An Averaging Theorem for Perturbed KdV Equation

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Abstract. We consider a perturbed KdV equation:
\[ \dot{u} + u_{xxx} - 6uu_x = \epsilon f(x, u(\cdot)), \quad x \in \mathbb{T}, \quad \int_{\mathbb{T}} u dx = 0. \]

For any periodic function \( u(x) \), let \( I(u) = (I_1(u), I_2(u), \cdots) \in \mathbb{R}_+^\infty \) be the vector, formed by the KdV integrals of motion, calculated for the potential \( u(x) \). Assuming that the perturbation \( \epsilon f(x, u(\cdot)) \) is a smoothing mapping (e.g. it is a smooth function \( \epsilon f(x) \), independent from \( u \)), and that solutions of the perturbed equation satisfy some mild a-priori assumptions, we prove that for solutions \( u(t, x) \) with typical initial data and for \( 0 \leq t \leq \epsilon^{-1} \), the vector \( I(u(t)) \) may be well approximated by a solution of the averaged equation.

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0. Introduction

We consider a perturbed Korteweg-de Vries (KdV) equation with zero mean-value periodic boundary condition:

\[ \dot{u} + u_{xxx} - 6uu_x = \epsilon f(x, u(\cdot)), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \int_{\mathbb{T}} u(x, t) dx = 0. \quad (0.1) \]

Here \( \epsilon f(x, u(\cdot)) \) is a nonlinear perturbation, specified below. For any \( p \in \mathbb{R} \) we denote by \( H^p \) the Sobolev space of order \( p \), formed by real-valued periodic functions with zero mean-value, provided with the homogeneous norm \( \| \cdot \|_p \). Particularly, if \( p \in \mathbb{N} \) we have

\[ H^p = \left\{ u \in L^2(\mathbb{T}) : \|u\|_p < \infty, \int_{\mathbb{T}} u dx = 0 \right\}, \quad \|u\|^2_p = \int_{\mathbb{T}} \left\| \frac{\partial^p u}{\partial x^p} \right\|^2 dx. \]

For any \( p \), the operator \( \frac{\partial}{\partial x} \) defines a linear isomorphism: \( \frac{\partial}{\partial x} : H^p \to H^{p-1} \). Denoting by \((\frac{\partial}{\partial x})^{-1}\) its inverse, we provide the spaces \( H^p, p \geq 0 \), with a symplectic structure by means of the 2-form \( \Omega \):

\[ \Omega(u_1, u_2) = -\left\langle (\frac{\partial}{\partial x})^{-1}u_1, u_2 \right\rangle, \quad (0.2) \]

where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( L^2(\mathbb{T}) \). Then in any space \( H^p, p \geq 1 \), the KdV equation \((\text{I})_{\epsilon=0}\) may be written as a Hamiltonian system with the Hamiltonian \( \mathcal{H} \), given by

\[ \mathcal{H}(u) = \int_{\mathbb{T}} \left( \frac{1}{2} u_x^2 + u^3 \right) dx. \]

That is, KdV may be written as

\[ \dot{u} = \frac{\partial}{\partial x} \nabla \mathcal{H}(u). \]

It is well-known that KdV is integrable. It means that the function space \( H^p \) admits analytic symplectic coordinates \( v = (v_1, v_2, \cdots, v_j) = \Psi(u(\cdot)) \), where \( v_j = (v_j, v_{-j}) \in \mathbb{R}^2 \), such that the quantities \( I_j = \frac{1}{2} |v_j|^2, j \geq 1 \), are actions (integrals of motion), while \( \varphi_j = \text{Arg} v_j, j \geq 1 \), are angles. In the \((I, \varphi)\)-variables, KdV takes the integrable form

\[ \dot{I} = 0, \quad \dot{\varphi} = W(I), \quad (0.3) \]

where \( W(I) \in \mathbb{R}^{\infty} \) is the frequency vector (see [1, 2]). The integrating transformation \( \Psi \), called the nonlinear Fourier transform, for any \( p \geq 0 \) defines an analytic isomorphism \( \Psi : H^p \to h^p \), where

\[ h^p = \left\{ v = (v_1, v_2, \cdots) : |v|^2_p = \sum_{j=1}^{+\infty} (2\pi j)^{2p+1} |v_j|^2 < \infty, v_j \in \mathbb{R}^2, j \in \mathbb{N} \right\}. \]

It is well established that for a perturbed integrable finite-dimensional system,

\[ \dot{I} = \epsilon f(I, \varphi), \quad \dot{\varphi} = W(I) + \epsilon g(I, \varphi), \quad \epsilon << 1, \]

where \( I \in \mathbb{R}^n, \varphi \in \mathbb{T}^n \), on time intervals of order \( \epsilon^{-1} \) the actions \( I(t) \) may be well approximated by solutions of the averaged equation:

\[ \dot{J} = \epsilon \langle f(J) \rangle, \quad f(J) = \int_{\mathbb{T}^n} f(J, \varphi) d\varphi, \]

provided that the initial data \((I(0), \varphi(0))\) are typical (see [3, 4, 5, 6]). This assertion is known as the \textit{averaging principle}. But in the infinite dimensional case, there is no
similar general result. In [7, 8], S. Kuksin and A. Piatniski proved that the averaging principle holds for the randomly perturbed KdV equation of the form:

$$\dot{u} - \epsilon u_{xx} + u_{xxx} - 6uu_x = \sqrt{\epsilon} \eta(t, x), \quad x \in S^1, \quad \int u dx = \int \eta dx = 0, \quad (0.4)$$

where the force $\eta$ is a white noise in $t$, is smooth in $x$ and is non-degenerate. Our goal in this work is to justify the averaging principle for the KdV equation with deterministic perturbations, using the Anosov scheme (see [3]), exploited earlier in the finite dimensional situation. The main technical difficulty to achieve this goal comes from the fact that to perform the scheme one has to use a measure in the function space which is quasi-invariant under the flow of the perturbed equation (it is needed to guarantee that a small 'bad' set which we have to prohibit for a solution of the perturbed equation at a time $t > 0$ corresponds to a small set of initial data). For a reason, explained in Section 3, to construct such a quasi-invariant measure we have to assume that the perturbation $\epsilon f$ is smoothing. More precisely, we assume that:

**Assumption A.** (i) For any $p \geq 0$, the mapping defined by the perturbation in (0.1):

$$\mathcal{P} : H^p \rightarrow H^{p+\zeta_0}, \quad u \mapsto f(x, u(\cdot)), \quad (0.5)$$

is analytic. Here $\zeta_0 > 1$ is a constant.

(ii) For any $p \geq 3$ and $T > 0$, the perturbed KdV equation (0.1) with initial data $u(0) = u_0 \in H^p$,

has a unique solution $u(t, x) \in H^p$ in the time interval $[-T\epsilon^{-1}, T\epsilon^{-1}]$, and

$$||u(t)||_p \leq C(p, ||u_0||_p, T), \quad |t| \leq T\epsilon^{-1}.$$

We are mainly concerned with the behavior of the actions $I(u(t)) \in \mathbb{R}^\infty$ for $|t| \lesssim \epsilon^{-1}$. For this end, it is convenient to pass to the slow time $\tau = \epsilon t$ and write the perturbed KdV equation (0.1) in the action-angle coordinates $(I, \varphi)$:

$$\frac{dI}{d\tau} = F(I, \varphi), \quad \frac{d\varphi}{d\tau} = \epsilon^{-1} W(I) + G(I, \varphi). \quad (0.6)$$

Here $I \in \mathbb{R}^\infty$, $\varphi \in T^\infty$ and $T^\infty := \{\theta = (\theta_i)_{i \geq 1}, \theta_i \in T\}$ is the infinite-dimensional torus, endowed with the Tikhonov topology. The two functions $F(I, \varphi)$ and $G(I, \varphi)$ are the perturbation term $\epsilon f$, written in action-angle variables, see below (1.3) and (1.4). The corresponding averaged equation is

$$\frac{dJ}{d\tau} = \langle F \rangle(J), \quad \langle F \rangle(J) = \int_{T^\infty} F(J, \varphi)d\varphi, \quad (0.7)$$

where $d\varphi$ is the Haar measure on $T^\infty$. It turns out that the (0.7) is a Lipschitz equation, see below (4.17). We denote by $h^p_{I+}$ the image of the space $h^p$ under the action-mapping

$$\pi_I : v \mapsto I, \quad I_j(v) = \frac{1}{2} |v_j|^2, \quad j \geq 1.$$  

Clearly, $I = \pi_I(v) \in h^p_{I+} \subset h^p_I$, where $h^p_I$ is the weighted $l^1$-space

$$h^p_I = \left\{ I \in \mathbb{R}^\infty : |I|_{h^p_I} = |I|_p = 2 \sum_{j=1}^\infty (2\pi j)^{2p+1} |I_j| < \infty \right\}.$$
and $h^p_{I^+}$ is its positive octant, $h^p_{I^+} = \{ I \in h^p_I : I_j \geq 0, \forall j \}$. This is a closed subset of $h^p_I$.

For any $\theta = (\theta_i)_{i \geq 1} \in \mathbb{T}^\infty$, let us denote by $\Phi_\theta$ the linear operator on the space of sequences $(v_1, v_2, \cdots) \in h^p$ which rotates each component $v_j \in \mathbb{R}^2$ by the angle $\theta_j$.

**Definition 0.1** A Gaussian measure $\mu$ on the Hilbert space $h^p$ is said to be $\zeta_0$-admissible (where $\zeta_0 > 1$ is the same as in assumption A), if the following conditions are fulfilled:

(i) It is non-degenerate and has zero mean value.

(ii) It has a diagonal correlation operator $(v_1, v_2, \cdots) \mapsto (\sigma_1 v_1, \sigma_2 v_2, \cdots)$, where every $\sigma_j > 0$, $\sum_{j \geq 1} \sigma_j < \infty$ and $j^{-\zeta_0}/\sigma_j = O(1)$. In particular, $\mu$ is invariant under the rotations $\Phi_\theta$.

Such measures can be written as:

$$
\prod_{j=1}^{+\infty} \left( \frac{2\pi j}{2\pi \sigma_j} \right)^{1+2p} \exp \left\{ -\frac{\left(2\pi j\right)^{1+2p}|v_j|^2}{2\sigma_j} \right\} dv_j, \quad (0.8)
$$

where $dv_j, j \geq 1$, is the Lebesgue measure on $\mathbb{R}^2$ (see [9 10]). Clearly, they are invariant under the KdV flow $(0.3)$.

The main result of this work is the following theorem:

**Theorem 0.2.** Fix any $p \geq 3$ and $\bar{T} > 0$. Let the curve $u^\epsilon(t) \in H^p, |t| \leq \epsilon^{-1}\bar{T}$ be a solution of equation $(0.1)$ and $v^\epsilon(\tau) = \Psi(u^\epsilon(\epsilon^{-1}\tau)), \tau = ct, |\tau| \leq \bar{T}$. If assumption A is fulfilled and $\mu$ is a $\zeta_0$-admissible Gaussian measure on $h^p$, then

(i) For any $\rho > 0$, there exists a Borel subset $\Gamma^\rho_\epsilon$ of $h^p$ and $\epsilon_\rho > 0$ such that $\lim_{\epsilon \to 0} \mu(h^p \setminus \Gamma^\rho_\epsilon) = 0$, and for $\epsilon \leq \epsilon_\rho$ we have

$$
|I(v^\epsilon(\tau)) - J(\tau)|_\rho \leq \rho, \quad \text{for } |\tau| \leq \bar{T}, \quad v^\epsilon(0) \in \Gamma^\rho_\epsilon, \quad (0.9)
$$

where $J(\tau), |\tau| \leq \bar{T}$, is a solution of the averaged equation $(0.7)$ with the initial data $J(0) = \pi_1(v^\epsilon(0))$.

(ii) There is a full measure subset $\Gamma_\varphi$ of $h^p$ with the following property:

If $v^\epsilon(0) \in \Gamma_\varphi$, then for any $0 \leq \bar{T}_1 < \bar{T}_2 \leq \bar{T}$ the image $\mu_{\bar{T}_1, \bar{T}_2}$ of the probability measure $(\bar{T}_2 - \bar{T}_1)^{-1} d\tau$ on $[\bar{T}_1, \bar{T}_2]$ under the mapping $\tau \mapsto \varphi(v^\epsilon(\tau)) \in \mathbb{T}^\infty$ converges weakly, as $\epsilon \to 0$, to the Haar measure $d\varphi$ on $\mathbb{T}^\infty$.

The assertion (ii) of the theorem means that for any bounded continuous function $g(\varphi)$ on $\mathbb{T}^\infty$,

$$
\frac{1}{\bar{T}_2 - \bar{T}_1} \int_{\bar{T}_1}^{\bar{T}_2} g(\varphi(v^\epsilon(\tau)))d\tau \to \int_{\mathbb{T}^\infty} g(\varphi)d\varphi, \quad \epsilon \to 0.
$$

In particular, we have

**Proposition 0.3.** The assumption A holds if in $(0.1)$ $f = f(x)$ is a smooth function, independent from $u$.

It is unknown for us that if the result of Theorem 0.2 remains true for equation $(0.1)$ with non-smoothing perturbations, e.g. if the right hand side of equation $(0.1)$ is $\epsilon u_{xx}$ or $-\epsilon u$. So we do not know whether a suitable analogy of the result in [7 8] holds true if in equation $(0.4)$ the noise $\eta$ vanishes.
An Averaging Theorem for Perturbed KdV Equation

The paper has the following structure: Section 1 is about the transformation which integrates the KdV and its Birkhoff normal form. In Section 2 we discuss the averaged equation. We prove that the \( \zeta_0 \)-admissible Gaussian measures are quasi-invariant under the flow of equation (0.1) in Section 3. Finally in Section 4 and Section 5 we establish the main theorem and Proposition 0.3.

Agreements. Analyticity of maps \( B_1 \to B_2 \) between Banach spaces \( B_1 \) and \( B_2 \), which are the real parts of complex spaces \( B_c^i \) and \( B_c^2 \), is understood in the sense of Fréchet. All analytic maps that we consider possess the following additional property: for any \( R \), a map extends to a bounded analytical mapping in a complex \((\delta_R > 0)\)-neighborhood of the ball \( \{|u|_{B_1} < R\} \) in \( B_c^i \).

Notation. We use capital letters \( C \) or \( C(a_1, a_2, \ldots) \) to denote positive constants that depend on the parameters \( a_1, a_2, \ldots \) but not on the unknown function \( u \). We denote \( Z_{\geq 0} = \{n \in Z, n \geq 0\} \). For an infinite-dimensional vector \( w = (w_1, w_2, \ldots) \) and any \( n \in \mathbb{N} \) we denote \( w^n = (w_1, \ldots, w_n, 0, 0, \ldots) \). We often identify \( w^n \) with a corresponding \( n \)-vector.

1. Preliminaries on the KdV equation

In this section we discuss integrability of the KdV equation (0.1)\(_{\varepsilon=0}\).

1.1. Nonlinear Fourier transform for KdV

We provide the \( L^2 \)-space \( H^0 \) with the Hilbert basis \( \{e_s, s \in Z \setminus \{0\}\} \),

\[
e_s = \begin{cases} 
\sqrt{2} \cos(2\pi sx) & s > 0, \\
\sqrt{2} \sin(2\pi sx) & s < 0.
\end{cases}
\]

Theorem 1.1. There exists an analytic diffeomorphism \( \Psi : H^0 \mapsto h^0 \) and an analytic functional \( K \) on \( h^0 \) of the form \( K(v) = \tilde{K}(I(v)) \), where the function \( \tilde{K}(I) \) is analytic in a suitable neighborhood of the octant \( h^0_{I+} \) in \( h^0_I \), with the following properties:

(i) The mapping \( \Psi \) defines an analytic diffeomorphism \( \Psi : H^p \mapsto h^p \), for any \( p \in Z_{\geq 0} \). This is a symplectomorphism of the spaces \((H^p, \Omega) \) (see (0.2) and \((h^p, \omega_2)\), where \( \omega_2 = \sum dv_k \wedge dv_{-k} \).

(ii) The differential \( d\Psi(0) \) takes the form \( \sum u_s e_s \mapsto v, v_s = |2\pi s|^{-1/2} u_s \).

(iii) A curve \( u \in C^1(0, T; H^0) \) is a solution of the KdV equation (0.1)\(_{\varepsilon=0}\) if and only if \( v(t) = \Psi(u(t)) \) satisfies the equation

\[
\dot{v}_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial \tilde{K}}{\partial I_j}(I)v_j, \quad v_j = (v_j, v_{-j}) \in \mathbb{R}^2, \quad j \in \mathbb{N}.
\]

Since the maps \( \Psi \) and \( \Psi^{-1} \) are analytic, then for \( m = 0, 1, 2 \ldots \), we have

\[
||d^j\Psi(u)||_m \leq P_m(||u||_m), \quad ||d^j\Psi^{-1}(v)||_m \leq Q_m(||v||_m), \quad j = 0, 1, 2,
\]

where \( P_m \) and \( Q_m \) are continuous functions (cf. the agreements).
We denote
\[ W(I) = (W_1, W_2, \ldots), \quad W_k(I) = \frac{\partial \tilde{K}}{\partial I_k}(I), \quad k = 1, 2, \ldots. \]

**Lemma 1.2.** For any \( n \in \mathbb{N} \), if \( I_{n+1} = I_{n+2} = \cdots = 0 \), then
\[
\text{det}
\left(
\left(\frac{\partial W_i}{\partial I_j}\right)_{1 \leq i, j \leq n}
\right) \neq 0.
\]

Let \( l_\infty^{-1} \) be the Banach space of all real sequences \( l = (l_1, l_2, \ldots) \) with the norm
\[
|l|_{-1} = \sup_{n \geq 1} n^{-1}|l_n| < \infty.
\]
Denote \( \kappa = (\kappa_n)_{n \geq 1} \), where \( \kappa_n = (2\pi n)^3 \).

**Lemma 1.3.** The normalized frequency map \( \tilde{W} : I \mapsto \tilde{W}(I) = W(I) - \kappa \)
is real analytic as a map from \( h_1^1 \) to \( l_\infty^{-1} \).

The coordinates \( v = \Psi(u) \) are called the Birkhoff coordinates, and the form (1.1) of KdV is its Birkhoff normal form. See [1] for Theorem 1.1 and Lemma 1.3. A detailed proof of Lemma 1.2 can be found in [2].

### 1.2. Equation (0.1) in the Birkhoff coordinates.

For \( k = 1, 2, \ldots \) we denote:
\[
\Psi_k : H^m \to \mathbb{R}^2, \quad \Psi_k(u) = v_k,
\]
where \( \Psi(u) = v = (v_1, v_2, \ldots) \). Let \( u(t) \) be a solution of equation (0.1). We get
\[
\dot{v}_k = d\Psi_k(u)(\varepsilon f(x, u) + V(u)), \quad k \geq 1,
\]
where \( V(u) = -u_{xxx} + 6uu_x \). Since \( I_k(v) = \frac{1}{2}|\Psi_k|^2 \) is an integral of motion of KdV equation (0.1), we have
\[
\dot{I}_k = \varepsilon(d\Psi_k(u)f(x, u), v_k) := \varepsilon F_k(v),
\]
where \( v_k = (v_{-k}, v_k) \). Denoting for brevity, the vector field in equation (1.4) by \( \dot{I}_k + \varepsilon G_k(v) \), we rewrite the equation for the pair \( (I_k, \varphi_k)(k \geq 1) \) as
\[
\dot{I}_k(t) = \varepsilon F_k(v) = \varepsilon F_k(I, \varphi), \quad \dot{\varphi}_k(t) = W_k(I) + \varepsilon G_k(v).
\]
We set
\[
F(I, \varphi) = (F_1(I, \varphi), F_2(I, \varphi), \ldots).
\]
In the following lemma \( P_k \) and \( P_k^j \) are some fixed continuous functions.

**Lemma 1.4.** For \( k, j \in \mathbb{N} \), we have for any \( p \geq 0 \)
(i) The function \( F_k(v) \) is analytic in each space \( h^p \).

(ii) For any \( p \geq 0, \delta > 0 \), the function \( G_k(v)\chi_{\{I_k>\delta\}} \) is bounded by \( \delta^{-1/2}P_k(|v|_p) \).

(iii) For any \( \delta > 0 \), the function \( \frac{\partial F}{\partial I_j}(I, \varphi)\chi_{\{I_j>\delta\}} \) is bounded by \( \delta^{-1/2}P^j_k(|v|_p) \).

(iv) The function \( \frac{\partial F}{\partial \varphi_j}(I, \varphi) \) is bounded by \( P^j_k(|v|_p) \), and for any \( n \in \mathbb{N} \) and \((I_1, \ldots, I_n) \in \mathbb{R}^n_+ \), the function \( F_k(I_1, \varphi_1, \ldots, I_n, \varphi_n, 0, \ldots) \) is analytic on \( \mathbb{T}^n \).

**Proof:** Items (i) and (ii) follow directly from Theorem 1.1. Items (iii) and (iv) follow from item (i) and the chain-rule:

\[
\frac{\partial F_k}{\partial \varphi_j} = \sqrt{2I_j} \left( \frac{\partial F_k}{\partial v_j} \cos(\varphi_j) - \frac{\partial F_k}{\partial v_j} \sin(\varphi_j) \right),
\]

\[
\frac{\partial F_k}{\partial I_j} = (\sqrt{2I_j})^{-1} \left( \frac{\partial F_k}{\partial v_j} \cos(\varphi_j) + \frac{\partial F_k}{\partial v_j} \sin(\varphi_j) \right). \quad \square
\]

From this lemma we know that equation (1.5) may have singularities at \( \partial h^p_{I_+} \). We denote

\[
\Pi_I : h^p \to h^p_I, \quad \Pi_I(v) = I(v),
\]

\[
\Pi_{I,\varphi} : h^p \to h^p_I \times \mathbb{T}^\infty, \quad \Pi_{I,\varphi}(v) = (I(v), \varphi(v)).
\]

Abusing notation, we will identify \( v \) with \((I, \varphi) = \Pi_{I,\varphi}(v)\).

**Definition 1.5.** For \( p \geq 3 \), we say that a curve \((I(t), \varphi(t)), |t| \leq T\), is a regular solution of equation (1.5), if there exists a solution \( u(t) \in H^p \) of equation (0.1) such that \( u(t) \in H^p \) and

\[
\Pi_{I,\varphi}(\Psi(u(t))) = (I(t), \varphi(t)), \quad |t| \leq T.
\]

If \((I(t), \varphi(t))\) is a regular solution of (1.5) and \( |I(0)|_p \leq M_0 \), then by assumption A we have

\[
|I(t)|_p = |v(t)|_p^2 \leq C(p, M_0, T), \quad |t| \leq T\epsilon^{-1}. \quad (1.6)
\]

**2. Averaged equation**

For a function \( f \) on a Hilbert space \( H \), we write \( f \in Lip_{loc}(H) \) if

\[
|f(u_1) - f(u_2)| \leq P(R)||u_1 - u_2||, \quad ||u_1||, ||u_2|| \leq R,
\]

for a suitable continuous function \( P \) which depends on \( f \). Clearly, the set of functions \( Lip_{loc}(H) \) is an algebra. By the Cauchy inequality, any analytic function on \( H \) belongs to \( Lip_{loc}(H) \) (see agreements). In particularly, for any \( k \geq 1, \)

\[
W_k(I) \in Lip_{loc}(h^p_I), \quad p \geq 1, \quad \text{and} \quad F_k(v) \in Lip_{loc}(h^p), \quad p \geq 0.
\]

In the further analysis, we systematically use the fact that the functional \( F_k(v) \) only weakly depends on the tail of the vector \( v \). Now we state the corresponding results. Let \( f \in Lip_{loc}(h^p) \) and \( v \in h^{p_1}, p_1 > p \). Denoting by \( \Pi^M, M \geq 1 \) the projection

\[
\Pi^M : h^0 \to h^0, \quad (v_1, v_2, \ldots) \mapsto (v_1, \ldots, v_M, 0, \ldots),
\]

\[
\Pi^M : h^0 \to h^0, \quad (v_1, v_2, \ldots) \mapsto (v_1, \ldots, v_M, 0, \ldots),
\]
we have $|v - \Pi^M v|_p \leq (2\pi M)^{-(p_1 - p)}|v|_{p_1}$. Accordingly,
$$
|f(v) - f(\Pi^M v)| \leq P(|v|_{p_1})(2\pi M)^{-(p_1 - p)}. 
$$
(2.2)

The torus $\mathbb{T}^M$ acts on the space $\Pi^M h^0$ by linear transformations $\Phi_{\theta M}, \theta M \in \mathbb{T}^M$, where $\Phi_{\theta M} : (I_M, \varphi_M) \mapsto (I_M, \varphi_M + \theta M)$. Similarly, the torus $\mathbb{T}^\infty$ acts on $h^0$ by linear transformations $\Phi_{\theta} : (I, \varphi) \mapsto (I, \varphi + \theta)$ with $\theta \in \mathbb{T}^\infty$.

For a function $f \in \text{Lip}_{loc}(h^p)$ and a positive integer $N$ we define the average of $f$ in the first $N$ angles as the function
$$
\langle f \rangle_N(v) = \int_{\mathbb{T}^N} f((\Phi_N \oplus \text{Id})(v))d\theta_N,
$$
and define the averaging in all angles as
$$
\langle f \rangle(v) = \int_{\mathbb{T}^\infty} f(\Phi_{\theta}(v))d\theta,
$$
where $d\theta$ is the Haar measure on $\mathbb{T}^\infty$. The estimate (2.2) readily implies that
$$
|\langle f \rangle_N(v) - \langle f \rangle(v)| \leq P(R)(2\pi N)^{-(p_1 - p)}, \quad |v|_{p_1} \leq R.
$$

Let $v = (I, \varphi)$, then $\langle f \rangle_N$ is a function independent of $\varphi_1, \cdots, \varphi_N$, and $\langle f \rangle$ is independent of $\varphi$. Thus $\langle f \rangle$ can be written as $\langle f \rangle(I)$.

**Lemma 2.1.** (See [7]). Let $f \in \text{Lip}_{loc}(h^p)$, then

(i) The functions $\langle f \rangle_N(v)$ and $\langle f \rangle(v)$ satisfy (2.1) with the same function $P$ as $f$ and take the same value at the origin.

(ii) These two functions are smooth (analytic) if $f$ is. If $f$ is smooth, then $\langle f \rangle(I)$ is a smooth function with respect to vector $(I_1, \cdots, I_M)$, for any $M$. If $f(v)$ is analytic in the space $h^p$, then $\langle f \rangle(I)$ is analytic in the space $h^p$.

We recall that a vector $\omega \in \mathbb{R}^n$ is non-resonant if
$$
\omega \cdot k \neq 0, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}.
$$

Denote by $C^{0+1}(\mathbb{T}^n)$ the set of all Lipschitz functions on $\mathbb{T}^n$.

**Lemma 2.2.** Let $f \in C^{0+1}(\mathbb{T}^n)$ for some $n \in \mathbb{N}$. Then for any non-resonant vector $\omega \in \mathbb{R}^n$ we have
$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x_0 + \omega t)dt = \langle f \rangle,
$$
uniformly in $x_0 \in \mathbb{T}^n$. The rate of convergence depends on $n, \omega$ and $f$.

**Proof.** Let us write $f(x)$ as the Fourier series $f(x) = \sum f_k e^{ik \cdot x}$. Since the Fourier series of a Lipschitz function converges uniformly (see [11]), for any $\epsilon > 0$ we may find $R = R_\epsilon$ such that $\sum_{|k| > R} f_k e^{ik \cdot x} \leq \frac{\epsilon}{2}$ for all $x$. Now it is enough to show that
$$
\left| \frac{1}{T} \int_0^T f_R(x_0 + \omega t)dt - f_0 \right| \leq \frac{\epsilon}{2}, \quad \forall T \geq T_\epsilon,
$$
(2.3)
for a suitable \( T_\varepsilon \), where \( f_R(x) = \sum_{|k| \leq R} f_k e^{ik \cdot x} \). Observing that

\[
\left| \frac{1}{T} \int_0^T e^{ik \cdot (x_0 + \omega t)} dt \right| \leq \frac{2}{T|k \cdot \omega|},
\]

for each nonzero \( k \). Therefore the l.h.s of (2.3) is smaller than

\[
2T \left( \inf_{|k| \leq R} |k \cdot \omega| \right)^{-1} \sum_{|k| \leq R} |f_k|.
\]

The assertion of the lemma follows. \( \square \)

3. Quasi-invariance of Gaussian measures

Fix any integer \( p \geq 3 \), and let \( \mu \) be a \( \zeta_0 \)-admissible Gaussian measure on the Hilbert space \( h^p \). In this section we will discuss how this measure evolves under the flow of the perturbed KdV equation (0.1). We follow a classical procedure based on finite-dimensional approximations (see e.g. [12, 10]).

We suppose the assumption A holds. Let us write the equation (0.1) in the Birkhoff normal form, using the slow time \( \tau = \epsilon t \):

\[
\frac{d}{d\tau} v_j = \epsilon^{-1} \mathcal{J} W_j(I) v_j + X_j(v), \quad j \in \mathbb{N},
\]

where \( X_j = (X_j, X_{-j})^t \in \mathbb{R}^2 \) and \( \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \).

For any \( n \in \mathbb{N} \), we consider the \( 2n \)-dimensional subspace \( \pi_n(h^p) \) of \( h^p \) with coordinates \( v^n = (v_1, \ldots, v_n, 0, \ldots, 0) \). On \( \pi_n(h^p) \), we define the following finite-dimensional systems:

\[
\frac{d}{d\tau} \bar{\omega}_j = \epsilon^{-1} \mathcal{J} W_j(I(\omega^n)) \bar{\omega}_j + X_j(\omega^n), \quad 1 \leq j \leq n,
\]

where \( \bar{\omega}_j = (\omega_j, \omega_{-j})^t \in \mathbb{R}^2 \) and \( \omega^n = (\bar{\omega}_1, \ldots, \bar{\omega}_n, 0, \ldots) \in \pi_n(h^p) \).

We denote \( X^n(v^n) = (X_1(v^n), \ldots, X_n(v^n), 0, \ldots) \) and \( X(v) = (X_1(v), \ldots) \). By assumption A and Theorem 1.1, for any \( p \geq 0 \) the mapping

\[
X : h^p \to h^{p+\zeta_0}, \quad v \mapsto X(v) \text{ is analytic.}
\]

Theorem 3.2. For any \( T > 0 \), \( \omega^n(\cdot) \) converges to \( v(\cdot) \) as \( n \to \infty \) in \( C([-T, T]; h^p) \), where \( v(\cdot) \) and \( \omega^n(\cdot) \) are, respectively, solutions of (3.1) and (3.2) with initial data \( v(0) \in h^p \) and \( \omega^n(0) = v^n(0) \in \pi_n(h^p) \).

Proof. Fix any \( M_0 > 0 \). From (1.6) we know that there exists a constant \( M_1 \) such that if \( |v(0)|_p \leq M_0 \), then

\[
|v(\tau)|_p \leq M_1, \quad \tau \in [0, T].
\]

The equation (3.2) yields that

\[
\frac{d}{d\tau} |\omega^n|_p^2 = 2 \sum_{j=1}^n j^{1+2p} \bar{\omega}_j \cdot X_j(\omega^n) := \chi^n(\omega^n).
\]
We define
\[ \chi(v) := 2 \sum_{j=1}^{\infty} j^{1+2p} v_j \cdot X_j(v). \]

By (3.3), we know that there exists a constant \( C_1 > 0 \) such that
\[ |\chi^n(\omega^n)| \leq C_1, \quad |\omega^n|_p \leq 2M_1, \quad \forall n \in \mathbb{N}. \] (3.6)

Denote \( \bar{\tau} = M_1/C_1 \), then if \( |\omega^n(0)|_p \leq M_0 \), then
\[ |\omega^n(\tau)|_p \leq 2M_1, \quad \tau \in [-\bar{\tau}, \bar{\tau}], \quad \forall n \in \mathbb{N}. \] (3.7)

**Lemma 3.3.** In the space \( C([-\bar{\tau}, \bar{\tau}], h^{p-1}) \), we have the convergence
\[ \omega^n(\cdot) \to v(\cdot) \quad \text{as} \quad n \to \infty. \]

**Proof:** Denote \( \bar{\xi}_j = v_j - \bar{\omega}_j \), \( I_v = I(v) \) and \( I_{\omega^n} = I(\omega^n) \). Since \( \mathcal{J}v_j = v_j^\perp \), using equations (3.1) and (3.2), for \( 1 \leq j \leq n \), we get
\[
\frac{d}{d\tau} |\bar{\xi}_j|^2 = 2(\bar{\xi}_j)^t \mathcal{J}(W_j(I_v)v_j - W_j(I_{\omega^n})\bar{\omega}_j) + X_j(v) - X_j(\omega^n))
= 2\epsilon^{-1}[W_j(I_v) - W_j(I_{\omega^n})]v_j \cdot (\bar{\omega}_j)^\perp + 2(\bar{\xi}_j)^t \cdot (X_j(v) - X_j(\omega^n)).
\]

By Lemma 1.3 and Cauchy’s inequality, we know that
\[ |W_j(I(v)) - W_j(I(\omega^n))| \leq C_2(M_1)j|v - \omega^n|_{p-1}. \]

Using (3.3) we get that
\[
\frac{d}{d\tau} |v - \omega^n|_{p-1} \leq C_3(\epsilon, M_1)|v - \omega^n|_{p-1} + a_n(v), \quad \tau \in [-\bar{\tau}, \bar{\tau}],
\]
where
\[ a_n(v) = \sum_{j=n+1}^{\infty} j^{2p-1} v_j \cdot X_j(v). \]

Obviously, \( a_n(v) \to 0 \) as \( n \to \infty \) uniformly for \( |v|_p \leq M_1 \).

The lemma now follows directly from Gronwall’s Lemma. \( \square \)

**Lemma 3.4.** If \( \omega^n(0) \to v(0) \) strongly in \( h^p \) and \( \tau_n \to \tau \), \( \tau_n \in [-\bar{\tau}, \bar{\tau}] \), as \( n \to \infty \), then
\[ \lim_{n \to \infty} |v(\tau) - \omega^n(\tau_n)|_p = 0. \]

**Proof:** From (3.5) we know that for any \( \tau_n \in [-\bar{\tau}, \bar{\tau}] \),
\[ |\omega^n(\tau_n)|_p^2 - |\omega^n(0)|_p^2 = \int_{0}^{\tau_n} \chi^n(\omega(s))ds. \]

Since \( \omega^n(0) \to v(0) \) strongly in \( h^p \), then using (3.3) and Lemma 3.3 we get
\[ |v(\tau)|_p^2 \leq \liminf_{n \to \infty} |\omega^n(\tau_n)|_p^2 \leq \limsup_{n \to \infty} |\omega^n(\tau_n)|_p^2 \]
\[ = \limsup_{n \to \infty} \left( |\omega^n(0)|_p^2 + \int_{0}^{\tau_n} \chi^n(\omega(s))ds \right) = |v(0)|_p^2 + \int_{0}^{\tau} \chi(v(s))ds \]
\[ = |v(\tau)|_p^2. \]
Therefore, \( \lim_{n \to \infty} |\omega^n(\tau_n)|_p = |v(\tau)|_p \). Since \( \omega^n(\tau_n) \to v(\tau) \) in the space \( h^{p-1} \) as \( n \to \infty \), then the required convergence follows. \( \square \)

**Lemma 3.5.** In the space \( C([-\tau, \tau], h^p) \), \( \omega^n(\cdot) \to v(\cdot) \) as \( n \to \infty \).

**Proof.** Suppose this statement is invalid. Then there exists \( \delta > 0 \) and a sequence \( \{\tau^n\}_{n \in \mathbb{N}} \subset [-\tau, \tau] \) such that

\[
|\omega^n(\tau^n) - v(\tau^n)|_p \geq \delta.
\]

Let \( \{\tau^m\}_{m \in \mathbb{N}} \) be a subsequence of the sequence \( \{\tau^n\}_{n \in \mathbb{N}} \) converging to some \( \tau^0 \in [-\tau, \tau] \). But \( v(\tau^m_k) \to v(\tau^0) \) in \( h^p \) as \( k \to \infty \), and using Lemma 3.4, we can get \( \omega^{mk}(\tau^m_k) \to v(\tau^0) \) as \( k \to \infty \) in \( h^p \). So we get a contradiction, and Lemma 3.5 is proved. \( \square \)

If \( T \leq \bar{\tau} \), the theorem is proved, otherwise we iterate the above procedure. This finishes the proof of Theorem 3.2. \( \square \)

Let \( \mathcal{S}_n^\tau \) denote the flow determined by equations (3.1) in the space \( h^p \), and

\[
B_p^v(M) := \{v \in h^p : |v|_p \leq M\}.
\]

**Theorem 3.6.** For any \( M_0 > 0 \) and \( T > 0 \), there exists a constant \( C > 0 \) which depends only on \( M_0 \) and \( T \), such that if \( A \) is a open subset of \( B_p^v(M_0) \), then for \( \tau \in [0, T] \), we have

\[
e^{-C\tau} \mu(A) \leq \mu(\mathcal{S}_n^\tau(A)) \leq e^{C\tau} \mu(A).
\]

**Proof.** From (1.6) we know that there is constant \( M_1 \) which only depends on \( M_0 \) and \( T \), such that if \( v(0) \in B_p^v(M_0) \), then

\[
v(\tau) \in B_p^v(M_1), \quad |\tau| \leq T.
\]

(3.8)

For any \( n \in \mathbb{N} \), consider the measure \( \mu_n = \pi_n \circ \mu \) on the subspace \( \pi_n(h^p) \). Since \( \mu \) is a \( \zeta_0 \)-admissible Gaussian measure, by (0.8) \( \mu_n \) has the following density with respect to the Lebesgue measure:

\[
b_n(v^n) := (2\pi)^{-n} \prod_{j=1}^{n} (2\pi j)^{1+2p}\sigma_j^{-1} \exp\{-\frac{1}{2} \sum_{j=1}^{n} \frac{j^{1+2p}|v_j|^2}{\sigma_j}\}.
\]

Let \( \mathcal{S}_n^\tau \) be the flow determined by equations (3.2) on subspace \( \pi_n(h^p) \). For any open set \( A_n \subset \pi_n(B_p^v(M_0)) \), due to Theorem A in the appendix, we have

\[
\frac{d}{d\tau} \mu_n(\mathcal{S}_n^\tau(A_n)) = \int_{\mathcal{S}_n^\tau(A_n)} \sum_{j=1}^{n} \left( \frac{\partial (b_n(v^n)X_j(v^n))}{\partial v_j} + \frac{\partial (b_n(v^n)X_{-j}(v^n))}{\partial v_{-j}} \right) dv^n
\]

\[
= \int_{\mathcal{S}_n^\tau(A_n)} \sum_{j=1}^{n} j^{2p+1} \left( \frac{v_j X_j + v_{-j} X_{-j}}{\sigma_j} + \frac{\partial X_j}{\partial v_j} + \frac{\partial X_{-j}}{\partial v_{-j}} \right) b_n(v^n) dv^n
\]

\[
:= \int_{\mathcal{S}_n^\tau(A_n)} e^n(v^n) b_n(v^n) dv^n
\]
An Averaging Theorem for Perturbed KdV Equation

Since \( j^{-\zeta_0/\sigma} = O(1) \), using (3.3) and the Cauchy’s inequality, there exists a constant \( C \) which depends only on \( M_1 \), such that
\[
|c_n(v^n)| \leq C, \quad v^n \in \pi_n(B^\nu_p(M_1)), \quad \forall n \in \mathbb{N}.
\] (3.9)

We have
\[
e^{-C\tau} \mu_n(A_n) \leq \mu_n(S^\tau_n(A_n)) \leq e^{C\tau} \mu_n(A_n),
\] (3.10)
as long as \( S^\tau_n(A_n) \subset \pi_n(B^\nu_p(M_1)) \).

Since \( \mu_n \) converges weakly to \( \mu \), the theorem follows from (3.8), (3.10) and Theorem 3.2 (see [12, 10]). □

4. Proof of the main theorem

In this section we prove Theorem 0.2 by developing a suitable infinite-dimensional version of the Anosov scheme (see [3, 4, 5, 6]), and by studying the behavior of the regular solutions of equation (1.5) and the corresponding solutions of (0.1). We fix \( p \geq 3 \). Assume \( u(0) = u_0 \in H^p \). So
\[
\Pi_{I,\varphi}(\Psi(u_0)) = (I_0, \varphi_0) \in h^p_{I_+} \times \mathbb{T}^\infty, \quad p \geq 3.
\] (4.1)

4.1. Proof of the assertion (i)

We denote
\[
B^I_p(M) = \{ I \in h^p_{I_+} : |I|_p \leq M \}.
\]
Without loss of generality, we assume that \( \bar{T} = 1 \) and \( t \geq 0 \).

Fix any \( M_0 > 0 \). Let
\[
(I_0, \varphi_0) \in B^I_p(M_0) \times \mathbb{T}^\infty := \Gamma_0,
\]
that is,
\[
v_0 = \Psi(u_0) \in B^\nu_p(\sqrt{M_0}).
\]

Let \((I(t), \varphi(t))\) be a regular solution of the system (1.5) with \((I(0), \varphi(0)) = (I_0, \varphi_0)\). Then by (1.6), there exists \( M_1 \geq M_0 \) such that
\[
I(t) \in B^I_p(M_1), \quad t \in [0, \epsilon^{-1}].
\] (4.2)

By the definition of the perturbation we know that
\[
|F(I, \varphi)|_1 \leq C_{M_1}, \quad \forall (I, \varphi) \in B^I_p(M_1) \times \mathbb{T}^\infty,
\] (4.3)
where the constant \( C_{M_1} \) depends only on \( M_1 \).

We denote \( I^m = (I_1, \ldots, I_m, 0, 0, \ldots) \), \( \varphi^m = (\varphi_1, \ldots, \varphi_m, 0, 0, \ldots) \), and \( W^m(I) = (W_1(I), \ldots, W_m(I), 0, 0, \ldots) \), for any \( m \in \mathbb{N} \).

Fix \( n_0 \in \mathbb{N} \). By (2.2), for any \( \rho > 0 \), there exists \( m_0 \in \mathbb{N} \), depending only on \( n_0 \) and \( \rho \), such that if \( m \geq m_0 \), then
\[
|F_k(I, \varphi) - F_k(I^m, \varphi^m)| \leq \rho, \quad \forall (I, \varphi) \in B^I_p(M_1) \times \mathbb{T}^\infty,
\] (4.4)
where $k = 1, \cdots, n_0$.

From now on, we always assume that

$$(I, \varphi) \in B^l_p(M_1) \times \mathbb{T}^\infty, \quad \text{i.e.} \quad v \in B^o_p(\sqrt{M_1}).$$

By Lemma 1.4, we have

$$|G_j(I, \varphi)| \leq \frac{C_0(j, M_1)}{\sqrt{T_j}},$$

$$\left| \frac{\partial F_k}{\partial I_j}(I, \varphi) \right| \leq \frac{C_0(k, j, M_1)}{\sqrt{T_j}},$$

$$\left| \frac{\partial F_k}{\partial \varphi_j}(I, \varphi) \right| \leq C_0(k, j, M_1).$$

From Lemma 1.3 and Lemma 2.1, we know that

$$|W_j(I) - W_j(\bar{I})| \leq C_1(j, M_1)|I - \bar{I}|_1,$$

$$|\langle F_k \rangle(I) - \langle F_k \rangle(\bar{I})| \leq C_1(k, j, M_1)|I - \bar{I}|_1. \quad (4.6)$$

By (2.1) we get

$$|F_k(I^m, \varphi^m) - F_k(\bar{I}^m, \bar{\varphi}^m)| \leq C_2(k, m_0, M_1)|v^m - \bar{v}^m|, \quad (4.7)$$

where $| \cdot |$ is the maximum norm.

We denote

$$C^m_{M_1} = m_0 \cdot \max\{C_0, C_1, C_2 : 1 \leq j \leq m_0, 1 \leq k \leq n_0\}.$$ 

Below we define a number of sets, depending on various parameters. All of them also depend on $m_0$ and $n_0$, but this dependence is not indicated. For any $\delta > 0$, and $T_0 > 0$, we define a subset $E(\delta, T_0) \subset B^l_p(M_1)$ as the collection of all $I \in B^l_p(M_1)$ such that for every $\varphi \in \mathbb{T}^\infty$ and any $T \geq T_0$, we have

$$\left| \frac{1}{T} \int_0^T [F_k(I^m, \varphi^m + W^m(I)t) - \langle F_k \rangle(I^m)]dt \right| \leq \delta, \quad (4.8)$$

for $k = 1, \cdots, n_0$. Let $S^t_\epsilon$ be the flow generated by regular solutions of the system (1.5). We define two more groups of sets.

$$S(t) = S(t, \epsilon, \delta, T_0, I, \varphi) := \{t_1 \in [0, t] : S^t_{\epsilon_1}(I, \varphi) \notin E(\delta, T_0) \times \mathbb{T}^\infty\}.$$

$$N(\bar{T}) = N(\bar{T}, \epsilon, \delta, T_0) := \{(I, \varphi) \in \Gamma_0 : \text{Mes}[S(\epsilon^{-1}, \epsilon, \delta, T_0, I, \varphi)] \leq \bar{T}\}.$$

Here and below $\text{Mes}[\cdot]$ stands for the Lebesgue measure in $\mathbb{R}$.

Clearly, $E(\delta, T_0)$ is a closed subset of $B^l_p(M_1)$ and $S(t, \delta, T_0, I, \varphi)$ is an open subset of $[0, t]$. The following result is the main lemma of this work:

**Lemma 4.1.** For $k = 1, \cdots, n_0$, the $I_k$-component of any regular solution of (1.5) with initial data in $N(\bar{T}, \epsilon, \delta, T_0)$ can be written as:

$$I_k(t) = I_k(0) + \epsilon \int_0^t \langle F_k \rangle(I(s))ds + \Xi(t),$$
An Averaging Theorem for Perturbed KdV Equation

where for any $\gamma \in (0, 1)$ the function $|\Xi(t)|$ is bounded on $[0, \frac{1}{\epsilon}]$ by

$$4\epsilon C_{M_1}^{\rho_0, \rho_0} \left\{ \left[ 2(\gamma + 2T_0C_{M_1}\epsilon)^{1/2} \right] (T_0 + \tilde{T} + \epsilon^{-1}) \right. \left. + \left[ \frac{T_0C_{M_1}\epsilon}{\gamma^{1/2}} + T_0C_{M_1}\epsilon + \left( \frac{T_0\epsilon}{2\gamma^{1/2}} + \frac{\epsilon C_{M_1}T_0^2}{3} \right) \right] (T_0 + \tilde{T} + \epsilon^{-1}) \right\}$$

$$+ 2\epsilon C_{M_1} \tilde{T} + 2\rho + 2\delta + 2\epsilon C_{M_1}(T_0 + \tilde{T}).$$

**Proof:** For any $(I, \varphi) \in N(\tilde{T})$, we consider the corresponding set $S(t)$. It is composed of open intervals of total length less than $\min\{\tilde{T}, t\}$. Thus at most $[\tilde{T}/T_0]$ of them have length greater than or equal to $T_0$. We denote these long intervals by $(a_i, b_i), 1 \leq i \leq d, d \leq [\tilde{T}/T_0]$ and denote by $C(t)$ the complement of $\cup_{1 \leq i \leq d}(a_i, b_i)$ in $[0, t]$.

By (4.1), we have

$$\int_0^t F_k(I(s), \varphi(s))dt = \int_{C(t)} F_k(I^{\rho_0}(s), \varphi^{\rho_0}(s))ds + \xi_1(t),$$

where $|\xi_1(t)| \leq C_{M_1}\tilde{T} + pt$.

The set $C(t)$ is composed of segments $[b_{i-1}, a_i]$ (if necessary, we set $b_0 = 0$, and $a_{d+1} = t$). We proceed by dividing each segment $[b_{i-1}, a_i]$ into shorter segments by points $t^i_j$, where $b_i = t^i_1 < t^i_2 < \cdots < t^i_{n_i} = a_i$. The points $t^i_j$ lie outside the set $S(t)$ and $T_0 \leq t^i_{j+1} - t^i_j \leq 2T_0$ except for the terminal segment containing the end points $a_i$, which may be shorter than $T_0$.

This partition is constructed as follows:

---

- If $a_{i-1} - b_{i-1} \leq 2T_0$, then we keep the whole segment with no subdivisions. ($t^i_1 = b_{i-1}$, $t^i_2 = a_i$).

- If $a_{i-1} - b_{i-1} > 2T_0$, we divide the segment in the following way:
  a) If $b_{i-1} + 2T_0$ does not belong to $S(t)$, we chose $t^i_2 = b_{i-1} + 2T_0$, and continue by subdividing $[t^i_2, a_i]$;
  b) if $b_{i-1} + 2T_0$ belongs to $S(t)$, then there are points in $[b_{i-1} + T_0, b_{i-1} + 2T_0]$ which do not, by definition of $b_{i-1}$. We set $t^i_2$ equal to one of these points and continue by subdividing $[t^i_2, a_i]$.

We will adopt the notation: $h^i_j = t^i_{j+1} - t^i_j$ and $s(i, j) = [t^i_j, t^i_{j+1}]$. So

$$C(t) = \bigcup_{i=1}^d \bigcup_{j=1}^{n_i-1} s(i, j), T_0 \leq h^i_j = |s(i, j)| \leq 2T_0, j \leq n_i - 2.$$

By its definition, $C(t)$ contains at most $[\tilde{T}/T_0] + 1$ segments $[b_{i-1}, a_i]$, thus $C(t)$ contains at most $[\tilde{T}/T_0] + 1$ terminal subsegments of length less than $T_0$. Since all other segments have length no less than $T_0$ and $t \leq \frac{1}{\epsilon}$, the number of these segments is not greater than $[\epsilon T_0]^{-1}$. So the total number of subsegments $s(i, j)$ is bounded by

$$1 + [(\epsilon T_0)^{-1} + [\tilde{T}/T_0]].$$
For each segment \( s(i, j) \) we define a subset \( \Lambda(i, j) \) of \( \{1, 2, \ldots, m_0\} \) in the following way:

\[
l \in \Lambda(i, j) \iff \exists t \in s(i, j), \; I_t < \gamma.
\]

If \( l \in \Lambda \), then by (4.3) we have

\[
|I_t| < 2T_0C_{M_1} \epsilon + \gamma, \quad t \in s(i, j).
\]  

(4.9)

For \( I = (I_1, I_2, \ldots) \) and \( \varphi = (\varphi_1, \varphi_2, \ldots) \) we set

\[
\lambda_{i,j}(I) = \hat{I}, \quad \lambda_{i,j}(\varphi) = \hat{\varphi},
\]

where \( \hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, \ldots) \) and \( \hat{I} = (\hat{I}_1, \hat{I}_2, \ldots) \) are defined by the following relation:

If \( l \in \Lambda(i, j) \), then \( \hat{I}_l = 0, \; \hat{\varphi}_l = 0 \); else \( \hat{I}_l = I_l, \; \hat{\varphi}_l = \varphi_l \).

We also denote \( \lambda_{i,j}(I, \varphi) = (\lambda_{i,j}(I), \lambda_{i,j}(\varphi)) \) and when the segment \( s(i, j) \) is clearly indicated, we write for short \( \lambda_{i,j}(I, \varphi) = (\hat{I}, \hat{\varphi}) \).

Then on \( s(i, j) \), using (4.7) and (4.9) we obtain

\[
\int_{s(i, j)} \left| F_k\left(I_{ma}(s), \varphi_{ma}(s)\right) - F_k\left(\lambda_{i,j}(I_{ma}(s), \varphi_{ma}(s))\right) \right| ds \\
\leq \int_{s(i, j)} \left| C_{M_1}^{ma, ma} \right| I_{ma}(s) - \lambda_{i,j}(I_{ma}(s)) \right|^{1/2} ds \\
\leq 2T_0C_{M_1}^{ma, ma} (\gamma + 2T_0C_{M_1} \epsilon)^{1/2}.
\]

In Proposition 1-5 below, \( k = 1, \ldots, n_0 \).

**Proposition 1.**

\[
\int_{C(t)} F_k\left(I_{ma}(s), \varphi_{ma}(s)\right) ds = \sum_{i,j} \int_{s(i, j)} F_k\left(I_{ma}(t^i_j), \varphi_{ma}(s)\right) ds + \xi_2(t),
\]

where

\[
|\xi_2| \leq 4C_{M_1}^{ma, ma} \left(\gamma + 2T_0C_{M_1} \epsilon\right)^{1/2} + \gamma^{-1/2} T_0C_{M_1} \epsilon \left(T_0 + \tilde{T} + \epsilon^{-1}\right).
\]

(4.11)

**Proof:** We may write \( \xi_2(t) \) as

\[
\xi_2(t) = \sum_{i,j} \int_{s(i, j)} \left[ F_k\left(I_{ma}(s), \varphi_{ma}(s)\right) - F_k\left(I_{ma}(t^i_j), \varphi_{ma}(s)\right) \right] ds
\]

\[
:= \sum_{i,j} I(i, j).
\]

For each \( s(i, j) \), we have

\[
\int_{s(i, j)} \left| F_k\left(\hat{I}_{ma}(s), \hat{\varphi}_{ma}(s)\right) - F_k\left(\hat{I}_{ma}(t^i_j), \hat{\varphi}_{ma}(s)\right) \right| ds \\
\leq \int_{s(i, j)} \gamma^{-1/2} C_{M_1}^{ma, ma} \left| \hat{I}_{ma}(s) - \hat{I}_{ma}(t^i_j) \right| ds \\
\leq 2\gamma^{-1/2} T_0^2 C_{M_1} \epsilon.
\]  

(4.12)
We replace the integrand $F_k(I^{mo}, \varphi^{mo})$ by $F_k(I^{mo}, \varphi^{mo})$. Using (4.10) and (4.12) we obtain that

$$I(i, j) \leq 4T_0C_{M_1}^{mo, mo} \left[ (\gamma + 2T_0C_{M_1}\epsilon)^{1/2} + \gamma^{-1/2}T_0C_{M_1}\epsilon \right].$$

The inequality (4.11) follows. □

On each subsegment $s(i, j)$, we now consider the unperturbed linear dynamics $\varphi_j(t)$ of the angles $\varphi^{mo} \in \mathbb{T}^{mo}$:

$$\varphi_j(t) = \varphi^{mo}(t_i^j) + W^{mo}(I(t_i^j))(t - t_i^j) \in \mathbb{T}^{mo}, \quad t \in s(i, j).$$

**Proposition 2.**

$$\sum_{i,j} \int_{s(i,j)} F_k \left( I^{mo}(t_i^j), \varphi^{mo}(s) \right) ds = \sum_{i,j} \int_{s(i,j)} F_k \left( I^{mo}(t_i^j), \varphi(s) \right) ds + \xi_3(t),$$

where

$$|\xi_3(t)| \leq 4C_{M_1}^{mo, mo} (\gamma + 2T_0C_{M_1}\epsilon)^{1/2} (T_0 + \tilde{T} + \epsilon^{-1})$$

$$+ (C_{M_1}^{mo, mo})^2 \left( \frac{2T_0\epsilon}{\gamma} + \frac{4\epsilon C_{M_1}T_0^2}{3} \right) (T_0 + \tilde{T} + \epsilon^{-1}). \quad (4.13)$$

**Proof:** For each $s(i, j)$ we have

$$\int_{s(i,j)} \left| \lambda_{i,j} \left( \varphi^{mo}(s) - \varphi(s) \right) \right| ds$$

$$\leq \int_{s(i,j)} \int_{t_i^j}^s \left| \lambda_{i,j} \left( \epsilon G^{mo}(I(s'), \varphi(s')) + W^{mo}(I(s')) - W^{mo}(I(t_i^j)) \right) \right| ds' ds$$

$$\leq \int_{s(i,j)} \int_{t_i^j}^s C_{M_1}^{mo, mo} \left[ \epsilon \gamma^{-1/2} + |I(s') - I(t_i^j)|_1 \right] ds' ds$$

$$\leq \int_{s(i,j)} C_{M_1}^{mo, mo} \left[ \gamma^{-1/2} \epsilon (s - t_i^j) + \frac{1}{2} C_{M_1}\epsilon(s - t_i^j)^2 \right] ds$$

$$\leq C_{M_1}^{mo, mo} \left( \frac{2T_0^2\epsilon}{\sqrt{\gamma}} + \frac{4\epsilon C_{M_1}T_0^3}{3} \right).$$

Here the first inequality comes from equation (1.4), and using (4.5) and (4.6) we can get the second inequality. The third one follows from (4.3).

Using again (4.5), we get

$$\int_{s(i,j)} \left[ F_k \left( \lambda_{i,j} \left( I^{mo}(t_i^j), \varphi^{mo}(s) \right) \right) - F_k \left( \lambda_{i,j} \left( I^{mo}(t_i^j), \varphi(s) \right) \right) \right] ds$$

$$\leq \int_{s(i,j)} C_{M_1}^{mo, mo} \left| \lambda_{i,j} \left( \varphi^{mo}(s) - \varphi(s) \right) \right| ds$$

$$\leq (C_{M_1}^{mo, mo})^2 \left( \frac{2T_0^2\epsilon}{\sqrt{\gamma}} + \frac{4\epsilon C_{M_1}T_0^3}{3} \right).$$

Therefore (4.13) holds for the same reason as (4.11). □

We will now compare the integral $\int_{s(i,j)} F_k(I^{mo}(t_i^j), \varphi(s)) ds$ with the average value $\langle F_k(I^{mo}(t_i^j)) \rangle h_i^j$. **An Averaging Theorem for Perturbed KdV Equation**
Proposition 3.

\[ \sum_{i,j} \int_{s(i,j)} F_k \left( I^{ma}(t_j^i), \varphi_j^i(s) \right) ds = \sum_{i,j} h_j^i(F_k) \left( I^{ma}(t_j^i) \right) + \xi_4(t), \]

where

\[ |\xi_4(t)| \leq \frac{2\delta}{\epsilon} + 2C_{M_1}(T_0 + \tilde{T}). \]  

(4.14)

Proof: We divide the set of segments \( s(i, j) \) into two subsets \( \Delta_1 \) and \( \Delta_2 \). Namely, \( s(i, j) \in \Delta_1 \) if \( h_j^i \geq T_0 \) and \( s(i, j) \in \Delta_2 \) otherwise.

(i) \( s(i, j) \in \Delta_1 \). In this case, by (4.8), we have

\[ \left| \int_{s(i,j)} \left[ F_k \left( I^{ma}(t_j^i), \varphi_j^i(s) \right) - \langle F_k \rangle \left( I^{ma}(t_j^i) \right) \right] ds \right| \leq \delta h_j^i. \]

So

\[ \sum_{s(i,j) \in \Delta_1} \left| \int_{s(i,j)} F_k \left( I^{ma}(t_j^i), \varphi_j^i(s) \right) ds - \langle F_k \rangle \left( I^{ma}(t_j^i) \right) h_j^i \right| \leq \delta \sum_{s(i,j) \in \Delta_1} h_j^i \leq \frac{2\delta}{\epsilon}. \]

(ii) \( s(i, j) \in \Delta_2 \). Now, using (4.3) we get

\[ \left| \int_{s(i,j)} F_k \left( I^{ma}(t_j^i), \varphi_j^i(s) \right) ds - \langle F_k \rangle \left( I^{ma}(t_j^i) \right) h_j^i \right| \leq 2C_{M_1} h_j^i \leq 2C_{M_1} T_0. \]

Since \( \text{Card}(\Delta_2) \leq (1 + \tilde{T}/T_0) \), then

\[ \sum_{s(i,j) \in \Delta_2} \left| \int_{s(i,j)} F \left( I^{ma}(t_j^i), \varphi_j^i(s) \right) ds - \langle F_k \rangle \left( I^{ma}(t_j^i) \right) h_j^i \right| \leq 2C_{M_1}(\tilde{T} + T_0). \]

This implies the inequality (4.14). \( \square \)

Proposition 4.

\[ \sum_{i,j} h_j^i(F_k) \left( I^{ma}(t_j^i) \right) = \int_{C(t)} \langle F_k \rangle \left( I^{ma}(s) \right) ds + \xi_5(t), \]

where

\[ |\xi_5(t)| \leq 4\epsilon C_{M_1} C^{ma}_{M_1} T_0(T_0 + \tilde{T} + \epsilon^{-1}). \]

(4.15)

Proof: Indeed, as

\[ |\xi_5(t)| = \left| \sum_{i,j} \int_{s(i,j)} \left[ \langle F_k \rangle(I^{ma}(s)) - \langle F_k \rangle(I^{ma}(t_j^i)) \right] ds \right|, \]

using (4.3) and (4.6) we get

\[ |\xi_5(t)| \leq \epsilon \sum_{i,j} \int_{s(i,j)} C_{M_1}^{ma} |I^{ma}(s) - I^{ma}(t_j^i)| ds \]

\[ \leq \epsilon \sum_{i,j} C_{M_1} C_{M_1}^{ma} (h_j^i)^2 \leq 4\epsilon C_{M_1} C^{ma}_{M_1} T_0(T_0 + \tilde{T} + \epsilon^{-1}). \] \( \square \)

Finally,
An Averaging Theorem for Perturbed KdV Equation

Proposition 5.

$$\int_{C(t)} \langle F_k \rangle \left( I^{m_o}(s) \right) ds = \int_0^t \langle F_k \rangle \left( I(s) \right) ds + \xi_6(t),$$

and $|\xi_6(t)|$ is bounded by $C_{M_1}\tilde{T} + \rho t$. □

Gathering the estimates in Propositions 1-5, we obtain

$$I_k(t) = I_k(0) + \epsilon \int_0^t F_k \left( I(s), \varphi(s) \right) ds$$

$$= I_k(0) + \epsilon \int_0^t \langle F_k \rangle \left( I(s) \right) ds + \Xi(t),$$

where

$$|\Xi(t)| \leq \epsilon \sum_{i=1}^6 |\xi_i(t)|$$

$$\leq 4\epsilon C_{M_1}^{r_0, r_0} \left[ 2(\gamma + 2T_0C_{M_1}\epsilon)^{1/2} + \frac{T_0C_{M_1}\epsilon}{\gamma^{1/2}} + T_0C_{M_1}\epsilon \right]$$

$$+ \left( \frac{T_0\epsilon}{2\gamma^{1/2}} + \frac{\epsilon C_{M_1}T_0^2}{3} \right) (T_0 + \tilde{T} + \epsilon^{-1}) + 2\epsilon C_{M_1}T_1$$

$$+ 2\rho + 2\delta + 2\epsilon C_{M_1}(T_0 + \tilde{T}), \quad t \in [0, \frac{1}{\epsilon}].$$

Lemma 4.1 is proved. □

Corollary 4.2. For any $\bar{\rho} > 0$, with a suitable choice of $\rho$, $\gamma$, $\delta$, $T_0$, $\tilde{T}$, the function $|\Xi(t)|$ in Lemma 4.1 can be made smaller than $\bar{\rho}$, if $\epsilon$ is small enough.

Proof: We choose

$$\gamma = \epsilon^\alpha, \quad T_0 = \epsilon^{-\sigma}, \quad \tilde{T} = \frac{\bar{\rho}}{9C_{M_1}\epsilon}, \quad \delta = \rho = \frac{\bar{\rho}}{9}$$

with

$$1 - \frac{\alpha}{2} - \sigma > 0, \quad 0 < \sigma < \frac{1}{2}.$$

Then for $\epsilon$ sufficiently small we have

$$|\Xi(t)| < \bar{\rho}. \quad \Box$$

On the Hilbert space $h^p$, we adopt a $\zeta_0$-admissible Gaussian measure $\mu$. Define corresponding measures $\mu_I = \Pi_1 \circ \mu$ and $\mu_{I, \varphi} = \Pi_{I, \varphi} \circ \mu$ in the spaces $h^p_{I^+}$ and $h^p_{I^+} \times \mathbb{T}^\infty$.

Lemma 4.3. The measure $\mu_{I, \varphi}$ is a product measure $d\mu_{I, \varphi} = d\mu_I d\varphi$, where $d\varphi$ is the Haar measure on $\mathbb{T}^\infty$.

Proof: Since the measure $\mu$ is invariant under rotations $\Phi_\theta$, the $\Pi_\varphi \circ d\mu$ is a measure on $\mathbb{T}^\infty$, invariant under the rotations. So this is the Haar measure $d\varphi$. Consequently the image of the measure $\mu_{I, \varphi}$ under the natural projection $(I, \varphi) \mapsto \varphi$ is $d\varphi$. Since the spaces $h^p_{I^+}$ and $\mathbb{T}^\infty$ are separable, then for $\varphi \in \mathbb{T}^\infty$ there exists a Borel probability
measure \( \pi_\varphi(dI) \) on \( h_{1+}^p \) such that \( \mu_{I, \varphi} = \pi_\varphi(dI) d\varphi \). That is, for any bounded continuous function \( f(I, \varphi) \), we have
\[
\langle \mu_{I, \varphi}, f \rangle = \int_{\mathbb{T}^\infty} \left( \int_{h_{1+}^p} f(I, \varphi) \pi_\varphi(dI) \right) d\varphi.
\]
(see e.g. [9]). For any \( \theta \in \mathbb{T}^\infty \) we have
\[
\langle \mu_{I, \varphi}, f \rangle = \langle \mu_{I, \varphi}, f \circ \Phi_\theta \rangle \quad = \quad \int \int f(I, \varphi + \theta) \pi_\varphi(dI) d\varphi = \int \int f(I, \varphi) \pi_{\varphi - \theta}(dI) d\varphi.
\]
Integrating in \( d\theta \) we see that
\[
\mu_{I, \varphi}(dId\varphi) = d\mu'(dI)d\varphi,
\]
where \( d\mu'(dI) = \int_{\mathbb{T}^\infty} \pi_\theta(dI) d\varphi \). We must have \( d\mu' = d\mu_I \), and the assertion of the lemma is proved. \( \square \)

The two lemmas below deal with the sets \( E \) and \( N \), defined at the beginning of this section.

**Lemma 4.4.** For any \( \delta > 0 \), \( \lim_{T_0 \to \infty} \mu_I(B_{p}^I(M_1) \setminus E(\delta, T_0)) = 0 \).

**Proof.** From the definition of \( E(\delta, T_0) \), we know that
\[
E(\delta, T_0) \subset E(\delta, T_0'), \quad \text{if} \quad T_0 \leq T_0'.
\]
Let \( E_{\infty}(\delta) := \bigcup_{T_0 > 0} E(\delta, T_0) \). Due to the inclusion above we have to check that
\[
\mu_I(B_{p}^I(M_1) \setminus E_{\infty}(\delta)) = 0.
\]
Denote
\[
\mathcal{R}(N) := \bigcup_{L \in \mathbb{Z}^{2m_0} \setminus \{0\}, |L| \leq N} \{ I \in B_{p}^I(M_1) : \quad W^{m_0}(I) \cdot L = 0 \},
\]
where \( W^{m_0}(I) = (W_1(I), \ldots, W_{m_0}(I)) \). Let us write \( F_k(I^{m_0}, \varphi^{m_0}) \) as a Fourier series
\[
F_k(I^{m_0}, \varphi^{m_0}) = \sum_{L \in \mathbb{Z}^{m_0}} F_k^L e^{iL \cdot \varphi^{m_0}},
\]
where \( F_k^L = F_k^L(I^{m_0}) \). Then there exists \( N_0 > 0 \) such that
\[
\left| F_k(I^{m_0}, \varphi^{m_0}) - \sum_{|L| \leq N_0} F_k^L e^{iL \cdot \varphi^{m_0}} \right| < \frac{\delta}{2}, \quad k = 1, \ldots, n_0.
\]
Arguing as in the proof of Lemma 2.2, we see that if \( I \notin \mathcal{R}(N_0) \), then
\[
\left| \sum_{0 \neq |L| \leq N_0} \frac{1}{T_0} \int_0^{T_0} F_k^L e^{iL \cdot W^{m_0}t} dt \right| \leq \frac{2}{T_0} \left( \inf_{0 \neq |L| \leq N_0} |L \cdot W^{m_0}| \right)^{-1} \sum_{|L| \leq N_0} |F_k^L|.
\]
where \( W^{m_0} = W^{m_0}(I) \). The r.h.s of the above inequality can be made smaller than \( \delta/2 \) by choosing \( T_0 \) large enough. So we have
\[
B_{p}^I(M_1) \setminus \mathcal{R}(N_0) \subset E_{\infty}(\delta),
\]
and it remains to show that
\[
\mu_I(\mathcal{R}(N_0)) = 0.
\]
By Lemma 1.2,
\[ W^m(I) \cdot L \neq 0, \quad \forall L \in \mathbb{Z}^m \setminus \{0\}. \]
Since \( W^m(I) \) is analytic with respect to \( I \) and \( \mu_I \) is a non-degenerated Gaussian measure, then due to Theorem 1.6 in [13], for any \( L \in \mathbb{Z}^m \), we have
\[ \mu_I(\{ I \in h^p : W^m(I) \cdot L = 0 \}) = 0. \]
Therefore,
\[ \mu_I(\mathcal{R}(N)) = 0. \]

**Lemma 4.5.** Fix any \( \delta > 0, \bar{\rho} > 0 \). Then for every \( \nu > 0 \) we can find \( T_0 > 0 \) such that
\[ \mu_{I, \varphi}(\Gamma_0 \setminus N) < \nu, \]
where \( N = N(\rho, \delta, T_0). \)

**Proof.** Let us denote \( \Gamma_E = E(\delta, T_0) \times \mathbb{T}^\infty \), \( \Gamma_1 = B^t_{\bar{\rho}}(M_1) \times \mathbb{T}^\infty \) and \( \Gamma^\infty_E := \bigcup_{T_0 > 0} \Gamma_E(\delta, T_0) \). Since the sets \( \Gamma_E(\delta, T_0) \) are increasing with \( T_0 \), then from Lemmas 4.3 and 4.4 we know that
\[ \lim_{T_0 \to \infty} \mu_{I, \varphi}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)) = \mu_{I, \varphi}(\Gamma_1 \setminus \Gamma^\infty_E) = 0. \] (4.16)

Let \( d\mu_I \) be the measure \( d\mu dt \) on \( h^p \times \mathbb{R} \), and \( S^t_{v, \varepsilon} \) be the flow of the perturbed KdV equation [12] on \( h^p \). We now define following subset of \( h^p \times \mathbb{R} \):
\[ B' = \{(v, t) : S^t_{v, \varepsilon}(v) \in \Pi_{I, \varphi}^{-1}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)), v \in B^u_{\rho}(\sqrt{M_0}), t \in [0, \frac{1}{\varepsilon}] \}. \]

By Theorem 3.6, there exists a constant \( C_2(M_1) \) depending only on \( M_1 \) such that
\[ \mu_1(B') = \int_0^{\varepsilon^{-1}} \mu\left(S_{v, \varepsilon}^{-t}\left(\Pi_{I, \varphi}^{-1}(\Gamma_1 \setminus \Gamma_E(\delta, T_0))\right) \cap \Pi_{I, \varphi}^{-1}(\Gamma_0)\right)dt \leq \frac{1}{\varepsilon} e^{C_2(M_1)} \mu_1\left(\Pi_{I, \varphi}^{-1}(\Gamma_0)\right) \]
\[ = \frac{1}{\varepsilon} e^{C_2(M_1)} \mu_{I, \varphi}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)). \]

For \( v \in \Pi_{I, \varphi}^{-1}(\Gamma_0) \), we define
\[ S(I, \varphi) = S(v) = \{ t \in [0, \varepsilon^{-1}] : S^t_{v, \varepsilon}(v) \in B^u_{\rho}(\sqrt{M_1}) \cap \Pi_{I, \varphi}^{-1}(\Gamma_E(\delta, T_0)) \}. \]

By the Fubini theorem, we have
\[ \mu_1(B') = \int_{\Pi_{I, \varphi}^{-1}(\Gamma_0)} \text{Mes}(S(v))\mu(dv), \]

Thus
\[ \mu_{I, \varphi}(\Gamma_0 \setminus N) = \mu_{I, \varphi}\left(\{ (I, \varphi) \in \Gamma_0 : \text{Mes}(S(I, \varphi) > \frac{\bar{\rho}}{9C_2(M_1)}\} \right) \leq \frac{9C_2(M_1)}{\bar{\rho}} \mu_{I, \varphi}(\Gamma_1 \setminus \Gamma_E(\delta, T_0)), \]

by the Chebyshev inequality. In view of (4.16), the term on the right hand side becomes arbitrary small when \( T_0 \) is large enough. The statement of Lemma 4.5 follows. □
An Averaging Theorem for Perturbed KdV Equation

We pass to the slow time $\tau = \epsilon t$. Let $v^e(\tau)$, $\tau \in [0, 1]$, be a solution of the equation (3.1) and $(I^e(\tau), \varphi^e(\tau)) = \Pi_{t,\varphi}(v^e(\tau))$.

By Lemma 2.1 and (3.3), we know that for any $p \geq 0$, the mapping

$$F_J : h^0_\tau \to h^{l+\alpha}_\tau, \quad J \mapsto \langle F \rangle(J),$$

where $\langle F \rangle(J) = (\langle F_1 \rangle(J), \langle F_2 \rangle(J), \ldots)$ is analytic. Hence, there exists $C_3(M_1)$ such that

$$|F_J(J_1) - F_J(J_2)|_p \leq C_3(M_1)|J_1 - J_2|_p, \quad J_1, J_2 \in B_p^l(2M_1). \quad (4.17)$$

Using Picard’s theorem, for any $J_0 \in B_p^l(M_1)$ there exists a unique solution $J(t)$ of the averaged equation (0.7) with $J(0) = J_0$. We denote

$$T(J_0) := \inf \{ \tau > 0 : |J(\tau)|_p > 2M_1 \}.$$

Now we are in a position to prove the assertion (i) of Theorem 0.2.

For any $\bar{\rho} > 0$, there exist $n_1$ such that

$$|F(I, \varphi) - F^{n_1}(I, \varphi)|_p < \frac{\bar{\rho}}{8} e^{-C_3(M_1)}, \quad (I, \varphi) \in B_p^l(2M_1) \times \mathbb{T}^\infty,$$

$$|\langle F \rangle(J) - \langle F \rangle^{n_1}(J)|_p < \frac{\bar{\rho}}{8} e^{-C_3(M_1)}, \quad J \in B_p^l(2M_1).$$

Choose $\rho_0$ such that

$$\sum_{j=1}^{n_1} j^{1+2p} \rho_0 = \bar{\rho} e^{-C_3(M_1)}.$$

By Lemmata 4.1 and 4.2, there is a set $\Gamma_\rho = N(\frac{\rho_0}{9C_3(M_1)}, \epsilon, \frac{\rho_0}{9}, \epsilon^{-\sigma})$, $\sigma < 1/2$, such that if $\epsilon$ is small enough and $(I^e(0), \varphi^e(0)) \in \Gamma_\rho$, then

$$I_k(\tau) = I_k(0) + \int_0^\tau \langle F_k \rangle(I(s)) ds + \xi_k(\tau), \quad |\xi_k(\tau)| < \rho_0, \quad \tau \in [0, 1],$$

for $k = 1, \ldots, n_1$. Therefore, by (4.17) and (4.18),

$$|I(\tau) - J(\tau)|_p \leq \int_0^\tau C_3(M_1)|I(\tau) - J(\tau)|_p ds + \xi_0(\tau), \quad |\xi_0(\tau)| \leq \frac{\bar{\rho}}{2} e^{-C_3(M_1)},$$

for $(I(0), \varphi(0)) \in \Gamma_\rho$, $I(0) = J(0)$ and $|\tau| \leq \min\{1, T(J(0))\}$. By Gronwall’s lemma,

$$|I(\tau) - J(\tau)|_p \leq \bar{\rho}, \quad |\tau| \leq \min\{1, T(J(0))\}.$$

Assuming that $\bar{\rho} << M_1$, we get from the definition of $T(J(0))$ that $T(J(0))$ is bigger than 1. This establishes inequality (0.9). From Lemma 4.5 we know that $\lim_{\epsilon \to 0} \mu_{t,\varphi}(\Gamma_0 - \Gamma_\rho) = 0. \quad \Box$

4.2. Proof of the assertion (ii)

It is not hard to see that the assertion for any $0 \leq T_1 < T_2 \leq 1$ would follow if we can prove it for $\bar{T}_1 = 0, \bar{T}_2 = 1$. So we assume that $\bar{T}_1 = 0$, and $\bar{T}_2 = 1$. For any $(m, n) \in \mathbb{N}^2$, we fix $\alpha < 1/8$, and denote

$$\mathcal{B}_m(\epsilon) := \left\{ I \in B_p^l(M_1) : \inf_{k \leq m} |I_k| < \epsilon^\alpha \right\},$$

$$\mathcal{R}_{m,n}(\epsilon) := \bigcup_{|L| \leq n, L \in \mathbb{Z}^m \setminus \{0\}} \left\{ I \in B_p^l(M_1) : |W(I) \cdot L| < \epsilon^\alpha \right\}.$$
Then let
\[ \Upsilon_{m,n}(\epsilon) = \left( \bigcup_{m_0 \leq m} \mathcal{R}_{m_0,n}(\epsilon) \right) \cup \mathcal{B}_m(\epsilon). \]

Denote
\[ S(\epsilon, m, n, I_0, \varphi_0) = \{ \tau \in [0, 1] : I^\epsilon(\tau) \in \Upsilon_{m,n}(\epsilon) \} \]
and fix any \( \nu > 0 \). Then using Theorem 3.6 and arguing as in Lemma 4.4 and Lemma 4.5, we get that, for any \((m, n) \in \mathbb{N}^2\), there exists open subset \( \Gamma_{m,n} \subset \Gamma_0 \), \( \epsilon_{m,n} > 0 \) and a positive function \( \rho_{m,n}(\epsilon) \), converging to zero as \( \epsilon \to 0 \), such that
\[ \mu_{I,\varphi}(\Gamma_0 - \Gamma_{m,n}^\epsilon) < \frac{\nu}{2mn} \quad \text{and} \quad \operatorname{Mes}(S(\epsilon, m, n, I_0, \varphi_0)) \leq \rho_{m,n}(\epsilon), \]
if \((I_0, \varphi_0) \in \Gamma_{m,n}^\epsilon \) and \( \epsilon \leq \epsilon_{m,n} \). Let
\[ \Gamma_{\nu} = \bigcap_{(m,n) \in \mathbb{N}^2} \Gamma_{m,n}^\nu, \]
then
\[ \mu_{I,\varphi}(\Gamma_0 - \Gamma_{\nu}) < \nu. \quad \text{(4.19)} \]

The sets \( \Gamma_{\nu} \) may be chosen in such a manner that
\[ \Gamma_{\nu_1} \subset \Gamma_{\nu_2}, \quad \text{if} \quad \nu_2 < \nu_1. \quad \text{(4.20)} \]

For any \((I_0, \varphi_0) \in \Gamma_{\nu}\), consider a solution \((I^\epsilon(\tau), \varphi^\epsilon(\tau))\) such that
\[ (I^\epsilon(0), \varphi^\epsilon(0)) = (I_0, \varphi_0). \]

Fix \( m \in \mathbb{N} \), take a bounded Lipschitz function \( g \) defined on the torus \( T^m \subset T^\infty \) such that \( \text{Lip}(g) \leq 1 \) and \( |g|_{L^\infty} \leq 1 \). Let \( \sum_{s \in \mathbb{Z}^m} g_s e^{is \cdot \varphi} \) be its Fourier series. Then for any \( \rho > 0 \), there exists \( n \), such that if we denote \( \bar{g}_n = \sum_{|s| \leq n} g_s e^{is \cdot \varphi} \), then
\[ \left| g(\varphi) - \bar{g}_n(\varphi) \right| < \frac{\rho}{2}, \quad \forall \varphi \in T^m. \]

For any \((I_0, \varphi_0) \in \Gamma_{\nu}\), we consider the set \( S(\epsilon, m, n, I_0, \varphi_0) \). It is composed of open intervals of total length less than \( \bar{T} = \rho_{m,n}(\epsilon) \). Proceeding as in Lemma 4.1 and Corollary 4.2, we find that for \( \epsilon \) small enough we have
\[ \left| \int_{0}^{1} g(\varphi^\epsilon,m(\tau))d\tau - \int_{T^m} g(\varphi)d\varphi \right| < \rho. \]
That is,
\[ \left| \int g(\varphi) \mu_{T_1, T_2}(d\varphi) - \int g(\varphi)d\varphi \right| \to 0 \quad \text{as} \quad \epsilon \to 0, \quad \text{(4.21)} \]
for any Lipschitz function as above. Hence, \( \mu_{T_1, T_2} \) converges weakly to \( d\varphi \) (see [9]). This proves the required assertion with \( \Gamma_{\varphi} \) replaced by \( \Gamma_{\nu} \). Let us choose
\[ \Gamma_{\varphi} = \bigcup_{\nu > 0} \Gamma_{\nu}. \]
Then
\[ \mu_{I,\varphi}(\Gamma_0 - \Gamma_{\varphi}) = 0, \]
by (4.19) and (4.20), and for any \((I_0, \varphi_0) \in \Gamma_{\varphi}\) the required convergence of measures holds. This proves the second assertion of Theorem 0.2. \( \square \)
5. Application to a special case

In this section we prove Proposition 0.3. Clearly, we only need to prove the statement (ii) of assumption A. Let $F : H^m \to \mathbb{R}$ be a smooth functional (for some $m \geq 0$). If $u(t)$ is a solution of (0.1), then

$$\frac{d}{dt} F(u(t)) = \langle \nabla F(u(t)), -V(u) + \epsilon f(x) \rangle.$$ 

In particular, if $F(u)$ is an integral of motion for the KdV equation, then we have $\langle \nabla F(u(t), V(u)) = 0$, so

$$\frac{d}{dt} F(u(t)) = \epsilon \langle \nabla F(u(t)), f(x) \rangle.$$ 

Since $\|u(0)\|^2_0$ is an integral of motion, then

$$\frac{d}{dt} \|u(t)\|^2_0 = 2\epsilon \langle u, f(x) \rangle \leq \epsilon (\|u\|^2_0 + \|f(x)\|^2_0).$$

Thus we have

$$\|u(t)\|^2_0 \leq e^{\epsilon t} (\|u(0)\|^2_0 + \epsilon t \|f(x)\|^2_0). \quad (5.1)$$

The KdV equation has infinitively many integral of motion $J_m(u)$, $m \geq 0$. The integral $J_m$ can be written as

$$J_m(u) = \|u\|^2_m + \sum_{r=3}^{m} \sum_{m} \int C_{r,m} u^{(m_1)} \cdots u^{(m_r)} dx,$$

where the inner sum is taken over all integer $r$-vectors $m = (m_1, \ldots, m_r)$, such that $0 \leq m_j \leq m - 1$, $j = 1, \ldots, r$ and $m_1 + \cdots + m_r = 4 + 2m - 2r$. Particularly, $J_0(u) = \|u\|^2_0$.

Let consider

$$I = \int u^{(m_1)} \cdots f^{(m_i)} \cdots u^{(m_r)} dx, \quad m_1 + \cdots + m_r = M,$$

where $r \geq 2$, $M \geq 1$, and $0 \leq m_j \leq \mu - 1$. Then, by Hölder’s inequality,

$$|I| \leq \|u^{(m_1)}\|_{L_{\mu_1}} \cdots \|f(x)\|_{L_{\mu_i}} \cdots \|u^{(m_r)}\|_{L_{\mu_r}}, \quad p_j = \frac{M}{m_j} \leq \infty.$$ 

Applying next the Gagliardo-Nirenberg and the Young inequalities, we obtain that

$$|I| \leq \delta \|u\|^2_m + C_\delta \|u\|^C_0, \quad \forall \delta > 0, \quad (5.2)$$

where $C_\delta$ and $C_1$ do not depend on $u$. Below we denote $C$ a positive constant independent of $u$, not necessary the same in each inequality. Let

$$I_1 := \langle \nabla J_m(u), f \rangle = \langle u^{(m)}, f^{(m)} \rangle + \sum_{r=3}^{m} \sum_{m} C'_{r,m} u^{(m_1)} \cdots f^{(m_i)} \cdots u^{(m_r)} dx,$$

where $m_1 + \cdots + m_r = 6 + 2m - 2r$. Using (5.2) with a suitable $\delta$, we get

$$I_1 \leq \|u\|^2_m + C \|u\|^C_0 \leq \|u\|^2_m + C (1 + \|u\|^4_0) + \|f\|^2_m. \quad (5.3)$$
If \( u(t) = u(t, x) \) is a solution of equation (0.1), then
\[
\frac{d}{dt} J_m(u) = \langle \nabla J_m(u), \epsilon f \rangle \leq \epsilon \|u\|_{m}^2 + \epsilon C(1 + \|u\|_{0}^{4m}) + \epsilon \|f\|_{m}^2,
\]
ad
\[
\frac{1}{2} \|u\|_{m}^2 - C(1 + \|u\|_{0}^{4m}) \leq J_m(u) \leq 2 \|u\|_{m}^2 + C(1 + \|u\|_{0}^{4m}).
\]
Denote \( C_m = C(1 + \|u(0)\|_{0}^{4m}) + C\|f\|_{m}^2 \), then from (5.1) and above, we deduce
\[
\frac{d}{dt}(J_m(u) - C_m) \leq \frac{1}{2} \epsilon (J_m(u) - C_m),
\]
thus
\[
J_m(u) - C_m \leq e^{\frac{1}{2} \epsilon t}[J_m(u(0)) - C_m],
\]
so
\[
\|u(t)\|_{m}^2 \leq 4\|u(0)\|_{m}^2 e^{\frac{1}{2} \epsilon t} + C_m.
\]
This prove Proposition 0.3.  \( \square \)

**Appendix**

Consider the following system of ordinary differential equations:
\[
\dot{x} = Y(x), \quad x(0) = x_0 \in \mathbb{R}^n,
\]
where \( Y(x) = (Y_1(x), \cdots, Y_n(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuously differentiable map. Let \( F(t, x) \) be a (local) flow determined by this equation.

**Theorem A** (Liouville). Let \( B(x_1, \cdots, x_n) \) be a continuous differentiable function on \( \mathbb{R}^n \). For the Borel measure \( d\mu = B(x)dx \) in \( \mathbb{R}^n \) and any bounded open set \( A \subset \mathbb{R}^n \), we have
\[
\frac{d}{dt} \mu(F(t, A)) = \int_{F(t, A)} \left[ \sum_{i=1}^{n} \frac{\partial (B(x)Y_i(x))}{\partial x_i} \right] dx,
\]
where \( T > 0 \) is such that \( F(t, x) \) is well defined and bounded for any \( t \in (-T, T) \) and \( x \in A \).

For \( B = \text{const} \) this result is well known. For its proof for a non-constant density \( B \) see e.g. [14] [10].

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