Spectral properties of statistical mechanics models

Hendrik Meyer, Jean-Christian Anglès d’Auriac, and Henrik Bruus
Centre de Recherches sur les Très Basses Températures, BP 166, 38042 Grenoble, France
(April 5, 1996)

The full spectrum of transfer matrices of the general eight-vertex model on a square lattice is obtained by numerical diagonalization. The eigenvalue spacing distribution and the spectral rigidity are analyzed. In non-integrable regimes we have found eigenvalue repulsion as for the Gaussian orthogonal ensemble in random matrix theory. By contrast, in integrable regimes we have found eigenvalue independence leading to a Poissonian behavior, and, for some points, level clustering. These first examples from classical statistical mechanics suggest that the conjecture of integrability successfully applied to quantum spin systems also holds for classical systems.

PACS numbers: 05.50.+q, 05.20.-y, 05.45+b

Since the work of Wigner [1] random matrix theory (RMT) has been applied successfully in various domains of physics [2]. Recently several quantum spin Hamiltonians have been investigated from this point of view. It has been found [3,4] that 1D systems for which the Bethe ansatz applies have a level spacing distribution close to a Poissonian (exponential) distribution, \( P(s) = \exp(-s) \), whereas if the Bethe ansatz does not apply, the level spacing distribution is described by the Wigner surmise for the Gaussian orthogonal ensemble (GOE), \( P(s) = \frac{2}{\pi s} \exp(-\pi s^2/4) \). Similar results have been found for 2D systems [5]. This suggests that the GOE describes properly some properties of the spectrum of complex quantum systems. In this letter we extend the RMT analysis from quantum spin systems to models of classical statistical mechanics. In particular, we look at possible consequences of integrability on the spectral properties of the transfer matrices.

At first sight it seems natural to start with the Ising model in two dimensions without magnetic field as an example of an integrable model, and then to add a magnetic field. It turns out that the spectrum of transfer matrices of the Ising model for finite size leads to numerical difficulties as explained below. We then have chosen the case of the general eight-vertex model on a square lattice (which contains the zero-field Ising model as a special case) [6]. Moreover, it is known that zero-field eight-vertex transfer matrices commute with the Hamiltonian of the anisotropic XYZ quantum spin chain for certain relations between the parameters of the two models [7]. We shall use the notation of Ref. [6] to designate the eight admissible vertices and their respective Boltzmann weights \( a, a', b, b', c, c', d, \) and \( d' \). We consider the row-to-row transfer matrices \( T_N \) to build iteratively a periodic rectangular lattice by adding rows of length \( N \) with periodic boundary conditions. Therefore the partition function of a periodic rectangle of \( n \) rows of \( N \) sites is \( Z_{n,N} = \text{Tr}(T_N^n) \). Note that there can be different expressions for the matrix \( T_N \), but all these expressions have the same value of the trace of the \( n \)th power for any \( n \), and therefore they can be deduced from each other by a similarity transformation; the spectrum is indeed an intrinsic quantity which does not depend on any particular choice of the transfer matrix. To perform the usual statistical analysis of the spectrum we need to have real eigenvalues. However, in general this is not the case and we will restrict ourselves to cases where the transfer matrix is symmetric. It is well known that the eight-vertex model can be mapped onto an anisotropic Ising model on a square lattice with diagonal interactions and four spin interactions around each plaquette. We again use the notations of Ref. [6] and introduce the five coupling constants \( J_h, J_v, J, J' \) and \( J'' \). The transfer matrix of the spin model can be chosen symmetric if \( J = J' \). In terms of the Boltzmann weights of the transfer matrix it requires only that \( c = d = c' = d' \) (\( c = c' \) and \( d = d' \) is not a restriction). This condition is verified for models without electrical field (i.e. when \( a = a' \) and \( b = b' \)) and also for models with a field. So we are able to build symmetric transfer matrices for integrable cases with \( a = a' \), \( b = b' \), and \( c = c' = d = d' \), as well as for non-integrable cases with \( a \neq a' \), \( b \neq b' \), and \( c = c' = d = d' \).

Before presenting our results, we briefly recall some features of the RMT analysis, which is a statistical analysis of the eigenvalues of a given matrix regarded as an ordered set. Firstly, one has to sort the eigenvalues according to the symmetry of the corresponding eigenstate. In contrast to quantum spin systems, transfer matrices possess a priori no continuous symmetry (as e.g. the SU(2) spin symmetry), but only space symmetries. For row-to-row transfer matrices these are given by the automorphism group of a single row (and not of the full lattice). This group is the set of permutations \( g \) of sites such that \( g(i) \) and \( g(j) \) are neighbours if and only if \( i \) and \( j \) are neighbours. To each \( g \) acting on the set of vertices one can easily associate a linear operator \( \hat{g} \) acting in the configuration space. Obviously \( \hat{g} \) and \( T_N \) commute. It is then possible to construct a set of projectors onto invariant subspaces of \( T_N \). This amounts to block-diagonalize \( T_N \). This is not only a useful way of lowering the size of the matrices to diagonalize, but also the manner to sort the eigenvalues. The automorphism group of the periodic ring...
of length $N$ is the dihedral group $D_N$ generated by a translation and a reflection. Elementary group theoretical analysis can be performed to build the $(N/2 + 3$ if $N$ is even, or $(N − 1)/2 + 2$ if $N$ is odd) projectors onto the invariant subspaces. The transfer matrix of the zero-field model is also invariant under the reversal of all arrows of the vertices and another projector has to be applied. Secondly, to find universal behavior within each invariant subspace, one needs to ‘rescale’ the eigenvalues $E_i$ in order to have a local density of eigenvalues equal to one. This operation is called the “unfolding” and produces the unfolded eigenvalues $\epsilon_i$. The aim is to remove the non-universal or system specific large scale variations of the integrated density of states, and to study only the presumably universal short scale fluctuations. It amounts to compute an average integrated density of states $N_{av}(E)$ which is the smooth part of the actual integrated density of states. We then have $\epsilon_i = N_{av}(E_i)$. In the generic case, several methods can be used to compute $N_{av}(E)$: running average unfolding (local averaging of eigenvalues followed by interpolation), Gaussian unfolding (Gaussian broadening of each delta peak in the density of states), and Fourier unfolding (removal of short scale wave lengths using Fourrier transformation).

The simplest quantity one calculates in RMT analysis is the distribution $P(s)$ of the differences between two consecutive unfolded eigenvalues $s_1 = \epsilon_{i+1} − \epsilon_i$. For integrable systems a Poissonian distribution is expected, since the $\epsilon_i$ are expected to be independent. In contrast, for the simplest non-integrable systems the Wigner surmise is expected. Another quantity of interest is the spectral rigidity $\Delta_3$:

$$\Delta_3(L) = \left\langle \frac{1}{L} \min_{a,b} \int_{-L/2}^{L/2} (N(e) − a\epsilon − b)^2 de \right\rangle_{\alpha},$$

where $\langle \ldots \rangle_{\alpha}$ denotes an average over $\alpha$. This quantity measures the deviation from equal spacing. For a totally rigid spectrum, as that of the harmonic oscillator, one has $\Delta_3^{\text{osc}}(L) = 1/12$, for an integrable (Poissonian) system one has $\Delta_3^{\text{Poi}}(L) = L/15$, while for the Gaussian Orthogonal Ensemble one has $\Delta_3^{\text{GOE}}(L) = \frac{1}{12}(\log(L) − 0.0687) + O(L^{-1})$. It has been found that the spectral rigidity of quantum spin systems follows $\Delta_3^{\text{Poi}}(L)$ in the integrable case and $\Delta_3^{\text{GOE}}(L)$ in the non-integrable case. However, in both cases, even though $P(s)$ is in good agreement with RMT, deviations from RMT occur for $\Delta_3(L)$ at some system dependent point $L^*$. This stems from the fact that the rigidity $\Delta_3(L)$ probes correlations beyond nearest neighbours in contrast to $P(s)$. This is probably why the rigidity is much more sensitive to the parameters of the unfolding than the spacing distribution.

We have generated transfer matrices for different values of the Boltzmann weights and linear size $N = 16$ (resp. $N = 14$). This leads to matrix sizes of $65536^2$ (resp. $16384^2$). Because of the eight-vertex condition, the transfer matrix couples only configurations with the numbers of up (or down) arrows having the same parity, so the matrix trivially separates into two blocks. After projection the matrix splits up into 18 (resp. 16) symmetry invariant blocks of which the largest has a size of $2062^2$ (resp. $594^2$). We have tried several methods of unfolding. The results presented here are obtained using either a Gaussian unfolding with a local broadening over five states, or a running average unfolding over ten states. In Fig. 1 we present a typical integrated density

![Fig. 1](image)

**FIG. 1.** The integrated density of eigenvalues $N(E)$ (circles) for the case $a = 2.5$, $a' = 1.6$, $b = b' = 3$, $c = c' = d = d' = 1/\sqrt{6}$ for a single symmetry invariant block. The insert shows how complex $N(E)$ is even at a very fine scale. The full line is $N_{av}(E)$.
free-fermion point within the generally non-integrable region where an electrical field is present. The Boltzmann weights are \( a = 0.8, a' = 1/a, b = b' = \sqrt{2c^2 - 1}, \) and \( c = c' = d = d' = 1/\sqrt{6} \). The full line is the Poissonian distribution while the dashed line is the Wigner surmise.

In summary, we have numerically calculated the spectrum of transfer matrices of the 2D eight-vertex model for various parameters. After having sorted and unfolded difficulties in the unfolding. A possible explanation of the peak at \( s = 0 \) could be that the free-fermion model is a trivial non-generic model as for example the Hubbard model at zero Coulomb repulsion. However, for some other values of the Boltzmann weights also obeying the free-fermion condition, the spacing distribution is much closer to the Poissonian distribution. This suggests as another possible explanation the existence of quasi-degeneracy leading to a Shnirelman peak at the origin \( b \) for this specific set of Boltzmann weights. This will be studied in detail in a forthcoming publication. From the above results we conjecture that the spacing distribution of eigenvalues of non-integrable models is close to the Wigner surmise corresponding to level repulsion, while for integrable models there is no level repulsion. In integrable systems there is level independence leading to a Poissonian spacing distribution, but with a tendency to level attraction in some cases.

To go further in the RMT analysis, we present in Fig. 4 the spectral rigidity \( \Delta_3(L) \) for the same points in parameter space corresponding to integrability and to non-integrability as in Fig. 3. The two limiting cases corresponding to the Poissonian distributed eigenvalues (solid line) and to GOE distributed eigenvalues (dashed line) are also shown. For the integrable point the agreement between the numerical data and the expected rigidity is good up to a value \( L \approx 5 \). For larger values of \( L \) a saturation occurs showing the limitation of the model of independent eigenvalues. For the non-integrable case the departure of the rigidity from the expected behavior appears at \( L \approx 2 \), indicating that the RMT is only valid at short scales. Such behavior has already been seen in quantum spin systems \( b \). We stress that these numerical results depend much more on the unfolding than the results concerning the spacing distribution.

In summary, we have numerically calculated the spectrum of transfer matrices of the 2D eight-vertex model for various parameters. After having sorted and unfolded difficulties in the unfolding. A possible explanation of the peak at \( s = 0 \) could be that the free-fermion model is a trivial non-generic model as for example the Hubbard model at zero Coulomb repulsion. However, for some other values of the Boltzmann weights also obeying the free-fermion condition, the spacing distribution is much closer to the Poissonian distribution. This suggests as another possible explanation the existence of quasi-degeneracy leading to a Shnirelman peak at the origin \( b \) for this specific set of Boltzmann weights. This will be studied in detail in a forthcoming publication. From the above results we conjecture that the spacing distribution of eigenvalues of non-integrable models is close to the Wigner surmise corresponding to level repulsion, while for integrable models there is no level repulsion. In integrable systems there is level independence leading to a Poissonian spacing distribution, but with a tendency to level attraction in some cases.

To go further in the RMT analysis, we present in Fig. 4 the spectral rigidity \( \Delta_3(L) \) for the same points in parameter space corresponding to integrability and to non-integrability as in Fig. 3. The two limiting cases corresponding to the Poissonian distributed eigenvalues (solid line) and to GOE distributed eigenvalues (dashed line) are also shown. For the integrable point the agreement between the numerical data and the expected rigidity is good up to a value \( L \approx 5 \). For larger values of \( L \) a saturation occurs showing the limitation of the model of independent eigenvalues. For the non-integrable case the departure of the rigidity from the expected behavior appears at \( L \approx 2 \), indicating that the RMT is only valid at short scales. Such behavior has already been seen in quantum spin systems \( b \). We stress that these numerical results depend much more on the unfolding than the results concerning the spacing distribution.

In summary, we have numerically calculated the spectrum of transfer matrices of the 2D eight-vertex model for various parameters. After having sorted and unfolded difficulties in the unfolding. A possible explanation of the peak at \( s = 0 \) could be that the free-fermion model is a trivial non-generic model as for example the Hubbard model at zero Coulomb repulsion. However, for some other values of the Boltzmann weights also obeying the free-fermion condition, the spacing distribution is much closer to the Poissonian distribution. This suggests as another possible explanation the existence of quasi-degeneracy leading to a Shnirelman peak at the origin \( b \) for this specific set of Boltzmann weights. This will be studied in detail in a forthcoming publication. From the above results we conjecture that the spacing distribution of eigenvalues of non-integrable models is close to the Wigner surmise corresponding to level repulsion, while for integrable models there is no level repulsion. In integrable systems there is level independence leading to a Poissonian spacing distribution, but with a tendency to level attraction in some cases.

To go further in the RMT analysis, we present in Fig. 4 the spectral rigidity \( \Delta_3(L) \) for the same points in parameter space corresponding to integrability and to non-integrability as in Fig. 3. The two limiting cases corresponding to the Poissonian distributed eigenvalues (solid line) and to GOE distributed eigenvalues (dashed line) are also shown. For the integrable point the agreement between the numerical data and the expected rigidity is good up to a value \( L \approx 5 \). For larger values of \( L \) a saturation occurs showing the limitation of the model of independent eigenvalues. For the non-integrable case the departure of the rigidity from the expected behavior appears at \( L \approx 2 \), indicating that the RMT is only valid at short scales. Such behavior has already been seen in quantum spin systems \( b \). We stress that these numerical results depend much more on the unfolding than the results concerning the spacing distribution.

In summary, we have numerically calculated the spectrum of transfer matrices of the 2D eight-vertex model for various parameters. After having sorted and unfolded difficulties in the unfolding. A possible explanation of the peak at \( s = 0 \) could be that the free-fermion model is a trivial non-generic model as for example the Hubbard model at zero Coulomb repulsion. However, for some other values of the Boltzmann weights also obeying the free-fermion condition, the spacing distribution is much closer to the Poissonian distribution. This suggests as another possible explanation the existence of quasi-degeneracy leading to a Shnirelman peak at the origin \( b \) for this specific set of Boltzmann weights. This will be studied in detail in a forthcoming publication. From the above results we conjecture that the spacing distribution of eigenvalues of non-integrable models is close to the Wigner surmise corresponding to level repulsion, while for integrable models there is no level repulsion. In integrable systems there is level independence leading to a Poissonian spacing distribution, but with a tendency to level attraction in some cases.
the spectrum we have computed the eigenvalue spacing distribution and the spectral rigidity averaged over all representations. To our knowledge this is the first RMT analysis of transfer matrices of models in classical statistical mechanics. We have found that the non-integrable cases are well described at short scales by a the Gaussian orthogonal ensemble, while in the integrable cases the eigenvalues are mostly independent. We speculate that this is a general results.

ACKNOWLEDGMENTS

We would like to thank J.M. Maillard for many discussions concerning integrability of vertex models. H.B. is supported by the European Commission under grant no. ERBFMBICT 950414.

E-mail:
hmeyer@crtbt.polycnrs-gre.fr
dauriac@crtbt.polycnrs-gre.fr
bruus@crtbt.polycnrs-gre.fr

[1] E.P. Wigner. Ann. Math. 53, 36 (1953).
[2] M.L. Mehta. Random matrices (2nd. ed., Academic Press, New York, N.Y.) 1991.
[3] G. Montambaux, D. Poilblanc, J. Bellisard, and C. Sire. Phys. Rev. Lett. 70, 497 (1993).
[4] T.C. Hsu and J.-C. Anglès d’Auriac. Phys. Rev. B47, 14291 (1993).
[5] H. Bruus and J.-C. Anglès d’Auriac. in Proc. XXXIst rencontre de Moriond, Les Arcs (Ed. Frontière, Paris) 1996 and SISSA cond-mat/9507012
[6] R.J. Baxter. Exactly solved models in statistical mechanics. Acad. Press, London 1982.
[7] B. Sutherland. J. Math. Phys. 11, 3183 (1970).
[8] C. Fan and F.Y. Wu. Phys. Rev. B2, 723 (1970).
[9] B.V. Chirikov and D.L. Shpelyansky. Phys. Rev. Lett. 74, 518 (1995).