Bulk and surface plane electromagnetic waves in anisotropic media

Abstract

A new analytical approach to description of electromagnetic waves in nonmagnetic anisotropic media is presented. Amplitudes of their reflection and refraction at interfaces and also reflection and transmission of plane parallel plates are derived. Beam splitting at reflection, and creation of surface waves at the interfaces are studied. A simple laboratory demonstration of the beam splitting is proposed. D'yakonov surface waves, their description and observation are discussed.

—Reading the manuscript, this reviewer decided that the authors must be new to electromagnetics. Otherwise, they would not have wasted their time reinventing the wheel. Everything that could be known about propagation in an uniaxial material has been known since at least 1841. So there is no sense wasting time there.

Referee of Opt.Technol.

1 Introduction

Description of electromagnetic waves in homogeneous anisotropic media did not change since Fresnel times more than 160 years ago. Here the first time since then we present a new approach, which makes physics here very simple and transparent. We will not waist time describing the standard approach, which can be found with only slight variation in all the textbooks on electrodynamics or optics [1-13] containing chapters on anisotropic media. Instead we directly start with our approach.

An anisotropic medium is characterized by some direction, called axis which we will denote by a unit vector \( \mathbf{a} \). In a plane electromagnetic wave

\[
\mathbf{E} \exp(i\mathbf{k}\mathbf{r} - i\omega t) \tag{1}
\]

propagating in such a medium at an arbitrary direction with respect to \( \mathbf{a} \) the wave vector \( \mathbf{k}(\omega) \) and polarization vector \( \mathbf{E} \) depend on angle \( \theta \) between \( \mathbf{a} \) and direction of propagation \( \mathbf{k} = \mathbf{k}/k \).

The main feature of our approach (similar to the one used for elastic waves [14]) is a special representation of the dielectric permittivity tensor \( \mathbf{\varepsilon} \). It was proven by Fedorov [12] that in a uniaxial anisotropic medium the tensor \( \mathbf{\varepsilon} \) can be represented as

\[
\varepsilon_{ij} = \varepsilon_1 \delta_{ij} + \varepsilon' a_i a_j, \tag{2}
\]

where \( \varepsilon_1 \) is isotropic part, and anisotropy is characterized by the unit vector \( \mathbf{a} \) with components \( a_i \) and by anisotropy parameter \( \varepsilon' \).

In the next section we find \( \mathbf{E} \) and \( k(\omega) \) in anisotropic media with one and two mutually orthogonal axes. In the third section we discuss reflection of these waves from interfaces between anisotropic and isotropic media, study beam splitting at reflection, conditions for creation of
surface waves, and propose a device to demonstrate the beam splitting at reflections in a laboratory. In section 4 we calculate reflection and transmission for plane parallel anisotropic plates. In 5-th section we calculate speed of the D’yakonov surface waves [18], which are an analog of Rayleigh elastic waves on a free surface of elastic media. We correct some defect of the derivation of these waves in [18] and propose an experiment to generate and to observe the D’yakonov waves. In 6-th section we summarize our results. In sections 3-5 we limit ourselves only to uniaxial media and only explain the ideas. An approach to biaxial media and most heavy mathematics are shifted to appendices.

2 Plane waves in anisotropic media

— I don’t get the goal of this paper
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The wave equation is derived from Maxwell equations, which in the absence of currents and charges are

\[-[\nabla \times E(r, t)] = \frac{\partial}{\partial t} B(r, t), \quad [\nabla \times H(r, t)] = \frac{\partial}{\partial t} D(r, t), \quad \nabla B = 0, \quad \nabla D = 0, \quad (3)\]

where

\[B = \mu H, \quad D = \varepsilon E, \quad (4)\]

and \(\mu, \varepsilon\) are magnetic and dielectric permittivities. In the following we take \(\mu = 1\), and then the equations (3) are simplified to

\[-[\nabla \times E(r, t)] = \frac{\partial}{\partial t} H(r, t), \quad [\nabla \times H(r, t)] = \frac{\partial}{\partial t} \varepsilon E(r, t), \quad \nabla H = 0, \quad \nabla \varepsilon E = 0. \quad (5)\]

Differentiation of the second equation of (5) over time and after that substitution of the first equation leads to the wave equation

\[-[\nabla \times [\nabla \times E(r, t)]] = \frac{\partial^2}{\partial t^2} \varepsilon E(r, t). \quad (6)\]

2.1 Electric vector of the wave

Substitution of the plane wave (1) into this equation reduces (6) to

\[k^2 \mathcal{E} - k(k \cdot \mathcal{E}) = k_0^2 \varepsilon \mathcal{E}, \quad (7)\]

where \(k_0 = \omega/c\). Eq. (7) is valid only in homogeneous media. If we have an interface between two homogeneous media we have two different equations of the type (2) in them, and matching of waves in two media is performed via boundary conditions, which follow not from the wave equation itself, like in quantum mechanics, but from Maxwell equations.

In uniaxial anisotropic media we use tensor \(\varepsilon\) in the form (2). Therefore for a plane wave (1) we have

\[\varepsilon \mathcal{E} = \varepsilon_1 \mathcal{E} + \varepsilon'(a \cdot \mathcal{E}), \quad (8)\]

and the last equation in (5) leads to

\[\varepsilon_1(k \cdot \mathcal{E}) + \varepsilon'(k \cdot a)(a \cdot \mathcal{E}) = 0. \quad (9)\]
we obtain
\( k^2 \mathbf{E} - k(k \cdot \mathbf{E}) - k_0^2 \varepsilon \mathbf{E} \equiv (k^2 - k_0^2) \mathbf{E} - k(k \cdot \mathbf{E}) - k_0^2 \varepsilon' a(a \cdot \mathbf{E}) = 0. \) (10)

To find \( \mathbf{E} \) we need to solve (10) with account of (9).

The 3-dimensional vector \( \mathbf{E} \) can be represented by coordinates in some basis. If \( \mathbf{k} \) is not parallel to \( \mathbf{a} \), we can use as a basis three independent vectors \( \mathbf{a}, \mathbf{\kappa} = \mathbf{k}/k \) and
\[ e_1 = [a \times \kappa]. \] (11)

In this basis \( \mathbf{E} \) looks
\[ \mathbf{E} = \alpha \mathbf{a} + \beta \mathbf{\kappa} + \gamma e_1 \] (12)
with coordinates \( \alpha, \beta \) and \( \gamma \), which are not independent, because of Eq. (9).

Substitution of (12) into (9) gives
\[ \epsilon_1 [k \beta + (k \cdot a) \alpha] + \epsilon'(k \cdot a) [\alpha + \beta (\kappa \cdot a)] = 0, \] (13)
from which it follows that
\[ \beta = -\frac{(\kappa \cdot a)(1 + \eta)}{1 + \eta(\kappa \cdot a)^2} \alpha, \] (14)
where \( \eta = \epsilon'/\epsilon_1 \). Substitution of (14) into (12) gives
\[ \mathbf{E} = \alpha \left( \mathbf{a} - \kappa \frac{(\kappa \cdot a)(1 + \eta)}{1 + \eta(\kappa \cdot a)^2} \right) + \gamma e_1 = \alpha \mathbf{e}_2 + \gamma e_1, \] (15)
which shows that \( \mathbf{E} \) lies in a plane of two independent vectors \( e_1 = [a \times \kappa] \) and the orthogonal to it
\[ e_2 = a - \kappa(\kappa \cdot a) \frac{1 + \eta}{1 + \eta(\kappa \cdot a)^2} \equiv a - \kappa(\kappa \cdot a) \epsilon_2(\theta)/\epsilon_1, \] (16)
where \( \cos \theta = \kappa \cdot a \) and we introduced anisotropic dielectric permittivity
\[ \epsilon_2(\theta) = \epsilon_1 \frac{1 + \eta}{1 + \eta \cos^2 \theta}. \] (17)

To find coordinates \( \alpha \) and \( \beta \) we substitute (15) into (10) and multiply it by \( e_1 \). As a result we obtain
\[ (k^2 - k_0^2 \varepsilon_1) \gamma e_1^2 = 0. \] (18)
It shows that if \( \gamma \neq 0 \), then (18) can be satisfied only when
\[ k^2 = k_0^2 \varepsilon_1. \] (19)

Multiplying (10) by \( \mathbf{a} \) and taking into account that
\[ a \cdot e_2 = \frac{1 - (\kappa \cdot a)^2}{1 + \eta(\kappa \cdot a)^2}, \quad \kappa \cdot e_2 = -\eta(\kappa \cdot a) \frac{1 - (\kappa \cdot a)^2}{1 + \eta(\kappa \cdot a)^2} = -\eta(\kappa \cdot a)(\mathbf{a} \cdot e_2), \] (20)
we obtain
\[ \left( k^2 - k_0^2 \epsilon_2(\theta) \right) \alpha(a \cdot e_2) = 0. \] (21)
Therefore, if \( \alpha \neq 0 \) and \( \mathbf{a} \neq \mathbf{\kappa} \), Eq. (21) can be satisfied only, when
\[ k^2 = k_0^2 \epsilon_2(\theta), \] (22)
where \( \epsilon_2(\theta) \) is given in (17). Since the length of \( k \) is different for two polarization vectors, therefore a single plain wave can exist only with a single polarization along either \( e_2 \) or \( e_1 \).

We will call “transverse” the mode with polarization \( \mathbf{E}_1 = e_1 \), and “mixed” the mode with polarization along \( \mathbf{E}_2 = e_2 \). The mixed mode according to (20) contains a longitudinal component along the wave vector \( \mathbf{\kappa} \). We think that such a nomenclature is better than common names: “ordinary” for \( \mathbf{E}_1 = e_1 \), and “extraordinary” for \( \mathbf{E}_2 = e_2 \), because our names point to physical features of these waves.
2.2 Magnetic fields

Every electromagnetic wave besides electric contains also magnetic field. From the equation \( \nabla \cdot \mathbf{H} = 0 \), which is equivalent to \( \mathbf{k} \cdot \mathbf{H} = 0 \), it follows that the field \( \mathbf{H} \) is orthogonal to \( \mathbf{k} \). It is also orthogonal to \( \mathbf{E} \), which follows from the first equation of (5). After substitution of Eq. (1) into (5) and the field \( \mathbf{H} \) in the plane wave form

\[
\mathbf{H}(\mathbf{r}, t) = \mathcal{H} \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t),
\]

with polarization vector \( \mathcal{H} \) we obtain

\[
\mathcal{H} = \frac{k}{k_0} [\mathbf{\kappa} \times \mathbf{E}].
\]

For transverse and mixed modes in uniaxial media, respectively, we therefore obtain

\[
\mathcal{H}_1 = \frac{k}{k_0} [\mathbf{\kappa} \times \mathbf{e}_1] = \frac{k}{k_0} [\mathbf{\kappa} \times [\mathbf{a} \times \mathbf{\kappa}]], \quad \mathcal{H}_2 = \frac{k}{k_0} [\mathbf{\kappa} \times \mathbf{e}_2] = \frac{k}{k_0} [\mathbf{\kappa} \times \mathbf{a}],
\]

and the total plain wave field looks

\[
\Psi(\mathbf{r}, t) = \psi_j \exp(i\mathbf{k}_j \cdot \mathbf{r} - i\omega t),
\]

where \( \psi_j = \mathcal{E}_j + \mathcal{H}_j \), and \( j \) denotes mode 1 or 2. In isotropic media we also can choose, say \( \mathcal{E} = [\mathbf{a} \times \mathbf{\kappa}] \) and \( \mathcal{H} = [\mathbf{\kappa} \times [\mathbf{a} \times \mathbf{\kappa}] \). However, there \( \mathbf{a} \) can have arbitrary direction, therefore the couple of orthogonal vectors \( \mathcal{E} \) and \( \mathcal{H} \) can be rotated any angle around the wave vector \( \mathbf{k} \).

3 Reflection from an interface between uniaxial and isotropic medium

—I spent a few separate sittings reading this paper but cannot convince myself to sit with pencil and paper and follow the mathematics presented here, since I know I can open Jackson,... or Griffith and get really all I would need to know about the interaction of E\&M waves with the surface of a dielectric medium— in a more streamlined, concise and easy to follow manner.

Referee of Am.J.Phys.

A note Anisotropic media are not considered in [15,16].

Imagine that our space is split into two half spaces. The part at \( z < 0 \) is a uniaxial anisotropic medium, and the part at \( z > 0 \) is vacuum with \( \epsilon_1 = 1, \eta = 0 \). We have two different wave equations in these parts, and waves go from the reign of one equation into the reign of another one through the interface where they must obey boundary conditions imposed by Maxwell equations.

Let’s look for reflection of the two possible modes incident onto the interface from within the anisotropic medium.

3.1 Nonspecularity and mode transformation at the interface

First we note that reflection of the mixed mode is not specular. Indeed, since direction of \( \mathbf{k} \) after reflection changes, therefore the angle \( \theta \) between \( \mathbf{a} \) and \( \mathbf{\kappa} \) does also change, and \( k \),
where $x$.

In the following we will present such differences in dimensionless variables $k$-modes. We see that the difference of the normal components of reflected and incident waves of mixed nonspecularity of the reflection.

Let’s calculate the change of $k_{\perp}$ for the incident mixed mode with wave vector $k_{2r}$, where the index $r$ means that the mode 2 propagates to the right toward the interface. For a given angle $\theta$ between $k_{2r}$ and $a$ we can write

$$k_{2r\perp} = \frac{\epsilon_1 k_0^2 (1 + \eta)}{1 + \eta \cos^2 \theta} - k_{2r\parallel}^2,$$  \hspace{1cm} (27)

however the value of $k_{2r\perp}$ enters implicitly into $\cos \theta$, so to find explicit dependence of $k_{2r\perp}$ on $a$ it is necessary to solve the equation

$$k_{2r\parallel}^2 + x^2 + \eta (k_{\parallel} (l \cdot a) + x(n \cdot a))^2 = k_0^2 \epsilon_1 (1 + \eta),$$ \hspace{1cm} (28)

where $x$ denotes $k_{2r\perp}$, $n$ is a unit vector of normal, directed toward isotropic medium, and $l$ is a unit vector along $k_{\parallel}$, which together with $n$ constitutes the plane of incidence. Solution of this equation is

$$x = \frac{-\eta k_{\parallel} (n \cdot a) (l \cdot a) + \sqrt{\epsilon_1 k_0^2 (1 + \eta) (1 + \eta (n \cdot a)^2) - k_{2r\parallel}^2 (1 + \eta (l \cdot a)^2 + \eta (n \cdot a)^2)}}{1 + \eta (n \cdot a)^2}. \hspace{1cm} (29)$$

The sign chosen before square root provides the correct asymptotics at $\eta = 0$ equal to isotropic value $\sqrt{\epsilon_1 k_0^2 - k_{2r\parallel}^2}$.

In general vector $a$ is representable as $a = \alpha n + \beta l + \gamma t$, where $t = [nl]$ is a unit vector perpendicular to the plane of incidence. The normal component $k_{2r\perp}$ depends only on part of this vector $a' = \alpha n + \beta l$, which lies in the incidence plane. If we denote $\alpha = |a'| \cos(\theta_a)$, $\beta = |a'| \sin(\theta_a)$, where $|a'|$ is projection of $a$ on the incidence plane, and introduce new parameter $\eta' = \eta |a'|^2 \leq \eta$, then formula (29) is simplified to

$$k_{2r\perp} = \frac{-\eta' k_{\parallel} \sin(2\theta_a) + 2\sqrt{\epsilon_1 k_0^2 (1 + \eta') [1 + \eta' \cos^2(\theta_a)] - k_{2r\parallel}^2 (1 + \eta')}}{2[1 + \eta' \cos^2(\theta_a)]}. \hspace{1cm} (30)$$

For the reflected mixed mode (mode 2, propagating to the left from the interface) an equation similar to (28) looks

$$k_{2r\parallel}^2 + x^2 + \eta (k_{\parallel} (l \cdot a) - x(n \cdot a))^2 = k_0^2 \epsilon_1 (1 + \eta),$$ \hspace{1cm} (31)

where $x = k_{2l\perp}$, and its solution is

$$k_{2l\perp} = \frac{\eta' k_{\parallel} \sin(2\theta_a) + 2\sqrt{\epsilon_1 k_0^2 (1 + \eta') [1 + \eta' \cos^2(\theta_a)] - k_{2l\parallel}^2 (1 + \eta')}}{2[1 + \eta' \cos^2(\theta_a)]}. \hspace{1cm} (32)$$

We see that the difference of the normal components of reflected and incident waves of mixed modes $k_{2l\perp} - k_{2r\perp}$ is

$$k_{2l\perp} - k_{2r\perp} = \frac{\eta' k_{\parallel} \sin(2\theta_a)}{1 + \eta' \cos^2(\theta_a)}. \hspace{1cm} (33)$$

In the following we will present such differences in dimensionless variables

$$\Delta_{22} = \frac{k_{2l\perp} - k_{2r\perp}}{k_0 \sqrt{\epsilon_1}} = \frac{\eta' q \sin(2\theta_a) + 2\sqrt{(1 + \eta)[1 + \eta' \cos^2(\theta_a)] - q^2 (1 + \eta')}}{2[1 + \eta' \cos^2(\theta_a)]},$$ \hspace{1cm} (34)
Figure 1: Variation of change $\Delta$ of normal components for reflected and incident waves in dependence of angle $\theta = \theta_a$ of anisotropy vector with respect to normal $n$. Vector $a$ is supposed to lie completely in the incidence plane. The curves $\Delta_{ij}$ represent dimensionless ratio $\Delta_{ij}(\theta)$ given by (34), (35) and (36). The curves were calculated for $\eta = \eta' = 0.4$ and $q = k_\parallel/k_0\sqrt{\epsilon_1} = 0.7$.

where $q^2 = k_\parallel^2/k_0^2\epsilon_1$. The reflection angle depends on orientation of anisotropy vector $a$ and it can be both larger than the specular one, when $\theta_a > 0$, or smaller, when $\theta_a < 0$.

In the case of transverse incident mode the length $k = |k|$ of the wave vector, according to (19), does not depend on orientation of $a$, therefore this wave is reflected specularly.

Every incident mode after reflection creates another one, because without another mode it is impossible to satisfy the boundary conditions. Let's look what will be the normal component of the wave vector of other mode. If the incident is the wave of mode 2, reflected transverse mode (mode 1 propagating to the left, away from the interface) will have $k_{1\perp} = \sqrt{\epsilon_1 k_0^2 - k_\parallel^2}$. Therefore according to (30) the difference $\Delta_{12} = (k_{1\perp} - k_{2r\perp})/k_0\sqrt{\epsilon_1}$ is

$$\Delta_{12} = \sqrt{1 - q^2} - \frac{-\eta' q \sin(2\theta_a) + 2\sqrt{(1 + \eta)(1 + \eta' \cos^2(\theta_a))} - q^2(1 + \eta')}{2[1 + \eta' \cos^2(\theta_a)]}.$$  \hfill (35)

In the opposite case, when the incident mode is transverse one, the reflected mixed mode will have $k_{2\perp}$ shown in (32). Therefore the difference $\Delta_{21} = (k_{2\perp} - k_{1r\perp})/k_0\sqrt{\epsilon_1}$ is

$$\Delta_{21} = \frac{\eta' q \sin(2\theta_a) + 2\sqrt{(1 + \eta)(1 + \eta' \cos^2(\theta_a))} - q^2(1 + \eta')}{2[1 + \eta' \cos^2(\theta_a)]} - \sqrt{1 - q^2}.$$  \hfill (36)

The changes of normal components with variation of $\theta_a$ according to (34), (35) and (36) for some values of dimensionless parameters $\eta$ and $q$ and vector $a$ lying completely in the incidence plane, are shown in Fig. 1. From this figure it is seen that the strongest deviation of reflected wave from specular direction is observed for reflection of mixed to mixed mode.

Since reflection of mode 2 is in general nonspecular, it can happen that the wave vectors of reflected and transmitted waves will be arranged as shown in fig. 2, and it follows that there are two critical angles for $\varphi$. The first critical angle $\varphi_{c1}$ \hfill (37)

$$1 < q^2 < \frac{(1 + \eta)(1 + \eta' \cos^2(\theta_a))}{1 + \eta'}.\hfill (37)$$

and
Figure 2: Arrangement of wave vectors of all the modes created by the incident wave of mode 2, i.e. of polarization $\vec{E}_2$, when the anisotropy vector $a$ has the direction shown here. The grazing angle of the reflected mode 2, $\vec{E}_2$, is less than specular one (specular direction is shown by the broken arrow), and the grazing angle $\varphi_1$ of the reflected mode 1, $\vec{E}_1$, is even lower. The grazing angle $\varphi_0$ of the transmitted wave $\vec{E}_0$ is even lower than $\varphi_1$. We can imagine than at some critical value $\varphi = \varphi_{c1}$ the angle $\varphi_0$ becomes zero. It means that at $\varphi < \varphi_{c1}$ transmitted wave becomes evanescent and all the incident energy is totally reflected in the form of two modes. More over, there exists a second critical angle $\varphi_{c2}$, when $\varphi_1 = 0$. Below this angle at $\varphi < \varphi_{c2}$ the mode $\vec{E}_1$ also becomes evanescent. In this case all the incident energy is totally reflected nonspecularly in the form of the mode 2. At the same time the two evanescent waves $\vec{E}_0$ and $\vec{E}_2$ combine into a surface wave, propagating along the interface. The arrows over $\vec{E}$ show direction of waves propagation with respect to the interface. In the figure there is also shown the basis which is used along the paper. It consists of unit normal vector $n$ along normal (z-axis), unit tangential vector $l$ (x-axis) which together with $n$ defines the incidence plane, and the vector $t$ (y-axis) looking toward the reader, which is normal to the incidence plane.

Figure 3: Dependence of dimensionless normal components of incident and reflected waves on $q = k \cos \varphi / k_0 \sqrt{\epsilon_1}$. The solid curve corresponds to the incident wave moving to the right $kr(q) = k_{2r\perp} / k_0 \sqrt{\epsilon_1}$. The dotted curve corresponds to the reflected wave of mode 2 moving to the left $kl(q) = k_{2l\perp} / k_0 \sqrt{\epsilon_1}$. And the broken curve corresponds to the reflected wave of mode 1 moving to the left $k1(q) = k_{1l\perp} / k_0 \sqrt{\epsilon_1}$. It is seen that at $q > 1$ the mode 1 ceases to propagate. Its normal component $k1(q) = \sqrt{1 - q^2} = -i \sqrt{q^2 - 1}$ becomes imaginary, therefore the reflected mode 1 becomes an evanescent wave. Together with transmitted wave, which becomes evanescent at $q^2 = 1 / \epsilon_1$, the mode 1 constitute the surface electromagnetic wave.
the reflected mode 1 also becomes evanescent. Together with evanescent transmitted wave the mode 1 constitutes a surface wave, propagating along the interface. In that case we have nonspecular total reflection of the single mode $E_2$ and the surface wave tied to it.

In figure 3 it is shown how do the normal components of wave vectors change with increase of $q$, which is equivalent to decrease of $\varphi$. For $\epsilon_1 = 1.6$ the first critical angle corresponds to $q \approx 0.8$. The second critical angle corresponds to $q = 1$.

3.2 Demonstration of the beam splitting with the help of a birefringent cone

— In order to ensure that our journal continues to publish articles that cater for our broad readership, every paper submitted must meet our stringent editorial criteria. We believe that your article does not meet these criteria, so it has been withdrawn from consideration.

Editor of J.Phys.A Optics

Figure 4: Demonstration of the beam splitting of light in a birefringent cone. Bright spots on a vertical screen change their position and brightness when the cone is rotated around vertical axis.

Beam splitting at interfaces of an anisotropic medium can be spectacularly demonstrated with the help of birefringent cone as shown in fig.4. In the geometrical optics approximation a narrow incident beam of light after refraction on the side surface of the cone is split into two rays of two different modes 1 and 2. Both modes are further split into two components at reflection from basement of the cone. Four resulting beams after refraction at the side surface go out of the cone and produce on a vertical screen four bright spots. Their positions and brightness depend on direction of the anisotropy axis inside the cone and vary with the cone rotation.

The direct numerical calculations for parameters $\epsilon_1 = 1.6$, $\eta = 0.8$, $a$ in the figure plane, and $\sin \alpha = 0.5$, $\sin \beta = 0.3$, $\sin \gamma = 0.5$ show that outgoing beams from below to top have directions characterized by $\tan \delta_1 = 0.2$, $\tan \delta_2 = 0.4$, $\tan \delta_3 = 0.6$ and $\tan \delta_4 = 0.7$ respectively.

4 Calculation of refraction at interfaces and scattering on plates

— Too much of a mathematical exercise and too little physics for Physica Scripta.

Referee of Phys.Scripta

To calculate reflection and transmission of a plane parallel anisotropic plate, placed in isotropic (for instance, vacuum) medium, it is necessary to know reflections and refractions at
interfaces from inside and outside anisotropic medium, which is obtained by imposing boundary conditions stemmed from Maxwell equations. Knowledge of everything at interfaces permits to write directly reflection and transmission of the plate by the method which is explained in [17] and will be shortly described below.

4.1 Reflection and refraction from inside of the anisotropic medium

The wave function in the full space is

$$\Psi(r) = \Theta(z < 0) \left( e^{i \vec{k}_j \cdot r} \psi_j + \sum_{j' = 1, 2} e^{i \vec{k}_{j'} \cdot r} \psi_{j'} \right) + \Theta(z > 0) e^{i k_0 \cdot r} \left( \psi_e \tau_e + \psi_m \tau_m \right), \quad (38)$$

where $\Theta$ is a step function equal to unity, when inequality in its argument is satisfied, and to zero in opposite case, half space $z < 0$ is occupied by anisotropic medium, and the half space $z > 0$ is vacuum, $\psi = E + H$, arrows show direction of waves propagation, $\psi_j$ denotes the incident wave of mode $j$ ($j = 1, 2$), $\psi_{j'}$ ($l = 1, 2$) denotes reflected wave of mode $j'$, $\vec{k}_j = (k_\|, k_{jr})$, $\vec{\psi}_{j'} = (k_\|, -k_{j'r})$, $k_0 = (k_\|, \sqrt{k_0^2 - k_{jr}^2})$, $\psi_e, \psi_m$, $\tau_e, \tau_m$ are the refracted fields and refraction amplitudes of TE- and TM-modes respectively for the incident $j$-mode. To find reflection and refraction amplitudes (the arrow over them shows the direction of propagation of the incident wave toward the interface), we need to impose on (38) the following boundary conditions.

4.2 General equations from boundary conditions

Every incident wave field can be decomposed at the interface into TE- and TM-modes. In TE-mode electric field is perpendicular to the incidence plane, $E \propto \hat{t}$, therefore contribution of $j$-th mode into TE-mode is $(E_j \cdot \hat{t})$. In TM-mode magnetic field is perpendicular to the incidence plane, $H \propto \hat{t}$, therefore contribution of $j$-th mode into TM-mode is $(H_j \cdot \hat{t})$. For refracted field in TE-mode we accept $\vec{E}_e = t$, $\vec{H}_e = [\kappa_0 \times t]$, and for refracted field in TM-mode we accept $\vec{H}_m = t$, $\vec{E}_m = -[\kappa_0 \times t]$.

4.2.1 TE-boundary conditions

In TE-mode for incident j-mode we have the following three equations from boundary conditions:

1. continuity of electric field

$$(t \cdot \vec{E}_j) + (t \cdot \vec{E}_1) \tau_{1j} + (t \cdot \vec{E}_2) \tau_{2j} = (t \cdot \vec{E}_e) \tau_e, \quad (39)$$

2. continuity of magnetic field parallel to the interface

$$(l \cdot \vec{H}_j) + (l \cdot \vec{H}_1) \tau_{1j} + (l \cdot \vec{H}_2) \tau_{2j} = (l \cdot [\kappa_0 \times t]) \tau_e \equiv -\kappa_0 \tau_e, \quad (40)$$

3. and continuity of the normal component of magnetic induction, which for $\mu = 1$ looks

$$(n \cdot \vec{H}_j) + (n \cdot \vec{H}_1) \tau_{1j} + (n \cdot \vec{H}_2) \tau_{2j} = (n \cdot [\kappa_0 \times t]) \tau_e \equiv \kappa_0 \tau_e. \quad (41)$$

The last eq. (41) is, in fact, not needed, because it coincides with (39).
4.2.2 TM-boundary conditions

In TM-mode we have the equations

1. continuity of magnetic field

\[
(t \cdot \mathbf{H}_j) + (t \cdot \mathbf{H}_1) \mathbf{p}_{1j} + (t \cdot \mathbf{H}_2) \mathbf{p}_{2j} = \mathbf{\tau}_{mj}, \quad (42)
\]

2. continuity of electric field parallel to the interface

\[
(l \cdot \mathbf{E}_j) + (l \cdot \mathbf{E}_1) \mathbf{p}_{1j} + (l \cdot \mathbf{E}_2) \mathbf{p}_{2j} = -(l \cdot [\kappa_0 \times t]) \mathbf{\tau}_{mj} \equiv \kappa_0 \perp \mathbf{\tau}_{mj}, \quad (43)
\]

3. and continuity of the normal component of field \( D \)

\[
(n \cdot \mathbf{E}_j) + (n \cdot \mathbf{E}_1) \mathbf{p}_{1j} + (n \cdot \mathbf{E}_2) \mathbf{p}_{2j} = (n \cdot [\kappa_0 \times t]) \mathbf{\tau}_{mj} \equiv \kappa_0 \parallel \mathbf{\tau}_{mj}. \quad (44)
\]

Again we can neglect Eq. (44), because it coincides with (42). In the following we will not show third equations like (41) and (44), because they are useless. Solution of all the equation is presented in Appendix B. Here for simplicity we limit ourselves only to a particular case of normal incidence of the waves.

— The authors should not have allowed themselves to use the phrase "for simplicity".
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4.2.3 A particular case of normal incidence

In the case of normal incidence reflection and refraction are especially simple, because there is no splitting at reflection. We define geometry by three basis vectors \( \mathbf{n}, \mathbf{l} \) and \( \mathbf{t} \), where \( \mathbf{n} \) denotes normal directed along \( z \) axis toward vacuum, while vectors \( \mathbf{l} \) and \( \mathbf{t} \) lie in the interface and define \( x \) and \( y \) axes respectively. The anisotropy vector \( \mathbf{a} \) is supposed to lie in \((x,z)\) plane at angle \( \theta \) with respect to \( \mathbf{n} \).

A plane wave propagating along \( \mathbf{n} (\kappa = k/k = \mathbf{n}) \) can have only two types of polarizations. It can be of a transverse mode with \( \mathbf{E}_1 = \mathbf{e}_1 \equiv \mathbf{t} \) and \( \mathbf{H}_1 = -n_1 \mathbf{l} \), where \( n_1 = \sqrt{\epsilon_1} \) (see (2)), or it can be of mixed mode with \( \mathbf{E}_2 = \mathbf{e}_2 (16) \), and \( \mathbf{H}_2 = n_2(\theta) [\mathbf{n} \times \mathbf{a}] \), where \( n_2(\theta) = \sqrt{\epsilon_2(\theta)} \) (see (17)).

Since there are no splitting at normal incidence the boundary conditions are simplified. For the mode 1 (39) and (40) are reduced to

\[
[1 + \mathbf{p}_{11}] = \mathbf{\tau}_{e1}, \quad n_1[1 - \mathbf{p}_{11}] = \mathbf{\tau}_{e1}, \quad (45)
\]

and transmitted wave has field polarization \( \mathbf{E}_{e1} = \mathbf{t}, \mathbf{E}_{e1} = \mathbf{l} \) identical to that of the incident field.

Solution of (45) is

\[
\mathbf{p}_{11} = \frac{n_1 - 1}{n_1 + 1}, \quad \mathbf{\tau}_{e1} = \frac{2n_1}{n_1 + 1}, \quad n_1 = \sqrt{\epsilon_1}. \quad (46)
\]

With these formulas we can immediately find reflection and transmission of a plane parallel plate of thickness \( D \) for an incident from vacuum electromagnetic wave with polarization \( \mathbf{E}_{1e} = \mathbf{t} \):

\[
R_1 = -\mathbf{p}_{11} \frac{1 - \exp(2ik_1D)}{1 - \mathbf{p}_{11}^2 \exp(2ik_1D)}, \quad T_1 = \exp(ik_1D) \frac{1 - \mathbf{p}_{11}^2}{1 - \mathbf{p}_{11} \exp(2ik_1D)}, \quad (47)
\]
where \( k_1 = k_0 n_1 \). We see that the incident wave with linear polarization \( \mathbf{E}_{e_1} = t \) parallel to that of mode 1 inside the plate does not change polarization after transmission through the plate.

Now let’s apply boundary conditions to the mode \( \mathbf{E}_2 \) incident normally on to the interface from inside the plate. Now the boundary conditions are reduced to

\[
(l \cdot a)[1 + \rho_{22}(\theta)] = \tau e_2, \quad k_2(\theta)(l \cdot a)[1 - \rho_{22}(\theta)] = k_0 \tau e_2, \tag{48}
\]

where factor \((l \cdot a)\) appears because of projection of the polarization vector onto the interface. Therefore, since \( k_2 = k_0 n_2(\theta) \), and \( n_2(\theta) = \sqrt{\varepsilon(\theta)} \) solution of (48) is

\[
\rho_{22}(\theta) = \frac{n_2(\theta) - 1}{n_2(\theta) + 1}, \quad \tau e_2(\theta) = \frac{2n_2(\theta)(l \cdot a)}{n_2(\theta) + 1}. \tag{49}
\]

From symmetry consideration we can immediately find reflection and transmission amplitudes for outside incident waves with unit polarization along \( l \):

\[
\rho_{2e}(\theta) = \frac{1 - n_2(\theta)}{1 + n_2(\theta)}, \quad \tau_{2e}(\theta) = \frac{2(l \cdot a)(1 + n_2(\theta))}{(l \cdot a)(1 + n_2(\theta))}, \tag{50}
\]

and therefore we can immediately find reflection and transmission of a plane parallel plate of thickness \( D \) for an incident from vacuum electromagnetic wave with polarization \( \mathbf{E}_{2e} = l \):

\[
R_2(\theta) = -\rho_{22}(\theta) \frac{1 - \exp(2ik_2(\theta)D)}{1 - \rho_{22}(\theta) \exp(2ik_2(\theta)D)},
\]

\[
T_2(\theta) = \exp(ik_2(\theta)D) \frac{1 - \rho_{22}^2(\theta)}{1 - \rho_{22}(\theta) \exp(2ik_2(\theta)D)}, \tag{51}
\]

where \( k_2(\theta) = k_0 n_2(\theta) \). We see that the incident wave with linear polarization \( \mathbf{E}_{2e} = l \), which lies in the plane \((n, a)\), does not change polarization after transmission through the plate.

Now let’s consider transmission through the plate of a plane wave \( \exp(ik_0z - i\omega_0t)\mathbf{E}_e \) with intermediate polarization: \( \mathbf{E}_e = \alpha t + \beta l \), where \( |\alpha|^2 + |\beta|^2 = 1 \). The transmitted electrical part of the wave will be

\[
\mathbf{E}_t(z, t) = \exp(ik_0(z - D) - i\omega_0t) [\alpha T_1 t + \beta T_2(\theta)l]. \tag{52}
\]

Since \( \alpha T_1 = |\alpha T_1| \exp(i\gamma_1) \), and \( \beta T_2(\theta) = |\beta T_2(\theta)| \exp(i\gamma_2(\theta)) \), the real part of the wave (52) at some point \( z \) chosen for convenience so that \( k_0(z - D) + \gamma_1 = 2\pi N \) with an integer \( N \), looks

\[
\text{Re}(\mathbf{E}_t(z, t)) = |\alpha T_1| \cos(\omega_0t) t + |\beta T_2(\theta)| \sin(\omega_0t - \varphi) l,
\]

where \( \varphi = \gamma_2(\theta) - \gamma_1 - \pi/2 \). So, in this case the transmitted field has elliptical polarization.

### 4.3 Reflection and refraction from outside an anisotropic medium

Let’s consider the case, when the half space at \( z < 0 \) is vacuum, and that at \( z > 0 \) is an anisotropic medium. The incident wave goes from the left in the vacuum. The wave function in the full space now looks

\[
\Psi(r) = \Theta(z < 0) \left( e^{i\mathbf{k}_0 r} \psi_j + e^{i\mathbf{k}_0 r} \sum_{j' = e, m} \psi_{j'} \rho_{j'j} \right) + \Theta(z > 0) \left( e^{i\mathbf{k}_0 r} \psi_1 r_{1j} + e^{i\mathbf{k}_0 r} \psi_2 r_{2j} \right), \tag{54}
\]
where \( j, j' \) denote \( e \) or \( m \) for TE- and TM-modes respectively, the term \( \exp(i\mathbf{k} \cdot \mathbf{r}) \) with the wave vector \( \mathbf{k}_0 = (k_\parallel, k_{0\perp} = \sqrt{k_0^2 - k_\parallel^2}) \) describes the plain wave incident on the interface from vacuum. In TE-mode factor \( \mathbf{\psi}_e = \mathbf{\hat{E}}_e + \mathbf{\hat{H}}_e \) contains \( \mathbf{\hat{E}}_e = t \) and \( \mathbf{\hat{H}}_e = [\mathbf{\hat{K}}_0 t] \). In TM-mode factor \( \mathbf{\psi}_m = \mathbf{\hat{E}}_m + \mathbf{\hat{H}}_m \) contains \( \mathbf{\hat{H}}_m = t \) and \( \mathbf{\hat{E}}_m = -[\mathbf{\hat{K}}_0 t] \).

The reflected wave has the wave vector \( \mathbf{k}_0 = (k_\parallel, -k_{0\perp}) \), and fields \( \mathbf{\hat{E}}_e = t \), \( \mathbf{\hat{H}}_e = [\mathbf{\hat{K}}_0 t] \), \( \mathbf{\hat{E}}_m = t \), and \( \mathbf{\hat{H}}_m = -[\mathbf{\hat{K}}_0 t] \). The refracted field contains two wave modes with wave vectors \( \mathbf{k}_1 = (k_\parallel, k_{1\perp}) \), \( \mathbf{k}_2 = (k_\parallel, k_{2\perp}) \) and electric fields \( \mathbf{\hat{E}}_1 = e_1 = [\mathbf{a} \mathbf{\hat{K}}_1] \) and \( \mathbf{\hat{E}}_2 = e_2 = a - \mathbf{k}_2(a \mathbf{k}_2)\epsilon_2(\mathbf{\hat{K}}_2) / \epsilon_1 \). Here \( \mathbf{k} = k/k_\parallel \), \( k_{1\perp} = \sqrt{\epsilon_1 k_0^2 - k_\parallel^2} \), and \( k_{2\perp} \) is given by \( (30) \). For incident TE-mode reflection \( \rho_{ee}, \rho_{me} \) and refraction \( \tau_{je} \) amplitudes \((j = 1, 2)\) are found from boundary conditions

\[
(t \mathbf{\hat{E}}_1 \mathbf{\hat{r}}_{1e} + t \mathbf{\hat{E}}_2 \mathbf{\hat{r}}_{2e}) = 1 + \rho_{ee},
\]

\[
(l \mathbf{\hat{H}}_1 \mathbf{\hat{r}}_{1e} + l \mathbf{\hat{H}}_2 \mathbf{\hat{r}}_{2e}) = -\kappa_{0\perp}(1 - \rho_{ee}),
\]

\[
(t \mathbf{\hat{H}}_1 \mathbf{\hat{r}}_{1e} + t \mathbf{\hat{H}}_2 \mathbf{\hat{r}}_{2e}) = \rho_{me},
\]

\[
(l \mathbf{\hat{E}}_1 \mathbf{\hat{r}}_{1e} + l \mathbf{\hat{E}}_2 \mathbf{\hat{r}}_{2e}) = -\kappa_{0\perp} \rho_{me}.
\]

Solution of these equations is elementary and is given in Appendix C.

### 4.4 Reflection and transmission amplitudes for a plane parallel plate of thickness \( L \)

Now, when we understand what happens at interfaces, we can construct [17] expressions for reflection, \( \mathbf{\hat{R}}(L) \), and transmission, \( \mathbf{\hat{T}}(L) \), matrices for a whole anisotropic plane parallel plate of some thickness \( L \), when the state of the incident wave is described by a general vector \( |\mathbf{\xi}_0\rangle \). To do that let’s denote the state of field of the modes \( e_1 \) and \( e_2 \) incident from inside the plate onto the second interface at \( z = L \) by unknown 2-dimensional vector \( |\mathbf{\xi}\rangle \). If we were able to find \( |x\rangle \) we could immediately write the state of transmitted field

\[
|\mathbf{\xi}_0\rangle = \mathbf{\hat{T}}|\mathbf{\xi}\rangle,
\]

and the state of the field, reflected from the whole plate

\[
\mathbf{\hat{R}}(L)|\mathbf{\xi}_0\rangle = \mathbf{\hat{R}}|\mathbf{\xi}_0\rangle + \mathbf{\hat{T}}(L)\mathbf{\hat{E}}(L)\mathbf{\hat{R}}|\mathbf{\xi}\rangle,
\]

where \( \mathbf{\hat{E}}(L), \mathbf{\hat{E}}(L) \) denote diagonal matrices

\[
\mathbf{\hat{E}}(L) = \left( \begin{array}{cc} \exp(ik_{1\perp}L) & 0 \\ 0 & \exp(ik_{2\perp}L) \end{array} \right),
\]

\[
\mathbf{\hat{E}}(L) = \left( \begin{array}{cc} \exp(ik_{1\perp}L) & 0 \\ 0 & \exp(ik_{2\perp}L) \end{array} \right).
\]

which describe propagation of two modes between two interfaces. Here \( k_{1\perp} = \sqrt{\epsilon_1 k_0^2 - k_\parallel^2} \), while \( k_{2\perp} \) and \( k_{2\perp} \) are calculated according to \( (29) \) or \( (30) \) and \( (32) \), respectively.

It is very easy to put down a self consistent equation for determination of \( |\mathbf{\xi}\rangle \):

\[
|\mathbf{\xi}\rangle = \mathbf{\hat{E}}(L)|\mathbf{\xi}_0\rangle + \mathbf{\hat{E}}(L)\mathbf{\hat{R}}\mathbf{\hat{E}}(L)\mathbf{\hat{R}}|\mathbf{\xi}\rangle.
\]

The first term at the right hand side describes the incident state transmitted through the first interface and propagated up to the second one. The second term describes contribution to the
Figure 5: Dependence of reflectivities $|R_{ee}|^2$ and $|R_{me}|^2$ of an anisotropic plate with $\epsilon_1 = 1.6$, $\eta = 0.8$ and dimensionless thickness $L\omega/c = 10$ on angle $\phi$ of the plate rotation around its normal, when the anisotropy vector $a$ is parallel to interfaces and at $\phi = 0$ is directed along $k_\parallel$. The incidence angle $\theta$ is given by $\sin \theta = 0.9$.

state $|\overrightarrow{F}\rangle$ of the $|\overrightarrow{F}\rangle$ itself. After reflection from the second interface this state propagates to the left up to the first interface, and after reflection from it propagates back to the point $z = L$. Two terms at the right hand side of (60) add together, which results to some new state. But we denoted it $|\overrightarrow{F}\rangle$, and it explains derivation of the equation (60).

From (60) we can directly find

$$|\overrightarrow{F}\rangle = \left[ I - \overrightarrow{E}(L)\overrightarrow{R}\overrightarrow{E}(L)\overrightarrow{R} \right]^{-1} \overrightarrow{E}(L)\overrightarrow{T}|\xi_0\rangle,$$

and substitution into (59) and (60) gives

$$\overrightarrow{T}(L) \equiv \left( \begin{array}{cc} T_{ee} & T_{em} \\ T_{me} & T_{mm} \end{array} \right) = \overrightarrow{T} \left[ I - \overrightarrow{E}(L)\overrightarrow{R}\overrightarrow{E}(L)\overrightarrow{R} \right]^{-1} \overrightarrow{E}(L)\overrightarrow{T},$$

$$\overrightarrow{R}(L) \equiv \left( \begin{array}{cc} R_{ee} & R_{em} \\ R_{me} & R_{mm} \end{array} \right) = \overrightarrow{R} + \overrightarrow{T} \overrightarrow{E}(L)\overrightarrow{R} \left[ I - \overrightarrow{E}(L)\overrightarrow{R}\overrightarrow{E}(L)\overrightarrow{R} \right]^{-1} \overrightarrow{E}(L)\overrightarrow{T}.$$ (65)

With these formulas we can easily calculate all the reflectivities and transmissivities

$$\left( \begin{array}{cc} |R_{ee}|^2 & |R_{em}|^2 \\ |R_{me}|^2 & |R_{mm}|^2 \end{array} \right), \quad \left( \begin{array}{cc} |T_{ee}|^2 & |T_{em}|^2 \\ |T_{me}|^2 & |T_{mm}|^2 \end{array} \right),$$ (66)

for arbitrary parameters, arbitrary incidence angles, arbitrary incident polarizations and arbitrary direction of the anisotropy vector $a$. In fig.5 we present, for example, reflectivities of TE-mode wave from a plate of thickness $L$ such, that $L\omega/c = 10$. The anisotropy vector is parallel to interfaces. Therefore, its orientation with respect to wave vector $k_0$ of the incident wave varies with rotation of the plate by an angle $\phi$ around its normal. The transmissivities of the same plate in dependence on the angle $\phi$ are presented in fig.6.

5 D’yakonov surface waves (Dsw)

Above we found that a surface wave can appear on the interface at total reflection of a mixed mode at some incident angles. This surface wave, however, is tied to the incident and reflected
wave and does not exist without them. D'yakonov in 1988 had discovered \[18\] (see also \[18-21\]) that on the surface of a uniaxial anisotropic medium there can exist free surface waves, analogous to elastic Rayleigh waves on a free surface. We will derive them with our tensor (2) and a little bit correct previous derivation by D'yakonov \[18\].

Let’s again consider the space separated by a plane at \(z = 0\) to two halves, as shown in fig.2. The left part \((z < 0)\) corresponds to anisotropic medium with dielectric permittivity \(\epsilon_1\) and the right part is an isotropic medium with dielectric permittivity \(\epsilon_i\).

The surface wave is characterized by the wave function

\[
\Psi(r, t) = [\Theta(z < 0) (\psi_1 \exp(p_1 z) + \psi_2 \exp(p_2 z)) + \Theta(z > 0) \psi_i \exp(-p_i z)] e^{i k_\parallel r - i \omega t},
\]

where \(\psi = \mathbf{E} + \mathbf{H}\),

\[
\mathbf{H}_{1,2} = \frac{k_\parallel}{k_0} ([l \times \mathbf{E}_{1,2}] - i q_{1,2}[n \times \mathbf{E}_{1,2}]), \quad \mathbf{H}_i = \frac{k_\parallel}{k_0} ([l \times \mathbf{E}_i] + i q_i[n \times \mathbf{E}_i]),
\]

parameters \(p_{1,2,i}\) provide exponential decay of the surface wave away from the interface. In (68) we also introduced dimensionless parameters \(q_{1,2,i} = p_{1,2,i}/k_\parallel\). For \(\mathbf{E}_{1,i}\) we have

\[
q_{1,i} = \sqrt{1 - \epsilon_{1,i} z},
\]

respectively, where we denoted \(z = k_0^2/k_\parallel^2\). Parameters \(q_{1,i}\) are positive reals when \(\epsilon_{1,i} z < 1\).

To find \(q_2\) for the field \(\mathbf{E}_2\) we need to solve the equation

\[
1 - x^2 + \eta(l \cdot a - i x(n \cdot a))^2 = z\epsilon_1(1 + \eta),
\]

where \(x\) denotes \(q_2\), \(\mathbf{n}\) is a unit vector of normal, directed toward isotropic medium, and \(\mathbf{l}\) is a unit vector along \(k_\parallel\), which together with \(\mathbf{n}\) constitutes the plane of incidence. From this equation it is seen, that \(q_2\) can be real only if vector \(\mathbf{a}\) is perpendicular to \(\mathbf{n}\) or to \(\mathbf{l}\). In the first case the axis of anisotropy is parallel to the interface [18]:

\[
\mathbf{a} = a_l \mathbf{l} + a_t \mathbf{t} = \cos \theta \mathbf{l} + \sin \theta \mathbf{t},
\]
and solution of (70) is

\[ q_2 = \sqrt{1 + \eta \cos^2 \theta - \epsilon_1 z (1 + \eta)}. \]  

(72)

It is seen that \( q_2(\theta) \) is positive real when

\[ z \epsilon_1 \frac{1 + \eta}{1 + \eta \cos^2 \theta} < 1. \]  

(73)

In the second case

\[ a = a_n n + a_t = \cos \phi n + \sin \phi t, \]  

(74)

and solution of (70) is

\[ q_2(\phi) = \frac{\sqrt{1 - \epsilon_1 (1 + \eta) z}}{\sqrt{1 + \eta \cos^2 \phi}}. \]  

(75)

Below we will show that in the second case free surface waves do not exist.

5.1 Anisotropy axis is parallel to the interface

When \( a \) is parallel to the interface, vectors \( \mathbf{E}_{1,2} \), according to section (2), can be represented as

\[ \mathbf{E}_1 = -\frac{C_1}{k_\|}[a \times k] = C_1 (\sin \theta n + iq_1[a \times n]) = C_1 (\sin \theta n + iq_1 \sin \theta l - iq_1 \cos \theta t), \]  

(76)

\[ \mathbf{E}_2 = \tilde{C}_2 \left[ a - \frac{k(ak)}{k_\|^2} \frac{1 + \eta}{1 - q_2^2 + \eta(l \cdot a)^2} \right] = \tilde{C}_2 \left[ a - (l - iq_2 n)(al) \frac{1}{1 - q_2^2} \right] = \]  

\[ = C_2 \left( iq_2 \cos \theta n - q_1^2 \cos \theta l \right) j \sin \theta t, \]  

(77)

where \( C_2 = \tilde{C}_2/(1 - q_1^2) \), \( C_{1,2} \) are some complex coefficients, and we used relation (72).

In the basis \( n, l, t \), shown in fig.2, polarization \( \vec{E}_i \) in isotropic medium can be represented as

\[ \mathbf{E}_i = \alpha n + \beta l + \gamma t \]  

(78)

with coordinates \( \alpha, \beta \) and \( \gamma \). Because of equation \( \epsilon_i \nabla \cdot \mathbf{E}_i = 0 \), which is equivalent to

\[ iq_i \alpha + \beta = 0, \]  

(79)

vector (78) is reduced to

\[ \mathbf{E}_i = \alpha (n - iq_i l) + \gamma t. \]  

(80)

From continuity of \( t \)- and \( l \)-components of electric field at the interface we obtain two equations

\[ iC_1 q_1 \sin \theta - C_2 q_1^2 \cos \theta = -iq_i \alpha, \]  

\[ -iC_1 q_1 \cos \theta + C_2 (1 - q_1^2) \sin \theta = \gamma. \]  

(81)

To get another two equations we need to use continuity of tangential components of magnetic fields. Substituting (76), (77) and (80) into (68) and neglecting common factor \( k_\|/k_0 \) we obtain

\[ \mathbf{H}_1 = C_1 \left( -iq_1 \cos \theta n + q_1^2 \cos \theta l - [1 - q_1^2] \sin \theta t \right), \]  

\[ \mathbf{H}_2 = C_2 (1 - q_1^2) (\sin \theta n + iq_2 \sin \theta l - iq_2 \cos \theta t), \]  

\[ \mathbf{H}_i = (\gamma n - iq_i \gamma l - \alpha (1 - q_1^2) t). \]  

(82)
Figure 7: Dependence $v(\theta) = C_D(x)/c$ on angle $\theta$ between anisotropy axis $a$ and direction $k_\parallel$ of the surface wave propagation. The curve was calculated for $\epsilon_1 = 1.6$, $\eta = 0.4$ and $\epsilon_i = 1$.

Continuity of $l$ and $t$ components gives

$$q_1^2 C_1 \cos \theta + i C_2 q_2 (1 - q_1^2) \sin \theta = -i q_1 \gamma, \quad (83)$$

$$C_1 \sin \theta + i q_2 C_2 \cos \theta = \alpha \varepsilon, \quad (84)$$

where we denoted $\varepsilon = \epsilon_i / \epsilon_1$. If we exclude $\gamma$ and $\alpha$ from these equations, we obtain a homogeneous system of 2 equations for $C_{1,2}$

$$q_1 \cos \theta(q_1 + q_i) C_1 + i C_2(q_2 + q_i) \epsilon_1 z \sin \theta = 0,$$

$$i C_1 \sin \theta[q_1 \varepsilon + q_i] - C_2 \cos \theta \left(\varepsilon q_1^2 + q_i q_2\right) = 0. \quad (85)$$

The system of the two linear equations (85) has solution only if the determinant of its coefficients is equal to zero, which gives an equation for $z = k_0^2 / k_\parallel^2$:

$$f(z) = q_1(q_1 + q_i) \left(\varepsilon q_1^2 + q_i q_2\right) \cos^2 \theta - \epsilon_1 z(q_2 + q_i) [q_1 \varepsilon + q_i] \sin^2 \theta = 0. \quad (86)$$

Solution of this equation gives the speed of the Dsw $c_D(z) = c \sqrt{z}$.

We derived equation (86) so scrupulously, to show that the result (86) slightly differs from the one presented by equation (8) of [18], and it is not reducible to Eq.(9) of [18], because solution of (86) exists even for $\epsilon_i < \epsilon_1$ ($\epsilon_1$ is denoted $\epsilon_{\perp}$ in [18]). Moreover it follows from (86) that the surface wave exists in much larger range of angles $\theta$ than was obtained in [18]. For instance, in fig. 7 it is shown the dependence of ratio $v(\theta) = c_D(z)/c$ on $\theta$, it is seen that solution of (86) exists in the full range $0 < \theta < \pi/2$ for $\epsilon_1 = 1.6$, $\eta = 0.4$ and $\epsilon_i = 1$.

5.2 Anisotropy axis is perpendicular to the propagation direction

When $a \bot l$ then vectors $\mathbf{E}_{1,2}$, according to section (2), and $\mathbf{H}_{1,2}$ according to (68) can be represented as

$$\mathbf{E}_1 = -\frac{C_1}{k_\parallel} [a \times k] = C_1 (\sin \phi n + i q_1 \sin \phi l - \cos \phi t), \quad (87)$$

$$\mathbf{E}_2 = C_2 \left(\left[1 - q_1^2 + q_2^2\right] \cos \phi n + i q_2 \cos \phi l + \sin \theta t\right), \quad (88)$$

$$\mathbf{H}_1 = C_1 \left(-\cos \phi n - i q_1 \cos \phi l - [1 - q_1^2] \sin \phi t\right), \quad (89)$$

$$\mathbf{H}_2 = C_2 \left(\left[1 - q_2^2 + q_1^2\right] \cos \phi n + i q_1 \cos \phi l + \sin \theta t\right). \quad (90)$$
\[ H_2 = C_2 (1 - q_1^2) (\sin \phi n + iq_2 \sin \phi t - \cos \phi t), \]  
\[ (89) \]
and \( E_i, H_i \) are the same as (80) and in (82) respectively. After performing the same procedure as above we obtain an equation for \( z = k_0^2/k_\| \) in the form

\[ f_1(z) = \left( \frac{\epsilon_i}{\epsilon_1} q_2 + q_i \right) \cos^2 \phi + (q_2 + q_i)(1 - q_1 q_i) \sin^2 \phi = 0, \]  
\[ (90) \]
which has no solution because all the terms in it are positive. Therefore the surface waves do not exist at such orientations of axis \( a \), as was correctly pointed out in [18].

6 Conclusion-summary

—I have no confidence that these authors can form professionally written, carefully researched work with a decent literature search.

Referee of Am.J.Phys.

In the case of uniaxial or biaxial anisotropic media we used for the tensor of dielectric permittivity \( \epsilon_{ij} \) in the form

\[ \epsilon_{ij} = \epsilon_1 [\delta_{ij} + \eta a_i a_j], \quad \epsilon_{ij} = \epsilon_1 [\delta_{ij} + \eta_a a_i a_j + \eta_b b_i b_j], \]  
\[ (91) \]
where \( \epsilon_1 \) is a parameter of isotropic part of the tensors, \( a, b \) are the unit vectors along axes of anisotropy, and \( \eta, \eta_{a,b} \) are respective anisotropy parameters. With such tensors we can immediately find for a plain wave \( E \exp(ikr - i\omega t) \) with an arbitrary direction \( \kappa = k/k_0 \) of propagation analytical expressions for the polarization vector \( E \) and wave number \( k(\omega) \). In the case of uniaxial anisotropic medium we found that only two modes of linear polarizations can propagate inside it. One is transverse mode with

\[ E_1 = [a \times \kappa], \quad k_1 = (\omega/c)\sqrt{\epsilon_1}, \]  
\[ (92) \]
and another one is the mixed mode (it has a component of polarization parallel to the wave vector)

\[ E_2 = a - \kappa (\kappa \cdot a) \frac{\epsilon_2(\theta)}{\epsilon_1}, \quad k_2 = (\omega/c)\sqrt{\epsilon_2(\theta)}, \quad \epsilon_2(\theta) = \frac{\epsilon_1(1 + \eta)}{1 + \eta \cos^2 \theta}, \quad \cos \theta = (a \cdot \kappa). \]  
\[ (93) \]

Next we considered reflection of obtained plain waves from an interface with an isotropic medium and had shown that reflection of every mode is accompanied by beam splitting, that the wave of mode 2 is in general reflected nonspeculally, and at some incident angles reflection of mode 2 can create a surface wave, which is bound to the incident and reflected waves of mode 2. The beam splitting at reflection can be spectacularly demonstrated with the help of light transmission through an anisotropic cone.

After calculation of reflection from interfaces from inside and outside anisotropic medium we had shown an algorithm to calculate reflection and transmission of plane parallel plates without matching of the wave field at two interfaces. In the case of normal incidence on the plate of a plane wave with linear polarization transmitted wave in general has elliptical polarization. The form of ellipse changes with rotation of the birefringent plate around its normal and at two distinct orthogonal direction the ellipse reduces to linear polarization identical to that of the incident wave.

Next we considered the free Dsw on the surface of anisotropic media. We corrected some error in derivation presented in [18] and have shown that Dsw exist in larger range of variation of
dielectric constants and in a larger range of angles between direction of surface wave propagation and direction of anisotropy vector \( \mathbf{a} \).

To observe the D'yakonov surface waves it is possible to use the experimental scheme shown in fig. 8, which is different comparing to the one used in [22]. Disc of a uniaxial crystal with anisotropy axis \( \mathbf{a} \) parallel to the surface can be pivoted around its axis to change angle between direction of Dsw propagation \( \mathbf{\kappa} = \mathbf{k}_\parallel / k_\parallel \) and the vector \( \mathbf{a} \). The Dsw is excited at frustrated total reflection in an anisotropic cone similar to that one shown in fig. 4 (here for simplicity we draw only one transmitted ray). Excitation takes place only when speed of the incident or reflected wave inside cone matches the speed of Dsw. Rotation of the cone around its axis permits some tuning of the speed.

The second anisotropic cone identical to the first one detects Dsw, and the light transmitted into it through the small gap should be visible on a screen, as shown in fig. 8.

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**A Waves in a two axes anisotropic medium**

A two-axial anisotropic medium is characterized by two unit vectors \( \mathbf{a} \) and \( \mathbf{b} \), and two anisotropy parameters \( \epsilon'_a \) and \( \epsilon'_b \). Therefore the tensor \( \mathbf{\varepsilon} \) has matrix elements

\[
\varepsilon_{ij} = \epsilon_1 \delta_{ij} + \epsilon'_a a_i a_j + \epsilon'_b b_i b_j , \tag{94}
\]

and equations (8), (9) and (10) take the respective forms

\[
\varepsilon \mathbf{E} = \epsilon_1 \mathbf{E} + \epsilon'_a \mathbf{a} (\mathbf{a} \cdot \mathbf{E}) + \epsilon'_b \mathbf{b} (\mathbf{b} \cdot \mathbf{E}) , \tag{95}
\]

\[
(\mathbf{\kappa} \cdot \mathbf{E}) + \eta_a (\mathbf{\kappa} \cdot \mathbf{a}) (\mathbf{a} \cdot \mathbf{E}) + \eta_b (\mathbf{\kappa} \cdot \mathbf{b}) (\mathbf{b} \cdot \mathbf{E}) = 0 . \tag{96}
\]

\[
(k^2 - k_0^2 \epsilon_1) \mathbf{E} - k^2 \mathbf{\kappa} (\mathbf{\kappa} \cdot \mathbf{E}) - k_0^2 \epsilon_1 \eta_a \mathbf{a} (\mathbf{a} \cdot \mathbf{E}) - k_0^2 \epsilon_1 \eta_b \mathbf{b} (\mathbf{b} \cdot \mathbf{E}) = 0 . \tag{97}
\]

In the last two equations we introduced notations \( \eta_a = \epsilon'_a / \epsilon_1 \), and \( \eta_b = \epsilon'_b / \epsilon_1 \). For simplicity, we assume that \( \mathbf{a} \perp \mathbf{b} \), introduce the orthonormal basis \( \mathbf{a}, \mathbf{b}, \mathbf{c} = [\mathbf{a} \times \mathbf{b}] \) and in this basis represent

\[
\mathbf{E} = \alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c} \tag{98}
\]
with coordinates $\alpha$, $\beta$, $\gamma$, which are not completely independent, because (98) should satisfy (96). Substituting (98) into (96) gives
\[
\alpha(k \cdot a) + \beta(k \cdot b) + \gamma(k \cdot c) + \eta_a(k \cdot a)\alpha + \eta_b(k \cdot b)\beta = 0.
\] (99)

Therefore
\[
\gamma(k \cdot c) = -\alpha(k \cdot a)(1 + \eta_a) - \beta(k \cdot b)(1 + \eta_b).
\] (100)

From (98) we also obtain that
\[
(E \cdot a) = \alpha, \quad (E \cdot b) = \beta.
\] (101)

Now let’s substitute (101) and $(k \cdot E)$ from (96) into (97) and multiply (97) consecutively by $a$ and $b$. As a result we obtain a system of two linear equations
\[
\begin{align*}
(k^2[1 + \eta_a(k \cdot a)^2] - k_0^2\epsilon_1(1 + \eta_a))\alpha + \eta_b k^2(k \cdot a)(k \cdot b)\beta &= 0 \quad (102) \\
(k^2[1 + \eta_b(k \cdot b)^2] - k_0^2\epsilon_1(1 + \eta_b))\beta + \eta_a k^2(k \cdot a)(k \cdot b)\alpha &= 0.
\end{align*}
\]

The solution to this system exists if the determinant is equal to zero. This condition can be written as
\[
(k^2 - \epsilon_a(\theta_a)k_0^2)(k^2 - \epsilon_b(\theta_b)k_0^2) = \frac{\eta_\alpha \eta_\beta k^2(k \cdot a)^2(k \cdot b)^2}{[1 + \eta_\alpha(k \cdot a)^2][1 + \eta_\beta(k \cdot b)^2]}.
\] (103)

where
\[
\epsilon_{a,b}(\theta_{a,b}) = \frac{\epsilon_1(1 + \eta_{a,b})}{1 + \eta_{a,b}\cos^2 \theta_{a,b}}, \quad \cos \theta_a = (k \cdot a), \quad \cos \theta_b = (k \cdot b).
\] (104)

The solution to (103) provides two different values of $k_{1,2}$, for which we find $\alpha$, $\beta$. After substituting the latter coordinates into (100), we obtain the last coordinate $\gamma$. Thus we find two different plain waves with wave vectors $k_{1,2} = k_{1,2}\kappa$ and linear polarizations $E_{1,2}$ (98).

## B Reflection from an interface from inside of the anisotropic medium

Exclusion of $\overrightarrow{r}_{ej}$ from (39) and (40), and exclusion of $\overrightarrow{r}_{mj}$ from (42) and (43) gives two equations for $\overrightarrow{r}_{1j}$, $\overrightarrow{r}_{2j}$, which is convenient to represent in the matrix form
\[
\begin{pmatrix}
(l \cdot \overrightarrow{H}_1) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_1) \\
\kappa_{0\perp}(t \cdot \overrightarrow{H}_1) - (l \cdot \overrightarrow{E}_1)
\end{pmatrix}
\begin{pmatrix}
(l \cdot \overrightarrow{H}_2) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_2) \\
\kappa_{0\perp}(t \cdot \overrightarrow{H}_2) - (l \cdot \overrightarrow{E}_2)
\end{pmatrix}
\begin{pmatrix}
\overrightarrow{r}_{1j} \\
\overrightarrow{r}_{2j}
\end{pmatrix}
= -\begin{pmatrix}
(l \cdot \overrightarrow{H}_j) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_j) \\
\kappa_{0\perp}(t \cdot \overrightarrow{H}_j) - (l \cdot \overrightarrow{E}_j)
\end{pmatrix},
\] (105)

Solution of this equation is
\[
\begin{pmatrix}
\overrightarrow{r}_{1j} \\
\overrightarrow{r}_{2j}
\end{pmatrix}
= -\frac{1}{D}
\begin{pmatrix}
(l \cdot \overrightarrow{H}_1) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_1) \\
\kappa_{0\perp}(t \cdot \overrightarrow{H}_1) - (l \cdot \overrightarrow{E}_1)
\end{pmatrix}
\begin{pmatrix}
(l \cdot \overrightarrow{H}_2) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_2) \\
\kappa_{0\perp}(t \cdot \overrightarrow{H}_2) - (l \cdot \overrightarrow{E}_2)
\end{pmatrix}
\begin{pmatrix}
(l \cdot \overrightarrow{H}_j) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_j) \\
\kappa_{0\perp}(t \cdot \overrightarrow{H}_j) - (l \cdot \overrightarrow{E}_j)
\end{pmatrix},
\] (106)

where
\[
D = \left((l \cdot \overrightarrow{H}_1) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_1)\right)
\left(\kappa_{0\perp}(t \cdot \overrightarrow{H}_2) - (l \cdot \overrightarrow{E}_2)\right) - \\
- \left((l \cdot \overrightarrow{H}_2) + \kappa_{0\perp}(t \cdot \overrightarrow{E}_2)\right)
\left(\kappa_{0\perp}(t \cdot \overrightarrow{H}_1) - (l \cdot \overrightarrow{E}_1)\right).
\] (107)

Substitution of these expressions into (39) and (42) gives refraction amplitudes $\overrightarrow{r}_{e,mj}$
\[
\begin{pmatrix}
\overrightarrow{r}_{ej} \\
\overrightarrow{r}_{mj}
\end{pmatrix}
= \begin{pmatrix}
(t \cdot \overrightarrow{E}_j) \\
(t \cdot \overrightarrow{H}_j)
\end{pmatrix}
+ \begin{pmatrix}
(t \cdot \overrightarrow{E}_1) \\
(t \cdot \overrightarrow{H}_1)
\end{pmatrix}
\begin{pmatrix}
\frac{1}{D} \\
\frac{1}{D}
\end{pmatrix}
\begin{pmatrix}
\overrightarrow{r}_{1j} \\
\overrightarrow{r}_{2j}
\end{pmatrix},
\] (108)
B.0.1 The most general case

Above we considered the case when the incident wave has polarization vector \( e_j \) with unit amplitude. (We remind that vectors \( e_j \) can be not normalized to unity.) To find later reflections from plane parallel plates we will need a more general case, when the incident wave has both modes with amplitudes \( x_{1,2} \). To find amplitudes of reflected and transmitted waves in the general case it is convenient to represent the state of the incident wave in the form of 2 dimensional vector

\[
|\overrightarrow{x}\rangle = \left( \begin{array}{c} \overrightarrow{x}_1 \\ \overrightarrow{x}_2 \end{array} \right)
\] (109)

then the states of reflected and refracted waves are also described by 2-dimensional vectors, which can be represented as

\[
|\overrightarrow{\psi}\rangle = \left( \begin{array}{c} \overrightarrow{\psi}_1 \\ \overrightarrow{\psi}_2 \end{array} \right) = \widehat{\mathcal{R}} |\overrightarrow{x}\rangle, \\
|\overrightarrow{\psi}_0\rangle = \left( \begin{array}{c} \overrightarrow{\psi}_e \\ \overrightarrow{\psi}_m \end{array} \right) = \widehat{\mathcal{T}} |\overrightarrow{x}\rangle,
\] (110)

where \( \widehat{\mathcal{R}} \) and \( \widehat{\mathcal{T}} \) are two dimensional matrices

\[
\widehat{\mathcal{R}} = \left( \begin{array}{cc} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{array} \right), \\
\widehat{\mathcal{T}} = \left( \begin{array}{cc} \tau_{e1} & \tau_{e2} \\ \tau_{m1} & \tau_{m2} \end{array} \right).
\] (111)

We introduced the prime here and below to distinguish refraction and reflection from inside the medium and the similar matrices obtained for incident waves outside the medium.

These formulas will be used later for calculation of reflection and transmission of plane parallel anisotropic plates. In the case of a plate we have two interfaces, therefore we need also reflection and refraction at the left interface from inside the plate. They can be easily found from symmetry considerations. Their representation is obtained from (106) — (108) by reverse of arrows and change of the sign before \( \kappa_{0\perp} \). After this action we find

\[
\overleftarrow{\widehat{\mathcal{R}}} = \left( \begin{array}{cc} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{array} \right), \\
\overleftarrow{\widehat{\mathcal{T}}} = \left( \begin{array}{cc} \overleftarrow{\tau}_{e1} & \overleftarrow{\tau}_{e2} \\ \overleftarrow{\tau}_{m1} & \overleftarrow{\tau}_{m2} \end{array} \right).
\] (112)

Reflection from outside the medium is to be considered separately.

B.0.2 Energy conservation

It is always necessary to control correctness of the obtained formulas. One of the best controls is the test of energy conservation. One should always check whether the energy density flux of incident wave along the normal to interface is equal to the sum of energy density fluxes of reflected and refracted waves, and the most important in such tests is the correct definition of the energy fluxes. In isotropic media it is possible to define energy flux along a vector \( n \) as

\[
(J \cdot n) = \frac{(k \cdot n)}{k} \frac{c E^2 + H^2}{\sqrt{\varepsilon} 8\pi},
\] (113)
or

\[
(J \cdot n) = c \frac{\langle n \cdot [E \times H]\rangle}{4\pi}.
\] (114)

In isotropic media both definitions are equivalent, because \( \mathcal{H} = \|k \times E\|/k_0 \), and \( (k \cdot E) = 0 \). The first definition looks even more preferable since the second one can be written even for stationary fields, where there are no energy flux.

In anisotropic media only the second definition is valid, and because in mode 2 the field \( E \) is not orthogonal to \( k \), the direction of the energy density flux is determined not only by wave vector, but also by direction of the field \( E \) itself.
C  Reflection from an interface from outside of the anisotropic medium

Exclusion of $\vec{p}_{ee}$ and $\vec{p}_{me}$ leads to

$$\begin{pmatrix} (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_1) - (\mathbf{l}\mathbf{H}_1)) & (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2)) \\ (\kappa_{0\perp}(\mathbf{t}\mathbf{H}_1) + (\mathbf{l}\mathbf{E}_1)) & (\kappa_{0\perp}(\mathbf{t}\mathbf{H}_2) + (\mathbf{l}\mathbf{E}_2)) \end{pmatrix} \begin{pmatrix} \vec{p}_{1e} \\ \vec{p}_{2e} \end{pmatrix} = \begin{pmatrix} 2\kappa_{0\perp} \\ 0 \end{pmatrix} \quad (115)$$

and the solution

$$\begin{pmatrix} \vec{p}_{1e} \\ \vec{p}_{2e} \end{pmatrix} = \frac{1}{D_e} \begin{pmatrix} (\kappa_{0\perp}(\mathbf{t}\mathbf{H}_2) + (\mathbf{l}\mathbf{E}_2)) - (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2)) & (2\kappa_{0\perp}) \\ (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_1) - (\mathbf{l}\mathbf{H}_1)) & (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2)) \end{pmatrix} \begin{pmatrix} \vec{p}_{1e} \\ \vec{p}_{2e} \end{pmatrix} = \begin{pmatrix} (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_1) - (\mathbf{l}\mathbf{H}_1)) \left( \kappa_{0\perp}(\mathbf{t}\mathbf{H}_2) + (\mathbf{l}\mathbf{E}_2) \right) - (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2)) \left( \kappa_{0\perp}(\mathbf{t}\mathbf{H}_1) + (\mathbf{l}\mathbf{E}_1) \right) \end{pmatrix} \quad (116)$$

where $D_e$ is determinant

$$D_e = \left( \kappa_{0\perp}(\mathbf{t}\mathbf{E}_1) - (\mathbf{l}\mathbf{H}_1) \right) \left( \kappa_{0\perp}(\mathbf{t}\mathbf{H}_2) + (\mathbf{l}\mathbf{E}_2) \right) - \left( \kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2) \right) \left( \kappa_{0\perp}(\mathbf{t}\mathbf{H}_1) + (\mathbf{l}\mathbf{E}_1) \right). \quad (117)$$

Substitution of $\vec{p}_{je}$ into (55) and (57) gives

$$\begin{pmatrix} \vec{p}_{ee} \\ \vec{p}_{me} \end{pmatrix} = \begin{pmatrix} (\mathbf{t}\mathbf{E}_1) & (\mathbf{t}\mathbf{E}_2) \\ (\mathbf{t}\mathbf{H}_1) & (\mathbf{t}\mathbf{H}_2) \end{pmatrix} \begin{pmatrix} \vec{p}_{1e} \\ \vec{p}_{2e} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (118)$$

In the case of incident TM-mode we have boundary conditions

$$\begin{aligned} (\mathbf{t}\mathbf{H}_1)\vec{p}_{1m} + (\mathbf{t}\mathbf{H}_2)\vec{p}_{2m} &= 1 + \vec{p}_{mm}, \\
(\mathbf{l}\mathbf{E}_1)\vec{p}_{1m} + (\mathbf{l}\mathbf{E}_2)\vec{p}_{2m} &= \kappa_{0\perp}(1 - \vec{p}_{mm}), \\
(\mathbf{t}\mathbf{E}_1)\vec{p}_{1m} + (\mathbf{t}\mathbf{E}_2)\vec{p}_{2m} &= \vec{p}_{em}, \\
(\mathbf{l}\mathbf{H}_1)\vec{p}_{1m} + (\mathbf{l}\mathbf{H}_2)\vec{p}_{2m} &= \kappa_{0\perp}\vec{p}_{em}. \end{aligned} \quad (119)-(122)$$

Exclusion of $\vec{p}_{me}$ and $\vec{p}_{mm}$ leads to

$$\begin{pmatrix} (\kappa_{0\perp}(\mathbf{t}\mathbf{H}_1) + (\mathbf{l}\mathbf{E}_1)) & (\kappa_{0\perp}(\mathbf{t}\mathbf{H}_2) + (\mathbf{l}\mathbf{E}_2)) \\ (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_1) - (\mathbf{l}\mathbf{H}_1)) & (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2)) \end{pmatrix} \begin{pmatrix} \vec{p}_{1m} \\ \vec{p}_{2m} \end{pmatrix} = \begin{pmatrix} 2\kappa_{0\perp} \\ 0 \end{pmatrix}. \quad (123)$$

Therefore

$$\begin{pmatrix} \vec{p}_{1m} \\ \vec{p}_{2m} \end{pmatrix} = \frac{1}{D_m} \begin{pmatrix} (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2)) - (\kappa_{0\perp}(\mathbf{t}\mathbf{H}_2) + (\mathbf{l}\mathbf{E}_2)) & (2\kappa_{0\perp}) \\ (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_1) - (\mathbf{l}\mathbf{H}_1)) & (\kappa_{0\perp}(\mathbf{t}\mathbf{E}_2) - (\mathbf{l}\mathbf{H}_2)) \end{pmatrix} \begin{pmatrix} \vec{p}_{1m} \\ \vec{p}_{2m} \end{pmatrix} = \begin{pmatrix} 2\kappa_{0\perp} \\ 0 \end{pmatrix}. \quad (124)$$

where $D_m = -D_e$ (117). Substitution of $\vec{p}_{jm}$ into (119) and (121) gives

$$\begin{pmatrix} \vec{p}_{em} \\ \vec{p}_{mm} \end{pmatrix} = \begin{pmatrix} (\mathbf{t}\mathbf{E}_1) & (\mathbf{t}\mathbf{E}_2) \\ (\mathbf{t}\mathbf{H}_1) & (\mathbf{t}\mathbf{H}_2) \end{pmatrix} \begin{pmatrix} \vec{p}_{1m} \\ \vec{p}_{2m} \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (125)$$

In the general case, when the incident wave has an amplitude $\xi e$ in TE-mode and amplitude $\xi m$ in TM-mode, the state of the incident wave can be described by two-dimensional vector

$$|\vec{\xi}_0\rangle = \begin{pmatrix} \xi e \\ \xi m \end{pmatrix}. \quad (126)$$
and the states of reflected and transmitted waves can be represented as
\[
|\xi_0\rangle = \begin{pmatrix} \xi_e \\ \xi_m \end{pmatrix} = \mathbf{R} |\xi_0\rangle, \quad |\xi\rangle = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \mathbf{T} |\xi_0\rangle, \quad (127)
\]
where \(\mathbf{R}\) and \(\mathbf{T}\) are the two dimensional matrices
\[
\mathbf{R} = \begin{pmatrix} \rho_{ee} & \rho_{em} \\ \rho_{me} & \rho_{mm} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \tau_{1e} & \tau_{1m} \\ \tau_{2e} & \tau_{2m} \end{pmatrix}. \quad (128)
\]

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