A fast algorithm for solving diagonally dominant symmetric quasi-pentadiagonal Toeplitz linear systems

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Abstract

In this paper, we develop a new algorithm for solving diagonally dominant symmetric quasi-pentadiagonal Toeplitz linear systems. Numerical experiments are given in order to illustrate the validity and efficiency of our algorithm.

Keywords: Quasi-pentadiagonal Toeplitz matrix, Diagonally dominant, LU decomposition.

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1. Introduction

In this paper, we will focus on the problem of solving

\begin{equation}
Tx = f
\end{equation}

where $T$ is a quasi-pentadiagonal Toeplitz matrix.

An $n \times n$ matrix $T = (t_{ij})$ is said to be Toeplitz if $t_{i,j} = t_{i-j}$. $T$ is said to be banded Toeplitz if there are positive integers $p$ and $q$ such that $p + q = k < n$ and $t_{v} = 0$ if $v > q$ or $v < -p$. A banded quasi-Toeplitz matrix is defined to be a banded Toeplitz matrix where there are at most $p$ altered rows among the first $p$ rows and at most $q$ altered rows among the last $q$ rows. For example, when $p = q = 1$, and only the first row and the last row of $T$ are perturbed, then $T$ is said to be quasi-tridiagonal Toeplitz matrix, the numerical solution of $Tx = f$ for this kind of linear equations was studied by [1], and more general, the numerical solution of block quasi-tridiagonal Toeplitz matrix was studied by [2]. Here we will study the case when $p = q = 2$, and only the first two rows and the last two rows of $T$ are perturbed i.e., when $T$ is a quasi-pentadiagonal Toeplitz matrix.

Pentadiagonal matrices and quasi-pentadiagonal matrices frequently arise in many application areas, such as computational physics, scientific and engineering computations [3, 4, 5, 6], as well as in the wave-function formalism [7] and density functional theory [8] in quantum chemistry. The importance of these applications motivated an extensive theoretical study of these kinds of matrices, such as determinant evaluation, eigenvalues computing and pentadiagonal linear systems solving in the last decades, see for example [9, 10] and a large literature therein.

In this work, we will present a fast algorithm for the numerical solution of an $n \times n$, nonsingular, diagonally dominant, symmetric quasi-pentadiagonal Toeplitz linear system. In other words, the coefficient matrix of (1) is
and

$$|a| > 2(|b| + |c|), \ c \neq 0.$$  \hfill (2)

When \(x = q = k = h = a, \ s = z = t = g = c, \) and \(d = w = e = p = r = y = b,\) the matrix \(T\) becomes a symmetric pentadiagonal Toeplitz matrix. This case was studied in [11, 12, 13]. For the general case of nonsymmetric pentadiagonal linear systems, algorithms have been introduced in [14, 15].

In the following sections, we will introduce an algorithm for solving the diagonally dominant symmetric quasi-pentadiagonal Toeplitz linear systems (1). Then present the numerical results.

2. An algorithm for solving quasi-pentadiagonal Toeplitz linear systems

In general, we can deal with the quasi-pentadiagonal Toeplitz linear systems (1) as an usual linear systems, and solve it by the \(LU\) decomposition without pivoting. Here we give an alternative choice, we factor the quasi-pentadiagonal Toeplitz into the following form

$$T = LU + SV + PQ$$  \hfill (3)

where

$$L = \begin{bmatrix} 1 & l_2 & 1 & l_1 & l_2 & \cdots & l_1 & l_2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_1 & u_2 & c & u_1 & u_2 & c & \cdots & u_1 & u_2 & c \end{bmatrix}, \quad l_1, l_2, u_1, u_2 \in \mathbb{R},$$

\(S = \begin{bmatrix} e_1 & e_2 \end{bmatrix}\), \(P = \begin{bmatrix} e_{n-1} & e_n \end{bmatrix}\) and \(e_i\) is the \(i\)th column of the identity matrix \(I_n\).

$$V = \begin{bmatrix} x - u_1 & y - u_2 & z - c & 0 & \cdots & 0 \\ p - l_2 u_1 & q - (l_2 u_2 + u_1) & r - (d_2 + u_2) & s - c & \cdots & 0 \\ 0 & \cdots & 0 & t - u_1 & w - (l_1 u_2 + l_2 u_1) & k - (c l_1 + l_2 u_2 + u_1) & e - (u_2 + c l_2) \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 0 & \cdots & 0 & t - u_1 & w - (l_1 u_2 + l_2 u_1) & k - (c l_1 + l_2 u_2 + u_1) & e - (u_2 + c l_2) \\ 0 & \cdots & 0 & g - l_1 u_1 & d - (l_1 u_2 + l_2 u_1) & h - (c l_1 + l_2 u_2 + u_1) & 0 \end{bmatrix}.$$  \hfill (4)

By (3), the system (1) becomes

$$(LU + SV + PQ)x = f.$$  \hfill (5)

Multiplying (4) by \((LU)^{-1}\), where it is assumed to be non-singular, we obtain the following system

$$(I + ZV + WQ)x = x'$$

where \(Z = (LU)^{-1}S, \ W = (LU)^{-1}P\) and \(x' = (LU)^{-1}f\). The matrices \(Z, W\) and \(x'\) can be calculated by Algorithm 1.
From (5), the final solution of (1) is given by
\[ x = (I + ZV + WQ)^{-1}x'. \]  
(6)

Next step, we will use Sherman-Morrison-Woodbury inversion formula to give the inverse of \((I + ZV + WQ)\).

Let \( G = [Z \ W] \) and \( H = \begin{bmatrix} V \\ Q \end{bmatrix} \), then \( ZV + WQ = GH \), now apply Sherman-Morrison-Woodbury inversion formula directly to \((I + GH)^{-1}\), we have that
\[ (I + GH)^{-1} = I - G(I + HG)^{-1}H = I - [Z, W]N^{-1} \begin{bmatrix} V \\ Q \end{bmatrix} \]
where
\[ N = \begin{bmatrix} I + VZ & VW \\ QZ & I + QW \end{bmatrix} \]
is a matrix of the order \(4 \times 4\), which is assumed to be non-singular and its inverse is very easy to get.

Finally we can obtain the solution \(x\) of (1) as
\[ x = x' - [Z, W]N^{-1} \begin{bmatrix} V \\ Q \end{bmatrix} x' \]  
(7)

Now all we need is to determine \(u_1, u_2, l_1\) and \(l_2\), once these values are determined, we may go to Algorithm 2 to solve our equation.

2.1. Determination of parameters \(u_1, u_2, l_1\) and \(l_2\)

In this section we discuss how to determine the parameters \(u_1, u_2, l_1\) and \(l_2\).

By (3) we have the four equations
\[ \begin{align*}
    l_1u_1 &= c \\
    cl_2 + u_2 &= b \\
    l_1u_2 + l_2u_1 &= b \\
    cl_1 + l_2u_2 + u_1 &= a.
\end{align*} \]
(8) (9) (10) (11)

By (9), \(u_1 = \frac{c}{l_1}\), and by (10), \(u_2 = b - cl_2\). Replacing \(u_1\) and \(u_2\) into equations (11) and (12), respectively, we obtain two quadratic equations
\[ \begin{align*}
    (b - l_2c)l_1^2 - bl_1 + cl_2 &= 0 \\
    cl_1l_2^2 - bl_1l_2 - (c - al_1 + cl_1^2) &= 0.
\end{align*} \]
(12) (13)

By solving equation (12) we have
\[ l_1 = 1 \text{ or } l_1 = \frac{cl_2}{b - cl_2}. \]

**Case 1:** When \(l_1 = 1\).

In this case we have that \(u_1 = c\),
\[ l_2 = \frac{b + \sqrt{b^2 - 4c(a - 2c)}}{2c}, \]
and
\[ u_2 = \frac{b + \sqrt{b^2 - 4c(a - 2c)}}{2}. \]
These solutions are not stable in digital tests.

**Case 2:** When \( l_1 = \frac{cl_2}{b} \). (Here we assume that \( b \neq 0 \), if not \( l_1 = 1 \). )

First we assume that \( b - cl_2 \neq 0 \). Replacing \( l_1 = \frac{cl_2}{b} \) in (13), we obtain the following quadratic equation

\[
l_2^2 + m_1l_2^3 + m_2l_2^2 + m_3l_2 + m_4 = 0,
\]

where \( m_1 = -\frac{2b}{c} \), \( m_2 = \frac{b^2 + ac}{c^2} \), \( m_3 = -\frac{2bc + ab}{c^2} \), \( m_4 = \left(\frac{b}{c}\right)^2 \).

Let \( l_2 = \gamma - \frac{2b}{c} \), then (14) becomes

\[
\gamma^4 + \xi \gamma^2 + \eta = 0,
\]

with

\[
\xi = m_2 - \frac{3m_2^2}{8} = \frac{1}{2c^2}(4c^2 + 2ac - b^2),
\]

\[
\eta = m_4 - \frac{3m_4}{256} + \frac{m_2^2m_2}{16} - \frac{m_1m_3}{4} = \frac{1}{16c^2}(b^4 + 8b^3c^2 - 4acb^2).
\]

After a simple calculation, we get the solutions of (15). Furthermore, we get the four roots of (14), they are

\[
l_2^{(1)} = \frac{1}{2c} \left( b - \sqrt{-2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2} \right)
\]

\[
l_2^{(2)} = \frac{1}{2c} \left( b + \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2} \right)
\]

\[
l_2^{(3)} = \frac{1}{2c} \left( b - \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2} \right)
\]

\[
l_2^{(4)} = \frac{1}{2c} \left( b + \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2} \right).
\]

Knowing \( l_2 \), we may calculate \( l_1, u_1 \) and \( u_2 \) easily.

When \( b - cl_2 = 0 \), that is \( u_2 = 0 \), we may get \( u_1 = c, l_1 = 1, \) and \( l_2 = \frac{b}{c} \).

### 2.2. Selection of \( l_2 \)

In this section, we will discuss the choice of \( l_2 \). There are four \( l_2 \)s, the selected \( l_2 \) must guarantee the inverse of \( L \) and \( U \) exist. Let’s look at the structure of \( L^{-1} \) and \( U^{-1} \). (The inverse can be calculated by \( A^{-1} = \text{adj}(A)/\text{det}(A) \).)

Let

\[
L = \begin{bmatrix}
1 & l_2 & 1 \\
l_1 & l_2 & \cdots \\
\vdots & \ddots & \ddots \\
l_1 & l_2 & 1
\end{bmatrix}, \quad \text{then } L^{-1} = \begin{bmatrix}
1 & \pi_1 & 1 \\
\pi_2 & \pi_1 & \cdots \\
\vdots & \vdots & \ddots \\
\pi_n & \ldots & \ldots & \pi_2 & \pi_1 & 1
\end{bmatrix}
\]

where \( \pi_1 = -l_2, \pi_2 = \begin{bmatrix} l_2 & 1 & \ldots \\
l_1 & l_2 & \ldots \\
\vdots & \ddots & \ddots \\
0 & \ldots & l_1 & l_2
\end{bmatrix}, \pi_n = (-1)^{n-1} \begin{bmatrix} l_2 & 1 & \ldots \\
l_1 & l_2 & \ldots \\
\vdots & \ddots & \ddots \\
0 & \ldots & l_1 & l_2
\end{bmatrix} \)
From here, we can see that, if $|l_2| > 1$, with the increase of $n$, the down-left corner of $L^{-1}$ will become larger and larger, and at the end, tends infinity. So $|l_2| < 1$ is a sufficient condition to guarantee our process going on. On the other hand, $U = \begin{bmatrix} u_1 & u_2 & c & \cdots & \cdots & \cdots & \phi_1 \\ u_1 & u_2 & c & \cdots & \cdots & \cdots & \phi_2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ u_1 & u_2 & c & \cdots & \cdots & \cdots & \phi_n \end{bmatrix}$, then $U^{-1} = \begin{bmatrix} u_2 & c & \cdots & 0 \\ u_1 & u_2 & c & \cdots & \cdots & \cdots & \phi_1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & u_1 & u_2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{bmatrix}$.

where $\phi_1 = \frac{1}{u_1}$, $\phi_2 = -\frac{u_2}{u_1}$, \ldots, $\phi_n = \frac{(-1)^{n-1}}{u_1}$, $\phi_n = (-1)^{n-1} * + \cdots + (n-1) cu_1 u_2^{n-3} + u_2^{n-1} \frac{u_2}{u_1} = 0$.

so if $\frac{|u_2|}{u_1} < 1$, $\phi_n$ is convergent.

From $u_2 = b - cl_2$ and $l_1 = \frac{c_2}{b - cl_2} = \frac{cl_2}{u_2}$, then we have that $u_1 = c \frac{l_1}{l_2} = \frac{u_2}{l_2}$, which gives $l_2 = \frac{u_2}{u_1}$, so $|\frac{u_2}{u_1}| < 1$ is equivalent to $|l_2| < 1$.

Therefore $|l_2| < 1$ is sufficiently to guarantee that $L^{-1}$ and $U^{-1}$ converge.

In the next, we take $a, b, c$ all positive as an example to show that there exists an $l_2$ which satisfies the required conditions. For the other cases, the arguments are similar.

We take $l_2^{(2)}$ as an example to prove that $l_2 \in \mathbb{R}$.

**Theorem 1.** Suppose that $a, b$ and $c$ are all positive, and let $l_2 = l_2^{(2)}$, i.e.,

$$l_2 = \frac{1}{2c}(b - \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2 - 2ac + b^2 - 4c^2}}).$$

Then $l_2$ is real.

**Proof.** We first show $4ac + a^2 - 4b^2 + 4c^2 > 0$. By our hypothesis, the matrix $T$ is diagonally dominant i.e., $|a| > 2(b + |c|)$, or $|a| > 2|c| > |b|$.

So $4ac + a^2 - 4b^2 + 4c^2 = (a + 2c)^2 - (2b)^2 > (2b)^2 - (2b)^2 = 0$. 5
Next, we will show
\[ 2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2 \geq 0. \]

When \(-2ac + b^2 - 4c^2 \geq 0\), the inequality holds true. We consider only
\[ 2ac - b^2 + 4c^2 > 0. \]

In fact,
\[ 2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2 \geq 0 \]
\[ \iff 4c^2(4ac + a^2 - 4b^2 + 4c^2) \geq (2ac - b^2 + 4c^2)^2 \]
\[ \iff 4ac - 8c^2 - b^2 \geq 0 \]

By \(a > 2(b + c)\), we have
\[ 4ac - 8c^2 - b^2 > 4(2(b + c))c - 8c^2 - b^2 = 8bc - b^2. \]

When \(c > b\), then
\[ 8bc - b^2 > 8b^2 - b^2 > 0, \]
the inequality holds true.

When \(c < b\), we consider in two cases.

(i) \(c \leq b \leq 2c\). In this case, \(a > 2(b + c) \geq 4c\).
Then we have \(b^2 \leq 4c^2, b^2 + 8c^2 \leq 4c^2 + 8c^2 = 12c^2\) and \(4ac > 4(4c)c = 16c^2\).

So that
\[ 4ac - 8c^2 - b^2 > 16c^2 - (8c^2 + b^2) > 16c^2 - 12c^2 > 0 \]

(ii) \(b > 2c\). In this case, \(a > 2(b + c) \geq 6c\) and \(2ac > 2(6c)c = 12c^2\). Since we consider only \(2ac > b^2 - 4c^2\),
so we have \(b^2 + 8c^2 = b^2 - 4c^2 + 12c^2 < 2ac + 12c^2\).

Then
\[ 4ac - 8c^2 - b^2 > 4ac - (2ac + 12c^2) = 2ac - 12c^2 > 0. \]

So \(l_2\) is real, and we end the proof.

**Theorem 2.** Under the assumption of Theorem 1, we have that \(|l_2| < 1\).

**Proof.** We first prove that \(|l_2| < 1\), i.e., \(-1 < l_2 < 1\). We begin by proving the left side, that is \(-1 < l_2\).

\[ -2c < b - \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2} \]
\[ \implies (2c + b)^2 > \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} - 2ac + b^2 - 4c^2} \]
\[ \implies 2ac + 4bc + 8c^2 > 2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} \]
\[ \implies 2a + 4b + 8c > 2\sqrt{4ac + a^2 - 4b^2 + 4c^2} \]
\[ \implies (2a + 4b + 8c)^2 > (2\sqrt{4ac + a^2 - 4b^2 + 4c^2})^2 \]
\[ \implies 4a^2 + 16ab + 32ac + 16b^2 + 64bc + 64c^2 > 4a^2 + 16ac - 16b^2 + 16c^2 \]
\[ 4a^2 + 16ab + 32ac + 16b^2 + 64bc + 64c^2 - (4a^2 + 16ac - 16b^2 + 16c^2) > 0 \]
\[ \Rightarrow 32b^2 + 64bc + 16ab + 48c^2 + 16ac > 0 \]

Since \( a, b, c \) are positive, so the left side holds true. Now we prove the right side, that is \( t_2 < 1 \).

\[ b - \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2}} < 2c \]

gives

\[ b - 2c < \sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2}} - 2ac + b^2 - 4c^2 < 2c \]

If \( b - 2c < 0 \), then the inequality holds true. In the following, we suppose that \( b > 2c \).

\[ (b - 2c)^2 < (\sqrt{2c\sqrt{4ac + a^2 - 4b^2 + 4c^2}} - 2ac + b^2 - 4c^2)^2 \]
\[ \Rightarrow -4bc + 4c^2 < 2c\sqrt{4ac + a^2 - 4b^2 + 4c^2} < 2ac - 4c^2 \]
\[ \Rightarrow (2\sqrt{a^2 + 4ac - 4b^2 + 4c^2})^2 > (-4b + 8c + 2a)^2 \]
\[ \Rightarrow 4a^2 + 16ac - 16b^2 + 16c^2 > 4a^2 - 16ab + 32ac + 16b^2 - 64bc + 64c^2 \]
\[ \Rightarrow -2b^2 + 4bc + ab - 3c^2 - ac > 0 \]

Since \( b > 2c \) and \( a > 2(b + c) \), so
\[ -2b^2 + 4bc + ab - 3c^2 - ac = -2b^2 + 4bc - 3c^2 + ab - ac = -2b^2 + 4bc - 3c^2 + a(b - c) > \]
\[ -2b^2 + 4bc - 3c^2 + 2(b + c)(b - c) = 4bc - 5c^2 > 4(2c)c - 5c^2 > 0. \]

Therefore \( |t_2| < 1 \) is true. So the proof of this theorem is concluded.

According to the signs of \( a, b, c \), we give the following table for the selection of \( t_2 \).

| a | b | c | \( t_2 \) |
|---|---|---|---|
| + | + | + | \( t_2^{(1)} \) |
| - | + | + | \( t_2^{(2)} \) |
| + | - | + | \( t_2^{(1)} \) |
| + | + | - | \( t_2^{(2)} \) |
| - | - | + | \( t_2^{(3)} \) |
| + | - | - | \( t_2^{(4)} \) |
| - | + | - | \( t_2^{(1)} \) |
| - | - | - | \( t_2^{(2)} \) |

2.3. Case \( b = 0 \)

In the previous section, we assume that \( b \neq 0 \). Here we study what happens when \( b = 0 \). By solving equations (8)-(11), we get 6 solutions of the system.
(1) \( l_1 = \frac{1}{c} \left( \frac{1}{2}a + \frac{1}{2} \sqrt{a^2 - 4c^2} \right) \), \( l_2 = 0 \), \( u_1 = \frac{1}{2}a - \frac{1}{2} \sqrt{a^2 - 4c^2}, u_2 = 0 \).

(2) \( l_1 = -\frac{1}{c} \left( -\frac{1}{2}a + \frac{1}{2} \sqrt{a^2 - 4c^2} \right) \), \( l_2 = 0 \), \( u_1 = \frac{1}{2}a + \frac{1}{2} \sqrt{a^2 - 4c^2}, u_2 = 0 \).

(3) \( l_1 = 1, l_2 = -\frac{1}{c} \sqrt{-c(a - 2c)}, u_1 = c, u_2 = \sqrt{-c(a - 2c)} \).

(4) \( l_1 = 1, l_2 = \frac{1}{c} \sqrt{-c(a - 2c)}, u_1 = c, u_2 = -\sqrt{-c(a - 2c)} \).

(5) \( l_1 = -1, l_2 = -\frac{1}{c} \sqrt{c(a + 2c)}, u_1 = -c, u_2 = \sqrt{-c(a + 2c)} \).

(6) \( l_1 = -1, l_2 = \frac{1}{c} \sqrt{c(a + 2c)}, u_1 = -c, u_2 = -\sqrt{-c(a + 2c)} \).

Let’s look at solution (1).

\[
L^{-1} = \begin{bmatrix}
1 & 0 & 1 \\
-l_1 & 0 & \\
0 & -l_1 & \\
\vdots & \ddots & \ddots \\
(-1)^n l_1^{(n-k)} & \ldots & 0 & l_1^{-1} & 0 & -l_1 & 0 & 1
\end{bmatrix}
\]

and

\[
U^{-1} = \begin{bmatrix}
\frac{1}{w_1} & 0 & -\frac{c}{w_1^2} & 0 & \ldots & \ldots & \ldots & (-1)^n \frac{c^{(n-k)}}{w_1^{n-k}} \\
\frac{1}{w_1} & 0 & -\frac{c}{w_1^2} & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\frac{1}{w_1} & 0 & -\frac{c}{w_1^2} & 0 & \ddots & \ddots & \ddots & \frac{1}{w_1} \\
\frac{1}{w_1} & 0 & -\frac{c}{w_1^2} & 0 & \ddots & \ddots & \ddots & \frac{1}{w_1} \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{1}{w_1} \\
\frac{1}{w_1} & 0 & -\frac{c}{w_1^2} & 0 & \ddots & \ddots & \ddots & \frac{1}{w_1}
\end{bmatrix}
\]

we need \(|l_1| \leq 1\) and \(|u_1| \geq 1\) to guarantee the convergence of \(L^{-1}\) and \(U^{-1}\). We consider first \(|l_1| \leq 1\), that is equivalent to

\[
|a + \sqrt{a^2 - 4c^2}| \leq 2|c|.
\]

When \(c > 0\),

\[
|a + \sqrt{a^2 - 4c^2}| \leq 2|c| \iff -2c \leq a + \sqrt{a^2 - 4c^2} \leq 2c.
\]
When $c < 0$,
\[ |a + \sqrt{a^2 - 4c^2}| \leq 2|c| \iff 2c \leq a + \sqrt{a^2 - 4c^2} \leq -2c. \]

By a straightforward calculation, we get the solution for $|l_1| \leq 1$, which is $a < 0$.

Now we consider $|u_1| \geq 1$. Again, by a straightforward calculation, we get the solution for $|u_1| \geq 1$, which is $a \leq -2$. So when $a \leq -2$, we have that $|l_1| \leq 1$ and $|u_1| \geq 1$.

By analogous arguments, we get that when $a \geq 2$, $|l_1| \leq 1$ and $|u_1| \geq 1$ are guaranteed by solution (2), i.e.,
\[
(2) \quad l_1 = -\frac{1}{c} \left( -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4c^2} \right), \quad l_2 = 0, \quad u_1 = \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4c^2}, \quad u_2 = 0.
\]

So when $a \leq -2$, we choose solution (1) and when $a \geq 2$ we choose solution (2). And when $a \in (-2, 2)$, we may simply solve the system $\frac{2}{a}Tx = \frac{2}{a}f$.

Remark 1. The other four solutions are not suitable for the case $b = 0$.

2.4. The algorithm

In this subsection we give an algorithm for solving (1). We first give the Algorithm 1 to solve $LUy = f$, then Algorithm 2 to solve the equation (1), i.e., $Tx = f$.

**Algorithm 1** An algorithm for solving $LUy = f$

**Input**: $l_1, l_2, u_1, u_2, e$ and $f$

1. (solving $Lz = f$) $z_1 = f_1$, $z_2 = f_2 - l_1z_1$, $z_i = f_i - l_1z_{i-1} - l_2z_{i-2}$, $i = 3$ to $n$
2. (solving $Uy = z$) $y_n = \frac{z_n}{u_1}$, $y_{n-1} = \frac{2n-1w_{n-1}}{u_1}$, $y_i = \frac{z_i - u_2y_{i+1} - cy_{i-1}}{u_1}$, $i = n - 2$ to 1

**Output**: $y = [y_1, y_2, \ldots, y_n]^T$.

**Algorithm 2** : An algorithm for solving $Tx = f$

**Input**: $a, b, c, d, e, f, y, z, p, q, r, s, h, w, k, g, t$ and $f$;

1. Find the parameters $l_i$ and $u_i$ ($i = 1, 2$);
2. Solve linear systems $LUZ = S$, $LUW = P$, and $LUx' = f$ by using Algorithm 1;

**Output**: Compute $x$ by (7).

For the computational cost, when $n$ is large this algorithm takes about $\approx 8n + O(1)$ flops. An advantage of our algorithm is that it needs less data transmission since both the subdiagonal and supernodal of $L$ and $U$ have constant values, respectively. It only reads one vector (the right-hand side vector) and writes one vector (the solution). The stability of Algorithm 2 depends on the step that solves the upper and lower pentadiagonal linear systems $LDU[Z, W, f] = [S, P, x']$. More precisely, two recursive iteration steps such as $z_i = f_{i-1} - l_1z_{i-1} - l_2z_{i-2}$ and $x_{n+1-i} = z_{n+1-i} - u_2y_{n+2-i} - cy_{n+3-i}$ for $i = 2, \ldots, n$ corresponding to the forward and backward substitutions as in Algorithm 1 are essential for Algorithm 2. When using finite precision arithmetic, we should avoid roundoff error propagation. If all the roots of their characteristic equations $\lambda^2 + l_1\lambda + l_2 = 0$ and $\lambda^2 + u_2\lambda + e = 0$ are all less than unity in magnitude, errors of $z_i$ and $x_{n+1-i}$ will smaller than errors of previous values $z_{i-1}$ and $x_{n+2-i}$, respectively, in which case the Algorithm 2 is stable.
3. Numerical examples

In this section, numerical results are presented to confirm the effectiveness of our algorithm. All algorithms are implemented in MATLAB R2018a and the computations are done on an Intel PENTIUM computer, (2.2 GHz), 6 GB memory. We fix the exact solution to be \( x^* = [1, 1, \ldots, 1]^T \) and the right-hand side vector is set to be \( f = Ax^* \).

3.1. Experiment 1: Kuramoto Sivashinsky equation

In this experiment, we will take the pentadiagonal matrices which appear in the numerical solution of Kuramoto Sivashinsky (KS) equation as an example. KS equation is a nonlinear partial differential equation first derived for the study of chemical reaction system, see [16, 17].

In paper [17], the initial vector is calculated by solving the following linear equations

\[
\begin{bmatrix}
12a - 3b + c & d - b & e - a & 0 & 0 & 0 & 0 \\
-3b - 2 & c - a & d & e & 0 & 0 & 0 \\
-3a & -2 & c & d & e & 0 & 0 \\
0 & -2 & c & d & e & 0 & 0 \\
0 & 0 & -2 & c & d & e & 0 \\
0 & 0 & 0 & -2 & c & d & e \\
0 & 0 & 0 & 0 & -2 & c & d & e & d - 3e \\
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{N-2} \\
c_{N-1} \\
c_N \\
\end{bmatrix}
= \begin{bmatrix}
u(x_0) \\
u(x_1) \\
\vdots \\
u(x_{N-2}) \\
u(x_{N-1}) \\
u(x_N) \\
\end{bmatrix}.
\]

By applying von-Neumann boundary conditions [17], the coefficient matrix becomes

\[
\begin{bmatrix}
54 & 60 & 6 & 0 & 0 & 0 & 0 & 0 \\
101/4 & 135/2 & 105/4 & 1 & 0 & 0 & 0 & 0 \\
1 & 26 & 66 & 26 & 1 & 0 & 0 & 0 \\
0 & 1 & 26 & 66 & 26 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 26 & 66 & 26 & 1 \\
0 & 0 & 0 & 0 & 1 & 105/4 & 135/2 & 101/4 \\
0 & 0 & 0 & 0 & 0 & 6 & 60 & 54 \\
\end{bmatrix}
\]

By applying second order mixed boundary conditions [16], the coefficient matrix becomes

\[
\begin{bmatrix}
121 & -2 & 1 & 0 & 0 & \cdots & \cdots & 0 \\
28 & 65 & 26 & 1 & 0 & \cdots & \cdots & 0 \\
1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0 \\
0 & 1 & 26 & 66 & 26 & 1 & \cdots & 0 \\
0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
0 & \cdots & 0 & 1 & 26 & 66 & 26 & 1 \\
0 & \cdots & \cdots & 0 & 1 & 26 & 65 & 28 \\
0 & \cdots & \cdots & \cdots & 0 & 1 & -1 & 121 \\
\end{bmatrix}
\]

We first compare these two examples, then we arbitrary choose some \( b, c, d, e, x, y, z, p, q, r, s, t, w, k, h, \) and \( g \) to test our algorithm.
Table 1: Numerical results of Experiment 1 (with von-Neumann boundary conditions in [17])

| Algorithm | \( n = 10^4 \) | \( n = 10^5 \) | \( n = 10^6 \) | \( n = 10^7 \) |
|-----------|----------------|----------------|----------------|----------------|
| \[ \|x^* - x\|_2 \] | \( 1.7593e-14 \) | \( 5.5524e-14 \) | \( 1.7555e-13 \) | \( 5.5511e-13 \) |
| Our algo. | \( 4.4402e-14 \) | \( 1.4043e-13 \) | \( 4.4409e-13 \) | \( 1.4043e-12 \) |
| CPU(s)    | \( 4.24e-3 \)  | \( 6.97e-2 \)   | \( 0.45 \)      | \( 4.63 \)      |
| Our algo. | \( 2.18e-3 \)  | \( 2.49e-2 \)   | \( 0.29 \)      | \( 2.25 \)      |

Table 2: Numerical results of Experiment 1 (with mixed boundary conditions in [16])

| Algorithm | \( n = 10^4 \) | \( n = 10^5 \) | \( n = 10^6 \) | \( n = 10^7 \) |
|-----------|----------------|----------------|----------------|----------------|
| \[ \|x^* - x\|_2 \] | \( 1.7579e-14 \) | \( 5.5519e-14 \) | \( 1.7554e-13 \) | \( 5.5511e-13 \) |
| Our algo. | \( 4.3822e-14 \) | \( 1.4042e-13 \) | \( 4.4409e-13 \) | \( 1.4043e-12 \) |
| CPU(s)    | \( 5.57e-3 \)  | \( 5.48e-2 \)   | \( 0.53 \)      | \( 5.87 \)      |
| Our algo. | \( 3.61e-3 \)  | \( 3.94e-2 \)   | \( 0.39 \)      | \( 2.69 \)      |

In Figures 1, and 2 a comparison of our algorithm with \( LU \) method are presented.

Figure 1: CPU time [s] comparison for Experiment 1 with von-Neumann boundary conditions in [17]

Figure 2: CPU time [s] comparison for Experiment 1 with mixed boundary conditions in [16]

It can be seen that our proposed algorithm takes less CPU time than the \( LU \) method.

3.2. Experiment 2

Some artificial examples were used in this experiment. The values of \( a, b, c, d, e, x, y, z, p, q, r, s, t, w, k, h, \) and \( g \) corresponding to each examples are presented in Table 3.

| \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( x \) | \( y \) | \( z \) | \( p \) | \( q \) | \( r \) | \( s \) | \( t \) | \( w \) | \( k \) | \( h \) | \( g \) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Example 1 | -62 | -10 | -19 | -2 | -5 | -2.3 | 4 | 3.5 | 10 | 2 | -4 | 3 | -1 | -1.7 | 4.2 | -3.5 | 10 |
| Example 2 | 66 | 10 | 15 | 1 | 1 | 8 | 2 | -1.5 | -0.7 | -1 | -2.3 | 7 | 2.5 | 1.6 | -4 | -3.2 | 4 |
| Example 3 | 2.5 | -0.8 | 0.8 | 2.2 | 1 | 1.3 | 0.4 | -0.2 | 3 | 1 | -4 | 3 | -2 | -1.2 | 1 | -1 | 1.3 |
| Example 4 | 246 | 30 | -56 | -2 | 1.6 | 0.5 | -2 | 2.4 | 2.6 | -7.2 | 2 | 1 | -1 | 2.6 | 5 | 1 | 1 |
| Example 5 | -5 | 0 | 2 | 1 | 2.4 | 1 | 2 | 1 | -5 | 5 | -26 | -2 | 0.6 | -25 | -6.5 | 0.6 | 2 |
| Example 6 | 6.5 | 0 | 1.3 | 1 | 4.5 | 1.5 | -3.2 | -1.3 | -3.2 | 5 | -19 | -7 | -1 | -2 | -1.5 | 0.7 | 1 |
From tables 4 to table 9 we show the absolute accuracy $\Delta x = \|x^* - x\|_2$ of the approximate solution of (1) where $x^* = [1, \ldots, 1]^T$ is the exact solution and $x$ is the approximate solution computed by LU method, and our algorithm, respectively. We display also, in the same tables the CPU time [s] of the computed solution of our algorithm.

### Table 4: Numerical results of Example 1

| Algorithm | $n = 10^4$ | $n = 10^5$ | $n = 10^6$ | $n = 10^7$ |
|-----------|------------|------------|------------|------------|
| $\|x^* - x\|_2$ |            |            |            |            |
| Algo. LU  | 2.2181e-14 | 7.0208e-14 | 2.2204e-13 | 7.0217e-13 |
| Our algo. | 6.0168e-15 | 6.0168e-15 | 6.0168e-15 | 6.0168e-15 |
| CPU(s)    |            |            |            |            |
| Algo. LU  | 3.49e-3    | 8.1e-2     | 1.11       | 10.02      |
| Our algo. | 3.36e-3    | 3.2e-2     | 0.34       | 3.55       |

### Table 5: Numerical results of Example 2

| Algorithm | $n = 10^4$ | $n = 10^5$ | $n = 10^6$ | $n = 10^7$ |
|-----------|------------|------------|------------|------------|
| $\|x^* - x\|_2$ |            |            |            |            |
| Algo. LU  | 1.3038e-14 | 2.6820e-14 | 7.8182e-14 | 2.4847e-13 |
| Our algo. | 3.9339e-14 | 8.3841e-14 | 3.8316e-13 | 7.8512e-13 |
| CPU(s)    |            |            |            |            |
| Algo. LU  | 4.66e-3    | 6.27e-2    | 0.87       | 9.80       |
| Our algo. | 4.49e-3    | 2.77e-2    | 0.35       | 3.38       |

### Table 6: Numerical results of Example 3

| Algorithm | $n = 10^4$ | $n = 10^5$ | $n = 10^6$ | $n = 10^7$ |
|-----------|------------|------------|------------|------------|
| $\|x^* - x\|_2$ |            |            |            |            |
| Algo. LU  | 3.5346e-14 | 1.1110e-13 | 3.5111e-13 | 1.1102e-12 |
| Our algo. | 2.3572e-14 | 7.0342e-14 | 2.2208e-13 | 7.0218e-13 |
| CPU(s)    |            |            |            |            |
| Algo. LU  | 4.35e-3    | 4.46e-2    | 0.98       | 9.84       |
| Our algo. | 4.09e-3    | 2.84e-2    | 0.34       | 3.46       |

### Table 7: Numerical results of Example 4

| Algorithm | $n = 10^4$ | $n = 10^5$ | $n = 10^6$ | $n = 10^7$ |
|-----------|------------|------------|------------|------------|
| $\|x^* - x\|_2$ |            |            |            |            |
| Algo. LU  | 1.7668e-13 | 1.7668e-13 | 1.7668e-13 | 1.7668e-13 |
| Our algo. | 8.5199e-14 | 9.1478e-14 | 1.3956e-13 | 3.6100e-13 |
| CPU(s)    |            |            |            |            |
| Algo. LU  | 8.0e-3     | 4.99e-2    | 0.99       | 9.95       |
| Our algo. | 4.69e-3    | 3.03e-2    | 0.34       | 3.48       |

### Table 8: Numerical results of Example 5

| Algorithm | $n = 10^4$ | $n = 10^5$ | $n = 10^6$ | $n = 10^7$ |
|-----------|------------|------------|------------|------------|
| $\|x^* - x\|_2$ |            |            |            |            |
| Algo. LU  | 1.7624e-14 | 5.5537e-14 | 1.7555e-13 | 5.5511e-13 |
| Our algo. | 2.0742e-14 | 3.9238e-14 | 1.1240e-13 | 3.5152e-13 |
| CPU(s)    |            |            |            |            |
| Algo. LU  | 4.71e-3    | 6.31e-2    | 1.11       | 9.78       |
| Our algo. | 3.62e-3    | 2.79e-2    | 0.34       | 3.45       |
Table 9: Numerical results of Example 6

| Algorithm | \( n = 10^4 \) | \( n = 10^5 \) | \( n = 10^6 \) | \( n = 10^7 \) |
|----------|---------------|---------------|---------------|---------------|
| \( \|x^* - x\|_2 \) | | | | |
| Algo. LU | 8.1259e-15 | 2.4914e-14 | 7.8533e-14 | 2.4826e-13 |
| Our algo. | 1.3822e-15 | 1.3822e-15 | 1.3822e-15 | 1.3822e-15 |
| CPU(s) | | | | |
| Algo. LU | 5.36e-3 | 6.06e-2 | 1.11 | 10.03 |
| Our algo. | 4.30e-3 | 3.48e-2 | 0.36 | 3.52 |

Thus, comparing with the initial \( LU \) method for a sparse matrix, our algorithm improves the computational cost of the numerical solution remarkably. For the accuracy, our algorithm is similar to the other well-known existing methods.

4. Conclusion

In this paper, we have proposed a new algorithm for solving diagonally dominant symmetric quasi-pentadiagonal Toeplitz linear systems. We discussed possible choices for the parameters for each situation. We implemented our method in Matlab with respect to computational costs. The numerical results show the robustness of our method. The required memory and the computational time of our algorithm are lower than those of other well-known existing methods. The effectiveness of our algorithm is confirmed by numerical experiments.

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