An algebraic method of obtaining of symplectic coordinates in a rigid body dynamics

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Abstract

An algebraic procedure of getting of canonical variables in a rigid body dynamics is presented. The method is based on using a structure of an algebra of Lie—Poisson brackets with which a Hamiltonian dynamics is set. In a particular case of a problem of a top in a homogeneous gravitation field the method leads to well-known Andoyer—Deprit variables. Earlier, the method of getting of them was based on geometric approach (Arhangelski, 1977).

Introduction

Poisson brackets play a key role in Hamiltonian mechanics. According to the theorem of Darboux (Olver, 1986), degenerated Poisson manifolds are stratified to symplectic manifolds (leaves). A reduction of a Hamiltonian system on common level of all Casimir functions leads to usual Hamiltonian mechanics. The reduction is especially algebraic problem and can be made without referring to a concrete physical problem. In the present paper an algebraic method of finding of symplectic coordinates of Lie—Poisson brackets in some important for physical application cases is demonstrated.

1 A symplectic basis of the Lie—Poisson brackets with an algebra \( \mathfrak{e}(3) \)

The equations of motion of a rigid body with a fixed point in a homogeneous gravitation field are traditionally written in variables \( \mathbf{M}, \gamma \), where \( \mathbf{M} \) is a vector of an angular momentum of the body and \( \gamma \) is unit vector parallel to the vector of gravitational field strength. The coordinate system is toughly connected with the body.

The equations of motion are Hamiltonian with a degenerated Poisson brackets (Olver, 1986)

\[
\{M_i, M_j\} = -\epsilon_{ijk} M_k, \quad \{\gamma_i, \gamma_j\} = 0, \quad \{M_i, \gamma_j\} = -\epsilon_{ijk} \gamma_k \quad (1.1)
\]
The algebra (1.1) is a semi-direct sum of an algebra of rotation \( so(3) \) and an algebra of translation \( \text{R}^3 : e(3) = so(3) \oplus e_{\text{R}^3}. \) Under reduction of the algebra (1.1) to Casimir functions \( H = (\textbf{M}, \gamma) \) and \( (\gamma, \gamma) = 1 \), the Poisson brackets (1.1) become non-degenerated and the equations of motion can be written in canonical form. In the capacity of canonical coordinates, most convenient for qualitative analysis, the canonical variables of Andoyer—Deprit \((l, L; g, G; h, H)\) are used.

For describing of mechanical sense of these variables we denote by \( OXYZ \) a stationary trihedron with the origin in the point of fixation of the body, \( Oxyz \) is connected with the body the frame of coordinates, \( \Sigma \) is a plane passing through the point of fixation perpendicular to the vector of angular momentum \( \textbf{M} \). At last, in the admitted notations

- \( L \) is a projection of the angular momentum to the moving axis \( Oz \);
- \( G \) is a value of the angular momentum;
- \( H \) is a projection of the angular momentum to the fixed axis \( OZ \);
- \( l \) is an angle between the axis \( Ox \) and a line of intersection of \( \Sigma \) with \( Oxy \);
- \( g \) is an angle between the line of intersection of \( \Sigma \) with planes \( Oxy \) and \( OXY \);
- \( h \) is an angle between the axis \( OX \) and the line of intersection of \( \Sigma \) with the plane \( OXY \).

The essence of the method consists in extracting from (1.1) of the subalgebra \( so(3) \) and getting of canonical variables there. The canonical variables \( l, L \)

\[
M_1 = \sqrt{G^2 - L^2 \sin l}, \quad M_2 = \sqrt{G^2 - L^2 \cos l}, \quad M_3 = L,
\]  

are cylindrical coordinates on two-dimensional concentric spheres (orbits of the \( so(3) \)) (Olver, 1986). The angular momentum \( G = \sqrt{M_1^2 + M_2^2 + M_3^2} \) is the Casimir function of the considered algebra. The Poisson brackets between \( l, L, G, \gamma_1, \gamma_2, \gamma_3 \) are following:

\[
\{l, L\} = 1, \quad \{G, L\} = \{G, l\} = 0, \quad \{\gamma_i, \gamma_j\} = 0,
\]  

\[
\{L, \gamma_1\} = -l \gamma_2, \quad \{L, \gamma_2\} = \gamma_1, \quad \{L, \gamma_3\} = 0,
\]  

\[
\{l, \gamma_1\} = -\frac{\sin l \gamma_3}{\sqrt{G^2 - L^2}}, \quad \{l, \gamma_2\} = -\frac{\cos l \gamma_3}{\sqrt{G^2 - L^2}},
\]  

\[
\{l, \gamma_3\} = \frac{H - L \gamma_3}{G^2 - L^2},
\]  

\[
\{G, \gamma_1\} = \frac{1}{G}(\sqrt{G^2 - L^2} \cos l \gamma_3 - L \gamma_2),
\]  

\[
\{G, \gamma_2\} = \frac{1}{G}(L \gamma_1 - \sqrt{G^2 - L^2} \sin l \gamma_3),
\]
\[
\{G, \gamma_3\} = \frac{1}{G} \sqrt{G^2 - L^2} (\sin l \gamma_2 - \cos l \gamma_1),
\]
where \(H\) is a projection of the angular momentum to the fixed axis. In addition, \(H = (M, \gamma)\) is the Casimir function of the algebra \(e(3)\).

Now we solve step by step the systems of partial differential equations (1.2)–(1.6), assuming \(\gamma_i\) as functions of \((l, L; g, G; H)\). As a result, we get the required relations with the Andoyer—Deprit variables:

\[
\gamma_1 = \left(\frac{H}{G} \sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g\right) \sin l + \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \cos l,
\]

\[
\gamma_2 = \left(\frac{H}{G} \sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g\right) \cos l - \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \sin l,
\]

\[
\gamma_3 = \left(\frac{H}{G}\right) \left(\frac{L}{G}\right) - \sqrt{1 - \left(\frac{L}{G}\right)^2} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g.
\]

The canonical variables of the four–dimensional symplectic leaf of the Poisson brackets of the algebra \(e(3)\) are \((l, L; g, G)\). They are enumerated by the variable \(H\).

## 2 A symplectic basis of the Lie—Poisson brackets with an algebra \(l(7)\)

The very convenient variables for description of the rigid body dynamics instead of the angles \((\alpha, \beta, \gamma)\) that form the orthogonal matrix of rotations are quaternions (Koshlyakov, 1985). The real variables \(\lambda_0, \lambda_1, \lambda_2, \lambda_3\) normalized by the condition

\[
\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1,
\]
and connected with the vector \(\gamma\) by the formulae (Koshlyakov, 1985):

\[
\gamma_1 = 2(\lambda_1 \lambda_3 - \lambda_0 \lambda_2),
\]

\[
\gamma_2 = 2(\lambda_0 \lambda_1 + \lambda_2 \lambda_3),
\]

\[
\gamma_3 = \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2,
\]
are named the Hamilton—Rodrigues parameters.

A point \(M(\mathbf{r})\) under some rotation around a vector \(\mathbf{e}(\alpha', \beta', \gamma')\) is displaced to \(M'(\mathbf{r}')\) on an angle \(\chi\). The vector \(\mathbf{\theta} = 2\tan(\chi/2)\mathbf{e}\) is a vector of finite rotation. Instead of projections \(\theta_1, \theta_2, \theta_3\) one can introduce variables \(\lambda_k\), by \(\lambda_k = 1/2 \lambda_0 \theta_k, k = 1, 2, 3\) are subjected to the condition (2.1). So we get

\[
\lambda_0 = \cos \chi/2, \quad \lambda_1 = \cos \alpha' \sin \chi/2, \quad \lambda_2 = \cos \beta' \sin \chi/2, \quad \lambda_3 = \cos \gamma' \sin \chi/2.
\]
Now, it is not difficult to obtain the algebra of Poisson brackets of variables $\mathbf{M}, \lambda_0, \lambda_1, \lambda_2, \lambda_3$. The Lagrangian of a free rigid body with a diagonal tensor of inertia $\mathbf{I} = \text{diag}(A, B, C)$ is

$$L = \frac{1}{2}(\mathbf{I}\omega, \omega),$$  \hspace{1cm} (2.4)

where $\omega$ is a vector of angular velocity. In capacity of quasimomenta we take components of the angular momenta

$$\mathbf{M} = \partial L / \partial \omega.$$  \hspace{1cm} (2.5)

The Hamiltonian $\mathcal{H}$ is defined by the Lagrange transformation

$$\mathcal{H} = \left( \omega, \frac{\partial L}{\partial \omega} \right) - L \big|_{\omega \rightarrow \mathbf{M}}.$$  \hspace{1cm} (2.6)

Using (2.5), (2.6) and kinematic Euler relations (Koshlyakov, 1985):

$$\omega_1 = \dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi,$$

$$\omega_2 = \dot{\psi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi,$$

$$\omega_3 = \dot{\psi} \cos \theta + \dot{\varphi},$$

where $\theta, \varphi, \psi$ are the Euler angles, we obtain the formulae:

$$M_1 = \frac{\sin \varphi}{\sin \theta}(p_\psi - p_\varphi \cos \theta) + p_\theta \cos \varphi,$$

$$M_2 = \frac{\cos \varphi}{\sin \theta}(p_\psi - p_\varphi \cos \theta) - p_\theta \sin \varphi,$$

$$M_3 = p_\varphi.$$  \hspace{1cm} (2.8)

Taking under consideration canonical brackets between generalized coordinates —Euler angles and corresponding to them canonical momenta $p_\theta, p_\varphi, p_\psi$, it is possible to obtain the algebra

$$\{M_i, M_j\} = -\epsilon_{ijk}M_k,$$  \hspace{1cm} (2.9)

$$\{M_i, \lambda_j\} = -\frac{1}{2}(\epsilon_{ijk}\lambda_k + \delta_{ij}\lambda_0),$$  \hspace{1cm} (2.10)

where $\epsilon_{ijk}$ is the permutation symbol.

Let us obtain by the developed method canonical coordinates of the six-dimensional symplectic leaf of the Poisson brackets with a seven-dimensional Lie algebra $\mathfrak{l}(7) = \mathfrak{so}(3) \oplus \mathfrak{R}^4$. Also, as in the first section, we solve step by step the following systems of partial differential equations corresponding to the algebras.

1. $\{L, \lambda_0\} = \frac{1}{2}\lambda_3$,  \hspace{1cm} $\{L, \lambda_1\} = -\frac{1}{2}\lambda_2,$

$$\{L, \lambda_2\} = \frac{1}{2}\lambda_1, \hspace{1cm} \{L, \lambda_3\} = -\frac{1}{2}\lambda_0;$$
2. \{H, \lambda_0\} = \frac{1}{2} \lambda_3, \quad \{H, \lambda_1\} = \frac{1}{2} \lambda_2, \quad \{H, \lambda_2\} = \frac{1}{2} \lambda_1, \quad \{H, \lambda_3\} = \frac{1}{2} \lambda_0; \quad (2.11)

\{l, \lambda_0\} = \frac{\lambda_1 \cos l - \lambda_2 \sin l}{2\sqrt{G^2 - L^2}}, \quad \{l, \lambda_1\} = \frac{\lambda_3 \cos l + \lambda_0 \sin l}{2\sqrt{G^2 - L^2}}, \quad \{l, \lambda_2\} = \frac{\lambda_0 \sin l - \lambda_3 \cos l}{2\sqrt{G^2 - L^2}}, \quad \{l, \lambda_3\} = \frac{\lambda_1 \sin l + \lambda_2 \cos l}{2\sqrt{G^2 - L^2}}; \quad (2.12)

4. \{G, \lambda_0\} = \frac{\sqrt{G^2 - L^2}}{2G} (\lambda_1 \sin l + \lambda_2 \cos l) + \frac{L}{2G} \lambda_3,
\{G, \lambda_1\} = \frac{\sqrt{G^2 - L^2}}{2G} (-\lambda_0 \sin l + \lambda_3 \cos l) - \frac{L}{2G} \lambda_2, \quad (2.13)
\{G, \lambda_2\} = -\frac{\sqrt{G^2 - L^2}}{2G} (\lambda_0 \cos l + \lambda_3 \sin l) + \frac{L}{2G} \lambda_1,
\{G, \lambda_3\} = \frac{\sqrt{G^2 - L^2}}{2G} (\lambda_2 \sin l - \lambda_1 \cos l) - \frac{L}{2G} \lambda_0.

Using the condition of the norm (2.1) and relations (2.2) we get the following solutions:

\lambda_0 = \frac{1}{\sqrt{2}} (\sin(g/2) \sin(y_+) \cos(x_-) + \sin(g/2) \cos(y_+) \cos(x_-) + 
\cos(g/2) \sin(y_+) \sin(x_+) - \cos(g/2) \cos(y_+) \sin(x_+)),
\lambda_1 = \frac{1}{\sqrt{2}} (\sin(g/2) \cos(y_-) \sin(x_-) - \sin(g/2) \sin(y_-) \sin(x_-) - 
\cos(g/2) \sin(y_-) \cos(x_-) - \cos(g/2) \cos(y_-) \cos(x_-)),
\lambda_2 = \frac{1}{\sqrt{2}} (-\sin(g/2) \sin(y_-) \sin(x_-) - \sin(g/2) \cos(y_-) \sin(x_-) + 
\cos(g/2) \sin(y_-) \cos(x_-) - \cos(g/2) \cos(y_-) \cos(x_-)),
\lambda_3 = \frac{1}{\sqrt{2}} (\sin(g/2) \sin(y_+) \cos(x_-) - \sin(g/2) \cos(y_+) \cos(x_-) - 
\cos(g/2) \sin(y_+) \sin(x_+) - \cos(g/2) \cos(y_+) \sin(x_+)). \quad (2.14)

In the formulae (2.14) there are introduced the angles \(\zeta, \tau:\)
\[\zeta = \arcsin \frac{H}{G}, \quad \tau = \arcsin \frac{L}{G}\]
and the combinations:
\[x_+ \equiv \frac{1}{2}(\zeta + \tau), \quad x_- \equiv \frac{1}{2}(\zeta - \tau), \quad y_+ \equiv \frac{1}{2}(l + h), \quad y_- \equiv \frac{1}{2}(l - h)\]

The obtained formulae (2.14) can be used for applications of methods of the theory of perturbation to the problem of the rigid body rotations in superposition of some potential strength fields.
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