On geometric Brauer groups and Tate-Shafarevich groups

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1 Introduction

For any noetherian scheme $X$, the cohomological Brauer group

$$\text{Br}(X) := H^2(X, \mathbb{G}_m)_{\text{tor}}$$

is defined to be the torsion part of the etale cohomology group $H^2(X, \mathbb{G}_m)$.

1.1 Tate conjecture and Brauer group

Let us first recall the Tate conjecture for divisors over finitely generated fields.

**Conjecture 1.1.** (Conjecture $T^1(X, \ell)$) Let $X$ be a projective and smooth variety over a finitely generated field $k$ of characteristic $p \geq 0$, and $\ell \neq p$ be a prime number. Then the cycle class map

$$\text{Pic}(X) \otimes \mathbb{Z} \mathbb{Q}_\ell \rightarrow H^2(X^s, \mathbb{Q}_\ell(1))^G_k$$

is surjective.

By a well-known result (see Proposition 2.1), $T^1(X, \ell)$ is equivalent to the finiteness of $\text{Br}(X^s)^G_k(\ell)$.

1.2 Main theorems

**Theorem 1.2.** Let $X$ be a smooth projective variety over a finitely generated field $K$ of characteristic $p > 0$. Assuming that $\text{Br}(X^s)^G_K(\ell)$ is finite for some prime $\ell \neq p$, then $\text{Br}(X^s)^G_K(\text{non}-p)$ is finite.

In the case that $K$ is finite, the above theorem was proved by Tate [Tat3, Thm. 5.2] and Lichtenbaum [Lic]. The general case was proved by Cadoret-Hui-Tamagawa[CHT]. Skorobogatov-Zarkhin [SZ1, SZ2] proved the finiteness of $\text{Br}(X^s)^G_K(\text{non}-p)$ for abelian varieties and $K3$ surfaces($\text{char}(k) \neq 2$). In this paper, we will give an elementary proof of the theorem based on the idea of Tate and Lichtenbaum’s proof for the case over finite fields.

**Theorem 1.3.** Let $Y$ be a smooth irreducible variety over a finite field of characteristic $p > 0$ with function field $K$. Let $A$ be an abelian variety over $K$. Define

$$\Pi_Y(A) := \text{Ker}(H^1(K, A) \rightarrow \prod_{s \in Y^1} H^1(K^s_{sh}, A)),$$

where $Y^1$ denotes the set of points of codimension $1$ and $K^s_{sh}$ denotes the quotient field of a strict local ring at $s$. Assuming that $\Pi_Y(A)(\ell)$ is finite for some prime $\ell \neq p$, then $\Pi_Y(A)(\text{non}-p)$ is finite.

In the case that $K$ is a global function field, the above theorem was proved by Tate [Tat3, Thm. 5.2] and Schneider[Sch]. In [Qin2], we proved that the finiteness of $\Pi_Y(A)(\ell)$ for a single $\ell$ is equivalent to the BSD conjecture for $A$ over $K$ formulated by Tate (cf. [Tat1]).
1.3 Notation and Terminology

Fields

By a finitely generated field, we mean a field which is finitely generated over a prime field. For any field $k$, denote by $k^s$ the separable closure. Denote by $G_k = \text{Gal}(k^s/k)$ the absolute Galois group of $k$.

Henselization

Let $R$ be a noetherian local ring, denote by $R^h$ (resp. $R^{sh}$) the henselization (resp. strict henselization) of $R$ at the maximal ideal. If $R$ is a domain, denote by $K^h$ (resp. $K^{sh}$) the fraction field of $R^h$ (resp. $R^{sh}$).

Varieties

By a variety over a field $k$, we mean a scheme which is separated and of finite type over $k$. For a smooth proper geometrically connected variety $X$ over a field $k$, we use $\text{Pic}^0_{X/k}$ to denote the identity component of the Picard scheme $\text{Pic}_{X/k}$.

Cohomology

The default sheaves and cohomology over schemes are with respect to the small étale site. So $H^i$ is the abbreviation of $H^i_{\text{ét}}$.

Brauer groups

For any noetherian scheme $X$, denote the cohomological Brauer group

$$\text{Br}(X) := H^2(X, \mathbb{G}_m)_{\text{tor}}.$$ 

Abelian group

For any abelian group $M$, integer $m$ and prime $\ell$, we set

$$M[m] = \{ x \in M | mx = 0 \}, \quad M_{\text{tor}} = \bigcup_{m \geq 1} M[m], \quad M(\ell) = \bigcup_{n \geq 1} M[\ell^n],$$

$$M \otimes \mathbb{Z}_\ell = \lim_M M/\ell^n, \quad T_\ell M = \text{Hom}_\mathbb{Z}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M) = \lim_M M[\ell^n], \quad V_\ell M = T_\ell(M) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$ 

For any torsion abelian group $M$ and prime number $p$, we set

$$M(\text{non-}p) = \bigcup_{p|m} M[m].$$
A torsion abelian group $M$ is of cofinite type if $M[m]$ is finite for any positive integer $m$. We use $M_{\text{div}}$ to denote the maximal divisible subgroup of $M$. For a group $G$, we use $|G|$ to denote the order of $G$.

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2 Preliminary reductions

2.1 The Kummer sequence

Proposition 2.1. Let $X$ be a smooth projective geometrically connected variety over a field $k$. Let $\ell$ be a prime different from $\text{char}(k)$, then the exact sequence of $G_k$-representations

$$0 \rightarrow \text{NS}(X_{\overline{k}})/\ell^n \rightarrow H^2(X_{\overline{k}}, \mathbb{Z}/\ell^n(1)) \rightarrow \text{Br}(X_{\overline{k}})[\ell^n] \rightarrow 0$$

is split for sufficient large $\ell$. And for any $\ell \neq \text{char}(k)$, the exact sequence of $G_k$-representations

$$0 \rightarrow \text{NS}(X_{\overline{k}}) \otimes \mathbb{Z}/\ell \rightarrow H^2(X_{\overline{k}}, \mathbb{Q}_{\ell}(1)) \rightarrow V_\ell \text{Br}(X_{\overline{k}}) \rightarrow 0$$

is split. Taking $G_k$-invariant, there is an exact sequence

$$0 \rightarrow \text{NS}(X) \otimes \mathbb{Z}/\ell \rightarrow H^2(X_{k^s}, \mathbb{Q}_{\ell}(1))^{G_k} \rightarrow V_\ell \text{Br}(X_{k^s})^{G_k} \rightarrow 0.$$

Proof. Let $d$ denote the dimension of $X$. Let $Z_1(X_{k^s})$ denote the group of 1-cycles on $X_{k^s}$, it admits a natural $G_k$ action. Since $\tau$-equivalence is same as the numerical equivalence for divisors (cf., e.g. SGA 6 XIII, Theorem 4.6), thus the intersection pairing

$$\text{NS}(X_{k^s}) \otimes \mathbb{Q} \rightarrow H^2(X_{k^s}, \mathbb{Q}_{\ell}(1))$$

is left non-degenerate. Since $\text{NS}(X_{k^s})$ is finitely generated, so there exists a finite dimensional $G_k$ invariant subspace $W$ of $Z^1(X_{k^s})_{\mathbb{Q}}$ such that the restriction of the intersection pairing to $\text{NS}(X_{k^s})_{\mathbb{Q}} \times W$ is left non-degenerate. Since $G_k$-actions factor through a finite quotient of $G_k$, we can choose $W$ such that the pairing is actually perfect. Let $W_{\mathbb{Q}}$ denote the subspace of $H^{2d-2}(X_{k^s}, \mathbb{Q}_{\ell}(d-1))$ generated by cycle classes of $W$. Then the restriction of

$$H^2(X_{k^s}, \mathbb{Q}_{\ell}(1)) \times H^{2d-2}(X_{k^s}, \mathbb{Q}_{\ell}(d-1)) \rightarrow H^{2d}(X_{k^s}, \mathbb{Q}_{\ell}(d)) \cong \mathbb{Q}_{\ell}$$

to $\text{NS}(X_{k^s})_{\mathbb{Q}} \times W_{\mathbb{Q}}$ is also perfect. So we have

$$H^2(X_{k^s}, \mathbb{Q}_{\ell}(1)) = \text{NS}(X_{k^s})_{\mathbb{Q}} \oplus W^\perp_{\mathbb{Q}}.$$
This proves the second claim.

There exists a finitely generated $G_k$ invariant subgroup $W_0$ of $Z^1(X_{k^s})$ such that $W = W_0 \otimes \mathbb{Z}/\ell \mathbb{Q}$. Therefore, there exists a positive integer $N$, such that the base change of the pairing $\text{NS}(X_{k^s}) \times W_0 \rightarrow \mathbb{Z}$ to $\mathbb{Z}/N\mathbb{Z}$ is perfect. So for any $\ell \nmid N$, the intersection pairing

$$\text{NS}(X_{k^s})/\ell^n \times W_0/\ell^n \rightarrow \mathbb{Z}/\ell^n$$

is perfect. Since it is compatible with

$$H^2(X_{k^s}, \mathbb{Z}/\ell^n(1)) \times H^{2d-2}(X_{k^s}, \mathbb{Z}/\ell^n(d-1)) \rightarrow H^{2d}(X_{k^s}, \mathbb{Z}/\ell^n(d)) \cong \mathbb{Z}/\ell^n.$$

Thus we have

$$H^2(X_{k^s}, \mathbb{Z}/\ell^n(1)) = \text{NS}(X_{k^s})/\ell^n \oplus (W_0/\ell^n)^\perp.$$

This proves the first claim. □

2.2 Reduce to the vanishing of Galois cohomology

Let $X$ be a smooth projective geometrically connected variety over a finitely generated field $K$ of characteristic $p > 0$. Assuming that $\text{Br}(X_{K^s})^{G_K}(\ell)$ is finite for some prime $\ell \neq p$, by [Qin2, Cor. 1.7], $\text{Br}(X_{K^s})^{G_K}(\ell)$ is finite for all primes $\ell \neq p$. Thus, to prove Theorem 1.2, it suffices to prove $\text{Br}(X_{K^s})^{G_K}[\ell] = 0$ for $\ell \gg 0$ under the assumption of the finiteness of $\text{Br}(X_{K^s})^{G_K}(\ell)$. By Proposition 2.1, for $\ell \gg 0$, there is an exact sequence

$$0 \rightarrow (\text{NS}(X_{K^s})/\ell)^{G_K} \rightarrow H^2(X_{K^s}, \mathbb{Z}/\ell(1))^{G_K} \rightarrow \text{Br}(X_{K^s})^{G_K}[\ell] \rightarrow 0.$$

Therefore, it suffices to show that $(\text{NS}(X_{K^s})/\ell)^{G_K} \rightarrow H^2(X_{K^s}, \mathbb{Z}/\ell(1))^{G_K}$ is surjective for $\ell \gg 0$. Consider the following exact sequence of $\ell$-adic sheaves on $X$

$$0 \rightarrow \mathbb{Z}_\ell(1) \rightarrow \mathbb{Z}_\ell(1) \rightarrow \mathbb{Z}/\ell(1) \rightarrow 0.$$

It gives a long exact sequence

$$H^2(X_{K^s}, \mathbb{Z}_\ell(1)) \rightarrow H^2(X_{K^s}, \mathbb{Z}_\ell(1)) \rightarrow H^2(X_{K^s}, \mathbb{Z}/\ell(1)) \rightarrow H^3(X_{K^s}, \mathbb{Z}_\ell(1))[\ell] \rightarrow 0.$$

By a theorem of Gabber (cf.[Gab]), $H^3(X_{K^s}, \mathbb{Z}_\ell(1))[\ell] = 0$ for $\ell \gg 0$. Thus, for $\ell \gg 0$, there is a canonical isomorphism

$$H^2(X_{K^s}, \mathbb{Z}_\ell(1))/\ell \cong H^2(X_{K^s}, \mathbb{Z}/\ell(1)).$$

Assuming that $\text{Br}(X_{K^s})^{G_K}(\ell)$ is finite, by the exact sequence

$$0 \rightarrow (\text{NS}(X_{K^s}) \otimes \mathbb{Z}_\ell)^{G_K} \rightarrow H^2(X_{K^s}, \mathbb{Z}_\ell(1))^{G_K} \rightarrow T_\ell \text{Br}(X_{K^s})^{G_K},$$

we have

$$(\text{NS}(X_{K^s}) \otimes \mathbb{Z}_\ell)^{G_K} \cong H^2(X_{K^s}, \mathbb{Z}_\ell(1))^{G_K}.$$
Thus, it suffices to show that the natural map
\[ H^2(X_{K^s}, \mathbb{Z}_\ell(1))^{G_K}/\ell \longrightarrow (H^2(X_{K^s}, \mathbb{Z}_\ell(1)))/\ell^{G_K} \]
is surjective for \(\ell \gg 0\). By Gabber’s theorem (cf. [Gab]), \(H^2(X_{K^s}, \mathbb{Z}_\ell(1))[\ell] = 0\) for \(\ell \gg 0\). Thus, for \(\ell \gg 0\), there is an exact sequence
\[ 0 \longrightarrow H^2(X_{K^s}, \mathbb{Z}_\ell(1))^{G_K} \overset{\ell}{\longrightarrow} H^2(X_{K^s}, \mathbb{Z}_\ell(1)) \overset{\ell}{\longrightarrow} (H^2(X_{K^s}, \mathbb{Z}_\ell(1)))/\ell^{G_K} \]
\[ \longrightarrow H^1_{cts}(G_K, H^2(X_{K^s}, \mathbb{Z}_\ell(1)))[\ell] \longrightarrow 0. \]
Therefore, the question is reduced to show
\[ H^1_{cts}(G_K, H^2(X_{K^s}, \mathbb{Z}_\ell(1)))[\ell] = 0 \text{ for } \ell \gg 0 \]
under the assumption of the finiteness of \(\text{Br}(X_{K^s})^{G_K}(\ell)\).

## 3 Proof of Theorem 1.2

Let \(X\) be a smooth projective geometrically connected variety over a finitely generated field \(K\) of characteristic \(p > 0\). Assuming that \(\text{Br}(X_{K^s})^{G_K}(\ell)\) is finite for all prime \(\ell \neq p\), we will show
\[ (1) \quad H^1_{cts}(G_K, H^2(X_{K^s}, \mathbb{Z}_\ell(1)))[\ell] = 0 \text{ for } \ell \gg 0. \]

### 3.1 The spreading out trick

**Lemma 3.1.** Let \(Y\) be an affine smooth geometrically connected variety over a finite field \(k\) with function field \(K\). Let \(\pi : X \longrightarrow Y\) be a smooth projective morphism with a generic fiber \(X\) geometrically connected over \(K\). Let \(\ell \neq \text{char}(k)\) be a prime. Then the following statements are true.

(a) The spectral sequence
\[ E_2^{p,q} = H^p(Y_{k^s}, R^q\pi_*\mathbb{Z}_\ell(1)) \Rightarrow H^{p+q}(X_{k^s}, \mathbb{Z}_\ell(1)) \]
degenerates at \(E_2\) for \(\ell \gg 0\).

(b) There is an exact sequence
\[ 0 \longrightarrow H^1(Y, R^2\pi_*\mathbb{Z}_\ell(1)) \longrightarrow H^1_{cts}(K, H^2(X_{K^s}, \mathbb{Z}_\ell(1))) \longrightarrow \prod_{y \in Y} H^1_{cts}(K^s_{y^\text{sh}}, H^2(X_{K^s}, \mathbb{Z}_\ell(1))). \]

(c) For \(\ell \gg 0\), we have
\[ H^1_{cts}(K, H^2(X_{K^s}, \mathbb{Z}_\ell(1)))[\ell] \cong H^1(Y, R^2\pi_*\mathbb{Z}_\ell(1))[\ell]. \]
(d) There is an exact sequence

$$0 \longrightarrow H^0(Y_k^\ast, R^2\pi_*\mathbb{Z}_\ell(1))_{G_k} \longrightarrow H^1(Y, R^2\pi_*\mathbb{Z}_\ell(1)) \longrightarrow H^1(Y_k^\ast, R^2\pi_*\mathbb{Z}_\ell(1))_{G_k}.$$ 

Proof. To prove (a), we will use Deligne’s Lefschetz criteria for degeneration of spectral sequences (cf.[Del1, Prop. 2.4]). Let $u \in H^2(X, \mathbb{Z}_\ell(1))$ be the $\ell$-adic chern class of the ample line sheaf $\mathcal{O}(1)$ associated to the projective morphism $\pi$. It suffices to check $u$ and $\mathbb{Z}_\ell(1)$ satisfies the Lefschetz condition. Since $R^q\pi_*\mathbb{Z}_\ell(1)$ is lisse for all $q$, by the proper base change theorem, it suffices to check the Lefschetz condition for a single fiber of $\pi$ over a closed point $y \in Y$. Thus, we might assume that $Y = \text{Spec } k$. Let $m$ denote $\dim(X)$.

By Gabber’s Theorem [Gab, §6], the Lefschetz morphism

$$H^{m-i}(X_k^\ast, \mathbb{Z}_\ell(1)) \longrightarrow H^{m+i}(X_k^\ast, \mathbb{Z}_\ell(i+1))$$

induced by $u^i$ is an isomorphism for $0 \leq i \leq m$ and $\ell \gg 0$. This proves the first claim.

For (b), let $j : \text{Spec } K \rightarrow Y$ be the generic point. Since $\pi$ is smooth and proper, $R^2\pi_*\mathbb{Z}/\ell^n(1)$ is a locally constant sheaf for all $n \geq 1$. Thus, we have

$$R^2\pi_*\mathbb{Z}/\ell^n(1) \cong j_*j^* R^2\pi_*\mathbb{Z}/\ell^n(1).$$

By the spectral sequence

$$E_2^{p,q} = H^p(Y, R^q j_* j^* R^2 \pi_* \mathbb{Z}/\ell^n(1)) \Rightarrow H^{p+q}(\text{Spec } K, j_* R^2 \pi_* \mathbb{Z}/\ell^n(1)),$$

we get an exact sequence

$$0 \longrightarrow H^1(Y, j_* j^* R^2 \pi_* \mathbb{Z}/\ell^n(1)) \longrightarrow H^1(K, j_* R^2 \pi_* \mathbb{Z}/\ell^n(1)) \longrightarrow H^0(Y, R^1 j_* j^* R^2 \pi_* \mathbb{Z}/\ell^n(1)).$$

Thus, there is an exact sequence

$$0 \longrightarrow H^1(Y, R^2 \pi_* \mathbb{Z}/\ell^n(1)) \longrightarrow H^1(K, H^2(X_{K^s}, \mathbb{Z}/\ell^n(1))) \longrightarrow \prod_{y \in Y} H^1(K_{y}^{sh}, H^2(X_{K^s}, \mathbb{Z}/\ell^n(1))).$$

Taking limit, we get (b).

For (c), it suffices to show that the third term of the sequence in (b) has no torsion for $\ell \gg 0$. Since $R^2\pi_*\mathbb{Z}_\ell(1)$ is lisse, the action of $\text{Gal}(K^s/K_{y}^{sh})$ on $H^2(X_{K^s}, \mathbb{Z}_\ell(1))$ is trivial. Thus,

$$H^1_{cts}(K_{y}^{sh}, H^2(X_{K^s}, \mathbb{Z}_\ell(1))) = \text{Hom}_{cts}(\text{Gal}(K^s/K_{y}^{sh}), H^2(X_{K^s}, \mathbb{Z}_\ell(1))).$$

So the claim follows from the fact $H^2(X_{K^s}, \mathbb{Z}_\ell(1))|\ell = 0$ for $\ell \gg 0$.

(d) follows from the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{cts}(G_k, H^q(Y_k^\ast, R^2\pi_*\mathbb{Z}_\ell(1))) \Rightarrow H^{p+q}(Y, R^2\pi_*\mathbb{Z}_\ell(1))$$

and the fact $H^1_{cts}(G_k, M) = M_{G_k}$. 

\qed
3.2 Compatible system of \( G_k \)-modules

By Lemma 3.1 (c) and (d), to prove (1), it suffices to show

\[
H^0(\mathcal{Y}_k, R^2\pi_*\mathbb{Z}_\ell(1))_{G_k}[\ell] = 0 \quad \text{and} \quad H^1(\mathcal{Y}_k, R^2\pi_*\mathbb{Z}^\ell(1))_{G_k}[\ell] = 0 \quad \text{for } \ell \gg 0.
\]

Let \( k \) be a finite field of characteristic \( p > 0 \) and \( I \) be the set of primes different from \( p \). Let \( F \in G_k \) be the geometric Frobenius element. Let \( M = (M_\ell, \ell \in I) \) be a family of finitely generated \( \mathbb{Z}_\ell \)-modules equipped with continuous \( G_k \)-actions. We say that \( M = (M_\ell, \ell \in I) \) is a compatible system of \( G_k \)-modules if there exists a polynomial \( P(T) \in \mathbb{Q}[T] \) with \( P(1) \neq 0 \) such that \( (F - 1)P(F) \) kills all \( M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \). Let \( N = (N_\ell, \ell \in I) \) be a family of \( G_k \)-modules. We say that \( N \) is a system of submodules (resp. quotient modules) of \( M \) if \( N_\ell \) is a \( G_k \)-submodule (resp. \( G_k \)-quotient modules) of \( M_\ell \) for all \( \ell \in I \). It is obvious that a system of submodules (resp. quotient modules) of a compatible system of \( G_k \)-modules is also a compatible system.

We will show that \( H^0(\mathcal{Y}_k, R^2\pi_*\mathbb{Z}_\ell(1)), \ell \in I \) and \( H^1(\mathcal{Y}_k, R^2\pi_*\mathbb{Z}_\ell(1)), \ell \in I \) are compatible systems of \( G_k \)-modules.

**Lemma 3.2.** Let \( M \) be a compatible system of \( G_k \)-modules and \( f_\ell : M_\ell^{G_k} \to (M_\ell)_{G_k} \) be the map induced by the identity. Then \( f_\ell \) has a finite kernel and a finite cokernel and

\[
|\ker(f_\ell)| = |\coker(f_\ell)| \quad \text{for } \ell \gg 0.
\]

Moreover, if 1 is not an eigenvalue of \( F \) on \( M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \) for all \( \ell \in I \), then

\[
|M_\ell^{G_k}| = |(M_\ell)_{G_k}| < \infty \quad \text{for } \ell \gg 0.
\]

**Proof.** Let \( \theta_\ell \) denote the endomorphism \( F - 1 \) on \( M_\ell \) and \( \theta_\ell \otimes 1 \) denote the corresponding endomorphism of \( M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \). Write \( \det(T - \theta_\ell \otimes 1) = T^{n_\ell} R_\ell(T) \) with \( R_\ell(0) \neq 0 \). By the assumption, there exists a polynomial \( P(T) \in \mathbb{Q}[T] \) with \( P(1) \neq 0 \) such that \( (F - 1)P(F) \) kills all \( M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \). Thus, the generalized 1-eigenspace of \( F \) on \( M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \) is equal to \( \ker(F - 1) \). Thus, \( \rho_\ell = \text{rank}_{\mathbb{Z}_\ell}(\ker(\theta_\ell)) \). By \[\text{Tat3, } \S 5, \text{ Lem. } z.4\], \( f_\ell \) has a finite kernel and a finite cokernel and

\[
|\ker(f_\ell)|/|\coker(f_\ell)| = |R_\ell(0)|^\ell.
\]

\( R_\ell(0) \) is equal to the product of roots of \( R_\ell(T) \) and all roots of \( R_\ell(T) \) are roots of \( P(T + 1) \). Since all roots of \( P(T + 1) \) have \( \ell \)-adic absolute values 1 for \( \ell \gg 0 \), so \(|R_\ell(0)|^\ell = 1 \) for \( \ell \gg 0 \). This completes the proof of the first statement.

For the second statement, under the assumption, \( f_\ell \) is a map between two finite groups. Then the claim follows from the first statement and the following exact sequence of finite abelian groups

\[
0 \to \ker(f_\ell) \to M_\ell^{G_k} \to (M_\ell)_{G_k} \to \coker(f_\ell) \to 0.
\]

\( \square \)
Lemma 3.3. Assume that \( \text{Br}(X_{k^s})^{G_k}(\ell) \) is finite for all \( \ell \in I \). Then \( H^0(\mathcal{Y}_{k^s}, R^2\pi_\ast\mathbb{Z}_\ell(1)), \ell \in I \) is a compatible system of \( G_k \)-modules. Moreover,

\[
H^0(\mathcal{Y}_{k^s}, R^2\pi_\ast\mathbb{Z}_\ell(1))_{G_k}[\ell] = 0 \text{ for } \ell \gg 0.
\]

Proof. Let \( K' \) denote \( Kk^s \). Since \( R^2\pi_\ast\mathbb{Q}_\ell(1) \) is lisse, we have

\[
H^0(\mathcal{Y}_{k^s}, R^2\pi_\ast\mathbb{Q}_\ell(1)) = H^2(X_{k^s}, \mathbb{Q}_\ell(1))^{G_k}.
\]

By Proposition 2.1, there is a split exact sequences of \( G_k \)-representations

\[
0 \longrightarrow (\text{NS}(X_{k^s}) \otimes \mathbb{Q}_\ell)^{G_{K'}} \longrightarrow H^2(X_{k^s}, \mathbb{Q}_\ell(1))^{G_{K'}} \longrightarrow V_\ell \text{Br}(X_{k^s})^{G_{K'}} \longrightarrow 0.
\]

Write \( W_\ell \) for \( H^2(X_{k^s}, \mathbb{Q}_\ell(1))^{G_{K'}} \). By the assumption, \( V_\ell \text{Br}(X_{k^s})^{G_k} = 0 \). Thus, \( W_\ell^{G_k} \) is equal to \( (\text{NS}(X_{k^s}) \otimes \mathbb{Q}_\ell)^{G_k} \) which is a direct summand of \( W_\ell \) as \( G_k \)-representations. This implies that the generalized 1-eigenspace of \( F \) on \( W_\ell \) is equal to \( \text{Ker}(F - 1) \). By Lemma 3.1 (a), \( W_\ell \) is a quotient of \( H^2(X_{k^s}, \mathbb{Q}_\ell(1)) \). By Proposition 4.3 below, there exists a non-zero polynomial \( P(T) \in \mathbb{Q}[T] \) such that \( P(F) \) kills all \( W_\ell \). Write \( P(T) = (T - 1)^n P_1(T) \) with \( P(1) \neq 0 \). Then \( (F - 1)P_1(F) \) kills all \( W_\ell \). This proves the first claim.

For the second claim, set \( M_\ell = H^0(\mathcal{Y}_{k^s}, R^2\pi_\ast\mathbb{Z}_\ell(1)) \). \( M_\ell \cong H^2(X_{k^s}, \mathbb{Z}_\ell(1))^{G_{K'}} \), so \( (M_\ell)_{\text{tor}} = 0 \) for \( \ell \gg 0 \). Let \( f_\ell : M_\ell^{G_k} \to (M_\ell)_{G_\ell} \) be the map in Lemma 3.2. By Lemma 3.2, \( \text{Ker}(f_\ell) \) is a finite subgroup of \( M_\ell \), thus \( \text{Ker}(f_\ell) = 0 \) for \( \ell \gg 0 \). By Lemma 3.2, this implies \( \text{Coker}(f_\ell) = 0 \) for \( \ell \gg 0 \). So \( f_\ell \) is an isomorphism for \( \ell \gg 0 \). Thus, \( (M_\ell)_{G_k}[\ell] \cong M_\ell^{G_k}[\ell] = 0 \) for \( \ell \gg 0 \).

\[
\square
\]

Lemma 3.4. Let \( H^1(\mathcal{Y}_{k^s}, R^2\pi_\ast\mathbb{Z}_\ell(1)) \) as in Lemma 3.1. Then

\[
H^1(\mathcal{Y}_{k^s}, R^2\pi_\ast\mathbb{Z}_\ell(1))^{G_k}[\ell] = 0 \text{ for } \ell \gg 0.
\]

Proof. Write \( H^3_\ell \) for \( H^3(X_{k^s}, \mathbb{Z}_\ell(1)) \) in Lemma 3.1 (a). And let \( 0 \subseteq (H^3_\ell)^3 \subseteq (H^3_\ell)^2 \subseteq (H^3_\ell)^1 \subseteq (H^3_\ell)^0 = H^3_\ell \) be the filtration induced by the spectral sequence in Lemma 3.1 (a).

Since the spectral sequence degenerates at \( E_2 \), we have

\[
H^1(\mathcal{Y}_{k^s}, R^2\pi_\ast\mathbb{Z}_\ell(1)) \cong (H^3_\ell)^1/(H^3_\ell)^2.
\]

By Proposition 4.3 below, \( H^3_\ell, \ell \in I \) is a compatible system of \( G_k \)-modules. Thus, \( (H^3_\ell)^1, \ell \in I \) and \( (H^3_\ell)^2, \ell \in I \) are also compatible systems of \( G_k \)-modules. Since \( H^3_\ell \otimes \mathbb{Q}_\ell \) is of weight \( \geq 1 \), 1 is not an eigenvalue for \( F \). By Lemma 3.2, we have

\[
|(H^3_\ell)^2| = |(H^3_\ell)^3| = 0 \text{ for } \ell \gg 0.
\]

By Theorem 4.4 below,

\[
(H^3_\ell)^G_k = 0 \text{ for } \ell \gg 0.
\]

It follows that

\[
|(H^3_\ell)^2| = 0 \text{ for } \ell \gg 0.
\]
Thus,

\[ ((H^3_\ell)^2)_{G_k} = 0 \text{ for } \ell \gg 0. \]

By the exact sequence

\[ 0 \rightarrow ((H^3_\ell)^2)_{G_k} \rightarrow ((H^3_\ell)^1)_{G_k} \rightarrow ((H^3_\ell)^1/(H^3_\ell)^2)_{G_k} \rightarrow ((H^3_\ell)^2)_{G_k}, \]

we can conclude

\[ ((H^3_\ell)^1/(H^3_\ell)^2)_{G_k} = 0 \text{ for } \ell \gg 0. \]

\[ \square \]

### 3.3 Proof of Theorem 1.2

Lemma 3.3 and Lemma 3.4 completes the proof of Theorem 1.2.

### 4 Torsions of cohomology of varieties over finite fields

In this section, we will study the torsion part of \( l \)-adic cohomology for smooth varieties over finite fields.

#### 4.1 Alteration

By de Jong’s alteration theorem (cf. [deJ]), every integral variety \( X \) over an algebraic closed field \( k \) admits a proper and generic finite morphism \( f : X_1 \rightarrow X \) such that \( X_1 \) is an open subvariety of a smooth projective connected variety over \( k \). The alteration theorem will allow us to reduce questions about general non-compact smooth varieties to smooth varieties with smooth projective compactification.

**Lemma 4.1.** Let \( f : Y \rightarrow X \) be a proper and generic finite morphism between smooth irreducible varieties over an algebraic closed field \( k \). Let \( \ell \neq \text{char}(k) \) be a prime and \( d \) denote \( [K(Y) : K(X)] \). Then the kernels of the pullbacks

\[ H^i(X, \mathbb{Z}/\ell^n) \rightarrow H^i(Y, \mathbb{Z}/\ell^n) \text{ and } H^i_c(X, \mathbb{Z}/\ell^n) \rightarrow H^i_c(Y, \mathbb{Z}/\ell^n) \]

are killed by \( d \) for all integers \( i \geq 0 \) and \( n \geq 1 \).

**Proof.** Let \( m \) denote the dimension of \( X \). Since \( X \) is smooth and irreducible, there is a perfect Poincaré duality pairing

\[ H^i(X, \mathbb{Z}/\ell^n) \times H^{2m-i}(X, \mathbb{Z}/\ell^n(m)) \rightarrow H^{2m}_c(X, \mathbb{Z}/\ell^n(m)). \]

Since the pullback \( f^* \) map is compatible with the above pairing, it suffices to show that the kernel of the pullback map

\[ H^{2m}_c(X, \mathbb{Z}/\ell^n(m)) \rightarrow H^{2m}_c(Y, \mathbb{Z}/\ell^n(m)) \]
is killed by \(d\). Thus, without loss of generality, we can shrink \(X\) such that \(f\) is finite flat. Then the above map can be identified with the multiplication by \(d\)

\[
\mathbb{Z}/\ell^n \to \mathbb{Z}/\ell^n
\]

through trace maps for \(X\) and \(Y\). This proves the claim.

\[\square\]

**Proposition 4.2.** Let \(X\) be a smooth variety over an algebraic closed field \(k\). Then, for \(0 \leq i \leq 2\),

\[H^i(X, \mathbb{Z}_\ell)_{\text{tor}} = 0\] for \(\ell \gg 0\).

**Proof.** In the case that \(X\) is smooth and projective over \(k\), this is Gabber’s theorem. We might assume that \(X\) is irreducible. Then \(X\) admits an alteration \(f : Y \to X\) such that \(Y\) is an open subvariety of a smooth projective connected variety \(\bar{Y}\). By Lemma 4.1, the kernel of

\[H^i(X, \mathbb{Z}_\ell) \to H^i(Y, \mathbb{Z}_\ell)\]

is killed by the degree of \(f\) and therefore, is injective for \(\ell \gg 0\). So it suffices to prove the claim for \(Y\). For \(i = 0\), the claim is obvious. Let \(Z\) denote \(\bar{Y} - Y\) and \(Z_i\) denote the smooth locus of irreducible components of \(Z\) with codimension 1 in \(\bar{Y}\). By the theorem of purity (cf. [Fuj]), there is an exact sequence

\[
0 \to H^1(\bar{Y}, \mathbb{Z}_\ell) \to H^1(Y, \mathbb{Z}_\ell) \to \bigoplus_i H^0(Z_i, \mathbb{Z}_\ell(-1)).
\]

Since the first and the third term are torsion-free for \(\ell \gg 0\), so \(H^1(Y, \mathbb{Z}_\ell)_{\text{tor}} = 0\) for \(\ell \gg 0\). This proves the claim for \(i = 1\). For \(i = 2\), consider the exact sequence

\[
0 \to \text{Pic}(Y) \hat{\otimes} \mathbb{Z}_\ell \to H^2(Y, \mathbb{Z}_\ell(1)) \to T_\ell \text{Br}(Y) \to 0.
\]

Since the third term is always torsion free, it suffices to show that \(\text{Pic}(Y) \hat{\otimes} \mathbb{Z}_\ell\) is torsion free for \(\ell \gg 0\). Let \(a : \text{Pic}(\bar{Y}) \to \text{Pic}(Y)\) be the restriction map. Since \(\text{Pic}^0(\bar{Y})\) is \(\ell\)-divisible for all \(\ell \neq \text{char}(k)\), so \(a(\text{Pic}^0(\bar{Y}))\) is also \(\ell\)-divisible. Thus

\[
\text{Pic}(Y) \hat{\otimes} \mathbb{Z}_\ell = \text{Pic}(Y)/a(\text{Pic}^0(\bar{Y})) \hat{\otimes} \mathbb{Z}_\ell.
\]

Since \(\text{NS}(Y) \to \text{Pic}(Y)/a(\text{Pic}^0(\bar{Y}))\) is surjective, \(\text{Pic}(Y)/a(\text{Pic}^0(\bar{Y}))\) is a finitely generated abelian group. Thus, the torsion part of \(\text{Pic}(Y)/a(\text{Pic}^0(\bar{Y})) \hat{\otimes} \mathbb{Z}_\ell\) is isomorphic to the \(\ell\)-primary torsion part of \(\text{Pic}(Y)/a(\text{Pic}^0(\bar{Y}))\), which is zero for \(\ell \gg 0\). This completes the proof.

\[\square\]
4.2 $\ell$-independence

**Proposition 4.3.** Let $X$ be a separated scheme of finite type over a finite field $k$. Let $F \in G_k$ be the geometric Frobenius element. Then there exists a non-zero polynomial $P(T) \in \mathbb{Q}[T]$ such that $P(F)$ kills $H_c^i(X_{k^s}, \mathbb{Q}_\ell)$ for all primes $\ell \neq \text{char}(k)$ and $i \geq 0$. And the same claim holds for $H^i(X_{k^s}, \mathbb{Q}_\ell)$ when $X$ is smooth over $k$.

**Proof.** Note that the second claim follows from the first one and Poincaré duality. For the first claim, in the case that $X$ is smooth proper over $k$, this is a consequence of Deligne’s theorem in Weil I. For the general case, we will prove it by induction on the dimension of $X$. Firstly, we note that replacing $k$ by a finite extension will not change the question and we can assume that $X$ is integral. Let $U$ be the regular locus of $X$ and $Y = X - U$. There is a long exact sequence

$$H_c^i(U_{k^s}, \mathbb{Q}_\ell) \rightarrow H_c^i(X_{k^s}, \mathbb{Q}_\ell) \rightarrow H_c^i(Y_{k^s}, \mathbb{Q}_\ell) \rightarrow H_c^{i+1}(U_{k^s}, \mathbb{Q}_\ell)$$

By induction, the claim is true for $Y$. Thus, we can assume that $X$ is smooth. Let $X_1 \rightarrow X$ be an alteration such that $X_1$ admits a smooth projective compactification. By Lemma 4.1, the pullback map

$$H_c^i(X_{k^s}, \mathbb{Q}_\ell) \rightarrow H_c^i((X_1)_{k^s}, \mathbb{Q}_\ell)$$

is injective. It suffices to prove the claim for $X_1$. Thus, we can assume that $X$ is an open subvariety of smooth projective irreducible variety $\tilde{X}$ over $k$. Let $Y$ denote $\tilde{X} - X$. Consider the long exact sequence

$$H_c^{i-1}(Y_{k^s}, \mathbb{Q}_\ell) \rightarrow H_c^{i}(X_{k^s}, \mathbb{Q}_\ell) \rightarrow H_c^{i}(\tilde{X}_{k^s}, \mathbb{Q}_\ell) \rightarrow H_c^{i}(Y_{k^s}, \mathbb{Q}_\ell).$$

By induction, the claim is true for $Y$. Since the claim is true for $\tilde{X}$, it follows that the claim holds for $X$. This completes the proof. \qed

**Theorem 4.4.** Let $X$ be a smooth variety over a finite field $k$ of characteristic $p$. Then

$$H^3(X_{k^s}, \mathbb{Z}_\ell(1))^{G_k} = 0$$

for $\ell \gg 0$.

**Proof.** Note that replacing $k$ by a finite extension will not change the question. Thus, we can assume that $X$ is geometrically connected. There is an alteration (extending $k$ if necessary) $X_1 \rightarrow X$ such that $X_1$ is an open subvariety of a smooth projective geometrically connected variety over $k$. By Lemma 4.1, the pullback map

$$H^3(X_{k^s}, \mathbb{Z}_\ell(1)) \rightarrow H^3((X_1)_{k^s}, \mathbb{Z}_\ell(1))$$

is injective for $\ell \gg 0$. So it suffices to prove the claim for $X_1$. Thus, we can assume that $X$ is open subvariety of a smooth projective geometrically connected variety $\tilde{X}$ over $k$. There is an open subset $U$ of $\tilde{X}$ such that $U$ contains $X$ with $Y = U - X$ a smooth divisor in $U$ and $Z = \tilde{X} - U$ is of codimension $\geq 2$ in $\tilde{X}$. There is an exact sequence

$$H^3_Z(X_{k^s}, \mathbb{Z}_\ell(1)) \rightarrow H^3(X_{k^s}, \mathbb{Z}_\ell(1)) \rightarrow H^3(U_{k^s}, \mathbb{Z}_\ell(1)) \rightarrow H^3_Z(X_{k^s}, \mathbb{Z}_\ell(1)).$$
By the theorem of purity (cf. [Fuj]), $H^2_\ell(\bar{X}_{k^s}, \mathbb{Z}_\ell(1)) = 0$. By the theorem of semi-purity (cf. [Fuj]), removing a close subset of codimension $\geq 3$ in $\bar{X}$ from $Z$ will not change these groups. So we can assume that $Z$ is smooth and of pure codimension 2 in $\mathcal{Y}$. Then, by the theorem of purity,

$$H^1_\ell(\bar{X}_{k^s}, \mathbb{Z}_\ell(1)) \cong H^0(Z_{k^s}, \mathbb{Z}_\ell(-1)) \cong \oplus \mathbb{Z}_\ell(-1).$$

It follows

$$H^3(\bar{X}_{k^s}, \mathbb{Z}_\ell(1))^{G_k} \cong H^3(U_{k^s}, \mathbb{Z}_\ell(1))^{G_k}.$$  

Since $H^3(\bar{X}_{k^s}, \mathbb{Q}_\ell(1))$ is of pure weight 1, so $H^3(\bar{X}_{k^s}, \mathbb{Z}_\ell(1))^{G_k}$ is a torsion group and therefore, vanishes for $\ell \gg 0$ by Gabber’s theorem. Thus, the claim holds for $U$. There is an exact sequence

$$0 \longrightarrow H^3(U_{k^s}, \mathbb{Z}_\ell(1)) \longrightarrow H^3(U_{k^s}, \mathbb{Z}_\ell(1)) \overset{\alpha_k}{\longrightarrow} H^3(\bar{X}_{k^s}, \mathbb{Z}_\ell(1)) \longrightarrow H^4_\ell(U_{k^s}, \mathbb{Z}_\ell(1)).$$

By the theorem of purity, $H^4_\ell(U_{k^s}, \mathbb{Z}_\ell(1)) \cong H^2(Y_{k^s}, \mathbb{Z}_\ell)$. Thus, $H^4_\ell(U_{k^s}, \mathbb{Z}_\ell(1))^{G_k} \cong H^2(Y_{k^s}, \mathbb{Z}_\ell)$ is a torsion abelian group and therefore, vanishes for $\ell \gg 0$ by Proposition 4.2. Thus, it suffices to prove $\text{Im}(\alpha_k)^{G_k} = 0$ for $\ell \gg 0$. Since $H^3(U_{k^s}, \mathbb{Z}_\ell(1))$ is of weight $\geq 1$, by Proposition 3.2, $H^3(U_{k^s}, \mathbb{Z}_\ell(1)), \ell \in I$ and $\text{Im}(\alpha_k), \ell \in I$ are compatible systems of $G_k$-modules. Let $M$ be one of these systems. By Lemma 3.2, the vanishing of $M^G_k$ and $(M^\ell)^{G_k}$ are equivalent for $\ell \gg 0$. Since $H^3(U_{k^s}, \mathbb{Z}_\ell(1))^{G_k} = 0$ for $\ell \gg 0$, we have $H^3(U_{k^s}, \mathbb{Z}_\ell(1))^{G_k} = 0$ for $\ell \gg 0$. Thus, $\text{Im}(\alpha_k)^{G_k} = 0$ for $\ell \gg 0$. It follows that $\text{Im}(\alpha_k)^{G_k} = 0$ for $\ell \gg 0$. This completes the proof. \hfill \Box

5 Proof of Theorem 1.3

5.1 Theorem 1.2 for non-compact varieties

To prove Theorem 1.3, we need a version of Theorem 1.2 for non-compact smooth varieties over finite fields.

Lemma 5.1. Let $X$ be a smooth irreducible variety over a finite field $k$ of characteristic $p$. Assuming that $X$ admits a finite flat alteration $X_1 \longrightarrow X$ such that $X_1$ is an open subvariety of a smooth projective variety over $k$, then the natural map

$$\text{Br}(X)(\text{non-p}) \longrightarrow \text{Br}(X_{k^s})^{G_k}(\text{non-p})$$

has a cokernel of finite exponent.

Proof. Let $f : X_1 \rightarrow X$ be an alteration as in the assumption. One can define a canonical norm map $\text{Nm} : f_* \mathbb{G}_m \longrightarrow \mathbb{G}_m$ such that the composition

$$\mathbb{G}_m \longrightarrow f_* \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

is equal to the multiplication by $[K(X_1) : K(X)]$. Then $\text{Nm}$ induces a natural map

$$\text{Nm} : H^2(X_1, \mathbb{G}_m) = H^2(X, f_* \mathbb{G}_m) \longrightarrow H^2(X, \mathbb{G}_m).$$
By definition, the composition of $N_m$ and $f^*$

$$H^2(X, \mathbb{G}_m) \xrightarrow{f^*} H^2(X_1, \mathbb{G}_m) \xrightarrow{N_m} H^2(X, \mathbb{G}_m)$$

is the multiplication by $[K(X_1) : K(X)]$. It follows that $N_m$ has a cokernel of finite exponent. Consider the following diagram

$$\begin{array}{ccc}
Br(X_1) & \longrightarrow & Br((X_1)_{k^s})^{G_k} \\
\downarrow_{N_m} & & \downarrow_{N_m} \\
Br(X) & \longrightarrow & Br(X_{k^s})^{G_k}
\end{array}$$

The diagram commutes (cf., e.g., [ABBG, Prop. 2.2]). Since all vertical maps have cokernels of finite exponent, it suffices to show the first row has a cokernel of finite exponent up to $p$-torsion. Thus, we can assume that $X$ is an open subvariety of a smooth projective variety $\bar{X}$. Consider the following commutative diagram

$$\begin{array}{ccc}
Br(\bar{X})^{(non-p)} & \longrightarrow & Br(\bar{X}_{k^s})^{G_k,(non-p)} \\
\downarrow & & \downarrow \\
Br(X)^{(non-p)} & \longrightarrow & Br(X_{k^s})^{G_k,(non-p)}
\end{array}$$

It is well-known that the first row has a cokernel of finite exponent (cf. [Yua]). By [Qin1, Prop. 2.3], the second column has a cokernel of finite exponent. Thus, the second row has a cokernel of finite exponent. This completes the proof.

**Theorem 5.2.** Let $X$ be a smooth geometrically connected variety over a finite field $k$ of characteristic $p$. Assuming that $Br(X_{k^s})^{G_k}(\ell)$ is finite for some prime $\ell \neq p$, then $Br(X_{k^s})^{G_k,(non-p)}$ is finite.

**Proof.** By [Qin2, Thm. 3.5], $Br(X_{k^s})^{G_k}(\ell)$ is finite for all $\ell \neq p$. By [Qin1, Prop. 2.3], shrinking $X$ will not change the question. Thus, we can assume that $X$ satisfies the assumption in Lemma 5.1. So the natural map $Br(X)(\ell) \to Br(X_{k^s})^{G_k}(\ell)$ is surjective for $\ell \gg 0$. It suffices to show that $Br(X)(\ell)$ is divisible for $\ell \gg 0$. Consider the exact sequence

$$0 \longrightarrow Br(X)/\ell^n \longrightarrow H^3(X, \mathbb{Z}/\ell^n(1)) \longrightarrow H^3(X, \mathbb{G}_m)[\ell^n] \longrightarrow 0.$$ 

Taking limit, we get an injection

$$Br(X)\otimes\mathbb{Z}_\ell \hookrightarrow H^3(X, \mathbb{Z}_\ell(1)).$$

Since $Br(X)$ is a torsion abelian group of cofinite type, we have

$$Br(X)\otimes\mathbb{Z}_\ell \cong Br(X)(\ell)/(Br(X)(\ell))_{div}.$$
Thus, it suffices to show \( H^3(X, \mathbb{Z}_\ell(1))_{\text{tor}} = 0 \) for \( \ell \gg 0 \). By the Hochschild-Serre spectral sequence
\[
E_2^{p,q} = H^p_{CR}(G_k, H^q(X_{k^s}, \mathbb{Z}_\ell(1))) \Rightarrow H^{p+q}(X, \mathbb{Z}_\ell(1))
\]
and the vanishing of \( E_2^{p,q} \) for \( p \geq 2 \), we get an exact sequence
\[
0 \to H^2(X_{k^s}, \mathbb{Z}_\ell(1))_{G_k} \to H^3(X, \mathbb{Z}_\ell(1)) \to H^3(X_{k^s}, \mathbb{Z}_\ell(1))_{G_k} \to 0.
\]
By Theorem 4.4, the third term vanishes for \( \ell \gg 0 \). It suffices to show
\[
H^2(X_{k^s}, \mathbb{Z}_\ell(1))_{G_k}[\ell] = 0 \text{ for } \ell \gg 0.
\]

By [Qin2, Lem. 3.4], there is a split exact sequence of \( G_k \)-representations
\[
0 \to \text{Pic}(X_{k^s}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2(X_{k^s}, \mathbb{Q}_\ell(1)) \to V_\ell \text{Br}(X_{k^s}) \to 0.
\]
Notice that \( H^2(X_{k^s}, \mathbb{Z}_\ell(1))_{\text{tor}} = 0 \) for \( \ell \gg 0 \) by Lemma 4.2. Set \( M_\ell = H^2(X_{k^s}, \mathbb{Z}_\ell(1)) \), by a same argument as in the proof of Lemma 3.3, \( M = (M_\ell, \ell \in I) \) is a compatible system of \( G_k \)-modules. By the exact same argument of the proof of the second claim in Lemma 3.3, we have \( (M_\ell)_{\text{tor}} = 0 \) for \( \ell \gg 0 \). This completes the proof.

Remark 5.3. It is possible that Theorem 5.2 still holds for any finitely generated field \( k \) of positive characteristic. The tricky part is to show the \( \ell \)-independence of the finiteness of \( \text{Br}(X_{k^s})_{G_k}(\ell) \). Assuming resolution of singularity for varieties over fields of positive characteristic, then Theorem 5.2 can be derived directly from Theorem 1.2.

5.2 Weil restriction

To reduce Theorem 1.3 to Theorem 5.2, we need a technique of Weil restriction for abelian varieties. Following Milne’s paper [Mil2], we will show that the Tate-Shafarevich group for an abelian variety does not change under Weil restriction. In fact, \( L \)-function for an abelian variety does not change under Weil restriction either. So the same idea can give a new proof of Theorem 1.9 in [Qin2].

Proposition 5.4. Let \( f : \mathcal{Z} \to \mathcal{Y} \) be a finite etale morphism between smooth geometrically connected varieties over a finite field \( k \) of characteristic \( p \). Write \( K \) (resp. \( L \)) for the function field of \( \mathcal{Y} \) (resp. \( \mathcal{Z} \)). Let \( B \) be an abelian variety over \( L \) and \( A \) be its Weil restriction to \( K \). Assuming that \( A \) and \( B \) extend to abelian schemes over \( \mathcal{Y} \) and \( \mathcal{Z} \) respectively. Then
\[
\text{III}_{\mathcal{Y}}(A)(\ell) \cong \text{III}_{\mathcal{Y}}(B)(\ell)
\]
for any \( \ell \neq p \).

Proof. Let \( \mathcal{B} \) (resp. \( \mathcal{A} \)) denote the etale sheaf on \( \text{Spec} L \) associated to the Galois module \( B(L^s) \) (resp. \( B(K^s) \)). Let \( f \) denote the induced morphism \( \text{Spec} L \to \text{Spec} K \). Then the sheaf \( f_* \mathcal{B} \) is isomorphic to the sheaf \( \mathcal{A} \). This can be seen by checking the equivalence between Frobenius reciprocity and adjoint property of \( f^* \) and \( f_* \). Let \( \mathcal{A} \to \mathcal{Y} \) (resp.
$B \rightarrow \mathcal{Z}$) be an abelian scheme extending $A \rightarrow \text{Spec} \, K$ (resp. $B \rightarrow \text{Spec} \, L$). Let $i$ (resp. $j$) denote the morphism $\text{Spec} \, K \rightarrow \mathcal{Y}$ (resp. $\text{Spec} \, L \rightarrow \mathcal{Z}$). By [Kel1, Thm. 3.3], we have $\mathcal{A} \cong i_* \mathcal{A}$ and $\mathcal{B} \cong j_* \mathcal{B}$. Thus, $f_* \mathcal{B} \cong \mathcal{A}$. By [Qin2, Lem. 5.2], we have
\[ \Pi_{\mathcal{Y}}(A)(\ell) \cong H^1(\mathcal{Y}, \mathcal{A})(\ell) \text{ and } \Pi_{\mathcal{Z}}(B)(\ell) \cong H^1(\mathcal{Z}, \mathcal{B})(\ell). \]
Since $f$ is finite, we have $H^1(\mathcal{Z}, \mathcal{B}) \cong H^1(\mathcal{Y}, f_* \mathcal{B})$. Then, the claim follows from
\[ H^1(\mathcal{Z}, \mathcal{B}) \cong H^1(\mathcal{Y}, f_* \mathcal{B}) \cong H^1(\mathcal{Y}, \mathcal{A}). \]

\[ \square \]

**Remark 5.5.** By the same argument, we have $f_* V_t \mathcal{B} \cong V_t \mathcal{A}$. Thus, $f_* V_t \mathcal{B}(-1) \cong V_t \mathcal{A}(-1)$. It follows $L(\mathcal{Y}, V_t \mathcal{A}(-1), s) = L(\mathcal{Y}, f_* V_t \mathcal{B}(-1), s) = L(\mathcal{Z}, V_t \mathcal{B}(-1), s)$. Since $A(K) = B(L)$ by definition, so the BSD conjecture for $A$ is equivalent to the BSD conjecture for $B$.

### 5.3 Proof of Theorem 1.3

**Lemma 5.6.** Let $K = k(t_1, \ldots, t_m)$ be a purely transcendental extension of a finite field $k$. Let $A$ be an abelian variety over $K$. Then Theorem 1.3 holds for $A$.

**Proof.** Let $\mathcal{Y}$ be a smooth geometrically connected variety over $k$ with a function field $K$. Let $A^t$ denote $\text{Pic}^0_{A/K}$. Then $A$ can be identified with $\text{Pic}^0_{A^t/K}$. By [Qin2, Lem. 5.2], without loss of generality, we can shrink $\mathcal{Y}$ such that $A^t$ extends to an abelian scheme $\pi : \mathcal{A}^t \rightarrow \mathcal{Y}$. By [Qin2, Thm. 1.11], there is an exact sequence
\[ 0 \rightarrow V_t \text{Br}((\mathcal{Y}_{k^s})^G_k) \rightarrow \text{Ker}(V_t \text{Br}(\mathcal{A}^t_{k^s})^G_k \rightarrow V_t \text{Br}(A^t_{k^s})^G_k) \rightarrow V_t \Pi_{\mathcal{Y}}(A) \rightarrow 0. \]
Since $T^1(X, \ell)$ holds for rational varieties and abelian varieties (Zarhin’s theorem), we have $V_t((\mathcal{Y}_{k^s})^G_k) = 0$ and $V_t \text{Br}(A^t_{k^s})^G_k = 0$. Assuming that $\Pi_{\mathcal{Y}}(A)(\ell)$ is finite, then $V_t \Pi_{\mathcal{Y}}(A) = 0$. Thus, $V_t \text{Br}(\mathcal{A}^t_{k^s})^G_k = 0$. So $\text{Br}(\mathcal{A}^t_{k^s})^G_k(\ell)$ is finite. By Theorem 5.2, $\text{Br}(\mathcal{A}^t_{k^s})^G_k(\ell) = 0$ for $\ell \gg 0$. In fact, the proof of [Qin2, Thm. 1.11] can imply that there is an exact sequence for $\ell \gg 0$
\[ 0 \rightarrow \text{Br}((\mathcal{Y}_{k^s})^G_k(\ell)) \rightarrow \text{Ker}(\text{Br}(\mathcal{A}^t_{k^s})^G_k(\ell) \rightarrow \text{Br}(A^t_{k^s})^G_k(\ell)) \rightarrow \Pi_{\mathcal{Y}_{k^s}}(A)^G_k(\ell) \rightarrow 0. \]
It follows that $\Pi_{\mathcal{Y}_{k^s}}(A)^G_k(\ell) = 0$ for $\ell \gg 0$. Next, we will show that the natural map
\[ \Pi_{\mathcal{Y}}(A) \rightarrow \Pi_{\mathcal{Y}_{k^s}}(A)^G_k \]
has a finite kernel. Let $K'$ denote $Kk^s$. There is an exact sequence
\[ 0 \rightarrow H^1(G_k, A(K')) \rightarrow H^1(K, A) \rightarrow H^1(K', A). \]
It suffices to show that $H^1(G_k, A(K'))$ is finite. By Lang-Néron Theorem, the quotient
\[ A(K')/\text{Tr}_{K/k}(A)(k^s) \]

is a finitely generated abelian group. Thus, $H^1(G_k,A(K'))/\text{Tr}_{K/k}(A)(k^s)$ is finite. By Lang’s theorem, $H^1(G_k,\text{Tr}_{K/k}(A)(k^s)) = 0$. Thus, $H^1(G_k,A(K'))$ is finite. So

$$\Pi Y(A)(\ell) \rightarrow \Pi Y_k(A)^G_k(\ell)$$

is injective for $\ell \gg 0$. It follows that $\Pi Y(A)(\ell) = 0$ for $\ell \gg 0$. This completes the proof. \qed

**Proof of Theorem 1.3.**

Since $k$ is perfect, so $K/k$ is separably generated, i.e. there exist algebraic independent elements $t_1,...,t_m$ in $K$ such that $K/k(t_1,...,t_m)$ is a finite separable extension. Let $M$ denote $k(t_1,...,t_m)$. Let $B$ be the Weil restriction of $A$ to $M$. By Proposition 5.4, it suffices to prove the claim for $B$. By Lemma 5.6, the claim holds for $B$. This completes the proof. \qed

**References**

[ABBG] Auel, A., Bigazzi, A., Bölling, C., & Graf von Bothmer, H. C. . (2019) Universal Triviality of the Chow Group of 0-cycles and the Brauer Group, International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–18

[Čes] Česnavičius, K., 2019. Purity for the Brauer group. Duke Mathematical Journal, 168(8), pp.1461-1486.

[Cha] F. Charles, The Tate conjecture for K3 surfaces over finite fields, Invent. Math. 194, 119-145 (2013).

[CHT] A. Cadoret, C.Y. Hui and A. Tamagawa, $\mathbb{Q}_\ell$ versus $\mathbb{F}_\ell$-coefficients in the Grothendieck-Serre-Tate conjectures. Available at https://webusers.imj-prg.fr/~anna.cadoret/GST.pdf

[Con] Conrad, B. (2006). Chow’s $K/k$-image and $K/k$-trace, and the Lang-Néron theorem. Enseignement Mathématique, 52(1/2), 37.

[deJ] De Jong, A. J. (1996). Smoothness, semi-stability and alterations. Publications Mathématiques de l’Institut des Hautes Études Scientifiques, 83(1), 51-93.

[Del1] Deligne, P. (1968). Lefschetz’s theorem and criteria of degeneration of spectral sequences. Mathematical Publications of the Institut des Hautes Études Scientialités, 35 (1), 107-126.

[Del2] Deligne, Pierre. “Weil’s conjecture: II.” IHÉS Mathematical Publications 52 (1980): 137-252.

[Fu] Fu, Lei. Étale cohomology theory. Vol. 13. World Scientific, 2011.
[Fuj] Fujiwara, K. (2002). A proof of the absolute purity conjecture (after Gabber). In Algebraic geometry 2000, Azumino (pp. 153-183). Mathematical Society of Japan.

[Gab] GABBER, O. 1983. Sur la torsion dans la cohomologie l-adique d’une variété. C. R. Acad. Sci. Paris Ser. I Math. 297:179–182.

[Gro1] A. Grothendieck, Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 46–66.

[Gro2] A. Grothendieck, Le groupe de Brauer. II. Théorie cohomologique, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 67–87.

[Gro3] A. Grothendieck, Le groupe de Brauer. III. Exemples et compléments, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188.

[Kel1] Keller T. On the Tate–Shafarevich group of Abelian schemes over higher dimensional bases over finite fields. Manuscripta Mathematica. 2016 May 1;150(1-2):211-45.

[Kel2] Keller T. On an analogue of the conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher dimensional bases over finite fields. Documenta Mathematica. 2019;24:915-93.

[Lic] Lichtenbaum, S. (1983). Zeta-functions of varieties over finite fields at s= 1. In Arithmetic and geometry (pp. 173-194). Birkhäuser, Boston, MA.

[Mil1] Milne, James S. Etale cohomology (PMS-33). Vol. 5657. Princeton university press, 1980.

[Mil2] Milne, J. S. (1972). On the arithmetic of abelian varieties. Inventiones mathematicae, 17(3), 177-190.

[MP] K. Madapusi Pera, The Tate conjecture for K3 surfaces in odd characteristic, Invent. Math. 201, 625-668 (2015).

[Ser] Serre JP. Zeta and L functions. In Arithmetical Algebraic Geometry, Proc. of a Conference held at Purdue Univ., Dec. 5-7, 1963 1965. Harper and Row.

[Qin1] Qin, Y. On the Brauer groups of fibrations. arXiv preprint arXiv:2012.01324.

[Qin2] Qin, Y. Comparison of different Tate conjectures. arXiv preprint arXiv:2012.01337.

[Sch] Schneider, P. (1982). On the conjecture of Birch and Swinnerton-Dyer about global function fields. Mathematische Annalen , 260 (4), 495-510.

[Tat1] Tate J. Algebraic cycles and poles of zeta functions. Arithmetical algebraic geometry. 1965:93-110.
[Tat2] Tate, John. Conjectures on algebraic cycles in \( \ell \)-adic cohomology. Motives (Seattle, WA, 1991) 55 (1994): 71-83.

[Tat3] Tate, J. (1965). On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Séminaire Bourbaki, 9(306), 415-440.

[SZ1] Skorobogatov, A. N., & Zarkhin, Y. G. (2008). A finiteness theorem for the Brauer group of Abelian varieties and KS surfaces. Journal of Algebraic Geometry, 17(3), 481-502.

[SZ2] Skorobogatov, Alexei N., and Yuri G. Zarhin. A finiteness theorem for the Brauer group of K3 surfaces in odd characteristic. International Mathematics Research Notices 2015.21 (2015): 11404-11418.

[Ulm] D. Ulmer, Curves and Jacobians over function fields, Arithmetic geometry over global function fields, 2014, pp.281-337.

[Yua] Yuan, Xinyi. Comparison of arithmetic Brauer groups with geometric Brauer group. arXiv preprint arXiv:2011.12897v2.

[Zar] Yu.G. Zarhin, Abelian varieties in characteristic \( p \). Mat. Zametki 19 (1976), 393-400; Math. Notes 19 (1976), 240-244.