On the canonical representation of curves in positive characteristic

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Abstract

Given a smooth curve, the canonical representation of its automorphism group is the space of global holomorphic differential 1-forms as a representation of the automorphism group of the curve. In this paper, we study an explicit set of curves in positive characteristic with irreducible canonical representation whose genus is unbounded. Additionally, we study the implications this has for the de Rham hypercohomology as a representation of the automorphism group.

1 Introduction

1.1 Motivation

Consider a smooth projective curve $X$ of genus $g$ over an algebraically closed field $k$. The canonical representation of $G := \text{Aut}(X)$ is the $g$-dimensional $k$-representation of $G$ induced by the natural action of $G$ on the space of global holomorphic differential 1-forms on $X$. When $k = \mathbb{C}$ the canonical representation has been extensively studied in [9], [8] and [10]. These methods extend to positive characteristic if the characteristic of $k$ does not divide $|G|$, but little is known in the case where it does.

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In this paper, we focus on the natural question of when the canonical representation is irreducible. If it is and \( k \) has characteristic zero, then \( g^2 \leq |G| \) since the sum of the squares of the dimensions of the irreducible representations is equal to the order of the group. On the other hand, for \( g \geq 2 \), the Hurwitz bound gives \( |G| \leq 84(g - 1) \). Together, these imply that \( g \leq 82 \). This situation is studied in §19 of [3], which includes a list of all possible genera and automorphism groups. The Klein quartic \( X(7) \) is a particularly important example of this phenomenon in characteristic zero (see [5]).

The situation is quite different in positive characteristic. The Hurwitz bound continues to hold for characteristic \( p > 0 \), provided that \( 2 \leq g < p \), with the sole exception of the curve \( y^2 = x^p - x \) (see [11]). However, in the general case, the bounds on irreducible representations are weaker, and thus we get no upper bound independent of \( |G| \) on the genera of curves with irreducible canonical representation. This presents the possibility of having curves with arbitrarily large genus and an irreducible canonical representation. This paper shows that this holds for the projective smooth curve corresponding to \( y^2 = x^p - x \), which has genus \((p - 1)/2\). The question remains open whether for fixed \( p \) there exist curves over \( k \) with irreducible canonical representation of arbitrary large dimension. The curve studied in this paper is also interesting because it is one of the four projective curves that arise in the semistable reduction at \( p \) of the tower of modular curves \( \{ X(Np^r) \}_r \), whose (étale) cohomology realizes the local Langlands correspondence for \( GL_2(\mathbb{Q}_p) \) (see [14]).

### 1.2 Summary of Results

Let \( k \) be an algebraically closed field of characteristic \( p > 2 \), and let \( X \) be the unique smooth projective curve with affine equation \( y^2 = x^p - x \). Set \( G := Aut_k(X) \), which is a degree 2 central extension of \( PGL_2(\mathbb{F}_p) \) (discussed in [11] §5). Explicitly, let \( \tilde{G} \subseteq GL_2(\mathbb{F}_p) \times \mathbb{F}_p^\times \) be the subgroup of elements \((\sigma, u_\sigma)\) with \( \det \sigma = u_\sigma^2 \). Then \( \mathbb{F}_p^\times \) acts diagonally on \( \tilde{G} \) via

\[
\lambda(\sigma, u_\sigma) = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \sigma \end{array} \right) \sigma, \lambda^{(p+1)/2} u_\sigma 
\]

and \( G = \mathbb{F}_p^\times \backslash \tilde{G} \). We will consistently represent elements of \( G \) by chosen lifts in \( \tilde{G} \). Then if \( \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\mathbb{F}_p) \), and \( u_\sigma \) is a choice of square root of
det $\sigma$, we can write the action of $G$ on the affine coordinates of $X$ as

$$(x, y) \mapsto \left( \frac{ax + b}{cx + d}, u_\sigma \frac{y}{(cx + d)^{(g+1)/2}} \right).$$

In this paper, we show that $H^0(X, \Omega^1_{X/k})$ is irreducible as a $k[G]$-module. We do this by considering a small index subgroup $H$ of $G$ that is a quotient of $\text{SL}_2(\mathbb{F}_p)$. The restriction of the canonical representation to this subgroup is the $(g - 1)^{st}$ symmetric power of the standard representation of $\text{SL}_2(\mathbb{F}_p)$ on $k^2$, which is well-known to be irreducible. It is not difficult to then describe $H^0(X, \Omega^1_{X/k})$ as a $k[G]$-submodule of the induced representation of $\text{Sym}^{g-1}(k^2)$.

In the second half of this paper, we consider the de Rham hypercohomology as a $k[G]$-module. In characteristic 0, Maschke’s Theorem implies that any representation that is not irreducible must be semisimple (i.e. must split as the direct sum of its irreducible submodules). However, in characteristic $p$, the theorem does not apply if $p$ divides $|G|$ and it is possible to find representations that are reducible but not semisimple. If a nonzero representation is not the direct sum of two proper subrepresentations, call it indecomposable. The de Rham hypercohomology of a curve is clearly reducible, since $H^0(X, \Omega^1_{X/k})$ embeds as a $k[G]$-submodule, but we show that in the case of the curve $X$, the de Rham cohomology is indecomposable. As explained above, this phenomena is unique to positive characteristic.

## 2 The action of $G$ on $H^0(X, \Omega^1_{X/k})$

As $H^0(X, \Omega^1_{X/k})$ is functorial in $X$, it has a $k$-linear action of $G$. To determine the structure of of $H^0(X, \Omega^1_{X/k})$ as a $k[G]$-module, we begin by studying it as a $k$-vector space.

**Lemma 2.1.** Let

$$(2.1) \tilde{\tau}_i = \frac{x^i dx}{y}.$$

The set $\{\tilde{\tau}_i \mid i = 0, 1, \ldots, g - 1\}$ is then a basis for $H^0(X, \Omega^1_{X/k})$.

**Proof.** The set $S = \{x^i y^j dx \mid i, j \in \mathbb{Z}\}$ certainly spans the set of all meromorphic differentials (since $2ydy = dx$), but will include differentials with
poles. To determine which of these are holomorphic, we first note that an easy calculation shows that the divisors of $x$, $y$, and $dx$ are

$$\text{div}(x) = 2P_0 + (-2)P_\infty$$
$$\text{div}(y) = P_0 + P_1 + \ldots + P_{p-1} + (-p)P_\infty$$
$$\text{div}(dx) = P_0 + P_1 + \ldots + P_{p-1} + (-3)P_\infty.$$  

Where $P_t = (t, 0)$ for $t = 0, 1, \ldots, p-1$ and $P_\infty$ the “point at infinity” on $X$. We compute:

$$\text{div}(x^iy^jdx) = (2i+j+1)P_0 + (j+1)P_1 + (j+1)P_2 + \ldots + (-2i+(-p)j-3)P_\infty,$$

so an element of $S$ is holomorphic if and only if the following conditions hold:

$$2i + j + 1 \geq 0, j + 1 \geq 0 \text{ and } (-2)i + (-p)j - 3 \geq 0.$$

The only elements of $S$ that fulfill these conditions are $\bar{\tau}_i$ for $i = 0, 1, \ldots, g-1$. This gives a spanning set of $g$ elements and hence a basis, as $H^0(X, \Omega^1_X/k)$ is $g$-dimensional. □

To study $H^0(X, \Omega^1_X/k)$ as a $G$-representation, we begin by considering a specific subgroup of small index. Consider the morphism $\theta : SL_2(\mathbb{F}_p) \to G$ that sends $\sigma \in SL_2(\mathbb{F}_p)$ to the equivalence class of $(\sigma, 1)$, and let $H = \text{im } \theta$.

One can check that

$$\ker \theta = \begin{cases} I & \text{if } p \equiv 1 \text{ mod } 4 \\ \pm I & \text{if } p \equiv 3 \text{ mod } 4 \end{cases}$$

As $|G| = 2(p-1)p(p+1)$ and $|SL_2(\mathbb{F}_p)| = (p-1)p(p+1)$, we conclude $|G : H| = 2$ if $p \equiv 1 \text{ mod } 4$, and that $|G : H| = 4$ if $p \equiv 3 \text{ mod } 4$.

Let $\text{Sym}^{g-1}(k^2)$ be the $(g-1)^{st}$ symmetric power of the standard representation of $SL_2(\mathbb{F}_p)$ on $k^2$. We identify $\text{Sym}^{g-1}(k^2)$ with the $k$-subspace of $k[u, v]$ consisting of homogeneous polynomials with degree $g-1$. This space has basis $\{v^{g-1}, v^{g-2}u, \ldots, u^{g-1}\}$ and $SL_2(\mathbb{F}_p)$-action given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}.$$ 

Note that if $p \equiv 3 \text{ mod } 4$, then $g - 1$ is even, and the action of $SL_2(\mathbb{F}_p)$ on $\text{Sym}^{g-1}(k^2)$ is trivial on $\pm I$. Because of this, we can define an $H$-action on $\text{Sym}^{g-1}(k^2)$ where if $h \in H$ such that $\theta(\sigma) = h$, then $h$ acts as $\sigma$ does. This is well-defined since the kernel of $\theta$ acts trivially. Let $V$ be this $H$-representation.
Proposition 2.2. The map

\[ \varphi : \text{res}_H(H^0(X, \Omega^1_{X/k})) \to V \quad \text{given by} \quad \frac{x^i dx}{y} \mapsto u^i v^{g-i-1} \]

is a \( k[H] \)-module isomorphism.

Proof. Clearly \( \varphi \) is an isomorphism of vector spaces over \( k \), so we need only check that \( \varphi \) is \( H \)-equivariant. Let \( h = (\gamma, 1) \in H \) for \( \gamma = (a \ b \ c \ d) \in SL_2(\mathbb{F}_p) \). We calculate

\[
h \circ \varphi \left( x^iy^{-1}dx \right) = h(u^iv^{g-i-1}) = (au + bv)^i(cu + dv)^{g-i-1}
\]

and

\[
\varphi \circ h(x^iy^{-1}dx) = \varphi \left( \left( \frac{ax + b}{cx + d} \right)^i \left( \frac{y}{(cx + d)^{g+1}} \right)^{-1} \right) d \left( \frac{ax + b}{cx + d} \right)
\]

\[
= \varphi((ax + b)^i(cx + d)^{g-i-1}y^{-1}dx)
\]

\[
= (au + bv)^i(cu + dv)^{g-i-1},
\]

since \( d \left( \frac{ax + b}{cx + d} \right) = (cx + d)^{-2}dx \).

Corollary 2.3. As a \( G \)-representation, \( H^0(X, \Omega^1_{X/k}) \) is irreducible. Moreover, \( H^1(X, \mathcal{O}_X) \) is naturally the contragredient of \( H^0(X, \Omega^1_{X/k}) \), hence irreducible as well.

Proof. It is well-known that \( \text{Sym}^{g-1}(k^2) \) is irreducible as a representation of \( SL_2(\mathbb{F}_p) \) (see the discussion following Corollary 3 of Chapter 3 in [1] on pages 14-16). It is clear from the way we defined \( V \) that it is thus irreducible as an \( H \)-representation, and this implies it is irreducible as a \( G \)-representation. The Serre duality pairing \( H^0(X, \Omega^1_{X/k}) \times H^1(X, \mathcal{O}_X) \to k \) is \( G \)-equivariant, so \( H^1(X, \mathcal{O}_X) \) is canonically the contragredient of \( H^0(X, \Omega^1_{X/k}) \), and thus irreducible.

Recall that the induced representation \( \text{Ind}^G_H(V) \) is \( k[G] \otimes_{k[H]} V \). Frobenius reciprocity tells us that

\[ \text{Hom}_G(H^0(X, \Omega^1_{X/k}), \text{Ind}^G_H(V)) = \text{Hom}_H(\text{res}_H(H^0(X, \Omega^1_{X/k})), V) \]
However, since Corollary 2.3 tells us that res$_H(H^0(X, \Omega^1_{X/k}))$ and $V$ are isomorphic irreducible $H$-representations, this means that the right hand side is one dimensional. Thus there is a nonzero $k[G]$-linear map from $H^0(X, \Omega^1_{X/k})$ into Ind$_H^G(V)$ and it is unique up to scaling. (For more on irreducible and induced representations, see sections 1 and 8 respectively of [1].)

In the case where $p \equiv 1 \mod 4$, $|G : H| = 2$ so as a vector space the induced representation is $V \oplus V$. If $\alpha \in G$ corresponds to the equivalence class of $((1, 0) , -1) \in \tilde{G}$, then $G = H \sqcup H\alpha$. Note in particular that if $\omega \in H^0(X, \Omega^1_{X/k})$, the $\alpha \omega = -\omega$ and if $(v_1, v_2) \in \text{Ind}_H^G(V)$, then $\alpha(v_1, v_2) = (v_2, v_1)$. Note that we can easily calculate the $G$-action on the induced representation from this.

In particular the map

$$T : H^0(X, \Omega^1_{X/k}) \to \text{Ind}_H^G(V)$$

$$\omega \mapsto (\varphi(\omega), -\varphi(\omega))$$

is $G$-equivariant. Thus $H^0(X, \Omega^1_{X/k})$ embeds into Ind$_H^G(V)$ as the $k[G]$-submodule $\{(v, -v) \in V \oplus V\}$.

If $p \equiv 3 \mod 4$, then $|G : H| = 4$ so as a vector space the induced representation is $V \oplus V \oplus V \oplus V$. If $\beta \in G$ corresponds to the equivalence class of $((-1, 0), i)$ where $i \in k$ such that $i^2 = -1$, then $G = H \sqcup H\beta \sqcup H\beta^2 \sqcup H\beta^3$. Note in particular that if $\tilde{\tau}_j$ is a basis element of $H^0(X, \Omega^1_{X/k})$ as in Lemma 2.1, then $\beta \tilde{\tau}_j = (-1)^j i \tilde{\tau}_j$. If $(v_1, v_2, v_3, v_4) \in \text{Ind}_H^G(V)$, then $\beta(v_1, v_2, v_3, v_4) = (v_4, v_1, v_2, v_3)$; note we can easily calculate the $G$-action on the induced representation from this. In particular, the map

$$T : H^0(X, \Omega^1_{X/k}) \to \text{Ind}_H^G(V)$$

$$\tau_j \mapsto (\varphi(\tau_j), (-1)^j i \varphi(\tau_j), -\varphi(\tau_j), (-1)^{j+1} i \varphi(\tau_j))$$

if $G$-equivariant. Thus $H^0(X, \Omega^1_{X/k})$ embeds in Ind$_H^G(V)$ as the $k[G]$-submodule spanned by

$$\{(\varphi(\tau_j), (-1)^j i \varphi(\tau_j), -\varphi(\tau_j), (-1)^{j+1} i \varphi(\tau_j)) \in V^4 \mid j = 0, \ldots, g - 1\}.$$

**Remark:** This suggests a new proof that the $p$-rank of $X$ is 0, different from more computational approaches such as that of [13]. The Cartier operator is the $k[G]$-linear map on $H^1(X, \mathcal{O}_X)$ induced by $f \mapsto f^p$. Since Serre duality is $G$-equivariant, this induces a $k[G]$-linear map on $H^0(X, \Omega^1_{X/k})$, the
matrix of which with respect to any basis is the Hasse-Witt matrix. The rank of the Hasse-Witt matrix of a curve is the $p$-rank of the curve. Since the image must be $G$-stable, and $H^0(X, \Omega^1_{X/k})$ is irreducible, it follows that the $p$-rank is either 0 or $g$. It is then a simple check to see that the image of a basis element vanishes, indicating it must be 0.

3 The action of $G$ on $H^1_{dR}(X/k)$

For any smooth proper curve $Y$ over a field $L$, $H^0(Y, \Omega^1_{Y/L})$ embeds as strict subrepresentation of $H^1_{dR}(Y/L)$. If $L$ is of characteristic zero, it follows from Maschke’s Theorem that $H^0(Y, \Omega^1_{Y/L})$ is a $L[\text{Aut}(Y)]$-module direct summand of $H^1_{dR}(Y/L)$. However, Maschke’s Theorem does not apply in positive characteristic if the characteristic of $L$ divides $|\text{Aut}(Y)|$ (as in our case), so it need not be the case that reducible implies decomposable. In this section we will show that, in fact, $H^1_{dR}(X/k)$ is indecomposable as a $k[G]$-module.

For these computations we will use the Čech cohomology with the open affine cover $U = \{U_1, U_2\}$, where $U_1 = X - P_0$ and $U_2 = X - P_\infty$ (for background on Čech hypercohomology, see [6] 0III §12 and [2] Exp V). The method used here is largely based on that in [7]. By definition, $\check{H}^1_{dR}(U)$ is the quotient of the $k$-vector space

$$\{ (\omega_1, \omega_2, f_{12}) \mid \omega_i \in \Omega^1_{X/k}(U_i), f_{12} \in \mathcal{O}_X(U_1 \cap U_2), df_{12} = \omega_1 - \omega_2 \}$$

by the $k$-subspace

$$\{ (df_1, df_2, f_1 - f_2) \mid f_i \in \mathcal{O}_X(U_i) \}.$$

Theorem 3.1. The canonical exact sequence of $k[G]$-modules

$$0 \to H^0(X, \Omega^1_{X/k}) \to H^1_{dR}(X/k) \to H^1(X, \mathcal{O}_X) \to 0$$

does not split.

To prove this, we will first find a $k$-basis for $\check{H}^1_{dR}(U)$ and study the action of $G$ on this basis.

Lemma 3.2. For $i = 0, 1, \ldots, g - 1$, let $\tau_i$ be the class of

$$\{ (x^iy^{-1}dx, x^iy^{-1}dx, 0) \}$$

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in $\hat{H}^1_{dR}(U)$. For $i = 1, 2, \ldots, g$, let $\eta_i$ be the class of

$$
(3.3) \quad ((1 - 2i)x^{1-g-i}d(yx^{-g-1}), -2ix^{2g-i}dy, yx^{-i})
$$

in $\hat{H}^1_{dR}(U)$. Together, these form a $k$-basis for $\hat{H}^1_{dR}(U)$.

Proof. We need to first determine that these are well-defined elements in $\hat{H}^1_{dR}(U)$. It is clear the $\tau_i$ are well defined in $H^1_{dR}(X/k)$ since $x^iy^{-1}dx$ is holomorphic for $i = 0, 1, \ldots, g - 1$ (note: the $\tau_i$ are the images of the $\tilde{\tau}_i \in \tilde{H}^1(U, \Omega^1_{X/k})$ under the canonical map). To show that $\eta_i$ is well defined, we first calculate that $d(yx^{-g-1}) = x^{g-1}dy$. So

$$
(1 - 2i)x^{1-g-i}d(yx^{-g-1}) - (-2ix^{2g-i}dy) = (1 - 2i)x^{-i}dy + 2ix^{2g-i}dy
$$

$$
= d(yx^{-i}).
$$

The requirement that $\omega_1 - \omega_2 = df_{12}$ for an element $(\omega_1, \omega_2, f_{12}) \in \hat{H}^1_{dR}(U)$ follows from this, but we still need to ensure that the $\omega_1, \omega_2$ and $f_{12}$ are holomorphic on the appropriate open sets. To do this, it is enough to calculate their divisors, which are:

$$
\text{div}((1 - 2i)x^{1-g-i}d(yx^{-g-1})) = (-2i)P_0 + (2g + 2i - 2)P_\infty
$$

$$
\text{div}(-2ix^{2g-i}dy) = 2(2g - i)P_0 + (2i - 2g - 2)P_\infty
$$

$$
\text{div}(yx^{-i}) = (1 - 2i)P_0 + \sum_{j=1}^{p-1} P_j + (2i - p)P_\infty
$$

Notice that the first equation refers to a differential holomorphic on $U_1$, the second a differential holomorphic on $U_2$, and the third a function holomorphic on $U_1 \cap U_2$; we conclude that the $\eta_i$s are well-defined in $\hat{H}^1_{dR}(U)$.

Certainly, the exact sequence in Equation (3.1) splits as a sequence of $k$-vector spaces. Since by Lemma 2.1 the $\tilde{\tau}_i$s form a basis for $\tilde{H}^0(U, \Omega^1_{X/k})$, and the $\tau_i$s are their images in $\tilde{H}^1_{dR}(U)$, it suffices to show that the image of $\{\eta_i | i = 1, \ldots, g\}$ in $\tilde{H}^1(U, \mathcal{O}_X)$ is a basis.

Recall that $H^1(X, \mathcal{O}_X)$ is the dual of $H^0(X, \Omega^1_{X/k})$, and that under the canonical Serre-duality pairing there is a map $H^0(X, \Omega^1_{X/k}) \times H^1(X, \mathcal{O}_X) \to k$ given by the residue map (see [12] or Appendix B of [4] for a complete discussion). We can use this to show that $-2yx^{-i}$ is the dual element to $y^{-1}x^{i-1}dx$. 

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For $i = 0, \ldots, g - 1$ and $j = 1, \ldots, g$ the pairing gives

\begin{equation}
\langle y^{-1}x^i dx, yx^{-j} \rangle = \text{res}(x^{i-j}dx).
\end{equation}

This can be calculated by summing the residues at points $P$ where $P \in U_1$, or equivalently (by applying the Residue Theorem), the negative of the residue at $P_0$. Note in particular this means for the inner product to be non-zero it must have a pole somewhere on $U_1$ and a pole at $P_0$. By the calculations in Lemma 2.1

\[ \text{div}(x^{i-j}dx) = (2i - 2j + 1)P_0 + P_0 + P_1 + \cdots + P_{p-1} + (2j - 2i - 3)P_{\infty}. \]

This has a pole at $P_0$ if $j \geq i + 1$ and a pole at $P_{\infty}$ if $j \leq i + 1$. It follows that $\langle y^{-1}x^i dx, yx^{-j} \rangle = 0$ if $i \neq j - 1$. If $i = j - 1$, then $\langle y^{-1}x^i dx, yx^{-j} \rangle = \text{res}_{P_{\infty}}(x^{-1}dx)$. To calculate this residue, note that $t = \frac{y}{x^{(p+1)/2}}$ is a uniformizer at $P_{\infty}$ (using the proof of Lemma 2.1). Since

\[ t^2 = \frac{y^2}{x^{p+1}} = \frac{x^p - x}{x^{p+1}} = \frac{1}{x} - \frac{1}{x^p} \]

we find

\[ \frac{1}{x} = t^2 + \left( \frac{1}{x} \right)^p, \]

from which it follows (by repeatedly substituting) that $x^{-1} = \sum_{i=0}^{\infty} t^{2^i}$. It is straightforward to show from this that $\text{res}_{P_{\infty}}(x^{-1}dx) = -2$.

Thus $\{-2yx^{-i} \mid i = 1, \ldots, g\}$ is the dual basis of the $\{\tilde{\tau}_{i-1} \mid i = 1, \ldots, g\}$, and in particular a basis for $H^1(X, \mathcal{O}_X)$.  

**Proof of Theorem 3.1** Now that we have a basis for $H^1_{dR}(X/k)$ as a split vector space, we need to consider how our group $G$ acts on these basis vectors. Specifically, we are going to consider $\sigma \in G$, the equivalence class of $((0 \ 1), 1)$, and its action on the span of $\{\eta_i \mid i = 1, \ldots, g\}$, the subspace (non-canonically) isomorphic to $H^1(X, \mathcal{O}_X)$. We will calculate the image of each $\eta_i$ under $\sigma$, and use this to show that there is no splitting of the sequence as $k[G]$-modules.

Here we run into the problem that our cover $U$ is not preserved under the group action. While $\sigma U_2 = U_2$, we get that $\sigma U_1 = X - P_{p-1}$. To account for
this, we will refine our cover. Let $U_3 = X - P_1$. Note that then $U_3 = \sigma^{-1}U_1$. Define the covers $\mathcal{U}' = \{U_2, U_3\}$ and $\mathcal{U}'' = \mathcal{U} \cup \mathcal{U}'$.

We can then calculate the action of $\sigma$ on $\tilde{H}^1_{dR}(\mathcal{U})$ using the following commutative diagram:

\[
\begin{array}{ccc}
H^1_{dR}(X/k) & \cong & \tilde{H}^1_{dR}(\mathcal{U}) \\
\downarrow \sigma & & \downarrow \rho \\
H^1_{dR}(X/k) & \cong & \tilde{H}^1_{dR}(\mathcal{U}') \\
\end{array}
\]

where $\rho, \rho'$ are the restriction maps, which give isomorphisms of the spaces. As we did with $\tilde{H}^1_{dR}(\mathcal{U})$, we have a similar definition for $\tilde{H}^1_{dR}(\mathcal{U}'')$, namely as the quotient of

\[
\left\{ (\omega_1, \omega_2, \omega_3, f_{12}, f_{13}, f_{23}) \mid \omega_j \in \Omega_{X/k}(U_j), f_{jk} \in \mathcal{O}_X(U_j \cap U_k) \right. \\
\left. df_{jk} = \omega_j - \omega_k, f_{23} - f_{13} + f_{12} = 0, \right\}
\]

by the subspace spanned by

\[
\left\{ (df_1, df_2, df_3, f_1 - f_2, f_1 - f_3, f_2 - f_3) \mid f_j \in \mathcal{O}_X(U_j) \right\}.
\]

Then $\rho$ and $\rho'$ are respectively the projections to $(\omega_1, \omega_2, f_{12})$ and $(\omega_2, \omega_3, f_{23})$.

**Lemma 3.3.** For $i = 1, \ldots, g$, let

\[
\begin{align*}
\omega_{1i} &= (1 - 2i)x^{-g-i+1}d(yx^{-g-1}) \\
\omega_{2i} &= -2i x^p y^{i-1}dy \\
\omega_{3i} &= \sum_{m=1}^{p-i} \left( \binom{p-i}{m} (1 + 2m)(x-1)^{m-3g}d(y(x-1)^{-g-1}) \\
f_{12i} &= yx^{-i} \\
f_{23i} &= \sum_{m=1}^{p-i} \left( \binom{p-i}{m} y(x-1)^{m-p} = \left( \frac{x^m - 1}{x^p - 1} \right) y \\
f_{13i} &= yx^{-i} + \left( \frac{x^m - 1}{x^p - 1} \right) y.
\end{align*}
\]

Let $\nu_i = (\omega_{1i}, \omega_{2i}, \omega_{3i}, f_{12i}, f_{13i}, f_{23i})$. If $i = 1, \ldots, g$, then $\nu_i$ is a well-defined hyper 1-cocycle for the covering $\mathcal{U}''$. Moreover, under the canonical refinement map $\tilde{H}^1_{dR}(\mathcal{U}'') \sim \tilde{H}^1_{dR}(\mathcal{U})$, the image of each $\nu_i$ is $\eta_i$. 

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Proof. Since it is clear that the projection of $\nu_i$ to $\check{H}^1_{dR}(U)$ is $\eta_i$, it suffices to show that each $\nu_i$ is well-defined.

Since we know from Lemma 3.2 that the $\eta_i$ are well-defined, we can conclude that $\omega_{1i} \in \Omega_{X/k}(U_1), \omega_{2i} \in \Omega_{X/k}(U_2), f_{12i} \in \mathcal{O}_X(U_1 \cap U_2)$, and $df_{12i} = \omega_{1i} - \omega_{2i}$ for $i = 1, \ldots, g$. It then remains to verify the following conditions:

1. $\omega_{3i} \in \Omega_{X/k}(U_2)$
2. $f_{23i} \in \mathcal{O}_X(U_2 \cap U_3)$
3. $f_{13i} \in \mathcal{O}_X(U_1 \cap U_3)$
4. $df_{23i} = \omega_{2i} - \omega_{3i}$
5. $df_{13i} = \omega_{1i} - \omega_{3i}$
6. $f_{23i} - f_{13i} + f_{12i} = 0$

Note that 6 is clear from the definition, and that with the other conditions it will imply 5. Next we will show that conditions 1, 2 and 4 on $f_{23i}$ and $\omega_{3i}$ follow from previous calculations. We know from Lemma 3.2 that $(1 - 2i)x^{1-g-i}d(yx^{-g-1}) \in \Omega_{X/k}(U_1), -2ix^{2g-i}dy \in \Omega_{X/k}(U_2)$, and that their difference is $d(yx^{-i}) \in \Omega_{X/k}(U_1 \cap U_2)$. Changing coordinates by replacing $x$ with $x - 1$, we can conclude that differentials that previously were holomorphic on $U_1$ will now be holomorphic on $U_3$, while those holomorphic on $U_2$ remain so, and that the equality still holds. Thus we get that

$$(3.5) \ (1 - 2i)(x - 1)^{1-g-i}d(y(x-1)^{-g-1}) + 2i(x-1)^{2g-i}dy = d(y(x-1)^{-i})$$

where

$$(1 - 2i)(x - 1)^{1-g-i}d(y(x-1)^{-g-1}) \in \Omega_{X/k}(U_3)$$
$$-2i(x - 1)^{2g-i}dy \in \Omega_{X/k}(U_2)$$
$$y(x - 1)^{-i} \in \mathcal{O}_X(U_2 \cap U_3).$$

If we substitute $i = p - m$ into (3.5) (recalling that $p = 2g + 1$), it becomes

$$(3.6) \ (1+2m)(x-1)^{m-3g}d(y(x-1)^{-g-1}) - 2m(x-1)^{m-1}dy = d(y(x-1)^{m-p}).$$
Now take (3.6), multiply it by \( \binom{p - i}{m} \), take the sum of these from \( m = 1 \) to \( m = p - i \). Referencing back to the definitions of \( \omega_{3i} \) and \( f_{23i} \), we find

\[
-\omega_{3i} + \sum_{m=1}^{p-1} \binom{p - 1}{m} 2m(x - 1)^{m-1}dy = df_{23i}.
\]

It follows from this that conditions 1 and 2 hold, that is \( \omega_{3i} \in \Omega_{X/k}(U_2) \) and \( f_{23i} \in \mathcal{O}_{X/k}(U_2 \cap U_3) \). Now note that

\[
\omega_{2i} = -2ix^{p-i-1}dy = -2i \sum_{\ell=0}^{p-i-1} \left( \binom{p - i - 1}{\ell} (x - 1)^{\ell} dy \right) = \sum_{m=1}^{p-i} \left( \binom{p - i}{m} 2m(x - 1)^{m-1} dy. \right.
\]

So we can substitute this into (3.7), and it follows that \( \omega_{2i} - \omega_{3i} = df_{23} \), which is condition 4.

The only thing that remains is to check that \( f_{13i} \in \mathcal{O}_X(U_1 \cap U_3) \). Since \( f_{23i} \) and \( f_{12i} \) are holomorphic on \( U_1 \cap U_2 \cap U_3 \), we need only check that \( f_{13i} \) is holomorphic at \( P_\infty \). We can do this by calculating the image of \( f_{13i} \) in \( \text{Frac}(\mathcal{O}_{X,P_\infty}) \). Recall from the proof of Lemma 3.2 that \( t = \frac{y}{x^{(p+1)/2}} \) is a uniformizer at \( P_\infty \). Using this and the definition of \( f_{13i} \), we calculate that \( f_{13i} = O(t) \). Since this means \( f_{13i} \) has no pole at \( P_\infty \), we can conclude \( f_{13i} \) is holomorphic on \( U_1 \cap U_3 \). So \( f_{13i} \in \mathcal{O}_X(U_1 \cap U_3) \). \( \Box \)

Now we conclude the proof of Theorem 3.1. By Lemma 3.3 each \( \nu_i \) is the inverse image of \( \eta_i \) under the canonical restriction map \( \tilde{H}^1_{dR}(U') \rightarrow \tilde{H}^1_{dR}(U) \). The projection of \( \nu_i \) onto \( \tilde{H}^1_{dR}(U') \) gives us \( (\omega_{2i}, \omega_{3i}, f_{23i}) \), and to understand \( \sigma_\eta \), we need only calculate \( \sigma(\omega_{2i}, \omega_{3i}, f_{23i}) \).
So we need to consider the image of \((\omega_{3i}, \omega_{2i}, f_{23i})\) under \(\sigma\).

\[
\sigma \omega_{3i} = -\frac{i}{p-i} \sum_{j=1}^{p-i} \binom{p-i}{j} (1 + 2j)x^j y^j (yx^{-g-1})
\]

\[
\sigma \omega_{2i} = -2i(x+1)^{p-i-1} dy = -\frac{i}{p-i} \sum_{j=1}^{p-i} \binom{p-i}{j} 2jx^j dy
\]

\[
\sigma f_{23i} = -\frac{i}{p-i} \sum_{j=1}^{p-i} \binom{p-i}{j} yx^j
\]

We can restate this as

\[
\sigma(\omega_{3i}, \omega_{2i}, f_{23i}) = -\frac{i}{p-i} \sum_{j=1}^{p-i} \binom{p-i}{j} \tilde{\eta}_{p-j}
\]

where

\[
\tilde{\eta}_\ell = ((1 - 2\ell)x^{1-g-i} d(yx^{-g-1}), -2\ell x^{2g-\ell} dy, yx^{-\ell}).
\]

We would like to be able to write this in terms of the basis from Lemma 3.2. If \(g + 1 \leq j \leq p - 1\), then each \(\tilde{\eta}_{p-j} = \eta_{p-j}\), which is such a basis element. However, for \(1 \leq j \leq g\), this is not the case and so we need to rewrite \(\tilde{\eta}_{p-j}\) in terms of the basis. We find that for \(1 \leq j \leq g\),

\[
\tilde{\eta}_{p-j} - (d(yx^{j-p}), 0, yx^{j-p}) = (2jx^{j-1} dy, 2jx^{j-1} dy, 0) = -2j \tau_{j-1}.
\]

Since the above is in the same equivalence class as \(\tilde{\eta}_{p-j}\) within \(\tilde{H}_{dR}^1(U)\), for \(1 \leq j \leq g\), \(\eta_{p-j}\) is in the image of \(H^0(X, \Omega^1_{X/k})\). This shows that the space spanned by the \(\eta_i\) is not stable under the action of \(\sigma\).

Suppose there is a \(k[G]-linear map \alpha : H^1(X, \mathcal{O}_X) \to H^1_{dR}(X/k)\) that splits the exact sequence in Equation (3.1). Let \(\{\tilde{\tau}_\ell \in H^1(X, \mathcal{O}_X) \mid \ell = 0, \ldots, g-1\}\) be a dual basis of \(\{\tilde{\tau}_\ell \in H^0(X, \Omega^1_{X/k}) \mid \ell = 0, \ldots, g-1\}\) under the Serre duality pairing. Then define \(\tau^*_\ell = \alpha(\tilde{\tau}_\ell) \in H^1_{dR}(X/k)\). We would like to know how to write \(\tau^*_\ell\) in terms of the \(\tau_i\) and \(\eta_i\), the basis given in Lemma 3.2.

Fix \(t \in \mathbb{F}_p^\times\), and let \(T\) indicate the element of \(G\) such that

\[
(x, y) \mapsto (t^2 x, t y).
\]
It is easy to calculate that \( T\tau_i = t^{2i+1} \tau_i \) and \( T\eta_j = t^{1-2j} \eta_j \). Since the pairing is \( G \)-equivariant, as is the map \( \alpha \), it has to be the case that \( T\tau^{*}_i = t^{-2\ell-1} \tau^{*}_i \).

This implies that \( \tau^{*}_i \) is in the \( t^{-2\ell-1} \)-eigenspace of \( T \), which is spanned by \( \tau_{g-\ell-1} \) and \( \eta_{\ell+1} \). So there exist unique nontrivial \( a_\ell, b_\ell \in k \) such that \( \tau^{*}_i = a_\ell \tau_{g-\ell-1} + b_\ell \eta_{\ell+1} \).

Since \( \alpha \) is \( G \)-equivariant, \( \sigma \tau^{*}_i = \sigma \alpha(\tau^{*}_i) = \alpha(\sigma \tau^{*}_i) \), so it must be that

\[
a_\ell \sigma \tau_{g-\ell-1} + b_\ell \sigma \eta_{\ell+1} = \sigma \tau^{*}_i \in \text{im} \alpha.
\]

However, we know from the proof of Theorem 2.2 and Equation 3.8 that

\[
\sigma \tau_{g-\ell-1} = \sum_{i=0}^{g-\ell-1} \binom{g-\ell-1}{i} \tau_i
\]

and

\[
\sigma \eta_{\ell+1} = -\frac{\ell+1}{p-\ell-1} \left( -2 \sum_{i=1}^{g} \binom{p-\ell-1}{i} i \tau_{i-1} + \sum_{i=g+1}^{p-\ell-1} \binom{p-\ell-1}{i} \eta_{p-i} \right).
\]

It is simple to show, by plugging this into Equation (3.9), that no non-trivial \( a_\ell, b_\ell \) satisfy this, giving a contradiction. Thus the exact sequence in Equation (3.1) does not split under the action of \( G \).

\[\square\]

**Corollary 3.4.** \( H^1_{dR}(X/k) \) is a non-projective indecomposable \( k[G] \)-module.

**Proof.** Corollary 7 on page 33 of [1] states that the order of a Sylow \( p \)-group of \( G \) divides the dimension of any projective \( k[G] \)-module. A Sylow \( p \)-group of \( G \) has order \( p \) and the dimension of \( H^1_{dR}(X/k) \) is \( 2g = p-1 \). So \( H^1_{dR}(X/k) \) is non-projective.

Suppose there exist nontrivial \( k[G] \)-submodules \( M \) and \( N \) such that

\[ H^1_{dR}(X/k) = M \oplus N. \]

Consider the sequence from Theorem 3.1

\[
0 \to H^0(X, \Omega^1_{X/k}) \xrightarrow{\psi} H^1_{dR}(X/k) \xrightarrow{\psi'} H^1(X, \mathcal{O}_X) \to 0
\]

where \( \psi, \psi' \) are the canonical maps. Suppose that \( \text{im} \psi \cap M = 0 \). Then \( \ker \psi' \cap M = 0 \), so \( \psi'|M \) gives an isomorphism between \( M \) and some submodule of \( H^1(X, \mathcal{O}_X) \). However, since \( H^1(X, \mathcal{O}_X) \) is irreducible by Corollary 2.3.
this means that $M$ is isomorphic to $H^1(\mathcal{O}_X)$ as a $k[G]$-module. However, this gives a splitting of the exact sequence in (3.10), which contradicts Theorem 3.1.

So suppose that $\text{im} \psi \cap M$ is nonzero. Since $\text{im} \psi \cong H^0(X, \Omega^1_{X/k})$ as a $k[G]$-module, and $H^0(X, \Omega^1_{X/k})$ is irreducible by Corollary 2.3, we can conclude that $\text{im} \psi \subseteq M$. So $\ker \psi' \subseteq M$, and thus $\ker \psi' \cap N$ is zero. Thus, $N$ is isomorphic to a $k[G]$-submodule of $H^1(X, \mathcal{O}_X)$ through $\psi'$. Since $N$ is nontrivial and $H^1(X, \mathcal{O}_X)$ is irreducible by Corollary 2.3, this means that $\psi'$ induces an isomorphism between $N$ and $H^1(X, \mathcal{O}_X)$. However, this gives a splitting of the exact sequence in (3.10), which contradicts Theorem 3.1. □

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