CLASSIFYING p-GROUPS VIA THEIR MULTIPLIER

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Abstract. The author in (On the order of Schur multiplier of non-abelian p-groups. J. Algebra (2009).322: 4479–4482) showed that for any p-group \( G \) of order \( p^n \) there exists a nonnegative integer \( s(G) \) such that the order of Schur multiplier of \( G \) is equal to \( p^{\frac{1}{2}(n-1)(n-2)}+1-s(G) \). Furthermore, he characterized the structure of all non-abelian p-groups \( G \) when \( s(G) = 0 \). The present paper is devoted to characterization of all p-groups when \( s(G) = 2 \).

The concept of Schur multiplier, \( \mathcal{M}(G) \), have been studied by several authors, initiated by Schur in 1904. It is known that the order of Schur multiplier of a given finite p-group of order \( p^n \) is equal to \( p^{\frac{1}{2}n(n-1)-t(G)} \) for some \( t(G) \geq 0 \) by a result of Green [5]. It is of interest to know which p-groups have the Schur multiplier of order \( p^{\frac{1}{2}n(n-1)-t(G)} \), when \( t(G) \) is in hand.

Historically, there are several papers trying to characterize the structure of \( G \) by just the order of its Schur multiplier. In [1] and [13], Berkovich and Zhou classified the structure of \( G \) when \( t(G) = 0, 1 \) and \( 2 \), respectively.

Later, Ellis in [2] showed that having a new upper bound on the order of Schur multiplier of groups reduces characterization process of structure of \( G \). He reformulated the upper bound due to Gaschütz at. al. [4] and classified with a quite way to that of [1, 13] the structure of \( G \) when \( t(G) = 3 \).

The result of [9] shows that there exists a nonnegative integer \( s(G) \) such that \( |\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)}+1-s(G) \) which is a reduction of Green’s bound for any given non-abelian p-group \( G \) of order \( p^n \). One can check that the structure of \( G \) can be characterized by using [9, Main Theorem], when \( t(G) = 1, 2, 3 \). Moreover, characterizing non-abelian p-groups by \( s(G) \) can be significant since for instance the result of [9] and [12] emphasize that the number of groups with a fixed \( s(G) \) is more than that with fixed \( s(G) \). Also the results of [10] and [11] show handling the p-groups characterized by \( s(G) = 0, 1 \) may be caused to characterize the structure of \( G \) by \( t(G) \).

In the present paper, we intend to classify the structure of all non-abelian p-groups when \( s(G) = 2 \).

Throughout this paper we use the following notations.

\( Q_8 \): quaternion group of order 8,
\( D_8 \): dihedral group of order 8,
\( E_1 \): extra special p-group of order \( p^3 \) and exponent \( p \),
\( E_2 \): extra special p-group of order \( p^3 \) and exponent \( p^2 \) (\( p \neq 2 \)),
\( \mathbb{Z}_{p^m} \): direct product of \( m \) copies of the cyclic group of order \( p^n \),
\( G_{ab} \): the abelianization of group \( G \),
\( H \cdot K \): the central product of \( H \) and \( K \).

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\[ E(m) : E \cdot Z(E), \] where \( E \) is an extra special \( p \)-group and \( Z(E) \) is cyclic group of order \( p^m \) \((m \geq 2)\),

\( \Phi(G) \): the Frattini subgroup of group \( G \).

Also, \( G \) has the property \( s(G) = 2 \) or briefly with \( s(G) = 2 \) means the order of its Schur multiplier is of order \( p^{2(n-1)(n-2)-1} \).

The following lemma is a consequence of \( [9] \) Main Theorem.

**Lemma 1.** There is no \( p \)-group with \( |G'| \geq p^3 \) and \( s(G) = 2 \).

**Lemma 2.** There is no \( p \)-group of order \( p^n \) \((n \geq 5)\) when \( G^{ab} \) is not elementary abelian and \( s(G) = 2 \).

**Proof.** First suppose that \( n = 5 \). By virtue of \( [11] \) Theorem 3.6, the result follows.

In case \( n \geq 6 \), by invoking \( [9] \) Lemma 2.3, we have \( |\mathcal{M}(G/G')| \leq p^{2(n-2)(n-3)} \), and since \( G/Z(G) \) is capable the rest of proof is obtained by using \( [3] \) Proposition 1. □

**Lemma 3.** Let \( G \) be a \( p \)-group and \( |G'| = p \) or \( p^2 \) with \( s(G) = 2 \). Then \( Z(G) \) is of exponent at most \( p^2 \) and \( p \), respectively.

**Proof.** Taking a cyclic central subgroup \( K \) of order \( p^k \) \((k \geq 3)\) and using \( [5] \) Theorem 2.2, we should have

\[
|\mathcal{M}(G)| \leq |G/K \otimes K|p^{\frac{k}{2}(n-k)(n-k-1)} \leq p^{n-k-1-p^{\frac{k}{2}(n-k)(n-k-1)}} \leq p^{\frac{k}{2}(n-1)(n-2)-2},
\]

which is a contradiction. In case \( |G'| = p^2 \), the result obtained similarly. □

Lemma \( [11] \) indicates when \( G \) has the property \( s(G) = 2 \), then \( |G'| \leq p^2 \). First we suppose that \( |G'| = p \).

**Theorem 4.** Let \( G \) be a \( p \)-group with centre of order at most \( p^2 \) and \( G^{ab} \) be elementary abelian of order \( p^{n-1} \) and \( s(G) = 2 \). Then \( G \cong E(2), E_2 \times \mathbb{Z}_p, Q_8 \) or \( H \), where \( H \) is an extra special \( p \)-group of order \( p^{2m+1} \) \((m \geq 2)\).

**Proof.** First assume that \( |Z(G)| = p \). Hence \( G \cong Q_8 \) or \( G \cong H \), where \( H \) is an extra special \( p \)-group of order \( p^{2m+1} \) \((m \geq 2)\) by a result of \( [3] \) Theorem 3.3.6. Now, assume that \( |Z(G)| \geq p^2 \). Lemma \( [3] \) and assumption show that \( Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \) or \( \mathbb{Z}_{p^2} \).

In case for which \( Z(G) \) is of exponent \( p \), \( [9] \) Lemma 2.1] follows that \( G \cong H \times \mathbb{Z}_p \). It is easily checked that \( H \cong E_2 \) by using \( [8] \) Theorems 2.2.10 and 3.3.6.

In case \( Z(G) \) is of exponent \( p^2 \), since \( \Phi(G) = G' \), \( [4] \) Theorem 3.1] shows that

\[
p^{\frac{k}{2}(n-1)(n-2)} = |\mathcal{M}(G/\Phi(G))| \leq p \cdot |\mathcal{M}(G)|,
\]

and hence \( p^{\frac{k}{2}(n-1)(n-2)-1} \leq |\mathcal{M}(G)| \). On the other hand, Main Theorems of \( [9] \) and \( [2] \) imply that \( |\mathcal{M}(G)| = p^{\frac{k}{2}(n-1)(n-2)-1} \) since \( Z(G) \) is cyclic of order \( p^2 \). Moreover, \( G \cong E(2) \) by appealing \( [3] \) Lemma 2.1, as required. □

**Theorem 5.** Let \( G \) be a \( p \)-group, \( G^{ab} \) be elementary abelian, \( |G'| = p \) and \( |Z(G)| \geq p^3 \) be of exponent \( p^2 \). Then \( G = H \cdot Z(G) \) and \( H \cap Z(G) = G' \) by virtue of \( [9] \) Lemma 2.1.

Now, for the sake of clarity, we consider two cases.

Case 1. First assume that \( G' \) lies in a central subgroup \( K \) of exponent \( p^2 \). Therefore, one can check that there exists a central subgroup \( T \) such that \( G = H \cdot K \times T \cong \).
Thus, when $T$ is an elementary abelian $p$-group by using [8] Theorem 2.2.10 and Theorem [3] we have
\[ |\mathcal{M}(G)| = |\mathcal{M}(E(2))||\mathcal{M}(T)||E(2)^{ab} \otimes T| \]
\[ = 2m^2 + m - 1 + \frac{1}{p}(n - 2m - 2)(n - 2m - 3) + 2m + 1(n - 2m - 2) \]
\[ = \frac{1}{p}(n - 1)(n - 2) - 1. \]

In the case $T$ is not elementary abelian, a similar method and [9] Lemma 2.2 asserts that
\[ |\mathcal{M}(G)| \leq p^{\frac{1}{2}(n^2 - 5n + 4)} \leq p^{\frac{1}{2}(n - 1)(n - 2) - 2}. \]

Case 2. $G'$ has a complement $T$ in $Z(G)$, and hence $G = H \times T$ where $T$ is not elementary abelian, and so by invoking [9] Lemma 2.2 and [8] Theorems 2.2.10 and 3.3.6, $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n - 1)(n - 2) - 2}$. □

**Theorem 6.** Let $G$ be a $p$-group of order $p^n$, $G^{ab}$ be elementary abelian of order $p^{n-1}$ and $Z(G)$ be of exponent $p$. Then $G$ has the property $s(G) = 2$ if and only if it is isomorphic to one of the following groups.
\[ Q_8 \times \mathbb{Z}_2^{(n-3)}, E_2 \times \mathbb{Z}_p^{(n-3)} \text{ or } H \times \mathbb{Z}_p^{(n-2m-1)}, \]
where $H$ is extra special of order $p^{2m+1}$. $m \geq 2$.

**Proof.** It is obtained via Theorem [3][8] Theorems 2.2.10 and 3.3.6 and assumption. □

**Lemma 7.** Let $G$ be a $p$-groups of order $p^n$ and $|G'| = p$. Then $G$ has the property $s(G) = 2$ if and only if $G$ is isomorphic to the one of the following groups.
\[
(1) \quad Q_8 \times \mathbb{Z}_2,
(2) \quad \langle a, b | a^4 = 1, b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2b^2 \rangle
(3) \quad E_2 \cong E_2(2),
(4) \quad E_2 \otimes \mathbb{Z}_2,
(6) \quad \langle a, b | a^{p^2} = 1, b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle,
\]

**Proof.** It is obtained by using Theorems [1][2] and a result of [10] Lemma 3.5. □

The structure of all $p$-group of order $p^n$ is characterized with the property $s(G) = 2$ and $|G'| = p^2$. Now, we may suppose that $|G'| = p^2$.

**Lemma 8.** There is no $p$-group of order $p^n$ $(n \geq 5)$ with $s(G) = 2$, where $|G'| = p^2$ and $G' \not\subseteq Z(G)$.

**Proof.** First assume that $|Z(G)| = p^2$, since $Z(G)$ is elementary by Lemma [4] Let $K$ be a central subgroup of order $p$, such that $|(G/K)'| = p^2$. It is seen that
\[ |\mathcal{M}(G)| \leq |\mathcal{M}(G/K)||K \otimes G/(K \times G')| \leq |\mathcal{M}(G/K)| \leq p^{n-3} \]
by [11] Theorem 4.1. On the other hand, [9][12] Main Thorems imply that
\[ |\mathcal{M}(G/K)| \leq p^{\frac{1}{2}(n-2)(n-3)-1}, \]
and hence $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)-2}$.

In case $|Z(G)| = p$, there exists a central subgroup $K$ of order $p^2$ such that $G' \cap K = 1$. The rest of proof is obtained similar to the pervious case. When $|Z(G)| = p$, since $G$ is nilpotent of class 3, the result is deduced by [8] Proposition 3.1.11]. □
Theorem 9. Let $G$ be a $p$-group of order $p^n$ ($n \geq 5$) and $|G'| = p^2$ with $s(G) = 2$. Then

$$G \cong \mathbb{Z}_p \times (\mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p) \ (p \neq 2).$$

By the results of Lemmas 4 and 8 we may assume that $G' \subseteq Z(G)$ and $Z(G)$ is of exponent $p$. We consider three cases relative to $|Z(G)|$.

Case1. Assume that $|Z(G)| = p^4$, there exists a central subgroup $K$ of order $p^2$ such that $K \cap G' = 1$. [9] Main Theorem implies that $|\mathcal{M}(G/K)| \leq p^{\frac{1}{2}(n-3)(n-4)}$, and so $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)-1}$ due to [7] Theorem 4.1.

Case2. In the case $|Z(G)| = p^2$, we have $G' = Z(G)$. Moreover [12] Main Theorem deduces that $n \geq 6$ and so there exists a central subgroup $K$ such that $G/K \cong H \times Z(G/K)$ where $H$ is a extra special $p$-groups of order $p^{2m+1}$ $m \geq 2$, thus

$$|\mathcal{M}(G)| \leq p^{n-3} |\mathcal{M}(G/K)| \leq p^{n-3} p^{\frac{1}{2}(n-1)(n-4)} \leq p^{\frac{1}{2}(n-1)(n-2)-2}.$$ 

Case3. Now, we may assume that $|Z(G)| = p^3$. Let $K$ be a complement of $G'$ in $Z(G)$, so [11] Main Theorem asserts that $|\mathcal{M}(G/K)| \leq p^{\frac{1}{2}(n-2)(n-3)}$. On the other hand, [7] Theorem 4.1 and assumption imply that

$$p^{\frac{1}{2}(n-1)(n-2)-1} = |\mathcal{M}(G)| \leq |\mathcal{M}(G/K)||K \otimes G/Z(G)| \leq |\mathcal{M}(G/K)|p^{n-3},$$

so we should have $|\mathcal{M}(G/K)| = p^{\frac{1}{2}(n-2)(n-3)}$ and $G/Z(G)$ is elementary abelian. Now, since $|\mathcal{M}(G/K)| = p^{\frac{1}{2}(n-2)(n-3)}$ and $|Z(G')| = p^2$, by using [12] Main Theorem], $G/K \cong \mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p (p \neq 2)$. Moreover, [3] Proposition 1 and assumption show that $G^{ab}$ is elementary abelian. Hence, it is readily shown that

$$G \cong \mathbb{Z}_p \times (\mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p) \ (p \neq 2).$$

Theorem 10. Let $G$ be a group of order $p^4$ with $s(G) = 2$ and $|G'| = p^2$. Then $G$ is isomorphic to the one of the following groups.

1. \(a, b \mid a^4 = b^2 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1\),
2. \(a, b \mid a^p = 1, b^p = 1, [a, b, a] = [a, b, b] = [a, b, b, b] = 1\) ($p \neq 3$).

Proof. The structure of these groups has been characterized in [10] Lemma 3.6. □

We summarize all results as follows

Theorem 11. Let $G$ be a group of order $p^n$. Then $G$ has a property $s(G) = 2$ if and only if isomorphic to the one of the following groups.

1. $E(2) \times \mathbb{Z}_p^{(n-2m-2)}$,
2. $E_2 \times \mathbb{Z}_p^{(n-3)}$,
3. $Q_8 \times \mathbb{Z}_p^{(n-3)}$,
4. $H \times \mathbb{Z}_p^{(n-2m-1)}$, where $H$ is an extra special $p$-group of order $p^{2m+1}$ ($m \geq 2$),
5. \(a, b \mid a^4 = 1, b^4 = 1, [a, b, a] = 1, [a, b] = a^2 b^2\)
6. \(a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\).
7. \(a, b \mid a^{p^2} = 1, b^p = 1, [a, b, a] = [a, b, b] = 1\),
8. $\mathbb{Z}_p \times (\mathbb{Z}_p^{(4)} \rtimes_\theta \mathbb{Z}_p) \ (p \neq 2)$,
9. \(a, b \mid a^4 = b^2 = 1, [a, b, a] = 1, [a, b] = a^6, [a, b, b, b] = 1\),
10. \(a, b \mid a^p = 1, b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1\) ($p \neq 3$).
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