Exact $L_2$-small ball asymptotics of Gaussian processes
and the spectrum of boundary value problems
with "non-separated" boundary conditions

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February 2, 2008

Abstract

We sharpen a classical result on the spectral asymptotics of the boundary value problems for self-adjoint ordinary differential operator. Using this result we obtain the exact $L_2$-small ball asymptotics for a new class of zero mean Gaussian processes. This class includes, in particular, integrated generalized Slepian process, integrated centered Wiener process and integrated centered Brownian bridge.

Introduction

The problem of small ball behavior for norms of Gaussian processes was actively studied in recent years, see, for example, the reviews [1] and [2]. We discuss the most explored case of $L_2$-norm. Suppose we have a Gaussian process $X(t), 0 \leq t \leq 1$, with zero mean and covariance function $G_X(t, s) = EX(t)X(s), t, s \in [0, 1]$. Let $\|X\| = \|X\|_{L_2(0,1)}$ and consider

$$Q(X; \varepsilon) = P\{\|X\| \leq \varepsilon\}.$$ 

The problem to define the behavior of $Q(X; \varepsilon)$ as $\varepsilon \to 0$ was solved in [3], but in an implicit way. Therefore, a number of papers provided the simplification of the expression for $Q(X; \varepsilon)$ under various assumptions (see, e.g., the references in [2] and in [4]).

According to the classical Karhunen-Loève expansion one has for the process $X$ the equality in distribution

$$\|X\|^2 = \int_0^1 X^2(t) dt = \sum_{n=1}^{\infty} \lambda_n \eta_n^2,$$

where $\eta_n, n \in \mathbb{N}$, are independent standard Gaussian random variables while $\lambda_n = \lambda_n(X) > 0, n \in \mathbb{N}, \sum_n \lambda_n < \infty$ are the eigenvalues of the integral equation

$$\lambda y(t) = \int_0^1 G_X(t, s)y(s) ds, \quad 0 \leq t \leq 1.$$ 

*Partially supported by RFFR grant No.07-01-00159
Thus we are led to the equivalent problem of studying the asymptotics of \( P\{ \sum_{n=1}^{\infty} \lambda_n \eta^2_n \leq \varepsilon^2\} \) as \( \varepsilon \to 0 \). Unfortunately, explicit formulas for eigenvalues can be obtained only in a limited number of examples.

A new approach developed in the paper [5] gives exact (up to a constant) small ball asymptotics for Gaussian process \( X \) under assumption that \( G_X \) is the Green function of a boundary value problem (BVP) for ordinary differential operator with "separated" boundary conditions (Sturm-type conditions). This work was completed by the paper [6] where sharp constants in the small ball asymptotics were calculated for many Gaussian processes. A part of results of [6] was independently obtained in [7], [8]. In [9] these results were transferred to a class of weighted processes.

The approach of [5] is based on classical Birkhoff’s results on the spectral asymptotics of BVPs to ordinary differential operators. It is well-known, see, e.g., [10, §4], [11, Ch.XIX], that eigenvalues \( \mu_n \) of regular (in particular, self-adjoint) BVPs can be expanded into asymptotic series in powers of \( n \). The first term of this expansion is completely determined by the main coefficient of the operator while the formulas for other terms are rather complicated. It turned out that in the case of "separated" boundary conditions the second term of the asymptotics is completely determined by the sum of orders of boundary functionals. Therefore, it can be derived in explicit form without additional assumptions. Having in hands two-term asymptotics for \( \mu_n \) (and consequently for \( \lambda_n = \mu_n^{-1} \)) we can apply the approach from [4] and comparison theorem [12] to obtain the final result.

In general case, where the boundary conditions are non-separated, the eigenvalues of BVP can be split into two subsequences, and the formulas for the second terms of the asymptotics of these subsequences, generally speaking, cannot be simplified. However, the Lifshits lemma (see below), combined with [5, Theorem 6.2], shows that the \( L_2 \)-small ball behavior up to constant for the corresponding Gaussian process depends only on the sum of these second terms. In this paper we show that this sum, as before, is completely determined by the sum of orders of boundary functionals. This allows us to generalize the results of [5] to considerably larger class of processes.

As for sharp constants in the small ball asymptotics, one can write down explicit formulas for them if the eigenfunctions of the covariance kernel can be expressed in terms of elementary or special functions. In this case the asymptotics of corresponding Fredholm determinants can be calculated by the complex variable methods, see [6], [7]. In this paper we show this by example of several well-known processes generating boundary value problems with non-separated boundary conditions.

The paper is organized as follows. In Section 1 we prove the theorem on the second terms of spectral asymptotics to BVPs with non-separated boundary conditions. Also the small ball asymptotics up to constant for corresponding Gaussian processes is given. The sharp small deviation constants for multiply integrated generalized Slepian process, for some variants of integrated centered Brownian bridge and for some kinds of integrated centered Wiener process are calculated, respectively, in Sections 2, 3 and 4. We note that the \( L_2 \)-small ball asymptotics for some centered processes was derived in [13].

Let us recall some notation. For any zero mean Gaussian process \( X(t), 0 \leq t \leq 1 \), we introduce the centered process \( \overline{X}(t) = X(t) - \int_0^1 X(s) \, ds \) and the \( m \)-times integrated process

\[
X^{[\beta_1, \ldots, \beta_m]}_m(t) = (-1)^{\beta_1 + \ldots + \beta_m} \int_{t_0}^t \cdots \int_{t_1}^{t_0} X(s) \, ds \, dt_1 \ldots t_m
\]
(here any index $\beta_j$ equals either zero or one, $0 \leq t \leq 1$). For the sake of brevity the upper index is sometimes omitted.

The function $G(t, s)$ is called the Green function of (self-adjoint) boundary value problem for differential operator $L$ if it satisfies the equation $LG = \delta(t - s)$ in the sense of distributions and satisfies the boundary conditions. The existence of Green function is equivalent to the invertibility of operator $L$ with given boundary conditions, and $G(t, s)$ is the kernel of the integral operator $L^{-1}$. If homogeneous BVP has a non-trivial solution $\varphi_0$ (without loss of generality it can be assumed to be normalized in $L^2(0, 1)$) then the Green function obviously does not exist. If $\varphi_0$ is the unique solution up to a constant multiplier then the function $G(t, s)$ is called generalized Green function provided it satisfies the equation $LG = \delta(t - s) - \varphi_0(t)\varphi_0(s)$ in the sense of distributions, satisfies the boundary conditions and the orthogonality condition

$$\int_0^1 G(t, s)\varphi_0(s)\, ds = 0 \quad \text{for all} \quad 0 \leq t \leq 1. \quad (0.1)$$

The generalized Green function is the kernel of the integral operator which is inverse to $L$ on the subspace of functions orthogonal to $\varphi_0$ in $L^2(0, 1)$. The reader is referred to [14, Chapter 2, §1] for more detailed properties of the Green function and the generalized Green function (for the second order operators).

The space $W^m_p(0, 1)$ is the Banach space of functions $u$ having continuous derivatives up to $(m - 1)$-th order when $u^{(m-1)}$ is absolutely continuous on $[0, 1]$ and $u^{(m)} \in L^p(0, 1)$. If $p = 2$ it is a Hilbert space.

We set $z_\ell = \exp(i\pi/\ell)$ while $\mathfrak{V}(\ldots)$ stands for the Vandermond determinant:

$$\mathfrak{V}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \det \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \ldots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \ldots & \alpha_n^{n-1} \end{bmatrix} = \prod_{1 \leq j < k \leq n} (\alpha_k - \alpha_j).$$

We cite the statement due to M.A. Lifshits, see [13]. This statement is repeatedly used in our paper.

**Lemma 0.1.** Let $V_1, V_2 > 0$ be two independent random variables with given small ball behavior; namely, let as $r \to 0$

$$\mathbb{P}\{V_1 \leq r\} \sim K_1 r^{a_1} \exp(-D_1^{d+1} r^{-d}), \quad \mathbb{P}\{V_2 \leq r\} \sim K_2 r^{a_2} \exp(-D_2^{d+1} r^{-d}).$$

Then their sum has the following small ball asymptotics:

$$\mathbb{P}\{V_1 + V_2 \leq r\} \sim K r^a \exp(-D^{d+1} r^{-d}),$$

where

$$D = D_1 + D_2, \quad a = a_1 + a_2 - \frac{d}{2}, \quad K = K_1 K_2 \sqrt{\frac{2\pi d}{d + 1}} \cdot \frac{D_1^{a_1+\frac{d}{2}} D_2^{a_2+\frac{d}{2}}}{D^{a+\frac{d}{2}}}.$$  

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1In a similar way one can consider the case of the multiple zero eigenvalue but we do not need it.
1 Eigenvalues asymptotics for BVPs and small ball asymptotics

Let \( \mathcal{L} \) be a self-adjoint differential operator of order \( 2\ell \) generated by a differential expression

\[
\mathcal{L} u \equiv (-1)^\ell \left( p_{\ell} u^{(\ell)} \right)^{(\ell)} + \left( p_{\ell-1} u^{(\ell-1)} \right)^{(\ell-1)} + \ldots + p_0 u,
\]

\((p_{\ell}(x) > 0)\) and by \( 2\ell \) boundary conditions

\[
U_\nu(u) \equiv U_{\nu 0}(u) + U_{\nu 1}(u) = 0, \quad \nu = 1, \ldots, 2\ell,
\]

where

\[
U_{\nu 0}(u) = \alpha_\nu u^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu-1} \alpha_{\nu j} u^{(j)}(0),
\]

\[
U_{\nu 1}(u) = \gamma_\nu u^{(k_\nu)}(1) + \sum_{j=0}^{k_\nu-1} \gamma_{\nu j} u^{(j)}(1),
\]

and for any index \( \nu \) at least one of coefficients \( \alpha_\nu \) and \( \gamma_\nu \) is not equal to zero.

It is well known, see, e.g., [10, §4], that the system of boundary conditions (1.2) can be reduced to the **normalized form** by equivalent transformations. In what follows we always assume that this reduction is realized. This form is specified by the minimal sum of orders of all boundary conditions. Since this quantity is of great importance in our arguments, we introduce the notation \( \kappa = \sum_{\nu=1}^{2\ell} k_\nu \). We remark also that the inequalities

\[
2\ell - 1 \geq k_1 \geq k_2 \geq \cdots \geq k_{2\ell} \geq 0, \quad k_\nu > k_{\nu+2}
\]

hold true for the normalized boundary conditions.

For simplicity we assume \( p_j \in W^j_\infty[0,1], j = 0, \ldots, \ell \). Then the domain \( \mathcal{D}(\mathcal{A}) \) consists of the functions \( u \in W^{2\ell}_2(0,1) \) satisfying boundary conditions (1.2).

Consider the eigenvalue problem

\[
\mathcal{L} u = \mu u \quad \text{on } [0,1], \quad u \in \mathcal{D}(\mathcal{L}).
\]

(1.3)

It is well known, see, e.g., [10, §4, Theorem 2], that for \( p_{\ell} \equiv 1 \) the eigenvalues of (1.3) counted according to their multiplicities can be split into two subsequences \( \mu'_n, \mu''_n, n \in \mathbb{N} \), such that, as \( n \to \infty \),

\[
\mu'_n = (2\pi n + \rho' + O(n^{-1/2}))^{2\ell}, \quad \mu''_n = (2\pi n + \rho'' + O(n^{-1/2}))^{2\ell},
\]

(1.4)

where \( \xi' = \exp(i \rho') \), \( \xi'' = \exp(i \rho'') \) are the roots of quadratic equation

\[
\begin{align*}
\theta_1 \xi + \theta_0 + \theta_{-1} \xi^{-1} & \equiv \\
\equiv \det \begin{bmatrix}
(\alpha_1 + \xi_1 \gamma_1) & \alpha_1 \omega^k_1 & \ldots & \alpha_1 \omega^k_{\ell-1} & \omega^k_1 (\alpha_1 + \xi_1^{-1} \gamma_1) & \gamma_1 \omega^k_{\ell+1} & \cdots & \gamma_1 \omega^k_{2\ell-1} \\
(\alpha_2 + \xi_2 \gamma_2) & \alpha_2 \omega^k_2 & \ldots & \alpha_2 \omega^k_{\ell-1} & \omega^k_2 (\alpha_2 + \xi_2^{-1} \gamma_2) & \gamma_2 \omega^k_{\ell+1} & \cdots & \gamma_2 \omega^k_{2\ell-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(\alpha_{2\ell} + \xi_{2\ell} \gamma_{2\ell}) & \alpha_{2\ell} \omega^k_{2\ell} & \ldots & \alpha_{2\ell} \omega^k_{2\ell-1} & \omega^k_{2\ell} (\alpha_{2\ell} + \xi_{2\ell}^{-1} \gamma_{2\ell}) & \gamma_{2\ell} \omega^k_{2\ell+1} & \cdots & \gamma_{2\ell} \omega^k_{2\ell+2\ell-1}
\end{bmatrix} & = 0
\end{align*}
\]
expression (1.1) and by boundary conditions (1.2). Let \( \kappa \) be weakened.

be the Green function of a self-adjoint positive definite operator of separated boundary conditions.

assumption of regularity of the system (1.2). The requirements on the coefficients \( \alpha \) and \( \beta \) as long as boundary conditions (1.2) are regular. To check this fact we write

\[
\theta_1 = \det \begin{bmatrix}
\gamma_1 & \alpha_1 \omega_1^{k_1} & \ldots & \alpha_1 \omega_{\ell-1}^{k_1} & \alpha_1 \omega_{\ell}^{k_1} & \gamma_1 \omega_1^{k_1} & \ldots & \gamma_1 \omega_{2\ell-1}^{k_1} \\
\gamma_2 & \alpha_2 \omega_1^{k_2} & \ldots & \alpha_2 \omega_{\ell-1}^{k_2} & \alpha_2 \omega_{\ell}^{k_2} & \gamma_2 \omega_1^{k_2} & \ldots & \gamma_2 \omega_{2\ell-1}^{k_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{2\ell} & \alpha_1 \omega_1^{k_{2\ell}} & \ldots & \alpha_2 \omega_{\ell-1}^{k_{2\ell}} & \alpha_2 \omega_{\ell}^{k_{2\ell}} & \gamma_2 \omega_1^{k_{2\ell}} & \ldots & \gamma_2 \omega_{2\ell-1}^{k_{2\ell}}
\end{bmatrix}
\]

Taking the common multiplier \( \omega_{1}^{k_j} \) over from the \( j \)-th row in the first determinant we obtain

\[
\theta_1 = -\omega_1^{k_j} \theta_{-1}
\]

and, therefore,

\[
\rho + \rho' = -\frac{\pi \kappa}{\ell} + (2N - 1)\pi, \quad N \in \mathbb{Z}
\]  

(1.6)

Since the eigenvalues of the problem (1.3) depend on \( \alpha \) and \( \gamma \), continuously, the parameter \( N \) in (1.6) does not depend on \( \alpha \) and \( \gamma \). However, Theorem 7.1 [5] shows that if the boundary conditions are separated, i.e. \( U_{\nu1} \equiv 0 \) for \( \nu = 1, \ldots, \ell \) and \( U_{\nu0} \equiv 0 \) for \( \nu = \ell + 1, \ldots, 2\ell \) then in (1.4)

\[
\rho' = \pi (\ell - 1) - \frac{\pi \kappa}{2\ell}, \quad \rho'' = \pi (\ell - 2) - \frac{\pi \kappa}{2\ell}
\]

we note that in this case two sequences \( \mu'_n \) and \( \mu''_n \) can be naturally merged into \( \mu_n = \left( \pi (n + \ell - 1 - \frac{\kappa}{2\ell}) + O(n^{-1/2}) \right) 2\ell \). Thus, in this case (1.5) holds true, and the statement follows.

Remark 2. In fact the assumption of self-adjointness of the operator can be relaxed to the assumption of regularity of the system (1.2). The requirements on the coefficients \( p_j \) also can be weakened.

The next theorem generalizes [5, Theorem 7.2] where this result was obtained for the case of separated boundary conditions.

Theorem 1.2. Let the covariance \( G_X(t, s) \) of a zero mean Gaussian process \( X(t) \), \( 0 \leq t \leq 1 \), be the Green function of a self-adjoint positive definite operator \( \mathcal{L}_X \) generated by a differential expression (1.1) and by boundary conditions (1.2). Let \( \kappa < 2\ell^2 \). Then, as \( \varepsilon \to 0 \),

\[
P\{\|X\| \leq \varepsilon\} \sim C(X) \cdot \varepsilon^3 \exp \left( -\frac{2\ell - 1}{2} \left( \frac{\eta_\ell}{2\ell \sin \pi/2\ell} \right)^{2\ell} \varepsilon^{-2} \right).
\]  

(1.7)
Here we denote
\[ \gamma = -\ell + \frac{\kappa + 1}{2\ell - 1}, \quad \vartheta_\ell = \int_0^1 p_\ell^{-1/(2\ell)}(x) \, dx, \]
and the constant \( C(X) \) is given by
\[ C(X) = C_{\text{dist}}(X) \cdot \frac{(2\pi)^{\ell/2} (\pi/\vartheta_\ell)^{\ell \gamma} \left( \sin \frac{\pi}{2\ell} \right)^{1+\gamma}}{(2\ell - 1)^{1/2} \left( \frac{\pi}{2\ell} \right)^{1+\frac{\ell}{2}} \Gamma(\ell - \frac{\kappa}{2\ell})}, \]
where \( C_{\text{dist}}(X) \) is the so-called distortion constant
\[ C_{\text{dist}}(X) \equiv \prod_{n=1}^\infty \left( \frac{\mu_n}{\pi/\vartheta_\ell \cdot \left[ n + \ell - 1 - \frac{\gamma}{2\ell} \right]} \right)^{\ell/2}, \tag{1.8} \]
and \( \mu_n = (\lambda_n(X))^{-1} \) are the eigenvalues of the problem (1.3).

**Proof.** According to comparison principle [12] and to formulas (1.4), taking into account Remark 1 we have
\[ \mathbb{P}\{\|X\| \leq \varepsilon\} = \mathbb{P}\left\{ \sum_{n=1}^\infty \lambda_n \eta_n^2 \leq \varepsilon^2 \right\} = \mathbb{P}\left\{ \sum_{n=1}^\infty \frac{\eta'_n^2}{\mu_n} + \sum_{n=1}^\infty \frac{\eta''_n^2}{\mu'_n} \leq \varepsilon^2 \right\} = \prod_{n=1}^\infty \left( \frac{\mu_n'}{\mu_n} \cdot \frac{\mu''_n}{\mu''_n} \right)^{1/2} \cdot \mathbb{P}\{V' + V'' \leq \varepsilon^2\}, \tag{1.9} \]
where
\[ V' = \sum_{n=1}^\infty \frac{\eta'_n^2}{\mu_n'}, \quad V'' = \sum_{n=1}^\infty \frac{\eta''_n^2}{\mu''_n}, \tag{1.10} \]
\( \eta'_n \) and \( \eta''_n \) are two independent sequences of independent standard Gaussian r.v.’s,
\[ \tilde{\mu}'_n = \left( \frac{2\pi n + \rho'}{\vartheta_\ell} \right)^{2\ell}, \quad \tilde{\mu}''_n = \left( \frac{2\pi n + \rho''}{\vartheta_\ell} \right)^{2\ell}. \tag{1.11} \]
The asymptotic behavior of small ball probabilities for the infinite sums (1.10) with coefficients of the form (1.11) was derived in [5, Theorem 6.2]. The asymptotics of \( \mathbb{P}\{V' + V'' \leq \varepsilon^2\} \) as \( \varepsilon \to 0 \) can be deduced from the asymptotics of \( \mathbb{P}\{V' \leq \varepsilon^2\} \) and \( \mathbb{P}\{V'' \leq \varepsilon^2\} \) by Lemma 0.1. Finally, the infinite product in (1.9) differs from (1.8) by the multiplier which converges due to (1.5). After simplification, we arrive at (1.7). \( \square \)

## 2 Slepian process and related processes

Consider the generalized Slepian process \( S^{(c)} \), that is a stationary zero mean Gaussian process with covariance function
\[ G_{S^{(c)}}(t, s) = c - |t - s|, \quad t, s \in [0, 1]. \]
It is easy to check that \( G_{S^{(c)}} \) is indeed a covariance for \( c \geq 1/2 \). Remark that for \( c \geq 1 \) we have the distributional equality
\[ S^{(c)}(t) \overset{d}{=} W(t + c) - W(t), \quad 0 \leq t \leq 1, \]
where $W(t)$ is a standard Wiener process. The conventional Slepian process [15] corresponds to $c = 1$; the small ball asymptotics for this process was derived in [16].

The direct calculation shows, see [17], that $G_{S(c)}$ is the Green function of the BVP
\[ L_{S(c)} u \equiv -\frac{1}{2} u'' = \mu u \quad \text{on} \quad [0, 1], \quad u(0) + \mu(1) = 0, \quad (2c - 1) u'(0) - (u(0) + u(1)) = 0. \quad (2.1) \]

**Theorem 2.1.** 1. Let $c = 1/2$. Then, as $\varepsilon \to 0$,
\[ P\{\|S^{(1/2)}\| \leq \varepsilon \} \sim \frac{4 \varepsilon}{\sqrt{\pi}} \cdot \exp\left(-\frac{1}{4} \varepsilon^{-2}\right). \quad (2.2) \]

2. Let $c > 1/2$. Then, as $\varepsilon \to 0$,
\[ P\{\|S^{(c)}\| \leq \varepsilon \} \sim \frac{4\sqrt{2} \varepsilon^2}{\sqrt{\pi} (2c - 1)} \cdot \exp\left(-\frac{1}{4} \varepsilon^{-2}\right). \quad (2.3) \]

**Remark 3.** When $c > 1/2$, the problem (2.1) does not satisfy assumptions of Theorem 1.2, since $\kappa = 2 = 2\ell^2$. This is only a formal difficulty, and the ways to avoid it are well known, see, e.g., [5, Proposition 6.4] and [16]. In this particular case, however, it is more simple to use the available results.

**Proof.** 1. Set $\zeta = \sqrt{2\mu}$. Substituting the general solution of the equation (2.1) $u(t) = c_1 \sin(\zeta t) + c_2 \cos(\zeta t)$ into boundary conditions we deduce that $\mu_n = \frac{1}{2} r_n^2$, where $r_1 < r_2 < \ldots$ are positive roots of the equation
\[ F^{(c)}(\zeta) \equiv 2 + 2 \cos(\zeta) - (2c - 1) \zeta \sin(\zeta) = 0. \]

It is easy to see that for $c = 1/2$ the spectrum of the problem (2.1) consists of double eigenvalues $\mu_n' = \mu_n'' = 2(\pi n - \frac{\pi}{2})$, $n \in \mathbb{N}$. Thus, we have the distributional equality
\[ \|S^{(1/2)}\|^2 \overset{d}{=} \frac{1}{2} (\|W_1\|^2 + \|W_2\|^2), \]
where $W_1$ and $W_2$ are independent standard Wiener processes. The asymptotics of $P\{\|W\| \leq \varepsilon\}$ as $\varepsilon \to 0$ is well known. Applying Lemma 0.1 we arrive at (2.2).

2. When $c > 1/2$ it is obvious that $\left|\frac{F^{(c)}(0)}{F^{(c)}(\zeta)}\right| = 1$ and $\left|\frac{F^{(c)}(\zeta)}{F^{(c)}(0)}\right| \Rightarrow 2c - 1$ as $|\zeta| = \pi(N + \frac{1}{2})$, $N \to \infty$. Let us apply the comparison theorem [12] to the processes $S^{(c)}$ and $S^{(1)}$. Then we apply Jensen’s Theorem, see [18, §3.6.1], to the functions $F^{(c)}$ and $F^{(1)}$ and obtain
\[ P\{\|S^{(c)}\| \leq \varepsilon\} \sim \frac{1}{\sqrt{2c - 1}} \cdot P\{\|S^{(1)}\| \leq \varepsilon\}, \quad \varepsilon \to 0. \]

The asymptotics of the last expression, as we mentioned, was established in [16]. This gives (2.3).

Now we consider $m$-times integrated process $(S^{(c)})_{m_1, \ldots, m_m}(t)$. Following [6] we introduce the notation
\[ \zeta_{t} = \left(\varepsilon \sqrt{\ell \sin \frac{\pi}{2\ell}} \right)^{\frac{1}{m-1}}; \quad \Omega_{t} = \frac{2\ell - 1}{2\ell \sin \frac{\pi}{2\ell}}, \]
for $j = 1, \ldots, m$
\[ k_j = \begin{cases} m - j, & \text{if } \beta_j = 0, \\ m + 1 + j, & \text{if } \beta_j = 1, \end{cases} \quad k_j' = 2m + 1 - k_j. \quad (2.5) \]
**Theorem 2.2.** Let $m \in \mathbb{N}$. Then, as $\varepsilon \to 0$,

1. for $c = 1/2$

\[
\mathbb{P}\{\|S^{(1/2)}_{m}[\beta_1, \ldots, \beta_m]\| \leq \varepsilon\} \sim \frac{(2m + 2)^{\frac{m+1}{2}}}{|\mathcal{W}(z_{m+1}^{k_1}, z_{m+1}^{k_2}, \ldots, z_{m+1}^{k_m})|} \cdot \frac{\prod_{j=1}^m (1 + z_{m+1}^{k_j})^2 + \prod_{j=1}^m (1 + z_{m+1}^{k_j})^2}{\sqrt{\pi D_{m+1}}} \cdot \frac{2\varepsilon_{m+1}}{\sqrt{\pi D_{m+1}}} \exp \left( -\frac{D_{m+1}}{2\varepsilon_{m+1}^2} \right). \tag{2.6}
\]

2. for $c > 1/2$

\[
\mathbb{P}\{\|S^{(c)}_{m}[\beta_1, \ldots, \beta_m]\| \leq \varepsilon\} \sim \frac{(2m + 2)^{\frac{m+1}{2}}}{|\mathcal{W}(z_{m+1}^{k_1}, z_{m+1}^{k_2}, \ldots, z_{m+1}^{k_m})|} \cdot \frac{\prod_{j=1}^m (1 + z_{m+1}^{k_j})^2 + \prod_{j=1}^m (1 + z_{m+1}^{k_j})^2}{\sqrt{\pi D_{m+1}}} \cdot \frac{2\varepsilon_{m+1}}{\sqrt{\pi D_{m+1}}} \exp \left( -\frac{D_{m+1}}{2\varepsilon_{m+1}^2} \right). \tag{2.7}
\]

**Proof.** The BVP generated by integrated process can be expressed in terms of BVP generated by original process due to [5, Theorem 2.1]. Applying this theorem to (2.1) we get

\[
\begin{align*}
\mathcal{L}_{s_{(c)}} u &\equiv (-1)^{m+1} \cdot \frac{1}{2} u^{(2m+2)} = \mu u \quad \text{on} \quad [0, 1], \\
 u(\beta_m) &= u'(\beta_{m-1}) = \cdots = u^{(m-1)}(\beta_1) = 0, \\
 u^{(m+1)}(0) + u^{(m+1)}(1) &= 0, \\
 u^{(m+2)}(1 - \beta_1) &= u^{(m+3)}(1 - \beta_2) = \cdots = u^{(2m+1)}(1 - \beta_m) = 0. 
\end{align*}
\tag{2.8}
\]

Using (2.5) we can rewrite the second and the fourth rows in (2.8) as follows:

\[
u^{(k_j)}(0) = u^{(k_j)}(1) = 0, \quad j = 1, \ldots, m.
\]

Since the BVP (2.8) satisfies all assumptions of Theorem 1.2 (with $\ell = m + 1$), to prove (2.6)-(2.7) we only need to calculate the distortion constants (1.8). Note that $\vartheta_{\ell} = 2\varepsilon_{m+1}^2$.

Put $\zeta = (2\mu)^{m+1}$. Then the general solution of the equation (2.8) is

\[
u(t) = \sum_{j=0}^{2m+1} c_j \exp(\omega_j \zeta t), \tag{2.9}
\]

where $\omega_j = z_{m+1}^{j}$.  

1. If $c = \frac{1}{2}$ then $\varpi = (2m + 1)(m + 1)$. Substituting (2.9) into boundary conditions we deduce that $\mu_n = \frac{1}{2} r_{1n}^{2m+2}$, where $r_1 < r_2 < \ldots$ are positive roots of the entire function

\[
\mathcal{F}(\zeta) \equiv \begin{bmatrix}
1 & \omega_1^{k_1} & \ldots & \omega_m^{k_1} & (-1)^{k_1} & \ldots & (-\omega_m)^{k_1} \\
1 & \omega_1^{k_m} & \ldots & \omega_m^{k_m} & (-1)^{k_m} & \ldots & (-\omega_m)^{k_m} \\
1 + e^i\zeta & \omega_1^{m+1}(1 + e^{i\omega_m}) & \ldots & \omega_m^{m+1}(1 + e^{i\omega_m}) & (-1)^{m+1}(1 + e^{-i\zeta}) & \ldots & (-\omega_m)^{m+1}(1 + e^{-i\zeta}) \\
e^i\zeta & \omega_1^{m+1}e^{i\omega_m} & \ldots & \omega_m^{m+1}e^{i\omega_m} & (-1)^{k_1}e^{-i\zeta} & \ldots & (-\omega_m)^{k_1}e^{-i\omega_m} \\
e^i\zeta & \omega_1^{k_m}e^{i\omega_m} & \ldots & \omega_m^{k_m}e^{i\omega_m} & (-1)^{k_m}e^{-i\zeta} & \ldots & (-\omega_m)^{k_m}e^{-i\omega_m}
\end{bmatrix}
\]
Therefore,
\[ C_{\text{dist}}(S_m^{(1/2)}) = \prod_{n=1}^{\infty} \left( \frac{r_n}{\pi n - \frac{\pi}{2}} \right)^{m+1}. \]

Since \(|F(\zeta)| = |F(\omega_1 \zeta)|\), the set of all nonzero roots of the function \(F\) consists of \(2m + 2\) sequences \(\omega_j r_n, j = 0, \ldots, 2m + 1, n \in \mathbb{N}\).

According to [10, §4, Theorem 2], the relation
\[ F(\zeta) = \exp(-i \omega_1 \zeta) \exp(-i \omega_2 \zeta) \cdots \exp(-i \omega_m \zeta) \cdot \left( \Phi(\zeta) + O(|\zeta|^{-1}) \right), \quad (2.10) \]
holds true for \(|\zeta| \to \infty\) and \(|\arg(\zeta)| \leq \frac{\pi}{2m+2}\). Here
\[ \Phi(\zeta) = \det \begin{pmatrix} 1 & \omega_{k_1} & \cdots & \omega_{k_1}^{m+1} & \cdots & 0 & \cdots & 0 \\ 1 & \omega_{k_1} & \cdots & \omega_{k_1}^{m+1} & \cdots & 0 & \cdots & 0 \\ 1 + e^{i \zeta} & \omega_{m} & \cdots & \omega_{m}^{m+1} & \cdots & 0 & \cdots & 0 \\ e^{i \zeta} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 1 + e^{i \zeta} & \omega_{1} & \cdots & \omega_{1}^{m+1} & \cdots & 0 & \cdots & 0 \\ e^{i \zeta} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \]

Expanding this determinant in the elements of the first and the \((m + 2)\)-nd columns we obtain
\[ |\Phi(\zeta)| = \mathcal{M} \cdot | \exp(i \zeta) + \exp(-i \zeta) + R|, \quad (2.11) \]
where
\[ \mathcal{M} = | \Psi(\omega_{1}^{k_1}, \ldots, \omega_{1}^{k_m}, \omega_{1}^{m+1}) \cdot \Psi(\omega_{1}^{m+1}, \omega_{1}^{k_1}, \ldots, \omega_{1}^{k_m}) + \\ + \Psi(\omega_{1}^{k_1}, \ldots, \omega_{1}^{k_m}, \omega_{1}^{m+1}) \cdot \Psi(\omega_{1}^{m}, \omega_{1}^{k_1}, \ldots, \omega_{1}^{k_m}) |, \]
while \(R\) is a constant which is inessential for us.

Following [6], for arbitrary \(\delta > -1\) we introduce the function
\[ \Psi_\delta(\zeta) = \psi_\delta(\omega_1 \zeta) \psi_\delta(\omega_2 \zeta) \cdots \psi_\delta(\omega_m \zeta), \quad (2.12) \]
where
\[ \psi_\delta(\zeta) = \frac{\Gamma^2(1 + \delta)}{\Gamma(1 + \delta + \frac{\pi}{n}) \Gamma(1 + \delta - \frac{\pi}{n})} = \prod_{n=1}^{\infty} \left( 1 - \frac{\zeta^2}{(\pi(n + \delta))^2} \right). \]
It is easy to check, see [6, Lemma 1.3)], that
\[ \psi_\delta(\zeta) \sim \Gamma^2(1 + \delta) \pi^{2\delta} \zeta^{-2\delta - 1} \cos(\zeta - \pi(\delta + 1/2)), \quad (2.13) \]
as \(\zeta \to \infty\), \(|\arg(\zeta)| \leq \phi_0 < \pi\). Moreover, the convergence is uniform in \(\arg(\zeta)\).

Setting \(\delta = -1/2\) we deduce from (2.10)-(2.13)
\[ \frac{|F(\zeta)|}{|\Psi_\delta(\zeta)|} \to 2^{m+1} \mathcal{M}, \quad (2.14) \]
as \(\zeta \to \infty\), \(\arg(\zeta) \neq \frac{\pi j}{2m+2}, j \in \mathbb{Z}\).
By Jensen’s Theorem
\[ C_{\text{dist}}^2(S_m^{(1/2)}) = \left| \mathcal{F}(0) \right| \cdot \exp \left\{ \lim_{\epsilon \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \ln \left| \Psi_{\delta}(\rho e^{i\theta}) \right| \, d\theta \right\}. \]
The integrand obviously has a summable majorant. The Lebesgue Dominated Convergence Theorem gives, in view of (2.14),
\[ C_{\text{dist}}^2(S_m^{(1/2)}) = \frac{\left| \mathcal{F}(0) \right|}{2^{m+1}M} = \frac{4|\mathfrak{M}(1, \omega_1, \omega_1^2, \ldots, \omega_1^{2m+1})|}{2^{m+1}M} = \frac{4(m + 1)^{m+1}}{M}. \]
It remains to take into account that, due to (2.5),
\[ \mathcal{M} = |\mathfrak{M}(\omega_1^{k_1}, \ldots, \omega_1^{k_m})|^2 \cdot \left( \prod_{j=1}^{m} |1 + \omega_1^{k_j}|^2 + \prod_{j=1}^{m} |1 + \omega_1^{k_j}|^2 \right). \]
After some simplification we arrive at (2.6).

2. If \( c > \frac{1}{2} \) then \( \kappa = (2m + 1)(m + 1) + 1 \). Substituting (2.9) into boundary conditions we deduce that
\[ C_{\text{dist}}(S_m^{(c)}) = \prod_{n=1}^{\infty} \left( \frac{r_n}{\pi(n - \frac{m+2}{2m+2})} \right)^{m+1}, \]
where \( r_1 < r_2 < \ldots \) are positive roots of the function \( \mathcal{F}_{c}(\zeta) \), which arises if we change in the determinant \( \mathcal{F}(\zeta) \) the row
\[ [1 + e^{i\zeta} \omega_1^{m+1} (1 + e^{i\omega_m})] \ldots \omega_m^{m+1} (1 + e^{i\omega_m}) \]
by the row
\[ [1 - \frac{i\zeta}{\tau}(1 + e^{i\zeta}) \omega_1^{m+1} (1 - \frac{i\zeta}{\tau}e^{i\zeta})(1 + e^{i\omega_m})] \ldots \omega_m^{m+1} (1 - \frac{i\zeta}{\tau}(1 + e^{i\zeta}) \ldots \ldots \] (here \( \tau = \frac{1}{2c-1} \)).
Similarly to part 1, the relation
\[ \mathcal{F}_{c}(\zeta) = \exp(-i\omega_1\zeta) \exp(-i\omega_2\zeta) \ldots \exp(-i\omega_m\zeta) \cdot \left( \Phi(\zeta) + O(|\zeta|^{-1}) \right), \]
holds true as \( |\zeta| \to \infty \) and \( |\arg(\zeta)| \leq \frac{\pi}{2m+2} \). Here
\[
\Phi(\zeta) = \begin{detmatrix}
1 & \omega_1^{k_1} & \ldots & \omega_1^{k_m} & (-1)^{k_1} & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \omega_1^{k_m} & \ldots & \omega_1^{k_m} & (-1)^{k_m} & 0 & \ldots & 0 \\
1 & \omega_1^{m+1} & \ldots & \omega_1^{m+1} & (-1)^{m+1} & 0 & \ldots & 0 \\
1 + e^{i\zeta} & \omega_1^{m+1} & \ldots & \omega_1^{m+1} & (-1)^{m+1} e^{i\zeta} & (-\omega_1)^{m+1} & \ldots & (-\omega_1)^{m+1} \\
e^{i\zeta} & 0 & \ldots & 0 & (-1)^{k_1} e^{-i\zeta} & (-\omega_1)^{k_1} & \ldots & (-\omega_1)^{k_1} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
e^{i\zeta} & 0 & \ldots & 0 & (-1)^{k_m} e^{-i\zeta} & (-\omega_1)^{k_m} & \ldots & (-\omega_1)^{k_m}
\end{detmatrix}
\] Let us subtract \( (m + 1) \)-st row from \( (m + 2) \)-nd one. Then, expanding the determinant in the elements of the first column we obtain
\[ |\Phi(\zeta)| = |\mathfrak{M}(\omega_1^{k_1}, \ldots, \omega_1^{k_m}, \omega_1^{m+1})| \cdot |\mathfrak{M}(\omega_1^{m+1}, \omega_1^{k_1}, \ldots, \omega_1^{k_m})| \cdot |\exp(-i\zeta) - \omega_1^{k_1} \exp(i\zeta)| = |\mathfrak{M}(\omega_1^{k_1}, \ldots, \omega_1^{k_m})| \cdot |\mathfrak{M}(\omega_1^{k_1}, \ldots, \omega_1^{k_m})| \cdot 
\prod_{j=1}^{m} |(\omega_1^{m+1} - \omega_1^{k_j})(\omega_1^{m+1} - \omega_1^{k_j})| \cdot |\exp(-i\zeta) + \omega_1 \exp(i\zeta)| = 
= |\mathfrak{M}(\omega_1^{k_1}, \ldots, \omega_1^{k_m})|^2 \cdot \frac{2m + 2}{|1 - \omega_1|} \cdot |\exp(-i\zeta) + \omega_1 \exp(i\zeta)|.
Hence, setting $\delta = -\frac{m+2}{2m+2}$ and taking into account $|\mathcal{F}(\zeta)| \equiv |\mathcal{F}(\omega_1 \zeta)|$, we obtain

$$\frac{|\zeta \mathcal{F}(\zeta)|}{|\Psi(\zeta)|} \to \frac{2^{m+1} |\mathcal{W}(\omega_1^k, \ldots, \omega_1^m)|^2}{\Gamma^{m+2}(1 + \delta)\pi^{2m+2}\delta} \cdot \frac{2m + 2}{1 - \omega_1}$$

as $\zeta \to \infty$, arg$(\zeta) \neq \frac{\pi j}{2m+2}, j \in \mathbb{Z}$.

Applying Jensen’s Theorem similarly to part 1, we arrive at

$$C^2_{\text{dist}}(S^{(c)}_m) = \frac{\Gamma^{m+2}(1 + \delta)\pi^{2m+2}\delta |1 - \omega_1|}{2m+1 |\mathcal{W}(\omega_1^k, \ldots, \omega_1^m)|^2 (2m + 2)} \cdot |\zeta \mathcal{F}(\zeta)|_{\zeta=0} =$$

$$= \frac{2\tau (m + 1)^m \Gamma^{2m+2}(1 + \delta) |1 - \omega_1|}{\pi^{m+2} |\mathcal{W}(\omega_1^k, \ldots, \omega_1^m)|^2}.$$

Since $|1 - \omega_1| = 2 \sin \frac{\pi}{2m+2}$, this gives (2.7) after some simplification. \(\square\)

**Remark 4.** Let $c > 1/2$. Using (2.7) and the extremal properties of the Vandermonde determinants [19] one can see that among all $m$-times integrated processes $(S^{(c)})^{[\beta_1, \ldots, \beta_m]}_m$ the processes $(S^{(c)})^{[0, \ldots, 0]}_m$ and $(S^{(c)})^{[1, \ldots, 1]}_m$ have the largest small ball constant while the Euler integrated processes $(S^{(c)})^{[0, 1, 0, \ldots]}_m$ and $(S^{(c)})^{[1, 0, 1, \ldots]}_m$ have the smallest one. We conjecture that this is true also for $c = 1/2$ but this problem is open yet.

We also point out a curious relation arising when one compares the small ball asymptotics for the process $(S^{(c)})_m, c > 1/2$, and for integrated Ornstein–Uhlenbeck process, see [6, Theorem 2.2]. Since corresponding BVPs have the same parameters $\kappa$, these asymptotics differ only by a constant. Rather unexpected is the fact that this constant equals $\sqrt{\frac{e}{2(2e-1)}}$ and therefore depends neither on $\beta_j$ nor even on $m$.

### 3 Integrated centered Brownian bridge and related processes

The most famous process generating the BVP with non-separated boundary conditions is the centered Brownian bridge $\overline{B}(t)$; its spectrum was derived for the first time in [20]. Note that this BVP

$$L_{\overline{B}}u \equiv -u'' = \mu u \quad \text{on} \quad [0, 1], \quad u(0) - u(1) = 0, \quad u'(0) - u'(1) = 0 \quad (3.1)$$

has a zero eigenvalue with constant eigenfunction $\varphi_0(t) \equiv 1$. Hence the covariance $G_{\overline{B}}(t, s)$ is the generalized Green function of the problem (3.1). We show later that this is a typical situation for centered processes. In this case Theorem 2.1 [5] is not applicable. To study integrated processes we need two auxiliary statements.

**Theorem 3.1.** 1. Let the BVP (1.3) have a zero eigenvalue with constant eigenfunction $\varphi_0(t) \equiv 1$. Let the kernel $G(t, s)$ be the generalized Green function of the problem (1.3). Then the integrated kernel

$$G_1(t, s) = \int_0^t \int_0^s G(x, y) \, dx \, dy \quad (3.2)$$

is (conventional) Green function of the BVP

$$L_1 u \equiv -(Lu')' = \mu u \quad \text{on} \quad [0, 1], \quad u \in \mathcal{D}(L_1), \quad (3.3)$$
where the domain \( \mathcal{D}(\mathcal{L}_1) \) consists of the functions \( u \in W_{2}^{2\ell+2}(0,1) \) satisfying the boundary conditions
\[
    u(0) = 0; \quad u(1) = 0; \quad u' \in \mathcal{D}(\mathcal{L}).
\]  

2. Let the kernel \( G(t,s) \) be the Green function of the problem (3.3)-(3.4). Then the centered kernel
\[
    \overline{G}(t,s) = G(t,s) - g(t) - g(s) + \overline{g}
\]
(here \( g(t) = \frac{1}{0} \int G(t,s) ds, \overline{g} = \frac{1}{0} \int g(t) dt \)) is the generalized Green function of the BVP
\[
    \tilde{\mathcal{L}} u \equiv - (\mathcal{L}u)' = \mu u \quad \text{on} \quad [0,1], \quad u \in \mathcal{D}(\tilde{\mathcal{L}}_1),
\]  
where the domain \( \mathcal{D}(\tilde{\mathcal{L}}_1) \) consists of the functions \( u \in W_{2}^{2\ell+2}(0,1) \) satisfying the boundary conditions
\[
    u(0) - u(1) = 0; \quad u' \in \mathcal{D}(\mathcal{L}); \quad (\mathcal{L}u')(0) - (\mathcal{L}u')(1) = 0.
\]  

**Remark 5.** Let \( X(t), 0 \leq t \leq 1 \) be a Gaussian process with zero mean. It is well known that the covariance of the integrated process \( G_{X_1} \) can be expressed in terms of the original covariance \( G_X \) by formula (3.2). Note that under assumptions of part 1 of our Theorem the processes \( X_1^0 \) and \( X_1^1 \) coincide almost surely. It is easy to show that the covariance of the centered process \( G_{X} \) can be expressed in terms of the original covariance \( G_X \) by formula (3.5).

**Remark 6.** It is easy to see that the differential expression (1.1) can be represented in the form (3.3) iff \( p_0 \equiv 0 \).

**Proof.** 1. The first boundary condition in (3.4) is trivially satisfied for the function \( G_1 \) while the second one is satisfied due to (0.1). Further, differentiating (3.2) w.r.t. \( t \) we obtain
\[
    (G_1)'_t(t,s) = \int_0^t G(t,y) dy,
\]
whence the other boundary conditions follow by linearity of the set \( \mathcal{D}(\mathcal{L}) \). Since \( \mathcal{L}G(t,s) = \delta(t-s) - 1 \), we obtain consequently
\[
    \mathcal{L}(G_1)'_t(t,s) = \chi_{\mathbb{R}_+}(t-s) - s; \quad (\mathcal{L}(G_1)'_t)_t = -\delta(t-s),
\]
and the statement follows.

2. The orthogonality condition (0.1) follows from the definition of \( \overline{G} \):
\[
    \int_0^1 \overline{G}(t,s) ds = g(s) - g(s) - \overline{g} + \overline{g} = 0.
\]
The first two conditions in (3.4) provide
\[
    \overline{G}(0,s) = -g(s) + \overline{g} = \overline{G}(1,s).
\]
Differentiating (3.5) w.r.t. \( t \) we obtain
\[
    \overline{G}'_t(t,s) = G'_t(t,s) - \int_0^1 G'_t(t,y) dy,
\]
\[
\]
2The problem (3.6)-(3.7) obviously has a zero eigenvalue with constant eigenfunction \( \varphi_0(t) \equiv 1 \).
that gives $\mathcal{G}_t \in \mathcal{D}(\mathcal{L})$. Since $\mathcal{L}_1 G(t, s) = \delta(t - s)$, we have
\[
\mathcal{L}_1 \mathcal{G}(t, s) = \delta(t - s) - \int_0^1 \delta(t - y) \, dy = \delta(t - s) - 1.
\]
Finally, the last boundary condition follows from
\[
(\mathcal{L} \mathcal{G}_t)(0, s) - (\mathcal{L} \mathcal{G}_t)(1, s) = \int_0^1 \mathcal{L}_1 \mathcal{G}(t, s) \, dt = \int_0^1 (\delta(t - s) - 1) \, dt = 0,
\]
and the second statement is also proved. \hfill \Box

Now we define the sequence of integrated centered analogues of Brownian bridge. We set
\[
B_{(0)}(t) = B(t); \quad B_{(l)}(t) = \int_0^t \overline{B_{(l-1)}}(s) \, ds, \quad l \in \mathbb{N}.
\]

Theorem 3.1 allows us to write down the BVPs generated by processes $B_{(l)}$ and $\overline{B_{(l)}}$. We begin from the second process because its eigenvalues can be derived explicitly. This permits us to derive the small ball asymptotics without using Theorem 1.2.

**Theorem 3.2.** Let $l \in \mathbb{N}_0$. Then, as $\varepsilon \to 0$,
\[
\mathbb{P}\{|\overline{B_{(l)}}| \leq \varepsilon\} \sim \sqrt{2l + 2} \cdot \frac{\varepsilon^{-(2l+1)}_{l+1}}{\sqrt{\pi \mathcal{D}_{l+1}}} \exp \left(-\frac{\mathcal{D}_{l+1}}{2\varepsilon^{l+1}_{l+1}}\right), \tag{3.8}
\]
where $\varepsilon_{l} = \left(\varepsilon \sqrt{2l \sin \frac{\pi}{2l}}\right)^{2l+1}$ while the quantity $\mathcal{D}_{l}$ was defined in (2.4).

**Remark 7.** The multiplier before the exponent in (3.8) equals $\sqrt{\frac{2l+2}{2l+1}} \cdot \frac{\varepsilon}{\sqrt{\pi}}$. For $l = 0$ (3.8) coincides with formula obtained in [13, §3].

**Proof.** Applying $l$ times in turns the first and the second statements of Theorem 3.1 to the problem (3.1) we deduce that the covariance $\mathcal{G}_{\overline{B_{(l)}}}(t, s)$ is the generalized Green function of the BVP with periodic boundary conditions
\[
\begin{align*}
\mathcal{L}_{\overline{B_{(l)}}} u &\equiv (-1)^{l+1} u^{(2l+2)} = \mu u \quad \text{on} \quad [0, 1], \quad u^{(j)}(0) - u^{(j)}(1) = 0, \quad j = 0, 1, \ldots, 2l + 1.
\end{align*}
\]

Whence the operator $\mathcal{L}_{\overline{B_{(l)}}}$ coincides with $(\mathcal{L}_{\overline{B_{(l)}}})^{l+1}$. Therefore, its spectrum is double, excluding zero eigenvalue which is inessential for us due to the orthogonality condition (0.1): $\mu'_{n} = \mu''_{n} = (2\pi n)^{2l+2}$. Thus, we have distributional equality
\[
|\overline{B_{(l)}}| \overset{d}{=} \sum_{n=1}^{\infty} \frac{\eta^{2}_{n}}{(2\pi n)^{2l+2}} + \sum_{n=1}^{\infty} \frac{\eta^{''2}_{n}}{(2\pi n)^{2l+2}},
\]
where $\eta'_{n}$ and $\eta''_{n}$ are two independent sequences of independent standard Gaussian r.v.’s. Using [5, Theorem 6.2] and Lemma 0.1 we arrive at (3.8). \hfill \Box

**Theorem 3.3.** Let $l \in \mathbb{N}_0$. Then, as $\varepsilon \to 0$,
\[
\mathbb{P}\{|B_{(l)}| \leq \varepsilon\} \sim (2l + 2) \sqrt{\sin \frac{\pi}{2l+2}} \, \frac{\varepsilon^{2l}_{l+1}}{\sqrt{\pi \mathcal{D}_{l+1}}} \exp \left(-\frac{\mathcal{D}_{l+1}}{2\varepsilon^{l+1}_{l+1}}\right), \tag{3.9}
\]
with the same notation as in Theorem 3.2.
Proof. Similarly to Theorem 3.2, the covariance $G_{B(t)}(t, s)$ is the Green function of the BVP

$$
\begin{align*}
\mathcal{L}_{B(t)} u &\equiv (-1)^{l+1} u^{(2l+2)} = \mu u \quad \text{on} \quad [0, 1], \\
u(0) = u(1) = 0, \quad u^{(j)}(0) - u^{(j)}(1) = 0, \quad j = 1, \ldots, 2l.
\end{align*}
(3.10)
$$

Since the problem (3.10) satisfies all assumptions of Theorem 1.2 (with $\ell = l + 1$), to prove (3.9) we only need to calculate the distortion constant. Note that $\partial \ell = 1$ and $\kappa = (2l + 1)l$.

Put $\zeta = \mu^{1/(2l+2)}$. Substituting the general solution of the equation (3.10) into boundary conditions, we deduce that

$$
C_{\text{dist}}(B_{(t)}) = \prod_{n=1}^{\infty} \left( \frac{r_n}{\pi (n + \frac{1}{2l+2})} \right)^{l+1},
$$

where $r_1 < r_2 < \ldots$ are positive roots of the entire function

$$
\mathfrak{f}(\zeta) \equiv \det \begin{bmatrix}
1 & e^{i\zeta} & e^{i\zeta} & \cdots & 1 & e^{i\zeta} & e^{i\zeta} & \cdots & 1 \\
1 - e^{i\zeta} & \omega_1 (1 - e^{i\zeta}) & \cdots & \cdots & (1)(1 - e^{i\zeta}) & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_1^2 (1 - e^{i\zeta}) & \cdots & \cdots & (1)^2(1 - e^{i\zeta}) & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots & \ddots & \cdots \\
1 - e^{i\zeta} & \omega_1^{2l} (1 - e^{i\zeta}) & \cdots & \cdots & (1)^{2l}(1 - e^{i\zeta}) & \cdots & \cdots
\end{bmatrix},
$$

while $\omega_j = z_{j+1}^i$.

Subtracting the first row from the second one, similarly to Theorem 2.2 we obtain for $|\zeta| \to \infty$ and $|\arg(\zeta)| \leq \frac{\pi}{2l+2}$

$$
\mathfrak{f}(\zeta) = (-1)^{l+1} \exp(-i\omega_1 \zeta) \exp(-i\omega_2 \zeta) \ldots \exp(-i\omega_l \zeta) \cdot \left( \Phi(\zeta) + O(|\zeta|^{-1}) \right),
$$

where

$$
\Phi(\zeta) = \det \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\
1 - e^{i\zeta} & 1 & \cdots & 1 & 1 - e^{-i\zeta} & 1 & \cdots & 1 \\
1 - e^{i\zeta} & \omega_1 & \cdots & \omega_l & (-1)(1 - e^{-i\zeta}) & -\omega_1 & \cdots & -\omega_l \\
\vdots & \vdots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_1^{2l} & \cdots & \omega_l^{2l} & (-1)^{2l}(1 - e^{-i\zeta}) & (-\omega_1)^{2l} & \cdots & (-\omega_l)^{2l}
\end{bmatrix}.
$$

Expanding this determinant in the elements of the first row we derive

$$
|\Phi(\zeta)| = \frac{2|\mathfrak{g}(1, \omega_1, \ldots, \omega_2l)|}{|1 - \omega_1|} \cdot |(1 - \exp(i\zeta))(\exp(-i\zeta) - \omega_1)| = \frac{2(2l + 2)^l}{|1 - \omega_1|} \cdot |\exp(-i\zeta) + \omega_1 \exp(i\zeta) - (1 + \omega_1)|.
$$

Setting $\delta = \frac{l}{2l+2}$ we obtain in view of $|\mathfrak{f}(\zeta)| \equiv |\mathfrak{f}(\omega_1 \zeta)|$

$$
\frac{|\mathfrak{f}(\zeta)|}{|\zeta^{2l+1} \prod_{j=0}^{l} \psi_\delta(\omega_j \zeta)|} \to \frac{2^{l+2}(2l + 2)^l}{\Gamma^{2l+2}(1 + \delta) \pi^l |1 - \omega_1|^l},
$$

for $\zeta \to \infty$, $\arg(\zeta) \neq \frac{\pi j}{2l+2}, j \in \mathbb{Z}$. 
This implies, similarly to the proof of Theorem 2.2,

\[
C_{\text{dist}}^2(B_{\{t\}}) = \frac{\Gamma(2l+1)\pi^l}{2^l(2l+2)^l} \frac{\|\mathbf{F}(\zeta)\|_{\zeta=0}}{\mathfrak{F}(\zeta)}.
\]

Since

\[
-\frac{\mathfrak{F}(\zeta)}{\zeta^{2l+1}} = \det \begin{bmatrix}
\frac{1}{\zeta} & \frac{1}{\zeta} & \ldots & \frac{1}{\zeta} & \frac{1}{\zeta} & \ldots & \frac{1}{\zeta} \\
\frac{1}{1-e^{i\zeta}} & \frac{1}{\zeta} & \ldots & \frac{1}{1-e^{i\zeta}} & \frac{1}{\zeta} & \ldots & \frac{1}{1-e^{i\zeta}} \\
\frac{1}{1-e^{i\zeta}} & \frac{1}{\zeta} & \ldots & \frac{1}{1-e^{i\zeta}} & \frac{1}{\zeta} & \ldots & \frac{1}{1-e^{i\zeta}} \\
\frac{1}{1-e^{i\zeta}} & \frac{1}{\zeta} & \ldots & \frac{1}{1-e^{i\zeta}} & \frac{1}{\zeta} & \ldots & \frac{1}{1-e^{i\zeta}} \\
\end{bmatrix},
\]

we have

\[
\left| \frac{\mathfrak{F}(\zeta)}{\zeta^{2l+1}} \right|_{\zeta=0} = \left| \mathfrak{A}(1, \omega_1, \ldots, \omega_{2l+1}) \right| = (2l+2)^{l+1}.
\]

Since \(|1-\omega_1| = 2\sin \frac{\pi}{2l+2}\), this gives (3.9) after some simplification. \(\square\)

**Remark 8.** For \(l = 0\) (3.9) gives the classical formula for small ball asymptotics of Brownian bridge under \(L_2\)-norm. For \(l = 1\) the formula (3.9) was given in [13, §6], but the distortion constant there was calculated only numerically.

Now we consider \(m\)-times integrated process \((B_{\{t\}})^{[\beta_1, \ldots, \beta_m]}(t)\). Due to [5, Theorem 2.1], its covariance is the Green function of the BVP

\[
\begin{align*}
\mathcal{L}_{(B_{\{t\}}^{[m]})} u &\equiv (-1)^l + m^2 u(2l+2) = \mu u \quad \text{on} \quad [0, 1], \\
u(\beta_m) = u'(\beta_{m-1}) = \ldots = u'(m-1)(\beta_1) = 0, \\
u^{(m)}(0) = u^{(m)}(1) = 0, \\
u^{(m+j)}(0) - u^{(m+j)}(1) = 0, \quad j = 1, \ldots, 2l, \\
u^{(m+2l+2)}(1 - \beta_1) = u^{(m+2l+3)}(1 - \beta_2) = \ldots = u^{(2m+2l+3)}(1 - \beta_m) = 0.
\end{align*}
\]

The problem (3.11) satisfies all assumptions of Theorem 1.2 (with \(\ell = m + l + 1\)). This gives us the small ball asymptotics for the processes \((B_{\{t\}})^{[\beta_1, \ldots, \beta_m]}(t)\) up to a constant (note that \(\eta = (2m + 2l + 1)(m + l + 1) - (2l + 1)\)). As for the distortion constant, the only problem for its calculation is the length of explicit representation of the corresponding Fredholm determinant. We restrict ourselves to the case \(l = 1\).

**Theorem 3.4.** Let \(m \in \mathbb{N}\). Then, as \(\varepsilon \to 0\),

\[
P\{\|B_{\{t\}}^{[\beta_1, \ldots, \beta_m]}\| \leq \varepsilon\} \sim \frac{(2m + 4)^{\frac{m+2}{2}}}{2} \frac{2\sin \frac{3\pi}{2m+4}}{\mathfrak{M}(\zeta_{m+2}, \zeta_{m+2}, \ldots, \zeta_{m+2})} \cdot \sqrt{\prod_{j=1}^{m} |1 + \zeta_{m+2}|} \cdot \sqrt{\prod_{j=1}^{m} |1 + \zeta_{m+2}|} \cdot \exp \left( - \frac{\mathfrak{M}_{m+2}}{2\varepsilon_{m+2}} \right).
\]

where \(\mathfrak{M}_{m+2}\) is defined in (2.4), \(\varepsilon_{m+2}\) is introduced in Theorem 3.2, and for \(j = 1, \ldots, m\)

\[
k_j = \begin{cases} 
\frac{m-j}{2}, & \text{if } \beta_j = 0, \\
\frac{m+3-j}{2}, & \text{if } \beta_j = 1,
\end{cases}
\]

\[
k_j' = 2m + 3 - k_j.
\]
Proof. Put \( \zeta = \mu^{\frac{1}{2m+4}} \). Substituting the general solution of the equation (3.11) into boundary conditions, we deduce that

\[
C_{\text{dist}}((B_{11})_m) = \prod_{n=1}^{\infty} \left( \frac{r_n}{\pi(n - \frac{m-1}{2m+1})} \right)^{m+2},
\]

where \( r_1 < r_2 < \ldots \) are positive roots of the entire function

\[
\tilde{\mathcal{F}}_1(\zeta) \equiv \det \begin{bmatrix}
1 & \omega_1^{k_1} & \ldots & \omega_1^{k_{m+1}} & (-1)^{k_1} & \ldots & (-\omega_{m+1})^{k_1} \\
1 & \omega_1^{k_m} & \ldots & \omega_1^{k_{m+1}} & (-1)^{k_m} & \ldots & (-\omega_{m+1})^{k_m} \\
1 & \omega_1^{m} & \ldots & \omega_1^{m+1} & (-1)^{m} & \ldots & (-\omega_{m+1})^{m} \\
1 - e^{i\zeta} \omega_1^{m+1}(1 - e^{i\omega_1 \zeta}) & \ldots & \ldots & (1 - e^{i\omega_1 \zeta})^{m+1} & \ldots & \ldots \\
e^{i\zeta} \omega_1^{m} e^{i\omega_1 \zeta} & \ldots & \ldots & \omega_1^{m+1} e^{i\omega_m^{m+1} \zeta} & (1) & \ldots & \ldots \\
e^{i\zeta} \omega_1^{k_1} e^{i\omega_1 \zeta} & \ldots & \ldots & \omega_1^{m+1} e^{i\omega_m^{m+1} \zeta} & (1) & \ldots & \ldots \\
e^{i\zeta} \omega_1^{k_m} e^{i\omega_1 \zeta} & \ldots & \ldots & \omega_1^{m+1} e^{i\omega_m^{m+1} \zeta} & (1) & \ldots & \ldots \\
\end{bmatrix},
\]

while \( \omega_j = \nu_{j+2}^m \).

Similarly to Theorem 2.2 we obtain for \(|\zeta| \to \infty\) and \(|\arg(\zeta)| \leq \frac{\pi}{2m+4}\)

\[
\tilde{\mathcal{F}}_1(\zeta) = \exp(-i\omega_1 \zeta) \exp(-i\omega_2 \zeta) \ldots \exp(-i\omega_{m+1} \zeta) \cdot \left( \Phi(\zeta) + O(|\zeta|^{-1}) \right),
\]

where

\[
\Phi(\zeta) = \det \begin{bmatrix}
1 & \omega_1^{k_1} & \ldots & \omega_1^{k_{m+1}} & (-1)^{k_1} & 0 & \ldots & 0 \\
1 & \omega_1^{k_m} & \ldots & \omega_1^{k_{m+1}} & (-1)^{k_m} & 0 & \ldots & 0 \\
1 & \omega_1^{m} & \ldots & \omega_1^{m+1} & (-1)^{m} & 0 & \ldots & 0 \\
1 - e^{i\zeta} \omega_1^{m+1}(1 - e^{i\omega_1 \zeta}) & \ldots & \ldots & (1 - e^{i\omega_1 \zeta})^{m+1} & \ldots & \ldots \\
- e^{i\zeta} 0 & \ldots & \ldots & (1) & \ldots & \ldots \\
- e^{i\zeta} 0 & \ldots & \ldots & (1) & \ldots & \ldots \\
- e^{i\zeta} 0 & \ldots & \ldots & (1) & \ldots & \ldots \\
\end{bmatrix},
\]

Expanding this determinant in the elements of the first and \((m + 3)\)-rd columns we derive

\[
|\Phi(\zeta)| = \mathcal{M} \cdot |\exp(-i\zeta) + \omega_1^{-3} \exp(i\zeta) + R|,
\]

where

\[
\mathcal{M} = |\mathcal{W}(\omega_1^{k_1}, \ldots, \omega_1^{k_m}, \omega_1^{m}, \omega_1^{m+1}) \cdot \mathcal{W}(\omega_1^{m+2}, \omega_1^{m}, \omega_1^{m+1} + k_1, \ldots, \omega_1^{k_m}) + \\
+ \mathcal{W}(\omega_1^{k_1}, \ldots, \omega_1^{k_m}, \omega_1^{m}, \omega_1^{m+2}) \cdot \mathcal{W}(\omega_1^{m+1}, \omega_1^{m}, \omega_1^{m+2}, \ldots, \omega_1^{k_m})|,
\]

while \( R \) is a constant which is inessential for us.
Setting $\delta = -\frac{m-1}{2m+4}$ we obtain in view of $|\mathfrak{F}_1(\zeta)| \equiv |\mathfrak{F}_1(\omega_1 \zeta)|$

$$\frac{|\mathfrak{F}_1(\zeta)|}{\zeta^3 \prod_{j=0}^{m+1} \psi_3(\omega_j \zeta)} \to \frac{2^{m+2} \pi^{m-1} \mathfrak{M}}{\Gamma^{2m+4}(1+\delta)}.$$ 

for $\zeta \to \infty$, $\arg(\zeta) \neq \frac{\pi j}{2m+4}$, $j \in \mathbb{Z}$.

This implies, similarly to the proof of Theorem 2.2,

$$C^2_{\text{dist}}((B_{\{1\}})m) = \frac{\Gamma^{2m+4}(1+\delta)}{2^{m+2} \pi^{m-1} \mathfrak{M}} \left| \frac{\mathfrak{F}_1(\zeta)}{\zeta^3} \right|_{\zeta=0}.$$ 

Subtracting in the determinant $(m+1)$-st row from $(m+4)$-th one we obtain in view of (3.13)

$$\left| \frac{\mathfrak{F}_1(\zeta)}{\zeta^3} \right|_{\zeta=0} = |\mathfrak{W}(1, \omega_1, \omega_1^2, \ldots, \omega_1^{2m+3})| = (2m+4)^{m+2}.$$ 

It remains to take into account that due to (3.13)

$$\mathfrak{M} = |\mathfrak{W}(\omega_1^{k_1}, \ldots, \omega_1^{k_m})|^2 \cdot \frac{2m+4}{|1-\omega_3|} \cdot \left( \prod_{j=1}^{m} |1+\omega_1^{k_j}|^2 + \prod_{j=1}^{m} |1+\omega_1^{k_j}|^2 \right).$$

Since $|1-\omega_3| = 2 \sin \frac{3\pi}{2m+4}$, we arrive at (3.12) after some simplification.

**4 Integrated centered Wiener process and related processes**

Similarly to Section 3, we define the sequence of integrated centered analogues of Wiener process. We set

$$W_{\{0\}}(t) = W(t); \quad W_{\{l\}}(t) = \int_0^t W_{\{l-1\}}(s) ds, \quad l \in \mathbb{N}.$$ 

The spectrum of the process $W_{\{1\}}$ and its $L_2$-small ball asymptotics was studied in [13, §7]. In [5, Example 5.4] it is pointed out that the covariance $G_{W_{\{1\}}}$ is the Green function of the BVP

$$L_{W_{\{1\}}} u \equiv u^{IV} = \mu u \quad \text{on} \quad [0, 1], \quad u(0) = u(1) = u''(0) = u''(1) = 0. \quad (4.1)$$

Theorem 3.1 allows us to write down the BVPs generated by processes $W_{\{l\}}$ and $\overline{W_{\{l\}}}$. We begin from the second process and prove an unexpected relation.

**Theorem 4.1.** Let $l \in \mathbb{N}_0$. Then the following distributional equality holds true:

$$\|\overline{W_{\{l\}}}\| \overset{d}{=} \|B_{\{l\}}\|. \quad (4.2)$$

**Proof.** For $l = 0$ the equality (4.2) is well known, see, e.g., [21] and [13, §3]. Let $l \geq 1$. Applying in turns the second and the first statements of Theorem 3.1 to the problem (4.1) we deduce that the covariance $G_{\overline{W_{\{l\}}}}(t, s)$ is the Green function of the BVP

$$\left\{ \begin{array}{l}
L_{\overline{W_{\{l\}}}} u \equiv (-1)^{l+1} u^{(2l+2)} = \mu u \quad \text{on} \quad [0, 1], \\
u^{(l+1)}(0) = u^{(l+1)}(1) = 0, \quad u^{(j)}(0) - u^{(j)}(1) = 0, \quad j = 0, \ldots, l-1, l+2, \ldots, 2l+1.
\end{array} \right. \quad (4.3)$$
It is easy to check that \((l+1)\)-times differentiation maps mutually the eigenfunctions of BVPs (3.10) and (4.3) (if the corresponding eigenvalue \(\mu \neq 0\)). Hence nonzero eigenvalues of these BVPs coincide pairwise, and therefore nonzero eigenvalues of the covariances also coincide. This gives (4.2).

**Theorem 4.2.** Let \(l \in \mathbb{N}\). Then, as \(\varepsilon \to 0\),

\[
P\{\|W_{(l)}\| \leq \varepsilon\} \sim (2l + 2)^{\frac{3}{4}} \sin \frac{\pi}{2l+2} \langle \cos \frac{\pi}{2l+2} \rangle \cdot \frac{\varepsilon^{-(2l+1)}}{\sqrt{\pi}D_{l+1}} \exp \left(-\frac{D_{l+1}}{2\varepsilon^{2l+1}}\right) \tag{4.4}
\]

with the same notation as in Theorem 3.2 (the angle brackets must be omitted if \(l\) is even).

**Remark 9.** For \(l = 1\) (4.4) coincides with formula obtained in [13, §6], see also [6, Proposition 1.7].

**Proof.** Similarly to Theorem 4.1, the covariance \(G_{W_{(l)}}(t, s)\) is the Green function of the BVP

\[
\begin{aligned}
\mathcal{L}_{W_{(l)}}u &\equiv (-1)^{l+1}u^{(2l+2)} = \mu u \quad \text{on} \quad [0, 1], \\
u(0) = u(1) = 0, \quad u^{(l+1)}(0) = u^{(l+1)}(1) = 0, \\
u^{(j)}(0) - u^{(j)}(1) = 0, \quad j = 1, \ldots, l-1, l+2, \ldots, 2l.
\end{aligned} \tag{4.5}
\]

Since the problem (4.5) satisfies all assumptions of Theorem 1.2 (with \(\ell = l+1\)), to prove (4.4) we only need to calculate the distortion constant. Note that \(\vartheta_\ell = 1\) and \(\varkappa = (2l + 1)l + 1\).

Put \(\zeta = \mu^{\frac{1}{2l+2}}\). Substituting the general solution of the equation (4.5) into boundary conditions, we deduce that

\[
C_{\text{dist}}(W_{(l)}) = \prod_{n=1}^{\infty} \left(\frac{r_n}{\pi(n + \frac{(l+1)}{2l+2})}\right)^{l+1},
\]

where \(r_1 < r_2 < \ldots\) are positive roots of the entire function

\[
\Phi(\zeta) \equiv \det \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_1(1 - e^{i\omega_1\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) \\
1 - e^{i\zeta} & \omega_2(1 - e^{i\omega_2\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) \\
1 - e^{i\zeta} & \omega_3(1 - e^{i\omega_3\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_{l-1}(1 - e^{i\omega_{l-1}\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) & \cdots & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_{l}^{l-1}(1 - e^{i\omega_{l}^{l-1}\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) & \cdots & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_{l+1}(1 - e^{i\omega_{l+1}\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) & \cdots & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_{l+2}(1 - e^{i\omega_{l+2}\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 - e^{i\zeta} & \omega_{2l+2}(1 - e^{i\omega_{2l+2}\zeta}) & \cdots & (1)^{l+1}(1 - e^{-i\zeta}) & \cdots & \cdots & \cdots \\
\end{bmatrix},
\]

while \(\omega_j = \zeta_{l+1}^{j}\).

Subtracting the first row from the second one and \((l+2)\)-nd row from \((l+3)\)-rd one, similarly to Theorem 2.2 we obtain for \(|\zeta| \to \infty\) and \(|\arg(\zeta)| \leq \frac{\pi}{2l+2}\)

\[
\Phi(\zeta) = \exp(-i\omega_1\zeta) \exp(-i\omega_2\zeta) \ldots \exp(-i\omega_l\zeta) \cdot \left(\Phi(\zeta) + O(|\zeta|^{-1})\right),
\]
where

\[ \Phi(\zeta) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 - e^{i\zeta} & 1 & \cdots & 1 & 1 - e^{-i\zeta} & -1 & \cdots & -1 \\ 1 - e^{i\zeta} & \omega_1 & \cdots & \omega_l & (-1)(1 - e^{-i\zeta}) & \omega_1 & \cdots & \omega_l \\ 1 - e^{i\zeta} & \omega_1^{-1} & \cdots & \omega_l^{-1} & (1 - e^{-i\zeta}) & (-\omega_1)^{-1} & \cdots & (-\omega_l)^{-1} \\ -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 - e^{i\zeta} & -1 & \cdots & (-1)^l & (1 - e^{-i\zeta}) & (-1)^l & \cdots & 1 \\ 1 - e^{i\zeta} & -\omega_1 & \cdots & (-1)^l & (1 - e^{-i\zeta}) & (1)^l + 2(1 - e^{-i\zeta}) & \cdots & -\omega_l \\ 1 - e^{i\zeta} & -\omega_1^{-1} & \cdots & (-1)^l & (1 - e^{-i\zeta}) & (1)^l + 3(1 - \omega_1) & \cdots & (-\omega_l)^{-1} \end{bmatrix} \]

Adding the upper half of the matrix to the lower one we get \(|\Phi(\zeta)| = 2^{l+1}\Delta_1(\zeta) \cdot |\Delta_2(\zeta)|\), where

for even \(l\)

\[ \Delta_1(\zeta) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 - e^{i\zeta} & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 - e^{i\zeta} & \omega_2 & \cdots & \omega_l & \omega_{l+2} & \cdots & \omega_{2l} \\ 1 - e^{i\zeta} & \omega_2^{-1} & \cdots & \omega_l^{-1} & \omega_{l+2}^{-1} & \cdots & \omega_{2l}^{-1} \end{bmatrix} \]

\[ \Delta_2(\zeta) = \det \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 - e^{-i\zeta} & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 - e^{-i\zeta} & \omega_2 & \cdots & \omega_l & \omega_{l+2} & \cdots & \omega_{2l} \\ 1 - e^{-i\zeta} & \omega_2^{-1} & \cdots & \omega_l^{-1} & \omega_{l+2}^{-1} & \cdots & \omega_{2l}^{-1} \end{bmatrix} \]

for odd \(l\)

\[ \Delta_1(\zeta) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 - e^{i\zeta} & 1 & \cdots & 1 & 1 - e^{-i\zeta} & 1 & \cdots & 1 \\ 1 - e^{i\zeta} & \omega_2 & \cdots & \omega_{l-1} & (-1)(1 - e^{-i\zeta}) & -\omega_2 & \cdots & -\omega_{l-1} \\ 1 - e^{i\zeta} & \omega_2^{-1} & \cdots & \omega_{l-1}^{-1} & (1 - e^{-i\zeta}) & (-\omega_2)^{-1} & \cdots & (-\omega_{l-1})^{-1} \end{bmatrix} \]

\[ \Delta_2(\zeta) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \omega_1 & \omega_3 & \cdots & \omega_l & -\omega_1 & \cdots & -\omega_l \\ \omega_1^{-1} & \omega_3^{-1} & \cdots & \omega_l^{-1} & (-\omega_1)^{-1} & \cdots & (-\omega_l)^{-1} \end{bmatrix} \]

(note that in this case \(\Delta_1\) coincides with the determinant from Theorem 3.3).

Expanding these determinants in the elements of the first row we obtain:

for even \(l\)

\[ |\Phi(\zeta)| = \frac{2^{l+1} |\Theta(1, \omega_2, \ldots, \omega_{2l-2})|^2}{|1 - \omega_1|^2} \cdot |1 - \omega_1 \exp(i\zeta)|^2 = \]

\[ = \frac{4(2l + 2)^{l-1}}{|1 - \omega_1|^2} \cdot |\exp(-i\zeta) + \omega_1^2 \exp(i\zeta) - 2\omega_1|; \]
for odd \( l \)
\[
|\Phi(\zeta)| = \frac{2^{l+3}|\mathfrak{B}(1, \omega_2, \ldots, \omega_{2l-2})|^2}{|1 - \omega_2|^2} \cdot |(1 - \exp(i\zeta))(\exp(-i\zeta) - \omega_1^2)| = \\
= \frac{16(2l + 2)^{l-1}}{|1 - \omega_2|^2} \cdot |\exp(-i\zeta) + \omega_1^2 \exp(i\zeta) - (1 + \omega_1^2)|.
\]

Setting \( \delta = \frac{l-1}{2l+2} \), we obtain in view of \( |\Phi(\zeta)| \equiv |\Phi(\omega_1\zeta)| \)
\[
\frac{|\Phi(\zeta)|}{|\zeta^{l} \prod_{j=0}^{l-1} \psi_{j}(\omega_{j}\zeta)|} \rightarrow \frac{2^{l+3}(2l + 2)^{l-1}}{\Gamma^{2l+2}(1 + \delta)\pi^{l-1} \mathbb{M}}.
\]

for \( \zeta \to \infty \), \( \arg(\zeta) \neq \frac{\pi j}{2l+2}, j \in \mathbb{Z} \) (here \( \mathbb{M} = |1 - \omega_1|^2 \) for even \( l \) and \( \mathbb{M} = |1 - \omega_2|^2/4 \) for odd \( l \)).
This implies, similarly to the proof of Theorem 2.2,
\[
C^{2}_{\text{dist}}(W_{(l)}) = \frac{\Gamma^{2l+2}(1 + \delta)\pi^{l-1} \mathbb{M}}{2^{l+3}(2l + 2)^{l-1}} \cdot \left| \frac{\Phi(\zeta)}{\zeta^{2l}} \right|_{\zeta=0}.
\]

Since
\[
\frac{\Phi(\zeta)}{\zeta^{2l}} = \det
\begin{bmatrix}
1 & 1 & \ldots & 1 & 1 \\
\frac{1 - e^{i\zeta}}{\zeta} & \frac{1 - e^{i\omega_1 \zeta}}{\zeta} & \ldots & \frac{1 - e^{i\omega_l \zeta}}{\zeta} & \frac{1 - e^{-i\zeta}}{\zeta} \\
\frac{1 - e^{-i\zeta}}{\zeta} & \frac{1 - e^{-i\omega_1 \zeta}}{\zeta} & \ldots & \frac{1 - e^{-i\omega_l \zeta}}{\zeta} & \frac{1 - e^{i\zeta}}{\zeta} \\
\frac{1 - e^{i\zeta}}{\zeta} & \frac{1 - e^{i\omega_1 \zeta}}{\zeta} & \ldots & \frac{1 - e^{i\omega_l \zeta}}{\zeta} & \frac{1 - e^{-i\zeta}}{\zeta} \\
\frac{1 - e^{-i\zeta}}{\zeta} & \frac{1 - e^{-i\omega_1 \zeta}}{\zeta} & \ldots & \frac{1 - e^{-i\omega_l \zeta}}{\zeta} & \frac{1 - e^{i\zeta}}{\zeta} \\
\frac{1 - e^{i\zeta}}{\zeta} & \frac{1 - e^{i\omega_1 \zeta}}{\zeta} & \ldots & \frac{1 - e^{i\omega_l \zeta}}{\zeta} & \frac{1 - e^{-i\zeta}}{\zeta} \\
\frac{1 - e^{-i\zeta}}{\zeta} & \frac{1 - e^{-i\omega_1 \zeta}}{\zeta} & \ldots & \frac{1 - e^{-i\omega_l \zeta}}{\zeta} & \frac{1 - e^{i\zeta}}{\zeta}
\end{bmatrix},
\]

we have
\[
\left| \frac{\Phi(\zeta)}{\zeta^{2l}} \right|_{\zeta=0} = |\mathfrak{B}(1, \omega_1, \ldots, \omega_{2l+1})| = (2l + 2)^{l+1},
\]

Since \( |1 - \omega_j| = 2 \sin \frac{j\pi}{2l+2} \) this gives (4.4) after some simplification. \( \Box \)

**Remark 9.** In our proof we in fact use that the operator of the BVP (4.5) is the square of the BVP operator of order \( l + 1 \)
\[
\left\{ \begin{array}{l}
\mathcal{L}u \equiv i^{l+1}u^{(l+1)} = \mu u \quad \text{on } [0,1], \\
u(0) = u(1) = 0, \quad u^{(j)}(0) - u^{(j)}(1) = 0, \quad j = 1, \ldots, l - 1.
\end{array} \right.
\]

For odd \( l \) the operator \( \mathcal{L} \) coincides with the operator \( \mathcal{L}^{\beta_1, \ldots, \beta_m}_{B_{(l)}} \), see (3.10).

Now we consider \( m \)-times integrated process \( (W_{(l)})^{\beta_1, \ldots, \beta_m}_{m}(t) \). According to [5, Theorem 2.1], its covariance is the Green function of the BVP

\[
\left\{ \begin{array}{l}
\mathcal{L}(W_{(l)})_{m} u \equiv (-1)^{l+m+1}u^{(2m+2l+2)} = \mu u \quad \text{on } [0,1], \\
u(\beta_m) = u'(\beta_{m-1}) = \cdots = u^{(m-1)}(\beta_1) = 0, \\
u^{(m)}(0) = u^{(m)}(1) = 0, \quad u^{(m+j)}(0) - u^{(m+j)}(1) = 0, \quad j = 1, \ldots, l - 1, \\
u^{(m+l+1)}(0) = u^{(m+l+1)}(1) = 0, \quad u^{(m+j)}(0) - u^{(m+j)}(1) = 0, \quad j = l + 2, \ldots, 2l, \\
u^{(m+2l+2)}(1 - \beta_1) = u^{(m+2l+3)}(1 - \beta_2) = \cdots = u^{(2m+2l+1)}(1 - \beta_m) = 0.
\end{array} \right.
\]
The problem (4.6) satisfies all assumptions of Theorem 1.2 (with \( \ell = m + l + 1 \)). This gives us the small ball asymptotics for the processes \((W(t))_{m}^{(n)}(t)\) up to constant (note that \( \vartheta_\ell = 1 \) and \( \varkappa = (2m + 2l + 1)(m + l + 1) - 2l \)). For \( l = 1 \) the problem (4.6) has separated boundary conditions; distortion constants in this case were calculated in [6, Proposition 1.7]. We restrict ourselves to the case \( l = 2 \).

**Theorem 4.3.** Let \( m \in \mathbb{N} \). Then, as \( \varepsilon \to 0 \),

\[
\mathbb{P}\{(\| W_{(2)}^{[\beta_1, ..., \beta_m]} \| \leq \varepsilon) \sim \frac{4(2m + 6)^{\frac{m+2}{2}} \sin \frac{\pi}{m+3} \sqrt{\sin \frac{\pi}{2m+6} \sin \frac{5\pi}{2m+6}}}{|\mathcal{D}(z_{m+3}, z_{m+3}, ..., z_{m+3})| \cdot \sqrt{\prod_{j=1}^{m} |z_{m+3} + z_{m+3}|^2 + \prod_{j=1}^{m} |z_{m+3} + z_{m+3}|^2}} \cdot \frac{\varepsilon^{-3}}{\sqrt{\pi \mathcal{D}_{m+3}}} \exp \left(- \frac{\mathcal{D}_{m+3}}{2\varepsilon_{m+3}} \right), \tag{4.7}
\]

where \( \mathcal{D}_{m} \) is defined in (2.4), \( \mathcal{D}_{ \ell } \) is introduced in Theorem 3.2, and for \( j = 1, \ldots, m \)

\[
k_j = \begin{cases} \ m - j, & \text{if } \beta_j = 0, \\ \ m + 5 + j, & \text{if } \beta_j = 1, \\ \ k_j = 2m + 5 - k_j. \end{cases} \tag{4.8}
\]

**Proof.** Put \( \zeta = \mu^{\frac{1}{m+6}} \). Substituting the general solution of the equation (4.6) into boundary conditions, we deduce that

\[
C_{\text{dist}}(W_{(2)}^m) = \prod_{n=1}^{\infty} \left( \frac{r_n}{\pi(n - \frac{m+1}{2m+6})} \right)^{m+3},
\]

where \( r_1 < r_2 < \ldots \) are positive roots of the entire function

\[
\mathbb{F}_1(\zeta) \equiv \begin{vmatrix} \ 1 & \omega_{m+3} & \ldots & \omega_{m+2} & (-1)^{k_1} & \ldots & (-\omega_{m+2})^{k_1} \\
1 & \omega_{m+3} & \ldots & \omega_{m+2} & (-1)^{k_2} & \ldots & (-\omega_{m+2})^{k_2} \\
1 & \omega_{m+3} & \ldots & \omega_{m+2} & (-1)^{k_3} & \ldots & (-\omega_{m+2})^{k_3} \\
1 & \omega_{m+3} & \ldots & \omega_{m+2} & (-1)^{k_4} & \ldots & (-\omega_{m+2})^{k_4} \\
1 & \omega_{m+3} & \ldots & \omega_{m+2} & (-1)^{k_5} & \ldots & (-\omega_{m+2})^{k_5} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
1 & \omega_{m+3} & \ldots & \omega_{m+2} & (-1)^{k_l} & \ldots & (-\omega_{m+2})^{k_l} \end{vmatrix}.
\]

While \( \omega_j = z_{m+3}^j \).

Similarly to Theorem 2.2 we obtain for \( |\zeta| \to \infty \) and \( |\arg(\zeta)| \leq \frac{\pi}{2m+6} \)

\[
\mathbb{F}_1(\zeta) = \exp(-i\omega_1\zeta) \exp(-i\omega_2\zeta) \ldots \exp(-i\omega_{m+2}\zeta) \cdot \left( \Phi(\zeta) + O(|\zeta|^{-1}) \right),
\]
where

\[
\Phi(\zeta) = \det \begin{bmatrix}
1 & \omega_1^{k_1} & \ldots & \omega_{m+1}^{k_1} & (-1)^{k_1} & 0 & \ldots & 0 \\
1 & \omega_1^{k_m} & \ldots & \omega_{m+2}^{k_m} & (-1)^{k_m} & 0 & \ldots & 0 \\
1 & \omega_1^{m} & \ldots & \omega_{m+2}^{m} & (-1)^{m} & 0 & \ldots & 0 \\
1 & \omega_1^{m+3} & \ldots & \omega_{m+5}^{m+3} & (-1)^{m+3} & 0 & \ldots & 0 \\
1 - e^{i\zeta} & \omega_1^{m+1} & \ldots & \omega_{m+2}^{m+1} & \frac{1}{2}m+1 & (-1)^{m+1} & \ldots & \frac{1}{2}(m+2) \\
1 - e^{i\zeta} & \omega_1^{m+4} & \ldots & \omega_{m+2}^{m+4} & \frac{1}{2}m+4 & (-1)^{m+4} & \ldots & \frac{1}{2}(m+2) \\
- e^{i\zeta} & 0 & \ldots & 0 & \frac{1}{2}m+3 & (-1)^{m+3} & \ldots & \frac{1}{2}(m+2) \\
- e^{i\zeta} & 0 & \ldots & 0 & \frac{1}{2}m+1 & (-1)^{m+1} & \ldots & \frac{1}{2}(m+2) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}.
\]

Expanding this determinant in the elements of the first and \((m+4)\)-rd columns we derive

\[|\Phi(\zeta)| = M_1 \cdot |\exp(-i\zeta) + \omega_1^{-4}\exp(i\zeta) + R|,
\]

where

\[M_1 = |\mathfrak{F}(\omega_1^{k_1}, \ldots, \omega_1^{k_m}, \omega_1^{m}, \omega_1^{m+3}, \omega_1^{m+1}) \cdot \mathfrak{F}(\omega_1^{m+4}, \omega_1^{m}, \omega_1^{m+3}, \omega_1^{k_1}, \ldots, \omega_1^{k_m}) + \mathfrak{F}(\omega_1^{k_1}, \ldots, \omega_1^{k_m}, \omega_1^{m}, \omega_1^{m+3}, \omega_1^{m+4}) \cdot \mathfrak{F}(\omega_1^{m+1}, \omega_1^{m}, \omega_1^{m+3}, \omega_1^{k_1}, \ldots, \omega_1^{k_m})|,
\]

while \(R\) is a constant which is inessential for us.

Setting \(\delta = -\frac{m-1}{2m+6}\) we obtain in view of \(|F_1(\zeta)| \equiv |F_1(\omega_1 \zeta)|\)

\[
\frac{|F_1(\zeta)|}{\zeta^4 \prod_{j=0}^{m-2} \psi_j(\omega_j \zeta)} \rightarrow \frac{2^{m+3} \pi^{m-1} M_1}{\Gamma^{2m+6}(1 + \delta)},
\]

for \(\zeta \rightarrow \infty, \arg(\zeta) \neq \frac{\pi j}{2m+6}, j \in \mathbb{Z}.
\]

This implies, similarly to the proof of Theorem 2.2,

\[
C_{\text{dist}}(\{W(2)\}) = \frac{\Gamma^{2m+6}(1 + \delta)}{2^{m+3} \pi^{m-1} M_1} \left| \frac{|F_1(\zeta)|}{\zeta^4} \right|_{\zeta=0}.
\]

Let us subtract in the determinant \((m+1)\)-st row from \((m+5)\)-th one and \((m+2)\)-nd row from \((m+6)\)-th one. In view of (4.8) we obtain

\[
\left| \frac{F_1(\zeta)}{\zeta^4} \right|_{\zeta=0} = |\mathfrak{F}(1, \omega_1, \omega_1^2, \ldots, \omega_1^{2m+5})| = (2m+6)^{m+3}.
\]

It remains to take into account that due to (4.8)

\[
M_1 = |\mathfrak{F}(\omega_1^{k_1}, \ldots, \omega_1^{k_m})|^2 \cdot \frac{(2m+6)^2}{|1 - \omega_1| |1 - \omega_2| |1 - \omega_5|} \cdot \left( \prod_{j=1}^{m} |\omega_1 + \omega_1^{k_j}|^2 + \prod_{j=1}^{m} |\omega_1 + \omega_k^{k_j}|^2 \right).
\]

Since \(|1 - \omega_j| = 2 \sin \frac{j\pi}{2m+6}\) we arrive at (4.7) after some simplification.

I am grateful to Professor Ya.Yu. Nikitin for some comments and references, and also for his constant encouragement.
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