RELAXATION OF OPTIMAL CONTROL PROBLEMS DRIVEN BY NONLINEAR EVOLUTION EQUATIONS

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Dedicated to Professor Meir Shillor
on the occasion of his 70th birthday

Abstract. We consider a nonlinear optimal control problem with dynamics described by a nonlinear evolution equation defined on an evolution triple of spaces. Both the dynamics and the cost functional are not convex and so an optimal pair need not exist. For this reason using tools from multivalued analysis and from convex analysis, we introduce a relaxed version of the problem. No Young measures are involved in our relaxation method. We show that the relaxed problem is admissible.

1. Introduction. In this paper we study the following infinite dimensional optimal control problem governed by a nonlinear evolution equation defined on an evolution triple of spaces (see Section 2):

\[
\begin{cases}
\int_0^b L(t, x(t), u(t)) \, dt \rightarrow \inf = m, \\
\text{subject to} \\
x'(t) + A(t, x(t)) = G(t, x(t))u(t) \text{ for a.a. } t \in T = [0, b], \\
x(0) = x_0, \ u(t) \in U(t, x(t)) \text{ for a.a. } t \in T.
\end{cases}
\]  

(1.1)

It is well known that in order to guarantee existence of optimal state-control pairs for problem (1.1), we need to have some convex structure in the dynamics and the cost functional of problem (1.1). We infer to Ahmed-Teo [1], Cesari [4], Fattorini [7, 8, 9], Hou [16], Liu-Liu-Fu [21], Papageorgiou [22, 23] for such existence results. When this convex structure is not available, a minimizing sequence of admissible state-control pairs need not converge to an admissible pair. To overcome this difficulty, we need to enlarge the system in such a way, that the cost functional and the dynamics of the augmented system capture the asymptotic behaviour of the minimizing sequence of the original system. The new enlarged problem is known as the “relaxed problem” and it has to satisfy the following basic three requirements:
(a) Every admissible state-control pair of the original system is admissible for the relaxed system, that is, the original system is embedded in the relaxed one.

(b) Every relaxed state can be obtained as the uniform limit of a sequence of original states. This requirement guarantees that we did not augment the original dynamics too much.

(c) The value of the two problems are equal and the relaxed problem admits optimal pairs.

If these basic requirements are satisfied, then we say that the relaxation method is "admissible". The relaxation method is not unique. For finite dimensional systems various admissible relaxations and their relations can be found in Papageorgiou-Papalini [26] and Papageorgiou-Vetro-Vetro [33]. For infinite dimensional systems (distributed parameter systems), we refer to Buttazzo [2], Buttazzo-Dal Maso [3], Fattorini [10, 11], Papageorgiou [24, 25], Roubíček [34]. Here we develop a relaxation method which does not use transition measures (Young measures) and accommodates a large class of infinite dimensional optimal control problems.

Our work here can be extended to systems monitored by different types of evolution equations. We refer to the works of Kreulich [19] (equations driven by dissipative operators using the theory of nonlinear semigroups) and of Papageorgiou-Rădulescu [27], Papageorgiou-Rădulescu-Repovš [28, 29, 30, 31, 32] (evolutions driven by subdifferential operators, second order evolutions with viscosity, implicit evolutions).

2. Mathematical background - hypotheses. We start with some basic definitions and facts from multivalued analysis which we will need in the sequel. For details we refer to Hu-Papageorgiou [17] and Kuttler-Li-Shillor [20].

Let $(\Omega, \Sigma)$ be a measurable space and $X$ a separable Banach space. We introduce the following families of subsets of $X$:

- $P_{f(c)}(X) = \{ E \subseteq X : E \text{ is nonempty, closed (and convex)} \}$,
- $P_{(w)k(c)}(X) = \{ E \subseteq X : E \text{ is nonempty, } w\text{-compact (and convex)} \}$.

A multifunction $F: \Omega \to P_f(X)$ is said to be "measurable", if for all $x \in X$, the map

$$\omega \mapsto d(x, F(\omega)) = \inf_{u \in F(\omega)} \|x - u\|_X$$

is $\Sigma$-measurable. A multifunction $G: \Omega \to 2^X \setminus \{\emptyset\}$ is said to be "graph measurable", if

$$\text{Gr } G = \{(\omega, u) \in \Omega \times X : u \in G(\omega)\} \in \Sigma \otimes B(X),$$

with $B(X)$ being the Borel $\sigma$-field of $X$. For multifunctions with values in $P_f(X)$, measurability implies graph measurability and the converse is true if there is a $\sigma$-finite measure $\mu$ on $\Sigma$ and $\Sigma$ is $\mu$-complete. Now suppose that $(\Omega, \Sigma, \mu)$ is a finite measure space, $F: \Omega \to 2^X \setminus \{\emptyset\}$ is a multifunction and $1 \leq p \leq +\infty$. We introduce the following set

$$S^p_F = \{ f \in L^p(\Omega; X) : f(\omega) \in F(\omega) \text{ } \mu\text{-a.e. on } \Omega \}.$$

This set may be empty. A straightforward application of the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [17, Theorem 2.14, p. 158] or Gasiński-Papageorgiou [12, Theorem 1.2.3, p. 23]) shows that for a graph measurable multifunction $F$, $S^p_F \neq \emptyset$ if and only if the map

$$\omega \mapsto \inf_{u \in F(\omega)} \|u\|_X$$

is in $L^p(\Omega)$. 

On $P_f(X)$ we can define a generalized metric, known as the “Hausdorff metric” by

$$h(E, C) = \max \left\{ \sup_{e \in E} d(e, C), \sup_{c \in C} d(c, E) \right\}. $$

We know that $(P_f(X), h)$ is a complete metric space and $P_{f\epsilon}(X)$ is a closed subspace of it (see Gasiński-Papageorgiou [14, Problem 1.178, p. 62]). If $Y$ is a Hausdorff topological space and $G : Y \rightarrow P_f(X)$, then we say that $G$ is “Hausdorff continuous”, if it is continuous from $Y$ into the metric space $(P_f(X), h)$. Suppose that $Y$, $Z$ are Hausdorff topological spaces. A multifunction $G : Y \rightarrow 2^Z \setminus \{\emptyset\}$ is said to be “upper semicontinuous”, if for all $C \subseteq Z$ closed, the set $G^{-1}(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$ is closed. We say that $G$ is closed if $\text{Gr} G = \{(y, z) \in Y \times Z : z \in G(y)\}$ is closed. If $Y$ is regular and $G$ is upper semicontinuous, then $G$ is closed. For the converse to be true we need for $G$ to have closed values and to be locally compact, that is, every $y \in Y$ has a neighbourhood $U$ such that

$$\overline{G(U)} = \bigcup_{v \in U} G(v) \text{ is compact in } Z.$$

If $Y$ is a Banach space and $E \subseteq Y$, $E \neq \emptyset$, then

$$|E| = \sup_{u \in E} \|u\|_Y.$$

Let $T = [0, b]$. By $L^1_w(T; X)$ we denote the Lebesgue-Bochner space $L^1(T; X)$ equipped with the weak norm

$$\|f\|_w = \sup_{0 \leq t \leq b} \left\| \int_t^b f(s) \, ds \right\|_X.$$

Equivalently we can define the weak norm as

$$\|f\|_w = \sup_{0 \leq t \leq b} \left\| \int_0^t f(s) \, ds \right\|_X$$

(see Gasiński-Papageorgiou [15, p. 234]). This norm is equivalent to the Pettis norm (see Egghe [5]). From Lemma 2.8 of Hu-Papageorgiou [18, p. 24], we have the following result.

**Lemma 2.1.** If $X$ is a separable reflexive Banach space, $\{f_n\}_{n \geq 1} \subseteq L^p(\Omega; X)$ is a bounded sequence, with $1 < p < +\infty$ and $f_n \xrightarrow{w} f$, then $f_n \xrightarrow{w} f$ in $L^p(\Omega; X)$.

Now let $H$ be a separable Hilbert space and $X$ a separable, reflexive Banach space which is embedded continuously and densely into $H$. We identify $H$ with its dual (that is, $H = H^*$ by the Riesz-Fréchet theorem). Then $H$ is embedded continuously and densely into $X^*$ (see e.g., Gasiński-Papageorgiou [13, Lemma 2.2.27, p. 141]). Then the triple of spaces $(X, H, X^*)$ is known in the literature as an “evolution triple”. In this work we will also assume that $X \subseteq H$ compactly. Then by Schauder’s theorem (see e.g., Gasiński-Papageorgiou [13, Theorem 3.1.22, p. 275]) we also have that $X \subseteq X^*$ compactly. By $|\cdot|$ (respectively $\|\cdot\|$, $\|\cdot\|_*$) we denote the norm of $H$ (respectively of $X$, $X^*$). Also by $(\cdot, \cdot)$ we denote the inner product of $H$ and by $(\cdot, \cdot)$ the duality brackets for the dual pair $(X, X^*)$. We know that $(\cdot, \cdot)_{X \times H} = (\cdot, \cdot)$. We define

$$W_p(0, b) = \{u \in L^p(T; X) : u' \in L^{p'}(T; X^*)\},$$
Lemma 2.2. \cite[Proposition 2.2.34, p. 148]{13}.

with \( \frac{1}{p} + \frac{1}{p'} = 1 \). The time derivative involved in this definition is understood in the sense of vector valued distributions (see \cite{13}). The space \( W^p_0(0, b) \) is equipped with the norm

\[
\|u\|_{W^p_0} = \|u\|_{L^p(T; X)} + \|u'\|_{L^{p'}(T; X')}.
\]

Furnished with this norm, \( W^p_0(0, b) \) becomes a separable, reflexive Banach space. We know (see e.g., Gasiński-Papageorgiou \cite{13}), that

\[
W_0^p(0, b) \subseteq C(T; H) \quad \text{continuously and densely,}
\]

\[
W^p_0(0, b) \subseteq L^p(T; H) \quad \text{compactly.}
\]

Also we have the integration by parts formula (see e.g., Gasiński-Papageorgiou \cite[Proposition 2.2.34, p. 148]{13}).

**Lemma 2.2.** If \((X, H, X^*)\) is an evolution triple, \(1 < p < +\infty\) and \(u, v \in W^p_0(0, b)\), then

\[
(u(t), v(t)) - (u(s), v(s)) = \int_s^t \left( \langle u'\tau, v(\tau) \rangle + \langle u(\tau), v'(\tau) \rangle \right) \, d\tau,
\]

for all \(0 \leq s \leq t \leq b\).

We recall that for every \(r \in [1, +\infty)\) and a separable reflexive Banach space \(V\), we have \(L^r(T; V) = L^r(T; V^*)\), with \(\frac{1}{r} + \frac{1}{r'} = 1\). By \(\langle \cdot, \cdot \rangle_V\), we denote the duality brackets for the pair \((L^p([0, t]; X), L^{p'}([0, t]; X^*))\) and if \(t = b\), then we simply write \((\cdot, \cdot)_V\). So, if \(h^* \in L^p(T; V^*)\) and \(h \in L^p(T; V)\), then

\[
\langle h^*, h \rangle V = \int_0^b \langle h^*(s), h(s) \rangle_V \, ds,
\]

with \(\langle \cdot, \cdot \rangle_V\) being the duality brackets for the pair \((V, V^*)\). By \(V^*_w\) (respectively \(V_w\)) we denote the reflexive Banach space \(V^*\) (respectively \(V\)) furnished with the weak topology. A map \(A: V \to V^*\) is said to be “hemicontinuous”, if for all \(u, h \in V\), the map \(\lambda \mapsto A(u + \lambda h)\) is continuous form \([0, 1]\) into \(V^*_w\). We say that \(A\) is “monotone”, if

\[
\langle A(u) - A(v), u - v \rangle_V \geq 0 \quad \forall u, v \in V.
\]

We know that a hemicontinuous, monotone map is “maximal monotone”, that is

if \(\langle A(u) - v^*, u - v \rangle_V \geq 0\) for all \(u \in V\), then \((v, v^*) \in \text{Gr } A\).

A maximal monotone map \(A\) has a graph which is closed in \(V \times V^*_w\) and in \(V_w \times V^*\). Moreover, a maximal monotone map which is also coercive, that is, \(\|A(u)\|_{V^*} \to +\infty\) as \(\|u\|_V \to +\infty\), is surjective. For details on these and related issues, we refer to Gasiński-Papageorgiou \cite{13} and Hu-Papageorgiou \cite{17}.

Now we introduce our conditions on the data of problem \((1.1)\). So we are working on an evolution triple \((X, H, X^*)\) in which the embedding \(X \subseteq H\) is compact. In particular, there exists \(\beta > 0\) such that

\[
|x| \leq \beta \|x\| \quad \forall x \in X.
\]

Also, \(Y\) is a separable reflexive Banach space and represents the control space.

**H(A):** \(A: T \times X \to X^*\) is a map such that:

(i): for all \(x \in X\), \(t \mapsto A(t, x)\) is measurable;

(ii): for a.a. \(t \in T\), \(x \mapsto A(t, x)\) is hemicontinuous, monotone;

(iii): there exist \(a_1 \in L^p(T)\), \(c_1 > 0\) and \(2 \leq p < +\infty\) such that

\[
\|A(t, x)\| \leq a_1(t) + c_1\|x\|^{p-1} \quad \text{for a.a. } t \in T, \forall x \in X.
\]
(iv): there exists $c_0 > 0$ such that
\[ (A(t, x), x) \geq c_0 \| x \|^p \] for a.a. $t \in T$, all $x \in X$.

**Remark 2.3.** Evidently for a.a. $t \in T$, $A(t, \cdot)$ is maximal monotone, coercive.

$H(G): G : T \times H \longrightarrow \mathcal{L}(Y; H)$ is a map such that:

(i): for all $x \in H$ and $u \in Y$, $t \mapsto G(t, x)u$ is measurable;

(ii): for every $\eta > 0$, there exists $k_\eta \in L^1(T)$ such that
\[ \| G(t, x) - G(t, y) \|_\mathcal{L} \leq k_\eta(t) |x - y| \] for a.a. $t \in T$, all $x, y \in H$, $|x|, |y| \leq \eta$;

(iii): there exist $a \in L^p(T)$ and $c > 0$ such that
\[ \| G(t, x) \|_\mathcal{L} \leq a(t) + c|x|^{\frac{p}{p-1}} \] for a.a. $t \in T$, all $x \in H$.

If $p = 2$ we assume additionally that $2c\beta < c_0$ (see (2.1) and hypothesis $H(A)(iv)$).

For the control constraint multifunction we assume:

$H(U): U : T \times H \longrightarrow P_{wk}(Y)$ is a multifunction such that:

(i): for all $x \in H$, $t \mapsto U(t, x)$ is measurable;

(ii): there exists $\vartheta \in L^\infty(T)$ such that
\[ h(U(t, x), U(t, y)) \leq \vartheta(t) |x - y| \] for a.a. $t \in T$, all $x, y \in H$;

(iii): there exists $a_0 \in L^\infty(T)$ such that
\[ |U(t, x)| \leq a_0(t) \] for a.a. $t \in T$, all $x \in \mathbb{R}$.

**Remark 2.4.** So the control constraint multifunction is state dependent and thus the system has a priori feedback.

Above we have introduced the hypotheses on the items which determine the dynamics of the system. Next we introduce the conditions on the cost integrand $L(t, x, u)$.

$H(L): L : T \times H \times Y \longrightarrow \mathbb{R}$ is a function such that:

(i): for all $(x, u) \in H \times Y$, $t \mapsto L(t, x, u)$ is measurable;

(ii): for every $\eta > 0$, there exists $\beta_\eta \in L^1(T)$ such that
\[ |L(t, x, u) - L(t, y, v)| \leq \beta_\eta(t) (|x - y| + \| u - v \|_Y) \] for a.a. $t \in T$, all $x, y \in H$, $u, v \in Y$, $|x|, |y|, \| u \|_Y, \| v \|_Y \leq \eta$;

(iii): for every $\eta > 0$, there exists $a_\eta \in L^1(T)$ such that
\[ |L(t, x, u)| \leq a_\eta(t) \] for a.a. $t \in T$, all $x \in H$, $u \in Y$, $|x|, \| u \|_Y \leq \eta$.

$H_0: x_0 \in H$.

We consider also the “convexified” control system, that is, the control system we get by convexifying the dynamics of (1.1).

\[
\begin{aligned}
 & x'(t) + A(t, x(t)) = G(t, x(t))u(t) \quad \text{for a.a. } t \in T, \\
 & x(0) = x_0, \quad u(t) \in \overline{R(x)} U(t, x(t)) \quad \text{for a.a. } t \in T, \quad u \in L^\infty(T; Y).
\end{aligned}
\]  

(2.2)

We introduce the following two sets:

\[
\begin{align*}
P &= \{ (x, u) \in W_p(0, b) \times L^1(T; Y) : \text{ (x, u) is admissible for (1.1)} \}, \\
P_c &= \{ (x, u) \in W_p(0, b) \times L^1(T; Y) : \text{ (x, u) is admissible for (2.2)} \}.
\end{align*}
\]
We set
\[ \mathcal{Y} = \text{proj}_{W_p(0,b)} P \quad \text{and} \quad \mathcal{Y}_c = \text{proj}_{W_p(0,b)} P_c. \]
Evidently \( \mathcal{Y} \) is the set of states of the original system, while \( \mathcal{Y}_c \) is the set of states of the relaxed (convexified) system. Clearly \( \mathcal{Y} \subseteq \mathcal{Y}_c \).

3. **Admissible relaxation.** In this section we construct an admissible relaxation using notions and techniques from multivalued analysis and from convex analysis, avoiding the use of Young measures.

We start by proving the nonemptiness of \( \mathcal{Y} \subseteq W_p(0,b) \) and also by establishing useful topological properties for \( \mathcal{Y}_c \).

**Proposition 3.1.** If hypotheses \( H(A), H(G), H(U) \) and \( H_0 \) hold, then \( \mathcal{Y} \neq \emptyset \) and \( \mathcal{Y}_c \subseteq W_p(0,b) \subseteq C(T; H) \) is \( w \)-compact in \( W_p(0,b) \) and compact in \( C(T; H) \).

**Proof.** We introduce the multifunction \( F: T \times H \to 2^H \setminus \{\emptyset\} \) defined by
\[ F(t,x) = G(t,x)U(t,x) = \bigcup_{u \in U(t,x)} G(t,x)u \in P_{wk}(H). \]

On account of hypotheses \( H(U)(i), (ii) \), the multifunction \( (t,x) \mapsto U(t,x) \) is measurable (see Hu-Papageorgiou [17, Proposition 7.9, p. 229]). So, we can find measurable functions \( u_n: T \times H \to Y, n \in \mathbb{N} \) such that
\[ U(t,x) = \overline{\{u_n(t,x)\}}_{n \geq 1} \]
(see Hu-Papageorgiou [17, Proposition 2.3, p. 155]). Therefore, we have
\[ F(t,x) = G(t,x)\overline{\{u_n(t,x)\}}_{n \geq 1} = \{G(t,x)u_n(t,x)\}_{n \geq 1}. \]

Hypotheses \( H(G)(i), (ii) \) imply that the map \( (t,x) \mapsto G(t,x)u_n(t,x) \) is jointly measurable (for all \( n \in \mathbb{N} \)) and so we infer that
\[ (t,x) \mapsto F(t,x) \] is measurable. \([3.1]\)

Let \( \eta > 0 \) and consider \( x,y \in H \) with \( |x|, |y| \leq \eta \). Let \( h \in F(t,x) \). Then
\[ h = G(t,x)u \quad \text{with some} \quad u \in U(t,x). \]

Given \( \varepsilon > 0 \), we choose \( v \in U(t,y) \) such that
\[ \|u-v\|_Y \leq d(u,U(t,y)) + \varepsilon \leq h(U(t,x),U(t,y)) + \varepsilon \leq d(t)|x-y| + \varepsilon \quad \text{for a.a.} \quad t \in T \]
(see hypothesis \( H(U)(ii) \)). Then we have
\[ d_H(h,F(t,y)) = d_H(G(t,x)u,F(t,y)) \]
\[ \leq |G(t,x)u - G(t,y)v| \]
\[ \leq \|G(t,x)\|_{L^p} \|u-v\|_Y + \|G(t,x) - G(t,y)\|_{L^p} \|v\|_Y \]
\[ \leq a_\eta(t)\|u-v\|_Y + k_\eta(t)\|u_0\|_{\infty} |x-y| \]
\[ \leq \xi_\eta(t)(|x-y| + \varepsilon) \quad \text{for a.a.} \quad t \in T, \]

with \( a_\eta(t) = a(t)(1 + \eta^{\beta_\eta}) \) (see hypotheses \( H(G)(ii), H(U)(iii) \)) and \( \xi_\eta \in L^p(T) \) (see \([3.2]\)), so
\[ h(F(t,x),F(t,y)) \leq \xi_\eta(t)(|x-y| + \varepsilon) \quad \text{for a.a.} \quad t \in T. \]

Since \( \varepsilon > 0 \) is arbitrary, we let \( \varepsilon \to 0^+ \) to conclude that
\[ h(F(t,x),F(t,y)) \leq \xi_\eta(t)|x-y| \]
for a.a. \( t \in T \), all \( x, y \in H \), \(|x|, |y| \leq \eta \), \( \xi_{\eta} \in L^{p'(T)} \), so
\[
F(t, \cdot) \text{ is locally } h\text{-Lipschitz.} \tag{3.3}
\]

We consider the following evolution inclusion
\[
\begin{cases}
    x'(t) + A(t, x(t)) \in F(t, x(t)) \text{ for a.a. } t \in T, \\
    x(0) = x_0,
\end{cases}
\tag{3.4}
\]

We show that (2.2) and (3.4) are equivalent. Evidently, if \( x \in \mathcal{Y} \), then \( x \) also solves (3.4). On the other hand, let \( S(x_0) \) be the solution set of the multivalued Cauchy problem (3.4). If \( x \in S(x_0) \), then
\[
\begin{cases}
    x'(t) + A(t, x(t)) = f(t) \text{ for a.a. } t \in T, \\
    x(0) = x_0,
\end{cases}
\]
with \( f \in S^{p'}_{F',(x(\cdot))} \) (on account of (3.1), \( t \mapsto F(t, x(t)) \) is measurable). We consider the multifunction \( L: T \rightarrow 2^{\mathcal{Y}} \setminus \emptyset \) defined by
\[
L(t) = \{ u \in U(t, x(t)) : f(t) \in G(t, x(t))u \}.
\]

Clearly \( (t, u) \mapsto G(t, x(t))u \) is a Carathéodory function (that is, measurable in \( t \) and continuous in \( u \)). Therefore it is jointly measurable (see Hu-Papageorgiou [17, Proposition 1.6, p. 142] or Gasiński-Papageorgiou [14, Theorem 4.166, p. 676]). Also hypotheses \( H(U)(i), (ii) \) imply that \( \text{Gr}\ U(\cdot, x(\cdot)) \in \mathcal{L}_T \otimes B(Y) \), with \( \mathcal{L}_T \) being the Lebesgue \( \sigma \)-field of \( T \) and \( B(Y) \) is the Borel \( \sigma \)-field of \( Y \). It follows that
\[
\text{Gr } L \in \mathcal{L}_T \otimes B(Y).
\]

Applying the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [17, Theorem 2.14, p. 158] or Gasiński-Papageorgiou [12, Theorem 1.2.3, p. 23]), we can find a measurable function \( u: T \rightarrow Y \) such that \( u(t) \in L(t) \) for all \( t \in T \). Then
\[
f(t) = G(t, x(t))u(t) \quad \text{for a.a. } t \in T,
\]
so \( x \in \mathcal{Y} \) and thus \( S(x_0) \subseteq \mathcal{Y} \).

Then (3.1), (3.3) and Theorem 2.41 of Hu-Papageorgiou [18, p. 48] imply that \( S(x_0) = \mathcal{Y} \neq \emptyset \). Finally note that
\[
\text{conv } F(t, x) = G(t, x)\text{conv } U(t, x) \quad \text{for all } (t, x) \in T \times H.
\]

So, invoking Theorem 2.37 of Hu-Papageorgiou [18, p. 42], we infer that \( \mathcal{Y}_c \subseteq W_p(0, b) \) is \( w \)-compact and \( \mathcal{Y}_c \subseteq C(T, H) \) is compact.

Let
\[
\hat{c} = \sup_{x \in \mathcal{Y}_c} \|x\|_{C(T, H)} \quad \text{and} \quad \eta_0 = \hat{c} + \|a_0\|_\infty.
\]

We consider the multifunction \( \hat{\Gamma}: T \times H \rightarrow 2^{\mathcal{Y} \times \mathbb{R}} \setminus \emptyset \) defined by
\[
\hat{\Gamma}(t, x) = \{ (v, \eta) \in \mathcal{Y} \times \mathbb{R} : v \in U(t, x), \ L(t, x, v) = \eta \leq a_{\eta_0}(t) \}.
\]

**Proposition 3.2.** If hypotheses \( H(U) \) and \( H(L) \) hold, then \( \hat{\Gamma} \) is graph measurable and for all \( t \in T \), \( \hat{\Gamma}(t, \cdot) \) is closed.
so a sequence \((3.5)\). According to Lemma 8.20 of Hu-Papageorgiou \[17, p. 253\], we can find closed.

\[ \Gamma(t, x) = \{ (v, \eta) \in Y \times \mathbb{R} : d(v, U(t, x)) = 0, \ L(t, x, v) = \eta \}, \]

so

\[ \text{Gr} \hat{\Gamma}(\cdot, \cdot) \in L_T \otimes B(H \times Y \times \mathbb{R}) = L_T \otimes B(H) \otimes B(Y) \otimes B(\mathbb{R}), \]

thus \( \hat{\Gamma} \) is graph measurable.

Next suppose that \( x_n \to x \) in \( H \), \( v_n \to v \) in \( Y \), \( \eta_n \to \eta \in \mathbb{R} \) and \( (x_n, v_n, \eta_n) \in \text{Gr} \hat{\Gamma}(t, \cdot) \). We have

\[ v_n \in U(t, x_n), \quad \eta_n = L(t, x_n, v_n) \quad \text{for all } n \in \mathbb{N}, \]

so

\[ v \in U(t, x), \quad \eta = L(t, x, v) \]

(see hypotheses \( H(U)(ii) \) and \( H(L)(ii) \)), thus \( (x, v, \eta) \in \text{Gr} \hat{\Gamma}(t, \cdot) \), hence \( \hat{\Gamma}(t, \cdot) \) is closed.

We set

\[ \Gamma(t, x) = \hat{\Gamma}(t, x) \cap \{(v, \eta) \in Y \times \mathbb{R} : |\eta| \leq a_{\eta_0}(t)\}. \]

On account of Proposition 3.2, we have

\begin{itemize}
  \item \((t, x) \mapsto \Gamma(t, x)\) is graph measurable;
  \item for all \( t \in T \), \( x \mapsto \Gamma(t, x) \) is closed;
  \item \(|\Gamma(t, x)| \leq a_{\eta_0}(t)\) for a.a. \( t \in T \), all \( x \in H \).
\end{itemize}

We introduce the following modification of the cost integrand \( L \):

\[ \hat{L}(t, x, v) = \begin{cases} 
  L(t, x, v) & \text{if } v \in U(t, x), \\
  +\infty & \text{otherwise.}
\end{cases} \]

Evidently \( \hat{L} \) is measurable and for all \( t \in T \), \( \hat{L}(t, \cdot, \cdot) \) is lower semicontinuous. In what follows \( \hat{L}^{**}(t, x, u) \) denotes the second conjugate of the function \( u \mapsto \hat{L}(t, x, u) \) (see Gasiński-Papageorgiou \[13, Definition 4.4.1, p. 512\]).

**Proposition 3.3.** If hypotheses \( H(A), H(G), H(U), H(L), H_0 \) hold and \( (\hat{x}, \hat{u}) \in P_c \), then we can find \( \hat{\nu}_n \in S_{U(\cdot, \hat{x}(\cdot))}^1 \) such that

\[ \hat{\nu}_n \xrightarrow{\text{w}} \hat{u} \quad \text{in } L^1(T; Y) \quad \text{as } n \to +\infty \]

and

\[ \hat{L}(\cdot, \hat{x}(\cdot), \hat{\nu}_n(\cdot)) \xrightarrow{\text{w}} \hat{L}^{**}(\cdot, \hat{x}(\cdot), \hat{u}(\cdot)) \quad \text{as } n \to +\infty. \]

**Proof.** According to Proposition 3.2 of Ekeland-Témam \[6, Proposition 3.2, p. 16\], we have

\[ \hat{L}^{**}(t, x, v) = \min \{ \eta \in \mathbb{R} : (v, \eta) \in \text{conv} \Gamma(t, x)\}. \quad (3.5) \]

Let

\[ \hat{\xi}(t) = (\hat{u}(t), \hat{L}^{**}(t, \hat{x}(t), \hat{u}(t))). \]

Then

\[ \hat{\xi} \in S_{\text{conv} \Gamma(\cdot, \hat{x}(\cdot))}^1 \]

(see (3.5)). According to Lemma 8.20 of Hu-Papageorgiou \[17, p. 253\], we can find a sequence \( \{\hat{\xi}_n\}_{n \geq 1} \subseteq S_{\Gamma(\cdot, \hat{x}(\cdot))}^1 \) such that

\[ \hat{\xi}_n \xrightarrow{\text{w}} \hat{\xi} \quad \text{as } n \to +\infty. \]
Proposition 3.4.

We have
\[ \hat{L}_n(.):= (\hat{v}_n(.), \hat{L}(., \hat{x}(.), \hat{v}_n(.))) \]
with \( \hat{v}_n \in S^p_{U(., \hat{x}(.) )} \) for all \( n \in \mathbb{N} \). We see that
\[ \hat{u}_n \xrightarrow{\| \cdot \|_w} \hat{u} \quad \text{in} \ L^1(T; Y) \quad \text{and} \quad \hat{v}_n \xrightarrow{w} \hat{u} \quad \text{in} \ L^1(T; Y) \quad \text{as} \ n \to +\infty \]
(see Lemma 2.1) and
\[ \hat{L}(., \hat{x}(.), \hat{v}_n(.)) \xrightarrow{\| \cdot \|_w} \hat{L}^{**}(., \hat{x}(.), \hat{u}(.)) \quad \text{as} \ n \to +\infty. \]

Using this proposition, we can generate a sequence of admissible original state-control pairs which approximate \( (\hat{x}, \hat{u}) \in P_c \) and the same is also true for the corresponding costs in \( L^1_w(T) \).

**Proposition 3.4.** If hypotheses \( H(A), H(G), H(U), H(L), H_0 \) hold and \( (\hat{x}, \hat{u}) \in P_c \), then we can find a sequence \( \{(\hat{x}_n, \hat{u}_n)\}_{n \geq 1} \subseteq P \) such that
\[ \hat{x}_n \to \hat{x} \quad \text{in} \ C(T; H) \quad \text{and} \quad \hat{u}_n \xrightarrow{\| \cdot \|_w} \hat{u} \quad \text{in} \ L^1(T; Y) \]
and
\[ \hat{L}(., \hat{x}_n(.), \hat{v}_n(.)) \xrightarrow{\| \cdot \|_w} \hat{L}^{**}(., \hat{x}(.), \hat{u}(.)) \]

**Proof.** According to Proposition 3.3 we can find \( \hat{v}_n \in S^p_{U(., \hat{x}(.) )} \) for all \( n \in \mathbb{N} \) such that
\[ \hat{v}_n \xrightarrow{\| \cdot \|_w} \hat{u} \quad \text{and} \quad \hat{v}_n \xrightarrow{w} \hat{u} \quad \text{in} \ L^p(T; Y) \quad \text{as} \ n \to +\infty. \]

For every \( n \in \mathbb{N} \) we introduce the multifunction \( E_n : T \times H \to 2^Y \setminus \{\emptyset\} \) defined by
\[ E_n(t, x) = \{h \in U(t, x) : \|\hat{v}_n(t) - h\|_Y \leq \vartheta(t)|\hat{x}(t) - x| + \frac{1}{n}\}. \]

Using this multifunction as control set, we consider the following nonlinear control system
\[ \begin{align*}
 x'(t) + A(t, x(t)) &= G(t, x(t))u(t) \quad \text{for a.a. } t \in T, \\
 x(0) &= x_0, \quad u \in S^p_{E_n(., x(0))}. 
\end{align*} \tag{3.6} \]

As in the proof of Proposition 3.1, we show that we can find an admissible state-control pair \( (\hat{x}_n, \hat{u}_n), n \in \mathbb{N} \), for problem (3.6). Evidently
\[ (\hat{x}_n, \hat{u}_n) \in P \quad \forall n \in \mathbb{N}. \]

From Proposition 3.1 we know that \( \{\hat{x}_n\}_{n \geq 1} \subseteq C(T; H) \) (recall that \( W_p(0, b) \subseteq C(T; H) \) is relatively compact. So, by passing to a subsequence if necessary, we may assume that
\[ \hat{x}_n \to \hat{x}^* \quad \text{in} \ C(T; H) \quad \text{as} \ n \to +\infty. \tag{3.7} \]

We have
\[ \begin{align*}
 \hat{x}'_n(t) + A(t, \hat{x}_n(t)) &= G(t, \hat{x}_n(t))\hat{u}_n(t) \quad \text{for a.a. } t \in T, \\
 \hat{x}_n(0) &= x_0, \quad n \in \mathbb{N}, \\
 \hat{x}'(t) + A(t, \hat{x}(t)) &= G(t, \hat{x}(t))\hat{u}(t) \quad \text{for a.a. } t \in T, \\
 \hat{x}(0) &= x_0. 
\end{align*} \tag{3.8} \]

From (3.8) we have
\[ ((\hat{x}'_n - \hat{x}'(t), \hat{x}_n - \hat{x})_t + \int_0^t \langle A(s, \hat{x}_n) - A(s, \hat{x}), \hat{x}_n - \hat{x} \rangle ds \]
Moreover, we have for all $t \in T$.

$$\int_0^t (G(s, \tilde{x}_n)\tilde{u}_n - G(s, \tilde{x})\tilde{u}, \tilde{x}_n - \tilde{x}) \, ds \quad \forall t \in T. \tag{3.9}$$

We estimate the right hand side of (3.9):

$$\int_0^t (G(s, \tilde{x}_n)\tilde{u}_n - G(s, \tilde{x})\tilde{u}, \tilde{x}_n - \tilde{x}) \, ds$$

$$= \int_0^t (G(s, \tilde{x}_n)\tilde{u}_n - G(s, \tilde{x})\tilde{u}, \tilde{x}_n - \tilde{x}) \, ds$$

$$+ \int_0^t (G(s, \tilde{x}_n)\tilde{v}_n - G(s, \tilde{x})\tilde{u}, \tilde{x}_n - \tilde{x}) \, ds. \tag{3.10}$$

In the right hand side of (3.10), we examine the first summand:

$$\int_0^t (G(s, \tilde{x}_n)\tilde{u}_n - G(s, \tilde{x}_n)\tilde{v}_n, \tilde{x}_n - \tilde{x}) \, ds$$

$$= \int_0^t (\tilde{u}_n - \tilde{v}_n, G(s, \tilde{x}_n)^*(\tilde{x}_n - \tilde{x})) \, ds \tag{3.11}$$

(by $\langle \cdot, \cdot \rangle_Y$ we denote the duality brackets for the pair $(Y, Y^*)$).

From the definition of the multifunction $E_n(t, x)$, we have

$$\|\tilde{v}_n(s) - \tilde{u}_n(s)\|_Y \leq \vartheta(s) |\tilde{x}(s) - \tilde{x}_n(s)| + \frac{1}{n} \quad \forall n \in \mathbb{N}.$$  

Moreover, we have

$$\|G(s, \tilde{x}_n(s))^*\|_L = \|G(s, \tilde{x}_n(s))\|_L \leq c_2 \quad \forall n \in \mathbb{N},$$

for some $c_2 > 0$. Using these estimates in (3.11), we obtain

$$\int_0^t (G(s, \tilde{x}_n)\tilde{u}_n - G(s, \tilde{x}_n)\tilde{v}_n, \tilde{x}_n - \tilde{x}) \, ds \leq \int_0^t c_2 \vartheta(s) |\tilde{x}_n - \tilde{x}|^2 \, ds + \frac{c_3}{n} \tag{3.12}$$

for all $n \in \mathbb{N}$ and some $c_3 > 0$ (see (3.7)).

Now we examine the second summand in the right hand side of (3.10). We have

$$\int_0^t (G(s, \tilde{x}_n)\tilde{v}_n - G(s, \tilde{x})\tilde{u}, \tilde{x}_n - \tilde{x}) \, ds$$

$$= \int_0^t (G(s, \tilde{x}_n)(\tilde{v}_n - \tilde{u}), \tilde{x}_n - \tilde{x}) \, ds + \int_0^t (G(s, \tilde{x}_n) - G(s, \tilde{x}))\tilde{u}, \tilde{x}_n - \tilde{x}) \, ds$$

$$= \int_0^t \langle \tilde{v}_n - \tilde{u}, G(s, \tilde{x}_n)^*(\tilde{x}_n - \tilde{x}) \rangle_Y \, ds + \int_0^t k_\eta(s) |\tilde{x}_n - \tilde{x}|^2 \, ds \tag{3.13}$$

(see hypothesis $H(G)(ii)$). From Proposition 3.3, we have

$$\int_0^t \langle \tilde{v}_n - \tilde{u}, G(s, \tilde{x}_n)^*(\tilde{x}_n - \tilde{x}) \rangle_Y \, ds \longrightarrow 0 \quad \text{as } n \rightarrow +\infty. \tag{3.14}$$

We return to (3.10), use (3.12), (3.13), pass to the limit as $n \rightarrow +\infty$ and then exploit (3.7) and (3.14). We obtain

$$\limsup_{n \rightarrow +\infty} \int_0^t (G(s, \tilde{x}_n)\tilde{u}_n - G(s, \tilde{x})\tilde{u}, \tilde{x}_n - \tilde{x}) \, ds \leq \int_0^t \tilde{k}(s)|\tilde{x}^* - \tilde{x}|^2 \, ds,$$

for some $\tilde{k} \in L^1(T)$. 

From (3.9), Lemma 2.2 (the integration by parts formula) and the monotonicity of \( A(t, \cdot) \) (see hypothesis \( H(A)(ii) \)), in the limit as \( n \to +\infty \), we have
\[
|\hat{x}^*(t) - \bar{x}(t)|^2 \leq 2 \int_0^t \hat{k}(s)|\hat{x}^*(s) - \bar{x}(s)|^2 \, ds \quad \forall t \in T,
\]
so \( \hat{x}^* = \bar{x} \) (by Gronwall’s inequality).

So, for the original sequence we have
\[
\hat{x}_n \longrightarrow \bar{x} \quad \text{in } C(T; H) \quad \text{as } n \to +\infty. \tag{3.15}
\]
Then from the definition of the multifunction \( E_n(t, x) \), we have
\[
\|\hat{v}_n(t) - \hat{u}_n(t)\|_Y \longrightarrow 0 \quad \text{for a.a. } t \in T \quad \text{as } n \to +\infty,
\]
so
\[
\|\hat{v}_n - \hat{u}_n\|_{L^1(T; Y)} \longrightarrow 0 \quad \text{as } n \to +\infty,
\]
thus
\[
\hat{u}_n \overset{\|w\|}{\longrightarrow} \bar{u} \quad \text{in } L^1(T; Y)
\]
(see Proposition 3.3).

Recall that from Proposition 3.3, we have
\[
\hat{L}(\cdot, \hat{x}(\cdot), \hat{v}_n(\cdot)) \overset{\|w\|}{\longrightarrow} \hat{L}^{**}(\cdot, \bar{x}(\cdot), \hat{u}(\cdot)) \quad \text{as } n \to +\infty. \tag{3.16}
\]
On the other hand hypothesis \( H(L)(ii) \), the definition of the multifunction \( E_n(t, x) \) and (3.15), imply that
\[
\hat{L}(\cdot, \hat{x}_n(\cdot), \hat{u}_n(\cdot)) - \hat{L}(\cdot, \bar{x}(\cdot), \hat{u}(\cdot)) \overset{\|w\|}{\longrightarrow} 0,
\]
so
\[
\hat{L}(\cdot, \hat{x}_n(\cdot), \hat{u}_n(\cdot)) \overset{\|w\|}{\longrightarrow} \hat{L}^{**}(\cdot, \bar{x}(\cdot), \hat{u}(\cdot))
\]
(see (3.16)).

So, we introduce the following relaxation of the optimal control problem (1.1):
\[
\begin{cases}
\int_0^b \hat{L}^{**}(t, x(t), u(t)) \, dt \longrightarrow \inf = m_r, \\
\text{subject to} \\
x'(t) + A(t, x(t)) = G(t, x(t))u(t) \text{ for a.a. } t \in T, \\
x(0) = x_0, \quad u(t) \in \text{conv } U(t, x(t)) \text{ for a.a. } t \in T.
\end{cases} \tag{3.17}
\]

**Theorem 3.5.** If hypotheses \( H(A), H(G), H(U), H(L) \) and \( H_0 \) hold, then there exists \((\bar{x}, \bar{u}) \in P_r\) such that
\[
\int_0^b \hat{L}^{**}(t, \bar{x}(t), \bar{u}(t)) \, dt = m_r,
\]
with \( m = m_r \) and there exists a sequence \( \{ (\hat{x}_n, \hat{u}_n) \}_{n \geq 1} \subseteq P \) such that
\[
\hat{x}_n \longrightarrow \bar{x} \quad \text{in } C(T; H), \quad \hat{u}_n \overset{\|w\|}{\longrightarrow} \bar{u} \quad \text{weakly in } L^1(T; Y)
\]
and
\[
\int_0^b L(t, \hat{x}_n, \hat{u}_n) \, dt \longrightarrow m_r.
\]
Proof. Let \( \{(y_n, v_n)\}_{n \geq 1} \subseteq P_c \) be a minimizing sequence for the relaxed problem (3.17); that is
\[
\int_0^b \tilde{L}^{**}(t, y_n(t), v_n(t)) \, dt \leq m_r.
\]
From Proposition 3.1 and hypothesis \( H(U)(iii) \), we see that \( \{y_n\}_{n \geq 1} \subseteq C(T; H) \) is relatively compact and \( \{v_n\}_{n \geq 1} \subseteq L^1(T; Y) \) is relatively \( w \)-compact. Therefore, passing to a subsequence if necessary, we may assume that
\[
y_n \to \hat{x} \quad \text{in} \quad C(T; H) \quad \text{and} \quad v_n \overset{w}{\to} \hat{u} \quad \text{in} \quad L^1(T; Y),
\]
with \((\hat{x}, \hat{u}) \in P_c\). Since \( \tilde{L}^{**}(t, x, \cdot, \cdot) \) is convex, using Theorem 2.24 of Hu-Papageorgiou [18, p. 31], we have
\[
\int_0^b \tilde{L}^{**}(t, \hat{x}, \hat{u}) \, dt \leq \liminf_{n \to +\infty} \int_0^b \tilde{L}^{**}(t, y_n, v_n) \, dt = m_r,
\]
so
\[
\int_0^b \tilde{L}^{**}(t, \hat{x}, \hat{u}) \, dt = m_r
\]
(since \((\hat{x}, \hat{u}) \in P_c\)).

According to Proposition 3.4, we can find a sequence \( \{\hat{x}_n, \hat{u}_n\}_{n \geq 1} \subseteq P \) such that
\[
\hat{x}_n \to \hat{x} \quad \text{in} \quad C(T; H), \quad \hat{u}_n \overset{\|\cdot\|_Y}{\to} \hat{u} \quad \text{in} \quad L^1(T; Y)
\]
and
\[
L(\cdot, x_n(\cdot), u_n(\cdot)) \overset{\|\cdot\|_{Y}}{\to} \tilde{L}^{**}(\cdot, \hat{x}(\cdot), \hat{u}(\cdot)).
\]
From the last convergence, it follows that
\[
\left| \int_0^b (\tilde{L}^{**}(t, \hat{x}, \hat{u}) - L(t, \hat{x}_n, \hat{u}_n)) \, dt \right| \to 0,
\]
so \( m \leq m_r \). Since we always have \( m_r \leq m \), we conclude that \( m = m_r \). \( \square \)

Remark 3.6. The above theorem proves that the relaxation (3.17) of problem (1.1), that we have proposed, is admissible.

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