A generalization of $\lambda$-symmetry reduction for systems of ODEs: $\sigma$-symmetries

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Abstract

We consider a deformation of the prolongation operation, defined on sets of vector fields and involving a mutual interaction in the definition of prolonged ones. This maintains the 'invariants by differentiation' property, and can hence be used to reduce ODEs satisfying suitable invariance conditions in a fully algorithmic way, similarly to what happens for standard prolongations and symmetries.

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Introduction

Consider a (scalar) ODE $\mathcal{E}$ of order $n > 1$ in $J^nM$ (the jet bundle of order $n$ over the manifold $M$ of independent and dependent variables); it is well known that if this admits a (Lie-point) symmetry, then it can be reduced to an equation of order $(n - 1)$. More generally, if it admits a $p$-dimensional Lie group of symmetries $G$, then it can be reduced to an equation of order $(n - q)$, with $q \leq p$ depending on the algebraic structure of the operation of $G$ and its Lie algebra $\mathfrak{g}$ [1, 6, 13, 23, 24, 27].

The approach to symmetry reduction for ODEs goes roughly speaking as follows: if $\mathcal{E}$ admits the vector field $X$ (on $M$) as a symmetry (generator), then $\mathcal{E}$ is invariant under $Y = X^{(n)}$, the prolongation of $X$ to $J^nM$. Thus $\mathcal{E}$ can be expressed in terms of the differential invariants of $Y$. Or, this can be recursively generated starting from those of orders 0 and 1 thanks to the 'invariants by differentiation' property (see [1, 6, 13, 23, 24, 27]).

Using symmetry adapted coordinates, say $w$ and $y$ with $y$ being the independent variable, the equation will not depend explicitly on the coordinate along the vector field $X$, say $w$, and thus its order can be reduced by 1 passing to consider it as an equation for $z = w_y$ and its derivatives.
It was observed by Muriel and Romero back in 2001 that the same scheme works if $Y$ is not the (standard) prolongation of $X$ but instead some kind of ‘twisted’ prolongation, depending on an arbitrary smooth (by this we will always mean $C^\infty$) function $\lambda : J^1M \to R$ and hence called ‘$\lambda$-prolongation’ [15–17].

The key to this result is that $\lambda$-prolonged vector fields still have the property of ‘invariants by differentiation’, so the scheme working in the case of invariance under standardly prolonged vector fields can still be applied.

The same holds for systems of ODEs, except that here a symmetry will lead to reduction by one of the order of one of the equations; in the case of systems of $n$ first-order equation, a symmetry will lead to reduction to a system of $n-1$ equations. For a multi-generator symmetry group, as usual the attainable reduction will depend on the algebraic structure of the operation of the group.

It was shown by Pucci and Saccomandi [25] that for scalar ODEs, the $\lambda$-prolonged vector fields are essentially the only ones sharing the ‘invariants by differentiation’ property; they also provided a neat geometrical understanding of this fact.

In this paper we want to generalize the Muriel–Romero approach, and consider cases where the equation (or system) is invariant under a set of vector fields $Y_{(i)}$ defined on $J^pM$, and obtained from vector fields in $M$ under a further modified version of $\lambda$-prolongations. More precisely, in this case the modified prolongation operation will not act on the single vector field, but rather on the set of vector fields (that is, prolongation of each of them will involve the other ones). This ‘joint-$\lambda$’ prolongation will depend on a matrix $\sigma$ defined by a set of smooth functions $\sigma_{ij}$ on $J^1M$, and will be therefore also denoted as $\sigma$-prolongation (in the same way as $\lambda$-prolongations took their name from the function $\lambda$).

We will show that even in this case the standard approach to reduction sketched above is still valid, and hence the equation can be reduced. More precisely, we will show that the ‘invariants by differentiation’ property still holds. This will follow by explicit algebraic computation, but later on we will also discuss the geometrical meaning of our result, generalizing the result by Pucci and Saccomandi [25].

Finally, we note that the same construction with ‘joint $\lambda$-prolongation’ is considered in a companion paper [9]; however, there the assumptions, in particular on the vector fields and the functions, are different: the two papers are related, but deal with different situations and lead to different results.

1. Preliminaries and notation

We will firstly recall some basic notion, also in order to set some general notation to be used in the following.

1.1. Equations, solutions and symmetries

We will only consider ordinary differential equation(s); the independent variable will be denoted as $x \in R$, the dependent one(s) as $u \in U \equiv R$ or $u^\alpha \in U \subseteq R^p$ in the multi-dimensional case. We denote by $M = X \times U$ the phase bundle, and by $(J^kM, \pi^k, M)$, or $J^kM$ for short, the associated jet bundle of order $k$.

A differential equation $\mathcal{E}$ of order $n$ is a map $F : J^pM \to R$, and is naturally identified with the solution manifold $S(\mathcal{E}) = F^{-1}(0) \subset J^pM$. In the case of $\ell$-dimensional systems $\mathcal{E}$, we have $\ell$ maps $F^j : J^pM \to R$ (or equivalently a map $F : J^pM \to R^\ell$) and a solution manifold $S(\mathcal{E}) = F^{-1}(0) = (F^1)^{-1}(0) \cap \ldots \cap (F^\ell)^{-1}(0) \subset J^pM$. Note that for $p$-dimensional
dependent variables, \( J^o M \) has dimension \( (np + 1) \), and a system of \( \ell \) independent equations identifies therefore a solution manifold of dimension \( (np + 1 - \ell) \).

A function \( f : X \to U \) is a solution to the differential equation(s) \( \mathcal{E} \) under study if and only if its \( n \)th prolongation lies entirely in the solution manifold \( S(\mathcal{E}) \).

Let us now consider a vector field \( Y \) on \( J^o M \); we say that \( \mathcal{E} \) is invariant under \( Y \) if and only if its solution manifold is; that is, \( Y : S(\mathcal{E}) \to TS(\mathcal{E}) \). This can also be cast as the condition \( [Y(\mathcal{E})]_{S(\mathcal{E})} = 0 \).

If \( Y \) is the prolongation of a (Lie-point) vector field \( X \) on \( M \), \( Y = X^{(n)} \), we say that \( X \) is a symmetry for \( \mathcal{E} \) (more precisely, this would be a symmetry generator; we will adopt this standard abuse of notation for ease of language). The condition for \( X \) to be a symmetry is therefore \( [X^{(n)}(\mathcal{E})]_{S(\mathcal{E})} = 0 \).

1.2. Local coordinates

We will consider local coordinates \((x, u^a)\) in \( M = X \times U \), and correspondingly local coordinates \((x, u^a_k)\) (with \( k = 0, \ldots, n \)), where \( u^a_k := (\partial^a u^a/\partial x^a), \) in \( J^o M \).

A general vector field on \( J^o M \) will be written in local coordinates (here and below we will use the Einstein summation convention; the notation \( u^{(k)} \) denotes \( u \) together with its derivatives of order up to \( k \)) as

\[
Y = \xi(x, u) \frac{\partial}{\partial x} + \psi^a_k(x, u^{(k)}) \frac{\partial}{\partial u^a_k};
\]

this is the prolongation of

\[
X = \xi(x, u) \frac{\partial}{\partial x} + \psi^a(x, u) \frac{\partial}{\partial u^a};
\]

if and only if the coefficients \( \psi^a_k \) satisfy the (standard) prolongation formula

\[
\psi^a_{k+1} = D_x \psi^a_k - u^a_{k+1} D_x \xi; \quad \psi^a_0 = \psi^a.
\]

The notation \( D_x \) identifies the total derivative with respect to \( x \),

\[
D_x = \frac{\partial}{\partial x} + u^a_{k+1} \frac{\partial}{\partial u^a_k};
\]

1.3. \( \lambda \)-prolongations

After the work of Muriel and Romero [15, 16] it is common to also consider \( \lambda \)-symmetries of ODEs (see also [17–21] for their later work, and the bibliography given in [11]).

A vector field \( Y \) written in local coordinates in the form (1) is a \( \lambda \)-prolonged vector field if its coefficients satisfy

\[
\psi^a_{k+1} = D_x \psi^a_k - u^a_{k+1} D_x \xi + \lambda (\psi^a_k - u^a_{k+1} \xi)
\]

with \( \lambda \) being a smooth function \( \lambda : J^o M \to R \). We also say that \( Y \) is the \( n \)th \( \lambda \)-prolongation of \( X \) (see (2)) if \( \psi^a_0 = \psi^a \), i.e. if \( X \) is the restriction of \( Y \) to \( M \).

If the \( \lambda \)-prolongation \( \psi^a \) of \( Y \) leaves an equation (or system) \( \mathcal{E} \) invariant, we say that \( X \) is a \( \lambda \)-symmetry for \( \mathcal{E} \).

1.4. The ‘invariants by differentiation’ property

When considering a vector field in \( J^o M \), it is of interest to know its invariants, i.e. the functions \( \zeta : J^o M \to R \) such that \( Y(\zeta) = 0 \); these are also called differential invariants (to distinguish them from ‘geometrical invariants’, i.e. functions defined on \( M \) rather than on \( J^o M \).
For prolonged vector fields, it is well known that once we know differential invariants of orders 0 and 1, say $\eta$ and $\zeta_0$, respectively, we can generate recursively differential invariants of any order. (Note that starting from differential invariants of order 1 will also be needed in the case of $\lambda$-prolongations in order to take into account the properties of $\lambda_i$.) In fact, the functions

$$\zeta_{k+1} := \frac{D_k \zeta_k}{D_k \eta}$$

(6)

turn out to be automatically invariant under $Y$ if this is a prolongation and if $\eta$ and $\zeta_k$ were $[1, 6, 13, 23, 24, 27]$. This is also known as the invariants by differentiation property (IBDP).

This property remains true in the case of vector fields which are not standard prolongations but are $\lambda$-prolongations, and this fact is at the basis of the approach by Muriel and Romero; see e.g. their papers $[15, 16]$ for an algebraic proof.

The validity of IBDP was understood in geometrical terms by Pucci and Saccomandi $[25]$; their argument can be recast in the light of the approach by some of us $[8]$ follows.

We consider a set $\mathcal{X} = \{X_{(i)}, i = 1, \ldots, r\}$ of vector fields $X_{(i)}$ in involution on $M$, i.e. such that

$$[X_{(i)}, X_{(j)}] = \mu_{ij}^k(x) X_k$$

(7)

with $\mu_{ij}^k = -\mu_{ji}^k$ smooth functions on $M$. We will also consider vector fields $Y_{(i)}$ on $J^k M$, which will be some kind (to be specified in a moment) of generalized prolongation of the $X_{(i)}$.

In local coordinates, and with $u^a_1 := (\partial^a \partial^a / \partial x^i)$, these will be written as

$$X_{(i)} = \xi_{(i)}(x, u) \frac{\partial}{\partial x} + \psi^a_{(i)}(x, u) \frac{\partial}{\partial u^a},$$

(8)

$$Y_{(i)} = \xi_{(i)}(x, u) \frac{\partial}{\partial x} + \psi^a_{(i), k}(x, u^{(k)}) \frac{\partial}{\partial u^a_k},$$

(9)

where we set $\psi^a_{(i), 0} = \psi^a_{(i)}$.

In the following we will also use the shorthand notation

$$\partial_a^k := (\partial / \partial u^a_k);$$

(10)

with this equation (9) reads

$$Y_{(i)} = \xi_{(i)} \partial_x + \psi^a_{(i), k} \partial_a^k.$$  

(11)

**Definition 1.** Let $\sigma = \{\sigma_i^j =: \sigma_{ij}, i, j = 1, \ldots, r\}$ be a set of $r^2$ smooth real functions on $J^k M$. The vector fields $\mathcal{Y} = \{Y_{(i)}, i = 1, \ldots, r\}$ on $J^k M (i = 1, \ldots, r)$, written in local coordinates as in (9), are said to be jointly $\lambda$-prolonged (or $\sigma$-prolonged) if the coefficients $\psi^a_{(i), k}$ satisfy, for $k \geq 0$,

$$\psi^a_{(i), k+1} = (D_x \psi^a_{(i), k} - u^a_{k+1} D_x \xi_{(i)}) + \sigma_{ij} (\psi^a_{(j), k} - u^a_{k+1} \xi_{(j)}).$$

(12)
If $\psi^a_{(i)} = \psi^a_{(i)}$, we say that $\mathcal{Y} = \{Y_{(i)}\}$ are the joint $\lambda$-prolongation (or $\sigma$-prolongation for short) of $\mathcal{X} = \{X_{(i)}\}$.

In the following we use the notation with lower indices ($\sigma_{ij}$) for ease of writing when no confusion arises, resorting to the one with upper and lower ones ($\sigma_{i}^j$) when it becomes convenient to fully keep track of covariant and contravariant indices.

Remark 2.1. Note that the notion of $\sigma$-prolongation refers to a set of vector fields, not to a single one.

Remark 2.2. As $\sigma_{ij}: J^1\mathcal{M} \rightarrow \mathbb{R}$, this notion indeed represents a generalization of $\lambda$-symmetries. At the same time, the fact that we introduce matrices rather than scalar functions to describe the ‘twisting’ of the prolongation operation makes this similar to $\mu$-symmetries (in their ODE version) [11]. Remark 2.1 above guarantees that we are considering a really new notion, and we will see below that it gives new results.

Lemma 1. If $\mathcal{Y}$ are $\sigma$-prolonged, they satisfy

$$[Y_{(i)}, D_x] = \sigma_{ij} Y_{(j)} - (D_x \xi_{(i)} + \sigma_{ij} \xi_{(j)}) D_x.$$  \hspace{1cm} (13)

Proof. This follows from explicit computation. In fact,

$$D_x = \partial_x + u^a_{k+1} \partial_a^k;$$ \hspace{1cm} (14)

recalling (11) we have immediately

$$[Y_{(i)}, D_x] = (\psi^a_{(i),k+1} - D_x \psi^a_{(i),k}) \partial_a^k - (D_x \xi_{(i)}) \partial_a.$$ \hspace{1cm} (15)

Using (12), the rhs of this can be rewritten as

$$[-u^a_{k+1} D_x \xi_{(i)} + \sigma_{ij}(\psi^a_{(i),k} - u^a_{k+1} \xi_{(j)})] \partial_a^k - (D_x \xi_{(i)}) \partial_a$$

$$= -(D_x \xi_{(i)}) D_x + \sigma_{ij}(Y_{(i)} - \xi_{(j)}) \partial_a - \sigma_{ij} \xi_{(j)} (D_x - \partial_a)$$

$$= -[(D_x \xi_{(i)}) + \sigma_{ij} \xi_{(j)}] D_x + \sigma_{ij} Y_{(j)};$$

this completes the proof.

Remark 2.3. It follows easily from (15) that, conversely, if $Y_{(i)}$ satisfy (13), then they satisfy (12), i.e. are a $\sigma$-prolonged set.

Theorem 1. Let $\mathcal{Y}$ be a set of $\sigma$-prolonged vector fields, and let $\eta, \xi$ be independent common differential invariants of order $k$ for all of them. Then

$$\Theta := \frac{D_x \xi}{D_x \eta}$$ \hspace{1cm} (16)

is a common differential invariant of order $k + 1$ for all of them.

Proof. It is obvious that $\Theta$ is of order $k + 1$. To show that it is invariant under any of the $Y_{(i)}$, we just proceed by straightforward computation; first of all, by

$$Y_{(i)}(\Theta) = \frac{[Y_{(i)}(D_x \xi)] \cdot (D_x \eta) - (D_x \xi) \cdot [Y_{(i)}(D_x \eta)]}{(D_x \eta)^2} := \frac{\chi}{(D_x \eta)^2},$$
It is clear that we just have to show that the numerator $\chi$ vanishes. On the other hand, we have (recalling that by assumption $Y_i(\xi) = Y_{ij}(\eta) = 0$, and using Lemma 1)

$$\chi = [Y_i(D_\xi \eta) \cdot (D_\eta \eta)] = ([D_\xi(Y_i(\xi))] + [Y_i, D_\eta \xi] \cdot (D_\eta \eta)$$

This shows indeed $\chi = 0$ and hence the theorem.

**Corollary 1.** If a complete basis of (independent) invariants of orders 0 and 1 for $\mathcal{Y}$ is known, we can successively generate a basis for invariants of all orders.

**Proof.** One should only check that the independence of $\eta_a$ and $\xi_{\beta}$ (differential invariants of orders 0 and 1) implies independence of derived invariants. This only concerns functional independence of $\Theta_{a\beta} := \{D_\xi \eta_{a} / D_\eta \xi_{\beta}\}$, and is shown e.g. in [23]. Note here (and there) that we are using the fact there is only one independent variable.

Note that in theorem 1 $\sigma_{ij}$ are arbitrary (smooth) functions. It is clear that the relations between $X_i$ will not be shared by $Y_i$ for arbitrary $\sigma_{ij}$; the condition under which $Y_i$ have the same commutation properties as $X_i$ (i.e. the commutator of $\sigma$-prolonged vector fields in $\mathcal{Y}$ is the same as the $\sigma$-prolongation of the commutator of vector fields in $\mathcal{X}$) is discussed in section 3 (under the simplifying assumptions that the vector fields do not act on the independent variable).

### 3. Involutivity of $\sigma$-prolonged sets of vector fields

The IBDP for $\sigma$-prolonged vector fields, and hence the reduction procedure to be described in the following, is based on the condition that the $\sigma$-prolonged vector fields are in involution. In this section we discuss conditions guaranteeing that this is the case, based on the preliminary condition that $\{X_i\}$ are in involution.

**Theorem 2.** Assume the vector fields $X_i = \xi_i \partial_i + \psi_{ij} \partial_j$ are in involution, with $[X_i, X_j] = \mu_{ij}^k X_k$ for $\mu_{ij}^k$ smooth functions on $M$. Then their $\sigma$-prolongations $Y_i$ satisfy the same involution relations, i.e. $[Y_i, Y_j] = \mu_{ij}^k Y_k$, if and only if $\sigma_{ij}$ satisfy, for all $k = 1, \ldots, n$, the equations

$$\{[Y_i(\sigma_j^k) - Y_j(\sigma_j^k)] + ([D_\mu \mu_{ij}^k] + \sigma_{ij} \mu_{mj}^k - \sigma_{ij} \mu_{ik}^m - \mu_{ij}^m \sigma_{mk}^k) \psi_{ij}^k = 0. \tag{17}$$

**Proof.** This follows from an explicit computation, described hereafter; we proceed by induction on the order of the prolongation. Denoting by $Z$ the $(q - 1)$th $\sigma$-prolongation of $X_i$, we have

$$[Y_i, Y_j] = [Z_i, Z_j] + \{Y_i(\psi_{ij}^q) - Y_j(\psi_{ij}^q)\} D_q^a := [Z_i, Z_j] + F_{ij}^a \psi_{ij}^q.$$

Thus, assuming $[Z_i, Z_j] = \mu_{ij}^k Z_k$ (i.e. the involution relations are satisfied for $(q - 1)$th prolongations), the requirement that $[Y_i, Y_j] = \mu_{ij}^k Y_k$ is equivalent to the requirement that

$$F_{ij}^a = \mu_{ij}^k \psi_{ij}^k. \tag{18}$$
needs to be rewritten for easier comparison with the rhs of (18). In the course of this computation we will write, for a short notation, $\Phi^a$ for $\psi^{a}_{(i),q-1}$; note that this entails $Y_i(\Phi^a_j) = Z_i(\Phi^a)$; with this, (18) reads

$$F^a_{i,j} = \mu^a_{ij} (D_k \Phi^a_k + \sigma_{lk} \Phi^a_k),$$

(19)

Using also lemma 1, with standard algebra we obtain (omitting the vector indices $a$ for ease of writing—and reading)

$$F_{i,j} = Y_{(i)}(\psi_{(j),q}) - Y_{(j)}(\psi_{(i),q}) = Y_i(D_k \Phi_k) - \sigma_{lk} \Phi_k = Y_i(D_k \Phi_k) - \sigma_{lk} \Phi_k = Y_i(D_k \Phi_k) - \sigma_{lk} \Phi_k.$$

Comparing this with (19), we must require

$$F_{i,j} = (D_k \mu^a_{ij} \Phi_k + \mu^a_{ij} D_k \Phi_k + Y_i(\sigma_{jk}) - \sigma_{jk} \Phi_k) \Phi_k = \sigma_{lk} \mu^a_{ij} \Phi_k = \mu^a_{ij} (D_k \Phi_k).$$

That is, eliminating equal terms on both sides and renaming the summation indices, and finally obtain (reinserting the index $a$)

$$\{Y_i(\sigma_{jk}) - \sigma_{jk} \Phi_k\} \Phi_k + \sigma_{lk} \mu^a_{ij} \Phi_k + \mu^a_{ij} D_k \Phi_k = 0.$$  

(20)

We should now recall that $\Phi^a_k := \psi^{a}_{(k),q-1}$, and that (20) was the condition under which the commutation properties holding up to order $q - 1$ are also holding at order $q$.

On the other hand, since by definition $1 \sigma_{ij}$ are functions on $J^1 M$, and $D_k \mu^a_{ij}$ are also functions on $J^1 M$ since $\mu^a_{ij}$ are functions on $M$, all terms within the curly brackets in (20) only depend on $J^1 M$. That is, if (20) is satisfied for first prolongations (i.e. for $q = 1$), it will be automatically satisfied for all higher order prolongations (i.e. for $q > 1$) as well.

For $q = 1$ we have $\Phi^a_k = \psi^{a}_{(k),q-1} = \psi^{a}_{(k),1}$; hence, (20) reduces precisely to (17), and the proof is now complete.

**Corollary 2.** If, in the hypotheses and with the notation of theorem 2, it is

$$Y_i(\sigma^a_{jk}) - \sigma_{jk} \Phi_k = -[(D_k \mu^a_{ij}) + \sigma_{lm} \mu^a_{m} - \sigma_{jm} \mu^a_{k} - \mu^a_{ij} \sigma^a_{jk}],$$

(21)

then $[Y_i(\sigma^a_{jk})]$ satisfy the same involution relations as $[X_{(i)}]$.

**Proof.** In this case equation (17) is automatically satisfied. We stress that (21) is a sufficient but not necessary condition for (17) to hold.
Proof. Let us consider the set $X$ of vector fields in $M$ which, upon $\sigma$-prolongation, yield the set $Y$. We have supposed the number of dependent variables is the same as the rank of the set of vector fields in the set $Y$, and therefore also in the set $X$. This means that the distribution $D(X)$ spanned by $X$ in $TM$ has dimension $n$, and hence that we can introduce local coordinates $(y; w^1, \ldots, w^n)$ in $M$ such that $X_{(i)}(y) = 0$ for all $i$. (Note that this will in general

Remark 3.1. Equation (21) actually involves only the first $\sigma$-prolongations, the $\sigma_{ij}$ being functions on $J^1M$. Thus the rhs of relation (21) is rewritten as

$$\psi_{(i)}^a \frac{\partial \sigma^k}{\partial u^a} + (D_i \psi_{(i)}^a + \sigma_i^\ell \psi_{(i)}^\ell) \frac{\partial \sigma^k}{\partial u^a} - \psi_{(j)}^a \frac{\partial \sigma^k}{\partial u^a} + (D_j \psi_{(j)}^a + \sigma_j^\ell \psi_{(j)}^\ell) \frac{\partial \sigma^k}{\partial u^a}.$$  \hspace{1cm} (22)

Note that (21) are nonlinear in $\sigma$; thus determining suitable $\sigma$ preserving the involution properties of a given set of vector fields (under $\sigma$-prolongation) is in general a nontrivial task.

Remark 3.2. If $X_{(i)}$ commute, so that $\mu_{ij} = 0$, then (17) reduces simply to $[Y_i(\sigma^k_j) - Y_j(\sigma^k_i)] \Phi_{(k)}^a = 0$, and (21) to

$$Y_i(\sigma^k_j) - Y_j(\sigma^k_i) = 0.$$ \hspace{1cm} (23)

Remark 3.3. We stress that the preservation of involution relations is not required by the definition of $\sigma$-prolonged sets of vector fields, nor it will be required for the reduction of (systems of) differential equations based on $\sigma$-symmetries. On the other hand, $Y_{(i)}$ should be an involution system, and this is not guaranteed a priori by the fact $X_{(i)}$ are in involution, unless equation (21), or at least (17), is satisfied.

4. Reduction and $\sigma$-symmetries

We will now consider the case where a given differential equation, or rather a system of differential equations, is invariant under a set of $\sigma$-prolonged vector fields $Y = \{Y_1, \ldots, Y_N\}$. We will denote, as above, by $X = \{X_1, \ldots, X_N\}$ the set of vector fields in $M$ which, upon $\sigma$-prolongation, yield the set $Y$.

We will start by considering a (rather special) situation where the construction needed for reduction is particularly transparent, and then proceed to consider generalizations; we trust that this way of proceeding will help the reader.

4.1. Full reduction

Let us consider a system of ODEs $E = \{E^1 = 0, \ldots, E^m = 0\}$ of order $q$ for dependent variables $u = (u^1, \ldots, u^m)$, i.e. written as $F^h(x, u, \ldots, u^{(q)}) = 0$, where $h = 1, \ldots, m$.

We recall preliminarily that if $E$ is invariant under a set $Y$ of vector fields in $J^1M$, then it is equivalent to an equation which can be written in terms of the joint differential invariants (of order up to $q$) of $Y$, see e.g. [4]; we will thus deal with equations written in this form. (One also says that the equation admitting $Y$ as a system of symmetries is equivalent to equations admitting it as a system of strong symmetries. We will systematically use the equivalent form, also in section 7.)

Theorem 3. Let $X$ be an involution system of rank $n$ of vector fields over $M = \mathbb{R} \times \mathbb{R}^n$; let $Y$ be a set, also of rank $n$, of their $\sigma$-prolongations. Let the system of $m$ ordinary differential equations $E(x, u^{(q)})$ of order $q > 1$ in $n$ dependent variables be invariant under the set $Y$, i.e. admit $X$ as $\sigma$-symmetries. Then $E$ can be reduced to a system of $m$ differential equations of order $q = 1$.

Proof. Let us consider the set $X$ of vector fields in $M$ which, upon $\sigma$-prolongation, yield the set $Y$. We have supposed the number of dependent variables is the same as the rank of the set of vector fields in the set $Y$, and therefore also in the set $X$. This means that the distribution $D(X)$ spanned by $X$ in $TM$ has dimension $n$, and hence that we can introduce local coordinates $(y; w^1, \ldots, w^n)$ in $M$ such that $X_{(i)}(y) = 0$ for all $i$. (Note that this will in general

\[ Y_{(i)}(\sigma^k_j) - Y_{(j)}(\sigma^k_i) = 0. \]
mix x and u, i.e. dependent and independent coordinates; this is analogous to the standard situation [23, 27].)

Consider now the action of \( \mathcal{Y} \) in \( J^1M \), which has dimension \( d_1 = (2n + 1) \); again the distribution generated by \( \mathcal{Y} \) has dimension \( n \), and hence there exist \( n \) differential invariants of order 1 (in addition to \( \eta = y \), invariant of order 0). We denote these by \( (\zeta_1, \ldots, \zeta_n) \).

When considering the action of \( \mathcal{Y} \) in \( J^1M \), we note that the latter has dimension \( d_q = [(q + 1)n + 1] \), while \( D(\mathcal{Y}) \) still has dimension \( n \); hence, we have \( (qn + 1) \) invariants; one of these is \( \eta \), and we have \( n \) differential invariants of each order.

The latter can be built using theorem \( 1 \): in the \( (y, w) \) coordinates the higher order differential invariants \( \zeta^{(k)}_a \) of order \( k + 1 \leq q \) (thus \( 0 \leq k \leq q - 1 \)) will simply be

\[
\zeta^{(k)}_a = D^k \zeta_a, \quad (a = 1, \ldots, n).
\]  

As the system is supposed to be invariant under \( \mathcal{Y} \), the differential equations can be written in terms of the common differential invariants of \( \mathcal{Y} \), i.e. in view of our discussion as

\[
F^h(y; \xi, \xi^{(1)}, \ldots, \xi^{(q-1)}) = 0.
\]  

Introducing now new coordinates \( z_i = \zeta_i \) \( (i = 1, \ldots, n) \), and recalling (24)—which implies that \( \zeta^{(k)}_a = D^k \zeta_a = D^k \zeta^{(k)}_a \)—the system (25) will be written as

\[
F^h(y; z, z^{(1)}, \ldots, z^{(q-1)}) = 0,
\]  

where of course \( z^{(k)} \) denotes the set of \( k \)th order derivatives \( (d^k z_i / dy^k) \).

We have thus constructively obtained the reduction of the initial set of equations \( F^h(x, u, \ldots, u^{(q)}) = 0 \) to a set \( \hat{E} \) of equations (26) of order \( q - 1 \).

Finally we note that the requirement \( q > 1 \) guarantees that all invariants of order \( q \) are built recursively from those of orders 0 and 1 (see corollary \( 1 \)); we could not say the same for \( q = 1 \) (as also obvious from the fact the action at order 0 has no trace of \( \sigma_{ij} \) functions).

**Remark 4.1.** Needless to say, the situation we are considering in theorem \( 3 \), i.e. as many symmetries (or more precisely the rank of their system) as dependent variables in the system, is highly special and rarely met in practice. As anticipated at the beginning of the section, we are discussing this case first as it helps in visualizing the mechanism under our general reduction results, which will hold under much less exacting conditions. The geometrical meaning of theorem \( 3 \) is quite transparent and shows easily the direction of generalization: indeed we are using the fact the system \( \mathcal{Y} \) on \( J^1M \) admits an invariant distribution (of dimension \( n \)), and hence common integral manifolds, in \( T(J^1M) \), and use this information to reduce the equations. The fact that \( n \) is precisely the number of dependent variables allows to implement this reduction by lowering the order of the full system, while in general we will have less uniform reduction, as discussed below.

**Remark 4.2.** Note that if we have solutions of the reduced equations \( \hat{E} \), i.e. explicit functions \( z_i(y) \), in order to have solutions of the initial equations \( E \) we must solve the reconstruction equation system

\[
z_i(y) = \zeta_i(y, u, w)
\]  

as a differential equation system for \( w(y) = \{ w_1(y), \ldots, w_n(y) \} \). After this, we will also have to invert the change of coordinates \( (x, u) \rightarrow (y, w) \) in order to express the solution in the original coordinates in which \( E \) were set.

**Remark 4.3.** We stress that the reconstruction equations (27) are a (generally) nonlinear system of non-autonomous equations. Thus, we are definitely not guaranteed to be able to solve them.
Remark 4.4. As the differential invariants thus built are common to all the vector fields, they are in particular differential invariants for any one of them. Suppose one of them, say $Y_{(0)}$, represents the differential equation we want to reduce. The differential invariants can then be used to perform the reduction along the lines of standard reduction for ODEs with symmetry; see again [23]. Note also that if we wish $Y_{(0)}$ to be the standard prolongation of $X_{(0)}$, we should just require that $\sigma_{0j} = 0$ for all $j$.

4.2. Partial reduction

In theorem 3, the set $\mathcal{Y}$ was of the same rank $n$ as the number of dependent variables. Needless to say, this is a very special and fortunate case; in general, we will have (or be able to determine) a set of rank $r < n$.

It should be clear by inspection that the approach followed in the proof of theorem 3 can be followed also in this case, leading of course to somewhat different conclusions. We state this as a variant to theorem 3.

**Theorem 4.** Let the system of $m$ ordinary differential equations $\mathcal{E}(x, u^{(q)})$ in $n$ dependent variables be invariant under the set $\mathcal{Y}$, of rank $r < n$, of $\sigma$-prolonged vector fields. Then $\mathcal{E}$ can be reduced to a system of $m$ differential equations, depending on derivatives of order up to $q > 1$ for $(n - r)$ variables and on derivatives of order up to $(q - 1)$ for $r$ variables. In particular, if the Jacobian $(\delta F/\delta u^{(q)})$ is nonsingular, it can be reduced to a system of $m$ equations, $r$ of them of order $q - 1$, and $(m - r)$ of them still of order $q$.

**Proof.** Proceeding as in the proof to theorem 3, we determine $r$ independent invariant functions $\zeta_i(x, u, u_i), i = 1, \ldots, r$; up to relabeling of variables we can then pass to new variables $(v_i, z_j)$ defined by

$$v_i = u_i \text{ for } 1 \leq i \leq s = (n - r), \quad z_j = \zeta_j \text{ for } 1 \leq i \leq r.$$

The functions $F^h$ defining the ODEs under study can then be written as

$$F^h(x; v_1, \ldots, v_s; z_1, \ldots, z_r; v_1^{(q-1)}, \ldots, v_s^{(q-1)}; z_1^{(q-1)}, \ldots, z_r^{(q-1)}; v_1^{(q)}, \ldots, v_s^{(q)}).$$

In other words the system is of order $q$ in the variables $v$ and of order $(q - 1)$ in the variables $z_j$; we can thus rewrite it as in the statement. □

Remark 4.5. Needless to say, one could as well relax the assumption that all equations are of the same order, obtaining a corresponding special result by the same procedure. The necessary modifications are clear and we leave the details to readers, possibly with a specific application in view.

5. Geometrical features of $\sigma$-prolongations

The result of theorem 1 can appear rather surprising: in fact, the ‘invariants by differentiation’ property is intimately related to the standard prolongation structure, and one should think that if it works for a somewhat arbitrary (or apparently so at first sight) modified structure of vector fields, there is some geometrical reason. This is indeed the case, as we discuss in this section.

Let us consider a set of $\sigma$-prolonged vector fields $\{Y_i\}$ in $J^pM$. As remarked before (see section 1) following Pucci and Saccomandi, the invariants only depend on the distribution spanned by these vector fields; in other words, passing to a different set of vector fields $\{Z_i\}$ which are pointwise in the linear span of $Y_i$, we have the same distribution and hence in particular the same set of invariant functions in $J^pM$. That is, invariants are related to the
module (over $C^\infty(J^0M, R)$ function) generated by $\{Y_i\}$, and are the same for any choice of the module generators.

We want to discuss if the same module with $\sigma$-prolonged generators $\{Y_i\}$ also admits generators $\{Z_i\}$ which are \textit{standardly prolonged} vector fields. We will look for these in the form

$$Z_i = A_{ij} Y_j,$$

(28)

where $A_{ij}$ are smooth functions on $M$, and $A$ is nonsingular at all points; note that as $A$ is a function on $M$, the restriction of $Z_i$ to $J^0M$ is a proper vector field on $J^0M$ (since this is the case for $Y_i$). It clearly suffices to discuss the situation for first prolongations, as $k$th prolongations are obtained as first prolongations of $(k-1)$th prolongations.

\textbf{Theorem 5.} Let $\{X_i\}$ be vector fields in $M$, and $\{Y_i\}$ be their $\sigma$-prolongations. The module generated by the set of $\sigma$-prolonged vector fields $Y_i$ obtained for $\sigma = \sigma_{ij}$ coincides with the module generated by the standard prolongations $Z_i$ of the vector fields $W_i = A_{ij} X_j$ with $A_{ij}$ smooth functions on $M$ satisfying

$$D_x A = A \sigma.$$  

(29)

\textbf{Proof.} The argument is especially transparent if we deal with vertical vector fields in $M$, i.e. with vector fields acting only on dependent variables, and we will start by considering this case. We will work in local coordinates, and write

$$X_i = \psi^a_i \frac{\partial}{\partial u^a}; \quad Y_i = \psi^a_i \frac{\partial}{\partial u^a} + (D_x \psi^a_i + \sigma_{ij} \psi^a_j) \frac{\partial}{\partial u^a}.$$  

(30)

We look for vector fields $W_i$ of the form

$$W_i = \chi^a_i \frac{\partial}{\partial x^a};$$  

(31)

the standard prolongation of these will be

$$Z_i = (A_{ij} \psi^a_j) \frac{\partial}{\partial u^a} + \left[(D_x A_{ij}) \psi^a_j + A_{ij} (D_x \psi^a_j)\right] \frac{\partial}{\partial u^a}.$$  

(32)

In view of (30)–(32), it is clear that (28) is satisfied if and only if

$$(D_x A_{ij}) \psi^a_j + A_{ij} (D_x \psi^a_j) = A_{ij} (D_x \psi^a_j + \sigma_{jk} \psi^a_k);$$  

(33)

this is readily simplified getting rid of the $AD_x \psi$ factors, yielding (after rearrangement of summation indices)

$$(D_x A_{ik}) \psi^a_i = A_{ij} \sigma_{jk} \psi^a_k.$$  

(34)

We therefore find that this is true for any choice of $\psi^a_k$ provided the matrix function $A$ satisfies (29).

The computation is just slightly more involved in the case of general vector fields. In this case (30) is replaced by

$$X_i = \xi_i \partial_k + \psi^a_i \partial_a, \quad Y_i = X_i + \psi^a_i \partial_a$$  

(35)

with

$$\psi^a_i = (D_x \psi^a_i - u^a_i D_x \xi_i) + \sigma_{ij} (\psi^a_j - u^a_j \xi_i);$$

and

$$W_i = \chi^a_i \partial_k + \eta^a_i \partial_a, \quad Z_i = W_i + \Theta^a_i \partial_a,$$

(36)
Remark 5.1. As $A$ is invertible we can also write relation (29) in the form
\[ \sigma = A^{-1} D_x A. \] (36)
Note that the solution to equation (29), equivalently to (36), will in general not be unique; see example 4.

Remark 5.2. It should be stressed that, as also appearing from (36), the function $A$ can be a nonlocal one; in particular,
\[ A = \exp \left[ \int \sigma \, dx \right] \] (37)
and unless $\sigma = D_x S(x, u)$ for some local function $S$ we obtain a nonlocal function.

Remark 5.3. It should be stressed that the set $\{Z_i\}$ and the set $\{Y_i\}$ will in general not have the same invariants. Moreover, they could have different involution properties; see example 5.

Theorem 5 has a converse, which we state as a lemma (lemma 2) in view of its interest to build concrete examples. We also stress that this construction, if applied to vector fields which are in involution, will produce vector fields which are again in involution (lemma 3).

Lemma 2. Let $\{W_i\}$ be a set of vector fields on $M$, and $\{Z_i\}$ their standard prolongations. Consider linear combinations of these with a nonsingular matrix function $A : M \rightarrow \text{Mat}(n)$, $Y_i = A_{ij}Z_j$. Then $\{Y_i\}$ are a set of $\sigma$-prolonged vector fields with $\sigma$-prolongation coefficients $\sigma_{ij} = A^{-1}(D_x A)$.

Proof. Obvious by construction. □

Lemma 3. Let $\{Y\}$ be a set of vector fields on $J^kM$, and assume they are in involution, so that $[Y_i, Y_j] = \mu_{ij}^kY_k$ for some functions $\mu_{ij}^k : J^kM \rightarrow R$. Consider linear combinations of these with a nonsingular matrix function $A : M \rightarrow \text{Mat}(n)$, $Z_i = A_{ij}Y_j$. Then $\{Z_i\}$ are in involution, $[Z_i, Z_j] = \theta_{ij}^kZ_k$, with $\theta_{ij}^k : J^kM \rightarrow R$ explicitly given by $\theta_{ij}^k = [A_{m\ell}A^{-1}A_{jk} + A_{m\ell}Y_m(A_{jk}) - A_{m\ell}Y_m(A_{jk})A_{h\ell}^{-1}]$.\n
Proof. For $Z_i$ as in the statement, it follows by standard algebra that
\[
[Z_i, Z_j] = A_{m\ell}A_{jk}[Y_m, Y_k] + [(A_{m\ell}Y_m(A_{jk}))Y_k - [A_{jk}Y_m(A_{im})]Y_m = [A_{m\ell}A_{jk}A_{im}A_{jk} + A_{m\ell}Y_m(A_{jk}) - A_{m\ell}Y_m(A_{jk})A_{h\ell}^{-1}]Z_k.
\]
This proves that $Z_i$ are in involution, and moreover provides the explicit form for the functions $\theta_{ij}^k$ in terms of the functions $\mu_{ij}^k$ and of the transformation matrix $A$, invertible by assumption. □
Remark 5.4. Theorem 5 and its converse (lemma 2) can also be stated as the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\sigma \text{-prol}} & \mathcal{Y} \\
\downarrow A & & \downarrow A \\
\downarrow A & & \downarrow A \\
\mathcal{V} & \xrightarrow{\text{prol}} & \mathcal{Z}
\end{array}
\]

(where of course \(\mathcal{X} = \{X_i\}, \mathcal{Y} = \{Y_i\}\), and so on) if \(A\) and \(\sigma\) are related by

\[
\sigma = A (D_x A^{-1}) = -(D_x A) A^{-1}.
\]

Remark 5.5. Suppose we operate with a second (invertible) linear transformation \(B\), so that the vector fields \(X_i = \phi_i^a \partial_a\) and \(Y_i = X_i + \psi_i^a \partial_a\) are mapped into \(\hat{X}_i = B_j^i X_j = \hat{\phi}_i^a \partial_a\) and \(\hat{Y}_i = B_j^i Y_j = \hat{X}_i + \hat{\psi}_i^a \partial_a\). Then

\[
\hat{\psi}_i = B_j^i \psi_j = B_j^i \left( D_x \phi_j + \sigma_j^k \phi_k \right);
\]

writing \(\varphi = B^{-1} \hat{\varphi}\), we obtain readily that the vector fields \(\{Y_i\}\) are the \(\hat{\sigma}\)-prolongation of \(\{X_i\}\) (that is, satisfy \(\hat{\psi}_i = D_t \hat{\phi}_i + \hat{\sigma}_i^k \hat{\phi}_k\)) with

\[
\hat{\sigma} = B \sigma B^{-1} + B (D_x B^{-1}).
\]

Note that this reproduces (36) if we start from \(\sigma = 0\). Needless to say, this states that \(\sigma\) transform as gauge coefficients [11].

Remark 5.6. At first sight one could think that theorem 5 and lemma 2 state the triviality of \(\sigma\)-prolongations, in the sense that sets of \(\sigma\)-prolonged vector fields can be mapped into (and hence are equivalent to) sets of standardly prolonged ones. However, a little reflection shows that this equivalence is in general only local; in particular if the distribution spanned by \(Y_i\) is singular at some points (we recall this can mean either that some vector fields become singular, or that the rank becomes smaller, at specific points) so that the domain on which our procedures are well defined is not simply connected, then some cohomological effects will appear, and the equivalence will hold only on contractible subsets of \(M\) and \(J^k M\). The situation here is quite similar to the one met in discussing \(\mu\)-prolongations [8] (for other similarities—and differences—between \(\sigma\) and \(\mu\)-prolongations; see appendix A).

Remark 5.7. It should be stressed again that the results of this section are a direct generalization of those obtained by Pucci and Saccomandi [25] for standard \(\lambda\)-prolongations. In fact, they showed that—in the scalar ODE case—the most general class of vector fields in \(J^k M\) having the same characteristics as a standardly prolonged one (and as a consequence, sharing with them the IBDP) is precisely that of \(\lambda\)-prolonged vector fields (except for a degenerate case, \(\rho_1 = \rho_2 = 0\) in their classification, corresponding to contact symmetries). Here we have shown that—in the case of vector ODEs—the most general class of sets of vector fields in \(J^k M\) having the same integral manifolds as a set of standardly prolonged ones (and as a consequence, sharing with them the IBDP) is precisely that of \(\sigma\)-prolonged sets of vector fields. If we agree that our technique provides the proper generalization of their ‘telescopic vector fields’ (which were defined as those having the same integral lines as some standardly prolonged vector field) to the case where one has a system vector field in involution (rather than single ones) and correspondingly looks at integral manifolds (rather than integral lines), our discussion confirms their statement that ‘telescopic vector fields seem to be the natural framework for the study of reduction methods based on differential invariants’ (see [25], p 6151).
6. Discussion and possible generalizations

We have introduced a new modified prolongation operator, called $\sigma$-prolongation or joint $\lambda$-prolongation; this does not apply to vector fields individually, but instead to a given set of vector fields in involution. We have shown that the $\sigma$-prolonged vector fields still possess the ‘invariant by differentiation property’, and hence if they have suitable relations with the vector field describing a system of ODEs, they can be used for reducing the ODE system.

We have also discussed the geometrical meaning of $\sigma$-prolonged vector fields; we found that these generalize to higher dimensions the property of $\lambda$-prolongations—discovered by Pucci and Saccomandi [25]—of having the same integral lines as some other standardly prolonged vector field, and being the most general vector fields in $J^hM$ (projectable to $J^hM$ for all $0 \leq h \leq m$) with this property; needless to say, as we are in a higher dimension, integral lines are here replaced by integral manifolds.

Sets of vector fields which, after being $\sigma$-prolonged (with a specific matrix $\sigma$), leave invariant a given system of ODEs have been christened $\sigma$-symmetries. In general, vector fields which are $\sigma$-symmetries of a system are not ordinary symmetries as well; thus, our construction really gives new weapons to the arsenal of useful and structurally interesting procedures for studying nonlinear systems.

As already mentioned, the same procedure is studied from a slightly different perspective and with some differences in a companion paper [9]. In particular, in that paper one allows vector fields which are not symmetries of the evolution equation but which generate an involution system with the vector field representing it; and one also allows vector fields in $J^hM$ which are not necessarily prolongations of vector fields in $M$, in the same way as the functions $\sigma_{ij}$ are not necessarily depending only on variables in $J^1M$. From the present point of view, these represent generalized $\sigma$-symmetries (in the same sense as one usually speaks of generalized symmetries in standard symmetry analysis [1, 6, 13, 23, 24, 27]); discussing these would be outside the limits of this work.

The careful reader has probably noted that throughout this paper, we discussed how to use $\sigma$-symmetries if we know them for a given equation, but avoided to discuss how to determine these. The reason for this is that the determining equations are in this case (as also the case for $\lambda$-symmetries) functional equations—and in this case matrix functional equations; fully solving them is thus in general far beyond reach, and one has to rely on educated guesswork (or sheer luck) in order to determine convenient special solutions (see also appendix B in this respect), i.e. assume a given functional form for $\sigma$. Luckily, one does not need to know the most general $\sigma$-symmetries to be able to use them, and special solutions can be enough to reduce the equations under study (this feature is again in common with $\lambda$-symmetries).

Note that on the other hand, determining the equations or systems which admit a given set of vector fields as a $\sigma$-symmetry (with assigned $\sigma$) is a more tractable problem; actually, once we have determined differential invariants of orders 0 and 1, the IBDP makes it immediate to determine differential invariants of higher orders and hence invariants equations and systems.

Some other directions of further research can be foreseen, and we very sketchily discuss them before passing to discuss a number of examples illustrating our results; these are left for future investigations, by ourselves or by some readers of this paper.

(1) Here we mainly discussed the general case, i.e. equations of arbitrary order $q > 1$ with $n$ dependent variables and an involution system of rank $d \leq n$. The case of dynamical systems, i.e. $q = 1$, would be of obvious interest, and deserves further investigation along the lines of this paper; as already mentioned, it is considered from a slightly different point of view in the companion paper [9]. A forthcoming paper [10] contains a study along the lines of the present approach.
(2) Due to obvious physical reasons, one is especially interested in systems of ODEs which arise from a variational principle—i.e. systems of Euler–Lagrange equations. These will present special features, and it is an easy guess that $\sigma$-symmetries will be especially effective in symmetry reducing this kind of systems, exactly as it happens for standard symmetries (Noether theorem) and for $\lambda$- and $\mu$-symmetries [7, 21].

(3) $\lambda$-symmetries are naturally related to nonlocal (standard) symmetries [5, 22, 18]. It is natural to expect that, as $\sigma$-symmetries are a generalization of $\lambda$-symmetries, some relations exist between $\sigma$-symmetries and some type of nonlocal symmetries (possibly based on the construction of section 5).

(4) Finally, we would like to mention that an extension of the classical symmetry reduction is based on so-called solvable structures; see e.g. [2, 3, 12, 14, 26] for details. We expect a corresponding generalization, along the lines discussed here, would be possible; and actually quite natural as that theory is naturally set in terms of distributions of vector fields rather than of single ones. In this respect it should be noted that already Pucci and Saccomandi suggested that ‘a more geometric theory of telescopic vector fields and $\lambda$-symmetries is surely possible by means of the theory of solvable structures or the theory of coverings’ (see [25], p 6154), so we are again suggesting a generalization of their approach.

7. Examples

Example 1

Let us consider $X = \mathbb{R}$ with coordinate $x$, $U = \mathbb{R}^2$ with coordinates $(u, v)$. In $M = X \times U$ the system

$$\begin{cases}
u_{xt} = u_x v_t (1 + e^{-u}) \\ v_{xt} = u_x v_t (1 - e^{-v})
\end{cases}$$

is $\sigma$-symmetric under the vector fields

$$X_1 = \partial_u, \quad X_2 = \partial_v$$

with the functions $\sigma_{ij}$ given in the matrix form by

$$\sigma = \begin{pmatrix} 0 & v_x \\ u_x & 0 \end{pmatrix}.$$

The $\sigma$-prolonged vector fields $Y_i$ in $J^2 M$ are then given by

$$Y_1 = \partial_u + v_x \partial_{v_t} + u_x v_t \partial_{u_{xt}} + v_{xt} \partial_{v_{xt}},$$
$$Y_2 = \partial_v + u_x \partial_{u_t} + u_{xt} \partial_{u_{xt}} + u_x v_t \partial_{v_{xt}}.$$

Note that these are in involution, and actually commute, $[Y_1, Y_2] = 0$. Note also, in passing, that in this case equation (21) is satisfied, as easily checked; thus, we are in the situation claimed by theorem 2.

The only common geometrical invariant (differential invariant of order 0) for these vector fields is obviously $\eta = t$; common differential invariants of order 1 for $\mathcal{Y} = \{Y_1, Y_2\}$ are provided by

$$\xi_1^{(0)} = e^{-u} v_x, \quad \xi_2^{(0)} = e^{-u} u_t.$$

According to theorem 1, $\xi_i^{(1)} = D_t \xi_i^{(0)}$ should be differential invariants of order 2. In fact, we have

$$\xi_1^{(1)} = D_t \xi_1^{(0)} = e^{-u} (v_{xt} - u_x v_t),$$
$$\xi_2^{(1)} = D_t \xi_2^{(0)} = e^{-u} (u_{xt} - u_x v_t);$$
it is immediate to check that these are indeed invariant under both $Y_1$ and $Y_2$, as claimed by theorem 1.

Let us now consider, to give an illustration of theorem 3, any second-order differential equation(s) invariant under $Y_1$; these are necessarily written in the form

$$F^h(t, z_1, z_2, w_1, w_2) = 0,$$

where we have set, for ease of writing,

$$z_1 = \xi_1^{(0)}, \quad z_2 = \xi_2^{(0)}; \quad w_1 = \xi_1^{(1)}, \quad w_2 = \xi_2^{(1)}.$$

The above system corresponds to the choice

$$F^1 = w_1 + z_1 z_2, \quad F^2 = w_2 - z_1 z_2.$$

Through the change of coordinates

$$z_1 = e^{-u} v_x, \quad z_2 = e^{-v} u_x,$$

the system (38) will reduce, eliminating the common exponential factors, to the first-order system

$$\frac{dz_1}{dx} = -z_1 z_2, \quad \frac{dz_2}{dx} = z_1 z_2. \quad (39)$$

**Example 2**

Let us again consider $X = R$ with coordinate $x$ and $U = R^2$ with coordinates $(u, v)$. We now take the vector fields

$$X_1 = u \partial_u, \quad X_2 = -u \partial_v;$$

these are in involution, and actually satisfy $[X_1, X_2] = X_2$.

We consider the functions $\sigma_{ij}$ given in the matrix form by

$$\sigma = \begin{pmatrix} 0 & u_x \\ u_x & 0 \end{pmatrix}.$$  

The $\sigma$-prolonged vector fields $Y_i$ in $J^2 M$ are then given by

$$Y_1 = u \partial_u + u_x \partial_u + u^2 \partial_{u_x} + (u^3 u_x + u_{xx}) \partial_{u_{xx}} + (3u u_x) \partial_{v_x};$$

$$Y_2 = -u \partial_v - u x \partial_u + u_x \partial_{v_x} - (u u_{xx} + 2 u_x^2) \partial_{u_{xx}} - (u_x + u^2 u_x) \partial_{v_{xx}}.$$

Note that these are in involution, and actually satisfy $[Y_1, Y_2] = Y_2$; these are the same involution relations satisfied by $X_1$ and $X_2$, and also in this case one can check that equation (21) is indeed satisfied (and thus theorem 2 holds). The only common geometrical invariant (differential invariant of order 0) for $X = \{X_1, X_2\}$ is obviously $\eta = x$; common differential invariants of order 1 for $Y = \{Y_1, Y_2\}$ are provided by

$$\xi_1^{(0)} = e^{-v} \frac{u_x}{u}, \quad \xi_2^{(0)} = 2 v_x - u^3 - 2(1 - e^{-v}) \frac{u_x}{u}.$$

According to theorem 1, $\xi_i^{(1)} = D_x[\xi_i^{(0)}]$ should be differential invariants of order 2. In fact, we have

$$\xi_1^{(1)} = D_x[\xi_1^{(0)}] = (e^{-v}/u^2)(u u_{xx} - u_x^2 - u u_x v_x),$$

$$\xi_2^{(1)} = D_x[\xi_2^{(0)}] = (2/u^2)(u_{xx} u_x - u u_{xx} + u_x^2 v_x) + (u u_{xx} - u_x^2 - u u_x v_x).$$
It is immediate to check that these are indeed invariant under both $Y_1$ and $Y_2$. Let us now consider, with a view at theorem 3, the second-order equations

$$
\begin{align*}
\eta_{\xi}\left(1/\mu\right)\left[\left(4e^{-x}-7\right)\eta_{\xi}^2 + \left(9\mu v_x^2 - 4\eta^2\right)\eta_{\xi} + e_i^i\left(u^i_0 + 4\left(u^i_2 + \eta^2 v_x^2 + \eta^2 u_x - u^i v_x - 2\mu u_x v_x\right)\right)\right],
\end{align*}
\begin{align*}
\frac{\partial}{\partial x} \left[\left(1/\mu^2\right) \left[ u^2 \eta_{\xi}^2 + \left(\eta^2 - 1\right) u^2 \eta_{\xi}^2 + \mu \eta_{\xi} - e_i^i u \eta_{\xi} - 2\mu u_x \eta_{\xi} + \left(1/2\right) u^2 u_x\right]\right].
\end{align*}
$$

(40)

These are invariant under both $Y_1$ and $Y_2$, as can be checked by explicit computations; passing to the $Y$-adapted coordinates $\xi_1, \xi_2$ require to determine the inverse change of coordinates, which is (writing $w_1 = d\xi_1/dx$)

$$
\begin{align*}
\eta_{\xi} = u \xi_1, & & \eta_{\xi\xi} = \left(u^2/2\right) [2e_i^i w_1 + \xi_1 (u^2 + 2\xi_1 + \xi_2)], \\
\eta_x = (u^2 + \xi_2)/2, & & \eta_{xx} = \left(1/2\right) [2(e_i^i - 1) w_1 + w_2 + (3u^2 + \xi_2) \xi_2].
\end{align*}
$$

Plugging these into equations (40), these reduce to

$$
\begin{align*}
w_1 := d\xi_1/dx = \xi_1^2, & & w_2 := d\xi_2/dx = \xi_1 \xi_2,
\end{align*}
$$

(41)
i.e. to a system of first-order equations.

Example 3

In the previous two examples we have just remarked that equation (21) was satisfied; in order to show this is not always the case, and that failing to satisfy these equations leads in general to $\sigma$-prolonged vector fields which are not in involution, we consider again the vector fields of examples 1 and 2 but with different matrices $\sigma$.

Case 1. In the case of example 1, i.e. $X_1 = \partial_u$ and $X_2 = \partial_s$, and a matrix $\sigma$ which is just the transpose of that used in there, i.e.

$$
\sigma = \begin{pmatrix} 0 & u_x \\ v_x & 0 \end{pmatrix}.
$$

In this case the first joint prolongation is given by $Y_1 = \partial_u + u_x \partial_{u_x}, Y_2 = \partial_s + v_x \partial_{v_x};$ thus, the involution relations between $X$ are not mapped into corresponding relations between $Y$; more precisely, $[Y_1, Y_2] = Y_3 := u_x \partial_{u_x} - v_x \partial_{v_x}.$ Moreover, in order to close the involution relations, we also have to introduce two other new vector fields, $Y_4 := u_x \partial_{v_x}$ and $Y_5 := v_x \partial_{u_x}.$ With these, the involution (actually, algebraic—due to the constant coefficients in $X_1$ and $X_2$) relations are then

$$
\begin{align*}
[Y_1, Y_2] = Y_3, & & [Y_1, Y_3] = -2Y_4, & & [Y_1, Y_4] = 0, & & [Y_1, Y_5] = Y_3, \\
[Y_2, Y_3] = 2Y_5, & & [Y_2, Y_4] = -Y_3, & & [Y_2, Y_5] = 0, \\
[Y_3, Y_4] = 2Y_3, & & [Y_3, Y_5] = -2Y_5, & & [Y_4, Y_5] = Y_3.
\end{align*}
$$

Thus, the involution system $\mathcal{Y}$ is actually an algebra; the vector field we have to add is an ideal in $\mathcal{Y}$.

As for equations (21) and (17), it is immediate to check that these are not satisfied: the rhs of both vanishes (in this case $\mu_{ij} = 0$ for all $i, j, k$); and the lhs of (21) is easily checked to be nonzero. More precisely, let us define $Q_{ij}^k = Y_i(\sigma_j^k) - Y_j(\sigma_i^k)$ (it is obvious that $Q_{ij}^k = -Q_{ji}^k$, and hence $Q_{ii}^k = 0$ for all $i$); then $Q_{ij} = (u_x, -v_x)$. Applying this on vectors $\varphi_k$, i.e. computing the lhs of (17), we obtain again (the matrix built with $\varphi_i$ as columns are just the identity)

$$
\begin{align*}
Q_{ij} \varphi_k^i = (u_x, -v_x) \neq 0.
\end{align*}
$$
In the present case, using Case 2. Similar considerations apply if we consider the vector fields of example 2, i.e. \( X_1 = u \partial_u \) and \( X_2 = -u \partial_v \), and a matrix which is the transpose of the one used there, 

\[
\sigma = \begin{pmatrix} 0 & u_v \\ u & 0 \end{pmatrix},
\]

(42)

In this case \( Y_1 = u \partial_u + u_v \partial_u - u u_v \partial_v \); \( Y_2 = u \partial_v + u^2 \partial_u - u_v \partial_v \). We have then to introduce a vector field \( Y_3 = u^3 \partial_v \) and have

\[
[Y_1, Y_2] = Y_2 + Y_3, \quad [Y_1, Y_3] = 3Y_3, \quad [Y_2, Y_3] = 0.
\]

In this case we obtain \( Q_{12} = (u, -u^2) \); \( Q_{12}^\delta \omega_3^\delta = (u^2, u^3) \).

Case 3. Finally, let us consider the vector fields of example 1, \( X_1 = \partial_u \) and \( X_2 = \partial_v \), and the matrix \( \sigma \) just considered above; see (42). In this case the first joint prolongation is given by \( Y_1 = \partial_u + u_v \partial_v \), \( Y_2 = \partial_v + u \partial_u \); in order to close the involution relations, we have to add two new vector fields in \( J^3 M \), i.e. \( Y_3 = \partial_u - u \partial_v \), \( Y_4 = \partial_v \). With these, the involution (actually, again algebraic) relations are then

\[
[Y_1, Y_3] = Y_3, \quad [Y_1, Y_4] = -2 Y_4, \quad [Y_1, Y_4] = 0, \\
[Y_2, Y_3] = 0, \quad [Y_2, Y_4] = 0, \quad [Y_3, Y_4] = 0.
\]

As for equations (21) and (17), again these are not satisfied; the rhs of both vanishes for \( \mu_{ij}^a = 0 \), and as for the lhs we have \( Q_{12} = (1, -u) \), \( Q_{12}^\delta \omega_3^\delta = (1, -u) \neq (0, 0) \).

Example 4

We aim now at illustrations of theorem 5. Let us consider the (obviously commuting) vector fields \( X_1 = \partial_u \), \( X_2 = (1/u) \partial_u \); we \( \sigma \)-prolong them with the (nearly trivial) matrix \( \sigma = (u_v/u) I \) (\( I \) being the two-dimensional identity matrix); in this way we obtain \( Y_1 = \partial_u + (u_v/u) \partial_v \), \( Y_2 = (1/u) \partial_u \).

We note that these are not in involution: in fact, \( [Y_1, Y_2] = (u_v/u^3) \partial_v \) \( := Y_3 \); this satisfies \( [Y_1, Y_3] = 0, [Y_2, Y_3] = -2 Y_3 \), hence closes the algebra.

With the notation introduced in example 3, we have \( Q_{12} = (u_v/u^3, 0) \); \( Q_{12}^\delta \omega_3^\delta = (0, u_v/u^3) \).

We now look for a set of standardly prolonged vector fields which, as stated by theorem 5, are in the module generated by \( (Y_1, Y_2) \). As suggested by theorem 5, we look for \( A(x, u, v) \) solution to (29), i.e. in this case to

\[
D_A A = (u_v/u) A.
\]

The solution to this equation is not unique: any matrix of the form

\[
K = \begin{pmatrix} k_1 u & k_2 u \\ k_3 u & k_4 u \end{pmatrix}
\]

with \( k_i \) constants is a solution. Let us look in particular at the matrices

\[
A = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}, \quad B = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}.
\]

In the present case, using \( A \) we obtain

\[
Z^{(A)}_1 = u \partial_u + u_v \partial_u, \quad Z^{(A)}_2 = \partial_v;
\]

using \( B \) we obtain instead

\[
Z^{(B)}_1 = u \partial_v + u_v \partial_v, \quad Z^{(B)}_2 = \partial_u.
\]
These sets are transformed one into the other by the matrix

\[ M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

which also maps \( A \) and \( B \) one into the other; more precisely, writing

\[ Z_i^{(A)} = \eta_i^a \partial_a + \beta_i^a \partial^a, \quad Z_i^{(B)} = \gamma_i^a \partial_a + \theta_i^a \partial^a, \]

we have

\[ \gamma_i^a = M^a_b \eta_b^i, \quad \theta_i^a = M^a_b \beta_b^i. \]

We stress that

\[ \{ Z_1^{(A)}, Z_2^{(A)} \} = 0; \quad [Z_1^{(B)}, Z_2^{(B)}] = -\partial_b \notin \text{span}(Z_1^{(B)}, Z_2^{(B)}). \]

On the other hand, \( Z_3^{(B)} = -\partial_b \) commutes with both \( Z_1^{(B)} \) and \( Z_2^{(B)} \).

Finally, let us look at invariants; a set of common invariants (besides \( x \)) for \( \{ Z_i^{(A)} \} \) is provided by \( \{ v, u_x/u \} \). As for the vector fields \( Z_i^{(B)} \), we need to add the third vector field \( Z_3^{(B)} \) and hence we expect (as indeed the case) that only one common differential invariant of order 1 exists (besides \( x \)); this is \( u_x \).

**Example 5**

We now illustrate lemma 2, i.e. the converse of theorem 5. Let us consider the vector fields

\[ W_1 = u \partial_u + v \partial_v, \quad W_2 = -v \partial_u + u \partial_v; \]

their (standard) prolongations are of course

\[ Z_1 = u \partial_u + v \partial_v + u_x \partial_{u_x} + v_x \partial_{v_x}, \quad Z_2 = -v \partial_u + u \partial_v - v_x \partial_{u_x} + u_x \partial_{v_x}. \]

According to lemma 2, if we operate on these with a linear transformation \( A(x, u, v) \) depending on variables in \( M \), we should obtain a set of vector fields which are a \( \sigma \)-prolonged set, with \( \sigma \) corresponding to \( \sigma = A^{-1}(D_x A) \); more precisely, \( \{ Y_i \} \) should be the \( \sigma \)-prolongation of \( X_i = A_{ij} W_j \).

We will write \( \Phi = (1 - uv)^{-1} \), and choose

\[ A = \begin{pmatrix} u & 1 \\ 1 & v \end{pmatrix}, \quad A^{-1} = \Phi \begin{pmatrix} v & -1 \\ -u & u \end{pmatrix}, \]

which yields

\[ \sigma = \Phi \begin{pmatrix} -vu_x & v_x \\ u_x & -vu_x \end{pmatrix}. \]

With simple computations, we obtain

\[ Y_1 = \Phi[(1 - u)v \partial_u + (1 - v)u \partial_v + (v_x - vu_x) \partial_{u_x} + (u_x - u u_v) \partial_{v_x}], \]
\[ Y_2 = \Phi[(u + v^2) \partial_u - (u^2 + v) \partial_v + (u_x + v v_x) \partial_{u_x} - (v_x + u u_v) \partial_{v_x}]. \]

These are indeed the \( \sigma \)-prolongation, with \( \sigma \) given above, of the vector fields

\[ X_1 = A_{ij} W_j = \Phi[(1 - u)v \partial_u + (1 - v)u \partial_v], \]
\[ X_1 = A_{ij} W_j = \Phi[(u + v^2) \partial_u - (u^2 + v) \partial_v]. \]

Let us now consider first-order differential invariants; it is easily checked that \( Z_1 \) admits as invariants \( \{ x, u/v, u_x/v_x \} \), while for \( Z_2 \) we obtain \( \{ x, u^2 + v^2, u_x^2 + v_x^2 \} \); common differential invariants are

\[ P = \frac{u_x^2 + v_x^2}{u^2 + v^2}, \quad Q = \arctan(u_x/v_x) - \arctan(u/v). \]

Applying \( Y_1 \) and \( Y_2 \) on these, we obtain a nonzero result in all cases, as stated in remark 5.3.
Example 6

We give another illustration of lemma 2, this time for vector fields which are not defined in
$u = v = 0$. We write $\rho = (u^2 + v^2)$ and choose

$$W_1 = \rho^{-1} h(x, u) \partial_u, \quad W_2 = \rho^{-1} h(x, v) \partial_v,$$

with $h$ being an arbitrary smooth function; this yields as first standard prolongation the vector
fields

$$Z_1 = W_1 + \rho^{-2} (\rho [D_u h(x, u)] - h(x, u) (D_u \rho)) \partial_u,$$

$$Z_2 = W_2 + \rho^{-2} (\rho [D_v h(x, v)] - h(x, v) (D_v \rho)) \partial_v.$$

Let us now consider a transformation

$$A = -\begin{pmatrix} v & u \\ -u & v \end{pmatrix}, \quad A^{-1} = -\frac{1}{\rho} \begin{pmatrix} v & u \\ -u & v \end{pmatrix};$$

and the vector fields

$$X_i = A_i^j W_j, \quad Y_i = A_i^j Z_j.$$ 

Writing $P = h(x, u)$, $Q = h(x, v)$; moreover $\mathcal{P} = (\rho (D_u P) - 2 (D_u \rho) P)$, $\mathcal{Q} = (\rho (D_v Q) - 2 (D_v \rho) Q)$, these are given explicitly by

$$X_1 = -\frac{1}{\rho} [vP \partial_u + uQ \partial_v],$$

$$X_2 = \frac{1}{\rho} [uP \partial_u - vQ \partial_v];$$

$$Y_1 = X_1 - \rho^{-2} [vP \partial_u - uQ \partial_v],$$

$$Y_2 = X_2 + \rho^{-2} [uP \partial_u - vQ \partial_v].$$

It is easily checked that $\{Y_i\}$ are the $\sigma$-prolongation of $\{X_i\}$ with

$$\sigma = \frac{1}{\rho} \begin{pmatrix} -(uv_x + vv_x) & uv_x - vu_x \\ -uv_x + vu_x & -(uu_x + vv_x) \end{pmatrix} = A (D_{\mathcal{A}} A^{-1}).$$

Example 7

In the previous example, both the $Y_i$ and the $Z_i$ fields were singular. Situations where the
standardly prolonged fields are singular but the $\sigma$-prolonged ones are regular would be of
interest; here we deal with such a situation, obtained through a small variation on the setting
of example 6. We use the same notation introduced there.

We consider the (singular) vector fields

$$X_1 = (P/\rho) \partial_u, \quad X_2 = (Q/\rho) \partial_v;$$

these have standard prolongations

$$Z_1 = X_1 + [(\rho (D_u P) - P (D_u \rho))/\rho^2] \partial_u,$$

$$Z_2 = X_2 + [(\rho (D_v Q) - Q (D_v \rho))/\rho^2] \partial_v.$$

Acting now with

$$A = \rho^2 \begin{pmatrix} v & -u \\ u & v \end{pmatrix}$$

we obtain the (regular) vector fields

$$X_1 = \rho [(Pv) \partial_u - (Qu) \partial_v], \quad X_2 = \rho [(Pu) \partial_u + (Qv) \partial_v];$$

$$Y_1 = X_1 + v[\rho (D_u P) - P (D_u \rho)] \partial_u - u[\rho (D_v Q) - Q (D_v \rho)] \partial_v,$$

$$Y_2 = X_2 + u[\rho (D_u P) - P (D_u \rho)] \partial_u + v[\rho (D_v Q) - Q (D_v \rho)] \partial_v.$$

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One can check that \( \{Y_i\} \) are the \( \sigma \)-prolongation of \( \{X_i\} \), with
\[
\sigma = A(D_xA)^{-1} = -(D_xA)A^{-1} = -\frac{1}{\rho} \left( \begin{array}{cc}
5(uu_x + vv_x) & uv_x - vu_x \\
-uv_x + vu_x & 5(uu_x + vv_x)
\end{array} \right).
\]

**Example 8**

Let us now see an example of the situation considered in theorem 4; that is, we will have a set of three differential equations of second order, admitting as symmetries a set of two \( \sigma \)-prolonged vector fields in involution (actually, commuting). (A number of trivial examples are also easily obtained by adding to any \( n \)-dimensional example for theorem 3 (with \( \mathcal{Y} \) of rank \( n \)) a new equation for a new dependent variable \( w(x) \); this is not acted upon by, nor entering in, the coefficients of the considered vector fields and hence no reduction is possible on it. We will thus end up with a system of one second-order and \( n \) first-order equations.)

We consider the equations
\[
E_1 := e^{-(u+w)}(u_x - v_x) (u_x - v_x - w_x) - [u_x + w_x]^{-1}(u - v) \\
\times (-u_x^2 + u_x + v_x + v_xw_x + w_x^2 + w_x) = 0;
\]
\[
E_2 := e^{-(u-v-w)}(u_x - v_x) (u_x + w_x - u_x^2 + u_xv_x + v_xw_x + w_x^2) + \\
\times [u_x - v_x - w_x]^{-1}(u - v) \\
\times (-u_x^2 + u_x + v_x - v_xw_x + v_xw_x + w_x^2 - w_x) = 0;
\]
\[
E_3 := (u - v) - e^{-(u+w)} (u_x - v_x) \\
\times (u_x - v_x - w_x - u_x^2 + u_xv_x + v_xw_x + w_x^2) = 0;
\]

and the (autonomous) commuting vector fields
\[
X_1 = \partial_u + \partial_v - \partial_w, \quad X_2 = \partial_u + \partial_v.
\]

As for their (second) \( \sigma \)-prolongation, we set
\[
\sigma = \left( \begin{array}{ccc}
0 & u_x + w_x & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right)
\]

and hence we obtain, with standard computations,
\[
Y_1 = (\partial_u + \partial_v - \partial_w) + (u_x + w_x)(\partial_u) + (u_x + w_x)(\partial_v) + \\
+ [u_x + w_x + (u_x - v_x - w_x)](\partial_{u_x} + \partial_{v_x}) \\
= (u_x - v_x - w_x)(u_x + w_x)\partial_{u_x} + \\
\partial_{u_x} + \partial_{v_x}.
\]
\[
Y_2 = (\partial_u + \partial_v) + (u_x - v_x - w_x)(\partial_u) + (u_x - v_x - w_x)(\partial_v) + \\
+ [u_x - v_x + (u_x - v_x - w_x)](u_x + w_x) - w_x(\partial_{u_x} + \partial_{v_x}) \\
= (u_x - v_x - w_x)(\partial_{u_x} + \partial_{v_x} + \partial_{v_x}).
\]

Now, by trivial dimension counting, there must be two invariants of order 0, i.e. on \( M \); these are
\[
\eta_1 = x, \quad \eta_2 = u - v.
\]

The common differential invariants of order 1 for these vectors turn out to be
\[
\zeta_1^{(0)} = e^{-(u-v-w)} (u_x + v_x), \quad \zeta_2^{(0)} = e^{-(u+w)} (u_x - v_x - w_x), \quad \zeta_3^{(0)} = u_x - v_x.
\]
The second-order differential invariants can be readily computed by the IBDP (using $\eta_1$) as $\xi^{(1)}_t = D_3\xi^{(0)}_t$, which yields

$$
\begin{align*}
\xi^{(1)}_1 &= e^{-(w^2-w)}(u_{xx} + w_{xx} + u_x v_x + v_x w_x - u_x^2 + w_x^2), \\
\xi^{(1)}_2 &= e^{-(w^2+w)}(u_{xx} - v_{xx} + u_x v_x + v_x w_x - u_x^2 + w_x^2), \\
\xi^{(1)}_t &= u_{xx} - v_{xx}.
\end{align*}
$$

these are easily checked to be indeed invariant under $Y_t$.

We can then perform the change of variables

$$\xi = u - v, \quad z_1 = \xi^{(0)}_1, \quad z_2 = \xi^{(0)}_2;$$

this of course entails a corresponding change for derivatives, which we do not write down explicitly. In the new variables, the equations are written as

$$
E_i := \xi ((z_1)_x/z_1) - z_2\xi_{xx} = 0; \\
E_2 := \xi ((z_2)_x/z_2) - \xi_x(z_1)_x = 0; \\
E_3 := ((z_2)_x/z_2) - ((z_1)_x/z_1) = 0.
$$

The system is now of first order in the $z_i$ variables, and of second order in $\xi$; it can be rewritten as

$$
\frac{d^2\xi}{dx^2} = \frac{\xi^3}{z_1 z_2^2}, \quad \frac{dz_1}{dx} = \frac{\xi^2}{z_1^2}, \quad \frac{dz_2}{dx} = \frac{\xi}{z_2}.
$$

Needless to say, the same procedure—at least up to rewriting the system as first order in the $z_i$ variables, and second order in $\xi$—would work for any starting system of equations in the form $E_i := F[\eta_1, \eta_2; \xi^{(0)}_1, \xi^{(0)}_2, \xi^{(0)}_3, \xi^{(1)}_1, \xi^{(1)}_2, \xi^{(1)}_3, \xi^{(1)}] = 0$, leading to equations written in the new variables as $F[x, \xi; z_1, z_2, \xi_x; (z_1)_x, (z_2)_x, \xi_{xx}] = 0$.

**Example 9**

Let us now consider, again in order to illustrate theorem 4, a case with three dependent variables $(u, v, w)$ and two (non-commuting) independent vector fields in involution,

$$X_1 = u\partial_u + w\partial_w; \quad X_2 = -u\partial_u; \quad [X_1, X_2] = X_2.
$$

We will use the $(2 \times 2)$ matrix

$$
\sigma = \begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix}.
$$

With this choice, the $\sigma$-prolongations of the vector fields $\{X_1, X_2\}$ to $J^2M$ are

$$Y_1 = u\partial_u + w\partial_w + u_x\partial_{u_x} - uu_x\partial_{v_x} + w_x\partial_{w_x} + u_{xx}\partial_{u_{xx}} - (2u_x^2 + uu_{xx})\partial_{v_x} + w_{xx}\partial_{w_{xx}},$$

$$Y_2 = -u\partial_u - u_x\partial_{u_x} - u_{xx}\partial_{v_x}.$$ 

These still satisfy the same commutation (hence involution) relation: $[Y_1, Y_2] = Y_2$.

With trivial algebra one finds the common invariants of order up to 2 for these two vector fields; by trivial dimension counting they must be 8, one of them being the trivial one $J_{00} = x$.

A simple basis for the (seven) nontrivial ones is given by $J_{01} = (w/u), J_{11} = (w_x/w), J_{21} = (w_{xx}/u); J_{12} = (u_x/u), J_{22} = (u_{xx}/u); J_{13} = [v + (uu_x/2) - (u_xv/u)], J_{23} = \ldots$
\[v_{xx} + u_t^2 + (uu_{xx}/2) - (u_xw/u)].\] Another possible, maybe slightly less simple but more convenient, basis is provided by
\[
\begin{align*}
\zeta_1^{(0)} &= w/w; & \zeta_1^{(1)} &= (uw_x - u_x w)/u^2; & \zeta_1^{(2)} &= u_x/u, \\
\zeta_2^{(1)} &= v_x + uu_x/2 - u_x v/u; & \zeta_1^{(2)} &= (u^2 w_{xx} - uw_x u_x - 2uu_x u_x + 2uw_x^2)/u^3, \\
\zeta_2^{(2)} &= (uu_x - u_x^2)/u^2; & \zeta_3^{(2)} &= (v_x + uu_x/2 - v u_x/u).
\end{align*}
\]

This is convenient in that (wherever applicable)
\[
\zeta_i^{(i+1)} = D_x \zeta_i^{(i)}.
\]
Thus any system of equations \(E^{(i)} : F^{(i)}[x; u^{(2)}, v^{(2)}, w^{(2)}] = 0\) of order not higher than 2 written in terms of these joint invariants will admit \([X_1, X_2]\) as \(\sigma\)-symmetries (with \(\sigma\) given above), and conversely. Note all of these (if nontrivial) will necessarily be singular for \(u = 0\).

In order to consider a concrete example, let us look at the system
\[
\begin{align*}
u_{xx} &= 2uw_x^2 + u^2(2vu_x - 2uw_x + w_x - w) - uu_x(u^3 + w + w_x)/uw, \\
v_{xx} &= A + B + C, \\
w_{xx} &= -u(u + u_x)(w - w_x); \\
A &= 2u^2v_x - ((4v + u^2)u_x - w + w_x)u^4 + (u_x(w + w_x) - 4vu_x)u^3, \\
B &= \left[-3u^2w_x + 4v^2u_x + 2v(u_x - w)\right]u^2, \\
C &= 2(w_w_x - u_x(v(w + w_x) - v_x w))u + 2u_x^2vw.
\end{align*}
\]

In terms of the invariants, this system reads simply
\[
\begin{align*}
\zeta_2^{(2)} - \zeta_1^{(1)} + \zeta_1^{(0)} &= 0, \\
\zeta_3^{(2)} - \zeta_2^{(1)} &= 0, \\
\zeta_1^{(2)} - 2\zeta_3^{(1)} &= 0.
\end{align*}
\]

In order to reduce it to a system of one second-order equation and two first-order ones, we should change (dependent) coordinates. We choose
\[
\xi = \zeta_1^{(0)}, \quad \eta = \zeta_2^{(1)}, \quad \rho = \zeta_1^{(1)};
\]
and recall (43). The equations are then readily expressed in the new coordinates, providing
\[
\frac{d^2 \xi}{dx^2} = 2\rho, \quad \frac{d\rho}{dx} = \eta, \quad \frac{d\eta}{dx} = \frac{d\xi}{dx} - \xi;
\]
i.e. as claimed, a system of two first-order and one second-order equations.

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Appendix A. Relations between $\sigma$- and $\mu$-prolongations

The picture emerging from the discussion in section 5, see in particular theorem 5 and lemma 2, is quite reminiscent of the one holding for $\mu$-prolongations [8, 11] (here we assume the reader to be familiar with them; see [11] for details and a bibliography on twisted prolongations in general): in this appendix we will discuss similarities and differences between the two, and their relations when applied to a given set of vector fields.

A.1. Similarities and differences

First of all, let us stress the analogies between $\sigma$- and $\mu$-prolongations: in both cases,

(a) the vector fields $Y_i$ obtained from $X_i$ through the modified ($\sigma$ or $\mu$) prolongation operation are the standard prolongations $Z_i$ of different vector fields $W_i$;

(b) the relation between $X_i$ and $Y_i$ on the one hand, and $W_i$ and $Z_i$ on the other, can be summarized by the action of a linear transformation $A$;

(c) this $A$ then determines the relevant matrix ($\sigma$ in our case, a single $\Lambda$ in the case of $\mu$-prolongations for ODEs) entering in the definition of the modified prolongation operation—and reducing to zero for standard prolongations.

An essential difference between the two cases should however be emphasized: in the case of $\mu$-prolongations, the linear transformation acts on the basis vectors in $T^\times M$ and more generally in $T(J^kM)$ [11]; here instead we are changing the generators of the module of vector fields—i.e. act on a structure superimposed to $T(J^kM)$ with no action on the space itself.

In other words, in the case of $\mu$-prolongations all vector fields on $M$ and on $J^kM$ are affected by the action of the linear transformation—as obvious since this is a point-dependent change of basis, i.e. a gauge transformation [11]—while here we are just changing our choice for the generators of a given module of vector fields, or equivalently the generators for a given distribution on $TM$ and $T(J^kM)$, with of course no action on the general set of vector fields in $M$ or $J^kM$.

As a consequence, the $\mu$-prolongation operation acts individually on each vector field; they are thus well defined, and indeed different from standard ones, also for single vector fields. Moreover—in the framework of ODEs, which is the only interest here—$\mu$-prolongations reduce to $\lambda$-prolongations only if the linear transformation $A$ is a multiple of the identity, the proportionality factor being a smooth function on $M$.

On the other hand, the $\sigma$-prolongation operation is defined on finite sets of vector fields, and reduces to a somewhat trivial one in the case where the set mentioned above is made of a single vector field. They again reduce to usual $\lambda$-prolongations only if the linear transformation $A$ is a multiple of the identity, the proportionality factor being a smooth function on $M$.

We summarize the situation in the following commutative diagram:

\[
\begin{align*}
\{\hat{X}_i\} & \xleftarrow{M} \{W_i\} & \xrightarrow{S} \{X_i\} \\
\downarrow_{\mu\text{-prol}} & \downarrow_{\text{prol}} & \downarrow_{\sigma\text{-prol}} \\
\{\hat{Y}_i\} & \xleftarrow{M} \{Z_i\} & \xrightarrow{S} \{Y_i\}
\end{align*}
\]  

(A.1)

Here $\hat{Y}_i$ is the $\mu$-prolongation of $\hat{X}_i$, $[Y_i]$ are the $\sigma$-prolongation of $[X_i]$ and the invertible linear transformations $S$ and $M$ are related to $\sigma$ and to $\mu = \Lambda dx$ via

\[
\Lambda = M(D_\mu M^{-1}), \quad \sigma = S(D_\sigma S^{-1}).
\]  

(A.2)
A.2. Relations

It is quite clear that albeit the two operations are conceptually different in general, a relation exists between the two points of view when we consider a given transformation on a given set of vector fields; this is quite similar to the relation between ‘active’ and ‘passive’ points of view in fluid mechanics.

We discuss it for vertical vector fields only (they can be thought as evolutionary representatives of general vector fields in M; see [1, 6, 13, 23, 24, 27]), and refer to (A.1) for notation.

Consider vector fields \( W_i \) in \( M \) and their prolongations \( Z_i \) in \( JM \); consider also a linear (point-dependent) transformation in \( M \), under which \( W_i \) are mapped into \( X_i \) and \( Z_i \) into \( Y_i \); we know that \( \{ Y_i \} \) are then the \( \sigma \)-prolongation of \( \{ X_i \} \). We want to discuss if there is a linear map \( M \) such that these particular \( W_i \) and \( Z_i \) are mapped into \( X_i \) and \( Y_i \) (each of them being the \( \mu \)-prolongation of the corresponding \( \tilde{X}_i \)), with the additional properties that (a) \( \tilde{X}_i = X_i \); (b) \( \tilde{Y}_i = Y_i \). It will suffice to work on first prolongations, as prolongations of order \( k + 1 \) can be seen as first prolongations of prolongations of order \( k \).

It is convenient to work in coordinates, writing \( W_i = \phi_i^a \partial_a \), and embodying the components of the different vector fields into a single matrix \( \Phi \) with elements

\[
\Phi_{ij} = \phi_i^a. \tag{A.3}
\]

Note that if we have \( r \) vector fields with \( n \) components, the matrix \( \Phi \) is \((n \times r)\). In particular, it is a square—and thus possibly invertible—matrix if and only if \( r = n \); this is precisely the case we have considered in the main body of this paper. Note also that if \( r = n \), the condition for \( \Phi \) to be invertible (at all points) is precisely that the vector fields \( X_i \) are independent (at all points).

Coming back to our vector fields, \( X_i \) has components \( \tilde{\phi}_i^a = (S_i^j \phi_j^a) \), while \( \tilde{X}_i \) has components \( \tilde{\phi}_i^a = M^b_i \phi_j^b \). Requiring the two to be the same, i.e. \( \tilde{\phi} = \tilde{\phi} \), amounts to requiring

\[
S_i^j \phi_j^a = M^b_i \phi_j^b;
\]
or, acting with \( M^{-1} \) from the left,

\[
\phi_j^b = (M^{-1})^b_i S_i^j \phi_j^a = (M^{-1})^b_i \phi_j^a (S^T)^j_i.
\]

Rearranging the summation indices and using the notation (A.3), this is also rewritten as the matrix equation

\[
\Phi = M^{-1} \Phi S^T. \tag{A.4}
\]

It should be recalled that in our discussion of \( \sigma \)-prolongations, we have supposed the vector fields \( X_i \) to be as many as the dependent variables, and to be independent at all points of \( M \); in view of the requirement that also \( S \) and \( M \) are invertible at all points, this means that \( W_i \) will also be independent, and hence \( \Phi \) will be nonsingular, at all points of \( M \).

Thus if we require \( M \) which satisfies (A.4) for given \( \Phi \) and \( S \), the answer is that this is given by

\[
M = \Phi S^T \Phi^{-1}; \tag{A.5}
\]
similarly, if \( \Phi \) and \( M \) are given and we ask if there is \( S \) such that (A.4) is satisfied, we obtain

\[
S = \Phi^T M^T (\Phi^{-1})^T, \quad S^T = \Phi^{-1} M \Phi. \tag{A.6}
\]

Finally, if \( M \) and \( S \) are given and we wonder which are the vector fields \( W_i \) such that \( X_i = \tilde{X}_i \), the answer is provided directly by (A.4).

Our discussion so far only concerns vector fields in \( M \); let us now also consider vector fields in \( JM \). We write \( Z_i \) as
\[ Z_i = \phi_i^a (\partial/\partial u^a) + \psi_i^a (\partial/\partial u^a); \]
note{here of course \[ \psi_i^a = D_a \phi_i^a. \] We will embody the coefficients \[ \psi_i^a \] into a matrix \[ \Psi \] with entries \[ \Psi_{ij} = \psi_i^a; \] this satisfies \[ \Psi = D_a \Phi. \]}

Proceeding in exactly the same way as above, we easily obtain that
\[ Y_i = X_i + \{ S^i_j (D_a \phi_j^a) \} (\partial/\partial u^a); \]
\[ \hat{Y}_i = \hat{X}_i + \{ M^a_b (D_a \phi^b_i) \} (\partial/\partial u^a); \]

thus—assuming \[ X_i = \hat{X}_i—\] we have to require
\[ \Psi S^T = M \Psi. \]

Recalling now that \[ \Psi = D_a \Phi, \] this yields with trivial manipulations
\[ (D_a M) = \Phi (D_a S^T) \Phi^{-1}; \]
recalling also (A.5), we arrive at the condition
\[ \Phi^{-1} (D_a \Phi) S^T = S^T \Phi^{-1} (D_a \Phi). \quad (A.7) \] In other words, the equality \[ X_i = \hat{X}_i \] will be lifted to a corresponding equality \[ Y_i = \hat{Y}_i \] between prolonged vector fields if and only if (A.7) is satisfied.

### A.3. Combining \( \mu \) - and \( \sigma \) -prolongations

The previous discussion also suggests how to combine \( \mu \) - and \( \sigma \) -prolongations into a single prolongation operation; we will call it \( \chi \)-prolongation (where \( \chi \) stands for ‘combined’): working with matrices \( \Phi \) and \( \Psi = D_a \Phi, \) and writing \[ \Phi = M^{-1} \Phi R, \quad \Psi = M^{-1} \Psi R; \] the operator mapping \( \hat{\Phi} \) into \( \hat{\Psi} \) will be that of the prolongation combining \( \mu \) and \( \sigma \) ones. Inverting the first of relations (A.8), we have
\[ \Phi = M \hat{\Phi} R^{-1}, \]
hence, the second of (A.8) reads
\[ \hat{\Psi} = M^{-1} [D_a (M \hat{\Phi} R^{-1})] R \]
\[ = M^{-1} [D_a M] \hat{\Phi} R^{-1} + M (D_a \hat{\Phi}) R^{-1} + M (D_a M^{-1}) R \]
\[ = (D_a \hat{\Phi}) + [M^{-1} (D_a M)] \hat{\Phi} - \hat{\Phi} [R^{-1} (D_a R)] \]
\[ = (D_a \hat{\Phi}) + \Lambda \hat{\Phi} - \hat{\Phi} \Theta; \]

here we have of course defined
\[ \Lambda := M^{-1} (D_a M), \quad \Theta := R^{-1} (D_a R). \]

Reintroducing indices and passing to the vector notation, we rewrite the above relation as
\[ \hat{\psi}_i^a = (D_a \phi_i^a) + \Lambda^a_b \psi_i^b - (\Theta^T)^i_j \psi_j^a; \quad (A.9) \]

this defines the (first, and then recursively higher order) prolongation operator.

This can be described in terms of commutative diagrams by complementing (not substituting!) the diagram (A.1) with the diagram (A.10)
One could thus think of investigating $\chi$-symmetries of differential equations, defined as sets of vector fields which, when $\chi$-prolonged, give sets of vector fields which leave the solution manifold of a given set of ordinary differential equations invariant.

However, it is well known that while $\mu$-symmetries are ‘as useful as standard symmetries’ in the search for special solutions to PDEs or systems of PDEs, and they reduce to usual $\lambda$-symmetries for scalar ODEs—and are thus in a way again as useful as standard ones in their reduction—they are of no (known) use in the case of systems of ODEs. Thus at the moment we cannot describe any use of such $\chi$-prolongations, and they remain at this stage a mathematical curiosity.

Appendix B. Determining equations for $\sigma$-symmetries

In section 6 we have mentioned that the task of determining all $\sigma$-symmetries for a given equation or system is in general well beyond reach. In this appendix we will illustrate this statement by a concrete example, i.e. the system considered in example 1. This reads

$$u_{xx} = u_x v_x(1 + e^{-u}),$$
$$v_{xx} = u_x v_x(1 + e^{-v}).$$

We will look for a set of two vector fields in involution $X_{(i)}$ giving a $\sigma$-symmetry, and in order to keep formulas not too long, we will search for these with $\xi_{(i)} = 0$. In this case the second prolongations will be written as

$$Y_{(i)} = \psi^{a}_{(i)} \partial_a + \chi^{a}_{(i)} \sigma^{(a)} + X^{a}_{(i)} \sigma^{(a)},$$

where we have denoted $\delta^{a}_{(i)} := (\partial / \partial u^{a}_{(i)})$ and $\sigma^{(a)} := (\partial / \partial \sigma^{a}_{(i)})$; and the coefficients entering in first and second prolongations are now denoted as $\psi$ and $\chi$ to reduce the number of indexes. An explicit computation shows that

$$\psi^{a}_{(i)} = D_{a} \phi_{(i)} + \sigma_{ij} \psi_{(j)},$$
$$\chi^{a}_{(i)} = D_{a} \psi_{(i)} + \sigma_{ij} \psi_{(j)} - D_{b} \psi_{(i)} + 2 \sigma_{ij} (D_{a} \phi_{(j)}),$$
$$\chi^{a}_{(i)} = D_{a} \phi_{(i)} + \sigma_{ij} \chi_{(j)}.$$

In order to obtain the determining equations for $\sigma$-symmetries, we should apply both prolonged vector fields to both equations, obtaining in all cases zero upon restriction to the solution manifold of the system (B.1).

The determining equations—under the present simplifying assumption $\xi_{(i)} = 0$—are thus in this case (here and below $i = 1, 2, a = 1, 2, S$ represents the solution manifold to the system)

$$\chi^{a}_{(i)} - \left[[\psi^{1}_{(i)} u_{x} + u_{x} \phi^{2}_{(i)} (1 + e^{-u}) + (u_{x} v_{x}) \phi^{a}_{(i)} e^{-u}] \right]_{S} = 0.$$  \hspace{1cm} (B.2)

Substituting according to the above for $\psi$ and $\chi$, and making explicit the derivative terms, we obtain (with $\phi^{a}_{(i),b} = \partial \phi^{a}_{(i)}/\partial u^{b}$, etc; sum over repeated indices is implied) for the left-hand side of (B.2) before restriction to $S$

$$\phi^{a}_{(i),xx} + 2u_{x} \phi^{a}_{(i),bx} + u_{xx} \phi^{a}_{(i),b} + u_{x}^2 \phi^{a}_{(i),bc} + 2 \sigma_{ij} (\phi^{a}_{(i),x} + u_{x} \phi^{a}_{(i),b}),$$
$$\phi_{(i),x} + u_{x} \phi_{(i),x} + u_{xx} \phi_{(i),x} + \sigma_{ij} \sigma_{mj} .$$

Using now (B.1) to restrict to $S$, we obtain

$$\phi^{a}_{(i),xx} + 2u_{x} \phi^{a}_{(i),bx} + (u_{x}^2 \phi^{a}_{(i)} (1 + e^{-u}))(\phi^{a}_{(i),b} + u_{x} \phi^{a}_{(i),bc} + 2 \sigma_{ij} (\phi^{a}_{(i),x} + u_{x} \phi^{a}_{(i),b}),$$
$$\phi_{(i),x} + u_{x} \phi_{(i),x} + (u_{x}^2 \phi^{a}_{(i)} (1 + e^{-u})) \phi_{(i),x} + \sigma_{ij} \phi_{(ij)} .$$  \hspace{1cm} (B.3)
These are the determining equations we were looking for. It should be noted that they cannot be solved according to the standard procedure for determining equations of Lie-point symmetries, as they depend on the unknown matrix function $\sigma = \sigma(x, u, v)$. 

The best one can do is to look for special solutions of these; e.g., if we look for solutions such that $\phi_{(ij)}$, $\phi_{(i)}$ are constant, and $\sigma$ only depends on $u_i$ and $v_i$, this reduces to

$$
\left[\left(u_1^1 u_1^2 (1 + e^{-u'})^\sigma_{ij} + \sigma_{im} \sigma_{mj}\right)\phi_{(ij)}\right] = 0.
$$

Despite its innocent-looking shape (due to compact notation) this is still a system of four coupled nonlinear PDEs.

We will now look for $\sigma$ in the form

$$
\sigma = \begin{pmatrix} 0 & A(u, v) \\ B(u, v) & 0 \end{pmatrix};
$$

we also write $\phi_{(1)} = c_1, \phi_{(2)} = c_2, \phi_{(i)} = k_1, \phi_{(i)} = k_2$; in this way the determining equations (B.4) read

$$
A e^v (-Bc_1 e^v + (1 + e^v)(k_2 u_i + k_1 v_i))
$$

$$
- (A v_i e^v (1 + e^v) k_2 + A u_i (1 + e^v) k_1) u_i v_i = 0,
$$

$$
A e^v (-Bc_2 e^v + (1 + e^v)(k_2 u_i + k_1 v_i))
$$

$$
- (c_2 e^v + A u_i (1 + e^v) e^v k_2 + A u_i e^v (1 + e^v) k_2) u_i v_i = 0;
$$

$$
B e^v (-A e^v k_1 + (1 + e^v)(c_2 u_i + c_1 v_i))
$$

$$
- (B c_1 e^v (1 + e^v) + e^v (B u_i c_1 (1 + e^v) + k_1)) u_i v_i = 0,
$$

$$
B e^v (-A e^v k_2 + (1 + e^v)(c_2 u_i + c_1 v_i))
$$

$$
- (B u_i c_2 (1 + e^v) e^v + B c_i c_2 e^v (1 + e^v) + e^v k_2) u_i v_i = 0.
$$

Determining the general solutions of these is not easy despite the several ansatzes considered to simplify them from the original form. A further reduction is obtained e.g. looking for $A, B$ as linear functions, $A(u, v) = p_1 u_i + p_2 v_i, B(u, v) = q_1 u_i + q_2 v_i$. In any case, it can be checked that the vector fields $X_1, X_2$ and the matrix $\sigma$ considered in example 1 satisfy these equations.

Note that even assuming $\sigma = \sigma(x, u)$ (i.e. $\sigma_{ij}$ are functions on $M$ and not on the full $\mathcal{M}$), albeit in principle one obtains a more tractable problem, in that all dependences on derivatives are explicit and the determining equations can be decomposed into equation for the vanishing of different monomials in $\sigma_{ij}$, the resulting PDEs are nonlinear in $\sigma_{ij}$, and hence in general cannot be fully solved. For example, in the case of equations (B.1) even with the simplifying assumptions $\phi_{(i)} = \phi_{(i)}(u, v)$ and $\sigma_{ij} = \sigma_{ij}(u, v)$, after easily determining that $X_{(i)}$ must be of the form

$$
X_{(i)} = \left[\frac{\exp((1 + e^{-u})v)}{1 + e^{-u}} P_{(i)}(u) + Q_{(i)}(u)\right] \partial_u + \left[\frac{\exp((1 + e^{-v})u)}{1 + e^{-v}} R_{(i)}(u) + S_{(i)}(u)\right] \partial_v,
$$

one remains with PDEs which are nonlinear in $\sigma_{ij}(u, v)$ and appear to be untractable.

References

[1] Alekseevsky D V, Vinogradov A M and Lychagin V V 1991 Basic Ideas and Concepts of Differential Geometry (Berlin: Springer).

[2] Barco A and Prince G E 2001 Solvable symmetry structures in differential form Acta Appl. Math. 66 89–121

[3] Basarab-Horwath P 1992 Integrability by quadratures for systems of involutive vector fields Ukr. Math. J. 43 1236–42
[4] Carinena J F, Del Olmo M and Winternitz P 1993 On the relation between weak and strong invariance of differential equations Lett. Math. Phys. 29 151–63
[5] Catalano-Ferraioli D 2007 Nonlocal aspects of lambda-symmetries and ODEs reduction J. Phys. A: Math. Theor. 40 5479–89
[6] Cicogna G and Gaeta G 1999 Symmetry and Perturbation Theory in Nonlinear Dynamics (Berlin: Springer)
[7] Cicogna G and Gaeta G 2007 Noether theorem for \( \mu \)-symmetries J. Phys. A: Math. Theor. 40 11899–921
[8] Cicogna G, Gaeta G and Morando P 2004 On the relation between standard and \( \mu \)-symmetries for PDEs J. Phys. A: Math. Gen. 37 9467–86
[9] Cicogna G, Gaeta G and Walcher S 2012 Orbital reducibility and a generalization of lambda symmetries J. Lie Theory submitted
[10] Cicogna G, Gaeta G and Walcher S 2012 Dynamical systems and \( \sigma \)-symmetries (in preparation)
[11] Gaeta G 2009 Twisted symmetries of differential equations J. Nonlinear Math. Phys. 16 S107–36
[12] Hartl T and Athorne C 1994 Solvable structures and hidden symmetries J. Phys. A: Math. Gen. 27 3463–71
[13] Krasil’schik I S and Vinogradov A M 1999 Symmetries and Conservation Laws for Differential Equations of Mathematical Physics (Providence, RI: American Mathematical Society)
[14] Morando P and Sammarco S 2012 Variational problems with symmetries: a Pfaffian system approach Acta Appl. Math. 120 255–74
[15] Muriel C and Romero J L 2001 New methods of reduction for ordinary differential equations IMA J. Appl. Math. 66 111–25
[16] Muriel C and Romero J L 2001 \( C^\infty \) symmetries and nonsolvable symmetry algebras IMA J. Appl. Math. 66 477–98
[17] Muriel C and Romero J L 2002 Prolongations of vector fields and the invariants-by-derivation property Theor. Math. Phys. 113 1565–75
[18] Muriel C and Romero J L 2007 \( C^\infty \)-symmetries and nonlocal symmetries of exponential type IMA J. Appl. Math. 72 191–205
[19] Muriel C and Romero J L 2008 Integrating factors and lambda-symmetries J. Nonlinear Math. Phys. 15S3 300–9
[20] Muriel C and Romero J L 2009 First integrals, integrating factors and \( \lambda \)-symmetries of second-order differential equations J. Phys. A: Math. Theor. 42 365207
[21] Muriel C, Romero J L. and Olver P J 2006 Variational \( C^\infty \) symmetries and Euler–Lagrange equations J. Diff. Eqns. 222 164–84
[22] Nucci M C and Leach P G L 2000 The determination of nonlocal symmetries by the technique of reduction of order J. Math. Anal. Appl. 251 871–84
[23] Olver P J 1995 Application of Lie Groups to Differential Equations (Berlin: Springer)
[24] Olver P J 1995 Equivalence, Invariants and Symmetry (Cambridge: Cambridge University Press)
[25] Pucci E and Saccomandi G 2002 On the reduction methods for ordinary differential equations J. Phys. A: Math. Gen. 35 6145–55
[26] Sherring J and Prince G 1992 Geometric aspects of reduction of order Trans. Am. Math. Soc. 334 433–53
[27] Stephani H 1989 Differential Equations: Their Solution Using Symmetries (Cambridge: Cambridge University Press)