ITERATING THE PIMSNER CONSTRUCTION

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Abstract. For $A$ a $C^*$-algebra, $E_1, E_2$ two Hilbert bimodules over $A$, and a fixed isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$, we consider the problem of computing the $K$-theory of the Cuntz-Pimsner algebra $\mathcal{O}_{E_2 \otimes_A E_1}$ obtained by extending the scalars and by iterating the Pimsner construction.

The motivating examples are a commutative diagram of Douglas and Howe for the Toeplitz operators on the quarter plane, and the Toeplitz extensions associated by Pimsner and Voiculescu to compute the $K$-theory of a crossed product. The applications are for Hilbert bimodules arising from rank two graphs and from commuting endomorphisms of abelian $C^*$-algebras.

§0. INTRODUCTION

In his seminal paper [Pi], Pimsner introduced a large class of $C^*$-algebras generalizing both the Cuntz-Krieger algebras and the crossed products by an automorphism. The central notion is that of a Hilbert bimodule or $C^*$-correspondence, which appeared also in the theory of subfactors. His construction was modified by Katsura (see [Ka]) to include bimodules defined from graphs with sinks, or more generally, from topological graphs.

In the same way that a crossed product by $\mathbb{Z}^2$ could be thought as an iterated crossed product by $\mathbb{Z}$, we iterate the Pimsner construction for two “commuting” Hilbert bimodules over a $C^*$-algebra. A basic example comes from a rank two graph of Kumjian and Pask (see [KP]), where the two bimodules correspond to the horizontal and vertical edges, and the commutation relation is given by the unique factorization property. We get a particular case of the product systems defined by Fowler (see [F2]), but our approach has the advantage that it generates some exact sequences of $K$-theory.

After some preliminaries about the algebras $T_E$ and $\mathcal{O}_E$, we describe how from a Hilbert module $E$ over $A$ and a map $A \to B$ we can get a Hilbert module over $B$, using a tensor product. Given two Hilbert bimodules $E_1, E_2$ over $A$ and an isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$ we consider $E_2 \otimes_A T_{E_1}$ and $E_2 \otimes_A \mathcal{O}_{E_1}$ as Hilbert bimodules over $T_{E_1}$ and $\mathcal{O}_{E_1}$ and repeat the Pimsner construction. The last section deals with a $3 \times 3$ diagram involving the iterated Cuntz-Pimsner algebras, inspired from the work of Douglas and Howe (see [DH]) and Pimsner and Voiculescu (see [PV]). As a corollary, we obtain some exact sequences of $K$-theory, including information about the $K$-groups of the $C^*$-algebra $\mathcal{O}_{E_2 \otimes_A E_1}$. Several examples are considered, including commuting endomorphisms of abelian $C^*$-algebras.

Since this paper was completed, we discovered that J. Lindiarni and I. Raeburn ([LR]) already used the diagram which appears in our Lemma 4.2.
§1. PRELIMINARIES

Recall that a (right) Hilbert $A$-module is a Banach space $E$ with a right action of a $C^*$-algebra $A$ and an $A$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ linear in the second variable such that
$$\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a, \quad \langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*, \quad \langle \xi, \xi \rangle \geq 0, \quad ||\xi|| = ||\langle \xi, \xi \rangle||^{1/2}.$$ A Hilbert module is called full if the closed linear span of the inner products coincides with $A$. We denote by $\mathcal{L}(E)$ the $C^*$-algebra of adjointable operators on $E$, and by $\theta_{\xi,\eta} \in \mathcal{L}(E)$ the rank one operator
$$\theta_{\xi,\eta}(\zeta) = \langle \eta, \zeta \rangle.$$

The closed linear span of rank one operators is the ideal $\mathcal{K}(E)$. We have $\mathcal{L}(E) \cong M(\mathcal{K}(E))$, the multiplier algebra. Also, $\mathcal{K}(E)$ can be identified with the balanced tensor product $E \otimes_A E^*$, where $E^*$ is the dual of $E$, a left Hilbert $A$-module.

A Hilbert bimodule over $A$ (sometimes called a $C^*$-correspondence from $A$ to $A$) is a Hilbert $A$-module with a left action of $A$ given by a homomorphism $\varphi : A \to \mathcal{L}(E)$. A Hilbert bimodule is called faithful if $\varphi$ is injective. The left action is nondegenerate if $\overline{\varphi(A)E} = E$. For $n \geq 0$ we denote by $E^\otimes_n$ the Hilbert bimodule obtained by taking the tensor product of $n$ copies of $E$, balanced over $A$ (for $n = 0, E^\otimes_0 = A$). Recall that for $n = 2$, the inner product is given by
$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \eta, \varphi(\langle \xi, \xi' \rangle)\eta' \rangle,$$
and it is inductively defined for general $n$.

1.1 Definition. A Toeplitz representation of a Hilbert bimodule $E$ over $A$ in a $C^*$-algebra $C$ is a pair $(\tau, \pi)$ with $\tau : E \to C$ a linear map and $\pi : A \to C$ a $*$-homomorphism, such that
$$\tau(\xi a) = \tau(\xi) \pi(a), \quad \tau(\xi)^* \tau(\eta) = \pi(\langle \xi, \eta \rangle), \quad \tau(\varphi(a)\xi) = \pi(a) \tau(\xi).$$

Note that the first property actually follows from the second. Indeed,
$$||\tau(\xi a) - \tau(\xi) \pi(a)||^2 = (\tau(\xi a) - \tau(\xi) \pi(a))^* (\tau(\xi a) - \tau(\xi) \pi(a)) =$$
$$= \pi(\langle \xi a, \xi a \rangle) - \pi(a)^* \pi(\langle \xi, \xi a \rangle) - \pi(\langle \xi a, \xi \rangle) \pi(a) + \pi(a)^* \pi(\langle \xi a, \xi \rangle) \pi(a) = 0.$$

The corresponding universal $C^*$-algebra is called the Toeplitz algebra of $E$, denoted by $\mathcal{T}_E$. If $E$ is full, then $\mathcal{T}_E$ is generated by elements $\tau^n(\xi) \tau^m(\eta)^*, m, n \geq 0$, where $\tau^0 = \pi$ and for $n \geq 1, \tau^n(\xi_1 \otimes \cdots \otimes \xi_n) = \tau(\xi_1) \cdots \tau(\xi_n)$ is the extension of $\tau$ to $E^\otimes n$. If $E$ is also faithful, then $A \subset \mathcal{T}_E$.

There is a homomorphism $\psi : \mathcal{K}(E) \to C$ such that $\psi(\theta_{\xi,\eta}) = \tau(\xi) \tau(\eta)^*$. A representation $(\tau, \pi)$ is Cuntz-Pimsner covariant if $\pi(a) = \psi(\varphi(a))$ for all $a$ in the ideal
$$I_E = \varphi^{-1}(\mathcal{K}(E)) \cap (\ker \varphi)^{-1}.$$ The Cuntz-Pimsner algebra $\mathcal{O}_E$ is universal with respect to the covariant representations, and it is a quotient of $\mathcal{T}_E$.

There is a gauge action of $\mathbb{T}$ on $\mathcal{T}_E$ and $\mathcal{O}_E$ defined by
$$z \cdot (\tau^n(\xi) \tau^m(\eta)^*) = z^{n-m} \tau^n(\xi) \tau^m(\eta)^*, \quad z \in \mathbb{T},$$
and using the universal properties. The core $\mathcal{F}_E$ is the fixed point algebra $\mathcal{O}_E^\alpha$, generated by the union of the algebras $\mathcal{K}(E^{\otimes n})$.

The Toeplitz algebra $\mathcal{T}_E$ can be represented by creation operators $T_\xi(\eta) = \xi \otimes \eta$ on the Fock bimodule

$$\ell^2(E) = \bigoplus_{n \geq 0} E^{\otimes n},$$

and there is an ideal $I = \mathcal{K}(\ell^2(E)I_E)$ in $\mathcal{L}(\ell^2(E))$ such that

$$0 \to I \to \mathcal{T}_E \to \mathcal{O}_E \to 0$$

is exact. In particular, if $\mathcal{L}(E) = \mathcal{K}(E)$ or $\varphi(A) \subset \mathcal{K}(E)$, then

$$0 \to \mathcal{K}(\ell^2(E)) \to \mathcal{T}_E \to \mathcal{O}_E \to 0.$$

For more details about the algebras $\mathcal{F}_E, \mathcal{T}_E$, and $\mathcal{O}_E$ we refer to the original paper of Pimsner ([PV]) and to [Ka1].

If $E$ is finitely generated, then it has a basis $\{u_i\}$, in the sense that for all $\xi \in E$,

$$\xi = \sum_{i=1}^n u_i \langle u_i, \xi \rangle.$$

In this case, $\mathcal{L}(E) = \mathcal{K}(E)$, and the Cuntz-Pimsner algebra $\mathcal{O}_E$ is generated by $S_i = S_{u_i}$, with relations

$$\sum_{i=1}^n S_i S_i^* = 1, \quad S_i^* S_j = \langle u_i, u_j \rangle, \quad a \cdot S_j = \sum_{i=1}^n S_i \langle u_i, a \cdot u_j \rangle.$$

The Toeplitz algebra $\mathcal{T}_E$ is generated by $T_i = T_{u_i}$, which satisfy only the last two relations (see [KPW]).

### 1.2 Examples

1) For $A = \mathbb{C}$ and $E = H$ a Hilbert space with orthonormal basis $\{\xi_i\}_{i \in I}$, the Toeplitz algebra $\mathcal{T}_H$ is generated by $\{S_i\}_{i \in I}$ satisfying $S_i S_j = \delta_{ij} \cdot 1$, $i, j \in I$. In $\mathcal{O}_H$ we also have $\sum_{i \in I} S_i S_i^* = 1$ if the dimension of $H$ is finite. In particular, for $E = \mathbb{C} = A$ we get $\mathcal{T}_E \cong \mathcal{T}$, the classical Toeplitz algebra generated by the unilateral shift, $\mathcal{O}_E \cong C(\mathbb{T})$, the continuous functions on the unit circle, and $\mathcal{F}_E \cong \mathbb{C}$. For $E = \mathbb{C}^n$, we get $\mathcal{T}_E \cong \mathcal{E}_n$, the Cuntz-Toeplitz algebra, $\mathcal{O}_E \cong \mathcal{O}_n$, the Cuntz algebra, and $\mathcal{F}_E \cong UHF(n\infty)$. For $H$ infinite dimensional and separable, $\mathcal{T}_H \cong \mathcal{O}_H \cong \mathcal{O}_\infty$.

2) Let $E = A^n$ with the usual structure:

$$\langle (a_1, ..., a_n), (b_1, ..., b_n) \rangle = \sum_{i=1}^n b_i^* a_i, \quad (a_1, ..., a_n) \cdot a = (a_1 a, ..., a_n a), \quad a \cdot (a_1, ..., a_n) = (aa_1, ..., aa_n).$$

We get $\mathcal{T}_E \cong A \otimes \mathcal{E}_n$, $\mathcal{O}_E \cong A \otimes \mathcal{O}_n$, $\mathcal{F}_E \cong A \otimes UHF(n\infty)$.

3) Let $\alpha : A \to A$ be an automorphism of a unital $C^*$-algebra, and let $E = A(\alpha)$ be the Hilbert bimodule obtained from $A$ with the usual inner product and right multiplication, and with left action $\varphi(a)x = \alpha(a)x$. Then $\mathcal{T}_E$ is isomorphic to the Toeplitz extension $\mathcal{T}_\alpha$ used by Pimsner and Voiculescu in [PV], and $\mathcal{O}_E$ is isomorphic to the crossed product $A \rtimes_{\alpha} \mathbb{Z}$. Indeed, let $\hat{a}$ denote the element in $E$ obtained from $a \in A$. Then $S = \tau(1)$ is an isometry in any unital Toeplitz representation $(\tau, \pi)$, since $\langle 1, 1 \rangle = 1$, and it is an unitary in any unital covariant representation, since the rank one operator $\theta_{1, \hat{1}}$ is the identity. We also have

$$\pi(\alpha(a)) = \pi(\hat{1}, \alpha(\hat{a}))) = \tau(1)^* \pi(\alpha(\hat{a})) = \tau(1)^* \tau(\varphi(a)\hat{1}) = \tau(1)^* \pi(a) \tau(1) = S^* \pi(a) S,$$
therefore $\mathcal{O}_E \cong A \rtimes_\alpha \mathbb{Z}$. In the paper mentioned above, the Toeplitz extension $\mathcal{T}_\alpha$ was defined as the $C^*$-subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{T}$ generated by $A \otimes 1$ and $u \otimes S_+$, where $\alpha(a) = uau^*$ and $S_+$ is the unilateral shift. It is easy to see that $\mathcal{T}_\alpha \cong \mathcal{T}_E$ by the map which takes $a \otimes 1$ into $a$ and $u \otimes S_+$ into $S$. Since $\mathcal{K}(\ell^2(E)) \cong A \otimes \mathcal{K}$, we recover the short exact sequence

$$0 \to A \otimes \mathcal{K} \to \mathcal{T}_\alpha \to A \rtimes_\alpha \mathbb{Z} \to 0.$$  

4) Graph $C^*$-algebras. For an oriented countable graph $G = (G^0, G^1, r, s)$, $C^*(G)$ is defined as the universal $C^*$-algebra generated by mutually orthogonal projections $\{p_v\}_{v \in G^0}$ and partial isometries $\{s_e\}_{e \in G^1}$ with orthogonal ranges such that $s_e^*s_e = p_{r(e)}$, $s_es_e^* \leq p_{s(e)}$ and

$$(*) \quad p_v = \sum_{s(e) = v} s_es_e^* \text{ if } 0 < |s^{-1}(v)| < \infty.$$  

We can set $A = C_0(G^0)$, and denote by $E$ the Hilbert module we obtain after we complete $C_c(G^1)$ in the norm given by the inner product

$$\langle \xi, \eta \rangle(v) = \sum_{r(e) = v} \overline{\xi(e)}\eta(e)$$

with the right action defined by $(\xi f)(e) = \xi(e)f(r(e))$. The left action is defined by

$$\varphi : A \to \mathcal{L}(E), \varphi f \xi(e) = f(s(e)) \xi(e), \xi \in C_c(G^1) \subset E.$$  

We have

$$\varphi^{-1}(\mathcal{K}(E)) = C_0(\{v \in G^0 : |s^{-1}(v)| < \infty\}) \cap \ker(\varphi) = C_0(\{v \in G^0 : |s^{-1}(v)| = 0\}) \subset E.$$  

hence $I_E = C_0(\{v \in G^0 : 0 < |s^{-1}(v)| < \infty\})$. Define

$$\tau(f) = \sum_{v \in G^0} f(v)p_v, \quad \tau(\xi) = \sum_{e \in G^1} \xi(e)s_e, \xi \in C_c(G^1) \subset E.$$  

The pair $(\tau, \pi)$ is a Toeplitz representation into $C^*(G)$ iff $s_e^*s_e = p_{r(e)}$ and $s_es_e^* \leq p_{s(e)}$, which is covariant iff $(*)$ is satisfied. This proves that $C^*(G) \cong \mathcal{O}_E$ (see [Ka1]). The core $\mathcal{F}_E$ is an AF-algebra. For an irreducible oriented finite graph with no sinks, we obtain the Cuntz-Krieger algebras $\mathcal{O}_A$ as Cuntz-Pimsner algebras.

5) For a $C^*$-algebra $A$ and an injective unital endomorphism $\alpha \in \text{End}(A)$ such that there is a conditional expectation $P$ onto the range $\alpha(A)$, one can define a Hilbert bimodule $E = A(\alpha, P)$, using the transfer operator $L = \alpha^{-1} \circ P : A \to A$ as in [EV]. We complete the vector space $A$ with respect to the inner product

$$\langle \xi, \eta \rangle = L(\xi^*\eta),$$

and define the right and left multiplications by $\xi \cdot a = \xi_\alpha(a), a \cdot \xi = a\xi$. We have

$$\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta_\alpha(a) \rangle = L(\xi^*\eta_\alpha(a)) = \alpha^{-1}(P(\xi^*\eta_\alpha(a))) = \alpha^{-1}(P(\xi^*\eta)(a)) = \alpha^{-1}(P(\xi^*\eta))(a) = \langle \xi, \eta \rangle a.$$  

For $\alpha$ an automorphism and $P = \text{id}$, the corresponding $C^*$-algebra $\mathcal{O}_E$ is isomorphic to the crossed product $A \rtimes_{\alpha^{-1}} \mathbb{Z}$. Indeed, let $F$ be the Hilbert bimodule $A(\alpha^{-1})$ with the structure as in example 3. The map $\alpha^{-1} : A \to A$ induces an isomorphism of Hilbert bimodules $h : E \to F$. For $A = C(X)$ with $X$ compact, and $\alpha$ induced by a surjective local homeomorphism $\sigma : X \to X$, we take

$$(Pf)(x) = \frac{1}{\nu(x)} \sum_{\sigma(y) = \sigma(x)} f(y),$$
where \( \nu(x) \) is the number of elements in the fiber \( \sigma^{-1}(x) \). It was proved in \( \square \) that the corresponding algebra \( \mathcal{O}_{A(\alpha,\rho)} \) is isomorphic to \( C^*(\Gamma(\sigma)) \), where \( \Gamma(\sigma) \) is the Renault groupoid
\[
\Gamma(\sigma) = \{(x, p - q, y) \in X \times \mathbb{Z} \times X \mid \sigma^p(x) = \sigma^q(y)\}.
\]

\section{Extending the Scalars}

Let \( E \) be a Hilbert module over \( A \) and let \( \rho : A \to B \) be a \( C^* \)-algebra homomorphism (we will be interested mostly in the case when \( \rho \) is an inclusion). Then \( B \) is a left \( A \)-module with multiplication \( a \cdot b = \rho(a)b \), and \( E \otimes_A B \) becomes a Hilbert module over \( B \), with the inner product given by
\[
\langle \xi_1 \otimes b_1, \xi_2 \otimes b_2 \rangle = b_1^\dagger \rho((\xi_1, \xi_2)) b_2
\]
and right multiplication \((\xi \otimes b_1) \cdot b_2 = \xi \otimes b_1 b_2\). We have \( \mathcal{K}(E \otimes_A B) \cong (E \otimes_A B)_B (E \otimes_A B)^* \cong E \otimes_A B \otimes_A E^* \), which is strongly Morita equivalent to \( B \) in the case \( E \) is full. Also, we have an inclusion \( \mathcal{K}(E) \subset \mathcal{K}(E \otimes_A B) \) for \( B \) unital, given by \( \xi \otimes \eta^* \mapsto \xi \otimes 1 \otimes \eta^* \). If \( E \) is a Hilbert bimodule over \( A \), and if there is a \(*\)-morphism \( B \to \mathcal{L}(E) \) which extends the left multiplication of \( A \) on \( E \), then \( E \otimes_A B \) becomes a Hilbert bimodule over \( B \), and one can form the tensor powers \( (E \otimes_A B) \otimes^n \). Assuming that the left action of \( B \) on \( E \otimes_A B \) is nondegenerate, we get
\[
(E \otimes_A B) \otimes^n \cong E \otimes^n \otimes_A B.
\]
In particular, the Toeplitz algebra \( T_{E \otimes_A B} \) is represented on \( \ell^2(E \otimes_A B) \cong \ell^2(E) \otimes_A B \), and depends on the left multiplication of \( B \). An interesting question is to relate the \( C^* \)-algebras \( T_{E \otimes_A B} \) and \( \mathcal{O}_{E \otimes_A B} \) to \( T_E, \mathcal{O}_E \), and \( B \).

\subsection{Example}

Let \( A = \mathbb{C} \), let \( E = H \) be a separable infinite dimensional Hilbert space, and let \( B \) be a separable unital \( C^* \)-algebra. Then \( H \otimes B \) is a Hilbert module over \( B \), and \( \mathcal{K}(H \otimes B) \cong \mathcal{K}(H) \otimes B \). If \( B \) is faithfully represented on \( H \), then \( H \otimes B \) becomes a Hilbert bimodule over \( B \). Assuming, in addition, that the intersection of \( B \) with \( \mathcal{K}(H) \) is trivial, Kunjian (see \( \square \)) showed that \( \mathcal{T}_{H \otimes B} \cong \mathcal{O}_{H \otimes B} \) is simple and purely infinite, with the same \( K \)-theory as \( B \).

\subsection{Example}

Let \( A = C_0(X) \) and let \( E \) be a Hilbert module given by a continuous field of elementary \( C^* \)-algebras over \( X \). Then for an abelian \( C^* \)-algebra \( B \) containing \( C_0(X) \), the tensor product \( E \otimes_A B \) is obtained by a pull-back. In particular, let \((\tilde{G}^0, G^1, r, s)\) be a (topological) graph, and consider a covering map \( p : \tilde{G}^0 \to G^0 \) which gives an inclusion \( A = C_0(G^0) \subset C_0(\tilde{G}^0) = B \). The Hilbert module \( E \otimes_A B \), where \( E \) is obtained from \( C_0(\tilde{G}^1) \) as in example 4 §1, is associated to a "pull-back graph" in which the set of vertices is \( G^0 \) and the set of edges is \( \tilde{G}^1 := \{(x, e, y) \in \tilde{G}^0 \times G^1 \times G^0 \mid s(e) = p(x), r(e) = p(y)\} \). The new range and source maps are \( s(x, e, y) = x, r(x, e, y) = y \). It is known that \( \mathcal{K}(E) \) is isomorphic to \( C^*(R) \), where \( R \) is the equivalence relation
\[
R = \{(e_1, e_2) \in G^1 \times G^1 \mid r(e_1) = r(e_2)\}.
\]
Then \( \mathcal{K}(E \otimes_A B) \) is isomorphic to \( C^*(p^*(R)) \), where
\[
p^*(R) = \{((x_1, e_1, y_1), (x_2, e_2, y_2)) \in \tilde{G}^1 \times \tilde{G}^1 \mid p(y_1) = p(y_2)\}.
\]
2.3 Example. Let \( \alpha : A \to A \) be an automorphism of a \( C^* \)-algebra \( A \), which extends to \( \tilde{\alpha} : B \to B \), where \( A \subset B \). Consider \( E = A(\alpha) \) as in example 1.2.3. Then \( E \otimes_A B \cong B(\tilde{\alpha}) \), \( T_{E \otimes_A B} \cong T_\alpha \) and \( O_{E \otimes_A B} \cong B \times_\alpha \mathbb{Z} \), which contains \( O_E \cong A \times_\alpha \mathbb{Z} \).

2.4 Example. For a Hilbert bimodule \( E \) over \( A \), Pimsner used the Hilbert module \( E_\infty = E \otimes_A \mathcal{F}_E \) (see [P], section 2) in order to get an inclusion \( T_E \subset T_{E_\infty} \), an isomorphism \( O_E \cong O_{E_\infty} \), and a completely positive map \( \phi : O_E \to T_{E_\infty} \) which is a cross-section to the quotient map \( T_{E_\infty} \to O_E \).

§3. ITERATING THE PIMSNER CONSTRUCTION

Consider now two full finitely generated Hilbert \( A \)-bimodules \( E_1 \) and \( E_2 \) such that \( A \) is unital and the left actions \( \varphi_i : A \to \mathcal{L}(E_i) \) are injective and nondegenerate. We assume that there is an isomorphism of Hilbert \( A \)-bimodules \( \chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1 \). This isomorphism should be understood as a kind of commutation relation. The most interesting cases are when \( E_1 \) and \( E_2 \) are independent, in the sense that no tensor power of one is isomorphic to the other.

Note that the isomorphism \( \chi \) induces and isomorphism \( E_1^* \otimes_A E_2^* \to E_2^* \otimes_A E_1^* \) because \( (E_1 \otimes_A E_2)^* \cong E_2^* \otimes_A E_1^* \).

Since \( A \subset T_{E_1} \), the tensor product \( E_2 \otimes_A T_{E_1} \) becomes a Hilbert module over \( T_{E_1} \) as in §2, with the inner product given by \( \langle \xi \otimes x, \eta \otimes y \rangle = x^* \langle \xi, \eta \rangle y \), and the right multiplication by \( (\xi \otimes x)y = \xi \otimes xy \). Since \( T_{E_1} \) is generated by \( E_1 \), in order to define a left multiplication of \( T_{E_1} \) on \( E_2 \otimes_A T_{E_1} \), it is sufficient to define the left multiplication by elements in \( E_1 \). This is done via the composition

\[
E_1 \otimes_A E_2 \otimes_A T_{E_1} \to E_2 \otimes_A E_1 \otimes_A T_{E_1} \to E_2 \otimes_A T_{E_1},
\]

where the first map is \( \chi \otimes id \), and the second is given by absorbing \( E_1 \) into \( T_{E_1} \). For the adjoint, note that there is a map

\[
E_1^* \otimes_A E_2 \otimes_A E_1 \to E_1^* \otimes_A E_1 \otimes_A E_2 \to E_2,
\]

where the first map is \( id \otimes \chi^{-1} \), and the second is given by left multiplication with the inner product in \( E_1 \).

We will denote by \( E_2 \otimes_A T_{E_1} \) the bimodule obtained using this left multiplication, when \( \chi \) is not understood. In the same way, we may consider the bimodule \( E_1 \otimes_A T_{E_2} \) with the left multiplication induced by \( \chi^{-1} \).

3.1 Lemma. With the above structure, \( E_2 \otimes_A T_{E_1} \) is a Hilbert bimodule over \( T_{E_1} \), and we can consider the Toeplitz algebra \( T_{E_2 \otimes_A T_{E_1}} \). We have

\[
T_{E_2 \otimes_A T_{E_1}} \cong T_{E_1 \otimes_A T_{E_2}}.
\]

Proof. Both algebras \( T_{E_2 \otimes_A T_{E_1}} \) and \( T_{E_1 \otimes_A T_{E_2}} \) are represented on the Fock space \( \ell^2(E_2) \otimes_A \ell^2(E_1) \cong \ell^2(E_1) \otimes_A \ell^2(E_2) \), and are generated by \( T_{E_1} \) and \( T_{E_2} \) with the commutation relation given by the isomorphism \( \chi \).

Similarly, we can construct the Hilbert bimodules \( E_2 \otimes_A O_{E_1} \) and \( E_1 \otimes_A O_{E_2} \). We get

3.2 Lemma. With the above notation,

\[
T_{E_2 \otimes_A O_{E_1}} \cong O_{E_1 \otimes_A T_{E_2}}, \quad O_{E_2 \otimes_A O_{E_1}} \cong O_{E_1 \otimes_A O_{E_2}}.
\]

Proof. The first two algebras are quotients of \( T_{E_2 \otimes_A T_{E_1}} \) by the ideal generated by \( K(\ell^2(E_1)) \), and the last two are quotients by the ideal generated by \( K(\ell^2(E_1)) \) and \( K(\ell^2(E_2)) \).
Note that there is a gauge action of $T^2$ on $O_{E_2 \otimes _A C_{E_1}} \simeq O_{E_1 \otimes _A C_{E_2}}$.

3.3 Remark. Given $E_1, E_2$ as above, we can define a product system $E = E^x$ of Hilbert bimodules over the semigroup $(\mathbb{N}^2, +)$ (see [F2]), as follows. Define the fibers by $E_{(m,n)} = E_1^{\otimes m} \otimes _A E_2^{\otimes n}$ for $(m,n) \in \mathbb{N}^2$, and the multiplication induced by the isomorphism $\chi$. It is easy to see that we get associativity, and therefore we may consider the $C^*$-algebras $T_E, O_E$ as defined by Fowler. Note that if we change $\chi$, the product system also changes. In particular, a path of isomorphisms $\chi_t$ will determine a family of product systems $E^t \simeq E^{t'}$.

Recall that a Toeplitz representation of a product system $E$ in a $C^*$-algebra is obtained from a family of Toeplitz representations of the fibers, compatible with the product. In this particular case, it is sufficient to consider two Toeplitz representations $(\tau_1, \pi), (\tau_2, \pi)$ of the generators $E_1$ and $E_2$, respectively, and to define $\tau(\xi_1 \otimes ... \otimes \xi_m \otimes \eta_1 \otimes ... \otimes \eta_n) = \tau_1(\xi_1) ... \tau_1(\xi_m)\tau_2(\eta_1) ... \tau_2(\eta_n)$ for $\xi_i \in E_1, i = 1, ..., m$ and $\eta_j \in E_2, j = 1, ..., n$.

The Toeplitz algebra $T_E$ is represented on the Fock space $\ell^2(E) \cong \ell^2(E_1) \otimes _A \ell^2(E_2)$ by creation operators. The covariance condition requires that each $(\tau_i, \pi)$ is covariant, $i = 1, 2$. This means that $\psi_i(\varphi_i(a)) = \pi(a)$ for $a \in \varphi_i^{-1}(K(E_i)), i = 1, 2$, where $\psi_i(\theta \xi \eta) = \tau_i(\xi)\tau_i(\eta)*$. We get that

$$T_{E_2 \otimes _A T_{E_1}} \cong T_{E_1 \otimes _A T_{E_2}} \cong T_E$$ and $O_{E_2 \otimes _A O_{E_1}} \cong O_{E_1 \otimes _A O_{E_2}} \cong O_E$.

3.4 Example. Let $A$ be a unital $C^*$-algebra, and $\alpha, \beta$ two commuting automorphisms of $A$. We assume that $\alpha$ and $\beta$ generate $\mathbb{Z}^2$ as a subgroup of $Aut(A)$. Denote by $A(\alpha), A(\beta)$ the Hilbert bimodules as in example 1.2.3, which are independent. We have $A(\alpha) \otimes _A A(\beta) \cong A(\beta) \otimes _A A(\alpha)$ by the map $\chi(\hat{a} \otimes \hat{b}) = \beta(\alpha^{-1}(a)) \otimes \hat{b}$. Indeed,

$$\langle \chi(\hat{a}_1 \otimes \hat{b}_1), \chi(\hat{a}_2 \otimes \hat{b}_2) \rangle = \langle \beta(\alpha^{-1}(a_1)) \otimes \hat{b}_1, \beta(\alpha^{-1}(a_2)) \otimes \hat{b}_2 \rangle = \langle \hat{b}_1, \beta(\alpha^{-1}(a_1)) \otimes \beta(\alpha^{-1}(a_2)) \rangle \cdot \hat{b}_2 =$$

$$b_1^* \alpha(\beta(\alpha^{-1}(a_1)) \otimes \beta(\alpha^{-1}(a_2))) b_2 = b_1^* \beta(\alpha^{-1}(a_1)) b_2 = \langle \hat{b}_1, \beta(\alpha^{-1}(a_1)) \otimes \hat{b}_2 \rangle = \langle \hat{a}_1 \otimes \hat{b}_1, \hat{a}_2 \otimes \hat{b}_2 \rangle,$$

and

$$\chi(a' \cdot (\hat{a} \otimes \hat{b}) \cdot b') = \chi(\alpha(a') \hat{a} \otimes b b') = \beta(\alpha^{-1}(\alpha(a')a)) \otimes b b' =$$

$$\beta(a' \alpha^{-1}(a) \otimes b b') = \beta(a' \alpha^{-1}(a)) \otimes b b' = \chi(a' \otimes b) \cdot b'.$$

Denote by $T_\alpha, T_\beta$ the corresponding Toeplitz algebras, each generated by $A$ and an isometry as in example 3, §1. Then $A(\beta) \otimes _A T_\alpha$ becomes a Hilbert bimodule over $T_\alpha$, isomorphic to $T_\alpha(\beta)$, where $\beta$ is extended to $T_\alpha$ in the natural way, fixing the isometry. Similarly, $A(\alpha) \otimes _A T_\beta \simeq T_\beta(\alpha)$. We have

$$T_{\alpha(\beta)} \cong T_{\beta(\alpha)} \cong T_E,$$

where $E$ is the product system over $\mathbb{N}^2$ constructed from $A(\alpha), A(\beta)$, and the isomorphism $\chi$. Also, we may consider the Hilbert bimodules $A(\beta) \otimes _A (A \rtimes \alpha \mathbb{Z}) \cong (A \rtimes \alpha \mathbb{Z})(\beta)$, and $A(\alpha) \otimes _A (A \rtimes \beta \mathbb{Z}) \cong (A \rtimes \beta \mathbb{Z})(\alpha)$, where again $\alpha$ and $\beta$ are extended to $A \rtimes \alpha \mathbb{Z}$ and $A \rtimes \beta \mathbb{Z}$, respectively, by fixing the unitaries implementing the actions of $\mathbb{Z}$. We have

$$T_{(A \rtimes \alpha \mathbb{Z})(\beta)} \cong T_{\beta \rtimes \alpha \mathbb{Z}}, \quad T_{(A \rtimes \beta \mathbb{Z})(\alpha)} \cong T_{\alpha \rtimes \beta \mathbb{Z}},$$

$$O_{(A \rtimes \alpha \mathbb{Z})(\beta)} \cong O_{(A \rtimes \beta \mathbb{Z})(\alpha)} \cong O_E \cong A \rtimes _{\alpha, \beta} \mathbb{Z}^2.$$

3.5 Example. Let $E = F = \mathbb{C}^2$ with the usual structures of Hilbert bimodules over $A = \mathbb{C}$. Then $O_E \cong O_2$, and $F \otimes O_2$ becomes a Hilbert module over $O_2$ with the usual operations. The left action will depend on a fixed isomorphism $\chi : E \otimes F \rightarrow F \otimes E$. If $\{e_1, e_2\}$ and $\{f_1, f_2\}$ are the canonical bases
in $E$ and $F$ respectively, and $\chi_1(e_i \otimes f_j) = f_j \otimes e_i$, $i,j = 1,2$, then the left multiplication is given by $S_i(f_j \otimes S_k) = f_j \otimes S_i S_k$, where $O_2$ has generators $\{S_1, S_2\}$, and $O_{F \otimes T} \cong O_2 \otimes O_2 \cong O_2$. On the other hand, if $\chi_2(e_i \otimes f_j) = f_i \otimes e_j$, $i,j = 1,2$, then the left action is given by $S_i(f_j \otimes S_k) = f_i \otimes S_j S_k$, and the $O_2$-bimodule $F \otimes^{X^2} O_2$ is degenerate. The corresponding Cuntz-Pimsner algebra $O_{F \otimes T \otimes O_2}$ is isomorphic to $C(T) \otimes O_2$. Indeed, if we interpret $E$ and $F$ as being each associated to the 1-graph $\Gamma$ defining $O_2$ ($\Gamma$ has two edges and one vertex), then the isomorphism $\chi_2$ is defining the rank 2 graph $\phi^*(\Gamma)$ where $\phi : N^2 \to \mathbb{N}, \phi(m,n) = m + n$, and the last assertion follows from Example 6.1 and Proposition 2.10 in [KP].

Note that an arbitrary isomorphism $\chi$ does not necessarily define a rank 2 graph. For example, $\chi$ could be given by the unitary matrix $U = (u_{kl})$, where

$$u_{11} = \cos \alpha, u_{12} = -\sin \alpha, u_{21} = \sin \alpha, u_{22} = \cos \alpha,$$

$$u_{33} = \cos \beta, u_{34} = -\sin \beta, u_{43} = \sin \beta, u_{44} = \cos \beta,$$

for some angles $\alpha$ and $\beta$, and the rest of the entries equal to zero. For other $C^*$-algebras defined by a product system of finite dimensional Hilbert spaces over the semigroup $\mathbb{N}^2$, see [FI].

3.6 Example. Consider two star-commuting onto local homeomorphisms $\sigma_1$ and $\sigma_2$ of a compact space $X$. By definition, that means that for every $x, y \in X$ such that $\sigma_1(x) = \sigma_2(y)$, there exists a unique $z \in X$ such that $\sigma_2(z) = x$ and $\sigma_1(z) = y$. This condition ensures that the associated conditional expectations $P_1, P_2$ commute (see [ER]). Unfortunately, this condition was omitted in the Proposition on page 8 in [D2]. I would like to thank Ruy and Jean for pointing this to me.

We can define the Hilbert bimodules $E_i = A(\alpha_i, P_i), i = 1,2$ over $A = C(X)$ as in example 1.25. Then there is an isomorphism $h : A(\alpha_1, P_1) \otimes_A A(\alpha_2, P_2) \to A(\alpha_1 \circ \alpha_2, P_1 \circ P_2)$, given by $h(\hat{a} \otimes \hat{b}) = \hat{a} \alpha_1(\hat{b})$, with inverse $h^{-1}(\hat{x}) = \hat{x} \otimes \hat{1}$, which induces an isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$. The resulting $C^*$-algebra $O_{E_2 \otimes A E_1} \cong O_{E_1 \otimes_A E_2}$ can be understood as a crossed product of $C(X)$ by the semigroup $\mathbb{N}^2$. For other examples of semigroups of local homeomorphisms, see [ER].

§4. EXACT SEQUENCES IN K-THEORY

To study the $C^*$-algebra of Toeplitz operators on the quarter plane, Douglas and Howe (see [DH]) considered the commutative diagram with exact rows and columns, where $j$ is the inclusion map, and $\pi$ is the quotient map:

$$
\begin{array}{cccc}
0 & \rightarrow & K \otimes K & \rightarrow & \pi \otimes 1 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T \otimes K & \rightarrow & C(T) \otimes K & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T \otimes T & \rightarrow & C(T) \otimes T & \rightarrow 0.
\end{array}
$$

$$
\begin{array}{cc}
0 & \rightarrow \\
\downarrow & \downarrow \\
0 & \rightarrow \\
\downarrow & \downarrow \\
0 & \rightarrow \\
\downarrow & \downarrow \\
0 & 0
\end{array}
$$
4.1 Corollary. We have the short exact sequences

\[ 0 \rightarrow T \otimes K + K \otimes T \xrightarrow{1 \otimes j + j \otimes 1} T \otimes T \xrightarrow{\pi \otimes \pi} C(T) \otimes C(T) \rightarrow 0, \]

\[ 0 \rightarrow K \otimes K \xrightarrow{j \otimes 1 + 1 \otimes j} T \otimes K + K \otimes T \xrightarrow{\pi \otimes 1 + 1 \otimes \pi} C(T) \otimes K \otimes K \otimes C(T) \rightarrow 0. \]

The above 3 \times 3 diagram and the exact sequences in the corollary are particular cases of a more general situation, for which we provide a proof.

4.2 Lemma. Let \( A \) be a \( C^* \)-algebra and \( I, J \) two closed two-sided ideals of \( A \). Then we have the commutative diagram with exact rows and columns, where the maps are the canonical ones:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & I \cap J \\
\downarrow{\lambda_1} & \rightarrow & I \\
\downarrow{\lambda_2} & \rightarrow & J \\
\downarrow{\omega_2} & \rightarrow & J/(I \cap J) \\
0 & \rightarrow & A/I \\
\downarrow & & \downarrow \\
0 & \rightarrow & A/J \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

From this we get the exact sequences

\[ 0 \rightarrow I + J \xrightarrow{\lambda_1 + \omega_2} A \xrightarrow{\pi} A/(I + J) \rightarrow 0, \]

where \( \pi = \rho_1 \circ \pi_1 = \rho_2 \circ \pi_2 \), and

\[ 0 \rightarrow \lambda_1 \xrightarrow{1 + 1} I + J \rightarrow \omega_1 + \omega_2 \rightarrow \left(I/(I \cap J) \oplus J/(I \cap J)\right) \rightarrow 0. \]

Applying the \( K \)-theory functor, we get

\[ K_0(I + J) \rightarrow K_0(A) \rightarrow K_0(A/(I + J)) \]

\[ \uparrow \quad \quad \downarrow \]

\[ K_1(A/(I + J)) \leftarrow K_1(A) \leftarrow K_1(I + J), \]

\[ K_0(I \cap J) \rightarrow K_0(I + J) \rightarrow K_0(I/(I \cap J)) \oplus K_0(J/(I \cap J)) \]

\[ \uparrow \quad \quad \downarrow \]

\[ K_1(I/(I \cap J)) \oplus K_1(J/(I \cap J)) \leftarrow K_1(I + J) \leftarrow K_1(I \cap J). \]

Proof. Note that the first two rows and the first two columns are obviously exact. For the third row, the map \( \pi_1 \circ \omega_2 \) has kernel \( I \cap J \). This defines a map \( \sigma_2 : J/(I \cap J) \rightarrow A/I \) such that \( \sigma_2 \circ \omega_2 = \pi_1 \circ \omega_2 \).

By the second isomorphism theorem, \( (I + J)/I \cong J/(I \cap J) \), and by the third isomorphism theorem, \( (A/I)/((I + J)/I) \cong A/(I + J) \), hence the third row is exact. The exactness of the third column is proved similarly. Consider now the diagonal morphism \( \pi = \rho_1 \circ \pi_1 = \rho_2 \circ \pi_2 : A \rightarrow A/(I + J) \). If \( a \in \ker \pi \), then \( \pi_2(a) \in \ker \rho_2 = \sigma_1(I/(I \cap J)) = \sigma_1(\omega_1(I)) \), hence there is \( b \in I \) with \( \pi_2(a) = \sigma_1(\omega_1(b)) = \pi_2(\pi_1(b)) \). It
follows that \( a - \iota_1(b) \in \ker \pi_2 = \iota_2(J) \), and there is \( c \in J \) with \( a = \iota_1(b) + \iota_2(c) \). This gives the first exact sequence. For the second, we use the map \( \omega_1 + \omega_2 : I + J \to I/(I \cap J) \oplus J/(I \cap J) \) which has kernel \( I \cap J \).

We generalize the above diagram of Douglas and Howe to certain iterated Toeplitz and Cuntz-Pimsner algebras. By the lemma, we get some exact sequences which we hope will help to do \( K \)-theory computations in some particular cases.

### 4.3 Theorem

Consider a \( C^* \)-algebra \( A \) and two finitely generated Hilbert bimodules \( E_1, E_2 \) with a fixed isomorphism \( \chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1 \). We assume that \( E_2 \otimes_A \mathcal{T}_{E_1}, E_2 \otimes_A \mathcal{O}_{E_1}, E_1 \otimes_A \mathcal{T}_{E_2}, E_1 \otimes_A \mathcal{O}_{E_2} \) are nondegenerate as Hilbert bimodules, with the structure described in \( \S 3 \). Then we have the following commuting diagram (depending on \( \chi \)), with exact rows and columns, where the maps are canonical:

\[
\begin{array}{cccccc}
0 & \to & \mathcal{K}(\ell^2(E_1 \otimes_A E_2)) & \to & \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1})) & \to & \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{O}_{E_1})) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2})) & \to & \mathcal{O}_{E_1 \otimes A \mathcal{T}_{E_2}} & \cong & \mathcal{O}_{E_2 \otimes A \mathcal{T}_{E_1}} & \to & \mathcal{O}_{E_1 \otimes A \mathcal{O}_{E_2}} \cong \mathcal{O}_{E_2 \otimes A \mathcal{O}_{E_1}} & \to & 0 . \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{O}_{E_2})) & \to & \mathcal{O}_{E_1 \otimes A \mathcal{O}_{E_2}} & \cong & \mathcal{O}_{E_2 \otimes A \mathcal{T}_{E_1}} & \to & \mathcal{O}_{E_1 \otimes A \mathcal{O}_{E_2}} \cong \mathcal{O}_{E_2 \otimes A \mathcal{O}_{E_1}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

**Proof.** Recall that the algebra \( \mathcal{T}_{E_i} \) is represented on the Fock space \( \ell^2(E_i) \) and we have a short exact sequence

\[
0 \to \mathcal{K}(\ell^2(E_i)) \to \mathcal{T}_{E_i} \to \mathcal{O}_{E_i} \to 0,
\]

for \( i = 1, 2 \). We apply the above lemma for the \( C^* \)-algebra \( \mathcal{T}_{E_1 \otimes A \mathcal{T}_{E_2}} \cong \mathcal{T}_{E_2 \otimes A \mathcal{T}_{E_1}} \) with ideals \( I = \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1})) \) and \( J = \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2})) \). The nondegeneracy assumption implies that \( \ell^2(E_2 \otimes_A \mathcal{T}_{E_1}) = \ell^2(E_2) \otimes_A \mathcal{T}_{E_1} \) and \( \ell^2(E_2 \otimes_A \mathcal{O}_{E_1}) = \ell^2(E_2) \otimes_A \mathcal{O}_{E_1} \). Note that the map \( \chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1 \) induces isomorphisms

\[
\mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{K}(\ell^2(E_1)))) \cong \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{K}(\ell^2(E_2)))) \cong \mathcal{K}(\ell^2(E_1 \otimes E_2)),
\]

therefore \( I \cap J = \mathcal{K}(\ell^2(E_1 \otimes_A E_2)) \).

\( \square \)

### 4.4 Corollary

Under the same assumptions as in the theorem, we get the short exact sequences

\[
0 \to \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1})) + \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2})) \to \mathcal{T}_{E_1 \otimes A \mathcal{T}_{E_2}} \to \mathcal{O}_{E_1 \otimes A \mathcal{O}_{E_2}} \to 0
\]

and

\[
0 \to \mathcal{K}(\ell^2(E_1 \otimes_A E_2)) \to \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{T}_{E_1})) + \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{T}_{E_2})) \to \mathcal{K}(\ell^2(E_2 \otimes_A \mathcal{O}_{E_1})) \oplus \mathcal{K}(\ell^2(E_1 \otimes_A \mathcal{O}_{E_2})) \to 0.
\]

The corresponding six-term exact sequences of \( K \)-theory give us information about the \( K \)-theory of \( \mathcal{O}_{E_1 \otimes A \mathcal{O}_{E_2}} \), once we identify the maps between the various \( K \)-groups. Note that the Toeplitz algebras are \( KK \)-equivalent to the \( C^* \)-algebra \( A \).
4.5 Example. For $\alpha_1, \alpha_2$ two commuting independent automorphisms of a $C^*$-algebra $A$, the diagram is

\[
\begin{array}{cccccc}
0 & \rightarrow & K \otimes A \otimes K & \rightarrow & T_{\alpha_1} \otimes K & \rightarrow & A \rtimes_{\alpha_1} Z \otimes K & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K \otimes T_{\alpha_2} & \rightarrow & T_{\alpha_2}(\alpha_1) \cong T_{\alpha_1}(\alpha_2) & \rightarrow & T_{\alpha_2} \rtimes_{\alpha_1} Z \cong T_{(A \rtimes_{\alpha_1} Z)(\alpha_2)} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K \otimes A \rtimes_{\alpha_2} Z & \rightarrow & T_{\alpha_1} \rtimes_{\alpha_2} Z \cong T_{(A \rtimes_{\alpha_1} Z)(\alpha_1)} & \rightarrow & A \rtimes_{\alpha_1, \alpha_2} Z^2 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & &
\end{array}
\]

We get the exact sequences

\[
K_0(A) \rightarrow K_0(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}) \rightarrow K_0(A \rtimes_{\alpha_1} Z) \oplus K_0(A \rtimes_{\alpha_2} Z)
\]

\[
\uparrow \partial^1 + \partial^2 \quad \quad \quad \quad \quad \quad \partial^1 + \partial^2 \downarrow
\]

\[
K_1(A \rtimes_{\alpha_1} Z) \oplus K_1(A \rtimes_{\alpha_2} Z) \leftarrow K_1(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}) \leftarrow K_1(A)
\]

\[
K_0(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}) \rightarrow K_0(A) \rightarrow K_0(A \rtimes_{\alpha_1, \alpha_2} Z^2)
\]

\[
\uparrow \quad \quad \quad \quad \quad \quad \downarrow
\]

\[
K_1(A \rtimes_{\alpha_1, \alpha_2} Z^2) \leftarrow K_1(A) \leftarrow K_1(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}).
\]

For $A = C(X)$ with $X$ a Cantor set, and $\sigma, \tau : X \rightarrow X$ two commuting local homeomorphisms. Denote by $\alpha, \beta$ the induced endomorphisms of $A$, and assume that the associated conditional expectations commute. Applying the theorem for the Hilbert bimodules $E = A(\alpha, P)$, $F = A(\beta, Q)$ and the canonical isomorphism $E \otimes_A F \cong F \otimes_A E$, we get

\[
C(X, \mathbb{Z}) \rightarrow K_0(T_E \otimes K + K \otimes T_F) \rightarrow C(X, \mathbb{Z})/im(1 - \alpha) \oplus C(X, \mathbb{Z})/im(1 - \beta)
\]

\[
\uparrow \quad \quad \quad \quad \quad \quad \downarrow
\]

\[
ker(1 - \alpha) \oplus ker(1 - \beta) \leftarrow K_1(T_E \otimes K + K \otimes T_F) \leftarrow 0
\]

\[
K_0(T_E \otimes K + K \otimes T_F) \rightarrow C(X, \mathbb{Z}) \rightarrow K_0(O_{F \otimes_A E})
\]

\[
\uparrow \quad \quad \quad \quad \quad \quad \downarrow
\]

\[
K_1(O_{F \otimes_A E}) \leftarrow 0 \leftarrow K_1(T_E \otimes K + K \otimes T_F).
\]

Here we used the fact that $K_0(A) = C(X, \mathbb{Z})$ and $K_1(A) = 0$. 

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4.6 Example. Let $E = \mathbb{C}^m, F = \mathbb{C}^n$ and fix $\chi : E \otimes F \to F \otimes E$ an isomorphism. The corresponding diagram (depending on $\chi$) is

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & K \otimes K & \mathcal{E}_m \otimes K & \mathcal{O}_m \otimes K & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & K \otimes \mathcal{E}_n & T_{\mathcal{C}^n} \otimes \mathcal{E}_n & \mathcal{O}_m \otimes \mathcal{E}_n & T_{\mathcal{C}^m} \otimes \mathcal{O}_m & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & K \otimes \mathcal{O}_n & T_{\mathcal{C}^m} \otimes \mathcal{O}_n & \mathcal{O}_m \otimes \mathcal{O}_n & \mathcal{O}_m \otimes \mathcal{O}_m & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
$$

The exact sequences (depending on $\chi$) are

$$
\begin{array}{c}
\mathbb{Z} \to K_0(\mathcal{E}_m \otimes K + K \otimes \mathcal{E}_n) \to \mathbb{Z}_{m-1} \oplus \mathbb{Z}_{n-1} \\
\uparrow & \downarrow \\
0 & K_1(\mathcal{E}_m \otimes K + K \otimes \mathcal{E}_n) & \leftarrow & 0
\end{array}
$$

and

$$
\begin{array}{c}
K_0(\mathcal{E}_m \otimes K + K \otimes \mathcal{E}_n) \to \mathbb{Z} \to K_0(\mathcal{O}_{\mathcal{C}^m \otimes \mathcal{O}_n}) \\
\uparrow & \downarrow \\
K_1(\mathcal{O}_{\mathcal{C}^m \otimes \mathcal{O}_n}) & \leftarrow & 0 & \leftarrow & 0.
\end{array}
$$

In particular, if $\chi$ is just the flip, we have

$$
T_{\mathcal{C}^m \otimes \mathcal{E}_n} \cong T_{\mathcal{C}^n \otimes \mathcal{E}_m} \cong \mathcal{E}_n \otimes \mathcal{E}_m, \quad \mathcal{O}_{\mathcal{C}^m \otimes \mathcal{E}_n} \cong T_{\mathcal{C}^n \otimes \mathcal{O}_m} \cong \mathcal{O}_m \otimes \mathcal{E}_n,
$$

$$
T_{\mathcal{C}^m \otimes \mathcal{O}_n} \cong \mathcal{O}_{\mathcal{C}^n \otimes \mathcal{E}_m} \cong \mathcal{E}_m \otimes \mathcal{O}_n, \quad \mathcal{O}_{\mathcal{C}^m \otimes \mathcal{O}_n} \cong \mathcal{O}_{\mathcal{C}^n \otimes \mathcal{O}_m} \cong \mathcal{O}_m \otimes \mathcal{O}_n,
$$

and we recover the $K$-theory of $\mathcal{O}_m \otimes \mathcal{O}_n$.

4.7 Example. Let $A = C(T)$ and $\sigma_i(x) = x^{p_i}, i = 1, 2$ with $p_1, p_2$ relatively prime. Then the associated conditional expectations commute, and using the $K$-theory computations in [D1], we get

$$
\begin{array}{c}
\mathbb{Z} \to K_0(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}) \to \mathbb{Z}^2 \oplus \mathbb{Z}_{p_1-1} \oplus \mathbb{Z}_{p_2-1} \\
\uparrow & \downarrow \\
\mathbb{Z} \oplus \mathbb{Z} & \leftarrow & K_1(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}) & \leftarrow & \mathbb{Z}
\end{array}
$$

$$
\begin{array}{c}
K_0(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}) \to \mathbb{Z} \to K_0(\mathcal{O}_{\mathcal{E}_2} \otimes \mathcal{O}_{\mathcal{E}_1}) \\
\uparrow & \downarrow \\
K_1(\mathcal{O}_{\mathcal{E}_2} \otimes \mathcal{O}_{\mathcal{E}_1}) & \leftarrow & \mathbb{Z} & \leftarrow & K_1(T_{\alpha_1} \otimes K + K \otimes T_{\alpha_2}).
\end{array}
$$

More generally, we may consider coverings of the $n$-torus $T^n$. 
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