Representation-theoretic properties of balanced big Cohen–Macaulay modules

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Abstract
Let \((R, m, k)\) be a complete Cohen–Macaulay local ring. In this paper, we assign a numerical invariant, for any balanced big Cohen–Macaulay module, called \(h\)-length. Among other results, it is proved that, for a given balanced big Cohen–Macaulay \(R\)-module \(M\) with an \(m\)-primary cohomological annihilator, if there is a bound on the \(h\)-length of all modules appearing in \(CM\)-support of \(M\), then it is fully decomposable, i.e. it is a direct sum of finitely generated modules. While the first Brauer–Thrall conjecture fails in general by a counterexample of Dieterich dealing with multiplicities to measure the size of maximal Cohen–Macaulay modules, our formalism establishes the validity of the conjecture for complete Cohen–Macaulay local rings. In addition, the pure-semisimplicity of a subcategory of balanced big Cohen–Macaulay modules is settled. Namely, it is shown that \(R\) is of finite \(CM\)-type if and only if \(R\) is an isolated singularity and the category of all fully decomposable balanced big Cohen–Macaulay modules is closed under kernels of epimorphisms. Finally, we examine the mentioned results in the context of Cohen–Macaulay artin algebras admitting a dualizing bimodule \(\omega\), as defined by Auslander and Reiten. It will turn out that, \(\omega\)-Gorenstein projective modules with bounded \(CM\)-support are fully decomposable. In particular, a Cohen–Macaulay algebra \(\Lambda\) is of finite \(CM\)-type if and only if every \(\omega\)-Gorenstein projective module is of finite \(CM\)-type, which generalizes a result of Chen for Gorenstein algebras. Our main tool in the proof of results is Gabriel–Roiter (co)measure, an invariant assigned to modules of finite length, and defined by Gabriel and Ringel. This, in fact, provides an application of the Gabriel–Roiter (co)measure in the category of maximal Cohen–Macaulay modules.

Keywords Balanced big Cohen–Macaulay modules · Finite Cohen–Macaulay type · First Brauer–Thrall conjecture · Gabriel–Roiter (co)measure · Maximal Cohen–Macaulay modules · \(h\)-length

Mathematics Subject Classification 13C14 · 16G60 · 13H10 · 13D07 · 16E65

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1 Introduction

In representation theory of artin algebras, there is a large body of work on the connections between representation-theoretic properties of the category of finitely generated A-modules and global structural properties of the algebra A. In this direction, the first Brauer–Thrall conjecture asserts that if a finite-dimensional algebra A over a field k is of bounded representation type (meaning that there is a bound on the length of the indecomposable finitely generated A-modules), then A is of finite representation type, i.e. the set of isomorphism classes of indecomposable finitely generated modules is finite; see [26]. This conjecture was proved by Roiter [38] and it is proved by Ringel [34,35] over artin algebras. Another instance of this connection is the pure-semisimple conjecture which predicts that every left pure-semisimple ring (a ring over which every left module is a direct sum of finitely generated ones) is of finite representation type. Left pure-semisimple rings are known to be left artinian by a result of Chase [11, Theorem 4.4]. The validity of the pure-semisimple conjecture for artin algebras comes from a famous result of Auslander [2,4] (see also Ringel–Tachikawa [37, Corollary 4.4]), where they have shown that an artin algebra A is of finite representation type if and only if every left A-module is a direct sum of finitely generated ones. Motivated by Auslander’s result, studying decomposition of Gorenstein projective modules over artin algebras into finitely generated ones has been the subject of several expositions (see [9,13,27,39,40]). In particular, a result of Beligiannis [9, Theorem 4.10] asserts that a virtually Gorenstein algebra A is of finite Gorenstein representation type, in the sense that there are only finitely many isomorphism classes of indecomposable finitely generated modules. Gorenstein projective A-modules if and only if any left Gorenstein projective A-module is a direct sum of finitely generated ones. This solves a problem raised by Chen [13], who proved it for Gorenstein artin algebras.

On the other hand, over the past several decades Cohen–Macaulay rings and maximal Cohen–Macaulay modules have achieved a great deal of significance in commutative algebra and algebraic geometry. Hochster and Huneke [19] write that for many theorems “the Cohen–Macaulay condition (possibly on the local rings of a variety) is just what is needed to make the theory work.” Let \((R, m, k)\) be a commutative noetherian local ring. Hochster [21] defines a not necessarily finitely generated \(R\)-module \(M\) is big Cohen–Macaulay, if there exists a system of parameters of \(R\) which is an \(M\)-regular sequence. Sharp [41] called a big Cohen–Macaulay \(R\)-module \(M\) is (weak) balanced big Cohen–Macaulay, ((weak) balanced big CM, for short), provided that every system of parameters of \(R\) is an (a weak) \(M\)-regular sequence. A finitely generated \(R\)-module \(M\) is maximal Cohen–Macaulay (abbreviated, MCM), if it is either balanced big Cohen–Macaulay or zero.
Motivated by the above mentioned results, the major issues considered in this paper are when a given balanced big CM module is a direct sum of finitely generated modules; when every balanced big CM module is so; analogues of the first Brauer–Thrall conjecture for modules and analogues result for ω-Gorenstein projective modules over Cohen–Macaulay artin algebras in the sense of Auslander and Reiten [6,7].

A natural interpretation of the first Brauer–Thrall conjecture in this context, states that a commutative noetherian local ring \((R, m)\) is of finite Cohen–Macaulay type, provided that there is a bound on the multiplicities of indecomposable MCM modules. Recall that \(R\) is said to be of finite Cohen–Macaulay type (finite CM-type, for short), if there are only finitely many non-isomorphic indecomposable MCM \(R\)-modules. An example discovered by Dieterich [14], disproved the conjecture in general. However, over several classes of rings, this conjecture is known to be true. Namely, it has been answered affirmatively for complete, equicharacteristic CM local rings [43]. This result was extended by Leuschke and Wiegand [29, Theorem 3.4] to the case where the ring is equicharacteristic excellent with algebraically closed residue field \(k\). On the other hand, inspired by the pure-semisimplicity conjecture, Beligiannis [9, Theorem 4.20] has shown that a commutative noetherian Gorenstein complete local ring \(R\) being of finite CM-type is tantamount to saying that any Gorenstein projective \(R\)-module is a direct sum of finitely generated modules.

In this paper, we focus our attention on modules of finite type. In fact, we will treat the support of a module, instead of all finitely generated indecomposable modules. Recall that the support of a module \(M\) over an artin algebra \(\Lambda\), denoted by \(\text{supp}_\Lambda(M)\), is the set of all indecomposable finitely generated \(\Lambda\)-modules \(N\) such that \(\text{Hom}_\Lambda(N, M) \neq 0\). It is a consequence of nice results of Auslander [3, Theorem B] and also Ringel [35, Theorem 1] that, for a given \(\Lambda\)-module \(M\), if \(\text{supp}_\Lambda(M)\) is of bounded representation type (meaning that there is a bound on the length of modules in \(\text{supp}_\Lambda(M)\)), then \(M\) is of finite type. Recall that a \(\Lambda\)-module \(M\) is said to be of finite type, provided it is the direct sum of (arbitrarily many) copies of a finite number, up to isomorphism, of indecomposable MCM modules of finite length; see [35]. The main tool in Ringel’s proof is Gabriel–Roiter (co)measure, an invariant assigned to any module of finite length, and defined by Gabriel and Ringel [17,33,36] based on Roiter’s induction scheme in his proof of the first Brauer–Thrall conjecture.

In order to state our results precisely, let us recall some notions.

From now on, assume that \((R, m, k)\) is a commutative noetherian complete Cohen–Macaulay local ring with a canonical module \(\omega\). We say that a balanced big CM \(R\)-module \(M\) is of finite CM-type, if it is a direct sum of (arbitrarily many) copies of a finite number, up to isomorphisms, of indecomposable MCM modules and it is said to be fully decomposable, provided it is a direct sum of finitely generated modules. The class of all fully decomposable modules will be denoted by FD.

Moreover, by CM-support of a balanced big CM \(R\)-module \(M\), denoted by \(\text{CM-supp}_R(M)\), we mean the set of all indecomposable MCM \(R\)-modules \(N\) such that \(\text{Hom}_R(N, M) \neq 0\). For a (not necessarily finitely generated) balanced big CM \(R\)-module \(M\), we set \(h(M) = \text{Hom}_R(M, M \oplus G)\), where \(\alpha : G \rightarrow k\) is a right minimal MCM-approximation. We say that \(M\) has finite \(h\)-length, provided that \(I_R(h(M)) < \infty\). Also, \(M\) is said to have an \(m\)-primary cohomological annihilator, if \(m^i h(M) = 0\), for \(i \gg 0\). One should observe that, this is equivalent to saying that \(m^i \text{Ext}_R^1(M, -) = 0\), by a theorem of Hilton–Rees [25].

Section 2 of the paper, is devoted to comparing the length of the stable \(\text{Hom}\) and \(h\)-length of maximal Cohen–Macaulay modules with classical invariants such as multiplicity and Betti number.
The main result in Sect. 3 enables us to demonstrate the utilization of the Gabriel–Roiter (co)measure for the category of balanced big Cohen–Macaulay modules; see Theorem 3.3. The purpose of Sect. 4 is to study balanced big CM modules with bounded CM-support. In particular, we prove the result below; see Theorems 4.7 and 4.9.

Theorem 1.1 A balanced big CM $R$-module having an $m$-primary cohomological annihilator with bounded $h$-length on CM-support is fully decomposable. In particular, any balanced big CM modules with an $m$-primary cohomological annihilator and of bounded $h$-length on CM-support, satisfies complements direct summands.

In Sect. 5, we investigate balanced big CM modules with large (finite) $h$-length, for instance, we have the following result; see Theorem 5.2.

Theorem 1.2 Let $R$ be an isolated singularity and let $M$ be a balanced big CM $R$-module with an $m$-primary cohomological annihilator. If $M$ is not of finite CM-type, then there are indecomposable MCM $R$-modules of arbitrarily large (finite) $h$-length.

It should be noted that this result provides a kind of the first Brauer–Thrall theorem for modules. In particular, it guarantees the validity of the first Brauer–Thrall conjecture for complete Cohen–Macaulay local rings, considering $h$-length as an invariant to measure the size of MCM modules. Indeed, we have the result below; see Corollary 5.3.

Corollary 1.3 Let the category of all indecomposable MCM $R$-modules be of bounded $h$-length. Then $R$ is of finite CM-type.

We would like to point out that, as already mentioned previously, the first Brauer–Thrall conjecture fails in general when multiplicity is used as the size, by an example of Dieterich [14].

In addition, it will be observed that the representation-theoretic properties of balanced big CM modules have important consequences for the structural properties of the ring. Actually, Theorem 6.3 asserts that:

Theorem 1.4 If any balanced big CM $R$-module $M$ admitting a right resolution by modules in $\text{Add} \omega$, is fully decomposable, then $R$ is an isolated singularity.

It seems that this result is a generalization of a result of Chase [11] for the category of MCM modules. Furthermore, we prove a variant of a celebrated theorem of Auslander [2,4], Ringel–Tachikawa [37], Chen [13] and Beligiannis [9] for Cohen–Macaulay local rings. In fact, our main result in Sect. 6 reads as follows.

Theorem 1.5 A complete Cohen–Macaulay local ring $R$ is of finite CM-type if and only if the category of balanced big CM $R$-modules with $m$-primary cohomological annihilators coincides with the category of fully decomposable balanced big CM modules. Equivalently; $R$ is an isolated singularity and the category of all fully decomposable balanced big CM modules is closed under kernels of epimorphisms.

The precise statement of the above result is Theorem 6.7.

In the paper’s final section, we are concerned with Cohen–Macaulay artin algebras and Cohen–Macaulay modules in the sense of Auslander and Reiten [6,7]. Recall that an artin algebra $\Lambda$ is said to be a Cohen–Macaulay algebra, if there is a pair of adjoint functors $(G, F)$ on the category of finitely generated (left) $\Lambda$-module, $\text{mod}\Lambda$, which induce mutually
inverse equivalences between the full subcategories of \( \text{mod}\Lambda \) consisting of the \( \Lambda \)-modules of finite injective dimension and the \( \Lambda \)-modules of finite projective dimension. It is known that an artin algebra \( \Lambda \) is Cohen–Macaulay if and only if there is a \( \Lambda \)-bimodule \( \omega \) such that the pair of adjoint functors \((\omega \otimes_{\Lambda} -, \text{Hom}_{\Lambda}(\omega, -))\) has the desired properties. In this case, \( \omega \) is called a dualizing module for \( \Lambda \). A (not necessarily finitely generated) \( \Lambda \)-module \( M \) is said to be \( \omega \)-Gorenstein projective, provided that it admits a right resolution by modules in \( \text{Add}\omega \). Following Auslander and Reiten [6], a finitely generated \( \omega \)-Gorenstein projective module will be called a Cohen–Macaulay module. The notion of Cohen–Macaulay artin algebras (and also Cohen–Macaulay modules) is generalizations of commutative complete Cohen–Macaulay local rings as well as Gorenstein artin algebras (Gorenstein projective modules). Recall that an artin algebra \( \Lambda \) is said to be a Gorenstein algebra, provided the injective dimension of \( \Lambda \) as well as of \( \Lambda \) is finite. The main goal of Sect. 7 is to study the decomposition properties of \( \omega \)-Gorenstein projective modules in connection with the property that \( \Lambda \) is of finite CM-type. In this direction, it is proved that any \( \omega \)-Gorenstein projective \( \Lambda \)-module \( M \) in which CM-supp\( \Lambda \)(\( M \)) is of bounded length, is fully decomposable; see Theorem 7.5. Using this result, we prove that there exist indecomposable CM \( \Lambda \)-modules of (arbitrarily) large finite length, if there is an \( \omega \)-Gorenstein projective \( \Lambda \)-module which is not of finite CM-type; see Theorem 7.6. This is fruitful from the point of view that it is an analog of the first Brauer–Thrall theorem for modules over Cohen–Macaulay artin algebras; see Corollary 7.7. In addition, we extend Chen’s result [13, Main theorem] to Cohen–Macaulay artin algebras. Namely, it is shown that \( \Lambda \) is of finite CM-type if and only if every \( \omega \)-Gorenstein projective module is fully decomposable; see Theorem 7.9.

We would like to emphasize that in proving our results, we strongly use the notion of Gabriel–Roiter (co)measure; see 3.2 for the definition of Gabriel–Roiter (co)measure. So our method is totally different from the previous ones which are based on functorial approach; see for example [9,13]. It is well understood that the Gabriel–Roiter (co)measure is a helpful invariant dealing with representations of an artin algebra; see [12,33,36]. So it seems worthwhile to unfold the use of this notion in the setting of commutative noetherian rings. In this direction, our point of view gives another nice feature of the paper which brings the use of Gabriel–Roiter (co)measure in the context of MCM modules; see also Theorem 3.3.

2 Preliminary results

This section is devoted to stating the definitions and basic properties of notions which we will freely use in the later sections. We also define length of the stable Hom, h-length, of balanced big Cohen–Macaulay modules and study its relationship with well-known invariants, such as multiplicity and Betti number. Let us start with our convention.

Convention 2.1 Throughout the paper, unless otherwise specified, \((R, m, k)\) is a \( d \)-dimensional commutative complete Cohen–Macaulay local ring with a dualizing (or canonical) module \( \omega \). The category of all (finitely generated) \( R \)-modules will be denoted by \((\text{mod}\, R) \, \text{Mod} \, R\).

2.2. An \( R \)-homomorphism \( f : X \to Y \) is called right minimal, provided that any \( R \)-homomorphism \( g : X \to X \) satisfying \( fg = f \), is an isomorphism.

An \( R \)-homomorphism \( f : M \to X \) with \( M \) is MCM is called a right MCM-approximation, if the map \( \text{Hom}_{R}(L, f) : \text{Hom}_{R}(L, M) \to \text{Hom}_{R}(L, X) \) is surjective for any MCM \( R \)-module \( L \); and a right minimal MCM-approximation if, in addition, \( f \) is right minimal.
It should be noted that by [5, Theorem A], every finitely generated $R$-module admits a right minimal MCM-approximation. In the rest of this paper, we assume that $\alpha : G \to k$ is a right minimal MCM-approximation of the residue field $k$.

**Definition 2.3** (1) A local ring $(R, m)$ is called an isolated singularity, if $R_p$ is a regular ring for all nonmaximal prime ideals $p$ of $R$.

(2) A finitely generated $R$-module $M$ is said to be locally free on the punctured spectrum of $R$, if $M_p$ is a free $R_p$-module for all nonmaximal prime ideals $p$ of $R$.

(3) A system of parameters $x = x_1, \ldots, x_d$ of $R$ is said to be a faithful system of parameters, if it annihilates $\text{Ext}_1^R(M, N)$ for any $M$ in MCM modules and $N \in \text{mod} R$, [30, Definition 14.8]. If $R$-modules $M$ are taken from a subcategory $C$ of MCM modules, then we will say that $x$ is a faithful system of parameters for $C$. In the remainder of this paper, $x^t$, where $t > 0$ is an integer, stands for the ideal $(x_1^t, \ldots, x_d^t)$.

(4) Let $M$ be an $R$-module. A sequence of elements $x = x_1, \ldots, x_n \in R$ is called a weak $M$-regular sequence, provided that $x_i$ is a non-zerodivisor on $M/(x_1, \ldots, x_{i-1})M$ for any $1 \leq i \leq n$ (for $i = 1$, we mean that $x_1$ is a non-zerodivisor on $M$). If, in addition, $(x_1, \ldots, x_n)M \neq M$, then $x$ is said to be an $M$-regular sequence. It is worth remarking that if $M$ is a non-zero finitely generated $R$-module, then it follows from Nakayama’s lemma that any weak $M$-regular sequence is an $M$-regular sequence, as well. It is also known that over local rings, any permutation of $M$-regular sequence, is again an $M$-regular sequence.

2.4. (1) We use $\mathcal{X}_\omega$ to denote the subcategory consisting of all $R$-modules $M$ admitting a right resolution by modules in $\text{Add}_\omega$, that is, an exact sequence of $R$-modules;

$$0 \to M \to w_0 \xrightarrow{d_0} w_1 \xrightarrow{d_1} \cdots \xrightarrow{d_i-1} w_i \xrightarrow{d_i} \cdots,$$

with $w_i \in \text{Add}_\omega$. By $\text{Add}_\omega$ (resp. $\text{add}_\omega$) we mean the full subcategory of $\text{Mod}_R$ (resp. $\text{mod}_R$) consisting of all modules isomorphic to direct summands of direct sums (resp. finite direct sums) of copies of $\omega$. It is known that $\text{MCM} = \mathcal{X}_\omega \cap \text{mod}_R$. To see this, according to [10, Theorem 3.3.10], a given module $M$ is MCM if and only if $\text{Ext}_1^R(M, \omega) = 0 = \text{Ext}_i^R(M^*, \omega)$ for all $i \geq 1$ and the natural homomorphism $\delta : M \to M^{**}$ is an isomorphism, where $M^* = \text{Hom}_R(M, \omega)$. Now assume that $M$ is an arbitrary $R$-module and $0 \to P_1 \to P_0 \to M^* \to 0$ is an exact sequence in $\text{mod}_R$ such that each $P_i$ is projective. So, applying the functor $\text{Hom}_R(-, \omega)$, implies that $M$ admits a right resolution by modules in $\text{add}_\omega$, giving the containment $\text{MCM} \subseteq \mathcal{X}_\omega \cap \text{mod}_R$. For the opposite containment, take a finitely generated $R$-module $M$ in $\mathcal{X}_\omega$. Since $\omega$ has finite injective dimension and $\text{Ext}_1^R(\omega, \omega) = 0$ for all $i \geq 1$, one may deduce that $\text{Ext}_i^R(M, \omega) = 0$ for all $i \geq 1$. This, in turn, implies that $M^* \in \mathcal{X}_\omega$ and so $\text{Ext}_i^R(M^*, \omega) = 0$ for all $i \geq 1$. Now, one may use the fact that $\delta_\omega : \omega \to \omega^{**}$ is an isomorphism, and conclude that the same is true for $\delta_M : M \to M^{**}$. Hence $M$ will be a MCM module.

(2) Recall that for a subcategory $\mathcal{X}$ of $\text{mod}_R$, we let $\widehat{\mathcal{X}}$ denote the category whose objects are the modules $M$ for which there is an exact sequence of $R$-modules; $0 \to X_n \to \cdots \to X_0 \to M \to 0$ with $X_i \in \mathcal{X}$.

Now we introduce the notion of $h$-length, an invariant to measure the size of balanced big CM $R$-modules.

**Definition 2.5** (i) For a given balanced big CM $R$-module $M$, we set $h(M) = \text{Hom}_R(M, M \oplus G)$ and define $h$-length of $M$ as $l_R(h(M))$.

Assume that $R$ is an isolated singularity. So in view of [44, Lemma 3.3], $\text{Hom}_R(M, N)$ is an artinian $R$-module, for all MCM $R$-modules $M$ and $N$. In particular, any MCM...
Let $\mathcal{C}$ be a subcategory of MCM $R$-modules. We say that $\mathcal{C}$ is of bounded $\mathfrak{h}$-length, if there is an integer $b > 0$ such that $|h(C)| = \sup |l_R(h(M))| |M \in \mathcal{C}| < b$.

**Proposition 2.6** Let $\mathcal{C}$ be a subcategory of MCM $R$-modules of bounded $\mathfrak{h}$-length. Then there is a system of parameters $x$ such that $xh(M) = 0$ for all $M \in \mathcal{C}$. In particular, $\mathcal{C}$ admits a faithful system of parameters.

**Proof** Take an integer $b > 0$ such that $|h(C)| < b$. So for any $M \in \mathcal{C}, l_R(h(M)) < b$ implying that $m^b h(M) = 0$. Now choosing a system of parameters $x \in \mathfrak{m}^b$, one gets that $xh(M) = 0$. In particular, $x \text{Hom}_R(M, M) = 0$, for any $M \in \mathcal{C}$. Hence by a theorem of Hilton–Rees [25], we infer that $x \text{Ext}_R^1(M, -) = 0$. So the proof is finished.

For a given finitely generated $R$-module $M$, by the Betti number of $M$, $\beta(M)$, we mean the minimal number of generators for $M$.

**Lemma 2.7** Let $\mathcal{C}$ be a subcategory of MCM $R$-modules. Then $\mathcal{C}$ has a bound on multiplicities if and only if it has a bound on Betti numbers.

**Proof** Assume that there exists an integer $b > 0$ such that for any $M \in \mathcal{C}$, $\beta(M) < b$. So for each $M \in \mathcal{C}$, there is an $R$-epimorphism $f : R^b \rightarrow M$. Take a system of parameters $x = x_1, x_2, \ldots, x_d$ of $R$. Tensoring $f$ with $R/[x]R$ over $R$, gives rise to the epimorphism $\bar{f} : R^b/[x]R^b \rightarrow M/[x]M$, implying that $l_R(M/[x]M) < l_R(R/[x]R)b$. According to [44, Proposition 1.7], the multiplicity of $M$, $e(M)$, is less than or equal to $l_R(M/[x]M)$. Consequently, for any $M \in \mathcal{C}$, $e(M) < l_R(R/[x]R)b$. The other direction follows from the well-known fact that, for any MCM module $M$, $\beta(M)$ is less than or equal to $e(M)$. The proof then is completed.

The next result shows that there is a tight connection between the invariants $\mathfrak{h}$-length and multiplicity of MCM modules.

**Lemma 2.8** Let $\mathcal{C}$ be a subcategory of MCM $R$-modules and let $x$ be a faithful system of parameters for $C$. If there is a bound on the multiplicities of modules in $C$, then $\mathcal{C}$ is of bounded $\mathfrak{h}$-length.

**Proof** We first claim that there is an integer $b > 0$ such that for any $M \in \mathcal{C}$, $l_R(\text{Hom}_{R/[x]}(M/[x]M, (M \oplus G)/(x(M \oplus G)))) < b$. To do this, one should note that according to Lemma 2.7, there is a bound on the Betti numbers of modules in $\mathcal{C}$, say $n$. Assume that $M$ is an arbitrary object of $\mathcal{C}$. So there exist $R$-epimorphisms, $f : R^n \rightarrow M$ and $g : R^n \rightarrow M \oplus G$. Tensoring $f$ and $g$ with $R/[x]R$ over $R$, gives rise to the epimorphisms $\bar{f} : R^n/[x]R^n \rightarrow M/[x]M$ and $\bar{g} : R^n/[x]R^n \rightarrow (M \oplus G)/[x](M \oplus G)$. Now, the $R/[x]R$ (and also $R$)-monomorphism;

$$\text{Hom}_{R/[x]}(M/[x]M, (M \oplus G)/(x(M \oplus G))) \rightarrow \text{Hom}_{R/[x]}(R^n/[x]R^n, (M \oplus G)/(x(M \oplus G))),$$

together with $R/[x]R$ (and also $R$)-epimorphism;

$$\text{Hom}_{R/[x]}(R^n/[x]R^n, R^n/[x]R^n) \rightarrow \text{Hom}_{R/[x]}(R^n/[x]R^n, (M \oplus G)/(x(M \oplus G))),$$

lead us to obtain the inequality

$$l_R(\text{Hom}_{R/[x]}(M/[x]M, (M \oplus G)/(x(M \oplus G)))) \leq l_R(\text{Hom}_{R/[x]}(R^n/[x]R^n, R^n/[x]R^n)).$$
Lemma 2.9 Let $C$ be a subcategory of indecomposable MCM $R$-modules. If $C$ is of bounded $h$-length, then it has a bound on multiplicities.

Proof Take an integer $t > 0$ such that for any object $M$ in $C$, $I_R(\mathcal{H}(M)) < t$. Assume that $M$ is an arbitrary non-projective object of $C$. Applying the functor $\text{Hom}_R(M, -)$ to the short exact sequence of $R$-modules; $0 \rightarrow m \rightarrow R \rightarrow k \rightarrow 0$, gives rise to the following exact sequence;

$$0 \rightarrow \text{Hom}_R(M, m) \rightarrow \text{Hom}_R(M, R) \rightarrow \text{Hom}_R(M, k) \rightarrow \text{Hom}_R(M, k) \rightarrow 0.$$ 

Since $M$ is non-projective, it can be easily seen that the isomorphism of $R$-modules $\text{Hom}_R(M, k) \cong \text{Hom}_R(M, k)$ holds true. On the other hand, $\alpha : G \rightarrow k$ being a right minimal MCM-approximation forces $\text{Hom}_R(M, G) \rightarrow \text{Hom}_R(M, k)$ to be an epimorphism. All of these facts together enable us to infer that $I_R(\text{Hom}_R(M, k)) < t$, because $I_R(\mathcal{H}(M)) < t$. This indeed means that there is a bound on the Betti numbers of modules in $C$. Now Lemma 2.7 gives the desired result. □

Remark 2.10 As we have mentioned in the introduction, a given $R$-module $M$ has an $m$-primary cohomological annihilator if and only if $m^t \text{Ext}_R^1(M, -) = 0$ for some integer $t \gg 0$. Assume that $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of $R$-modules. So by applying the functor $\text{Ext}_R(-, N)$, where $N$ is an arbitrary $R$-module, one may infer that the class consisting of modules with $m$-primary cohomological annihilators, is closed under extensions and kernels of epimorphisms.

Lemma 2.11 Let $f : M \rightarrow \bigoplus_{i \in I} M_i$ be an $R$-homomorphism, where each $M_i$ is finitely generated. If for a sequence $x = x_1, \ldots, x_n \in m$, $\bar{f} : M/xM \rightarrow \bigoplus_{i \in I} M_i/xM_i$ is an epimorphism, then $f$ is so.

Proof In order to obtain the desired result, it suffices to show that for any finite subset $J$ of $I$, the composition map $M \xrightarrow{f} \bigoplus_{i \in J} M_i \xrightarrow{h} \bigoplus_{i \in J} M_i$ is an epimorphism, where $h$ is the projection map. Consider the composition map $M/xM \xrightarrow{f \otimes R/xR} \bigoplus_{i \in J} M_i/xM_i \xrightarrow{h \otimes R/xR} \bigoplus_{i \in J} M_i/xM_i$, which evidently is an epimorphism. Thus letting $\text{Coker}(hf) = Z$, we have $Z/\text{Coker}(f) = Z$ and so Nakayama’s lemma implies that $Z = 0$, meaning that $hf$ is an epimorphism. Consequently, $f$ is an epimorphism, as needed. □

Lemma 2.12 Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a pure exact sequence of $R$-modules. If $M', M$ are weak balanced big CM module, then $M''$ is so.
Proof Assume that \( x = x_1, \ldots, x_d \) is a system of parameters of \( R \). We must show that for any \( 1 \leq i \leq d \), \( x_i \) is a non-zerodivisor on \( M''/(x_1, \ldots, x_{i-1})M'' \). Since the sequence \( 0 \to M' \to M \to M'' \to 0 \) is pure exact, tensoring this sequence with \( R/xR \), for any \( x \in R \), gives us again a pure exact sequence. This fact allows us to prove the case only for \( i = 1 \). So assume that \( x_1 = x \) is a regular element of \( R \). Consider the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
& & x & \downarrow & x & \downarrow & x & \downarrow & \\
0 & \longrightarrow & M'/xM' & \longrightarrow & M/xM & \longrightarrow & M''/xM'' & \longrightarrow & 0.
\end{array}
\]

Since \( M', M \) are weak balanced big CM, the left multiplicative maps are monomorphism. So, one may apply the snake lemma and deduce that the right multiplicative map is also monomorphism, giving the desired result. \( \square \)

3 Using Gabriel–Roiter (co)measure in the category of MCM modules

This section is devoted to bring the use of Gabriel–Roiter (co)measure in the category of MCM \( R \)-modules. The notion of Gabriel–Roiter (co)measure, an invariant assigned to any module of finite length, was defined by Gabriel and Ringel \([17, 33, 36]\). Since this notion is a basic tool in proving the results of the paper, we recall it and some of its properties which will be used later.

3.1 Gabriel–Roiter (co)measure

Let \( \Lambda \) be an artin algebra and \( M \) a finitely generated \( \Lambda \)-module. The Gabriel–Roiter measure of \( M \), denoted by \( \mu(M) \), was defined in \([33]\) by induction on the length of modules as follows: let \( \mu(0) = 0 \). Given a non-zero module \( M \), we may assume by induction that \( \mu(N) \) is already defined for any proper submodule \( N \) of \( M \). Set

\[
\mu(M) = \max\{\mu(N)\} + \begin{cases} 0, & \text{if } M \text{ is decomposable}, \\
\frac{1}{\ell\Lambda(M)}, & \text{if } M \text{ is indecomposable}, 
\end{cases}
\]

here maximum is taken over all proper submodules \( N \) of \( M \) and \( \ell\Lambda(M) \) denotes the length of \( M \) over \( \Lambda \). Note that the maximum always exists. We should refer the reader to \([36]\) for an equivalent definition using subsets of natural numbers, which reformulates Gabriel’s definition. The Gabriel–Roiter comeasure of \( M \), denoted by \( \mu^*(M) \), is defined as \( -\mu(D(M)) \), where \( D(M) = \text{Hom}_\Lambda(M, \bigsqcup (E(S))) \) in which \( E(S) \) runs over all injective envelope of simple \( \Lambda \)-modules.

3.2. Let us make a list of several basic properties of Gabriel–Roiter (co)measure, which have been proved by Ringel in \([33]\) and \([36]\).

Property 1 Let \( Y \) be a \( \Lambda \)-module of finite length and \( X \subseteq Y \) a submodule. Then \( \mu(X) \leq \mu(Y) \). If \( Y \) is indecomposable and \( X \) is a proper submodule \( Y \), then \( \mu(X) < \mu(Y) \).
Property 2 If $M$ is an indecomposable $\Lambda$-module of length $n$, then $\mu(M) = a/2^n$ where $a$ is an odd natural number such that $2^{n-1} \leq a < 2^n$.

It follows from this property that any subcategory of $\text{ind}(\text{mod}\Lambda)$ with bounded length has only finitely many Gabriel–Roiter measures. Moreover, in view of the equality $l_{\Lambda}(M) = l_{\Lambda}(D(M))$ and the definition of Gabriel–Roiter comeasure, such a subcategory has also only finitely many Gabriel–Roiter comeasures. In particular, assume that $\{M_1\}$ is a family of indecomposable $\text{MCM} R$-modules of bounded $h$-length. By Proposition 2.6, there is a faithful system of parameters $x$ for the family $\{M_1\}$ and so $[30, \text{Corollary 15.11}]$ yields that $\{M_1/x^2 M_1\}$ is a family of indecomposable $R/x^2 R$-modules. Moreover, according to Lemma 2.9, the family $\{M_1\}$ is of bounded multiplicity. Now one may use $[44, \text{Proposition 1.7}]$, and conclude that the family $\{M_1/x^2 M_1\}$ is of bounded length. Consequently, there are only finitely many Gabriel–Roiter (co)measures for the family $\{M_1/x^2 M_1\}$.

Main Property [36, Main property*] Let $Y_1, \ldots, Y_t, Z$ be indecomposable $\Lambda$-modules of finite length and assume that there is an epimorphism $g : \oplus_{i=1}^t Y_i \rightarrow Z$.

(a) Then $\min \mu^*(Y_i) \leq \mu^*(Z)$.
(b) If $\mu^*(Z) = \min \mu^*(Y_i)$, then $g$ splits.

The dual version of the main property (for Gabriel–Roiter measure) has been also appeared in [36].

The next result and the method of its proof, will play an essential role throughout the paper.

Theorem 3.3 Let $F = \{M_j\}_{j \in I}$ be an infinite set of pairwise non-isomorphic indecomposable $\text{MCM} R$-modules of bounded $h$-length. Then there exists an infinite subset $I' \subseteq I$ such that for any $i \in I'$, there is a non-zero $R$-homomorphism $f_i : M_i \rightarrow k$ such that for any $i \neq j \in I'$, any composition map $M_j \rightarrow M_i \xrightarrow{f_j} k$ is zero.

Proof Let us divide the proof into three steps.

Step 1. Since $F$ is of bounded $h$-length, by Proposition 2.6, there is a faithful system of parameters $x$ for $F$. In view of Property 2 of 3.2, there are only finitely many Gabriel–Roiter comeasures for $R/x^2 R$-modules $M/x^2 M$, $\mu^*(M/x^2 M)$, where $M \in F$ and since $F$ is infinite, one may choose an infinite subset $F'$ of $F$ consisting of all indecomposable $R$-modules $M_i$ with the same Gabriel–Roiter comeasure $\mu^*(M_i/x^2 M_i)$.

Step 2. Set $I' = \{i \in I \mid M_i \in F'\}$. Fix $M_i \in F'$. We would like to show that any $R$-homomorphism $\oplus_{i \neq j \in I} M_j^{(a_j)} \rightarrow M_i$, where $\Lambda_j$ is a set for each $j$, is not an epimorphism. Assume on the contrary that there is an epimorphism $\oplus_{i \neq j \in I} M_j^{(a_j)} \rightarrow M_i$. As $M_i$ is finitely generated, one may find a finite subset $J$ of $I'$, say $J = \{1, 2, \ldots, s\}$ such that the $R$-homomorphism $\phi = (\phi_j)_{j=1}^s : \oplus_{j=1}^s M_j^{(a_j)} \rightarrow M_i$ is an epimorphism, implying that $\tilde{\phi} : \oplus_{j=1}^s (M_j/x^2 M_j)^{a_j} \rightarrow M_j/x^2 M_j$ is an epimorphism as well. In view of $[30, \text{Corollary 15.11}]$, each quotient module is indecomposable. Now since $\mu^*(M_j/x^2 M_j) = \mu^*(M_j/x^2 M_j)$ for any $1 \leq j \leq s$, by part (b) of Main property of 3.2, $\tilde{\phi}$ is a split epimorphism and so the Krull–Remak–Schmidt theorem gives rise to the isomorphism $\tilde{\phi}_j : M_j/x^2 M_j \rightarrow M_j/x^2 M_j$, for some $j \in J$. Consequently, by $[10, \text{Lemma 3.3.2}]$, $\phi_j : M_j \rightarrow M_j$ will be an isomorphism. But this contradicts the hypothesis that modules in $F$ (and so $F'$) are non-isomorphic.

Step 3. We prove that for any $i \in I'$, there is a non-zero $R$-homomorphism $f_i : M_i \rightarrow k$ such that for any $i \neq j \in I'$, any composition map $M_j \rightarrow M_i \xrightarrow{f_j} k$ is zero. Set
K = ⟨Imϕ⟩ = Σϕ Imϕ, where ϕ runs over all R-homomorphisms ϕ : ⊕ₙ≠₀ M(Aₙ) → Mᵢ.
According to the proof of the previous step, Mᵢ/K is non-zero and so there is a non-zero homomorphism g : Mᵢ/K → k. Therefore, fᵢ = gπᵢ : Mᵢ → k is a non-zero R-homomorphism, where πᵢ : Mᵢ → Mᵢ/K is the natural epimorphism. Moreover, it is obvious from the construction of R-homomorphisms fᵢ’s that Mⱼ → Mᵢ → fᵢ = k is zero, for any i ≠ j ∈ I’. So the proof is completed.

4 Balanced big CM modules with bounded CM-support

The main theme of this section is to show that every balanced big CM module with an m-primary cohomological annihilator and of bounded CM-support, is fully decomposable.

It follows from the definition that balanced big CM modules need not be closed under direct summands in general. The next result leads us to provide a criterion to fix this restriction; see Corollary 4.3.

Lemma 4.1 Let M be a weak balanced big CM R-module and x = x₁, …, xₜ ∈ m an R-sequence such that x₁Ext₁(R, M) = 0 over ModR. If M/x₁M is a projective R/x₁R-module, then M is projective as an R-module. In particular, if M/x₁M = 0, then M = 0.

Proof We prove by induction on t. Assume that t = 1. Since M/x₁M is a projective R/x₁R-module, we have pd_R M/x₁M ≤ 1 and so Ext₁(R, M/x₁M, −) = 0 over ModR, for any i ≥ 2.

So by applying the functor Hom_R(−, N), where N is in ModR, to the short exact sequence of R-modules: 0 → M[x₁] → M → M/x₁M → 0, one obtains an exact sequence Ext₁(R, M, N) → Ext₁(R, M/x₁M, N) → 0. This means that Ext₁(R, M, N) = x₁Ext₁(R, M, N). By the hypothesis, the right hand side vanishes, implying that Ext₁(R, M, −) = 0 and then M is projective over R. Now suppose that t > 1 and the result has been proved for all values smaller than t. Setting S = R/x₁…xₜ−₁R, where x₁…xₜ−₁ = x₁, …, xₜ−₁, we have pd_S M/x₁…xₜ−₁M ≤ 1, implying that pd_S M/x₁…xₜ−₁M ≤ 1, as well. Considering the following exact sequence of S-modules:

0 → M/x₁…xₜ−₁M → M/x₁…xₜ−₁M → M/x₁…xₜ−₁M → 0,

one gets an isomorphism x₁Ext₁_S(M/x₁…xₜ−₁M, −) = Ext₁_S(M/x₁…xₜ−₁M, −) over ModS. Now one may apply [31, Lemma 2 (ii) page 140], in order to conclude that the isomorphism Ext₁_S(M/x₁…xₜ−₁M, −) ≃ Ext₁_R(−, −) holds true over ModS. On the other hand, by the hypothesis, x₁Ext₁_R(M, −) = 0. All of these facts enable us to deduce that Ext₁_S(M/x₁…xₜ−₁M, −) = 0 over ModS, meaning that M/x₁…xₜ−₁M is projective over S. Therefore, induction hypothesis would imply that M is indeed projective over R, as desired. Next assume that M/x₁M = 0. So, M will be a projective R-module. Indeed M is a free R-module. Now, since M = x₁M, one may infer that M = 0. The proof is completed.

Lemma 4.2 Let M be a weak balanced big CM R-module and x = x₁, …, xₜ a weak M-regular sequence. If M = x₁M, then for any integer t > 1, M = xₜM.

Proof If there is an integer 1 ≤ i ≤ t such that M = xᵢM, then it is evident that for any integer t > 1, M = xᵢM and so, the equality M = xₜM follows. Suppose that, for any i, M ≠ xᵢM. Letting xᵢ−₁ = x₁, …, xᵢ−₁, the hypothesis gives rise to the isomorphism; M/xᵢ−₁M → M/xᵢ−₁M. In particular, the composition map M/xᵢ−₁M → M/xᵢ−₁M → M/xᵢ−₁M is again an isomorphism. Indeed, by continuing
this for \( t \) times, we conclude that \( M/x_{d-1}M \xrightarrow{x_d} M/x_{d-1}M \) is an isomorphism, meaning that \( M = (x_1, \ldots, x_{d-1}, x_d)M \). Since any permutation of \( x \) is again a weak \( M \)-regular sequence, continuing this manner for any \( i \), will complete the proof. \( \square \)

**Corollary 4.3** Let \( M \) be a balanced big CM \( R \)-module with an \( m \)-primary cohomological annihilator. Then any non-zero direct summand of \( M \) is balanced big CM.

**Proof** Assume that \( M' \) is a non-zero direct summand of \( M \). Since \( M \) is balanced big CM, \( M' \) is clearly a weak balanced big CM \( R \)-module. Take an arbitrary system of parameters \( x \) of \( R \). If \( M' \neq xM' \), then we are done. So assume that \( M' = xM' \). As \( M \) has an \( m \)-primary cohomological annihilator, there is an integer \( t > 0 \) such that \( x^t \operatorname{Ext}_R^1(M, -) = 0 \) and so \( x^t \) annihilates the functor \( \operatorname{Ext}_R^1(M', -) \), as well. Since \( M'/xM' = 0 \), by Lemma 4.2, \( M'/x^tM' = 0 \) and thus Lemma 4.1 ensures that \( M' \) is a projective \( R \)-module. Hence the proof is finished. \( \square \)

**Theorem 4.4** Let \( M \) be a weak balanced big CM \( R \)-module with bounded \( h \)-length on \( \operatorname{CM-supp}_R(M) \). Let \( x \) be a faithful system of parameters for \( \operatorname{CM-supp}_R(M) \) such that \( M/x^2M \) is non-zero. Then the following hold.

1. There exists an indecomposable MCM \( R \)-module \( X \) and a non-zero pure monomorphism \( \varphi : X \rightarrow M \). In particular, \( \tilde{\varphi} : X/x^2X \rightarrow M/x^2M \) is a split monomorphism.
2. If, in addition, \( x \operatorname{Ext}_R^1(M, N) = 0 \) for all MCM modules \( N \), then \( \varphi \) will be a split monomorphism.

**Proof** (1) Since \( M \) is a weak balanced big CM \( R \)-module, by [22, Theorem B] there is a direct system \( \{M_i, \varphi^i_j\}_{i,j \in I} \) of MCM \( R \)-modules such that \( M = \varinjlim M_i \). By our assumption, \( \varinjlim M_i/x^2M_i = M/x^2M \) is non-zero. In what follows, \( - \otimes_R R/x^2R \) for simplicity will be denoted by \( (\_\_\_) \). Thus, we may take an index \( j \in I \) and an indecomposable MCM direct summand \( X_j \) of \( M_j \) such that the morphism \( \tilde{\varphi}^j_j : X_j \rightarrow M \) is non-zero, where \( \varphi^j_j = \varphi^i_j |_{X_j} \) and \( \varphi_j : M_j \rightarrow M \) is the natural morphism such that for any \( i \leq j \), the equality \( \varphi_i = \varphi_j \varphi^j_j \) holds. Let \( k_1 \in I \) be an index with \( k_1 > j \), so we have the morphism \( \tilde{\varphi}^{k_1}_{k_1} : M_j \rightarrow M_{k_1} \). In view of the equality \( \varphi_j = \varphi^{k_1}_{k_1} \varphi^{k_1}_{j} \), one may find an indecomposable MCM direct summand \( X_{k_1} \) of \( M_{k_1} \) such that the composition map

\[
\tilde{X}_j \xrightarrow{\tilde{\varphi}^k_{k_1} |_{\tilde{X}_j}} \tilde{M}_{k_1} \xrightarrow{\pi} \tilde{X}_{k_1} \xrightarrow{\tilde{\varphi}^{k_1}_{k_1} |_{\tilde{X}_{k_1}}} \tilde{M}
\]

is non-zero, where \( \pi : M_{k_1} \rightarrow X_{k_1} \) is the canonical projection. We denote the composition map

\[
X_j \xrightarrow{\varphi^{k_1}_{k_1} |_{X_j}} M_{k_1} \xrightarrow{\pi} X_{k_1}
\]

by \( \psi_{k_1} \). Now apply the induction argument to obtain a chain of morphisms of indecomposable finitely generated modules

\[
\tilde{X}_j \xrightarrow{\tilde{\varphi}^{k_1}_{k_1}} \tilde{X}_{k_1} \xrightarrow{\tilde{\varphi}^{k_2}_{k_1}} \tilde{X}_{k_2} \xrightarrow{\tilde{\varphi}^{k_3}_{k_2}} \tilde{X}_{k_3} \rightarrow \cdots,
\]

such that the compositions have non-zero images in \( \tilde{M} \). This, in particular, means that all \( X_j \)'s belong to \( \operatorname{CM-supp}_R(M) \) and so they are of bounded \( h \)-length. Thus by applying Lemmas 2.7
and 2.9, we infer that there is a non-negative integer $b$ such that $l_R(\bar{X}_i) < b$. Now Harada–Sai Lemma [20, Lemma 11], guarantees the existence of an index $k_t \in I$ such that for each $k_s > k_t$ the induced morphism $\psi_{k_t}^{k_s}: \bar{X}_{k_t} \longrightarrow \bar{X}_{k_s}$ needs to be an isomorphism. So by making use of [10, Lemma 3.3.2], we conclude that $\psi_{k_t}^{k_s}: X_{k_t} \longrightarrow X_{k_s}$ is an isomorphism, as well. This yields that, for any $k_s > k_t$ the morphism $\psi_{k_t}^{k_s}|_{X_{k_t}}: X_{k_t} \longrightarrow M_{k_t}$ is a split monomorphism. This, in turn, would imply that $\phi_{k_t}^{k_s}: X_{k_t} \longrightarrow M$ is a pure monomorphism.

As $\bar{X}_{k_t}$ is a finitely generated module over the artinian ring $\bar{R}$, it will be pure injective, enforcing $\bar{\phi}_{k_t}^{k_s}$ to be a split monomorphism, giving the desired result.

(2) In view of part (1), there is an indecomposable MCM module $X$ and a pure monomorphism $\phi: X \longrightarrow M$. So $\bar{\phi}: \bar{X} \longrightarrow \bar{M}$ is a split monomorphism. Suppose that $g: \bar{M} \longrightarrow \bar{X}$ is a $\bar{R}$-homomorphism with $g\bar{\phi} = \text{id}_{\bar{X}}$. By our assumption, $x\text{Ext}^1_{\bar{R}}(M, X) = 0$ and so a verbatim pursuit of the argument given in the proof of [30, Proposition 14.9] (see also [44, Proposition 6.15]), yields that there exists an $R$-homomorphism $h: M \longrightarrow X$ such that $g \otimes R/xR = h \otimes R/xR$. Therefore, it is fairly easy to see that $h\phi \otimes R/xR = \text{id}_{\bar{X}}/\bar{xX}$ and so by [10, Lemma 3.3.2], $h\phi$ is an isomorphism. Hence $\phi$ will be a split monomorphism. The proof now is finished. 

**Remark 4.5** Let $M$ be as in the above theorem. The proof of Theorem 4.4, reveals that for any non-zero element $z \in M/x^2M$, there is some indecomposable direct summand $X$ of $M$ such that $X/x^2X$ is a direct summand of $M/x^2M$ and $z$ has non-zero component in $X/x^2X$, where $x$ is a faithful system of parameters for $X$’s.

4.6. Let $\Lambda$ be an artin algebra. A result due to Ringel [33, Theorem 4.2] asserts that an indecomposable $\Lambda$-module $X$ of finite length with $\mu(X) = \gamma$ is relative $\Sigma$-injective in $D(\gamma)$, where $D(\gamma)$ is the full subcategory consisting of all $\Lambda$-modules $M$ in which any indecomposable submodule $M'$ of $M$ of finite length satisfies $\mu(M') \leq \gamma$. That is to say, any submodule $M'$ of a module $M \in D(\gamma)$ which is a direct sum of copies of $X$ will be a direct sum of $M$.

By the aid of a counterexample, he realized that the hypothesis $M'$ being a direct sum of a finite number of non-isomorphic indecomposable modules of finite length with a fixed Gabriel–Roiter measure $\gamma$, is essential. Indeed, he showed that there are infinitely many isomorphism classes of submodules $M_i$ of a module $M \in D(\gamma)$ such that for any $i$, $\mu(M_i) = \gamma$, but the embedding $\phi: \oplus_i M_i \longrightarrow M$ is not split. The argument given in the proof of the next result reveals that if we impose the hypothesis that $\text{supp}_\Lambda(M)$ is of bounded length, then $\phi$ will be split.

**Theorem 4.7** Let $M$ be a weak balanced big CM $R$-module with bounded $h$-length on $\text{CM-supp}_R(M)$. Let $x$ be a faithful system of parameters for $\text{CM-supp}_R(M)$. Then the following statements hold.

(1) If $M/x^2M$ is non-zero, then there is a non-zero fully decomposable balanced big CM module $Y$ and a pure monomorphism $\phi: Y \longrightarrow M$ such that $\bar{\phi}: Y/x^2Y \longrightarrow M/x^2M$ is an isomorphism.

(2) If, in addition, $x\text{Ext}^1_R(M, -) = 0$, then $\phi$ is an isomorphism.

**Proof** (1) By Theorem 4.4(1), there is a pure monomorphism $i_X: X \longrightarrow M$, where $X$ is an indecomposable MCM module. Assume that $\Sigma$ is the set of all pure submodules of $M$ which are direct sums of indecomposable MCM modules. For any two objects $N, L \in \Sigma$, we write $N \leq L$ if and only if $N$ is a pure submodule of $L$ and the following diagram is commutative;
By virtue of part (1), the morphism $(2)$ is an isomorphism, because $g$ is an isomorphism. Hence, one may deduce that $Y \oplus N \xrightarrow{[i_Y \psi]} M$ is indeed a pure monomorphism and $Y \oplus N$ contains $Y$ properly, but this contradicts the maximality of $Y$. Thus $K/x^2K = 0$ and so $i_Y \otimes R/x^2R$ is an isomorphism. Now we set $\varphi$ to be $i_Y$, and the desired result is obtained.

(2) By virtue of part (1), the morphism $\varphi \otimes R/x^2R : Y/x^2Y \rightarrow M/x^2M$ is an isomorphism. Assume that $\rho : M/x^2M \rightarrow Y/x^2Y$ is the inverse of $\varphi \otimes R/x^2R$. By the hypothesis, $x\text{Ext}^1_R(M, Y) = 0$, and so we may find a morphism $g : M \rightarrow Y$ such that $g \otimes_R R/xR = \rho \otimes_R R/xR$. Now we show that $g$ is an isomorphism. Since $\tilde{g} : M/x^2M \rightarrow Y/x^2Y$ is an isomorphism, Lemma 2.11 ensures that $g$ is an epimorphism. Taking the exact sequence of $R$-modules; $0 \rightarrow L \rightarrow M \xrightarrow{g} Y \rightarrow 0$, and using the fact that both modules $M, Y$ have $m$-primary cohomological annihilators, Remark 2.10 yields that the same is true for the weak balanced big CM module $L$. On the other hand, by [10, Proposition 1.1.5], this sequence remains exact after applying the functor $- \otimes_R R/x^2R$. Consequently, $L/x^2L = 0$, and so, Lemma 4.1 forces $L$ to be zero, implying that $g$ is an isomorphism. Hence, we will have the equality $g\varphi \otimes R/xR = id_Y \otimes R/xR$, and then, the argument appeared just above, yields that $g\varphi$ is an isomorphism. In particular, $\varphi$ is an isomorphism, because $g$ is so. Thus the proof is completed.

\[\square\]

4.8. Anderson and Fuller [1] posed the problem of determining over which rings does every module has a decomposition $M = \oplus_{i \in I} M_i$ that complements direct summands in the sense that whenever $K$ is a direct summand of $M$, $M = K \oplus (\oplus_{j \in J} M_j)$ for some $J \subseteq I$. This problem has been settled for artin rings of finite representation type by Tachikawa [42]. The
result below indicates that Tachikawa type theorem satisfies for Cohen–Macaulay rings of finite CM-type.

**Theorem 4.9** Any balanced big CM module with an m-primary cohomological annihilator and bounded h-length on CM-support, satisfies complements direct summands.

**Proof** Take a balanced big CM R-module M with an m-primary cohomological annihilator such that its CM-support is of bounded h-length. By Theorem 4.7, M is fully decomposable and so, M = ⊕_{i∈I} M_i, where each M_i is an indecomposable finitely generated submodule of M. Now assume that K is a direct summand of M. In view of Corollary 4.3, any direct summand of M is again balanced big CM which has an m-primary cohomological annihilator with bounded h-length on CM-support, so another use of Theorem 4.7 forces it to be fully decomposable. This, in particular, gives rise to another decomposition of M. Hence the Krull–Schmidt–Azumaya theorem [20, pages 331–332] gives the desired result. □

**Corollary 4.10** Let R be of finite CM-type. Then any balanced big CM R-module M with an m-primary cohomological annihilator, is of finite CM-type. In particular, each balanced big CM module with an m-primary cohomological annihilator, satisfies complements direct summands.

**Corollary 4.11** Let R be an isolated singularity containing its residue field k and let M be a balanced big CM R-module with an m-primary cohomological annihilator such that CM-supp_R(M) is of bounded multiplicity. If k is perfect, then M is fully decomposable.

**Proof** According to [30, Theorem 14.19] (see also [43, Corollary 2.8]), there is a faithful system of parameters x for the class of all indecomposable MCM R-modules. In particular, x is a faithful system of parameters for CM-supp_R(M). So by making use of Lemma 2.8, CM-supp_R(M) is of bounded h-length. Now Theorem 4.7 gives the desired result. □

### 5 MCM modules of large finite h-length

This section reveals that any balanced big CM module with an m-primary cohomological annihilator is of finite CM-type, whenever R is Gorenstein or it is an isolated singularity and the class of all indecomposable MCM R-modules is of bounded h-length. Our results provide the first Brauer–Thrall type theorem for rings, concerning the invariant h-length.

**Proposition 5.1** Let \( F = \{ M_i \}_{i \in I} \) be a set of pairwise non-isomorphic indecomposable MCM R-modules and let \( (f_i : M_i \rightarrow k)_{i \in I} \) be a family of non-zero R-homomorphisms such that any composition map \( M_j \rightarrow M_i \xrightarrow{f_j} k \) with \( j \neq i \), is zero. Let, for any \( i \), \( g_i : M_i \rightarrow G \) be an induced homomorphism by \( f_i \) (i.e. \( \alpha g_i = f_i \)) and set \( g = (g_i)_{i \in I} : \oplus_{i \in I} M_i \rightarrow G \). Assume that \( \beta : R^n \rightarrow G \) is a homomorphism such that \( (\oplus_{i \in I} M_i) \oplus R^n \xrightarrow{[g \beta]} G \) is an epimorphism. If the kernel of this epimorphism is fully decomposable, then \( F \) is a finite set.

**Proof** Assume on the contrary that \( F \) is an infinite set. Consider the short exact sequence of R-modules, \( 0 \rightarrow K \rightarrow (\oplus_{i \in I} M_i) \oplus R^n \xrightarrow{[g \beta]} G \rightarrow 0 \). By the hypothesis, \( K = \oplus_{i \in J} K_i \), where each \( K_i \) is an indecomposable finitely generated module. Since \( K \) is weak balanced big CM, for any \( i \), \( K_i \) is a MCM R-module. Therefore, for each \( i \), there is an R-monomorphism
\[ \epsilon_i : K_i \rightarrow \omega^{n_i} \] and so we may obtain the following commutative diagram;

\[
\begin{array}{c}
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow (\bigoplus_{i \in I} M_i) \oplus R^n \xrightarrow{[g \beta]} G \rightarrow 0 \\
\downarrow \text{id} \quad \downarrow u \quad \downarrow \phi \\
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow \bigoplus_{i \in J} \omega^{n_i} \xrightarrow{\varphi} \bigoplus_{i \in J} \Omega^{-1} K_i \rightarrow 0.
\end{array}
\tag{5.1}
\]

in which the morphism \( u \) is induced by the identity map. As \( G \) is finitely generated, the image of \( \varphi \) is non-zero only in a finite number of \( \Omega^{-1} K_i \)'s. This, in turn, allows us to decompose the morphism \( \varphi \) into the direct sum of \( \varphi' : G \rightarrow \bigoplus_{i \in J'} \Omega^{-1} K_i \) and \( 0 \rightarrow \bigoplus_{i \in J''} \Omega^{-1} K_i \), where \( J' \) is a finite subset of \( J \) and \( J'' = J - J' \). Set, for simplicity, \( \epsilon' := \bigoplus_{i \in J'} \epsilon_i \), \( \phi' := \bigoplus_{i \in J'} \phi_i \), and we define the morphisms \( \epsilon'' \) and \( \phi'' \), similarly. Take the following pull-back diagram;

\[
\begin{array}{c}
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow M' \rightarrow h \rightarrow G \rightarrow 0 \\
\downarrow \text{id} \quad \downarrow v \quad \downarrow \psi' \\
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow \bigoplus_{i \in J} \omega^{n_i} \xrightarrow{\phi'} \bigoplus_{i \in J} \Omega^{-1} K_i \rightarrow 0.
\end{array}
\tag{5.2}
\]

Consider the following commutative diagram;

\[
\begin{array}{c}
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow (\bigoplus_{i \in I} M_i) \oplus R^n \xrightarrow{[g \beta]} G \rightarrow 0 \\
\downarrow \pi' \quad \downarrow u' \quad \downarrow \psi' \\
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow \bigoplus_{i \in J} \omega^{n_i} \xrightarrow{\phi'} \bigoplus_{i \in J} \Omega^{-1} K_i \rightarrow 0,
\end{array}
\]

where \( \pi' \) is the projection and \( u' \) is the composition map \( (\bigoplus_{i \in I} M_i) \oplus R^n \xrightarrow{u} \bigoplus_{i \in J} \omega^{n_i} \xrightarrow{\pi_1} \bigoplus_{i \in J} \omega^{n_i} \), in which \( \pi_1 \) is the natural projection. By using the property of pull-back diagram, we may find \( R \)-homomorphisms \( \psi : \bigoplus_{i \in I} M_i \oplus R^n \rightarrow M' \) and \( t : \bigoplus_{i \in J} K_i \rightarrow \bigoplus_{i \in J} K_i \) such that the following diagram;

\[
\begin{array}{c}
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow (\bigoplus_{i \in I} M_i) \oplus R^n \xrightarrow{[g \beta]} G \rightarrow 0 \\
\downarrow i \quad \downarrow \psi \quad \downarrow \text{id} \\
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow M' \rightarrow h \rightarrow G \rightarrow 0,
\end{array}
\tag{5.3}
\]

is commutative. Another use of the property of pull-back diagram, gives rise to the equality \( u' = v\psi \). This, in conjunction with the commutativity of the left squares in the above two diagrams, leads us to obtain the equality \( \epsilon' \pi' = \epsilon' t \). Now \( \epsilon' \) being monomorphism, yields that \( \pi' = t \). Since \( \phi'' u = 0 \), the commutativity of (5.1) yields that there exists an \( R \)-homomorphism \( \theta : (\bigoplus_{i \in I} M_i) \oplus R^n \rightarrow \bigoplus_{i \in J'} K_i \) such that \( \epsilon'' \theta = u'' \), where \( u'' \) stands for the composition map \( (\bigoplus_{i \in I} M_i) \oplus R^n \xrightarrow{u} \bigoplus_{i \in J} \omega^{n_i} \xrightarrow{\pi''} \bigoplus_{i \in J''} \omega^{n_i} \), in which \( \pi'' \) is the natural projection. Also, another use of the commutativity of (5.1) gives rise to the equality \( u'' s_{\bigoplus_{i \in J'} K_i} = \epsilon'' \), and so \( \epsilon'' \theta s_{\bigoplus_{i \in J'} K_i} = \epsilon'' \). As \( \epsilon'' \) is a monomorphism, \( \theta s_{\bigoplus_{i \in J'} K_i} = id_{\bigoplus_{i \in J'} K_i} \). Therefore, we have the following commutative diagram;

\[
\begin{array}{c}
0 \rightarrow \bigoplus_{i \in J} K_i \rightarrow (\bigoplus_{i \in I} M_i) \oplus R^n \xrightarrow{[g \beta]} G \rightarrow 0 \\
\downarrow \text{id} \quad \downarrow \left[ \begin{array}{c} \psi \\ \theta \end{array} \right] \downarrow \text{id} \\
0 \rightarrow \bigoplus_{i \in J'} K_i \rightarrow \bigoplus_{i \in J''} K_i \xrightarrow{h} G \rightarrow 0.
\end{array}
\]
It should be observed that the commutativity of the right-hand side square follows from the equality $h\psi = [g, \beta]$, however the left-hand side square is commutative, because of the definition of $\theta$ and the commutativity of the left square in (5.3), and so, $[\frac{\psi}{\theta}]$ will be an isomorphism with inverse $\eta$. In particular, one obtains the next commutative square:

$$
\begin{array}{ccc}
(\oplus_{i \in I} M_i) \oplus R^n & \xrightarrow{[f, \alpha \beta]} & k \\
\downarrow \psi & & \downarrow id \\
M' \oplus (\oplus_{i \in J''} K_i) & \xrightarrow{[\alpha h, 0]} & k,
\end{array}
$$

where $f = (f_i) : \oplus_{i \in I} M_i \rightarrow k$ and $\alpha : G \rightarrow k$ is the right minimal MCM-approximation of $k$. As $M'$ is finitely generated, there are only finitely many $M_i$s; say $\{M_{i_1}, \ldots, M_{i_t}\}$, such that under $\eta$, $M'$ may have non-zero image in $\{M_{i_1}, \ldots, M_{i_t}, R^n\}$. Since $\mathcal{F}$ is assumed to be infinite, one may take a non-projective indecomposable module $M_s$ in $\mathcal{F}$ such that $s \notin \{i_1, \ldots, i_t\}$. As $f_s : M_s \rightarrow k$ is non-zero, there is an element $x \in M_s$ such that $f_s(x) \neq 0$ and so the image of $x$ under the composition map $M_s \xrightarrow{i} \oplus_{i \in I} M_i \oplus R^n \xrightarrow{[f, \alpha \beta]} k$ will be non-zero, where $i$ is the injection map. Thus the commutativity of the above square enables us to conclude that the image of $x$, say $x'$, under the morphism

$$
M_s \xrightarrow{i} \oplus_{i \in I} M_i \oplus R^n \xrightarrow{\psi} M' \oplus (\oplus_{i \in J''} K_i) \xrightarrow{\pi} M'
$$
is non-zero. Consequently, $(x', 0)$ is a non-zero element of $M' \oplus (\oplus_{i \in J''} K_i)$ and in particular, $\alpha h(x')$ is non-zero in $k$, as well. Therefore, the composition map;

$$
M_s \xrightarrow{i} \oplus_{i \in I} M_i \oplus R^n \xrightarrow{\psi} M' \xrightarrow{\eta |_{M'}} \oplus_{j=1}^{t'} M_{i_j} \oplus R^n \xrightarrow{f'} k,
$$
is non-zero, where $f' = f_{|\oplus_{j=1}^{t'} M_{i_j} \oplus R^n}$. On the other hand, the construction of morphisms $f_i$s, indicates that the composition map

$$
M_s \xrightarrow{\psi i} M' \xrightarrow{\eta'} \oplus_{j=1}^{t'} M_{i_j} \xrightarrow{+} \oplus_{i \in I} M_i \xrightarrow{f} k,
$$
is zero. Here $\eta'$ stands for the composition map $M' \xrightarrow{\eta |_{M'}} \oplus_{j=1}^{t'} M_{i_j} \oplus R^n \xrightarrow{=} \oplus_{j=1}^{t'} M_{i_j}$.

Consequently, the composition map $M_s \xrightarrow{\psi i} M' \xrightarrow{\eta''} R^n \xrightarrow{\alpha h} k$ will be non-zero, where $\eta''$ is the composition map $M' \xrightarrow{\eta |_{M'}} \oplus_{j=1}^{t'} M_{i_j} \oplus R^n \rightarrow R^n$. This implies that $M_s$ is indecomposable, which contradicts the choice of $M_s$. Hence $\mathcal{F}$ will be a finite set. The proof then is completed.

Now, we are in a position to state the main theorem of this section, which provides the local version of the first Brauer–Thrall conjecture, i.e. for modules instead of the base ring.

**Theorem 5.2** Let $R$ be an isolated singularity and let $M$ be a balanced big $CM$-$R$-module having an $m$-primary cohomological annihilator. If $M$ is not of finite CM-type, then there are indecomposable MCM $R$-modules of arbitrary large (finite) $h$-length.

**Proof** Suppose on the contrary that the class of all indecomposable MCM $R$-modules, is of bounded $h$-length. Since the same will be true for $CM$-$\text{supp}_R(M)$, in view of Theorem 4.7, $M$ is fully decomposable. So, we may write $M = \oplus_{i \in I} M_i^{(i)}$, where each $M_i$ is an indecomposable MCM $R$-module. As $M$ is not of finite CM-type, $\mathcal{F} = \{M_i\}_{i \in I}$ is an infinite set of pairwise non-isomorphic indecomposable MCM $R$-modules. In addition, $\mathcal{F}$ is of bounded $h$-length,
because $\text{CM-supp}_R(M)$ is so. According to Theorem 3.3, there is an infinite subset $I'$ of $I$ in which for any $i \in I'$, there exists a non-zero $R$-homomorphism $f_i : M_i \to k$ such that for each $j \neq i \in I'$, any composition map $M_j \to M_i \to k$ is zero. As $\alpha : G \to k$ is a right minimal $\text{MC}$-approximation, for any $i \in I'$, one may find an $R$-homomorphism $g_i : M_i \to G$ such that $\alpha g_i = f_i$. Set $g = (g_i)_{i \in I'} : (\oplus_{i \in I'} M_i) \to G$. Consider the exact sequence of $R$-modules; $0 \to K \xrightarrow{\theta} (\oplus_{i \in I'} M_i) \oplus R^n [g \beta] \to G \to 0$. We claim that $K$ is a balanced big CM module. To this end, suppose that $y$ is an arbitrary system of parameters of $R$. Evidently, $y$ is a weak $R$-regular sequence, because it is regular sequence for both modules $(\oplus_{i \in I'} M_i) \oplus R^n$ and $G$. In addition, $K \neq y K$. Indeed, if this is not the case, we will obtain an isomorphism of $R$-modules; $(\oplus_{i \in I'} M_i) \oplus R^n / y((\oplus_{i \in I'} M_i) \oplus R^n) \cong G / yG$, and this would be contradiction, because $G / yG$ is finitely generated whereas $(\oplus_{i \in I'} M_i) \oplus R^n / y((\oplus_{i \in I'} M_i) \oplus R^n)$ is not so, and thus the claim follows. Next $M$ having an $m$-primary cohomological annihilator, yields that $m^t \text{Ext}^t_R((\oplus_{i \in I'} M_i) \oplus R^n, -) = 0$ for some integer $t > 0$. On the other hand, as $G$ is locally free on the punctured spectrum, there is an integer $t' > 0$ such that $m^t \text{Ext}^t_R(G, -) = 0$. Therefore $m^{t+t'} \text{Ext}^{t+t'}_R(K, -) = 0$, meaning the balanced big CM module $K$ has an $m$-primary cohomological annihilator. Hence, another use of Theorem 4.7 yields that $K$ is fully decomposable. Namely, $K = \oplus_{i \in I} K_i$, where each $K_i$ is an indecomposable finitely generated $R$-module. Now, Proposition 5.1 forces $I'$ to be finite, which contradicts with the fact that $I'$ is infinite. The proof then is completed.

Here we include several corollaries of Theorem 5.2. First of all, this theorem enables us to prove the first Brauer–Thrall type theorem for complete Cohen–Macaulay local rings with considering the invariant $h$-length. It is worth noting that, this conjecture fails in general by the aid of counterexamples of Dieterich [14] and Leuschke and Wiegand [29], dealing with multiplicity.

**Corollary 5.3** Let the category of all indecomposable $\text{MC}$ $R$-modules be of bounded $h$-length. Then $R$ is of finite $\text{CM}$-type.

**Proof** By the hypothesis, any indecomposable $\text{MC}$ $R$-module $X$ has finite $h$-length and so $h(X)$ is an artinian $R$-module. Consequently, for any $\text{MC}$ $R$-module $M$, $h(M)$ will be also an artinian module, implying that $h(M)_p = 0$ for all nonmaximal prime ideals $p$ of $R$. In particular, we have $(\text{Hom}_R(M, M))_p \cong \text{Hom}_{R_p}(M_p, M_p) = 0$, and so $M_p$ is a free $R_p$-module, for all nonmaximal prime ideals $p$ of $R$ and so by [44, Lemma 3.3], $R$ is an isolated singularity. Now, Suppose for the contradiction that there is an infinite set $\{M_i\}_{i \in I}$ of pairwise non-isomorphic indecomposable $\text{MC}$ $R$-modules. So $M = \oplus_{i \in I} M_i$ is not of finite $\text{CM}$-type. On the other hand, in view of Proposition 2.6, there is a system of parameters $x$ of $R$ such that $x \text{Ext}^1_R(M_i, -) = 0$ for any $i \in I$. Consequently, $x \text{Ext}^1_R(\oplus_{i \in I} M_i, -) \cong \prod_{i \in I} x \text{Ext}^1_R(M_i, -) = 0$, meaning that the balanced big CM module $M$ has an $m$-primary cohomological annihilator. Therefore, by virtue of Theorem 5.2, there exist indecomposable $\text{MC}$ $R$-modules of arbitrary large (finite) $h$-length, which is a contradiction. The proof hence is completed. \[\blacksquare\]

The above corollary leads us to deduce a result of Dieterich [14], Leuschke and Wiegand [29] and Yoshino [43]. Indeed we have the next result.

**Corollary 5.4** Let $(R, m, k)$ be a complete equicharacteristic Cohen–Macaulay local ring with algebraically closed residue field $k$. Then $R$ is of finite $\text{CM}$-type if and only if $R$ is an isolated singularity and there is a bound on the multiplicities of the indecomposable $\text{MC}$ $R$-modules.
Proof Since the ‘only if’ part is evident, we prove only the ‘if’ part. To do this, according to Corollary 5.3, it suffices to show that the category of all indecomposable MCM $R$-modules is of bounded $h$-length. By [30, Theorem 14.19], $R$ admits a faithful system of parameters $x$. Moreover, by the hypothesis, there is an integer $b > 0$ such that $e(M) < b$ for any indecomposable MCM $R$-module $M$. Now Lemma 2.8 finishes the proof. \[\square\]

The result below, can be proved similarly to the above corollary.

Corollary 5.5 Let $R$ be a $d$-dimensional complete Cohen–Macaulay local ring containing the residue field that is perfect. Let $M$ be a balanced big CM $R$-module having an $m$-primary cohomological annihilator which is not of finite CM-type. Then there are indecomposable MCM $R$-modules of arbitrarily large multiplicity.

In the remainder of this section, we want to show that over Gorenstein local rings of finite CM-type, in Theorems 4.7 and 4.9, the hypothesis $M$ having an $m$-primary cohomological annihilator, is redundant; see Theorem 5.12 and Corollary 5.13.

Lemma 5.6 Let $N$ be a MCM $R$-module and $x$ a system of parameters of $R$ such that \(x \operatorname{Ext}^1_R(N^*, -) = 0\). Then \(x \operatorname{Ext}^1_R(M, N) = 0\) for any module $M \in \mathcal{X}_{\omega^*}$, where $N^* = \operatorname{Hom}_R(N, \omega)$.

Proof Since $\operatorname{Ext}^1_R(N^*, \omega) = 0$, applying the functor $\operatorname{Hom}_R(-, \omega)$ to a free resolution $P_\bullet : \cdots \to P_1 \to P_0 \to N^* \to 0$ of $N^*$ gives rise to the exact sequence of $R$-modules; $0 \to N \to \operatorname{Hom}_R(P_0, \omega) \to \operatorname{Hom}_R(P_1, \omega) \to \cdots$. Thus, for a given object $M \in \mathcal{X}_{\omega^*}$, we have the following isomorphisms:

\[
\begin{align*}
\operatorname{Ext}^1_R(M, N) & \cong \operatorname{H}^1(\operatorname{Hom}_R(M, \operatorname{Hom}_R(P_\bullet, \omega))) \\
& \cong \operatorname{H}^1(\operatorname{Hom}_R(P_\bullet, M^*)) \\
& \cong \operatorname{Ext}^1_R(N^*, M^*),
\end{align*}
\]

giving the desired result. \[\square\]

Lemma 5.7 Let $R$ be of finite CM-type and \(\{N_i\}_{i \in I}\) a family of MCM $R$-modules. Then there is an integer $t > 0$ such that \(m^t \operatorname{Ext}^1_R(M, \oplus_{i \in I} N_i) = 0\), for any module $M \in \mathcal{X}_{\omega^*}$.

Proof Assume that \(\{X_1, X_2, \ldots, X_t\}\) is the set of all pairwise non-isomorphic indecomposable MCM $R$-modules. Take cardinal numbers $s_1, s_2, \ldots, s_t$ such that $\oplus_{i \in I} N_i = \oplus_{i=1}^t X_i^{(s_i)}$ and assume that $s$ is a non-negative integer with $m^s h(X_i^{*}) = 0$, for any $1 \leq i \leq t$. Suppose that for each $i$, $P_{X_i^*}$ is a projective resolution of $X_i^*$. So considering an arbitrary $R$-module $M$ in $\mathcal{X}_{\omega^*}$, analogues to the proof of Lemma 5.6, we have the following isomorphisms:

\[
\begin{align*}
\operatorname{Ext}^1_R(M, \oplus_{i \in I} N_i) & \cong \operatorname{H}^1(\operatorname{Hom}_R(M, \oplus_{i=1}^t \operatorname{Hom}_R(P_{X_i^*}, \omega^{(s_i)}))) \\
& \cong \oplus_{i=1}^t \operatorname{H}^1(\operatorname{Hom}_R(M, \operatorname{Hom}_R(P_{X_i^*}, \omega^{(s_i)}))) \\
& \cong \oplus_{i=1}^t \operatorname{H}^1(\operatorname{Hom}_R(P_{X_i^*}, \operatorname{Hom}_R(M, \omega^{(s_i)}))) \\
& \cong \oplus_{i=1}^t \operatorname{Ext}^1_R(X_i^*, \operatorname{Hom}_R(M, \omega^{(s_i)})),
\end{align*}
\]

giving the desired result. \[\square\]

Lemma 5.8 Let $0 \to M \to F \to L \to 0$ be an exact sequence of $R$-modules such that $F$ is free. If $L$ admits a non-projective indecomposable MCM direct summand, then $M$ has an indecomposable MCM direct summand.
Proof Assume that $L'$ is a non-projective indecomposable MCM direct summand of $L$. Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & K & \rightarrow & R^n & \rightarrow & L' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & F & \rightarrow & L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \pi & & \\
0 & \rightarrow & K & \rightarrow & R^n & \overset{f}{\rightarrow} & L' & \rightarrow & 0,
\end{array}
$$

where $R^n \rightarrow L'$ is a projective cover. Since $L'$ is non-projective, the finitely generated $R$-module $K$ is non-zero. Now using the fact that the right column is identity and the middle one is an isomorphism, we infer that the left column will be an isomorphism. This means that the MCM $R$-module $K$ is a direct summand of $M$, as required. □

Recall that a commutative noetherian local ring $R$ is said to be Gorenstein, if it has finite self-injective dimension. A (not necessarily finitely generated) module $M$ over a Gorenstein ring $R$ is called Gorenstein projective, whenever it admits a right resolution of projective modules, i.e. $M \in \mathcal{X}_R$. For the basic properties of these modules, we refer the reader to [15]. In the setting of artinian rings, the result below is [32, Corollary 6].

**Proposition 5.9** Let $R$ be a Gorenstein ring of finite $CM$-type. Then any non-zero Gorenstein projective $R$-module has an indecomposable MCM direct summand.

**Proof** Assume that $M$ is a non-zero Gorenstein projective $R$-module and $x$ is a faithful system of parameters for the class of MCM $R$-modules, which exists by Proposition 2.6. Moreover, Lemma 5.7 allows us to further assume that $x\text{Ext}^1_R(N, \oplus_{i \in I} Y_i) = 0$, for any Gorenstein projective module $N$ and any family of MCM modules $\{Y_i\}_{i \in I}$. We prove the result in two steps.

Step 1: We show that if $M/x^2M \neq 0$, then $M$ admits an indecomposable MCM direct summand. Since $M$ is a Gorenstein projective $R$-module and so it belongs to $\mathcal{X}_R$, one may infer that $M$ is a weak balanced big CM module. So in view of Theorem 4.4(1), there is a non-zero pure monomorphism $\varphi : X \rightarrow M$, where $X$ is an indecomposable MCM module. As $X^* = \text{Hom}_R(X, R)$ is a MCM $R$-module, $x\text{Ext}^1_R(X^*, -) = 0$. Now, by applying Lemma 5.6, we get that $x\text{Ext}^1_R(M, X) = 0$. Hence, the argument given in the proof of Theorem 4.4(2), reveals that $\varphi$ is a split monomorphism.

Step 2: We prove that $M/x^2M \neq 0$. Suppose for the contradiction that $M/x^2M = 0$. Take a short exact sequence of $R$-modules; $0 \rightarrow M \rightarrow F \rightarrow L \rightarrow 0$, in which $F$ is free and $L$ is Gorenstein projective. As $F/x^2F \neq 0$, we conclude that the same will be true for $L/x^2L$. So, by Theorem 4.7(1), there is a pure monomorphism $\varphi : Y = \oplus X_i \rightarrow L$, where each $X_i$ is an indecomposable MCM module, such that $\tilde{\varphi} : Y/x^2Y \rightarrow L/x^2L$ is an isomorphism. It is evident that, $X_i \rightarrow Y \rightarrow L$, for any $i$, is also pure monomorphism. So, the argument given in the proof of step (1), indicates that these pure monomorphisms are split. By virtue of Lemma 5.8, we can assume that any indecomposable MCM direct summand of $L$ is projective. Consequently, each $X_i$, and then $Y$, will be projective $R$-modules. Assume that $\rho : L/x^2L \rightarrow Y/x^2Y$ is the inverse of $\tilde{\varphi}$. Since $x\text{Ext}^1_R(L, Y) = 0$, one may find a morphism $g : L \rightarrow Y$ such that $g \otimes_R R/xR = \rho \otimes_R R/xR$. By Lemma 2.11, $g$ is an epimorphism. Take a short exact sequence of $R$-modules, $0 \rightarrow T \rightarrow L \overset{g}{\rightarrow} Y \rightarrow 0$. As $g \otimes R/xR$ is an isomorphism and by [10, Proposition 1.1.5], this sequence remains exact.
after applying the functor $- \otimes_R R/xR$, we deduce that $T/xT = 0$ and so, the same is true for $T/x^2T$, thanks to Lemma 4.2. Consider the following pull-back diagram:

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & M & \to & F' & \to & T & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & M & \to & F & \to & L & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
Y & \to & Y & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & 0 & \\
\end{array}
$$

Since $Y$ is projective, the short exact sequence $0 \to F' \to F \to Y \to 0$ is split, and then, $F'$ will be a projective $R$-module. On the other hand, as $M/x^2M = 0 = T/x^2T$, one may infer that $F'/x^2F' = 0$, implying that $F' = 0$ and so the same is true for $M$, which is a contradiction. The proof now is finished. \qed

As a direct consequence of Proposition 5.9, we include the following result.

**Corollary 5.10** Let $R$ be a Gorenstein ring of finite CM-type. Then any Gorenstein projective $R$-module is balanced big CM.

**Corollary 5.11** Let $R$ be a Gorenstein ring and $M$ a Gorenstein projective $R$-module such that $\text{CM-supp}_R(M)$ is of bounded $\mathfrak{h}$-length. If $M$ is balanced big CM, then $M$ has an indecomposable $\text{CM}$ direct summand.

**Proof** According to Proposition 2.6, we may find a faithful system of parameters $x$ for $\text{CM-supp}_R(M)$ such that $x\text{Hom}_R(X, X) = 0$ for any object $X$ in $\text{CM-supp}_R(M)$. Set $X = \{X^* | X \in \text{CM-supp}_R(M)\}$, where $X^* = \text{Hom}_R(X, R)$. Take an arbitrary object $X$ of $\text{CM-supp}_R(M)$. Since $(-)^*$ is a duality on the category of finitely generated Gorenstein projective modules, and in particular, on its stable category modulo projectives, we get the isomorphism $\text{Hom}_R(X, X) \cong \text{Hom}_R(X^*, X^*)$, and then, $x\text{Hom}_R(X^*, X^*) = 0$. Now one may apply a theorem of Hilton–Rees [25] and deduce that $x\text{Ext}^1_R(X^*, -) = 0$. This, indeed, means that $x$ is a faithful system of parameters for $\text{CM-supp}_R(M) \cup X$. Now, repeating the proof of step 1 of Proposition 5.9, will give the desired result. \qed

**Theorem 5.12** Let $R$ be a complete Gorenstein local ring. If $R$ is of finite CM-type, then every Gorenstein projective $R$-module is fully decomposable. In particular, any Gorenstein projective $R$-module satisfies complements direct summands.

**Proof** Take an arbitrary Gorenstein projective $R$-module $M$. By Proposition 2.6 there is a faithful system of parameters $x$ for $M$ $R$-modules. According to Corollary 5.10, $M$ is a balanced big CM module, and in particular, $M/x^2M$ is non-zero. Now Theorem 4.7(1) guarantees the existence of a pure monomorphism $\varphi : Y = \oplus_{i \in I} X_i \to M$, where each $X_i$ is indecomposable MCM, such that $\overline{\varphi} : Y/x^2Y \to M/x^2M$ is an isomorphism. Assume that $\rho : M/x^2M \to Y/x^2Y$ is the inverse of $\overline{\varphi}$. In view of Lemma 5.7, $x\text{Ext}^1_R(M, \oplus_{i \in I} X_i) = 0$, and so, one may find an $R$-homomorphism $g : M \to Y$ such that $\rho \otimes_R R/xR = g \otimes_R R/xR$. By making use of Lemma 2.11, we infer that $g$ is an epimorphism. Take a short exact sequence $0 \to L \to M \xrightarrow{g} Y \to 0$. As $M, Y$ are Gorenstein projective, so is $L$. In particular,
L is balanced big CM. Since by [10, Proposition 1.1.5], the functor $- \otimes_R R/x^2R$ leaves this sequence exact and $g \otimes_R R/x^2R$ is an isomorphism, we infer that $L/x^2L = 0$. This, in turn, would imply that $L = 0$, meaning that $g$ is an isomorphism. So we are done.

**Corollary 5.13** Let $R$ be of finite CM-type. Then a given module $M$ is a direct sum of MCM modules if and only if $M \in \mathcal{X}_\omega$ and any non-zero direct summand of $M$ is balanced big CM.

**Proof** By making use of Lemma 5.7 in the proof of Theorem 4.7, one can deduce the ‘if’ part of the result. For the ‘only if’ part, assume that $M'$ is an arbitrary non-zero direct summand of $M$. We would have nothing to prove, whenever $M'$ is projective. So assume that $M'$ is not projective. As $M$ is fully decomposable, it belongs to $\mathcal{X}_\omega$, and then one may easily infer that $M'$ is weak balanced big CM. Take an arbitrary system of parameters $x$ of $R$. Since $R$ is of finite CM-type, it will admit a faithful system of parameters, and so we may assume further that $x$ is also a faithful system of parameters for MCM modules. On the other hand, the assumption $M$ being fully decomposable leads us to deduce that $x\text{Ext}^1_R(M',-) = 0$, because the same is true for $M$. Now by making use of Lemma 4.1, we obtain that $M'$ is a balanced big CM module, as needed.

6 Representation properties of balanced big CM modules

The aim of this section is to show that the representation-theoretic properties of balanced big CM modules have important consequences for the structural shape of the ring. It will turn out that any balanced big CM $R$-module which belongs to $\mathcal{X}_\omega$, being fully decomposable forces $R$ to be an isolated singularity. Moreover, it is proved that $R$ is of finite CM-type if and only if it is an isolated singularity and the category of all fully decomposable modules is closed under kernels of epimorphisms. First we state a lemma.

**Lemma 6.1** Let $A$ be a noetherian ring and let $\{M_i, \phi_i^j\}_{i,j \in I}$ be a direct system of indecomposable finitely generated $A$-modules, over a totally ordered set $I$. If $\lim M_i$ is a direct sum of finitely generated modules, then $\lim M_i = 0$ or there exists an index $t \in I$ such that for any $i \geq t$, $\phi_i^j$ is an isomorphism.

**Proof** If $\lim M_i = 0$, then there is nothing to prove. So assume that $\lim M_i$ is non-zero. By our assumption, $\lim M_i = \bigoplus_{j \in J} C_j$, where each $C_j$ is a finitely generated $A$-module.

Since by [18, Corollary 1.2.7], $\eta : 0 \to L \to \bigoplus_{i \in I} M_i \xrightarrow{\psi = (\psi_i)_i} \lim M_i \to 0$ is a pure exact sequence, the functor $\text{Hom}_A(C_j, -)$ leaves the previous sequence exact, for any $j$, implying that $\text{Hom}_A(\bigoplus_{j \in J} C_j, -)$ leaves this sequence exact as well. This indeed means that $\eta$ is split. Take an $A$-homomorphism $\psi = (\psi_i)_i : \lim M_i \to \bigoplus_{i \in I} M_i$ with $\phi \psi = \Sigma_{i \in I} \psi_i \psi_i = id_{\lim M_i}$. Take a non-zero finitely generated module $C_j$ such that, under $\psi$, it has non-zero image in only finitely many of $M_i$'s, say $M_1, \ldots, M_t$. So, we may define an $A$-homomorphism $\psi'_i : C_j \xrightarrow{i} \lim M_i \xrightarrow{\psi} \bigoplus_{i \in I} M_i \xrightarrow{\beta_i} M_i$, where $i$ is injection and $\beta_i((y_i)_{i \in I}) = \Sigma_{i \in I} \psi_i(y_i)$, for any $(y_i)_{i \in I} \in \bigoplus_{i \in I} M_i$. Considering an $A$-homomorphism $\phi'_i : M_i \xrightarrow{\phi_i} \lim M_i \xrightarrow{\rho} C_j$, where $\rho$ is the projection map, we have $\phi_i'(\psi'_i) = \psi'(\beta_i \psi_i) =
\[ \rho(\varphi_t \beta_t \psi)t \]. Now, by making use of the following commutative diagram;

\[
\begin{array}{ccc}
\bigoplus_{i=1}^t M_i & \xrightarrow{\varphi^t_{i-1} M_i} & \lim M_i \\
\beta_t & \downarrow \varphi_t & \downarrow \psi_t \\
M_t & & \\
\end{array}
\]

and the fact that \( \varphi \psi = id_{\lim M_t} \), one may infer that \( \varphi^t \psi^t = id_{C_j} \). Therefore, \( \psi^t \) is a split monomorphism and so it will be an isomorphism, because \( M_t \) is indecomposable. Since for any \( s \geq t \) we have the following commutative diagram of \( A \)-modules;

\[
\begin{array}{ccc}
M_t & \xrightarrow{\psi_t} & \lim M_i \\
\varphi^t_s & \downarrow \psi_s & \downarrow \varphi_s \\
M_s & & \\
\end{array}
\]

One may have the equalities; \( id_{C_j} = \psi^t_s \psi^t_i = \rho \psi_t \psi^t = (\rho \psi_s)(\psi^t_s \psi^t_i) \). Thus we have that \( \psi^t_s \psi^t_i : C_j \rightarrow M_t \rightarrow M_s \) is a split monomorphism, for any \( s \geq t \). Now \( C_j \) and \( M_s \) being indecomposable, forces \( \psi^t_s \psi^t_i \) to be an isomorphism. This, in turn, implies that \( \psi^t_s \) is indeed an isomorphism. The proof then is completed.

\[ \square \]

**Lemma 6.2** Let \( \{M_i, \varphi^t_j\}_{i \in \mathbb{N}_0} \) be a direct system of MCM \( R \)-modules such that for any \( i \leq j \), \( \varphi^t_j : M_i \rightarrow M_j \) is a monomorphism with Coker \( \varphi^t_j \) is MCM. Then \( \varinjlim M_i \) belongs to \( \mathcal{X}_\omega \).

**Proof** First one should note that, as we have mentioned in 2.4(1), each MCM \( R \)-module belongs to \( \mathcal{X}_\omega \). For any \( R \)-module \( M_i \), we inductively construct a right resolution by modules in \( \text{add } \omega \) forming a direct system. Set \( M_{-1} = 0 \). For \( i = 0 \), take an exact sequence of \( R \)-modules; \( 0 \rightarrow M_0 \rightarrow \omega_0 \rightarrow \omega_1 \rightarrow \cdots \), where for any \( j \geq 0 \), \( \omega_j \in \text{add } \omega \). Now assume that \( i \geq 0 \) and we have constructed such a resolution for \( M_i \) with the following commutative diagram of \( R \)-modules;

\[
\begin{array}{cccccccc}
0 & \rightarrow & M_{i-1} & \rightarrow & \omega_{i-1} \rightarrow & \omega_{i-1} \rightarrow & \omega_{i-1} \rightarrow & \cdots \\
& & \searrow \varphi^t_{i-1} & \downarrow & \omega_{i-1} & \downarrow & \omega_{i-1} & \\
0 & \rightarrow & M_i & \rightarrow & \omega_i & \rightarrow & \omega_i & \rightarrow & \cdots \\
\end{array}
\]

in which, all columns, except the left-hand side, are split monomorphism. Now we construct the diagram for the case \( i + 1 \). By our assumption, Coker \( \varphi^t_{i+1} \) is a MCM \( R \)-module and so, there is an exact sequence of \( R \)-modules; \( 0 \rightarrow \text{Coker } \varphi^t_{i+1} \rightarrow \omega_{i+1} \rightarrow \omega_{i+1} \rightarrow \cdots \), such that each \( \omega_{i+1} \) lies in \( \text{add } \omega \). Since the functor \( \text{Hom}_R(\cdot, \omega) \) leaves any short exact sequence in MCM modules, exact, one may construct the following commutative diagram of \( R \)-modules

\[ \text{Springer} \]
with exact rows and columns;

\[
\begin{array}{cccc}
0 & 0 & 0 & \\
\downarrow & \downarrow & \downarrow & \\
M_i & \omega_i & \omega_{i+1} & \\
\downarrow & \downarrow & \downarrow & \\
M_{i+1} & \omega_{i+1} & \omega_{i+1} & \\
\downarrow & \downarrow & \downarrow & \\
\text{Coker} \phi & \omega'_{i+1} & \omega'_{i+1} & \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & \\
\end{array}
\]

such that any column, except the left-hand side, is split. Hence applying the direct limit functor, gives rise to the exact sequence of \( R \)-modules; \( 0 \rightarrow \lim M_i \rightarrow \omega_n \rightarrow \omega_{n+1} \rightarrow \cdots \), where \( \omega_n \in \text{Add}\omega \), meaning that \( \lim M_i \in \mathcal{X}_\omega \). So we are done. \( \square \)

For given two \( R \)-modules \( M, N \), \( \text{Hom}_R(M, N) \) stands for \( \text{Hom}_R(M, N) / \mathfrak{I}(M, N) \), where \( \mathfrak{I}(M, N) \) is the \( R \)-submodule of \( \text{Hom}_R(M, N) \) consisting of all homomorphisms factoring through a module in \( \text{add}\omega \).

As we have mentioned in the introduction, it has been proved by Chase [11] that every pure-semisimple ring is artinian. From this point of view, the following result can be seen as a generalization of Chase’s result for the category of MCM modules. Indeed, the result below asserts that if every balanced big CM module which belongs to \( \mathcal{X}_\omega \), is fully decomposable, then \( \text{Hom}_R(-, -) \) is artinian over MCM modules.

**Theorem 6.3** Let any balanced big CM \( R \)-module belonging to \( \mathcal{X}_\omega \) be fully decomposable. Then \( R \) is an isolated singularity.

**Proof** If any MCM \( R \)-module belongs to \( \text{add}\omega \), then \( R \) will be of finite CM-type and so \( R \) is known to be an isolated singularity; see [4, 24]. Hence, in this case the desired result is obtained. So assume that there are some MCM \( R \)-modules which are not in \( \text{add}\omega \). First we show that for any MCM module \( M \), \( \text{Hom}_R(M, M) \) is an artinian \( R \)-module. To this end, clearly we only need to treat with those (indecomposable) modules which do not belong to \( \text{add}\omega \). Suppose that \( M_0 \) is an arbitrary indecomposable MCM \( R \)-module which does not belong to \( \text{add}\omega \). Taking an arbitrary \( R \)-regular element \( x \), we may have the following pull-back diagram;

\[
\begin{array}{cccc}
0 & 0 & \\
\downarrow & \downarrow & \\
\omega^1 & \omega^1 & \\
\downarrow & \downarrow & \\
0 & M_0 & x & M_0/\langle x \rangle M_0 \\
\downarrow & \downarrow & \downarrow & \\
0 & M_0 & x & M_0/\langle x \rangle M_0 \\
\end{array}
\]
where $G' \longrightarrow M_0/xM_0$ is a right minimal MCM-approximation. As $\text{Ext}_R^1(M_0, \omega^\mathfrak{n}1) = 0$, the left column will be split, implying that $M_1 \cong M_0 \oplus \omega^\mathfrak{n}1$. Evidently $M_1$ is also a MCM $R$-module which does not belonging to add$\omega$. Thus replacing $M_0$ by $M_1$ in the above argument, gives rise to the existence of $R$-homomorphism $M_1 \xrightarrow{x} M_2$ such that $M_2 \cong M_1 \oplus \omega^\mathfrak{n}2$. By repeating this procedure, we obtain a chain of $R$-homomorphisms of MCM modules;

$$M_0 \xrightarrow{x} M_1 \xrightarrow{x} M_2 \xrightarrow{x} \cdots ,$$

(6.1)

such that for any $i \geq 1$, there is an isomorphism $M_i \cong M_0 \oplus \omega^\mathfrak{n}i$, for some non-negative integer $n_i$. Applying the functor $\text{Hom}_R(M_0, -)$ to (6.1), yields the following chain of $R$-modules;

$$\text{Hom}_R(M_0, M_0) \xrightarrow{x} \text{Hom}_R(M_0, M_0) \xrightarrow{x} \text{Hom}_R(M_0, M_0) \longrightarrow \cdots ,$$

(6.2)

because $\text{Hom}_R(M_0, M_i) \cong \text{Hom}_R(M_0, M_0)$, for any $i \geq 1$. According to our construction, $M_i \xrightarrow{x} M_j$, where $j > i$, is an $R$-monomorphism such that its cokernel is MCM and so it can be easily seen that $\lim M_i$ is a balanced big CM $R$-module. In view of Lemma 6.2, $\lim M_i \in \chi_\omega$ and so by the hypothesis, $\lim M_i = \bigoplus_{j \in J} C_j$, where each $C_j$ is finitely generated. Therefore, we have the following isomorphisms;

$$\lim \text{Hom}_R(M_0, M_i) \cong \text{Hom}_R(M_0, \text{lim} M_i) \cong \text{Hom}_R(M_0, \bigoplus_{j \in J} C_j) \cong \bigoplus_{j \in J} \text{Hom}_R(M_0, C_j).$$

Since $M_0$ is an indecomposable $R$-module, $\text{Hom}_R(M_0, M_0)$ is indecomposable as an $\text{End}_R(M_0)$-module. Now one may apply Lemma 6.1 and conclude that after some steps, all of morphisms in (6.2) are isomorphism or $\lim \text{Hom}_R(M_0, M_0) = 0$. The former one cannot take place. Otherwise, we will have the isomorphism $\text{Hom}_R(M_0, M_0) \cong x^t \text{Hom}_R(M_0, M_0)$, for some integer $t > 0$, and so by Nakayama’s lemma $\text{Hom}_R(M_0, M_0) = 0$, guaranteeing that $M_0$ lies inside add$\omega$, which is a contradiction. Hence the latter one will take place, meaning that there exists an integer $t > 0$ such that $x^t \text{Hom}_R(M_0, M_0) = 0$. Next suppose that $x = x_1, \ldots , x_d$ is a system of parameters of $R$. As any permutation of $x$ is again $R$-regular sequence, one may find an integer $n > 0$ such that $x^n \text{Hom}_R(M_0, M_0) = 0$, for any $1 \leq i \leq d$, that is to say, $x^n \text{Hom}_R(M_0, M_0) = 0$. This would imply that $m^n \text{Hom}_R(M_0, M_0) = 0$ for some integer $u > 0$, and so $\text{Hom}_R(M_0, M_0)$ is an artinian $R$-module, as claimed. Next we show that the module $M_0$ is locally free on the punctured spectrum of $R$. As we have already showed, $\text{Hom}_R(M_0, M_0)_p = 0$, for all nonmaximal prime ideals $p$ of $R$. Now, if $R$ is Gorenstein, i.e. $R = \omega$, then the equality $\text{Hom}_R(M_0, M_0) = \text{Hom}_R(M_0, M_0)$ gives the desired result. In the case $R$ is not necessarily Gorenstein, it will not belong to add$\omega$. Thus, by repeating the above argument for $R$ instead of $M_0$, we deduce that $\omega_p = R_p$, for all nonmaximal prime ideals $p$ of $R$, meaning that $R$ is locally Gorenstein and consequently, $M_0$ is a free $R_p$-module. Hence any MCM $R$-module is locally free on the punctured spectrum of $R$, and so, [44, Lemma 3.3] yields that $R$ is an isolated singularity. The proof is now completed. □

The result below, is an immediate consequence of Theorem 6.3.

**Corollary 6.4** Let $(R, \mathfrak{m})$ be a complete Gorenstein local ring. If every Gorenstein projective $R$-module is fully decomposable, then $R$ is an isolated singularity.

Let $M$, $N$ be two MCM $R$-modules. Recall that rad$(M, N)$ is a submodule of $\text{Hom}_R(M, N)$ consisting of those homomorphisms $\varphi : M \longrightarrow N$ such that, when we decompose $M =
⊕_j M_j and N = ⊕_i N_i into indecomposable modules, and accordingly decompose φ = (φ_ij : M_j → N_i), no φ_ij is an isomorphism. Moreover, rad^2(M, N) is a submodule of Hom_R(M, N) consisting of those homomorphisms φ : M → N for which there is a factorization

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & N \\
\downarrow{\alpha} & & \downarrow{\beta} \\
X & & \\
\end{array}
\]

with X is an MCM R-module, α ∈ rad(M, X) and β ∈ rad(X, N). For n > 2, rad^n(M, N) is defined inductively; see [30, Definition 12.20].

On the other hand, suppose that F/x^2 F consists of all indecomposable MCM modules that is closed under isomorphism. Let A be the subcategory consisting of those homomorphisms φ : M → N for which there is a surjective h ∈ Hom_R(N, X) such that φ is an isomorphism. Moreover, rad^2(M, N) is isomorphic indecomposable objects of A:

\[
\text{rad}^2(M, N) = \{ M \text{ such that for any } X \in \text{add} A \text{ and a faithful system of parameters } x \text{ for } A, \text{ there is an } R\text{-homomorphism } f : X \xrightarrow{f_1} M \oplus N, \text{ where } M \in \text{add} C \text{ and } N \in \text{add} A \text{ such that for any } L \in \text{add} C, \text{ Hom}_R(f, L) \text{ is surjective and } f_2 \otimes_R R/x^2 R = 0. \]

**Proof** We show by induction that for any n ≥ 0 and each X ∈ add A, there is a morphism f : X → M ⊕ N, where M ∈ add C and N ∈ add A such that for any L ∈ add C, Hom_R(f, L) is surjective and f_2 ∈ rad^n(X, N). In case, n = 0, set f = id_X. Now assume that n > 0 and the result has been proved for values smaller than n. By the induction hypothesis, there exists a morphism f' : X → M' ⊕ N', in which f'_1 ∈ rad^{n-1}(X, N'), N' ∈ add A, M' ∈ add C and for any K ∈ add C, the morphism Hom_R(f', K) is surjective. Since the category of MCM modules has left almost split morphisms, there is an R-homomorphism g : N' → Z = Z_1 ⊕ Z_2, with Z_1 ∈ add A, Z_2 ∈ add C and g_1 ∈ rad(N', Z_1) such that ImHom(g, L) = Hom_R(N', L) for any L ∈ add C. Assuming h as the following composition morphism

\[
h : X \xrightarrow{f_1'} M' \oplus N' \xrightarrow{id_M \oplus g} M' \oplus Z
\]

where g_1 f'_2 ∈ rad^n(X, Z_1), we have that Hom(h, K) is surjective for any K ∈ add C.

On the other hand, suppose that F := \{X_1, \ldots, X_t\} is the set of all pairwise non-isomorphic indecomposable objects of A. Since x is a faithful system of parameters for A, F/x^2 F = \{X_1/x^2 X_1, \ldots, X_t/x^2 X_t\} is a set of indecomposable modules of finite length. By virtue of Corollary to [20, Lemma 12], there is a non-negative integer n such that rad^n(X_i/x^2 X_i, X_j/x^2 X_j) = 0 for all X_i, X_j ∈ F. Consequently, f = f_2 ⊗_R R/x^2 R ∈ rad^n(X_i/x^2 X_i, X_j/x^2 X_j) and so f = 0, which gives the desired result. □
Theorem 6.6 Let $R$ be an isolated singularity which is not of finite CM-type. Then there is an infinite set of pairwise non-isomorphic indecomposable MCM modules $\{M_i\}_{i \in \mathbb{N}}$ and non-zero $R$-homomorphisms $f_i : M_i \to k$ such that any composition map $M_j \to M_i \xrightarrow{f_j} k$ is zero, for all $j > i$.

Proof In order to obtain the desired result, we will first construct a pairwise disjoint infinite family of finite type subcategories of indecomposable MCM modules $A_1, A_2, A_3, \ldots$, inductively as follows: (i) We set $A_1$ to be the class of all projective $R$-modules that are isomorphic to $R$. (ii) Suppose $j > 1$ is an integer and assume that we have already constructed $A_1, \ldots, A_{j-1}$. Letting $C_j = \text{ind}(\text{MCM}) - \bigcup_{i=1}^{j-1} A_i$ and $x$ a faithful system of parameters for $\bigcup_{i=1}^{j-1} A_i$, by Proposition 6.5, there is an $R$-homomorphism $f : R \xrightarrow{[f_1, f_2]} K_j \oplus N$, where $K_j \in \text{add}C_j$ and $N \in \text{add}(\bigcup_{i=1}^{j-1} A_i)$ such that for any $L \in \text{add}A_j$, $\text{Hom}_R(f, L)$ is surjective and $f_2 \otimes_R R/x^2R = 0$. By the Krull–Remak–Schmidt theorem, $K_j = \bigoplus_{i=1}^{l_j} X_i$, where each $X_i$ is an indecomposable finitely generated submodule of $K_j$. We put $A_j$ to be the class of all MCM modules that are isomorphic to one of $X_1, \ldots, X_l$.

So we have constructed a pairwise disjoint infinite family of finite type subcategories of indecomposable MCM modules $A_1, A_2, \ldots$

Let us divide the remainder of the proof into three steps:

Step 1: We show that, for any $j, A_j$ is a generator for $C_s$, for any $s \geq j$, namely, for each $L \in C_s$, there exists an $R$-epimorphism $Y \to L$, where $Y \in \text{add}A_j$. To see this, take an arbitrary object $L \in C_s$ and consider an epimorphism $\alpha : R^n \to L$. By part (ii), there exists an $R$-homomorphism $f : R \xrightarrow{[f_1, f_2]} K_j \oplus N$ such that $K_j \in \text{add}C_j$ and $N \in \text{add}(\bigcup_{i=1}^{j-1} A_i)$. Since for any $s \geq j, C_s \subseteq C_j, L \in C_j$ and by construction of $f$ in part (ii), $\text{Hom}_R(f, L)$ is surjective. Thus there is an $R$-homomorphism $(\psi_1, \psi_2) : K_j^n \oplus N^n \to L$ such that the diagram

\[
\begin{array}{ccc}
R^n & \xrightarrow{[f_1^n, f_2^n]} & K_j^n \oplus N^n \\
\downarrow{\alpha} & & \downarrow{(\psi_1, \psi_2)} \\
L & & L
\end{array}
\]

is commutative. Take a faithful system of parameters $x$ for $\bigcup_{i=1}^{j-1} A_i$, which has been used in the construction of $A_j$ in part (ii). Now, by applying the functor $- \otimes_R R/x^2R$ and using the fact that $f_2 \otimes_R R/x^2R = 0$, we infer that $\psi_1 : K_j^n/x^2K_j^n \to L/x^2L$ is an epimorphism and so by Nakayama’s lemma $\psi_1$, will be an epimorphism, as well. We set $Y := K_j^n$.

Step 2: Next we show that for any $j$, there is an object $X_j$ of $A_j$ such that there is not any epimorphism $\bigoplus_{i=1}^{l_j} M_i^{n_i} \to X_j$ where $M_i$’s are objects of $A_i$’s and $s_i$ is a set for each $i$. Suppose that for some $j$, this is not the case. Let $A_j = \{X_1, X_2, \ldots, X_l\}$, up to isomorphism. As $\bigoplus_{i=1}^{l_j} X_i$ is finitely generated, we may assume that there is an epimorphism $\varphi : \bigoplus_{i=1}^{l_j} M_i^{n_i} \to \bigoplus_{i=1}^{l_j} X_i$ for some integer $n > 0$, where $M_i$’s belong to $A_i$’s and $n_i > 0$ is an integer for each $i$. Since $R$ is an isolated singularity, there exists a faithful system of parameters $y$ for $\{M_j+1, \ldots, M_n, X_1, \ldots, X_l\}$, by Proposition 2.6. Considering the epimorphism $\tilde{\varphi} : \bigoplus_{i=1}^{l_j} (M_i^{n_i}/y^2M_i^{n_i}) \to (\bigoplus_{i=1}^{l_j} X_i)/y^2(\bigoplus_{i=1}^{l_j} X_i)$, Main property (a) of 3.2 yields that

$$\min\{\mu^*(X_i/y^2X_i)|1 \leq i \leq t\} \geq \min\{\mu^*(M_i/y^2M_i)|1 \leq i \leq n\}.$$
It should be observed that if the equality takes place, then by part (b) of Main property of 3.2, \( \bar{\varphi} \) is a split epimorphism. On the other hand by [30, Corollary 15.11], quotient modules, for any \( 1 \leq i \leq t \), \( X_i/y^2X_i \) and for any \( j+1 \leq i \leq n \), \( M_i/y^2M_i \) are indecomposable. Thus by the Krull–Remak–Schmidt theorem, for some \( 1 \leq i \leq t \), \( X_i/y^2X_i \) is isomorphism with a direct summand of \( \oplus_{i=j+1}^{n} M_j/y^2M_j \), and so, \( X_i/y^2X_i \cong M_i/y^2M_i \) for some integer \( j+1 \leq s \leq n \). Hence, applying [10, Lemma 3.3.2] gives rise to the isomorphism \( X_i \cong M_i \), which contradicts with our construction of \( A_i \)'s.

Proof (1) \( \Rightarrow \) (2): Theorem 4.7 gives the desired result.

(2) \( \Rightarrow \) (3): Take an arbitrary MCM \( R \)-module \( M \). As \( M \) lies in FD, by the hypothesis, \( M \) has an \( m \)-primary cohomological annihilator. So, \( R \) is an isolated singularity. Next consider a short exact sequence of \( R \)-modules; \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \), where \( M, M'' \) belong to FD. By the hypothesis, \( M, M'' \) are balanced big CM modules with \( m \)-primary cohomological annihilators, implying that \( M' \) is a weak balanced big CM \( R \)-module, and by Remark 2.10, we get that \( M' \) has an \( m \)-primary cohomological annihilator. Hence invoking Lemma 4.1, yields that either \( M' \) is zero or it is balanced big CM. Consequently, \( M' \) is in FD.

(3) \( \Rightarrow \) (1): Assume on the contrary that \( R \) is not of finite CM-type. So in view of Theorem 6.6, there is an infinite set of pairwise non-isomorphic indecomposable MCM \( R \)-modules \( \{M_i\}_{i \in I} \) and non-zero \( R \)-homomorphisms \( f_i : M_i \rightarrow k \) such that any composition map \( M_j \rightarrow M_i \xrightarrow{f_i} k \) with \( j > i \), is zero. Here \( I \) is a subset of \( \mathbb{N} \). As \( G \xrightarrow{\varphi} k \) is a right minimal MCM-approximation, there is a non-zero homomorphism \( \varphi_i : M_i \rightarrow G \), for any \( i \) such that \( \varphi \varphi_i = f_i \). Setting \( G = (g_i)_{i \in I} : \oplus_{i \in I} M_i \rightarrow G \), we have a short exact sequence of \( R \)-modules; \( 0 \rightarrow K \xrightarrow{\theta} (\oplus_{i \in I} M_i) \oplus R^n \xrightarrow{(g_i, 0)} G \). Hence, the hypothesis FD being closed under kernels of epimorphisms, yields that \( K \) is in FD. Now Proposition 5.1 forces \( I \) to be a finite set, which is a contradiction. Therefore, \( R \) is of finite CM-type.

\( \square \)
(1) ⇒ (4): According to Corollary 2 of [24], $R$ is an isolated singularity. Next, consider a short exact sequence of $R$-modules; $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, in which $M', M'' \in FD$. By the hypothesis, $M'$ and $M''$ are balanced big CM modules with $m$-primary cohomological annihilators. So it is fairly easy to see that $M$ is balanced big CM with an $m$-primary cohomological annihilator. Now Theorem 4.7 would imply that $M \in FD$. This means that FD is closed under extensions. Moreover, Theorem 4.9 indicates that FD is closed under direct summand.

(4) ⇒ (3). Take a short exact sequence of $R$-modules; $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, in which $M, M'' \in FD$. We would like to show that $M' \in FD$. By the hypothesis, $M'' = \bigoplus_{i \in I} X_i$, where each $X_i$ is finitely generated. For any $i \in I$, take a short exact sequence of finitely generated $R$-modules, $0 \rightarrow L_i \rightarrow P_i \rightarrow X_i \rightarrow 0$, in which $P_i \rightarrow X_i$ is a projective cover. In particular, one may have the short exact sequence of $R$-modules, $0 \rightarrow L \rightarrow P \rightarrow M'' \rightarrow 0$, where $P = \bigoplus_{i \in I} P_i$ and $L = \bigoplus_{i \in I} L_i$. Considering the following commutative diagram with exact rows;

$\begin{array}{cccccc}
0 & \rightarrow & L & \rightarrow & P & \rightarrow & M'' & \rightarrow & 0 \\
& & u & & \downarrow & & \downarrow & & id & \\
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0
\end{array}$

we obtain the exact sequence $0 \rightarrow L \rightarrow P \oplus M' \rightarrow M \rightarrow 0$. Since $L, M \in FD$, by our assumption, $P \oplus M' \in FD$. Consequently, $M' \in FD$, because by the hypothesis FD is closed under direct summand. So the proof is completed.

Here we recover a notable result of Beligiannis [9, Theorem 4.20].

**Theorem 6.8** Let $R$ be a Gorenstein complete local ring. Then the following conditions are equivalent:

1. $R$ is of finite CM-type.
2. Every Gorenstein projective $R$-module is fully decomposable.
3. The subcategory of Gorenstein projective $R$-modules with $m$-primary cohomological annihilators coincides with FD.
4. The category of all indecomposable finitely generated Gorenstein projective $R$-modules is of bounded h-length.

**Proof** (1) ⇒ (2): This is Theorem 5.12.

(2) ⇒ (1): In view of Corollary 6.4, $R$ is an isolated singularity. Now by applying the proof of the implication (3 ⇒ 1) of Theorem 6.7 and using the fact that the category of Gorenstein projective modules is closed under kernels of epimorphisms, we deduce that $R$ is of finite CM-type.

(3) ⇒ (1): By the assumption, every MCM $R$-module has an $m$-primary cohomological annihilator, implying that $R$ is an isolated singularity. Moreover, it follows from the hypothesis that FD is closed under kernels of epimorphisms. Now the implication (3 ⇒ 1) of Theorem 6.7 yields the required result.

(1) ⇒ (3): This follows from the implication (1) ⇒ (2).

(4) ⇔ (1): The implication (4) ⇒ (1) follows from Corollary 5.3, whereas the reverse implication holds trivially.

□
7 Representation properties of CM modules over artin algebras

Motivated by (commutative) complete Cohen–Macaulay local rings, Auslander and Reiten in [6,7] have introduced and studied Cohen–Macaulay artin algebras. Recall that an artin algebra $\Lambda$ is said to be Cohen–Macaulay if there exists a pair of adjoint functors $(G, F)$ between $\text{mod}\Lambda$ and $\text{mod}\Lambda$, inducing mutually inverse equivalences;

$$
\mathcal{T}^\infty(\Lambda) \xrightarrow{F} \mathcal{P}^\infty(\Lambda),
$$

where $\mathcal{P}^\infty(\Lambda)$ (resp. $\mathcal{T}^\infty(\Lambda)$) denotes the category of all finitely generated modules of finite projective (resp. injective) dimension.

As we have noted in the introduction, it is well-known that if $\Lambda$ is a Cohen–Macaulay artin algebra, then there is a finitely generated $\Lambda$-bimodule $\omega$ such that the functors $F$ and $G$ are presented by $\text{Hom}_\Lambda(\omega, -)$ and $\omega \otimes_\Lambda -$, respectively; see [6]. In this case, $\omega$ is called a dualizing module for $\Lambda$.

Remark 7.1 There is a tight connection between dualizing modules and strong cotilting modules over artin algebras. Precisely, a $\Lambda$-bimodule $\omega$ is dualizing if and only if $\omega$ is strong cotilting viewed both as left and right modules and the natural map $\Lambda \rightarrow \text{End}(\Lambda\omega)$ is an isomorphism; see [6, Proposition 3.1]. This connection gives an interesting interplay between cotilting theory for artin algebras and module theory for commutative Cohen–Macaulay rings. A selforthogonal $\Lambda$-module $\omega$ is cotilting if $\text{id}_{\Lambda} \omega < \infty$ and all injective $\Lambda$-modules are in $\text{add}\omega$, and it is said to be strong cotilting if, moreover, the equality $\mathcal{T}^\infty(\Lambda) = \text{add}\omega$ holds. Recall that $\omega$ is said to be selforthogonal, provided that $\text{Ext}^i_\Lambda(\omega, \omega) = 0$ for all $i > 0$.

We emphasize that the results of this section remain true even if $\omega$ is assumed to be a cotilting $\Lambda$-module and the natural map $\Lambda \rightarrow \text{End}(\Lambda\omega)$ is an isomorphism. We are indebted to Professor Osamu Iyama for pointing us this fact.

7.2. Throughout this section, $\Lambda$ is always a Cohen–Macaulay artin algebra and $\omega$ is a dualizing $\Lambda$-bimodule. We say that a $\Lambda$-module $M$ is $\omega$-Gorenstein projective, if it admits a right resolution by modules in $\text{Add}\omega$, that is, an exact sequence of $\Lambda$-modules;

$$
0 \rightarrow M \rightarrow w_0 \xrightarrow{d_0} w_1 \xrightarrow{d_1} \cdots \xrightarrow{d_i} w_i \xrightarrow{d_{i+1}} \cdots,
$$

with $w_i \in \text{Add}\omega$. So finitely generated $\omega$-Gorenstein projective modules are Cohen–Macaulay in the sense of Auslander and Reiten [7] and we also call them Cohen–Macaulay modules (CM modules). It should be noted that since $\omega$ is a selforthogonal $\Lambda$-module of finite injective dimension, $\text{Ext}^i_{\Lambda}(W, W') = 0$ for all modules $W, W' \in \text{Add}\omega$. Indeed, this follows from the isomorphisms $\text{Ext}^i_{\Lambda}(\oplus_{j \in J} \omega, W') \cong \prod_{j \in J} \text{Ext}^i_{\Lambda}(\omega, W')$ and $\text{Ext}^i_{\Lambda}(\omega, \oplus_{j \in J} \omega) \cong \oplus_{j \in J} \text{Ext}^i_{\Lambda}(\omega, \omega)$. One should observe that, as $\omega$ admits a projective resolution of finitely generated projective modules, [15, Exercise 2(a), page 16] ensures the validity of the latter isomorphism. So it is easily seen that our notion of $\omega$-Gorensteiness coincides with the one given by Holm and Jørgensen in [23]. We say that an $\omega$-Gorenstein projective $\Lambda$-module $M$ is fully decomposable (resp. of finite CM-type) if it is a direct sum of arbitrarily many copies (resp. of a finite number up to isomorphisms) of indecomposable CM modules.

Moreover, by CM-support of an $\omega$-Gorenstein projective module $M$, denoted by $\text{CM-supp}_\Lambda(M)$, we mean the set of all indecomposable CM $\Lambda$-modules $N$ such that $\text{Hom}_\Lambda(N, M) \neq 0$. 

[Springer]
Our aim in this section is to examine results in the previous sections in the context of Cohen–Macaulay artin algebras. It is proved that any ω-Gorenstein projective Λ-module with bounded length on CM-support must be fully decomposable; see Theorem 7.5. In particular, it will be observed in Theorem 7.6 that if an ω-Gorenstein projective module \( M \) is not of finite CM-type, then there are indecomposable CM-Λ-modules of arbitrarily large (finite) length, guaranteeing the validity of the first Brauer-Thrall conjecture for the category of Cohen-Macaulay modules over Cohen-Macaulay artin algebras. Moreover, our results extend a result of Chen [13, Main Theorem] for Cohen-Macaulay artin algebra, that is, we specify Cohen-Macaulay artin algebras of finite CM-type in terms of the decomposition properties of ω-Gorenstein projective modules.

Let \( \Lambda \ltimes \omega \) denote the trivial extension of \( \Lambda \) by \( \omega \). Then according to ring homomorphisms; \( \Lambda \rightarrow \Lambda \ltimes \omega \rightarrow \Lambda \), any \( \Lambda \)-module can be viewed as a \( \Lambda \ltimes \omega \)-module and vise versa, and in this section we shall do so freely.

Proposition 7.3 Every ω-Gorenstein projective \( \Lambda \)-module is a direct limit of CM-modules.

Proof Take an arbitrary \( \omega \)-Gorenstein projective \( \Lambda \)-module \( M \). Because of [28, Proposition 2.1], it suffices to show that any \( \Lambda \)-homomorphism \( f : N \rightarrow M \), where \( N \) is finitely generated, factors through a CM-\( \Lambda \)-module, say \( C \). Assume that \( \text{id}_\Lambda \omega = n \). In view of [23, Proposition 2.13], \( M \) is Gorenstein projective over \( \Lambda \ltimes \omega \), so one may take the following exact sequence of \( \Lambda \ltimes \omega \)-modules;

\[
0 \rightarrow M \rightarrow Q^0 \rightarrow \cdots \rightarrow Q^{n-1} \rightarrow L \rightarrow 0,
\]

in which for any \( i \), \( Q^i \) is projective and \( L \) is Gorenstein projective. Another use of [23, Proposition 2.13] yields that as a \( \Lambda \)-module, \( L \) is \( \omega \)-Gorenstein projective. Since \( N \) is a finitely generated \( \Lambda \)-module, evidently it is finitely generated over \( \Lambda \ltimes \omega \). Consider the following sequence of finitely generated \( \Lambda \ltimes \omega \)-modules;

\[
N \xrightarrow{d^0} P^0 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} P^n \rightarrow K \rightarrow 0,
\]

where \( N \rightarrow P^0 \) and \( \text{Coker}(d^i) \rightarrow P^{i+1} \), for any \( i \), are projective preenvelopes. It should be noted that these preenvelopes exist because of [15, page 247]. According to [16, Theorem 4.32], \( \Lambda \ltimes \omega \) is a Gorenstein algebra with injective dimension \( n \), where \( n = \text{id}_\Lambda \omega \). Hence by using [15, Theorem 10.2.14], we have the following exact sequence of finitely generated \( \Lambda \ltimes \omega \)-modules;

\[
0 \rightarrow C \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow K \rightarrow 0,
\]

such that each \( P_i \) is projective and \( C \) is Gorenstein projective. Consequently, one obtains the following commutative diagram of \( \Lambda \ltimes \omega \) (and also \( \Lambda \))-modules; which is similar to diagram appeared in the proof of [15, Lemma 10.3.6];
One should note that the morphisms $h_i$’s are lifted from $id: K \rightarrow K$, whereas, the existence of $f_i$’s follows from the construction of upper row. Finally, the morphisms $g_i$’s exist, because they are lifted from $f_n$. Now chasing diagram enables us to deduce that $f$ factors through $C \oplus P^0$ which is CM over $\Lambda$, thanks to [23, Proposition 2.13]. Clearly $f$ factors from this module as a $\Lambda$-homomorphism. So the proof is finished.

We need the following result for later use.

**Lemma 7.4** Let $M$ be a non-zero $\omega$-Gorenstein projective $\Lambda$-module. If $\text{CM-supp}_\Lambda(M)$ is of bounded length, then $M$ has an indecomposable CM direct summand.

**Proof** According to Proposition 7.3, there is a direct system of $\text{CM} \Lambda$-modules $\{M_i, \varphi_j^i|_{i, j \in I}\}$ such that $M = \varprojlim M_i$. As $M$ is non-zero, we can take an index $j \in I$ and an indecomposable CM direct summand $X_j$ of $M_j$ such that the morphism $\varphi_{j|X_j} : X_j \rightarrow M$ is non-zero, where $\varphi_j : M_j \rightarrow M$ is the natural morphism such that for any $i \leq j$, $\varphi_i = \varphi_j \varphi_{i|j}$. Let $k_1 \in I$ be an index with $k_1 > j$. So we have an indecomposable CM direct summand $X_{k_1}$ of $M_{k_1}$ such that

$$X_j \xrightarrow{\varphi_{k_1|X_j}} M_{k_1} \xrightarrow{\pi} X_{k_1} \xrightarrow{\varphi_{k_1|X_{k_1}}} M$$

is non-zero, where $\pi : M_{k_1} \rightarrow X_{k_1}$ is the canonical projection. We denote the composition map $X_j \xrightarrow{\varphi_{k_1|X_j}} M_{k_1} \xrightarrow{\pi} X_{k_1} \xrightarrow{\psi_{k_1}^j}$ by $\psi_{k_1}^j$. One can use the induction argument to obtain a chain of morphisms of indecomposable CM $\Lambda$-modules

$$X_j \xrightarrow{\psi_{k_1}} X_{k_1} \xrightarrow{\psi_{k_2}} X_{k_2} \xrightarrow{\psi_{k_3}} X_{k_3} \rightarrow \cdots,$$

such that any composite of finite number of morphisms has non-zero image in $M$. Since all $X_j$’s belong to $\text{CM-supp}_\Lambda(M)$, they are of bounded length. Hence, Harada-Sai Lemma [20, Lemma 11], guarantees the existence of an index $k_s \in I$ such that for each $k_s > k_t$, the induced morphism $\psi_{k_s}^j$ needs to be an isomorphism. This implies that, for any $k_s > k_t$, the morphism $\varphi_{k_s|X_{k_t}} : X_{k_t} \rightarrow M_{k_s}$ is a split monomorphism. This, in turn, would imply that $\varphi_{k_s|X_{k_t}} : X_{k_t} \rightarrow M$ is a pure monomorphism. As $X_{k_t}$ is a finitely generated module over the artinian ring $\Lambda$, it will be pure injective, enforcing $\varphi_{k_t|X_{k_t}}$ to be a split monomorphism. Hence, $M$ has an indecomposable CM direct summand $X_{k_s}$. So the proof is finished. □

The next result indicates that, for a given $\omega$-Gorenstein projective module $M$, the boundedness of its CM-support forces $M$ to be fully decomposable.

**Theorem 7.5** Let $M$ be an $\omega$-Gorenstein projective $\Lambda$-module. If $\text{CM-supp}_\Lambda(M)$ is of bounded length, then $M$ is fully decomposable.

**Proof** According to Lemma 7.4, $M$ has an indecomposable CM direct summand $X$. Put $\Sigma$ to be the set of all fully decomposable pure submodules of $M$. For any two objects $N, L \in \Sigma$, we write $N \leq L$ if and only if the following diagram is commutative;

$$\begin{array}{ccc}
N & \xrightarrow{i_{NL}} & L \\
\downarrow{i_N} & & \downarrow{i_L} \\
M & \xrightarrow{i_L} & L
\end{array}$$

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where \( i_N, i_L, i_{NL} \) are pure monomorphism, and \( i_{NL} \) is the inclusion map. Assume that \( Y = \oplus X_i \) is a pure submodule of \( M \), where each \( X_i \) is an indecomposable \( CM \) direct summand of \( M \), and \( Y \) is maximal with respect to this property. Take the pure exact sequence of \( \Lambda \)-modules;

\[
\eta : 0 \longrightarrow Y \xrightarrow{i_Y} M \longrightarrow K \longrightarrow 0.
\]

Let \( f : N \longrightarrow K \) be a non-zero \( \Lambda \)-homomorphism, where \( N \) is finitely generated. As \( \eta \) is a pure exact sequence, \( f \) will factor through \( M \). In view of Proposition 7.3, \( M = \lim_{i \in I} M_i \), where each \( M_i \) is a \( CM \) module. Consequently, for some index \( i \), the morphism \( f \) factors through \( M_i \), and so, [28, Proposition 2.1], enables us to infer that \( K = \lim_{i \in I} K_i \), where each \( K_i \) is a \( \Lambda \)-module. On the other hand, it is evident that any element of \( CM-\text{supp}_\Lambda(K) \) belongs to \( CM-\text{supp}_\Lambda(M) \), and so \( CM-\text{supp}_\Lambda(K) \) will be of bounded length. Therefore, by virtue of Lemma 7.4, \( K \) has an indecomposable \( CM \) direct summand \( X \). Thus \( Y \oplus X \) is a pure submodule \( M \), containing \( Y \) properly. However, this contradicts the maximality of \( Y \). Hence, \( K = 0 \), and then we get the isomorphism \( Y \cong M \). So the proof is completed. \( \square \)

**Theorem 7.6** Let \( M \) be an \( \omega \)-Gorenstein projective \( \Lambda \)-module which is not of finite \( CM \)-type. Then there are indecomposable \( CM \) \( \Lambda \)-modules of arbitrarily large finite length.

**Proof** Assume for the contradiction that the class of all indecomposable \( CM \) \( \Lambda \)-modules is of bounded length. So by Theorem 7.5, we deduce that \( M \) is fully decomposable. Suppose that \( M = \oplus_{i \in I} M_i^{(i)} \), in which for any \( i \), \( M_i \) is an indecomposable \( CM \) \( \Lambda \)-module. Put \( \mathcal{F} = \{ M_i \mid i \in I \} \). By our assumption, \( \mathcal{F} \) is of bounded length. By Property 2 of 3.2, there are only finitely many Gabriel–Roiter comeasures for \( \mathcal{F} \). Thus it is not a restriction if we additionally assume that all modules in \( \mathcal{F} \) have a fixed Gabriel–Roiter comeasure. Suppose that \( \{ S_1, \ldots, S_n \} \) is the complete list of non-isomorphic simple \( \Lambda \)-modules. Putting \( S = \bigoplus_{j=1}^n S_j \), analogous to the proof of Theorem 3.3 (steps 2 and 3), for each \( i \), there is a \( \Lambda \)-homomorphism \( f_i : M_i \longrightarrow S \) such that for any \( j \in I \) with \( i \neq j \), any composition map \( M_i \longrightarrow M_j \xrightarrow{f_j} S \) is zero. In view of [6, Proposition 1.4], there exists a right \( CM \)-approximation \( \alpha' : G' \longrightarrow S \), and so for any \( i \in I \), one may find a \( \Lambda \)-homomorphism \( g_i : M_i \longrightarrow G' \) such that \( \alpha' g_i = f_i \). Set \( g = (g_i)_{i \in I} : \oplus_{i \in I} M_i \longrightarrow G' \). Consider the exact sequence of \( \Lambda \)-modules:

\[
0 \longrightarrow K \xrightarrow{\theta} \bigoplus_{i \in I} M_i \longrightarrow \Lambda^n [\oplus_{i \in I} G'] \longrightarrow 0.
\]

Evidently, \( K \) is \( \omega \)-Gorenstein projective and hence any direct summand of \( K \) is again \( \omega \)-Gorenstein projective. Consequently, by Theorem 7.5, \( K = \oplus_{i \in I} K_i \), where for any \( i \), \( K_i \) is an indecomposable \( \omega \)-Gorenstein projective \( \Lambda \)-module. Now a similar result to Proposition 5.1 leads us to infer that \( I \) is a finite set, meaning that \( M \) is of finite \( CM \)-type. \( \square \)

The result below, which is an immediate consequence of Theorem 7.6, should be seen as the first Brauer–Thrall theorem for \( CM \) \( \Lambda \)-modules.

**Corollary 7.7** Let the category of all indecomposable \( CM \) \( \Lambda \)-modules be of bounded length. Then \( \Lambda \) is of finite \( CM \)-type.

7.8. According to [6], the category of \( CM \) \( \Lambda \)-modules admits almost split sequences. Moreover, for a given object \( M \in \text{mod}\Lambda \), by [6, Proposition 1.4] there is a \( CM \)-approximation \( X \longrightarrow M \). Hence one may deduce that the category of \( CM \) modules has left almost split morphisms. Assume that \( \mathcal{A} \) is a finite type subcategory of \( CM \) \( \Lambda \)-modules. Then the same argument given in the proof of Proposition 6.5 (see also [8, Proposition 3.13]) indicates that for any \( X \in \text{add}\mathcal{A} \), there is a \( \Lambda \)-homomorphism \( f : X \longrightarrow M \) with \( M \in \text{add}\mathcal{C} = CM - \mathcal{A} \).
such that for any \( L \in \text{add}C \), \( \text{Hom}(f, L) \) is surjective, that is to say, \( f : X \rightarrow M \) is a \( C \)-preenvelope.

**Theorem 7.9** Every \( \omega \)-Gorenstein projective module is fully decomposable if and only if \( \Lambda \) is of finite \( \text{CM} \)-type.

**Proof** The ‘if’ part is Theorem 7.5. For the ‘only if’ part, assume that \( \Lambda \) is not of finite \( \text{CM} \)-type. Analogously to the proof of Theorem 6.6, we obtain a pairwise disjoint infinite family of finite type subcategories of indecomposable \( \text{CM} \) modules \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots \) as follows:

(i) Assume that \( \mathcal{A}_1 \) is the class of all indecomposable \( \Lambda \)-modules that are isomorphic to indecomposable projective \( \Lambda \)-modules.

(ii) Suppose that for any \( j \geq 1 \), we have already constructed \( \mathcal{A}_1, \ldots, \mathcal{A}_{j-1} \). Set \( \mathcal{C}_j = \text{indCM} - (\bigcup_{i=1}^{j-1} \mathcal{A}_i) \). For a given \( Q \in \text{Add} \mathcal{A}_1 \), take a \( \mathcal{C}_j \)-preenvelope \( f : Q \rightarrow K_j \), which exists by 7.8. By the Krull-Remark-Schmidt theorem, \( K_j = \bigoplus_{i=1}^{r} X_i \), where each \( X_i \) is a finitely generated indecomposable submodule of \( K_j \). Now put \( \mathcal{A}_j \) to be the class of all \( \text{CM} \) modules that are isomorphic to one of \( X_1, \ldots, X_r \).

Let us divide the remainder of the proof into three steps:

**Step 1:** We show that, for any \( j \), \( \mathcal{A}_j \) is a generator for \( \mathcal{C}_j \), for any \( s \geq j \). To see this, take an arbitrary object \( L \in \mathcal{C}_s \) and consider an epimorphism \( \alpha : Q^n \rightarrow L \), where \( Q \) is projective \( \Lambda \)-module. By part (ii), there exists a \( \mathcal{C}_j \)-preenvelope \( f : Q^n \rightarrow K_j \). Since for any \( s \geq j \), \( \mathcal{C}_s \subseteq \mathcal{C}_j \) and \( L \in \mathcal{C}_j \) and in particular, there is the following commutative diagram:

\[
\begin{array}{ccc}
Q^n & \xrightarrow{f^n} & K_j^n \\
\downarrow{\alpha} & & \downarrow{\psi} \\
L & & \\
\end{array}
\]

because \( f \) is \( \mathcal{C}_j \)-preenvelope. We set \( Y := K_j^n \).

**Step 2:** We show that for any \( j \), there exists an object \( X_j \) of \( \mathcal{A}_j \) such that there is no any epimorphism \( \bigoplus_{i=1}^{r} M_{i}^{(s_i)} \rightarrow X_j \), in which \( M_{i}^{(s_i)} \) are objects of \( \mathcal{A}_j \)’s and \( s_i \) is a set for any \( i \). Assume that this is not the case. By our construction, \( \mathcal{A}_j = \{X_1, \ldots, X_r\} \), up to isomorphism. As \( \bigoplus_{i=1}^{r} X_i \) is finitely generated, we could assume that there exists a \( \Lambda \)-epimorphism \( \bigoplus_{i=1}^{n} M_i^{(n_i)} \rightarrow \bigoplus_{i=1}^{n} X_j^{(m_j)} \) for some positive integers \( n, m_i \). Therefore, Main property (a) of 3.2 gives rise to the inequality \( \min\{\mu^*(X_j) \mid 1 \leq j \leq t\} \geq \min\{\mu^*(M_i) \mid j+1 \leq i \leq n\} \). Since by construction of \( \mathcal{A}_j \)’s, none of modules \( X_j \) is not a direct summand of \( \bigoplus_{i=1}^{n} M_i \), the equality may not be accomplished. Letting \( \mu^*(M_i) = \min\{\mu^*(M_i) \mid j+1 \leq i \leq n\} \), by step 1, there is a \( \Lambda \)-epimorphism \( \bigoplus_{i=1}^{n} X_j^{(m_i)} \rightarrow M_i \), for some integers \( m_i > 0 \), implying that \( \mu^*(M_i) > \mu^*(X_j) \), for some \( 1 \leq j \leq t \), and so we derive a contradiction.

**Step 3:** As we have seen in step 2, for any \( i \), there exists an object \( M_i \in \mathcal{A}_i \) such that there does not exist any epimorphism \( \varphi : \bigoplus_{j=i}^{r} M_j^{(s_j)} \rightarrow M_i \), where \( M_j^{(s_j)} \) are objects of \( \mathcal{A}_j \)’s and \( s_j \) is a set for any \( j \). Now, for any \( i \), we take only one of such modules \( M_i \) and denote the class consisting of all these modules by \( \mathcal{F} \). Since \( \mathcal{A}_j \)’s are pairwise disjoint infinite family, \( \mathcal{F} \) will be an infinite set of indecomposable pairwise non-isomorphic \( \text{CM} \) \( \Lambda \)-modules. Therefore, similar to the argument given in the proof of Theorem 3.3, we get \( \Lambda \)-homomorphisms \( f_i : M_i \rightarrow S \) such that for any \( j > i \), each composition map \( M_j \rightarrow M_i \xrightarrow{f_j} S \) is zero, where \( S = \bigoplus_{j=1}^{n} S_j \) and \( \{S_1, \ldots, S_n\} \) is the complete list of non-isomorphic simple \( \Lambda \)-modules. As \( \alpha' : G' \rightarrow S \) is a right \( \text{CM} \)-approximation, one may obtain a \( \Lambda \)-homomorphism
$g_i : M_i \rightarrow G'$, for any $i$. Setting $g = (g_i)_{i \in I} : \oplus_{i \in I} M_i \rightarrow G'$, where $I$ is a subset of $\mathbb{N}$, we have an exact sequence of $\Lambda$-modules; $0 \rightarrow K \rightarrow \oplus_{i \in I} M_i \oplus \Lambda^n \xrightarrow{g} G' \rightarrow 0$. Clearly, $K$ is $\omega$-Gorenstein projective and so, by the hypothesis, it can be written as a direct sum of indecomposable finitely generated modules, say $K = \oplus_{j \in J} K_j$. Now the remainder of the proof goes along the same lines of the method given in the proof of Theorem 5.2, by replacing $\Lambda$ and $S$ with $R$ and $k$, respectively. So we omit it.

Since over Gorenstein algebras, $\omega$-Gorenstein projective $\Lambda$-modules are just Gorenstein projective modules, as a direct consequence of Theorem 7.9 together with Corollary 7.7, we recover Chen’s theorem [13, Main Theorem].

**Corollary 7.10** Let $\Lambda$ be a Gorenstein artin algebra. Then the following conditions are equivalent:

1. $\Lambda$ is of finite CM-type.
2. Any Gorenstein projective $\Lambda$-module is fully decomposable.
3. The category of all indecomposable finitely generated Gorenstein projective $\Lambda$-modules is of bounded length.

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