ABSTRACT: We calculate homological blocks for Seifert manifolds from the exact expression for the $G = SU(N)$ Witten-Reshetikhin-Turaev invariants of Seifert manifolds obtained by Lawrence, Rozansky, and Mariño. For $G = SU(2)$ case, it is possible to express them in terms of false theta functions and their derivatives. For $G = SU(N)$, we calculate them as a series expansion and also discuss some properties of contributions from reducible flat connections to the Witten-Reshetikhin-Turaev invariants for general $N$. We also provide an expected form of the $S$-matrix for general cases and the structure of the Witten-Reshetikhin-Turaev invariants in terms of homological blocks.
1 Introduction

The Chern-Simons partition function on knot complement in 3-sphere $S^3$ with boundary conditions or on $S^3$ with Wilson loops supported on knots provide knot polynomial invariants, e.g. the Jones or the HOMFLY polynomial with colors or with refinement. The Jones polynomial, which is a polynomial with integer powers and integer coefficients, can be understood as the graded Euler characteristic of Khovanov homology, and this homology provides the categorification of the Jones polynomial. Those integer coefficients can be understood as the dimension of the vector space. Such categorification of knot polynomials have been studied in many literature, but it has not been much studied on closed
3-manifolds. For example, a mathematical definition or construction that categorifies the Chern-Simons (CS) partition function or the Witten-Reshetikhin-Turaev (WRT) invariant on closed 3-manifolds is not available yet.

The WRT invariants for closed 3-manifolds have been calculated for a number of 3-manifolds including Seifert manifolds. From what is originally expressed, it was not obvious to see whether such WRT invariant can be expressed in terms of $q$-series with integer powers and integer coefficients, which is the property suitable for categorification.

Meanwhile, in [1], it has been found that the WRT invariant on a Poincaré homology sphere $\Sigma(2,3,5)$ can be expressed as (linear combination of) false theta functions with modular parameter $\tau$ being related to the Chern-Simons level as

$$q = e^{\frac{2\pi i}{K}} = e^{2\pi i \tau}$$

where $K$ is the quantum corrected Chern-Simons level. More precisely, such expression is actually equal to the WRT invariant upon $\tau \searrow \frac{1}{K}$ meaning that $\tau$ approaches to $1/K$ from the upper half plane of complex $\tau$-plane. As the Poincaré homology sphere $\Sigma(2,3,5)$ is an integer homology sphere, in this case it can be seen rather easily that the WRT invariant is expressed as $q$-series with integer powers and integer coefficients. After [1], a number of examples including the case of rational homology spheres have been calculated in [2–6]. However, the integrality of the WRT invariant was not obvious in those examples on rational homology spheres.

Recently, it has been conjectured in [7, 8] that the WRT invariant for closed 3-manifold can be expressed in terms of so called homological block, which is a $q$-series with integer powers and integer coefficient so that it may admit the categorification. The conjecture for the case of $G = U(N)$ was also discussed in [8].

Here, we briefly summarize the conjecture of [7, 8]. The conjecture states that the partition function of Chern-Simons theory with $G = SU(2)$ can be decomposed into homological blocks, $\hat{Z}_b(q)$,

$$Z_{SU(2)}(M_3) = \sum_{a,b \in \text{Tor} H_1(M_3,\mathbb{Z})/\mathbb{Z}_2} e^{\pi i Klk(a,a)} S_{ab} \hat{Z}_b(q) \bigg|_{q^{\frac{2\pi i}{K}}}$$

with the $S$-matrix

$$S_{ab} = \frac{e^{2\pi ilk(a,b)} + e^{-2\pi ilk(a,b)}}{|W_a| |\text{Tor} H_1(M_3,\mathbb{Z})|^{1/2}}$$

where $\hat{Z}_b(q)$, which is defined on $|q| < 1$, takes a form of $q^{\Delta_b}Z[[q]]$ with rational number $\Delta_b$, $K \in \mathbb{Z}$ is a quantum-corrected level, and $W_a$ denotes the stabilizer subgroup $\text{Stab}_{\mathbb{Z}_2}(a)$ for $a$ in Weyl group $\mathbb{Z}_2$ of $SU(2)$. The label $a$ denotes the reducible flat connections or equivalently the abelian flat connections, when $G = SU(2)$. Here, $lk(\cdot, \cdot)$ denotes the linking form on Tor $H_1(M_3,\mathbb{Z})$. More specifically, given two elements $a, b \in \text{Tor} H_1(M_3,\mathbb{Z})$, there is a 2-chain $B'$ such that $s b = \partial B'$ for some $s \in \mathbb{Z}_{\neq 0}$. Then the linking form is
defined as
\[ \text{lk}(a, b) = \frac{\#(a \cap B')}{s} \mod \mathbb{Z} \] (1.4)
where \# denotes the intersection number. Also, the linking form \( \text{lk}(a, a) \) can be interpreted as the Chern-Simons invariant of abelian flat connection \( a \). In addition, the WRT invariant \( Z_{SU(2)}(M) \) can be written as
\[ Z_{SU(2)}(M) = \sum_{a \in \text{Tor} H_1(M_3, \mathbb{Z})/\mathbb{Z}_2} e^{\pi i \text{lk}(a, a)} Z_a. \] (1.5)
Here, \( Z_a \) means the contribution from the abelian flat connections to the WRT invariant, which can be obtained from Borel resummation of perturbative expansion with respect to abelian flat connection \( a \). Several examples including Lens space, \( \mathbb{O}(1) \to \Sigma_g \), 3-manifolds from plumbing graphs, and several rational homology Seifert manifolds were worked out and they supported the conjecture in [7, 8].

Interestingly, the homological blocks are labelled by reducible flat connections, and this can be understood in the context of resurgent analysis [9]. According to the resurgent analysis on \( G = SU(2) \) case, the exact partition function is expressed in terms of Borel resummation of the perturbative expansion around abelian (or reducible) flat connections, while the contributions from non-abelian (or irreducible) flat connections are encoded in transseries expansions of the Borel resummation of the perturbative expansion with respect to abelian flat connections.

The homological blocks can be understood in the context of the M-theory configuration,
\[
\begin{align*}
\text{space-time} & : \mathbb{R} \times TN \times T^*M_3 \\
\text{M5 branes} & : \mathbb{R} \times D^2 \times M_3
\end{align*}
\] (1.6)
where \( D^2 \) is a disc, \( TN \) is Taub-NUT space, and \( D^2 \subset TN \). There are two \( U(1) \) symmetries in this system, which are the rotational symmetry \( U(1)_q \) on \( D^2 \) and the \( U(1)_R \) symmetry. These two symmetries provide two gradings \( \mathbb{Z} \times \mathbb{Z} \) in the homological invariants that lead to homological blocks. When \( M_3 \) is a Seifert manifold, which is the 3-manifold that is considered in this paper, there is an additional symmetry \( U(1)_\beta \) that arises due to the existence of semi-free \( U(1) \) action on the Seifert manifold. This will lead another extra grading \( \mathbb{Z} \) in the homological invariants.

Via the 3d-3d correspondence\(^1\), the homological invariants for generic \( M_3 \) can be realized in the Hilbert space \( \mathcal{H}_T[M_3, G] \) of 3d \( \mathcal{N} = 2 \) theory, \( T[M_3, G] \), on \( D^2 \times \mathbb{R} \) where \( \mathbb{R} \) is regarded as the time direction. Since there is a boundary \( \partial D = S^1 \), the Hilbert space \( \mathcal{H}_T[M_3, G] \) is specified by the boundary condition and this is labelled by \( a \in \text{Tor} H_1(M_3, \mathbb{Z})/\mathbb{Z}_2 \). Thus, the Hilbert space \( \mathcal{H}_T[M_3, G] \) can be decomposed into
\[ \mathcal{H}_T[M_3, G] = \bigoplus_{a \in \text{Tor} H_1(M_3, \mathbb{Z})/\mathbb{Z}_2} \mathcal{H}_a \quad \text{with} \quad \mathcal{H}_a = \bigoplus_{i \in \mathbb{Z} + \Delta_a} \mathcal{H}^i \bigoplus_{j \in \mathbb{Z}} (1.7)
\]
\(^1\)Some aspects of the 3d-3d correspondence for Seifert manifolds have been discussed in [10–13].
where $i$ and $j$ denote the grading for $U(1)_q$ and $U(1)_R$. Then the homological block, $\hat{Z}_a$, is given by

$$\hat{Z}_a(q) = \text{Tr}_{\mathcal{H}_a} q^i (-1)^j$$

and this is a partition function or the half index of $\mathcal{T}[M_3, G]$ on $D^2 \times_q S^1$ with a boundary condition $a$. If $M_3$ is a Seifert manifold, there is an additional grading due to $U(1)_\beta$,

$$\mathcal{H}_a = \bigoplus_{i \in \mathbb{Z}_+ \Delta_a} \mathcal{H}_{i,a}^{i,j}$$

and the homological block is given by

$$\hat{Z}_a(q,t) = \text{Tr}_{\mathcal{H}_a} q^i (-1)^j t^j$$

where $t$ denotes the fugacity of $U(1)_\beta$. In this paper, we only consider the case $t = 1$ of (1.10).

The $S$-transform in (1.2) can be understood from several dualities from the M-theory configuration (1.6), now with $S^1$ instead of $\mathbb{R}$. The Hilbert space $\mathcal{H}_b$ encodes the spectrum of massless BPS particles of $\mathcal{T}[M_3, G]$, which are realized as M2 branes ending on M5 branes. If considering $G = SU(2)$ case, at the boundary, M2 branes are wrapped on non-trivial 1-cycle $(b',-b')$ on $M_3$ where $[b'] = b \in \text{Tor}H_1(M_3, \mathbb{Z})/\mathbb{Z}_2$ and this $b$ is interpreted as charge of the spectrum. Meanwhile, taking the type IIA limit of (1.6) on $S^1$ of $D^2$ and then $T$-dualizing along $S^1$ of (1.6), the resulting configuration becomes D3-D5 system in the type IIB string theory. Taking the $S$-duality of type IIB string, it becomes D3-NS5 system and the boundary condition of the system at the infinity of $\mathbb{R}_+$ on which D3’s are supported is provided by the connected component of $SL(2, \mathbb{C})$ flat connections $[14]$. Meanwhile, since $Z_{SU(2)}(M_3)$ is written only in terms of the contributions from abelian flat connections $a$, $Z_a$, in (1.2), the boundary conditions labelled only by the abelian flat connections $a$ of $SU(2)$ are taken into account.\footnote{The reason why only reducible flat connections are taken into account, not all flat connections, is not known yet, though resurgent analysis provide some explanation on it.} Then, the subscript $a$ of $S$-matrix $S_{ab}$ corresponds to the connected component of abelian flat connections, $\text{Hom} (\text{Tor}H_1(M_3, \mathbb{Z}), U(1))/\mathbb{Z}_2$, while the subscript $b$ of $S_{ab}$ denotes the M2 brane charges in the configuration (1.6). We note that the linking form provides a pairing or isomorphism between $b \in \text{Tor}H_1(M_3, \mathbb{Z})/\mathbb{Z}_2$ and $a \in (\text{Tor}H_1(M_3, \mathbb{Z}))^*/\mathbb{Z}_2 := \text{Hom}(\text{Tor}H_1(M_3, \mathbb{Z}), U(1))/\mathbb{Z}_2$,

$$\text{Tor}H_1(M_3, \mathbb{Z})/\mathbb{Z}_2 \cong_k (\text{Tor}H_1(M_3, \mathbb{Z}))^*/\mathbb{Z}_2 \cong \pi_0 \mathcal{M}_{flat}^a(M_3, SU(2)) .$$

In this paper, we consider the $G = SU(N)$ WRT invariant on general Seifert manifolds $X(P_1/Q_1, \ldots, P_F/Q_F)$ where $P_j$ and $Q_j$ are coprime for each $j = 1, \ldots, F$ and $P_j$’s are pairwise coprime. We provide a formula that calculates $Z_a$’s from which we can calculate the homological blocks exactly for $G = SU(2)$ case and as a $q$-series expansion for $G = SU(N)$ cases from the exact expression given by Lawrence and Rozansky $[15]$ and Mariño.
We see that examples calculated in this paper fit into the expected structure of the WRT invariant of Seifert manifolds in terms of homological blocks which is discussed in section 3.2.

The organization is as follows. In section 2, we calculate homological blocks from the integral expression of the $G = SU(2)$ WRT invariant of Seifert manifolds with three singular fibers obtained by Lawrence and Rozansky, which serves as a basic example in this paper. The calculation covers any values of $|\text{Tor} H_1(M_3, \mathbb{Z})|$. Considering the case that $|\text{Tor} H_1(M_3, \mathbb{Z})|$ is even number, we see that the formula for the $S$-transform (1.3) can be generalized. From the calculation, we see that the homological block can be expressed in terms of false theta functions that are available in literature. We also discuss resurgent analysis and see that when $H = 1$ the exact expression of Lawrence and Rozansky can be understood as the exact Borel sum of perturbative expansion of the analytically continued partition function around the trivial flat connections. Then, we move on to the case of higher rank $G = SU(N)$ in section 3. We discuss some properties of the formula that we obtained and provide some number of examples. From those examples, we provide a general form of the $S$-matrix for $G = SU(N)$ and also the structure of the WRT invariant in terms of homological blocks. In Appendix A, we perform similar calculation for the case of four singular fibers with $G = SU(2)$. In this case, the homological block is expressed in terms of the false theta function of a different type that is discussed in [17]. We also calculate homological blocks for Seifert manifolds with more singular fibers and with higher genus base surface when $G = SU(2)$ in Appendix B. In these cases, the homological blocks are expressed in terms of false theta functions discussed in section 2 and Appendix A, and their derivatives with respect to $q$.

**Note added:** While preparing the manuscript, we found that there are some partial overlaps with [18] on the $G = SU(2)$ case.

## 2 $G = SU(2)$ and three singular fibers

We consider the WRT invariant for $G = SU(2)$ on Seifert manifolds with three singular fibers with genus of base surface being zero, $X(P_1/Q_1, P_2/Q_2, P_3/Q_3)$, where $P_j$'s are coprime to $Q_j$'s for each $j$ and also $P_j$'s are pairwise coprime. $b$ of the Seifert invariant is set to zero. The order of the torsion of the first homology group $|\text{Tor} H_1(M, \mathbb{Z})|$ is given by

$$H := P \sum_{i=1}^{3} \frac{Q_i}{P_i} = \pm |\text{Tor} H_1(M, \mathbb{Z})|$$

(2.1)

where $P = P_1 P_2 P_3$. Due to conditions that $P_j$'s and $Q_j$'s are coprime for each $j$ and $P_j$'s are pairwise coprime, $P$ and $H$ are coprime, which is also the case for arbitrary number of singular fibers. We denote quantum corrected level as $K = k + 2$ and

$$q := e^{2\pi i K}.$$  

(2.2)

According to [15], contributions of reducible flat connections including the trivial flat connection of $G = SU(2)$ to the WRT invariant on Seifert manifolds described above can
be written as

\[ Z_{SU(2)}^{ab}(M_3) = \frac{B}{2\pi i} q^{-\phi_3/4} \sum_{t=0}^{H-1} \int_{\Gamma_t} dy \, e^{-\frac{K}{2\pi i} y^2 - 2Kty} \prod_{j=1}^{3} e^{\frac{y_j}{P_j} - e^{-\frac{y_j}{P_j}}} \] (2.3)

where

\[ B = -\frac{\text{sign}P}{4\sqrt{|P|}} e^{\frac{3}{4}\pi i \text{sign}(\frac{H}{P})} \] (2.4)

\[ \phi_3 = 3 \text{sign} \left( \frac{H}{P} \right) + \sum_{j=1}^{3} \left( 12s(Q_j, P_j) - \frac{Q_j}{P_j} \right) \] (2.5)

and \( s(P, Q) \) is the Dedekind sum

\[ s(P, Q) = \frac{1}{4Q} \sum_{l=1}^{Q-1} \cot \left( \frac{\pi l}{Q} \right) \cot \left( \frac{\pi P l}{Q} \right) \] (2.6)

The integration cycle \( \Gamma_t \) is chosen in such a way that for each \( t \) the integrand is convergent on both ends of infinity, e.g. when \( K \in \mathbb{Z}_+ \) and \( \frac{P}{H} \) is positive, \( \Gamma_0 \) is a line from \(- (1 + i)\infty\) through the origin and \( \Gamma_t \) is parallel to \( \Gamma_0 \) and passes through \( y = -2\pi i \frac{P}{H} t \), which is a stationary phase point of the integrand. When \( \frac{P}{H} \) is negative, the contour is given by clockwise rotation of \( \Gamma_0 \) by \( \frac{\pi}{2} \) and similarly for \( \Gamma_t \). Also, we note that reducible flat connections in \( SU(2) \) case is abelian flat connections, so we use both interchangeably for \( G = SU(2) \). Here, \( t \) labels abelian flat connections where \( t = 0 \) corresponds to the trivial flat connection.

In (2.3) and in the rest of paper, we use the physics normalization

\[ Z_{SU(2)}(S^1 \times S^2) = 1, \quad Z_{SU(2)}(S^3) = \sqrt{\frac{2}{K}} \sin \left( \frac{\pi}{K} \right) \] (2.7)

Also we put additional 1/2 to the expression in [15] to have a same overall coefficient with the result of [16]. We would like to express (2.3) in terms of convergent \( q \)-series for the analytically continued theory.

### 2.1 Calculation of the partition function

The last factor in the integral formula (2.3) can be expanded as

\[ \prod_{j=1}^{3} e^{\frac{y_j}{P_j} - e^{-\frac{y_j}{P_j}}} = \sum_{n=0}^{\infty} \chi_{2P}(n) e^{-\frac{n}{2P}} \] (2.8)

when \( \sum_{j=1}^{3} \frac{1}{P_j} < 1 \) and

\[ \prod_{j=1}^{3} e^{\frac{y_j}{P_j} - e^{-\frac{y_j}{P_j}}} = e^{y(\sum_{j=1}^{3} \frac{1}{P_j} - 1)} + e^{-y(\sum_{j=1}^{3} \frac{1}{P_j} - 1)} + \sum_{n=0}^{\infty} \chi_{2P}(n) e^{-\frac{n}{2P}} \] (2.9)
when $\sum_{j=1}^{3} \frac{1}{P_j} > 1$. Here, $\chi_{2P}(n)$ is a periodic function with period $2P$

$$\chi_{2P}(n) = \begin{cases} \mp \epsilon_1 \epsilon_2 \epsilon_3 & \text{if } n = P \left( 1 \pm \sum_{j=1}^{3} \frac{\epsilon_j}{P_j} \right) \\ 0 & \text{otherwise} \end{cases} \quad (2.10)$$

and $\epsilon_j = \pm 1$. This depends on the choice of $P_j$, but for simplicity of notation we denote it as $\chi_{2P}(n)$. Also, we assumed $Re y > 0$ and $P > 0$. We will discuss the case of other possible ranges in section 2.7. Then for $\sum_{j=1}^{3} \frac{1}{P_j} < 1$, (2.3) is given by

$$Z_{SU(2)}^{ab}(M_3) = \frac{B}{2\pi i} q^{-\phi_3/4} \left( \frac{2iP}{K/H} \right)^{1/2} \sum_{t=0}^{H-1} 2^{2\pi i K \frac{t}{H}} \int_{\Gamma_1} e^{-\frac{t}{2\pi i H} \left( y + 2\pi i \frac{t}{H} \right)^2} \sum_{n=0}^{\infty} \chi_{2P}(n) e^{-\frac{2\pi i n}{H} y} \sum_{j=0}^{\infty} \frac{e^{2\pi i n t}}{\sum_{j=1}^{3} \frac{1}{\tau_j} - 1} \frac{q}{\pi} (\sum_{j=1}^{3} \frac{1}{\tau_j} - 1)^2 (2.11)$$

Here we didn’t move the integration cycle from $\Gamma_1$ to $\Gamma_0$ but just changed integration variable from $y + 2\pi i \frac{t}{H}$ to $y$ for each $t$. We analytically continue $K$ and take $Im K < 0$, i.e. $|q| < 1$. We also take $H > 0$. Then we choose an integration contour $\gamma$ as a line parallel to the imaginary axis of $y$-plane that passes through $Re y > 0$. Calculating the integral, we obtain the partition function of the analytically continued $SU(2)$ theory for abelian flat connections,

$$Z_{SU(2)}^{ab}(M_3) = \frac{B}{2\pi i} q^{-\phi_3/4} \left( \frac{2iP}{K/H} \right)^{1/2} \sum_{t=0}^{H-1} 2^{2\pi i K \frac{t}{H}} \sum_{n=0}^{\infty} \chi_{2P}(n) e^{2\pi i \frac{n}{H} y} q^{\frac{2\pi i n}{H}}. \quad (2.12)$$

When $\sum_{j=1}^{3} \frac{1}{P_j} > 1$, there is an additional term from $(e^{y(\sum_{j=1}^{3} \frac{1}{\tau_j} - 1)} + e^{-y(\sum_{j=1}^{3} \frac{1}{\tau_j} - 1)})$ in (2.9) and this contributes to the WRT invariant as

$$\frac{B}{2\pi i} q^{-\phi_3/4} \left( \frac{2iP}{K/H} \right)^{1/2} \sum_{t=0}^{H-1} 2^{2\pi i K \frac{t}{H}} \left( e^{-2\pi i (\sum_{j=1}^{3} \frac{1}{\tau_j} - 1) \frac{t}{H}} + e^{2\pi i (\sum_{j=1}^{3} \frac{1}{\tau_j} - 1) \frac{t}{H}} \right) q^{\frac{2\pi i n}{H} (\sum_{j=1}^{3} \frac{1}{\tau_j} - 1)^2} \quad (2.14)$$

We note that though $Z_{SU(2)}^{ab}(M_3)$ in (2.13) seemingly contains contributions only from abelian flat connections, it is expected to be the full partition function $Z_{SU(2)}(M_3)$ of analytically continued theory, which contains the contributions from nonabelian flat connections as transseries. This will be discussed in section 2.6. So in the following discussion we will denote (2.13) by $Z_{SU(2)}(M_3)$ instead of $Z_{SU(2)}^{ab}(M_3)$.

### 2.2 The case $H = 1$

When $H = 1$, there is only a trivial flat connection and the partition function is

$$Z_{SU(2)}(M_3) = \frac{B}{2\pi i} q^{-\phi_3/4} \left( \frac{2iP}{K/H} \right)^{1/2} \times \begin{cases} \sum_{n=0}^{\infty} \chi_{2P}(n) q^{\frac{n^2}{H}} & \text{when } \sum_{j=1}^{3} \frac{1}{\tau_j} < 1 \\ 2q^{\frac{P}{\tau} (\sum_{j=1}^{3} \frac{1}{\tau_j} - 1)^2} + \sum_{n=0}^{\infty} \chi_{2P}(n) q^{\frac{n^2}{H}} & \text{when } \sum_{j=1}^{3} \frac{1}{\tau_j} > 1 \end{cases} \quad (2.15)$$
It is possible to decompose \( \chi_{2P}(n) \) into another periodic function \( \psi_{2P}^{(l)}(n) \),

\[
\psi_{2P}^{(l)}(n) = \begin{cases} 
\pm 1 & \text{if } n = \pm l \pmod{2P} \\
0 & \text{otherwise},
\end{cases}
\]

which satisfies \( \psi_{2P}^{(l)}(n) = -\psi_{2P}^{(2P-l)}(n) \). Then we have

\[
\chi_{2P}(n) = \psi_{2P}^{(P(1-(1/p_1+1/p_2+1/p_3)))}(n) + \psi_{2P}^{(P(1-(1/p_1-1/p_2-1/p_3)))}(n) + \psi_{2P}^{(P(1-(1/p_1-1/p_2-1/p_3)))}(n)
\]

\[
= \sum_{s=0}^3 \psi_{2P}^{(R_s)}(n)
\]

where, for simplicity, we denote \( P(1-(1/p_1+1/p_2+1/p_3)) \), \( P(1-(1/p_1-1/p_2-1/p_3)) \), \( P(1-(1/p_1+1/p_2+1/p_3)) \), and \( P(1-(1/p_1-1/p_2-1/p_3)) \) by \( R_0, R_1, R_2, \) and \( R_3 \) in (2.17), respectively. Then, when \( \sum_{j=1}^3 \frac{1}{p_j} < 1 \), the partition function can be written as

\[
Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_3/4} \left( \frac{2i}{K} \right)^{1/2} \sum_{s=0}^3 \tilde{\Psi}_P^{(R_s)}(q)
\]

where

\[
\tilde{\Psi}_P^{(l)}(q) := \sum_{n=0}^\infty \psi_{2P}^{(l)}(n) q^{\frac{n^2}{2}}.
\]

It is known that \( \tilde{\Psi}_P^{(l)}(q) \) is a false theta function, which is the Eichler integral of modular form \( \Psi_P^{(l)}(q) := \sum_{n=0}^\infty n \psi_{2P}^{(l)} q^{\frac{n^2}{2}} \) of half-integer weight 3/2 [1].

The only coprime \( P_j \)'s that satisfy \( \sum_{j=1}^3 \frac{1}{p_j} > 1 \) is \( (P_1, P_2, P_3) = (2, 3, 5) \) for the case of three singular fibers. Thus, in this case, the partition function is given by

\[
Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_3/4} \left( \frac{60i}{K} \right)^{1/2} \left( 2q^{\frac{1}{20}} - \tilde{\Psi}_3^{(1)}(q) - \tilde{\Psi}_3^{(11)}(q) - \tilde{\Psi}_3^{(19)}(q) - \tilde{\Psi}_3^{(29)}(q) \right)
\]

The \( q \)-series in parenthesis takes a form of \( q^{\frac{1}{20}} \mathbb{Z}[q] \). This agrees with the result of [1] on the Poincaré homology sphere \( \Sigma(2,3,5) \). As it is an integer homology sphere, there is one homological block from the trivial flat connection, which is (2.20).

### 2.3 The case \( H \geq 2 \)

When \( H \geq 2 \), we have rational homology Seifert manifolds and there are contributions from other abelian flat connections in addition to the trivial flat connection. The periodic function \( \psi_{2P}^{(l)}(n) \) can be decomposed in terms of \( \psi_{2P_H}^{(l)}(n) \)'s as

\[
\psi_{2P}^{(l)}(n) = \psi_{2P_H}^{(l)}(n) - \psi_{2P_H}^{(2P-l)}(n) + \cdots - \psi_{2P_H}^{((H-1)P-l)}(n) + \psi_{2P_H}^{((H-1)P+l)}(n)
\]

\[
= \sum_{h=0}^{H-1} \psi_{2P_H}^{(2hP+l)}(n) - \sum_{h=0}^{H-1} \psi_{2P_H}^{(2h+1)P-l}(n)
\]


when $H$ is odd

$$
\psi_{2H}^{(l)}(n) = \psi_{2H}^{(l)}(n) - \psi_{2H}^{(2P-l)}(n) + \cdots + \psi_{2H}^{((H-2)P+l)}(n) - \psi_{2H}^{(H P-l)}(n)
$$

when $H$ is even. By introducing the floor and the ceiling function

$$
[x] = \max\{m \in \mathbb{Z} \mid m \leq x\},
$$

$$
[x] = \min\{m \in \mathbb{Z} \mid m \geq x\},
$$

(2.21) and (2.22) can be written as

$$
\psi_{2H}^{(l)}(n) = \sum_{h=0}^{[\frac{H}{2}-1]} \psi_{2H}^{(2hP+l)}(n) - \sum_{h=0}^{[\frac{H}{2}-1]} \psi_{2H}^{(2(h+1)P-l)}(n).
$$

Therefore from (2.17) and (2.25),

$$
\chi_{2P}(n) = \sum_{s=0}^{3} \left( \sum_{h=0}^{[\frac{H}{2}-1]} \psi_{2H}^{(2hP+R_s)}(n) - \sum_{h=0}^{[\frac{H}{2}-1]} \psi_{2H}^{(2(h+1)P-R_s)}(n) \right).
$$

Then, when $\sum_{j=1}^{3} \frac{1}{M_j} < 1$, the partition function is

$$
Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\frac{1}{2}} \left( \frac{2i}{K} \right)^{1/2} \left[ \sum_{n=0}^{\infty} \left( \sum_{s=0}^{3} \left( \sum_{h=0}^{[\frac{H}{2}-1]} \psi_{2H}^{(2hP+R_s)}(n) - \sum_{h=0}^{[\frac{H}{2}-1]} \psi_{2H}^{(2(h+1)P-R_s)}(n) \right) \right) q^{n^2} \right] 
$$

$$
+ \sum_{t=1}^{H-1} e^{2\pi i K \frac{t^2}{4}} \sum_{n=0}^{\infty} \left( \sum_{s=0}^{3} \psi_{2H}^{(2hP+R_s)}(n) - \sum_{h=0}^{[\frac{H}{2}-1]} \psi_{2H}^{(2(h+1)P-R_s)}(n) \right) q^{n^2} \right] 
$$

(2.27)

where we put the contribution from the trivial flat connection ($t = 0$) separately.

In (2.27), we see that $e^{2\pi i \frac{t}{4}n}$ in the summation over $n$ can be taken out of summation over $n$ when $K \in \mathbb{Z}$. Indeed, consider

$$
\sum_{t=1}^{H-1} e^{2\pi i K \frac{t^2}{4}} \sum_{n=0}^{\infty} e^{2\pi i \frac{n}{m} \psi_{2H}^{(l)}(n)} q^{n^2} 
$$

(2.28)

in (2.27). It is nonzero when $n = 2HPm + l$ and $2HPm' - l$ with $m, m' \in \mathbb{Z}_{\geq 0}$. Since $e^{2\pi i \frac{t}{4}(2HPm+l)} = e^{2\pi i \frac{t}{4}}$, the $n = 2HPm + l$ part of (2.28) is given by

$$
\sum_{t=1}^{H-1} e^{2\pi i K \frac{t^2}{4}} \sum_{n=2HPm+l \atop m \in \mathbb{Z}_{\geq 0}} e^{2\pi i \frac{n}{m} \psi_{2H}^{(l)}(n)} q^{n^2} = \sum_{t=1}^{H-1} e^{2\pi i K \frac{t^2}{4} \psi_{2H}^{(l)}(n) q^{n^2}} 
$$

(2.29)
While, for the $n = 2 HPm - l$ part of (2.28), we have

$$\sum_{t=1}^{H-1} e^{2\pi iK\frac{P}{2}t^2} e^{\infty \sum_{n=2HPm-l} \psi(2l)_{2HP}(n)q^{2\pi\frac{P}{2}n^2}} = \sum_{t=1}^{H-1} e^{2\pi iK\frac{P}{2}t^2} e^{-\sum_{n=2HPm-l} \psi(2l)_{2HP}(n)q^{2\pi\frac{P}{2}n^2}}.$$

(2.30)

Renaming $t'$ to $t = H - t$, (2.30) can be written as

$$\sum_{t'=1}^{H-1} e^{2\pi iK\frac{P}{2}t'^2 - 2Pt'(H+P)} e^{\infty \sum_{n=2HPm-l} \psi(2l)_{2HP}(n)q^{2\pi\frac{P}{2}n^2}}.$$

(2.31)

If we want to obtain the usual WRT invariant, the limit $q \downarrow e^{\frac{2\pi i}{K}}$ with $K \in \mathbb{Z}$ is taken, so $e^{2\pi iK\frac{P}{2}t'^2}$ becomes $e^{2\pi iK\frac{P}{2}t'}$. Then, we see that (2.29) and (2.31) share same $e^{2\pi i\frac{l}{l}}$, so (2.28) can be written as

$$\sum_{t=1}^{H-1} e^{2\pi iK\frac{P}{2}t^2} e^{\infty \sum_{n=0}^{\infty} \psi(2l)_{2HP}(n)q^{2\pi\frac{P}{2}n^2}}.$$

(2.32)

Thus, from (2.27), the WRT invariant is given by

$$Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\frac{3}{4}} \left( \frac{2i}{K_H} \right)^{1/2} \left[ \sum_{n=0}^{\infty} \left( \sum_{s=0}^{3} \left( \sum_{h=0}^{\frac{H-1}{2}} \psi(2hP + R_s)_{2HP}(n) - \sum_{h=0}^{\frac{H-1}{2}} \psi(2(h+1)P - R_s)_{2HP}(n) \right) \right) \right] q^{2\pi\frac{P}{2}n^2}$$

(2.33)

In terms of the false theta function $\Psi_{HP}(q)$, when $H$ is odd, (2.33) becomes

$$Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\frac{3}{4}} \left( \frac{2i}{K_H} \right)^{1/2} \left[ \sum_{n=0}^{\infty} \left( \sum_{s=0}^{3} \left( \sum_{h=0}^{\frac{H-1}{2}} \Psi(2hP + R_s)_{HP}(q) - \sum_{h=0}^{\frac{H-1}{2}} \Psi(2(h+1)P - R_s)_{HP}(q) \right) \right) \right] q^{2\pi\frac{P}{2}n^2}$$

(2.34)
When \( H \) is even,
\[
Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_3/4} \left( \frac{2i}{\mathcal{K} \mathcal{H}} \right)^{1/2} \left[ \sum_{s=0}^{\mathcal{H}/2} \sum_{h=0}^{\mathcal{H}/2} \left( \widetilde{\Psi}_{HP}^{2(h+1)P-R_+} - \widetilde{\Psi}_{HP}^{(2hP+R_+)} \right) \right] \\
+ \sum_{s=0}^{\mathcal{H}/2} \sum_{h=0}^{\mathcal{H}/2} \left( \sum_{t=1}^{\mathcal{H}/2} e^{2\pi i \frac{P}{\mathcal{H}} t^2} \left( e^{-2\pi i \frac{P}{\mathcal{H}}} (2h+1)^P R_+ + e^{2\pi i \frac{P}{\mathcal{H}}} (2hP+R_+) \right) \right) \\
+ e^{\frac{2\pi i}{\mathcal{H}}} e^{\frac{\pi i}{\mathcal{K} \mathcal{H}}} \left( e^{i \pi \frac{P}{\mathcal{H}}} (2h+1)^P R_+ - e^{i \pi \frac{P}{\mathcal{H}}} (2hP+R_+) \right) \right] \\
\left. \left| q \right| e^{-\frac{2\pi i}{\mathcal{K}}} \right| (2.35)
\]

Here, we write the expression as sum over distinct Weyl orbits where Weyl reflection is given by \( t \leftrightarrow -t \equiv H - t \mod H \). Meanwhile, for \( \sum_{j=1}^{3} \frac{1}{p_j} > 1 \), there is an additional term (2.14), which is
\[
B \left( \frac{2i}{\mathcal{K} \mathcal{H}} \right)^{1/2} \left[ \sum_{s=0}^{\mathcal{H}/2} \sum_{h=0}^{\mathcal{H}/2} \left( \sum_{j=1}^{3} \frac{1}{p_j} - 1 \right)^2 \right] \\
+ \sum_{t=1}^{\mathcal{H}/2} e^{2\pi i \frac{P}{\mathcal{H}} t^2} \left( e^{-2\pi i \frac{P}{\mathcal{H}}} (\sum_{j=1}^{3} \frac{1}{p_j} - 1)^2 + e^{2\pi i \frac{P}{\mathcal{H}}} (\sum_{j=1}^{3} \frac{1}{p_j} - 1)^2 \right) q^{\frac{P}{\mathcal{H}}} \left( \sum_{j=1}^{3} \frac{1}{p_j} - 1 \right)^2 \right] \\
\left. \left| q \right| e^{-\frac{2\pi i}{\mathcal{K}}} \right| (2.36)
\]

when \( H \) is odd, and
\[
B \left( \frac{2i}{\mathcal{K} \mathcal{H}} \right)^{1/2} \left[ \sum_{s=0}^{\mathcal{H}/2} \sum_{h=0}^{\mathcal{H}/2} \left( \sum_{j=1}^{3} \frac{1}{p_j} - 1 \right)^2 \right] \\
+ \sum_{t=1}^{\mathcal{H}/2} e^{2\pi i \frac{P}{\mathcal{H}} t^2} \left( e^{-2\pi i \frac{P}{\mathcal{H}}} (\sum_{j=1}^{3} \frac{1}{p_j} - 1)^2 + e^{2\pi i \frac{P}{\mathcal{H}}} (\sum_{j=1}^{3} \frac{1}{p_j} - 1)^2 \right) q^{\frac{P}{\mathcal{H}}} \left( \sum_{j=1}^{3} \frac{1}{p_j} - 1 \right)^2 \right] \\
+ e^{\frac{2\pi i}{\mathcal{H}}} e^{\frac{\pi i}{\mathcal{K} \mathcal{H}}} \left( e^{i \pi \frac{P}{\mathcal{H}}} (\sum_{j=1}^{3} \frac{1}{p_j} - 1) + e^{i \pi \frac{P}{\mathcal{H}}} (\sum_{j=1}^{3} \frac{1}{p_j} - 1) \right) q^{\frac{P}{\mathcal{H}}} \left( \sum_{j=1}^{3} \frac{1}{p_j} - 1 \right)^2 \right] \\
\left. \left| q \right| e^{-\frac{2\pi i}{\mathcal{K}}} \right| (2.37)
\]

when \( H \) is even.

2.4 Properties of the formula

Before providing some examples, we study properties of the formula (2.13) or (2.33).

Variable or label \( t \) in (2.27) is regarded as \( t \) of the diagonal matrix \( \text{diag}(t, -t) \in (\mathbb{Z}_H)^2 \), which gives \( \text{diag}(e^{2\pi i \frac{P}{\mathcal{H}} t}, e^{-2\pi i \frac{P}{\mathcal{H}} t}) \) of \( SU(2) \) holonomy where we note that \( H \) and \( P \) are coprime. So the Weyl group action on \((t, -t)\) gives \((-t, t)\). The Weyl orbit of \((t, -t)\) corresponds to abelian flat connections where \( t = 0 \), in particular, corresponds to the trivial flat
connection [15], c.f. section 3.1. Given a $t$, we see that the summand in (2.34) and (2.35) is invariant under $t \leftrightarrow -t$. We note that $t \leftrightarrow -t$ provide complex conjugate of $(t,-t)$ at the level of holonomy. Thus, it can be said that the contributions from the abelian flat connection corresponding to the Weyl orbit of $(t,-t)$ and from the conjugate abelian flat connection corresponding to the Weyl orbit of $(-t,t)$ are same though in the case of $SU(2)$ those two abelian flat connections are equivalent so that contributions from them are obviously same from the beginning.

In addition, the abelian flat connections that are related by the action of the center of $SU(2)$ give the same contribution. The center of $SU(2)$ is given by $e^{2\pi i/2}I_2$, $c \in \mathbb{Z}_2$, which we denote by $(c,-c)$ at the level of $(\mathbb{Z}_2)^2/\mathbb{Z}_2$. As we see below, there are cases that elements in $(\mathbb{Z}_H)^2/\mathbb{Z}_2$ are related by the nontrivial center, $(1,-1) \equiv (1,1) \mod (\mathbb{Z}_2)^2$ and it can only be possible when $H$ is even. That is because in order to relate them the center should also be expressed as $(m,-m) \in (\mathbb{Z}_H)^2$, which gives $\text{diag}(e^{2\pi i/m}, e^{-2\pi i/m})$ of $SU(2)$ holonomy, and the nontrivial center is given by $(\frac{H}{2}, -\frac{H}{2}) \in (\mathbb{Z}_H)^2$. Therefore, if $H$ is odd then $m = \frac{H}{2}$ is not an integer, so $H$ should be even.

When $H$ is even, upon $t \rightarrow t + \frac{H}{2}$, $e^{2\pi i/\pi t} + e^{-2\pi i/\pi t}$ get an additional factor $e^{\pi i l}$. From the assumption that $P_j$'s and $Q_j$'s are coprime for each $j$ and $P_j$'s are coprime to each other, $P_j$ should be all odd when $H$ is even, so $l$'s are always even for the case of three singular fibers $F = 3$. $l$ is also even for other number of singular fibers and higher genus case when $H$ is even considering (B.21) and (B.22) in section B. Hence, $e^{\pi i l} = 1$, so the abelian flat connections that are related by the center have same $e^{2\pi i/\pi t} + e^{-2\pi i/\pi t}$, so contributions from them are same.

For $e^{2\pi i K/\pi t^2}$, upon $t \rightarrow t + \frac{H}{2}$, we have an additional factor $e^{\pi i K \frac{PH}{2}}$ when $K \in \mathbb{Z}$. Therefore, when $H$ is a multiple of 2 but not of 4, there is an additional factor $e^{\pi i K}$ for the abelian flat connection $(t + \frac{H}{2}, -t - \frac{H}{2})$ compared to the case of $(t,-t)$. There is no such factor when $H$ is a multiple of 4. Thus, the contributions from abelian flat connections that are related by the action of the center can have a different coefficient by $e^{\pi i K}$.

We denote Weyl orbit of $(t,-t)$ in $(\mathbb{Z}_H)^2/\mathbb{Z}_2$ as $W_t$. When $H$ is even, elements in distinct Weyl orbits, say $W_t$ and $W_{t+\frac{H}{2}}$, which are related by the center, give the same contribution to the WRT invariant up to the overall coefficient $e^{\pi i K}$, so we group $W_t$ and $W_{t+\frac{H}{2}}$ by orbits under the action of the center, which are denoted by $C_a$ where $a$ is a label for the abelian flat connection. The range of $a$ is $a = 0, 1, \ldots, \frac{H-2}{4}$ when $H$ is multiple of 2 but not of 4, and $a = 0, 1, \ldots, \frac{H}{4}$ when $H$ is multiple of 4. We also denote elements in the Weyl orbit $W_t$ by $\tilde{t}$ and a representative of any $W_t$ in $C_b$ by $\tilde{b}$.

With the setup above, the $S$-matrix can be written as

$$S_{ab} = \frac{1}{\sqrt{\gcd(2,H)}} \sum_{W_t \in C_a} \sum_{\tilde{t} \in W_t} e^{2\pi i k (\tilde{t}, \tilde{b})} \frac{1}{|\text{Tor} H_1(M_3, \mathbb{Z})|^\frac{1}{2}} (2.38)$$
with

$$lk(t, t') = \frac{P}{H} \sum_{j=1}^{2} t_j t'_j = \frac{2P}{H} t_1 t'_1$$

(2.39)

where \( t = (t_1, -t_1) \) and \( t' = (t'_1, -t'_1) \).

We often use the notation that \( lk(a, b) := lk(\tilde{a}, \tilde{b}) \).

When \( H \) is odd, there is only a single Weyl orbit \( W_t \) in \( C_a \), which we denote by \( W_a \), as there is no non-trivial center that relates \( (t, -t) \)'s. So in that case, we simply have

$$S_{ab} = \sum_{i \in W_a} e^{2\pi i lk(i, \tilde{b})} |\text{Tor } H_1(M_3, \mathbb{Z})|^{1/2},$$

(2.41)

which agrees with the \( S \)-matrix provided in [7, 8] for odd \( H \).

Then, the WRT invariant is given by

$$Z_{SU(2)}(M_3) = \frac{B}{2t} q^{-\phi/4} (-2K)^{1/2} b_1(M_3) \left( \frac{2i}{K} \right)^{1/2} \sqrt{\text{gcd}(2, H)} \sum_{a, b} e^{\pi i K k(a, a)} S_{ab} Z_b(q) \bigg|_{q \rightarrow e^{2\pi i \frac{t}{H}}}$$

(2.42)

when \( H \) is odd or multiple of 4. Here, \( b_1(M_3) \) is the first Betti number, which is 2 when the genus of base surface is \( g \). The linking form \( lk(a, a) \) of \( a \) is the Chern-Simons invariant for abelian flat connection \( a \), so we call it \( \frac{1}{2} lk(a, a) = CS_a \). We also note that a factor \( e^{2\pi i K \frac{t}{H^2}} \) in (2.34) and (2.35) can be written as \( e^{\pi i K k(t, t)} \).

When \( H \) is multiple of 2 but not of 4, the WRT invariant can be written as

$$Z_{SU(2)}(M_3) = \frac{B}{2t} (-2K)^{1/2} b_1(M_3) q^{-\phi/4} \left( \frac{2i}{K} \right)^{1/2} \sqrt{\frac{H}{2}} \sum_{a, b} e^{\pi i K k(\tilde{a}, \tilde{a})} (I_2 \otimes S_{ab})_{ab} Z_b(q) \bigg|_{q \rightarrow e^{2\pi i \frac{t}{H}}}$$

(2.43)

where \( \tilde{a}, \tilde{b} = 0, \ldots, \frac{H}{2}, (I_2 \otimes S_{ab})_{ab} = \begin{pmatrix} S_{ab} & S_{ab} \\ S_{ab} & S_{ab} \end{pmatrix} \), and \( e^{\pi i K k(\tilde{a}, \tilde{a})} = (e^{\pi i K k(0, 0)}, \ldots, e^{\pi i K k(\frac{H}{2}, \frac{H}{2})}) \), \( e^{\pi i K (lk(0, 0) + 1)}, \ldots, e^{\pi i K (lk(\frac{H}{2} + 1, \frac{H}{2} + 1))} \). Some arrangement of the order of entries can be necessary to put the WRT invariant in the form of (2.43). We also note that \( Z_2(q) = Z_{\tilde{a} + \frac{H}{2} + 1}(q), \tilde{a} = 0, \ldots, \frac{H}{2} \) where \( Z_2(q) = (I_2 \otimes S_{ab})_{ab} Z_b(q) \) up to arrangement of the order of entries. If we ignore the issue about the center, one can also obtain \( (I_2 \otimes S_{ab})_{ab} \) in (2.43) from (2.41) and (2.39) with some possible arrangement of the order of entries.

### 2.5 Some examples with homological blocks

From the expression (2.34) and (2.35) above, we provide some concrete examples with homological blocks. We omit \( q \rightarrow e^{2\pi i \frac{t}{H}} \) in the expression from now on.

\( ^3 \)Or this can also be written as

$$S_{ab} = \frac{1}{|\text{Tor } H_1(M_3, \mathbb{Z})|^{1/2}} \sum_{W_t \in C_a} \frac{e^{2\pi i k(i, \tilde{b})} + e^{-2\pi i k(i, \tilde{b})}}{|\text{Stab}_{Z_2}(a)||\text{Tor } H_1(M_3, \mathbb{Z})|^{1/2}}.$$

(2.40)

where \( \text{Stab}_{Z_2}(a) \) is \( Z_2 \) if \( a = -a \) in \( Z_H \) or is 1 otherwise.
\( H = 2 \)

We can have \( H = 2 \), for example, from a choice \((P_1, P_2, P_3) = (3, 5, 7)\) and \((Q_1, Q_2, Q_3) = (1, 2, -5)\). Then the WRT invariant is given by

\[
Z_{SU(2)}(M_3) = \frac{B}{2t} q^{-\phi_3/4} \left( \frac{105i}{K} \right)^{1/2} \left[ \sum_{s=0}^{3} (\tilde{\Psi}_{210}^{(R_s)}(q) - \tilde{\Psi}_{210}^{(210-R_s)}(q)) \right. \\
+ e^{\pi iK} \sum_{s=0}^{3} (\tilde{\Psi}_{210}^{(R_s)}(q) - \tilde{\Psi}_{210}^{(210-R_s)}(q)) \right]
\]

(2.44)

where \( R_0 = 34, R_1 = 106, R_2 = 134, \) and \( R_3 = 146 \). Contributions come from the trivial flat connection, \((0,0)\), and the abelian flat connection, \((1,-1)\), which is central, in \((\mathbb{Z}_2)^3/\mathbb{Z}_2\). Thus, in this case, we have \( W_0 = \{(0,0)\} \) and \( W_1 = \{(1,-1)\} \), and \( C_0 = \{W_0, W_1\} \). From (2.44), their contributions, \( Z_0(q) \) and \( Z_1(q) \), are same,

\[
Z_0(q) = Z_1(q) = \sum_{s=0}^{3} (\tilde{\Psi}_{210}^{(R_s)}(q) - \tilde{\Psi}_{210}^{(210-R_s)}(q))
\]

(2.45)

where we denote \( \tilde{\Psi}^{(a)}(q) \) by \( \tilde{\Psi}_{H_P}^{(a)} \) for simplicity. Then (2.44) is written as

\[
Z_{SU(2)}(M_3) = \frac{B}{2t} q^{-\phi_3/4} \left( \frac{105i}{K} \right)^{1/2} (1 + e^{\pi iK}) Z_0(q).
\]

(2.47)

Homological blocks are

\[
\hat{Z}_0(q) = \tilde{\Psi}_{210}^{(34)} + \tilde{\Psi}_{210}^{(106)} + \tilde{\Psi}_{210}^{(134)} + \tilde{\Psi}_{210}^{(146)},
\]

\[
\hat{Z}_1(q) = -\tilde{\Psi}_{210}^{(64)} - \tilde{\Psi}_{210}^{(76)} - \tilde{\Psi}_{210}^{(104)} - \tilde{\Psi}_{210}^{(176)}
\]

(2.48) \( (2.49) \)

where \( \hat{Z}_0 = q^{Z_{\pi K}} Z[q] \) and \( \hat{Z}_1 = q^{Z_{\pi K}} Z[q] \). So in terms of homological blocks, the WRT invariant can be expressed as

\[
Z_{SU(2)}(M) = \frac{B}{2t} q^{-\phi_3/4} \left( \frac{105i}{K} \right)^{1/2} (1 + e^{\pi iK}) (\hat{Z}_0 + \hat{Z}_1).
\]

(2.50)

Or we put it in the form of

\[
Z_{SU(2)}(M) = \frac{B}{2t} q^{-\phi_3/4} \left( \frac{105i}{K} \right)^{1/2} \sum_{a,b=0}^{1} e^{2\pi i K C S_a} S_{ab} \hat{Z}_b,
\]

(2.51)

where \( S_{ab} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and \((CS_0, CS_1) = (0, \frac{1}{2})\). This \( "S" \)-matrix and \( CS_a \)'s can also be calculated from (2.41) and (2.39) with \( W_0 \) and \( W_1 \). However, it doesn’t satisfy \( S^2 = 1 \) and actually there is no \( S \)-matrix giving \( Z_a = \sum_b S_{ab} \hat{Z}_b \) that satisfy \( S^2 = 1 \) at the same time.

This case \( H = 2 \), \( \text{Tor} H_1(M_3, \mathbb{Z}) = \mathbb{Z}_2 \), was not a part of conjecture in [7, 8] where it was assumed that \( \mathbb{Z}_2 \) doesn’t appear in \( \text{Tor} H_1(M_3, \mathbb{Z}) \).\(^4\)

\(^4\)Recently, some examples with even \( H \) were also considered with issue on the center in \( SU(2) \) in [18].
• $H = 3$

$H = 3$ can be obtained, for example, by choosing $(P_1, P_2, P_3) = (2, 5, 7)$ and $(Q_1, Q_2, Q_3) = (1, 2, -6)$. Then, the WRT invariant is given by

$$Z_{SU(2)}(M_3) = \frac{B}{2} q^{-\phi_3/4} \left( \frac{140iK}{3} \right)^{1/2} \sum_{a=0}^{3} \left( \sum_{h=0}^{1} \tilde{\Psi}_{210}^{(140h+R_a)}(q) - \tilde{\Psi}_{210}^{(140-R_a)}(q) \right) + e^{\frac{2\pi i}{3} K} \sum_{a=0}^{3} \left( \sum_{h=0}^{1} e^{-\frac{2\pi i}{3} (140h+R_a)} + e^{\frac{2\pi i}{3} (140h+R_a)} \right) \tilde{\Psi}_{210}^{(140h+R_a)}(q) - (e^{\frac{2\pi i}{3} (140-R_a)} + e^{\frac{2\pi i}{3} (140-R_a)} \tilde{\Psi}_{210}^{(140-R_a)}(q) \right)$$

(2.52)

where $R_0 = 11, R_1 = 59, R_2 = 101,$ and $R_3 = 109$. There are two abelian flat connections; the trivial flat connection $W_0 = \{(0, 0)\}$ and the abelian flat connection $W_1 = \{(1, -1), (2, -2) \equiv (-1, 1)\}$ in $(\mathbb{Z}_3)^2/\mathbb{Z}_2$. From above expression, $Z_{\tilde{a}}$’s are, respectively,

$$Z_0(q) = \tilde{\Psi}_{210}^{(11)} - \tilde{\Psi}_{210}^{(31)} - \tilde{\Psi}_{210}^{(39)} + \tilde{\Psi}_{210}^{(59)} - \tilde{\Psi}_{210}^{(81)} + \tilde{\Psi}_{210}^{(101)}$$

$$+ \tilde{\Psi}_{210}^{(109)} - \tilde{\Psi}_{210}^{(129)} + \tilde{\Psi}_{210}^{(151)} + \tilde{\Psi}_{210}^{(199)} + \tilde{\Psi}_{210}^{(241)} + \tilde{\Psi}_{210}^{(249)},$$

(2.53)

$$Z_1(q) = -\tilde{\Psi}_{210}^{(11)} + \tilde{\Psi}_{210}^{(31)} - 2\tilde{\Psi}_{210}^{(39)} - \tilde{\Psi}_{210}^{(59)} - 2\tilde{\Psi}_{210}^{(81)} - \tilde{\Psi}_{210}^{(101)}$$

$$- \tilde{\Psi}_{210}^{(109)} - 2\tilde{\Psi}_{210}^{(129)} - \tilde{\Psi}_{210}^{(151)} - \tilde{\Psi}_{210}^{(199)} - \tilde{\Psi}_{210}^{(241)} + 2\tilde{\Psi}_{210}^{(249)}.$$  

(2.54)

There are two homological blocks, $\tilde{Z}_0$ and $\tilde{Z}_1$,

$$\tilde{Z}_0 = -\tilde{\Psi}_{210}^{(39)} - \tilde{\Psi}_{210}^{(81)} - \tilde{\Psi}_{210}^{(129)} + \tilde{\Psi}_{210}^{(249)},$$

$$\tilde{Z}_1 = \tilde{\Psi}_{210}^{(11)} - \tilde{\Psi}_{210}^{(31)} + \tilde{\Psi}_{210}^{(59)} + \tilde{\Psi}_{210}^{(101)} + \tilde{\Psi}_{210}^{(109)} + \tilde{\Psi}_{210}^{(151)} + \tilde{\Psi}_{210}^{(199)} + \tilde{\Psi}_{210}^{(241)}.$$  

(2.55)

(2.56)

where $\tilde{Z}_0 = q^{220} \mathbb{Z}[[q]]$ and $\tilde{Z}_1 = q^{12} \mathbb{Z}[[q]]$. From this, we obtain the $S$-matrix

$$S_{ab} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$  

(2.57)

Or the $S$-matrix can also be calculated from (2.38) and (2.39) and it agrees with (2.57). Therefore, the WRT invariant in this case can be written as

$$Z_{SU(2)}(M_3) = \frac{B}{2} q^{-\phi_3/4} \left( \frac{70iK}{3} \right)^{1/2} \sum_{a,b=0}^{1} e^{2\pi i KCS_a} S_{ab} \tilde{Z}_b$$

(2.58)

with $(CS_0, CS_1) = (0, \frac{1}{3})$, which can also be calculated from (2.39) with $C_a = \{W_a\}, a = 0, 1$.

• $H = 4$

We can have $H = 4$ by taking, for example, $(P_1, P_2, P_3) = (3, 5, 7)$ and $(Q_1, Q_2, Q_3) = (1, -4, -3)$. In this case, there are three abelian flat connections, $W_0 = \{(0, 0)\}, W_1 = \{(1, -1), (3, -3)\}$, and $W_2 = \{(2, -2)\}$ in $(\mathbb{Z}_4)^2/\mathbb{Z}_2$. 
The WRT invariant is given by

\[
Z_{SU(2)}(M) = \frac{B}{2i} q^{-\phi_{3}/4} \left(\frac{105i}{2K}\right)^{1/2} (Z_0 + e^{\Psi K} Z_1 + Z_2).
\] (2.59)

Here we denoted the contributions from \(W_0, W_1, \) and \(W_2\) by \(Z_0, Z_1, \) and \(Z_2,\) respectively, which are

\[
Z_0 = \sum_{s=0}^{3} \sum_{h=0}^{1} \left( \tilde{\Psi}_{420}^{(210h+R_s)}(q) - \tilde{\Psi}_{420}^{(210(h+1)-R_s)}(q) \right),
\] (2.60)

\[
Z_1 = \sum_{s=0}^{3} \sum_{h=0}^{1} \left( 2 \cos \left( \frac{\pi}{2} (210h + R_s) \right) \tilde{\Psi}_{420}^{(210h+R_s)}(q) - 2 \cos \left( \frac{\pi}{2} (210(h+1) - R_s) \right) \tilde{\Psi}_{420}^{(210(h+1)-R_s)}(q) \right),
\] (2.61)

\[
Z_2 = \sum_{s=0}^{3} \sum_{h=0}^{1} \left( \tilde{\Psi}_{420}^{(210h+R_s)}(q) - \tilde{\Psi}_{420}^{(210(h+1)-R_s)}(q) \right) = Z_0
\] (2.62)

where \(R_0 = 34, R_1 = 106, R_2 = 134, \) and \(R_3 = 146.\) More specifically,

\[
Z_0 = Z_2
\]

\[
= \tilde{\Psi}_{420}^{(34)} - \tilde{\Psi}_{420}^{(64)} - \tilde{\Psi}_{420}^{(76)} - \tilde{\Psi}_{420}^{(104)} + \tilde{\Psi}_{420}^{(106)} + \tilde{\Psi}_{420}^{(134)} + \tilde{\Psi}_{420}^{(146)} - \tilde{\Psi}_{420}^{(176)}
\] (2.63)

\[
+ \tilde{\Psi}_{420}^{(244)} - \tilde{\Psi}_{420}^{(274)} - \tilde{\Psi}_{420}^{(286)} - \tilde{\Psi}_{420}^{(314)} + \tilde{\Psi}_{420}^{(316)} + \tilde{\Psi}_{420}^{(344)} + \tilde{\Psi}_{420}^{(356)} - \tilde{\Psi}_{420}^{(386)},
\]

\[
\frac{1}{2} Z_1 = - \tilde{\Psi}_{420}^{(34)} - \tilde{\Psi}_{420}^{(64)} - \tilde{\Psi}_{420}^{(76)} - \tilde{\Psi}_{420}^{(104)} - \tilde{\Psi}_{420}^{(106)} - \tilde{\Psi}_{420}^{(134)} - \tilde{\Psi}_{420}^{(146)} - \tilde{\Psi}_{420}^{(176)}
\] (2.64)

\[
+ \tilde{\Psi}_{420}^{(244)} + \tilde{\Psi}_{420}^{(274)} + \tilde{\Psi}_{420}^{(286)} + \tilde{\Psi}_{420}^{(314)} + \tilde{\Psi}_{420}^{(316)} + \tilde{\Psi}_{420}^{(344)} + \tilde{\Psi}_{420}^{(356)} + \tilde{\Psi}_{420}^{(386)}
\]

\[
W_0 \text{ and } W_2 \text{ are in the same orbit by the center, } C_0 = \{W_0, W_2\}, \text{ and we see that } Z_0 \text{ and } Z_2 \text{ are same as discussed in section 2.4.}
\]

From \(Z_0,\) the homological blocks are given by

\[
\tilde{Z}_0 = - \tilde{\Psi}_{420}^{(64)} - \tilde{\Psi}_{420}^{(76)} - \tilde{\Psi}_{420}^{(104)} - \tilde{\Psi}_{420}^{(176)} + \tilde{\Psi}_{420}^{(244)} + \tilde{\Psi}_{420}^{(274)} + \tilde{\Psi}_{420}^{(286)} - \tilde{\Psi}_{420}^{(314)} - \tilde{\Psi}_{420}^{(316)} - \tilde{\Psi}_{420}^{(344)} - \tilde{\Psi}_{420}^{(356)} + \tilde{\Psi}_{420}^{(386)},
\] (2.65)

\[
\tilde{Z}_1 = \tilde{\Psi}_{420}^{(34)} + \tilde{\Psi}_{420}^{(104)} + \tilde{\Psi}_{420}^{(134)} + \tilde{\Psi}_{420}^{(146)} - \tilde{\Psi}_{420}^{(274)} - \tilde{\Psi}_{420}^{(286)} - \tilde{\Psi}_{420}^{(314)} + \tilde{\Psi}_{420}^{(386)}
\] (2.66)

where \(\tilde{Z}_0 = q^{\frac{a}{2}a} Z[[q]] \) and \(\tilde{Z}_1 = q^{\frac{b}{2}b} Z[[q]].\) So we have \(Z_0 = \tilde{Z}_0 + \tilde{Z}_1\) and \(\frac{1}{2} Z_1 = \tilde{Z}_0 - \tilde{Z}_1.\) Thus, the S-matrix is given by

\[
S_{ab} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\] (2.67)

This can also be calculated from (2.38) and (2.39) with \(C_0 = \{W_0, W_2\} \) and \(C_1 = \{W_1\}.\) Thus, (2.59) can be written by

\[
Z_{SU(2)}(M_3) = \frac{B}{i} q^{-\phi_{3}/4} \left(\frac{105i}{K}\right)^{1/2} \sum_{a,b=0}^{1} e^{2\pi i KCS_a S_{ab} \tilde{Z}_b}
\] (2.68)

where \((CS_0, CS_1) = (0, \frac{1}{3}),\) which can also be obtained from (2.39).
\( \bullet H = 5 \)

We take, for example, \((P_1, P_2, P_3) = (2, 3, 7)\) and \((Q_1, Q_2, Q_3) = (1, 1, -5)\) to have \(H = 5\). There are three abelian flat connections \(W_0 = \{(0, 0)\}, W_1 = \{(1, -1), (4, -4)\}, \) and \(W_2 = \{(2, -2), (3, -3)\}\) and we denote their contributions by \(Z_0, Z_1, \) and \(Z_2, \) respectively. Then the WRT invariant is

\[
Z_{SU(2)}(M_3) = \frac{B}{2\pi} \sum_{0}^{3} \left( \frac{84^{i}}{5K} \right)^{1/2} \left( Z_0 + \frac{e^{4piK}}{5} Z_1 + \frac{e^{6piK}}{5} Z_2 \right) \tag{2.69}
\]

with

\[
Z_0 = \sum_{s=0}^{3} \left( \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) - \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.70}
\]

\[
Z_1 = \sum_{s=0}^{3} \left( \sum_{h=0}^{2} (e^{-\frac{4pi}{5}(84+h-R_s)} + e^{\frac{4pi}{5}(84+h-R_s)}) \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) \right.
\]

\[
\left. - \sum_{h=0}^{1} (e^{-\frac{4pi}{5}(84(h+1)-R_s)} + e^{\frac{4pi}{5}(84(h+1)-R_s)}) \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.71}
\]

\[
Z_2 = \sum_{s=0}^{3} \left( \sum_{h=0}^{2} (e^{-\frac{4pi}{5}(84+h-R_s)} + e^{\frac{4pi}{5}(84+h-R_s)}) \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) \right.
\]

\[
\left. - \sum_{h=0}^{1} (e^{-\frac{4pi}{5}(84(h+1)-R_s)} + e^{\frac{4pi}{5}(84(h+1)-R_s)}) \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.72}
\]

where \(R_0 = 1, R_1 = 41, R_2 = 55, \) and \(R_3 = 71.\) More explicitly,

\[
Z_0(q) = \sum_{s=0}^{3} \left( \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) - \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.73}
\]

\[
Z_1(q) = B \sum_{s=0}^{3} \left( \sum_{h=0}^{2} (e^{-\frac{4pi}{5}(84+h-R_s)} + e^{\frac{4pi}{5}(84+h-R_s)}) \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) \right.
\]

\[
\left. - \sum_{h=0}^{1} (e^{-\frac{4pi}{5}(84(h+1)-R_s)} + e^{\frac{4pi}{5}(84(h+1)-R_s)}) \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.74}
\]

\[
Z_2(q) = A \sum_{s=0}^{3} \left( \sum_{h=0}^{2} (e^{-\frac{4pi}{5}(84+h-R_s)} + e^{\frac{4pi}{5}(84+h-R_s)}) \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) \right.
\]

\[
\left. - \sum_{h=0}^{1} (e^{-\frac{4pi}{5}(84(h+1)-R_s)} + e^{\frac{4pi}{5}(84(h+1)-R_s)}) \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.75}
\]

where \(A := \frac{1}{\pi}(-\sqrt{5} - 1)\) and \(B := \frac{1}{\pi}(\sqrt{5} - 1)\) for convenience. The homological blocks are given by

\[
\hat{Z}_0 = \sum_{s=0}^{3} \left( \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) - \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.76}
\]

\[
\hat{Z}_1 = \sum_{s=0}^{3} \left( \sum_{h=0}^{2} (e^{-\frac{4pi}{5}(84+h-R_s)} + e^{\frac{4pi}{5}(84+h-R_s)}) \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) \right.
\]

\[
\left. - \sum_{h=0}^{1} (e^{-\frac{4pi}{5}(84(h+1)-R_s)} + e^{\frac{4pi}{5}(84(h+1)-R_s)}) \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.77}
\]

\[
\hat{Z}_2 = \sum_{s=0}^{3} \left( \sum_{h=0}^{2} (e^{-\frac{4pi}{5}(84+h-R_s)} + e^{\frac{4pi}{5}(84+h-R_s)}) \sum_{h=0}^{2} \sum_{h=0}^{2} (84+h-R_s)(q) \right.
\]

\[
\left. - \sum_{h=0}^{1} (e^{-\frac{4pi}{5}(84(h+1)-R_s)} + e^{\frac{4pi}{5}(84(h+1)-R_s)}) \sum_{h=0}^{2} (84(h+1)-R_s)(q) \right), \tag{2.78}
\]
where \( \tilde{Z}_0(q) = q^{\frac{266}{385}}Z[[q]] \), \( \tilde{Z}_1(q) = q^{\frac{169}{505}}Z[[q]] \), and \( \tilde{Z}_2(q) = q^{\frac{461}{507}}Z[[q]] \). Therefore, the S-matrix is given by

\[
S_{ab} = \frac{1}{\sqrt{5}} \begin{pmatrix}
1 & 1 & 1 \\
2 & \frac{1}{2}(\sqrt{5} - 1) & \frac{1}{2}(\sqrt{5} - 1) \\
2 & \frac{1}{2}(\sqrt{5} - 1) & \frac{1}{2}(\sqrt{5} - 1)
\end{pmatrix} \quad (2.79)
\]

This can also be calculated from (2.38) and (2.39) with \( C_a = \{W_a\} \), \( a = 0, 1, 2 \). Thus, (2.69) can be written as

\[
Z_{SU(2)}(M_3) = \frac{B}{2} q^{-\phi_3/4} \left( \frac{42i}{K} \right)^{1/2} \sum_{a,b=0}^2 e^{2\pi i K S_{ab}} S_{ab} \tilde{Z}_b(q) \quad (2.80)
\]

where \((CS_0, CS_1, CS_2) = (0, \frac{2}{5}, \frac{3}{5})\), which also agree with the calculation from (2.39).

\bullet \quad H = 6

We take, for example, \((P_1, P_2, P_3) = (5, 7, 11)\) and \((Q_1, Q_2, Q_3) = (-2, 8, -8)\) to have \( H = 6 \). There are four abelian flat connections, \( W_0 = \{(0,0)\}, W_1 = \{(1,-1), (5, -5)\}, W_2 = \{(2, -2), (4, -4)\}, \) and \( W_3 = \{(3, -3)\} \). We denote their contributions by \( Z_0, Z_1, Z_2, \) and \( Z_3 \), respectively. Then, the WRT invariant is given by

\[
Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_3/4} \left( \frac{385i}{3K} \right)^{1/2} \left( Z_0 + e^{2\pi i K} \hat{Z}_1 + e^{\pi i K} Z_2 + e^{2\pi i K} \hat{Z}_3 \right) \quad (2.81)
\]

where

\[
Z_0 = \sum_{s=0}^3 \sum_{h=0}^2 \left( \tilde{\Psi}_{2310}^{(770 h + R_s)} (q) - \tilde{\Psi}_{2310}^{(770 (h+1) - R_s)} (q) \right), \quad (2.82)
\]

\[
Z_1 = \sum_{s=0}^3 \sum_{h=0}^2 \left( e^{-2\pi i \frac{1}{6} (770 h + R_s)} - e^{2\pi i \frac{1}{6} (770 h + R_s)} \right) \tilde{\Psi}_{2310}^{(770 h + R_s)} (q) - \left( e^{-2\pi i \frac{1}{6} (770 (h+1) - R_s)} + e^{2\pi i \frac{1}{6} (770 (h+1) - R_s)} \right) \tilde{\Psi}_{2310}^{(770 (h+1) - R_s)} (q), \quad (2.83)
\]

\[
Z_2 = \sum_{s=0}^3 \sum_{h=0}^2 \left( e^{-2\pi i \frac{1}{6} (770 h + R_s)} + e^{2\pi i \frac{1}{6} (770 h + R_s)} \right) \tilde{\Psi}_{2310}^{(770 h + R_s)} (q) - \left( e^{-2\pi i \frac{1}{6} (770 (h+1) - R_s)} + e^{2\pi i \frac{1}{6} (770 (h+1) - R_s)} \right) \tilde{\Psi}_{2310}^{(770 (h+1) - R_s)} (q), \quad (2.84)
\]

\[
Z_3 = \sum_{s=0}^3 \sum_{h=0}^2 \left( e^{\pi i (770 h + R_s)} \tilde{\Psi}_{2310}^{(770 h + R_s)} (q) - e^{\pi i (770 (h+1) - R_s)} \tilde{\Psi}_{2310}^{(770 (h+1) - R_s)} (q) \right) \quad (2.85)
\]
with $R_0 = 218$, $R_1 = 398$, $R_2 = 442$, and $R_3 = 482$. From them, we have

\[
Z_0 = Z_3 = 0(218)_{2310} - 0(288)_{2310} - 0(328)_{2310} - 0(372)_{2310} + 0(398)_{2310} + 0(442)_{2310} + 0(482)_{2310} - 0(552)_{2310} + 0(528)_{2310} - 0(1058)_{2310} - 0(1098)_{2310} - 0(1142)_{2310} + 0(1168)_{2310} + 0(1212)_{2310} + 0(1252)_{2310} - 0(1322)_{2310} + 0(1758)_{2310} - 0(1828)_{2310} - 0(1868)_{2310} - 0(1912)_{2310} + 0(1938)_{2310} + 0(1982)_{2310} + 0(2022)_{2310} + 0(2092)_{2310}.
\]

\[
Z_1 = Z_2 = -0(218)_{2310} - 2\Psi(288)_{2310} + 0(328)_{2310} - 2\Psi(372)_{2310} - 2\Psi(398)_{2310} - 2\Psi(442)_{2310} - 2\Psi(482)_{2310} - 2\Psi(552)_{2310} - 2\Psi(1058)_{2310} - 2\Psi(1098)_{2310} - 2\Psi(1142)_{2310} + 2\Psi(1168)_{2310} + 2\Psi(1212)_{2310} + 2\Psi(1252)_{2310} + 2\Psi(1322)_{2310} + 2\Psi(1758)_{2310} + 2\Psi(1828)_{2310} + 2\Psi(1868)_{2310} + 2\Psi(1912)_{2310} + 2\Psi(1938)_{2310} + 2\Psi(1982)_{2310} + 2\Psi(2022)_{2310} + 2\Psi(2092)_{2310}.
\]

As explained in section 2.4, $W_0$ and $W_3$ (resp. $W_1$ and $W_2$) are in the same orbit $C_0$ (resp. $C_1$) under the action of center and give same contribution up to $e^{\pi iK}$. Therefore, (2.81) can be written

\[
Z_{SU(2)}(M_3) = B 2^q q^{-\phi_3/4} \left( \frac{385i}{3K} \right)^{1/2} \left( 1 + e^{\pi iK} \right) \left( Z_0 + e^{2\pi iK} \hat{Z}_1 \right).
\]

Homological blocks are given by

\[
\hat{Z}_0 = -0(288)_{2310} - 0(372)_{2310} - 0(552)_{2310} + 0(1212)_{2310},
\]

\[
\hat{Z}_1 = -0(328)_{2310} + 0(288)_{2310} + 0(1168)_{2310} + 0(1252)_{2310} - 0(1828)_{2310} - 0(1868)_{2310} - 0(1912)_{2310} - 0(1938)_{2310} + 0(1982)_{2310} + 0(2022)_{2310} + 0(2092)_{2310},
\]

\[
\hat{Z}_2 = -0(1098)_{2310} + 0(1758)_{2310} + 0(1938)_{2310} + 0(2022)_{2310},
\]

\[
\hat{Z}_3 = 0(218)_{2310} + 0(372)_{2310} + 0(442)_{2310} - 0(1058)_{2310} - 0(1142)_{2310} - 0(1322)_{2310} + 0(1938)_{2310} + 0(1982)_{2310} + 0(2022)_{2310}.
\]

where $\hat{Z}_0 = q^{2850}Z[[q]]$, $\hat{Z}_1 = q^{4344}Z[[q]]$, $\hat{Z}_2 = q^{4944}Z[[q]]$, and $\hat{Z}_3 = q^{1344}Z[[q]]$. So $Z_0 = Z_3 = \hat{Z}_0 + \hat{Z}_1 + \hat{Z}_2 + \hat{Z}_3$ and $Z_1 = Z_2 = 2\hat{Z}_0 - \hat{Z}_1 + 2\hat{Z}_2 - \hat{Z}_3$. We put them in a form of

\[
\frac{1}{\sqrt{3}} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} Z_0 \\ Z_0 \\ Z_0 \end{pmatrix} \begin{pmatrix} \hat{Z}_0 \\ \hat{Z}_1 \\ \hat{Z}_2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \hat{Z}_0 \\ \hat{Z}_1 \\ \hat{Z}_2 \end{pmatrix}.
\]

The S-matrix

\[
\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}
\]

also appeared in the case $H = 3$ where we note that $Z_6 = Z_2 \times Z_3$. It can also be calculated from (2.38) and (2.39) with $C_0 = \{W_0, W_3\}$ and $C_1 = \{W_1, W_2\}$. Or $(I_2 \times S_{ab})_{\alpha \beta}$ can also be calculated from (2.41) and (2.39) where indices run over 0, 1, 3, and 2. Thus, the WRT invariant can be written as

\[
Z_{SU(2)}(M_3) = B 2^q q^{-\phi_3/4} \left( \frac{385i}{3K} \right)^{1/2} \sum_{\alpha, \beta = 0}^{3} e^{2\pi iKCS_a} \left( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes S_{ab} \right)_{\alpha \beta} \hat{Z}_a(q).
\]
2.6 Resurgent analysis

We discuss the resurgent analysis on the expression we obtained. We consider the case \( H = 1 \). The overall factor of the analytically continued Chern-Simons partition function or the WRT invariant is proportional to \( 1/\sqrt{K} \). Since the partition functions with \( H = 1 \) are given by linear combination of \( \tilde{\Psi}_P^{(l)}(q) \), we consider first the resurgent analysis for \( \frac{1}{\sqrt{K}} \tilde{\Psi}_P^{(l)}(q) \) [9].

Perturbative expansion of \( \frac{1}{\sqrt{K}} \tilde{\Psi}_P^{(l)}(q) \) is \( \tilde{\Psi}_P^{(l)\text{pert}} := \frac{1}{\sqrt{K}} \tilde{\Psi}_P^{(l)} \), so its Borel transform is given by

\[
B\tilde{\Psi}_P^{(l)\prime}(\xi) = \sum_{n=0}^{\infty} \psi_2^{(l)}(n) \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{i\pi n^2}{2P} \right)^m \frac{1}{\Gamma(m+\frac{1}{2})} \xi^{m-\frac{1}{2}},
\]

(2.94)

so

\[
\frac{1}{\sqrt{\pi\xi}} \sinh(P-l)(\frac{2\pi\xi}{P})^{\frac{1}{2}}
\]

(2.95)

Here, \( \text{Re } y > 0 \) and also \( P > 0 \) are assumed and

\[
\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi} (2m)!}{2^{2m} m!}
\]

(2.96)

and

\[
\frac{\sinh(P-l)y}{\sinh Py} = \sum_{n=0}^{\infty} \psi_2^{(l)}(n) e^{-ny}
\]

(2.97)

are used. We note that we have seen (2.97) in (2.8) with (2.17).

After taking \( y \to y/P \) in (2.97) and then multiplying \( e^{-\frac{2\pi \xi}{P} y^2} \) at each side of equality of (2.97), we integrate them over \( \epsilon + i\mathbb{R} \) with \( \epsilon > 0 \),

\[
\int_{\epsilon+i\mathbb{R}} dy \frac{\sinh(P-l)y}{\sinh y} e^{-\frac{2\pi \xi}{P} y^2} = \int_{\epsilon+i\mathbb{R}} dy \sum_{n=0}^{\infty} \psi_2^{(l)}(n) e^{-ny} e^{-\frac{2\pi \xi}{P} y^2},
\]

(2.98)

where we note that the RHS is the integral we evaluated in section 2.1 and it gives \( \frac{1}{\sqrt{K}}(2\pi^2 iP)^{\frac{1}{2}} \tilde{\Psi}_2^{(l)}(q) \). Integrating the LHS after changing of variable \( \xi = \frac{2\pi i}{P} y^2 \), we finally have [9]

\[
\frac{1}{\sqrt{K}} \tilde{\Psi}_P^{(l)}(q) = \frac{1}{2} \left( \int_{\epsilon+i\mathbb{R}^+} + \int_{\epsilon-i\mathbb{R}^+} \right) d\xi \frac{\sinh(P-l)(\frac{2\pi\xi}{P})^{\frac{1}{2}}}{\sqrt{\pi\xi}} \sinh(P(\frac{2\pi\xi}{P})^{\frac{1}{2}}) e^{-K\xi}
\]

(2.99)

\[
= \frac{1}{2} \left( \int_{\epsilon+i\mathbb{R}^+} + \int_{\epsilon-i\mathbb{R}^+} \right) d\xi B\tilde{\Psi}_P^{(l)\prime}(\xi) e^{-K\xi}.
\]

(2.100)

where \( \delta \) is a small real number and direction of contour is from the origin to infinity. Therefore, the original \( \tilde{\Psi}_P^{(l)}(q) \) can be recovered as the average of the Borel sum of \( B\tilde{\Psi}_P^{(l)\prime}(\xi) \).
As the analytically continued CS partition function on Seifert manifolds with three singular fibers and with $H = 1$ for abelian flat connections are given by linear combination of $\tilde{\Psi}^{(a)}_P(q)$, we have

$$
Z_{SU(2)}^{ab}(M_3) = \frac{B}{2i} q^{-\phi_3/4} (2iP)^{1/2} \sum_{s=0}^{3} \frac{1}{2} \left( \int_{\mathbb{C}^+} + \int_{\mathbb{C}^-} \right) d\xi B \tilde{\Psi}^{(R_s)}_P(\xi) e^{-K\xi}. \tag{2.101}
$$

The residues at poles of this integral are the contribution from the irreducible or nonabelian flat connections in the resurgent analysis [9]. That is, by deforming the integral contour for $K \in \mathbb{Z}_+$, the integral picks the poles and calculation of the residue agrees with the known results on the contributions from non-abelian flat connections. Thus, the $\tilde{\Psi}^{(a)}_P(q)$ or their linear combinations, which are obtained from Borel sum of the perturbative expansion around the trivial flat connection and are convergent $q$-series, contain the contribution from non-abelian flat connections as transseries, so $Z_{SU(2)}^{ab}(M_3)$ in (2.101), which is expressed in the context of the resurgent analysis, provide a full exact partition function $Z_{SU(2)}(M_3)$ when $H = 1$. Meanwhile, as we noted earlier, (2.101) or (2.98) in the resurgent analysis are basically the integral formula (2.3) with $H = 1$ from which we obtained (2.13). So, we obtained a convergent $q$-series (2.13) from the exact expression (2.3) for reducible flat connections with $H = 1$, which is basically (2.101) or (2.98) in the resurgent analysis, and the resurgent analysis tells that (2.13) is indeed a full exact partition function. In the context of the exact formula of the WRT invariants in [15], contributions from nonabelian flat connections also appear as the residue part and we can see, for example, that the residues in [15] agree with the result of [9, 19] on $F = 3$ and $H = 1$.

Also when $H \geq 2$, residues are interpreted as the contribution from nonabelian flat connections in [15] as well as in the resurgent analysis. We expect that the residues corresponding to nonabelian flat connections attached to abelian flat connections in the resurgent analysis agree with the residues calculated in [15]. In other words, we expect that (2.27) with $H \geq 2$ also provide full exact partition function. Similarly, we also expect that the partition functions that we calculate for arbitrary number of singular fibers $F$, $H$, and genus $g$ in section A and section B provide a full exact partition function.

### 2.7 Other ranges of $P$ and $H$ and reversed orientation of $M_3$

So far, we have considered the case that $P$ and $H$ are both positive. We also assumed that $\text{Im} K < 0 \iff |q| < 1$. We consider other ranges of their values.

In the expansion (2.8), it was assumed that $\text{Re}(y/P) > 0$. This is obtained when $\text{Re} y > 0$ and $P > 0$ and also $\text{Re} y < 0$ and $P < 0$. The former case with $H > 0$ has been considered in previous sections with $|q| < 1$.

#### 2.7.1 Reversing orientation of $M_3$

In [15], though the residue part looks different as it contains additional factor in the denominator, they agree with the result of [9, 19] on $F = 3$ and $H = 1$. More specifically, when the number of singular fibers is three, we can show that $\sum_{m=1}^{2P-1} \left( \sum_{j=1}^{3} \frac{1}{P_j} \cot(\frac{m\pi}{P_j}) \right) \prod_{j=1}^{3} \sin(\frac{m\pi}{P_j})$ in [15] vanishes, while the other part $\left( \frac{2}{P} - \frac{1}{P_j} \right) \prod_{j=1}^{3} \sin(\frac{2\pi}{P_j})$ in the residue is nonzero. When $H = 1$, this agrees with the result of [9, 19] on Brieskorn spheres. We found that recently resurgent analysis on arbitrary $F$ with $H = 1$ was considered in [20].
When $H < 0$ with Re $y > 0$ and $P > 0$, the coefficient of $y^2$, which is $-\frac{K}{2\pi i H}$, in the exponential factor in (2.12) needs to be positive so that we can choose a contour that passes through a fixed Re $y$ and extends along the imaginary axis of $y$-plane for the consistency of the calculation. This requires that Im $K > 0 \Leftrightarrow |q| > 1$. Then the integral gives $q^{\frac{q^2}{4\pi i H}}$ in the expression and $|q^{\frac{q^2}{4\pi i H}}| < 1$. Therefore, $\sum_{n=0}^{\infty} \psi^{(a)}_{2P} q^{\frac{q^2}{4\pi i H}}$ is convergent. In sum, if Re $y > 0$ and $P > 0$, the calculation is consistent when $H > 0$ and Im $K < 0 \Leftrightarrow |q| < 1$ or $H < 0$ and Im $K > 0 \Leftrightarrow |q| > 1$.

We can do similar analysis with Re $y < 0$ and $P < 0$. The calculation is consistent if $H > 0$ and Im $K > 0 \Leftrightarrow |q| > 1$ or $H < 0$ and Im $K < 0 \Leftrightarrow |q| < 1$.

When Re$(y/P) < 0$, the RHS of the expansion (2.8) should be $\sum_{n=0}^{\infty} \chi_{2P}(n)e^{\frac{\pi i}{H} n^2}$. This change only leads to the change of the exponent, $e^{\frac{2\pi i n}{H}} \rightarrow e^{-\frac{2\pi i n}{H}}$ in (2.13). But as we saw in (2.34) and (2.35), this doesn’t affect the final result. This case Re$(y/P) < 0$ is obtained when Re $y > 0$ and $P < 0$ and also when Re $y < 0$ and $P > 0$. Summarizing the analysis, if Re $y > 0$ and $P < 0$, $H \geq 0$ and Im $K \geq 0 \Leftrightarrow |q| \geq 1$, respectively, provide consistent calculations. Similarly, if Re $y < 0$ and $P > 0$, the calculation is consistent when $H \geq 0$ and Im $K \leq 0 \Leftrightarrow |q| \leq 1$, respectively.

In summary, whatever Re $y$ is, the calculation is consistent when

\begin{align*}
P > 0, H > 0, \text{ Im } K < 0 &\Leftrightarrow |q| < 1, \quad P > 0, H < 0, \text{ Im } K > 0 \Leftrightarrow |q| > 1, \quad (2.102) \\
P < 0, H > 0, \text{ Im } K > 0 &\Leftrightarrow |q| > 1, \quad P < 0, H < 0, \text{ Im } K < 0 \Leftrightarrow |q| < 1. \quad (2.103)
\end{align*}

The results stay same in those ranges just with a few differences. For example, when $P > 0$ and $H < 0$ so $|q| > 1$, we obtain

\begin{equation}
Z_{SU(2)}(M_3) = \frac{B}{2\pi} q^{-\phi_3/4} \left( \frac{2\pi}{K H} \right)^{1/2} \sum_{t=0}^{[H]^{-1}} e^{2\pi i K t^2} \sum_{n=0}^{\infty} \chi_{2P}(n) e^{2\pi i \frac{t^2}{H} n^2} q^{\frac{n^2}{4\pi i H}}. \tag{2.104}
\end{equation}

with $\chi_{2P}(n)$ is expressed in terms of $\psi^{(a)}_{2P}(n)$. Other cases are also similar with $P$ or/and $H$ are replaced with $|P|$ or/and $|H|$ appropriately.

This has an implication on the reverse of orientation of $M_3$ and sign of $K$. In above calculation and also below where we have set $b$ of Seifert invariant to be zero, the reverse of orientation of Seifert manifold of $M_3 = X(P_1/Q_1, \cdots, P_F/Q_F)$ is realized by the change of signs of all $Q_j$’s,

\begin{equation}
M_3 = X(P_1/Q_1, \cdots, P_F/Q_F) \rightarrow -M_3 = X(-P_1/Q_1, \cdots, -P_F/Q_F). \tag{2.105}
\end{equation}

This also leads to $H \rightarrow -H$ as $H := P \sum_{j=1}^F \frac{Q_j}{P_j}$. We see from (2.102) and (2.103) that given a $P$ when $H$ changes its sign, then also the sign of Im $K$ should be changed. This is consistent with the expectation that when the orientation of $M_3$ is reversed to $-M_3$ and $q$ is inverted to $q^{-1}$ it gives the same partition function [8, 18], i.e. $Z_G(M_3, K) = Z_G(-M_3, -K)$. 

\[ -22 - \]
3 Higher rank gauge group

The expression of the WRT invariants of Seifert manifolds for the ADE gauge group with arbitrary number of singular fibers \( F \) was obtained in [16]. The contributions from the reducible flat connections for gauge group \( G \) are given by

\[
\frac{(-1)^{\Delta_+}}{|\mathcal{W}|(2\pi i)^r} \left( \frac{\text{Vol } \Lambda_w}{\text{Vol } \Lambda_r} \right)^{|\Delta_+|} \frac{\text{sign}(P)^{|\Delta_+|}}{|P|^{r/2}} e^{\frac{\pi i}{2} \text{sign}(H/P) - \frac{\pi d d_{\phi}}{4}} \sum_{t \in \Lambda_r/H \Lambda_r} \int_{\Gamma^r} d\beta \, e^{-\frac{K}{2} \beta \cdot \phi} \prod_{j=1}^F \Pi_{\alpha > 0} \frac{2 \sinh \frac{2 \pi \alpha}{\phi}}{2 \sinh \frac{2 \pi \alpha}{\phi}} \prod_{\alpha > 0} \left( 2 \sinh \frac{2 \pi \alpha}{\phi} \right)^{-2}
\]

(3.1)

where \( r, d, \) and \( c_0 \) denote the rank, the dimension of group \( G, \) and the dual Coxeter number of Lie algebra \( g \) of \( G, \) respectively. Also \( |\Delta_+|, \mathcal{W}, \Lambda_r, \) and \( \Lambda_w \) denote, respectively, the number of positive roots, Weyl group, the volume of root and weight lattice. \( K \) is quantum corrected CS level \( K = k + c_0. \) Here \( \Gamma^r = \Gamma \times \cdots \times \Gamma \) is a multiple contour of \( \Gamma \) in \( \mathbb{C}^r \) where \( \Gamma \) is the contour discussed in previous section. Elements \( t \in \Lambda_r/H \Lambda_r \) correspond to reducible flat connections where those related by Weyl reflections are regarded as equivalent reducible flat connections. Also, the case \( t = 0 \) corresponds to the trivial flat connection.

As done in the \( SU(2) \) case, we would like to express (3.1) in terms of the \( q \)-series with integer coefficients and integer powers. In this paper, we consider the case \( G = SU(N). \) We leave the cases of other gauge groups as future work.

3.1 Calculation of the partition function

The integral and summation part\(^6\) of (3.1) for \( G = SU(N) \) is

\[
\sum_{t \in \Lambda_r/H \Lambda_r} \int_{\Gamma^r} d\beta_1 \ldots d\beta_{N-1} \, e^{\frac{K}{2} \beta_1^2 + \sum_{i<j} \beta_i \beta_j} - K(2 \sum_{i=1}^{N-1} \beta_i + \sum_{i \neq j}^{N-1} \beta_i) + K(2 \sum_{i=1}^{N-1} \beta_i + \sum_{i \neq j}^{N-1} \beta_j) \times \prod_{j=1}^F \prod_{i<j} \frac{2 \sinh \frac{1}{2} (\beta_i - \beta_j)}{2 \sinh \frac{1}{2} (\beta_i - \beta_j)} \prod_{i<j} (2 \sinh \frac{1}{2} (\beta_i - \beta_j))^2
\]

(3.3)

where \( \beta_N = -\sum_{i=1}^{N-1} \beta_i. \) We consider the case \( F = 3 \) with \( \sum_j \frac{1}{\gamma_j} < 1 \) for simplicity. For larger number of fibers, higher genus, or \( \sum_j \frac{1}{\gamma_j} > 1, \) we can just use the formula in section B in the following calculations.

Given a pair \( i \) and \( j \) with \( i < j \) and \( j \neq N, \) we have

\[
\prod_{j=1}^3 \frac{e^{\frac{1}{2} (\beta_i - \beta_j)} - e^{-\frac{1}{2} (\beta_i - \beta_j)}}{e^{\frac{1}{2} (\beta_i - \beta_j)} - e^{-\frac{1}{Z} (\beta_i - \beta_j)}} = \sum_{n_{i,j}=0}^{\infty} \chi_{2P}(n_{i,j}) e^{\frac{1}{2} \Pi_{n_{i,j}} (\beta_i - \beta_j)},
\]

(3.4)

\(^6\) When \( G = SU(N), \) the overall factor to the integral is

\[
\frac{(-1)^{(N-1)/2}}{(2\pi i)^{N-1} N!} \frac{1}{N} \frac{\text{sign}(P)^{n_{\phi}}}{|P|^{r/2}} e^{\frac{\pi i (N^2 - 1)}{4} \text{sign}(H/P) - \frac{\pi i (N^2 - 1) c_0}{4}} (N^2 - 1) \phi.
\]

(3.2)
and for \( j = N \),
\[
\prod_{j=1}^{3} \left( e^{\frac{1}{\beta_j} (\beta_i \cdot \beta_j - \beta_i \cdot \beta_N)} - e^{\frac{1}{\beta_j} (\beta_i \cdot \beta_N)} \right) = \sum_{n_{i,N}=0}^{\infty} \chi_{2P}(n_{i,N}) e^{-\frac{1}{\beta_j N_{i,N}} (\beta_i + \sum_{l=1}^{N-1} \beta_l)}
\]

(3.5)

where we chose \( 0 < \text{Re} \beta_1 < \text{Re} \beta_2 < \cdots < \text{Re} \beta_{N-1} \) and \( P > 0 \) for convergence.\(^7\) Then, (3.3) is expressed as

\[
\sum_{t \in \Lambda_t/H \Lambda_t} \int_{\Gamma_{N-1}} d\beta_1 d\beta_2 \cdots d\beta_{N-1} \exp \left( - \frac{K}{2 \pi i} \frac{H}{P} \left( \sum_{i=1}^{N-1} \beta_i^2 + \sum_{i<j} \beta_i \beta_j \right) \right) \exp \left( -K \left( 2 \sum_{i=1}^{N-1} \beta_i t_i + \sum_{i \neq j} t_i \beta_j \right) \right)
\]
\[
\times \sum_{n_{i,j}=0}^{\infty} \left( \prod_{i<j} \chi_{2P}(n_{i,j}) \right) \exp \left( \frac{1}{2} \sum_{i=1}^{N-1} \left( - \sum_{j=1}^{N-1} n_{i,j} + \sum_{j=m+1}^{N-1} n_{i,j} - n_{i,N} + \sum_{j=1}^{N-1} n_{j,N} \right) \right)
\]

(3.6)

We do similar calculation as in section 2.1. Expressing the integral in such a way that the contours pass at the origin as in section 2.1, we analytically continue \( K \) to complex number with \( \text{Im} K < 0 \) and take integral contour as \( \gamma_j, j = 1, \ldots, N-1 \) that passes through \( \text{Re} \beta_j > 0 \) and extends parallel to the imaginary axis of \( \beta_j \)-plane. Then we obtain the partition function of analytically continued \( SU(N) \) CS theory,

\[
\frac{2^{N-1} \pi^{N-1}}{\sqrt{N}} \left( \frac{2iP}{KH} \right)^{N-1} \sum_{t \in \Lambda_t/H \Lambda_t} \left[ \exp \left( \frac{2\pi i KP}{H} \left( \sum_{m=1}^{N-1} t_m^2 + \sum_{m<n} t_m t_n \right) \right) \right]
\]
\[
\times \sum_{n_{i,j}=0}^{\infty} \left( \prod_{i<j} \chi_{2P}(n_{i,j}) \right) \exp \left( -\frac{\pi i K}{H} \sum_{m=1}^{N-1} t_m \left( - \sum_{j=1}^{m-1} n_{j,m} + \sum_{j=m+1}^{N-1} n_{m,j} - n_{m,N} - \sum_{j=1}^{N-1} n_{j,N} \right) \right)
\]
\[
\times \exp \left( \frac{\pi i}{2KH} \left( \sum_{1 \leq i < j \leq N} n_{i,j}^2 + \sum_{1 \leq i < j \leq N-1} n_{i,j} n_{i,l} - \sum_{1 \leq i < j \leq N} n_{i,j} n_{i,N} \right) \right)
\]
\[
\sum_{1 \leq i < j \leq N} n_{i,j} n_{i,l} \sum_{1 \leq i < j \leq N} n_{i,j} n_{j,N} + \sum_{1 \leq i < j \leq N} n_{i,l} n_{j,l} \right) \right] \right]
\]

(3.7)

Or this can be written as

\[
\frac{2^{N-1} \pi^{N-1}}{\sqrt{N}} \left( -2iK \right)^{N-1} \left( \frac{\pi i}{K} \right) \sum_{t \in \Lambda_t/H \Lambda_t} \left[ e^{\pi i \frac{P}{\pi} \sum_{m=1}^{N-1} t_m^2} \right]
\]
\[
\times \sum_{n_{i,j}=0}^{\infty} \left( \prod_{1 \leq i < j \leq N} \chi_{2P}(n_{i,j}) \right) e^{-\frac{\pi i}{\pi} \sum_{m=1}^{N-1} t_m b_m(\vec{n})} \sum_{m=1}^{N-1} b_m(\vec{n})^2 \right]
\]

(3.8)

\(^7\) We can choose other ordering of \( \text{Re} \beta_i \) in the calculations, which gives some of expansions in (3.4) to have \(-n_{i,j}\). But by renaming \( n_{i,j} \)'s and considering \( \sum_{t \in \Lambda_t/H \Lambda_t} \) we can see that the final expressions are all same.
Here, \( b_i(\vec{n}) := - \sum_{j=1}^{i-1} n_{j,i} + \sum_{j=i+1}^{N-1} n_{i,j} - n_{i,N} \) for \( i = 1, \ldots, N-1 \) and \( b_N(\vec{n}) = - \sum_{i=1}^{N-1} b_i(\vec{n}) = \sum_{j=1}^{N-1} n_{j,N} \) with \( \vec{n} := (n_{1,2}, n_{1,3}, \ldots, n_{2,3}, \ldots, n_{N-1,N}) \) and \( t_N = - \sum_{j=1}^{N-1} t_j \). When \( N = 2 \), i.e. \( SU(2) \), we can recover (2.34) and (2.35) from (3.8).

Contributions from irreducible flat connections would appear as residues of the integral (3.1) [16]. Since all irreducible flat connections are attached to the reducible flat connections in the case of \( G = SU(2) \) (when \( H = 1 \)), we may also expect that similar phenomena happen in the \( G = SU(N) \) case. So we expect that the partition function (3.8) provides a full exact partition function.

Regarding other possible ranges of \( P, H, \text{Im} K \), we can do similar analysis as we discussed in section 2.7, and obtain same result also for the case of \( G = SU(N) \). So we will only consider the case \( P > 0, H > 0 \), and \( \text{Im} K < 0 \).

**Reducible flat connections**

Before moving on to the properties of (3.8), we discuss reducible flat connections for \( G = SU(N) \) on Seifert manifolds.

If the stabilizer subgroup for the holonomy \( \text{Hol}_A \) of the flat connection \( A \) or \( \text{Hom}(\pi_1(M_3), G)/\text{conj.} \), is a center of \( G \), it is called the irreducible flat connection. If not, it is called the reducible flat connection. In the case of \( G = SU(2) \), reducible and irreducible flat connections are given by the abelian and nonabelian \( SU(2) \) flat connection, respectively. So abelian or nonabelian flat connections were used interchangeably with reducible or irreducible flat connections, respectively, in section 2.

In the case of higher rank, reducible flat connections are not necessarily abelian flat connections but there are reducible flat connections that are nonabelian. For example, when \( G = SU(3) \), holonomy of reducible flat connection can be \( S(U(2) \times U(1)) \).

In order to sort out the type of flat connections that contribute to the partition function in \( G = SU(N) \) case, we briefly review the case of \( G = SU(2) \) [15, 21–23]. Seifert manifold \( M_3 \) is obtained by \( (P_j, Q_j) \) surgery, \( j = 1, \ldots, F \), on the link \( p_j \times S^1 \) in \( \Sigma_g \times S^1 \) where \( p_j \)'s are points on \( \Sigma_g \). Let \( h \) be the loop that wraps \( S^1 \), \( x_j \) be the loops around each punctures \( p_j \), and \( c_l \), \( d_l \), \( l = 1, \ldots, g \), be the standard generators of \( \pi_1(\Sigma_g) \). Then the fundamental group \( \pi_1(M_3) \) of Seifert manifold is generated by \( h, x_j, c_l \), and \( d_l \) that satisfies \( x_j^{P_j} h Q_j = 1 \) and \( x_1 \ldots x_F [c_1, d_1] \ldots [c_g, d_g] = 1 \) where \( h \) commutes with all \( x_j \), \( c_l \), and \( d_l \),

\[
\pi_1(M_3) = \{ h, x_j^s, c_l^s, d_l^s \mid x_j^{P_j} b^{Q_j} = 1, \prod_{j=1}^{F} x_j \prod_{l=1}^{g} [c_l, d_l] = 1, [h, x_j] = [h, c_l] = [h, d_l] = 1 \}
\]

(3.9)

---

**Footnote:** The result is similar for \( G = U(N) \). Instead of above \( b_i(\vec{n}) \)'s and \( t_i \)'s, if we choose \( b_i(\vec{n}) = - \sum_{j=1}^{i-1} n_{j,i} + \sum_{j=1}^{N} n_{i,j} \) and take \( t_N \) to be an independent variable, we obtain the partition function of analytically continued \( G = U(N) \) CS theory.
Introducing a map $\phi : \pi_1 \to [0, \frac{1}{2}]$ such that $e^{2\pi i \alpha_3(h)}$ is in the same conjugacy class of $\text{Hol}_A(h)$ in $SU(2)$, one can see that holonomies $\text{Hol}_A(h), \text{Hol}_A(x_1), \ldots, \text{Hol}_A(x_F)$ depend on the choice of $\phi(b)$ and some other integers. When $\phi(b) \neq 0, \frac{1}{2}$, $\text{Hol}_A(h)$ is $U(1)$, and since $h$ commutes with $x_j$'s, $\text{Hol}_A(x_j)$'s are also $U(1)$. Also, $\text{Hol}_A(c_1)$ and $\text{Hol}_A(d_1)$ are $U(1)$. Stabilizer subgroup with respect to them is $U(1)$, so this choice gives reducible flat connection which are abelian. In the notation of $(2.3)$, $e^{2\pi i \alpha_3(h)} \cong \text{Hol}_A(h)$ is equal to $e^{2\pi i \alpha_3(h)}$, $1 \leq t \leq H - 1$ [15]. When $\phi(b) = 0, \frac{1}{2}$, $\text{Hol}_A(h)$ is a center of $SU(2)$, so flat connections can be irreducible. However, in the special case of this choice, one can have the trivial or central flat connections, and one can also find that this particular case gives the contribution from the trivial or central flat connections to the WRT invariant. So $\phi(b) = 0, \frac{1}{2}$ case can be regarded as trivial and central flat connection in the integral formula $(2.3)$.

So far, we have seen that $\text{Hol}_A(h)$ determines the type of flat connections in the case of $G = SU(2)$. So we perform similar analysis in the higher rank case. As in the case of $G = SU(2)$, $t \in \Lambda_r/\Lambda_r$ determines $\text{Hol}_A(h) \cong e^{2\pi i K \Phi \sum_{m=1}^N t_m^2}$ when $K \in \mathbb{Z}$ and also $e^{-\frac{\pi}{H} \sum_{m=1}^N t_m b_m(\bar{n})}$. It is easy to see that such shift only leads to change of $\sum_{m=1}^N t_m^2$ by multiple of $2H$. For $\sum_{m=1}^N t_m b_m(\bar{n})$, we first note that the values that $n_{i,j}$'s take are all odd or all even, which depends on $P_j$'s, $j = 1, \ldots, F$. We also note that $b_m(\bar{n})$ contains $N - 1 n_{i,j}$'s. If we take $t_1 \to t_1 + H$, the difference in $\sum_{m=1}^N t_m b_m(\bar{n})$ is $H(2b_1(\bar{n}) + b_2(\bar{n}) + \cdots + b_{N-1}(\bar{n}))$, which contains even number of $n_{i,j}$'s. So whatever $N$ is even or odd, the difference is multiple of $2H$. Hence, such shift by $H$ doesn’t affect $e^{2\pi i K \Phi \sum_{m=1}^N t_m^2}$ when $K \in \mathbb{Z}$ and $e^{-\frac{\pi}{H} \sum_{m=1}^N t_m b_m(\bar{n})}$.

Orbit under the action of the center

We will see in examples below that there are cases that some elements $t \in \Lambda_r/\Lambda_r$ are related by the action of the center of $G = SU(N)$, $e^{2\pi i \vec{K}} I_N$, $c = 0, 1, \ldots, N - 1 \mod N$
as in SU(2) cases. As elements \( t \in \Lambda_r / H \Lambda_r \) give \( \text{diag}(e^{2\pi i H}, \ldots, e^{2\pi i H}) \) at the level of holonomy, the center that can relate elements in \( \Lambda_r / H \Lambda_r \) take, for example, a form of \( e^{2\pi i H} \) with \( c := (c, \ldots, c - H, \ldots, c - H) \), \( c \in \mathbb{Z}_H \) where \( m \) is an integer such that \( Nc = mH \) which comes from the condition \( \sum_{i=1}^{N} t_i = 0 \). For instance, when \( N = 2 \) discussed in the previous section, elements can be related by the nontrivial center when \( H \) is even number, and given such an \( H \) we can see from \( Nc = mH \) that \( c \) can be 0 or \( \frac{H}{2} \). Also, when \( N = 2 \) and if \( H \) is an odd number, only possible \( c \) that satisfies \( Nc = mH \) is zero, which gives identity \( I_2 \).

Given a \( t = (t_1, t_2, \ldots, t_N) \), action of the center, for example \( c \), on \( t \) doesn’t affect \( e^{-\sum_{m=1}^{N} t_m b_m(i) \phi} \). With \( t + c = (t_1 + c, \ldots, t_{N-m} + c, t_{N-m+1} + c - H, \ldots, t_N + c - H) \), difference of \( e^{-\sum_{m=1}^{N} t_m b_m(i) \phi} \) from the case with \( (t_1, \ldots, t_N) \) is \( e^{m\pi i \sum_{j=N-m+1}^{N} b_j(i)} \). We check whether \( m \sum_{j=N-m+1}^{N} b_j(i) \) is even. As noted earlier, given an \( F \) and \( P_j \)'s, \( n_{i,j}'s \) take value in all odd numbers or all even numbers and the total number of \( n_{i,j}'s \) is \( b_j(i) \) for \( G = SU(N) \) is \( N - 1 \). Therefore, when \( N \) is odd, \( N - 1 \) is even, so \( b_j(i) \)'s are always even. Thus, \( m \sum_{j=N-m+1}^{N} b_j(i) \) is even when \( N \) is even. Meanwhile, when \( N \) is even, \( N - 1 \) is odd. When \( H \) is even, from the assumption on \( P_j \)'s and \( Q_j \)'s, \( P_j \)'s should be all odd, which makes all \( n_{i,j}'s \) to take values in even number by considering \( P \alpha = P \sum_{j=1}^{P} \phi_j \), (B.21), and (B.22) in section B. Therefore \( b_j(i) \)'s are all even, so \( m \sum_{j=N-m+1}^{N} b_j(i) \) is even. For the case that \( H \) is odd, from \( Nc = mH \), as the LHS is even but \( H \) is odd \( m \) should be even. Accordingly, \( m \sum_{j=N-m+1}^{N} b_j(i) \) is even. Therefore, whatever \( N, H, \) and \( F \) are, elements that are related by the action of the center give same \( e^{-\sum_{m=1}^{N} t_m b_m(i) \phi} \). This implies that the elements or the reducible flat connections related by the center give same contribution to the analytically continued CS partition function or the WRT invariant up to \( e^{\pi i K} \sum_{m=1}^{N} t_m^2 \phi \). Therefore it is natural to group those related by the center under the same label.

However, such action can change \( e^{\pi i K} \sum_{m=1}^{N} t_m^2 \phi \) for some cases. Here, we consider the case with \( K \in \mathbb{Z} \). After some calculation, possible difference is given by \( e^{\pi i K P(N+m)c} \). If \( P(N+m)c \) is even, then the elements related by such action, which give same contribution to the WRT invariant, have same coefficient \( e^{\pi i K} \sum_{m=1}^{N} t_m^2 \phi \). However, if \( P(N+m)c \) is odd, those, which give same contributions to the WRT invariant, have different coefficients by \( e^{\pi i K} \). When \( N \) is odd, in order for \( P(N+m)c \) to be odd, both \( c \) and \( P \) should be odd and \( m \) should be even. However, this leads that, from \( Nc = mH \), the LHS is odd but the RHS is even, which is a contradiction. Thus, \( N \) cannot be odd for \( P(N+m)c \) to be odd and extra factor \( e^{\pi i K} \) doesn’t appear in the case of odd \( N \). Meanwhile, if \( N \) is even, both \( c \) and \( m \) should be odd to make \( P(N+m)c \) to be odd. From \( Nc = mH \), we see that \( H \) should be even. Therefore, if there are even \( N \), even \( H \), odd \( m \), and odd \( c \) that satisfy \( Nc = mH \), there is an extra factor \( e^{\pi i K} \). For example, if \( N \) is not divisible by \( m \), from \( H = \frac{Nc}{m} \), \( c \) should be divisible by \( m \). Since \( c \) and \( m \) are both odd, \( c/m \) is also an odd number, so we find that there can be an extra factor \( e^{\pi i K} \) when \( N \) is even and \( H \) is an odd integer times \( N \). When \( N = 2 \), we saw in previous section that the extra factor \( e^{\pi i K} \) appeared when \( H \) is multiple of 2 but not multiple of 4, which is consistent with above discussion applied to
the $SU(2)$ case. When $N$ is even and $H$ is odd, there is no extra factor $e^{\pi i K}$.

Weyl orbit of $t$ and of $-t$

There are some cases that element $t$ and $-t$, which are complex conjugate to each other at the level of holonomy, are not in the same Weyl orbit. For example, when $N = 3$ and $H = 4$, $(2, -1, -1)$ and $(-2, 1, 1)$ are in different Weyl orbits. We can see that contributions from Weyl orbit of $t$ and from Weyl orbit of $-t$ to the analytically continued CS partition function or the WRT invariant are same. This means that contributions from the reducible flat connection corresponding to Weyl orbit of $t$ and from the conjugate reducible flat connection corresponding to Weyl orbit of $-t$ are same.

In (3.8), we consider renaming or permuting $n_{i,j}$’s. If we permute the indices of $n_{i,j}$’s with a convention $i < j$ in such a way that $i \leftrightarrow (N - 1) - i$, $i = 1, \ldots, \frac{N - 1}{2}$, and $N - 1 \leftrightarrow N$ with $\frac{N - 1}{2}$ staying same when $N$ is odd, or $i \leftrightarrow (N - 1) - i$, $i = 1, \ldots, \frac{N - 2}{2}$ and $N - 1 \leftrightarrow N$ when $N$ is even, we find $b_i \leftrightarrow -b_{(N-1)-i}$ and $b_{N-1} \leftrightarrow -b_N$. Under this permutation, we see that $q\sum_{i=1}^{\frac{N}{2}} \frac{1}{t_i} \sum_{m=1}^{\frac{N}{2}} b_m (i \alpha_i)^2$ stays same. Since the Weyl orbit of $t$ contains all possible permutations of $t_i$’s in $\Lambda_r/H \Lambda_r$, $e^{-\frac{i}{N} \sum_{m=1}^{N} \frac{1}{t_m} b_m (i \alpha_i)}$ on a given Weyl orbit of $t$ in $\Lambda_r/H \Lambda_r$ is same as $e^{-\frac{i}{N} \sum_{m=1}^{N} \frac{1}{t_m} b_m (i \alpha_i)}$ on Weyl orbit of $-t$ in $\Lambda_r/H \Lambda_r$. Also, obviously $e^{\pi i K} \sum_{m=1}^{N} t_m$ is same for $t$ and $-t$. Thus, the contributions from the reducible flat connection corresponding to Weyl orbit of $t$ and from the conjugate reducible flat connection corresponding to Weyl orbit of $-t$ to the partition function of analytically continued CS theory or the WRT invariant are same.

General structure

In order to write an expected general expression for the $S$-matrix, we introduce some notations. From now on, we take $K \in \mathbb{Z}$. We group elements in $\Lambda_r/H \Lambda_r$ by Weyl orbits where elements in Weyl orbit are considered as equivalent reducible flat connections. Among Weyl orbits, there are cases that the orbit contains both $t$ and $-t$. We denote such Weyl orbit of $t$ in $\Lambda_r/H \Lambda_r$ by $W_t$ where $t$ is a label for reducible flat connections. There are also cases that Weyl orbit containing $t$ is distinct from the one containing $-t$. We denote such orbits as $W_t$ and $W_{-t}$, respectively.

In some cases, it happens that some elements $t$’s are related by the action of the centers. As they give same contributions to the WRT invariant up to overall coefficient $e^{\pi i K}$, we group the Weyl orbits by orbits of them under the action of centers. We denote such orbits by $C_a$ where $a$ is a label for reducible flat connections in $C_a$. There are cases that $W_t$ and $W_{-t}$ are in the same $C_a$. For example, when $N = 4$ and $H = 4$, $C_0 = \{W_0, W_2, W_7, W_{-7}\}$ where $W_0, W_2, W_7$, and $W_{-7}$ are Weyl orbits of $(0, 0, 0, 0), (2, -2, 2, -2), (3, -1, -1, -1)$, and $(-3, 1, 1, 1)$ in $\Lambda_3/4 \Lambda_3$, respectively. There are also cases that orbits $W_t$ and $W_{-t}$ are not related by the action of the center. For instance, when $N = 4$ and $H = 6$, $W_12$ and $W_{14}$ are Weyl orbits of $(3, -1, -1, -1)$ and $(0, 2, 2, -4)$ in $\Lambda_3/6 \Lambda_3$, respectively, and they are related by the action of center, i.e. $(-3, 3, 3, -3)$ in $\mathbb{Z}_6^4$. We see that $\{W_{12}, W_{14}\} \neq \{W_{-12}, W_{-14}\}$ but they are complex conjugate to each other at the level of holonomy. As we discussed above, contributions from $W_t$ and from $W_{-t}$ to the WRT
invariant are same, so we put them in the same class, which we denote by $C_{\pm a}$.

We denote elements in the Weyl orbit $W_t$ in $\Lambda_r/H\Lambda_r$ by $\tilde{t}$. Also, we denote any representative of any of $W_t$ in $C_b$ or $C_{\pm b}$ by $\tilde{b}$. Then from examples we worked out, we expect that when $G = SU(N)$ the $S$-matrix for Seifert manifolds is given by

$$S_{ab} = \frac{1}{\sqrt{\gcd(N,H)}} \sum_{\tilde{t} \in C_a} \frac{\sum_{i \in W_t} e^{2\pi i \text{lk}(\tilde{t}, \tilde{b})}}{|\text{Tor} H_1(M_3, \mathbb{Z})|^{\frac{N-2}{2}}}.$$  \hspace{1cm} (3.10)

and

$$\text{lk}(a,b) = \frac{P}{H} \sum_{j=1}^{N} a_j b_j$$ \hspace{1cm} (3.11)

with $a_N = -\sum_{j=1}^{N-1} a_j$ and similarly for $b_N$. Here, $C_a$ in the summation can be $C_a$'s or $C_{\pm a}$'s depending on $a$ under consideration. For example, in the calculation below, “$C_b$” in the case of $N = 4$ and $H = 6$ is $C_{\pm b}$ and $W_t \in C_{\pm b}$ are $W_{12}$, $W_{14}$, $W_{-12}$, and $W_{-14}$ in $\Lambda_3/6\Lambda_3$. We also expect that (3.10) holds for general closed 3-manifold.

Given the $S$-matrix above, the general form of the WRT invariant is expected to take

$$Z_{SU(N)}(M_3) = C \gcd(N,H)^\frac{1}{2} H^{\frac{N-1}{2}} \sum_{a,b} e^{\pi i \text{lk}(a,a)} S_{ab} \tilde{Z}_b(q)$$ \hspace{1cm} (3.12)

when $N$ is odd or $N$ is even such that there is no odd $m$, odd $c$, and even $H$ that satisfy $Nc = mH$. Here the overall factor $C$ is

$$C = (NK^{-1}N^{N-1}i^{N(N-1)}) \frac{b_1(M_3)}{N! N^{3/2}} (-1)^{(N-1)(N-2)} \left( \frac{i}{P} \frac{\text{sign}(P)}{K/H} \frac{N-1}{2} \text{sign}(H/P) q^{-\frac{1}{H}} \right) \quad (3.13)$$

where $b_1(M_3)$ is the first Betti number of $M_3$. Also, $\frac{1}{2}\text{lk}(a,a) = CS_a$ is interpreted as Chern-Simons invariant for reducible flat connection $a$.

Meanwhile, when $N$ is even and there are odd $m$, odd $c$, and even $H$ that satisfy $Nc = mH$, the WRT invariant is expected to take a form of

$$Z_{SU(N)}(M_3) = \frac{1}{2} C \gcd(N,H)^\frac{1}{2} H^{\frac{N+1}{2}} \sum_{a,b} e^{\pi i \text{lk}(\tilde{a}, \tilde{b})} (I_2 \otimes S_{ab})_{\tilde{a} \tilde{b}} \tilde{Z}_b(q)$$ \hspace{1cm} (3.14)

where $(I_2 \otimes S_{ab})_{\tilde{a} \tilde{b}} = \left( \begin{array}{cc} S_{ab} & S_{ab} \\ S_{ab} & S_{ab} \end{array} \right)_{\tilde{a} \tilde{b}}$. If we denote the total number of distinct classes of orbits under the action of center as $L$, we have $\tilde{a}, \tilde{b} = 0, 1, \ldots, 2L - 1$, $\tilde{Z}_b(q) = (\tilde{Z}_0, \ldots, \tilde{Z}_{2L-1})^T$ and $\text{lk}(\tilde{a}, \tilde{b}) = (\text{lk}(0,0), \ldots, \text{lk}(L-1,L-1), \text{lk}(0,0) + 1, \ldots, \text{lk}(L-1,L-1) + 1)$. We note that $Z_\tilde{a}(q) = Z_{\tilde{a} + L}(q)$, $\tilde{a} = 0, \ldots, L - 1$ where $Z_\tilde{a}(q) = (I_2 \otimes S_{ab})_{\tilde{a} \tilde{b}} \tilde{Z}_b(q)$. We also expect that the WRT invariant for general closed 3-manifold would take a form of (3.12) or (3.14).

In next sections, we provide a number of examples. We omit overall factor (3.13) in the following examples.
3.3 The case $H = 1$

For the integer homology Seifert manifolds, $H = 1$, there is a contribution only from the trivial flat connection, $t_i = 0$ for $i = 1, \cdots, N - 1$ to the WRT invariant, which is

$$Z_{SU(N)}(M_3) = \sum_{n_{i,j}=0}^{\infty} \left( \prod_{1 \leq i < j \leq N} \chi_{2P}(n_{i,j}) \right) q^{\frac{1}{2N} \sum_{m=1}^{N} b_m(n)^2}$$  \hspace{1cm} (3.15)

Or by using (2.17), $Z_{SU(N)}(M_3)$ can be written as

$$Z_{SU(N)}(M_3) = \sum_{s_{i,j}=0}^{3} \Psi_{2P}^{(s)}(q)$$ \hspace{1cm} (3.16)

in terms of a building block $\Psi_{2P}^{(s)}(q)$,

$$\Psi_{2P}^{(s)}(q) := \sum_{n_{i,j}=0}^{\infty} \psi_{2P}^{(s)}(n) q^{\frac{1}{2P} \sum_{m=1}^{N} b_m(n)^2}$$  \hspace{1cm} (3.17)

where $\psi_{2P}^{(s)}(n) := \psi_{2P}^{(s_{1,2})}(n_{1,2})\psi_{2P}^{(s_{1,3})}(n_{1,3})\cdots\psi_{2P}^{(s_{2,3})}(n_{2,3})\cdots\psi_{2P}^{(s_{N-1,N})}(n_{N-1,N})$. We provide examples as series expansion to make them explicit. From examples, we see that the analytically continued CS partition function or the WRT invariant is indeed expressed as a single homological block $\tilde{Z}_0$

$$Z_{SU(3)}(M_3) = \tilde{Z}_0(q).$$  \hspace{1cm} (3.18)

- $G = SU(3)$ and $(P_1, P_2, P_3) = (2, 3, 7)$

$$\tilde{Z}_0(q) = q^{\frac{1}{5}} (1 - 2q + 2q^3 + q^4 - 2q^5 - 2q^7 - q^{10} - q^{11} + 2q^{13} - 6q^{14} + \cdots).$$  \hspace{1cm} (3.19)

- $G = SU(3)$ and $(P_1, P_2, P_3) = (2, 5, 7)$

$$\tilde{Z}_0(q) = q^{1 + \frac{5}{2}} (1 - 2q^3 - 2q^5 + 2q^6 + 2q^9 + q^{12} - 2q^{14} + 2q^{15} - 2q^{18} - 3q^{20} + 6q^{21} + \cdots).$$  \hspace{1cm} (3.20)

- $G = SU(3)$ and $(P_1, P_2, P_3) = (5, 11, 13)$

$$\tilde{Z}_0(q) = q^{285 + \frac{29}{2N}} (1 - 2q^{39} - 2q^{47} + 2q^{81} + 2q^{96} + 2q^{117} - 2q^{119} + 2q^{141} + 2q^{147} + 2q^{153} + \cdots).$$  \hspace{1cm} (3.21)

- $G = SU(4)$ and $(P_1, P_2, P_3) = (2, 5, 7)$

$$\tilde{Z}_0(q) = q^{2 + \frac{5}{2}} (1 - 3q^3 - 3q^5 + 5q^6 - q^7 + 2q^8 + 3q^9 - q^{10} - q^{12} + 2q^{13} - 6q^{14} + 2q^{15} - q^{16} + \cdots).$$  \hspace{1cm} (3.22)
• $G = SU(4)$ and $(P_1, P_2, P_3) = (3, 5, 7)$

$$
\hat{Z}_0(q) = q^{27+\frac{11}{15}}(1 - 3q^7 - 3q^{11} + q^{14} + 4q^{15} + 2q^{18} - q^{19} + 4q^{21} + q^{22} - 4q^{23} + 3q^{24} + \cdots).
$$

(3.23)

3.4 The case $H \geq 2$

When $H \geq 2$, we have sum over $t \in \Lambda_r/H\Lambda_r$ in (3.8). We denote simple roots by $\alpha_{i,i+1} = e_i - e_{i+1}, i = 1, \cdots N - 1$ where $\{e_i\}_{i=1,\ldots,N}$ are the standard orthonormal basis. We provide some examples for $H \geq 2$.

3.4.1 $G = SU(3)$

When $G = SU(3)$, we obtain

$$
\sum_{t \in \Lambda_2/H\Lambda_2} \left[ e^{2\pi i K} \sum_{m=1}^{N} (t_1^2 + t_2^2 + t_1 t_2) \sum_{n_{i,j}=0}^\infty \prod_{1 \leq i < j \leq 3} \chi_{2P}(n_{i,j}) e^{-\frac{\pi i}{H} \sum_{i=1}^{3} t_i b_i(\vec{n})} q^{\sum_{i=1}^{3} b_i(\vec{n})^2} \right]
$$

(3.24)

up to an overall factor where

$$
b_1(\vec{n}) = n_{1,2} - n_{1,3}, \quad b_2(\vec{n}) = -n_{1,2} - n_{2,3}, \quad b_3(\vec{n}) = n_{1,3} + n_{2,3}
$$

(3.25)

and $t_3 = -t_1 - t_2$. In the following examples, we write the WRT invariant in terms of homological blocks.

• $H = 2$

We begin with the case $H = 2$ from $(P_1, P_2, P_3) = (3, 5, 7)$. There are two Weyl orbits in $\Lambda_2/2\Lambda_2$,

$$
W_t \quad \Lambda_2/2\Lambda_2
$$

$$
W_0 \quad \{0\}
$$

$$
W_1 \quad \{\alpha_{12}, \alpha_{23}, \alpha_{12} + \alpha_{23}\}
$$

where $\alpha_{12} = (1, -1, 0), \alpha_{13} = (1, 0, -1),$ and $\alpha_{23} = (0, 1, -1).$ $W_0$ corresponds to the trivial flat connection and $W_1$ to the reducible flat connection of type $S(U(2) \times U(1))$, which is nonabelian. Contributions from $W_0$ and $W_1$ are denoted by $Z_0$ and $Z_1$, which are written in terms of two homological blocks $\hat{Z}_0$ and $\hat{Z}_1$,

$$
Z_0 = \hat{Z}_0 + \hat{Z}_1,
$$

(3.26)

$$
Z_1 = 3\hat{Z}_0 - \hat{Z}_1
$$

(3.27)

where

$$
\hat{Z}_0 = q^{5+\frac{11}{105}}(1 + 2q^{12} - q^{14} - 2q^{16} - 3q^{22} + 2q^{24} - 2q^{28} + 2q^{36} - 2q^{38} + \cdots),
$$

(3.28)

$$
\hat{Z}_1 = -2q^{9+\frac{1}{15}}(1 + q^2 - q^4 - q^7 + q^8 - q^9 - q^{10} - q^{13} + q^{14} + 2q^{17} - q^{19} + \cdots).
$$

(3.29)
Thus, the WRT invariant is given by

$$Z_{SU(3)}(M_3) = \sum_{a,b=0}^{1} e^{2\pi i KCS_a} S_{ab} \tilde{Z}_b(q) \bigg|_{q^\frac{1}{6}}$$

(3.30)

where \((CS_0, CS_1) = (0, \frac{1}{2})\) and

$$S_{ab} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}.$$  (3.31)

This \(S\)-matrix and \(CS_a\)’s can also be calculated from (3.10) and (3.11) with \(C_0 = \{W_0\}\) and \(C_1 = \{W_1\}\), and they agree.

\(\bullet\) \(H = 3\)

We consider \(H = 3\), for example, by choosing \((P_1, P_2, P_3) = (2, 5, 7)\). In the case of \(H = 3\), there are four Weyl orbits

\[
\begin{align*}
W_t & : \Lambda_2/3\Lambda_2 \\
W_0 & : \{0\} \\
W_1 & : \{\alpha_{12}, \alpha_{13}, \alpha_{23}, 2\alpha_{12}, 2\alpha_{13}, 2\alpha_{23}\} \\
W_2 & : \{\alpha_{12} + \alpha_{13}\} \\
W_3 & : \{\alpha_{13} + \alpha_{23}\}
\end{align*}
\]

in \(\Lambda_2/3\Lambda_2\). Here, \(W_0\) corresponds to the trivial flat connection, \(W_2\) and \(W_3\) to central flat connections, and \(W_1\) to abelian flat connection. So we have \(C_0 = \{W_0, W_2, W_3\}\) and \(C_1 = \{W_1\}\). Contributions from \(W_0\), \(W_2\), and \(W_3\) are all same. We denote contribution from \(W_0\) and \(W_1\) by \(Z_0\) and \(Z_1\), respectively. Then, the WRT invariant is written as

$$Z_{SU(3)}(M_3) = 3Z_0 + e^{\frac{2\pi i K}{3}} Z_1.$$  (3.32)

In terms of homological blocks, these \(Z_a\)’s are given by

$$Z_0 = \tilde{Z}_0 + \tilde{Z}_1,$$  (3.33)

$$\frac{1}{3} Z_1 = 2\tilde{Z}_0 - \tilde{Z}_1$$  (3.34)

with

$$\tilde{Z}_0 = -q^{\frac{5}{12}} \left(2 + 2q^3 + 3q^5 + 4q^6 + 4q^9 + 4q^{11} + 2q^{12} + 4q^{14} + 8q^{18} + \cdots\right),$$

(3.35)

$$\tilde{Z}_1 = q^{\frac{25}{12}} \left(1 - 2q + 2q^2 + 2q^3 + q^4 + 2q^5 - 2q^6 + 6q^7 - 2q^8 + 2q^9 + 2q^{10} + \cdots\right).$$

(3.36)

Thus, we have

$$Z_{SU(3)}(M_3) = 3\sqrt{3} \sum_{a,b=0}^{1} e^{2\pi i KCS_a} S_{ab} \tilde{Z}_b(q) \bigg|_{q^\frac{1}{6}}$$

(3.37)
Therefore, the WRT invariant can be written as

\[
S_{ab} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}
\]  

(3.38)

which satisfies \( S_{ab}S_{bc} = \delta_{ac} \). This S-matrix and \( CS_a \)'s can also be calculated from (3.10) with (3.11) and they agree.

\bullet \ H = 4

We can choose, for example, \((P_1, P_2, P_3) = (3, 5, 7)\) to have \( H = 4 \). There are five Weyl orbits for the elements in \( \Lambda_2/4\Lambda_2 \),

\[
\begin{align*}
W_i & \quad \Lambda_2/4\Lambda_2 \\
W_0 & \quad \{0\} \\
W_1 & \quad \{\alpha_{12}, \alpha_{23}, \alpha_{12} + \alpha_{23}, 3\alpha_{12}, 3\alpha_{23}, 3\alpha_{12} + 3\alpha_{23}\} \\
W_2 & \quad \{2\alpha_{12}, 2\alpha_{23}, 2\alpha_{12} + 2\alpha_{23}\} \\
W_3 & \quad \{2\alpha_{12} + \alpha_{23}, 3\alpha_{12} + \alpha_{23}, 3\alpha_{12} + 2\alpha_{23}\} \\
W_{-3} & \quad \{\alpha_{12} + 2\alpha_{23}, \alpha_{12} + 3\alpha_{23}, 2\alpha_{12} + 3\alpha_{23}\}.
\end{align*}
\]

\( W_0 \) corresponds to the trivial flat connection, \( W_1 \) to abelian flat connection, and \( W_2, W_3, \) and \( W_{-3} \) to reducible flat connections of type \( S(U(2) \times U(1)) \), which are nonabelian. Contributions from \( W_0, W_1, \) and \( W_2 \) are denoted by \( Z_0, Z_1, \) and \( Z_2 \), respectively. Reducible flat connections corresponding to \( W_3 \) and \( W_{-3} \) are complex conjugate to each other, and their contributions are same. We denote sum of their contributions by \( Z_3 \). \( Z_a \)'s are written in terms of homological blocks

\[
\begin{align*}
Z_0 & = \hat{Z}_0 + \hat{Z}_1 + \hat{Z}_2 + \hat{Z}_3, \\
Z_1 & = 6\hat{Z}_0 - 2\hat{Z}_1 - 2\hat{Z}_2 + 2\hat{Z}_3, \\
Z_2 & = 3\hat{Z}_0 - \hat{Z}_1 + 3\hat{Z}_2 - \hat{Z}_3, \\
Z_3 & = 6\hat{Z}_0 + 2\hat{Z}_1 - 2\hat{Z}_2 - 2\hat{Z}_3.
\end{align*}
\]

(3.39)  (3.40)  (3.41)  (3.42)

where

\[
\begin{align*}
\hat{Z}_0 & = -q^{6 + \frac{2}{10}}(1 + 3q^4 + 2q^{12} + 3q^{16} + 2q^{28} + 2q^{48} + 2q^{52} + q^{64} + 4q^{68} + \cdots), \\
\hat{Z}_1 & = -2q^{8 + \frac{1}{10}}(1 + q + q^3 - 2q^5 + q^6 - q^7 + 2q^9 + q^{10} + 2q^{12} + q^{13} - q^{14} + \cdots), \\
\hat{Z}_2 & = q^{2 + \frac{2}{10}}(1 + 2q^6 - 2q^8 + 2q^{12} - 2q^{14} + 2q^{18} - 2q^{20} + q^{24} - 2q^{26} + 2q^{28} + \cdots), \\
\hat{Z}_3 & = -2q^{2 + \frac{4}{10}}(1 + q - q^2 + q^4 - q^5 + q^7 + q^{10} + q^{13} - 2q^{14} - q^{15} + 2q^{16} + \cdots).
\end{align*}
\]

(3.43)  (3.44)  (3.45)  (3.46)

Therefore, the WRT invariant can be written as

\[
Z_{SU(3)}(M_3) = \sum_{a,b} e^{2\pi i KCS_a} Z_a(q) = 4 \sum_{a,b} e^{2\pi i KCS_a} S_{ab} \hat{Z}_b(q) \bigg|_{q \leq e^{\frac{2\pi i}{h}}}
\]

(3.47)
where

\[ S_{ab} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & -2 & -2 & 2 \\ 3 & -1 & 3 & -1 \\ 6 & 2 & -2 & -2 \end{pmatrix} \] (3.48)

and \((C_{S0}, C_{S1}, C_{S2}, C_{S3}) = (0, \frac{1}{2}, 0, \frac{1}{2})\). This \(S\)-matrix and \(C_{Sa}\)’s can also be calculated from (3.10) and (3.11) with \(C_a = \{W_a\}\) for \(a = 0, 1, 2\) and \(C_{\pm 3} = \{W_3, W_{-3}\}\).

We may rearrange terms in \(e^{\pi iK}Z_1\) and \(e^{\pi iK}Z_3\) as they have a same exponential factor. We find that after rearrangement, it is possible to put them in a form

\[
\begin{pmatrix} Z_0 \\ Z'_1 \\ Z_2 \\ Z'_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & -1 \\ 3 & 3 & -1 & -1 \\ 9 & -3 & -3 & 1 \end{pmatrix} \begin{pmatrix} \hat{Z}_0 \\ \hat{Z}_2 \\ \hat{Z}_3 \\ \hat{Z}_1 \end{pmatrix}
\] (3.49)

where \(Z_1 + Z_3 = Z'_1 + Z'_3\). The \(S\)-matrix above is a tensor product of \(S\)-matrix of \(H = 2\), \(i.e.\ \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\), where we note that \(Z_4 = Z_2 \times Z_2\).

\(\bullet\)\(H = 5\)

We consider \(H = 5\), for example, from \((P_1, P_2, P_3) = (2, 3, 7)\). In \(A_2/5A_2\), there are seven Weyl orbits

\begin{align*}
W_t & \quad A_2/5A_2 \\
W_0 & \quad \{0\} \\
W_1 & \quad \{\alpha_{12}, \alpha_{13}, \alpha_{23}, 4\alpha_{12}, 4\alpha_{13}, 4\alpha_{23}\} \\
W_2 & \quad \{2\alpha_{12}, 2\alpha_{13}, 2\alpha_{23}, 3\alpha_{12}, 3\alpha_{13}, 3\alpha_{23}\} \\
W_3 & \quad \{\alpha_{12} + \alpha_{13}, 3\alpha_{12} + \alpha_{13}, \alpha_{12} + 3\alpha_{13}\} \\
W_{-3} & \quad \{\alpha_{13} + \alpha_{23}, 3\alpha_{13} + \alpha_{23}, \alpha_{13} + 3\alpha_{23}\} \\
W_4 & \quad \{2\alpha_{12} + 2\alpha_{13}, \alpha_{12} + 2\alpha_{13}, 2\alpha_{12} + \alpha_{13}\} \\
W_{-4} & \quad \{2\alpha_{13} + 2\alpha_{23}, \alpha_{13} + 2\alpha_{23}, +2\alpha_{13} + \alpha_{23}\}.
\end{align*}

\(W_0\) corresponds to trivial flat connection, \(W_1\) and \(W_2\) to abelian flat connections, and \(W_3, W_{-3}, W_4,\) and \(W_{-4}\) to reducible flat connections of type \(S(U(2) \times U(1))\). Contributions from \(W_0, W_1,\) and \(W_2\) are denoted by \(Z_0, Z_1,\) and \(Z_2,\) respectively. Reducible flat connections corresponding to \(W_3\) and \(W_{-3}\) (resp. \(W_4\) and \(W_{-4}\)) are complex conjugate to each other and contributions from them are same. We denote their sum by \(Z_3\) (resp. \(Z_4\)). Then the WRT invariant is written as

\[ Z_{SU(3)}(M_3) = Z_0 + e^{\frac{\pi iK}{3}}Z_1 + e^{\frac{2\pi iK}{3}}Z_2 + e^{\frac{2\pi iK}{3}}Z_3 + e^{\frac{\pi iK}{3}}Z_4 \] (3.50)
Each $Z_a$'s are

\[
Z_0 = \tilde{Z}_0 + \tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3 + \tilde{Z}_4, \\
Z_1 = 6\tilde{Z}_0 + \frac{1}{2}(-\sqrt{5} - 3)\tilde{Z}_1 + \frac{1}{2}(\sqrt{5} - 3)\tilde{Z}_2 + (\sqrt{5} + 1)\tilde{Z}_3 + (1 - \sqrt{5})\tilde{Z}_4, \\
Z_2 = 6\tilde{Z}_0 + \frac{1}{2}(\sqrt{5} - 3)\tilde{Z}_1 + \frac{1}{2}(-\sqrt{5} - 3)\tilde{Z}_2 + (1 - \sqrt{5})\tilde{Z}_3 + (\sqrt{5} + 1)\tilde{Z}_4, \\
Z_3 = 6\tilde{Z}_0 + (\sqrt{5} + 1)\tilde{Z}_1 + (1 - \sqrt{5})\tilde{Z}_2 + \frac{1}{2}(\sqrt{5} - 3)\tilde{Z}_3 + \frac{1}{2}(-\sqrt{5} - 3)\tilde{Z}_4, \\
Z_4 = 6\tilde{Z}_0 + (1 - \sqrt{5})\tilde{Z}_1 + (\sqrt{5} + 1)\tilde{Z}_2 + \frac{1}{2}(-\sqrt{5} - 3)\tilde{Z}_3 + \frac{1}{2}(\sqrt{5} - 3)\tilde{Z}_4
\]

where

\[
\tilde{Z}_0 = -q^{14 + \frac{17}{10}}(2 + 4q^5 + q^{10} + 4q^{15} + 6q^{25} + \cdots), \\
\tilde{Z}_1 = q^{\frac{36}{10}}(1 + 4q - 6q^2 + 2q^3 - 2q^5 - 3q^6 + 2q^{10} - 4q^{11} + 2q^{12} + 6q^{13} + 2q^{14} + \cdots), \\
\tilde{Z}_2 = q^{\frac{24}{10}}(1 - 2q + 2q^2 + 2q^3 - q^4 + 2q^5 + 6q^6 - 2q^7 + q^8 + 6q^9 - 2q^{11} + \cdots), \\
\tilde{Z}_3 = q^{\frac{24}{10}}(1 - q + q^2 + 2q^3 - 2q^4 + 3q^5 - q^6 - q^7 + 2q^8 + q^{11} - q^{12} + q^{13} + q^{14} + \cdots), \\
\tilde{Z}_4 = -q^{\frac{35}{10}}(1 + 2q^2 + q^3 - 2q^4 + 2q^5 + q^6 + 2q^8 - q^9 + q^{10} + 3q^{11} + 3q^{14} + \cdots).
\]

Hence, we obtain

\[
Z_{SU(3)}(M_3) = 5\sum_{a,b} e^{2\pi i KCS_a S_{ab} \tilde{Z}_b(q)} \bigg|_{q \rightarrow e^{\frac{2\pi i}{3}}}
\]

with $(CS_0, CS_1, CS_2, CS_3, CS_4) = (0, \frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5})$ and

\[
S_{ab} = \frac{1}{5} \begin{pmatrix}
1 & 1 & 1 & 1 \\
6 & \frac{1}{2}(-\sqrt{5} - 3) & \frac{1}{2}(\sqrt{5} - 3) & \sqrt{5} + 1 \\
6 & \frac{1}{2}(-\sqrt{5} - 3) & \frac{1}{2}(\sqrt{5} - 3) & 1 - \sqrt{5} \\
6 & \frac{1}{2}(\sqrt{5} + 1) & 1 - \sqrt{5} & \frac{1}{2}(\sqrt{5} - 3) \\
6 & \frac{1}{2}(\sqrt{5} + 1) & \sqrt{5} + 1 & \frac{1}{2}(\sqrt{5} - 3)
\end{pmatrix}.
\]

which satisfies $S_{ab}S_{bc} = \delta_{ac}$. This $S$-matrix and $CS_a$'s can also be calculated from (3.10) and (3.11) with $C_a = \{W_a\}$, $a = 0, 1, 2$, and $C_{\pm a} = \{W_a, W_{-a}\}$, $a = 3, 4$, and they agree.
\( H = 6 \)

It is possible to have \( H = 6 \), for example, from \((P_1, P_2, P_3) = (5, 7, 11)\). The elements in \( \Lambda_2/6\Lambda_2 \) are grouped by Weyl orbits,

\[
\begin{align*}
W_0 & \quad \{0\} \\
W_1 & \quad \{\alpha_{12}, \alpha_{23}, \alpha_{12} + \alpha_{23}, 5\alpha_{12}, 5\alpha_{23}, 5\alpha_{12} + 5\alpha_{23}\} \\
W_2 & \quad \{2\alpha_{12}, 2\alpha_{23}, 2\alpha_{12} + 2\alpha_{23}, 4\alpha_{12}, 4\alpha_{23}, 4\alpha_{12} + 4\alpha_{23}\} \\
W_3 & \quad \{3\alpha_{12}, 3\alpha_{23}, 3\alpha_{12} + 3\alpha_{23}\} \\
W_4 & \quad \{2\alpha_{12} + \alpha_{23}, 5\alpha_{12} + \alpha_{23}, 5\alpha_{12} + 4\alpha_{23}\} \\
W_5 & \quad \{3\alpha_{12} + \alpha_{23}, 3\alpha_{12} + 2\alpha_{23}, 4\alpha_{12} + \alpha_{23}, 4\alpha_{12} + 3\alpha_{23}, 5\alpha_{12} + 2\alpha_{23}, 5\alpha_{12} + 3\alpha_{23}\} \\
W_{-4} & \quad \{\alpha_{12} + 2\alpha_{23}, \alpha_{12} + 5\alpha_{23}, 4\alpha_{12} + 5\alpha_{23}\} \\
W_{-5} & \quad \{\alpha_{12} + 3\alpha_{23}, 2\alpha_{12} + 3\alpha_{23}, \alpha_{12} + 4\alpha_{23}, 3\alpha_{12} + 4\alpha_{23}, 2\alpha_{12} + 5\alpha_{23}, 3\alpha_{12} + 5\alpha_{23}\} \\
W_6 & \quad \{4\alpha_{12} + 2\alpha_{23}\} \\
W_{-6} & \quad \{2\alpha_{12} + 4\alpha_{23}\}. 
\end{align*}
\]

\( W_0 \) corresponds to the trivial flat connection, \( W_6 \) and \( W_{-6} \) to central flat connections, \( W_1, W_2, W_5, \) and \( W_{-5} \) to abelian flat connections, \( W_3, W_4 \) and \( W_{-4} \) to reducible flat connections of type \( SU(2) \times U(1) \). We denote contributions from \( W_0 \), \( W_1 \), \( W_2 \), and \( W_3 \) by \( Z_0', Z_1', Z_2', \) and \( Z_3' \), respectively. Contributions from \( W_4 \) and \( W_{-4} \), which are in the same orbit of \( W_3 \) under the action of center, are same and equal to \( Z_3' \). Similarly, we can see that \( W_5 \) and \( W_{-5} \) (resp. \( W_6 \) and \( W_{-6} \)) are related to \( W_1 \) (resp. \( W_0 \)) by the center, and their contributions are same and equal to \( Z_1' \) (resp. \( Z_0' \)). In sum, we have \( C_0 = \{W_0, W_6, W_{-6}\} \), \( C_1 = \{W_1, W_5, W_{-5}\} \), \( C_2 = \{W_2\} \), and \( C_3 = \{W_3, W_4, W_{-4}\} \). Thus, the WRT invariant can be written by

\[
Z_{SU(3)}(M_3) = 3Z_0' + 3e^{i\pi k}Z_1' + e^{2i\pi k}Z_2' + 3e^{i\pi k}Z_3' = 3(Z_0 + e^{\pi k}Z_1 + e^{2\pi k}Z_2 + e^{i\pi k}Z_3) \tag{3.63}
\]

where \( Z_2' = 3Z_2 \). Each \( Z_a \)'s are written in terms of homological blocks,

\[
3 \begin{pmatrix}
Z_0 \\
Z_1 \\
Z_2 \\
Z_3
\end{pmatrix} = 3 \begin{pmatrix}
1 & 1 & 1 & 1 \\
6 & -2 & -1 & -3 \\
2 & 2 & -1 & -1 \\
3 & -1 & -1 & 3
\end{pmatrix} \begin{pmatrix}
\tilde{Z}_0 \\
\tilde{Z}_1 \\
\tilde{Z}_2 \\
\tilde{Z}_3
\end{pmatrix} \tag{3.64}
\]

where

\[
\begin{align*}
\tilde{Z}_0 &= -q^{35} + \frac{q^{42}}{44} (1 + 2q^6 + 2q^8 + 2q^{14} + 2q^{18} + 3q^{24} + 2q^{32} + 2q^{42} + \cdots), \tag{3.65} \\
\tilde{Z}_1 &= -q^{24} + \frac{q^{15}}{16} (1 + q^6 + q^{15} + q^{16} + q^{18} + q^{21} + q^{22} + q^{24} + q^{30} + q^{31} + \cdots), \tag{3.66} \\
\tilde{Z}_2 &= -q^{27} + \frac{q^{16}}{24} (1 - q^3 - q^5 + q^6 - q^8 - q^{13} + q^{14} - q^{17} - q^{21} - q^{22} - 2q^{23} + \cdots), \tag{3.67} \\
\tilde{Z}_3 &= q^{20} + \frac{q^{15}}{11} (1 + 2q^{12} + 2q^{16} + 2q^{20} - q^{26} + 3q^{48} + 2q^{52} + 2q^{56} + \cdots). \tag{3.68}
\end{align*}
\]
Therefore, the WRT invariant can be expressed as

\[ Z_{SU(3)}(M_3) = 6\sqrt{3} \sum_{a,b} e^{2\pi i KCS_a S_{ab} \hat{Z}_b(q)} \bigg|_{q = e^{2\pi i}} \]  

(3.69)

where \((CS_0, CS_1, CS_2, CS_3) = (0, \frac{1}{6}, \frac{2}{3}, \frac{1}{2})\) and the S-matrix is

\[ S_{ab} = \frac{1}{2\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 & 1 \\
6 & -2 & 1 & -3 \\
2 & 2 & -1 & -1 \\
3 & -1 & -1 & 3
\end{pmatrix} , \]  

(3.70)

which satisfies \(S_{ab}S_{bc} = \delta_{ac}\). This S-matrix and \(CS_a\)'s can also be obtained from (3.10) and (3.11) and they agree.

We also find that relation between \(Z_a\) and \(\hat{Z}_a\) can be put in a form of

\[
\begin{pmatrix}
Z_0 \\
Z_2 \\
Z_3 \\
Z_1
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 \\
2 & -1 & 2 & -1 \\
3 & 3 & -1 & -1 \\
6 & -3 & -2 & 1
\end{pmatrix} \begin{pmatrix}
\hat{Z}_0 \\
\hat{Z}_3 \\
\hat{Z}_1 \\
\hat{Z}_2
\end{pmatrix} \]  

(3.71)

where the matrix is a tensor product of the S-matrix of the case \(H = 2\) and \(H = 3\), i.e. \(\begin{pmatrix}
1 & 1 \\
3 & -1
\end{pmatrix} \otimes \begin{pmatrix}
1 & 1 \\
2 & -1
\end{pmatrix}\), which is expected as \(Z_6 = Z_2 \times Z_3\).

\(\bullet H = 7\)

We take \((P_1, P_2, P_3) = (2, 5, 11)\) for \(H = 7\) as an example. In \(\Lambda_2/7\Lambda_2\), there are 12 Weyl orbits

\[
\begin{align*}
W_t &\quad \Lambda_2/7\Lambda_2 \\
W_0 &\quad \{0\} \\
W_1 &\quad \{\alpha_{12}, \alpha_{13}, \alpha_{23}, 6\alpha_{12}, 6\alpha_{13}, 6\alpha_{23}\} \\
W_2 &\quad \{2\alpha_{12}, 2\alpha_{13}, 2\alpha_{23}, 5\alpha_{12}, 5\alpha_{13}, 5\alpha_{23}\} \\
W_3 &\quad \{3\alpha_{12}, 3\alpha_{13}, 3\alpha_{23}, 4\alpha_{12}, 4\alpha_{13}, 4\alpha_{23}\} \\
W_4 &\quad \{\alpha_{12} + \alpha_{13}, 5\alpha_{12} + \alpha_{13}, \alpha_{12} + 5\alpha_{13}\} \\
W_5 &\quad \{\alpha_{13} + \alpha_{23}, 5\alpha_{13} + \alpha_{23}, \alpha_{13} + 5\alpha_{23}\} \\
W_6 &\quad \{2\alpha_{12} + 2\alpha_{13}, 3\alpha_{12} + 2\alpha_{13}, 2\alpha_{12} + 3\alpha_{13}\} \\
W_{-5} &\quad \{2\alpha_{12} + 2\alpha_{13}, 3\alpha_{12} + 2\alpha_{13}, 2\alpha_{12} + 3\alpha_{13}\} \\
W_6 &\quad \{3\alpha_{12} + 3\alpha_{13}, \alpha_{12} + 3\alpha_{13}, 3\alpha_{12} + \alpha_{13}\} \\
W_{-6} &\quad \{3\alpha_{12} + 3\alpha_{23}, \alpha_{12} + 3\alpha_{23}, 3\alpha_{12} + \alpha_{23}\} \\
W_7 &\quad \{2\alpha_{12} + \alpha_{13}, \alpha_{12} + 2\alpha_{13}, 4\alpha_{12} + 2\alpha_{13}, 4\alpha_{12} + \alpha_{13}, \alpha_{12} + 4\alpha_{13}, 2\alpha_{12} + 4\alpha_{13}\} \\
W_{-7} &\quad \{2\alpha_{13} + \alpha_{23}, \alpha_{13} + 2\alpha_{23}, 4\alpha_{13} + 2\alpha_{23}, 4\alpha_{13} + \alpha_{23}, \alpha_{13} + 4\alpha_{23}, 2\alpha_{13} + 4\alpha_{23}\}.
\end{align*}
\]
$W_0$ corresponds to the trivial flat connections, $W_1$, $W_2$, $W_3$, $W_7$, and $W_{-7}$ to abelian flat connections, $W_4$, $W_{-4}$, $W_5$, $W_{-5}$, $W_6$, and $W_{-6}$ to reducible flat connections of type $SU(2) \times U(1)$. Contributions from $W_0$, $W_1$, $W_2$, and $W_3$ are denoted by $Z_0$, $Z_1$, $Z_2$, and $Z_3$, respectively. Also, the contributions from reducible flat connections corresponding to $W_4$ and $W_{-4}$, which are complex conjugate to each other, are same and sum of their contributions is denoted by $Z_4$. This is similar to $W_5$ and $W_{-5}$, $W_6$ and $W_{-6}$, and $W_7$ and $W_{-7}$ and we denote their sum by $Z_5$, $Z_6$, and $Z_7$, respectively. Then the WRT invariant is written as

$$Z_{SU(3)}(M_3) = \sum_a e^{\pi iK S_a} Z_a = 7 \sum_{a,b} e^{2\pi iKCS_a} S_{ab} \widehat{Z}_b$$

(3.72)

where

$$\widehat{Z}_0 = q^{22+\frac{177}{110}} (1 - 2q^7 + 4q^{21} + 4q^{12} + 2q^{63} + 4q^{70} - q^{84} + \cdots),$$

(3.73)

$$\widehat{Z}_1 = q^{\frac{123}{60}} (1 - 2q^6 - 4q^9 + 4q^{10} - 2q^{18} + 6q^{19} + 2q^{30} - 2q^{33} + 4q^{39} - q^{40} + \cdots),$$

(3.74)

$$\widehat{Z}_2 = -q^{\frac{5}{8}} \frac{19}{70} (3 - 2q^2 + 2q^3 - q^4 + 4q^9 + 2q^{12} - 2q^{15} + 2q^{19} + 2q^{21} + 2q^{28} + \cdots),$$

(3.75)

$$\widehat{Z}_3 = -q^{2+\frac{109}{110}} (1 - 4q^7 + 4q^{12} - 2q^{15} + 4q^{21} + 2q^{25} + q^{28} - 2q^{30} - 2q^{33} + \cdots),$$

(3.76)

$$\widehat{Z}_4 = -2q^{1+\frac{117}{110}} (1 - q + q^3 - 2q^7 + 2q^9 + 2q^{12} + q^{33} - q^{34} + q^{36} + q^{37} + \cdots),$$

(3.77)

$$\widehat{Z}_5 = 2q^{4+\frac{109}{110}} (1 - q^3 + 2q^{10} + 2q^{13} + q^{19} + q^{22} + 2q^{33} + q^{39} + 2q^{45} + 2q^{48} + \cdots),$$

(3.78)

$$\widehat{Z}_6 = -2q^{3+\frac{109}{110}} (1 - q + 2q^3 + q^{21} - q^{22} + q^{24} - q^{33} - q^{36} - q^{37} + 2q^{39} + \cdots),$$

(3.79)

$$\widehat{Z}_7 = 2q^{2+\frac{177}{110}} (1 - q^2 + 2q^3 - q^5 + 2q^9 + 2q^{12} - 2q^{14} - 3q^{17} + q^{18} + q^{23} + \cdots),$$

(3.80)

$CS_a = (0, \frac{5}{7}, \frac{6}{7}, \frac{1}{7}, \frac{4}{7}, \frac{2}{7}, 0)$ and the $S$-matrix is

$$S_{ab} = \frac{1}{7} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 -2 \cos \left( \frac{\pi}{4} \right) + 2 \sin \left( \frac{\pi}{4} \right) & 2 \sin \left( \frac{\pi}{4} \right) - 2 \cos \left( \frac{\pi}{4} \right) & 4 \sin \left( \frac{\pi}{4} \right) - 2 \sin \left( \frac{\pi}{4} \right) & 4 \sin \left( \frac{\pi}{4} \right) + 2 & 2 - 4 \sin \left( \frac{\pi}{4} \right) & 2 - 4 \cos \left( \frac{\pi}{4} \right) & -1 \\
6 & 4 \sin \left( \frac{\pi}{4} \right) - 2 \sin \left( \frac{\pi}{4} \right) & 2 - 4 \cos \left( \frac{\pi}{4} \right) & -2 \cos \left( \frac{\pi}{4} \right) + 2 \sin \left( \frac{\pi}{4} \right) & 2 - 4 \sin \left( \frac{\pi}{4} \right) & 4 \sin \left( \frac{\pi}{4} \right) + 2 & -1 \\
6 & 4 \sin \left( \frac{\pi}{4} \right) - 2 \sin \left( \frac{\pi}{4} \right) & -2 \cos \left( \frac{\pi}{4} \right) + 2 \sin \left( \frac{\pi}{4} \right) & 2 - 4 \sin \left( \frac{\pi}{4} \right) & 4 \sin \left( \frac{\pi}{4} \right) + 2 & -1 \\
6 & 4 \sin \left( \frac{\pi}{4} \right) - 2 \sin \left( \frac{\pi}{4} \right) & 2 - 4 \cos \left( \frac{\pi}{4} \right) & 4 \sin \left( \frac{\pi}{4} \right) - 2 \sin \left( \frac{\pi}{4} \right) & 2 - 4 \sin \left( \frac{\pi}{4} \right) & 4 \sin \left( \frac{\pi}{4} \right) + 2 & -1 \\
12 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
\end{pmatrix}$$

(3.81)

which satisfies $S_{ab}S_{bc} = \delta_{ac}$. This $S$-matrix and $CS_a$'s can also be calculated from (3.10) and (3.11) with $C_a = \{W_a\}$, $a = 0, 1, 2, 3$, and $C_{\pm a} = \{W_a, W_{-a}\}$, $a = 4, 5, 6, 7$, and they agree.

3.4.2 $G = SU(4)$

When $G = SU(4)$, (3.8) becomes

$$\sum_{t \in A_3 / H A_3} \left[ e^{2\pi iK \frac{1}{2}(t_1^2+t_2^2+t_3^2+t_4+t_5+t_6+t_7+t_8+t_9)} \sum_{n_{i,j} \geq 0} \prod_{1 \leq i < j \leq 4} \chi_2 P(n_{i,j}) e^{-\frac{\pi^2}{110} \sum_{1 \leq i < j \leq 4} t_i t_j} q^{\frac{1}{110} \sum_{i=1}^4 b_i (n_{i,j})^2} \right]$$

(3.82)

- 38 -
up to an overall factor where

\[ b_1(\vec{n}) = n_{1,2} + n_{1,3} - n_{1,4}, \quad b_2(\vec{n}) = -n_{1,2} + n_{2,3} - n_{2,4}, \]
\[ b_3(\vec{n}) = -n_{1,3} - n_{2,3} - n_{3,4}, \quad b_4(\vec{n}) = n_{1,4} + n_{2,4} + n_{3,4} \]  

(3.83)

and \( t_4 = -t_1 - t_2 - t_3 \). In the following examples, we express the WRT invariant in terms of homological blocks.

- **H = 2**

We consider an example for \( H = 2 \). This can be obtained, for example, from \((P_1, P_2, P_3) = (3, 5, 7)\). The elements in \( \Lambda_3/2\Lambda_3 \) can be put in three Weyl orbits,

\[
W_t \Lambda_3/2\Lambda_3 \\
W_0 \{0\} \\
W_1 \{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{12} + \alpha_{23}, \alpha_{23} + \alpha_{12} \alpha_{34} + \alpha_{34}\} \\
W_2 \{\alpha_{12} + \alpha_{34}\}
\]

where \( W_0 \) corresponds to the trivial flat connections, \( W_2 \) to central flat connection, and \( W_1 \) to reducible flat connections of type \( SU(2) \times U(2) \), which is nonabelian. So we have \( C_0 = \{W_0, W_2\} \) and \( C_1 = \{W_1\} \). Contributions from \( W_0 \) and \( W_2 \) are same, and we denote their sum by \( 2Z_0 \). Contribution from \( W_1 \) is denoted by \( Z_1 \).

These \( Z_a \)'s are expressed as

\[
Z_0 = \hat{Z}_0 + \hat{Z}_1, \\
\frac{1}{2} Z_1 = 3\hat{Z}_0 - \hat{Z}_1
\]

(3.84, 3.85)

where

\[
\hat{Z}_0 = q^{13+16} (1 + q^7 + 2q^9 + q^{11} + 3q^{12} - 2q^{13} - 2q^{14} + 2q^{15} + 4q^{16} - 2q^{17} + \cdots),
\]
\[
\hat{Z}_1 = q^{17+12} (3 + 3q^2 - 4q^4 + q^6 - 4q^7 + 4q^8 - 4q^9 - 3q^{10} + 4q^{11} + 5q^{12} + \cdots)
\]

(3.86, 3.87)

Thus, the WRT invariant can be written as

\[
Z_{SU(3)}(M_3) = 4 \sum_{a,b} e^{2\pi i KCS_a S_{ab}} \hat{Z}_b(q) \bigg|_{q \leq e^{-\frac{\pi i}{K}}}
\]

(3.88)

where \( CS_a = (0, \frac{1}{2}) \) and the \( S \)-matrix is

\[
S_{ab} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}
\]

(3.89)

which satisfies \( S_{ab} S_{bc} = \delta_{ac} \). This \( S \)-matrix and \( CS_a \)'s can also be calculated from (3.10) and (3.11) and they agree.
\*\* $H = 3$

We consider, for example, $(P_1, P_2, P_3) = (2, 5, 7)$ to have $H = 3$. In $\Lambda_3/3\Lambda_3$, there are five Weyl orbits

\begin{align*}
W_l & \Lambda_3/3\Lambda_3 \\
W_0 & \{0\} \\
W_1 & \{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{12} + \alpha_{23}, \alpha_{23} + \alpha_{34}, \alpha_{12} + \alpha_{23} + \alpha_{34}, 2\alpha_{12}, 2\alpha_{23}, 2\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 2\alpha_{34}\} \\
W_2 & \{\alpha_{12} + \alpha_{34}, \alpha_{12} + 2\alpha_{23} + \alpha_{34}, 2\alpha_{12} + \alpha_{23} + \alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 2\alpha_{34}\} \\
W_3 & \{2\alpha_{12} + \alpha_{23} + \alpha_{34}, 2\alpha_{12} + \alpha_{23} + \alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + \alpha_{34}\} \\
W_{-3} & \{\alpha_{12} + 2\alpha_{23}, \alpha_{23} + 2\alpha_{34}, \alpha_{12} + \alpha_{23} + 2\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 2\alpha_{34}\}
\end{align*}

Here, $W_0$ corresponds to the trivial flat connections, $W_1$ and $W_2$ to reducible flat connections of type $S(U(2) \times U(1) \times U(1))$ and $S(U(2) \times U(2))$, respectively, and $W_3$ and $W_{-3}$ to reducible flat connections of type $S(U(3) \times U(1))$. We denote contributions from $W_0$, $W_1$, and $W_2$ by $Z_0$, $Z_1$, and $Z_2$, respectively. Reducible flat connections corresponding to $W_3$ and $W_{-3}$ are complex conjugate to each other and their contributions are same. We denote their sum by $Z_3$. These $Z_a$’s are written in terms of homological blocks

\begin{align*}
Z_0 & = \tilde{Z}_0 + \tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3, \quad (3.90) \\
Z_1 & = 12\tilde{Z}_0 - 3\tilde{Z}_1 + 0\tilde{Z}_2 + 3\tilde{Z}_3, \quad (3.91) \\
Z_2 & = 6\tilde{Z}_0 + 0\tilde{Z}_1 + 3\tilde{Z}_2 - 3\tilde{Z}_3, \quad (3.92) \\
Z_3 & = 8\tilde{Z}_0 + 2\tilde{Z}_1 - 4\tilde{Z}_2 - \tilde{Z}_3 \quad (3.93)
\end{align*}

with

\begin{align*}
\tilde{Z}_0 & = -q^{3+\frac{7}{6}}(1 + 2q^2 + 2q^6 + 3q^9 - 2q^{12} - 10q^{15} + 7q^{18} - 4q^{21} - 14q^{24} + \cdots), \quad (3.94) \\
\tilde{Z}_1 & = q^{1+\frac{7}{6}}(1 - 3q + 5q^2 - 3q^3 - q^4 + 2q^5 - 4q^6 + 19q^7 - 9q^8 - q^{10} + \cdots), \quad (3.95) \\
\tilde{Z}_2 & = -q^{3+\frac{5}{6}}(1 + q - 2q^2 + q^3 + q^4 + 2q^5 - 2q^6 - 7q^7 + 4q^8 - q^{10} - 5q^{11} + \cdots), \quad (3.96) \\
\tilde{Z}_3 & = -2q^{3+\frac{1}{6}}(1 - q + 2q^2 + 3q^5 + q^6 - 3q^7 + q^9 + 3q^9 - q^{10} + 5q^{11} + \cdots). \quad (3.97)
\end{align*}

Thus, the WRT invariant can be written as

\begin{equation}
Z_{SU(4)}(M_3) = 3^{3/2} \sum_{a,b} e^{2\pi i KCS_a} S_{ab} \tilde{Z}_b(q) \bigg|_{q^\frac{1}{3} = 2^{\frac{1}{3}}} \quad (3.98)
\end{equation}

where $CS_a = (0, \frac{1}{3}, \frac{2}{3}, 0)$ and $S_{ab}$ is

\begin{equation}
S_{ab} = \frac{1}{3^{3/2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 12 & -3 & 0 & 3 \\ 6 & 0 & 3 & -3 \\ 8 & 2 & -4 & -1 \end{pmatrix} \quad (3.99)
\end{equation}

which satisfies $S_{ab} S_{bc} = \delta_{ac}$. The $S$-matrix and $CS_a$’s can also be obtained from (3.10) and (3.11) with $C_a = \{W_a\}$, $a = 0, 1, 2$, and $C_{\pm 3} = \{W_3, W_{-3}\}$.
\* $H = 4$

The case $H = 4$ can be obtained, for example, from $(P_1, P_2, P_3) = (3, 5, 7)$. The elements in $\Lambda_3/4\Lambda_3$ are put in 10 Weyl orbits,

$W_1$ $\Lambda_3/4\Lambda_3$

$W_0$ $\{0\}$

$W_1$ $\{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{12} + \alpha_{23}, \alpha_{23} + \alpha_{34}, \alpha_{12} + \alpha_{23} + \alpha_{34}, 3\alpha_{12}, 3\alpha_{23}, 3\alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}\}$

$W_2$ $\{2\alpha_{12}, 2\alpha_{23}, 2\alpha_{34}, 2\alpha_{12} + 2\alpha_{23}, 2\alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 2\alpha_{34}\}$

$W_3$ $\{2\alpha_{12} + \alpha_{23}, 2\alpha_{23} + \alpha_{34}, 2\alpha_{12} + \alpha_{23} + \alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + \alpha_{34}, 3\alpha_{12} + \alpha_{23}, 3\alpha_{23} + \alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}\}$

$W_{-3}$ $\{\alpha_{12} + 2\alpha_{23}, \alpha_{23} + 2\alpha_{34}, \alpha_{12} + \alpha_{23} + 2\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 2\alpha_{34}, \alpha_{12} + 3\alpha_{23}, \alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}\}$

$W_{-4}$ $\{\alpha_{12} + \alpha_{23} + 2\alpha_{34}\}$

$W_5$ $\{2\alpha_{12} + \alpha_{23}, \alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + \alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + \alpha_{23} + 3\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 2\alpha_{34}\}$

$W_6$ $\{2\alpha_{12} + \alpha_{23}, \alpha_{23} + \alpha_{34}\}$

$W_7$ $\{3\alpha_{12} + 2\alpha_{23} + \alpha_{34}\}$

$W_{-7}$ $\{\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}\}$.

Here, $W_0$ corresponds to the trivial flat connection, $W_6$, $W_7$, and $W_{-7}$ to central flat connections, $W_1$, $W_3$, $W_{-3}$, and $W_5$ to reducible flat connections of type $SU(2) \times U(1)$, and $W_2$ and $W_4$ to reducible flat connections of type $SU(2) \times U(2)$. Contributions from $W_0$, $W_6$, $W_7$, and $W_{-7}$, which are in the same orbit by the action of the center, are same but have different overall exponential factors by $e^{\pi iK}$. We denote their sum by $2(1 + e^{\pi iK})Z_0$. Similarly, contributions from $W_1$, $W_3$, $W_{-3}$, and $W_5$ (resp. from $W_2$ and $W_4$), which are related by centers, are same up to an overall exponential factor, and their sum is denoted by $2(1 + e^{\pi iK})e^{\pi iK}Z_1$ (resp. $2(1 + e^{\pi iK})Z_2$). So here we have $C_0 = \{W_0, W_6, W_7, W_{-7}\}$, $C_1 = \{W_1, W_3, W_{-3}, W_5\}$, $C_2 = \{W_2, W_4\}$. Then, the WRT invariant can be written as

$$Z_{SU(4)}(M_3) = 2(1 + e^{\pi iK})Z_0 + 2(1 + e^{\pi iK})e^{\pi iK}Z_1 + 2(1 + e^{\pi iK})Z_2$$

$$= 2(1 + e^{\pi iK})(Z_0 + e^{\pi iK}Z_1 + Z_2)$$

(3.100) (3.101)
and $Z_a$’s are expressed in terms of homological blocks,

\begin{align*}
Z_0 &= \left( \hat{Z}_0 + \hat{Z}_3 \right) + \left( \hat{Z}_1 + \hat{Z}_4 \right) + \left( \hat{Z}_2 + \hat{Z}_5 \right), \\
Z_1 &= 12 \left( \hat{Z}_0 + \hat{Z}_3 \right) - 4 \left( \hat{Z}_2 + \hat{Z}_5 \right), \\
Z_2 &= 3 \left( \hat{Z}_0 + \hat{Z}_3 \right) - \left( \hat{Z}_1 + \hat{Z}_4 \right) + 3 \left( \hat{Z}_2 + \hat{Z}_5 \right)
\end{align*}

where

\begin{align*}
\hat{Z}_0 &= -2q^{13+\frac{1}{12}} (1 + 3q^4 + 2q^8 + 6q^{12} + 6q^{16} + 4q^{20} + 9q^{24} + 9q^{28} + 11q^{32} + \cdots), \\
\hat{Z}_1 &= -q^{8+\frac{1}{12}} (3 + 3q - 4q^2 + q^3 + 4q^4 - 3q^5 + 5q^6 + 8q^7 - 2q^9 + 7q^{10} + \cdots), \\
\hat{Z}_2 &= q^{6+\frac{27}{12}} (1 + 3q^6 - 4q^8 + 7q^{12} - 2q^{14} - q^{16} + 6q^{18} + 3q^{20} + 7q^{22} + 4q^{24} + \cdots), \\
\hat{Z}_3 &= q^{10+\frac{27}{12}} (1 + q^2 + 2q^4 + 3q^8 + q^{14} + 6q^{16} + 8q^{18} + 6q^{20} + 3q^{22} + 11q^{24} + \cdots), \\
\hat{Z}_4 &= 2q^{12+\frac{87}{12}} (2 + 2q - 2q^2 + 2q^3 + q^4 - 2q^5 + 6q^6 + 2q^7 + 8q^8 + 6q^9 + \cdots), \\
\hat{Z}_5 &= q^{11+\frac{87}{12}} (2 - 2q^2 - 2q^4 - 3q^6 - 2q^8 - 5q^{10} - 14q^{12} - 5q^{14} - 4q^{16} + \cdots).
\end{align*}

Therefore, the WRT invariant can be written as

\begin{equation}
Z_{SU(4)}(M_3) = \sum_a e^{2\pi i CS_a} Z_a = 8 \sum_{a,b} e^{2\pi i CS_a} (I_2 \otimes S_{ab})_{ab} \hat{Z}_b(q) \bigg|_{q^\frac{e^{2\pi i}}{}},
\end{equation}

where $Z_a = 2(Z_0, Z_1, Z_2, Z_0, Z_1, Z_2)^T$, $CS_a = (0, \frac{1}{3}, 0, \beta, \frac{2}{3}, \frac{1}{3})$, and $S_{ab}$ is

\begin{equation}
S_{ab} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 \\ 12 & 0 & -4 \\ 3 & -1 & 3 \end{pmatrix},
\end{equation}

which satisfies $S_{ab} S_{bc} = \delta_{ac}$. The $S$-matrix $S_{ab}$ and $CS_a$’s can also be calculated from (3.10) and (3.11) and they agree with calculation above.
• $H = 5$

The case $H = 5$ can be obtained, for example, from $(P_1, P_2, P_3) = (2, 3, 7)$. Elements in $\Lambda_3/5\Lambda_3$ are grouped by 14 Weyl orbits

\begin{align*}
W_t & \quad \Lambda_3/5\Lambda_3 \\
W_0 & \quad \{0\} \\
W_1 & \quad \{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{12} + \alpha_{23}, \alpha_{23} + \alpha_{34}, \alpha_{12} + \alpha_{23} + \alpha_{34}, \\
& \quad \quad 4\alpha_{12}, 4\alpha_{23}, 4\alpha_{34}, 4\alpha_{12} + 4\alpha_{23} + 4\alpha_{34}, 4\alpha_{12} + 4\alpha_{23} + 4\alpha_{34}\} \\
W_2 & \quad \{2\alpha_{12}, 2\alpha_{23}, 2\alpha_{34}, 2\alpha_{12} + 2\alpha_{23}, 2\alpha_{12} + 2\alpha_{23}, 2\alpha_{23} + 2\alpha_{34}, \\
& \quad \quad 3\alpha_{12}, 3\alpha_{23}, 3\alpha_{34}, 3\alpha_{12} + 3\alpha_{23}, 3\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}\} \\
W_3 & \quad \{2\alpha_{12} + \alpha_{23}, 2\alpha_{23} + \alpha_{34}, 2\alpha_{12} + \alpha_{23} + \alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + \alpha_{34}, 4\alpha_{12} + \alpha_{23}, \\
& \quad \quad 4\alpha_{23} + \alpha_{34}, 4\alpha_{12} + 3\alpha_{23}, 4\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, \\
& \quad \quad 4\alpha_{12} + \alpha_{23} + \alpha_{34}, 4\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}\} \\
W_{-3} & \quad \{\alpha_{12} + 2\alpha_{23}, 2\alpha_{23} + \alpha_{34}, \alpha_{12} + \alpha_{23} + 2\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 2\alpha_{34}, \alpha_{12} + 4\alpha_{23}, \\
& \quad \quad \alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 4\alpha_{23}, 3\alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, \\
& \quad \quad \alpha_{12} + \alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 4\alpha_{34}, \alpha_{12} + 4\alpha_{23} + 4\alpha_{34}\} \\
W_4 & \quad \{\alpha_{12} + \alpha_{34}, \alpha_{12} + 2\alpha_{23} + \alpha_{34}, 4\alpha_{12} + \alpha_{34}, \alpha_{12} + 4\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, \alpha_{12} + 4\alpha_{23} + 4\alpha_{34}\} \\
W_5 & \quad \{2\alpha_{12} + 2\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 2\alpha_{34}, 2\alpha_{12} + 3\alpha_{34}, 3\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 3\alpha_{34}\} \\
W_6 & \quad \{2\alpha_{12} + \alpha_{34}, \alpha_{12} + 2\alpha_{23}, 2\alpha_{12} + \alpha_{23} + 2\alpha_{34}, \alpha_{12} + 3\alpha_{23} + \alpha_{34}, \\
& \quad \quad 2\alpha_{12} + 3\alpha_{23} + \alpha_{34}, \alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, \alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, \\
& \quad \quad \alpha_{12} + 3\alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, \alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + \alpha_{23} + 4\alpha_{34}, \\
& \quad \quad 4\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}, 2\alpha_{23} + 4\alpha_{34}, 4\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}, 2\alpha_{23} + \alpha_{23} + 4\alpha_{34}, 2\alpha_{23} + \alpha_{23} + 4\alpha_{34}\} \\
W_7 & \quad \{3\alpha_{12} + \alpha_{23}, 3\alpha_{23} + \alpha_{34}, 3\alpha_{12} + 2\alpha_{23}, 3\alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + \alpha_{34}, \alpha_{12} + 3\alpha_{23} + \alpha_{34}, \\
& \quad \quad 3\alpha_{12} + 2\alpha_{23} + 3\alpha_{23} + 3\alpha_{23} + \alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, \\
& \quad \quad 4\alpha_{12} + 2\alpha_{23}, 4\alpha_{23} + 2\alpha_{34}, 4\alpha_{12} + 2\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 4\alpha_{34}\} \\
W_{-7} & \quad \{\alpha_{12} + 3\alpha_{23}, \alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 3\alpha_{23}, 2\alpha_{23} + 3\alpha_{34}, \alpha_{12} + \alpha_{23} + 3\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, \alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, \\
& \quad \quad 2\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, \alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, \\
& \quad \quad 2\alpha_{12} + 4\alpha_{23}, 2\alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}\} \\
W_8 & \quad \{3\alpha_{12} + 2\alpha_{23} + \alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, 4\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}\} \\
W_{-8} & \quad \{\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, \alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 4\alpha_{34}\} \\
W_9 & \quad \{3\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 4\alpha_{34}\} \\
W_{-9} & \quad \{2\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}\}
\end{align*}
\( W_4 \) and \( W_5 \) to reducible flat connections of type \( S(U(2) \times U(2)) \), and \( W_6, W_{-8}, W_9, \) and \( W_{-9} \) to reducible flat connection of type \( S(U(3) \times U(1)) \). We denote contributions from \( W_0, W_1, W_2, W_4, W_5, \) and \( W_6 \) by \( Z_0, Z_1, Z_2, Z_4, Z_5, \) and \( Z_6 \), respectively. Contributions from reducible flat connections corresponding to \( W_3 \) and \( W_{-3} \), which are complex conjugate to each other, are same and we denote their sum by \( Z_3 \). The case \( W_7 \) and \( W_{-7}, W_8 \) and \( W_{-8}, \) and \( W_9 \) and \( W_{-9} \) are also similar and we denote sum of their contributions as \( Z_7, Z_8, \) and \( Z_9 \), respectively. These \( Z_a \)'s can be written in terms of homological blocks, \( \hat{Z}_a \), and the WRT invariant can be written as

\[
Z_{SU(4)}(M_3) = \sum_{a,b} e^{2\pi i KCS_a} Z_a(q) = 5^{3/2} \sum_{a,b} e^{2\pi i KCS_a} S_{ab} \hat{Z}_b(q) \bigg|_{q= e^{2\pi i}} \tag{3.113}
\]

where

\[
\hat{Z}_0 = -q^{3/8} (q + 7q^6 + 16q^{11} + 33q^{16} + 11q^{21} + 10q^{26} + 17q^{31} + 3q^{36} + 6q^{41} + \cdots), \tag{3.114}
\]

\[
\hat{Z}_1 = q^{173} (1 - 3q - q^2 + 15q^3 + 4q^4 - 15q^5 + 9q^6 + 6q^7 + 3q^8 + 4q^9 + \cdots), \tag{3.115}
\]

\[
\hat{Z}_2 = q^{287} (1 - 3q + 9q^2 + 2q^3 - 4q^4 - 3q^5 + 25q^6 + 10q^7 - 19q^8 + 16q^9 + \cdots), \tag{3.116}
\]

\[
\hat{Z}_3 = -2q^{89} (1 + q + 3q^2 - q^3 - 2q^5 + 11q^6 + 9q^7 - 15q^8 - 8q^9 + 25q^{10} + \cdots), \tag{3.117}
\]

\[
\hat{Z}_4 = -q^{91} (1 - q + q^2 + 4q^3 - 3q^4 - q^5 + 8q^6 - 3q^7 + 2q^9 + 5q^{10} + \cdots), \tag{3.118}
\]

\[
\hat{Z}_5 = -q^{41} (1 - q + 5q^2 + 6q^3 - 8q^4 + 7q^5 + 3q^8 + 7q^{10} + 6q^{11} - 6q^{12} + \cdots), \tag{3.119}
\]

\[
\hat{Z}_6 = q^{19} (1 - 2q + 5q^2 - 8q^3 + 6q^4 + 12q^5 + 8q^6 - 16q^7 + 6q^8 + 24q^9 + \cdots), \tag{3.120}
\]

\[
\hat{Z}_7 = 2q^{144} (8 - 8q - 6q^2 - 3q^3 + 10q^4 - 9q^5 - 15q^6 - 3q^7 + q^8 - 2q^9 + \cdots), \tag{3.121}
\]

\[
\hat{Z}_8 = q^{173} (1 + 2q^2 - 3q^3 - 2q^4 + 5q^5 + 2q^6 - q^7 - 5q^8 + 2q^9 + 8q^{10} + 5q^{11} + \cdots), \tag{3.122}
\]

\[
\hat{Z}_9 = 2q^{425} (1 - q + q^2 + 2q^3 - 5q^4 + 6q^5 + 6q^6 - q^7 - 3q^8 - 2q^9 + 10q^{10} + \cdots), \tag{3.123}
\]

and

\[
CS_a = (0, \frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5}, \frac{1}{5}, \frac{0}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5}), \tag{3.124}
\]

\[
S_{ab} = \begin{pmatrix}
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}
\end{pmatrix} \tag{3.125}
\]

which satisfies \( S_{ab} S_{bc} = \delta_{ac} \). The S-matrix and \( CS_a \)'s can also be calculated from (3.10) and (3.11) with \( C_a = \{ W_a \} \) for \( a = 0, 1, 2, 4, 5, 6 \) and \( C_{\pm a} = \{ W_{\pm a} \} \) for \( a = 3, 7, 8, 9 \).
• $H = 6$

The case $H = 6$ can be obtained, for example, from $(P_1, P_2, P_3) = (5, 7, 11)$. In $\Lambda_3/6\Lambda_3$, there are 22 Weyl orbits,

\[
\begin{align*}
W_1 & \Lambda_3/6\Lambda_3 \\
W_0 & \{0\} \\
W_1 & \{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{12} + \alpha_{23}, \alpha_{23} + \alpha_{34}, \alpha_{12} + \alpha_{23} + \alpha_{34}, \\
& 5\alpha_{12}, 5\alpha_{23}, 5\alpha_{34}, 5\alpha_{12} + 5\alpha_{23}, 5\alpha_{23} + 5\alpha_{34}, 5\alpha_{12} + 5\alpha_{23} + 5\alpha_{34}\} \\
W_2 & \{2\alpha_{12}, 2\alpha_{23}, 2\alpha_{34}, 2\alpha_{12} + 2\alpha_{23}, 2\alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 2\alpha_{34}, \\
& 4\alpha_{12}, 4\alpha_{23}, 4\alpha_{34}, 4\alpha_{12} + 4\alpha_{23}, 4\alpha_{23} + 4\alpha_{34}, 4\alpha_{12} + 4\alpha_{23} + 4\alpha_{34}\} \\
W_3 & \{3\alpha_{12}, 3\alpha_{23}, 3\alpha_{34}, 3\alpha_{12} + 3\alpha_{23}, 3\alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}\} \\
W_4 & \{2\alpha_{12} + \alpha_{23}, 2\alpha_{23} + \alpha_{34}, 2\alpha_{12} + \alpha_{23} + \alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + \alpha_{34}, 5\alpha_{12} + \alpha_{23}, \\
& 5\alpha_{12} + 4\alpha_{23}, 5\alpha_{23} + 4\alpha_{34}, 5\alpha_{23} + 5\alpha_{34}, 5\alpha_{12} + 5\alpha_{23} + 5\alpha_{34}, \\
& 5\alpha_{12} + 5\alpha_{23} + 4\alpha_{34}, 5\alpha_{12} + 4\alpha_{23} + 4\alpha_{34} + 4\alpha_{23} + 4\alpha_{34}\} \\
W_{-4} & \{\alpha_{12} + 2\alpha_{23}, \alpha_{23} + 2\alpha_{34}, \alpha_{12} + \alpha_{23} + 2\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 2\alpha_{34}, \alpha_{12} + 5\alpha_{23}, \\
& 4\alpha_{12} + 5\alpha_{23}, 4\alpha_{23} + 5\alpha_{34}, \alpha_{23} + 5\alpha_{34}, 4\alpha_{12} + 5\alpha_{23} + 5\alpha_{34}, \\
& \alpha_{12} + 5\alpha_{23} + 5\alpha_{34}, \alpha_{12} + \alpha_{23} + 5\alpha_{34}, 4\alpha_{12} + 4\alpha_{23} + 5\alpha_{34}\} \\
W_5 & \{2\alpha_{12} + \alpha_{34}, \alpha_{12} + 2\alpha_{34}, 2\alpha_{12} + \alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + \alpha_{34}, \\
& \alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, \alpha_{12} + 3\alpha_{23} + \alpha_{34}, 4\alpha_{12} + \alpha_{34}, \\
& \alpha_{12} + 4\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, 5\alpha_{12} + 2\alpha_{34}, 2\alpha_{12} + 5\alpha_{34}, 5\alpha_{12} + 3\alpha_{23} + 5\alpha_{34}, \\
& 5\alpha_{12} + 4\alpha_{34}, 4\alpha_{12} + 5\alpha_{34}, 5\alpha_{12} + \alpha_{23} + 5\alpha_{34}, 5\alpha_{12} + \alpha_{23} + 2\alpha_{34}, \\
& 5\alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + \alpha_{23} + 5\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 5\alpha_{34}, \alpha_{12} + 5\alpha_{23} + 5\alpha_{34}, \\
& \alpha_{12} + 5\alpha_{23} + 4\alpha_{34}, \alpha_{12} + 5\alpha_{23} + 4\alpha_{34} + 4\alpha_{23} + \alpha_{34}, \alpha_{12} + 5\alpha_{23} + 4\alpha_{34}\} \\
W_6 & \{\alpha_{12} + \alpha_{34}, \alpha_{12} + 2\alpha_{23} + \alpha_{34}, 5\alpha_{12} + \alpha_{34}, \alpha_{12} + 5\alpha_{34}, \\
& 5\alpha_{12} + 4\alpha_{23} + 5\alpha_{34}, 5\alpha_{12} + 5\alpha_{34}\} \\
W_7 & \{3\alpha_{12} + \alpha_{34}, \alpha_{12} + 3\alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, \alpha_{12} + 4\alpha_{23} + \alpha_{34}, \\
& \alpha_{12} + 4\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + \alpha_{34}, \alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 5\alpha_{12} + 3\alpha_{34}, \\
& 3\alpha_{12} + 5\alpha_{34}, \alpha_{12} + 5\alpha_{23} + 5\alpha_{34}, \alpha_{12} + 2\alpha_{23} + \alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 5\alpha_{34}\} \\
W_8 & \{3\alpha_{12} + 2\alpha_{34}, 2\alpha_{12} + 3\alpha_{34}, 3\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 3\alpha_{34}, 3\alpha_{12} + 4\alpha_{34}, \\
& 4\alpha_{12} + \alpha_{23} + 4\alpha_{34}, 4\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + \alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + 5\alpha_{23} + 2\alpha_{34}, \\
& 2\alpha_{12} + 5\alpha_{23} + 3\alpha_{34}, 3\alpha_{12} + 5\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 5\alpha_{23} + 3\alpha_{34}\} \\
W_9 & \{2\alpha_{12} + 4\alpha_{23} + 2\alpha_{34}, 2\alpha_{12} + 2\alpha_{34}, 4\alpha_{12} + 2\alpha_{34}, 2\alpha_{12} + 4\alpha_{34}, \\
& \alpha_{12} + 2\alpha_{23} + 4\alpha_{34}, 4\alpha_{12} + 4\alpha_{34}\} \\
W_{10} & \{3\alpha_{12} + 3\alpha_{34}\}
\end{align*}
\]
$W_{11} \{3\alpha_{12} + \alpha_{23}, 3\alpha_{12} + 2\alpha_{23}, 3\alpha_{23} + \alpha_{34}, 3\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 2\alpha_{34},$
$3\alpha_{12} + \alpha_{23} + \alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + \alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, 4\alpha_{12} + \alpha_{23},$
$4\alpha_{12} + 3\alpha_{23}, 4\alpha_{23} + \alpha_{34}, 4\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 4\alpha_{23} + \alpha_{34},$
$4\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 5\alpha_{12} + 2\alpha_{23},$
$5\alpha_{12} + 3\alpha_{23}, 5\alpha_{23} + 2\alpha_{34}, 5\alpha_{23} + 3\alpha_{34}, 5\alpha_{12} + 5\alpha_{23} + 2\alpha_{34},$
$5\alpha_{12} + 5\alpha_{23} + 3\alpha_{34}, 5\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, 5\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}\}$

$W_{-11} \{\alpha_{12} + 3\alpha_{23}, 2\alpha_{12} + 3\alpha_{23} + \alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 3\alpha_{34},$
$\alpha_{12} + \alpha_{23} + \alpha_{34}, \alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, \alpha_{12} + 4\alpha_{23},$
$3\alpha_{12} + 4\alpha_{23}, \alpha_{23} + 4\alpha_{34}, 3\alpha_{23} + 4\alpha_{34}, \alpha_{12} + 4\alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 4\alpha_{34},$
$\alpha_{12} + \alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + 5\alpha_{23}, 3\alpha_{12} + 5\alpha_{23},$
$2\alpha_{23} + 5\alpha_{34}, 3\alpha_{23} + 5\alpha_{34}, 2\alpha_{12} + 5\alpha_{23} + 5\alpha_{34}, 3\alpha_{12} + 5\alpha_{23} + 5\alpha_{34},$
$2\alpha_{12} + 2\alpha_{23} + 5\alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 5\alpha_{34}\}$

$W_{12} \{3\alpha_{12} + 2\alpha_{23} + \alpha_{34}, 5\alpha_{12} + 2\alpha_{23} + \alpha_{34}, 5\alpha_{12} + 4\alpha_{23} + \alpha_{34}, 5\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}\}$

$W_{-12} \{\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, \alpha_{12} + 2\alpha_{23} + 5\alpha_{34}, \alpha_{12} + 4\alpha_{23} + 5\alpha_{34}, \alpha_{12} + 4\alpha_{23} + 5\alpha_{34}\}$

$W_{13} \{3\alpha_{12} + \alpha_{23} + 2\alpha_{34}, \alpha_{12} + 4\alpha_{23} + 2\alpha_{34}, 3\alpha_{12} + 4\alpha_{23} + 2\alpha_{34},$
$4\alpha_{12} + \alpha_{23} + 2\alpha_{34}, 4\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, \alpha_{12} + 5\alpha_{23} + 2\alpha_{34},$
$\alpha_{12} + 5\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 5\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 5\alpha_{23} + 3\alpha_{34},$
$3\alpha_{12} + 5\alpha_{34}, 4\alpha_{12} + 2\alpha_{23} + 5\alpha_{34}, 4\alpha_{12} + \alpha_{23} + 5\alpha_{34}\}$

$W_{-13} \{2\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + \alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 3\alpha_{34},$
$2\alpha_{12} + \alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + 5\alpha_{23} + \alpha_{34},$
$2\alpha_{12} + 5\alpha_{23} + 4\alpha_{34}, 3\alpha_{12} + 5\alpha_{23} + \alpha_{34}, 3\alpha_{12} + 5\alpha_{23} + 4\alpha_{34},$
$5\alpha_{12} + \alpha_{23} + 3\alpha_{34}, 5\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}, 5\alpha_{12} + \alpha_{23} + 4\alpha_{34}\}$

$W_{14} \{4\alpha_{12} + 2\alpha_{23} + 4\alpha_{34} + 4\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}\}$

$W_{-14} \{2\alpha_{12} + 4\alpha_{23}, 2\alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 3\alpha_{34}, 2\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}\}$

$W_{15} \{4\alpha_{12} + 2\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 4\alpha_{12} + 3\alpha_{23} + 2\alpha_{34},$
$5\alpha_{12} + 3\alpha_{23} + 3\alpha_{34}, 5\alpha_{12} + 3\alpha_{23} + 2\alpha_{34}, 5\alpha_{12} + 4\alpha_{23} + 2\alpha_{34}\}$

$W_{-15} \{\alpha_{12} + 2\alpha_{23} + 4\alpha_{34}, \alpha_{12} + 3\alpha_{23} + 4\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 4\alpha_{34},$
$\alpha_{12} + 3\alpha_{23} + 5\alpha_{34}, 2\alpha_{12} + 3\alpha_{23} + 5\alpha_{34}, 2\alpha_{12} + 4\alpha_{23} + 5\alpha_{34}\}$

Here, $W_0$ corresponds to the trivial flat connection, $W_{10}$ to central flat connection, $W_5$, $W_{11}$, and $W_{-11}$ to abelian flat connection $U(1)^3$, $W_1$, $W_2$, $W_4$, $W_{-4}$, $W_7$, $W_8$, $W_{13}$, and $W_{-13}$ to reducible flat connections of type $SU(2) \times U(1) \times U(1)$, $W_3$, $W_6$, $W_9$, $W_{15}$, and $W_{-15}$ to reducible flat connections of type $SU(2) \times U(2)$, and $W_{12}$, $W_{-12}$, $W_{14}$ and $W_{-14}$ to reducible flat connections of type $SU(3) \times U(1)$). We can see that contributions from reducible flat connections corresponding to $W_t$ and $W_{-t}$, $t = 4, 11, 12, 13, 14, 15$, which are complex conjugate to each other, are separately same. Sums of their contributions are denoted by $Z'_t$, $t = 4, 11, 12, 13, 14, 15$, respectively. Contributions from $W_{t}$’s are denoted by $Z'_t$’s, $t = 0, 1, 2, 3, 5, 6, 7, 8, 9, 10$, respectively. In addition, $W_0$ and $W_{10}$ are related by
center, and their contributions are same, $Z'_0 = Z'_{10}$. This is similar to $W_1$ and $W_8$, $W_2$ and $W_7$, $W_4$ and $W_{13}$, $W_6$ and $W_9$, and $W_{12}$ and $W_{14}$, and their contributions are same, $Z'_1 = Z'_{11}$, $Z'_2 = Z'_{13}$, $Z'_4 = Z'_{13}$, $Z'_6 = Z'_0$, and $Z'_{12} = Z'_{14}$, respectively. Thus, we have $C_0 = \{W_0, W_{10}\}$, $C_1 = \{W_1, W_8\}$, $C_2 = \{W_2, W_7\}$, $C_3 = \{W_3\}$, $C_{\pm 4} = \{W_4, W_{-4}, W_{13}, W_{-13}\}$, $C_5 = \{W_5\}$, $C_6 = \{W_6, W_9\}$, $C_{\pm 7} = \{W_{11}, W_{-11}\}$, $C_{\pm 8} = \{W_{12}, W_{-12}, W_{14}, W_{-14}\}$, $C_{\pm 9} = \{W_{15}, W_{-15}\}$. After renaming $Z'_i$'s as $Z_i = Z'_i$, $t = 0, 1, \ldots, 6$, $Z_7 = Z'_{11}$, $Z_8 = Z'_{12}$, and $Z_9 = Z'_{15}$, the WRT invariant can be written as

$$Z_{SU(4)}(M_3) = \sum_a e^{2\pi i KCS_a} Z_a = \sqrt{2} 6^q \sum_{a,b} e^{2\pi i KCS_a} S_{ab} \hat{Z}_b \left| q = e^{2\pi i} \right. (3.126)$$

where

$$\hat{Z}_0 = -q^{74+40/17} (2 + 2q^6 - q^{15} - q^{18} - 3q^{21} + 2q^{24} - q^{27} - 4q^{30} - 4q^{33} + \cdots),$$

$$\hat{Z}_1 = -q^{75+40/17} (1 - 2q^6 - q^{15} - q^{18} - 4q^{27} - 2q^{30} - 2q^{33} + 2q^{36} + \cdots),$$

$$\hat{Z}_2 = -q^{51+100/17} (1 + 2q^6 + q^{15} + 3q^{18} - 2q^{21} + 2q^{24} + 3q^{29} - 3q^{33} + 3q^{36} + \cdots),$$

$$\hat{Z}_3 = -q^{51+100/17} (1 + q^6 + 2q^{15} + q^{18} + q^{21} + 2q^{24} + q^{30} + 2q^{33} + 3q^{36} + \cdots),$$

$$\hat{Z}_4 = -q^{55+40/17} (1 + q^6 + q^{12} + q^{15} + 3q^{18} + 2q^{21} - 3q^{24} + 2q^{27} - q^{30} + q^{33} + q^{36} + \cdots),$$

$$\hat{Z}_5 = -q^{60+227/9} (1 + 2q^6 + 2q^{10} + q^{15} - q^{18} - q^{21} + 2q^{24} - 3q^{27} + 3q^{30} + q^{33} + q^{36} + \cdots),$$

$$\hat{Z}_6 = q^{59+227/9} (1 + q^6 + q^{12} - 2q^{14} + 2q^{16} - q^{18} + q^{19} + q^{23} + 2q^{24} - q^{26} + \cdots),$$

$$\hat{Z}_7 = -q^{57+40/17} (1 - q^3 - 2q^5 - 2q^6 - q^{12} - q^{13} + 2q^{14} - q^{15} + 2q^{16} + \cdots),$$

$$\hat{Z}_8 = 2q^{61+227/9} (1 - q^5 - 2q^{11} - q^{13} - q^{18} - q^{21} - 3q^{22} - 2q^{23} + 2q^{24} - 3q^{26} + \cdots),$$

$$\hat{Z}_9 = -2q^{68+227/9} (2 + q^3 - q^8 - 2q^9 - 2q^{13} - q^{14} + q^{16} + q^{17} - q^{19} + \cdots),$$

and

$$CS_a = (0, 1/6, 2/3, 1/2, 1/2, 1/3, 1/3, 1/3, 1/6, 5/6, 0),$$

$$S_{ab} = \frac{1}{2 \cdot 3^2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 12 & 5 & -3 & -4 & -1 & -4 & 0 & -1 & 3 & 8 \\ 12 & -3 & -3 & 12 & 3 & 0 & 3 & 3 & 0 & 3 \\ 3 & -1 & 3 & -1 & -1 & 3 & -1 & 3 & -1 \\ 24 & -2 & 6 & -8 & 1 & 4 & -12 & -2 & -3 & 4 \\ 12 & -4 & 0 & -4 & 2 & -4 & 6 & 2 & -6 & 2 \\ 6 & 0 & 0 & 6 & -3 & 3 & 3 & 0 & -3 & 3 \\ 24 & -2 & -6 & -8 & -2 & 4 & 0 & 4 & 6 & -8 \\ 8 & 2 & 2 & 8 & -1 & -4 & -4 & 2 & -1 & -4 \\ 6 & 4 & 0 & -2 & 1 & 1 & 3 & -2 & -3 & -5 \end{pmatrix} (3.138)$$

which satisfies $S_{ab} S_{bc} = \delta_{ac}$. This S-matrix and $CS_a$'s can also be obtained from (3.10) and (3.11) and they agree with results above.
4 Discussion

We calculated homological blocks for general $SU(N)$ gauge group and general Seifert manifolds with $P_j$’s being pairwise coprime from the exact expression of Lawrence, Rozansky, and Mariño. We firstly calculated the partition function of the analytically continued $SU(N)$ Chern-Simons theory or the WRT invariant and obtained the exact expression for $Z_a$’s that are labelled by reducible flat connections. From $Z_a$’s, we extracted homological blocks $\hat{Z}_b$, which are related to $Z_a$ by the $S$-matrix. We proposed a formula to calculate the $S$-matrix in general cases and checked that the $S$-matrix calculated from the formula by using the linking form agrees with the result that was obtained from the calculation of $Z_a$’s and $\hat{Z}_b$’s. Also, we discussed general expression for the WRT invariant and checked that examples that we discussed fit into the expected form. In addition, we discussed some properties of the $SU(N)$ WRT invariant. We found a symmetry that the reducible flat connections in the same orbit under the action of center give same contribution $Z_a$ to the WRT invariant up to overall exponential factor $e^{\pi i K}$. We also discussed a symmetry that contributions from the reducible flat connection and from the conjugate reducible flat connection are same. In addition, we also saw that the exact expression of Lawrence and Rozansky can be understood in the context of resurgent analysis when $H = 1$.

There are several interesting directions to consider. In [8], the superconformal index and topologically twisted index can be calculated from homological blocks via

\begin{align}
I(q,t) &= \sum_a |\text{Stab}_{SN}(a)| \hat{Z}_a(q,t) \hat{Z}_a(q^{-1},t^{-1}), \\
I_{\text{top}}(q,t) &= \sum_a |\text{Stab}_{SN}(a)| \hat{Z}_a(q,t) \hat{Z}_a(q^{-1},t)
\end{align}

and several examples supported it. However, as we saw in some examples, reducible flat connections in the same orbit under the action of center were grouped under a single label. So this should be taken into account when calculating the indices from the homological blocks obtained in this paper. If we consider the multiplicity from the orbit under the action of centers, it is expected that the indices are given by

\begin{align}
I(q,t) &= \sum_a |C_a||\text{Stab}_{SN}(a)| \hat{Z}_a(q,t) \hat{Z}_a(q^{-1},t^{-1}) \\
I_{\text{top}}(q,t) &= \sum_a |C_a||\text{Stab}_{SN}(a)| \hat{Z}_a(q,t) \hat{Z}_a(q^{-1},t)
\end{align}

whereas there would be another overall $1/2$ factor when considering the case that involves the matrix $((1, 1), (1, 1))$. In the calculation discussed above, it was not obvious to calculate $\hat{Z}(q^{-1})$ on $M_3$ with $|q| < 1$, so we didn’t calculate (4.3) or (4.4) with $t = 1$. It would be interesting to calculate the indices for general Seifert manifolds.

We have obtained various homological blocks. When $G = SU(2)$ and the number of singular fibers $F$ is 3 or 4 with genus zero, homological blocks are expressed in terms of false theta functions, which are known in literature. By varying the number of singular
fibers or genus, we found that homological blocks are given by false theta functions $\tilde{\Phi}_{2HP}(q)$ and $\tilde{\Psi}_{2HP}(q)$, and their derivatives. Moreover, considering the higher rank case $SU(N)$, $N \geq 3$, it is expected that the homological blocks in those cases would provide new false theta functions. It would be interesting to study modular properties of homological blocks obtained in this paper including their properties upon $q \to q^{-1}$ [18].

Resurgent analysis was discussed in the case of $SU(2)$ Chern-Simons theory with $H = 1$ in literature. We considered it in the context of the exact formula in [15] when $H = 1$. It would be interesting to study resurgent analysis for more general cases.

A mathematical construction or definition for Khovanov-type homology for closed 3-manifolds is not available yet. But if it is constructed, from the analogy, it would be expected that there is also a homology theory for closed 3-manifolds analogous to the HOMFLY homology for knots and links. It will be also interesting to study properties of such expected homology theory from homological blocks for $G = SU(N)$ obtained in this paper.

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Appendix A  \hspace{1em} G = SU(2) and four singular fibers

We can proceed similarly also for the case of four singular fibers as in the previous sections. The contributions of abelian flat connections to the WRT invariant of Seifert manifolds with four singular fibers is

$$Z_{SU(2)}^{ab}(M_3) = \frac{B}{2\pi i} q^{-\phi_4/4} \sum_{t=0}^{H-1} \int_{\Gamma_t} dy \, e^{-\frac{\psi}{2\pi i} y^2 - 2Ky} \prod_{j=1}^{4} e^{\frac{\psi_j}{2}} e^{-\frac{\psi_j}{2}}$$

(A.1)

where

$$P = \prod_{j=1}^{4} P_j, \hspace{1em} H = \sum_{j=1}^{4} \frac{Q_j}{P_j}, \hspace{1em} \phi_4 = 3 \text{sign} \left( \frac{H}{P} \right) + \sum_{j=1}^{4} \left( 12s(Q_j, P_j) - \frac{Q_j}{P_j} \right)$$

(A.2)

with the integration cycle $\Gamma_t$ as in section 2. The last factor can be expressed as

$$\frac{\prod_{j=1}^{4} e^{\frac{\psi_j}{2}} e^{-\frac{\psi_j}{2}}}{(e^y - e^{-y})^2} = \sum_{s=0}^{7} (-1)^s \cosh \frac{R_s y}{2(\sinh y)^2}$$

(A.3)

where $R_s$’s with even (resp. odd) $s$ are $P\left( \sum_{j=1}^{4} \frac{e_j}{P_j} \right)$ with $\prod_{j=1}^{4} e_j = 1$ (resp. $-1$) up to overall sign where $e_j = \pm 1$.

\footnote{For example,}
Let $\varphi_{2P}^{(l)}(n)$ be a periodic function

\[
\varphi_{2P}^{(l)}(n) = \begin{cases} 
1 & \text{if } n \equiv \pm l \mod 2P \\
0 & \text{otherwise}, 
\end{cases}
\] (A.4)

so it satisfies $\varphi_{2P}^{(l)}(n) = \varphi_{2P}^{(2P-l)}(n)$. Then we have

\[
\frac{\cosh(P + l)y}{\sinh Py} = \sum_{m=0}^{M_l} (e^{(l-2mP)y} - e^{-(l-2mP)y}) + \sum_{n=0}^{\infty} \varphi_{2P}^{(l-2M_lP)}(n)e^{-ny} \tag{A.5}
\]

where $M_l = -1, 0, 1, \cdots$ such that $2M_lP < l < 2(M_l + 1)P$. By taking derivative with respect to $y$, it leads to

\[
\frac{\cosh ly}{(\sinh Py)^2} = \frac{1}{P} \left( \sum_{m=0}^{M_l} 2mP(e^{(l-2mP)y} + e^{-(l-2mP)y}) + \sum_{n=0}^{\infty} n\varphi_{2P}^{(l-2M_lP)}(n)e^{-ny} - l \sum_{n=0}^{\infty} \psi_{2P}^{(l-2M_lP)}(n)e^{-ny} \right). \tag{A.6}
\]

We plug (A.3) into (A.1) and calculate the integral as in section 2.1. Evaluating the integral with the contour $\gamma$ extending along the imaginary axis of $y$-plane that passes through $\Re{y} > 0$ with analytically continued $K$, we have the partition function of analytically continued $SU(2)$ theory

\[
Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_4/4} \left( \frac{2iP}{KH} \right)^2 \sum_{l=0}^{H-1} e^{2\pi i K P l^2} \sum_{s=0}^{7} (-1)^s \left( \sum_{m=0}^{M_{R_s}} 2m(e^{-\frac{2\pi i l}{H}(R_s-2mP)} + e^{\frac{2\pi i l}{H}(R_s-2mP)})q^{\frac{1}{4MP}(R_s-2mP)^2} \right.
\]

\[
+ \frac{1}{P} \sum_{n=0}^{\infty} \left( n\varphi_{2P}^{(R_s-2M_{R_s}P)}(n) - R_s\psi_{2P}^{(R_s-2M_{R_s}P)}(n) \right) e^{2\pi i \frac{1}{H} n^2 q^{\frac{2}{4MP}} K} \right)
\] (A.7)

where $M_{R_s}$’s, $M_{R_s} = 0, 1, \cdots$, satisfy $2M_{R_s}P < R_s < 2(M_{R_s} + 1)P$ given $R_s$ for each $s$. As $P_s$’s are coprime in our setup, the maximum value that $R_s$ can take is less than $2P$. Thus, when $F = 4$, $M_{R_s}$ is 0, so the terms in the second line vanish. Therefore, we obtain

\[
Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_4/4} \left( \frac{2iP}{KH} \right)^2 \frac{1}{P} \sum_{l=0}^{H-1} e^{2\pi i K P l^2} \sum_{s=0}^{7} (-1)^s \sum_{n=0}^{\infty} (n\varphi_{2P}^{(R_s)}(n) - R_s\psi_{2P}^{(R_s)}(n)) e^{2\pi i \frac{1}{H} n^2 q^{\frac{2}{4MP}} K} \right)
\] (A.8)

\[
R_0 = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} + \frac{1}{P_4}, \quad R_1 = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} + \frac{1}{P_4}, \quad R_2 = \frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} - \frac{1}{P_4}, \quad R_3 = \frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} + \frac{1}{P_4}, \quad R_4 = \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} - \frac{1}{P_4}, \quad R_5 = \frac{1}{P_1} - \frac{1}{P_2} + \frac{1}{P_3} + \frac{1}{P_4}, \quad R_6 = \frac{1}{P_1} - \frac{1}{P_2} - \frac{1}{P_3} + \frac{1}{P_4}, \quad R_7 = \frac{1}{P_1} - \frac{1}{P_2} - \frac{1}{P_3} - \frac{1}{P_4}.
\]

\[ \boxed{ -50 - } \]
A.1 The case $H = 1$

For the integer Seifert homology sphere, the partition function is given by

$$Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_4/4} \left( \frac{2i}{K} P \right)^{1/2} \frac{1}{\Gamma} \sum_{s=0}^{7} (-1)^s \sum_{n=0}^{\infty} \left( n \varphi_{2H_P}^{(R_s)}(n) - R_s \tilde{\Phi}_{2H_P}^{(R_s)}(q) \right) q^{\frac{n^2}{4H}} \quad (A.9)$$

Denoting

$$\tilde{\Phi}_{2H_P}^{(l)}(q) := \sum_{n=0}^{\infty} n \varphi_{2H_P}^{(l)}(n) q^{\frac{n^2}{4H}} \quad (A.10)$$

(A.9) can be written as

$$Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_4/4} \left( \frac{2i}{K} P \right)^{1/2} \frac{1}{\Gamma} \sum_{s=0}^{7} (-1)^s \left( \tilde{\Phi}_{2H_P}^{(R_s)}(q) - R_s \tilde{\Phi}_{2H_P}^{(R_s)}(q) \right) q^{\frac{n^2}{4H}} \quad (A.11)$$

It is known that $\tilde{\Phi}_{2H_P}^{(l)}(q)$ is a false theta function, which is the Eichler integral of modular form $\Phi_{2H_P}^{(l)}(q) := \sum_{n=0}^{\infty} \varphi_{2H_P}^{(l)}(n) q^{\frac{n^2}{4H}}$ of weight $1/2$ [17]. (A.11) agrees with the result in [3] where only the case $H = 1$ has been discussed.\(^\text{10}\)

A.2 The case $H \geq 2$

When $H \geq 2$, $\varphi_{2H_P}^{(l)}(n)$ can also be decomposed in terms of $\varphi_{2H_P}^{(l)}(n)$ (c.f. (2.25))

$$\varphi_{2H_P}^{(l)}(n) = \sum_{h=0}^{[\frac{H}{2}]-1} \varphi_{2H_P}^{(2hP+l)}(n) + \sum_{h=0}^{[\frac{H}{2}]-1} \varphi_{2H_P}^{(2(h+1)P-l)}(n). \quad (A.12)$$

Therefore, by using (2.25) and (A.12) the partition function can be written as

$$Z_{SU(2)}(M_3) = \frac{B}{2i} q^{-\phi_4/4} \left( \frac{2i}{K} P \right)^{1/2} \frac{1}{\Gamma} \sum_{s=0}^{7} (-1)^s \sum_{n=0}^{\infty} \left( n \varphi_{2H_P}^{(2hP+R_s)}(n) + \sum_{h=0}^{[\frac{H}{2}]-1} \varphi_{2H_P}^{(2(h+1)P-R_s)}(n) \right)$$

$$\quad - R_s \left( \sum_{h=0}^{[\frac{H}{2}]-1} \psi_{2H_P}^{(2hP+R_s)}(n) - \sum_{h=0}^{[\frac{H}{2}]-1} \psi_{2H_P}^{(2(h+1)P-R_s)}(n) \right) q^{\frac{n^2}{4H}}$$

$$+ \sum_{t=1}^{H-1} e^{2\pi K t^2} \left( \frac{2i}{K} P \right)^{1/2} \sum_{s=0}^{7} (-1)^s \left( n \varphi_{2H_P}^{(2hP+R_s)}(n) + \sum_{h=0}^{[\frac{H}{2}]-1} \varphi_{2H_P}^{(2(h+1)P-R_s)}(n) \right)$$

$$\quad - R_s \left( \sum_{h=0}^{[\frac{H}{2}]-1} \psi_{2H_P}^{(2hP+R_s)}(n) - \sum_{h=0}^{[\frac{H}{2}]-1} \psi_{2H_P}^{(2(h+1)P-R_s)}(n) \right) e^{2\pi i n^2 \frac{q^{\frac{n^2}{4H}}}} \right) \quad (A.13)$$

\(^{10}\)Recently, the case of $F = 4$ with $H \geq 2$ was also discussed in [18].
As before, when $H$ is odd, the WRT invariant is given by

$$Z_{SU(2)}(M_3) = \frac{B}{2t} q^{-\frac{\phi_4}{4}} \left( \frac{2i}{K \bar{H}} \right)^{\frac{1}{2}} \frac{1}{P} \times \left[ \sum_{s=0}^{7} (-1)^s \left( \sum_{h=0}^{\frac{H-1}{2}} \left( \Phi_{HP}^{(2hP+R_s)} - R_s \tilde{\Psi}_{HP}^{(2hP+R_s)} \right) + \sum_{h=0}^{\frac{H-1}{2}} \left( \Phi_{HP}^{(2(h+1)P-R_s)} - R_s \tilde{\Psi}_{HP}^{(2(h+1)P-R_s)} \right) \right) + \sum_{s=0}^{7} (-1)^s \left( \sum_{h=0}^{\frac{H-1}{2}} \left( e^{2\pi i \frac{K}{P} h} \Phi_{HP}^{(2hP+R_s)} - e^{2\pi i \frac{K}{P} (2hP+R_s)} \tilde{\Psi}_{HP}^{(2hP+R_s)} \right) \right) \right] \left| q^{\frac{1}{2}} e^{\frac{2\pi i}{K}} \right| .$$

(A.14)

When $H$ is even,

$$Z_{SU(2)}(M_3) = \frac{B}{2t} q^{-\frac{\phi_4}{4}} \left( \frac{2i}{K \bar{H}} \right)^{\frac{1}{2}} \frac{1}{P} \times \left[ \sum_{s=0}^{7} (-1)^s \left( \sum_{h=0}^{\frac{H-2}{2}} \left( \Phi_{HP}^{(2hP+R_s)} - R_s \tilde{\Psi}_{HP}^{(2hP+R_s)} \right) + \sum_{h=0}^{\frac{H-2}{2}} \left( \Phi_{HP}^{(2(h+1)P-R_s)} - R_s \tilde{\Psi}_{HP}^{(2(h+1)P-R_s)} \right) \right) + \sum_{s=0}^{7} (-1)^s \left( \sum_{h=0}^{\frac{H-2}{2}} \left( e^{2\pi i \frac{K}{P} h} \left( e^{-2\pi i \frac{K}{P} (2hP+R_s)} + e^{2\pi i \frac{K}{P} (2hP+R_s)} \right) \right) \right) \right] \left| q^{\frac{1}{2}} e^{\frac{2\pi i}{K}} \right| .$$

(A.15)

We note that the structure of the WRT invariant, i.e. the coefficients to the contributions from abelian flat connections including the trivial one are same as in the case of $F = 3$, (2.34) or (2.35). Therefore the WRT invariant for Seifert manifolds with four singular fibers can also be written similarly as in section 2. We provide one example.

• $F = 4$ and $H = 3$

The structure is same as in the case of $F = 3$. $H = 3$ can be obtained, for example, from $(P_1, P_2, P_3, P_4) = (2, 5, 7, 11)$ and $(Q_1, Q_2, Q_3, Q_4) = (-1, 2, 2, -2)$. In this case, the
homological blocks are given by

$$
\hat{Z}_0 = -\Psi^{(51)}_{2310} + 51\Psi^{(51)}_{2310} - \Psi^{(411)}_{2310} + 411\Psi^{(411)}_{2310} - \Phi^{(579)}_{2310} + 579\Psi^{(579)}_{2310} + \Phi^{(1041)}_{2310} - 499\Psi^{(1041)}_{2310} + \Phi^{(1269)}_{2310} + 271\Psi^{(1269)}_{2310} + \Phi^{(1731)}_{2310} - 191\Psi^{(1731)}_{2310},
$$

(A.16)

$$
\hat{Z}_1 = \Psi^{(191)}_{2310} - 191\Psi^{(191)}_{2310} + \Phi^{(271)}_{2310} - 271\Psi^{(271)}_{2310} + \Phi^{(259)}_{2310} - 359\Psi^{(259)}_{2310} - \Phi^{(499)}_{2310} + 499\Psi^{(499)}_{2310} + \Phi^{(1129)}_{2310} - 411\Psi^{(1129)}_{2310} + \Phi^{(1181)}_{2310} - 359\Psi^{(1181)}_{2310} + \Phi^{(1439)}_{2310} + 191\Psi^{(1439)}_{2310} - \Phi^{(1489)}_{2310} - 51\Psi^{(1489)}_{2310} - 51\Phi^{(1501)}_{2310} + 51\Psi^{(1501)}_{2310} + 499\Psi^{(2039)}_{2310} - 499\Phi^{(2039)}_{2310} - \Phi^{(2119)}_{2310} + 579\Psi^{(2119)}_{2310},
$$

(A.17)

where $\hat{Z}_0 = q^{\frac{6\pi i}{K}} Z[q]$ and $\hat{Z}_1 = q^{\frac{6\pi i}{K}} Z[q]$. In terms of homological blocks, the WRT invariant is written as

$$
Z_{SU(2)}(M_3) = \frac{B}{770i} q^{-\phi_4/4} \left( \frac{770i}{K} \right)^{\frac{1}{2}} \sum_{a,b=0} e^{2\pi i KCS_ab} S_{ab} \psi_b(q) \bigg|_{q=e^{2\pi i K}}
$$

(A.18)

with $(CS_0, CS_1) = (0, \frac{1}{3})$ and $S_{ab} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. The S-matrix and $CS_a$'s can also be calculated from (2.38) and (2.39).

**Appendix B  $G = SU(2)$, more singular fibers, and the base with higher genus**

We would like to consider Seifert manifolds with more singular fibers and with higher genus of the base surface. The case of $H = 1$ with arbitrary number of singular fibers and with genus zero was discussed in [4]. Here we consider general Seifert manifolds with more than 4 singular fibers and also with higher genus case where $P_j$'s are pairwise coprime.

As before, we consider the integral expression for the WRT invariant of Seifert manifolds with $F$ singular fibers and with genus $g$ according to [15, 21],

$$
Z_{SU(2)}(M_3) = \frac{B}{2\pi i} (-2K)^g q^{-\phi_F/4} \sum_{i=0}^{H-1} \int_{\Gamma_1} dy \ e^{-\frac{2\pi i}{2Kt} q^2 - 2Kty} \prod_{j=1}^{F} e^{\frac{\phi}{2Kt}} - e^{-\frac{\phi}{2Kt}}
$$

(B.1)

where

$$
P = \prod_{j=1}^{F} P_j, \quad H = \sum_{j=1}^{F} \frac{Q_j}{P_j}, \quad \phi_F = 3 \text{sign} \left( \frac{H}{P} \right) + \sum_{j=1}^{F} \left( 12s(Q_j, P_j) - \frac{Q_j}{P_j} \right)
$$

(B.2)

with same integration cycle as in section 2.

We will see that calculation for arbitrary genus $g$ can be done easily if calculation for genus zero with arbitrary number of singular fibers is done. So from now on, we proceed
calculation with \( g = 0 \).

As in previous sections, we expand the last factor in (B.1). The numerator can be expressed in terms of hyperbolic sine or cosine function when \( F \) is odd or even, respectively,

\[
\prod_{j=1}^{F} \frac{e^{\frac{\pi}{F} \epsilon_j} - e^{-\frac{\pi}{F} \epsilon_j}}{(e^{y} - e^{-y})^{F-2}} = \begin{cases} 
\frac{1}{2F} \sum_{\epsilon_j = \pm 1} \sinh \left( \frac{\sum_{j=1}^{F} \frac{\pi}{F} \epsilon_j}{(\sinh y)^{F-2}} \right) y & \text{if } F \text{ is odd} \\
\frac{1}{2F} \sum_{\epsilon_j = \pm 1} \cosh \left( \frac{\sum_{j=1}^{F} \frac{\pi}{F} \epsilon_j}{(\sinh y)^{F-2}} \right) y & \text{if } F \text{ is even}
\end{cases}
\]

(B.3)

where \( \epsilon = \prod_{j=1}^{F} \epsilon_j \). By \( \sum_{\epsilon_j = \pm 1} \), we mean the summation over \( \epsilon_j = \pm 1 \) such that \( \epsilon = +1 \). We find that (B.1) can be expressed as false theta functions, \( \tilde{\Psi}^{(a)}_H(q) \), \( \tilde{\Phi}^{(a)}_H(q) \), and their derivatives.

For arbitrary number of singular fibers \( F \), we see that types of functions appearing in the integrand are \( \frac{\sinh \alpha y}{(\sinh y)^{2m+1}} \) or \( \frac{\cosh \alpha y}{(\sinh y)^{2m}} \) where \( \alpha = \sum_{j=1}^{F} \epsilon_j \). After some calculations, we have

\[
\frac{\sinh \alpha y}{(\sinh y)^{2m+1}} = \frac{1}{2m(2m-1)} \left( \frac{d^2}{dy^2} - (2m - 1 - \alpha)^2 \right) \frac{\sinh \alpha y}{(\sinh y)^{2m-1}} + \frac{2\alpha \cosh(1 - \alpha) y}{2m} \frac{\sinh \alpha y}{(\sinh y)^{2m-1}},
\]

(B.4)

\[
\frac{\cosh \alpha y}{(\sinh y)^{2m}} = \frac{1}{(2m-1)(2m-2)} \left( \frac{d^2}{dy^2} - (2m - 2 - \alpha)^2 \right) \frac{\cosh \alpha y}{(\sinh y)^{2m-2}} - \frac{2\alpha}{2m-1} \frac{\sinh(1 - \alpha) y}{(\sinh y)^{2m-1}}.
\]

(B.5)

For notational convenience, we denote

\[
S(\alpha, m) := \frac{\sinh \alpha y}{(\sinh y)^{2m+1}}, \quad C(\alpha, m) := \frac{\cosh \alpha y}{(\sinh y)^{2m}},
\]

(B.6)

and

\[
s_1(\alpha, m) := \frac{1}{2m(2m-1)} \left( \frac{d^2}{dy^2} - (2m - 1 - \alpha)^2 \right), \quad s_2(\alpha, m) := \frac{2\alpha}{2m},
\]

(B.7)

\[
c_1(\alpha, m) := \frac{1}{(2m-1)(2m-2)} \left( \frac{d^2}{dy^2} - (2m - 2 - \alpha)^2 \right), \quad c_2(\alpha, m) := -\frac{2\alpha}{2m-1},
\]

(B.8)

then (B.4) and (B.5) are written as

\[
S(\alpha, m) = s_1(\alpha, m)S(\alpha, m - 1) + s_2(\alpha, m)C(1 - \alpha, m), \quad m \geq 1 \quad \text{(B.9)}
\]

\[
C(\alpha, m) = c_1(\alpha, m)C(\alpha, m - 1) + c_2(\alpha, m)S(1 - \alpha, m - 1), \quad m \geq 2. \quad \text{(B.10)}
\]

**Odd \( F \)**

Firstly, we consider the case of odd \( F \). (B.9) and (B.10) lead to a recursion relation for \( S(\alpha, m) \),

\[
S(\alpha, m) = \left( s_1(\alpha, m) + s_2(\alpha, m)s_2(1 - \alpha, m) + \frac{s_2(\alpha, m)}{s_2(\alpha, m - 1)}c_1(1 - \alpha, m) \right) S(\alpha, m - 1)
\]

\[
- \frac{s_2(\alpha, m)}{s_2(\alpha, m - 1)}c_1(1 - \alpha, m)s_1(\alpha, m - 1)S(\alpha, m - 2), \quad m \geq 2.
\]

(B.11)
Again, for simplicity, denoting the coefficient of $S(\alpha, m - 1)$ and $S(\alpha, m - 2)$ as

$$X_1(\alpha, m - 1) := s_1(\alpha, m) + s_2(\alpha, m)s_2(1 - \alpha, m) + \frac{s_2(\alpha, m)}{s_2(\alpha, m - 1)}c_1(1 - \alpha, m)$$

$$= \frac{1}{2m(2m - 1)}\left(2\frac{d^2}{dy^2} - 2(\alpha - 1)^2 - 8(m - 1)^2\right) + \frac{\alpha(1 - \alpha)}{m^2} \quad \text{(B.12)}$$

$$X_2(\alpha, m - 2) := -\frac{s_2(\alpha, m)}{s_2(\alpha, m - 1)}c_1(1 - \alpha, m)s_1(\alpha, m - 1)$$

$$= -\frac{1}{2m(2m - 1)(2m - 2)(2m - 3)}\left(\frac{d^4}{dy^4} - 2(2m - 3)^2 + \alpha^2\right)\frac{d^2}{dy^2} + ((2m - 3)^2 - \alpha^2)^2 \quad \text{(B.13)}$$

respectively, (B.11) is written as

$$S(\alpha, m) = X_1(\alpha, m - 1)S(\alpha, m - 1) + X_2(\alpha, m - 2)S(\alpha, m - 2), \quad m \geq 2. \quad \text{(B.14)}$$

By expanding some of terms, we obtain

$$S(\alpha, m) = \left(\sum_{\#s=0}^{[m/2]} \sum_{3 \leq j_1 < \ldots < j_s \leq m-1} X_1(\alpha, m - 1) \cdots X_{1,2}(\alpha, m - (j_1 - 2))X_2(\alpha, m - j_1)X_1(\alpha, m - (j_1 + 1)) \right. \quad \text{(B.15)}$$

$$\cdots X_{1,2}(\alpha, m - (j_s - 2))X_2(\alpha, m - j_s)X_1(\alpha, m - (j_s + 1)) \cdots X_{1,2}(\alpha, 1)\right)S(\alpha, 1)$$

$$+ \left(\sum_{\#s=1}^{[m/2]} \sum_{4 \leq j_1 < \ldots < j_s \leq m-2} X_2(\alpha, m - 2) \cdots X_{1,2}(\alpha, m - (j_1 - 2))X_2(\alpha, m - j_1)X_1(\alpha, m - (j_1 + 1)) \right.$$  

$$\cdots X_{1,2}(\alpha, m - (j_s - 2))X_2(\alpha, m - j_s)X_1(\alpha, m - (j_s + 1)) \cdots X_{1,2}(\alpha, 2) \right) X_2(\alpha, 0)S(\alpha, 0)$$

for $m \geq 3$ where $\#s$ denotes the total number of terms $X_{2}(\alpha, j_s)$ appearing in the expression. Some explanation is needed for (B.15). In (B.15), $X_1(\alpha, m - 1)$ and $X_2(\alpha, m - 2)$ are always there if the summation containing them is valid and there is no term whose argument is greater than or equal to $m - 1$ and $m - 2$, respectively. Also, summation is such that $m - j$ in parentheses is strictly decreasing. We mean terms $X_{1,2}(\alpha, j)$ by that
they can be either $X_1$ or $X_2$ depending on the summation, and the last terms $X_{1,2}(\alpha, 1)$ or $X_{1,2}(\alpha, 2)$ are always there. For example,

\begin{align}
S(\alpha, 2) &= X_1(\alpha, 1)S(\alpha, 1) + X_2(\alpha, 0)S(\alpha, 0) \quad (B.16) \\
S(\alpha, 3) &= (X_1(\alpha, 2)X_1(\alpha, 1) + X_2(\alpha, 1))S(\alpha, 1) + X_1(\alpha, 2)X_2(\alpha, 0)S(\alpha, 0) \quad (B.17) \\
S(\alpha, 4) &= (X_1(\alpha, 3)X_1(\alpha, 2)X_1(\alpha, 1) + X_1(\alpha, 3)X_2(\alpha, 1) + X_2(\alpha, 2)X_1(\alpha, 1))S(\alpha, 1) \\
&\quad + (X_1(\alpha, 3)X_1(\alpha, 2)X_2(\alpha, 0) + X_2(\alpha, 2)X_2(\alpha, 0))S(\alpha, 0) \quad (B.18) \\
S(\alpha, 5) &= (X_1(\alpha, 4)X_1(\alpha, 3)X_1(\alpha, 2)X_1(\alpha, 1) + X_1(\alpha, 4)X_1(\alpha, 3)X_2(\alpha, 1) + X_1(\alpha, 4)X_2(\alpha, 2)X_1(\alpha, 1) \\
&\quad + X_2(\alpha, 3)X_1(\alpha, 2)X_1(\alpha, 1) + X_2(\alpha, 3)X_2(\alpha, 1))S(\alpha, 1) + (X_1(\alpha, 4)X_1(\alpha, 3)X_1(\alpha, 2)X_2(\alpha, 0) \\
&\quad + X_1(\alpha, 4)X_2(\alpha, 2)X_2(\alpha, 0) + X_2(\alpha, 3)X_1(\alpha, 2)X_2(\alpha, 0))S(\alpha, 0) \\
&\quad (B.19) \\
S(\alpha, 6) &= (X_1(\alpha, 5)X_1(\alpha, 4)X_1(\alpha, 3)X_1(\alpha, 2)X_1(\alpha, 1) + X_1(\alpha, 5)X_1(\alpha, 4)X_1(\alpha, 3)X_2(\alpha, 1) \\
&\quad + X_1(\alpha, 5)X_2(\alpha, 2)X_1(\alpha, 1) + X_1(\alpha, 5)X_2(\alpha, 3)X_1(\alpha, 2)X_1(\alpha, 1) \\
&\quad + X_2(\alpha, 4)X_1(\alpha, 3)X_2(\alpha, 1) + X_2(\alpha, 4)X_2(\alpha, 2)X_1(\alpha, 1))S(\alpha, 1) \\
&\quad + (X_1(\alpha, 5)X_1(\alpha, 4)X_1(\alpha, 3)X_1(\alpha, 2)X_2(\alpha, 0) + X_1(\alpha, 5)X_1(\alpha, 4)X_2(\alpha, 2)X_2(\alpha, 0) \\
&\quad + X_1(\alpha, 5)X_2(\alpha, 3)X_1(\alpha, 2)X_2(\alpha, 0) + X_2(\alpha, 4)X_1(\alpha, 3)X_1(\alpha, 2)X_2(\alpha, 0) \\
&\quad + X_2(\alpha, 4)X_2(\alpha, 2)X_2(\alpha, 0))S(\alpha, 0). \quad (B.20)
\end{align}

Therefore, $S(\alpha, m)$ with $m \geq 3$ is determined from $S(\alpha, 0)$ and $S(\alpha, 1)$. By using

\begin{equation}
\frac{\sinh P \alpha y}{\sinh P y} = \sum_{m=0}^{M} (e^{(P\alpha-(2m+1)P)y} + e^{-(P\alpha-(2m+1)P)y}) - \sum_{n=0}^{\infty} \psi_{2P}^{(Pa-(2M+1)P)}(n)e^{-ny} \quad (B.21)
\end{equation}

where $M$ is such that $(2M+1)P < P\alpha < (2M+3)P$, $M = -1, 0, 1, \ldots$, and

\begin{equation}
\frac{\cosh P \alpha y}{(\sinh P y)^2} = \sum_{m=0}^{M} 2m(e^{(P\alpha-2mP)y} + e^{-(P\alpha-2mP)y}) + \frac{1}{P} \sum_{n=0}^{\infty} \phi_{2P}^{(Pa-2MP)}(n)e^{-ny} - \alpha \sum_{n=0}^{\infty} \psi_{2P}^{(Pa-2MP)}(n)e^{-ny} \quad (B.22)
\end{equation}

where $M$ is such that $2MP < P\alpha < (2M+2)P$, $M = 0, 1, \ldots$, $S(\alpha, 0)$ and $S(\alpha, 1)$ are obtained

\begin{equation}
S(\alpha, 0) = \frac{\sinh \alpha y}{\sinh y} = \sum_{m=0}^{M} (e^{(\alpha-(2m+1))y} + e^{-(\alpha-(2m+1))y}) - \sum_{n=0}^{\infty} \psi_{2P}^{(Pa-(2M+1)P)}(n)e^{-ny}, \quad (B.23)
\end{equation}
\[ S(\alpha, 1) = \frac{\sinh \alpha y}{(\sinh y)^3} = \frac{1}{2} \left( \frac{d^2}{dy^2} - (1 - \alpha)^2 \right) \frac{\sinh \alpha y}{\sinh y} + \alpha \frac{\cosh(1 - \alpha)y}{(\sinh y)^2} \]

\[ = \sum_{m=0}^{M} 2m(m+1)(e^{\alpha-(2m+1)y} + e^{-(\alpha-(2m+1))y}) \]

\[ - \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{n^2}{p^2} + \alpha^2 - 1 \right) \psi_{2p}^{(P \alpha-(2M+1)P)}(n) e^{-\frac{n}{p}y} + \frac{\alpha}{p} \sum_{n=0}^{\infty} n \psi_{2p}^{(P \alpha-(2M+1)P)}(n) e^{-\frac{n}{p}y} \]

(B.24)

for \((2M + 1)P < P\alpha < (2M + 3)P, M = -1, 0, 1, \cdots\). Therefore when the number of singular fibers is odd, the partition function of analytically continued \(SU(2)\) theory can be obtained from

\[ Z_{SU(2)}(M) = B_{\frac{\phi_F}{4}} \frac{2}{2\pi i} \sum_{t=0}^{H-1} \int dy e^{-\frac{K}{2H}y^2 - 2Ky} \frac{1}{2^{F-1}} \sum_{\epsilon_j = \pm 1} S \left( \sum_{j=1}^{F} \epsilon_j P_j, \frac{F - 3}{2} \right). (B.25) \]

By doing similar calculation as done in section 2, we have similar results as previous cases and the difference comes from coefficients of \(e^{-ny/P}\) in the integrand. For example, when \(F = 5\) and \(H = 1\), by integrating \(S(\alpha, 1)\) from (B.24) we have

\[ 4 \sum_{m=0}^{M} m(m+1)q^{(P\alpha-(2m+1)P)^2} - \frac{1}{2} \left( \frac{4}{P^2} \frac{\partial}{\partial q} + \frac{1}{P^2} (P\alpha)^2 - 1 \right) \tilde{\psi}_{P}^{(P\alpha-(2M+1)P)}(q) + \frac{P\alpha}{P^2} \tilde{\Phi}_{P}^{(P\alpha-(2M+1)P)}(q) \]

(B.26)

For general \(H\), denoting \(P \sum_{j=1}^{5} \frac{\epsilon_j}{P_j}\) with \(\epsilon = +1\) by \(R_s, \) \(\psi_{2p}^{(R_s-(2M+1)P)}\) in (B.24) is decomposed to

\[ \psi_{2p}^{(R_s-(2M+1)P)} = \sum_{h=0}^{\frac{H-1}{2}} \psi_{2HP}^{(2kP+R_s-(2M+1)P)}(n) - \sum_{h=0}^{\frac{H-1}{2}} \psi_{2HP}^{(2(h+1)P-R_s+(2M+1)P)} (B.27) \]

\(^{11}\)When \(R_s < 0\), we use the formula by considering, for example, \(\sinh R_s y = -\sinh(-R_s y).\)
and also similarly for $\varphi^{(R_s-(2M+1)P)}$. Then the WRT invariant can be written as

$$Z_{SU(2)}(M_3) = \frac{B}{32\pi} q^{-\phi/4} \left(\frac{2i}{R H} \right)^{1/2} \sum_{t=0}^{H-1} e^{2\pi i K t^2} \times 15 \sum_{s=0}^{M_s} \left[ \sum_{m=0}^{M_s} m(m+1)(e^{-2\pi i} \mathcal{R}_{R_s-(2m+1)P} t + e^{2\pi i} \mathcal{R}_{R_s-(2m+1)P} t) q^{\mathcal{R}_{R_s-(2m+1)P}^2} + \sum_{h=0}^{\left[\frac{H}{2}\right]-1} e^{2\pi i \mathcal{R}_{(2hP+R_s-(2M_s+1)P)}} \left( -\frac{1}{2} \left( \frac{4H}{P} \frac{\partial}{\partial q} + \frac{R_s^2}{P^2} - 1 \right) \tilde{\psi}^{(2hP+R_s-(2M_s+1)P)}(q) + \frac{R_s}{P^2} \tilde{\psi}^{(2hP+R_s-(2M_s+1)P)}(q) \right) + \sum_{h=0}^{\left[\frac{H}{2}\right]-1} e^{2\pi i \mathcal{R}_{(2h+1)P-R_s+(2M_s+1)P}} \left( \frac{1}{2} \left( \frac{4H}{P} \frac{\partial}{\partial q} + \frac{R_s^2}{P^2} - 1 \right) \tilde{\psi}^{(2h+1)P-R_s+(2M_s+1)P)}(q) + \frac{R_s}{P^2} \tilde{\psi}^{(2h+1)P-R_s+(2M_s+1)P)}(q) \right) \right] \bigg|_{q^e^{2\pi i}}$$

where $M_s = -1, 0, 1, \ldots$ is such that $(2M_3+1)P < R_s < (2M_3+3)P$. But when $F = 5$, the maximum value that $R_s$ can have is greater than $P$ but smaller than $2P$. So $M_{R_s}$ can be $-1$ or $0$, and terms in the second line of (B.28) vanishes. We provide an example for $H = 3$ in section B.1.

Even $F$

We perform similar calculation for even $F$. From (B.9) and (B.10), the recursion relation for $C(\alpha, m)$ is given by

$$C(\alpha, m) = \left( c_1(\alpha, m) j + c_2(\alpha, m) s_2(1-\alpha, m-1) + \frac{c_2(\alpha, m)}{c_2(\alpha, m-1)} s_1(1-\alpha, m-1) \right) C(\alpha, m-1) - \frac{c_2(\alpha, m)}{c_2(\alpha, m-1)} s_1(1-\alpha, m-1) c_1(\alpha, m-1) C(\alpha, m-2), \quad m \geq 3. \tag{B.29}$$

Also, for simplicity, we denote the coefficient of $C(\alpha, m-1)$ and $C(\alpha, m-2)$ by

$$V_1(m-1) := c_1(\alpha, m) + c_2(\alpha, m) s_2(1-\alpha, m-1) + \frac{c_2(\alpha, m)}{c_2(\alpha, m-1)} s_1(1-\alpha, m-1) \tag{B.30}$$

$$V_2(m-2) := -\frac{c_2(\alpha, m)}{c_2(\alpha, m-1)} s_1(1-\alpha, m-1) c_1(\alpha, m-1). \tag{B.31}$$

Then (B.29) is written as

$$C(\alpha, m) = V_1(m-1) C(\alpha, m-1) + V_2(m-2) C(\alpha, m-2), \quad m \geq 3. \tag{B.32}$$
So we have

\[
C(a, m) = \left( \sum_{s=0}^{m-3} \sum_{3 \leq j_1 < \ldots < j_s \leq m-2} V_1(a, m-1) \cdots V_{1,2}(a, m-(j_1-2))V_2(a, m-j_1)V_1(a, m-(j_1+1)) \right. \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 2) \\
+ \left( \sum_{s=1}^{m-2} \sum_{j_1=2}^{\lfloor \frac{m}{2} \rfloor} V_2(a, m-2) \cdots V_{1,2}(a, m-(j_1-2))V_2(a, m-j_1)V_1(a, m-(j_1+1)) \right. \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 2)C(a, 2) \\
+ \left( \sum_{s=1}^{m-3} \sum_{j_1=2}^{\lfloor \frac{m}{3} \rfloor} V_1(a, m-1) \cdots V_{1,2}(a, m-(j_1-2))V_2(a, m-j_1)V_1(a, m-(j_1+1)) \right. \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3)C(a, 2) \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3) \\
+ \left( \sum_{s=1}^{m-1} \sum_{j_1=2}^{\lfloor \frac{m}{4} \rfloor} V_2(a, m-2) \cdots V_{1,2}(a, m-(j_1-2))V_2(a, m-j_1)V_1(a, m-(j_1+1)) \right. \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3) \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3) \\
+ \sum_{s=1}^{m-4} \sum_{j_1=2}^{\lfloor \frac{m}{5} \rfloor} V_3(a, m-3) \cdots V_{1,2}(a, m-(j_1-2))V_2(a, m-j_1)V_1(a, m-(j_1+1)) \right. \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3) \\
+ \sum_{s=1}^{m-3} \sum_{j_1=2}^{\lfloor \frac{m}{6} \rfloor} V_2(a, m-2) \cdots V_{1,2}(a, m-(j_1-2))V_2(a, m-j_1)V_1(a, m-(j_1+1)) \right. \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3) \\
+ \sum_{s=1}^{m-2} \sum_{j_1=2}^{\lfloor \frac{m}{7} \rfloor} V_3(a, m-3) \cdots V_{1,2}(a, m-(j_1-2))V_2(a, m-j_1)V_1(a, m-(j_1+1)) \right. \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3) \\
\cdots V_{1,2}(a, m-(j_s-2))V_2(a, m-j_s)V_1(a, m-(j_s+1)) \cdots V_{1,2}(a, 3) \\
\left. \right)C(a, 1)
\]  

(B.33)

for \( m \geq 4 \). This is similar to the case of odd \( F \) where explanations on this sum and notations are available. We provide some examples,

\[
C(a, 3) = V_1(a, 2)C(a, 2) + V_2(a, 1)C(a, 1),
\]  

(B.34)

\[
C(a, 4) = (V_1(a, 3)V_1(a, 2) + V_2(a, 2))C(a, 2) + V_1(a, 3)V_2(a, 1)C(a, 1),
\]  

(B.35)

\[
C(a, 5) = (V_1(a, 4)V_1(a, 3)V_1(a, 2) + V_1(a, 4)V_2(a, 2) + V_2(a, 3)V_1(a, 2))C(a, 2)
\]  

+ \((V_1(a, 4)V_1(a, 3)V_2(a, 1) + V_2(a, 3)V_2(a, 1))C(a, 1),
\]  

(B.36)

\[
C(a, 6) = (V_1(a, 5)V_1(a, 4)V_1(a, 3)V_1(a, 2) + V_1(a, 5)V_1(a, 4)V_2(a, 2) + V_1(a, 5)V_2(a, 3)V_1(a, 2)
\]  

+ \(V_2(a, 4)V_1(a, 3)V_1(a, 2) + V_2(a, 4)V_2(a, 2))C(a, 2) + (V_1(a, 5)V_1(a, 4)V_1(a, 3)V_2(a, 1)
\]  

+ \(V_1(a, 5)V_2(a, 3)V_2(a, 1) + V_2(a, 4)V_1(a, 3)V_2(a, 1))C(a, 1),
\]  

(B.37)

\[
C(a, 7) = (V_1(a, 6)V_1(a, 5)V_1(a, 4)V_1(a, 3)V_1(a, 2) + V_1(a, 6)V_1(a, 5)V_1(a, 4)V_2(a, 2)
\]  

+ \(V_1(a, 6)V_1(a, 5)V_2(a, 3)V_1(a, 2) + V_1(a, 6)V_2(a, 4)V_1(a, 3)V_1(a, 2)
\]  

+ \(V_1(a, 6)V_2(a, 4)V_2(a, 2) + V_2(a, 5)V_1(a, 4)V_1(a, 3)V_1(a, 2)
\]  

+ \(V_2(a, 5)V_1(a, 4)V_2(a, 2) + V_2(a, 5)V_2(a, 3)V_1(a, 2))C(a, 2)
\]  

+ \((V_1(a, 6)V_1(a, 5)V_1(a, 4)V_1(a, 3)V_2(a, 1) + V_1(a, 6)V_1(a, 5)V_2(a, 3)V_2(a, 1)
\]  

+ \(V_1(a, 6)V_2(a, 4)V_1(a, 3)V_2(a, 1) + V_2(a, 5)V_1(a, 4)V_1(a, 3)V_2(a, 1)
\]  

+ \(V_2(a, 5)V_2(a, 3)V_2(a, 1))C(a, 1).
\]  

(B.38)
Also, $C(\alpha, m)$ with $m \geq 4$ can be calculated from $C(\alpha, 1)$ and $C(\alpha, 2)$. By using (B.21) and (B.22), they are given by

$$C(\alpha, 1) = \frac{\cosh \alpha y}{(\sinh y)^2} = \sum_{m=0}^{M} 2m (e^{(\alpha - 2m)y} + e^{-(\alpha - 2m)y}) + \frac{1}{P} \sum_{n=0}^{\infty} n \varphi_{2P}^{(P\alpha - 2MP)}(n)e^{-\frac{\alpha}{P}y} - \alpha \sum_{n=0}^{\infty} \psi_{2P}^{(P\alpha - 2MP)}(n) e^{-\frac{\alpha}{P}y},$$

(B.39)

$$C(\alpha, 2) = \frac{\cosh \alpha y}{(\sinh y)^2} = \frac{1}{6} \left( \frac{d^2}{dy^2} - (2 + \alpha)^2 \right) \frac{\cosh \alpha y}{(\sinh y)^2} + \frac{2\alpha \sinh \alpha y}{3} \left( \frac{\cosh \alpha y}{\sinh y} \right)^3$$

$$+ \frac{4}{3} \sum_{m=0}^{M} m(m - 1)(m + 1)(e^{(\alpha - 2m)y} + e^{-(\alpha - 2m)y})$$

$$- \alpha \sum_{n=0}^{\infty} \left( \frac{1}{2} \frac{n^2}{P^2} + \frac{1}{6} (\alpha^2 - 4) \right) \varphi_{2P}^{(P\alpha - 2MP)}(n) e^{-\frac{\alpha}{P}y} + \frac{1}{6P} \sum_{n=0}^{\infty} \left( 3\alpha^2 - 4 \right) \varphi_{2P}^{(P\alpha - 2MP)}(n) e^{-\frac{\alpha}{P}y},$$

(B.40)

for $2MP < P\alpha < 2(M + 1)P$, $M = 0, 1, \cdots$. Thus, when the number of singular fibers is even, the partition function of analytically continued $SU(2)$ theory is obtained from

$$Z_{SU(2)}(M_3) = B \frac{q^{-\phi/4}}{2\pi^4} \sum_{t=0}^{H-1} \int dy e^{-\frac{K}{2\pi} y^2 - 2Kty} \left( \frac{\alpha}{2} \sum_{\epsilon=\pm 1} \epsilon C \left( \sum_{j=1}^{F} \epsilon_j \frac{F - 2}{2} \right) \right).$$

(B.41)

Also, by doing similar calculation as before, we have similar results as previous cases but with different coefficient of $e^{-ny/P}$. For example, when $F = 6$ and $H = 1$, by integrating $C(\alpha, 2)$ from (B.40), we obtain

$$\frac{8}{3} \sum_{m=0}^{M} m(m - 1)(m + 1) q^{\frac{1}{4} P^{(P\alpha - 2MP)^2}}$$

$$- \frac{\alpha}{2} \left( \frac{4}{P^2} \frac{\partial}{\partial q} + \frac{1}{3} (\alpha^2 - 4) \right) \tilde{\Psi}_{P}^{(P\alpha - 2MP)}(q) + \frac{1}{6P} \left( \frac{4}{P} \frac{\partial}{\partial q} + (3\alpha^2 - 4) \right) \tilde{\Phi}_{P}^{(P\alpha - 2MP)}(q).$$

(B.42)
For general $H$, decomposing $\psi_{2P}^{(P\alpha-2MP)}(n)$ and $\varphi_{2P}^{(P\alpha-2MP)}(n)$ as before, from (B.41) we obtain

$$Z_{SU(2)}(M_3) = B \frac{q^{-\phi/4}}{K'H} \prod_{t=0}^{H-1} e^{2\pi iK_H t^2} \times \sum_{s=0}^{31} (-1)^{s} \left[ \frac{4}{3} \sum_{m=0}^{M_s} m(m-1)(m+1) \right]$$

$$\times \sum_{t=0}^{31} (-1)^{t} \left[ \frac{M_s}{3} \sum_{m=0}^{M_s} \left( e^{-2\pi iK_H (R_s-2mP) t} + e^{2\pi iK_H (R_s-2mP) t} \right) q^{4HHP(R_s-2mP)^2} \right]$$

$$\times \sum_{t=0}^{31} (-1)^{t} \left[ \frac{M_s}{3} \sum_{m=0}^{M_s} \left( e^{-2\pi iK_H (R_s-2mP) t} + e^{2\pi iK_H (R_s-2mP) t} \right) q^{4HHP(R_s-2mP)^2} \right]$$

$$\times \sum_{t=0}^{31} (-1)^{t} \left[ \frac{M_s}{3} \sum_{m=0}^{M_s} \left( e^{-2\pi iK_H (R_s-2mP) t} + e^{2\pi iK_H (R_s-2mP) t} \right) q^{4HHP(R_s-2mP)^2} \right]$$

$$\times \sum_{t=0}^{31} (-1)^{t} \left[ \frac{M_s}{3} \sum_{m=0}^{M_s} \left( e^{-2\pi iK_H (R_s-2mP) t} + e^{2\pi iK_H (R_s-2mP) t} \right) q^{4HHP(R_s-2mP)^2} \right]$$

where $R_s$ with even (odd) $s$ are $P \sum_{j=1}^{6} \epsilon_j$ with $\epsilon = +1 (-1)$ up to an overall sign. Also, $M_{R_s}$ is such that $2M_{R_s} < R_s < 2(M_{R_s}+1)P$, $M_s = 0, 1, \cdots$. When $F = 6$, the maximum value that $R_s$ can take is less than $2P$, so $M_{R_s}$ is zero. Therefore, the second line of (B.43) vanishes. We provide an example for $H = 3$ in section B.1.

As we saw above, given an $H$, the structure of the WRT invariant is same for arbitrary number of singular fibers. That is, given an $H$ the $S$-matrix is same but the homological blocks are different depending on the number of singular fibers, $P_j$’s and $Q_j$’s.

Higher genus

For the case of higher genus, the power of the denominator increases by $2g$. Thus, we can use the formula for $\sinh_{\alpha}y_{\alpha}^{2g+2}$ or $\cosh_{\alpha}y_{\alpha}^{2g+2}$ given an $F$ to calculate the homological blocks. Then we obtain the partition function of analytically continued theory or the WRT invariant with the same structure as before but with different homological blocks.
For example, when $F = 3$ and $g = 1$, the WRT invariant is given by

$$Z_{SU(2)}(M_3) = \frac{B}{32\pi i} (-2K)^{q^{-\phi/4}} \left( \frac{2\pi i}{K H} \right)^{1/2} \sum_{s=0}^{H-1} e^{2\pi i K s^2}$$

$$\times \sum_{h=0}^{3} \left[ \left( \sum_{s=0}^{H-1} e^{2\pi i \pi (2h+P-R_s-(2M_s+1)P)} \left( -\frac{1}{2} \left( \frac{4H}{P} \right)^2 \frac{\partial}{\partial q} + \frac{R_s^2}{P^2} + 1 \right) \tilde{\Psi}_{2HP}^{(2h+P-R_s-(2M_s+1)P)}(q) \right.ight.$$

$$\left. + \frac{R_s}{P^2} \tilde{\Psi}_{2HP}^{((2h+P-R_s-(2M_s+1)P)}(q) \right) \bigg] \right|_{q^{\epsilon} \sim \frac{2\pi i}{3}} \ (B.44)$$

where $R_s$'s are given by $P \sum_{j=1}^{3} 1/p_j$ with $\epsilon = 1$ and $M_s = -1, 0$ such that $(2M_s + 1)P < R_s < (2M_s + 3)P$. Also, since $M_s = -1, 0$, there is no additional term in the expression as in the case of (B.28). Therefore, also when $g \neq 0$, given an $H$ we have same S-matrix. The explicit expression of homological blocks for $H = 3$ with a choice $(P_1, P_2, P_3) = (2, 5, 7)$ is available in section B.1.

**B.1 Some Examples**

- **F = 5 and H = 3**

When the number of singular fibers is 5, $H = 3$ can be obtained, for example, by choosing $(P_1, P_2, P_3, P_4, P_5) = (2, 5, 7, 11, 13)$ and $(Q_1, Q_2, Q_3, Q_4, Q_5) = (-1, 4, -2, -1, 1)$. The WRT invariant can be written as

$$Z_{SU(2)}(M_3) = \frac{B}{i} q^{\phi_5/4} \left( \frac{10010 i}{K} \right)^{1/2} \frac{1}{2^4 \cdot 2 \cdot 7^2} \sum_{a,b=0}^{1} e^{2\pi i KCS_0 S_{ab} \tilde{Z}_b(q)} \bigg|_{q^{\epsilon} \sim \frac{2\pi i}{3}} \ (B.45)$$

where $(CS_0, CS_1) = (0, \frac{2}{3})$ and $S_{ab} = \frac{1}{\sqrt{3}} \left( \begin{array}{c} 1 \\ 1 \\ 2 \\ -1 \end{array} \right)$. The homological blocks, $\tilde{Z}_b(q)$, are given by

$\tilde{Z}_0 = -16594 \tilde{\Psi}_{30030}^{(1713)} + 12226 \tilde{\Psi}_{30030}^{(3897)} + 1434 \tilde{\Psi}_{30030}^{(4293)} + 5506 \tilde{\Psi}_{30030}^{(7257)} - 2866 \tilde{\Psi}_{30030}^{(8577)} - 214 \tilde{\Psi}_{30030}^{(9903)} + 6506 \tilde{\Psi}_{30030}^{(13263)} + 4146 \tilde{\Psi}_{30030}^{(15835)}$

$+ 10874 \tilde{\Psi}_{30030}^{(15447)} + 13514 \tilde{\Psi}_{30030}^{(16767)} + 20234 \tilde{\Psi}_{30030}^{(20127)} - 17154 \tilde{\Psi}_{30030}^{(21543)} - 14514 \tilde{\Psi}_{30030}^{(22773)} - 7794 \tilde{\Psi}_{30030}^{(26133)} - 3426 \tilde{\Psi}_{30030}^{(28317)} + 8586 \tilde{\Psi}_{30030}^{(34233)}$

$+ (31359891 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(1713)} + (62831331 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(3897)} + 3(-22505337 + 4004q\partial/\partial q) \tilde{\Psi}_{30030}^{(4293)}$

$+ (92621091 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(7257)} + (98146611 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(8577)} - 3(-33396217 + 4004q\partial/\partial q) \tilde{\Psi}_{30030}^{(9903)}$

$+ (89618901 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(11263)} + (79287771 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(14543)} + (70665213 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(15447)}$

$+ (54543051 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(16767)} + 3(717863 + 4004q\partial/\partial q) \tilde{\Psi}_{30030}^{(20127)} + (26635171 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(21453)}$

$+ (47536051 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(22773)} + (85013491 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(26133)} + (97265731 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(28317)}$

$+ (81770251 - 120120q\partial/\partial q) \tilde{\Psi}_{30030}^{(34233)}$.  

(B.46)
\[ \hat{Z}_1 = 20234 \Phi^{(107)}_{30030} - 17154 \Phi^{(143)}_{30030} - 14514 \Phi^{(275)}_{30030} + 13514 \Phi^{(425)}_{30030} + 10874 \Phi^{(4537)}_{30030} + 9146 \Phi^{(5437)}_{30030} + 8586 \Phi^{(5717)}_{30030} 
- 7794 \Phi^{(6113)}_{30030} + 650 \Phi^{(6757)}_{30030} - 3426 \Phi^{(8297)}_{30030} - 21 \Phi^{(10117)}_{30030} - 286 \Phi^{(11443)}_{30030} - 3426 \Phi^{(17263)}_{30030} - 5506 \Phi^{(17263)}_{30030} 
- 7794 \Phi^{(19913)}_{30030} + 8586 \Phi^{(21733)}_{30030} + 11434 \Phi^{(21733)}_{30030} - 12226 \Phi^{(21733)}_{30030} - 14514 \Phi^{(275)}_{30030} - 15594 \Phi^{(1807)}_{30030} - 17154 \Phi^{(1587)}_{30030} 
+ 2034 \Phi^{(33283)}_{30030} - 1659 \Phi^{(21733)}_{30030} - 12226 \Phi^{(23917)}_{30030} - 5506 \Phi^{(2757)}_{30030} - 2866 \Phi^{(2857)}_{30030} - 214 \Phi^{(29921)}_{30030} + 650 \Phi^{(33283)}_{30030} 
+ 9146 \Phi^{(34603)}_{30030} + 10874 \Phi^{(35467)}_{30030} + 11434 \Phi^{(35747)}_{30030} + 13514 \Phi^{(36787)}_{30030} 
- 3(717863 + 40040q\theta/\partial q)\Phi^{(107)}_{30030} + (26635171 - 120120q\theta/\partial q)\Phi^{(1433)}_{30030} + (47536051 - 120120q\theta/\partial q)\Phi^{(275)}_{30030} 
+ 3(-18181017 + 40040q\theta/\partial q)\Phi^{(3325)}_{30030} + 3(-25346377 + 40040q\theta/\partial q)\Phi^{(4573)}_{30030} + 3(-26429257 + 40040q\theta/\partial q)\Phi^{(5437)}_{30030} 
+ (-81770251 + 120120q\theta/\partial q)\Phi^{(5717)}_{30030} + (85013491 - 120120q\theta/\partial q)\Phi^{(6113)}_{30030} + 3(-29872697 + 40040q\theta/\partial q)\Phi^{(6757)}_{30030} 
+ (97265731 - 120120q\theta/\partial q)\Phi^{(8297)}_{30030} + 3(-33396217 + 40040q\theta/\partial q)\Phi^{(10117)}_{30030} + 3(-32715537 + 40040q\theta/\partial q)\Phi^{(11443)}_{30030} 
+ (-97265731 + 120120q\theta/\partial q)\Phi^{(11723)}_{30030} + 3(-30873697 + 40040q\theta/\partial q)\Phi^{(12763)}_{30030} + 3(-85013491 + 120120q\theta/\partial q)\Phi^{(13907)}_{30030} 
+ (81770251 - 120120q\theta/\partial q)\Phi^{(14303)}_{30030} + (67516011 - 120120q\theta/\partial q)\Phi^{(15727)}_{30030} + 3(-20943777 + 40040q\theta/\partial q)\Phi^{(16123)}_{30030} 
+ (-47536051 + 120120q\theta/\partial q)\Phi^{(17263)}_{30030} + 3(-10453297 + 40040q\theta/\partial q)\Phi^{(1807)}_{30030} + 3(-26635171 + 120120q\theta/\partial q)\Phi^{(18547)}_{30030} 
+ 3(717863 + 40040q\theta/\partial q)\Phi^{(19913)}_{30030} + (31359891 - 120120q\theta/\partial q)\Phi^{(21733)}_{30030} + 3(62831331 - 120120q\theta/\partial q)\Phi^{(23917)}_{30030} 
+ (92621091 - 120120q\theta/\partial q)\Phi^{(2757)}_{30030} + (98146611 - 120120q\theta/\partial q)\Phi^{(2857)}_{30030} - 3(-33396217 + 40040q\theta/\partial q)\Phi^{(29921)}_{30030} 
+ (89618091 - 120120q\theta/\partial q)\Phi^{(33283)}_{30030} + (79287771 - 120120q\theta/\partial q)\Phi^{(35467)}_{30030} + (70639131 - 120120q\theta/\partial q)\Phi^{(35467)}_{30030} 
+ (67516011 - 120120q\theta/\partial q)\Phi^{(35747)}_{30030} + (54543051 - 120120q\theta/\partial q)\Phi^{(36787)}_{30030} 
\]

where \( \hat{Z}_0 = q^{355597}Z[q] \) and \( \hat{Z}_1 = q^{1747201} \hat{Z}[q] \).

\( \bullet \) \( F = 6 \) and \( H = 3 \)

When \( F = 6 \), we can have \( H = 3 \), for example, by choosing \((P_1, P_2, P_3, P_4, P_5, P_6) = (2, 5, 7, 11, 13, 17)\) and \((Q_1, Q_2, Q_3, Q_4, Q_5) = (-1, 2, 4, -2, -3, -1)\). The WRT invariant can be written by

\[
Z_{SU(2)}(M_3) = \frac{B}{i} q^{-\phi_6/4} \left( \frac{170170i}{K} \right)^{1/2} \frac{1}{2^5 \cdot 6 \cdot 170170} \sum_{a,b=0}^1 e^{2\pi i KCS_a} S_{ab} \hat{Z}_b(q) \bigg| q^x e^{2\pi i} \]

(B.48)
where \((CS_0, CS_1) = (0, \frac{1}{2})\) and \(S_{ab} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\). The homological blocks, \(\tilde{Z}_b(q)\), are given by

\[
\begin{align*}
\tilde{Z}_b &= (115111539877 - 2042400q\partial_q/\partial q)\Phi_{11829} + (11282781677 - 2042400q\partial_q/\partial q)\Phi_{24371} + (10967749157 - 2042400q\partial_q/\partial q)\Phi_{45291} \\
&+ (106148864437 - 2042400q\partial_q/\partial q)\Phi_{56811} + (10296855011 - 2042400q\partial_q/\partial q)\Phi_{77731} + (9542649017 - 2042400q\partial_q/\partial q)\Phi_{92449} \\
&+ (82845685957 - 2042400q\partial_q/\partial q)\Phi_{104859} + (78135643797 - 2042400q\partial_q/\partial q)\Phi_{121059} + (76869049717 - 2042400q\partial_q/\partial q)\Phi_{11931} \\
&+ (63747851157 - 2042400q\partial_q/\partial q)\Phi_{118221} + (61719726277 - 2042400q\partial_q/\partial q)\Phi_{151059} + (42992633317 - 2042400q\partial_q/\partial q)\Phi_{117510} \\
&+ (62461442677 - 2042400q\partial_q/\partial q)\Phi_{208991} + (64317657037 + 2042400q\partial_q/\partial q)\Phi_{246429} + (939030781477 - 2042400q\partial_q/\partial q)\Phi_{267249} \\
&+ (111237610117 - 2042400q\partial_q/\partial q)\Phi_{301209} + (111770582557 + 2042400q\partial_q/\partial q)\Phi_{250519} + (115213641997 + 2042400q\partial_q/\partial q)\Phi_{272599} \\
&+ (10393275077 - 2042400q\partial_q/\partial q)\Phi_{403311} + (103834753437 + 2042400q\partial_q/\partial q)\Phi_{405651} + (92736060597 + 2042400q\partial_q/\partial q)\Phi_{428901} \\
&+ (-84350694437 + 2042400q\partial_q/\partial q)\Phi_{442779} + (-77280526657 + 2042400q\partial_q/\partial q)\Phi_{45699} + (-69047026177 + 2042400q\partial_q/\partial q)\Phi_{460219} \\
&+ (-60507210937 + 2042400q\partial_q/\partial q)\Phi_{476139} + (-16460407597 + 2042400q\partial_q/\partial q)\Phi_{522339} + (-106342840237 + 2042400q\partial_q/\partial q)\Phi_{524441} \\
&+ (-11475624637 + 2042400q\partial_q/\partial q)\Phi_{561869} + (-11563838157 + 2042400q\partial_q/\partial q)\Phi_{572489} + 35487(-115758537153 + 2042400q\partial_q/\partial q)\Phi_{11829} \\
&+ 34371(-346312581159 + 6126120q\partial_q/\partial q)\Phi_{34371} + 45291(-345442672119 + 6126120q\partial_q/\partial q)\Phi_{45291} \\
&+ 56811(-344266457079 + 6126120q\partial_q/\partial q)\Phi_{56811} + 67731(-34296458449 + 6126120q\partial_q/\partial q)\Phi_{67731} \\
&+ 82419(-34070155529 + 6126120q\partial_q/\partial q)\Phi_{82419} + 314577(-11216178973 + 2042400q\partial_q/\partial q)\Phi_{104859} \\
&+ 453177(-108225041773 + 2042400q\partial_q/\partial q)\Phi_{121059} + 485937(-10708583453 + 2042400q\partial_q/\partial q)\Phi_{1216979} \\
&+ 133379(-32970398159 + 6126120q\partial_q/\partial q)\Phi_{2151059} + 131039(-33032727279 + 6126120q\partial_q/\partial q)\Phi_{216979} \\
&+ 93919(-33867467879 + 6126120q\partial_q/\partial q)\Phi_{2513161} + 82991(-340606440719 + 6126120q\partial_q/\partial q)\Phi_{257349} \\
&- 8719(-339893768759 + 6126120q\partial_q/\partial q)\Phi_{257349} + 76259(-34167811719 + 6126120q\partial_q/\partial q)\Phi_{264641} \\
&- 39319(-345062711639 + 6126120q\partial_q/\partial q)\Phi_{301209} + 36791(-34614369191 + 6126120q\partial_q/\partial q)\Phi_{303549} \\
&+ 14351(-34728995599 + 6126120q\partial_q/\partial q)\Phi_{329899} + 62971(-343528599095 + 6126120q\partial_q/\partial q)\Phi_{403311} \\
&+ 65311(-343328420079 + 6126120q\partial_q/\partial q)\Phi_{405651} + 87751(-339793780799 + 6126120q\partial_q/\partial q)\Phi_{428901} \\
&- 102439(-33700198079 + 6126120q\partial_q/\partial q)\Phi_{428901} + 54479(-33464689319 + 6126120q\partial_q/\partial q)\Phi_{453699} \\
&+ 135799(-32902578393 + 6126120q\partial_q/\partial q)\Phi_{460219} + 181999(-34139870199 + 6126120q\partial_q/\partial q)\Phi_{510519} \\
&+ 25629(-34307669179 + 6126120q\partial_q/\partial q)\Phi_{587249} + 19111(-347128716479 + 6126120q\partial_q/\partial q)\Phi_{510519} + 8191(-347426854319 + 6126120q\partial_q/\partial q)\Phi_{510519}.
\end{align*}
\]
where $\hat{Z}_0 = q \frac{35507}{100000} Z[q]$ and $\hat{Z}_1 = \frac{1747201}{100000} Z[q]$.
• $F = 3$, $H = 3$, and $g = 1$

When $F = 3$ and $g = 1$, we can have $H = 3$, for example, from $(P_1, P_2, P_3) = (2, 5, 7)$ and $(Q_1, Q_2, Q_3) = (1, 2, -6)$. The WRT invariant is given by

$$Z_{SU(2)}(M_3) = \frac{B}{t}(-2K)q^{-\phi_3/4}\left(\frac{70t}{K}\right)^{1/2} \frac{1}{2^4} \cdot \frac{1}{2 \cdot 70^2} \sum_{a,b=0}^1 e^{\pi i Ks_a} S_{ab} \hat{Z}_b(q) \bigg|_{q^2}.$$  \hspace{1cm} (B.51)

where $(CS_0, CS_1) = (0, \frac{1}{3})$, and $S_{ab} = \frac{1}{\sqrt{3}} \left(\frac{1}{2} \hspace{1cm} -1\right)$. The homological blocks, $\hat{Z}_b(q)$, are given by

$$\hat{Z}_0 = -62\Phi_2(30)_{210} + 22\Phi_2(81)_{210} + 118\Phi_2(129)_{210} - 78\Phi_2(171)_{210}$$

$$+ (3939 - 840q\partial/\partial q)\Psi_{210}^{(39)} + (4779 - 840q\partial/\partial q)\Psi_{210}^{(81)}$$

$$+ (1419 - 840q\partial/\partial q)\Psi_{210}^{(129)} + (3379 - 840q\partial/\partial q)\Psi_{210}^{(171)},$$

$$\hat{Z}_1 = 118\Phi_2(11)_{210} - 78\Phi_2(31)_{210} + 22\Phi_2(59)_{210} - 62\Phi_2(101)_{210} - 78\Phi_2(109)_{210} - 62\Phi_2(179)_{210} + 22\Phi_2(221)_{210} + 118\Phi_2(269)_{210}$$

$$+ 3(-473 + 280q\partial/\partial q)\Psi_{210}^{(11)} + (3379 - 840q\partial/\partial q)\Psi_{210}^{(31)} + (-4779 + 840q\partial/\partial q)\Psi_{210}^{(59)}$$

$$+ (-3939 + 840q\partial/\partial q)\Psi_{210}^{(101)} + (-3379 + 840q\partial/\partial q)\Psi_{210}^{(109)} + (3939 - 840q\partial/\partial q)\Psi_{210}^{(179)}$$

$$+ (4779 - 840q\partial/\partial q)\Psi_{210}^{(221)} + (1419 - 840q\partial/\partial q)\Psi_{210}^{(269)}.$$  \hspace{1cm} (B.53)
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