THE CONDITIONING OF BLOCK KRONECKER ℓ-IFICATIONS OF MATRIX POLYNOMIALS

JAVIER PÉREZ*

Abstract. A strong ℓ-ification of a matrix polynomial \( P(λ) = \sum A_i λ^i \) of degree \( d \) is a matrix polynomial \( L(λ) \) of degree \( ℓ \) having the same finite and infinite elementary divisors, and the same number of left and right minimal indices as \( P(λ) \). Strong ℓ-ifications can be used to transform the polynomial eigenvalue problem associated with \( P(λ) \) into an equivalent polynomial eigenvalue problem associated with a larger matrix polynomial \( L(λ) \) of lower degree. Of most interest in applications is \( ℓ = 1 \), for which \( L(λ) \) receives the name of strong linearization. However, there exist some situations, e.g., the preservation of algebraic structures, in which it is more convenient to replace strong linearizations by other low degree matrix polynomials. In this work, we investigate the eigenvalue conditioning of ℓ-ifications from a family of matrix polynomials recently identified and studied by Dopico, Pérez and Van Dooren, the so-called block Kronecker companion forms. We compare the conditioning of these ℓ-ifications with that of the polynomial itself, and show that they are about as well conditioned as the polynomial, provided we scale \( P(λ) \) so that \( \max\{∥A_i∥_2\} = 1 \), and the quantity \( \min\{∥A_0∥_2,∥A_d∥_2\} \) is not too small. Moreover, under the scaling assumption \( \max\{∥A_i∥_2\} = 1 \), we show that any block Kronecker companion form, regardless of its degree or block structure, is about as well-conditioned as the well-known Frobenius companion forms. Our theory is illustrated by numerical examples.

Key words. matrix polynomial, polynomial eigenvalue problem, linearization, quadratification, ℓ-ification, companion form, conditioning, condition number, accuracy

AMS subject classifications. 65F15, 65F30, 65F35

1. Introduction. Finding eigenvalues of matrix polynomials is an important task in scientific computation. In the present paper, we study the conditioning of solving a polynomial eigenvalue problem by using ℓ-ifications, and its implications on the accuracy of computed eigenvalues.

An \( n \times n \) matrix polynomial takes the form

\[
P(λ) = \sum_{i=0}^{d} A_i λ^i, \quad \text{with} \quad A_0, A_1, \ldots, A_d \in \mathbb{C}^{n \times n}.
\]

(1.1)

We assume throughout that the polynomial \( P(λ) \) is regular, this is, the scalar polynomial \( \det P(λ) \) is not identically equal to the zero polynomial. When \( A_d \neq 0 \), we say that \( P(λ) \) has degree \( d \), otherwise we say that \( P(λ) \) has grade \( d \). The polynomial eigenvalue problem (PEP) associated with \( P(λ) \) consists in finding scalars \( λ \in \mathbb{C} \) and nonzero vectors \( x, y \in \mathbb{C}^n \) satisfying

\[
P(λ)x = 0 \quad \text{and} \quad y^* P(λ) = 0,
\]

where \( (\cdot)^* \) denotes the complex conjugate transpose. The scalar \( λ \) is called an eigenvalue of \( P(λ) \), and the vectors \( x \) and \( y \) are, respectively, the right and left eigenvectors of \( P(λ) \) associated with the eigenvalue \( λ \). The pairs \( (λ, x) \) and \( (y, λ) \) are called, respectively, right and left eigenpairs of \( P(λ) \). The triple \( (y, λ, x) \) is called an eigentriple of \( P(λ) \). We assume that the reader has some familiarity with matrix polynomials and polynomial eigenvalue problems. For those readers not familiar with these concepts, we refer to the classical works [15, 18], or to the recent reference [8] and the references therein.

*Department of Mathematical Sciences, University of Montana, USA. Email: javier.perez-alvaro@mso.umt.edu.
Any numerical algorithm for computing eigenvalues of matrix polynomials is affected by roundoff errors due to the limitations of floating point arithmetic. Ideally, we would like to use polynomial eigenvalue solvers that are at least forward stable, which is, algorithms that are able to find well-conditioned eigenvalues with high relative accuracy

\[
\frac{|\lambda - \tilde{\lambda}|}{|\lambda|} = O(u)\kappa_P(\lambda) \quad \lambda: \text{exact eigenvalue}, \quad \tilde{\lambda}: \text{computed eigenvalue.} \tag{1.2}
\]

where \( u \) is the unit roundoff, and we use the notation \( O(u) \) for any quantity that is upper bounded by \( u \) times a modest constant. In (1.2) \( \kappa_P(\lambda) \) denotes the conditioning of the eigenvalue \( \lambda \) of the matrix polynomial \( P(\lambda) \). The condition number \( \kappa_P(\lambda) \) measures how sensitive is the eigenvalue \( \lambda \) to perturbations of the matrix coefficients of \( P(\lambda) \) [28]. More precisely, if \( \lambda \) is a simple, finite, nonzero eigenvalue of a matrix polynomial \( P(\lambda) \) as in (1.1), with corresponding right and left eigenvectors \( x \) and \( y \), then the condition number of \( \lambda \) is defined by

\[
\kappa_P(\lambda) := \limsup_{\epsilon \to 0} \left\{ \frac{\|\Delta \lambda\|}{\epsilon |\lambda|} : \sum_{i=0}^{d} (A_i + \Delta A_i)(\lambda + \Delta \lambda)^i (x + \Delta x) = 0, \right. \\
\left. \quad \text{with } \|\Delta A_i\|_2 \leq \epsilon \omega_i \right\},
\]

where the tolerances \( \omega_i \) provide some freedom in how perturbations are measured. An explicit formula for \( \kappa_P(\lambda) \) was given by Tisseur (see [28, Theorem 5]):

\[
\kappa_P(\lambda) = \frac{\sum_{i=0}^{d} |\lambda|^i \omega_i \|x\|_2 \|y\|_2}{|\lambda| \cdot |y^* P'(\lambda)x|}. \tag{1.3}
\]

Then, we say that a simple, finite, nonzero eigenvalue \( \lambda \) of a matrix polynomial \( P(\lambda) \) is well-conditioned when \( \kappa_P(\lambda) \approx 1 \).

Of most interest are the choices \( \omega_i = \|A\|_2 \), for \( i = 0, 1, \ldots, d \), for which we call \( \kappa_P(\lambda) \) the coefficientwise condition number, and \( \omega_i = \|[A_0 A_1 \cdots A_d]\|_2 \), for \( i = 0, 1, \ldots, d \), for which we call \( \kappa_P(\lambda) \) the normwise condition number. We denote the coefficientwise condition number of \( \lambda \) by

\[
\text{coeff cond}_P(\lambda) = \frac{\sum_{i=0}^{d} |\lambda|^i \|A_i\|_2 \|x\|_2 \|y\|_2}{|\lambda| \cdot |y^* P'(\lambda)x|},
\]

and the normwise condition number of \( \lambda \) by

\[
\text{norm cond}_P(\lambda) = \frac{\|[A_0 \cdots A_d]\|_2 \left( \sum_{i=0}^{d} |\lambda|^i \right) \|x\|_2 \|y\|_2}{|\lambda| \cdot |y^* P'(\lambda)x|}.
\]

Readily from their definitions, it follows \( \text{norm cond}_P(\lambda) \geq \text{coeff cond}_P(\lambda) \). So, a well-conditioned eigenvalue in the normwise sense is also well-conditioned in the coefficientwise sense, but not the other way around.

A common approach to solving a polynomial eigenvalue problem associated with a matrix polynomial \( P(\lambda) \) starts by transforming \( P(\lambda) \) into a larger matrix polynomial of lower degree:

\[
\mathcal{L}(\lambda) = \sum_{i=0}^{\ell} \mathcal{L}_i \lambda^i \quad \text{with } \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_\ell \in \mathbb{C}^{m \times m}
\]
The matrix polynomial \( \mathcal{L}(\lambda) \) receives the name of \( \ell \)-ification of \( P(\lambda) \) [8]. More specifically, an \( \ell \)-ification of \( P(\lambda) \) is a matrix polynomial \( \mathcal{L}(\lambda) \) of degree at most \( \ell \) such that

\[
U(\lambda) \mathcal{L}(\lambda) V(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_s \end{bmatrix},
\]

for some unimodular matrices \( U(\lambda) \) and \( V(\lambda) \), and where \( I_s \) denotes the \( s \times s \) identity matrix. This definition implies that \( P(\lambda) \) and \( \mathcal{L}(\lambda) \) have the same finite elementary divisor and, thus, the same finite eigenvalues with the same multiplicities [8].

The matrix polynomial obtained by reversing the order of the matrix coefficients of \( P(\lambda) \), i.e.,

\[
\text{rev} P(\lambda) := \lambda^d P \left( \frac{1}{\lambda} \right) = \sum_{i=0}^{d} A_{d-i} \lambda^i,
\]

receives the name of the reversal matrix polynomial of \( P(\lambda) \). If an \( \ell \)-ification \( \mathcal{L}(\lambda) \) of \( P(\lambda) \) satisfies additionally

\[
W(\lambda) \text{rev} \mathcal{L}(\lambda) Z(\lambda) = \begin{bmatrix} \text{rev} P(\lambda) & 0 \\ 0 & I_s \end{bmatrix},
\]

for some unimodular matrices \( W(\lambda) \) and \( Z(\lambda) \), the matrix polynomial \( \mathcal{L}(\lambda) \) is called a strong \( \ell \)-ification of \( P(\lambda) \). The definition of strong \( \ell \)-ification implies that \( P(\lambda) \) and \( \mathcal{L}(\lambda) \) have the same finite and infinite elementary divisor and, thus, the same finite and infinite eigenvalues with the same multiplicities [8].

In numerical computations, the most common \( \ell \)-ifications used are those with \( \ell = 1 \) [17, 23, 29]. When \( \ell = 1 \), (strong) \( \ell \)-ifications receive the name of (strong) linearizations [16]. However, there are situations in which it is more convenient to transform \( P(\lambda) \) into a matrix polynomial of degree larger than 1. For instance, it is known that there exist matrix polynomials with some important algebraic structures for which there are not strong linearizations preserving those structures [8]. Since the preservation of algebraic structures has been recognized as a key factor for obtaining better and physically more meaningful numerical results [25], linearizations have been replaced by other low-degree matrix polynomials in some numerical computations. For example, in [22], 2-ifications (better known as quadratifications) are used in combination with doubling algorithms for solving even-degree structured PEPs.

The goal of this paper is to analyze the influence of the \( \ell \)-ification process on the eigenvalue conditioning, and on the accuracy of computed eigenvalues. The key point is that the eigenvalues of \( P(\lambda) \) are computed usually by applying to an \( \ell \)-ification \( \mathcal{L}(\lambda) \) a backward stable algorithm—like the QZ algorithm in the case of linearizations, or doubling algorithms in the case of quadratifications\(^2\). So, the computed eigenvalues are the exact eigenvalues of

\[
\mathcal{L}(\lambda) + \Delta \mathcal{L}(\lambda) = \sum_{i=0}^{\ell} (\mathcal{L}_i + \Delta \mathcal{L}_i) \lambda^i \quad \text{with} \quad \| \Delta \mathcal{L}_i \|_2 \leq O(\delta) \| \mathcal{L}_i \|_2.
\]

\(^1\)A matrix polynomial is unimodular if its determinant is a nonzero constant (independent of \( \lambda \)).

\(^2\)No backward stability proof exists so far for doubling algorithms, but in practice they produce small backward errors.
As a consequence of the backward stability, the forward error of a computed eigenvalue can be bounded as
\[
\frac{|\lambda - \overline{\lambda}|}{|\lambda|} \leq O(u) \operatorname{coeff\,cond}_L(\lambda),
\tag{1.4}
\]
where we recall that \(\operatorname{coeff\,cond}_L(\lambda)\) denotes the coefficientwise condition number of \(\lambda\) as an eigenvalue of the \(\ell\)-ification \(\mathcal{L}(\lambda)\). However, in view of (1.2), it is natural to look for a bound of the relative error \(|\lambda - \overline{\lambda}|/|\lambda|\) in terms of the conditioning of the original polynomial \(P(\lambda)\). This bound can be obtained by rewriting (1.4) as
\[
\frac{|\lambda - \overline{\lambda}|}{|\lambda|} \leq O(u) \frac{\operatorname{coeff\,cond}_L(\lambda)}{\kappa_P(\lambda)}. 
\]
Hence, the ratio \(\operatorname{coeff\,cond}_L(\lambda)/\kappa_P(\lambda)\) controls how far the eigensolver based on \(\ell\)-ification may be from being forward stable. In view of this, one should use \(\ell\)-ifications such that \(\operatorname{coeff\,cond}_L(\lambda) \approx \kappa_P(\lambda)\), since, in this ideal situation, one would be able to compute well-conditioned eigenvalues with high relative accuracy.

In this work, we will focus both on coefficient and normwise condition numbers. Specifically, we will study the ratios
\[
\frac{\kappa_L(\lambda)}{\kappa_P(\lambda)} = \frac{\operatorname{coeff\,cond}_L(\lambda)}{\operatorname{coeff\,cond}_P(\lambda)} \quad \text{and} \quad \frac{\kappa_L(\lambda)}{\kappa_P(\lambda)} = \frac{\operatorname{coeff\,cond}_L(\lambda)}{\operatorname{norm\,cond}_P(\lambda)},
\]
when the \(\ell\)-ification \(\mathcal{L}(\lambda)\) belongs to the family of block Kronecker companion forms (introduced in Section 2.2). We recall that this family includes the very well-known Frobenius companion forms (1.7) and (1.8), permuted versions of the famous Fiedler pencils, and generalized Fiedler pencils [4, 6, 7]. Assuming \(P(\lambda)\) is scaled so that \(\max_{i=0:d} \{\|A_i\|_2\} = 1\), we will show that there exist modest constants \(c_1\) and \(c_2\) such that
\[
\frac{\operatorname{coeff\,cond}_L(\lambda)}{\operatorname{norm\,cond}_P(\lambda)} \leq c_1 \quad \text{and} \quad \frac{\operatorname{coeff\,cond}_L(\lambda)}{\operatorname{coeff\,cond}_P(\lambda)} \leq \frac{c_2}{\min\{\|A_0\|_2, \|A_d\|_2\}}. \tag{1.5}
\]
Additionally, if \(\mathcal{R}(\lambda)\) is another block Kronecker companion form, then we will show that there exist modest constants \(c_3\) and \(c_4\) such that
\[
c_3 \leq \frac{\operatorname{coeff\,cond}_R(\lambda)}{\operatorname{coeff\,cond}_L(\lambda)} \leq c_4. \tag{1.6}
\]
These results are stated in Theorems 5.4 and 6.2, which are the main contributions of this work.

Notice that (1.5) implies that well-conditioned eigenvalues in the normwise sense can be computed with high relative accuracy as the eigenvalues of any block Kronecker companion form if we apply a backward stable algorithm to the \(\ell\)-ification. A similar conclusion holds for well-conditioned eigenvalues in the coefficientwise sense, provided that \(\min\{\|A_0\|_2, \|A_d\|_2\}\) is not too small. Moreover, we want to emphasized that (1.6) covers the case when \(\mathcal{L}(\lambda)\) is one of the well-known Frobenius companion forms:
\[
C_1(\lambda) := \begin{bmatrix}
\lambda A_d + A_{d-1} & A_{d-2} & \cdots & A_1 & A_0 \\
-I_n & \lambda I_n & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & -I_n & \lambda I_n & 0 \\
0 & \cdots & 0 & -I_n & \lambda I_n
\end{bmatrix}. \tag{1.7}
\]
and

$$C_2(\lambda) := \begin{bmatrix} \lambda A_d + A_{d-1} & -I_n & 0 & \cdots & 0 \\ A_{d-2} & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ A_1 & 0 & \cdots & \cdots & \lambda I_n - I_n \\ A_0 & 0 & \cdots & 0 & \lambda I_n \end{bmatrix},$$

(1.8)

since $C_1(\lambda)$ and $C_2(\lambda)$ are particular instances of block Kronecker companion forms. This is quite a surprising result, since (1.6) implies that any block Kronecker companion form, regardless of its degree or block structure, is about as well-conditioned as the Frobenius companion forms (1.7) and (1.8). Hence, block Kronecker companion forms can be used in the polynomial eigenvalue problem with similar reliability than Frobenius companion forms.

We begin in Section 2 by introducing the families of block Kronecker matrix polynomials, block Kronecker $\ell$-ifications (Section 2.1) and block Kronecker companion forms (Section 2.2). In Section 3 we establish one-sided factorizations and obtain eigenvector formulas needed for studying effectively the conditioning of block Kronecker $\ell$-ifications. Section 4 studies the conditioning of block Kronecker $\ell$-ifications relative to that of the matrix polynomial. Then, in Section 5, we prove that block Kronecker companion forms are about as well conditioned as the original polynomial, provided we scale $P(\lambda)$ so that $\max_{i=1:d} \{\|A_i\|_2\} = 1$, and the quantity $\min \{\|A_0\|_2, \|A_d\|_2\}$ is not too small. In Section 6, we show that under the scaling assumption $\max_{i=1:d} \{\|A_i\|_2\} = 1$ no block Kronecker companion form is better or worse conditioned than any other block Kronecker companion form. Finally, we present in Section 7 extensive numerical experiments that support our theoretical results.

Throughout the paper, we use the following notation. We denote by $\mathbb{C}[\lambda]$ the ring of polynomials in the variable $\lambda$ with complex coefficients. The set of $m \times n$ matrix polynomials, this is, the set of $m \times n$ matrices with entries in $\mathbb{C}[\lambda]$, is denoted by $\mathbb{C}[\lambda]^{m \times n}$. We denote by $I_n$ the $n \times n$ identity matrix, and by 0 the matrix with all its entries equal to zero, whose size should be clear from the context. By $A \otimes B$, we denote the Kronecker product of the matrices $A$ and $B$.

The next lemma will be useful when taking norms of block matrices (see [21, Lemma 3.5] and [5, Proposition 3.1]).

**Lemma 1.1.** For any $p \times q$ block matrix $A = [A_{ij}]$, we have $\max_{ij} \{\|A_{ij}\|_2\} \leq \|A\|_2 \leq \sqrt{pq} \max_{ij} \{\|A_{ij}\|_2\}$.

**2. Block Kronecker matrix polynomials.** A PEP can be transformed into an equivalent PEP associated with a larger matrix polynomial of lower degree (typically of degree 1 or 2) by using strong $\ell$-ifications [8]. Several approaches to constructing strong $\ell$-ifications have been introduced in the last years [3, 8, 11, 14, 26]. In this work, we focus on the strong $\ell$-ifications that belong to the family of block Kronecker matrix polynomials [14].

We recall the family of block Kronecker matrix polynomials in Definition 2.1. But, first, we introduce the following two matrix polynomials which are important for
In this section, we are given an arbitrary grade-\(k\)-block Kronecker matrix polynomial, or simply a block Kronecker matrix polynomial, is a degree-\(\ell\)-ification of a certain matrix polynomial. Then, from Theorem 2.1 [14, Theorem 5.5] the block Kronecker matrix polynomial (2.11) is a strong degree-\(\ell\)-ification of the \(m \times n\) matrix polynomial

\[
Q(\lambda) := (\Lambda_\eta(\lambda^d) \otimes I_n)M(\lambda)(\Lambda_\epsilon(\lambda^\ell) \otimes I_n),
\]

considered as a matrix polynomial of degree \(\ell(\epsilon + \eta + 1)\).

In practice, the matrix polynomial \(Q(\lambda)\) in the left-hand-side of (2.3) is given, and one would like to find a matrix polynomial \(M(\lambda)\) satisfying (2.3). This inverse problem has been addressed in [14]. We summarize the main results in the following section, focusing on the square case, that is, the case when \(m = n\).

### 2.1. Block Kronecker \(\ell\)-ifications for a prescribed matrix polynomial.

In this section, we are given an \(n \times n\) matrix polynomial \(P(\lambda)\) as in (1.1) of degree \(d\), and a nonzero natural number \(\ell\). We assume that \(d\) is divisible by \(\ell\) (notice that this assumption is automatically satisfied when \(\ell = 1\)). Our goal is to construct strong \(\ell\)-ifications for \(P(\lambda)\) by using block Kronecker matrix polynomials.

The starting point of the construction is to write \(k := d/\ell\) as \(k = \epsilon + \eta + 1\), for some nonnegative integers \(\epsilon\) and \(\eta\). Then, from Theorem 2.2, we see that to get a strong \(\ell\)-ification for \(P(\lambda)\) from a \((\epsilon, n, \eta, n)\)-block Kronecker matrix polynomial, we must solve

\[
(\Lambda_\eta(\lambda^d) \otimes I_n)M(\lambda)(\Lambda_\epsilon(\lambda^\ell) \otimes I_n) = P(\lambda),
\]

for the matrix polynomial \(M(\lambda)\) of degree \(\ell\). Solving (2.4) is always possible because this equation is consistent for every \(n \times n\) matrix polynomial \(P(\lambda)\). The consistency of (2.4) can be easily established as follows. Based on the coefficients of \(P(\lambda)\), we introduce the following degree-\(\ell\) matrix polynomials

\[
B_1(\lambda) := A_1 \lambda^\ell + A_{\ell -1} \lambda^{\ell-1} + \cdots + A_1 \lambda + A_0,
\]

\[
B_j(\lambda) := A_{\ell j} \lambda^\ell + A_{\ell j -1} \lambda^{\ell-1} + \cdots + A_{\ell(j-1)+1} \lambda, \quad \text{for } j = 2, \ldots, k.
\]
Notice that if $P(\lambda)$ has degree $d$, i.e., $A_d \neq 0$, then $B_k(\lambda)$ has degree $\ell$. Moreover, the polynomials $B_i(\lambda)$ satisfy the equality

$$P(\lambda) = \lambda^{k(\ell-1)} B_k(\lambda) + \lambda^{k(\ell-2)} B_{k-1}(\lambda) + \cdots + \lambda^1 B_2(\lambda) + B_1(\lambda). \quad (2.6)$$

Then, from (2.6), we can easily verify that the matrix polynomial

$$M_{\epsilon, \eta}(\lambda; P) := \begin{bmatrix} B_k(\lambda) & B_{k-1}(\lambda) & \cdots & B_{\eta+1}(\lambda) \\ 0 & \cdots & 0 & \vdots \\ \vdots & \ddots & \ddots & B_2(\lambda) \\ 0 & \cdots & 0 & B_1(\lambda) \end{bmatrix} \quad (2.7)$$

is a solution of (2.4). Hence, we have established the consistency of (2.4). Observe that $M_{\epsilon, \eta}(\lambda; P)$ has degree $\ell$ when $P(\lambda)$ has degree $d$.

The matrix polynomial $M_{\epsilon, \eta}(\lambda; P)$ is not the only solution of (2.4); see [14, Theorem 5.9]. We recall in Theorem 2.3 two characterizations of the set of degree-$\ell$ solutions of (2.4). Part (ii) in Theorem 2.3 gives a close formula for any solution of (2.4), while part (iii) shows how the coefficients of $P(\lambda)$ are distributed along the block anti-diagonals of any solution of (2.4).

**Theorem 2.3.** Let $P(\lambda)$ as in (1.1) be an $n \times n$ matrix polynomial of degree $d$. Assume $d$ is divisible by $\ell$, and set $k = d/\ell$. Let $\epsilon$ and $\eta$ be nonnegative integers such that $\epsilon + \eta + 1 = k$. Then, the following conditions are equivalent.

(i) The degree-$\ell$ matrix polynomial $M(\lambda)$ satisfies (2.4).

(ii) The matrix polynomial $M(\lambda)$ is of the form

$$M(\lambda) = M_{\epsilon, \eta}(\lambda; P) + \lambda \begin{bmatrix} 0 \\ D(\lambda) \end{bmatrix} + \begin{bmatrix} I_n & \cdots & I_n \end{bmatrix} \left( \begin{bmatrix} \lambda^\epsilon \otimes I_n \\ \vdots \\ \lambda^\eta \otimes I_n \end{bmatrix} + \begin{bmatrix} 0 & -D(\lambda) \end{bmatrix} + C \right), \quad (2.8)$$

for some matrices $B \in \mathbb{C}^{(n+1) \times n}$ and $C \in \mathbb{C}^{n \times (\epsilon+1)n}$ and some matrix polynomial $D(\lambda) \in \mathbb{C}[\lambda]^{n \times n}$ of degree $\ell - 2$, and where $M_{\epsilon, \eta}(\lambda; P)$ has been defined in (2.7).

(iii) If we consider $M(\lambda) = \sum_{t=0}^\ell M_t \lambda^t$ as an $(\eta + 1) \times (\epsilon + 1)$ block matrix polynomial with $n \times n$ block entries, denoted by $[M(\lambda)]_{ij} = \sum_{t=0}^\ell [M_t]_{ij} \lambda^t$, then the matrix polynomial $M(\lambda)$ satisfies

$$\sum_{i+j=t+1} [M_t]_{ij} + \sum_{i+j=t} [M_0]_{ij} = A_{d-t}, \quad \text{for } t = 0, 1, \ldots, k, \quad (2.9)$$

and

$$\sum_{i+j=s+1} [M_{t-s}]_{ij} = A_{d-s-t} \quad \text{for } s = 0, 1, \ldots, k-1, \text{ and } t = 1, \ldots, \ell - 1. \quad (2.10)$$

Combining Theorems 2.2 and 2.3 allows us to obtain infinitely many strong $\ell$-ifications of the prescribed matrix polynomial $P(\lambda)$. This motivates the following definition.

**Definition 2.4 (Block Kronecker $\ell$-ification of $P(\lambda)$).** Given an $n \times n$ matrix polynomial $P(\lambda)$ as in (1.1) with degree $d$, we will refer to any block Kronecker matrix
polynomial

\[ \mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & L_\eta(\lambda^I)^T \otimes I_n \\ L_\epsilon(\lambda^I) \otimes I_n & 0 \end{bmatrix} \quad \text{with} \quad M(\lambda) = \sum_{i=0}^\ell M_i \lambda^i, \quad (2.11) \]

where \( M(\lambda) \) satisfies (2.4), as a block Kronecker \( \ell \)-ification of \( P(\lambda) \).

In the following section, we identify an interesting subset of the family of block Kronecker \( \ell \)-ifications.

### 2.2. Block Kronecker companion \( \ell \)-ifications and Frobenius-like companion forms.

In practice, the most important \( \ell \)-ifications for an \( n \times n \) matrix polynomial \( P(\lambda) \) as in (1.1) are the so-called companion \( \ell \)-ifications or companion forms [8]. These \( \ell \)-ifications are introduced in Definition 2.5.

**Definition 2.5 (Companion form).** Consider an \( n \times n \) matrix polynomial \( P(\lambda) \) as in (1.1) with degree \( d \). An \( \ell \)-ification \( \mathcal{L}(\lambda) = \sum_{i=0}^\ell L_i \lambda^i \) of the matrix polynomial \( P(\lambda) \) is called a companion form (or companion \( \ell \)-ification) of \( P(\lambda) \) if, considering the matrix coefficients \( L_i \) as block matrices, each block entry of \( L_i \) is either \( 0_n \), \( I_n \) or \( A_i \), for some \( i \in \{0, \ldots, d\} \).

We are aware that Definition 2.5 is not the most general definition of companion form considered in the literature (see, for example, [8, Definition 5.1]), but it is the easiest to work with. It is also worth observing that Definition 2.5 implies that companion forms can be constructed from the coefficients of \( P(\lambda) \) without performing any arithmetic operation, useful property in numerical computations.

The family of block Kronecker \( \ell \)-ifications contains many examples of companion forms; see [14, Section 5.4]. This motivates the following definition.

**Definition 2.6 (Block Kronecker companion form).** Given an \( n \times n \) matrix polynomial \( P(\lambda) \) as in (1.1) with degree \( d \), we will refer to any block Kronecker matrix polynomial

\[ \mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & L_\eta(\lambda^I)^T \otimes I_n \\ L_\epsilon(\lambda^I) \otimes I_n & 0 \end{bmatrix}, \]

where \( M(\lambda) \) satisfies (2.4) and each of its block entries is either \( 0_n \), \( I_n \) or \( A_i \), for some \( i \in \{0, \ldots, d\} \), as a block Kronecker companion form (or block Kronecker companion \( \ell \)-ification) of \( P(\lambda) \).

Let us see some examples of block Kronecker companion forms. Let \( P(\lambda) \) denote an \( n \times n \) matrix polynomial as in (1.1) with degree \( d \), and let \( \ell \) be any divisor of \( d \). Write \( k = d/\ell = \epsilon + \eta + 1 \), for some nonnegative integers \( \epsilon, \eta \). Consider the matrix polynomials \( B_i(\lambda) \) defined in (2.5). Then, from Theorem 2.2, we can easily check that the block Kronecker matrix polynomial

\[ \mathcal{L}_{\epsilon, \eta}(\lambda) := \begin{bmatrix} B_k(\lambda) & B_{k-1}(\lambda) & \cdots & B_{\eta+1}(\lambda) & -I_n & 0 & 0 \\ 0 & \cdots & 0 & \vdots & \lambda^\epsilon I_n & \ddots & 0 \\ \vdots & \ddots & \vdots & \vdots & 0 & \ddots & -I_n \\ 0 & \cdots & 0 & B_1(\lambda) & 0 & \ddots & 0 \\ -I_n & \lambda^\epsilon I_n & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & -I_n & \lambda^\epsilon I_n & 0 & \cdots & 0 \end{bmatrix} \]
is a block Kronecker companion form of $P(\lambda)$. Particularizing $\mathcal{L}_{\epsilon, \eta}(\lambda)$ to the pairs $(\epsilon, \eta) = (k - 1, 0)$ and $(\epsilon, \eta) = (0, k - 1)$ yields the so-called Frobenius-like companion forms

$$C_1^\ell(\lambda) := \begin{bmatrix}
B_k(\lambda) & B_{k-1}(\lambda) & \cdots & B_2(\lambda) & B_1(\lambda) \\
-I_n & \lambda^0 I_n & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & -I_n & \lambda^1 I_n & 0 \\
0 & \cdots & 0 & -I_n & \lambda^\ell I_n
\end{bmatrix}$$

and

$$C_2^\ell(\lambda) := \begin{bmatrix}
B_k(\lambda) & -I_n & 0 & \cdots & 0 \\
B_{k-1}(\lambda) & \lambda^0 I_n & \ddots & \ddots & \ddots \\
\vdots & \ddots & -I_n & 0 \\
B_2(\lambda) & \ddots & \lambda^1 I_n & -I_n \\
B_1(\lambda) & 0 & \cdots & 0 & \lambda^\ell I_n
\end{bmatrix}$$

introduced and thoroughly analyzed in [8].

3. One-sided factorizations and eigenvector formulas for block Kronecker $\ell$-ifications. One key property of block Kronecker $\ell$-ifications is that they satisfy simple left- and right-sided factorizations. Obtaining such factorization is the subject of the following section. These one-sided factorizations will allow us to obtain in Section 3.2 eigenvector formulas for block Kronecker $\ell$-ifications.

3.1. One-sided factorizations for block Kronecker $\ell$-ifications. The goal of this section is to show that every block Kronecker $\ell$-ification $\mathcal{L}(\lambda)$ of a matrix polynomial $P(\lambda)$ satisfies one-sided factorizations of the form

$$\mathcal{L}(\lambda)H(\lambda) = g \otimes P(\lambda) \quad \text{and} \quad G(\lambda)\mathcal{L}(\lambda) = h^T \otimes P(\lambda),$$

where $\otimes$ denotes the Kronecker product, for some nonzero vectors $g$ and $h$, and some matrix polynomials $H(\lambda)$ and $G(\lambda)$. The importance of one-sided factorizations is widely recognized, since they are a useful tool for analyzing nonlinear eigenvalue problems [19].

We begin with the auxiliary matrix polynomials that appear repeatedly throughout the following development.

Definition 3.1. For any nonnegative integer $k$ and nonzero natural number $n$, we define the following two block-Toeplitz matrix polynomials:

$$R_k(\lambda) := \begin{bmatrix}
I_n & 0 & \cdots & 0 & 0 \\
\lambda I_n & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & I_n & 0 & \vdots \\
\lambda^{k-1} I_n & \cdots & \lambda I_n & I_n & 0
\end{bmatrix} \in \mathbb{C}[\lambda]^{kn \times (k+1)n}, \quad (3.1)$$

and

$$S_k(\lambda) := \begin{bmatrix}
0 & \lambda^{k-1} I_n & \lambda^{k-2} I_n & \cdots & I_n \\
\vdots & 0 & \lambda^{k-1} I_n & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \lambda^{k-2} I_n \\
0 & 0 & \cdots & 0 & \lambda^{k-1} I_n
\end{bmatrix} \in \mathbb{C}[\lambda]^{kn \times (k+1)n}, \quad (3.2)$$
with the convention that when \( k = 0 \), both \( R_k(\lambda) \) and \( S_k(\lambda) \) denote the empty matrix.

For any nonnegative integers \( p \) and \( q \), nonzero natural numbers \( n \) and \( \ell \), and \( n(q + 1) \times n(p + 1) \) matrix polynomial \( M(\lambda) \), we define the following two matrix polynomials:

\[
H(\lambda; p, q, M) := \begin{bmatrix}
\Lambda_p(\lambda^\ell) \otimes I_n \\
R_q(\lambda^\ell) M(\lambda)(\Lambda_p(\lambda^\ell) \otimes I_n)
\end{bmatrix},
\]

and

\[
G(\lambda; p, q, M) := \begin{bmatrix}
\lambda^q(\Lambda_p(\lambda^\ell) \otimes I_n) \\
-S_q(\lambda^\ell) M(\lambda)(\Lambda_p(\lambda^\ell) \otimes I_n)
\end{bmatrix},
\]

where \( \Lambda_k(\lambda) \), \( S_k(\lambda) \) and \( R_k(\lambda) \) are defined, respectively in (2.2), (3.2) and (3.1).

Observe that \( H(\lambda; p, 0, M) = G(\lambda; p, 0, M) = \Lambda_p(\lambda^\ell) \otimes I_n \).

Lemma contains some results on the norms of the matrices introduced in Definition 3.1 needed for proving the main results of this work.

**Lemma 3.2.** Consider the matrix polynomials \( \Lambda_k(\lambda) \), \( R_k(\lambda) \) and \( S_k(\lambda) \) defined in (2.2), (3.1) and (3.2), respectively. If \( |\lambda| \leq 1 \), then

(a1) \( \|\Lambda_k(\lambda^\ell) \otimes I_n\|_2 \leq \sqrt{k+1} \),

(b1) \( \|R_k(\lambda^\ell)\|_2 \leq k \), and

(c1) \( \|S_k(\lambda^\ell)\|_2 \leq k \).

If \( |\lambda| > 1 \), then

(a2) \( |\lambda|^{-k}\|\Lambda_k(\lambda^\ell) \otimes I_n\|_2 \leq \sqrt{k+1} \),

(b2) \( |\lambda|^{-(k-1)}\|R_k(\lambda^\ell)\|_2 \leq k \), and

(c2) \( |\lambda|^{-(k-1)}\|S_k(\lambda^\ell)\|_2 \leq k \).

**Proof.** Since \( \|A [0 \ A] \|_2 = \|A [A \ 0]\|_2 = \|A\|_2 \) for any matrix \( A \), notice, first, that for computing the 2-norm of the matrices \( S_k(\lambda^\ell) \) and \( R_k(\lambda^\ell) \) we can ignore their zero columns. Recall that the 2-norm is an absolute norm and, thus, a monotone norm. Then, the six bounds follow from the bound \( \|A\|_2 \leq \sqrt{m\max_i \{||a_{i,j}||\}} \), which is true for any \( m \times n \) matrix \( A = (a_{i,j}) \), the fact that \( \|A \otimes B\|_2 = \|A\|_2 \otimes \|B\|_2 \), and the fact that the modulus of the entries of the matrices are all upper bounded by 1.

In Theorem 3.3 we establish two different right-sided factorizations for block Kronecker \( \ell \)-ifications.

**Theorem 3.3.** (right-sided factorizations) Let \( P(\lambda) \) be an \( n \times n \) matrix polynomial as in (1.1) of degree \( d \). Assume \( d \) is divisible by \( \ell \), and let \( \mathcal{L}(\lambda) \) as in (2.11) be an \( (\epsilon, n, \eta, n) \)-block Kronecker \( \ell \)-ification of \( P(\lambda) \). Then, the following right-sided factorizations hold:

\[
\mathcal{L}(\lambda) H(\lambda; \epsilon, \eta, M) = \mathcal{L}(\lambda) \begin{bmatrix}
\Lambda_\epsilon(\lambda^\ell) \otimes I_n \\
R_\eta(\lambda^\ell) M(\lambda)(\Lambda_\epsilon(\lambda^\ell) \otimes I_n)
\end{bmatrix} = e_{n+1} \otimes P(\lambda),
\]

and

\[
\mathcal{L}(\lambda) G(\lambda; \epsilon, \eta, M) = \mathcal{L}(\lambda) \begin{bmatrix}
\lambda^\ell(\Lambda_\epsilon(\lambda^\ell) \otimes I_n) \\
-S_\eta(\lambda^\ell) M(\lambda)(\Lambda_\epsilon(\lambda^\ell) \otimes I_n)
\end{bmatrix} = e_1 \otimes P(\lambda),
\]

where \( e_j \) denotes the \( j \)-th column of the \( d/\ell \times d/\ell \) identity matrix.

**Proof.** Recall that the matrix polynomial \( M(\lambda) \) in the (1,1) block of \( \mathcal{L}(\lambda) \) satisfies \( (\Lambda_\eta(\lambda^\ell) \otimes I_n) M(\lambda)(\Lambda_\epsilon(\lambda^\ell) \otimes I_n) = P(\lambda) \), as this will be important. The proofs of (3.5) and (3.6) consist of direct verifications.
We begin by proving (3.5). First, notice \((L_ε(λ^T) \otimes I_n)(Λ_ε(λ^T) \otimes I_n) = 0\). This implies that the bottom \(ne\) rows of \(L(λ)H(λ; ε, η, M)\) are all equal to zero. The remaining rows (upper \((η + 1)\) rows) are equal to
\[
M(λ)(Λ_ε(λ^T) \otimes I_n) + (L_η(λ^T) \otimes I_n)R_η(λ^T)M(λ)(Λ_ε(λ^T) \otimes I_n) = \\
(I_{(n+1)n} + (L_η(λ^T) \otimes I_n)R_η(λ^T)) M(λ)(Λ_ε(λ^T) \otimes I_n) = \\
\left(I_{(η+1)n} - I_{(η+1)n}\right) + \left[Λ_η(λ^T) \otimes I_n\right] \left[0 \Lambda_η(λ^T) \otimes I_n\right] = \\
\left[Λ_η(λ^T) \otimes I_n\right] M(λ)(Λ_ε(λ^T) \otimes I_n) = \\
\left[0 \Lambda_η(λ^T) \otimes I_n\right] = \left[0 P(λ)\right],
\]
from where the result readily follows.

Next, we prove (3.6). From \((L_ε(λ^T) \otimes I_n)(Λ_ε(λ^T) \otimes I_n) = 0\), we obtain that the bottom \(ne\) rows of \(L(λ)G(λ; ε, η, M)\) are all equal to zero. The remaining rows (upper \((η + 1)\) rows) are equal to
\[
λ^εM(λ)(Λ_ε(λ^T) \otimes I_n) - (L_η(λ^T) \otimes I_n)S_η(λ^T)M(λ)(Λ_ε(λ^T) \otimes I_n) = \\
λ^ε(I_{(η+1)n} - (L_η(λ^T) \otimes I_n)S_η(λ^T)) M(λ)(Λ_ε(λ^T) \otimes I_n) = \\
\left(λ^ηI_{(η+1)n} - λ^ηI_{(η+1)n}\right) + \left[Λ_η(λ^T) \otimes I_n\right] \left[0 \Lambda_η(λ^T) \otimes I_n\right] = \\
\left[Λ_η(λ^T) \otimes I_n\right] M(λ)(Λ_ε(λ^T) \otimes I_n) = \\
\left[0 \Lambda_η(λ^T) \otimes I_n\right] = \left[0 P(λ)\right],
\]
which implies the desired result. □

We obtain in Theorem 3.4 left-sided factorizations for block Kronecker \(ℓ\)-ifications, analogues of the right-sided factorizations in Theorem 3.3.

**Theorem 3.4.** (left-sided factorizations) Let \(P(λ)\) be an \(n × n\) matrix polynomial as in (1.1) of degree \(d\). Assume \(d\) is divisible by \(ℓ\), and let \(L(λ)\) as in (2.11) be an \((ε, n, η, n)\)-block Kronecker \(ℓ\)-ification of \(P(λ)\). Then, the following left-sided factorizations hold:
\[
H(λ; ε, M^T) L(λ) = \left[Λ_η(λ^T) \otimes I_n \right] R_ε(λ^T) M(λ) \Lambda_η(λ^T) = e_{ε+1}^T \otimes P(λ), \quad (3.7)
\]
and
\[
G(λ; ε, M^T) L(λ) = \left[λ^ε(Λ_η(λ^T) \otimes I_n) \right] - S_ε(λ^T) M(λ) \Lambda_η(λ^T) = e_1^T \otimes P(λ), \quad (3.8)
\]
where \(e_j\) denotes the \(j\)th column of the \(d/ℓ × d/ℓ\) identity matrix.

**Proof.** Notice that \(L(λ)^T\) is a block Kronecker matrix polynomial with the roles of \(ε\) and \(η\) interchanged, and with (1,1) block equal to \(M(λ)^T\). Hence, \(L(λ)^T\) is an \((η, n, ε, n)\)-block Kronecker \(ℓ\)-ification of the matrix polynomial
\[
(Λ_ε(λ^T) \otimes I_n)M(λ)^T(Λ_η(λ^T) \otimes I_n) = P(λ)^T.
\]
Then, the left-sided factorizations (3.7) and (3.8) can be obtained by applying Theorem 3.3 to \(L(λ)^T\). □

The one-sided factorizations in Theorems 3.3 and 3.4 allow us in the following section to obtain simple formulas for the eigenvectors of block Kronecker \(ℓ\)-ifications in terms of the eigenvectors of the matrix polynomial \(P(λ)\). These formulas are a key feature of block Kronecker \(ℓ\)-ifications, and they will play a key role in all of our results.
3.2. Eigenvector formulas for block Kronecker $\ell$-ifications. Theorem 3.5 establishes two relations between right eigenvectors of $P(\lambda)$ and right eigenvectors of a block Kronecker $\ell$-ification of $P(\lambda)$.

**Theorem 3.5.** (right eigenvector formulas) Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1) of degree $d$. Assume $d$ is divisible by $\ell$, and let $\mathcal{L}(\lambda)$ as in (2.11) be an $(\epsilon, n, \eta, n)$-block Kronecker $\ell$-ification of $P(\lambda)$. The following statements hold.

(a) Let $\lambda_0$ be a finite eigenvalue of $P(\lambda)$. A vector $z$ is a right eigenvector of $\mathcal{L}(\lambda)$ associated with $\lambda_0$ if and only if

$$z = H(\lambda_0; \epsilon, \eta, M)x = \begin{bmatrix} \Lambda_\epsilon(\lambda_0^\ell) \otimes I_n \\ R_\eta(\lambda_0^\ell)M(\lambda_0)(\Lambda_\epsilon(\lambda_0^\ell) \otimes I_n) \end{bmatrix} x,$$

for some right eigenvector $x$ of $P(\lambda)$ associated with $\lambda_0$.

(b) Let $\lambda_0$ be a finite nonzero eigenvalue of $P(\lambda)$. A vector $z$ is a right eigenvector of $\mathcal{L}(\lambda)$ associated with $\lambda_0$ if and only if

$$z = G(\lambda_0; \epsilon, \eta, M)x = \begin{bmatrix} \lambda_0^{\eta \ell}(\Lambda_\epsilon(\lambda_0^\ell) \otimes I_n) \\ -S_\eta(\lambda_0^\ell)M(\lambda_0)(\Lambda_\epsilon(\lambda_0^\ell) \otimes I_n) \end{bmatrix} x,$$

for some right eigenvector $x$ of $P(\lambda)$ associated with $\lambda_0$.

**Proof.** The eigenvector formulas are an immediate consequence of the right-sided factorizations (3.5) and (3.6). We only prove part (a), since part (b) follows from a similar argument using (3.6) instead of (3.5).

Let $x$ be a right eigenvector of $P(\lambda)$ with eigenvalue $\lambda_0$, and let $z$ be the vector (3.9). Notice that if $x$ is a nonzero vector, so is $z$. Evaluating (3.5) at the eigenvalue $\lambda_0$, multiplying from the right by $x$, and using $P(\lambda_0)x = 0$, yields

$$\mathcal{L}(\lambda_0) \begin{bmatrix} \Lambda_\epsilon(\lambda_0^\ell) \otimes I_n \\ R_\eta(\lambda_0^\ell)M(\lambda_0)(\Lambda_\epsilon(\lambda_0^\ell) \otimes I_n) \end{bmatrix} x = \mathcal{L}(\lambda_0)z = e_{\eta+1} \otimes (P(\lambda_0)x) = 0.$$

Hence, $z$ is a right eigenvector of $\mathcal{L}(\lambda)$ with eigenvalue $\lambda_0$.

Let $z$ be a right eigenvector of $\mathcal{L}(\lambda)$ with eigenvalue $\lambda_0$. Assume $\lambda_0$ as an eigenvalue of $\mathcal{L}(\lambda)$ has geometric multiplicity $m$. Since $\mathcal{L}(\lambda)$ is a strong $\ell$-ification of $P(\lambda)$, the geometric multiplicity of $\lambda_0$ as an eigenvalue of $P(\lambda)$ is also $m$. Let $x_1, \ldots, x_m$ be linearly independent eigenvectors of $P(\lambda)$ with eigenvalue $\lambda_0$, and define $z_i = H(\lambda_0; \epsilon, \eta, M)x_i$, for $i = 1, \ldots, m$. Since $H(\lambda_0)$ has full column rank, the vectors $z_1, \ldots, z_m$ are linearly independent eigenvectors for $\mathcal{L}(\lambda)$. Hence, the vector $z$ must be a linear combination of $z_1, \ldots, z_m$, that is,

$$z = \sum_{i=1}^m c_i z_i = \sum_{i=1}^m c_i H(\lambda_0; \epsilon, \eta, M)x_i = H(\lambda_0; \epsilon, \eta, M) \sum_{i=1}^m c_i x_i.$$

Therefore, $z$ is of the form $H(\lambda_0; \epsilon, \eta, M)x$ for some eigenvector $x$ for $P(\lambda)$ with eigenvalue $\lambda_0$.

Theorem 3.6 establishes two relations between left eigenvectors of $P(\lambda)$ and left eigenvectors of a block Kronecker $\ell$-ification of $P(\lambda)$. Here, the scalar $\overline{\lambda}$ denotes the complex conjugate of the complex number $\lambda$.

**Theorem 3.6.** (left eigenvector formulas) Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1). Assume its degree $d$ is divisible by $\ell$, and let $\mathcal{L}(\lambda)$ as in (2.11) be an $(\epsilon, n, \eta, n)$-block Kronecker $\ell$-ification of $P(\lambda)$. The following statements hold.
denote right and left eigenvectors of \( P(\lambda) \) associated with \( \lambda_0 \).

(b) Let \( \lambda_0 \) be a simple, nonzero eigenvalue of \( P(\lambda) \). A vector \( \mathbf{w} \) is a left eigenvector of \( L(\lambda) \) associated with \( \lambda_0 \) if and only if

\[
\mathbf{w} = G(\lambda_0; \eta, \epsilon, M^*) \mathbf{y} = \begin{bmatrix}
\frac{\lambda_0^*}{\lambda_0} (\Lambda_0^* \otimes I_n) \\
-S_\epsilon(\lambda_0^*) M(\lambda_0)^* (\Lambda_\eta^* \otimes I_n)
\end{bmatrix} \mathbf{y},
\]

for some left eigenvector \( \mathbf{y} \) of \( P(\lambda) \) associated with \( \lambda_0 \).

Proof. Parts (a) and (b) follow from the left-sided factorizations in Theorem 3.4. The proof is nearly identical to the proof of Theorem 3.5, so it is omitted. \( \square \)

4. The conditioning of block Kronecker \( \ell \)-ifications. Let \( \lambda_0 \) be a simple, finite, nonzero eigenvalue of an \( n \times n \) matrix polynomial \( P(\lambda) \) as in (1.1), and let

\[
L(\lambda) = \sum_{i=0}^\ell \mathcal{L}_i \lambda^i
\]

be a block Kronecker \( \ell \)-ification of \( P(\lambda) \). Note that \( \lambda_0 \) as an eigenvalue of \( L(\lambda) \) is also simple, because strong \( \ell \)-ifications preserve geometric multiplicities. Let \( x \) and \( y \) denote right and left eigenvectors of \( P(\lambda) \), and let \( z \) and \( w \) denote right and left eigenvectors of \( L(\lambda) \), all corresponding to the eigenvalue \( \lambda_0 \). By (1.3), we have eigenvalue condition numbers given by

\[
\begin{align*}
\text{norm cond}_P(\lambda_0) &= \frac{\| [A_0 \cdots A_d] \|_2 \sum_{i=0}^d |\lambda_0|^i \| \Lambda_0^* \|_2}{|\lambda_0| \cdot |y^* P'(\lambda_0) x|}, \\
\text{coeff cond}_P(\lambda_0) &= \frac{\sum_{i=0}^d |\lambda_0|^i \| A_i \|_2}{|\lambda_0| \cdot |y^* P'(\lambda_0) x|}, \\
\text{coeff cond}_L(\lambda_0) &= \frac{\sum_{i=0}^\ell |\lambda_0|^i \| \mathcal{L}_i \|_2}{|\lambda_0| \cdot \| w^* \mathcal{L}'(\lambda_0) z \|_2}.
\end{align*}
\]

It is known that an \( \ell \)-ification \( L(\lambda) \) may alter the conditioning of the eigenvalue problem quite drastically (see the numerical experiments in Section 7). For this reason, our aim in this section is to study the ratios

\[
\begin{align*}
\frac{\text{coeff cond}_L(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} &= \frac{\sum_{i=0}^\ell |\lambda_0|^i \| \mathcal{L}_i \|_2}{\sum_{i=0}^d |\lambda_0|^i \| A_i \|_2} \| y^* P'(\lambda_0) x \|_2 \| z \|_2 \| w \|_2, \\
\frac{\text{coeff cond}_L(\lambda_0)}{\text{norm cond}_P(\lambda_0)} &= \frac{\sum_{i=0}^\ell |\lambda_0|^i \| \mathcal{L}_i \|_2}{\| [A_0 \cdots A_d] \|_2 \sum_{i=0}^d |\lambda_0|^i} \| y^* P'(\lambda_0) x \|_2 \| z \|_2 \| w \|_2.
\end{align*}
\]

More specifically, our immediate goals are, first, to obtain upper bounds for (4.1) and (4.2), and, second, to study under which conditions these upper bounds are moderate for all the eigenvalues of \( P(\lambda) \).

We start with Lemma 4.1, which implies a close relation between the conditioning of \( P(\lambda) \) and the conditioning of its block Kronecker \( \ell \)-ifications.
Lemma 4.1. Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1) of degree $d$. Assume $d$ is divisible by $\ell$, and let $\mathcal{L}(\lambda)$ as in (2.11) be an $(\epsilon, n, \eta, n)$-block Kronecker $\ell$-ification of $P(\lambda)$. If $\lambda_0$ is a simple and finite eigenvalue of $P(\lambda)$, with right and left eigenvectors $x$ and $y$, respectively, then the following statements hold.

(a) The vectors $z = H(\lambda_0; \epsilon, \eta, M)x$ and $H(\overline{\lambda_0}; \eta, \epsilon, M^*)y$ are, respectively, right and left eigenvectors of $\mathcal{L}(\lambda)$ with eigenvalue $\lambda_0$, and

$$|w^* \mathcal{L}'(\lambda_0)z| = |y^* P'(\lambda_0)x|, \quad (4.3)$$

where $H(\lambda; p, q, M)$ is defined in (3.3).

(b) Assume $\lambda_0$ is, in addition, nonzero. The vectors $z = G(\lambda_0; \epsilon, \eta, M)x$ and $G(\overline{\lambda_0}; \eta, \epsilon, M^*)y$ are, respectively, right and left eigenvectors of $\mathcal{L}(\lambda)$ with eigenvalue $\lambda_0$, and

$$|w^* \mathcal{L}'(\lambda_0)z| = |\lambda_0|^d |y^* P'(\lambda_0)x|, \quad (4.4)$$

where $G(\lambda; p, q, M)$ is defined in (3.4).

Proof. We first prove (4.3). Observe that part (a) of Theorems 3.5 and 3.6 say that $z = H(\lambda_0; \epsilon, \eta, M)x$ and $w = H(\overline{\lambda_0}; \eta, \epsilon, M^*)y$ are, respectively, right and left eigenvectors of $\mathcal{L}(\lambda_0)$ associated with $\lambda_0$. Now, differentiating (3.5) with respect to $\lambda$ gives

$$\mathcal{L}'(\lambda)H(\lambda; \epsilon, \eta, M) + \mathcal{L}(\lambda)H'(\lambda; \epsilon, \eta, M) = e_{\eta+1} \otimes P'(\lambda).$$

Evaluating the equation above at the eigenvalue $\lambda_0$, multiplying from the left by $w^*$, multiplying from the right by $x$, and using $w^* \mathcal{L}(\lambda_0) = 0$ produces

$$w^* \mathcal{L}'(\lambda_0)z = w^*(e_{\eta+1} \otimes (P'(\lambda_0)x)) = y^* P'(\lambda_0)x,$$

where we have used that $y = w^*H(\lambda_0) = 0$.

Next, we prove (4.4). From part (b) of Theorems 3.5 and 3.6, it follows that $z := G(\lambda_0; \epsilon, \eta, M)x$ and $w := G(\overline{\lambda_0}; \eta, \epsilon, M^*)y$ are, respectively, right and left eigenvectors of $\mathcal{L}(\lambda)$ associated with $\lambda_0$. Differentiating (3.6) with respect to $\lambda$ gives

$$\mathcal{L}'(\lambda)G(\lambda; \epsilon, \eta, M) + \mathcal{L}(\lambda)G'(\lambda; \epsilon, \eta, M) = e_1 \otimes P'(\lambda).$$

Evaluating the equation above at the eigenvalue $\lambda_0$, multiplying from the left by $w^*$, multiplying from the right by $x$, and using $w^* \mathcal{L}(\lambda_0) = 0$ yields

$$w^* \mathcal{L}'(\lambda_0)z = w^*(e_1 \otimes (P'(\lambda_0)x)) = \overline{\lambda_0}^{\ell-\ell} y^* P'(\lambda_0)x,$$

where we have used that $\lambda_0$ is a simple, finite, nonzero eigenvalue of $\mathcal{L}(\lambda)$.

The expressions (4.3) and (4.4) can now be used to investigate the size of the ratios (4.1) and (4.2).

Theorem 4.2. Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1) of degree $d$. Assume $d$ is divisible by $\ell$, and let $\mathcal{L}(\lambda) = \sum_{i=0}^\ell \mathcal{L}_i \lambda^i$ as in (2.11) be an $(\epsilon, n, \eta, n)$-block Kronecker $\ell$-ification of $P(\lambda)$. If $\lambda_0$ is a simple, finite, nonzero eigenvalue of $P(\lambda)$, then

$$\frac{\text{coeff cond}_{\ell}(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} \leq \frac{2 \max\{1, \max_{i=0, \ell} \{\|M_i\|_2\}\}}{\min\{\|A_0\|_2, \|A_d\|_2\}} \left(\ell + 1\right)^{1/2} \left(\epsilon + 1\right)^{1/2} \left(\eta + 1\right)^{1/2} \times

\left(1 + \epsilon^2 (\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2\right)^{1/2} \left(1 + \eta^2 (\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2\right)^{1/2}, \quad (4.5)$$
and
\[
\frac{\text{coeff cond}_L(\lambda_0)}{\text{norm cond}_P(\lambda_0)} \leq \frac{2 \max\{1, \max_{i=0,\ldots,\ell} \|M_i\|_2\}}{\| [A_0 \cdots A_d] \|_2} (\ell + 1)(\epsilon + 1)^{1/2}(\eta + 1)^{1/2} \times
\]
\[
\left(1 + \epsilon^2(\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2 \right)^{1/2} \left(1 + \eta^2(\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2 \right)^{1/2}
\]

(4.6)

**Proof.** We only prove (4.5), since the proof of (4.6) is entirely analogous.

Let \( x \) and \( y \) be, respectively, right and left eigenvectors of \( P(\lambda) \) associated with the eigenvalue \( \lambda_0 \). We need to distinguish two cases, namely, the case when \( |\lambda_0| \leq 1 \) and the case when \( |\lambda_0| > 1 \).

Let us assume, first, \( |\lambda_0| \leq 1 \). From part (a) in Lemma 4.1, we obtain that the ratio (4.1) equals
\[
\frac{\text{coeff cond}_L(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} = \frac{\sum_{i=0}^\ell |\lambda_0|^i \|L_i\|_2}{\sum_{i=0}^d |\lambda_0|^i \|A_i\|_2} \frac{\|H(\lambda_0; \epsilon, \eta, M)x\|_2 \|H(\lambda_0; \eta, \epsilon, M^*)y\|_2}{\|x\|_2 \|y\|_2},
\]
where \( H(\lambda; p, q, M) \) is defined in (3.3). Now, since \( |\lambda_0| \leq 1 \), we have
\[
\frac{\sum_{i=0}^\ell |\lambda_0|^i \|L_i\|_2}{\sum_{i=0}^d |\lambda_0|^i \|A_i\|_2} \leq \frac{\sum_{i=0}^\ell |\lambda_0|^i \|L_i\|_2}{\|A_0\|_2} \leq \frac{(\ell + 1) \max_{i=0,\ldots,\ell} \{\|L_i\|_2\}}{\min\{\|A_0\|_2, \|A_d\|_2\}} \leq \frac{2(\ell + 1) \max_{i=0,\ldots,\ell} \{\|L_i\|_2\}}{\min\{\|A_0\|_2, \|A_d\|_2\}}
\]
where the last inequality follows from Lemma 1.1. Then, notice
\[
\frac{\|H(\lambda_0; \epsilon, \eta, M)x\|_2^2}{\|x\|_2^2} = \frac{1}{\|x\|_2^2} \left\| \left[ \Lambda(\lambda_0^i) \otimes x \right] \right\|_2^2 \leq \frac{1}{\|x\|_2^2} \left( \|\Lambda(\lambda_0^i)\|_2^2 \|x\|_2^2 + \|R_\eta(\lambda_0^i)\|_2^2 \|M(\lambda_0)\|_2^2 \|\Lambda(\lambda_0^i)\|_2 \|x\|_2^2 \right) \leq
\]
\[
(\epsilon + 1) + \eta^2(\epsilon + 1) \left( \sum_{i=0}^\ell |\lambda_0|^i \|M_i\|_2 \right)^2 \leq (\epsilon + 1) \left(1 + \eta^2(\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2 \right),
\]
where we have used the inequality \((\sum_{i=0}^k |a_i|)^2 \leq (k + 1) \sum_{i=0}^k |a_i|^2\) for obtaining the last inequality above. An identical argument gives the upper bound
\[
\frac{\|H(\lambda_0; \eta, \epsilon, M^*)y\|_2^2}{\|y\|_2^2} \leq (\eta + 1) \left(1 + \epsilon^2(\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2 \right). \quad (4.9)
\]
Combining the previous three inequalities, yields the desired result.

Next, let us assume \( |\lambda_0| > 1 \). By part (b) in Lemma 4.1, the ratio (4.1) can be rewritten as
\[
\frac{\text{coeff cond}_L(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} = \frac{\sum_{i=0}^\ell |\lambda_0|^i \|L_i\|_2}{\sum_{i=0}^d |\lambda_0|^i \|A_i\|_2} \frac{\|G(\lambda_0; \epsilon, \eta, M)x\|_2 \|G(\lambda_0; \eta, \epsilon, M^*)y\|_2}{\|\lambda_0^d - \epsilon\|_2 \|x\|_2 \|y\|_2},
\]
(4.10)
Since $|\lambda_0| > 1$, we easily see that

$$\sum_{i=0}^d |\lambda_0|^i \|L_i\|_2 \leq \sum_{i=0}^d |\lambda_0|^d \|A_d\|_2 \leq (\ell + 1) \max_{i=0:d} \{ \|L_i\|_2 \} \left( \frac{1}{|\lambda_0|^{d-\ell}} \min \{ \|A_0\|_2, \|A_d\|_2 \} \right) \leq \frac{2(\ell + 1) \max_{i=0:d} \{ \|M_i\|_2 \}}{|\lambda_0|^{d-\ell}} \min \{ \|A_0\|_2, \|A_d\|_2 \}.$$ 

Hence, we have the upper bound

$$\frac{\text{coeff cond}_\ell(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} \leq \frac{2(\ell + 1) \max \{1, \max_{i=0:d} \{ \|M_i\|_2 \} \}}{\min \{ \|A_0\|_2, \|A_d\|_2 \}} \frac{\|G(\lambda_0; \epsilon, \eta, M)x\|_2}{|\lambda_0|^{d-\ell}} \frac{\|G(\lambda_0; \epsilon, \eta, M^*)y\|_2}{|\lambda_0|^{d-\ell}}.$$ 

Then, recall $d - \ell = \epsilon \ell + \eta \ell$ and the bounds in Lemma 3.2, and notice

$$\frac{\|G(\lambda_0; \epsilon, \eta, M)x\|_2^2}{|\lambda_0|^{2(d-\ell)} \|x\|_2^2} = \frac{1}{\|x\|_2^2} \left\| \left[ -\lambda_0^{-\ell} S_\eta(\lambda_0^\epsilon) M(\lambda_0) (A_\eta(\lambda_0^\epsilon) \otimes x) \right] \right\|_2 \leq \frac{1}{\|x\|_2^2} \left( |\lambda_0|^{-\ell} |\lambda_0|^{\frac{\epsilon}{2}} \|\lambda_0|^{\frac{\eta}{2}} \|\lambda_0^{\frac{\epsilon}{2}} M(\lambda_0)\|_2^2 \right) \leq (\epsilon + 1) + \eta^2 (\epsilon + 1) \left( \sum_{i=0}^\ell \|M_i\|_2 \right)^2 \leq (\epsilon + 1) \left( 1 + \eta^2 (\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2 \right).$$

(4.11)

Analogously, we can obtain the upper bound

$$\frac{\|G(\lambda_0; \epsilon, \eta, M^*)y\|_2^2}{|\lambda_0|^{2(d-\ell)} \|y\|_2^2} \leq (\eta + 1) \left( 1 + \epsilon^2 (\ell + 1) \sum_{i=0}^\ell \|M_i\|_2^2 \right).$$

(4.12)

The desired result now follows.

We refer the bounds in Theorem 4.2 moderate, the block Kronecker $\ell$-ification $L(\lambda)$ would be as well conditioned as the polynomial $P(\lambda)$ itself. In Proposition 4.3, we show that a necessary condition for having moderate bounds (4.1) and (4.2) is

$$\max_{i=0:d} \{ \|A_i\|_2 \}$$

to be moderate.

**Proposition 4.3.** Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1) of degree $d = k(\epsilon + \eta + 1)$, and let $M(\lambda) = \sum_{i=0}^\ell M_i \lambda^i$ be an $(\eta + 1)n \times (\epsilon + 1)n$ grade-$\ell$ matrix polynomial satisfying (2.4). Then,

$$\max_{i=0:d} \{ \|M_i\|_2 \} \geq \frac{1}{2 \max \{ \epsilon + 1, \eta + 1 \}} \max_{i=0:d} \{ \|A_i\|_2 \}.$$ 

Hence, the upper bounds (4.1) and (4.2) can potentially show a cubic dependence on

$$\max_{i=0:d} \{ \|A_i\|_2 \}.$$ 

**Proof.** By part (iii) in Theorem 2.3, notice that each matrix coefficient $A_i$ satisfies either an equation of the form $A_i = \sum_{i+j = \epsilon} [M_i]_{ij} + \sum_{i+j = \epsilon - 1} [M_0]_{ij}$, for some
constant $c$, or of the form $A_i = \sum_{i+j=c_1} [M_{c2}]_{ij}$, for some constants $c_1$ and $c_2$. In the former case, we have
\[
\|A_i\|_2 \leq \sum_{i+j=c} \|[M_{c}]_{ij}\|_2 + \sum_{i+j=c-1} \|[M_0]_{ij}\|_2 \leq \max\{\epsilon + 1, \eta + 1\} \left( \max_{ij} \|[M_{c}]_{ij}\|_2 + \max_{ij} \|[M_0]_{ij}\|_2 \right) \leq \max\{\epsilon + 1, \eta + 1\} (\|M_{c}\|_2 + \|M_0\|_2) \leq 2 \max\{\epsilon + 1, \eta + 1\} \max_{i=0:\ell} \{\|M_i\|_2\}.
\]
In the latter case, we have
\[
\|A_i\|_2 \leq \sum_{i+j=c_1} \|[M_{c2}]_{ij}\| \leq \max_{ij} \|[M_{c2}]_{ij}\|_2 \leq \max\{\epsilon + 1, \eta + 1\} \max_{i=0:\ell} \{\|M_i\|_2\}.
\]
Hence, $\max_{i=0:d} \{\|A_i\|_2\} \leq 2 \max\{\epsilon + 1, \eta + 1\} \max_{i=0:\ell} \{\|M_i\|_2\}$, as we wanted to show.

As a consequence of Theorem 4.2 and Proposition 4.3, a necessary, but not sufficient, condition under which the upper bounds (4.1) and (4.2) could be moderate is the condition
\[
\max_{i=0:d} \{\|A_i\|_2\} \approx 1. \tag{4.13}
\]
Notice that (4.13) is very mild, since it can be always achieved by dividing the original matrix polynomial by a number.

In the following section, we particularize the upper bounds (4.1) and (4.2) to the case when $L(\lambda)$ is a block Kronecker companion form. We will show that (4.13) is sufficient to guarantee a moderate upper bound (4.2). In other words, scaling $P(\lambda)$ so that (4.13) is satisfied guarantees all block Kronecker companion forms to be about as well conditioned in the normwise sense as the polynomial $P(\lambda)$ itself. Additionally, we will show that the conditions
\[
\max_{i=0:d} \{\|A_i\|_2\} \approx 1 \quad \text{and} \quad \min\{\|A_0\|_2, \|A_d\|_2\} \approx 1 \tag{4.14}
\]
are sufficient to guarantee a moderate upper bound (4.1). In other words, assuming the conditions in (4.14), block Kronecker companion forms are optimally conditioned in the more stringent coefficientwise sense.

5. The conditioning of companion $\ell$-ifications. This section contains one of the main results of this work, Theorem 5.4. We show that block Kronecker companion forms (recall their definition given in Section 2.2) are optimally conditioned in the normwise sense if condition (4.13) holds, and optimally conditioned in the coefficientwise sense if the conditions in (4.14) hold.

Before stating the main theorems, we present Lemma 5.1.

**Lemma 5.1.** Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1) of degree $d$. Assume $d$ is divisible by $\ell$, and let $L(\lambda) = \sum_{i=0}^{\ell} L_i \lambda^i$ as in (2.11) be a block Kronecker companion form of $P(\lambda)$. If we consider $M_t$, for $t = 0, \ldots, \ell$, as an $(\eta + 1) \times (\epsilon + 1)$ block-matrix with $n \times n$ blocks $[M_t]_{ij}$, then
\[
\max_{i,j,t} \{\|M_t\|_2\} \leq \max\{1, \max_{i=0:d} \{\|A_i\|_2\}\}. \tag{5.1}
\]
Proof. The proof follows immediately from the fact that each block entry \([M_i]_{ij}\) equals either 0, \(I_n\) or \(A_i\), for some \(i \in \{0, \ldots, d\}\). □

When \(L(\lambda)\) is a block Kronecker companion form, we can obtain upper bounds on the ratios (4.1) and (4.2) that depend essentially on the norms of the matrix coefficients of the polynomial \(P(\lambda)\).

**Theorem 5.2.** Let \(P(\lambda)\) be an \(n \times n\) matrix polynomial as in (1.1) of degree \(d\). Assume \(d\) is divisible by \(\ell\), and let \(L(\lambda)\) as in (2.11) be a block Kronecker companion form of \(P(\lambda)\). If \(\lambda_0\) is a simple, finite, nonzero eigenvalue of \(P(\lambda)\), then

\[
\frac{\text{coeff cond}_L(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} \leq 16d^3(\epsilon + 1)^{3/2}(\eta + 1)^{3/2} \frac{\max\{1, \max_{i=0:d}\{\|A_i\|_2\}\}}{\min\{\|A_0\|_2, \|A_d\|_2\}},
\]

(5.2)

and

\[
\frac{\text{coeff cond}_L(\lambda_0)}{\text{norm cond}_P(\lambda_0)} \leq 16d^3(\epsilon + 1)^{3/2}(\eta + 1)^{3/2} \frac{\max\{1, \max_{i=0:d}\{\|A_i\|_2\}\}}{\max_{i=0:d}\{\|A_i\|_2\}}.
\]

(5.3)

**Proof.** We only prove (5.2). The upper bound (5.3) follows from (4.6) using a similar argument.

First, observe that Lemmas 1.1 and 5.1 imply

\[
\|M_i\|_2 \leq \sqrt{(\epsilon + 1)(\eta + 1)} \max\{1, \max_{i=0:d}\{\|A_i\|_2\}\},
\]

(5.4)

for \(i = 0, 1, \ldots, \ell\). So, we have

\[
\frac{\max\{1, \max_{i=0:d}\{\|M_i\|_2\}\}}{\min\{\|A_0\|_2, \|A_d\|_2\}} \leq \sqrt{(\epsilon + 1)(\eta + 1)} \frac{\max\{1, \max_{i=0:d}\{\|A_i\|_2\}\}}{\max_{i=0:d}\{\|A_i\|_2\}}.
\]

(5.5)

Then, using (5.4), we get the inequality

\[
\left(1 + \epsilon^2(\ell + 1) \sum_{i=0}^{\ell} \|M_i\|_2^2\right)^{1/2} \leq \left(1 + \epsilon^2((\ell + 1)^2(\epsilon + 1)(\eta + 1) \max\{1, \max_{i=0:d}\{\|A_i\|_2^2\}\}\right)^{1/2}
\]

(5.6)

\[
(\epsilon + 1)^{1/2}(\eta + 1)^{1/2}(1 + \epsilon^2((\ell + 1)^2)^{1/2} \max\{1, \max_{i=0:d}\{\|A_i\|_2^2\}\}) \leq 2d(\epsilon + 1)^{1/2}(\eta + 1)^{1/2} \max\{1, \max_{i=0:d}\{\|A_i\|_2^2\}\},
\]

where, to get the last inequality, we have used \(\ell \leq d\) and some elementary inequalities. An analogous argument yields

\[
\left(1 + \eta^2(\ell + 1) \sum_{i=0}^{\ell} \|M_i\|_2^2\right)^{1/2} \leq 2d(\epsilon + 1)^{1/2}(\eta + 1)^{1/2} \max\{1, \max_{i=0:d}\{\|A_i\|_2^2\}\}. \tag{5.7}
\]

Finally, inserting the inequalities (5.5), (5.6) and (5.7) in (4.5), and using \((\ell + 1)(\epsilon + 1)^{1/2}(\eta + 1)^{1/2} \leq 2d\), we obtain the desired result. □

**Remark 5.3.** The factor \(16d^3(\epsilon + 1)^{3/2}(\eta + 1)^{3/2}\) in the upper bounds (5.2) and (5.3) may be pessimistic for some block Kronecker companion forms. This factor
takes into account the worst case scenario in which the matrices \( M_i \) are dense block-matrices. One can obtain tighter constants, for example, particularizing the analysis to block Kronecker companion forms such that the matrices \( M_i \) are of low block-bandwidth. In this case, the constant reduces essentially to \( d^3 \), result that is coherent with other analyses; see [5, Theorem 5.1].

As an immediate corollary of Theorem 5.2, we obtain Theorem 5.4, which is one of the main results of this work. Theorem 5.4 gives conditions on the coefficients of \( P(\lambda) \) that guarantee that all block Kronecker companion forms of \( P(\lambda) \) are about as well conditioned as the polynomial itself.

**Theorem 5.4.** Let \( P(\lambda) \) be an \( n \times n \) matrix polynomial as in (1.1) of degree \( d \). Assume \( d \) is divisible by \( \ell \), and let \( L(\lambda) \) as in (2.11) be a block Kronecker companion form of \( P(\lambda) \). If \( \lambda_0 \) is a simple, finite, nonzero eigenvalue of \( P(\lambda) \), then the following statements hold.

(a) If \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \), then

\[
\frac{\text{coeff cond}_L(\lambda_0)}{\text{norm cond}_P(\lambda_0)} \lesssim 1.
\]

In other words, under the scaling \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \) assumption, block Kronecker companion forms are optimally conditioned in the normwise sense.

(b) If \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \) and \( \min\{\|A_0\|_2, \|A_d\|_2\} = 1 \), then

\[
\frac{\text{coeff cond}_L(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} \lesssim 1.
\]

In other words, under the scaling \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \) assumption, block Kronecker companion forms are optimally conditioned in the coefficientwise sense provided \( \min\{\|A_0\|_2, \|A_d\|_2\} \) is not too small.

The result in part (a) of Theorem 5.4 is entirely consistent with the results in [12, 13] on the backward stability of solving PEPs by using block Kronecker companion linearizations, as we now explain. The analyses in [12, 13] show that solving a PEP by applying a backward stable algorithm to a block Kronecker companion linearization is backward stable for the PEP under the scaling condition \( \| [A_0 \cdots A_d] \|_F \approx 1 \). This means that the computed eigenvalues are the exact eigenvalues of a perturbed matrix polynomial

\[
P(\lambda) + \Delta P(\lambda) = \sum_{i=0}^d (A_i + \Delta A_i)\lambda^i
\]

where \( \| [\Delta A_0 \cdots \Delta A_d] \|_F = \mathcal{O}(u)\| [A_0 \cdots A_d] \|_F \). Hence, the eigenvalue relative errors can be bounded as

\[
\frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|} = \mathcal{O}(u \cdot \text{norm cond}_P(\lambda_i)),
\]

where \( \lambda_i \) are the exact eigenvalues of \( P(\lambda) \) and \( \tilde{\lambda}_i \) are the computed eigenvalues. In conclusion, [12, 13] show that well-conditioned eigenvalues in normwise sense (i.e., \( \text{norm cond}_P(\lambda) \approx 1 \)) can be computed with high relative accuracy if we apply a backward stable algorithm to the block Kronecker companion linearization. The same conclusion can be drawn from part (a) in Theorem 5.4.
6. Comparing the coefficientwise conditioning of different block Kronecker companion forms. In the coefficientwise sense, even after scaling the matrix polynomial $P(\lambda)$ so that $\max_{i=0:d}\{\|A_i\|_2\} = 1$, block Kronecker companion forms may be potentially much worse conditioned than the polynomial $P(\lambda)$ when the quantity $\min\{\|A_0\|_2, \|A_d\|_2\}$ is much smaller than one. For this reason, we investigate in this section whether some block Kronecker companion forms are preferable to others, from an eigenvalue conditioning point of view.

**Theorem 6.1.** Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1) of degree $d$. Assume $d$ is divisible both by $\ell$ and $r$. Let $L(\lambda) = \sum_{i=0}^{\ell} (\lambda - \lambda_i) L_i$ be an $(\ell, n, \eta_1, n)$-block Kronecker companion $\ell$-ification of $P(\lambda)$, and let $R(\lambda) = \sum_{i=0}^{\ell} (\lambda - \lambda_i) R_i$ be an $(\epsilon_2, n, \eta_2, n)$-block Kronecker companion $r$-ification of $P(\lambda)$. If $\lambda_0$ is a finite, nonzero and simple eigenvalue of $P(\lambda)$, then

$$\frac{1}{16d^3(\epsilon_1 + 1)^{3/2}(\eta_1 + 1)^{3/2} \max\{1, \max_{i=0:d}\{\|A_i\|_2^3\}\}} \leq \frac{\text{coeff cond}_R(\lambda_0)}{\text{coeff cond}_L(\lambda_0)} \leq 16d^3(\epsilon_2 + 1)^{3/2}(\eta_2 + 1)^{3/2} \max\{1, \max_{i=0:d}\{\|A_i\|_2^3\}\}. \tag{6.1}$$

**Proof.** Throughout the proof, we assume $\ell \geq r$. The case when $\ell < r$ is completely analogous and left to the reader.

Let $(y, \lambda_0, x)$, $(w_L, \lambda_0, z_L)$ and $(w_R, \lambda_0, z_R)$ be eigentriples of, respectively, the polynomial $P(\lambda)$, the $\ell$-ification $L(\lambda)$ and the $r$-ification $R(\lambda)$. We have

$$\frac{\text{coeff cond}_R(\lambda_0)}{\text{coeff cond}_L(\lambda_0)} = \frac{\sum_{i=0}^{r-1} |\lambda_0|^i \|R_i\|_2}{\sum_{i=0}^{\ell} |\lambda_0|^i \|L_i\|_2} \cdot \frac{\|z_R\|_2 \|w_R\|_2}{\|z_L\|_2 \|w_L\|_2} \cdot \frac{|w_R^* L'(\lambda_0) z_L|}{|w_R^* R'(\lambda_0) z_R|}. \tag{6.1}$$

To upper bound the ratio (6.1), we need to distinguish two cases, namely, the cases $|\lambda_0| > 1$ and $|\lambda_0| \leq 1$.

Assume first $|\lambda_0| > 1$. Notice that Lemma 1.1 implies $\|L_i\|_2 \geq 1$, because the leading matrix coefficient $L_0$ has at least one block entry equal to $I_n$. If we denote by $N(\lambda) = \sum_{i=0}^{r} N_i \lambda^i$ the (1,1) block of $R(\lambda)$, then, from Lemmas 1.1 and 5.1, we obtain

$$\frac{\sum_{i=0}^{r} |\lambda_0|^i \|R_i\|_2}{\sum_{i=0}^{\ell} |\lambda_0|^i \|L_i\|_2} \leq \frac{\sum_{i=0}^{r} |\lambda_0|^i \|R_i\|_2}{|\lambda_0|^r \|L_r\|_2} \leq \frac{1}{|\lambda_0|^{r-r}} \sum_{i=0}^{r} |\lambda_0|^{i-r} \|R_i\|_2 \leq \frac{1}{|\lambda_0|^{r-r}} (r+1) \max_{i=0:r}\{\|R_i\|_2\} \leq \frac{2}{|\lambda_0|^{r-r}} (r+1) \max_{i=0:r}\{\|N_i\|_2\} \leq \frac{2}{|\lambda_0|^{r-r}} \sqrt{(\epsilon_2 + 1)(\eta_2 + 1)} \max_{i=0:d}\{\|A_i\|_2\}. \tag{6.2}$$

Next, recall the definition of the matrix polynomial $G(\lambda; p, q, M)$ introduced in (3.4), and let us denote by $M(\lambda)$ the (1,1) block of $L(\lambda)$. From part (b) of Theorems 3.5
and 3.6, we have

\[
\frac{\|z_R\|_2 \|w_R\|_2}{\|z_L\|_2 \|w_L\|_2} = \frac{\|G(\lambda_0; \epsilon_2, \eta_2, N)x\|_2 \|G(\lambda_0; \eta_2, \epsilon_2, N^*)y\|_2}{\|G(\lambda_0; \epsilon_1, \eta_1, M)x\|_2 \|G(\lambda_0; \eta_1, \epsilon_1, M^*)y\|_2} \leq \frac{|\lambda_0|^{2(d-\ell)} \|x\|_2 \|y\|_2}{|\lambda_0|^{d-r} \|y\|_2} = \frac{|\lambda_0|^{2(d-\ell)}}{|\lambda_0|^{d-r}} \cdot 4d^2(\epsilon_2 + 1)^{3/2}(\eta_2 + 1)^{3/2} \max_{i=0:d}\{\|A_i\|_2^2\}, \tag{6.3}
\]

where the first inequality above follows from \(\|G(\lambda_0; \epsilon_1, \eta_1, M)x\|_2 \geq |\lambda_0|^{d-\ell} \|x\|_2\) and \(\|G(\lambda_0; \eta_1, \epsilon_1, M^*)y\|_2 \geq |\lambda_0|^{d-r} \|y\|_2\), and the second inequality follows from similar arguments to the ones used in the proofs of (4.11) and (4.12). Finally, observe that part (b) in Lemma 4.1, implies

\[
\frac{\|w^*_R z_L'(\lambda_0)z_L\|_2}{\|w^*_R R'(\lambda_0)z_R\|_2} = \frac{|\lambda_0|^{d-\ell} \|y^* P'(\lambda_0)x\|_2}{|\lambda_0|^{d-r} \|y^* P'(\lambda_0)x\|_2} = \frac{1}{|\lambda_0|^{\ell-r}}. \tag{6.4}
\]

Using the inequalities (6.2)–(6.4), we obtain from (6.1)

\[
\frac{\text{coeff cond}_R(\lambda_0)}{\text{coeff cond}_L(\lambda_0)} \leq 8d^2(r + 1)(\epsilon_2 + 1)^2(\eta_2 + 1)^2 \max_{i=0:d}\{\|A_i\|_2^2\} \leq 16d^3(\epsilon_2 + 1)^{3/2}(\eta_2 + 1)^{3/2} \max_{i=0:d}\{\|A_i\|_2^2\},
\]

which is the desired upper bound.

Assume now \(|\lambda_0| \leq 1\). Observe that Lemma 1.1 implies \(|L_0\|_2 \geq 1\), since the trailing matrix coefficient \(L_0\) has at least one block entry equal to \(I_n\). Then, from Lemmas 1.1 and 5.1, we easily obtain

\[
\sum_{i=0}^{r-1} |\lambda_0|^i \|R_i\|_2^2 \leq \sum_{i=0}^{r-1} |\lambda_0|^i \|z_L\|_2^2 \leq \sum_{i=0}^{r-1} |\lambda_0|^i \|R_i\|_2 \leq (r + 1) \max_{i=0:r}\{\|R_i\|_2\} \leq (r + 1) \sqrt{(\epsilon_2 + 1)(\eta_2 + 1) \max_{i=0:d}\{\|A_i\|_2\}}. \tag{6.5}
\]

Next, from part (a) in Theorems 3.5 and 3.6, we have

\[
\frac{\|z_R\|_2 \|w_R\|_2}{\|z_L\|_2 \|w_L\|_2} = \frac{\|H(\lambda_0; \epsilon_2, \eta_2, N)x\|_2 \|H(\lambda_0; \eta_2, \epsilon_2, N^*)y\|_2}{\|H(\lambda_0; \epsilon_1, \eta_1, M)x\|_2 \|H(\lambda_0; \eta_1, \epsilon_1, M^*)y\|_2} \leq \frac{\|x\|_2 \|y\|_2}{4d^2(\epsilon_2 + 1)^{3/2}(\eta_2 + 1)^{3/2} \max_{i=0:d}\{\|A_i\|_2^2\}}, \tag{6.6}
\]

where the first inequality follows from the inequalities \(\|H(\lambda; \epsilon_1, \eta_1, M)x\|_2 \geq \|x\|_2\) and \(\|H(\lambda_0; \eta_1, \epsilon_1, M^*)y\|_2 \geq \|y\|_2\), and the second inequality follows from similar arguments to the ones used in the proofs of (4.8) and (4.9). Finally, we obtain from Lemma 4.1

\[
\frac{|w^*_R z_L'(\lambda_0)z_L|}{|w^*_R R'(\lambda_0)z_R|} = \frac{\|y^* P'(\lambda_0)x\|_2}{\|y^* P'(\lambda_0)x\|_2} = 1. \tag{6.7}
\]
Using the inequalities (6.5)—(6.7) to bound (6.1), the desired upper bound readily follows.

We finally observe that the lower bound follows from applying the just established upper bound to \( \text{coeff cond}_L(\lambda_0) / \text{coeff cond}_R(\lambda_0) \).

As an immediate corollary of Theorem 6.1, we obtain the second main result of this work. From the conditioning point of view, Theorem 6.2 establishes that no block Kronecker companion form is more preferable to other block Kronecker companion form, provided we scale the polynomial \( P(\lambda) \) so that \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \).

**Theorem 6.2.** Let \( P(\lambda) \) be an \( n \times n \) matrix polynomial as in (1.1) of degree \( d \). Assume \( d \) is divisible both by \( \ell \) and \( r \), and let \( L(\lambda) \) be a block Kronecker companion \( \ell \)-ification of \( P(\lambda) \), and let \( R(\lambda) \) be a block Kronecker companion \( r \)-ification of \( P(\lambda) \). Assume that \( P(\lambda) \) has been scaled so that \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \). If \( \lambda_0 \) is a finite, nonzero and simple eigenvalue of \( P(\lambda) \), then

\[
\frac{\text{coeff cond}_R(\lambda_0)}{\text{coeff cond}_L(\lambda_0)} \approx 1.
\]

In other words, under the scaling \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \) assumption, no block Kronecker companion form is much better or much worse conditioned than other block Kronecker companion form.

7. Numerical examples. We illustrate the theory on some random and on benchmark matrix polynomials from the NLEVP collection [2]. Our experiments were performed in MATLAB 8, for which the unit roundoff is \( 2^{-53} \approx 10^{-16} \). To obtain condition numbers, we took as exact eigenvalues and eigenvectors the ones computed in MATLAB’s VPA arithmetic at 40 digit precision (except in Section 7.2, which was not possible due to the large size of the problem). The x-axis in all our figures represents eigenvalue index. The eigenvalues are always sorted in increasing order of absolute value.

7.1. Experiment 1: well-conditioned eigenvalues. One of the main predictions of our theory is that well-conditioned eigenvalues can be computed with high relative accuracy as the eigenvalues of block Kronecker companion forms. The goal of this experiment is to verify this prediction. Due to the lack of software for computing eigenvalues of low, but larger than one, degree matrix polynomials, we will focus only on companion linearizations.

We generate a random \( n \times n \) matrix polynomial \( P(\lambda) \) as in (1.1) with degree \( d = 3 \) and size \( n = 30 \). We construct each matrix coefficient \( A_i \) with the MATLAB line command

\[
\text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n);
\]

Then, we compute the eigenvalues of \( P(\lambda) \) as the eigenvalues of the Frobenious companion form

\[
L_1(\lambda) = \begin{bmatrix}
\lambda A_3 + A_2 & A_1 & A_0 \\
-I_n & \lambda I_n & 0 \\
0 & -I_n & \lambda I_n
\end{bmatrix},
\]

(7.1)
a (permuted) Fiedler pencil \( L_2(\lambda) \), a (permuted) generalized Fiedler pencil \( L_3(\lambda) \), and
another block Kronecker pencil \( L_4(\lambda) \), where

\[
\begin{align*}
L_2(\lambda) &= \begin{bmatrix} \lambda A_3 + A_2 & A_1 & -I_n \\ 0 & A_0 & \lambda I_n \\ -I_n & \lambda I_n & 0 \end{bmatrix}, \\
L_3(\lambda) &= \begin{bmatrix} \lambda A_3 + A_2 & 0 & -I_n \\ 0 & \lambda A_1 + A_0 & \lambda I_n \\ -I_n & \lambda I_n & 0 \end{bmatrix}, \quad \text{and} \\
L_4(\lambda) &= \begin{bmatrix} \lambda A_3 - A_2 & \lambda A_2 + A_1 & -I_n \\ \lambda A_2 + A_1 & -\lambda A_1 + A_0 & \lambda I_n \\ -I_n & \lambda I_n & 0 \end{bmatrix}
\end{align*}
\] (7.2)

Observe that the four pencils in (7.1)-(7.2) are block Kronecker companion forms of the matrix polynomial \( P(\lambda) = \sum_{i=0}^{4} A_i \lambda^i \).

All the eigenvalues of the matrix polynomial \( P(\lambda) \) are well-conditioned in the normwise sense [1]. Since \( \max_{i=0:3} \| A_i \|_2 \) is approximately equal to 1, our theory predicts that all the eigenvalues of \( P(\lambda) \) must be computed with high relative accuracy, regardless of the block Kronecker companion employed. This is confirmed in Figure 7.1, where we plot the relative forward errors

\[
\frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|} \quad \lambda_i: \text{exact eigenvalue}, \quad \tilde{\lambda}_i: \text{computed eigenvalue}, \quad \text{(7.3)}
\]

for \( i = 1, 2, \ldots, 90 \), for the four linearizations in (7.1)-(7.2).

![Fig. 7.1. Relative forward errors (7.3) of the computed eigenvalues of a random matrix polynomial \( P(\lambda) \). The eigenvalues of \( P(\lambda) \) were computed as the eigenvalues of the following block Kronecker companion forms: a Frobenius companion form \( L_1(\lambda) \), a (permuted) Fiedler pencil \( L_2(\lambda) \), a (permuted) generalized Fiedler pencil \( L_3(\lambda) \), and a block Kronecker pencil \( L_4(\lambda) \). The pencils \( L_1(\lambda), L_2(\lambda), L_3(\lambda) \) and \( L_4(\lambda) \) are as in (7.1)-(7.2). Observe that all the eigenvalues are computed with high relative accuracy, as predicted by our theory.](image)

7.2. Experiment 2: the condition of block Kronecker \( \ell \)-ifications relative to that of the Frobenius companion form. Another key prediction of our
theory is that under the scaling \( \max_{i=0:d}\{\|A_i\|_2\} = 1 \) assumption the Frobenius companion forms (1.7) and (1.8) are not better (or worse) conditioned than any other block Kronecker companion form. The goal of the following three experiments is to verify this prediction.

In the first experiment, we will compare the conditioning of the block Kronecker companion forms \( L_2(\lambda), L_3(\lambda), L_4(\lambda) \) in (7.2) with that of the Frobenius companion form (7.1). We consider the “plasma drift” matrix polynomial from the NLEVP collection [2]. This is a matrix polynomial with degree \( d = 3 \), size \( n = 128 \) and \( \max_{i=0:d}\{\|A_i\|_2\} \approx 1.2 \times 10^3 \). In Figure 7.2, we plot the ratios

\[
\frac{\text{coeff cond}_{L_i}(\lambda)}{\text{coeff cond}_{L_1}(\lambda)} \quad \text{for } i = 2, 3, 4.
\] (7.4)

for the scaled polynomial (lower figure) and the unscaled polynomial (upper figure). Notice that the results in Figure 7.2 are in complete accordance with our theory: a potentially large or small ratios in the unscaled case, and ratios approximately equal to 1 in the scaled case.

In the second experiment, we will compare the conditioning of the block Kronecker quadratification (Frobenius-like quadratification)

\[
Q(\lambda) = \begin{bmatrix}
\lambda^2 A_4 + \lambda A_3 & \lambda^2 A_2 & \lambda A_1 + A_0 \\
-I_n & \lambda^2 I_n \\
0 & -I_n & \lambda I_n \\
0 & 0 & -I_n & \lambda I_n
\end{bmatrix},
\] (7.5)

and the Frobenius companion form

\[
L(\lambda) = \begin{bmatrix}
\lambda A_4 + A_3 & A_2 & A_1 & A_0 \\
-I_n & \lambda I_n & 0 & 0 \\
0 & -I_n & \lambda I_n & 0 \\
0 & 0 & -I_n & \lambda I_n
\end{bmatrix},
\] (7.6)

both associated with a matrix polynomial of degree 4. We will consider the “Orr-Sommerfeld” matrix polynomial from the NLEVP collection [2]. This polynomial has degree \( d = 4 \), size \( n = 64 \), and \( \max_{i=0:d}\{\|A_i\|_2\} \approx 2^{12} \). In Figure 7.3, we plot the ratio

\[
\frac{\text{coeff cond}_{L}(\lambda_0)}{\text{coeff cond}_{Q}(\lambda_0)}.
\] (7.7)

for the scaled polynomial (red dashed line) and the unscaled polynomial (blue solid line). Figure 7.3 confirms the prediction of our theory: scaling the polynomial guarantees the ratio (7.7) to be moderate.

In the last experiment of this section, we consider a random matrix polynomial \( P(\lambda) \) as in (1.1) with degree \( d = 6 \), size \( n = 10 \), and with badly-scaled matrix coefficients. This matrix polynomial is constructed as follows:

\[
\begin{align*}
A_0 &= \text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n); \\
A_1 &= 1e3 \ast (\text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n)); \\
A_2 &= \text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n); \\
A_3 &= 1e4 \ast (\text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n)); \\
A_4 &= 1e4 \ast (\text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n)); \\
A_5 &= 1e2 \ast (\text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n)); \\
A_6 &= \text{randn}(n) + \text{sqrt}(-1) \ast \text{randn}(n);
\end{align*}
\]
In the experiment, we study the conditioning of the following three block Kronecker companion ℓ-ifications: the linearization (1-ification)

\[
\mathcal{F}(\lambda) = \begin{bmatrix}
\lambda A_6 & \lambda A_5 & \lambda A_4 & -I_n & 0 & 0 \\
0 & 0 & \lambda A_3 & \lambda I_n & -I_n & 0 \\
0 & 0 & \lambda A_2 & 0 & \lambda I_n & -I_n \\
0 & 0 & \lambda A_1 + A_0 & 0 & 0 & \lambda I_n \\
-I_n & \lambda I_n & 0 & 0 & 0 & 0 \\
0 & -I_n & \lambda I_n & 0 & 0 & 0
\end{bmatrix},
\] (7.8)

the quadratification (2-ification)

\[
Q(\lambda) = \begin{bmatrix}
\lambda^2 A_6 + \lambda A_5 & A_2 & -I_n \\
\lambda^2 A_4 + \lambda A_3 & \lambda A_1 + A_0 & \lambda^2 I_n \\
-I_n & \lambda^2 I_n & 0
\end{bmatrix},
\] (7.9)
and the cubification (3-ification)

\[ C(\lambda) = \begin{bmatrix} \lambda^3 A_6 + \lambda^2 A_5 + \lambda A_4 & \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0 \\ -I_n & \lambda I_n \end{bmatrix}, \]  

relative to the conditioning of the Frobenius companion form

\[ L(\lambda) = \begin{bmatrix} \lambda A_6 + A_5 & A_4 & A_3 & A_2 & A_1 & A_0 \\ -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 \\ 0 & 0 & -I_n & \lambda I_n & 0 \\ 0 & 0 & 0 & -I_n & \lambda I_n \end{bmatrix}. \]  

In Figure 7.4, we plot the ratios

\[ \frac{\text{coeff cond}_F(\lambda_0)}{\text{coeff cond}_C(\lambda_0)}, \quad \frac{\text{coeff cond}_Q(\lambda_0)}{\text{coeff cond}_C(\lambda_0)} \quad \text{and} \quad \frac{\text{coeff cond}_C(\lambda_0)}{\text{coeff cond}_C(\lambda_0)}, \]  

for the scaled polynomial (lower figure) and the unscaled polynomial (upper figure). Figure 7.3 validates our theory: scaling the polynomial makes the ratios (7.12) close to 1, regardless of how badly-scaled is the polynomial (i.e., regardless of how small is the quantity \( \min\{\|A_0\|_2, \|A_d\|_2\}\)).

7.3. Experiment 3: the conditioning of block Kronecker companion forms relative to that of the polynomial. Ideally, we would like the block Kronecker companion form \( L(\lambda) \) that we use to solve a PEP to be as well conditioned as the original polynomial \( P(\lambda) \). Our theory predicts that the coefficientwise conditioning of \( L(\lambda) \) is within a factor

\[ \rho(P) := \frac{\max_{i=0:d}\{\|A_i\|_2\}}{\min\{\|A_0\|_2, \|A_d\|_2\}}, \]  

Fig. 7.3. Condition numbers ratio (7.7) for the unscaled (blue solid line) and scaled (red dashed line) “Orr-Sommerfeld” matrix polynomial.
Fig. 7.4. Condition numbers ratios (7.12) for a random matrix polynomial with badly scaled matrix coefficients. Results for the unscaled polynomial are in the upper figure, and results for the scaled polynomial are in the lower figure. The matrix polynomials $F(\lambda)$, $Q(\lambda)$ and $C(\lambda)$ are as in (7.11)–(7.10).

of the coefficientwise conditioning of $P(\lambda)$. Hence, if we scale the matrix polynomial so that $\max_{i=0:d} \{ \|A_i\|_2 \} = 1$, then $L(\lambda)$ and $P(\lambda)$ are guaranteed to have similar condition numbers, provided $\min(\|A_0\|_2, \|A_d\|_2)$ is not too small. The goal of the following two examples is to illustrate this fact, and to show the benefits of scaling the polynomial.

In the first experiment, we consider again the “plasma drift” matrix polynomial from the NLEVP collection [2], the Frobenius companion form $L_1(\lambda)$ in (7.1) and the block Kronecker companion forms $L_2(\lambda)$, $L_3(\lambda)$ and $L_4(\lambda)$ in (7.2). In Figure 7.5, we plot the ratios

$$\frac{\text{coeff cond}_{L_i}(\lambda_0)}{\text{coeff cond}_{P}(\lambda_0)} \quad \text{for } i = 1, 2, 3, 4,$$

(7.14)

for the scaled polynomial (lower figure) and the unscaled polynomial (upper figure).
The unscaled matrix polynomial $\rho(P)$ factor (7.13) of order $10^8$, which explains the large ratios in Figure 7.5. Notice how scaling brings a considerable improvement in the conditioning of the four block Kronecker linearizations. This improvement is predicted by our theory, since the scaled polynomial has a $\rho(P)$ factor (7.13) approximately equal to 100.

![Condition numbers ratios](image)

**Fig. 7.5.** Condition numbers ratios (7.14) for the unscaled (upper figure) and scaled (lower figure) “plasma drift” matrix polynomial. The pencils $L_1(\lambda)$, $L_2(\lambda)$, $L_3(\lambda)$ and $L_4(\lambda)$ are as in (7.1)-(7.2).

In the second experiment, we consider again the “Orr-Sommerfeld” matrix polynomial from the NLEVP collection [2], the Frobenius companion linearization $L(\lambda)$ in (7.6), and the Frobenius-like quadratification $Q(\lambda)$ in (7.5). In Figure 7.6, we plot the ratios

$$\frac{\text{coeff cond}_L(\lambda_0)}{\text{coeff cond}_P(\lambda_0)} \quad \text{and} \quad \frac{\text{coeff cond}_Q(\lambda_0)}{\text{coeff cond}_P(\lambda_0)},$$

for the scaled polynomial (lower figure) and the unscaled polynomial (upper figure). We observe that both the scaled and unscaled matrix polynomials have very large
factors (7.13), which explains the large ratios in Figure 7.6. However, notice the significant improvement that scaling the polynomial brings on the conditioning of the Frobenius companion form.

Fig. 7.6. Condition numbers ratio (7.15) for the “Orr-Sommerfeld” matrix polynomial. Results for the unscaled polynomial are in the upper figure, and results for the scaled polynomial are in the lower figure. The matrix polynomials $Q(\lambda)$ and $L(\lambda)$ are as in (7.5)-(7.6).

8. Conclusions. Several recent papers have systematically addressed the task of broadening the menu of available $\ell$-ifications [3, 7, 11, 14, 24]. Unfortunately, this explosion of new classes of $\ell$-ifications has not been followed by the corresponding analyses of the influence of the $\ell$-ification process on the accuracy and stability of the computed eigenvalues and/or eigenvectors. Only the influence of some classes of linearizations (the Frobenius companion forms and a block tridiagonal linearization [5], linearizations in the $\mathbb{D}L(P)$ vector space [7, 20, 21, 25, 27], and Fiedler matrices [9, 10]) has been studied in the last years. In this work, we have started a systematic study of the numerical influence of $\ell$-ifications. Focusing on the recent family of block Kronecker companion forms [12], we have analyzed the influence of $\ell$-ifications on the
conditioning of the polynomial eigenvalue problem. Our findings lead to two main conclusions. First, block Kronecker companion forms are about as well conditioned as the polynomial itself, provided we scale $P(\lambda)$ so that $\max_{i=0,d}\{\|A_i\|_2\} = 1$, and the quantity $\min_{i=0,d}\{\|A_0\|_2,\|A_d\|_2\}$ is not too small. Second, under the scaling assumption $\max_{i=0,d}\{\|A_i\|_2\} = 1$, any block Kronecker companion form, regardless of its degree or block structure, is about as well-conditioned as the Frobenius companion forms. We hope that the theoretical findings of this work will help to gain confidence on companion forms other than the Frobenius companion forms, and that this will lead to new algorithmic developments.

REFERENCES

[1] D. Armentano and C. Beltrán. The polynomial eigenvalue problem is well conditioned for random inputs. Available as arXiv:1706.00025 (2018).
[2] T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder and F. Tisseur. NLEVP: A Collection of Nonlinear Eigenvalue Problems. ACM Trans. Math. Software 39(2) (2013), Art. 7.
[3] D. A. Bini and L. Robol. On a class of matrix pencils and $\ell$-ifications equivalent to a given matrix polynomial. Linear Algebra Appl., 502 (2016), pp. 275–298.
[4] M. I. Bueno and F. De Terán. Eigenvectors and minimal bases for some families of Fiedler-like linearizations. Lin. Multilin. Algebra, 62(1) (2014), pp. 39–62.
[5] M. I. Bueno, F. M. Dopico, S. Furtado and L. Medina. A block-symmetric linearization of odd degree matrix polynomials with optimal eigenvalue condition number and backward error. To appear in Calcolo, 2018.
[6] M. I. Bueno, F. M. Dopico, J. Pérez, R. Saavedra and B. Zykoski. A simplified approach to Fiedler-like pencils via block minimal bases pencils. Linear Algebra Appl., 547 (2018), pp. 45–104.
[7] F. De Terán, F. M. Dopico and D. S. Mackey. Fiedler companion linearizations and the recovery of minimal indices. SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2181–2204.
[8] F. De Terán, F. M. Dopico and D. S. Mackey. Spectral equivalence of matrix polynomials and the Index Sum Theorem. Linear Algebra Appl., 459 (2014), pp. 264–333.
[9] F. De Terán, F. M. Dopico and J. Pérez. Eigenvalue condition numbers and pseudospectra of Fiedler matrices. Calcolo, 54(1) (2017), pp. 319–365.
[10] F. De Terán, F. M. Dopico and J. Pérez. Backward stability of polynomial root-finding using Fiedler companion matrices. IMA J. Numer. Anal., 36 (2016), pp. 133–173.
[11] F. De Terán, F. M. Dopico, and P. Van Dooren. Constructing $\ell$-ifications from dual minimal bases. Linear Algebra Appl., 495 (2016), PP. 344–372.
[12] F. M. Dopico, P. Lawrence, J. Pérez and P. Van Dooren. Block Kronecker linearizations of matrix polynomials and their backward errors. To appear in Numer. Math. (2018).
[13] F. M. Dopico, J. Pérez and P. Van Dooren. Structured backward error analysis of linearized structured polynomial eigenvalue problems. To appear in Math. Comp. (2018).
[14] F. M. Dopico, J. Pérez and P. Van Dooren. Block minimal bases $\ell$-ifications. To appear in Linear Algebra Appl. (2018).
[15] F. R. Gantmacher. The Theory of Matrices, Vol. I and II (transl.). Chelsea, New York (1959).
[16] I. Gohberg, M. A. Kaashoek and P. Lancaster. General theory of regular matrix polynomials and band Toeplitz operators. Integr. Eq. Oper. Theory, 11 (1988), pp. 776-882.
[17] S. Gätta, R. Van Beeumen, K. Meerbergen and W. Michiels. NLEIGS: A class of fully rational Krylov methods for nonlinear eigenvalue problems. SIAM J. on Sci. Comput., 36 (2014), pp. A2842–A2864.
[18] I. Gohberg, P. Lancaster and L. Rodman. Matrix Polynomials. Academic Press, New York-London (1982).
[19] L. Grammont, N. J. Higham, and F. Tisseur. A framework for analyzing nonlinear eigenproblems and parametrized linear systems. Linear Algebra Appl., 435(3) (2011), pp. 623–640.
[20] N. J. Higham, D. S. Mackey and F. Tisseur. The conditioning of linearizations of matrix polynomials. SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1005–1028.
[21] N. J. Higham, R. -C. Li and F. Tisseur. Backward error of polynomial eigenproblems solved by linearizations. SIAM J. Matrix Anal. Appl., 29 (2006), pp. 143–159.
[22] T. -H. Huang, W. -W. Lin and W. -S. Su. Palindromic quadratization and structure-preserving algorithm for palindromic matrix polynomials of even degree. Numer. Math., 118 (2011), pp. 713–735.
[23] P. Lietaert, J. Pérez, B. Vandereycken and K. Meerbergen. Automatic Rational Approximation and Linearization of Nonlinear Eigenvalue Problems. Submitted for publications. Available as arXiv:1801.08622.

[24] D. S. Mackey, N. Mackey, C. Mehl and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4) (2006), pp. 971–1004.

[25] D. S. Mackey, N. Mackey, C. Mehl and V. Mehrmann. Structured polynomial eigenvalue problems: good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28 (2006), pp. 1029–1051.

[26] A. Mehlman. Polynomial eigenvalue bounds from companion forms. *Lin. Multilin. Algebra*, DOI: 10.1080/03081087.2018.1430118 (2018).

[27] Y. Nakatsukasa, V. Noferini and A. Townsend. Vector spaces of linearizations for matrix polynomials: a bivariate polynomial approach. *SIAM J. Matrix Analysis Appl.*, 38(1) (2017), pp. 1–29.

[28] F. Tisseur. Backward error and condition of polynomial eigenvalue problems. *Linear Algebra Appl.*, 309(1–3) (2000), pp. 339–361.

[29] R. Van Beeumen, K. Meerbergen and W. Michiels. Compact rational Krylov methods for nonlinear eigenvalue problems. *SIAM J. Matrix Analysis Appl.*, 36 (2015), pp. 820–838.