Imperfection sensitivity of the post-buckling characteristics of functionally gradient plates using higher-order shear and normal deformation theory

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Abstract: The aim of the present paper is to investigate the post-buckling responses of geometrically imperfect gradient plate using hybrid deformation plate theory (HHDT). This theory is accountable for realistic transverse shear distribution along with thickness stretching effect. The geometric imperfection is incorporated using various imperfection functions in the transverse direction only. Parametric studies have been carried out to present new results using finite element method with C⁰ continuous element. The consequences of geometric nonlinearity, various geometric imperfection, and geometric configuration on the Post-buckling characteristics of FGM plate is investigated.

1. Introduction: Functionally gradient materials (FGMs) are microscopically heterogeneous unconventional composite material having mechanical properties vary gradually along the preferred direction [1]. Its suitability in structural application has been the focus of the intense investigation since last two decades. Several reports dealing with the structural response of gradient structure are available in the literature. Praveen and Reddy [2] explored the large amplitude vibration and bending response of gradient plates under thermal and transverse mechanical loads using FSDT. Woo et al. [3] used mixed Fourier series to find the post-buckling characteristics of gradient structure under thermo-mechanical loading. Talha and Singh [4,5] used modified HSDT to examine the dynamic characteristics of gradient plate with random material properties. Gupta et al. [6,7] extended their work to find the effect of uncustomary boundary constraints on the free vibration response of gradient plate.

In the current study, the Post-buckling characteristics of geometrically imperfect gradient plate is investigated using recently proposed HHSNDT by the authors’ [8,9]. Finite element formulation is done using C⁰ continuous nine-noded elements with seventy-two degrees of freedom per element. The convergence and validation study of the present solution have been effectuated to confirm the efficacy of the present solution. The effects of various geometric imperfection, amplitude ratio, geometric configuration and volume fraction exponent on the Post-buckling response of gradient plate have been examined in detail.

2. Displacement field

The structural kinematics used in the present paper is given as [8–10],
The field variables can be given as

\[
\begin{align*}
U_x &= u_{xo} - z \left( \phi_x + \frac{3.4M}{l} \delta_x \right) + M \sinh^{-1} \left( \frac{3.4z}{l} \right) \delta_x \\
U_y &= u_{yo} - z \left( \phi_y + \frac{3.4M}{l} \delta_y \right) + M \sinh^{-1} \left( \frac{3.4z}{l} \right) \delta_y \\
U_z &= u_{zo} + 3.4 \cosh \left( \frac{3.4z}{l} \right) \delta_z
\end{align*}
\]

(1)

The field variables can be given as \( \{ \mathbf{D} \} = \{ u_{xo}, u_{yo}, u_{zo}, \phi_x, \phi_y, \delta_x, \delta_y, \delta_z \}^T \).

Where, \( U_x, U_y \), and \( U_z \) denotes the displacements of a point along the \((x, y, z)\) coordinates. The detailed description of the presently used theory along with the finite element formulation can be found in Gupta and Talha [8].

**2.1 Strain-displacement relations**

In the present formulation, the von-Karman sense of nonlinearity has been incorporated for large deflections and small strains. The effective strain components related with the displacement field in Eq. (1) along with geometric nonlinearity and initial geometric imperfection is represented as [11],

\[
\{ \mathbf{\varepsilon} \} = \{ \mathbf{\varepsilon}_l \} + \{ \mathbf{\varepsilon}_{nl} \}
\]

(2)

Where, \( \{ \mathbf{\varepsilon}_l \}_{16 \times 1} \) and \( \{ \mathbf{\varepsilon}_{nl} \}_{9 \times 1} \) are the vectors consisting of generalized linear and nonlinear strains components respectively. It is noteworthy that only vertical geometric imperfection is considered for the analysis of imperfect plate. Three modes of geometric imperfection are considered in the present study as shown in Fig 1 and Table 1.

**Table 1 Various modes of geometric imperfection**

| Geometric Imperfection ‘z’ | Case-1 | Case-2 | Case-3 |
|-----------------------------|--------|--------|--------|
| \( z \) \sin \left( \frac{X_1 \pi}{l_1} \right) \sin \left( \frac{X_2 \pi}{l_2} \right) \) | \( z \) \cos \left( \frac{X_1 \pi}{l_1} \right) \cos \left( \frac{X_2 \pi}{l_2} \right) \) | \( z \) \sin \left( \frac{X_1 \pi}{l_1} \right) \cos \left( \frac{X_2 \pi}{l_2} \right) \) |

Where, ‘\( z \)’ is termed as the imperfection size whereas, \( t, l_1 \) and \( l_2 \) are the thickness, width and length of the plate.
The effective strain components developed along with geometric nonlinearity and geometric imperfection is represented as,

\[ \{\mathbf{e}_i\} = \{\mathbf{e}_l\} + \{\mathbf{e}_{nl}\} \]

\[ \{\mathbf{e}_l\}_{6\times1} = [\Gamma]_{6\times6} \{\mathbf{e}_l\}_{16\times1} \]

\[ \{\mathbf{e}_{nl}\}_{6\times1} = [\Gamma_{nl}]_{6\times9} \{\mathbf{e}_{nl}\}_{9\times1} \]  \hspace{1cm} (3)

Where, \( \{\mathbf{e}_l\}_{16\times1} \) and \( \{\mathbf{e}_{nl}\}_{9\times1} \) are the vectors consisting of generalized linear and nonlinear strains components respectively. The various non-zero strain terms are given below,

\[ \{\mathbf{e}_l\} = \left\{ \frac{\partial U_x}{\partial x}, \frac{\partial U_y}{\partial y}, \frac{\partial U_z}{\partial z}, \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z}, \frac{\partial U_z}{\partial z} + \frac{\partial U_x}{\partial x}, \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} \right\} \]

\[ \{\mathbf{e}_{nl}\} = \left\{ \frac{1}{2} \left( \frac{\partial \nu_x}{\partial x} \right)^2 + \frac{\partial \nu_y}{\partial x} \frac{\partial \nu_z}{\partial x}, \frac{1}{2} \left( \frac{\partial \nu_y}{\partial y} \right)^2 + \frac{\partial \nu_z}{\partial y} \frac{\partial \nu_x}{\partial y}, 0, 0, 0, \frac{\partial \nu_z}{\partial y} \frac{\partial \nu_x}{\partial x} \right\} \]  \hspace{1cm} (4)

\[ \{\mathbf{e}_l\} = \left\{ \varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{yz}^0, \gamma_{xz}^0, \gamma_{xy}^0, \gamma_{xx}, \gamma_{yy}, \gamma_{yy}^1, \gamma_{xy}^1, \gamma_{yy}^2, \gamma_{xy}^2, \varepsilon_{xx}^2, \varepsilon_{yy}^2, \gamma_{xx}, \gamma_{yy}, \gamma_{yy}^1, \gamma_{xy}^1, \gamma_{yy}^2, \gamma_{xy}^2, \varepsilon_{xx}^2, \varepsilon_{yy}^2, \gamma_{xx}, \gamma_{yy}, \gamma_{yy}^1, \gamma_{xy}^1, \gamma_{yy}^2, \gamma_{xy}^2 \right\} \]

\[ \{\mathbf{e}_{nl}\} = \left\{ \varepsilon_{xx}^{nl/0}, \varepsilon_{yy}^{nl/0}, \gamma_{xy}^{nl/0}, \gamma_{xx}^{nl/1}, \gamma_{yy}^{nl/1}, \varepsilon_{xx}^{nl/2}, \varepsilon_{yy}^{nl/2}, \gamma_{xy}^{nl/2}, \gamma_{xx}^{nl/2}, \gamma_{yy}^{nl/2}, \gamma_{xy}^{nl/2} \right\} \]

\[ \varepsilon_{xx} = \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 - C(z)\varepsilon_{xx}^2, \]

\[ \varepsilon_{yy} = \varepsilon_{yy}^0 + z\varepsilon_{yy}^1 - C(z)\varepsilon_{yy}^2, \]

\[ \varepsilon_{zz} = \varepsilon_{zz}^0, \]

\[ \gamma_{xy} = \gamma_{xy}^0 + z\gamma_{xy}^1 - C(z)\gamma_{xy}^2 \]
\[ \gamma_{xx} = \gamma_{xx}^0 + M(3.4/t) \left( \frac{1}{\sqrt{1-(3.4/z)^2}} \right)^2 \gamma_{xx}^1 + N(z) \beta^2 \gamma_{xx}^2, \]

\[ \gamma_{xz} = \gamma_{xz}^0 + M(3.4/t) \left( \frac{1}{\sqrt{1-(3.4/z)^2}} \right)^2 \gamma_{xz}^1 + N(z) \beta^2 \gamma_{xz}^2, \]

\[ \varepsilon_{xx}^{nl} = \varepsilon_{xx}^{nl0} + N(z) \left( \varepsilon_{xx}^{nl1} + N(z) \varepsilon_{xx}^{nl2} \right), \]

\[ \varepsilon_{yy}^{nl} = \varepsilon_{yy}^{nl0} + N(z) \left( \varepsilon_{yy}^{nl1} + N(z) \varepsilon_{yy}^{nl2} \right), \]

\[ \gamma_{xx}^{nl} = \gamma_{xx}^{nl0} + N(z) \left( \gamma_{xx}^{nl1} + N(z) \gamma_{xx}^{nl2} \right), \]

\[ C(z) = M \left( \sinh^{-1} \left( \frac{3.4}{t} \right) \right) \left( \frac{3.4}{t} \right)^2, \quad n(z) = 3.4 \cosh^2 \left( \frac{3.4}{t} \right) \]

3. Finite element formulation and energy equations

3.1 Element selection

In the elemental formulation, a C\(^0\) nine-noded element is used to discretize the plate geometry.

The displacement vector and element in terms of shape functions is given as

\[ \{ \mathbf{D} \} = \sum_{j=1}^{n_{nod}} \mathbf{N}_j \{ \mathbf{D}_j \} \]

(5)

Where, \{ \mathbf{D}_j \} and \{ \mathbf{N}_j \} are the displacement vector and shape function of \( i^{th} \) node respectively.

3.2 Expression of Strain energy

The strain energy of \( i^{th} \) element of gradient plate is given by:

\[ S.E^i = \frac{1}{2} \int_{V} \{ \mathbf{\varepsilon}^{i} \}^T \{ \mathbf{\sigma} \}_{ijkl} \{ \mathbf{\varepsilon}^{i} \} dV = \frac{1}{2} \int_{V} \{ \mathbf{\varepsilon}^{i} \}^T \{ \mathbf{Q}_q \}_{ijkl} \{ \mathbf{\varepsilon}^{i} \} dV \]

By substituting the stress and strain from Eq. (3), the expression of strain energy is written as

\[ S.E^i = \frac{1}{2} \int_{V} \{ \mathbf{\varepsilon}^{i} + \mathbf{\varepsilon}^{nl} \}^T \{ \mathbf{\varepsilon}^{i} + \mathbf{\varepsilon}^{nl} \} dV \]

(6)

Expanded form of strain energy expression is given as

\[ S.E^i = \frac{1}{2} \int_{A} \left[ \frac{1}{2} \left( \{ \mathbf{\varepsilon}^{i} \}^T \{ \mathbf{\varepsilon}^{i} \} \right) + \frac{1}{2} \left( \{ \mathbf{\varepsilon}^{nl} \}^T \{ \mathbf{\varepsilon}^{nl} \} \right) \right] dA \]

Where,

\[ \{ \varepsilon_{1} \} = \int_{\Gamma_{1/2}} \left[ \mathbf{Q}_q \right]^T \left[ \mathbf{\Gamma}_{1/2} \right] d\Gamma, \quad \{ \varepsilon_{2} \} = \int_{\Gamma_{1/2}} \left[ \mathbf{Q}_q \right]^T \left[ \mathbf{\Gamma}_{1/2} \right] d\Gamma \]

\[ \{ \varepsilon_{3} \} = \int_{\Gamma_{1/2}} \left[ \mathbf{Q}_q \right]^T \left[ \mathbf{\Gamma}_{1/2} \right] d\Gamma, \quad \{ \varepsilon_{4} \} = \int_{\Gamma_{1/2}} \left[ \mathbf{Q}_q \right]^T \left[ \mathbf{\Gamma}_{1/2} \right] d\Gamma \]

(7)
The matrix \( [\mathcal{N}_t] \) and \( [\mathcal{N}_n] \) are the function of thickness coordinate and the shape function parameter.

### 3.3. Expression of Kinetic energy of the plate

The kinetic energy of plate is written as:

\[
K.E = \frac{1}{2} \int \rho_{\text{eff}} \{\dot{R}\}^T \{\dot{R}\} \, dV
\]  
(8)

Where \( \{\dot{R}\} \) and \( \rho_{\text{eff}} \) are global displacement vector and effective density of the plate. The global displacement field is written as:

\[
\{R\} = [\mathcal{N}] \{\mathcal{D}\}
\]  
(9)

Where, \([\mathcal{N}]\) is thickness coordinate function. The elemental kinetic energy is given as:

\[
K.E^e = \frac{1}{2} \int_A \{\mathcal{B}\}^T [\mathcal{I}] \{\mathcal{D}\} \, dA
\]  
(10)

Kinetic energy of vibrating plate for total number of element \('ne'\) may be given as:

\[
K.E = \sum_{e=1}^{ne} K.E^e
\]  
(11)

### 3.4 Expression of Work done due to in-plane load

The work done on the plate by applied load \( Q \) is as follows

\[
W.D_{\text{ext}}^e = Q(x, y) \{w\} \, dA
\]

For the discretised domain, the above equation is given as,

\[
W.D_{\text{ext}} = \sum_{e=1}^{ne} W_{\text{ext}}^e
\]

The potential energy due to the imposed load is given as,

\[
P.E = -W.D_{\text{ext}}^e = -\int_A \{\dot{R}\}^T \{Q\} \, dA = -\{\dot{R}\}^{(e)T} \{Q\}^{(e)}
\]  
(12)

### 3.5 Governing Equation

The required governing equations for stability analysis is given as:

\[
\lambda_{cr} \left[ K_e \right] + \left[ K \right] \{R\} = 0
\]  
(13)
4. Result and discussion:

4.1 Convergence and validation study

The effectiveness and efficiency of the present methodology is ascertained through convergence and validation. As shown in Fig. 2a, the buckling load parameter of gradient plate made of SUS303/Si₃N₄ is plotted against the different mesh size (2x2-6x6) for various volume fraction exponents. It is found from the convergence study, that present solution shows good convergence rate. Furthermore, a (5 x 5) mesh size has been used for numerical computation in the present study. In Fig. 2b, the comparison of the buckling load parameter of gradient plate calculated from the present approach with the results reported by Thai and Kim [12] is shown. They have used TSDT with the conjunction of Navier solution to compute the buckling load of gradient plate. The material properties are $E_m = 70$ GPa, $\rho_m = 2702$ kg/m$^3$ for aluminium (Al), and $E_c = 380$ GPa, $\rho_c = 3800$ kg/m$^3$ for Alumina (Al$_2$O$_3$). The comparison of the results has been shown in Fig 2b, which are in good agreement.

![Figure 2. Convergence and comparison study of present solution](image)

4.2 New Results

The influence of geometric imperfection, volume fraction exponent ($n$) and amplitude ratio on the post-buckling response of gradient plate has been examined in this section. The material properties are $E_m = 105.70$ GPa, $\rho_m = 4429$ kg/m$^3$ for Ti-6Al-4V, and $E_c = 200$ GPa, $\rho_c = 5700$ kg/m$^3$ for ZrO$_2$. The results are obtained under uniaxial compression of the plate. The various modes of geometric imperfections are shown in Fig. 1, whereas the concerned mathematical expressions are given in Table 1.

Fig. 3a demonstrates the change of $\lambda' / \lambda_{cr}$ with ‘$n$’ for various types of geometric imperfection. The $l/t$ is considered as 100, whereas the imperfection size ($\zeta$) and amplitude ratio ($P_0$) is taken as 0.2 and 0.1 respectively. It can be observed that at the lower value of volume fraction exponent ($n<6$), the difference in the $\lambda' / \lambda_{cr}$ between the perfect plate and imperfect plate (Case-1 and Case-2) is not significant. But in the case of Case-3, the $\lambda' / \lambda_{cr}$ has increased considerably in comparison to the perfect plate. This variation in the load ratio is also due to the fact that, in geometric nonlinearity, the strain-displacement relation is also nonlinear.

In Figs 3b-e, the variation of buckling load ratio is plotted against ‘$n$’ with amplitude ratio as 0.3, 0.5, 0.8 and 1.0 respectively. It can be noticed that as the amplitude ratio increases, the influence
of various geometric imperfection becomes irregular. For example, in Fig. 3c, the buckling load ratio increased due to the presence of Case-3 geometric imperfection when \( P_0 \) is 0.5, on the other hand in Figs 3d-e, the load ratio decreased due to Case-3 geometric imperfection when \( P_0 \) increased to 0.8 and 1.0. Therefore, it can be concluded that in the presence of geometric imperfection, the Post-buckling response reflects highly nonlinear behaviour at the higher value of amplitude ratio.

Fig. 3f shows the variation of buckling load ratio of FGM plate with \( l/t \) ratio when the plate is under biaxial compression. It is observed that no considerable change occurs in the buckling load when the plate is subjected to Case-1 geometric imperfection for \( a/h=10 \) (Moderately thick). But at \( l/t=20 \), a notable change in the load ratio is observed. Henceforth its value is kept on decreases as the \( l/t \) ratio increase. Therefore, it can be concluded that influence of geometric imperfection on the post-buckling response is more in the thin plate in comparison to the thick plate.
5. Conclusions: In the present article, the Post-buckling response of geometric imperfect gradient plates has been presented in this article. The mathematical formulation was based on HHDT developed by the authors’. Geometric nonlinearity is employed using Von-Karman assumption. Convergence and validation studies have been done to authenticate the efficacy of the present theory. The influence of geometric imperfection, amplitude ratio, and geometric configuration and volume fraction exponent on the Post-buckling characteristics is investigated. It is concluded that the geometric imperfection and amplitude ratio have a considerable influence on the Post-buckling behaviour of the gradient plates.

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