A new approach on the stability analysis in ELKO cosmology

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Abstract

In this work it has been developed a new approach to study the stability of a system composed by an ELKO field interacting with dark matter, which could give some contribution in order to alleviate the cosmic coincidence problem. It is assumed that the potential that characterizes the ELKO field is not specified, but it is related to a constant parameter \( \delta \). The strength of the interaction between the matter and the ELKO field is characterized by a constant parameter \( \beta \) and it is also assumed that both the ELKO field as the matter energy density are related to their pressures by equations of state parameters \( \omega_{\phi} \) and \( \omega_{m} \), respectively. The system of equations is analysed by a dynamical system approach. It was found out the conditions of stability between the parameters \( \delta \) and \( \beta \) in order to have stable fixed points for the system for different values of the equation of state parameters \( \omega_{\phi} \) and \( \omega_{m} \), and the results are presented in form of tables. The possibility of decay of Elko field into dark matter or vice versa can be read directly from the tables, since the parameters \( \delta \) and \( \beta \) satisfy some inequalities. This opens the possibility to constrain the potential in order to have a stable system for different interactions terms between the Elko field and dark matter. The cosmic coincidence problem can be alleviated for some specific relations between the parameters of the model.

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I. INTRODUCTION

As so many segments in science, there are still open questions in modern cosmology to be answered. One of the greatest puzzles concerns the composition of the universe, which can be roughly divided into three components, namely the ordinary baryonic matter ($\approx 5\%$), the Dark Matter (DM) ($\approx 25\%$) and the Dark Energy (DE) ($\approx 70\%$), according to the most accepted models [1–3]. Until now, we only know the former component, but there are many attempts to detect DM particles, since it behaves exactly like the usual baryonic matter, although not interacting electromagnetically [4, 5]. DE is something even more mysterious, since their behavior is gravitationally repulsive [6, 7]. From a theoretical point of view it is very common to interpret these entities as being some kind of field (scalar fields [8, 9] or spinor fields [10], for example). Other models also consider the possibility of interaction between DM and DE [11–21], which could alleviate the coincidence problem for instance.

Besides the usual scalar fields, some recent works have shown that there is some class of Non-Standard Spinors with interesting properties that could be useful in order to describe both DM as DE. One of these spinors is called ELKO, from German *Eigenspinoren des Ladungskonjugationsoperators*, which has the property to be an eigenspinor of change conjugation and parity, possessing non-locality [22–35]. It satisfies $(CPT)^2 = -I$ and also has some other properties, as a spin one half and a mass dimension 1, which makes it a good candidate to a particle with small chances of interaction with Standard Model particles, exactly as desired for the DM particles and maybe also to DE. The scalar part of a ELKO spinor plays the role of a scalar field with a much richer structure, hence the recent interest in this kind of field.

Due to the complexity in dealing with the exact equations involving the ELKO field into cosmology, dynamical systems analyses have been developed in order to deal with ELKO field as a possible candidate to an inflaton field, playing the role of DM or DE. An interesting aspect concerning the ELKO field dynamics is that the choice of variables is an important question when one is looking for stable points of the dynamical system. Such search for available variables depends in general on the structure of some physical quantities, like the energy density, pressure and Friedmann constraints. In some recent works [36–38], different choices of variables for interacting systems concerning the ELKO field have shown that there are no stable points in order to explain the cosmic coincidence problem. In [39] it was proposed a new method of analysis based on a constant parameter that leads to stable points under some conditions. In the present work we extend the method of [39] for a new set of dynamics variables and stable fixed points have also been found for this system. This opens the possibility to alleviate the cosmological coincidence problem considering the ELKO field interacting with dark matter.

The new set of dynamical variables for the ELKO field used in this work is independent of the choice of potential. Besides that it is assumed that the pressure and energy density of the ELKO field satisfies an equation of state of the form $p_\phi = \omega_\phi \rho_\phi$, where $p_\phi$ and $\rho_\phi$ are the
ELKO field pressure and energy density, respectively. The dark matter content is described by an energy density $\rho_m$ that satisfies an equation of state of the type $p_m = \omega_m \rho_m$. It is also assumed an interaction between ELKO field and dark matter. The conditions of stability depending on the type of the thermodynamic equation of state parameter (radiation, dust, vacuum or ultra-relativistic matter) associated to them have been studied.

The paper is organized as follows. In Section II we introduce the basic ELKO field equations related to our cosmological applications in a flat FRW background, namely the pressure and energy density expressions, as well as the Friedmann equations and the conservation equations including an interaction between dark matter and ELKO field. We also define the variables concerning the dynamical system equations to be analyzed. Section III contains the main results. We study the stability of the dynamical system by imposing the restriction that the potential of the ELKO field is related to a constant parameter $\delta$. This analysis allows us to study under which conditions satisfied by the equation of state parameters of DM and ELKO field presents stability. For each interaction we present the results in form of tables, specifying the equation of state parameter and conditions of stability to be satisfied. For DM and ELKO field we restrict the equation of state parameters to vacuum, dust, radiation and ultra-relativistic matter. In Section IV we finish with some concluding remarks.

II. ELKO COSMOLOGY AS A DYNAMICAL SYSTEM

The ELKO Lagrangian density can be written as

$$\mathcal{L} = \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} (\nabla_\mu \tilde{\lambda} \nabla_\nu \lambda) - V(\lambda, \bar{\lambda}) \right].$$

(1)

The equation of motion follows from a principle of least action for $\mathcal{L}$.

As has been done in recent works [29, 31, 34, 36], we restrict the ELKO spinor field to the form $\lambda \equiv \phi(t) \xi$ and $\bar{\lambda} \equiv \phi(t) \bar{\xi}$, where $\xi$ and $\bar{\xi}$ are constant spinors. In [35] it has been presented exact solutions to ELKO spinor in spatially flat Friedmann-Robertson-Walker expanding space times, and it has been shown that such factorisation of the time component of the ELKO field is possible for some types of scale factors.

Due to the homogeneity of the field ($\partial_i \phi = 0$), the equation of motion that follows from (1) is substantially simplified to,

$$\ddot{\phi} + 3H \dot{\phi} - \frac{3}{4} H^2 \phi + V_{,\phi} = 0,$$

(2)

where $H = \dot{a}/a$ and $V_{,\phi} \equiv dV/d\phi$. The pressure and energy density of spinor dark energy are, according to [31], respectively given by

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) - \frac{3}{8} H^2 \phi^2 - \frac{1}{4} \dot{\phi}^2 - \frac{1}{2} H \dot{\phi} \phi,$$

(3)
\[ \rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{3}{8} H^2 \phi^2. \] (4)

It is assumed that the Universe is filled with only two components, namely a matter energy density \( \rho_m \) representing the DM and a ELKO energy density \( \rho_{\phi} \), which could represent the DE for the late time acceleration or the inflaton field for the inflationary epoch. The Friedmann equations in a spatially flat background are given by:

\[ H^2 = \frac{\kappa^2}{3} (\rho_m + \rho_{\phi}), \] (5)
\[ \dot{H} = -\frac{\kappa^2}{2} (\rho_m + p_m + \rho_{\phi} + p_{\phi}), \] (6)

where \( \kappa^2 \equiv 8\pi G \) and we have assumed \( c = 1 \).

We will assume that the DM and ELKO field satisfy equations of state of the form \( p_m = \omega_m \rho_m \) and \( p_{\phi} = \omega_{\phi} \rho_{\phi} \), respectively.

The continuity equations for DM and ELKO field are:

\[ \dot{\rho}_m + 3H(1 + \omega_m)\rho_m = Q, \] (7)
\[ \dot{\rho}_{\phi} + 3H(1 + \omega_{\phi})\rho_{\phi} = -Q, \] (8)

where \( Q \) stands for a possible interaction term between the DM and the ELKO field. If \( Q = 0 \) there is no interaction and the two components evolve separately. If \( Q > 0 \) there is the decay of ELKO field into DM, an interesting scenery at the inflation, and if \( Q < 0 \) we have DM decaying into Elko field (or DE).

We define new variables:

\[ x = \frac{\kappa \dot{\phi}}{\sqrt{6}H}, \quad y = \frac{\kappa \sqrt{V}}{\sqrt{3}H}, \quad v = \frac{\kappa \sqrt{\rho_m}}{\sqrt{3}H}, \] (9)

where we have redefined the potential as \( \tilde{V} = V + \frac{3}{8} H^2 \phi^2 \). The Friedmann equation (5) can be written as a constraint equation

\[ x^2 + y^2 + v^2 = 1, \] (10)

or in terms of the densities parameters, \( \Omega_{\phi} + \Omega_m = 1 \), where

\[ \Omega_{\phi} = \frac{\kappa^2 \rho_{\phi}}{3H^2} = x^2 + y^2, \quad \Omega_m = \frac{\kappa^2 \rho_m}{3H^2} = v^2. \] (11)

The equations (6), (7) and (8) can be written as a dynamical system of the form:

\[ x' = \frac{3}{2} (\omega_m - \omega_{\phi}) v^2 x - \left[ \frac{3}{2} (1 + \omega_{\phi}) + \frac{\lambda}{2H} \right] \frac{y^2}{x} - \frac{Q_1}{x}, \] (12)
\[ v' = \frac{3}{2} (\omega_{\phi} - \omega_m) (1 - v^2) v + \frac{Q_1}{v}, \] (13)
\[ y' = \left( \frac{3}{2} (1 + \omega_{\phi}) + \frac{3}{2} (\omega_m - \omega_{\phi}) v^2 + \frac{\lambda}{2H} \right) y \] (14)

where \( \lambda = \dot{\tilde{V}}/\tilde{V}, \) \( Q_1 = \frac{\kappa^2 \dot{Q}}{\kappa^2 \dot{H}} \) and \( ' \) stands for the derivative with respect to \( N \equiv \ln a \), such that \( f' = \dot{f}/H \) for any function \( f \).
III. STABILITY ANALYSIS

The stability analysis of the above set of dynamical equations consists to find out fixed points $\bar{x}$, $\bar{v}$ and $\bar{y}$ that makes $x'$, $v'$ and $y'$ equal to zero. In the last section we have defined a three dimensional system according to our variable choice. However, due to the Friedmann constraint (10) the system can be reduced to a two dimensional one.

Before we proceed, let us examine carefully the dynamical system (12)-(14). We see that, in addition to the dynamical variables $x$, $v$ and $y$ we also have the factor $\frac{\lambda}{2H} = \frac{\dot{\tilde{V}}}{2\tilde{V}}$. The presence of the derivative shows that such term is also a dynamical variable, which should also be taken into account. However $\tilde{V}$ depends on the potential but the potential is not specified in our analysis, thus we can not deal with this new variable. In order to avoid this problem we set the additional supposition related to the potential, $-\frac{\lambda}{2H} \equiv \delta$, where $\delta$ is a constant parameter. Such imposition just reflects our ignorance on the potential $V(\phi)$.

Now it is easy to see that the resulting dynamical system, for a given interaction $Q_1(x,v,y)$, is written in terms of the dynamical variables $x$, $v$, $y$ and the constants $\omega_m$, $\omega_\phi$ and $\delta$. In order to analyse the stability of this system around fixed points $\bar{x}$, $\bar{v}$ and $\bar{y}$ we must to study the system satisfying $x' = 0$, $v' = 0$ and $y' = 0$. Notice that the equation (14) is independent of the interaction $Q_1$, thus the condition $y' = 0$ can be achieved only if $\bar{y} = 0$ or $-\frac{\lambda}{2H} = \delta = \frac{3}{2} (1 + \omega_\phi) + \frac{3}{2} (\omega_m - \omega_\phi) \bar{v}^2$. The first condition can be satisfied only if $\dot{V} = 0$ (see (9)), but from the definition of $\lambda$ this leads to a divergent $\lambda$. So, we restrict ourselves to the second condition, namely

$$\delta = \frac{3}{2} (1 + \omega_\phi) + \frac{3}{2} (\omega_m - \omega_\phi) \bar{v}^2,$$

(15)

where $y' = 0$ even for $\bar{y} \neq 0$. By imposing the above condition on the dynamical system (12)-(14) we are left with a $2 \times 2$ system, since that $y' = 0$ is always satisfied:

$$x' = \left[ \frac{3}{2} (1 + \omega_\phi) - \delta + \frac{3}{2} (\omega_m - \omega_\phi) \bar{v}^2 \right] x + \left[ \delta - \frac{3}{2} (1 + \omega_\phi) \right] \frac{1 - v^2}{x} - \frac{Q_1}{x},$$

(16)

$$v' = \frac{3}{2} (\omega_\phi - \omega_m) (1 - v^2) v + \frac{Q_1}{v},$$

(17)

where we have also used the Friedmann constraint (10).

In order to study such dynamical system it is worth to define its linearized matrix, with which one can determine the stability of a fixed point by just analysing its determinant and trace. Such mechanism is ensured by Hartmann-Grobman theorem [41]. Thus, in the neighborhood of the fixed points we take infinitesimal displacements of variables from its fixed points, $x \rightarrow \bar{x} + \delta x$ and $y \rightarrow \bar{v} + \delta v$, so that

$$\left( \begin{array}{c} \delta x' \\ \delta v' \end{array} \right) = M \left( \begin{array}{c} \delta x \\ \delta v \end{array} \right),$$

(18)
where $M$ is given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$  \hspace{1cm} (19)$$

and

$$a = -\left[ \delta - \frac{3}{2}(1 + \omega) \right] \frac{1 - \bar{v}^2}{\bar{x}^2} + \frac{Q_1}{\bar{x}} \frac{\partial Q_1}{\partial \bar{x}},$$

$$b = \frac{\partial x'}{\partial v} = 3(\omega_m - \omega) \bar{x} \bar{v} + 3(1 + \omega) - 2\delta \frac{\bar{v}}{\bar{x}} - \frac{1}{\bar{x}} \frac{\partial Q_1}{\partial \bar{v}},$$  \hspace{1cm} (20)

$$c = \frac{\partial v'}{\partial x} = \frac{1}{\bar{v}} \frac{\partial Q_1}{\partial \bar{v}},$$  \hspace{1cm} (21)

$$d = \frac{\partial v'}{\partial v} = \frac{3}{2}(\omega - \omega_m)(1 - 3\bar{v}^2) - \frac{Q_1}{\bar{v}^2} + \frac{1}{\bar{v}} \frac{\partial Q_1}{\partial \bar{v}}.$$  \hspace{1cm} (22)

All variables carry a bar over them to show that the matrix $M$ is taken at the stable points that solve the system.

There is a simple way to know if the system described by the matrix $M$ is stable or not. It depends on the values of the determinant ($\Delta$) and also on the trace ($\tau$) of such matrix. When $\Delta > 0$ both eigenvalues have the same sign, and if they are positive the solution tends to increase with the time evolution. That means that the solutions diverges from the fixed points and consequently this point is considered an unstable one. On the other hand when both eigenvalues are negative the solution approximates to zero and the fixed point is stable.

In order to know what kind of fixed point we are dealing with, it is necessary to check the value of the trace of matrix $M$. When $\tau > 0$, it means that both eigenvalues are bigger than zero, describing unstable points. However, when $\tau < 0$ we have that they are negative and the point is stable. For the case where $\Delta < 0$ we have that both eigenvalues have opposite signs and then this fixed point is in fact a saddle point. The last possible case corresponds to take $\Delta = 0$. In this case we have that at least one of the eigenvalues is zero and consequently nothing can be said about the stability of system.

Let us return to the dynamical system. Together with the equations (16) and (17), the supposition $-\frac{\lambda}{2\hbar} = \delta$ leads to the new constraint for the fixed point $\bar{v}$, according to (15):

$$\bar{v} = \sqrt{\frac{2\delta - 3(1 + \omega)}{3(\omega_m - \omega)}}.$$  \hspace{1cm} (24)

But it is easy to see that such constraint already determines the value of the fixed point $\bar{v}$, since it depends only on the fixed parameters $\omega_m$, $\omega$ and $\delta$. Thus, in order to also satisfy the dynamical equation (17), we have verified that this restriction tells us that $Q_1$, which represents the interaction, could not assume an arbitrary value, since that Eq.(17) would not
be always solved for an arbitrary $Q_1$. In other words, we have found out that when $Q_1$ does not depend on $x$, not all fixed $\bar{v}$ that makes the rhs of Eq.(17) equal to zero also satisfy (24), except for some very specific relations among the parameters $\omega_m$, $\omega_\phi$ and $\delta$. Such restriction on the interaction term $Q_1$ is not so strong, since the variable $x$ is proportional to $\dot{\phi}$, which characterizes the time variation of the field $\phi$, which is reasonable for an interacting theory.

In which follows it will be analysed the stability conditions for different interaction terms between DM and ELKO field. The interaction terms are characterized by a dimensionless coupling constant $\beta$. We will search for stability conditions between the parameters $\delta$ and $\beta$ for different equation of state parameters $\omega_m$ and $\omega_\phi$. Besides stability conditions characterized by negative eigenvalues of the matrix of perturbation $M$, we impose the additional reality condition on the parameters (9), namely we will impose $\bar{x}^2 > 0$, $\bar{v}^2 > 0$ and $\bar{y}^2 > 0$. As particular cases, we will discuss the physical content concerning the present time, characterized by $\dot{\bar{v}}^2 = \Omega_m \simeq 0.315$, $\bar{x}^2 + \bar{y}^2 = \Omega_\phi \simeq 0.685$ and $\omega_m = 0$ according to recent observations based on the $\Lambda$CDM model [40]. We will also analyse the inflationary phase, where we believe there is no matter contribution, corresponding to $\bar{v}^2 \rightarrow 0$.

A. $Q_1 = 0$ and $Q_1 = \beta$

For the case $Q_1 = 0$ there is no interaction between the DM and the Elko field, thus they evolve independently. The fixed points that follows from the analysis of the system (12) to (14) and satisfy $x' = 0$, $v' = 0$ and $y' = 0$ are given just by $\bar{x} = 1$, $\bar{v} = 0$ and $\bar{y} = 0$, which does not represents a scaling solution, in the sense that does not admits a mixture of fluids. Beyond that our model is valid just for $\bar{y} \neq 0$ and $\bar{v}$ given by (24).

For the case $Q_1 = \beta$, a constant interaction term, we have scaling solutions of the form $\bar{x} \neq 0$, $\bar{v} \neq 0$ and $\bar{y} = 0$, which could admit a mixture of the fluids, but the condition $\bar{y} = 0$ shows that the potential part of the Elko field is null according to (9), leading to $\tilde{V} = 0$, and as discussed earlier, this leads to a divergence in the $\lambda$ term, but we have defined it as proportional to the constant $\delta$, so we will discard such kind of fixed point in our analysis. We are interested only in fixed points that satisfies $\bar{x} \neq 0$, $\bar{v} \neq 0$ and $\bar{y} \neq 0$, which are scaling solutions and does not have null potential contribution.

B. $Q_1 = \beta x^2$

Such interaction between DM and ELKO field corresponds to $Q = \beta H \dot{\phi}^2$, where $\beta$ is a dimensionless parameter.

From the analysis of the system of equations (16) and (17), the fixed points are given by $\left[\bar{x} = \sqrt{\frac{(3+3\omega_m-2\delta)(3+3\omega_m-2\delta)}{6\beta(\omega_\phi-\omega_m)}}, \quad \bar{y} = \sqrt{\frac{(2\delta-3\omega_m-3)(2\beta-2\delta+3\omega_\phi+3)}{6\beta(\omega_\phi-\omega_m)}}, \quad \bar{v}\right]$, where $\bar{v}$ is given by (24), which is valid for all interactions.
|                  | ELKO Spinor |                  |                  |                  |
|------------------|-------------|------------------|------------------|------------------|
|                  | Vacuum      | Dust             | Radiation        | Ultrarelativistic|
|                  | ($\omega_\phi = -1$) | ($\omega_\phi = 0$) | ($\omega_\phi = 1/3$) | ($\omega_\phi = 1$) |
| Matter           |             | No stable point  | No stable point  | No stable point   |
|                  | Vacuum      | Dust             | Radiation        | Ultrarelativistic|
|                  | ($\omega_m = 0$) | ($\omega_m = 0$) | ($\omega_m = 1/3$) | ($\omega_m = 1$) |
|                  | $\delta \leq 3/2$ if $\beta \geq 3/2$ or $\delta < \beta$ if $\beta < 3/2$ | No stable point | No stable point | No stable point |
| Radiation        | $\delta \leq 3$ if $\beta \geq 1$ or $\delta < \beta + 3/2$ if $\beta < 1$ | $\delta \leq 3$ if $\beta \geq 3/2$ or $\delta < \beta + 3/2$ if $\beta < 1$ | $\delta \leq 3$ if $\beta \geq 1$ or $\delta < \beta + 3/2$ if $\beta < 1$ | $\delta \leq 3$ if $\beta \geq 1$ or $\delta < \beta + 3/2$ if $\beta < 1$ |
| Ultrarelativistic| $\delta \leq 3$ if $\beta \geq 3$ or $\delta < \beta$ if $\beta < 3$ | $\delta \leq 3$ if $\beta \geq 3/2$ or $\delta < \beta + 3/2$ if $\beta < 1$ | $\delta \leq 3$ if $\beta \geq 1$ or $\delta < \beta + 3/2$ if $\beta < 1$ | $\delta \leq 3$ if $\beta \geq 1$ or $\delta < \beta + 3/2$ if $\beta < 1$ |

**TABLE I:** Stability conditions for some equation of state parameters of DM and ELKO field, corresponding to the interaction $Q_1 = \beta x^2$.

The determinant $\Delta$ and the trace $\tau$ of the matrix of the linearized system \(^{[18]}\) to \(^{[23]}\) are given by $\Delta = 4\delta^2 - 2\delta(2\beta + 6 + 3(\omega_m + \omega_\phi)) + 6\beta(1 + \omega_m) + 9(1 + \omega_m\omega_\phi + \omega_m + \omega_\phi)$ and $\tau = 4\delta - 2\beta - 6 - 3(\omega_m + \omega_\phi)$.

The stability of the fixed points, namely $\Delta > 0$ and $\tau < 0$, is related with the values of $\delta$, $\beta$, $\omega_m$ and $\omega_\phi$. Table II presents the above conditions plus to reality conditions for the fixed points, namely $\bar{x}^2 > 0$, $\bar{v}^2 > 0$ and $\bar{y}^2 > 0$ for some specific values of equation of state parameter for both DM ($\omega_m$) and ELKO field ($\omega_\phi$).

Now let us analyse the Table II. We are interested in two different epochs, namely the inflation and the late time acceleration. The first one corresponds to an universe without DM and totally filled with ELKO field. This means $\bar{v}^2 = \Omega_m = 0$ and $\omega_m = 0$, which leads to $\delta = \frac{3}{2}(1 + \omega_\phi)$. By substituting into $\bar{x}$ and $\bar{y}$ we obtain $\bar{x}^2 = 0$ and $\bar{y}^2 = 1$, which shows that all the contribution should come from the potential part $\tilde{V}$ and also we should have $\dot{\phi} = 0$ from \(^{[9]}\), but our interaction term $Q \sim \dot{\phi}$, thus we conclude that such condition can not be applied to the inflation.
|                | ELKO Spinor          |        |
|----------------|----------------------|--------|
|                | Vacuum $(\omega_\phi = -1)$ | Dust $(\omega_\phi = 0)$ | Radiation $(\omega_\phi = 1/3)$ | Ultrarelativistic $(\omega_\phi = 1)$ |
| Matter $(\omega_m = -1)$ | ——                  | No stable point         | No stable point         | No stable point         |
| Dust $(\omega_m = 0)$     | $\frac{18}{2\beta+6} < \delta < \frac{3}{2}$ if $\beta > \frac{4}{3}$ | ——                  | No stable point         | No stable point         |
| Radiation $(\omega_m = 1/3)$ | $\frac{16}{\beta+6} < \delta < 2$ if $\beta > 2$ | $\frac{(6\beta+13)}{4\beta+6} < \delta < 2$ if $\beta > \frac{1}{2}$ | ——                  | No stable point         |
| Ultrarelativistic $(\omega_m = 1)$ | $\frac{36}{\beta+9} < \delta < 3$ if $\beta > 3$ | $\frac{3\beta+21}{\beta^2+9} < \delta < 3$ if $\beta > \frac{3}{2}$ | $\frac{2(\beta+5)}{\beta+3} < \delta < 3$ if $\beta > 1$ | ——                  |

TABLE II: Stability conditions for some equation of state parameters of DM and ELKO field, corresponding to the interaction $Q_1 = \beta v^2 x^2$

On the other hand, for the present time such variables are given by $\bar{v}^2 = \Omega_m = 0.315$ and $\bar{x}^2 + \bar{y}^2 = \Omega_\phi = 0.685$ according to the $\Lambda$CDM model, and beside that we must have $\omega_m = 0$. From (24) we conclude that: (i) $\delta \simeq 0.47$ if $\omega_\phi = -1$. From the corresponding cell in the Table I ($\omega_\phi = -1, \omega_m = 0$), we see that if $\beta \gtrsim 0.47$ the system is stable around the fixed points. Such positive value of $\beta$ corresponds to decay of Elko field into DM; (ii) $\delta \simeq 1.84$ if $\omega_\phi = 1/3$, but there is no solution for $\beta$ from the Table I in this case; and (iii) $\delta \simeq 2.53$ if $\omega_\phi = 1$, which has no solution for $\beta$ too. We conclude that, for the present time, the only possibility in order to have stable fixed points is $\beta \gtrsim 0.47$, leading to decay of Elko field into DM for an equation of state parameter $\omega_\phi = -1$, that is, the Elko field behaviour must be of vacuum type. Notice that other stability conditions are possible if the equation of state parameter of dark matter is of radiation or ultra-relativistic type.

The case $\omega_m = \omega_\phi$ has no physical meaning since both fluids has the same equation of state parameter, thus they are thermodynamically identical.

**C. $Q_1 = \beta v^2 x^2$**

For this interaction we have $Q = \frac{1}{3} \kappa^2 H \rho_m \dot{\phi}^2$.  

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The fixed points are \( \bar{x} = \sqrt{\frac{3+3\omega_m-2\delta}{2\beta}}, \quad \bar{y} = \sqrt{\frac{(2\delta-3\omega_m-3)(2\delta-3\omega_m+3\omega_\phi)}{6\beta(\omega_\phi-\omega_m)}} \), \( \bar{v} \). From the linearized matrix one finds \( \Delta = (4 - \frac{8\beta}{3(\omega_m-\omega_\phi)})\delta^2 + \frac{1}{\omega_m-\omega_\phi}[(8 + 4\omega_m + 4\omega_\phi)\beta - 6(\omega_m - \omega_\phi + 2\omega_m - 2\omega_\phi)]\delta + \frac{1}{\omega_m-\omega_\phi}[-6\beta(1 + \omega_\phi + \omega_m + \omega_\phi\omega_m) - 9(\omega_\phi - \omega_m + \omega_\phi^2 - \omega_m^2 + \omega_m\omega_\phi^2 - \omega_\phi^2\omega_m)] \) and \( \tau = (2 - \frac{4\beta}{3(\omega_m-\omega_\phi)})\delta + \frac{1}{\omega_m-\omega_\phi}[2\beta(1 + \omega_\phi) - 3(\omega_m - \omega_\phi - \omega_\phi^2 + \omega_m\omega_\phi)]. \)

The Table II shows the stability conditions for some specific values of \( \omega_m \) and \( \omega_\phi \).

For the inflationary epoch (\( \dot{v}^2 = 0 \) and \( \omega_m = 0 \)), we have the same condition for \( \delta \), namely \( \delta = \frac{2}{3}(1 + \omega_\phi) \). From the corresponding cell in the Table II it is easy to see that \( \omega_\phi = -1 \) is the only possible condition of stability, which leads to \( \delta = 0 \), but such value of \( \delta \) is not possible from the Table II if \( \omega_m = 0 \).

For the present time we have \( \dot{v}^2 = \Omega_m = 0.315 \). As the previous case, from (24) we have: (i) \( \delta \simeq 0.47 \) if \( \omega_\phi = -1 \), which is possible if \( \beta \gtrsim 14.6 \) and implies a positive value of \( \beta \), consequently the decay of Elko field into DM. The other cases: (ii) \( \delta \simeq 1.84 \) if \( \omega_\phi = 1/3 \), and (iii) \( \delta \simeq 2.53 \) if \( \omega_\phi = 1 \) does not presents stable solutions if \( \omega_m = 0 \).

**D.** \( Q_1 = \beta(x^2 + y^2)x^2 \)

For this interaction we have \( Q = \frac{1}{2} \kappa^2 \beta \rho_\phi \phi^2 \).

The fixed points are \( \bar{x} = \sqrt{\frac{2\delta-3\omega_\phi}{2\beta}}, \quad \bar{y} = \sqrt{\frac{2\delta(2\beta+3\omega_m-3\omega_\phi)-6\beta(1+\omega_m)+9(\omega_\phi^2-\omega_m\omega_\phi\omega_\phi+\omega_m)}{6\beta(\omega_\phi-\omega_m)}}, \quad \bar{v} \). From the linearized matrix one finds \( \Delta = (4 + \frac{8\beta}{3(\omega_m-\omega_\phi)})\delta^2 - \frac{1}{\omega_m-\omega_\phi}[8\beta(1 + \omega_m) + 9\omega_\phi^2 - \omega_m^2 - 2\omega_m - 2\omega_\phi + 2\omega_m - 2\omega_\phi)]\delta + \frac{1}{\omega_m-\omega_\phi}[6\beta(1 + \omega_\phi + \omega_m + \omega_\phi\omega_m) + 27(\omega_\phi\omega_m^2 + \omega_m^2 - \omega_\phi^2\omega_m + \omega_m - \omega_\phi^2 - \omega_m^2)] \) and \( \tau = (2 + \frac{4\beta}{3(\omega_m-\omega_\phi)})\delta + \frac{1}{\omega_m-\omega_\phi}[-2\beta(1 + \omega_m) - 3(\omega_m - \omega_\phi + \omega_\phi^2 - \omega_m\omega_\phi)]. \) Table III shows the stability conditions for some types of \( \omega_m \) and \( \omega_\phi \).

For the inflationary epoch (\( \dot{v}^2 = 0 \) and \( \omega_m = 0 \)) we have \( \bar{x}^2 = 0 \) and \( \bar{y}^2 = 1 \), which shows that the contribution come only from the potential part and the kinetic part is null, but the interaction is proportional to \( \phi^2 \), thus this interaction does not applies to inflation.

Considering the present time (\( \dot{v}^2 = 0.315 \)) we have: (i) \( \delta \simeq 0.47 \) if \( \omega_\phi = -1 \). We have verified that for \( \beta \gtrsim 0.69 \) we have \( \bar{x}^2 > 0 \) and \( \bar{y}^2 > 0 \), which represents stable fixed points. The cases (ii) \( \delta \simeq 1.84 \) if \( \omega_\phi = 1/3 \), and (iii) \( \delta \simeq 2.53 \) if \( \omega_\phi = 1 \) does not presents stable fixed points for \( \omega_m = 0 \).

**E.** \( Q_1 = \beta(v^2 - y^2) \)

The corresponding interaction is \( Q = 2\beta H(\rho_m - \rho_\phi + \frac{1}{2}\phi^2) \).

The fixed points are: \( \bar{x} = \sqrt{\frac{4\delta^2-2\beta-6-3(\omega_\phi + \omega_m)}{6\beta(\omega_\phi-\omega_m)}}, \quad \bar{y} = \sqrt{\frac{(2\delta-3\omega_\phi)(2\delta+3\omega_m)}{6\beta(\omega_\phi-\omega_m)}}, \quad \bar{v} \). The determinant and trace are given by: \( \Delta = 4\delta^2 + 2\delta(2\beta - 6 - 3(\omega_\phi + \omega_m)) - 6\beta(1 + \omega_\phi) + 9(1 + \omega_m + \omega_\phi + \omega_m\omega_\phi) \) and \( \tau = 4\delta - 6 + 2\beta - 3(\omega_m + \omega_\phi) \). Table IV shows the stability conditions for some specific values of \( \omega_m \) and \( \omega_\phi \).
|                         | Vacuum \( (\omega_m = -1) \) | Dust \( (\omega_m = 0) \) | Radiation \( (\omega_m = 1/3) \) | Ultrarelativistic \( (\omega_m = 1) \) |
|------------------------|-------------------------------|--------------------------|-------------------------------|----------------------------------|
| **Matter**             |                               |                          |                               |                                  |
| Vacuum \( (\omega_m = 0) \) | ——                           | No stable point          | No stable point               | No stable point                  |
| Dust \( (\omega_m = 1/3) \) | \(0 < \delta < \frac{3\beta}{3+4\beta}\) if \(\beta > 0\) | ——                      | No stable point               | No stable point                  |
| Radiation \( (\omega_m = 1) \) | \(0 < \delta < \frac{2\beta}{2+\beta}\) if \(\beta > 0\) | \(\frac{3}{2} < \delta < \frac{3+8\beta}{2+4\beta}\) if \(\beta > 0\) | ——                            | No stable point                  |
| Ultrarelativistic \( (\omega_m = 1) \) | \(0 < \delta < \frac{3\beta}{3+2\beta}\) if \(\beta > 0\) | \(\frac{3}{2} < \delta < \frac{9+12\beta}{6+4\beta}\) if \(\beta > 0\) | \(2 < \delta < \frac{2+3\beta}{1+\beta}\) if \(\beta > 0\) | ——                                    |

TABLE III: Stability conditions for some equation of state parameters of DM and ELKO field, corresponding to the interaction \(Q_1 = \beta(x^2 + y^2)x^2\).

For the inflationary epoch \((\bar{v}^2 = 0 \text{ and } \omega_m = 0)\), we have the same condition for \(\delta\), namely \(\delta = \frac{3}{2}(1 + \omega_m)\). Contrary to previous cases we see that \(\omega_m = -1\) is not a stable solution. For \(\omega_m = 1/3\) we have \(\delta = 2\), which is a stable solution corresponding to \(\beta = -1/2\), representing the decay of DM into ELKO field. But we have inferred \(\bar{v}^2 = 0\), thus there is no matter to decay at the inflation epoch. For \(\omega_m = 1\) we have \(\delta = 3\), but the stability condition requires \(\delta < 3\) from the corresponding cell in Table IV. We conclude that such interaction does not present stable points for the inflation.

For the present time we have \(\bar{v}^2 = 0.315\) and \(\omega_m = 0\). As the previous case, we have:
(i) \(\delta \approx 0.47\) if \(\omega_m = -1\), which has no stable solution; (ii) \(\delta \approx 1.84\) if \(\omega_m = 1/3\), and it is easy to see from Table IV that such value of \(\delta\) is possible for a negative value of \(\beta\). For instance, if \(\beta = -1/2\) the condition for \(\delta\) is \(1.69 \lesssim \delta \lesssim 2\); and (iii) \(\delta \approx 2.53\) if \(\omega_m = 1\), and for this condition we also have stable solution for a negative \(\beta\), as can be see from the Table IV. If \(\beta = -3/2\) for instance, the condition on \(\delta\) is \(2.07 \lesssim \delta \lesssim 3.0\), which include \(\delta \approx 2.53\). Thus, contrary to the previous cases, if the ELKO equation of state parameter is of radiation or ultrarelativistic type, the system presents stable solutions if \(\beta\) is negative, which corresponds to the decay of DM into ELKO field. The present acceleration of the
TABLE IV: Stability conditions for some equation of state parameters of DM and ELKO field, corresponding to the interaction $Q_1 = \beta(v^2 - y^2)$. We have defined the following parameters:

$\delta_1 = \frac{3}{4} - \beta - \frac{1}{4} \sqrt{9 + 16\beta^2}; \delta_2 = 1 - \beta - \sqrt{1 + \beta^2}; \delta_3 = \frac{3}{4} - \beta - \frac{1}{2} \sqrt{9 + 4\beta^2}; \delta_4 = \frac{7}{4} - \beta - \frac{1}{4} \sqrt{1 + 16\beta^2};$

$\delta_5 = \frac{9}{4} - \beta - \frac{1}{4} \sqrt{9 + 16\beta^2}; \delta_6 = \frac{5}{4} - \beta - \frac{1}{2} \sqrt{1 + 4\beta^2};$

universe can be understood in this model as the decay of dark matter into ELKO particles. That is a very interesting scenery that also alleviate the cosmic coincidence problem.

IV. CONCLUDING REMARKS

In this work we have developed a new approach to study the stability of a system composed by an ELKO field interacting with DM, which could give some contribution in order to alleviate the cosmic coincidence problem. Since that recent works $[36,39]$ have not found stable points for such system for different dynamic variables and interactions terms, we are
led to believe (without to demonstrate) that the system ELKO-DM does not allows stable points. Based on these results, we have supposed an additional constraint to the dynamical system, namely that the potential of the ELKO field is related to a constant parameter $\delta$, then we have analysed the stability conditions for such new system. We have also supposed that both the ELKO field as the dark matter energy density are related to the pressure by equations of state parameters $\omega_{\phi}$ and $\omega_{m}$, respectively. We have found different stability conditions relating the parameter $\delta$ and the interaction parameter $\beta$, which states if the decay is from DM to ELKO ($\beta < 0$) or from ELKO to DM ($\beta > 0$). Different values of $\omega_{\phi}$ and $\omega_{m}$ corresponding to vacuum, dust, radiation and ultra-relativistic equation of state parameter are presented in the Tables I to IV for different interactions terms, showing the conditions for the existence of stable points.

For the first three tables, corresponding to the interactions $Q_{1} = \beta x^{2}$, $Q_{1} = \beta v^{2} x^{2}$ and $Q_{1} = \beta (x^{2} + y^{2}) x^{2}$, the conditions for stable fixed points in order to satisfy the present stage of acceleration (with $\omega_{m} = 0$) are given by positive $\beta$ and $\delta$ parameters and also requires an equation of state parameter for the ELKO field of vacuum type ($\omega_{\phi} = -1$). Positive values of $\beta$ means the decay of ELKO field into DM particles. Such conditions could alleviate the cosmological coincidence problem. The inflationary phase cannot be driven for these interactions if we set $\omega_{m} = 0$ and $\bar{v}^{2} = 0$. Other possibilities are allowed if the equation of state parameters of DM and ELKO field are of radiation or ultra-relativistic type. Another interesting aspect that follows from Tables I to III is that there is no stable fixed points if $\omega_{\phi} > \omega_{m}$.

For the last interaction, namely $Q_{1} = \beta (v^{2} - y^{2})$, we have the opposite. The conditions for stable fixed points in order to satisfy the present stage of acceleration are given by negative $\beta$ and positive $\delta$ parameters. The equation of state parameter for the ELKO field must be of radiation ($\omega_{\phi} = 1/3$) or ultra-relativistic ($\omega_{\phi} = 1$) type. The case $\omega_{\phi} = -1$ does not present stable fixed points. Negative values of $\beta$ means the decay of DM articles into ELKO field. Such conditions also could alleviate the cosmological coincidence problem, and the equation of state parameter of ELKO field is not of exotic type. The inflationary phase cannot be driven for this interaction too. Other possibilities are allowed if the equation of state parameters of DM and ELKO field are of radiation or ultra-relativistic type. Another interesting aspect that follows from Table IV is that, contrary to the previous cases, there is no stable fixed points if $\omega_{\phi} < \omega_{m}$.

In such analysis the interaction $Q$ must be proportional to the variable $x$, otherwise the relations among the parameters must be very restrictive. But such condition is not so strong, since that the variable $x$ is proportional to $\dot{\phi}$, which characterizes a time dependence of the field $\phi$, and it is reasonable for an interacting theory. Notice that all the interactions studied are proportional to $\dot{\phi}^2$. For all the interactions analysed here there are conditions of stability in order to alleviate the cosmic coincidence problem. Such kinds of interactions and conditions on the parameters $\beta$ and $\delta$ opens the possibilities for future searches concerning the interaction between DM and ELKO field for specific potentials satisfying the conditions.
presented in the tables. The stability analysis presented here ensures the stability of these systems under some conditions.

Acknowledgments

SHP is grateful to CNPq - Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazilian research agency, for the financial support, process number 477872/2010-7.

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