Abstract

We consider the multi-armed bandit problems in which a player aims to accrue reward by sequentially playing a given set of arms with unknown reward statistics. In the classic work, policies were proposed to achieve the optimal logarithmic regret order for some special classes of light-tailed reward distributions, e.g., Auer et al.’s UCB1 index policy for reward distributions with finite support. In this paper, we extend Auer et al.’s UCB1 index policy to achieve the optimal logarithmic regret order for all light-tailed (or equivalently, locally sub-Gaussian) reward distributions defined by the (local) existence of the moment-generating function.

I. INTRODUCTION

In the classic MAB, there are $N$ independent arms offering random rewards to a player. At each time, the player chooses one arm to play and obtains a reward drawn i.i.d. over time from a distribution with unknown mean. Different arms may have different reward distributions. The design objective is a sequential arm selection policy that maximizes the total expected reward over a long but finite horizon $T$.

Each received reward plays two roles: increasing the wealth of the player, and providing one more observation for learning the reward statistics of the arm. The tradeoff is thus between exploration and exploitation: which role should be emphasized in arm selection—an arm less explored thus holding potentials for the future or an arm with a good history of rewards. In 1952, Robbins addressed the two-armed bandit problem [1]. He showed that the same maximum average reward achievable under a known model can be obtained by dedicating two arbitrary
sublinear sequences for playing each of the two arms. In 1985, Lai and Robbins proposed a finer performance measure, the so-called regret, defined as the expected total reward loss with respect to the ideal scenario of known reward models (under which the arm with the largest reward mean is always played) \[2\]. Regret not only indicates whether the maximum average reward under known models is achieved, but also measures the convergence rate of the average reward, or the effectiveness of learning. Lai and Robbins showed that the minimum regret has a logarithmic order in \( T \). They also constructed explicit policies to achieve this minimum regret for Gaussian, Bernoulli, Poisson and Laplacian distributions assuming the knowledge of the distribution type \[1\]. In 1995, Agrawal developed index-type policies that achieve \( O(\log T) \) regret for Gaussian, Bernoulli, Poisson, Laplacian, and exponential distributions \[3\]. In 2002, Auer et al. proposed a simpler index policy, referred to as UCB1, with \( O(\log T) \) regret for reward distributions with finite support \[4\]. UCB1 policy does not require the knowledge of the distribution type; it only requires an upper bound on the finite support.

These classic policies focus on finite-support reward distributions and several specific infinite-support light-tailed distributions. In this paper, we generalize Auer et al.’s index to achieve \( O(\log T) \) regret for all light-tailed reward distributions. Light-tailed distributions, also referred to as locally sub-Gaussian distributions, are defined by the (local) existence of the moment-generating function. This work thus provides a simple index policy that achieves the optimal regret order for a broader class of reward distributions.

MAB with general and unknown reward distributions was also considered in our prior work \[5\], where a Deterministic Sequencing of Exploration and Exploitation (DSEE) approach was proposed to achieve the logarithmic regret order for all light-tailed reward distributions. DSEE also achieves sublinear regret orders for heavy-tailed reward distributions. Specifically, for any \( p > 1 \), \( O(T^{1/p}) \) regret can be achieved by DSEE when the moments of the reward distributions exist (only) up to the \( p \)th order. The advantage of DSEE is its simple deterministic structure that can handle variations of MAB including general objectives, decentralized MAB with partial reward observations, and rested/restless Markovian reward models \[6\]. However, compared to the extended UCB1 policy that adaptively adjusts the number of plays on each arm based on

\[1\] For the existence of an optimal policy in general, Lai and Robbins established a sufficient condition on the reward distributions. However, the condition is difficult to check and is only verified for the specific distributions mentioned above.
observations, DSEE spends equal amount of time during the exploration phase for learning the reward statistics. Simulation results indicate that the extended UCB1 policy can have a better leading constant in the logarithmic regret order.

Other work on extensions of the UCB1 policy includes [7]–[11]. In [7]–[9], UCB1 was extended to handle decentralized MAB with multiple distributed players. In [10], [11], UCB1 was extended to the rested and restless Markovian reward models, respectively.

II. THE CLASSIC MAB

In this section, we present the non-Bayesian formulation of the classic MAB and Auer et al. ’s UCB1 policy.

A. Problem Formulation

Consider an $N$-arm bandit and a single player. At each time $t$, the player chooses one arm to play. Playing arm $n$ yields i.i.d. random reward $X_n(t)$ drawn from an unknown distribution $f_n(s)$. Let $\mathcal{F} = (f_1(s), \cdots, f_N(s))$ denote the set of the unknown distributions. We assume that the reward mean $\theta_n \triangleq \mathbb{E}[X_n(t)]$ exists for all $1 \leq n \leq N$.

An arm selection policy $\pi$ is a function that maps from the player’s observation and decision history to the arm to play. Let $\sigma$ be a permutation of $\{1, \cdots, N\}$ such that $\theta_{\sigma(1)} \geq \theta_{\sigma(2)} \geq \cdots \geq \theta_{\sigma(N)}$. The system performance under policy $\pi$ is measured by the regret $R_T^\pi(\mathcal{F})$ defined as

$$R_T^\pi(\mathcal{F}) \triangleq T \theta_{\sigma(1)} - \mathbb{E}_{\pi}[\sum_{t=1}^{T} X_{\pi}(t)],$$

where $X_{\pi}(t)$ is the random reward obtained at time $t$ under policy $\pi$, and $\mathbb{E}_{\pi}[\cdot]$ denotes the expectation with respect to policy $\pi$. The objective is to minimize the rate at which $R_T^\pi(\mathcal{F})$ grows with $T$ under any distribution set $\mathcal{F}$ by choosing an optimal policy $\pi^\ast$. Although all policies with sublinear regret achieve the maximum average reward, the difference in their total expected reward can be arbitrarily large as $T$ increases. The minimization of the regret is thus of great interest.

B. UCB1 Policy

In Auer et al. ’s UCB1 policy [4], an index $I(t)$ is computed for each arm and the arm with the largest index is chosen. Assume that the support of the reward distributions is normalized
to $[0,1]$. The index (referred to as the upper confidence bound) has the following simple form:

$$I(t) = \bar{\theta}(t) + \sqrt{\frac{2\log t}{\tau(t)}}.$$  

(1)

This index form is intuitive in the light of Lai and Robbins’s result on the logarithmic order of the minimum regret which indicates that each arm needs to be explored on the order of $\log t$ times. For an arm sampled at a smaller order than $\log t$, its index, dominated by the second term, will be sufficient large for large $t$ to ensure further exploration.

Based on the Chernoff-Hoeffding bound on the convergence of the sample mean for distributions with finite support [12], Auer et al. established a regret growing at the logarithmic order with time. Furthermore, an upper bound on the regret accumulated up to any finite time was also established.

III. EXTENSION OF UCB1 POLICY

In this section, we generalize UCB1 for the class of light-tailed reward distributions.

A. Light-Tailed Reward Distributions

The class of light-tailed reward distributions are defined by the (local) existence of the moment-generating function. Such reward distributions are also referred to as locally sub-Gaussian distributions (see [13]).

**Definition 1:** The moment-generating function $M(u) = \mathbb{E}[\exp(uX)]$ of a random variable $X$ exists if there exists a $u_0 > 0$ such that

$$M(u) < \infty \quad \forall \ u \leq |u_0|.$$  

(2)

By the mean-value theorem, the function $M(u)$ is infinitely differentiable. A direct application of Taylor’s theorem leads to the following upper bound on $M(u)$ (see Theorem 1 in [13]). Without loss of generality, assume that $\mathbb{E}[X] = 0$. We have

$$M(u) \leq \exp(\zeta u^2/2), \quad \forall \ u \leq |u_0|, \ \zeta \geq \sup\{M^{(2)}(u), \ -u_0 \leq u \leq u_0\},$$  

(3)

where $M^{(2)}(\cdot)$ denotes the second derivative of $M(\cdot)$ and $u_0$ the parameter specified in (2). We observe that the upper bound in (3) is the moment-generating function of a zero-mean Gaussian random variable with variance $\zeta$. The distributions satisfying (3) (i.e., with a finite moment-generating function around 0) are thus called locally sub-Gaussian distributions. If there is no
constraint on the parameter \( u \) in (3), the corresponding distributions are referred to as sub-Gaussian.

Based on (3), we show a Bernstein-type bound on the convergence rate of the sample mean as given in the lemma below.

**Lemma 1:** Consider i.i.d. light-tailed random variables \( \{X(t)\}_{t=1}^{\infty} \) with a finite moment-generating function over range \([-u_0, u_0]\). Let \( \overline{X}_t = (\sum_{k=1}^{t} X(k))/t \) and \( \theta = \mathbb{E}[X(1)] \). We have, for all \( \epsilon > 0 \),

\[
\Pr\{\overline{X}_t - \theta \geq \epsilon\} \leq \begin{cases} 
\exp\left(-\frac{t\epsilon^2}{2\zeta}\right), & \epsilon < \zeta u_0 \\
\exp\left(-\frac{tu_0\epsilon}{2}\right), & \epsilon \geq \zeta u_0
\end{cases}
\]

where \( \zeta, u_0 \) are the parameters specified in (3). The same bound holds for \( \Pr\{\overline{X}_t - \theta \leq -\epsilon\} \) by symmetry.

**Proof:** The proof follows a similar line of arguments as given in [14]. We provide it below for completeness. By Markov’s inequality, \( \forall u \in [0, u_0] \),

\[
\Pr\{\overline{X}_t - \theta \geq \epsilon\} = \Pr\{ut(\overline{X}_t - \theta) \geq u\epsilon\} \leq \frac{\mathbb{E}[\exp(ut(\overline{X}_t - \theta))]}{\exp(u\epsilon)} \leq \frac{\mathbb{E}[\exp(\sum_{k=1}^{t} u(X(k) - \theta))]\exp(u\epsilon)}{\exp(u\epsilon)} = \frac{\mathbb{E}[\prod_{k=1}^{t} \exp(u(X(k) - \theta))]\exp(u\epsilon)}{\exp(u\epsilon)} \leq \exp(t\zeta u_0^2/2 - u\epsilon)
\]

It is not difficult to show that if \( \epsilon \geq \zeta u_0 \), then

\[
\frac{t\zeta u_0^2}{2} - tu_0\epsilon \leq \frac{-tu_0}{2}\epsilon.
\]

Based on (6) and (7), we arrive at (4).

Note that for a small sample mean deviation \( \epsilon < \zeta u_0 \), the bound has a similar form to the classical Chernoff-Hoeffding bound for finite-support distributions. Although the bound for large sample mean deviations has a different form (linear in the deviation \( \epsilon \) rather than quadratic in
the exponent), it preserves the exponential decaying rate in terms of both the sample size and the
deviation $\epsilon$. These properties of light-tailed reward distributions lead to the following extension
of Auer et al.’s index (UCB1) policy while preserving the logarithmic regret order.

B. Extended UCB1

As mentioned in Sec. II-B, the second term in Auer et al.’s index (1) is used for specifying
the upper confidence bound to ensure sufficient but bounded explorations on each arm, given that
the Chernoff-Hoeffding bound holds. To adapt to the Bernstein-type bound given in Lemma 1,
we consider two upper confidence bounds and determine which one to use at each time based
on their values. The detailed algorithm is shown in Fig. 1. Note that for sub-Gaussian reward
distributions (i.e., $u_0 = \infty$), the extended UCB1 is reduced to the case in which only one upper
confidence bound is used as the index function of each arm. This upper confidence bound, as
given in (8), has the same form of (1) except for a difference in choosing the parameter $a_1$.

**Theorem 1:** For all light-tailed arm reward distributions, the regret of the extended UCB1
policy for any $T > 1$ is bounded by

$$ R_T^* (\mathcal{F}) \leq \sum_{n: \theta_n < \theta_{\sigma(1)}} (\theta_{\sigma(1)} - \theta_n) \left( \max \left\{ \frac{4a_1}{(\theta_{\sigma(1)} - \theta_n)^2}, \frac{2a_2}{\theta_{\sigma(1)} - \theta_n} \right\} \log T + 1 + \frac{\pi^2}{3} \right). $$

**Proof:** Define

$$ c(t, s) = \left\{ \begin{array}{ll}
\sqrt{\frac{a_1 \log t}{s}} & , \quad \sqrt{\frac{a_1 \log t}{s}} < \zeta u_0 \\
\sqrt{\frac{a_2 \log t}{s}} & , \quad \sqrt{\frac{a_2 \log t}{s}} \geq \zeta u_0
\end{array} \right. $$

Following a similar procedure as in [4], for any integers $L \geq 0$ and $n$ such that $\theta_n < \theta_{\sigma(1)}$, we have

$$ \mathbb{E}[\tau_n(T)] \leq L + \sum_{t=1}^{T} \Pr \{ \bar{\theta}_n(\tau_n(t)) + c(t, \tau_n(t)) \geq \bar{\theta}_{\sigma(1)}(\tau_{\sigma(1)}(t)) + c(t, \tau_{\sigma(1)}(t)) \} $$

$$ \leq L + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{k=L}^{t-1} \Pr \{ \bar{\theta}_n(k) + c(t, k) \geq \bar{\theta}_{\sigma(1)}(s) + c(t, s) \} $$

$$ \leq L + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{k=L}^{t-1} \left( \Pr \{ \bar{\theta}_n(k) \geq \theta_n + c(t, k) \} + \Pr \{ \bar{\theta}_{\sigma(1)}(s) \leq \theta_{\sigma(1)} - c(t, s) \} \right) $$

$$ + \Pr \{ \theta_n + 2c(t, k) > \theta_{\sigma(1)} \} \}. $$

Choose

$$ L_0 = \left\lceil \max \left\{ \frac{4a_1 \log T}{(\theta_{\sigma(1)} - \theta_n)^2}, \frac{2a_2 \log T}{\theta_{\sigma(1)} - \theta_n} \right\ \right\rceil. $$
The UCB1-LT Policy $π^*$

- Notations and Inputs: Let $τ_n(t)$ denote the number of plays on arm $n$ up to (but excluding) time $t$ and $θ_n(τ_n(t))$ the sample mean of arm $n$ at time $t$. Choose $a_1 ≥ 8ζ$ and $a_2 ≥ a_1/(ζu_0)$. Define two index functions

\[ I_n^{(1)} = \bar{θ}_n(τ_n(t)) + \sqrt{\frac{a_1 \log t}{τ_n(t)}}, \]

\[ I_n^{(2)} = \bar{θ}_n(τ_n(t)) + \frac{a_2 \log t}{τ_n(t)}. \]

- Initialization: In the first $N$ steps, play all arms once to obtain the initial sample means.

- At time $t > N$,

1. for each arm $n$, if $\sqrt{\frac{a_1 \log t}{τ_n(t)}} < ζu_0$, compute its index according to $I_n^{(1)}$, otherwise compute its index according to $I_n^{(2)}$.
2. play the arm with the largest index.

For any $k ≥ L_0$, we have

\[ c(t, k) ≤ \max \left\{ \sqrt{\frac{a_1 \log t}{k}}, \frac{a_2 \log t}{k} \right\} \]

\[ ≤ \max \left\{ \sqrt{\frac{a_1 \log t}{L_0}}, \frac{a_2 \log t}{L_0} \right\} \]

\[ ≤ \max \left\{ \sqrt{a_1 \log t \cdot \frac{(θ_{σ(1)} - θ_n)^2}{4a_1 \log T}}, \frac{a_2 \log t \cdot \frac{θ_{σ(1)} - θ_n}{2a_2 \log T}}{2} \right\} \]

\[ = \frac{θ_{σ(1)} - θ_n}{2}. \]

From (11) and (12), we have

\[ \mathbb{E}[τ_n(T)] ≤ L_0 + \sum_{t=1}^{∞} \sum_{s=1}^{t-1} \sum_{k=L_0}^{t-1} (\Pr\{θ_n(k) ≥ θ_n + c(t, k)\} + \Pr\{θ_{σ(1)}(s) ≤ θ_{σ(1)} - c(t, s)\}). \]
Next, we bound the probabilities in (13) by the Bernstein-type bound (4). If
\[ \sqrt{\frac{a_1 \log t}{k}} < \zeta u_0, \]
then
\[ \Pr\{ \bar{\theta}_n^{(k)} \geq \theta_n + c(t, k) \} = \Pr\left\{ \bar{\theta}_n^{(k)} \geq \theta_n + \sqrt{\frac{a_1 \log t}{k}} \right\} \]
\[ \leq \exp \left( -\frac{k}{2\zeta} \left( \sqrt{\frac{a_1 \log t}{k}} \right)^2 \right) \]
\[ \leq t^{-4}; \]
otherwise
\[ c(t, k) = \frac{a_2 \log t}{k} \geq \frac{a_1 \log t}{\zeta u_0 k} \geq \zeta u_0 \]
and we have
\[ \Pr\{ \bar{\theta}_n^{(k)} \geq \theta_n + c(t, k) \} = \Pr\left\{ \bar{\theta}_n^{(k)} \geq \theta_n + \frac{a_2 \log t}{k} \right\} \]
\[ \leq \exp \left( -\frac{k u_0}{2} \cdot \frac{a_2 \log t}{k} \right) \]
\[ \leq \exp \left( -\frac{k u_0}{2} \cdot \frac{a_1 \log t}{\zeta u_0 k} \right) \]
\[ \leq t^{-4}. \]
The same bound also applies on
\[ \Pr\{ \bar{\theta}_{\sigma(1)}(s) \leq \theta_{\sigma(1)} - c(t, s) \}. \]
We thus have
\[ \mathbb{E}[\tau_n(T)] \leq L_0 + 2 \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{k=L_0}^{t-1} t^{-4} \]
\[ \leq \max\left\{ \frac{4a_1 \log T}{(\theta_{\sigma(1)} - \theta_n)^2}, \frac{2a_2 \log T}{\theta_{\sigma(1)} - \theta_n} \right\} + 1 + \frac{\pi^2}{3}, \]
as desired.
IV. Conclusion

In this paper, we have considered a broader class of reward distributions for MAB problems. Auer et al. ’s UCB1 policy was extended to achieve a uniform logarithmic regret bound over time for all light-tailed reward distributions.

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