HILBERT SCHEMES OF POINTS ON SMOOTH PROJECTIVE SURFACES AND GENERALIZED KUMMER VARIETIES WITH FINITE GROUP ACTIONS

SAILUN ZHAN

Abstract. Göttscbe and Soergel gave formulas for the Hodge numbers of Hilbert schemes of points on a smooth algebraic surface and the Hodge numbers of generalized Kummer varieties. When a smooth projective surface \( S \) admits an action by a finite group \( G \), we describe the action of \( G \) on the Hodge pieces via point counting. Each element of \( G \) gives a trace on \( \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i H^i(S^{[n]}, \mathbb{C}) q^n \). In the case that \( S \) is a K3 surface or an abelian surface, the resulting generating functions give some interesting modular forms when \( G \) acts faithfully and symplectically on \( S \).

1. Introduction

Let \( S \) be a smooth projective surface over \( \mathbb{C} \). In [GS93], the Hodge numbers of the Hilbert scheme of points of \( S \) are computed via perverse sheaves/mixed Hodge modules:

\[
\sum_{n=0}^{\infty} h(S^{[n]}, u, v) t^n = \prod_{m=1}^{\infty} \prod_{p,q} \left( \sum_{i=0}^{\infty} (-1)^i (p+q+1) \binom{h_{pq}}{i} u^{i(p+m-1)} v^{i(q+m-1)} t^{mi} \right)^{(-1)^{p+q+1}},
\]

where \( S^{[n]} \) is the Hilbert scheme of \( n \) points of \( S \), \( h(S^{[n]}, u, v) = \sum_{p,q} h_{pq}(S^{[n]}) u^p v^q \) is the Hodge-Deligne polynomial, and \( h_{pq} \) are the dimensions of the Hodge pieces \( H^{p,q}(S, \mathbb{C}) \). The Hodge numbers of the higher order Kummer varieties (generalized Kummer varieties) of an abelian surface are also computed:

\[
h(K_n(A), -u, -v) = \frac{1}{((1-u)(1-v))^2} \sum_{\alpha \in P(n)} \gcd(\alpha)^4 (uv)^{n-|\alpha|} \left( \prod_{i=1}^{\infty} \sum_{\beta^i \in P(\alpha_i)} \prod_{j=1}^{\beta^i} \frac{1}{j^{\beta^i} \beta^i!}((1-u^j)(1-v^j))^{2\beta^i} \right),
\]

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where $\alpha = (1^{\alpha_1}2^{\alpha_2}...) \text{ is a partition of } n$, $|\alpha|$ is the number of parts, and $gcd(\alpha) := gcd\{i \in \mathbb{Z}|\alpha_i \neq 0\}$.

In this paper $G$ will always be a finite group. We will consider a smooth projective K3 surface $S$ over $\mathbb{C}$ with a $G$-action, and ask whether we can prove similar equalities for $G$-representations. We use an equivariant version of the idea in Göttsche [Göt90], which studies the cohomology groups by counting the number of rational points over finite fields. Then we lift the results to the Hodge level by $p$-adic Hodge theory.

We will consider the $G$-equivariant Hodge-Deligne polynomial for a smooth projective variety $X$

$$E(X; u, v) = \sum_{p,q} (-1)^{p+q} [H^{p,q}(X, \mathbb{C})] u^p v^q,$$

where the coefficients lie in the ring of virtual $G$-representations $R(\mathbb{C})(G)$, of which the elements are the formal differences of isomorphism classes of finite dimensional $\mathbb{C}$-representations of $G$. The addition is given by direct sum and the multiplication is given by tensor product.

**Theorem 1.1.** Let $S$ be a smooth projective surface over $\mathbb{C}$ with a $G$-action. Let $S^{[n]}$ be the Hilbert scheme of $n$ points of $S$. Then we have the following equality as virtual $G$-representations.

$$\sum_{n=0}^{\infty} E(S^{[n]}) t^n = \prod_{m=1}^{\infty} \prod_{p,q} \left( \sum_{i=0}^{h_{p,q}} (-1)^i [\wedge^{i}H^{p,q}(S, \mathbb{C})] u^i(p+m-1) v^i(q+m-1) t^m \right)^{-1}.$$

where $h_{p,q}$ are the dimensions of the Hodge pieces $H^{p,q}(S, \mathbb{C})$.

**Remark 1.2.** Theorem 1.1 has been proved in [Zha21, Theorem 1.1], where the proof uses Nakajima operators. We give a new proof here using the Weil conjecture and $p$-adic Hodge theory.

For a complex K3/abelian surface $S$ with an automorphism $g$ of finite order $n$, $H^0(S, K_S) = \mathbb{C}\omega_S$ has dimension 1, and we say $g$ acts symplectically on $S$ if it acts trivially on $\omega_S$, and $g$ acts non-symplectically otherwise, namely, $g$ sends $\omega_S$ to $\zeta_n^k \omega_S$, $0 < k < n$, where $\zeta_n$ is a primitive $n$-th root of unity.

Denote by $[e(X)]$ the virtual graded $G$-representation $\sum_{i=0}^{\infty} (-1)^i [H^i(X, \mathbb{C})]$ for a smooth projective variety $X$ over $\mathbb{C}$ with a $G$-actoin.
Theorem 1.3. Let $G$ be a finite group which acts faithfully and symplectically on a smooth projective K3 surface $S$ over $\overline{\mathbb{F}_q}$. Suppose $p \nmid |G|$. Then

$$\sum_{n=0}^{\infty} \text{Tr}(g, [e(S^{[n])}) t^n = \exp \left( \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\epsilon(\text{ord}(g^k)) t^{mk}}{k} \right)$$

for all $g \in G$, where $\epsilon(n) = 24 \left( n \prod_{p|n} \left(1 + \frac{1}{p}\right) \right)^{-1}$. In particular, if $G$ is generated by a single element $g$ of order $N$, then we deduce that

| $N$ | $\sum_{n=0}^{\infty} \text{Tr}(g, [e(S^{[n])}) t^n$ |
|-----|----------------------------------|
| 1   | $t/\eta^{24}(t)$                 |
| 2   | $t/\eta^{8}(t)\eta^{8}(t^2)$    |
| 3   | $t/\eta^{6}(t)\eta^{6}(t^3)$   |
| 4   | $t/\eta^{4}(t)\eta^{2}(t^2)\eta^{4}(t^4)$ |
| 5   | $t/\eta^{4}(t)\eta^{4}(t^5)$   |
| 6   | $t/\eta^{2}(t)\eta^{2}(t^2)\eta^{2}(t^3)\eta^{2}(t^6)$ |
| 7   | $t/\eta^{3}(t)\eta^{3}(t^7)$   |
| 8   | $t/\eta^{2}(t)\eta(t^2)\eta(t^4)\eta^2(t^8)$ |

where $\eta(t) = t^{1/24} \prod_{n=1}^{\infty} (1 - t^n)$.

Remark 1.4. If $g$ acts symplectically on $S$, then $g$ has order $N \leq 8$ by [DK09 Theorem 3.3] since the $G$-action is tame. These eta quotients coincide with the results in the characteristic zero case. See [BG19], [BO18, Lemma 3.1], or [Zha21].

Theorem 1.5. Let $g$ be a symplectic automorphism (fixing the origin) of order $N$ on an abelian surface $S$ over $\mathbb{C}$. Then

| $N$ | $\sum_{n=0}^{\infty} \text{Tr}(g, [e(S^{[n])}) t^n$ |
|-----|----------------------------------|
| 1   | 1                                |
| 2   | $\eta^{8}(t^2)/\eta^{16}(t)$    |
| 3   | $\eta^{3}(t^3)/\eta^{9}(t)$     |
| 4   | $\eta^{4}(t^4)/\eta^{4}(t)\eta^{6}(t^2)$ |
| 6   | $\eta^{4}(t^6)/\eta(t^2)\eta^{4}(t^2)\eta^{5}(t^3)$ |

Remark 1.6. If $g$ is a symplectic automorphism on a complex abelian surface, then $g$ has order 1, 2, 3, 4 or 6 by [Fuj88, Lemma 3.3]. These eta quotients coincide with the results of [Pie21 Theorem 1.1] when $G$ is cyclic.
Define a multiplication $\odot$ on the ring of power series $R_C(G)[[u, v, w]]$ by $u^{n_1}v^{m_1}w^{l_1} \odot u^{n_2}v^{m_2}w^{l_2} := u^{n_1+n_2}v^{m_1+m_2}w^{gcd(l_1,l_2)}$.

**Theorem 1.7.** Let $A$ be an abelian surface over $\mathbb{C}$ with a $G$-action. Let $K_n(A)$ be the generalized Kummer variety. Then we have the following equality as virtual $G$-representations:

$$\sum_{n=0}^{\infty} E(K_n(A); u, v)t^n = \frac{(w \frac{d}{dw})^4}{E(A)}$$

$$\odot \left( 1 + w^m \left( -1 + \prod_{p,q} \left( \sum_{i=0}^{b_{p,q}} (-1)^i [\wedge^i H^{p,q}(S, \mathbb{C})] u^{i(p+m-1)}v^{i(q+m-1)}t^{mi} \right)(-1)^{p+q+1} \right) \right).$$

When we say $S$ is a surface with a $G$-action over a field $K$, we mean that both $S$ and the $G$-action can be defined over $K$.

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**2. Preliminaries**

Let $X$ be a smooth projective variety over $\mathbb{C}$. Then we can choose a finitely generated $\mathbb{Z}$-subalgebra $R \subset \mathbb{C}$ such that $X \cong \mathcal{X} \times_S \text{Spec}\mathbb{C}$ for a regular projective scheme $\mathcal{X}$ over $S = \text{Spec}R$, and we can choose a maximal ideal $q$ of $R$ such that $\mathcal{X}$ has good reduction modulo $q$. Since there are comparison theorems between étale cohomology and singular cohomology, we focus on characteristic $p$.

Now let $X$ be a quasi-projective variety over $\mathbb{F}_p$ with an automorphism $\sigma$ of finite order. Suppose $X$ and $\sigma$ can be defined over some finite field $\mathbb{F}_q$. Let $F_q^n \circ \sigma$ be the corresponding geometric Frobenius. Then for $n \geq 1$, the composite $F_q^n \circ \sigma$ is the Frobenius map relative to some new way of lowering the field of definition of $X$ from $\mathbb{F}_p$ to $\mathbb{F}_{q^n}$ ([DL76, Prop.3.3] and [Car85, Appendix(h)]). Then the Grothendieck trace formula implies that $\sum_{k=0}^{\infty} (-1)^k \text{Tr}((F_q^n \circ \sigma)^k, H^k_c(X, \mathbb{Q}_l))$ is the number of fixed points of $F_q^n \circ \sigma$, where $H^k_c(X, \mathbb{Q}_l)$ are the compactly supported $l$-adic cohomology groups.

**Lemma 2.1.** Let $X$ and $Y$ be two smooth projective varieties over $\mathbb{F}_p$ with finite group $G$-actions. Suppose $X, Y$ and the actions of $G$ can be defined over $\mathbb{F}_q$, where
$q$ is a $p$ power. If $|X(\mathbb{F}_p)_{\mathbb{F}_q}| = |Y(\mathbb{F}_p)_{\mathbb{F}_q}|$ for every $n \geq 1$ and $g \in G$, then $H^i(X, \mathbb{Q}_l) \cong H^i(Y, \mathbb{Q}_l)$ as $G$-representations for every $i \geq 0$.

Proof. Fix $g \in G$. Denote by $F_q$ the geometric Frobenius over $\mathbb{F}_q$. Since the finite group action is defined over $\mathbb{F}_q$, the action $g$ commutes with $F_q$ and the action of $g$ on the cohomology group is semisimple. There exists a basis of the cohomology group such that the actions of $g$ and $F_q$ are in Jordan normal forms simultaneously.

Let $\alpha_{i,j}, j = 1, 2, \ldots, a_i$ (resp. $\beta_{i,j}, j = 1, 2, \ldots, b_i$) denote the eigenvalues of $F_q$ acting on $H^i(X, \mathbb{Q}_l)$ (resp. $H^i(Y, \mathbb{Q}_l)$) in such a basis, where $a_i$ (resp. $b_i$) is the $i$-th betti number. Let $c_{i,j}, j = 1, 2, \ldots, a_i$ (resp. $d_{i,j}, j = 1, 2, \ldots, b_i$) denote the eigenvalues of $g$ acting on the same basis of $H^i(X, \mathbb{Q}_l)$ (resp. $H^i(Y, \mathbb{Q}_l)$). Then the Grothendieck trace formula ([DL76 Prop.3.3] and [Car85 Appendix(h)]) implies that

$$|X(\mathbb{F}_p)_{\mathbb{F}_q}| = \sum_{i=0}^{\infty} (-1)^i \text{Tr}((gF_q)^i, H^i(X, \mathbb{Q}_l))$$

Since $|X(\mathbb{F}_p)_{\mathbb{F}_q}| = |Y(\mathbb{F}_p)_{\mathbb{F}_q}|$ for every $n \geq 1$, we have

$$\sum_{i=0}^{\infty} (-1)^i \sum_{j=1}^{a_i} c_{i,j} \alpha_{i,j}^n = \sum_{i=0}^{\infty} (-1)^i \sum_{j=1}^{b_i} d_{i,j} \beta_{i,j}^n$$

for every $n \geq 1$. By linear independence of the characters $\chi_\alpha : \mathbb{Z}^+ \to \mathbb{C}, n \mapsto \alpha^n$ and the fact that $\alpha_{i,j}, \beta_{i,j}, j = 1, 2, \ldots$ all have absolute value $q^{i/2}$ by Weil’s conjecture, we deduce that $a_i = b_i$ and $\sum_{j=1}^{a_i} c_{i,j} = \sum_{j=1}^{b_i} d_{i,j}$ for each $i$. But since $g$ is arbitrary, this implies that the $G$-representations $H^i(X, \mathbb{Q}_l)$ and $H^i(Y, \mathbb{Q}_l)$ are the same. □

Proposition 2.2. Let $X$ be a smooth projective variety with a $G$-action over $\mathbb{F}_q$. Denote the dimension of $X$ by $N$. Then

$$\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i [H^i(X^{(k)}_{\mathbb{F}_p}, \mathbb{Q}_l)] z^i t^k = 2N \prod_{j=0}^{b_j} \left( \sum_{i=0}^{\infty} (-1)^i [\wedge^i H^j(X^{(k)}_{\mathbb{F}_p}, \mathbb{Q}_l)] z^i t^i \right)^{(-1)^{j+1}},$$

where the coefficients lie in $R_{\mathbb{Q}_l}(G)$.

Proof. By the Weil conjectures, we have

$$\exp(\sum_{r=1}^{\infty} |X(\mathbb{F}_q)| t^r) = \prod_{k=0}^{\infty} |X^{(k)}(\mathbb{F}_q)| t^k = \prod_{k=0}^{\infty} |X^{(k)}(\mathbb{F}_p)| t^k = 2N \prod_{j=0}^{b_j} \left( \prod_{i=1}^{\infty} (1 - \alpha_{j,i} t) \right)^{(-1)^{j+1}},$$

where $\alpha_{j,i}$ are the eigenvalues of $F_q$ on $H^j(X^{(k)}_{\mathbb{F}_p}, \mathbb{Q}_l)$.
By the discussion at the beginning of the section and the Grothendieck trace formula, we deduce that

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \sum_{i} h_{k,m,i} \beta_{k,m,i}^n t^k = 2N \prod_{j=0}^{b_j} \left( \prod_{i=1}^{(1-g_{j,i}) \alpha_{j,i}^n t} \right)^{(-1)^{j+1}}$$

where $h_{k,m,i}$ (resp. $\beta_{k,m,i}$) are the eigenvalues of $g$ (resp. $F_q$) on $H^m(X_{\mathbb{P}^p}, \mathbb{Q}_l)$, and $g_{j,i}$ are the eigenvalues of $g$ on $H^j(X_{\mathbb{P}^p}, \mathbb{Q}_l)$. Hence we deduce that the trace of $g$ on the left hand side equals the trace of $g$ on the right hand side for each graded piece in the equality in Proposition 2.2 by the proof of Lemma 2.1. \qed

We obtain the information of Hodge pieces via $p$-adic Hodge theory by using an equivariant version of the method in [Ito03, §4].

**Proposition 2.3.** [Ser68, I. 2.3] Let $K$ be a number field, $m,m' \geq 1$ be integers, and $l$ be a prime number. Let

$$\rho: \text{Gal}({\overline{K}}/K) \rightarrow GL(m, \mathbb{Q}_l), \quad \rho': \text{Gal}({\overline{K}}/K) \rightarrow GL(m', \mathbb{Q}_l)$$

be continuous $l$-adic Gal($\overline{K}/K$)-representations such that $\rho$ and $\rho'$ are unramified outside a finite set $S$ of maximal ideals of $\mathcal{O}_K$. If

$$\text{Tr}(\rho(Frob_p)) = \text{Tr}(\rho'(Frob_p)) \quad \text{for all maximal ideals } p \notin S;$$

then $\rho$ and $\rho'$ have the same semisimplifications as Gal($\overline{K}/K$)-representations. Here $Frob_p$ is the geometric Frobenius at $p$.

Let $p$ be a prime number and $F$ be a finite extension of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be a $p$-adic completion of an algebraic closure $\overline{F}$ of $F$. Define $\mathbb{Q}_p(0) = \mathbb{Q}_p$, $\mathbb{Q}_p(1) = \lim_{\leftarrow} \mu_p^n \otimes \mathbb{Q}_p$, and for $n \geq 1$, $\mathbb{Q}_p(n) = \mathbb{Q}_p(1)^{\otimes n}$, $\mathbb{Q}_p(-n) = \text{Hom}(\mathbb{Q}_p(n), \mathbb{Q}_p)$. Moreover, we define $\mathbb{C}_p(n) = \mathbb{C}_p \otimes \mathbb{Q}_p \mathbb{Q}_p(n)$, on which $\text{Gal}(F/F)$ acts diagonally. It is known that $(\mathbb{C}_p)^{\text{Gal}(F/F)} = F$ and $(\mathbb{C}_p(n))^{\text{Gal}(F/F)} = 0$ for $n \neq 0$.

Let $B_{HT} = \oplus_{n \in \mathbb{Z}} \mathbb{C}_p(n)$ be a graded $\mathbb{C}_p$-module with an action of $\text{Gal}(\overline{F}/F)$. For a finite dimensional $\text{Gal}(\overline{F}/F)$-representation $V$ over $\mathbb{Q}_p$, we define a finite dimensional graded $F$-module $D_{HT}(V)$ by $D_{HT}(V) = (V \otimes \mathbb{Q}_p B_{HT})^{\text{Gal}(F/F)}$. The graded module structure of $D_{HT}(V)$ is induced from that of $B_{HT}$. In general, it is known that

$$\dim_F D_{HT}(V) \leq \dim_{\mathbb{Q}_p} V.$$

If the equality holds, $V$ is called a Hodge-Tate representation.
Theorem 2.4. [Fal88] [Tsu99] (Hodge-Tate decomposition) Let $X$ be a proper smooth variety over $F$ and $k$ be an integer. The $p$-adic étale cohomology $H^k_{\text{ét}}(X_F, \mathbb{Q}_p)$ of $X_F = X \otimes_F \bar{F}$ is a finite dimensional $\text{Gal}(\bar{F}/F)$-representation over $\mathbb{Q}_p$. Then, $H^k_{\text{ét}}(X_F, \mathbb{Q}_p)$ is a Hodge-Tate representation. Moreover, there exists a canonical and functorial isomorphism
\[
\bigoplus_{i+j=k} H^i(X, \Omega^j_X) \otimes_F \mathbb{C}_p(-j) \cong H^k_{\text{ét}}(X_F, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p
\]
of $\text{Gal}(\bar{F}/F)$-representations, where $\text{Gal}(\bar{F}/F)$ acts on $H^i(X, \Omega^j_X)$ trivially and the right hand side diagonally.

Now for a finite dimensional $\text{Gal}(\bar{F}/F)$-representation $V$ over $\mathbb{Q}_p$, suppose it is also a $G$-representation such that the $G$-action commutes with the $\text{Gal}(\bar{F}/F)$-action. In this case, we call it a $\text{Gal}(\bar{F}/F)$-$G$-representation and we define a $G$-representation over $F$:
\[
[h^n(V)] := (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n))^{\text{Gal}(\bar{F}/F)}.
\]

Lemma 2.5. Let $W_2$ be a Hodge-Tate $\text{Gal}(\bar{F}/F)$-$G$-representation and
\[
0 \to W_1 \to W_2 \to W_3 \to 0
\]
be an exact sequence of finite dimensional $\text{Gal}(\bar{F}/F)$-$G$-representations over $\mathbb{Q}_p$. Then $W_1$ and $W_3$ are Hodge-Tate representations and
\[
[h^n(W_2)] = [h^n(W_1)] \oplus [h^n(W_3)] = [h^n(W_1 \oplus W_3)]
\]
as $G$-representations for all $n$.

Proof. It follows from [Ito03, Lemma 4.4] that $W_1$ and $W_3$ are Hodge-Tate representations and we have the following short exact sequence of $G$-representations
\[
0 \to D_{HT}(W_1) \to D_{HT}(W_2) \to D_{HT}(W_3) \to 0,
\]
which implies that
\[
[h^n(W_2)] = [h^n(W_1)] \oplus [h^n(W_3)] = [h^n(W_1 \oplus W_3)].
\]

Corollary 2.6. Let $X$ be a proper smooth variety over $F$ with a $G$-action. Then
\[
H^i(X, \Omega^j_X) = [h^i(H^{i+j}(X_F, \mathbb{Q}_p)^{ss})] 
\]
as $G$-representations for all $i, j$. 

\[\square\]
where \( H^{i+j}(X, \mathbb{Q}_p) \) denotes the semisimplification of \( H^{i+j}(X, \mathbb{Q}_p) \) as a \( \text{Gal}(\overline{F}/F) \)-representation.

**Proof.** By theorem 2.4, if we take the \( \text{Gal}(\overline{F}/F) \)-invariant of \( H^{i+j}(X, \mathbb{Q}_p) \), we have

\[
H^i(X, \Omega^j_X) = [h^j(H^{i+j}(X, \mathbb{Q}_p))].
\]

On the other hand, since \( H^{i+j}(X, \mathbb{Q}_p) \) is a \( \text{Gal}(\overline{F}/F) \)-Hodge-Tate representation,

\[
[h^j(H^{i+j}(X, \mathbb{Q}_p))] = [h^j(H^{i+j}(X, \mathbb{Q}_p))]
\]

by lemma 2.5. Hence we are done. \( \Box \)

**Theorem 2.7.** Let \( X \) and \( Y \) be \( n \)-dimensional smooth projective varieties over a number field \( K \) with \( G \)-actions. Suppose for all but finitely many good reductions, we have

\[
|X(F_p)^{gF_q^n}| = |Y(F_p)^{gF_q^n}| \quad \text{for every } n \geq 1 \text{ and } g \in G,
\]

where \( X, Y \) are the good reductions over \( F_q \). Then

\[ H^{p,q}(\mathcal{X}_\mathbb{C}) \cong H^{p,q}(\mathcal{Y}_\mathbb{C}). \]

for all \( p, q \) as \( G \)-representations.

**Proof.** By the proof of Lemma 2.1 and Proposition 2.3, we deduce that \( H^i(\mathcal{X}_K, \mathbb{Q}_l) \) and \( H^i(\mathcal{Y}_K, \mathbb{Q}_l) \) have the same semisimplifications as \( \text{Gal}(\overline{K}/K) \)-\( G \)-representations.

Now take a maximal ideal \( \mathfrak{q} \) of \( \mathcal{O}_K \) dividing \( l \). Let \( F \) be the completion of \( K \) at \( \mathfrak{q} \). Fix an embedding \( \overline{K} \rightarrow \overline{F} \). Then we have an inclusion \( \text{Gal}(\overline{F}/F) \subset \text{Gal}(\overline{K}/K) \). Therefore, \( H^i(\mathcal{X}_F, \mathbb{Q}_l) \) and \( H^i(\mathcal{Y}_F, \mathbb{Q}_l) \) have the same semisimplifications as \( \text{Gal}(\overline{F}/F) \)-\( G \)-representations. By Corollary 2.6 we conclude that

\[ H^q(\mathcal{X}_\mathbb{C}, \Omega^p_{\mathcal{X}_\mathbb{C}}) \cong H^q(\mathcal{Y}_\mathbb{C}, \Omega^p_{\mathcal{Y}_\mathbb{C}}) \]

for all \( p, q \) as \( G \)-representations. \( \Box \)

### 3. Hilbert scheme of points

We denote by \( X^{[n]} \) the component of the Hilbert scheme of a projective scheme \( X \) parametrizing subschemes of length \( n \) of \( X \). For properties of Hilbert scheme of points, see references [Iar77], [Göt94] and [Nak99].
Lemma 3.1. Let $S$ be a smooth projective surface with a $G$-action over $\mathbb{F}_q$. Suppose $g \in G$ and let $F_q$ be the geometric Frobenius. Then

$$\sum_{n=0}^{\infty} |S^{[n]}(\mathbb{F}_q)^{gF_q}| t^n = \prod_{r=1}^{\infty} \left( \sum_{n=0}^{\infty} |\text{Hilb}^n(\hat{O}_{S_{\mathbb{F}_q}})(\mathbb{F}_q)^{g^r F_q})| t^{nr} \right)^{|P_r(S,gF_q)|},$$

where $\text{Hilb}^n(\hat{O}_{S_{\mathbb{F}_q}})$ is the punctual Hilbert scheme of $n$ points at some $g^r F_q$-fixed point $x \in S(\mathbb{F}_q)$, and $P_r(S,gF_q)$ is the set of primitive $0$-cycles of degree $r$ of $gF_q$ on $S$, whose elements are of the form $\sum_{i=0}^{r-1} g^i F_q(x)$ with $x \in S(\mathbb{F}_q)^{g^r F_q} \setminus (\cup_{j<r} S(\mathbb{F}_q)^{g^j F_q})$.

Proof. Let $Z \in S^{[n]}(\mathbb{F}_q)^{gF_q}$. Suppose $(n_1, \ldots, n_r)$ is a partition of $n$ and $Z = (Z_1, \ldots, Z_r)$ with $Z_i$ being the closed subscheme of $Z$ supported at a single point with length $n_i$. Then $\text{Supp} Z$ decomposes into $gF_q$ orbits. We can choose an ordering $\leq$ on $S(\mathbb{F}_q)$. In each orbit, we can find the smallest $x_j \in S(\mathbb{F}_q)$. Suppose $Z_j$ which is supported on $x_j$ has order $k$. Then the component of $Z$ which is supported on the orbit of $x_j$ is determined by $Z_j$, namely, it is $\cup_{i=0}^{k-1} g^i F_q(Z_j)$ with length $kl$. Also notice that $Z_j$ is fixed by $g^k F_q$. Hence, to give an element of $S^{[n]}(\mathbb{F}_q)^{gF_q}$ is the same as choosing some $gF_q$ orbits and for each orbit choosing some element in $\text{Hilb}^n(\hat{O}_{S_{\mathbb{F}_q}})(\mathbb{F}_q)^{g^k F_q}$ for some $g^k F_q$-fixed point $x$ in this orbit such that the final length altogether is $n$. Combining all of these into power series, we get the desired equality. 

The idea we used above is explained in detail in [Göt90, lemma 2.7]. We implicitly used the fact that $\pi : (S^{[n]}(\mathbb{F}_q[[s,t]])_{\text{red}} \to S$ is a locally trivial fiber bundle in the Zariski topology with fiber $\text{Hilb}^n(\mathbb{F}_q[[s,t]])_{\text{red}}$ [Göt94, Lemma 2.1.4], where $S^{[n]}(\mathbb{F}_q)$ parametrizes closed subschemes of length $n$ that are supported on a single point.

We need the following key lemma.

Lemma 3.2. Let $S$ be a smooth projective surface with a $G$-action over $\mathbb{F}_q$. If $x \in S(\mathbb{F}_q)^{gF_q}$, where $g \in G$ and $F_q$ is the geometric Frobenius, then

$$|\text{Hilb}^n(\hat{O}_{S_{\mathbb{F}_q}})(\mathbb{F}_q)^{gF_q}| = |\text{Hilb}^n(\mathbb{F}_q[[s,t]])(\mathbb{F}_q)^{F_q}|.$$

We will prove this lemma later in this section.

From Lemma 3.2, we observe that $|\text{Hilb}^n(\hat{O}_{S_{\mathbb{F}_q}})(\mathbb{F}_q)^{gF_q}|$ is a number independent of the choice of the $gF_q$-fixed point $x$. 
We denote $\text{Hilb}^n(F_q[[s, t]])$ by $V_n$. Combining Lemma 3.1 and Lemma 3.2, we deduce that

$$\sum_{n=0}^{\infty} \left| S^{[n]}(F_q) q^{F_q} | t^n \right| = \prod_{r=1}^{\infty} \left( \sum_{n=0}^{\infty} \left| V_n(F_q) F_q \right| t^{nr} \right)^{|P_r(S, gF_q)|}.$$ 

Recall the following structure theorem for the punctual Hilbert scheme of points.

**Proposition 3.3.** [ES87, Prop 4.2] Let $k$ be an algebraically closed field. Then $\text{Hilb}^n(k[[s, t]])$ over $k$ has a cell decomposition, and the number of $d$-cells is $P(d, n - d)$, where $P(x, y) := \#\{\text{partition of } x \text{ into parts } \leq y\}$.

Denote by $p(n, d)$ the number of partitions of $n$ into $d$ parts. Then $p(n, d) = P(n - d, d)$. Now we can proceed similarly as in the proof of [Göt90, Lemma 2.9].

**Proof of Theorem 1.1.** Since we have

$$\prod_{i=1}^{\infty} \left( \frac{1}{1 - z^{-1} t^i} \right) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} p(n, n - i) t^n z^i,$$

by Proposition 3.3 we get

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \#\{m\text{-dim cells of } \text{Hilb}^n(F_p[[s, t]])\} t^n z^m = \prod_{i=1}^{\infty} \frac{1}{1 - z^{-1} t^i}.$$

Fix $N \in \mathbb{N}$. Then by choosing sufficiently large $q$ powers $Q$ such that the cell decomposition of $V_{n, F_q}$ is defined over $F_Q$ for $n \leq N$, we deduce that

$$\sum_{n=0}^{\infty} \left| V_{n, F_q}(F_Q) \right| t^{nr} \equiv \prod_{i=1}^{\infty} \frac{1}{1 - Q_r(i-1) t^{ri}} \mod t^N.$$
Now consider a good reduction of \( S \) over \( \mathbb{F}_q \).
\[
\sum_{n=0}^{\infty} \left| S(n) \right|^{qF_q} |t^n| \equiv \prod_{r=1}^{\infty} \prod_{i=1}^{\infty} \left( \frac{1}{1 - Q^{r(i-1)}t^{-i/r}} \right) \mod t^N
\]
\[
= \exp \left( \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \sum_{h=1}^{\infty} |P_r(S, gF_Q)|Q^{hr(i-1)}t^{hri/h}/h \right)
\]
\[
= \exp \left( \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{r|m} r|P_r(S, gF_Q)|Q^{m(i-1)}t^{mi/m} \right) \right)
\]
\[
= \prod_{i=1}^{\infty} \exp \left( \sum_{m=1}^{\infty} |S(\mathbb{F}_q)|^{qF_q} |Q^{m(i-1)}t^{mi/m} \right)
\]
\[
= \prod_{i=1}^{\infty} \sum_{n=0}^{\infty} |S(n)\mathbb{F}_q|^{qF_q} |Q^{n(i-1)}t^{ni}.
\]

By replacing \( Q \) by \( Q \)-powers and using the proof of Proposition 2.7 and Theorem 2.7, we obtain
\[
\sum_{n=0}^{\infty} E(S^{[n]})t^n = \prod_{m=1}^{\infty} \prod_{p,q} \left( \sum_{i=0}^{b_{p,q}} (-1)^i [\wedge^i H^p,q(S, \mathbb{C})]u^{i(p+m-1)}v^{i(q+m-1)}t^{mi} \right)^{(-1)^{p+q+1}},
\]
since we can reduce to the case where everything is defined over a number field \( K \) as in [Ito03, Prop. 5.1].

**Corollary 3.4.** For a smooth projective surface \( S \) over \( \mathbb{F}_p \) or \( \mathbb{C} \), we have
\[
\sum_{n=0}^{\infty} \left[ e(S)^{[n]} \right] t^n = \prod_{m=1}^{\infty} \prod_{j=0}^{4} \left( \sum_{i=0}^{b_j} (-1)^i [\wedge^i H^j(S, \mathbb{Q}_l)][-2i(m-1)]t^{mi} \right)^{(-1)^{j+1}},
\]
where the coefficients lie in \( R_{\mathbb{Q}_l}(G) \), and \([-2i(m-1)]\) indicates shift in degrees.

**Remark 3.5.** Notice that the generating series of the topological Euler characteristic of \( S^{[n]} \) is \( \sum_{n=0}^{\infty} e(S^{[n]})t^n = \prod_{m=1}^{\infty} (1 - t^m)^{-e(S)} \). But this is not the case if we consider \( G \)-representations and regard \( \prod_{m=1}^{\infty} (1 - t^m)^{-e(S)} \) as
\[
\exp(\sum_{m=1}^{\infty} [e(S)](-\log(1 - t^m))) = \exp(\sum_{m=1}^{\infty} [e(S)|\sum_{k=1}^{\infty} t^{mk}/k])).
\]

What we have is actually
\[
\sum_{n=0}^{\infty} \text{Tr}(g, [e(S^{[n]})])t^n = \prod_{m=1}^{\infty} \left( \frac{\prod_{i=1}^{b_2} (1 - g_{1,i}t^m)\prod_{i=1}^{b_3} (1 - g_{3,i}t^m)}{(1 - t^m)(\prod_{i=1}^{b_2} (1 - g_{2,i}t^m))(1 - t^m)} \right)^{(-1)^{j+1}},
\]
\[ \exp \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{t^{nk}}{k} \left( 1 - \sum_{i=1}^{b_1} g_{1,i}^k + \sum_{i=1}^{b_2} g_{2,i}^k - \sum_{i=1}^{b_3} g_{3,i}^k + 1 \right) \right) \].

We will use this expression to determine the \( G \)-representation \( \epsilon(S^{[n]}) \) later when \( S \) is a K3 surface or an abelian surface.

Now we start to prove lemma \( \text{3.2} \).

Let \( S \) be a smooth projective surface over \( \mathbb{F}_q \) with an automorphism \( g \) over \( \mathbb{F}_q \) of finite order. If \( x \in S(\mathbb{F}_q)^{gF_q} \) where \( F_q \) is the geometric Frobenius, then \( x \) lies over a closed point \( y \in S \). Denote the residue degree of \( y \) by \( N \). Hence \( x \in S(\mathbb{F}_q^N) \) and there are \( N \) geometric points \( x, F_q(x), ..., F_q^{N-1}(x) \) lying over \( y \).

Let us study the relative Hilbert scheme of \( n \) points at a closed point.

\[ \text{Hilb}^n(\text{Spec}(\mathcal{O}_{S,y})/\text{Spec}\mathbb{F}_q) \cong \text{Hilb}^n(\text{Spec}(\mathbb{F}_q^N[[s,t]])/\text{Spec}\mathbb{F}_q). \]

Since \( g \) and \( \mathbb{F}_q \) fix \( y \), they act on this Hilbert scheme. Over \( \mathbb{F}_q \), we have

\[ \text{Hilb}^n(\text{Spec}(\mathcal{O}_{S,y})/\text{Spec}\mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \cong \text{Hilb}^n(\text{Spec}(\mathbb{F}_q^N[[s,t]])/\text{Spec}\mathbb{F}_q) \]

by the base change property of the Hilbert scheme. Denote by \( u \) a primitive element of the field extension \( \mathbb{F}_q^N/\mathbb{F}_q \) and denote by \( f(x) \) the irreducible polynomial of \( u \) over \( \mathbb{F}_q \). Since we have an \( \mathbb{F}_q \)-algebra isomorphism

\[ \mathbb{F}_q \otimes_{\mathbb{F}_q} \mathbb{F}_q^N \cong \mathbb{F}_q \otimes_{\mathbb{F}_q} (\mathbb{F}_q[x]/(f(x))) \cong \mathbb{F}_q[x]/(x-u) \times ... \times \mathbb{F}_q[x]/(x-u^{N-1}) \]

by the Chinese Remainder Theorem, we deduce that

\[ \text{Hilb}^n(\text{Spec}(\mathcal{O}_{S,y})/\text{Spec}\mathbb{F}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q \cong \text{Hilb}^n(\text{Spec}((\mathbb{F}_q \times ... \times \mathbb{F}_q)[[s,t]])/\text{Spec}\mathbb{F}_q) \]

\[ \cong \text{Hilb}^n(\coprod \text{Spec}\mathbb{F}_q[[s,t]]/\text{Spec}\mathbb{F}_q). \]

Hence the \( \mathbb{F}_q \)-valued points of \( \text{Hilb}^n(\text{Spec}(\mathcal{O}_{S,y})/\text{Spec}\mathbb{F}_q) \) correspond to the closed subschemes of degree \( n \) of \( \coprod \text{Spec}\mathbb{F}_q[[s,t]] \), i.e. the closed subschemes of degree \( n \) of \( S \) whose underlying space is a subset of the points \( x, F_q(x), ..., F_q^{N-1}(x) \).

Since \( F_q \) acts on \( \mathbb{F}_q^N[[s,t]] \) by sending \( s \) to \( s^q \), \( t \) to \( t^q \) and \( c \in \mathbb{F}_q^N \) to \( c^q \), we deduce from the above discussion that \( F_q \) acts on \( (\mathbb{F}_q \times ... \times \mathbb{F}_q)[[s,t]] \) by sending \( s \) to \( s^q \), \( t \) to \( t^q \) and \( (\alpha_0, \alpha_1, ..., \alpha_{N-2}, \alpha_{N-1}) \in \mathbb{F}_q \times ... \times \mathbb{F}_q \) to \( (\alpha_1, \alpha_2, ..., \alpha_{N-1}, \alpha_0) \). This is actually an algebraic assertion, which can also be seen geometrically. For example, \( F_q \) is a \( \mathbb{F}_q \)-morphism from \( \mathcal{O}_{\mathcal{S}_{\mathbb{F}_q}, x} \) to \( \mathcal{O}_{\mathcal{S}_{\mathbb{F}_q}, x} \cong (\{0\} \times \mathbb{F}_q \times ... \times \{0\})[[s,t]] \) to \( (\mathbb{F}_q \times \{0\} \times ... \times \{0\})[[s,t]]. \)
Let $\sigma$ be an element of $\text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$. Recall that for an $\mathbb{F}_q$-vector space $V$, a $\sigma$-linear map $f : V \to V$ is an additive map on $V$ such that $f(\alpha v) = \sigma(\alpha)f(v)$ for all $\alpha \in \mathbb{F}_q$ and $v \in V$.

**Lemma 3.6.** Let $H = \langle g \rangle$. Suppose $p \nmid |H|$. Then we can choose $s$ and $t$ such that $g$ acts on $\mathbb{F}_q[[s, t]]$ $\sigma$-linearly, where $\sigma$ is the inverse of the Frobenius automorphism of $\text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$.

**Proof.** The automorphism $g$ acts as an $\mathbb{F}_q$-automorphism on $\mathbb{F}_q[[s, t]]$ fixing the ideal $(s, t)$ and sending $\mathbb{F}_q$ to $\mathbb{F}_q$. Since we know $F_q$ sends $(\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1}) \in \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ to $(\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_0)$ and $gF_q$ fixes the geometric points $x, F_q(x), \ldots, F_q^{N-1}(x)$, we deduce that $g$ sends $(\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1}) \in \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ to $(\alpha_{N-1}, \alpha_0, \ldots, \alpha_{N-3}, \alpha_{N-2})$. Hence $g(\alpha) = \sigma(\alpha)$ for all $\alpha \in \mathbb{F}_q$ where $\sigma$ is the inverse of the Frobenius automorphism.

For any element $h \in H$, we write $h(s) = as + bt + \ldots$ and $h(t) = cs + dt + \ldots$ where $a, b, c, d \in \mathbb{F}_q$ since $h$ commutes with $F_q$. Define an automorphism $\rho(h)$ of $\mathbb{F}_q[[s, t]]$ by $\rho(h)(s) = as + bt$, $\rho(h)(t) = cs + dt$ and the action of $\rho(h)$ on $\mathbb{F}_q$ is the same as the action of $h$. Then we denote the $\mathbb{F}_q$-automorphism $\frac{1}{|H|} \sum_{h \in H} h\rho(h)^{-1}$ by $\theta$. Notice that $\theta$ is an automorphism because the linear term of $\theta$ is an invertible matrix, and here is the only place we use the assumption that $p \nmid |G|$. We deduce that $g\theta = \theta\rho(g)$, which implies $\theta^{-1}g\theta = \rho(g)$. Hence we are done. \hfill \Box

The above discussion implies that the $g$-action on $(\mathbb{F}_q \times \cdots \times \mathbb{F}_q)[[s, t]]$ is given by sending $s$ to $(a, \ldots, a)s + (b, \ldots, b)t$, $t$ to $(c, \ldots, c)s + (d, \ldots, d)t$ and $(\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1}) \in \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ to $(\alpha_{N-1}, \alpha_0, \ldots, \alpha_{N-3}, \alpha_{N-2})$.

Hence the action of $gF_q$ on $(\mathbb{F}_q \times \cdots \times \mathbb{F}_q)[[s, t]]$ is given by sending $s$ to $(a, \ldots, a)s^q + (b, \ldots, b)t^q$, $t$ to $(c, \ldots, c)s^q + (d, \ldots, d)t^q$ and $(\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1}) \in \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ to itself. This implies that $gF_q$ acts on each complete local ring, which is what we expected since $gF_q$ fixes each geometric point over $y$. In particular, it acts on $\overline{\mathcal{O}_{\mathbb{F}_q[x]}} \cong (\mathbb{F}_q \times \{0\} \times \cdots \times \{0\})[[s, t]] \cong \mathbb{F}_q[[s, t]]$.

Recall that $\text{Hilb}^n(\overline{\mathcal{O}_{\mathbb{F}_q[x]}}(\mathbb{F}_q))$ parametrizes closed subschemes of degree $n$ of $\mathcal{S}_{\mathbb{F}_q}$ supported on $x$.

**Proof of Lemma 3.2** First we define an $\mathbb{F}_q$-automorphism $\tilde{g}$ on $\mathbb{F}_q[[s, t]]$ by $\tilde{g}(s) = as + bt$ and $\tilde{g}(t) = cs + dt$.\hfill \Box
Recall that the action of $F_q$ on $\mathbb{F}_q[[s, t]]$ is an $\mathbb{F}_q$-endomorphism sending $s$ to $s^q$ and $t$ to $t^q$. By the above discussion, we observe that the action of $gF_q$ on $\mathbb{F}_q[[s, t]]$ on the left is the same as the action of $\tilde{g}F_q$ on $\mathbb{F}_q[[s, t]]$ on the right. Hence we have

$$|\text{Hilb}^n(\mathcal{O}_{\mathbb{S}_\mathbb{F}_q})^{gF_q}| = |\text{Hilb}^n(\mathbb{F}_q[[s, t]])^{\tilde{g}F_q}|.$$ 

Now for the right hand side, $\tilde{g}$ is an automorphism of finite order and $F_q$ is the geometric Frobenius. Then by the Grothendieck trace formula, we have

$$|\text{Hilb}^n(\mathbb{F}_q[[s, t]])^{\tilde{g}F_q}| = \sum_{k=0}^{\infty} (-1)^k \text{Tr}((\tilde{g}F_q)^*, H^k_c(\text{Hilb}^n(\mathbb{F}_q[[s, t]]), \mathbb{Q}_l)).$$

But the action of $\tilde{g}$ factors through $GL_2(\mathbb{F}_q)$. Now we use the fact that if $G$ is a connected algebraic group acting on a separated and finite type scheme $X$, then the action of $g \in G$ on $H^*_c(X, \mathbb{Q}_l)$ is trivial [DL76, Corollary 6.5]. Hence we have

$$|\text{Hilb}^n(\mathbb{F}_q[[s, t]])^{\tilde{g}F_q}| = \sum_{k=0}^{\infty} (-1)^k \text{Tr}((F_q)^*, H^k_c(\text{Hilb}^n(\mathbb{F}_q[[s, t]]), \mathbb{Q}_l))$$

$$= |\text{Hilb}^n(\mathbb{F}_q[[s, t]])^{F_q}|.$$ 

□

Suppose $S$ is a smooth projective K3 surface over $\mathbb{F}_p$ with a $G$-action. Recall that a Mathieu representation of a finite group $G$ is a 24-dimensional representation on a vector space $V$ over a field of characteristic zero with character

$$\chi(g) = \epsilon(\text{ord}(g)),$$

where

$$\epsilon(n) = 24(n \prod_{p|n} (1 + \frac{1}{p}))^{-1}.$$ 

**Proposition 3.7.** [DK09, Proposition 4.1] Let $G$ be a finite group of symplectic automorphisms of a K3 surface $X$ defined in characteristic $p > 0$. Assume that $p \nmid G$. Then for any prime $l \neq p$, the natural representation of $G$ on the $l$-adic cohomology groups $H^*(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24}$ is Mathieu.

**Proof of theorem 1.3.** By Remark 3.5 we deduce that

$$\sum_{n=0}^{\infty} \text{Tr}(g, [e(S^{[n]})]) t^n = \exp \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\text{Tr}(g^k, [e(S)]) t^{nk}}{k} \right).$$

Then by Proposition 3.7, we obtain the equality we want.
When $G$ is a cyclic group of order $N$, we know that $N \leq 8$ by \cite[Theorem 3.3]{DK09}. Then the proof is the same as the proof in \cite{Zha21} in the characteristic zero case. □

**Proof of theorem 1.5.** If $g$ is a symplectic automorphism (fixing the origin) on a complex abelian surface, then $g$ has order 1, 2, 3, 4 or 6 by \cite[Lemma 3.3]{Fuj88}. We will do the case when the order $N = 4$, and the calculation for other cases are similar. By \cite[Page 33]{Fuj88}, we know the explicit action of $g$ on the torus $S = \mathbb{C}^2/\Lambda$ in each case. If $N = 4$, then the action on $H^1(S, \mathbb{C})$ is given by

$$
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
$$

Hence we deduce that $\text{Tr}(g, [e(S)]) = \text{Tr}(g^3, [e(S)]) = 4$, and $\text{Tr}(g^2, [e(S)]) = 16$. Now

$$
\sum_{n=0}^{\infty} \text{Tr}(g, [e(S^{[n]})]) t^n = \exp \left( \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\text{Tr}(g^k, [e(S)]) t^{mk}}{k} \right) 
$$

$$
= \exp \left( \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{4t^{mk}}{k} + \sum_{m=1}^{\infty} \sum_{k=2}^{\infty} \frac{16t^{mk}}{k} \right) 
$$

$$
= \exp \left( \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{4t^{mk}}{k} - \sum_{k=1}^{\infty} \frac{4t^{2mk}}{2k} \right) + \sum_{m=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{16t^{2mk}}{2k} - \sum_{k=1}^{\infty} \frac{16t^{4mk}}{4k} \right) \right) 
$$

$$
= \prod_{m=1}^{\infty} \frac{(1 - t^m)^{-4}}{(1 - t^{2m})^{-2}} \prod_{m=1}^{\infty} \frac{(1 - t^{2m})^{-8}}{(1 - t^{4m})^{-4}} 
$$

$$
= \frac{\eta^4(t)}{\eta^4(t^2)} \eta^6(t^2). 
$$

□

**Remark 3.8.** Fix a smooth projective surface $S$ and an automorphism $g$ of finite order. From the proof of Theorem 1.3 or Theorem 1.5, we notice that if $\text{Tr}(g^k, [e(S)])$ only depends on the order of $g^k$ in the cyclic group $\langle g \rangle$ for $k \geq 0$, then the generating function $\sum_{n=0}^{\infty} \text{Tr}(g, [e(S^{[n]})]) t^n$ is an eta quotient by the inclusion-exclusion principle.
4. Generalized Kummer varieties

Let $A$ be an abelian surface over $\mathbb{C}$. Let $\omega_n : A[n] \to A^{(n)}$ be the Hilbert-Chow morphism and let $g_n : A^{(n)} \to A$ be the addition map. The generalized Kummer variety of $A$ is defined to be

$$K_n(A) := \omega_n^{-1}(g_n^{-1}(0)).$$

This is a smooth projective holomorphic symplectic variety. We follow the strategy in [Göt94].

Now suppose $A$ is an abelian surface with a $G$-action over $\mathbb{F}_q$. Define the map $\gamma_n$ by

$$\gamma_n : \prod_{\alpha \in P(n)} \left( \prod_{i=1}^{\infty} S^{(\alpha_i)}(\mathbb{F}_q)^{g_F q} \right) \times A^{n-|\alpha|}(\mathbb{F}_q) \to S^{(n)}(\mathbb{F}_q)^{g_F q},$$

$$((\zeta_i), v) \mapsto \sum_i i \cdot \zeta_i.$$

Lemma 4.1. For any $\zeta \in S^{(n)}(\mathbb{F}_q)^{g_F q}$, we have $|\gamma_n^{-1}(\zeta) = |\omega_n^{-1}(\zeta)|$.

Proof. Let $\zeta = \sum_{i=1}^r n_i \zeta_i \in S^{(n)}(\mathbb{F}_q)^{g_F q}$, where $\zeta_i$ are distinct primitive cycles of degree $d_i$. Then

$$|\omega_n^{-1}(\zeta)| = \prod_{i=1}^r |V_{n_i}(\mathbb{F}_q)^{(g_F q) d_i}|$$

$$= \prod_{i=1}^r |V_{n_i}(\mathbb{F}_q)^{(F_q) d_i}|$$

$$= \prod_{i=1}^r \sum_{\beta_j \in P(n_i)} q^{d_i(n_i-|\beta_j|)},$$

where $V_n = \text{Hilb}^n(\mathbb{F}_q[[s, t]])$. Here we use the key Lemma 3.2.

For $i = 1, \ldots, r$, let $\beta^i = (1^{\beta_1^i}, 2^{\beta_2^i}, \ldots)$ be a partition of $n_i$, and let $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ be the union of $d_i$ copies of each $\beta^i$, where $\alpha_j = \sum_i d_i \beta_j^i$. Let

$$\eta_j = \sum_{i=1}^r \beta_j^i \zeta_i.$$

Let $\eta$ be the sequence $(\eta_1, \eta_2, \eta_3, \ldots)$. Then for all $w \in A^{n-|\alpha|}$ we have

$$\gamma_n((\eta, w)) = \zeta,$$
and in this way we get all the elements of $\gamma_n^{-1}(\zeta)$. Hence

$$|\gamma_n^{-1}(\zeta)| = \sum_{\beta_1 \in P(n_1)} \sum_{\beta_2 \in P(n_2)} \cdots \sum_{\beta_r \in P(n_r)} q^{n - \sum d_\beta} = |\omega_n^{-1}(\zeta)|.$$  

\[\square\]

**Lemma 4.2.** Denote by $h_n : A(n)(\mathbb{F}_q)\rightarrow A(\mathbb{F}_q)$ the restriction of $g_n$. Then $h_n$ is onto and $|h_n^{-1}(x)|$ is independent of $x \in A(\mathbb{F}_q)$.

**Proof.** Since $gF_q$ is the Frobenius map of some twist of $A$, we can replace $gF_q$ by $F_q$ in the statement, and this is true by [Göt94, Lemma 2.4.8].\[\square\]

For each $l \in \mathbb{N}$, let $A(F_q)^{gF_q}_l$ be the image of the multiplication $(l) : A(F_q)^{gF_q} \rightarrow A(F_q)^{gF_q}$.

**Lemma 4.3.** Let $\mu = (n_1, \ldots, n_t)$ be a partition of a number $n \in \mathbb{N}$. Then

$$\sigma_\mu : (A(F_q)^{gF_q})^t \rightarrow A(F_q)^{gcd(\mu)}$$

$$(x_1, \ldots, x_t) \mapsto \sum_{i=1}^t n_i x_i$$

is onto and $|\sigma_\mu^{-1}(x)|$ is independent of $x \in A(F_q)^{gF_q}$.

**Proof.** As the above lemma, we can replace $gF_q$ by $F_q$, and this is true by [Göt94 Lemma 2.4.9].\[\square\]

We denote $(\prod_{i=1}^\infty S^{(\alpha_i)}(\mathbb{F}_q)^{gF_q}) \times A^{n-|\alpha|}(\mathbb{F}_q) \times A[\alpha]$. Denote the restriction map of $\gamma_n$ on $A[\alpha]$ by $\gamma_n,\alpha : A[\alpha] \rightarrow S^{(n)}(\mathbb{F}_q)^{gF_q}$.

**Lemma 4.4.**

$$|K_n(A)(\mathbb{F}_q)^{gF_q}| = \frac{1}{|A(F_q)^{gF_q}|} \sum_{\alpha \in P(n)} \gcd(\alpha)^4 q^{n-|\alpha|} \prod_{i=1}^\infty |A^{(\alpha_i)}(\mathbb{F}_q)^{gF_q}|.$$  

**Proof.** By Lemma 4.1, we have

$$|K_n(A)(\mathbb{F}_q)^{gF_q}| = |\gamma_n^{-1}(h_n^{-1})| = \sum_{\alpha \in P(n)} |\gamma_n^{-1,\alpha}(h_n^{-1}(0))|.$$  

Suppose $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$. Let

$$\mu = (m_1, \ldots, m_t) = (1^{\mu_1}, 2^{\mu_2}, \ldots),$$
where \( \mu_i = \min(1, \alpha_i) \) for all \( i \). Let

\[
\begin{align*}
f_\alpha : S[\alpha] & \to (A(\mathbb{F}_q)^g_{F_q})^t \\
((\zeta_1, \ldots, \zeta_t), w) & \mapsto (g_{\alpha_{m_1}}(\zeta_1), \ldots, g_{\alpha_{m_t}}(\zeta_t)).
\end{align*}
\]

Then the following diagram commutes:

\[
\begin{array}{ccc}
A[\alpha] & \xrightarrow{g_{\nu}} & A(\mathbb{F}_q)^g_{F_q} \\
\downarrow f_\alpha & & \downarrow h_\nu \\
(A(\mathbb{F}_q)^g_{F_q})^t & \xrightarrow{\sigma} & A(\mathbb{F}_q)^g_{F_q}.
\end{array}
\]

By Lemma 4.2 and Lemma 4.3, \( \sigma \circ f_\alpha \) maps \( S[\alpha] \) onto \( A(\mathbb{F}_q)^g_{F_q} = A(\mathbb{F}_q)^g_{F_q} \), and \( |f_\alpha^{-1}(\sigma^{-1}(x))| \) is independent of \( x \in A(\mathbb{F}_q)^g_{F_q} \). Since the multiplication with \( \text{gcd}(\alpha) \) is an étale morphism of degree \( (\text{gcd}(\alpha))^4 \), we have

\[
|K_n(A)(\mathbb{F}_q)^g_{F_q}| = \sum_{\alpha \in P(n)} |f_\alpha^{-1}(\sigma^{-1}(x))| = \sum_{\alpha \in P(n)} \frac{|A[\alpha]|}{|A(\mathbb{F}_q)^g_{\text{gcd}(\alpha)}|} = \frac{1}{|A(\mathbb{F}_q)^g_{F_q}|} \sum_{\alpha \in P(n)} \left( \text{gcd}(\alpha)^4q^{n-|\alpha|} \prod_{i=1}^\infty |A(\alpha_i)(\mathbb{F}_q)^g_{F_q}| \right).
\]

\[\Box\]

**Proof of theorem 1.7.** By Lemma 4.4, we have

\[
\sum_{n=0}^\infty |K_n(A)(\mathbb{F}_q)^g_{F_q}| t^n = \sum_{n=0}^\infty \frac{1}{|A(\mathbb{F}_q)^g_{F_q}|} \sum_{\alpha \in P(n)} \left( \text{gcd}(\alpha)^4q^{n-|\alpha|} \prod_{i=1}^\infty |A(\alpha_i)(\mathbb{F}_q)^g_{F_q}| \right) t^n = \frac{(w^{d_{Dw}})^4}{|A(\mathbb{F}_q)^g_{F_q}|} \sum_{n=0}^\infty \sum_{\alpha \in P(n)} w^{\text{gcd}(\alpha)} \prod_{i=1}^\infty \left( |A(\alpha_i)(\mathbb{F}_q)^g_{F_q}|q^{(i-1)\alpha_i}t^{\alpha_i} \right) = \frac{(w^{d_{Dw}})^4}{|A(\mathbb{F}_q)^g_{F_q}|} \sum_{n=1}^\infty \left( 1 + w^m(-1 + \sum_{n=0}^\infty |A(n)(\mathbb{F}_q)^g_{F_q}|q^{m-1n}t^{mn}) \right).
\]

Then by the proof of Proposition 2.2 and Theorem 2.7, the theorem follows. \[\Box\]

**Remark 4.5.** It is calculated in [Göt94, Corollary 2.4.13] that \( \sum_{n=1}^\infty e(K_n(A))q^n = \frac{(\frac{d_{Dw}}{24})^2}{24} E_2 \), where \( E_2 := 1 - 24 \sum_{n=1}^\infty \sigma_1(n)q^n \) is a quasi-modular form. As in the case of Hilbert schemes of points, we can calculate \( \sum_{n=0}^\infty \text{Tr}(g, [e(K_n(A))])t^n \), where \( g \) is a symplectic automorphism of finite order on the abelian surface \( A \). But it is not
obvious to the author whether or not the sum can be expressed by quasi-modular forms.

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**Department of Mathematical Sciences, Binghamton University, Binghamton, NY, 13902, U.S.A.**

*Email address:* zhans@binghamton.edu