ON THE NORMAL HOLONOMY REPRESENTATION OF
SPACELIKE SUBMANIFOLDS IN PSEUDO-RIEMANNIAN
SPACE FORMS

KORDIAN LÄRZ

Abstract. In this paper we study weakly irreducible holonomy representations of the normal connection of a spacelike submanifold in a pseudo-Riemannian space form. We associate screen representations to weakly irreducible normal holonomy groups and classify the screen representations having the Borel-Lichnerowicz property. In particular, we derive a classification of Lorentzian normal holonomy representations.

1. Introduction

Any spacelike submanifold $M \subset \tilde{M}$ in a pseudo-Riemannian space form $\tilde{M}$ induces an orthogonal decomposition $TM|_M = TM \oplus NM$. The projections define the induced connection $\nabla^\perp$ on the normal bundle $NM$ by

$$\nabla^\tilde{M}X = -A_X + \nabla^\perp_X$$

for $\xi \in \Gamma(NM)$, $X \in \Gamma(TM)$,

where $A_X := -pr_{TM}(\nabla^\tilde{M}_X \xi)$ is the shape operator of $M \subset \tilde{M}$. Using $\nabla^\perp$ we derive a (restricted) holonomy representation which we call the normal holonomy representation and whose Lie algebra is denoted by $\mathfrak{hol}^\perp \subset \mathfrak{so}(N_pM)$. In [Olm90] irreducible normal holonomy representations of submanifolds in Euclidean space forms have been shown to act as the isotropy representation of a semisimple Riemannian symmetric space.

$\mathfrak{hol}^\perp$ is said to act weakly irreducible if all invariant proper subspaces are degenerate. If $A \subset N_pM$ is invariant then $A^\perp$ is invariant. Hence, $\Xi := A \cap A^\perp$ is an invariant, isotropic subspace of $N_pM$. Moreover, we conclude $\mathfrak{hol}^\perp \subset \text{Stab}_{\mathfrak{so}(N_pM)}(\Xi)$. In the following we consider normal holonomy representations where $\Xi$ has maximal dimension. We identify $N_pM$ with $\mathbb{R}^{p,q+p}$ by choosing a pseudo-orthonormal basis $(v_1, \ldots, v_p, e_1, \ldots, e_q, w_1, \ldots, w_p)$ of $N_pM$ where $\Xi = \text{span}(v_1, \ldots, v_p)$, $\Xi^\perp = \text{span}(v_1, \ldots, v_p, e_1, \ldots, e_q)$ and

$$\langle v_i, v_j \rangle = \langle w_1, w_p \rangle = \langle v_i, e_j \rangle = \langle w_i, e_j \rangle = 0, \quad \langle e_i, e_j \rangle = \langle v_i, w_j \rangle = \delta_{ij}.$$

2000 Mathematics Subject Classification. 53C29, 53C50.

Key words and phrases. pseudo-Riemannian space forms, submanifolds, normal holonomy group.

1In fact, throughout this paper we may substitute space forms by arbitrary pseudo-Riemannian spaces of constant curvature.
With respect to this basis we derive the identification

\[
\text{Stab}_{\text{so}(N_p,M)}(\Xi) = \begin{cases}
-X_I^T & A \\
A^T & * \\
-X_p^T & : B \\
0 & B \ 
X_1 \ldots X_p \\
0 & 0 \ -A^T \\
\end{cases}
\]

The projection \( pr_{\text{so}(q)} : \text{Stab}_{\text{so}(N_p,M)} \to \text{so}(q) \) defines the normal screen holonomy algebra \( g := pr_{\text{so}(q)}(\mathfrak{hol}^1) \). In this paper we classify the representations of \( g \) having the Borel-Lichnerowicz property (see below). In particular, if \( g \) acts irreducibly we show the connected Lie subgroup \( G \subset SO(q) \) with \( \text{Lie}(G) = g \) to act as the isotropy representation of a semisimple Riemannian symmetric space. As an application we derive a classification of all normal holonomy representations of spacelike submanifolds in Lorentzian space forms.

2. ON THE CLASSIFICATION OF NORMAL SCREEN HOLONY REPRESENTATIONS

Define the curvature tensor \( R^\perp : \Gamma(TM) \times \Gamma(TM) \times \Gamma(NM) \to \Gamma(NM) \) of \( \nabla^\perp \) by \( R^\perp(X,Y) := [\nabla^\perp_X,\nabla^\perp_Y] - \nabla^\perp_{[X,Y]} \). The key observation is that associated to \( R^\perp \) there is an algebraic curvature tensor \( \mathcal{R} \) on \( NM \) generating the same endomorphism of \( NM \). This idea has already been used in [Olm90]. Its generalization to the pseudo-Riemannian case is straightforward since the Ricci equation

\[
\langle R^\perp(X_1,X_2)\xi_1,\xi_2 \rangle = \langle [A_{\xi_1},A_{\xi_2}]X_1,X_2 \rangle
\]

and the self-adjointness of \( A_\xi \) are still obeyed. Hence, we state it without proof:

**Lemma 2.1.**

1. Let \((e_1,\ldots,e_{\dim M})\) be an orthonormal basis of \( T_pM \). Then

\[
\mathcal{R}_p(\xi_1,\xi_2)\xi_3 := \sum_{i=1}^{\dim M} R^\perp_p(A_{\xi_1}e_i, A_{\xi_2}e_i)\xi_3
\]

is an algebraic curvature tensor on \( N_pM \), i.e.,

\[
\mathcal{R}(\xi_1,\xi_2) = -\mathcal{R}(\xi_2,\xi_1),
\]

\[
\langle \mathcal{R}(\xi_1,\xi_2)\xi_3,\xi_4 \rangle = -\langle \xi_3, \mathcal{R}(\xi_1,\xi_2)\xi_4 \rangle,
\]

\[
\langle \mathcal{R}(\xi_1,\xi_2)\xi_3,\xi_4 \rangle = \langle \mathcal{R}(\xi_3,\xi_4)\xi_1,\xi_2 \rangle,
\]

\[
\mathcal{R}(\xi_1,\xi_2)\xi_3 + \mathcal{R}(\xi_2,\xi_3)\xi_1 + \mathcal{R}(\xi_3,\xi_1)\xi_2 = 0.
\]

Moreover, \( \langle \mathcal{R}(\xi_1,\xi_2)\xi_3,\xi_4 \rangle = -\frac{1}{2} Tr([A_{\xi_1},A_{\xi_2}] \circ [A_{\xi_3},A_{\xi_4}]) \).

2. \( \text{span}\{\mathcal{R}_p(\xi_1,\xi_2) : \xi_1,\xi_2 \in N_pM\} = \text{span}\{R^\perp_p(X,Y) : X,Y \in T_pM\} \).

Let \( \mathcal{R}_{\gamma}^\perp(\xi_1,\xi_2) := \tau^{-1}_{\gamma} \circ \mathcal{R}_{\gamma(1)}(\tau_{\gamma}(\xi_1), \tau_{\gamma}(\xi_2)) \circ \tau_{\gamma} \), where \( \tau_{\gamma} \) denotes the parallel displacement with the normal connection \( \nabla^\perp \) along the piecewise
smooth curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$. Using the Ambrose-Singer theorem we conclude

$$\mathfrak{so}_p^\perp = \text{span}\{\mathcal{R}_{\gamma}(\xi_1, \xi_2) : \xi_1, \xi_2 \in N_pM, \gamma(0) = p\}.$$  

Using the basis $(v_1, \ldots, v_p, e_1, \ldots, e_q, w_1, \ldots, w_p)$ of $N_pM$ and the identification from the introduction we observe

$$\mathfrak{g} = \text{span}\{\text{pr}_\mathcal{E} \circ \mathcal{R}_{\gamma}(\xi_1, \xi_2) |_{\mathcal{E}} : \xi_1, \xi_2 \in N_pM\},$$

where $\mathcal{E} := \text{span}\{e_1, \ldots, e_q\}$. For $1 \leq i, j \leq p$ we define

$$\mathcal{P}_0 := \text{pr}_\mathcal{E} \circ \mathcal{R}_{\gamma}|_{\mathcal{E} \times \mathcal{E} \times \mathcal{E}} \in \Lambda^2\mathcal{E}^* \otimes \mathfrak{g},$$

$$\mathcal{P}_i := \text{pr}_\mathcal{E} \circ \mathcal{R}_{\gamma}(w_i, \cdot)|_{\mathcal{E} \times \mathcal{E}} \in \mathcal{E}^* \otimes \mathfrak{g},$$

$$Q_{ij} := \text{pr}_\mathcal{E} \circ \mathcal{R}_{\gamma}(w_i, w_j)|_{\mathcal{E}} \in \mathfrak{g}.$$

**Definition 2.2.** Let $\mathfrak{h} \subset \mathfrak{so}(\mathcal{E}, h)$ for some Euclidean vector space $(\mathcal{E}, h)$.

1. The space of algebraic curvature tensors with values in $\mathfrak{h}$ is given by

$$\mathcal{K}(\mathfrak{h}) := \{R \in \Lambda^2\mathcal{E}^* \otimes \mathfrak{h} : R(x, y)z + R(y, z)x + R(z, x)y = 0\}.$$  

2. The space of algebraic weak curvature tensors with values in $\mathfrak{h}$ is given by

$$\mathcal{B}_h(\mathfrak{h}) := \{Q \in \mathcal{E}^* \otimes \mathfrak{h} : h(Q(x)y, z) + h(Q(y)z, x) + h(Q(z)x, y) = 0\}.$$  

We will need the following simple observation:

**Proposition 2.3.** $\mathfrak{g} = \text{span}\{\mathcal{P}_0(Y_1, Y_2), \mathcal{P}_k(Y_k), Q_{ij} : Y. \in \mathcal{E}\}.$

**Proof.** Fix $\xi_1, \xi_2 \in N_pM$ and $E \in \mathcal{E}$. Then we have

$$\text{pr}_\mathcal{E}(\mathcal{R}_{\gamma}(\xi_1, \xi_2)E) = \sum_{k=1}^{q} \langle \mathcal{R}_{\gamma}(\xi_1, \xi_2)E, e_k \rangle e_k.$$  

For $i \in \{1, 2\}$ we may write $\xi_i = \alpha_i^j v_j + Y_i + \beta_i^j Z_j$ with $Y_i \in \mathcal{E}$. Then

$$\langle \mathcal{R}_{\gamma}(\xi_1, \xi_2)E, e_k \rangle = \langle \mathcal{R}(\tau_\gamma(\xi_1), \tau_\gamma(\xi_2))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$= \alpha_1^j \alpha_2^k \langle \mathcal{R}(\tau_\gamma(v_j), \tau_\gamma(v_i))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \alpha_1^j \langle \mathcal{R}(\tau_\gamma(v_j), \tau_\gamma(Y_2))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \alpha_2^j \langle \mathcal{R}(\tau_\gamma(Y_1), \tau_\gamma(v_i))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \alpha_1^j \beta_2^k \langle \mathcal{R}(\tau_\gamma(v_j), \tau_\gamma(w_i))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \beta_1^j \alpha_2^k \langle \mathcal{R}(\tau_\gamma(w_j), \tau_\gamma(v_i))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \beta_1^j \langle \mathcal{R}(\tau_\gamma(w_j), \tau_\gamma(Y_2))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \beta_2^j \langle \mathcal{R}(\tau_\gamma(Y_1), \tau_\gamma(w_i))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \beta_1^j \beta_2^k \langle \mathcal{R}(\tau_\gamma(w_j), \tau_\gamma(w_i))\tau_\gamma(E), \tau_\gamma(e_k) \rangle$$

$$+ \langle \mathcal{R}(\tau_\gamma(Y_1), \tau_\gamma(Y_2))\tau_\gamma(E), \tau_\gamma(e_k) \rangle.$$  

Using Lemma 2.1 we derive

$$\langle \mathcal{R}(\tau_\gamma(v_j), \tau_\gamma(v_i))\tau_\gamma(E), \tau_\gamma(e_k) \rangle = -\langle \overbrace{\tau_\gamma(E), \tau_\gamma(e_k)}^{\in \mathcal{E}}, \overbrace{\tau_\gamma(v_j), \tau_\gamma(v_i)}^{\in \mathcal{E}} \rangle = 0.$$
and \( \langle R(\tau_Y(v_j), \tau_Y(Y_2)) \rangle = \langle R(\tau_Y(Y_1), \tau_Y(v_i)) \rangle = \langle R(\tau_Y(Y_2), \tau_Y(v_j)) \rangle = 0 \). Moreover, the Bianchi identity for \( R \) implies
\[
\langle R(\tau_Y(v_j), \tau_Y(w_i)) \rangle = - \langle R(\tau_Y(w_j), \tau_Y(v_i)) \rangle \tau_Y(\tau_Y(v_j)) \\
- \langle R(\tau_Y(E), \tau_Y(v_j)) \rangle \tau_Y(w_i) \tau_Y(\tau_Y(v_j)) \\
= \langle R(\tau_Y(w_j), \tau_Y(e_k)) \rangle \tau_Y(\tau_Y(v_j)) \tau_Y(v_j)
\]
and \( \langle R(\tau_Y(w_j), \tau_Y(v_i)) \rangle = 0 \). Therefore we conclude
\[
\langle R_Y(\xi_1, \xi_2) E, e_k \rangle = \langle P_0(Y_1, Y_2)(E), e_k \rangle \\
+ \langle P_j(\beta^1_1 Y_2 - \beta^1_2 Y_1)(E), e_k \rangle \\
+ \langle \beta^1_1 \beta^1_2 Q_{ij}(E), e_k \rangle.
\]

Remark 2.4. Using the definition and Lemma 2.3 we derive \( P_0 \in \mathcal{K}(\mathfrak{g}) \) and \( P_k \in \mathcal{B}_{h_0}(\mathfrak{g}) \). A Lie algebra \( \mathfrak{h} \) with \( \mathfrak{h} = \text{span}\{Q(x) : x \in \mathcal{E}, Q \in B_{h_0}(\mathfrak{h})\} \) is called weak Berger algebra. The computations above imply that \( \mathfrak{g} \) is a weak Berger algebra if the \( Q_{ij} \) are generated by \( P_0 \) and \( P_k \). This happens, e.g., if \( NM \) has Lorentzian signature. Moreover, representations of weak Berger algebras have been classified in [Lei07]. There each weak Berger algebra is shown to act as the holonomy representation of a Riemannian manifold.

We would like to restrict to irreducible normal screen holonomy representations. For submanifolds in Lorentzian space forms this approach is justified by

**Proposition 2.5.** Let \( \mathfrak{g} \subset \mathfrak{so}(\mathcal{E}) \) be the normal screen holonomy algebra of a submanifold in a Lorentzian space form. Then there is an orthogonal decomposition \( \mathcal{E} = E_0 \oplus \ldots \oplus E_\ell \) such that \( E_0 \) is a trivial submodule and \( E_i \) are irreducible. Moreover, there is a decomposition \( \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\ell \) into commuting ideals such that \( \mathfrak{g}_j \) acts irreducibly on \( E_j \) and trivially on \( E_i \) for \( i \neq j \).

**Proof.** We may use a similar approach as in [BBI93]. If \( V \subset \mathcal{E} \) is invariant then \( V \) decomposes into \( \mathcal{E} = V \oplus V^\perp \) since \( (E, h) \) is Euclidean. Let \( g_V \) be the subalgebra of \( \mathfrak{so}(\mathcal{E}) \) which leaves \( V \) invariant and annihilates \( V^\perp \). Define \( \mathfrak{g}_1 = g_V \cap g_V \) and \( \mathfrak{g}_2 = \mathfrak{g}_1 g_V \). We need to show \( \mathfrak{g} \subset g_V \oplus g_V^\perp \). Let \( Y_1, \bar{Y}_1 \in V \) and \( Y_2, \bar{Y}_2 \in V^\perp \). If the ambient space has Lorentzian signature we conclude \( Q_{ij} = 0 \) for all \( i, j \). Moreover, using the algebraic curvature properties for \( R \) we observe
\[
\mathcal{P}_0(Y_1, Y_2) = 0, \quad \mathcal{P}_0(Y_1, \bar{Y}_1) = 0, \quad \mathcal{P}_0(Y_2, \bar{Y}_2) = 0, \quad \mathcal{P}_k(Y_1) = 0, \quad \mathcal{P}_k(Y_2) = 0.
\]
Hence, the proof follows using Proposition 2.3.

**Definition 2.6.** Let \( \mathfrak{g} \subset \mathfrak{so}(\mathcal{E}) \). We say \( \mathfrak{g} \) has the Borel-Lichnérowicz property if there are decompositions \( \mathcal{E} = E_0 \oplus \ldots \oplus E_\ell \) and \( \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\ell \) such that each \( \mathfrak{g}_j \) acts irreducibly on \( E_j \) and trivially on \( E_i \) for \( i \neq j \).
Remark 2.7. According to Proposition 2.5 the normal screen holonomy algebra \( \mathfrak{g} \) has the Borel-Lichnerowicz property if the ambient space is a Lorentzian space form. It is not known if this is true for ambient spaces with arbitrary signature. In the proof we have only used that the \( \mathcal{K} \) is generated by \( \mathcal{P}_0 \) and \( \mathcal{P}_k \).

Lemma 2.8. Let \( \mathfrak{g} \subset \mathfrak{so}(\mathcal{E}) \) be the normal screen holonomy algebra of a submanifold in a pseudo-Riemannian space form. Assume \( \mathfrak{g} \) has the Borel-Lichnerowicz property with \( \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_\ell \). Then \( \mathcal{K}(\mathfrak{g}_j) \neq 0 \).

Proof. If \( \mathcal{K}(\mathfrak{g}_j) = 0 \) we conclude

\[
\mathfrak{g}_j = \text{span}\{\mathcal{P}_k|_{\mathcal{E}_j}, \mathcal{Q}|_{\mathcal{E}_j} : k \neq 0\}.
\]

using Proposition 2.3. Let \( e_1, e_2, e_3 \in \mathcal{E}_j \) and write \( \tau_{\gamma}e_i = \tilde{e}_i + V_i \) for \( \tilde{e}_i \in \mathcal{E}_j \subset \Xi^\perp \) and \( V_i \in \Xi \). By Lemma 2.1 we have

\[
0 = \langle \mathcal{P}_0(e_2, e_3)e_2, e_3 \rangle = \langle R_{\gamma}(e_2, e_3)e_2, e_3 \rangle
\]

\[
= -\frac{1}{2} \text{Tr}(\mathcal{A}_{\gamma_1}(e_2), \mathcal{A}_{\gamma_3}(e_3)) \circ [\mathcal{A}_{\gamma_1}(e_2), \mathcal{A}_{\gamma_3}(e_3)]).
\]

Since \( M \) is spacelike \( \text{Tr}(\mathcal{A} \circ B^T) \) defines an inner product on skewsymmetric operators. Therefore, we derive \( [\mathcal{A}_{\gamma_1}(e_2), \mathcal{A}_{\gamma_3}(e_3)] = 0 \). On the other hand

\[
\langle \mathcal{P}_k(e_1)e_2, e_3 \rangle = \langle R_{\gamma}(w_k, e_1)e_2, e_3 \rangle
\]

\[
= \langle R(\tau_{\gamma}(w_k), \tau_{\gamma}(e_1))\tau_{\gamma}(e_2), \tilde{e}_3 \rangle + \langle R(\tau_{\gamma}(w_k), \tau_{\gamma}(e_1))\tau_{\gamma}(e_2), V_3 \rangle
\]

\[
= \langle R(\tau_{\gamma}(w_k), \tau_{\gamma}(e_1))\tilde{e}_2, \tilde{e}_3 \rangle + \langle R(\tau_{\gamma}(w_k), \tau_{\gamma}(e_1))V_2, \tilde{e}_3 \rangle
\]

\[
= -\frac{1}{2} \text{Tr}(\mathcal{A}_{\gamma_1}(w_k), \mathcal{A}_{\gamma_1}(e_1) \circ [\mathcal{A}_{\gamma_2}, \mathcal{A}_{\gamma_3}])
\]

and

\[
\langle \mathcal{Q}_{ij}e_2, e_3 \rangle = \langle R_{\gamma}(w_j, e_1)e_2, e_3 \rangle
\]

\[
= -\frac{1}{2} \text{Tr}(\mathcal{A}_{\gamma_1}(w_j), \mathcal{A}_{\gamma_1}(e_1) \circ [\mathcal{A}_{\gamma_2}, \mathcal{A}_{\gamma_3}]).
\]

However, by the Ricci equation we have

\[
0 = \langle R^\perp(X, Y)V_i, \tilde{e}_j \rangle = \langle [\mathcal{A}_{\gamma_1}, \mathcal{A}_{\gamma_2}]X, Y \rangle,
\]

i.e., \( [\mathcal{A}_{\gamma_1}, \mathcal{A}_{\gamma_2}] = 0 \). Hence, we conclude

\[
\langle \mathcal{P}_k(e_1)e_2, e_3 \rangle = -\frac{1}{2} \text{Tr}(\mathcal{A}_{\gamma_1}(w_k), \mathcal{A}_{\gamma_1}(e_1) \circ [\mathcal{A}_{\gamma_2}, \mathcal{A}_{\gamma_3}]) = 0,
\]

\[
\langle \mathcal{Q}_{ij}e_2, e_3 \rangle = -\frac{1}{2} \text{Tr}(\mathcal{A}_{\gamma_1}(w_j), \mathcal{A}_{\gamma_1}(e_1) \circ [\mathcal{A}_{\gamma_2}, \mathcal{A}_{\gamma_3}]) = 0.
\]

Therefore \( \mathfrak{g}_j = 0 \) and we have a contradiction. \( \blacksquare \)

Observe that we did not use any irreducibility conditions in the proof of Lemma 2.3. However, we need the normal screen holonomy to act irreducibly for the proof of our main result:
Theorem 2.9. Let $g \subset \mathfrak{so}(\mathcal{E})$ be the normal screen holonomy algebra of a submanifold in a pseudo-Riemannian space form. Assume $g$ has the Borel-Lichnerowicz property and let $G_j \subset SO(E_j)$ be the connected Lie subgroup with $\text{Lie}(G_j) = g_j$. Then $G_j$ acts irreducibly on $E_j$ as the isotropy representation of a semisimple Riemannian symmetric space if $j > 0$.

Proof. Given the observations from Proposition 2.3 and Lemma 2.8 we can find

$$R_j := \text{pr}_{E_j} \circ \mathcal{R}^{\tau_\gamma}|_{E_j \times E_j \times E_j} \neq 0.$$  

Moreover, $g_j$ and therefore $G_j \subset SO(E_j)$ act irreducibly on $E_j$, i.e., $G_j$ is compact and $(E_j, R_j, G_j)$ is an irreducible holonomy system in the sense of [Sim62]. Hence, the statement follows from [Sim62] [Thm. 5] once we have shown $\text{scal}(R_j) \neq 0$ where $\text{scal}(R_j) = \sum_{k, \ell} \dim E_j \langle R_j(e_k, e_\ell) e_\ell, e_k \rangle$ is the scalar curvature of $R_j$. We compute

$$\dim E_j \sum_{k, \ell=1}^{\dim E_j} \langle R_j(e_k, e_\ell) e_\ell, e_k \rangle = \sum_{k, \ell=1}^{\dim E_j} \langle \mathcal{R}^{\tau_\gamma} e_k, e_\ell \rangle e_\ell, e_k \rangle$$

$$= \sum_{k, \ell=1}^{\dim E_j} \langle \mathcal{R}^\tau (\tau_\gamma(e_k)), \tau_\gamma(e_\ell) \rangle \tau_\gamma(e_\ell), \tau_\gamma(e_k) \rangle$$

$$= -\frac{1}{2} \sum_{k, \ell=1}^{\dim E_j} \text{Tr}([A_{\tau_\gamma(e_k)}, A_{\tau_\gamma(e_\ell)}] \circ [A_{\tau_\gamma(e_\ell)}, A_{\tau_\gamma(e_k)}])$$

$$= -\frac{1}{2} \sum_{k, \ell=1}^{\dim E_j} \{ [A_{\tau_\gamma(e_k)}, A_{\tau_\gamma(e_\ell)}], [A_{\tau_\gamma(e_\ell)}, A_{\tau_\gamma(e_k)}] \}$$

$$< 0,$$

where $\{A, B\} = \text{Tr}(A \circ B^T)$ is the usual inner product on skewsymmetric operators. The last inequality follows since $\langle R_j(e_k, e_\ell) e_\ell, e_k \rangle \neq 0$ for some $k, \ell$ unless $R_j = 0$. \hfill \blacksquare

3. Applications

3.1. Lorentzian normal holonomy representations.

As an application of the previous results we want to derive a classification of all normal holonomy representations of spacelike submanifolds $M \subset \bar{M}$ in Lorentzian space forms. Using the same approach as in [Olm90] we get an orthogonal holonomy invariant splitting

$$NM = NM_0 \oplus NM_L \oplus NM_1 \oplus \ldots \oplus NM_r$$

of the normal bundle and a splitting

$$\text{Hol}^\perp_0(M) = \text{Hol}^\perp_L \times \text{Hol}^\perp_1 \times \ldots \times \text{Hol}^\perp_r$$

of the restricted normal holonomy group such that

$^2$NM_0$ may have Lorentzian signature and $\text{Hol}^\perp_L$ may vanish. E.g., if $M \subset \mathbb{H}^n \subset \mathbb{R}^{1,n}$ where $\mathbb{H}^n$ is the hyperbolic space then using the position vector field we get a timelike $\nabla^\perp$-parallel normal vector field on $M \subset \mathbb{R}^{1,n}$.\footnote{NM_0 may have Lorentzian signature and Hol^\perp_L may vanish. E.g., if M \subset \mathbb{H}^n \subset \mathbb{R}^{1,n} where \mathbb{H}^n is the hyperbolic space then using the position vector field we get a timelike \nabla^\perp-parallel normal vector field on M \subset \mathbb{R}^{1,n}.}$
• \(NM_0\) is a maximal flat and non-degenerate subspace and \(NM_j\) has Riemannian signature for \(j \geq 1\),

• \(Hol_j^\bot\) acts irreducibly on \(NM_j\) for \(j \geq 1\) and trivially on \(NM_i\) for \(i \neq j\),

• \(Hol_L^\bot\) acts weakly irreducibly on \(NM_L\) and trivially on \(NM_i\) for \(i \neq L\).

The actions of \(Hol_j^\bot\) have been classified in [Olm90]. Moreover, if \(Hol_L^\bot \subset SO_0(1,m+1)\) acts irreducibly then by a well known result [DSO01] we conclude \(Hol_L^\bot = SO_0(1,m+1)\). Hence, we have to classify the weakly irreducible but non-irreducible representations. For such representations we have

**Theorem 3.1** (Bérard-Bergery & Ikemakhen [BBI93]). Let \(h \subset \mathfrak{so}(1,m+1)\) be the Lie algebra of the connected Lie group \(H \subset SO(1,m+1)\) acting weakly irreducibly but non-irreducibly. Then \(h \subset (\mathbb{R} \oplus \mathfrak{so}(m)) \times \mathbb{R}^m\). Moreover, using \(g := \text{pr}_{\mathfrak{so}(m)}(h)\) the Lie algebra \(h\) belongs to one of the following types:

- **Type 1**: \(h = (\mathbb{R} \oplus g) \times \mathbb{R}^m\)
- **Type 2**: \(h = g \times \mathbb{R}^m\)
- **Type 3**:

\[
h = \left\{ \begin{pmatrix} \varphi(A) & w^T & 0 \\ 0 & A & w \\ 0 & 0 & -\varphi(A) \end{pmatrix} : A \in g, \ w \in \mathbb{R}^m \right\}
\]

where \(\varphi : g \rightarrow \mathbb{R}\) is an epimorphism satisfying \(\varphi|_{[g,g]} = 0\).

- **Type 4**: There is \(0 < \ell < m\) such that \(\mathbb{R}^m = \mathbb{R}^\ell \oplus \mathbb{R}^{m-\ell}\), \(g \subset \mathfrak{so}(\ell)\) and

\[
h = \left\{ \begin{pmatrix} \psi(A)^T & w^T & 0 \\ 0 & 0 & -\psi(A) \\ 0 & 0 & A & -w \\ 0 & 0 & 0 & 0 \end{pmatrix} : A \in g, \ w \in \mathbb{R}^\ell \right\}
\]

for some epimorphism \(\psi : g \rightarrow \mathbb{R}^{m-\ell}\) satisfying \(\psi|_{[g,g]} = 0\).

In our case the Lie algebra \(g\) in the previous theorem is the normal screen holonomy algebra. Combining the Theorems 3.1, 2.9 and Proposition 2.5 we derive

**Corollary 3.2.** Let \(M\) be a spacelike submanifold in a Lorentzian space form. Then the weakly irreducible part of the normal holonomy representation is given by one of the types in Thm. 3.1 and its normal screen holonomy representation acts as the isotropy representation of a Riemannian symmetric space.

### 3.2. Submanifolds with \(\nabla^\bot\)-parallel second fundamental form.

Let \(M \subset \mathbb{R}^{1,N}\) be a full spacelike submanifold and \(\Pi(X,Y) := \nabla_X^{1,N} Y - \nabla^M_X Y\) its second fundamental form. Moreover, assume \(\nabla^\bot \Pi = 0\). In order
to study the possible normal holonomy groups we will apply ideas from [Fer74]. Let $\xi \in \Gamma(NM)$ and $X, Y, Z \in \Gamma(TM)$. Then

$$\langle (\nabla_X^\perp \Pi)(Y, Z), \xi \rangle = \langle (\nabla_X A_\xi)Y, Z \rangle - \langle A_{\nabla_X^\perp \xi}Y, Z \rangle.$$  

The mean curvature vector field $H := \frac{1}{\dim M} \sum_{i=1}^{\dim M} \Pi(e_i, e_i) = Tr(\Pi)$ is $\nabla^\perp$-parallel and therefore $\nabla_X A_H = 0$. Using [Fer74][Lemma 1] we conclude that $A_H$ has constant eigenvalues $$(\lambda_1, \ldots, \lambda_r)$$ and parallel eigendistributions. Since $M$ is full we may apply [Kat08][Lemma 3.2] and Moore’s Lemma. Hence, $M$ is locally a product of immersions

$$M = M_1 \times \ldots \times M_r \to \mathbb{R}^{1,n} \times \mathbb{R}^{n_2} \ldots \mathbb{R}^{n_r}.$$  

Consider the local normal holonomy of the full immersion $M_1 \to \mathbb{R}^{1,n}$. By construction $\nabla^\perp \Pi_1 = 0$ and $A_H = \lambda \cdot id_{TM}$. The following cases may occur:

Case 1: $H$ is timelike, i.e., the Lorentzian part of the local normal holonomy representation vanishes.

Case 2: $H \neq 0$ and $H \in NM_L$. Then $\|H\| = 0$ and

$$\lambda \langle X, Y \rangle = \langle A_H X, Y \rangle = \langle \Pi_1(X, Y), H \rangle,$$

i.e., $\lambda = \langle H, H \rangle = 0$ and therefore $A_H = 0$. However, this would imply that $M_1$ is not full.

Case 3: $H \notin NM_L$ or $H = 0$ and moreover $Hol_{L,loc}^\perp(M_1) \neq 0$ is of type 2 or 4. Then $\Xi$ is locally spanned by a $\nabla^\perp$-parallel lightlike vector field $\xi \notin 0$. In the same way as for $H$ we conclude $\nabla^\perp(A_\xi) = 0$ and $A_\xi = \mu \cdot id_{TM}$. Since $\mu = \langle \xi, H \rangle = 0$ we derive a contradiction.

Observe that the $Hol^\perp_L(M)$ is only of type 2 or of type 4 if $Hol^\perp_{L,loc}(M_1)$ is somewhere. Hence, we conclude

**Proposition 3.3.** Let $M \subset \mathbb{R}^{1,N}$ be a full spacelike submanifold with $\nabla^\perp \Pi = 0$. Then the normal holonomy is reducible and if $Hol^\perp_L \neq 0$ then it is of type 1 or 3.

I do not know if $M$ can have a normal holonomy representation of type 1 or 3.

### 3.3. Submanifolds of the light cone.

For any spacelike submanifold $M \subset \mathbb{R}^{1,N}$ with $Hol^\perp_L \neq 0$ the distribution $\Xi$ is locally spanned by a $\nabla^\perp$-parallel lightlike vector field $\xi$ if $Hol^\perp_L$ is of type 2 or 4. Now we assume that $\xi$ is globally defined on $M$ and moreover $A_\xi = \lambda \cdot id_{TM}$ for some $\lambda \neq 0$. Consider the position vector field $V : \mathbb{R}^{1,N} \to \mathbb{R}^{1,N}$ with $V(p) = p$. Then $\nabla_{\mathbb{R}^{1,N}} V = id_{\mathbb{R}^{1,N}}$ and therefore

$$\nabla_{\mathbb{R}^{1,N}}^X (\xi + \lambda V) = 0 \quad \text{for} \quad X \in TM.$$  

Hence, we can find $V_0 \in \mathbb{R}^{1,N}$ such that $V = V_0 - \frac{1}{\lambda} \xi$, i.e., after a translation by $V_0$ we have $M \subset L^N$. Here $L^N$ is the light cone in $\mathbb{R}^{1,N}$. Conversely, let $M \subset L^N$ and $V$ be the restriction of the position vector field to $M$. Then $V \in \Gamma(NM)$ and

$$\langle V, V \rangle = 0, \quad \nabla^\perp V = 0, \quad A_V = -id_{TM}.$$  

This follows from $\nabla_{\mathbb{R}^{1,N}}^X V = id_{\mathbb{R}^{1,N}}$ and $0 = X(\langle V, V \rangle) = 2\langle \nabla_{\mathbb{R}^{1,N}}^X V, V \rangle = 2\langle X, V \rangle$ for $X \in TM$.  

Finally, if $M \subset L^N$ has codimension 3 in $\mathbb{R}^{1,N}$ the normal holonomy is either vanishing or of type 2, since $V$ is a lightlike $\nabla^\perp$-parallel vector field, i.e., $1 \times SO(2)$, $SO(1,1) \times 1$ and $SO_0(1,2)$ cannot be the normal holonomy group.

References

[BBI93] L. Bérard-Bergery and A. Ikemakhen, *On the holonomy of Lorentzian manifolds*, Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 27–40. MR MR1216527 (94d:53106)

[DSO01] Antonio J. Di Scala and Carlos Olmos, *The geometry of homogeneous submanifolds of hyperbolic space*, Math. Z. 237 (2001), no. 1, 199–209. MR MR1836778 (2002d:53064)

[Fer74] Dirk Ferus, *Produkt-Zerlegung von Immersionen mit paralleler zweiter Fundamentalform*, Math. Ann. 211 (1974), 1–5. MR MR0367876 (51 #4118)

[Kat08] Ines Kath, *Indefinite extrinsic symmetric spaces I*, http://arxiv.org/abs/0809.4713, 2008.

[Lei07] Thomas Leistner, *On the classification of Lorentzian holonomy groups*, J. Differential Geom. 76 (2007), no. 3, 423–484. MR MR2331527 (2008j:53085)

[Olm90] Carlos Olmos, *The normal holonomy group*, Proc. Amer. Math. Soc. 110 (1990), no. 3, 813–818. MR MR1023346 (91b:53069)

[Sim62] James Simons, *On the transitivity of holonomy systems*, Ann. of Math. (2) 76 (1962), 213–234. MR MR0148010 (26 #5520)

KORDIAN LÄRZ,
HUMBOLDT-UNIVERSITÄT BERLIN,
INSTITUT FÜR MATHEMATIK,
RUDOWER CHAUSSEE 25,
D-12489 BERLIN.
E-mail address: laerz@math.hu-berlin.de