Exact Solutions of Homogeneous Partial Differential Equation by A New Adomian Decomposition Method

Bingquan Zhang, Jun Lu

Department of Mathematics
Zhijiang College of Zhejiang University of Technology, Hanghou City, P.R.China
bssdlk@hotmail.com, lujun@163.com

Abstract
Solving homogeneous partial differential equation (PDE) by using the traditional Adomian decomposition method (ADM), we are not able to obtain the zeroth component in most cases. This paper proposes a new ADM to overcome these perplexities. We illustrate advantages of the new ADM by solving some homogeneous PDEs.

Keywords: homogeneous partial differential equation; standard ADM; modified ADM; A Two-step ADM; A new ADM.

1. Introduction
The ADM has been proved has many advantages in solving algebraic, differential, differential-delay, and partial differential equations [1-4]. In recent years, Wazwaz e.t., see [5-9], improved the ADM and expanded fields of its application. Further, Luo [10] presented “A two-step Adomian decomposition method (TSADM)” based on a modification of the Adomian decomposition method (MADM) proposed by Wazwaz [6]. Zhang, Wu and Luo [11] applied the TSADM to solve some evolution models. Especially, Ray [12] used the TSADM to solve fractional diffusion equations by showing its efficiency.

However, as demonstrated in the following, all the decomposition methods of the ADM mentioned above have perplexities about how to determine the zeroth component $U_0$ in case of dealing with the homogeneous partial differential equations. In this paper, we discuss the standard ADM, Wazwaz’s MADM and Luo’s TSADM, and propose a new ADM. Several homogeneous partial differential equations are solved to illustrate the advantages of the new ADM.

2. ADM, MADM and TSADM
Because the standard ADM is well known and all the details can be seen in [3] and [4], we will focus on main features of the method here. The most important equations are written below:
\( Lu + Ru + Nu = g \quad (1) \)
\( u = \Phi + L^{-1}g - L^{-1}(Ru + Nu) \), \quad (2)

where \( L^{-1} \) is the inverse operator of the highest order differential operator \( L \), which is assumed to be easily invertible, and the function \( \Phi \) represents the terms arising from using the given conditions. In addition, means of other characters that are not explained in this paper seen from reference articles.

In the standard ADM, the iteration relations are following:
\( u_0 = \Phi + L^{-1}g \), \quad (3)
\( u_1 = -L^{-1}(Ru_0) - L^{-1}(A_0) \), \quad (4)
\( u_{k+1} = -L^{-1}(Ru_k) - L^{-1}(A_k), k \geq 0 \). \quad (5)

The exact solution of the equation can be expressed in series:
\( u = \sum_{n=0}^{\infty} u_n \). \quad (6)

The modified decomposition method: In [7], Wazwaz assume that \( f = \Phi + L^{-1}g \) can be divided into two parts: \( f_1 \) and \( f_2 \). Then, the first part \( f_1 \) is assigned to the zeroth component \( u_0 \). The iteration relations are written as [7]:
\( u_0 = f_1 \), \quad (7)
\( u_1 = f_2 - L^{-1}(Ru_0) - L^{-1}(A_0) \), \quad (8)
\( u_{k+1} = -L^{-1}(Ru_k) - L^{-1}(A_k), k \geq 1 \). \quad (9)

The two-step Adomian decomposition method: In [10], Luo improved the MADM to a new method, which named two-step Adomian decomposition method. The main ideas of TSADM are written below [10]:

**Step1** In equation (1), using the given conditions, we obtain
\( \varphi = \Phi + L^{-1}g \), \quad (10)
where the function \( \Phi \) represents the terms arising from using the given conditions, all are assumed to be prescribed. We set
\( \varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_m \), \quad (11)
where \( \varphi_0, \varphi_1, \cdots, \varphi_m \) are the terms arising from integrating the source term \( g \) and from using the given conditions. Based on this, we define
\( u_0 = \varphi_s + \cdots + \varphi_{s+k} \), \quad (12)
where \( k = 0, 1, \ldots, m \), \( s = 0, 1, \ldots, m - k \). Then we verify that \( u_0 \) satisfies the original equation (1) and the given conditions by substitution. Once the exact solution is obtained we finish. Otherwise, we go to following step two.

**Step2** We set \( u_0 = \varphi \) and continue with the standard Adomian recursive relation (5).

3. A new ADM

From the articles in reference [3, 4, 6, 10, 11, 12], we can see that all the methods about Adomian decomposition method presented above have their convenience in their special applications.

However, after analyzing these methods, and considering the conditions given by the equation, we find that there are many perplexities on how to determine the zeroth component \( u_0 \) in some cases. Moreover,
the $u_0$ cannot be obtained when the function $g=0$ in the original equation (1), for example:

**Case 1** Using the given conditions, we obtain that $\varphi = \varphi_0 = \varphi_1 = \cdots = \varphi_m = 0$;

**Case 2** The given conditions are not enough, that is, at least one of the terms in $\varphi$ cannot be obtained.

### A New Adomian Decomposition Method

Under these cases mentioned above, we can abandon $\varphi$ in all the iteration steps in equations (3), (7), (8) and (12). We demonstrate the new method in the following three steps:

**Step (1)** We select $u_0$ via two ways depending the two cases respectively:

**First way:** In the case 1, we apply another inverse operator to both sides of the equation, and select $u_0 = c_i$, which is evolved from the given conditions, and the function $c_i$ satisfy the most of other given conditions;

**Second way:** In the case 2, for instance as $L^{-1} = L_{xx}^{-1}$, then $\varphi = h + xφ = \varphi_0 + \cdots + \varphi_m$. We Select $u_0 = h$, or $u_0 = xφ$, or $u_0 = \varphi_k + \cdots + \varphi_{k+s}$, depending on which can be calculated from using the given conditions. And also the $u_0$ must satisfy most of the given conditions.

**Step (2)** Substitute the zeroth component to the iteration relations to calculate the other component of the series of the solution. For convenient, all the equations are written below.

\[
Lu + Ru + Nu = 0, \quad (a)
\]
\[
u = \varphi - L^{-1}(Ru + Nu), \quad (b)
\]
\[
u_0 = c_i \quad (or \ h \ ectl.), \quad (c)
\]
\[
u_1 = -L^{-1}(Ru_0) - L^{-1}(A_0), \quad (d)
\]
\[
u_{k+1} = -L^{-1}(Ru_k) - L^{-1}(A_k), k \geq 0 \quad (e)
\]

where we abandon the terms

\[
\varphi = f_1 + f_2 = \varphi_0 + \varphi_1 + \cdots + \varphi_m
\]

in the iteration (e). The solution $\nu$ is given as a series form:

\[
\nu = \sum_{n=0}^{\infty} u_n.
\]

**Step (3)** Substitute the solution $\nu$ to the original equation (a). If this $\nu$ is not the exact solution, go step 1 to re-select $u_0$.

### 4. Examples

In this section, we apply the new method to some examples to illustrate the advantages of the method.

**Example 1.** Consider homogeneous PDE

\[
u_{tt} - \nu_{xx} + \left(\frac{\pi^2}{4} + \frac{2}{x^2}\right)u = 0, \quad x > 0, \ 0 < t < 1
\]

\[
s.t. \quad \nu(x,0) = x^2, \ \nu(x,1) = 0, \quad u(0,t) = 0, \ \nu_x(0,t) = 0.
\]

Obviously, if apply the inverse operator $L_{xx}^{-1}$ to both sides of the equation, then we can obtain
\[ \varphi = \varphi_0 = \varphi_1 = \cdots = \varphi_m = 0. \]

It cannot be solved by Adomian decomposition method, because the zeroth component \( u_0 \) is not given. Therefore, we apply the inverse operator \( \mathbb{L}_\mu^{-1} \) to both sides of Eq. (13) and obtain

\[ u = h(x) + t \phi(x) + \mathbb{L}_\mu^{-1} \left( \frac{\pi^2}{4} + \frac{2}{x^2} \right) u \]

Taking notice of the conditions, we can obtain that the \( h(x) = x^2 \), but the function \( \varphi(x) \) is not given. We select \( u_0 = x^2 \), which satisfies most of the conditions, although does not satisfy the original equation. Substituting \( u_0 \) to the iteration relation (e), We obtain:

\[ u_1(x,t) = \mathbb{L}_\mu^{-1} L_{xx} u_0 - \mathbb{L}_\mu^{-1} \left( \frac{\pi^2}{4} + \frac{2}{x^2} \right) u_0 \]

\[ u_2(x,t) = \mathbb{L}_\mu^{-1} L_{xx} u_1 - \mathbb{L}_\mu^{-1} \left( \frac{\pi^2}{4} + \frac{2}{x^2} \right) u_1 \]

\[ u_3(x,t) = -\frac{\pi^2}{2} x^2 t^2, \quad u_4(x,t) = \frac{\pi^4}{2} x^2 t^4, \quad u_5(x,t) = \frac{\pi^6}{2} x^2 t^6. \]

In the same manner, the rest components of the series of solution can be easily obtained. Then the solution in a series form is

\[ u(x,t) = x^2 \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right). \]

Therefore, the solution in a closed form is

\[ u(x,t) = x^2 \cos \frac{\pi}{2} t, \]

which is the exact solution of the equation (13).

Example 2. Consider the homogeneous PDE, (this equation is evolved from Eq.(3.1) of [13])

\[ u_{tt} - u_{xx} + 2u = 0, \quad x > 0, \quad 0 < t < \frac{\pi}{2} \]

\[ \text{s.t.} \quad u_x(0,t) = \sin t, \quad u_t(x,0) = e^x. \]

Applying the inverse operator \( \mathbb{L}_\mu^{-1} \) to both sides of Eq. (14) yields

\[ u = h(x) + t \phi(x) + \mathbb{L}_\mu^{-1} \left( L_{xx} u - 2u \right). \]

Taking notice of the condition, we can see the \( \varphi(x) = e^x \) and the \( h(x) \) are not given. Considering the boundary conditions above, we can presume \( u_0 = e^x t \). Obviously, the \( u_0 \) satisfies the boundary conditions of Eq. (14), with approximation \( \sin x \approx x \). Substituting the \( u_0 \) into the iteration relation
(e), we obtain:

\[ u_1(x,t) = L_{tt}^{-1}(L_{xx}u_0 - 2u_0) = -e^x \frac{t^3}{3!}, \]

\[ u_2(x,t) = L_{tt}^{-1}(L_{xx}u_1 - 2u_1) = e^x \frac{t^5}{5!}, \]

\[ u_3(x,t) = L_{tt}^{-1}(L_{xx}u_2 - 2u_2) = -e^x \frac{t^7}{7!}. \]

It is obvious that the rest components of the series solution can be easily obtained. Then the exact solution in series form is

\[ u(x,t) = e^x \left(t - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right). \]

Further, we obtain

\[ u(x,t) = e^x \sin t. \]

It is the exact solution of equation (14).

**Example 3.** Consider the homogeneous PDE

\[ u_t + \frac{1}{2} x^2 u_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \quad (15) \]

Subject to the boundary conditions

\[ u(0,t) = 0, \quad u_x(0,t) = 0, \]

\[ u(1,t) = \sin t, \quad u_t(x,t) = x^2. \]

As before applying the inverse operator \( L_{xx}^{-1} \), then we can obtain

\[ \varphi = \varphi_0 = \varphi_1 = \cdots = \varphi_m = 0. \]

So, we apply the inverse operator \( L_{tt}^{-1} \), then we obtain

\[ u = h(x) + tx^2 - \frac{1}{2} x^2 L_{tt}^{-1} u_{xx} \]

where \( h(x) \) is not given. We select \( u_0 = x^2 t \), which satisfies most of the given conditions.

Proceeding as before, the beginning components of the solution series are

\[ u_1 = -\frac{1}{2} x^2 L_{tt}^{-1}(L_{xx}u_0) = -\frac{x^2 t^3}{3!}, \]

\[ u_2 = -\frac{1}{2} x^2 L_{tt}^{-1}(L_{xx}u_1) = \frac{x^2 t^5}{5!}, \]

\[ u_3 = -\frac{1}{2} x^2 L_{tt}^{-1}(L_{xx}u_2) = -\frac{x^2 t^7}{7!}. \]

We can easily obtain the rest components of the solution series, and the solution in a series form is
Further, we obtain the exact solution

\[ u = x^2 \sin t. \]

**Example 4.** Consider the two-dimensional heat-like IBVP[10]

\[ u_t = \frac{1}{2} \left( y^2 u_{xx} + x^2 u_{yy} \right), \]

\[ 0 < x, y < 1, t > 0, \]

subject to the Neumann boundary conditions

\[ u_x (0, y, t) = 0, \quad u_x (1, y, t) = 2 \sinh t, \]
\[ u_y (x, 0, t) = 0, \quad u_y (x, 1, t) = 2 \cosh t. \]

Here we ignore the initial condition \( u(x, y, 0) \), so the function \( \varphi \) is not given. Considering all the given conditions, we can presume zeroth component \( u_0 = x^2 t \), which is evolved from condition \( u_x (1, y, t) \). Proceeding as before, we obtain

\[ u_0 = x^2 t. \]

\[ u_1 = \frac{1}{2} \left( y^2 L_{xx} u_0 + x^2 L_{yy} u_0 \right) = y^2 \frac{t^2}{2!}, \]
\[ u_2 = \frac{1}{2} \left( y^2 L_{xx} u_1 + x^2 L_{yy} u_1 \right) = x^2 \frac{t^3}{3!}, \]
\[ u_3 = \frac{1}{2} \left( y^2 L_{xx} u_2 + x^2 L_{yy} u_2 \right) = y^2 \frac{t^4}{4!}. \]

The rest component of the series are easily obtained. Hence, the solution in a series form is

\[ u = x^2 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \]
\[ + y^2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}, \]

and therefore, the solution in closed form is

\[ u = x^2 \sinh t + y^2 (\cosh t - 1). \]

However, this \( u \) is not the exact solution of the original equation (16). Go back to the beginning and consider all the process, we reselect the zeroth component \( u_0 = y^2 \). Then, we obtain the solution in a series form

\[ u = x^2 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \]
\[ + y^2 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}, \]

and therefore, the solution in closed form is

\[ u = x^2 \sinh t + y^2 \cosh t. \]

It is the exact solution of the original equation (16).
5. Conclusion

To solve homogeneous partial differential equation by using traditional Adomian decomposition method, we find that, in most cases, it is not possible to select the zeroth component \( u_0 \). The main reasons are \( g = 0 \) in the original equation (1) and the given conditions are not enough to match. In this paper, based on the features of the standard ADM, the modified ADM and the two-step ADM, we propose a new ADM to overcome the difficulties.

By employing the new ADM to solve some homogeneous PDEs, we demonstrate that the new ADM is convenient and efficient. While we attempt to solve a class of homogeneous PDEs here, further effort can be conducted to improve the ADM for other applications. For example, how to solve homogeneous nonlinear equations, and how to select the exact zeroth component \( u_0 \) so as to reduce the repeat work.

Acknowledgment

This work has been supported by Science Foundation of Education of Zhejiang Province Foundation (No. 2007314).

Reference

[1] G. Adomian, “Stochastic Systems”, Academic Press, London, 1983.
[2] G. Adomian, “A new approach to the heat equation – an application of the decomposition method”, J. Math. Anal. Appl. 113(1) (1986) 202-209.
[3] G. Adomian, “A review of the decomposition method in applied mathematics”, J. Math. Anal. Appl. 135(1988) 501-544.
[4] G. Adomian, “A review of the decomposition method and some recent results for nonlinear equation”, Math. Comput. Model, 13(7) (1990) 17-43.
[5] A.M. Wazwaz, “A new method for solving singular initial value problems in the second-order ordinary differential equations”, Appl. Math. Comput. 128 (2002) 45-57.
[6] A.M. Wazwaz, “A reliable modification of Adomian decomposition method”, Appl. Math. Comput. 102 (1999) 77-86.
[7] A.M. Wazwaz, “The modified decomposition method and Padé approximants for solving Thomas-Fermi equation”, Appl. Math. Comput. 105 (1999) 11-19.
[8] A.M. Wazwaz, S.M. El-Sayed, “A new modification of the Adomian decomposition method for linear and nonlinear operators”, Appl. Math. Comput.122 (2001) 393-405.
[9] A.M. Wazwazm, “A new algorithm for calculating adomian polynomials for nonlinear operators”, Appl. Math. Comput. 111 (2000) 53-69.
[10] X.G. Luo, “A two-step Adomian decomposition method”, Appl. Math. Comput, 170 (2005) 570-583.
[11] B Q. Zhang, Q B. Wu, X G. Luo, “Experimentation with two-step Adomian decomposition method to solve evolution models”, Appl. Math. Comput. 175 (2006) 1495-1502.
[12] Santanu Saha Ray, “Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method”, Commu. Non. Sci. Num. Simu. 14 (2009) 1295-1306
[13] K.C. Basak, P.C. Ray, R.K. Bera, “Solution of non-linear Klein-Gordon equation with a quadratic non-linear term by Adomian decomposition method”, Commu. Non. Sci and Num. Simu. 14(2009) 718-723.