On the one-dimensional family of Riemann surfaces of genus $q$ with $4q$ automorphisms

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Automorphism groups

Automorphism groups of Riemann surfaces have been extensively studied, going back to Wiman, Klein and Hurwitz, among others.

Let $S$ denote a compact Riemann surface of genus $g$. Classically known:

- $|\text{Aut}(S)| \leq 84(g - 1)$.
- In the abelian case $|\text{Aut}(S)| \leq 4g + 4$.
- In the cyclic case $|\text{Aut}(S)| \leq 4g + 2$.

General problem: to understand the extent to which the order of the full automorphism group determines the Riemann surface.

Examples:
1. Hurwitz curves are $(2, 3, 7)$-branched coverings of the projective line.
2. ∃! Riemann surface genus $g$ admitting an automorphism of order $4g$.
3. ∃! Riemann surface genus $g$ with $8(g + 1)$ automorphisms.
Automorphism groups
A very special family

Theorem (Bujalance-Costa-Izquierdo, 2017). Assume

\[ g \neq 3, 6, 12, 15, 30. \]

The Riemann surfaces of genus \( g \) admitting **exactly** \( 4g \) automorphisms form an equisymmetric one-dimensional family, denoted by \( \mathcal{F}_g \).

Moreover, if \( S \) is a Riemann surface in \( \mathcal{F}_g \) then

- its full automorphism group \( G \) is isomorphic to \( \mathbb{D}_{2q} \), and
- the corresponding quotient \( S/G \) has genus zero.

**Remark:** This is the second possible largest order (next talk!).
Let $q \geq 5$ be a prime number. For each Riemann surface $S$ in $\mathcal{F}_q$ we study:

- an algebraic description of $S$ and of its automorphisms,
- a decomposition of the Jacobian variety $JS$,
- the possible fields of definitions of $S$ and of $JS$, and
- the Shimura family associated to $S$.

Let $S$ denote a Riemann surface in the family $\mathcal{F}_q$ and let

$$G = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle \cong \mathbf{D}_{2q}$$

denote its full automorphism group.
Algebraic description

The quotient Riemann surface $S/G$ has genus zero, and the associated $4q$-fold branched regular covering map

$$
\pi_G : S \to S/G \cong \mathbb{P}^1
$$

ramifies over four values; three ramification values marked with 2 and one ramification value marked with $2q$.

**Assumption.** The branch values are $\infty, 0, 1$ marked with 2 and $\lambda \in \mathbb{C} - \{0, 1\}$ marked with $2q$.

Let

$$
\Omega := \mathbb{C} - \{0, \pm 1, \frac{1}{2}, 2, \gamma, \gamma^2\} \text{ where } \gamma^3 = -1
$$

denote the set of *admissible* parameters.
Algebraic description

Then $\mathcal{F}_q$ can be understood by means of an everywhere maximal rank holomorphic map

$$h : \mathcal{F}_q \rightarrow \Omega$$

in such a way that the fibers of $h$ agree with the Riemann surfaces in $\mathcal{F}_q$. We denote by $S_\lambda$ the Riemann surface $h^{-1}(\lambda)$.

Theorem. Let $\lambda \in \Omega$. Then $S_\lambda$ is isomorphic to the Riemann surface defined by the normalization of the hyperelliptic algebraic curve

$$y^2 = x(x^{2q} + 2\frac{1+\lambda}{1-\lambda}x^q + 1).$$

The full automorphism group of $S_\lambda$ is generated by the transformations

$$r(x, y) = (\omega_q x, \omega_{2q} y) \quad \text{and} \quad s(x, y) = \left(\frac{1}{x}, \frac{y}{x^{q+1}}\right)$$

where $\omega_t = \exp\left(\frac{2\pi i}{t}\right)$. 
The Jacobian variety

It is well-known that the dihedral group

\[ G = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle \]

has, up to equivalence, 4 complex irreducible representations of degree one; namely,

- \( V_1 : \begin{cases} r \to 1 \\ s \to 1 \end{cases} \)
- \( V_2 : \begin{cases} r \to 1 \\ s \to -1 \end{cases} \)
- \( V_3 : \begin{cases} r \to -1 \\ s \to 1 \end{cases} \)
- \( V_4 : \begin{cases} r \to -1 \\ s \to -1 \end{cases} \)

and \( q - 1 \) complex irreducible representations of degree two; namely,

- \( V_{k+4} : r \mapsto \text{diag}(\omega_{2q}^k, \bar{\omega}_{2q}^k), \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

for \( 1 \leq k \leq q - 1 \) and \( \omega_t = \exp(\frac{2\pi i}{t}) \).
The Jacobian variety

Lemma.

(1) The rational irreducible representations of \( G \), up to equivalence, are:

(a) four of degree 1; namely \( W_i := V_i \) for \( 1 \leq i \leq 4 \) and

(b) two of degree \( q - 1 \); namely

\[
W_5 = \bigoplus_{\sigma \in G_5} V_5^\sigma \quad \text{and} \quad W_6 = \bigoplus_{\sigma \in G_6} V_6^\sigma
\]

where \( G_5 \) and \( G_6 \) denote the Galois group associated to the extensions \( \mathbb{Q} \leq \mathbb{Q}(\omega_2 q + \bar{\omega}_2 q) \) and \( \mathbb{Q} \leq \mathbb{Q}(\omega_q + \bar{\omega}_q) \) respectively, and \( \omega_t = \exp(\frac{2\pi i t}{t}) \).

(2) The group algebra decomposition of \( J S_\lambda \) with respect to \( G \) is

\[
J S_\lambda \sim_G B_1 \times B_2 \times B_3 \times B_4 \times B_5^2 \times B_6^2
\]

where \( B_j \) stands for the factor associated to the representation \( W_j \).
The Jacobian variety

To compute the dimension of the factors $B_j$ (which may be zero) we need to choose a generating vector representing the action of $G$ on $S_\lambda$.

**Lemma.** Let $\sigma$ be a generating vector of $G$ of type $(2, 2, 2, 2q)$. Then there exist integers $e_1, e_2$ with $e_1 - e_2$ even and not congruent to 0 modulo $2q$, such that

$$ \sigma = (s^{r_{e_1}}, s^{r_{e_2}}, r^q, r^{e_1-e_2+q}) $$

up to the action of the symmetric group $S_3$ over the first three entries.

**Remark.** The family $F_q$ is equisymmetric: every generating vector of $G$ of the desired type can be chosen to represent the action of $G$ on $S_\lambda$. 
The Jacobian variety

Problem. To analyze how such a choice changes the dimension of the factors arising in the group algebra decomposition of $JS_{\lambda}$.

Definition. Two generating vectors $\sigma_1$ and $\sigma_2$ are termed essentially equal with respect to the action of $G$ on $S_{\lambda}$ if

$$\dim_{\tau_1}(B_j) = \dim_{\tau_2}(B_j)$$

for all $j$, where $\tau_i$ is the geometric signature associated to $\sigma_i$.

Lemma. Each generating vector of $G$ of type $(2, 2, 2, 2q)$ is essentially equal to

$$\sigma_0 = (s, sr^{-2}, rq, rq^{+2}) \quad \text{or to} \quad \sigma_1 = (sr, sr^{-1}, rq, rq^{+2}).$$
The Jacobian variety

Proposition. Let \( \lambda \in \Omega \), and consider the group algebra decomposition of \( J_{S_{\lambda}} \) with respect to \( G \)

\[
J_{S_{\lambda}} \sim_{G} B_1 \times B_2 \times B_3 \times B_4 \times B_5^2 \times B_6^2.
\]

If \( \tau_0 \) denotes the geometric signature associated to \( \sigma_0 \), then

\[
\dim_{\tau_0}(B_j) = \begin{cases} 
0 & \text{if } j = 0, 1, 2, 3, 6 \\
1 & \text{if } j = 4 \\
\frac{q-1}{2} & \text{if } j = 5
\end{cases}
\]

If \( \tau_1 \) denotes the geometric signature associated to \( \sigma_1 \), then

\[
\dim_{\tau_0}(B_j) = \begin{cases} 
0 & \text{if } j = 0, 1, 2, 4, 6 \\
1 & \text{if } j = 3 \\
\frac{q-1}{2} & \text{if } j = 5
\end{cases}
\]

In particular, \( J_{S_{\lambda}} \) contains an elliptic curve.
The Jacobian variety

**Theorem.** Let $\lambda \in \Omega$. The group algebra decomposition of $JS_\lambda$ with respect to $G$ does not depend on the choice of the generating vector.

**Proof** We only need to compare the decompositions associated to $\sigma_0$ and $\sigma_1$. These decompositions are

$$JS_\lambda \sim_{G,\sigma_0} B_4 \times B_5^2 \quad \text{and} \quad JS_\lambda \sim_{G,\sigma_1} B_3 \times B_5^2$$

respectively, showing that $B_3$ and $B_4$ are isogenous. We claim that, in addition, $B_4$ and $B_5$ are equal: the outer automorphism $\Phi$ of $G$

$$r \mapsto r, \quad s \mapsto sr$$

identifies $\sigma_0$ and $\sigma_1$ and identifies $W_3$ and $W_4$. 

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The Jacobian variety

Remark The independence of the group algebra decomposition on the choice of the generating vector is not new: it was

1. noticed by Rojas when she considered the Weyl group $\mathbb{Z}_2^3 \rtimes S_3$ acting on a Riemann surface of genus three with signature $(2, 4, 6)$.

2. noticed by Izquierdo, Jiménez and Rojas when they studied a two-dimensional family of Riemann surfaces of genus $2n - 1$ with action of $D_{2n}$ with signature $(2, 2, 2, 2, n)$.

The existence of outer automorphisms of the group is the key ingredient... however it has not been proved a general result on this respect!!

From now on, we assume the action of $G$ on $S_\lambda$ to be determined by the generating vector $\sigma_0$ and

$$JS_\lambda \sim_G B_4 \times B_5^2.$$. 
The Jacobian variety

**Theorem.** Let \( \lambda \in \Omega \). Consider the subgroups

\[
H_4 = \langle r^{-2}, sr^{-1} \rangle \quad \text{and} \quad H_5 = \langle s \rangle
\]

of \( G \), and the quotient Riemann surfaces \( E_\lambda \) and \( C_\lambda \) given by the action of \( H_4 \) and of \( H_5 \) on \( S_\lambda \), respectively. Then

\[
B_4 \sim JE_\lambda \quad \text{and} \quad B_5 \sim JC_\lambda.
\]

In particular, \( JS_\lambda \) decomposes into a product of Jacobians as follows:

\[
JS_\lambda \sim_G JE_\lambda \times JC_\lambda^2.
\]

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**Remark.** \( C_\lambda \) is an irregular \( 2q \)-gonal Riemann surface of genus \( \frac{q-1}{2} \). The elliptic curve \( E_\lambda \) is algebraically represented by

\[
y^2 = x(x - 1)(x - \lambda).
\]
Fields of definition

Let $k$ be a subfield of $\mathbb{C}$ and let $X$ be an algebraic variety.

**Definition.** The field $k$ is a field of definition of $X$ if there exists $Y \cong X$ such that $Y$ is the zero locus of polynomials with coefficients in $k$.

Interesting fields of definition are:

1. the field of the reals,
2. the algebraic closure of $\mathbb{Q}$, and
3. the field of moduli of $X$.

**Real Riemann surfaces** An algebraic variety is called *real* if it can be defined over the field of the real numbers; equivalently, if it admits an anticonformal involution.

**Remark.** $\mathcal{F}_q \subset \mathcal{M}_q$ admits an anticonformal involution whose fixed point set consists of points representing real Riemann surfaces.
Fields of definition

**Theorem.** Let $\lambda \in \Omega$. Then the following statements are equivalent:

(a) $S_\lambda$ is a real Riemann surface.
(b) $JS_\lambda$ is a real algebraic variety.
(c) $\lambda \in \{\bar{\lambda}, 1 - \bar{\lambda}, 1/\bar{\lambda}, \bar{\lambda}/(1 - \bar{\lambda})\}$

**Remark.** The real Riemann surfaces in the family $\mathcal{F}_q$ form three one-real-dimensional arcs.

To compactify the union of these arcs in the Deligne-Mumford compactification of $\mathcal{M}_g$, it is enough to add to $\mathcal{F}_q$ three points:

1. two nodal Riemann surfaces, and
2. the *Wiman surface* of type II
Fields of definition

We can recover part of these results:

The Riemann surfaces $S_{\lambda_1}$ and $S_{\lambda_2}$ are isomorphic if and only if $\lambda_2 = T(\lambda_1)$ for some $T \in \mathbb{G} = \langle z \mapsto \frac{1}{z}, z \mapsto \frac{1}{1-z} \rangle \cong S_3$. \hspace{1cm} (1)

Observe that for the exceptional values $-1, \frac{1}{2}, 2, \gamma$ and $\gamma^2$ where $\gamma^3 = -1$, the Riemann surface $S_{\lambda}$ has more than $4q$ automorphisms.

Thus, the family $\mathcal{F}_q$ is isomorphic to the quotient of the parameter space

$$\Omega = \mathbb{C} - \{0, \pm1, \frac{1}{2}, 2, \gamma, \gamma^2\}$$

up to the action of $\mathbb{G}$. Namely: $\Omega \rightarrow \Omega/\mathbb{G} \cong \mathcal{F}_q \cong \mathbb{C} - \{0, 1\}$. 
Fields of definition

The complex numbers $\lambda \in \Omega$ representing Riemann surfaces $S_\lambda$ which are real can be represented in the diagram below; the colored red points represent Riemann surfaces with more than $4q$ automorphisms.
Fields of definition

A fundamental region for the action of $\mathbb{G}$ on $\Omega$ is given by

$$\{z \in \mathbb{C} : |z| < 1, \text{Re}(z) < \frac{1}{2}\}$$

and, consequently, the subsets of $\mathcal{F}_q$ given by

$$\Pi(\{e^{i\theta} : \pi < \theta < \frac{\pi}{2}\}), \quad \Pi(\{z : |z - 1| = 1, |z| < 1\}) \quad \text{and} \quad \Pi([1, 0])$$

are the three arcs in $\mathcal{F}_q$ (denoted by $a_2, a_1$ and $b$ respectively)

The limit point of $\mathcal{F}_q$ which connects the arcs $a_2$ and $b$ correspond to $S_{-1}$ and therefore can be algebraically described by

$$y^2 = x(x^{2q} + 1).$$
Fields of definition

The map \((x, y) \mapsto (-\omega_{4q} x, \omega_{8q} y)\) where \(\omega_t = \exp(\frac{2\pi i}{t})\), induces an isomorphism between \(S_{-1}\) and the curve

\[ y^2 = x(x^{2q} - 1); \]

this is the Wiman surface of type II.

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**Arithmetic Riemann surfaces.** An algebraic variety is called *arithmetic* if it can be defined over a number field.

**Equivalence.** (Belyi’s theorem)

1. A Riemann surface \(S\) is arithmetic.
2. \(S\) admits a non-constant meromorphic function with three critical values.

As in the case of real Riemann surfaces, arithmetic Riemann surfaces among the Riemann surfaces in the family \(\mathcal{F}_q\) can be easily identified.
Fields of definition

**Theorem.** Let $\lambda \in \Omega$. Then the following statements are equivalent:

(a) $S_\lambda$ is an arithmetic Riemann surface.
(b) $JS_\lambda$ is an arithmetic algebraic variety.
(c) $\lambda$ is an algebraic complex number.

**Corollary.** Let $\lambda \in \Omega$ be an algebraic complex number. Then $JS_\lambda$ is an arithmetic algebraic variety admitting a group algebra decomposition in which each factor is arithmetic as well.

Riemann surfaces defined over the field of moduli. The field of moduli $\mathcal{M}(S)$ of a compact Riemann surface $S$ is by definition the fixed field of the group

$$
\mathbb{I}(S) = \{\sigma \in \text{Gal}(\mathbb{C}) : S^\sigma \cong S\}.
$$
Fields of definition

Proposition. Let $\lambda \in \Omega$. Then

$$\mathbb{Q}(j(\lambda)) \leq \mathcal{M}(S) \leq \mathbb{Q}(\lambda)$$

where $j$ denotes the Legendre invariant function for elliptic curves.

- Weil: necessary conditions for $S$ to admit its field of moduli as a field of definition.
- these conditions hold trivially if $S$ does not have non-trivial automorphisms.
- Wolfart: if $S/\text{Aut}(S)$ is an orbifold with signature of type $(a, b, c)$ then $S$ can be defined over its field of moduli.
- Dèbes-Emsalem: there is a field of definition of $S$ which is an extension of finite degree of its field of moduli.

By a result of Huggins follows directly that:

Proposition. The field of moduli of $S_\lambda$ is a field of definition for $S_\lambda$. 

Shimura family

Let $S$ be a compact Riemann surface of genus $g \geq 2$, and let

$$JS = (\mathcal{H}^{1,0}(S, \mathbb{C}))^*/H_1(S, \mathbb{Z})$$

be its Jacobian variety.

After fixing a symplectic basis of $H_1(S, \mathbb{Z})$ we have:

1. a period matrix $(I_g Z_S)$ with $Z_S \in \mathcal{H}_g$ for $JS$, and
2. a rational representation of $L_S := \text{End}_\mathbb{Q}(JS) = \text{End}(JS) \otimes_{\mathbb{Z}} \mathbb{Q}$

If $S$ is hyperelliptic, then the symplectic representation

$$\rho_r : G \to \text{Sp}(2g, \mathbb{Z})$$

of the automorphism group $G$ of $S$ induces an isomorphism

$$G \cong G := \{R \in \text{Sp}(2g, \mathbb{Z}) : R \cdot Z_S = Z_S\}.$$
We can now consider the complex submanifold of $\mathcal{H}_g$

$$\mathcal{H}_g(G) = \{ Z \in \mathcal{H}_g : R \cdot Z = Z \text{ for all } R \in G \}$$

consisting of those period matrices $Z$ representing ppavs of dimension $g$ admitting the given action of $G$. Clearly, $Z_S \in \mathcal{H}_g(G)$.

In the case of the action of $D_{10}$ on the Riemann surfaces in family $\mathcal{F}_5$, we can be much more explicit.

**Theorem.** Consider the action of $D_{10}$ with generating vector $\sigma_0$. There exists a three-dimensional family

$$\mathcal{A}_5(D_{10}) \subset \mathcal{A}_5$$

of principally polarized abelian varieties of dimension five admitting the given group action; it is given by the period matrices in $\mathcal{H}_5$ of the following form:
Shimura family

\[
\begin{pmatrix}
2(u+v+u) & -w-u & -2v & -v-w-u & -v+u \\
-w-u & -v-\frac{1}{2}w+\frac{5}{4}u & v-\frac{1}{2}u & w+\frac{1}{2}u & v-u \\
-2v & v-\frac{1}{2}u & u & v & w \\
-v-w-u & w+\frac{1}{2}u & v & u & -w \\
v+u & v-u & w & -w & 2(u-v-w)
\end{pmatrix}
\]

for complex numbers \(u, v\) and \(w\).
Furthermore, \(A_5(D_{10})\) contains the one-dimensional family \(F_5\).

The automorphism group \(G\) of \(S\) can be canonically seen as a subgroup of \(L_S\). Thus, the variety \(\mathcal{H}_g(G)\) contains the complex submanifold

\[\mathbb{H}(L_S) = \text{the Shimura domain of } S\]

whose points are matrices representing ppavs containing \(L_S\) in their endomorphism algebras (the Shimura family).

**Proposition.** Let \(\lambda \in \Omega\). The dimension of the Shimura family of each Riemann surface \(S_\lambda\) in \(F_q\) is \(\frac{q+1}{2}\).
Shimura family

Given a Riemann surface $S$, to provide an explicit description of the elements of $\mathbb{H}(L_S)$ seems to be a difficult task.

As a simple consequence of the previous theorem, we obtain:

**Corollary.** Each element of the Shimura family associated to every member of the family $\mathcal{F}_5$ admits a period matrix of the form

$$\begin{pmatrix}
2(u+v+u) & -w-u & -2v & -v-w-u & -v+u \\
-w-u & -v-\frac{1}{2}w+\frac{5}{4}u & v-\frac{1}{2}u & w+\frac{1}{2}u & v-u \\
-2v & v-\frac{1}{2}u & u & v & w \\
-v-w-u & w+\frac{1}{2}u & v & u & -w \\
-v+u & v-u & w & -w & 2(u-v-w)
\end{pmatrix}$$

for some $u, v, w \in \mathbb{C}$. 

(2)
Thanks!