec Q'_n}\)-normalized. Take any representatvie \(Q''_0 < \ldots < Q''_n < Q''_{n+1}\) in \([Q'_0 < \ldots < Q'_n < \cap_{i=0}^n N_S(Q'_i)]\). If it were the case that \(Q''_{n+1} < \cap_{i=0}^{n+1} N_S(Q''_i)\) then we would have
\[ \cap_{i=0}^n N_S(Q'_i) \cong Q''_{n+1} < \cap_{i=0}^{n+1} N_S(Q''_i) \leq \cap_{i=0}^n N_S(Q''_i), \]
which is in contradiction with \(Q'_0 < \ldots < Q'_n\) being fully \(A_{Q'_0 \prec \ldots \prec Q'_n}\)-normalized.
c) \( \psi \) is injective. Suppose we have \([Q_0 < \ldots < Q_n]\) and \([R_0 < \ldots < R_n]\) with
\[
[Q_0 < \ldots < Q_n < \bigcap_{i=0}^{n-n_0} N_S(Q_i')] = [R_0 < \ldots < R_n < \bigcap_{i=0}^{n-n_0} N_S(R_i')].
\]
Then \([R_0 < \ldots < R_n] = [R_0' < \ldots < R_n'] = [Q_0 < \ldots < Q_n'] = [Q_0 < \ldots < Q_n].

d) \( \psi \) is surjective. Take \([Q_0 < \ldots < Q_n < Q_{n+1}]\) in \(K_{n+1}\). We check that
\[
\psi([Q_0 < \ldots < Q_n]) = [Q_0 < \ldots < Q_n < Q_{n+1}]
\]
Take a representantive \(Q_0' < \ldots < Q_n'\) in \([Q_0 < \ldots < Q_n]\) which is fully \(A_{Q_0' < \ldots < Q_n'}\)-normalized. Then \([Q_0 < \ldots < Q_n] \in \text{Ob}_n([S_\alpha(F)]^\text{op}) \setminus K_n\) and \(\psi([Q_0 < \ldots < Q_n]) = [Q_0' < \ldots < Q_n' < \bigcap_{i=0}^{n-n_0} N_S(Q_i')].\)

Then, by [10] A.2(c), there is
\[
\varphi \in \text{Hom}_F(N^A_{Q_0 < \ldots < Q_n}(Q_n), N^A_{Q_0' < \ldots < Q_n'}(Q_n))
\]
with \(\varphi(Q_i) = Q_i'\) for \(i = 0, \ldots, n.\) As \(Q_{n+1} = \bigcap_{i=0}^{n+1-n_0} N_S(Q_i)\) then \(Q_{n+1} \leq \bigcap_{i=0}^{n-n_0} N_S(Q_i)\) and \(\varphi(Q_{n+1}) \leq \bigcap_{i=0}^{n-n_0} N_S(Q_0').\) If it were the case that \(\varphi(Q_{n+1}) < \bigcap_{i=0}^{n-n_0} N_S(Q_i')\) then we would have
\[
\varphi(Q_{n+1}) < N_{\bigcap_{i=0}^{n-n_0} N_S(Q_i')}\varphi(Q_{n+1}) = \bigcap_{i=0}^{n+1-n_0} N_S(\varphi(Q_i)) = \varphi(\bigcap_{i=0}^{n+1-n_0} N_S(Q_i)) = \varphi(Q_{n+1}),
\]
a contradiction. Thus, \(\varphi(Q_{n+1}) = \bigcap_{i=0}^{n-n_0} N_S(Q_0')\) and the proof is finished. 

**Lemma 6.1.3.** The family \(K = \{K_n\}_{n \geq 0}\) defined in 6.0.17 is a global covering family for \([S_\alpha(F)]^\text{op}\).

**Proof.** For \(n = 0\) we have to prove that the map
\[
\varprojlim_{n} c Z \to \prod_{i \in K_0} F_0(i) = \prod_{i \in \{s\}} F_0(i) = Z
\]
is a pure monomorphism. In fact, as \([S_\alpha(F)]\) is connected, this map is an isomorphism.

For \(n \geq 1\) we have to prove that
\[
\varprojlim_{n} F_n \to \prod_{i \in K_n} F_n(i)
\]
is a monomorphism and the map
\[
\prod_{i \in \text{Ob}_{n-1}(P) \setminus K_{n-1}} F_{n-1}(i) \to \prod_{i \in K_n} F_n(i)
\]
is pure. We begin proving the injectivity. Take \(\psi \in \varprojlim F_p\) such that \(\psi(i) = 0\) for each \(i \in K_n.\) If there is no object of degree greater than \(n,\) i.e. \(\text{Ob}_{\{n+2,n+3\}}([S_{\alpha}(F)]^\text{op}) = \emptyset\) then \(K_n = \text{Ob}_n([S_\alpha(F)]^\text{op})\) and we are done. If not, we prove that \(\psi(j) = 0\) for each \(j\) of degree \(n+1\) by induction on \(|Q_{n+1}|.\) This is enough to see that \(\psi\) is zero as \(F_n\) is \(n\)-condensed. We shall use the (local) covering family \(F\) defined in 5.4 for the simplex-like category \([S_\alpha(F)]^\text{op}.\)

The case base is \(j = [Q_0 < \ldots < Q_{n+1}]\) with \(|Q_{n+1}|\) maximal. This implies that \(J_{n}^{j} \subseteq K_{n}.\) Then \(\psi(j)\) goes to zero by the monomorphism
\[
F_n(j) \to \prod_{i \in J_{n}^{j}} F_n(i),
\]
and thus \( \psi(j) = 0 \). For the induction step consider \( j = [Q_0 < \ldots < Q_{n+1}] \) and \( j' = [Q_0 < \ldots < \tilde{Q}_l < \ldots < Q_{n+1}] \in J^l_n \) with \( 0 \leq l < n \). Then, either \( j' \in K_n \) and \( \psi(j') = 0 \), or \( j' \notin K_n \) and there is an arrow in \([\mathcal{S}_n(\mathcal{F})]^{op}\).

\[ j'' = [Q_0 < \ldots < \tilde{Q}_l < \ldots < Q_{n+1} < \cap_{i=0; i \neq l}^m N_S(Q'_i)] \rightarrow j' = [Q_0 < \ldots < \tilde{Q}_l < \ldots < Q_{n+1}]. \]

In the latter case \( \psi(j'') = 0 \) by the induction hypothesis, and thus \( \psi(j') = 0 \) too. As before, the map \( F_n(j) \rightarrow \prod_{i \in J^l_n} F_n(i) \) is a monomorphism, \( \psi(j) = 0 \).

Now we prove that the map

\[ \omega : \prod_{i \in Ob_{n-1}(\mathcal{P}) \setminus K_{n-1}} F_{n-1}(i) \rightarrow \prod_{i \in K_n} F_n(i) \]

is pure. Take \( y \in \prod_{i \in K_n} F_n(i), \ n \geq 1 \) and \( x \in \prod_{i \in Ob_{n-1}(\mathcal{P}) \setminus K_{n-1}} F_{n-1}(i) \) with

\[ n \cdot y = \omega(x). \]

We want to find \( x' \) with \( n \cdot x' = x \). We prove that \( x_i \) is divisible by \( n \) for each \( i = [Q_0 < \ldots < Q_{n-1}] \in Ob_{n-1}(\mathcal{P}) \setminus K_{n-1} \) by induction on \( |Q_{n-1}| \).

The case base is \( i = [Q_0 < \ldots < Q_{n-1}] \) with \( |Q_{n-1}| \) maximal. Consider the arrow in \([\mathcal{S}_n(\mathcal{F})]^{op}\).

\[ j = [Q_0 < \ldots < Q_{n-1} < \cap_{i=0}^{n-1} N_S(Q'_i)] \rightarrow [Q_0 < \ldots < Q_{n-1}]. \]

As \( Q_{n-1} \) is maximal then \( J^i_{n-1} \subseteq K_{n-1} \). Then \( n \cdot y_j = \omega(x)_j \) is the image of \((x_i, 0, \ldots, 0)\) by the map

\[ \text{Ker}_{F_{n-1}}(j) = \prod_{l \in (j \downarrow [\mathcal{S}_n(\mathcal{F})])_{n-1}} F_{n-1}(i) \xrightarrow{\pi_j} F_n(j). \]

As \( F_n(j) \cong \prod_{l \in (j \downarrow [\mathcal{S}_n(\mathcal{F})])_{n-1} \setminus J^i_{n-1}} F_{n-1}(l) = F_{n-1}(i) \) by Remark 5.2.5, then \( n \) divides \( x_i \).

For the induction step consider \( i = [Q_0 < \ldots < Q_{n-1}] \in Ob_{n-1}(\mathcal{P}) \setminus K_{n-1} \) and \( j = [Q'_0 < \ldots < Q'_{n-1} < \cap_{i=0}^{n-1} N_S(Q'_i)] \). As before, \( n \cdot y_j = \omega(x)_j \) is the image of \( \tilde{x} = x_0(\tilde{j} \downarrow [\mathcal{S}_n(\mathcal{F})])_{n-1} \) by the map

\[ \text{Ker}_{F_{n-1}}(j) = \prod_{l \in (j \downarrow [\mathcal{S}_n(\mathcal{F})])_{n-1}} F_{n-1}(i) \xrightarrow{\pi_j} F_n(j). \]

By Remark 5.2.5 again,

\[ n \cdot y_j = \pi_j(\tilde{x} - (\lambda_j \circ s_j)(\tilde{x})) = \pi_j((x_i - (\lambda_j \circ s_j)(x|J^j_n), 0, \ldots, 0)). \]

Now, by the induction hypothesis, for each \( l \in J^j_n \) either \( l \in K_{n-1} \) and \( x_i = 0 \), either \( l \notin K_{n-1} \) and thus, by the induction hypothesis, \( n \) divides \( x_l \). Then \( n \) divides \( x|J^j_n \) and so, by the equation above and the isomorphism \( F_n(j) \cong F_{n-1}(i) \), \( n \) divides \( x_i \) too. \( \square \)
CHAPTER 7

Application: homotopy colimit

In this section we deal with the problem of when the natural map from the homotopy colimit of a diagram of nerves of groups to the nerve of the colimit of the groups is an isomorphism. By hocolim and hocolim$_*$ we denote the unpointed and pointed homotopy colimits in the sense of Bousfield-Kan respectively. For any discrete group $G_0 \in \text{Grp}$ we denote by $BG_0 \in SSet$ the nerve of the category with one object and with automorphism group $G_0$, which is a classifying space for $G_0$, and by $I_{G_0} \subseteq \mathbb{Z}[G_0]$ the augmentation ideal of $G_0$. $\mathcal{P}$ always denote a small category in this section.

More precisely, for a functor $G : \mathcal{P} \to \text{Grp}$ with takes as values discrete groups, and a cone $\tau : G \Rightarrow G_0$, we have an induced cone $B\tau : BG \Rightarrow BG_0$ which gives a map $\text{hocolim } BG \to BG_0$.

We are interested in finding conditions such that this map induces a weak homotopy equivalence

$$\text{hocolim } BG \cong BG_0$$

in case $G_0 = \lim_{\rightarrow} G$. We have the following preliminary result, which proof is left to Section 7.1:

**Theorem 7.0.4.** Let $G : \mathcal{P} \to \text{Grp}$ be a functor and $\tau : G \Rightarrow G_0$ be a cone, and call $F$ the homotopy fiber

$$F \to \text{hocolim } BG \to BG_0.$$ 

Assume $\mathcal{P}$ is contractible and the cone $\tau : G \Rightarrow G_0$ is monic. Then

1. $\pi_1(F) = \text{Ker}(\lim_{\rightarrow} G \to G_0)$
2. $\pi_0(F) = \text{Coker}(\lim_{\rightarrow} G \to G_0)$
3. $H_j(F) = \lim_{\rightarrow} H_{j-1}$ for each $j \geq 2$, where $H : \mathcal{P} \to \text{Ab}$ is a monic functor.

Thus, if $G_0 = \lim_{\rightarrow} G$, then $F$ is simply connected. Some examples follow:

**Example 7.0.5.** Consider a monic functor $G : \mathcal{P} \to \text{Grp}$ where $\mathcal{P}$ is the “telescope category” $\mathcal{P}$ with shape

$$a_0 \overset{f_1}{\longrightarrow} a_1 \overset{f_2}{\longrightarrow} a_2 \overset{f_3}{\longrightarrow} a_3 \overset{f_4}{\longrightarrow} a_4 \ldots$$
Then $\mathcal{P}$ is contractible and the cone $G \Rightarrow G_0 = \bigcup_{l \geq 0} G(a_l)$ is monic. Then by part Theorem 7.0.4 and Example 3.3.13
\[
\text{hocolim } BG \cong BG_0.
\]

**Example 7.0.6.** Consider a functor $G : \mathcal{P} \to \text{Grp}$ where $\mathcal{P}$ is the “pushout category” $\mathcal{P}$ with shape

\[
\begin{array}{ccc}
a & \xrightarrow{f} & b \\
g \downarrow & & \downarrow \\
c & \xleftarrow{g} & d
\end{array}
\]

Then $\mathcal{P}$ is contractible and, if $G(f)$ and $G(g)$ are injective, then the cone $\tau : G \Rightarrow \lim G$ is monic $[37]$. Then, by Theorem 7.0.4 and Example 3.3.13 we have
\[
\text{hocolim } BG \cong BG_0.
\]

Thus we obtain the classical result of Whitehead that states that
\[
\text{hocolim } BG \cong B(G(b) \ast_{G(a)} G(c))
\]
if both $G(f)$ and $G(g)$ are monomorphisms.

**Remark 7.0.7.** Assume $G(f)$ is a monomorphism. Then $[8]$ $BG(f)$ is a cofibration, the diagram $BG(b) \leftarrow BG(a) \to BG(c)$ is a cofibrant object for some structure of closed model category on $\text{SSet}^\mathcal{P}$ and so
\[
\text{hocolim } BG \cong \lim BG.
\]

This has the following consequence

**Claim 7.0.7.1.** If $G(f)$ and $G(g)$ are monomorphisms then
\[
\text{hocolim } BG \cong B(\lim (G(b) \leftarrow G(a) \to G(c))) \cong \lim BG.
\]

Recall that the $n$-simplices in $\lim BG$ are given by
\[
(\lim BG)_n = \lim_{\mathcal{S}^\mathcal{P}} BG_n.
\]

This is a finite set if the groups $G(b)$ and $G(c)$ are, but the $n$-simplices in $B(\lim (G(b) \leftarrow G(a) \to G(c)))$ are in general infinite as $\lim (G(b) \leftarrow G(a) \to G(c))$ is in general infinite.

Thus the geometric realization of $\lim BG$ gives a dimension-wise finite $CW$-complex as classifying space for the (possibly infinite) group $G(b) \ast_{G(a)} G(c)$.

**Example 7.0.8.** Consider a monic functor $G : \mathcal{P} \to \text{Grp}$ where $\mathcal{P}$ is a graded poset which is a tree. Then $\mathcal{P}$ is contractible and the cone $\tau : G \Rightarrow \lim G$ is monic $[37]$. Then, by Theorem 7.0.4 and Corollary 3.4.11 we have
\[
\text{hocolim } BG \cong B \lim G.
\]
Example 7.0.9. Recall that a filtered category is a category $\mathcal{P}$ such that any two objects $i, j \in \text{Ob}(\mathcal{P})$ can be joined

\[
\begin{array}{ccc}
  i & \rightarrow & k \\
  \downarrow & & \downarrow \\
  j & \rightarrow & \\
\end{array}
\]

and such that any two parallel arrows $u, v : i \rightarrow j$ can be co-equalized by $w : j \rightarrow k$ with $wu = wv$.

Any functor $F : \mathcal{P} \rightarrow \text{Ab}$ with $\mathcal{P}$ filtered is $\lim\rightarrow$-acyclic as $\lim\rightarrow : \text{Ab}^{\mathcal{P}} \rightarrow \text{Ab}$ is exact (see [42, 2.6.15]).

A finite poset which is filtered has a terminal object, and so direct limits become trivial. The notion of pseudo-injectivity gives an alternative condition to being filtered that also implies $\lim\rightarrow$-acylicity but that does not impose the existence of a terminal object when $\mathcal{P}$ is a finite poset.

Consider a contractible filtered category $\mathcal{P}$ and a monic functor $G : \mathcal{P} \rightarrow \text{Grp}$. Then

- The limiting cone $\tau : G \Rightarrow \lim\rightarrow G$ is monic (use that the forgetful functor $\text{Grp} \rightarrow \text{Set}$ creates filtered colimits, [29, p.208]).
- $\lim\rightarrow H = 0$ for $j \geq 1$ because $\lim\rightarrow : \text{Ab}^{\mathcal{P}} \rightarrow \text{Ab}$ is exact (see [42, 2.6.15]).

Thus, by the Theorem 7.0.4, we have that

$$
\text{hocolim}\, BG \cong B\lim\rightarrow G.
$$

Example 7.0.10. The last example applies to any locally finite group $G_0$: call $\mathcal{P}$ to the poset category of its finite subgroups and $G : \mathcal{P} \rightarrow \text{Grp}$ to the monic functor which takes as values the finite subgroups of $G_0$ and inclusion among them. Then $G_0 = \lim\rightarrow_{\mathcal{P}} G$ and $\mathcal{P}$ is filtered and contractible (the trivial group is an initial object). Thus:

$$
\text{hocolim}_{G \subseteq G_0, G \text{ finite}} BG \cong BG_0
$$

Example 7.0.11. Consider a group $G_0$ and the poset $\mathcal{P}$ (with the inclusion as relation) of its normal finite $p$-subgroups for a fixed prime $p$. If $H$ and $K$ are normal finite $p$-subgroups of $G_0$ then $HK$ is a $p$-normal subgroup of $G_0$ too. This implies that $\mathcal{P}$ is directed. Moreover, $\mathcal{P}$ is contractible as $\{1\} \in \mathcal{P}$. Let $G : \mathcal{P} \rightarrow \text{Grp}$ the monic functor which takes each subgroup of $\mathcal{P}$ to itself and inclusions to inclusions. Then by Example 7.0.9

$$
\text{hocolim}_{G \subseteq G_0, G \text{ normal finite } p\text{-subgroup}} BG \cong B\lim\rightarrow G.
$$

If $G$ is finite then $\lim\rightarrow G = O_p(G)$. 

Example 7.0.12. Consider a finite group $G_0$ and the poset $\mathcal{P}$ (with the inclusion as relation) of the normal subgroups of $G_0$ which have $p'$-quotient for a fixed prime $p$. Then $\Omega^p(G_0)$ is an initial object of $\mathcal{P}$ and thus $\mathcal{P}$ is contractible. Moreover, $G_0$ itself is a terminal object of $\mathcal{P}$ and thus $\mathcal{P}$ is filtered. Let $G : \mathcal{P} \to \text{Grp}$ the monic functor which takes each subgroup of $\mathcal{P}$ to itself and inclusions to inclusions. Then $\limleftarrow G = G_0$ and, by Example 7.0.9,

\[
\text{hocolim}_{G \subseteq G_0, G \text{ normal subgroup with } p' \text{ quotient}} BG \cong BG_0.
\]

7.1. Proof of the Theorem

Theorem 7.1.1. Let $G : \mathcal{P} \to \text{Grp}$ be a functor and $\tau : G \Rightarrow G_0$ be a cone, and call $F$ the homotopy fiber

\[
F \to \text{hocolim} BG \to BG_0.
\]

Assume $\mathcal{P}$ is contractible and the cone $\tau : G \Rightarrow G_0$ is monic. Then

1. $\pi_1(F) = \text{Ker}(\limleftarrow G \to G_0)$
2. $\pi_0(F) = \text{Coker}(\limleftarrow G \to G_0)$
3. $H_j(F) = \limleftarrow_{\pi_{j-1}^* H}$ for each $j \geq 2$, where $H : \mathcal{P} \to \text{Ab}$ is a monic functor.

Proof. Notice that because the spaces $BG_i$ are connected and we have a pointed diagram $BG : \mathcal{P} \to S\text{Sets}$, we can apply the Van Kampen’s spectral sequence to obtain that $\pi_1(\text{hocolim}_* BG) = \limleftarrow \pi_1(BG) = \limleftarrow G$ and $\pi_0(\text{hocolim}_* BG) = 0$.

The fibration

\[
\mathcal{P} \to \text{hocolim} BG \to \text{hocolim}^* BG
\]
gives, as $\pi_1(\mathcal{P}) = \pi_0(\mathcal{P}) = 0$, $\pi_1(\text{hocolim} BG) = \limleftarrow G$ and $\pi_0(\text{hocolim} BG) = 0$. Then the fibration

\[
F \to \text{hocolim} BG \to BG_0
\]
gives $\pi_1(F) = \text{Ker}(\limleftarrow G \to G_0)$ and $\pi_0(F) = \text{Coker}(\limleftarrow G \to G_0)$.

It is readily checked that the usual construction of the homotopy fiber $F_f$ of a map $f : A \to B$ is functorial on $A$ and $B$. So the maps $B\tau_i : BG_i \to BG_0$ give a diagram $F : \mathcal{P} \to S\text{Set}$ which take values the fibers $F_i$ of $B\tau_i$:

\[
F_i \to BG_i \to BG_0.
\]

By [12], the fiber $F$ is the homotopy colimit of the diagram of the fibers $F : \mathcal{P} \to S\text{Set}$:

\[
F = \text{hocolim}(F_i).
\]

Consider the homology type first quadrant spectral sequence (Bousfield-Kan) that converges to $H_*(F) = H_*(\text{hocolim} F_i)$ and such that $E^2_{p,q} = \limleftarrow_{p} H_q(F_i)$. This spectral sequence describes the homology of the fiber $F$. Consider again the diagram $F$. It can be pointed taking $(1, e_1) \in F_i$ for each $i \in \text{Ob}(\mathcal{P})$. Then we have natural transformations of functors from $\mathcal{P}$ to $S\text{Set}_*$

\[
F_i \Rightarrow BG_i \Rightarrow BG_0.
\]
For each \( i \in \text{Ob}(\mathcal{P}) \) the pointed fibration
\[
F_i \to BG_i \to BG_0
\]
gives the homotopy long exact sequence
\[
\ldots \to \pi_1(F_i) \to \pi_1(BG_i) \to \pi_1(BG) \to \pi_0(F_i) \to \pi_0(BG_i) \to \pi_0(BG_0) \to 0.
\]
As the spaces \( G_i \) and \( \varinjlim G \) are discrete we obtain that \( \pi_j(F_i) = 0 \) for every \( j \geq 2 \).
Moreover, because \( BG_i \) and \( BG_0 \) are connected we have:
\[
0 \to \pi_1(F_i) \to G_i \xrightarrow{\tau_i} \varinjlim G \xrightarrow{\pi} \pi_0(F_i) \to 0.
\]
This is a exact sequence with three groups and a set. We can identify \( \pi_1(F_i) = \pi_1(F_i, (1, c_1)) \) with \( \text{Ker}(\tau_i) \) and write
\[
(34) \quad 0 \to \text{Ker}(\tau_i) \to G_i \xrightarrow{\tau_i} \varinjlim G \xrightarrow{\pi} \pi_0(F_i) \to 0.
\]
If \( \xi_i \in \pi_0(F_i) \) denotes the connected component of \( (1, c_1) \in F_i \) then the exactness at \( \varinjlim G \) states that
\[
p_{i-1}(\xi_i) = \text{Im}(\tau_i).
\]
Because the long homotopy exact sequence is natural
\[
\begin{CD}
0 @>>> \text{Ker}(\tau_i) @>>> G_i @>{\tau_i}>> \varinjlim G @>{\pi}>> \pi_0(F_i) @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> \text{Ker}(\tau_j) @>>> G_j @>{\tau_j}>> \varinjlim G @>{\pi}>> \pi_0(F_j) @>>> 0
\end{CD}
\]
then we have in fact a exact sequence of functors
\[
0 \Rightarrow \text{Ker}(\tau_i) \Rightarrow G_i \xrightarrow{\tau_i} \varinjlim G \xrightarrow{\pi} \pi_0(F_i) \Rightarrow 0
\]
If the cone \( \tau : G \Rightarrow \varinjlim G \) is monic then we have a short exact sequence of functors
\[
0 \Rightarrow G_i \xrightarrow{\tau_i} \varinjlim G \xrightarrow{\pi} \pi_0(F_i) \Rightarrow 0
\]
and the spaces \( F_i \) are discrete. The Bousfield-Kan homology spectral sequence reduces to
\[
H_p(F) = \varinjlim \pi_0(F_i)
\]
for \( p \geq 0 \).
Applying the functor free abelian group \( \mathbb{Z} \) to the short exact sequence above we obtain a sequence
\[
\mathbb{Z}[G] \Rightarrow \mathbb{Z}[\varinjlim G] \xrightarrow{\mathbb{Z}[\pi]} \mathbb{Z}[\pi_0(F_i)] \Rightarrow 0
\]
and taking kernels we have a short exact sequence of functors
\[
0 \Rightarrow H \Rightarrow \mathbb{Z}[\varinjlim G] \Rightarrow H_0(F) \Rightarrow 0
\]
in \( \text{Ab}^P \), where \( H(i) = \text{Ker}(\mathbb{Z}[\varinjlim G] \Rightarrow H_0(F_i)) \), which is a subgroup of the free abelian group \( \mathbb{Z}[\varinjlim G] \) and so it is free abelian too. Notice that the functor \( H : \mathcal{P} \to \text{Ab} \) is monic: the arrow \( H(i_1 \to i_2) \) is the inclusion of the subgroup \( H(i_1) \) into the subgroup \( H(i_2) \). Also it is clear that the functor \( H_0(F) \) is epic: it is deduced from the short
exact sequence above where the middle functor $\mathbb{Z}[\lim G]$ is constant and takes the value $1_{\mathbb{Z}[\lim G]}$ on morphisms.

The long exact sequence of derived limits $\lim$ for the short exact sequence above gives

$$\ldots \rightarrow \lim H \rightarrow \lim Z[\lim G] \rightarrow \lim H_0(F) \rightarrow \lim H \rightarrow \lim Z[\lim G] \rightarrow \lim H_0(F) \rightarrow 0$$

and recalling the definition of homology of a simplicial set

$$\ldots \rightarrow \lim H \rightarrow H_1(\mathcal{P}, Z[\lim G]) \rightarrow \lim H_0(F) \rightarrow \lim H \rightarrow H_0(\mathcal{P}, Z[\lim G]) \rightarrow \lim H_0(F) \rightarrow 0$$

Assume $\mathcal{P}$ is contractible. The we obtain

$$H_p(F) = \lim_{p-1} H$$

for $p \geq 2$ by using the long exact sequence above.

For convenience we give an explicit description of the functor $H : \mathcal{P} \rightarrow \text{Ab}$. Consider again the short exact sequence with two groups and one set for any $i \in \text{Ob}(\mathcal{P})$

$$0 \rightarrow G_i \overset{\xi_i}{\rightarrow} \lim G \overset{p_0}{\rightarrow} \pi_0(F_i) \rightarrow 0.$$

There is an action of $\lim G = \pi_1(B \lim G)$ on $\pi_0(F_i)$ such that $p_i$ is $\lim G$-equivariant when $\lim G$ is given the left action on itself, i.e., such that

$$p_i(g \cdot g') = g \cdot p_i(g')$$

for any $g, g' \in \lim G$. Moreover, this action is natural in the sense that for any arrow $i \rightarrow j$ in $\mathcal{P}$ we have

$$\pi_0(i \rightarrow j)(g \cdot \eta) = g \cdot \pi_0(i \rightarrow j)(\eta)$$

for each $g \in \lim G$ and $\eta \in \pi_0(F_i)$.

As $BG_i$ is path-connected the action of $\lim G$ on $\pi_0(F_i)$ is transitive. Fix, for each $\eta \in \pi_0(F_i)$, an element $g_\eta \in \lim G$ such that

$$g_\eta \cdot \xi_i = \eta$$

and $g_\xi_i = 1$, where $\xi_i \in \pi_0(F_i)$ is the connected component of $(1, c_1)$. Notice that $p_i(g_\eta) = \eta$. Recall that the exactness of the short exact sequence above means that $p_i^{-1}(\xi_i) = G_i \subseteq \lim G$. Then

$$p_i^{-1}(\eta) = g_\eta \cdot G_i$$

and

$$\lim G = \bigcup_{\eta \in \pi_0(F_i)} g_\eta \cdot G_i$$

where the union is disjoint. In fact, $\pi_0(F_i)$ can be identified with the set of cosets $G_i \backslash G$ [\text{p.73}], but we do not do it here. From this it is straightforward that

$$H(i) = \bigoplus_{\eta \in \pi_0(F_i)} g_\eta \cdot IG_i,$$
where $IG_i$ is the augmentation ideal of $G_i$, which is a free abelian group with basis $\{g - 1\}_{1 \neq g \in G_i}$. Then the set

$$B_i = \{g_\eta \cdot g - g_\eta, 1 \neq g \in G_i, \eta \in \pi_0(F_i)\}$$

is a basis for the free abelian group $H(i)$.

### 7.2. Another example.

A. Libman proposed the following example as one in which $\text{hocolim} \ BG \cong B \lim G$ does not hold in spite the category $P$ is contractible and the cone $\tau$ is monic: Consider the product category of the “pushout category” $c \leftarrow a \rightarrow b$ with itself and denote it by $P$. Then consider for any fixed group $G_0$ the functor $G : P \rightarrow \text{Grp}$ which takes value the trivial group 1 on $(a, a)$ and value $G_0$ on the rest.

We compute the fiber $F$

$$F \rightarrow \text{hocolim} \ BG \rightarrow B \lim G = BG_0$$

by means of the tools developed in earlier sections:

**Example 7.2.1.** Consider the contractible category $P$ and the functor $G : P \rightarrow \text{Grp}$ defined above. By Theorem 7.0.4 the fiber $F$

$$F \rightarrow \text{hocolim} \ BG \rightarrow BG_0$$

is simply connected and $H_j(F) = \lim_{\rightarrow} H$ for $j \geq 2$, where $H : P \rightarrow \text{Ab}$ is a monic functor. By dimensional reasons $\lim_{\rightarrow} H = 0$ for $j \geq 3$. The graded poset $P$ has shape:

```
(a, b) ----> (c, b)
|          |
(a, a) ----> (b, a) ----> (b, b)
|          |
(a, c) ----> (b, c)
|          |
(c, a) ----> (c, c)
```
and the functor $G : \mathcal{P} \to \text{Grp}$ takes values

$$
\begin{array}{c}
1 \\
G_0 \rightarrow G_0 \\
G_0 \rightarrow G_0 \\
G_0 \\
G_0 \\
\end{array}
$$

The functor $H : \mathcal{P} \to \text{Ab}$ is

$$
\begin{array}{c}
IG_0 \rightarrow IG_0 \\
IG_0 \rightarrow IG_0 \\
IG_0 \\
IG_0 \\
\end{array}
$$

The functor $K_1$ (Section 4.3) is isomorphic to

$$
\begin{array}{c}
0 \\
0 \rightarrow IG_0 \oplus IG_0 \\
0 \rightarrow 0 \rightarrow IG_0 \oplus IG_0 \\
0 \rightarrow IG_0 \oplus IG_0 \\
0 \rightarrow IG_0 \oplus IG_0 \\
\end{array}
$$

This implies that the functor $K_2$ is identically zero, and thus $H_3(F) = \lim_{\rightarrow} H = \ker\{\lim_{\rightarrow} K_2 \rightarrow \lim_{\rightarrow} K_1\} = 0$. The higher limit $\lim_{\rightarrow} H$ corresponds to pairs $(x_i, y_i) \in IG_0 \oplus IG_0$ for $i = 1, 2, 3, 4$ with

$$x_i + y_i = 0$$

and

$$x_i + x_{i+1} = 0$$
for \( i = 1, 2, 3, 4 \). Then it is clear that \( \lim_{\to 1} H \cong IG_0 \) and thus we have a fibration

\[
\bigvee_{\alpha \in G_0 \setminus \{1\}} (S^2)_\alpha \to \hocolim BG \to BG_0
\]
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