Conventional and Improved Inclusion-Exclusion Derivations of Symbolic Expressions for the Reliability of a Multi-State Network

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Authors' contributions

This work was carried out in collaboration between the two authors. Author AMAR wrote the entire draft of the manuscript, conducted the mathematical and conceptual analyses and managed the basic literature survey. Author MHA participated in the literature search, performed the computational work, and constructed the table of results. Both authors read and approved the final manuscript.

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ABSTRACT

This paper deals with an emergent variant of the classical problem of computing the probability of the union of n events, or equivalently the expectation of the disjunction (ORing) of n indicator variables for these events, i.e., the probability of this disjunction being equal to one. The variant considered herein deals with multi-valued variables, in which the required probability stands for the reliability of a multi-state delivery network (MSDN), whose binary system success is a two-valued function expressed in terms of multi-valued component successes. The paper discusses a simple method for handling the afore-mentioned problem in terms of a standard example MSDN, whose success is known in minimal form as the disjunction of prime implicants or minimal paths of the pertinent network. This method utilizes the multi-state inclusion-exclusion (MS-IE) principle associated with a multi-state generalization of the idempotency property of the ANDing operation. The method discussed is illustrated with a detailed symbolic example of a real-case study, and it produces a more precise version of the same numerical value that was obtained earlier. The example demonstrates the notorious shortcomings and the extreme inefficiency that the MS-IE

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1. INTRODUCTION

The Inclusion-Exclusion Principle is a very useful principle of enumeration in combinatorics and discrete probability [1-20]. This principle computes the cardinality of the union of \( n \) sets, through a finite repetition of alternation between a usually over-generous inclusion and a usually over-compensating exclusion. This principle remains valid when set cardinalities are replaced by probabilities. In the probability context, the IE Principle is used for computing the probability of the union of \( n \) events. The IE principle proceeds to achieve this computation by first including (adding) the probabilities of the \( n \) events (corresponding to a number of terms that equals \( n \) choose \( 1 = n \)). This is followed by excluding (subtracting) the probabilities of the (\( n \) choose 2) pairwise intersections of these events. Next, the probabilities of the (\( n \) choose 3) triple-wise intersections of the \( n \) events are included (added), the probabilities of the (\( n \) choose 4) quadruple-wise intersections of the \( n \) events are excluded (subtracted), the probabilities of the (\( n \) choose 5) quintuple-wise intersections of the \( n \) events are included (added). This alternation of addition (inclusion) and subtraction (exclusion) is continued until the sole intersection of all the \( n \) events (corresponding to \( n \) choose \( n = 1 \)) is included (added) (if \( n \) is odd) or excluded (subtracted) (if \( n \) is even).

This paper deals with a fundamental application of the IE principle to the computation of multi-state reliability, specifically the computation of the expectation of the logical expression of a multi-state disjunctive normal form (DNF). Currently, the most computationally efficient method for handling this problem is an automated implementation of the method of the recursive sum of disjoint products (RSDP) [21,22]. We present a tutorial discussion and exposition of the conventional IE method and an improved IE approach for solving this problem. This paper is a part of an on-going activity [23-31] that strives to provide a pedagogical treatment and exposition of multi-state reliability problems. We aspire to establish a clear and insightful interrelationship between the two-state modeling and the multi-state one by stressing that multi-valued concepts are natural and simple extensions of two-valued ones. Moreover, we hope to mitigate the notorious shortcomings of the conventional MSIE procedure by combining it with a 'shellable' version of the concept of the sum of disjoint products (SDP), or the more encompassing concept of a probability-ready expression (PRE) in the multi-state domain.

The organization of the remainder of this paper is as follows. Section 2 introduces the running example used herein of a multi-state delivery network (MSDN) with multiple suppliers, borrowed from Lin et al. [22]. Section 3 introduces the multi-state inclusion-exclusion (MSIE) principle, and hints at its special cases and improved variants. Section 4 symbolically applies MSIE, in its standard or conventional form, to the running example. Section 5 introduces a 'shellable' version of the concept of a multi-state probability-ready expression (MSPRE), uses it in conjunction with the MSIE principle, and demonstrates its applicability in terms of the running example. Section 6 reports and discusses numerical results for the two solutions given in Sections 4 and 5. Section 7 concludes the paper. Appendix A presents the python listing of a program that obtains the conventional IE solution of the running example.

1.1 Specifications for a Running Example

Lin et al. [22] studied a specific multi-state delivery network (MSDN) with multiple suppliers,
one market, multiple transfer centers and eight branches. They derived an expression of system success for specific data of delivery costs, probability distributions of all branches, available capacities, suppliers' production capacities, deterioration rate vector for the minimal paths obtained, demand, and budget. They presented the final multi-state success in their Table 2, which is expressed in our following formula with an appropriate translation of notation.

\[
S = X_3(\geq 3) X_5(\geq 3) X_6(\geq 3)
\]

\[
\lor X_3(\geq 3) X_7(\geq 3)
\]

\[
\lor X_2(\geq 3) X_4(\geq 3) X_6(\geq 3)
\]

\[
\lor X_2(\geq 2) X_3(\geq 2) X_4(\geq 2) X_5(\geq 3) X_6(\geq 2)
\]

\[
\lor X_2(\geq 3) X_7(\geq 3)
\]

\[
\lor X_1(\geq 2) X_3(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
\]

\[
\lor X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
\]

\[
\lor X_1(\geq 2) X_2(\geq 2) X_3(\geq 2) X_4(\geq 2) X_7(\geq 3)
\]

Note that the expression of system success \( S \) in (1) reveals clearly that it pertains to a coherent system that enjoys causality, monotonicity and component relevancy [23-29]. The expression comprises eight distinct prime implicants, none of which subsumes (can be absorbed) in another. Each prime implicant is a product of solely upper values \( X_k(\geq j) \) of various variables. The numerical values for the expectations of various variable instances, computed from the data given in Lin et al. [22] and used in [30,31] are listed in Table 1.

### 1.2 Inclusion-Exclusion Principle for Multi-State Probabilities

We note that computing the probability of the union of \( n \) events is equivalent to calculating the expectation of the disjunction (ORing) of the \( n \) indicator variables of such events. Usually these indicator variables are products of instances of the underlying variables. These products usually stand for the minimal paths of the system, which are the prime implicants \( P_i \) of system success, or for the minimal cutsets of the system, which are the prime implicants \( C_i \) of system failure. Note that the expectations of system success and failure are the reliability and unreliability of the system. With this interpretation, an application of the IE principle results in the following expression of reliability [32-36].

\[
R = E[\prod_{i=1}^{n_p} P_i] = \sum_{i=1}^{n_p} E(P_i) - \sum_{i=1}^{n_p} \sum_{j=i+1}^{n_p} E(P_i \land P_j) + \sum_{i=1}^{n_p} \sum_{j=i+1}^{n_p} \sum_{k=j+1}^{n_p} E(P_i \land P_j \land P_k) - \ldots + (-1)^{n_p-1} E[\prod_{i=1}^{n_p} P_i]. \tag{2}
\]

A formal proof of the IE formula is available in Hall [37], Feller [38] and Trivedi [39]. Cerasoli and Fedullo [9] discuss and compare the various available proofs for the IE principle. The number of terms in (2) is given by

\[
\binom{n_p}{1} + \binom{n_p}{2} + \binom{n_p}{3} + \ldots + \binom{n_p}{n_p} = 2^{n_p} - 1, \tag{3}
\]

i.e., it is exponential in the number of minimal paths. To apply the IE principle to (1), which has \( n_p = 8 \), we need 255 terms as we will see in the sequel.

### Table 1. Numerical values for the expectations of various variable instances, computed from data given in [22], and used in [30,31]

| \(X_1(\geq 2)\) | 0.897 | \(X_4(\geq 3)\) | 0.905 |
| \(X_1(\geq 3)\) | 0.892 | \(X_5(\geq 2)\) | 0.953 |
| \(X_2(<3)\) | 0.108 | \(X_3(2)\) | 0.048 |
| \(X_2(\geq 2)\) | 0.965 | \(X_4(<3)\) | 0.095 |
| \(X_3(2)\) | 0.073 | \(X_4(\geq 2)\) | 0.863 |
| \(X_4(<2)\) | 0.137 | \(X_5(\geq 3)\) | 0.903 |
| \(X_5(<3)\) | 0.097 | \(X_6(\geq 2)\) | 0.943 |
| \(X_7(\geq 2)\) | 0.945 | \(X_7(\geq 3)\) | 0.884 |
| \(X_7(2)\) | 0.061 | \(X_8(<3)\) | 0.116 |
| \(X_8(\geq 3)\) | 0.906 | \(X_8(\geq 2)\) | 0.965 |
| \(X_8(2)\) | 0.059 | \(X_8(<3)\) | 0.094 |
The IE principle is valid and applicable whether the
implicants $P_i$ and their constituting variables
are two-valued or multi-valued. However, the
implementation of the IE formula (2) in the multi-
state case needs to be aided by simplification
rules for various products of the underlying
variables. The IE simplicity is manifested in the
fact that the simplification rules it requires (when
handling coherent success) is just a single rule,
namely, the following domination rule (which
generalizes the idempotency rule of AND for an
uncomplemented literal $(X_k \land X_k = X_k)$ in the two-
valued case).

$$X_k(\geq j_2) \ X_k(\geq j_2) = X_k(\geq j_2) \quad \text{for} \quad j_2 \geq j_1,$$  \hspace{1cm} (4a)

A similar simplification required by IE (when
handling coherent failure) is the following
domination rule (which is another generalization
of the idempotency rule of AND for a
complemented literal $(\bar{X}_k \land \bar{X}_k = \bar{X}_k)$ in the two-
valued case).

$$X_k(\leq j_2) \ X_k(\leq j_2) = X_k(\leq j_2) \quad \text{for} \quad j_2 \leq j_1,$$  \hspace{1cm} (4b)

Despite the great importance of the IE principle
in combinatorics and probability theory, and
despite its genuine unrivalled conceptual
simplicity, it does not seem to be the method of
choice for evaluation of system reliability. It
produces an exponential number of terms that
have to be reduced subsequently via addition
and cancellation. Moreover, it involves so many
subtractions that make it highly sensitive to round-off
errors in the ultra-reliable regime [36,40-42]. For the problem of the running
example, the symbolic computations are tedious,
indeed.

In passing, we note that if the paths $P_i$ in
equation (2) are mutually disjoint ($P_i \land P_j = 0 \ \text{for} \ \ 1 \leq i < j \leq n_p$ ), then the complexity of this
equation reduces from exponential to linear, viz.

$$R = E\{V_{i=1}^{n_p} P_i\} = \sum_{i=1}^{n_p} E(P_i).$$  \hspace{1cm} (5)

When many of the products $P_i$ are mutually
disjoint, then many of the pair-wise and k-tuple-
wise intersections in (2) vanish, and the number
of non-zero terms in (2) might decrease
dramatically, thereby allowing one of the most
effective IE improvements.

On the other hand, if the paths $P_i$ in equation (2)
are statistically independent, then (2) reduces to

$$R = E(V_{i=1}^{n_p} R) = \sum_{i=1}^{n_p} E(P_i) - \sum \sum_{i<j \leq n_p} E(P_i E(R)) + \sum \sum \sum_{i<j<k} E(P_i E(R)) E(P_k) \ldots + (-1)^{n^2-1} \prod_{i=1}^{n_p} E(P_i) + 1 - \prod_{i=1}^{n_p} (1 - E(P_i)).$$  \hspace{1cm} (6)

which might be obtained through an application
of De Morgan law to the complement of the union
of products $P_i$ so as to obtain the intersection
of complemented products $\bar{P}_i$ which is a
probability-ready-expression since the
complemented products $\bar{P}_i$ are statistically
independent. Kessler [10] points out that when
the products $P_i$ are, in some sense, close to
being independent, then there are many useful
results bounding $E\{V_{i=1}^{n_p} P_i\}$, so that one might
hope to obtain good estimates for it.

In short, we note that the IE complexity
decreases dramatically for the two extreme
cases of the products $P_i$ being either statistically
independent or mutually exclusive. There is also
some appreciable improvements if these
products $P_i$ are, in some sense, close to either
being statistically independent or mutually
exclusive. To understand why these two cases
are opposite extremes, and how to effectively
utilize one of them, the interested reader might
consult some of the References [43-47].

1.3 Application of the Conventional
Inclusion-Exclusion Principle to the
Running Example

Equation (1) might be rewritten as a disjunction
of eight paths, namely

$$S = P_1 \lor P_2 \lor P_3 \lor P_4 \lor P_5 \lor P_6 \lor P_7 \lor P_8,$$  \hspace{1cm} (7)

where

$$P_1 = X_3(\geq 3) \ X_5(\geq 3) \ X_8(\geq 3)$$

$$P_2 = X_3(\geq 3) \ X_7(\geq 3)$$

$$P_3 = X_4(\geq 3) \ X_5(\geq 3) \ X_8(\geq 3)$$

$$P_4 = X_4(\geq 3) \ X_5(\geq 3) \ X_7(\geq 3) \ X_8(\geq 3)$$

$$P_5 = X_4(\geq 3) \ X_7(\geq 3)$$

$$P_6 = X_4(\geq 3) \ X_5(\geq 3) \ X_7(\geq 3) \ X_8(\geq 3)$$

$$P_7 = X_4(\geq 3) \ X_5(\geq 3) \ X_7(\geq 3) \ X_8(\geq 3)$$

$$P_8 = X_4(\geq 3) \ X_5(\geq 3) \ X_7(\geq 3)$$

The symbolic application of the IE formula (2) to
the disjunction in (7) is very tedious, indeed.
Hopefuly, the reader would bear with this
cumbersome computation, in which the
derivation of 255 terms is involved, and through
which repeated use is made of the domination
rule (4a).
\[P_1P_2 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_7 \geq 3) X_8 \geq 3)\]

\[P_1P_3 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_3 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]

\[P_1P_4 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 3) X_8 \geq 2)) = X_2 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_2P_3 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_5 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]

\[P_2P_4 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)) = X_1 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_3P_4 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_5 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]

\[P_4P_5 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)) = X_1 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_4P_6 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)) = X_1 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_5P_6 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_5 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]

\[P_5P_7 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)) = X_1 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_6P_7 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_5 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]

\[P_6P_8 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)) = X_1 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_7P_8 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_5 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]

\[P_7P_9 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)) = X_1 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_8P_9 = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_5 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]

\[P_8P_{10} = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)) = X_1 \geq 2) X_3 \geq 2) X_6 \geq 2) X_7 \geq 2) X_8 \geq 2)\]

\[P_9P_{10} = (X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)) X_2 \geq 3) X_5 \geq 3) X_6 \geq 3)) X_3 \geq 3) X_5 \geq 3) X_8 \geq 3)\]
\[ P_1P_6 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 3) X_6 \geq 2)) (X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_5 \geq 2) X_7 \geq 3) ] = X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_6 \geq 2) X_8 \geq 2) X_7 \geq 2) \]

\[ = X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]

\[ P_2P_3 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 2) X_8 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]

\[ P_3P_4 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 2) X_8 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]

\[ P_4P_5 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 2) X_8 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]

\[ P_5P_6 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 2) X_8 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]

\[ P_2P_3P_5 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 2) X_8 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]

\[ P_3P_4P_6 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 2) X_8 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]

\[ P_4P_5P_6 = (X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_7 \geq 2) X_8 \geq 2) X_9 \geq 2) X_1 \geq 2) X_2 \geq 2) X_3 \geq 2) X_4 \geq 2) X_6 \geq 2) X_7 \geq 2) X_9 \geq 2) \]
\[P_1P_2P_3 = X_1(\geq 2) X_2(\geq 3) X_3(\geq 2) X_4(\geq 3) X_5(\geq 3) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 2) X_4(\geq 2) X_5(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 3)
\]

\[P_1P_2P_4 = X_1(\geq 2) X_2(\geq 3) X_3(\geq 2) X_4(\geq 2) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 3)
\]

\[P_1P_2P_5 = X_1(\geq 2) X_2(\geq 3) X_3(\geq 2) X_4(\geq 2) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 3)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 3)
\]

\[P_1P_3P_4 = X_1(\geq 2) X_2(\geq 3) X_3(\geq 3) X_4(\geq 3) X_6(\geq 2) X_7(\geq 3) X_8(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 3) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
\]

\[P_1P_3P_5 = X_1(\geq 2) X_2(\geq 3) X_3(\geq 3) X_4(\geq 2) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 3) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
\]

\[P_2P_3P_4 = (X_1(\geq 2) X_2(\geq 3) X_3(\geq 3) X_4(\geq 2) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 3) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
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\[P_2P_3P_5 = (X_1(\geq 2) X_2(\geq 3) X_3(\geq 3) X_4(\geq 2) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 3) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
\]

\[P_2P_4P_5 = (X_1(\geq 2) X_2(\geq 3) X_3(\geq 3) X_4(\geq 2) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 3) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
\]

\[P_3P_4P_5 = (X_1(\geq 2) X_2(\geq 3) X_3(\geq 3) X_4(\geq 2) X_6(\geq 3)(X_1(\geq 2) X_2(\geq 2) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)) = X_1(\geq 2) X_2(\geq 2) X_3(\geq 3) X_4(\geq 2) X_6(\geq 2) X_7(\geq 2) X_8(\geq 2)
\]
\[ P_2P_3p_0 = (X_2(\geq 3)X_3(\geq 3)X_7(\geq 3))(X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_6(\geq 2)X_7(\geq 3) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) ) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]

\[ P_2P_3p_3 = (X_1(\geq 2)X_3(\geq 3)X_4(\geq 2)X_6(\geq 2)X_7(\geq 3)) (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]

\[ P_2P_3p_8 = (X_1(\geq 2)X_3(\geq 3)X_4(\geq 2)X_6(\geq 2)X_7(\geq 3)) (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]

\[ P_2P_3p_9 = (X_1(\geq 2)X_3(\geq 3)X_4(\geq 2)X_6(\geq 2)X_7(\geq 3)) (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]

\[ P_2P_3p_3 = (X_1(\geq 2)X_3(\geq 3)X_4(\geq 2)X_6(\geq 2)X_7(\geq 3)) (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]

\[ P_2P_3p_3 = (X_1(\geq 2)X_3(\geq 3)X_4(\geq 2)X_6(\geq 2)X_7(\geq 3)) (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]

\[ P_2P_3p_3 = (X_1(\geq 2)X_3(\geq 3)X_4(\geq 2)X_6(\geq 2)X_7(\geq 3)) (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]

\[ P_2P_3p_3 = (X_1(\geq 2)X_3(\geq 3)X_4(\geq 2)X_6(\geq 2)X_7(\geq 3)) (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)) = X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_6(\geq 2)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3) \]
\[ P_1P_4P_5 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) X_2 \geq 3 X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) = X_2 \geq 3 X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]

\[ P_1P_4P_6 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]

\[ P_1P_4P_7 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]

\[ P_1P_4P_8 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]

\[ P_2P_3P_4P_5 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) X_2 \geq 3 X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]

\[ P_2P_3P_5P_6 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) X_2 \geq 3 X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]

\[ P_2P_3P_7P_8 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]

\[ P_3P_4P_5P_6 = (X_2 \geq 3) X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) X_2 \geq 3 X_3 \geq 3 X_4 \geq 2 X_5 \geq 3 X_7 \geq 3 X_8 \geq 3) \]
\[ P_1 P_3 P_7 = (X_1 \geq 2) X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_2 P_3 P_7 = (X_1 \geq 2) X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_3 P_5 P_7 = (X_1 \geq 2) X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_4 P_5 P_7 = (X_1 \geq 2) X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_5 P_5 P_7 = (X_1 \geq 2) X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_6 P_5 P_7 = (X_1 \geq 2) X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_7 P_5 P_7 = (X_1 \geq 2) X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_8 P_5 P_7 = (X_1 \geq 2) X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]

\[ P_9 P_5 P_7 = (X_1 \geq 2) X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2) \Rightarrow X_1 \geq 2 X_2 \geq 2 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 \]
\[ P_{2}P_{4}P_{6}^{b} \]
\[ = (X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 2)) (X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 2) X_{4}(\geq 2) X_{5}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 2)) \]
\[ = X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 2) \]
\[ P_1 P_2 P_3 P_4 = (X_1 \geq 2 \quad X_2 \geq 3 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_5 \geq 2 \quad X_6 \geq 1 \quad X_7 \geq 1 \quad X_8 \geq 1) \quad (X_1 \geq 2 \quad X_2 \geq 2 \quad X_3 \geq 2 \quad X_4 \geq 2) \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2
\]

\[ P_1 P_2 P_3 P_4 = (X_1 \geq 2 \quad X_2 \geq 3 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_5 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2) \quad (X_1 \geq 2 \quad X_2 \geq 2 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2)
\]

\[ P_1 P_2 P_3 P_4 = (X_1 \geq 2 \quad X_2 \geq 3 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_5 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2) \quad (X_1 \geq 2 \quad X_2 \geq 2 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2)
\]

\[ P_1 P_2 P_3 P_4 = (X_1 \geq 2 \quad X_2 \geq 3 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_5 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2) \quad (X_1 \geq 2 \quad X_2 \geq 2 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2)
\]

\[ P_1 P_2 P_3 P_4 = (X_1 \geq 2 \quad X_2 \geq 3 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_5 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2) \quad (X_1 \geq 2 \quad X_2 \geq 2 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2)
\]

\[ P_1 P_2 P_3 P_4 = (X_1 \geq 2 \quad X_2 \geq 3 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_5 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2) \quad (X_1 \geq 2 \quad X_2 \geq 2 \quad X_3 \geq 2 \quad X_4 \geq 2 \quad X_6 \geq 2 \quad X_7 \geq 2 \quad X_8 \geq 2)
\]
\[ P_1P_2P_3P_6P_8 = (X_1(\geq 2)X_2(\geq 2)X_3(\geq 3)X_4(\geq 2)X_5(\geq 3)X_6(\geq 3)X_7(\geq 3)X_8(\geq 2)X_9(\geq 3)) = \sum_{X_1(\geq 2)}\sum_{X_2(\geq 2)}\sum_{X_3(\geq 3)}\sum_{X_4(\geq 2)}\sum_{X_5(\geq 3)}\sum_{X_6(\geq 3)}\sum_{X_7(\geq 3)}\sum_{X_8(\geq 2)}\sum_{X_9(\geq 3)} \]

\[ P_1P_2P_3P_6P_8 = (X_1(\geq 2)X_2(\geq 2)X_3(\geq 3)X_4(\geq 2)X_5(\geq 3)X_6(\geq 3)X_7(\geq 3)X_8(\geq 3)) = \sum_{X_1(\geq 2)}\sum_{X_2(\geq 2)}\sum_{X_3(\geq 3)}\sum_{X_4(\geq 2)}\sum_{X_5(\geq 3)}\sum_{X_6(\geq 3)}\sum_{X_7(\geq 3)}\sum_{X_8(\geq 3)} \]

\[ P_1P_2P_3P_7P_8 = \sum_{X_1(\geq 2)}\sum_{X_2(\geq 2)}\sum_{X_3(\geq 3)}\sum_{X_4(\geq 2)}\sum_{X_5(\geq 3)}\sum_{X_6(\geq 3)}\sum_{X_7(\geq 3)}\sum_{X_8(\geq 2)}\sum_{X_9(\geq 3)} \]

\[ P_1P_2P_3P_8P_9 = \sum_{X_1(\geq 2)}\sum_{X_2(\geq 2)}\sum_{X_3(\geq 3)}\sum_{X_4(\geq 2)}\sum_{X_5(\geq 3)}\sum_{X_6(\geq 3)}\sum_{X_7(\geq 3)}\sum_{X_8(\geq 2)}\sum_{X_9(\geq 3)} \]
\[ P_2P_3P_4P_6 = (X_1(\geq 2)X_2(\geq 3)X_3(\geq 3)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3))X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)
\]

\[ P_2P_3P_4P_7 = (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3))X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)
\]

\[ P_2P_3P_4P_8 = (X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)X_7(\geq 3))X_1(\geq 2)X_2(\geq 2)X_3(\geq 2)X_4(\geq 2)X_5(\geq 2)X_6(\geq 2)
\]
\[
P_I P_2 P_3 P_4 P_5 P_6 P_7 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]

\[
P_I P_2 P_3 P_4 P_5 P_6 = (X_1 \geq 2)X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_8 \geq 2 X_9 \geq 2 X_{10} \geq 2) = X_1 \geq 2 X_2 \geq 3 X_3 \geq 2 X_4 \geq 2 X_5 \geq 2 X_6 \geq 2 X_7 \geq 2 X_{10} \geq 2
\]
\[ P_{2}P_{3}P_{4}P_{5}P_{7} = (X_{1} \geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 3) X_{6}(\geq 3)(X_{1}(\geq 2) X_{2}(\geq 2) X_{4}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 2) X_{8}(\geq 2)) = X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 3) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3) \]

\[ P_{2}P_{3}P_{4}P_{5}P_{6} = (X_{1} \geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 3) X_{6}(\geq 3)(X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)) = X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 3) X_{5}(\geq 2) X_{6}(\geq 3) X_{7}(\geq 3) X_{8}(\geq 3) \]

\[ P_{2}P_{3}P_{5}P_{6}P_{7} = (X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)(X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 2) X_{4}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)) = X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 3) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3) \]

\[ P_{2}P_{3}P_{4}P_{6}P_{7} = (X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)(X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 2) X_{4}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)) = X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 3) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3) \]

\[ P_{2}P_{3}P_{4}P_{5}P_{7} = (X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)(X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 2) X_{4}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)) = X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 3) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3) \]

\[ P_{2}P_{3}P_{4}P_{5}P_{6} P_{7} = (X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)(X_{1}(\geq 2) X_{2}(\geq 2) X_{3}(\geq 2) X_{4}(\geq 2) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3)) = X_{1}(\geq 2) X_{2}(\geq 3) X_{3}(\geq 3) X_{4}(\geq 2) X_{5}(\geq 3) X_{6}(\geq 2) X_{7}(\geq 3) X_{8}(\geq 3) \]
1.4 Improving IE with a ‘Shellable’ Version of a PRE

In this section, we deal with the introduction of orthogonality in a given sum-of-products formula (disjunctive normal form). If neither of the two terms \( A \) and \( B \) in the sum \( (A \lor B) \) subsumes the other \( (A \lor B \neq A \land A \lor B \neq B) \) and the two terms are not already disjoint \((A \land B \neq 0)\), then \( B \) can be disjointed with \( A \) by using the formula [24,36,43,48-57]

\[
A \lor B = A \lor (\overline{Y}_1 \lor Y_1 \overline{Y}_2 \lor Y_1 Y_2 \overline{Y}_3 \lor \cdots \lor Y_1 Y_2 \cdots Y_{e-1} \overline{Y}_e) B
\]

(8)

Where the first term \( A \) still remains intact, while the second term \( B \) is replaced by \( e \) terms which are each disjoint with \( A \) and are also disjoint among themselves. Note that each \( Y_k \) \((1 \leq k \leq e)\) is a literal that appears in the product \( A \) and does not appear in the product \( B \). It stands for a disjunction of certain instances of some variable \( x_{1(k)} \). We are interested herein in the particular case for which \( e = 1 \), i.e., when the two products \( A \) and \( B \) are such that there is a single literal that appears in the product \( A \) and does not appear in the product \( B \). For this case, the disjointing formula (8) simplifies to the Reflection law [36].

\[
A \lor B = A \lor \overline{Y}_1 B
\]

(9)

Which is conveniently referred to as a case of ‘shellable’ disjointing. We coin the name of ‘shellable disjointing’ to mimic the well-known term of a ‘shellable disjunctive normal form (DNF)’ that designates a DNF for which orthogonalization can be (most) efficiently performed, without any increase in the number of terms [58-61]. In the sequel, we will not strive to achieve complete orthogonality in a given sum-of-products formula (disjunctive normal form). Instead, we will apply shellable disjointing as much as we can. In comparison with schemes for producing a probability-ready expression, this scheme enjoys the advantages of simplicity and avoidance of increase in the number of terms, at the expense of still requiring the further use of the inclusion-exclusion (IE) principle. However, the IE use might simplify dramatically. Therefore, the net complexity of this scheme, which borrows ideas partially from the PRE and IE procedures, seems to be much faster and less error-prone, than a scheme based on the IE scheme alone or another using the PRE scheme solely.

To demonstrate the proposed scheme, we rearrange the 8 prime implicants in (1) as shown in (10), and further introduce as much orthogonality as possible through shellable disjointing (9), as shown by the bold literals in (10). For example, the first prime implicant in (1) has two literals \( X_2 \{\geq 3\} \) and \( X_7 \{\geq 3\} \), and none of the succeeding implicants share both these literals (for otherwise, the succeeding implicant would subsume the first implicant and get absorbed by it, contradicting the fact that it is prime). Three out of these succeeding implicants (the fourth, the seventh, and the eighth) do not share any of the aforementioned literals with the first implicant, and we deliberately abstain from disjointing any of them with the first implicant, since such an action would be complicated and would split each of the disjointed terms into several (here two) terms. The third implicant (alone) shares \( X_2 \{\geq 3\} \) with the first implicant, so that only the literal \( X_7 \{\geq 3\} \) appears in the first implicant but not in the third one. Therefore, we achieve shellable disjointing of the third implicant with the first by multiplying the third implicant with the complement of \( X_2 \{\geq 3\} \), which is \( X_7 \{\geq 3\} \) (shown in bold). Likewise, we attain shellable disjointing of the second, fifth, and sixth prime implicants with the first by multiplying each of them with the complement of \( X_2 \{\geq 3\} \), which is \( X_7 \{\geq 3\} \) (again, highlighted in bold). Shellable disjointing is also possible for the fourth implicant with the third by multiplying the fourth implicant with \( X_2 \{\geq 3\} \) (shown in bold). As a result, the fourth implicant immediately becomes disjoint with the first, and turns capable of shellable disjointing with the second implicant through further multiplication with \( X_7 \{\geq 3\} \) (shown in bold). Each of the fifth and sixth implicants is capable of shellable disjointing with the second implicant through multiplication with \( X_2 \{\geq 3\} \) (again, shown in bold). The eighth implicant is capable of shellable disjointing with the seventh implicant through multiplication with \( X_2 \{\geq 2\} \) (shown in bold). As an offshoot, the eighth implicant becomes orthogonal with four other implicants (the first, the third, the fifth, and the sixth).

\[
S = X_2 \{\geq 3\} X_7 \{\geq 3\} \lor X_2 \{\geq 3\} X_3 \{\geq 3\} X_7 \{\geq 3\} \lor X_2 \{\geq 3\} X_5 \{\geq 3\} X_7 \{\geq 3\} \lor X_2 \{\geq 3\} X_3 \{\geq 3\} X_4 \{\geq 3\} X_8 \{\geq 3\} \lor X_2 \{\geq 3\} X_5 \{\geq 3\} X_4 \{\geq 3\} X_8 \{\geq 3\} \lor X_2 \{\geq 3\} X_3 \{\geq 3\} X_4 \{\geq 3\} X_8 \{\geq 3\} \lor X_2 \{\geq 3\} X_5 \{\geq 3\} X_4 \{\geq 3\} X_8 \{\geq 3\} \lor X_2 \{\geq 3\} X_3 \{\geq 3\} X_4 \{\geq 3\} X_8 \{\geq 3\} \lor X_2 \{\geq 2\} X_2 \{\geq 2\} X_4 \{\geq 2\} X_6 \{\geq 2\} X_7 \{\geq 2\} X_8 \{\geq 2\} \lor X_1 \{\geq 2\} X_1 \{\geq 2\} X_3 \{\geq 2\} X_4 \{\geq 2\} X_6 \{\geq 2\} X_7 \{\geq 2\} X_8 \{\geq 2\}
\]

(10)
Employing the relation \((X_i \geq 2) \wedge X_i \geq 3 = X_i(2)\), we simplify Equation (10) to
\[ R_1 = X_2(3) X_7(3) \]
\[ R_2 = X_2(3) X_7(3) X_7(3) \]
\[ R_3 = X_2(3) X_3(3) X_4(3) X_9(3) \]
\[ R_4 = X_2(3) X_3(3) X_5(3) X_7(3) X_9(3) \]
\[ R_5 = X_2(3) X_3(3) X_4(3) X_7(3) X_9(3) \]
\[ R_6 = X_2(3) X_3(3) X_4(3) X_7(3) X_9(3) \]
\[ R_7 = X_2(3) X_3(3) X_4(3) X_7(3) X_9(3) \]
\[ R_8 = X_2(3) X_3(3) X_4(3) X_9(3) \]
\[ R_9 = X_2(3) X_3(3) X_4(3) X_9(3) \]
where
\[ (11a) \]

\[ E[S] = E[R] + E[R_2] + E[R_3] + E[R_4] + E[R_5] + E[R_6] - E[R_7] - E[R_8] - E[R_9] \]

where the various intersections in (12) are
\[ R_1 R_2 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]
\[ R_1 R_3 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]
\[ R_1 R_4 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]
\[ R_1 R_5 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]
\[ R_1 R_6 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]
\[ R_1 R_7 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]
\[ R_1 R_8 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]
\[ R_1 R_9 = \{X_2(2) X_3(3) X_5(3) X_7(3) X_9(3) = X_2(2) X_3(3) X_5(3) X_7(3) X_9(3)\} \]

2. REPORT AND DISCUSSION OF NUMERICAL RESULTS

We presented two IE methods, the conventional one and an improved one, for solving the problem of our running example. Our two methods agreed on a value of 0.9819022224313, which is in agreement with the solution of Rushdi and Amashah [30,31], and also in agreement (albeit more precise) with the numerical value (0.981902) that was obtained earlier by Lin et al. [22]. Table 2 details the
computations performed by the two IE methods. The table indicates clearly the improvements brought about by the second method, both in decreasing the number of operations and in diminishing the effect of round-off errors. We note that the round-off error in the first method would have been more pronounced, had we implemented the IE formula as is. Actually, we decreased the round-off significantly by performing an actual subtraction only once, as we summed all positive terms, summed all negative terms, and only then took the difference. We have to admit that we could not fix our symbolic computations via the conventional IE formula from the outset. The manual computation of 255 terms were too tedious and error-prone to be completed correctly in one trial. We evaluated the IE formula correctly via the python program in Appendix A, and then used the results of this program to fix bugs in the symbolic computations.

In passing, we note that the present work inspired us to apply the MS-IE Principle to the union of fewer (factored or composite) paths that is subsequently converted (at minimal cost) to PRE form [62]. We augmented the resulting procedure with another that uses the multi-state Boole-Shannon expansion [23,24,27,30,42,48,54,63-68]. Consequently, we were in a position to point out a liaison among inclusion-exclusion, probability-ready expressions and Boole-Shannon expansion for multi-state reliability [62].

| Item | Conventional IE | IE improved with PRE |
|------|-----------------|----------------------|
| Sum of 8 expectations of single indicators | + 5.7002040911313365 | + 1.638655916679425 |
| − Sum of 28 expectations of pairwise ANDing of indicators | − 16.183321233651064 | − 0.6589357136472941 |
| + Sum of 56 expectations of triple-wise ANDing of indicators | + 28.38786762461678 | + 0.0021820193991679074 |
| − Sum of 70 expectations of quadruple-wise ANDing of indicators | − 32.6648909833673 | 0 |
| + Sum of 56 expectations of quintuple-wise ANDing of indicators | + 24.82589302827858 | 0 |
| − Sum of 28 expectations of sextuple-wise ANDing of indicators | − 12.067134436416122 | 0 |
| + Sum of 8 expectations of septuple-wise ANDing of indicators | + 3.4094675792097604 | 0 |
| − expectation of octuple-wise ANDing of indicators | − 0.42618344740122005 | 0 |
| Sum of all positive terms | 62.32343232364569 | 1.6408379360785929074 |
| − Sum of all negative terms | − 61.34153010080513605 | − 0.6589357136472941 |
| Net required value | 0.9819022243132085 | 0.981902224312988074 |

3. CONCLUSIONS

This paper is a continuation of our ongoing efforts to extend concepts of reliability computations in the binary domain to the multi-state domain. The paper serves as an exposition of the inter-relationships between the multi-state concepts MS-IE and MS-PRE. This exposition was obtained by using the standard MS-IE approach and an improved MS-IE approach preceded by an efficient shellable PRE preprocessing. The two approaches were applied to the same problem of multi-state network reliability. Each of the two approaches recovered the same result obtained by the conventional RSDP method. Hopefully, the present work can guide more useful applications of the Inclusion-Exclusion Principle and its improved variants to other real-life problems.

COMPETING INTERESTS

The authors have declared that no competing interests exist.

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APPENDIX A

Listing in Python of a Program Computing the IE Solution for the Running Example

```python
from itertools import combinations
from functools import reduce

# the lists below are from Lin et al. [21, Table 2, p.6689]
P1 = [0, 0, 3, 0, 3, 0, 0, 3]
P2 = [0, 0, 3, 0, 0, 3, 0, 0]
P3 = [0, 3, 0, 3, 0, 0, 3, 0]
P4 = [2, 2, 0, 0, 3, 2, 2, 2]
p5 = [2, 2, 0, 0, 0, 3, 0, 0]
p6 = [2, 2, 0, 2, 2, 2, 2, 2]
p7 = [2, 2, 2, 0, 0, 2, 2, 2]
p8 = [2, 2, 2, 2, 2, 2, 0, 3]

total = 0
temp = []  # temporary list to store digits
multi = 1  # multiplier counter
Full_List = [P1, P2, P3, P4, p5, p6, p7, p8]

for r in range(1, 9, 1):  # r number of tuples to be compared starting from 1
    Combinations = combinations(Full_List, r)
    print("Finding maximum for a combination size of %d" % r)
    for eachCombination in Combinations:
        # print("Finding combination for tuples ", *eachCombination)
        temp = list(map(max, zip(*eachCombination)))
        print(temp)
        if (temp[0] == 2): multi = multi * 0.897  # Enter the data from Table 1 p-6688
        if (temp[1] == 2): multi = multi * 0.965  # 0 for X_1, 1 for X_2 ...etc
        if (temp[1] == 3): multi = multi * 0.892
        if (temp[2] == 2): multi = multi * 0.953
        if (temp[2] == 3): multi = multi * 0.905
        if (temp[3] == 2): multi = multi * 0.863
        if (temp[3] == 3): multi = multi * 0.903
        if (temp[4] == 2): multi = multi * 0.943
        if (temp[4] == 3): multi = multi * 0.945
        if (temp[5] == 2): multi = multi * 0.884
        if (temp[5] == 3): multi = multi * 0.965
        if (temp[6] == 2): multi = multi * 0.906
        print(multi)

    if (r % 2 == 1): total += multi  # if r odd we add it otherwise subtract from total
    else: total -= multi
    multi = 1  # reset the counter
    print("the total = ", total)
```

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