UTILITY MAXIMIZATION WITH HABIT FORMATION OF INTERACTION

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Abstract. In this paper, we analytically solve the utility maximization problem for a consumption set with multiple habit formation of interaction. Consumption is here composed of habitual and nonhabitual components, where habitual consumption represents the effect of past consumption. We further assume that the individual seeks to maximize his/her expected utility from nonhabitual consumption. Although there is usually no explicit solution of linear dynamic systems in the habit formation model, we study the functional minimum of habitual consumption. To solve the optimization problem with a general utility function, we adopt the convex dual martingale approach to construct the optimal consumption strategy and provide an economic interpretation for nearly every object throughout the solution process.

1. Introduction. The consumption-portfolio optimization problem, a basic application of stochastic control theory, has received considerable attention in financial mathematics and microeconomics. As the investigation of this problem is based on utility maximization, it is vital to consider the impact of habit formation. Current preferences should depend in part on past consumption, which might considerably affect current consumption. For example, a higher level of current consumption must increase short-term welfare, which in turn entails a higher level of habit formation, with the long-term result being a decrease in welfare. Hicks[9] introduced the living standard, which is determined by past consumption, to reflect habit formation. Sundaresan[20] introduced habit formation into the utility function and extended the results of the consumption-portfolio optimization problem. Constantinides[2] used habit formation to explain the equity premium puzzle and concluded that individuals are more sensitive to short-term consumption.

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Since Merton\cite{14, 15} pioneered the analysis of consumption-portfolio optimization in the continuous-time model, numerous studies have proposed substantial extensions to this problem. Karatzas et al.\cite{11, 12} and Cvitanić and Karatzas\cite{3, 4} adopted the convex dual martingale approach (see Bismut\cite{1}) to maximize the utility from consumption and terminal wealth under a finite time horizon. Based on those works, Englezos and Karatzas\cite{8} considered habit formation with a single consumption channel and studied the equivalence of the forward-backward stochastic differential equation (FBSDE) and backward stochastic partial differential equation (BSPDE). Detemple and Zapatero\cite{6, 7} adopted Malliavin calculus to study the risk premium in the CAPM model and proved the existence of an optimal consumption portfolio in a model with habit formation. Detemple and Karatzas\cite{5} also studied the optimal consumption-portfolio policy with non-additive habits, finding that consumption can fall below the living standard. Schroder and Skiadas\cite{18} demonstrated the isomorphism of consumption-portfolio optimization models with and without linear habit formation. Subsequent works on habit formation include Munk\cite{16} on stochastic investment opportunities, Muraviev\cite{17} on incomplete markets, and Kakeu and Nguimkeu\cite{10} on exhaustible resource risk-pricing.

To extend this problem for additional dimensions, we seek to maximize the total expected utility of nonhabitual consumption and terminal wealth in a continuous-time framework within a finite time horizon, with the endogenous multiple habit formation of interaction embedded into individual preferences; cf. (6) and (15). That is, each type of consumption (such as that of food, education and medical care) is assumed to be composed of a habitual and a nonhabitual component. It is reasonable to assume that consumers have different preferences with respect to food and education and that except for habitual consumption, welfare is derived entirely from nonhabitual consumption, for example improved diet quality and additional education. It is also reasonable to assume that household members’ habits and consumption interact.

The difficulty of solving this problem arises from the linear dynamic system represented by multiple habit formation of interaction. In contrast to the one-dimensional case (cf. Englezos and Karatzas\cite{8}), how to obtain an explicit solution is unclear, despite that we adopt the same approach to study the global minimum of future consumption, namely subsistence consumption, which can be represented as a proportion of current habitual consumption (thus the living standard) and then removed from the budget. The inability to disentangle the consumption habits for individual categories in the multiple habit formation case means that the utility from all the nonhabitual consumption should be considered as a whole when we follow the approach in, for example, Karatzas et al.\cite{12} to solve the utility maximization problem.

This paper is organised as follows: Section 2 introduces the model dynamics and construct the feasible sets; Section 3 analyses the case with multiple habit formation; Section 4 adopts a general utility function to establish the optimization problem and solves it via the convex duality martingale approach; Section 5 provides an example of the two-dimensional case under CRRA utility and constant parameters.

2. Basic model.

2.1. Complete market and risk-neutral measure. Denote the finite time horizon by $T$, with $T \triangleq [0, T]$. Assume that the financial market is complete, where there is no trading constraint (e.g. a minimum unit or maximum amount). Without
loss of generality, we assume that there is a risk-free asset, the interest rate of which is denoted by \{r_t\}_{t\in\mathcal{T}}}, and a risky asset, the value process of which satisfies the following geometric Brownian motion

\[dS_t = (r_t + \sigma_t \vartheta_t)S_t\,dt + \sigma_t S_t\,dW_t.\]

Here, \{W_t\}_{t\in\mathcal{T}} is the one-dimensional standard Brownian motion defined on a complete filtered probability space \((\Omega, \mathcal{F}_t, \mathbb{P})\) with the real-world probability measure \(\mathbb{P}\). The right-continuous, \(\mathbb{F}\)-completed, natural filtration \(\mathcal{F}_t\) is generated by this Brownian motion. The bounded and strictly positive processes \(\{\sigma_t\}_{t\in\mathcal{T}}\) and \(\{\vartheta_t\}_{t\in\mathcal{T}}\) represent the instantaneous volatility and Sharpe ratio, respectively, which are both \(\mathbb{F}^W\)-progressively measurable. As well, \(\{r_t\}_{t\in\mathcal{T}}\) is assumed to be bounded and \(\mathbb{F}^W\)-progressively measurable.

Denote the discount factor by \(\beta_t \triangleq e^{-\int_0^t r_s\,ds}\). Referring to Shreve [19], by Girsanov’s theorem and Novikov’s condition, the probability measure \(\tilde{\mathbb{P}}\) lead by Radon-Nikodym derivative

\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = \Lambda_t \triangleq e^{-\int_0^t \vartheta_s\,dW_s - \frac{1}{2} \int_0^t \vartheta_s^2\,ds}
\]

is an equivalent martingale measure, while \(\{\tilde{W}_t \triangleq W_t + \int_0^t \vartheta_s\,dW_s\}_{t\in\mathcal{T}}\) is the standard Brownian motion under \(\tilde{\mathbb{P}}\). Since the discounted value process \(\{\beta_t S_t\}_{t\in\mathcal{T}}\) is a \((\mathbb{F}^W, \tilde{\mathbb{P}})\)-martingale, \(\tilde{\mathbb{P}}\) is named the risk-neutral measure.

### 2.2. Survival model and filtration.

Denote the individual lifetime random variable by \(\tau\). Assume that \(\tau\) is independent of \(\mathbb{F}^W\), and the survival probability \(\mathbb{P}(\tau > s|\tau > t) = e^{-\int_t^s \mu_v\,dv}\) for \(s \in (t, T]\), where the mortality force \(\{\mu_t\}_{t\in\mathcal{T}}\) is bounded and positive.

In terms of filtration, let us denote \(\mathbb{F} \triangleq \{\mathcal{F}_t\}_{t\in\mathcal{T}}\), where \(\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \sigma\{\tau > t\}\) contains all past information up to time \(t\), until which point the individual survives.

### 2.3. Wealth dynamics.

The wealth process \(\{X_t\}_{t\in\mathcal{T}}\) is obviously \(\mathbb{F}\)-adapted. The level of investment in the risky asset, denoted by \(\{\pi_t\}_{t\in[0, T]}\), is of the class \(L_2^2(0, T; L^2(\Omega))\); thus, \(\mathbb{E} \int_0^T |\pi_t \sigma_t|^2\,dt < \infty\), and it is \(\mathbb{F}\)-progressively measurable. Denote the consumption rate of the total \(N\) kinds at time \(t\) by the vector\(^\dagger\) \(c_t = (c_{1,t}, \ldots, c_{N,t})^{\text{Tr}}\), where the \(k\)-th opponent \(c_{k,t}\) means the rate of the \(k\)-th consumption, such that the consumption rate process \(\{c_{k,t}\}_{t\in[0, T]}\) is non-negative and locally bounded, continuous on the positive part, and of the class \(L_2^2[0, T; L(\Omega)]\); thus \(\mathbb{E} \int_0^T c_{k,t}\,dt < \infty\), and it is \(\mathbb{F}\)-progressively measurable. Hence, \(\mathbb{E} \int_0^T \langle 1, c_t\rangle\,dt < \infty\), where \(\langle \cdot, \cdot \rangle\) denotes the scalar product, and every opponent of the column vector \(1 \in \mathbb{R}^N\) is 1. In addition, we also consider an exogenous income process \(\{I_t\}_{t\in\mathcal{T}}\) that is non-negative, bounded, and \(\mathbb{F}\)-progressively measurable.

Hence, the wealth process \(\{X_t\}_{t\in\mathcal{T}}\) satisfies the SDE:

\[dX_t = X_t r_t\,dt + \pi_t \sigma_t (\vartheta_t\,dt + dW_t) + I_t\,dt - \langle 1, c_t\rangle\,dt,
\]

thus the discounted wealth process \(\{X_t\}_{t\in\mathcal{T}}\) satisfies the SDE:

\[
d(\beta_t X_t) = \beta_t \pi_t \sigma_t \vartheta_t\,dW_t + \beta_t I_t\,dt - \beta_t \langle 1, c_t\rangle\,dt. \tag{1}
\]

\(^\dagger\)It is common to consider column vectors. The notation \(\text{Tr}\) is used to represent the transpose.
2.4. Feasible set. Let $\beta_s^t \triangleq e^{-\int_t^s r_u \, du}$, $E^t_s \triangleq \beta_s^t \mathbb{P}(\tau > s|\tau > t)$ (namely the actuarial discount factor), $E^t_s(\cdot) \triangleq \mathbb{E}(\cdot \mid \mathcal{F}^W_t \vee \sigma(\tau > t))$ under the measure $\mathbb{P}$. It follows (1) that
\[
\mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} d(\beta_u^t X_u) = \mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} \beta_u^t \pi_u \sigma_u dW_u \\
+ \mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} \beta_u^t I_u du - \mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} \beta_u^t (1, c_u) du. \tag{2}
\]
The left side of (2) is
\[
\mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} d(\beta_u^t X_u) = \int_t^s e^{-\int_t^u r_v \, dv} d(\beta_v^u X_u) = E^s_t X_s - X_t - \int_t^s \mu_u E^s_t X_u du,
\]
where the first equation holds due to the independence of $\tau$ and $\mathbb{F}^W$, and the definition of $\mathbb{E}^t_s$, and the second equation follows from the integration by part. From the definition of $\mathbb{E}^t_s$, we have also $\mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} \beta_u^t \pi_u \sigma_u dW_u = \int_t^s E_u^t \pi_u \sigma_u dW_u$. Substituting $\mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} d(\beta_u^t X_u)$ and $\mathbb{E}^t_s \int_t^s 1_{\{\tau > u\}} \beta_u^t \pi_u \sigma_u dW_u$ into (2) leads to
\[
E^s_t X_s - X_t + \int_t^s \mu_u E^s_t X_u du - \int_t^s E^s_t I_u du + \int_t^s E^s_t (1, c_u) du = \int_t^s E^s_u \pi_u \sigma_u dW_u. \tag{3}
\]
Obviously, the left-hand side is a ($\mathbb{F}^W, \hat{\mathbb{P}}$)-local martingale, which is dependent on the portfolio process. In fact, in addition to the stopping time for boundedness of the stochastic integral, the ruin time and limited debt remaining are also both involved. That is, if $\tau > T$, then $X_s + \int_t^T \beta_u^s I_u du \geq 0$; if $\tau \leq T$, then $X_\tau > -\infty$. However, because of randomness, at any time, the individual can only make an estimation about the future. Under the risk-neutral measure $\hat{\mathbb{P}}$, the point-by-point budget constraint on $[t, T] \times \Omega$ might be\(^2\)
\[
X_s + \hat{\mathbb{E}}_s \int_t^{T \land \tau} \beta_u^s I_u du \geq \hat{\mathbb{E}}_s(1_{\{\tau \leq T\}} \beta_s^\tau X_\tau) > -\infty, \tag{4}
\]
where $\hat{\mathbb{E}}_s(\cdot) \triangleq \hat{\mathbb{E}}(\cdot | \mathcal{F}_s) = \hat{\mathbb{E}}^W_s[\mathbb{E}^W_s(\cdot)]$ and $\hat{\mathbb{E}}^W_s(\cdot) \triangleq \hat{\mathbb{E}}(\cdot | \mathcal{F}^W_s \vee \sigma(\tau > t))$. In the case of no mortality risk, the point-by-point budget constraint degenerates to
\[
X_s + \hat{\mathbb{E}}^W_s \int_t^T \beta_u^s I_u du \geq 0.
\]
The feasible set can be established with a simplification of (4) due to independence.

**Definition 2.1.** $\mathcal{A}(t, X_t)$ is the set of all the consumption-portfolio pairs satisfying the point-by-point budget constraint on $[t, T] \times \Omega$; thus,
\[
\left\{ (\pi, c) : X_s + \hat{\mathbb{E}}^W_s \int_s^T E^s_u I_u du \geq \hat{\mathbb{E}}^W_s \int_s^T \mu_u E^s_u X_u du > -\infty, \forall s \in [t, T] \right\} .
\]

Let us consider the feasible set contingent on a global budget constraint.

\(^2\)If the conditional expectation operator $\hat{\mathbb{E}}_s$ disappears, (4) will satisfy the ruin constraint and limited debt remaining. If only $\hat{\mathbb{E}}_t^W$ disappears, (4) also satisfies the result of non-negative consumption.
Lemma 2.3. \( B(t, X_t) \) is the set of the consumption policies on \([t, T) \times \Omega\), the total discounted value of which falls below the current wealth plus the actuarial present value of income:

\[
\left\{ c : X_t + \mathbb{E}^W_t \int_t^T E^*_s I_s ds - \mathbb{E}^W_t \int_t^T E^*_s (1, c_s) ds \geq \mathbb{E}^W_t \int_t^T \mu_s E^*_s X_s ds > -\infty \right\}.
\]

The requirement of limited debt remaining is also involved in Definition 2.2. If mortality risk were not considered, the global budget constraint would degenerate to the classical form:

\[
X_t + \mathbb{E}^W_t \int_t^T \beta^*_s I_s ds \geq \mathbb{E}^W_t \int_t^T \beta^*_s (1, c_s) ds.
\]

Since all the contingent claims can be hedged in the complete market, referring to Karatzas and Shreve\cite{karatzast:1998}, we have the following lemma:

Lemma 2.3 (Existence of the portfolio hedging against financial risk).

\[
\forall (\pi, c) \in A(t, X_t), \quad s.t. \quad c \in B(t, X_t);
\]
\[
\forall c \in B(t, X_t), \quad \exists \pi \in L^2(t, T; L^2(\Omega)), \quad s.t. \quad (\pi, c) \in A(t, X_t).
\]

Moreover, for a problem under a finite time horizon, it is not necessary to consider mortality risk in the global budget constraint:

Lemma 2.4. If \( \mathbb{P}(\tau > T|\tau > t) > 0 \), then the feasible set

\[
B(t, X_t) = \left\{ c : X_t + \mathbb{E}^W_t \int_t^T \beta^*_s I_s ds \geq \mathbb{E}^W_t \int_t^T \beta^*_s (1, c_s) ds \right\}.
\]

We provide the proof of Lemma 2.3 and Lemma 2.4 in Section A.1.

3. Multiple habit formation.

3.1. Definition and dynamics. As noted above, the \( k \)-th type of consumption \( c_{k,t} \) should be composed of the habitual component \( H_{k,t} \geq 0 \) and the nonhabitual component \( c_{k,t} - H_{k,t} \geq 0 \), where the habitual component is endogenously formulated by the weighted average of all kinds of past consumption:

\[
H_{k,t} = H_{k,0} e^{-\int_0^t a_{kk,v} dv} + \int_0^t b_{kk,s} c_{k,s} e^{-\int_0^s a_{kk,v} dv} ds
\]

\[
+ \sum_{j \neq k} \int_0^t [b_{kj,s}(c_{j,s} - H_{j,s}) + (b_{kj,s} - a_{kj,s}) H_{j,s}] e^{-\int_0^s a_{kk,v} dv} ds
\]

\[
= H_{k,0} e^{-\int_0^t a_{kk,v} dv} + \int_0^t \left( \sum_{j=1}^N b_{kj,s} c_{j,s} - \sum_{j \neq k} a_{kj,s} H_{j,s} \right) e^{-\int_0^s a_{kk,v} dv} ds.
\]

For \( k = 1, \ldots, N \), \( H_{k,0} \geq 0 \) denotes the initial value of the habitual component of the \( k \)-th consumption. \( a_{kk,v} \geq 0 \) denotes the habit attenuation rate (namely, the living standard) of the \( k \)-th consumption. \( b_{kk,v} \geq 0 \) reflects the positive impact of the \( k \)-th past consumption on the \( k \)-th habitual consumption, which has been adjusted by the attenuation. For \( j \neq k \), \( b_{kj,v} \geq 0 \) and \( b_{kj,v} - a_{kj,v} \geq 0 \) reflects the positive interaction among the different types of consumption, both habitual and
nonhabitual\(^3\). Obviously, when all the processes \(\{a_{k,j,t}|t \in \mathcal{T}\}\) of \(j \neq k\) vanish, the effects of the habitual and nonhabitual components of the \(j\)-th past consumption on the \(k\)-th current consumption become the same. Then, (6) degenerates into \(N\) relatively independent components, as all \(\{b_{k,j,t}|t \in \mathcal{T}\}\) of \(j \neq k\) vanish. On the other hand, when the processes \(\{b_{k,j,t}|t \in \mathcal{T}\}\) and \(\{b_{k,j,t}|t \in \mathcal{T}\}\) vanish, the endogeneity and interaction disappear. In that case, embedded habit formation becomes trivial, as all the initial values \(H_{k,0} = 0\), meaning that the utility maximization problem degenerates to the classical Merton model.

3.2. Properties. Let us investigate the vector \(\hat{H}_t \equiv (H_{1,t}, \ldots, H_{N,t})^T\). By (6), we have the linear dynamic system\(^4\):

\[
d\hat{H}_t = (B_t c_t - A_t \hat{H}_t) dt,
\]

where the matrix processes \(B_t = [b_{k,j}; t, j = 1, \ldots, N]\) and \(A_t = [a_{k,j}; t, j = 1, \ldots, N]\) on \(\mathcal{T} \times \Omega\) are both continuous, bounded, \(\mathbb{F}^W\)-progressively measurable and independent of lifetime. Hence, under the filtration \(\mathbb{F}^W\), (6) is the implicit unique solution of (7). The functional fixed point \(\check{H} \equiv \hat{c} \equiv H(\hat{c})\) exists and satisfies the linear ordinary differential equation (ODE):

\[
d\check{H}_t = (B_t - A_t)\check{H}_t dt.
\]

Lemma 3.1. If \(B_t \equiv B\) and \(A_t \equiv A\) are constant matrix processes, (8) has the following explicit solution:

\[
\check{H}_t = e^{(B-A)t} \check{H}_0 \equiv \sum_{k=0}^{\infty} \frac{t^k}{k!} (B - A)^k \check{H}_0.
\]

Although the right-hand side is a series, it can be calculated by finite algebraic operations (see the proof in Section A.2). Denote the solution of (8) subject to \(\check{H}_t = h \equiv (h_1, \ldots, h_N)^T\) by \(\{\check{H}_s^{[H]}|s \in [t,T]\}\).

Lemma 3.2. Given the initial condition \(\check{H}_t = H_t\), if \(c \gg H\), holds\(^5\) on \([t, s] \times \Omega\), then \(c_s \gg \check{H}_s\); thus, \(\{c_s : \forall u \in [t, s], c_s \gg H_u\} \subset \{c_s : c_s \gg \check{H}_s^{[H]}\}\).

Lemma 3.3. Assume that \(c \gg H\) on \([t, T] \times \Omega\). Then \(c \gg \check{H}_s^{[H]}\) on \([t, T] \times \Omega\).

There is a simple proof of the one-dimensional case (see Englezos and Karatzas [8]). We provide the proof of this high-dimensional version in Section A.3, with the proof of Lemmas 3.4 and 3.5 therein. It is clear that Lemma 3.3 is a direct inference from Lemma 3.2. That is, as nonhabitual consumption is non-negative, the functional fixed point \(\check{c} \equiv \check{H}\) of the initial condition \(\check{H}_t = H_t\) is the point-by-point minimum of consumption during period \([t, T]\). This is the reason that the minimum consumption is also called “subsistence consumption”\(^6\).

\(^3\)For example, when having spent a relatively high amount on education in the past, including the cost of higher education \(H_{j,s}\) and additional professional training \(c_{j,s} - H_{j,s}\), an individual usually has a relatively high habitual cost \(H_{k,t}\) of medical care.

\(^4\)The habit formation parameters can take the form of stochastic processes driven by \(\{W_t|t \in \mathcal{T}\}\). They are assumed to be independent of consumption \(c\) and its habitual component \(H\).

\(^5\)Here, \(\gg\) denotes a partial order of two vectors, where each opponent of the former is strictly greater than the corresponding opponent of the latter. Similarly, \(\geq\) means that each opponent of the former is no less than the corresponding opponent of the latter.

\(^6\)Note that subsistence consumption only exists in the consumer’s expectation. Once realized, it will become habitual consumption.
3.3. Adjustment factor and feasible budget. Crucially, the subsistence consumption provided by Lemma 3.3 implies that the feasibility of the budget should be considered. That is, there exists a feasible set of the initial condition triplet \((t, X_t, H_t)\) such that subsistence consumption \(\hat{c} \equiv \hat{H}\) is admissible.

By Lemma 2.4, we have the following necessary condition:

\[
X_t + \tilde{E}_t^W \int_t^T \beta_s W_s ds - \tilde{E}_t^W \int_t^T \beta_s^T (1, \hat{H}_s^{t,h}) ds \geq \tilde{E}_t^W \int_t^T \beta_s^T (1, c_s - \hat{H}_s^{t,h}) ds \geq 0.
\]

(9)

In particular, if and only if the nonhabitual consumption vanishes after time \(t\), i.e. \(c = H\) over \([t, T]\) happens to be the subsistence consumption, does the second inequality hold as an equality. To verify the sufficiency of (9) for the feasibility condition that \(c - H \geq 0\), we make the following transform.

**Lemma 3.4.** For \(0 \leq t \leq s \leq T\), Jacobi matrix\(^7\) \(\partial_h \hat{H}_s^{t,h} \triangleq \frac{\partial (\hat{H}_1^{t,h}, \ldots, \hat{H}_N^{t,h})}{\partial (h_1, \ldots, h_N)}\) is independent of the initial condition \(h\) and has the decomposition \(\partial_h \hat{H}_s^{0,h} = (\partial_h \hat{H}_s^{1,h})(\partial_h \hat{H}_s^{0,h})\), so that

\[
\hat{H}_s^{t,h} = (\partial_h \hat{H}_s^{1,h}) h.
\]

(10)

**Lemma 3.5.** The adjustment factor of habit formation \(\{\Gamma_t | t \in T\}\) is the bounded vector process of strictly positive opponents:

\[
\Gamma_t \triangleq 1 + B_t^T \tilde{E}_t^W \int_t^T \beta_s^T (\partial_h \hat{H}_s^{t,h}, 1) ds,
\]

(11)

so that

\[
\tilde{E}_t^W \int_t^T \beta_s^T (\Gamma_s, c_s - H_s) ds = \tilde{E}_t^W \int_t^T \beta_s^T (1, c_s - \hat{H}_s^{t,h}) ds.
\]

(12)

Lemma 3.4 implies that Jacobi matrix \(\partial_h \hat{H}_s^{t,h}\) acts as the exponential factor (cf. Lemma 3.1). Since \(\{\beta_t \Lambda_t | t \in T\}\) is the state price density process (see Shreve[19]), while in the one-dimensional case, the scalar process \(\{\beta_t \Lambda_t \Gamma_t | t \in T\}\) is named the “adjusted state price density process” (see Englezos and Karatzas[8]), the vector process \(\{\beta_t \Lambda_t \Gamma_t | t \in T\}\) provided by Lemma 3.5 is a further extension for multiple habit formation of interaction.

Hence, over the feasible set \(B(t, X_t)\), (9) is equivalent to

\[
X_t + \tilde{E}_t^W \int_t^T \beta_s W_s ds - (H_t, \tilde{E}_t^W \int_t^T \beta_s^T (\partial_h \hat{H}_s^{t,h}, 1) ds)
\]

\[
\geq \tilde{E}_t^W \int_t^T \beta_s^T (\Gamma_s, c_s - H_s) ds \geq 0.
\]

(13)

Lemma 3.4 implies that the left-hand side of (13) is a linear combination of the initial condition \((X_t, H_t)\). Specifically, when (13) becomes an equality, the admissible consumption is unique (that is, all nonhabitual consumption must vanish) almost for certain, and we do not consider this trivial case further.

**Definition 3.6.** For \(t \in T\), the budget for nonhabitual consumption is defined by

\[
g_t^{X_t,H_t} \triangleq X_t + \tilde{E}_t^W \int_t^T \beta_s W_s ds - (H_t, \tilde{E}_t^W \int_t^T \beta_s^T (\partial_h \hat{H}_s^{t,h}, 1) ds).
\]

(14)

\(^7\)If the gradient operator \(\nabla h \triangleq (\frac{\partial}{\partial h_1}, \ldots, \frac{\partial}{\partial h_N})^T\) is used, then \((\partial_h \hat{H}_s^{t,h})^T = \nabla h (\hat{H}_s^{t,h})^T\).
while the feasible set is defined by the convex open set
\[ D_t \triangleq \{(X_t, H_t) \in \mathbb{R} \times \mathbb{R}_+^N : g_t^{X_t, H_t} > 0\}. \]

Obviously, for \((X_t, H_t) \in D_t\), there exist admissible consumption policies such that the nonhabitual components are non-negative, e.g.
\[ c_{k,s} - H_{k,s} = \frac{g_t^{X_t, H_t}}{\Gamma_{k,s} \mathbb{E}_t^W \int_t^T \beta_u^t du} > 0, \quad s \in [t,T], \quad k = 1, \ldots, N, \]
where \(\Gamma_{k,t}\) is the \(k\)-th opponent of \(\Gamma_t\).

4. Utility maximization problem and solution.

4.1. Utility function and convex dual. Adopting the optimal consumption-portfolio pair is assumed to be to maximize total expected utility under von Neumann-Morgenstern preferences
\[ J(t, X_t, H_t; \pi, c) \triangleq \mathbb{E}_t \left[ \sum_{k=1}^N \int_t^{T \wedge \tau} e^{-\int_t^s \delta_t \, du} u_k(s, c_{k,s} - H_{k,s}) \, ds \right. \]
\[ \left. + \mathbb{I}_{\{T > \tau\}} e^{-\int_t^T \delta_t \, du} u_0(T, X_T) \right], \quad (15) \]
with the non-negative subjective intertemporal utility discount factor \(\{\delta_t| t \in \mathcal{T}\}\).

For each \(k = 0, 1, \ldots, N\), the general utility function \(u_k: \mathcal{T} \times \mathbb{R}_+ \to \mathbb{R}\) satisfies the criterion that \(u_k(t, \cdot) \in C^3(\mathbb{R}_+)\) is monotonically increasing and strictly concave, with the marginal utility function\(^8\) satisfying the Inada condition:
\[ \lim_{c \searrow 0} \mathcal{D}_c u_k(t, c) = +\infty, \quad \lim_{c \to +\infty} \mathcal{D}_c u_k(t, c) \equiv 0. \]

Denote the inverse of the marginal utility function \(\mathcal{D}_c u_k(t, \cdot)\) by \(\mathcal{I}_k(t, \cdot)\). Furthermore, we make some technical assumptions:
\[ \exists l_{k,1}, p_{k,1}, q_{k,1} > 0, \quad \text{s.t.} \quad \sup_{t \in \mathcal{T}} \mathcal{I}_k(t, y) \leq l_{k,1} + p_{k,1} y^{-q_{k,1}}; \quad (16) \]
\[ \exists l_{k,2}, p_{k,2}, q_{k,2}, p_{k,3}, q_{k,3} > 0, \quad \text{s.t.} \quad \sup_{t \in \mathcal{T}} |u_k(t, \mathcal{I}_k(t, y))| \leq l_{k,2} + p_{k,2} y^{-q_{k,2}} + p_{k,3} y^{q_{k,3}}; \quad (17) \]
\[ u_k(t, c_{k,t} - H_{k,t}) = -\infty, \quad \text{if } c_{k,t} \leq H_{k,t}. \quad (18) \]

In addition, referring to Bismut [1] and Cvitanic and Karatzas [3], we define the convex dual of the utility function \(u_k(t, \cdot)\) by
\[ \hat{u}_k(t, y) \triangleq \max_{c > 0} \{u_k(t, c) - yc\}, \quad (t, y) \in \mathcal{T} \times \mathbb{R}_+. \quad (19) \]
This is the conjugate function of \(-u_k(t, -c)\) of the negative variable \(c\), namely a Legendre-Fenchel transform. Under the assumption of the strict concavity of the utility function and \(\mathcal{D}_c u_k(t, \cdot) \rightharpoonup \mathbb{R}_+, (19)\) can be represented by
\[ \hat{u}_k(t, y) = u_k(t, \mathcal{I}_k(t, y)) - y \mathcal{I}_k(t, y), \quad (t, y) \in \mathcal{T} \times \mathbb{R}_+. \quad (20) \]
with \(\mathcal{D}_y \hat{u}_k(t, y) = -\mathcal{I}_k(t, y)\).

---

\(^8\)In an effort to introduce as few new notations as possible, we use the operator \(\mathcal{D}\) of the Jacobi matrix to also denote the partial derivative.
4.2. Problem formulation and decomposition. Since the integrability of the negative part is required, i.e.
\[
\mathbb{E}_t \int_t^{T \wedge \tau} e^{-\int_t^s \delta_r dw_r} u_k(s, c_{k,s} - H_{k,s}) ds < \infty, \quad k = 1, \ldots, N,
\] (21)
based on \((X_t, H_t) \in \mathcal{D}_t\), feasible sets should be introduced, where
\[
\mathcal{A}'(t, X_t, H_t) \triangleq \{(\pi, c) \in \mathcal{A}(t, X_t) : (21) \text{ is satisfied.}\}
\]
\[
\mathcal{B}'(t, X_t, H_t) \triangleq \{c \in \mathcal{B}(t, X_t) : (21) \text{ is satisfied.}\}
\]
Under assumption (18) and continuity on the positive part of consumption, \(c \gg H\) almost for certain is involved in determining the integrability of the negative part (due to the strict concavity of the utility function). Moreover, the feasible sets \(\mathcal{A}'(t, X_t, H_t)\) and \(\mathcal{B}'(t, X_t, H_t)\) are both convex\(^9\) and satisfy the relation described by Lemma 2.3 to guarantee the existence of the hedging portfolio. Hence, we investigate the utility maximization problem over the feasible set \(\mathcal{B}'(t, X_t, H_t)\). The dynamic programming value function that maximizes (15) is
\[
V(t, X_t, H_t) \triangleq \max_{(\pi, c) \in \mathcal{A}'(t, X_t, H_t)} J(t, X_t, H_t; \pi, c) \equiv \max_{c \in \mathcal{B}'(t, X_t, H_t)} J(t, X_t, H_t; \pi^*(c), c),
\] (22)
where \(\pi^*(c)\) is the hedging portfolio for the consumption strategy \(c\).

By the independence of the Brownian motion and lifetime, the randomness in (15) caused by \(\tau\) can be eliminated. Thus, we have
\[
J_1(t, X_{1,t}, H_t; \pi_1, c) \triangleq \mathbb{E}_t^W \sum_{k=1}^N \int_t^T e^{-\int_t^s (\mu_u + \delta_r) dw_r} u_k(s, c_{k,s} - H_{k,s}) ds,
\] (23)
\[
J_2(t, X_{2,t}; \pi_2) \triangleq \mathbb{E}_t^W \left[ e^{-\int_t^T (\mu_u + \delta_r) dw_r} u_0(T, X_{2,T}) \right],
\] (24)
where \((X_{1,t}, H_t) \in \mathcal{D}_t\) and \(X_{2,t} > 0\). The wealth processes \(\{X_{1,t} | t \in T\}\) and \(\{X_{2,t} | t \in T\}\) follow the dynamics below:
\[
\begin{align*}
\tilde{d}X_{1,t} &= X_{1,t} r_1 dt + \pi_{1,t} \sigma_1 dW_2 + I_t dt - (1, c_t) dt, \\
\tilde{d}X_{2,t} &= X_{2,t} r_2 dt + \pi_{2,t} \sigma_2 d\tilde{W}_1.
\end{align*}
\]

Then, the primal utility maximization problem can be decomposed into two subproblems, and we define their respective value functions as follows:

**Definition 4.1.** The value function when maximizing the utility of nonhabitual consumption is denoted by
\[
V_1(t, X_{1,t}, H_t) \triangleq \max_{(\pi_1, c) \in \mathcal{A}'(t, X_{1,t}, H_t)} J_1(t, X_{1,t}, H_t; \pi_1, c) \equiv \max_{c \in \mathcal{B}'(t, X_{1,t}, H_t)} J_1(t, X_{1,t}, H_t; \pi_1^*(c), c).
\]

**Definition 4.2.** The value function when maximizing the utility of terminal wealth is denoted by
\[
V_2(t, X_{2,t}) \triangleq \max_{\pi_2 \in \mathbb{L}^2_{d}(t, T, L^2(\Omega))} J_2(t, X_{2,t}; \pi_2) \equiv J_2(t, X_{2,t}; \pi_2^*).
\]

---

\(^9\)This is because of the linearity of the budget with respect to the initial condition and of habit formation in consumption; see Englezos and Karatzas\([8]\). We omit a similar proof in this paper.
Lemma 4.3. For \((X_t, H_t) \in \mathcal{D}_t\), given \(c \in B'(t, X_t, H_t)\) with its hedging portfolio \(\pi^*(c)\), there exists such a pair \((\pi_1^*(c), \pi_2^*)\) that

\[
\forall s \in [t, T), \quad X_s = X_{1,s} + X_{2,s}, \quad \pi_s^* = \pi_{1,s}^* + \pi_{2,s}^*,
\]

\[
(\pi_1^*(c), c) \in \mathcal{A}'(s, X_{1,s}, H_s), \quad X_T = \frac{X_{2,s}}{E_T^2}.
\]

Lemma 4.3 is the dynamic version of Karatzas et al.\[12\], which guarantees the existence of wealth decomposition in the dynamics, so that for all \(s \in [t, T)\) and \((X_s, H_s) \in \mathcal{D}_s\), there exists a pair \((X_{1,s}, X_{2,s})\) for consumption and termination such that

\[
J(s, X_s, H_s; \pi^*(c), c) = J_1(s, X_{1,s}, H_s; \pi_1^*(c), c) + J_2(s, X_{2,s}; \pi_2^*).
\]

Conversely, for \((X_{1,t}, H_t) \in \mathcal{D}_t\) and \(X_{2,t} > 0\), the optimal investment policies of the two subproblems are additive, as is the wealth process. The total portfolio must also be the hedging strategy, while the total consumption \(c \in B'(t, X_{1,t} + X_{2,t}, H_t)\). Then, set \(X_t = X_{1,t} + X_{2,t}\). Denote the optimal consumption in the primal problem by \(c^*\) and the optimal consumption in the first subproblem in Definition 4.1 by \(c^{**}\). Hence, we have

\[
V(t, X_t, H_t) = J(t, X_t, H_t; \pi^*(c^*), c^*)
\]
\[
= J_1(t, X_{1,t}, H_t; \pi_1^*(c^*), c^*) + J_2(t, X_{2,t}; \pi_2^*)
\]
\[
\leq V_1(t, X_{1,t}, H_t) + V_2(t, X_{2,t})
\]
\[
= J_1(t, X_{1,t}, H_t; \pi_1^*(c^{**}), c^{**}) + J_2(t, X_{2,t}; \pi_2^*)
\]
\[
= J(t, X_t, H_t; \pi^*(c^{**}), c^{**})
\]
\[
\leq V(t, X_t, H_t)
\]

This means that the primal optimization problem can be decomposed into two subproblems for functional optimization (e.g., Definitions 4.1 and 4.2) and a real variable optimization subproblem for the initial division of wealth; thus,

\[
V(t, X_t, H_t) = \max_{x_{1,t} + x_{2,t} \in \mathcal{D}_t, x_{2,t} > 0} [V_1(t, X_{1,t}, H_t) + V_2(t, X_{2,t})]. \quad (25)
\]

4.3. Solution of subproblems. Since the interaction of habit formation terms cannot be decomposed into a sequence of individual types of habitual consumption as is usually done, we have to solve for the optimal consumption policy as a whole. Denote \(u_k(t, c) \triangleq u_k(t, c_k)\), the function \(\tilde{u}_k(t, y) \triangleq \tilde{u}_k(t, y_k)\) and the function vector \(\mathcal{I}(t, y) \triangleq (\mathcal{I}_1(t, y_1), \ldots, \mathcal{I}_N(t, y_N))^{\text{Tr}}\) of \(y \triangleq (y_1, \ldots, y_N)^{\text{Tr}} \in \mathbb{R}_+^N\). Note that

\[
V_1(t, X_{1,t}, H_t) = \mathbb{E}_t^W \int_t^T e^{-\int_t^s (\mu_+ + \delta_s)ds} \sum_{k=1}^N u_k(s, c^{**}_s - H_s)ds. \quad (26)
\]

By Walras’ law, the optimal nonhabitual consumption satisfies the necessary condition

\[
g_t^{X_{1,t}, H_t} = \mathbb{E}_t^W \int_t^T \beta_t^s (\mathcal{I}_s, c^{**}_s - H_s)ds. \quad (27)
\]
Since \( u_k(t, c_{k,t} - H_{k,t}) \leq \tilde{u}_k(t, y) + y(c_{k,t} - H_{k,t}) \) holds for all \((t, y) \in \mathcal{T} \times \mathbb{R}_+\) by (19), using the notation \( K_s^t \triangleq e^{\int_t^s (\mu_x + \delta_x) ds} \beta_s \) we obtain
\[
\mathbb{E}^W_t \int_t^T e^{-f'(\mu_x + \delta_x) ds} \sum_{k=1}^N u_k(s, c_s - H_s) ds
\]
\[
\leq \mathbb{E}^W_t \int_t^T e^{-f'(\mu_x + \delta_x) ds} \sum_{k=1}^N \tilde{u}_k(s, y_{1,t} K_s^t \Gamma_s) ds + \frac{y_{1,t}}{N} \mathbb{E}^W_t \int_t^T \beta_s \Lambda_s \langle \Gamma_s, c_s - H_s \rangle ds
\]
\[
\leq \mathbb{E}^W_t \int_t^T e^{-f'(\mu_x + \delta_x) ds} \sum_{k=1}^N u_k(s, \mathcal{I}(s, y_{1,t} K_s^t \Gamma_s)) ds
\]
\[
+ y_{1,t} \left[ g_k(x, H_s) - \mathbb{E}^W_t \int_t^T \beta_s \langle \Gamma_s, \mathcal{I}(s, y_{1,t} K_s^t \Gamma_s) \rangle ds \right].
\]
(28)

This holds for all \( c \in \mathcal{B}'(t, X_{1,t}, H_t) \), meaning that \( c^*_s - H_s = \mathcal{I}(s, y_{1,t} K_s^t \Gamma_s) \) conditional on \( \mathcal{F}_t \); thus, \( c^*_s | \mathcal{F}_t = H_{k,s} + \tilde{I}_k(s, y_{1,s} K_s^t \Gamma_{k,s}). \)

To evaluate \( y_{1,t} \), substituting \( c^* \) back into (27), let us introduce the auxiliary function \( \mathcal{X}_1 : \mathcal{T} \times \mathbb{R}_+ \to \mathbb{R}_+ \):
\[
\mathcal{X}_1(t, y) \triangleq \mathbb{E}_t^W \int_t^T \beta_s \langle \Gamma_s, \mathcal{I}(s, y K_s^t \Gamma_s) \rangle ds.
\]
(29)

By (16) and the existence of the moment generating function of the Gaussian distribution, the uniform convergence of (29) can be verified. Then, due to the property of the marginal utility function, \( \mathcal{X}_1(t, \cdot) \) is monotonically decreasing and continuously differentiable, which guarantees the existence of the inverse function \( \mathcal{Y}_1(t, \cdot) \in C^1(\mathbb{R}_+) \).

**Theorem 4.4.** The optimal consumption of the subproblem in Definition 4.1 is
\[
c^*_s | \mathcal{F}_t = H_s + \mathcal{I}(s, y_{1,t} K_s^t \Gamma_s), \quad y_{1,t} = \mathcal{Y}_1(t, g_t^{X_s^{1,t}, H_t}),
\]
(30)
while the optimal portfolio is a hedging strategy such that \( X_{1,T} = 0 \), a.s.\(^{10}\)

In fact, \( y_{1,t} \) coincides with the shadow price corresponding to current wealth \( X_{1,t} \). Let us introduce another auxiliary function \( G_1 : \mathcal{T} \times \mathbb{R}_+ \to \mathbb{R} :\)
\[
G_1(t, y) \triangleq \mathbb{E}_t^W \int_t^T e^{-f'(\mu_x + \delta_x) ds} \sum_{k=1}^N u_k(s, \mathcal{I}(s, y K_s^t \Gamma_s)) ds.
\]
(31)
The uniform convergence of (31) can be provided by (17), similar to that of (29). Combining this with (26) and (30), we obtain
\[
V_1(t, X_{1,t}, H_t) = G_1(t, y_{1,t}) = G_1(t, \mathcal{Y}_1(t, g_t^{X_s^{1,t}, H_t})).
\]
(32)
From (20), we have \( \partial_y G_1(t, y) = y \partial_y \mathcal{X}_1(t, y). \) Hence,
\[
\partial_{X_1} V_1(t, X_{1,t}, H_t) = \mathcal{Y}_1(t, g_t^{X_s^{1,t}, H_t}) = y_{1,t}.
\]
(33)

\(^{10}\)Because the utility evaluation of terminal wealth and investment is zero in the subproblem from (23), due to the “ruin constraint” \( X_{1,T} \geq 0 \) from (13), Walras’ law shows that all the wealth should be consumed, even though in the final infinitesimal period (namely the left neighborhood of the finite time-horizon \( T \)). Consequently, it’s optimal that no wealth remains at \( T \) almost for certain.
In terms of the subproblem in Definition 4.2, we have similar results as follows:

**Theorem 4.5.** Define the auxiliary function $X_2 : \mathcal{T} \times \mathbb{R}_+ \xrightarrow{ontos} \mathbb{R}_+$ by

$$X_2(t, y) \triangleq \mathbb{E}^W_t \left[ \beta_T \mathcal{I}_0(T, yK_T^y) \right],$$

meaning that $X_2(t, \cdot)$ is monotonically decreasing and continuously differentiable, with the inverse function $Y_2(t, \cdot) \in C^1(\mathbb{R}_+)$. Then, the optimal portfolio of the subproblem in Definition 4.2 is a hedging strategy such that

$$X_{2,T} = \mathcal{I}_0(T, y_{2,T}K_T^y) \quad a.s., \quad y_{2,T} = Y_2(t, X_{2,t}).$$

Furthermore, define $G_2 : \mathcal{T} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$G_2(t, y) \triangleq \mathbb{E}^W_t \left[ e^{-\int_t^T (\mu_v + \delta_v) dv} u_0(T, \mathcal{I}_0(T, yK_T^y)) \right],$$

then the value function of this subproblem is

$$V_2(t, X_{2,t}) = G_2(t, y_{2,t}) = G_2(t, Y_2(t, X_{2,t})), \quad (37)$$

with the corresponding shadow price

$$\mathcal{D}_{X_2}, V_2(t, X_{2,t}) = Y_2(t, X_{2,t}) = y_{2,t}. \quad (38)$$

### 4.4. Solution of the primal problem

Since the feasible budget set is an open set in the non-trivial case in which the Inada condition is satisfied, if the optimal solution of the primal optimization problem exists, the optimal division of wealth $(X_{1,t}^*, X_{2,t}^*)$ must be an interior solution. $(33)$ and $(38)$ prove that $y_{1,t}$ and $y_{2,t}$ are the respective shadow prices of the two subproblems. The concavity of the value function\(^\text{11}\) can be deduced by the monotonicity of $X_1(t, \cdot)$ and $X_2(t, \cdot)$. By the equal marginal principle, $y_{1,t} = y_{2,t}$ is the sufficient and necessary condition for the optimal division of wealth. Then the optimal pair $(X_{1,t}^*, X_{2,t}^*)$ can be determined by $(29)$ and $(34)$. Substitute these back into Theorem 4.4. The optimal consumption $c^*$ of the primal problem is clear.

Denote $y_{1,t} = y_{2,t} = y_t$. Based on $(29)$ and $(34)$, let us introduce the auxiliary function $\mathcal{X} : \mathcal{T} \times \mathbb{R}_+ \xrightarrow{ontos} \mathbb{R}_+$:

$$\mathcal{X}(t, y) \triangleq X_1(t, y) + X_2(t, y),$$

meaning that $\mathcal{X}(t, \cdot)$ is monotonically decreasing and continuously differentiable, with the inverse function $\mathcal{Y}(\cdot, \cdot) \in C^1(\mathbb{R}_+)$. By

$$\mathcal{X}(t, y_t) = X_1(t, y_{1,t}) + X_2(t, y_{2,t}) = g_{t}^H + X_{1,t}^*, \quad X_{2,t}^* = g_{t}^H,$$

we have $y_t = \mathcal{Y}(t, g_{t}^H)$. Similarly, let us define the auxiliary function $G : \mathcal{T} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$G(t, y) \triangleq G_1(t, y) + G_2(t, y),$$

satisfying $\mathcal{D}_y G(t, y) = y \mathcal{D}_y \mathcal{X}(t, y)$. From $(25)$, $(32)$ and $(37)$, we finally obtain

$$V(t, X_t, H_t) = G(t, y_t), \quad \mathcal{D}_X V(t, X_t, H_t) = y_t.$$

Consequently, conditioned on $\mathcal{F}_t$, the optimal consumption strategy and the corresponding terminal wealth satisfy

$$\begin{align*}
\mathcal{C}^s_{\mathcal{F}_t} &= H_t + \mathcal{I}(s, y_t K_t^y) ; \\
X_{2,T} &= \mathcal{I}_0(T, y_t K_T^y), \quad a.s.
\end{align*}$$

\(^{11}\) More strictly, the joint concavity of $V_1(t, \cdot, \cdot)$ should be verified; see Englezos and Karatzas[8].
It following from (7) that
\[ H_s = H_t + \int_t^s [B_v (c_v - H_v) + (B_v - A_v) H_v] dv. \] (39)
Conditioned on \( F_t \) on both sides, similar to the derivation in Section A.3, substituting \( c_v^* \) into (39) leads to the following habitual consumption therein
\[ H_s = (\mathcal{D}_h \hat{H}_s^{t,h}) H_t + \int_t^s (\mathcal{D}_h \hat{H}_s^{t,h}) B_v T(v, y_t K_v T_v) dv, \]
where the first term is the subsistence consumption, and the second term completely relies on the nonhabitual consumption.

5. Example: 2-dimensional consumption and CRRA utility. Following the assumption of constant model parameters as in Lemma 3.1, here we investigate the 2-dimensional case. Assume that \( |\text{Tr}(B - A)|^2 > 4 \det |B - A| \), meaning that the eigenvalues are different real numbers\(^{12}\):
\[ \lambda = \frac{1}{2} \text{Tr} (B - A) + \frac{1}{2} \sqrt{[\text{Tr}(B - A)]^2 - 4 \det |B - A|}. \]
The two eigenvalues are placed in a suitable order \((\lambda_1, \lambda_2)\) such that the matrix of eigenvectors becomes
\[ (v_1, v_2) = \begin{bmatrix} (b_{22} - a_{22}) - \lambda_1 & -(b_{12} - a_{12}) \\ -(b_{21} - a_{21}) & (b_{11} - a_{11}) - \lambda_2 \end{bmatrix}, \]
\[ \det |(v_1, v_2)| = 2 \det |B - A| - [(b_{11} - a_{11}) \lambda_1 + (b_{22} - a_{22}) \lambda_2] \neq 0. \]
Referring to the proof of Lemma 3.1, we denote the solution of \( \mathcal{D}_h \hat{H}_s^{t,h} \) by
\[ \mathcal{D}_h \hat{H}_s^{t,h} = (v_1, v_2)[\text{diag}(e^{\lambda_1(s-t)}, e^{\lambda_2(s-t)})]Q, \]
with the decomposition of the \( N \)-th order identity matrix \( E = (v_1, v_2)Q \). Hence, \( \mathcal{D}_h \hat{H}_s^{t,h} = [\det |(v_1, v_2)|]^{-1}(v_1, v_2)[\text{diag}(e^{\lambda_1(s-t)}, e^{\lambda_2(s-t)})]|B - A - \text{diag}(\lambda_2, \lambda_1)|]. \)
Substitute back into Lemma 3.5, and assume the constant interest rate \( r_i \equiv r \).
Then, the adjustment factor identifies
\[ \Gamma_t = 1 + [\det |(v_1, v_2)|]^{-1}B^T [B - A - \text{diag}(\lambda_2, \lambda_1)]^\text{Tr} \left[ \text{diag} \left( \frac{e^{(\lambda_1-r)(T-t)} - 1}{\lambda_1 - r}, \frac{e^{(\lambda_2-r)(T-t)} - 1}{\lambda_2 - r} \right) \right] (v_1, v_2)^\text{Tr} 1, \]
In the same way, by Definition 3.6, we obtain the budget for nonhabitual consumption, i.e. the risk-neutral discounting adjusted wealth
\[ g_t X_t, H_t = X_t + \tilde{E}_t^W \int_t^T \beta_s I_s ds - [\det |(v_1, v_2)|]^{-1}H_t^s B [B - A - \text{diag}(\lambda_2, \lambda_1)]^\text{Tr} \left[ \text{diag} \left( \frac{e^{(\lambda_1-r)(T-t)} - 1}{\lambda_1 - r}, \frac{e^{(\lambda_2-r)(T-t)} - 1}{\lambda_2 - r} \right) \right] (v_1, v_2)^\text{Tr} 1. \]
In the optimization problem, we adopt the utility function (for \( k = 0, 1, 2 \)):
\[ u_k(t, c) = \begin{cases} \frac{w_k}{1+\gamma} c^{1-\gamma}, & \gamma \in \mathbb{R}_+ \setminus \{1\}; \\ w_k \ln c, & \gamma = 1, \end{cases} \]
\(^{12}\)The operator \( \text{Tr}(\cdot) \) denotes the trace of the matrix, and \( \det |\cdot| \) denotes the determinant.
where \( \gamma \) is the constant relative risk aversion (CRRA) coefficient, and \( \{w_k|k = 0, 1, 2\} \) are weighted coefficients. Hence, \( \mathcal{I}_k(t, y) = w_k^\frac{1}{\gamma} y^{-\frac{1}{\gamma}} \). Assume that \( \vartheta \equiv \bar{\vartheta} \) over \( T \). Note that

\[
(K^t_k)^{-\frac{1}{\gamma}} = e^{-\frac{1}{\gamma} \int_t^T (\mu_s + \delta_s) dv + \frac{1}{2} \sigma^2 (s-t) + \frac{1}{2} \vartheta (W_t - \bar{W}_t)} - \frac{1}{\gamma} \vartheta (W_t - \bar{W}_t)
\]

where \( \{e^{\frac{1}{\gamma} \vartheta W_t - \frac{1}{2} \sigma^2 t} | t \in T \} \) is an exponential martingale under \( \tilde{\mathbb{P}} \). Denote

\[
C(t, T) \equiv w_0^\frac{1}{\gamma} e^{-\frac{1}{\gamma} \int_t^T (\mu_s + \delta_s) dv + \frac{1}{2} \sigma^2 (T-t) + \frac{1}{2} \vartheta^2) + \int_t^T \left( w_1^\frac{1}{\gamma} \Gamma_{1,s}\vartheta^2 + w_2^\frac{1}{\gamma} \Gamma_{2,s}\vartheta^2 \right) e^{-\frac{1}{\gamma} \int_t^s (\mu_v + \delta_v) dv + \frac{1}{2} \sigma^2 (s-t) + \frac{1}{2} \vartheta^2) ds,
\]

then, \( \mathcal{X}(t, y) = y^{-\frac{1}{\gamma}} C(t, T), y_t = C^\gamma (t, T)(g_t^{X_t, H_t})^{-\gamma} \). By Theorems 4.4 and 4.5, the optimal consumption and the terminal wealth at this optimum must be

\[
c_{k,s}^* x_t = H_{k,s} + g_t^{X_t, H_t, w_0^\frac{1}{\gamma} (K^t_k)^{-\frac{1}{\gamma}} C(t, T)}, \quad X_T = g_t^{X_t, H_t, w_0^\frac{1}{\gamma} (K^t_k)^{-\frac{1}{\gamma}} C(t, T)}, \quad a.s.,
\]

where

\[
H_{s,x_t} = (\mathcal{D}_s \mathcal{H}_s^{X_t,H_t}) H_t + g_t^{X_t, H_t, w_0^\frac{1}{\gamma} (K^t_k)^{-\frac{1}{\gamma}} C(t, T)} \int_t^s (K^t_k)^{-\frac{1}{\gamma}} (\mathcal{D}_s \mathcal{H}_s^{X_t,H_t}) B(w_1^\frac{1}{\gamma} \Gamma_{1,s}, w_2^\frac{1}{\gamma} \Gamma_{2,s})^T dv.
\]

In comparison with the result of the classical Merton model, the optimal non-habitual consumption \( c_{k,s}^* - \bar{H}_{k,s} \) is a proportion of the risk-neutral discounting adjusted wealth \( g_t^{X_t, H_t} \), instead of the wealth \( X_t \). Furthermore, the difference between the habitual consumption and the subsistence consumption \( H_{k,s} - \bar{H}_{k,s} \) is also a proportion of the risk-neutral discounting adjusted wealth.

Appendix A. Proof of lemmas and theorems.

A.1. Proof of Lemmas 2.3 and 2.4.

Proof of Lemmas 2.3. For \( (\pi, c) \in \mathcal{A}(t, X_t) \), rewrite (3) as

\[
E^s_t \left( X_s - \tilde{E}_s^W \int_t^T \mu_u E_u^s X_u du + \tilde{E}_s^W \int_t^T E_u^s I_u^s du \right) + \int_t^s E_u^s (1, c_u) du
\]

\[
= \left( X_t - \tilde{E}_t^W \int_t^T \mu_u E_u^t X_u du + \tilde{E}_t^W \int_t^T E_u^t I_u^s du \right) + \int_t^s E_u^t \pi_u c_u d\tilde{W}_u.
\]

The left-hand side is a \( (\tilde{\mathbb{W}}, \tilde{\mathbb{P}}) \)-supermartingale since it is continuous, non-negative and integrable. Because of non-negative terminal wealth \( X_T \), over \( \mathcal{A}(t, X_t) \) the inequality

\[
X_t - \tilde{E}_t^W \int_t^T \mu_s E_s^t X_s ds + \tilde{E}_t^W \int_t^T E_s^t I_s^t ds \geq \tilde{E}_t^W (E^t_T X_T) + \tilde{E}_t^W \int_t^T E_s^t (1, c_s) ds
\]

holds; thus, \( \mathcal{E} \in \mathcal{B}(t, X_t) \), which proves the first half of Lemma 2.3.
Conversely, for $c \in \mathcal{B}(t, X_t)$, consider the $\mathbb{F}^W$-adapted process $\{Y_s|s \in [t, T]\}$ defined by

$$Y_s = X_s + \tilde{E}_s^W \int_t^T E_u^tI_u du - \tilde{E}_s^W \int_t^T E_u^t(1, c_u) du - \tilde{E}_s^W \int_t^T \mu_u E_u^t X_u du, \quad Y_t \geq 0.$$ 

By the martingale representation theorem, there exists a square-integrable $\mathbb{F}^W$-predictable process $\{\Psi_s|s \in [t, T]\}$ such that $E^W_t[Y_s - E^W_sX_s] = Y_t - X_t - \int_t^s \Psi_u d\tilde{W}_u$. Set the portfolio $\pi$ satisfying $E^W_t \pi_u \sigma_u = \Psi_u$ such that

$$\tilde{E}_s^W \int_t^T E_u^t(1, c_u) du - \tilde{E}_s^W \int_t^T E_u^tI_u du + \tilde{E}_s^W \int_t^T \mu_u E_u^t X_u du$$

$$= X_t - Y_t + \int_t^T E^W_t \pi_u \sigma_u d\tilde{W}_u.$$ 

Compared with (3),

$$X_s - \tilde{E}_s^W \int_t^T \mu_u E_u^t X_u du + \tilde{E}_s^W \int_t^T E_u^tI_u du = \tilde{E}_s^W \int_t^T E_u^t(1, c_u) du + Y_t \tilde{E}_s^W \geq 0.$$ 

Thus, $X_T = (E^W_T)^{-1}Y_T$ almost for certain implies that $\pi$ is the hedging portfolio.

For $\mathbb{F}^W$-adapted portfolio, by the independence of Brownian motion and lifetime,

$$\tilde{E}_t^W \int_t^T \mu_s E_s^t X_s du = \tilde{E}_t^W \int_t^T \mu_s e^{-\int_t^s \mu_u du} \left[ X_t + \int_t^s \beta_u^s I_u du - \int_t^s \beta_u^s(1, c_u) du \right] ds$$

$$= X_t \mathbb{P}(\tau \leq T|\tau > t) + \tilde{E}_t^W \int_t^T \mathbb{P}(s < \tau \leq T|\tau > t) \beta_s^t I_s ds$$

$$- \tilde{E}_t^W \int_t^T \mathbb{P}(s < \tau \leq T|\tau > t) \beta_s^t(1, c_s) ds$$

$$= X_t + \tilde{E}_t \int_t^{T\wedge \tau} \beta_s^t I_s ds - \tilde{E}_t \int_t^{T\wedge \tau} \beta_s^t(1, c_s) ds$$

$$- \mathbb{P}(\tau > T|\tau > t) \left[ X_t + \tilde{E}_t^W \int_t^T \beta_s^t I_s ds - \tilde{E}_t^W \int_t^T \beta_s^t(1, c_s) ds \right].$$

With the boundedness of income, integrability of consumption and the property of a local martingale, $\tilde{E}_t^W \int_t^T \mu_s E_s^t X_u du > -\infty$. Replace $t$ by $s \in [t, T]$ to arrive at the second half of Lemma 2.3.

This also shows that when $\mathbb{P}(\tau > T|\tau > t) > 0$ holds, Lemma 2.4 becomes valid.

A.2. Proof of Lemma 3.1 and finite algebraic operations.

Proof of Lemma 3.1. Because of the finite eigenvalues of matrix $B - A$ on the complex plain and the existence of the Jordan canonical form, the real matrix of power series $e^{(B-A)t} \triangleq \sum_{k=0}^{\infty} \frac{t^k}{k!} (B - A)^k$ uniformly converges. By the rule of term-by-term differentiation, $de^{(B-A)t} = (B - A)e^{(B-A)t} dt$ verifies that $\tilde{H}_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} (B - A)^k \tilde{H}_0$ is the solution of the dynamic system $d\tilde{H}_t = (B - A)\tilde{H}_t dt$.

Let us temporarily set aside consideration of the initial value and assume that the basic solution of $d\tilde{H}_t = (B - A)\tilde{H}_t dt$ can be represented by $\tilde{H}_t = e^{\lambda t} v$ with $(B - A - \lambda E)v = 0$. Denote the set of eigenvalues of $B - A$ by $\{\lambda_k|k = 1, \ldots, l\}$,
where the multiplicity of \( \lambda_k \in \mathbb{C} \) is \( n_k \) such that \( \sum_{k=1}^{l} n_k = N \). The solution of the system of linear equations \((B - A - \lambda_k E)^{n_k} v = 0\) spans the \( n_k \)-dimensional linear subspace of \( \mathbb{C}^N \). Note that \( \mathbb{C}^N \) coincides with the direct sum of \( \{S_k|k = 1, \ldots, l\} \); thus, \( \mathbb{C}^N = S_1 \oplus \cdots \oplus S_l \).

For the initial condition \( \hat{H}_0 = h \), there must exist a decomposition \( h = \sum_{k=1}^{l} v_k \) such that \( v_k \in S_k \) and \((B - A - \lambda_k E)^j v_k = 0\) for all the integers \( j \geq n_k \). Then, the simple fact that \( e^{\lambda k t} e^{-\lambda k t} = E \) provides

\[
e^{(B-A)^j} v_k = e^{\lambda k t} e^{(B-A-\lambda k E)^j} v_k = e^{\lambda k t} \sum_{j=0}^{n_k-1} \frac{t^j}{j!} (B-A-\lambda k E)^j v_k,
\]

\[
e^{(B-A)^j} h = \sum_{k=1}^{l} e^{\lambda k t} \sum_{j=0}^{n_k-1} \frac{t^j}{j!} (B-A-\lambda k E)^j v_k.
\]

The right-hand side can be calculated by finite algebraic operations. In particular, when each eigenvalue in \( \{\lambda_k|k = 1, \ldots, N\} \) is a simple root, \( v_k \) must be the eigenvector of \( \lambda_k \); with each \( n_k = 1 \), the formula degenerates to \( e^{(B-A)^j} h = \sum_{k=1}^{l} e^{\lambda k t} v_k \).

### A.3. Proof of Lemmas 3.2, 3.4 and 3.5.

**Proof of Lemma 3.4.** Because \( \hat{H}^{t,h}_s = h + \int_t^s (B_u - A_u) \hat{H}^{t,h}_u du \) is differentiable with respect to \( h \),

\[
\mathcal{D}_h \hat{H}^{t,h}_s = E + \int_t^s (B_u - A_u) \mathcal{D}_h \hat{H}^{t,h}_u du.
\] (40)

Note that subject to the initial condition \( Z_t = E \), \( dZ_s = (B_s - A_s) Z_s ds \) has a unique solution. It can be deduced that the Jacobi matrix \( \mathcal{D}_h \hat{H}^{t,h}_s \) is independent of \( \hat{H}_t = h \). This means that \( \mathcal{D}_h \hat{H}^{t,h}_s \) can be replaced by \( \mathcal{D}_h \hat{H}^{t,h}_0 \). Then, due to the semi-group property, differentiate \( \hat{H}^{0,h}_s = \hat{H}^{t,h}_s \hat{H}^{0,0}_s \) with respect to \( h \) to arrive at the decomposition

\[
\mathcal{D}_h \hat{H}^{0,h}_s = \left[ \frac{\partial}{\partial h_j} \hat{H}^{t,h}_s \right]_{i,j=1,\ldots,N} = \left[ \sum_{k=1}^{N} \frac{\partial}{\partial h_j} \hat{H}^{t,h}_k \right]_{i,j=1,\ldots,N} = (\mathcal{D}_h \hat{H}^{t,h}_s)(\mathcal{D}_h \hat{H}^{0,0}_s).
\]

On the other hand, from (40), \( (\mathcal{D}_h \hat{H}^{t,h}_s) h = h + \int_t^s (B_u - A_u)(\mathcal{D}_h \hat{H}^{t,h}_u) h du \). The uniqueness of solution of the dynamic system provides \( \hat{H}^{t,h}_s = (\mathcal{D}_h \hat{H}^{t,h}_s) h \). \( \square \)

**Proof of Lemma 3.2.** The proof is divided into two parts. The first is to prove the Jacobi matrix \( \mathcal{D}_h \hat{H}^{t,h}_s \geq 0 \). Investigate the opponent \( z_{kj,s} \) of the matrix \( Z_s \) driven by the dynamic system \( Z_s = E + \int_t^s (B_u - A_u) Z_u du \), which follows the ODE:

\[
dz_{kj,s} = \sum_{l=1}^{N} (b_{kl,s} - a_{kl,s}) z_{lj,s} ds = (b_{kk,s} - a_{kk,s}) z_{kj,s} ds + \sum_{l \neq k} (b_{kl,s} - a_{kl,s}) z_{lj,s} ds.
\]
Hence, for all $k, j = 1, \ldots, N$,
\[
z_{kj,s} = 1_{\{k=j\}}e^{\int_t^s (b_{kk,u} - a_{kk,u})du} + \sum_{l \neq k} \int_t^s (b_{kl,u} - a_{kl,u})z_{lj,u}e^{\int_u^s (b_{kk,v} - a_{kk,v})dv}du,
\]
with the non-negative initial value. Note that $b_{kl,u} - a_{kl,u} \geq 0$ for all $k \neq l$. Therefore, $z_{kj,s}$ is monotonically increasing, and thus, $\mathcal{D}_h \hat{H}_s^{u,h} = Z_s \geq 0$.

The second part is to prove that $H_s = \hat{H}_s^{t,H_t} + \int_t^s (\mathcal{D}_h \hat{H}_s^{u,h})B_u(c_u - H_u)du$, using the uniqueness of the solution of the dynamic system. Note that the dynamic of (40) and $\lim_{u \uparrow s} \mathcal{D}_h \hat{H}_s^{u,h} = \mathcal{D}_h h = E$ hold. We have
\[
d(H_s - \hat{H}_s^{t,H_t}) = B_s(c_s - H_s)ds + (B_s - A_s)(H_s - \hat{H}_s^{t,H_t})ds
\]
\[
d \int_t^s (\mathcal{D}_h \hat{H}_s^{u,h})B_u(c_u - H_u)du = B_s(c_s - H_s)ds
\]
\[
+ (B_s - A_s) \int_t^s (\mathcal{D}_h \hat{H}_s^{u,h})B_u(c_u - H_u)duds.
\]
Hence, the vector processes $\{H_s - \hat{H}_s^{t,H_t} | s \in [t, T]\}$ and $\{ \int_t^s (\mathcal{D}_h \hat{H}_s^{u,h})B_u(c_u - H_u)du | s \in [t, T]\}$ are both the solution of the dynamic system
\[
Y_s = \int_t^s B_u(c_u - H_u)du + \int_t^s (B_u - A_u)Y_u du.
\]
They must be the same. With the non-negative opponents of $\mathcal{D}_h \hat{H}_s^{u,h}$ and $B_u$, as well as $c \gg H$, over $[t, s]$, Lemma 3.2 is proven by
\[
c_s \gg H_s = \hat{H}_s^{t,H_t} + \int_t^s (\mathcal{D}_h \hat{H}_s^{u,h})B_u(c_u - H_u)du \geq \hat{H}_s^{t,H_t}.
\]
\[
\square
\]

**Proof of Lemma 3.5.** It has been deduced that
\[
c_s - \hat{H}_s^{t,H_t} = c_s - H_s + \int_t^s (\mathcal{D}_h \hat{H}_s^{u,h})B_u(c_u - H_u)du.
\]
Using the tower rule (namely the law of total expectation),
\[
\mathbb{E}_t^W \int_t^T \beta_s^t(1, c_s - \hat{H}_s^{t,H_t})ds
\]
\[
= \mathbb{E}_t^W \int_t^T \beta_s^t(1, c_s - H_s)ds + \mathbb{E}_t^W \int_t^T \beta_s^t(1, (\mathcal{D}_h \hat{H}_s^{u,h})B_u(c_u - H_u))duds
\]
\[
= \mathbb{E}_t^W \int_t^T \beta_s^t \left\{ 1 + B_s^T \mathbb{E}_t^W \int_t^T \beta_s^u(\mathcal{D}_h \hat{H}_s^{u,h}, 1)du, c_s - H_s \right\} ds.
\]
Because of the boundedness of the parameter processes,
\[
\Gamma_t = 1 + B_t^T \mathbb{E}_t^W \int_t^T \beta_s^t(\mathcal{D}_h \hat{H}_s^{t,h}, 1)ds
\]
must be continuous and bounded, with the opponents being no less than 1. \[
\square
\]
A.4. Proof of Lemma 4.3.

Proof of Lemma 4.3. For $c \in \mathcal{B}'(t, X_t, H_t)$ and the hedging portfolio $\pi^*$,

$$X_s = \tilde{E}_s^W \int_s^T \beta^s_u(1, c_u) du - \tilde{E}_s^W \int_s^T \beta^s_u I_u du + \tilde{E}_s^W (\beta^s_T X_T).$$

Set $X_{1,s} = \tilde{E}_s^W \int_s^T \beta^s_u(1, c_u) du - \tilde{E}_s^W \int_s^T \beta^s_u I_u du$. Then,

$$\tilde{E}_s^W \int_s^T \mu_u E^s_u X_{1,u} du = \left[ \tilde{E}_s^W \int_s^T \beta^s_u(1, c_u) du - \tilde{E}_s^W \int_s^T \beta^s_u I_u du \right] - \left[ \tilde{E}_s^W \int_s^T E^s_u(1, c_u) du - \tilde{E}_s^W \int_s^T E^s_u I_u du \right].$$

Hence $X_{1,s} = \tilde{E}_s^W \int_s^T E^s_u(1, c_u) du - \tilde{E}_s^W \int_s^T E^s_u I_u du + \tilde{E}_s^W \int_s^T \mu_u E^s_u X_{1,u} du$. Similar to Lemma 2.3, by the martingale representation theorem, there exists a square-integrable $\mathbb{P}^W$-predictable process $\{\Psi_{1,s}, s \in [t, T]\}$ such that

$$\tilde{E}_s^W \int_s^T E^s_u(1, c_u) du - \tilde{E}_s^W \int_s^T E^s_u I_u du + \tilde{E}_s^W \int_s^T \mu_u E^s_u X_{1,u} du = X_{1,t} + \int_t^s \Psi_{1,u} d\tilde{W}_u.$$

Take the $\mathbb{P}^W$-progressively measurable process $\pi_1^*$ satisfying $E^w_{u} \pi_1^* u \sigma_u = \Psi_{1,u}$. The forward integral equation

$$E^s_u X_{1,s} + \int_s^t \mu_u E^s_u X_{1,u} du + \int_s^t \int_s^t E^s_u(1, c_u) du - \int_s^t \int_s^t E^s_u I_u du = X_{1,t} + \int_t^s E^s_u \pi_1^* u \sigma_u d\tilde{W}_u$$

becomes the wealth dynamic of the first subproblem, cf. Definition 4.1. Since $X_{1,T} = 0$ a.s., $\pi_1^*$ must be a hedging portfolio that $(\pi_1^*, c) \in \mathcal{A}'(t, X_t, H_t)$.

Take the portfolio $\pi_2^* = \pi^* - \pi_1^*$ for the second subproblem (cf. Definition 4.2). From (1), $X_s = X_{1,s} + X_{2,s}$. Hence, $X_{2,s} = \tilde{E}_s^W (\beta^s_T X_T)$, and $X_{2,T} = X_T$ almost for certain, which means that $\pi_2^*$ is a hedging portfolio such that $X_{2,T} = (E^s_T)^{-1} X_t$. □

A.5. Proof of Theorem 4.5.

Proof of Theorem 4.5. From $u_0(T, X_{2,T}) \leq \tilde{u}_0(T, y_{2,t} K_{2,T}^t)$, we have

$$E^W_t \left[ e^{-\int_t^T (\mu_u + \delta_u) du} u_0(T, X_{2,T}) \right] \leq E^W_t \left[ e^{-\int_t^T (\mu_u + \delta_u) du} u_0(T, y_{2,t} K_{2,T}^t) \right] + y_{2,t} \tilde{E}_t^W (\beta^r_T X_{2,T})$$

$$\leq E^W_t \left[ e^{-\int_t^T (\mu_u + \delta_u) du} u_0(T, \mathcal{I}_0(T, y_{2,t} K_{2,T}^t)) \right] + y_{2,t} \left\{ X_{2,t} - \tilde{E}_t^W [\beta^r_T \mathcal{I}_0(T, y_{2,t} K_{2,T}^t)] \right\},$$

where the second inequality is due to the supermartingale property of the discounted wealth process; thus, $\tilde{E}_t^W (\beta^r_T X_{2,T}) \leq X_{2,t}$. Set a hedging portfolio $\pi_2^*$ such that $X_{2,T} = \mathcal{I}_0(T, y_{2,t} K_{2,T}^t)$ almost for certain. By (34), $\tilde{E}_t^W (\beta^r_T X_{2,T}) = \tilde{E}_t^W [\beta^r_T \mathcal{I}_0(T, y_{2,t} K_{2,T}^t)] = X_{2,t}$, and the discounted wealth process becomes a $(\mathbb{F}^W, \mathbb{P})$-martingale. This also proves

$$\max_{\pi_2} E^W_t \left[ e^{-\int_t^T (\mu_u + \delta_u) du} u_0(T, X_{2,T}) \right] = E^W_t \left[ e^{-\int_t^T (\mu_u + \delta_u) du} u_0(T, \mathcal{I}_0(T, y_{2,t} K_{2,T}^t)) \right],$$

where $y_{2,t} = \mathcal{I}_2(t, X_{2,t})$.

The remaining results of Theorem 4.5 are similar to the solution of the first subproblem (cf. Section 4.3) and are omitted here. □
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