On the problem of relativistic particles motion in a strong magnetic field and dense matter

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Abstract

We consider a problem of electron motion in different media and magnetic fields. It is shown that in the case of an immovable medium and constant homogenous magnetic field the electron energies are quantized. We also discuss the general problem of eigenvectors and eigenvalues of a given class of Hamiltonians. We examine obtained exact solutions for the particular case of the electron motion in a rotating neutron star which account for matter and magnetic field effects. We argue that all of these considerations can be useful for astrophysical applications, in particular for the description of electrons' and neutrinos motion in different environments.

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1. Introduction

Exact solutions of quantum field equations of motion provide an effective tool in studies of different phenomena of particle interactions in high-energy physics. They supply with particular applications in solving problems of charged particles motion in electromagnetic fields of terrestrial experimental devices, as well as in astrophysics and cosmology. Exact solutions were first applied in quantum electrodynamics for development the quantum theory of the synchrotron radiation, i.e. for studies of motion and radiation of the electron in a magnetic field (see, for instance, [1]), and also for studies of the electrodynamics and weak interaction in different other configurations of external electromagnetic fields [2]. This method is based on the Furry representation [3] in quantum electrodynamics (for more detailed discussion on this item see [4]) widely used for the description of particles interactions in the presence of external electromagnetic fields. Recently, it has been shown that the method of exact solutions can also be applied for the problem of neutrinos and electron motion in the presence of dense matter (see [4] for a review on this topic). Most pronoucedly, this possibility was pointed
out in [5, 6] where the exact solution for the modified Dirac equation for a neutrino moving in matter was derived and discussed in detail. The corresponding exact solution for an electron moving in matter was obtained in [7], the problem of neutrino propagation in transversally moving matter was first solved in [8], and in our recent paper [9] we considered neutrino propagation in a rotating matter accounting for the effect of non-zero neutrino mass.

Note that the modified effective Dirac equations for a neutrino interacting with various background environments within different models were previously used [10] in a study of the neutrino dispersion relations, neutrino mass generation and for derivation of the neutrino oscillation probabilities in matter. On the same basis, the neutrino decay into an antineutrino and a light scalar particle (majoron), as well as the corresponding process of the majoron decay into two neutrinos or antineutrinos, was studied in the presence of matter [11].

It should also be mentioned that a type of the modified Dirac equations with an anomalous vector and pseudovector interactions was considered within a framework for treatment of low-energy effects of spontaneous CPT violation and Lorentz breaking (see [12] and references therein).

Below we further develop the method of exact solutions for the problem of charged leptons propagating in matter and strong magnetic fields. The paper is organized as follows. In section 2, we discuss the general form of the modified Dirac equation for an electron moving in the matter and magnetic field, and the corresponding spin operator is derived in section 3. In sections 4–7, the exact wavefunctions and energy spectrum are obtained. The general problems of eigenvectors and eigenvalues of a given class of Hamiltonians are discussed. In section 8, we examine the exact solutions obtained in sections 2–7 for the particular case of the charged particle motion in a rotating neutron star with account for matter and magnetic field effects.

2. Modified Dirac equation for electron moving in the matter and magnetic field

We consider an electron propagating in an immovable magnetized medium composed of neutrons and suppose that the magnetic field is homogenous and constant. This can be regarded as the first approach to modeling of an electron propagation inside a neutron star. For distinctness, we consider here the case of an electron, whereas generalization for other charged particles is just straightforward. We start with the modified Dirac equation for the electron wavefunction exactly accounting for the electron interaction with matter in the absence of the magnetic field [5] (see also [6]):

$$\left\{ \gamma_\mu p^\mu + \frac{1}{2} \gamma_\mu (1 - 4 \sin^2 \theta_W + \gamma^5) f^\mu - m \right\} \Psi(x) = 0. \quad (2.1)$$

This is the most general form of the equation for the electron wavefunction in which the effective potential $V^\mu = \frac{1}{2} (1 - 4 \sin^2 \theta_W + \gamma^5) f^\mu$ includes both the neutral and charged current interactions of the electron with the background particles, and which can also account for the effects of the motion and polarization of the matter.

In order to also include the effect of an external electromagnetic field, we replace in equation (2.1) the electron momentum $p^\mu$ by the ‘extended’ momentum $p^\mu \rightarrow p^\mu + e_0 A^\mu$:

$$\left\{ \gamma_\mu (p^\mu + e_0 A^\mu) + \frac{1}{2} \gamma_\mu (1 - 4 \sin^2 \theta_W + \gamma^5) f^\mu - m \right\} \Psi(x) = 0, \quad (2.2)$$

where $e_0$ is a module of the electron charge.

Note that in the general case it is not a trivial task to find solutions of this equation. In what follows, we consider the particular case of a constant magnetic field and an immovable uniform matter, so that for the electromagnetic field and effective matter potential we obtain

$$A^\mu = \left( 0, -\frac{y B}{2}, \frac{x B}{2}, 0 \right), \quad f^\mu = -Gn(1, 0, 0, 0). \quad (2.3)$$
where $G = \frac{G_L}{\sqrt{2}}$, and $n$ is the matter’s number density. We rewrite equation (2.2) in the Hamiltonian form and obtain

\[ i \frac{\partial}{\partial t} \Psi(x) = \hat{H} \Psi(x), \]

\[ \hat{H} = \gamma^0 \gamma(p + e_0 A) + m \gamma^0 + \frac{i}{2} (1 - 4 \sin^2 \theta_W + \gamma^5) G n, \]

where $A = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 0)$. Using the chiral representation of the $\gamma$-matrixes, we obtain the Hamiltonian in the block-matrix form:

\[ \hat{H} = \begin{pmatrix} -\sigma'(\hat{p} + e_0 A) + G n (1 - 2 \sin^2 \theta_W) & m \\ m & \sigma'(\hat{p} + e_0 A) - 2 G n \sin^2 \theta_W \end{pmatrix}. \]

where $\sigma'$ are the Pauli matrixes. This form of the Hamiltonian makes the solution describing spin properties of the electron quite transparent.

3. Spin operator

It is obvious from (2.5) that the longitudinal polarization operator $\hat{T}^0 = \frac{1}{m} \sigma' (\hat{p} + e_0 A)$ [1], where $\sigma = (\sigma'\sigma)$ are the Dirac matrixes, can be written in the form

\[ \hat{T}^0 = \begin{pmatrix} \frac{1}{m} \sigma' (\hat{p} + e_0 A) & 0 \\ 0 & \frac{1}{m} \sigma' (\hat{p} + e_0 A) \end{pmatrix} \]

and commutes with the Hamiltonian, $[\hat{T}^0, \hat{H}] = 0$. Therefore, for any of its eigenvectors the Hamiltonian can be presented as the matrix

\[ \hat{H} = \begin{pmatrix} -m \hat{T}^0 - 2 G n \sin^2 \theta_W + G n \\ m \end{pmatrix} \begin{pmatrix} m \\ m \hat{T}^0 - 2 G n \sin^2 \theta_W \end{pmatrix}, \]

where $T^0$ is one of the eigenvalues of the spin operator $\hat{T}^0$. The matrix (3.2) is still $4 \otimes 4$ one, and each of its element is actually a product of the number by the $2 \otimes 2$ unit matrix. Note that in the presence of the matter potential proportional to $G n$ the transverse polarization operator does not commute with the Hamiltonian. This is a consequence of $\gamma^5$, present in (2.4b).

4. Energy spectrum of an electron in the matter and magnetic field

In order to find the electron energy spectrum $p_0$ in the matter and constant magnetic field, $\hat{H} \Psi = p_0 \Psi$, we should solve the equation

\[ \begin{vmatrix} -m \hat{T}^0 + G n - \hat{p}_0 & m \\ m & m \hat{T}^0 - \hat{p}_0 \end{vmatrix} = 0, \]

where $\hat{p}_0 = p_0 + 2 G n \sin^2 \theta_W$. The solutions can be written in the form

\[ p_0 = \frac{G n}{2} - 2 G n \sin^2 \theta_W + \varepsilon \sqrt{\left( m \hat{T}^0 - \frac{G n}{2} \right)^2 + m^2}, \]

where $\varepsilon = \pm 1$ is the ‘sign’ of the energy.

It is significant to note an interesting feature of the electron energy spectrum in the magnetized matter following from (4.2). It is well known that the energy spectrum of the electron in the magnetic field is degenerated in respect of the spin quantum number
The presence of the matter (of any non-vanishing density \( n \neq 0 \)) removes the spin degeneracy. This phenomenon can be attributed to the parity violation in weak interactions, but the infinite degeneracy of the Landau levels associated with a quantum number \( p_2 \) is still preserved.

Let us emphasize one important relation between \( p_0 \) and \( T_0 \) that immediately follows from the spectrum (4.2):

\[
\left( p_0 - \frac{Gn}{2} + 2Gn \sin^2 \theta_W \right)^2 = \left( mT_0 - \frac{Gn}{2} \right)^2 + m^2,
\]

where \( T_0 \) is one of the eigenvalues of the spin operator \( \hat{T}_0 \). Note that this formula can also be obtained by using the concept of \( \ast \)-spin introduced in [13]. We use this relation in section 7.

5. The electron wavefunctions

Note that we can considerably simplify the problem of finding wavefunctions if take into account some obvious facts. The solution of equation (2.4a) due to symmetries can be sought in the form

\[
\Psi(t, x, y, z) = e^{-ip_0 t + ip_3 z} \begin{pmatrix} \psi_1(x, y) \\ \psi_2(x, y) \\ \psi_3(x, y) \\ \psi_4(x, y) \end{pmatrix},
\]

Substituting (5.1) into (2.4a), we arrive at a system of linear equations for the electron wavefunction components:

\[
(Gn - p_3)\psi_1 + i \left\{ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e_0 B}{2} (x - iy) \right\} \psi_2 + m\psi_3 = \tilde{p}_0 \psi_1,
\]

\[
i \left\{ \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) - \frac{e_0 B}{2} (x + iy) \right\} \psi_1 + (p_3 + Gn)\psi_2 + m\psi_4 = \tilde{p}_0 \psi_2,
\]

\[
\frac{m}{2} \psi_4 + i \left\{ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{e_0 B}{2} (x - iy) \right\} \psi_3 - p_3 \psi_4 = \tilde{p}_0 \psi_3.
\]

Note that the following calculations can be performed in different ways. The shortest way is to abbreviate this system of equations by using the chiral creation and annihilation operators of the two-dimensional harmonic oscillator. This approach has been discussed in detail in papers [14, 15]. Nevertheless, we would like to choose a longer procedure because it resembles the initial technics developed for the derivation of the exact solution of the Dirac equation in a constant magnetic field, which may be familiar and simpler for the reader.

In the polar coordinates \( x + iy = r e^{i\phi}, x - iy = r e^{-i\phi} \) one obtains

\[
\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = e^{i\phi} \left( \frac{\partial}{\partial r} + i \frac{1}{r} \frac{\partial}{\partial \phi} \right), \quad \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} = e^{-i\phi} \left( \frac{\partial}{\partial r} - i \frac{1}{r} \frac{\partial}{\partial \phi} \right),
\]

and the system of equations (5.2a)–(5.2d) reads now

\[
(-p_3 + Gn)\psi_1 + i e^{-i\phi} \left\{ \frac{\partial}{\partial r} - i \frac{1}{r} \frac{\partial}{\partial \phi} + \frac{e_0 B}{2} \frac{1}{r} \right\} \psi_2 + m\psi_3 = \tilde{p}_0 \psi_1.
\]
The solutions are the eigenvectors of the total momentum operator \( J^z \) and obtain the Hamiltonian in the following form:

\[
\hat{H} = \begin{pmatrix}
-m\chi_1 + (p_3 + G\chi_2)\chi_3 + m\chi_4 = \tilde{p}_0\chi_1, \\
\frac{d}{dr} - \frac{l - 1}{r} - \frac{e_0B}{2r} \chi_1 + (p_3 + G\chi_2 + m\chi_4 = \tilde{p}_0\chi_2, \\
m\chi_2 - \frac{d}{dr} - \frac{l - 1}{r} - \frac{e_0B}{2r} \chi_3 - p_3\chi_4 = \tilde{p}_0\chi_4.
\end{pmatrix}
\]

Now, we define the creation and annihilation operators

\[
R^+ = \frac{d}{dr} - \frac{l - 1}{r} - \frac{e_0B}{2r}, \quad R^- = \frac{d}{dr} + \frac{l}{r} + \frac{e_0B}{2r}
\]

and obtain the Hamiltonian in the following form:

\[
\hat{H} = \begin{pmatrix}
-p_3 + G\chi_1 & -R^- & m & 0 \\
R^+ & p_3 + G\chi_2 & 0 & m \\
m & 0 & p_3 & R^- \\
0 & m & -R^+ & -p_3
\end{pmatrix} - 2Gn\sin^2\theta_w \hat{I},
\]

where \( \hat{I} \) is the unit matrix. Note that the derived form of the Hamiltonian is spectacularly transparent that significantly simplifies the problem of obtaining the explicit form of the eigenvalues of the Hamiltonian (2.4a). For forthcoming applications, we take into consideration the properties of the operators \( R^+ \) and \( R^- \):

\[
R^+ L^l_{-1} \left( \frac{e_0B}{2r^2} \right) = -\sqrt{2e_0B(s + l)} L^l_{-1} \left( \frac{e_0B}{2r^2} \right),
\]

\[
R^- L^l_{-1} \left( \frac{e_0B}{2r^2} \right) = \sqrt{2e_0B(s + l)} L^l_{-1} \left( \frac{e_0B}{2r^2} \right),
\]

where \( L^l_{-1} \) are the Laguerre functions [1].
The solution of the system (5.6a)–(5.6d) (the eigenvector of the Hamiltonian (5.8)) can be written in the form

\[
\begin{pmatrix}
\chi_1 \\
\chi_2 \\
\chi_3 \\
\chi_4
\end{pmatrix} = \sqrt{e_0 B} \begin{pmatrix}
C_1 \mathcal{L}_s^{l-1} \left( \frac{e_0 B}{2} \right) \\
C_2 \mathcal{L}_s \left( \frac{e_0 B}{2} r^2 \right) \\
C_3 \mathcal{L}_s^{l-1} \left( \frac{e_0 B}{2} \right) \\
C_4 \mathcal{L}_s \left( \frac{e_0 B}{2} r^2 \right)
\end{pmatrix}.
\] (5.10)

Now we can obtain the eigenvalues of the Hamiltonian (the energy spectrum) and of the spin operator \(T_0\),

\[
p_0 = \frac{Gn}{2} - 2Gn \sin^2 \theta_W + \varepsilon \sqrt{\left( \frac{Gn}{2} \pm \sqrt{p_3^2 + 2e_0 B(l + s)} \right)^2 + m^2}, \quad \varepsilon = \pm 1,
\] (5.11)

\[
T_0^0 = \frac{s'}{m} \sqrt{p_3^2 + 2e_0 B(l + s)}, \quad s' = \pm 1.
\] (5.12)

It is easy to see that the spectrum (4.2) obtained above is in agreement with expressions (5.11) and (5.12). From this energy spectrum, it is straightforward that the well-known energy spectrum in the magnetic field (the Landau levels) is modified by interaction of the electron with matter. However, the radius of the classical orbits corresponding to a certain level (5.11) does not depend on the matter density:

\[
\langle R \rangle = \int_0^\infty \Psi^* r \Psi \, dr = \sqrt{\frac{2N}{e_0 B}}.
\] (5.13)

Note that this result is a simple consequence of the fact that the orbital part of the wavefunctions (5.10) is not altered by the matter potential. To conclude this section, we would like to note that the effect of electron trapping on circular orbits in magnetized matter exists, and this can be important for astrophysical applications.

To finalize the section, we note that it is not obvious from the matrix form of the Hamiltonian (5.8) that it is a self-adjoint operator. To prove this, the properties of creation and annihilation operators should be considered more carefully. The functions \(F^i_s = \sqrt{e|BL_s} \left( \frac{\mu B}{2} r^2 \right)\) constitute a basis in the Hilbert space with a scalar product defined as

\[
\langle F^i_s, F^j_s' \rangle = \int_0^\infty F^i_s F^j_s' r \, dr.
\] (5.14)

So that, we obtain for each \(s\) and \(l\):

\[
\langle F^{l-1}_s, R^- F^l_s \rangle = -\langle F^l_s, R^* F^{l-1}_s \rangle.
\] (5.15)

Hence for operators (5.7) we obtain \((R^-)^* = -R^+\) and \((R^+)^* = -R^-\), where the symbol * implies the Hermitian conjugation of operators. From the matrix form of the Hamiltonian (5.8), we see that \(H^* = H\).

6. General problem of eigenvectors and eigenvalues of a given class of Hamiltonians

We argue in this section that the problem considered above is a particular case of eigenvectors and eigenvalues evaluation for more general class of Hamiltonians. We would like to note
that the discussed problem can be attributed to Noether’s theorem, according to which the symmetry and the algebraic form of the Hamiltonian determine the solvability. The examples of these particular cases can be found in papers [1, 8, 9].

Taking into account the increasing interest to such Hamiltonians, we expect that their profound consideration would be important for applications. Let us explicitly specify conditions under which the problem can be solved for different Hamiltonians. For the further discussion, it is useful to introduce the creation and annihilation operators determined as

\[
\hat{a}\psi_{\{n\}} = f_-(\{n\})\psi_{\{n-1\}}, \quad \hat{a}^\dagger\psi_{\{n-1\}} = f_+(\{n\})\psi_{\{n\}},
\]

where \(f_-(\{n\})\) and \(f_+(\{n\})\) are known functions that depend on the set of quantum numbers \(\{n\}\). A Hamiltonian should be of the following structure:

\[
\hat{H} = \begin{pmatrix}
-m & 0 & p & \hat{a} \\
0 & -m & \hat{a}^\dagger & -p \\
p & \hat{a}^\dagger & m & 0 \\
\hat{a}^\dagger & m & 0 & 0
\end{pmatrix}
\text{ or } \hat{H}' = \begin{pmatrix}
-p & \hat{a}^\dagger & m & 0 \\
\hat{a} & p & 0 & m \\
m & 0 & p & -\hat{a} \\
0 & m & -\hat{a}^\dagger & -p
\end{pmatrix},
\]

where \(m\) is the mass of a particle, \(p\) is (often a third) component of the momentum. Note that these two Hamiltonians are unitary equivalent. Indeed, \(HU = UH'\) with

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
i & 0 & -i & 0 \\
0 & -i & 0 & i \\
-i & 0 & -i & 0 \\
0 & i & 0 & i
\end{pmatrix}
\]

and so that we shall use \(\hat{H}\) only.

The approach is also valid for definite replacements within the matrixes (6.2), in particular, \(\hat{a} \leftrightarrow \hat{a}^\dagger\), \(\hat{a} \rightarrow \hat{i}\), \(\hat{a}^\dagger \rightarrow -i\hat{a}^\dagger\), \(p \leftrightarrow -p\), \(m \leftrightarrow -m\), etc. Nevertheless, the disposition of operators \(\hat{a}, \hat{a}^\dagger\) and zeros in the matrixes is important here. There should be only one operator (\(\hat{a}\) or \(\hat{a}^\dagger\)) and only one zero in any string and in any column. Remarkably, for any string or column the operators and zeros occupy only even or only odd positions. It is also known [16] that the considered Hamiltonians (6.2) and (5.8) can be reduced to a Jaynes–Cummings model that is a general integrable model including all other possible solvable extensions.

The eigenvector now is given by

\[
\Psi_{\{n\}} = \begin{pmatrix}
C_1\psi_{\{n-1\}} \\
C_2\psi_{\{n\}} \\
C_3\psi_{\{n-1\}} \\
C_4\psi_{\{n\}}
\end{pmatrix},
\]

and the equation for the spectrum (for the Hamiltonian \(\hat{H}\)) takes the form of determinant

\[
\begin{vmatrix}
m & -E & 0 & p & f_-(\{n\}) \\
0 & m & p & f_+(\{n\}) & -E \\
p & f_-(\{n\}) & -m & p & 0 \\
f_+(\{n\}) & -p & 0 & m & -E
\end{vmatrix} = 0.
\]

The spectrum can be obtained from (6.4) explicitly in the form

\[
E = \varepsilon \sqrt{m^2 + p^2 + f_+(\{n\})f_-(\{n\})}, \quad \varepsilon = \pm 1.
\]
The spin operator (one of the possible variants) can be constructed from the Hamiltonian. It is a matrix of operators, zeros, integrals of motion and other parameters \((m)\). A structure of the spin operator corresponding to the Hamiltonians in \((6.2)\) can take one of the following variants:

\[
\hat{S} = \begin{pmatrix}
* & 0 & * & \hat{a} \\
0 & * & \hat{a}^* & *
\end{pmatrix} \quad \text{or} \quad \hat{S} = \begin{pmatrix}
* & \hat{a} & * & 0 \\
\hat{a}^* & * & 0 & *
0 & * & \hat{a}^* & *
\end{pmatrix}, \quad (6.6)
\]

where all free positions * should be filled in using the main condition \([S, H] = 0\). Of course, some refinement of the structure of spin operator (or any of its blocks) is possible, for example, \(\hat{a} \to c\hat{a}, \hat{a}^* \to c^*\hat{a}^*\), where \(c\) is a complex number, \(|c| = 1\). Note that many of the known spin operators (see for example [1]) have such a structure.

The approach described above has many useful applications; some of them were presented in our previous papers \([8, 9]\). Now we consider one more. Let us return to the Hamiltonian \((2.4b)\). It differs from \(\hat{H}'\) in equation \((6.2)\) by the presence of the term \(B\gamma^5\) with \(B\) constant. Applying the above prescription, we can construct the spin operator in the following form:

\[
\hat{S} = \begin{pmatrix}
* & \hat{a} & 0 & 0 \\
\hat{a}^* & * & 0 & 0 \\
0 & 0 & * & \hat{a} \\
0 & 0 & \hat{a}^* & *
\end{pmatrix}. \quad (6.7)
\]

The energy spectrum depends on a spin quantum number similar to \((5.11)\).

At last, we can generalize the method to the Hamiltonians of more complicated structure, namely,

\[
\hat{H} = \begin{pmatrix}
h_{11} & 0 & h_{13} & \hat{a} \\
0 & h_{22} & \hat{a}^* & h_{24} \\
h_{13} & \hat{a} & h_{33} & 0 \\
\hat{a}^* & h_{24} & 0 & h_{44}
\end{pmatrix} \quad \text{or} \quad \hat{H} = \begin{pmatrix}
h_{11} & \hat{a} & h_{13} & 0 \\
\hat{a}^* & h_{22} & 0 & h_{24} \\
h_{13} & 0 & h_{33} & \hat{a} \\
0 & h_{24} & \hat{a}^* & h_{44}
\end{pmatrix}. \quad (6.8)
\]

The equation for the spectrum of this model is

\[
\begin{vmatrix}
h_{11} - E & 0 & h_{13} & f_-((n)) \\
0 & h_{22} - E & f_+((n)) & h_{24} \\
h_{13} & f_+((n)) & h_{33} - E & 0 \\
f_+((n)) & h_{24} & 0 & h_{44} - E
\end{vmatrix} = 0. \quad (6.9)
\]

It is well known that the solution of such an equation can be explicitly given in radicals (the Ferrari–Cardano formula).

7. Spin coefficients

We return to the Hamiltonian \((3.2)\). In order to find its eigenvectors, we consider the following chain of transformations:

\[
(-mT^0 + \tilde{p}_0)C_1 + mC_3 = 0, \quad (7.1)
\]
Hence, we obtain

$$\left( mT^0 - \frac{Gn}{2} + \frac{Gn}{2} \right) C_1 = mC_3,$$

$$\left( \frac{p_0 - Gn}{2} \right) \left( 1 + \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}} \right) C_1 = \left| \frac{p_0 - Gn}{2} \right| \sqrt{1 - \left( \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}} \right)^2} C_3, \quad (7.2)$$

\[ \varepsilon \sqrt{1 + \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} C_1 = \sqrt{1 - \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} C_3. \]

Hence, we obtain

$$C_1 = \sqrt{1 - \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} A, \quad C_3 = \varepsilon \sqrt{1 + \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} A. \quad (7.3)$$

In the same way, we obtain the following expressions for \( C_2 \) and \( C_4 \):

$$C_2 = \sqrt{1 - \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} B, \quad C_4 = \varepsilon \sqrt{1 + \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} B, \quad (7.4)$$

where \( A \) and \( B \) are new constants.

Getting the relation between \( C_1 \) and \( C_2 \) (and between \( C_3 \) and \( C_4 \)) is similar. Namely, we should use the equations

$$(p_3 - mT^0)C_{1,3} + \sqrt{2e_0B(l + s)}C_{2,4} = 0. \quad (7.5)$$

We finally obtain

$$A = \sqrt{1 + \frac{p_3}{mT^0} C}, \quad B = s \sqrt{1 - \frac{p_3}{mT^0} C}, \quad (7.6)$$

with the only coefficient \( C \) which has to be defined from the normalization condition

$$C_1^2 + C_2^2 + C_3^2 + C_4^2 = 1.$$ We obtain \( C = \frac{1}{2} \). Finally, we obtain the wavefunction

$$\Psi(t, x, y, z) = e^{-ip_0t} \frac{1}{\sqrt{L}} e^{ip_3z} \sqrt{\frac{e_0B}{2\pi}} \left( \begin{array}{c} C_1 L_{l}^{\frac{1}{2}} \left( \frac{e_0B}{2} r^2 \right) e^{i(l+1)\phi} \\ iC_2 L_{l}^{\frac{1}{2}} \left( \frac{e_0B}{2} r^2 \right) e^{i\phi} \\ C_3 L_{l}^{\frac{1}{2}} \left( \frac{e_0B}{2} r^2 \right) e^{i(l-1)\phi} \\ iC_4 L_{l}^{\frac{1}{2}} \left( \frac{e_0B}{2} r^2 \right) e^{i\phi} \end{array} \right), \quad (7.7)$$

where

$$C_1 = \frac{1}{2} \sqrt{1 - \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} \sqrt{1 + \frac{p_3}{mT^0}}, \quad C_2 = \frac{s'}{2} \sqrt{1 - \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} \sqrt{1 - \frac{p_3}{mT^0}}, \quad (7.8)$$

$$C_3 = \frac{\varepsilon}{2} \sqrt{1 + \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} \sqrt{1 + \frac{p_3}{mT^0}}, \quad C_4 = \frac{s'\varepsilon}{2} \sqrt{1 + \frac{mT^0 - \frac{Gn}{2}}{p_0 - \frac{Gn}{2}}} \sqrt{1 - \frac{p_3}{mT^0}} \quad (7.9)$$

and \( L \) is a normalizing factor.

Equations (7.7)–(7.9) represent the exact solution of (2.4a) with the Hamiltonian (2.5) that describes the electron moving in the matter and magnetic field. Note that in the case \( n = 0 \), these formulas are reduced to well-known solutions for the electron wavefunctions in a constant homogenous magnetic field [1].

9
8. Application

It is important to point out that the obtained exact solution for the electron motion in the matter and magnetic field can be used as the first approximation of description of particles moving in an external environment of more complicated configuration. As an example, we demonstrate how the problem of an electron’s (or another charged particle) motion in a rotating matter with the magnetic field can be solved. This problem is of interest in different astrophysical contexts.

If the angular velocity $\omega$ is small compared to the magnetic field, we can calculate the spectrum using a standard perturbation theory with the small parameter $\frac{G_{\text{ne}}}{e_0B} \ll 1$. If, for example, we choose for the matter density, angular velocity and magnetic field the values peculiar for a rotating neutron star ($n = 10^{37}$ sm$^{-3}$ = 7.72 $\times$ 10$^{22}$ (eV)$^3$, $\omega = 2 \pi \times 10^3$ s$^{-1}$ = $2\pi \times 0.66 \times 10^{-12}$ eV, $B = 10^{10}$ Gs = $7 \times 10^8$ (eV)$^2$), then the parameter is really small:

$$\frac{G_{\text{ne}}}{e_0B} = 6.3 \times 10^{-20} \ll 1.$$  (8.1)

Now, we can take the spectrum and wavefunctions found above (i.e. without rotation) as the lowest order of perturbation series and find the correction term.

We consider the particular case of the constant magnetic field and rotating uniform matter so that the electromagnetic field and effective matter potential are given by

$$A^\mu = \left(0, -\frac{yB}{2}, \frac{xB}{2}, 0\right), \quad f^\mu = -Gn(1, \omega y, \omega x, 0),$$  (8.2)

where $G = \frac{G_{\text{ne}}}{\sqrt{2}}$. Let us rewrite equation (2.2) in the Hamiltonian form

$$i \frac{\partial}{\partial t} \Psi(x) = \hat{H} \Psi(x),$$  (8.3)

$$\hat{H} = \gamma^0 \gamma(p + e_0A) + m \gamma^0 \left(1 - 4 \sin^2 \theta_W + \gamma^5\right) + \frac{Gn}{2} \gamma^0 \gamma^1 \left(1 - 4 \sin^2 \theta_W + \gamma^5\right) \omega y - \frac{Gn}{2} \gamma^0 \gamma^2 \left(1 - 4 \sin^2 \theta_W + \gamma^5\right) \omega x,$$  (8.4)

with $\tilde{A} = \left(-\frac{yB}{2}, \frac{xB}{2}, 0\right)$. Using again the chiral representation of the $\gamma$-matrixes we obtain the Hamiltonian in the block-matrix form:

$$\hat{H} = \left(-\sigma(\hat{p} + e_0A) + \frac{Gn}{m} \sigma(\hat{p} + e_0A)\right)$$

$$-2Gn \sin^2 \theta_W - G_{\text{ne}} \begin{pmatrix} \sigma_1 y - \sigma_2 x & 0 \\ 0 & 0 \end{pmatrix}.$$  (8.5)

It is obvious from (8.5) that $H = H_0 + H_1$, where $H_1$ is the last term of (8.5) and takes the following form in the polar coordinates:

$$H_1 = \begin{pmatrix} 0 & -i \rho e^{-i\phi} & 0 & 0 \\ i \rho e^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho = G_{\text{ne}}.$$  (8.6)

So, we obtain for the first correction to the energy spectrum (4.2) of the electron

$$\Delta E_0^N = \int \Psi_N^* \hat{H}_1 \Psi_N \, dV = \sqrt{2e_0BC_1C_2} \frac{G_{\text{ne}}}{e_0B} \int_0^\infty \mathcal{L}_{N-\frac{1}{2}}(\xi) \sqrt{\xi} \mathcal{L}_{N-\frac{1}{2}}'(\xi) \, d\xi.$$  (8.7)
where $C_1$ and $C_2$ are the coefficients from equation (7.8). The integral can be calculated explicitly, and finally we obtain

$$\Delta p_{0N}^N = \sqrt{2\varepsilon_0 B C_1 C_2^2 \frac{2G\hbar\alpha}{\varepsilon_0 B}} \sqrt{N} = 2G\hbar\alpha C_1 C_2 \sqrt{\frac{2N}{\varepsilon_0 B}} = 2G\hbar\alpha C_1 C_2 \omega \langle R \rangle. \quad (8.8)$$

This shift of levels in the energy spectrum depending on the energy quantum number $N = 0, 1, 2 \ldots$ leads to a corresponding shift in a frequency of synchrotron radiation of electron inside of dense rotating matter, that can be registered. Calculation of the next perturbation terms (higher order) is an interesting task that could reveal new effects. However, we pointed out that the small parameter (8.1) is really very small.

9. Conclusion

In this paper, we found a class of exact solutions of the modified Dirac equation which describes the charged leptons propagating in the uniform matter and strong constant magnetic field. We also pointed out, how this approach can be generalized to a given class of Dirac Hamiltonians. The obtained solution for the particular case of the electron motion in a rotating neutron star which account for matter and magnetic field effects can be used as the first approximation in more complicated models. All of these considerations can be useful in studies of Dirac particles interactions in different astrophysical and cosmology environments. The obtained solutions of the Dirac equation can be applied in studies of gamma-rays production during collapse or coalescence processes of neutron stars (the one predicted within the fireball models of GRBs [17]), as well as during a neutron star being ‘eaten up’ by the black hole, or in investigations of other processes in environments discussed in [12]. The behavior of hypothetical electrically millicharged neutrino in external fields [18] provides another interesting application of the performed studies.

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