Tensionless string in the notoph background

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\textbf{Abstract}

We study the interaction between a tensionless (null) string and an antisymmetric background field $B_{ab}$ using a 2-component spinor formalism. A geometric condition for the absence of such an interaction is formulated. We show that only one gauge-invariant degree of freedom of the field $B_{ab}$ does not satisfy this condition. Identification of this degree of freedom with the notoph field $\phi$ of Ogievetskii-Polubarinov-Kalb-Ramond is suggested. Application of a 2-component spinor formalism allows one a reduction of the complete system of non-linear partial differential equations and constraints governing the interacting null string dynamics to a system of linear differential equations for the basis spinors of the spin-frame. We find that total effect of the interaction is contained in a single derivation coefficient which is identified with the notoph field.

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I. INTRODUCTION

Recently, there has been considerable interest in the investigation of the string and null string equations of motion in different backgrounds of external fields and in curved spacetimes (e.g. [1–3] and references therein). Finding the exact solutions for the equations of motion in such systems is a rather difficult task, mainly due to the non-linear character of the equations of motion, so it seems interesting to study those situations in which these equations possess exact solutions. It is known that the equations of motion for strings in 4D Minkowski spacetime, null strings in some backgrounds and for particular types of curved spacetimes can be solved exactly [2–6]. Being a zero-tension limit of strings [7], null strings possess simpler equations of motion than those of strings and can be considered as a zero approximation with the string tension as the perturbation parameter [8–10]. This explains our current interest in this problem.

Geometrically worldsheets of null strings are lightlike (null) 2-surfaces which generalize worldlines of massless particles [7,11–13]. By convention, null string interactions with various fields can be analysed into two types. To the first type we attribute the interactions which violate the lightlike character of the worldsheets and can lead to generation of non-zero tension [14–18]. All other interactions fall into the second type. In particular, null string interactions with antisymmetric background tensor fields are of the second type. As shown in [14], 2-component spinors of 3D Minkowski spacetime provide a particularly convenient framework for studying the general solutions of null string equations of motion in arbitrary antisymmetric background fields $B_{ab}(x)$. On the other hand, a convenience of adopting 2-component spinor description for spacelike and timelike strings in 4D Minkowski spacetime was demonstrated in [19] and for free null (super) strings and p-branes in [20]. This formalism is based on the fundamental results of Penrose’s twistor programme [21] and is a part of the so-called twistor approach in the theory of supersymmetric objects [22–31]. From this viewpoint the goal of this article is to study further the utility of such an approach in describing exact solutions of non-linear equations of motion for extended objects in external fields. The first signs as to the efficiency of this approach were found in [14] where the dynamics of null strings in 3D Minkowski spacetime and null membranes in 4D Minkowski spacetime in background antisymmetric fields $B_{ab}(x)$ and $B_{abc}(x)$ was investigated. It has been shown that the contribution of these fields into the null string and null membrane equations of motion can be annihilated by reparametrization of the corresponding null worldsheets.

On the other hand, from the point of view of gauge field theory for $B_{ab}(x)$ in 3D Minkowski spacetime and $B_{abc}(x)$ in 4D Minkowski spacetime these results are natural ones since $B_{ab}(x)$ and $B_{abc}(x)$ are the pure gauge fields in the given dimensions of spacetimes. Duality relations $B_{ab} = \varepsilon_{abc} \phi^c$ and $B_{abc} = \varepsilon_{abc} \phi^d$ allow exclusion of the vector field $\phi^a$ from equations of motion by fixing gauge parameters $\Lambda_a$ and $\Lambda_{ab}$ of the corresponding U(1) internal group of gauge symmetry. Nevertheless, the non-trivial nature of results [14] lie in the fact that they were derived without using
the gauge symmetry described above. Utilization of worldsheet Virasoro symmetry together with new gauge symmetries of the 2-component spinor formalism were sufficient for elimination of these pure gauge fields. These symmetries allow reduction of the null string and null membrane non-linear equations of motion to linear and exactly solvable ones. Taking a pragmatical point of view it could be advantageous to exclude all non-physical degrees of freedom of the external fields interacting with extended objects by fixing gauge conditions corresponding to the internal symmetry group from the very beginning. However, this leads to the Lorentz non-covariant description which is disadvantageous on the quantum level.

Our investigation suggests that 2-component spinor formalism allows one to simplify the non-linear dynamics of strings and membranes propagating in external fields or curved spacetimes without breaking the Lorentz covariance and internal gauge symmetry invariance of the description explicitly. We hope that simplifications achieved in the framework of such a description could provide the most relevant choice of effective variables which would permit integration of non-linear motion equations for p-branes in various spacetime dimensions.

Assuming the correctness of the above-stated hypothesis we shall study non-linear equations of motion of a tensionless string embedded into the background of an antisymmetric field $B_{ab}(x)$ in 4D Minkowski spacetime. In this case the field $B_{ab}(x)$ has physical degrees of freedom which cannot be eliminated by fixing of U(1) gauge symmetry. In what follows we have to keep in mind that the results discussed in this paper depend on the dimension of spacetime [32].

We show below that the use of Virasoro reparametrization symmetry and local symmetries inherent in the 2-component spinor formalism provide a gauge invariant possibility to exclude all non-physical degrees of freedom of the gauge field $B_{ab}(x)$. It turns out that only one gauge-invariant physical degree of freedom influences the null string dynamics, and we identify this degree of freedom with the notoph field of Ogievetskii-Polubarinov-Kalb-Ramond [33,34].

We also formulate a geometric condition for the absence of interaction between the tensionless string and background antisymmetric fields which is equivalent to the condition that $B_{ab}(x)$ is a pure gauge field. This condition consists in the vanishing of the scalar product of the 4-vector velocity vector $\dot{x}^a$ with field strength $Q^a(x)$ of the gauge field $B_{ab}(x)$.

Similar to the case of the Lund-Regge geometrical approach [35–40] we show that null string equations of motion can be reduced to algebraic conditions on the coefficients of decomposition (the so-called derivation coefficients) of the first derivatives of the basis spinors of a 2-component spin-frame (dyad) with respect to worldsheet coordinates $\tau$ and $\sigma$. Then integrability conditions for the representation of the first derivatives of $x^a(\tau, \sigma)$ with respect to $\tau$ and $\sigma$ start playing the role of the dynamical equations.

We establish that the effect of interaction with the antisymmetric field is contained in one of the derivation coefficients of $\dot{x}^a$, and we interpret it as the notoph field $\phi$. From the point of view of the gauge field theory approach [37–39] these derivation coefficients have physical meaning of 2D Yang-Mills or Higgs
fields. Therefore the notoph field finds interpretation as an additional component of 2D Yang-Mills-Higgs field.

The resulting equations of motion of the null string in a background of antisymmetric fields can be written as a system of linear second-order partial differential equations for the basis spinors of the dyad. This shows that the 2-component spinor formalism can be successfully employed for the purpose of linearization of non-linear equations for the basis spinors of the dyad. This shows that the 2-component spinor metric fields can be written as a system of linear second-order partial differential equations of motion of tensionless strings interacting with background antisymmetric gauge fields in 4D Minkowski spacetime.

II. EQUATIONS OF MOTION

Let us consider the action of a null string in an external antisymmetric field \( B_{ab}(x) \). In the spinor form it can be rewritten as follows:

\[
S = \int \left[ \rho^\mu \partial_\mu x^{AA'} o_A \bar{\omega}_{A'} - \kappa \varepsilon^{\mu_1 \mu_2} \partial_{\mu_1} x^{A_1 A'_1} \partial_{\mu_2} x^{A_2 A'_2} B_{A_1 A'_1 A'_2} B_{A_2 A'_2} \right] d^2 \xi, \tag{2.1}
\]

where \( B_{A_1 A'_1 A'_2}, x^{A A'} \) and \( d^2 \xi \) represent, respectively, the field \( B_{ab} \), the coordinates of the null string \( x^a(\tau, \sigma) \) and the area element \( d\tau d\sigma \) of the worldsheet. We assume that \( \xi^\mu = (\tau, \sigma) \) is a smooth parametrization of the null-string worldsheet, choose \( \varepsilon^{\tau \sigma} = -\varepsilon^{\sigma \tau} = -1 \), \( \eta_{ab} = \text{diag}(+-+) \) and use the notation \( \partial_\mu = \partial / \partial \xi^\mu \). The coordinates \( \xi \) on the worldsheet are dimensionless and we assume the worldsheet vector density \( \rho^\mu(\xi) \) to have the dimensions of inverse length. The density \( \rho^\mu \) ensures invariance of action (2.1) under arbitrary non-degenerate reparametrisations of the null string worldsheet [18]. Interaction constant \( \kappa \) has the dimensions of inverse length. Spinor fields \( o^A \) and \( \iota^A \) form a basis for 2-dimensional complex vector space and obey the normalization conditions

\[
o_A (\iota^A) = \chi, \quad o_A o^A = \iota_A \iota^A = 0, \tag{2.2}
\]

where \( \chi(x^{AA'}) \) represents a possibility of rescaling of the dyad element \( \iota^A \) at each spacetime point \( x^{AA'} \). The relation

\[
B_{A_1 A'_1 A'_2} B_{A_1 A'_1 A'_2} = -B_{A_2 A'_1 A'_2 A'_1} \tag{2.3}
\]

holds for the antisymmetric tensor field \( B_{ab} \) and we introduce \( \partial_{AA'} = \partial / \partial x^{AA'} \), \( \dot{x}^{AA'} = \partial_x x^{AA'} \), \( \dot{\sigma}^{AA'} = \partial_\sigma x^{AA'} \) and

\[
3 \dot{\partial}_{[AA'} B_{A_1 A'_1 A'_2]} = \partial_{AA'} B_{A_1 A'_1 A'_2} + \partial_{A_1 A'_1} B_{A_2 A'_2 A'_1} + \partial_{A_2 A'_2} B_{A_1 A'_1 A'_1}. \tag{2.4}
\]

Variation of action (2.1) results in the equations describing the null string dynamics

\[
\rho^\mu \partial_\mu x^{AA'} o_A = 0, \tag{2.5}
\]

\[
\partial_{\mu} x^{AA'} o_A = 0,
\]

\[
\partial_{\mu} (\rho^\mu o_A \bar{\omega}_{A'}) + 3 \kappa \varepsilon^{\mu_1 \mu_2} \partial_{\mu_1} x^{A_1 A'_1} \partial_{\mu_2} x^{A_2 A'_2} \partial_{[AA'} B_{A_1 A'_1 A'_2]} = 0,
\]

where \( \bar{\omega}_{A'} \) is the dual of \( \omega_{A'} \). The relation

\[
\rho^\mu \partial_\mu x^{AA'} o_A \bar{\omega}_{A'} = 0 \tag{2.6}
\]

is a consequence of the normalization conditions (2.2) and the notoph field finds interpretation as an additional component of 2D Yang-Mills-Higgs field.
and complex conjugate of them. The first equation in (2.5) and its complex conjugate imply
\[ \rho^\mu \partial_\mu x^{AA'} = e o^A \bar{o}^{A'}, \] (2.6)
where \( e(\xi) \) is an arbitrary real-valued function with transformation properties of a scalar worldsheet density. Assuming that \( \rho^\tau \) is a nowhere-zero function we rewrite this equation as
\[ \dot{x}^{AA'} = \frac{e}{\rho^\tau} o^A \bar{o}^{A'} - \frac{\rho^\sigma}{\rho^\tau} \dot{x}^{A A'}. \] (2.7)
Taking into account the second equation in (2.5) we obtain
\[ \dot{x}^{AA'} o^{A} \bar{o}^{A'} = 0. \] (2.8)
This equation yields the representation for the spin-tensor \( \dot{x}^{AA'} \) in the form
\[ \dot{x}^{AA'} = o^A \bar{r}^{A'} + r^A \bar{o}^{A'}. \] (2.9)
where condition \( o_A r^A \neq 0 \) is imposed on the spinor field \( r^A \).

Action (2.1) is invariant with respect to the following transformations:
\[ \bar{o}^A = o^A, \quad \bar{\iota}^A = \iota^A + u o^A, \] (2.10)
and
\[ \bar{o}^A = e^u o^A, \quad \bar{\iota}^A = e^{-u} \iota^A, \quad \bar{\rho}^\mu = e^{-(\nu + \bar{\nu})} \rho^\mu \] (2.11)
with complex-valued functions \( u(\xi) \) and \( v(\xi) \). Transformations (2.10) and (2.11) are usually referred to as null rotations about the null direction corresponding to \( o^A \) and as boost-rotations, respectively. Expanding spinor field \( r^A \) in the basis \( (o^A, \iota^A) \) as \( r^A = p o^A + q \iota^A \), where functions \( p(\xi) \) and \( q(\xi) \) are complex-valued, representing \( q \) in the polar form \( q = |q| \exp(i\varphi) \) and using (2.9) we find
\[ \dot{x}^{AA'} = (p + \bar{p}) o^A \bar{o}^{A'} + |q|(e^{-i\varphi} o^A \bar{r}^{A'} + e^{i\varphi} \iota^A \bar{o}^{A'}). \] (2.12)
Carrying out successive transformations (2.10) and (2.11) with parameters \( u = |q|^{-1} \bar{p} \exp(-i\varphi) \) and \( v = -[\ln(\rho^\tau/e) + i\varphi]/2 \) we obtain
\[ \dot{x}^{AA'} = o^A \bar{o}^{A'} - \frac{\rho^\sigma}{\rho^\tau} \dot{x}^{A A'}, \] \[ \dot{x}^{A A'} = |q|(o^A \bar{r}^{A'} + \iota^A \bar{o}^{A'}). \] (2.13)

\[^1\text{Representations (2.7) and (2.9) imply } \dot{x}^2 = (\rho^\sigma/\rho^\tau)^2 \dot{x}^2 \text{ and } \dot{x} \dot{x} = -(\rho^\sigma/\rho^\tau) \dot{x}^2. \text{ Hence, the determinant of the induced metric on the null string worldsheet vanishes identically } (\dot{x}^2 \dot{x}^2 - (\dot{x} \dot{x})^2 = 0). \text{ This verifies that the action principle (2.1) provides a description of a null string worldsheet.}\]
Here we have not put any tildes over the basis spinors, because motion equations (2.5) are invariant with respect to transformations (2.10) and (2.11). Using the possibility of rescaling the dyad element

\[ \tilde{\iota}^A = \lambda \iota^A, \quad \tilde{\chi} = \lambda \chi, \]

we incorporate factor \( |q| \) into \( \iota^A \) and write (2.13) in the form

\[ \dot{x}^{AA'} = o^A \bar{o}^{A'} - \frac{\rho^A}{\rho} \bar{\chi}^{AA'}, \quad \dot{x}^{AA'} = o^A \bar{\iota}^{A'} + \iota^A \bar{o}^{A'}. \]

Using (2.15) we represent the last equation in (2.5) as

\[ \partial_\mu (\rho^\mu o_A \bar{o}_{A'}) = i \kappa (\chi Q_{AB} \bar{o}^{B'} \bar{o}_{A'} - \bar{\chi} o_A Q_{BA'} o^B), \]

where spin-tensor \( Q^{AA'} \) corresponds to vector \( Q^a = \varepsilon^{abcd} \partial_0 B_{cd} \) with \( 3! \partial_0 B_{bc} = \varepsilon_{abcd} \varepsilon^{defg} \partial_0 B_{fg} \) and we take \( \varepsilon_{0123} = -\varepsilon^{0123} = 1 \). For this duality relation we shall call \( Q^{AA'} \) the field strength of the gauge potential \( B_{ab} \).

Invariance of action (2.1) under arbitrary reparametrizations of the worldsheet implies (via the second Noether’s theorem [41,40]) that the solutions of the motion equations depend upon two arbitrary real-valued functions. We fix one of them using the condition

\[ \rho^\sigma = 0. \]

The null string equations of motion take the form

\[ \dot{x}^{AA'} = o^A \bar{o}^{A'}, \quad \dot{x}^{AA'} = o^A \bar{\iota}^{A'} + \iota^A \bar{o}^{A'}, \quad (\rho^\sigma o^A \bar{o}_{A'}) = \mathcal{F}^{AA'}, \]

where we have introduced the notation

\[ \mathcal{F}_{AA'} = i \kappa (\chi Q_{AB} \bar{o}^{B'} \bar{o}_{A'} - \bar{\chi} o_A Q_{BA'} o^B). \]

By analogy with the case of a charged particle in a background electromagnetic field we call \( \mathcal{F}_{AA'} \) the Lorentz force produced by the antisymmetric gauge potential \( B_{ab} \).

For the purposes of future analysis we write the second equation of system (2.18) as

\[ \dot{x}^{AA'} + (\ln|\rho^\sigma|) \dot{x}^{AA'} = (\rho^\sigma)^{-1} \mathcal{F}^{AA'}. \]

Representation (2.18) for \( \dot{x}^{AA'} \) and \( \dot{x}^{AA'} \) results in the Virasoro constraints

\[ \dot{x}^2 = 0 \quad \text{and} \quad \dot{x} \dot{x} = 0 \]

appearing in the standard theory of null strings.
Let us analyse null string equations of motion (2.18) – (2.20) and constraints (2.21). The presence of the Lorentz force \( F_{\mathcal{A}\mathcal{A}'} \) makes them different from that of the free null string \([11–13]\).

We start by showing that projections of \( F_{\mathcal{A}\mathcal{A}'} \) on vectors \( \dot{x}_{\mathcal{A}\mathcal{A}'} \) and \( \dot{x}_{\mathcal{A}'\mathcal{A}} \) tangent to the null string worldsheet vanish. Taking into account normalization (2.2) we find the projections of the force \( F_{\mathcal{A}\mathcal{A}'} \) onto the spin-tensor basis elements \( o_A \bar{o}_{\mathcal{A}'} \), \( o_A \bar{\iota}_{\mathcal{A}'} \) and \( \bar{o}_{\mathcal{A}} \iota_{\mathcal{A}'} \):

\[
\begin{align*}
F_{\mathcal{A}\mathcal{A}'} o_A \bar{o}_{\mathcal{A}'} &= \dot{x}_{\mathcal{A}\mathcal{A}'} F_{\mathcal{A}\mathcal{A}'} = 0, \\
F_{\mathcal{A}\mathcal{A}'} o_A \bar{\iota}_{\mathcal{A}'} &= -i\kappa |\chi|^2 (o_A \bar{\iota}_{\mathcal{A}'} - \bar{o}_{\mathcal{A}'} \iota_{\mathcal{A}'}) Q^{\mathcal{A}\mathcal{A}'}, \\
F_{\mathcal{A}\mathcal{A}'} o_A \iota_{\mathcal{A}'} &= i\kappa |\chi|^2 o_A \bar{o}_{\mathcal{A}'} Q^{\mathcal{A}\mathcal{A}'}.
\end{align*}
\] (3.1)

Therefore, we obtain

\[
(o_A \bar{\iota}_{\mathcal{A}'} + \bar{o}_{\mathcal{A}} \iota_{\mathcal{A}'}) F_{\mathcal{A}\mathcal{A}'} = \dot{x}_{\mathcal{A}\mathcal{A}'} F_{\mathcal{A}\mathcal{A}'} = 0.
\] (3.2)

This shows that the force \( F_{\mathcal{A}\mathcal{A}'} \) possesses only two non-zero projections onto the moving tetrad associated with each point of the null string worldsheet. For the sake of brevity, we define the second spacelike member of the moving tetrad by the condition

\[
k_{\mathcal{A}\mathcal{A}'} = i(o_A \bar{\iota}_{\mathcal{A}'} - \bar{o}_{\mathcal{A}'} \iota_{\mathcal{A}'}).
\] (3.3)

Evidently \( k_{\mathcal{A}\mathcal{A}'} \) is orthogonal to \( \dot{x}_{\mathcal{A}\mathcal{A}'} \) and \( \dot{x}_{\mathcal{A}'\mathcal{A}} \). Then, subtracting the third and fourth equations in (3.1), the second non-zero projection of \( F_{\mathcal{A}\mathcal{A}'} \) is given by

\[
k_{\mathcal{A}\mathcal{A}'} F_{\mathcal{A}\mathcal{A}'} = -2\kappa |\chi|^2 o_A \bar{o}_{\mathcal{A}'} Q^{\mathcal{A}\mathcal{A}'}.
\] (3.4)

Now we are in a position to find the number of physical degrees of freedom taking part in the interaction between the gauge field \( B_{ab} \) and the tensionless string. As we show below, this number is equal to one.

First, we consider the case when the field strength \( Q^{\mathcal{A}\mathcal{A}'} \) is restricted by the condition

\[
Q^{\mathcal{A}\mathcal{A}'} o_A \bar{o}_{\mathcal{A}'} = Q^{\mathcal{A}\mathcal{A}'} \dot{x}_{\mathcal{A}\mathcal{A}'} = 0.
\] (3.5)

In this case, the right-hand side of equation (3.4) vanishes and we are left with only one non-zero component of the Lorentz force, \( F_{\mathcal{A}\mathcal{A}'} o_{\mathcal{A}'\mathcal{A}} \). Taking into account the first pair of equations in system (2.18) and equation (3.3) we can write the general solution of equation (3.5) for \( Q^{\mathcal{A}\mathcal{A}'} \) as follows:

\[
Q^{\mathcal{A}\mathcal{A}'} = s \dot{x}_{\mathcal{A}\mathcal{A}'} + r_1 \dot{x}_{\mathcal{A}'\mathcal{A}} + r_2 k_{\mathcal{A}\mathcal{A}'}.
\] (3.6)

Here \( s(\tau, \sigma) \), \( r_1(\tau, \sigma) \) and \( r_2(\tau, \sigma) \) are arbitrary real-valued functions. Equation (3.6) shows that under condition (3.3) the field strength \( Q^{\mathcal{A}\mathcal{A}'} \) expands into the triple of
vectors $\dot{x}^{AA'}$, $\dot{x}^{AA'}$ and $k^{AA'}$ of the moving tetrad. At a given point on the null string worldsheet vectors $\dot{x}^{AA'}$ and $\dot{x}^{AA'}$ define the flag plane associated with the flagpole $\dot{x}^{AA'}$ [21]. We note in passing that this flag plane lies in the tangent plane of the null string worldsheet. The vector $k^{AA'}$ can be obtained from $\dot{x}^{AA'}$ by rotating the latter through a right angle about the flagpole $\dot{x}^{AA'}$. This can also be achieved as a result of performing transformations (2.11) with parameter $v = i\pi/4$. Under such transformations $o^A \mapsto \exp(i\pi/4) o^A$ and $\iota^A \mapsto \exp(-i\pi/4)\iota^A$ which means that $\dot{x}^{AA'} \mapsto k^{AA'}$. Rotations of the flag plane through arbitrary angles sweep out a null hypersurface spanned by the three mutually orthogonal vectors $\dot{x}^{AA'}$, $\dot{x}^{AA'}$ and $k^{AA'}$. The null character of this hypersurface follows immediately from the null property of vector $\dot{x}^{AA'}$ which is also normal to the hypersurface. Expansion (3.6) means that the field strength $Q^{AA'}$ restricted by condition (3.5) belongs to this hypersurface at each point of the null string worldsheet. Using the expression for $F^{AA'}\iota_A\bar{\iota}_A$ from system (3.1) together with representation (3.6) and equation (2.20) we obtain

$$\ddot{x}^{AA'}\ell_A\bar{\ell}_A = -(\rho^\tau)^{-1}|\chi|^2(\dot{\rho}^\tau - 2\kappa r_2).$$  (3.7)

We eliminate the right-hand side of this equation by fixing the second gauge condition corresponding to the remaining reparametrization symmetry [14] of the Virasoro constraints (2.21)

$$\bar{\tau} = \bar{\tau}(\tau, \sigma) \quad \text{and} \quad \bar{\sigma} = \bar{\sigma}(\sigma).$$  (3.8)

The required gauge choice is given by the following condition on the worldsheet density component $\rho^\tau$:

$$\dot{\rho}^\tau - 2\kappa r_2 = 0 \Rightarrow \rho^\tau = \rho^\tau(0, \sigma) + 2\kappa \int_0^\tau r_2 d\tau.$$  (3.9)

The null string equations of motion in the background of antisymmetric fields restricted by condition (3.5) take the form of free null string equations of motion

$$\ddot{x}^a = 0.$$  (3.10)

Therefore, external antisymmetric fields of the form (3.6) can be excluded from the interaction with the null string by fixing suitable gauge conditions. The possibility of fixing such gauge conditions corresponds to the local symmetries of the null string action functional (2.1). These local symmetries describe the reparametrization freedom of the null string worldsheet and arbitrariness (2.10) and (2.11) inherent in the choice of the dyad $o^A$, $\iota^A$. We conclude that antisymmetric background fields $B_{ab}$ obeying condition (3.3) can be treated as pure gauge fields in the present formalism.

In the general case the vector $Q^{AA'}$ is given by expansion in the basis of moving tetrad

$$Q^{AA'} = s\dot{x}^{AA'} + r_1\dot{x}^{AA'} + r_2k^{AA'} + \phi\iota^A\bar{\iota}^A.$$  (3.11)

The tetrad is produced by addition of the null vector $\iota^A\bar{\iota}^A$ to the vectors $\dot{x}^{AA'}$, $\dot{x}^{AA'}$, $k^{AA'}$. In the view of the statement proved above the components $s$, $r_1$ and $r_2$ of the
field strength $Q^{AA'}$ does not contribute to the interaction with the null string. Hence, only one component, $\phi$, of the field strength $Q^{AA'}$ is essential for the interaction and cannot be excluded by any gauge transformation. This implies that for the purposes of interaction with a null string the behaviour of antisymmetric gauge field $B_{ab}$ is classically equivalent to that of the real-valued scalar field. The scalar field $\phi$ describes single gauge-invariant physical degree of freedom which is naturally associated with the notoph – a massless particle of zero helicity [33]. Our conclusion agrees with results of Kalb and Ramond [34]. They found that tensor interactions between closed tensile strings are effectively equivalent to those of between a string and an antisymmetric background field $B_{ab}$ which has only one physical degree of freedom on its mass-shell. This conclusion was derived in the framework of Wheeler-Feynman action-at-a-distance approach [13] generalized to the case of strings. Under such an approach physical fields appear as secondary effective variables composed of worldline or worldsheet coordinates of the interacting particles and strings. In spite of this fact Kalb and Ramond had proven that their results are equivalent to the results of the field theory for the field $B_{ab}$.

In this connection let us briefly recall that the theory of an antisymmetric gauge field $B_{ab}$ is characterized by the following gauge symmetry:

$$B'_{ab} = B_{ab} + \partial_a \Lambda_b - \partial_b \Lambda_a. \quad (3.12)$$

This is also one of the symmetries of the interacting null string action (2.1). The equation of motion for $B_{ab}$ is

$$\partial^a \tilde{F}_a = 0, \quad (3.13)$$

where $\tilde{F}_a = \varepsilon^{abcd} \partial_c B_{db}$. Defining a real-valued scalar field $\tilde{\phi}$ by the relation $\tilde{F}_a = \partial_a \tilde{\phi}$ we can represent (3.13) in the form

$$\Box \tilde{\phi} = 0. \quad (3.14)$$

Thus, we see that antisymmetric field $B_{ab}$ has only one gauge-invariant degree of freedom which can be described by a real-valued massless scalar field.

Comparison of this description with the picture of the null string interaction considered above shows that we can identify the vector $Q^a$ and the corresponding real-valued scalar $\phi$ with $\tilde{F}_a$ and $\tilde{\phi}$, respectively. This identification is possible because of the fact that null string equations of motion (2.18) together with the remaining reparametrization symmetry (3.8) play a similar role to that of equation (3.13) in the theory of propagating antisymmetric field $B_{ab}$. In our analysis we do not derive the Klein-Gordon equation for the field $\phi$. It was, however, shown

\footnote{It is interesting to observe at this point that compatibility of the action-at-a-distance approach with global supersymmetry in Minkowski spacetime has been recently shown in [45] (see also [44]).}
in [13] that wave equations for Yang-Mills and other types of background fields which belong to the string spectrum appear as conditions ensuring the absence of a conformal anomaly on the quantum level. Since the quantum field theory of tensionless (super)strings and (super)membranes is free of conformal anomalies [20], the Klein-Gordon equation for the field $\phi$ must appear in the quantum picture as a consequence of this result.

IV. ANALYSIS OF EQUATIONS OF MOTION

Having established the number of the physical degrees of freedom for the anti-symmetric field interacting with the null string we can now continue the analysis of the null string motion equations (2.18).

Firstly, the representations of $\dot{x}^{AA'}$ and $\dot{x}^{A'A}$ must obey some compatibility conditions. Since $(\dot{x}^{AA'})$ is equal to $(\dot{x}^{A'A})$ the spinors $o^A$ and $\iota^A$ satisfy the following relation:

$$\dot{o}^A o^{A'} + o^A \dot{o}^{A'} = o^A \dot{\iota}^{A'} + \dot{o}^A \iota^{A'} + i^A \dot{o}^{A'} = 0.$$ (4.1)

Multiplying both sides of equation (4.1) by $o_A \bar{o}_{A'}$ we get

$$\bar{\chi} \dot{o}^A o_A + \chi \dot{o}^{A'} \bar{o}_{A'} = 0,$$ (4.2)

which yields

$$\dot{o}^A o_A + i\chi \psi = 0,$$ (4.3)

where $\psi(\xi)$ is an arbitrary real-valued function. Allowing for (2.2), one can write

$$\dot{o}^A = i\omega o^A - i\psi \iota^A.$$ (4.4)

Here $\omega(\xi)$ is an arbitrary complex-valued function and the multiplier $i$ is introduced for future convenience. Substituting (4.4) into (4.1) and projecting the resulted equation on $o_A \iota_{A'}$ we find

$$(\dot{\iota}^A - i\dot{\omega} \iota^A - \dot{o}^A) o_A = 0$$ (4.5)

and it follows that

$$\dot{\iota}^A - i\dot{\omega} \iota^A - \dot{o}^A = \dot{\mu} o^A.$$ (4.6)

Substituting again (4.4) and (1.6) into (1.1) we obtain

$$\dot{\mu} = i\dot{\mu},$$ (4.7)

where $\dot{\mu}(\xi)$ is an arbitrary real-valued function. Thus, compatibility condition (1.1) results in the pair of equations for the basis spinors $o^A$ and $\iota^A$. 

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\[
\dot{\mathbf{A}} = i\mathbf{w}_\mathbf{A} - i\psi_\mathbf{A},
\]
\[
i^\mathbf{A} = i\bar{\mu}_\mathbf{A} + i\bar{\omega}_\mathbf{A} + \dot{\alpha}_\mathbf{A}.
\] (4.8)

Secondly, we find that projection of both sides of the last equation in (2.18), or equivalently, of equation (2.20) on the flagpole direction \(o_\mathbf{A}\bar{0}_\mathbf{A}\) vanishes and its projection onto \(\bar{\tau}_\mathbf{A}\) gives equation (1.2) by virtue of using (3.2). Projection of (2.20) onto the remaining members of the tetrad, \(k^\mathbf{A}\) and \(\iota^\mathbf{A}\) gives
\[
\bar{\chi}\dot{\mathbf{A}} + \chi_\mathbf{A}\bar{\tau} = 0,
\]
\[
\bar{\chi}\hat{A}^\mathbf{A} - \chi_\mathbf{A}\bar{\tau}^\mathbf{A} = -2\kappa|\chi|^4(\rho_{\text{fix}})^{-1}\phi,
\] (4.9)

where we took into account equations (3.1) – (3.4) and gauge condition (3.9). Using representation (4.8) for \(\dot{\mathbf{A}}\) in system (4.9) we derive that \(\omega(\xi)\) is a real-valued function and \(\psi(\xi)\) is given by
\[
\psi = -\kappa|\chi|^2(\rho_{\text{fix}})^{-1}\phi.
\] (4.10)

We observe that in this approach the null string motion equations (2.20) result in simple algebraic conditions which reduce arbitrariness in the definition of the component \(\omega\) in the dyad decomposition of the spinor fields \(\dot{\mathbf{A}}\) and \(i^\mathbf{A}\) and identify the component \(\psi\) with the notoph field \(\phi\). These components define the dynamics of the dyad moving frame on the null string worldsheet. The same situation arises in the Lund-Regge geometric approach to the dynamics of strings as embedded surfaces in spacetime [35–39,42]. In this approach the string equations of motion are transformed into algebraic relations for the first and second fundamental forms of the string worldsheet, and the integrability conditions for the moving frames associated with the worldsheet start playing a main dynamical role. In our case the integrability conditions are given by equations (4.8). Allowing for reality of \(\omega\) and the interpretation of \(\psi\) given above we can re-express them as
\[
\dot{\mathbf{A}} = i\omega_\mathbf{A} - i\psi_\mathbf{A},
\]
\[
i^\mathbf{A} = i\bar{\mu}_\mathbf{A} + i\bar{\omega}_\mathbf{A} + \dot{\alpha}_\mathbf{A}.
\] (4.11)

From geometric point of view these equations are similar to the well known Maurer-Cartan equations for the special case of isotropic 2-surfaces embedded into 4D Minkowski spacetime in the background of antisymmetric fields. The formulation of the Lund-Regge approach in terms of Cartan’s theory of moving frames was considered in [37,39], where a connection between strings and 2D Yang-Mills-Higgs theories was also established. According to this gauge formulation the notoph field finds interpretation as some additional component of the worldsheet Yang-Mills-Higgs field.

We can further simplify equations (4.11) by choosing spinor variables \(u^\mathbf{A}(\tau,\sigma)\) and \(v^\mathbf{A}(\tau,\sigma)\)
\[
\bar{u}^\mathbf{A} = e^{i\omega}u^\mathbf{A}, \quad v^\mathbf{A} = e^{i\omega}v^\mathbf{A},
\] (4.12)

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where the scalar products of new spinors are
\[ u_A v^A = e^{-2i\omega \chi}, \quad \bar{u}_{A'} \bar{v}^{A'} = e^{2i\omega \bar{\chi}}. \] (4.13)

Using these spinors we rewrite equations (4.11) in the form
\[ \dot{u}^A = -i\psi v^A, \]
\[ \dot{v}^A = i\mu u^A + \dot{u}^A, \] (4.14)

where \( \mu = \tilde{\mu} + \dot{\omega} \) is a real-valued function of \( \tau \) and \( \sigma \). If we re-express \( v^A \) through \( \dot{u}^A \) using the first equation in system (4.14) and substitute the result into the second equation then we obtain a homogeneous second-order partial differential equation for the basis spinor \( u^A \)
\[ \ddot{u}_A - (\ln|\psi|) \dot{u}_A + i\psi \dot{u}_A - \mu \psi u_A = 0. \] (4.15)

Solutions of this equation define the second basis spinor \( v^A \) via the first equation in system (4.14). Equation (4.15) must be accompanied by corresponding initial data and appropriate periodicity conditions in the case of closed strings. Introducing real \( \zeta_A \) and imaginary \( \eta_A \) parts of the spinor field \( u_A \)
\[ u_A = \zeta_A + i\eta_A \] (4.16)

together with linear differential operators \( L_\tau \) and \( L_\sigma \)
\[ L_\tau = \partial_\tau^2 - (\ln|\psi|) \partial_\tau - \mu, \quad L_\sigma = \psi \partial_\sigma \] (4.17)

allows to write the null string equations of motion in the matrix form,
\[ \begin{pmatrix} L_\tau & -L_\sigma \\ L_\sigma & L_\tau \end{pmatrix} \begin{pmatrix} \zeta_A \\ \eta_A \end{pmatrix} = 0. \] (4.18)

Equation (4.15), or its matrix representation (4.18), is final equation which completely determines evolution of the tensionless string in the background of antisymmetric fields. Solutions of these equations will be studied in another paper.

V. CONCLUSION

We studied the dynamics of the null string embedded in an arbitrary background of antisymmetric fields and formulated a condition for the absence of interaction with such fields. According to this condition the null string identifies the field strength of the antisymmetric field as the notoph – a massless particle of zero helicity. We show that non-linear equations of motion of the null string in background antisymmetric fields can be reduced to algebraic constraints and integrability conditions for 2-component spinor representation of \( \dot{x}^{AA'} \) and \( \dot{x}^{AA''} \), and the integrability conditions play the role of dynamical equations. The total effect of the null string interaction with such fields is contained in a single derivation coefficient of the decomposition.
of $\dot{o}^A$ into the dyad basis. This derivation coefficient is given by projection $\dot{o}^A$ on the dyad basis spinor $o_A$, and we interpret this coefficient as the notoph field. In the case of free null string this derivation coefficient vanishes. Therefore, we can detect a notoph field with the aid of a tensionless string. Under the gauge field theory interpretation of the Lund-Regge geometric approach suggested in [37,38] these derivation coefficients play the role of the components of the 2D Yang-Mills-Higgs field associated with the holonomy group of the string worldsheet. Thus from the point of view of 2D worldsheet physics the appearance of the notoph field means the presence of an additional component of the 2D Yang-Mills field associated with the null string embedded into the background of antisymmetric fields.

We also find that equations of motion defining the dynamics of the null string can be transformed into a linear homogeneous system of second-order partial differential equations. Hence, 2-component spinor formalism provides a natural geometric framework for reduction of non-linear null string equations of motion in the background of antisymmetric fields to linear ones. At this point, it would be interesting to consider generalizations of these results to the case of tensionless p(D)-branes propagating in external antisymmetric fields of higher rank in higher dimensions of spacetime.

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