Iteration of Functions $f : X^k \to X$ and their Periodicity

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Abstract

We propose a notion of iterating functions $f : X^k \to X$ in a way that represents recurrence relations of the form $a_{n+k} = f(a_n, a_{n+1}, \ldots, a_{n+k-1})$. We define a function as $n$-involutory when its $n$th iterate is the identity map, and discuss elementary group-theoretic properties of such functions along with their relation to cycles of their corresponding recurrence relations. Further, it is shown that a function $f : X^k \to X$ that is 2-involutory in each of its $k$ arguments (holding others fixed) is $(k+1)$-involutory.

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1 Introduction

This paper proposes a means to extend the notion of function iteration to functions $f : X^k \to X$ for some set $X$ and integer $k \geq 1$, and aims to explore such functions that obey a certain iterative periodic property.

Function iteration is well defined when functions are self-maps. For a self-map $f : X \to X$, the $n$th iterate of $f$, denoted by $f^n$ for some nonnegative integer $n$, is defined recursively by

$$f^0 \equiv \text{id}_X$$

and

$$f^{n+1} \equiv f \circ f^n,$$

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where \( id_X \) is the identity map on \( X \). If \( f \) is invertible with inverse \( f^{-1} \), then this definition extends to negative iterates, where the \(-n\)th iterate of \( f \) is the \( n\)th iterate of \( f^{-1} \). The associativity of function composition immediately gives the following properties for integers \( m \) and \( n \):

1. Addition rule: \( f^m \circ f^n = f^n \circ f^m = f^{m+n} \)  
2. Multiplication rule: \((f^m)^n = (f^m)^n = f^{mn}\).  

The multiplication rule (2) provides a natural means of extension to fractional iterates; for integers \( n \neq 0 \) and \( m \), with \( \gcd(m, n) = 1 \), an \( \frac{m}{n} \)th iterate of \( f \) is any function \( g \) such that \( g^n = f^m \) (c.f. Isaacs (1950)).

Function iteration for self-maps may be understood as a representation of a recurrence relation. For a sequence \( \{a_n\}_{n \in \mathbb{N}} \) defined recursively through some self-map \( f \) by

\[
  a_{n+1} = f(a_n)
\]

given some seed value \( a_0 \), the \( n\)th term of the sequence may be computed as

\[
  a_n = f^n(a_0).
\]

In extending the notion of function iteration to a function \( f : X^k \to X \), it is desirable for the iterates to likewise represent the state of a recurrence relation. One way to define its iterates is by defining its first iterate, \( f^1 \), as the self-map on \( X^k \) given by

\[
  f^1 : (x_1, x_2, ..., x_k) \to (f(x_1, x_2, ..., x_k), ..., f(x_1, x_2, ..., x_k)),
\]

and defining other iterates of \( f \) as typical iterates of \( f^1 \). Iterates thus defined treat the \( k \) arguments in a symmetric manner and consequently feature redundancies in that each of their \( k \) component functions are identical. However, it is not immediately clear what distinct application this definition of function iteration serves.\(^1\) Instead, the definition of function iteration for a function \( f : X^k \to X \) that is offered in this paper has the natural interpretation of representing \( k \)th order recurrence relations of the form

\[
  a_{n+k} = f(a_n, a_{n+1}, ..., a_{n+k-1}).
\]

\(^1\)Defining function iteration as in (3) can be understood as representing a recurrence relation \( a_{n+1} = f(a_n, ..., a_n) \), \( k \) times. However, such a system can be written succinctly as \( a_{n+1} = g(a_n) \) for self-map \( g \) and thus can be represented by usual function iteration.
This representation is done by defining \( f^1 \) as a self-map on \( X^k \) that produces \( k \) consecutive terms of the recurrence relation, and defining iterates of \( f \) as typical iterates of \( f^1 \). A formal definition with examples is presented in section 2.

A self-map \( f : X \rightarrow X \) is said to be involutory (or is an involution) if it is its own inverse: \( f^2 = id_X \) (Aczel (1948)). The immediate generalization of this property is that a self-map’s \((n - 1)\)th iterate is identical to its own inverse for positive integer \( n \):

\[
f^n = id_X.
\]

Treating \( f \) as an unknown, (5) defines a functional equation known as Babbage’s functional equation (Babbage (1815), Babbage (1816), Babbage (1820), Babbage and Gergonne (1822)) and its solutions have been well studied in the literature. Lojasiewicz (1951) provides the general construction of the solution (see also Bogdanov (1961), and for certain real solutions, see the earlier work Ritt (1916)). Much of the work of the twentieth century relating to Babbage’s equation—and more broadly, the theory of functional equations—can be found in the monograph Kuczma et al. (1990) (c.f. also Baron and Jarczyk (2001) for a further development of this work).

A solution \( f \) of (5) is generally referred to as an \( n \)th iterative root of identity (Kuczma et al. (1990)), and when \( n \) is the smallest positive integer such that \( f \) satisfies (5), \( f \) is variously known as a function of order \( n \) or said to circulate with period \( n \) (Ritt (1916)), or is said to be periodic with period \( n \) (McShane (1961)). In this paper, we introduce and mostly use the terminology involutory of order \( n \), or \( n \)-involutory. As defined in section 3, a function \( f : X^k \rightarrow X \) is said to be \( n \)-involutory when its \( n \)th iterate is the identity map. That is, \( f \) is \( n \)-involutory when \( f^1 \), a self-map on \( X^k \), satisfies Babbage’s equation throughout its domain, \( X^k \). More generally, a function \( f : X^k \rightarrow X \) is \( n \)-involutory at point \( x \) when \( f^1 \) satisfies Babbage’s equation at the specified point \( x \in X^k \).

In section 3.1, I discuss elementary group-theoretic properties of self-maps that are \( n \)-involutory. In this context, I also provide a concise proof that continuous self-maps on \( \mathbb{R} \) that are involutory of an integral order must in fact be involutory. This latter result, Proposition 1, is well-known in the literature on functional equations (c.f. Ewing and Utz (1953), Vincze (1959), McShane (1961)), and is stated in Theorem 11.7.1 in Kuczma et al. (1990). Proposition 1 supplies general motivation in the sequel for approaching the question of when multivariate functions \( f : X^k \rightarrow X \) have a common involutory order (only possibly depending on \( k \)). In section 3.2, I consider elementary properties of

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2 This terminology may seem redundant, although there are some subtle distinctions between being \( n \)-involutory and being periodic with period \( n \). To begin with, I apply the term “\( n \)-involutory” to describe \( f \), which need not be a self-map (if \( k > 1 \)), since the focus of the paper is on a sufficient condition on \( f \) under which the self-map \( f^1 \) satisfies Babbage’s equation. I also do not restrict the order \( n \) to be minimal in stating that a function is \( n \)-involutory. More conceptually, however, the term “\( n \)-involutory” reminds the reader that the property it signifies is ultimately a generalization of being involutory, and further, as will be made clear in Proposition 2, that involutory self-maps on \( X \) directly give rise to functions \( f : X^k \rightarrow X \) having this property.
functions \( f : X^k \to X \) that are \( n \)-involutory. In the spirit of Proposition 1, the main result obtained in this section is Proposition 2, which asserts that when multivariate functions are involutory in each of their \( k \) arguments (a property I refer to as being \textit{induced involutory}, defined more precisely in the sequel), they are involutory of common order \( k+1 \). Section 3.3 discusses the relation between functions that are \( n \)-involutory at a point and cycles of the recurrence relations that they represent. Section 4 concludes with discussion on possible extensions and some research questions.

2 Definition of iteration of functions \( f : X^k \to X \)

Consider the recurrence relation in (4). Observe that the state of the system is characterized by a \( k \)-tuple of the elements of the sequence itself. Our definition of function iteration for functions \( f : X^k \to X \) is based on representing such systems and is given as follows:

**Definition 1.** For a set \( X \) and function \( f : X^k \to X \), the \( n \)th iterate of \( f \), denoted by \( f^n \) for nonnegative integer \( n \), is defined as the \( n \)th iterate of the self-map \( f^1 : X^k \to X^k \), given by

\[
\begin{align*}
\text{for } \ell = 1, 2, \ldots, k: \\
(f_1^{\ell}) &= f(x_1, x_2, \ldots, x_k), \\
(f_2^{\ell}) &= f(x_2, x_3, \ldots, x_k, f_1^{\ell}), \\
&\hspace{1cm} \vdots \\
(f_j^{\ell}) &= f(x_j, x_{j+1}, \ldots, x_k, f_1^{\ell}, f_2^{\ell}, \ldots, f_{j-1}^{\ell}), \\
&\hspace{1cm} \vdots \\
(f_k^{\ell}) &= f(x_k, f_1^{\ell}, f_2^{\ell}, \ldots, f_{k-1}^{\ell}).
\end{align*}
\]

Note that while \( f \) itself is not a self-map, its iterates thus defined are self-maps and satisfy the addition and multiplication rules, (1) and (2). This definition hence also allows for an immediate extension to negative and fractional iterates in the way described before. In the event the function \( f \) is a self-map \((k = 1)\), this definition naturally concurs with the usual definition of function iteration. The graphical construction of \( f^1 \) for \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is illustrated in Figure (1).

Definition (1) allows us to represent recurrence relations described by (4) as an iterated function:

\[
(a_{nk+1}, a_{nk+2}, \ldots, a_{nk+k}) = f^n(a_1, a_2, \ldots, a_k).
\] (6)
2.1 Examples

**Example 1.** Consider the function \( f : \mathbb{C}^2 \to \mathbb{C} \) given by
\[
f(x_1, x_2) = x_1 + x_2.
\]
Its iterates encode terms of the recurrence relation \( a_{n+2} = a_n + a_{n+1} \), and are given by
\[
f^n(x_1, x_2) = (F_{2n-1}x_1 + F_{2n}x_2, F_{2n}x_1 + F_{2n+1}x_2),
\]
where \( F_n \) is the \( n \)th Fibonacci number, where \( F_0 = 0 \) and \( F_1 = 1 \).

**Example 2.** Consider the function \( f : X^k \to X \) for some set \( X \) given by
\[
f(x_1, x_2, ..., x_k) = g(x_j), \quad j \in \{1, 2, ..., k\}
\]
for some function \( g : X \to X \). The iterates of \( f \) encode terms of the recurrence relation \( a_{n+k} = g(a_{n+j-1}) \). In the case \( j = 1 \), the iterates are given by
\[
f^n(x_1, x_2, ..., x_k) = (g^n(x_1), g^n(x_2), ..., g^n(x_k)),
\]
while in the case $j = k$, the iterates (for positive integer $n$) are given by

$$f^n(x_1, x_2, \ldots, x_k) = (g^{nk-k+1}(x_k), g^{nk-k+2}(x_k), \ldots, g^{nk}(x_k)).$$

**Example 3.** Consider the function $f : \mathbb{C}^k \to \mathbb{C}$ given by

$$f(x_1, x_2, \ldots, x_k) = A - \sum_{j=1}^k x_j,$$

for some constant $A \in \mathbb{C}$, whose iterates encode terms of the recurrence relation $a_{n+k} = A - \sum_{i=0}^{k-1} a_{n+i}$.

Its iterates are cyclical, such that for any integer $m$, they are given by

$$f^{m(k+1)}(x_1, x_2, \ldots, x_k) = (x_1, x_2, \ldots, x_k)$$

$$f^{m(k+1)+1}(x_1, x_2, \ldots, x_k) = (A - \sum_{j=1}^k x_j, x_1, x_2, \ldots, x_{k-1})$$

$$\vdots$$

$$f^{m(k+1)+p}(x_1, x_2, \ldots, x_k) = (x_{k-p+2}, x_{k-p+3}, \ldots, x_k, A - \sum_{j=1}^k x_j, x_1, x_2, \ldots, x_{k-p})$$

$$\vdots$$

$$f^{m(k+1)+k}(x_1, x_2, \ldots, x_k) = (x_2, x_3, \ldots, x_k, A - \sum_{i=1}^k x_i).$$

Cycles such as those in the third example supply the motivation for exploring functions $f : X^k \to X$ featuring such iterative periodicity, as pursued in section 3.

### 3 Periodicity

#### 3.1 Periodicity of self-maps

Based on the notion of function iteration established in Definition (1), I offer the following definition as a generalization of involutory functions:

**Definition 2.** A function $f : X^k \to X$ is *involutory of order* $n$, or $n$-*involutory*, for integer $n$ when $f^n = id_{X^k}$. The integer $n$ is referred to as an *involutory order* of $f$.

A function that is $n$-involutory is thus one whose $(n-1)$th iterate is the inverse map of its first iterate. More generally, such a function can be characterized as one whose $j$th iterate is the inverse map
of its \((n - j)\)th iterate for any integer \(j\). Note that our definition is based on integral involutory orders to ensure well-definedness. Of course, every function is 0-involutory. If \(f\) has a positive involutory order, then when \(n\) is the smallest positive involutory order of \(f\) (equivalently, when \(f^1\) is periodic with period \(n\) as per McShane (1961)), the set

\[
\{id_X^k, f^1, f^2, ..., f^{n-1}\},
\]

endowed with the composition operation, \(\circ\), comprises a cyclical group of order \(n\) generated by \(f^1\).

The following are immediate properties of such functions when they are self-maps:

**Lemma 1.** Let \(f : X \to X\) be \(n\)-involutory for \(n \in \mathbb{Z}\). Then we have the following:

1. \(f\) is \(nm\)-involutory for any \(m \in \mathbb{Z}\).
2. For \(m \in \mathbb{Z}\), \(f^m\) is \(\frac{n}{\gcd(n, m)}\)-involutory.
3. The identity map, \(id_X\), is \(m\)-involutory for any \(m \in \mathbb{Z}\).
4. If \(f\) is also \(m\)-involutory for \(m \in \mathbb{Z}\), and \(\gcd(m, n) = 1\), then \(f = id_X\).
5. For any invertible function \(g : Y \to X\), the map \(g^{-1} \circ f \circ g\) is \(n\)-involutory.

The properties of Lemma 1 are apparent from the aforementioned group structure. The fifth property is a conjugacy property that has been noted as early as Babbage’s work; in particular, when \(Y = X\), it reveals that any self-map on set \(X\) that is \(n\)-involutory gives rise to a conjugacy class of other functions that are also \(n\)-involutory within the symmetric group on \(X\). For example, the involutory self-map \(h\) defined on \(X \equiv \mathbb{R}_{>0}\), given by \(h : x \to \ln\left(\frac{e^x + 1}{e^{x/2} + 1}\right)\), is conjugate to the linear self-map \(-id_X\) since \(h = g^{-1} \circ (-id_X) \circ g\), where \(g = g_1 \circ g_2 \circ g_3 \circ g_4\), with \(g_1 : x \to x - \frac{1}{2}, g_2 : x \to \log_2(x), g_3 : x \to x - 1, g_4 : x \to e^x\). In fact, it turns out all strictly decreasing involutions on a real interval are conjugate to the negative identity map and hence are conjugate to each other (c.f. Theorem 11.7.3 in Kuczma et al. (1990)). Another result, perhaps more surprising, is that all complex rational self-maps that are \(n\)-involutory fall into one of three simple explicit conjugacy classes (c.f. Theorem 11.7.4 in Kuczma et al. (1990)). Exploring this conjugacy property, and particularly determining when a function is linearizable (or conjugate to a linear map), has given rise to a fruitful area of research within the theory of functional equations (see, for instance, the recent work Homs-Dones (2020), which gives a review of several main results in this area for periodic functions, and looks at linearization of functions satisfying a generalization of Babbage’s equation (5)).

While Lemma 1 applies to functions that are involutory of arbitrary order \(n\), for continuous self-maps on \(\mathbb{R}\), a function that is involutory of some integral order must be involutory of order 2 (i.e. involutory), as the next proposition establishes.
**Proposition 1.** Let \( f : I \to I \) be a continuous self map on interval \( I \subset \mathbb{R} \). Then \( f \) is \( n \)-involutory for some integer \( n \) if and only if either 1. \( f = id_I \) or 2. \( n \) is even and \( f \) is a strictly decreasing involution.

**Proof.**

(\( \iff \)) Sufficiency is trivial by properties 1 and 3 of Lemma 1.

(\( \Rightarrow \)) Necessity is established in steps as follows:

Step 1: \( f \) is strictly monotone

By continuity, it suffices to note that \( f \) is injective, which follows from the invertibility of \( f \).

For the remaining steps, assume that \( f \neq id_I \) i.e. \( \exists z \in I \) s.t. \( f(z) \not\geq z \); we will show \( n \) is even and \( f \) is a strictly decreasing involution.

Step 2: \( f \) is strictly decreasing

Assume by contradiction that \( f \) is strictly increasing. Then \( f(z) \not\geq z \Rightarrow f^2(z) \not\geq f(z) \not\geq z \Rightarrow \ldots \Rightarrow z = f^n(z) \not\geq f^{n-1}(z) \not\leq \ldots \not\leq f(z) \not\geq z = f^n(z) \). The first inequality in this chain thus asserts that \( z \not\geq f^{n-1}(z) \), while the last inequality in this chain asserts \( f(z) \not\geq f(f^{n-1}(z)) \), contradicting that \( f \) is strictly decreasing.

Step 3: \( n \) is even

Assume by contradiction \( n \) is odd. By repeated application of step 2, we have \( f(z) \not\geq z \Rightarrow z = f^n(z) \not\geq f^{n-1}(z) \not\leq \ldots \not\leq f(z) \not\geq z = f^n(z) \). The first inequality in this chain thus asserts that \( z \not\geq f^{n-1}(z) \), while the last inequality in this chain asserts \( f(z) \not\geq f(f^{n-1}(z)) \), contradicting that \( f \) is strictly decreasing.

Step 4: \( f^2 = id_I \)

Assume by contradiction that \( \exists z' \in I \) s.t. \( f^2(z') \not\geq f^n(z') = z' \). By step 2, applying \( f^{-1} \) to both sides of this inequality implies \( f(z') \not\leq f^{n-1}(z') \). Applying the inverse again yields \( f^n(z') = z' \not\geq f^{n-2}(z') \). Repeatedly applying the inverse and noting \( n \) is even (step 3) implies \( f^2(z') \not\geq f^n(z') \not\geq f^{n-2}(z') \not\geq \ldots \not\geq f^2(z') \), a contradiction.

As mentioned in section 1, the result given in Proposition 1 is well established in the literature. Presenting the result as such supplies some motivation in the sequel, where, in the general spirit of this proposition, I obtain a sufficient condition under which functions \( f : X^k \to X \) are involutory of a common order.

### 3.2 Periodicity of functions \( f : X^k \to X \)

In this section, we consider properties of functions \( f : X^k \to X \) that are \( n \)-involutory. A function \( f \) being \( n \)-involutory is equivalent to the self-map \( f^1 \) being \( n \)-involutory, so the properties of Lemma 1 extend quite naturally to such functions, as stated in the following lemma:

**Lemma 2.** Let \( f : X^k \to X \) be \( n \)-involutory for \( n \in \mathbb{Z} \). Then we have the following:

1. Properties 1-2 of Lemma 1 hold.
2. The map \( \hat{id}_{X^k} : X^k \to X \), defined as \( \hat{id}_X : (x_1, x_2, \ldots, x_k) \to x_1 \), is \( m \)-involuntary for any \( m \in \mathbb{Z} \).

3. If \( f \) is also \( m \)-involuntary for \( m \in \mathbb{Z} \), and \( \gcd(m, n) = 1 \), then \( f = \hat{id}_{X^k} \).

4. For \( \tilde{f} : X \to X \) any involutory function, the map \( \tilde{f} \circ \hat{id}_{X^k} \) is 2-involutory.

5. For \( g : Y \to X \) any invertible function, let \( \tilde{g} : Y^k \to X^k \) be given by

\[
\tilde{g} : (y_1, y_2, \ldots, y_k) \to (g(y_1), g(y_2), \ldots, g(y_k)).
\]

Then the map \( g^{-1} \circ \tilde{f} \circ \tilde{g} : Y^k \to Y \) is \( n \)-involutory.

It is worth noting that properties 2-4 reveal that \( \hat{id}_{X^k} \) acts as a kind of identity map for functions \( f : X^k \to X \), since \( \hat{id}_{X^k} \) has its first iterate given by \( \hat{id}_{X^k} \). However, a map \( (x_1, x_2, \ldots, x_k) \to x_j \) for \( j \neq 1 \) generally need not be involutory of an integral order. This contrast results from the asymmetric manner in which the iterates of \( f \) treat their arguments. While the properties of Lemma 1 thus extend to functions \( f : X^k \to X \), Proposition 1 does not immediately extend in that continuous functions in Euclidean space that are involutory of an integral order need not be involutory of some common order (depending only possibly on \( k \)). For example, consider the function \( f : \mathbb{R}^k \to \mathbb{R} \) given by \( f : (x_1, x_2, \ldots, x_k) \to A - x_1 \) for some constant \( A \in \mathbb{R} \), which is 2-involutory by property 4 of Lemma 2. In contrast, the function \( f : \mathbb{R}^k \to \mathbb{R} \) given by \( f : (x_1, x_2, \ldots, x_k) \to A - \sum_{i=1}^{k} x_i \) is \( (k + 1) \)-involutory, as per Example 3. However, Example 3 displays certain properties that suggest sufficient conditions under which one may obtain a notion of a common involutory order, in the spirit of Proposition 1. We define a few terms to facilitate understanding these properties:

**Definition 3.** Given function \( f : X^k \to X \), let \( f_j(\cdot|x_{-j}) : X \to X \) denote the induced function

\[
f_j(x_j|x_{-j}) \equiv f(x_1, x_2, \ldots, x_k)
\]

where \( j \in \{1, 2, \ldots, k\} \) and \( x_{-j} \equiv \{x_1, x_2, \ldots, x_k\} \setminus \{x_j\} \in X^{k-1} \) is fixed. The function \( f : X^k \to X \) is **induced involutory of order** \( n \) **in argument** \( j \) (written II-\( n \{j\} \)) when the induced function \( f_j(\cdot|x_{-j}) \) is \( n \)-involutory for any \( x_{-j} \in X^{k-1} \). The function is **induced involutory of order** \( n \) **(written II-n)** when it is II-\( n \{j\} \) for every \( j \in \{1, 2, \ldots, k\} \). The function is **induced involutory in argument** \( j \) (written II-\{\( j \} \)) when it is II-\( 2 \{j\} \). The function is **induced involutory (II)** when it is II-2.

In other words, a function is II-\( n \{j\} \) when it is \( n \)-involutory in argument \( j \), holding fixed the other arguments. Also, recall that a function \( f : X^k \to X \) is said to be **symmetric** when its value is unchanged by any permutation of its \( k \) arguments. The following lemma establishes the relation between II-\( n \) functions and symmetry.
Lemma 3. If \( f : X^k \to X \) is symmetric and \( II-n\{j\} \) for integer \( n \) and \( j \in \{1, 2, ..., k\} \), then \( f \) is \( II-n \).
If \( f : X^k \to X \) is \( II \), then it is symmetric.

Proof. The first claim is immediate since symmetry permits the arguments of \( f \) to be transposed so that for any \( x_1, x_2, ..., x_k \in X \) and any \( j' \in \{1, 2, ..., k\} \), we have \( x_j = f^n_j(x_j|x_{j'}) = f^n_j(x_j|x_{j'}) \).

It suffices to show the second claim for \( k = 2 \) since any permutation of \( k > 2 \) arguments is a composition of pairwise transpositions. Note that \( f \) being \( II \) implies \( f \) is invertible with respect to each argument. Observe that for any \( x_1, x_2 \in X \), we have

\[
x_1 = f(f(x_1, f(x_2, x_1)), f(x_2, x_1))
\]

since \( f \) is \( II-\{1\} \), while we also have

\[
x_1 = f(x_2, f(x_2, x_1))
\]

since \( f \) is \( II-\{2\} \). Equating the two expressions for \( x_1 \) and applying the inverse of \( f(\cdot, f(x_2, x_1)) \) to both implies

\[
x_2 = f(x_1, f(x_2, x_1)).
\]

We also have

\[
x_2 = f(x_1, f(x_1, x_2))
\]

since \( f \) is \( II-\{2\} \). Equating the two expressions for \( x_2 \) and applying the inverse of \( f(x_1, \cdot) \) to both implies \( f(x_1, x_2) = f(x_2, x_1) \). \( \square \)

Note that the second claim in Lemma 3 does not apply to \( II-n \) functions for integers \( n \geq 3 \). Consider the following example:

Example 4. Suppose \( f : \mathbb{C}^2 \to \mathbb{C} \) is given by \( f : (x_1, x_2) \to ax_1 + bx_2 \), where \( a \) and \( b \) are distinct \( n \)th roots of unity for integer \( n \geq 3 \), neither of which is unity itself. Then the \( n \)th iterates of \( f \) with respect to each argument are given as

\[
f^n_1(x_1|x_2) = a^n x_1 + b \frac{1-a^n}{1-a} x_2 = x_1,
\]

\[
f^n_2(x_2|x_1) = b^n x_2 + a \frac{1-b^n}{1-b} x_1 = x_2,
\]

so that \( f \) is \( II-n \) but not symmetric.

The following proposition asserts that \( II \) functions are involutory of a common order.

Proposition 2. If \( f : X^k \to X \) is \( II \), then \( f \) is \((k+1)-\)involutory.\(^3\)

\(^3\)In fact, \( k+1 \) is the minimal (positive integral) involutory order of \( f \), so that \( f^1 \) is periodic with period \( k+1 \).
Proof. Let $x$ denote $(x_1, x_2, ..., x_k) \in X^k$, and let $x_{-j}$ denote $\{x_1, x_2, ..., x_k\}\{x_j\} \in X^{k-1}$. By Lemma 3, $f$ is symmetric, so for any $j \in \{1, 2, ..., k\}$, we have $f(f(x), x_{-j}) = x_j$ (independent of the order of arguments). Consequently, the first $k+1$ iterates are computed as follows:

\[
f^1(x_1, x_2, ..., x_k) = (f(x), x_1, x_2, ..., x_{k-1})
\]

\[
\vdots
\]

\[
f^p(x_1, x_2, ..., x_k) = (x_{k-p+2}, x_{k-p+3}, ..., x_k, f(x), x_1, x_2, ..., x_{k-p})
\]

\[
\vdots
\]

\[
f^k(x_1, x_2, ..., x_k) = (x_2, x_3, ..., x_k, f(x))
\]

\[
f^{k+1}(x_1, x_2, ..., x_k) = (x_1, x_2, ..., x_k)
\]

Proposition 2 thus permits a way to generate functions that are involutory of any order $k+1$ by defining a function $f : X^k \rightarrow X$ that is a direct analogue of an involutory self-map on $X$, as in Example 3, which is the analogue of the self-map $f : x \rightarrow A - x$. One can see how Proposition 2 breaks down when the II assumption is violated. Consider the following examples:

Example 5. Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ be given as $f : (x_1, x_2) \rightarrow ax_1 + bx_2$, where $a = \frac{-1 + i\sqrt{3}}{2}$ and $b = \frac{-1 - i\sqrt{3}}{2}$.

By Example 4, $f$ is II-3, but it is not involutory of any integral order, since its iterates are given by

\[
f^n(x_1, x_2) = (F_{2n-1}a^n x_1 + F_{2n}a^{n+1} x_2, F_{2n}a^{n+2} x_1 + F_{2n+1}a^n x_2),
\]

where $F_n$ is the $n$th Fibonacci number, with $F_0 = 0$ and $F_1 = 1$.

Example 6. Consider the symmetric function $f : X^2 \rightarrow X$ specified below, where $X \equiv \{x_1, x_2, x_3\}$.

|   | $x_1$ | $x_2$ | $x_3$ |
|---|-------|-------|-------|
| $x_1$ | $x_1$ | $x_2$ | $x_3$ |
| $x_2$ | $x_2$ | $x_3$ | $x_1$ |
| $x_3$ | $x_3$ | $x_1$ | $x_2$ |

The value of $f(x_i, x_j)$ for $(x_i, x_j) \in X^2$ is read as the table element corresponding to row $x_i$ and column $x_j$. It is easily verified that $f$ is II-3 since fixing any row or column, the elements form a 3-cycle. For instance, the 3-cycle corresponding to the third row is $x_3 \rightarrow x_2 \rightarrow x_1 \rightarrow x_3$. Moreover,
$f$ is 4-involutory since all the elements of the set $X^2$ endowed with the self-map $f^1$ are generated as part of two 4-cycles and one singleton cycle:

$$(x_1, x_2) \rightarrow f^1 : (x_2, x_3) \rightarrow f^2 : (x_1, x_3) \rightarrow f^3 : (x_3, x_2) \rightarrow f^4 : (x_1, x_2);$$

$$(x_2, x_1) \rightarrow f^1 : (x_2, x_2) \rightarrow f^2 : (x_3, x_1) \rightarrow f^3 : (x_3, x_3) \rightarrow f^4 : (x_2, x_1);$$

$$(x_1, x_1) \rightarrow f^1 : (x_1, x_1).$$

**Example 7.** Consider the function $f : X^2 \rightarrow X$ specified below, where $X \equiv \{x_1, x_2, x_3, x_4\}$.

| $f$   | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-------|-------|-------|-------|-------|
| $x_1$ | $x_1$ | $x_3$ | $x_4$ | $x_2$ |
| $x_2$ | $x_3$ | $x_1$ | $x_2$ | $x_4$ |
| $x_3$ | $x_4$ | $x_2$ | $x_1$ | $x_3$ |
| $x_4$ | $x_2$ | $x_4$ | $x_3$ | $x_1$ |

Here, $f$ is both symmetric and persymmetric (i.e. symmetric along the antidiagonal) and II-3. Moreover, $f$ is 15-involutory since all the elements of the set $X^2$ endowed with the self-map $f^1$ are generated as part of a 15-cycle and singleton cycle:

$$(x_1, x_2) \rightarrow f^1 : (x_3, x_2) \rightarrow f^2 : (x_2, x_1) \rightarrow f^3 : (x_3, x_4)$$

$\rightarrow f^4 : (x_3, x_3) \rightarrow f^5 : (x_1, x_4) \rightarrow f^6 : (x_2, x_4) \rightarrow f^7 : (x_4, x_1)$$

$\rightarrow f^8 : (x_2, x_3) \rightarrow f^9 : (x_2, x_2) \rightarrow f^{10} : (x_1, x_3) \rightarrow f^{11} : (x_4, x_3)$$

$\rightarrow f^{12} : (x_3, x_1) \rightarrow f^{13} : (x_4, x_2) \rightarrow f^{14} : (x_4, x_4) \rightarrow f^{15} : (x_1, x_2);$$

$$(x_1, x_1) \rightarrow f^1 : (x_1, x_1).$$

Examples 6 and 7 show how functions $f : X^k \rightarrow X$ that are II-$n$ for $n \geq 3$ may be involutory of some order, but the involutory order is sensitive to the cardinality of $X$, unlike II functions.

### 3.3 Fixed points and recursive cycles

A natural question to consider is what the interpretation is of functions that are $n$-involutory in terms of the recurrence relations that they represent. We can approach this question more generally in light of a far less restrictive condition than being $n$-involutory:

**Definition 4.** A function $f : X^k \rightarrow X$ is $n$-involutory at point $x \in X^k$ when $f^n(x) = x$. 


Functions that are \( n \)-involutory at some point reveal specific cycles of recurrence relations that they represent. For instance, the function in Example 3 is \((k + 1)\)-involutory, and relatedly, any seed value in \( C^k \) generates a \((k + 1)\)-cycle of the recurrence relation represented by it, \( a_{n+k} = A - \sum_{i=0}^{k-1} a_{n+i} \).

This relationship is to be expected, since the iterates of a function \( f : X^k \to X \) encode every \( k \) terms of the recurrence relation it represents. The relationship is formalized in the following claim:

**Claim 1.** If a recurrence relation given by \((4)\) has a \( j \)-cycle, then \( f \) is \( \frac{j}{\gcd(j,k)} \)-involutory at a point. If \( f \) is \( n \)-involutory at a point, then the recurrence relation has a \( j \)-cycle for some \( j \) such that \( j \mid n \).

In the special case in which a recurrence relation given by \((4)\) has a \( k \)-cycle characterized by some \( x^0 \in X^k \), then \( f \) is 1-involutory at \( x^0 \); that is, \( x^0 \) is a fixed point of \( f^1 \). By definition, such a fixed point \( x^0 = (x^0_1, x^0_2, ..., x^0_k) \in X^k \) satisfies

\[
\begin{align*}
 f(x^0_1, x^0_2, ..., x^0_k) &= x^0_1 \\
 f(x^0_2, x^0_3, ..., x^0_k, x^0_1) &= x^0_2 \\
 &\vdots \\
 f(x^0_k, x^0_1, x^0_2, ..., x^0_{k-1}) &= x^0_k.
\end{align*}
\]

Conversely, such a fixed point of \( f^1 \) corresponds with a \( j \)-cycle of the recurrence relation \((4)\), where \( j \mid k \). In the special case in which the recurrence relation has a 1-cycle, so that the recurrence relation is constant when seeded by the element \( x \in X \) of the 1-cycle, then the corresponding fixed point of \( f^1 \), \( x^0 \in X^k \), is symmetric in the sense that \( x_i^0 = x \) \( \forall i \in \{1, 2, ..., k\} \). Of course, if \( f \) is a symmetric function, then all the fixed points of \( f^1 \) must be symmetric as such and the recurrence relation it represents can only have 1-cycles.

While Claim 1 asserts that \( j \)-cycles in a recurrence relation correspond with its representative function \( f : X^k \to X \) being involutory of some order at a point, in fact, any such \( j \)-cycle can be understood as corresponding with a point at which a function is 1-involutory even when \( j \nmid k \), by a trivial redefinition of \( f \) that augments its domain. By defining the map \( \tilde{f} : X^{k'} \to X \) for \( k' > k \) such that \( j \mid k' \), given as

\[
\tilde{f} : (x_1, x_2, ..., x_{k'}) \to f(\tilde{x}_{k'-k+1}, \tilde{x}_{k'-k+2}, ..., \tilde{x}_{k'}) | \tilde{x}_m = f(\tilde{x}_{m-k}, \tilde{x}_{m-k+1}, ..., \tilde{x}_{m-1}) \text{ for } m \geq k+1; \tilde{x}_m = x_m \text{ for } m \leq k,
\]

then a \( j \)-cycle of the recurrence relation represented by \( f \) corresponds with a fixed point of \( \tilde{f}^1 \). For instance, the recurrence relation in Example 3 that was characterized by \( f : C^k \to C \) —and for which
every point in $\mathbb{C}^{k+1}$ is a $(k+1)$-cycle—satisfies

$$a_{n+k+1} = \tilde{f}(a_n, a_{n+1}, \ldots, a_{n+k}),$$

where $\tilde{f} : \mathbb{C}^{k+1} \to \mathbb{C}$ is defined as

$$\tilde{f} : (x_1, x_2, \ldots, x_{k+1}) \to A - \tilde{x}_{k+1} - \sum_{j=2}^{k} x_{m\mid \tilde{x}_{k+1}=A} - \sum_{q=1}^{k} x_q = x_1.$$

Thus, $\tilde{f}$ is simply $\hat{id}_{\mathbb{C}^{k+1}}$ (as defined in Lemma 2), which is, of course, 1-involutory at every point in $\mathbb{C}^{k+1}$. In general, we can thus understand attractors of a recurrence relation (4) through fixed point iteration of the self-map $f^1$ given function $f$ (appropriately augmented) that characterizes behavior of the recurrence relation.

4 Discussion

This paper offers an introduction to a proposed notion of defining function iteration for maps $f : X^k \to X$ that is based on representing recurrence relations of the kind given in (4). There are several questions that can be examined based on the definitions posed in section 3 that are not considered in this paper but that can likely be addressed within the scope of combinatorics and group theory. For instance, for finite sets $X$, one can consider enumerating the possible functions that are II-$n\{j\}$ for a specified number of arguments $j$ (and possibly also involutory of some integral order) and determining when they are symmetric. This problem extends the line of research started in 1800 by Heinrich August Rothe, who obtained a recursive formula defining the telephone numbers, which enumerate the involutions in the symmetric groups (c.f. Chowla et al. (1951) and Knuth (1973)). Moreover, as per the fifth property of Lemma 2, one may consider the conjugacy classes of functions that are involutory of some order, as well as the class number associated with various sets.

Likewise, one may also consider how the second claim in Lemma 3 may extend to II-$n$ functions. We can heuristically consider a possible extension by further examining Example 4. Observe that the function in Example 4 can afford to be asymmetric while still satisfying the restriction of being II-3 because there are fewer than three arguments ($k = 2$). Suppose more generally that $f$ in this setup is constrained to be II-$n$ for integer $n \geq 2$ and is defined as a linear map in $k$ arguments, i.e. $f : \mathbb{C}^k \to \mathbb{C}$, where $f : (x_1, x_2, \ldots, x_k) \to \sum_{i=1}^{k} a_i x_i$. Note that all coefficients $a_i$ must be $n$th roots of unity but none can be unity itself, allowing the coefficients $n - 1$ degrees of freedom. However, if $k \geq m(n - 1) + 1$ for an integer $m \geq 1$, one can see that at least $m + 1$ coefficients must be identical by the pigeonhole principle. One can thus consider more generally if a function $f : X^k \to X$ that is II-$n$ for integer $n \geq 2$
with \( k \geq m(n - 1) + 1 \) must be symmetric in at least \( m + 1 \) of its arguments.

Another question that may be considered is how Proposition 2 might be extended. Examples 6 and 7 demonstrate functions defined on finite sets that are II-n (and symmetric) and violate Proposition 2 insofar as not having a common involutory order (depending only on \( k \)). Nonetheless, they are involutory of some order. One can thus consider for finite sets \( X \) under what conditions functions that are II-n are involutory of some order, and how this order varies with the cardinality of \( X \). One may also consider how many distinct kinds of cycles are associated with such functions (e.g. recall all elements in \( X \) in Example 6 were part of one of two 4-cycles or a singleton cycle). Moreover, while Proposition 2 gives sufficient conditions for being \( n \)-involuntary, one may consider what the necessary conditions are, including whether there are examples of such functions that are asymmetric outside of the kind described by property 4 of Lemma 2.

A final point to consider is whether we may likewise suitably define function iteration for functions \( f : X^k \rightarrow X^\ell \). The multiplicity of outputs may be reasonably treated as representing distinct recursive equations, but there is no unique suitable way to treat the inputs in this case. For instance, we may define function iteration of a function \( f : X^3 \rightarrow X^2 \) in a way that represents the recursive system

\[
a_{n+2} = f_1(a_n, b_n, a_{n+1})
\]

\[
b_{n+2} = f_2(a_n, b_n, b_{n+1}),
\]

or alternatively

\[
a_{n+2} = f_1(a_n, b_n, a_{n+1})
\]

\[
b_{n+2} = f_2(a_n, b_n, a_{n+1}),
\]

or in terms of other combinations of three inputs. The state of the first system is characterized by \((a_n, a_{n+1}, b_n, b_{n+1})\) and function iterates may thus be defined as a self-map over \( X^4 \), while the state of the second system is characterized by \((a_n, a_{n+1}, b_n)\) and function iterates may thus be defined as a self-map over \( X^3 \). Consequently, it would be more appropriate to define function iteration in such cases in a way that suits the particular application considered.
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