Chiral Quark-Soliton Model

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Abstract

The Chiral Quark-Soliton Model of nucleons is based on two ideas: 1) the major role of spontaneous chiral symmetry breaking in hadron physics and 2) the relevance of the large $N_c$ (= number of colours) limit for the real world. In these lectures I review the theoretical foundations of the model, the physics involved, and some of applications.

Contents

1 How do we know chiral symmetry is spontaneously broken? 2
2 Low-energy limit of QCD from instantons 3
  2.1 Some of the results 3
  2.2 Instanton-induced interactions 5
2.3 Bosonization 9
2.4 Chiral lagrangian 12
3 Properties of the effective chiral lagrangian (EChL) 14
  3.1 Derivative expansion and interpolation formula 15
  3.2 The Wess–Zumino term and the baryon number 17
4 The nucleon 21
  4.1 Physical motivations 21
  4.2 Nucleon mass: a functional of the pion field 23
  4.3 Nucleon profile 25
  4.4 Quantum numbers of baryons 30
  4.5 Some applications 36
  4.6 Nucleon structure functions 37
5 Conclusions 39

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1 How do we know chiral symmetry is spontaneously broken?

The QCD lagrangian with $N_f$ massless flavours is known to posses a large global symmetry, namely a symmetry under $U(N_f) \times U(N_f)$ independent rotations of left- and right-handed quark fields. This symmetry is called chiral. Instead of rotating separately the 2-component Weyl spinors corresponding to left- and right-handed components of quark fields, one can make independent vector and axial $U(N_f)$ rotations of the full 4-component Dirac spinors – the QCD lagrangian is invariant under these transformations too. Meanwhile, axial transformations mix states with different P-parities. Therefore, were that symmetry exact, one would observe parity degeneracy of all states with otherwise the same quantum numbers. In reality the splittings between states with the same quantum numbers but opposite parities are huge. For example, the splitting between the vector $\rho$ and the axial $a_1$ meson is $(1260 - 770) \simeq 500 \text{ MeV}$; the splitting between the nucleon and its parity partner is even larger: $(1535 - 940) \simeq 600 \text{ MeV}$.

The splittings are too large to be explained by the small bare or current quark masses which break the chiral symmetry from the beginning. Indeed, the current masses of light quarks are: $m_u \simeq 4 \text{ MeV}$, $m_d \simeq 7 \text{ MeV}$, $m_s \simeq 150 \text{ MeV}$. The only conclusion one can draw from these numbers is that the chiral symmetry of the QCD lagrangian is broken down spontaneously, and very strongly. Consequently, one should have light (pseudo) Goldstone pseudoscalar hadrons – their role is played by pions which indeed are by far the lightest hadrons.

The order parameter associated with chiral symmetry breaking is the so-called chiral or quark condensate:

$$\langle \bar{\psi} \psi \rangle \simeq - (250 \text{ MeV})^3.$$  \hspace{1cm} (1.1)

It should be noted that this quantity is well defined only for massless quarks, otherwise it is somewhat ambiguous. By definition, this is the quark Green function taken at one point; in momentum space it is a closed quark loop. If the quark propagator has only the ‘slash’ term, the trace over the spinor indices implied in this loop would give an identical zero. Therefore, chiral symmetry breaking implies that a massless (or nearly massless) quark develops a non-zero dynamical mass (i.e. a ‘non-slash’ term in the propagator). There are no reasons for this quantity to be a constant independent of the momentum; moreover, we understand that it should anyhow vanish at large momentum. The value of the dynamical mass at small virtuality can be estimated as one half of the $\rho$ meson mass or one third of the nucleon mass, that is about

$$M(0) \simeq 350 - 400 \text{ MeV};$$  \hspace{1cm} (1.2)

this scale is also related to chiral symmetry breaking and should emerge together with the condensate (1.1).

One could imagine a world without confinement but with chiral symmetry breaking: it would not be drastically different from what we meet in reality. There would be a tightly

\footnote{The word was coined by Lord Kelvin in 1894 to describe molecules not superimposable on its mirror image.}
bound light Goldstone pion, and relatively loosely bound \( \rho \) meson and nucleon with approximately correct masses, which, however, would be possible to ‘ionize’ from time to time. Probably the spectrum of the highly excited hadrons would be wrong, though even that is not so clear \(^1\). We see, thus, that the spontaneous chiral symmetry breaking is the main dynamical happening in QCD, which determines the face of the strong interactions world.

If one understands the microscopic mechanism of spontaneous chiral symmetry breaking and knows how to get the quantities (1.1,1.2) from the only dimensional parameter there is in massless QCD, namely \( \Lambda_{\text{QCD}} \), one gets to the heart of hadron physics. My sense is that it is achieved by ways of the QCD instanton vacuum, see \(^2\) for recent reviews. Therefore, I start by showing in section 2 how instantons lead to a low-energy theory exhibiting chiral symmetry breaking and the appearance of a momentum-dependent constituent quark mass \( M(k) \). In section 3 the resulting effective chiral lagrangian is theoretically studied and in section 4 it is applied to build the Chiral Quark-Soliton Model.

## 2 Low-energy limit of QCD from instantons

The idea that the QCD partition function is dominated by instanton fluctuations of the gluon field, with quantum oscillations about them, has successfully confronted the majority of facts we know about the hadronic world (for a review see ref.\(^3\)). Instantons have been reliably identified in lattice simulations (for a review see ref.\(^4\)), and their relevance to hadronic observables clearly demonstrated \(^5\) \(^6\). Thus, the effective low-energy theory coming from instantons seems to be well-motivated. The aim of this section is to derive this effective theory to which QCD is reduced at low momenta.

### 2.1 Some of the results

There is a well-known general statement that to get chiral symmetry breaking one needs a finite spectral density \( \nu(\lambda) \) of the quark Dirac operator at zero eigenvalues, since the chiral condensate is proportional to exactly this quantity: \( \langle \bar{\psi} \psi \rangle = -\pi \nu(0)/V^{(4)} \). A natural way to get \( \nu(0) \neq 0 \) is to have a finite density of instantons and antiinstantons (\( I \)'s and \( \bar{I} \)'s for short) in the 4-dimensional space-time, \( N/V^{(4)} \). Indeed in the presence of the topologically non-trivial gluon fluctuations fermions necessarily have an exact zero mode \(^7\), as it follows from the Atiah–Singer index theorem. In the ensemble of \( I \)'s and \( \bar{I} \)'s the would-be zero modes in the background field of individual \( I \)'s and \( \bar{I} \)'s are smeared into a band with a finite spectral density at zero eigenvalues \(^8\), leading to \( \nu(0) \neq 0 \). The instanton vacuum provides thus a beautiful mechanism of chiral symmetry breaking \(^9\).

There are two mathematically equivalent ways to treat quarks in the instanton vacuum. One is to calculate an observable in a given instanton backgound and then to average over the collective coordinates of \( I \)'s and \( \bar{I} \)'s and sum over their total numbers, \( N_+ \) and \( N_- \). This approach has been developed in refs. \(^10\), \(^11\). The quark propagator in the instanton vacuum

\(^3\)It can be added that in the solvable \( N = 2 \) supersymmetric version of QCD it is instantons — and nothing besides them — that seem to be sufficient to reproduce the expansion of the exact Seiberg–Witten prepotential \(^{12}\).
takes the form of a massive propagator with a dynamically generated momentum-dependent mass (in Euclidean space, hence the factor $i$ in the ‘non-slash’ term):

$$S(p) = \frac{p + iM(p^2)}{p^2 + M^2(p^2)}, \quad M(p^2) = \text{const} \cdot \sqrt{\frac{N\pi^2\bar{\rho}^2}{VN_c}} F^2(p\bar{\rho}),$$

(2.1)

Here $N/V$ is the instanton density at equilibrium and $\bar{\rho}$ is the average instanton size, $F(p\bar{\rho})$ is a combination of modified Bessel functions and is related to the Fourier transform of the would-be zero fermion mode of individual instantons,

$$F(p\rho) = 2z \left[ I_0(z)K_1(z) - I_1(z)K_0(z) - \frac{1}{z} I_1(z)K_1(z) \right]_{z=p\rho/2} \rightarrow \frac{6}{(p\rho)^3}, \quad F(0) = 1. \quad (2.2)$$

The numerical constant in eq. (2.1) is of the order of unity and is determined by the self-consistency or gap equation:

$$4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^2(p)}{M^2(p) + p^2} = \frac{N}{V}. \quad (2.3)$$

The chiral condensate $\langle \bar{\psi}\psi \rangle$ is the quark propagator taken at one point; in momentum space it is given by a quark loop:

$$-\langle \bar{\psi}\psi \rangle_{\text{Mink}} = i\langle \psi^\dagger\psi \rangle_{\text{Eucl}} \approx 4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M(p)}{M^2(p) + p^2} = \text{const}' \cdot \sqrt{\frac{NN_c}{V\pi^2\bar{\rho}^2}}. \quad (2.4)$$

To get the numerical estimates of the condensate and of the constituent quark mass one may rely on the variational calculation of the instanton vacuum characteristics [9, 13], which relates them to the only dimensional parameter in QCD, $\Lambda_{QCD}$. Taking $\Lambda_{QCD}^{(3)} = 280 \text{ MeV}$, one finds from refs. [9, 13] the basic characteristics of the instanton vacuum, namely the average distance between neighbouring instantons, $\bar{R} \equiv (N/V)^{-1/4}$ and their average mean square radius, $\bar{\rho}$, to be

$$\bar{R} \approx 1 \text{ fm}, \quad \bar{\rho} \approx 0.35 \text{ fm}. \quad (2.5)$$

Using these basic quantities one gets from eqs. (2.3, 2.4):

$$M(0) \approx 350 \text{ MeV}, \quad -\langle \bar{\psi}\psi \rangle \approx (250 \text{ MeV})^3. \quad (2.6)$$

Another quantity closely associated with chiral symmetry breaking is the pion decay constant which in the instanton vacuum is given by [11]:

$$F^2_\pi \approx 4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^2(p)}{[M^2(p) + p^2]^2} = \text{const}'' \cdot \frac{N}{V} \rho^2 \ln \left( \frac{\bar{R}}{\bar{\rho}} \right) \approx (100 \text{ MeV})^2. \quad (2.7)$$

All these quantities appear to be close to their phenomenological values. I would say that I don’t know of any other approach to non-perturbative QCD (except, of course, brute-force lattice calculations) which would relate observables directly to $\Lambda_{QCD}$, and with such an accuracy. Personally, I conclude that the idea that the (Euclidean) QCD partition function is saturated by relatively dilute instantons with quantum fluctuations of gluon field about them, works quite satisfactory.
2.2 Instanton-induced interactions

As I have mentioned, these results have been obtained from considering the motion of light quarks in a given instanton background and then averaging it over the instanton ensemble [13]. There is a mathematically equivalent technique to rederive these results, namely one first averages over the instanton ensemble [14]. This averaging induces many-quark interactions whose simplified version was first suggested by ’t Hooft [8]. Using the effective quark interaction theory one can calculate various observables. According to the derivation of ref. [14] (recently reviewed in refs. [2, 13]) averaging over instanton ensemble leads to a specific form of the QCD partition function valid at low momenta, \( p \leq 1/\bar{\rho} \). In what follows we shall use the Euclidean formulation of the theory; \( N_f \) is the number of light fermion flavours whose masses are put to zero for simplicity. The effects of non-zero current quark masses have been considered in ref. [13].

It is convenient to decompose the 4-component Dirac bi-spinors describing quark fields into left- and right-handed Weyl spinors which we denote as

\[
\psi^{f\alpha}_{L(R)}, \quad \psi^\dagger_{L(R)f\alpha}, \tag{2.8}
\]

where \( f = 1...N_f \) are flavour, \( \alpha = 1...N_c \) are colour and \( i = 1, 2 \) are spinor indices. Let us introduce the ’t Hooft-like \( 2N_f \)-fermion vertices generated by \( I \)'s and \( \bar{I} \)'s, which we denote by \( Y_{N_f}^{(\pm)} \), respectively. These vertices are obtained by explicit averaging over (anti)instanton orientation matrices \( U^\alpha_\beta \) and over the instanton size distribution \( \nu(\rho) \). Averaging over instanton positions in \( d = 4 \) Euclidean space–time produces the overall conservation of momenta of quarks entering the vertex \( Y \), hence it is convenient to write down the quark interaction vertex in the momentum space. There are formfactor functions \( F(k\rho) \) (2.2) associated with the Fourier transform of the fermion zero modes of one instanton, attached to each quark line entering the vertex. The \( 2N_f \)-fermion vertex induced by an instanton is, in momentum space,

\[
Y_{N_f}^{+} = \int d\rho \nu(\rho) \int dU \prod_{f=1}^{N_f} \left\{ \int \frac{d^4k_f}{(2\pi)^4} 2\pi \rho F(k_f\rho) \int \frac{d^4l_f}{(2\pi)^4} 2\pi \rho F(l_f\rho) \right. \\
\cdot (2\pi)^4 \delta(k_1 + ... + k_{N_f} - l_1 - ... - l_{N_f}) \cdot U^{\alpha_f}_{i_f} U^\dagger_{\beta_f} \epsilon^{i_f} \epsilon^{j_f} \left[ i\psi^\dagger_{L,f\alpha_f,i_f}(k_f) \psi^{f\beta_j}_{L,i_f}(l_f) \right] \right\}; \tag{2.9}
\]

for the \( Y^- \) vertices induced by \( \bar{I} \)'s one has to replace left-handed Weyl spinors \( \psi_L, \psi_L^\dagger \) by right-handed ones, \( \psi_R, \psi_R^\dagger \). Using these vertices one can write down the partition function to which QCD is reduced at low momenta, as a functional integral over quark fields [14, 2, 13]:

\[
\mathcal{Z} = \int D\psi^D\psi^\dagger \exp \left( \int d^4x \sum_{f=1}^{N_f} \bar{\psi}_f i\partial_\mu \psi^\mu \right) \left( \frac{Y_{N_f}^+}{VM_1^{N_f}} \right)^{N_+} \left( \frac{Y_{N_f}^-}{VM_1^{N_f}} \right)^{N_-} \tag{2.10}
\]

where \( N_\pm \) are the number of \( I \)'s and \( \bar{I} \)'s in the whole \( d = 4 \) volume \( V \). The volume factors in the denominators arise because of averaging over individual instanton positions, and certain mass factors \( M_1^{N_f} \) are put in to make eq. (2.10) dimensionless. Actually, the mass parameter \( M_1 \) plays the role of separating high-frequency part of the fermion determinant in the instanton background from the low-frequency part considered here. Its concrete value
is irrelevant for the derivation of the low-energy effective action performed below; in fact it is established from smooth matching of high- and low-frequency contributions to the full fermion determinant in the instanton vacuum [10].

Having fermion interactions in the pre-exponent of the partition function is not convenient: one should rather have the interactions in the exponent, together with the kinetic energy term. This can be achieved by rewriting eq. (2.10) with the help of additional integration over ‘Lagrange multipliers’ \( \lambda_{\pm} \):

\[
Z = \int \frac{d\lambda_+}{2\pi} \int D\psi D\psi^\dagger \exp \left\{ N_+ \left( \ln \frac{N_+}{\lambda_+ VM_{1\gamma}} - 1 \right) + N_- \left( \ln \frac{N_-}{\lambda_- VM_{1\gamma}} - 1 \right) + \int d^4x \psi_j \partial_\mu \psi^j + \lambda_+ Y_{N_f}^+ + \lambda_- Y_{N_f}^- \right\}.
\]

(2.11)

Since \( N_{\pm} \sim V \to \infty \) integration over \( \lambda_{\pm} \) can be performed by the saddle-point method; the result is eq. (2.10) we started from.

As seen from eq. (2.11) \( \lambda_{\pm} \) plays the role of the coupling constant in the many-quark interactions. It is very important that their strength is not pre-given but is, rather, determined self-consistently from the fermion dynamics itself; in particular, the saddle-point values of \( \lambda_{\pm} \) depend on the phase quarks assume in the instanton vacuum. As shown below, in the chiral symmetry broken phase the values of \( \lambda_{\pm} \), as determined by a saddle-point equation, appear to be real.

To get the \( 2N_f \)-fermion vertices (2.9) in a closed form one has to explicitly integrate over instanton orientations in colour space. For the \( N_f \)-fermion vertex one has to average over \( N_f \) pairs of \((U, U^\dagger)\). In particular, one has:

\[
\left. \right| \int dU = 1, \quad \int dU U_i^\alpha U_j^\beta = \frac{1}{N_c} \delta_\beta^\alpha \delta_i^j, \quad \int dU U_i^\alpha_1 U_j^\beta_1 U_i^\alpha_2 U_j^\beta_2 = \frac{1}{N_c^2 - 1} \left[ \delta_\beta_1^\alpha \delta_\beta_2^\alpha \left( \delta_i_1^j \delta_i_2^j - \frac{1}{N_c} \delta_i_2^j \delta_i_1^j \right) + \delta_\beta_1^\alpha \delta_\beta_2^\alpha \left( \delta_i_2^j \delta_i_1^j - \frac{1}{N_c} \delta_i_1^j \delta_i_2^j \right) \right], \quad \text{etc.} \quad (2.12)
\]

We present below the resulting vertices for \( N_f = 1, 2, 3 \) and for any \( N_f \) but \( N_c \to \infty \).

\( N_f = 1 \)

In this case the “vertex” (2.3) is just a mass term for quarks,

\[
Y_1^\pm = \frac{i}{N_c} \int \frac{d^4k}{(2\pi)^4} \int d\rho \nu(\rho) [2\pi \rho F(k\rho)]^2 \left[ \psi^\dagger_\alpha(k) \frac{1}{2} \left[ \psi_\alpha(k) \right] \frac{1}{2} \gamma_5 \psi_\alpha(k) \right],
\]

(2.13)

with a momentum dependent dynamically-generated mass \( M(k) \) given by

\[
M(k) = \frac{\lambda}{N_c} \int d\rho \nu(\rho) [2\pi \rho F(k\rho)]^2 \approx \frac{\lambda}{N_c} [2\pi \bar{\rho} F(k\bar{\rho})]^2.
\]

(2.14)

In all our previous work on the instanton vacuum we have assumed that the distribution in the sizes of instantons, \( \nu(\rho) \), is a sharp function peaked at certain \( \bar{\rho} \), and replaced \( \rho \) by this \( \bar{\rho} \) in the argument of the formfactor functions \( F(k\rho) \). However, there is a subtlety
here: if the size distribution for large $\rho$ behaves as $\nu(\rho) \sim 1/\rho^3$ (corresponding to the linear potential between heavy quarks)\(^1\) the dynamical quark mass logarithmically diverges at small momenta implying, in a sense, the confinement of light quarks. We shall not pursue this interesting topic here but replace each time $\rho$ by its average value $\bar{\rho}$.

In order to find the overall scale $\lambda$ of the dynamical mass one has to put (2.13) into eq. (2.11), integrate over fermions, and find the minimum of the free energy in respect to $\lambda_{\pm}$. At $\theta = 0$ the QCD vacuum is $CP$ invariant so that $N_+ = N_- = N/2$ and consequently $\lambda_+ = \lambda_- = \lambda$\(^1\). In this case the $\gamma_5$ term in $Y^\pm$ gets cancelled, and the exponent of the partition function (2.11) reads:

\[-N \ln \lambda + \int d^4 x \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \ln \left\{ \frac{\lambda}{N_c} [2\pi \bar{\rho} F(k\bar{\rho})]^2 \right\} \]

\[-N \ln \lambda + 2N_c V \int \frac{d^4 k}{(2\pi)^4} \ln \left\{ k^2 + \left( \frac{\lambda}{N_c} [2\pi \bar{\rho} F(k\bar{\rho})]^2 \right)^2 \right\}. \tag{2.15} \]

Differentiating it in respect to $\lambda$ and using eq. (2.14) one gets the gap eq. (2.3) or the self-consistency condition which is in fact a requirement on the overall scale of the constituent quark mass $M(k)$; its momentum dependence is anyhow given by eq. (2.14). Since the momentum integration in eq. (2.3) is well convergent and is actually cut at momenta $k \sim 1/\bar{\rho}$, the saddle-point value of the ‘Lagrange multiplier’ $\lambda$ is of the order of $\sqrt{N_c N/V/\bar{\rho}}$. The steepness of the saddle-point integration is proportional to the volume $V$, hence the use of the saddle-point method is absolutely justified.

Note that eq. (2.13) reproduces the massive quark propagator (2.1), hence the chiral condensate is given by eq. (2.4). It is very important that, initially, one does not know the strength of the quark interactions (represented by the ‘Lagrange multiplier’ $\lambda$): it is fixed only after integration over the quark fields is performed. We also stress that the basic quantities associated with spontaneous chiral symmetry breaking, such as $\langle \bar{\psi} \psi \rangle$, $M$ or $F_\pi$ are non-analytic in the instanton density, $N/V$: such a behaviour is characteristic of spontaneous breaking of continuous symmetry.

$N_f = 2$

In this case averaging eq. (2.13) over the instanton orientations with the help of eq. (2.12) gives a nontrivial 4-fermion interaction. It is, of course, non-local: a formfactor function $F(k\rho)$ is attributed to each fermion entering the vertex; in addition it should be averaged over the sizes of instantons. The non-locality is thus of the order of the average instanton size in the vacuum. One has\(^4\):

\[ Y^+_2 = \frac{i^2}{N_c^2 - 1} \int \frac{d^4 k_1 d^4 k_2 d^4 l_1 d^4 l_2}{(2\pi)^{12}} \delta(k_1 + k_2 - l_1 - l_2) \]

\[ \cdot \int d\rho \nu(\rho) (2\pi \rho)^4 F(k_1 \rho) F(k_2 \rho) F(l_1 \rho) F(l_2 \rho) \]

\(^{4}\)Fluctuations of the topological charge, $N_+ - N_-$, leading to the so-called topological susceptibility (related to the solution of the $U(1)$ problem) has been considered in refs.\(^{14},^{13}\).
\[
\frac{1}{2!} \epsilon^f_1 \epsilon_ f_2 \epsilon_{g_1 g_2} \left\{ \left( 1 - \frac{1}{2N_c} \right) [\psi^\dagger_{L, f_1}(k_1)\psi_L^{g_1}(l_1)][\psi^\dagger_{L, f_2}(k_2)\psi_L^{g_2}(l_2)] \\
+ \frac{1}{8N_c} [\psi^\dagger_{L, f_1}(k_1)\sigma_{\mu\nu}\psi_L^{g_1}(l_1)][\psi^\dagger_{L, f_2}(k_2)\sigma_{\mu\nu}\psi_L^{g_2}(l_2)] \right\},
\]
(2.16)

For the \( \bar{I} \)-induced vertex \( Y^- \) one has to replace left-handed components by right-handed ones. In all square brackets summation over colour is understood. Note that the last-line (tensor) term is suppressed at large \( N_c \); it, however, is crucial at \( N_c = 2 \) to support the actual \( SU(4) \) chiral symmetry in that case \[13\]. The antisymmetric \( \epsilon^f_1 \epsilon_ f_2 \epsilon_{g_1 g_2} \) structure demonstrates that the interactions have a determinant form in the two flavours. Using the identity

\[
2\epsilon^f_1 \epsilon_ f_2 \epsilon_{g_1 g_2} = \delta_{g_1}^{f_1} \delta_{g_2}^{f_2} - (\tau^A)^{f_1}_{g_1} (\tau^A)^{f_2}_{g_2}
\]
(2.17)

and adding the \( \bar{I} \)-induced vertex \( Y_2^- \) one can rewrite the leading-\( N_c \) (first) term of eq. (2.16) as

\[
(\psi^\dagger \gamma_5 \psi)^2 + (\psi^\dagger \gamma_5 \psi)^2 - (\psi^\dagger \tau^A \psi)^2 - (\psi^\dagger \tau^A \gamma_5 \psi)^2
\]
(2.18)

which resembles closely the Vaks–Larkin \[17\] / Nambu–Jona-Lasinio \[18\] model. It should be stressed though that in contrast to that at \textit{hoc} model the interaction (2.16) \( i) \) violates explicitly the \( U_A(1) \) symmetry, \( ii) \) has a fixed interaction strength related to the density of instantons (see below) and \( iii) \) contains an intrinsic ultraviolet cutoff due to the formfactor functions \( F(k\rho) \). In addition, at \( N_c = 2 \) it correctly preserves the actual \( SU(4) \) chiral symmetry. We shall show in the next section that the four-fermion interaction (2.16) leads to the spontaneous chiral symmetry breaking, with the appearance of the constituent quark mass \( M(k) \) satisfying the same gap equation (2.3) as we obtained above in the case \( N_f = 1 \).

\( N_f = 3 \)

In this case one gets a 6-fermion vertex of the following structure \[17\]:

\[
Y^+_3 = \frac{i^3}{N_c(N_c^2 - 1)} \int \frac{d^4k_1 d^4k_2 d^4k_3 d^4l_1 d^4l_2 d^4l_3}{(2\pi)^{20}} \delta(k_1 + k_2 + k_3 - l_1 - l_2 - l_3) \cdot \int d\rho \nu(\rho) (2\pi\rho)^6 F(k_1\rho) F(k_2\rho) F(k_3\rho) F(l_1\rho) F(l_2\rho) F(l_3\rho) \]

\[
\frac{1}{3!} \epsilon^{f_1 f_2 f_3} \epsilon_{g_1 g_2 g_3} \left\{ \left( 1 - \frac{3}{2(N_c + 2)} \right) [\psi^\dagger_{L, f_1}(k_1)\psi_L^{g_1}(l_1)][\psi^\dagger_{L, f_2}(k_2)\psi_L^{g_2}(l_2)][\psi^\dagger_{L, f_3}(k_3)\psi_L^{g_3}(l_3)] \\
+ \frac{3}{8(N_c + 2)} [\psi^\dagger_{L, f_1}(k_1)\sigma_{\mu\nu}\psi_L^{g_1}(l_1)][\psi^\dagger_{L, f_2}(k_2)\sigma_{\mu\nu}\psi_L^{g_2}(l_2)][\psi^\dagger_{L, f_3}(k_3)\sigma_{\mu\nu}\psi_L^{g_3}(l_3)] \right\}
\]
(2.19)

Again, it is of a determinant structure, this time in 3 flavours, and again the tensor term is suppressed at \( N_c \to \infty \).

\( \text{Any} \ N_f \)

For arbitrary \( N_f \) the \textit{leading} term at \( N_c \to \infty \) can be written as a determinant of \( N_f \times N_f \) matrices composed of non-local chiral quark bilinears \( J_{fg} \) \[14, 15, 13\]:
\[ Y_{N_f}^{\pm} \xrightarrow{N_c \to \infty} \left( \frac{1}{N_c} \right)^{N_f} \int d^4x \int d\rho \nu(\rho) \det_{N_f} \left[ iJ^{\pm}(x, \rho) \right] \]  

where

\[ J^{\pm}_{fg}(x, \rho) = \int \frac{d^4k}{(2\pi)^4} e^{i(k-l,x)} [2\pi \rho F(k\rho)] \frac{1}{2} \psi^{\dagger}_{f\alpha}(k) \frac{1 \pm \gamma_5}{2} \psi_{g\alpha}(l) \]  

If one neglects formfactors, that is puts \( F = 1 \), these chiral currents become local,

\[ J^{\pm}_{fg}(x, \rho) \approx (2\pi\rho)^2 \left[ \psi^{\dagger}_{f\alpha}(x) \frac{1 \pm \gamma_5}{2} \psi_{g\alpha}(x) \right] \]  

### 2.3 Bosonization

One can linearize the many-fermion vertices induced by instantons by introducing auxiliary boson fields. This formal procedure is called bosonization of the theory. Roughly speaking, when one has a theory with 4-fermion interactions, it can be viewed as a limit of a one-boson exchange when the mass of intermediate boson tends to infinity. This is the meaning of the bosonization.

In case \( N_f = 2 \) the instanton-induced interactions are 4-fermion ones (see eqs. (2.16, 2.18)) and it is very easy to bosonize them by introducing scalar and pseudoscalar fields. Note that the non-leading second term in eq. (2.16) requires additional tensor fields for the bosonization. In case \( N_f \geq 3 \) the instanton-induced interactions become 6-fermion and so forth, so that the bosonization becomes less trivial. However, it is still possible to perform it using \( N_c \) as an algebraically large parameter \([14, 15, 13]\). Indeed, introducing \( N_f \times N_f \) matrices \( \mathcal{M} \) the following equation becomes true in the saddle-point approximation (justified at large \( N_c \)):

\[ \exp \left( \lambda \det \left[ \frac{iJ}{N_c} \right] \right) = \int d\mathcal{M} \exp \left\{ i\text{Tr}(\mathcal{M}J) - (N_f - 1) \left( \frac{\det[\mathcal{M}N_c]}{\lambda} \right)^{\frac{1}{N_f-1}} \right\} \]
This remarkable formula enables one to bosonize many-fermion interactions of the determinant type, like in eq. (2.20). It should be stressed however that the procedure is justified only at large $N_c$, otherwise:

i) the fermion interactions have not the simple determinant form and

ii) the saddle-point evaluation of the integral in eq. (2.23) is not justified. We notice that at $N_f = 2$ eq. (2.23) becomes exact since in this case the integral over $\mathcal{M}$ is Gaussian.

Indeed, in this particular case the power of $\det \mathcal{M}$ is unity while the determinant of a $2 \times 2$ matrix itself is quadratic in matrix entries.

We are now prepared to write down the partition function (2.11) by introducing auxiliary integration over ‘meson’ fields $\mathcal{M}_{L,R}(x)$ coupled linearly to the quark chiral currents $J^{\pm}(x)$; both quantities are $N_f \times N_f$ matrices in flavour. We have [15, 13]:

$$
\mathcal{Z} = \int \frac{d\lambda}{2\pi} \exp(-N \ln \lambda) \cdot \int D\mathcal{M}_{L,R} \exp \int d^4x \left\{- (N_f - 1) \left[ \left( \frac{\det[\mathcal{M}_L N_c]}{\lambda} \right)^{1/N_f} + \left( \frac{\det[\mathcal{M}_R N_c]}{\lambda} \right)^{1/N_f} \right] \right\} \\
\cdot \int D\psi^\dagger D\psi \exp \int d^4x \left\{ \psi^\dagger i \partial_\mu \psi + i \text{Tr} [\mathcal{M}_L J^+] + i \text{Tr} [\mathcal{M}_R J^-] \right\}.
$$

The last line presents a theory of quarks interacting with external chiral ‘meson’ fields, $\mathcal{M}_{L,R}$. Integration over quarks can be performed by expanding the resulting functional fermion determinant in powers of $\mathcal{M}_{L,R}$ and / or their derivatives. A concrete example of such expansion will be given below.

The second line in eq. (2.24) presents the potential energy of the $\mathcal{M}_{L,R}$ fields: it has a rather peculiar form of a power of the determinant composed of these fields. In the particular case $N_f = 1$ these terms vanish, as it should be since at $N_f = 1$ instantons induce just a mass term for fermions, see the previous subsection. In the particular case of $N_f = 2$ the potential energy of the $\mathcal{M}_{L,R}$ fields becomes quadratic in the fields, that is to say, we have just a mass term for the ‘meson’ fields. Notice that there is no kinetic energy term at the tree level: the kinetic energy (as well as higher derivative terms) is generated dynamically after one integrates over quarks, that is through quark loops.

The fermion action (last line in eq. (2.24)) is invariant under full chiral rotations with arbitrary $U(N_f)$ matrices $A, B$:

$$
\begin{align*}
\begin{cases}
\psi_L \to A \psi_L, & \psi^\dagger_L \to \psi^\dagger_L B^\dagger, & \mathcal{M}_L \to B \mathcal{M}_L A^\dagger, \\
\psi_R \to B \psi_R, & \psi^\dagger_R \to \psi^\dagger_R A^\dagger, & \mathcal{M}_R \to A \mathcal{M}_R B^\dagger.
\end{cases}
\end{align*}
$$

However, the potential energy of the ‘meson’ fields (the second line in eq. (2.24)) has a smaller invariance. Indeed the determinants transform as

$$
\det[\mathcal{M}_L] \to \det[A^\dagger B] \det[\mathcal{M}_L], \quad \det[\mathcal{M}_R] \to \det[B^\dagger A] \det[\mathcal{M}_R],
$$

therefore they acquire a $U(1)$ phase factor of the relative $A^\dagger B$ transformation. For that reason eq. (2.24) breaks explicitly the axial $U_A(1)$ symmetry, as expected on general grounds from instantons [8].

Let us show that eq. (2.24) leads to the spontaneous breaking of the chiral symmetry. To that end, let us parametrize the ‘meson’ fields by

$$
\begin{align*}
\begin{cases}
\psi_L \to A \psi_L, & \psi^\dagger_L \to \psi^\dagger_L B^\dagger, & \mathcal{M}_L \to B \mathcal{M}_L A^\dagger, \\
\psi_R \to B \psi_R, & \psi^\dagger_R \to \psi^\dagger_R A^\dagger, & \mathcal{M}_R \to A \mathcal{M}_R B^\dagger.
\end{cases}
\end{align*}
$$
\[ \mathcal{M}_L(x) = [\sigma(x) + \eta(x)]U(x)V(x), \]
\[ \mathcal{M}_R(x) = [\sigma(x) - \eta(x)]V(x)U^\dagger(x) \] (2.27)

where \( \sigma(x) \) is the scalar flavour-singlet field and \( \eta(x) \) is the pseudoscalar flavour-singlet field. The \( SU(N_f) \) matrix fields \( U(x) \) and \( V(x) \) can be further on parametrized by scalar \( \sigma^A(x) \) and pseudoscalar \( \pi^A(x) \) fields belonging to the adjoint representation of the flavour \( SU(N_f) \) group:

\[ U(x) = \exp i\pi^A(x)\lambda^A, \]
\[ V(x) = \exp i\sigma^A(x)\lambda^A \] (2.28)

where \( \lambda^A \) are the Hermitean generators of the \( SU(N_f) \).

The vacuum state of the theory given by the partition function (2.24) corresponds to a non-zero value of the flavour-singlet \( \sigma \) field. This is equivalent to the spontaneous breakdown of chiral symmetry, and the \( \pi^A \) fields become Goldstone particles.

To reveal the nonzero value of the \( \sigma \) field let us calculate the effective potential for it. Putting

\[ \sigma = \text{const}, \quad \eta = 0. \quad U = V = 1 \] (2.29)

and integrating over quarks in the constant background scalar field we obtain the effective potential \( V_{\text{eff}}(\sigma, \lambda) \) to which we add the ‘Lagrange multiplier’ piece, \( N \ln \lambda \) (see eq. (2.24)):

\[ V_{\text{eff}}(\sigma, \lambda) = \frac{N}{V} \ln \lambda + 2(N_f - 1)(\sigma N_c)^{N_f - 1} \lambda - 2N_f N_c \int \frac{d^4k}{(2\pi)^4} \ln \left\{ k^2 + \sigma^2 [2\pi \bar{\rho} F(k\bar{\rho})]^4 \right\}. \] (2.30)

We have now to minimize (2.30) in both quantities, \( \sigma \) and \( \lambda \). The extremum condition gives the following equations for the saddle-point values \( \sigma_0, \lambda_0 \):

\[ \frac{N}{V} = 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{k^2 + M^2(k)}, \quad M(k) = \sigma_0 [2\pi \bar{\rho} F(k\bar{\rho})]^2, \] (2.31)
\[ \lambda_0 = \frac{N}{2V} \left( \frac{2\sigma_0 VN_c}{N} \right)^{N_f}. \] (2.32)

These equations demonstrate that the scalar field develops a non-zero v.e.v. \( \sigma_0 \), quarks get a dynamical mass \( M(k) \) and chiral symmetry is broken. The first equation is the familiar self-consistency or gap equation (2.3) which we have also obtained in the \( N_f = 1 \) case: it determines the overall scale of the dynamically generated momentum-dependent constituent quark mass \( M(k) \). The second eq. (2.32) relates the value of the ‘Lagrange multiplier’ \( \lambda \) to the v.e.v. of the \( \sigma \) field. We find that these quantities have the following parametric dependence on the basic characteristics of the instanton vacuum, their density, \( N/V \), and their average size, \( \bar{\rho} \):

\[ M(0) \approx \sigma_0(2\pi \bar{\rho})^2 \sim \sqrt{\frac{N\pi^2 \bar{\rho}^2}{VN_c}}, \quad \lambda_0 \sim \frac{N}{V} \left( \frac{VN_c}{N\pi^2 \bar{\rho}^2} \right)^{N_f}. \] (2.33)
It is important that the strength $\lambda$ of the $2N_f$-fermion instanton-induced interactions is not fixed beforehand but is, rather, determined by the phase the fermion system assumes. In a given phase, like the chirality broken phase, the coupling constant $\lambda$ is defined unambiguously through the extremum conditions \((2.31, 2.32)\). In another phase, say, chiral invariant or in a phase where diquarks condense \([15]\), the saddle-point value of $\lambda$ would be, generally speaking, different. It means that the formalism presented here is different from the ‘quenched approximation’: it incorporates the back reaction of fermions on the instanton medium. The main assumption in the starting formula for this derivation, eq. \((2.10)\), is that one can average independently over the collective coordinates of $I$’s and $\bar{I}$’s. It implies that correlations between pseudoparticles coming from the gauge sector are neglected or, better to say, treated \(\text{`la variational principle resulting in an effective size distribution $\nu(\rho)$ [9, 13], however correlations arising from fermions are taken into account.}

In getting the equation for the v.e.v. of the scalar field $\sigma$ we have used the mean-field approximation, eq. \((2.31)\). Theoretically speaking, its accuracy is of the order of $O(N_f/N_c)$ since the meson loops have been disregarded. In principle, they could be taken into account. It should be reminded, though, that eq. \((2.24)\) itself has been derived in the limit of $N_c \to \infty$; therefore, if one wishes to take into account higher order corrections in $N_f/N_c$ one should rather work with the unabridged vertices \((2.16)\) or \((2.19)\).

### 2.4 Chiral lagrangian

The low-momentum QCD partition function \((2.24)\), after integration over quarks in the given background ‘meson’ fields $\mathcal{M}_{L,R}$, gives an effective action for $N_f^2$ scalar and $N_f^2$ pseudoscalar fields which one can parametrize according to eqs.\((2.27, 2.28)\). The scalar flavour-singlet field $\sigma(x)$ should be counted from its mean-field value $\sigma_0$ given by eq. \((2.31)\), and the ‘Lagrange multiplier’ $\lambda$ should be put to its saddle-point value \((2.32)\).

The $2N_f^2$ fields introduced by eqs.\((2.27, 2.28)\) are not properly normalized: to get the correct normalization one has to extract their kinetic energy (= two derivative) terms from the quark functional determinant and redefine the fields so that the kinetic energy is standard, $\frac{1}{2} (\partial_{\mu} \phi)^2$. The cubic, quartic,... terms in the boson fields arising from eq. \((2.24)\) will then give the meson interactions. It can be shown from $N_c$ power counting \([14]\) that the cubic coupling constants for the properly normalized meson fields are of the order of $1/\sqrt{N_c}$, the quartic couplings are $\sim 1/N_c$, and so on. This is as it should be from the general $N_c$ counting rules.

We have already explained in the previous subsection that the axial $U_A(1)$ symmetry is explicitly (not spontaneously!) broken in eq. \((2.24)\), therefore the pseudoscalar flavour-singlet $\eta$ meson is not a Goldstone boson, the $U_A(1)$ problem is solved. Moreover, in the limit $N_f/N_c \to 0$ we recover from eq. \((2.24)\) \([14, 13]\) the theoretical Witten–Veneziano formula for the singlet $\eta'$ mass, as given by

$$m_{\eta'}^2 = \frac{2N_f}{F_\pi^2} \frac{<Q_T^2>/V}{>2}$$

(2.34)

where $<Q_T^2>/V = <(N_+ - N_-)^2>/V$ is the topological susceptibility of the vacuum.

As to the non-singlet pseudoscalar fields $\pi^A(x)$ introduced by eqs.\((2.27, 2.28)\) which we shall call pions for short, they appear to be massless Goldstone fields. Indeed, the constant fields $\pi^A$ correspond to global chiral rotations \((2.25)\) with $A^\dagger = B = \exp(i\pi^A A^A)$. Eq. \((2.24)\)
is invariant under such rotations, therefore the lagrangian for the \( \pi^A(x) \) fields contain only derivatives, \( i.e. \) pions are massless – in accordance with the Goldstone theorem.

One can check from eq. (2.24) that scalar meson fields get the mass of the order of \( 1/\bar{\rho} \); numerically it is in the \( 1 \, GeV \) range. Though the \( \eta' \) mass, algebraically, is given by a different formula (2.34), numerically it also turns out to be about \( 1 \, GeV \). Strictly speaking, in that range of momenta the low-energy QCD partition function (2.24) is not justifiable as it has been derived above from the partition function (2.10) valid for momenta \( k \leq 1/\bar{\rho} \). Therefore, in a consistent approach to the low-momentum theory at \( k \leq 1/\bar{\rho} \) one has to freeze out all the meson fields except massless pions. [In the academic limit of \( N_f/N_c \to 0 \) one would also need to keep the \( \eta' \) degree of freedom.] It is very important that the quark masses are, parametrically speaking, much less than \( 1/\bar{\rho} \): the dimensionless quantity

\[
(M\bar{\rho})^2 \sim \frac{1}{N_c} \pi^2 \rho^4 \frac{N}{V} \ll 1
\]

is suppressed by the packing fraction of instantons in the vacuum. The whole approach to the instanton vacuum implies that instantons are on the average relatively dilute and that this packing fraction is numerically small. Theoretically, the smallness of the parameter (2.35) can be traced back to the “accidentally” large coefficient in the Gell-Mann–Low function, the famous \( 11/3 \) [9, 13].

We, thus, arrive to the conclusion that at low momenta \( k \leq 1/\bar{\rho} \) there are exactly two degrees of freedom left: quarks with a dynamical mass \( M \ll 1/\bar{\rho} \) and the massless Goldstone pions. In principle, one could think of a “soft” instanton size distribution going at large sizes as \( \nu(\rho) \sim 1/\rho^3 \). Such a distribution is peculiar because it automatically leads to a linear potential between heavy quarks [16] and could therefore reproduce confinement. Moreover, there are certain reasons to believe that this particular distribution is, effectively, realized in nature [Diakonov and Petrov, in preparation]. As explained above in such a case \( M(k) \) would logarithmically diverge at \( k \to 0 \) (see eq. (2.14)) so that free quarks would not show up. Amusingly, such a possibility would not invalidate the use of the effective chiral theory for bound state problems, like inside the nucleons, since for virtualities \( k \sim (\text{nucleon radius})^{-1} \) the estimate (2.35) would still hold true. In these lectures, however, we shall not pursue this interesting possibility of marrying confinement with the chiral theory but assume that averaging over instanton sizes merely replaces \( \rho \) by its peak or average value \( \bar{\rho} \).

Having made these preliminary remarks, let us write down the effective partition function to which QCD is reduced at low momenta, \( k \leq 1/\bar{\rho} \). It follows from the instanton-induced partition function (2.24) where we freeze out all meson fields except pions and put \( \sigma \) and \( \lambda \) to their saddle-point values:

\[
Z = \int D\pi^A \int D\psi^A D\psi \exp \int d^4x \left\{ \bar{\psi}^f(x) i \partial \psi^f(x) + i \frac{d^4k d^4l}{(2\pi)^8} e^{i(k-l,x)} \sqrt{M(k)M(l)} \cdot \left[ \psi^\dagger_{f\alpha}(k) \left( U_{g}^f(x) \frac{1 + \gamma_5}{2} + U_{g}^f(x) \frac{1 - \gamma_5}{2} \right) \psi^{g\alpha}(l) \right] \right\}, \quad U_{g}^f(x) = \left( \exp i\pi^A(x) \lambda^A \right)_g^f. \tag{2.36}
\]

This effective theory has been first derived in refs. [11, 14]. Eq. (2.36) shows quarks interacting with chiral fields \( U(x) \), with formfactor functions equal to the square root of the dynamical quark mass attributed to each vertex where \( U(x) \) applies. The matrix entering in
the parentheses is actually a $N_f \times N_f$ matrix in flavour and a $4 \times 4$ matrix in Dirac indices. It can be identically rewritten as

$$U(x) \frac{1 + \gamma_5}{2} + U^\dagger(x) \frac{1 - \gamma_5}{2} = \exp \left( i\pi A(x) \lambda^A \gamma_5 \right) \equiv U^{\gamma_5}(x), \quad (2.37)$$

the industrious final abbreviation being due to Pavel Pobylitsa.

The formfactor functions $\sqrt{M(k\bar{p})}$ for each quark line attached to the chiral vertex automatically cut off momenta at $k \geq 1/\bar{p}$. In the range of quark momenta $k \ll 1/\bar{p}$ (which we shall be mostly interested in) one can neglect this non-locality, and the partition function (2.36) is simplified to a local field theory:

$$Z = \int D\pi A \int D\psi^\dagger D\psi \exp \int d^4x \psi^\dagger(x) [i\partial / + i\mu U^{\gamma_5}(x)] \psi(x). \quad (2.38)$$

One should remember, however, to cut the quark loop integrals at $k \approx 1/\bar{p} \approx 600 \text{MeV}$. Notice that there is no kinetic energy term for pions: it appears only after one integrates over the quark loop, see below. Summation over colour is assumed in the exponent of eq. (2.38).

Eq. (2.38) defines a simple and elegant local field theory though it is still a highly non-trivial one. Its main properties will be established in the next section.

3 Properties of the effective chiral lagrangian (EChL)

Properties of effective theories of quarks interacting with various meson fields have been studied by several authors in the 80’s, most notably by Volkov and Ebert [21] and Dhar, Shankar and Wadia [22]. The fact that integrating over quarks one gets, in particular, the so-called Wess–Zumino term has been first established by Eides and myself [20], though in somewhat different settings, see also below.

Integrating over the quark fields in eq. (2.38) one gets the effective chiral lagrangian (EChL):

$$S^{\text{eff}}[\pi] = -N_c \ln \det (i\partial + i\mu U^{\gamma_5}). \quad (3.1)$$

We have been asked about the relation of this low-energy theory with that suggested by Manohar and Georgi [19]. One can redefine the quark fields

$$\psi \rightarrow \psi' = \exp(i\pi A^A \gamma_5/2)\psi, \quad \psi^\dagger \rightarrow \psi'^\dagger = \psi^\dagger \exp(i\pi A^A \gamma_5/2), \quad (2.39)$$

and rewrite the lagrangian in (2.38) as

$$\mathcal{L} = \psi'^\dagger (i\partial / + V + A^A \gamma_5 + i\mu) \psi' \quad (2.40)$$

with

$$V_\mu = \frac{i}{2}(\xi \partial_\mu \xi^\dagger + \xi^\dagger \partial_\mu \xi), \quad A_\mu = \frac{i}{2}(\xi \partial_\mu \xi^\dagger - \xi^\dagger \partial_\mu \xi), \quad \xi = \exp(i\pi A^A \gamma_5/2) = U^{1/2}. \quad (2.41)$$

which resembles closely the effective lagrangian of Manohar and Georgi (the effective chiral lagrangian in a similar form has been independently suggested in ref. [20]).

The crucial difference is that Manohar and Georgi have added an explicit kinetic energy term $F_2^2 \text{Tr} (\partial_\mu U^\dagger \partial_\mu U)/4$ on top of eq. (2.40). This is a typical double counting as the kinetic energy term arises from quark loops, see the next section.
The Dirac operator entering eq. (3.1) is not hermitean:

\[ D = i\partial + iMU^\gamma_5, \quad D^\dagger = i\partial - iMU^\gamma_5, \quad (3.2) \]

to therefore the effective action has an imaginary part. The real part can be defined as

\[ \text{Re} S_{\text{eff}}[\pi] = -\frac{N_c}{2} \ln \det \left( \frac{D^\dagger D}{D^\dagger_0 D_0} \right), \]

\[ D^\dagger D = -\partial^2 + M^2 - M(\partial U^\gamma_5), \quad D^\dagger_0 D_0 = -\partial^2 + M^2. \quad (3.3) \]

In the next two subsections we establish the properties of the real and imaginary parts of the EChL separately, following ref. [23].

### 3.1 Derivative expansion and interpolation formula

There is no general expression for the functional (3.1) for arbitrary pion fields. For certain pion fields the functional determinant (3.1) can be estimated numerically, see section 4. However, one can make a systematic expansion of the EChL in increasing powers of the derivatives of the pion field, \( \partial U \). It is called long wave-length or derivative expansion. Moreover, one can do even better and expand the real part of the EChL in powers of \( \frac{pM}{p^2 + M^2} (U - 1) \) (3.4) where \( p \) is the characteristic momentum of the pion field. This quantity becomes small in three limiting cases: \( i) \) small pion fields, \( \pi^A(x) \ll 1 \), with arbitrary momenta, \( ii) \) arbitrary pion fields but with small gradients or momenta, \( p \ll M \), \( iii) \) arbitrary pion fields and large momenta, \( p \gg M \). We see thus that expanding the EChL in this parameter one gets accurate results in three corners of the Hilbert space of pion fields. For that reason we call it interpolation formula [23]. Our experience is that its numerical accuracy is quite good for more or less arbitrary pion fields, even if one uses only the first term of the expansion in (3.4), see below.

The starting point for both expansions is the following formal manipulation with the real part of the EChL (3.3). The first move is to use the well-known formula, \( \ln \det[\text{operator}] = \text{Sp} \ln[\text{operator}] \), where \( \text{Sp} \) denotes a functional trace. One can write:

\[ \text{Re} S_{\text{eff}}[\pi] = -\frac{N_c}{2} \ln \left[ 1 - (-\partial^2 + M^2)^{-1}M(\partial U^\gamma_5) \right] \]

\[ = -\frac{N_c}{2} \text{Sp} \ln \left[ 1 - (-\partial^2 + M^2)^{-1}M(\partial U^\gamma_5) \right] \]

\[ = -\frac{N_c}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \text{Tr} \ln \left[ 1 - (-\partial^2 + M^2)^{-1}M(\partial U^\gamma_5) \right] e^{ik\cdot x} \]

\[ = -\frac{N_c}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{Tr} \ln \left[ 1 - (k^2 + M^2 - 2ik\cdot \partial - \partial^2)^{-1}M(\partial U^\gamma_5) \right] \cdot 1, \quad (3.5) \]

In going from the second to the third line we have written down explicitly what does the functional trace \( \text{Sp} \) mean: take matrix elements of the operator involved in a complete basis.
(here: plane waves, exp(ik·x)), sum over all states (here: integrate over d⁴k/(2π)⁴) and take the trace in x. ‘Tr’ stands for taking not a functional but a usual matrix trace, in our case both in flavour and Dirac bispinor indices. In going from the third to the last line we have dragged the factor exp(ik·x) through the operator, thus shifting all differential operators ∂ → ∂ + ik. We have put a unity at the end of the equation to stress that the operator is acting on unity, in particular, it does not differentiate it. The above is a standard procedure for dealing with functional determinants [24].

The last expression in eq. (3.5) can be now easily expanded in powers of the derivatives of the pion field: it arises from expanding (3.5) in powers of ∂/Uγ⁵ and of 2ik·∂ + ∂². The first non-zero term has two derivatives,

$$\text{Re} S^{(2)}_{eff}[\pi] = \frac{N_c}{4} \int d^4 x \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left( \frac{M \partial U^{\gamma_5}}{(k^2 + M^2)} \right)^2$$

$$= \frac{1}{4} \int d^4 x \text{Tr} \left( \partial_\mu U^\dagger \partial_\mu U \right) \cdot 4N_c \int \frac{d^4 k}{(2\pi)^4} \frac{M^2}{(k^2 + M^2)^2}. \quad (3.6)$$

It is the kinetic energy term for the pion field or, better to say, the Weinberg chiral lagrangian: actually it contains all powers of the pion field if one substitutes U(x) = exp(iπA(x)λ⁴). The proportionality coefficient (the last factor in eq. (3.4)) is called $F_\pi^2$, experimentally, $F_\pi \approx 94 \text{MeV}$. The last factor in eq. (3.6) is logarithmically divergent; to make it meaningful we have to recall that we have actually simplified the theory when writing it in the local form (2.38). Actually, the dynamical quark mass M is momentum-dependent (see eq. (2.36)); it cuts the logarithimically divergent integral at $k \approx 1/\bar{\rho}$. Using the numerical values of $\bar{\rho} \approx 600 \text{MeV}$ and $M \approx 350 \text{MeV}$ we find

$$F_\pi^2 = 4N_c \int \frac{d^4 k}{(2\pi)^4} \frac{M^2}{(k^2 + M^2)^2} \approx \frac{N_c}{2\pi^2} M^2 \ln \frac{1}{M\bar{\rho}} \approx (100 \text{MeV})^2 \quad (3.7)$$

being not in a bad approximation to the experimental value of $F_\pi$. Actually, the two-derivative term is the only divergent quantity in the EChL: higher derivative terms are all finite.

A more standard way to present the two-derivative term is by using hermitean $N_f \times N_f$ matrices $L_\mu = iU^\dagger \partial_\mu U$. One can rewrite eq. (3.6) as

$$\text{Re} S^{(2)}_{eff}[\pi] = \frac{F_\pi^2}{4} \int d^4 x \text{Tr} L_\mu L_\mu, \quad L_\mu = iU^\dagger \partial_\mu U. \quad (3.8)$$

The next, four-derivative term in the expansion of Re$S_{eff}$ is (note that the metric is Euclidean)

$$\text{Re} S^{(4)}_{eff}[\pi] = -\frac{N_c}{192\pi^2} \int d^4 x \left[ 2\text{Tr} (\partial_\mu L_\mu)^2 + \text{Tr} L_\mu L_\nu L_\mu L_\nu \right]. \quad (3.9)$$

These terms describe, in particular, the $d$-wave $\pi\pi$ scattering lengths, and other observables. They can be compared with the appropriate phenomenological terms in the Gasser–Leutwyler chiral perturbation theory [23]: eq. (3.3) with the concrete coefficients like $N_c/192\pi^2$ appears to be in good agreement with the phenomenological analysis [26].
The next-to-next-to-leading six derivative terms following from eq. (3.5) have been computed in refs. [27, 28]; however, a detailed comparison with phenomenology is still lacking here.

I would like to mention an interesting paper [29] where the derivative expansion of the EChL has been obtained from dual resonance models. For reasons not fully appreciated the dual resonance model for the $\pi\pi$ scattering gives (numerically) very similar coefficients as those following from eq. (3.9), however there is a discrepancy at the 6-derivative level.

I now turn to the interpolation formula promised in the beginning of this subsection. One can start from the last line in eq. (3.5) and expand in powers of $M(\partial/U\gamma^5)$. It is clear that the actual expansion parameter will be (3.4). In the first non-zero order we get [23]

$$\text{Re}S_{\text{eff}}^\text{interpol}[\pi] = \frac{N_c}{4} \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \frac{1}{(k+i\partial)^2 + M^2} M(\partial U\gamma^5) \frac{1}{(k+i\partial)^2 + M^2} M(\partial U\gamma^5) \right].$$

(3.10)

It will be convenient now to pass to the Fourier transform of the $U(x)$ field understood as a matrix,

$$U(p) = \int d^4x \, e^{ipx} [U(x) - 1].$$

(3.11)

The partial derivatives appearing in eq. (3.10) act on the exponents of the Fourier transforms of $U, U^\dagger$ and become corresponding momenta. As a result we get

$$\text{Re}S_{\text{eff}}^\text{interpol}[\pi] = \frac{1}{4} \int \frac{d^4p}{(2\pi)^4} \, p^2 \text{Tr} \left[ (U^\dagger(p)U(p)) \cdot 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2}{(k-i\xi)^2 + M^2} \frac{M^2}{(k+i\xi)^2 + M^2} \right].$$

(3.12)

At $p \to 0$ the last factor becomes $F_+^2$ (see eq. (3.7)), and eq. (3.12) in nothing but the first term in the derivative expansion, eq. (3.6). However, eq. (3.12) also describes correctly the functional $S_{\text{eff}}[\pi]$ for rapidly varying pion fields (with momenta $p \gg M$) and for small pion fields of any momenta, when one can anyhow expand eq. (3.5) in terms of $\pi^A(x)$ and hence in $U(x) - 1$. The logarithmically divergent loop integral in eq. (3.12) should be regularized, as in eq. (3.7).

Similarly, one can get the next term in the ‘interpolation’ expansion which will be quartic in $U(p)$, however our experience tells us that already eq. (3.12) gives a good approximation to the EChL for most pion fields.

### 3.2 The Wess–Zumino term and the baryon number

We now consider the imaginary part of $S_{\text{eff}}[\pi]$. The first non-zero term in the derivative expansion of $\text{Im}S_{\text{eff}}[\pi]$ is [20, 22] the Wess-Zumino term [30]. It is known [31] that it cannot be written as a $d = 4$ integral over a local expression made of the unitary $U(x)$ matrices, however the variation of the Wess–Zumino term is local. For this reason let us consider the variation of $\text{Im}S_{\text{eff}}[\pi]$ in respect to the pion matrix $U(x)$. We have [23]

$$\delta \text{Im}S_{\text{eff}}[\pi] = -N_c \delta \text{Im} \ln \det D = \frac{iN_c}{2} \text{Sp} \left( \frac{1}{D} \delta D - \frac{1}{D^\dagger} \delta D^\dagger \right).$$

17
\[
\frac{iN_c}{2} \text{Sp} \left[ (D^\dagger D)^{-1} D^\dagger D - (DD^\dagger)^{-1} D^\dagger D^\dagger \right].
\]

(3.13)

Now one can put in explicit expressions for \(D, D^\dagger\) from eqs. (3.2, 3.3). The aim of this exercise is to get \(\partial U\) in the denominators so that an expansion in this quantity similar to that of the previous subsection could be used.

Using the Dirac algebra relations (in Euclidean space)

\[
\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma_\mu, \quad \gamma_5^2 = \gamma_5, \quad \text{tr} (\gamma_5 \gamma_\alpha \gamma_\beta \gamma_\gamma) = 4\epsilon_{\alpha\beta\gamma\delta},
\]

(3.14)

one gets after expanding eq. (3.13) in powers of \(\partial U\) the first non-zero term

\[
\delta \text{Im} S_{\text{eff}}[\pi] = \frac{iN_c}{48\pi^2} \int d^4x \epsilon_{\alpha\beta\gamma\delta} \text{Tr} \left( \partial_\alpha U^\dagger \partial_\beta U \partial_\gamma U^\dagger \partial_\delta U \right).
\]

(3.15)

It can be easily checked that this expression coincides with the variation of the Wess–Zumino term written in the form

\[
\text{Im} S_{\text{eff}}[\pi] = \frac{iN_c}{240\pi^2} \int d^3x \epsilon_{ijk} \text{Tr} \left( U^\dagger \partial_i U \right) \left( U^\dagger \partial_j U \right) \left( U^\dagger \partial_k U \right) + \text{higher derivative terms}.
\]

(3.16)

In fact the integrand in eq. (3.16) is a full derivative, however, to write it explicitly one would need a parametrization of the unitary matrix \(U\). The expansion of eq. (3.16) starts from the fifth power of \(\pi^A(x)\), and it is non-zero only if \(N_f \geq 3\). It is important that, similar to \(\text{Re} S_{\text{eff}}\), the imaginary part is also an infinite series in the derivatives.

The EChL (3.1) or, more generally, the low-energy partition function (2.24) from where eq. (3.1) has been derived, is invariant under vector flavour-singlet transformations. Therefore, there should be a corresponding conserved Noether baryon current, \(B_\mu\). This current is associated with the imaginary part of \(S_{\text{eff}}\) only; since \(\text{Im} S_{\text{eff}}\) is an infinite series in the derivatives so is the associated Noether current \(B_\mu\). For the first Wess–Zumino term (3.16) the corresponding charge is

\[
B = -\frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \text{Tr} \left( U^\dagger \partial_i U \right) \left( U^\dagger \partial_j U \right) \left( U^\dagger \partial_k U \right) + \text{higher derivative terms}.
\]

(3.17)

The explicitly written term is the winding number of the field \(U(x)\). Let me briefly explain this notion.

If \(\pi^A(x) \to 0\) at spatial infinity so that \(U(x) \to 1\) in all directions, one can say that the spatial infinity is just one point. Eq. (3.17) gives then the winding number for the mapping of the three-dimensional sphere \(S^3\) (to which the flat \(d = 3\) space is topologically equivalent when \(\infty\) is one point) to the parameter space of the \(SU(N_f)\) group. In case \(N_f = 2\) the parameter space is also \(S^3\) so that the mapping is \(S^3 \to S^3\). The topologically non-equivalent mappings \(U(x)\), \(i.e.\) those which can not be continuously deformed one to another, are classified by their winding number, an integer analytically given by eq. (3.17). In case of \(N_f > 2\) mathematicians prove that mappings are also classified by integers given by the same eq. (3.17).
There exists a prejudice that the baryon number carried by quarks in the external pion field coincides with the winding number of that field: generally speaking it is not so because of the higher derivative terms omitted in eq. (3.17). Only if the pion field is spatially large and slowly varying so that one can neglect the higher derivative terms in eq. (3.17) one can say that the two coincide. Otherwise, for arbitrary pion fields, the baryon number is not related to the winding number: the former may be zero when the latter is unity, and vice versa.

To see what is going on here, let us calculate directly the baryon number carried by quarks in an external time-independent pion field \( U(\mathbf{x}) \). The definition of the baryon charge operator in the Minkowski space is

\[
\hat{B} = \frac{1}{N_c} \int d^3 \mathbf{x} \, \bar{\psi} \gamma_0 \psi. \tag{3.18}
\]

Passing to Euclidean space (which we prefer to work with since functional integrals are more readily defined in Euclidean) one has to make a substitution \( \bar{\psi} \rightarrow -i\psi^\dagger, \gamma_0 \rightarrow \gamma_4 \), so that

\[
\hat{B} = -i \frac{N_c}{\gamma_4} \int d^3 \mathbf{x} \, \psi^\dagger \gamma_4 \psi. \tag{3.19}
\]

The baryon charge in the path integral formulation of the theory given by eq. (2.38) is then

\[
B = \langle \hat{B} \rangle = -i \frac{N_c}{\gamma_4} \int d^3 \mathbf{x} \, \psi^\dagger \gamma_4 \psi = -i \int d^3 \mathbf{x} \, \text{Tr} \left\langle x | \gamma_4 (i\partial + iMU^\gamma_5)^{-1} | x \right\rangle
\]

\[
= -i \int d^3 \mathbf{x} \, \text{Tr} \left\langle x_4, x | \frac{1}{i\partial + i\gamma_4 \gamma_k \partial_k + iM\gamma_4 U^\gamma_5} | x_4, x \right\rangle
\]

\[
= -i \int d^3 \mathbf{x} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, \text{Tr} \left\langle x | \frac{1}{\omega + iH} | x \right\rangle
\]

\[
= \text{Sp} \theta(-H) = \text{number of levels with } E < 0. \tag{3.20}
\]

Here

\[
H = \gamma_4 \gamma_k \partial_k + M\gamma_4 U^\gamma_5 \tag{3.21}
\]

is the Dirac hamiltonian in the external time-independent pion field \( U(\mathbf{x}) \) and \( \theta \) is a step function.

Eq. (3.20) is divergent since it sums up the baryon charge of the whole negative-energy Dirac continuum. This divergence can be avoided by subtracting the baryon charge of the free Dirac sea, i.e. with the pion field switched out, \( H_0 = \gamma_4 \gamma_k \partial_k + M\gamma_4 \):

\[
B = -i \int d^3 \mathbf{x} \int \frac{d\omega}{2\pi} \, \text{Tr} \left\langle x | \frac{1}{\omega + iH} - \frac{1}{\omega + iH_0} | x \right\rangle = \text{Sp} \left[ \theta(-H) - \theta(-H_0) \right]. \tag{3.22}
\]

In performing the integration over \( \omega \) we have closed the \( \omega \) integration contour in the upper semiplane. Had we closed it in the lower semiplane we would obtain \(-\text{Sp} [\theta(H) - \theta(H_0)]\) which is the same result since \( \text{Sp} [\theta(H) + \theta(-H) - \theta(H_0) - \theta(-H_0)] = 0. \)
We have thus obtained a most natural result: the baryon charge of quarks in the external pion field is the number of negative-energy levels of the Hamiltonian \((3.21)\) (the number of the levels of the free Hamiltonian subtracted).

One can perform the gradient expansion for the baryon number similarly to that of the real part of the EChL. To that end let us write

\[
B = - \int d^3x \int \frac{d\omega}{2\pi} \text{Tr} \langle x \mid \frac{H}{\omega^2 + H^2} - \frac{H_0}{\omega^2 + H_0^2} \mid x \rangle
\]

where

\[
H^2 = -\partial_k^2 + M^2 - M\gamma_4(\partial_k U^\gamma_5), \quad H_0^2 = -\partial_k^2 + M^2.
\]

Calculating the matrix element in the plane-wave basis one gets

\[
B = - \int d^3x \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ \frac{1}{\omega^2 - (\partial + ik)^2 + M^2 - M\gamma \cdot (\partial U^\gamma_5)} \right] \cdot 1.
\]

For slowly varying fields \(U(x)\) eq. \((3.25)\) can be expanded in powers of \(\partial U^\gamma_5\) and \(\partial\) (applied ultimately to \(U^\gamma_5\)). Because of the \(\text{Tr} \gamma_5...\) the first non-zero contribution arises from expanding the denominator in eq. \((3.25)\) to the third power of \(\gamma \cdot (\partial U^\gamma_5)\). Integrals over \(\omega\) and \(k\) should be explicitly performed. After some simple algebra one gets

\[
B = - \frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \text{Tr} (\partial_i U^\dagger \partial_j U \partial_k U^\dagger U) + \text{higher derivative terms}
\]

coinciding with eq. \((3.17)\) derived from the Noether current corresponding to the Wess–Zumino term \((3.16)\) \[32, 33, 34, 35\].

It should be stressed that the baryon number carried by quarks in the background pion field is equal to the topological winding number of the field only if it is a slowly varying one. The deep reason for it is the following \[23\]. Imagine we start from a pion field \(U(x)\) whose winding number is one but whose spatial size is tending to zero. Such a field would have no impact on the spectrum of the Dirac Hamiltonian \((3.21)\): it would remain the same as that of the free Hamiltonian, namely it would have the upper \((E > M)\) and lower \((E < -M)\) Dirac continua separated by the mass gap of \(2M\).

We now (adiabatically) increase the spatial size of the pion field preserving its winding number equal to unity. Since the winding number is dimensionless this can always be done. At certain critical spatial size the potential well for quarks formed by the external pion field is wide enough so that a bound-state level emerges from the upper continuum. With the increase of the width of the potential well the bound-state level goes down towards the lower Dirac continuum. Asymptotically, as one blows up the spatial size of the pion field (always remaining in the winding number equal unity sector) the bound-state level travels all the way through the mass gap separating the two continua and joins the lower Dirac sea – this is a theorem proven in ref. \[23\]. At this point one would discover that there is an extra state
close to the lower Dirac continuum (as compared to the free, that is no-field case). Therefore, one would say that the baryon number is now unity, – in correspondence to eqs. (3.17, 3.20).

In a general case, however, the baryon number of the quark system is the number of eigenstates of the Dirac Hamiltonian (3.21) one bothers to fill in. The role of the winding number of the background pion field is only to guarantee that, if the spatial size of the field is large enough, the additional bound-state level emerging from the upper continuum is a deep one: asymptotically it goes all the way to the lower continuum.

4 The nucleon

4.1 Physical motivations

All constituent quark models start from assuming that the nearly massless light quarks of the QCD lagrangian obtain a non-zero dynamical quark mass $M \approx 350 - 400$ MeV. This is due to the spontaneous breaking of chiral symmetry, its microscopic driving force being, to my belief, instantons, as explained in section 2. Even if one does not believe in instantons as the microscopic mechanism of spontaneous chiral symmetry breaking one has to admit that once a constituent quark mass is introduced such quarks inevitably have to interact with Goldstone pions. The lagrangian $\bar{\psi}(i \not{\partial} - M)\psi$ is not invariant under axial rotation $\psi \rightarrow \exp(i \alpha A^A \gamma_5)\psi$, it is the chiral lagrangian (2.38),

$$\mathcal{L} = \bar{\psi}(i \not{\partial} - MU^\gamma)\psi,$$

(4.1)

which is, since the rotation of the quark fields can be compensated in this lagrangian by renaming of the pion fields.

Expanding $U^\gamma = \exp(i \pi A^A \gamma_5 / F_\pi)$ in powers of the properly normalized pion fields (that is why we have inserted the $F_\pi$ constant) we see that the dimensionless constant of the linear coupling of pions to constituent quarks is, numerically, quite large:

$$g_{\pi qq} = \frac{M}{F_\pi} \simeq 4.$$  

(4.2)

I would like to emphasize that this is a model-independent consequence of saying that quarks get a constituent mass.

Actually, the coupling (4.2) is so strong that one may wonder how some people manage to get along without taking it into account. Not surprisingly, baryon models which do take into account pion exchange between constituent quarks give much more realistic predictions than, say, the old simple-minded Isgur–Karl model (for a review see ref. [36]).

Moreover, at distances between quarks of the order of 0.5 fm typical for interquark distances inside nucleons, neither the one-gluon exchange nor the supposed linear potential are as large as the chiral forces. Therefore, it is worthwhile to investigate whether the chiral forces alone are able to bind the constituent quarks inside nucleons. Such approach may or may not be successful for describing high nucleon excitations where, according to the standard logic, the confining forces become crucial. To that we can remark the following:
1. The constituent quark mass is momentum-dependent; the behaviour of $M(k)$ at low virtualities $k$ may well be divergent (see section 2) and it may thus play the role of confining forces. [To my knowledge, this line of thought has not been pursued in the literature.]

2. Highly excited baryons with large angular momenta, if understood as chiral solitons, lie on linear Regge trajectories with a realistic slope related to the $F_\pi$ constant [1, 37].

Therefore, it may be expected that even highly excited baryons can be incorporated into the chiral theory. After all, as emphasized by Witten [31], the theory of all hadrons can be, in principle, formulated completely in terms of the EChL. For example, there should exist an EChL corresponding to the Lovelace–Shapiro dual resonance amplitudes for $\pi\pi$ scattering exhibiting the correct Regge behaviour [28]. A (unsolved) problem is to formulate an appropriate EChL in the field-theoretic language.

Leaving aside these interesting problems, we concentrate on the lowest state with baryon number one, i.e. the nucleon. As mentioned above the interquark separations in the ground-state nucleon are moderate (order of 0.5 fm) and it is worthwhile asking whether the simple EChL (2.38) is capable of explaining the basic properties of the ground-state nucleon. Notice that the expected typical momenta of quarks inside the nucleon are of the order of $M \approx 350 – 400 \text{ MeV}$, that is perfectly inside the domain of applicability of the low-momentum effective theory (2.38), according to its derivation in section 2.

The chiral interactions of constituent quarks in the 3-quark nucleon, as induced by the effective theory (2.38), are schematically shown in Fig. 2, where quarks are denoted by solid and pions by dashed lines. Notice that, since there is no tree-level kinetic energy for pions in eq. (2.38), the pion propagation occurs only through quark loops. Quark loops induce also many-quark interactions indicated in Fig. 2 as well. We see that the emerging picture is, unfortunately, rather far from a simple one-pion exchange between the constituent quarks: the non-linear effects in the pion field are not at all suppressed.

At this point one may wonder: isn’t the resulting theory as complicated as the original
QCD itself? The answer is no, the effective low-energy theory is an enormous simplification as compared to the original quark-gluon theory, because it deals with adequate degrees of freedom. Let us imagine that one would like to describe ‘low-energy’ properties of solid states, superconductivity for example. Would working with the underlying theory (QED) be helpful? Not at all. We know that the microscopic theory leads, under certain conditions, to the rearrangement of atoms into a lattice, so that translational symmetry is spontaneously broken. As a result the Goldstone bosons appear (here: phonons), and electrons get a dynamical mass different from the input one. The most important forces are due to phonon exchange between electrons: in fact they are driving superconductivity in the BCS theory. Playing with this analogy, nucleon is like a polaron (a bound state of electrons in the phonon field), rather than a positronium state in the vacuum. After chiral symmetry is broken we deal with a ‘metal’ phase rather than with the vacuum one, and one has to use adequate degrees of freedom to face this new situation.

The instanton vacuum plays the role of the bridge between the microscopic theory (QCD) and the low-energy theory where one neglects all degrees of freedom except the Goldstone bosons and fermions with the dynamically-generated mass. Instantons do the most difficult part of the job: they explain why atoms in metals are arranged into a lattice and what is the effective mass of the electron and what is the strength of the electron-phonon interactions. However, one can take an agnostic stand and say: I don’t care how 350 MeV is obtained from the microscopic ΛQCD and why do atoms form a lattice in the metals: I just know it happens. To such a person I would advise to take the low-energy theory (2.38) at face value and proceed to the nucleon.

A considerable technical simplification is achieved in the limit of large $N_c$. For $N_c$ colours the number of constituent quarks in a baryon is $N_c$ and all quark loop contributions are also proportional to $N_c$, see section 3. Therefore, at large $N_c$ one can speak about a classical self-consistent pion field inside the nucleon: quantum fluctuations about the classical self-consistent field will be suppressed as $1/N_c$. The problem of summing up all diagrams of the type shown in Fig.2 is thus reduced to finding a classical pion field pulling $N_c$ massive quarks together to form a bound state.

### 4.2 Nucleon mass: a functional of the pion field

Let us imagine that there is a classical time-independent pion field which is strong and spatially wide enough to make a bound-state level of the Dirac hamiltonian (3.21) for massive quarks, call its energy $E_{\text{level}}$. We fill in this level by $N_c$ quarks in the antisymmetric state in colour, thus obtaining a baryon number one state, as compared to the vacuum. The interactions with the background chiral field are, naturally, colour-blind, so one can put $N_c$ quarks on the same level; the fact that one has to put them in an antisymmetric state in colour, i.e. in a colour-singlet state, follows from Fermi statistics.

One has to pay for the creation of this trial pion field, however. Call this energy $E_{\text{field}}$. Since there are no direct terms depending on the pion field in the low-momentum theory (2.38) the only origin of $E_{\text{field}}$ is the fermion determinant (3.1) which should be calculated for time-independent field $U(x)$. It can be worked out with a slight modification of section 3. We have [23]:

23
\[ S_{\text{eff}}[\pi] = -N_c \ln \det \left( \frac{D}{D_0} \right) \]

\[ = -N_c \text{Sp} \left[ \ln(i\partial_t + iH) - \ln(i\partial_t + iH_0) \right] \]

\[ = -TN_c \int \frac{d\omega}{2\pi} \text{Sp} \left[ \ln(\omega + iH) - \ln(\omega + iH_0) \right] , \quad (4.3) \]

where \( H \) is the Dirac hamiltonian (3.21) in the stationary pion field, \( H_0 \) is the free hamiltonian and \( T \) is the (infinite) time of observation. Using an important relation

\[ \text{Sp}(H - H_0) = 0 \quad (4.4) \]

(telling us that the sum of all energies, with their signs, of the Dirac hamiltonian (3.21) is the same as for the free case) one can integrate in eq. (4.3) by parts and get

\[ S_{\text{eff}}[\pi] \equiv T E_{\text{field}} = TN_c \int \frac{d\omega}{2\pi} \text{Sp} \left[ \frac{\omega}{\omega + iH} - \frac{\omega}{\omega + iH_0} \right] \]

\[ = TN_c \sum_{E_n^{(0)} < 0} \left( E_n - E_n^{(0)} \right) . \quad (4.5) \]

Going from the first to the second line we have closed the \( \omega \) integration contour in the upper semiplane; owing to the trace relation (4.4) closing it in the lower semiplane would produce the same result.

We see that the energy cost \( E_{\text{field}} \) one pays for a creation of the time-independent pion field coincides with the aggregate energy of the lower Dirac continuum in that field. The energy of the additional level emerging from the upper continuum, which one has to fill in to get the baryon number one state, \( E_{\text{level}} \), should be added to get the total nucleon mass. This simple scheme [38, 23] is depicted in Fig.3. Naturally, the mass of the nucleon should be counted from the vacuum state corresponding to the filled levels of the free lower Dirac continuum. Therefore, the (divergent) aggregate energy of the free continuum should be subtracted, as in eq. (4.5).

We have thus for the nucleon mass:

\[ M_N = \min_{\{\pi^A(x)\}} \left( N_c E_{\text{level}}[\pi] + E_{\text{field}}[\pi] \right) . \quad (4.6) \]

Both quantities, \( E_{\text{level}} \) and \( E_{\text{field}} \), are functionals of the trial pion field \( \pi^A(x) \). The classical self-consistent pion field is obtained from minimizing the nucleon mass (4.6) in \( \pi^A(x) \). It is called the soliton of the non-linear functional (4.6), hence the Chiral Quark-Soliton Model. An accurate derivation of eq. (4.6) from the path-integral representation for the nucleon-current correlation function is presented in ref. [23]. It solves the problem of summing up all diagrams of the type shown in Fig.2 in the large-\( N_c \) limit.

6It follows from taking the matrix trace of the hamiltonian (3.21). It should be kept in mind, though, that such a naive derivation can be potentially dangerous because of anomalies in infinite sums over levels. However, it can be checked that in this particular case there are no anomalies and the naive derivation is correct.
The idea that a sigma model with pions coupled directly to constituent quarks, can be used to build the nucleon soliton has been first suggested by Kahana, Ripka and Soni [39] and independently by Birse and Banerjee [40]. I would call them the authors of the Chiral Quark-Soliton Model. Technically, however, in these refs. an additional ad hoc kinetic energy term for pion fields has been used, leading to a vacuum instability paradox. The present formulation of the model has been given in ref. [38], together with the discussion of its domain of applicability and its physical contents. A detailed theory based on the path integral approach paving the way for calculating nucleon observables has been presented in ref. [23].

4.3 Nucleon profile

To find the classical pion field minimizing the nucleon mass (4.6) one has first of all decide on the symmetry of the pion field. Had the field been a singlet one would take a spherically-symmetric ansatz. However, the pion field has flavour indices $A = 1, ..., N_f^2 - 1$. At $N_f = 2$ the three components of the pion field can be married with the three space axes. This is called the hedgehog ansatz; it is the minimal generalization of spherical symmetry to incorporate the $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$ and $\pi^0 = \pi^3$ fields:

$$\pi^A(x) = n^A P(r), \quad n^A = \frac{x^A}{r}, \quad r = |x|, \quad (4.7)$$

where $P(r)$ is called the soliton profile.

The choice of the ansatz is not at all innocent: baryons corresponding to different choices would have qualitatively different properties, see the next subsection. The maximally-symmetric ansatz (4.7) will have definite consequences in applications.
In the $N_f = 3$ case there are 8 components of the ‘pion’ field, and there are several possibilities to marry them to the space axes. The commonly used ansatz is the ‘left upper corner’ ansatz \[41\]:

$$U(\mathbf{x}) \equiv \exp \left( i \pi^A(\mathbf{x}) \lambda^A \right) = \begin{pmatrix} \exp [i(\mathbf{n} \cdot \mathbf{\tau}) P(\mathbf{r})] & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{n} = \frac{\mathbf{x}}{r}. \quad (4.8)$$

As we shall see in the next subsection, the quantization of rotations for this ansatz leads to the correct spectrum of the lowest baryons.

Another SU(3) ansatz \[42\] discussed in the literature is ($f,g,h=1,2,3$ are the flavour indices):

$$U_{fg} = e^{iP_2/3} \left[ \cos P_1 \delta_{fg} + \left( e^{-iP_2} - \cos P_1 \right) n_f n_g + \sin P_1 \epsilon_{fgh} n_h \right], \quad (4.9)$$

where $P_{1,2}(r)$ are spherically-symmetric profile functions. This ansatz is used to describe strangeness -2 dibaryons \[43\]. We shall not consider it here but concentrate on the usual baryons for which the hedgehog ansatz (4.7) or (4.8) is appropriate.

Let us first discuss restrictions on the best profile function $P(r)$ which should minimize the nucleon mass (4.6). What is the asymptotics of $P(r)$ at large $r$? To answer this question one has to know the behaviour of $E_{\text{field}}[\pi]$ for slowly varying pion fields. Using eq. (4.5) as a starting point one can work out the derivative expansion of the functional $E_{\text{field}}[\pi]$ similar to that for the full EChL, see subsection 3.1. We have \[38, 23\]

$$E_{\text{field}}[\pi] = \frac{F_\pi^2}{4} \int d^3 \mathbf{x} \text{Tr} L_i L_i - \frac{N_c}{192 \pi^2} \int d^3 \mathbf{x} \left[ 2 \text{Tr} (\partial_i L_i)^2 + \text{Tr} L_i L_j L_i L_j \right] + \text{higher derivative terms}, \quad L_i = iU^\dagger \partial_i U. \quad (4.10)$$

Substituting here the hedgehog ansatz one gets a functional of the profile function $P(r)$; varying it one finds the Euler–Lagrange equation valid for slowly varying profiles, in particular, for the tail of $P(r)$ at large $r$. It follows from this equation that $P(r) = A/r^2$ at large $r$. The second contribution to the nucleon mass, $E_{\text{level}}[\pi]$, does not alter this derivation since the bound-state wave function has an exponential, not power behaviour at large $r$. Actually we get the pion tail inside the nucleon, and the constant $A$ is related to the nucleon axial constant. This relation is identical to the one found in the Skyrme model \[44\]:

$$g_A = \frac{8\pi}{3} A F_\pi^2. \quad (4.11)$$

The exponentially decreasing wave function of the bound-state level does not change this derivation, as well as the Goldberger–Treiman relation for the pion-nucleon coupling constant,

$$g_{\pi NN} = \frac{g_A M_N}{F_\pi} = \frac{8\pi A M_N}{3 F_\pi}. \quad (4.12)$$

Furthermore, it follows from the next four-derivative term in eq. (4.10) that the $1/r^4$ correction to $P(r)$ at large $r$ is absent \[38, 23\]! It means probably that the pion tail inside nucleon is unperturbed to rather short distances.
The second important question is what should we choose for $P(0)$? As explained in subsection 3.2, a quantity which guarantees a deeply-bound state in the background pion field is the winding number of the field, eq. (3.17). Substituting the hedgehog ansatz into eq. (3.17) one gets

$$N_{\text{wind}} = -2 \frac{\pi}{\int_0^\infty dr \sin^2 P(r) \frac{dP(r)}{dr}} = -\frac{1}{\pi} \left[ P(r) - \frac{\sin 2P(r)}{2} \right]_0^\infty. \quad (4.13)$$

Since $P(\infty) = 0$ the way to make this quantity unity is to choose $P(0) = \pi = 3.14...$

An example of a one-parameter variational function satisfying the above requirements is

$$P(r) = 2 \arctg \left( \frac{r_0}{r} \right)^2, \quad A = 2r_0^2. \quad (4.14)$$

The Dirac hamiltonian (3.21) in the hedgehog pion field (4.17) commutes neither with the isospin operator $T$ nor with the total angular momentum $J = L + S$ but only with their sum $K = T + J$ called the ‘grand spin’. The eigenvalue Dirac equations for given value of $K^2, K_3$ have been derived in refs. [38, 23]. Generally speaking, there appears a bound-state level with the $K^P = 0^+$ quantum numbers whose energy can be found from solving the Dirac equations for two spherically-symmetric functions $j, h$

$$\frac{dh}{dr} = -Mh \sin P + (E_{\text{level}} + M \cos P) j, \quad (4.15)$$
$$\frac{dj}{dr} + \frac{2}{r} j = Mj \sin P + (-E_{\text{level}} + M \cos P) h \quad (4.16)$$

with the boundary conditions $h(0) = 1, j(0) = Cr, h(\infty) = j(\infty) = 0$.

These equations determine one of the two contributions to the nucleon mass, $E_{\text{level}}$. The second contribution, namely that of the aggregate energy of the lower Dirac continuum in the trial pion field, which we have called $E_{\text{field}}$, can be found in several different ways. One way is to find the phase shifts in the lower continuum, arising from solving the Dirac equation for definite grand spin $K$ [23, 45]. Another method is to diagonalize the Dirac hamiltonian (3.21) in the so-called Kahana–Ripka basis [46] written for a finite-volume spherical box. Both methods are, numerically, rather involved. There exists a third (approximate) method [38, 23] allowing one to make an estimate of $E_{\text{field}}$ in a few minutes on a PC. It is based on the interpolation formula for the EChL, see eq. (3.12) and ref. [23] for details.

Let us discuss the qualitative behaviour of $E_{\text{level}}$ and $E_{\text{field}}$ with the soliton scale parameter $r_0$ assuming for definiteness that the profile is given by eq. (4.14).

The trial pion field plays the role of the (relativistic) potential well for massive quarks. The ‘depth’ of this potential well is fixed by the condition $P(0) = \pi$ and cannot be made infinite: this is related to the fact that the pion field has the meaning of angles. The spatial size of the trial pion field $r_0$ plays the role of the ‘width’ of the potential well. It is well known that in three dimensions the condition for the appearance of a bound state is $MVr_0^2 > \text{const}$ where $V$ is the depth of well and ‘const’ is a numerical constant of the order of unity depending on the concrete shape of the potential well. In our case $V \approx M$, so the condition that the bound state appears is $Mr_0 \sim 1$. Therefore, at small sizes $r_0$ there
is no bound state for the Dirac hamiltonian (3.21), so that \( E_{\text{level}} \) coincides with the border of the upper Dirac continuum, \( E_{\text{level}} = +M \). At certain critical value of \( r_0 \) a weakly bound state emerges from the upper continuum. [For the concrete ansatz (1.14) the threshold value is \( r_0M \simeq 0.5 \).] As one increases \( r_0 \) the bound state goes deeper and \( E_{\text{level}} \) monotonously decreases. At very large spatial sizes, \( r_0 \to \infty \), \( E_{\text{level}} \) approaches the lower continuum, its difference from \( -M \) falling as \( 1/r_0^2 \) [23]. The behaviour of \( E_{\text{level}} \) as function of \( r_0 \) is plotted in Fig.4.

The monotonous decrease of \( E_{\text{level}} \) with the increase of \( r_0 \) is a prerogative of the trial pion field with winding number 1. Had it been zero, \( E_{\text{level}} \) would first go down and then start to go up, asymptotically joining back the upper continuum. In the case of \( N_{\text{wind}} = -1 \) the bound state would travel in the opposite direction: from the lower towards the upper continuum. At \( N_{\text{wind}} = n \) as much as \( n \) levels would emerge, one by one, from the upper continuum and travel all the way through the mass gap towards the lower one. For the trial pion field of the hedgehog form all these things happen exclusively for states with grand spin \( K = 0 \) [23].

Turning now to \( E_{\text{field}} \) we first notice that for large spatial sizes of the trial pion field one can use the first term in the derivative expansion for \( E_{\text{field}} \), see eq. (4.10). On dimension grounds one immediatelly concludes that \( E_{\text{field}} \sim F_\pi^2 r_0 \) for large \( r_0 \), i.e. is infinitely linearly rising in \( r_0 \). At small \( r_0 \) a slightly more complicated analysis [23] shows that \( E_{\text{field}} \sim r_0^3 \). On the whole, \( E_{\text{field}} \) is a monotonously rising function of \( r_0 \) shown in Fig.4.

The nucleon mass, \( \mathcal{M}_N = 3E_{\text{level}} + E_{\text{field}} \) (for \( N_c = 3 \)) is also plotted in Fig.4 taken from refs. [88, 23]. One observes a non-trivial minimum for \( \mathcal{M}_N \) corresponding to \( r_0 \simeq 0.98/M \simeq 0.57 \, \text{fm} \). This is, phenomenologically, a very reasonable value, since from eqs. (1.11, 1.12) one immediatelly gets \( g_A \simeq 1.15 \) versus 1.25 (exp.) and \( g_{\pi NN} \simeq 13.6 \) versus 13.5 (exp.). The nucleon mass appears to be \( \mathcal{M}_N \simeq 1100 \, \text{MeV} \) with \( E_{\text{level}} \simeq 123 \, \text{MeV}, \ E_{\text{field}} \simeq 730 \, \text{MeV} \). Figure 4: The classical nucleon mass and its constituents as function of the soliton size. The short-dash line shows \( 3E_{\text{level}} \), the long-dash line shows \( E_{\text{field}} \), the solid line is their sum, \( \mathcal{M}_N \).
Note that the ‘valence’ quarks (sitting on the bound-state level) come out to be very strongly bound: their wave function falls off as \(\exp(-r/0.6 \text{ fm})\), and about 2/3 of the quark mass \(M \simeq 350 \text{ MeV}\) is eaten up by interactions with the classical pion field. Relativistic effects are thus essential.

Though the nucleon bound state appears to be somewhat higher than the free-quark threshold, \(3M \simeq 1050 \text{ MeV}\), there are several known corrections to it which all seem to be negative. The largest correction to the nucleon mass is due to taking into account explicitly the one-gluon exchange between both ‘valence’ and ‘sea’ quarks; this correction is \(O(N_c)\) as is the nucleon mass itself. Numerically, it turns out to be about \(-200 \text{ MeV}\) \([17]\) and seems to move the nucleon mass just into the right place \(7\).

We see thus that the ‘valence’ quarks in the nucleon get bound by a self-consistent pion field whose energy is given just by the aggregate energy of the negative Dirac continuum distorted by the presence of the external field. This picture of the nucleon interpolates between the old non-relativistic quark models (which would correspond to a shallow bound-state level and an undistorted negative continuum) and the Skyrme model (which would correspond to a spatially very large pion soliton so that the bound-state level would get close to the lower continuum and the field energy \(E_{\text{field}}\) would be given just by a couple of terms in its derivative expansion). The reality is somewhere in between: the bound-state level is a deep one but not as deep as to say that all the physics is in the lower Dirac continuum.

Ideologically, this picture of the nucleon at large \(N_c\) is somewhat similar to the Thomas–Fermi picture of the atom at large \(Z\). In that case quantum fluctuations of the self-consistent electrostatic field binding the electrons are suppressed by large \(Z\), however corrections go as powers of \(Z^{-2/3}\). Therefore, the Chiral Quark-Soliton Model is in a slightly better position in respect to quantum corrections than the Thomas–Fermi approximation.

Apart from using the large \(N_c\) approximation (which is in fact just a technical device needed to justify the use of the classical pion field) the Chiral Quark-Soliton Model makes use of the small algebraic parameter \((M\bar{\rho})^2\) where \(\bar{\rho}\) is the average size of instantons in the vacuum. This \(\bar{\rho}\) is, roughly, the size of the constituent quark, while the size of the nucleon is, parametrically, \(1/M\). The fact that the constituent quark picture works so well in the whole hadron physics finds its explanation in this small numerical parameter being due to the relative diluteness of the instanton vacuum, which in its turn is related to the ‘accidentally’ large number \((11/3)\) in the asymptotic freedom law \([4, 13]\). The small parameter \((M\bar{\rho})^2 \ll 1\) makes it possible to use only quarks with dynamically generated mass and chiral fields as the only essential degrees of freedom in the range of momenta \(k \sim M \ll 1/\bar{\rho}\), and that is exactly the range of interest in the nucleon binding problem.

The above numerics have been obtained from the interpolation formula for \(E_{\text{field}}\) \([38, 23]\). Exact calculations of \(E_{\text{field}}\) performed in \([45]\) as well as taking more involved profiles with three variational parameters did not lead to any significant changes in the numerics.

Following refs. \([38, 23]\) there had been many calculations of the nucleon mass and of the ‘best’ profile using various regularization schemes and parameters of the chiral model, see \([48]\) for a review. The effective theory derived from the instanton vacuum comes with

\[\text{There exist also numerous quantum corrections to the nucleon mass of different origin, which are of the order of } O(N_c^0). \text{ Unfortunately, it is difficult today to treat them in a systematic fashion; see, however, the next subsection.}\]
an intrinsic ultraviolet cutoff, in the form of a momentum dependence of the constituent quark mass, \( M(k) \). It can be shown on general grounds that this is a rapidly falling function at momenta of the order of the inverse average instanton size, \( 1/\bar{\rho} \approx 600 \text{ MeV} \). However, the present ‘state of the art’ does not allow one to determine this function accurately at all values of momenta – to do so, one would need a very detailed understanding of the instanton vacuum. This places certain restrictions on the kinds of quantities which can sensibly be computed using the effective theory. Those are either finite ones, which do not require an UV cutoff at all, or quantities at most logarithmically divergent. Both type of quantities are dominated by momenta much smaller than the UV cutoff, \( k \ll 1/\bar{\rho} \), so one can compute them mimicking the fall–off of \( M(k) \) by an external UV cutoff \( \Lambda \approx 1/\bar{\rho} \), using some regularization scheme. Fortunately, almost all nucleon observables belong to these two classes. The uncertainty related to the details of the ultra-violet regularization leads to a 15-20% numerical uncertainty of the results, and that is the expected accuracy of the model today.

### 4.4 Quantum numbers of baryons

The picture of the nucleon outlined in the previous subsection is “classical”: the quantum fluctuations of the self-consistent pion field binding \( N_c \) quarks are totally ignored. Among all possible quantum corrections to the nucleon mass a special role belongs to the zero modes. Fluctuations of the pion field in the direction of the zero modes cannot be considered small, and one has to treat them exactly. Zero modes are always related to continuous symmetries of the problem at hand. In our case there are 3 zero translational modes and a certain number of zero rotational modes. The latter determine the quantum numbers of baryons; it is here that the hedgehog (or whatever) symmetry of the ansatz taken for the self-consistent pion field becomes crucial.

A general statement is that if the chiral field \( U_{cl}(x) \) minimizes the nucleon mass functional (4.6), a field corresponding to rotated spatial axes, \( x_i \rightarrow O_{ij} x_j \), or to a unitary-rotated matrix in flavour space, \( U_{cl} \rightarrow RU_{cl}R^\dagger \), has obviously the same classical mass. This is because the functional \( (4.6) \) to be minimized is isotropic both in flavour and ordinary spaces.

Specifically for the hedgehog ansatz [see eq. (4.7) for the flavour \( SU(2) \) and eq. (4.8) for the \( SU(3) \)] any spatial rotation is equivalent to a flavour rotation. We show it for a more complicated case of \( SU(3) \). Indeed, the space-rotating \( 3 \times 3 \) matrix \( O_{ij} \) can be written as

\[
O_{ij} = \frac{1}{2} \text{Tr} (S \tau_i S^\dagger \tau_j)
\]

(4.17)

where \( S \) is an \( SU(2) \) \( 2 \times 2 \) matrix and \( \tau_i \) are the three Pauli matrices. One can immediately check that \( O_{ij} \) are real orthogonal 3-parameter matrices with \( O_{ij} O_{kj} = \delta_{ik} \) and \( O_{ij} O_{ik} = \delta_{jk} \), as it should be.

When one rotates the space putting \( n'_i = O_{ij} n_j \) the \( 2 \times 2 \) matrix standing in the left upper corner of the ansatz \( (4.8) \) can be written as

\[
\exp \left[ i(n' \cdot \tau) P(r) \right] = \cos P(r) + i(n' \cdot \tau) \sin P(r)
\]

\[
= S [\cos P(r) + i(n \cdot \tau) \sin P(r)] S^\dagger.
\]

(4.18)
Therefore, if one considers the hedgehog ansatz (4.8) rotated both in flavour and usual 
spaces, the latter can be completely absorbed into the former one:

\[ RU_{\alpha}(Ox) R^\dagger = \tilde{R} U_{\alpha}(x) \tilde{R}^\dagger \] (4.19)

with

\[ \tilde{R} = R \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}. \] (4.20)

For that reason it is sufficient to consider rotations only in the flavour space. Hence there 
are 3 zero rotational modes in the \( SU(2) \) and 8-1=7 in the \( SU(3) \) flavour case. The rotation 
of the form \( R = \exp(i\alpha \lambda^8) \) commutes with the left-upper-corner ansatz and therefore does 
not correspond to any zero mode. This will have important consequences in getting the 
correct spectrum of hyperons.

The general strategy [23] is to consider a slowly rotating ansatz

\[ \tilde{U}(x,t) = R(t) U_{\alpha}(x) R^\dagger(t) \] (4.21)

and to expand the energy of the bound-state level and of the negative Dirac continuum in 
‘right’ \( \Omega_A \) and ‘left’ \( \tilde{\Omega}_A \) angular velocities

\[ \Omega_A = -i \text{Tr} (R^\dagger \dot{R} \lambda^A), \quad \tilde{\Omega}_A = -i \text{Tr} (\dot{R} R^\dagger \lambda^A), \quad \Omega^2 = \tilde{\Omega}^2 = 2 \text{Tr} R^\dagger \dot{R}. \] (4.22)

Taking into account only the lowest terms in the time derivatives of the rotation matrix 
\( R(t) \) one gets [23, 49] the following form of the rotation lagrangian:

\[ L^{\text{rot}} = \frac{1}{2} I_{AB} \Omega_A \Omega_B - \frac{N_c}{2\sqrt{3}} \Omega_8. \] (4.23)

Here \( I_{AB} \) is the \( SU(3) \) tensor of the moments of inertia,

\[ I_{AB} = \frac{N_c}{4} \int \frac{d\omega}{2\pi} \text{Tr} \left( \frac{1}{\omega + iH} \lambda^A \frac{1}{\omega + iH} \lambda^B \right) \] (4.24)

where the \( \omega \) integration contour should be drawn above the bound-state energy \( E_{\text{level}} \) to incorporate the ‘valence’ quarks.

The appearance of a linear term in \( \Omega_8 \) is an important consequence of the presence of an 
extra bound-state level emerging from the upper Dirac continuum, which fixes the baryon 
charge to be unity. We remind the reader that in the Skyrme model this linear term arises 
from the Wess-Zumino term [50]. For simplicity we have written unregularized moments of 
inertia, though eq. (4.24) should be regularized in some way, see e.g. [49].

Owing to the left-upper-corner ansatz for the static soliton (being essentially \( SU(2) \)) the 
tensor \( I_{AB} \) is diagonal and depends on two moments of inertia, \( I_{1,2} \):

\[ I_{AB} = \begin{cases} I_1 \delta_{AB}, & A, B = 1, 2, 3, \\ I_2 \delta_{AB}, & A, B = 4, 5, 6, 7, \\ 0, & A, B = 8. \end{cases} \] (4.25)
Therefore, the rotational lagrangian (4.23) can be rewritten as

\[ L^{\text{rot}} = \frac{I_1}{2} \sum_{A=1}^{3} \Omega_A^2 + \frac{I_2}{2} \sum_{A=4}^{7} \Omega_A^2 - \frac{N_c}{2\sqrt{3}} \Omega_8. \] (4.26)

To quantize this rotational Lagrangian one can use the canonical quantization procedure, same as in the Skyrme model [50, 32, 51, 52, 53]. Introducing eight angular momenta canonically conjugate to ‘right’ angular velocities \( \Omega_A \),

\[ J_A = \frac{\partial L^{\text{rot}}}{\partial \Omega_A}, \] (4.27)

and writing the Hamiltonian as

\[ H^{\text{rot}} = \Omega_A J_A - L^{\text{rot}} \] (4.28)

one gets

\[ H^{\text{rot}} = \frac{1}{2I_1} \sum_{A=1}^{3} J_A^2 + \frac{1}{2I_2} \sum_{A=4}^{7} J_A^2 \] (4.29)

with the additional quantization prescription following from eq. (4.27),

\[ J_8 = -\frac{N_c}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}. \] (4.30)

In the Skyrme model this quantization rule follows from the Wess-Zumino term. In our approach it arises from filling in the bound-state level, i.e. from the ‘valence’ quarks. It is known to lead to the selection rule: not all possible spin and \( SU(3) \) multiplets are allowed as rotational excitations of the \( SU(2) \) hedgehog. Eq. (4.30) means that only those \( SU(3) \) multiplets are allowed which contain particles with hypercharge \( Y = 1 \); if the number of particles with \( Y = 1 \) is denoted as \( 2J + 1 \), the spin of the allowed \( SU(3) \) multiplet is equal to \( J \).

Therefore, the lowest allowed \( SU(3) \) multiplets are:

- octet with spin 1/2 (since there are two baryons in the octet with \( Y = 1 \), the \( N \))
- decuplet with spin 3/2 (since there are four baryons in the decuplet with \( Y = 1 \), the \( \Delta \))
- anti-decuplet with spin 1/2 (since there are two baryons in the anti-decuplet with \( Y = 1 \), the \( N^* \))

The next are 27-plets with spin 1/2 and 3/2 but we do not consider them here.

We see that the lowest two rotational excitations are exactly the lowest baryon multiplets existing in reality. The third predicted multiplet, the anti-decuplet, contains exotic baryons which cannot be made of three quarks, most notably an exotic \( Z^+ \) baryon having spin 1/2, isospin 0 and strangeness +1. A detailed study of the anti-decuplet performed recently in [54] predicts that such a baryon can have a mass as low as 1530 MeV and be very narrow. Several experimental searches of this exotic baryon are now under way.

It is easy to derive the splittings between the centers of the multiplets listed above. For the representation \((p, q)\) of the \( SU(3) \) group one has
\[
\sum_{A=1}^{8} J_{A}^2 = \frac{1}{3} [p^2 + q^2 + pq + 3(p + q)],
\]
\(\text{Therefore the eigenvalues of the rotational hamiltonian } (4.29) \text{ are}
\]
\[
E_{(p,q)}^{\text{rot}} = \frac{1}{6I_2} [p^2 + q^2 + pq + 3(p + q)] + \left( \frac{1}{2I_1} - \frac{1}{2I_2} \right) J(J+1) - \frac{3}{8I_2}.
\]
\(\text{We have the following three lowest rotational excitations:}
\]
\[
(p, q) = (1, 1), \ J = 1/2 : \ \text{octet, spin } 1/2,
\]
\[
(p, q) = (3, 0), \ J = 3/2 : \ \text{decuplet, spin } 3/2,
\]
\[
(p, q) = (0, 3), \ J = 1/2 : \ \text{anti-decuplet, spin } 1/2.
\]
\(\text{The splittings between the centers of these multiplets are determined by the moments of inertia, } I_{1,2}:
\]
\[
\Delta_{10-8} = E_{(3,0)}^{\text{rot}} - E_{(1,1)}^{\text{rot}} = \frac{3}{2I_1},
\]
\[
\Delta_{10-8} = E_{(0,3)}^{\text{rot}} - E_{(1,1)}^{\text{rot}} = \frac{3}{2I_2}.
\]
\(\text{The appropriate rotational wave functions describing members of these multiplets are given by Wigner finite-rotation functions } D_{8,10,10}^{8,10,10}(R) \ [49, 54].
\)
\(\text{When dealing with the flavour } SU(3) \text{ case neglecting the strange quark mass } m_s \text{ is an oversimplification. In fact, it is easy to incorporate } m_s \neq 0 \text{ in the first order. As a result one gets very reasonable splittings inside the } SU(3) \text{ multiplets, as well as mass corrections to different observables } [49, 48, 54].
\)
\(\text{In general, the idea that all light baryons are rotational excitations of one object, the ‘classical’ nucleon, leads to numerous relations between properties of the members of octet and decuplet, which follow purely from symmetry considerations and which are all satisfied up to a few percent in nature. The } SU(3) \text{ symmetry by itself says nothing about the relation between different multiplets, of course. Probably the most spectacular is the Guadagnini formula } [50] \text{ which relates splittings inside the decuplet with those in the octet,}
\]
\[
8(m_{\Xi^*} + m_{\Sigma^*}) + 3m_{\Sigma} = 11m_{\Lambda} + 8m_{\Sigma^*},
\]
\(\text{which is satisfied with better than one-percent accuracy!}
\)

\(N_f = 2 \text{ case}
\)
\(\text{If one is interested in baryons predominantly ‘made of’ } u, d \text{ quarks, the flavour group is } SU(2) \text{ and the quantization of rotations is more simple.}
\)
\(\text{In this case the rotational lagrangian is just}
\]
\[
L^{\text{rot}} = \frac{I_1}{2} \sum_{i=1}^{3} \Omega_i^2 = \frac{I_1}{2} \sum_{A=1}^{3} \tilde{\Omega}_A^2,
\]
where the ‘right’ ($\Omega_i$) and ‘left’ ($\tilde{\Omega}_A$) angular velocities are defined by eq. (4.22). This is the lagrangian for the spherical top: the two sets of angular velocities have the meaning of those in the ‘lab frame’ and ‘body fixed frame’. The quantization of the spherical top is well known from quantum mechanics. One has to introduce two sets of angular momenta, $S_i$ (canonically conjugate to $\Omega_i$) and $T_A$ (conjugate to $\tilde{\Omega}_A$). Both sets of operators act on the coordinates of the spherical top, say, the Euler angles. It will be more convenient for us to say that the coordinates of the spherical top are just the entries of the unitary matrix $R$ defining its finite-angle rotation \[23\].

The angular momenta operators $S_i, T_A$ act on $R$ as generators of right (left) multiplication,

\[
e^{i(\alpha S)} R e^{-i(\alpha S)} = R e^{i(\alpha \sigma)}, \quad e^{i(\alpha S)} R e^{-i(\alpha S)} = e^{-i(\alpha \tau)} R,
\]

and satisfy the commutation relations

\[
[T_A, T_B] = i \epsilon_{ABC} T_C, \quad [S_i, S_j] = i \epsilon_{ijk} S_k, \quad [T_A, S_i] = 0, \quad (T_A)^2 = (S_i)^2.
\] (4.42)

A realization of these operators is

\[
S_i = R_{pq} \left( \frac{\sigma_i}{2} \right)_{kq} \frac{\partial}{\partial R_{pq}}, \quad T_A = - \left( \frac{\tau_A}{2} \right)_{pk} R_{kq} \frac{\partial}{\partial R_{pq}}.
\] (4.43)

The rotational hamiltonian is

\[
H^{rot} = \Omega_i S_i - L^{rot} = \tilde{\Omega}_A T_A - L^{rot} = \frac{S_i^2}{2I_1} = \frac{T_A^2}{2I_1}.
\] (4.44)

Comparing the definition of the generators (4.40, 4.41) with the ansatz (4.21) we see that $T_A$ is the flavour (here: isospin) operator and $S_i$ is the spin operator, since the former acts to the left from $R$ and the latter acts to the right.

The normalized eigenfunctions of the mutually commuting operators $S_3, T_3$ and $S^2 = T^2$ with eigenvalues $S_3, T_3$ and $S(S + 1) = T(T + 1)$ are \[23\]

\[
\psi_{T_3S_3}^{(S-T)}(R) = \sqrt{2S + 1}(-1)^{T+T_3} D_{T_3S_3}^{(S-T)}(R)
\] (4.45)

where $D(R)$ are Wigner finite-rotation matrices. For example, in the $S = T = 1/2$ representation $D_{pq}^{1/2}(R) = R_{pq}$, i.e. coincides with the unitary matrix $R$ itself.

The rotational energy is thus

\[
E^{rot} = \frac{S(S + 1)}{2I_1} = \frac{T(T + 1)}{2I_1}
\] (4.46)

and is $(2S + 1)^2 = (2T + 1)^2$-fold degenerate. The wave functions (4.43) describe at $S = T = 1/2$ four nucleon states (proton, neutron, spin up, spin down) and at $S = T = 3/2$ the sixteen $\Delta$-resonance states, the splitting between them being
\[ m_\Delta - m_N = \frac{3}{2I_1} = O(N_c^{-1}) \]  

(4.47)

(coinciding in fact with the splitting between the centers of decuplet and octet in the more general \( SU(3) \) case, see eq. (4.36)).

It is remarkable that the nucleon and its lowest excitation, the \( \Delta \), fits into this spin-equal-isospin scheme, following from the quantization of the hedgehog rotation. Moreover, since \( N \) and \( \Delta \) are, in this approach, just different rotational states of the same object, the ‘classical nucleon’, there are certain relations between their properties. These relations are identical to those found first in the Skyrme model [44] since they follow from symmetry considerations only and do not depend on concrete dynamics which is of course different in the naive Skyrme model. For example, one gets for the dynamics-independent ratio of magnetic moments and pion couplings [44]

\[ \frac{\mu_\Delta N}{\mu_p - \mu_n} = \frac{1}{\sqrt{2}} \simeq 0.71 \quad \text{vs.} \quad 0.70 \pm 0.01 \quad (\text{exp.}), \]

\[ \frac{g_{\pi N \Delta}}{g_{\pi NN}} = \frac{3}{2} = 1.5 \quad \text{vs.} \quad 1.5 \pm 0.12 \quad (\text{exp.}). \]  

(4.48)

I should mention that there might be interesting implications of the ‘baryons as rotating solitons’ idea to nuclear physics. The low-energy interactions between nucleons can be viewed as interactions between spherical tops depending on their relative orientation \( R_1 R_2^\dagger \) in the spin-isospin spaces [53, 54]. It leads to an elegant description of \( NN \) and \( N\Delta \) interactions in a unified fashion, and it would be very interesting to check its experimental consequences (as far as I know this has not been done yet). A nuclear medium is then a medium of interacting quantum spherical tops with extremely anisotropic interactions depending on relative orientations of the tops both in the spin-isospin and in ordinary spaces.

This unconventional point of view is strongly supported by the observation [55] that one can get the correct value of the so-called symmetry energy of the nucleus, \( 25 \text{MeV} \cdot (N - Z)^2/A \), with the coefficient \( 25 \text{MeV} \) appearing as \( 1/8I_1 \) where \( I_1 \) is the \( SU(2) \) moment of inertia; from the \( \Delta - N \) splitting (4.47) one finds \( I_1 \simeq (200 \text{MeV})^{-1} \). I do not know whether the language of spherical tops is fruitful to describe ordinary nuclear matter (probably it is but nobody tried), however it is certainly useful to address new questions, for example whether nuclear matter at high densities can be in a strongly correlated antiferromagnet-type phase [56].

Finally, let us ask what the next rotational excitations could be? If one restricts oneself to only two flavours, the next state should be a \((5/2, 5/2)\) resonance; in the three-flavour case the third rotational excitation is the anti-decuplet with spin 1/2, see above. Why do not we have any clear signal of the exotic \((5/2, 5/2)\) resonance? The reason is that the angular momentum \( J = 5/2 \) is numerically comparable to \( N_c = 3 \). Rotations with \( J \approx N_c \) cannot be considered as slow: the centrifugal forces deform considerably the spherically-symmetric profile of the soliton field [1, 57]; simultaneously at \( J \approx N_c \) the radiation of pions by the rotating body makes the total width of the state comparable to its mass [1, 37, 58]. In order to survive strong pion radiation the rotating chiral solitons with \( J \geq N_c \) have to stretch into cigar-like objects; such states lie on linear Regge trajectories with the slope \( \alpha' \approx 1/8\pi^2 F^2 \pi \) [1, 57].
The situation, however, might be somewhat different in the three-flavour case. First, the rotation is, roughly speaking, distributed among more axes in flavour space, hence individual angular velocities are not necessarily as large as when we consider the two-flavour case with \( J = 5/2 \). Actually, the \( SU(2) \) baryons with \( J = 5/2 \) belong to a very high multiplet from the \( SU(3) \) point of view. Second, the radiation by the soliton includes now \( K \) and \( \eta \) mesons which are substantially heavier than pions, and hence such radiation is suppressed. Actually, the anti-decuplet seems to have moderate widths \(^5\) and it is worthwhile searching for the predicted exotic states.

### 4.5 Some applications

There exists by now a rather vast literature studying baryon observables in the Chiral Quark-Soliton Model. Baryon formfactors (electric, magnetic and axial), mass splittings, the nucleon sigma term, magnetic moments, weak decay constants, tensor charges and many other characteristics of nucleons and hyperons have been calculated in the model. I address the reader to an extensive review \(^{13}\) on these matters.

Here I would like to point out several developments of the Chiral Quark-Soliton Model interesting from the theoretical point of view. The list below is, of course, very subjective.

The study of the spin content of the nucleon in the model has been pioneered by Wakanmatsu and Yoshiki \(^{61}\). They showed that the fraction of the nucleon spin carried by the spin of quarks is about 50\% (and could be made less): the rest is carried by the interquark orbital moment, the Dirac sea contribution to it being quite essential.

An important question is \( 1/N_c \) corrections to baryon observables. These can be classified in two groups: one comes, for example, from meson loops and is therefore accompanied by an additional small factor \( \sim 1/8\pi^2 \), the second arises from a more accurate account for the quantization of the zero rotational modes. The second-type corrections are not accompanied by small loop factors, and may be quite substantial: after all in the real world \( N_c = 3 \) so a 30\% correction is not so small. Such corrections for certain quantities have been fished out in refs. \(^{62, 63}\) in the two-flavour case and in ref. \(^{64}\) for three flavours. These corrections

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\(^5\)Sometimes the model has been called ‘the solitonic sector of the Nambu–Jona-Lasinio model’. I have many objections to this title. First, Vaks and Larkin have suggested independently and at the same time a 4-fermion model to illustrate how symmetry can be dynamically broken in field theory. Therefore, in any case I would call it the VL/NJL model. Second, both VL \(^1\) and NJL \(^8\) were talking about nucleons as fundamental fermions (there were no quarks in 1961), and this is rather far from what we consider now. Third, as discussed in section 2, the instanton-induced interactions in contrast to the \( \text{ad hoc} \) 4-fermion interactions correctly reproduce the symmetries: \( U_A(1) \) is explicitly broken, while in the \( N_c = 2 \) case they possess a more wide \( SU(4) \times U(4) \) symmetry \(^8\); at \( N_f > 2 \) they necessarily are \( 2N_f \)-fermion interactions and not at all 4-fermion. Fourth, instanton-induced interactions provide a natural UV cutoff, as given by the momentum-dependence of the constituent quark mass. For the success of the Chiral Quark-Soliton Model it is extremely important that this UV cutoff is much larger than the constituent quark mass itself (actually squared), meaning that the size of the constituent quark is much less than the size of the nucleon. In an \( \text{ad hoc} \) 4-fermion model having no obvious relation to QCD one has to impose the UV cutoff by hands. Fifth, in an arbitrarily introduced 4-quark interaction model there are no \( a \ priori \) reasons to freeze out all degrees of freedom except the chiral ones. Meanwhile, if one includes the \( \sigma \) field into the minimization of the nucleon mass the soliton collapses \(^{65, 66}\). In short, when one knows results coming from instantons, it is possible to mimic some of them by imposing certain rules of the game with the 4-quark interactions. But why then should it be called the ‘NJL model’?
work in a welcome direction: they lower the fraction of nucleon spin carried by quark spins and increase the flavour non-singlet axial constants.

A recent development of the model deals with the parton distributions in nucleon. This topic deserves however a special subsection.

### 4.6 Nucleon structure functions

The distribution of quarks, antiquarks and gluons, as measured in deep inelastic scattering of leptons, provides us probably with the largest portion of quantitative information about strong interactions. Until recently only the evolution of the structure functions from a high value of the momentum transfer $Q^2$ to even higher values has been successfully compared with the data. This is the field of perturbative QCD, and its success has been, historically, essential in establishing the validity of the QCD itself. However, the initial conditions for this evolution, namely the leading-twist distributions at a relatively low normalization point, belong to the field of non-perturbative QCD. If we want to understand the vast amount of data on unpolarized and polarized structure functions we have to go into non-perturbative physics.

The Chiral Quark-Soliton Model presents a non-perturbative approach to the nucleons, and it is worthwhile looking into the parton distributions it predicts. Contrary to several models of nucleons on the market today, it is a relativistic field-theoretical model. This circumstance is of crucial importance when one deals with parton distributions. It is only with a relativistic field-theoretical model one can preserve general properties of parton distributions such as

- relativistic invariance,
- positivity of parton distributions,
- partonic sum rules which hold in full QCD.

There are two seemingly different ways to define parton distributions. The first, which I would call the Fritsch–Gell-Mann definition, is a nucleon matrix element of quark bilinears with a light-cone separation between the quark $\psi$ and $\bar{\psi}$ operators. According to the second, which I would call the Feynman–Bjorken definition, parton distributions are given by the number of partons carrying a fraction $x$ (the Bjorken variable) of the nucleon momentum in the nucleon infinite-momentum frame. See Feynman’s book [65] for the discussion of both definitions. In perturbative QCD only the Fritsch–Gell-Mann definition has been exploited as one has no idea how to write down the nucleon wave function in the infinite-momentum frame, which is necessary for the Feynman–Bjorken definition.

Despite the apparent difference in wording, it has been shown for the first time, within the field-theoretical Chiral Quark-Soliton Model, that the two definitions are, in fact, equivalent and lead to identical working formulae for computing parton distributions: in ref. [66] the first definition has been adopted while in ref. [67] the second was used. The deep reason for that equivalence is that the main hypothesis of the Feynman–Bjorken parton model, namely that partons transverse momenta do not grow with $Q^2$ [63], is satisfied in the model.
Let me point out some key findings of refs. [66, 67].

(i) **Classification of quark distributions in $N_c$**

Since the nucleon mass is $O(N_c)$ all parton distributions are actually functions of $xN_c$. Combining this fact with the known large-$N_c$ behaviour of the integrals of the distributions over $x$ one infers that all distributions can be divided in ‘large’ and ‘small’. The ‘large’ distributions are, for example, the unpolarized singlet and polarized isovector distributions, which are of the form

$$D_{\text{large}}(x) \sim N_c^2 f(xN_c),$$

where $f(y)$ is a stable function in the large-$N_c$ limit. On the contrary, the polarized singlet and unpolarized isovector distributions give an example of ‘small’ distributions, having the form

$$D_{\text{small}}(x) \sim N_c f(xN_c).$$

One, indeed, observes in experiment that ‘large’ distributions are substantially larger than the ‘small’ ones.

(ii) **Antiquark distributions**

In the academic limit of a very weak mean pion field in the nucleon the Dirac continuum reduces to the free one (and should be subtracted to zero) while the bound-state level joins the upper Dirac continuum. In such a limit there are no antiquarks, while the distribution of quarks becomes $q(x) = N_c^2 \delta(xN_c - 1)$. In reality there is a non-trivial mean pion field which a) creates a bound-state level, b) distorts the negative-energy Dirac continuum. As the result, the above $\delta$-function is smeared significantly, and a non-zero antiquark distribution appears.

An inevitable consequence of the relativistic invariance is that the bound-state level makes a *negative*-definite contribution to the antiquark distribution [9]. The antiquark distribution becomes positive only when one includes the contribution of the Dirac continuum. Numerically, the antiquark distribution appears to be sizeable even at a low normalization point, in accordance with phenomenology.

(iii) **Sum rules**

The general sum rules holding in full QCD are automatically satisfied in the Chiral Quark-Soliton Model: in refs. [66, 67] the validity of the baryon number, isospin, total momentum and Bjorken sum rules has been checked. In fact, it is for the first time that

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[9] This is also true for any nucleon model with valence quarks, for example for any variant of bag models. Bag models are essentially non-relativistic, so they fail to resolve this paradox. In order to cure it, one has to take into account contributions to parton distributions from *all* degrees of freedom involved in binding the quarks in the nucleon. That can be consistently done only in a relativistic field-theoretical model, like the one under consideration.
nucleon parton distributions at a low normalization point have been consistently calculated in a relativistic model preserving all general properties.

(iv) **Smallness of the gluon distribution**

As many times stressed in these lectures, the whole approach of the Chiral Quark-Soliton Model is based on the smallness of the algebraic parameter \((M\bar{\rho})^2\) where \(\bar{\rho}\) is the average size of instantons in the vacuum. This \(\bar{\rho}\) is the size of the constituent quark, while the size of the nucleon is, parametrically, \(1/M\). Computing parton distribution in the model one is restricted to momenta \(k \ll 1/\bar{\rho} \approx 600\,\text{MeV}\), so that the internal structure of the constituent quarks remains unresolved. There are no gluons in the nucleon at this resolution scale; indeed, the momentum sum rule is satisfied with quarks and antiquarks only. However, when one moves to the resolution scale of 600 MeV or higher, the constituent quarks cease to be point-like, and that is at this scale that a non-zero gluon distribution emerges. Having a microscopic theory of how quarks get their dynamical masses one can compute the non-perturbative gluon distribution in the constituent quarks\(^{10}\). What can be said on general grounds is that the fraction of momentum carried by gluons is of the order of \((M\bar{\rho})^2 \approx 1/3\), which seems to be the correct portion of gluons at a low normalization point of about 600 MeV where the normal perturbative evolution sets in.

(v) **Comparison with phenomenology**

There are several parametrizations of the nucleon parton distributions at a relatively low normalization point, which, after their perturbative evolution to higher momentum transfer \(Q^2\), fit well the numerous data on deep inelastic scattering. The most daring (and convenient for our purpose) parametrization is that of Glück, Reya *et al.*\(^{69, 70}\) who pushed the normalization point for their distributions to as low as 600 MeV, starting from the perturbative side. In refs. \(^{66, 67}\) parton distributions following from the Chiral Quark-Soliton Model have been compared with those of refs. \(^{69, 70}\). There seems to be a good qualitative agreement though the constituent quark and antiquark distributions appear to be systematically ‘harder’ than those of \(^{69, 70}\). This deviation is to be expected since the structure of the constituent quarks themselves has not been yet taken into account, see above.

5 **Conclusions**

The Chiral Quark-Soliton Model is a simple and elegant reduction of the full-scale QCD at low energies, however preserving its main ingredients, namely spontaneous chiral symmetry breaking, and the appearance of the dynamical (or constituent) quark mass. Personally, I prefer the word ‘dynamical’: first, because it is, indeed, dynamically generated, second, because it is momentum-dependent.

The momentum dependence of the dynamical quark mass \(M(k)\) is the key to understanding why the notion of constituent quarks have worked so remarkably well over 30 years in hadron physics. The point is, the scale \(\Lambda\) at which the function \(M(k)\) falls off appears to

\(^{10}\)Steps in that direction has been taken in refs. \(^{13, 18}\).
be much larger than $M(0)$; the former parameter determines the size of constituent quarks while the latter parameter determines the size of hadrons. These two distinctive scales come neatly from instantons, as described in section 2.

The Chiral Quark-Soliton Model fully exploits the existence of the two distinctive scales: it is because of them it makes sense to restrict oneself to just two degrees of freedom in the nucleon problem, namely, to massless or nearly massless (pseudo) Goldstone pions and to the constituent quarks with a momentum-dependent dynamical mass $M(k)$. The scale $\Lambda$ actually plays the role of the physical ultra-violet cutoff for the low-energy theory; its domain of applicability is thus limited to the range of momenta $k \sim M < \Lambda$. This is precisely the domain of interest for the nucleon binding problem.

A technical tool simplifying considerably the nucleon problem is the use of the large $N_c$ logic. At large $N_c$ the nucleon is heavy, and one can speak of the classical self-consistent pion field binding the $N_c$ valence quarks of the nucleon together. The classical pion field (the soliton) is found from minimizing the energy of the bound-state level plus the aggregate energy of the lower Dirac continuum in a trial pion field. The valence quarks (sitting on the bound-state level) appear to be strongly bound by the classical pion field.

By quantizing the slow rotations of the soliton field in flavour and ordinary spaces one gets baryon states which are rotational excitations of the static ‘classical nucleon’. The classification of the rotational excitations depends on the symmetry properties of the soliton field, but not on the details of dynamics. Taking the hedgehog ansatz one gets the following lowest baryon multiplets: octet with spin 1/2, decuplet with spin 3/2 (these are, indeed, the lowest multiplets observed in nature) and antidecuplet with spin 1/2. This last multiplet contain baryons with exotic quantum numbers (in the sense that they cannot be composed of only three quarks); some of them are predicted to be relatively light and narrow resonances, and it would be of great interest to search for such states.

By saying that all lightest baryons are nothing but rotational excitations of the same object, the ‘classical nucleon’, we get many relations between members of baryon multiplets which are all realized with astonishing accuracy in nature. Especially successful are predictions which do not depend on dynamical quantities (like the values of moments of inertia) but follow from symmetry considerations only, and are therefore shared, e.g., by the Skyrme model. Predictions of the Chiral Quark-Soliton Model which do depend on concrete dynamics are, in general, also in good accordance with reality: the typical accuracy for numerous baryon observables computed in the model is about 15-20\%, coinciding with the expected theoretical accuracy of the model. To get a better accuracy one needs a better understanding of the underlying QCD vacuum and of the resulting effective low-energy theory. The developed theory of the instanton vacuum of QCD seems to do the job of explaining the hadron world pretty well already, however if one wants to improve the accuracy of predictions one has to make the theory more precise.

To my knowledge, the Chiral Quark-Soliton Model is the only relativistic field-theoretical model of the nucleon on the market today, and this advantage of the model becomes crucial when one turns to the numerous parton distribution functions. It is impossible to get a consistent description of parton distributions satisfying positivity and sum rules restrictions, without having a relativistic theory at hand and without taking into account the complete set of forces which bind quarks together. The leading-twist parton distributions computed so far in the Chiral Quark-Soliton Model refer to a very low normalization point where the structure
of the constituent quarks is not resolved yet. Nevertheless, they seem to be in qualitative agreement with parametrizations of the DIS data at low $Q^2$ though, not unnaturally, they appear to be more ‘hard’.

I think that it is the field of parton distributions where the Chiral Quark-Soliton Model will be used most of all in the near future. We know how to (perturbatively) evolve parton distributions from high to still higher values of $Q^2$ but we do not really understand how to explain the initial conditions for that evolution, that is the leading-twist parton distributions at a low normalization point. This is where a relativistic model satisfying all general requirements could be of great use. Also in the years to come there will be much experimental activity involving numerous spin and off-forward parton distributions, as well as non-leading-twist distributions. Practically nothing is known about these numerous distributions from the theoretical side, and the predictions of the Chiral Quark-Soliton Model can be very valuable, see refs. [68, 71, 72] for the first predictions.

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