Effective field theory approach to Bose–Einstein condensation

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Abstract

We consider the low-energy collective excitations at finite temperature of Bose–Einstein condensed gases (and liquids as well). A most general model-independent effective Lagrangian is written down according to a prescription obtained from the breakdown of the global symmetry $U(1)$. To show how the theory predicts easily, we derive the momentum and temperature dependence of the damping of excitations by means of power counting as an example.

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1 Introduction

Bose–Einstein condensation (BEC) was achieved in 1995 in a remarkable series of experiments on atomic vapors [1, 2], and has tremendously boosted both theoretical and experimental studies (for an up-to-date review, see, e. g., Dalfovo et al. [3]). Among others, the study of collective excitations in BEC gives us a great test on the finite-temperature many-body theory of interacting Bose gases that has been developed over last several decades. It is well known that excitations associated with BEC have been well studied in the context of liquid $^4$He [4]. The semi-phenomenological Landau hydrodynamic theory is very successful, but it essentially leans on an ad hoc postulated Hamiltonian and the roles of Bose condensation and broken symmetry are not clear there [5]. This theory was justified in a sense of microscopic arguments by Feynman [6]. On the other hand, beginning with the work of London [7], a microscopic theory of superfluid $^4$He using field theoretical methods has also been extensively investigated through a weakly interacting gas model [4]. The most important results obtained from this model are based on the presence of a Bose condensate, which breaks down the global symmetry $U(1)$ of the system. In fact, the concept of broken symmetry originated in particle physics has been brought into the low temperature phenomena of superfluidity and superconductivity first by Anderson [8], and now it is believed that it controls much of dynamics of many-body systems.

In the microscopic study of a Bose condensed gas, one usually needs a detailed model like that of Bogoliubov, Beliaev [9], or Gross-Pitaevskii to explain the mechanism for a Bose condensate, and as a basis for approximate quantitative calculations, but not to derive the most important exact consequences of the spontaneous symmetry breakdown. (Of course, such a microscopic model itself is an approximation.) Yet, if the general features are in fact model independent consequences of the spontaneous breakdown of the $U(1)$ symmetry, why can’t we derive them directly from this breakdown by finding the most general effective Lagrangian of the system? An answer to this question is provided by a standard technique of modern quantum field theory [10], called the effective field theory approach to the symmetry breaking. It was applied to superconductivity [11, 12] where the fundamental properties of conventional superconductors such as Meissner effect and flux quantization are derived directly from the spontaneous breakdown of electromagnetic gauge invariance without using a detailed model like that of Bardeen-Cooper-Schrieffer (BCS) theory.

This paper aims to show that many important properties of low-energy excitations in a generic homogeneous Bose system can be essentially determined as the exact consequence of spontaneously broken symmetry without introducing unnecessary approximations. Assuming the $U(1)$ invariance is spontaneously broken, we shall derive the most general effective Lagrangian of the system by performing procedures very much analogous to that of pion-nucleon effective field theory [13] in quantum chromodynamics. The present paper gives an alternative derivation of the Popov effective Lagrangian that we used for calculating the damping rate of the excitation [14, 15]. As we shall see, the whole procedure is incredibly simple and straightforward, but also gives a feasible way of predicting some experimental observables. An example on the damping rate shall be given to show how the theory works easily. The purpose of the present work is twofold. First, it offers a deep way of justifying the Landau theory of quantum hydrodynamics and explaining why a weakly interacting gas
model can yield some meaningful results for strongly interacting superfluid $^4$He. Second, it shows a simple, powerful effective field theory approach to investigating the excitation dynamics of a homogeneous Bose condensed system.

2 Effective Lagrangian

We start by considering a simple nonrelativistic many-body problem of spinless bosonic particles at finite temperature. In units of $\hbar = 1$ and $k_B = 1$, its Euclidean action functional is given by [16, 17]

$$I[\psi, \psi^\dagger] = \int_0^\beta d\tau \int_{-\infty}^{\infty} d^3x \mathcal{L}$$

with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \left[ \psi^\dagger(x) \partial_\tau \psi(x) - \psi(x) \partial_\tau \psi^\dagger(x) \right] - \frac{1}{2m} \nabla \psi^\dagger(x) \cdot \nabla \psi(x) - \lambda \left[ \psi(x)^\dagger \psi(x) \right]^2$$

where $\beta = 1/T$ denotes the inverse of temperature. Here, we write $x = (x, \tau)$ and $\tau$ is the imaginary “time”. All fields fulfill the boundary conditions as of $\psi(x, \tau + \beta) = \psi(x, \tau)$. Obviously, the action (1) is Galilean-invariant, and is invariant under the global phase transformations of group $U(1)$,

$$\psi(x) \to \exp(i\Lambda)\psi(x),$$

with $\Lambda$ an arbitrary constant. This symmetry is known to be completely broken below some critical temperature $T_c$, i.e., a Bose–Einstein condensate appears with $\langle \psi \rangle \neq 0$ [18]. The particle density and current associated with the symmetry are $J = (i/2m)(\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger)$, $J_4 = i\psi^\dagger \psi \equiv i\rho$, with the current conservation $i\partial_\tau \rho + \nabla \cdot J = 0$.

According to our general understanding of spontaneously broken symmetries [10], any system described by an action with symmetry group $G$, when in a phase in which $G$ is spontaneously broken to a subgroup $H$, will possess a set of Goldstone modes, described by fields that transform under $G$ like the coordinates of the coset space $G/H$. In our case, there will be a single Goldstone mode described by a (massless) real scalar field $\phi(x)$ that transforms under $G = U(1)$ like the phase $\Lambda$ itself. The group $U(1)$ has the multiplication rule $g(\Lambda_1)g(\Lambda_2) = g(\Lambda_1 + \Lambda_2)$, so under a phase transformation with parameter $\Lambda$ the field $\phi(x)$ will undergo the transformation

$$\phi(x) \to \phi(x) + \Lambda.$$  

In low temperature physics, the Goldstone mode is accompanied with another excitation, known as the density (order parameter) fluctuation, which we will see to have nearly zero frequency in the long-wavelength limit. Both together form a non-trivial irreducible linear representation of the group $U(1)$. To see the theory must involve the Goldstone fields, we may write all ordinary complex fields as [19]

$$\psi(x) = \sqrt{\rho(x)} \exp(i\phi(x))$$
where the $\rho$ is the density field. Under phase transformation, the $\rho$ is invariant while $\phi$ transforms according to the rule (3). Now, rewritten in terms of fields $\phi$ and $\rho$, the Lagrangian (4) becomes

$$L = \rho \dot{\partial}_t \phi \left( - \frac{(\nabla \rho)^2}{8m\rho} - \frac{\rho(\nabla \phi)^2}{2m} - \lambda \rho^2 \right). \quad (5)$$

If the $U(1)$ symmetry is broken, we have $\rho_0 \equiv |\langle \psi \rangle_0|^2 \neq 0$ where $\langle \cdots \rangle_0$ indicates the expectation value over the ground state. It follows that we may write $\rho(x) = \rho_0 + \sigma(x)$, where the $\sigma$ field describes density fluctuations, known also as collective modes. The effective Lagrangian for $\phi$ and $\sigma$ then reads

$$L_{\text{eff}} = \sigma \dot{\partial}_t \phi \left( - \frac{(\nabla \sigma)^2}{8m(\rho_0 + \sigma)} - \frac{(\rho_0 + \sigma)(\nabla \phi)^2}{2m} - \lambda(\rho_0 + \sigma)^2 \right), \quad (6)$$

where we have ignored all total derivatives. For later use, we record the current conservation in terms of $\phi$ and $\sigma$ explicitly:

$$i \dot{\partial}_t \sigma - \frac{\nabla \cdot [(\rho_0 + \sigma)\nabla \phi]}{m} = 0. \quad (7)$$

Here is an important point: to derive Eq. (6) it was not really necessary to start with the model Lagrangian (4). Indeed, according to our understanding of spontaneously broken symmetry, we did not need to start with any specific theory. A familiar example in particle physics is the effective theory of pion-nucleon interaction with $SU(2) \times SU(2)$ spontaneously broken to $SU(2)$. The important thing is that Eq. (6) is invariant under the $U(1)$ transformation. The general theory of broken symmetries (see, for example, Weinberg [10]) tells us that, for symmetry group $U(1)$, the most general form of the effective Lagrangian for the density fluctuation field and the Goldstone field must be constructed solely from the ingredients $\sigma$, $\nabla \sigma$, $\partial_t \sigma$, $\nabla \phi$ and $\partial_t \phi$ together with higher derivatives of these objects. There are also two additional rules that the Lagrangian must obey. The first rule is the Galilean invariance for a non-relativistic system such that the combination of $i \partial_t \phi - (\nabla \phi)^2/2m$ must always appear together in the effective Lagrangian and likewise $i \partial_t \sigma - (1/m)\nabla \phi \cdot \nabla \sigma$ [20]. This has been understood through investigations by Takahashi [21] and Greiter et al. [22]. The second rule is the time-reversal symmetry such that the action is invariant under the transformation

$$\phi \rightarrow -\phi, \quad \tau \rightarrow -\tau. \quad$$

This is observed from the action (4). This symmetry requires that only the even powers of the Galilean invariant $[i \partial_t \sigma - (1/m)\nabla \phi \cdot \nabla \sigma]$ be included in the Lagrangian. For instance, the possible lowest power of it is $[i \partial_t \sigma - (1/m)\nabla \phi \cdot \nabla \sigma]^2$. This is equivalent to have terms of $[(\rho_0 + \sigma)(\nabla^2 \phi)^2/m^2]$ by making use of the current conservation (7) [23].

Hence, according to the above prescription the most general $U(1)$-invariant action functional takes the following form

$$I_{\text{eff}}[\sigma, \phi] = \int_0^\beta d\tau \int_{-\infty}^\infty d^3x L_{\text{eff}} \quad (8)$$
with
\[
\mathcal{L}_{\text{eff}} = \sigma \left[ i\partial \phi - \frac{(\nabla \phi)^2}{2m} \right] - \frac{F_\phi}{2m} (\nabla \phi)^2 - \frac{F'_m}{2} \left[ i\partial \phi - \frac{(\nabla \phi)^2}{2m} \right]^2 \\
- \frac{F''}{3!} \left[ i\partial \phi - \frac{(\nabla \phi)^2}{2m} \right]^3 - \frac{F_\sigma}{2m} (\nabla \sigma)^2 - \frac{c_2}{2m} \sigma^2 - \frac{c_3}{3!} \sigma^3 \\
- \frac{c'_m}{2} \sigma \left[ i\partial \phi - \frac{(\nabla \phi)^2}{2m} \right]^2 - \frac{c''}{2} \sigma^2 \left[ i\partial \phi - \frac{(\nabla \phi)^2}{2m} \right] + \cdots. \tag{9}
\]

The terms indicated by \( \cdots \) will contain higher powers and/or derivatives of the \( \sigma \) and/or \( \phi \) fields. Any term of a total derivative has been ignored. The coefficients \( F_\phi, F'_\phi, F''_\phi, F_\sigma, c_2, c_3, c'_m \) and \( c''_m \) have the dimensions of \( K^3, K, K^{-1}, K^{-3}, K^{-1}, K^{-4}, K^{-3} \), respectively, where \( K \) represents a typical momentum scale that shall be discussed in the next section. (Here, we adopt a normalization such that \( \phi \) is dimensionless and \( \sigma \) has the dimension of \( K^3 \).) Similar effective Lagrangians appeared in different contexts, but were all based on some microscopic model. For instance, Popov \[19\] derived it by means of the power expansion of the pressure (or equivalently grand potential) in terms of inhomogeneity for Bose gases, Aitchison et al. \[24\] found it equivalent to a time-dependent non-linear Schrödinger Lagrangian for BCS superconductors at \( T = 0 \), and Demircan et al. \[25\] implied that it could be obtained from the Feynman wave function of superfluids. If \(-\frac{1}{m} \nabla \phi \) and \( \sigma \) are identified with the phonon velocity field \( v \) and the density variation \( \rho' \) of Ref. \[5\], respectively, one finds the action (8) corresponds to a Hamiltonian
\[
\int d^3x \left\{ \frac{F_\phi}{2m} v^2 + \frac{F_\sigma}{2m} (\nabla \rho')^2 + \frac{c_2}{2m} \rho'^2 + \frac{c_3}{3!m} \rho'^3 + \frac{1}{2} m \rho' v^2 + \cdots \right\}, \tag{10}
\]
which is one form of the Landau-Khalatnikov hydrodynamic Hamiltonian \[26\]. Hence, the Landau quantum hydrodynamics is the exact consequence of the breakdown of the \( U(1) \) symmetry.

### 3 Power Counting

Many-body properties can be studied through propagators (or Green's functions). In the following, we write \( \phi \) and \( \sigma \) fields into a real 2-component scalar
\[
\Phi = \begin{pmatrix} \phi \\ \sigma \end{pmatrix}
\]
with Greek indices \( \alpha, \beta, \cdots (= 1, 2) \) labeling its components. The propagators are defined by the matrix
\[
\Delta_{\alpha\beta}(x - x') = \langle T \{ \Phi_\alpha(x) \Phi_\beta(x') \} \rangle = \frac{\int [\Pi x d\phi(x)d\sigma(x)] \Phi_\alpha(x) \Phi_\beta(x') \exp I_{\text{eff}}[\phi, \sigma]}{\int [\Pi x d\phi(x)d\sigma(x)] \exp I_{\text{eff}}[\phi, \sigma]},
\]

\[5\]
where \( T \) denotes a time-ordered product on \( \tau, \tau' \). Consider the quadratic part of the effective action

\[
I_{\text{quad}}^{\text{eff}} = \int_0^\beta d\tau \int_{-\infty}^{\infty} d^3x \left[ \sigma i\partial_\tau \phi - \frac{F_\phi}{2m}(\nabla \phi)^2 + \frac{F_\phi'}{2m}(\partial_\tau \phi)^2 - \frac{F_\sigma}{2m}(\nabla \sigma)^2 - \frac{c_2}{2m} \sigma^2 \right]
\]

\[\equiv -\frac{1}{2} \int d^4x d^4x' \Phi^\dagger(x) \mathcal{D}(x,x') \Phi(x'),\]  

(11)

where

\[
\mathcal{D}(x,x') = \left( \begin{array}{cc}
\frac{F_\phi}{m} \nabla_x^2 + F_\phi' m \partial_\tau^2 \delta^4(x - x') & i\partial_\tau \delta^4(x - x') \\
-i\partial_\tau \delta^4(x - x') & \frac{F_\sigma}{m} \nabla_x^2 + \frac{c_2}{m} \delta^4(x - x')
\end{array} \right). \quad (12)
\]

The free propagators are given by the inverse of the matrix \( \mathcal{D} \):

\[
\Delta(x,x') = \mathcal{D}^{-1}(x,x'). \quad (13)
\]

The calculation of propagators is simplified by transforming to momentum basis via the following Fourier transformation

\[
\Delta(x,x') = \frac{1}{\beta(2\pi)^3} \sum_\nu \int d^3k \Delta(k) e^{i\mathbf{k} \cdot (x - x')} e^{-i\omega_\nu (\tau - \tau')}, \quad (14)
\]

where Matsubara frequencies \( \omega_\nu \equiv 2\pi \nu/\beta \) \((\nu = 0, \pm 1, \pm 2, \cdots)\) and the 4-momentum notation \( k = (k, \omega_\nu) \) is used. We then have

\[
\Delta^{-1}(k) = \mathcal{D}(k) = \left( \begin{array}{cc}
\frac{F_\phi}{m} k^2/m - F_\phi' m \omega_\nu^2 & \omega_\nu \\
-\omega_\nu & (F_\sigma k^2/m - \omega_\nu) (F_\sigma k^2/m - c_2)/m
\end{array} \right). \quad (15)
\]

By finding its inverse matrix, the free propagators (see Fig. 1) are

\[
\Delta(k) = \left[ 1 - \frac{F_\phi' (F_\sigma k^2 + c_2)}{\omega_\nu^2 + \epsilon^2(k)} \right]^{-1} \left( \begin{array}{cc}
(F_\sigma k^2 + c_2)/m & -\omega_\nu \\
\omega_\nu & F_\phi k^2/m - F_\phi' m \omega_\nu^2
\end{array} \right), \quad (16)
\]

with the energy spectrum

\[
\epsilon(k) = \frac{1}{m} \sqrt{\frac{F_\phi k^2 (F_\sigma k^2 + c_2)}{1 - F_\phi' (F_\sigma k^2 + c_2)}}. \quad (17)
\]

We shall consider the low momentum-energy region such that \( k \equiv |k| \ll k_0 \equiv \sqrt{c_2/F_\sigma} \), in which the spectrum reduces to the phonon type

\[
\epsilon(k) \simeq ck \quad \text{with the phonon velocity} \quad c \equiv (1/m) \sqrt{F_\phi c_2/(1 - F_\phi' c_2)}. \quad \text{Notice that the energy spectrum is linear in} \ k, \ \text{vanishing as} \ k \to 0.
\]

In the calculation of Feynman diagrams at finite temperatures in Euclidean field theory, one encounters the summation over discrete Matsubara frequencies. A standard technique (see, for example, [27]) is available to perform such a Matsubara summation. The trick is
to use a contour integral in the complex energy plane. Let \( h(\omega) \) be a function of complex variable \( \omega \), analytical on the line \( \text{Re}\omega = 0 \), which decreases faster than \( 1/|\omega| \) as \( |\omega| \to \infty \). We then have

\[
\frac{1}{\beta} \sum_{\nu} h(i\omega_{\nu}) = -\oint_{C} \frac{d\omega}{2\pi i} f(\omega) h(\omega),
\]  

(18)

where

\[
f(\omega) = \frac{1}{e^{\beta \omega} - 1},
\]

and the C is a contour in complex \( \omega \)-plane encircling all poles of function \( h(\omega) \) in a positive sense (but those of function \( f(\omega) \) in a negative sense).

Now consider a general process involving arbitrary numbers of the Goldstone field \( \phi \) and the density fluctuation field \( \sigma \). We suppose that their energies and momenta and the thermal energy (\( \sim T \)) are all at most of some order \( K \), which is small compared with \( k_0 \) defined above. Even though Lagrangians like (9) are not renormalizable in the usual sense, we saw in, for example, the pion-nucleon theory \([10]\) that such Lagrangians can yield finite results as long as they contain all possible terms allowed by symmetries, for then there will be a counterterm available to cancel every infinity. If we define the renormalized values of the constants \( F_{\phi}, F'_{\phi}, F''_{\phi}, F_{\sigma}, c_2, c_3, c'_3, \cdots \) in \( \mathcal{L}_{\text{eff}} \) by specifying the values at energies of order \( K \), then the integrals in momentum-space Feynman diagrams will be dominated by contributions from virtual momenta which are also of order \( K \) (because renormalization makes them finite, and there is no other possible effective cut-off in the theory). We can then develop perturbation theory as a power series expansion in \( K \).

Each derivative in each interaction vertex contributes one factor of \( K \) to the order of magnitude of the diagram; internal propagators \( \Delta_{11}(k), \Delta_{12}(k) (= -\Delta_{21}(k)) \), and \( \Delta_{22}(k) \) contribute factors of \( K^{-2}, K^{-1} \) and a unit, respectively; and each integration volume \( d^4k \) (\( \equiv d^3kd\omega \)) associated with the loops of the diagram contributes a factor of \( K^4 \). So a general connected diagram make a contribution of order \( K^\nu \), where

\[
\nu = -2I_{\phi} - I_\times + \sum_i d_i V_i + 4L.
\]

(19)

Here \( d_i \) is the number of derivatives in an interaction of type \( i \), \( V_i \) is the number of interaction vertices of type \( i \) in the diagram, \( I_{\phi} \) and \( I_\times \) are the numbers of internal \( \phi \) lines and \( \phi-\sigma \) cross lines, respectively, and \( L \) is the number of loops. There is a familiar topological relation for connected graphs:

\[
2I_{\phi} + I_\times + E_{\phi} = \sum_i \phi_i V_i,
\]

(20)

where \( \phi_i \) is the number of Goldstone fields \( \phi \) in interactions of type \( i \) and \( E_{\phi} \) is the number of external \( \phi \) lines. Eliminating the quantity \( I_{\phi} \), the two topological equations above give

\[
\nu = \sum_i (d_i - \phi_i) V_i + E_{\phi} + 4L.
\]

(21)

The important point here is that the coefficient \( d_i - \phi_i \) in the first term is always positive or zero. Hence, with the numbers of loops and external \( \phi \)-lines fixed, the leading terms are those graphs of \( d_i - \phi_i = 0 \). The interactions that satisfy this condition are of no derivatives
Figure 1: Feynman diagrams of propagators, vertices, and the one-loop selfenergy \( \Pi \). Notations are \( k = (k, \omega_\nu) \) and \( \Phi^\dagger_\alpha = (\phi, \sigma) \) with Greek indices \( \alpha, \beta, \cdots = 1, 2 \).

of the \( \sigma \) field, noticing that the Goldstone fields always appear with derivatives. Next, we shall see the power counting (21) provides us a tool that ensures that only finite number of terms of \( \mathcal{L}_\text{eff} \) are necessary.

4 Low-energy Excitation Spectrum

Let us now apply this method to the calculation of higher order corrections to the propagator,

\[
\Delta'(k) = \Delta(k) + \Delta(k)\Pi(k)\Delta'(k)
\]

where the matrix \( \Pi(k) \) denotes the self-energy connected graphs. The lowest order correction shall be from the graphs of one loop with \( L = 1 \) in Eq. (21), to which only cubic interactions of the Lagrangian (9) contribute. The following are all cubic interactions that concern us:

\[
-\frac{\sigma (\nabla \phi)^2}{2m}, \quad \frac{F'_\phi i\partial_\nu \phi(\nabla \phi)^2}{2}, \quad -\frac{F''_\phi m^2}{3!} (i\partial_\nu \phi)^3, \quad -\frac{c_3m}{2}\sigma^3, \quad \frac{c'_3 m^2}{2}\sigma (\partial_\nu \phi)^2, \quad \text{and} \quad \frac{c''_3}{2}\sigma^2 i\partial_\nu \phi,
\]

which give the following vertices

\[
\delta^4(k + k' + k'') \left\{ \sum_{P\{\alpha \to \alpha', \alpha'' \to \alpha\}} \left[ \delta_{\alpha,1}\delta_{\alpha',1}\delta_{\alpha'',2} \left( \frac{(k \cdot k')}{m} - \frac{c'_3 m \omega_\nu \omega_{\nu'} \omega_{\nu''}}{2} \right) - \delta_{\alpha,1}\delta_{\alpha',2}\delta_{\alpha'',2} \frac{c''_3 \omega_{\nu'}}{2} \right] \right. \\
-\delta_{\alpha,1}\delta_{\alpha',1}\delta_{\alpha'',1} \left. \left[ \sum_{P\{k \to k' \to k'' \to k\}} F'_\phi (k \cdot k')\omega_{\nu''} + F''_\phi m^2 \omega_\nu \omega_{\nu'} \omega_{\nu''} \right] - \delta_{\alpha,2}\delta_{\alpha',2}\delta_{\alpha'',2} \frac{c_3}{m} \right\}, \tag{22}
\]

where we define \( \delta^4(k + k' + k'') = \delta^3(k + k' + k'')\delta_{\nu + \nu' + \nu'',0} \), and \( \sum_{P\{\ldots\}} \) indicates the sum over three cyclic permutations. (The vertices and the one-loop diagram of \( \Pi \) are depicted in Fig. 1.) In the following, we will show how some fundamental low temperature properties of low energy excitations can be easily derived without complicated calculations.

The spectrum of excitations is given by the poles of the exact propagators (Green’s functions) that are solutions to the following equation

\[
\det \Delta'^{-1}(k) = \det(\Delta^{-1}(k) - \Pi(k)) = 0. \tag{23}
\]
From this expression, the spectrum can be obtained by analytically continuing it to values of \( \omega \) that are not Matsubara frequencies by doing \( i \omega \nu \rightarrow \omega + i \eta \) \((\eta \equiv 0^+)\), but of course this shall only be done after the Matsubara frequency sum. From the free propagator expression (13), the (real) leading term of the spectrum is obviously given by \( \omega = \epsilon(k) \), and we shall approximate \( \text{Re} \omega = \epsilon(k) \) as far as only the lowest corrections are concerned. Thus the spectrum is given in the following form

\[
\omega = \epsilon(k) - i \gamma(k).
\] (24)

Now our task is to find the leading order of the imaginary part of the spectrum, called the damping rate \( \gamma \). After some elementary algebras, keeping only leading order in \( k \), Eqs. (15) and (23) yield

\[
\gamma(k) = \left\{ \frac{c_2}{2m\epsilon(k)} \text{Im}\Pi_{11}(k, -i\omega + \eta) + \frac{F_\phi k^2 + F'_\phi m^2 \epsilon^2(k)}{2m\epsilon(k)} \text{Im}\Pi_{22}(k, -i\omega + \eta) \right. \\
- \left. \frac{1}{2} \text{Re}[\Pi_{12}(k, -i\omega + \eta) - \Pi_{21}(k, -i\omega + \eta)] \right\} (1 - F'_\phi c_2)^{-1}
\] (25)

with \( \Pi_{12}(k) = -\Pi_{21}(k) \). One can verify that Eq. (25) can reduce to the specific forms previously presented in Refs. [19, 14]. Rather than go on to really calculate all elements of the self-energy matrix, we shall try to quickly derive the temperature and momentum dependences of the damping by using the power counting formula (21).

At \( T = 0 \), the typical momentum scale \( K \) is just the momentum \( k \) of a excitation that is carried into \( \Pi \) graphs through external lines. From the expression (21), we find

\[
\Pi_{11}(k) \sim k^6, \quad \Pi_{12}(k) = -\Pi_{21}(k) \sim k^5, \quad \text{and} \quad \Pi_{22}(k) \sim k^4,
\]

so that Eq. (25) immediately yields

\[
\gamma_{T=0}(k) \propto k^5.
\] (26)

We therefore have determined \( \gamma(k) \) up to some coefficient that only depends on something other than \( k \) (e.g., the interaction strength). Expression (25) agrees with the well-known result first derived by Beliaev in 1958 [4] for a weakly interacting dilute gas model by calculating the Green’s function till the second order approximation. However, our result has been derived without assuming a weak coupling, so it is valid for general Bose condensed liquids including superfluid \(^4\)He.

For temperature such that \( ck \ll T \ll ck_0 \), we can follow the same arguments as we had for \( T = 0 \), but some additional examination on diagrams is demanded in order to get the correct \( k \) and \( T \) dependences of \( \gamma \). Each self-energy element \( \Pi_{\alpha\beta} \) carrying a momentum \( k \) yields a contribution of order \( K^\nu \), which for \( T \neq 0 \) can be decomposed into \( K^\nu = k^l T^{\nu-l} \) \((l = 0, 1, \cdots, \nu)\). For temperature regime concerned here, the \( k \)-dependence of the \( \Pi \) arises only from those vertices that are associated with external (input) momenta after taking off the \( \delta \)-function dependence from the vertex expression (22). Hence, \( k^l \) can be determined by simply counting the power of external (input) momenta from each vertex. In other words, \( l \) is equal to the number of external \( \phi \) lines attached to each \( \Pi \) graph. Thus,

\[
\Pi_{11}(k) \sim k^2 T^4, \quad \Pi_{12}(k) = -\Pi_{21}(k) \sim k T^4, \quad \text{and} \quad \Pi_{22}(k) \sim T^4,
\]
in comparison with those of $T = 0$. From Eq. (25), we find the damping rate satisfying

$$\gamma(k) \propto kT^4. \quad (27)$$

Again, in the context of a weakly interacting dilute Bose gas model, this remarkable result was first implied by Mohling and Morita [28] and explicitly obtained by Hohenberg and Martin [29] and by Popov [19]. However, our result is universally true for a generic system of scalar bosons in low temperature and low energy, free of the assumption of weak coupling and low density. That implies that the result (27) also holds for liquid $^4$He.

The low-energy excitation spectrum (24) is called phonon spectrum and has an important property: the excitation frequency is equal to zero for zero momentum to all orders in interactions, which means that the spectrum does not exhibit an energy gap. This is known as the Hugenholtz-Pines theorem [30] that was proved in perturbation field theoretical analysis. It has played a crucial role in the early study of superfluidity. Here we gain another understanding of this property. It is known from both microscopic theories [31, 32] and superfluid hydrodynamics [29] that, if $T < T_c$, the quasiparticle excitations have exactly the same spectra as collective excitations (described here by the $\sigma$ field) in the long wavelength limit. Further, the effective Lagrangian (9) dictates that the Goldstone mode $\phi$ and the collective mode $\sigma$ possess the same pole structure up to all orders in all interactions allowed by symmetries. This statement is implied by Eq. (23). For the Goldstone theorem protects all kinds of excitations in low temperature and low momentum limit from having energy gap, their spectrum must be of phonon type. In this point of view, the Hugenholtz-Pines theorem can be simply understood as the equivalent statement of the nonrelativistic version of the Goldstone theorem [33, 34] for the special case of a dilute Bose gas.

We conclude here that the effective field theory approach to the study of collective excitations in BEC makes the role of the Bose–Einstein condensate (or broken $U(1)$ symmetry) evident, and gives us a model-independent effective action that can immediately predict results for experiments. We expect the approach to be very productive and fundamental if one can conduct it for trapped alkali vapors that are currently under extensive investigation. Yet, the inhomogeneity of the systems due to a trap potential shall alter our results.

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