DeepCodec: Adaptive Sensing and Recovery via Deep Convolutional Neural Networks

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Abstract—In this paper we develop a novel computational sensing framework for sensing and recovering structured signals. When trained on a set of representative signals, our framework learns to take undersampled measurements and recover signals from them using a deep convolutional neural network. In other words, it learns a transformation from the original signals to a near-optimal number of undersampled measurements and the inverse transformation from measurements to signals. This is in contrast to traditional compressive sensing (CS) systems that use random linear measurements and convex optimization or iterative algorithms for signal recovery. We compare our new framework with \( \ell_1 \)-minimization from the phase transition point of view and demonstrate that it outperforms \( \ell_1 \)-minimization in the regions of phase transition plot where \( \ell_1 \)-minimization cannot recover the exact solution. In addition, we experimentally demonstrate how learning measurements enhances the overall recovery performance, speeds up training of recovery framework, and leads to having fewer parameters to learn.

I. INTRODUCTION

Signal recovery is a fundamental problem that appears in a number of important applications. Whenever a signal has some type of structure or redundancy, the signal recovery problem can be solved with a small number of measurements, as compressing the signal appropriately does not lose any information. A widely applicable and well-studied form of signal structure is sparsity. In this case, signal recovery is the problem of recovering a sparse signal \( x \in \mathbb{R}^N \) from a set of random linear measurements \( y = \Phi x \in \mathbb{R}^M \), where \( \Phi \) is an \( M \times N \) measurement matrix, and \( M \) is typically smaller than \( N \). By sparse, we mean that we can write \( x = \Psi s \), where \( \Psi \) is a basis and only \( K \ll N \) of the coefficients \( s \) are nonzero. A natural estimate of the original \( x \) may be obtained by solving the \( \ell_0 \) minimization problem, i.e., \( \min \| \hat{x} \|_0, \text{ s.t. } y = \Phi \hat{x} \). However, since solving this optimization program is believed to have combinatorial complexity, several approaches have been proposed to “relax” it. For instance, arguably the most successful approach is solving the tightest convex relaxation of the above optimization program, i.e., \( \ell_1 \)-norm minimization

\[
\min \| \hat{x} \|_1, \text{ s.t. } y = \Phi \hat{x} \text{ [1, 2].}
\]

This price we pay for using \( \ell_1 \)-minimization instead of \( \ell_0 \)-minimization is reduced recovery performance, namely that \( \ell_1 \)-minimization requires more measurements \( M \) to recover a \( K \)-sparse signal than \( \ell_0 \)-minimization. Let \( \delta = \frac{M}{N} \) denote the undersampling ratio and let \( \rho = \frac{K}{N} \) indicate the normalized sparsity level. An instructive way of studying these phenomena is to visualize the probability of successful recovery for each value of \( (\delta, \rho) \in [0, 1]^2 \). Given that there is a sharp phase transition in the recovery performance of \( \ell_1 \)-minimization, such plots have come to be known as phase transition plots [3] (see Figure 3 for an example of such a plot). The two-dimensional phase transition plot has two phases: a success phase and a failure phase, where \( \ell_1 \)-minimization can and cannot recover the exact signal, respectively. In other words, \( \ell_1 \)-minimization successfully recovers the sparse signal if its normalized sparsity level is less than a certain threshold.

Our goal in this paper is to show that by using deep learning techniques, we can design computational sensing frameworks that can overcome the limitations of \( \ell_1 \)-minimization. To achieve this goal, we make improvements to \( \ell_1 \)-minimization in 2 separate directions:

- First, instead of traditional schemes that use random undersampled measurements we learn a transformation from original signals to undersampled measurements. The critical issue with random undersampled measurements is that they are universal and do not use the structure that is specific to the set of signals that we care about for the particular problem at hand. However, one can use these structures to find a better way to compress signals and derive measurements.
- Second, instead of using iterative or convex optimization algorithms, we learn the inverse transformation from undersampled measurements to original signals using a deep convolutional network (DCN) and training it on several examples.

Our new framework is thus able to learn an efficient and compressed representation of training signals, and also the corresponding inverse map from this representation onto the original signal space. We call our framework DeepCodec, where codec of course is the well known portmanteau of “coder-decoder”. DeepCodec is closely related to the DeepInverse
Fig. 1: **DeepInverse** learns an approximate inverse transformation from measurement vectors $y$ to signals $x$ using a deep convolutional network.

framework [4] proposed by a subset of the current authors. The major difference between **DeepInverse** and **DeepCodec** is that **DeepInverse** recovers original signals from random linear undersampled measurements while **DeepCodec** learns to take nonlinear undersampled measurements and recovers original signals from them.

In this paper, we study the sparse recovery performance of **DeepCodec** and show that it significantly outperforms $\ell_1$-minimization in this context. In particular, **DeepCodec** succeeds, with overwhelmingly high probability, even when the normalized sparsity of the problem at hand is larger than the threshold that comes from $\ell_1$-minimization’s phase transition (i.e., in regimes where $\ell_1$-minimization almost always fails). We show that **DeepInverse** also has this feature. We also compare these algorithms in terms of their runtime and show that **DeepCodec** and **DeepInverse** are orders of magnitude faster than the conventional algorithms. The tradeoff for the ultrafast runtime is a one-time, computationally intensive, off-line training procedure typical of deep networks. This makes our approach applicable to real-time sparse recovery problems. Finally, we show how taking adaptive measurements and learning a transformation from signals to their undersampled measurements improves recovery performance compared to simply using random measurements. In other words, we show adaptivity helps **DeepCodec** to outperform **DeepInverse**.

The rest of this paper is organized as follows: Section II summarizes prior art in using deep learning frameworks for structured signal recovery. Section III introduces the network architecture we have used to take adaptive measurements from signals and to recover signals using adaptive measurements. Section IV summarizes our experimental results and the comparison of our method with previous works. We make some concluding remarks in Section V.

### II. Prior Work

In this section we briefly describe previous works on designing structured signal recovery algorithms. These algorithms span a wide spectrum but may be itemized as the follows:

- **Physics-driven approaches.** These methods are conventional recovery algorithms inspired by the concept of sparsity. They are mainly based on enforcing sparsity (or other types of structure) in the signal approximation/estimation process. The most well-known methods from these approaches are greedy algorithms [5], convex optimization algorithms [2], and iterative algorithms [6], [7].

- **Mixtures of physics-driven and data-driven approaches.** This class of algorithms mixes concepts from physics-driven approaches that give them interpretability and the power of using training data that gives them adaptivity and performance enhancement for specific applications. In most cases, these approaches are a combination of conventional sparse recovery algorithms and deep learning frameworks [8]–[12].

- **Data-driven approaches.** These methods are mainly based on designing frameworks that can use training data to learn a representation of signals and a transformation from undersampled measurements to original signals [4], [13]–[15]. The main benefit of using these approaches is that they provide ultrafast signal recovery. However, this ultrafast runtime comes at a price that is a computationally intensive, off-line training procedure typical
to deep networks. Since this training procedure needs to be completed only once, data-driven approaches are applicable for real-time signal recovery problems.

The first data-driven signal recovery approach that was based on deep learning frameworks was introduced in [13]. Authors in [13] used stacked denoising autoencoders (SDA) as an unsupervised feature learner. The major drawback of using SDA is that it consists of fully-connected layer. This means that as signal size grows, we need to tune a larger set of network parameters and this could cause overfitting if we do not have enough training data. One solution that authors introduced in [13] was dividing signals into smaller blocks and performing blocky reconstruction. This approach is not applicable for certain applications like compressive sensing and performing blocky reconstruction. This approach is not for real-time signal recovery problems.

As an alternative to the SDA approach, a subset of the authors of this paper in [4] introduced DeepInverse that is a signal recovery framework based on DCNs. The main motivation for replacing the SDA approach with DCNs is their two distinctive features: first, sparse connectivity of neurons in each layer. Second, having shared weights across the entire receptive fields. These two features significantly reduce computational complexity of DCNs compared to the SDA approach and make DCNs distinctly applicable for structured signal recovery problem. In the following, we briefly describe the DeepInverse framework [4] for sparse signal recovery (see Figure 1).

DeepInverse takes as input a set of measurements \( y \) in \( \mathbb{R}^M \) and outputs the signal estimate \( \hat{x} \) in \( \mathbb{R}^N \). To increase the dimensionality of the input from \( \mathbb{R}^M \) to \( \mathbb{R}^N \), it applies the adjoint operator \( \Phi^T \) in the first layer. To preserve the dimensionality of the processing in \( \mathbb{R}^N \), it dispenses with the downsampling max-pooling operations made popular in modern DCNs [16]. We assume that the measurement matrix \( \Phi \) is fixed. Therefore, each \( y_i \) (1 ≤ \( i \) ≤ \( M \)) is a linear combination of \( x_j \)s (1 ≤ \( j \) ≤ \( N \)). By training a DCN, we learn a nonlinear mapping from the signal proxy \( \tilde{x} = \Phi^T y \) to the original sparse signal \( x \).

Among the many possibilities for the deep network architecture, DeepInverse uses one layer to implement the adjoint operator \( \Phi^T \) and five convolutional layers with their corresponding batch normalization [17] layers. Each convolutional layer applies a leaky-ReLU [18] nonlinearity to its output. The \( i \)-th entry of the \( t \)-th feature map in the first convolutional layer receives the signal proxy \( \tilde{x} \) as its input and outputs \( (x_{c_i})^t_i = \mathcal{S}(\text{L-ReLU}((W^t_i \ast \tilde{x})_i + (b^t_i)_i)), \) where \( W^t_i \in \mathbb{R}^{P \times P} \) and \( b^t_i \in \mathbb{R}^{K} \) denote the filter and bias values corresponding to the \( t \)-th feature map of the first layer and L-ReLU(x) = x if x > 0 and = 0.01x if x ≤ 0. Finally, the subsampling operator \( \mathcal{S}(\cdot) \) takes the output of L-ReLU(·) to the original signal size by ignoring the borders created by zero-padding the input. The feature maps for the other convolutional layers are processed in a similar manner. If we denote the set of weights and biases in the DCN by \( \Omega \), then we can define a nonlinear mapping from the measurements to the original signal by \( \tilde{x} = \mathcal{M}(y, \Omega) \). To learn the weights and biases, we employ backpropagation algorithm to minimize the mean-squared error (MSE) of the estimate \( \tilde{x} \).

### III. DeepCodec

Having reviewed the DeepInverse framework, we are now ready to introduce our new framework DeepCodec (See Figure
Similar to DeepInverse, its sequel DeepCodec learns the inverse transformation from undersampled measurements vectors to signals using a special form of DCNs. However, there is a major difference between DeepInverse and DeepCodec. DeepInverse uses adjoint operator $\Phi^T$ in the first layer to make a proxy of original signals from their random linear undersampled measurements. However, DeepCodec does not use random linear undersampled measurements of original signals. Instead, it learns to take nonlinear undersampled measurements from original signals and recover original signals from these learned measurements. Learning measurements from original signals helps to preserve more information compared to taking random measurements and we experimentally show this in Section IV.

For taking undersampled measurements from original signals, one should reduce their dimensionality. In DCNs, the conventional form of dimensionality reduction is by using pooling layers that are a form of downsampling and provide translation invariance. There are different ways of implementing pooling layers such as max pooling where we take the maximum value from each cluster of neurons; or average pooling where we take the average of values in each cluster. In addition to these traditional implementations of a pooling layer, there are new approaches for implementing it such as stochastic pooling [19] and spatial pyramid pooling [20].

 Almost all implementations of a pooling layer are hand-designed in the sense that either they are designed for a special goal like achieving translation invariance or they use some sort of expert knowledge. Since they are hand-designed, they do not optimally preserve signal information while reducing its dimensionality. Our argument here is that one can learn dimensionality reduction from data instead of using a hand-designed transformation. In other words, learning dimensionality reduction from data itself, if done correctly, will preserve more information compared to conventional ways that do not use training data for downsampling.

In this paper, instead of using conventional implementations of pooling layers, we learn a transformation for dimensionality reduction. The major benefit of learning this transformation is that we preserve more information. The tradeoff for the preservation of information is a computationally more complex pooling layer compared to conventional ones such as max-pooling or average-pooling. The design of our pooling layer is inspired by the sub-pixel convolution layer initially introduced in [21]. Sub-pixel convolution layer was initially designed for aggregating feature maps from low-resolution image space in order to build a high-resolution image. In other words, it was initially designed for increasing the dimensionality rather than reducing it. However, as we show in the following, with some modifications it can be used for dimensionality reduction as well. Once we reduce signal dimensionality and take undersampled measurements, we can use convolutional layers plus a sub-pixel convolution layer to reconstruct signals from their undersampled measurements. Having said this preliminary explanation of DeepCodec, we are now ready to describe it in more detail.

DeepCodec takes an input that is an original signal $x \in \mathbb{R}^N$. For the sake of brevity we assume that DeepCodec input is a single channel and one-dimensional signal. However, it is straightforward to extend it to multi-channel and multi-dimensional signals like RGB images. In addition, we assume that the length of original signals $N$ is divisible by the length of undersampled measurements vector we are interested in $M$, i.e., $N = rM$. We should note that it is straightforward to extend DeepCodec to cases where $N$ is not divisible by $M$. For DeepCodec the first task is to take undersampled measurements from its input. To do so, the first layer of DeepCodec rearranges its input. Let the size of input layer be $N \times 1 \times 1$ where $N$ denotes the length, the first 1 denotes the width, and the second 1 denotes the number of input channels. The rearranging layer takes this input and turns it into an $M \times 1 \times r$ signal where $M = N/r$. In other words, this rearranging layer divides the length of output signal by a factor of $r$ while increase the number of channels in output signal by a factor of $r$. Mathematically, we can describe a rearranging layer as the following:

$$\hat{x}(x, r)_{l, 1.c} = x_{i \times r + \text{mod}(c, r), 1, 1},$$

(1)

where $\hat{x}$ denotes the output of the rearranging layer and $x$ denotes its input. This rearranging layer clearly reduces its input dimensionality; however, its output is still not an undersampled measurements vector of the original signal since the total number of output neurons is the same as input layer. Therefore, we employ several convolutional layers to adjust the total number of neurons such that it equals to $M$ that is the number we are interested in. We can mathematically formulate the $i$-th entry of the $l$-th feature map of the $l$-th convolutional layer as the following:

$$(x_l)_{i, 1, t} = S(\text{ReLU}(W_i \ast x_{i-1} + b_i))_{i, 1, t},$$

(2)

where $x_{i-1} \in \mathbb{R}^{M \times 1 \times c_{i-1}}$ denotes the input of the $l$-th convolutional layer and $c_{i-1}$ denotes the number of channels in $x_{i-1}$. $W_i \in \mathbb{R}^{h_i \times 1 \times c_{i-1} \times c_i}$ and $b_i \in \mathbb{R}^{c_i}$ denote the filter and bias values corresponding to the $l$-th convolutional layer where $h_i$ is the length of the $l$-th convolutional layer’s filter. Finally, $x_i \in \mathbb{R}^{M \times 1 \times c_l}$ denotes the output of the $l$-th convolutional layer. If the $p$-th layer’s output corresponds to undersampled measurements, then $W_p \in \mathbb{R}^{h_{lp} \times 1 \times r_{l-1} \times 1}$ and $x_p \in \mathbb{R}^{M \times 1 \times 1}$. Finally, ReLU$(x) = \max(0, x)$ and the subsampling operator $S()$ takes the output of ReLU(·) to the original signal size by ignoring the borders created by zero-padding the input.

Once we have the undersampled measurements vector (i.e., the output of $p$-th layer), we employ several convolutional layers to extract feature maps from it. The main advantage of this method compared to DeepInverse is that in DeepInverse we were reconstructing signals from their proxies which had the same size as original signals. However, in DeepCodec we are reconstructing signals in measurement domain by employing several convolutional layers that receive undersampled measurements as their initial input.
The output of the $p$-th layer (i.e., undersampled measurements vector) lies in $\mathbb{R}^{M \times 1 \times 1}$. However, the reconstructed signal should lie in $\mathbb{R}^{N \times 1 \times 1}$ where $N = r \times M$. Therefore, we have to boost the dimensionality from $\mathbb{R}^{M \times 1 \times 1}$ to $\mathbb{R}^{(r \times M) \times 1 \times 1}$. We initially employ several convolutional layers as we introduced in (2). These convolutional layers will help us to boost the dimensionality from $\mathbb{R}^{M \times 1 \times 1}$ to $\mathbb{R}^{M \times 1 \times r}$ through employing several convolutional layers in measurements domain. Once we have boosted the dimensionality of the measurements vector to $\mathbb{R}^{M \times 1 \times r}$, we employ a sub-pixel convolution layer (21) to rearrange neurons and produce an output which lies in $\mathbb{R}^{N \times 1 \times 1}$ from its input which lies in $\mathbb{R}^{M \times 1 \times r}$. Mathematically, we can describe our sub-pixel convolution layer as the following:

$$\hat{x}(x, r)_{i, 1, 1} = x_{i/r, 1, \text{mod}(i, r)};$$

where $\hat{x}$ denotes the output of the rearranging layer and $x$ denotes its input.

Figure 2 shows the schematic of DeepCodec framework. Note that we can consider DeepCodec as a special form of convolutional autoencoder. Here is the summary of how DeepCodec works:

- Receiving an input signal.
- Rearranging input’s components and reduce its dimensionality through convolutional layers.
- Taking undersampled measurements.
- Boosting measurements dimensionality through convolutional layers.
- Transforming the output to a reconstructed signal through a sub-pixel convolution layer.

If we denote the output of DeepCodec by $\hat{x}$ and assume that DeepCodec has overall of $d$ convolutional layers, then we can denote its set of parameters by $\Omega = \{W_i, b_i\}_{i=1}^{d}$ and define a nonlinear mapping from original signals to reconstructed signals as $\hat{x} = F(x, \Omega)$. Now if we have a training set $D_{\text{train}} = \{x^{(i)}\}_{i=1}^{s}$ that consists of $s$ original signals, we can use the mean squared error (MSE) as a loss function over the training set $D_{\text{train}}$:

$$\mathcal{L}(\Omega) = \frac{1}{s} \sum_{i=1}^{s} \|F(x^{(i)}, \Omega) - x^{(i)}\|_2^2.$$  

we can employ either stochastic gradient descent (SGD) or ADAM optimizer (22) to minimize $\mathcal{L}(\Omega)$ and learn weights and biases.

IV. EXPERIMENTAL RESULTS

In this section, we compare the performance of DeepCodec to DeepInverse and to the LASSO (23) $\ell_1$-solver (implemented using the coordinate descent algorithm of (24)) over a grid of regularization parameters. In all experiments, we assume that the optimal regularization parameter of LASSO is given by an oracle.

Our DeepCodec framework has eight layers. The first layer is a rearranging layer for dimensionality reduction. We assume that the output of the first layer lies in $\mathbb{R}^{M \times 1 \times r}$. The second to seventh layers are all convolutional layers and the size of filters in all of these layers are 49 × 1. The second layer has 8 filters each having $r$ channels of size 49 × 1. The third layer has 4 filters each having 8 channels of size 49 × 1. The fourth layer has 1 filter that has 4 channels of size 49 × 1. The output of the fourth layer is our undersampled measurements vector. The fifth layer has 4 filters each having 1 channel of size 49 × 1. The sixth layer has 8 filters each having 4 channels of size 49 × 1. The seventh layer has $r$ filters each having 8 channels of size 49 × 1. Finally, the eighth layer is a sub-pixel convolution layer for aggregating feature maps of the seventh layer and boosting the dimensionality. The eighth layer gets an input which lies in $\mathbb{R}^{M \times 1 \times r}$ and converts it into a signal reconstruction that lies in $\mathbb{R}^{N \times 1 \times 1}$.

Our DeepInverse network has five layers. The first and third layers have 32 filters, each having 1 and 16 channels of size 125 × 1, respectively. The second and fourth layers have 16 filters, each having 32 channels of size 125 × 1. The fifth layer has 1 filter that has 16 channels of size 125 × 1. We trained and tested DeepCodec and DeepInverse using wavelet sparsified versions of 1D signals of size $N = 512$ extracted from random rows of CIFAR-10 images (25). The training set contains 100,000 signals, and the test set contains 20,000 signals.

The blue curve in Figure 3 is the $\ell_1$ phase transition curve. The circular points denote the problem instances, i.e., $(\delta, \rho)$, on which we study the performance of DeepInverse and the LASSO. By design, these problem instances are on the “failure” side of the $\ell_1$ phase transition. The undersampling ratios, i.e., $\delta$ for these instances are 0.3, 0.5, and 0.7 and the normalized sparsity, i.e., $\rho$ for these instances are 0.42, 0.56, and 0.72 respectively.

We have used the same set of sparse signals for training and testing DeepCodec framework to compare its performance with DeepInverse and the LASSO. The square points in Figure 3 denote the problem instances, i.e., $(\delta, \rho)$, on which we have trained and tested DeepCodec. As we mentioned, signals that we have used for these problem instances are the same ones we have used for training and testing DeepInverse with settings denoted by circular points. However, for DeepCodec we have made recovery problems harder by reducing undersampling ratios. The arrows between square points and circular points in Figure 3 denote correspondence between problem instances in DeepCodec and DeepInverse. As an example, the set of $k$-sparse signals in $\mathbb{R}^N$ where $k = 64$ and $N = 512$ corresponds to $(\delta, \rho) = (0.3, 0.42)$ for DeepInverse (with undersampling ratio 0.3) while corresponds to $(\delta, \rho) = (0.125, 1)$ for DeepCodec (with undersampling ratio 0.125).

Table I shows the average normalized mean squared error (NMSE) for the test set signals using all three methods. As we can see in this table, even though we have made recovery of the same signals significantly harder for DeepCodec by reducing the undersampling ratio, it still outperforms DeepInverse and
the LASSO (with the optimal regularization parameter) in all of the configurations determined in Figure 3.

Table III compares the number of parameters learned in each setting for DeepInverse and DeepCodec. As we can see, in all of the problem instances DeepCodec has significantly fewer number of parameters compared to DeepInverse while outperforming it in recovery performance. This is mainly due to the fact that DeepCodec learns a transformation for taking measurements from signals while DeepInverse uses random measurements.

Table III compares the computational complexity of all three methods. If we use a fast iterative algorithm for solving the LASSO like AMP \(^26\), then the runtime of every iteration would be \(O(MN)\) where \(N\) is the size of signal and \(M\) is the size of undersampled measurements vector. If we let \(M = ck\log(N)\), then this runtime would be \(O(kN\log(N))\).

In DeepInverse every convolutional layer’s input and output has the same size as the input signal that is \(N\) and hence, runtime of computing every layer’s output is \(O(N)\). Therefore, computational cost corresponding to one layer of DeepInverse is significantly less than the one for an iteration of AMP for solving LASSO. In DeepCodec since we are recovering signals in the measurements space, computing the output of typical middle layers will cost \(O(M)\). As we can see, not only DeepCodec gives us a better recovery performance compared to DeepInverse, but also it has a faster runtime and needs fewer parameters to learn. It is noteworthy to mention that usually the number of iterations an iterative algorithm such as AMP needs for recovering a signal is tens of times more than the number of convolutional layers needed by either DeepInverse or DeepCodec to recover the same signal. This is another factor that makes signal recovery by DeepInverse and DeepCodec significantly faster than iterative algorithms like AMP.

Figure 4 compares the effect of training on DeepInverse and DeepCodec. It shows the MSE of recovering test signals by DeepInverse and DeepCodec in different training epochs. The set of training signals are the same for both DeepInverse and DeepCodec. However, the problem instance for DeepInverse is \((\delta, \rho) = (0.7, 0.72)\) while for DeepCodec is \((\delta, \rho) = (0.5, 1.003)\) which means we have given DeepCodec a more difficult recovery problem. However, as we can see in Figure 4 training is significantly faster for DeepCodec compared to DeepInverse. DeepCodec outperforms the LASSO (for the problem instance \((\delta, \rho) = (0.7, 0.72)\)) after only 4 training epochs while for DeepInverse it takes 138 epochs to outperform the LASSO. This fast training has two major reasons: First, DeepCodec has fewer number of parameters to learn. Second, DeepCodec learns adaptive measurements instead of using random measurements.

Figure 5 compares the probability of successful recovery by DeepCodec and LASSO as measured by 20000 Monte Carlo samples (that are test set signals). For each undersampling ratio \(\delta\) and for the \(j\)-th Monte Carlo sample, we define the success variable \(\varphi_{\delta,j} = \mathbb{I}\left(\frac{\|\hat{\mathbf{x}}^{(j)} - \mathbf{x}^{(j)}\|_2^2}{\|\mathbf{x}^{(j)}\|_2^2} \leq 0.01\right)\), where \(\mathbf{x}^{(j)}\) is the \(j\)-th Monte Carlo sample, \(\hat{\mathbf{x}}^{(j)}\) is the recovered signal from measurements of \(j\)-th Monte Carlo sample, and \(\mathbb{I}(\cdot)\) is the indicator function. We define the empirical successful recovery probability as \(P_\delta = \frac{1}{q} \sum_{j=1}^{q} \varphi_{\delta,j}\), where \(q\) is the total number of Monte Carlo samples. In Figure 5, our test set signals are \(k\)-sparse in \(\mathbb{R}^N\) where \(k = 34\) and \(N = 512\) and we have considered three different configurations:

- \(M = 64\) measurements which corresponds to \((\delta, \rho) = (0.125, 0.53)\) that is a setting above the \(\ell_1\) phase transition, i.e., failure side.
- \(M = 128\) measurements which corresponds to \((\delta, \rho) = (0.25, 0.26)\) that is a setting on the \(\ell_1\) phase transition curve.
- \(M = 256\) measurements which corresponds to \((\delta, \rho) = (0.5, 0.13)\) that is a setting below the \(\ell_1\) phase transition, i.e., success side.

As we can see in Figure 5 DeepCodec significantly outperforms LASSO when the problem configuration lies above or on the \(\ell_1\) phase transition curve. Only when the problem configuration lies below the \(\ell_1\) phase transition curve LASSO slightly outperforms DeepCodec (probability of successful recovery equals to 1 for LASSO vs. 0.99 for DeepCodec). This is expected since for a setting below \(\ell_1\) phase transition curve, we expect \(\ell_1\) minimization to behave the same as \(\ell_0\) minimization. In other words, we expect to have a successful recovery based on physics of \(\ell_1\) minimization. However, DeepCodec should learn a transformation for transforming measurements back to the original signals. In this setting, we have trained DeepCodec for only 5000 epochs. We conjecture that if we train DeepCodec for more than 5000 epochs or use more number of parameters, it can achieve probability of successful recovery equals to 1. We leave this topic, i.e., understanding the number of parameters and training epochs needed for achieving certain recovery quality, as an avenue for the future research.

Figure 6 shows examples of signal recoveries using DeepCodec and LASSO. The original signal is a \(k\)-sparse signal in \(\mathbb{R}^N\) where \(k = 64\) and \(N = 512\). DeepCodec recovers this signal from \(M = 64\) measurements, i.e., \((\delta, \rho) = (0.125, 1)\) while LASSO recovers it from \(M = 154\) measurements, i.e., \((\delta, \rho) = (0.3, 0.42)\). Therefore, as we can see in this Figure, DeepCodec has solved a more challenging recovery problem significantly better than the LASSO with optimal regularization parameter.

V. CONCLUSIONS

In this paper we have developed DeepCodec that is a novel computational sensing framework for sensing and recovering structured signals. We have shown that DeepCodec can learn to take undersampled measurements and recover signals from undersampled measurements using convolutional and sub-pixel convolution layers. We compared DeepCodec with \(\ell_1\)-minimization from the phase transition point of view and showed that it significantly outperforms \(\ell_1\)-minimization in the failure side of phase transition. In addition, compared to using
Fig. 3: The blue curve is the $\ell_1$ sparse recovery phase transition. The circular and square points denote problem configurations for DeepInverse and DeepCodec respectively. Arrows between circular and square points show different configurations for the same set of signals. As shown in Table I, DeepCodec outperforms DeepInverse in recovering the same signals even when the configuration is more challenging.

Fig. 4: Test MSE of DeepInverse and DeepCodec during training epochs for $(\delta, \rho) = (0.7, 0.72)$. DeepCodec outperforms LASSO after only 4 epochs while DeepInverse outperforms LASSO after 138 epochs.

Fig. 5: Average probability of successful signal recovery for different undersampling ratios when we use DeepCodec and LASSO. This plot studies three different configurations: below, above, and on the $\ell_1$ phase transition curve.

Fig. 6: Recovery example by DeepCodec and LASSO (with optimal regularization parameter). The problem configuration, i.e., $(\delta, \rho)$ lies above $\ell_1$ phase transition. DeepCodec significantly outperforms LASSO in this setting.

TABLE I: Average NMSE of test set signals for all three methods. DeepCodec outperforms DeepInverse and the LASSO in all cases in spite of the fact that it is solving a significantly more difficult recovery problem in each case.

| $(\delta, \rho)$ | DeepCodec | DeepInverse | LASSO |
|------------------|------------|-------------|-------|
| 0.3, 0.42        | 0.0140     | 0.0466      |      |
| 0.5, 0.56        | 0.0112     | 0.0312      |      |
| 0.7, 0.72        | 0.0052     | 0.0164      |      |

TABLE II: Number of parameters that DeepCodec and DeepInverse learn in each problem instance. DeepCodec uses significantly fewer number of parameters while outperforming DeepInverse in all the cases.

| $(\delta, \rho)$ | DeepInverse | DeepCodec |
|------------------|-------------|-----------|
| 0.3, 0.42        | 198000      | 6664      |
| 0.5, 0.56        | 198000      | 5096      |
| 0.7, 0.72        | 198000      | 4312      |

random measurements we experimentally showed how learning undersampled measurements enhances the overall recovery performance, speeds up training of recovery framework, and could lead to having fewer parameters to learn.

TABLE III: Computational complexity for every typical iteration/layer of different methods. Since DeepCodec recovers signals mainly in the measurements space, it is computationally cheaper than DeepInverse.

| Computational Complexity |
|--------------------------|
| LASSO | DeepInverse | DeepCodec |
| $O(MN)$ | $O(N)$ | $O(M)$ |
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