RESEARCH ARTICLE

Piecewise Constant Parameters Identification Under Finite Excitation Condition: Time Alertness Preservation, Exponential Convergence, Robustness and Applications

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Summary

The scope of this research is the identification of piecewise constant parameters of linear regression equations under the finite excitation condition. Such an equation is considered as a switched system, which identification usually consists of three main steps: a switching time instant detection, choice of the most appropriate model from the known set or generation of a new one, online adjustment of the chosen model parameters. Compared to the known methods, to make the computational burden lower and simplify the stability analysis, we use only one model to identify all switching states of the regression. So, the proposed identification procedure includes only two main approaches. The first one is a new estimation algorithm to detect switching time and preserve time alertness, which is based on a well-known DREM procedure and ensures adjustable detection delay. Unlike existing solutions, it does not involve an offline operation of data monitoring and stacking. The second one is the adaptive law, which provides element-wise monotonous exponential convergence of the regression parameters to their true values over the time range between two consecutive switches. Its convergence condition is that the regressor is finitely exciting somewhere inside such time interval. The robustness of the proposed identification procedure to the influence of external disturbances is analytically proved. Its effectiveness is demonstrated via numerical experiments, in which both abstract regressions and a second-order plant model are used.

KEYWORDS:
identification, linear regression, piecewise constant parameters, switching, finite excitation, monotonicity, extension and mixing

INTRODUCTION

In recent years, the problem of identification of regression equation (RE) unknown parameters has received considerable attention from the scientific community in the field of control theory. The mentioned problem is of high importance as, using special parameterizations, most of the adaptive control theory problems can be reduced to the identification of the RE unknown parameters. For example, in a parametrization is proposed, in which the plant dynamics identification problem is converted into the estimation of the unknown initial conditions vector. In an approach is developed to represent the classical problems of the
model reference adaptive control as the identification of unknown ideal parameters of a control law, which is written as RE. The known major studies related to the solution of the RE parameters identification problem, in general, can be divided into two main groups with the following objectives:

- ensure exponential convergence of the parameter error to zero under weak regressor excitation requirements \([1,2,9] \).  
- improve the quality of identification for RE with over-parametrization \([1,7,24,27] \).

The present work is devoted to the development of methods, which belong to the first group.

It is well known that the classical adaptive laws, which are based on the gradient descent and least squares methods, provide exponential convergence of the parameter error to zero only if the strict requirement of the regressor persistent excitation (PE) is met. This means that the regressor is to be sufficiently rich \([25] \), i.e. it is expected to contain as many different spectral lines as the number of RE unknown parameters is. But that is not the truth for many practical problems as far as the normal functioning mode of most plants is concerned. Therefore, in order to obtain the true values of the unknown parameters, it becomes necessary to artificially provide the required regressor richness by injection of test signals into the plant control signal. For example, the procedures \([25,31] \) can be used for that. On the other hand, in many applications, such injection can lead to damage of actuators, energy consumption increase, and cause situations when the plant functions out of standard operating mode.

As a result, new adaptive laws have been proposed in the literature \([5,15] \) to guarantee exponential convergence of the unknown parameters estimates to their true values in case of the regressor initial (IE) or finite (FE) excitation, which are strictly weaker than PE. Most of these approaches are based on the composite/combined adaptive laws that use previously stored and current values of the regression function and the regressor concurrently. Some off-line algorithms are applied to store data in a special data stack and ensure that they are sufficiently rich \([21,5] \). Thus, using some kind of memory, such laws provide a parameter error exponential convergence to zero even when the regressor excitation has already vanished. Another way to provide exponential convergence is to filter the RE using different filters with the integral or strong inertial properties \([7,14] \). Also, recently the energy pumping-and-damping injection principle \([22] \) has been successfully applied to generate the PE regressor from the IE one \([18,19] \).

In contrast to the above-considered adaptive laws, which require the data stack processing, this approach does not use offline operations, but, like the schemes \([7,14] \), has strong inertial properties. In \([20,23] \) adaptive laws are proposed, which, unlike the previously considered ones, provide finite time identification of the RE unknown parameters if IE or FE condition is met. A more detailed review of some modern methods to relax the PE condition is given in \([11] \).

However, the laws \([3,32] \) have been developed only to solve the identification problems of the RE time-invariant parameters. Their applicability to estimate the piecewise constant parameters is usually tested in numerical experiments only and, with a number of exceptions like \([13,28] \), does not have a sufficiently rigorous analytical proof.

Adaptive laws \([32,22] \) have difficulties in solving problems of piecewise constant unknown parameters identification due to imperfect data storage/filtering procedures and inertial characteristics of filters, which are used for the RE processing. The point is that when the data on RE, which parameters are not time-invariant, are stored in the data stack or the regression with piecewise constant parameters is filtered with integral-based filters, then the combined/composite adaptive laws lose their ability to track true values of the unknown parameters. The reason is the superpositional mixing of data on the regressions with different values of parameters. In this case only estimations boundedness could be guaranteed \([31,13] \). A more detailed analytical and numerical comparison of some modern combined adaptive laws to solve the problem of piecewise constant unknown parameters identification is given in \([21] \) (see Fig.4).

Many authors \([11,14,16,20,23,30] \) have pointed out the vital necessity to derive adaptive laws that could also provide the identification of piecewise constant parameters with exponential or finite-time convergence. In \([13] \) it is mentioned that a reinitialization procedure is required for a scheme with the excited regressor generation. It should be applied each time when the unknown parameters of RE change their values. In \([15,18] \) various methods are proposed to overcome superpositional mixing of data in the data stack, which is caused by the unknown parameters switch. In \([17] \) the dynamic regressor extension and mixing scheme sensitivity to the unknown parameters switching is discussed. In \([25] \) the applicability of the adaptive law with finite time convergence is studied to identify the piecewise constant unknown parameters. Thus, the problem of identification of RE with switched parameters is relevant and deserves a stand-alone detailed discussion.

In the machine learning theory, the classification methods \([38] \), principal component analysis \([10] \), ARX (SNARX) regression models \([40] \) are well known to be able to find the unknown piecewise constant parameters. Such models are trained offline on the basis of a data array, which includes the measured values of the regressor and function at some time points. However, the application of these and other machine-learning-based techniques to solve the identification and adaptive control problems online
faces difficulties, which are mainly caused by high computational costs in the case of an increasing number of measurements and higher system dimensions.

On the other hand, the problem of piecewise constant unknown parameters identification is well-developed as a part of the switched systems adaptive control theory. The general formulation of the identification problem for switched systems includes: (1) the identification of a discrete function, which defines the time points when the unknown parameters switch their current values to new ones, (2) the choice of the current model of the system from a known set or a generation of a new one, (3) the estimation of the current values of the unknown piecewise constant parameters of the system (usually they include the state and control matrices of the piecewise-linear plant). As a rule, classical solutions of such a complex identification problem require the restrictive PE requirement to be met to provide convergence.

In a recent paper, an approach, which is based on a combination of earlier results, has been proposed to solve the switched system identification problem under consideration quite effectively when the regressor is finitely exciting.

In order to identify the function, which defines the switching time instants, the above-mentioned research proposes to compare the regression equations stored in the data stack with the current regression, which is formed using the measured data from the plant. Such a comparison is based on the data-driven projection subspace method. Using the principles from, a residual signal is formed to be used as an indicator of the system unknown parameters change (switch). Then, applying the numerical robust algorithm of residual analysis, the switching signal function is estimated with almost arbitrary accuracy. Then the residual value is calculated for each model, the information about which is stored in the data stack. If such value is less than the preset threshold for some model, then it is assumed that the plant is described by such model. And it is marked as ‘active’. If no model with the residual below the threshold has been found, then a new one is generated. A data stack is formed and processed for each model, which was active for at least one time. This makes it possible to identify the parameters of each model all the time, even when the model is not active.

To identify the parameters of a switched system, according to the number of the adaptive laws to be introduced coincide with the number of detected models. And, as the estimation of the unknown parameters is performed on basis of the data from the data stack, this method requires only a finite excitation of the regressor over the time range when the model is active to provide exponential convergence of the parameter error to zero.

Thus, the method ensures the piecewise constant unknown parameters tracking, which is its main advantage over other procedures to relax the PE requirement. However, over time, the number of models used can increase indefinitely, and the necessity to adjust simultaneously all parameters of all models, as well as store, monitor and process a high number of data stacks, requires high computational power of the hardware in use. In addition, the presence of several models makes it difficult to analyze the stability of a closed-loop system, when such identification method is applied as a part of the adaptive control system. The problem is that the control law is discontinuous when several models are in use, and the proof of stability requires the application of a multiple Lyapunov function.

In this regard, the actual problem is to develop a law of the unknown piecewise constant parameters estimation, which does not use several separate adaptive laws, as well as offline operations of data monitoring and stacking, but is able to identify piecewise constant unknown parameters and ensure the exponential rate of convergence under PE condition.

Thus we propose a new procedure to identify unknown piecewise parameters with the following contribution:

C1) on the basis of the dynamic regressor extension and mixing procedure, a new online algorithm to estimate switching time instants is proposed, which provides an adjustable value of the detection lag. In contrast to the solution, the proposed algorithm does not involve an offline operation of data monitoring and stacking;

C2) based on the proposed estimation algorithm, the adaptive law is derived. It ensures exponential convergence of the unknown parameters estimates to their true values over the time interval between two consecutive changes of the regression parameters if the regressor is finitely exciting somewhere inside such interval. In contrast to, only one model with adjustable parameters is used to solve the identification problem, and the proposed law provides element-wise monotonicity of the parameter error vector, which, in addition, is not discontinuous in the course of the adjustment process.

To the best of the authors’ knowledge, the proposed method of piecewise constant unknown parameters identification is the first solution that ensures all the above-stated properties simultaneously.

The further part of the manuscript is arranged as follows. Section 2 gives a rigorous mathematical problem statement, Section 3 is to propose (1) an algorithm to estimate the switching time instants, and (2) an adaptive law that provides exponential convergence of the error of piecewise constant unknown parameters estimation. Section 4 presents the results of the numerical experiments.
The following notation is used throughout the paper. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote the sets of $n$-dimensional real vectors and $n \times m$-dimensional real matrices respectively, $|.|$ represents the absolute value, $\| \|$ denotes Euclidean norm of a vector, the identity and nullity $n \times m$-dimensional matrices are denoted as $I_{n \times m}$ and $0_{n \times m}$ respectively. $E\{.\}$ is the operator to calculate the mean value, $\text{var}(.\text{)}$ is the operator to calculate the variance. $\text{det}\{.\}$ stands for a matrix determinant, $\text{adj}\{.\}$ – for an adjoint matrix, $L_{\infty}$ is the space of all essentially bounded functions. We also use the fact that for all (possibly singular) $n \times n$ matrices $M$ the following holds: $\text{adj}\{M\} \cdot M = \text{det}\{M\} \cdot I_{n \times n}$.

The following definition from Glushchenko et al. is introduced.

**Definition 1.** A regressor $\phi(t) \in \mathbb{R}^n$ is finitely exciting ($\phi \in FE$) over the time range $[t_0^+; t_e]$ if there exist $t_e \geq t_e^+ \geq 0$ and $\alpha > 0$ such that the following inequality holds:

$$\int_{t_e^+}^{t_e} \phi(\tau) \phi^T(\tau) \, d\tau \geq \alpha I_{n \times n},$$

where $\alpha$ is an excitation level.

## 2 PROBLEM STATEMENT

The identification problem of unknown piecewise constant parameters of a linear regression equation is considered:

$$\forall t \geq t_0^+, \quad y(t) = \phi^T(t) \Theta_{\kappa(t)},$$

where $\phi(t) \in \mathbb{R}^{n \times m}$, $y(t) \in \mathbb{R}^{m \times p}$ are measurable regressor and function, $\Theta_{\kappa(t)} \in \mathbb{R}^{n \times p}$ is a matrix of unknown piecewise constant ($\forall t \geq t_0^+ \quad \Theta_{\kappa(t)} \equiv 0$) parameters, $\kappa(t) \in \mathbb{N} = \{1, 2, \ldots, N\}$ is an unknown discrete function, which defines the time points when the regression parameters switch to their new values, $t_0^+$ is a known initial time instant, $N$ is a number of different regression switching states (sets of parameters $\Theta_{\kappa(t)}$). To be explicit, $\kappa(t)$ and $\Theta_{\kappa(t)}$ are assumed to be right-continuous:

$$\forall t \geq t_0^+, \quad \Theta_{\kappa(t)} = \lim_{t \to t_0^+} \Theta_{\kappa(t)}.$$  

(3)

In general case, the signal $\kappa(t)$ is used to represent the switching sequence:

$$\Sigma = \left\{ (j_0, t_0^+), (j_1, t_1^+), \ldots, (j_{i-1}, t_{i-1}^+), (j_i, t_i^+), \ldots \right\}, \quad j_0, j_i \in \mathbb{N}, \quad j_i \neq j_{i+1}, \quad i \in \mathbb{N}.$$  

(4)

This means that $\forall t \in [t_i^+, t_{i+1}^+]$, $\kappa(t) = j_i$, $\Theta_{\kappa(t)} = \Theta_{j_i}$. For the sake of brevity, the element of the set $\Sigma$, which corresponds to $[t_i^+, t_{i+1}^+]$, is denoted as $\theta_i$ ($\forall t \in [t_i^+, t_{i+1}^+]$ $\theta_i = \Theta_{\kappa(t)} = \Theta_{j_i}$). So, the equation (2) is rewritten as follows:

$$\forall t \geq t_0^+, \quad y(t) = \phi^T(t) \theta_i,$$

$$\theta_i = \sum_{q=0}^{\|} \Lambda_q h\left(t - t_q^+\right).$$  

(5)

where $\Lambda_i = \theta_i - \theta_{i-1} = \Theta_{j_i} - \Theta_{j_{i-1}}$ is an amplitude of $\theta_{i-1}$ value change at time point $t_q^+$, $h(t)$ is a Heaviside step function. The parameters $\theta_i$ are considered to be differentiable ($\forall t \in [t_i^+, t_{i+1}^+]$ $\dot{\theta}_i \equiv 0$) $\forall t \in [t_i^+, t_{i+1}^+]$ because of (3).

Additionally, the time points, at which switching occurs in (5), are written as a time sequence:

$$\mathcal{S} = \left\{ t_0^+, t_1^+, \ldots, t_{i-1}^+, t_i^+, \ldots \right\},$$  

(6)

The following assumption is introduced with respect to $\theta_i$, the time range $[t_i^+, t_{i+1}^+]$ and the regressor $\phi(t)$.

**Assumption 1.** Let $\exists \Delta_\theta > 0$, $T_{\min} \geq \min_{\forall i \in \mathbb{N}} T_i > 0$ such that $\forall i \in \mathbb{N}$ simultaneously:

1) $t_{i+1}^+ - t_i^+ \geq T_{\min}$, $\| \theta_i - \theta_{i-1} \| \leq \Delta_\theta$;
2) the regressor $\phi(t)$ is finitely exciting ($\phi(t) \in FE$) over the time range $[t_i^+, t_i^+ + T_i]$ with the excitation level $\alpha_i$;
3) the regressor $\phi(t)$ is finitely exciting ($\phi(t) \in FE$) over the time range $[t_i^+, t_i^+ + T_i]$ with the excitation level $\Delta_\theta$, where $\alpha_i > \Delta_\theta > 0$, $\hat{t} = t_i^+ + T_i$;
4) $\phi(t) \in L_{\infty}$. 


Considering the regression (5) and Assumption 1, the following goals are to be achieved \( \forall i \in \mathbb{N} \):

\[
\begin{align*}
\hat{t}_i^+ &\leq T_i, \\
\lim_{T_{\text{min}} \to \infty} \| \tilde{\theta} \| = 0, \\
\forall t \in [t_i^+ + T_i; t_{i+1}^+), \quad \tilde{\theta}(t) &= e^{-\eta (t_i^++T_i)} (\hat{t}_i^+ + T_i),
\end{align*}
\]

where \( \hat{\theta}_i \left( t_{i+1}^+ \right), \hat{\theta}_i = \hat{\theta}_i \left( t_{i+1}^+ \right) - \theta_i \) are the parameters estimates and error respectively at time point \( t_{i+1}^+ \), \( \hat{\theta}_i = \hat{\theta}_i \left( t_i^+ \right) - \theta_i \) is the parameter error for each \( i \) from the time range \( [t_i^+; t_{i+1}^+] \). \( \hat{t}_i^+, \tilde{t}_i^+ = \hat{t}_i^+ - t_i^+ \) are the estimate of \( i^{\text{th}} \) element of the sequence (6) and the error of such estimation.

All notions, which are introduced in Assumption 1 and the problem statement (7), are graphically explained in Fig. 1.

![FIGURE 1 Illustration of introduced notation](image_url)

According to the stated goal (7) and explanations in Fig.1, in this research the update laws for \( \hat{\theta}_i \) and \( \hat{t}_i^+ \) are to be derived. They must ensure: 1) the bounded value of the estimation error of each element of the sequence (6), 2) zero error of \( \theta_i \) identification when the length of the range \( [t_i^+; t_{i+1}^+] \) tends to infinity (the next switch will never happen); 3) the exponential convergence of the parameter error \( \tilde{\theta}(t) \) over the time range \( [t_i^+ + T_i; t_{i+1}^+] \).

Remark 1: The first part of Assumption 1 requires a finite frequency and amplitude of the step change of the unknown parameters, which are classical requirements for the switched systems and identification theories respectively. The second and third parts of the assumption present a necessary and sufficient condition to identify the true values of all elements of the \( i^{\text{th}} \) matrix of unknown parameters. The fourth part of the assumption can be satisfied by the multiplication of the regression with the normalizing coefficient \( n_\tau(t) = \frac{1}{1 + \varphi(t)^2 \varphi(t)} \). The requirements of Assumption 1 are not restrictive and usually satisfied in practical scenario.

3 | MAIN RESULT

It is proposed to solve the problem (7) in two steps. At the first one, it is necessary to estimate the elements of the sequence (6), i.e. to propose an estimation algorithm. Using the obtained estimates \( \hat{t}_i^+ \), the second step is to derive an adaptive law to track piecewise constant parameters of the regression (2) and ensure that the stated goal (7) is achieved. At the same time, the estimation algorithm and adaptive law should function in parallel in online mode and be robust to the possible presence of an external bounded disturbance in the regression (2).

In Section 3.1 the estimation algorithm to detect time instants when the unknown parameters of regression switch to new values is proposed, which ensures the required boundedness \( \hat{t}_i^+ \leq T_i \). In Section 3.2 an adaptive law is proposed, which is based on the obtained estimates \( \hat{t}_i^+ \) and guarantees that the second and third goals from (7) are achieved.
3.1 | Switching detection algorithm

To introduce the algorithm of switching time instants detection, first of all, the dynamic regressor extension and mixing (DREM) procedure is applied. In order to do that, the regression equation \( \text{(5)} \) is extended as:

\[
\varphi (t) \ y(t) = \varphi (t) \varphi^T (t) \theta_i.
\]  

\( (8) \)

The filters with exponential forgetting and resetting at time point \( \hat{t}_i^+ \) are introduced:

\[
Y (t) = \int_{\hat{t}_i}^{t} e^{-\int_{\hat{t}_i}^{\tau} \varphi (\tau) \varphi^T (\tau) d\tau} \omega (\tau) Y (\tau) d\tau, \quad Y (\hat{t}_i^+) = 0,
\]

\( (9) \)

\[\omega (t) = \int_{\hat{t}_i}^{t} e^{-\int_{\hat{t}_i}^{\tau} \varphi (\tau) \varphi^T (\tau) d\tau} \omega (\tau) \varphi (\tau) d\tau, \quad \omega (\hat{t}_i^+) = 0,\]

where \( Y (t) \in \mathbb{R}^{n \times p}, \omega (t) \in \mathbb{R}^{n \times n}. \) The time point \( \hat{t}_i^+ \) will be precisely defined further.

The dynamic regressor extension and mixing procedure \( \text{(10)} \) is applied to the function \( Y (t) \) to obtain:

\[ Y (t) : = ad \int \{ \omega (t) \} Y (t), \quad Y (0) = \Omega (0) \theta_0 , \quad \Omega (t) : = \int \{ \omega (t) \} \omega (t) = det \{ \omega (t) \},\]

\( (11) \)

where \( \Omega (t) \in \mathbb{R}^{n \times n}. \)

It follows from the definition of \( Y (t) \) that if the vector of the unknown parameters \( \theta_{i-1} \) has changed its values at time instant \( t_i^+ > \hat{t}_{i-1}^+ \), then the filtration \( \text{(9)} \) mixes data on the regression equations with the parameters \( \theta_i \) and \( \theta_{i-1}. \) So, to avoid such mixture, we need to detect the time point when the parameters of the regression \( \text{(2)} \) has changed their values and reset the filters \( \text{(9)} \) at time instant \( \hat{t}_i^+ \geq t_i^+ \).

To do that, owing to the inertial characteristics of the filter \( \text{(9)} \), the parameters of the regressions \( \text{(8)} \) and \( \text{(11)} \) are solved with the help of the direct least-squares method to obtain:

\[
\theta_0 \triangleq \Omega^{-1} (t) Y (t) = \left( \varphi (t) \varphi^T (t) \right)^{-1} \varphi (t) y(t).
\]  

\( (12) \)

When \( \theta_i = \theta_0 \ \forall t \geq t_0^+ \), the unknown parameters of the regressions \( \text{(8)} \) and \( \text{(11)} \) coincide to each other. Hence, the fact that the equality \( \text{(12)} \) does not hold is considered as an indicator that the parameters of the regression \( \text{(2)} \) have already changed their values. However, to check such indicator, the calculation of \( \Omega^{-1} (t) \) and \( \left( \varphi (t) \varphi^T (t) \right)^{-1} \) is needed, which may become infinite in the course of the estimation process. To avoid that, the indicator \( \text{(12)} \) is represented in the form, which allows one to calculate it without matrix inversion at each time instance:

\[
\forall t \geq t_0^+, \quad e (t) = \varphi (t) \varphi^T (t) Y (t) - \Omega (t) \varphi (t) y(t) = 0,
\]  

\( (13) \)

where \( e (t) \in \mathbb{R}^{n \times p} \) is a residual.

The fact that the equation \( \text{(13)} \) is derived in the right way can be proved by substitution of the functions \( y(t) \) and \( Y (t) \) into it under the temporarily introduced assumption that \( \theta_i = \theta_0 \ \forall t \geq t_0^+. \)

If the parameters of the regression \( \text{(2)} \) are not time-invariant, then, in general case, the equation \( \text{(13)} \) does not equal to zero over some time ranges and can be rewritten as:

\[
e (t) : = \varphi (t) \varphi^T (t) \omega \{ t \} Y (t) - \Omega (t) \varphi (t) y(t), \quad e (\hat{t}_i^+) = 0.
\]  

\( (14) \)

Then, the fact that equation \( \text{(13)} \) does not hold is an indicator of the regression \( \text{(2)} \) parameters change. However, contrary to \( \text{(12)} \), the analytical solution of the equations \( \text{(8)} \) and \( \text{(11)} \) with respect to \( \theta_i \) is not needed to calculate the residual \( \text{(13)}, \) and only measurable signals are used for such calculation. The following proposition is introduced on the basis of \( \text{(13)} \) and \( \text{(14)} \).
Proposition 1. If \( \tilde{t}_i^+ \geq \tilde{t}_i^- \), then:

\[
\epsilon(t) = \begin{cases} 
\phi(t) \phi^T(t) \text{adj} \{ \omega(t) \} \int_{\tilde{t}_i^-}^{\tilde{t}_i^+} e^{-\int_0^s \sigma ds} \varphi(\tau) \phi^T(\tau) d\tau \left( \theta_{i-1} - \theta_i \right), \forall t \in \left[ t_{i-1}^+, t_{i+1}^+ \right), \\
0, \forall t \in \left[ t_{i-1}^+, t_{i+1}^+ \right)
\end{cases}
\]  

(15)

Proof of Proposition 1 is postponed to Appendix.

The results obtained in Proposition 1 make it reasonable to choose the function \( \epsilon(t) \) as an indicator to find \( \tilde{t}_i^+ \) value. The following estimation algorithm is introduced to solve the problem under consideration:

\[
\text{initialize : } i \leftarrow 1, \ t_{up} = \tilde{t}_{i-1}^+
\]

\[
\begin{cases} 
\text{IF } t - t_{up} \geq \Delta_{pr} \text{ AND } \| \epsilon(t) \| > 0 \text{ THEN } \tilde{t}_i^+ := t + \Delta_{pr}, \ t_{up} \leftarrow t, \ i \leftarrow i + 1, \\
\end{cases}
\]

(16)

where \( 0 \leq \Delta_{pr} < \min_{\forall i \in \mathbb{N}} T_i \) is the parameter of the estimation algorithm, which value can be chosen arbitrarily in accordance with the written inequality.

Then the estimation error (14) takes the form:

\[
\text{Remark 3: The estimation algorithm (16) detects the switching time instant with the precision } \Delta_{pr}. \text{ The properties of the algorithm (16) are described in the following proposition.}
\]

Proposition 2. Let Assumption 1 be satisfied and \( \tilde{t}_i^+ \) be estimated using (16). Then \( \tilde{t}_i^+ \geq \tilde{t}_i^- \), and the fact that the condition \( \tilde{t}_i^+ \leq T_i \) from (7) holds is ensured by appropriate choice of \( \Delta_{pr} \).

Proof of Proposition 2 is presented in Appendix.

So, the estimation algorithm (16) detects the time points when the parameters switch their values to new ones in online mode. As a result, it allows one to obtain the estimation of the sequence (6):

\[
\hat{\Delta}_i = \left\{ \tilde{t}_0^+, \tilde{t}_1^+, \ldots, \tilde{t}_{i-1}^+, \tilde{t}_i^+, \ldots \right\} \in \mathbb{N}.
\]

The estimation error between each element of (17) and (6) is equal or lower than the value of \( T_i \), and could be adjusted by the appropriate choice of \( \Delta_{pr} \). So, according to Proposition 2, it is concluded that, using the algorithm (16), the first goal from (7) is achieved.

Remark 2: The parameter \( \Delta_{pr} \), first of all, defines a small neighborhood of the time point \( t_i^+ \), in which no change of the regression (2) parameters can happen (\( t_i^+ + \Delta_{pr} \ll t_i^+ + T_{min} \)). Secondly, \( \Delta_{pr} \) separates the time point of switch detection (16) and the one to reset the filters (9) in order to make the computational procedures more stable. In practice, even when \( \Delta_{pr} = 0 \), there is some time delay between the detection of the parameters switch and the filters (9) resetting. Therefore, as far as the mathematical analysis is concerned, it is considered that the parameter \( \Delta_{pr} \) allows one to take into account, in some way, what effect such delay has on the quality of the estimates. However, it should be noted that the above-stated reasoning is also valid when \( \Delta_{pr} = 0 \), and in such case the algorithm (16) provides the best detection accuracy \( \hat{t}_i^+ \rightarrow t_i^+ \).

Remark 3: It is of high importance for practice to ensure that the estimation algorithm (16) can function in the presence of the external bounded disturbances in the regressions (2), (5), and (9). It is easy to show that if \( \forall t \geq t_0^+ \ y(t) = \varphi^T(t) \theta_i + w(t) \), then the error (14) takes the form:

\[
\epsilon(t) = \phi(t) \varphi^T(t) \text{adj} \{ \omega(t) \} \int_{\tilde{t}_i^-}^{\tilde{t}_i^+} e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \left( \theta_{i-1} - \theta_i \right) + \varphi(t) \varphi^T(t) \text{adj} \{ \omega(t) \} \int_{\tilde{t}_i^-}^{\tilde{t}_i^+} e^{-\int_0^s \sigma ds} \varphi(\tau) w(t) d\tau - \Omega(t) \varphi(t) w(t), \ni \epsilon(t) = 0
\]

(18)

where \( w(t) \in \mathbb{R}^{m \times p} \) is a bounded \( \| w(t) \| \leq w_{max} \) external disturbance, which is added to the right hand side of (2).

Hence, \( \forall t \neq t_i^+ \| \epsilon(t) \| > 0 \). As a consequence, according to the algorithm (16), the filters (9) reset their state periodically – each \( \Delta_{pr} \) seconds. Such result is inappropriate from the practical point of view. So, using the mathematical statistics theory (21), in case of external disturbances a modified (robust) version of the algorithm (16) should be used:

\[
\text{initialize : } i \leftarrow 1, \ t_{up} = \tilde{t}_{i-1}^+
\]

\[
\begin{cases} 
\text{IF } t - t_{up} \geq \Delta_{pr} \text{ AND } E \{ \epsilon(t) \} > 0.9 \sqrt{\text{var} \{ \epsilon(t) \} + c(t)}, \\
\text{THEN } \tilde{t}_i^+ := t + \Delta_{pr}, \ t_{up} \leftarrow t, \ i \leftarrow i + 1,
\end{cases}
\]

(19)

where \( c(t) \) is the arbitrary parameter of the robust algorithm.
The choice of the algorithm (19) parameter \( c(t) \) value allows one to adjust the estimation accuracy and adapt to each specific class of the external disturbances. For example, if the disturbance is a noise with zero mean, then, according to the results \([11,14]\), it is enough to choose \( c = 0 \). In the general case it is recommended to choose the function \( c(t) \) as follows:

\[
c(t) = E \left\{ w_{\text{max}} \phi(t) \varphi^T(t) d \, j \{ \omega(t) \} \int_{t_i}^{t} e^{-\int_{s}^{t} \phi(t') \varphi(t') d t'} I_{\text{nom}} d t \right\} .
\]  

(20)

In the disturbance free case, the properties of the robust algorithm (19) completely coincide with the ones described in Proposition 2. However, if the external disturbance is added to the right hand side of (2), the algorithm (19), in contrast to (16), does not face problems of false detections in case the parameter \( c(t) \) value is chosen correctly. It does not also cause periodic reset of the filter (9) and provides sufficient accuracy in terms of error \( \hat{t}_i^+ \). More details about the robust algorithm (19) can be found in \([11,14]\).

Remark 4: The regression equation (2) is usually a parameterized form of representation of some particular adaptive control problems. It is often obtained using various stable minimum-phase filters (see \([15]\) for details). Therefore, when the change of the regression (2) parameters is detected according to (16) or (19), in addition to the filters (9), all filters previously used to obtain the regression (2) must be set to their initial zero states.

3.2 Adaptive law

The required adaptive law of the unknown parameters \( \theta \), will be introduced on the basis of the regression function (10), which is formed by the dynamic regressor extension and mixing procedure \([17]\). But before that the properties of the function \( Y(t) \) and the regressor \( \Omega(t) \) have been studied. The result of this analysis is presented in the form of a proposition.

Proposition 3. Let the requirements of Assumption 1 be met, and \( \hat{t}_i^+ \) be formed according to the algorithm (16), then the regression function \( Y(t) \) can be represented in the form:

\[
Y(t) := \Omega(t) \theta_i + d(t),
\]

\[
d(t) := \begin{cases} \int_{t_i}^{t_i^+} e^{-\int_{s}^{t} \phi(t') \varphi(t') d t'} \varphi(t) \varphi^T(t') d t' \left( \theta_{i-1} - \theta_i \right), & \forall t \in \left[ t_i^+; t_{i+1}^+ \right), \\
0, & \forall t \in \left[ t_i^+; t_{i+1}^+ \right],
\end{cases}
\]  

(21)

where \( \|d(t)\| \leq d_{\text{max}} \), and the regressor \( \Omega(t) \) is such that \( \forall t \in \left[ t_i^+ + T_i; t_{i+1}^+ \right) \Omega(t) \geq \Omega_{LB} > 0 \).

Proof of Proposition 3 and the definitions of \( d_{\text{max}}, \Omega_{LB} \) are presented in Appendix.

Taking into account that the function \( Y(t) \) can be represented as (21), as well as the properties of the regressor \( \Omega(t) \), the unknown parameters adaptive law is chosen as:

\[
\dot{\theta}(t) = \ddot{\theta}(t) := -\gamma \Omega(t) \left( \Omega(t) \ddot{\theta}(t) - \Omega(t) \theta_i \right) = -\gamma \Omega^2(t) \ddot{\theta}(t),
\]

\[
\gamma = \left\{ \begin{array}{ll} 0, & \text{if } \Omega(t) \leq \rho, \\
\frac{\gamma_0}{\Omega^2(t)}, & \text{otherwise}, \end{array} \right.
\]  

(22)

where \( \gamma_0 > 0 \) is the adaptive gain, \( \rho \in \left( 0; \Omega_{LB} \right] \) is the arbitrary parameter of the adaptive law.

The properties of the errors \( \dot{\theta}(t) \) and \( \ddot{\theta}(t) \) are analyzed in Theorem.

Theorem 1. Let the requirements of Assumption 1 be satisfied, \( \hat{t}_i^+ \) be obtained in accordance with the algorithm (16), and \( \dot{\theta}(t) \) be calculated using (22), then:

a) \( \forall t_a, t_b \in \left[ t_i^+ + T_i; t_{i+1}^+ \right), \quad t_a \geq t_b \Rightarrow \| \ddot{\theta}_k(t_a) \| \leq \| \ddot{\theta}_k(t_b) \| ; \)

b) \( \forall t \in \left[ t_i^+; t_i^+ + T_i \right) \Rightarrow \| \ddot{\theta}(t) \| \leq \| \ddot{\theta}(t_i^+) \| + \rho^{-1} \| d(t) \| ; \)

c) \( \forall t \in \left[ t_i^+ + T_i; t_{i+1}^+ \right) \Rightarrow \| \ddot{\theta}(t) \| \leq e^{-\gamma_0 \left( t_i^+ - t_i^+ - T_i \right)} \left( \| \ddot{\theta}(t_i^+) \| + \rho^{-1} \| d(t) \| \right) ; \)

d) \( \lim_{t_{\text{min}} \to -\infty} \| \ddot{\theta} \| = 0, \quad \lim_{\gamma_0 \to \infty} \| \ddot{\theta} \| = 0, \)
where \( \hat{\theta}_k(t), \hat{\theta}(t) \), \( k = 1, n \) is the \( k \)th element of the vector \( \hat{\theta}(t) \).

The proof of the theorem is given in Appendix.

According to Theorem, the adaptive law (22) ensures that the stated goal (7) is achieved. Additionally, the monotonicity of the estimation process transient curve is guaranteed for each element of the error vector \( \hat{\theta}(t) \) over the time range \( [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \).

Due to the fact that a disturbance \( d(t) \) affects the regression (21) over the time range \( [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \), in general case, the law (22) does not provide monotonicity of \( \hat{\theta}_k(t) \) transients over such time range. Moreover, such transients may suffer sufficient oscillations.

The proposed way to improve their quality over the interval \( [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \) is presented as the following proposition, which is based on the choice of the filter (9) parameter \( \sigma \).

**Proposition 4.** If all conditions of Theorem are satisfied and \( \sigma \to \infty \), then:

a) \( \forall t_a, t_b \in [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \quad t_a \geq t_b \quad \| \hat{\theta}_k(t_a) - \theta_{i-1,k} \| \leq \| \hat{\theta}_k(t_b) - \theta_{i-1,k} \| \),

b) \( \forall t \in [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \quad \| \hat{\theta}(t) - \theta_{i-1} \| \leq e^{-0.5\gamma_0(t-t^*_i)} \| \hat{\theta}(t^*_i) - \theta_{i-1} \| \).

**Proof of Proposition 4** is shown in Appendix.

In other words, if the parameter \( \sigma \) value tends to infinity, then the time point \( t^*_i \) turns out to be inside a dead zone of the filter (9), i.e. when it does not react to the received input data. So, in this case, considering the time range \( [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \), the law (22) allows one to estimate the vector \( \theta_{i-1,k} \) of the previous (before the last switch) values of the RE parameters. As a consequence, the above-mentioned oscillations of the transients of \( \hat{\theta}_k(t) \) and \( \hat{\theta}(t) \) do not occur over the time range \( [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \). And, as a result, the monotonicity of the parameter errors \( \hat{\theta}_k(t) \) is provided almost everywhere.

Then, the following corollary can be derived from Theorem and Proposition 4.

**Corollary:** If all conditions of Theorem are satisfied and \( \sigma \to \infty \), then:

1. \( \forall t_a, t_b \in [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \quad t_a \geq t_b \quad \| \hat{\theta}_k(t_a) - \theta_{i-1,k} \| \leq \| \hat{\theta}_k(t_b) - \theta_{i-1,k} \| \).

2. \( \forall t \in [t^*_i; t^*_i + T_1; t^*_i + T_{i+1}] \quad \| \hat{\theta}(t) - \theta_{i-1} \| \leq e^{-0.5\gamma_0(t-t^*_i)} \| \hat{\theta}(t^*_i) - \theta_{i-1} \| \).

3. \( \forall t_a, t_b \in [t^*_i + T_1; t^*_i + T_{i+1}] \quad t_a \geq t_b \quad \| \hat{\theta}_k(t_a) \| \leq \| \hat{\theta}_k(t_b) \| \).

4. \( \forall t \in [t^*_i + T_1; t^*_i + T_{i+1}] \quad \| \hat{\theta}(t) \| \leq e^{-\gamma_0(t-t^*_i)} \| \hat{\theta}(t^*_i) \| + \rho^{-1} \| d \| \).

The correctness of Corollary follows from the fact that it is a combination of Theorem and Proposition 4 statements.

Thus, if Assumption 1 requirements are met, the obtained identifier provides exponential convergence of the parameter error for each element of the set (4) with the rate, which is adjusted with the help of \( \gamma_0 \). In this case, the value of the convergence rate does not depend on the level of the regressor excitation. So, when the value of the parameter \( \sigma \) tends to infinity, it is possible to improve the quality of the transients of the estimator, which are calculated using (22).

**Remark 4:** The operation of division by \( \Omega^2 \) (which is used in (22)), is "safe" one as, if Assumption 1 requirements are met, then, in accordance with Proposition 3, \( \forall t \in [t^*_i + T_1; t^*_i + T_{i+1}] \quad \Omega(t) \geq \Omega_{LB} > 0 \). Also, only one switch of the nonlinear operator (22) happens over the time range \( [t^*_i + T_1; t^*_i + T_{i+1}] \), particularly, at the time point \( t^*_i + T_1 \). Such division by \( \Omega^2 \) allows one to provide the desired rate (defined by \( \gamma_0 \)) of the exponential convergence of the identification error \( \hat{\theta}(t) \). However, the value of the parameter \( \rho \) is required to be chosen in accordance with \( \rho \in (0; \Omega_{LB}) \).

Actually, adaptive law (22) can be rewritten in the following well-known\(^{[12]}\) form:

\[
\hat{\theta}_{FT}(t) := \begin{cases} 
\hat{\theta}(t), & \text{if } \Omega(t) \leq \rho \\
\frac{\hat{\theta}(t)}{\Omega(t)}, & \text{otherwise}.
\end{cases}
\]

(23)

where \( \hat{\theta}_{FT}(t) \) is the finite time \( t^*_i + T_1 \) estimation of the unknown parameters, and the parameter \( \gamma_0 \) has the sense of an aperiodic filter constant.
Remark 5: Let a bounded external disturbance be added to the regression equation (2), so that \( \forall t \geq t_0^+ \ y(t) = \varphi^T(t) \theta_t + w(t). \) Then, in accordance with the results of (30) and due to boundedness of \( w(t) \) and \( \varphi(t) \), the filtration procedure (9) forms a function \( Y(t) \) with the bounded external disturbance \( W(t) \):  

\[
Y(t) = ad\{\omega(t)\} Y(t) + W(t),
\]

where \( W(t) = ad\{\omega(t)\} \int_{t}^{\hat{t}^+} e^{-\int_{t}^{\tau} \sigma ds} \varphi(\tau) w(\tau) d\tau \). \( W(\hat{t}^+) = 0. \)

In such a case, the adaptive law (22) is rewritten as:

\[
\hat{\theta}(t) = \hat{\theta}(t) = -\gamma \Omega^2(t) \hat{\theta}(t) + \gamma \Omega(t) W(t),
\]

\[
\gamma = \begin{cases} 0, & \text{if } \Omega(t) \leq \rho \\ \frac{\rho}{\Omega(t)}, & \text{otherwise} \end{cases}
\]

and instead of the exponential convergence of \( \hat{\theta}(t) \) to zero, it provides \( \forall t \in [t_i^+ + T_i; t_{i+1}^+] \) the exponential convergence of \( \hat{\theta}(t) \) to a compact set:

\[
\lim_{t \to \infty} \|\hat{\theta}(t)\| \leq \lim_{t \to \infty} \|W(t)\| \leq \lim_{t \to \infty} \|\Omega(t)\| \leq \frac{1}{\gamma_0}
\]

The right-hand side of (26) can be made lower if an appropriate value of the parameter \( \sigma \) is chosen. The transients quality of \( \hat{\theta}(t) \) and \( \hat{\theta}_k(t) \) can be improved by reduction of the convergence rate (considering the interpretation (23), it means that the time constant \( \frac{1}{\gamma_0} \) of the aperiodic filter is improved). It should be noted that in case of external disturbance \( w(t) \), the law (22) does not ensure the monotonicity of the transients of \( \hat{\theta}_k(t) \), and the choice of high values of the parameter \( \sigma \) is undesirable, since in such case the right-hand side value of (26) may increase significantly.

4 | NUMERICAL EXPERIMENTS

To prove the efficiency of the developed procedure of piecewise constant unknown parameters identification, in this section the results of numerical experiments are presented. They were conducted in Matlab/Simulink using numerical integration with the help of the Euler method with the constant step size \( \tau = 10^{-4} \) seconds.

Section 4.1 presents the results of parameters identification of the regression equation (2) with and without the external disturbance. Section 4.2 is to show the results of identification of switched system parameters.

4.1 | Simple Example

4.1.1 | Noise Free Scenario

The regression (2) was chosen as:

\[
\forall t \geq 0, \ y(t) = \left[ \begin{array}{c} 1 \\ e^{-t} \end{array} \right] \Theta_\kappa(t), \ \kappa(t) \in \{1, 2\}, \\
\Theta_1 = \left[ \begin{array}{c} -2 \\ 1 \end{array} \right]^T, \ \Theta_2 = \left[ \begin{array}{c} -4 \\ 2 \end{array} \right]^T,
\]

where \( \kappa(t) \) was to define the following switching sequence:

\[
\Sigma = \{(1, 0), (2, 0.5), (1, 1)\}, \\
\Theta_\kappa(t) = \begin{cases} \Theta_1, & \forall t \in [0; 0.5), \\
\Theta_1 = \Theta_1, & \forall t \geq 1, \\
\Theta_2, \forall t \in [0.5; 1), \\
\Theta_2 = \Theta_1 \end{cases}.
\]

The regression (27), (28) met all requirements of Assumption 1. As we considered the disturbance free scenario, then the algorithm (16) was used to identify the elements of \( \Sigma \). The parameters of filters (9), estimation algorithm (16) and adaptive law
were chosen as:

\[ \sigma = 5, \ \Delta_{pr} = 0.1, \ \rho = 10^{-19}, \ \gamma_0 = 10. \]  \tag{29}

Figure 2 demonstrates the obtained transients of the regressor \( \Omega(t) \) and residual \( e(t) \).

\[ \begin{array}{c}
\begin{array}{c}
\Omega(t) \\
\times 10^{-4}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
e(t) \\
\times 10^{-4}
\end{array}
\end{array} \]

\[ \begin{array}{c}
0 \quad 0.5 \quad 1 \quad 1.5
\end{array} \]

\[ \begin{array}{c}
t, [s.]
\end{array} \]

\[ \begin{array}{c}
0 \quad 1 \quad 2
\end{array} \]

\[ \begin{array}{c}
\Omega(t) \\
e(t)
\end{array} \]

FIGURE 2 Transients of \( \Omega(t) \) and \( e(t) \)

The results, which are shown in Figure 2, corroborated the theoretical conclusions, which were made in Propositions 1 and 3. The regressor \( \Omega(t) \) was bounded from below \( \Omega(t) \geq \Omega_{LB} > 0 \) over the time range \([t_i^+ + T_i; t_i^{+1}]\), and the function \( e(t) \) was equal to zero \( \forall t \in [t_i^+; t_i^{+1}] \).

Figure 3 depicts the comparison of: 1) \( t_i^+ \) and its estimate \( \hat{t}_i^+ \), which was obtained with the help of the algorithm (16), and 2) \( \theta_i \) and its estimate \( \hat{\theta}(t) \), which was calculated using the adaptive law (22).

\[ \begin{array}{c}
\begin{array}{c}
\hat{t}_i^+ \\
\tilde{t}_i^+
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\hat{\theta}(t) \\
\theta(t)
\end{array}
\end{array} \]

\[ \begin{array}{c}
0 \quad 0.5 \quad 1 \quad 1.1 \quad 1.5
\end{array} \]

\[ \begin{array}{c}
t, [s.]
\end{array} \]

\[ \begin{array}{c}
0 \quad 0.5 \quad 1 \quad 1.1 \quad 1.5
\end{array} \]

\[ \begin{array}{c}
t, [s.]
\end{array} \]

FIGURE 3 Transients of \( \hat{t}_i^+ \) and \( \hat{\theta}(t) \)

The transients of \( \hat{t}_i^+ \) and \( \hat{\theta}(t) \) demonstrated all properties, which were theoretically shown in Proposition 2 and Theorem. The dependence of the value of the estimation error \( \tilde{t}_i^+ \) from the value of \( \Delta_{pr} \) was demonstrated. Transients of \( \hat{\theta}_k(t) \) \( k = 1, 2 \) were monotonous \( \forall t \in [t_i^+ + T_i; t_i^{+1}] \), and the error \( \tilde{\theta}(t) \) converged to zero exponentially \( \forall t \in [t_i^+ + T_i; t_i^{+1}] \).

Transients of \( \hat{t}_i^+ \) and \( \hat{\theta}(t) \) for different values of \( \Delta_{pr} \) and \( \sigma \) respectively are depicted in Figure 4.

The obtained results shown in Fig. 4 confirmed that the error \( \tilde{t}_i^+ \) value could be adjusted by the choice of the parameter \( \Delta_{pr} \). Also, they proved that the estimates of elements of \( \hat{\theta}(t) \) would be monotonous almost everywhere in case \( \sigma \to \infty \) as it was noted in Corollary.

Thus, the numerical experiments confirmed all the theoretically stated properties of the estimation algorithm (16) and the adaptive law (22). The obtained transients of \( \hat{t}_i^+ \) and \( \hat{\theta}(t) \) indicated that all goals from (7) were achieved.
4.1.2 | Simple Noised Scenario

Then the estimation algorithm (19) and the adaptive law (22) were tested in the presence of an external disturbance, which was caused by the measurement noise added to the regression equation (2).

The regression (2) was chosen as:

$$\forall t \geq 0, \ y(t) = \varphi^T(t) \Theta_{\kappa(t)} + w(t), \ k(t) \in \{1, 2\},$$

where the upper bound of $w(t)$ was considered to be known $w_{\text{max}} = 0.15$. The signal $k(t)$ was used to define the switching sequence (28).

The estimation algorithm (19) was used to identify the elements of the set $\mathcal{I}$. The parameters of filters (9), estimation algorithm (19) and adaptive law (22) were set as follows:

$$\sigma = 25, \ \Delta_{pr} = 0.01, \ \rho = 2.5 \cdot 10^{-8}, \ y_0 = 10,$n$$

$$c(t) = E \left\{ w_{\text{max}} \varphi(t) \varphi^T(t) d\int \{\omega(t)\} \int_{t_i}^{t} e^{-\int_{t_i}^{\tau} \varphi(\tau)d\tau} \right\}. \quad (31)$$

Figure 5 is to show the comparison of a) $\varphi^T(t) \Theta_{\kappa(t)}$ and the function $y(t)$; b) $E\{e(t)\}$ and the function $0.9\sqrt{\text{var}\{e(t)\}} + c(t)$. The results shown in Figure 5a demonstrated how the perturbation $w(t)$ affected the regression $\varphi^T(t) \Theta_{\kappa(t)}$. Figure 5b proves that the robust algorithm (19) accurately detected the time instants when the unknown parameters switched to their new values.

Figure 6 is to compare a) $t_i^+$ and the estimates $\hat{t}_i^+$ obtained using the algorithm (19), and b) the parameters $\hat{\theta}_{i}$ and estimates $\hat{\theta}(t)$ obtained with the help of the adaptive law (22).

The results shown in Figure 6 confirmed that the robust algorithm (19) detected the elements of the set $\mathcal{I}$ in the presence of an external disturbance. The transients of $\hat{\theta}(t)$ demonstrated exponential convergence of the parameter error $\hat{\theta}(t)$ to the compact set, which was described in Remark 5. At the same time, the monotonicity of the transients was retained in the presence of this type of perturbation.

Then the estimation algorithm (19) and the adaptive law (22) were tested in the presence of a disturbance, which was caused by both the measurement noise and a harmonic function added to the regression equation (2). The regression and parameters were chosen as:

$$\forall t \geq 0, \ y(t) = \varphi^T(t) \Theta_{\kappa(t)} + w(t), \ k(t) \in \{1, 2\},$$

where the upper bound of $w(t)$ was considered to be known $w_{\text{max}} = 0.25$. The signal $k(t)$ was used to define the switching sequence (28).

The estimation algorithm (19) was used to detect the elements of the set $\mathcal{I}$. The parameters of filters (9), estimation algorithm (19) and adaptive law (22) were set in accordance with (31).

Figure 7 demonstrates the comparison of (a) $\varphi^T(t) \Theta_{\kappa(t)}$ and the function $y(t)$; (b) $E\{e(t)\}$ and the function $0.9\sqrt{\text{var}\{e(t)\}} + c(t)$. 

![FIGURE 4 Transient curves of $\hat{t}_i^+ (\Delta_{pr})$ and $\hat{\theta}(\sigma)$](image-url)
The results shown in Figure 7a demonstrates how the disturbance $w(t)$ affected the regression $\varphi^T(t) \Theta_{\kappa(t)}$. Figure 7b proves correctness of the conclusions made in the propositions for the robust algorithm (19).

Figure 8 shows the comparison of a) $t_i^+$ and the estimate $\hat{t}_i^+$, b) $\|\hat{\theta}(t)\|$ and its asymptotic upper bound (UB), which was calculated using (26).

The results shown in Figure 8 confirmed that the robust algorithm (19) detected the elements of the set (6) in case of a bounded disturbance. The transients of $\|\hat{\theta}(t)\|$ converged exponentially to the compact set (26) in full accordance with Remark 5. The value of UB over the intervals $[0.5; 1)$ and $[1; 1.5)$ was substantially higher than the one over the time range $[0; 0.5)$, because the level of the regressor $\varphi(t)$ excitation was vanishing with time and, as a consequence, the denominator of (26) significantly decreased.

The transients of the estimates $\hat{\theta}(t)$, which were obtained using different values of $\sigma$, are shown in Figure 9.
Glushchenko A. et al.

**FIGURE 7** Comparison of (a) $\varphi^T(t) \Theta_{x(t)}$ and $y(t)$; (b) $E\{e(t)\}$ and $0.9\sqrt{\text{var}\{e(t)\}} + c(t)$

**FIGURE 8** Transients of $\hat{t}_i^+$ and $\|\hat{\theta}(t)\|$.

**FIGURE 9** Transients of $\hat{\theta}(t)$ for different values of $\sigma$.

It follows from Fig. 9 that, in the presence of an external disturbance $w(t)$, it is possible to improve the transient quality of the estimates $\hat{\theta}(t)$ by choice of the filter (9) parameter $\sigma$.

Thus, the conducted experiments demonstrated the performance and efficiency of the developed estimation algorithm (19) and adaptive law (22) when the regression equation (2) is subjected to external bounded disturbances.
4.2  Switched System Identification

In this subsection it is shown how the developed procedure can be applied to identify Switched System parameters.

The following plant is considered:

\[
\forall t \geq t_0^+, \quad \dot{x}(t) = \bigotimes_{i}^{T} I(t) \Phi(t) = A_{k(t)}x(t) + B_{k(t)}u(t), \quad x(t_0^+) = x_0, \quad \Phi(t) = \left[ x(t) \quad u(t) \right]^T, \quad \bigotimes_{i}^{T} I(t) = \left[ A_{k(t)} \quad B_{k(t)} \right],
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector with the initial condition \( x_0, u(t) \in \mathbb{R}^m \) is the control vector, \( A_{k(t)} \in \mathbb{R}^{nxn} \) is the unknown state matrix, \( B_{k(t)} \in \mathbb{R}^{nxm} \) is the unknown input matrix, \( \kappa(t) \in \Xi = \{1, 2, \ldots, N\} \) is the unknown discrete function, which defines the switching time instants. The pair \( (A_{k(t)}, B_{k(t)}) \) is considered to be controllable, the vector \( \Phi(t) \in \mathbb{R}^{nxm} \) is measurable \( \forall t > t_0^+ \), and the parameter matrix \( \bigotimes_{i}^{T} I(t) \in \mathbb{R}^{nx(n+m)} \) is unknown \( \forall t \geq t_0^+ \).

It is assumed that the control signal \( u(t) \) for (33) is formed by the following law:

\[
u(t) = K_x x(t) + K_r r(t),
\]

where \( K_x \in \mathbb{R}^{nxn} \) is the matrix of feedback parameters, \( K_r \in \mathbb{R}^{nxm} \) is the matrix of feedforward parameters, \( r(t) \in \mathbb{R}^m \) is the reference signal.

The derivative of the state vector \( \dot{x}(t) \) is considered to be unknown. So, to represent the plant (33) as a linear regression (2) with measurable function \( y(t) \), the filtration procedure on the base of a stable filter is applied to (33):

\[
y(t) = x(t) - l \Phi(t) = \varphi(t) + \epsilon(r(t)), \quad \varphi(t) = \left[ \Theta^T_{k(t)} e^{-lt}, \Theta^T_{k(t)} x(t) \right]^T, \quad \Theta^T_{k(t)} = \left[ A_{k(t)} \quad B_{k(t)} \quad x(t_0^+) \right],
\]

where \( l > 0 \) is the filter constant, \( \Phi(t) \in \mathbb{R}^{m+n} \) is the filtered regressor, \( x(t) \in \mathbb{R}^n \) is an element of the vector \( \Phi(t), \Theta^T_{k(t)} \in \mathbb{R}^{nx(n+m+1)} \) is the extended vector of unknown parameters, \( \varphi(t) \in \mathbb{R}^{m+n+1} \) is the extended regressor vector, \( \epsilon(r(t)) \in \mathbb{R}^{nx1} \) is the disturbance, which is caused by the plant (33) parameters switch and inertial characteristics of the filter from (35). According to the recommendations given in Remark 4, as far as the parameterization (35) is considered, the filter is reset at time points \( t_1^+ \). More details on how to obtain (35) from (33) can be found in [10].

We assumed that \( \kappa(t) \in \Xi = \{1, 2\} \) and the matrices \( A_{k(t)}, B_{k(t)} \) and \( x(t_0^+) \) defined as follows:

\[
A_1 = \begin{bmatrix} 0 & 1 \\ -6 & -8 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix},
\]

and \( \kappa(t) \) was to represent the following switching sequence:

\[
\Sigma = \{(1, 0), (2, 5), (1, 10)\}.
\]

The disturbance \( \epsilon(r(t)) \) was caused by the plant (33) parameters switch and \( \forall t \in [t_1^+; t_{1+1}^+] \) \( \epsilon(r(t)) = 0 \), whereas \( \forall t \in [t_1^+; t_{1+1}^+] \) \( \epsilon(r(t)) \neq 0 \). It is easy to prove the correctness of the above equations by analogy with the proof of Propositions 1 and 3. Therefore, in spite of the disturbance \( \epsilon(r(t)) \) in (35), the estimation algorithm (16) was applied to identify the sequence \( \Sigma \) in the course of the experiment.

The parameters of the control law (34), filters (9), (35), estimation algorithm (16), and adaptive law (19) were chosen as follows:

\[
K_x = [ -5 \quad -4 ]^T, \quad K_r = 8, \quad r(t) = 1, \quad \sigma = 5, \quad \Delta_{pr} = 0.1, \quad \rho = 10^{-17}, \quad \gamma_0 = 10.
\]

Figure 10 presents the comparison of \( t^+_1 \) and estimate \( \hat{t}^+_1 \) obtained with the help of algorithm (16), as well as comparison of parameters \( \theta \) and estimates \( \hat{\theta}(t) \) calculated with the help of the adaptive law (22).

The shown transients demonstrates that the goal (7) was achieved, and the estimation algorithm (16) and the adaptive law (22) could be used to identify the parameters of the Switched System (33).

Thus, the experiments have demonstrated that the estimation algorithm (16) or (19) and adaptive law (22) can be applied to solve various identification and adaptive control problems, which can be reduced to the identification problem of unknown piecewise constant parameters of the regression equation (2).
5 | CONCLUSION

A procedure to identify piecewise constant unknown parameters of a linear regression equation has been proposed, which provides: 1) elementwise monotonicity of the parameter error almost everywhere in the disturbance free case, 2) interval exponential convergence of the unknown parameters estimates to their true values if the regressor finite excitation condition is met, and 3) adjustable accuracy of estimates of the switching time instants. Robustness of the proposed identification procedure to external disturbances has been analytically proved. Recommendations on the choice of the arbitrary parameters of the procedure are given to improve the quality of transients of the unknown parameters estimates for both cases: with and without external disturbances.

All theoretically proved properties of the developed identification procedure are experimentally confirmed. Particularly, the procedure has been applied to estimate the parameters of the Switched System.

The scope of future research is to use the proposed procedure to develop adaptive control systems for Switched Systems.

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Conflict of interest

The authors declare no potential conflict of interests.

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APPENDIX

A PROOF OF PROPOSITION 1

As \( \kappa (t) = j \), holds \( \forall t \in [\hat{t}^+_i; t^+_{i+1}) \), when \( \hat{t}^+_i \geq t^+_i \), then the unknown parameters of the regressions (8) and (11) coincide to each other \( \forall t \in [\hat{t}^+_i; t^+_{i+1}) \). In its turn, according to (9), it is obtained:

\[
\forall t \in [\hat{t}^+_i; t^+_{i+1}) \), \( Y(t) = \int_{\hat{t}^+_i}^{t} e^{-\int_{\hat{t}^+_i}^{\tau} \sigma d\tau} \varphi(\tau) y(\tau) d\tau = \int_{\hat{t}^+_i}^{t} e^{-\int_{\hat{t}^+_i}^{\tau} \sigma d\tau} \varphi(\tau) \varphi^T(\tau) d\tau \theta_i = \omega(t) \theta_i. \quad (A1)
\]
The function \( \mathbf{(A1)} \) is substituted into \((14)\) to obtain:
\[
\forall t \in [t_i^*; t_{r_i}^*], \quad e(t) = \varphi(t) \varphi^T(t) \text{adj} j \{\omega(t)\} \omega(t) \theta_i - \Omega(t) \varphi(t) \varphi^T(t) \theta_i = 0,
\]
(A2)
as it was to be proved for the first time interval.

So, the next time range is considered: \([t_i^*; t_{r_i}^*]\). The equation for the function \( Y(t) \) from \((9)\) is rewritten in the following form:
\[
\forall t \in [t_i^*; t_{r_i}^*], \quad Y(t) = \int_{t_{r_i}^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) y(\tau) d\tau = \int_{t_{r_i}^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \theta_{i-1} +
\]
\[
+ \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \theta_i.
\]
(A3)

Then \((A3)\) is substituted into \((14)\) to obtain:
\[
\forall t \in [t_i^*; t_{r_i}^*], \quad e(t) = \varphi(t) \varphi^T(t) \text{adj} j \{\omega(t)\} \int_{t_{r_i}^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \theta_{i-1} +
\]
\[
+ \varphi(t) \varphi^T(t) \text{adj} j \{\omega(t)\} \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \pm \Omega(t) \varphi(t) y(t) \pm
\]
\[
\int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \theta_i =
\]
\[
= \varphi(t) \varphi^T(t) \text{adj} j \{\omega(t)\} \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \left(\theta_{i-1} - \theta_i\right),
\]
(A4)
as it was to be proved for the second time interval.

The combination of \((A2)\) and \((A4)\) is the equation \((15)\), which completes the proof of Proposition 1.

**B PROOF OF PROPOSITION 2**

Following Proposition 2 statement and proof of Proposition 1, it is written:
\[
\forall t \geq t_i^*, \quad \|e(t)\| = \left\| \varphi(t) \varphi^T(t) \text{adj} j \{\omega(t)\} \int_{t_{r_i}^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \left(\theta_{i-1} - \theta_i\right) \right\| 
\]
\[
\leq \left\| \varphi(t) \varphi^T(t) \right\| \cdot \|\text{adj} j \{\omega(t)\}\| \cdot \left\| \int_{t_{r_i}^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau \right\| \cdot \left\| \theta_{i-1} - \theta_i \right\|. 
\]
(B5)

According to Problem Statement section, Assumption 1 and Definition 1 (eq. \(1\)), the lower bounds for the second and third multipliers of \((B5)\) are obtained:
\[
\int_{t_{r_i}^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau =
\]
\[
\int_{t_{i-1}^*+T_{i-1}}^{t_{r_i}^*+T_{i-1}} e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau + \int_{t_{i-1}^*+T_{i-1}}^{t_{r_i}^*} e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) d\tau 
\]
\[
\geq \int_{t_{i-1}^*+T_{i-1}}^{t_{r_i}^*+T_{i-1}} e^{-\sigma(-t_{i-1}^*+T_{i-1})} \varphi(\tau) \varphi^T(\tau) d\tau \geq e^{-\sigma(-t_{i-1}^*+T_{i-1})} \int_{t_{i-1}^*+T_{i-1}}^{t_{r_i}^*} \varphi(\tau) \varphi^T(\tau) d\tau 
\]
\[
\geq \overline{a}_{i-1} e^{-\sigma(-t_{i-1}^*+T_{i-1})} I > 0
\]
(B6)
\( \text{adj} \{ \omega(t) \} = \text{adj} \left\{ \int_{t_{i}^-}^{t_{i}^+ + T_{i}} e^{-\int_{\hat{t}_{i}^-}^{t} s ds} \varphi(t) \varphi^T(t) \, d\tau \right\} = \]
\[= \text{adj} \left\{ \int_{t_{i}^-}^{t_{i}^+ + T_{i}} e^{-\int_{\hat{t}_{i}^-}^{t} s ds} \varphi(t) \varphi^T(t) \, d\tau + \int_{t_{i}^+ + T_{i}}^{t_{i}^+ + T_{i} + T_{i}} e^{-\int_{\hat{t}_{i}^-}^{t} s ds} \varphi(t) \varphi^T(t) \, d\tau \right\} \geq \]
\[\geq \text{adj} \left\{ \int_{t_{i}^+ + T_{i}}^{t_{i}^+ + T_{i} + T_{i}} e^{-\int_{\hat{t}_{i}^-}^{t} s ds} \varphi(t) \varphi^T(t) \, d\tau \right\} \geq \text{adj} \left\{ \overline{a}_{i-1} e^{-\sigma(-\hat{t}_{i}^- + T_{i})} \right\} > 0 \]  
(B7)

As \( \varphi(t) \in F_{E} \) over the time range \([t_{i}^+; t_{i}^+ + T_{i}]\), then \( \exists \left[t_{a_i}; t_{b_i}\right] \subset [t_{i}^+; t_{i}^+ + T_{i}] \) such that:
\[\forall t \in \left[t_{a_i}; t_{b_i}\right] \quad \| \varphi(t) \varphi^T(t) \| > 0 \]  
(B8)

The equations (B6), (B7), (B8) are substituted into (B5) to obtain \( \forall t \in \left[t_{a_i}; t_{b_i}\right] \subset [t_{i}^+; t_{i}^+ + T_{i}] \quad \| \epsilon(t) \| > 0 \). Then, according to the algorithm (16), for a general case it is obtained that \( t_{a_{up}} = t_{c} \in \left[t_{a_i}; t_{b_i}\right] \) and \( \hat{t}_{i}^+ = t_{c} + \Delta_{pr} \). As \( \hat{t}_{i}^+ = t_{i}^+ + \Delta_{pr} \), then:
\[\hat{t}_{i}^+ = t_{c} + \Delta_{pr} - t_{i}^+ \]  
(B9)

Taking into consideration \( t_{c} \in \left[t_{a_i}; t_{b_i}\right] \subset [t_{i}^+; t_{i}^+ + T_{i}] \), the following is obtained from (B9):
\[\Delta_{pr} \leq \hat{t}_{i}^+ \leq T_{i} + \Delta_{pr} \]  
(B10)

It is concluded from (B10) that the fact that \( \hat{t}_{i}^+ \geq t_{i}^+ \) and appropriate choice of \( \Delta_{pr} \) value ensure achievement of the first goal from (7) \( \hat{t}_{i}^+ \leq T_{i} \).

**C PROOF OF PROPOSITION 3**

As \( \kappa (t) = j_{i} \) holds \( \forall t \in [\hat{t}_{i}^+; t_{i}^+ + T_{i}] \) when \( \hat{t}_{i}^+ \geq t_{i}^+ \), then, using (A1), it is written:
\[\forall t \in [\hat{t}_{i}^+; t_{i}^+ + T_{i}], \quad \gamma(t) : = \text{adj} \{ \omega(t) \} \int_{\hat{t}_{i}^+}^{t} e^{-\int_{\hat{t}_{i}^-}^{\tau} s ds} \varphi(\tau) y(\tau) \, d\tau = \]
\[= \text{adj} \{ \omega(t) \} \int_{\hat{t}_{i}^-}^{t} e^{-\int_{\hat{t}_{i}^-}^{\tau} s ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \theta_{i} = \Omega(t) \theta_{i}. \]  
(C11)

It is concluded from (C11) that, when \( \forall t \in [\hat{t}_{i}^+; t_{i}^+ + T_{i}] \), then \( d(t) = 0 \), as it was to be proved for the first time interval.

Then the second time range is considered: \([t_{i}^+; \hat{t}_{i}^+ + T_{i}]\). The equation (A3) is substituted into (10):
\[\forall t \in [t_{i}^+; \hat{t}_{i}^+ + T_{i}], \quad \gamma(t) : = \text{adj} \{ \omega(t) \} \int_{t_{i}^+}^{\hat{t}_{i}^+} e^{-\int_{t_{i}^-}^{\tau} s ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \theta_{i-1} + \]
\[+ \text{adj} \{ \omega(t) \} \int_{t_{i}^+}^{\hat{t}_{i}^+} e^{-\int_{t_{i}^-}^{\tau} s ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \theta_{i} + \text{adj} \{ \omega(t) \} \int_{\hat{t}_{i}^-}^{t_{i}^+} e^{-\int_{\hat{t}_{i}^-}^{\tau} s ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \theta_{i} = \]
\[= \Omega(t) \theta_{i} + \text{adj} \{ \omega(t) \} \int_{t_{i}^+}^{\hat{t}_{i}^+} e^{-\int_{t_{i}^-}^{\tau} s ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \left( \theta_{i-1} - \theta_{i} \right). \]  
(C12)

It follows from (C12) that \( \forall t \in [t_{i}^+; \hat{t}_{i}^+ + T_{i}], \quad d(t) = \text{adj} \{ \omega(t) \} \int_{t_{i}^+}^{\hat{t}_{i}^+} e^{-\int_{t_{i}^-}^{\tau} s ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \left( \theta_{i-1} - \theta_{i} \right), \) as was to be proved for the second time range.

The combination of (C11) and (C12) is (21).
The following notations are introduced to obtain the upper bound of the disturbance \( d(t) \), which is the next aim of the proof:

\[
\forall t \in [t_i^*; t_f^*], \quad \omega(t) = \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau = \omega_1(t) + \omega_2(t),
\]

(C13)

\[
\omega_1(t) = \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau, \quad \omega_2(t) = \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau.
\]

The following estimates are correct for the exponentially decaying multipliers of the integrands from (C13):

\[
\forall \tau \in [t_i^*; t_f^*] : e^{-\sigma(t_f^* - t_i^*)} \leq e^{-\int_0^s \sigma ds} \leq 1,
\]

(C14)

Considering \( \varphi(t) \in L_\infty \), (C14) and mean value theorem, the upper bounds of \( \omega_1(t) \) and \( \omega_2(t) \) are obtained:

\[
\omega_1(t) = \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \leq \int_{t_i^*}^t \varphi(\tau) \varphi^T(\tau) \, d\tau \leq \bar{\bar{\delta}}_1 (t_f^* - t_i^*) I,
\]

(C15)

\[
\omega_2(t) = \int_{t_i^*}^t e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \leq e^{-\sigma(t_f^* - t_i^*)} \int_{t_i^*}^t \varphi(\tau) \varphi^T(\tau) \, d\tau \leq e^{-\sigma(t_f^* - t_i^*)} \bar{\bar{\delta}}_2 (t_f^* - t_i^*) I.
\]

The equality \( ad_j \{ A + B \} = ad_j \{ A \} + ad_j \{ B \} \forall A, B \) is applied to the definition of \( d(t) \). As a result:

\[
d(t) = ad_j \{ \omega(t) \} \omega_1 (\theta_{t_i^*} - \theta_t) = (ad_j \{ \omega_1(t) \} \omega_1(t) + ad_j \{ \omega_2(t) \} \omega_1(t)) \Delta \theta.
\]

The upper bounds from (C15) are substituted into (C16). Then \( ad_j \{ c I_n \} = c \cdot ad_j \{ I_n \} \), \( det \{ c I_n \} = c^n \cdot det \{ I_n \} \) \forall c are applied to the obtained result. Finally, we obtain:

\[
\| d(t) \| \leq \left( \bar{\bar{\delta}}_1 (t_f^* - t_i^*) a_n + \bar{\bar{\delta}}_2 (t_f^* - t_i^*) (t_f^* - t_i^*) e^{-\sigma(t_f^* - t_i^*)} \right) \Delta \theta.
\]

(C17)

It follows from (C17) that the disturbance \( d(t) \) is bounded.

The final aim is to analyze the regressor \( \Omega(t) \) properties. The first step is, considering \( \Omega(t) = det \{ \omega(t) \} \), to obtain the regressor \( \omega(t) \) lower bound.

To achieve this, the bounds of the exponentially decaying multiplier of the integrand from the definition of \( \omega(t) \) is written:

\[
\forall \tau \in [t_i^*; t_f^*], \quad e^{-\sigma(\theta_{t_i^*} - \theta_{t_f^*})} \leq e^{-\int_0^s \sigma ds} \leq 1
\]

(C18)

Considering (C18), the mean value theorem and the third requirement of Assumption 1, the lower bound of the regressor \( \omega(t) \) is obtained:

\[
\forall t \geq t_i^* + T_i, \quad \omega(t) = \int_{t_i^*}^{t_f^* + T_i} e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau = \int_{t_i^*}^{t_f^* + T_i} e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau + \int_{t_i^*}^{t_f^* + T_i} e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \geq \int_{t_i^*}^{t_f^* + T_i} e^{-\int_0^s \sigma ds} \varphi(\tau) \varphi^T(\tau) \, d\tau \geq \Omega(t_f^* + T_i) \geq \Omega(t_f^* - T_i) e^{-\sigma(\theta_{t_i^*} - \theta_{t_f^*})} I > 0
\]

(C19)
Now this result is used to obtain bounds of the regressor $\Omega(t)$. As $\det \{ e^{t_n \alpha_n} \} = e^n \cdot \det \{ I_n \alpha_n \} \ \forall c$, then the lower bound of the regressor $\Omega(t)$ is:

$$\forall t \geq t^+_i + T_i, \ \Omega(t) \geq \left( \hat{a}_i e^{-\sigma(t^- + T_i)} \right)^n \geq \min_{\hat{a}_i \in \Omega_{LB}} \left( \hat{a}_i e^{-\sigma(t^- + T_i)} \right)^n > 0,$$

which completes the proof of Proposition 3.

**D PROOF OF THEOREM**

Proving the statement (a) of Theorem, the adaptive law \[22\] is written in the element-wise form:

$$\hat{\theta}_k(t) = -\gamma \Omega^2(t) \hat{\theta}_k (t). \quad \text{(D21)}$$

Considering the definition of $\gamma$ and the fact that $\rho \in (0; \Omega_{LB}]$, the solution of (D21) is written as:

$$\hat{\theta}_k (t) = e^{-\gamma_0(t^- + T_i)} \hat{\theta}_k (t_i) + \chi \left( \frac{d}{dt} \Omega(t) \right). \quad \text{(D22)}$$

As $\gamma_0 > 0$, it follows from (D22) that

$$\forall t \geq t^+_i + T_i, \ \hat{\theta}_k(t) \leq \left| \hat{\theta}_k(t_i) \right| \forall t_a, \ t_b 
\in \left[ t^+_i + T_i; t^+_i + T_i \right], \ t_a \geq t_b, \ \text{as was to be proved in the statement (a) of Theorem.}$$

Two time ranges $[t^+_i; t^+_i]$ and $[t^+_i; t^+_i + T_i]$ are considered one after another to prove the statement (b) of Theorem. The quadratic function is introduced to analyze the local properties of the parameter error $\hat{\theta}(t)$ over the time range $[t^+_i; t^+_i + T_i]$

$$V = (\hat{\theta} - \hat{\theta}_i)^T (\hat{\theta} - \hat{\theta}_i) = \hat{\theta}^T \hat{\theta}. \quad \text{(D23)}$$

The derivative of (D23) is written as:

$$V = 2 \hat{\theta}^T \dot{\hat{\theta}} = 2 \hat{\theta}^T (\gamma \Omega^2 \hat{\theta} + \gamma \Omega d) = -2 \gamma \Omega^2 \hat{\theta} + 2 \gamma \Omega^2 d. \quad \text{(D24)}$$

Considering the definition of $\gamma$ and the inequalities $-a^2 + ab \leq -\frac{1}{2}a^2 + \frac{1}{2}b^2$ and $\forall t \geq t^+_i, \ \Omega(t) \geq \rho > 0$, the upper bound of (D24) over the time range $[t^+_i; t^+_i + T_i]$ takes the form:

$$\forall t \in [t^+_i; t^+_i + T_i] \quad \dot{V} \leq -2 \gamma_0 \left\| \hat{\theta} \right\|^2 + 2 \gamma_0 \rho^{-1} \left\| \hat{\theta} \right\| d \leq -\gamma_0 \left\| \hat{\theta} \right\|^2 + \gamma_0 \rho^{-1} \left\| d \right\|^2. \quad \text{(D25)}$$

The solution of the differential equation (D25) is obtained as:

$$\forall t \in [t^+_i; t^+_i] \quad \left\| \hat{\theta}(t) \right\| \leq e^{-\gamma_0(t^- + T_i)} \left\| \hat{\theta}(t_i^+) \right\| + \rho^{-1} \left\| d \right\|. \quad \text{(D26)}$$

Then the second time interval is considered. Taking into account the most conservative case, the properties of the regressor $\Omega(t)$ allows to make a conclusion that $\Omega(t) \equiv 0$ over the time interval $[t^+_i; t^+_i + T_i]$. This means that $\dot{V} = 0$. In its turn, as a consequence, the following inequality holds:

$$\forall t \in [t^+_i; t^+_i + T_i] \quad \left\| \hat{\theta}(t) \right\| \leq \left\| \hat{\theta}(t_i^+) \right\| + \rho^{-1} \left\| d \right\|. \quad \text{(D27)}$$

which completes the proof of the statement (b) of Theorem.

Then the time range $[t^+_i + T_i; t^+_i + T_i]$ is considered. Using the fact that $\rho \in (0; \Omega_{LB}]$ and the properties of the regressor $\Omega(t)$, the solution of equation (D22) over the time interval $[t^+_i + T_i; t^+_i + T_i]$ takes the form:

$$\forall t \in [t^+_i + T_i; t^+_i + T_i] \quad \hat{\theta}(t) = e^{-\gamma_0(t^- + T_i)} \hat{\theta}(t_i^+) + \chi \left( \frac{d}{dt} \Omega(t) \right). \quad \text{(D28)}$$

The identification error bound from (D27) is substituted into (D28) to obtain:

$$\forall t \in [t^+_i + T_i; t^+_i + T_i] \quad \left\| \hat{\theta}(t) \right\| \leq e^{-\gamma_0(t^- + T_i)} \left( \left\| \hat{\theta}(t_i^+) \right\| + \rho^{-1} \left\| d \right\| \right). \quad \text{(D29)}$$

which completes the proof of the statement (c) of Theorem.

To prove the statement (d) of Theorem, first of all, the upper bound of $\hat{\theta}_i$ is written on the basis of (D29):

$$\left\| \hat{\theta}_i \right\| = \left\| \hat{\theta}(t_i^+) \right\| \leq e^{-\gamma_0(t_i^+ - T_i)} \left( \left\| \hat{\theta}(t_i^+) \right\| + \rho^{-1} \left\| d \right\| \right) \leq e^{-\gamma_0(t_i^+ - T_i)} \left( \left\| \hat{\theta}(t_i^+) \right\| + \rho^{-1} \left\| d \right\| \right). \quad \text{(D30)}$$
The following upper bound for (D30) is introduced:
\[
\|\tilde{\vartheta}\| = \|\tilde{\vartheta} (t^*_i) - \theta_i\| \leq e^{-\gamma_0\Delta T} \left( \|\tilde{\vartheta} (t^*_i)\| + \rho^{-1} \|d\| \right),
\]
(D31)

Taking into consideration:
\[
\|\tilde{\vartheta} (t^*_i)\| = \|\tilde{\vartheta} (t^*_i) - \theta_i\| = \|\tilde{\vartheta} (t^*_i) + \theta_{i-1} - \theta_i\| \leq \|\tilde{\vartheta} (t^*_i) - \theta_{i-1}\| + \Delta_\theta,
\]
(D32)

the equation (D31) is rewritten as:
\[
\|\tilde{\vartheta} (t^*_i)\| = e^{-\gamma_0\Delta T} \|\tilde{\vartheta} (t^*_i) - \theta_{i-1}\| + \Delta_\theta + \rho^{-1} \|d\|.
\]
(D33)

The inequalities (D32) and (D33) are applied to (D30) recursively \(i_{\text{max}}\) times. Then the equation of sum of a geometric progression is used to transform the obtained result. Finally, the following is written:
\[
\|\tilde{\vartheta}\| \leq e^{-\gamma_0\Delta T} \|\tilde{\vartheta} (t^*_i) - \theta_{i-1}\| + e^{-\gamma_0\Delta T} (\Delta_\theta + \rho^{-1} \|d\|) \leq \\
\leq e^{-2\gamma_0\Delta T} \left( \|\tilde{\vartheta} (t^*_i) - \theta_{i-1}\| + \rho^{-1} \|d\| \right) + e^{-\gamma_0\Delta T} (\Delta_\theta + \rho^{-1} \|d\|) \leq \\
\leq e^{-2\gamma_0\Delta T} \left( \|\tilde{\vartheta} (t^*_i) - \theta_{i-1}\| + (1 + e^{-\gamma_0\Delta T}) \Delta_\theta + \rho^{-1} \|d\| \right) \leq \\
\leq e^{-3\gamma_0\Delta T} \left( \|\tilde{\vartheta} (t^*_i) - \theta_{i-1}\| + \rho^{-1} \|d\| \right) + \Delta_\theta.
\]
(D34)

It is concluded from (D34) that the following holds:
\[
\lim_{T_{\text{max}} \to \infty} \|\tilde{\vartheta}\| = 0, \quad \lim_{T_{\text{max}} \to \infty} \|\hat{\vartheta}\| = 0,
\]
(D35)

which completes the proof of the statement (d) of Theorem.

**E PROOF OF PROPOSITION 4**

First of all, to prove the proposition, the function \(Y (t)\) is represented in the following form:
\[
\forall t \in [t^*_i; \tilde{t}^*_i), \quad Y (t) = a \int_{t^*_i}^{t^*_i\gamma} e^{-\int_{t^*_i}^s \sigma ds} \varphi (\tau) \varphi^T (\tau) d\tau + a \int_{t^*_i}^{t^*_i\gamma} e^{-\int_{t^*_i}^s \sigma ds} \varphi (\tau) \varphi^T (\tau) d\tau = \Omega (t)\theta_{i-1} + a \int_{t^*_i}^{t^*_i\gamma} e^{-\int_{t^*_i}^s \sigma ds} \varphi (\tau) \varphi^T (\tau) d\tau (\theta_{i} - \theta_{i-1}) = \]
\[
\Omega (t)\theta_{i-1} + a \int_{t^*_i}^{t^*_i\gamma} e^{-\int_{t^*_i}^s \sigma ds} \varphi (\tau) \varphi^T (\tau) d\tau (\theta_{i} - \theta_{i-1}) =
\]
(E36)

Following (E36), the upper bounds of \(\omega_2 (t)\) and \(\omega (t)\) are written as:
\[
e^{-\sigma(t^*_i; \tilde{t}^*_i)} \int_{t^*_i}^{\tilde{t}^*_i} \varphi (\tau) \varphi^T (\tau) d\tau \leq \omega_2 (t) \leq e^{-\sigma(t^*_i; \tilde{t}^*_i)} \int_{t^*_i}^{\tilde{t}^*_i} \varphi (\tau) \varphi^T (\tau) d\tau,
\]
\[
e^{-\sigma(t^*_i; \tilde{t}^*_i)} \int_{t^*_i}^{\tilde{t}^*_i} \varphi (\tau) \varphi^T (\tau) d\tau + \omega_2 (t) \leq \omega (t) \leq \int_{t^*_i}^{\tilde{t}^*_i} \varphi (\tau) \varphi^T (\tau) d\tau.
\]
(E37)
The upper and lower bounds of the functions \( \omega_2(t) \) and \( \omega(t) \) are compared under the condition that \( \sigma \to \infty \):

\[
\lim_{\sigma \to \infty} \frac{e^{-(t_{j}^*-t_{j-1})} \int_{t_{j-1}}^{t_{j}} q(t) \varphi'(t) \, dt}{e^{-\sigma(t)} \int_{t_{j-1}}^{t_{j}} q(t) \varphi'(t) \, dt} = 0,
\]

\[
\lim_{\sigma \to \infty} \frac{\alpha_{2}(t)}{\alpha(t)} = \lim_{\sigma \to \infty} \frac{\alpha_{2}(t)}{\alpha(t) + o_{2}(t)} = e^{-\sigma(t)} \frac{\int_{t_{j-1}}^{t_{j}} q(t) \varphi'(t) \, dt}{\int_{t_{j-1}}^{t_{j}} q(t) \varphi'(t) \, dt} = 0,
\]

where the last limit holds because of the inequality \( t_{j}^* - \hat{t}_{j-1}^* < \hat{t}_{j}^* - \hat{t}_{j-1}^* \).

It follows from (E38) that \( \omega_2(t) = o(\omega(t)) \). Then the equation (36) can be rewritten as:

\[
\forall t \in [t_{j}^*; \hat{t}_{j}^*], \quad Y(t) = adj \{ \omega(t) \} \left( \omega(t) \vartheta_{j-1} + o(\omega(t)) \left( \vartheta_{j} - \vartheta_{j-1} \right) \right) = \Omega(t) \vartheta_{j-1}.
\]

In its turn, the differential equation with respect to the error \( \hat{\vartheta}(t) - \theta_{j-1} \) is obtained on the basis of (E39) and (22):

\[
\dot{\hat{\vartheta}}(t) = \Omega_{j}(t) \dot{\vartheta}(t) - \Omega(t) \vartheta_{j-1} = -\gamma \Omega(t) \left( \hat{\vartheta}_{j}(t) - \theta_{j-1} \right).
\]

Hence, using the reasoning (D21)-(D22) from the Theorem proof, the correctness of the statement (a) of Proposition 4 is obtained.

To prove the statement (b), a quadratic function is introduced:

\[
V = \frac{1}{2} (\hat{\vartheta} - \vartheta_{j-1})^T (\hat{\vartheta} - \vartheta_{j-1}).
\]

The derivative of (E41) with respect to (E40) is written as:

\[
\dot{V} = (\hat{\vartheta} - \vartheta_{j-1})^T \dot{\vartheta} = (\hat{\vartheta} - \vartheta_{j-1})^T (\gamma \Omega \hat{\vartheta}) = -\gamma \Omega^2 (\hat{\vartheta} - \vartheta_{j-1})^T (\hat{\vartheta} - \vartheta_{j-1}).
\]

The solution of the differential equation (E42) is obtained as:

\[
\forall t \in [t_{j}^*; \hat{t}_{j}^*], \quad \left\| \dot{\vartheta}(t) - \vartheta_{j-1} \right\| \leq e^{-0.5\gamma(t-t_{j}^*)} \left\| \hat{\vartheta}(t_{j}^*) - \vartheta_{j-1} \right\|.
\]

which completes the proof of the statement (b) of Proposition 4.

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