HYPERBOLICITY OF VARIETIES WITH BIG LINEAR REPRESENTATION OF $\pi_1$

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Abstract. We show the following algebraicity result for a complex projective variety $X$ with big representation of $\pi_1$ into a semi-simple algebraic group: There exists a proper subvariety $Z \subset X$ such that for any algebraic curve $C$, any holomorphic map $\gamma : C \to X$ with $\gamma(C) \not\subset Z$ is induced from an algebraic morphism. As an application, we prove pseudo-Brody hyperbolicity of certain varieties with big reductive representations of $\pi_1$.

1. Introduction

For some complex manifolds it is known that the “size” of their fundamental group has influence on geometric properties. In complex dimension one this is clear from the uniformization theorem: a closed Riemann surface is hyperbolic if and only if it has an infinite and non-abelian fundamental group. Things become more complicated in higher dimensions. Hypersurfaces with high degree in projective spaces provide examples of Kobayashi hyperbolic complex manifolds with trivial $\pi_1$. The product of a hyperbolic compact Riemann surface with $\mathbb{P}^n$ is not hyperbolic in any sense, albeit with infinite and non-abelian $\pi_1$.

In his study of the Shafarevich conjecture, Kollár introduced the notion of large fundamental group (cf. [Kol93]), which turns out to be a suitable substitution of infinity of $\pi_1$ in higher dimension. A complex projective variety $X$ is said to have large fundamental group if for every positive dimensional subvariety $Y \subset X$, the image $\text{Im}\[\pi_1(Y) \to \pi_1(X)\]$ is infinite. People are also interested in varieties with linear $\pi_1$ or linear representations of $\pi_1$, since in this case tools from non-abelian Hodge theory can be applied. In a similar manner, we have

**Definition 1.1.** Let $G$ be a linear algebraic group defined over $\bar{\mathbb{Q}}$. A Zariski dense representation $\rho : \pi_1(X) \to G(\mathbb{C})$ is called big if for a sufficiently general point $x \in X$ and every positive dimensional subvariety $Y \subset X$ containing $x$, the image $\rho(\text{Im}[\pi_1(Y) \to \pi_1(X)])$ is infinite.

Note that if we take $G$ to be semi-simple, then the bigness of $\rho$ implies that $\pi_1(X)$ is infinite and non-abelian. Then similar to the complex dimension one case, one can expect hyperbolicity results for these varieties. In [Zuo96] Zuo proved

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Theorem 1.2. Suppose that $X$ is a smooth projective variety over $\mathbb{C}$ and $\rho : \pi_1(X) \to G(\mathbb{C})$ is a Zariski-dense representation into a semi-simple algebraic group. If $\rho$ is a big representation, then $X$ is Chern hyperbolic, i.e. there exists a proper subvariety $Z \subset X$, such that for any algebraic curve $C$ of genus $g(C) \leq 1$, the image of every non-constant morphism $f : C \to X$ is contained in $Z$.

Zuo stated the above theorem for almost simple algebraic group $G$ (cf. [Zuo96, Theorem 2]). His argument can be easily generalized to the semi-simple case since one can replace $X$ by a finite étale cover and thus $G$ can be assumed to be a direct product of almost simple algebraic groups.

Yamanoi revisited these varieties with Zariski-dense linear representation in [Yam10], with interest in the distribution of entire curves (i.e. non-constant holomorphic maps from $\mathbb{C}$ to these varieties). Motivated by Campana’s abelianity conjecture (cf. [Cam04, Conjecture 9.8]), in [Yam10, Proposition 2.1] Yamanoi proved that for $X$ admitting a Zariski-dense representation of $\pi_1(X)$ into an almost simple algebraic group, every entire curve $\gamma : \mathbb{C} \to X$ is degenerate (i.e. the image curve is not Zariski-dense in $X$). A key ingredient of Yamanoi’s proof is the combination of the value distribution theory with some construction from non-abelian Hodge theory (i.e. harmonic maps into Bruhat-Tits buildings and spectral coverings, cf. §4.1 and [Yam10, §3]).

Inspired by Yamanoi’s work, we obtain the following algebraicity theorem in this paper:

Main Theorem. Suppose that $X$ is a smooth projective variety over $\mathbb{C}$ and $\rho : \pi_1(X) \to G(\mathbb{C})$ is a Zariski-dense representation into a semi-simple algebraic group. If $\rho$ is big, then $X$ is pseudo Borel hyperbolic. That is, there exists a proper subvariety $Z \subset X$ such that for any algebraic curve $C$, any holomorphic map $\gamma : C \to X$ with $\gamma(C) \not\subset Z$ is induced from an algebraic morphism.

Remark 1.3. Note that if $X$ admits such a big representation $\rho$, then for any birational model $\hat{X} \to X$, the pull-back of $\rho$ to $\pi_1(\hat{X})$ is again a big representation. That means the above result cannot be strengthened to $Z = \emptyset$.

The reader is referred to [JK20] for the generalities of the notion of Borel hyperbolicity. The above algebraicity result implies pseudo Brody hyperbolicity:

Corollary 1.4. Let $X$ and $\rho : \pi_1(X) \to G(\mathbb{C})$ be the same as in Main Theorem. If $\rho$ is big, then $X$ is pseudo Brody hyperbolic. That is, there exists a proper subvariety $Z \subset X$ such that the image of every entire curve $\gamma : \mathbb{C} \to X$ is contained in $Z$.

Proof. Suppose that there exists an entire curve $\gamma : \mathbb{C} \to X$ with $\gamma(\mathbb{C}) \not\subset Z$. Note that one can always replace $\gamma$ by the following transcendental holomorphic map

$$\gamma_{\text{new}} : \mathbb{C} \xrightarrow{\exp} \mathbb{C} \xrightarrow{\gamma} X$$

and we still have $\gamma_{\text{new}}(\mathbb{C}) \not\subset Z$.

On the other hand, by the Main Theorem we know that $\gamma_{\text{new}}$ is induced from a non-constant morphism $\mathbb{P}^1 \to X$. This gives a contradiction. \qed
We are now interested in studying Lang’s conjecture on varieties with linear fundamental groups. Recall that for a projective variety $X$ the special set of $X$, $\text{Sp}(X)$, is defined as the Zariski closure of the union of images of all non-constant rational maps from abelian varieties to $X$ (cf. [Lan91] or [Yam15b, §2.4]). Then Lang’s conjecture predicts that $X$ is of general type if and only if $X \neq \text{Sp}(X)$.

In [Yam15b, Theorem 2.17], Yamanoi considered a normal projective variety $X = S/\Gamma$ where $S$ is a Stein space and $\Gamma \subset \text{GL}_n(\mathbb{C})$ is a discrete linear group which acts freely on $S$. Motivated by the hyperbolic version of Lang’s conjecture, he proved that $X$ is Kobayashi hyperbolic if and only if $\text{Sp}(X) = \emptyset$.

Inspired by Lang’s conjecture and Yamanoi’s theorem, we show the following

**Theorem 1.5.** Let $X$ be a complex projective variety with a big reductive representation $\rho : \pi_1(X) \to \text{GL}_n(\mathbb{C})$. If the special set of $X$ is a proper subset, then $X$ is pseudo-Brody hyperbolic.

The strategy of the proof is to consider the abelian part and the semi-simple part of this reductive representation respectively. We use Yamanoi’s theorem [Yam15a] to deal with the abelian part, and the semi-simple part follows from the Main Theorem of this paper. Details will be given in Section 5.

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2. **Dichotomy of representations and the strategy of the proof of Main Theorem**

We first investigate the moduli space of representations. Recall that $\text{Hom}(\pi_1(X), G)$ is an affine variety. Then the *Betti moduli space* is defined as the categorical quotient

$$M_B(X, G) := \text{Hom}(\pi_1(X), G) / G$$

which is a quasi-projective scheme over $\bar{\mathbb{Q}}$ (cf. [Sim92]).

A representation $\rho : \pi_1(X) \to G(\mathbb{C})$ is said to be rigid if $[\rho] \in M_B(X, G)(\mathbb{C})$ is an isolated point. Then the rigidity of $\rho$ gives the factorization $\pi_1(X) \to G(K) \subset G(\mathbb{C})$ after conjugation, where $K$ is some number field (cf. [Rag72], p.90, Proposition 6.6).

Let $p$ be a prime ideal of the ring of integers of $K$ and $K_p$ be the $p$-adic field. We say $\rho : \pi_1(X) \to G(K)$ is $p$-bounded if $\text{Im}[\pi_1(X) \xrightarrow{\rho} G(K) \xrightarrow{\iota} G(K_p)]$ is contained in some maximal compact subgroup (see for instance [Zim84, §6]).

Following the strategy in [Zuo96], we consider the dichotomy of representations:

**Type A:** $\rho$ is rigid and the factorization $\pi_1(X) \to G(K)$ is $p$-bounded for each $p \in \text{Spec} \mathcal{O}_K$;

**Type B:** the rest cases, that is, either $\rho : \pi_1(X) \to G(\mathbb{C})$ is non-rigid, or $\rho : \pi_1(X) \to G(K)$ is $p$-unbounded for some $p \in \text{Spec} \mathcal{O}_K$. 


We will see in Section 3 that the big representation $\rho$ of Type A induces a variation of Hodge structures over $X$ with \textit{generically injective} Higgs map. In Section 4 a pluriharmonic map from $\tilde{X}$, the universal covering of $X$, into some Bruhat-Tits building will be constructed from a Type B representation $\rho$. One can consider the spectral covering $X^s$ of $X$ associated to this pluriharmonic map, and the bigness of $\rho$ guarantees that $X^s$ has \textit{maximal Albanese dimension}.

We will prove the Main Theorem respectively in these two cases.

3. Type A representations and variations of Hodge structures

In this section we prove the Main Theorem for the case that $\rho$ is a Type A representation.

3.1. Generalities about Type A representations. We first recall a lemma about algebraic groups ([Zim84], p.120-121).

\textbf{Lemma 3.1.} Suppose $H \subset G(K)$ is a subgroup which is $p$-bounded for every $p \in \text{Spec}\, \mathcal{O}_K$. Then we have $[H : H \cap G(\mathcal{O}_K)] < +\infty$.

In our case, this means that $\rho(\pi_1(X)) \cap G(\mathcal{O}_K)$ is a finite index subgroup for a Type A representation $\rho$. Therefore, after replacing $X$ by some finite étale covering, we can assume further that $\rho : \pi_1(X) \to G(\mathcal{O}_K)$.

Next we shall construct a \textit{real semi-simple discrete} representation from $\rho$. We consider the restriction of scalars. Let $\sigma_i : K \hookrightarrow \mathbb{C}$, $i = 1, 2, \ldots, d$ be different embeddings. Define

$$R_{K/\mathbb{Q}}(G) := \prod_{i=1}^d G_{\sigma_i},$$

which is an algebraic $\mathbb{Q}$-group with the diagonal embedding

$$\alpha : G(K) \to R_{K/\mathbb{Q}}(G)(\mathbb{Q}).$$

Now we consider the composition map

$$\pi_1(X) \overset{\rho}{\to} G(\mathcal{O}_K) \overset{\alpha}{\to} R_{K/\mathbb{Q}}(G)(\mathbb{Z}).$$

Following Zuo’s argument one can find a noncompact factor $G_0$ of the Zariski closure of the image of $\alpha \circ \rho$ in $R_{K/\mathbb{Q}}(G)$, such that the induced map

$$\rho_0 : \pi_1(X) \to G_0(\mathbb{Z})$$

is a discrete big representation of $\pi_1(X)$ into $G_0(\mathbb{R})$, a semi-simple real Lie group of noncompact type.

From Simpson correspondence [Sim92], we know that the rigid representation $\rho$, as well as the induced representation $\rho_0$, have more structures. First note that the original representation

$$\rho : \pi_1(X) \to G \subset \text{GL}_n(\mathbb{C})$$
gives us a Higgs bundle \((E, \theta)\) together with a harmonic metric \(u\). The rigidity of \(\rho\) implies that \((E, \theta)\) is the fixed point of the \(C^*\)-action on the moduli space of Higgs bundles. Then by Simpson’s ubiquity theorem, we know that \((E, \theta)\) is a Hodge bundle associated to a \(C\)-VHS, i.e., there exists a bigrading \(E = \bigoplus_{p+q=k} E^{p,q}\) with

\[ \theta|_{E^{p,q}} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X. \]

For each embedding \(\sigma_i: K \hookrightarrow \mathbb{C}\), we know that \(\rho^\sigma_i: \pi_1(X) \to G(K) \xrightarrow{\simeq} G(\mathbb{C})\) is still rigid, and thus carries a structure of \(C\)-VHS by Simpson’s theorem. This means that the induced representation \(\rho_0: \pi_1(X) \to G_0(\mathbb{Z}) \subset R_{K/\mathbb{Q}}(G)(\mathbb{Z})\) is equipped with a structure of \(\mathbb{Z}\)-VHS.

Denote by \((E_0, \theta_0, u_0)\) the Hodge bundle associated to \(\rho_0\) with the harmonic metric. By using the theory of harmonic maps and Mok’s factorization theorem \([\text{Mok92, Main Theorem}]\), Zuo proved the following

**Theorem 3.2** ([Zuo96, §3]). Suppose \(\rho\) is a big representation of Type A. Then the induced representation \(\rho_0\) is also big and the Higgs map

\[ \theta_0: T_X \to \text{End}(E_0) \]

is generically injective.

We shall use this construction to prove the pseudo hyperbolicity of \(X\).

### 3.2. Griffiths line bundle and the big Picard theorem.

Recall that we have a Hodge bundle

\[
\left( E_0 = \bigoplus_{p+q=k} E_0^{p,q}, \theta_0 = \bigoplus_{p+q=k} \theta_0^{p,q} \right)
\]

coming from a \(\mathbb{Z}\)-VHS together with a harmonic metric \(u_0\) (the Hodge metric).

We consider the Griffiths line bundle on \(X\)

\[
K(E_0) := \left( \det E_0^{k,0} \right)^{\otimes k} \otimes \left( \det E_0^{k-1,1} \right)^{\otimes (k-1)} \otimes \cdots \otimes \left( \det E_0^{1,k-1} \right).
\]

Then the curvature form of \(K(E_0)\) with respect to the induced Hodge metric can be written as

\[
\Theta(K(E_0)) = k \text{Tr} \Theta(E_0^{k,0}) + (k-1) \text{Tr} \Theta(E_0^{k-1,1}) + \cdots + \text{Tr} \Theta(E_0^{1,k-1}).
\]

**Proposition 3.3** (Griffiths, [Gri70, Proposition(7.15)]). For a tangent vector \(\eta \in T_X\), one has

\[ \Theta(K(E_0))(\eta \wedge \bar{\eta}) \geq 0 \]

with equality if and only if \(\theta_{0,\eta} = 0\).

In our situation, it is easy to show

**Lemma 3.4.** The Griffiths line bundle \(K(E_0)\) is big and nef.
Proof. Note that $K(E_0)$ is nef since $\Theta(K(E_0))$ is semi-positive. To prove the bigness, one only needs to check that

$$\int_X \Theta(K(E_0))^{\dim X} > 0.$$ 

This follows from the fact that $\theta$ is generically injective and thus $\theta_0, \eta \neq 0$ for any tangent vector $\eta$ at a general point of $X$. □

Now we can use the Second Main Theorem of Brotbek-Brunebarbe (cf. [BB20, Theorem 1.1]) to obtain the following

**Theorem 3.5.** Let $(E_0, \theta_0)$ and $K(E_0)$ be the same as above. Let $\gamma$ be a holomorphic map from an algebraic curve $C$ to $X$. For any ample line bundle $A$ on $X$, there exists a constant $\epsilon > 0$ such that

$$T(r, \gamma, K(E_0)) \leq \epsilon \cdot (\log r + \log T(r, \gamma, A)),$$

(3.1)

**Remark 3.6.** Readers are referred to [BB20, §2.4] for notations in value distribution theory. Note that in the general form of Brotbek-Brunebarbe’s Second Main Theorem, the source space of $\gamma$ is assumed to be a parabolic Riemann surface and the weighted Euler characteristic (cf. [PS14, Definition 1.2]) of it appears in the right hand side of (3.1). In our setting, since $C$ is an algebraic curve, we know that the weighted Euler characteristic of $C$ has logarithmic growth (see p.4, (2) of [PS14]).

**Proof of Main Theorem.** From Lemma 3.4 we know that the Griffiths line bundle $K(E_0)$ is big. Then by Kodaira’s lemma one can find a positive integer $m$ such that there exists a nonzero section $s$ of $K(E_0)^{\otimes m} \otimes A^{-1}$.

Now take $Z$ to be the zero locus of $s$. For any holomorphic map $\gamma : C \to X$ with $\gamma(C) \not\subset Z$, we apply (3.1) and obtain

$$\frac{1}{m} \cdot T(r, \gamma, A) \leq T(r, \gamma, K(E_0)) \leq \epsilon \cdot (\log r + \log T(r, \gamma, A)),$$

This means that $T(r, \gamma, A) = O(\log r)$, which implies the algebraicity of $\gamma$ (see e.g. [Dem97, 2.11. cas local] or [NW14, Remark 4.7.4.(ii)]).

□

4. **Type B representations and harmonic maps into Bruhat-Tits buildings**

In this section we deal with Type B representations.

4.1. **Harmonic maps into buildings and spectral coverings.** As we mentioned in the introduction, after replacing $X$ by some finite étale cover, we can assume that $G \cong G_1 \times \cdots \times G_k$, where $G_i$ are almost simple algebraic groups. We will consider the induced representations $\pi_1(X) \to G \to G_i$.

We first consider the $p$-unbounded representations. By the theory of harmonic maps into Bruhat-Tits buildings due to Gromov and Schoen [GS92], there exists a non-constant $\rho$-equivariant pluriharmonic map

$$u_i : \tilde{X} \to \Delta(G_i(K_p))$$
Next we consider the case that \( \rho : \pi_1(X) \to G(\mathbb{C}) \) is non-rigid. In this case we know that \( M_B(X,G) \) has positive dimensional component and thus one can find an affine curve contained in \( M_B(X,G)(\mathbb{Q}) \). Denote by \( T \) the modulo-p reduction of this affine curve to a finite field \( k \). Choose a compactification \( \bar{T} \) and a smooth point \( \infty \in \bar{T} \setminus T \). Then we can define the \( \infty \)-adic valuation \( \nu_\infty(\bullet) \) on the function field \( k(T) \) of \( T \), where for any function \( f \in k(T) \) the valuation \( \nu_\infty(f) \) is the vanishing order of \( f \) at \( \infty \).

Now the deformation of representations along \( T \) will induce the following representation

\[
\rho_{T,i} : \pi_1(X) \to G(k(T)_\infty) \rightarrow G_i(k(T)_\infty)
\]

where \( k(T)_\infty \) is the completion of \( k(T) \) under the \( \infty \)-valuation \( \nu_\infty(\bullet) \). Note that \( \rho_{T,i} \) is an unbounded representation with respect to the non-archimedean norm induced by the valuation \( \nu_\infty(\bullet) \). Then again by the theorem of Gromov-Schoen, we can construct a \( \rho_{T,i} \)-equivariant non-constant pluriharmonic map

\[
u_i : \bar{X} \to \Delta(G_i(k(T)_\infty))
\]

from the universal covering to the Bruhat-Tits building of \( G_i(k(T)_\infty) \) for \( i = 1, 2, \ldots, k \).

Thus in both aforementioned cases of Type B representations, we obtain an equivariant pluriharmonic map \( \nu \) from \( \bar{X} \) to product of Bruhat-Tits buildings \( (\prod_{i=1}^k \Delta(G_i(K_p)) \) or \( \prod_{i=1}^k \Delta(G_i(k(T)_\infty))) \). From the complexified differential \( \partial \nu \) one can extract a multi-valued holomorphic one-form \( \omega \) on \( X \). Following [Zuo96, §1, p.146-147] we consider a finite ramified Galois covering \( \pi : X^s \to X \), the spectral covering, such that \( \pi^*\omega \) splits into \( l \) single-valued holomorphic one-forms \( \omega_1, \ldots, \omega_l \in H^0(X^s, \pi^*\Omega_X^1) \). Here \( l \) is the number of roots of the Weyl group of the algebraic group \( G \). Note that the spectral covering \( \pi \) is unramified outside the union of zero loci \( \bigcup_{i \neq j} (\omega_i - \omega_j) \).

Now we consider the Albanese map \( a : X^s \to \text{Alb}(X^s) \). Note that all \( \omega_i \)'s are pull-back from the Albanese variety. Thus one can find holomorphic one-forms \( \tilde{\omega}_i \) on \( \text{Alb}(X^s) \) such that \( \omega_i = a^*\tilde{\omega}_i \) for \( i = 1, \ldots, l \). Let \( B \subset \text{Alb}(X^s) \) be the maximal abelian subvariety such that all \( \tilde{\omega}_i \) vanish on it. We set \( A := \text{Alb}(X^s)/B \) and consider the induced morphism

\[
\Phi : X^s \xrightarrow{a} \text{Alb}(X^s) \to A.
\]

**Proposition 4.1.** If \( \rho : \pi_1(X) \to G \) is a big representation of Type B, then

1) \( \Phi : X^s \to A \) is generically finite onto its image.

2) \( X^s \) is of general type.

**Proof.** See §1 of [Zuo96]. \( \square \)

4.2. **Yamanoi’s Second Main Theorem and algebraicity.** Now we prove the Main Theorem in the case that \( \rho \) is a Type B representation. First notice that the
general case can be easily reduced to the following situation: $C$ is a smooth quasi-projective curve which is a finite ramified covering of $\mathbb{C}$. Denote by $p_C : C \to \mathbb{C}$ the covering map. We suppose that $\gamma : C \to X$ is a holomorphic map such that the image curve is not contained in the branched locus of $\pi : X^s \to X$.

Now we take $Y$ to be the normalization of the fiber product $C \times_X X^s$. Note that a priori $Y$ is only a Riemann surface. Denote by $\tilde{\gamma} : Y \to X^s$ the induced holomorphic map. Then we have the following diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\gamma} & X^s \\
\downarrow{p_Y} & & \downarrow{\pi} \\
C & \xrightarrow{\gamma} & X \\
\end{array}
$$

where the composed map $p_Y : Y \to \mathbb{C}$ is a finite surjective holomorphic map (thus $Y$ is a parabolic Riemann surface).

Next we employ tools from Nevanlinna theory (cf. §3 of [Yam10] for a brief introduction to the notations). For $r > 0$, we set $Y^r := p_Y^{-1}(D_r)$. Let $\text{Ram}(p_Y)$ be the (analytic) ramification divisor of $p_Y$. We define

$$
N_{\text{ram}}(r) := \frac{1}{\deg(p_Y)} \int_0^r \# (\text{Ram}(p_Y) \cap Y(t)) \frac{dt}{t}.
$$

Denote by $R$ the ramification divisor of the spectral covering $\pi$. Denote by $R_C$ the ramification divisor of $p_C : C \to \mathbb{C}$. Then according to the pull-back divisors $\tilde{\gamma}^* R$ and $\pi_Y^* R_C$ we have the following partition of $\text{Ram}(p_Y)$:

$$
\text{Ram}(p_Y) = R_1 + R_2
$$

where $R_1 = \tilde{\gamma}^* R$ and $R_2 = \text{Ram}(p_Y) - R_1$. Since $\pi_Y$ is étale outside $R_1$, we know that the ramification over $R_2$ comes from the ramification divisor of $p_C$ and therefore $R_2 \leq \pi_Y^* R_C$.

Thus we have

$$
N_{\text{ram}}(r) = N(r, R_1) + N(r, R_2)
$$

where $N(r, R_i) := \frac{1}{\deg(p_Y)} \int_0^r \# (R_i \cap Y(t)) \frac{dt}{t}$. Since $p_C : C \to \mathbb{C}$ is algebraic, we have

$$
N(r, R_2) \leq N(r, \pi_Y^* R_C) := \frac{1}{\deg(p_Y)} \int_0^r \# (\pi_Y^* R_C \cap Y(t)) \frac{dt}{t} = \frac{\deg(\pi_Y)}{\deg(p_Y)} \int_0^r \# (R_C \cap C(t)) \frac{dt}{t} = O(\log r).
$$

Next we want to determine the growth rate of $N(r, \tilde{\gamma}, R)$. Let $L$ be an ample line bundle on $X^s$. Yamanoi proved the following

**Lemma 4.2.** For any $\varepsilon > 0$, we have

$$
N(r, \tilde{\gamma}, R) \leq \varepsilon \cdot T(r, \tilde{\gamma}, L),
$$
Proof. It is known from [Zuo96] that \( \text{Supp} R \subset \bigcup_{i \neq j} (\omega_i - \omega_j)_0 \). In [Yam10, p.557, CLAIM], Yamanoi proved that for \( i \neq j \),
\[
N(r, \tilde{\gamma}, (\omega_i - \omega_j)_0) \leq \varepsilon \cdot T(r, \tilde{\gamma}, L),
\]
\[
\square
\]
Thus we have proved
\[
(4.1) \quad N_{\text{ram}}(r) \leq \varepsilon \cdot T(r, \tilde{\gamma}, L) + O(\log r),
\]
Now we can apply Yamanoi’s Second Main Theorem [Yam15a, Theorem 1] for varieties with maximal Albanese dimension and obtain the following

**Theorem 4.3.** Let \( L \) be an ample line bundle on \( X^s \) and let \( \varepsilon \) be a positive constant. Then there exist a proper Zariski closed subset \( \Sigma \subsetneq X^s \) and a positive constant \( \alpha \) satisfying the following property: for any holomorphic map \( \tilde{\gamma}: Y \to X^s \) from any parabolic Riemann surface \( p_Y: Y \to \mathbb{C} \) such that the image of \( \tilde{\gamma} \) is not contained in \( \Sigma \), we have
\[
T(r, \tilde{\gamma}, K_{X^s}) \leq \alpha \cdot N_{\text{ram}}(r) + \varepsilon \cdot T(r, \tilde{\gamma}, L),
\]

**Proof of Main Theorem.** Let \( Z \) be the union of \( \pi(R) \) and \( \pi(\Sigma) \), which is a proper subvariety of \( X \). Let \( \gamma: C \to X \) be a holomorphic map with \( \gamma(C) \not\subset Z \). Denote by \( \tilde{\gamma}: Y \to X^s \) the induced holomorphic map as above.

Recall that \( X^s \) is of general type. That means, we can find some positive integer \( m \) such that \( L \hookrightarrow K_{X^s}^\otimes m \). By Theorem 4.3, we can find some constant \( C > 0 \) such that
\[
T(r, \gamma, H) \leq C \cdot N_{\text{ram}}(r),
\]
Combing this with the previous estimate (4.1) of \( N_{\text{ram}}(r) \), we finally proved
\[
T(r, \tilde{\gamma}, L) = O(\log r).
\]

Note that we can choose the line bundle \( L \) on \( X^s \) to be sufficiently positive such that there is a nonzero map \( \pi^*H \to L \), where \( H \) is an ample line bundle on \( X \). Then we have
\[
(4.2) \quad T(r, \gamma \circ \pi_Y, H) = T(r, \tilde{\gamma}, \pi^*H) \leq T(r, \tilde{\gamma}, L) = O(\log r),
\]
where the first equality comes from the commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\tilde{\gamma}} & X^s \\
\pi_Y \downarrow & & \pi \\
C & \xrightarrow{\gamma} & X
\end{array}
\]
By definition of the Nevanlinna order function, we have
\[
T(r, \gamma \circ \pi_Y, H) = \int_0^r \frac{dt}{t} \int_{\pi_Y^{-1}(\pi C)} \pi_Y^* \gamma^* c_1(H)
\]
\[
= \int_0^r \frac{dt}{t} \deg(\pi_Y) \int_{\pi C} \gamma^* c_1(H)
\]
\[
= \deg(\pi_Y) \cdot T(r, \gamma, H).
\]
Combining this with (4.2), we know that \( T(r, \gamma, H) = O(\log r) \). Therefore \( \gamma: C \to X \) is algebraic. \( \square \)
5. Proof of Theorem 1.5

Denote by $H$ the Zariski closure of the image of the reductive representation $\rho : \pi_1(X) \to \text{GL}_n(\mathbb{C})$. After replacing $X$ by some finite etale covering of it, we can assume that $H \cong T \times G_1 \times \cdots \times G_k$, where $T$ is an algebraic torus and $G_i$ are almost simple groups. We first show Theorem 1.5 in two extreme cases: the abelian case $H = T$ and the semi-simple case $H = G := G_1 \times \cdots \times G_k$.

**Abelian case $H = T$.** We can consider the Stein factorization $X \to Y \to A$ of the Albanese map $X \to A$ induced by the abelian representation $\rho : \pi_1(X) \to T$. Then the bigness of $\rho$ implies that $X \to Y$ is birational. So the assumption on the special set of $X$ also holds on the special set of $Y$. Then we know that $Y$ is a finite covering of an abelian variety with $\text{Sp}(Y) \subseteq Y$. By Kawamata-Ueno fibration theorem, we know that $Y$ is of general type. Then by [Yam15a, Corollary 1, (1)] we know that $Y$ (and thus $X$) is pseudo-Brody hyperbolic.

**Semi-simple case $H = G$.** This case follows from Corollary 1.4.

Now we start to prove Theorem 1.5 for the general case (i.e. both the abelian part $T$ and the semi-simple part $G$ are non-trivial). We will consider the induced representations

$$
\rho_T : \pi_1(X) \to H \cong T \times G \to T
$$

and

$$
\rho_G : \pi_1(X) \to H \cong T \times G \to G.
$$

By the result of Kollár (cf. [Kol93, §3] or [Zuo99, §5.1]), we know that for the representation $\rho$ we have the Shafarevich map

$$
\text{sh}_\rho : X \dashrightarrow \text{Sh}_\rho(X)
$$

which is a rational map with connected fibers from $X$ to a normal algebraic variety $\text{Sh}_\rho(X)$ (see also [Cam94] for the Kähler case). Note that the bigness of $\rho$ implies that $\text{sh}_\rho$ is birational.

For $\rho_T$ and $\rho_G$, we have the corresponding Shafarevich maps:

$$
\text{sh}_{\rho_T} : X \dashrightarrow \text{Sh}_{\rho_T}(X), \quad \text{sh}_{\rho_G} : X \dashrightarrow \text{Sh}_{\rho_G}(X).
$$

**Lemma 5.1.** The product Shafarevich map

$$
g := (\text{sh}_{\rho_T}, \text{sh}_{\rho_G}) : X \dashrightarrow \text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X)
$$

is birational onto its image in $\text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X)$.

**Proof.** Denote by $W$ the Zariski closure of $g(X)$ in $\text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X)$. Note that Shafarevich maps have connected fibers. Thus we only need to show that the general fiber of $g : X \dashrightarrow W$ is of zero dimension.

Let $F := g^{-1}(w)$ be a general fiber of $g : X \dashrightarrow W$. Since $\text{sh}_{\rho_T}(F) = \text{point}$ (resp. $\text{sh}_{\rho_G}(F) = \text{point}$), by the property of Shafarevich maps we know that $\text{Im}[\pi_1(F) \to \pi_1(X) \overset{\rho_T}{\to} T]$ (resp. $\text{Im}[\pi_1(F) \to \pi_1(X) \overset{\rho_G}{\to} G]$) is finite. That means $\text{Im}[\pi_1(F) \to \pi_1(X) \overset{\rho}{\to} H]$ is finite. Since $\rho : \pi_1(X) \to H \subset \text{GL}_n(\mathbb{C})$ is a big representation, we know that $F$ is of zero dimension. □

Moreover, since $\rho_T$ is an abelian representation, we know that $\text{Sh}_{\rho_T}(X)$ is an abelian variety (in fact a quotient of the Albanese variety of $X$).

Now we apply the Factorization theorem to the semi-simple representation $\rho_G$ (cf. [Mok92] for Type A representations and [Zuo99, §4] for Type B representations). Then after replacing $X$ by some finite étale covering and blowing up it if necessary, there exists a projective variety $Y$ and a surjective morphism $f : X \to Y$ such that $\rho_G : \pi_1(X) \to G$ factors through a big representation $\rho_Y : \pi_1(Y) \to G$. By the semi-simple case ($H = G$), we know that $Y$ is pseudo-Brody hyperbolic.

Now we have the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & \text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X) \\
\downarrow f & & \downarrow \text{Sh}_{\rho_G}(X) \\
Y & \to & \text{Sh}_{\rho_G}(X)
\end{array}
$$

where the general fiber of $Y \to \text{Sh}_{\rho_G}(X)$ is of zero dimension.

Let $Y_0$ be the Zariski open subset of $Y$ where the rational map $Y \dashrightarrow \text{Sh}_{\rho_G}(X)$ is a well-defined morphism. Let $X_0 := f^{-1}(Y_0)$. Denote by $\mathcal{A} := Y_0 \times_{\text{Sh}_{\rho_G}(X)} (\text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X))$ the fiber product. Then $g : X_0 \dashrightarrow \mathcal{A} \to \text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X)$ factors through $X_0 \dashrightarrow \mathcal{A} \to \text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X)$. Note that $h : X_0 \dashrightarrow \mathcal{A}$ is generically finite onto its image in $\mathcal{A}$ since $\mathcal{A} \to \text{Sh}_{\rho_T}(X) \times \text{Sh}_{\rho_G}(X)$ is generically finite and $g$ is birational onto its image. We have the commutative diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{h} & \mathcal{A} \\
\downarrow f & & \downarrow \mathcal{A} \\
Y_0 & \xrightarrow{\text{id}} & Y_0
\end{array}
$$

such that for a general $y \in Y_0$, the restriction of the birational map $h : X_0 \dashrightarrow \mathcal{A}$ on the fiber $F := f^{-1}(y)$ coincides with the restriction of the Shafarevich map of $\rho_T$ on $F$:

$$
F \hookrightarrow X \xrightarrow{\rho_T} \text{Sh}_{\rho_T}(X).
$$

This restriction of Shafarevich map is induced from the following abelian representation

$$
\pi_1(F) \to \pi_1(X) \xrightarrow{\rho_T} H \cong T \times G \to T.
$$

Therefore, the restriction $h|_F : F \to \text{Sh}_{\rho_T}(X)$ is a well-defined morphism. We can assume that $F$ is not contained in $\text{Sp}(X)$ by our assumption on the special set. Then by the abelian case ($H = T$), we know that $F$ is also pseudo-Brody hyperbolic.

Now we consider an entire curve $\gamma : \mathbb{C} \to X$ and the composed holomorphic map $\gamma_Y : \mathbb{C} \to X \xrightarrow{f} Y$. Since $Y$ is pseudo-Brody hyperbolic, we know that outside a proper subvariety of $Y$ all the maps $\gamma_Y$ are constant. Thus we only need to consider these entire curves $\gamma$ which are contained in the general fiber $F$. Thus we have

$$
\mathbb{C} \xrightarrow{\gamma} F \xrightarrow{h|_F} \text{Sh}_{\rho_T}(X)
$$
where $h|_F$ is generically finite onto its image in $\text{Sh}_{\text{pyt}}(X)$.

Now by [Yam15a, Corollary 1, (2)], we know that every entire curve in $F$ is contained in $E \subset F$, where $E$ is the union of $\text{Sp}(F)$ and the exceptional locus of $h|_F$. Note that the special set of $F$ is contained in the special set of $X$ (which is a proper subset by our assumption), and the exceptional locus of $h|_F$ is contained in the exceptional locus of $h$ (which is a proper subset since $h : X_0 \to \mathcal{A}$ is generically finite onto its image). Therefore, there exists a proper subset of $X$ such that every entire curve is contained in it.

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