Periodic Solutions of 2D Isothermal Euler-Poisson

Equations with Possible Applications to Spiral

and Disk-like Galaxies

MAN KAM KWONG*

Department of Applied Mathematics,

The Hong Kong Polytechnic University,

Hung Hom, Kowloon, Hong Kong

MANWAI YUEN†

Department of Mathematics and Information Technology,

The Hong Kong Institute of Education,

10 Lo Ping Road, Tai Po, New Territories, Hong Kong

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Abstract

Compressible Euler-Poisson equations are the standard self-gravitating models for stellar dynamics in classical astrophysics. In this article, we construct periodic solutions to the isothermal ($\gamma = 1$) Euler-Poisson equations in $R^2$ with possible applications to the formation of plate, spiral galaxies and the evolution of gas-rich, disk-like galaxies. The results complement Yuen’s solutions without rotation (M.W. Yuen, Analytical Blowup Solutions to the 2-dimensional Isothermal Euler-Poisson Equations of Gaseous Stars, J. Math. Anal. Appl.

*E-mail address: mankwong@polyu.edu.hk
†Corresponding author and E-mail address: nevetsyuen@hotmail.com
Here, the periodic rotation prevents the blowup phenomena that occur in solutions without rotation. Based on our results, the corresponding 3D rotational results for Goldreich and Weber’s solutions are conjectured.

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1 Introduction

The evolution of self-gravitating galaxies or gaseous stars in astrophysics can be described by the compressible Euler-Poisson equations:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0 \\
\rho (\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}) + \nabla P &= -\rho \nabla \Phi \\
\Delta \Phi(t, \vec{x}) &= \alpha(N) \rho,
\end{align*}
\]

where \(\alpha(N)\) is a constant related to the unit ball in \(\mathbb{R}^N\), such that \(\alpha(1) = 2\), \(\alpha(2) = 2\pi\), and for \(N \geq 3\)

\[
\alpha(N) = N(N - 2)V(N) = N(N - 2) \frac{\pi^{N/2}}{\Gamma(N/2 + 1)},
\]

where \(V(N)\) is the volume of the unit ball in \(\mathbb{R}^N\) and \(\Gamma\) is the Gamma function. The unknown functions \(\rho = \rho(t, \vec{x})\) and \(\vec{u} = \vec{u}(t, \vec{x}) = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N\) are the density and the velocity, respectively. The \(\gamma\)-law is usually imposed on the pressure term:

\[
P = P(\rho) = K \rho^\gamma
\]

with the constant \(\gamma \geq 1\). In addition, the ideal fluid is called isothermal if \(\gamma = 1\). The Poisson equation (1) can be solved as

\[
\Phi(t, \vec{x}) = \int_{\mathbb{R}^N} Green(\vec{x} - \vec{y}) \rho(t, \vec{y}) d\vec{y},
\]

with the Green’s function

\[
Green(\vec{x}) = \begin{cases} 
\log |\vec{x}| & \text{for } N = 2 \\
\frac{-1}{|\vec{x}|^{N-2}} & \text{for } N \geq 3.
\end{cases}
\]
For $N = 3$, the Euler-Poisson equations are the classical models in stellar dynamics given in [2], [3], [7] and [9]. Some results on local existence of the system can be found in [10], [1], and [5].

If we seek solutions with radial symmetry, the Poisson equation is transformed to

$$r^{N-1} \Phi_{rr}(t, r) + (N-1)r^{N-2}\Phi_r(t, r) = \alpha(N)\rho r^{N-1}$$

(6)

$$\Phi_r = \frac{\alpha(N)}{r^{N-1}} \int_0^r \rho(t, s)s^{N-1}ds.$$  

(7)

In particular, radially symmetric solutions without rotation can be expressed as

$$\rho(t, \vec{x}) = \rho(t, r), \quad \vec{u}(t, \vec{x}) = \vec{x}V(t, r)$$

(8)

with the radius $r := \left(\sum_{i=1}^N x_i^2\right)^{1/2}$. In 1980, Goldreich and Weber first constructed analytical blowup (collapsing) solutions of the 3D Euler-Poisson equations for $\gamma = 4/3$ for the non-rotating gas spheres [6]. In 1992, Makino [11] provided a rigorous proof of the existence of these kinds of blowup solutions. In 2003, Deng, Xiang and Yang [4] generalized the solutions to higher dimensions, $R^N (N \geq 3)$. In 2008, Yuen constructed the corresponding solutions (without compact support) in $R_2$ with $\gamma = 1$ [12]. In summary, the family of the analytical solutions is as follows:

for $N \geq 3$ and $\gamma = (2N-2)/N$, in [4]

$$\rho(t, r) = \begin{cases} 
\frac{1}{a(t)^N}f\left(\frac{r}{a(t)}\right)^{N/(N-2)} & \text{for } r < a(t)S_\mu, \\
0 & \text{for } a(t)S_\mu \leq r 
\end{cases}, \quad V(t, r) = \frac{\dot{\alpha}(t)}{a(t)}r$$

(9)

where the finite $S_\mu$ is the first zero of $f(s)$ and

$$\ddot{f}(s) + \frac{N-1}{s}\dot{f}(s) + \frac{\alpha(N)}{(2N-2)K}f(s)^{N/(N-2)} = \frac{2\alpha}{\lambda}K, \quad f(0) = \alpha > 0, \quad \dot{f}(0) = 0,$$

(10)

for $N = 2$ and $\gamma = 1$, in [12]

$$\rho(t, r) = \frac{1}{a(t)^2}e^{f\left(\frac{r}{a(t)}\right)}, \quad V(t, r) = \frac{\dot{\alpha}(t)}{a(t)}r$$

(10)

$$\ddot{f}(s) + \frac{1}{s}\dot{f}(s) + \frac{2\pi}{K}e^{f(s)} = \frac{2\alpha}{\lambda}, \quad f(0) = \alpha, \quad \dot{f}(0) = 0.$$

Similar solutions exist for other similar systems, see, for example, [13] and [14]. All the above known solutions are without rotation.
For the 2D Euler equations with $\gamma = 2$,
\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0 \\
\rho (\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}) + \nabla P &= 0,
\end{align*}
\] (11)

Zhang and Zheng [16] in 1995 constructed the following explicitly spiral solutions:
\[
\begin{align*}
\rho &= \frac{r^2}{8Kt^2}, \ u_1 = \frac{1}{2t} (x + y), \ u_2 = \frac{1}{2t} (x - y)
\end{align*}
\] (12)
in $r \leq 2t \sqrt{P_0}$, and
\[
\begin{align*}
\rho &= \rho_0, \\
u_1 &= \frac{(2t \dot{P}_0 \cos \theta + \sqrt{2P_0 \sqrt{r^2 - 2t^2 P_0 \sin \theta}})}{r}, \\
u_2 &= \frac{(2t \dot{P}_0 \sin \theta - \sqrt{2P_0 \sqrt{r^2 - 2t^2 P_0 \cos \theta}})}{r}
\end{align*}
\] (13)
in $r > 2t \sqrt{P_0}$, where $\rho_0 > 0$ is an arbitrary parameter, $\dot{P}_0 = \dot{P}(\rho_0)$, $x = r \cos \theta$ and $y = r \sin \theta$.

In this article, we combine the above results to construct solutions with rotation for the 2D isothermal Euler-Poisson equations. Our main contribution is in applying the isothermal pressure term to balance the potential force term to generate novel solutions.

**Theorem 1** For the isothermal ($\gamma = 1$) Euler-Poisson equations [1] in $R^2$, there exists a family of global solutions with rotation in radial symmetry,
\[
\begin{align*}
\rho(t, \vec{x}) &= \rho(t, r) = \frac{r}{a(t)} e^{f(\frac{r}{a(t)})}, \ u_1 = \frac{a(t)}{a(t)^2} x - \frac{\xi}{a(t)^2} y, \ u_2 = \frac{\xi}{a(t)^2} x + \frac{\dot{a}(t)}{a(t)} y, \\
\dot{a}(t) &= \frac{-\lambda}{a(t)} + \frac{\xi^2}{a(t)^2}, \ a(0) = a_0 > 0, \ \dot{a}(0) = a_1 \\
f(s) + \frac{1}{s} f(s) + \frac{2\pi}{K} e^{f(s)} - \frac{2\lambda}{K}, \ f(0) = \alpha, \ \dot{f}(0) = 0,
\end{align*}
\] (14)
with arbitrary constants $\xi \neq 0, a_0, a_1$ and $\alpha$.

(I) With $\lambda > 0$,
(a) solutions (14) are non-trivially time-periodic, except for the case $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$;
(b) if $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$, solutions (14) are steady.

(II) With $\lambda \leq 0$, solutions (14) are global in time.

Here, 2D rotational solutions (14) of the Euler-Poisson equations (1) may be reference examples for modeling the formation of plate and spiral galaxies or gaseous stars in the non-relativistic
content, because most of the matter is gas at the early stage of their evolution. Readers can refer to [16] for the detail description of astrophysical situations. In addition, solutions (14) may also be applied to the development of gas-rich and disk-like (dwarf) galaxies [2].

Remark. By taking \( \xi = 0 \) for solutions (14) in Theorem 1, we obtain Yuen’s non-rotational solutions (10), which blow up in a finite time \( T \) if \( \lambda > 0 \). However, the rotational (when \( \xi \neq 0 \)) term in (14) prevents the blowup phenomena.

2 Periodic and Spiral Solutions

Our main work is to design the relevant functions with rotation to fit the 2D mass equation (1).

Lemma 2 For the 2D equation of conservation of mass

\[
\rho_t + \nabla \cdot (\rho \vec{u}) = 0, \tag{15}
\]

there exist the following solutions:

\[
\rho(t, \vec{x}) = \rho(t, r) = \frac{f \left( \frac{r}{a(t)} \right)}{a(t)^2}, \quad u_1 = \frac{\dot{a}(t)}{a(t)} x - \frac{G(t, r)}{r} y, \quad u_2 = \frac{G(t, r)}{r} x + \frac{\dot{a}(t)}{a(t)} y \tag{16}
\]

with arbitrary \( C^1 \) functions \( f(s) \geq 0 \) and \( G(t, r) \) and \( a(t) > 0 \) \( \in C^1 \).

Proof. We plug the following functional form

\[
\rho(t, \vec{x}) = \rho(t, r) = \frac{f \left( \frac{r}{a(t)} \right)}{a(t)^2}, \quad u_1 = \frac{F(t, r)}{r} x - \frac{G(t, r)}{r} y, \quad u_2 = \frac{G(t, r)}{r} x + \frac{F(t, r)}{r} y \tag{17}
\]

with arbitrary \( C^1 \) functions \( f(s) \geq 0, F(t, r), G(t, r) \) and \( a(t) > 0 \) \( \in C^1 \), into the 2D mass equation (15) to have

\[
\rho_t + \nabla \cdot (\rho \vec{u}) = \rho_t + \frac{\partial}{\partial x} \left( \rho \frac{F_x}{r} - \rho \frac{G_y}{r} \right) + \frac{\partial}{\partial y} \left( \rho \frac{F_y}{r} + \rho \frac{G_x}{r} \right) \tag{18}
\]

\[
= \rho_t + \left( \frac{\partial}{\partial x} \rho \right) \frac{F_x}{r} + \rho \left( \frac{\partial}{\partial x} \frac{F_x}{r} \right) - \left( \frac{\partial}{\partial x} \rho \right) \frac{G_y}{r} - \rho \left( \frac{\partial}{\partial x} \frac{G_y}{r} \right)
+ \left( \frac{\partial}{\partial y} \rho \right) \frac{F_y}{r} + \rho \left( \frac{\partial}{\partial y} \frac{F_y}{r} \right) + \left( \frac{\partial}{\partial y} \rho \right) \frac{G_x}{r} + \rho \left( \frac{\partial}{\partial y} \frac{G_x}{r} \right) \tag{19}
\]
\[ = \rho_t + \rho \frac{x F_x}{r} + \rho \left( F_t + \frac{F}{r} \right) \frac{F_x}{r^3}
\]

\[ - \frac{x G_y}{r} - \rho G_r x + \rho G y + \rho \left( G_t + \frac{F}{r} \right) y \frac{y}{r^3}
\]

\[ + \rho \frac{F}{r} - \rho \frac{G_y y}{r^3} + \rho G x \frac{x}{r} - \rho G x y \frac{y}{r^3} \]  \hspace{1cm} (21)

\[ = \rho_t + \rho \frac{x F_x}{r} + \rho \left( F_t + \frac{F}{r} \right) \frac{F_x}{r^3}
\]

\[ + \rho \frac{G_y}{r} y + \rho \left( G_t + \frac{F}{r} \right) y - \rho G y \frac{y}{r^3} \]  \hspace{1cm} (22)

\[ = \rho_t + \rho \left( F + \frac{F}{r} \right) \frac{y}{r} - \rho F y \frac{y}{r^3} \]  \hspace{1cm} (23)

Then we take the self-similar structure for the density function

\[ \rho(t, \vec{x}) = \rho(t, r) = f \left( \frac{r}{a(t)} \right) \frac{a(t)}{r^2}, \]  \hspace{1cm} (24)

and \( F(t, r) = \frac{\dot{a}(t)}{a(t)} r \) for the velocity \( \vec{u} \) to balance equation (23):

\[ = \frac{\partial}{\partial t} f \left( \frac{r}{a(t)} \right) \frac{\dot{a}(t)}{a(t)} + \frac{\partial}{\partial r} f \left( \frac{r}{a(t)} \right) \frac{\dot{a}(t)}{a(t)} \frac{r}{a(t)} + f \left( \frac{r}{a(t)} \right) \frac{\dot{a}(t)}{a(t)} + f \left( \frac{r}{a(t)} \right) \frac{\dot{a}(t)}{a(t)} \]  \hspace{1cm} (25)

\[ = \frac{-2 \dot{a}(t) f \left( \frac{r}{a(t)} \right)}{a(t)^3} - \frac{\dot{a}(t) r f \left( \frac{r}{a(t)} \right)}{a(t)^2} \]

\[ + \frac{f \left( \frac{r}{a(t)} \right)}{a(t)} \frac{a(t) r}{a(t)} + f \left( \frac{r}{a(t)} \right) \frac{a(t)}{a(t)^2} \frac{\dot{a}(t)}{a(t)} + f \left( \frac{r}{a(t)} \right) \frac{a(t)}{a(t)^2} \frac{\dot{a}(t)}{a(t)} \]  \hspace{1cm} (26)

\[ = 0. \]  \hspace{1cm} (27)

The proof is completed. \( \blacksquare \)

The following Lemma is required to show the cyclic phenomena of the rotational solutions (14).

**Lemma 3** With \( \xi \neq 0 \), for the Emden equation

\[ \ddot{a}(t) = -\lambda a(t) + \frac{\xi^2}{a(t)^3}, \ a(0) = a_0 > 0, \ \ddot{a}(0) = a_1, \]  \hspace{1cm} (28)

(I) with \( \lambda > 0 \), the solution is non-trivially periodic, except for the case with \( a_0 = \frac{\xi}{\sqrt{\lambda}} \) and \( a_1 = 0 \);

(II) with \( \lambda \leq 0 \), the solution is global.
**Proof.** The proof is standard and similar to Lemma 3 in [14] for the Euler-Poisson equations with a negative cosmological constant.

(I) For equation (28), we could multiply $\dot{a}(t)$ and integrate it in the following manner:

$$\frac{\dot{a}(t)^2}{2} + \lambda \ln a(t) + \frac{\xi^2}{2a(t)^2} = \theta$$

with the constant $\theta = \frac{a_0^2}{2} + \lambda \ln a_0 + \frac{\xi^2}{2a_0^2}$.

Then, we could define the kinetic energy as

$$F_{\text{kin}} = \frac{\dot{a}(t)^2}{2}$$

and the potential energy as

$$F_{\text{pot}} = \lambda \ln a(t) + \frac{\xi^2}{2a(t)^2}.$$  

(31)

Here, the total energy is conserved such that

$$\frac{d}{dt}(F_{\text{kin}} + F_{\text{pot}}) = 0.$$  

(32)

The potential energy function has only one global minimum at $\ddot{a} = \frac{|\xi|}{\sqrt{\lambda}}$ for $a(t) \in (0, +\infty)$. Therefore, by the classical energy method (in Section 4.3 of [8]), the solution for equation (28) has a closed trajectory. The time for traveling the closed orbit is

$$T = 2 \int_{t_1}^{t_2} dt = 2 \int_{a_{\min}}^{a_{\max}} \frac{da(t)}{\sqrt{2 \left[ \theta - \left( \lambda \ln a(t) + \frac{\xi^2}{2a(t)^2} \right) \right]}}.$$  

(33)

where $a(t_1) = a_{\min} = \inf_{t \geq 0} a(t))$ and $a(t_2) = a_{\max} = \sup_{t \geq 0} a(t)$ with some constants $t_1, t_2$ such that $t_2 \geq t_1 \geq 0$.

We let $H(t) = \theta - \left( \lambda \ln a(t) + \frac{\xi^2}{2a(t)^2} \right)$, $H_0 = \left| \theta - \left( \lambda \ln (a_{\min} + \epsilon) + \frac{\xi^2}{2(a_{\min} + \epsilon)^2} \right) \right|$, and $H_1 = \left| \theta - \left( \lambda \ln (a_{\max} - \epsilon) + \frac{\xi^2}{2(a_{\max} - \epsilon)^2} \right) \right|$. Except for the case with $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$, the time in equation (33) can be estimated by

$$T = \int_{a_{\min}}^{a_{\min} + \epsilon} \frac{2da(t)}{\sqrt{2 \left[ \theta - \left( \lambda \ln a(t) + \frac{\xi^2}{2a(t)^2} \right) \right]}} + \int_{a_{\min} + \epsilon}^{a_{\max} - \epsilon} \frac{2da(t)}{\sqrt{2 \left[ \theta - \left( \lambda \ln a(t) + \frac{\xi^2}{2a(t)^2} \right) \right]}}$$

$$+ \int_{a_{\max} - \epsilon}^{a_{\max} + \epsilon} \frac{2da(t)}{\sqrt{2 \left[ \theta - \left( \lambda \ln a(t) + \frac{\xi^2}{2a(t)^2} \right) \right]}}.$$  

(34)
with a sufficient small constant $\epsilon > 0$,

\[
\leq \sup_{a_{\min} \leq a(t) \leq a_{\min} + \epsilon} \left| \frac{1}{a(t)} \right| \left( \int_0^{H_0} \frac{\sqrt{2dH(t)}}{H(t)} \right) + \int_{a_{\min} + \epsilon}^{a_{\max} - \epsilon} \left( 2da(t) \sqrt{2 \left[ \theta - \left( \lambda \ln a(t) + \frac{\epsilon^2}{2a(t)^2} \right) \right]} \right)
\]

\[
+ \sup_{a_{\max} - \epsilon \leq a(t) \leq a_{\max}} \left| \frac{1}{a(t)} \right| \left( \int_0^{H_1} \frac{\sqrt{2dH(t)}}{H(t)} \right)
\]

\[
= \sup_{a_{\min} \leq a \leq a_{\min} + \epsilon} \left| \frac{1}{a(t)} \right| \left( 2\sqrt{2H_0} + \int_{a_{\min} + \epsilon}^{a_{\max} - \epsilon} 2da(t) \sqrt{2 \left[ \theta - \left( \lambda \ln a(t) + \frac{\epsilon^2}{2a(t)^2} \right) \right]} \right)
\]

\[
+ \sup_{a_{\max} - \epsilon \leq a(t) \leq a_{\max}} \left| \frac{1}{a(t)} \right| \left( 2\sqrt{2H_1} \right) (36)
\]

\[
< \infty. (37)
\]

Therefore, we have (a) the solutions to the Emden equation (28) are non-trivially periodic except for the case with $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$.

Figure 2 below shows a particular solution for the Emden equation:

\[
\begin{cases}
\dot{a}(t) = \frac{1}{a(t)} + \frac{1}{a(t)^3}, \\
a(0) = 1, \dot{a}(0) = 1.
\end{cases} (38)
\]

It is clear to see (b) if $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$, the solutions (28) are steady.

By applying the similar analysis, we can show that (II) with $\lambda \leq 0$, the solutions are global.

The proof is completed. ■

After obtaining the above two lemmas, we can construct the periodic and spiral solutions with rotation to the 2D isothermal Euler-Poisson system (11) as follows.

**Proof of Theorem 1**. The procedure of the proof is similar to the proof for the non-rotational fluids (12). It is clear that our functions (14) satisfy Lemma 2 for the mass equation (11). For the first momentum equation (11), we get

\[
\rho \left[ \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right] + \frac{\partial}{\partial x} P + \rho \frac{\partial}{\partial x} \Phi = 0
\]

\[
= \rho \left[ \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right] + \frac{\partial}{\partial x} K e^{f \left( \frac{|\tau|}{r} \right)} + \rho \frac{\partial}{\partial x} \Phi. (40)
\]
By defining the variable \( s = \frac{1}{\tau(t)} \) with \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \), we have

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + K \frac{e^{f(s)}}{a(t)^2} \frac{1}{r} \frac{\partial}{\partial \tau} \left( \frac{r}{a(t)} \right) f(s) + \frac{x}{r} \frac{\partial}{\partial \tau} \Phi \\
= \rho \left[ \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + K \frac{x}{r} \frac{\partial}{\partial \tau} f(s) + \frac{x}{r} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right]
\end{align*}
\]

Equation (41)

\[
\frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) + \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) \\
+ \left( \frac{\hat{a}(t)}{\tau(t)} + \frac{x(t)}{\tau(t)^2} \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) + x \frac{a(t)}{a(t)^2} \left( K \hat{f}(s) + \frac{2\pi}{a(t)} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right)
\]

Equation (43)

\[
\frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) + \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) \\
- \left( \frac{\xi}{\tau(t)^2} x + \frac{x(t)}{\tau(t)^2} \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x - \frac{\xi}{\tau(t)^2} y \right) + \frac{x(t)}{\tau(t)^2} \left( K \hat{f}(s) + \frac{2\pi}{a(t)} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right)
\]

Equation (44)

\[
\begin{align*}
\frac{\partial}{\partial \tau} \left( \hat{a}(t) - \frac{\xi^2}{a(t)^2} \right) x + 2\xi \frac{\hat{a}(t)}{a(t)^2} y + \frac{\hat{a}(t)}{a(t)^2} y \\
- \left( \frac{\xi}{\tau(t)^2} x + \frac{x(t)}{\tau(t)^2} \right) \frac{\partial}{\partial \tau} \left( \hat{a}(t) - \frac{\xi^2}{a(t)^2} \right) y + \frac{x(t)}{\tau(t)^2} \left( K \hat{f}(s) + \frac{2\pi}{a(t)} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right)
\end{align*}
\]

Equation (45)

\[
\frac{\partial}{\partial \tau} \left( \hat{a}(t) - \frac{\xi^2}{a(t)^2} \right) x + \frac{\hat{a}(t)}{a(t)^2} y + \frac{2\pi}{a(t)} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta
\]

Equation (46)

with the Emden equation

\[
\begin{align*}
\hat{a}(t) &= -\frac{\lambda}{a(t)} + \frac{\xi^2}{a(t)^2} \\
a(0) &= a_0 > 0, \quad \hat{a}(0) = a_1,
\end{align*}
\]

Equation (47)

with an arbitrary constant \( \xi \neq 0 \).

Similarly, we obtain the corresponding result for the second momentum equation \( \Omega_{2,2} \) in the following manner with \( \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \).

\[
\begin{align*}
\rho \left[ \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + \frac{\partial}{\partial y} P + \rho \frac{\partial}{\partial \tau} \phi \right] \\
= \rho \left[ \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K \frac{e^{f(s)}}{a(t)^2} \frac{1}{r} \frac{\partial}{\partial \tau} \left( \frac{r}{a(t)} \right) f(s) + \frac{y}{r} \frac{\partial}{\partial \tau} \phi \right]
\end{align*}
\]

Equation (48)

\[
\begin{align*}
\rho \left[ \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + \frac{K}{a(t)^2} \frac{y}{r} \frac{\partial}{\partial \tau} f(s) + \frac{y}{r} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right] \\
= \rho \left[ \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) + \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) \\
+ \left( \frac{\xi}{\tau(t)^2} x + \frac{x(t)}{\tau(t)^2} \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) + \frac{x(t)}{\tau(t)^2} \left( K \hat{f}(s) + \frac{2\pi}{a(t)} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right) \right]
\end{align*}
\]

Equation (49)

\[
\begin{align*}
\rho \left[ \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + \frac{K}{a(t)^2} \frac{y}{r} \frac{\partial}{\partial \tau} f(s) + \frac{y}{r} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right] \\
= \rho \left[ \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) + \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) \\
+ \left( \frac{\xi}{\tau(t)^2} x + \frac{x(t)}{\tau(t)^2} \right) \frac{\partial}{\partial \tau} \left( \frac{\hat{a}(t)}{\tau(t)} x + \frac{\xi}{\tau(t)^2} y \right) + \frac{x(t)}{\tau(t)^2} \left( K \hat{f}(s) + \frac{2\pi}{a(t)} \int_0^r \frac{e^{f(s)}}{a(t)^2} \eta d\eta \right) \right]
\end{align*}
\]
\[
\frac{\partial \rho}{\partial t} = \rho \left[ -2\frac{\dot{a}(t)}{a(t)} x + \left( \frac{\dot{a}(t)}{a(t)} - \frac{\dot{a}(t)^2}{a(t)^2} \right) y + \left( \frac{\dot{a}(t)}{a(t)} x - \frac{\xi}{a(t)} y \right) \frac{\xi}{a(t)^2} \right] + \left( \frac{\xi}{a(t)} x + \frac{\dot{a}(t)}{a(t)} y \right) \frac{\dot{a}(t)}{a(t)} r + \frac{\nu}{a(t) r} \left( K \dot{f}(s) + \frac{2\pi}{rs} \int_0^r e^{f(s)} \left( \frac{n}{a(t)} \right) d \left( \frac{n}{a(t)} \right) \right)
\]

(52)

\[
\frac{\partial \rho}{\partial t} = \frac{y \rho}{a(t) r} \left[ \left( \ddot{a}(t) - \frac{\xi^2}{a(t)^2} \right) + K \dot{f}(s) + \frac{2\pi}{s} \int_0^s e^{f(s)} \tau d\tau \right]
\]

(53)

\[
\frac{\partial \rho}{\partial t} = \frac{y \rho}{a(t) r} \left[ -\lambda s + K \dot{f}(s) + \frac{2\pi}{s} \int_0^s e^{f(s)} \tau d\tau \right]
\]

(54)

To make equations (46) and (54) equal zero, we may require the Liouville equation from differential geometry:

\[
\begin{cases}
\dot{f}(s) + \frac{f(s)}{s} + \frac{2\pi}{K} e^{f(s)} = \frac{2\lambda}{K} \\
f(0) = \alpha, \quad \dot{f}(0) = 0
\end{cases}
\]

(55)

We note that the global existence of the initial value problem of the Liouville equation (55) has been shown by Lemma 10 in [12]. Thus, we confirm that functions (14) are a family of classical solutions for the isothermal ($\gamma = 1$) Euler-Poisson equations (1) in $\mathbb{R}^2$.

With Lemma 3, it is clear that

(I) With $\lambda > 0$,

(a) solutions (14) are non-trivially time-periodic, except for the case $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$;

(b) if $a_0 = \frac{|\xi|}{\sqrt{\lambda}}$ and $a_1 = 0$, solutions (14) are steady.

(II) With $\lambda \leq 0$, solutions (14) are global in time.

Therefore all of the rotational solutions (14) with $\xi \neq 0$, are global in time.

We complete the proof.

3 Conclusion and Discussion

Our results confirm that there exists a class of periodic solutions which can be found by choosing a sufficiently small constant $a_0 << 1$ in solutions (14), in the Euler-Poisson equations (11) in $\mathbb{R}^2$, even without a negative cosmological constant [14]. Here, the periodic rotation prevents the blowup phenomena that occur in solutions without rotation [12].
It is open to show the existences of solutions and their stabilities for the small perturbation of these solutions \cite{14}. Numerical simulation and mathematical proofs for the perturbational solutions are suggested for understanding their evolution.

As our solutions in this paper works for the 2D case, the corresponding rotational solutions in \( R^3 \) are conjectured. We conjecture that the corresponding rotational solutions to Goldreich and Weber’s solutions \cite{9} for the 3D Euler-Poisson equations with \( \gamma = 4/3 \) \cite{6} exist, such as the ones for the Euler equations \cite{15}. Further research is expected to shed more light on the possibilities.

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