The maximum sizes of large scale structures in alternative theories of gravity

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Abstract. The maximum size of a cosmic structure is given by the maximum turnaround radius — the scale where the attraction due to its mass is balanced by the repulsion due to dark energy. We derive generic formulae for the estimation of the maximum turnaround radius in any theory of gravity obeying the Einstein equivalence principle, in two situations: on a spherically symmetric spacetime and on a perturbed Friedman-Robertson-Walker spacetime. We show that the two formulae agree. As an application of our formula, we calculate the maximum turnaround radius in the case of the Brans-Dicke theory of gravity. We find that for this theory, such maximum sizes always lie above the ΛCDM value, by a factor $1 + \frac{1}{3\omega}$, where $\omega \gg 1$ is the Brans-Dicke parameter, implying consistency of the theory with current data.

Keywords: gravity, modified gravity

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1 Introduction

The Λ-cold dark matter (ΛCDM) model is widely considered to be the simplest and most successful theoretical description of our universe, and finds support from a wide range of cosmological observations. Despite its success, this model is unfortunately not without problems. While certain observational glitches have been reported from time to time \[^1\]–\[^4\], the biggest challenge the ΛCDM model has to face is the cosmological constant problem.\[^1\]

The tiny value of the observed cosmological constant, $\Lambda \sim O(10^{-3} \text{eV})^4$, that is needed for the model to be observationally viable, finds no compelling explanation from a quantum field theoretical point of view. There had been numerous attempts to explain the value of $\Lambda$ by relating it to vacuum energy density of quantum fields, but all such attempts have either theoretical or observational inconsistencies \[^5\],\[^6\]. A related problem is that de Sitter space may also be unstable to quantum corrections \[^7\]–\[^10\].

These conceptual and observational problems with the cosmological constant $\Lambda$ have triggered in recent years vigorous research in alternatives to the ΛCDM model. The chief agenda of these alternative models is to generate the effect of the dark energy through additional matter fields (for instance quintessence \[^11\]), or, by replacing the theory of gravity on which ΛCDM rests, i.e. General Relativity, by a different theory \[^4\],\[^12\] (see also \[^13\] for a recent critique of the current status of cosmology). In order to discriminate between such alternative theories of gravity and GR, it is necessary to test all their possible observable consequences with cosmological observations. The next generation of cosmological surveys will offer a huge boost in precision making such tests possible \[^14\]–\[^18\].

In this paper, we are particularly interested in one possible test of ΛCDM and alternative theories of gravity, namely, the stability of the large scale cosmic structures \[^19\].\[^2\] The

\[^1\] The cosmological constant problem is not manifest only for ΛCDM but for any other theory with dynamical dark energy or alternative to GR.

\[^2\] See also \[^20\] for a different approach.
maximum size of a large scale cosmic structure with a given mass $M$ can be estimated using the maximum turnaround radius (or simply the turnaround radius $R_{TA}$ for short). More precisely, $R_{TA}$ is the point where for radially moving test particles the attraction due to normal matter is balanced by the repulsion due to the dark energy. Specific theories are expected to lead to estimates for the turnaround radius, which depend on the theory parameters. If a certain theory predicts a maximum possible size smaller than the actual observed size of structures of mass $M$, the latter are expected to be unstable in the framework of that theory. Thus, parameter ranges resulting to maximum possible sizes smaller than what we observe are ruled out.

The turnaround radius was calculated in [21, 22] in the wider context of geodesics of the Schwarzschild-de Sitter spacetime. In a cosmological context, the turnaround radius for spherical structures was calculated for ΛCDM in [19], and in [23] for smooth dark energy. The turnaround radius as a cosmological observable was investigated in [24, 25]. In [26] it was proposed to look for the violation of the maximum upper bound of $R_{TA}$ using the zero velocity surfaces of a large scale structure, by observing the peculiar velocity profiles of its members. It turns out that for structures as massive as $10^{15} M_{\odot}$ (e.g. the Virgo supercluster), the actual sizes lie very close and below the theoretical prediction of ΛCDM [19]. The structures studied in the references above are at sufficiently low redshifts ($z \sim 10^{-2}$), and hence $R_{TA}$ measurements could provide a local indication and check for dark energy. In other words, it does not require any data coming from the high redshift Supernovae or from the early universe.

Measuring the turnaround radius offers yet another way of putting constraints on alternative gravity models. For instance, the maximum turnaround radius has recently been calculated for a cubic galileon model [27]. A method to calculate the turnaround radius in generic gravitational theories was put forward in [28, 29] by considering timelike geodesics, in the framework of alternative gravity theories admitting McVittie-like [30–32] solutions. We agree with the general formula for the turnaround, eq. (21), of [28] but disagree in other results of that article which appear to be in conflict with ours.4

This article is organized as follows. In section 2 we derive a general formula for the calculation of the turnaround radius, valid in any metric theory of gravity obeying the Einstein Equivalence Principle (EEP), a necessary assumption as the geodesic equation is used in our derivation. We perform our derivation in steps: (i) we first calculate the maximum turnaround radius in the case of the ΛCDM model using static coordinates, (ii) we extend the static metric calculation to arbitrary theories (arbitrary static metrics), (iii) we re-calculate the turnaround radius using the McVittie metric and finally (iv) we relate the two types of calculation (static and McVittie) using cosmological perturbation theory in a general theory of gravity. Our result is (2.21). Our derivation makes it clear why the standard formula for the turnaround radius (which in fact agrees with eq. (21) of [28]) is valid for any theory of gravity obeying the EEP, once the solution for the potential $\Psi$ is known (see eq. (2.15)).

3The turnaround radius was called the “static radius” in [21, 22].

4In [28] it is assumed that the potentials $\Phi$ and $\Psi$ have solutions $\sim \frac{Gm}{r}$ under the assumption of spherical symmetry (even in ΛCDM), where the constant $m$ is the mass of the source. This however is incorrect as the correct solution (as can be verified by inspecting the McVittie solution) is $\sim \frac{Gm}{ar}$. Indeed, eq. (29) in [28] gives a time-dependent turnaround radius, which is in disagreement with the known result for ΛCDM. Our second source of disagreement is the recasting of the turnaround radius in terms of the areal radius. While we agree with the reasoning and with the relation between the comoving and areal radius, the contribution to the turnaround formula is of higher order in perturbation theory and should be neglected unless higher order in perturbation theory solutions are also used.
and (2.23) below). In section 3 we consider a specific theory, the Brans-Dicke theory of gravity, as an example to demonstrate the use of our formula. Within the Brans-Dicke theory, we perform the calculation of the turnaround radius in two coordinate systems, arriving (as expected) at the same result. We firstly determine the solution to the field equations around a static spherically mass distribution and secondly around a spherical solution in an expanding universe and show that the two solutions are equivalent, related by a coordinate transformation. Our calculation yields $R_{\text{TA}}^{(BD)} \approx R_{\text{TA}}^{(\Lambda CDM)} \left(1 + \frac{1}{3\omega}\right)$ for large Brans-Dicke parameter $\omega$ and hence it is always larger than $\Lambda CDM$, implying that the Brans-Dicke theory is also consistent with current data. We conclude finally in section 4.

Throughout this article we work with mostly positive signature of the metric, $(-,+,+,+)$ and use the greek alphabet for spacetime indices and latin alphabet for spatial indices. We use units where the speed of light is equal to unity.

2 The turnaround radius

2.1 The turnaround radius in GR with a cosmological constant

Let us first briefly present the case of GR with a cosmological constant, where the derivation of the turnaround radius is well known. This will be useful further below when we generalize the result to arbitrary metric theories of gravity.

In [23] the turnaround radius for a spherical mass $M$ in the $\Lambda CDM$ model was defined in the following way. Consider a stationary probe in a Schwarzschild-de Sitter (SdS) spacetime with metric

$$ds_{\text{SdS}}^2 = -\left(1 - \frac{2G_NM}{R} - \frac{\Lambda}{3} R^2\right) dT^2 + \frac{dR^2}{1 - \frac{2G_NM}{R} - \frac{\Lambda}{3} R^2} + R^2 d\Omega$$

following a trajectory in spacetime with four-velocity

$$u^\mu = \left(\sqrt{1 - \frac{2G_NM}{R} - \frac{\Lambda}{3} R^2}, 0, 0, 0\right).$$

Here $G_N$ is the measured Newtonian gravitational constant. The maximum turnaround radius is the point along a radial trajectory where the four-acceleration $a^\nu = u^\mu \nabla_\mu u^\nu$ of the probe vanishes. Using the SdS metric (2.1) yields $a^1 = \left(1 - \frac{2G_NM}{R} - \frac{\Lambda}{3} R^2\right) \left(\frac{G_NM}{R^2} - \frac{\Lambda}{3} R\right)$ and setting it to zero gives the turnaround radius,

$$R_{\text{TA}} = \left(\frac{3G_NM}{\Lambda}\right)^{1/3}$$

in the case of GR with a cosmological constant.

In a different theory of gravity, the SdS metric (2.1) need not be a solution. However, assuming that a static solution exists of the form

$$ds^2 = -f(R) dT^2 + h(R) dR^2 + R^2 d\Omega$$

one can follow the same line of thought to define the turnaround radius by the vanishing of the four-acceleration for a stationary probe. This leads to the condition

$$f'(R) \equiv \frac{\partial f}{\partial R} \quad \text{at} \quad R = R_{\text{TA}}$$

(2.5)
supplying us with an algebraic equation for \( R \), which must be solved in order to obtain the maximum turnaround radius \( R_{TA} \). The definition (2.5) is valid in any theory of gravity which obeys the weak equivalence principle and can be used to calculate the turnaround radius once the solution \( f(R) \) is known. Let us also note that even if the spacetime is spherically symmetric but not static, the metric may still be brought into a diagonal form, in which case the condition (2.5) still holds, although the resulting turnaround radius will in general be time dependent.

The above definition (2.5) of the turnaround radius is not formulated in a covariant language, but can be made so. In particular, the turnaround radius corresponds to the locus where \( u^\mu \nabla_\mu u^\nu = 0 \) for a stationary observer in a spherically symmetric spacetime. With this definition one can calculate the turnaround radius in any coordinate system of choice, although, the definition depends on this particular choice of observer.

Our goal is to find a definition of the turnaround radius, suited for cosmology, equivalent to the definition above. Consider the McVittie metric \[30\text{–}32\]

\[
ds_{\text{McV}}^2 = - \left( \frac{1 - \mu}{1 + \mu} \right)^2 dt^2 + (1 + \mu)^4 a^2 (dr^2 + r^2 d\Omega) \quad (2.6)
\]

where \( \mu = \frac{G N M}{2 a r} \), describing the exterior of a spherical mass in an expanding Universe evolving with scale factor \( a(t) \). The field equations are

\[
3H^2 = 8\pi G_N \rho \quad (2.7)
\]

\[
-2 \frac{1 + \mu}{1 - \mu} \dot{H} - 3H^2 = 8\pi G_N P \quad (2.8)
\]

where \( H(t) \equiv \dot{a}/a \) is the Hubble parameter, \( \rho = \rho(t) \) is the energy density and \( P = P(t, r) \) the (inhomogeneous) pressure.\(^5\) If \( 8\pi G_N \rho = \Lambda \) is a constant then this spacetime reduces to the Schwarzschild-de Sitter spacetime in a different coordinate system to (2.1). To see this (and remembering always that \( H \) is a constant in Schwarzschild-de Sitter) define new coordinates \( T(t, r) \) and \( R(t, r) \) via

\[
t = T - Q(R), \quad R = (1 + \mu)^2 a r \quad (2.9)
\]

with \( Q(R) \) the solution to

\[
\frac{\partial Q}{\partial R} = \frac{\sqrt{\frac{2}{3} R}}{1 - \frac{2G_N M}{R} - \frac{\Lambda}{3} R^2} \sqrt{1 - \frac{2G_N M}{R}} \quad (2.11)
\]

so that one recovers (2.1).

How does the turnaround condition look-like from the McVittie’s point of view? Since we already know the result in the case of the static Schwarzschild-de Sitter coordinate system, we can simply transform the conditions leading to that result, to the McVittie coordinate system. In particular, we need to transform the velocity vector field (2.2) of the stationary

\(^5\)Having a homogeneous density, yet, inhomogeneous pressure seems somewhat unnatural.
observer, into the new system\textsuperscript{6} and apply the condition \( u^\mu \nabla_\mu u^\nu = 0 \). For this we need the inverse transformation of \((2.10)\), i.e.
\[
r(t, R) = \frac{R - G_N M + \sqrt{R^2 - 2G_N M R}}{2a},
\]
where we have chosen the positive sign of the square root.\textsuperscript{7}

With the above transformation, the observer’s velocity \((2.2)\) becomes
\[
u^\mu = 1 + \mu \sqrt{(1 - \mu)^2 - H^2(1 + 6a^2r^2)} (1, -rH, 0, 0).
\]
Using the condition \( u^\mu \nabla_\mu u^\nu = 0 \) we find (remember \( H \) is constant)
\[
2\mu = (1 + \mu)^6 H^2 a^2 r^2
\]
which translates to \((2.3)\) using \((2.10)\).

2.2 New definition of the turnaround radius

We now present a new definition of the turnaround radius, valid in any theory of gravity obeying the EEP. In generic alternative theories of gravity that we deal with in this article, the Schwarzschild-de Sitter metric will in general not be a solution. Neither will some general static spherically symmetric metric have an equivalent form, which resembles the McVittie metric. However, our interest is in cosmology, where a perturbed FRW metric always exists. Let us then consider the perturbed version of the McVittie construction of the previous subsection.

In the Newtonian gauge, the perturbed FRW metric takes the form
\[
ds^2 = -(1 + 2\Psi) dt^2 + a^2 (1 - 2\Phi) \gamma_{ij} dx^i dx^j
\]
where \( \Psi \) and \( \Phi \) are the two metric potentials and where we have assumed that \( \gamma_{ij} \) is flat, so that \( \gamma_{ij} dx^i dx^j = dr^2 + r^2 d\Omega \) in spherical coordinates.

By inspection, when \( \mu \ll 1 \), the McVittie metric \((2.6)\) may be interpreted as a perturbation on FRW sourced by a point-mass by identifying \( \Psi = \Phi = -2\mu = -\frac{G_N M}{ar} \). We exploit this fact and re-cast the definition of the turnaround radius using cosmological perturbation theory. Starting from \((2.13)\), we rotate into an arbitrary spatial direction, using \( r^i = (x, y, z) = \frac{1}{2} \vec{\nabla}_i r^2 \), where \( \vec{\nabla}_i = \gamma^{ij} \nabla_j \). The 3-vector \( r^i \) has components \((-rH, 0, 0)\) in the original coordinate system used in \((2.13)\). We also use the Friedman equation, \( \Lambda = 3H^2 \), so that the equivalent version of \((2.13)\) albeit in an arbitrary direction is
\[
u^\mu = \frac{1 + \mu}{\sqrt{(1 - \mu)^2 - (Har)^2(1 + \mu)^6}} \left(1, \frac{1}{2} H \vec{\nabla}_i r^2 \right).
\]
This is the four-velocity of a test particle at rest in a coordinate system which is equivalent to \((2.15)\). Taking the limit \( \mu \ll 1 \) and \( aHr \ll 1 \), corresponding to regions far away from both horizons, leads to
\[
u^\mu = \left(1 - \Psi + Ha\Theta, -\frac{1}{a} \vec{\nabla}_i \Theta \right)
\]
\textsuperscript{6}It is easy to show that using a stationary observer in the McVittie coordinate system fails. Indeed a stationary observer in one coordinate system is no longer stationary in the other.

\textsuperscript{7}The negative sign also works, however, issues arise when one considers a perturbative analogue of the McVittie metric as we do further below.
where we have defined the scalar function
\[ \Theta = \frac{1}{2} a H r^2. \]  
We have assigned the perturbation orders \( \mathcal{O}(\Psi) \sim \mathcal{O}(H^2) \sim \mathcal{O}(\Theta^2) \sim \mathcal{O}(\dot{\Theta}) \), which are reminiscent of the Parametrized Post-Newtonian formalism. Indeed the vector field \( \vec{\nabla} \Theta \) has all the properties of a spatial curl-less velocity field.

We have managed to create a covariant definition of the turnaround radius, which is adapted to cosmology. In particular one starts from the observer moving with velocity given by (2.17) and impose the EEP. The EEP implies the geodesic equation
\[ u^\mu \nabla_\mu u^\nu = 0 \]  
which in turn leads to
\[ \nabla_i \left[ \dot{\Theta} - \frac{1}{a} \Psi + H \Theta \right] - \frac{1}{a} \nabla^j \Theta \nabla_j \nabla_i \Theta = 0. \]  
Since the last term can be written as \( \nabla^j \Theta \nabla_j \nabla_i \Theta = \frac{1}{2} \nabla_i |\nabla \Theta|^2 \), we finally get the general turnaround equation
\[ \nabla_i \left[ a \left( \dot{\Theta} + H \Theta \right) - \frac{1}{2} |\nabla \Theta|^2 \right] = \nabla_i \Psi. \]  
The above equation is valid in any theory of gravity obeying the EEP. Despite appearances the above equation is fully consistent in perturbation theory (remember the assignment of perturbation orders above). One should not treat (2.21) as a differential equation for \( \Theta \). Rather, one should assume a specific functional form for \( \Theta(x, t) \) and then given that functional form, as well as the solution for \( \Psi \) from the field equations of the theory, one should determine the 3-surface \( F(x^i) = \text{const} \). such that the equation holds. In the case of spherical symmetry \( \Theta \) is given by (2.18), however, (2.21) may be used as a starting point for generalizing the turnaround radius calculation into a turnaround surface when the shape of the bound object is non-spherical. One possibility would be to consider a non-spherical function \( \Theta(t, \vec{x}) \) corresponding to some non-spherical surface.

Let us now return to our spherically-symmetric ansatz, i.e. \( \Theta = \frac{1}{2} a H r^2 \). In this case we have that \( |\nabla \Theta|^2 = a^2 H^2 r^2 = 2 a H \Theta \), hence the l.h.s. of (2.21) leads to
\[ a \frac{\partial \dot{\Theta}}{\partial r} = a^2 [H^2 + \dot{H}] r \]  
and the turnaround equation simplifies to
\[ a^2 [H^2 + \dot{H}] r = \frac{\partial \Psi}{\partial r}. \]  

The above equation which we name the reduced turnaround equation (due to spherical symmetry) can then be used to calculate the turnaround radius \( R_{\text{TA}} = ar \) given a Hubble parameter \( H(t) \) and the solution to the potential \( \Psi \), both of which are specified in a given theory, including a theory beyond GR.

From (2.23) a quick calculation gives the turnaround radius for the case of a cosmological constant as dark energy and for the case of a dark energy fluid with equation of state parameter \( w \) both within the GR framework. In both models the solution to the potential
is $\Psi = -\frac{G_{N}M}{ar}$ [23]. What is different between the two models is the Hubble parameter. In the first case it is a constant given by $H = \sqrt{\frac{\Lambda}{3}}$ so that (2.23) leads to (2.3), while in the second case it is given by $\dot{a}H = H_0 a^{-(1+3w)/2}$ so that (since for dark energy $1 + 3w < 0$)

$$R_{TA} = \left[ -\frac{2G_{N}M}{(1+3w)H^2} \right]^{1/3} = \left[ -\frac{2G_{N}M}{(1+3w)H_0^2} \right]^{1/3} a^{1+w}.$$  

(2.24)

We observe that when $w \neq -1$ the maximum turnaround radius is time-dependent. In the limit $w \rightarrow -1$, i.e. $\Lambda$CDM, the maximum turnaround radius agrees with the time-independent $\Lambda$CDM formula (2.3).

3 The turnaround radius of Brans-Dicke theory of gravity

The Brans-Dicke theory [33] can be thought of as a prototype alternative theory of gravity. Its action in the presence of a cosmological constant is given by

$$S = \frac{1}{16\pi G} \int \sqrt{-g} d^4 x \left[\phi R - 2\Lambda - \frac{\omega}{\phi} (\nabla \phi)^2 \right] + S_M,$$

(3.1)

where the scalar $\phi$ is the Brans-Dicke field, the constant $\omega$ is the Brans-Dicke parameter and $S_M$ is the collective action for all matter fields present, which depends on the metric $g_{\mu\nu}$ but not on the scalar field. The shift of conceptual paradigm from GR in this theory is certainly the scalar field $\phi$, whose non-minimal coupling with the Ricci scalar indicates a spacetime dependent gravitational coupling. In the limit $\omega \rightarrow \infty$ the scalar field $\phi$ must be a constant $\phi \rightarrow \phi_0$ in which case GR is recovered.

Solar system data severely constrain $\omega \gtrsim 40000$ [34, 35], thereby making it practically indistinguishable from General Relativity in our local neighbourhood. However, any test of gravity should be accompanied by a specification of the curvature and potential regime it is performed in [36]. In this sense cosmological constraints on Brans-Dicke theory should be treated independently from solar system tests as they lie in different regions of the gravitational parameter space.

Let us exemplify. As shown in [37], the Brans-Dicke theory arises as a specific limit of Horndeski theory [38, 39], the most general Lorentz-invariant scalar-tensor theory, having second order field equations in four dimensions. The Horndeski theory offers the possibility of realizing screening mechanisms such as the Vainshtein [40], the chameleon [41] and the symmetron [42] mechanisms. These mechanisms restore GR around the high-curvature/high-density environments of astrophysical bodies, such as the sun. Hence, it is possible that certain subsets of Horndeski theory which realize these mechanisms tend to Brans-Dicke theory in the low curvature environment of the cosmological regime but acquire corrections which send it back to GR in regions of high curvature. As such, cosmological constraints on Brans-Dicke theory give different information than solar system tests. In [37] the lower bound $\omega > 890$ at the 99% confidence level was placed (see also [43, 44]), using the latest Cosmic Microwave Background data from Planck. Future photometric and spectroscopic cosmological surveys are expected to increase this by a factor of 20–30 [45, 46], making cosmological tests comparable to solar system tests.

In [47], the no hair theorems for the Brans-Dicke theory with $\Lambda > 0$ for stationary axisymmetric black holes and stars were discussed. It was shown there that no matter how large the Brans-Dicke parameter $\omega$ is, unless it is infinite (i.e., the theory coincides exactly...
with the General Relativity), there can exist no regular such solutions if asymptotic de Sitter boundary condition is imposed. The Brans-Dicke theory has also been investigated in the context of galactic dark matter in [48].

In order to pave the way for the calculation of the turnaround radius we construct solutions in Brans-Dicke theory with a cosmological constant. We consider two types of solutions, i.e. static spherically symmetric solutions and cosmological solutions, in order to apply both formulae (2.5) and (2.23) for the determination of the turnaround radius.

### 3.1 Stationary spherically symmetric point-mass solutions

Adopting a static spherically symmetric ansatz as in (2.4) and in addition that \( \phi = \phi(R) \), the field equations are

\[
\frac{1}{R} \left( \frac{h'}{h} + \frac{h - 1}{R} \right) - \frac{1}{2} \omega \left( \frac{\phi'}{\phi} \right)^2 + \frac{f'}{f} \frac{\phi'}{\phi} = \frac{8\pi G h}{\phi} \left[ \rho + \frac{-\rho + 3P}{2\omega + 3} \right] \tag{3.2a}
\]

\[
\frac{1 - h}{R^2} + \frac{1}{R} \frac{f''}{f} + \frac{2}{R} \frac{\phi'}{\phi} + \frac{f'}{f} \frac{\phi'}{\phi} - \frac{1}{2} \omega \left( \frac{\phi'}{\phi} \right)^2 = \frac{8\pi G P}{\phi} - \frac{1}{h} \tag{3.2b}
\]

\[
\frac{f''}{f} - \frac{(f')^2}{2f^2} + \left( \frac{f'}{2f} - \frac{h'}{2h} \right) \left( \frac{f'}{2f} + \frac{1}{R} \right) + \frac{1}{2} \omega \left( \frac{\phi'}{\phi} \right)^2 - \frac{1}{R} \frac{\phi'}{\phi} = \frac{8\pi G h}{\phi} \left[ \rho + \frac{-\rho + 3P}{2\omega + 3} \right] \tag{3.2c}
\]

and

\[
\left[ \sqrt{\frac{f}{h}} \frac{R^2 \phi'}{\sqrt{h}} \right]' = \frac{8\pi G (\rho - 3P)}{2\omega + 3} \sqrt{f h} R^2 \tag{3.2d}
\]

where \( \rho \) and \( P \) are the total density and pressure of matter respectively, including the cosmological constant. Consistency requires that the matter velocity has components

\[
u_{\mu} = \left( \frac{1}{\sqrt{f}}, 0, 0, 0 \right). \]

In the Einstein equations above, we have used the scalar equation (3.2d) to eliminate the \( \Box \phi \) terms.

A complete analytic solution of (3.2a) is impossible. Indeed, as we discussed above, it has been shown that the Brans-Dicke theory with a cosmological constant does not admit stationary and spherically symmetric solutions, which are exterior solutions to a compact object and which have a cosmological horizon where the Brans-Dicke field is regular [47]. Clearly then, any spherically symmetric solution in this theory (in the presence of \( \Lambda \)) must be necessarily time-dependent. However, we expect this time-dependence to become more and more manifest only when we approach the cosmological horizon. As the turnaround radius is on much smaller scales, we take a different approach: perturbation theory.

Physical systems of interest are those where the Schwarzschild horizon \( R_s \), the turnaround radius \( R_{TA} \) and the de Sitter horizon \( R_h \) are widely separated. To be more precise, in standard GR we have \( R_s / R_{TA} = 2G_N M (\frac{\Lambda}{3G_{N M^2}})^{1/3} \lesssim 10^{-8} - 10^{-4} \) for the most massive galaxy clusters in the range \( M \sim 10^{11} - 10^{17} M_\odot \) while \( R_{TA} / R_h = (\frac{3G_N M}{\Lambda})^{1/3} \sqrt{\Lambda} / \sqrt{3} \lesssim 10^{-4} - 10^{-2} \). It thus seems like a good first approximation that \( 2G_N M / R \ll 1 \) and \( AR^2 / 3 \ll 1 \), so that the Scharzschild-de Sitter spacetime may be considered as a perturbation around Minkowski for the scales of interest.\(^8\)

\(^8\)One may instead perturb around a de Sitter, or even, a Schwarzschild-de Sitter spacetime. However, this introduces tremendous complication in solving the scalar equation and in the end, the Minkowski space approximation used here, where \( 2G_N M / R \ll 1 \) and \( AR^2 / 3 \ll 1 \), is recovered.
We expand our variables as

\[ f = 1 + U \] (3.3)
\[ h = 1 + V \] (3.4)
\[ \phi = \bar{\phi}_0 (1 + \varphi) \] (3.5)

so that \( U, V \) and \( \varphi \) are small compared to unity and \( \bar{\phi}_0 \) is a background value for \( \phi \). We consider a point-mass source in a spacetime filled with a cosmological constant so that the energy-density and pressure entering (3.2d) take the form

\[ 8\pi G \rho = \frac{2GM}{R^2} \delta(R) + \Lambda \] (3.6)
\[ 8\pi G P = -\Lambda . \] (3.7)

Consistently with our approximation both the point mass and \( \Lambda \) are treated as small perturbations. We start from (3.2a), linearize and then integrate to get

\[ V = \frac{1}{\phi_0 (2\omega + 3)} \left[ 2(\omega + 1) \frac{2GM}{R} \right] \left( \frac{2\omega - 1}{3} \frac{\Lambda R^2}{\bar{\phi}_0} \right) + \left( 2(\omega + 2) \frac{2GM}{R} \right) \left( \frac{2\omega + 1}{3} \frac{\Lambda R^2}{\bar{\phi}_0} \right) \right] . \] (3.8)

The above solution is then used in the linearized version of (3.2d), which when integrated gives

\[ \varphi = -\frac{1}{\phi_0 (2\omega + 3)} \left[ \frac{2}{3} \frac{\Lambda R^2}{\bar{\phi}_0} - \frac{2GM}{R} \right] . \] (3.9)

Finally, the expressions for \( V \) and \( \varphi \) are used in the linearized version of (3.2b), leading after integration to

\[ U = -\frac{1}{\phi_0 (2\omega + 3)} \left[ 2(\omega + 2) \frac{2GM}{R} \right] \left( \frac{2\omega + 1}{3} \frac{\Lambda R^2}{\bar{\phi}_0} \right) \right] . \] (3.10)

so that the metric is

\[ ds^2 = \left[ 1 - \frac{2(\omega + 2)}{2\omega + 3} \frac{2GM}{\phi_0 R} - \frac{2\omega + 1}{2\omega + 3} \frac{\Lambda R^2}{\phi_0 \bar{\phi}_0} \right] dT^2 \]
\[ + \left[ 1 + \frac{2(\omega + 1)}{2\omega + 3} \frac{2GM}{\phi_0 R} + \frac{2\omega - 1}{2\omega + 3} \frac{\Lambda R^2}{\phi_0 \bar{\phi}_0} \right] dR^2 + R^2 d\Omega . \] (3.11)

### 3.2 Cosmological solutions with a point-mass source

Let us now construct cosmological solutions for the metric and the Brans-Dicke field with a point-mass source. By “cosmological” we mean that in the limit \( M \to 0 \), the metric becomes the Friedman-Robertson-Walker metric and so these solutions are the analogue of the McVittie solution in the case of GR. We construct our solution by first considering a background FRW solution and then adding the perturbation due to the mass (see also [49] for cosmological perturbation theory equations with an array of point masses).

#### 3.2.1 FRW solutions

The FRW metric is

\[ ds^2 = -dt^2 + a^2 \gamma^{(r)}_{ij} dx^i dx^j , \] (3.12)
where \( a(t) \) is the scale factor of cosmic time \( t \), \( \gamma^{(\kappa)}_{ij} \) is the 3-metric (used to raise and lower three-dimensional indices) of constant spatial curvature \( \kappa \).

The Friedman equation in Brans-Dicke theory takes the form

\[
3 \left( H + \frac{1}{2} \frac{\dot{\phi}}{\phi} \right)^2 + \frac{3 \kappa}{a^2} = \frac{8\pi G}{\phi} \bar{\rho} + \frac{2\omega + 3}{4} \left( \frac{\ddot{\phi}}{\phi} \right)^2
\]

(3.13)

where \( \bar{\rho} \) is the background energy-density of matter (including the cosmological constant), \( H = \dot{a}/a \) is the time-dependent Hubble parameter and \( \ddot{\phi} \) is the homogeneous part of the scalar field adopted to the FRW symmetries. The scalar evolves according to

\[
\ddot{\phi} + 3H \dot{\phi} = \frac{8\pi G}{2\omega + 3} (\bar{\rho} - 3\bar{P})
\]

(3.14)

where \( \bar{P} \) is the background pressure of matter (including the cosmological constant).

It is straightforward to verify that an exact analytical solution is, in general, impossible, even if \( 8\pi G \bar{\rho} = \Lambda \) is a constant. Indeed, it can be shown that the de Sitter spacetime in no longer an exact solution of the field equations as it is in GR.\(^9\) This is equivalent to the non-existence of static-spherically symmetric solutions in the presence of a cosmological constant [47], as we have discussed in the previous subsection. Hence, we proceed using perturbation theory. Both the Friedman equation (3.13) and the scalar equation (3.14) suggest that the small parameter to use is

\[
\epsilon = \frac{1}{2\omega + 3}.
\]

(3.15)

We are interested in the case of a flat universe filled with cosmological constant so that \( 3H_0^2 \bar{\phi}_0 = 8\pi G \bar{\rho} = \Lambda = -8\pi G \bar{P} \). We construct the perturbative solution as a power series in \( \epsilon \) which yields

\[
\bar{\phi} = \bar{\phi}_0 \left[ 1 + 4\epsilon \ln a + \ldots \right] = \bar{\phi}_0 \left[ 1 + 4\epsilon H_0 t + \ldots \right]
\]

(3.16)

\[
H = H_0 \left[ 1 - \frac{4}{3} \epsilon - 2\epsilon \ln a + \ldots \right] = H_0 \left[ 1 - \frac{4}{3} \epsilon - 2\epsilon H_0 t + \ldots \right]
\]

(3.17)

as can be verified by direct substitution. A more formal derivation which is valid for a generic matter field and curvature can be found in the appendix. The dependence of the scale factor on time \( t \) is found by integrating (3.17) so that to \( O(\epsilon) \) we find

\[
a = \bar{a} \left[ 1 - \frac{4}{3} \epsilon H_0 t - \epsilon (H_0 t)^2 + \ldots \right]
\]

(3.18)

where \( \bar{a} = e^{H_0 t} \). The solutions found above are of course only valid close to \( \ln a \sim 1 \), i.e. for all times \( t \) such that \( \epsilon H_0 t \ll 1 \).

---

\(^9\)It may be shown that an exact solution exists for \( 8\pi G \rho = \Lambda = \text{const.} \) with \( a = (t/t_0)^{2\omega + 1}/2\omega + 3 \) and \( \phi = 4\Lambda t^2/(2\omega + 3)/(6\omega + 5) \). However, this requires that initially both the scalar and its first derivative vanish, i.e. \( \phi_0 = \dot{\phi}_0 = 0 \), and therefore this is a spurious solution of no physical significance and must be discarded.
3.2.2 Perturbed FRW solutions

Including the point-mass in our system inevitably introduces spatial dependence in the solutions. Assuming that the point-mass is not too massive as to overclose the universe, we may treat its contribution as a perturbation on top of the FRW solution we have constructed. This requires perturbing the FRW metric to linear order as in (2.15) by adopting the Newtonian gauge. Likewise we perturb the scalar field

\[ \phi = \bar{\phi}(1 + \varphi) \]  

(3.19)

where \( \bar{\phi} \) is the background value and \( \bar{\phi}\varphi \) the perturbation.

Before proceeding into solving the system, caution is warranted. Our background solution was arbitrarily close to de Sitter. We may then reinterpret the background FRW solution as being exact de Sitter plus small time-dependent perturbations. In other words set \( a = \bar{a}(1 + \delta_a) \) where from (3.18) we get

\[ \delta_a = -\epsilon \left[ \frac{4}{3} \Omega_0 + (H_0 t)^2 \right]. \]

This also implies \( H = H_0 + \dot{\delta}_a \), which may be checked for consistency with (3.17). Then we may define a new potential as

\[ a^2 (1 - 2\Phi) = \bar{a}^2 (1 - 2\bar{\Phi}) \]

so that \( \bar{\Phi} = \Phi - \delta_a \). The background field equations can only be satisfied under this transformation, if and only if a further transformation is also implemented: by observing that \( \bar{\phi} = \bar{\phi}_0(1 + \delta_\phi) \) with \( \delta_\phi = 4\epsilon H_0 t \) from (3.16), we may transform \( \delta_\phi \) away via

\[ \phi = \bar{\phi}(1 + \varphi) = \bar{\phi}_0(1 + \tilde{\varphi}) \]

so that \( \tilde{\varphi} = \varphi + \delta_\phi \).

Consistency of this line of thought requires that \( O(\Phi) \sim O(\bar{\Phi}) \sim O(\delta_a) \sim O(\delta_\phi) \) so that when considering linearized perturbations we ignore terms like \( \Phi \delta_a \) or \( \delta_a^2 \), etc. This means that in the perturbation equations we may replace \( a \rightarrow \bar{a}, H \rightarrow H_0 \) and \( \phi \rightarrow 0 \) resulting in great simplification. A further consistency requirement is that since after the transformation the background scalar field is constant, the scalar field equation must be treated entirely perturbatively. With these considerations and letting \( \hat{\nabla}_i \) to be the covariant derivative of \( \gamma_{ij} \), the perturbed Einstein equations sourced by matter with density perturbation \( \delta \rho = M \delta(3) (a\vec{r}) \) are as follows. Using the identity \( \delta(3) (a\vec{r}) = \delta(r)/(4\pi a^3 r^2) \), the 0–0 perturbed Einstein equation is

\[ -6H_0 \left( \hat{\Phi} + H_0 \Psi \right) + 3H_0 \left( \hat{\varphi} + H_0 \bar{\varphi} \right) + \frac{2}{a^2} \hat{\nabla}^2 \left( \hat{\Phi} - \frac{1}{2} \hat{\varphi} \right) = \frac{2GM}{\phi_0 a^3 r^2} \delta(r) \]  

(3.20)

and the 0–i-Einstein equation is

\[ 2\hat{\nabla}_i \left( \hat{\Phi} + H_0 \Psi \right) = \hat{\nabla}_i \left( \hat{\varphi} - H_0 \bar{\varphi} \right). \]  

(3.21)

We combine (3.20) and (3.21), assume the quasistatic limit where \( H_0^2 \bar{\varphi} \ll \hat{\nabla}^2 \bar{\varphi} \) and integrate to get

\[ \bar{\Phi} - \frac{1}{2} \bar{\varphi} = -\frac{GM}{\phi_0 R} \]  

(3.22)

where we have defined

\[ R = \bar{a}r. \]  

(3.23)

The perturbed scalar field equation is

\[ \ddot{\varphi} + 3H_0 \dot{\varphi} - \frac{1}{a^2} \hat{\nabla}^2 \bar{\varphi} = \frac{\epsilon}{\phi_0} \left[ 4\Lambda + \frac{2GM}{a^3 r^2} \delta(r) + 8\Lambda \Psi \right] \]  

(3.24)
and after assuming the quasistatic limit and integrating gives
\[ \tilde{\phi} = 2\epsilon \left[ \frac{GM}{\phi_0 R} - H_0^2 \tilde{R}^2 \right]. \] (3.25)
Hence, \( \tilde{\Phi} = -\frac{GM(1-\epsilon)}{\phi_0 R} - \epsilon H_0^2 \tilde{R}^2 \) while \( \Psi = -\frac{GM(1+\epsilon)}{\phi_0 R} + \epsilon H_0^2 \tilde{R}^2 \) after using the traceless-\( ij \)-Einstein equation \( D_{ij}(\Phi - \Psi - \phi) = 0 \) and ignoring the kernel which results to pure gauge-solutions.

Therefore, the metric to \( \mathcal{O}(\epsilon) \) is
\[ ds^2 = -\left[ 1 - \frac{2GM}{\phi_0 R} (1 + \epsilon) + 2\epsilon H_0^2 \tilde{R}^2 \right] dt^2 + \tilde{a}^2 \left[ 1 + \frac{2GM}{\phi_0 R} (1 - \epsilon) + 2\epsilon H_0^2 \tilde{R}^2 \right] \gamma_{ij} dx^i dx^j. \] (3.26)
Setting \( \epsilon \to 0 \) recovers the perturbed McVittie metric as expected, i.e. it recovers (2.6) in the limit \( \mu \ll 1 \).

### 3.3 The turnaround radius in Brans-Dicke theory

Having found the two types of solutions let us return to our original goal: the turnaround radius. A quick calculation using (2.5) along with the static spherically symmetric solution (3.11) yields
\[ R_{TA}^3 = \frac{3GM}{\lambda} \frac{2\omega + 4}{2\omega + 1} \] (3.27)
and taking the large \( \omega \) (small \( \epsilon \)) limit
\[ R_{TA} \approx \left( \frac{3GM}{\lambda} \right)^{1/3} (1 + \epsilon) \approx \left( \frac{3GM}{\lambda} \right)^{1/3} \left( 1 + \frac{1}{2\omega} \right) \] (3.28)
to \( \mathcal{O}(\epsilon) \sim \mathcal{O}(1/\omega) \).

Similarly, another quick calculation using (2.23) along with \( H = H_0 \) and the cosmological solution (3.26) yields once again (3.28). This should not come as a surprise. After all the two solutions (3.11) and (3.26) are in fact one and the same, after a coordinate transformation. This may be checked using the general form of such coordinate transformations between a static spherically symmetric space time and a perturbed FRW spacetime [50].

Note that we may also transform the cosmological solution back to the original FRW background given by (3.16), (3.17) and (3.18). In that case, the potential \( \Phi \) acquires a pure time-dependence, which is in turn eliminated by a gauge-transformation. This introduces a time-dependence into \( \Psi \) and in order to use (2.23) we must determine the canonical form of \( \Psi \) as in [50]. This is found to be \( \Psi = -\frac{GM(1+\epsilon)}{\phi_0 R} - \epsilon H_0^2 (\frac{4}{3} + 2H_0 t) \tilde{R}^2 \) so that (2.23) along with (3.17) gives back (3.28).

In (3.28) we have found the turnaround radius in terms of the bare parameters of the theory, \( G \) and \( \Lambda \). However, as is well known, the bare \( G \) in the Brans-Dicke action is not the actual measured Newtonian gravitational constant \( G_N \). Indeed, the latter is defined as [35, 51, 52]
\[ G_N = \frac{2(\omega + 2)}{\phi_0 (2\omega + 3)} G \approx \frac{1 + \epsilon}{\phi_0} G, \] (3.29)
so that \( g_{00} \approx -1 + 2G_NM/R \) as \( R \to 0 \). Hence, \( 3GM/\Lambda = (1 - \epsilon)G_NM/H_0^2 \). Furthermore, we should consider how we measure the cosmological constant. The Friedman equation
(under the assumption that $\phi \approx \text{const}$.) is $3H^2 \approx \Lambda/\bar{\phi}_0 + 8\pi G_N (1 - \epsilon) \rho_{\text{matter}}$. Hence, using cosmological observations one would measure $\Lambda_{\text{eff}} = \Lambda/\bar{\phi}_0$ rather than the bare $\Lambda$ and we call this the effective cosmological constant. With these considerations the expression (3.28) should be adjusted accordingly to

$$R_{\text{TA}} \approx \left(\frac{3G_NM}{\Lambda_{\text{eff}}}\right)^{1/3} \left(1 + \frac{2}{3} \epsilon\right) \approx \left(\frac{3G_NM}{\Lambda_{\text{eff}}}\right)^{1/3} \left(1 + \frac{1}{3\omega}\right)$$

which is our final result.

4 Conclusions

In this article we have calculated the effect of generic alternative theories of gravity obeying the Einstein Equivalence Principle on the maximum size of large scale cosmic structures. The maximum size of a structure is given by the maximum turnaround radius $R_{\text{TA}}$ — the point where the attraction due to the central mass gets balanced with the repulsion due to the dark energy, beyond which no compact mass distribution is possible. Thus any model predicting a maximum size of a structure with a given mass smaller than its actual observed size, gets ruled out on the basis of the stability of the structure. Conversely, if a given theory predicts a maximum size larger than the actual or observed size, the theory certainly persists. The theoretical prediction of $\Lambda$CDM on $R_{\text{TA}}$ was shown to be absolutely consistent with the observed astrophysical data [19, 23], and it is only about 10% larger than the observed ones for large scale structures with $M \geq 10^{13}M_\odot$ which are yet to virialize and much larger for masses below that [24]. Thus, it is clear that in order to have a meaningful phenomenology with the maximum turnaround radius to constrain various models, we must consider large scale objects with $M \geq 10^{13}M_\odot$. In particular, such consideration completely rules out dark energy models with equation of state parameter $w < -2$ [23].

We have introduced a new definition of the maximum turnaround radius, given by the turnaround equation (2.21), valid in any theory of gravity obeying the EEP and for any non-spherical bound object. We have further adopted (2.21) under the simplified assumptions of a spherically symmetric setup and a time-dependent cosmological setup with spherically symmetric perturbations arriving at the same conclusions. In both cases we deal with spherical symmetry. As we discussed above, since the large scale structures we should apply the turnaround calculation to are yet to virialize, spherical symmetry seems to be a very good approximation for our current purpose. The members of such a structure would redistribute their kinetic energy in order to reach virialization and the structure would get smaller in size. Thus, non-sphericity would eventually be created, but at a later time. In particular, it was argued in [19] that even the maximum departure from non-sphericity is not very large for most of those structures, except that of the Corona-Borialis supercluster — which may not be a single structure at all. Nevertheless, it is quite instructive and interesting to extend the current formalism to include non-sphericity as well. One possibility is to start from the general turnaround equation (2.21) and consider a non-spherical function $\Theta(t, \vec{x})$, possibly corresponding to some non-spherical surface. Another possible way to do this without adhering to perturbation theory, would be to consider an axisymmetric generalization of the McVittie solution we investigated by putting in a rotation and also to consider the Sheth-Tormen statistical mass function instead of the Press-Schechter statistical mass function (see e.g. [53]) in the analysis of [24].

The most important point we have demonstrated is that the turnaround radii predicted by both spherically symmetric and cosmological spacetimes are the same — establishing it...
as a purely geometric, coordinate invariant quantity. Such equality was earlier established for $\Lambda$CDM in \cite{19, 23}. As an application, we used the formalism in the context of the Brans-Dicke theory with a positive cosmological constant. Owing to the severe constraint of the Brans-Dicke parameter from the solar system data, $\omega \gtrsim 40000$ \cite{35}, we used a perturbative expansion in the Brans-Dicke parameter in terms of $\epsilon = 1/(2\omega + 3)$ and showed that the maximum turnaround radius is always larger than that of the $\Lambda$CDM, eq. (3.30) since our formula is only valid for $\omega \gg 1$. The increment of $R_{TA}$ from the $\Lambda$CDM is apparent from eq. (3.30) — depicting the increment of the term $G_N M$ for a finite and positive $\omega$, keeping $\Lambda$ fixed. The physical meaning behind this is related to the fact that since the gravitational attraction in Brans-Dicke is increased compared to GR (due to the additional scalar mediating gravity), we should move further radial distance away than $\Lambda$CDM in order to get it balanced by the repulsion of the dark energy whose value is being fixed. In other words, the maximum size of a structure with given mass should be regarded as the maximum length scale up to which it can hold itself against the repulsion due to the ambient dark energy. If we specify the latter, certainly $R_{TA}$ would increase with increasing mass or gravitational coupling.

Another important point to note here that we have used the definition of the mass and the cosmological constant as that of the General Relativity in eq. (3.30). Certainly, this should not be the case in general and such parameters should be defined within the framework of the theory itself. However, as long as we are doing perturbation theory over $\Lambda$CDM, such notion seems practically reasonable. Similar considerations within the Brans-Dicke theory in the context of the Parameterized Post-Newtonian formalism can be found in \cite{51, 52}. In any case, our result shows that the Brans-Dicke theory is perfectly consistent with the mass versus observed maximum sizes and hence the stability of structures.

It would be highly interesting to go beyond the first order perturbation theory considered here, in order to further investigate the stability issues. We hope to return to this in a future work.

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A  Perturbative solution of the background FRW in Brans-Dicke theory

In this appendix we give a formal derivation of the perturbative background FRW solution presented in section 3.2.1. We give the derivation for a general matter source in the presence of curvature and specialize at the end to a constant-$w$ component in a flat universe.
We eliminate the \( t \)-dependence in the background field equations by changing variables from \( t \) to \( \ln a \) so that the Friedman equation (3.13) becomes
\[
H^2 = \frac{8\pi G \bar{\phi} - \frac{3\kappa}{a^2}}{3 \left( 1 + \frac{1}{2} \frac{d\ln \phi}{d\ln a} \right)^2 - \frac{1}{4} \left( \frac{d\ln \phi}{d\ln a} \right)^2} \tag{A.1}
\]
while the scalar equation (3.14) can be formally integrated to
\[
\bar{\phi} = \bar{\phi}_0 + \epsilon \frac{8\pi G}{\phi_0} \int d\ln a \frac{1}{a^3} \frac{1}{H} \int d\ln a (\bar{\rho} - 3\bar{P}) \frac{a^3}{H}. \tag{A.2}
\]
We have set the initial condition \( \dot{\bar{\phi}}(\text{in}) \) to zero as it leads to a decaying solution.

The calculation now proceeds by expanding the fields as
\[
\bar{\phi} = \bar{\phi}_0 + \sum_{n=1}^{\infty} \bar{\phi}_n \epsilon^n \tag{A.3}
\]
\[
H = \bar{H} \left( 1 + \sum_{n=1}^{\infty} h_n \epsilon^n \right) \tag{A.4}
\]
where \( \bar{H} \) is the time-dependent Hubble parameter in the limit \( \epsilon \to 0 \) (not to be confused with the Hubble constant \( H_0 \)) and is given by
\[
3\bar{H}^2 = \frac{8\pi G \bar{\phi} - \frac{3\kappa}{a^2}}{3 \left( 1 + \frac{1}{2} \frac{d\ln \phi}{d\ln a} \right)^2 - \frac{1}{4} \left( \frac{d\ln \phi}{d\ln a} \right)^2}. \tag{A.5}
\]

Let us define the operator \( S[A, B] \) acting on functions \( A \) and \( B \) by
\[
S[A, B] = \frac{8\pi G}{\phi_0} \int d\ln a \frac{1}{a^3} \frac{1}{H} \int d\ln a (\bar{\rho} - 3\bar{P}) \frac{a^3}{H} B. \tag{A.6}
\]
This operator is then used to construct the perturbed variables \( \bar{\phi}_n \) from the scalar integral (A.2). The first three expansion coefficients are
\[
\bar{\phi}_1 = S[1, 1] \tag{A.6a}
\]
\[
\bar{\phi}_2 = -S[1, h_1] - S[h_1, 1] \tag{A.6b}
\]
\[
\bar{\phi}_3 = S[1, h_1^2 - h_2] + S[h_1, h_1] + S[h_1^2 - h_2, 1] \tag{A.6c}
\]
\[
\ldots
\]

The Friedman equation (A.1) is also perturbed to give
\[
h_1 = -\frac{1}{2} \bar{\chi}_1 \left( 1 - \frac{1}{12} \bar{\chi}_1 \right) - (1 - \Omega_K) \frac{1}{2} \bar{\phi}_1 \tag{A.7a}
\]
\[
h_2 = -\frac{1}{2} \left( \bar{\chi}_2 - \bar{\phi}_1 \bar{\chi}_1 + \frac{1}{4} \bar{\chi}_1^2 \right) + \frac{1}{12} \bar{\chi}_1 \left( \bar{\chi}_2 - \bar{\phi}_1 \bar{\chi}_1 \right) + \frac{3}{8} \bar{\chi}_1^2 \left( 1 - \frac{1}{12} \bar{\chi}_1 \right)^2 + \frac{1}{2} (1 - \Omega_K) \left[ \frac{3}{4} \bar{\phi}_1^2 - \bar{\phi}_2 + \frac{1}{2} \bar{\phi}_1 \bar{\chi}_1 \left( 1 - \frac{1}{12} \bar{\chi}_1 \right) + \frac{1}{4} \Omega_K \bar{\phi}_1^2 \right] \tag{A.7b}
\]
\[
\ldots
\]
where \( \bar{\chi}_n = d\bar{\phi}_n/d\ln a \) and \( \Omega_K = \kappa a^{-2}/\bar{H} \). The final solution is constructed from (A.6a)–(A.6c) and (A.7a)–(A.7b) with the help of (A.5). In particular one proceeds as \( \bar{\phi}_1 \to h_1 \to \bar{\phi}_2 \to h_2 \to \ldots \) and so forth.
A particular case of interest is a flat universe with $\Omega_K = 0$ and matter with constant equation of state $w$. Then

$$\bar{\phi} = \bar{\phi}_0 \left[ 1 + 2(\alpha + \alpha^2 + \alpha^3) \ln a + (2\alpha^2 + 4\alpha^3) \ln^2 a + \frac{4}{3} \alpha^3 \ln^3 a + \ldots \right]$$

(A.8)

and

$$H = \bar{H} \left[ 1 - \frac{1}{6} \frac{3 - 3w}{1 - w} \frac{1 - 3w(3 - w)}{24(1 - w)^2} \alpha^2 - \left( \frac{1 - 3w}{6(1 - w) \alpha^2} \right) \ln a + \frac{1}{2} \alpha^2 \ln^2 a + \ldots \right]$$

(A.9)

where

$$\alpha = \frac{1 - 3w}{1 - w} .$$

(A.10)

Clearly, in a radiation dominated Universe, the solution is $\bar{\phi} =$ constant and $H = \bar{H}$ as we would expect from the fact that the scalar couples to the trace of the energy-momentum tensor.

Imposing $w = -1$ in (A.8) and (A.9) and keeping terms to $O(\epsilon)$ gives (3.16) and (3.17) after letting $\bar{H} = H_0 = \sqrt{\frac{\Lambda}{3\bar{\phi}_0}}$.

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