Large spin corrections in $\mathcal{N} = 4$ SYM sl(2): still a linear integral equation

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Abstract

Anomalous dimension and higher conserved charges in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM for generic spin $s$ and twist $L$ are described by using a novel kind of non-linear integral equation (NLIE). The latter can be derived under typical situations of the SYM sectors, i.e. when the scattering need not depend on the difference of the rapidities and these, in their turn, may also lie on a bounded range. Here the non-linear (finite range) integral terms, appearing in the NLIE and in the dimension formula, go to zero as $s \to \infty$. Therefore they can be neglected at least up to the $O(s^0)$ order, thus implying a linear integral equation (LIE) and a linear dimension/charge formula respectively, likewise the 'thermodynamic' (i.e. infinite spin) case. Importantly, these non-linear terms go faster than any inverse logarithm power $(\ln s)^{-n}$, $n > 0$, thus extending the linearity validity.

Keywords: AdS-CFT correspondence; Lattice Integrable Models; Bethe Ansatz.
1 Introduction

The AdS/CFT duality [1] conjectures the equivalence between a string theory on the curved space-time $\text{AdS}_5 \times S^5$ in the strong coupling and a conformal quantum field theory on the boundary of $\text{AdS}_5$ in the weak coupling. In particular, type IIB superstring theory should be dual to $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) in four dimensions. In particular, it would relate energies of string states to anomalous dimensions of local gauge invariant operators of the quantum field theory. In this context, the discovery of integrability in both free string theory and planar field theory was a great achievement, both from the conceptual and the practical (i.e. calculative) point of view, being the 't Hooft coupling, $\lambda = 8\pi^2 g^2$, the only non-running constant on stage (the string tension is proportional to $g$). Actually, integrable models appear as spin chain like Bethe equations, satisfied by 'rapidities' which parametrise on the one side the quantum string states (and their energies) and on the other side the corresponding composite operators (and their anomalous dimensions) in SYM, respectively. Actually, the initial result, which re-launched attention on integrability in maximally SYM, identified the one-loop dilatation operator of scalar gauge-invariant fields (of bare dimension $L$) with a genuine $\mathfrak{so}(6)$ integrable hamiltonian of a spin chain (with $L$ sites) [2]. In the next turmoil, integrable structures were hinted and found in all the sectors of $\mathcal{N} = 4$ SYM and at all loops (cf. for instance [3]), bearing in mind the convergence of the anomalous dimension (weak coupling) expansion. Yet, all these integrable (Bethe, asymptotic) scattering equations have the form of a deformation of the one-loop spin chain case, plus an additional universal (string) scattering factor; the deformation is such that the interaction range increases with the number of loops. Therefore, starting from a certain loop order (generically $L$ or higher), they are plagued by the 'wrapping' problem (cf. for instance the third of [3]), which was solved only in the $SU(2)$ sector, where a mapping to the Hubbard model [4] was possible, however without incorporating the dressing factor. In a parallel way, integrability in superstring theory was discovered at classical level [5] and then extended to semiclassical and quantum level.

In this paper, we want contain ourselves within the (non-compact) $sl(2)$ sector, viz. to the local composite operators

$$\text{Tr}(\mathcal{D}^s \mathcal{Z}^L) + \ldots, \quad (1.1)$$

where $\mathcal{D}$ is the (symmetrised, traceless) covariant derivative acting in all possible ways on the $L$ bosonic fields $\mathcal{Z}$. The spin of these operators is $s$ and $L$ is the so-called 'twist'. Proper superpositions of operators (1.1) have definite anomalous dimension $\Delta$ depending on $L$, $s$ and the 't Hooft coupling $\lambda = 8\pi^2 g^2$:

$$\Delta = L + s + \gamma(g, s, L), \quad (1.2)$$

where the anomalous part $\gamma$ is easily related to the integrable chain energy, $E$ (not to
be confused with the string energy) via the well-known proportionality

\[ \gamma(g, s, L) = g^2 E(g, s, L). \] (1.3)

Actually, at one loop the integrable problem is equivalent to the analogue in planar one-loop QCD for various types of quasi-partonic operators under specific circumstances \[6, 23\].

In the context of Bethe Ansatz like equations, a useful tool to perform calculations is indeed the so-called non-linear integral equation (NLIE) in its 'excited state' version \[8\]. The NLIE allows to write exact expressions for the eigenvalues of the observables for arbitrary values of the system length and of the Bethe root number. Actually, this equation turns out to be more efficient for numerical computations as well as for analytic evaluations in some particular conditions, e.g. large number of Bethe roots. Implementing this idea we previously found and discussed finite size effects to the anomalous dimensions in the \(su(2)\) sector of \(\mathcal{N} = 4\) SYM \[9, 10, 11\]. In this case, the NLIE and the exact expressions for the eigenvalues of the charges have the same structure as in models studied in the past. This is a consequence of two simple facts concerning this specific case. First one, the scattering matrix between two magnons, which appears in the r.h.s. of the Bethe equations, depends only on the difference of their rapidities (Bethe roots), provided the so-called dressing factor (cf. below) is neglected. Secondly, the Bethe roots completely fill the real axis and only permit the presence of a finite number of holes (or complex roots). However, at least one of these properties fails when considering other sectors of \(\mathcal{N} = 4\) SYM, or when the \(S\)-matrix is suitably equipped by a string theory CDD factor\(^2\) the dressing term \[12\]: this is indeed the situation in the \(sl(2)\) sector. In general, for dealing with this intricated structure of the Bethe equations appearing in the whole \(\mathcal{N} = 4\) SYM, we proposed in \[13\] a path to a NLIE substantially different from the original idea of \[8\]. The different strategy is to perform the integrations just on the region (generally intervals of the real axis) in which Bethe roots concentrate and, consequently, to avoid the use of the Fourier transform in order to write the equation. It follows that this new procedure is effective when the magnon scattering matrix has a general dependence on the rapidities and the Bethe roots are concentrated on intervals of the real axis or even complex lines, i.e. in all the relevant cases of \(\mathcal{N} = 4\) SYM. As a consequence, we obtain a simplification that we will explain here in the \(sl(2)\) case: since nonlinear integral terms enjoy an integration just on the region where the \(s\) Bethe roots actually lie, they become depressed more than any inverse power \((\ln s)^{-n}\), \(n = 1, 2, \ldots\), for any fixed value of the twist \(L\). Therefore,

\(^{1}\)Albeit QCD is in the whole not a conformal quantum field theory, it still behaves like one at one loop and as far as the anomalous dimensions are concerned. Even under these circumstances, integrability is not complete since it requires additional constraints as, for instance, aligned helicities of the partonic degrees of freedom.

\(^{2}\)Although originally thought of as a correction to the scattering coming from string effects, in the end it revealed its effects already at four loops.
stated in advance the logarithmic scaling

$$\gamma(g, s, L) = f(g) \ln s + O(s^0), \quad (1.4)$$

after the very important BES' paper [12] on $f(g)$ (cf. also e.g. [14] for preceding literature), we are left with a Linear Integral Equation (LIE) which allows us to compute the sub-leading corrections, $O(s^0)$, of the conformal dimensions \(^3\) and, of the other charges, both the leading and the subleading terms. In this respect, our approach is clearly different from that of [10], which uses the full real axis NLIE presentation by [8], because this needs to take into account and to evaluate the non-linear integrals (on the whole real axis) as well.

Very interestingly, our LIE does not differ from the BES one [12], but for the inhomogeneous part, which consists in an integral on the one loop root density and a hole depending term (apart from a known function) \(^4\). As the equation is non-perturbative (i.e. for any $g$) and linear and it drives the cross-over between weak (small $g$) to strong (large $g$) coupling in a rather intelligible way. Nevertheless, for evaluating and checking dominant string effects (like for instance the dressing phase) the aforementioned wrapping effects ought to be negligible or known.

Eventually, these facts have furnished us the stimulus to investigate the next-to-leading-order (nlo) term – although coming from an asymptotic Bethe Ansatz – in that the leading order $f(g)$ has been conjectured to be independent of $L$ or universal [12], after the one loop proof by [15].

In this paper we will give an explicit application of this new type of (N)LIE to the $(L,s)$-vacua of the $sl(2)$ sector with the following plan. In Section 2, we will outline the formalism suitable for writing the NLIE on an interval in the most general case. In Section 3 we will apply this technique to the case of the spin $-1/2$ XXX chain, which describes the one loop $sl(2)$ sector of $\mathcal{N} = 4$ SYM and in Section 4 we will discuss the many loop case and end up with our main object, the linear integral equation. By means of the latter, we will compute up to three loops in the 't Hooft coupling the leading and sub-leading corrections of the eigenvalues of the conserved charges in the large $s$ limit, and check our results vs. the anomalous dimensions in [16] (for what concerns the other charges, our findings are new at the best of our knowledge).

\(^3\)Then, we could also consider the limit $L = j \ln s \to \infty$ for fixed $j$, which is indeed the relevant scaling of this theory [15] and entails an improvement of the previous formula into $\gamma(g, s, L) = f(g, j) \ln s + O((\ln s)^{-\infty})$ [16]; but this would be the subject of some future publications.

\(^4\)This structure encourages us to proceed further in the direction of the fixed $j$ expansion and suggests the preservation of itself: but this way may only be the topic of future publications.
2 A new approach: the NLIE on the interval(s)

In almost all the cases considered up to now, the NLIE was written for counting functions defined as

$$Z(u) = \Phi(u) - \sum_{k=1}^{s} \phi(u - u_k),$$

(2.1)

and when the Bethe roots distribute on the real axis, allowing the presence of only a finite number of holes and possibly complex roots. Even if this case is relevant for the study of the fundamental state and the first excitations of many models, it does not cover many of the Bethe Ansatz systems proposed in the context of $\mathcal{N} = 4$ SYM.

For this reason we want to write the NLIE (and the expression for the eigenvalues of the observables in terms of its solution) for the more general case in which the counting function is defined as

$$Z(u) = \Phi(u) - \sum_{k=1}^{s} \phi(u, u_k),$$

(2.2)

(i.e. the function $\phi(x, y)$ does not depend only on the difference $x - y$: this happens, for instance, when the dressing factor is present). We suppose also that the $s$ Bethe roots $\{u_k\}_{k=1,...,s}$ are concentrated in an interval $[A, B]$ of the real axis and that a finite number of holes is present. We call $u_h^{(i)}$ the holes (in number $H_i$) lying inside the interval and $u_h^{(o)}$ the holes (in number $H_o$) lying outside the interval. This particular distribution of roots is peculiar, for instance, of states in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM.

On this state we consider a sum over the Bethe roots $\{u_k\}_{k=1,...,s}$ of a function (observable) $O(u)$ analytic in a strip around the real axis. Without loss of generality, we can put ourselves in the case in which on both roots and holes the condition $e^{iZ(u_k)} = e^{iZ(u_h^{(i)})} = -1$ holds. In this case, this sum can be written as

$$2\pi i \sum_{k=1}^{s} O(u_k) = \lim_{\epsilon \to 0^+} \left[ \int_{A}^{B} du O(u - i\epsilon) \frac{e^{iZ(u - i\epsilon)}iZ'(u - i\epsilon)}{1 + e^{iZ(u - i\epsilon)}} + \int_{A}^{B} du O(u + i\epsilon) \frac{e^{iZ(u + i\epsilon)}iZ'(u + i\epsilon)}{1 + e^{iZ(u + i\epsilon)}} - 2\pi i \sum_{h=1}^{H_i} O(u_h^{(i)}) \right].$$

(2.3)

Supposing $Z'(u) < 0$ and supposing that the values $Z(A)$ and $Z(B)$ are known, we can

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5The case in which the Bethe roots are concentrated on a finite number of intervals on the real axis follows straightforwardly from the results of this Section. Moreover, even the case when the roots lie on complex lines can be treated as follows.
rearrange this expression as follows,

\[
\sum_{k=1}^{s} O(u_k) = - \int_{A}^{B} \frac{dv}{2\pi} O(v) Z'(v) + \int_{A}^{B} \frac{dv}{\pi} O(v) \frac{d}{dv} \text{Im} \ln \left[ 1 + e^{iZ(v-i0)} \right] - \\
- \sum_{h=1}^{H_i} O(u^{(i)}_h) = - \frac{1}{2\pi} \left[ O(B)Z(B) - O(A)Z(A) \right] + \\
+ \frac{1}{\pi} \left\{ O(B)\text{Im} \ln \left[ 1 + e^{iZ(B)} \right] - O(A)\text{Im} \ln \left[ 1 + e^{iZ(A)} \right] \right\} + \\
+ \int_{A}^{B} \frac{dv}{2\pi} O'(v) Z(v) - 2 \int_{A}^{B} \frac{dv}{2\pi} O'(v) \text{Im} \ln \left[ 1 + e^{iZ(v-i0)} \right] - \sum_{h=1}^{H_i} O(u^{(i)}_h). \tag{2.4}
\]

In brief, what we are doing is to evaluate a sum on the Bethe roots by integrating just on the interval containing them. Therefore, this method is alternative and complementary to the idea proposed in the first of [8] which consists in first integrating on all the real axis and then subtracting the contributions coming from the real holes. The convenience of this new method is that the non-linear terms present in (2.4) are strongly suppressed in the limit of large number of Bethe roots. We will come back on this point in the next sections, when we shall apply this technique to the \( sl(2) \) sector of \( \mathcal{N} = 4 \) SYM.

We now use (2.4) in the sum over the Bethe roots appearing in the definition (2.2) and obtain the following equation

\[
Z(u) = \Phi(u) + \int_{A}^{B} \frac{dv}{2\pi} \phi(u, v) \frac{d}{dv} Z(v) + \sum_{h=1}^{H_i} \phi(u, u^{(i)}_h) - \\
- 2 \int_{A}^{B} \frac{dv}{2\pi} \phi(u, v) \frac{d}{dv} \text{Im} \ln \left[ 1 + e^{iZ(v-i0)} \right] = \\
= f(u) - \int_{A}^{B} \frac{dv}{2\pi} \frac{d}{dv} \phi(u, v) Z(v) + \\
+ 2 \int_{A}^{B} \frac{dv}{2\pi} \phi(u, v) \text{Im} \ln \left[ 1 + e^{iZ(v-i0)} \right], \tag{2.5}
\]

where

\[
f(u) = \Phi(u) + \frac{1}{2\pi} \left[ \phi(u, B)Z(B) - \phi(u, A)Z(A) \right] + \sum_{h=1}^{H_i} \phi(u, u^{(i)}_h) - \\
- \frac{1}{\pi} \left\{ \phi(u, B)\text{Im} \ln \left[ 1 + e^{iZ(B)} \right] - \phi(u, A)\text{Im} \ln \left[ 1 + e^{iZ(A)} \right] \right\}. \tag{2.6}
\]

We can now write a NLIE for the counting function by inserting in an iterative way (2.5) for \( Z \) in the right hand side of the same equation. Using the notation

\[
(\varphi \ast f)(u) = \int_{A}^{B} dv \varphi(u, v) f(v), \tag{2.7}
\]
We eventually gain the NLIE in the final form

\[ Z(u) = F(u) + 2(G \ast L)(u) , \]  

where

\[ F(u) = f(u) + \sum_{k=1}^{\infty} (-1)^k (\varphi^k \ast f)(u) , \quad G(u, v) = \varphi(u, v) + \sum_{k=2}^{\infty} (-1)^{k-1} (\varphi^k)(u, v) . \]  

We used the simplified notations

\[ L(u) = \text{Im } \ln \left[ 1 + e^{i\pi(u-i0)} \right] , \quad \varphi(u, v) = \frac{1}{2\pi} d \frac{d}{dv} \phi(u, v) . \]  

More explicitly,

\[ F(u) = f(u) + \sum_{k=1}^{\infty} (-1)^k \int_{A}^{B} dv_1 \varphi(u, v_1) \int_{A}^{B} dv_2 \varphi(v_1, v_2) \ldots \int_{A}^{B} dv_k \varphi(v_{k-1}, v_k)f(v_k) , \]  

\[ G(u, v) = \varphi(u, v) + \sum_{k=1}^{\infty} (-1)^k \int_{A}^{B} dv_0 \varphi(u, v_0) \int_{A}^{B} dv_1 \varphi(v_0, v_1) \ldots \int_{A}^{B} dv_{k-1} \varphi(v_{k-2}, v_{k-1}) \varphi(v_{k-1}, v) . \]  

These expressions are quite formal and very difficult to handle. More easily, both the forcing term \( F(u) \) and the kernel \( G(u, v) \) may be found by solving respectively the linear integral equation, which easily follow from (2.11, 2.12),

\[ F(u) = f(u) - \int_{A}^{B} dv \varphi(u, v) F(v) , \]  

\[ G(u, v) = \varphi(u, v) - \int_{A}^{B} dw \varphi(u, w) G(w, v) , \]  

and are linked the one to the other via

\[ F(u) = f(u) - \int_{A}^{B} dv G(u, v)f(v) . \]  

Eventually, inserting (2.8) in (2.4) we obtain an expression for the eigenvalues of an
observable as
\[ \sum_{k=1}^{s} O(u_k) = -\frac{1}{2\pi} [O(B)Z(B) - O(A)Z(A)] + \]
\[ + \frac{1}{\pi} \{ O(B)\text{Im} \ln [1 + e^{iZ(B)}] - O(A)\text{Im} \ln [1 + e^{iZ(A)}] \} + \]
\[ + \int_{A}^{B} \frac{dv}{2\pi} O'(v)F(v) - \sum_{h=1}^{H} O(u_h^{(i)}) + \]
\[ + 2 \int_{A}^{B} \frac{dv}{2\pi} O'(v) \int_{A}^{B} dw [G(v,w) - \delta(v-w)]\text{Im} \ln [1 + e^{iZ(w-i\theta)}] . \]

We remark that all the already known NLIEs can be reproduced in this way, without Fourier transforming. In this sense the method sketched in this section is more general. It seems natural to use formulæ (2.8, 2.9) and (2.17) in order to write, respectively, the NLIE and the eigenvalues of the observables on states appearing in models relevant for \( \mathcal{N} = 4 \) SYM. In the next two sections we will apply this new method to the widely studied \([3, 17, 12]\) \( \mathfrak{sl}(2) \) sector of the theory.

3 An example: the XXX -1/2 spin chain

We want to apply the techniques developed in the previous Section to the XXX -1/2 spin chain, as the latter clearly gives a representation of the \( \mathfrak{sl}(2) \) sector at one loop, as stated in the 'Introduction'.

The spectrum of this non-compact spin chain may be described by the Bethe equations,
\[ \left( \frac{u_k - \frac{i}{2}}{u_k + \frac{L}{2}} \right)^L = \prod_{j=1}^{s} \frac{u_k - u_j + i}{u_k - u_j - i}, \]

where we have indicated with \( L \) the length of the chain and \( s \) the number of Bethe roots. In this case the \( s \) Bethe roots concentrate in an the interval of the real axis symmetric with respect to zero. \( L \) holes are present: two holes \( u_h^{(o)}, h = 1, 2 \), lie outside, \( H_i = L - 2 \) holes \( u_h^{(i)}, h = 1, \ldots, L - 2 \), lie inside this interval. Let us define the counting function (for reasons that will be clear in the following, we will put an index 0 to all the functions, e.g. \( Z, F, G \), related to the XXX -1/2 spin chain) as
\[ Z_0(u) = iL \ln \left( \frac{\frac{i}{2} - u}{\frac{i}{2} + u} \right) - i \sum_{j=1}^{s} \ln \left( \frac{i + u - u_j}{i - u + u_j} \right) . \]

From the Bethe equations (3.1), we have the conditions
\[ iL \ln \left( \frac{u_k - \frac{i}{2}}{u_k + \frac{L}{2}} \right) - i \sum_{j=1}^{s} \ln \left( \frac{u_k - u_j + i}{u_k - u_j - i} \right) = 2\pi n_k, \quad n_k \in \mathbb{Z} . \]
Using the property
\[
i \ln \left( \frac{x - i\xi}{x + i\xi} \right) - i \ln \left( \frac{i\xi - x}{i\xi + x} \right) = \pi, \quad \xi > 0,
\] (3.4)
we have that, on both Bethe roots \( u_k \) and holes \( u_h \),
\[
Z_0(u_k) = \pi(2n_k - L - s + 1), \quad Z_0(u_h) = \pi(2n_h - L - s + 1).
\] (3.5)
We choose \( L + s \) even, in such a way that \( e^{iZ(u_k)} = e^{iZ(u_h^{(i)})} = e^{iZ(u_1^{(o)})} = -1 \), as in the previous Section. If we now go from the smallest to the biggest root of the interval (passing also \( L - 2 \) internal holes) the (decreasing) counting function \( Z_0 \) varies of \(-2\pi(s + L - 3)\). Since \( Z_0 \) is an odd function, this means that
\[
Z_0(u_s) = -\pi(s + L - 3), \quad Z_0(u_1^{(o)}) = -\pi(s + L - 1),
\] (3.6)
where \( u_s \) is the biggest (most positive) root of the interval and \( u_1^{(o)} \) is the positive hole outside the interval. We can choose the separator \( b_0 \), which defines the interval of integration in our formulæ, i.e. \( A = -b_0, \quad B = b_0 \), such that
\[
Z_0(-b_0) = -Z_0(b_0) = \pi(s + L - 2).
\] (3.7)
With this position, the relevant functions defined in the previous section take the form
\[
\phi_0(u) = -2L \arctan 2u, \quad \phi_0(u, v) = 2 \arctan(u - v), \quad \varphi_0(u, v) = -\frac{1}{\pi}\frac{1}{1 + (u - v)^2};
\]
\[
f_0(u) = -2L \arctan 2u - (s + L - 2)[\arctan(u - b_0) + \arctan(u + b_0)] + \sum_{h=1}^{L-2} \arctan(u - u_h^{(i)}).
\] (3.8)
Since \( s + L \) is even, we have that \( \text{Im} \ln \left[ 1 + e^{iZ_0(\pm b_0)} \right] = 0 \). In addition, one easily shows that in this case the functions \( F_0 \) and \( G_0 \) enjoy the parity properties
\[
F_0(u) = -F_0(-u), \quad G_0(u, v) = -G_0(-u, v) = G_0(-u, -v).
\] (3.9)
They may be determined by solving the linear equations
\[
F_0(u) = -2L \arctan 2u - (s + L - 2)[\arctan(u - b_0) + \arctan(u + b_0)] + \sum_{h=1}^{L-2} \arctan(u - u_h^{(i)}) + \int_{-b_0}^{b_0} \frac{dv}{\pi} \frac{1}{1 + (u - v)^2} F_0(v),
\] (3.10)
\[
G_0(u, v) = -\frac{1}{\pi}\frac{1}{1 + (u - v)^2} + \int_{-b_0}^{b_0} \frac{dw}{\pi} \frac{1}{1 + (u - w)^2} G_0(w, v).
\] (3.11)
At the leading \( s \to \infty \) order these equations become simpler and therefore we will start with this case.

\[6\] The number of derivatives \( s \) has to be even and in any case, there is no loss of generality, in that, if \( L + s \) is odd, the only modification in all the formulæ is the replacement of the logarithmic indicator \( L_0(u) \) by \( \text{Im} \ln \left[ 1 - e^{iZ_0(u - i0)} \right] \).
3.1 Determination of $F_0$ when $s \to +\infty$

Let us consider equation (3.10) for the function $F_0(u)$. It is convenient to rescale the variable $u = \bar{u} s$ and the extreme of the interval $b_0 = \bar{b}_0 s$, in view of the limit $s \to +\infty$:

$$F_0(\bar{u}s) = -2L \arctan 2\bar{u}s - (s + L - 2)[\arctan(\bar{u}s - \bar{b}_0s) + \arctan(\bar{u}s + \bar{b}_0s)] +$$

$$+ \sum_{h=1}^{L-2} \arctan(\bar{u}s - u_h^{(i)}) + \int_{-b_0}^{b_0} \frac{d\bar{v}}{\pi} \frac{1}{1 + s^2(\bar{u} - \bar{v})^2} F_0(\bar{v}s).$$

(3.12)

Using the asymptotic expansions ($O(s^{-n})$ means “terms of order $s^{-n}$”)

$$\arctan \bar{u}s = \frac{\pi}{2} \text{sgn}(\bar{u}) - \frac{1}{\bar{u}s} + O(s^{-3})$$

(3.13)

$$\frac{s}{\pi \left( 1 + s^2(\bar{u} - \bar{v})^2 \right)} = \delta(\bar{u} - \bar{v}) - \frac{1}{s\pi} \frac{d}{d\bar{u}} \frac{1}{\bar{u} - \bar{v}} + O(s^{-3}),$$

(3.14)

inside the integral and the fact that

$$\lim_{s \to +\infty} u_h^{(i)} = 0,$$

(3.15)

we obtain in this approximation

$$F_0(\bar{u}s) = -2\pi \text{sgn}(\bar{u}) + \frac{4 - L}{\bar{u}s} + \frac{s + L - 2}{s} \frac{2\bar{u}}{\bar{u}^2 - \bar{b}_0^2} + F_0(\bar{u}s) +$$

$$+ \int_{-b_0}^{b_0} \frac{d\bar{v}}{s\pi} \left( \frac{d}{d\bar{v}} \frac{1}{\bar{u} - \bar{v}} \right) F_0(\bar{v}s).$$

(3.16)

Integrating by part gives

$$0 = -2\pi \text{sgn}(\bar{u}) + \frac{4 - L}{\bar{u}s} + \frac{s + L - 2}{s} \frac{2\bar{u}}{\bar{u}^2 - \bar{b}_0^2} - \frac{F_0(-\bar{b}_0s)}{s\pi} \left( \frac{1}{\bar{u} + b_0} + \frac{1}{\bar{u} - b_0} \right) -$$

$$- \int_{-b_0}^{b_0} \frac{d\bar{v}}{s\pi} \left( \frac{1}{\bar{u} - \bar{v}} \right) \frac{d}{d\bar{v}} F_0(\bar{v}s).$$

(3.17)

Now, condition (3.7) at large $s$ imposes on the function $F_0$ the constraint

$$F_0(\bar{b}_0s) = -F_0(-\bar{b}_0s) = -\pi(s + L - 2) + o(s^0),$$

(3.18)

where $o(s^0)$ means “terms of order smaller than $s^0$”. Therefore, equation (3.17) simplifies into

$$0 = -2\pi \text{sgn}(\bar{u}) + \frac{4 - L}{\bar{u}s} - \int_{-b_0}^{b_0} \frac{d\bar{v}}{s\pi} \left( \frac{1}{\bar{u} - \bar{v}} \right) \frac{d}{d\bar{v}} F_0(\bar{v}s).$$

(3.19)

This is indeed a subtle exchange of two limits.
This equation can be solved by finite Hilbert transform techniques. Its solution reads

\[- \frac{1}{2\pi s} \frac{d}{d\bar{u}} F_0(\bar{u}s) = \frac{1}{\pi} \ln \left( \frac{\bar{b}_0 + \sqrt{\bar{b}_0^2 - \bar{u}^2}}{\bar{u}} \right)^2 - \frac{2 - \frac{L}{2}}{s} \delta(\bar{u}) \Rightarrow \]  

\[ F_0(\bar{u}s) = -4\bar{b}_0 s \arcsin \frac{\bar{u}}{\bar{b}_0} - 2\bar{u} s \ln \left( \frac{\bar{b}_0 + \sqrt{\bar{b}_0^2 - \bar{u}^2}}{\bar{u}} \right)^2 + \pi \left( 2 - \frac{L}{2} \right) \text{sgn} \bar{u}. \]  

(3.20)

Remember, however, that also \( \bar{b}_0 \) depends on \( s \) through (3.7), which at large \( s \) implies (3.18) - a normalization condition for (3.20) in this approximation

\[ \int_{-\bar{b}_0}^{\bar{b}_0} d\bar{u} \left[ \frac{1}{\pi} \ln \left( \frac{\bar{b}_0 + \sqrt{\bar{b}_0^2 - \bar{u}^2}}{\bar{u}} \right)^2 - \frac{2 - \frac{L}{2}}{s} \delta(\bar{u}) \right] = 1 + \frac{L - 2}{s}. \]  

(3.21)

From this equation we deduce, after integration,

\[ 2\bar{b}_0 - \frac{2 - \frac{L}{2}}{s} = 1 + \frac{L - 2}{s}, \]  

\[ \Rightarrow \bar{b}_0 = \frac{1}{2} \left( 1 + \frac{L}{2s} \right). \]  

(3.22)

Inserting this expansion in (3.20), we obtain the following behaviour of \( F_0 \) and its derivative when \( s \to \infty \):

\[- \frac{1}{2\pi s} \frac{d}{d\bar{u}} F_0(\bar{u}s) = \frac{1}{\pi} \ln \left( \frac{\frac{1}{2} + \sqrt{\frac{1}{4} - \bar{u}^2}}{\bar{u}} \right)^2 + \frac{1}{\pi s} \frac{1}{\sqrt{\frac{1}{4} - \bar{u}^2}} - \frac{2 - \frac{L}{2}}{s} \delta(\bar{u}), \]

\[ F_0(\bar{u}s) = -2s \left[ \arcsin 2\bar{u} + \bar{u} \ln \left( \frac{\frac{1}{2} + \sqrt{\frac{1}{4} - \bar{u}^2}}{\bar{u}} \right)^2 \right] - 2 \arcsin 2\bar{u} + \]

\[ + \pi \left( 2 - \frac{L}{2} \right) \text{sgn} \bar{u}. \]  

(3.23)

### 3.2 The leading order density equation from the NLIE

Once we have determined the functions \( F_0(u) \) and \( G_0(u, v) \), as well as the value of the extreme \( b_0 \), one can write, according to the general formula (2.8) the nonlinear integral equation on a finite interval satisfied by the counting function of the \( XXXX_{-1/2} \) spin chain. It is not difficult to relate such NLIE with the linear equation [17] satisfied by the density of roots in the limit \( s \to +\infty \). In this respect, it is convenient to use the first equation in the chain (2.5)

\[ Z_0(u) = -2L \arctan 2u + \int_{-b_0}^{b_0} \frac{dv}{2\pi} 2\arctan(u - v) \frac{d}{dv} [Z_0(v) - 2L_0(v)] + \]

\[ + \sum_{h=1}^{L-2} 2 \arctan(u - u_h^{(i)}), \]  

(3.24)
where $L_0(v) = \text{Im} \ln[1 + e^{iZ_0(v-i\alpha)}]$. As before, we rescale the variable $u = \bar{u}s$ and the extreme of the interval $b_0 = \bar{b}_0 s$ and then we let $s \to +\infty$. Using the asymptotic expansion

$$\arctan \bar{u}s = \frac{\pi}{2} \text{sgn} \bar{u} - \frac{1}{\bar{u}s} + O(s^{-3})$$

(3.25)

*inside the integral* and the expansion (3.22) for $\bar{b}_0$, we obtain, *in this approximation*, the equation

$$Z_0(\bar{u}s) = -\pi L \text{sgn} \bar{u} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d\bar{v}}{2\pi} \left[ \pi \text{sgn}(\bar{u} - \bar{v}) - \frac{2}{s(\bar{u} - \bar{v})} \right] \frac{d}{d\bar{v}}[Z_0(\bar{v}s) - 2L_0(\bar{v}s)] +$$

$$+ \sum_{h=1}^{L-2} \pi \text{sgn}(\bar{u} - \bar{u}_h^{(i)}).$$

(3.26)

Performing the integral involving the sgn function and using the fact that, when $s \to +\infty$, $\bar{u}_h^{(i)} \to 0$, we are left with

$$2L_0(\bar{u}s) = -2\pi \text{sgn} \bar{u} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d\bar{v}}{2\pi} \left[ -\frac{2}{s(\bar{u} - \bar{v})} \right] \frac{d}{d\bar{v}}[Z_0(\bar{v}s) - 2L_0(\bar{v}s)].$$

(3.27)

As the term involving $L_0(\bar{v}s)$ approaches zero (cf also the Appendix), we can neglect it and obtain

$$0 = -2\pi \text{sgn} \bar{u} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d\bar{v}}{2\pi} \frac{2}{s(\bar{u} - \bar{v})} \frac{d}{d\bar{v}}Z_0(\bar{v}s).$$

(3.28)

Defining the density

$$\bar{\rho}_0(\bar{u}) = -\frac{1}{2\pi s} \frac{d}{d\bar{u}}Z_0(\bar{u}s),$$

(3.29)

which, because of the condition $Z_0(b_0) - Z_0(-b_0) = -2\pi(s + L - 2)$, is normalised as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{u} \bar{\rho}_0(\bar{u}) = 1 + o(s^0).$$

(3.30)

our equation is written as,

$$0 = -2\pi \text{sgn} \bar{u} + 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{v} \frac{1}{\bar{u} - \bar{v}} \bar{\rho}_0(\bar{v}),$$

(3.31)

which coincides with (52) of [17]. Therefore, the NLIE on a finite interval for the $XXX_{-1/2}$ spin chain links itself in a simple way to the linear equation for the density of roots in the limit $s \to +\infty$. 

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3.3 Determination of $G_0$ when $s \to \infty$

On the other hand, $G_0(u, v)$ satisfies the integral equation

$$G_0(u, v) = \frac{1}{\pi} \frac{1}{1 + (u - v)^2} + \int_{-b_0}^{b_0} \frac{dw}{\pi} \frac{1}{1 + (w - v)^2} G_0(w, v),$$

(3.32)

which, in terms of rescaled variables reads as

$$G_0(\bar{u}, \bar{v}) = \frac{1}{\pi} \frac{1}{1 + s^2(\bar{u} - \bar{v})^2} + \int_{-\bar{b}_0}^{\bar{b}_0} \frac{d\bar{w}}{\pi} \frac{s}{1 + s^2(\bar{u} - \bar{w})^2} G_0(s\bar{w}, s\bar{v}),$$

(3.33)

Using the expansions (3.13), we obtain the equation

$$0 = -\frac{1}{s} \delta(\bar{u} - \bar{v}) + \frac{1}{\pi s} \int_{-\bar{b}_0}^{\bar{b}_0} d\bar{w} \left( \frac{d}{d\bar{w}} P \frac{1}{\bar{u} - \bar{w}} \right) G_0(s\bar{w}, s\bar{v}) + o(s^{-1}).$$

(3.34)

We now define the order $s^0$ antisymmetric combination

$$X_0(\bar{u}, \bar{v}) = G_0(s\bar{u}, s\bar{v}) - G_0(s\bar{u}, -s\bar{v}) + o(s^0),$$

(3.35)

which satisfies the equation

$$\delta(\bar{u} + \bar{v}) - \delta(\bar{u} - \bar{v}) + \frac{1}{\pi} \frac{1}{\bar{u} - \frac{1}{2}} X_0 \left( \frac{1}{2}, \bar{v} \right) - \frac{1}{\pi} \frac{1}{\bar{u} + \frac{1}{2}} X_0 \left( -\frac{1}{2}, \bar{v} \right) -$$

$$-\frac{1}{\pi} \frac{d}{d\bar{w}} X_0(\bar{w}, \bar{v}) = 0.$$ 

(3.36)

We have set $\bar{b}_0 = 1/2$, since $X_0$ takes into account only the leading $O(s^0)$ contribution. The solution to this equation is given by

$$\frac{d}{d\bar{u}} X_0(\bar{u}, \bar{v}) = \left( -\frac{1}{\pi} \frac{1}{\bar{v} - \bar{u}} - \frac{1}{\pi} \frac{1}{\bar{v} + \bar{u}} \right) \sqrt{\frac{1}{4} - \bar{u}^2} +$$

$$+ 2 \left[ \delta \left( \bar{u} - \frac{1}{2} \right) + \delta \left( \bar{u} + \frac{1}{2} \right) \right] X_0 \left( \frac{1}{2}, \bar{v} \right).$$

(3.37)

Integrating this function we may write down

$$X_0(\bar{u}, \bar{v}) = -\frac{1}{\pi} \ln \left| \frac{\bar{v} + \bar{u}}{\bar{v} - \bar{u}} \right| \sqrt{\frac{1}{4} - \bar{v}^2} \sqrt{\frac{1}{4} - \bar{u}^2} - \frac{2\bar{v}}{\pi} \sqrt{\frac{1}{4} - \bar{u}^2} \arcsin 2\bar{u} +$$

$$+ \left[ \text{sgn} \left( \bar{u} - \frac{1}{2} \right) + \text{sgn} \left( \bar{u} + \frac{1}{2} \right) \right] X_0 \left( \frac{1}{2}, \bar{v} \right),$$

(3.38)

where we took into account the fact that $X_0(u, v) = -X_0(-u, v)$ in order to fix the undetermined function of $v$. 
We also define the order $s^0$ symmetric combination
\[ Y_0(\bar{u}, \bar{v}) = G_0(s\bar{u}, s\bar{v}) + G_0(s\bar{u}, -s\bar{v}) + o(s^0), \] (3.39)
which satisfies the equation
\[ -i\theta(\bar{u} - \bar{v}) + i\theta(-\bar{u} - \bar{v}) = \frac{1}{i\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{w} \left( P \frac{1}{\bar{w} - \bar{u}} \right) Y_0(\bar{w}, \bar{v}). \] (3.40)

This equation can be solved as
\[
Y_0(\bar{u}, \bar{v}) = \frac{1}{i\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{w} \left( P \frac{1}{\bar{w} - \bar{u}} \right) \left[ -i\theta(\bar{w} - \bar{v}) + i\theta(-\bar{w} - \bar{v}) \right] \sqrt{\frac{1}{4} - \bar{u}^2} = \frac{1}{\sqrt{4 - \bar{w}^2}}. \] (3.41)

We remark that this solution satisfies the parity properties $Y_0(\bar{u}, \bar{v}) = Y_0(\bar{u}, -\bar{v}) = Y_0(-\bar{u}, \bar{v})$, as it follows also from properties (3.9). As boundary condition, we find
\[
Y_0 \left( \pm \frac{1}{2}, \bar{v} \right) = Y_0 \left( \bar{u}, \pm \frac{1}{2} \right) = 0. \] (3.42)

### 3.4 Evaluation of the charges when $s \to \infty$

Using (2.17) we want to compute the eigenvalues of the energy and of all the charges when $s \to \infty$. We are interested in the leading terms in the large $s$ expansion, i.e. in the terms proportional to $\ln s$ and to $s^0$. We have to specialise formula (2.17) to the case in which
\[
A = -b_0, \quad B = b_0, \quad Z_0(-b_0) = -Z_0(b_0) = \pi(s + L - 2), \] (3.43)
\[
O(v) = q_r(v) = \frac{i}{r-1} \left[ \frac{1}{(\frac{1}{2} + v)^{r-1}} - \frac{1}{(-\frac{1}{2} + v)^{r-1}} \right]. \] (3.44)

For parity reasons, the eigenvalues of the charges $Q_r$, with $r$ odd, are zero. Therefore, we restrict to even $r$. One easily sees also that the first two lines of (2.17) give contributions at most $O(s^{-1})$. More importantly, we have strong evidence from numerical simulations that the nonlinear terms in the fourth line of (2.17) go to zero as $s \to \infty$. This peculiar behaviour of the nonlinear terms is due to the fact (see for example the discussion in Appendix A of [16]) that in our approach we are integrating only on the interval in which the Bethe roots and the internal holes are present and not - as in the approach of [8] used by [16] - on the whole real line. It follows that, differently from [16], at least up to the order $o(s^0)$ at which non-linear terms in the fourth line of (2.17) start
contributing, the eigenvalues of the charges are given by only the linear term in the third line (cf also Appendix)

\[
Q_r = \int_{-b_0}^{b_0} \frac{dv}{2\pi} \frac{d}{dv} q_r(v) F_0(v) - \sum_{h=1}^{H_0} q_r(u_h^{(i)}) + o(s^0) =
- \int_{-b_0}^{b_0} \frac{dv}{2\pi} q_r(v) \frac{d}{dv} F_0(v) - \frac{(-1)^{1-\frac{r}{2}}}{r - 1} (L - 2)^2 + o(s^0),
\]

(3.45)

which involves \( F_0 \), i.e. the solution of the linear integral equation (3.12).

One could now insert for \( F_0 \) the solution at large \( s \) given by formula (3.20). However, we will now show that this procedure is accurate only for the energy \( (r = 2) \) and only for the (coefficient of the) leading \( \ln s \) term. Indeed, let us use (3.20). In fact, we obtain

\[
Q_r = \int_{-b_0}^{b_0} \frac{dv}{2\pi} \frac{2i}{r - 1} \left[ \frac{1}{(\frac{r}{2} + v)^r} - \frac{1}{(-\frac{r}{2} + v)^r} \right] \left[ \ln \left( \frac{b_0 + \sqrt{b_0^2 - v^2}}{v} \right) \right]^2 - \pi \left( 2 - \frac{2 - L}{2} \right) \delta(v) - \frac{(-1)^{1-\frac{r}{2}}}{r - 1} (L - 2)^2 =
= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \frac{4b_0^i}{(r - 1)} \frac{1}{(2b_0 x + \frac{1}{2})^{r-1}} \tilde{\rho}_0(x) - \frac{2r(-1)^{1-\frac{r}{2}} L}{(r - 1)\frac{2}{2}},
\]

(3.46)

identifying the Korchemsky density of Bethe roots \( (s \to \infty) \) \([18]\) as

\[
\tilde{\rho}_0(x) = \frac{1}{\pi} \ln \left( \frac{\frac{r}{2} + \sqrt{\frac{r}{4} - x^2}}{x} \right)^2.
\]

(3.47)

Introducing the resolvent

\[
G(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \frac{\tilde{\rho}_0(y)}{y - x} = i \ln \frac{\sqrt{1 - 4x^2} + 1}{\sqrt{1 - 4x^2} - 1},
\]

(3.48)

we see that

\[
Q_r = \frac{2}{i} \int \frac{(2b_0)^{2-r} dx}{(r - 1)!} G(x)|_{x = i/4b_0} = \frac{2r(-1)^{1-\frac{r}{2}} L}{(r - 1)\frac{2}{2}} + o(s^0).
\]

(3.49)

Explicit computation would give, for \( r = 2 \),

\[
Q_2 = E = 4 \ln s + 4 \ln 2 - 2L + o(s^0),
\]

(3.50)

and, for \( r \geq 4 \),

\[
Q_r = \frac{2r(-1)^{1-\frac{r}{2}}}{(r - 1)(r - 2)} - \frac{2r(-1)^{1-\frac{r}{2}} L}{(r - 1)\frac{2}{2}} + o(s^0) = \frac{2r(-1)^{1-\frac{r}{2}}}{(r - 1)(r - 2)} \left[ 1 - \frac{L}{2}(r - 2) \right] + o(s^0).
\]

(3.51)
One can check for \( L = 2, 3 \) that the order \( s^0 \) terms in the energy and in the higher charges are not the correct ones (cf. for instance [18, 17, 19, 20]). Therefore, the large spin solution for \( F_0, \) (3.20), is a good approximation only if we are interested in the leading \( O(\ln s) \) term of the energy, although \( F_0 \) would be exact up to the \( O(s^0) \) order, i.e. neglecting (the going to zero) \( o(s^0) \) terms. Yet, this failure reflects the subtlety of considering the \( s \to +\infty \) limit of the integral equation (3.10).

Nevertheless, a sufficiently accurate approximation for \( F_0 \), efficient when neglecting in the charges the \( o(s^0) \) terms \( \delta \), comes out by solving the one loop density equation, i.e. the derivative of (both members of) equation (3.10), upon approximating \( u_h^{(i)} = 0 \), by means of Fourier transform technique

\[
\hat{F}_0(k) = -4\pi k^2 \frac{-e^{-ik} \cos(ks/\sqrt{2})}{2 \sinh(k/2)} + 2\pi (L-2) \frac{-e^{-ik} \cos(kL/2)}{2 \sinh(k/2)} - (4\pi \ln 2)^2 \delta(k) + o(s^0). \tag{3.52}
\]

Using (3.52) and the Fourier transform of the charges functions \( \hat{Q}_r(k) \), the eigenvalues of \( Q_r \) at order \( \ln s \) and \( s^0 \) are given by

\[
Q_r = -\int_{-\infty}^{+\infty} \frac{dk}{4\pi^2} \hat{Q}_r(k) i k \hat{F}_0(k) - \frac{-1}{r-1} (L-2)^2 + o(s^0). \tag{3.53}
\]

From (3.53) we obtain the correct results for the eigenvalues of the charges

\[
\begin{align*}
Q_2 &= E = 4 \ln s + 4\gamma_E - 4(L-2) \ln 2 + o(s^0), \\
Q_r &= 2(-1)^{1-\frac{r}{2}} \zeta(r-1) [(2-2r^{-1})L - 2(1-2r^{-1})] + o(s^0), \quad r \geq 4, \tag{3.54}
\end{align*}
\]

where \( \gamma_E \) is the Euler-Mascheroni constant and \( \zeta(x) \) is the Riemann zeta function.

## 4 All loops

Let us now consider the Bethe ansatz like equations

\[
\left( \frac{u_k + i/2}{u_k - i/2} \right) \left( \frac{1 + g^2 x}{1 + g^2 x} \right)^L = \prod_{j=1}^{s} \left( \frac{u_k - u_j - i}{u_k - u_j + i} \right)^2 \left( \frac{1 - g^2 x_j}{1 + g^2 x_j} \right)^2 e^{2i\theta(u_k, u_j)}, \tag{4.1}
\]

where we used the notations

\[
x^\pm_k = x^\pm(u_k) = x(u_k \pm i/2), \quad x(u) = \frac{u}{2} \left[ 1 + \sqrt{1 - \frac{2g^2}{u^2}} \right], \quad \lambda = 8\pi^2 g^2, \tag{4.2}
\]

\( \lambda \) being the 't Hooft coupling. It is believed that configurations of Bethe roots, i.e. solutions of (4.1), and the corresponding eigenvalues of the energy are related respectively

\[8\text{More precisely, the first order we are neglecting is } O(1/\ln s) \text{ and comes from the approximation of all the internal hole positions } u_h^{(i)} = 0, \text{ which is strictly valid only if } s = +\infty.\]
to composite operators and their anomalous dimensions in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM. This correspondence, however, breaks at the so-called wrapping order, i.e. at order $g^{2L-2}$, and higher. Therefore, results in this section are relevant for $\mathcal{N} = 4$ SYM only until the order $g^{2L-4}$.

In a fashion similar to the one loop case, Bethe roots concentrate in an interval $[-b, b]$ of the real axis. Inside this interval, $L - 2$ holes are present, while outside it two external holes lie. We use for them the same notations as in the one loop case.

The counting function is

$$Z(u) = -2L \arctan 2u - iL \ln \left( \frac{1 + \frac{g^2}{2x^- (u)^2}}{1 + \frac{g^2}{2x^+ (u)^2}} \right) - 2 \sum_{j=1}^{s} \arctan(u - u_j) +$$

$$+ 2i \sum_{j=1}^{s} \ln \left( \frac{1 - \frac{g^2}{2x^+ (u) x_j}}{1 - \frac{g^2}{2x^- (u) x_j}} \right) - 2 \sum_{j=1}^{s} \theta(u, u_j),$$

where the so-called dressing factor is given by

$$\theta(u_k, u_j) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r, r+1+2\nu}(g) [q_r(u_k)q_{r+1+2\nu}(u_j) - q_r(u_j)q_{r+1+2\nu}(u_k)],$$

the functions $\beta_{r, r+1+2\nu}(g)$ being

$$\beta_{r, r+1+2\nu}(g) = 2 \sum_{\mu=\nu}^{\infty} \frac{g^{2r+2\nu+2\mu}}{2r+\mu+\nu} (-1)^{r+\mu+1} \frac{(r-1)(r+2\nu)}{2\mu+1} \cdot \left( \frac{2\mu+1}{\mu-r-\nu+1} \right) \left( \frac{2\mu+1}{\mu-\nu} \right) \zeta(2\mu+1).$$

and $q_r(u)$ being the density of the $r$-th charge

$$q_r(u) = \frac{i}{r-1} \left[ \left( \frac{1}{x^+(u)} \right)^{r-1} - \left( \frac{1}{x^-(u)} \right)^{r-1} \right].$$
4.1 Equations for $F$, $F_0$ and $F^H$.

The counting function (4.3) can be treated according to the general formalism given in Section 2, provided we choose $A = -b$, $B = b$ and

$$
\Phi(u) = -2L \arctan 2u - iL \ln \left( 1 + \frac{g^2}{2x^-(u)^2} \right),
$$

$$
\phi(u,v) = 2 \arctan(u-v) - 2i \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u,v),
$$

$$
f(u) = -2L \arctan 2u - iL \ln \left( 1 + \frac{g^2}{2x^+(u)^2} \right) +
$$

$$
+ 2 \sum_{h=1}^{L-2} \left[ \arctan(u - u_h^{(i)}(g)) - i \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(u_h^{(i)}(g))}}{1 - \frac{g^2}{2x^-(u)x^+(u_h^{(i)}(g))}} \right) + \theta(u, u_h^{(i)}(g)) \right],
$$

$$
+ \frac{1}{\pi} Z(b) \left\{ \arctan(u-b) - i \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(b)}}{1 - \frac{g^2}{2x^-(u)x^+(b)}} \right) + \theta(u,-b) \right\} +
$$

$$
+ \left[ \arctan(u+b) - i \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(b)}}{1 - \frac{g^2}{2x^-(u)x^+(b)}} \right) + \theta(u,-b) \right],
$$

with the explicit $g$-dependence of the internal all loop holes $u_h^{(i)}(g)$. We have supposed that $\text{Im} \ln[1 + e^{iZ(\pm b)}] = 0$, as in the one loop case. It follows that the function $F(u)$ entering the NLIE satisfies the linear equation (2.13) with the function $f(u)$ and $\varphi(u,v) = \frac{1}{2\pi} \frac{d}{dv} \phi(u,v)$ obtained from (4.7). We can split $F(u)$ into its one loop contribution $F_0$ and its higher loop contribution $F^H(u)$:

$$
F(u) = F_0(u) + F^H(u).
$$

Of course, the one loop contribution satisfies the LIE (3.10)

$$
F_0(u) = -2L \arctan 2u - (s + L - 2)[\arctan(u - b_0) + \arctan(u + b_0)] +
$$

$$
+ 2 \sum_{h=1}^{L-2} \arctan(u - u_h^{(i)}(0)) + \int_{-b_0}^{b_0} dv \frac{1}{\pi} \frac{1}{1 + (u-v)^2} F_0(v),
$$

where $u_h^{(i)}(0)$ are indeed the internal one loop holes (as $g = 0$ value of the (internal) all loop holes). On the contrary, the LIE obeyed by the higher loop $F^H(u)$ contains
additionally the $g$-depending holes in the form

\[ F^H(u) = -iL \ln \left( \frac{1 + \frac{g^2}{2x^+(u)x^-}}{1 + \frac{g^2}{2x^+(u)^2}} \right) - 2i \sum_{h=1}^{L-2} \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(u_h)}(g)}{1 - \frac{g^2}{2x^-(u)x^+(u_h)(g)}} \right) + i\theta(u, u_h(i)(g)) \right] + \
+ \left( \arctan(u - u_h(i)(g)) - i \arctan(u - u_h(i)(0)) \right) - \
- \frac{i}{\pi} Z(b) \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-}(b)}{1 - \frac{g^2}{2x^+(u)x^+(b)}} \right) + \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-}(-b)}{1 - \frac{g^2}{2x^-(u)x^+}(-b)} \right) + i\theta(u, b) + i\theta(u, -b) \right] + \
+ \frac{1}{\pi} Z(b) [\arctan(u - b) + \arctan(u + b)] - \frac{1}{\pi} [\arctan(u - b_0) + \arctan(u + b_0)] + \
+ \int_{-b}^{b} \frac{dv}{v} \left[ \frac{1}{\pi (1 + (u - v)^2)} F^H(v) + \int_{-b}^{b} \frac{dv}{1 + (u - v)^2} F_0(v) + \int_{-b}^{b} \frac{dv}{1 + (u - v)^2} F_0(v) + \int_{-b}^{b} \frac{dv}{1 + (u - v)^2} F_0(v) + \right] + \
+ \frac{i}{\pi} \int_{-b}^{b} dv \left[ \frac{d}{dv} \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\frac{d}{dv} \theta(u, v) \right] \left[ F_0(v) + F^H(v) \right]. \tag{4.10}

Now, we proceed to the limit $s \to +\infty$ keeping, of course, only the non-vanishing terms. Upon treating by parts the last integral, the term in the square brackets in the third line of (4.10) gets multiplied by the factor $Z(b) - F(b) = o(s^0)\|$. Therefore it can be neglected. Furthermore, since $u_h(i)(g) = o(s^0)$, we can set everywhere $u_h(i)(g) = 0$. Moreover, as $s \to +\infty$, one has that $Z(b) = O(s)$, $Z_0(b_0) = O(s)$, but also that $\arctan(u - b) + \arctan(u + b) = O(s^{-2})$, $\arctan(u - b_0) + \arctan(u + b_0) = O(s^{-2})$. So, the terms in the fourth line can be neglected as well. Since $b - b_0$ is $O(s^0)$ and $F_0(b)$ is $O(s)$, the second and the third integral in the fifth line can also be removed. For the same reason, in the integrals in the sixth line involving $F_0$ we can set the extremes to their one loop value $\mp b_0$. Finally, in the integrals involving $F^H(v)$ we may replace the extreme, $b$, — which in principle is an unknown of the integral equation (4.10) as well — with $+\infty$, since as $v \to \infty$ $F^H(v)$ becomes constant, while the rest of the integrands vanishes as $1/v^2$ or faster. Rearranging the terms in the right hand side, we have

\[ F^H(u) = -iL \ln \left( \frac{1 + \frac{g^2}{2x^+(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) - 2i (L - 2) \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(0)}}{1 - \frac{g^2}{2x^-(u)x^+(0)}} \right) + i\theta(u, 0) \right] - \
- \frac{i}{\pi} \int_{-b_0}^{b_0} dv \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right] \frac{d}{dv} F_0(v) + \
+ \int_{-\infty}^{+\infty} \frac{dv}{\pi (1 + (u - v)^2)} F^H(v) - \
- \frac{i}{\pi} \int_{-\infty}^{+\infty} dv \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right] \frac{d}{dv} F^H(v) + o(s^0). \tag{4.11}

\[ \text{This is certainly true at one loop, as follows by comparing (3.7) and (3.18). We suppose that it stays true also when considering all the loop corrections.} \]
With such rearrangement, we have collected in the first two lines of the right hand side the forcing terms of our equation. In the last two lines, we have the integral terms, i.e. terms involving integral of a kernel function with the unknown $F^H$.

Several comments are now in order. The LIE (4.11) constraining the density of roots $-\frac{1}{2\pi} \frac{d}{du} F^H(u)$ may be thought of as an improvement of the BES equation [12], which in fact takes into account not only the leading $O(\ln s)$ term, but also the subleading $O(s^0)$ (constant) corrections. This equation is exact and linear and therefore drives the interpolation between weak (small $g$) to strong (large $g$) coupling in a non-perturbative way. In particular, it may be useful for studying the strong coupling regime where the string effects, and in particular the dressing phase, dominate. However, in view of possible comparisons and checks vs. string results, one should focus the attention in cases where the aforementioned wrapping effects are negligible or known.

As in the one loop case, in order to compute the energy and the eigenvalues of the charges in the limit $s \to \infty$ and including constant terms, in our framework it is sufficient to consider the third line of the general formula (2.17). Explicitly,

$$E(g,s) = \int_{-b_0}^{b_0} \frac{dv}{2\pi} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] F_0(v) - (L-2) \left[ \frac{i}{x^+(0)} - \frac{i}{x^-(0)} \right] + \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] F^H(v) + o(s^0).$$

This means that, in contrast to the approach of [16], where the [8] method is used, we have to cope only with linear equations. In respect to (4.12) and for future convenience, we introduce the function

$$h(g,s) = -\int_{-b_0}^{b_0} \frac{dv}{2\pi} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] \frac{d}{dv} F_0(v) - (L-2) \left[ \frac{i}{x^+(0)} - \frac{i}{x^-(0)} \right] = \int_{-b_0}^{b_0} \frac{dv}{2\pi} \frac{2i}{x^-(v)} \frac{d}{dv} F_0(v) + 2i(L-2) \frac{1}{x^-(0)} ,$$

in such a way that the energy reads

$$E(g,s) = h(g,s) - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] \frac{d}{dv} F^H(v) + o(s^0).$$

Direct calculations show that

$$h(g,s) = 4 \ln s + 4\gamma_E - 4(L-2) \ln 2 + h(g),$$

$$h(g) = 2g^2[3L\zeta(3) - 7\zeta(3)] + g^4(62 - 30L)\zeta(5) + O(g^6) + o(s^0).$$

In addition, we remark that equation (4.11) has the same kernel as the BES equation. They differ only in their forcing terms, since in the BES case, in contrast to (4.11), the
forcing terms are simply
\[
\left\{ \frac{ig^2}{2\pi} \left[ \frac{1}{x^+ (u)} + \frac{1}{x^- (u)} \right] + \frac{2i}{\pi} \sum_{\nu=0}^{\infty} \beta_{2,3+2\nu} (g) q_{3+2\nu} (u) \right\} \int_{-b_0}^{b_0} dv \frac{1}{v - \frac{i}{2}} dF_0 (v) =
\]
\[
= 2 g^2 \ln s \left[ \frac{1}{x^+ (u)} + \frac{1}{x^- (u)} \right] + 8 \ln s \sum_{\nu=0}^{\infty} \beta_{2,3+2\nu} (g) q_{3+2\nu} (u) + O (s^0),
\] (4.16)

where we used the one loop results at order \( \ln s \):
\[
\int_{-b_0}^{b_0} dv \frac{1}{2\pi v + \frac{i}{2}} dF_0 (v) = - \int_{-b_0}^{b_0} dv \frac{1}{2\pi v - \frac{i}{2}} dF_0 (v) = 2 i \ln s + O (s^0).
\] (4.17)

Our forcing terms differ from the ones in the BES equation for two reasons. Firstly, in addition to the BES terms, we have genuine new terms coming from the first line of the right hand side of (4.11). If we expand them in powers of \( g^2 \), we see that they are structurally different from the BES forcing terms, with the exception of the term
\[
2 i (L - 2) \left[ \frac{g^2}{2x^+ (u) x^- (0)} - \frac{g^2}{2x^- (u) x^+ (0)} \right] - 2 (L - 2) \sum_{\nu=0}^{\infty} \beta_{2,3+2\nu} (g) q_{3+2\nu} (u) q_2 (0) =
\]
\[
\left\{ \frac{ig^2}{2\pi} \left[ \frac{1}{x^+ (u)} + \frac{1}{x^- (u)} \right] + \frac{2i}{\pi} \sum_{\nu=0}^{\infty} \beta_{2,3+2\nu} (g) q_{3+2\nu} (u) \right\} \frac{2\pi (L - 2)}{x^- (0)},
\] (4.18)

which is proportional to the BES forcing terms. Secondly, for what concerns the terms in the second line of the right hand side of (4.11), in their expansions in powers of \( g^2 \) all the terms have to be kept, since we want to be precise in \( s \) up to the order \( s^0 \). However, some of these terms, namely
\[
\left\{ \frac{ig^2}{2\pi} \left[ \frac{1}{x^+ (u)} + \frac{1}{x^- (u)} \right] + \frac{2i}{\pi} \sum_{\nu=0}^{\infty} \beta_{2,3+2\nu} (g) q_{3+2\nu} (u) \right\} \int_{-b_0}^{b_0} dv \frac{1}{x^- (v)} dF_0 (v),
\]
as functions of \( u \), are also proportional to the BES ones. We conclude that in our equation (4.11) a part of the forcing terms,
\[
\left\{ \frac{ig^2}{2\pi} \left[ \frac{1}{x^+ (u)} + \frac{1}{x^- (u)} \right] + \frac{2i}{\pi} \sum_{\nu=0}^{\infty} \beta_{2,3+2\nu} (g) q_{3+2\nu} (u) \right\} \cdot \left[ \int_{-b_0}^{b_0} dv \frac{1}{x^- (v)} dF_0 (v) + \frac{2\pi (L - 2)}{x^- (0)} \right] =
\]
\[
= \left\{ \frac{ig^2}{2\pi} \left[ \frac{1}{x^+ (u)} + \frac{1}{x^- (u)} \right] + \frac{2i}{\pi} \sum_{\nu=0}^{\infty} \beta_{2,3+2\nu} (g) q_{3+2\nu} (u) \right\} \frac{\pi}{i} h (g, s),
\] (4.19)

is proportional to the forcing terms in the BES equation (the forcing term in the BES equation is given by (4.19) in which \( h (g, s) \) is replaced by \( 4 \ln s \)). Therefore we are
Let us define the perturbative expansion in order to solve equation (4.11) and to compute the energy.

\[ F^H(u) = \frac{h(g,s)}{4\ln s} F^{BES}(u) + F^{extra}(u) + o(s^0). \]  

(4.20)

This means that in the expression for the energy \( E(g,s) \) we expect that

\[ E(g,s) = h(g,s) + \frac{1}{4} h(g,s)[f(g) - 4] + E^{extra}(g,s) + o(s^0) = \]

\[ = \frac{1}{4} h(g,s)f(g) + E^{extra}(g,s) + o(s^0), \]  

(4.21)

where \( f(g) \) is the universal scaling function of \( N = 4 \) SYM and \( E^{extra}(g,s) \) indicates contributions coming from \( F^{extra}(u) \).

We can find the structure (4.21) for the energy \( E(g,s) \), by performing a brute force perturbative expansion in order to solve equation (4.11) and to compute the energy. Let us define

\[ F^H(u) = g^2 F^H_1(u) + g^4 F^H_2(u) + O(g^6). \]  

(4.22)

At the order \( g^2 \) we have the equation

\[ F^H_1(u) = \frac{L}{2i} \left[ \frac{1}{(u - \frac{i}{2})^2} - \frac{1}{(u + \frac{i}{2})^2} \right] + \int_{-b}^{b} dv \frac{1}{\pi} \frac{1}{1 + (u - v)^2} F^H_1(v) - \]

\[ - 2(L - 2) \left( \frac{1}{u + \frac{i}{2}} + \frac{1}{u - \frac{i}{2}} \right) + \frac{1}{2i\pi} \int_{-b}^{b} dv \frac{1}{v + \frac{i}{2}} \frac{d}{dv} F_0(v) - \]

\[ - \frac{1}{2i\pi} \frac{1}{u + \frac{i}{2}} \int_{-b}^{b} dv \frac{1}{v + \frac{i}{2}} \frac{d}{dv} F_0(v) + o(s^0). \]  

(4.23)

We can now use the one loop results and then pass to the Fourier transform

\[ \hat{F}^H_1(k) = \frac{\pi L \zeta}{2} e^{-\frac{|k|}{2}} + e^{-|k|} \hat{F}^H_1(k) - \]

\[ - 4\pi i \ln s + \gamma_E - (L - 2) \ln 2 \right] \text{sgn}(k) e^{-\frac{|k|}{2}} \]  

(4.24)

Solving this equation and going back to the coordinate space we reach

\[ F^H_1(u) = 2\pi \ln s + \gamma_E - (L - 2) \ln 2 \right] \tanh u\pi + \frac{L}{2i} \left[ \psi' \left( \frac{1}{2} - iu \right) - \psi' \left( \frac{1}{2} + iu \right) \right]. \]

In a similar fashion, one computes \( F^H_2(u) \). We omit the details and give only the final result:

\[ F^H_2(u) = \frac{L}{16} \frac{d^3}{du^3} \left[ \psi \left( \frac{1}{2} - iu \right) + \psi \left( \frac{1}{2} + iu \right) \right] - \]

\[ - \frac{\pi^2}{12} (L - 3) \frac{d}{du} \left[ \psi \left( \frac{1}{2} - iu \right) + \psi \left( \frac{1}{2} + iu \right) \right] + \]

\[ + \frac{\pi}{2} \ln s + \gamma_E - (L - 2) \ln 2 \right] \frac{d^2}{du^2} \tanh \pi u - \]

\[ - \pi \left\{ \frac{\pi^2}{3} \ln s + \gamma_E - (L - 2) \ln 2 \right] + 7\zeta(3) - 2L\zeta(3) \right\} \tanh \pi u. \]  

(4.25)
Therefore, we are allowed to write that

\[ F^H(u) = \left[ \ln s + \gamma_E - (L - 2) \ln 2 \right] \left( 2\pi g^2 \tanh \pi u + \frac{\pi}{2} g^4 \frac{d^2}{du^2} \tanh \pi u - \frac{\pi^3}{3} g^4 \tanh \pi u \right) - \pi (7\zeta(3) - 2L\zeta(3)) g^4 \tanh \pi u + \left[ \frac{L}{2} g^2 - \frac{\pi^2}{12} (L - 3) g^4 \right] \frac{d}{du} \left[ \psi \left( \frac{1}{2} - iu \right) + \psi \left( \frac{1}{2} + iu \right) \right] + g^4 \frac{L}{16} \frac{d^3}{dv^3} \left[ \psi \left( \frac{1}{2} - iu \right) + \psi \left( \frac{1}{2} + iu \right) \right] + O(g^6). \] (4.26)

After obtaining these results we can evaluate the energy \( E(g, s) \) up to the order \( g^4 \) and at the orders \( \ln s \) and \( s^0 \). From the general formula (2.17) we may write that

\[ E(g, s) = \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] [F_0(v) + F^H(v)] - (L - 2) \left[ \frac{i}{x^+(0)} - \frac{i}{x^-(0)} \right] + o(s^0). \] (4.27)

Defining the coefficients of the expansion in powers of \( g^2 \) as

\[ E(g, s) = E_0(s) + E_1(s) g^2 + E_2(s) g^4 + O(g^6), \] (4.28)

we have that

\[ E_0(s) = \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{i}{v + \frac{i}{2}} - \frac{i}{v - \frac{i}{2}} \right] F_0(v) - 4(L - 2) + O(s^{-1}) = 4 \ln s + 4\gamma_E - 4(L - 2) \ln 2 + o(s^0). \] (4.29)

On the other hand the order \( g^2 \) of \( E(g, s) \) is given by

\[ E_1(s) = \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{i}{v + \frac{i}{2}} - \frac{i}{v - \frac{i}{2}} \right] F_1^H(v) + \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{i}{\left( v + \frac{i}{2} \right)^3} - \frac{i}{\left( v - \frac{i}{2} \right)^3} \right] F_0(v) + 8(L - 2) + o(s^0). \] (4.30)

The first line of (4.30) gives

\[ \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{i}{v + \frac{i}{2}} - \frac{i}{v - \frac{i}{2}} \right] F_1^H(v) = - \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{i}{v + \frac{i}{2}} - \frac{i}{v - \frac{i}{2}} \right] \frac{d}{dv} F_1^H(v) + o(s^0) = \]

\[ = -\pi \ln s + \gamma_E - (L - 2) \ln 2 \int_{-b}^{b} \frac{1}{v + \frac{i}{2}} + \frac{1}{v - \frac{i}{2}} \frac{1}{\cosh^2 \pi v} - \frac{L}{2} \left[ \psi \left( \frac{1}{2} - iv \right) + \psi \left( \frac{1}{2} + iv \right) \right] + o(s^0) = \]

\[ = -\frac{2}{3} \pi^2 \ln s + \gamma_E - (L - 2) \ln 2 - 2L\zeta(3) + o(s^0). \] (4.31)
On the other hand, the second line of (4.30) gives
\[
\int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{i}{2} \left( \frac{v + i/2}{v - i/2} \right)^3 - \frac{i}{2} \left( \frac{v - i/2}{v + i/2} \right)^3 \right] F_0(v) + 8(L - 2) = \]
\[
= \frac{L}{2} \psi''(1) - \frac{L - 2}{2} \psi'' \left( \frac{3}{2} \right) + 8(L - 2) + o(s^0) = 6L\zeta(3) - 14\zeta(3) + o(s^0).
\]
Adding up the two contributions we obtain
\[
E_1(s) = -\frac{2}{3} \pi^2 \ln s + \gamma_E - (L - 2) \ln 2] + 4L\zeta(3) - 14\zeta(3) + o(s^0). \tag{4.32}
\]

Analogously the third order in the energy, \(E_2(s)\), given by
\[
E_2(s) = \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{1}{2-i v} + \frac{1}{2+i v} \right] F_2^H(v) - \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{1}{(1/2-i v)^3} + \frac{1}{(1/2+i v)^3} \right] F_1^H(v) + \int_{-b}^{b} \frac{dv}{2\pi} \frac{d}{dv} \left[ \frac{1}{(1/2-i v)^5} + \frac{1}{(1/2+i v)^5} \right] F_0(v) - 32(L - 2) + o(s^0), \tag{4.33}
\]
after similar and lengthy calculations, equals
\[
E_2(s) = \frac{11}{45} \pi^4 \ln s + \gamma_E - (L - 2) \ln 2] + \]
\[
+ \frac{\pi^2}{3} (4 - L)\zeta(3) + (62 - 21L)\zeta(5) + o(s^0). \tag{4.34}
\]
Collecting all these terms we have
\[
E(g, s) = [\ln s + \gamma_E - (L - 2) \ln 2] \left( 4 - \frac{2}{3} \pi^2 g^2 + \frac{11}{45} \pi^4 g^4 \right) + g^2 [4L\zeta(3) - 14\zeta(3)] + \]
\[
+ g^4 \frac{\pi^2}{3} (4 - L)\zeta(3) + g^4 (62 - 21L)\zeta(5) + O(g^6) + o(s^0). \tag{4.35}
\]
We recognize in (4.35) the expansion of the universal scaling function \(f(g)\),
\[
f(g) = 4 - \frac{2}{3} \pi^2 g^2 + \frac{11}{45} \pi^4 g^4 + O(g^6), \tag{4.36}
\]
and remark that, consistently with (4.21), relation (4.35) can be written also
\[
E(g, s) = \frac{1}{4} h(g, s) f(g) + E^{extra}(g, s) + O(g^6) + o(s^0), \tag{4.37}
\]
where \(h(g, s)\) is given by (4.15) and
\[
E^{extra}(g, s) = -2 g^2 L\zeta(3) + \frac{2}{3} \pi^2 g^4 L\zeta(3) - \pi^2 g^4 \zeta(3) + 9 Lg^4 \zeta(5) + O(g^6). \tag{4.38}
\]
We remark also that expansion (4.35) agrees with result (3.16) of [16].

Finally, using similar techniques one computes, up to the order \( g^6 \), the eigenvalues of all the conserved charges,

\[
Q_r(g, s) = \frac{i}{r - 1} \sum_{k=1}^{s} \left[ \left( \frac{1}{x^+(u_k)} \right)^{r-1} - \left( \frac{1}{x^-(u_k)} \right)^{r-1} \right],
\]

with \( r, r' \geq 4 \). Defining

\[
Q_r(g, s) = Q_{r,0}(s) + Q_{r,1}(s)g^2 + Q_{r,2}(s)g^4 + O(g^6),
\]

we obtain the following results

\[
Q_{r,0}(s) = \frac{2(-1)^{s-1} \zeta(r-1)}{r-1}[(2 - 2^{r-1})L - 2(1 - 2^{r-1})] + o(s^0),
\]

\[
Q_{r,1}(s) = 4(-1)^{s} \zeta(r)[\ln s + \gamma_E - (L - 2) \ln 2] + L(-1)^{\frac{r}{2}}(r + 2 - 2^{r+1})\zeta(r + 1) + 2(-1)^{\frac{r}{2}}(2^{r+1} - 1)\zeta(r + 1) + o(s^0),
\]

\[
Q_{r,2}(s) = \frac{1}{8}(-1)^{s+1}L(r + 2)(r + 1)\zeta(r + 3) + r(-1)^{s+1}(L - 3)\zeta(2)\zeta(r + 1) + 2(-1)^{s+1}(r + 1)\zeta(r + 1) + 2(-1)^{s+1}(r + 1)L\zeta(r + 3) + \frac{1}{2}(-1)^{s+1}(r + 1)^2\zeta(r + 2) + L\zeta(r + 3) + \frac{1}{2}(-1)^{s+1}(r + 1)^2\zeta(r + 2) + L\zeta(r + 3) + o(s^0).
\]

As far as we know these expansions are new results.

### 4.2 The NLIE in the \( s \to +\infty \) limit

As in the one loop case, in the limit \( s \to +\infty \) the NLIE satisfied by the counting function reduces to the linear equation satisfied by the density of roots, i.e. the BES equation.

The counting function satisfies the NLIE

\[
Z(u) = -2L \arctan 2u - iL \ln \left( \frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) + 2 \int_{-b}^{b} \frac{dv}{2\pi} \arctan(u - v) \frac{d}{dv}[Z(v) - 2L(v)] - 2i \int_{-b}^{b} \frac{dv}{2\pi} \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)^2}}{1 - \frac{g^2}{2x^-(u)^2}} \right) + i\theta(u, v) \right] \frac{d}{dv}[Z(v) - 2L(v)] + 2 \sum_{h=1}^{L-2} \arctan(u - u_h^{(i)}) - 2i \sum_{h=1}^{L-2} \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)^2}}{1 - \frac{g^2}{2x^-(u)^2}} \right) + i\theta(u, u_h^{(i)}) \right],
\]

(4.41)
where \( u_h^{(i)} \) refers to the holes present inside the interval \([-b, b]\).

As in the one loop case, we go to the limit \( s \to +\infty \). In this limit, at the leading order in \( s \) (i.e. \( \ln s \)) we can drop the second and the last term in the rhs of the NLIE, as well as also the terms containing \( L(v) = \text{Im} \ln[1 + e^{iZ(u-i\theta)}] \). We end up with the linear equation

\[
Z(u) = -2L \arctan 2u + 2 \int_{-b}^{b} \frac{dv}{2\pi} \arctan(u-v) \frac{d}{dv} Z(v) - 2i \int_{-b}^{b} \frac{dv}{2\pi} \left[ \ln \left( \frac{1 - \frac{q^2}{2x^+(u)x^-(v)}}{1 - \frac{q^2}{2x^-(u)x^+(v)}} \right) + i\theta(u,v) \right] \frac{d}{dv} Z(v) + 2 \sum_{h=1}^{L-2} \arctan(u - u_h^{(i)}).
\]

It seems natural to split the solution \( Z(u) \) as

\[
Z(u) = Z_0(u) + Z^H(u),
\]

where \( Z_0(u) \) is the solution of the one loop part of (4.42), already written in (3.24) and \( Z^H(u) \) is the solution of the higher than one loop part of (4.42),

\[
Z^H(u) = -2i \int_{-b_0}^{b_0} \frac{dv}{2\pi} \left[ \ln \left( \frac{1 - \frac{q^2}{2x^+(u)x^-(v)}}{1 - \frac{q^2}{2x^-(u)x^+(v)}} \right) + \theta(u,v) \right] \frac{d}{dv} Z_0(v) + 2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[ \arctan(u-v) - i \ln \left( \frac{1 - \frac{q^2}{2x^+(u)x^-(v)}}{1 - \frac{q^2}{2x^-(u)x^+(v)}} \right) + \theta(u,v) \right] \frac{d}{dv} Z^H(v).
\]

Let us now define the densities

\[
\rho_0(u) = -\frac{1}{2\pi s} \frac{d}{du} Z_0(u) = \frac{1}{s} \rho_0(\bar{u}), \quad \sigma^H(u) = -\frac{1}{2\pi sg^2} \frac{d}{du} Z^H(u),
\]

In terms of them, (4.44) reads as follows

\[
0 = 2\pi \sigma^H(u) - 2 \int_{-\infty}^{+\infty} dv \left[ \frac{1}{1 + (u-v)^2} - i \frac{d}{du} \ln \left( \frac{1 - \frac{q^2}{2x^+(u)x^-(v)}}{1 - \frac{q^2}{2x^-(u)x^+(v)}} \right) + \frac{d}{du} \theta(u,v) \right] \sigma^H(v) + 2i \int_{-b_0}^{b_0} dv \frac{d}{du} \left[ \ln \left( \frac{1 - \frac{q^2}{2x^+(u)x^-(v)}}{1 - \frac{q^2}{2x^-(u)x^+(v)}} \right) + i\theta(u,v) \right] \rho_0(v).
\]

This equation coincides with the BES equation. In particular, if we drop the dressing factor, it reduces to the ES equation - see (65) of [17].

### 4.3 An alternative derivation of the anomalous dimension

In the large spin limit, for the leading \( O(\ln s) \) contribution, \( f(g) \), an elegant proportionality holds between the all-loops energy (anomalous dimension) and the Fourier zero
mode of the higher-than-one-loop density of roots [21]. Here, we are going to show that the connection extends up to the $O(s^0)$ order.

Regardless the normalisation, we may define the higher than one loop and one loop densities as

$$
\sigma_H(u) = \frac{du}{F_H(u)}, \quad \sigma_0(u) = \frac{du}{F_0(u)},
$$

respectively, and write down their specific linear integral equations in the Fourier space. For we may move from (4.11) and use the fundamental Fourier transforms

$$
\int_{-\infty}^{\infty} du e^{-iku} \left[ \frac{1}{x^\pm(u)} \right]^r = \pm r \left( \frac{\sqrt{2}}{i \theta} \right)^r \theta(\pm k) \frac{2\pi}{k} e^{\mp \frac{\pi}{2} r k} J_r(\sqrt{2} g k), \tag{4.47}
$$

to obtain

$$
\hat{\sigma}_H(k) = \pi L \frac{1 - J_0(\sqrt{2} g k)}{\sinh \frac{|k|}{2}} + \frac{1}{2 \sinh \frac{|k|}{2}} \int_{-\infty}^{\infty} dh \sum_{r=1}^{\infty} \frac{r}{h} (-1)^{r+1} J_r(\sqrt{2} g k) J_r(\sqrt{2} g h) (1 - \text{sgn}(kh)) e^{-\frac{|h|}{h}} + 2 \sum_{\nu=0}^{\infty} \sum_{r=2}^{\infty} c_{r,r+2\nu}(g)(-1)^{r+\nu} e^{-\frac{|h|}{h}} \left( J_{r-1}(\sqrt{2} g k) J_{r+2\nu}(\sqrt{2} g h) - J_{r-1}(\sqrt{2} g h) J_{r+2\nu}(\sqrt{2} g k) \right) [\hat{\sigma}_H(h) + \hat{\sigma}_0(h) + 2\pi(L-2)] + o(s^0). \tag{4.51}
$$

Performing now the limit $k \to 0^\pm$, we gain

$$
\lim_{k \to 0^+} \hat{\sigma}_H(k) = \lim_{k \to 0^-} \hat{\sigma}_H(k) = -\frac{g}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{dh}{h} J_1(\sqrt{2} g h) e^{-\frac{|h|}{h}} [\hat{\sigma}_H(h) + \hat{\sigma}_0(h) + 2\pi(L-2)] + o(s^0). \tag{4.48}
$$

On the other hand, if we re-write the energy (4.12) up to the order $s^0$ by means of the Fourier transforms (4.47), we also obtain

$$
E(g, s) = -\frac{1}{\sqrt{2\pi g}} \int_{-\infty}^{\infty} \frac{dh}{h} J_1(\sqrt{2} g h) e^{-\frac{|h|}{h}} [\hat{\sigma}_H(h) + \hat{\sigma}_0(h) + 2\pi(L-2)] + o(s^0). \tag{4.49}
$$

Upon comparing (4.48) and (4.49), we size the desired relation

$$
\hat{\sigma}_H(0) = \pi g^2 E(g, s) + o(s^0). \tag{4.50}
$$

Now, in order to check this result we can repeat the perturbative expansion (4.22) and extract from it the quantities $E_0(s)$ and $E_1(s)$. In fact, (4.24) yields, at order $g^2$, the Fourier transform

$$
\hat{\sigma}_1^H(k) = \frac{1}{2 \sinh(k/2)} \left[ \pi L k^2 + 4\pi |k| (\ln s + \gamma_E - (L-2) \ln 2) \right], \tag{4.51}
$$

which just verifies

$$
\hat{\sigma}_1^H(0) = \pi E_0(s). \tag{4.52}
$$
Similarly, (4.25) implies
\[
\hat{\sigma}^H_2(k) = -\frac{1}{2\sinh(\frac{|k|}{2})}\left\{ \frac{\pi L k^4}{8} + \frac{\pi |k|^3 E_0(s)}{4} + \frac{\pi^3 (L - 3) k^2}{6} + \frac{\pi^3 |k| E_0(s)}{6} + 2\pi |k| [7\zeta(3) - 2L\zeta(3)] \right\}.
\]

(4.53)

Finally, for \( k = 0 \)
\[
\hat{\sigma}^H_2(0) = -\frac{\pi^3 E_0(s)}{6} - 2\pi [7\zeta(3) - 2L\zeta(3)] = \pi E_1(s) .
\]

(4.54)

Of course, these results, (4.52, 4.54), agree with the general one (4.50).

5 Summary

In this paper we have developed and applied the technique of the NLIE on intervals sketched in [13]. This new formalism allows to treat magnon scattering matrices with general dependence on the rapidities and states with roots on intervals of the real line (or even of complex lines). Therefore, it seems perhaps more indicated than the historical method presented in [8], if we want to study Bethe equations appearing in the context of \( \mathcal{N} = 4 \) SYM.

We have given an explicit application of the NLIE on interval for the Bethe Ansatz type equations describing the \( sl(2) \) sector of \( \mathcal{N} = 4 \) SYM. In particular, we have written the exact equations which allow us to define the relevant functions (i.e. the forcing term \( F \) and the kernel function \( G \)) entering the NLIE. Then, we passed on to studying the limit of large number of Bethe roots (or spin \( s \)). In this limit and at the leading order \( \ln s \), the NLIE as well as the equation for the derivative of the forcing term naturally becomes the BES equation. If we take into account also the sub-leading correction \( O(s^0) \), the forcing term satisfies a modification of the BES equation, with a different inhomogeneous term. Therefore, this equation is suitable also for a (non-perturbative and) strong coupling study, even if possible results in this direction should be corrected by eliminating the wrapping effects. Interestingly, we noticed that in the formalism of the NLIE on intervals the non-linear terms are negligible as \( s \to +\infty \). Therefore, in order to determine the eigenvalues of the conserved charges up to the order \( s^0 \), it is sufficient to consider in their general expressions only linear terms, involving the forcing term. This conceptually and practically enormously simplifies their calculations and could suggest applications of our formalism to cases in which wrapping effects are under control or absent.

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A Numerical evaluation of the non-linear integrals

In this appendix, we will analyse the large spin $s$ behaviour of the non-linear integral function in (2.5) and of the non-linear integral in (2.17). Both involve the logarithmic function (of $Z(u)$), $L(u)$, as defined in (2.10). In principle, we could pursue an analytic saddle point evaluation as in the sine-Gordon case [8], since anew, because of its forcing term, the counting function $Z$ scales like the size $\ln s$, thus implying, roughly speaking, a correction of order $\sim e^{-\ln s} = 1/s$: we will see that this easy conclusion should be not far from reality. Nevertheless, we find here more instructive for us to gain some flavour by performing a still easy numerical exercise. In fact, for simplicity’s sake we restrict ourselves to the one loop case and, in the aforementioned spirit, we approximate $Z_0(u)$ in $L_0(u)$ by the forcing term $F_0(u)$ as given by (3.52). In the numerical implementation, we must learn to write the integrals for finite $\epsilon$: thanks to the Cauchy theorem, we may move the integration contour in the complex plane as far as we do not meet a singularity

$$\delta Z_0(u; s) = -\text{Im} \int_{b_0(s)}^{b_0(s)} \frac{dv}{\pi} \frac{\phi_0(u, v - i\epsilon)}{\phi_0(u, v - i\epsilon)} \left[ \frac{d}{dv} \ln [1 + e^{iZ_0(u - i\epsilon)}] - \right]$$

$$- \text{Im} \int_0^\epsilon \frac{dy}{\pi} \left\{ \phi_0(u, -b_0(s) - iy) \frac{d}{dy} \ln [1 + e^{iZ_0(-b_0(s) - iy)}] - \phi_0(u, b_0(s) - iy) \frac{d}{dy} \ln [1 + e^{iZ_0(b_0(s) - iy)}] \right\},$$

with $\phi_0(u, v)$ defined in (3.8) and $b_0(s)$ determined by the condition (3.7), which gives $b_0(s) \simeq 0.472 s$. In fact, if we should assume exactly $b_0(s) = s/2$, we would observe an external hole jumping into the interval of integration at a certain value of $s$. Of course, this would give a deceiving discontinuous dependence of the integrals (A.1) on $s$. In other words, the value $s/2$ is not a good one as for the separator $11$ between the roots and the external hole. Once clarified this crucial point, we may evaluate the integrals with Mathematica. For instance, we report in Fig. 1 the behaviour of (A.1) with $s$ on the $x$-axis, for $u = 1$ and $\epsilon = 0.1$: the leading $\sim 1/s$ decrease can be easily spotted. Eventually, we want to numerically evaluate the non-linear integral in the one-loop energy, namely the last line of (2.17), once we specialise the observable $O(v) = q_2(v)$. We still need to take into account the ”lateral” contributions due to the integration over $(-i \epsilon, i \epsilon)$ and then, upon using the leading order solution $G_0(v, w)$ of Section 3.3,
we obtain

\[
\delta E_0(s) = \text{Im} \int_{-b_0(s)}^{b_0(s)} \frac{dv}{\pi} q_2(v) \int_{-b_0(s)}^{b_0(s)} dw \frac{d}{dv} \left[ G_0(v, w - i\epsilon) - \delta(v - b_0(s) + iy) \right] \ln \left[ 1 + e^{iZ_0(-b_0(s) - iy)} \right] - \delta(v + b_0(s) + iy) \frac{d}{dy} \ln \left[ 1 + e^{iZ_0(-b_0(s) - iy)} \right] - \frac{d}{dv} \left[ G_0(v, b_0(s) - iy) - \delta(v - b_0(s) + iy) \right] \frac{d}{dy} \ln \left[ 1 + e^{iZ_0(b_0(s) - iy)} \right].
\]

Actually, since the counting function is odd, we may substitute the \(v\)-derivative of \(G_0(v, w)\) directly the antisymmetric combination (3.37). The dependence of \(\delta E_0(s)\) on \(s\) is plotted in Fig. 2, where we still have \(\epsilon = 0.1\), and also in this case the behaviour seems to be suitably fitted by a polynomial of \(1/s\), yielding again a \(1/s\) leading contribution. Unfortunately, this one loop set-up cannot exhibit the appearance of the logarithms \(\ln s\), in agreement with the large \(s\) expansions of \([19, 20, 23]\).
Figure 2: Comparison between numerical evaluations (dots) of $\delta E_0(s)$ and their best fit as $b_1/s + b_2/s^2 + b_3/s^3$ (line), with $b_1 \simeq 0.675$, $b_2 \simeq -4.667$, $b_3 \simeq -39.390$.

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