CONDENSATE OSCILLATIONS, KINETIC EQUATIONS AND TWO-FLUID HYDRODYNAMICS IN A BOSE GAS

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1 Introduction

Trapped Bose-condensed atomic gases are remarkable because, in spite of the fact that these are very dilute systems, they exhibit robust coherent dynamic behaviour when perturbed. These quantum “wisps of matter” are a new phase of highly coherent matter. While binary collisions are very infrequent, the coherent mean field associated with the Bose condensate ensures that interactions play a crucial role in determining the collective response of these superfluid gases.

In our discussion of the theory of collective oscillations of atomic condensates, which is the main theme of these lectures, the macroscopic Bose wavefunction $\Phi(\mathbf{r}, t)$ will play a central role. This wavefunction is the BEC order parameter. The initial attempts at defining this order parameter began with the pioneering work of Fritz London in 1938, was further developed by Bogoliubov in 1947 and finally formalized in the general quantum field theoretic formalism of Beliaev in 1957. The first extension of these ideas to inhomogeneous Bose condensates was by Pitaevskii and, independently, by Gross in 1961, which led to the now famous Gross-Pitaevskii (GP) equation of motion for $\Phi(\mathbf{r}, t)$. Most of this early work was limited to $T = 0$ where, in a dilute Bose gas, one can assume all of the atoms are in the condensate.

In Section 2, I will first review the dynamics of a pure condensate at $T = 0$, based on solving the linearized GP equation. This limit is especially appealing since one can ignore all the complications which arise from the presence of non-condensate atoms (the thermal cloud). We will discuss the normal mode solutions of the $T = 0$ GP equation of motion using the “quantum hydrodynamic” formalism, which works in terms of the local condensate density $n_c(\mathbf{r}, t)$ and superfluid velocity $\mathbf{v}_c(\mathbf{r}, t)$. Within the Thomas-Fermi approximation, Stringari has shown that the equations of motion for these two variables can be combined to give a wave equation for oscillations of the condensate.
In Section 3, we derive a generalized form of the GP equation for $\Phi(r, t)$ which is valid at finite temperatures. It involves terms which are coupled to the non-condensate component (the thermal cloud) and thus its solution in general depends on knowing the equations of motion for the dynamics of the non-condensate. In the present lectures, we will restrict ourselves to finite temperatures where the non-condensate can be described by a quantum kinetic equation for the single-particle distribution function $f(p, r, t)$. In this Boltzmann equation, the relatively high energy non-condensate atoms are simply free atoms moving in a self-consistent Hartree-Fock mean field.

A unique feature of a Bose-condensed gas is that the kinetic equation for $f(p, r, t)$ involves a collision integral (denoted by $C_{12}[f, \Phi]$) describing collisions between condensate and non-condensate atoms. The generalized GP equation for $\Phi(r, t)$ also has a term which is related to the $C_{12}$ collisions. This gives rise to damping of condensate fluctuations. In Section 3, we derive a finite $T$ Stringari wave equation with damping using the static Popov approximation. This means the thermal cloud is treated as always being in static thermal equilibrium, with $f(p, r, t) = f_0(p, r)$ being given by the equilibrium Bose distribution.

In Section 4, we turn to a detailed treatment of the dynamics of the coupled condensate and non-condensate components, starting from the finite $T$ generalized GP equation for $\Phi(r, t)$ and our kinetic equation for $f(p, r, t)$. We derive a new set of two-fluid hydrodynamic equations, following the recent work by Zaremba, Nikuni and Griffin. This derivation assumes the non-condensate is in local hydrodynamic equilibrium, induced by rapid collisions between the atoms in the thermal cloud. As a result, the non-condensate is completely described in terms of the non-condensate density $\tilde{n}(r, t)$, velocity $\mathbf{v}_n(r, t)$ and local pressure $\tilde{P}(r, t)$. Both the condensate and non-condensate exhibit coupled coherent oscillations at the same frequency.

In Section 4, we review the famous two-fluid hydrodynamic equations first derived by Landau in 1941 and which form the basis of our understanding of the hydrodynamic behaviour of superfluid $^4$He. We discuss first sound and second sound and consider how they differ in superfluid $^4$He and in a uniform Bose gas. We also point out the appearance of a new hydrodynamic zero frequency mode which appears in our generalized two-fluid equations. This mode is associated with the relaxation time for the condensate and non-condensate atoms to come into “diffusive” equilibrium with each other. Inclusion of damping due to thermal conductivity of the thermal cloud shows that this mode is the analogue of the well-known thermal diffusion mode in a classical gas.

We close this Introduction with some general references which may be useful to the reader. The 1998 Varenna Summer School lectures on BEC.
has excellent articles on recent research on BEC in atomic gases, including
very detailed reviews of the experimental work carried out by the JILA \(^1\) and MIT \(^2\) groups. For a general introduction to the theory of trapped Bose
gases, we recommend the Reviews of Modern Physics \(^1\) article by a leading theoretical group at the University of Trento. This authoritative review
concentrates on thermodynamic properties and the collisionless dynamics. A
recent long paper by Zaremba, Nikuni and the author \(^7\) discusses the details
of the derivation of the coupled hydrodynamic equations for the condensate
and the thermal cloud. The present lectures may be viewed as an introduction
to this paper, with more emphasis on the general structure of the derivation
and the implications of this new set of superfluid equations.

2 Pure condensate dynamics (at \(T = 0\))

This section will be largely a review of standard material \(^1\) but will provide
the starting point for generalizations in the following sections. As we have
mentioned, at \(T = 0\), one can assume that all atoms in a dilute Bose gas
are described by the macroscopic wavefunction \(\Phi(r, t)\). This obeys the time-
dependent Hartree equation of motion first written down 40 years ago by
Pitaevskii and Gross,

\[
\frac{i\hbar}{\partial t} \Phi(r, t) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + V_H(r, t) \right] \Phi(r, t).
\]  

Here the trap harmonic potential is

\[
V_{ex}(r) = \frac{1}{2} m \omega_0^2 r^2 \quad \text{(isotropic)}
\]  

and the self-consistent condensate Hartree potential is

\[
V_H(r, t) = \int dr' v(r - r') n_c(r, t) = gn_c(r, t).
\]  

As usual, since we are interested in extremely low energy atoms, we use
the \(s\)–wave approximation and approximate the interatomic potential by the
pseudopotential

\[
v(r - r') = \frac{4\pi a \hbar^2}{m} \delta(r - r') = g \delta(r - r'),
\]  

where \(a\) is the correct \(s\)–wave scattering length. For further discussion of the
effective interaction to use in the Gross-Pitaevskii (GP) equation, we refer to
the lectures by Burnett in this volume. The condensate density is given by
\[ n_c(r,t) = |\Phi(r,t)|^2 \]
and hence (3) reduces to a NLSE for \( \Phi(r,t) \), namely
\[ i\hbar \frac{\partial \Phi(r,t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + g|\Phi(r,t)|^2 \right] \Phi(r,t). \tag{5} \]

Since all atoms are in the identical quantum state, there is no exchange mean field in (3).

The \( T = 0 \) GP equation in (5) has been the subject of literally hundreds of papers since the discovery of BEC in laser-cooled trapped atomic gases - and the equation appears in almost every chapter in this book. As discussed in recent reviews [1, 2], it gives an excellent quantitative description of both the static and dynamic (linear and non-linear) behaviour in trapped Bose gases below about \( T \lesssim 0.5T_{BEC} \). The accuracy can be of the order of a few percent, which is as much as one can expect since the non-condensate fraction of atoms at \( T = 0 \) is also estimated to be about 1% or so. The \( T = 0 \) GP equation has been extended to deal with two-component Bose gases (involving two different atomic hyperfine states) and the effect of perturbations related to laser and rf fields (used to manipulate the Bose condensates) is easily incorporated. For further discussion and references, see the lectures by Ballagh in this book.

Our main purpose in this section is to use the GP equation (5) to discuss collective oscillations of a pure condensate. In Section 3 we will discuss the extension of this equation at finite \( T \) to deal with the effect of non-condensate atoms. We first briefly review the static equilibrium solution \( \Phi_0(r) \) of the GP equation, given by
\[ \langle \hat{\psi}(r,t) \rangle \equiv \Phi(r,t) = \Phi_0(r)e^{-i\mu t/\hbar}, \tag{6} \]
where \( \mu \) is the chemical potential of the condensate. The physics behind this can be seen from
\[ \langle N - 1|\hat{\psi}(r,t)|N \rangle = e^{iE_{N-1}t/\hbar} \langle N - 1|\hat{\psi}(r)|N \rangle e^{-iE_{N}t/\hbar} = \langle N - 1|\sqrt{N}|N - 1 \rangle e^{-(E_{N}-E_{N-1})t/\hbar} = \sqrt{N}e^{-i\mu t/\hbar}. \tag{7} \]

Using (6) in (5) gives
\[ \mu_c\Phi_0(r) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + g|\Phi_0(r)|^2 \right] \Phi_0(r). \tag{8} \]
Assuming a vortex-free ground state, this GP equation for the static condensate wavefunction \( \Phi_0(r) = \sqrt{n_c(r)} \) leads to the following expression for the
equilibrium condensate chemical potential

\[ \mu_c = -\frac{\hbar^2 \nabla^2 \sqrt{n_c(r)}}{2m \sqrt{n_c(r)}} + V_{ex}(r) + g n_c(r). \]  

(9)

A standard approximation in solving (9) is to ignore the kinetic energy associated with the condensate amplitude \( \sqrt{n_c} \), i.e., neglect the \(-\frac{\hbar^2 \nabla^2}{2m}\) term. In this “Thomas-Fermi” (TF) approximation, the static GP equation for \( \Phi_0(r) \) reduces to

\[ [V_{ex}(r) + g |\Phi_0(r)|^2] = \mu_c, \]  

(10)

which is easily inverted to give the condensate density profile

\[ n_c(r) = \frac{1}{g} \left[ \mu_c - V_{ex}(r) \right] \]

\[ = \frac{1}{g} \left[ \mu_c - \frac{1}{2} m \omega_0^2 r^2 \right]. \]  

(11)

Clearly in the TF approximation, the size of the condensate is \( R_{TF} \), where

\[ \mu_c = \frac{1}{2} m \omega_0^2 R_{TF}^2. \]  

(12)

One finds \( \mu_c \) from the condition \( \int d^3r n_c(r) = N = N_c \), which gives

\[ \mu_c = \frac{\hbar \omega_0}{2} \left( 15 \frac{N_c a}{a_{HO}} \right)^{2/5}; \quad a_{HO} \equiv (\hbar/m \omega_0)^{1/2}. \]  

(13)

We recall that the oscillator length \( a_{HO} \) is the size of the ground state Gaussian wavefunction of an atom in a parabolic potential. Combining (12) and (13) gives

\[ R_{TF} = a_{HO} \left( 15 \frac{N_c a}{a_{HO}} \right)^{1/5} \]

\[ \gg a_{HO}, \]  

if \( N_c a / a_{HO} \gg 1. \)  

(14)

The TF approximation (11) for \( n_c(r) \) is very good for large \( N_c \), except for a small region near the edge of condensate (\( r \approx R_{TF} \)). In practice, for typical values of \( a_{HO} \) and \( a \), one finds the TF approximation is very good for \( N_c \gtrsim 10^4 \) atoms.

To discuss condensate fluctuations around the static equilibrium value of \( \Phi_0(r) \), we first reformulate the time-dependent GP equation in terms of the condensate density and phase variables

\[ \Phi(r,t) = \sqrt{n_c(r,t)} e^{i\theta(r,t)}. \]  

(15)
Inserting this into the GP equation (3), gives
\[ i\hbar \frac{\partial \sqrt{n_c}}{\partial t} - \sqrt{n_c} \frac{\hbar \partial \theta}{\partial t} = \mu_c(r, t) \sqrt{n_c} - i\hbar^2 \sqrt{n_c} \frac{\nabla^2 \theta}{2m} + \frac{\hbar^2}{2m} (\nabla \theta)^2 \]
\[ - i \frac{\hbar^2}{m} (\nabla \sqrt{n_c}) \cdot \nabla \theta, \] (16)
where we have defined
\[ \mu_c(r, t) \equiv -\frac{\hbar^2 \nabla^2 \sqrt{n_c}}{2m \sqrt{n_c}} + V_{ex}(r) + gn_c(r, t). \] (17)
Separating out the real and imaginary path gives two equations. The real part gives
\[ \hbar \frac{\partial \theta}{\partial t} = -(\mu_c + \frac{1}{2} m v^2_c), \] (18)
where the condensate velocity field is defined by
\[ m v_c(r, t) \equiv \hbar \nabla \theta(r, t). \] (19)
The imaginary part of (16) gives
\[ i \frac{\partial \sqrt{n_c}}{\partial t} = -\frac{\sqrt{n_c}}{2} \nabla \cdot v_c - v_c \nabla \sqrt{n_c}, \] (20)
which can be rewritten in the form
\[ \frac{\partial n_c}{\partial t} = -n_c \nabla \cdot v_c - v_c \cdot \nabla n_c = -\nabla \cdot (n_c v_c). \] (21)
From now on, we shall set \( \hbar = 1 \) except in some final formulas.

In summary, we have shown that the time-dependent GP equation for the two-component order parameter \( \Phi(r, t) \) is completely equivalent to the following two coupled equations for the condensate density \( n_c(r, t) \) and velocity field \( v_c(r, t) \):
\[ \frac{\partial n_c}{\partial t} = -\nabla \cdot (n_c v_c) \]
\[ m \left( \frac{\partial v_c}{\partial t} + \frac{1}{2} \nabla v^2_c \right) = -\nabla \mu_c, \] (22)
where the position and time-dependent generalized condensate chemical potential \( \mu_c \) is defined in (17). In this formulation, a complete description of the condensate dynamics is given in terms of two variables \( n_c(r, t) \) and \( v_c(r, t) \), reminiscent of the hydrodynamic equations for a classical fluid. In the recent BEC literature, the equations in (22) are often referred to as the “hydrodynamic” theory. A better description would be to call it the “quantum

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hydrodynamic” theory, since it is equivalent to the mean-field GP equation. In later sections, we shall see that an exact GP equation taking into account the dynamics of the non-condensate leads to a generalized set of equations quite analogous to (22). The key equation (19) defining the superfluid velocity field of the condensate will turn out to be quite general. Needless to say, all aspects related to superfluidity of a trapped Bose gas (as compared to Bose condensation) are tied to the fact that the condensate exhibits motion related to the gradient of a phase (which means that, ignoring vortices, the condensate motion is irrotational, \(\nabla \times \mathbf{v}_c(r, t) = 0\)).

Stringari first pointed out that within the dynamic TF approximation, one could combine the two equations in (22) into a single condensate wave equation. This approach will be the basis of our development in these lectures. Neglecting the kinetic energy term \(-\nabla^2 \sqrt{n_c}\) in (17) as small compared to the condensate interaction energy \(g n_c\), we linearize the resulting equations around the equilibrium values

\[
\begin{align*}
n_c &= n_{c0} + \delta n_c \\
\mathbf{v}_c &= \mathbf{v}_{c0} + \delta \mathbf{v}_c.
\end{align*}
\]

We obtain

\[
\begin{align*}
\frac{\partial \delta n_c}{\partial t} &= -\nabla \cdot (n_{c0} \delta \mathbf{v}_c) - \nabla \cdot (\mathbf{v}_{c0} \delta n_c) \\
\frac{\partial \delta \mathbf{v}_c}{\partial t} &= -\nabla (\mu'_c + g \delta n_c + m \mathbf{v}_{c0} \cdot \delta \mathbf{v}_c),
\end{align*}
\]

where \(\mu'_c \equiv \mu_{c0} + \frac{1}{2} m \mathbf{v}_{c0}^2\) is independent of position. Assuming that there is no vortex in the solution of the static GP equation (ie, \(\mathbf{v}_{c0} = 0\)), (24) reduces to the following coupled linearized equations for \(\delta n_c\) and \(\delta \mathbf{v}_c\):

\[
\begin{align*}
\frac{\partial \delta n_c}{\partial t} &= -\nabla \cdot (n_{c0}(\mathbf{r}) \delta \mathbf{v}_c) \\
\frac{\partial \delta \mathbf{v}_c}{\partial t} &= -\frac{g}{m} \nabla \delta n_c.
\end{align*}
\]

These can be combined to give the well known \(T = 0\) Stringari wave equation

\[
\frac{\partial^2 \delta n_c}{\partial t^2} = \frac{g}{m} \nabla \cdot [n_{c0}(\mathbf{r}) \nabla \delta n_c].
\]

Since the derivation used the TF approximation (where \(n_{c0}(\mathbf{r})\) is given by (11) for \(r \leq R_{TF}\)), one can rewrite (24) in the equivalent form (for an isotropic trap potential)
Figure 1. Plot of the lowest condensate mode frequencies (monopole and quadrupole) as a function of $N_c$. The full curves are the solutions of the coupled Bogoliubov equations\textsuperscript{14} and the dashed lines are the Stringari TF approximation Ref.\textsuperscript{6}. The data points are for a JILA trap with $\lambda = \sqrt{8}$. For these trap parameters, $N_c$ varies from approximately $10^3$ to $10^4$ atoms. For further details, see Fig.14 of Ref.\textsuperscript{1}. 

\begin{equation}
\frac{\partial^2 \delta n_c}{\partial t^2} = \frac{\mu_c \alpha}{m} \nabla \cdot \left\{ \left[ 1 - \frac{r^2}{R_{TF}^2} \right] \nabla \delta n_c \right\}, \quad r \leq R_{TF}, \quad (27)
\end{equation}

where $\mu_c \alpha$ is given by \textsuperscript{13}. It turns out that the normal mode solutions $\delta n_c(r, t) = \delta n_{\omega}(r) e^{-i\omega t}$ of (27) have frequencies which are independent of the interaction strength $g$ or the value of $N_c$. This is a feature of the underlying TF approximation, which typically starts to breakdown (as noted earlier) when $N_c \lesssim 10^4$ atoms. This is shown by explicit numerical solutions\textsuperscript{13,14} of
the coupled Bogoliubov equations of motion which describe the normal mode solutions of the linearized GP equation when we take the kinetic energy of the condensate amplitude \( \sqrt{n_c} \) fully into account. For values of \( N_c < 10^4 \), the normal mode frequencies depend significantly on the magnitude of \( N_c \), as shown in Fig. 1.

We also note that (26) can be equally well rewritten in terms of the superfluid velocity \( \delta v_c \) or, equivalently, the phase fluctuations \( \delta \theta \). This emphasizes that the measured condensate density fluctuations \( \delta n \) are directly related to the existence of phase fluctuations. The existence of collective modes of a pure condensate may thus be viewed already as “evidence” of superfluidity, the latter being always a consequence of the phase coherence of the macroscopic wavefunction given in (15) which gives rise to the irrotational velocity in (19).

We conclude this section with several examples of condensate normal modes based on solving the \( T = 0 \) Stringari equation (26). A wonderful aspect about the collective oscillations of a condensate in a trapped gas is you can “see” them. As Ketterle has remarked, these condensates are robust - one can kick them, shake them and these “wisps” of Bose-condensed matter keep their integrity.\(^{1,6,15}\) The uniform Bose-condensed gas is especially simple, since \( \delta n_c = \delta n_{k\omega} e^{i(k\cdot r - \omega t)} \). This gives

\[
-\omega^2 \delta n_{k\omega} = \frac{g n_{c0}}{m} (-k^2) \delta n_{k\omega},
\]

or \( \omega^2 = c_0^2 k^2 \), with \( c_0 \equiv \sqrt{gn_{c0}/m} \). This recovers the well-known Bogoliubov phonon oscillations of a uniform Bose condensate. The neglect of the kinetic energy in our TF approximation precludes us from obtaining the particle-like behaviour at large values of the wavevector \( k \).\(^{14}\)

The Kohn (or sloshing) mode corresponds to the oscillation of the centre-of-mass of the static condensate profile with the trap frequency \( \omega_0 \). This mode is described by

\[
n_c(r, t) = n_{c0}(r - \eta(t)),
\]

where \( \frac{d\eta(t)}{dt} \equiv v_c(t) \) and

\[
\frac{d^2 \eta(t)}{dt^2} = -\omega_0^2 \eta(t).
\]

The proof is simple. Linearizing (26), we find

\[
n_c(r, t) = n_{c0}(r) - \eta \cdot \nabla n_{c0}(r),
\]
which gives the explicit form for the condensate fluctuation,

$$\delta n_c(r, t) = \frac{1}{g} m \omega_0^2 r \cdot \eta(t). \quad (32)$$

Inserting this into the Stringari equation (26), we obtain

$$- \nabla n_c \cdot \frac{d^2 \eta(t)}{dt^2} = \omega_0^2 \nabla \cdot [n_{c0} \nabla (\eta(t) \cdot r)] = \omega_0^2 \nabla \cdot [n_{c0} \eta] = \omega_0^2 \eta \cdot \nabla n_c. \quad (33)$$

This confirms that the centre-of-mass position $\eta(t)$ of the static condensate distribution satisfies the SHO equation (30) with frequency $\omega_0$.

The breathing (or monopole) condensate normal mode corresponds to a velocity fluctuation of the form

$$\delta v_c(r, t) = Ar e^{-i\omega t} \quad (r \leq R_{TF}). \quad (34)$$

Using $\nabla \cdot r = 3$ and $\nabla \cdot (r^2 r) = 5r^2$, it is easy to verify from the continuity equation in (22) that the associated density fluctuation is

$$- i\omega \delta n_c = - \nabla \cdot [n_{c0}(r) \delta v_c] = - \frac{\mu_{c0}}{g} A \left[3 - 5 \frac{r^2}{R_{TF}^2}\right] e^{-i\omega t}, \quad (35)$$

or

$$\delta n_c(r) = B \left(1 - \frac{5}{3} \frac{r^2}{R_{TF}^2}\right), \quad (36)$$

where $\mu_{c0}$ is given by (12). Inserting this into the Stringari wave equation (27) gives

$$- \omega^2 \delta n_c(r) = \frac{\mu_{c0}}{m} \nabla \cdot \left[ \left(1 - \frac{r^2}{R_{TF}^2}\right) B \frac{5}{3} \frac{2r}{R_{TF}^2}\right]$$

$$= -5\omega_0^2 \delta n_c(r). \quad (37)$$

Thus the breathing mode for an isotropic parabolic trap has a frequency $\omega = \sqrt{5} \omega_0$. As noted earlier, this mode frequency is not explicitly dependent on the interaction strength $g$. However the underlying theory is very dependent on mean-field effects. Moreover, we recall that a non-interacting trapped Bose gas has a breathing mode with a frequency $\omega = 2\omega_0$ at all temperatures.

As a final example, we consider the so-called “surface” modes of a $T = 0$ condensate, as described by

$$\delta v_c(r, t) = A \nabla \left[r^l Y_{lm}(\theta, \phi)\right] e^{-i\omega t}. \quad (38)$$
This mode corresponds to phase fluctuations of the form $\delta\theta_{\omega} = mA_{l} Y_{lm}(\theta, \phi)$. One may easily verify that $\nabla \cdot \delta \mathbf{v}_c = 0$ and hence from (25) we obtain

$$\frac{\partial^2 \delta \mathbf{v}_c}{\partial t^2} = -\omega_0^2 \nabla (\delta \mathbf{v}_c \cdot \mathbf{r}).$$

This is equivalent to

$$\frac{\partial^2 \delta \theta}{\partial t^2} = -\omega_0^2 (\nabla \delta \theta) \cdot \mathbf{r}$$

or

$$-\omega^2 \delta \theta_{\omega}(\mathbf{r}) = -\omega_0^2 l \delta \theta_{\omega}(\mathbf{r}).$$

Thus the surface oscillations of the condensate phase have a frequency given by $\omega = \sqrt{l} \omega_0 (l = 1, 2, 3, \ldots)$.

One great advantage of the quantum hydrodynamic formalism (within the TF approximation) is that it is easy to also treat the $T = 0$ normal modes of an anisotropic parabolic well. We refer to the literature for further discussion. We also note that it is straightforward to derive a Stringari wave equation for the oscillations of a vortex state (this has been done by Svidzinsky and Fetter).

3 Generalized GP equation at finite temperatures

In this section, we generalize the $T = 0$ GP equation of Section 2 to finite $T$, so that it includes the effect of the coupling of the condensate to the non-condensate degrees of freedom. We concentrate in this Section on how the $T = 0$ condensate modes are renormalized and damped by the thermal cloud. We defer discussion of the appearance of new collective modes mainly associated with the thermal cloud to sections 4 and 5.

Up to the beginning of 2000, the study of the dynamics of a trapped Bose gas at finite temperatures has been largely ignored by experimentalists but actively studied by many theorists (especially in the last year or so). One of the reasons for the lack of finite temperature data is that there are so many interesting phenomena to study at $T \sim 0$! However, another reason seems to be the implicit belief that the presence of the non-condensate just complicates the behaviour of a pure $T = 0$ condensate - but is not the source of any interesting new physics. In this and subsequent sections, I hope to argue that this “belief” is quite wrong. The coupling of the condensate and non-condensate degrees of freedom at finite $T$ leads to a new two-component...
system in which both components can exhibit coherent behaviour, analogous to the well-known superfluid macroscopic phenomena appearing in liquid $^4$He.

We shall see that, as expected, the finite temperature GP equation of motion for $\Phi(r,t)$ is not closed. It’s general solution involves knowing the equations of motion for the non-condensate. We will work within an approximation where the non-condensate atoms can be described by a quantum Boltzmann transport equation for the single-particle distribution function $f(p,r,t)$, with

$$\hat{n}(r,t) = \int \frac{dp}{(2\pi)^3} f(p,r,t).$$

(42)

The details of this kinetic equation will be developed in Section 4. In this section, we concentrate on incorporating the effect of a static thermal cloud into the finite $T$ dynamics of the condensate.

The theory of interacting Bose-condensed fluids is most usefully discussed using quantum field operators. This procedure was formalized by Beliaev in 1957 and developed by Bogoliubov, Gavoret and Nozières, Martin and Hohenberg, and others in the early 1960’s. We recall:

$$\hat{\psi}^\dagger(r) = \text{creates atom at } r$$
$$\hat{\psi}(r) = \text{destroys atom at } r.$$  

(43)

These quantum field operators satisfy the usual Bose commutation relations, such as

$$\left[\hat{\psi}(r), \hat{\psi}^\dagger(r')\right] = \delta(r - r').$$

(44)

All observables can be written in terms of these quantum field operators, such as the density $\hat{n}(r) = \hat{\psi}^\dagger(r)\hat{\psi}(r)$ and interaction energy

$$\hat{V} = \frac{1}{2} \int dr \int dr' \hat{\psi}^\dagger(r')\hat{\psi}^\dagger(r)\psi(r - r')\psi(r')\psi(r)$$

$$= \frac{1}{2} g \int dr\hat{\psi}^\dagger(r)\hat{\psi}^\dagger(r)\psi(r)\psi(r).$$

(45)

The crucial idea due to Bogoliubov and later generalized by Beliaev is to separate out the condensate component of the field operators,

$$\hat{\psi}(r) = \langle \hat{\psi}(r) \rangle + \tilde{\psi}(r),$$

(46)

where

$$\langle \hat{\psi}(r) \rangle \equiv \Phi(r) = \text{Bose macroscopic wavefunction.}$$
This quantity plays the role of the “order parameter” for the Bose superfluid phase transition:

\[ \Phi(r) = 0 \quad T > T_c \]
\[ \neq 0 \quad T < T_c. \]

We note that \( \Phi(r) \equiv \sqrt{n_c} e^{i\theta} \) is a 2-component order parameter. Clearly, \( \Phi(r) \) is not simply related to the many-particle wavefunctions \( \Psi(r_1, r_2, \ldots, r_N) \). The thermal average in \( \langle \hat{\psi}(r) \rangle \) involves a small symmetry-breaking perturbation to allow \( \Phi(r) \) to be finite,

\[
\hat{H}_{SB} = \lim_{\eta \to 0} \int dr \left[ \eta(r) \hat{\psi}^\dagger(r) + \eta^*(r) \hat{\psi}(r) \right].
\] (47)

The philosophy behind the concept of symmetry-breaking was extensively discussed by Bogoliubov, for a variety of condensed matter systems, in an article which is still highly recommended.

It is useful to make a few comments on the physics behind \( \Phi(r, t) \). \( \Phi(r, t) \) is a coherent state, with a “clamped” value of phase - rather than a Fock-state of fixed \( N_c \), with no well-defined phase. \( \Phi(r, t) \) acts like a classical field, since quantum fluctuations are negligible when \( N_c \) is large. Probably P.W. Anderson deserves the greatest credit for understanding (in the period 1958-1963) the new physics behind working with a broken-symmetry state \( \Phi(r, t) \), both in BCS superconductors and in superfluid \( ^4\)He. It captures the physics of the new phase of matter (such as the occurrence of the Josephson effect) and the associated superfluidity. The symmetry-breaking perturbation (47) allows the system to internally set up off-diagonal symmetry-breaking fields, which persist even when the external symmetry-breaking perturbation in (47) is set to zero at the end (\( \eta \to 0 \)). The same sort of physics is the basis of the BCS theory of superconductors.

One can formulate the GP and Bogoliubov approximations directly in terms of a variational many-particle wavefunction (see the lectures by Leggett in this volume). However, such formulations are limited to simple mean-field approximations. The explicit introduction of the broken-symmetry order parameter \( \Phi(r, t) \) gives a systematic way of isolating the role of the Bose condensate in a general treatment of an interacting Bose-condensed fluid. As we shall see, the resulting formalism allows one to deal with questions related to damping as well as superfluidity in both the collisionless and hydrodynamic regions.
The exact Heisenberg equation of motion for the field operator is
\[
i \frac{\partial \hat{\psi}(r, t)}{\partial t} = \left[ -\frac{\nabla^2}{2m} + V_{ex}(r) + \delta U(r, t) \right] \hat{\psi}(r, t) + \eta(r) + g \hat{\psi}^\dagger(r, t) \hat{\psi}(r, t),
\]
where \(\delta U(r, t)\) is a small time-dependent driving potential. This gives an exact equation of motion for \(\Phi(r, t) \equiv \langle \hat{\psi}(r, t) \rangle\),
\[
i \frac{\partial \Phi(r, t)}{\partial t} = \left[ -\frac{\nabla^2}{2m} + V_{ex}(r) + \delta U(r, t) \right] \Phi(r, t) + \eta(r) + g \langle \hat{\psi}^\dagger(r, t) \hat{\psi}(r, t) \rangle,
\]
with \[\text{using the decomposition (46)}\]
\[
\hat{\psi}^\dagger \hat{\psi} \hat{\psi} = |\Phi|^2 \Phi + 2|\Phi|^2 \tilde{n} \Phi + |\Phi|^2 \tilde{m} \Phi^* + 2 \tilde{n} \Phi + \langle \hat{\psi}^\dagger \hat{\psi} \rangle. \tag{50}
\]

Taking the symmetry-breaking average of (50), one finds
\[
\langle \hat{\psi}^\dagger \hat{\psi} \rangle = n_c \Phi + \tilde{m} \Phi^* + 2 \tilde{n} \Phi + \langle \hat{\psi}^\dagger \hat{\psi} \rangle, \tag{51}
\]
where
\[
n_c(r, t) \equiv |\Phi(r, t)|^2 = \text{condensate density}
\]
\[
\tilde{n}(r, t) \equiv \langle \hat{\psi}^\dagger(r, t) \hat{\psi}(r, t) \rangle = \text{non-condensate density}
\]
\[
\tilde{m}(r, t) \equiv \langle \hat{\psi}(r, t) \hat{\psi}(r, t) \rangle = \text{off-diagonal (anomalous) density}. \tag{52}
\]

Using (51) in (49), our “exact” equation of motion for \(\Phi(r, t)\) is
\[
i \frac{\partial \Phi(r, t)}{\partial t} = \left[ -\frac{\nabla^2}{2m} + V_{ex}(r) + gn_c(r, t) + 2g\tilde{n}(r, t) \right] \Phi + g \tilde{m}(r, t) \Phi^* + g \langle \hat{\psi}^\dagger(r, t) \hat{\psi}(r, t) \rangle. \tag{53}
\]

We now consider various approximations to the generalized GP equation in (53):

(a) The Hartree-Fock-Bogoliubov (HFB) approximation for \(\Phi\) corresponds to neglecting the three-field correlation function \(\langle \hat{\psi}^\dagger \hat{\psi} \rangle\) but keeping the \(n_c, \tilde{n}\) and \(\tilde{m}\) fluctuations. This HFB has been exhaustively treated in many recent papers, in conjunction with the separate equations of motion for \(\tilde{n}\) and \(\tilde{m}\). This HFB approximation can be used to generate the same normal mode spectrum which the Beliaev second-order self-energy approximation gives at finite \(T\).
(b) The dynamic Popov approximation corresponds to ignoring both \( \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \) and \( \tilde{m} = \langle \tilde{\psi} \tilde{\psi} \rangle \) in (53). Theories of this kind involve coupled equations for \( \Phi \) and \( \tilde{n} \).

(c) The static Popov approximation involves even a further simplification, namely it ignores fluctuations in the density \( \tilde{n}(r,t) \) of the thermal cloud,

\[
\tilde{n}(r,t) \simeq n_{c0}(r).
\]

Using (54) in (53) corresponds to treating the dynamics of the condensate moving in the static mean field of the non-condensate thermal cloud, ie,

\[
i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\nabla^2}{2m} + V_{\text{ex}}(r) + 2g\tilde{n}_0(r) + gn_c(r,t) \right] \Phi(r,t).
\]

There is a considerable literature based on this static Popov-approximation. In a nutshell, in this section, we will discuss an extension of (55) which involves including the damping associated with the \( \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \) term, but again treating the thermal cloud statically.

As in Section 2, we first rewrite (53) in terms of the condensate amplitude and phase. Inserting (15) into (53) and following the same procedure which led to the \( T = 0 \) equations in (22), we obtain:

\[
\frac{\partial n_c}{\partial t} + \nabla \cdot n_c v_c = 2g \text{Im} \left[ \Phi^* m + \Phi^* \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \right]
\]

\[
\frac{\partial \theta}{\partial t} = - \left( \mu_c + \frac{(\nabla \theta)^2}{2m} \right),
\]

where now (compare with (17))

\[
\mu_c(r,t) = - \frac{\nabla^2 \sqrt{n_c}}{2m \sqrt{n_c}} + V_{\text{ex}} + gn_c + 2g\tilde{n} \\
+ \frac{2}{n_c} Re \left[ \Phi^* \tilde{m} + \Phi^* \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \right].
\]

Formally, (53) and hence (56) - (58) are \textit{exact} results. In these lectures, we will limit ourselves to finite temperatures where the dominant thermal excitations...
are well approximated by high energy non-condensate atoms moving in a self-consistent dynamic HF mean-field
\[ \tilde{\varepsilon}_p(r, t) = \frac{p^2}{2m} + V_{ex}(r) + 2g [n_c(r, t) + \tilde{n}(r, t)] \]
\[ \equiv \frac{p^2}{2m} + U(r, t). \] (59)

In this region, we shall neglect the anomalous pair correlations among the thermal atoms (i.e., \( \tilde{m} = 0 \)). One can show that both \( \tilde{m} \) and \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \) are of order \( g \). Thus one sees the correction terms in (56) and (58) involving these functions are of order \( g^2 \). In fact, calculation shows that to order \( g \), \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \) is imaginary. Our procedure is to ignore the \( O(g^2) \) terms in (56) and (58) except those which are imaginary and hence are a source of damping of condensate motion.

The end result of this approach is that we are left with
\[ \frac{\partial n_c}{\partial t} + \nabla \cdot n_c v_c = -\Gamma_{12}[f, \Phi] \]
\[ m \left( \frac{\partial v_c}{\partial t} + \frac{1}{2} \nabla v_c^2 \right) = -\nabla \mu_c, \] (60)

with
\[ \mu_c(r, t) = -\frac{\nabla^2 \sqrt{n_c}}{2m \sqrt{n_c}} + V_{ex}(r) + gn_c(r, t) + 2g\tilde{n}(r, t) \] (61)
and we have defined the new function
\[ \Gamma_{12}[f, \Phi] \equiv -2gIm \left[ \Phi^* \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \right]. \] (62)

We see that \( \Gamma_{12} \) plays the role of a source term in the condensate continuity equation in (60). It depends on both \( \Phi(r, t) \) and the single-particle distribution function \( f(p, r, t) \). It is useful to note that (60), and (61) are equivalent to the approximate GP equation (see also Refs. [28, 29])
\[ i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + gn_c(r, t) + 2g\tilde{n}(r, t) - i\hbar R(r, t) \right] \Phi, \] (63)
where the dissipative function is
\[ R(r, t) \equiv \frac{\Gamma_{12}[f, \Phi]}{2n_c(r, t)} \sim 0(g^2). \] (64)

To proceed, we need to calculate the field correlation \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \), which determines \( \Gamma_{12} \) in (62). This has been done in Appendix A of Ref. [28, working
to lowest order in $g$ at finite $T$ where the single-particle spectrum in (59) is adequate. The result is

$$\langle \tilde{\psi} \tilde{\psi} \rangle = -ig \frac{\Phi(r,t)}{(2\pi)^5} \int dp_1 \int dp_2 \int dp_3 \times \delta(p_c + p_1 - p_2 - p_3) \delta(\epsilon_c + \tilde{\epsilon}_1 - \tilde{\epsilon}_2 - \tilde{\epsilon}_3) \times [f_1(1 + f_2)(1 + f_3) - (1 + f_1)f_2f_3], \quad (65)$$

where $f_1 \equiv f(p, r, t), \epsilon_c \equiv \mu_c(r, t) + \frac{1}{2}mv_c^2(r, t)$ is the local condensate atom energy and $p_c \equiv mv_c$ is the condensate atom momentum. In Section 4, we shall see that (65) is closely related to the collision integral $C_{12}[f, \Phi]$ which enters the Boltzmann equation for $f(p, r, t)$ and describes collisions with the non-condensate atoms in which one condensate atom is involved. We note that to leading order, the expression in (65) is imaginary. Using this in (62), we find

$$\Gamma_{12}(r, t) = 2g^2n_c(r, t) \frac{\Phi(r,t)}{(2\pi)^5} \int dp_1 \int dp_2 \int dp_3 \delta(p_c + p_1 - p_2 - p_3) \times \delta(\epsilon_c + \tilde{\epsilon}_1 - \tilde{\epsilon}_2 - \tilde{\epsilon}_3) \times [f_1(1 + f_2)(1 + f_3) - (1 + f_1)f_2f_3]. \quad (66)$$

As a first application of our new equations in (60), we will ignore the dynamics of the non-condensate cloud. That is, we will assume that the condensate interacts with a static thermal cloud in thermal equilibrium

$$f(p, r, t) \simeq f^0(p, r) = \frac{1}{e^{\beta[p^2/2m + U_0(r) - \tilde{\mu}_0]} - 1}, \quad (67)$$

where $\tilde{\mu}_0$ is the equilibrium chemical potential of the non-condensate atoms. In Section 4, we will discuss the $C_{22}$ collisions which produce this static equilibrium Bose distribution. The value of $\tilde{\mu}_0$ is known since it must equal the condensate equilibrium chemical potential $\mu_{c0}$ given by the static solution of the generalized GP equation. Within the TF approximation, (61) reduces to

$$\tilde{\epsilon}_p(r) - \tilde{\mu}_0 = \frac{p^2}{2m} + U_0(r) - \tilde{\mu}_0$$

and hence we see that (65) reduces to

$$\tilde{\epsilon}_p(r) - \tilde{\mu}_0 = \frac{p^2}{2m} + gn_{c0}(r) = \frac{p^2}{2m} + gn_{c0}(r). \quad (69)$$
This result is consistent with the correct Bogoliubov “excitation energy.”

The static non-condensate density in (68) is given by

\[ \tilde{n}_0(r) = \frac{1}{\Lambda} g_{3/2} \left( z_0 = e^{-\beta g n_c(r)} \right). \]  

(70)

We can now calculate \( \Gamma_{12}(r, t) \) using the equilibrium Bose distribution (67) for the thermal cloud. One finds (for a related calculation, see Ref. 8). \( \Gamma^0_{12}(r, t) = \frac{n_c(r, t)}{\tau_{12}(r, t)} \left[ e^{-\beta(\tilde{\mu}_0 - \varepsilon_c(r, t) - \frac{1}{2} m v_c^2) - 1} \right], \)  

(71)

where we have defined a collision time

\[ \frac{1}{\tau_{12}(r, t)} = \frac{2g^2}{(2\pi)^3} \int dp_1 \int dp_2 \int dp_3 \delta(p_c + p_1 - p_2 - p_3) \times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3})(1 + f_0^0 f_2^0 f_3^0). \]  

(72)

\( \Gamma_{12}[f_0^0, \Phi] \) in (74) still depends on \( n_c(r, t) \) and \( v_c(r, t) \) of the condensate through \( \mu_c, p_c \) and \( \epsilon_c \). We note that when the condensate is in static equilibrium, \( \mu_c(r, t) \rightarrow \mu_{c0} \) and \( v_{c0} = 0 \). In this case, \( \mu_{c0} = \tilde{\mu}_0 \) and the expression in the square bracket in (71) is seen to vanish. Thus \( \Gamma_{12}(f_0^0, \Phi_0) = 0 \), as it should when both components are in static thermal equilibrium.

Summarizing, at this point we have a closed set of equations which can be used to describe the dynamics of the condensate in a trapped Bose gas which include the interactions with a static equilibrium thermal cloud. These equations are

\[ \frac{\partial n_c}{\partial t} + \nabla \cdot n_c v_c = -\Gamma_{12}[f_0^0, \Phi] \]  

(73)

\[ m \left( \frac{\partial}{\partial t} + v_c \cdot \nabla \right) v_c = -\nabla \mu_c, \]  

(74)

where \( \Gamma_{12}[f_0^0, \Phi] \) is given explicitly by the expression in (71) and

\[ \mu_c(r, t) = -\frac{\nabla^2 \sqrt{n_c}}{2m \sqrt{n_c}} + V_{ex}(r) + 2g \tilde{n}_0(r) + g n_c(r, t). \]  

(75)

From (73) and (74), we can obtain linearized equations of motion for the condensate fluctuations \( \delta n_c \) and \( \delta v_c \). We use the fact that, to lowest order in the fluctuations from static equilibrium, (71) reduces to

\[ \delta \Gamma^0_{12} = \frac{\beta n_{c0}(r)}{\tau_{12}^0(r)} \delta \mu_c(r, t), \]  

(76)
where the “equilibrium” $C_{12}$ collision rate [using (69) in (72)] is defined by

$$
\frac{1}{\tau_{12}^0(r)} = \frac{2g^2}{(2\pi)^3} \int dp_1 \int dp_2 \int dp_3 \delta(p_1 - p_2 - p_3) \times \delta(p_1^2 - p_2^2 - p_3^2 - gn_c)(1 + f_i^0)f_i^0 f_i^0.
$$

(77)

In the static TF approximation, we recall that the equilibrium distribution is given by

$$
f_{0i} = \left[ e^{\beta(p_i^2/2m + gn_c)} - 1 \right]^{-1}.
$$

In the present discussion, we further restrict ourselves to the dynamic Thomas-Fermi limit valid for large $N_c$,

$$
\frac{\partial \delta n_c}{\partial t} + \nabla \cdot (n_c \delta v_c) = -\frac{1}{\tau'} \delta n_c
$$

(78)

$$
m \frac{\partial \delta v_c}{\partial t} = -g \nabla \delta n_c.
$$

(79)

The collision time $\tau'(r)$ describes collisions between the condensate and non-condensate atoms when the condensate is perturbed away from equilibrium,

$$
\frac{1}{\tau'(r)} = \frac{gn_{c0}(r)}{k_B T} \frac{1}{\tau_{12}^0(r)}.
$$

(80)

The new term on the right-hand side of (78) causes damping of the condensate fluctuations due to the lack of collisional detailed balance between the condensate and the static thermal cloud. We note that this collision time $\tau'(r)$ is only a function of the position $r$ through its dependence on the static condensate density $n_{c0}(r)$.

We can easily combine (78) and (79) to obtain what we shall refer to as the finite $T$ Stringari wave equation

$$
\frac{\partial^2 \delta n_c}{\partial t^2} - \frac{g}{m} \nabla \cdot (n_{c0} \nabla \delta n_c) = \frac{1}{\tau'} \frac{\partial \delta n_c}{\partial t}.
$$

(81)

If we neglect the right-hand side, we obtain the undamped finite $T$ Stringari normal modes $\delta n_c(r,t) = \delta n_i(r)e^{-i\omega t}$ given by the solution of

$$
-\frac{g}{m} \nabla \cdot [n_{c0}(r) \nabla \delta n_i(r)] = \omega^2 \delta n_i(r).
$$

(82)

As has been noted by several authors in recent papers $^{13,24}$, $n_{c0}(r)$ at finite $T$ can be well approximated by the TF condensate profile at $T = 0$ but with the number of atoms in the condensate $N_c(T)$ now being a function of temperature. This is because the static mean field of the non-condensate plays a minor role. In the regions where the condensate density is finite, we effectively always have $n_{c0}(r) \gg \tilde{n}_0(r)$, as illustrated in Fig. 2. With this
Figure 2. Density of atoms as a function of position in an isotropic parabolic trap containing 5000 $^{87}$Rb atoms at $T = 100$ nK, with $T_{\text{BEC}} = 149$ nK. At this temperature, the total number of atoms in the condensate and non-condensate are equal. The solid line is the total density and the dashed line is the non-condensate density. The TF approximation is not used. The parameter $a_{\text{HO}}$ is the oscillator length. See Ref. 7.

approximation for $n_c(r)$, the solutions of the finite $T$ Stringari equation (81) will be identical to those at $T = 0$ (see Section 2), since the $T = 0$ Stringari frequencies do not depend on the magnitude of $N_c$. Of course, as shown by calculations solving the coupled Bogoliubov equations (14), when $N_c < 10^4$ the TF approximation breaks down. Thus the condensate collective mode frequencies will always become temperature dependent close to $T_{\text{BEC}}$, where the TF approximation is no longer valid. 13
Figure 3. Collision rates in a homogeneous Bose-condensed gas, normalized to the collision time $\tau_{cl}$ in a classical gas at the BEC transition temperature $\tau_{cl}^{-1}(T_{BEC})$. We have taken $\gamma_n = 0.1k_BT_{BEC}$, where $n$ is the total density. See also Ref. [8].

We can use the undamped Stringari modes given by (82) as a basis set to solve (81) and find the damping of these modes. Writing $\delta n_i(r) = \sum c_i \delta n_i(r)$, and using the orthonormality condition $\int d\mathbf{r} \delta n_i(\mathbf{r}) \delta n_j(\mathbf{r}) = \delta_{ij}$, one obtains the following algebraic equations for the coefficients $c_i$

$$\omega^2 c_i = \omega_\tau^2 c_i - i\omega \sum_j \gamma_{ij} c_j, \quad (83)$$

where

$$\gamma_{ij} \equiv \int d\mathbf{r} \delta n_i(\mathbf{r}) \delta n_j(\mathbf{r}) / \tau'(\mathbf{r}). \quad (84)$$
Assuming the damping is small (our present discussion is for the collisionless region), (83) is easily solved using perturbation theory by setting \( \gamma_{ij} = 0 \) for \( j \neq i \). This gives the damped Stringari frequency (to lowest order) \( \Omega_i = \omega_i - i\Gamma_i \), with

\[
\Gamma_i = \frac{\gamma_{ii}}{2} = \frac{1}{2} \int dr \frac{\delta n_i(r)^2}{\tau'(r)}.
\]  

This result for \( \Gamma_i \) is reasonable, namely it involves a spatial average over \( 1/\tau'(r) \) weighted with respect to the undamped density fluctuations of the Stringari wave equation (81). Calculation shows that the effect of coupling to other modes \( (\gamma_{ij} \neq 0) \) is extremely small.

Before discussing the trapped gas, it is useful to first apply our theory to a homogeneous gas, where \( \tau' \) is independent of position. In this case, (85) simply reduces to \( \Gamma_i = 1/2\tau' \). Although our model applies only to the collisionless region, it is useful to compare the inter-component collision time for a uniform gas in the collisionless and hydrodynamic regimes. In Section 5, we show that the inter-component collision time \( \tau_{\mu} \) in the hydrodynamic region is given by \( \tau_{\mu} = \sigma_H \tau' \), where the temperature-dependent factor \( \sigma_H \) depends on various thermodynamic functions \( \{\sigma_H \) is given explicitly by (165) in Section 4. In Fig. 4, we compare \( 1/\tau' \) and \( 1/\tau_{\mu} \) as functions of \( T \).

We briefly discuss some numerical calculations of the inter-component damping using the above formalism. We consider \(^{87}\text{Rb} \) atoms in a spherically symmetric trap with frequency \( \nu_0 = 10 \text{ Hz} \) and \( N = 2 \times 10^6 \). In the collisionless limit, we require \( \omega_i \tau_{cl} \gg 1 \). We obtain an upper limit on \( 1/\tau_{cl} \) by taking the density in the center of the trap \( n(r = 0) \), which gives \( 1/\tau_{cl} = 8\pi^2 N\omega_0^3 m/(\pi k_B T) \). For the parameters we use, \( \omega_{10} \tau_{cl} \approx 19 \) (compared to \( \omega_{10} \tau_{cl} \approx 20 \) for the JILA data and \( \omega_{10} \tau_{cl} \approx 2 \) for the MIT data on collective oscillations at finite \( T \)).

In Fig. 3 we plot the damping rate \( \Gamma_{10} \) for the breathing mode \((n = 1, l = 0)\) as a function of temperature up to \( T = 0.95T_{\text{BEC}} \), where \( N_c \approx 7 \times 10^4 \). At higher temperatures, the Thomas-Fermi approximation will start to break down and the mode frequencies become temperature dependent.

The damping of condensate modes we have discussed in this section is due to the fact that the condensate is out of diffusive equilibrium with the thermal cloud. At finite \( T \), the collective oscillations of the coupled condensate and non-condensate can be generally split into two classes. One mode
mainly involves (out-of-phase) motion of the condensate and is the finite $T$ generalization of the analogous $T = 0$ condensate mode. For the same symmetry, there is another mode which mainly involves the (in-phase) motion of the non-condensate thermal cloud. This mode is naturally viewed as a generalization of the $T > T_{BEC}$ thermal cloud oscillation to temperatures below $T_{BEC}$. These two types of modes (for a given symmetry) have been obtained in the collision-dominated hydrodynamic region, when the dynamics of the thermal cloud is fully allowed for. Treating (to first approximation) the thermal cloud (the non-condensate component) as always in static equilibrium in this section means that our theory is only applicable to condensate oscillations which involve motions which are out-of-phase with the thermal cloud.

As we discuss in Section 4, the possibility that the condensate may be out of diffusive equilibrium with the non-condensate is a general feature of the dynamics of trapped Bose-condensed gases. In the hydrodynamic two-fluid region, it leads to a new relaxational mode. This effect is missed in the
well-known form of the two-fluid hydrodynamics developed by Landau, where one (implicitly) assumes that the normal and superfluid components are always in local diffusive equilibrium. We will return to this question in Sections 4 and 5.

Of course, in addition to the inter-component relaxation discussed in this section, one also has Landau and Beliaev damping of condensate oscillations. These mechanisms are treated in more detail by Burnett in this book. Landau damping is briefly discussed in Section 4.

Clearly one can extend the formalism of this section to any problem based on the $T = 0$ GP equation. In particular, it can be used to discuss this kind of inter-component damping at finite $T$ of oscillations in two-component Bose gases, vortex dynamics, and Josephson oscillations between two traps.

4 Dynamics of the coupled condensate and non-condensate at finite temperatures

In contrast to Section 3, we now grapple with the dynamics of the thermal cloud. As we noted in Section 3, in dealing with this very complicated problem, we will be quite modest and treat the non-condensate using the simplest microscopic model which captures the important physics. We limit ourselves to finite $T$, where the non-condensate atoms can be described by the particle-like HF spectrum given by (59). For trapped Bose gases, this spectrum is probably adequate down to quite low temperatures, for reasons discussed in an important paper by the Trento group. To extend our present analysis to very low temperatures is much more involved since then the excitations of the thermal cloud take on a collective aspect (ie, a Bogoliubov-type quasiparticle spectrum must be used). The single-particle spectrum (59) is appropriate in the semi-classical limit, where we can use the single-particle distribution function $f(p, r, t)$ given by the solution of a kinetic equation. This procedure generalizes the approach of Boltzmann (1880’s) for describing binary collisions in a classical gas. Such a quantum Boltzmann equation for a trapped Bose-condensed gas at finite temperatures has been derived and extensively discussed by Zarembo, Nikuni and the author. The conditions of validity are

$$k_B T \gg g n_c, \quad k_B T \gg \hbar \omega_0,$$

where $\hbar \omega_0$ is the spacing of the energy levels of the harmonic trap. More general but less explicit discussions of kinetic equations are given in Refs. 28, 29. Related work of Gardiner and coworkers is discussed in the lectures by
Ballagh in this book. The quantum kinetic equation we use is given by

$$\frac{\partial f(p, r, t)}{\partial t} + \frac{p}{m} \cdot \nabla_r f(p, r, t) - \nabla_r U(r, t) \cdot \nabla_p f(p, r, t) = C_{22}[f] + C_{12}[f].$$  \hspace{1cm} (87)

The right hand side describes how binary collisions effect the value of the single-particle distribution function $f(p, r, t)$. The effective time-dependent HF potential $U(r, t)$ is defined in (59). The effect of collisions between excited atoms in the non-condensate is described by:

$$C_{22}[f] = \frac{2g^2}{(2\pi)^5} \int dp_2 \int dp_3 \int dp_4 \delta(p + p_2 - p_3 - p_4) \times \delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4}) \times [(1 + f)(1 + f_2)f_3f_4 - f f_2(1 + f_3)(1 + f_4)].$$  \hspace{1cm} (88)

We recall that creating a Boson gives a factor $(1 + f)$ and destroying a Boson gives $f$. In the classical high temperature limit, $f \ll 1$ and the collision integral $C_{22}$ in (88) becomes much simpler.

In addition to $C_{22}$ collisions, we also have collisions which involve one condensate atom:

$$C_{12}[f] = \frac{2g^2}{(2\pi)^6} \int dp_1 \int dp_2 \int dp_3 (m \nu_c + p_1 - p_2 - p_3) \times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) \times [n_c(1 + f_1)f_2f_3 - n_c f_1(1 + f_2)(1 + f_3)].$$  \hspace{1cm} (89)

For convenience, we recall that the condensate atom has

- energy: $\varepsilon_c = \mu_c + \frac{1}{2} m \nu_c^2$; $\mu_c = V_{ex} + gn_c + 2g \tilde{n}$
- momentum: $p_c = m \nu_c$

We note the key difference between $C_{12}$ and $C_{22}$ collisions:

- $C_{22}$ and $C_{12}$ both conserve energy and momentum in collisions.
- $C_{12}$ does not (but $C_{22}$ does) conserve the number of condensate atoms. $C_{12}$ describes how atoms are “kicked” in and out of the condensate.

It turns out in the generalized GP equation given in (63), the damping term $-iR(r, t)$ is closely related to $C_{12}[f]$. This makes sense, since the $C_{12}$ collisions modify the condensate described by $\Phi(r, t)$. One easily may verify that [comparing (66) and (89)]

$$\Gamma_{12}[f, \Phi] = \int \frac{dp}{(2\pi)^3} C_{12}[f(p, r, t)].$$  \hspace{1cm} (91)
More precisely, the three-field correlation function in the exact equation of motions [see (56) and (58)] is related to \( C_{12}[f] \) by

\[
\int \frac{dp}{(2\pi)^3} C_{12}[f] = 2g\sqrt{n_c} \text{Im} \langle \tilde{\psi}^\dagger \tilde{n} \tilde{\psi} \rangle.
\] (92)

We have to solve the equation of motions for \( f(p, r, t) \) and \( \Phi(r, t) \) treating \( C_{12}[f] \) very carefully. There will be an exchange of atoms between the \( \tilde{n}(r, t) \) and \( n_c(r, t) \) components through the \( C_{12} \) collisions. We can use these coupled equations for a variety of problems. Recently these coupled equations have been used in two problems:

(a) Derivation of a generalized set of two-fluid hydrodynamic equations.

(b) Discussion of the rate of growth of a condensate due to a sudden quench in which the high energy spectrum of the thermal cloud distribution is suddenly removed. This work has many points of contact with the formalism reviewed by Ballagh in this book for condensate growth.

In this section, we will linearize these equations and consider the collective oscillations of the combined system composed of condensate and non-condensate. It is useful to introduce two regimes to describe collective modes in interacting systems:

I. Collisionless (produced by mean fields)

\[ \omega \tau_R \gg 1 \text{ or } T \ll \tau_R \left( \omega \equiv \frac{2\pi}{T} \right) \] (93)

II. Hydrodynamic (produced by collisions)

\[ \omega \tau_R \ll 1 \text{ or } T \gg \tau_R, \] (94)

where \( \tau_R \) is some appropriate relaxation time. What should we use for \( \tau_R \)? For a uniform classical gas, this is the collision time

\[
\frac{1}{\tau_c} = \sqrt{2\tilde{n}\sigma \tilde{v}}
\] (95)

where

\[
\sigma = 8\pi a^2 \text{ (for Bose atoms); } a = s\text{-wave scattering length.}
\]

\[ \tilde{v} \simeq \text{average velocity of atoms } \sim \sqrt{\frac{k_B T}{m}}. \]

\[ \tilde{n} = \text{density of excited atoms.} \]
Even for a Bose-condensed gas, taking $\tau_R \sim \tau_c$ is a reasonable first estimate. To get into the interesting hydrodynamic region ($\omega \tau_R \ll 1$), we need a small value of $\tau_R$, i.e., a large density $\tilde{n}$ or a large collision cross-section $\sigma$ (perhaps by working near a Feshbach resonance). We note that current BEC experiments deal with $N \approx 10^6$ atoms or larger and thus the kind of density of thermal atoms needed to be in the hydrodynamic region now seem achievable.

Let us look at the kinetic equation (97), writing it in the schematic form:

$$\hat{L}f = C_{22}[f] + C_{12}[f].$$

(96)

In the collisionless region, we need only solve $\hat{L}f = 0$. For comparison with the collision-dominated hydrodynamic domain, we briefly discuss some features of this collisionless domain, limiting ourselves to a uniform gas for simplicity. This enables us to understand how Landau and Beliaev damping arise from the collisionless Boltzmann equation, in contrast to the inter-component collisional damping (discussed in Section 3) which has its origin in the $C_{12}[f, \Phi]$ collision integral.

We consider linear response theory for the deviations of $f(p, r, t)$ from thermal equilibrium

$$f(p, r, t) = f_0(p, r) + \delta f(p, r, t).$$

(97)

For a uniform normal Bose gas to first order in $\delta f$, the collisionless kinetic equation reduces to

$$\frac{\partial \delta f}{\partial t} + \frac{p}{m} \cdot \nabla_r \delta f - \nabla_r \left[2g\delta \tilde{n}(r, t) + \delta U(r, t)\right] \cdot \nabla_p f_0(p) = 0.$$  

(98)

Assuming $\delta U(r, t) = \delta U_{k\omega} e^{i(k \cdot r - \omega t)}$, then we have $\delta f(p, r, t) = \delta f_{k\omega}(p) e^{i(k \cdot r - \omega t)}$, where $\delta f_{k\omega}(p)$ satisfies

$$- i\omega \delta f(p) + \frac{p}{m} \cdot k \frac{\partial f_0}{\partial \epsilon_p} \delta f - [2g\delta n_{k\omega} + \delta U_{k\omega}] k \cdot \nabla_p f_0(p) = 0,$$

(99)

where

$$f_0(p) = \frac{1}{e^{\beta \frac{\epsilon_p}{m} + g n_0 - \mu_0} - 1} \equiv f_0(\epsilon_p)$$

(100)

and

$$\delta n_{k\omega} \equiv \int \frac{dp}{(2\pi)^3} \delta f_{k\omega}(p).$$

(101)

Solving (99), one finds

$$\delta n_{k\omega} = - \int \frac{dp}{(2\pi)^3} \frac{p \cdot k}{m} \frac{\partial f_0(\epsilon_p)}{\partial \epsilon_p} \frac{[2g\delta n_{k\omega} + \delta U_{k\omega}]}{\omega - \frac{p^2}{m}}$$

(102)
or

\[ \delta n_{k\omega} = \frac{\chi_{nn}^0(k,\omega)}{1 - 2g \chi_{nn}^0(k,\omega)} \delta U_{k\omega} \equiv \chi_{nn}(k,\omega) \delta U_{k\omega}. \quad (103) \]

Here

\[ \chi_{nn}^0(k,\omega) = -\int \frac{dp}{(2\pi)^3} \frac{p \cdot k}{m} \frac{\partial f_0(\epsilon_p)}{\partial \epsilon_p} \frac{1}{\omega - \frac{p \cdot k}{m}} \quad (104) \]

is the \( k \to 0 \) limit of the density response function of a non-interacting uniform Bose gas.

This shows how one can use the collisionless kinetic equation to find the density response function \( \chi_{nn}(k,\omega) \) of a uniform Bose gas. The density fluctuations are given by the poles of \( \chi_{nn}(k,\omega) \) in (103). In this RPA with exchange, the solutions of

\[ 1 - 2g \chi_{nn}^0(k,\omega) = 0 \quad (105) \]

are called “zero sound” modes (following Landau’s terminology in the analogous calculation for an interacting Fermi gas). It is clear that the solutions \( \omega(k) \) of (105) will be Landau damped, with a width related to

\[ -2g \text{Im} \chi_{nn}^0(k,\omega + i0^+) \]

\[ = 2g\pi \int \frac{dp}{(2\pi)^3} \left[ f_0(\epsilon_p) - f_0(\epsilon_{p+k}) \right] \delta[\omega(k) - (\epsilon_{p+k} - \epsilon_p)]. \quad (106) \]

Clearly this damping will come from the zero sound mode absorbing a thermal excitation of energy \( \epsilon_p \) to create a thermal excitation \( \epsilon_{p+k} \),

\[ \hbar \omega(k) = \epsilon_{p+k} - \epsilon_p \quad. \quad (107) \]

We might also remark that if this linear response calculation is extended to deal with a Bose-condensed gas, one finds a new damping mechanism because now \( \chi_{nn}^0(k,\omega) \) describes non-interacting Bogoliubov excitations \( E_p \). In addition to the contributions of the kind given by (107), one finds \( \chi_{nn}^0(k,\omega) \) now has additional poles at

\[ \hbar \omega(k) = E_{p+k} + E_p. \quad (108) \]

These give rise to “Beliaev” damping, where the zero sound mode can decay into two Bogoliubov excitations. In the limit of \( T \to 0 \), Beliaev damping dominates over Landau damping. It was first calculated at \( T = 0 \) by Beliaev in 1957 in his classic study of a weakly interacting Bose gas.

Several recent papers have discussed Landau and Beliaev damping of collective modes in a trapped Bose gas. This problem has been nicely
formulated directly in terms of response functions by Minguzzi and Tosi in conjunction with mean fields produced by the condensate and non-condensate density fluctuations. Giorgini has given a physically appealing discussion working directly with the coupled mean field equations of motion for $\Phi(r,t), \tilde{n}(r,t)$ and $\tilde{m}(r,t)$. The approach of the Oxford group is reviewed in Burnett’s lectures in this book.

In this connection, a simple treatment of Landau damping of condensate collective modes can be given starting from our generalized GP equation in (63). In Section 3 we ignored the fluctuations in the HF term $2g\tilde{n}(r,t)$ and concentrated on the damping associated with the $-iR(r,t)$ term in (69). A simple way of including the $\delta\tilde{n}(r,t)$ fluctuations induced by the mean field associated with the condensate fluctuations is to use (see Refs. 26, 8)

$$\delta\tilde{n}(r,\omega) = \int dr' \chi_{\tilde{n}\tilde{n}}^0(r,r',\omega)2g\delta n_c(r',\omega).$$

(109)

Here $\chi_{\tilde{n}\tilde{n}}^0$ is the density response function for a non-interacting gas of atoms with the equilibrium HF spectrum given by (69) and the chemical potential $\tilde{\mu}_0$ is equal to $\mu_{c0}$ in (68). For a uniform Bose gas, the condensate collective modes given by (63) are found to be

$$\omega^2 = c_0^2k^2[1 + 4g\chi_{\tilde{n}\tilde{n}}^0(k,\omega)] - \frac{i\omega}{\tau'}.\quad (110)$$

Here $c_0 = \sqrt{gn_{c0}/m}$ is the Bogoliubov phonon velocity [see (28)] and $\tau'$ is defined in (80). In the long wavelength limit, one finds

$$\lim_{k \to 0} -Im \chi_{\tilde{n}\tilde{n}}^0(k,\omega = c_0k) = \frac{mk_B T}{2\pi c_0^3},\quad (111)$$

where the $1/3$ factor arises from the exchange term in the single-particle self-energy. Thus one obtains

$$\omega = c_0k - i(\Gamma_L + \frac{1}{2\tau'}),\quad (112)$$

where the Landau damping is given by (RPA with exchange)

$$\Gamma_L = \frac{4}{3}ak_B T k.\quad (113)$$

The exact result for $\Gamma_L$ obtained from a careful evaluation of the Beliaev self-energies at finite $T$ is the same as (113), apart from a slightly smaller numerical coefficient ($4/3$ is replaced by $3\pi/8$). This difference is due to our neglect of the fluctuations associated with the anomalous pair density $\tilde{m}(r,t)$. Such terms are in the exact equation for $\Phi(r,t)$ given in (63) but were neglected in deriving (13).
To conclude this discussion of different kinds of damping in the collisionless limit, we can also include the effect of the collision terms in (87) as a perturbative correction to the solution determined by $\hat{L}f = 0$. For a Bose gas above $T_{BEC}$, one might use the simple “relaxation time” approximation

$$C_{22}[f] \simeq -(f - f_0)/\tau_{22} = -\delta f/\tau_{22}. \quad (114)$$

The inclusion of such a term in (98) simply involves the change

$$\frac{\partial \delta f}{\partial t} \rightarrow \frac{\partial \delta f}{\partial t} + \frac{\delta f}{\tau_{22}}, \quad (115)$$

and hence one finds (103) again, with the replacement $\omega \rightarrow \omega + i/\tau_{22}$ in the denominator of (104). The collision time $\tau_{22}$ is the analogue of the classical expression given in (95). One sees that including $C_{22}[f]$ in the simple relaxation time approximation (114) only slightly alters the Landau damping given by (106). This is quite different from the contribution arising from $C_{12}[f]$ in so far as this enters directly into the generalized GP equation (see Section 3).

We now leave the discussion of the collisionless domain and turn to the collision-dominated hydrodynamic region. In this case, the collisions between the non-condensate atom are assumed to be sufficiently rapid that the form of $f(p, \mathbf{r}, t)$ is largely determined by the requirement that

$$C_{22}[f, \Phi] = 0. \quad (116)$$

That is, the collisions are so strong they force the system to be in local equilibrium. The unique solution (denoted by $\tilde{f}$) of the equation (116) is well-known to be given by

$$\tilde{f}(p, \mathbf{r}, t) = \frac{1}{e^{\beta [\frac{(p - m v_n)^2}{m} + U(\mathbf{r}, t) - \tilde{\mu}(\mathbf{r}, t)]} - 1}, \quad (117)$$

where $v_n$ is the average local velocity and $\tilde{\mu}$ is the local chemical potential of the thermal atoms. This local equilibrium Bose distribution involves the local variables $\beta, v_n, \tilde{\mu}$ and $U$, all of which depend on $(\mathbf{r}, t)$. Why must $\tilde{f}$ have the form in (117)? To satisfy $C_{22}[f_1] = 0$, we must have [see (88)]

$$(1 + f_1)(1 + f_2)f_3f_4 - f_1f_2(1 + f_3)(1 + f_4) = 0, \quad (118)$$

and this requires that $f$ be given by the Bose distribution (117). Here we have used the fact that a Bose distribution satisfies

$$f(x) \equiv \frac{1}{e^x - 1} = -[f(-x) + 1] \quad (119)$$
and energy and momentum conservation
\[ \mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3 + \mathbf{P}_4 \]
\[ \tilde{\varepsilon}_{p_1} + \tilde{\varepsilon}_{p_2} = \tilde{\varepsilon}_{p_3} + \tilde{\varepsilon}_{p_4} \]  
(120)

However, while the fact that \( \tilde{f} \) is given by the local equilibrium Bose distribution in (117) ensures that \( C_{22}[\tilde{f}] = 0 \), one finds \[ C_{12}[\tilde{f}] \neq 0 \]. More precisely, the term in (89) reduces to
\[ \left[ (1 + \tilde{f}_1) \tilde{f}_2 \tilde{f}_3 - \tilde{f}_1 (1 + \tilde{f}_2) (1 + \tilde{f}_3) \right] \]
\[ \propto \left[ e^{-\beta[\mu - \mu_c - \frac{1}{2}m(v_n - v_c)^2]} - 1 \right] (1 + \tilde{f}_1) \tilde{f}_2 \tilde{f}_3, \]  
(121)

using energy and momentum conservation. The expression in the square bracket in (121) only vanishes if the condensate and non-condensate are in \textit{diffusive} equilibrium, which requires that the chemical potentials must be equal
\[ \tilde{\mu} = \mu_c + \frac{1}{2}m(v_n - v_c)^2. \]  
(122)

When we \textit{perturb} the system, this may not be true, i.e., the two components may be out of diffusive equilibrium.

We can now derive hydrodynamic equations for the non-condensate by taking moments of the Boltzmann equation, the standard procedure used in classical gases. \[ \text{[18]} \] The first moment gives a continuity equation with a source term:
\[ \int d\mathbf{p} \left\{ L \tilde{f} = C_{12}[\tilde{f}] \right\} \rightarrow \frac{\partial \tilde{n}}{\partial t} = -\nabla \cdot (\tilde{n} \mathbf{v}_n) + \Gamma_{12}[\tilde{f}], \]  
(123)

where we have used the fact that \( C_{22}[\tilde{f}] = 0 \) and
\[ \tilde{n} \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \tilde{f}(\mathbf{p}, \mathbf{r}, t) \]
\[ \tilde{n} \mathbf{v}_n \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\mathbf{p}}{m} \tilde{f}(\mathbf{p}, \mathbf{r}, t) \]
\[ \Gamma_{12}[\tilde{f}, \Phi] \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} C_{12}[\tilde{f}, \Phi]. \]  
(124)

More explicitly, we find \[ \text{[compare with (71)]} \]
\[ \Gamma_{12}[\tilde{f}] = \frac{2g^2n_c}{(2\pi)^3} \left[ e^{-\beta[\tilde{\mu} - \mu_c - \frac{1}{2}m(v_n - v_c)^2]} - 1 \right] \]
\[
\times \int dp_1 \int dp_2 \int dp_3 \delta (mv_c + p_1 - p_2 - p_3) \\
\times \delta (\tilde{\varepsilon}_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3)(1 + \tilde{f}_1 \tilde{f}_2 \tilde{f}_3) \\
\equiv \frac{n_c}{\tau_{12}} \left[ e^{-\beta [\tilde{\mu} - \mu_c - \frac{1}{2} m(v_n - v_c)^2]} - 1 \right].
\] (125)

We note that \(\tau_{12}\) is a collision time which describes the \(C_{12}\) collisions between the condensate and non-condensate atoms. Adding (123) to the continuity equation in (60)

\[
\frac{\partial n_c}{\partial t} = -\nabla \cdot (n_c v_c) - \Gamma_{12}[\tilde{f}, \Phi].
\] (126)

we see that the source term \(\Gamma_{12}\) cancels out to give

\[
\frac{\partial (n_c + \tilde{n})}{\partial t} = -\nabla \cdot (n_c v_c + \tilde{n} v_n). 
\] (127)

Thus our theory gives the correct continuity equation for the total local density \(n = n_c + \tilde{n}\).

Similarly, one finds

\[
\int dp \left\{ \hat{L} \dot{\tilde{f}} = C_{12}[\tilde{f}] \right\} \rightarrow m \tilde{n} \left( \frac{\partial v_n}{\partial t} + \frac{1}{2} \nabla v_n^2 \right) \\
= -\nabla \tilde{P}(r, t) - \tilde{n} \nabla U(r, t) - m(v_n - v_c) \Gamma_{12}[\tilde{f}],
\] (128)

where the kinetic pressure is given by

\[
\tilde{P}(r, t) = \frac{m}{3} \int \frac{dp}{(2\pi)^3} (p - m v_n)^2 \tilde{f}(p, r, t).
\] (129)

Finally, the second moment gives

\[
\int dp d^2 p \left\{ \mathcal{L} \dot{\tilde{f}} = C_{12}[\tilde{f}] \right\} \rightarrow \frac{\partial \tilde{P}}{\partial t} + \nabla \cdot (\tilde{P} v_n) \\
= -\frac{2}{3} \tilde{P} \nabla \cdot v_n + \frac{2}{3} \left[ \mu_c + \frac{1}{2} m(v_n - v_c)^2 - U \right] \Gamma_{12}[\tilde{f}].
\] (130)

The detailed derivation of these equations is not important here. It involves straightforward manipulations using the explicit form of \(\tilde{f}\) in (117).

The hydrodynamic equations (123), (128) and (130) describe the non-condensate in terms of three new “coarse-grained” local variables:

\(\tilde{n}(r, t), v_n(r, t)\) and \(\tilde{P}(r, t)\).
These are coupled to the two additional local variables which describe the condensate:

$$n_c(r, t), \ v_c(r, t).$$

We note that the two condensate equations of motion given by (60) are always “hydrodynamic” in form. In contrast, it is only in the collision-dominated region that the non-condensate dynamics can also be described in terms of a few collective variables. Both components exhibit coupled, coherent collective motions at the same frequency. This is the essence of two-fluid superfluid behaviour, familiar in liquid $^4$He studies but still an unexplored frontier in trapped Bose gases.

We will now discuss the linearized version of our condensate and non-condensate equations for local equilibrium as given by (60), (122), (128) and (130). We work to first order in the fluctuations around static equilibrium,

$$\hat{n} = \tilde{n}_0 + \delta \tilde{n}, \ v_n = \delta v_n, \ \hat{P} = \tilde{P}_0 + \delta \tilde{P}$$

$$n_c = n_{c0} + \delta n_c, v_c = \delta v_c .$$

(131)

What is new about the two-fluid hydrodynamic equations derived above is the role of the source term $\Gamma_{12}[\tilde{f}, \Phi]$. In a linearized theory expanded around the static equilibrium Bose distribution $f_0$ (where $\Gamma_{12}[f_0, \Phi_0]$ vanishes), one finds

$$\Gamma_{12}[\tilde{f}, \Phi] = \delta \Gamma_{12}[\tilde{f}, \Phi] = -\frac{\beta_0 n_{c0}}{\tau_{12}^{0}} \delta \mu_{diff},$$

(132)

where

$$\mu_{diff}(r, t) \equiv \tilde{\mu}(r, t) - \mu_c(r, t).$$

(133)

Here $\tau_{12}^{0}(r)$ is the $C_{12}$ collision time defined in (125) with both components being in static equilibrium (see also (77)). The linearized coupled hydrodynamic equations for the two components are given by the ZGN' equations

$$\frac{\partial \delta n_c}{\partial t} = -\nabla \cdot (n_{c0} \delta v_c) - \delta \Gamma_{12}$$

$$m \frac{\partial \delta v_c}{\partial t} = -\nabla \delta \mu_c$$

(134)

and

$$\frac{\partial \delta \tilde{n}}{\partial t} = -\nabla \cdot (\tilde{n}_0 \delta v_n) + \delta \Gamma_{12}$$

$$m\tilde{n}_0 \frac{\partial \delta v_n}{\partial t} = -\nabla \delta \tilde{P} - \delta \tilde{n} \nabla U_0(r) - 2g\tilde{n}_0 [\nabla \delta n_c + \nabla \delta \tilde{n}]$$

$$\frac{\partial \delta \tilde{P}}{\partial t} = -\frac{5}{3} \nabla \cdot (\tilde{P}_0 \delta v_n) + \frac{2}{3} \delta v_n \cdot \nabla \tilde{P}_0 - \frac{2}{3} gn_{c0} \delta \Gamma_{12},$$

(135)
with $\delta\Gamma_{12}$ given by (132) and (within the TF approximation)

$$\delta\mu_c = g\delta n_c + 2g\delta\tilde{n}. \quad (136)$$

We note that to lowest order, $\delta\Gamma_{12}$ does not appear in the second equation in (135). In addition, in the last term in the third equation in (135), we have used $\mu_{c0} - \bar{U}_0 = -gn_{c0}$. The condensate couples into the non-condensate equations in (135) directly via the mean field terms $g\delta n_c$ and, indirectly, through $\delta\Gamma_{12}$ (see Section 5).

Despite appearances, this coupled set of hydrodynamic equations form a closed set. We have 9 scalar equations in (134) and (135), while they appear to involve 10 fluctuating variables

$$\delta n_c, \delta v_c; \delta\tilde{n}, \delta v_n; \delta\tilde{P}, \delta\mu_{diff}. \quad (137)$$

However one can show that $\delta\mu_{diff}$ can be written as a linear combination of the variables $\delta\tilde{P}, \delta\tilde{n}$ and $\delta n_c$, and thus we are in fact left with only 9 variables. In Section 5, we prove that (134) and (135) are equivalent to the two-fluid hydrodynamic equations of Landau when applied to a trapped Bose gas, but only when the condensate and non-condensate are in diffusive local equilibrium, i.e., $\mu_c(r,t) = \bar{\mu}(r,t)$. It will turn out that the rate at which $\delta\mu_{diff}(r,t)$ relaxes to zero is controlled by a relaxation time $\tau_\mu$ related to the $C_{12}$ collisions. The Landau two-fluid hydrodynamics [4] is only valid for low frequency oscillations which satisfy $\omega\tau_\mu \ll 1$, as we discuss in Section 5.

5 Two-fluid hydrodynamics in Bose liquids and gases

To put our new two-fluid equations derived in Section 4 into some sort of context, we first briefly review the more general theory developed by Landau in 1941 to explain superfluidity in liquid $^4$He.

The original discovery of superfluidity in liquid $^4$He is associated with the famous 1938 papers of Kapitza in Moscow and Allen and Misener at Cambridge. (I cannot resist remarking that Allen and Misener had been graduate students at the University of Toronto where they carried out (with Burton) the pioneering studies on vanishing viscosity in the period 1935-1937). These and subsequent experiments in the next few years [5] showed that superfluid $^4$He could exhibit very bizarre behaviour compared to ordinary liquids. This led to the development of a two-fluid theory of the hydrodynamic behaviour of liquid $^4$He by Landau (1941). An earlier but less complete version of Landau’s hydrodynamic equations was developed by Tisza in the period 1938-40. For further discussion of this early history, I refer to a recent article of mine [6].
In this early work, superfluidity (the term was coined in 1938 by Kapitza) was entirely associated with the relative motion of the normal fluid and the superfluid components under a variety of conditions. The main point was that while the normal fluid exhibited finite viscosity and thermal conductivity typical of ordinary fluid, the superfluid component (which exhibited irrotational flow) did not. In more recent times, the aspect of superfluidity which has been emphasized (see Ch.4 of Ref. and Leggett’s lectures in this book, for example) are those most directly tied to the fact that the superfluid velocity is associated with the gradient of the phase of the macroscopic wavefunction $\Phi(r,t)$. Given that this point of view is indeed more fundamental, it is still crucial to understand why superfluidity persists even in the presence of a dissipative normal fluid. This question can be addressed using two-fluid hydrodynamic equations.

In essence, Landau developed his generic two-fluid hydrodynamics by generalizing the standard theory of classical hydrodynamics to include the equations of motion for a new “superfluid” degree of freedom. We recall that classical fluid dynamics was developed well before one knew about the existence of atoms. Since the work of Maxwell and Boltzmann in the 1880’s, we know the “coarse-grained” hydrodynamic description of a fluid in terms of a few quantities like $n(r,t)$ and $v(r,t)$ is only valid when the collisions between atoms are strong enough to produce local equilibrium. It only describes low frequency phenomena, where the condition in (94) is satisfied.

In his 1941 paper, Landau did not connect the superfluid component with the motion of a “Bose condensate.” Indeed, he rejected the efforts by Tisza and F. London to use a Bose-condensed gas to get some insight into superfluid $^4$He. However, since the period 1957-1965, Landau’s superfluid degree of freedom has been understood microscopically in terms of the complex order parameter $\Phi(r,t)$. As noted earlier, the superfluid velocity field $v_s(r,t)$ is related to the gradient of the phase of $\Phi(r,t)$, as given by (19). In the same period, it was also realized that there are two distinct kinds of quantum fluids:

(a) Bose fluids (associated with a Bose condensate wavefunction $\Phi$)

(b) Normal Fermi fluids (associated with the key role of a Fermi surface).

These two kinds of quantum fluids were magnificently described in two books by Nozières and Pines written around 1965, although the one on superfluid Bose liquids was only published in 1990 (it was in wide circulation as a preprint before then). I think the clearest account of the connection between $\Phi(r,t)$ and superfluidity in Bose fluids is still the discussion given in Chapters 4 and 5 of Vol II by Nozières and Pines. I highly recommend it to everyone in the BEC field.
Landau’s pioneering work on superfluid $^4\text{He}$ has two separate aspects which are logically distinct but sometimes confused with each other:

(a) The two-fluid equations describing hydrodynamic behaviour. These equations are generic and apply (under certain conditions) to trapped Bose gases as well as to superfluid $^4\text{He}$. In the period 1947-1950, Landau and Khalatnikov extended these equations to include hydrodynamic damping of the normal fluid (described by various kinds of viscosities, thermal conductivity, etc). This work is all described in the classic 1965 monograph by Khalatnikov.

(b) A “microscopic” theory of the elementary excitations describing the normal fluid of liquid $^4\text{He}$ - the famous phonon - roton spectrum. Within his picture of a weakly interacting gas of quasiparticles, Landau and coworkers could calculate the thermodynamic and transport properties of superfluid $^4\text{He}$. These quantities enter into the expressions given by the two-fluid equations for the first and second sound modes (velocity and damping). Of course, the roton part of the spectrum is not valid for a dilute Bose gas.

As we have noted, Landau’s original formulation of his two-fluid hydrodynamic equations was phenomenological in that the superfluid component was not given an explicit microscopic basis. The first derivation of the Landau hydrodynamic equations starting simply from the existence of the macroscopic order parameter $\Phi(\textbf{r},t)$ was given by Bogoliubov in 1963. This derivation built on Bogoliubov’s earlier work on deriving hydrodynamic equations for classical liquids without going through the intermediate stage of using Boltzmann-like kinetic equations. While Bogoliubov’s derivation is often viewed as being quite general, buried in his complex analysis is the assumption that the normal fluid and the superfluid are in local equilibrium with each other. Even today, probably the definitive account which formulates the various levels of theory for Bose superfluids is the classic paper by Hohenberg and Martin, published in 1965. It clearly shows (to the patient reader!) the central unifying role of $\Phi(\textbf{r},t)$, summarizes the collisionless and hydrodynamic domains, and finally gives criteria for developing and judging various approximation schemes using Green’s function techniques.

With the preceding discussion as a preamble, I will now summarize the linearized two-fluid hydrodynamic equations of Landau, following the approach given in Ch. 7 of Ref. ... As noted, these equations are valid for both superfluid liquids as well as gases. The differences come in only at the last stage when we evaluate the thermodynamic coefficients appearing in these
The first two (linearized) Landau equations are familiar from ordinary fluid dynamics

\[
\frac{\partial \delta n}{\partial t} + \nabla \cdot \delta j = 0
\] (138)

\[
m \frac{\partial \delta j}{\partial t} = -\nabla \delta P - \delta n \nabla V_{ex},
\] (139)

where we have included the effect of an external potential. Landau’s work incorporated two components and hence the total mass density and mass current fluctuations are given by

\[
\delta \rho \equiv m \delta n = \delta \rho_s + \delta \rho_n
\] (140)

\[
m \delta j \equiv \rho_s 0 \delta v_s + \rho_n 0 \delta v_n.
\] (141)

The new superfluid component was argued to only exhibit pure potential (irrotational) flow and carry no entropy. Thus Landau’s final two hydrodynamic equations were new,

\[
m \frac{\partial \delta v_s}{\partial t} = -\nabla \delta \mu
\] (142)

\[
\frac{\partial \delta s}{\partial t} + \nabla \cdot (s_0 \delta v_n) = 0 
\] (143)

Here the local equilibrium entropy density is \(s(r, t) = s_0 + \delta s\), the local equilibrium pressure is \(P(r, t) = P_0 + \delta P\) and the local chemical potential is \(\mu(r, t) = \mu_0 + \delta \mu(r, t)\). We note that the Landau hydrodynamic equations only involves 8 equations, in contrast to the 9 equations we derived at the end of Section 8. In particular, we see that Landau does not have separate continuity equations for the superfluid \(\rho_s(r, t)\) and normal fluid \(\rho_n(r, t)\) densities. Finally, it is assumed that these two components are always in local equilibrium with each other and hence the fluctuations are related by the thermodynamic identity

\[
n_0 \delta \mu = -s_0 \delta T + \delta P,
\] (144)

where \(n_0\) is the equilibrium total density.
For simplicity, we now specialize our analysis to a uniform superfluid, where \( \rho_s, \rho_n, s_0 \) and other thermodynamic quantities are all position-independent. Multiplying (142) by \( n_s \) gives [using (144)]
\[
\rho_s \frac{\partial \delta v_s}{\partial t} = -n_s \nabla \delta \mu = -\frac{n_s}{n_0} \nabla (-s_0 \delta T + \delta P) = -\frac{n_s}{n_0} \nabla \delta P + \frac{n_s}{n_0} s_0 \nabla \delta T.
\] (145)

Using this in (138) gives
\[
-\nabla \delta P = \rho_s \frac{\partial \delta v_s}{\partial t} + \rho_n \frac{\partial \delta v_n}{\partial t} = -\frac{n_s}{n_0} \nabla \delta P + \frac{n_s}{n_0} s_0 \nabla \delta T + \rho_n \frac{\partial v_n}{\partial t}
\]
or
\[
\rho_n \frac{\partial \delta v_n}{\partial t} = -\frac{n_n}{n_0} \nabla \delta P - \frac{n_s}{n_0} s_0 \nabla \delta T.
\] (146)

Combining (138) and (139) gives
\[
\frac{\partial^2 \delta \rho}{\partial t^2} = -m \nabla \cdot \frac{\partial \delta \mathbf{v}}{\partial t} = \nabla^2 \delta P.
\] (147)

Finally, (143) gives [using (146)]
\[
\frac{\partial^2 \delta s}{\partial t^2} = -s_0 \nabla \cdot \frac{\partial \delta \mathbf{v}}{\partial t}
\]
\[
= \frac{s_0}{\rho_0} \nabla^2 \delta P + \frac{s_0^2}{\rho_0} \left( \frac{\rho_s}{\rho_n} \right) \nabla^2 \delta T
\]
\[
= \frac{s_0}{\rho_0} \frac{\partial^2 \delta \rho}{\partial t^2} + \frac{s_0^2}{\rho_0} \left( \frac{\rho_s}{\rho_n} \right) \nabla^2 \delta T.
\] (148)

The last equation gives the local entropy fluctuations in terms of the local mass density \( \delta \rho \) and temperature \( \delta T \) fluctuations. Eq.(148) can be re-written [1] in terms of the local entropy per unit mass \( \bar{s}(\mathbf{r},t) \equiv s(\mathbf{r},t)/\rho(\mathbf{r},t) \). Using
\[
\delta \bar{s} = -\frac{s_0}{\rho_0} \delta \rho + \frac{1}{\rho_0} \delta s,
\] (149)

(148) takes on the simpler form
\[
\frac{\partial^2 \delta \bar{s}}{\partial t^2} = \frac{s_0^2}{\rho_0} \left( \frac{\rho_n}{\rho_s} \right) \nabla^2 \delta T
\] (150)
Expanding $\delta P$ and $\delta T$ in terms of $\delta \rho$ and $\delta \bar{s}$ fluctuations

$$\partial P = \left. \frac{\partial P}{\partial \rho} \right|_z \delta \rho + \left. \frac{\partial P}{\partial \bar{s}} \right|_\rho \delta \bar{s},$$

$$\delta T = \left. \frac{\partial T}{\partial \rho} \right|_z \delta \rho + \left. \frac{\partial T}{\partial \bar{s}} \right|_\rho \delta \bar{s},$$ \hspace{1cm} (151)

we see that (150) and (147) reduce to two coupled scalar equations for $\delta \rho$ and $\delta \bar{s}$. Inserting the normal mode solutions

$$\delta \rho, \delta \bar{s} \propto e^{i(k \cdot r - \omega t)},$$ \hspace{1cm} (152)

one finds these coupled algebraic equations have two phonon solutions $\omega^2 = u^2k^2$, where $u^2$ is the solution of the quadratic equation:

$$u^4 - u^2 \left[ \left. \frac{\partial P}{\partial \rho} \right|_T + \frac{T}{\bar{c}_v} \left( \frac{1}{\rho_0} \left. \frac{\partial P}{\partial T} \right|_{\rho_0} \right)^2 + \frac{\rho_{s0}}{\rho_{n0}} \frac{T\bar{s}_0^2}{\bar{c}_v} \right]$$

$$+ \frac{\rho_{s0}}{\rho_{n0}} \left( \frac{T\bar{s}_0^2}{\bar{c}_v} \right) \left. \frac{\partial P}{\partial \rho} \right|_T = 0.$$ \hspace{1cm} (153)

Here $\bar{c}_v = \left. \frac{\partial \bar{s}}{\partial T} \right|_\rho$ is the equilibrium specific heat per unit mass.

The coefficients in (153) are daunting, but only involve equilibrium thermodynamic quantities which can be calculated for any given superfluid. We emphasize that (153) is valid for both a uniform Bose-condensed gas and superfluid $^4$He, assuming that the superfluid and normal fluid are in local equilibrium with each other. However, as we shall show, the detailed characteristics and behaviour of the two phonon modes (first and second second) are quite different in a Bose gas and in superfluid $^4$He. \cite{48,49}

A key feature about superfluid $^4$He is that (typical of any liquid) $\partial P/\partial T|_\rho \approx 0$. This can be shown to be equivalent to $C_p \simeq C_v$, where $C_{p,v}$ are the specific heats at constant pressure or constant volume. In this case, (153) reduces to

$$u^4 - u^2(A + B) + AB = 0,$$ \hspace{1cm} (154)

with two solutions $u^2 = A, B$ :

$$u_1^2 = \left. \frac{\partial P}{\partial \rho} \right|_T : \text{first sound}$$ \hspace{1cm} (155)

$$u_2^2 = \frac{\rho_{s0}}{\rho_{n0}} \left( \frac{T\bar{s}_0^2}{\bar{c}_v} \right) : \text{second sound}.$$ \hspace{1cm} (156)
Working out the associated motions, one finds the first sound mode ($\omega = u_1 k$) involves the in-phase motion of the superfluid and normal fluid components. Moreover, one can show that first sound is essentially a pressure wave. In contrast, the second sound mode ($\omega = u_2 k$) involves the out-of-phase motion of the two components, with $\delta j \simeq 0$ or

$$\rho_{n0} \delta v_n = -\rho_{s0} \delta v_s, \ldots$$

(157)

This corresponds to an almost pure temperature wave. The successful detection in 1946 of a second sound mode (or temperature wave) was of tremendous significance in low temperature physics.\[21,47\] The good agreement of the measured second sound velocity $u_2$ with the Landau expression in (156), calculated using the postulated phonon-roton quasiparticle spectrum, vindicated both aspects of the Landau theory of superfluid $^4$He.

In contrast, in a (Bose-condensed) gas, the pressure and temperature fluctuations are strongly coupled and hence $\partial P/\partial T|_\rho$ in (153) plays a significant role. We recall that in a classical gas, one has $C_p/C_v = \frac{5}{3}$. Evaluating all the thermodynamic derivatives using our finite $T$ microscopic model (the single-particle HF spectrum in (59) replaces the roton spectrum of superfluid $^4$He) and working to lowest order in $gn_c/k_BT$, one finds (after very lengthy calculations)

$$u_1^2 = \frac{5}{3} \frac{k_B T}{m} g_{5/2}(z_0 = 1) + O(gn)$$

(158)

$$u_2^2 = \frac{gn_c}{m} + \ldots$$

(159)

It turns out that the first sound mode in a Bose gas is largely an oscillation of the thermal cloud (the normal fluid). The second sound mode is largely an oscillation of the condensate (the superfluid). Both involve density fluctuations and thus will have significant weight in the dynamic structure factor related to the density response function. The quite different features of first and second sound in Bose gases vs superfluid $^4$He should be remembered when reading standard texts about superfluid $^4$He. Results equivalent to (158) and (159) are given in Refs.\[48-51\].

Comparing (159) with (28), it is clear that in a uniform Bose-condensed gas, the hydrodynamic second sound mode at finite $T$ smoothly extrapolates to the Bogoliubov phonon mode in the collisionless region. The fact that the first sound velocity $u_1$ does not depend on the interaction strength $g$ to lowest order is typical of ordinary sound waves in any gas. However, one must remember that interactions (collisions) play a crucial indirect role in enforcing dynamic local equilibrium.
We now turn to a discussion of the generalized two-fluid hydrodynamic equations we derived at the end of Section 4. Again, for simplicity, we limit our analysis to a uniform Bose-condensed gas. If we can ignore vorticity in both fluids, we can introduce two velocity potentials
\[ \delta v_c \equiv \nabla \phi_c(r, t) \]
\[ \delta v_n \equiv \nabla \phi_n(r, t), \] (160)
and reduce the equations in (134) and (135) to three equations for \( \phi_c, \phi_n \) and \( \delta \mu_{\text{diff}} \) [the latter is defined in (133)]:
\[
m \frac{\partial^2 \phi_c}{\partial t^2} = g n_c 0 \nabla^2 \phi_c + 2 g \tilde{n}_0 \nabla^2 \phi_n + \frac{\sigma_H}{\tau_\mu} \delta \mu_{\text{diff}} \] (161)
\[
m \frac{\partial^2 \phi_n}{\partial t^2} = \left( \frac{5 \tilde{P}_0}{3 \tilde{n}_0} + 2 g \tilde{n}_0 \right) \nabla^2 \phi_n + 2 g n_c 0 \nabla^2 \phi_c - \frac{2}{3} \frac{\sigma_H}{\tau_\mu} \delta \mu_{\text{diff}} \] (162)
\[
\frac{\partial \delta \mu_{\text{diff}}}{\partial t} = \frac{2}{3} g n_c 0 \nabla^2 \phi_n - g n_c 0 \nabla^2 \phi_c - \frac{\delta \mu_{\text{diff}}}{\tau_\mu}. \] (163)
Here \( \tau_\mu \) is a new relaxation time governing how \( \delta \mu_{\text{diff}} \) relaxes to 0 (ie, \( \tilde{\mu} \to \mu_c \)). It is found to be related to the \( C_{12} \) collision time \( \tau' \) defined in (80) by
\[
\frac{1}{\tau_\mu} = \frac{1}{\sigma_H \tau'}. \] (164)
where the dimensionless hydrodynamic renormalization factor \( \sigma_H \) is given by
\[
\sigma_H \equiv \frac{5}{2} \tilde{\gamma}_0 \tilde{P}_0 - \frac{5}{2} g \tilde{n}_0^2 \frac{3}{2} \tilde{\gamma}_0 + \frac{3}{2} g \tilde{n}_0^2 - \frac{3}{2} g \tilde{n}_0^2. \] (165)
For completeness, we recall that
\[
\tilde{P}_0(z_0) = \frac{k_B T}{\Lambda_0^3} g_{5/2}(z_0) = e^{-\beta g_{n,0}} \]
\[
\tilde{n}_0(z_0) = \frac{1}{\Lambda_0^3} g_{3/2}(z_0) \]
\[
\tilde{\gamma}_0 = \frac{g}{k_B T \Lambda_0^3} g_{1/2}(z_0), \] (166)
where the Bose-Einstein functions are \( g_n(z) = \sum_{l=1}^{\infty} z^l / l^n \) and \( \Lambda_0 \) is the equilibrium thermal de Broglie wavelength. Calculation shows that \( \sigma_H \) becomes large as \( T \to T_{\text{BEC}} \) and thus \( \tau_\mu \) can become very large near the superfluid
Figure 5. Squares of the first and second sound velocities (normalized by the first sound velocity of the ideal gas at \( T = T_{BEC} \)) vs. \( T / T_{BEC} \). The solid lines are for the Landau limit \( \omega \tau \mu \ll 1 \) while the dashed lines are for \( \omega \tau \mu \gg 1 \). The curves were generated for \( g_n/k_B T_{BEC} = 0.2 \). The interesting crossover at low temperatures was first pointed out in Ref. [49]. It is reproduced by our equations, even though it occurs when \( g_n c_0 \sim k_B T \) which is outside the region of validity of our model. See Refs. [7], [48].

transition. This is not in contradiction with the fact that the present discussion is for the collision-dominated region \( [\omega \tau_2 \ll 1, \text{where } \tau_2 \text{ is some relaxation time associated with the } C_{22} \text{ collision integral in (88)}] \).

It is clear from the fact that we have 3 coupled equations for the 3 variables \( \phi_c, \phi_n \) and \( \delta \mu_{diff} \), we will obtain a new mode, in addition to the usual first and second sound oscillations discussed earlier. It is convenient to first eliminate \( \delta \mu_{diff} \) using the solution of (163),

\[
-i \omega \delta \mu_{diff} = \frac{2}{3} g_n c_0 (-k^2) \phi_n - g_n c_0 (-k^2) \phi_c - \frac{\delta \mu_{diff}}{\tau \mu}
\]
or
\[
\delta \mu_{\text{diff}} = \frac{g n_c \tau_\mu}{1 - i \omega \tau_\mu} (\phi_c - \frac{2}{3} \phi_n) k^2.
\]  
(167)

Using this in (161) and (162) gives
\[
m \omega^2 \phi_c = g n_c \sigma_H \left( 1 - \frac{\sigma_H}{1 - i \omega \tau_\mu} \right) k^2 \phi_c + 2 g n_0 \left( 1 + \frac{\sigma_H}{3(1 - i \omega \tau_\mu)} n_c \right) k^2 \phi_n
\]
(168)

\[
m \omega^2 \phi_n = \left( \frac{5}{3} \frac{\bar{P}_0}{n_0} + 2 g n_0 \left[ 1 - \frac{2 \sigma_H}{9(1 - i \omega \tau_\mu)} n_c \right] \right) k^2 \phi_n
\]
+ \[2 g n_c \sigma_H \left( 1 + \frac{\sigma_H}{3(1 - i \omega \tau_\mu)} n_c \right) k^2 \phi_c.
\]  
(169)

These two coupled equations for \( \phi_n \) and \( \phi_c \) are easily solved and we obtain (as expected) first and second sound modes. Clearly the velocities \( u_1 \) and \( u_2 \) will now depend on the value of \( \omega \tau_\mu \), although it turns out that this dependence is not very strong. This is shown by the results in Fig. 3.

The \( \omega \tau_\mu \to 0 \) limit is of special interest since the equations (168) and (169) can then be shown to be completely equivalent to the predictions of the Landau two-fluid equations discussed earlier in this section. To be precise, the first and second sound velocities \( u_{1,2} \) obtained from (168) and (169) in the limit \( \omega \tau_\mu \to 0 \) agree with those given by (153). This equivalence makes physical sense since in the limit of \( \tau_\mu \to 0 \), one sees that \( \delta \mu_{\text{diff}} \to 0 \) very rapidly [see (163) and (167)]. Thus we have proven that the Landau two-fluid hydrodynamic equations are correct if the condensate and thermal cloud are in diffusive local equilibrium. This, as noted earlier, is an (sometimes implicit) assumption in all previous derivations of the Landau equations. 17, 51

This Landau limit (\( \omega \tau_\mu \to 0 \)) is, in fact, very subtle in the context of our microscopic calculation. We see that in this limit, there are still correction terms in (168) and (169) which are proportional to the hydrodynamic renormalization factor \( \sigma_H \) as defined in (165). These terms are crucial in ensuring that (168) and (169) reproduce the results of the Landau two-fluid equations. Another way of seeing this is that even though \( \delta \mu_{\text{diff}} \to 0 \) when \( \tau_\mu \to 0 \), we note that \( \delta \Gamma_{12} \) in (132) is still finite. Using (167), one finds
\[
\delta \Gamma_{12} = -n_c \sigma_H \left( \phi_c - \frac{2}{3} \phi_n \right) k^2.
\]
(170)

In conclusion, one might say that the hydrodynamic renormalization factor \( \sigma_H \) contains a key part of the physics buried in the hydrodynamic two-fluid equations of Landau.
Figure 6. A schematic illustration of the predicted hydrodynamic mode spectrum for a uniform Bose gas. (a) Above $T_{\text{BEC}}$, one has the usual first sound mode and a zero-frequency thermal diffusion mode. (b) Below $T_{\text{BEC}}$, the standard Landau-Khalatnikov two-fluid hydrodynamics predicts first and second sound, but no remnant of any zero-frequency mode. (c) In the ZGN’ theory presented in Sections 4 and 5, one has first and second sound modes as well as a zero-frequency relaxational mode. Based on Ref. 52.
In the opposite limit $\omega \tau_\mu \gg 1$ (which can arise near $T_{BEC}$) we see all the terms proportional to $\sigma_H$ in (168) and (169) are negligible. This domain is missed in the Landau two-fluid equations. It describes situations in which the condensate and thermal cloud are out of diffusive equilibrium with each other. It would be of great interest to look for this kind of phenomenon in trapped Bose gases.

One can work out the frequency of the new mode associated with the dynamics of $\delta \mu_{\text{diff}}$ and for a uniform gas, it is well approximated by

$$\omega_R \simeq -i/\tau_\mu. \quad (171)$$

Thus, in general, our two-fluid hydrodynamic equations predict the existence of a relaxational mode peaked at zero frequency. One can improve the theory to include deviations from local equilibrium [i.e., deviations of $f$ from $\tilde{f}$ in (116)]. This involves a Chapman-Enskog kind of calculation familiar in the theory of classical gases and gives rise to hydrodynamic damping. The normal fluid (non-condensate) equations of motion have new terms corresponding to shear viscosity ($\eta$) and thermal conductivity ($\kappa$) transport coefficients, where we recall that $\kappa$ and $\eta$ are proportional to some $\tau_{22}$ collision time and hence go as $1/g^2$. The damping of first sound, second sound and the relaxational mode due to small deviations from local equilibrium among the thermal atoms $(C_{22}[f, \Phi])$ has been recently worked out in detail by Nikuni, Zaremba and the author.

Of particular interest is the effect on the relaxational mode, which is now described by [compare with (171)]

$$\omega_R \simeq -i \left[ \frac{1}{\tau_\mu} + A \kappa k^2 \right]. \quad (172)$$

Effectively the relaxation mode is strongly coupled into thermal conduction processes (or flow of heat). The expression for the coefficient $A$ in (172) is somewhat complicated but above $T_{BEC}$ ($\tau_\mu \to 0$), we find the mode reduces to the well-known thermal diffusion mode,

$$\omega_R = -i \frac{\kappa k^2}{n_0 C_p} = -iD_T k^2. \quad (173)$$

This result strongly suggests that our new relaxational mode below $T_{BEC}$ is the renormalized version of the usual thermal diffusion mode above $T_{BEC}$.

We recall that in the standard two-fluid hydrodynamic theory of Landau, the thermal diffusion mode below $T_{BEC}$ disappears below $T_{BEC}$, with its spectral weight going into the emerging second sound doublet. In Fig. 6, we schematically illustrate the different hydrodynamic mode spectra predicted for above and below $T_{BEC}$.
To illustrate the essential physics, the detailed discussion in this section has been limited to a uniform Bose-condensed gas. The analysis can, of course, be extended to trapped Bose gas. In particular, the out-of-phase dipole mode $E_5$ in the hydrodynamic limit is of special interest. This corresponds to centre-of-mass oscillation of the equilibrium condensate and non-condensate density profiles,

\begin{align}
    n_c(r,t) &= n_{c0}(r - \eta_c(t)) \\
    \dot{n}_c(r,t) &= n_{0c}(r - \eta_c(t)),
\end{align}

with

\begin{align}
    v_n &= \frac{d\eta_n}{dt} ; \quad v_c = \frac{d\eta_c}{dt}.
\end{align}

One finds that the hydrodynamic equations give (as expected) an undamped in-phase normal mode, with $v_n = v_c$ and $\omega = \omega_0$, where $\omega_0$ is the trap frequency. This is a generalized version of the $T = 0$ Kohn mode discussed at the end of Section 2. In addition, however, there is an out-of-phase mode satisfying

\begin{align}
    N_c v_c &= -\dot{N} v_n,
\end{align}

with a frequency different from the trap frequency $\omega_0$. The frequency of this mode is shown in Fig. 7. Calculation shows that this out-of-phase mode is only damped by its coupling to the relaxational mode given by (171). There is no hydrodynamic-type damping from the finite transport coefficients of the normal gas. Calculations are in reasonable agreement with the observed damping of such a out-of-phase mode. A careful study of this out-of-phase dipole mode as a function of the temperature would be a nice way of probing the unusual hydrodynamics of a Bose-condensed gas.

In these lectures, I have not made any attempt to analyze the available experimental investigations of collective modes at finite temperatures. These pioneering experimental studies are very promising but we need more systematic investigations, especially as a function of temperature and density. A key reason why such studies are perhaps a unique way of probing the many-body dynamics of superfluid gases is that one can measure collective mode frequencies very accurately (at $T = 0$, with errors of only a few percent).

In this section, we have put emphasis on deriving two-fluid hydrodynamic equations by starting from an approximate but still microscopic model. Such a derivation has been carried out recently by Zaremba, Nikuni and the author for a trapped Bose gas. The pioneering work on deriving two-fluid
hydrodynamic equations for a uniform dilute Bose gas was by Kirkpatrick and Dorfman. In this regard, it is useful to point out that the linearized Landau two-fluid hydrodynamic equations [as given by (138) - (143)] are expected to be exact as long as the two components are in local equilibrium. Thus, for a uniform system, the exact first and second sound velocities are given by the solutions of (153). The only question is how to calculate the various thermodynamic functions which are involved in this equation. The analogous Landau equations for a trapped gas allows us to go past the simplified microscopic models we have used in Section 4 to derive equations of this kind. In particular, one should be able to use the Landau two-fluid equa-

Figure 7. (a) Mode frequencies for the in-phase (solid dots) and out-of-phase (open dots) dipole modes vs. temperature for 2000 Rb atoms in an isotropic parabolic trap (see Ref. 13 for values of the physical parameters used). (b) Condensate ($\eta_c$) and non-condensate ($\eta_n$) amplitudes for the out-of-phase dipole mode. (c) Fraction of atoms in the condensate as a function of temperature. From Ref. 53.
tions as a direct probe of the superfluid density, just as one does in superfluid $^4$He. One might be able to detect small differences between the magnitude of the superfluid density $n_{s0}$ and the condensate density $n_{c0}$ (at the level of our model in Section 4, $n_{s0}$ and $n_{c0}$ are equal). In this connection, we note that in the formal zero temperature limit, the Landau two-fluid equations reduce to two coupled equations for a pure superfluid ($\rho_s = \rho, \rho_n = 0$ at $T = 0$). This limit has been used to find a more accurate version of the $T = 0$ quantum hydrodynamic equations discussed in Section 3.

In conclusion, I hope I have given some insight into why the two-fluid hydrodynamics of trapped Bose gases has so much potential interest. I hope some of the young experimentalists attending this Summer School will take up the challenge to study this new frontier. In a sense, the natural next step after understanding the dynamics of a pure condensate ($T = 0$) is to study the two-fluid hydrodynamics of trapped gases (at finite $T$) since now one has two components which can execute coupled coherent collective motions. In contrast, the collisionless region at finite $T$ as discussed in Section 3 does not seem as interesting since the thermal cloud has always such a low density (see Fig. 2). As a result, mean field effects produced by the non-condensate are never very important above or below $T_{\text{BEC}}$, and thus no new many-body dynamics emerges which is different than already found at $T = 0$. What is new is the different mechanisms of damping of collisionless condensate motion, as discussed in Sections 3 and 4.

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