Edit Errors with Block Transpositions: Deterministic Document Exchange Protocols and Almost Optimal Binary Codes

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Document exchange and error correcting codes are two fundamental problems regarding communications. In the first problem, Alice and Bob each holds a string, and the goal is for Alice to send a short sketch to Bob, so that Bob can recover Alice’s string. In the second problem, Alice sends a message with some redundant information to Bob through a channel that can add adversarial errors, and the goal is for Bob to correctly recover the message despite the errors. In both problems, an upper bound is placed on the number of errors between the two strings or that the channel can add, and a major goal is to minimize the size of the sketch or the redundant information. In this paper we focus on deterministic document exchange protocols and binary error correcting codes.

Both problems have been studied extensively. In the case of Hamming errors (i.e., bit erasures and substitutions), we have explicit constructions with asymptotically optimal parameters. However, other error types are still rather poorly understood. In a recent work [6], the authors constructed explicit deterministic document exchange protocols and binary error correcting codes for edit errors with almost optimal parameters. Unfortunately, the constructions in [6] do not work for other common errors such as block transpositions.

In this paper, we generalize the constructions in [6] to handle a much larger class of errors. These include bursts of insertions and deletions, as well as block transpositions. Specifically, we consider document exchange and error correcting codes where the total number of block insertions, block deletions, and block transpositions is at most \( k \leq \alpha n / \log n \) for some constant \( 0 < \alpha < 1 \). In addition, the total number of bits inserted and deleted by the first two kinds of operations is at most \( t \leq \beta n \) for some constant \( 0 < \beta < 1 \), where \( n \) is the length of Alice’s string or message. We construct explicit, deterministic document exchange protocols with sketch size \( O(k \log^2 n + t) \) and explicit binary error correcting code with \( O(k \log n \log \log n + t) \) redundant bits. As a comparison, the information-theoretic optimum for both problems is \( \Theta(k \log n + t) \). As far as we know, previously there are no known explicit deterministic document exchange protocols in this case, and the best known binary code needs \( \Omega(n) \) redundant bits even to correct just one block transposition [15].

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1 Introduction

In communications and more generally distributed computing environments, there often arise questions regarding the synchronization of files or messages. For example, a message sent from one party to another party through a channel may get modified by channel noise or adversarial errors, and files stored on distributed servers may become out of sync due to different edit operations by different users. In many situations, these questions can be formalized in the framework of the following two fundamental problems.

- **Document exchange.** In this problem, two parties Alice and Bob each holds a string $x$ and $y$, and the two strings are within distance $k$ in some metric space. The goal is for Alice to send a short sketch to Bob, so that Bob can recover $x$ based on his string $y$ and the sketch.

- **Error correcting codes.** In this problem, two parties Alice and Bob are linked by a channel, which can change any string sent into another string within distance $k$ in some metric space. Alice’s goal is to send a message to Bob. She does this by sending an encoding of the message through the channel, which contains some redundant information, so that Bob can recover the correct message despite any changes to the codeword.

These two problems are closely related. For example, in many cases a solution to the document exchange problem can also be used to construct an error correcting code, but the reverse direction is not necessarily true. In both problems, a major goal is to is to minimize the size of the sketch or the redundant information. For applications in computer science, we also require the computations of both parties to be efficient, i.e., in polynomial time of the input length. In this case we say that the solutions to these problems are explicit. Here we focus on deterministic document exchange protocols and error correcting codes with a binary alphabet, arguably the most important setting in computer science.

Both problems have been studied extensively, but the known solutions and our knowledge vary significantly depending on the distance metric in these problems. In the case of Hamming distance (or Hamming errors), we have a near complete understanding and explicit constructions with asymptotically optimal parameters. However, for other distance metrics/error types, our understanding is still rather limited.

An important generalization of Hamming errors is edit errors, which consist of bit insertions and deletions. These are strictly more general than Hamming errors since a bit substitution can be replaced by a deletion followed by an insertion. Edit errors can happen in many practical situations, such as reading magnetic and optical media, mutations in gene sequences, and routing packets in Internet protocols. However, these errors are considerably harder to handle, due to the fact that a single edit error can change the positions of all the bits in a string.

Non-explicitly, by using a greedy graph coloring algorithm or a sphere packing argument, one can show that the optimal size of the sketch in document exchange, or the redundant information in error correcting codes is roughly the same for both Hamming errors and edit errors. Specifically, suppose that Alice’s string or message has length $n$ and the distance bound $k$ is relatively small (e.g., $k \leq n/4$), then for both Hamming errors and edit errors, the optimal size in both problems is $\Theta(k \log(n/k))$ [12]. For Hamming errors, this can be achieved by using sophisticated linear Algebraic Geometric codes [9], but for edit errors the situation is quite different. We now describe some of the previous works regarding both document exchange and error correcting codes for edit errors.

**Document exchange.** Orlitsky [13] first studied the document exchange problem for generally correlated strings $x, y$. Using the greedy graph coloring algorithm mentioned before, he obtained a deterministic protocol with sketch size $O(k \log n)$ for edit errors, but the running time is exponential in $k$. Subsequent improvements appeared in [8], [10], and [11], achieving sketch size $O(k \log(n^2) \log n)$ [10] and
\[O(k \log^2 n \log^k n) \] \cite{11} with running time \( \tilde{O}(n) \). A recent work by Chakraborty et al. \cite{5} further obtained sketch size \( O(k^2 \log n) \) and running time \( \tilde{O}(n) \), by using a clever randomized embedding from the edit distance metric to the Hamming distance metric. Based on this work, Belazzougui and Zhang \cite{3} gave an improved protocol with sketch size \( O(k(\log^2 k + \log n)) \), which is asymptotically optimal for \( k = 2^{O(\sqrt{\log n})} \). The running time in \cite{3} is \( \tilde{O}(n + poly(k)) \).

Unfortunately, all of the above protocols, except the one in \cite{13} which runs in exponential time, are randomized. Randomized protocols are less useful than deterministic ones in practice, and are also not suitable for the applications in constructing error correcting codes. However, designing an efficient deterministic protocol appears quite tricky, and it was not until 2015 when Belazzougui \cite{2} gave the first deterministic protocol even for \( k > 1 \). The protocol in \cite{2} has sketch size \( O(k^2 + k \log^2 n) \) and running time \( \tilde{O}(n) \).

**Error correcting codes.** As fundamental objects in both theory and practice, error correcting codes have been studied extensively from the pioneering work of Shannon and Hamming. While great success has been achieved in constructing codes for Hamming errors, the progress on codes for edit errors has been quite slow despite much research. A work by Levenshtein \cite{12} in 1966 showed that the Varshamov-Tenengolts code \cite{14} corrects one deletion with an optimal redundancy of roughly \( \log n \) bits, but even correcting two deletions requires \( \Omega(n) \) redundant bits. In 1999, Schulman and Zuckerman \cite{15} gave an explicit asymptotically good code, that can correct up to \( \Omega(n) \) edit errors with \( O(n) \) redundant bits. However the same amount of redundancy is needed even for smaller number of errors.

For any fixed constant \( k \), a recent work by Brakensiek et. al \cite{4} constructed an explicit code that can correct \( k \) edit errors with \( O(k^2 \log k \log n) \) redundant bits. This is asymptotically optimal when \( k \) is a fixed constant, but the construction in \cite{4} only works for constant \( k \), and breaks down for larger \( k \) (e.g., \( k = \log n \)). Based on his deterministic document exchange protocol, Belazzougui \cite{2} also gave an explicit code that can correct up to \( k \) edit errors with \( O(k^2 + k \log^2 n) \) redundant bits.

In a very recent work by the authors \cite{6}, we significantly improved the situation. Specifically, we constructed an explicit document exchange protocol with sketch size \( O(k \log^2 n \frac{1}{\alpha}) \), which is optimal except for another \( \log \frac{1}{\alpha} \) factor. We also constructed explicit codes for \( k \) edit errors with \( O(k \log n) \) redundant bits, which is optimal for \( k \leq n^{1-\alpha} \), any constant \( 0 < \alpha < 1 \). These results bring our understanding of document exchange and error correcting codes for edit errors much closer to that of standard Hamming errors.

However, the constructions in \cite{6} do not work for other common types of errors, such as block transpositions. Given any string \( x \), a block transposition takes an arbitrary substring of \( x \), moves it to a different position, and insert it into \( x \). These errors happen frequently in distributed file systems and Internet protocols. For example, it is quite common that a user, when editing a file, copies a whole paragraph in the file and moves it to somewhere else; and in Internet routing protocols, packets can often get rearranged during the process. Block transpositions also arise naturally in computational biology, where a subsequence of genes can be moved in one operation. In the setting of document exchange or error correcting codes, it is easy to see that even a single transposition of a block with length \( t \) can result in \( 2t \) edit errors, thus a naive application of document exchange protocols or codes for edit errors will result in very bad parameters.

In this paper we consider document exchange protocols and error correcting codes for edit errors as well as block transpositions. In fact, even for edit errors we also consider a larger, more general class of errors. Specifically, we consider edit errors that happen in bursts. This kind of errors is also pretty common, as most errors that happen in practice, such as in wireless or mobile communications and magnetic disk readings, tend to be concentrated. We model such errors as block insertions and deletions, where in one operation the adversary can insert or delete a whole block of bits. It is again easy to see that this is indeed a generalization of standard edit errors. However, in addition to the bound \( k \) on such operations, we also need to put a bound on the total number of bits that the adversary can insert or delete, since otherwise the adversary can simply delete the whole string in one block deletion. Therefore, we model the adversary as follows.
Model of the adversary. An adversary considered in this paper is allowed to perform two kinds of operations: block insertion/deletion and block transposition. The adversary is allowed to perform $k_1$ block insertions/deletions and $k_2$ block transpositions, such that $k_1 + k_2 \leq k$ for a given parameter $k$. In addition, the total number of bits inserted/deleted by block insertions/deletions is at most $t$ for a given parameter $t$. We note that by the result of Schulman and Zuckerman [15], to correct $\Omega(n / \log n)$ block transpositions one needs at least $\Omega(n)$ redundant bits. Thus we only consider $k \leq \alpha n / \log n$ for some constant $0 < \alpha < 1$. Similarly, we only consider $t \leq \beta n$ for some constant $0 < \beta < 1$ since otherwise the adversary can simply delete the whole string.

Edit errors with block transpositions have been studied before in several different contexts. For example, Shapira and Storer [16] showed that finding the distance between two given strings under this metric is NP-hard, and they gave an efficient algorithm that achieves $O(\log n)$ approximation. Interestingly, a work by Cormode and Muthukrishnan [7] showed that this metric can be embedded into the $L_1$ metric with distortion $O(\log n \log^* n)$; and they used it to give a near linear time algorithm that achieves $O(\log n \log^* n)$ approximation for this distance, something currently unknown for the standard edit distance. Coming back to document exchange and error correcting codes, in our model, we show in the appendix that non-explicitly, the information optimum for both the sketch size of document exchange, and the redundancy of error correcting codes, is $\Theta(k \log n + t)$. However, as far as we know, there are no known explicit deterministic document exchange protocols; and the only randomized protocol which can handle edit errors as well as block transpositions is the protocol of [10], that has sketch size $O(k \log(\frac{2}{k}) \log n)$. Similarly, the only previous explicit code that can handle edit errors as well as block transpositions is the work of Schulman and Zuckerman [15], which needs $\Omega(n)$ redundant bits even to correct one block transposition. We note that however none of the previous works mentioned studied edit errors that can allow block insertions/deletions.

1.1 Our results

In this paper we construct explicit deterministic document exchange protocols, and error correcting codes for adversaries discussed above. We have the following theorems.

**Theorem 1.1.** There exist constants $\alpha, \beta \in (0, 1)$ such that for every $n, k, t \in \mathbb{N}$ with $k \leq \alpha n / \log n$, $t \leq \beta n$, there exists an explicit, deterministic document exchange protocol with sketch size $O(k \log n \log \frac{n}{k \log n + t} + t)$, for an adversary who can perform at most $k$ block insertions/deletions and block transpositions, where the total number of bits inserted/deleted is at most $t$.

This is the first explicit, deterministic document exchange protocol for edit errors with block transpositions. The sketch size matches the randomized protocol of [10], and is optimal up to an extra $\log \frac{n}{k \log n + t}$ factor. Using this protocol, we can construct the following error correcting code.

**Theorem 1.2.** There exist constants $\alpha, \beta \in (0, 1)$ such that for every $n, k, t \in \mathbb{N}$ with $k \leq \alpha n / \log n$, $t \leq \beta n$, there exists an explicit binary error correcting code with message length $n$ and codeword length $n + O(k \log n \log \frac{n}{k \log n + t} + t)$, that can correct up to $k$ block insertions/deletions and block transpositions, where the total number of bits inserted/deleted is at most $t$.

For small $k, t$ we can actually achieve the following result, which gives better parameters.

**Theorem 1.3.** There exist constants $\alpha, \beta \in (0, 1)$ such that for every $n, k, t \in \mathbb{N}$ with $k \leq \alpha n / \log n$, $t \leq \beta n$, there exists an explicit binary code with message length $n$ and codeword length $n + O(k \log n \log \log n + t)$, that can correct up to $k$ block insertions/deletions and block transpositions, where the total number of bits inserted/deleted is at most $t$.

In the case of small $k, t$, these results significantly improve the result of Schulman and Zuckerman [15], which needs $\Omega(n)$ redundant bits even to correct one block transposition. The redundancy here is also optimal up to an extra $\log \log n$ factor or $\log \frac{n}{k \log n + t}$ factor.
1.2 Overview of our techniques

In this section we provide an informal, high-level overview of our techniques. We start by describing our document exchange protocol.

**Document exchange.** Our construction is based on the deterministic document exchange protocol of [6], which is in turn based on the randomized protocol of Irmak et. al. [10]. We first briefly describe the construction of [6]. The protocol has $O(\log \frac{n}{k})$ levels where $k$ is the number of edit errors. Throughout the protocol, Bob always maintains a string $\hat{x}$, which is his current version of Alice’s string $x$. In the $i$-th level, both Alice and Bob parse their strings $x$ and $\hat{x}$ evenly into $O(2^i k)$ blocks, i.e., in each subsequent level they divide a block in the previous level evenly into two blocks. The following invariance is maintained: at the beginning of each level, at most $ck$ blocks are different between $x$ and $\hat{x}$, where $c$ is a universal constant.

This property is satisfied at the beginning of the protocol, and maintained for subsequent levels as follows: in each level Alice constructs an $\epsilon$-self-matching hash function (see [6] for the exact definition) using the parsed $x$. This function has a short description. Alice then hashes every block of $x$ and sends some redundancy of the hash values together with the description of the hash function to Bob. The redundancy here is computed by a systematic error correcting code that can correct $ck$ Hamming errors, whose alphabet corresponds to the output of the hash function. Bob, after receiving the redundancy of the hash values and the short description, first uses the hash function to hash every block of $\hat{x}$. Since $\hat{x}$ and $x$ differ in at most $ck$ blocks, Bob can use the redundancy to correctly recover all the hash values. He then uses dynamic programming to find a maximum monotone matching between $x$ and $y$ under the hash function and the hash values, and uses the matched blocks of $y$ to fill the corresponding blocks of his string $\hat{x}$. The analysis shows that the Bob can correctly recover all blocks of $x$ except at most $ck/2$ of them. Thus in the next level $\hat{x}$ is the same as $x$ except for at most $ck$ blocks. At the end of the protocol when the size of each block has become small enough (i.e., $O(\log \frac{n}{k})$), Alice can just send a sketch for $k$ hamming errors of the blocks to let Bob finally recover $x$.

Our starting point is to try to generalize the above protocol. However, one immediate difficulty is to handle block transposition. The protocol of [6] actually performs badly for such errors. To see this consider the following example: the adversary simply moves the first $0.4n$ bits of $x$ to the end. Since the protocol in [6] tries to find the maximum monotone matching in each level, Bob can only recover the last $0.6n$ bits of $x$ since this gives the maximum monotone matching. In this case, one single error has cost roughly half of the string; while as a comparison, for standard edit errors, the protocol in [6] lets Bob recover all except $O(1)$ blocks if there is only one edit error.

To solve this issue, we need to make several important changes to the protocol in [6]. The first major change is that, in each level, instead of having Bob find the maximum monotone matching between $x$ and $y$ using the hash values, we let Bob find the maximum non-monotone matching. However, the notion of $\epsilon$-self-matching hash function in [6] is not suitable for this purpose, since $\epsilon$-self-matching hash functions actually allow a small number of collisions in the hash values of blocks of $x$, and the use of these hash functions in [6] relies crucially on the property of a monotone matching. Instead, here we strengthen the hash function to ensure that there is no collision, by using a slightly larger output size. We call such hash functions good hash functions.

**Definition 1.4** (Good hash functions). Given $n, p, q \in \mathbb{N}, p \leq n$ and a string $x \in \{0, 1\}^n$, we say a function $h: \{0, 1\}^p \rightarrow \{0, 1\}^q$ is good (for $x$), if for every $i, j \in [n - p + 1]$, $h(x[i, i + p]) = h(x[j, j + p])$ if and only if $x[i, i + p] = x[j, j + p]$.

This definition guarantees that if the hash function we used is good, then substrings of $x$ cannot have the same hash value.
We show that a good hash function can be constructed by using a \( \frac{1}{\text{poly} n} \)-almost \( O(\log n) \)-wise independence generator with seed length \( O(\log n) \). This can work since for each pair of distinct substrings, their hash values are the same with probability \( 1/\text{poly}(n) \). Since here are at most \( O(n^2) \) pairs, a union bound shows the existence of good hash functions. To get a deterministic hash function, we check each possible seed to see if the corresponding hash function is good, which can be done by checking if every pair of different substrings of \( x \) have different hash values. Note that there are at most \( O(n^2) \) pairs and the seed length of the generator is \( O(\log n) \), so this can be done in polynomial time.

However, even a non-monotone matching is not enough for our purpose. The reason is that in the matching, we are trying to match every well divided block of \( x \) to every possible block of \( y \) (not necessarily the blocks obtained by dividing \( y \) evenly into disjoint blocks), because we have edit errors here. If we just do this in the naive way, then the matched blocks of \( y \) can be overlapping. Using these overlapping blocks of \( y \) to fill the blocks of \( \tilde{x} \) is problematic, since even a single edit error or block transposition can create many new (overlapping) blocks in \( y \) (which can be as large as the length of the block in each level). These new blocks are all possible to be matched, and then we won’t be able to maintain an upper bound of \( O(k) \) on the different blocks between \( x \) and \( \tilde{x} \).

To solve this issue, we need to insist on computing a non-overlapping, non-monotone matching.

**Definition 1.5** (Non-overlapping (non-monotone) matching). Given \( n, n', p, q \in \mathbb{N}, p \leq n, p \leq n' \), a function \( h : \{0, 1\}^p \rightarrow \{0, 1\}^q \) and two strings \( x \in \{0, 1\}^n \), \( y \in \{0, 1\}^n' \), a (non-overlapping) matching between \( x \) and \( y \) under \( h \) is a sequence of matches (pairs of indices) \( w = ((i_1, j_1), \ldots, (i_{|w|}, j_{|w|})) \) s.t.

- for every \( k \in [|w|] \),
  - \( i_k = 1 + p l_k \in [n] \) for some \( l_k \), i.e., each \( i_k \) is the starting index of some block of \( x \), when \( x \) is divided evenly into disjoint blocks of length \( p \),
  - \( j_k \in [n'] \),
  - \( h(x[i_k, i_k + p]) = h(y[j_k, j_k + p]) \),
- \( i_1, \ldots, i_{|w|} \) are distinct.
- Intervals \([j_k, j_k + p), k \in [|w|], \) are disjoint.

Under this definition, we can indeed show a similar upper bound on the number of different blocks between \( x \) and \( \tilde{x} \) in each level. However, a tricky question here is how to compute a maximum non-overlapping, non-monotone matching efficiently, since the standard algorithm to compute a maximum matching only gives a possibly overlapping matching, while the dynamic programming approach in [6] only works for a monotone matching.

We show how to compute a maximum non-overlapping, non-monotone matching in polynomial time, by using an integer programming based on semi-definite programming (SDP). Specifically, we model the objective as solving a max-flow problem with constraints. In each level, we first evenly divide \( x \) into a sequence of blocks with length \( b_i \), and for every block we create a node. Next create a node for every length \( b_i \) substring of \( y \) (these substrings can overlap). We create a tap which is connected to all \( x \)'s blocks with unlimited capacity, and a sink which is connected from every \( y \)'s blocks with unlimited capacity. For every potential match \((i, j)\) in the sense that \( h(x[i, i + b_i]) = h(y[j, j + b_j]) \), we create an edge \((i, j)\) from \( x[i, i + b_i] \) to \( y[j, j + b_j] \) with capacity \( 1 \).

We create two binary vectors \( u = \{u_{i,j} \mid i \in [n-p+1], j \in [n'-p+1]\} \), and \( v = \{v_j \mid j \in [n'-p+1]\} \). Intuitively, \( u_{i,j} \) denotes the flow (either 0 or 1) on edge \((i, j)\), and \( v_j \) denotes the total flow coming into node \( y[j, j + b_j] \). We introduce the constraints as follows.

By our definition of the non-overlapping matching, we require that each \( x[i, i + b_i] \) is matched to at most one interval \( y[j, j + b_j] \). So for \( j, j' \in [n], j \neq j' \) if there are edges \((i, j)\) and \((i, j')\) for some \( i \), we require
$u_{i,j} \cdot u_{i,j'} = 0$. Similarly, each $y[j, j + b_i]$ should be matched to at most one interval $x[i, i + b_i)$. Thus for $i, i' \in [n], i \neq i'$, if there are edges $(i, j)$ and $(i', j)$, then we require $u_{i,j} \cdot u_{i', j} = 0$.

To force the non-overlapping condition of the intervals $[j_l, j_l + b_i], l \in [|w|]$, where these intervals correspond to $v$'s substrings in the matching, we require $v_j \cdot v_{j'} = 0$ for every pair $j, j'$ s.t. $|j' - j| < b_i$.

We now have all the major constraints. Another natural constraint is to require that for any $i, j, u_{i,j}^2 \leq 1$.

The last constraint comes directly from the flow problem, i.e. for every $i, i'$, if there are edges $(i, j)$ and $(i', j)$, then we require $u_{i,j} \cdot u_{i', j} = 0$.

We now have all the major constraints. Another natural constraint is to require that for any $i,j \in [n], i \neq j$, if there are edges $(i, j)$ and $(i', j)$, then we require $u_{i,j} \cdot u_{i', j} = 0$.

The reason is that such errors are concentrated, and won’t cause too many errors in the maximum non-overlapping matching.

The total number of levels is $O(\log n \log \frac{n}{k \log n + t} + t)$. This is because the number of block differences in the $i$-th level is $O(k + \frac{b_i}{n})$, where $b_i$ is the block length in the $i$-th level. This again comes from the fact that the $t$ insertions/deletions are concentrated in at most $k$ blocks, and each block transposition affects at most $O(1)$ blocks. Thus the redundancy of the hash values in the $i$-th level is $O((k + \frac{b_i}{n}) \log n)$.

The total number of levels is $O(\log \frac{n}{k \log n + t})$. Note that in each level, $b_i$ decreases by a factor of 2, and in the last level the block length is $O(\log n)$, thus the summation of the redundancy in all levels gives $O(k \log n \log \frac{n}{k \log n + t} + t)$. The total length of other transmitted strings that contribute to the overall redundancy, is smaller than this term.

**Error Correcting Code.** We now describe how to construct an error correcting code from a document exchange protocol for block insertions/deletions and block transpositions. Similar to the construction in [6], our starting point is to first encode the sketch of the document exchange protocol using the asymptotically good code by Schulman and Zuckerman [15], which can resist edit errors and block transpositions. Then we concatenate the message with the encoding of the sketch. When decoding, we first decode the sketch, then apply the document exchange protocol on Bob’s side to recover the message using the sketch.

However, here we have an additional issue with this approach: a block transposition may move some part of the encoding of the sketch to somewhere in the middle of the message, or vice versa. In this case, we won’t be able to tell which part of the received string is the encoding of the sketch, and which part of the received string is the original message.

To solve this issue, we use a fixed string $buf = 0^{\ell_{buf}} \circ 1$ as buffer to mark the encoding of the sketch, for some $\ell_{buf} = O(\log n)$. More specifically, we evenly divide the encoding of the sketch into small blocks of length $\ell_{buf}$, and insert $buf$ before every block. Note that this only increase the length of the encoding of the sketch by a constant factor. The reason we use such a small block length is that, even if the adversary can forge or destroy some buffers, the total number of bits inserted or deleted caused by this is still small. In fact, we can bound this by $O(k)$ block insertions/deletions with at most $O(k \log n)$ bits inserted/deleted, for which both the sketch and the encoding of the sketch can handle. When decoding, we first recognize all the $buf$’s. Then we take the $\ell_{buf}$ bits after each $buf$ to form the encoding of the sketch, and take the remaining bits as the message.

Unfortunately, this approach introduces two additional problems here. The first problem is that the origi-
nal message may contain buf as a substring. If this happens then again we will be taking part of the message to be in the encoding of the sketch. The second problem is that the small blocks of the encoding of the sketch may also contain buf. In this case we will be deleting information from the encoding of the sketch, which causes too many edit errors.

To address the first problem, we turn the original message into a pseudorandom string by computing the XOR of the message with the output of a pseudorandom generator. Using a \( \frac{1}{\text{poly}} \)-almost \( O(\log n) \)-wise independence generator with seed length \( O(\log n) \), we can ensure that with high probability buf does not appear as a substring in the XOR. We can then exhaustively search for a fixed seed that satisfies this requirement, and append the seed to the sketch of the document exchange protocol.

To address the second problem, we choose the length of the buffer to be longer than the length of each block in the encoding of the sketch, so that buf doesn’t appear as a substring in any block. This is exactly why we choose the length of the buffer to be \( \ell_{\text{buf}} + 1 \) while we choose the length of each block to be \( \ell_{\text{buf}} \).

If we directly apply our document exchange protocol to the construction above, we would obtain an error correcting code with \( O(k \log n \log \frac{n}{k \log n + t} + t) \) redundant bits. However, by combining the ideas in [6], we can achieve redundancy size \( O(k \log n \log n \log n + t) \), which is better for small \( k \) and \( t \).

We first briefly describe the construction of the explicit binary indel code for \( k \) edit errors with redundancy \( O(k \log n) \) in [6]. The construction in [6] starts by observing that a uniform random string satisfies some nice properties. For example, with high probability, any two substrings of length some \( B = O(\log n) \) are distinct. [6] calls this property \( B\)-distinct. The construction in [6] goes by first transforming the message into a pseudorandom string, which is obtained by computing the XOR of the message with an appropriately designed pseudorandom generator. The construction then designs a document exchange protocol for a pseudorandom string with better parameters, and encodes the sketch of the document exchange protocol to give an error correcting code.

The document exchange protocol for a pseudorandom string in [6] actually consisted of two stages: in stage I, Alice uses a fixed pattern \( p = 1 \circ 0^{s-1} \) of length \( s \) to divide her string into blocks of size \( \text{poly}(\log n) \). [6] defines a \( p \)-split point to be the index that \( p \) appears as a substring, and defines its next \( p \)-split point to be the \( p \)-split point right after it. When dividing the string, Alice needs to ensure that the size of each block is larger than \( B \), so that the \( B \)-distinct property holds for the blocks. Hence she only chooses a \( p \)-split point if its next \( p \)-split point is at least \( 2^s/2 \) indices away. One can set the parameter \( s = \log \log n + O(1) \) so that \( 2^s/2 > B \). Furthermore, [6] shows that for a pseudorandom string, with high probability, the size of each block is not too large, i.e., at most \( \text{poly}(\log n) \).

Next, Alice sends a sketch of size \( O(k \log n) \) to help Bob recover the partition of her string. To achieve this, Bob also divides his string into blocks in the same way that Alice does, by using the pattern \( p = 1 \circ 0^{s-1} \). However, due to \( k \) edit errors, there are \( O(k) \) different blocks between Alice’s partition and Bob’s partition. So Alice creates a vector \( V \) of length \( 2^B = \text{poly}(n) \), where each entry of \( V \) is indexed by a binary string of length \( B \). Alice looks at each block in her partition, and stores the \( B \)-prefix of its next block and the length of the current block in the entry of \( V \) indexed by the \( B \)-prefix of the current block. This ensures each entry of the vector \( V \) has only \( O(\log n) \) bits. Similarly Bob will create a vector \( V’ \) in the same way. [6] shows that \( V \) and \( V’ \) differ in at most \( O(k) \) entries, thus Alice can send a sketch of size \( O(k \log n) \) using the Reed-Solomon Code to help Bob recover \( V \) from \( V’ \). Once this is done, Bob can use \( V \) to obtain a guess of Alice’s string, which we call \( \tilde{x} \), by using his blocks to fill the blocks of \( \tilde{x} \), if they have the same \( B \)-prefix.

Stage II of the construction in [6] consists of a constant number of levels. In each level, both parties divide each of their blocks evenly into \( O(\log^{0.4} n) \) smaller blocks, and Alice generates a sequence of special hash functions called \( \epsilon \)-synchronization hash functions (see [6] for the exact definition) with respect to her string. The nice properties of the \( \epsilon \)-synchronization hash functions guarantee that in each level Alice can send \( O(k \log n) \) bits to Bob, so that Bob can recover all but \( O(k) \) blocks of Alice’s string. This stage ends in \( O(\log^{0.4} n(\text{poly}(\log n))) = O(1) \) levels when the final block size reduces to \( O(\log n) \), at which point Alice can simply send a sketch of size \( O(k \log n) \) for Bob to recover her string \( x \).
Checking the two stages in [6], it turns out that stage I can actually be modified to work for block insertions/deletions and block transpositions as well. Intuitively, this is because both kinds of errors won’t cause too many different blocks in \( V \) and \( V' \). On the other hand, stage II becomes quite tricky, since the use of \( \varepsilon \)-synchronization hash functions crucially relies on the monotone property of edit errors. Allowing block transpositions will ruin this property, and it is not clear how to apply \( \varepsilon \)-synchronization hash functions.

To solve the issue, in stage II, we apply the deterministic document exchange protocol we developed earlier, based on the small blocks of size \( \text{poly}(\log n) \) produced by stage I. Thus our stage II has \( \log(\text{poly}(\log n)) = \Theta(\log \log n) \) levels. Note that after stage I, the block size can be as small as \( O(\log n) \). Hence, for \( k \) block insertions/deletions of \( t \) bits, the number of blocks that are affected by these errors can only be bounded by \( O(k + t/\log n) \). Thus we add an extra level before applying the deterministic document exchange protocol, to ensure that after this extra level, the number of bad blocks is at most \( O(k + t/\text{poly}(\log n)) \).

We show that in stage I, Alice sends a sketch with \( O(k \log n + t) \) bits; and in stage II, Alice sends a sketch with \( O(k \log n \log \log n + t) \) bits. So the total sketch size is still \( O(k \log n \log \log n + t) \). By using the asymptotically encoding of Schulman and Zuckerman [15] and the buffer \( \text{buf} \), the final redundancy of the error correcting code is also \( O(k \log n \log \log n + t) \).

1.3 Discussions and open problems.

In this paper we study document exchange protocols and error correcting codes for block insertions/deletions and block transpositions. We give the first explicit, deterministic document exchange protocol in this case, and significantly improved error correcting codes. In particular, for both document exchange and error correcting codes, our sketch size or redundant information is close to optimal.

The obvious open problem is to try to achieve truly optimal constructions, where an interesting intermediate step is to try to adapt the \( \varepsilon \)-self matching hash functions and the \( \varepsilon \)-synchronization hash functions in [6] to handle block transpositions. More broadly, it would be interesting to study document exchange protocols and error correcting codes for other more general errors.

Organization of the paper. The rest of the paper is organized as follows. In Section 3 we give a deterministic document exchange protocol for block insertions/deletions and block transpositions. In Section 4 we give a document exchange protocol for block insertions/deletions and block transpositions for uniformly random strings. Then in Section 5 we give constructions of codes correcting block insertions/deletions and block transpositions. Finally we give tight bounds of the sketch size or redundancy in Appendix A.

2 Preliminaries

2.1 Notations

Let \( \Sigma \) be an alphabet (which can also be a set of strings) and \( x \in \Sigma^* \) be a string. \( |x| \) denotes the length of the string. Let \( x[i,j] \) denote the substring of \( x \) from the \( i \)-th symbol to the \( j \)-th symbol (Both ends included). Similarly \( x[i,j) \) denotes the substring of \( x \) from the \( i \)-th symbol to the \( j \)-th symbol (not included). We use \( x[i] \) to denote the \( i \)-th symbol of \( x \). The concatenation of \( x \) and \( x' \) is \( x \circ x' \). The \( B \)-prefix of \( x \) is the first \( B \) symbols of \( x \). \( x^N \) is the concatenation of \( N \) copies of \( x \). For two sets \( A \) and \( B \), let \( A \Delta B \) denotes the symmetric difference of \( A \) and \( B \).

Usually we use \( U_n \) to denote the uniform distribution over \( \{0, 1\}^n \).
2.2 Edit errors

Consider two strings $x, x' \in \Sigma^*$.

**Definition 2.1** (Edit distance). The edit distance $ED(x, x')$ is the minimum number of edit operations (insertions and deletions) that is enough to transform $x$ to $x'$.

A subsequence of a vector $x$ is a vector $x'$ s.t. $x'_1 = x_{j_1}, x'_2 = x_{j_2}, \ldots, x'_l = x_{j_l}, l = |x'|, 1 \leq j_1 < j_2 < \cdots < j_l \leq |x|$.

**Definition 2.2** (Longest Common Subsequence). The longest common subsequence between $x$ and $x'$ is the longest subsequence which is the subsequence of both $x$ and $x'$, its length denoted by $\text{LCS}(x, x')$.

We have $ED(x, x') = |x| + |x'| - 2 \text{LCS}(x, x')$.

**Definition 2.3** (Block-Transposition). Given a string $x \in \Sigma^n$, the $(i, j, l)$-block-transposition operation for $1 \leq i \leq i + l \leq n$ and $j \in \{0, \cdots, i-1, i+l, \cdots, n\}$ is defined as an operation which removes $x[i, i+l)$ and inserts $x[i, i+l)$ right after $x[j]$ for the original string $x$ (if $j = 0$, then inserts $x[i, i+l)$ to the beginning of $x$).

**Definition 2.4** (Block-insertion/deletion). A block-insertion/deletion (or burst-insertion/deletion) of $b$ symbols to a string $x$ is defined to be inserting/deleting a block of consecutive $b$ symbols to $x$. When we do not need to specify the number of symbols inserted or deleted, we simply say a block-insertion/deletion.

We define $(k, t)$-block-insertions/deletions (to $x$) to be a sequence of $k$ block-insertions/deletions, where the total number of symbols inserted/deleted is at most $t$.

2.3 Almost k-wise independence

**Definition 2.5** ($\varepsilon$-almost $\kappa$-wise independence in max norm [1]). A series of random variables $X_1, \ldots, X_n \in \mathbb{R}$ are $\varepsilon$-almost $\kappa$-wise independent in Maximum norm if $\forall x \in \mathbb{R}^\kappa, \forall i_1, i_2, \ldots, i_\kappa \in [n], |\Pr[(X_{i_1}, X_{i_2}, \ldots, X_{i_\kappa}) = x] - 2^{-\kappa}| \leq \varepsilon$.

A function $g : \{0, 1\}^d \rightarrow \{0, 1\}^n$ is an $\varepsilon$-almost $\kappa$-wise independence generator in Maximum norm if $Y = g(U_d) = Y[1] \circ \cdots \circ Y[n]$ are $\varepsilon$-almost $\kappa$-wise independent in Maximum norm.

For simplicity, we neglect the term in Maximum norm when saying $\varepsilon$-almost $\kappa$-wise independence, unless specified.

**Theorem 2.6** ($\varepsilon$-almost $\kappa$-wise independence generator [1]). For every $n, \kappa \in \mathbb{N}, \varepsilon > 0$, there exists an explicit $\varepsilon$-almost $\kappa$-wise independence generator $g : \{0, 1\}^d \rightarrow \{0, 1\}^n$, where $d = O(\log \frac{\kappa \log n}{\varepsilon})$.

The construction is highly explicit in the sense that, $\forall i \in [n]$, the $i$-th output bit can be computed in time $\text{poly}(\kappa, \log n, \frac{1}{\varepsilon})$ given the seed and $i$.

2.4 Pseudorandom Generator (PRG)

**Definition 2.7** (PRG). A function $g : \{0, 1\}^r \rightarrow \{0, 1\}^n$ is a pseudorandom generator (PRG) for a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ with error $\varepsilon$ if

$$|\Pr[f(U_n) = 1] - \Pr[f(g(U_r)) = 1]| \leq \varepsilon$$

where $r$ is the seed length of $g$.

Usually this is also called that $g$ $\varepsilon$-fools function $f$. Similarly, if $g$ fools every function in a class $\mathcal{F}$ then we say $g$ $\varepsilon$-fools $\mathcal{F}$.
2.5 Error correcting codes (ECC)

An ECC $C$ is called an $(n, m, d)$-code (for hamming errors) if it has block length $n$, message length $m$, and distance $d$. The rate of the code is defined as $\frac{m}{n}$.

We utilize the following algebraic geometry codes in our constructions.

Theorem 2.8 ([9]). For every $n, m, q \in \mathbb{N}, m \leq n, d = n - m - O(1), q = \text{poly}(\frac{n}{m})$, there is an explicit $(n, m, d)$-ECC over $\mathbb{F}_q$ with polynomial-time unique decoding.

Moreover, $\forall n, m \in \mathbb{N}$, for every message $x \in \mathbb{F}_q^n$, the codeword is $x \circ z$ with redundancy $z \in \mathbb{F}_q^{n-m}$.

For an ECC $C \subseteq \{0,1\}^n$ for edit errors, with message length $m$, we usually regard it as having an encoding mapping $\text{Enc}: \{0,1\}^m \rightarrow \{0,1\}^n$ and a decoding mapping $\text{Dec}: \{0,1\}^n \rightarrow \{0,1\}^m \cup \{\text{Fail}\}$.

We say an ECC for edit errors is explicit (or has an explicit construction) if both encoding and decoding can be computed in polynomial time.

To construct ECCs in following sections, we use an asymptotically good binary ECC for edit errors by Schulman and Zuckerman [15].

Theorem 2.9 ([15]). For every $n \in \mathbb{N}$, there is an explicit ECC with codeword length $n$, message length $m = \Omega(n)$, which can correct up to $k_1 = \Omega(n)$ edit errors and $k_2 = \Omega(n/\log n)$ block-transpositions.

3 Deterministic document exchange protocol for block insertions/deletions and block transpositions

Given $n, p, q \in \mathbb{N}, p \leq n$ and a string $x \in \{0,1\}^n$, we say a hash function $h : \{0,1\}^p \rightarrow \{0,1\}^q$ is good (for $x$), if for every $i, j \in [n - p + 1], h(x[i, i + p]) = h(x[j, j + p])$ if and only if $x[i, i + p] = x[j, j + p]$.

Given $n, n', p, q \in \mathbb{N}, p \leq n, p \leq n'$, a function $h : \{0,1\}^p \rightarrow \{0,1\}^q$ and two strings $x \in \{0,1\}^n, y \in \{0,1\}^{n'}$, a (non-overlapping) matching between $x$ and $y$ under $h$ is a sequence of matches (pairs of indices) $w = (i_1, j_1, \ldots, i_{|w|}, j_{|w|})$ s.t.

- for every $k \in [|w|],$
  - $i_k = 1 + pl_k \in [n]$ for some $l_k$,
  - $j_k \in [n']$,  
  - $h(x[i_k, i_k + p]) = h(y[j_k, j_k + p])$,
- $i_1, \ldots, i_{|w|}$ are distinct.
- Intervals $[j_k, j_k + p), k \in [|w|],$ are disjoint.

In the rest of the paper, when we talk about matchings we always mean non-overlapping matchings.

A match $(i, j)$ in the matching is called a wrong match (or wrong pair) if $x[i, i + p] \neq y[j, j + p]$. Otherwise it is called a correct match (or correct pair). A pair of indices $(i, j)$ is called a potential match between $x$ and $y$ if $h(x[i, i + p]) = h(y[j, j + p])$. When $x, y$ are clear from the context we simply say $(i, j)$ is a potential match.

To find a monotone matching we can use the dynamic programming method in [6]. But our matching is not necessarily monotone. So this raises the question of whether these exists a polynomial time algorithm which can do that. Next we give a positive answer to this question by using semi-definite programming (SDP).

Construction 3.1. Given $n, n', p, q \in \mathbb{N}, p \leq n, p \leq n'$, a polynomial time computable function $h : \{0,1\}^p \rightarrow \{0,1\}^q$ and two strings $x \in \{0,1\}^n, y \in \{0,1\}^{n'}$, we have the following algorithm.
• Solve the following SDP.

Let \( u = \{ u_{i,j} \mid i \in [n - p + 1], j \in [n' - p + 1] \} \), \( v = \{ v_j \mid j \in [n' - p + 1] \} \) be two real vectors.

Let \( E \) denote the set of potential matches between \( x \) and \( y \).

\[
\begin{align*}
\max_{u,v} & \sum_{j=1}^{n-p+1} v_j^2 \\
\text{s.t.} & \text{ for every } i, \text{ every pair of distinct matches } (i, j), (i, j') \in E, & u_{i,j}u_{i,j'} = 0 \\
& \text{ for every } j, \text{ every pair of distinct matches } (i, j), (i', j) \in E, & u_{i,j}u_{i',j} = 0 \\
& \text{ for every pair of overlapping intervals } y[j, j + p], y[j', j' + p], & v_jv'_j = 0 \\
& \text{ for every } j, & v_j^2 = \sum_{(i,j) \in E} u_{i,j}^2 \\
& \text{ for every edge } (i, j), & u_{i,j}^2 \leq 1.
\end{align*}
\]

• For every pair \( (i, j) \in E, u_{i,j} \neq 0 \), put \( (i, j) \) into the matching \( w \).

• return \( w \).

Lemma 3.2. Given \( n, n', p, q \in \mathbb{N}, p \leq n, p \leq n' \), a polynomial time computable function \( h : \{0,1\}^p \to \{0,1\}^q \) and two strings \( x \in \{0,1\}^n, y \in \{0,1\}^n \), the maximum matching between \( x \) and \( y \) under \( h \) can be computed in polynomial time.

Proof. We model the algorithm as a max-flow with restrictions.

We evenly parse \( x \) to be a sequence of length \( p \) blocks. Every block corresponds to a node. Then we let every length \( p \) interval of \( y \) be a node (these intervals can overlap). Also, we create a tap which is connected to all \( x \)'s blocks with unlimited capacity, and a sink which is connected from every \( y \)'s blocks with unlimited capacity. For every potential match \( (i, j) \) where \( h(x[i, i + p]) = h(y[j, j + p]) \), we create an edge \( (i, j) \) from \( x[i, i + p] \) to \( y[j, j + p] \) with capacity 1.

For every \( (i, j) \in E \) in the model, let \( u_{i,j}^2 \) denote the flow from \( x[i, i + p] \) to \( y[j, j + p] \). For every \( j \in [n' - p + 1] \) let \( v_j^2 \) denote the flow from \( y[j, j + p] \) to the sink.

There are two issues to notice here.

First, by definition of the matching, we require that each \( x[i, i + p] \) is matched to one interval \( y[j, j + p] \) only. So for \( j, j' \in [n], j \neq j' \) if there are edges \( (i, j) \) and \( (i, j') \) for some \( i \), we let \( u_{i,j} \cdot u_{i,j'} = 0 \). Also, for \( i, i' \in [n] \) if there are edges \( (i, j) \) and \( (i', j) \), then \( u_{i,j} \cdot u_{i',j} = 0 \).

Second, we require that intervals \( [j_l, j_l + p], l \in [w] \) should be non-overlapping, where these intervals are \( y \)'s intervals in the matching. So we let \( v_j \cdot v'_j = 0 \) for every pair \( j, j' \) s.t. \( |j' - j| < p \).

In summary, we have the following SDP.

\[
\begin{align*}
\max_{u,v} & \sum_{j=1}^{n-p+1} v_j^2 \\
\text{s.t.} & \text{ for every } i, \text{ every pair of distinct edges } (i, j), (i, j') \in E, & u_{i,j}u_{i,j'} = 0 \\
& \text{ for every } j, \text{ every pair of distinct edges } (i, j), (i', j) \in E, & u_{i,j}u_{i',j} = 0 \\
& \text{ for every pair of overlapping intervals } y[j, j + p], y[j', j' + p], & v_jv'_j = 0 \\
& \text{ for every } j, & v_j^2 = \sum_{(i,j) \in E} u_{i,j}^2 \\
& \text{ for every edge } (i, j), & u_{i,j}^2 \leq 1.
\end{align*}
\]
This SDP is exactly the same as that in Construction 3.1. A standard ellipsoid method can solve it in polynomial time.¹

Note that the optimal solution is integral. Because if some \( u_{i,j} \) is not an integer, then we can round it to be \( \frac{u_{i,j}}{|u_{i,j}|} \). After rounding, we get another valid solution (all restrictions are still met). But it gives a larger flow, since it increases \( u_j^2 \). This shows that the returned optimal solution must be integral.

For every \( u_{i,j} = \pm 1 \), we put \( (i, j) \) into the matching. This would give us a maximum matching. We show this by establishing the correspondence between matchings and valid integral solutions of our SDP.

The reason that every valid integral solution corresponds to a matching, is that each integral flow in our flow model corresponds to a matching between \( x \) and \( y \) under \( h \), since we put \( (i, j) \) into the matching if \( |u_{i,j}| = 1 \). By definition of the edge, the first bullet of our matching definition is satisfied. The second bullet is satisfied since we have the first restriction in our SDP. The third bullet is satisfied because the second restriction in our SDP.

On the other hand, every matching \( w \) corresponds to an integral flow in our flow model. Note that for each pair \( (i, j) \) in the matching, we can let the corresponding \( u_{i,j} = 1 \). Other \( u_{i,j}, (i, j) \) being a potential match, are set to be 0. Thus the fifth restriction in our SDP is satisfied. Also, since it is a match, \( (i, j) \) is definitely a potential match. Note that \( w \) is a matching of the bipartite graph, it meets the first and second restrictions in our SDP. Moreover, our matching has its third bullet requirement which means that the intervals of \( y \) in the matching are non-overlapping. Hence the third and fourth restrictions in the SDP are satisfied.

As a result, the optimal solution of our SDP gives a maximum matching.

\[ \square \]

**Theorem 3.3.** There exists an algorithm which, on input \( n, p, q \in \mathbb{N}, p \leq n, q = c_n \log n \) for large enough constant \( c_0 \), \( x \in \{0, 1\}^n \), outputs a description of a hash function \( h : \{0, 1\}^p \rightarrow \{0, 1\}^q \) that is good for \( x \), in time \( \text{poly}(n) \), where the description length is \( O(\log n) \).

Also there is an algorithm which, given the same \( n \in \mathbb{N} \), the description of \( h \) and any \( u \in \{0, 1\}^p \), can output \( h(u) \) in time \( \text{poly}(n) \).

**Proof.** Let \( \varepsilon = 1/\text{poly}(n) \) be small enough. Let \( g : \{0, 1\}^d \rightarrow (\{0, 1\}^q)^{\{0, 1\}^p} \) be an \( \varepsilon \)-almost \( 2q \)-wise independence generator with \( d = O(\log \frac{2q \log (2p q)}{\varepsilon}) \), from Theorem 2.6. Here \( g \) outputs \( q 2^p \) bits and we view the output as an array indexed by elements in \( \{0, 1\}^p \), where each array entry is in \( \{0, 1\}^q \).

To construct \( h \), we try every seed \( v \in \{0, 1\}^d \). Let \( h(\cdot) = g(v)[\cdot] \) (meaning for every \( i, h(i) \) is the \( i \)-th entry of the array \( g(v) \)). For any \( i, j \in [n - p + 1] \), we check that \( h(x[i, i + p]) = h(x[j, j + p]) \) if and only if \( x[i, i + p] = x[j, j + p] \). If this is the case then the algorithm returns \( h \). The description of \( h \) is the corresponding seed \( v \).

There exists a \( v \) s.t. the corresponding \( h \) is good. This is because, if we let \( v \) be uniformly random, then by a union bound, the probability that there exists \( i, j \in [n - p + 1] \) s.t. \( h(x[i, i + p]) = h(x[j, j + p]) \) but \( x[i, i + p] \neq x[j, j + p] \) is at most \( 1/\text{poly}(n) \cdot n^2 = 1/\text{poly}(n) \).

The exhaustive search is in polynomial time because the seed length is \( d = O(\log n) \). The evaluation of \( h \) is in polynomial time by Theorem 2.6. Thus the overall running time of our algorithm is a polynomial in \( n \).

We now have the following document exchange protocol.

**Construction 3.4.** The protocol works for every input length \( n \in \mathbb{N} \), every \( (k_1, t) \) block-insertions/deletions \( k_2 \) block-transpositions, \( k_1, k_2 \leq \alpha n/\log n, t \leq \beta n \), for some constant \( \alpha, \beta \). (If \( k_1 \) or \( k_2 > \alpha n/\log n \), or \( t > \beta n \), we simply let Alice send her input string.) Let \( k = k_1 + k_2 \). ¹More specifically, it suffices to achieve an output within \( 1/2 \) of the optimal SDP value.
Both Alice’s and Bob’s algorithms have $L = O(\log \frac{n}{k \log n + t})$ levels.

Alice: On input $x \in \{0, 1\}^n$;

1. We set up the following parameters;
   - For every $i \in [L]$, in the $i$-th level,
     - The block size is $b_i = \frac{n}{2^{i+2}(i+1)^{\frac{k}{k+1}}}$, i.e., in each level we cut every block in the previous level evenly into two blocks. We choose $L$ properly s.t. $b_L = O(\log n)$;
     - The number of blocks $l_i = n/b_i$;
   
2. For the $i$-th level,
   2.1. Construct a hash function $h_i : \{0, 1\}^{b_i} \rightarrow \{0, 1\}^{b_i = \Theta(\log n)}$ for $x$ by Theorem 3.3.
   2.2. Compute the sequence $v[i] = (h_i(x[1, 1+b_i]), h_i(x[1+b_i, 1+2b_i]), \ldots, h_i(x[1+(l_i-1)b_i, l_ib_i]))$;
   2.3. Compute the redundancy $z[i] \in (\{0, 1\}^{b_i})^{\Theta(k+1)}$ for $v[i]$ by Theorem 2.8, where the code has distance $25(k + \frac{1}{b_i})$;
   3. Compute the redundancy $z_{final} \in (\{0, 1\}^{b_L})^{\Theta(k+1)}$ for the blocks of the $L$-th level by Theorem 2.8, where the code has distance $14(k + \frac{1}{b_L})$;
   4. Send $h = (h_1, \ldots, h_L)$, $z = (z[1], z[2], \ldots, z[L]), v[1], z_{final}$.

Bob: On input $y \in \{0, 1\}^{O(n)}$ and received $h, z, v[1], z_{final}$;

1. Create $\tilde{x} \in \{0, 1, \ast\}^n$ (i.e. his current version of Alice’s $x$), initiating it to be $(\ast, \ast, \ldots, \ast)$;
2. For the $i$-th level where $1 \leq i \leq L-1$,
   2.1. Apply the decoding of Theorem 2.8 on $h_i(\tilde{x}'[1, 1+b_i]), h_i(\tilde{x}'[1+b_i, 1+2b_i]), \ldots, h_i(\tilde{x}'[1+(l_i-1)b_i, l_ib_i]), z[\tilde{x}]$ to get the sequence of hash values $v[\tilde{x}]$. Note that $v[1]$ is received directly, thus Bob does not need to compute it;
   2.2. Compute $w_i = ((p_1, p'_1), \ldots, (p_{|w|}, p'_{|w|})) \in ([b_i] \times [|y|])^{w}$ which is the maximum matching (may not be monotone) between $x$ and $y$ under $h_i$, using $v[\tilde{x}]$, by Lemma 3.2;
   2.3. Evaluate $\tilde{x}$ according to the matching, i.e. let $\tilde{x}[p_j, p_j + b_i] = y[p'_j, p'_j + b_i], j \in [|w|]$;
3. In the $L$’th level, apply the decoding of Theorem 2.8 on the blocks of $\tilde{x}$ and $z_{final}$ to get $x$;
4. Return $x$.

Lemma 3.5. For every $i$, $|w_i| \geq l_i - 3(k + \frac{1}{b_i})$.

Proof. We show that there exists a matching $w'_i$ s.t. $|w'_i| \geq l_i - 3(k + \frac{1}{b_i})$.

Note that $k_1$ block-insertions/deletions which insert/delete $t$ symbols can delete at most $O(k_1 + \frac{1}{b_i})$ blocks of $x$ in the $i$-th level. This is because block-insertions does not delete symbols. For the $j$-th block-deletion of $t_j$ symbols. It can corrupt (delete a block totally or delete part of a block) at most $[t_j/b_i] + 1$ blocks. So the total number of corrupted blocks is at most $\sum_{j=1}^{k_1} ([t_j/b_i] + 1) \leq 2 k_1 + t/b_i$.

On the other hand, $k_2$ block-transpositions can corrupt at most $3 k_2$ blocks, because 1 block-transposition can only corrupt the two blocks at the end of the transposed substring and another block which contains the position that is the destination of the transposition.
As a result, the total number of corrupted blocks is at most $2k_1 + 3k_2 + t/b_i$. The remaining blocks can all be matched to distinct blocks of $x$, since they are not corrupted. This gives a matching $w_i'$ of size at least $l_i - (2k_1 + 3k_2 + t/b_1) \geq l_i - 3(k + \frac{1}{b_i})$.

\[ \square \]

**Lemma 3.6.** For every $i$, if $v[i]$ is correctly recovered, then in the $i$-th level the number of wrong matches in $w$ is at most $3(k + \frac{1}{b_i})$.

**Proof.** All wrong matches are between blocks of $x$ and newly created blocks by errors. This is because $h_i$ is good for $x$. So for any match $(p, p')$ in $w$, if $y[p', p' + b_i]$ is a substring of $x$, then $x[j, j + b_i] = y[p', p' + b_i]$. Thus it is a correct match.

By definition of matching, intervals $y[p_j, p_j + b_i], j \in |w|$ are non-overlapping. Each block of $y$ in a wrong match must have edit distance at least 1 from any block of $x$. Thus the $j$-th block-insertion of $t_j$ symbols can cause at most $[t_j/b_i] + 1$ wrong matches. Thus $k_1$ block-insertions can cause at most $\sum_{j=1}^{k_1} ([t_j/b_i] + 1) \leq 2k_1 + t/b_i$ wrong matches. Each block deletion can cause $2$ wrong matches, because it may create two new blocks. Also each block-transposition can create at most $3$ wrong matches, one created by removing that block, the other two from inserting it to a new position.

So the total number of wrong matches is at most $2k_1 + t/b_i + 3k_2 \leq 3(k + t/b_i)$.

\[ \square \]

**Lemma 3.7.** Both Alice and Bob’s algorithms are in polynomial time.

**Proof.** For Alice’s algorithm, let’s consider the $i$-th level. Constructing $h_i$ and evaluating $h_i$ takes polynomial time by Theorem 3.3. Computing the redundancy $z[i]$ takes polynomial time by Theorem 2.8. So the overall running time is polynomial.

For Bob’s algorithm, we still consider the $i$-th level. By Theorem 2.8, getting $v[i]$ takes polynomial time. By Lemma 3.2, computing the maximum matching takes polynomial time. So the overall running time is also polynomial.

\[ \square \]

**Lemma 3.8.** The communication complexity is $O(k \log n \log \frac{n}{k \log n + t} + t)$.

**Proof.** For the $i$-th level of Alice, $|z[i]| = \Theta(k + \frac{1}{b_i})b^* = \Theta((k + \frac{1}{b_i}) \log n)$. So

$$|	ext{z}| = \sum_{i=1}^{L} |z[i]|$$

$$= O(k \log n \log \frac{n}{k \log n + t}) + O(t(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_L}) \log n)$$

$$= O(k \log n \log \frac{n}{k \log n + t}) + O(t(\frac{k + t/\log n}{n}(1 + 2 + 4 + \cdots + O(\frac{n}{k + t/\log n}))))$$

$$= O(k \log n \log \frac{n}{k \log n + t} + t).$$

Also $|z_{final}| = O(k + \frac{1}{b_L}) \cdot O(\log n) = O(k \log n + t)$ by Theorem 2.8.

For every $i \in [L], |h_i| = O(\log n)$ by Theorem 3.3. So $|h| = O(\log n)L = O(\log n \log \frac{n}{k \log n + t})$.

The length of $v[i]$ is $l_i = O(\log n) = \frac{n}{b_i}O(\log n) = O(k + \frac{1}{\log n}) \cdot O(\log n) = O(k \log n + t)$.

So the overall communication complexity is $O(k \log n \log \frac{n}{k \log n + t} + t)$.

\[ \square \]

**Lemma 3.9.** Bob can recover $x$ correctly.
Proof. We first show that in the \( i \)-th level \( i \in [L] \), Bob can get \( v[i] \) correctly. We use induction.

For the first level, Bob gets \( v[1] \) directly by our construction.

For every \( i \in [L] \), if Bob gets \( v[i] \), then he can compute \( w_i \) where \( |w_i| \geq l_i - 3(k + \frac{k}{b_i}) \), by Lemma 3.5.

Excluding those wrong matches in \( w_i \), which has number \( 3(k + \frac{k}{b_i}) \) by Lemma 3.6, the remaining ones can recover \(|w_i| - 3(k + \frac{k}{b_i}) \geq l_i - 6(k + \frac{k}{b_i}) \) blocks of \( x \) correctly.

So the number of blocks of \( \hat{x} \) that are not recovered (different from the corresponding blocks of \( x \)), is at most \( l_i - (|w_i| - 3(k + \frac{k}{b_i})) \leq 6(k + \frac{k}{b_i}) \). Each of these uncovered blocks is cut into two blocks evenly in the next level. Note that we let the redundancy length, saying \(|z[i + 1]| = O(k + \frac{k}{b_i})\), to be large enough s.t. the distance of the corresponding code is \( 25(k + \frac{k}{b_i}) > 2 \times 12(k + \frac{k}{b_i}) \). Thus Bob can recover \( v[i + 1] \) by Theorem 2.8.

This finishes the induction.

As a result, for level \( L \), Bob can get \( v[L] \) correctly. So by Lemma 3.5, he can compute \( w_L \) where \(|w_L| \geq l_L - 3(k + \frac{k}{b_i}) \). Thus he can recover \( l_L - 6(k + \frac{k}{b_i}) \) number of blocks of \( x \), since the number of wrong matches in \( w_L \) is at most \( 3(k + \frac{k}{b_i}) \) by Lemma 3.6. Then by using \( z_{\text{final}} \), Bob can recover \( x \), by Theorem 2.8, since the distance of the corresponding code is \( 14(k + \frac{k}{b_i}) > 2 \times 6(k + \frac{k}{b_i}) \).

\[\square\]

Theorem 3.10. There exists a deterministic protocol for document exchange, having communication complexity (redundancy) \( O(k \log n \log \frac{n}{k \log n + t}) + t \), time complexity \( \text{poly}(n) \), where \( n \) is the input size and \( k = k_1 + k_2 \), for \((k_1, t)\) block-insertions/deletions and \( k_2 \) block-transpositions, \( k_1, k_2 \leq \alpha n / \log n, t \leq \beta n \), for some constant \( \alpha, \beta \).

Proof. It follows from Construction 3.4, Lemma 3.7, 3.8 and 3.9. \[\square\]

Corollary 3.11. If both parties know that Bob has a string \( x' \in \{0, 1\}^n \) which has at least \( l_i - 12(k + \frac{k}{b_i}) \) blocks the same as that of \( x \) (dividing both \( x \) and \( x' \) evenly into \( l_i \) blocks each of length \( b_i \)), then they can start from the \( i \)-th level of the protocol in Construction 3.4 by letting \( \hat{x} = x' \) and Bob can compute \( x \) correctly. This will have communication complexity \( O(k \log n (L - i + 1) + t) \) and polynomial running time, where \( n \) is the input size, \( k = k_1 + k_2 \), for \((k_1, t)\) block-insertions/deletions and \( k_2 \) block-transpositions, \( L = O(k \log n / \log n \log n) \), \( k_1, k_2 \leq \alpha n / \log n, t \leq \beta n \), for some constant \( \alpha, \beta \).

Proof. Since \( x' \) has at least \( l_i - O(k + \frac{k}{b_i}) \) blocks the same as that of \( x \). So he can compute the hash values of all blocks and get \( v[i] \) from these hash values and \( z[i] \) by Theorem 2.8. Then we can do an induction proof in the same way as that in Lemma 3.9 to show that Bob can compute \( x \) correctly.

The time complexity is still polynomial time since the whole protocol is in polynomial time by Lemma 3.7 and we are running from the \( i \)-th level. The communication complexity is at most

\[
\sum_{j=1}^{L} |h_j| + |v[1]| + |z_{\text{final}}| + \sum_{j=1}^{L} |z[j]| \\
= O(\log n)(L - i + 1) + O(k \log n + t) + O(k \log n + t) + \sum_{j=1}^{L} O(k + \frac{k}{b_i}) \cdot b^* \tag{4}
\]

\[\square\]
4 Document exchange for block-insertions/deletions and block-transpositions of a uniform random string

In this section we prove the following theorem.

**Theorem 4.1.** There is a polynomial time one round document exchange protocol for \((k, t)\)-block insertion/deletion and block transpositions for a uniform random string with communication cost \(O(k \log n \log \log n + t)\) bits and error probability \(1 - 1/\text{poly}(n)\).

**Definition 4.2** \((p\)-split point\([4]\)). For a string \(p \in \{0, 1\}^s\) and \(x \in \{0, 1\}^n\), a \(p\)-split point of \(x\) is an index \(1 \leq i \leq n - s + 1\) such that \(x|_{i, i + s} = p\).

**Definition 4.3** (next \(p\)-split point \([6]\)). Let \(p\) and \(x\) be two strings, and \(i\) be a \(p\)-split point of \(x\). The next \(p\)-split point of \(i\) is the smallest \(j\) such that \(j\) is a \(p\)-split point and \(j > i\). In the case that such \(j\) does not exist, we define the next \(p\)-split point of \(i\) to be \((n + 1)\).

We recall the following properties for a uniform random string from \([6]\). **Property 1** and **Property 2** will be used to ensure the block size is bounded by \(O(\text{poly}(\log n))\) in Lemma 4.7.

**Theorem 4.4** (Theorem 4.4 \([6]\)). Let \(p = 1 \circ 0^{s-1}\) be a string of length \(s\). For a uniform random string \(x \in \{0, 1\}^n\), there exist three integers \(B_1 = 2s2^s \log n, B_2 = 2^s \log n\) and \(B = 3 \log n\) such that the following properties hold with probability \(1 - 1/\text{poly}(n)\).

**Property 1** Any interval of \(x\) with length \(B_1\) contains a \(p\)-split point.

**Property 2** Any interval of \(x\) with length \(B_2\) starting at a \(p\)-split point contains a \(p\)-split point \(i\) such that its next \(p\)-split point \(j\) satisfies \(j - i > 2^s/2\).

**B-distinct** Every two substrings of length \(B\) at different positions of \(x\) are distinct.

4.1 Construction

Similar to \([6]\), our construction here has two stages. In fact, the protocol of Stage I in \([6]\) can be slightly modified to resist block insertions/deletions and block transpositions.

**Construction 4.5** (Stage I, modified from \([6]\)). Let \(n\) denote the length of Alice’s string \(x, s = \log \log n + 3, B = 3 \log n, T_0 = s2^s \log n,\) and \(p = 1 \circ 0^{s-1}\) be a fixed string of length \(s\). To make the representation simple, we assume \(n\) is a multiple of \(T_0\).

Alice: On input uniform random string \(x \in \{0, 1\}^n\).

1. Choose all \(p\)-split points \(i\) of \(x\) such that its next \(p\)-split point \(j\) satisfies \(j - i > 2^s/2\). Denote the chosen \(p\)-split points as \(i_1, i_2, \ldots, i_{n'}\). Partition the string \(x\) into blocks \(x|_{1, i_2}, x|_{i_2, i_3}, x|_{i_3, i_4}\) \(\ldots, x|_{i_{n'}, n}\), and index these blocks as 1, 2, 3, \(\ldots, n'\).

2. Create a set \(V = \{(\text{len}_b, \text{B-prefix}_b, \text{B-prefix}_{b+1}) \mid 1 \leq b \leq n' - 1\}\), where \(\text{len}_b\) is the length of the \(b\)-th block, and \(\text{B-prefix}_b\) and \(\text{B-prefix}_{b+1}\) are the B-prefix of the \(b\)-th block and the \((b + 1)\)-th block respectively.

3. Represent the set \(V\) as its indicator vector, which has size \(\text{poly}(n)\), and send the redundancy \(z_V\) being able to correct \(12k + 2\lceil t/B \rceil\) Hamming errors, using Theorem 2.8 (or simply using a Reed-Solomon code).
4 Partition the string \( x \) evenly into \( n/T_0 \) blocks, each of size \( T_0 \).

Bob: On the redundancy \( z_V \) sent by Alice, and the string \( y \) obtained from \( x \) by \((k_1, t)\)-block-insertions/deletions and \( k_2 \) block transpositions, where \( h_1 + k_2 \leq k \).

1 Choose the \( p \)-split points \( i \) of \( y \) such that its next \( p \)-split point \( j \) satisfies \( j - i > 2^s/2 \). Denote the chosen \( p \)-split points as \( i_1', i_2', \ldots, i_m' \). Partition the string \( y \) into blocks \( y[1, i_{j_1'}), y[i_{j_2}', i_{j_3'}), y[i_{j_3}', i_{j_4'}), \ldots, y[i_{m'}', n] \), and index these blocks as \( 1, 2, 3, \ldots, m' \).

2 Create a set \( V' = \{(\text{len}, B\text{-prefix}_{b}, B\text{-prefix}_{b+1}) \mid 1 \leq b \leq m' - 1\} \) using the partition of \( y \).

3 Use the indicator vector of \( V' \) and the redundancy \( z_V \) to recover Alice’s set \( V \).

4 Create an empty string \( \bar{x} \) of length \( n \), and partition \( \bar{x} \) according to the set \( V \) in the following way: first find the element \((\text{len}, B\text{-prefix}_{b}, B\text{-prefix}_{b})\) in \( V \) such that for all elements \((\text{len}, B\text{-prefix}, B\text{-prefix}')\) in \( V, B\text{-prefix}(1) \neq B\text{-prefix}'. Then partition \( \bar{x}[1, \text{len}(1)] \) as the first block, and fill \( \bar{x}[1, \text{B-prefix}] \) with \( B\text{-prefix}(1) \). Then find the element \((\text{len}(2), B\text{-prefix}(2), B\text{-prefix}'(2))\) such that \( B\text{-prefix}(2) = B\text{-prefix}'(1) \), and partition \( \bar{x}[\text{len}(1) + 1, \text{len}(1) + \text{len}(2)] \) as the second block, and fill \( \bar{x}[\text{len}(1) + 1, \text{len}(1) + \text{B-prefix}] \) with \( B\text{-prefix}'(2) \). Continue doing this until all elements in \( V \) are used to recover the partition of \( x \).

5 For each block \( b \) in \( \bar{x} \), if Bob finds a unique block \( b' \) in \( y \) such that the \( B\text{-prefix} \) of \( b' \) matches the \( B\text{-prefix} \) of \( b \) and the lengths of \( b \) and \( b' \) are equal, Bob fills the block \( b \) using \( b' \). If such \( b' \) doesn’t exist or Bob has multiple choices of \( b' \), then Bob just leaves the block \( b \) as blank.

6 Partition the string \( \bar{x} \) evenly into \( n/T_0 \) blocks, each of size \( T_0 \).

**Construction 4.6 (Stage II).** Stage II consists of \( L = O(\log \log n) \) levels.

Let \( i^* = \Theta(\log \log n) \) be s.t. \( b_{i^*} \geq T_0 \geq b_{i^*+1} \) where \( b_{i^*} = \Theta(\frac{n}{2^{k + \Theta(\log \log n)}}) \) is from Construction 3.4 (assuming that protocol has \( L^* \) levels). Let \( L = L^* - i^* + 1 \).

Alice does the following.

1. (An extra level)
   
   - Construct a hash function \( h_{i^*} : \{0, 1\}^{b_{i^*}} \rightarrow \{0, 1\}^{b^*} = O(\log n) \) for \( x \) by Theorem 3.3.
   
   - Compute the sequence \( v[i^*] = (h_{i^*}(x[1, 1 + b_{i^*}]), h_{i^*}(x[1 + b_{i^*}, 1 + 2b_{i^*}]), \ldots, h_{i^*}(x[1 + (l_{i^*} - 1)b_{i^*}, l_{i^*}b_{i^*}])) \).
   
   - Compute the redundancy \( z[i^*] \in \{(0, 1)^{b^*} \Theta(k + \frac{b^*}{\log n}) \text{ for } v[i^*] \} \text{ by Theorem 2.8, where the code has distance } c^* \text{ with } c^* \text{ being a large enough constant;} \)

2. Conduct Alice’s operations in Construction 3.4 from level \( i^* \) to the last level.

3. Send \( h = (h_{i^*}, h_{i^*+1}, \ldots, h_{L^*}), z[i^*], z[i^* + 1], \ldots, z[L^*], z_{\text{final}} \).

Bob conducts the following. Assume now his version of \( x \) is \( \bar{x} \) (which is the input for this stage).

1. (An extra level)
   
   - Apply the decoding of Theorem 2.8 on \( h_{i^*}(\bar{x}[1, 1 + b_{i^*}]), h_{i^*}(\bar{x}[1 + b_{i^*}, 1 + 2b_{i^*}]), \ldots, h_{i^*}(\bar{x}[1 + (l_{i^*} - 1)b_{i^*}, l_{i^*}b_{i^*}]), z[i^*] \) to get the sequence of hash values \( v[i^*] \).
   
   - Compute \( w^* = ((p_1, p_2'), \ldots, (p_{\lceil w^* \rceil}, p'_{\lceil w^* \rceil})) \in (\{b_{i^*}\} \times \{y\})^{\lceil w^* \rceil} \text{ which is the maximum matching (may not be monotone) between } x \text{ and } y \text{ under } h_{i^*}, \text{ using } v[i^*], \text{ by Lemma 3.2;} \)

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• Evaluate \( \bar{x} \) according to the matching, i.e. let \( \bar{x}[p_j, p_j + b_{i^*}] = y[p_j', p_j' + b_{i^*}], j \in [|w|] \);

2. Conduct Bob’s operations in Construction 3.4 from level \( i^* \) to the last level.

3. Apply the decoding of Theorem 2.8 on the blocks of \( \bar{x} \) and \( z_{\text{final}} \) to get \( x \);

4.2 Analysis

For Stage I, we first recall the following lemma about the upper and lower bound of the block size in Alice’s step 1.

Lemma 4.7 ([6] Lemma 4.14). For Stage I, in Alice’s partition of \( x \) and Bob’s partition of \( y \), the size of every block is greater than \( 2^s/2 \). If all properties in Theorem 4.4 hold, then in Alice’s partition of \( x \), the size of every block is at most \( B_1 + B_2 = O(s2^s \log n) \).

Lemma 4.8. If \( 2^s/2 > B \), then \( |V \Delta V'| \leq 12k + 2t/B \).

Proof. Suppose there are \( k \) block insertion, block deletion and block transposition operations \( op_1, op_2, \ldots, op_k \) transforming \( x \) into \( y \). Let \( x_i, i = 1, 2, \ldots, k \) denote the string obtained by applying \( op_1, op_2, \ldots, op_k \) to \( x \).

And let \( V_i, i = 1, 2, \ldots, k \), denote the set obtained by applying Bob’s step 1 and step 2 in Stage I to string \( x_i \), and index the blocks in \( x_i \) as \( 1, 2, \ldots, m_i \). To prove the lemma, it suffices to bound \( |V_i \Delta V_{i+1}| \). There are two cases: 1. \( op_i \) inserts or deletes \( t_i \) bits as a block. 2. \( op_i \) is a block transposition.

For case 1, if \( x_i \)’s \( b \)-th block and \( (b+1) \)-th block are not involved in the block insdel operation, then \( |V_i \Delta V_{i+1}| \leq 2|t_i/B| + 2 \). Hence \( |V_i \Delta V_{i+1}| \leq 2|t_i/B| + 2 \).

For case 2, suppose \( op_i \) is a block transposition \((i, j, l)\) transporting \( x_i[i, i + l] \) to the position right after the index \( j \). For any \( b \in [m_i - 1] \), if \( x_i \)’s \( b \)-th block and \( (b+1) \)-th block don’t contain the indexes \( i, i + l \) and \( j \), then the \( b \)-th and \( (b+1) \)-th block are still substrings in \( x_{i+1} \), and thus \( |V_i \Delta V_{i+1}| \leq 2|t_i/B| + 2 \).

Hence we derive \( |V \Delta V'| \leq \sum_{i \in [k-1]} |V_i \Delta V_{i+1}| \leq 12k + 2t/B \).

Theorem 4.9. If all three properties in Theorem 4.4 hold, then after Stage I, at most \( O(k + t/B) \) blocks of \( \bar{x} \) contains unfilled bits or incorrectly filled bits.

Proof. In this proof, we say a block is a bad block, if it contains unfilled bits or incorrectly filled bits. By Lemma 4.8, Bob can recover the set \( V \) correctly using \( z_V \). To prove the theorem, it suffices to show that in Bob’s step 5, Bob’s partition of \( \bar{x} \) has at most \( 6k + t/B \) bad blocks.

We prove by induction on \( k \). When \( k = 0 \), then Bob’s string \( y \) is the same with Alice’s string \( x \). The total number of unfilled blocks and incorrectly filled blocks is 0, so the theorem holds.

Let’s assume the theorem holds for \((k - 1, t)\)-block-insertion/deletions and block transpositions, i.e., the number of bad blocks is at most \( 6(k - 1) + t/B \). There are two cases: 1. the \( k \)-th operation is a block insdel. 2. the \( k \)-th operation is a \((i, j, l)\) block transposition.

For case 1, as \( 2^s/2 > B \), inserting or deleting a block of size \( t' \) can affect at most \([t'/B] + 2\) blocks in Bob’s partition of \( y \). Hence this operation can only increase the number of bad blocks by at most \([t'/B] + 2\). Hence, for \( k \) operations, the number of bad blocks is at most \( 6(k - 1) + t/B + [t'/B] + 2 < 6k + (t + t')/B \). So the theorem holds for \( k \) operations in this case.
For case 2, the block transposition can only affect 6 blocks in Bob’s partition of $y$. Hence, the number of bad blocks increases by at most 6. So the theorem holds for $k$ operations in this case.

Hence, the theorem holds for $k$ operations for both cases. □

Proof of Theorem 4.1. From Theorem 4.4, with probability $1 - 1/poly(n)$, all properties in Theorem 4.4 hold. From Theorem 4.9 and Corollary 3.11, The protocol in Construction 4.5 and Construction 4.6 is correct.

The size of $z_V$ is $(12k + 2\lceil t/B \rceil) \log n = O(k \log n + t)$ bits.

Note that since $b_i^\ast \geq T_0 \geq b_{i+1}^\ast$, $i^\ast = L^\ast - O(\log\log n)$. From Corollary 3.11, the later levels of Stage II send $O(k \log n \log \log n + t)$ bits. The first level (the extra level) of Stage II sends $|z[i^\ast]| = O(k + t/B) \log n = O(k \log n + t)$ bits by Theorem 2.8. So the total communication complexity is $O(k \log n \log \log n + t)$.

What remains to prove is that after the extra level, $\bar{x}$ has at least $l_{i^\ast+1} - 12(k + \frac{t}{b_{i^\ast+1}})$ blocks the same as that of $x$ (in order to use Corollary 3.11), where the block length is $b_{i+1}$. Before the extra level $\bar{x}$ has at least $l_i^\ast - O(k + \frac{t}{B})$ blocks the same as that of $x$ (i.e. being recovered). Since $z[i^\ast]$ is the redundancy of a code which has distance $c^\ast(k + t/B)$ with $c^\ast$ being large enough, $v[i^\ast]$ can be recovered by Bob using $h_i^\ast(\bar{x}[1, 1 + b_i^\ast]), h_i^\ast(\bar{x}[1 + b_i^\ast, 1 + 2b_i^\ast]), \ldots, h_i^\ast(\bar{x}[1 + (l_i^\ast - 1)b_i^\ast, l_i^\ast b_i^\ast]), z[i^\ast]$, by Theorem 2.8. Note that Lemma 3.2 guarantees that $w_i^\ast$ is the maximum matching computed using $v[i^\ast]$ and $y$. Then by Lemma 3.5, $|w^\ast| \geq l_i^\ast - 3(k + \frac{t}{b_i})$. The number of wrong matches in $w^\ast$ is at most $3(k + \frac{t}{b_i})$ by Lemma 3.6. So the total number of uncovered blocks is at most $6(k + \frac{t}{b_i})$ in level $i^\ast$ and thus at most $12(k + \frac{t}{b_i})$ in level $i^\ast + 1$.

As a result, after the extra level, by Corollary 3.11, Bob can finally recover $x$. □

Also we have the following corollary which will be used in later sections. It follows by a similar proof as that of Theorem 4.1.

Corollary 4.10. If the input strings for Alice have all three properties in Theorem 4.4, then there is a deterministic polynomial time one round document exchange protocol for $(k_1, t)$-block insertion/deletion and $k_2$ block transpositions with communication cost $O(k \log n \log \log n + t)$ bits and error probability $1 - 1/poly(n)$, where $k = k_1 + k_2$.

5 Binary codes for block insertions/deletions and block transpositions

5.1 Encoding and decoding algorithm

Given the document exchange protocol for block transposition, block insertions and deletions, we can now construct codes capable of correcting $(k_1, t)$ block insertions/deletions, and $k_2$ block transpositions, where $k_1 + k_2 = k$, and $t \leq \alpha n$ for some constant $\alpha$. The encoding and decoding algorithms are as follows:

Algorithm 5.1. Encoding algorithm

Let $\ell_{buf} = 2 \log n$ and $buf = 0^{\ell_{buf} - 1} \circ 1$.

Input: $msg$ of length $n$.

Ingredients:

- A pseudorandom generator PRG : $\{0, 1\}^{O(\log n)} \rightarrow \{0, 1\}^n$, from Theorem 5.3, s.t. there exists at least one seed $r$ for which $msg \oplus PRG(r)$ doesn’t contain $buf$ as a substring and having Property 1, Property 2 and $B$-distinct.

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An error correcting code \( C_1 \) from Theorem 2.9 which is capable of correcting \( O(k \log n + t) \) edit errors, as well as \( k_2 \) block transpositions. Denote the encoding map of \( C_1 \) as \( \text{Enc}_1 : \{0, 1\}^{m_1} = O(k \log^2 n + t) \rightarrow \{0, 1\}^{cl_1} = O(k \log^2 n + t) \) and the decoding map as \( \text{Dec}_1 : \{0, 1\}^{cl_1} \rightarrow \{0, 1\}^{ml_1} \).

**Operations:**

1. Find a seed \( r \) of PRG s.t. \( \text{msg} \oplus \text{PRG}(r) \) does not contain \( \text{buf} \), having Property 1, Property 2 and \( B \)-distinct. Let \( \text{msg}_P = \text{msg} \oplus \text{PRG}(r) \).

2. Compute the sketch \( \text{sk}_m \) for \( \text{msg}_P \) for \( \Omega(k) \) block insertions/deletions and \( \Omega(k) \) block transpositions, where the number of bits inserted and deleted is \( \Omega(k \log n + t) \) in total.

3. Let \( \text{sk} = \text{sk}_m \circ r \), and encode \( \text{sk} \) with \( C_1 \). Let the codeword be \( c_1 = \text{Enc}_1(\text{sk}) \).

4. Divide \( c_1 \) into blocks of length \( \log n \). Denote these blocks as \( c_1^{(1)}, c_1^{(2)}, \ldots, c_1^{(M)} \) where \( M \) is the number of blocks.

5. Insert \( \text{buf} \) to the beginning of each block \( c_1^{(i)} \), \( 1 \leq i \leq M \).

6. Let \( c = (\text{msg} \oplus \text{PRG}(r)) \circ \text{buf} \circ c_1^{(1)} \circ \text{buf} \circ c_1^{(2)} \circ \cdots \circ \text{buf} \circ c_1^{(M)} \).

**Output:** \( c \).

The construction of PRG is left to subsection 5.2. We call the concatenation \( \text{buf} \circ c_1^{(1)} \circ \text{buf} \circ c_1^{(2)} \circ \cdots \circ \text{buf} \circ c_1^{(M)} \) as the sketch part and \( \text{msg}_P = \text{msg} \oplus \text{PRG}(r) \) as the message part. Now we give the corresponding decoding algorithm.

**Algorithm 5.2. Decoding algorithm**

**Input:** the received codeword \( c' \).

**Operations:**

1. Find out all substrings \( \text{buf} \) in \( c' \). Number these buffers as \( \text{buf}_1, \ldots, \text{buf}_{M'} \).

2. Pick the \( \log n \) bits after \( \text{buf}_j \) as block \( c_1^{(j)} \), \( 1 \leq j \leq M' \). Then remove all the buffers \( \text{buf}_j \) and \( c_1^{(j)} \), \( 1 \leq j \leq M' \) from \( c' \). The rest of \( c' \) is regarded as the message part \( \text{msg}'_P \).

3. Let \( c_1' = c_1^{(1)} \circ c_1^{(2)} \circ \cdots \circ c_1^{(M')} \). Decode \( c_1' \) with the decoding algorithm \( \text{Dec}_1 \) for \( C_1 \) and get \( \text{sk} = \text{Dec}_1(c_1') \).

4. Get \( \text{sk}_m \) and \( r \) from \( \text{sk} \).

5. Use \( \text{sk}_m \) and \( \text{msg}'_P \) to recover \( \text{msg}_P \).

6. Compute \( \text{msg} = \text{msg}_P \oplus \text{PRG}(r) \).

**Output:** \( \text{msg} \).

**5.2 Analysis**

In this subsection we’ll give the construction of PRG and prove the correctness of the algorithms.
5.2.1 Building blocks: PRG

We recall the pseudorandom generator in Theorem 5.1 in [6].

**Theorem 5.3** (Theorem 5.1 in [6]). For every \( n \in \mathbb{N} \), there exists an explicit PRG \( g : \{0, 1\}^{\ell = O(\log n)} \to \{0, 1\}^n \) s.t. for every \( x \in \{0, 1\}^n \), with probability \( 1 - 1/\text{poly}(n) \), \( g(U_\ell) + x \) satisfies Property 1, Property 2 and \( B \)-distinct. (Let \( s \) in Property 1, Property 2 be \( \log \log n + O(1) \).)

**Theorem 5.4.** For every \( n \in \mathbb{N} \), \( x \in \{0, 1\}^n \), there exists an explicit PRG \( g : \{0, 1\}^{\ell = O(\log n)} \to \{0, 1\}^n \) s.t. for every \( x \in \{0, 1\}^n \), with probability \( 1 - 1/\text{poly}(n) \), the following two conditions hold simultaneously.

- \( \text{buf} \) is not a substring of \( \text{PRG}(U_\ell) \oplus x \).
- \( \text{PRG}(U_\ell) \oplus x \) satisfies Property 1, Property 2 and \( B \)-distinct.

**Proof.** Let \( \kappa = \ell_{\text{buf}} \) be the length of buf, \( \varepsilon = 1/n^2 \). From Theorem 2.6, there exists an explicit \( \varepsilon \)-almost \( \kappa \)-wise independence generator \( g' : \{0, 1\}^d \to \{0, 1\}^n \), where \( d = O(\log n \varepsilon /\kappa) = O(\log n) \). Then, for any \( x \in \{0, 1\}^n \),

\[
\Pr_{r' \leftarrow \{0, 1\}^d} \left[ \text{buf is a substring of } g'(r') \oplus x \right] \leq \sum_{i \in [n - \ell_{\text{buf}} + 1]} \Pr[\text{buf} = (g'(r') \oplus x)[i, i + \ell_{\text{buf}}]] \leq n \left( 1/2^{\ell_{\text{buf}}} + 1/n^2 \right) = 1/\text{poly}(n)
\]

Let \( g \) be the generator in Theorem 5.3 with seed length \( \ell' \). Let \( \ell = \max(\ell', d) \), and construct \( \text{PRG}(r) = g(r_1) \oplus g'(r_2) \) where \( r_1, r_2 \) are disjoint substrings of \( r \) of length \( \ell' \) and \( \ell \). Then by the union bound, the probability that at least one of the conditions fails is upper bounded by \( 1/\text{poly}(n) \).

5.2.2 Correctness of the construction

We show that a code \( C \) with encoding algorithm 5.1 and decoding algorithm 5.2 can correct \((k_1, t)\)-block insertions/deletions and \( k_2 \) block transpositions.

First, we prove the sketch \( \text{sk} \) can be correctly recovered.

**Lemma 5.5.** In the 4th step of decoding algorithm 5.2, the sketch \( \text{sk} \) is correctly recovered.

**Proof.** We show that \( c_1' \) can be obtained by applying at most \( 12k \log n + t \) edit errors and \( k \) block transpositions over \( c_1 \).

Note that after inserting buffers to the blocks of \( c_1 \), the total number of appearance of the buffer in the sketch part is equal to the number of buffers inserted, because the buffer length is longer than the block length of \( c_1 \). Also note that concatenating the message part and sketch part will not insert any buffers because by the choice of \( r \), \( \text{msg} \oplus \text{PRG}(r) \) does not contain buf. As a result, if there are no errors, by the decoding algorithm we can get the correct \( c_1 \) and thus get the correct \( \text{sk} \).

Next we consider the effects of block insertions/deletions and transpositions for the sketch part. Specifically, we consider how the sketch part changes after each of these operations.

- **block insertion**: Consider one block insertion of \( t_0 \) bits. We claim that after this operation, at most \( \lceil t_0/(3 \log n) \rceil \) new blocks can be introduced to the sketch part, because to insert one new block to the sketch, we only need to insert a new buffer and attach the new block to it. We also note that this operation may delete one block by damaging a buffer, or replace one block by damaging the block right after the buffer.

So \( k_1 \) block insertions of \( t \) bits inserted can insert at most \( k_1 + t/(3 \log n) \) new blocks. It can also delete at most \( k_1 \) blocks, and replace at most \( k_1 \) blocks.
• block deletion: we first consider a block deletion of $t_0$ bits. After this operation, at most $\lceil t_0/(3\log n) \rceil$ blocks of the sketch part can be deleted, since there are at most $\lceil t_0/(3\log n) \rceil$ blocks in the deleted substring. The operation may also create one extra block, since the remaining bits may combine together to be a buffer. It may also replace an existing block, since the remaining bits may combine together to be a new block after an original buffer.

So $k_1$ block deletions of $t$ bits deleted can delete at most $k_1 + t/(3\log n)$ blocks. It can insert at most $k_1$ blocks. It can also replace $k_1$ blocks.

• block transposition: After one block transposition $(i, j, l)$, at most 3 new blocks can be introduced to the sketch part, since a new block may be created at the original position $i$, and two new blocks may appear when inserting the block to the destination $j$. Also it may delete at most 3 blocks, since two buffers may be damaged when removing the transferred block, and one buffer can be damaged when inserting the transferred block. By a similar argument this operation can replace at most 3 blocks. Also, a block transposition can cause one block transposition for the sketch part.

As a result, $k_2$ block transpositions can insert or delete at most $O(k_2)$ blocks and cause $O(k_2)$ block transpositions.

In summary, there are at most $O(k + t/\log n)$ block insertions/deletions and $k_2$ block transpositions on $c_1$. Note that $O(k + t/\log n)$ block insertions/deletions, each of length $O(\log n)$ bits can be regarded as $O(k\log n+t)$ edit errors. Since our code $C_1$ can correct $O(k\log n+t)$ edit errors and $k_2$ block transpositions, we can decode $sk$ correctly.

Next, we show that the message output by the decoding algorithm is correct.

**Lemma 5.6.** *At the end of algorithm 5.2, the original message is correctly decoded.*

**Proof.** According to Lemma 5.5, we have correctly recovered $sk$. Thus we get $sk_m$ and $r$ correctly.

Note that if there are no errors, then by deleting the buffers and the blocks of $c_1$ appended to these buffers, the remaining string is exactly the original message part, since the original message part does not contain $buf$ as substrings.

Now we consider the effects of block insertions/deletions and transpositions for the message part. Specifically, we consider how the message part changes after each of these operations.

• block insertion: First consider one block insertion of $t_0$ bits. It can insert at most $t_0$ symbols to the message part if it does not damaging any original buffers. If it damages buffers, it may insert $O(\log n)$ more bits to the message part. It can also cause at most one block deletion of $O(\log n)$ bits since the rightmost buffer it inserts may cause our algorithm to delete the $O(\log n)$ bits following that buffer.

• block deletion: Consider a block deletion of $t_0$ bits. It can delete at most $t_0$ blocks of the message part. If it damages buffers, it can cause at most one block insertion of $O(\log n)$ bits, since the rightmost deleted buffer may cause our algorithm to regard the $O(\log n)$ bits following that buffer as part of the message part.

• block transposition: now we consider one block transposition. It may cause at most one block transposition of the message part. Also it may create at most 3 new buffers and thus delete $3\log n$ bits of the message part. Moreover, it may delete three buffers and thus insert $3\log n$ bits to the message part.
Thus \((k_1, t)\)-block insertions/deletions can cause inserting/deleting at most \(O(k_1)\) blocks of \(O(t + k_1 \log n)\) bits. Also \(k_2\) block transpositions can cause \(O(k_2)\) block insertions/deletions of \(O(k_2 \log n)\) bits in total and \(k_2\) block transpositions.

In summary, there are at most \(O(k)\) block insertions/deletions of \(O(k \log n + t)\) bits in total and \(k_2\) block transpositions. Since our sketch \(sk\) can be used to correct \((O(k), O(k \log n + t))\) block insertions/deletions and \(k\) block transpositions, we can get \(msg_P\) correctly. As a result we can compute \(msg = msg_P \oplus \text{PRG}(r)\) correctly.

\[\square\]

**Theorem 5.7.** For every \(n, k_1, k_2, t \in \mathbb{N}\) with \(k = k_1 + k_2 < \alpha n / \log n, t \leq \beta n\), for some constant \(\alpha, \beta\), there exists an explicit binary error correcting code for \((k_1, t)\)-block insertions/deletions and \(k_2\) block transpositions, having message length \(n\), codeword length \(n + O(k \log n \log n + t)\).

**Proof.** We show that, by using the document exchange protocol of Theorem 4.10 in Operation 2 of the Algorithm 5.1, we can have such a code.

The correctness of the decoding algorithm in 5.1 is shown by Lemma 5.6. The sketch length \(|sk|\) is \(O(k \log n \log n + t)\) by Theorem 4.10. The length of \(c_1\) is \(O(|sk|) = O(k \log n \log n + t)\) by Theorem 2.9. As there are \(|c_1| / \log n\) length \(l_{buf} = O(\log n)\) buffers, each followed by a length \(\log n\) block of \(c_1\), the total length of the sketch part is \(O(|c_1|) = O(k \log n \log n + t)\).

**Theorem 5.8.** For every \(n, k_1, k_2, t \in \mathbb{N}\) with \(k = k_1 + k_2 < \alpha n / \log n, t \leq \beta n\), for some constant \(\alpha, \beta\), there exists an explicit binary error correcting code for \((k_1, t)\)-block insertions/deletions and \(k_2\) block transpositions, having message length \(n\), codeword length \(n + O(k \log n \log n + t)\).

**Proof.** We show that, by using the document exchange protocol of Theorem 3.10 in Operation 2 of the Algorithm 5.1, we can have such a code.

The correctness of the construction is similar to Lemma 5.5, 5.6, the \((k_1, t)\)-block insertions/deletions and \(k_2\) block transpositions causes \((k, O(k \log n + t))\)-block insertions/deletions and transpositions on the message and sketch part. Hence, according to Theorem 3.10, a sketch of size \(O(k \log n \log \frac{n}{k \log n + t} + t)\) for the document exchange protocol is enough to correct the errors.

By Theorem 2.9, the size of \(c_1\) is \(O(k \log n \log \frac{n}{k \log n + t} + t)\). The total length of the buffer inserted is \(O(\log n) \cdot |c_1| / \log n = O(|c_1|)\). Hence, the total length of the redundancy is \(O(k \log n \log \frac{n}{k \log n + t} + t)\).

\[\square\]

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Appendices

A

Theorem A.1. Suppose there is a deterministic document exchange protocol for strings of length \( n \), and can resist \( k \) block insertions/deletions and block transposition errors, where the total number of bits inserted or deleted is bounded by \( t \), and \( t < n/2 \), then the sketch size is at least \( \Omega(k \log n + t) \).

Proof. Suppose Alice has string \( x \) and Bob has string \( y \), and Alice sends a sketch \( sk(x) \) to allow Bob recovering her string \( x \). For a fixed string \( y \), each different strings \( x_1, x_2 \) satisfy \( sk(x_1) \neq sk(x_2) \), otherwise the correctness of the document exchange protocol will be violated. Now suppose \( y \) is a fixed string of length \( n \) satisfying \( B \)-distinct property, where \( B = O(\log n) \), we give a lower bound on the number of possible strings of \( x \).

Consider the following adversarial tempering of the string \( x \): delete the last \( t/2 \) bits as a block, then insert arbitrary \( t/2 \) bits at the end as a block. Next, divide the \( (n - t/2) \)-prefix evenly to small blocks of length \( B \). Arbitrary choose \( k - 2 \) different small blocks and transpose them to the begining of the string in an arbitrary order. Then any differences in the \( t/2 \) bits inserted, the choice of the blocks or the ordering will result to different strings. Hence, the number of strings \( x \) is lower bounded by

\[
2^{t/2} \left( \frac{n-t/2}{B} \right)^k \geq 2^{t/2} \left( \frac{3n}{4(k-2)B} \right)^k \left( \frac{k-2}{2} \right)^k = 2^{t/2} \left( \frac{3n}{4eB} \right)^k
\]

Taking the log, we obtain \( |sk| \geq \Omega(k \log n + t) \). \( \square \)

Theorem A.2. Let \( n', n \) be two integers, if \( C \subseteq \{0, 1\}^{n'} \), \( |C| = 2^n \) is an Error Correcting Code for \( k \) block insertions/deletions and block transpositions, where the total number of bits inserted or deleted is bounded by \( t \), and \( t < n/100 \), then the redundancy size \( n' - n \geq \Omega(k \log n + t) \).

Proof. Denote \( n'' = n' - t/2 \). It suffices to consider the case \( n'' < 2n \). We evenly divide the interval \([1, n'']\) into smaller intervals of length \( 10 \log n'' \), and denote these intervals as \( I_1, I_2, \ldots, I_{n''/10 \log n''} \).

Let \( C' \) be a subset of \( C \) containing all the codewords \( c \) such that the number of distinct strings in \( \{c_{I_1}, c_{I_2}, \ldots, c_{I_{n''/10 \log n''}}\} \) is at least \( n''/1000 \log n'' \). We will show that \( C' \) contains a large fraction of the codewords.

For simplicity, we denote \( a = n''/1000 \log n'' \). Now we bound the size of the set \( C \setminus C' \). Note that any codewords \( c \in C \setminus C' \) satisfies that the number of distinct strings in \( \{c_{I_1}, c_{I_2}, \ldots, c_{I_{n''/10 \log n''}}\} \) is smaller than \( a \). Hence we have

\[
|C \setminus C'| \leq a^{n''/10 \log n''} (2^{10 \log n''})^{n} 2^{t/2} = 2^{n'' \log a/10 \log n'' + 10a \log n'' + t/2} \leq 2^{3n/5}
\]

Now we obtain the lower bound of \( |C'| \). When \( n \geq 2 \),

\[
|C'| = |C| - |C \setminus C'| \geq 2^n - 2^{3n/5} \geq 2^n/2
\]

For any codeword \( c \in C' \), define the ball \( B_t(k, t) \) to be the set containing all strings obtained by applying \( k \) block insertions/deletions and block transpositions to \( c \), where the total number of bits inserted or deleted is bounded by \( t \).

Consider the following adversarial tempering of the codeword \( c \): delete the last \( t/2 \) bits of \( c \) as a block deletion, then insert arbitrary \( t/2 \) bits at the ending of the tempered string as a block insertion. Next, arbitrary
choose \( k - 2 \) distinct strings from \( c_{I_1}, c_{I_2}, \ldots, c_{I_{n''/10 \log n''}} \), and transport them to the beginning of the string in an arbitrary order. Then, any differences in the \( t \) bits inserted, the choice of the \((k - 2)\) substrings or the order of transpositions will result in different strings in \( B_{c}(k, t) \). Hence,

\[
|B_{c}(k, t)| \geq 2^{t/2} \left( \frac{a}{k - 2} \right) (k - 2)! \geq 2^{t/2} \left( \frac{n''/1000 \log n''}{k - 2} \right)^{k-2} \left( \frac{k - 2}{e} \right)^{k-2} = 2^{t/2} \left( \frac{n''}{1000e \log n''} \right)^{k-2}
\]

As \( C \) is a code, the ball \( B_{c}(k, t) \) should be disjoint, so we have

\[
2^{n'} \geq \sum_{c \in C} |B_{c}(k, t)| \geq |C'| 2^{t/2} \left( \frac{n''}{1000e \log n''} \right)^{k-2} \geq 2^{n+t/2-1} \left( \frac{n''}{1000e \log n''} \right)^{k-2}
\]

Taking a log on both sides of the equation, we obtain \( n' \geq n + \Omega(k \log n'' + t) \geq n + \Omega(k \log n+t) \).

**Theorem A.3.** There exists a deterministic document exchange protocol running in exponential time in \( n \) with sketch size \( O(k \log n + t) \). Moreover, we can construct an Error Correcting Code with redundancy size \( O(k \log n + t) \) from the document exchange protocol. Hence the lower bounds in Theorem A.1 and Theorem A.2 are tight.

**Proof.** We build a graph. Each string with length smaller than \( n + t \) corresponds to a vertex in the graph. For every two different strings \( x \) and \( y \), if one can transform \( x \) to \( y \) using \( k \) block insertions/deletions and transpositions, and the total number of inserted and deleted bits is bounded by \( t \), then add an edge between \( x \) and \( y \). Now the degree of the graph is at most \((2n)^{O(k)2^t} = 2^{O(k \log n + t)}\), hence we can use \( 2^{O(k \log n + t)} \) colors to color the graph.

We construct the document exchange protocol as follows. Given the input string, Alice sends the color of the string as the sketch, so the sketch has size \( O(k \log n + t) \) bits. Then Bob looks at the strings connected to his string, and find the string whose color matches the sketch.

In fact, the construction of the Error Correcting Code in Section 5 can be applied to any document exchange protocol, so we obtain an Error Correcting Code of redundancy \( O(k \log n + t) \).