Derivation of Hamilton-like equations on a non-Cauchy hypersurface and their expected connection to quantum gravity theories

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Abstract

Recently it was found that quantum gravity theories may involve constructing a quantum theory on non-Cauchy hypersurfaces. However, this is problematic since the ordinary Poisson brackets are not causal in this case. We suggest a method to identify classical brackets that are causal on 2+1 non-Cauchy hypersurfaces and use it in order to show that the evolution of scalars and vectors fields in the 3rd spatial direction can be constructed by using a Hamilton-like procedure. Finally, we discuss the relevance of this result to quantum gravity.

Introduction

The correct way in order to obtain a gravitational theory from microscopic objects or quantum fields is not known but it is expected that it should be related to surfaces. The main examples are the holographic principle[1], first proposed by ’t Hooft [2], which states that in quantum gravity the description of a volume of space can be encoded on a lower-dimensional
boundary to the region, and the AdS/CFT [3] correspondence, which uses a non-perturbative formulation of string theory to obtain a realization of the holographic principle. As far as we know, in all these descriptions the holographic screen is a light-like surface.

However, recently it seems that non-Cauchy hypersurfaces can also be related to quantum gravity theories. Non-Cauchy hypersurface foliation was first used in the membrane paradigm[4] which models a black hole as a thin, classically radiating membrane, vanishingly close to the black hole's event horizon. This non-Cauchy hypersurface is useful for visualizing and calculating the effects predicted by quantum mechanics for the exterior physics of black holes.

The second example that a non-Cauchy hypersurface is useful for aspects of quantum gravity involves the surface density of space time degrees of freedom (DoF). These are expected to be observed by an accelerating observer in curved spacetime, i.e. whenever an external non-gravitational force field is introduced [5]. This DoF surface density was first derived by Padmanabhan for a static spacetime using thermodynamic considerations. We found that this DoF surface density can also be constructed from specific canonical conjugate pairs as long as they are derived in a unique way [6]. These canonical conjugate pairs must be obtained by foliating spacetime with respect to the direction of the external non-gravitational force field. Note that this aspect reinforces the importance of singling out a very unique spatial direction: the direction of a non-gravitational force.

The third example which suggests that a non-Cauchy hypersurface is useful for aspects of quantum gravity involves string theory excitation. It was found that some specific kind of singularities are obtained by string theory excitations of a $D1D5$ black hole [7, 8, 9, 10]. We found [11] that these singularities can also be explained using the uncertainty principle between canonical conjugate pairs which are obtained by singling out the radial direction. The radial direction can be regarded as the direction of a non-gravitational force that causes observers to “stand steel” in these coordinate frame. Thus, these singularities, which according to string theory are expected in quantum gravity theories, are derived by the uncertainty principle.
only by singling out the non-gravitational force direction.

The fourth example involves “holographic quantization”. The holographic quantization uses spatial foliation in order to quantize the gravitational fields for different backgrounds in Einstein theory. This is carried out by singling out one of the spatial directions in a flat background [12], and also singling out the radial direction for a Schwarzschild metric [13].

The fifth example involves the developing of a quantum black hole wave packet [14]. In this case, the gravitational foliation is used in order to obtain a quantum Schwarzschild black hole, at the mini super spacetime level, by a wave packet composed of plane wave eigenstates.

The sixth and final example involves the Wheeler-DeWitt metric probability wave equation. Recently, in [15], foliation in the radial direction was used to obtain Wheeler-DeWitt metric probability wave equation on the apparent horizon hypersurface of the Schwarzschild de Sitter black hole. By solving this equation, the authors found that a quantized Schwarzschild de Sitter black hole has a nonzero value for the mass in its ground state. This property of quantum black holes leads to stable black hole remnants.

All these suggest that in order to find a quantum gravity theory, one needs to obtain a proper way of constructing a quantum theory using non-Cauchy hypersurface foliations.

However, in general, one should expect problems when foliating with respect to hypersurfaces that are not Cauchy, since in this case the field evolution isn’t usually causal. Even constructing a quantum theory with causal commutation relation on a non-Cauchy hypersurface is expected to be challenging, since the usual Poisson brackets do not lead to causal commutation relation on the non-Cauchy hypersurface and thus are not relevant in this case.

In this paper we propose a method to identify a set of causal canonical classical brackets on hypersurfaces that are not Cauchy. We use this method to derive the field equations for scalar fields and find that the expected field equations can be derived only if the scalar field is physical on the non-Cauchy hypersurface. This means that as long as a scalar field is physical on the hypersurface one can use non-Cauchy foliation to derive causal evolution.
Since this can easily be extended to any vector field, our proof is expected to be relevant even for the gravitational field in Minkowski spacetime. Thus, when using spatial foliation, a causal evolution can be constructed for the gravitational field, as long as the gravitational field is physical on the surface. However, according to our method, in order to be able to use the spatial foliation we need to obtain the classical brackets from the quantum commutation relation on the non-Cauchy hypersurface. This is problematic since we do not have a proper quantum gravitational theory. In this paper we suggest a method in order overcome this problem, and derive a possible renormalized quantum gravity theory.

This paper is organized as follows: First we present the problem of using the Poisson brackets on a non-Cauchy hypersurface. Next we suggest a method for identifying causal classical brackets between the fields on the hypersurface. Then we use this method for free scalar field and find that the Klein-Gordon equation can be derived by a Hamiltonian-like formalism. Finally, we extend this to any vector field and discuss the relevance of this result to quantum gravity theories.

Presenting the problem and the suggested solution

In general, in order to quantize a field theory we usually start with the Lagrangian density of the theory $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$ and a Cauchy surface which can be defined as an equal time surface $x_0 = 0$. Then we use to define the momentum canonically conjugate to the field variable $\phi(x)$ as $\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$ where $\dot{\phi} = \partial_0 \phi$, and the Hamiltonian density as $\mathcal{H} = \Pi(x) \dot{\phi} - \mathcal{L}$. Next, we need to verify whether the dynamical equation derived by the Euler-Lagrange equations, can be written in the Hamiltonian form: $\dot{\phi}(x) = \{\phi(x), H\}$ and $\dot{\Pi}(x) = \{\Pi(x), H\}$, where $H = \int d^3x \mathcal{H}$. For this purpose we need to identify equal time canonical bracket relations between the field variable $\phi(x)$ and the conjugate momentum $\Pi(y)$. Usually, we assume the equal time canonical Poisson bracket relations: $\{\phi(x), \phi(y)\}_{x_0 = y_0} = \{\Pi(x), \Pi(y)\}_{x_0 = y_0} = 0$ and $\{\phi(x), \Pi(y)\}_{x_0 - y_0} = \delta^3(x - y)$. Then, if the fields equations can be written in the Hamiltonian form, we can treat $\phi(x)$ and $\Pi(x)$ as operators that sat-
isfy the same equal time commutation relations, i.e.: \( \{ A(x), B(y) \}_{x^0 = y^0} \to i\hbar [A(x), B(y)]_{x^0 = y^0} \), and the quantization is straightforward.

In this paper we want to repeat this process of quantization for a non-Cauchy hypersurface which we define as a hypersurface with an equal spatial coordinate \( x_1 = \text{const} \). The other coordinates on this hypersurface will be denoted by \( \tilde{x} = (x^0, x^2, x^3) \). In this case we propose that the quantization will be as follows: we will define a new canonically conjugate momentum to the field variable \( \phi(x) \): 

\[
\Pi_1(x) = \frac{\partial L}{\partial \phi'(x)} \quad \text{where} \quad \phi' = \partial_1 \phi.
\]

The new Hamiltonian-like density will be defined as \( \mathcal{H}_1 = \Pi_1(x)\phi' - \mathcal{L} \). Next we will need to verify that the dynamical equations derived by Euler-Lagrange equations could be written in a Hamiltonian-like form as 

\[
\phi'(x) = \{ \phi(x), H_1 \} \quad \text{and} \quad \Pi_1'(x) = \{ \Pi_1(x), H_1 \},
\]

where \( H_1 = \int d^3\tilde{x} \mathcal{H} \). For this purpose we will need to assume equal \( x_1 \) bracket relations between the field variable \( \phi(x) \) and the new canonical conjugate momentum \( \Pi_1(x) \). However, if we will use equal \( x_1 \) canonical Poisson bracket relations in the same way as equal \( x^0 \) canonical Poisson bracket relations, namely:

\[
\{ \phi(x), \phi(y) \}_{x^1 = y^1} = \{ \Pi_1(x), \Pi_1(y) \}_{x^1 = y^1} = 0 \quad \text{and} \quad \{ \phi(x), \Pi_1(y) \}_{x^1 = y^1} = \delta^3(\tilde{x} - \tilde{y}),
\]

we will find non causal relations. Thus, equal spatial coordinate brackets can not be defined by the ordinary canonical Poisson brackets. In order to identify correctly the equal \( x_1 \) classical bracket relations between the field variable \( \phi(x) \) and the new canonically conjugate momentum \( \Pi_1(x) \), we will need to have an extension of the canonical Poisson brackets in such a way that it will be causally defined even when foliating spacetime with respect to non-Cauchy hypersurfaces.

We suggest a way for identifying the equal \( x_1 \) canonical classical brackets without using the extended Poisson brackets: It seems that for any theory that we are able to quantize using the ordinary equal time canonical Poisson brackets, we can easily obtain the quantum commutation relation for any equal spatial coordinate. This can be done by using the field equations and calculating the commutation relation \([\phi(x), \phi(y)]\) for any two points \( x \) and \( y \) in space-time. This will enable us to calculate the equal spatial commutation relations \([\phi(x), \phi(y)]_{x^1 = y^1}, [\phi(x), \Pi_1(y)]_{x^1 = y^1}, \text{and} [\Pi_1(x), \Pi_1(y)]_{x^1 = y^1}\). Thus from the calculated quantum commutation relations we deduce the classical equal spatial coordinate brackets between the fields.
Example: the free scalar field in (1+3)D

1. Spatial foliation and Hamilton-like equation

We begin with the Lagrangian density of a free scalar field in 1+3 dimension:
\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{m^2}{2} \phi^2, \]
and a non-Cauchy equal hypersurface \( x_1 = \text{const.} \). Then we define
\[ \Pi_1(x) = \frac{\partial \mathcal{L}}{\partial \phi'(x)} = -\partial_1 \phi, \]
and the new Hamiltonian-like density becomes:
\[ \mathcal{H}_1 = \Pi_1(x) \phi' - \mathcal{L} = \frac{1}{2} \left( -\Pi_1^2 - (\partial_0 \phi)^2 + (\partial_2 \phi)^2 + (\partial_3 \phi)^2 + m^2 \phi^2 \right). \]

Then the new Hamiltonian-like equations are:
\[ \phi(x)' = \{ \phi(x), H_1 \} \]
\[ \Pi_1(x)' = \{ \Pi_1(x), H_1 \} \]

where \( H_1 = \int d^3 \tilde{y} \mathcal{H}_1(\tilde{y}, y_1) \).

Note that, since \( H_1 \) is \( y_1 \) independent, we can choose \( y_1 = x_1 \).

Obviously, when going to the quantum limit attention is needed in order to construct the new Hamiltonian and the apparent phase space on a time-like surface. For example, it is problematic to quantize such Hamiltonians since they are unbounded from below. Even operator ordering ambiguities are expected and must be redefined. Moreover, we used to expect that before going to the quantum limit one requires a geometric symplectic structure which will lead to the definition of the phase space and the conjugate relations. However, as we will demonstrate in the next section, this symplectic structure is not relevant on time-like surfaces.

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1This can easily be proven using the Hamilton formalism where we expect that for any \( A(x) : A(x)' = \{ A(x), H_1(x^1) \} \) thus for \( A(x) = \mathcal{H}_1(x) \) we find \( \mathcal{H}_1(x)' = \{ \mathcal{H}_1(x), H_1(x^1) \} \). Using \( H_1(x^1) = \int dx^0 dx^2 dx^3 \mathcal{H}_1 \) we find \( H_1(x^1)' = \{ H_1(x^1), H_1(x^1) \} = 0 \).
2. Finding the equal $x_1$ canonical classical brackets for a free scalar field

In order to identity the equal $x_1$ classical bracket relations, we use the commutation relation $[\phi(x), \phi(y)]$, derived by the ordinary quantization \cite{17}, for any two points $x$ and $y$ in space-time:

$$[\phi(x), \phi(y)] = -i \int \frac{d^3k}{(2\pi)^3} \frac{\sin[E_k(x^0 - y^0)]}{E_k} e^{ik(x-y)}$$  \hspace{1cm} (5)$$

where $E_k = \sqrt{k^2 + m^2} = \sqrt{k_0^2 + k_1^2 + k_2^2 + k_3^2 + m^2}$.

It is useful to write this commutation relation in a covariant way, using the Heaviside step function $\theta(x)$:

$$[\phi(x), \phi(y)] = \int \frac{d^4k}{(2\pi)^4} (\theta(k^0) - \theta(-k^0)) \delta(k^2 - m^2) e^{i\vec{k}(x-y)}. \hspace{1cm} (6)$$

Next, using $\delta(k^2 - m^2) = \delta ((k_1)^2 - P_x^2) = \frac{1}{2P_x} \left[ \delta(k^1 - P_x) + \delta(k^1 + P_x) \right]$ where $P_x \equiv \sqrt{(k_0)^2 - (k_1)^2 - (k_2)^2 - (k_3)^2 - m^2}$, we find that the same commutation relations equal also to:

$$[\phi(x), \phi(y)] = \int \frac{d^4k}{(2\pi)^4} (\theta(k^0) - \theta(-k^0)) \frac{1}{2P_x} \left[ \delta(k^1 - P_x) + \delta(k^1 + P_x) \right] e^{i\vec{k}(x-y)}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} (\theta(k^0) - \theta(-k^0)) \theta(P^2_x) \frac{e^{ip_x(x^1-y^1)} + e^{-ip_x(x^1-y^1)}}{2P_x} e^{i\vec{k}(\vec{x}-\vec{y})}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} (\pm k^0, P^2_x) \cos[P_x(x^1-y^1)] \frac{P_x}{P_x} e^{i\vec{k}(\vec{x}-\vec{y})}, \hspace{1cm} (7)$$

where $\vec{k} = (k^0, k^1, k^2, k^3), \vec{k} \cdot (\vec{x} - \vec{y}) = k^0(x^0 - y^0) - k^2(x^2 - y^2) - k^3(x^3 - y^3)$, and $\epsilon(\pm k^0, P^2_x) \equiv (\theta(k^0) - \theta(-k^0)) \theta(P^2_x)$. Note that the term $\theta(P^2_x)$ actually limits the possible values of $\vec{k}$ so that only physical modes are considered. The fact that we are only considering physical modes is important in our derivation of the field equations. From (7) we see that the equal spatial coordinate $(x^1 = y^1)$ commutation relations become:
\[ [\phi(x), \phi(y)]_{x^1=y^1} = \int \frac{d^3k}{(2\pi)^3} \epsilon(\pm k^0, P_x^2) \frac{1}{P_x} e^{ik \cdot (\tilde{x} - \tilde{y})}. \]  

(8)

Note that this commutation relations are indeed causal since they do not vanish for \( x \) and \( y \) which are causally connected.

Next we need to evaluate the equal spatial coordinate \((x^1 = y^1)\) commutation relations between \(\phi(x)\) and its new canonical conjugate momentum \(\Pi_1(y)\). Varying eq. (7) with respect to \(y^1\) and using: \(\Pi_1(y) = -\frac{\partial \phi(y)}{\partial y^1}\) gives:

\[ [\phi(x), \Pi_1(y)] = -\int \frac{d^3k}{(2\pi)^3} \epsilon(\pm k^0, P_x^2) \sin[P_x(x^1 - y^1)] e^{ik \cdot (\tilde{x} - \tilde{y})} \]  

(9)

and the equal spatial coordinate \(x^1 = y^1\) commutation relations become:

\[ [\phi(x), \Pi_1(y)]_{x^1=y^1} = 0. \]  

(10)

Thus we got that for a free scalar field \(\phi(x)\) and \(\Pi_1(y)\) commutes on \(x^1 = y^1\) and do not influence each other even when their coordinates are causally connected.

Finally, we evaluate the equal spatial coordinate \((x^1 = y^1)\) commutation relations between the two canonical conjugate momenta \(\Pi_1(x)\) and \(\Pi_1(y)\) . Varying eq. (9) with respect to \(x^1\) once again and using: \(\Pi_1(x) = -\frac{\partial \phi(x)}{\partial x^1}\) gives:

\[ [\Pi_1(x), \Pi_1(y)] = \int \frac{d^3k}{(2\pi)^3} \epsilon(\pm k^0, P_x^2) P_x \cos[P_x(x^1 - y^1)] e^{ik \cdot (\tilde{x} - \tilde{y})} \]  

(11)

and thus the equal spatial coordinate \((x^1 = y^1)\) commutation relations between the \(\Pi_1\) fields are:

\[ [\Pi_1(x), \Pi_1(y)]_{x^1=y^1} = \int \frac{d^3k}{(2\pi)^3} \epsilon(\pm k^0, P_x^2) P_x e^{ik \cdot (\tilde{x} - \tilde{y})}. \]  

(12)

Now we take a very naive assumption and suggest what the classical "Poisson-like" brackets for an equal spatial coordinate can be achieved just
by multiplying the commutation relation we have got by $-i$. We have:

\[
\{\phi(x), \phi(y)\}_{x^1=y^1} = -i \int \frac{d^3\widetilde{k}}{(2\pi)^3} \epsilon(\pm k^0, P_x) \frac{1}{P_x} e^{i\widetilde{k} \cdot (\widetilde{x} - \widetilde{y})} \tag{13}
\]

\[
\{\Pi_1(x), \Pi_1(y)\}_{x^1=y^1} = -i \int \frac{d^3\widetilde{k}}{(2\pi)^3} \epsilon(\pm k^0, P_x) P_x e^{i\widetilde{k} \cdot (\widetilde{x} - \widetilde{y})} \tag{14}
\]

\[
\{\phi(x), \Pi_1(y)\}_{x^1=y^1} = 0. \tag{15}
\]

As we mentioned before, we used to expect that in order to obtain the commutation relation in the quantum limit one requires a geometric symplectic structure of the phase space. The symplectic structure leads to the definition of phase space, conjugate relations and their evolution. However, as we see from eqs. (13,14,15) the symplectic structure is not relevant on time-like surfaces. Thus though it is not clear whether one can construct the phase space in this case, we suggest examining whether these seemingly canonical fields can lead to the expected evolution, i.e. the field equation.

3. The dynamical equation derived by the Hamiltonian-like equations

Finally, we must check whether the dynamical equation derived by the Euler-Lagrange equation; i.e. the Klein-Gordon equation: $(\partial_\mu \partial^\mu + m^2)\phi = 0$ can be constructed from the Hamiltonian-like equations (3) and (4) when the classical brackets (13)-(15) are assumed.

We start with (3), and use (2) and get

\[
\phi(x)' = \left\{ \phi(x), \int d^3\widetilde{y} \left( \frac{1}{2} (-\Pi_1(y)^2 - (\partial_0 \phi(y))^2 + (\partial_2 \phi(y))^2 + (\partial_3 \phi(y))^2 + m^2 \phi(y)^2) \right) \right\}
\]

Using the vanishing equal $x_1$ bracket relation $\phi$ and $\Pi_1$ we have:

\[
\phi(x)' = \int d^3\widetilde{y} \{\phi(x), \phi(y)\}_{x^1=y^1} \left( \partial_0^2 \phi(y) - \partial_2^2 \phi(y) - \partial_3^2 \phi(y) + m^2 \phi(y) \right) y_1=x_1, \tag{16}
\]
and after using the equal $x_1$ bracket relations (13) we get after integration by parts:

\[ \phi(x) = i \int d^3\tilde{y} \int \frac{d^3\tilde{k}}{(2\pi)^3} \epsilon(\pm k_0, P_x) P_x e^{i\tilde{k} \cdot (\tilde{x} - \tilde{y})} \phi(y)_{x_1 = y_1}. \quad (17) \]

Varying (17) with respect to $x_1$ once again we get:

\[ \phi(x)'' = i \int d^3\tilde{y} \int \frac{d^3\tilde{k}}{(2\pi)^3} \epsilon(\pm k_0, P_x) P_x e^{i\tilde{k} \cdot (\tilde{x} - \tilde{y})} \phi(y)_{x_1 = y_1}''. \quad (18) \]

and using (17) for $\phi(y)'$ we find:

\[ \phi(x)'' = - \int d^3\tilde{y}' \phi(y)'_{x_1 = y_1} \int \frac{d^3\tilde{k}}{(2\pi)^3} \epsilon^2(\pm k_0, P_x) P_x^2 e^{i\tilde{k} \cdot (\tilde{x} - \tilde{y}')} \phi(y)_{x_1 = y_1}''. \quad (19) \]

Since $\epsilon^2(\pm k_0, P_x) = \theta(P_x^2)$ we finally have:

\[ \phi(x)'' = - \frac{d\tilde{k}}{(2\pi)^3} \theta(P_x^2) ((k_0)^2 - (k^2)^2 - (k^3)^2 - m^2) \int d^3\tilde{y}' e^{i\tilde{k} \cdot (\tilde{x} - \tilde{y}')} \phi(y)_{x_1 = y_1}''. \quad (20) \]

Now, Fourier transforming this equation with respect to $\tilde{k}$ yields:

\[ (k^1)^2 = \left[ (k_0)^2 - (k^2)^2 - (k^3)^2 - m^2 \right] \cdot \theta \left[ (k_0)^2 - (k^2)^2 - (k^3)^2 - m^2 \right]. \quad (21) \]

Thus, if we limit ourselves to physical fields i.e.:

\[ (k_0)^2 - (k^2)^2 - (k^3)^2 - m^2 \geq 0, \]

we get:

\[ (k_0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2 - m^2 = 0 \quad (22) \]

which are the expected fields equation.

To conclude: we found that quantizing a scalar field by foliating space-
time with respect to a spacelike vector is possible.

The same procedure can be used in order to find an equation for the $\Pi_1$ field. Using (1) and (2) we have:

$$\Pi_1(x)' = \left\{ \Pi_1(x), \int d^3\tilde{y} \left( \frac{1}{2} \left( -\Pi_1(y)^2 - (\partial_0 \phi(y))^2 + (\partial_2 \phi(y))^2 + (\partial_3 \phi(y))^2 + m^2 \phi(y)^2 \right) \right) \right\}_{y_1=x_1}.$$  

With (15) we get:

$$\Pi_1(x)' = -\int d^3\tilde{y} \{\Pi_1(x), \Pi_1(y)\}_{x^1=y^1} \Pi_1(y)|_{y_1=x_1}$$

i.e.:

$$\Pi_1(x)' = i \int d^3\tilde{y} \int \frac{d^3k}{(2\pi)^3} \epsilon(\pm k^0, P_z^2) P_x e^{i\tilde{k} \cdot (\tilde{x} - \tilde{y})} \Pi_1(y)|_{y_1=x_1}$$  \hspace{1cm} (23)

which is the same equation which the field $\phi$ fulfills.

Thus, though the symplectic structure is not relevant on time-like surfaces, we got two Hamilton-like independent field equations for the fields $\phi$ and $\Pi_1$.

This result can easily extended for any vector fields. For vector fields, one can ignore complication associated with gauge invariance and work directly with physical components. In this case the action of each physical component will be the same as for a scalar field. For example, though in $(3+1)D$ the metric has 10 components 8 of them are non-physical, and each of the two remaining physical components has an effective action of a scalar field. Thus, quantizing a vector field by foliating spacetime with respect to a spacelike vector field is also possible whenever the vector field components have causal commutation relation on the non-Cauchy hypersurface.

However, the new Hamiltonian and phase space need much more attention in order to be upgraded to these commutation relations from first principles on any time-like surface. The encouragement to overcome the expected difficulties comes from the relevance of such foliations to quantum gravity theories.
Implication to quantum gravity theories

In order to find the way to a renormalized quantum gravity, we need to derive the true degrees of freedom of quantum gravity. Various reductions of the quantum gravity degrees of freedom were reported in the past in [18, 19, 20, 21]. All these works employed the usual (3+1)D splitting with the genuine time coordinate separated out. However, in light of the holographic picture, it is worthwhile considering the possible advantages of a non-Cauchy surface foliation on quantum gravity theories and its expected benefit to the renormalization problem of these theories. Park [12] separates the spatial directions. His strategy for reduction has been the removal of all of the nonphysical degrees of freedom from the external states. His key observation for the reduction was the fact that the residual 3D gauge symmetry - whose detailed analysis is given in [22] - can be employed to gauge away the non-dynamical fields. However, this kind of reduction is not always possible.

We suggest that, whenever it is possible to obtain a renormalized (2+1)D quantum gravitational theory only on a specific non-Cauchy hypersurface, we may use this non-Cauchy spatial foliation in order to obtain a (3+1)D quantum gravitational theory. In order to examine this option we will first find the conditions in order to obtain a renormalized (2+1)D quantum gravitational theory only on a specific non-Cauchy hypersurface. Next we will suggest a way to construct a causal (3+1)D quantum gravitational theory. Finally, we will discuss the implication of this procedure on the expected properties of quantum gravity theories and on the renormalization possibility of a (3+1)D quantum gravity theory.

Let us start by considering the standard foliation of spacetime with respect to some spacelike hypersurfaces whose directions are $n^a$. The lapse function $M$ and shift vector $W_a$ satisfy $r_a = M n_a + W_a$ where $r^a \nabla_a r = 1$ and $r$ is constant on $\Sigma_r$. The $\Sigma_r$ hyper-surfaces metric $h_{ab}$ is given by $g_{ab} = h_{ab} + n_a n_b$. The extrinsic curvature tensor of the hyper-surfaces is given by $K_{ab} = - \frac{1}{2} \mathcal{L}_n h_{ab}$ where $\mathcal{L}_n$ is the Lie derivative along $n^a$. Instead of

\[ R^{(3)}_{ab} = \frac{1}{2} h^{kl} \left( \frac{\partial h_{lk}}{\partial x^a} + \frac{\partial h_{ak}}{\partial x^l} - \frac{\partial h_{al}}{\partial x^k} \right) \] so that $R^{(3)}_{ab} = \frac{\partial \Gamma^k_{ab}}{\partial x^k} - \frac{\partial \Gamma^k_{ak}}{\partial x^b} + \Gamma^k_{ab} \Gamma^l_{kl} - \Gamma^l_{ak} \Gamma^k_{lb}$.\n
3The intrinsic curvature $R^{(3)}_{ab}$ is then given by the 2+1 Christoffel symbols: $\Gamma^k_{ab} = \frac{1}{2} h^{kl} \left( \frac{\partial h_{lk}}{\partial x^a} + \frac{\partial h_{ak}}{\partial x^l} - \frac{\partial h_{al}}{\partial x^k} \right)$ so that $R^{(3)}_{ab} = \frac{\partial \Gamma^k_{ab}}{\partial x^k} - \frac{\partial \Gamma^k_{ak}}{\partial x^b} + \Gamma^k_{ab} \Gamma^l_{kl} - \Gamma^l_{ak} \Gamma^k_{lb}$.\n
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the $(3 + 1)D$ Einstein equations

\[ R_{ab}^{(4)} = 8\pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right) \]

one finds [25]:

\[ R_{ab}^{(3)} + K K_{ab} - 2 K_i g^i_b - M^{-1} (\mathcal{L}_r K_{ab} + D_a D_b M) = 8\pi \left( S_{ab} - \frac{1}{2} (S - P) h_{ab} \right) , \]

\[ R^{(3)} + K^2 - K_{ab} K^{ab} = 16\pi P , \]

\[ D_b K_a^b - D_a K = 8\pi F_a . \]

Where \( D_a \) represent the 2+1 covariant derivatives, \( S_{ab} = h_{ac} h_{ad} T^{cd} \), \( P = n_c n_d T^{cd} \) and \( F_a = h_{ac} n_b T^{cb} \).

Let's assume that on some unique hypersurface \( r = r_0 \):

\[ K K_{ab} - 2 K_i g^i_b - M^{-1} (\mathcal{L}_r K_{ab} + D_a D_b M) = 0 \]

then we have:

\[ R_{ab}^{(3)} = 8\pi \left( S_{ab} - \frac{1}{2} (S - P) h_{ab} \right) , \]

which is just the Einstein equation in 2+1 dimension, when \( P \) serves as a cosmological constant.

From the quantum gravity point of view, this unique hypersurface is interesting. To see this note that though we don’t know how to obtain a renormalized quantum gravity theory in 3+1 dimensions, a renormalized quantum theory in 2+1 dimensions is possible [16]. Thus, when the \( (3 + 1)D \) Einstein equations reduce to a kind of 2+1 Einstein equations on some hypersurface \( r = r_0 \), we can quantize the gravitational fields on the hypersurface \( r = r_0 \) at least with respect to this foliation.

Now, as we have got this \( (2+1)D \) Einstein equation on this spacelike non-Cauchy hypersurface, let’s assume we have found a \( (2 + 1)D \) renormalized quantum gravitational theory on the hypersurface \( r = r_0 \). As we noted before, having the quantum theory on the non-Cauchy hypersurface means
one can find also a set of causal commutation relation on this hypersurface. But, as we showed in the last section by an example, knowing the causal commutation relation on a non-Cauchy hypersurface automatically enables us to know how the quantum gravity fields will evolve through the remaining spatial direction $r \neq r_0$, with the aid of the new Hamiltonian-like equations we mentioned. Thus, using a sort of Hamilton-like ADM equation may serve as a way of finding the evolution of the gravitational quantum fields in the third spatial direction and thus of producing, at least in principle, a $(3+1)D$ renormalized quantum gravity theory.

Note that this procedure is related to holography. To see this note that in this formalism, the evolution on the gravitational fields in the $4-th$ dimension is determined by the "initial" condition on the non-Cauchy hypersurface $r = r_0$. Thus in this formalism all the information needed to describe the evolution of the gravitational field in the $4-th$ dimension is encoded on this hyper-surface, as suggested by holography.

**Summary**

In this paper we argued that deriving a proper quantum gravity theory may involve quantization on a non-Cauchy hypersurface. We showed that since in this kind of hypersurfaces the ordinary Poisson brackets are not causal, constructing a quantum theory on a non-Cauchy hypersurface is expected to be problematic. We suggested a method to identify classical brackets that are causal even on a non-Cauchy hypersurface. We used this method to identify the (causal) brackets between free relativistic scalar fields and their (new) canonical conjugate momentum on a non-Cauchy hypersurface. Next we proposed a sort of classical Hamilton-like equations for the evolution along the direction perpendicular to the non-Cauchy hypersurface. We found that, as long as the field on the non-Cauchy hypersurface is physical, these equations leads to the expected free Klein-Gordon equation.

However, the new Hamiltonian and phase space need more attention in order to be upgraded to the commutation relations we obtained from first principles on any time-like surface. To begin with, the probability definition
can only be defined on a Cauchy surface. Thus, in this case, it seems that
in order to obtain the probability density one needs to consider unitarity
as a constraint. Moreover, the new Hamiltonians seem to be unbounded
from below and operator ordering ambiguities are expected and must be
redefined. Although this is only a partial list of the expected ambiguities
when quantum theory is derived by spatial foliation, the expected properties
of a quantum gravity theory lead us to suggest that such a theory may be
constructed using spatial foliation.

Thus, we considered the possible advantages of such unique foliation on
expected theories of quantum gravity. We considered a foliation of spacetime
with respect to spacelike hypersurfaces whose directions are \( n^a \), and the
conditions for the \((3+1)D\) Einstein equations to reduce to a kind of \((2+1)D\)
Einstein equations on a hypersurface \( r = r_0 \). Then, by using the argument
that a renormalized quantum gravity theory can be constructed in \((2 + 1)D\), we discussed the procedure needed in order to derive the renormalized
gravitational fields causal evolution in the 3rd spatial direction. However,
whether this suggested procedure is expected to leads to a renormalized
quantum gravity theory on \( r \neq r_0 \) remains to be seen.

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\(^4\)To see this note that any attempt to derive the state normalizability on a time-like
surface leads to temporal decay, both in past and future, making the usual norm of a state
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