A Note on the Modified Albertson Index

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Abstract

The modified Albertson index, denoted by $A^*$, of a graph $G$ is defined as $A^*(G) = \sum_{uv \in E(G)} |d_u - d_v|^2$, where $d_u$, $d_v$ denote the degrees of the vertices $u$, $v$, respectively, of $G$ and $E(G)$ is the edge set of $G$. In this note, a sharp lower bound of $A^*$ in terms of the maximum degree for the case of trees is derived. The $n$-vertex trees having maximal and minimal $A^*$ values are also characterized here. Moreover, it is shown that $A^*(G)$ is non-negative even integer for every graph $G$ and that there exist infinitely many connected graphs whose $A^*$ value is $2t$ for every integer $t \in \{0, 3, 4, 5\} \cup \{8, 9, 10, \cdots\}$.

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1 Introduction

All the graphs considered in this note are simple and finite. Sets of vertices and edges of a graph $G$ will be denoted by $V(G)$ and $E(G)$, respectively. Degree of a vertex $u$ and the edge connecting the vertices $u, v \in V(G)$ will be denoted by $d_u$ and $uv$, respectively. Let $N(u)$ be the set of all those vertices of $G$ which are adjacent to $u$. By an $n$-vertex graph, we mean a graph with $n$ vertices. The graph theoretical terminology not defined here, can be found from some standard books of graph theory, like [5][10].

For a graph $G$, the imbalance of the edge $uv \in E(G)$, denoted by $imb(uv)$, is defined as $|d_u - d_v|$. The idea of the imbalance of an edge was actually appeared implicitly in [2] within the study of Ramsey graphs. Using the concept of imbalance, Albertson [1] defined the following graph invariant

$$A(G) = \sum_{uv \in E(G)} imb(uv)$$
and named it as the \textit{irregularity} of $G$; however, several researchers [3,7,9,11,14] referred it as the \textit{Albertson index} and we do the same in this paper. Detail about the mathematical properties of the Albertson index can be found in the recent papers [4,6,12,13] and related references listed therein.

This note is devoted to establish some properties of the following modified version of the Albertson index

$$A'(G) = \sum_{uv \in E(G)} |(d_u)^2 - (d_v)^2|.$$ 

We propose to call the graph invariant $A'$ as the \textit{modified Albertson index}.

\section{Main Results}

Firstly, we prove two results concerning the modified Albertson index of trees; one of these results is related to a sharp lower bound of $A'$ in terms of maximum degree and the second one is an extremal result, in which we characterize the $n$-vertex trees having maximal and minimal $A'$ values.

\textbf{Proposition 2.1.} If $T$ is a tree with maximum degree $\Delta$ then $A'(T) \geq \Delta(\Delta^2 - 1)$ with equality if and only if $T$ is isomorphic to either a path or a tree containing only one vertex of degree greater than 2.

\textbf{Proof.} The result is obvious for $\Delta \leq 2$ and hence we assume that $\Delta \geq 3$. If $v \in V(T)$ has maximum degree then there are $d_v$ pendant vertices namely $w_1, w_2, \ldots, w_{d_v}$ in $T$ such that the paths $v - w_1, v - w_2, \ldots, v - w_{d_v}$ are pairwise internally disjoint. If the path $v-w_1$ has length greater than 1, suppose that $w_{1,1}, w_{1,2}, \ldots, w_{1,r}$ are the internal vertices of the path $v - w_1$. Then,

$$|\left(d_v\right)^2 - \left(d_{w_{1,1}}\right)^2| + \left|\left(d_{w_{1,1}}\right)^2 - \left(d_{w_{1,2}}\right)^2\right| + \cdots + \left|\left(d_{w_{1,r}}\right)^2 - \left(d_{w_1}\right)^2\right| \geq \left|\left(d_v\right)^2 - \left(d_{w_{1,1}}\right)^2\right| + \left|\left(d_{w_{1,1}}\right)^2 - \left(d_{w_{1,2}}\right)^2\right| + \cdots + \left|\left(d_{w_{1,r}}\right)^2 - \left(d_{w_1}\right)^2\right| = \Delta^2 - 1.$$ 

We note that the equality

$$|\left(d_v\right)^2 - \left(d_{w_{1,1}}\right)^2| + \left|\left(d_{w_{1,1}}\right)^2 - \left(d_{w_{1,2}}\right)^2\right| + \cdots + \left|\left(d_{w_{1,r}}\right)^2 - \left(d_{w_1}\right)^2\right| = \Delta^2 - 1$$ 

holds if and only if the degrees of successive vertices along the path from $v$ to $w_1$ decrease monotonously (not necessarily strictly). Similarly, for $i = 2, \ldots, r$, the sum of contributions of edges to $A'(T)$ along the path $v - w_i$ is at least $\Delta^2 - 1$ with equality if and only if the degrees of successive vertices along the path from $v$ to $w_i$ decrease monotonously (not necessarily strictly), and hence the desired result follows. \hfill $\square$

\textbf{Proposition 2.2.} For $n \geq 5$, if $T$ is an $n$-vertex tree different from the path $P_n$ and star $S_n$, then $A'(P_n) < A'(T) < A'(S_n)$.

\textbf{Proof.} The inequality $A'(P_n) < A'(T)$ follows from Proposition 2.1. To prove the inequality $A'(T) < A'(S_n)$, we note that for any two vertices $u, v \in V(T)$, it holds that $|\left(d_u\right)^2 - \left(d_v\right)^2| \leq |(n-1)^2 - 1|$ with equality if and only if one of the vertices $u, v$ has degree 1 and the other has degree $n-1$. But, $T$ does not contain any vertex of degree $n-1$ and hence

$$A'(T) = \sum_{uv \in E(T)} |(d_u)^2 - (d_v)^2| < (n-1)(n-1)^2 - 1 = A'(S_n).$$

\hfill $\square$
Let $u$ be a fixed vertex of $G$. We partition the set $N(u)$ as follows: $L(u) = \{v \in N(u) : d_v < d_u\}$, $E(u) = \{v \in N(u) : d_v = d_u\}$ and $G(u) = \{v \in N(u) : d_v > d_u\}$. The number of elements in $L(u)$, $E(u)$ and $G(u)$ ar denoted by $l_u$, $e_u$ and $g_u$, respectively. Clearly, $d_u = l_u + e_u + g_u$. Now, we will prove that the modified Albertson index $A^*$ is non-negative even integer for every graph; but, before proving this fact, we derive the following useful result first.

**Lemma 2.3.** If $u$ and $v$ are non-adjacent vertices in a graph $G$ such that $d_u \geq d_v$ then
\[
A'(G + uv) = A'(G) + 3d_u(d_u + 1) + d_v(d_v - 1) - 2[(2d_u + 1)g_u + (2d_v + 1)g_v].
\]

**Proof.** We consider the difference
\[
A'(G + uv) - A'(G) = (d_u + 1)^2 - (d_v + 1)^2
+ \sum_{x \in N(u)} \left( |(d_u + 1)^2 - (d_x)^2| - |(d_u)^2 - (d_x)^2| \right)
+ \sum_{y \in N(v)} \left( |(d_v + 1)^2 - (d_y)^2| - |(d_v)^2 - (d_y)^2| \right).
\]

Now, using the facts $N(u) = L(u) \cup E(u) \cup G(u)$, $N(v) = L(v) \cup E(v) \cup G(v)$ and then after simplifying, we arrive at
\[
A'(G + uv) - A'(G) = (d_u - d_v)(d_u + d_v + 2) + (2d_u + 1)(e_u + l_u - g_u)
+ (2d_v + 1)(e_v + l_v - g_v),
\]
which is equivalent to
\[
A'(G + uv) - A'(G) = 3d_u(d_u + 1) + d_v(d_v - 1) - 2[(2d_u + 1)g_u + (2d_v + 1)g_v].
\]

\[\square\]

**Proposition 2.4.** The modified Albertson index $A^*$ of every graph is a non-negative even integer.

**Proof.** Let $G$ be any graph. By definition, $A'(G) \geq 0$ with equality if and only if every component of $G$ is regular. The result obviously holds if $G$ is the complete graph and hence we assume that $G$ is not isomorphic to a complete graph. We prove the result by induction on the number of edges of $G$. If $G$ is the edgeless graph then $A'(G) = 0$ and hence the induction starts. Let $u$ and $v$ be non-adjacent vertices of $G$ such that $d_u \geq d_v$. Then, by Lemma 2.3 it holds that
\[
A'(G + uv) = A'(G) + 3d_u(d_u + 1) + d_v(d_v - 1) - 2[(2d_u + 1)g_u + (2d_v + 1)g_v]. \tag{1}
\]
By induction hypothesis, $A'(G)$ is even and hence from Equation (1), it follows that $A'(G + uv)$ is even. This completes the induction and hence the proof.

\[\square\]

**Transformation 1.** Let $uv$ be an edge of a graph $G$ satisfying $d_u = d_v = 3$. Let $G'$ be the graph obtained from $G$ by inserting a new vertex $x \not\in V(G)$ of degree 2 on the edge $uv$.

Finally, we prove that there exist infinitely many connected graphs whose modified Albertson index is $2t$ for every integer $t \in \{0, 3, 4, 5\} \cup \{8, 9, 10, \cdots\}$. For this, we need the following two lemmas whose proofs are straightforward.
Lemma 2.5. If $G$ and $G'$ are the two graphs specified in Transformation 1, then $A'(G') = A'(G) + 10$.

Lemma 2.6. Let $uv$ be an edge of a graph $G$ satisfying one of the following conditions
1. $d_u = 1$ and $d_v \geq 2$;
2. at least one of the vertices $u, v$ has degree 2.
If $G'$ is the graph obtained from $G$ by inserting a new vertex $x \not\in V(G)$ of degree 2 on the edge $uv$, then $A'(G') = A'(G)$.

Proposition 2.7. For every integer $t \in \{0, 3, 4, 5\} \cup \{8, 9, 10, \ldots\}$, there exist infinitely many connected graphs whose $A'$ value is $2t$.

Figure 1: The graphs $H_{0,0}$, $H_{0,1}$, $H_{0,2}$, $H_{0,3}$ and $H_{0,4}$, used in the proof of Proposition 2.7.

Proof. Let $H_{0,0}$ be the cubic graph shown in Figure 1. Obviously, $H_{0,0}$ has $3(t + 2)$ edges and its $A'$ value is 0. Also, we consider the graphs $H_{0,1}$, $H_{0,2}$, $H_{0,3}$ and $H_{0,4}$ (which are obtained from $H_{0,0}$) depicted in Figure 1; their $A'$ values are 32, 24, 16 and 8, respectively. For $j = 0, 1, 2, 3, 4$ and $1 \leq i < 3(t + 2)$, let $H_{i,j}$ be the graph obtained from $H_{i-1,j}$ by applying Transformation 1. Then,

$$A'(H_{i,j}) = \begin{cases} 10i & \text{if } j = 0; \\ 2(5i - 4j + 20) & \text{otherwise}. \end{cases}$$

We yet need to find the graphs with $A'$ values 22 and 6. The $A'$ value of the 3-vertex path graph $P_3$ is 6. Let $H$ be the graph obtained from the 5-vertex complete graph $K_5$ by inserting a new vertex $x \not\in V(K_5)$ of degree 2 on an edge of $K_5$. If $H'$ is the graph obtained from $H$ by attaching a new vertex $y \not\in V(H)$ to the vertex $x \in V(H)$, then $A'(H') = 22$. Until now, we have found a single graph having modified Albertson index $2t$ for each $t \in \{0, 3, 4, 5\} \cup \{8, 9, 10, \ldots\}$. Now, by using the transformation specified in Lemma 2.6, we get infinitely many graphs with the same $A'$ value, corresponding to each of the graphs $H_{i,j}$, $P_3$, $H'$. \qed
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