INITIAL IDEALS OF BOREL TYPE

Fabrizio Brienza, Anna Guerrieri

Abstract

In this paper we use some results related to regularity, Betti numbers and reduction of generic initial ideals, showing their stability in passing from an ideal to its initial ideal if the last has some simple properties.

1 Introduction

Throughout the paper $R = K[x_1, \ldots, x_n]$ is the polynomial ring in $n$ variables over an infinite field $K$, $<$ a monomial order on $R$ with $x_1 > x_2 > \cdots > x_n$ and $M$ a graded $R$–module. It is well know that for a graded ideal $I \subseteq R$ (an ideal generated by homogeneous elements) there exists a nonempty open set $U$ of linear automorphisms of $R$ such that $\text{in}_<(\alpha I)$ does not depend on $\alpha \in U$. The resulting initial ideal, $\text{gin}_<(I)$ is called the generic initial ideal of $I$ with respect to $<$. Generic initial ideals are Borel-fixed [3, Theorem 2.8]
and are even strongly stable if the base field is of characteristic 0 [see [3]]. Passing to the generic initial ideal with reverse lexicographic order preserves the extremal Betti numbers [2, Theorem 1.6] and the reduction number [18, Theorem 4.3]. However it is difficult to compute $g_{in}$, because one does not have much information about the open subset $U$, besides the fact that it is dense in $K^m$ in the standard topology ($m = n^2$) and therefore hard to avoid. Thus if we pick $x \in K^m$ randomly, i.e. “generically enough”, then most likely $x$ will belong to $U$ and this is how most computer algebra systems compute $g_{in}(I)$. An uncertainty though remains. In the wake of the works of Bermejo and Gimenez [1], Conca, Herzog and Hibi [10], and Trung [18], we avoid $g_{in}$s to show the same results on numerical invariants for Borel type ideals. Bayer and Stillman in [3] prove that ideals Borel-fixed are of Borel type [for the definition see Section 2] even though the converse is clearly not true. A monomial ideal is of Borel type if and only if all the annihilator modules associated to the sequence $\{x_n, x_{n-1}, \ldots, x_1\}$ are zero dimensional [Proposition 2.13]. Using this fact we prove that the extremal Betti numbers [Theorem 3.6] and the reduction number with respect the sequence $\{x_n, \ldots, x_{n-d+1}\}$, where $d = \dim(R/I)$ [see Section 4], are preserved in passing to the initial ideal if the latter is of Borel type. We also observe that the annihilator numbers are preserved [Corollary 3.2]. The annihilator numbers of a filter regular sequence are intimately related to the extremal Betti numbers, in the sense that the two diagrams are specular; in this way one attains also the information about extremal Betti numbers. We choose this approach because the annihilator numbers are easy to compute, since they are in fact colons. The present work is divided in four sections. In the first we recall some basic properties related to Borel-type ideals and to the annihilator numbers of a filter regular sequence. In the second section we prove
that the extremal Betti numbers and the annihilator numbers of $I$ and $\text{in}_<(I)$ are equal in the case $\text{in}_<(I)$ is a Borel-type ideal. Then we study the rigidity of resolutions of $I$ and $\text{in}_<(I)$, if $\text{in}_<(I)$ is of Borel-type and we show that, if $I$ is an ideal with an initial ideal of Borel-type, we don’t have necessarily the rigidity of the resolution. In the third section we see that also the reduction numbers of $I$ and $\text{in}_<(I)$ with respect the sequence $\{x_n, \ldots , x_{n-d+1}\}$ are the same, if the last ideal is of Borel-type. In the last section we compare the Borel-type ideals to the quasi stable ideals of Hashemi, Schweinfurter and Seiler. Recently Hashemi, Schweinfurter and Seiler in [13] proved, using the Pommaret bases (a special class of Gröbner basis with additional combinatorial properties), that, if $I$ has a finite Pommaret basis, $I$ and $\text{in}(I)$ have the same extremal Betti numbers. When $I$ has a finite Pommaret basis, it is called a quasi-stable ideal. It is possible to prove that monomial quasi-stable ideals are mirror images of Borel type ideals. It is however possible to give examples of monomial quasi-stable ideals that are not Borel type and viceversa, even though a suitable change of variables transforms the one in the others. Hashemi, Schweinfurter and Seiler also prove that given $I$ a polynomial ideal, there always exists a change of variables such that $I$ has a finite Pommaret basis. These variables are called $\delta$–regular. From their result and the observation above, it follows that after a suitable change of variables, $\text{in}(I)$ becomes an ideal of Borel type. It is now clear that, if $I$ has an initial ideal of Borel type, after a suitable change of basis, it means that $I$ has a finite Pommaret basis.
2 Preliminary notions and Borel type ideals

Given the assumptions described in the introduction, we recall some basic notions. Let
\[ \cdots \longrightarrow F_i = \bigoplus_{j \geq 0} R(-j)^{\beta_{ij}} \longrightarrow \cdots \longrightarrow F_0 = \bigoplus_{j \geq 0} R(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0, \]
be the minimal free resolution of \( M \), where \( \beta_{ij}(M) \geq 0 \) is the rank of the shift \(-j\) in \( i\)-th position. The minimum length of such a free resolution is called the projective dimension of \( M \) over \( R \) and it is written \( \text{pd}(M) \).

\( \beta_{ij}(M) \), for short \( \beta_{ij} \), are called the Betti graded numbers of \( M \). Betti numbers have been widely investigated and for the general theory we refer to [6]. It is well known, [see [5]], that \( \beta_{ij} = \dim_k \text{Tor}^R_i(M, k) = \dim_k H_i(F \otimes k) \),
and \( \text{pd}(M) = \max \{ i : \beta_{ij} \neq 0 \text{ for some } j \} \).

**Definition 2.1.** Let \( a_j \) be the maximum degree of the generators of \( F_j \).
Then \( \text{reg}(M) = \max \{ a_j - j : j > 0 \} \) is called the Castenuovo-Mumford regularity, or simply regularity, of \( M \).

The regularity is an important invariant which measures the complexity of the given module; for the theory of regularity see [11]. It is well known the connection with Betti numbers, in fact
\[ \text{reg}(M) = \max \{ j : \beta_{i,i+j}(M) \neq 0 \text{ for some } i \}. \]

Let now \( \mathfrak{m} \) be the maximal graded ideal of \( R \). Suppose that \( M \) is finitely generated, then we denote by \( H^i_{\mathfrak{m}}(M) \) the \( i\)-th local cohomology of \( M \) with respect to \( \mathfrak{m} \) [see [15]]. Since \( H^i_{\mathfrak{m}}(M) \) is artinian, we may consider the numbers \( a_i(M) = \max \{ n : (H^i_{\mathfrak{m}}(M))_n \neq 0 \} \) assuming that \( a_i(M) = -\infty \), if \( H^i_{\mathfrak{m}}(M) = 0 \). The Castelnuovo-Mumford regularity of \( M \) can be defined also as
\[ \text{reg}(M) = \max \{ a_i(M) + i : i \geq 0 \}. \]
Moreover the largest non-vanishing degree of local cohomology modules is defined as the number 
\[ a^*(M) = \max\{a_i(M) : i \geq 0\}. \]

The Castelnuovo-Mumford regularity and the largest non-vanishing degree for local cohomology modules can be viewed as special cases of the more general invariants:

\[ \text{reg}_t(M) = \max\{a_i(M) + i : i \leq t\}, \]

\[ a^*_t(M) = \max\{a_i(M) : i \leq t\}, \]

where \( t \in \{0, \ldots, d\} \), where \( d = \dim(M) \). These invariants have been studied in \[19\], \[20\], \[21\].

Now let \( I \subseteq R \) be a graded ideal. We define the ideals 
\[ I : (x_1, \ldots, x_i)^\infty = \bigcup_{k \geq 0} I : (x_1, \ldots, x_i)^k, \]

for \( i = 1, \ldots, n \). By abuse of notation \( I : x_j^\infty = I : (x_j)^\infty \).

Accordingly with \[15\], we give the following definition.

**Definition 2.2.** The ideal \( I \) is said to be of **Borel type**, or a weakly stable ideal, if \( I : x_j^\infty = I : (x_1, \ldots, x_j)^\infty \), for all \( j = 1, \ldots, n \).

We recall that an ideal \( I \subseteq R \) is said to be Borel-fixed if \( \alpha(I) = I \) for all \( \alpha \in B \), where \( B \) is the Borel subgroup of \( \text{GL}_n(K) \), that is the subgroup of all non-singular upper triangular matrices.

**Remark 2.3.** Let \( I \) be an ideal of \( R \). An example of ideal of Borel type is the generic initial ideal of \( I \), \( \text{gin}(I) \). This depends on the fact that \( \text{gin}(I) \) is Borel-fixed as showed in \[3\].

**Proposition 2.4.** \[15\], Proposition 4.2.9] Let \( I \subseteq R \) be a graded monomial ideal, that is an ideal generated by monomials. The following conditions are equivalent:
1. $I$ is of Borel type;

2. for each monomial $u \in I$ and all integers $i, j, s$ with $1 \leq j < i \leq n$ and $s > 0$ such that $x_i^s | u$, there exists an integer $t \geq 0$ such that $x_j^t (u/x_i^s) \in I$;

3. for each monomial $u \in I$ and all integers $i, j$ with $1 \leq j < i \leq n$, there exists an integer $t \geq 0$ such that $x_j^t (u/x_i^{\nu_i(u)}) \in I$, where $\nu_i(u)$ is the highest power of $x_i$ which divides $u$;

4. if $p \in \text{Ass}(R/I)$, then $p = (x_1, \ldots, x_j)$ for some $j$.

We show a class of ideals whose initial ideals are of Borel type in any characteristic [see [7]]:

**Example 2.5.** Let $I = ((ax + by)^2, (cx + dy)^2)$ be ideals in $K[x, y]$ with $a, b, c, d \in K^\times$. Let $x > y$ a monomial order. Then, using CoCoA, one can see that

\[
\text{in}_{<}(I) = \begin{cases} 
(x^2, xy, y^3) & \text{if char}(K) \neq 2 \text{ and } ab \neq 0; \\
(x^2, y^2) & \text{if char}(K) = 2.
\end{cases}
\]

We can easily see that $\text{in}_{<}(I)$ is Borel type in both cases.

**Definition 2.6.** Let $I \subseteq R$ be a monomial ideal. Then $I$ is strongly stable if one has $x_i(u/x_j) \in I$ for all monomials $u \in I$ and all $i < j$ such that $x_j$ divides $u$.

**Remark 2.7.** It easily follows that a strongly stable monomial ideal is always of Borel type, moreover a strongly stable ideal is Borel-fixed. Furthermore in a characteristic zero field an ideal is Borel-fixed if and only if it is strongly stable. [[15], Proposition 4.2.4].
We use this remark to give an example of an ideal that is of Borel type but not Borel-fixed.

**Example 2.8.** Let \( I = (x_1^3, x_1 x_2, x_1 x_2^2, x_1 x_3^2) \) be an ideal in \( K[x_1, x_2, x_3] \) with \( \text{char}(K) = 0 \). Since \( x_3 | x_1 x_3^2 \) but \( x_1 x_2 x_3 \notin I \), we know that \( I \) is not Borel-fixed, since it is not a strongly stable ideal. Checking the condition (3) of Proposition 2.4, we show that \( I \) is of Borel type. Since \( x_2 | x_1 x_2^3 \) and \( \nu_2(x_1 x_2^3) = 2 \) we have to show that there exists an integer \( t \geq 0 \) such that \( x_1^{t+1} \in I \). It is sufficient to pick \( t = 2 \). For the next generator we have that \( x_2 | x_1 x_2^3 \) with \( \nu_2(x_1 x_2^3) = 1 \) and so for \( t = 0 \) we know that \( x_1^{t+3} \in I \). For the last generator \( x_1 | x_1 x_3^2 \) and \( x_3 | x_1 x_3^2 \) with \( \nu_1(x_1 x_3^2) = 1 \) and \( \nu_3(x_1 x_3^2) = 2 \). If we pick \( t \geq 2 \) we have that \( x_1 x_2^t \in I \) and \( x_1^{t+1} \in I \). Hence \( I \) is of Borel type.

**Definition 2.9.** Let \( M \) be an \( R \)-module. An element \( y \in R_1 \) is said to be **filter regular** on \( M \) if the multiplication map \( y : M_{i-1} \to M_i \) is injective for all \( i \gg 0 \). The elements \( y_1, \ldots, y_r \in R_1 \) form a **filter regular sequence** on \( M \), if \( y_i \) is filter regular on \( M/(y_1, \ldots, y_{i-1})M \), for all \( i = 1, \ldots, r \).

**Remark 2.10.** Immediately follows that, \( y \in R_1 \) is filter regular on \( M \), if and only if the ideal \( (0 :_M y) \) has finite length. Herzög and Hibi in [15] show that if \( |K| = \infty \) always exists a \( K \)-basis of \( R_1 \) that is a filter regular sequence on \( M \).

**Definition 2.11.** Let \( M \) be a finitely generated graded \( R \)-module. Let \( y = y_1, \ldots, y_n \) elements in \( R_1 \). We denote by \( A_i(y; M) \) the graded \( R \)-module \( (0 :_{M/(y_1, \ldots, y_{i-1})M} y_i) \). The numbers \( \alpha_{ij}(y; M) = \dim_K A_i(y; M)_j \) are the **annihilator numbers** of \( M \) with respect to the sequence \( y \).

Clearly, if \( y \) is a filter regular sequence on \( M \), then for each \( i \) one has that \( \alpha_{ij}(y; M) \) are equal to zero for almost all \( j \).
Proposition 2.12. [15, Proposition 4.3.5] Let $M$ be a finitely generated graded $R$-module and $y$ a sequence of elements in $R_1$. The following conditions are equivalent:

1. $y$ is a filter regular sequence on $M$;
2. $H_j(y_1, \ldots, y_i; M)$ has finite length for all $j > 0$ and all $i$;
3. $H_1(y_1, \ldots, y_i; M)$ has finite length for all $i$.

Here $H_j(y_1, \ldots, y_i; M)$ denotes the $j$-th homology module of the Koszul complex $K.(y_1, \ldots, y_i; M)$.

In the case $M = R/I$ and the sequence $x = x_n, x_{n-1}, \ldots, x_1$ one may define some useful annihilator modules. Let $I_0 = I$ and $I_i = I_{i-1} + (x_{n-i+1})$ for all $i \in \{1, \ldots, n\}$, one defines

$$a_x^i(I) = \frac{I_{i-1} : (x_{n-i+1})}{I_{i-1}}.$$

We remark that $a_x^i(I)$ have finite length for all $i$ if and only if $x$ is a filter regular sequence on $R/I$.

Lemma 2.13. Let $S$ be a graded $K$-algebra and let $m$ be the irrelevant ideal. Let $x$ be a homogeneous element of $S$. Then $(0 : x)_j = 0$, for $j \gg 0$, if and only if $x \notin p$ for all $p \in \text{Ass}(S), p \neq m$.

Proof. We first suppose that $(0 : x)$ has finite length and suppose by contradiction that there exist a relevant associated prime $p$ such that $x \in p$ and an element $g \in S_i$ such that $(0 : g) = p$. Since $p \neq m$, there exists an element $x \in m - p$. Now since $p$ is a prime ideal, we may assume that for all integer $k$, $x^k \notin (0 : g)$. Further, for any integer $\nu$, we may choose $k$ sufficiently large. If we set $j = k \cdot \deg(x) + i$, then $j > \nu$ and so $0 \neq x^k g \in (0 : x)_j$.
since $x^k \notin (0 : g)$ and $x \in p$. Hence for any integer $\nu$ there exist elements of $(0 : x)$ of higher degree, a contradiction. Conversely, it is sufficient to show that every element of $(0 : x)$ is nilpotent. In fact we deduce that in high degree (for example higher then the product of nilpotent orders of a finite system of generators) there is no element different from zero. Then let $p_i$ for $i = 1, \ldots, k$ the associated primes of $S$ which are not in $m$ and $q_i$ for $i = 1, \ldots, k$ the related primary components. Let $J$ the primary component associated to the maximal ideal. If $y$ is an element of $(0 : x)$, then
\[ yx = 0 = \bigcap_{i=1}^{k} q_i \cap J. \]
So we have $y \in q_i$ for all $i = 1, \ldots, k$, otherwise we have that $x^n \in q_i$, that is $x \in p_i$, a contradiction. We may suppose $y$ a homogeneous element, in other words $y \in m$. Then there exist an integer $r$ such that $y^r \in m^r \subseteq J$. Hence
\[ y^r \in \bigcap_{i=1}^{k} q_i \cap J, \] that is $y^r = 0$. \□

We use this Lemma to prove the following result:

**Proposition 2.14.** Let $I$ be a monomial ideal in $R$ and let $l(\cdot)$ the length function. The following are equivalent

1. $l(a_i^i(I)) < \infty$ for all $i$;
2. $I$ is an ideal of Borel-type.

**Proof.** The implication (2) $\Rightarrow$ (1) was proved in [15], Proposition 4.3.3: in fact since $l(a_i^i(I))$ are finite for all $i$, $x$ is a filter regular sequence on $R/I$. So using an induction argument and Lemma 2.13 we are done. Conversely, if $n = 2$ there is nothing to prove. Now consider the case $n > 2$. We use Proposition 2.14(2). We may assume that $I + (x_n)/(x_n)$ is of Borel type by induction hypothesis. Hence for any monomial generator $u$ which is not divided by $x_n$,
if $x^i_s|u$, $1 \leq i < j \leq n - 1$, there exists $t \geq 0$ such that $x^i_s(u/x^i_s) \in I + (x^n)$ (and hence $x^i_s(u/x^i_s) \in I$). Now suppose $x^n_s|u = x^{s_1}_1 \cdots x^{s_n}_n$, $u \in I$, that is, $s = s_n \geq 1$. Then we first show that for any $i$ with $1 \leq i \leq n - 1$, there exists $k \geq 0$ such that $x^{s_1}_1 \cdots x^{s_i+k}_i \cdots x^{s_{n-1}}_{n-1} \in I$ by induction on $s$. (Indeed, if $s = 1$, the assertion follows from $l((I : x^n)/I) < \infty$). This implies that for any $1 \leq i < j \leq n - 1$, there exists $k_i \geq 0$ such that $x^{s_1}_1 \cdots x^{s_i+k_i}_i \cdots x^{s_j}_j \cdots x^{s_{n-1}}_{n-1} \in I + (x^n)$. In particular $x^{s_1}_1 \cdots x^{s_i+k}_i \cdots x^{s_j}_j \cdots x^{s_{n-1}}_{n-1} x^n_s \in I$, as required.

3 Preserving extremal Betti numbers and annihilator numbers

In this section we deal with the annihilator numbers of a graded $K$-algebra and the correspondence with extremal Betti numbers. We will use this approach to prove Theorem 3.6 based on the Theorem on extremal Betti numbers by Bayer, Charalambous, Popescu [see [2]]. This section is based on [15]. The following fundamental lemma will be useful later.

Lemma 3.1. Let $I \subseteq R$ be a homogeneous ideal such that $l(a^i_s(I))$ are finite for all $i$. Let $<$ be the reverse lexicographic order. Then:

$$\dim_K A_{i-1} \left( x_n, x_{n-1}, \ldots, x_1; \frac{R}{I} \right)_j = \dim_K A_{i-1} \left( x_n, x_{n-1}, \ldots, x_1; \frac{R}{{\text{in}}_<(I)} \right)_j,$$

for all $i, j$.

PROOF: It is suffices to show that the two modules above have the same Hilbert function. By properties of reverse lexicographic order, we have that the modules

$$\frac{R}{(I, x_n, x_{n-1}, \ldots, x_{n-i+1})}, \quad \frac{R}{{\text{in}}_<(I), x_n, x_{n-1}, \ldots, x_{n-i+1}}$$
have the same Hilbert function. Consider now the two exact sequences

$$0 \to A_{i-1}(x_n, \ldots, x_1; R/J) \to \frac{R}{(J, x_n x_{n-1}, \ldots, x_{n-i+1})}(-1) \xrightarrow{x_{n-i}} \frac{R}{(J, x_n x_{n-1}, \ldots, x_{n-i})} \to 0,$$

for $i = 0, \ldots, n$ and $J = I$ or $J = \text{in} < (I)$. Since the Hilbert function is additive on short exact sequences, we have that the Hilbert function of $A_{i-1}(x_n, \ldots, x_1; R/I)$ is determined by the Hilbert function of the modules $R/(I, x_n, x_{n-1}, \ldots, x_{n-i})$ and $R/(I, x_n, x_{n-1}, \ldots, x_{n-i+1})$. This two modules have the same Hilbert function respectively of $R/(\text{in} < (I), x_n, x_{n-1}, \ldots, x_{n-i})$ and $R/(\text{in} < (I), x_n, x_{n-1}, \ldots, x_{n-i+1})$, that determine the Hilbert function of the module $A_{i-1}(x_n, \ldots, x_1; R/\text{in} < (I))$. □

Annihilator numbers can also be defined for modules. To do it one just has to extend the concept of generic initial ideals to generic initial submodules [for details see [11]] and one can show that Theorem 3.1 holds in the general case of a finitely generated graded $R$-module.

We write $\alpha_{ij}(R/I)$ instead of $\alpha_{ij}(x_n, x_{n-1}, \ldots, x_1; R/I)$, which are the annihilator numbers on $R/I$ with respect $x_n, x_{n-1}, \ldots, x_1$.

**Corollary 3.2.** $\alpha_{ij}(R/I) = \alpha_{ij}(R/\text{in} < (I))$, for all $i, j$.

Annihilator numbers of a filtered regular sequence and Betti numbers are related to each other. We shall use the convention that

$$\binom{i}{-1} = \begin{cases} 0 & \text{if } i \neq -1, \\ 1 & \text{if } i = -1. \end{cases}$$

**Proposition 3.3.** [15], Proposition 4.3.12] Let $M$ be a finitely generated graded $R$-module and $y = y_1, \ldots, y_n$ a $K$-basis of $R_1$ which is a filter regular
sequence on \( M \). Then

\[
\beta_{i,i+j}(M) \leq \sum_{k=0}^{n-i} \binom{n-k-1}{i-1} \alpha_{kj}(y; M),
\]

for all \( i \geq 0 \) and all \( j \).

**Definition 3.4.** Let \( M \) be a finitely generated graded \( R \)-module and let \( y \) be a \( K \)-basis of \( R_1 \) which is a filter regular sequence on \( M \). Let \( \alpha_{ij} \) be the annihilator numbers of \( M \) with respect \( y \) and \( \beta_{ij} \) be the graded Betti numbers of \( M \).

a. An annihilator number \( \alpha_{ij} \neq 0 \) is called **extremal** if \( \alpha_{kl} = 0 \) for all \( (k, l) \neq (i, j) \) with \( k \leq i \) and \( l \geq j \);

b. A graded Betti number \( \beta_{i,i+j} \neq 0 \) is called **extremal** if \( \beta_{k,k+l} = 0 \) for all \( (k, l) \neq (i, j) \) with \( k \geq i \) and \( l \geq j \).

Using Proposition 2.12 and Proposition 3.3 one can prove the following result:

**Proposition 3.5.** [15, Theorem 4.3.15] Let \( M \) be a graded \( R \)-module and let \( y \) be a \( K \)-basis of \( R_1 \) which is a filter regular sequence on \( M \). Let \( \alpha_{ij} \) be the annihilator numbers of \( M \) with respect to \( y \) and \( \beta_{ij} \) be the graded Betti numbers of \( M \). Then \( \beta_{i,i+j} \) is an extremal Betti number of \( M \) if and only if \( \alpha_{n-i,j} \) is an extremal annihilator number of \( M \). Moreover, if the equivalent conditions hold, then

\[
\beta_{i,i+j} = \alpha_{n-i,j}.
\]

Combining Proposition 3.5 and Corollary 3.2 we immediately obtain the following:

**Theorem 3.6.** Let \( I \subseteq R \) be a graded ideal. Suppose that \( l(a^I_x(I)) \) are finite for all \( i \). Let \( \prec \) be the reverse lexicographic order. Then for any two numbers \( i, j \in \mathbb{N} \) one has:
1. $\beta_{i,i+j}(I)$ is extremal if and only if $\beta_{i,i+j}(\text{in}_<(I))$ is extremal;

2. if $\beta_{i,i+j}(I)$ is extremal, then $\beta_{i,i+j}(I) = \beta_{i,i+j}(\text{in}_<(I))$.

The Theorem above and the results in this section and in section 2 yield two results that recover what has been done respectively by Bayer and Stillman in generic coordinates in [3] and Trung in [18]:

**Corollary 3.7.** Let $I \subseteq R$ be a graded ideal such that $l(a^i_x(I))$ are finite for all $i$. Let $<$ be the reverse lexicographic order. Then

1. $\text{pd}(I) = \text{pd}(\text{in}_<(I))$;
2. $\text{depth}(R/I) = \text{depth}(R/\text{in}_<(I))$;
3. $R/I$ is Cohen-Macaulay if and only if $R/\text{in}_<(I)$ is Cohen-Macaulay;
4. $\text{reg}(I) = \text{reg}(\text{in}_<(I))$.

**Corollary 3.8.** [18, Theorem 1.3.] Let $I \subseteq R$ be a graded ideal such that $l(a^i_x(I))$ are finite for all $i$. Let $<$ be the reverse lexicographic order. Then

1. $\text{reg}_t(I) = \text{reg}_t(\text{in}_<(I))$;
2. $a^*_t(I) = a^*_t(\text{in}_<(I))$;

We conclude this section by investigating on a question about the rigidity of resolutions in a specific case. Conca in [8] raised the following question: let $I$ be a graded ideal and let $\text{Gin}(I)$ be his generic initial ideal with respect the reverse lexicographic order. If $\beta_i(I) = \beta_i(\text{Gin}(I))$ for some $i$, then it is true that $\beta_k(I) = \beta_k(\text{Gin}(I))$ for all $k \geq i$? This question has a positive answer as proved by Conca, Herzog and Hibi [10, Corollary 2.4]. For $i = 0$ this fact was first proved by Aramova, Herzog and Hibi in [1].

Consider now $I$ a graded ideal of $R$. We define $I_{(j)}$ the ideal generated by all homogeneous polynomial of degree $j$ belonging to $I$. 

13
Definition 3.9. A homogeneous ideal $I \subseteq R$ is componentwise linear ([1]), if $I_{(j)}$ has a linear resolution for all $j$.

It is known that $I$ is a componentwise linear ideal in $R$ if and only if $\beta_{ij}(R/I) = \beta_{ij}(R/\text{gin}_{<} I)$ for all $i, j$ [1, Theorem 1.1]. Here we show an example of an ideal with initial ideal of Borel-type, $\mu(I) = \mu(\text{in}_{<}(I))$ and with different resolution with respect his initial ideal. We use CoCoA for computations.

Example 3.10. Let $R = K[x_1, x_2, x_3]$ be the polynomial ring in 3 variables with the reverse lexicographic order and

\[
I = ((2x_1 + x_2)^3, (x_1 + x_3)^3, (x_1 + 3x_3)^3, (3x_1 + 2x_3)^3, (2x_1 - 3x_3)^3, (4x_1 + x_2)^3, (3x_1 - 5x_3)^3).
\]

Then $\text{in}_{<}(I) = (x_1^3, x_1^2x_2, x_2^2, x_1^2x_3, x_1x_3^2, x_3^3, x_2^2x_3, x_1x_2^2)$. It is easy to see that

\[
\text{Ass} \left( \frac{R}{\text{in}_{<} I} \right) = \{(x_1, x_2, x_3)\}
\]

and so we can conclude that $\text{in}_{<}(I)$ is an ideal of Borel-type. Further $\text{gin}_{<}(I) = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_1x_2x_3, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_3^4)$. The Betti tables of the three ideals are the following:

\[
\begin{array}{ccccc}
\text{BettiDiagram}(I); & & \text{BettiDiagram}(\text{LT}(I)); & \\
0 & 1 & 2 & 0 & 1 & 2 \\
\hline
3: & 8 & 9 & 1 & 3: & 8 & 9 & 2 \\
4: & - & 1 & 2 & 4: & - & 2 & 2 \\
\hline
\text{Tot}: & 8 & 10 & 3 & \text{Tot}: & 8 & 11 & 4
\end{array}
\]
We immediately notice that $\mu(I) = \mu(\text{in}_<(I)) \neq \mu(\text{gin}_<(I))$, $\beta_i(I) \neq \beta_i(\text{in}_<(I))$ for all $i = 1, 2$, and $\beta_i(I) \neq \beta_i(\text{gin}_<(I))$ for all $i$. Then this is an example of a monomial ideal, $\text{in}_<(I)$, that is Borel-type but not componentwise linear.

In particular if $I$ is a graded ideal with initial ideal of Borel-type such that $\mu(I) = \mu(\text{in}_<(I))$, it is not true in general that all the $\beta_i$'s are equal.

Remark 3.11. Conca in a private communication reported that if $I$ is a graded ideal such that $\text{in}_<(I)$ is componentwise linear and $\mu(I) = \mu(\text{in}_<(I))$, then $\beta_i(I) = \beta_i(\text{in}_<(I))$ for all $i$.

### 4 Preserving reduction number

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring, $K$ an infinite field and $I \neq 0$ a homogeneous ideal in $R$. We set $m = (x_1, \ldots, x_n)R/I$. A homogeneous ideal $J \subset m$ is called a reduction of $m$ if $m^{r+1} = Jm^r$ for some integer $r \geq 0$. $J$ is called a minimal reduction, if it is minimal with respect to inclusion.

The **reduction number** of $m$ with respect to a minimal reduction $J$ of $m$, denoted by $r_J(m)$ or $r_J(R/I)$, is the smallest $r \geq 0$ such that $m^{r+1} = Jm^r$. The reduction number of $m$, denoted by $r(m)$ or $r(R/I)$, is the infimum of $r_J(m)$ over all possible minimal reductions $J$ of $m$. For the reductions theory see [16]. Consider now $\text{in}(I)$, the initial ideal of $I$ with respect to some
admissible term order on the terms of $R$. Vasconcelos in [23] conjectured that

$$r(R/I) \leq r(R/in(I)).$$

Bresinsky and Hoa proved, in [5], that the conjecture is true for generic coordinates. Trung in [18] proved that the equality holds in generic coordinates with respect the reverse lexicographic order. Moreover the conjecture was proved by Conca in [9] and independently by Trung. We see now that if $I$ is an ideal such that the lengths $l(a^*_x)$ are finite, the equality of reduction numbers of $I$ and in$(I)$ is not necessarily reached. So assume that $\dim(R/I) = d$ and denote by $\overline{y}$ the image of $y \in R$ in $R/I$.

**Definition 4.1.** If $M$ is any graded $R$-module of finite length, we define

$$a(M) = \begin{cases} 
\max\{p : M_p \neq 0\} & \text{if } M \neq 0; \\
-\infty & \text{if } M = 0.
\end{cases}$$

By [10] a minimal reduction of $m$ can always be generated by a system of parameters. Furthermore worth the following

**Lemma 4.2.** [5, Lemma 3] The ideal $J = (\overline{y}_1, \ldots, \overline{y}_d) \subseteq R/I$ is a minimal reduction of $m$ if and only if $\{\overline{y}_1, \ldots, \overline{y}_d\}$ is a system of parameters (s.o.p.) of $R/I$ with $\overline{y}_i$ linear forms, $1 \leq i \leq d$. In this case

$$r_J(R/I) = a(R/(I, y_1, \ldots, y_d)).$$

**Proposition 4.3.** [5, Proposition 4] Let $I$ be a homogeneous ideal in $R = K[x_1, \ldots, x_n]$. Then,

$$r(R/I) \geq \min\{\deg(F) : F \in I, \ F \text{ homogeneous}\} - 1.$$
Lemma 4.4. [5, Lemma 5] Assume that $I$ is a monomial ideal of $R$ such that $x_{n-d+1}, \ldots, x_n$ is a s.o.p. of $R/I$. Then any minimal reduction $J$ of $m$ is generated by $d$ linear forms $y_1, \ldots, y_d$ with

$$y_i = x_{n-d+1} + a_{i,1}x_1 + \cdots + a_{i,n-d}x_{n-d}, \quad 1 \leq i \leq d,$$

where $a_{i,j} \in K$, $1 \leq j \leq n-d$.

Corollary 4.5. [5, Corollary 6] Assume that $I$ is a monomial ideal of $R$ such that $x_{n-d+1}, \ldots, x_n$ is a s.o.p. of $R/I$. Then for any minimal reduction $J$ of $m$ we have

$$r_J(R/I) \leq r_{(x_{n-d+1}, \ldots, x_n)}(R/I).$$

We notice that an ideal $I$ such that the lengths $l(a^i_x)$ are finite for $i = 1, \ldots, d$, satisfies the hypothesis of Corollary 4.5 but in general the equality may not be reached:

Example 4.6. [5, Example 7] Consider the ideal $I = (x_1^4, x_1x_2^3, x_1x_3^2)$ in $R = K[x_1, x_2, x_3]$. It is easy to see that $I$ is of Borel-type and $d = \dim(R/I) = 2$. Equivalently $\{x_2, x_3\}$ is a filter regular sequence in $R/I$ and so a s.o.p in $R/I$. Hence $(x_2, x_3)$ is a minimal reduction of $m$ in $R/I$ by Lemma 4.2. Again by Lemma 4.2 we get

$$r_J(R/I) = \begin{cases} 3 & \text{if } J = (x_2, x_3), \\ 2 & \text{if } J = (x_2, x_3 - x_1). \end{cases}$$

Since 3 is the least degree in the generating set of $I$, by Proposition 4.3 we have that $r(R/I) = 2 < r_{(x_2,x_3)}(R/I)$.

The following lemma generalize Lemma 4.1 in [18].

Lemma 4.7. Let $\text{in}(I)$ be the initial ideal of $I$ with respect revlex. Let $d = \dim(R/I)$. Let $J = (I, x_{n-d+1}, \ldots, x_n)/I$ and $K = (\text{in}(I), x_{n-d+1}, \ldots, x_n)/\text{in}(I)$. If $l(a^i_x)$ are finite for all $i \in \{1, \ldots, d\}$, then $r_J(R/I) = r_K(R/\text{in}(I))$. 

17
Proof: By hypothesis $x_{n-d+1}, \ldots, x_n$ is a filter regular sequence in $R/I$ and so it is a s.o.p in $R/I$. Then, using Lemma 4.2, $J$ is a minimal reduction of $m$ in $R/I$ such that $r_J(R/I) = a(R/(I, x_{n-d+1}, \ldots, x_n))$. Since $R/I$ and $R/\text{in}(I)$ share the same Hilbert function, $x_{n-d+1}, \ldots, x_n$ is also a s.o.p in $R/\text{in}(I)$ and so by Lemma 4.2, $K$ is a minimal reduction of the homogeneous maximal ideal in $R/\text{in}(I)$ such that $r_K(R/\text{in}(I)) = a(R/(\text{in}(I), x_{n-d+1}, \ldots, x_n))$. Since we use the reverse lexicographic order, we have the following identity:

$$\text{in}(I, x_{n-d+1}, \ldots, x_n) = (\text{in}(I), x_{n-d+1}, \ldots, x_n).$$

Hence $a(R/(\text{in}(I), x_{n-d+1}, \ldots, x_n)) = a(R/\text{in}(I), x_{n-d+1}, \ldots, x_n))$. Finally, since $R/I$ and $R/\text{in}(I)$ share the same Hilbert function, we obtain that $a(R/(\text{in}(I), x_{n-d+1}, \ldots, x_n)) = a(R/(I, x_{n-d+1}, \ldots, x_n))$, as required. □

Remark 4.8. Let $I$ be an ideal such that the lengths $l(a^i_\kappa)$ are finite for all $i = 1, \ldots, d$ and let $\text{in}(I)$ be the initial ideal of $I$ with respect the reverse lexicographic order. Under these hypothesis, we observe that the reduction numbers of $I$ and $\text{in}(I)$ are not necessarily equal. In fact, by hypothesis, $x_n, \ldots, x_{n-d+1}$ is a filter regular sequence in $R/I$ and so a s.o.p in $R/I$. Hence the ideal $K = (I, x_{n-d+1}, \ldots, x_n)/I$ is a minimal reduction of $m$ in $R/I$. We first suppose that for all minimal reductions $J$ in $R/I$, $r(R/I) = r_K(R/I) \leq r_J(R/I)$. Using Lemma 4.7 we obtain that

$$r(R/I) = r_K(R/I) = r_{K'}(R/\text{in}(I)) \geq r(R/I),$$

where $K' = (\text{in}(I), x_{n-d+1}, \ldots, x_n)/\text{in}(I)$ is a minimal reduction in $R/\text{in}(I)$. Since $r(R/I) \leq r(R/\text{in}(I))$ is true in general, we obtain that $r(R/I) = r(R/\text{in}(I))$. By Example 4.6 it might exist a minimal reduction $J$ in $R/I$ such that $r(R/I) = r_J(R/I) < r_K(R/I)$. By Corollary 4.5 we know that
\( r_{J'}(R/\text{in}(I)) \leq r_{K'}(R/\text{in}(I)) \) for all minimal reductions \( J' \) in \( R/\text{in}(I) \). In particular, we can pick \( J' \) such that \( r(R/\text{in}(I)) = r_{J'}(R/\text{in}(I)) \). Suppose now that \( r_{J'}(R/\text{in}(I)) = r_{K'}(R/\text{in}(I)) \). In this case using Lemma 4.7 we have

\[
r(R/I) < r_K(R/I) = r_K'(R/\text{in}(I)) = r(R/\text{in}(I)),
\]

that is \( r(R/I) < r(R/\text{in}(I)) \). Conversely suppose \( r_{J'}(R/\text{in}(I)) < r_{K'}(R/\text{in}(I)) \). In this case nothing can be said more than just the well known inequality \( r(R/I) \leq r(R/\text{in}(I)) \).

5 Quasi-stable versus Borel type ideals and Pommaret bases

Let \( R = K[x_1, \ldots, x_n] \) be the polynomial ring over a field \( K \) in \( n \) variables and \( \mu = [\mu_1, \ldots, \mu_n] \) be an exponent vector, with \( x^\mu \) we denote a monomial in \( R \) and with \( f \) a polynomial such that \( \text{in}_{<}(f) = x^\mu \) with respect to the reverse lexicographic order. The following definitions and Proposition 5.5 are in [13] and [17].

**Definition 5.1.** We define the class of \( \mu \) as the integer

\[
\text{cls}(\mu) = \min\{i : \mu_i \neq 0\}.
\]

If \( f \) is a polynomial with \( \text{in}_{<}(f) = x^\mu \), by \( \text{cls}(f) \) one means \( \text{cls}(\mu) \). Then the multiplicative variables of \( f \in R \) (or \( x^\mu \)) are

\[
X_R(f) = X_R(x^\mu) = \{x_1, \ldots, x_{\text{cls}(\mu)}\}.
\]

If we consider \( f = x_2^2x_3 + x_3x_4^2 + x_3^3 \in R = K[x_1, x_2, x_3, x_4] \), then we have that \( X_R(f) = \{x_1, x_2\} \), since \( \text{cls}(\text{in}_{<}(f)) = \text{cls}(x_2^2x_3) = 2 \).
Definition 5.2. We say that \( x^\mu \) is an involutive divisor of \( x^\nu \), with \( \nu \) another index vector, if \( x^\mu | x^\nu \) and \( x^{\nu - \mu} \in \mathbb{K}[X_R(x^\mu)] \).

For example \( x_2x_3^2 \) is not an involutive divisor of \( x_2^3x_3^3 \), since his class is two and \( x_2x_3 \notin \mathbb{K}[x_1, x_2] \). Instead \( x_2^3x_3^3 \) is an involutive divisor of \( x_2^3x_3^3 \), since his class is two and \( x_2 \in \mathbb{K}[x_1, x_2] \). In the same way one can see that also \( x_3^3 \) is an involutive divisor of \( x_2^3x_3^3 \).

Definition 5.3. Let \( \mathcal{H} \subseteq R \) be a finite subset of only terms. We say that \( \mathcal{H} \) is a Pommaret basis of the monomial ideal \( I = (\mathcal{H}) \), if

\[
\bigoplus_{h \in \mathcal{H}} \mathbb{K}[X_R(h)] \cdot h = I
\]

as \( K \)-linear space. (In this case each term \( x^\nu \in I \) has a unique involutive divisor \( x^\mu \in \mathcal{H} \)) A finite polynomial set \( \mathcal{H} \subseteq R \) is a Pommaret basis of a polynomial ideal \( I \) with respect a monomial order \(<\) (revlex in our case), if all elements of \( \mathcal{H} \) have distinct leading terms and these terms form a Pommaret basis of the ideal \( \text{lt}_{<}(I) \).

Definition 5.4. A monomial ideal \( I \) is quasi-stable, if possess a finite Pommaret basis.

Similarly to the Borel type case, we give here a characterization of quasi-stable ideal:

Proposition 5.5. [13, Theorem 11] Let \( I \subseteq R \) be a monomial ideal and \( d = \dim(R/I) \). Then the following are equivalent.

1. \( I \) is quasi-stable;
2. \( x_1 \) is a non zero divisor for \( R/I^{sat} \), where \( I^{sat} = I : m^k \) is the saturation of \( I \). Besides, for all \( 1 \leq k < d \), \( x_{k+1} \) is a non zero divisor for \( R/(I, x_1, \ldots, x_k)^{sat} \).
3. $I : x_1^\infty \subseteq I : x_2^\infty \subseteq \cdots \subseteq I : x_d^\infty$ and for all $d < k \leq n$, there exists an exponent $e_k \geq 1$ such that $x_i^{e_k} \in I$;

4. for all $1 \leq k \leq n$ the equality $I : x_k^\infty = I : (x_k, \ldots, x_n)^\infty$ holds;

5. if $p \in \text{Ass}(R/I)$, then $p = (x_k, \ldots, x_n)$ for some $k$;

6. if $x^\mu \in I$ and $\mu_i > 0$ for some $1 \leq i < n$, then for some $0 < r \leq \mu_i$ and $i < j \leq n$, there exists an integer $s \geq 0$ such that $x^s j x^\mu / x^r i \in I$.

Proof: for the proof see [Proposition 4.4, [17]]. □

Example 5.6. Consider the ideal $I = (x_1x_3, x_2x_3, x_2^3)$ in $K[x_1, x_2, x_3]$. We claim that $\mathcal{H} = [x_1x_3, x_2x_3, x_3^2]$ is a Pommaret basis of $I$. In fact one can easily see that $I = (x_1, x_2, x_3^2) \cap (x_3)$ satisfies the condition 5 of Proposition 5.5 (see also Example 5.7).

However $I$ is not an ideal of Borel type. In fact using (3) in Proposition 2.4 we have that $x_3 | x_1x_3$ but there exists no integer $t \geq 0$ such that $x_1^{t+1} \in I$, otherwise we have

$$x_1^{t+1} = x_1x_3f + x_2x_3g + x_3^2h$$

for some $f, g, h \in R$ and so $x_3 | x_1^{t+1}$, a contradiction.

Example 5.7. Let $R = K[x_1, x_2, x_3]$. If we consider $I = (x_1x_3, x_2x_3, x_3^2)$ the ideal of the previous example, we have that

$$\text{Ass}(R/I) = \{(x_3), (x_1, x_2, x_3)\}.$$ 

All the associated ideal of $I$ are of the form $(x_k, \ldots, x_3)$ for some $k \leq 3$ ($k = 3$ and $k = 1$). Using (5) in Proposition 5.5 we have that $I$ is quasi-stable but not Borel type, since there exists no $k$ such that $(x_3) = (x_1, \ldots, x_k)$. We show now an example of ideal $J$ that is Borel type but not quasi-stable. We
can transform $I$ using the change of variables $x_1 \rightarrow x_3$, $x_2 \rightarrow x_2$ and $x_3 \rightarrow x_1$ and we obtain the ideal $J = (x_1x_2, x_1x_3, x_1^2)$. In this case we have

$$\text{Ass}(R/J) = \{(x_1), (x_1, x_2, x_3)\}$$

and so the associated primes of $J$ are of the form $(x_1, \ldots, x_k)$ for some $k \leq 3$ ($k = 1$ and $k = 3$). So $I$ is Borel type but it is not quasi-stabile since there exists no $k \leq 3$ such that $(x_1) = (x_k, \ldots, x_3)$.

**Acknowledgment**

We thank Professor Maria Evelina Rossi for her support and guidance in investigating the subject and the Department of Mathematics of University of Genova for hospitality.
References

[1] A. Aramova, J. Herzog, T. Hibi, Ideals with stable Betti numbers, Adv. Math. 152 (2000), 72-77.

[2] D. Bayer, H. Charalambous, S. Popescu, Extremal Betti numbers and applications to monomial ideals, J. Algebra, 221 (1999), 497-512.

[3] D. Bayer, M. Stillman, A criterion for detecting m-regularity, Invent. Math. 87 (1987), no. 1, 1-11.

[4] I. Bermejo, P. Gimenez, Saturation and Castelnuovo-Mumford regularity, J. Algebra 303 (2006), 592-617.

[5] H. Bresinsky, L.T. Hoa, On the reduction number of some graded algebras, Proc. Amer. Math. Soc. 127 (1999), 1257-1263.

[6] W. Bruns, J. Herzog, Cohen Macaulay rings, Cambridge University Press, 59 (1993).

[7] A. Conca, Betti Numbers and Generic Initial Ideals (Lecture 2), International School on Computer Algebra: CoCoA 2007 RISC Hangenberg-Linz (Austria), June 2007.

[8] A. Conca, Koszul homology and extremal properties of Gin and Lex, Trans. Amer. Math. Soc. 356 (2004), 2945-2961.

[9] A. Conca Reduction numbers and initial ideals, Proc. Amer. Math. Soc. 131 (2003), 1015-1020.

[10] A. Conca, J. Herzog, T. Hibi Rigid resolutions and big Betti numbers, Comment. Math. Helv. 79 (2004), Issue 4, 826-839.
[11] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, 150, New York, Springer-Verlag, 1995.

[12] S. Eliahou, M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra, 129 (1990), 1-25

[13] A. Hashemi, M. Schweinfurter, W. M. Seiler, *Quasi-stability versus Genericity*, CASC, Springer 7442 (2012), 172-184.

[14] J. Herzog, T. Hibi, *Componentwise linear ideals*, Nagoya Math. J. 153 (1999), 141-153

[15] Jürgen Herzog, Takayuki Hibi, *Monomial Ideals*, Graduate Texts in Mathematics, 260, Springer, 2010.

[16] D. G. Northcott, D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. 50 (1954), 145-158.

[17] W. Seiler, *A combinatorial approach to involution and δ—regularity II: Structure analysis of polynomial modules with Pommaret basis*, Appl. Alg. Eng. Comm. Comp. 20 (2009), 261-338.

[18] N. Trung, *Gröbner bases, local cohomology and reduction number*, Proc. Amer. Math. Soc. 129 (2001), no. 1, 9-18.

[19] N Trung, *Reduction exponent and degree bounds for the defining equations of a graded ring*, Proc. Amer. Math. Soc. 350 (1998), 2813-2832.

[20] N Trung, *The Castelnuovo regularity of the Rees algebra and the associated graded ring*, Trans. Amer. Math. Soc. 350 (1998), 2813-2832
[21] N. Trung, *The largest non-vanishing degree of graded local cohomology modules*, J. Algebra 215 (1999), 481-499.

[22] N. Trung, *Constructive characterization of the reduction numbers*, Compositio Math. 137 (2003), 99-113.

[23] V. W. Vasconcelos, *The degrees of graded modules*, Proceedings of Summer School on Commutative Algebra, Bellaterra 1996, Vol. II, CRM Publication, 141-196.