Exceptional collections in surface-like categories

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Abstract. We provide a categorical framework for recent results of Markus Perling’s on the combinatorics of exceptional collections on numerically rational surfaces. Using it we simplify and generalize some of Perling’s results as well as Vial’s criterion for the existence of a numerical exceptional collection.

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§1. Introduction

A beautiful recent paper of Perling’s [1] proves that any numerically exceptional collection of maximal length in the bounded derived category of coherent sheaves on a numerically rational surface (that is, a surface with zero irregularity and geometric genus) can be transformed by mutations into a numerically exceptional collection consisting of objects of rank 1. In this note we provide a categorical framework that allows us to extend and simplify Perling’s result.

To do this we introduce the notion of a surface-like category. Roughly speaking, it is a triangulated category $\mathcal{T}$ whose numerical Grothendieck group $K_0^{\text{num}}(\mathcal{T})$, considered as an abelian group with a bilinear form (Euler form), behaves similarly to the numerical Grothendieck group of a smooth projective surface; see Definition 3.1. Of course, the derived category $D(X)$ of any smooth projective surface $X$ is surface-like. However, there are surface-like categories of a different kind, for instance, the derived categories of noncommutative surfaces and of even-dimensional Calabi-Yau varieties turn out to be surface-like (Example 3.4). Also, some subcategories of surface-like categories are surface-like. Thus, the notion of a surface-like category is indeed more general.

In fact, all the results in this paper have a numerical nature, so instead of considering categories, we pass directly to their numerical Grothendieck groups. These are free abelian groups $G$ (we assume them to be of finite rank), equipped with a bilinear form $\chi$ that is neither symmetric, nor skew-symmetric in general. We call such a pair $(G, \chi)$ a pseudolattice (since this is a nonsymmetric version of a lattice).

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We define and investigate the notion of a surface-like pseudolattice (Definition 3.1), and show that it has many features similar to numerical Grothendieck groups of surfaces. For instance, the rank function and the Néron-Severi lattice $\text{NS}(G)$ for such groups $G$ (in the case when $G = K_0^\text{num}(D(X))$ for a surface $X$ the latter is isomorphic to $\text{NS}(X)$, the Néron-Severi lattice of $X$) and the canonical class $K_G \in \text{NS}(G) \otimes \mathbb{Q}$ can be constructed (in general, the canonical class is only rational).

We also introduce some important properties of surface-like pseudolattices: geometricity (Definition 4.3), minimality (Definition 4.1), and define their defects (Definition 5.3). The main result of this paper is Theorem 5.10, which says that if a geometric surface-like pseudolattice with zero defect has an exceptional basis (Definition 2.2), then this basis can be transformed by mutations to a basis consisting of elements of rank 1.

To prove Theorem 5.10 we first classify all minimal geometric pseudolattices $G$ with an exceptional basis. It turns out that minimality implies that the pseudolattice is isometric either to $K_0^\text{num}(D(\mathbb{P}^2))$ or to $K_0^\text{num}(D(\mathbb{P}^1 \times \mathbb{P}^1))$; see Theorem 4.6. In particular, its rank is 3 or 4, and its defect is 0.

To obtain the general case from this we investigate a kind of minimal model program for surface-like pseudolattices: we introduce the notion of a contraction of a pseudolattice (with respect to an exceptional element of zero rank) and show that we can always pass from a general surface-like pseudolattice to a minimal one by means of a finite number of contractions. We verify that geometricity is preserved under contractions, and that the defect does not decrease. In particular, if we start with a geometric pseudolattice of zero defect with an exceptional basis, then defect does not change under these contractions. This allows us to deduce Theorem 5.10 from Theorem 4.6.

In most of the proofs we follow Perling’s original arguments. The main new feature we introduce is the notion of a surface-like pseudolattice, which allows us to define the contraction operation and gives us greater flexibility. In particular, it lets us easily get rid of exceptional objects of zero rank that are a headache in Perling’s approach. Also, the general categorical perspective we take simplifies some of the computations, especially those related to use of the Riemann-Roch theorem.

Besides Perling’s results, we also apply this technique to prove a generalization of a criterion due to Vial (see [2], Theorem 3.1) for the existence of a numerically exceptional collection in the derived category of a surface; see Theorem 5.12. In fact, in the proof we use a lattice-theoretic result (see [2], Proposition A.12), but apart from this, the proof is an elementary consequence of the minimal model program for surface-like pseudolattices.

Of course, it would be very interesting to find higher-dimensional analogues of this technique. To do this, we would need a clear understanding of the relation between the (numerical) Grothendieck group of higher dimensional varieties and their (numerical) Chow groups. An important result in this direction is proved in a recent paper by Gorchinskiy [3].

The paper is organized as follows. In §2 we discuss numerical Grothendieck groups of triangulated categories and define pseudolattices. We also discuss excep-
tional bases of pseudolattices and their mutations there. In §3 we define surface-like categories and pseudolattices, provide some examples and discuss their basic properties. In particular, we explain how to define the rank function, the Néron-Severi lattice and the canonical class of a surface-like pseudolattice. In §4 we define minimality and geometricity, and classify minimal geometric surface-like pseudolattices with an exceptional basis. In the course of this classification, we associate a toric system with an exceptional basis and construct a fan in a rank-2 lattice from it, giving rise to a toric surface. Finally, in §5 we define a contraction of a pseudolattice with respect to a zero rank exceptional vector, and deduce the main results of the paper from the classification results in §4 via a minimal model program. In addition, we define the defect of a pseudolattice and investigate its behaviour under contractions.

After the first version of this paper was published, I was informed about a paper by de Thanhoffer de Volcsey and Van den Bergh [4], where a very similar categorical framework was introduced. In particular, in [4] the notion of a Serre lattice of surface type was defined, which is almost equivalent to the notion of a surface-like pseudolattice (see Remark 3.2), and some numerical notions of algebraic geometry were developed on this basis. So, the content of §3 in this paper is very close to the content of [4], §3.

**Acknowledgements.** It should be clear from the above that the paper owes its existence to the work of Markus Perling. I would also like to thank Sergey Gorchinskiy for very useful discussions. I am very grateful to Pieter Belmans and Michel Van den Bergh for informing me about the paper [4].

**§2. Numerical Grothendieck groups and pseudolattices**

Let $\mathcal{T}$ be a saturated (that is, smooth and proper) $k$-linear triangulated category, where $k$ is a field. Let $K_0(\mathcal{T})$ be the Grothendieck group of the category $\mathcal{T}$ and $\chi: K_0(\mathcal{T}) \otimes K_0(\mathcal{T}) \to \mathbb{Z}$ the Euler bilinear form:

$$\chi(F_1, F_2) = \sum (-1)^i \dim \text{Hom}(F_1, F_2[i]).$$

In general, the form $\chi$ is neither symmetric nor skew-symmetric; however, it is symmetrized by the Serre functor $S: \mathcal{T} \to \mathcal{T}$ of $\mathcal{T}$, that is, we have

$$\chi(F_1, F_2) = \chi(F_2, S(F_1)).$$

Since the Serre functor is an autoequivalence, it follows that the left kernel of $\chi$ coincides with its right kernel. We denote the quotient by

$$K_0^{\text{num}}(\mathcal{T}) := K_0(\mathcal{T})/\text{Ker } \chi;$$

it is called the **numerical Grothendieck group** of $\mathcal{T}$.

The numerical Grothendieck group $K_0^{\text{num}}(\mathcal{T})$ is torsion-free (any torsion element would be in the kernel of $\chi$) and finitely generated\(^1\), hence it is a free abelian group of finite rank. The form $\chi$ induces a nondegenerate bilinear form on $K_0^{\text{num}}(\mathcal{T})$ which we also denote by $\chi$. This form $\chi$ is again neither symmetric nor skew-symmetric.

\(^1\)I know about the proof of this from A.I. Efimov.
2.1. Pseudolattices. For the purposes of this paper we want to axiomatize the above situation.

**Definition 2.1.** A *pseudolattice* is a finitely generated free abelian group $G$ equipped with a nondegenerate bilinear form $\chi: G \otimes G \rightarrow \mathbb{Z}$. A pseudolattice $(G, \chi)$ is *unimodular* if the form $\chi$ induces an isomorphism $G \rightarrow G^\vee$. An *isometry* of pseudolattices $(G, \chi)$ and $(G', \chi')$ is an isomorphism of abelian groups $f: G \rightarrow G'$ such that $\chi'(f(v_1), f(v_2)) = \chi(v_1, v_2)$ for all $v_1, v_2 \in G$.

For any $v_0 \in G$ we define

$$v_0^\perp := \{ v \in G \mid \chi(v_0, v) = 0 \} \quad \text{and} \quad ^\perp v_0 := \{ v \in G \mid \chi(v, v_0) = 0 \}$$

to be the right and left orthogonal complements of $v_0$ in $G$.

We say that a pseudolattice $(G, \chi)$ has a *Serre operator* if there exists an automorphism $S_G: G \rightarrow G$ such that $\chi(v_1, v_2) = \chi(v_2, S_G(v_1))$ for all $v_1, v_2 \in G$.

Of course, a Serre operator is unique, and if the pseudolattice $G$ is unimodular then $S_G = (\chi^{-1})^T \circ \chi$ is the Serre operator. It is also clear that if $G = K_{\text{num}}^0(\mathcal{T})$ and $\mathcal{T}$ admits a Serre functor then the induced operator on $G$ is the Serre operator. The notion of a pseudolattice with a Serre operator is equivalent to the notion of a Serre lattice in [4].

We let

$$\chi_+ := \chi + \chi^T \quad \text{and} \quad \chi_- := \chi - \chi^T$$

denote the symmetrization and skew-symmetrization of the form $\chi$ on $G$. If $\mathcal{T}$ has a Serre operator, these forms can be written as

$$\chi_+(v_1, v_2) = \chi(v_1, (1 + S_G)v_2) \quad \text{and} \quad \chi_-(v_1, v_2) = \chi(v_1, (1 - S_G)v_2). \quad (1)$$

2.2. Exceptional bases and mutations. In this section we discuss a pseudolattice version of exceptional collections and mutations.

**Definition 2.2.** An element $e \in G$ is *exceptional* if $\chi(e, e) = 1$. A sequence of elements $(e_1, e_2, \ldots, e_n)$ is *exceptional* if each element $e_i$ is exceptional and $\chi(e_i, e_j) = 0$ for all $i > j$ (semi-orthogonality). An *exceptional basis* is an exceptional sequence in $G$ that forms a basis for it.

Of course, when $G = K_{\text{num}}^0(\mathcal{T})$, the class of an exceptional object in $\mathcal{T}$ is exceptional, and the classes of elements of an exceptional collection in $\mathcal{T}$ form an exceptional sequence in $G$, which is an exceptional basis if the collection is full.

**Lemma 2.3.** If $G$ has an exceptional sequence $e_1, \ldots, e_n$ of length $n = \text{rk} G$ then $e_1, \ldots, e_n$ is an exceptional basis and $G$ is unimodular.

**Proof.** Consider the composition of maps

$$\mathbb{Z}^n \xrightarrow{(e_1,\ldots,e_n)} G \xrightarrow{\chi} G^\vee \xrightarrow{(e_1,\ldots,e_n)^T} \mathbb{Z}^n.$$
The composition is given by the Gram matrix of the form $\chi$ on the set of vectors $e_1, \ldots, e_n$ which by the definition of an exceptional sequence is upper-triangular with units on the diagonal, hence it is an isomorphism. It follows that the first map is injective and the last map is surjective. Since $\text{rk}(G^\vee) = \text{rk}(G) = n$ and $G^\vee$ is torsion-free, the last map is an isomorphism, hence the first is too. Thus $e_1, \ldots, e_n$ is an exceptional basis in $G$. Moreover, it follows that $\chi$ is an isomorphism, hence $G$ is unimodular.

Assume that $e \in G$ is exceptional.

**Definition 2.4.** The left and right mutations with respect to $e$ are endomorphisms of $G$ defined by

$$L_e(v) := v - \chi(e, v)e \quad \text{and} \quad R_e(v) := v - \chi(v, e)e. \quad (2)$$

In fact, $L_e$ is just the projection onto the right orthogonal $e^\perp$ (and similarly for $R_e$). In particular, the mutations ‘kill’ the vector $e$ and define mutually inverse isomorphisms of the orthogonals

$$\perp e \xleftrightarrow{L_e} e^\perp.$$ 

Moreover, it is easy to see that given an exceptional sequence $e_\bullet = (e_1, \ldots, e_n)$ in $G$, the sequences

$$L_{i,i+1}(e_\bullet) := (e_1, \ldots, e_{i-1}, L_{e_i}(e_{i+1}), e_i, e_{i+2}, \ldots, e_n),$$

$$R_{i,i+1}(e_\bullet) := (e_1, \ldots, e_{i-1}, R_{e_{i+1}}(e_i), e_{i+1}), e_{i+2}, \ldots, e_n)$$

are exceptional, and these two operations are mutually inverse.

If $G$ has a Serre operator, every exceptional sequence $e_1, \ldots, e_n$ can be extended to an infinite sequence $\{e_i\}_{i \in \mathbb{Z}}$ by the rule

$$e_i = S_G(e_{i+n}).$$

This sequence is called a helix. Its main property is that for any $k \in \mathbb{Z}$ the sequence $e_{k+1}, \ldots, e_{k+n}$ is an exceptional sequence generating the same helix (up to an index shift).

If $e_1, \ldots, e_n$ is an exceptional basis then

$$e_0 = (L_{e_1} \circ \cdots \circ L_{e_{n-1}})(e_n) \quad \text{and} \quad e_{n+1} = (R_{e_n} \circ \cdots \circ R_{e_2})(e_1),$$

and it follows that every element of the helix can be obtained from the original basis by mutations. Mutations $L_{i,i+1}$ and $R_{i,i+1}$ (where $i$ is now an arbitrary integer) can be also defined for helices.

§ 3. Surface-like categories and pseudolattices

3.1. Definition and examples. The next definition is the main one in the paper.
Definition 3.1 (cf. [4], Definition 3.2.1). We say that a pseudolattice \((G, \chi)\) is \textit{surface-like} if there is a primitive element \(p \in G\) such that

1) \(\chi(p, p) = 0\);
2) \(\chi_-(p, -) = 0\) (that is, \(\chi(p, v) = \chi(v, p)\) for any \(v \in G\));
3) the form \(\chi_-\) vanishes on the subgroup \(p^\perp = {}^\perp_p G \subseteq G\) (that is, \(\chi\) is symmetric on \(p^\perp\)).

An element \(p\) with the above properties is called a \textit{point-like element} in \(G\).

We say that a smooth and proper triangulated category \(\mathcal{T}\) is \textit{surface-like} if its numerical Grothendieck group \((K^0_{\text{num}}(\mathcal{T}), \chi)\) is surface-like (with some choice of a point-like element).

Remark 3.2. A Serre operator \(S_G\) of a surface-like pseudolattice, if it exists, is unipotent by Corollary 3.15, and by Lemma 3.3, equation (1) and the nondegeneracy of \(\chi\) the rank of \(S_G^{-1}\) does not exceed 2, hence a surface-like pseudolattice with a Serre operator is a Serre lattice of surface type as defined in [4], Definition 3.2.1. Conversely, as a combination of Lemma 3.3 below and Lemma 3.3.2 in [4] shows, a Serre lattice of surface* type is a surface-like pseudolattice (with a point-like element being a primitive generator of the smallest term of the numerical codimension filtration in [4], §3.3). However, again by Lemma 3.3, a Serre lattice of surface type which is not of surface* type is surface-like only if \(\chi\) has an isotropic vector.

If we choose a basis \(v_0, v_1, \ldots, v_{n-2}, v_{n-1}\) in a pseudolattice \(G\) such that \(v_{n-1} = p\) and \(p^\perp = \langle v_1, \ldots, v_{n-1} \rangle\) then Definition 3.1 is equivalent to the fact, that the Gram matrix of the bilinear form \(\chi\) takes the following form:

\[
\chi(v_i, v_j) = \begin{pmatrix}
    a & b_1 & \ldots & b_{n-2} & d \\
    b_1' & c_{11} & \ldots & c_{1,n-2} & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    b_{n-2}' & c_{n-2,1} & \ldots & c_{n-2,n-2} & 0 \\
    d & 0 & \ldots & 0 & 0
\end{pmatrix}
\]  

(3)

with the submatrix \((c_{ij})\) being symmetric.

The following is a useful reformulation of Definition 3.1.

Lemma 3.3. A pseudolattice \(G\) is surface-like if and only if one of the following two cases takes place:

(a) \(\text{either } \chi_- = 0\) (that is, the form \(\chi\) is symmetric) and \(\chi\) has an isotropic vector;
(b) \(\text{or the rank of } \chi_- \text{ equals } 2, \text{ and the restriction } \chi|_{\ker \chi_-} \text{ is degenerate.}\)

In case (a) an element \(p \in G\) is point-like if and only if it is isotropic, that is, \(\chi(p, p) = 0\). In case (b) an element \(p\) is point-like if and only if \(p \in \ker(\chi|_{\ker \chi_-})\).

Proof. Assume that \(G\) is surface-like. If \(\chi_- = 0\), we are in case (a); then \(p\) is isotropic by Definition 3.1, part 1). Otherwise the rank of \(\chi_-\) equals 2, since \(\chi_-\) vanishes on the hyperplane \(p^\perp \subseteq G\) by Definition 3.1, part 3), and moreover \(\ker \chi_- \subseteq p^\perp\). Furthermore, \(p \in \ker \chi_-\) by Definition 3.1, part 2), and since \(\ker \chi_- \subseteq p^\perp\) we have \(\chi(\ker \chi_-, p) = 0\); hence \(p\) lies in the kernel of the restriction \(\chi|_{\ker \chi_-}\).
Conversely, if (a) holds and \( p \) is isotropic, then \( \chi_- = 0 \) and Definition 3.1 clearly holds. Similarly, if (b) holds and \( p \in \text{Ker}(\chi|_{\text{Ker} \chi_-}) \), then parts 1) and 2) of Definition 3.1 hold, and since \( p^\perp \) contains \( \text{Ker} \chi_- \) as a hyperplane, part 3) also holds.

The above argument also shows that when \( \chi_- \neq 0 \), part 1) of Definition 3.1 follows from parts 2) and 3). We note that the restriction on the rank of \( \chi_- \) also appeared recently in [5].

We now give some examples.

**Example 3.4.** Let \( \mathcal{T} \) be a smooth and proper Calabi-Yau category of even dimension (that is, its Serre functor is a shift \( S_{\mathcal{T}} \cong [k] \) with even \( k \)) and let \( P \in \mathcal{T} \) be a point object, that is, \( \text{Ext}^\bullet(P, P) \) is an exterior algebra on \( \text{Ext}^1(P, P) \). Then \( \mathcal{T} \) is a surface-like category and \( G = K^\text{num}_0(\mathcal{T}) \) is a surface-like pseudolattice with \( [P] \) being a point-like element, since the Euler form is symmetric and so we are in case (a) of Lemma 3.3.

However, the main example for us is the next one.

**Example 3.5.** Let \( X \) be a smooth projective surface over a field, \( \mathcal{T} = \text{D}(X) \) its bounded derived category of coherent sheaves, and \( G = K^\text{num}_0(\mathcal{T}) = K^\text{num}_0(X) \). The topological filtration \( 0 \subset F^2K_0(X) \subset F^1K_0(X) \subset F^0K_0(X) = K_0(X) \) on the Grothendieck group \( K_0(X) \) (by codimension of support) induces a filtration \( 0 \subset G_2 \subset G_1 \subset G_0 = G \) on the numerical Grothendieck group. Consider the maps

\[
r : G_0/G_1 \longrightarrow \mathbb{Z}, \quad c_1 : G_1/G_2 \longrightarrow \text{NS}(X) \quad \text{and} \quad s : G_2 \longrightarrow \mathbb{Z},
\]

given by the rank, the first Chern class and the Euler characteristic, where \( \text{NS}(X) \) is the numerical Néron-Severi group of \( X \) (in particular, we quotient out the torsion in the Chow group \( \text{CH}^1(X) \)). The rank map is clearly an isomorphism, and so is \( c_1 \) (the surjectivity of \( c_1 \) follows from the surjectivity of \( F^1K_0(X) \to \text{CH}^1(X) \)), and for injectivity it is enough to note that if \( \mathcal{L} \) is a line bundle such that \( c_1(\mathcal{L}) \) is numerically equivalent to zero, then by Riemann-Roch \( [\mathcal{L}] = [\mathcal{O}_X] \) in \( K^\text{num}_0(\text{D}(X)) \)). It is also clear that \( s \) is injective, so if we normalize it by dividing by the minimal degree of a 0-cycle on \( X \), the map we obtain is an isomorphism. Considering \( c_1 \) as a linear map \( G \to \text{NS}(X) \) in the standard way and extending the normalized Euler characteristic map linearly to a map \( \tilde{s} : G \to \mathbb{Z} \) (if \( X \) has a 0-cycle of degree 1, we can take \( \tilde{s} \) to be the Euler characteristic map) we obtain an isomorphism

\[
G \xrightarrow{\sim} \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}, \quad v \mapsto (r(v), c_1(v), \tilde{s}(v)).
\]

A simple Riemann-Roch computation shows that the form \( \chi_- \) is given by

\[
\chi_-((r_1, D_1, s_1), (r_2, D_2, s_2)) = r_1(K_X \cdot D_2) - r_2(K_X \cdot D_1),
\]

where \( K_X \in \text{NS}(X) \) is the canonical class of \( X \) and \( \cdot \) stands for the intersection pairing on \( \text{NS}(X) \). The kernel of \( \chi_- \) is spanned by all \((0, D, s) \in \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z} \) such that \( K_X \cdot D = 0 \), in particular, the rank of \( \chi_- \) is at most 2. Furthermore, by the Riemann-Roch Theorem we have

\[
\chi((0, D_1, s_1), (0, D_2, s_2)) = -D_1 \cdot D_2.
\]
Therefore, $p_X := (0, 0, 1)$ is contained in the kernel of $\chi|_{\text{Ker} \chi_-}$, and since the intersection pairing on the numerical Néron-Severi group $\text{NS}(X)$ is nondegenerate, $p_X$ generates this kernel unless $K_X^2 = 0$. In the latter case, the kernel of $\chi|_{\text{Ker} \chi_-}$ is generated by $p_X$ and $K_X$ (unless $K_X = 0$). In particular, Lemma 3.3(b) shows that $G$ is surface-like with $p_X$ a point-like class, and that $p_X$ is the unique point-like class unless $K_X^2 = 0$.

We say that a surface-like category $\mathcal{T}$ (pseudolattice $G$) is standard if $(\mathcal{T}, p) = (D(X), p_X)$ for a surface $X$ ($(G, p) = (K^0_{\text{num}}(D(X)), p_X)$, respectively). The next example shows that even the standard surface-like pseudolattice in Example 3.5 may sometimes have a nonstandard surface-like structure.

**Example 3.6.** We again assume that $X$ is a smooth projective surface, $\mathcal{T} = D(X)$ and $G = K^0_{\text{num}}(\mathcal{T})$. In the notation of Example 3.5 set $p_K := (0, K_X, 0)$. If $K_X^2 = 0$ then $p_K$ is a point-like element, which gives a different surface-like structure on the pseudolattice $G$ and the category $D(X)$.

In what follows we will need an explicit form of the standard pseudolattices for the surfaces $X = \mathbb{P}^2$, $X = \mathbb{P}^1 \times \mathbb{P}^1$, and $X = F_1$ (the Hirzebruch ruled surface).

**Example 3.7.** If $X = \mathbb{P}^2$, the classes of the sheaves $(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$ form an exceptional basis in which the Gram matrix of the Euler form looks like

\[
\chi_{\mathbb{P}^2} = \begin{pmatrix}
1 & 3 & 6 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{pmatrix}.
\]

If $X = \mathbb{P}^1 \times \mathbb{P}^1$, for each $c \in \mathbb{Z}$ the classes of the sheaves

\[
(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c, 1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c + 1, 1))
\]

form an exceptional basis in which the Gram matrix of the Euler form looks like

\[
\chi_{\mathbb{P}^1 \times \mathbb{P}^1} = \begin{pmatrix}
1 & 2 & 2c + 2 & 2c + 4 \\
0 & 1 & 2c & 2c + 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

If $X = F_1$, for each $c \in \mathbb{Z}$ the classes of the sheaves

\[
(\mathcal{O}_{F_1}, \mathcal{O}_{F_1}(f), \mathcal{O}_{F_1}(s + cf), \mathcal{O}_{F_1}(s + (c + 1)f))
\]

(where $f$ is the class of a fibre and $s$ is the class of the $(-1)$-section) form an exceptional basis in which the Gram matrix of the Euler form looks like

\[
\chi_{F_1} = \begin{pmatrix}
1 & 2 & 2c + 1 & 2c + 3 \\
0 & 1 & 2c - 1 & 2c + 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Note that the left mutation of $O_{F_1}(s + (c + 1)f)$ through $O_{F_1}(s + cf)$ is isomorphic to $O_{F_1}(s + (c - 1)f)$ (up to a shift); hence all such exceptional collections are mutation-equivalent. Note also that when $c = 1$ the left mutation of $O_{F_1}(f)$ through $O_{F_1}(f)$ is isomorphic to the structure sheaf of the $(-1)$-section; in particular, its rank is zero.

The next examples come from noncommutative geometry.

Example 3.8. Let $\mathcal{I}$ be the derived category of a noncommutative projective plane (see [6], for instance). Then the numerical Grothendieck group $K_0^{\text{num}}(\mathcal{I})$ is isometric to $K_0^{\text{num}}(\mathbb{P}^2)$ (the numerical Grothendieck group of a commutative plane), hence $\mathcal{I}$ is a surface-like category. The same applies to noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ and other noncommutative deformations of surfaces.

From this perspective it would be interesting to classify directed quivers whose derived categories of representations are surface-like (for quivers with four vertices this was done in [4]). Note that each of these categories can be realized as a semi-orthogonal component of the derived category of a smooth projective variety (see [7] and also [8]). More generally, we could ask when a gluing [9]–[11] of two triangulated categories is surface-like.

Example 3.9. Let $X$ be a smooth projective complex K3 surface and $\beta \in \text{Br}(X)$ an element in the Brauer group. Let $\mathcal{I} = \mathbf{D}(X, \beta)$ be the twisted derived category. Its numerical Grothendieck group $K_0^{\text{num}}(X, \beta)$ can be described as follows (see [12], §1, for details). Analyzing the cohomology in the exponential sequence can write the Brauer group as an extension

$$0 \to \frac{H^2(X, \mathbb{Q})}{\text{NS}(X)_{\mathbb{Q}} + H^2(X, \mathbb{Z})} \to \text{Br}(X) \to H^3(X, \mathbb{Z})_{\text{torsion}} \to 0.$$ 

For a K3 surface the right term vanishes, hence a Brauer class $\beta$ can be represented by a rational cohomology class $B \in H^2(X, \mathbb{Q})$ (called a B-field). Then

$$K_0^{\text{num}}(X, \beta) = \left\{(r, D, s) \in \mathbb{Q} \oplus \text{NS}(X)_{\mathbb{Q}} \oplus \mathbb{Q} \mid r \in \mathbb{Z}, \ D + rB \in \text{NS}(X), \ s + D \cdot B + \frac{rB^2}{2} \in \mathbb{Z}\right\},$$

and the Euler form is given by the Mukai pairing

$$\chi((r_1, D_1, s_1), (r_2, D_2, s_2)) = r_1s_2 - D_1 \cdot D_2 + s_1r_2.$$ 

In this case, as before, $p_X = (0, 0, 1)$ is a point-like element.

Other examples of surface-like categories are given in [13] and [14].

3.2. The rank function and Néron-Severi lattice. Assume that $(G, \mathbf{p})$ is a surface-like pseudolattice and let $\mathbf{p} \in G$ be its point-like element. The linear functions $\chi(\mathbf{p}, -)$ and $\chi(-, \mathbf{p})$ on $G$ coincide (by Definition 3.1, part 2). We define the rank function associated with the point-like element $\mathbf{p}$ by

$$r(-) := \chi(\mathbf{p}, -) = \chi(-, \mathbf{p}).$$

Note that $\mathbf{p}^\perp = \perp \mathbf{p} = \text{Ker}r.$
Lemma 3.10. If $G$ is a surface-like pseudolattice and $p$ is its point-like element, there is a complex

$$Z \xrightarrow{p} G \xrightarrow{r} Z$$

(5)

where $p$ is injective and, if $G$ is unimodular, $r$ is surjective. Its middle cohomology

$$\text{NS}(G) := p^\perp / p$$

(6)

is a finitely generated free abelian group of rank $\text{rk}(G) - 2$.

Proof. We have $r(p) = \chi(p, p) = 0$ by Definition 3.1, part 1), hence (5) is a complex. The first map in (5) is injective since $G$ is torsion-free. The second map is nonzero since $\chi$ is nondegenerate on $G$. If its image is $dZ \subset Z$ with $d \geq 2$ (up to sign, this is the same $d$ as in (3)), then $\frac{1}{d}r$ is a well defined element of $G^\vee$. If $G$ is unimodular, then there is an element $v \in G$ such that $d \cdot \chi(v, -) = \chi(p, -)$. Therefore, $p - d \cdot v$ lies in the kernel of $\chi$, hence it is zero, so $p$ is not primitive. This contradiction shows that $r$ is surjective for unimodular $G$.

The group $p^\perp$ is torsion-free of rank $\text{rk}(G) - 1$ since $r \neq 0$, and the group $\text{NS}(G)$ is torsion-free of rank $\text{rk}(G) - 2$ since $p$ is primitive.

The unimodularity assumption is not necessary for $r$ to be surjective, however $r$ is not surjective in general. Indeed, if $(G, p\chi)$ is the surface-like pseudolattice in Example 3.9, the index of $r(G)$ in $Z$ equals the order of $\beta$ in the Brauer group $\text{Br}(X)$.

The group $\text{NS}(G)$ defined in (6) is called the Néron-Severi group of a surface-like pseudolattice $G$.

Lemma 3.11. If $G$ is a surface-like pseudolattice and $p$ is its point-like element, then the form $\chi$ induces a nondegenerate symmetric bilinear form on $\text{NS}(G)$. In other words, there exists a unique nondegenerate symmetric bilinear form $q: \text{Sym}^2(\text{NS}(G)) \to Z$ such that the diagram

$$\begin{array}{ccc}
p^\perp \otimes p^\perp & \xrightarrow{\chi} & \text{NS}(G) \otimes \text{NS}(G) \\
\downarrow & & \downarrow q \\
Z & & Z
\end{array}$$

is commutative. The pseudolattice $G$ is unimodular if and only if $r$ is surjective and $q$ is unimodular.

Proof. Recall that $\chi_-$ vanishes on $p^\perp$ by Definition 3.1, part 3), hence $\chi$ is symmetric on $p^\perp$. Furthermore, if $v \in p^\perp$ then $\chi(p, v) = r(v) = 0$, hence $p$ is in the kernel of $\chi$ on $p^\perp$, and thus $\chi$ induces a symmetric form $q$ on $\text{NS}(G)$ such that the above diagram commutes.
Finally, consider the diagram
\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{p} & G \\
\downarrow & & \downarrow r \\
\mathbb{Z} & \xrightarrow{r} & G^\vee \\
\end{array}
\]

which is commutative by the definition of \(r\), and hence it is a morphism of complexes. The map induced on the middle cohomologies of complexes \(\text{NS}(G) \to \text{NS}(G)^\vee\) coincides with the map \(-q\) by the definition of the latter. As \(\chi\) is nondegenerate, thus it is injective, hence \(q\) is nondegenerate.

If \(G\) is unimodular then the middle vertical arrow is an isomorphism, hence the induced map \(-q\) on the cohomologies is an isomorphism, hence \(q\) is unimodular. On the other hand, \(r\) is surjective by Lemma 3.10. Conversely, if \(r\) is surjective, then the Snake Lemma implies that the cokernels of \(\chi: G \to G^\vee\) and \(q: \text{NS}(G) \to \text{NS}(G)^\vee\) are isomorphic, so if \(q\) is unimodular so is \(\chi\).

The lattice \((\text{NS}(G), q)\) is called the *Néron-Severi lattice* of the surface-like pseudolattice \(G\). The filtration \(0 \subset Zp \subset p^\perp \subset G\) can be thought of as an analogue of the topological filtration on \(G\). In Corollary 3.15 below we show that the Serre operator of \(G\) (if it exists) preserves this filtration and acts on its factors as the identity.

### 3.3. The canonical class.

In this section we show how to define the canonical class of a surface-like pseudolattice. It is always well defined as a linear function on \(\text{NS}(G)\) or as an element of \(\text{NS}(G) \otimes \mathbb{Q}\), and under a unimodularity assumption, also as an element of \(\text{NS}(G)\).

The rank map induces a map
\[
\lambda: \bigwedge^2 G \to p^\perp, \quad v_1 \wedge v_2 \mapsto r(v_1)v_2 - r(v_2)v_1. \tag{7}
\]

Its kernel is \(\bigwedge^2(p^\perp)\), and if the rank map is surjective, the map \(\lambda\) is surjective too.

**Lemma 3.12.** There exists a unique element \(K_G \in \text{NS}(G) \otimes \mathbb{Q}\) such that for all \(v_1, v_2 \in G\)
\[
\chi_-(v_1, v_2) = \chi(v_1, v_2) - \chi(v_2, v_1) = -q(K_G, \lambda(v_1, v_2)). \tag{8}
\]

*If \(G\) is unimodular, then \(K_G \in \text{NS}(G)\) is integral.*

**Proof.** Let \(\overline{\lambda}: \bigwedge^2 G \to \text{NS}(G)\) denote the composition of \(\lambda\) with the projection \(p^\perp \to \text{NS}(G)\). Consider the diagram
\[
\begin{array}{ccc}
\bigwedge^2 G & \xrightarrow{\lambda} & \text{Im}(\lambda) \\
\downarrow \chi_- & & \downarrow \text{Im}(\overline{\lambda}) \\
\mathbb{Z} & \xleftarrow{} & \text{NS}(G)
\end{array}
\]

Our goal is to extend the map \(\chi_-\) to a linear map from \(\text{NS}(G)\). First, we will show that it extends to the dashed arrow. To do this it is enough to note that \(\chi_-\) vanishes
on $\text{Ker}(\lambda) = \bigwedge^2(p^\perp)$ by Definition 3.1, part 3). Next, to construct the dotted arrow it is enough to check that the dashed arrow vanishes on the kernel of the map $\text{Im}(\lambda) \to \text{Im}(\overline{\lambda})$. Clearly, this kernel is generated by an appropriate multiple of $p$. If $v \in G$ is such that $r(v) \neq 0$, then $\lambda(v, p) = r(v)p$, and $\chi_-(v, p) = 0$ by Definition 3.1, part 2), hence the dashed arrow vanishes on $r(v)p$. Finally, $\text{Im}(\lambda) \subset p^\perp$ is a subgroup of finite index, hence $\text{Im}(\lambda) \subset \text{NS}(G)$ is a subgroup of finite index, hence the linear map from $\text{Im}(\lambda)$ to $\mathbb{Z}$ extends uniquely to a linear map $\text{NS}(G) \to \mathbb{Q}$, and since the form $q$ on $\text{NS}(G)$ is nondegenerate, there is a unique element $K_G$ such that (8) holds.

If $G$ is unimodular, then $\text{Im}(\lambda) = p^\perp$, $\text{Im}(\overline{\lambda}) = \text{NS}(G)$, hence the dashed arrow is a map taking $\text{NS}(G)$ to $\mathbb{Z}$. By the unimodularity of $q$ it is given by a scalar product with an integral vector $K_G \in \text{NS}(G)$.

The sign convention in (8) is chosen in order to agree with the case of the standard pseudolattice. We call the element $K_G \in \text{NS}(G)$ the canonical class of $G$. If we have the inclusion $K_G \in \text{NS}(G) \subset \text{NS}(G) \otimes \mathbb{Q}$, we say that the canonical class of $G$ is integral. In terms of (3) the canonical class is given by the formula $K_G \cdot v_i = (b'_i - b_i)/d$.

3.4. The Serre operator. Assume that $G$ is a surface-like pseudolattice which has a Serre operator $S_G$.

Lemma 3.13. Any point-like element in a surface-like pseudolattice is fixed by the Serre operator, that is, $S_G(p) = p$. Analogously, $S_G$ fixes the corresponding rank function, that is, $r(S_G(-)) = r(-)$.

Proof. Since $\chi$ is nondegenerate, (1) implies $\text{Ker}(\chi_-) = \text{Ker}(1 - S_G)$. Therefore, it follows from Definition 3.1, part 2) that $S_G(p) = p$.

It follows that the Serre operator induces an automorphism of the complex (5), hence an automorphism of the Néron-Severi lattice $\text{NS}(G)$.

Lemma 3.14. The automorphism of the group $\text{NS}(G)$ induced by the Serre operator is the identity map.

Proof. We have to check that $1 - S_G$ acts by zero on $\text{NS}(G)$, or equivalently, that it takes $p^\perp$ to $\mathbb{Z}p$. To do this we note that if $v_1, v_2 \in p^\perp$ then

$$\chi(v_1, (1 - S_G)v_2) = \chi_-(v_1, v_2) = 0$$

by (1) and Definition 3.1, part 3). On the other hand, since $\chi$ is nondegenerate, its kernel on $p^\perp$ is generated by $p$.

Corollary 3.15. If $G$ is a surface-like pseudolattice then $S_G$ is unipotent and, moreover, $(1 - S_G)^3 = 0$.

Proof. By Lemma 3.13 the three-step filtration $0 \subset \mathbb{Z}p \subset p^\perp \subset G$ is fixed by $S_G$, and the induced action on its factors is the identity by Lemma 3.14.

The relation between the canonical class and the Serre operator is given by the following lemma.
Lemma 3.16. The relation $\lambda(v, S_G(v)) = r(v)^2 K_G \pmod{p}$ holds for any $v \in G$.

Proof. If $r(v) = 0$ then $r(S_G(v)) = 0$ as well (by Lemma 3.13), hence both sides are zero. So we can assume that $r(v) \neq 0$. Take an arbitrary $v' \in G$ and let $r = r(v)$, $r' = r(v')$. Then

$$q(\lambda(v, v'), \lambda(v, S_G(v))) = -\chi(\lambda(v, v'), \lambda(v, S_G(v))) = -\chi(rv' - r'v, rS_G(v) - rv)$$

$$= r^2(\chi(v', v) - \chi(v', S_G(v))) + rr'(\chi(v, S_G(v)) - \chi(v, v))$$

$$= r^2(\chi(v', v) - \chi(v, v')) = r^2q(K_G, \lambda(v, v')).$$

It follows that the difference $\lambda(v, S_G(v)) - r^2 K_G$ is orthogonal to all elements of the form $\lambda(v, v')$. If $v'' \in p^\perp$ then $\lambda(v, v' + v'') = \lambda(v, v') + rv''$, hence elements of this form span all $p^\perp$. It follows that $\lambda(v, S_G(v)) - r^2 K_G$ is in the kernel of $q$ on $p^\perp \otimes \mathbb{Q}$, hence it lies in $\mathbb{Q}p$.

Corollary 3.17. For any $v \in G$, the following holds:

$$\chi(S_G(v), v) - \chi(v, v) = r(v)^2 q(K_G, K_G).$$

Proof. Substitute $v_1 = S_G(v)$ and $v_2 = v$ into (8), take the equality $\chi(v, S_G(v)) = \chi(v, v)$ into account, and use Lemma 3.16 and the fact that the point-like element $p$ is in the kernel of $q$.

The rational number $q(K_G, K_G)$ should be thought of as the canonical degree of a surface-like pseudolattice.

3.5. Exceptional sequences in surface-like pseudolattices. The computations in this section are analogues of those in [1], §3. However, the categorical approach enables us to simplify them considerably.

Assume that $G$ is a surface-like pseudolattice, $p$ is its point-like element, $r$ is the corresponding rank function, $q$ is the induced quadratic form on $NS(G)$ and $\lambda$ is the map defined in (7).

Lemma 3.18 (cf. [1], §3.5 (ii)). Assume that $e_1, e_2 \in G$ are exceptional and have nonzero ranks. Then

$$\chi(e_1, e_2) + \chi(e_2, e_1) = \frac{1}{r(e_1)r(e_2)} \left(q(\lambda(e_1, e_2)) + r(e_1)^2 + r(e_2)^2\right). \quad (9)$$

Proof. For brevity, we write $r_i = r(e_i)$. By definition we have $\lambda(e_1, e_2) = r_1 e_2 - r_2 e_1$, hence

$$-q(\lambda(e_1, e_2)) = \chi(\lambda(e_1, e_2), \lambda(e_1, e_2)) = \chi(r_1 e_2 - r_2 e_1, r_1 e_2 - r_2 e_1)$$

$$= r_1^2 \chi(e_2, e_2) + r_2^2 \chi(e_1, e_1) - r_1 r_2 (\chi(e_1, e_2) + \chi(e_2, e_1))$$

(the first equality follows from Lemma 3.11, the second from the definition of $\lambda$ and the third from the bilinearity of $\chi$). Using the exceptionality of $e_1$ and $e_2$, that is, $\chi(e_1, e_1) = \chi(e_2, e_2) = 1$, we easily deduce the required formula.
Lemma 3.19 (cf. [1], Proposition 3.8). Assume that \((e_1, e_2)\) and \((e_2, e_3)\) are exceptional pairs in \(G\) and \(r(e_2) \neq 0\). Then \((e_1, e_3)\) is an exceptional pair if and only if
\[
q(\lambda(e_1, e_2), \lambda(e_2, e_3)) = r(e_1)r(e_3).
\]

Proof. Set \(r_i = r(e_i)\). As in Lemma 3.18, we have
\[
q(\lambda(e_2, e_3), \lambda(e_1, e_2)) = -\chi(\lambda(e_2, e_3), \lambda(e_1, e_2)) = -\chi(r_2e_3 - r_3e_2, r_1e_2 - r_2e_1)
\]
\[
= -r_1r_2\chi(e_3, e_2) - r_3r_2\chi(e_2, e_1) + r_3r_1\chi(e_2, e_2) + r_2^2\chi(e_3, e_1).
\]
By assumption, the first two terms on the right-hand side vanish, and the third term equals \(r_1r_3\). Hence \(r_2\chi(e_3, e_1) = q(\lambda(e_1, e_2), \lambda(e_2, e_3)) - r_1r_3\). Since \(r_2\) is nonzero, \(\chi(e_3, e_1)\) vanishes if and only if \(q(\lambda(e_1, e_2), \lambda(e_2, e_3)) = r_1r_3\).

Lemma 4.2. If \((e_1, e_2, e_3, e_4)\) is an exceptional sequence in \(G\) then
\[
q(\lambda(e_1, e_2), \lambda(e_3, e_4)) = 0.
\]

Proof. Set \(r_i = r(e_i)\). Then, as before,
\[
q(\lambda(e_1, e_2), \lambda(e_3, e_4)) = q(\lambda(e_3, e_4), \lambda(e_1, e_2))
\]
\[
= -\chi(\lambda(e_3, e_4), \lambda(e_1, e_2)) = -\chi(r_3e_4 - r_4e_3, r_1e_2 - r_2e_1)
\]
\[
= -r_1r_3\chi(e_4, e_2) - r_2r_4\chi(e_3, e_1) + r_2r_3\chi(e_4, e_1) + r_1r_4\chi(e_3, e_2).
\]
By assumption all the terms on the right-hand side vanish.

Lemma 3.21 (cf. [1], §3.3 (i) and §3.4 (iv)). If \(r(e) = 0\) then \(e\) is exceptional in \(G\) if and only if \(q(e) = -1\). If \(r(e_1) = r(e_2) = 0\) then \(\chi(e_1, e_2) = 0\) if and only if \(q(e_1, e_2) = 0\).

The proof follows immediately from Lemma 3.11.

§ 4. Minimal surface-like pseudolattices

Let \(G\) be a surface-like pseudolattice. We use the theory developed in §3 keeping the point-like element \(p\) implicit. Also, to simplify notation we write \(D_1 \cdot D_2\) instead of \(q(D_1, D_2)\) for any \(D_1, D_2 \in \text{NS}(G)\) and call this pairing the intersection form. We always denote the rank of \(G\) by \(n\).

4.1. Minimality, geometricity and norm-minimality. We start by introducing some useful concepts.

Definition 4.1. A surface-like pseudolattice \(G\) is minimal if it has no exceptional elements of zero rank.

There is a simple characterization of minimality in terms of Néron-Severi lattices.

Lemma 4.2. A surface-like pseudolattice \(G\) is minimal if and only if the intersection form on its Néron-Severi lattice \(\text{NS}(G)\) does not represent \(-1\).

The proof follows immediately from Lemma 3.21.
Definition 4.3. We say that a lattice \( L \) with a vector \( K \in L \) is geometric if \( L \) has signature \((1, \text{rk} L - 1)\) and \( K \) is characteristic, that is, \( D^2 \equiv K \cdot D \pmod{2} \) for any \( D \in L \). A surface-like pseudolattice \( G \) is geometric if its canonical class is integral and \((\text{NS}(G), K_G)\) is a geometric lattice.

Remark 4.4. In terms of the matrix representation (3), a unimodular geometric pseudolattice can be represented by a matrix with \( b'_1 = \cdots = b'_{n-2} = 0, \, d = 1 \) and the symmetric matrix \((c_{ij})\) representing (up to sign) the unimodular intersection pairing of its Néron-Severi lattice, \((b_1, \ldots, b_{n-2})\) being the intersection products of the basis vectors with the anticanonical class. Indeed, \( d = \pm 1 \) by the proof of Lemma 3.10, and changing the sign of \( v_0 \) if necessary we can ensure that \( d = 1 \). Similarly, using the unimodularity of \((c_{ij})\) and adding an appropriate linear combination of \( v_1, \ldots, v_{n-2} \) to \( v_0 \) we can make \( b'_i = 0 \). Then \( b_i = -K_G \cdot v_i \), in view of an observation at the end of §3.3.

The name ‘geometric’ is motivated by the next result.

Lemma 4.5. If \( X \) is a smooth projective surface, the standard surface-like pseudolattice \((G, p) = (K_0^{\text{num}}(D(X)), p_X)\) is geometric.

Proof. The canonical class in the Néron-Severi lattice \( \text{NS}(X) \) of a smooth projective surface \( X \) is characteristic since by the Riemann-Roch Theorem we have \( D^2 - K_X \cdot D = 2(\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X)) \), and the signature of \( \text{NS}(X) \) is equal to \((1, \text{rk} \text{NS}(X) - 1)\) by the Hodge Index Theorem. We conclude by \( \text{NS}(G) = \text{NS}(X) \).

In this paper we are mostly interested in pseudolattices with an exceptional basis.

Let \( G \) be a surface-like pseudolattice, and \( e_{\bullet} = (e_1, \ldots, e_n) \) an exceptional basis in \( G \). We define its norm as

\[
\|e_{\bullet}\| = \sum_{i=1}^{n} r(e_i)^2.
\]

(10)

We say that a basis is norm-minimal if the norm of any exceptional basis obtained from \( e_{\bullet} \) by a sequence of mutations is greater than or equal to the norm of \( e_{\bullet} \). Clearly, any exceptional basis can be transformed to a norm-minimal basis by mutations.

The main result in this section is the following theorem.

Theorem 4.6. Let \( G \) be a geometric surface-like pseudolattice of rank \( n \) with a norm-minimal exceptional basis \( e_{\bullet} \). If \( r(e_i) \neq 0 \) for all \( 1 \leq i \leq n \) (for instance if \( G \) is minimal), then \( G \) is isometric either to \( K_0^{\text{num}}(D(\mathbb{P}^2)) \) or to \( K_0^{\text{num}}(D(\mathbb{P}^1 \times \mathbb{P}^1)) \), and the basis \( e_{\bullet} \) corresponds to one of the standard exceptional collections of line bundles in these categories (see Example 3.7). In particular, \( n = 3 \) or \( n = 4 \), \( K_G^2 = 12 - n \), and \( r(e_i) = \pm 1 \) for all \( i \).

The proof takes up the rest of this section, and makes essential use of Perling’s arguments. It is obtained as a combination of Corollaries 4.22, 4.24 and 4.26. Note that \( n \geq 3 \) by geometricity, since the rank of \( \text{NS}(G) \) is at least 1 as its signature is equal to \((1, n - 3)\). We will assume this inequality implicitly from now on.
4.2. The toric system associated with an exceptional collection. We start with a general important definition, which is a slight modification of Perling’s definition. The history, in fact, goes back to [15], where the authors associated a fan of a smooth toric surface to an exceptional collection of line bundles on any rational surface. The concept below is used to generalize this construction to exceptional collections consisting of objects of arbitrary nonzero ranks.

**Definition 4.7** (cf. [1], Definition 5.5). Let \((L, K)\) be a lattice of rank \(n - 2\) with a vector \(K \in L\). Let \(\lambda_{i,i+1} \in L \otimes \mathbb{Q}, 1 \leq i \leq n\), be a collection of \(n\) vectors. Put \(\lambda_{i+n,i+n+1} = \lambda_{i,i+1}\) for all \(i \in \mathbb{Z}\) and

\[
\lambda_{i,j} = \lambda_{i,i+1} + \cdots + \lambda_{j-1,j} \quad \text{for all } i < j < i + n.
\]

We say that the collection \(\lambda_{\bullet, \bullet}\) is a toric system in \(L\) if

1) there exist integers \(r_i \in \mathbb{Z}\) such that for all \(i \in \mathbb{Z}\) we have

\[
\lambda_{i-1,i} \cdot \lambda_{i,i+1} = \frac{1}{r_i^2};
\]

2) for all \(i < j < i + n - 2\) we have \(\lambda_{i-1,i} \cdot \lambda_{j,j+1} = 0\);

3) for all \(i < j < i + n\) the vectors \(r_i r_j \lambda_{i,j}\) are integral, hence their squares are integers, that is,

\[
 r_i r_j \lambda_{i,j} \in L \quad \text{and} \quad a_{i,j} := (r_i r_j \lambda_{i,j})^2 \in \mathbb{Z};
\]

4) for all \(i < j < i + n\) the ratio

\[
n_{i,j} := \frac{a_{i,j} + r_i^2 + r_j^2}{r_i r_j} \in \mathbb{Z}
\]

is an integer;

5) \(\lambda_{1,2} + \cdots + \lambda_{n,n+1} = -K\);

6) \(\gcd(r_1, \ldots, r_n) = 1\).

Note that if \(\lambda_{\bullet, \bullet}\) is a toric system, the integers \(r_i\) are determined up to sign, and whether the other conditions hold or not is independent of the choice of these signs. Furthermore, if the signs of \(r_i\) are fixed, the other integers \(a_{i,j}\) and \(n_{i,j}\) are determined unambiguously.

**Remark 4.8.** The main difference between Perling’s definition and Definition 4.7 is that we demand that the \(n_{i,j}\) are integers.

For the reader’s convenience we write down the Gram matrix of the scalar product in \(L \otimes \mathbb{Q}\) on the set \(\lambda_{i,i+1}\) for \(0 \leq i \leq n - 1\) (note, however, that this set is not
a basis in \( L \otimes \mathbb{Q} \) since \( n > n - 2 = \dim L \otimes \mathbb{Q} \), so the matrix below is degenerate:

\[
(\lambda_{i,i+1} \cdot \lambda_{j,j+1}) = \begin{pmatrix}
\frac{a_{0,1}}{r_0^2 r_1^2} & \frac{1}{r_1^2} & 0 & 0 & \ldots & 0 & \frac{1}{r_0^2} \\
\frac{1}{r_1^2} & \frac{a_{1,2}}{r_1^2 r_2^2} & \frac{1}{r_2^2} & 0 & \ldots & 0 & 0 \\
0 & \frac{1}{r_2^2} & \frac{a_{2,3}}{r_2^2 r_3^2} & \frac{1}{r_3^2} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \frac{a_{n-2,n-1}}{r_{n-2}^2 r_{n-1}^2} & \frac{1}{r_{n-1}^2} \\
\frac{1}{r_{n-1}^2} & 0 & 0 & 0 & \ldots & \frac{a_{n-1,n}}{r_{n-1}^2 r_n^2} & \frac{1}{r_n^2}
\end{pmatrix}
\] (11)

Note that the matrix is cyclically tridiagonal and symmetric (since \( r_n = r_0 \) by the periodicity of \( \lambda_{\bullet, \bullet} \)). The submatrix of (11) obtained by deleting the last row and last column is used extensively in [2].

Recall the map \( \lambda : \wedge^2 G \to p^\perp \) defined in (7). We implicitly compose it with the projection \( p^\perp \to p^\perp / p = \mathrm{NS}(G) \).

**Proposition 4.9.** Let \( e_1, \ldots, e_n \) be an exceptional basis in a surface-like pseudolattice \( G \) with \( r(e_i) \neq 0 \) for all \( i \). Extend it to an infinite sequence of vectors in \( G \) (called a helix) by setting \( e_{i+n} = S_G^{-1}(e_i) \) for all \( i \in \mathbb{Z} \), where \( S_G \) is the Serre operator. Then the collection of vectors

\[
\lambda_{i,i+1} := \frac{1}{r(e_i) r(e_{i+1})} \lambda(e_i, e_{i+1}) = \frac{e_{i+1}}{r(e_{i+1})} - \frac{e_i}{r(e_i)} \in \mathrm{NS}(G) \otimes \mathbb{Q}
\] (12)

(where \( 1 \leq i \leq n \)) is a toric system in \( (\mathrm{NS}(G), K_G) \) with \( r_i = r(e_i) \).

Note that the equality \( r_i = r(e_i) \) is one of the results of the proposition—we claim that the integers \( r_i \) determined from the toric sequence agree (up to sign, of course) with the ranks of the original vectors \( e_i \).

**Proof of Proposition 4.9.** Set \( r_i := r(e_i) \). Then the equalities in Definition 4.7, part 1), follow from Lemma 3.19, and those in Definition 4.7, part 2), follows from Lemma 3.20. Furthermore, we have

\[
\lambda_{i,j} = \lambda_{i,i+1} + \cdots + \lambda_{j-1,j} = \left( \frac{e_{i+1}}{r_{i+1}} - \frac{e_i}{r_i} \right) + \cdots + \left( \frac{e_j}{r_j} - \frac{e_{j-1}}{r_{j-1}} \right) = \frac{e_j}{r_j} - \frac{e_i}{r_i} = \frac{1}{r_i r_j} \lambda(e_i, e_j),
\]

hence the vector \( r_i r_j \lambda_{i,j} = \lambda(e_i, e_j) \) is integral; this proves Definition 4.7, part 3). By Lemma 3.18 we have

\[
n_{i,j} = \chi(e_i, e_j)
\]
provided that \( i < j < i + n \); this proves Definition 4.7, part 4). Furthermore,

\[
\sum_{i=1}^{n} \lambda_{i,i+1} = \frac{1}{r_1 r_{n+1}} \lambda(e_1, e_{n+1}) = \frac{1}{r_{n+1}^2} \lambda(S_G(e_{n+1}), e_{n+1})
\]

and, by Lemma 3.16, in \( NS(G) \) this is equal to \(-K_G\); this proves Definition 4.7, part 5). Finally, Definition 4.7, part 6), follows from the fact that the \( e_i \) form a basis in \( G \), and the rank map is surjective since \( G \) is unimodular.

**Remark 4.10.** Mutations of toric systems can also be defined in a way that is compatible with mutations of exceptional collections. However, we will not need this in our paper, so we skip the construction.

Below we discuss some properties of toric systems. We will always denote by \( r_i, a_{i,j} \) and \( n_{i,j} \) the integers determined by the toric system (for some choice of the signs of the \( r_i \)). We consider the index set of a toric system as \( \mathbb{Z}/n\mathbb{Z} \) with its natural cyclic order. We say that indices \( i \) and \( j \) are cyclically distinct if \( i - j \neq 0 \) (mod \( n \)) and are adjacent if \( i - j = \pm 1 \) (mod \( n \)).

**Lemma 4.11.** If \( \lambda_{\bullet, \bullet} \) is a toric system, then for any integer \( i \in \mathbb{Z} \) and any \( k, 0 \leq k \leq n - 2 \), the sequence of \( k \) vectors \( (\lambda_{i,i+1}, \ldots, \lambda_{i+k-1,i+k}) \) is linearly independent. In particular, \( \lambda_{i,i+1} \neq 0 \) for all \( i \).

**Proof.** We prove this by induction on \( k \). When \( k = 0 \) there is nothing to prove. Assume now that \( k \geq 1 \) and \( \lambda_{i+k,i+k+1} = x_1 \lambda_{i,i+1} + \cdots + x_{i+k-1} \lambda_{i+k-1,i+k} \). Consider the scalar product of this equality with \( \lambda_{i+k+1,i+k+2} \). The left-hand side equals \( 1/r_{i+k+1} \neq 0 \) by Definition 4.7, part 1), while the right hand side is zero by Definition 4.7, part 2). This contradiction proves the induction step.

The next lemma uses the signature assumption on the Néron-Severi lattice.

**Lemma 4.12.** Let \( \lambda_{\bullet, \bullet} \) be a toric system in a lattice \( L \) of signature \((1, n - 3)\). If both \( a_{i,i+1} \geq 0 \) and \( a_{j,j+1} \geq 0 \) (that is, \( \lambda_{i,i+1} \geq 0 \) and \( \lambda_{j,j+1}^2 \geq 0 \)) for cyclically distinct \( i \) and \( j \), then

- either \( i \) and \( j \) are adjacent;
- or \( n = 4, j = i + 2 \) (mod \( n \)), \( \lambda_{i,i+1} \) and \( \lambda_{j,j+1} \) are proportional, and finally, \( a_{i,i+1} = a_{j,j+1} = 0 \).

**Proof.** We can assume that \( i < j < i + n \). Suppose \( i \) and \( j \) are not adjacent. Then the intersection form on the sublattice of \( L_\mathbb{Q} \) spanned by \( \lambda_{i,i+1} \) and \( \lambda_{j,j+1} \) is

\[
\begin{pmatrix}
 a_{i,i+1} & 0 \\
 0 & a_{j,j+1}
\end{pmatrix},
\]

and hence is nonnegative definite. But by the signature assumption \( L_\mathbb{Q} \) does not contain nonnegative definite sublattices of rank 2, hence the vectors \( \lambda_{i,i+1} \) and \( \lambda_{j,j+1} \) are proportional. By Lemma 4.11 it follows that \( j - i \geq n - 2 \) and \( (i+n) - j \geq n - 2 \). Adding, we deduce that \( n \geq 2n - 4 \), hence \( n \leq 4 \). Since for \( n = 3 \) any two cyclically distinct \( i \) and \( j \) are adjacent, we conclude that \( n = 4 \), and nonadjacency means that \( j = i + 2 \). Finally, \( \lambda_{i,i+1} \cdot \lambda_{j,j+1} = 0 \) by Definition 4.7, part 2), and since the vectors are proportional and nonzero, we also have \( \lambda_{i,i+1}^2 = \lambda_{j,j+1}^2 = 0 \), hence \( a_{i,i+1} = a_{j,j+1} = 0 \).
4.3. The fan associated with a toric system. Following [1], §6, we associate a fan in $\mathbb{Z}^2$ with a toric system $\lambda_{\bullet, \bullet}$ in the Néron-Severi lattice $\text{NS}(G)$ of a surface-like pseudolattice $G$. As is customary in toric geometry, we consider a pair of mutually dual abelian groups

$$M \cong \mathbb{Z}^2 \quad \text{and} \quad N := M^\vee.$$ 

We define the map $M \to \mathbb{Z}^n$ to be the kernel of the map $\mathbb{Z}^n \to \text{NS}(G)_\mathbb{Q}$ defined by taking the base vectors to $\lambda_{i, i+1}$. Then we consider the dual map $(\mathbb{Z}^n)^\vee \to N$ and denote the images of the base vectors by $\ell_{i, i+1} \in N$. The definition of $\ell_{i, i+1}$ implies that

$$\sum_{i=1}^n \ell_{i, i+1} \otimes \lambda_{i, i+1} = 0 \quad \text{in} \quad N \otimes \text{NS}(G)_\mathbb{Q} \quad \text{and} \quad N \text{ is generated by } \ell_{i, i+1}. \quad (13)$$

Indeed, the toric system $(\lambda_{i, i+1})$ defines an element of

$$\text{NS}(G)_\mathbb{Q} \otimes (\mathbb{Z}^n)^\vee \cong \text{Hom}(\mathbb{Z}^n, \text{NS}(G)_\mathbb{Q}),$$

while the collection of vectors $(\ell_{i, i+1})$ defines an element of

$$\mathbb{Z}^n \otimes N \cong \text{Hom}(M, \mathbb{Z}^n),$$

and the sum in (13) is an expression for the composition of these maps.

Remark 4.13. Perling considers the fan in $N_\mathbb{R}$ generated by the vectors $\ell_{i, i+1}$, proves that it defines a projective toric surface (see [1], Proposition 10.7) and gives some nice results for it. For instance, he proves that its singularities are T-singularities (which gives a nice connection to Hacking’s results [16]), and relates mutations of exceptional collections and their toric systems (see Remark 4.10) to degenerations of toric surfaces. However, we shall not need this material and refer the interested reader to [1]. So, toric geometry will not be explicitly discussed below, but the experienced reader will notice it always lurking in the background.

The general tensor relation (13) implies many linear relations.

**Proposition 4.14.** For every $i \in \mathbb{Z}$

$$r_{i+1}^2 \ell_{i-1, i} + a_{i, i+1} \ell_{i, i+1} + r_i^2 \ell_{i+1, i+2} = 0. \quad (14)$$

**Proof.** Take the scalar product of the tensor relation (13) with $r_i^2 r_{i+1} \lambda_{i, i+1}$ and use Definition 4.7.

**Corollary 4.15.** For every $i \in \mathbb{Z}$ the vectors $\ell_{i-1, i}$ and $\ell_{i, i+1}$ are linearly independent.

**Proof.** Assume that $\ell_{i-1, i}$ and $\ell_{i, i+1}$ are linearly dependent. Then there exists a nonzero element $m \in M$ such that $m(\ell_{i-1, i}) = m(\ell_{i, i+1}) = 0$. Evaluating it on (13) we see that

$$m(\ell_{i+1, i+2})\lambda_{i+1, i+2} + \cdots + m(\ell_{i+n-2, i+n-1})\lambda_{i+n-2, i+n-1} = 0.$$
But by Lemma 4.11 the vectors $\lambda_{i+1,i+2}, \ldots, \lambda_{i+n-2,i+n-1}$ are linearly independent. It follows that $m(\ell_{j,j+1}) = 0$ for all $j$. This contradicts the fact that the $\ell_{j,j+1}$ generate $N$.

On the other hand, we have relations of a completely different sort. Let $\det: N \times N \to \mathbb{Z}$ denote a skew-symmetric bilinear form on $N$ which induces an isomorphism $\wedge^2 N \sim \to \mathbb{Z}$ (that is, a volume form).

**Proposition 4.16.** For every $i \in \mathbb{Z}$

$$
\det(\ell_{i,i+1},\ell_{i+1,i+2})\ell_{i-1,i} + \det(\ell_{i+1,i+2},\ell_{i-1,i})\ell_{i,i+1} + \det(\ell_{i-1,i},\ell_{i,i+1})\ell_{i+1,i+2} = 0. 
$$

(15)

**Proof.** This is standard linear algebra.

With an appropriate choice of the volume form, relations (14) almost coincide with (15).

**Proposition 4.17.** There exists a choice of the volume form $\det$ on $N$ and a positive integer $h \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$

$$
\det(\ell_{i-1,i},\ell_{i,i+1}) = hr_i^2 \quad \text{and} \quad \det(\ell_{i+1,i+2},\ell_{i-1,i}) = ha_i,i+1. 
$$

(16)

For this choice $\det(\ell_{i-1,i},\ell_{i,i+1}) > 0$ for all $i \in \mathbb{Z}$.

**Proof.** Now, by Corollary 4.15 the space of linear relations between the vectors $\ell_{i-1,i}$, $\ell_{i,i+1}$ and $\ell_{i+1,i+2}$ is one-dimensional. Since both relations (14) and (15) are nontrivial (the first because $r_i \neq 0$ and the second because $\det(\ell_{i-1,i},\ell_{i,i+1}) \neq 0$), they are proportional, hence for every $i \in \mathbb{Z}$ there exists unique $h_i \in \mathbb{Q} \setminus 0$ such that

$$
\det(\ell_{i-1,i},\ell_{i,i+1}) = h_i r_i^2, \\
\det(\ell_{i,i+1},\ell_{i+1,i+2}) = h_i r_{i+1}^2, \\
\det(\ell_{i+1,i+2},\ell_{i-1,i}) = h_i a_{i,i+1}.
$$

Comparing the relations for $i$ and $i+1$, we see that $h_i = h_{i+1}$. Hence $h_i = h$ for one nonzero rational number $h$, so (16) holds. Furthermore, since all the $r_i$ are mutually coprime (by Definition 4.7, part 6)), it follows that $h$ is a nonzero integer. Finally, changing the volume form on $N$ if necessary, we can assume that $h > 0$.

Now we will deduce some consequences about the geometry of vectors $\ell_{i,i+1}$ on the plane $N_\mathbb{R} \cong \mathbb{R}^2$. Consider the polygon defined as the convex hull of the vectors $\ell_{i,i+1}$:

$$
P := \text{Conv}(\ell_{1,2},\ell_{2,3},\ldots,\ell_{n,n+1}) \subset N_\mathbb{R}.
$$

**Lemma 4.18.** The point $0 \in N$ is contained in the interior of the polygon $P$.

**Proof.** If not, all the $\ell_{i,i+1}$ are contained in a closed half-plane of $N_\mathbb{R}$. On the other hand, by Proposition 4.17 we have $\det(\ell_{i-1,i},\ell_{i,i+1}) > 0$ for all $i \in \mathbb{Z}$, hence the oriented angle between the vectors $\ell_{i-1,i}$ and $\ell_{i,i+1}$ lies in the interval $(0,\pi)$. Evidently, a periodic sequence of vectors with this property cannot be contained in a half-plane. This contradiction proves the lemma.
We say that a vector $\ell_{i,i+1}$ is extremal if it is a vertex of the polygon $P$. In other words, if it does not lie in the convex hull of the other vectors.

**Corollary 4.19.** There are at least three extremal vectors among $\ell_{i,i+1}$.

**Proof.** The polygon $P$ has a nonempty interior by Lemma 4.18, hence it has at least three vertices.

### 4.4. The fan of a norm-minimal basis.

Throughout this section we assume that $\lambda_{*,*}$ is the toric system of a norm-minimal exceptional basis $e_{*}$ in a surface-like pseudolattice $G$, that is, the sum $\sum_{i=1}^{n} r_i^2$ of the squares of the ranks of the basis vectors is the least possible among all mutations of the basis.

**Lemma 4.20.** Assume that the basis is norm-minimal. Then for every $i \in \mathbb{Z}$

- if $a_{i,i+1} \geq 0$ then $a_{i,i+1} \geq |r_i^2 - r_{i+1}^2|$;
- if $a_{i,i+1} < 0$ then $a_{i,i+1} \leq -(r_i^2 + r_{i+1}^2)$.

**Proof.** We have $\chi(e_i, e_{i+1}) = n_{i,i+1}$ (see the proof of Proposition 4.9), hence by (2) the rank of the left mutation $\mathbb{L}_{e_i}(e_{i+1})$ of $e_{i+1}$ through $e_i$ equals

$$|r'| = |n_{i,i+1}r_i - r_{i+1}| = \frac{|a_{i,i+1} + r_i^2|}{|r_{i+1}|}.$$ 

Norm-minimality implies $|r'| \geq |r_{i+1}|$, that is,

$$|a_{i,i+1} + r_i^2| \geq r_{i+1}^2.$$ 

Analogously, considering the right mutation $\mathbb{R}_{e_{i+1}}(e_i)$, we deduce that

$$|a_{i,i+1} + r_{i+1}^2| \geq r_i^2.$$ 

Analyzing the case of nonnegative and of negative $a_{i,i+1}$, we easily deduce the required inequalities.

Note that in the proof we only use local norm-minimality of the basis, that is, the norm of the basis does not decrease only under elementary mutations.

**Lemma 4.21.** Assume that the basis is norm-minimal. If $a_{i,i+1} < 0$ then we have $\ell_{i,i+1} \in \text{Conv}(0, \ell_{i-1,i}, \ell_{i+1,i+2})$. In particular, $\ell_{i,i+1}$ is not extremal.

**Proof.** If $a_{i,i+1} < 0$ then relation (14) can be rewritten as

$$\ell_{i,i+1} = \frac{r_i^2}{|a_{i,i+1}|} \ell_{i-1,i} + \frac{r_{i+1}^2}{|a_{i,i+1}|} \ell_{i+1,i+2}.$$ 

By Lemma 4.20 we have $|a_{i,i+1}| \geq r_i^2 + r_{i+1}^2$, hence the coefficients on the right-hand side are nonnegative and their sum does not exceed 1. This proves the claim about the convex hull.

Now, since $0$ is in the interior of the polygon $P$ (Lemma 4.18), we see that $\text{Conv}(0, \ell_{i-1,i}, \ell_{i+1,i+2}) \subset P$, so the only possibility for $\ell_{i,i+1}$ to be extremal is if it coincides with one of the vertices of the triangle $\text{Conv}(0, \ell_{i-1,i}, \ell_{i+1,i+2})$. But this is impossible by Corollary 4.15.
Combining Lemmas 4.20 and 4.21 we see that there are at least three indices \( i \) such that \( a_{i,i+1} \geq 0 \). This already gives the required restriction on \( n \).

**Corollary 4.22.** If a geometric surface-like pseudolattice \( G \) has a norm-minimal exceptional basis that consists of elements of nonzero rank, then its rank is equal to \( n = 3 \) or \( n = 4 \).

**Proof.** Each vertex of the convex polygon \( P \) corresponds to an extremal vector \( \ell_{i,i+1} \), hence by Lemma 4.21 the corresponding integer \( a_{i,i+1} \) is nonnegative. Thus, if there is a pair of nonadjacent vertices \( i \) and \( j \) such that the vectors \( \ell_{i,i+1} \) and \( \ell_{j,j+1} \) are extremal, then \( n = 4 \) by Lemma 4.12. On the other hand, if all \( i \) such that \( \ell_{i,i+1} \) is extremal are pairwise adjacent, then clearly \( n = 3 \).

### 4.5. Norm-minimal bases for \( n = 3 \) and \( n = 4 \).

It remains to consider two cases. As before, we assume that \( e_\bullet \) is a norm-minimal exceptional basis of a geometric pseudolattice \( G \) of rank \( n \) consisting of elements of nonzero rank and that \( \lambda_{\bullet,\bullet} \) is its toric system, constructed in Proposition 4.9.

**Lemma 4.23.** If \( n = 3 \), then \( r_{i} = \pm 1 \), \( a_{i,i+1} = 1 \) and \( K_G^2 = 9 \).

**Proof.** Replacing \( e_i \) by \(-e_i\), we can assume that all the ranks are positive. Let \( H \) denote a generator of \( \text{NS}(G) \); by the unimodularity (Lemma 2.3) and geometricity of \( G \), we have \( H^2 = 1 \). By the definition of a toric system \( \lambda_{ij} = c_{ij}H \), \( c_{ij} \in \mathbb{Q} \). Accordingly, by Definition 4.7, part 1) we have

\[
\frac{c_{12}c_{31}}{r_1^2}, \quad \frac{c_{12}c_{23}}{r_2^2}, \quad \text{and} \quad \frac{c_{31}c_{23}}{r_3^2}.
\]

Solving this system for \( c_{ij} \), we see that

\[
c_{12} = \frac{r_3}{r_1r_2}, \quad c_{31} = \frac{r_2}{r_3r_1}, \quad \text{and} \quad c_{23} = \frac{r_1}{r_2r_3}
\]

up to a common sign, which can be fixed by replacing \( H \) with \(-H \) if necessary. By Definition 4.7, part 5)

\[
-K_G = \lambda_{12} + \lambda_{23} + \lambda_{31} = \left( \frac{r_3}{r_1r_2} + \frac{r_2}{r_3r_1} + \frac{r_1}{r_2r_3} \right) H = \frac{r_1^2 + r_2^2 + r_3^2}{r_1r_2r_3} H.
\]

Since \( G \) is unimodular, \( K_G \) is integral by Lemma 3.12, that is, there exists \( \gamma \in \mathbb{Z} \) such that \(-K_G = \gamma H \) and hence

\[
r_1^2 + r_2^2 + r_3^2 = \gamma r_1r_2r_3.
\]

Since all the integers \( r_i \) are positive, so is \( \gamma \). Moreover, \( \gcd(r_1, r_2, r_3) = 1 \) by Definition 4.7, part 6), and Proposition 4.9. But the only positive \( \gamma \) for which the above equation has an integral indivisible solution is \( \gamma = 3 \) (see [17], §2.1). Therefore, \( K_G^2 = \gamma^2 = 9 \).

Furthermore, in case \( \gamma = 3 \), equation (17) is the Markov equation, and its positive norm-minimal solution (with respect to the standard braid group action) is \( r_1 = r_2 = r_3 = 1 \). We obtain \( c_{i,i+1} = 1 \) and \( a_{i,i+1} = r_i^2r_{i+1}^2c_{i,i+1}^2 = 1 \).
Thus, the Néron-Severi lattice of a geometric surface-like pseudolattice with an exceptional basis of length \( n = 3 \) can be written in the form
\[
\text{NS}(G) = \mathbb{Z}H, \quad H^2 = 1, \quad K_G = -3H, \tag{18}
\]
and the toric system of a norm-minimal exceptional collection in \( G \) is
\[
\lambda_{1,2} = \lambda_{2,3} = \lambda_{1,3} = H.
\]
Since \( \chi(e_i, e_j) = n_{i,j} = (a_{i,j} + r_i^2 + r_j^2)/r_i r_j \) and \( a_{i,j} = (r_i r_j \lambda_{i,j})^2 \), we see that the Gram matrix of the form \( \chi \) in the basis \( e_i \) is equal to the form \( \chi_{\mathbb{P}^2} \) from Example 3.7.

**Corollary 4.24.** If a geometric surface-like pseudolattice \( G \) of rank \( n = 3 \) has an exceptional basis, then \( G \) is isometric to \( K_0^\mathrm{num}(\mathbb{P}^2) \) and its norm-minimal basis corresponds to the exceptional collection \((\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))\) in \( \mathbb{D}(\mathbb{P}^2) \).

Now we go over to the case \( n = 4 \). The results in this case are quite close to the results in [4].

**Lemma 4.25.** If \( n = 4 \), then \( r_i = \pm 1 \) and \( K_G^2 = 8 \).

**Proof.** As in the proof of Lemma 4.23, we can assume that all the ranks are positive. By Corollary 4.19 there exist at least three extremal vectors \( \ell_{i,i+1} \), and by Lemma 4.21 the corresponding integers \( a_{i,i+1} \) are nonnegative. After a cyclic shift of indices we can assume that
\[
a_{12}, a_{23}, a_{34} \geq 0.
\]
Applying Lemma 4.12 we conclude that \( a_{12} = a_{34} = 0 \), \( \lambda_{12} \) and \( \lambda_{34} \) are proportional, and \( \lambda_{12}^2 = \lambda_{34}^2 = 0 \). By Lemma 4.20
\[
 r_1^2 = r_2^2 \quad \text{and} \quad r_3^2 = r_4^2.
\]
Since a multiple of \( \lambda_{12} \) is integral, we have
\[
\lambda_{12} = c_{12} f \quad \text{and} \quad \lambda_{34} = c_{34} f, \quad \text{where} \quad f \in \text{NS}(G) \text{ is primitive with} \ f^2 = 0,
\]
with \( c_{12}, c_{34} \in \mathbb{Q} \). Since \( \text{NS}(G) \) is unimodular, there exists an element \( s \in \text{NS}(G) \) such that \((f, s)\) is a basis of \( \text{NS}(G) \) and
\[
 f \cdot s = 1 \quad \text{and} \quad d := s^2 \in \{0, -1\}.
\]
We have
\[
\lambda_{23} = c_{23} s + c'_{23} f \quad \text{and} \quad \lambda_{41} = c_{41} s + c'_{41} f.
\]
Here all the \( c \) and \( c' \) are rational numbers, and by Lemma 4.11 all the \( c \) are nonzero. We have
\[
c_{23} c_{34} = \lambda_{23} \cdot \lambda_{34} = \frac{1}{r_3^2} = \frac{1}{r_4^2} = \lambda_{34} \cdot \lambda_{41} = c_{34} c_{41},
\]
hence \( c_{23} = c_{41} \). Further,
\[
 -K_G = \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{41} = (c_{12} + c'_{23} + c_{34} + c'_{41}) f + 2c_{23}s.
\]
Since $K_G$ is characteristic and $f^2 = 0$, the intersection product $-K_G \cdot f = 2c_{23}$ is even and hence $c_{23} \in \mathbb{Z}$. On the other hand, $r_1r_2\lambda_{12} = r_2^2c_{12}f$ and $r_3r_4\lambda_{34} = r_3^2c_{34}f$ are integral, hence $r_2^2c_{12}$ and $r_3^2c_{34}$ are both integral. But
\[
\frac{1}{r_2^2} = \lambda_{12}\lambda_{23} = c_{12}c_{23} \quad \text{and} \quad \frac{1}{r_3^2} = \lambda_{23}\lambda_{34} = c_{23}c_{34},
\]
hence $(r_2^2c_{12})c_{23} = (r_3^2c_{34})c_{23} = 1$, and so (changing the signs of $f$ and $s$ if necessary) we can write
\[
c_{12} = \frac{1}{r_2^2}, \quad c_{34} = \frac{1}{r_3^2} \quad \text{and} \quad c_{23} = c_{41} = 1.
\]
Finally, from $0 = \lambda_{23} \cdot \lambda_{41} = d + c'_{23} + c'_{41}$ we can deduce that $c'_{23} + c'_{41} = -d \in \mathbb{Z}$, hence from the integrality of $K_G$ we obtain
\[
\frac{1}{r_2^2} + \frac{1}{r_3^2} = c_{12} + c_{34} \in \mathbb{Z},
\]
and it follows from this that $r_2^2 = r_3^2 = 1$. We finally see that all the $r_i$ are equal to 1, all $c_{i,i+1} = 1$ and $-K_G = (2 - d)f + 2s$, and hence $K_G^2 = 4(2 - d) + 4d = 8$.

Thus, the Néron-Severi lattice of a geometric surface-like pseudolattice of rank $n = 4$ with an exceptional basis can be written as
\[
\text{NS}(G) = \mathbb{Z}f \oplus \mathbb{Z}s, \quad f^2 = 0, \quad f \cdot s = 1, \quad s^2 = d, \quad K_G = (2 - d)f - 2s \quad (19)
\]
and $d \in \{0, -1\}$.

Assume first that $d = 0$. As we showed in the proof of Lemma 4.25, the toric system of a norm-minimal exceptional basis in $G$ has the form $\lambda_{12} = f$, $\lambda_{23} = s + c'f$, $\lambda_{34} = f$ for some $c' \in \mathbb{Z}$. This enables us to compute all the $a_{i,j}$ and $n_{i,j}$, and to show that the Gram matrix of the form $\chi$ is equal to the form $\chi_{\mathbb{P}^1 \times \mathbb{P}^1}$ in Example 3.7 with $c = c' + 1$.

Similarly, if $d = -1$ the toric system of a norm-minimal exceptional basis in $G$ must have the form $\lambda_{12} = f$, $\lambda_{23} = s + c'f$, $\lambda_{34} = f$ for some $c' \in \mathbb{Z}$, and computing the integers $n_{i,j}$ we check that $G$ is isometric to $K_0^{\text{num}}(D(\mathbb{F}_1))$, where $\mathbb{F}_1$ is the Hirzebruch surface; see Example 3.7. On the other hand, again by Example 3.7 the corresponding exceptional collection in $D(\mathbb{F}_1)$ can be transformed by mutations into a collection of objects of ranks $(1, 1, 1, 0)$, and therefore the original exceptional basis is not norm-minimal, and this case does not fit into our assumptions.

**Corollary 4.26.** If a geometric surface-like pseudolattice $G$ with $n = 4$ has a norm-minimal exceptional basis all of whose ranks are nonzero, then $G$ is isometric to $K_0^{\text{num}}(D(\mathbb{P}^1 \times \mathbb{P}^1))$. A norm-minimal exceptional basis in such a $G$ corresponds to one of the collections
\[
(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(c, 1), \mathcal{O}(c + 1, 1))
\]
in $D(\mathbb{P}^1 \times \mathbb{P}^1)$ for some $c \in \mathbb{Z}$.

A combination of Corollary 4.22, Corollary 4.24 and Corollary 4.26 proves Theorem 4.6.
§ 5. The minimal model program

Assume that $G$ is a surface-like pseudolattice.

5.1. Contraction. Let $e \in G$ be an exceptional element of zero rank. We consider the right orthogonal

$$G_e := \perp e = \{ v \in G \mid \chi(v, e) = 0 \} \subset G.$$  

Since $\chi(e, e) = 1$, we have a direct sum decomposition

$$G = Ze \oplus G_e.$$  \hspace{1cm} (20)

Note that $r(e) = 0$ means that $\chi(p, e) = 0$; hence $p \in G_e$ and also $e \in p^\perp$. Abusing notation, we also denote the projection of $e$ onto $\text{NS}(G) = p^\perp / p$ by $e$.

**Lemma 5.1.** The pseudolattice $G_e$ is surface-like and the element $p \in G_e$ is point-like. Moreover, the rank function on $G_e$ is the restriction of the rank function of $G$; there is an orthogonal direct sum decomposition

$$\text{NS}(G) = \text{NS}(G_e) \perp Ze$$  \hspace{1cm} (21)

and a relation between the canonical classes

$$K_G = K_{G_e} + (-K_G \cdot e) e.$$  \hspace{1cm} (22)

If $G$ is unimodular or geometric, then so is $G_e$.

**Proof.** Clearly, $p$ is primitive in $G_e$, $\chi(p, p) = 0$ and $\chi_-(p, -)$ is zero on $G_e$. Moreover, it is clear that the orthogonal of $p$ in $G_e$ is the intersection $p^\perp \cap G_e \subset G$, hence the form $\chi_-$ vanishes on it. This means that $G_e$ is surface-like with point-like element $p$.

Further, the direct sum decomposition (20) by restriction gives the direct sum

$$p^\perp = (p^\perp \cap G_e) \oplus Ze,$$

and then, taking the quotient with respect to $Zp$ gives the direct sum (21). Its summands are mutually orthogonal by definition.

Furthermore, equality (8) shows that the orthogonal projection of the canonical class $K_G$ onto $\text{NS}(G_e)$ is the canonical class for $G_e$. It fits into (22) since, by Lemma 3.21, we have $e^2 = -1$ as $e$ is exceptional of zero rank. Finally, the unimodularity of $G_e$ is clear, and geometricity follows from (21) and (22).

**Lemma 5.2.** If $G$ is a surface-like pseudolattice, then there is an exceptional sequence $e_1, \ldots, e_k$ of rank zero elements such that the iterated contraction $G_{e_1, \ldots, e_k}$ is a minimal surface-like pseudolattice.

The proof evidently follows by induction on the rank of $G$.

The pseudolattice obtained from $G$ by the iterated contraction from Lemma 5.2 is called a **minimal model** of $G$. As we have seen above, minimal geometric surface-like pseudolattices admitting an exceptional basis are isometric to numerical Grothendieck groups of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, so Lemma 5.2 can be thought of as a categorical minimal model program for surface-like pseudolattices of this type.
5.2. The defect. In this section we define the defect of a lattice and of a surface-like pseudolattice.

**Definition 5.3.** Let $L$ be a lattice with a vector $K$. The **defect** of $(L, K)$ is defined as the integer

$$\delta(L, K) := K^2 + \text{rk } L - 10.$$ 

If $G$ is a unimodular surface-like pseudolattice, we define its defect as

$$\delta(G) := \delta(\text{NS}(G), K_G).$$

**Remark 5.4.** It follows from Van der Blij’s Lemma (see [18], Lemma II.5.2) that if $K$ is characteristic then $K^2$ is congruent to the signature of the form modulo 8. This implies that the defect of a geometric surface-like pseudolattice is divisible by 8.

It is easy to see that the defect is zero for numerical Grothendieck groups of surfaces with zero irregularity and geometric genus.

**Lemma 5.5.** Let $X$ be a smooth projective surface over an algebraically closed field of zero characteristic with $q(X) = p_g(X) = 0$. The corresponding pseudolattice $G = \text{K_0^{num}(D(X))}$ has zero defect $\delta(G) = 0$.

**Proof.** Since $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, we have $\text{rk } \text{NS}(X) = e(X) - 2$, where $e(X)$ is the topological Euler characteristic of $X$, and the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$ equals 1. By Noether’s formula we have $e(X) = 12\chi(\mathcal{O}_X) - K^2_X$, hence $\text{rk } \text{NS}(G) = 10 - K^2_G$, hence $\delta(G) = 0$.

In general, the defect of a surface-like pseudolattice can be either positive or negative.

**Example 5.6.** Let $X \rightarrow C$ be a ruled surface over a genus $g$ curve $C$ (we are still assuming that the base field is algebraically closed of zero characteristic) and let $G = \text{K_0^{num}(D(X))}$. Then $\text{rk } \text{NS}(X) = 2$, while $K^2_X = 8(1 - g)$, and therefore $\delta(G) = 8(1 - g) + 2 - 10 = -8g$.

Assume further that $X$ has a section $i: C \rightarrow X$ with normal bundle of degree $-1$ (for instance, such a section exists if $X$ is the projectivization of a direct sum $\mathcal{O}_C \oplus \mathcal{L}$, where $\text{deg } \mathcal{L} = -1$). Set $e$ to be the class of the sheaf $i_* \mathcal{O}_C$ in the numerical Grothendieck group $\text{K_0^{num}(D(X))}$. It is then numerically exceptional of rank 0, and $K_X \cdot e = 2g - 1$. Consequently, by Lemma 5.8 below, the contraction $G_e$ has defect

$$\delta(G_e) = \delta(G) - (1 - (K_X \cdot e)^2) = -8g - (1 - (2g - 1)^2) = 4g(g - 3).$$

In particular, it is negative for $g = 1$ and $g = 2$, zero for $g = 0$ and $g = 3$ and positive for $g > 3$.

As a result of the classification in Theorem 4.6 (more precisely, see Lemma 4.23 and Lemma 4.25) we have the following.

**Corollary 5.7.** If $G$ is a minimal geometric surface-like pseudolattice admitting an exceptional basis, then $\delta(G) = 0$. 
An important property of the defect of geometric pseudolattices is that it does not decrease under contractions.

**Lemma 5.8.** Assume that $G$ is a surface-like pseudolattice and $e \in G$ is an exceptional vector of zero rank. Then

$$\delta(G) = \delta(G_e) + (1 - (K_G \cdot e)^2).$$

(23)

In particular, if $G$ is geometric, then $\delta(G) \leq \delta(G_e)$ and this becomes equality if and only if $K_G \cdot e = \pm 1$.

**Proof.** Equality (23) easily follows from Lemma 5.1. Furthermore, $K_G \cdot e \equiv e^2 \pmod{2}$ since $K_G$ is characteristic, so $K_G \cdot e$ is an odd integer, hence the second summand on the right-hand side of (23) is nonpositive. This proves the required inequality for defects.

### 5.3. Exceptional bases in geometric surface-like pseudolattices.

Combining the minimal model program of §5.1 with the classification result in Theorem 4.6 we get the following results.

**Theorem 5.9** (cf. [1], Corollary 10.8). Let $G$ be a geometric surface-like pseudolattice. Any exceptional basis in $G$ can be transformed by mutations into an exceptional basis consisting of three or four elements of rank 1 and all other elements of rank 0.

**Proof.** First, we mutate the exceptional basis to a norm-minimal basis $e_1, \ldots, e_n$. Next, we apply a sequence of right mutations to this collection according to the following rule: if a pair $(e_i, e_{i+1})$ is such that $r(e_i) \neq 0$ and $r(e_{i+1}) = 0$, we mutate it to $(e_{i+1}, \Re_{e_{i+1}} e_i)$. Then $r(\Re_{e_{i+1}} e_i) = r(e_i)$ by (2), so the new collection is still norm-minimal. It is also clear that after a number of such operations we will have the following property: there exists $k$, $1 \leq k \leq n$, such that

$$r(e_1) = \cdots = r(e_k) = 0 \quad \text{and} \quad r(e_i) \neq 0 \text{ for } i > k.$$ 

(24)

We consider the iterated contraction $G' = G_{e_1, \ldots, e_k}$ of $G$. Clearly, $e_{k+1}, \ldots, e_n$ is an exceptional basis in $G'$. Furthermore, it is norm-minimal. Indeed, if there exists a mutation in $G'$ decreasing the norm of the collection, then the same mutation in $G$ would also decrease the norm in the same way. Since the ranks of all elements in the collection of $G'$ are nonzero and $G'$ is geometric by Lemma 5.1, we conclude from Theorem 4.6 that the ranks of the elements $e_{k+1}, \ldots, e_n$ are $\pm 1$ and $n - k$ is equal to 3 or 4. Changing the signs of elements with negative ranks we deduce that the collection we have obtained in $G$ consists of elements of rank 0 and 1 only.

**Theorem 5.10** (cf. [1], Theorem 10.9). Let $G$ be a geometric surface-like pseudolattice with $\delta(G) = 0$. Any exceptional basis in $G$ can be transformed by mutations into an exceptional basis consisting of elements of rank 1.

**Proof.** By Theorem 5.9 there exists an exceptional basis in $G$ satisfying (24) with $r(e_i) = 1$ for $i > k$. Note also that the pseudolattice $G' = G_{e_1, \ldots, e_k}$ has zero defect
by Theorem 4.6. Since the defect does not change under the above contractions, we deduce from Lemma 5.8 that $K_G \cdot e_i = \pm 1$ for all $1 \leq i \leq k$.

We apply a sequence of right mutations to this basis according to the following rule: if a pair $(e_i, e_{i+1})$ is such that $r(e_i) = 0$ and $r(e_{i+1}) = 1$, we mutate it to $(e_{i+1}, e_i)$. We claim that $r(R_{e_{i+1}} e_i) = \pm 1$. In fact, by (8) we have

$$\chi(e_i, e_{i+1}) = \chi_-(e_i, e_{i+1}) = -K_G \cdot \lambda(e_i, e_{i+1}).$$

By the assumptions on the rank and (7) we have $\lambda(e_i, e_{i+1}) = -e_i$. It follows that $\chi(e_i, e_{i+1}) = K_G \cdot e_i = \pm 1$. Therefore, by (2)

$$r(R_{e_{i+1}} e_i) = \chi(e_i, e_{i+1}) r(e_{i+1}) - r(e_i) = \pm 1.$$

If the rank is $-1$, we change the sign of the element to make the rank equal 1.

Clearly, after a finite number of such mutations we obtain an exceptional basis consisting of rank-1 elements only.

5.4. A criterion for the existence of an exceptional basis. It is easy to show that the criterion for the existence of a numerical exceptional collection in the derived category of a surface proved by Vial in [2] also works for surface-like pseudolattices. We start with a simple lemma.

**Lemma 5.11.** Let $(G, \chi)$ be a pseudolattice with integral and characteristic canonical class $K_G$. If $r(v_1) = r(v_2) = 1$, then $\chi(v_1, v_1) \equiv \chi(v_2, v_2) \pmod{2}$. Moreover, if $\chi(v, v)$ is odd for some $v \in G$ with $r(v) = 1$, then $\chi(v, v) = 1$ for an appropriate choice of $v$.

**Proof.** We have $v_2 = v_1 + D$, where $r(D) = 0$. Therefore,

$$\chi(v_2, v_2) = \chi(v_1, v_1) + \chi(v_1, D) + \chi(D, v_1) + \chi(D, D).$$

Furthermore, by (8) we have

$$\chi(v_1, D) + \chi(D, v_1) \equiv K_G \cdot \lambda(v_1, D) = K_G \cdot D \pmod{2},$$

and by Lemma 3.11 we have $\chi(D, D) \equiv D^2 \pmod{2}$. Therefore,

$$\chi(v_2, v_2) - \chi(v_1, v_1) \equiv K_G \cdot D + D^2 \equiv 0 \pmod{2}$$

since $K_G$ is characteristic.

For the second part note that $\chi(v, p) = \chi(p, v) = r(v) = 1$; hence

$$\chi(v + tp, v + tp) = \chi(v, v) + 2t,$$

so if $\chi(v, v)$ is odd, an appropriate choice of $t$ ensures that $\chi(v + tp, v + tp) = 1$.

The condition that $\chi(v, v)$ is odd for a rank-1 vector $v \in G$ is thus independent of the choice of $v$ and is equivalent to the existence of a rank-1 vector $v$ with $\chi(v, v) = 1$. If this holds, we say that $(G, \chi)$ represents 1 by a rank-1 vector. This is a pseudolattice analogue of Vial’s condition $\chi(\theta X) = 1$ for a surface $X$. In terms of the matrix representation (3) of a pseudolattice this condition can be rephrased as $a \equiv 1 \pmod{2}$. 

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Theorem 5.12 (cf. [2], Theorem 3.1). Suppose that \((G, \chi)\) is a unimodular geometric pseudolattice of rank \(n \geq 3\) and zero defect such that \((G, \chi)\) represents 1 by a rank-1 vector. Then the following three statements hold:

1) \(n = 3\) and \(K_G = -3H\) for some \(H \in \text{NS}(G)\) if and only if \(G\) is isometric to \(K_0^{\text{num}}(\mathcal{D}(\mathbb{P}^2))\);

2) \(n = 4\), \(\text{NS}(G)\) is even and \(K_G = -2H\) for some \(H \in \text{NS}(G)\) if and only if \(G\) is isometric to \(K_0^{\text{num}}(\mathcal{D}(\mathbb{P}^1 \times \mathbb{P}^1))\);

3) \(n \geq 4\), \(\text{NS}(G)\) is odd and \(K_G\) is primitive if and only if \(G\) is isometric to \(K_0^{\text{num}}(\mathcal{D}(X_{n-3}))\), where \(X_{n-3}\) is the blowup of \(\mathbb{P}^2\) in \(n - 3\) points.

Furthermore, a geometric surface-like pseudolattice \(G\) with zero defect has an exceptional basis if and only if it represents 1 by a rank-1 vector and one of the possibilities 1)–3) is satisfied.

Proof. The fact that for the surfaces \(\mathbb{P}^2\), \(\mathbb{P}^1 \times \mathbb{P}^1\) and \(X_{n-3}\) (with the standard surface-like structure) the numerical Grothendieck group has the properties listed in 1)–3) is evident. Now we check the converse.

First, assume that \(n = 3\) and \(K_G = -3H\). By the zero defect assumption we have \(9H^2 = K_G^2 = 12 - 3 = 9\), hence, using the signature assumption we see that \(H^2 = 1\) and \(H\) is primitive. By Lemma 5.11 and Remark 4.4, for an appropriate choice of the vector \(v_0 \in G\), in the basis \((v_0, H, p)\) the matrix of \(\chi\) has the form

\[
\chi = \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

The above matrix is equal to the Gram matrix of \(\chi\) in \(K_0^{\text{num}}(\mathcal{D}(\mathbb{P}^2))\) in the basis \((\mathcal{O}(-2H), \mathcal{O}_L, \mathcal{O}_P)\), where \(L\) is a line and \(P\) is a point. Hence \(G\) is isometric to \(K_0^{\text{num}}(\mathcal{D}(\mathbb{P}^2))\).

Next assume that \(n = 4\), \(\text{NS}(G)\) is even and \(K_G = -2H\). In view of the zero defect we have \(4H^2 = K_G^2 = 12 - 4 = 8\), hence \(H^2 = 2\) and \(H\) is primitive. By the parity and signature assumption \(\text{NS}(G)\) is a hyperbolic lattice, hence we can write \(H = H_1 + H_2\), where \((H_1, H_2)\) is the standard hyperbolic basis. By Lemma 5.11 and Remark 4.4, for an appropriate choice of \(v_0 \in G\) in the basis \((v_0, H_1, H_2, p)\) the matrix of \(\chi\) has the form

\[
\chi = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]

The above matrix is equal to the Gram matrix of \(\chi\) in \(K_0^{\text{num}}(\mathcal{D}(\mathbb{P}^1 \times \mathbb{P}^1))\) in the basis \((\mathcal{O}(-H_1 - H_2), \mathcal{O}_{L_1}, \mathcal{O}_{L_2}, \mathcal{O}_P)\), where \(L_1\) and \(L_2\) are two rulings and \(P\) is a point, hence \(G\) is isometric to \(K_0^{\text{num}}(\mathcal{D}(\mathbb{P}^1 \times \mathbb{P}^1))\).

Finally, assume that \(n \geq 4\), \(\text{NS}(G)\) is odd and \(K_G\) is primitive. Then, by [2], Proposition A.12, there exists a basis \(e_1, \ldots, e_{n-3}, H\) in \(\text{NS}(G)\) such that

\[
e_i \cdot e_j = -\delta_{ij}, \quad H^2 = 1, \quad e_i \cdot H = 0 \quad \text{and} \quad K_G = -3H + \sum e_i.
\]
By Lemma 5.11 and Remark 4.4, for an appropriate choice of $v_0 \in G$, in the basis $(v_0, H, e_1, \ldots, e_{n-3}, p)$ the matrix of $\chi$ has the form

$$\chi = \begin{pmatrix} 1 & 3 & 1 & \ldots & 1 & 1 \\ 0 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$ 

This matrix is equal to the Gram matrix of $\chi$ in $\mathbb{K}_{0}^{\text{num}}(D(X_{n-3}))$ in the basis

$$(\mathcal{O}(-2H), \mathcal{O}_{L}, \mathcal{O}_{E_1}, \ldots, \mathcal{O}_{E_{n-3}}, \mathcal{O}_{p}),$$

where $L$ is the preimage of a general line on $\mathbb{P}^2$ and $E_1, \ldots, E_{n-3}$ are the exceptional divisors, so $G$ is isometric to $\mathbb{K}_{0}^{\text{num}}(D(X_{n-3}))$.

Now we prove the second part of the theorem.

Assume that $G$ has an exceptional basis. If $G$ is minimal, then we know from Theorem 4.6 that $G$ is isometric to $D(\mathbb{P}^2)$ or $D(\mathbb{P}^1 \times \mathbb{P}^1)$, hence either 1) or 2) holds. If $G$ is not minimal, then by Theorem 5.9 the exceptional basis can be transformed by mutations into a norm-minimal exceptional collection such that $r(e_i) = 0$ for $i \leq k$ and $r(e_i) = 1$ for $i \geq k + 1$ and $n - k \leq 4$, $k \geq 1$. We have $\chi(e_n, e_n) = 1$, hence $\chi$ represents 1 by a rank-1 vector. Moreover, $e_2^2 = -1$, hence NS$(G)$ is odd, and as we showed in the proof of Theorem 5.10, we have $K_G \cdot e_1 = \pm 1$; hence $K_G$ is primitive. Thus 3) holds.

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