Phase separation in effective hard-core boson and triplet models in one and two dimensions

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Effective models of hard-core hole pair bosons and triplets are derived from the t–J model in ladders and on the square lattice by performing a change from site to dimer basis. The Hilbert space is truncated by projecting out single-occupied electron states and only nearest neighbor interactions are retained. The resulting effective models in one and two dimensions are studied by numerical techniques. In both spatial dimensions, the main result is that each hole pair is surrounded by a singlet cloud expelling triplet excitations from its vicinity. It is suggested an interpretation of this feature as a phase separated state between a hole-pair rich singlet phase and an undoped triplet phase with antiferromagnetic correlations. The possible relevance of this result to other theoretical scenarios and experimental results is discussed.

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I. INTRODUCTION

The SO(5) theory is a very appealing framework in which antiferromagnetic (AF) and superconducting (SC) phases of high-\(T_c\) superconductors are naturally related. This theory not only gives an explanation of the resonant peak observed in the SC phase but also predicts that it smoothly connects with the magnon peak of the AF phase as it was recently observed experimentally. Another of the predictions of this theory, i.e. the presence of AF order inside the vortices present when a magnetic field is applied on the SC phase, has recently motivated a flurry of experimental activity confirming partially this prediction. At the realm of the SO(5) theory lies the fact of singlet pairing of electrons. This is a deeply strongly correlated electrons feature which shares with resonant valence bond (RVB) theories and excludes an explanation of the most important features of SC cuprates, not only the AF and SC phases but also the intervening pseudogap phase, via Fermi liquid or Fermi liquid instability concepts. Although there have been many attempts of formulating a SO(5) symmetric model on lattices, a connection between the continuum theory and a more microscopic model like the \(t–J\) model is still missing.

In this sense, one of the motivations of the present study is to help bridge the gap between the more phenomenological SO(5) theory and the microscopic \(t–J\) model. More specifically, our goal is to obtain and study an effective model obtained by a change of basis from the site, spin-1/2 electrons, basis to the dimer basis, and on the square lattice by performing a change from site to dimer basis. The Hilbert space is truncated by projecting out single-occupied electron states and only nearest neighbor interactions are retained. The resulting effective models in one and two dimensions are studied by numerical techniques. In both spatial dimensions, the main result is that each hole pair is surrounded by a singlet cloud expelling triplet excitations from its vicinity. It is suggested an interpretation of this feature as a phase separated state between a hole-pair rich singlet phase and an undoped triplet phase with antiferromagnetic correlations. The possible relevance of this result to other theoretical scenarios and experimental results is discussed.

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been initially studied, among various reasons, as a real-

problems as well as experimental results is discussed.

square lattice, which are obtained in Section III. Finally,
techniques. The main results are common to the corre-

effective model allows its study by exact diagonalization

In addition, the one-dimensional (1D) character of this

various strongly correlated electron phases in 2D, it is in-

equations) has been derived previously (e.g. Ref. 8, and

The Hamiltonian of the $t$-$J$ model on this lattice has

away from half-filling, the states with a single hole

$P_{\alpha,i}$ is the projection operator on the subspace of the

for a singlet dimer. This splitting of the Hilbert space

operators: $\Psi_{E_0} = \Psi_0$. If $\Psi_0$ is the ground state

where $t_{\alpha,i}^\dagger$ is a creation operator of a triplet with

where the notation is standard. On ladders, $t_{ij} = t$, $J_{ij} =

Fig. I. Change from a site to a dimer basis in (a) a chain

At half-filling, the exact change of basis from the site to

the dimer basis (Fig. I(b)) leads to the following Hamil-

II. LADDERS

Although the main interest in connection with high-

Tc superconductivity is to study the interplay between

various strongly correlated electron phases in 2D, it is in-

structive to start with the analysis of two-leg ladders. It

should be noticed that the $t$–$J$ model on this lattice has

been initially studied, among various reasons, as a real-

ization of an RVB state.[4] In addition, extensive studies

have revealed strong similarities between the behaviors

obtained on ladders and on the 2D square lattice.

The Hamiltonian of the $t$–$J$ model is:

$$\mathcal{H} = - \sum_{\langle ij \rangle, \sigma} t_{ij} \bar{c}_{i \sigma}^\dagger c_{j \sigma} + h.c. + \sum_{\langle ij \rangle} J_{ij} (S_i \cdot S_j - \frac{1}{4} n_i n_j)$$

(1)

where the notation is standard. On ladders, $t_{ij} = t$, $J_{ij} = J$

along the legs, and $t_{ij} = J_{\perp}$ on the rungs. On chains and on the square lattice, the isotropic and

homogeneous case ($t_{ij} = t$, $J_{ij} = J$) will be considered.

At half-filling, the exact change of basis from the site to

the dimer basis (Fig. I(b)) leads to the following Hamiltonian:

$$\mathcal{H}_{ladder}^{(dimer)} = 2J \sum_{\langle ij \rangle} \left( t_{\alpha,i}^\dagger t_{\alpha,j} + h.c. \right) + J_{\perp} \sum_i (n_{i,i} - 1)$$

(2)

$$+ 2J \sum_{\langle ij \rangle} \left( t_{\alpha,i}^\dagger t_{\alpha,j} + h.c. \right) + 2J \sum_i S_i \cdot S_j$$

where $t_{\alpha,i}^\dagger$ is a creation operator of a triplet with $S^z = 0, 1, -1$ (with $\alpha = 0, +, -$) at dimer $i$, $n_{i,i} = n_{0,i} + n_{+,i} + n_{-,i}$, $n_{i,i} = t_{\alpha,i}^\dagger t_{\alpha,i}$. The first term corresponds to the spontaneous creation of two triplets out of the vacuum with the constraint of keeping total $S^z = 0$ and the second term is a triplet "hopping". It has been noted that this constraint is not satisfied in the pSO(5) Hamiltonian. The third term is the usual spin-1 Heisenberg exchange Hamiltonian.

This Hamiltonian (or equivalent expressions) has been derived previously (e.g. Ref. 3 and references therein) and used in restricted Hilbert space diagonalizations. The Hilbert space of the effective model turns out to be disconnected into two pieces: the set with even and the set of odd number of triplets. For example, in the subspace of total $S^z = 0$ the first set is generated by successive applications of the Hamiltonian to the initial state (...$ssssss$...), and the second one to the initial state (...$ssst00ss$...), where $s$ stands for a singlet dimer. This splitting of the Hilbert space does not appear in the effective model obtained for chains (Fig. I(a)).

Away from half-filling, the states with a single hole occupancy on a dimer are projected out. If $\mathcal{P}$ is the projection operator on the subspace of retained states and $\mathcal{Q}$ is the projection operator on the subspace of the eliminated states, then the effective Hamiltonian is given by the standard formula:

$$\mathcal{H}_{eff} = \mathcal{P}\mathcal{H}\mathcal{P} - \mathcal{P}\mathcal{Q}\frac{1}{\mathcal{Q}\mathcal{H}\mathcal{Q} - E_0}\mathcal{Q}\mathcal{H}\mathcal{P}$$

(3)

where $\mathcal{H}\Psi_0 = E_0\Psi_0$, and $\mathcal{H}_{eff}\mathcal{P}\Psi_0 = E_0\mathcal{P}\Psi_0$. If $\mathcal{H}$ is the one given by Eq. (1) on the $2 \times 2$ ladder with 2 holes, the effective Hamiltonian will contain nearest neighbor (NN) terms only. In this way, the hopping term of the effective Hamiltonian, $\mathcal{H}_{t,eff}$ results:

$$\mathcal{H}_{t,eff}^{(ladder)} = - t_s \sum_{\langle ij \rangle} \left( b_{i,j}^\dagger b_i + h.c. + n_{p,j} n_{s,i} + n_{s,j} n_{p,i} \right)$$

$$- t_t \sum_{\langle ij \rangle, s} \left( b_{i,j,s}^\dagger b_i + h.c. + n_{p,j} n_{t,i} + n_{t,j} n_{p,i} \right)$$

$$+ J_{\perp} \sum_i (n_{p,i} - 1)$$

(4)

where $b_{i,j}^\dagger$ is a creation operator of a hole pair in the dimer $i$, $n_{p,i} = b_{i,j}^\dagger b_i$. $n_{s,i} = 1, 0$ if the site is empty (i.e., a singlet) or occupied (by a triplet or a pair) respectively. The first term corresponds to the hopping of a pair to a singlet site, while the second term corresponds to the hopping to a site occupied by a triplet. The coupling constants $t_s, t_t$ are complicated functions of the original
parameters \( \{J, J_{\perp}, t, t_{\perp}\} \) but always \( t_{s} > t_{t} \), as shown in Fig. 2 for two values of the lattice anisotropy \( a = t_{\perp}/t \) \( (t_{s} = 2t_{t}, \) for the isotropic ladder \( a = 1 \) \). This is true even in the presence of a NN pair-pair Coulomb repulsion. This relation between the hopping parameters already suggests one of the most important results of this work. That is, the pairs would tend to be surrounded by singlets rather than by triplets in order to gain kinetic energy. In addition, the Heisenberg term in Eq. (2) would favor the clustering of triplets. These two combined effects would imply the phase separation between a pair-doped singlet phase and an undoped antiferromagnetics phase.

Finally, the full effective Hamiltonian in the projected dimer basis is given by:

\[
\mathcal{H}^{(e\text{ff})}_{\text{ladder}} = \mathcal{H}^{(dimer)}_{\text{ladder}} + \mathcal{H}^{(e\text{ff})}_{\text{1, ladder}}
\]

The 1D effective model can be studied by exact diagonalization techniques (Lanczos algorithm). Most of the results below were obtained on a \( L = 12 \) chain with periodic boundary conditions. Some computations for a \( L = 16 \) chain show that finite size effects are not important. All results shown below correspond to fixed number of pairs \( N_{p} \), and were obtained for the ground state, \( \mathbf{k} = (0,0) \). In the following, all energies and coupling constants are expressed in units of \( t \) of the original \( t-J \) Hamiltonian Eq. (1).

In Fig. 3 the correlation between a pair and a triplet, \( \langle n_{s}n_{p}\rangle \) at distance \( r = 1 \) (NN sites) is shown for the \( L = 12 \) chain with one pair, as a function of \( J/t \) and for three ladder anisotropy ratios. It can be seen that the probability of finding a triplet next to a pair is much smaller than the probability of finding a pair next to a singlet. The later, in the one pair system, is simply \( \langle n_{s}n_{p}\rangle = 1 - \langle n_{t}n_{p}\rangle \). It is more important the result that, as it can be easily seen in Fig. 3, \( \langle n_{t}n_{p}\rangle \) \( (r = 1) \) is always smaller than the triplet density \( \langle n_{t}\rangle \) (inset) for the same \( J/t \). The normalization of the later is such that \( n_{s} + n_{t} + n_{p} = 1 \), \( n_{p} = 0.0833 \) (one pair), and \( n_{p} = 0.1667 \) (two pairs). This result suggests that triplets are expelled from the vicinity of a pair.

![Fig. 2](image2.png)

**Fig. 2.** Hopping amplitudes between pairs \( p \) and singlets \( s \) \( (t_{s}) \) and between hole pairs and triplets \( t \) \( (t_{t}) \), as defined in Eq. (4), as a function of \( J \) and for \( a = 1 \) and 2. Basis change on a ladder (Fig. 1b).

![Fig. 3](image3.png)

**Fig. 3.** Triplet-pair correlation at \( r = 1 \), \( L = 12 \), \( N_{p} = 1 \) (open symbols), as a function of \( J \) and for the values of the anisotropy ratio \( a \) indicated on the plot. The inset shows the triplet density for the same cluster and parameters. Results for \( N_{p} = 2 \) (filled circles), \( a = 1 \) are also included.

![Fig. 4](image4.png)

**Fig. 4.** Diagonal correlations as a function of distance obtained for \( L = 12 \), \( N_{p} = 1 \), \( a = 1 \), and various values of \( J \), as indicated on the plot. (a) Triplet-triplet (dot lines) and triplet-pair (dashed lines) correlations; (b) singlet-singlet (full symbols, divided by 2) and singlet-triplet (dashed lines) correlations.
The complete picture can be inferred from Fig. 4 by looking at various correlations \( \langle n_a n_b \rangle, (a, b = t_0, t_+, t_-, p) \) as a function of distance. From now on, the study will be limited to the isotropic case \( (t = t_\perp, J = J_\perp \text{ in the original } t-J \text{ model}) \) but similar results were also found for all \( a > 1 \) investigated. In this Figure, which corresponds also to the system with one pair, it can be observed that the triplet-triplet correlation is maximum at \( r = 0 \), while the triplet-pair correlation is maximum at the largest distance on the chain. That is, triplets try to stay as far apart as possible form a pair. On the other hand, although singlet-singlet correlations are also maximum at \( r = 0 \), the singlet-pair correlations are maximum at \( r = 0 \). The final piece is that the singlet-triplet correlation is maximum at the maximum distance. Similar results were obtained for the larger \( L = 16 \) site chain.

The picture emerging from these correlations is that pairs move preferentially in a background of singlets, and that both pairs and singlets try to keep themselves away from triplets. Thus, the system is separated between a pair-rich singlet phase (it is tempting to consider this from triplets. Thus, the system is separated between a pair-rich singlet phase and a pair-poor triplet-rich phase which can be identified as an undoped AF phase.

Essentially the same behavior is observed in the case of two pairs present in the system. Results for the triplet-pair correlation in NN sites and the triplet density as a function of \( J/t \) are also included in Fig. 3 for comparison with the one pair case. Both the probability of finding a pair next to a triplet and the triplet density are smaller than for the one pair case. A likely explanation of this behavior is that the introduction of more pairs in the system increases the volume of the pair-doped “RVB” phase leaving less room for the AF phase.

In Fig. 3, diagonal correlations are shown for \( L = 12 \) and same parameters as in Fig. 4. The new feature in Fig. 3 with respect to Fig. 4 is the fact that two pairs repel themselves as can be read from the fact that pair-pair correlations are maximum at the largest distance. Then, the same qualitative behavior found in the one-pair case holds: the system separates into a pair-doped singlet region, here formed by two islands and a pair-poor triplet-rich region, presumably with short-range AF order, in this case filling the space between those two islands. Notice also in Fig. 3(b) the singlet-singlet correlations falling down to its bulk value within a lattice spacing.

### III. TWO DIMENSIONS

The most important situation is that of the square lattice, which corresponds to the CuO\(_2\) planes in superconducting cuprates. There are again infinitely many different ways in which a change from the site to the dimer basis can be performed. One of them is shown in Fig. 5. A well-known feature of choosing a dimer basis like the one depicted in Fig. 5 is that the rotation invariance of the square lattice is broken and it is not simple to restore it at the level of the effective Hamiltonian. In the ladder case, examined in the previous Section, this is not important since the lattice itself is spatially anisotropic. In the case of the square lattice, the purpose of the present study is to provide indications of the presence, in a rotationally-broken effective model for the square lattice, of the singlet-AF phase separation already observed in ladders and to suggest that this feature should be also present in a rotational invariant formulation.

![FIG. 5. Diagonal correlations as a function of distance obtained for \( L = 12, N_p = 2, a = 1 \), and various values of \( J \) as indicated on the plot. (a) Triplet-triplet (full symbols), pair-pair (open symbols) and triplet-pair (dashed lines) correlations; (b) singlet-singlet (full symbols, divided by 2) and singlet-triplet (dashed lines) correlations.](image)

![FIG. 6. A possible change from a site to a dimer basis in the square lattice. Dotted and dash-dotted lines indicate effective magnetic interactions in the new basis.](image)
given by the following Hamiltonian:

\[
\mathcal{H}_{\text{J,chain}}^{(\text{dimer})} = -J \sum_{\langle i,j \rangle} (t_{0,i}^{\dagger} t_{0,j} - t_{i}^{\dagger} t_{j} - t_{-i}^{\dagger} t_{-j} + h.c.) + J_\perp \sum_{i} (n_{i,\perp} - 1)
\]

where \( \beta = \pm, 0, -\), \( \gamma = -, +, 0 \) for \( \alpha = 0, +, - \) respectively. The sign of the second term depends on the definition of the singlet. It should be noticed that this term, as in the ladder case, locally conserves the total \( S^z \). This Hamiltonian can be read, with a slightly different notation though, in Ref. \[10\] and alternative or similar derivations can be found in several other places.[12][13] Finally, the exchange part of the Hamiltonian in the dimer basis is:

\[
\mathcal{H}_{\text{J,square}}^{(\text{dimer})} = \mathcal{H}_{\text{J,ladder}}^{(\text{dimer})} + \mathcal{H}_{\text{J,chain}}^{(\text{dimer})}
\]

with the ladder (chain) term acting on the horizontal (vertical) direction as indicated in Fig. [1].

\[
\mathcal{H}^{(\text{eff})}_{\text{J,chain}} = -t'_{0} \sum_{\langle i,j \rangle} (b_{0,i}^{\dagger} b_{0,j} + h.c.)
\]

\[
- t_{s0} \sum_{\langle i,j \rangle} (n_{p,j} s_{i} + n_{s,j} n_{p,i})
\]

\[
- t_{t} \sum_{\langle i,j \rangle} (b_{0,j}^{\dagger} b_{0,i} + h.c.)
\]

\[
- t_{t0} \sum_{\langle i,j \rangle} (n_{p,j} n_{t,i} + n_{t,j} n_{p,i})
\]

\[
+ J_\perp \sum_{i} (n_{p,i} - 1)
\]

As in the case of Eq. (4), the hopping amplitudes satisfy \( t'_{0} > t'_{i} \) and \( t_{s0} > t_{t0} \) (Fig. [1]) again favoring the movement of pairs away from triplet-rich regions.

Coming back to the square lattice, the correct procedure is to to take \( \mathcal{H} \) as the \( t-J \) Hamiltonian on the eight-site cluster indicated by a dashed box in Fig. [1]. As a result, the effective Hamiltonian contains three- and four-site terms in addition to NN interactions. In order to keep the Hamiltonian as close as possible with the proposed pSO(5) model, only NN hopping interactions as given by Eqs. (3) and (6) in the horizontal and vertical directions respectively are retained. The signs of these hopping terms coming from the eight-site cluster calculation are the same as in Eqs. (3) and (6). To compensate for neglecting three- and four-site terms, the hopping amplitudes in (1) are re-scaled by a single constant \( \alpha \) and the amplitudes in (8) by another constant \( \beta \). A reasonable fit of the energies of the effective model on the \( 4 \times 2 \) cluster to the exact energies of the \( t-J \) model on the \( 4 \times 4 \) cluster in the whole range studied, \( 0 \leq J \leq 2.5 \), is achieved with \( \alpha = 1 \) and \( \beta = 0.5 \). In this range of \( J \), the relative difference between these two energies is less than 0.01. The exchange part of the effective model is given by Eq. (8).

A first insight on the properties of the effective model can be gained by studying the \( 4 \times 4 \) cluster. Due to the large dimension of the Hilbert space (\( \approx 7.76 \times 10^{7} \) for \( N_p = 1 \) and \( \approx 1.50 \times 10^{8} \) for \( N_p = 2 \) taking into account translational invariance), conventional exact diagonalization techniques cannot be applied, except by resorting to massive computers. Alternatively, a diagonalization in a systematically expanded Hilbert space (SEHS)[12] is used. With a number of states \( \approx 5 \times 10^{6} \), variational energies within 1% of the exact energies, estimated by extrapolating to the full dimension of the Hilbert space, are obtained. In order to study larger clusters, a quantum Monte Carlo (QMC) technique with the conventional worldline checkerboard decomposition is used. Although there are no fermions involved, there is a “minus sign problem” which makes impossible the study at low temperatures. There are several terms in the effective Hamiltonian which lead to “minus sign” configurations in the \( 2+1 \)-dimensional space. An important reduction of this problem is achieved by not generating those configurations with interacting cubes which do not
conserve the parity of the number of triplets on its top and bottom plaquettes. Of course, the QMC algorithm is no longer exact but nevertheless it provides a reasonable approximation to its exact behavior. By eliminating certain transitions, it might be possible that certain regions of the phase space are disconnected. To cope with this problem, at each temperature, results were averaged over at least four independent runs starting from different initial states. Variations in the values of the energy from different runs were somewhat larger than the statistical error of each run but nevertheless smaller than 1% in all cases. In addition, since the present study concerns zero temperature properties, the simulations are restricted to the subspace of zero total magnetization and only local moves are included in the algorithm. Overall simulation error bars are approximately twice the size of the symbols used.

Evolution of various quantities with temperature in the \( S^z = 0 \) subspace on the 4 \( \times 4 \) cluster, \( J/t = 0.8 \), with one and two pairs are shown in Fig. 8. The corresponding zero temperature results obtained by diagonalization in an expanded Hilbert space are also included for comparison. As expected the energies obtained by the approximated QMC technique are higher than the obtained by diagonalization (which are virtually exact), while the triplet densities are smaller than the exact ones.

The same correlations previously studied on ladders are shown in Fig. 9(a) for the 4 \( \times 4 \) triplet densities are smaller than the exact ones. By diagonalization (which are virtually exact), while the approximated QMC technique are higher than the obtained comparison. As expected the energies obtained by the algorithm are also included for comparison. Overall statistical error of each run but nevertheless smaller than 1% in all cases. In addition, since the present study concerns zero temperature properties, the simulations are restricted to the subspace of zero total magnetization and only local moves are included in the algorithm. Overall simulation error bars are approximately twice the size of the symbols used.

A more systematic study as a function of \( J/t \) leads to the results depicted in Fig. 10(a). In all cases, the probability of finding a triplet near a pair (open symbols) is smaller than the average probability of finding a triplet on a given site (full symbols). Consistently with the idea of pairs expelling triplet excitations, the probability of finding a triplet near a pair (open symbols) is smaller than the average probability of finding a triplet on a given site (full symbols). Consistently with the idea of pairs expelling triplet excitations, the probability of finding a triplet near a pair (open symbols) is smaller than the average probability of finding a triplet on a given site (full symbols). Consistently with the idea of pairs expelling triplet excitations, the probability of finding a triplet near a pair (open symbols) is smaller than the average probability of finding a triplet on a given site (full symbols).

The superfluid density is of great importance for studying the superfluid properties of the system. The procedure to calculate the superfluid density \( \chi \) starts by computing the correlation:

\[
C(\tau) = \langle \Delta p_x(\tau)\Delta p_x(0) \rangle
\]

where \( \Delta p_x(\tau) = p_x(\tau + 1) - p_x(\tau) \) and \( p_x(\tau) = \sum x(i,\tau) \), where \( x(i,\tau) \) is the \( x \)-coordinate of pair \( i \) at imaginary
time \( \tau \) and the sum extends over all the pairs in the system. The superfluid density follows from:

\[
\rho_s \approx \lim_{\omega \to 0} \int_0^\beta d\tau C(\tau) \exp(-i\omega \tau)
\]

(10)

It is well-known that this quantity vanishes if the simulation is carried in a zero-winding number subspace which is the case in the present study. The way out of this problem stems from the fact that the winding number may be nonzero in half of the Trotter, imaginary time, direction, being also nonzero but with an opposite sign in the other half. The superfluid density then would come out by taking the Fourier transform in (10) between zero and \( \beta/2 \). This procedure should be exact in the limit of infinite Trotter number which is the limit in which on the other hand the whole worldline algorithm is valid. This procedure has been thoroughly checked in the hard-core boson model where exact and numerical results are available.\(^2\)

\[\text{FIG. 10. Results obtained by QMC on the 8} \times \text{8 cluster, T} = 0.09t\text{ at various fillings and as a function of } J/t. \text{ (a) Triplet density (full symbols) and triplet-pair correlations at NN sites (open symbols). The triplet-pair correlation at the maximum distance for } N_p = 1 \text{ are shown with stars(b) AF structure factor (triangles: } N_p = 4, \text{ diamonds: } N_p = 8) \text{ and superfluid density (plus: } N_p = 4, \text{ stars: } N_p = 8).} \]

The superfluid density, \( \rho_s \), and the magnetic structure factor at (\( \pi, \pi \)), \( \chi_{AF} \), on the 8 \times 8 cluster with four and eight pairs, are shown in Fig. 10(b) as a function of \( J/t \). Although the ultimate dominance of SC or AF, or eventually the coexistence of both SC and AF orders can be determined only by a finite size extrapolation which is out of the scope of the present study, two qualitative features are apparent. First, as a pair doping is increased there is an enhancement of SC and a suppression of AF. Second, as \( J/t \) is increased \( \chi_{AF} \) also increases while \( \rho \) decreases. Similar results were obtained with an even simplified model which mimics the spin triplets by spin doublets,\(^2\) and can be obtained by projecting out the \( S^z = 0 \) component of the triplets, \( \epsilon_0 \), again by using \( \{ \} \). The mutually exclusive behavior of AF and superconductivity, which has been shown both in the present work and previously in Ref. \(^2\), can now be understood in terms of this real space phase separation between SC-singlet and AF regions. The growth of one of each phases reduces the space available for the other phase.

**IV. CONCLUSIONS**

In the first place, the present work is concerned with the mapping of the \( t-J \) model into an effective model of pairs and triplets moving on a “sea” of singlets. The strategy adopted was first to map exactly the magnetic interactions in the undoped system and then to determine the effective hopping interactions in the two-hole sector of the original Hamiltonian. This second procedure is an approximate one because it implies projecting out single-occupied dimers and truncating the range of the interactions nearest neighbors. In the case of ladders this procedure is reasonably clean. The choice of dimers corresponds to the strong coupling limit which extends virtually to the anisotropic case. An effective model in one dimension is obtained in which the hopping couplings between pairs and singlets are larger than between pairs and triplets.

In the square lattice the procedure to obtain the effective model is more complicated. In the first place, the simple dimer covering adopted breaks rotational invariance of the lattice. Still, the mapping of the magnetic interactions at half-filling is exact. In the second place, even by restricting the range of hoppings to nearest neighbors, three- and four-site interactions appear at the effective level. To eliminate these interactions another fit to the energies of a small \( t-J \) cluster was performed to rescale the effective hopping constants. The resulting effective hopping couplings show the same behavior as those obtained for the ladder case.

The resulting effective models were studied by numerical techniques, exact diagonalization in the case of the model obtained for the ladder lattice and approximate diagonalization and Quantum Monte Carlo for the square lattice case. Again the results obtained in the effective model for ladders are more clear than the ones obtained for the square lattice.

From the study of several correlations the following picture emerges. Pairs are surrounded predominantly by singlets, and triplet excitations are located as far apart as possible. This is a kind of phase separation between a pair-rich “RVB” phase and a pair-poor triplet-rich phase which would correspond to a phase with at least short-range interactions (since triplet excitations restore AF order from a RVB state) and would quite likely be insulating. Pair-pair correlation functions indicate that
pairs try to be situated at the maximum distance (this behavior is more definite in ladders than in 2D). This behavior might correspond to pairs moving in a singlet phase macroscopically separated from the triplet phase. Alternatively, the phase separation might be microscopic: pair-singlet islands moving in a triplet background. The behavior of triplet-pair correlations in 1D could favor this second scenario. In any case, these PS scenarios provide an explanation for the AF-SC mutual exclusion shown in the previous Section.

Further studies are necessary to distinguish between both types of phase separation. In the macroscopic PS scenario it is expected a superfluid density comparable with that of the hard-core boson model while on the microscopic PS scenario it would be considerably reduced. On theoretical grounds, the emerging PS picture could be realted to the AF-SC coexistence phase predicted by SO(5) theories. The exclusion of triplets and pairs is highly nontrivial taking into account that previous studies on the $t$-$J$ model\cite{29} have suggested a bound state between a $d_{x^2-y^2}$ pair and a triplet. However, the internal structure of a pair, essential in the analysis of Ref.\cite{29}, is lost in the present study. Finally, it is also tempting to relate this PS state to recent observations of inhomogeneities in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ (Ref.\cite{30}). It might be possible that out-of-plane negative Coulomb centers could attract and pin pair-singlet islands in their surroundings.

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