Entanglement recovered periodically

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Abstract. We study the entanglement dynamics of two not interacting atoms coupled weak and far from the resonance with the same single mode of the field. We consider two special initial states of the atoms. Specifically we study the cases when the atoms start in a pure entangled state and in a mixed state. When the field mode is initially in a coherent or thermal state, we find that in general in the high energy limit there is no entanglement between the two atoms, however at well defined moments the initial entanglement is as suddenly recovered as removed.

1. Introduction
The non locality effect or entanglement is considered as a resort for manipulation of quantum information which does not have a classical counterpart \cite{1, 2}. During the last two decades a major research effort has been conducted in the emerging field of quantum information theory \cite{3} based on the renewed non locality effect. With this motivation, there has been a lot of interest in understanding and quantifying entanglement of pure and mixed states \cite{4, 5, 6, 7}.

The entanglement of two systems can arise through direct interaction between them as well as through the coupling of the systems with a common quantum bus in the form of an auxiliary system or environment \cite{8, 9, 10, 11, 12, 13, 14}.

In this work we investigate how the entanglement between two atoms can be modified when they interact dispersively with the same single mode of the electromagnetic field.

2. The effective Hamiltonian
We consider two noninteracting two-level atoms, labelled by \(a\) and \(b\), each one coupled dispersively with a single mode characterized by the frequency \(\omega\). The unitary dynamics in the whole tensorial product Hilbert space, \(H = H_a \otimes H_b \otimes H_{\text{mode}}\), is driven by the Jaynes-Cummings Hamiltonian \cite{15}.

Taking into account that the excitation number operator is a constant of motion and defining the small parameters: \(\epsilon_j = g_j / \Delta_j \ll 1 / \sqrt{\langle n \rangle} \ll 1\), \((j = a, b)\), where \(\langle n \rangle\) is the average photon number, \(\Delta_a = \omega_a - \omega\) and \(\Delta_b = \omega_b - \omega\), \(\omega_a\) (\(\omega_b\)) is the energy transition on the atoms \(a\) (\(b\)), and \(\omega\) is the single mode frequency.

Making use of the small rotation method \cite{16, 17} to obtain the effective Hamiltonian which approximately describes the interaction process, we can eliminate the terms which do
not represent the resonance interaction but represent rapid oscillations in the rotating frame. Considering terms up to first order in $\epsilon_a$ and $\epsilon_b$ we obtain the following effective Hamiltonian:

$$\hat{H}_{\text{eff}} = \frac{\Delta_a}{2} \sigma_z^{(a)} + \frac{\Delta_b}{2} \sigma_z^{(b)} + \left(b^\dagger b + \frac{1}{2}\right) \left(\frac{g_a^2}{\Delta_a} \sigma_z^{(a)} + \frac{g_b^2}{\Delta_b} \sigma_z^{(b)}\right) + \frac{g_a g_b}{2} \left(\frac{1}{\Delta_a} + \frac{1}{\Delta_b}\right) \left(\sigma_z^{(a)} \sigma_-^{(b)} + \sigma_-^{(a)} \sigma_z^{(b)}\right),$$

(1)

where $b$ and $b^\dagger$ are the annihilation and creation single mode operators, $\sigma_z^{(j)} = |1\rangle_j \langle 0|$, $\sigma_z^{(j)} = |0\rangle_j \langle 1|$, and $\sigma_z^{(j)}$ is the $z-$ component of the effective angular spin-half operator whose eigenstates are $\{|0\rangle_j, |1\rangle_j\}$, for $j = a, b$.

The first two terms in the above effective Hamiltonian represents the free evolution of the non-resonant atoms with renormalized transition frequencies. The third term is the so-called, dynamical Stark shifts, which describes an additional intensity dependent detuning of non-resonant atoms from the mode frequency. The last term represents an effective dipole-dipole interaction between the non-resonant atoms which appears as a consequence of a collective nature of the interaction of non-resonant atoms with the quantized mode.

For the sake of simplicity, we suppose that both atoms are identical in a way such that $g = g_a = g_b$ and $\Delta = \Delta_a = \Delta_b$. Under those conditions the unitary evolution operator is given by

$$\hat{U} = e^{-i\left(\Delta + g^2(2b^\dagger b + 1)/\Delta\right)\lambda_1} |1\rangle_a \langle 1|_b \langle 1|_b \langle 1| + e^{-i\frac{g^2}{\Delta} |\psi^+\rangle \langle \psi^+| + e^{i\frac{g^2}{\Delta} |\psi^-\rangle \langle \psi^-|} + e^{i\left(\Delta + g^2(2b^\dagger b + 1)/\Delta\right)\lambda_1} |0\rangle_a \langle 0|_b \langle 0|_b \langle 0|,$n

(2)

here $|\psi^\pm\rangle = (|0\rangle_a |1\rangle_b \pm |1\rangle_a |0\rangle_b)/\sqrt{2}$ are two Bell states.

From Eqs. (1) and (2) one realizes that the dressed states $|1\rangle_a |1\rangle_b |n\rangle$, $|0\rangle_a |0\rangle_b |n\rangle$, and $|\psi^\pm\rangle |n\rangle$ ($n = 0, 1, 2, \ldots$) are stationary. Thus, by their form, independent of the initial mode state, the $a - b$ bipartite system does not evolve when starting at one of the states $|1\rangle_a |1\rangle_b$, $|0\rangle_a |0\rangle_b$, or $|\psi^\pm\rangle$, and hence preserves the initial entanglement.

3. Atom-atom entanglement of formation

In order to quantify the entanglement between the two atoms we make use of the C concurrence expression found by W. Wootters [7]. $C = 1$ means that the two atoms are in a maximal entangled state whereas $C = 0$ stands for that the two atoms are in a separable state, otherwise means partially entangled state.

First we study the entanglement of formation between the two atoms when each of them is initially in a pure state. After that we consider the two atoms in a mixed states. In both cases we study how the entanglement evolves as a function of time and energy mode, under the weak and dispersive interaction (1). We consider the initial mode state to be a Fock state $|n\rangle$, a coherent state $|\alpha\rangle$, and a thermal state $P_T = \sum_n|n\rangle_T^n/(1 - (n)T)^{n+1}|n\rangle$, where $(n)T = 1/(e^{\epsilon/k_B T} - 1)$ is the average photon number of the mode, $T$ the absolute temperature and $k_B$ the Boltzmann constant ($\hbar = 1$).

Now we consider the atoms to be initially in the entangled pure state

$$|\psi\rangle = \cos \varphi |0\rangle_a |0\rangle_b + \sin \varphi |1\rangle_a |1\rangle_b.$$
Figure 1. Concurrence as a function of the dimensionless time $\tau/\pi$ and of the average photon number of the single mode when the initial mode state is: (a) a coherent state and (b) a thermal state. In both cases the atoms start in the $|\psi\rangle$ state. White color means zero entanglement of formation whereas black color stand for their maximum entanglement value $8/11$.

When the single mode is initially in a Fock state $|n\rangle$, the two atoms-system does not tangle with it at all time, and they do not change the initial entanglement amount. However when the single mode begins in a coherence state $|\alpha\rangle$ the concurrence evolves as follows:

$$C = |\sin(2\varphi)| e^{-|\alpha|^2[1-e^{4ig^2t/\Delta}]}.$$  \hspace{1cm} \text{(3)}$$

On the other hand, when the single mode is initially in a thermal state the concurrence between the two atoms is given by:

$$C = \frac{|\sin(2\varphi)|}{\sqrt{1 + \langle n \rangle_T(1 - e^{4ig^2t/\Delta})}}.$$  \hspace{1cm} \text{(4)}$$

where $\langle n \rangle_T$ is the average thermal photons of the single mode. From Eqs. (3) and (4) we can see that at high energy the entanglement is practically zero, however independently of the energy regime the initial concurrence value $C = |\sin(2\varphi)|$ is recovered at each $t = t_k$, where $t_k \equiv k\pi\Delta/(2g^2)$ with $k$ being an integer.

Figures (1) shows a linear black-white degradation of the concurrences given by: (a) the Eq. (3) and (b) the Eq. (4), as a function of the dimensionless time $\tau = 4g^2t/\Delta$ and of the average photon number of the single mode. White color means zero entanglement of formation whereas black color stand for their maximum entanglement value $C = |\sin(2\varphi)| = 8/11$. In both figures (1) we considered the qubits starting in the $|\psi\rangle$ state.

From the figures (1) we see that maximal entanglement arises periodically at high energy. Those maximal entanglement peak are separated by the so called entanglement dead valleys.
[12]. At low energy the entaglement practically do not change. This effect is a reminiscence of the entanglement generated by the dispersive vacuum state [18].

Now we study the case when both atoms are initially in a Werner state [19],[20] type:

$$\rho(0) = \frac{1-\gamma}{4} I + \gamma |X\rangle\langle X|,$$

with $I$ being the identity of the two atoms Hilbert space, $|X\rangle$ being one of the four Bell states [20], and $0 \leq \gamma \leq 1$ a physical real parameter.

One can prove easily that when the single mode starts in a Fock state $|n\rangle$ the concurrence does not change and remains in $(3\gamma - 1)/2$ for $\gamma \geq 1/3$ and zero otherwise.

However, when the field mode starts in a coherent state $|\alpha\rangle$ the concurrence at time $t$ is given by

$$C = \max\{0, \frac{1 + 2e^{-2|\alpha|^2 \sin^2 \frac{2\gamma^2 t}{\Delta}}}{2} \gamma - 1\}. \quad (6)$$

Clearly we see that for a high intensity coherent state, in general, there is not entanglement between the two atoms, however at each time $t = t_k$ the initial entanglement amount, $\max\{0, (3\gamma - 1)/2\}$, is suddenly recovered and is independent of the intensity of the coherent state. Here we have consider $|X\rangle = |\phi^\pm\rangle = (|0_a\rangle|0_b\rangle \pm |1_a\rangle|1_b\rangle)/\sqrt{2}$. When one consider the other two Bell states $|X\rangle = |\psi^\pm\rangle$, the (5) density operator does not evolve.

On the other hand, when the mode state is initially in a thermodynamic equilibrium at absolute temperature $T$ the concurrence of the two atoms at time $t$ becomes:

$$C(\rho) = \max\{0, \frac{1 + 2e^{-2|\alpha|^2 (1-e^{-4g^2 t/\Delta})}}{2} \gamma - 1\}. \quad (7)$$

Once again we find the entanglement is in general zero at high intensity or equivalently in the high temperature regime and the initial entanglement amount is suddenly recovered just at each times $t = t_k$. We can also seen from Eqs. (3), (4), (6), and (7) that when the field mode is initially in the vacuum state, $|\alpha|^2 = \langle n \rangle_T = 0$, the entanglement does not change. The expression (7) was calculated considering the state $|X\rangle = |\phi^\pm\rangle$ in Eq. (5). When $|X\rangle = |\psi^\pm\rangle$ the (5) density operator does not evolve.

4. Conclusions

We have studied the dynamics of the entanglement between two non interacting two-level atoms weakly coupled and far from the resonance with the same single mode field. We emphasize that in the dispersive regime, the atomic energy is not exchanged with the single mode, so the single mode is required to be only the mediator between the two two-level atoms effective interaction. This effect can not be generated by classical field because classical fields can not couple the two atoms, at any intensity. When they are initially entangled, the initial entanglement amount is periodically recovered in a sudden manner, only for a short moments separated by the time scale $\pi\Delta/g^2$. The wide of a peak is inversely proportional to the energy of the single mode. Besides, in that atomic initial condition the entanglement does not change when the single mode is initially in the vacuum state.

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