Activated resistivities in the integer quantum Hall effect

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Abstract

We have determined the off-diagonal and diagonal conductivities for a quantum Hall effect system at exactly integer filling at finite temperatures and in the presence of weak short ranged disorder potential within the self consistent Born approximation. We find that there is a finite temperature contribution to off-diagonal conductivity $\sigma_{xy}$ which is ‘anomalous’ in nature as it survives even in the zero impurity limit. The diagonal conductivity $\sigma_{xx}$ survives only when both temperature and disorder is non zero. At low temperatures, $\sigma_{xx}$ activates with a temperature dependent prefactor. Inverting the conductivity matrix, we determine the resistivities. The deviation of the off-diagonal resistivity $\rho_{xy}$ from its zero temperature value and the diagonal resistivity $\rho_{xx}$ activate with a temperature dependent prefactor at low temperatures, in agreement with experiments. Further, we find two physical regimes both of which are at low temperatures and low broadening, which provide the experimentally observed linear relationship between the deviation of $\rho_{xy}$ and the $\rho_{xx}$ with different signs. We have also estimated the effective masses from the experimental data of $\rho_{xy}$ and find them to be reasonable. Finally, our result on compressibility as a function of temperature shows that there is no phase transition involved in the system as far as the temperature is concerned.

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I. INTRODUCTION

It was first observed by von Klitzing et al. [1] that two dimensional electron gas at low temperatures and in the presence of high magnetic field $B$ perpendicular to the plane can exhibit quantization of the Hall resistivity at integer fillings, with a high accuracy (one part in $10^5$). Subsequently, it has been observed that the precision of this quantization could be very high (one part in $10^9$) [2]. The essence of this integer quantum Hall effect (IQHE) is that the quantization of Hall resistivity $\rho_{xy} = \frac{1}{i}(2\pi/e^2)$ (we have set the unit $\hbar = c = 1$) at integer filling factor $\nu = i$ exists for a wide range of physical parameters, viz, of $B$ and of the carrier density $\rho$. At the same time the diagonal resistivity $\rho_{xx}$ shows a sharp minimum.

Prange [3], Laughlin [4], and Halperin [5] have argued that as long as the Fermi level lies in the region of localized states between two current carrying regions of extended states, the Hall conductivity $\sigma_{xy}$ is quantized and $\sigma_{xx}$ vanishes. It has been observed that the measured value of $\rho_{xy}$ approaches the universal value $\frac{1}{i}(2\pi/e^2)$ as the temperature is lowered.

In his pioneering work, Laughlin [4] has shown that the edge effects are not important for the accuracy of quantization. He further speculates that the only significant source of error in quantization is the thermal activation. In fact, there exists a fairly good number of experiments [3-17] which study the effect of temperature on the quantization. Speaking broadly, there are two important aspects regarding these studies, viz, (i) the flatness of the plateaus formed in $\rho_{xy}$ and the critical transition between the plateaus, (ii) the value of quantization at the central point of the plateaus, i.e., the points at which $\rho_{xx}$ show minima and for which $\nu$ are exactly integers.

The present paper is principally concerned with the latter aspect, i.e., the behaviour of $\rho_{xy}$ and $\rho_{xx}$ with changing temperature and disorder. We shall study this at the centre of the plateau, i.e., we look at the point at which the filling factor $\nu$ is an integer.

Let us now briefly review the pertinent experimental situation. Early studies by Yoshihiro et al [6] and Cage et al [7] show that $\rho_{xy}$ decreases with increase in temperature at the minimum of $\rho_{xx}$. They also find a linear relationship $\Delta \rho_{xy} = -S \rho_{xx}^{\text{min}}$ between the deviation
\[\Delta \rho_{xy} \equiv \rho_{xy}(T) - \rho_{xy}(0)\] from the zero temperature value of \(\rho_{xy}\), and the minimum value of \(\rho_{xx}\). Cage et al \[7\] have further found that the value of \(S\) varies from 0.06 to 0.51 for different GaAs samples. In Si MOSFETs, Yoshihiro et al \[6\] have found \(S \simeq 0.1\). Although the values of \(S\) are device dependent and also on how the system is cooled, the linear relationship is itself universal. Cage \[18\] has pointed out that sample size effects which have been calculated by Rendell and Girvin \[19\], Hall probe misalignment, and variable range hopping conduction are not responsible for the linear relation between \(\Delta \rho_{xy}\) and \(\rho_{xx}^{\text{min}}\). Moreover, Cage et al \[7\] have surprisingly observed that even when the Hall steps are flat to within 0.01 ppm instrumental resolution, the temperature dependent error \(\Delta \rho_{xy}\) is still quite large. This suggests the loss of universality at finite temperatures. However, it is not always true that \(\rho_{xy}\) decreases with the increase of temperature, as Wel et al \[8\] have observed a positive slope between \(\Delta \rho_{xy}\) and \(\rho_{xx}^{\text{min}}\) in their GaAs samples. Finally, Weiss et al \[9\] have fitted the temperature dependent prefactor in the activation of \(\rho_{xx}^{\text{min}}\) by a form \(\rho_{xx}^{\text{min}} \sim \frac{1}{T} \exp[-\omega_c/2T]\) in the leading order, where \(\omega_c\) is the cyclotron frequency.

We remark here the fact that the value of \(S\), in general, depends not only on \(T\) but also on its prior history, hinders any explicit comparison between theory and experiment. Nevertheless, we show that the present model yields a linear relationship between \(\Delta \rho_{xy}\) and \(\rho_{xx}^{\text{min}}\) and can accommodate both positive and negative values of \(S\). The linearity is in a smaller range of temperatures for the case of positive values of \(S\). A further estimation of effective mass \(m^*\) which follows thereof is also not unrealistic, and is of the same order of magnitude as the experimental number.

The experimental results are quite ambiguous regarding the prefactor of \(\sigma_{xx}\). Clark et al \[10\] have measured the prefactor of \(\sigma_{xx}\) for fractional quantum Hall states and find a universal value \(e^2/2\pi\), independent even of the filling factors. In contrast, in a recent experiment by Katayama et al \[11\], the prefactor is found to be proportional to \(1/T\). On the theoretical side, Fogler and Shklovskii \[20\] have found that for the case of long range random potential, the prefactor is indeed universal and is given by \(e^2/2\pi\), but only above a certain critical temperature, and that it decays according to a power law below the critical
temperature. On the other hand, Polyakov and Shklovskii [21] obtain the prefactor to be $2e^2/2\pi$. Here we study the case of short range potential and we find the prefactor to be temperature dependent, in agreement with that of Katayama et al [11].

To be sure, there is more experimental information available largely related to the aspect (i) mentioned above and not studied in the present paper. For instance, Wei et al [12] have observed a similarity between $\rho_{xx}$ and $\frac{d\rho_{xy}}{dB}$ with the only difference that while the maximum value of $\rho_{xx}$ decreases, the maximum value of $\frac{d\rho_{xy}}{dB}$ increases with decreasing temperature over the range of temperatures from 0.1 K to 4.2 K. They have also found the power law behaviour $(\frac{d\rho_{xy}}{dB})_{\text{max}} \propto T^{-k}$ and the width of the $\rho_{xx}$ peak $\Delta B \propto T^k$ with $k = 0.42 \pm 0.04$. Further measurements by Wei et al [13] show that the extrema of $\frac{d^2\rho_{xy}}{dB^2}$ and $\frac{d^3\rho_{xy}}{dB^3}$ diverge like $T^{-2k}$ and $T^{-3k}$ respectively. Huckestein et al [14] also have measured the temperature dependence of the plateau and obtain a value $k = 0.42$ in agreement with Wei et al [12]. The above observations are for fully polarized quantum Hall states. When the Landau levels are spin degenerate, Wei et al [13] and Hwang et al [15] have reported that $(\frac{d\rho_{xy}}{dB})_{\text{max}}$ and $(\Delta B)^{-1}$ diverge like $T^{-k/2}$. However, Wakabayashi [16] and Koch et al [17] find no evidence for the universality of the exponent $k$. All these aspects are discussed in a recent review by Huckestein [22]. These mutual conflicting experimental results lie outside the scope of the present paper.

On the theoretical side, Ando et al [23] computed $\sigma_{xx}$ and $\sigma_{xy}$ at $T = 0$ using a simple Lorentzian density of states (DOS) approximated from the self consistent Born approximation (SCBA) density of states; they have not considered the frequency dependent imaginary part of the SCBA self energy of the single particle Green function. In this paper, we make the full use of frequency dependent self energy.

Finally, there is yet another novel aspect that emerges from the present study, viz., the temperature evolution of $\sigma_{xy}$, even for a pure system! Recall the classical argument [3] that the translationally invariant system does not lead to any temperature dependence on $\sigma_{xy}$. Therefore the Maxwell gauge interactions that are at play here belie such a naive expectation.
The plan of the paper is as follows. In the next section we have discussed the formalism. In section III, single particle Green’s function is determined within the SCBA. In section IV, we have evaluated response function and subsequently the off-diagonal and diagonal conductivities. Section V is devoted for determining the resistivities and comparison with experiments. We conclude the paper in section VI. Finally, we have computed the compressibility for the integer quantum Hall states in the appendix B.

II. FORMALISM

Consider a system of (weakly) interacting electrons in two space dimensions in the presence of a uniform external magnetic field of strength $B$, confined to the direction perpendicular to the plane. The electrons also experience a short ranged impurity potential $U(X)$. The strength of the magnetic field is fine tuned such that $N$ Landau levels (LL) are exactly filled. In the presence of sufficiently high magnetic fields (as is relevant to our case), the spins of the fermions would be ‘frozen’ in the direction of magnetic field. Therefore, one may treat the fermions as spinless. The study of such a spinless system can be accomplished with the Lagrangian density [24],

$$\mathcal{L} = \psi^* iD_0 \psi - \frac{1}{2m^*} |D_k \psi|^2 + \psi^* \mu \psi - \psi^* U \psi - eA_0^{\text{in}} \rho + \frac{1}{2} \int d^3x' A_0^{\text{in}}(x) V^{-1}(x - x') A_0^{\text{in}}(x') .$$  \hspace{1cm} (2.1)

Here $D_\nu = \partial_\nu - ie(A_\nu + A_0^{\text{in}} \delta_\nu,0)$ (where $A_\nu$ is the external Maxwell gauge field and $A_0^{\text{in}}$ is identified as internal scalar potential), $\mu$ is the chemical potential, and $m^*$ and $\rho$ are the effective mass and the mean density of electrons respectively. The fifth term in Eq. (2.1) describes the charge neutrality of the system. Finally, $V^{-1}(x - x')$ represents the inverse of the instantaneous charge interaction potential (in the operator sense). The above Lagrangian density is equivalent to the usual interaction term with quartic form of fermi fields, which can be obtained by an integration of $A_0^{\text{in}}$ field in Eq. (2.1). This form of the Lagrangian is obtained by Hubbard-Stratonovich transformation using an auxiliary field $A_0^{\text{in}}$. Note also
that the electrons interact with each other via $1/r$ or some other short range potential, i.e., the internal dynamics is governed by the $(3+1)$-dimensional Maxwell Lagrangian as is appropriate for the medium.

We therefore construct the partition function ($\beta = 1/T$ being the inverse temperature) of the system,

$$\mathcal{Z} = \int [dA_\tau^\text{in}] [d\psi] [d\psi^*] \exp \left[-\int_0^{\beta} d\tau \int d^2X L^{(E)}\right], \quad (2.2)$$

which on integration over the fermionic fields, (by fixing of the saddle point at the uniform background magnetic field $B$), factors into the form $\mathcal{Z} = \mathcal{Z}_B \mathcal{Z}_I$. Here $L^{(E)}$ is the Euclidean version of $L$ in Eq. (2.1). For the transformation into the Euclidean space, we make a substitution $t \to -i\tau$ and consider the real parameter $\tau$ as a coordinate on a circle of circumference $\beta$. Fermionic fields are antiperiodic on this circle while the bosonic fields are periodic. The time component of the vector field $A_\mu$ is redefined as $A_0 \to iA_\tau$. The background part of the partition function is given by

$$\mathcal{Z}_B = Tr e^{-\beta(H-\mu)}, \quad (2.3)$$

which is obtained from the fermion determinant, found by the integration of fermionic fields. The finite temperature background properties of the system can be studied using $\mathcal{Z}_B$. Here

$$H = -\frac{1}{2m^*}D^2_k + U(X) \equiv H_0 + U(X) \quad (2.4)$$

is the single particle Hamiltonian. $\mathcal{Z}_I$ is the partition function corresponding to the external probe, which will be determined later below.

III. SINGLE PARTICLE GREEN’S FUNCTION

A. Disorder free Green’s function

In the absence of disorder, $U(X) = 0$, the spectrum of $H$ for the system is a set of Landau levels (LL) with energy eigen value for the $n$-th LL,
\[ \epsilon_n = (n + \frac{1}{2})\omega_c ; \ n = 0, 1, 2, \cdots , \]  
\[ \]  
where \( \omega_c = eB/m^* \) is the cyclotron frequency. In the Landau gauge  
\[ \bar{A} = (-BX_2, 0) , \]  
the eigen function corresponding to the eigen value \( \epsilon_n \) is  
\[ \psi_{nk}(X) = \frac{1}{\sqrt{l}} e^{ikX_1} v_n \left( \frac{X_2}{l} + kl \right) , \]  
where \( v_n(x) \) is the appropriate harmonic oscillator wave function. Here \( l = (eB)^{-1/2} \) is the magnetic length of the system and is also the classical cyclotron radius in the lowest LL \( (n = 0) \). Each level is infinitely degenerate with a degeneracy \( \rho_l = 1/2\pi l^2 \) per unit area.

The single particle Green function \( G_0(x, x') \) for the pure system (disorderless) can be obtained by solving  
\[ (\delta_\tau + H_0 - \mu)G_0(x, x') = \delta^{(3)}(x - x') \]  
subject to the requirement of antiperiodicity under the translation \( \tau \rightarrow \tau + \beta \). \( G_0(x, x') \) can be expanded in the normal modes of discrete frequencies \( \xi_s = (2s + 1)\pi/\beta; \ s \in \mathbb{Z} \), as  
\[ G_0(x, x') = -\frac{1}{\beta} \sum_s e^{-i\xi_s(\tau - \tau')} \langle X|G_0(i\xi_s)|X'\rangle , \]  
where the operator \( G_0(i\xi_s) \) is given by  
\[ G_0(i\xi_s) = \frac{1}{i\xi_s + \mu - H_0} . \]  
with eigen value \( G_0^n(i\xi_s) = [i\xi_s + \mu - \epsilon_n]^{-1} \). By analytical continuation \( i\xi_s \rightarrow \epsilon \pm i\eta \), we obtain this eigen value in the real space as  
\[ G_0^n(\epsilon) = \frac{1}{\epsilon + \mu - \epsilon_n \pm i\eta} . \]  
Here +(-) refers to retarded(advanced) Green’s function.

The density of the particles is obtained in terms of the Green’s function as
\[ \rho(X) = -G_0(x, x')|_{x'=X, \tau'=	au+0}. \]  

(3.8)

The density is independent of the spatial coordinate, i.e., the density is uniform, and is given by (see Ref. [25] for details)

\[ p \equiv \frac{\rho}{\rho_l} = \sum_n f_n, \]  

(3.9)

where the Fermi function corresponding to \( n \)-th LL, \( f_n = [1 + \exp(\beta(\epsilon_n - \mu))]^{-1} \) and \( p \) is the number of fully filled LL at \( T = 0 \).

At low temperatures characterized by \( \beta\omega_c \gg 1 \), we shall determine the chemical potential for fixed density. To this end, we write [25]

\[ e^{\beta(\epsilon_n - \mu)} = w^{\mu/\omega_c - 1/2 - n}, \]  

(3.10)

where \( w = \exp[-\beta\omega_c] \) which is the perturbative expansion parameter. We expand the right-hand side of Eq. (3.9) to obtain

\[ \sum_n (1 + w^{\mu/\omega_c - 1/2 - n})^{-1} = \left[ \frac{\mu}{\omega_c} + \frac{1}{2} \right] + \sum_{r \geq 1} (-1)^r \frac{w^{r\delta} - w^{r(1-\delta)} - w^{r(\mu/\omega_c+1/2)}}{1 - w^r}, \]  

(3.11)

where \( [\mu/\omega_c + 1/2] \) denotes the (positive) integer part:

\[ \frac{\mu}{\omega_c} + \frac{1}{2} = \left[ \frac{\mu}{\omega_c} + \frac{1}{2} \right] + \delta, \quad 0 < \delta < 1. \]  

(3.12)

Since \( w \) is very small, Eq. (3.9) takes the form

\[ p = \left[ \frac{\mu}{\omega_c} + \frac{1}{2} \right] - w^\delta + w^{1-\delta} + w^{[\mu/\omega_c+1/2]+\delta} + \ldots. \]  

(3.13)

The solution of (3.13) in the leading order is obtained as

\[ \left[ \frac{\mu}{\omega_c} + \frac{1}{2} \right] = p, \quad \delta = \frac{1}{2} + \frac{w^p}{2 \ln w}. \]  

(3.14)

Therefore, the chemical potential is given by

\[ \mu = \frac{2\pi\rho}{m^*} - \frac{1}{2\beta} e^{-p\beta\omega_c} + \ldots. \]  

(3.15)

On substitution of the value of \( \mu \) on Eq. (3.10) we obtain

\[ e^{\beta(\epsilon_n - \mu)} = \exp [\beta\omega_c(n - p + 1/2)] \left( 1 + \frac{1}{2} e^{-\beta\omega_c(p+1/2)} + \ldots \right). \]  

(3.16)
B. Impurity averaged Green’s function

We assume a random distribution of white noise disorder potential \( U(X) \) all over the space with probability distribution

\[
P[U] = \mathcal{N} \exp \left[ -\frac{1}{2\lambda^2} \int d^2X |U(X)|^2 \right], \tag{3.17}
\]

where \( \mathcal{N} \) is the renormalization constant. The average potential

\[
\overline{U(X)} = 0. \tag{3.18}
\]

Hereafter overbar denotes the average with respect to the distribution \( P[U] \) (Eq. (3.17)) of the disorder potential. The correlation between the potentials at two points is given by

\[
\overline{U(X)U(X')} = \lambda^2 \delta(X - X'). \tag{3.19}
\]

The scattering potentials are short ranged.

The single particle Green’s function for the disordered system in terms of the free Green’s function (3.5) is given by

\[
G(x, x') \equiv \langle x | G | x' \rangle = \langle x | G_0 + G_0UG | x' \rangle
\]

\[
= \langle x | G_0 | x' \rangle + \langle x | G_0UG_0 | x' \rangle + \langle x | G_0UG_0UG_0 | x' \rangle + \cdots. \tag{3.20}
\]

When we average over the distribution in disorder, the terms with odd number of \( U \) operators vanish. Further, due to uncorrelated potentials (3.19), each scattering centre is responsible for an even number of scatterings.

We consider here a weak disorder, and assume that \( U/\omega_c \ll 1 \) \([5]\). Further, the parameter \( \lambda \ll 1 \). We may thus neglect the contribution of the off-diagonal matrix elements \( \langle m | U | n \rangle \) and consider only the diagonal terms \( \langle n | U | n \rangle \) in the determination of \( G \). As a consequence, the full Green’s function given in Eqn(3.20) will also be diagonal in the Landau level basis. If \( G_0^n(\epsilon) \) be the free Green’s function for \( n \)-th LL, then the corresponding impurity averaged Green’s function is given by
\[
\overline{G^n}(\epsilon) = G^n_0(\epsilon) + G^n_0(\epsilon) \Sigma_n(\epsilon) \overline{G^n}(\epsilon).
\]

(3.21)

In other words,
\[
\overline{G^n}(\epsilon) = \frac{1}{\epsilon + \mu - \epsilon_n - \Sigma_n(\epsilon)}.
\]

(3.22)

\(\overline{G^n}(\epsilon)\) has the diagramatic representation as in Fig. 1(a). Here \(\Sigma_n\) is the corresponding self energy. For the evaluation of \(\Sigma_n(\epsilon)\), we make the self consistent Born approximation (SCBA).

In SCBA, one neglects multiple scattering at any point and the cross scatterings which are shown in Fig. 1(b). The diagrams which contribute to SCBA are shown in Fig. 1(c).

We find that \(\Sigma_n\) (see Fig. 1(a)) is
\[
\Sigma_n(\epsilon) = \lambda^2 \rho_l \sum_{m \geq -n} \frac{1}{\epsilon + \mu - \epsilon_{n+m} - \Sigma_{n+m}(\epsilon)}.
\]

(3.23)

The solution of Eq. (3.23) can be obtained self consistently [26]. This is given by
\[
\Sigma_n(\epsilon) = -\frac{1}{2}(\epsilon + \mu - \epsilon_n) + \frac{1}{2} \sqrt{(\epsilon + \mu - \epsilon_n)^2 - \Gamma^2},
\]

(3.24)

where \(\Gamma^2 = 4\lambda^2 \rho_l\).

We now make a few remarks [27] on SCBA and its limitations. In SCBA, the Landau levels are broadened but the broadening is very small compared to Landau level spacings, i.e., \(\Gamma/\omega_c \ll 1\). This is, in fact, an outcome of the assumption that the magnetic field is very high, and that one has a weak disorder potential caused by a small impurity concentration.

The contributions of multiple and cross scatterings to the electron self energy are negligible as they are of higher order in impurity concentration [28]. The SCBA does not lead to any localized state. As a consequence, it cannot describe localization-delocalization transitions between two successive IQHE states. Considering all the scattering diagrams as well as the impurity induced mixing of Landau levels, a more rigourous analysis would presumably give rise to localized states and explain the critical behaviour of localization-delocalization transitions. The scaling law, which was derived field theoretically earlier by Pruisken [29], in critical transitions between the plateaus has, in fact, been obtained by Huo, Hetzel, and Bhatt [30] in an exact numerical simulation. However, we remark that SCBA is good
enough for obtaining the transport properties at integer filling factors, in particular. Further, as Pruisken [26] has shown, it can be improved by a renormalization group analysis to understand the formation of plateaus. This point is elaborated upon somewhat more in section IV-A.

For a disorderless system, $\Gamma = 0$ and hence the self energy $\Sigma_n$ also vanishes. In the range of energy $\epsilon$ for which $|\epsilon + \mu - \epsilon_n| > \Gamma$, $\Sigma_n$ is real; it merely changes the energy of the particles. On the other hand, for the range $|\epsilon + \mu - \epsilon_n| < \Gamma$, $\Sigma_n$ has both real and imaginary parts, and is given by

$$
\Sigma_n^\pm (\epsilon) = -\frac{1}{2}(\epsilon + \mu - \epsilon_n) \pm \frac{i}{2}\sqrt{\Gamma^2 - (\epsilon + \mu - \epsilon_n)^2}.
$$

Therefore in this range, the impurity averaged retarded and advanced Green’s functions are given by

$$
G_{r,a}^n (\epsilon) = \frac{2}{\epsilon + \mu - \epsilon_n \pm i\sqrt{\Gamma^2 - (\epsilon + \mu - \epsilon_n)^2}}.
$$

The DOS for the $n$-th Landau band,

$$
D_n(\epsilon) \equiv -\frac{\rho_l}{\pi} \text{Im} G_{r}^n(\epsilon)
= \frac{1}{2\pi l^2} \frac{2}{\pi \Gamma} \left[ 1 - \left( \frac{\epsilon + \mu - \epsilon_n}{\Gamma} \right)^2 \right]^{1/2}.
$$

This is the well known SCBA DOS which is semi circular in nature. This will be used to determine the linear response of the system.

Alternatively, one may also consider the Gaussian DOS which is given by

$$
D_n(\epsilon) = \frac{1}{2\pi l^2} \sqrt{\frac{2}{\pi \Gamma}} e^{-2(\epsilon+\mu-\epsilon_n)^2/\Gamma^2}.
$$

A brief comparison between the above two DOS will be made later.

**IV. RESPONSE FUNCTION**

The partition function corresponding to the external electromagnetic probe $A_{\mu}$ is
\[
Z_I[A_\mu] = \int [dA^\text{in}_\tau] \exp[-S^E],
\]

where \(S^E\) is identified as the Euclidean one-loop effective action of the gauge fields. It is given by

\[
\begin{align*}
S^E &= \frac{1}{2} \int d^3x \int d^3x' \left( A_\mu(x) + A^\text{in}_\mu(x') \right) \Pi^E_{\mu\nu}(x, x') \left( A_\nu(x') + A^\text{in}_\nu(x') \right) \\
&\quad - \frac{1}{2} \int d^3x \int d^3x' A^\text{in}_\tau(x) V^{-1}(x - x') A^\text{in}_\tau(x').
\end{align*}
\]

Here the integration over the imaginary time \(\tau\) runs from 0 to \(\beta\). The effective action is obtained by expanding the fermionic determinant up to quadratic terms in powers of fields \(A_\mu\) and \(A^\text{in}_\tau\). The polarization tensor \(\Pi^E_{\mu\nu}(x, x')\) which is evaluated at the saddle point mentioned above is impurity averaged. Since \(\Pi^E_{\mu\nu}\) is evaluated in the path integral formalism, \(\tau\)-ordering is implied for the correlations, i.e.,

\[
\Pi^E_{\mu\nu}(x, x') = -\left\langle T_{\tau} j_\mu(x) j_\nu(x') \right\rangle - \left\langle \frac{\delta j_\mu(x)}{\delta A_\nu(x')} \right\rangle.
\]

The overbar implies the average over the distribution of random disorder in Eq. (3.17). In terms of the thermal single particle Green’s function, the components of \(\Pi^E_{\mu\nu}\) are given by

\[
\begin{align*}
\Pi^E_{\tau\tau}(x, x') &= -e^2 \overline{G(x, x') G(x', x)} , \\
\Pi^E_{k\tau}(x, x') &= \frac{e^2}{2m^*} \left[ D_k G(x, x') D_k^* G(x', x) - G(x, x') D_k^* D_k G(x', x) \right] , \\
\Pi^E_{kl}(x, x') &= -\frac{e^2}{4m^*} \left[ D_k G(x, x') D_l^* G(x', x) - (D_k D_l^* G(x, x')) G(x', x) \right] \\
&\quad + D_k^* G(x, x') D_l^* G(x', x) - G(x, x') D_l^* D_k^* G(x', x) \\
&\quad + \frac{e^2}{m^*} \delta_{kl} \delta(x - x') \overline{G(x, x')} \bigg|_{x' = x, \tau' = \tau + 0^+}.
\end{align*}
\]

We remark here that once we average over the distribution of the impurity potential, the translational invariance gets restored. Therefore, the polarization tensor can be represented by Fourier expansion

\[
\Pi^E_{\mu\nu}(x, x') = \frac{1}{\beta} \sum_j \int \frac{d^2q}{(2\pi)^2} e^{-i\mathbf{q} \cdot (x - x')} e^{-i\omega_j (\tau - \tau')} \Pi^E_{\mu\nu}(\omega_j, \mathbf{q}),
\]

where the Matsubara frequencies are given by
\[ \omega_j = \frac{2\pi}{\beta} j ; \ j \in \mathbb{Z} . \]  

(4.6)

Again, due to rotational symmetry and gauge invariance, there are three independent form factors. One thus obtains the polarization tensor in the form (in momentum space)

\[
\Pi^E_{\mu\nu}(\omega_j, \mathbf{q}) = \Pi_0(\omega_j, \mathbf{q}) (q^2 \delta_{\mu\nu} - q_\mu q_\nu) + (\Pi_2 - \Pi_0)(\mathbf{q}^2 \delta_{ij} - q_i q_j)(\omega_j, \mathbf{q})
\]

\[
\times \delta_{\mu i} \delta_{\nu j} + \Pi_1(\omega_j, \mathbf{q}) \varepsilon_{\mu\nu\lambda} q^\lambda ,
\]

(4.7)

where \( q^2 = \omega_j^2 + \mathbf{q}^2 \). Here \( \Pi_0, \Pi_1 \) and \( \Pi_2 \) are the form factors. Note that \( \Pi_1 \) is a \( \mathcal{P}, \mathcal{T} \) violating form factor. The one loop effective action then acquires the form

\[
S_{\text{effective}}^{E} = \frac{1}{2\beta} \sum_j \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \left[ (A_\mu(q) + A^{in}_\mu(q))\delta_{\mu\nu} \right] \Pi^E_{\mu\nu}(\omega_j, \mathbf{q}) \left( A_\nu(-q) + A^{in}_\nu(-q)\delta_{\nu\tau} \right)
\]

\[
- A^{in}_\nu(q)V^{-1}(\mathbf{q})A^{in}_\tau \right] .
\]

(4.8)

A. Off-diagonal Conductivity

A straightforward linear response analysis from Eqs. (4.7) and (4.8), yields the expression for off-diagonal conductivity to be

\[
\sigma_{xy} = \lim_{\omega \to 0} \text{Re} \Pi_1(\omega, 0) ,
\]

(4.9)

subject to the condition \( \lim_{\mathbf{q} \to 0} V(\mathbf{q})\mathbf{q}^2 = 0 \). \( \Pi_1(\omega, 0) \) is obtained by the analytical continuation: \( i\omega_j \to \omega + i\delta \).

The form factor \( \Pi_1 \) can be determined from the evaluation of the component, say \( \Pi^{E}_{\tau\tau} \) of the polarization tensor (4.7). The impurity average over two Green’s functions in Eq. (4.4a) is not equal to the product of two averaged Green’s functions (Fig. 2(a)), in general. It has an additional contribution coming from vertex corrections (see Fig. 2(b)). We shall determine the contributions from Figs. 2(a) and 2(b) separately.

Considering figure 2(a), we get

\[
\Pi^{(0)}_{\tau}(\omega_j, 0) = \frac{e^2}{2m^*} \frac{1}{2\pi l^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} I_{nm}(\omega_j) [(n + 1)\delta_{m,n+1} + n\delta_{m,n-1} - (2n + 1)\delta_{mn}] .
\]

(4.10)
where

\[ I_{nm}(\omega_j) = -\frac{1}{\beta} \sum_{s=-\infty}^{\infty} G^n(i\xi_s) G^m(i\xi_s - i\omega_j). \tag{4.11} \]

The evaluation of \( I_{nm} \) is done in the Appendix A. Performing analytical continuation: \( i\omega_j \rightarrow \omega + i\delta \), we obtain the real part of \( I_{nm} \) (see Eq. (A9)),

\[ \text{Re} I_{nm}(\omega) = 2 \int_{\epsilon_n-\mu-\Gamma}^{\epsilon_n+\mu+\Gamma} \frac{d\epsilon}{2\pi} n_F(\epsilon) \text{Im} G^n(\epsilon) \text{Re} G^m(\epsilon - \omega) + 2 \int_{\epsilon_m-\mu-\Gamma}^{\epsilon_m+\mu+\Gamma} \frac{d\epsilon}{2\pi} n_F(\epsilon) \text{Im} G^m(\epsilon) \text{Re} G^n(\epsilon + \omega). \tag{4.12} \]

Here \( n_f(\epsilon) \) represents the Fermi function,

\[ n_F(\epsilon) = \frac{1}{1 + e^{\beta\epsilon}}. \tag{4.13} \]

By Taylor expansion of Eq. (4.12) in powers of \( \Gamma/\omega_c \), we obtain, with the use of Eqs. (4.9) – (4.13), the off-diagonal conductivity (without vertex correction)

\[ \sigma_{xy}^{(0)} = -\frac{e^2}{2\pi} \left[ \sum_{n=0}^{\infty} f_n - \frac{1}{2} \beta\omega_c \sum_{n=0}^{\infty} (2n + 1) f_n (1 - f_n) \right] + \mathcal{O} \left( \frac{\Gamma}{\omega_c} \right)^2. \tag{4.14} \]

**Vertex correction:** We next consider the contribution to \( \sigma_{xy} \) due to vertex corrections shown in Fig. 2(b). For the lowest order vertex correction, the contribution \( \sigma_{xy}^{(1)} \) is suppressed by a factor \( (\Gamma/\omega_c)^2 \) in the limit \( \Gamma/\omega_c \ll 1 \), since two more scatterings take place in the virtual process. The vertex corrections due to more and more number of scatterings lead to the contributions which are further suppressed by a factor \( \Gamma/\omega_c \) for each scattering. Therefore, in the limit \( \Gamma/\omega_c \ll 1 \), the total off-diagonal conductivity is essentially given by

\[ \sigma_{xy} \simeq \sigma_{xy}^{(0)} = -\frac{e^2}{2\pi} \left[ \sum_{n=0}^{\infty} f_n - \frac{1}{2} \beta\omega_c \sum_{n=0}^{\infty} (2n + 1) f_n (1 - f_n) \right] + \mathcal{O} \left( \frac{\Gamma}{\omega_c} \right)^2. \tag{4.15} \]

In agreement with Ando et al [23], the impurity contribution to \( \sigma_{xy} \) is of the order of \( (\Gamma/\omega_c)^2 \) which is small in the strong field and small broadening approximation.

Starting with the result of Ando et al [23] which was obtained at \( T = 0 \) using a simple Lorentzian DOS, Pruisken [26] has shown that after successive renormalizations, there is a
renormalization group flow towards $(\sigma_{xx}, \sigma_{xy}) = (0, \text{integer}) \frac{e^2}{2\pi}$, corresponding to the widening of the plateaus. He finds another flow towards an unstable fixed point which corresponds to $\sigma_{xy} = (p + 1/2) \frac{e^2}{2\pi}$ where the transition between the plateaus occur. The group flows have been observed experimentally [31]. Thus at $T = 0$, the stable group flow is towards the point for which filling factor $\nu$ is an integer and $(\sigma_{xx}, \sigma_{xy}) = (0, \nu) \frac{e^2}{2\pi}$ which is the value of $(\sigma_{xx}, \sigma_{xy})$ at $\Gamma = 0$. Recall that we are considering those specific values of $B$ (or $\rho$) for which $\nu = p$ is an integer. We have computed here $\sigma_{xy}$ more rigorously within SCBA and also at $T \neq 0$. It is therefore reasonable to expect the present improved treatment to be capable of explaining the behaviour of plateaus, \textit{a la} the Pruisken analysis. It would indeed be interesting to examine whether the stable renormalization group flow at finite $T$ occurs towards the point $(\sigma_{xx}, \sigma_{xy})$, which we evaluate here, for which $\nu$ is an integer and $\Gamma = 0$. This analysis would explain the behaviour of plateau at finite temperatures in $\sigma_{xy}$ given by Eq. (4.15). We hope to take this up in future.

Interestingly the contribution to $\sigma_{xy}$ in Eq. (4.15) is \textit{entirely} due to the temperature in the leading order, surviving even in the absence of impurities. At this juncture, we recall the lore [32] that a pure system may not be expected to show such a behaviour, in consequence of translation invariance. This classical argument clearly does not hold here; and the violation may be attributed to the fact that in the presence of an external magnetic field, the generators of translation group are anomalous. The co-ordinates of the centre of the classical orbit do not commute — a feature which was labeled by Chen, Wilczek, Witten, and Halperin [33] as \textit{violation of fact}. We note here that the formalism does \textit{not} violate translation invariance. Rather, it is analogous to the well-known field theoretic anomalies. Note that the expression for $\sigma_{xy}$ (4.15) is manifestly consistent with translation invariance. We may point out that such a temperature evolution has been noticed earlier in the finite temperature studies [25,34] of Chern-Simons superconductivity (CSS). In fact, Fradkin [35] has in his study of CSS drawn explicit analogy with the anomalous nature of the generators of the translation group with the anomaly arising in the Schwinger-Anderson model [36,37].
Should this analogy indeed hold perfectly, we may then look upon quantum Hall systems to provide a condensed matter laboratory to probe the new anomaly as much as the process $\Pi^0 \to 2\gamma$ provides, in order to probe the chiral anomaly. We also mention that Bellissard et al [38] also obtain a temperature dependence for a pure system, but they miss the crucial prefactor which is derived here. Finally, it may be noted that this ‘anomalous’ contribution owes its origin to intra LL transitions, which would contribute in the thermodynamic limit only if degeneracy grows with area.

It is interesting that the above mentioned intra Landau level transition which is unique to this case also contributes to the specific heat, as derived by Zawadzki and Lassnig [39]. This has been experimentally verified later by Gornik et al [40].

At low temperatures ($\beta \omega_c \gg 1$), we perturbatively evaluate $\sigma_{xy}$ with the expansion parameter $e^{-\beta \omega_c}$, as we have discussed in the previous section. We thus find

$$\sigma_{xy}(T) = -\frac{e^2 p}{2\pi} [1 - 4y] \, , \quad (4.16)$$

with

$$y = \frac{T_0}{T} \exp \left[ -\frac{T_0}{T} \right] \, , \quad (4.17)$$

where $T_0 = \omega_c/2$. The temperature correction to the $\sigma_{xy}$ is exponentially weak. This leads to some interesting consequences as we shall show in the next section.

**B. Diagonal Conductivity**

Again, a linear response analysis of Eq. (1.8) provides the diagonal conductivity to be

$$\sigma_{xx} = -\lim_{\omega \to 0} \text{Im} \, \Pi_{11}'(\omega, 0) \, . \quad (4.18)$$

Here $r$ represents the retarded part of the correlation function which is obtained from Eq. (1.7) by analytical continuation. Equation (1.18) is derived in the limit $\lim_{q \to 0} V(q)q^2 = 0$. Therefore Eq. (4.18) is true for all those potentials which are short ranged compared to $\ln r$. 

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We evaluate $\Pi_{11}^E(\omega_j, 0)$ using the same procedure employed in determining $\Pi_{1\tau}^F$ discussed above. It may again be shown that vertex corrections are suppressed by an extra factor of $\Gamma / \omega_c$. We therefore only need to evaluate the contribution arising from Fig. 2(a). We thus obtain

$$\Pi_{11}^E(\omega_j, 0) = \frac{e^2}{4 m^* 2\pi l^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} I_{nm}(\omega_j) [(n + 1)\delta_{m,n+1} + n\delta_{m,n-1} - (2n + 1)\delta_{mn}]$$

$$- \frac{e^2}{m^* 2\pi l^2} \sum_{n=0}^{\infty} I_n ,$$

(4.19)

where

$$I_n = \frac{1}{\beta} \sum_{s=-\infty}^{\infty} \frac{G_n(i\xi_s)}{\xi_s} .$$

(4.20)

The term $I_n$ is independent of $\omega_j$; it does not have any imaginary part. Using Eq. (A9), we write

$$\text{Im} I_{nm}(\omega) = -2 \int_{\epsilon_n - \mu + \Gamma}^{\epsilon_n - \mu - \Gamma} \frac{d\epsilon}{2\pi} n_F(\epsilon) \text{Im} G_n^m(\epsilon) \text{Im} G_m^m(\epsilon - \omega) + 2 \int_{\epsilon_m - \mu + \Gamma}^{\epsilon_m - \mu - \Gamma} \frac{d\epsilon}{2\pi} n_F(\epsilon) \text{Im} G_m^m(\epsilon) \text{Im} G_n^m(\epsilon + \omega) .$$

(4.21)

The diagonal conductivity is thus given by

$$\sigma_{xx} = -\lim_{\omega \to 0} \frac{e^2}{4 m^* 2\pi l^4} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \text{Im} I_{nm}(\omega) [(n + 1)\delta_{m,n+1} + n\delta_{m,n-1} - (2n + 1)\delta_{mn}] .$$

(4.22)

In the limit $\Gamma \to 0$ (disorder free), $\sigma_{xx}$ vanishes, as may easily be checked. For $\Gamma$ finite, there are some interesting consequences. Recall that according to SCBA, the imaginary part of $G_n^m(\epsilon)$ exists in the region of $\epsilon$ for which $\Gamma > |\epsilon_n + \epsilon - \mu|$. Therefore the integrals in Eq. (3.21) are non-zero only when $m = n$, i.e., the contribution is entirely due to transitions within the band of an LL. We obtain

$$\sigma_{xx} = \frac{2e^2}{3\pi^2} \left( \frac{\omega_c}{\Gamma} \right) \beta \omega_c \sum_{n=0}^{\infty} (2n + 1) f_n (1 - f_n) + \mathcal{O} \left( \frac{\Gamma}{\omega_c} \right) .$$

(4.23)

Notice that at finite temperatures, there is a singular contribution in $\sigma_{xx}$ as $\Gamma \to 0$. However, as we have seen above, $\sigma_{xx}$ vanishes for $\Gamma = 0$ at all temperatures. Further at $T = 0$, there is no possibility of intra band transition within an LL and so $\sigma_{xx}$ vanishes again. Therefore
the expression (4.23) has a pertinent role only at finite temperatures, and in the presence of disorder. In short, $\sigma_{xx}(\Gamma, T = 0) = 0; \sigma_{xx}(\Gamma = 0, T) = 0$. In the high field and low broadening limit, $\Gamma/\omega_c \ll 1$, and so we neglect the higher order terms in $\Gamma/\omega_c$. Note that $\sigma_{xx}(T)$ is evaluated for those specific value of $B$ (or $\rho$) for which $\nu = p$ is an integer.

A low temperature expansion of Eq. (4.23) yields

$$\sigma_{xx}(T) = \frac{16e^2 p}{3\pi^2} \left( \frac{\omega_c}{\Gamma} \right) y \equiv \frac{16e^2 p}{3\pi^2} \left( \frac{\omega_c}{\Gamma} \right) \frac{T_0}{T} \exp \left[ -\frac{T_0}{T} \right]. \quad (4.24)$$

It may be noted that there is now a competition between two energy scales — $T$ and $\Gamma$ in the theory. For very low temperatures, $y$ is exponentially small (see Eq. (4.17)) and the value of $\sigma_{xx}$ need not be large although $\omega_c/\Gamma \gg 1$. In fact, at $T = 0$, $y = 0$ leading to a vanishing of $\sigma_{xx}$. Further, we see that $\sigma_{xx}(T)$ is thermally activated with a temperature dependent prefactor, in agreement with the experiment of Katayama et al [11]. This prefactor is clearly not universal any more, since it depends on the integer filling $p$ and the broadening of an LL, as given by $\Gamma$. The mismatch between the results of Ref. [21] which predicts universality and that of ours may be due to consideration of different types of the disorder potential; whereas they have taken a long range scatterer, we have considered $\delta$-correlated short range disorder potential. Interestingly, Fogler et al [20] report that for long range scatterers, the prefactor is temperature dependent below a characteristic temperature. In concluding this section, we note that our results obey the phenomenological relation

$$\Delta \sigma_{xy} \equiv \sigma_{xy}(T) - \sigma_{xy}(0) \sim -\left( \frac{\Gamma}{\omega_c} \right) \sigma_{xx}, \quad (4.25)$$

which was proposed by Ando et al [23] based on general arguments.

1. $\sigma_{xx}$ for Gaussian density of states

For the Gaussian density of states (3.28), the limits of integration in Eq. (4.21) runs from $-\infty$ to $\infty$. There is a possibility of inter Landau band transition at finite temperatures apart from the intra Landau band transition, since the Gaussian density of states has a
long tail, unlike the SCBA density of states where there is a sharp cut-off. However, the 
contribution to $\sigma_{xx}$ due to inter Landau band transition is exponentially weak, and behaves 
like $\exp[-\omega_c^2/(2\Gamma^2)]$. Note that $\Gamma/\omega_c \ll 1$. The major contribution therefore again comes 
from the intra Landau band transition. The vertex corrections are suppressed by the higher 
opters of $\Gamma/\omega_c$. We thus obtain the diagonal conductivity for the Gaussian density of states

$$\sigma_{xx} \approx \frac{e^2}{8\sqrt{2\pi}} \left( \frac{\omega_c}{\Gamma} \right) \beta \omega_c \sum_{n=0}^{\infty} (2n + 1) f_n (1 - f_n) ,$$

which has the low temperature form

$$\sigma_{xx}(T) = \frac{e^2 p}{\sqrt{2\pi}} \left( \frac{\omega_c}{\Gamma} \right) y ,$$

which qualitatively agrees with $\sigma_{xx}(T)$ obtained from the SCBA density of states as in 
Eq. (4.24). This exercise serves to demonstrate that the crucial features of $\sigma_{xx}$ are of not 
artifacts of the SCBA.

V. RESISTIVITIES AND COMPARISON WITH EXPERIMENT

We now take up the discussion of resistivities of quantum Hall states which is of primary 
interest to us. Cage et al [7] and Yoshihiro et al [6] have found that $\rho_{xy}(T)$ decreases 
with increase in temperature and the deviation from its zero temperature value, $\Delta \rho_{xy}(T) \equiv \rho_{xy}(T) - \rho_{xy}(0)$ obeys a linear relationship with $\rho_{xx}^{\text{min}}$ as

$$\Delta \rho_{xy} = -S \rho_{xx}^{\text{min}} .$$

Cage et al [7] have further reported that the linearity remains for the temperature range 
1.2–4.2K for which the value of $\rho_{xx}^{\text{min}}$ changes up to four decades. The value of $S$ depends 
on the device and the cooling process and hence is not reproducible. However, the linear 
relationship (5.1) is universal. The sign of $S$ is not always positive as Wel et al [8] have 
observed that $\rho_{xy}(T)$ increases with temperature. We shall see that both the signs of $S$ 
along with the linearity (5.1) can be obtained in different physical regimes of temperature 
and broadening, (although in a narrow range of temperatures for positive $S$). Presumably,
these physical regimes correspond to the above mentioned experiments. Both the regimes
belong to low temperatures and low broadening, as we show below.

In the previous section, we have determined $\sigma_{xy}(T)$ and $\sigma_{xx}(T)$. Now by symmetry,
$\sigma_{yx}(T) = -\sigma_{xy}(T)$ and $\sigma_{yy}(T) = \sigma_{xx}(T)$. We then invert the conductivity matrix to obtain
the resistivity matrix with the components

$$
\rho_{xy}(T) = -\frac{\sigma_{xy}(T)}{\sigma_{xx}^2(T) + \sigma_{xy}^2(T)} = -\rho_{yx}(T) ;
\rho_{xx}(T) = \frac{\sigma_{xx}(T)}{\sigma_{xx}^2(T) + \sigma_{xy}^2(T)} = \rho_{yy}(T).
$$

(5.2)

Using the conductivities obtained from SCBA DOS (Eqs. 4.16 and 4.24), we get

$$
\rho_{xy}(T) = \frac{2\pi}{e^2p} \left(\frac{32}{3\pi} \right)^2 \frac{1 - 4y}{(32/3\pi)^2(\omega_c/\Gamma)^2 y^2 + (1 - 4y)^2}
$$

(5.3a)

$$
\rho_{xx}(T) = \left(\frac{2\pi}{e^2p} \right) \left(\frac{32}{3\pi} \right) \frac{(32/3\pi)(\omega_c/\Gamma)y}{(32/3\pi)^2(\omega_c/\Gamma)^2 y^2 + (1 - 4y)^2}.
$$

(5.3b)

We shall study these resistivities for two different situations below.

(i) If $\Gamma/\omega_c$ is so small that $(\Gamma/\omega_c)^2 = \alpha y$ with $\alpha \sim 1$, then $(\omega_c/\Gamma)^2 y^2 \sim y$. We then obtain

$$
\rho_{xy}(T) = \rho_{xy}(0) \left[1 - \left\{\alpha \left(\frac{32}{3\pi}\right)^2 - 4\right\} y\right]
$$

(5.4a)

$$
\rho_{xx}(T) = \frac{2\pi}{e^2p} \left(\frac{32}{3\pi} \right) \left[\frac{\omega_c}{\Gamma}\right] y,
$$

(5.4b)

where $\rho_{xy}(0) = 2\pi/e^2p$ is the Hall resistivity at $T = 0$.

First of all, $\rho_{xx}(T)$ which is thermally activated with the activation gap $T_0 = \omega_c/2$ with
a prefactor $\sim 1/T$, is in agreement with the best fit data of Weiss et al [9], who find that
in the lowest order of $y$, $\rho_{xx}^{\text{min}} \sim (1/T) \exp[-\omega_c/2T]$. Note that since we are at the centre
of the plateau ($\nu$ is an integer), Eqs. (5.3b) and (5.4b) do indeed correspond to the $\rho_{xx}^{\text{min}}$
which is measured. Secondly, the Hall resistivity decreases with increase in temperature,
with $\Delta \rho_{xy}(T)$ also being thermally activated, again in agreement with experiments of Cage
et al [7] and Yoshihiro et al [3]. Finally, the linear relation (5.1) for $S > 0$ holds in the rather
narrow range of temperatures over which $(\Gamma/\omega_c)^2 \sim y$. In such a case $S$ (which is effectively
temperature independent) is given by
\[ S = \frac{3\pi}{32} \left[ \alpha \left( \frac{32}{3\pi} \right)^2 - 4 \right] \left( \frac{\Gamma}{\omega_c} \right). \] (5.5)

The constancy of \( S \) remains over a decade in \( \rho_{xx} \), which is smaller than that of cage et al \[7\] who find \( S \) to be constant over four decades in \( \rho_{xx} \).

(ii) Consider the case where \( \Gamma/\omega_c \) is small \((\Gamma/\omega_c \ll 1)\) but \((\Gamma/\omega_c)^2 \gg y\). Then we obtain from Eq. (5.3),

\[
\begin{align*}
\rho_{xy}(T) & = \rho_{xy}(0) \left[ 1 + 4y \right] \quad \text{(5.6a)} \\
\rho_{xx}(T) & = \left( \frac{2\pi}{e^2 \rho} \right) \frac{32}{3\pi} \left( \frac{\omega_c}{\Gamma} \right) y. \quad \text{(5.6b)}
\end{align*}
\]

Here \( \rho_{xy}(T) \) increases with temperature and \( \rho_{xx}(T) \) behaves the same way as in case(i). Interestingly, in this case the linear relation (5.4) between \( \rho_{xx} \) and \( \Delta \rho_{xy} \) is more rigorous, with a positive slope given by

\[ S = -\frac{3\pi}{8} \left( \frac{\Gamma}{\omega_c} \right). \] (5.7)

Such a behaviour is seen by Wel et al \[8\].

We shall now proceed to estimate the effective masses \( m^* \) from the reported measurements \[6,7\] below. It is clear from the discussion above that we can hope only to get overall order of magnitude agreement. Given the accuracy of quantization

\[ R \equiv \left| \frac{\Delta \rho_{xy}}{\rho_{xy}(0)} \right| \] (5.8)

at temperature \( T \), the carrier density \( \rho \) and the filling factor \( p \), one can estimate \( m^* \) from Eq. (5.4). We find, for the choice \( \alpha = 1 \) in Eq. (5.4), with the reported accuracy of 4.2 ppm at \( T = 3\text{K} \) for \( p = 4 \) IQHE states in a GaAs sample having \( \rho = 5.61 \times 10^{11} \text{ cm}^{-2} \), \( m^* = 0.075 \text{ m}_e \). Their further measurement at \( T = 1.2\text{K} \) with accuracy 0.017 ppm gives the value of \( m^* = 0.14 \text{ m}_e \). In the measurement of Yoshihiro et al \[3\] in Si MOSFET with \( \rho = 1.0 \times 10^{12} \text{ cm}^{-2} \), they find \( R = 0.2 \text{ ppm} \) at \( T = 0.5\text{K} \) for \( p = 4 \) IQHE state. We estimate \( m^* = 0.67 \text{ m}_e \) from this measurement for Si MOSFET sample. We know that the values of \( m^* \) in GaAs and Si MOSFET samples are 0.07\text{ m}_e and 0.2\text{ m}_e respectively. Note
that it is possible to improve the agreement by further fine tuning the parameter $\alpha$. In short, the estimated effective masses agree with the known ones for the respective samples in the order of magnitude.

VI. CONCLUSION

We have considered a quantum Hall system in the presence of weak disorder and determined the diagonal conductivity $\sigma_{xx}$ and off-diagonal conductivity $\sigma_{xy}$ at finite temperatures, within the SCBA. We have considered the only integer value of filling factor $\nu$ which is in fact the central point of a given plateau in $\rho_{xy}$ and the point of minimum in $\rho_{xx}$. We have shown that $\sigma_{xy}(T)$ acquires a novel temperature dependence which, in fact, sustains in the limit of vanishing impurity. This ‘anomalous’ dependence, which is contrary to the classical expectation, is due to the anomalous nature of the generators of the translation group. $\sigma_{xx}$ vanishes in either of the zero temperature and zero impurity limits. i.e., $\sigma_{xx}$ is non-zero only when both temperature and impurity are non-zero. At low temperatures, $\sigma_{xx}(T)$ does activate with a temperature dependent prefactor. Inverting the conductivity matrix, we obtain $\rho_{xx}(T)$ and $\rho_{xy}(T)$. We have found that depending on the physical regimes of $T$ and $\Gamma$, $\rho_{xy}(T)$ either decreases or increases with increase in temperature. $\rho_{xx}(T)$ is thermally activated as $\rho_{xx} \sim (1/T) \exp [-\omega_c/2T]$. Further $\Delta \rho_{xy}$ is shown to obey a linear relationship with $\rho_{xx}$ in agreement with the experiments. An estimation of effective mass from the measured value of $\rho_{xy}(T)$ turns out to be reasonable, and finally, our determination of compressibility (appendix B) shows that there is no phase transition involved, again in agreement with experiments.

Finally, a remark on Fractional Quantum Hall states. For obtaining the temperature dependence of Hall and diagonal conductivities for these states, one may perform a similar analysis within the composite fermion model [12], since composite fermions fill an integer number of Landau levels. There is, however, an additional complication involving the integration over the Chern-Simons gauge field (which attaches an even number of flux tubes to
each particle) in the determination of the response functions. This is planned to be taken up in future.

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APPENDIX: A

The evaluation of $I_{nm}$

In this Appendix, we shall evaluate

$$I_{nm}(\omega_j) = -\frac{1}{\beta} \sum_{s=-\infty}^{\infty} G^n(i\xi_s) G^m(i\xi_s - i\omega_j),$$

(A1)

where

$$G^n(i\xi_s) = \frac{1}{i\xi_s + \mu - \epsilon_n - \Sigma_n(i\xi_s)}$$

(A2)

with

$$\Sigma_n(i\xi_s) = \frac{1}{2}(i\xi_s + \mu - \epsilon_n) \pm \frac{i}{2} \sqrt{\Gamma^2 - (i\xi_s + \mu - \epsilon_n)^2}.$$  

(A3)

We evaluate $I_{nm}$ by the contour integration method [28].

Let us consider the integral

$$S_{nm} = \int C \frac{dZ}{2\pi i} n_F(Z) G^n(Z) G^m(Z - i\omega_j),$$

(A4)

where

$$n_F(Z) = \frac{1}{1 + e^{\beta Z}}$$

(A5)

is the Fermi function. The contour is shown in Fig. 3. There are poles at $Z = i\frac{\pi}{\beta}(2s + 1)$. The integrand has also branch cuts on the two lines $Z = \epsilon$ and $Z = \epsilon + i\omega_j$, where $\epsilon$ is
real. To illustrate, for $G^m(Z)$, the range of $\epsilon$ is $\epsilon_n - \mu - \Gamma$ to $\epsilon_n - \mu + \Gamma$. The evaluation of residues at the poles of $S_{nm}$ reproduces $I_{nm}$. In other words, $I_{nm}$ is equivalent to $S_{nm}$ when $S_{nm}$ is evaluated along the branch cuts, where the contributions above and below the cuts are subtracted:

$$I_{nm}(\omega_j) = \int_{\epsilon_n - \mu - \Gamma}^{\epsilon_n - \mu + \Gamma} \frac{d\epsilon}{2\pi i} n_F(\epsilon) G^m(\epsilon - i\omega_j) \left[ G^n(\epsilon + i\delta) - G^n(\epsilon - i\delta) \right]$$

$$+ \int_{\epsilon_m - \mu - \Gamma}^{\epsilon_m - \mu + \Gamma} \frac{d\epsilon}{2\pi i} n_F(\epsilon) G^m(\epsilon + i\omega_j) \left[ G^m(\epsilon + i\delta) - G^m(\epsilon - i\delta) \right].$$

(A6)

Now the factor

$$G^n(\epsilon + i\delta) - G^n(\epsilon - i\delta) = G^n(\epsilon) - G^n(\epsilon) = 2i \text{Im} G^n(\epsilon).$$

Therefore

$$I_{nm}(\omega_j) = 2 \int_{\epsilon_n - \mu - \Gamma}^{\epsilon_n - \mu + \Gamma} \frac{d\epsilon}{2\pi i} n_F(\epsilon) G^m(\epsilon - i\omega_j) \text{Im} G^n(\epsilon)$$

$$+ 2 \int_{\epsilon_m - \mu - \Gamma}^{\epsilon_m - \mu + \Gamma} \frac{d\epsilon}{2\pi i} n_F(\epsilon) G^m(\epsilon + i\omega_j) \text{Im} G^n(\epsilon).$$

(A8)

We now analytically continue: $\omega_j \to \omega + i\delta$ to obtain

$$I_{nm}(\omega + i\delta) = 2 \int_{\epsilon_n - \mu - \Gamma}^{\epsilon_n - \mu + \Gamma} \frac{d\epsilon}{2\pi i} n_F(\epsilon) \text{Im} \left[ G^n(\epsilon) \right] G^m(\epsilon - \omega)$$

$$+ 2 \int_{\epsilon_m - \mu - \Gamma}^{\epsilon_m - \mu + \Gamma} \frac{d\epsilon}{2\pi i} n_F(\epsilon) \text{Im} \left[ G^m(\epsilon) \right] G^n(\epsilon + \omega),$$

(A9)

which is the required result.

**APPENDIX: B**

**Compressibility**

The compressibility of the system can be obtained from the density-density correlation $\Pi_{00}$ as

$$\kappa = \left( \frac{1}{\epsilon^2 \rho^2} \right) \lim_{\mathbf{q} \to 0} \text{Re} \Pi_{00}(0, \mathbf{q}).$$

(B1)
We now evaluate $\Pi_{\tau\tau}^{E}(\omega_j, q)$ which reduces to $\Pi_{00}(\omega, q)$ with the analytical continuation.

Without the vertex corrections, we find

$$
\Pi_{\tau\tau}^{(0)E} = -\frac{e^2}{2\pi l^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} I_{nm}(\omega_j) \left[ \delta_{mn} + \frac{q^2 l^2}{2} \left\{ (n + 1)\delta_{m,n+1} + n\delta_{m,n-1} - (2n + 1)\delta_{mn} \right\} \right] + \mathcal{O}(q^2). \tag{B2}
$$

Using the Eq. (A9), we evaluate $\kappa$ by the Taylor expansion in powers of $\Gamma/\omega_c$ as before. The vertex contributions will not be considered as those are in higher order of $\Gamma/\omega_c$. We, therefore, find

$$
\kappa = \left( \frac{1}{e^2 \rho^2} \right) \mathcal{K}, \tag{B3}
$$

where

$$
\mathcal{K} = \frac{e^2 m^* \beta \omega_c}{2\pi} \sum_{n=0}^{\infty} f_n (1 - f_n) + \mathcal{O} \left( \frac{\Gamma}{\omega_c} \right)^2. \tag{B4}
$$

Again, the thermal contribution to $\kappa$ is entirely due to intra Landau band transition.

A low temperature expansion of Eq. (B4) yields

$$
\kappa = \frac{2m^* T_0}{\pi \rho^2} \frac{T}{T_0} \exp \left( -\frac{T_0}{T} \right). \tag{B5}
$$

At $T=0$, $\kappa = 0$, i.e., the Hall fluid is incompressible, as we know. At low temperatures, $\kappa$ is non-zero but exponentially small and hence the fluid is effectively incompressible. Figure 4 shows how the value of $\kappa$ increases with increase in temperature for a given values of $\rho/m^*$ and $p$. The smooth behaviour of $\kappa$ with temperature up to a value 5K clearly shows that there is no phase transition involving the fluids. The same conclusion has been arrived at by Chang et al [41] by their measurement of $\rho_{xx}(T)$ for a wide range of temperatures.
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FIGURE CAPTIONS

FIG.1 : The thick (thin) lines with arrows represent disordered (disorderless) single particle Green’s functions. The symbol X denotes the scattering centre and the dashed lines represent the scatterings. (a) The propagator (impurity averaged) for \( n \)-th LL \( \mathcal{G}^n \) is shown. Here \( \Sigma_n \) is the corresponding self energy. (b) The diagrams which are not taken into account in the SCBA — the diagrams corresponding to multiple scatterings at a given scattering centre and the cross diagrams. (c) The diagrams which have been considered for the evaluation of self energy.

FIG.2 : The thick lines with the arrows represent the single particle Green’s function. The wiggly lines represent the external electromagnetic probe. (a) The response function \( \Pi_{\mu\nu} \) within the SCBA is shown. (b) The diagrams corresponding to the vertex corrections for the response function.

FIG.3 : The contour for \( S_{nm} \) in Eq. (A4) is shown. The wiggly lines represent the branch cuts along the lines \( Z = \epsilon \) and \( Z = \epsilon + i\omega_j \). The cross points are the poles located at \( Z = i\frac{\pi}{\beta}(2s + 1) \).

FIG.4 : The compressibility as a function of temperature for the integer quantum Hall state \( \nu = 1 \). We have chosen \( m^* = 0.07 \, m_e \) and \( \rho = 2 \times 10^{11} \, \text{cm.}^{-2} \).