THE INTERIOR $C^2$ ESTIMATE FOR PRESCRIBED GAUSS CURVATURE EQUATION IN DIMENSION TWO

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ABSTRACT. In this paper, we introduce a new auxiliary function, and establish the interior $C^2$ estimate for prescribed Gauss curvature equation in dimension two.

1. INTRODUCTION

Given a positive function $f(x) \in C^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$, to find a convex solution $u$, such that the Gauss curvature of the graph $(x, u(x))$ is $f(x)$, that is

$$\frac{\det \nabla^2 u}{(1 + |\nabla u|^2)^{\frac{n+2}{2}}} = f(x), \quad \text{in } \Omega.$$  

This is the classical prescribed Gauss curvature problem, and (1.1) is called prescribed Gauss curvature equation.

The a priori estimates are very important for fully nonlinear elliptic equations, especially the $C^2$ estimate. The interior $C^2$ estimate of $\sigma_2$ Hessian equation

$$\sigma_2(\nabla^2 u) = f(x), \quad \text{in } B_R(0) \subset \mathbb{R}^n$$  

in higher dimensions is a longstanding problem, where $\sigma_2(\nabla^2 u) = \sigma_2(\lambda(\nabla^2 u)) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \lambda_{i_2}$, $\lambda(\nabla^2 u) = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of $\nabla^2 u$, and $f > 0$. For $n = 2$, (1.2) is the Monge-Ampère equation, and Heinz [5] obtained the estimate by the convex hypersurface geometry method (see [2] for an elementary analytic proof). For Monge-Ampère equations with dimension $n \geq 3$, Pogorelov [7] constructed his

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famous counterexample, namely irregular solutions to Monge-Ampère equations. For \( n = 3 \) and \( f \equiv 1 \), \((1.2)\) can be reduced to a special Lagrangian equation after a Lewy transformation, and Warren-Yuan obtained the corresponding interior \( C^2 \) estimate in the celebrated paper [10]. Moreover, the problem is still open for general \( f \) with \( n \geq 4 \) and nonconstant \( f \) with \( n = 3 \). Also Urbas [9] generalized the counterexample for \( \sigma_k \) Hessian equations with \( k \geq 3 \).

Moreover, Pogorelov type estimates for the Monge-Ampère equations and \( \sigma_k \) Hessian equation \( (k \geq 2) \) were derived by Pogorelov [7] and Chou-Wang [3] respectively, and see [4] and [6] for some generalizations.

In this paper, we consider the convex solution of prescribed Gauss curvature equation in dimension \( n = 2 \) as follows

\[
(1.3) \quad \frac{\det \nabla^2 u}{(1 + |\nabla u|^2)^2} = f(x), \quad \text{in } B_R(0) \subset \mathbb{R}^2,
\]

and establish the interior \( C^2 \) estimate as follows

**Theorem 1.1.** Suppose \( u \in C^4(B_R(0)) \) be a convex solution of prescribed Gauss curvature equation \((1.3)\) in dimension \( n = 2 \), where \( 0 < m \leq f \leq M \) in \( B_R(0) \). Then

\[
(1.4) \quad |\nabla^2 u(0)| \leq C(m, M, R, \sup |\nabla f|, \sup |\nabla^2 f|, \sup |\nabla u|),
\]

where \( C \) is a positive constant depending only on \( m, M, R, \sup |\nabla f|, \sup |\nabla^2 f|, \) and \( \sup |\nabla u| \).

**Remark 1.2.** Heinz established an interior \( C^2 \) estimate for general Monge-Ampère equation with general conditions in [5], and the proof depends on the strict convexity of solutions and the geometry of convex hypersurface in dimension two. In this paper, our proof, which is based on a suitable choice of auxiliary functions, is elementary and avoids geometric computations on the graph of solutions. This technique is from [2].

**Remark 1.3.** By Trudinger’s gradient estimates of Hessian equations in [8] or a gradient estimate of convex function, that is

\[
\sup_{B_{\frac{R}{2}}(0)} |\nabla u| \leq \frac{2 \text{osc } u_{B_R(0)}}{R},
\]

we can bound \( |\nabla^2 u(0)| \) in terms of \( u \).
The rest of the paper is organized as follows. In Section 2, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. In Section 3, we introduce a new auxiliary function, and prove Theorem 1.1.

2. Derivatives of eigenvalues and eigenvectors

In this section, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. We think the following result is known for many people, for example see [1] for a similar result. For completeness, we give the result and a detailed proof.

**Proposition 2.1.** Let $W = \{W_{ij}\}$ is an $n \times n$ symmetric matrix and $\lambda(W) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the symmetric matrix $W$, and the corresponding continuous eigenvector field is $\tau^i = (\tau^i, 1, \cdots, \tau^i, n) \in \mathbb{S}^{n-1}$. Suppose that $W = \{W_{ij}\}$ is diagonal, $\lambda_i = W_{ii}$ and the corresponding eigenvector $\tau^i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{S}^{n-1}$ at the diagonal matrix $W$. If $\lambda_k$ is distinct with other eigenvalues, then we have at the diagonal matrix $W$

$$\frac{\partial \tau^k, k}{\partial W_{pq}} = 0, \forall p, q; \quad \frac{\partial \tau^k, i}{\partial W_{ik}} = \frac{1}{\lambda_k - \lambda_i}, \ i \neq k; \quad \frac{\partial \tau^k, i}{\partial W_{ip}} = 0, \text{ otherwise.}$$ (2.1)

$$\frac{\partial^2 \tau^k, k}{\partial W_{pk} \partial W_{pq}} = -\frac{1}{(\lambda_k - \lambda_p)^2}, \ p \neq k;$$ (2.2)

$$\frac{\partial^2 \tau^k, i}{\partial W_{ik} \partial W_{ii}} = \frac{1}{(\lambda_k - \lambda_i)^2}, \ i \neq k; \quad \frac{\partial^2 \tau^k, i}{\partial W_{ik} \partial W_{kk}} = -\frac{1}{(\lambda_k - \lambda_i)^2}, \ i \neq k;$$ (2.3)

$$\frac{\partial^2 \tau^k, i}{\partial W_{iq} \partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_q}, \ i \neq k, i \neq q, q \neq k;$$ (2.4)

$$\frac{\partial^2 \tau^k, i}{\partial W_{pq} \partial W_{rs}} = 0, \text{ otherwise.}$$ (2.5)

**Proof.** From the definition of eigenvalue and eigenvector of matrix $W$, we have

$$(W - \lambda_k I) \tau^k \equiv 0,$$

where $\tau^k$ is the eigenvector of $W$ corresponding to the eigenvalue $\lambda_k$. That is, for $i = 1, \cdots, n$, it holds

$$[W_{ii} - \lambda_k] \tau^k, i + \sum_{j \neq i} W_{ij} \tau^k, j = 0.$$ (2.6)
When \( W = \{W_{ij}\} \) is diagonal and \( \lambda_k \) is distinct with other eigenvalues, \( \lambda_k \) and \( \tau_k \) are \( C^2 \) at the matrix \( W \). In fact,

\[
(2.7) \quad \tau^k_{,k} = 1, \quad \tau^k_{,i} = 0, \quad i \neq k, \quad \text{at } W.
\]

Taking the first derivative of (2.6), we have

\[
\left[ \frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}} \right] \tau^k_{,i} + [W_{ii} - \lambda_k] \frac{\partial \tau^k_{,i}}{\partial W_{pq}} + \sum_{j \neq i} [\frac{\partial W_{ij}}{\partial W_{pq}} \tau^k_{,j} + W_{ij} \frac{\partial \tau^k_{,j}}{\partial W_{pq}}] = 0.
\]

Hence for \( i = k \), we get from (2.7)

\[
(2.8) \quad \frac{\partial \lambda_k}{\partial W_{pq}} = \frac{\partial W_{kk}}{\partial W_{pq}} = \left\{ \begin{array}{ll} 1, & p = k, q = k; \\ 0, & \text{otherwise}. \end{array} \right.
\]

And for \( i \neq k \),

\[
[W_{ii} - \lambda_k] \frac{\partial \tau^k_{,i}}{\partial W_{pq}} + \sum_{j \neq i} \frac{\partial W_{ij}}{\partial W_{pq}} \tau^k_{,j} = 0,
\]

then

\[
(2.9) \quad \frac{\partial \tau^k_{,i}}{\partial W_{pq}} = \frac{1}{\lambda_k - \lambda_i} \frac{\partial W_{ik}}{\partial W_{pq}} = \left\{ \begin{array}{ll} \frac{1}{\lambda_k - \lambda_i}, & p = i, q = k; \\ 0, & \text{otherwise}. \end{array} \right.
\]

Since \( \tau^k \in S^{n-1} \), we have

\[
(2.10) \quad 1 = |\tau^k|^2 = (\tau^k_{,1})^2 + \ldots + (\tau^k_{,n})^2.
\]

Taking the first derivative of (2.10), and using (2.7), it holds

\[
(2.11) \quad \frac{\partial \tau^k_{,k}}{\partial W_{pq}} = 0, \quad \forall (p, q).
\]

For \( i = k \), taking the second derivative of (2.6), and using (2.7), it holds

\[
\left[ \frac{\partial^2 W_{kk}}{\partial W_{pq} \partial W_{rs}} - \frac{\partial^2 \lambda_k}{\partial W_{pq} \partial W_{rs}} \right] \tau^k_{,k} + \sum_{j \neq k} \left[ \frac{\partial W_{kj}}{\partial W_{pq}} \frac{\partial \tau^k_{,j}}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}} \frac{\partial \tau^k_{,j}}{\partial W_{pq}} \right] = 0,
\]

hence

\[
(2.12) \quad \frac{\partial^2 \lambda_k}{\partial W_{pq} \partial W_{rs}} = \sum_{j \neq k} \left[ \frac{\partial W_{kj}}{\partial W_{pq}} \frac{\partial \tau^k_{,j}}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}} \frac{\partial \tau^k_{,j}}{\partial W_{pq}} \right] = \left\{ \begin{array}{ll} \frac{1}{\lambda_k - \lambda_q}, & p = k, q \neq k, r = q, s = k; \\ \frac{1}{\lambda_k - \lambda_s}, & r = k, s \neq k, p = s, q = k; \\ 0, & \text{otherwise}. \end{array} \right.
\]
For \( i \neq k \), it holds
\[
\left[ \frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}} \right] \frac{\partial \tau^k_{i,i}}{\partial W_{rs}} + \left[ \frac{\partial W_{ii}}{\partial W_{rs}} - \frac{\partial \lambda_i}{\partial W_{pq}} \right] \frac{\partial \tau^k_{i,i}}{\partial W_{pq}} + \left[ W_{ii} - \lambda_k \right] \frac{\partial^2 \tau^k_{i,i}}{\partial W_{pq} \partial W_{rs}}
\]
\[+ \sum_{j \neq i} \left( \frac{\partial W_{ij}}{\partial W_{pq}} \frac{\partial \tau^k_{j,i}}{\partial W_{rs}} + \frac{\partial W_{ij}}{\partial W_{rs}} \frac{\partial \tau^k_{j,i}}{\partial W_{pq}} \right) = 0,\]
then
\[
\frac{\partial^2 \tau^k_{i,i}}{\partial W_{ik} \partial W_{ii}} = \frac{1}{\lambda_k - \lambda_i}, \quad i \neq k;
\]
\[
\frac{\partial^2 \tau^k_{i,i}}{\partial W_{iq} \partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i}, \quad i \neq k, i \neq q, q \neq k;
\]
\[
\frac{\partial^2 \tau^k_{i,i}}{\partial W_{ik} \partial W_{kk}} = -\frac{1}{\lambda_k - \lambda_i}, \quad i \neq k;
\]
\[
\frac{\partial^2 \tau^k_{i,i}}{\partial W_{pq} \partial W_{rs}} = 0, \quad \text{otherwise.}
\]
From (2.10), we have
\[
2 \tau^k \frac{\partial^2 \tau^k_{i,i}}{\partial W_{pq} \partial W_{rs}} + 2 \sum_{i \neq k} \frac{\partial \tau^k_{i,i}}{\partial W_{pq}} \frac{\partial \tau^k_{i,i}}{\partial W_{rs}} = 0,
\]
then
\[
\frac{\partial^2 \tau^k_{i,i}}{\partial W_{pq} \partial W_{rs}} = -\sum_{i \neq k} \frac{\partial \tau^k_{i,i}}{\partial W_{pq}} \frac{\partial \tau^k_{i,i}}{\partial W_{rs}} = \begin{cases} \frac{1}{\lambda_k - \lambda_p}, & p \neq k, q = k, r = p, s = q; \\ 0, & \text{otherwise.} \end{cases}
\]
The proof of Proposition 2.1 is finished. \(\Box\)

Example 2.2. When \( n = 2 \), the matrix \( \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \) has two eigenvalues
\[
\lambda_1 = \frac{(u_{11} + u_{22}) + \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2},
\]
\[
\lambda_2 = \frac{(u_{11} + u_{22}) - \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2},
\]
with \( \lambda_1 \geq \lambda_2 \). If \( \lambda_1 > \lambda_2 \),
\[
\left[ \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,
\]
we can get
\[ \xi_1 = \left( u_{22} - u_{11} \right) - \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}; \]
\[ \xi_2 = -u_{21}. \]

Then the eigenvector \( \tau \) corresponding to \( \lambda_1 \) is
\[ \tau = -\frac{(\xi_1, \xi_2)}{\sqrt{\xi_1^2 + \xi_2^2}}. \]

We can verify Proposition 2.1.

3. Proof of Theorem 1.1

Now we start to prove Theorem 1.1.

Let \( \tau(x) = \tau(\nabla^2 u(x)) = (\tau_1, \tau_2) \in \mathbb{S}^1 \) be the continuous eigenvector field of \( \nabla^2 u(x) \) corresponding to the largest eigenvalue. Denote
\[ \Sigma = \{ x \in B_R(0) : r^2 - |x|^2 + \langle x, \tau(x) \rangle^2 > 0, r^2 - \langle x, \tau(x) \rangle^2 > 0 \}, \]
where \( r = \frac{1}{\sqrt{2}} R \). It is easy to know, \( \Sigma \) is an open set and \( B_r(0) \subset \Sigma \subset B_R(0) \). We introduce a new auxiliary function in \( \Sigma \) as follows
\[ \phi(x) = \eta(x)^{\beta / 2} g(1/2 |Du|^2)u_{\tau\tau}, \]
where \( \eta(x) = (r^2 - |x|^2 + \langle x, \tau(x) \rangle^2)(r^2 - \langle x, \tau(x) \rangle^2) \) with \( \beta = 4 \) and \( g(t) = e^{ct^2} \) with \( c_0 = \frac{32}{m} \). In fact, \( \langle x, \tau(x) \rangle \) is invariant under rotations of the coordinates, so is \( \eta(x) \).

From the definition of \( \Sigma \), we know \( \eta(x) > 0 \) in \( \Sigma \), and \( \eta = 0 \) on \( \partial \Sigma \). Assume the maximum of \( \phi(x) \) in \( \Sigma \) is attained at \( x_0 \in \Sigma \). By rotating the coordinates, we can assume \( \nabla^2 u(x_0) \) is diagonal. In the following, we denote \( \lambda_i = u_{ii}(x_0), \lambda = (\lambda_1, \lambda_2). \) Without loss of generality, we can assume \( \lambda_1 \geq \lambda_2 \). Then \( \tau(x_0) = (1, 0) \).

We will use notion \( h = O(f) \) if \( |h(x)| \leq Cf(x) \) for any \( x \in \Omega \) with a positive constant \( C \) depending only on \( m, M, R, \sup |\nabla f|, \sup |\nabla^2 f|, \) and \( \sup |\nabla u| \). Similarly we write \( h \geq O(f) \) if \( h(x) \geq -Cf(x) \) and \( h \leq O(f) \) if \( h(x) \leq Cf(x) \).

Now, we assume \( \eta \lambda_1 \) is big enough. Otherwise there is nothing to prove. Then we have from the equation (1.3),
\[ \lambda_2 = \frac{f(1 + |\nabla u|^2)^2}{\lambda_1} \leq \frac{M(1 + |\nabla u|^2)^2}{\lambda_1} < \lambda_1. \]
Hence $\lambda_1$ is distinct with the other eigenvalue, and $\tau(x)$ is $C^2$ at $x_0$. Moreover, the test function

\begin{equation}
\varphi = \beta \log \eta + \log \left( \frac{1}{2} |\nabla u|^2 \right) + \log u_{11}
\end{equation}

attains the local maximum at $x_0$. In the following, all the calculations are at $x_0$. Then, we can get

\[ 0 = \varphi_i = \beta \frac{\eta_i}{\eta} + \frac{g'}{g} \sum_k u_k u_{ki} + \frac{u_{11i}}{u_{11}}, \]

so we have

\begin{equation}
\frac{u_{11i}}{u_{11}} = -\beta \frac{\eta_i}{\eta} - \frac{g'}{g} u_{ii}, \quad i = 1, 2.
\end{equation}

At $x_0$, we also have

\[ 0 \geq \varphi_{ii} = \beta \left[ \frac{\eta_{ii}}{\eta} - \frac{\eta^2}{\eta^2} \right] + \frac{g''}{g^2} \sum_k u_k u_{ki} \sum_l u_l u_{li} \]
\[ + \frac{g'}{g} \sum_k (u_{ki} u_{ki} + u_{k} u_{kii}) + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}}{u_{11}} \]
\[ = \beta \left[ \frac{\eta_{ii}}{\eta} - \frac{\eta^2}{\eta^2} \right] + \frac{g'}{g} \left[ u_{ii}^2 \sum_k u_k u_{kii} \right] + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}, \]

since $g''g - g'^2 = 0$. Let

\[ F^{11} = \frac{\partial \det \nabla^2 u}{\partial u_{11}} = \lambda_2, \quad F^{22} = \frac{\partial \det \nabla^2 u}{\partial u_{22}} = \lambda_1, \]
\[ F^{12} = \frac{\partial \det \nabla^2 u}{\partial u_{12}} = 0, \quad F^{21} = \frac{\partial \det \nabla^2 u}{\partial u_{21}} = 0. \]

Then from the equation (1.3) we can get

\begin{equation}
\lambda_2 = \frac{f(1 + |\nabla u|^2)^2}{\lambda_1}.
\end{equation}

Differentiating (1.3) once, we can get

\begin{equation}
F^{11} u_{11i} + F^{22} u_{22i} = f_i (1 + |\nabla u|^2)^2 + f \cdot 2(1 + |\nabla u|^2) \cdot 2u_{i}, \quad u_{ii},
\end{equation}
then

\[ u_{22i} = \frac{1}{F_{22}} \left[ f_i(1 + |\nabla u|^2)^2 + f \cdot 2(1 + |\nabla u|^2) \cdot 2 u_i u_{ii} - F_{11}^1 u_{111} \right] \]
\[ = \frac{f_i(1 + |\nabla u|^2)^2}{\lambda_1} + f \cdot 4(1 + |\nabla u|^2) u_i \frac{u_{ii}}{\lambda_1} - \frac{f(1 + |\nabla u|^2)^2 u_{111}}{\lambda_1} u_{11} \]
\[ = - \frac{f(1 + |\nabla u|^2)^2 u_{111}}{\lambda_1} u_{11} + O(1). \]

(3.8)

Differentiating (1.3) twice, we can get

\[
F_{11}^1 u_{1111} + F_{22}^2 u_{2211} = \frac{\partial^2}{\partial x_1^2} [f(1 + |\nabla u|^2)^2] - 2 \frac{\partial^2 \det \nabla^2 u}{\partial u_{11} \partial u_{22}} u_{1111} u_{2211} - 2 \frac{\partial^2 \det \nabla^2 u}{\partial u_{12} \partial u_{21}} u_{1112}^2 \\
= f_{11}(1 + |\nabla u|^2)^2 + 2 f_1 \cdot 2(1 + |\nabla u|^2) \cdot 2 u_{1} u_{11} \\
+ f [2(2 u_{1} u_{111})^2 + 2(1 + |\nabla u|^2) \cdot (2 u_{1}^2 + 2 u_k u_{k11})] \\
- 2 u_{1111} u_{2211} + 2 u_{112}^2 \\
= f_{11}(1 + |\nabla u|^2)^2 + 8 f_1 \cdot (1 + |\nabla u|^2) u_{1} u_{11} + f [8 u_{1}^2 + 4(1 + |\nabla u|^2)] u_{11}^2 \\
+ 4 f(1 + |\nabla u|^2)[u_{1} u_{111} + u_{2} u_{211}] + 2 u_{112}^2 \\
- 2 u_{111}[f_{1}(1 + |\nabla u|^2)^2] + f \cdot 4(1 + |\nabla u|^2) u_{11} - \frac{f(1 + |\nabla u|^2)^2 u_{111}}{\lambda_1} u_{11} \\
= f_{11}(1 + |\nabla u|^2)^2 + 8 f_1 \cdot (1 + |\nabla u|^2) u_{1} u_{11} + f [8 u_{1}^2 + 4(1 + |\nabla u|^2)] u_{11}^2 \\
+ 4 f(1 + |\nabla u|^2)[-u_{1} u_{111} + u_{2} u_{211}] - 2 f_{1}(1 + |\nabla u|^2)^2 \frac{u_{111}}{u_{11}} \\
+ 2 u_{112}^2 + 2 f(1 + |\nabla u|^2)^2 (\frac{u_{111}}{u_{11}})^2 \\
= f [8 u_{1}^2 + 4(1 + |\nabla u|^2)] u_{11}^2 + O(\lambda_1) \\
+ 4 f(1 + |\nabla u|^2)[-u_{1} u_{111} + u_{2} u_{112}] - 2 f_{1}(1 + |\nabla u|^2)^2 \frac{u_{111}}{u_{11}} \\
+ 2 u_{112}^2 + 2 f(1 + |\nabla u|^2)^2 (\frac{u_{111}}{u_{11}})^2, \]

(3.9)
and

\[ F^{11}u_{1112} + F^{22}u_{2212} = \frac{\partial^2}{\partial x_1 \partial x_2} \left[ f(1 + |\nabla u|^2)^2 \right] \]

\[ - \frac{\partial^2 \det \nabla^2 u}{\partial u_{11} \partial u_{22}} u_{11} u_{222} - \frac{\partial^2 \det \nabla^2 u}{\partial u_{22} \partial u_{11}} u_{221} u_{112} - 2 \frac{\partial^2 \det \nabla^2 u}{\partial u_{12} \partial u_{21}} u_{121} u_{212} \]

\[ = f_{12} (1 + |\nabla u|^2)^2 + f_1 \cdot 2 (1 + |\nabla u|^2) \cdot 2u_{2}u_{22} + f_2 \cdot 2 (1 + |\nabla u|^2) \cdot 2u_{1}u_{11} \]

\[ + f_2 (2u_{1}u_{11})(2u_{2}u_{22}) + 2 (1 + |\nabla u|^2) \cdot 2u_{k}u_{k12} \]

\[ - u_{11} u_{222} - u_{112} u_{221} + 2u_{112} u_{221} \]

\[ = f_{12} (1 + |\nabla u|^2)^2 + 4 (1 + |\nabla u|^2) [f_1 \cdot u_{2}u_{22} + f_2 \cdot u_{1}u_{11}] \]

\[ + 8 f^2 u_{1}u_{2} (1 + |\nabla u|^2)^2 + 4 f (1 + |\nabla u|^2) u_{1}u_{112} \]

\[ + 4 f (1 + |\nabla u|^2) u_{2} \left[ - \frac{f (1 + |\nabla u|^2)^2 u_{111}}{\lambda_1} \right] + O(1) \]

\[ - u_{11} [f_2 (1 + |\nabla u|^2)^2 + f \cdot 4 (1 + |\nabla u|^2) u_{2} \frac{u_{22}}{\lambda_1} - \frac{f (1 + |\nabla u|^2)^2 u_{112}}{u_{11}}] \]

\[ + u_{112} [f_1 (1 + |\nabla u|^2)^2 + f \cdot 4 (1 + |\nabla u|^2) u_{1} \frac{u_{11}}{\lambda_1} - \frac{f (1 + |\nabla u|^2)^2 u_{111}}{u_{11}}] \]

\[ = - \frac{u_{111}^2 [f_2 (1 + |\nabla u|^2)^2 + f \cdot 8 (1 + |\nabla u|^2) u_{2}u_{22}] + u_{112} [f_1 (1 + |\nabla u|^2)^2 + f \cdot 8 (1 + |\nabla u|^2) u_{1}u_{11}] + O(\lambda_1)] \]
Hence we can get by (3.7) and (3.9),

\[
0 \geq \sum_{i=1}^{2} F_{ii} \varphi_{ii} \\
= \beta \sum_{i} F_{ii} \left[ \frac{\eta_{ii}}{\eta} - \frac{\eta_{ii}^2}{\eta^2} \right] + \frac{g'}{g} \sum_{i} F_{ii} u_{ii}^2 + \frac{g'}{g} \sum_{k} u_{kk} \sum_{i} F_{ii} u_{iik} \\
+ \frac{1}{u_{11}} \sum_{i} F_{ii} u_{11i} - \sum_{i} F_{ii} \left[ \frac{u_{11i}^2}{u_{11}} \right]^2 \\
= \beta \lambda_2 \left[ \frac{\eta_{11}}{\eta} - \frac{\eta_{11}^2}{\eta^2} \right] + \beta \lambda_1 \left[ \frac{\eta_{22}}{\eta} - \frac{\eta_{22}^2}{\eta^2} \right] + \frac{g'}{g} \left[ (u_1 f_1 + u_2 f_2) (1 + |\nabla u|^2)^2 + 4 f \cdot (1 + |\nabla u|^2) (u_1^2 u_{11} + u_2^2 u_{22}) \right] \\
+ \frac{1}{u_{11}} \left\{ f [8 u_1^2 + 4 (1 + |\nabla u|^2)] u_{11}^2 + O(\lambda_1) \right\} \\
+ \frac{1}{u_{11}} \left[ 4 f (1 + |\nabla u|^2) [-u_1 u_{111} + u_2 u_{211}] - 2 f_1 (1 + |\nabla u|^2) \frac{u_{111}}{u_{11}} \right] \\
+ 2 u_{112} + 2 f (1 + |\nabla u|^2) \left( \frac{u_{111}}{u_{11}} \right)^2 \right\} \\
- \lambda_2 \left[ \frac{u_{111}^2}{u_{11}} \right]^2 - \lambda_1 \left[ \frac{u_{112}^2}{u_{11}} \right]^2 \\
\geq \beta \left[ \frac{\eta_{11}}{\eta} + \lambda_1 \frac{\eta_{22}}{\eta} - \beta \lambda_2 \frac{\eta_{11}^2}{\eta^2} + \frac{g'}{g} (1 + |\nabla u|^2)^2 \lambda_1 \right] \\
+ \frac{1}{2} \frac{u_{111}^2}{u_{11}} - \left[ 4 f (1 + |\nabla u|^2) u_1 + 2 f_1 (1 + |\nabla u|^2)^2 \frac{u_{111}}{u_{11}} \right] \\
+ f [8 u_1^2 + 4 (1 + |\nabla u|^2)] u_{11} + \frac{1}{2} \frac{u_{112}^2}{u_{11}} + \frac{\lambda_1}{2} \left[ \frac{\eta_{22}}{\eta} + \frac{g'}{g} u_2 u_{22} \right]^2 \\
+ 4 f (1 + |\nabla u|^2) u_2 \frac{u_{112}}{u_{11}} + O(1) \\
\geq \beta \left[ \frac{\eta_{11}}{\eta} + \lambda_1 \frac{\eta_{22}}{\eta} - \beta \lambda_2 \frac{\eta_{11}^2}{\eta^2} + \frac{g'}{g} (1 + |\nabla u|^2)^2 \lambda_1 \right] \\
+ \frac{1}{2} \frac{u_{111}^2}{u_{11}} + \frac{\lambda_1}{2} \left[ \frac{u_{112}^2}{u_{11}} + \frac{\lambda_1}{2} \frac{u_{111}^2}{u_{11}} \right] + \frac{g'}{g} (1 + |\nabla u|^2)^2 \lambda_1 \eta_{22}^2 \\
+ \frac{1}{2} \frac{u_{112}^2}{u_{11}} + \frac{\lambda_1}{2} \left[ \frac{u_{111}^2}{u_{11}} + \frac{\lambda_1}{2} \frac{u_{112}^2}{u_{11}} \right] + \frac{g'}{g} (1 + |\nabla u|^2)^2 \lambda_1 \eta_{22}^2 \\
+ \frac{g'}{g} (1 + |\nabla u|^2)^2 \left( \frac{u_{111}}{u_{11}} \right)^2 \right\} \\
+ O(1),
\]
where we used the following inequalities
\[
\lambda_2 \frac{u_{111}^2}{u_{11}} - [4f(1 + |\nabla u|^2)u_1 + 2f_1(1 + |\nabla u|^2)^2]u_{111} \frac{u_{11}}{u_{11}} = f[8u_1^2 + 4(1 + |\nabla u|^2)]u_{11}
\]
\[
= \frac{\lambda_2}{2} \frac{u_{111}^2}{u_{11}} + \lambda_2 \left \{ \frac{u_{111}}{u_{11}} - \frac{1}{\lambda_2} [4f(1 + |\nabla u|^2)u_1 + 2f_1(1 + |\nabla u|^2)^2] \right \}^2
\]
\[
= \frac{1}{2\lambda_2} [4f(1 + |\nabla u|^2)u_1 + 2f_1(1 + |\nabla u|^2)^2] + f[8u_1^2 + 4(1 + |\nabla u|^2)]u_{11}
\]
\[
\Rightarrow \lambda_2 \frac{u_{111}^2}{u_{11}} - \frac{\lambda_1}{2} \frac{u_{111}}{u_{11}} + 2f_1(1 + |\nabla u|^2)^2
\]
\[
\Rightarrow \frac{\lambda_2}{2} \frac{u_{111}^2}{u_{11}} - 8f_1(1 + |\nabla u|^2)u_1 - 2f_1^2(1 + |\nabla u|^2)^2
\]
\[
\Rightarrow \frac{\lambda_2}{2} \frac{u_{111}^2}{u_{11}} - 8f_1(1 + |\nabla u|^2)u_1 - 2f_1^2(1 + |\nabla u|^2)^2.
\]

and
\[
4f(1 + |\nabla u|^2)u_2 \frac{u_{112}}{u_{11}} = 4f(1 + |\nabla u|^2)u_2 \left [ -\beta \frac{\eta_2}{\eta} \frac{g'}{g} u_2 + O(1) \right ]
\]

Now we have the following lemma.

Lemma 3.1. If \( \eta \lambda_1 \) is big enough, we have at \( x_0 \)

\[
\beta \lambda_2 \frac{\eta_2^2}{\eta^2} \leq \frac{\lambda_1}{4} \left ( \frac{u_{112}}{u_{11}} \right )^2 + O \left ( \frac{1}{\eta} \right );
\]
(3.12)

\[
\beta f(1 + |\nabla u|^2)[(1 + |\nabla u|^2) \frac{g'}{g} - 4] \frac{\eta_2}{\eta} u_2 = O \left ( \frac{1}{\eta} \right );
\]
(3.13)

and

\[
\beta \left [ \lambda_2 \frac{\eta_{11}}{\eta} + \frac{\eta_{22}}{\eta} \right ] \geq -\frac{1}{2} \frac{g'}{g} \lambda_1 - \beta \lambda \left [ \frac{\eta_2}{\eta} \right ]^2 - \frac{\lambda_2}{2} \left ( \frac{u_{111}}{u_{11}} \right )^2 - \frac{\lambda_1}{4} \left ( \frac{u_{112}}{u_{11}} \right )^2 + O \left ( \frac{1}{\eta} \right ).
\]
(3.14)

Proof. At \( x_0, \tau = (\tau_1, \tau_2) = (1, 0) \). Then from Proposition 2.1 we get

\[
\langle x, \partial \tau \rangle = \sum_{m=1}^{2} x_m \frac{\partial \tau_m}{\partial x} = \sum_{m=1}^{2} x_m \frac{\partial \tau_m}{\partial u_{pq}} u_{pq} = \sum_{m=1}^{2} x_m \frac{\partial \tau_m}{\partial u_{pq}} u_{pq}
\]
\[
= x_2 \frac{u_{12i}}{\lambda_1 - \lambda_2}, \quad i = 1, 2.
\]
From the definition of $\eta$, then we have at $x_0$
\begin{equation}
\eta = [r^2 - |x|^2 + \langle x, \tau \rangle^2][r^2 - \langle x, \tau \rangle^2] = (r^2 - x_2^2)(r^2 - x_1^2).
\end{equation}

Taking the first derivative of $\eta$, we can get
\begin{align*}
\eta_i &= \left\{ \begin{array}{l}
-2x_1(r^2 - x_3^2) + 2x_1 \langle x, \partial_i \tau \rangle (x_3^2 - x_3^2), \quad i = 1; \\
-2x_2(r^2 - x_1^2) + 2x_1 \langle x, \partial_2 \tau \rangle (x_3^2 - x_3^2), \quad i = 2.
\end{array} \right.
\end{align*}

Hence if $\eta \lambda_1$ is big enough, we can get
\begin{align*}
\frac{\beta \lambda_2 \eta_1^2}{\eta'} &= \beta \lambda_2 \left[ -\frac{2x_1(r^2 - x_3^2) + 2x_1x_2}{\eta} \frac{x_3^2 - x_3^2}{\lambda_1 - \lambda_2} u_{112} \right]^2 \\
&\leq \beta \lambda_2 \left[ \frac{8r^6}{\eta^2} + \frac{8r^8}{\eta^2} \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \right] \\
&\leq \frac{\lambda_1}{4} \left( \frac{u_{112}}{u_{11}} \right)^2 + O\left( \frac{1}{\eta} \right).
\end{align*}

Also we have
\begin{align*}
\frac{\eta_2}{\eta} &= \frac{-2x_2(r^2 - x_1^2) + 2x_1x_2}{\eta} \frac{x_3^2 - x_3^2}{\lambda_1 - \lambda_2} u_{221} \\
&= \frac{-2x_2(r^2 - x_1^2)}{\eta} + 2x_1x_2 \frac{x_3^2 - x_3^2}{\lambda_1 - \lambda_2} \frac{1}{u_{111} + O(1)} \\
&= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left[ 1 - x_1 \frac{(x_3^2 - x_3^2)(r^2 - x_3^2)}{\lambda_1 - \lambda_2} \cdot O(1) \right] \\
&\quad + \frac{2x_1x_2}{\eta} \frac{x_3^2 - x_3^2}{\lambda_1 - \lambda_2} \frac{1}{u_{111} + O(1)} \\
&= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left[ 1 + O\left( \frac{1}{\eta \lambda_1} \right) - x_1 \frac{(x_3^2 - x_3^2)(r^2 - x_3^2)}{\lambda_1 - \lambda_2} \right] \frac{1}{u_{111} + O(1)} \\
&\quad + \frac{2x_1x_2}{\eta} \frac{x_3^2 - x_3^2}{\lambda_1 - \lambda_2} \frac{1}{u_{111} + O(1)} \\
&= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left[ 1 + O\left( \frac{1}{\eta \lambda_1} \right) \right] \\
&\quad + \frac{2x_1x_2}{\eta} \frac{x_3^2 - x_3^2}{\lambda_1 - \lambda_2} \frac{1}{u_{111} + O(1)} \\
&= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left[ 1 + O\left( \frac{1}{\eta \lambda_1} \right) \right] \\
&\quad + \frac{[2x_1x_2 - x_3^2]^2}{\eta} \frac{f(1 + |\nabla u|^2)^2}{g} \frac{1}{u_{112}}.
\end{align*}
then we can get

\[
[1 + \beta^2 \frac{x_2^2 - x_1^2}{x_2^2 - x_1^2} \frac{f(1 + |\nabla u|^2)}{(\lambda_1 - \lambda_2)^2} \frac{\eta_2}{\eta} \frac{\eta}{\eta}]
\]

\[
= -\frac{2x_2(r^2 - x_1^2)}{\eta} [1 + O\left(\frac{1}{\eta \lambda_1}\right)] - \frac{2x_1x_2}{\eta} \frac{x_2^2 - x_1^2}{\eta} \frac{f(1 + |\nabla u|^2)^2}{(\lambda_1 - \lambda_2)^2} \frac{\beta g'}{g} u_2 u_{22}
\]

\[
= -\frac{2x_2(r^2 - x_1^2)}{\eta} [1 + O\left(\frac{1}{\eta \lambda_1}\right)] + 2x_1x_2 \frac{(x_2^2 - x_1^2)}{\eta^2} \frac{(r^2 - x_1^2)}{\eta} \frac{f(1 + |\nabla u|^2)^2}{(\lambda_1 - \lambda_2)^2} \frac{\beta g'}{g} u_2 u_{22}
\]

(3.18)

which implies

\[
\frac{\eta_2}{\eta} = -\frac{2x_2(r^2 - x_1^2)}{\eta} [1 + O\left(\frac{1}{\eta \lambda_1}\right)].
\]

That is \(\frac{\eta_2}{\eta} \approx -\frac{2x_2(r^2 - x_1^2)}{\eta}\) if \(\eta \lambda_1\) is big enough. Hence

\[
\beta f(1 + |\nabla u|^2)[(1 + |\nabla u|^2)^2 \frac{g'}{g} u_2 - 4] \frac{\eta_2}{\eta} u_2 = O\left(\frac{1}{\eta}\right).
\]

Taking second derivatives of \(\eta\), we can get

\[
\eta_{ii} = [-2 + 2 \langle x, \tau \rangle \langle x, \tau \rangle_{ii} + 2 \langle x, \tau \rangle \langle x, \tau \rangle_{i} \langle x, \tau \rangle_{i}] [r^2 - \langle x, \tau \rangle^2]
\]

\[
+ 2 [-2x_i + 2 \langle x, \tau \rangle \langle x, \tau \rangle] [-2 \langle x, \tau \rangle \langle x, \tau \rangle_{i}]
\]

\[
+ \langle r^2 - |x|^2 \rangle \langle x, \tau \rangle \langle x, \tau \rangle_{ii} - 2 \langle x, \tau \rangle \langle x, \tau \rangle_{i}]
\]

\[
= [-2 + 2x_1 \langle x, \tau \rangle_{ii} + 2(\delta_{i1} + \langle x, \partial_1 \tau \rangle)^2 [r^2 - x_1^2]
\]

\[
+ 2 [-2x_i + 2x_1(\delta_{i1} + \langle x, \partial_1 \tau \rangle)] [-2x_1(\delta_{i1} + \langle x, \partial_1 \tau \rangle)]
\]

\[
+ (r^2 - x_1^2) [-2x_1 \langle x, \tau \rangle_{ii} - 2(\delta_{i1} + \langle x, \partial_1 \tau \rangle)^2].
\]

so

\[
\eta_{11} = -2(r^2 - x_1^2) - 2x_1(x_1^2 - x_2^2) \langle x, \tau \rangle_{11}
\]

\[
+ (4x_2^2 - 12x_2^2) \langle x, \partial_2 \tau \rangle + (2x_2^2 - 10x_1^2) \langle x, \partial_1 \tau \rangle^2,
\]

(3.21)

\[
\eta_{22} = -2(r^2 - x_1^2) - 2x_1(x_1^2 - x_2^2) \langle x, \tau \rangle_{22}
\]

\[
+ 8x_1x_2 \langle x, \partial_2 \tau \rangle + (2x_2^2 - 10x_1^2) \langle x, \partial_2 \tau \rangle^2.
\]

(3.22)
Hence

\[
\beta \left[ \frac{\lambda_2 \eta_1}{\eta} + \lambda_1 \frac{\eta_2}{\eta} \right] = -2\beta \left[ \frac{\lambda_2 r^2 - x_2^2}{\eta} + \lambda_1 \frac{r^2 - x_1^2}{\eta} \right] \\
- 2\beta \frac{x_1(x_1^2 - x_2^2)}{\eta} [\lambda_2 \langle x, \tau \rangle_{11} + \lambda_1 \langle x, \tau \rangle_{22}] \\
+ \beta \frac{x_2(4x_2^2 - 12x_1^2)}{\eta} \frac{u_{112}}{\lambda_1 - \lambda_2} + \frac{x_2^3(2x_2^2 - 10x_1^2)}{\eta} \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \\
+ \beta \lambda_1 \frac{8x_1 x_2^2}{\eta} u_{221} + \frac{x_2^3(2x_2^2 - 10x_1^2)}{\eta} \left( \frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 .
\]

(3.23)

Direct calculations yield

\[
\langle x, \tau \rangle_{11} = \frac{\partial^2}{\partial x_1^2} \left[ \sum_{m=1}^{2} x_m \tau_m \right] = \frac{2}{\partial x_1} \left[ \sum_{m=1}^{2} x_m \frac{\partial \tau_m}{\partial x_1} \right] \\
= 2 \frac{\partial \tau_1}{\partial u_{pq}} u_{pq1} + \sum_{m=1}^{2} x_m \left[ \frac{\partial \tau_m}{\partial u_{pq}} u_{pq1} + \frac{\partial^2 \tau_m}{\partial u_{pq} \partial u_{rs}} u_{pq1} u_{rs1} \right] \\
= 0 + x_1 \frac{\partial^2 \tau_1}{\partial u_{pq} \partial u_{rs}} u_{pq1} u_{rs1} + x_2 \left[ \frac{\partial \tau_2}{\partial u_{pq}} u_{pq1} + \frac{\partial^2 \tau_2}{\partial u_{pq} \partial u_{rs}} u_{pq1} u_{rs1} \right] \\
= - x_1 \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 + x_2 \left( \frac{1}{\lambda_1 - \lambda_2} \right) u_{1211} + 2x_2 \left[ - \frac{u_{112} u_{111}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{112} u_{221}}{(\lambda_1 - \lambda_2)^2} \right] ,
\]

Similarly, we have

\[
\langle x, \tau \rangle_{22} = \frac{\partial^2}{\partial x_2^2} \left[ \sum_{m=1}^{2} x_m \tau_m \right] = \frac{2}{\partial x_2} \left[ \sum_{m=1}^{2} x_m \frac{\partial \tau_m}{\partial x_2} \right] \\
= 2 \frac{\partial \tau_2}{\partial u_{pq}} u_{pq2} + \sum_{m=1}^{2} x_m \left[ \frac{\partial \tau_m}{\partial u_{pq}} u_{pq2} + \frac{\partial^2 \tau_m}{\partial u_{pq} \partial u_{rs}} u_{pq2} u_{rs2} \right] \\
= 2 \frac{1}{\lambda_1 - \lambda_2} u_{221} - x_1 \frac{u_{221}}{\lambda_1 - \lambda_2} \left( \frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 \\
+ x_2 \frac{1}{\lambda_1 - \lambda_2} u_{1222} + 2x_2 \left[ - \frac{u_{112} u_{221}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{222} u_{221}}{(\lambda_1 - \lambda_2)^2} \right] ,
\]
From (3.23) and (3.24), we can get

\[
\begin{align*}
\lambda_2 (x, \tau)_{11} + \lambda_1 (x, \tau)_{22} &= - x_1 \lambda_2 \left[ \frac{u_{112}}{\lambda_1 - \lambda_2} \right]^2 + 2 \lambda_1 \left[ \frac{u_{221}}{\lambda_1 - \lambda_2} \right] - x_1 \lambda_1 \left[ \frac{u_{221}}{\lambda_1 - \lambda_2} \right]^2 \\
+ x_2 \left[ \frac{1}{\lambda_1 - \lambda_2} \right] \left( - \frac{u_{111}}{u_{11}} [f_2 (1 + |\nabla u|^2)^2 + f \cdot 8(1 + |D u|^2) u_{22}] \\
+ \frac{u_{112}}{u_{11}} [f_1 (1 + |\nabla u|^2)^2 + f \cdot 8(1 + |\nabla u|^2) u_{111}] + O(\lambda_1) \right) \\
- 2 x_2 \frac{u_{112}}{\lambda_1 - \lambda_2}^2 [f_1 (1 + |\nabla u|^2)^2 + f \cdot 2(1 + |\nabla u|^2) \cdot 2 u_{11}] \\
+ 2 x_2 \frac{u_{221}}{\lambda_1 - \lambda_2}^2 [f_2 (1 + |\nabla u|^2)^2 + f \cdot 2(1 + |\nabla u|^2) \cdot 2 u_{22}] \\
= \left[ \frac{u_{112}}{\lambda_1 - \lambda_2} \right]^2 \cdot O(\frac{1}{\lambda_1}) + \left[ \frac{u_{221}}{\lambda_1 - \lambda_2} \right]^2 \cdot O(\frac{1}{\lambda_1}) + O(1) \\
(3.24) + \left[ \frac{u_{221}}{\lambda_1 - \lambda_2} \right]^2 \cdot O(\lambda_1) + \left[ \frac{u_{112}}{\lambda_1 - \lambda_2} \right] \cdot O(\lambda_1) + \frac{u_{111}}{u_{11}} \cdot O(\frac{1}{\lambda_1}).
\end{align*}
\]

From (3.23) and (3.24), we can get

\[
\begin{align*}
\beta [\lambda_2 \frac{\eta_{11}}{\eta} + \lambda_1 \frac{\eta_{22}}{\eta}] &= O(\frac{1}{\eta \lambda_1}) - 2 \beta \lambda_1 \frac{r^2 - x_1^2}{\eta} + O(\frac{1}{\eta}) \\
+ \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \cdot O(\frac{1}{\eta \lambda_1}) + \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right) \cdot O(\frac{1}{\eta \lambda_1}) + \frac{u_{111}}{u_{11}} \cdot O(\frac{1}{\eta \lambda_1}) \\
+ \left( \frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 \cdot O(\frac{1}{\eta \lambda_1}) + \frac{u_{221}}{\lambda_1 - \lambda_2} \cdot O(\frac{1}{\eta}) \\
= - 2 \beta \lambda_1 \frac{r^2 - x_1^2}{\eta} + O(\frac{1}{\eta}) + \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \cdot O(\frac{1}{\eta \lambda_1}) + \left( \frac{u_{112}}{\lambda_1 - \lambda_2} \right) \cdot O(\frac{1}{\eta \lambda_1}) \\
+ \frac{u_{111}}{u_{11}} \cdot O(\frac{1}{\eta \lambda_1}) + [- \frac{f(1 + |\nabla u|^2)^2 u_{111}}{\lambda_1} + O(1)]^2 \cdot O(\frac{1}{\eta \lambda_1}) \\
+ [- \frac{f(1 + |\nabla u|^2)^2 u_{111}}{\lambda_1} + O(1)] \cdot O(\frac{1}{\eta}) \\
(3.25) \geq - 2 \beta \frac{r^2 - x_1^2}{\eta} + O(\frac{1}{\eta}) - \frac{\lambda_2}{2} \left( \frac{u_{111}}{u_{11}} \right)^2 - \frac{\lambda_1}{4} \left( \frac{u_{112}}{u_{11}} \right)^2.
\end{align*}
\]
Now we just need to estimate \(-2\beta \lambda_1 \frac{r^2-x_1^2}{\eta}\). If \(x_2^2 \leq \frac{r^2}{2}\), we can get

\[-2\beta \lambda_1 \frac{r^2-x_1^2}{\eta} = -\frac{8}{r^2-x_2^2} \lambda_1 \geq -\frac{16}{r^2} \lambda_1 \geq -\frac{1}{2} c_0 \frac{2}{r^2} f \lambda_1 = -\frac{1}{2} g' g f \lambda_1.
\]

If \(x_2^2 > \frac{r^2}{2}\), we can get

\[-2\beta \lambda_1 \frac{r^2-x_1^2}{\eta} = -\frac{8}{r^2-x_2^2} \lambda_1 \geq \frac{8}{r^2-x_2^2} \lambda_1 = -\beta \lambda_1 \frac{1}{2} \left(\frac{2x_2^2}{r^2-x_2^2}\right)^2 \
\geq -\beta \lambda_1 \left[\frac{n_2}{\eta}\right]^2,
\]

if \(\eta \lambda_1\) is big enough. Hence

\[(3.26) \quad -2\beta \lambda_1 \frac{r^2-x_1^2}{\eta} \geq -\frac{1}{2} g' f \lambda_1 - \beta \lambda_1 \left[\frac{n_2}{\eta}\right]^2,
\]

and

\[(3.27) \quad \beta \left[\frac{n_1}{\eta} + \lambda_1 \frac{n_2}{\eta}\right] \geq -\frac{1}{2} g' f \lambda_1 - \beta \lambda_1 \left[\frac{n_2}{\eta}\right]^2 - \frac{\lambda_2}{2} \left(\frac{u_{111}}{u_{11}}\right)^2 - \frac{\lambda_1}{4} \left(\frac{u_{112}}{u_{11}}\right)^2 + O\left(\frac{1}{\eta}\right).
\]

Now we continue to prove Theorem 1.1. From (3.11) and Lemma 3.1, we can get

\[0 \geq \sum_{i=1}^{2} F^{ii} \varphi_{ii} \geq \frac{1}{2} g' f \lambda_1 + O\left(\frac{1}{\eta}\right) + O(1)
\]

\[(3.28) \quad \frac{1}{2} g' f \lambda_1 - \frac{C_0}{\eta} - C_0,
\]

So we can get

\[\eta \lambda_1 \leq C_1.
\]

where \(C_0\) and \(C_1\) are positive constants depending only on \(m\), \(M\), \(R\), \(\sup |\nabla f|\), \(\sup |\nabla^2 f|\), and \(\sup |\nabla u|\). So we can easily get

\[u_{\tau(0)r(0)}(0) \leq \frac{1}{r^{4\beta}} \phi(0) \leq \frac{1}{r^{4\beta}} \phi(x_0) \leq C,
\]

and

\[|u_{\xi \xi}(0)| \leq u_{\tau(0)r(0)}(0) \leq C, \quad \forall \xi \in S^1.
\]

Then we have proved (1.4), and Theorem 1.1 holds.

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