Universal Codes as a Basis for Nonparametric Testing of Serial Independence for Time Series

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Abstract—We consider a stationary and ergodic source $p$ generated symbols $x_1 \ldots x_t$ from some finite set $A$ and a null hypothesis $H_0$ that $p$ is Markovian source with memory (or connectivity) not larger than $m$, $(m \geq 0)$. The alternative hypothesis $H_1$ is that the sequence is generated by a stationary and ergodic source, which differs from the source under $H_0$. In particular, for $m = 0$ we have the null hypothesis $H_0$, that the sequence is generated by Bernoulli source (or the hypothesis that $x_1 \ldots x_t$ are independent.) Some new tests which are based on universal codes and universal predictors, are suggested.

I. INTRODUCTION

Nonparametric testing for independence of time series is very important in statistical applications. There is an extensive literature dealing with nonparametric independence testing; a quite full review can be found in [10].

In this paper, we consider a source (or process), which generates elements from a finite set $A$ and two following hypotheses: $H_0$ is that the sequence is Markovian one, which memory (or connectivity) not larger than $m$, $(m \geq 0)$, and the alternative hypothesis $H_1$ that the sequence is generated by a stationary and ergodic source, which differs from the source under $H_0$. The testing should be based on a sample $x_1 \ldots x_t$ generated by the source.

For example, the sequence $x_1 \ldots x_t$ might be a DNA-string and one can consider the question about the depth of the statistical dependence.

We suggest a family of tests that are based on so called universal predictors (or universal data compression methods). The Type I error of the suggested tests is not larger than a given $\alpha$ ($\alpha \in (0, 1)$) for any source under $H_0$, whereas the Type II error for any source under $H_1$ tends to 0, when the sample size $t$ grows.

The suggested tests are based on results and ideas of Information Theory and, especially, those of the universal coding theory. Informally, the main idea of the tests can be described as follows. Suppose that the source generates letters from an alphabet $A$ and one wants to test $H_0$ (the source is Markovian with memory $m, m \geq 0$.) First we recall that there exist universal codes which, informally speaking can “compress” any sequence generated by a stationary and ergodic source, to the length $h_\infty$ bits, where $h_\infty$ is the limit Shannon entropy of any sequence generated by a stationary and ergodic source, to the length $t$, $t$ tends to infinity. Second, it is well known in information theory that $h_\infty$ equals $m$th-order (conditional) Shannon entropy $h_m$, if $H_0$ is true, and $h_\infty$ is strictly less than $h_m$ if $H_1$ is true. So, the following test looks like natural: Compress the sample sequence $x_1 \ldots x_t$ by a universal code and compare the lengths of the obtained file with $th_m^*$, where $h_m^*$ is an estimate of $h_m$. If the length of the compressed file is significantly less than $th_m^*$, then the hypothesis $H_0$ should be rejected.

This is no surprise that the results and ideas of a universal coding theory can be applied to some classical problems of mathematical statistics. In fact, methods of universal coding (and a closely connected universal prediction) are intended to extract information from observed data in order to compress (or predict) data efficiently in a case where the source statistics is unknown. Recently such a connection between universal coding and mathematical statistics was used in [4] for estimating the order of Markov sources and for constructing efficient tests for randomness, i.e. for testing the hypothesis $H_0$ that a sequence is generated by a Bernoulli source and all letters have equal probabilities against $H_1$ that the sequence is generated by a stationary and ergodic source, which differs from the source under $H_0$, see [19].

The outline of the paper is as follows. The next part contains definitions and necessary information from the theory of universal coding and universal prediction. Part three is devoted to the testing of the above described hypotheses. All proofs are given in the appendix.

II. DEFINITIONS AND PRELIMINARIES.

Consider an alphabet $A = \{a_1, \ldots, a_n\}$ with $n \geq 2$ letters and denote by $A^t$ the set of words $x_1 \ldots x_t$ of length $t$ from $A$. Let $\mu$ be a source which generates letters from $A$. Formally, $\mu$ is a probability distribution on the set of words of infinite length or, more simply, $\mu = (\mu_t)_{t \geq 1}$ is a consistent set of probabilities over the sets $A^t; \ t \geq 1$. By $M_\infty(A)$ we denote the set of all stationary and ergodic sources, which generate letters from $A$. Let $M_k(A) \subset M_\infty(A)$ be the set of Markov sources with memory (or connectivity) $k, k \geq 0$. More precisely, by definition $\mu \in M_k(A)$ if

$$
\mu(x_{t+1} = a_{i_1} / x_t = a_{i_2}, x_{t-1} = a_{i_3}, \ldots, x_{t-k+1} = a_{i_k+1}, \ldots)$$

$$= \mu(x_{t+1} = a_{i_1} / x_t = a_{i_2}, x_{t-1} = a_{i_3}, \ldots, x_{t-k+1} = a_{i_k+1})$$

for all $t \geq k$ and $a_{i_1}, a_{i_2}, \ldots \in A$. By definition, $M_0(A)$ is the set of all Bernoulli (or i.i.d.) sources over $A$. 


2.1 Universal prediction.

Now we briefly describe results and methods of universal coding and prediction, which will be used later. Let a source generate a message \( x_1 \ldots x_{t-1} x_t \ldots \) and let \( v^t(a) \) denote the count of letter \( a \) occurring in the word \( x_1 \ldots x_{t-1} x_t \). After the first \( t \) letters \( x_1, \ldots, x_{t-1}, x_t \) have been processed the following letter \( x_{t+1} \) needs to be predicted. By definition, a predictor is a set of non-negative numbers \( \pi(a_1 | x_1 \ldots x_t), \ldots, \pi(a_n | x_1 \ldots x_t) \) which are estimates of the unknown conditional probabilities \( p(a_1 | x_1 \ldots x_t), \ldots, p(a_n | x_1 \ldots x_t) \), i.e. of the probabilities \( p(x_{t+1} = a_i | x_1 \ldots x_t) ; i = 1, \ldots, n \).

Laplace suggested the following predictor:

\[
L(a| x_1 \ldots x_t) = (v^t(a) + 1)/(t + |A|),
\]

where \(|A|\) is the number of letters in the alphabet \( A \), see [8]. For example, if \( A = \{0, 1\}, x_1, \ldots, x_5 = 01010 \), then the Laplace prediction is as follows: \( L(x_5 = 0 | 01010) = (3+1)/(5+2) = 4/7 \), \( L(x_6 = 1 | 01010) = (2+1)/(5+2) = 3/7 \).

In Information Theory the error of prediction often is estimated by the the Kullback-Leibler (K-L) divergence between a distribution \( p \) and its estimation. Consider a source \( p \) and a predictor \( \gamma \). The error is characterized by the divergence

\[
\rho_{\gamma,p}(x_1 \ldots x_t) = \sum_{a \in A} p(a|x_1 \ldots x_t) \log \frac{p(a|x_1 \ldots x_t)}{\gamma(a|x_1 \ldots x_t)}.
\]

(2)

(Here and below \( \log \equiv \log_2 \)). It is well known that for any distributions \( p \) and \( \gamma \) the K-L divergence is nonnegative and equals 0 if and only if \( p(a) = \gamma(a) \) for all \( a \), see, for ex., [9], that is why the K-L divergence is a natural estimate of the prediction error. For fixed \( t \), \( \rho_{\gamma,p} \) is a random variable, because \( x_1, x_2, \ldots, x_t \) are random variables. We define the average error at time \( t \) by

\[
\rho^t(p|\gamma) = E(\rho_{\gamma,p}(\cdot)) = \sum_{x_1 \ldots x_t \in A^t} p(x_1 \ldots x_t) \rho_{\gamma,p}(x_1 \ldots x_t).
\]

(3)

It is known that the error of Laplace predictor goes to 0 for any Bernoulli source \( p \). More precisely, it is proven that

\[
\rho^t(p|\gamma) < (|A| - 1)/(t + 1)
\]

(4)

for any source \( p \); [18], [20].

Obviously, the convergence to 0 of a predictor’s error for any source from some set \( M \) is an important property. For example, we can see from [3] that it is true for the Laplace predictor and the set of Bernoulli sources \( M_0(A) \). Unfortunately, it is known that a predictor, which error \( \rho^t \) goes to 0 for any stationary and ergodic source, does not exist. More precisely, for any predictor \( \gamma \) there exists such a stationary and ergodic source \( p \), that \( \lim_{t \to \infty} \sup \rho_{\gamma,p}(x_1 \ldots x_t) \geq \text{const} > 0 \) with probability 1; [17]. (See also [1], [13], [14], where this result is generalized and a history of its discovery is described. In particular, they found out that such a result was described by Bailey [2] in his unpublished thesis). That is why it is difficult to use \( \rho^t \) for comparison of different predictors. On the other hand, it is shown in [16], [17] that there exists such a predictor \( R \), that the following average \( t^{-1} \sum_{i=1}^{t} \rho_{\gamma,p}(x_1 \ldots x_t) \) goes to 0 (with probability 1) for any stationary and ergodic source \( p \), where \( t \) goes to infinity. That is why we will focus our attention on such averages. First, we define for any predictor \( \pi \) the following probability distribution

\[
\pi(x_1 \ldots x_t) = \prod_{i=1}^{t} \pi(x_i | x_1 \ldots x_{i-1}).
\]

(5)

For example, we obtain for the Laplace predictor \( L \) that

\[
L(01010) = 4/7, L(01011) = 3/7.
\]

Then, by analogy with (2) we will estimate the error by K-L divergence and define

\[
\rho_{\gamma,p}(x_1 \ldots x_t) = t^{-1} (\log(p(x_1 \ldots x_t)/\gamma(x_1 \ldots x_t))
\]

(4)

For example, from those definitions and (3) we obtain the following estimation for Laplace predictor \( L \) and any Bernoulli source: \( \rho_{\gamma,L}(t,p) \rightarrow 0 \) as \( t \rightarrow \infty \).

The universal predictors will play a key role in suggested below tests. By definition, a predictor \( \gamma \) is called a universal (in average) for a class of sources \( M \), if for any \( p \in M \) the error \( \rho_{\gamma,p}(x_1 \ldots x_t) \) goes to 0 not only in average, but for almost all sequences \( x_1, x_2, \ldots \). For short, we will say that the predictor (or probability distribution) \( \gamma \) is universal, if \( \lim_{t \to \infty} \rho_{\gamma,p}(x_1 \ldots x_t) = 0 \) is valid with probability 1 for any stationary and ergodic source (i.e. for any \( p \in M_0(A) \)). Now there are quite many known universal predictors. One of the first such predictors is described in [16].

2.1 Universal coding.

This short subparagraphe is intended to give some explanation about why and how methods of data compression can be used for testing of independence. The point is that the prediction problem is deeply connected with the theory of universal coding. Moreover, practically used data compression methods (or so-called archivers) can be directly applied for testing.

Let us give some definitions. Let, as before, \( A \) be a finite alphabet and, by definition, \( A^* = \bigcup_{n=1}^{\infty} A^n \) and \( A^\infty \) is the set of all infinite words \( x_1 x_2 \ldots \) over the alphabet \( A \). A data compression method (or code) \( \varphi \) is defined as a set of mappings \( \varphi_n \) such that \( \varphi_n : A^n \rightarrow \{0, 1\}^* \), \( n = 1, 2, \ldots \) and for each pair of different words \( x, y \in A^n \) \( \varphi_n(x) \neq \varphi_n(y) \). Informally, it means that the code \( \varphi \) can be applied for compression of each message of any length \( n, n > 0 \) over alphabet \( A \) and the message can be decoded if its code is known. One more restriction is required in Information Theory. Namely, it is required that each sequence \( \varphi_n(x_1) \varphi_n(x_2) \ldots \varphi_n(x_r), r \geq 1, \) of encoded words from the set \( A^n, n \geq 1 \), can be uniquely decoded into \( x_1 x_2, \ldots x_r \). Such codes are called uniquely decodable. For example, let \( A = \{a, b\} \), the code \( \psi_1(a) = 0, \psi_1(b) = 00 \), obviously, is...
not uniquely decodable. (Indeed, the word 000 can be decoded in both ab and ba.) It is well known that if a code \( \varphi \) is uniquely decodable then the lengths of the codewords satisfy the following inequality (the Kraft inequality):

\[
\sum_{u \in A^n} 2^{-|\varphi(u)|} \leq 1,
\]

see, for ex., [9]. It will be convenient to reformulate this property as follows:

**Claim 1.** Let \( \varphi \) be a uniquely decodable code over an alphabet \( A \). Then for any integer \( n \) there exists a measure \( \mu_\varphi \) on \( A^n \) such that

\[
- \log \mu_\varphi(u) \leq |\varphi(u)|
\]

for any \( u \) from \( A^n \). (Obviously, it is true for the measure \( \mu_\varphi(u) = 2^{-|\varphi(u)|}/\sum_{u \in A^n} 2^{-|\varphi(u)|} \). It is known in Information Theory that sequences \( x_1 \ldots x_t \), generated by a stationary and ergodic source \( p \), can be "compressed" till the length \( - \log p(x_1 \ldots x_t) \) bits. There exist so-called universal codes, which, in a certain sense, are the best "compressors" for all stationary and ergodic sources. The formal definition is as follows: A code \( \varphi \) is universal if for any stationary and ergodic source \( p \)

\[
\lim_{t \to \infty} t^{-1}(- \log p(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)|) = 0
\]

with probability 1. So, informally speaking, the universal codes estimate the probability characteristics of the source \( p \) and use them for efficient "compression".

**III. The Tests.**

In this paragraph we describe the suggested tests. First, we give some definitions. Let \( v \) be a word \( v = v_1 \ldots v_k, k \leq t, v_i \in A \). Denote the rate of a word \( v \) occurring in the sequence \( x_1x_2 \ldots x_k, x_2x_3 \ldots x_{k+1}, x_3x_4 \ldots x_{k+2}, \ldots, x_{t-k+1} \ldots x_t \) as \( \nu^t(v) \). For example, if \( x_1, \ldots x_t = 000100 \) and \( v = 00 \), then \( \nu^5(00) = 3 \). Now we define for any \( k \geq 0 \) a so-called empirical Shannon entropy of order \( k \) as follows:

\[
h_k^t(x_1 \ldots x_t) = \frac{1}{t-k} \sum_{v \in A^k} \nu^t(v) \sum_{a \in A} \left( \nu^t(va)/\nu^t(v) \right) \log \left( \nu^t(va)/\nu^t(v) \right),
\]

where \( k < t \) and \( \nu^t(v) = \sum_{a \in A} \nu^t(va) \). In particular, if \( k = 0 \), we obtain

\[
h_0^t(x_1 \ldots x_t) = \frac{1}{t} \sum_{a \in A} \nu^t(a) \log(\nu^t(a)/t),
\]

The suggested test is as follows.

*Let \( \sigma \) be any probability distribution over \( A^t \). By definition, the hypothesis \( H_0 \) is accepted if

\[
(t - m)h_0^t(x_1 \ldots x_t) - \log(1/\sigma(x_1 \ldots x_t)) \leq \log(1/\alpha),
\]

where \( \alpha \in (0, 1) \). Otherwise, \( H_0 \) is rejected.* We denote this test by \( \Upsilon^t_{\alpha, \sigma, m} \).

**Theorem.** i) For any probability distribution (or predictor) \( \sigma \) the Type I error of the test \( \Upsilon^t_{\alpha, \sigma, m} \) is less than or equal to \( \alpha, \alpha \in (0, 1) \).

ii) If \( \sigma \) is a universal predictor (measure) (i.e., by definition, for any \( p \in M_\infty(A) \))

\[
\lim_{t \to \infty} (t - m)h_0^t(x_1 \ldots x_t) - \log(1/\sigma(x_1 \ldots x_t)) = 0
\]

with probability 1), then the Type II error goes to 0, where \( t \) goes to infinity.

The proof is given in Appendix.

**Comment.** Let \( \varphi \) be a uniquely decodable code (or a data compression method). Define the test \( \Upsilon^t_{\alpha, \varphi, m} \) as follows: The hypothesis \( H_0 \) is accepted if

\[
(t - m)h^*_{m}(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)| \leq \log(1/\alpha),
\]

where \( \alpha \in (0, 1) \). Otherwise, \( H_0 \) is rejected.

We immediately obtain from the theorem 1 and the claim 1 the following statement.

**Claim 2.** i) For any uniquely decodable code \( \varphi \) the Type I error of the test \( \Upsilon^t_{\alpha, \varphi, m} \) is less than or equal to \( \alpha, \alpha \in (0, 1) \).

ii) If \( \varphi \) is a universal code, then the Type II error goes to 0, where \( t \) goes to infinity.

**IV. Conclusion.**

The described above tests can be based on known universal codes (or so-called archivers) which are used for text compression everywhere. It is important to note that, on the one hand, the universal codes and archivers are based on results of Information Theory, the theory of algorithms and some other branches of mathematics; see, for example, probability [7], [11], [12], [15], [21]. On the other hand, the archivers have shown high efficiency in practise as compressors of texts, DNA sequences and many other types of real data. In fact, the archivers can find many kinds of latent regularities, that is why they look like a promising tool for independence testing and its generalizations.

The natural question is a possibility of generalization of the suggested tests for a case of an infinite source alphabet \( A \) (say, \( A \) is a metric space.) Apparently, such a generalization can be done for a case of independence testing, if we will use known methods of partitioning; [5], [6]. But we do not know how to generalize the suggested tests for a case where \( H_0 \) is the source is Markovian. The point is that the partitioning can increase the source memory. For example, even if the alphabet \( A \) contains three letters and we combine two of them in one subset (i.e., a new letter) the memory of the obtained source can increase till infinity. Hence, the generalization to Markov sources with infinite alphabet can be considered as an open problem.
V. APPENDIX.

Proof of Theorem. First we show that for any Bernoulli source $\tau^*$ and any word $x_1 \ldots x_t \in A^t$, $t > 1$, the following inequality is valid:

$$\tau^*(x_1 \ldots x_t) = \prod_{a \in A} \tau(a)^{\nu^*(a)} \leq \prod_{a \in A} \left( \frac{\nu^t(a)/t}{\nu(a)} \right)^{\nu^t(a)} \leq \prod_{a \in A} \nu(a) \log \left( \frac{\nu^t(a)/t}{\nu(a)} \right).$$  (11)

Indeed, the equality is true, because $\tau^*$ is a Bernoulli measure. The inequality follows from the well known inequality $\sum_{a \in A} p(a) \log(p(a)/q(a)) \geq 0$, for K-L divergence, which is true for any distributions $p$ and $q$ (see, for ex., [9]). So, if $p(a) = \nu^t(a)/t$ and $q(a) = \tau(a)$, then

$$\sum_{a \in A} \frac{\nu^t(a)}{t} \log \left( \frac{\nu^t(a)/t}{\tau(a)} \right) \geq 0.$$

From the last inequality we obtain (11).

Let now $\tau$ belong to $M_m(A)$, $m > 0$. We will prove that for any $x_1 \ldots x_t$

$$\tau(x_1 \ldots x_t) \leq \prod_{u \in A^m} \prod_{a \in A} (\nu^t(ua)/\nu(ua))^\nu^t(ua).$$  (12)

Indeed, we can present $\tau(x_1 \ldots x_t)$ as

$$\tau(x_1 \ldots x_t) = \tau_\infty(x_1 \ldots x_m) \prod_{u \in A^m} \prod_{a \in A} \tau(a/u)^{\nu^t(ua)},$$

where $\tau_\infty(x_1 \ldots x_m)$ is the limit probability of the word $x_1 \ldots x_m$. From the last equality we can see that

$$\tau(x_1 \ldots x_t) \leq \prod_{u \in A^m} \prod_{a \in A} \tau(a/u)^{\nu^t(ua)}.$$

Taking into account the inequality (11), we obtain

$$\prod_{a \in A} \tau(a)^{\nu^t(ua)} \leq \prod_{a \in A} (\nu^t(ua)/\nu(ua))^\nu(ua)$$

for any word $u$. So, from the last two inequalities we obtain (12).

It will be convenient to define an auxiliary measure on $A^t$ as follows:

$$\pi_m(x_1 \ldots x_t) = \Delta 2^{-(t-m)h^*_m(x_1 \ldots x_t)},$$  (13)

where $x_1 \ldots x_t \in A^t$ and

$$\Delta = (\sum_{x_1 \ldots x_t \in A^t} 2^{-(t-m)h^*_m(x_1 \ldots x_t)})^{-1}.$$

If we take into account that

$$2^{-(t-m)h^*_m(x_1 \ldots x_t)} = \prod_{u \in A^m} \prod_{a \in A} (\nu^t(ua)/\nu(ua))^\nu(ua),$$

we can see from (12) and (13) that, for any measure $\tau \in M_m(A)$ and any $x_1 \ldots x_t \in A^t$,

$$\tau(x_1 \ldots x_t) \leq \pi_m(x_1 \ldots x_t)/\Delta.$$  (14)

Let us denote the critical set of the test $\Upsilon_\alpha, \sigma, m$ as $C_\alpha$ i.e., by definition,

$$C_\alpha = \{ x_1 \ldots x_t : \{ t-m \} h^*_m(x_1 \ldots x_t) - \log(1/\sigma(x_1 \ldots x_t)) > \log(1/\alpha) \}. $$  (15)

From (14) and this definition we can see that for any measure $\tau \in M_m(A)$

$$\tau(C_\alpha) \leq \pi_m(C_\alpha)/\Delta.$$  (16)

From the definitions (15) and (13) we obtain

$$C_\alpha = \{ x_1 \ldots x_t : 2^{-(t-m)h^*_m(x_1 \ldots x_t)} > (\alpha \sigma(x_1 \ldots x_t))^{-1} \} = \{ x_1 \ldots x_t : (\pi_m(x_1 \ldots x_t)/\Delta)^{-1} > (\alpha \sigma(x_1 \ldots x_t))^{-1} \}.$$

Finally,

$$C_\alpha = \{ x_1 \ldots x_t : \sigma(x_1 \ldots x_t) > \pi_m(x_1 \ldots x_t)/(\alpha \Delta) \}. $$  (17)

The following chain of inequalities and equalities is valid:

$$1 \geq \sum_{x_1 \ldots x_t \in C_\alpha} \sigma(x_1 \ldots x_t) \geq \sum_{x_1 \ldots x_t \in C_\alpha} \pi_m(x_1 \ldots x_t)/(\alpha \Delta) = \pi_m(C_\alpha)/(\alpha \Delta) \geq \tau(C_\alpha)/(\alpha \Delta) = \tau(C_\alpha)/\alpha.$$  (18)

(Here both equalities and the first inequality are obvious, the second inequality and the third one follow from (17) and (16), correspondingly.) So, we obtain that $\tau(C_\alpha) \leq \alpha$ for any measure $\tau \in M_m(A)$. Taking into account that $C_\alpha$ is the critical set of the test, we can see that the probability of the Type I error is not greater than $\alpha$. The first claim of the theorem is proven.

The proof of the second statement of the theorem will be based on some results of Information Theory. The $t$-order entropic Shannon entropy is defined as follows:

$$h_t(p) = - \sum_{x_1 \ldots x_t \in A^t} p(x_1 \ldots x_t) \log p(x_1 \ldots x_t),$$  (18)

where $p \in M_\infty(A)$. It is known that for any $p \in M_\infty(A)$ firstly, $\log |A| \geq h_0(p) \geq h_1(p) \geq \ldots$, secondly, there exists the following limit Shannon entropy $h_\infty(p) = \lim_{t \to \infty} h_t(p)$, thirdly, $\lim_{t \to \infty} -t^{-1} \log p(x_1 \ldots x_t) = h_\infty(p)$ with the probability 1 and, finally, $h_m(p)$ is strictly greater than $h_\infty(p)$, if the memory of $p$ is larger $m$, i.e. $p \in M_m(A) \backslash M_m(A)$, see, for example, [3], [9].

Taking into account the definition of the universal predictor (see (2)), we obtain from the above described properties of the entropy that

$$\lim_{t \to \infty} -t^{-1} \log \sigma(x_1 \ldots x_t) = h_\infty(p)$$  (19)

with probability 1. It can be seen that $h^*_m$ is a consistent estimate for the $m$-order Shannon entropy $h_m$, i.e. $\lim_{t \to \infty} h_m(x_1 \ldots x_t) = h_m(p)$ with probability 1; see [3], [9]. Having taken into account that $h_m(p) > h_\infty(p)$ and (19) we obtain from the last equality that $\lim_{t \to \infty} (t - m) h^*_m(x_1 \ldots x_t) - \log(1/\sigma(x_1 \ldots x_t)) = \infty$. This proves the second statement of the theorem.
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