On the Structure Theory of Cubespace Fibrations

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Abstract
We study fibrations in the category of cubespaces/nilspaces. We show that a fibration of finite degree \( f : X \to Y \) between compact ergodic gluing cubespaces (in particular nilspaces) factors as a (possibly countable) tower of compact abelian Lie group principal fiber bundles over \( Y \). If the structure groups of \( f \) are connected then the fibers are (uniformly) isomorphic (in a strong sense) to an inverse limit of nilmanifolds. In addition we give conditions under which the fibers of \( f \) are isomorphic as subcubespaces. We introduce regionally proximal equivalence relations relative to factor maps between minimal topological dynamical systems for an arbitrary acting group. We prove that any factor map between minimal distal systems is a fibration and conclude that if such a map is of finite degree then it factors as a (possibly countable) tower of principal abelian Lie group extensions, thus achieving a refinement of both the Furstenberg’s and the Bronstein–Ellis structure theorems in this setting.

Keywords Cubespace · Fibration · Nilspace · Nilmanifold · Lie group · Translation · Cocycle on fibers · Relativized regionally proximal relation · Relative nilpotent regionally proximal relation

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1 Introduction

1.1 General Background

The theory of cubespaces and its important subclass of nilspaces originated in the work of Host and Kra [25] under the name of parallelepiped structures, and was developed further by Antolín Camarena and Szegedy [1]. A cubespace is a structure consisting of a compact metric space $X$, together with a closed collection of cubes $C_k^k(X) \subseteq X^{2k}$ for each integer $k \geq 0$, satisfying certain natural axioms. A nilspace is a cubespace satisfying an additional rigidity condition.\(^1\) The theory has already found applications in higher order Fourier analysis\(^2\) [35,36], in particular in relation to the inverse theorem for the Gowers norms [19], as well as in ergodic theory [9,20] in relation to the Host-Kra structure theorem [24,26]. Cubespaces and nilspaces also played an essential role in the structure theory of the higher order nilpotent regionally proximal relations introduced and developed by Glasner, Gutman and Ye in [17].

As the theory of cubespaces/nilspaces has matured, it has been observed that fibrations, cubespace morphisms satisfying an additional rigidity condition, are highly useful.\(^2\) This notion was introduced in [22] generalizing the previous notion of fiber-surjective morphism from [1]. Indeed in [21–23] Gutman, Manners and Varjú developed a weak structure theory for fibrations as an important step in the proof of the structure theorem for minimal topological dynamical systems of finite degree, i.e., such that the nilpotent regionally proximal relation of some degree is trivial. According to this theorem such a system may be represented as an inverse limit of nilsystems (subject to some mild assumptions on the acting group).

According to the weak structure theorem a fibration of finite degree factors as a finite tower of compact abelian group extensions. The groups appearing in this factorization are referred to as the structure groups of the fibration.

In this paper we give a finer structure theory for fibrations building on the Antolín Camarena-Szegedy fundamental structure theorems for nilspaces. We show that a fibration of finite degree between cubespaces obeying some natural conditions factors as a (possibly countable) tower of Lie-fibered fibrations, i.e., fibrations whose structure groups are (compact abelian) Lie groups.

Given a fibration of finite degree $f : X \to Y$ it is well known that the fibers $f^{-1}(y)$ are nilspaces. However this by itself does not elucidate the relation between different fibers.

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\(^1\) For the exact definitions of the class of cubespaces and the subclass of nilspaces see Definitions 1.1 and 1.6 respectively.

\(^2\) For the exact definition of the class of fibrations (of finite degree) see Definition 1.18 (Definition 1.21).
In this paper we are able to give conditions guaranteeing that all fibers are isomorphic as cubespaces. Moreover, taking advantage of the above-mentioned factorization into a tower of Lie-fibered fibrations, if the structure groups of the fibers are connected\(^3\) we show that the fibers are approximated uniformly as closely as desired by quotients by natural co-compact subgroups of a single Lie group associated with the factorization.

We relate our results to the theory of topological dynamical systems. We show that any factor map between minimal distal systems \(\pi : (G, X) \to (G, Y)\) where \(G\) is an arbitrary topological group, is a fibration between the associated dynamical cubespaces. This supplies an abundance of hitherto unknown new examples of fibrations.

The regionally proximal relations have a long history in topological dynamics [13,14,38]. Its relativization, the relativized regionally proximal relations play a fundamental role in the study of the structure of equicontinuous extensions [29,30]. Attesting to its importance is its central use in Bronstein’s proof [3,5]\(^4\) of the (relative) Furstenberg structure theorem for minimal distal extensions [15].\(^5\)

In this article we introduce the relative nilpotent regionally proximal relations for extensions between minimal systems and analyze its structure. In particular we show that when an extension between minimal distal systems has trivial relative nilpotent regionally proximal relation of some degree, then it factors as a (possibly countable) tower of principal abelian Lie (compact) group extensions.\(^6\)

### 1.2 Cubespaces and Nilspaces

In this subsection, we define the notions of cubespace and nilspace and survey their important properties. For more detailed information see [1,6–8,17,21–23].

**Definition 1.1** Let \(k, \ell \geq 0\) be two integers. A map \(f = (f_1, \ldots, f_\ell) : \{0, 1\}^k \to \{0, 1\}^\ell\) is called a morphism of discrete cubes if each coordinate function \(f_j(\omega_1, \ldots, \omega_k)\) is either identically 0, identically 1, or equals either \(\omega_i\) or \(\bar{\omega}_i = 1 - \omega_i\) for some \(1 \leq i = i(j) \leq k\). If \(\ell \leq k\), by an \(\ell\)-face of \(\{0, 1\}^k\) we mean a subset of \(\{0, 1\}^k\) obtained from fixing values of \((k - \ell)\) coordinates. In particular, a \((k - 1)\)-face is called a hyperface.

A **cubespace** is a metric space \(X\), associated with a sequence of closed subsets \(C^\bullet(X) := \{C^k(X)C^k(X) \subseteq X^{(0,1)^k} : k = 0, 1, \ldots\}\) satisfying:

1. \(C^0(X) = X\);
2. for every integer \(k, \ell \geq 0, c \in C^\ell(X), \) and morphism of discrete cubes \(\rho : \{0, 1\}^k \to \{0, 1\}^\ell\), one has that \(c \circ \rho \in C^k(X)\).

We call the elements of \(C^k(X)\) \(k\)-**cubes**. A map \(\{0, 1\}^k \to X\) is called a \(k\)-configuration.

A particular class of cubespaces arise from dynamical systems. By a (topological) dynamical system \((G, X)\), we mean a continuous action of a topological group \(G\) on a compact metric space \(X\).

**Definition 1.2** Let \((G, X)\) be a dynamical system. For every integer \(k \geq 0\) the **Host-Kra cube group** \(HK^k(G)\) is the subgroup of \(G^{(0,1)^k}\) generated by \([g]_F\) for all \(g \in G\) and hyperfaces

\(^3\) In Proposition 3.8 we show that the structure groups of all fibers are connected if the structure groups of \(f\) are connected.

\(^4\) See also [2, Chapter 7].

\(^5\) The relative Furstenberg structure theorem was also proven independently by Ellis in [12].

\(^6\) For the exact definitions of principal abelian group extensions see Definition 5.1.
An important property of fibrant cubespace is that they are gluing \cite[Proposition 6.2]{22}. Let us recall the definition as follows. Let \( c \) is fibrant \cite[Theorem 7.10]{17}. In general, it is not the case for nondistal systems \cite[Example 1.4]{37}. Let \( X \) be a cubespace, \( k \geq 0 \), and \( \lambda : L^k \rightarrow X \) a map. We call \( \lambda \) a \( k \)-corner if every lower face of \( \lambda \) is a \((k-1)\)-cubed, i.e. \( \lambda|_{\{\omega: \omega_i=0\}} \in C^{k-1}(X) \) for all \( i = 1, \ldots, k \).

**Definition 1.3** We say \( X \) has \( k \)-completion if every \( k \)-corner \( \lambda \) of \( X \) can be completed to a \( k \)-cubed of \( X \), i.e. \( \lambda = c|_{L^k} \) for some \( c \in C^k(X) \). A cubespace \( X \) is called fibrant if it has \( k \)-completion for every \( k \geq 0 \) (note that a 0-corner is the empty set).

Let \( d \) be the metric on a compact metric space \( X \). Recall that a dynamical system \((G, X)\) is called ergodic if every pair of points in \( X \) is a 1-cube, i.e. \( C^1(X) = X^{[0, 1]} \). Recall that a dynamical system \((G, X)\) is called minimal if the orbit of every point of \( X \) is dense in \( X \). A simple observation is that if \((G, X)\) is minimal then the induced dynamical cubespace \((X, G^X(X))\) is ergodic.

A cubespace \( X \) is called strongly connected when all \( C^k(X) \) are connected.

The first important property of cubespace is the (corner) completion property. Denote by \( \lambda \) the element \((1, 1, \ldots, 1) \in \{0, 1\}^k \) and \( L^k \) the set \( \{0, 1\}^k \setminus \{\lambda\} \). Let \( X \) be a cubespace, \( k \geq 0 \), and \( \lambda : L^k \rightarrow X \) a map. We call \( \lambda \) a \( k \)-corner if every lower face of \( \lambda \) is a \((k-1)\)-cubed, i.e. \( \lambda|_{\{\omega: \omega_i=0\}} \in C^{k-1}(X) \) for all \( i = 1, \ldots, k \).

**Definition 1.3** We say \( X \) has \( k \)-completion if every \( k \)-corner \( \lambda \) of \( X \) can be completed to a \( k \)-cubed of \( X \), i.e. \( \lambda = c|_{L^k} \) for some \( c \in C^k(X) \). A cubespace \( X \) is called fibrant if it has \( k \)-completion for every \( k \geq 0 \) (note that a 0-corner is the empty set).

Let \( d \) be the metric on a compact metric space \( X \). Recall that a dynamical system \((G, X)\) is called distal if \( \inf_{g \in G} d(gx, gx') > 0 \) for any distinct points \( x, x' \in X \).

**Example 1.4** Let \((G, X)\) be a minimal distal system. Then the associated Host-Kra cubespace is fibrant \cite[Theorem 7.10]{17}. In general, it is not the case for nondistal systems \cite[Example 3.10]{37} \cite[Example 9.3]{17}.

An important property of fibrant cubespaces is that they are gluing \cite[Proposition 6.2]{22}. Let us recall the definition as follows. Let \( c_1, c_2 : [0, 1]^k \rightarrow X \) be two configurations. The concatenation \( [c_1, c_2] : [0, 1]^{k+1} \rightarrow X \) of \( c_1 \) and \( c_2 \) is defined by sending \((\omega, 0)\) to \( c_1(\omega) \) and \((\omega, 1)\) to \( c_2(\omega)\) for every \( \omega \in [0, 1]^k \).

**Definition 1.5** We say a cubespace \( X \) has the gluing property or is gluing if for every integer \( k \geq 0 \) and every \( c_1, c_2, c_3 \in C^k(X) \), \([c_1, c_2], [c_2, c_3] \in C^{k+1}(X)\) implies that \([c_1, c_3] \in C^{k+1}(X)\).

Another important property of cubespaces is the uniqueness property.

**Definition 1.6** A cubespace \( X \) has \( k \)-uniqueness if for any \( c_1, c_2 \in C^k(X) \) such that \( c_1|_{L^k} = c_2|_{L^k} \), one has \( c_1 = c_2 \). Fix \( s \geq 0 \). We say \( X \) is a nilspace of degree at most \( s \) or simply an \( s \)-nilspace if it is fibrant and has \((s+1)\)-uniqueness. We say \( X \) is a nilspace if it is an \( s \)-nilspace for some integer \( s \).
Note that if $X$ has $k$-uniqueness then $X$ has $\ell$-uniqueness for every $\ell \geq k$ as one can apply the $k$-uniqueness to some suitable $k$-face of a given $\ell$-cube.

**Definition 1.7** For a cubespace $X$, we say a pair of points $(x, x')$ of $X$ are $k$-canonically related, denoted by $x \sim_k x'$ if there exists $c, c' \in C^{k+1}(X)$ such that $c|_{k+1} = c'|_{k+1}$, $c(\vec{1}) = x$, and $c'(\vec{1}) = x'$. The relation $\sim_k$ is called the $k$-th canonical relation.

For compact gluing cubespaces the $k$-th canonical relation is an equivalence relation ([22, Proposition 6.3]). This follows as compact gluing cubespaces satisfy the so-called universal replacement property [22, Proposition 6.3] (see also [1, Lemma 2.5] and [25, Proposition 3]):

**Proposition 1.8** Let $X$ be a compact gluing cubespace. Fix $s \geq 0$. Let $k \leq s + 1$ and $c \in C^k(X)$. Then if a configuration $c' \in X^{[0, 1]^k}$ has the same image as $c$ under the quotient map $X \to X/\sim_s$, then $c' \in C^k(X)$.

**Proposition 1.9** [22, Proposition 6.2] Fibrant cubespaces (in particular nilspaces) satisfy the gluing property.

Combining Propositions 1.8 and 1.9, we have that the universal replacement property for a fibrant cubespaces implies $(s + 1)$-uniqueness for the quotient cubespaces $X/\sim_s$. Indeed we have:

**Corollary 1.10** Let $X$ be a compact fibrant cubespace. Then $X/\sim_s$ is an $s$-nilspace for every $s \geq 0$.

The weak structure theorem for nilspaces of finite degree is an important factorization result into a finite tower of compact abelian group principal fiber bundles. We first recall the definition of a principal fiber bundle and then state the theorem.

**Definition 1.11** [28, Definition 2.2] Let $G$ be a topological group. A $G$-principal fiber bundle is a surjective continuous map $p: E \to B$ between two topological spaces $E$ and $B$ satisfying the following:

1. there exists a free continuous action of $G$ on $E$ such that for every $x \in E$
   $$p^{-1}(p(x)) = Gx;$$
2. there exists a homeomorphism $\varphi : B \to E/G$ such that $\varphi \circ p$ is the projection map $E \to E/G$.

**Theorem 1.12** [22, Theorem 5.4][1, Theorem 1] Let $X$ be a compact ergodic $s$-nilspace. Then $X$ factors as

$$X = X/\sim_s \to X/\sim_{s-1} \to \cdots \to X/\sim_0 \cong \{\ast\},$$

where $\{\ast\}$ is a singleton and each map $X/\sim_k \to X/\sim_{k-1}$ is an $A_k$-principal fiber bundle for some compact metrizable abelian group $A_k$.

The group $A_k$ is called the $k$-th structure group of $X$. When all $A_k$ are Lie groups, $X$ is called Lie-fibered.
1.3 Host-Kra Cubespaces

Given a topological group $G$, a sequence of decreasing closed subgroups

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{s+1} = \{e_G\} = G_{s+2} = \cdots$$

is an s-filtration or a filtration of degree s if $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \geq 0$ (here $[\cdot, \cdot]$ denotes the commutator subgroup). In such a case, we call $G$ a filtered group and write $G_*$ to emphasize that it is equipped with some s-filtration.

**Definition 1.13** For a filtered metric group $G_*$, the Host-Kra k-cube group $HK^k(G_*)$ is defined as the subgroup of $G\{0, 1\}_k$ generated by $[x]_F$ for every face $F \subseteq \{0, 1\}_k$ and $x \in G(k-\dim(F))$.

We remark that the Host-Kra k-cube group $HK^k(G)$ in Definition 1.2 can be recovered as $HK^k(G_*)$ with respect to the 1-filtration:

$$G = G_0 = G_1 \supseteq G_2 = \{e_G\} = \cdots.$$ 

When $G$ is a filtered topological group and $\Gamma$ is a discrete cocompact subgroup of $G$, $G/\Gamma$ carries a quotient cubespaces structure inherited from the Host-Kra cube group $HK^k(G_*)$, which we denote by $HK^k(G_*)/\Gamma$ (see [22, Definition 2.4] for more details).

We summarize the properties of Host-Kra cubes as follows [22, Appendix A.4, Propositions 2.5, 2.6].

**Proposition 1.14** $(G, HK^k(G_*))$ is a nilspace. Moreover, suppose that $\Gamma$ is compatible with $G_*$ in the sense that $\Gamma \cap G_i$ is discrete and cocompact in $G_i$ for all $i \geq 0$. Then for each $k \geq 0$, $HK^k(G_*)/\Gamma \subseteq (G/\Gamma)^{[0,1]^k}$ is a compact subset. Hence $(G/\Gamma, HK^k(G_*)/\Gamma)$ is a compact nilspace.

Given a Lie group $G$ and a discrete cocompact subgroup the quotient $G/\Gamma$ (which carries the structure of a manifold) is termed a nilmanifold. Using nilmanifolds the structure of a nilspace can be described as follows.

**Theorem 1.15** [23, Theorem 1.28] Suppose that $X$ is a compact ergodic strongly connected nilspace. Then $X$ is isomorphic as a cubespaces to an inverse limit $\lim \leftarrow X_n$ of nilmanifolds $X_n$ endowed with Host-Kra cubes.

1.4 Structure Theorems for Fibrations

**Definition 1.16** Suppose that $\varphi : X \to Y$ is a continuous map between two cubespaces $X$ and $Y$. We say $\varphi$ is a cubespaces morphism if $\varphi$ sends every cube of $X$ to a cube of $Y$. That is, the set $\{\varphi \circ c : c \in C^k(X)\}$ is contained in $C^k(Y)$ for every integer $k \geq 0$. We say $\varphi$ is a cubespaces isomorphism or simply an isomorphism if $\varphi$ is a bijection and both $\varphi$ and $\varphi^{-1}$ are cubespaces morphisms.

In the sequel, given an integer $n \geq 1$ and a map $\varphi : X \to Y$ between metric spaces, we will use the same notation $\varphi$ to denote the induced map $X^n \to Y^n$ by pointwise application of $\varphi$, when no confusion arises.

**Definition 1.17** A cubespaces morphism $\varphi : X \to Y$ is called relatively k-ergodic if for any $c \in C^k(Y)$ any configuration of $\varphi^{-1}(c)$ is a cube of $X$. 

\[ \text{Disclaimer: Springer} \]
It is clear that if \( \varphi \) is relatively \( k \)-ergodic, \( \varphi \) is relatively \( \ell \)-ergodic for each \( \ell \leq k \).

Relativizing the concept of corner-completion, fibrations are introduced in [22, Definition 7.1]:

**Definition 1.18** A cubespaces morphism \( f : X \rightarrow Y \) is called a **fibration** if \( f \) has \( k \)-completion for all \( k \geq 0 \). That is, given a \( k \)-corner \( \lambda \) in \( X \), if \( f(\lambda) \) can be completed to a cube \( c \) in \( Y \), then \( \lambda \) can be completed to a cube \( c_0 \) of \( X \) such that \( f(c_0) = c \).

**Example 1.19** In the setting of Definition 1.13, the quotient map \( G \rightarrow G/\Gamma \) induces a fibration \( HK^k(G_\bullet) \rightarrow HK^k(G_\bullet)/\Gamma \) [22, Proposition A.17].

It is clear that a composition of fibrations is a fibration. By a “corner-lifting” argument, we have the so-called **universal property** of fibrations [22, Lemma 7.8]:

**Proposition 1.20** Let \( f : X \rightarrow Y \) be a cubespaces morphism and \( g : Y \rightarrow Z \) a map between cubespaces. Then if \( f \) and \( g \circ f \) are fibrations, so is \( g \).

We recall the relative notion of uniqueness for a cubespaces morphism.

**Definition 1.21** Let \( f : X \rightarrow Y \) be a cubespaces morphism. We say \( f \) has \( k \)-uniqueness if for any \( k \)-cubes \( c, c' \) of \( X \) such that \( c|_k = c'|_k \) and \( f(c) = f(c') \) we have \( c = c' \). Moreover, we call \( f \) is a **fibration of degree at most** \( s \) or simply an \( s \)-**fibration** if \( f \) is a fibration and has \( (s + 1) \)-uniqueness.

**Definition 1.22** Given a fibration \( f : X \rightarrow Y \), two points \( x, x' \in X \) are called \( k \)-**canonically related relative to** \( f \), denoted by \( x \sim_{f,k} x' \), if \( x \sim_k x' \) and \( f(x) = f(x') \).

The following is a relative version of Corollary 1.10.

**Proposition 1.23** [22, Proposition 7.12] Let \( f : X \rightarrow Y \) be a fibration between compact gluing cubespaces. Fix \( s \geq 0 \). Then the relation \( \sim_{f,s} \) is a closed equivalence relation and the projection map \( \pi_{f,s} : X \rightarrow X/\sim_{f,s} \) is a fibration and factors through an \( s \)-fibration \( g : X/\sim_{f,s} \rightarrow Y \), that is, the relation induces a commutative diagram

![Commutative Diagram](image)

We remark that the dashed arrows in the above proposition emphasize that the underling maps are induced from the given map. This convention will be used throughout the paper.

The induced \( s \)-fibration \( g : X/\sim_{f,s} \rightarrow Y \) is maximal in the following sense.

**Proposition 1.24** Suppose that \( f : X \rightarrow Y \) is a fibration and \( s \geq 0 \). Then the induced fibration \( g : X/\sim_{f,s} \rightarrow Y \) is the **maximal** \( s \)-**fibration** in the sense that for each commutative diagram of cubespaces morphisms

![Diagram](image)
such that $\pi'$ is a fibration and $g'$ is an s-fibration. Then $\pi : X \to X/\sim_{f,s}$ factors through $\pi'$, i.e. there is a unique fibration $\varphi : X/\sim_{f,s} \to Z$ for which the following diagram is commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & X/\sim_{f,s} \\
\downarrow{\pi'} & & \downarrow{\varphi} \\
Z & \leftarrow & 
\end{array}
$$

Now we are ready to state the **relative weak structure theorem** for fibrations [22, Theorem 7.19, Corollary 7.20].

**Theorem 1.25** Let $f : X \to Y$ be an s-fibration between compact ergodic gluing cubespaces. Then $f$ factors as a finite tower of fibrations

$$
X = X/\sim_{f,s} \xrightarrow{f} X/\sim_{f,s-1} \xrightarrow{f} \cdots \xrightarrow{f} X/\sim_{f,1} \\
Y \cong X/\sim_{f,0}
$$

where for each $s \geq k \geq 1$ the fibration $X/\sim_{f,k} \to X/\sim_{f,k-1}$ is an $A_k(f)$-principal fiber bundle for a compact metrizable abelian group $A_k(f)$.

The group $A_k(f)$ in the above theorem is named the **k-th structure group** of $f$. In particular, $A_s(f)$ is called the **top structure group**. If all $A_k(f)$ are Lie groups, $f$ is called **Lie-fibered** as well as a **Lie fibration**.

Observe that a cubespaces $X$ is an s-nilspaces is equivalent to saying that the map $X \to \{\ast\}$ is an s-fibration. The following proposition elaborates on this point.

**Proposition 1.26** Let $f : X \to Y$ be an s-fibration. Then each fiber of $f$, as a subcubespace of $X$, is an s-nilspaces.

The main goal of this paper is to describe the structure of an s-fibration. Based on the relative weak structure theorem, our main endeavor is to relativize various techniques which appeared in the absolute setting, i.e., we first consider the Lie-fibered case and then deal with the general case.

One of the innovations in this article is to associate to a fibration $f : X \to Y$ the **k-th translation group** $\text{Aut}_k(f)$ of $f$ (Definition 2.2). Building on this concept, the main result we obtain is as follows.

**Theorem 1.27** Let $g : Z \to Y$ be a fibration of degree at most $s$ between compact ergodic gluing cubespaces. Then there exists a sequence of compact ergodic gluing cubespaces $\{Z_n\}_{n \geq 0}$, and an inverse system of fibrations $\{p_{m,n} : Z_n \to Z_m\}_{n \geq m}$ of degree at most $s$ satisfying the following properties:

1. $Z$ is isomorphic to the inverse limit $\lim \leftarrow Z_n$;
2. There exists a sequence of Lie fibrations $\{h_n : Z_n \to Y\}$ that are compatible with the connecting maps $p_{m,n}$;
3. Suppose that $g^{-1}(g(z))$ is strongly connected for some $z \in Z$. Define $z_n = p_n(z)$ as the image of the projection map $p_n : Z \to Z_n$. Then $g^{-1}(g(z))$ is isomorphic to the inverse limit

$$
\lim \leftarrow (\text{Aut}_1^\ast(h_n)/\text{Stab}(z_n)).
$$

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Moreover, for each \( k \geq 0 \), \( C^k(g^{-1}(g(z))) \) is isomorphic to the inverse limit

\[
\lim\left( HK^k(Aut^\bullet(h_n))/\text{Stab}(zn) \right).
\]

We remark that statements (1) and (2) of the above theorem imply that an \( s \)-fibration factors as an inverse limit of Lie fibrations. This is analogous to the statement in the absolute setting which says that an \( s \)-nilspace equals an inverse limit of Lie-fibered \( s \)-nilspaces.

By Proposition 1.26 the fiber \( g^{-1}(z) \) is a nilspace. As by assumption \( g^{-1}(z) \) is strongly connected, it holds by Theorem 1.15, that it is isomorphic as a cubespace to an inverse limit of nilmanifolds endowed with the Host-Kra cubes. However statement (3) is much stronger as it gives a simultaneous representation for all fibers of \( f \) in terms of the fixed sequence of Lie groups \( Aut^\bullet_1(h_n) \). Thus the fibers are approximated uniformly as closely as desired by quotients by natural co-compact subgroups of a single Lie group associated with the factorization.\(^7\) We note that the fact that if one fiber is strongly connected then all fibers are strongly connected follows from Proposition 3.8 in the sequel.

It is natural to ask what can be said for non strongly connected fibers. The general answer is not known, however in [32] Rees exhibited a certain minimal distal extension of a rotation on a solenoid for which all fibers are not homeomorphic. As such an extension is a fibration we see that necessarily its fibers are not isomorphic. Rees also proved that the fibers of a minimal distal extension of a path-connected system are homeomorphic. Inspired by this result we prove the following theorem, relying on a homotopy argument and the relative weak structure theorem.

**Theorem 1.28** Let \( g: Z \to Y \) be a fibration of degree at most \( s \) between compact ergodic gluing cubespaces. Suppose that \( Y \) is path-connected and \( g \) is Lie-fibered. Then for any \( y_0, y_1 \in Y \) there is a cubespace isomorphism between \( g^{-1}(y_0) \) and \( g^{-1}(y_1) \) as cubespaces.

### 1.5 Fibrations Arising from Dynamical Systems

Proving that a cubespace morphism is a fibration is in general non-trivial. Turning to topological dynamical systems we show they provide a hitherto unknown source of new examples of fibrations.

**Theorem 1.29** Let \( \pi: X \to Y \) be a factor map of minimal systems such that \( X \) is fibrant (e.g., \( X \) is minimal distal). Then \( \pi \) is a fibration between the associated dynamical cubespaces.

The proof given in Sect. 5 uses the relative Furstenberg structure theorem for distal extensions of minimal systems.

**Remark 1.30** Another rich source of \( s \)-fibrations is found in ergodic theory. In [20], Gutman and Lian studied the question when an ergodic abelian group extension of a strictly ergodic system admits a strictly ergodic distal model. Under some sufficient conditions they proved that the associated topological model map is actually an \( s \)-fibration.

\(^7\) This follows from the following lemma: For an inverse limit \( Z = \lim Z_n \) arising from continuous maps \( p_n: (Z, d) \to (Z_n, d_n) \) between compact metric spaces, we have \( \lim_{n \to \infty} \sup_{y \in Z_n} \text{diam}(p^{-1}(y), d) = 0 \). As each \( Z_n \) is homeomorphic to a disjoint union of quotients of \( Aut^\bullet_1(h_n) \), the approximation is uniform for all fibers simultaneously.
1.6 The Relative Nilpotent Regionally Proximal Relation

In [17] Glasner, Gutman, and Ye introduced the so-called nilpotent regionally proximal relations for actions of arbitrary groups. For minimal actions by abelian groups these relations coincide with the regionally proximal relations introduced by Host, Kra and Maass [27], generalizing the classical (degree 1) definition of Ellis and Gottschalk [13]. The regionally proximal relations are not equivalence relations in general. However [14,29,38] proposed various sufficient conditions under which the regionally proximal relations of degree 1 are equivalence relations. In particular this is the case for minimal abelian group actions [33]. In [17] it was proven unexpectedly that for any minimal action \((G, X)\), the nilpotent regionally proximal relation of degree \(s\), denoted by \(\text{NRP}^{[s]}(X)\), is an equivalence relation and moreover using [23, Theorem 1.4], under some restrictions on the acting group, it corresponds to the maximal dynamical nilspace factor (also known as pronilfactor) of order at most \(s\). Moreover \(\text{NRP}^{[1]}(X)\) corresponds to the maximal abelian group factor of \((G, X)\) for any acting group \(G\).

We recall the definition from [17]. Let \(x, x' \in X\) be two points of a metric space \(X\). Denote by \(\ell^k(x, x')\) the map \(\{0, 1\}^k \rightarrow X\) assigning the value \(x'\) at \(\overrightarrow{1}\) and \(x\) elsewhere.

**Definition 1.31** Let \((G, X)\) be a dynamical system. We say a pair of points \((x, x')\) of \(X\) are nilpotent regionally proximal of order \(k\), denoted by \((x, x') \in \text{NRP}^{[k]}(X)\), if \(\ell^{k+1}(x, x') \in C^+_G(X)\).

**Theorem 1.32** [17, Theorem 3.8] If \((G, X)\) is minimal, then \(\text{NRP}^{[k]}(X)\) is a closed \(G\)-invariant equivalence relation for every \(k \geq 0\).

**Definition 1.33** Let \(\pi : (G, X) \rightarrow (G, Y)\) be a factor map of dynamical systems. We define the relative nilpotent regionally proximal relation of order \(k\) w.r.t. \(\pi\), denoted by \(\text{NRP}^{[k]}(\pi)\), as the intersection of \(\text{NRP}^{[k]}(X)\) with \(R_\pi := \{(x, x') \in X^2 : \pi(x) = \pi(x')\}\).

**Proposition 1.34** [22, Lemma 7.17] For a factor map \(\pi : X \rightarrow Y\) of dynamical systems, we have \(\text{NRP}^{[k]}(\pi) = \sim_{\pi,k}\). In particular, \(\text{NRP}^{[1]}(X) = \sim_k\).

From Theorem 1.32, we see that for any factor map \(\pi\) between minimal systems \(\text{NRP}^{[k]}(\pi)\) is a closed \(G\)-invariant equivalence relation.

Let us discuss the relation between \(\text{NRP}^{[k]}(\pi)\) and several classical (relative) relations.

**Definition 1.35** Let \(\pi : (G, X) \rightarrow (G, Y)\) be a factor map. The relative proximal relation \(P(\pi)\) is defined as those pairs \((x, y) \in R_\pi\) such that \(d(g_i x, g_i y)\) approaches 0 for some sequence \(\{g_i\}\) of \(G\). The map \(\pi\) is called a distal extension if \(P(\pi)\) is trivial. In particular, \((G, X)\) is called a distal system if and only if \(P(\{G, X \rightarrow \{\ast\}\})\) is trivial. When \(G\) is abelian, for every positive integer \(k\), we define the relative \(k\)-th regionally proximal relation of \(\pi\), denoted by \(\text{RP}^{[k]}(\pi)\), by the collection of pairs \((x, y) \in R_\pi\) such that there exist sequences of elements \((x_i, y_i) \in R_\pi\) and \((g_1^i, \ldots, g_k^i) \in G^k\) satisfying that

\[
\lim_{i \rightarrow \infty} (x_i, y_i) = (x, y), \text{ and } \lim_{i \rightarrow \infty} d((\sum_{j=1}^{k} \epsilon_j g_i^j)x_i, (\sum_{j=1}^{k} \epsilon_j g_i^j)y_i) = 0
\]

for every \((\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d \setminus \{0\}\).
Penazzi gave some algebraic conditions under which the relative regionally proximal relation is an equivalence relation [31]. More about this topic may be found in [11, Chapter V.2].

Let $G$ be an abelian group and $\pi : (G, X) \to (G, Y)$ a factor map of minimal systems. From [17, Proposition 8.9], we have that $\text{RP}^{[k]}(X) \subseteq \text{NRP}^{[k]}(X)$. As a consequence, we have the following proposition.

**Proposition 1.36** Let $G$ be an abelian group and $\pi : (G, X) \to (G, Y)$ a factor map of minimal systems. Then $\text{RP}(\pi) \subseteq \text{RP}^{[1]}(\pi) \subseteq \text{RP}^{[k]}(\pi) \subseteq \text{NRP}^{[k]}(\pi)$. In particular, if $\text{NRP}^{[k]}(\pi)$ is trivial, $\pi$ is a distal extension.

**Remark 1.37** (1) For a factor map $\pi : (G, X) \to (G, Y)$ of minimal systems, since $\text{RP}^{[1]}(\pi) \subseteq \text{RP}^{[1]}(X) \cap R_\pi$ in general (even for $G = \mathbb{Z}$) [11, Remark (2.2.3), Page 411], we have in particular in general that $\text{RP}^{[1]}(\pi) \subseteq \text{NRP}^{[1]}(\pi)$.

(2) Recall that $\text{RP}^{[1]}(\pi)$ generates the maximal equicontinuous factor $(G, X/\text{RP}^{[1]}(\pi)) \to (G, Y)$ (see [11, Chapter V, Theorem 2.21]). One may wonder whether $\text{NRP}^{[1]}(\pi)$ corresponds to the maximal factor which is a principal abelian group extension of the base space. It turns out that this is not true. Indeed, from the famous Furstenberg counterexample [17, Remark 7.19], a principal abelian group extension of minimal (distal) systems is not necessarily of finite degree relative to the factor system. To see this, recall that Furstenberg’s example is given by the projection map $\pi : (\mathbb{T}^2, T) \to (\mathbb{T}, S)$. Here $T$ sends $(x, y)$ to $(x + \alpha, y + \varphi(x))$ for some continuous map $\varphi : \mathbb{T} \to \mathbb{T}$ and irrational number $\alpha$. Since $(\mathbb{T}, S)$ is a minimal abelian group system, by [17, Lemma 8.4], $\text{NRP}^{[k]}(\mathbb{T}) \subseteq \text{NRP}^{[1]}(\mathbb{T})$ are trivial for all $k \geq 1$. This implies that $\text{NRP}^{[k]}(\mathbb{T}^2) \subseteq R_\pi$ and hence $\text{NRP}^{[k]}(\mathbb{T}) = \text{NRP}^{[k]}(\mathbb{T}^2)$ is not trivial.

*Systems of finite degree* were introduced in [27] for $\mathbb{Z}$-actions. In [17] they were defined for general group actions and investigated from a structural point of view. Recall that a minimal system $(G, X)$ is called a system of degree at most $s$ if $\text{NRP}^{[s]}(X)$ equals the diagonal subset $\Delta$ of $X^2$ ([17, Definition 7.1]). It is thus natural to define:

**Definition 1.38** Let $\pi : X \to Y$ be a factor map of minimal systems. We say $\pi$ is an extension of degree at most $s$ (relative to $Y$) if $\text{NRP}^{[s]}(\pi) = \Delta$.

In Sect. 6 we develop the structure theory of extensions of finite degree. Our main structural theorem, proven in Sect. 6, is an application of Theorem 1.27 in the dynamical setting:

**Theorem 1.39** Let $\pi : (G, X) \to (G, Y)$ be an extension of degree at most $s$ between minimal distal systems. Then the following holds:

1. The map $\pi$ is an $s$-fibration.
2. There exists a sequence of dynamical systems $\{(G, Z_n)\}_{n \geq 0}$, and an inverse system of factor maps $\{p_{m,n} : (G, Z_n) \to (G, Z_m)\}_{n \geq m}$ such that $(G, X) = \lim(G, Z_n)$.
3. There exists a sequence of factor maps $\{h_n : (G, Z_n) \to (G, Y)\}$ which are Lie fibrations and are compatible with the connecting maps $p_{m,n}$.

**Question 1.40** Let $\pi : X \to Y$ be an extension of degree at most $s$ between minimal systems such that $\pi$ is a fibration. Is every fiber of $\pi$ isomorphic as a subspace to an inverse limit of nilmanifolds?

---

8 See Definition 5.1.

9 Also known as systems of finite order.
Remark 1.41 It is important to point out that in the relative setting, $G$ does not immerse into $\text{Aut}_1(\pi)$ because $G$ does not fix the fibers of $\pi$. This is a crucial obstruction not existing in the absolute setting, making Question 1.40 non-trivial.

Remark 1.42 It is interesting to compare Theorem 1.39 to a refinement of the relative Furstenberg structure theorem due to Bronstein (see [4, 3.17.8]) which states that a distal extension of metric minimal dynamical systems factors as (a possibly countable) tower of isometric extensions\(^{10}\) where the fibers are given by quotients of compact Lie groups by closed subgroups. Note however that these compact Lie groups are not necessarily abelian.

1.7 Conventions

Throughout the paper, all spaces are metric spaces and $G$ denotes a topological group. When we discuss an $s$-fibration, the underlying cubespaces are always assumed to be compact ergodic and have the gluing property. Unless specified otherwise $k$ always denotes a non-negative integer.

1.8 Structure of the Paper

Theorem 1.27 is proven in Sects. 2 and 3, where Sect. 2 is solely devoted to the Lie-fibered case of statement (3). In Sect. 4 we prove Theorem 1.28 and in Sect. 5 we prove Theorem 1.29. The final section, Sect. 6 is devoted to extensions of finite degree and to the proof of Theorem 1.39.

2 Strongly Connected Fibers

In this section, we prove the Lie-fibered case of Theorem 1.27(3). Recall that an $s$-fibration $g : Z \to Y$ between compact ergodic gluing cubespaces is called \textbf{Lie fibered} if all structure groups $A_k(g)$ of $g$ are Lie groups. In light of Theorem 1.25, we write $g_{s-1} : Z/\sim_{g,s-1} \to Y$ for the induced $(s-1)$-fibration, which we may call the \textbf{canonical $(s-1)$-th factor} of $g$.

2.1 The Main Steps of the Proof

Let us first recall the notion of the $k$-th translation groups of a cubespace. For a compact cubespace $X$, denote by $\text{Aut}(X)$ the collection of cubespace isomorphisms of $X$.

\textbf{Definition 2.1} Fix $k \geq 0$. We say $\varphi \in \text{Aut}(X)$ is a \textbf{$k$-translation} if for every $n \geq k$, $(n-k)$-face $F \subseteq \{0, 1\}^n$, and $c \in C^n(X)$, the map $(0, 1)^n \to X$ sending $\omega \in F$ to $\varphi(c(\omega))$ and $c(\omega)$ elsewhere is still a cube of $X$. The \textbf{$k$-th translation group} $\text{Aut}_k(X)$ of $X$ is defined as the collection of all $k$-translations of $X$.

Let $d$ be the metric on $X$. For each homeomorphism $\phi : X \to X$, define $||\phi|| := \max_{x \in X} d(x, \phi(x))$.

---

\(^{10}\) See Definition 5.3.
Then the compact-open topology of the group of homeomorphisms of \( X \), denoted by \( \text{Homeo}(X) \), can be induced from the metric

\[
d(\phi, \psi) := ||\phi \circ \psi^{-1}||.
\]

In particular, as a closed subgroup of \( \text{Homeo}(X) \), \( \text{Aut}(X) \) is a metric space. When we say \( \varphi \in \text{Homeo}(X) \) is (appropriately) small, we mean \(||\varphi||\) is (appropriately) small; if \( f \circ \varphi = f \), we say \( \varphi \) fixes the fibers of \( f \).

Now consider a fibration \( f : X \to Y \). To study the cubespaces structure of the fibers of \( f \), we introduce the notion of the \( k \)-th translation group of \( f \). Note that each fiber \( f^{-1}(y) \) is a subcubespace of \( X \).

**Definition 2.2** The \( k \)-th translation group of \( f \) is defined as

\[
\text{Aut}_k(f) := \{ \varphi \in \text{Homeo}(X) : \varphi|_{f^{-1}(y)} \in \text{Aut}_k(f^{-1}(y)) \text{ for every } y \in Y \}.
\]

Intuitively this means that each element \( \varphi \) of \( \text{Aut}_k(f) \) is a homeomorphism of \( X \) such that \( \varphi \) fixes the fibers of \( f \) and preserves the cubespaces structure of fibers in a strong sense.

**Remark 2.3** It is interesting to compare this definition with the definition of the fiber preserving automorphism group of a dynamical extension \( \phi : (G, X) \to (G, Y) \), used to characterize (weak) group extensions [11, Chapter V, Remark 4.2 (4)].

When \( f \) is an \( s \)-fibration, \( \text{Aut}_{s+1}(f) = \{1\} \) and we obtain a filtration \( \text{Aut}_\bullet(f) \) of degree \( s \) as follows:

\[
\text{Aut}_1(f) = \text{Aut}_1(f) \supseteq \text{Aut}_2(f) \supseteq \cdots \supseteq \text{Aut}_{s+1}(f) = \{1\}.
\]

We may therefore consider \( \text{Aut}_\bullet(f) \) as a cubespaces (furthermore an \( s \)-nilspace) endowed with Host-Kra cubes.

Denote by \( \text{Aut}_k^+(f) \) the connected component of \( \text{Aut}_k(f) \) containing the identity. We obtain a filtration of closed groups

\[
\text{Aut}_1^+(f) = \text{Aut}_1^+(f) \supseteq \text{Aut}_2^+(f) \supseteq \cdots \supseteq \text{Aut}_{s+1}^+(f) = 1.
\]

Similarly to [21, Proposition 3.1], we have

**Proposition 2.4** Let \( g : Z \to Y \) be an \( s \)-fibration between compact ergodic gluing cubespaces. Then \( A_s(g) \) embeds into \( \text{Aut}_s(g) \) via sending \( a \) to \( f_a \), where \( f_a : Z \to Z \) is given by \( f_a(z) = a.z \).

**Proof** Fix \( n \geq s, y \in Y \), and \( c \in C^n(g^{-1}(y)) \). Define \( A := A_s(g) \) and \( \pi := \pi_{g,s-1} \). Since \( C^n(\mathcal{D}_s(A))c = \pi^{-1}(\pi(c)) \) (see definition of \( \mathcal{D}_s(A) \) in [22, Section 5.1]), we have

\[
g([a]_{FC}) = g_{s-1} \circ \pi([a]_{FC}) = g_{s-1} \circ \pi(c) = g(c) = y
\]

for any face \( F \subseteq \{0,1\}^n \) of codimension \( s \). It follows that \( f_a|_{g^{-1}(y)} \in \text{Aut}_s(g^{-1}(y)) \) and thus \( f_a \in \text{Aut}_s(g) \).

As a relative analogue of [21, Proposition 2.17], the following proposition says that we can also use the evaluation map to construct the desired cubespaces morphism.

**Proposition 2.5** Let \( g : Z \to Y \) be an \( s \)-fibration. Fix \( z \in Z \) and a cube \( c \) of \( C^n(g^{-1}(g(z))) \). Then for any \( (\varphi_\omega)_{\omega \in \{0,1\}^n} \in \text{HK}^n(\text{Aut}_\bullet(g)) \), \( (\varphi_\omega(c(\omega)))_{\omega} \in C^n(g^{-1}(g(z))) \). In particular the induced natural map

\[
\text{Aut}_1(g) / \text{Stab}(z) \to g^{-1}(g(z))
\]

is a cubespaces morphism.
Proof The definition of $\text{Aut}_s(g)$ guarantees that the map is well defined. The rest of the argument stays the same as in [21, Proposition 2.17].

Recall that a cubespaces $X$ is called strongly connected if $C^k(X)$ is connected for every $k \geq 0$. In [21, Theorem 2.18], a strongly connected Lie-fibered nilspace $X$ is shown to be a nilmanifold using the translation groups $\text{Aut}_k(X)$. For the relative case, we are now ready to state the structure theorem for a Lie-fibered fibration of finite degree.

**Theorem 2.6** (The Lie-fibered case of Theorem 1.27(3)) Let $g : Z \to Y$ be a fibration of degree at most $s$ between compact ergodic cubespaces that obey the gluing condition. Fix a point $z \in Z$. Suppose that $g$ is Lie-fibered, then the following holds.

1. $\text{Aut}_1^s(g)$ is a Lie group;
2. the stabilizer $\text{Stab}(z)$ of $z$ in $\text{Aut}_1^s(g)$ is a discrete cocompact subgroup;
3. if $g^{-1}(g(z))$ is strongly connected as a subcubespace, then the fiber $g^{-1}(g(z))$ is homeomorphic to the nilmanifold $\text{Aut}_1^s(g)/\text{Stab}(z)$. Moreover, there are cubespaces isomorphisms

$$C^k(g^{-1}(g(z))) \cong \text{HK}^k(\text{Aut}_s^s(g))/\text{Stab}(z)$$

for all $k \geq 1$.

The proof of Theorem 2.6 uses the full strength of the relative weak structure theorem (Theorem 1.25) and will be given at the end of this subsection. Recall that in [21, Proposition 3.2], for an $s$-nispace $X$ a canonical group homomorphism $\text{Aut}_k(X) \to \text{Aut}_k(X/\sim_s)$ is exhibited. We adapt the statement and argument for our relative setting.

**Proposition 2.7** Fix $s, k \geq 1$. Let $g : Z \to Y$ be an $s$-fibration between compact ergodic gluing cubespaces. Then there is a canonical continuous group homomorphism $\pi_s : \text{Aut}_k(g) \to \text{Aut}_k(g_{s-1})$ such that for every $\phi \in \text{Aut}_k(g)$ the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & Z \\
\downarrow{\pi_{g,s-1}} & & \downarrow{\pi_{g,s-1}} \\
\pi_{g,s-1}(Z) & \xrightarrow{\pi_s(\phi)} & \pi_{g,s-1}(Z) \\
\uparrow{g_{s-1}} & & \uparrow{g_{s-1}} \\
Y & \xrightarrow{g} & Y
\end{array}
$$

commutes.

Proof We modify the proof in the absolute setting to the relative case. Denote by $\pi$ the projection map $\pi_{g,s-1}$. Given $\phi \in \text{Aut}_k(g)$ and $y \in \pi(Z)$, writing $y = \pi(x)$ for some $x \in Z$, we define $\pi_s(\phi)(y) := \pi \circ \phi(x)$.

We check that $\pi_s$ is well-defined. Note that $\phi \in \text{Aut}_k(g) \subseteq \text{Aut}_1(g)$. By Proposition 2.4, $A_s(g)$ embeds into $\text{Aut}_s(g)$. Since $\text{Aut}_s(g)$ is filtered, we conclude that $A_s(g)$ commutes with $\text{Aut}_k(g)$. For any $y \in \pi(Z)$ and $x, x' \in \pi^{-1}(y)$, there exists a unique $a \in A_s(g)$ such that $x' = ax$. Then

$$\pi_s(\phi)(y) := \pi \circ \phi(x') = \pi \circ \phi(ax) = \pi \circ (a\phi(x)) = \pi \circ \phi(x).$$

This shows that $\pi_s$ is independent of the choice of lifting. The statement that $\pi_s(\phi)$ is a $k$-translation follows from the facts that $\phi$ is a $k$-translation and $\pi$ is a cubespaces morphism. □
The first hard ingredient of the proof of Theorem 2.6, used for establishing the nilmanifold structure in statement (3), is to relativize [21, Proposition 3.3]: Every element of \( \text{Aut}_k(g_{s-1}) \) of sufficient small norm can be realized as the image of \( \pi_s \) of some element of \( \text{Aut}_s(g) \). The proof will be given in Section 4.2.

**Theorem 2.8** Fix \( s \geq 1 \). Let \( g : Z \to Y \) be an \( s \)-fibration between compact ergodic gluing cubespaces. Assume that \( A_s(g) \) is a Lie group. Then \( \pi_s : \text{Aut}_k(g) \to \text{Aut}_k(g_{s-1}) \) is open for all \( k \geq 1 \).

The next result is analogous to [21, Theorem 5.2]. As the proof is similar we omit it.

**Theorem 2.9** Let \( A \) be a compact abelian Lie group with a metric \( d_A \). Fix integers \( s \geq 0 \) and \( \ell \geq 1 \). Then there exists \( \epsilon \) depending only on \( s, \ell \) and \( A \) such that the following holds.

Let \( g : Z \to Y \) be an \( s \)-fibration between compact ergodic gluing cubespaces. Suppose that \( f : Z \to A \) is a continuous function such that

\[
d_A(f(z), f(z')) \leq \epsilon
\]

for every \( z, z' \in Z \) with \( g(z) = g(z') \), and the map \( \partial^\ell f : C^k_\ell(Z) \to A \) sending \( c \) to

\[
\sum_{\omega \in [0,1]^\ell} (-1)^{\omega(\ell)} f(c(\omega))
\]

is the zero function. Then \( f \) is a constant function.

The following is used in the the proof of Theorem 2.6.

**Lemma 2.10** Lie-fiberedness of \( g \) implies that \( \text{Stab}(z) \subseteq \text{Aut}_1(g) \) is discrete for every \( z \in Z \).

**Proof** In light of the relative weak structure theorem (Theorem 1.25), Theorem 2.9 guarantees that the argument in the absolute setting may be modified appropriately. \( \Box \)

The proof of the following lemma follows the argument of [21, Proposition 3.7] (with the help of Lemma 2.10) so we omit it.

**Lemma 2.11** \( A_s(g) \subseteq \ker(\pi_s) \) is open.

Based on Lemma 2.11, we can prove one of the statements of Theorem 2.6.

**Corollary 2.12** The 1-translation group \( \text{Aut}_1(g) \) has a Lie group structure.

**Proof** First note that \( \text{Aut}_1(g_0) = \text{Aut}_1(\text{id}_Y) = 1 \) and hence \( \ker(\text{Aut}_1(g_1) \to \text{Aut}_1(g_0)) = \text{Aut}_1(g_1) \). By Lemma 2.11, \( A_1(g) \) is an open subgroup of \( \ker(\text{Aut}_1(g_1) \to \text{Aut}_1(g_0)) \). Thus we can extend the differentiable structure of \( A_1(g) \) onto \( \text{Aut}_1(g_1) \).

Inductively assume that \( \text{Aut}_1(g_{s-1}) \) is Lie. By Theorem 2.8, \( \im(\pi_s) \) is an open subgroup of \( \text{Aut}_1(g_{s-1}) \) and hence Lie. Applying the extension theorem for Lie groups [18, Theorem 3.1], it suffices to show \( \ker(\pi_s) \) has a Lie group structure. By Lemma 2.11, \( A_s(g) \subseteq \ker(\pi_s) \) is an open subgroup. Thus we can extend the Lie group structure of \( A_s(g) \) onto \( \ker(\pi_s) \). \( \Box \)

The following is an analogue of [21, Lemma 3.10].

**Lemma 2.13** Let \( g : Z \to Y \) be a Lie-fibered s-fibration. Fix \( 0 \leq k < s \). Then for any \( \epsilon > 0 \) there exists \( \delta > 0 \) satisfying the following. For any \( z, z' \in Z \) such that \( z \sim_{g,k} z' \) and \( d(z, z') < \delta \), there is \( \varphi \in \text{Aut}_{k+1}(g) \) such that \( ||\varphi|| < \epsilon \) and \( \varphi(z) = z' \). In particular, for each \( z \in Z \), \( \text{Aut}_{k+1}(g)z \) is open in \( \pi_{g,k}^{-1}(\pi_{g,k}(z)) \).
Proof Based on Proposition 2.7 and Theorem 2.8, similarly to the proof of [21, Lemma 3.10], we can induct on \( s \) to obtain the first conclusion. We now prove that \( \text{Aut}_{k+1}(g)Z \) is open. Fix \( \varphi \in \text{Aut}_{k+1}(g) \). Let \( \varepsilon > 0 \) be small enough such that the \( \varepsilon \)-ball \( B_{\varepsilon}(\text{id}) \) at the identity is contained in \( \text{Aut}_{k+1}(g) \). There exists \( \delta > 0 \) satisfying that whenever \( y, y' \in \pi^{-1}_{g,k}(\pi_{g,k}(z)) \) such that \( d(y, y') < \delta \), we can find \( \varphi' \in B_{\varepsilon}(\text{id}) \) such that \( y = \varphi'(y') \).

Note that for \( y \in B_{\delta}(\varphi(z)) \cap \pi^{-1}_{g,k}(\pi_{g,k}(z)) \), we have

\[
\pi_{g,k}(y) = \pi_{g,k}(z) = \pi_{g,k} \circ \varphi(z).
\]

It follows that \( y = \varphi' \circ \varphi(z) \) for some \( \varphi' \in \text{Aut}_{k+1}(g) \). \( \Box \)

The following is an analogue of [21, Lemma 3.12]. Using the relative weak structure theorem, the proof follows a similar argument, inducting on \( s \).

Lemma 2.14 For each \( c \in C^n(g^{-1}(g(z))) \), the evaluation map \( \text{ev}_c : \text{HK}^n(\text{Aut}_k^*(g)) \to C^n(g^{-1}(g(z))) \) sending \( (\varphi_\omega)_\omega \) to \( (\varphi_\omega(c(\omega)))_\omega \) is open.

Proof of Theorem 2.6 Since \( \text{Aut}_k^*(g) \subseteq \text{Aut}_k(g) \subseteq \text{Aut}_1(g) \) are closed subgroups, and by Corollary 2.12, \( \text{Aut}_1(g) \) is Lie, we have \( \text{Aut}_k^*(g) \) is Lie.

To show that the evaluation map \( \text{HK}^n(\text{Aut}_k^*(g)) \to C^n(g^{-1}(g(z))) \) is surjective, it suffices to show that the action \( \text{HK}^n(\text{Aut}_k^*(g)) \curvearrowright C^n(g^{-1}(g(z))) \) is transitive. Since \( \text{Aut}_k^*(g) \) is open in \( \text{Aut}_k(g) \) for all \( k \geq 0 \), by a relative version of [21, Lemma 3.15], \( \text{HK}^n(\text{Aut}_k^*(g)) \) is open in \( \text{HK}^n(\text{Aut}_k(g)) \). Thus by Lemma 2.14, the orbits of \( \text{HK}^n(\text{Aut}_k^*(g)) \) are open and hence closed in \( C^n(g^{-1}(g(z))) \). As assumption, \( C^n(g^{-1}(g(z))) \) is connected, it implies that the action is transitive. In particular, the evaluation map \( \text{ev}_z : \text{Aut}_k^*(g)/\text{Stab}(z) \to Z \) and the induced maps

\[
\text{ev}_{\Box^p}(\omega) : \text{HK}^n(\text{Aut}_k^*(g))/\text{Stab}(z) \to C^n(g^{-1}(g(z)))
\]

by pointwise application are homeomorphisms. Here \( \Box^p(z) \) denotes the constant cube of \( C^n(Z) \) taking value \( z \).

To see \( \text{Stab}(z) \cap \text{Aut}_k^*(g) \) is co-compact in \( \text{Aut}_k^*(g) \) for all \( k \geq 0 \), we first note by Lemma 2.13 that \( \text{Aut}_k^*(g)z \) is open in \( \pi^{-1}_{g,k-1}(\pi_{g,k-1}(z)) \). Moreover, since orbits partition the space and \( \text{Aut}_k^*(g)z \) is an orbit, it must be closed and hence compact. From the homeomorphism between \( \text{Aut}_k^*(g)z \) and \( \text{Aut}_k^*(g)/\text{Stab}(z) \cap \text{Aut}_k^*(g) \), we see that \( \text{Stab}(z) \cap \text{Aut}_k^*(g) \) is co-compact. \( \Box \)

2.2 Lifting Relative Translations

In this subsection, we prove Theorem 2.8. There are two steps in the proof: (1) lift small translations \( \varphi \) in \( \text{Aut}_k(g_{s-1}) \) to a small homeomorphism \( \psi : Z \to Z \); (2) replacing \( \psi \) with a genuine \( k \)-translation of the form \( h(\cdot), \psi(\cdot) \) for some continuous map \( h : Z \to A_s(g) \) with small norm. We will refer to the operation of replacing \( \psi \) with a \( k \)-translation of the form \( h(\cdot), \psi(\cdot) \) as repairing. Step (2) will be based on a relative version of [21, Theorem 4.11] [1, Lemma 3.19].

By a careful local section argument for Lie-principal bundles as in [21, Lemma 4.2], we accomplish the first step by the following lemma.

Lemma 2.15 Let \( g : Z \to Y \) be an \( s \)-fibration between compact ergodic gluing cubespaces. Assume that \( A_s(g) \) is Lie. Then one can lift every small enough homeomorphism \( \varphi \) of \( \pi_{g,s-1}(Z) \) up to a small homeomorphism \( \psi : Z \to Z \) such that \( \psi \) is \( A_s(g) \)-equivariant.
Definition 2.16 For a cubespaces morphism \( f : X \to Y \), we define the \( k \)-th fiber cubes of \( f \) as the closed subset
\[
C_f^k(X) := \bigcup_{y \in Y} C_k(f^{-1}(y)).
\]

Definition 2.17 Let \( \ell \geq 1 \) and \( n \geq 0 \). Given \( c_1, c_2 \in C^n(X) \), the generalized \( \ell \)-corner \( \llcorner(c_1; c_2) : \{0, 1\}^{n+\ell} \to X \) is given by \( c_2(\omega_1, \ldots, \omega_n) \) for \( \omega = (\omega_1, \ldots, \omega_n, 1) \) and \( c_1(\omega_1, \ldots, \omega_n) \) elsewhere.

We need to repair the lift \( \psi \) to a \( k \)-translation of \( \text{Aut}_k(g) \). Unwrapping the definition, we want to find a small continuous function \( h : Z \to A_s(g) \) such that the map \( \tilde{\phi} : Z \to Z \) defined by sending \( z \) to \( h(z).\psi(z) \) is a \( k \)-translation of \( g \). To show \( \tilde{\phi} \in \text{Aut}_k(g) \), we need the following criterion.

Proposition 2.18 [21, Proposition 2.13] Let \( X \) be an \( s \)-nilspace. Fix \( 0 \leq k \leq s+1 \). Then a homeomorphism \( \phi : X \to X \) is a \( k \)-translation if and only if for any \( (s+1-k) \)-cube \( c \) of \( X \) the configuration \( \llcorner^k(c, \phi(c)) \) is an \((s+1)\)-cube.

Applying the above proposition to the cubespaces \( g^{-1}(y) \), we need to show that \( \llcorner^k(c, \tilde{\phi}(c)) \) is an \((s+1)\)-cube of \( g^{-1}(y) \) for every cube \( c \in C^{s+1-k}(g^{-1}(y)) \).

In order to measure the extent to which a configuration is failing to be a cube, let us introduce the definition of discrepancy in the setting of an \( s \)-fibration \( g : Z \to Y \). Let \( c : \{0, 1\}^{s+1} \to Z \) be a map such that \( \pi(c) \in C^{s+1}(\pi(Z)) \), say, \( \pi(c) = \pi(c_0) \) for some \( c_0 \in C^{s+1}(Z) \). By the relative weak structure theorem (Theorem 1.25), there exists a unique map \( \beta : \{0, 1\}^{s+1} \to A_s(g) \) such that \( c = \beta.c_0 \).

Definition 2.19 The discrepancy \( \Delta(c) \) of \( c \) is defined as
\[
\Delta(c) := \sum_{\omega \in \{0, 1\}^{s+1}} (-1)^{|\omega|} \beta(\omega).
\]
Here \( |\omega| \) denotes the sum \( \sum_{j=1}^{s+1} \omega_j \) for \( \omega = (\omega_1, \ldots, \omega_{s+1}) \).

Following the argument of [21, Proposition 4.5], we have that the discrepancy is well defined and \( \Delta(c) = 0 \) if and only if \( c \in C^{s+1}(Z) \). We now relativize the notion of cocycles and coboundaries of cubespaces:

Definition 2.20 Let \( f : X \to Y \) be a cubespace morphism and \( A \) an abelian group. Fix an integer \( \ell \geq 1 \). Consider a continuous map \( \rho : C_\ell^f(X) \to A \). We say \( \rho \) is an \( \ell \)-cocycle on fibers of \( f \) if it is additive in the sense that
\[
\rho([c_1, c_3]) = \rho([c_1, c_2]) + \rho([c_2, c_3])
\]
for any \( c_1, c_2, c_3 \in C_{f}^{\ell-1}(X) \) such that the three concatenations in the equation are in \( C_\ell^f(X) \). We say \( \rho \) is a coboundary if there exists a continuous map \( h : X \to A \) such that \( \rho \) can be written as
\[
\rho(c) = \partial h(c) := \sum_{\omega \in \{0, 1\}^\ell} (-1)^{|\omega|} h(c(\omega))
\]
for every \( c \in C_\ell^f(X) \).
In particular, $\ell$ is well defined and so is $\rho_\psi^{\text{nor}}$. We will show it is possible to "integrate $f/\Delta_1(\cdot)$ sending $c$ to $\text{Aut}_\rho$ sending $c$ to $\text{Aut}_\rho$. Since $\rho_\psi$ is an $(s + 1 - k)$-coboundary of some function with sufficient small norm.

**Proof** Fix $y \in Y$ and a cube $c \in C^{s+1-k}(g^{-1}(y))$. Since $\varphi \in \text{Aut}_k(g_{s-1})$, from
\[
\pi(\ell(c; \varphi(c))) = \ell(\pi(c); \pi(\varphi(c))) = \ell(\pi(c); \varphi(\pi(c))),
\]
we have $\pi(\ell(c; \varphi(c)))$ is a cube of $C^{s+1-k}(\pi \circ g^{-1}(y))$. Thus the discrepancy $\Delta(\ell(c; \varphi(c)))$ is well defined and so is $\rho_\psi$.

Assume $\rho_\psi = \partial^{s+1-k} f$ for some continuous function $f : Z \to A_s(g)$ with small norm. We will show it is possible to “integrate $f$ over $A_s(g)$” resulting with a function $F : Z \to A_s(g)$ such that $F(ax) = F(x)$ for every $a \in A_s(g)$ and $x \in Z$. In order to show that such a procedure is well-defined, let us recall a “lifting up & down” technique for defining integration. Since $A_s(g)$ is a compact abelian Lie group, there exists an isomorphism $\phi : A_s(g) \to (\mathbb{R}/\mathbb{Z})^d \times K$ for some finite-dimensional torus $(\mathbb{R}/\mathbb{Z})^d$ and some finite group $K$. If $f$ is of sufficiently small norm, there exists a sufficient small $\delta$ such that the image of $f$ belongs to a $\delta$-ball $B$ around the identity with respect to a given compatible metric. Notice that $\phi$ induces an embedding $p : B \to \mathbb{R}^d$. For every $x \in X$, define $F(x)$ as
\[
F(x) := p^{-1}\left(\int_{A_s(g)} p(f(ax)) dm(a)\right),
\]
where $m$ denotes the Haar measure on $A_s(g)$. It is easy to check that $F$ is well defined (see [21, Subsection 5.2] for further explanation).

Notice that the repaired map $\tilde{\varphi}(x) := F(x), \varphi(x)$ is a bijection and hence is a homeomorphism. By Lemma 1.26, $g^{-1}(y)$ is an $s$-nilspace. Applying Proposition 2.18 to it, we obtain that $\tilde{\varphi}|_{g^{-1}(y)} \in \text{Aut}_k(g^{-1}(y))$. Thus $\tilde{\varphi}$ is a $k$-translation of $\text{Aut}_k(g)$.

Let us introduce the notion of concatenation along the $k$-th axis.

**Definition 2.22** Let $1 \leq k \leq \ell + 1$. Given a cubespace $X$ and two maps: $c_1, c_2 : \{0, 1\}^\ell \to X$, the **concatenation along the $k$-th axis** $[c_1, c_2]_k : \{0, 1\}^{k+1} \to X$ is defined by sending $\omega$ to
\[
c_1(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_{\ell+1})
\]
if $\omega_k = 0$ and $c_2(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_{\ell+1})$ elsewhere.

In particular, $[c_1, c_2]_{\ell+1}$ is simply the concatenation as defined previously.

We need a variant of [21, Theorem 5.1] to prove Theorem 2.8.

**Theorem 2.23** Let $A$ be a compact abelian Lie group. Fix $s \geq 0$ and $\ell \geq 1$. Then there exists $\varepsilon > 0$ (depending only on $s$, $\ell$, and $A$) satisfying the following.

Let $\beta : Z \to Y$ be an $s$-fibration. Fix $0 < \delta < \varepsilon$. Suppose that $\rho : C^1(\beta)(Z) \to A$ is an $\ell$-cocycle on fibers of $\beta$ such that $d(\rho(c), \rho(c')) \leq \delta$ whenever $c, c'$ are cubes on the same fiber of $\beta$. Then
\[
\rho = \delta^\ell f
\]
for some continuous function $f : Z \to A$ which is almost constant on fibers of $\beta$, i.e. there exists a constant $c > 0$ (depending only on $s$ and $\ell$) such that $d(f(x), f(y)) \leq c\delta$ whenever $\beta(x) = \beta(y)$.  

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Proof The proof uses a similar argument to the one used in the proof of [21, Theorem 5.1]. To modify the proof of [21, Lemma 5.7], consider the set

$$T_1^\ell := \{ t : [0, 1]^{\ell-1} \rightarrow A_\beta(\beta) : [\overrightarrow{0}, t] \in C^\ell(D_\beta(A_\beta(\beta))) \}. $$

Then for any $c \in C_\beta^{\ell-1}(Z)$ and $t \in T_1^\ell$, we have $[c, t.c] \in C_\beta^\ell(Z)$. Therefore $\rho' : C_\beta^{\ell-1}(Z) \rightarrow A$ sending $c$ to

$$\rho'(c) := \int_{T_1^\ell} \rho([c, t.c])d\mu_{T_1^\ell}(t)$$

is well defined. Here $\mu_{T_1^\ell}$ denotes the measure on $T_1^\ell$ induced from the Lebesgue measure (see [21, Subsection 5.2] for more details).

Proof of Theorem 2.8 Without loss of generality, we may assume $k \leq s$. Since $\pi_s$ is a group homomorphism, it suffices to show it is open at the identity. Let $\varphi \in \text{Aut}_k(g_{s-1})$ be an element of small norm. We need to find a small $\tilde{\varphi}$ in $\text{Aut}_k(g)$ such that

$$\pi_{g, s-1} \circ \tilde{\varphi} = \varphi \circ \pi_{g, s-1}. $$

By Lemma 2.15, we can lift $\varphi$ to a small homeomorphism $\psi : Z \rightarrow Z$ fixing fibers of $g$. Since $\psi$ is of small norm, for any $c \in C_\beta^{s+1-k}(Z), \wedge^k(c, \psi(c))$ is close to the $(s+1)$-cube $\square^k(c) \in C_\beta^{s+1}(Z)$. Thus the discrepancy $\Delta(\wedge^k(c, \psi(c)))$ is close to $\Delta(\square^k(c)) = 0$, where $\square^k(c) : [0, 1]^{s+1} \rightarrow Z$ sends $\omega = (\omega_1, \ldots, \omega_{s+1})$ to $c(\omega_1, \ldots, \omega_{s+1-k})$. This implies that the image of the $(s+1-k)$-cocycle $\rho_\psi$ has small diameter. Thus we can apply Theorem 2.23 to write $\rho_\psi$ as

$$\rho_\psi = \partial^{s+1-k}f$$

for some continuous map $f : Z \rightarrow A_\beta(g)$ whose fibers are close to constant values. Then we can define $f$ as

$$f = \tilde{f} \circ g + h$$

for some continuous maps $\tilde{f} : Y \rightarrow A_\beta(g)$ and $h : Z \rightarrow A_\beta(g)$ such that $h$ is almost constant $0_{A_\beta(g)}$ (in the sense of Theorem 2.23).

Note that for any $c \in C_\beta^{s+1-k}(Z)$, $g(c)$ is a constant cube. It follows that $\rho_\psi = \partial^{s+1-k}h$. Since $\psi$ fixes the fibers of $\pi_{g, s-1}$ and $\varphi$ fixes the fibers of $g_{s-1}$, we have that $\psi$ fixes fibers of $g_{s-1} \circ \pi_{g, s-1} = g$. Since the image of $h$ is contained inside $A_\beta(g) \subseteq \text{Aut}_s(g)$, for each $z \in Z$, $h(z)$ also fixes fibers of $g$. Therefore, we have

$$g(h(z), \psi(z)) = g(\psi(z)) = g(z),$$

i.e. $\tilde{\varphi} := h \cdot \psi$ fixes the fibers of $g$. In summary, we prove that $\tilde{\varphi} \in \text{Aut}_k(g)$.

Finally, since $h$ is close to the constant function $0_{A_\beta(g)}$, the repairing $\tilde{\varphi}$ of small $\psi$ will also be small as desired. □

3 Approximating by Lie-Fibered Fibrations

In this section, we complete the proof of Theorem 1.27.
3.1 The Main Steps of the Proof

Let \( g : Z \to Y \) be an \( s \)-fibration between compact ergodic gluing cubespaces. To deal with the general case as in [23, Theorem 1.28], we need to first represent \( Z \) as an inverse limit \( \lim Z_n \) respecting the fibers of \( g \) as in [23, Theorem 1.26], and then endow the fibers of the fibrations \( Z_n \to Y \) with Host-Kra cube structure to obtain the cubespaces isomorphism.

The following is an analogue of [23, Theorem 1.26] and covers the statements (1) and (2) of Theorem 1.27.

**Theorem 3.1** Fix \( s \geq 1 \). Let \( g : Z \to Y \) be an \( s \)-fibration between compact ergodic gluing cubespaces. Then \( Z \) is isomorphic to an inverse limit

\[
\lim Z_n
\]

for an inverse system of \( s \)-fibrations between compact ergodic gluing cubespaces \( \{ p_{m,n} : Z_n \to Z_m \}_{0 \leq m, n \leq \infty} \) and compatible Lie-fibered \( s \)-fibrations \( \{ h_n : Z_n \to Y \} \). Here “compatible” means the following diagram commutes:

\[
\begin{array}{ccccccccc}
Z & \to & \cdots & \to & Z_n & \to & Z_{n-1} & \to & \cdots & \to & Z_1 \\
g & & & & & & & & & & h_1
\end{array}
\]

Let us start with some preliminary steps. Since every compact abelian group equals an inverse limit of compact abelian Lie groups [23, Lemma 2.1], we can write the top structure group \( As \) as an inverse limit of Lie groups \( A_n \) with \( A_0 = \{ 0 \} \), i.e.

\[
As(g) = \lim A_n.
\]

Denote by \( K_n \) the kernel of the quotient homomorphism \( As(g) \to A_n \) and denote the orbit space of \( Z \) under the action of by \( K_n \), by \( Z^{(n)}_\infty \), i.e. \( Z^{(n)}_\infty := Z/K_n \).

**Lemma 3.2** \( Z^{(n)}_\infty \) has the gluing property.

**Proof** Let \( c_1, c_2, c'_2, c_3 \in C^k(Z) \) such that \( [c_1, c_2], [c'_2, c_3] \in C^{k+1}(Z) \) and \( \overline{c_2} = \overline{c'_2} \) in \( C^k(Z^{(n)}_\infty) \). We want to show \( \overline{[c_1, c_3]} \in C^k(Z^{(n)}_\infty) \).

**Proposition 3.3** Let \( f : X \to Y \), \( g : Y \to Z \) and \( h : X \to Z \) be three cubespaces morphisms such that \( h = g \circ f \). Suppose that \( h \) has k-completion and \( g \) has k-uniqueness. Then \( f \) has k-completion.

**Proof** Assume that \( \lambda \) is a k-corner of \( X \) such that \( f(\lambda) = c \mid_{k} \) for some k-cube \( c \) of \( Y \). We want to show there exists \( x \in f^{-1}(c(\overline{1})) \) completing \( \lambda \) as a cube of \( X \).

Note that \( g(c) \) is a k-cube extending \( (g \circ f)(\lambda) = h(\lambda) \). Since \( h \) has k-completion, there exists \( x \in h^{-1}(g(c(\overline{1}))) \) completing \( \lambda \) to be a k-cube of \( X \). Thus \( (g \circ f)(x) = h(x) = g(c(\overline{1})) \). Thus \( f(x) \) completes \( f(\lambda) \) as another k-cube sharing the same \( g \)-image as \( c \). But since \( g \) has k-uniqueness, it implies that \( f(x) = c(\overline{1}) \). This finishes the proof.

Recall that for a compact abelian group \( A \), \( D_s(A) \) denotes the Host-Kra cubespaces with respect to the \( s \)-filtration:

\[
A = A_0 = A_1 = \cdots = A_s \supseteq A_{s+1} = \{ 0 \}.
\]
By the relative weak structure theorem (Theorem 1.25), there exists a unique \( \alpha \in C^k(D_\beta(K_n)) \) such that \( c_2' = \alpha.c_2 \). Note that \([\alpha, \alpha'] \in C^{k+1}(D_\beta(K_n))\). Thus \([\alpha.c_1, \alpha.c_2]\) is a \((k+1)\)-cube of \(X\). gluing with \([c_2', c_3]\), we obtain that \([\alpha.c_1, c_3] \in C^{k+1}(X)\). In particular, \([c_1, c_3] \in C^k(Z_{\infty}^{(n)})\). Thus \(Z_{\infty}^{(n)}\) has the gluing property.

Denote by \( \beta_n : Z \to Z_{\infty}^{(n)} \) the quotient map. By the definition of \(Z_{\infty}^{(n)}\), \(g\) uniquely factors through \(\beta_n\) via a map \(g^{(n)} : Z_{\infty}^{(n)} \to Y\). Since \(g\) has \((s+1)\)-uniqueness, applying the universal replacement property in a similar way to the proof of Proposition 1.23, we have \(g^{(n)}\) has \((s+1)\)-uniqueness again. Since \(g\) is a fibration, by Proposition 3.3, \(\beta_n : Z \to Z_{\infty}^{(n)}\) is fibrant for cubes of dimension greater than \(s\). Applying the universal replacement property of \(Z\), \(\beta\) is fibrant for cubes of dimension less than \(s+1\). Thus \(\beta\) is a fibration. By Proposition 1.20, \(g^{(n)}\) is again a fibration and hence is an \(s\)-fibration. Applying the \((s+1)\)-uniqueness of \(g\) again, we see that \(\beta\) has \((s+1)\)-uniqueness and hence is an \(s\)-fibration. Moreover, we have

\[
\pi_{g^{(n)},s-1}(Z_{\infty}^{(n)}) = \pi_{g,s-1}(Z) = Z_{\infty}^{(0)}.
\]

In summary, we have the commutative diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{\beta_n} & Z_{\infty}^{(n)} \\
\downarrow & & \downarrow \\
\pi_{g,s-1}(Z) & \xrightarrow{g^{(n)}} & Y
\end{array}
\]

By the inductive hypothesis, the \((s-1)\)-fibration \(g_{s-1} : \pi_{g,s-1}(Z) \to Y\) factors as an inverse limit of a sequence of Lie-fibered \((s-1)\)-fibrations \(\psi_{0,m} : Z_{m}^{(0)} \to Y\) along with compatible fibrations \(\psi_{m,\infty} : \pi_{g,s-1}(Z) \to Z_{m}^{(0)}\). In fact, since \(g_{s-1}\) is an \((s-1)\)-fibration, so are \(\psi_{0,m}\) and \(\psi_{m,\infty}\). Here it is convenient to denote \(Y\) by \(Z_{0}^{(0)}\) and \(Z\) by \(Z_{\infty}^{(0)}\).

To factor \(g^{(n)}\) properly based on the factorization of \(g_{s-1}\), we will introduce some key definitions stemming from the following lemma, which a variant version of Proposition 2.7.

**Lemma 3.4** Let \(X, Y, Z\) be three compact ergodic gluing cubspaces and \(\varphi : X \to Y\) a fibration. Let \(g : X \to Z\) and \(h : Y \to Z\) be two \(s\)-fibrations such that \(g = h \circ \varphi\). Then there exists a unique fibration \(\psi : \pi_{g,s-1}(X) \to \pi_{h,s-1}(Y)\) such that the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow & & \downarrow \\
\pi_{g,s-1}(X) & \xrightarrow{\psi} & \pi_{h,s-1}(Y) \\
\downarrow & & \downarrow \\
g_{s-1} & & g_{s-1}
\end{array}
\]

commutes.

**Proof** Using \(g = h \circ \varphi\), one can directly check that \(\pi_{h,s-1} \circ \varphi\) factors through \(\pi_{g,s-1}\) by a unique map \(\psi\) such that \(\pi_{h,s-1} \circ \varphi = \psi \circ \pi_{g,s-1}\). By the universal property of fibrations [22, Lemma 7.8], \(\psi\) is a fibration. From

\[
h_{s-1} \circ \psi \circ \pi_{g,s-1} = h_{s-1} \circ \pi_{h,s-1} \circ \varphi = h \circ \varphi = g = g_{s-1} \circ \pi_{g,s-1},
\]

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we obtain $h_{\ell-1} \circ \psi = g_{s-1}$. This shows that the diagram is commutative.

**Definition 3.5** With the setup in Lemma 3.4, we say that $\psi$ is the **shadow** of $\varphi$. Moreover, we say $\varphi$ is **horizontal** if it satisfies one of the following equivalent conditions:

1. $\varphi(x) \neq \varphi(x')$ for any $x \neq x' \in X$ with $x \sim_{g,s-1} x'$;
2. for any $x \in X$, the appropriate restriction of $\varphi$ induces a bijection between $\pi_{g,s-1}^{-1}(\pi_{g,s-1}(x))$ and $\pi_{h,s-1}^{-1}(\pi_{h,s-1}(\varphi(x)))$;
3. the equivalence relation $\sim_{g,s-1}$ is trivial.

Now we are ready to formulate the relative version of [23, Proposition 2.5].

**Proposition 3.6** Let $Z_{\infty}^{(n)}$, $\beta_n$, $g^{(n)}$, $\psi_{m,\infty}$, etc. be as above. There exists a strictly increasing sequence $M_1$, $M_2$, \ldots of positive integers satisfying the following. For each $n \in \mathbb{N}$ and $m \geq M_n$, there is a compact ergodic gluing cubespace $Z_m^{(n)}$ and an s-fibration $h_m^{(n)} : Z_m^{(n)} \to Y$ satisfying that:

1. $\psi_{0,m}^{(n)}$ is the canonical $(s-1)$-th factor of $h_m^{(n)}$ with top structure group $A_n$;
2. there is a horizontal s-fibration $\varphi_m^{(n)} : Z_{\infty}^{(n)} \to Z_m^{(n)}$ such that $g^{(n)} = h_m^{(n)} \circ \varphi_m^{(n)}$ and $\psi_{m,\infty}$ is the relative shadow of $\varphi_m^{(n)}$;
3. If $m_1 \leq m_2$ and $n_1 \leq n_2$ are such that $Z_{m_1}^{(n_1)}$ and $Z_{m_2}^{(n_2)}$ are both defined, then the fibers of $\varphi_m^{(n_2)} \circ \beta_n^{(n_2)}$ refine the fibers of $\varphi_m^{(n_1)} \circ \beta_n^{(n_1)}$.

In summary, we have a commutative diagram

**Proof of Theorem 3.1** Using the notation of Proposition 3.6, we define $Z_n = Z_m^{(n)}$ and the fibration $h_n = h_m^{(n)}$. Note that the top structure group of $h_n$ is the Lie group $A_n$. By the induction hypothesis, the canonical $(s-1)$-factor $\psi_{0,M_n}^{(n)}$ of $h_n$ is Lie-fibered. Combining these two facts, we have that $h_n$ is Lie-fibered.

Define $p_{n,\infty} = \psi_{M_n}^{(n)} \circ \beta_n$. Since both $\psi_{M_n}^{(n)}$ and $\beta_n$ are s-fibrations, so is $p_{n,\infty}$. Then for every $n < \ell < \infty$, the fibers of $p_{\ell,\infty}$ refine the fibers of $p_{n,\infty}$. Thus by Proposition 1.20, $p_{n,\infty}$ and $p_{\ell,\infty}$ induce a unique fibration $p_{n,\ell}$ such that $p_{n,\infty} = p_{n,\ell} \circ p_{\ell,\infty}$. Moreover, we obtain a commutative diagram
For every $0 \leq n < \ell < o \leq \infty$, the condition $p_{n,\ell} \circ p_{\ell,o} = p_{n,o}$ may be verified in a similar way to the absolute setting. We verify that the inverse system $\mathcal{S}_n$ separates points of $Z$. Let $z, z'$ be two distinct points of $Z$. If $\pi_{g,s-1}(z) \neq \pi_{g,s-1}(z')$, then $\psi_{M_n,\infty} \circ \pi_{g,s-1}(z) \neq \psi_{M_n,\infty} \circ \pi_{g,s-1}(z')$ as $n$ is large enough. Since $\psi_{M_n,\infty}$ is the relative shadow of $\varphi_{M_n}$, we have

$$\psi_{M_n,\infty} \circ \pi_{g,s-1} = \pi_{h_n,s-1} \circ \varphi_{M_n}^{(n)} = \pi_{h_n,s-1} \circ p_n, \infty.$$

It follows that $p_{n,\infty}(z) \neq p_{n,\infty}(z').$

If $\pi_{g,s-1}(z) = \pi_{g,s-1}(z')$, then there is a unique $a \in A_s(g)$ such that $z' = az$. Note that $a \notin K_n$ as $n$ is large enough. Thus $\beta_n(z) \neq \beta_n(z')$. Since $\varphi_{M_n}^{(n)}$ is horizontal, we get

$$p_{n,\infty}(z) = \varphi_{M_n}^{(n)} \circ \beta_n(z) \neq \varphi_{M_n}^{(n)} \circ \beta_n(z') = p_{n,\infty}(z').$$

□

The following is an analogue of [23, Theorem 1.27] [1, Theorem 4]. We will give the proof in Section 5.3.

**Theorem 3.7** Let $X, Y, Z$ be three compact ergodic gluing cubespaces. Suppose that $\varphi: X \to Y$ is a fibration, and $g: X \to Z$ and $h: Y \to Z$ are two Lie-fibered $s$-fibrations such that $g = h \circ \varphi$. Then for every $i \geq 1$, $\varphi$ induces a surjective continuous group homomorphism

$$\Phi: \text{Aut}^i_h(g) \to \text{Aut}^i_h(h)$$

such that $\Phi(u) \circ \varphi = \varphi \circ u$ for all $u \in \text{Aut}^i_h(g)$. In summary, the following diagram is commutative:

$$\begin{array}{ccc}
X & \xrightarrow{u} & X \\
\downarrow \varphi & & \downarrow \varphi \\
Y & \xrightarrow{\Phi(u)} & Y \\
\downarrow h & & \downarrow h \\
Z & & Z
\end{array}$$

**Proof of Theorem 1.27** Statement (1) and (2) have been proven in Theorem 3.1. Let us prove statement (3). The case $s = 0$ is trivial since $g$ is clearly a cubespace isomorphism and each fiber of $g$ is simply a singleton. We now assume $s \geq 1$. By Theorem 3.1, since the diagram commutes, we can first define $g^{-1}(g(z))$ as an inverse limit of $h_n^{-1}(g(z))$ by restricting the projection map $Z \to Z_n$. By Theorem 2.6, $h_n^{-1}(g(z)) = h_n^{-1}(h_n(z_n))$ is isomorphic to $\text{Aut}^i_h(h_n)/\text{Stab}(z_n)$. Thus

$$g^{-1}(g(z)) \cong \lim_{\to} h_n^{-1}(h_n(z_n)) \cong \lim_{\to} (\text{Aut}^i_h(h_n)/\text{Stab}(z_n)).$$

To see that the above isomorphism is a cubespaces isomorphism, apply Theorem 3.7 to the fibration $p_n-1,n$ and Lie-fibered $s$-fibrations $h_n$ and $h_{n-1}$, to obtain a surjective continuous homomorphism

$$\Phi_{n-1,n}: \text{Aut}^i_h(h_n) \to \text{Aut}^i_h(h_{n-1}).$$

This induces an inverse limit $\lim_{\to} (\text{Aut}^i_h(h_n)/\text{Stab}(z_n))$ for every $k \geq 0$. By Theorem 2.6, $C^k(h_n^{-1}(h_n(z_n)))$ is isomorphic to $\text{HK}^k(\text{Aut}^i_h(h_n))/\text{Stab}(z_n)$. Thus we obtain the desired cubespaces isomorphism.

□

The following proposition gives a a useful condition for verifying when a nilspace fiber is strongly connected.
Proposition 3.8 Let $f : X \to Y$ be a fibration of degree at most $d$ with structure groups $A_1, \ldots, A_d$, then for all $y \in Y$, the subcubes $f^{-1}(y)$ are nilspaces of degree at most $d$ with structure groups $A_1, \ldots, A_d$.

Proof Recall that for each $k \geq 0$ and $y \in Y$, $C^k(f^{-1}(y))$ is given by the restriction $C^k(X) \cap (f^{-1}(y))^{[0,1]^k}$. Since $f$ is a fibration, for each $k$-corner $\lambda$ of $f^{-1}(y)$, $f(\lambda)$ can be completed as a constant $k$-cube, we have $\lambda$ can be completed as a cube of $f^{-1}(y)$. Thus $f^{-1}(y)$ has $k$-completion. Since $f$ has $(d+1)$-uniqueness, it guarantees that $f^{-1}(y)$ has $(d+1)$-uniqueness. This proves that $f^{-1}(y)$ is a nilspace of degree at most $d$. Now we show the top structure group of $f^{-1}(y)$ is $A_d$. The case for other structure groups use the same argument which we omit. Set $A_d = A$ and $y = f(x)$ for some $x \in X$. By the construction of $A$ in the proof of [22, Theorem 7.19], for every $a \in A$, we have $f(ax) = f(x)$ and $ax \sim_{d-1} x$. Let $p : f^{-1}(y) \to f^{-1}(y)/\sim_{d-1}$ be the canonical projection map. It follows that $f^{-1}(y)$ is $A$-invariant and moreover $Ax \subseteq p^{-1}(p(x))$. On the other hand, if $x' \in f^{-1}(y)$ and $x' \sim_{d-1} x$, by the relative weak structure theorem, there exists a unique $a \in A$ such that $x' = ax$. So we have $Ax = p^{-1}(p(x))$. Thus $A$ is the top structure group of $f^{-1}(y)$. \qed

3.2 Straight Classes and Sections

To prove Proposition 3.6, we need an analogue of [23, Proposition 2.13]. Let us start with some preliminary steps.

Definition 3.9 Let $X$, $Z$, $W$ be three compact ergodic gluing cubespaces. Given an $s$-fibration $g : X \to Z$ and a fibration $\psi : \pi_{g,s-1}(X) \to W$, we call a subset $D \subseteq X$ a straight $\psi$-class if there exists $w \in W$ such that

1. $D \cap \pi_{g,s-1}^{-1}(u)$ is a singleton for every $u \in \psi^{-1}(w)$ and $D$ is the union of those singletons;
2. a configuration $c : \{0, 1\}^{s+1} \to D$ is a cube if and only if $\pi_{g,s-1}(c)$ is a cube of $W$.

In short, $D$ is a $\pi_{g,s-1}$-lifting of some fiber of $\psi$ respecting cube structure.

Consider a configuration $c : \{0, 1\}^{s+1} \to X$ inducing a cube $\pi_{g,s-1}(c)$. In light of the relative weak structure theorem (Theorem 1.25) and [22, Proposition 5.1], there exists a unique element $a \in A_s(g)$ such that the application of $a$ to $c$ at $(0, \ldots, 0) \in \{0, 1\}^{s+1}$ results with a cube of $X$. We call such an element $a \in A_s(g)$ the discrepancy of $c$ and denote it by $D(c)$.

Let $U \subseteq W$ be an open subset. We say a continuous map $\sigma : \psi^{-1}(U) \to Z$ is a straight section if

1. $\pi_{g,s-1} \circ \sigma = \text{id}_U$;
2. for any $c_1, c_2 \in C^{s+1}(\psi^{-1}(U))$ with $\psi(c_1) = \psi(c_2)$, we have $D(\sigma(c_1)) = D(\sigma(c_2))$.

We remark that the straightness of a section $\sigma$ implies that $\sigma$ maps every fiber of $\psi$ onto a straight $\psi$-class.

The following lemma is a relative version of [23, Lemma 2.7].

Lemma 3.10 Let $X$, $Y$, $Z$ be three compact ergodic gluing cubespaces and $\varphi : X \to Y$ be a fibration. Let $g : X \to Z$ and $h : Y \to Z$ be two $s$-fibrations such that $g = h \circ \varphi$. If $\varphi$ is horizontal, denoting by $\psi$ the shadow of $\varphi$, then each fiber of $\psi$ is a straight $\psi$-class.

The following lemma is a relative analogue of [23, Proposition 2.8], based on [23, Theorem 1.25] and the relative weak structure theorem (Theorem 1.25).
Lemma 3.11 Suppose that $g : X \to Z$ is an $s$-fibration between compact ergodic cubespaces such that the top structure group is a Lie group $A$. Then for any $\varepsilon > 0$ there is $\delta > 0$ satisfying the following property.

Let $\psi : \pi_{g,s-1}(X) \to W$ be an $(s-1)$-fibration to another compact ergodic cubespaces $W$ such that $\psi$ is a $\delta$-embedding in the sense that every fiber of $\psi$ has diameter less than $\delta$. Then for every $c \in C^{s+1}(\pi_{g,s-1}(X))$, there is an open set $U \subseteq W$ satisfying that (1) the image of $c$ is contained inside $\psi^{-1}(U)$; (2) there is a straight $\psi$-section $\sigma : \psi^{-1}(U) \to X$ with $\text{diam}(\sigma(\psi^{-1}(b))) \leq \varepsilon$ for every $b \in U$.

In particular, every $x \in X$ is contained in a $\psi$-class of small diameter (explicitly, $x \in a.\sigma \circ \psi^{-1}(\psi \circ \pi_{g,s-1}(x))$ for some $a \in A$).

The following is a relative analogue of [23, Proposition 2.9].

Lemma 3.12 Let $g : X \to Z$ be an $s$-fibration such that the top structure group is a Lie group $A$. Then there exists $\delta > 0$ depending only on $g$ satisfying the following.

Let $\psi : \pi_{g,s-1}(X) \to W$ be an $(s-1)$-fibration for another compact ergodic cubespaces $W$. Suppose that $D_1$ and $D_2$ are two straight $\psi$-classes with the same image of $\pi_{g,s-1}$ and the restriction $\pi_{g,s-1}|_{D_1 \cup D_2}$ is a $\delta$-embedding. Then $D_1 = aD_2$ for some $a \in A$.

Proof Since $\psi$ is a fibration, by Proposition 1.26, the space $B := \pi_{g,s-1}(D_1)$ is a compact ergodic nilspace. Applying [21, Theorem 5.2] to the nilspace $B$, we obtain $\varepsilon = \varepsilon(s, s+1, A)$ of [21, Theorem 5.2]. Define $\delta = \varepsilon/2$.

Denote by $\sigma_i : B \to D_i$ the inverses of $\pi_{g,s-1}$ restricted to $D_i$ for $i = 1, 2$. Let $f : B \to A$ be the continuous function determined by the equation $\sigma_2(y) = f(y), \sigma_1(y)$ for all $y \in B$. Since $D_1$ and $D_2$ are straight classes, for every $c \in C^{s+1}(B)$, $\sigma_1(c)$ and $\sigma_2(c)$ are cubes of $X$. Thus by the relative weak structure theorem (Theorem 1.25), $\partial^{s+1}(f(c)) = 0$. Since $\pi_{g,s-1}|_{D_1 \cup D_2}$ is a $\delta$-embedding, we have that the diameter of $\text{im}(f)$ is less than $\varepsilon$. Applying [21, Theorem 5.2], $f$ is constant. $\square$

Remark 3.13 We note that the assumption on $\psi$ in Lemmas 3.11 and 3.12 is weaker than the corresponding assumption in the absolute setting in [23].

Combining Lemmas 3.11 with 3.12, the straight $\psi$-classes induce an equivalence relation which we denote by $\equiv_{\psi}$. This equivalence relation allows us to construct the desired cubespaces and fibrations stated in Proposition 3.6.

Lemma 3.14 Let $g : X \to Z$ be an $s$-fibration such that the top structure group is a Lie group $A$. Write $\pi = \pi_{g,s-1}$. Then there exists $\delta > 0$ depending only on $g$ satisfying the following.

Let $\psi : \pi_{g,s-1}(X) \to W$ be an $(s-1)$-fibration such that $\psi$ is a $\delta$-embedding. Then the induced equivalence relation $\equiv_{\psi}$ from the straight $\psi$-classes is closed. Moreover, let $u : W \to Z$ be an $(s-1)$-fibrations with $g_{s-1} = u \circ \psi$. Then it holds:

1. the quotient map $\varphi : X \to X/\equiv_{\psi}$ is a fibration and induces a fibration $\pi' : X/\equiv_{\psi} \to W$ and an $s$-fibration $u' := u \circ \pi'$ such that $\varphi$ is horizontal and the diagram below commutes.

\[ \begin{array}{ccc} X & \longrightarrow & X/\equiv_{\psi} \\ \downarrow \ \varphi & & \downarrow \ \pi' \\ \pi_{g,s-1}(X) & \longrightarrow & W \\ \downarrow \ \psi & & \downarrow \ u \\ g_{s-1}(X) & \longrightarrow & Z \end{array} \]
(2) the top structure group of $u'$ is $A$ and $\pi' = \pi_{u' \cdot s^{-1}}$.

**Proof** Based on Lemma 3.12, the fact that the relation $\approx_{\psi}$ is closed follows from the argument of [23, Proposition 2.13]. It is clear from the definition of the equivalence relation that the induced map $\pi'$ is well-defined and $\psi \circ \pi = \pi' \circ \varphi$. From $g_{s-1} = u \circ \psi$, we have

$$g = g_{s-1} \circ \pi = u \circ \psi \circ \pi = u \circ \pi' \circ \varphi = u' \circ \varphi.$$ 

Thus $\psi$ is the shadow of $\varphi$.

We show $\varphi$ is a fibration. Since $g$ is an $s$-fibration and $u$ is an $(s-1)$-fibration, by a similar argument to the one in the proof of [23, Lemma 2.14], we have that $u'$ has $(s+1)$-uniqueness. By Proposition 3.3, $\varphi$ is fibrant for $k$-corners of every $k \geq s + 1$. From the fact that $\varphi$ is relatively $s$-ergodic (see Definition 1.17) and the fact that $\psi$ is a fibration, it follows that $\varphi$ is fibrant for corners of lower dimension. Note that $A$ respects straight $\psi$-classes. Thus $X/\approx_{\psi}$ inherits an $A$-action from the $A$-action on $X$. Moreover, the straightness guarantees that $W$ is exactly the orbit space induced and hence $\pi' = \pi_{u' \cdot s^{-1}}$ and $A$ is the top structure group of $u'$.

**Proof of Proposition 3.6** For each fixed $n$ applying Lemma 3.14 to the $s$-fibration $g^{(n)} : Z^{(n)}_{\varphi} \rightarrow Y$, we obtain a $\delta_n$ satisfying the desired property. Take $M_n$ large enough such that for every $m \geq M_n$, $\Psi_{m, \infty}$ is a $\delta_n$-embedding. Then we obtain the desired $s$-fibration $h^{(n)} : Z^{(n)}_m \rightarrow Y$ and a horizontal fibration $\varphi^{(n)}_m : Z^{(n)}_{\infty} \rightarrow Z^{(n)}_m$ such that $g^{(n)} = h^{(n)} \circ \varphi^{(n)}_m$ from Lemma 3.14.

In light of Lemma 12, following the argument as in the absolute setting, it holds that the fibers of $\varphi^{(n)}_{m_2} \circ \beta_{n_2}$ refine the fibers of $\varphi^{(n_1)}_{m_1} \circ \beta_{n_1}$ for $n_1 \leq n_2$, $M_{n_1} \leq m_1 \leq m_2$ with $M_{n_2} \leq m_2$.

### 3.3 Relations Between Relative Translations

For an $s$-fibration $g : Z \rightarrow Y$ between compact ergodic gluing cubespaces, denote by $\text{Aut}^\varepsilon_i(g)$ the $\varepsilon$-neighborhood of the identity in $\text{Aut}_i(g)$ under the metric

$$d(f,g) := \max_{x \in X} d(f(x), g(x)).$$

By Corollary 2.12, $\text{Aut}_i(g)$ is a Lie group. Thus $\text{Aut}^\varepsilon_i(g)$ is a group generated by $\text{Aut}^\varepsilon_i(g)$ as $\varepsilon$ is small enough. Then Theorem 3.7 is a consequence of the following general result.

**Theorem 3.15** Fix $i \geq 1$. Let $\varphi : X \rightarrow Y$, $g : X \rightarrow Z$ and $h : Y \rightarrow Z$ be three $s$-fibrations such that $g = h \circ \varphi$ and $g, h$ are Lie fibrations. Then for any $\varepsilon > 0$ there is $\delta > 0$ satisfying the following property. For any $u \in \text{Aut}^\varepsilon_i(g)$ there is $u' \in \text{Aut}^\varepsilon_i(h)$, and conversely for any $u' \in \text{Aut}^\varepsilon_i(h)$ there is $u \in \text{Aut}^\varepsilon_i(g)$, such that $u' \circ \varphi = \varphi \circ u$.

We introduce vertical fibrations [23, Definition 3.2] as follows.

**Definition 3.16** Let $\varphi : X \rightarrow Y$, $g : X \rightarrow Z$ and $h : Y \rightarrow Z$ be three $s$-fibrations such that $g = h \circ \varphi$. We say $\varphi$ is a **vertical fibration** if for any $x, x' \in X$ such that $\pi_{h,s-1} \circ \varphi(x) = \pi_{h,s-1} \circ \varphi(x')$, one obtains that $\pi_{g,s-1}(x) = \pi_{g,s-1}(x')$.

We can factor a fibration in a relative way in contrast with [23, Proposition 3.3].
Proof Define $W = X / \sim_{\varphi, s = 1}$. Denote by $\varphi_v$ the quotient map $X \to W$ and $\varphi_h$ the induced map $W \to Y$. Define $k := h \circ \varphi_h$. Then it is routine to check the desired properties by definition.

The following lemma is an analogue of [23, Proposition 3.5].

Lemma 3.18 Let $\varphi : X \to Y$, $g : X \to Z$ and $h : Y \to Z$ be three $s$-fibrations such that $g = h \circ \varphi$. Let $u \in \text{Aut}_i(g)$. Suppose that the fibers of $\varphi$ refine the fibers of $\varphi \circ u$. Then there is a unique relative translation $u' \in \text{Aut}_i(h)$ such that $u' \circ \varphi = \varphi \circ u$.

Proof By Proposition 1.20, there exists a unique fibration $u' : Y \to Y$ such that $u' \circ \varphi = \varphi \circ u$. We check that $u' \in \text{Aut}_i(h)$.

Firstly, since $u$ fixes the fibers of $g$, we have $u'$ fixes fibers of $h$. Let $n \geq i$, $y \in Y$, and $c \in C^n(h^{-1}(h(y)))$. Since fibrations are surjective [22, Corollary 7.6], we have that $c = \varphi(\tilde{c})$ for some $\tilde{c} \in C^n(X)$. Choose some $x \in X$ such that $\varphi(x) = y$. It follows that $\tilde{c} \in C^n(g^{-1}(g(x)))$. Let $F \subseteq [0, 1]^n$ be a face of codimension $i$. Since $u \in \text{Aut}_i(g)$, we have $[u]_F . \tilde{c} \in C^n(g^{-1}(g(x)))$. Thus $[u']_F . c = \varphi([u]_F . \tilde{c})$ is a cube of $h^{-1}(h(y))$.

The following lemma is an analogue of [23, Lemma 3.6].

Lemma 3.19 Let $\varphi : X \to Y$, $g : X \to Z$ and $h : Y \to Z$ be three $s$-fibrations such that $g = h \circ \varphi$ and $\varphi$ is vertical. Fix $u \in \text{Aut}_1(g)$. Then for any $x, x' \in X$ with $\varphi(x) = \varphi(x')$, one has $\varphi \circ u(x) = \varphi \circ u(x')$.

Proof Since $\varphi$ is vertical, we have $x \sim_{s = 1} x'$ and hence $\triangle^s(x, x')$ is a cube. Moreover, since $u \in \text{Aut}_1(g)$, we obtain an $(s + 1)$-cube $[\triangle^s(x, x'), \triangle^s(u(x), u(x'))]$. Hence $c := \varphi([\triangle^s(x, x'), \triangle^s(u(x), u(x'))])$ is also an $(s + 1)$-cube. On the other hand, by ergodicity of $Y$, we have another $(s + 1)$-cube $c' := \varphi([\square^s(x), \square^s(u(x))])$. Note that $c|_{i,s+1} = c'|_{i,s+1}$ and $h(c) = \square^{s+1}(g(x)) = h(c')$. By the $(s + 1)$-uniqueness of $h$, it holds that $\varphi(u(x)) = c'(\overrightarrow{1}) = c(\overrightarrow{1}) = \varphi(u(x'))$.

Combining Lemmas 3.18 with 3.19, we complete the pushing-forward part of Theorem 3.15 for vertical fibrations.
Proposition 3.20 Let \( \varphi : X \to Y \), \( g : X \to Z \) and \( h : Y \to Z \) be three \( s \)-fibrations such that \( g = h \circ \varphi \). Suppose that \( \varphi \) is vertical. Then there is a continuous homomorphism \( \Phi : \text{Aut}_i(g) \to \text{Aut}_i(h) \) such that 
\[
\Phi(u) \circ \varphi = \varphi \circ u
\]
for any \( u \in \text{Aut}_i(g) \).

Now we deal with the horizontal case.

Proposition 3.21 Let \( \varphi : X \to Y \), \( g : X \to Z \) and \( h : Y \to Z \) be three \( s \)-fibrations such that \( g = h \circ \varphi \). Suppose that \( \varphi \) is horizontal and \( g, h \) are Lie fibrations. Then for any \( \epsilon > 0 \) there exists \( \delta > 0 \) satisfying the following property. For any \( u \in \text{Aut}_i^\delta(g) \) there is \( u' \in \text{Aut}_i^\delta(h) \) such that \( u' \circ \varphi = \varphi \circ u \).

**Proof** By Proposition 3.18, it suffices to show as \( \delta \) is small enough every map in \( \text{Aut}_i^\delta(g) \) preserves \( \psi \)-fibers. We induct on \( s \) to prove this.

The case \( s = 0 \) is trivial as \( \psi \) is a cubespaces isomorphism. Denote by \( \psi : \pi_{g,s-1}(X) \to \pi_{h,s-1}(Y) \) the shadow of \( \varphi \). By the induction hypothesis, there exists \( \delta_0 > 0 \) such that any element of \( \text{Aut}_i^{\delta_0}(g_{s-1}) \) preserves \( \psi \)-fibers. Let \( u \in \text{Aut}_i^\delta(g) \) for \( \delta \) to be decided later. By Proposition 3.2, \( u \) induces a map \( v := \pi_s(u) \in \text{Aut}_i(g_{s-1}) \). As \( \delta \) is small enough, we have \( v \in \text{Aut}_i^{\delta_0}(g_{s-1}) \). By the induction hypothesis, we have that \( v \) preserves \( \psi \)-fibers.

Let \( y \in Y \) and fix a point \( x_0 \in \varphi^{-1}(y) \). Define \( z = \varphi(u(x_0)) \). We show that \( D_1 := u(\varphi^{-1}(y)) = D_2 := \varphi^{-1}(z) \) by applying Proposition 3.12 to the Lie fibration \( g \). Note that \( u(x_0) \in D_1 \cap D_2 \). By Lemma 3.10, \( D_2 \) is a straight \( \psi \)-class. Since \( v \) preserves \( \psi \)-fibers, we can deduce that \( D_1 \) is also a straight \( \psi \)-class. Denote by \( \delta' \) the number \( \delta \) in Proposition 3.12. Finally, one can check that \( \pi_{g,s-1}(D_1 \cup D_2) \) is a desired \( \delta' \)-embedding as \( \delta \) is small. This will force \( D_1 = D_2 \).

Following the argument of [23, Proposition 3.9], based on Propositions 3.20 and 3.21, and Lemmas 2.13 and 2.10, we obtain the following proposition.

Proposition 3.22 Let \( \varphi : X \to Y \), \( g : X \to Z \) and \( h : Y \to Z \) be three \( s \)-fibrations such that \( g = h \circ \varphi \) and \( g, h \) are Lie fibrations. Then for any \( \epsilon > 0 \) there exists \( \delta > 0 \) satisfying the following property. For any \( u' \in \text{Aut}_i^\delta(h) \) there is \( u \in \text{Aut}_i^\delta(g) \) such that \( u' \circ \varphi = \varphi \circ u \).

**Proof of Theorem 3.15** By Proposition 3.17, in order to prove the first statement, it is enough to consider the cases that \( \varphi \) is horizontal and vertical separately. These two cases are dealt with in Propositions 3.20 and 3.21 respectively. The second statement follows from Proposition 3.22.

4 Isomorphisms Between Fibers of a Fibration

In this section we prove Theorem 1.28 which gives a natural condition under which the fibers of a Lie-fibered \( s \)-fibration are isomorphic as subcubespaces.

Let us first recall the covering homotopy theorem [10, Theorem 2.1] [34, Theorem 11.3].
**Theorem 4.1** (Covering homotopy theorem) Let $p : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber $F$, where $B$ is compact. Let $h_0$ be a bundle map $E \overset{h_0}{\to} Z$. Let $H : B \times I \to Y$ be a homotopy of $h_0$, i.e. $h_0 = H|_{B \times \{0\}}$. Then there exists a covering $\tilde{H}$ of the homotopy $H$ by a bundle map $E \times I \overset{\tilde{H}}{\to} Z$.

**Proof of Theorem 1.28** We first modify the homotopy argument of [32, Theorem 5.1] to obtain a local homeomorphism and then repair it to a cubespace isomorphism. By compactness, the isomorphism can be built globally.

Denote by $I$ the interval $[0, 1]$. Since $Y$ is path-connected, there exists a continuous map $H_0 : \{y_0\} \times I \to Y$ such that $H_0(y_0, 0) = y_0$ and $H_0(y_0, 1) = y_1$. Define $R = \text{im}(H_0)$ and $y_i = H_0(y_0, t_i)$. For every $i = 0, 1, \ldots, s - 1$ abbreviate by $\pi_i$ the map $\pi_{g,i} : Z/\sim_{g,i+1} \to Z/\sim_{g,i}$. We first show for every $0 \leq i \leq s$ there exists an interval $I_i := [0, t_i]$ for some $t_i = t_i(y_0) > 0$ and a continuous map $G_i : g_i^{-1}(y_0) \times I_i \to g_i^{-1}(R)$ such that

1. $G_i(x, 0) = x$ for all $x \in g_i^{-1}(y_0)$, i.e. $G_i$ is a homotopy of inclusion map;
2. $G_i(t, t) : g_i^{-1}(y_0) \to g_i^{-1}(y_t)$ is a cubespace isomorphism for every $t \in I_i$.

The case $i = 0$ is clear since the fibers of $g_0$ are singletons and $G_0 := H_0$ works. Inductively, suppose that we have a continuous map $G_{s-1} : g_{s-1}^{-1}(y_0) \times I_{s-1} \to g_{s-1}^{-1}(R)$ with the desired properties. Consider the fiber bundle map given by the inclusion map

$$
\begin{array}{ccc}
g^{-1}(y_0) & \overset{i}{\longrightarrow} & (g_{s-1})^{-1}(R) \\
\downarrow\pi_{s-1} & & \downarrow\pi_{s-1} \\
(g_{s-1})^{-1}(y_0) & \overset{i}{\longrightarrow} & (g_{s-1})^{-1}(R)
\end{array}
$$

Clearly $G_{s-1}$ is a homotopy of the inclusion map $i : (g_{s-1})^{-1}(y_0) \to (g_{s-1})^{-1}(R)$. Applying Theorem 4.1 to this bundle map, we obtain a fiber bundle map $H_s$ such that the following diagram

$$
\begin{array}{ccc}
g^{-1}(y_0) \times I_{s-1} & \overset{H_s}{\longrightarrow} & g^{-1}(R) \\
\downarrow\pi_{s-1} \times \text{id} & & \downarrow\pi_{s-1} \\
(g_{s-1})^{-1}(y_0) \times I_{s-1} & \overset{G_{s-1}}{\longrightarrow} & (g_{s-1})^{-1}(R).
\end{array}
$$
commutes and \( H_s(x, 0) = x \) for all \( x \in g^{-1}(y_0) \). In particular, for every \( t \in I_{s-1} \), \( H_s \) restricts to a continuous map from \( g^{-1}(y_0) \times \{t\} \) to \( g^{-1}(y_t) \). As \( \pi_{s-1} \) is a principal \( A_s(g) \)-bundle map, it holds that this restriction map is a homeomorphism.

By the induction hypothesis \( G_{s-1}(\cdot, t) \) is a cubspace isomorphism for every \( t \in I_{s-1} \). Thus the discrepancy

\[
\rho_{H_s(\cdot, t)} : C^{s+1}(g^{-1}(y_0)) \to A_s(g)
\]

is well defined for every \( t \in I_{s-1} \). Since \( H_s \) is a homotopy, as \( t \) is small enough, say, \( t \in I_s := [0, t_s] \) for some \( 0 < t_s \leq t_{s-1} \), \( H_s(\cdot, t) \) is of sufficiently small norm. Thus by [21, Theorem 4.11] (or [1, Lemma 3.19]), \( \rho_{H_s(\cdot, t)} \) is a coboundary map. Furthermore, since \( H_s \) is continuous, it can be repaired to a (continuous) homotopy

\[
G_s : g^{-1}(y_0) \times I_s \to g^{-1}(R)
\]

which restricts to a cubspace isomorphism from \( g^{-1}(y_0) \times \{t\} \) to \( g^{-1}(y_t) \) for every \( t \in I_s \). This completes the inductive step.

Finally, for every \( y \in R \), the above argument shows in particular that for every \( t \in I \), there exists \( \delta_t > 0 \) such that \( g^{-1}(y_u) \) is isomorphic to \( g^{-1}(y_{u'}) \) as cubespaces for every \( u, u' \) in the ball \( (t - \delta_t, t + \delta_t) \). By compactness, we obtain a finite open cover of \( I \) consisting of open balls such that the \( g \)-fibers over each ball are isomorphic as cubespaces. Then a finite composition of isomorphisms give an isomorphism between \( g^{-1}(y_0) \) and \( g^{-1}(y_1) \).

\[\Box\]

**Remark 4.2** In the statement of Theorem 1.28, it is desirable to drop the assumption that \( g \) is a Lie fibration. Let us explain the obstruction. Note that the proof of Theorem 1.28 is based on an induction with a finite number of steps. To deal with the general case, one may apply the factorization of \( g \) as an inverse limit of Lie fibrations in Theorem 1.27. However, after an infinite steps of induction, one fails to obtain a homotopy for some strictly positive time interval. This gives rise to the obstruction to construct the desired cubspace isomorphism.

### 5 Factor Maps Between Minimal Distal Systems are Fibrations

In this section, we prove that every factor map between minimal distal systems is a fibration for the induced cubspace morphism.

**Definition 5.1** Let \( (G, X) \) be a dynamical system and \( K \) a compact group acting on \( X \) such that the action of \( K \) commutes with the action by \( G \). We say a factor map \( \pi : (G, X) \to (G, Y) \) is a (topological) group extension by \( K \) if

\[
R_\pi := \{(x, x') \in X^2 : \pi(x) = \pi(x')\} = \{(x, kx) : x \in X, k \in K\}.
\]

If furthermore \( K \) is an abelian group and \( K \) acts on \( X \) freely, we say \( \pi \) is a principal abelian group extension.\(^{11}\)

**Lemma 5.2** Let \( (G, X) \) be a dynamical system and \( \pi : X \to Y \) a group extension by a compact group \( K \). Then for every \( \ell \geq 0 \) we have

\[
KNRP_\ell[X] = NRP_\ell[X]
\]

where \( K \) acts on \( NRP_\ell[X] \) by diagonal action.

\(^{11}\) Comparing to Definition 1.11, it is easy to see that a group extension \( \pi : (G, X) \to (G, Y) \) by a compact group \( K \) which acts freely on \( X \) is a \( K \)-principal bundle.
\textbf{Proof} Let \( c \in C^\ell_G(X) \) and \( k \in K \). We check that \( kc \in C^\ell_G(X) \). By definition there exists \( g_n \in HK^\ell(G) \) and \( x_n \in X \) such that \( c = \lim g_n(x_n, x_n, \ldots, x_n) \). Thus
\[
k c = k(\lim g_n(x_n, x_n, \ldots, x_n)) = \lim g_n(k(x_n, x_n, \ldots, x_n)) \in C^\ell_G(X).
\]

Now let \((x, y) \in \text{NRP}^{\ell}(X)\). By definition we have \((x, x, \ldots, x, y) \in C^{\ell+1}_G(X)\). It follows that \( k(x, x, \ldots, x, y) \in C^{\ell+1}_G(X) \). In other words, \((kx, ky) \in \text{NRP}^{\ell}(X)\).

\textbf{Definition 5.3} A factor map \( \pi : X \to Y \) is an \textbf{isometric extension} if there exists a continuous function \( d : R_\pi \to \mathbb{R} \) such that
\begin{enumerate}
  \item for every \( y \in Y \) the restriction map \( d|_{\pi^{-1}(y) \times \pi^{-1}(y)} \) is a metric on \( \pi^{-1}(y) \);
  \item \( d(gx, gx') = d(x, x') \) for every \( g \in G \) and \((x, x') \in R_\pi\).
\end{enumerate}

The following lemma says that every isometric extension between minimal systems factors through group extensions.

\textbf{Lemma 5.4} [16, Page 15] A factor map \( \pi : X \to Y \) is an isometric extension of minimal systems if and only if there exists a compact group \( K \) and a closed subgroup \( H \) of \( K \) such that \( X \) admits a group extension \( \tilde{X} \) by \( H \) and \( Y \) admits a group extension \( \tilde{X} \) by \( K \) such that the diagram
\[
\begin{array}{ccc}
\tilde{X} & \rightarrow & X(\cong \tilde{X}/H) \\
\downarrow & & \downarrow \pi \\
Y(\cong \tilde{X}/K) & \rightarrow & \\
\end{array}
\]
commutes.

The following proposition says that fibrations are stable under inverse limits.

\textbf{Proposition 5.5} Let \( p_{i, i+1} : X_{i+1} \to X_i \) be a fibration for every \( i = 0, 1, 2, \ldots \) Denote by \( X \) the inverse limit of \( X_i \). Then the induced map \( f : X \to X_0 \) is a fibration. Similarly, the degree of fibrations is also preserved by the inverse limit operation.

\textbf{Proof} Denote by \( p_n \) the projection map \( X \to X_n \). Assume that \( \lambda \) is a \( k \)-corner of \( X \) such that \( f(\lambda) \) extends to a cube \( c \) of \( X_0 \). We need to complete \( \lambda \) as a cube via some point \((x_i)_i \) of \( X \) such that \( f((x_i)_i) = c(\overrightarrow{1}) \).

Note that \( p_1(\lambda) \) is a \( k \)-corner of \( X_1 \) and \( p_{0,1} \circ p_1(\lambda) = f(\lambda) \) extends to a cube of \( X_0 \) via \( c(\overrightarrow{1}) \). Since \( p_{0,1} \) is a fibration, there exists \( u_1 \in (p_{0,1})^{-1}(c(\overrightarrow{1})) \) completing \( p_1(\lambda) \) as a cube of \( X_1 \).

Now \( p_2(\lambda) \) is a \( k \)-corner of \( X_2 \) and \( p_{1,2} \circ p_2(\lambda) = p_1(\lambda) \) extends to a cube of \( X_1 \) via \( u_1 \). Since \( p_{1,2} \) is a fibration, there exists \( u_2 \in (p_{1,2})^{-1}(u_1) \) completing \( p_2(\lambda) \) as a cube of \( X_2 \).

Inductively, if \( \alpha \) is a limit ordinal, we obtain a point \((u_i)_{i<\alpha} \) in \( X_\alpha \) such that
\[
p_{0,\alpha}((u_i)_{i<\alpha}) = p_{0,1}(u_1) = c(\overrightarrow{1}).
\]
Here for every \( i < \alpha \), \( u_i \) completes \( p_1(\lambda) = p_{i,\alpha} \circ p_\alpha(\lambda) \) as a cube of \( X_i \). Thus \((u_i)_{i<\alpha} \) completes \( p_\alpha(\lambda) \) as a cube of \( X_\alpha \). We proceed by induction.

\textbf{Proof of Theorem 1.29} The relative Furstenberg structure theorem states that a distal extension of minimal systems is given by a (countable) transfinite tower\footnote{This is defined in [11, Appendix E14.3, E15.5].} of isometric extensions [11,
Chapter V, Theorem 3.34]. Applying Proposition 5.5, we reduce to the case where \( \pi \) is an isometric extension. By Lemma 5.4 and Proposition 1.20, we can further reduce to the case where \( \pi \) is a group extension by a compact group \( K \).

Fix \( k \geq 1 \). Suppose that \( \lambda \) is a \( k \)-corner of \( X \) such that \( \pi(\lambda) \) can be extended to a cube \( c \) of \( Y \). Since \( X \) is fibrant, we can extend \( \lambda \) to be a cube via some point \( x_0 \) in \( X \). It follows that \( \pi(x_0) \) and \( c(\overrightarrow{1}) \) are \((k-1)\)-canonically related. Since \((G, X)\) is minimal, from \[17, \text{Theorem 6.1}\], we have \( \text{NRP}^{k-1}(Y) = (\pi \times \pi)(\text{NRP}^{k-1}(X)) \). Thus by Proposition 1.34, we have

\[
(\pi(x_0), c(\overrightarrow{1})) \in \text{NRP}^{k-1}(Y) = (\pi \times \pi)(\text{NRP}^{k-1}(X)).
\]

Thus there exists \((x, z) \in \text{NRP}^{k-1}(X)\) such that \( \pi(x) = \pi(x_0) \) and \( \pi(z) = c(\overrightarrow{1}) \).

Now since \( \pi \) is a group extension by \( K \), there exists a unique \( a \in K \) such that \( x_0 = ax \). By Proposition 1.8, it suffices to show \((x_0, az) \in \text{NRP}^{k-1}(X)\). Indeed, in such a case, \( az \) completes \( \lambda \) as a cube and \( \pi(az) = \pi(z) = c(\overrightarrow{1}) \).

By Lemma 5.2, \( \text{K NRP}^{k-1}(X) = \text{NRP}^{k-1}(X) \). Since \((x, z) \in \text{NRP}^{k-1}(X)\), it follows that \((x_0, az) = (x, z) \in \text{NRP}^{k-1}(X)\).

\[\square\]

\section{6 Extensions of Finite Degree}

In this section we investigate extensions of finite degree (see Definition 1.38).

\textbf{Proposition 6.1} Let \( s \geq 1 \) and \( f : (G, X) \to (G, Y) \) an extension of degree at most \( s \) such that \( X \) is minimal distal. Then \( f \) factors as a tower of principal abelian group extensions:

\[
\begin{align*}
(G, X) & \longrightarrow (G, X/\text{NRP}^{s-1}(f)) & & \cdots & & \longrightarrow (G, X/\text{NRP}^{[1]}(f)) \\
\downarrow f & & & & \downarrow & & \\
(G, Y) & & & & & & \\
\end{align*}
\]

\textbf{Proof} Since \( X \) is minimal distal, by \[17, \text{Theorem 7.10}\], it is fibrant. From Theorem 1.29, \( f \) is a fibration. By Proposition 1.9, fibrant cubespaces have the gluing property. Thus applying Proposition 1.23, we conclude that \( f \) is an \( s \)-fibration. Since \( X \) is minimal, it is an ergodic cubespaces and hence so are the induced quotient spaces. From Proposition 1.34, \( \text{NRP}^{[s]}(f) = \sim_{f,s} \). By Theorem 1.25, \( f \) is factored as stated. It suffices to show every successive map in the tower is a principal abelian group extension. Let us show, for example, \( f_s : (G, X) \to (G, X/\text{NRP}^{s-1}(f)) \) is a principal abelian group extension by the structure group \( A_s \).

Since \( \text{NRP}^{[s]}(f) \) is a \( G \)-invariant closed equivalence relation, the quotient map \( f_s \) is a factor map. By Theorem 1.25, \( f_s \) is an \( A_s \)-principal fiber bundle. Thus we need only to show \( A_s \) commutes with the \( G \)-actions. Recall that in \[17, \text{Page 48}\], \( A_s \) is constructed as

\[
A_s = \text{NRP}^{s-1}(f)/\sim_f,
\]

where \((x, x') \sim_f (y, y')\) if and only if \( f(x) = f(x') \), \( f(y) = f(y') \) and \( [\alpha^s(x, x'), \alpha^s(y, y')] \in C_G^{s+1}(X) \). Denote by \([x, x']_f\) the equivalence class of \((x, x')\). Fix \( x \in X \), \( a \in A_s \) and \( t \in G \). We need to show \( a((tx) = t(ax) \). Set \( x' = ax \). By definition \[17, \text{Page 49}\], \((x, x') \in \text{NRP}^{s-1}(f)\) and \( a = [x, x']_f \). Applying \((\alpha^s(e), \alpha^s(t)) \in HK^{s+1}(G)\) to the
More generally, if for $s$ obtain a factorization of $\pi$. That is, for a factor map $f$ which are compatible with each other. It suffices to show that every $pm$ induct on the degree $X$. Thus $X$ coincides with $sx$. Note that $f(tx) = tf(x) = tf(x') = f(tx')$. It follows that $a = [tx, tx']_f = [tx, t(ax)]_f$.

In particular, by definition of $A_s$-actions, we obtain $a(tx) = t(ax)$.

Recall the definition of maximal $s$-fibration in Proposition 1.24. In [17, Theorem 7.15], it was shown that for a minimal system $(G, X)$, the maximal $s$-nilspace factor of $(X, C^*_G(X))$ coincides with $X/NRP^s(X)$. As a relative version of this, we have

**Proposition 6.2** Let $f : X \to Y$ be a factor map of minimal distal systems. Then $g : X/NRP^s(f) \to Y$ is the maximal $s$-fibration for every $s \geq 0$ and $g$ is an extension of degree at most $s$ relative to $Y$.

**Proof** From Proposition 1.34, we know $NRP^s(f) = \sim_{f,s}$. By Proposition 1.24, $g : X/NRP^s(f) \to Y$ is the maximal $s$-fibration. Moreover, $NRP^s(g) = \sim_{g,s} = \Delta$. Thus $g$ is an extension of degree at most $s$ relative to $Y$.

In general, given a factor map $f : X \to Y$ of minimal systems, from Proposition 1.36, the induced map $g : X/NRP^{k+1}(f) \to Y$ is a distal extension and has $(k+1)$-uniqueness.

**Question 6.3**

1. Is $g$ a fibration?
2. More generally, if $f$ is distal, can one conclude that $f$ is a fibration? In [37, Example 3.10], Tu and Ye considered the projection map of the Denjoy minimal system onto the unit circle and showed that it is not a fibration. However it is also not distal.

Recall that a minimal system $(G, X)$ is called a system of degree at most $s$ if $NRP^s(X) = \Delta$. In [17, Theorem 7.14], it is proved that $(G, X)$ is a system of degree at most $s$ if and only if it is an $s$-nilspace. As a relative analogue of this result, we remark that if the answer of Question 6.3 (1) is positive, then we will obtain a dynamical characterization of $s$-fibration. That is, for a factor map $f : X \to Y$ of minimal systems, $f$ is an extension of degree at most $s$ relative to $Y$ if and only if $f$ is an $s$-fibration.

**Proof of Theorem 1.39** By Proposition 6.2, $\pi$ is an $s$-fibration. Applying Theorem 1.27, we obtain a factorization of $\pi$ by $s$-fibrations $p_{m,n} : Z_n \to Z_m$ and Lie fibrations $h_n : Z_n \to Y$ which are compatible with each other. It suffices to show that every $p_{m,n}$ is a factor map. We induct on the degree $s$ in order to prove this.

The case $s = 0$ is trivial since then $\pi$ is an isomorphism. Assume that the statement is true for $s - 1$. Recall that in the proof of Theorem 1.27, the space $Z_n$ is constructed as a quotient space $X'/ \approx_{\psi_n}$ based on a fibration map $\psi_n : \pi_{\pi,s-1}(X) \to B_n$ for some cubespace $B_n$. To show $p_{m,n}$ is a factor map, it suffices to show that the equivalence relation $\approx_{\psi_n}$ is $G$-invariant. By the induction hypothesis, $\psi_n$ is $G$-equivariant. Let $x \approx_{\psi_n} x'$ for some $x, x' \in X$. Then for every $s \in G$

$$\psi(\pi_{\pi,s-1}(sx)) = \psi(s\pi_{\pi,s-1}(x)) = s\psi(\pi_{\pi,s-1}(x)) = s\psi(\pi_{\pi,s-1}(x')).$$

Thus $sx \approx_{\psi_n} sx'$.

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