A NOTE ON OPTIMIZATION FORMULATIONS OF MARKOV DECISION PROCESSES

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Abstract. This note summarizes the optimization formulations used in the study of Markov decision processes. We consider both the discounted and undiscounted processes under the standard and the entropy-regularized settings. For each setting, we first summarize the primal, dual, and primal-dual problems of the linear programming formulation. We then detail the connections between these problems and other formulations for Markov decision processes such as the Bellman equation and the policy gradient method.

1. Introduction

Most of the algorithms of Markov decision processes (MDPs) are derived from the fixed-point iteration of the Bellman equation [3]. Examples include value iteration [4,5,23], policy iteration [3,11], temporal difference (TD) learning [25], Q-learning [33], etc. The analyses of these algorithms in the tabular case and linear function approximation case often leverage the contraction property of the Bellman operator. In the past decade or so, nonlinear approximations such as neural networks have become more popular. However, for nonlinear function approximations, this contraction property no longer holds, often resulting in instability. Many variants and modifications have been proposed to stabilize the training, e.g., DQN [17], A3C [16]. However, theoretical guarantees for these algorithms are still missing.

A second perspective of studying MDPs is based on optimization. For nonlinear approximations, optimization formulations are often more convenient both for algorithmic design and mathematical analysis as they guarantee convergence to at least local minimums. Therefore in recent years, more attention has been given to the optimization framework. One major direction is based on linear programming (LP) [22] and some recent developments include [1,7,29,30]. Another direction is the Bellman residual minimization (BRM) [2], which includes algorithms based on the primal-dual form of the BRM [6,8,27], stochastic compositional gradient (SCGD) methods based on two-scale separation [31,32], algorithms based on the smoothness of the underlying transition dynamics [13,35,36]. The convergence properties of these algorithms have been studied in [14,15,18,28,34].

The contribution of this note is two-fold. First, we summarize the LP problems used in the study of MDPs. Many results in this note are well-known, but we were not able to find a place where these results are summarized in a uniform framework. Second, we point out the connections between the LP problems and other MDP formulations, including the equivalence between the dual problem and the policy gradient method and the equivalence between the primal problem and the Bellman equation.

Key words and phrases. Markov decision processes; Reinforcement learning; Optimization.

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1.1. **Notation.** A Markov decision process $\mathcal{M}$ with discrete state and action spaces is characterized by $\mathcal{M} = (\mathcal{S}, \mathcal{A}, P, r, \gamma)$. Here $\mathcal{S}$ is the discrete state space, with each state usually denoted by $s$. $\mathcal{A}$ is the discrete action space, with each action usually denoted by $a \in \mathcal{A}$. Throughout the note, $|\mathcal{S}|$ and $|\mathcal{A}|$ are used to denote the size of $\mathcal{S}$ and $\mathcal{A}$, respectively. $P$ is a third-order tensor where, for each action $a \in \mathcal{A}$, $P^a \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}$ is the transition matrix between the states, i.e., $P^a_{st}$ is the probability of arriving at state $t$ if action $a$ is taken at state $s$. $r$ is a second-order tensor where, for each action $a \in \mathcal{A}$, $r^a_s$ is the reward at state $s$ if action $a$ is taken. Finally, $\gamma \in [0, 1]$ is the discount factor.

Let $\Delta$ be the probability simplex over the space of actions, i.e.,

$$\Delta = \{ \eta = (\eta^a)_{a \in \mathcal{A}} : \sum_{a \in \mathcal{A}} \eta^a = 1 \text{ and } \eta^a \geq 0 \text{ for } \forall a \in \mathcal{A} \}.$$ 

The set of all valid policies is defined to be

$$\Delta^{|\mathcal{S}|} = \{ \pi = (\pi_s)_{s \in \mathcal{S}} : \pi_s \in \Delta \text{ for } \forall s \in \mathcal{S} \}.$$

For a policy $\pi \in \Delta^{|\mathcal{S}|}$, the transition matrix $P^\pi \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}$ under the policy $\pi$ is defined as

$$P^\pi_{st} = \sum_{a \in \mathcal{A}} P^a_{st} \pi^a_s,$$

i.e., $P^\pi_{st}$ is the probability of arriving at state $t$ from state $s$ if policy $\pi$ is taken. The reward $r^\pi_s \in \mathbb{R}^{|\mathcal{S}|}$ under the policy $\pi$ is

$$r^\pi_s = \sum_{a \in \mathcal{A}} r^a_s \pi^a_s,$$

i.e., $r^\pi_s$ represents the expected reward at state $s$ if policy $\pi$ is taken.

Each policy $\pi$ induces a discrete Markov process, where at each round $m$, an action $a_m$ is chosen at state $s_m$ according to a particular policy $\pi$, and then the agent arrives at state $s_{m+1}$ according to the distribution of the transition matrix $P^a_m$ and receive a reward $r^a_m$. The goal of an MDP problem is to maximize the cumulative reward among all possible policies. Depending on whether $\gamma$ is strictly less than one or not, an MDP can either be **discounted** ($\gamma < 1$) or **undiscounted** ($\gamma = 1$).

When solving the MDPs, entropy regularizer has been proved quite useful in terms of exploration and convergence [9, 16, 21, 24]. In this note, we adopt the following **negative conditional entropy** defined for non-negative $\rho \in \mathbb{R}^{|\mathcal{A}|}$:

$$h(\rho) := \sum_{a \in \mathcal{A}} \rho^a \log \frac{\rho^a}{\sum_{b \in \mathcal{A}} \rho^b}.$$ 

This entropy $h(\rho)$ is both convex and homogeneous of degree one in $\rho$ (see for example Appendix A.1 of [18]). This regularizer has been widely used in the literature [8, 10, 16, 21, 24]. Depending on whether this regularizer is used, we call an MDP either **standard** or **regularized**.

1.2. **Outline.** The rest of the note is organized as follows. In Section 2, we first derive the primal, dual, and primal-dual problems for the discounted standard MDP. We then show the equivalence between the policy gradient algorithm and the dual problem as well as the equivalence between the Bellman equation and the primal problem. Sections 3, 4, and 5 address the discounted regularized MDP, the undiscounted standard MDP, and the undiscounted regularized MDP, respectively, by following the same outline.
2. Discounted standard MDP

The discounted standard MDP is probably the most studied case in literature \cite{23,26}. For \(\gamma \in (0,1)\), the value function under policy \(\pi\) is a vector \(v^\pi \in \mathbb{R}^{|S|}\), where \(v^\pi_s\) represents the expected discounted cumulative reward starting from state \(s\) under the policy \(\pi\), i.e.,

\[
v^\pi_s = \mathbb{E} \left[ \sum_{m=0}^{\infty} \gamma^m r_{s_m} | s_0 = s \right],
\]

where the expectation is taken over \(a_m \sim \pi_{s_m}, s_{m+1} \sim P_{s_m}^a, \) for all \(m \geq 0\). The value function naturally satisfies the Bellman equation for any \(s \in S\):

\[
v^\pi_s = r^\pi_s + \gamma \mathbb{E}_\pi [v^\pi_{s_1} | s_0 = s] = r^\pi_s + \gamma \sum_{t \in S} P^\pi_{st} v^\pi_t.
\]

The goal of an MDP problem is to find the maximum value function among all possible policies.

2.1. LP problems.

**Primal problem.** The primal problem of finding the maximum value function reads

\[
\min_v \sum_{s \in S} e_s v_s, \text{ s.t. } \forall a, \forall s, r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t - v_s \leq 0,
\]

where \(e \in \mathbb{R}^{|S|}\) is an arbitrary vector with positive entries. An example with 2 states and 2 actions is illustrated in Figure 1, where the pink region represents the constraints and the pink arrow points to the minimization direction. The optimal solution of this minimization problem is the red point. As shall see later in Section 2.2, the optimal solution of the dual problem is the same red point, coming from the opposite direction.

**Primal-dual problem.** By introducing the Lagrangian multiplier \(\mu^a_s\) for the inequality constraints, we arrive at the primal-dual

\[
\min_{v_s} \max_{\mu^a_s \geq 0} \sum_{s \in S} e_s v_s + \sum_{s,a} (r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t - v_s) \mu^a_s,
\]

or equivalently in the matrix-vector notation

\[
\min_v \max_{\mu^a \geq 0} e^\top v + \sum_{a \in A} (\mu^a)^\top (r^a + \gamma P^a v - v),
\]

where \((\cdot)^\top\) stands for transpose. This primal-dual problem is, for example, used in \cite{30}.

**Dual problem.** Since the minimum of \(v_s\) is taken over a convex function and the maximum of \(\mu^a_s\) is over a concave function, one can exchange the order of minimum and maximum because of the minimax theorem \cite{19}. The primal-dual problem can also be written as

\[
\max_{\mu^a \geq 0} \min_v e^\top v + \sum_{a \in A} (\mu^a)^\top (r^a + \gamma P^a v - v).
\]

Taking derivative with respect to \(v\) and setting it to be zero gives rise to

\[
e = - \sum_{a \in A} (\gamma (P^a)^\top - I) \mu^a, \text{ i.e., } \sum_{a \in A} (I - \gamma (P^a)^\top) \mu^a = e.
\]

Hence the dual problem is

\[
\max_{\mu^a \geq 0} \sum_{a \in A} (r^a)^\top \mu^a, \text{ s.t. } \sum_{a \in A} (I - \gamma (P^a)^\top) \mu^a = e.
\]
This dual problem is mentioned, for example, in [34].

2.2. Equivalences.

**Dual problem and policy gradient.** The dual problem (6) is equivalent to the policy gradient method. To see this, let us parameterize $\mu^a_s = w_s \pi^a_s$ with $w_s = \sum_{a \in A} \mu^a_s$. This ensures that $\pi \in \Delta^{|S|}$ because $\sum_{a \in A} \pi^a_s = \sum_{a \in A} \frac{\mu^a_s}{w_s} = 1$ and $\pi^a_s \geq 0$. By this new parameterization, the constraints in dual become

$$(I - \gamma (P^\pi)^\top) w = e, \quad \text{or} \quad w = (I - \gamma (P^\pi)^\top)^{-1} e,$$

where $P^\pi$ is defined in (1) as the transition matrix under policy $\pi$. By denoting this $w$ as $w^\pi$ to indicate its $\pi$ dependence, we can write $\sum_{a \in A} (r^a)^\top \mu^a = (r^\pi)^\top w^\pi$. As a result, the dual problem (6) can be written as

$$\max_{\pi \in \Delta^{|S|}} r^\pi \cdot (I - \gamma (P^\pi)^\top)^{-1} e, \quad \text{or} \quad \max_{\pi \in \Delta^{|S|}} e^\top (I - \gamma P^\pi)^{-1} r^\pi.$$

It is clearly equivalent to the policy gradient method

$$(7) \quad \max_{\pi \in \Delta^{|S|}} e^\top v^\pi, \text{ s.t. } v^\pi = r^\pi + \gamma P^\pi v^\pi,$$

where we recall that $e \in \mathbb{R}^{|S|}$ is any vector with positive entries. Therefore, the policy gradient method can be viewed as a nonlinear reparameterization of the dual LP problem. This understanding is also illustrated in Figure 1 where the yellow region...
represents the constraints and the yellow arrow points to the maximum direction. Notice that both the primal and dual problems end up at the same red point from opposite directions.

**Primal problem and Bellman equation.** Next, we show that the primal problem is equivalent to solving the Bellman equation

\[
    v_s = \max_{a \in A} \left( r_s^a + \gamma \sum_{t \in S} P_{st}^a v_t \right),
\]

**Bellman equation to primal problem.** The derivation from (8) to (4) can be found for example in [23]. We provide a short derivation here for completeness. Let \( v^* \) be the solution to (8), then for each \( s \), there exists \( a^*_s \in A \) s.t.,

\[
    v^*_s = r^*_s + \gamma P^* a^*_s v^*.
\]

For any \( v \) that satisfies the constraints in the primal problem (4), the following inequality holds

\[
    v \geq r^*_s + \gamma P^* a^*_s v.
\]

Subtracting these two equations gives

\[
    v - v^* \geq \gamma P^* (v - v^*).
\]

Since \( P^* \) is a probability transition matrix, by maximum principle, one has \( v - v^* \geq 0 \), and thus

\[
    e^\top v \geq e^\top v^*
\]

for all \( v \) satisfying the constraints in (4). This proves that \( v^* \) is the minimizer of the primal problem (4).

**Primal problem to Bellman equation.** Let \( v^* \) be the minimizer of the primal problem (4). The KKT conditions for (4) read

\[
\begin{cases}
    r_s^a + \gamma \sum_{t \in S} P_{st}^a v_t^* \leq v_s^*, & \text{for } \forall s, a; \\
    \sum_{a \in A} (\mu_s^a - \gamma \sum_{t \in S} P_{ts}^a \mu_t^a) = e_s, & \text{for } \forall s; \\
    \mu_s^a (r_s^a + \gamma \sum_{t \in S} P_{st}^a v_t^* - v_s^*) = 0 & \text{for } \forall s, a.
\end{cases}
\]

First, we claim it is impossible that there exists \( s \) such that for \( \forall a \), \( r_s^a + \gamma \sum_{t \in S} P_{st}^a v_t^* - v_s^* > 0 \). If it were true, then from the last equation, one would have \( \mu_s^a = 0 \) for all \( a \). Let \( \mu_s^a = w_s \pi_s^a \), then \( w_s = 0 \) for this \( s \). Inserting it into the second equation, one has

\[
    w_s - \gamma \sum_{t \in S} P_{ts}^a w_t = -\gamma \sum_{t \in S} P_{ts}^a w_t = e_s.
\]

Since both \( P_{ts}^a, w_t \geq 0 \) for all \( t \), then LHS \( \leq 0 \). However, as the RHS \( e_s > 0 \), we reach a contradiction. Therefore, the claim is true, i.e., There does not exist any \( s \) such that \( \forall a \), \( r_s^a + \gamma \sum_{t \in S} P_{st}^a v_t^* - v_s^* > 0 \).

Therefore for any fixed \( s \), there exists \( a_s^* \), s.t.,

\[
    r_s^a + \gamma \sum_{t \in S} P_{st}^a v_t^* = v_s^*.
\]
For all $a \neq a^*_s$, by the first equation, one has

$$v^a_s + \gamma \sum_{t \in S} P^{a}_{st} v^*_t \leq v^*_s.$$  

Combining the above two equations leads to $v^*_s = \max_a (v^a_s + \gamma \sum_{t \in S} P^{a}_{st} v^*_t)$. Therefore, the minimizer $v^*$ also satisfies the Bellman equation \((\ref{eq:bellman})\) for all $s$.

3. DISCOUNTED REGULARIZED MDP

The discounted regularized MDP includes the negative conditional entropy

$$h(\mu_s) = \sum_{a \in A} \mu^a_s \log \frac{\mu^a_s}{\sum_{b \in A} \mu^b_s}$$

for each $s \in S$ in the objective function.

3.1. LP problems.

**Primal-dual problem.** Let us introduce first in the primal-dual problem:

\[
\min_{\nu_s} \sup_{\mu^a_s > 0} \sum_{s \in S} e_s \nu_s + \sum_{s,a} \left( r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t - v_s \right) \mu^a_s - \sum_{s \in S} h(\mu_s). \tag{9}
\]

As $h(\mu_s)$ is convex in $\mu_s$, this objective function is concave in $\mu^a_s$. In connection with the primal-dual problem \((\ref{eq:primal_dual})\) of the standard case, including the extra entropic term allows for replacing the condition $\mu^a_s \geq 0$ with $\mu^a_s > 0$. In the literature, it is common for the entropy term to have a prefactor $\eta > 0$. Here we simply assume $\eta = 1$ as one can always reduce to this case by rescaling the rewards $r^a_s$.

**Primal problem.** By introducing $\mu^a_s = w_s \pi^a_s$ with $w_s = \sum_{a \in A} \mu^a_s$ and $\pi \in \Delta^{|S|}$, \((\ref{eq:primal_dual})\) becomes

\[
\min_{\nu_s} \sum_{s \in S} e_s \nu_s + \sup_{w_s > 0} \sum_{s \in S} w_s \cdot \max_{\pi_s \in \Delta} \left( \sum_{a \in A} \left( r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t - v_s \right) \pi^a_s w_s - \sum_{s \in S} w_s h(\pi_s) \right),
\]

where we use the fact that $h(\cdot)$ is homogeneous of degree one. This is equivalent to

\[
\min_{\nu_s} \left( \sum_{s \in S} e_s \nu_s + \sup_{w_s > 0} \sum_{s \in S} w_s \cdot \max_{\pi_s \in \Delta} \left( \sum_{a \in A} \left( r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t - v_s \right) \pi^a_s - h(\pi_s) \right) \right).
\]

Since the inner optimal over $\pi_s$ cannot lie on the boundary, it is the same to write the optimal as $\max_{\pi_s \in \Delta}$ and $\sup_{\pi_s \in \Delta, \pi_s > 0}$. The primal problem of the above minimax problem is then given by

\[
\min_{\nu_s} e^\top \nu_s, \text{ s.t. } \forall s, \max_{\pi_s \in \Delta} \left( \sum_{a \in A} \left( r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t - v_s \right) \pi^a_s - h(\pi_s) \right) \leq 0,
\]

or equivalently,

\[
\min_v e^\top v, \text{ s.t. } \pi^\top \left( r^\pi + \gamma P^\pi v - h^\pi \right) \leq v.
\]

Note that the maximization in the constraint

\[
\max_{\pi_s \in \Delta} \left( \sum_{a \in A} \left( r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t - v_s \right) \pi^a_s - h(\pi_s) \right)
\]
is in the form of the Gibbs variational principle. Therefore, the optimizer for this maximization is

\[
\pi^\alpha_s = \frac{\exp(r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t - v_s)}{Z_s}
\]

where \( Z_s = \sum_{a \in A} \exp(r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t - v_s) \) is the normalization factor and the maximal value is \( \log Z_s \). Hence, the constraint is equivalent to \( \log Z_s \leq 0 \), i.e.,

\[
\sum_{a \in A} \exp(r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t - v_s) \leq 1
\]
i.e.

\[
e^{-v_s} \sum_{a \in A} \exp(r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t) \leq 1, \quad \text{or} \quad \sum_{a \in A} \exp(r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t) \leq e^{v_s}.
\]

Taking log on both sides leads to

\[
v_s \geq \log \left( \sum_{a \in A} \exp(r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t) \right).
\]

This implies that the primal problem \([10]\) is equivalent to

\[
\min_{v_s} e^T v, \quad \text{s.t.} \quad \forall s, v_s \geq \log \left( \sum_{a \in A} \exp(r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t) \right).
\]

**Dual problem.** The supremum and minimum in the primal-dual problem \([9]\) can be exchanged because the objective function is convex in \( v \) and concave in \( \mu \). Now one has,

\[
\sup_{\mu^\alpha > 0} \min_{v_s} \sum_{s \in S} e_s v_s + \sum_{s,a} (r^\alpha_s + \gamma \sum_{t \in S} P^\alpha_{st} v_t - v_s) \mu^\alpha_s - \sum_{s \in S} h(\mu_s).
\]

Taking derivative in \( v \) gives \( \sum_{a \in A} (I - \gamma (P^\alpha)^\top ) \mu^\alpha = e \). Therefore, the dual problem takes the form

\[
\sup_{\mu^\alpha > 0} \sum_{a \in A} (r^\alpha)^\top \mu^\alpha - \sum_{s \in S} h(\mu_s), \quad \text{s.t.} \quad \sum_{a \in A} (I - \gamma (P^\alpha)^\top ) \mu^\alpha = e.
\]

**3.2. Equivalences.**

**Dual problem and policy gradient.** We claim that the dual problem \([12]\) is again equivalent to the policy gradient method. As before, let us parameterize \( \mu^\alpha_s = w_s \pi^\alpha_s \) with \( w_s = \sum_{a \in A} \mu^\alpha_s \) and \( \pi \in \Delta^{|S|} \). Then the constraints in \([12]\) become

\[
\forall s, \sum_{a \in A} \pi^\alpha_s w_s - \gamma \sum_{a,t} P^\alpha_{ts} \pi^\alpha_t w_t = e_s, \quad \text{or} \quad w = (I - \gamma (P^\alpha)^\top )^{-1} e.
\]

By denoting the solution \( w \) by \( w^\pi \) to show its \( \pi \) dependence, one can rewrite \([12]\) as

\[
\max_{\pi \in \Delta^{|S|}} r^\pi \cdot w^\pi - \sum_{s \in S} w^\pi_s \left( \sum_{a \in A} \pi^\alpha_s \log \pi^\alpha_s \right)
\]

By further introducing \( h^\pi \in \mathbb{R}^{|S|} \) as the vector with entry \( h^\pi_s = h(\pi_s) = \sum_{a \in A} \pi^\alpha_s \log \pi^\alpha_s \), we transform \([12]\) to

\[
\max_{\pi \in \Delta^{|S|}} (r^\pi - h^\pi) \cdot w^\pi, \quad \text{or} \quad \max_{\pi \in \Delta^{|S|}} e^\top (I - \gamma P^\pi)^{-1} (r^\pi - h^\pi).
\]
We can also view $r^\pi - h^\pi$ as a regularized reward by subtracting the entropy function $h^\pi$. The value function $v^\pi = \mathbb{E}[\sum_{m>0}^\infty \gamma^m (r^\pi_{sa} - h(\pi_{sa}))]$ under the new reward satisfies the regularized Bellman equation $v^\pi = r^\pi - h^\pi + \gamma P^\pi v^\pi$. Hence, the policy gradient method of this regularized discounted MDP is

$$\max_{\pi \in \Delta(S)} e^v, \quad s.t., \quad v^\pi = r^\pi - h^\pi + \gamma P^\pi v^\pi,$$

which is clearly equivalent.

**Primal problem and Bellman equation.** The regularized Bellman equation is

$$v = \max_{\pi \in \Delta(S)} r^\pi + \gamma P^\pi v - h^\pi.$$ 

In each component,

$$v_s = \max_{\pi_s \in \Delta} \sum_{a \in A} (r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t) \pi^a_s - h(\pi_s).$$

By the Gibbs variational principle, the RHS is equal to $\log(\exp(\sum_{a \in A} (r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t)))$. Therefore, the regularized Bellman equation could also be written as the following log-sum-exp form

$$v_s = \log \left( \sum_{a \in A} \exp \left( r^a_s + \gamma \sum_{t \in S} P^a_{st} v_t \right) \right).$$

Below we show that the primal problem (10) is equivalent to solving (14).

**Bellman equation to primal problem.** Let $v^*$ be the solution of (14). Then there exists $\pi^*$ s.t.,

$$v^* = r^{\pi^*} + \gamma P^{\pi^*} v^* - h^{\pi^*}.$$ 

For any $v$ that satisfies the constraints of the primal problem, the following inequality holds

$$v \geq r^{\pi^*} + \gamma P^{\pi^*} v - h^{\pi^*}.$$ 

Subtracting these two equations gives rise to

$$v - v^* \geq \gamma P^{\pi^*} (v - v^*).$$ 

Again by maximum principle, one has $v - v^* \geq 0$ and hence

$$e^v \geq e^{v^*}$$

for all $v$ that satisfying the constraints in (10). This proves that $v^*$ is the minimizer of the primal problem (10).

**Primal problem to Bellman equation.** Let $v^*$ be the minimizer of the primal problem (10). We now prove $v^*$ is also the solution to the Bellman equation (14) by contradiction. Assume that $v^*$ does not satisfy (14). Then there must exist $\bar{s}$, s.t. for $\forall \pi$

$$v^*_{\bar{s}} \geq (r^{\pi} + \gamma P^{\pi} v^* - h^{\pi})_{\bar{s}} + \delta$$

with some constant $\delta > 0$. Let us define $\bar{v}$ s.t., $\bar{v}_{\bar{s}} = v^*_{\bar{s}} - \delta$ and $\bar{v}_s = v^*_s$ for $s \neq \bar{s}$. We claim that for $\forall \pi$

$$\bar{v}_s \geq (r^{\pi} + \gamma P^{\pi} \bar{v} - h^{\pi})_s, \quad \forall s.$$ 

First, for $s \neq \bar{s}$ the above inequality holds because $\bar{v}_s = v^*_s$ and $v^*_s$ satisfies the constraints in the primal problem (10). For $s = \bar{s}$, one has $\bar{v}_s = v^*_s - \delta \geq (r^{\pi} + \gamma P^{\pi} v^* - h^{\pi})_{\bar{s}}$. Since

$$(P^{\pi} v^*)_s = \sum_{t \in S} P^{\pi}_{st} v^*_t = \sum_{t \neq \bar{s}} P^{\pi}_{st} v^*_t + P^{\pi}_{\bar{s}t} v^*_\bar{s} - \delta + \delta = \sum_{t \in S} P^{\pi}_{st} \bar{v}_t + \delta,$$
\[ \bar{v}_s \geq (r^\pi + \gamma P^\pi \bar{v} - h^\pi) \bar{s} + \delta. \] This completes the proof of the claim. This means that \( \bar{v} \) also satisfies the constraints of the primal problem, but \( e^\top \bar{v} < e^\top v^* \) by construction, which contradicts with \( v^* \) being the minimizer of the primal problem. Therefore, the assumption is wrong and \( v^* \) satisfies (14).

4. Undiscounted standard MDP

In this section, we consider the MDP without discounts, i.e., \( \gamma = 1 \). Besides, we assume the MDP is unichain, i.e., for each policy \( \pi \), the MDP induced by policy \( \pi \) is ergodic [20].

Let \( \rho^\pi \in \mathbb{R} \) be the average reward under policy \( \pi \),

\[ \rho^\pi = \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{m=1}^{T} r_{s_m} \right], \]

where the expectation is taken over \( a_m \sim \pi_{s_m}, s_{m+1} \sim P_{s_m} \). Notice for an ergodic Markov process, the average-reward is the same for any initial states because the stationary distribution is unique invariant, strictly positive and independent of initial states (Chapter 6 of [12]). Let \( w^\pi \in \mathbb{R}^{|S|} > 0 \) be the stationary distribution induced by policy \( \pi \) satisfying

\[ w^\pi_s = \sum_{t \in S} P_{st}^\pi w^\pi_t, \]

with \( \sum_{s \in S} w^\pi_s = 1 \). Note that without this constrain, there are infinitely many \( v^\pi \), e.g., \( v^\pi + C \) for any constant \( C \) still satisfies the above equation. The goal here is to find the maximum average reward among all possible policies.

4.1. LP problems.

Primal problem. The primal problem is

\[ \min_{v_s, \rho} \ s.t. \ \forall a, \forall s, \ r^a_s + \sum_{t \in S} P_{st} v^\pi_t - v_s - \rho \leq 0. \]

This can be found for example in [18].

Primal-dual problem. By including the Lagrangian multiplier \( \mu^a_s \) for the inequality constraints, one obtains the primal-dual problem

\[ \min_{v_s, \rho} \ \max_{\mu^a_s \geq 0} \rho + \sum_{s, a} (r^a_s + \sum_{t \in S} P_{st}^a v^\pi_t - v_s - \rho) \mu^a_s, \]

or equivalently, in the matrix-vector notation

\[ \min_{v_s, \rho} \ \max_{\mu^a_s \geq 0} \rho + \sum_{a \in A} (\mu^a) \top (r^a + P^a v - v - \rho 1), \]

where \( 1 \) is the \( |S| \)-dimensional vector with all elements equal to 1.

Dual problem. To get the dual problem, we take the derivative with respect to \( \rho \) to get

\[ 1 - \sum_{s, a} \mu^a_s = 0. \]
Taking the derivative with respect to $v$ leads to
\[ \sum_{a \in A} (I - (P^a)\top) \mu^a = 0. \]

Hence the dual problem is
\[ \max_{\mu^a_s \geq 0} \sum_{a \in A} (r^a)\top \mu^a, \text{ s.t. } \sum_{a \in A} (I - (P^a)\top) \mu^a = 0, \quad 1 - \sum_{s,a} \mu^a_s = 0. \]

4.2. Equivalences.

**Dual problem and policy gradient.** The dual problem (19) is equivalent to the policy gradient. Let us again parameterize $\mu^a_s = w_s \pi^a_s$ with $w_s = \sum_{a \in A} \mu^a_s$ and $\pi \in \Delta(|S|)$. By the new parameterization, the constraints become
\[ \sum_{a \in A} (I - (P^a)\top) \mu^a = 0 \Rightarrow (I - (P^\pi)\top) w = 0. \]

$1 - \sum_{s,a} \mu^a_s = 0$ also implies that $1 - \sum_{s \in S} w_s = 0$. Together we conclude that $w$ is the stationary distribution induced by $\pi$. By denoting this $w$ by $w^\pi$, we can write the dual problem as
\[ \max_{\pi \in \Delta(|S|)} r^\pi \cdot w^\pi, \]
which is exactly the optimization formulation of the policy gradient method.

**Primal problem and Bellman equation.** Next, we show that the primal problem (17) is equivalent to the average reward Bellman equation for $v, \rho$
\[ v_s = \max_a \left( r^a_s - \rho + \sum_{t \in S} P^a_{st} v_t \right), \quad s \in S. \]

**Bellman equation to primal problem.** The derivation from (21) to (17) can be found for example in [23]. Here we provide a short proof for completeness. Let $v^*, \rho^*$ be the solution to the Bellman equation (21), then for all $s$, there exists $a^*_s$ s.t.,
\[ v^* = r^{a^*} - \rho^* 1 + P^{a^*} v^*, \]
where $r^{a^*}_s \equiv r^s_{a^*_s}, P^{a^*}_{st} \equiv P^{a^*_s}_{st}$. For any $v, \rho$ that satisfy the constraints in the primal problem (17), the following inequality holds
\[ v \geq r^{a^*} - \rho 1 + P^{a^*} v. \]

Subtracting these two equations gives
\[ v - v^* \geq P^{a^*} (v - v^*) - (\rho - \rho^*) 1. \]

Let $w^*$ be the stationary distribution induced by the policy $\pi^a_s = \begin{cases} 1, & a = a^*_s \\ 0, & a \neq a^*_s \end{cases}$, then $(w^*)\top = (w^*)\top P^{a^*}$. Multiplying $(w^*)\top$ to the last equation yields,
\[ (w^*)\top (\rho - \rho^*) 1 \geq 0 \]

Since we assume the MDP is unichain, the stationary distribution $w^*$ for any policy is strictly positive. This implies that
\[ \rho \geq \rho^* \]
holds for all \( v, \rho \) satisfying the constraints in [17]. This proves that \( v^*, \rho^* \) is the minimizer of the primal problem [17].

Primal problem to Bellman equation. Let \( (v^*, \rho^*) \) be the minimizer of the primal problem [17]. We now show that \( (v^*, \rho^*) \) also satisfies the average reward bellman equation (21). The KKT conditions of (17) are

\[
\begin{align*}
    & r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s \leq \rho^* \quad \text{for } \forall s, a; \\
    & 1 - \sum_{s, a} \mu^a_s = 0; \\
    & \sum_{a \in A} (\mu^a_s - \sum_{t \in S} P^a_{ts}w^*_t) = 0, \quad \text{for } \forall s; \\
    & \mu^a_s(r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s - \rho^*) = 0 \quad \text{for } \forall s, a.
\end{align*}
\]

First we claim that it is impossible that there exists \( s \) s.t. \( \forall a, r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s - \rho^* < 0 \). Let \( \mu^a_s = w^a_s\pi^a_s \) with \( w^a_s = \sum_{a \in A} \mu^a_s \) and \( \pi^a_s \in \Delta \) for all \( s \). Plugging it into the second and third equation gives,

\[1 - \sum_{s \in S} w^a_s = 0, \quad w - (P^\pi)^\top w = 0.\]

Therefore \( w \) is the unique stationary distribution induced by the policy \( \pi \). Since we assume the MDP is unichain, the stationary distribution is strictly positive. If there exists \( s \) s.t. for \( \forall a, r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s - \rho^* < 0 \), then by the last equation of the KKT condition, \( \mu^a_s = w^a_s\pi^a_s = 0 \) implies \( w^a_s = 0 \) for this \( s \), which contradicts with the unichain assumption. Therefore, the claim is true, i.e., there does not exist any \( s \) s.t. \( \forall a, r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s > 0 \).

Therefore, for \( \forall s \), there always exists \( a^*_s \), s.t.,

\[r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s = \rho^*.\]

By the first equation, for all \( a \neq a^*_s \),

\[r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s \leq \rho^*.\]

Combining the above two equations gives \( r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s \leq r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s = \rho^* \). This is equivalent to \( \rho^* = \max_a r^a_s + \sum_{t \in S} P^a_{st}v^*_t - v^*_s \), the average reward Bellman equation (21).

5. UNDISCOUNTED REGULARIZED MDP

5.1. LP problems.

Primal-dual problem. We again use the the negative conditional entropy

\[h(\mu_s) = \sum_{a \in A} \mu^a_s \log \frac{\mu^a_s}{\sum_b \mu^b_s}\]

as the regularizer. The primal-dual problem of the undiscounted regularized MDP is

\[
\min_{v_s, \rho} \sup_{\mu^a_s > 0} \rho + \sum_{s, a} (r^a_s + \sum_{t \in S} P^a_{st}v_t - v_s - \rho)\mu^a_s - \sum_{s \in S} h(\mu_s).
\]
Primal problem. By introducing $\mu_s^a = w_s \pi_s^a$, one obtains
\[ \min_{v_s, \rho} \sup_{\pi \in \Delta(S)} \rho + \sum_{s,a} (v_s^a + \sum_{t \in S} P_{st}^a v_t - v_s - \rho) \pi_s^a w_s - \sum_{s \in S} w_s h(\pi_s), \]
which is equivalent to
\[ \min_{v_s, \rho} \left( \rho + \sup_{w_s > 0} \sum_{s \in S} w_s \cdot \max_{\pi_s \in \Delta} \left( \sum_{a \in A} (r_s^a + \sum_{t \in S} P_{st}^a v_t - v_s - \rho) \pi_s^a - h(\pi_s) \right) \right). \]
The primal problem of the above minimax problem is
\[ \min_{v_s, \rho} \text{ s.t. } \forall \pi_s \in \Delta \left( \sum_{a \in A} (r_s^a + \sum_{t \in S} P_{st}^a v_t - v_s - \rho) \pi_s^a - h(\pi_s) \right) \leq 0, \]
or equivalently,
\[ \min_{v_s, \rho} \text{ s.t. } \max_{\pi} (r^\pi - \rho 1 + P^\pi v - h^\pi) \leq v. \tag{23} \]
Note that the constraint
\[ \max_{\pi_s \in \Delta} \left( \sum_{a \in A} (r_s^a + \gamma \sum_{t \in S} P_{st}^a v_t - v_s - \rho) \pi_s^a - h(\pi_s) \right) \]
is in the form of the Gibbs variational principle. The optimizer for this maximization is
\[ \pi_s^a = \frac{\exp(r_s^a + \sum_{a \in A} P_{st}^a v_t - v_s - \rho)}{Z_s}, \]
where $Z_s$ is the normalization factor, the optimal value is $\log Z_s$. Therefore, the constraint is equivalent to $\log Z_s \leq 0$, i.e.,
\[ \sum_{a \in A} \exp(r_s^a + \sum_{t \in S} P_{st}^a v_t - v_s - \rho) \leq 1 \]
i.e.
\[ e^{-v_s} \cdot \sum_{a \in A} \exp(r_s^a + \sum_{t \in S} P_{st}^a v_t - \rho) \leq 1, \quad \text{or} \quad \sum_{a \in A} \exp(r_s^a + \sum_{t \in S} P_{st}^a v_t - \rho) \leq e^{v_s}. \]
Taking log gives
\[ v_s \geq \log \left( \sum_{a \in A} \exp(r_s^a + \sum_{t \in S} P_{st}^a v_t - \rho) \right). \]
Hence the primal problem can also be written as
\[ \min_{v_s, \rho} \text{ s.t. } v_s \geq \log \left( \sum_{a \in A} \exp(r_s^a + \sum_{t \in S} P_{st}^a v_t - \rho) \right), \tag{24} \]
which is an alternative formulation of the primal problem. Notice that a similar problem with equality constraints was instead derived in [18]. In practice, the inequality constraints as in [24] are often preferred since the feasibility set is then convex.

Dual problem. The supremum and minimum in the primal-dual problem [9] can be exchanged because the objective function is convex in $v, \rho$ and concave in $\mu$. Then one has,
\[ \sup_{\mu_s^a > 0} \min_{v_s, \rho} \rho + \sum_{s,a} (r_s^a + \sum_{t \in S} P_{st}^a v_t - v_s - \rho) \mu_s^a - \sum_{s \in S} h(\mu_s). \]
Taking derivatives in $\rho$ and $v$ leads to

$$1 - \sum_{s,a} \mu_s = 0, \sum_{a \in A} (I - (P^a)^\top)\mu^a = 0.$$ 

Hence the dual problem in terms of $\mu^a$ is

$$(25) \sup_{\mu^a > 0} \sum_{a \in A} (r^a)^\top \mu^a - \sum_{s \in S} h(\mu_s) \text{ s.t. } \sum_{a \in A} (I - (P^a)^\top)\mu^a = 0, \quad 1 - \sum_{s,a} \mu^a_s = 0.$$ 

5.2. Equivalences.

**Dual problem and policy gradient.** The dual problem (25) is equivalent to the policy gradient method. Let us parameterize $\mu^a_s = w_s \pi^a_s$ with $w_s = \sum_{a \in A} \mu^a_s$ and $\pi \in \Delta^{|S|}$. Then the constraint in (25) becomes

$$1 - \sum_{s \in S} w_s = 0, \quad (I - (P^\pi)^\top)w = 0.$$ 

which indicates that $w$ is the stationary distribution of $P^\pi$. After denoting this solution by $w^\pi$ and plugging it into the objective function in (25), we transform the dual problem to

$$(26) \max_{\pi \in \Delta^{|S|}} (r^\pi - h^\pi)^\top w^\pi.$$ 

We can also view $r^\pi - h^\pi$ as a regularized reward by subtracting the entropy function $h^\pi$. The average-reward under the new reward becomes

$$\rho^\pi = \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{m=1}^T (r^a_{s_m} - h(\pi_{s_m})) \right].$$ 

In this way, (26) is exactly the policy gradient method of this undiscounted regularized MDP.

**Primal problem and Bellman equation.** The regularized average-reward Bellman equation for $v, \rho$ is

$$(27) v = \max_{\pi \in \Delta^{|S|}} r^\pi - \rho 1 + P^\pi v - h^\pi.$$ 

In each component,

$$v_s = \max_{\pi \in \Delta^{|S|}} \left( r^a_s - \rho + \sum_{t \in S} P^a_{st} v_t \right) \pi^a_s - h(\pi_s).$$ 

By the Gibbs variational principle, the RHS is equal to $\log(\exp(\sum_{a \in A} (r^a_s - \rho + \sum_{t \in S} P^a_{st} v_t)))$. Therefore, the regularized average-reward Bellman equation can be written as the log-sum-exp form for $v, \rho$

$$(28) v_s = \log \left( \sum_{a \in A} \exp \left( r^a_s - \rho + \sum_{t \in S} P^a_{st} v_t \right) \right).$$ 

Next, we show that the primal problem (23) is equivalent to solving (27).

**Bellman equation to primal problem.** Let $v^*, \rho^*$ be the solution to the Bellman equation. For $v^*, \rho^*$, there exists $\pi^*$ s.t.,

$$v^* = r^{\pi^*} - \rho^* 1 + P^{\pi^*} v^* - h^{\pi^*}.$$ 

Then, for any $v$ satisfying the above constraints, the following inequality holds,

$$v \geq r^{\pi^*} - \rho 1 + P^{\pi^*} v - h^{\pi^*}.$$
Subtracting these two equations gives
\[ v - v^* \geq P^{\pi^*} (v - v^*) - (\rho - \rho^*) \mathbf{1} \]
Again let \( w^* \) be the stationary distribution induced by the policy \( \pi^* \), then \((w^*)\dagger = (w^*)\dagger P^{\pi^*}\). Then multiplying \((w^*)\dagger\) to the last equation yields
\[ (w^*)\dagger (\rho - \rho^*) \mathbf{1} \geq 0 \]
Since we assume the MDP is unichain, the stationary distribution for any policy is strictly positive. This implies that
\[ \rho \geq \rho^* \]
holds for all \( v \) satisfying the constraints in (23). This proves that \( v^* \) is the minimizer of the primal problem (10).

Primal problem to Bellman equation. Let \( v^*, \rho^* = \text{argmin}_v \rho \) be the minimizer of the primal problem (23). Besides, there exists a policy \( \pi^* \) s.t. \( \rho^* = (w^*)\dagger (r^\pi^* - h^\pi^*) \). This can be seen from multiplying \((w^*)\dagger\) to the constraint \( v^* \geq r^\pi^* - \rho^* \mathbf{1} + P^{\pi^*} v^* - h^\pi^* \). Due to \((w^*)\dagger \mathbf{1} = 1\), \((w^*)\dagger P^\pi = (w^*)\dagger \), one has
\[ \rho^* \geq (w^*)\dagger (r^\pi^* - h^\pi^*) \]
for all \( \pi \). Let \( \pi^* = \text{argmin}_\pi (w^\pi)^\dagger (r^\pi - h^\pi) \), then the minimizer \( \rho^* = (w^*)\dagger (r^\pi^* - h^\pi^*) \).

We now show that \((v^*, \rho^*)\) is also the solution to the Bellman equation (27) by contradiction. Assume \((v^*, \rho^*)\) does not satisfy (27), then there must exist \( s \), s.t., for \( \forall \pi \)
\[ v^*_s \geq (r^\pi^* - \rho^* + P^{\pi^*} v^* - h^\pi^*) \delta + \delta \]
with some positive constant \( \delta > 0 \). The inequality (29) also holds for \( \pi^* \), so one can write it in vector form,
\[ v^* \geq r^\pi^* - \rho^* \mathbf{1} + P^{\pi^*} v^* - h^\pi^* + \tilde{\delta}, \]
where \( \tilde{\delta} \) is a vector with \( \delta \) on its \( \tilde{s} \)-th element and 0 on all other elements. Then multiplying \((w^\pi^*)\dagger\) to the above equation yields,
\[ 0 \geq \delta w^\pi^*_s. \]
Note that the RHS is always \( > 0 \) because the stationary distribution \( w^\pi^* \) is strictly positive, which leads to a contradiction. Hence we conclude that \( v^*, \rho^* \) is also the solution to the Bellman equation (27).

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