Primordial perturbation with a modified dispersion relation

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In this paper we study the generation of primordial perturbation with a modified dispersion relation in various cosmological evolutions. We stress that the formation of the power spectrum is strongly dependent on the background. Working in a bounce model with a matter-like contracting phase, we obtain a red tilt spectrum due to the modified dispersion relation.

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I. INTRODUCTION

The nature of primordial perturbation$^1$, responsible for the formation of large scale structure of the universe, has been intensively studied in the literature. The cosmological observations favors an adiabatic and nearly scale-invariant spectrum of primordial perturbation$^2$, as predicted by an inflation model$^3$ with a standard dispersion relation. However, advocated by the development of quantum gravity, one may question on the initial state of inflation and the perturbations generated in this model. As a possible consequence of quantum gravity, the dispersion relation of primordial perturbation may be modified$^4,5,6,7$. The studies around modified dispersion relations were earlier addressed in DBI inflation$^8$ by introducing a sound speed parameter, which has shown plentiful phenomenons$^9,10,11,12$. A modified dispersion relation can also be derived in the models of modified gravity, such as trans-Planckian physics$^{13}$, noncommutative field approach$^{14}$, loop quantum gravity$^{15}$, Lorentz-violating models$^{16,17,18}$, and so on. Recently, within the frame of a non-relativistic gravity model$^{19}$, several works have appeared to claim that a scale-invariant spectrum can be obtained without carefully considering the background cosmological evolution$^{20}$ and even without matter components$^{21,22}$; however, also see Refs. $^{23,24,25}$ for reexaminations. This is an interesting issue in which a modified dispersion relation was applied, but one ought to take the background cosmological evolution into account seriously. Assuming a mode of cosmological fluctuations emerges at the beginning of the universe, the form of its initial condition strongly depends on the background. Moreover, if this mode is responsible for the formation of the large scale structure at late times, it has to experience a process of exiting the Hubble scale in aim of freezing its oscillating behavior$^{26}$. This process is also determined by the background evolution.

In the current paper, we study the generation of the primordial perturbation by introducing a modified dispersion relation phenomenologically. Through following one mode of these perturbations along with the cosmological evolutions, we show that, a modified dispersion relation for primordial perturbations can not make their generations be independent on the background evolution, but changes the way of their freezing. Especially, if the universe is preceded by a bounce$^{27}$, the method of seeding a scale-invariant spectrum with a modified dispersion relation can only be applicable in a fine-tuned regime.

This paper is organized as follows. In Section II we show how the primordial fluctuations emerged inside the Hubble radius can seed the cosmological inhomogeneities when the dispersion relation is modified. In Section III we perform the detailed calculation of primordial perturbations by introducing a modified dispersion relation phenomenologically. Specifically, we study the spectrum in a pure expanding universe and a bouncing universe respectively. Our results show that the spectrum obtained in a pure expanding phase can be spoiled by a cosmological bounce$^{28}$. In Section IV we study the primordial perturbation with a modified dispersion relation in a specific bounce model and obtain a red tilt power spectrum. Section V is our conclusion and discussions.

II. SEEDING PRIMORDIAL PERTURBATIONS

We begin with a brief discussion on the cosmological evolution of primordial perturbations in the frame of an expanding universe. Due to the fact of cosmological inhomogeneities observed in the universe, it requires that the cosmological fluctuations initially emerge inside a Hubble radius, and then leave it in the primordial epoch, and finally reenter at late times. In an expanding universe, with a standard dispersion relation, this process can usually be realized by stretching the physical wavelength $\frac{2}{t}$ longer than the Hubble radius $\frac{1}{t}$. One may take the scale factor as

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^p, \text{ with } p = \frac{2}{3(1+w)}, \quad (1)$$

where $w$ is the background equation-of-state, and the subscript "0" denotes any fixing point along the cosmological evolution. Then the Hubble parameter takes the form

$$H = \frac{p}{t}. \quad (2)$$

\begin{equation}
\end{equation}
From the above equations, one can find that, for a mode of perturbation with the standard dispersion relation to exit the Hubble radius in an expanding universe, there has to be the background equation-of-state \( w < -\frac{1}{3} \), such as inflation. It may be characterized by an e-folding number parameter, which is defined as,

\[
N \equiv \ln \left( \frac{kf}{k_i} \right),
\]

where \( k \) is the comoving wave number, and the subscript "i" and "f" denotes the earliest and last modes generated in primordial epoch. For example, from current observations, this parameter is required to be around 60.

However, the dispersion relation may be modified at high energy scales, for example, due to trans-Planckian physics, Lorentz violating effects. If it happens, the scenario of the above generation for the primordial perturbation has to be changed correspondingly. Now we phenomenologically introduce a modified dispersion relation

\[
\nu = k f(k_{ph}) ,
\]

with

\[
f(k_{ph}) = \begin{cases} 
(k_{ph}/M)^\alpha, & k_{ph} > M \\
1, & k_{ph} \leq M 
\end{cases}
\]

where \( k_{ph} = \frac{k}{a} \) is a physical wave number and assume \( \alpha > 0 \). We suggest there is an energy scale \( M \) for the modification, so that the standard perturbation theory can be recovered at low energy scales.

Suppose we still work in the frame of an expanding universe. In order to realize the modes with a modified dispersion relation \( \nu \) to exit the Hubble radius, it requires that \( M^\alpha (\frac{k}{M})^{1+\alpha} \) grows faster than the Hubble radius \( \frac{1}{H} \). Therefore, there is a condition

\[
w < -\frac{1}{3} + \frac{2\alpha}{3},
\]

for the background equation-of-state. One may notice that, the modes are much easier to escape the Hubble radius when \( \alpha > 0 \), and thus it could be unnecessary to require an inflationary period at early times. For a recent gravity model, proposed by Hořava, \( \alpha \) is equal to 2 and the perturbations are able to exit the Hubble radius just requiring \( w < 1 \). In this case, it seems that requirements on the background evolution becomes quite loose, but this claim is not suitable. At least, we need the universe is expanding with all the perturbation modes emerge at a finite initial moment.

Moreover, if the initial big bang singularity is replaced by a big bounce and the universe has experienced a contracting phase, the evolution of primordial perturbations will be changed totally\(^1\). In this case, the scale factor is very large initially and so the modes of perturbation are almost in the infrared (IR) regime with \( k_{ph} \leq M \). Along with the contraction, these modes may firstly enter the ultraviolet (UV) regime with \( k_{ph} > M \), and then exit the Hubble radius; or, they exit the Hubble radius from the IR regime directly. We give a sketch plot to describe this scenario in Figure 1. From the figure, one can read that there are three critical moments,

\[
\begin{align*}
t_{IR}(k) &= t_0^I \left( \frac{k}{a_0} \right)^{\frac{1}{1+\alpha}}, \\
t_{UV}(k) &= t_0^I \left( \frac{k^{1+\alpha}}{pM^\alpha a_0^{1+\alpha}} \right)^{\frac{1}{p(p+1)\alpha}}, \\
t_{UI}(k) &= t_0 \left( \frac{k}{M_{a_0}} \right)^{\frac{1}{p}},
\end{align*}
\]

which are IR horizon crossing time \( t_{IR} \), UV horizon crossing time \( t_{UV} \), and UV/IR crossing time \( t_{UI} \) respectively. These three moments overlap at a triple point \( t_{tp} = \frac{k}{M} \), and correspondingly the comoving wave number of the critical mode takes the value \( k_{tp} = a_0 M^{1-p}pt_0^{p} \). Only for the modes with \( k < k_{tp} \), the initial conditions could be changed by the modified dispersion relation; while for those with \( k \geq k_{tp} \), their primordial spectrum is the same as that obtained with a standard dispersion relation\([32]\).

\( \text{FIG. 1: A sketch plot of the cosmological evolution for the primordial perturbations with a modified dispersion relation.} \)

The condition for the generation of perturbation in a contracting universe is exactly opposite to that in an expanding universe as shown in Eq. (6). It requires \( w > -\frac{1}{3} + \frac{2\alpha}{3} \) for the perturbation escaping the Hubble radius in a contracting phase. Therefore, the case of \( \alpha = 2 \) suggests an Ekpyrotic phase is needed for the generation of primordial spectrum. Besides, in order to make the modes with \( k > k_{tp} \) relevant for current observable ones, the evolution of the universe can not be symmetric to the bounce point. Namely, one may intro-

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\(^1\) A cosmological bounce may spoil the usual results obtained in an inflation model as studied in Refs. [21, 30, 31].
duce an inflationary stage after the bounce to stretch the UV modes into IR regime on super-Hubble scales.

III. THE SPECTRUM OF PRIMORDIAL PERTURBATION WITH A MODIFIED DISPERSION RELATION

To make the above analysis specific, we do the calculation of the spectrum of primordial perturbation with a modified dispersion relation. To seed the fluctuations, we consider a test scalar field $\phi$ which does not contribute to the background evolution, but only provides a degree of freedom for scalar perturbations $\delta \phi$. In the Fourier space, the equation of motion for its fluctuation $\delta \phi$ is given by

$$v_k'' + (\nu^2 - \frac{a''}{a})v_k = 0 , \quad (8)$$

where the variable $v_k$ is defined by $v_k \equiv a\delta \phi_k$, and the prime denotes the derivative with respect to the comoving time $\eta \equiv \int \frac{1}{a}$. For a constant equation-of-state $w$, there is

$$\frac{a''}{a} = \frac{\gamma}{\eta^2}, \quad \text{with } \gamma = \frac{p(2p - 1)}{(1-p)^2}. \quad (9)$$

Eq. (8) has an asymptotic solution when we neglect the last term $\frac{a''}{a}$ which implies $|\nu \eta| \gg 1$, and it is strongly oscillating like trigonometric functions. This feature coincides with an adiabatic condition $|\nu'/\nu^2| \gg 1$, which corresponds to the case that the effective physical wavelength is deep inside the Hubble radius. Therefore, the modes can be regarded as adiabatic when they are staying in the sub-Hubble regime with $|\nu \eta| \gg 1$, and we may impose a suitable initial condition in virtue of WKB approximation,

$$v_k^i \sim \frac{1}{\sqrt{2\nu(k, \eta)}} e^{-i \int^{\eta} \nu(k, \eta) d\eta}, \quad (10)$$

for cosmological fluctuations.

We would like to analyze the effective mass term $\frac{a''}{a}$ closely. As discussed in Sec. III the generation of primordial fluctuations strongly depends on the background evolution. In suitable environments, the variable $|\nu \eta|$ could decrease along with the expansion. Once there is $|\nu \eta| \ll 1$, the modes will exit the Hubble radius, and the equation of motion yields another asymptotic solution of which the leading term takes

$$v_k^i \sim \eta^{\frac{1}{2}} \left[ c(k) \nu^{-\frac{1}{2}} \int^{\nu \eta} \phi(k, \eta) d\eta \right]. \quad (11)$$

From the above expression, one may observe that this solution can seed appropriate perturbation spectrum at super-Hubble regime. Now we match two asymptotic solutions (10) and (11) at the moment of Hubble radius crossing $|\nu \eta| \sim 1$, and thus determine the form of $v_k$ at super-Hubble scales,

$$v_k(\eta) \sim \frac{1}{\sqrt{2\nu}} \left( \nu(k, \eta) \right)^{\frac{1}{2} - \frac{1}{2}(1-p)(1+\alpha)} . \quad (12)$$

Applying the solution (12) directly, we obtain the primordial spectrum as follows,

$$P_{\delta \phi} = \frac{k^3}{2\pi^2} \frac{v_k(\eta)^2}{a} \sim \left\{ \begin{array}{ll}
\frac{k^{1+\alpha}}{(1-p)(1+\alpha)}, & (1 - 3p)[1 - p(1 + \alpha)] \geq 0 \\
\frac{k^{1+\alpha}}{p(1-p)(1+\alpha)}, & (1 - 3p)[1 - p(1 + \alpha)] < 0 .
\end{array} \right. \quad (13)$$

Finally, we obtain the power spectrum of primordial perturbations without considering any specific models. From the result (13), we find out that there are four sufficient conditions for the spectrum to be scale-invariant. They are:

- (a) $p \to +\infty$ in the first case, which corresponds to an inflation model $[3]$;
- (b) $p \to -\infty$ in the first case, which may be realized in island cosmology $[33, 34]$;
- (c) $\alpha = 2$ in the first case, and this case coincides with the result obtained in the model of Horváth-Lifshitz cosmology $[35, 36]$;
- (d) $p = \frac{1 + \alpha}{1 + \alpha}$ in the second case.

In the following we will study the spectrum of primordial perturbation in concrete background evolutions. Specifically, we consider a purely expanding universe and a bouncing universe respectively.

A. In an expanding universe

For a pure expanding universe, a mode of cosmological perturbation emerge inside one Hubble radius when the universe was born, and then it is able to enter the super-Hubble regime when the condition (10) is satisfied. This condition implies $p > \frac{1}{1+\alpha}$ or $p < 0$. Obviously, the first three conditions can be realized in an expanding universe naturally. The last condition can also be satisfied when we choose $\alpha > 2$ and $\frac{1}{1+\alpha} < p \leq \frac{1}{3}$.

B. In a bouncing universe

Now we work in the frame of bouncing cosmology which may be realized by a model of non-relativistic gravity, namely in $[37]$. In this scenario the scale factor $a$ could be quite large at early times, and so all the modes of primordial perturbation stay in the IR regime initially, as shown in Figure I.
As is analyzed in Section III, a mode with comoving wave number \( k \leq k_{PI} \) can escape outside the Hubble radius in the IR regime directly if \( w > -\frac{1}{3} \). In this case, the dispersion relation is in standard form \( \nu = k \) and so the spectrum can be scale-invariant only in a model of matter bounce with \( p = \frac{3}{2} \) (see also \[40\]).

However, for the modes with \( k > k_{PI} \), the condition of generating primordial perturbation is \( w > -\frac{1}{3} + \frac{2\alpha}{3} \) and so requires \( 0 < p < \frac{1}{1+\alpha} \). Therefore, the sufficient condition (c) implies that \( \nu \) is scale-invariant spectrum in UV regime can be obtained when \( \alpha = 2 \) and \( w > 1 \). Moreover, we can take \( \alpha < 2 \) and \( \frac{1}{2} < p < \frac{1}{1+\alpha} \) to give a scale-invariant spectrum in virtue of the condition (d).

The conditions leading to scale-invariant power spectra in an expanding or contracting phase are summarized in Table I as shown in the following.

\[
P_{\delta\phi}^{IR} = \left( \frac{H_0}{4\pi} \right)^2,
\]

where we have placed the subscript “0” around the bounce point as shown in Figure 1.

Now we deal with the UV modes of which the comoving wave numbers satisfy \( k > k_{PI} \). In order for the UV modes leaving the Hubble radius, the parameter \( \alpha \) has to be less than \( \frac{1}{2} \). According to Table I, we can find that these modes can not give rise to an exactly scale-invariant spectrum. Consequently, we turn to resolve the basic equation of motion for the perturbation directly, and obtain the following solution

\[
\delta_k \approx \frac{1}{\sqrt{2\nu}} \left( \nu \eta \right)^{-\frac{1+\alpha}{1-\alpha}},
\]

in UV regime. From this solution, one can read that the modes at super-Hubble scales are growing. This growth would stop when the bounce takes place, and thus we can calculate the power spectrum around the bounce point, which is given by:

\[
P_{\delta\phi}^{UV} = \frac{1}{2} \frac{2^{2+2\alpha}}{\pi^2} H^{2(1+\alpha)} M^{3\alpha} \left( \frac{a_0}{k} \right)^{3\alpha}.\]

The corresponding spectral index can be derived as follows,

\[
n_{\phi} = 1 + \frac{d\ln P_{\delta\phi}}{d\ln k} = 1 - \frac{9\alpha}{1 - 2\alpha},
\]

which is red tilt in UV regime. In the following, we plot the spectral index of the primordial perturbation in this model in Figure 2. One may notice that, in order to fit the cosmological observations, for example Wilkinson Microwave Anisotropy Probe (WMAP) data \[44\], the parameter \( \alpha \) can not deviate from zero too much which provides a strong constrain on models of modified gravity. And these results indicate that a slightly red spectrum can be obtained by making a very small modification on the dispersion relation in a matter bounce model.

\section{IV. A FEATURED POWER SPECTRUM IN A BOUNCE MODEL}

In this section we will study the cosmological perturbation with a modified dispersion relation in a bounce model \[32\]. Specifically, we consider the model with an universe evolving from a matter-like contracting phase \[41\], of which the equation-of-state takes \( w = 0 \) and so gives \( p = \frac{3}{2} \). In this model, the perturbation with a standard dispersion relation has been studied in details as shown in Refs. \[31, 41\] and its non-Gaussianities were studied by Refs. \[42, 43\]. The modes in IR regime coincide with them, of which the spectrum is as follows \[42\],

\[
P_{\delta\phi}^{IR} = \left( \frac{H_0}{4\pi} \right)^2,
\]

where we have placed the subscript “0” around the bounce point as shown in Figure 1.

Now we deal with the UV modes of which the comoving wave numbers satisfy \( k > k_{PI} \). In order for the UV modes leaving the Hubble radius, the parameter \( \alpha \) has to be less than \( \frac{1}{2} \). According to Table I, we can find that these modes can not give rise to an exactly scale-invariant spectrum. Consequently, we turn to resolve the basic equation of motion for the perturbation directly, and obtain the following solution

\[
\delta_k \approx \frac{1}{\sqrt{2\nu}} \left( \nu \eta \right)^{-\frac{1+\alpha}{1-\alpha}},
\]

in UV regime. From this solution, one can read that the modes at super-Hubble scales are growing. This growth would stop when the bounce takes place, and thus we can calculate the power spectrum around the bounce point, which is given by:

\[
P_{\delta\phi}^{UV} = \frac{1}{2} \frac{2^{2+2\alpha}}{\pi^2} H^{2(1+\alpha)} M^{3\alpha} \left( \frac{a_0}{k} \right)^{3\alpha}.\]

The corresponding spectral index can be derived as follows,

\[
n_{\phi} = 1 + \frac{d\ln P_{\delta\phi}}{d\ln k} = 1 - \frac{9\alpha}{1 - 2\alpha},
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\section{V. CONCLUSIONS AND DISCUSSION}

Modifications on the dispersion relation of a cosmological scalar field have explored new approaches to generating the primordial perturbations in various background evolutions. In the current paper we have studied in details on the conditions for these perturbations to form scale-invariant spectra. Especially, we point out that the generation of primordial spectrum can not be independent on the background evolution even in the model of Hořava gravity with \( \alpha = 2 \) in our case. These perturbations may be transferred to the curvature perturbation at late times by certain mechanisms, such as curvaton \[46, 47\] or modulated reheating \[48, 49\], and so
TABLE I: Various possibilities of generating a scale-invariant primordial spectrum in an expanding or contracting universe.

| Parameter | Expanding | Contracting |
|-----------|-----------|-------------|
| $\alpha = 0$ IR | $p \rightarrow +\infty$ | $p = 4$ |
| $0 < \alpha < 2$ UV | $p \rightarrow +\infty$ | $p = \frac{2(1+\alpha)}{6(1+\alpha)}$ |
| $\alpha = 2$ UV | $p > \frac{1}{1+\alpha}$ or $p < 0$ | $0 < p < \frac{2}{1+\alpha}$ |
| $\alpha > 2$ UV | $p = \frac{1}{(1+\alpha)}$ or $p \rightarrow +\infty$ | N/A |

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