Regularity issues for Cosserat continua 
and $p$-harmonic maps

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Abstract. For minimizers in a geometrically nonlinear Cosserat model 
for micropolar elasticity of continua, we prove interior Hölder regularity, 
up to isolated singular points that may be possible if the exponent $p$ 
from the model is 2 or in $\left(\frac{22}{15},3\right)$. The obstacle to full continuity turns 
out to be the existence of certain minimizing homogeneous $p$-harmonic 
maps to $S^3$. For those, we slightly improve existing regularity theorems 
in order to achieve our result on the Cosserat model.

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1 Introduction and statement of results

Cosserat micropolar elasticity is a framework for theories of continua (as well as shells 
and rods) with some internal structure. The foundations have been laid out by the 
brothers Eugène and François Cosserat in 1909 [CC]. In addition to stresses responding 
to translational degrees of freedom of an elastic body, the framework also allows stresses 
coming from rotational degrees of freedom assigned to every point of the material.

On the other hand, $p$-harmonic maps are a well-established branch of geometric 
analysis. They are critical points of the $p$-Dirichlet integral $E^p(u) := \int_{\Omega} |Du|^p \, dx$ 
among mappings $u : \Omega \to N$, where $N$ is some fixed Riemannian manifold.

The current paper exploits relations between both theories. Since the $p$-Dirichlet 
integral, here for mappings to $SO(3)$, also appears in the energy functional for useful 
models within Cosserat theory, the equations for the latter couple the $p$-harmonic map 
equation with another one. The analytic difficulties of Cosserat theories have their 
origin, at least partially, in the geometric restriction of the rotational degrees of freedom 
to $SO(3)$. But this is exactly the kind of restriction that has to be understood in $p$-
harmonic map theory, which has been developed to quite some extent. We therefore 
aim at understanding the nonlinear aspects of Cosserat theory better by using methods 
that have been successfully established for $p$-harmonic maps. We address the question 
of partial regularity of minimizing weak solutions, and find out that methods invented 
by Luckhaus [Lu] (based on [SU1]) are particularly useful.

Many variants of Cosserat theory are available, and the author will not even try to 
give an overview about the different models and the vast body of results. Instead, we
restrict to a particular instance of that theory for micropolar elastic bodies, that has
been studied in the framework of the calculus of variations by Neff [Ne1] and others.
The elastic body exists over a reference configuration that can be thought of a subset
Ω of $\mathbb{R}^3$. From that configuration, the body can be deformed, shifting every point
$x \in \Omega$ to some point $\varphi(x) \in \mathbb{R}^3$, such that $\varphi(x) - x$ can be thought of its usually
dislocation. Additionally we assume some structure of the material that attaches
to every $x \in \Omega$ an orthonormal frame that is free to rotate in $\mathbb{R}^3$ by an orthogonal
matrix $R(x) \in SO(3)$. Both translations and rotations cause material stresses, which
are given by $R^t D\varphi - I$ and $R^t DR$, respectively. Here $I \in \mathbb{R}^{3 \times 3}$ is the identity
matrix, and we denote the transposed of the matrix $R$ by $R^t$. Note that $R^t DR$ is a
3-tensor rather than a 2-tensor, but since $R^t \partial_i R \in so(3)$ for $i \in \{1, 2, 3\}$, there are only
9 independent components. We will not bother about aspects of modelling, and refer
to the discussions in [Ne1] and [NBO] instead.

Now let us describe the energy functional summing up the energy stored in our elas-
tic body. The contribution of the translation should be measuring $R^t D\varphi - I$ somehow.
The usual choice is

$$\mu_1 \| \text{dev sym}(R^t D\varphi - I) \|_{L^2(\Omega)}^2 + \mu_c \| \text{skew}(R^t D\varphi - I) \|_{L^2(\Omega)}^2 + 3 \mu_2 \| \text{tr}(R^t D - I) \|_{L^2(\Omega)}^2. $$

Here $\text{dev sym} A$ is the deviatoric symmetric part $\frac{1}{2}(A + A^t) - (\text{tr} A) I$ of $A$, and skew $A := \frac{1}{2}(A - A^t)$ is the skew-symmetric part. Defining $P : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}$ to be the linear
operator given by

$$PA := \sqrt{\mu_1} \text{dev sym} A + \sqrt{\mu_c} \text{skew} A + \sqrt{\mu_2} (\text{tr} A) I,$$

the term is written more simply as

$$\| P(R^t D\varphi - I) \|_{L^2(\Omega)}^2.$$

The constants $\mu_1$, $\mu_c$, and $\mu_2$ will be assumed to be $> 0$. While this is completely usual
for $\mu_1$ and $\mu_2$, it would be desirable to allow the so-called “Cosserat constant” $\mu_c$ to
be 0. Remember that elasticity theory usually involves $\text{sym} D\varphi$ instead of $D\varphi$. This
would, however, combine the “geometric” difficulty for $R \in SO(3)$ with the “coercivity
issue” for $\varphi$, and we are currently not able to handle both. Anyway, the existence
of minimizers has been established in the $\mu_c > 0$ case in [Ne2] (where Cosserat is
a special case, discussed more explicitly in [Ne1]). For $\mu_c = 0$, on the other hand,
interesting cases are open, see the discussion in [Ne1]. And since we prefer dealing
with the regularity of minimizers in cases where they are known to exist, we have
another reason to restrict to $\mu_c > 0$ in this paper. We expect that the generalization
to $\mu_c \geq 0$ would provide us with some interesting additional problems.

The contribution of the rotational stresses to the energy is simply

$$\lambda \| R^t DR \|_{L^p(\Omega)}^p$$

for $\lambda > 0$ and some parameter $p \geq 2$. On first glimpse, $p = 2$ seems to be the natural
choice, but there are problems with the decoupling of linearized equations that suggest
that $p > 2$ might be better for many purposes. We shall see that even our regularity
theory works slightly better for \( p \) larger than (but close to) 2. Some more general terms, involving parameters like \( \mu_1, \mu_c, \mu_2 \) above, have been proposed, but they seem less natural, since \( R^t \partial_t R \) is always skew-symmetric. (One can use \( \text{Curl} \, R \) instead of \( DR \), though.) Anyway, most of our regularity theory would work for those more general energies, too, with the exception of those parts where point singularities are removed. We therefore restrict to the simple term above. Since \( |R^t DR| = |DR| \), and since we can make one of the constants to be 1, we can even work with the simpler term

\[
\|DR\|_{L^p(\Omega)}^p
\]

here, and of course this is the \( p \)-Dirichlet integral.

Our elastic body may be subject to exterior forces. Some of them, e.g. mechanical ones, will act on the boundary of the body, only. We need not consider them, because we are only concerned with interior regularity in this paper. Some boundary regularity may well be within reach of the methods presented here, but natural questions seem to be more involved, like what happens at the edge between regions with Dirichlet and Neumann boundary conditions. Other forces, like gravity or electromagnetic forces, will act on points in \( \Omega \), and we have to account for such forces in our functional. Exterior forces, given by a function \( f : \Omega \to \mathbb{R}^3 \), are accounted for in the term

\[
\int_{\Omega} (\varphi - x) \cdot f \, dx,
\]

while there may be also moments of force affecting the rotational degrees of freedom, given by \( M : \Omega \to \mathbb{R}^{3 \times 3} \). They contribute to the energy via

\[
\int_{\Omega} R \cdot M \, dx.
\]

For our domain \( \Omega \) we have to assume that it is bounded. In order to have existence of minimizers, we also assume that it is Lipschitz.

Summarizing, for a pair of functions \( \varphi : \Omega \to \mathbb{R}^3 \) and \( R : \Omega \to SO(3) \), we have the energy functional

\[
J(\varphi, R) := \int_{\Omega} \left( |P(R^t D\varphi - I)|^2 + |DR|^p + (\varphi - x) \cdot f + R \cdot M \right) \, dx.
\]

The topic of our paper is interior regularity of minimizers, which have been proven to exist by Neff, given suitable boundary conditions. Restricting to minimizers means we are only considering a static problem here, no dynamics. Minimizers are weak solutions of the Euler-Lagrange equations for \( J \), which are standard to derive. They read

\[
\text{div}(RP^2(R^t D\varphi - I)) = f; \quad (1)
\]

\[
\left( \text{div}(|DR|^{p-2} DR) - \frac{2}{p} D\varphi \cdot P^2((D\varphi)^t R - I) - \frac{1}{p} M \right)(x) \perp T_{R(x)} SO(3). \quad (2)
\]

The second one is an orthogonality relation, since variations of \( R \) can only be made in directions tangential to \( SO(3) \). Therefore, (2) represents only three independent
equations rather than the expected nine for the components of $R$. The “missing” six equations are simply the requirement $R(x) \in SO(3)$.

The terms involving $f$ and $M$ are of lower order and different scaling than the others, thus inflicting only minor complications to our study. For all of this paper except this introduction and the last section, we can therefore reasonably assume $f \equiv 0$ and $M \equiv 0$. The minor changes necessary for nonvanishing $f$ and $M$ will be hinted at in Section 7. For most of the paper, our functional therefore is

$$J(\varphi, R) := \int_{\Omega} \left( |P(R^t D \varphi - I)|^2 + |DR|^p \right) dx.$$ 

If we consider regularity for minimizers of $J$, we find that it cannot be any better than the regularity of minimizing $p$-harmonic maps $\Omega \to SO(3)$, i.e. minimizers of $E^p(R) := \int_{\Omega} |DR|^p dx$. For minimizing $p$-harmonic maps of an $n$-dimensional domain $\Omega$, partial regularity has been proven independently by Hardt/Lin [HL] and Fuchs [Fu], and with a more flexible proof by Luckhaus [Lu]. The result is that they are Hölder continuous (even $C^{1,\mu}$) in the interior of $\Omega$ away from a closed set $\text{Sing}(R)$ of Hausdorff dimension $\leq n - [p] - 1$, and that $\text{Sing}(R)$ is even discrete if $n - [p] - 1 = 0$, and empty if that is $<0$. For $n = 3$, that means that one has a discrete singular set for $p \in [2, 3)$, while for $p \geq 3$ we have full Hölder continuity. We will find the same for minimizers $(R, \varphi)$ of $J$, following Luckhaus’ techniques and modifying them for a functional where $\varphi$ and $R$ have different homogeneities and $\varphi$ is possibly unbounded, both of which are not allowed in [Lu].

Depending on the target, the singular set of $p$-harmonic maps may be even smaller, or empty altogether. In the $p = 2$ case, there are results for special targets, e.g. [SU1] for minimizing harmonic maps to spheres. For $p > 2$, there are some results, too, like [XY] and [Na].

In our study, $R$ does not map to a sphere, but to $SO(3)$, which however has $S^3$ as its universal cover. This enables us to prove the following. Whenever $p > 2$ and there is a point singularity in some minimizer $(\varphi, R)$ for our functional $J$, then there is also a minimizing $p$-harmonic $u : B^3 \to S^3$ having a point singularity. Unfortunately, the results in [XY] or [Na] are not strong enough to exclude the latter. But we can do so, at least for $p \in (2, \frac{32}{15}]$.

To do this, we partially follow recent progress by Chang, Chen, and Wei [CCW] for $p$-harmonic functions to $\mathbb{R}$, resulting in an “improved Kato inequality”. Our results strongly depend on the values of constants in estimates and therefore are almost certainly far from optimal. But anyway, this shows that for $p \in (2, \frac{32}{15}]$ (and for $p = 3$, and almost trivially for $p > 3$) minimizers of our Cosserat energy are Hölder continuous on the interior of the domain. This leaves the possibility of point singularities only for $p = 2$ or $p \in (\frac{32}{15}, 3)$.

Now, here are our precise results. The following theorem combines the statements of the Propositions 4.3, 5.2, 6.2, 6.4, and 7.2 formulated and proven below in this paper.

**Theorem 1.1 (interior (partial) regularity for minimizers)**

Assume $\mu_1, \mu_c, \mu_2 > 0$, and $p \geq 2$. Let $\Omega \subset \mathbb{R}^3$ be a bounded open domain, let functions $f \in C^{0,\mu}(\overline{\Omega}, \mathbb{R}^3)$ and $M \in C^0(\overline{\Omega}, \mathbb{R}^{3 \times 3})$ be prescribed, and let $(\varphi, R) \in$
be a minimizer of the functional $J$. Then there is a
discrete subset $\text{Sing}(\varphi,R)$ of $\Omega$ such that

$$(\varphi,R) \in C^{1,\mu}_\text{loc}(\Omega \setminus \text{Sing}(\varphi,R),\mathbb{R}^3) \times C^{0,\mu}_\text{loc}(\Omega \setminus \text{Sing}(\varphi,R),\text{SO}(3))$$

for every $\mu \in (0,\frac{2}{p})$ if $f \equiv 0$, and for every $\mu \in (0,\frac{1}{2p})$ if $f \not\equiv 0$.

Moreover, $\text{Sing}(\varphi,R)$ is empty, and therefore $\varphi$, $D\varphi$, and $R$ locally Hölder continuous on all of $\Omega$, if one of the following conditions holds.

(i) $p = 2$ and $\mu_1 = \mu_c = \mu_2$,
(ii) $p \in (2,\frac{32}{15}]$,
(iii) $p \geq 3$.

The obvious question this theorem provokes is if there can be singular points at all for minimizers of $J$. We cannot answer this question exactly, but, for every $p \in [2,3)$, we do find an explicit weak solution for the system of the Euler-Lagrange equations that has a singular point — one more motivation to study regularity theory. We do not know if our example is minimizing for any $p$.

Our observations are reflected by the regularity theory for $p$-harmonic maps, where, too, much more is known about minimizers than about more general weak solutions. In dimensions $> 2$, one cannot expect any good regularity in general, since Rivière [Ri] has constructed a weakly harmonic map $B^3 \to S^2$ that is discontinuous in every point of $B^3$. Under the additional assumption of stationarity, weakly $p$-harmonic maps are sometimes known to be Hölder continuous outside a closed set of vanishing $(n-p)$-dimensional Hausdorff measure, proven by Bethuel [Be] for $p = 2$ and any compact target, and by Toro and Wang [TW] for general $p$, but only if the target is a compact homogeneous space. Here, a weak solution is called stationary if it is also critical with respect to variations in the domain.

Assuming that our Cosserat system has regularity just as good as for harmonic maps, we would expect a singular set of vanishing $(3-p)$-dimensional Hausdorff measure for stationary weak solutions, and no good regularity theory at all for just weak solutions. In this paper, however, we only consider minimizers.

The paper is organized as follows. In Section 2, we demonstrate by our singular weak solution that regularity theory is an issue at all. In Section 3, we show that in the simple case $p = 2$ and $\mu_1 = \mu_c = \mu_2$ Luckhaus’ result from [Lu] already gives some partial regularity for the Cosserat body, but not as much regularity as we will achieve here. In Section 4, we adapt Luckhaus’ proof of partial regularity by blowing up in the target and domain in order to compare with a simplified system. This results in an $\varepsilon_0$-regularity theorem stating that singularities can only occur in points where enough energy concentrates. Section 5, also inspired by the techniques for ($p$-)harmonic maps, features a monotonicity formula saying that minimizers are automatically in some Morrey space rather than just in $W^{1,2} \times W^{1,p}$. We apply this in order to get regularity up to the possibility of isolated singularities. The obstruction to full regularity is identified to be the existence of certain $p$-harmonic maps to $S^3$. Therefore, in Section 6, we try to exclude their existence under suitable assumptions, which leads to full Hölder regularity for some exponents $p$. Finally, Section 7 discusses the changes necessary to
allow for exterior forces and moments, that in the sections before were assumed to vanish.

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2 A singular example

From now on, \( f \equiv 0 \) and \( M \equiv 0 \) will be assumed until we discuss the case of their nonvanishing in Section 7.

If all constants agree, by which we mean \( \mu_1 = \mu_c = \mu_2 = 1 \), the Euler-Lagrange equations simplify according to

\[
\Delta \varphi - \text{div} \ R = 0, \\
\text{div}(\|D\varphi\|^{p-2}D\varphi) + \frac{2}{p} D\varphi \perp T_RSO(3),
\]

where in the latter we have removed the term \( D\varphi D\varphi^t R \) since it is always orthogonal to \( T_RSO(3) \). In this simple case, we can write down an explicit weak solution that exhibits a singularity. It is given by \((\varphi, R): B^3 \to \mathbb{R}^3 \times SO(3), \)

\[
\varphi(x) := \frac{4}{3} x \log |x|, \\
R(x) := \frac{2}{|x|^2} x \otimes x - I = \frac{1}{|x|^2} \left( x_1^2 - x_2^2 - x_3^2 \ x_2x_1 \ x_1^2 - x_2^2 - x_3^2 \ x_3^2 - x_1^2 - x_2^2 \right).
\]

We perform a few calculations to see that this is a solution on \( B^3 \setminus \{0\} \). We have

\[
\partial_i \varphi(x) = \frac{4}{3} \left( e_i \log |x| + \frac{x_i x}{|x|^2} \right), \\
\partial_i^2 \varphi(x) = \frac{4}{3} \left( \frac{x_i e_i}{|x|^2} + \frac{x}{|x|^2} - 2 \frac{x_i}{|x|^4} \right), \\
\Delta \varphi(x) = 4 \frac{x}{|x|^2},
\]

and

\[
\text{div} \ R = 4 \frac{x}{|x|^2}, \\
\partial_k R_{ij} = \frac{2}{|x|^2} (\delta_{jk} x_i + \delta_{ik} x_j) - \frac{4x_i x_j x_k}{|x|^4}, \\
\|D \varphi\|^2 = 4 \frac{1}{|x|^2}, \\
\text{div}(\|D\varphi\|^{p-2} D\varphi) = \frac{2p}{|x|^p} \left( I - \frac{3}{|x|^2} x \otimes x \right).
\]
Now we immediately read off the first Euler-Lagrange equation. And for every \( x \in \mathbb{R}^3 \), we see that \( \text{div}(|D(R(x))|^{p-2}D(R(x))) \) and \( D\varphi(x) \) are linear combinations of \( I \) and \( x \otimes x \). A matrix \( A \in \mathbb{R}^{3\times3} \) is perpendicular to \( T_R SO(3) \) if \( R^t A \) is perpendicular to \( so(3) \), which holds if and only if \( R^t A \) is symmetric. But \( R^t(x)I = R^t(x) \) and \( R^t(x)(x \otimes x) = x \otimes x \) are both symmetric, which means that \( \text{div}(|D(R)|^{p-2}D(R)) + \frac{2}{p} D\varphi \) is perpendicular to \( T_R SO(3) \) for every \( x \in B^3 \setminus \{0\} \). Therefore also the second equation holds away from the origin.

Note that \((\varphi, R) \in W^{1,2} \times W^{1,p} \) if \( 2 \leq p < 3 \), and we easily find that, for those \( p \), \((\varphi, R) \) is a weak solution of the Euler-Lagrange equations. Of course, it is smooth on \( B^3 \setminus \{0\} \). In \( 0 \), \( R \) is not even continuous, while \( \varphi \) is Hölder continuous for every exponent \( < 1 \), but not differentiable.

The example shows that we must be ready to expect at least point singularities for weak solutions of our model, as long as \( p < 3 \). The solutions constructed by Neff, on the other hand, are better than just weak solutions, they are minimizers of \( J \). As often in the calculus of variations, we will find that sometimes minimizers have better regularity properties than other weak solutions.

### 3 A quick application of a result by Luckhaus

We first consider the case \( p = 2 \) and show that it is within the framework of a paper [Lu] by Luckhaus which was written with focus on \( p \)-harmonic maps.

Assume that \((\varphi, R) \in W^{1,2}(\Omega, \mathbb{R}^3 \times SO(3)) \) is a minimizer of \( J \) (subject to suitable boundary conditions). We check that assumptions from [Lu] are fulfilled. To this end, we introduce some notation. We write \( N := \mathbb{R}^3 \times SO(3) \), \( x \) for the independent variable in \( \Omega \) and \( y = (y_1, y_2) \in N \) for the dependent variable holding \((\varphi, R) \). Moreover, \( z = (z_1, z_2) \) stands for the variable that \((D\varphi, DR)\) take their values in. This means

\[
J(\varphi, R) = \int W((\varphi(x), R(x)), (D\varphi(x), DR(x))) \, dx,
\]

where here

\[
W(y, z) := \mu_1 \text{dev sym } y_2^t z_1^2 + \mu_c \text{skew } y_2^t z_1^2 + \mu_2 |\text{tr } y_2^t z_1 - 3|^2 + |z_2|^2.
\]

Since \( W \) does not depend explicitly on \( x \), the situation is even slightly simpler than Luckhaus’.

Luckhaus has the following conditions (adapted here for \( x \)-independent functionals)

\[
(A1a) \quad c^{-1}|z|^2 - 1 \leq W(y, z) \leq c|z|^2 + 1, \\
(A1b) \quad \lim_{y \to y_0, z} \sup_{z} (1 + |z|)^{-2} |W(y, z) - W(y_0, z)| = 0, \\
(A1c) \quad W(y, z) \text{ is convex in } z \text{ for all } y.
\]

Defining for \( y \in N \) the set

\[
\mathcal{H}_y := \{ F : \text{there exist } \alpha_i \to \infty \text{ s.t. } F(z) = \lim_{i} \alpha_i^{-2} W(y, \alpha_i z) \}
\]
as well as

\[ \mathcal{H}_\infty := \{ F : \text{there exist } \alpha_i \to \infty, |y_i| \to \infty \text{ s.t. } F(z) = \lim \alpha_i^{-2} W(y_i, \alpha_iz) \}, \]

Luckhaus has one more condition,

\[(A2) \text{ If } F \in \mathcal{H}_y \text{ or } F \in \mathcal{H}_\infty \text{ and } v \in W^{1,2}(B_1, T) \text{ for } T := T_yN \text{ or } T := \lim T_yN, \text{ and} \]

\[ \text{div}(D_zF(Dv)) = 0, \]

then \( v \) is \( \mu \)-Hölder continuous for some \( \mu \in (0, 1] \).

For the target manifold \( N \), in [Lu] it is enough for part (a) of his theorem, that the nearest point retraction when restricted to an \( \epsilon \)-neighborhood of \( N \) has a Lipschitz constant approaching 1 uniformly as \( \epsilon \to 0 \). This is clearly fulfilled for our \( N = \mathbb{R}^3 \times SO(3) \) because of its bounded curvatures.

Hence it remains to check the \((A1)\) and \((A2)\) assumptions. We see that \((A1a)\) clearly holds since for \( c := \max\{\mu_c, \mu_1, \mu_2, 1\} \) we have

\[ c^{-1}|z|^2 - 18 \leq W(y, z) \leq c|z|^2 + 18, \]

and 18 is certainly as good as 1 for our purposes. The condition \((A1b)\) is immediately read off for our functional, and the convexity required for \((A1c)\) also clearly holds.

We now observe that \( \mathcal{H}_y \) consists of the single function

\[ F(z) := \mu_1 |\text{dev sym } y_2^Tz_1|^2 + \mu_c |\text{skew } y_2^Tz_1|^2 + \mu_2 |\text{tr } y_2^Tz_1|^2 + |z_2|^2. \]

And \( \mathcal{H}_\infty \) consists of all those for all \( y_2 \in SO(3) \). Every such function is a homogeneous quadratic form in \( z \) which is positive definite. Therefore, \( \text{div } D_zF(Dv) = 0 \) is an elliptic equation for \( v \) with constant coefficients, whose solutions are of course Hölder continuous. This shows that \((A2)\) also holds.

We can therefore apply Theorem (a) from [Lu] and conclude that any minimizer \((\varphi, R)\) of \( J \) is Hölder continuous on the interior of \( \Omega \) outside a closed singular set \( \Sigma \) for which we have \( H^1(\Sigma) = 0 \), where \( H^1 \) means the 1-dimensional Hausdorff measure.

A second theorem from [Lu] states that the singular set consists of isolated points only, but it cannot be applied immediately, because it requires a compact target manifold \( N \), plus more assumptions. Our \( N = \mathbb{R}^3 \times SO(3) \) is not compact, but the noncompact factor is quite trivial, which allows us to work around this in the sections that follow.

### 4 A first partial regularity result for general \( p \geq 2 \)

If \( p > 2 \), the reasoning from the previous section does not work properly, and we have to use modifications of Luckhaus’ arguments from [Lu] to prove partial Hölder continuity. Since the unknown functions \( \varphi \in W^{1,2}(\Omega, \mathbb{R}^3) \) and \( R \in W^{1,p}(\Omega, \mathbb{R}^n) \) now are in Sobolev spaces of different scaling, [Lu] cannot be applied immediately. The overall strategy, however, continues to work, and we will be able to use the key ideas,
including the “Luckhaus Lemma”, without too much modification. Our first lemma is a discrete version of Morrey’s Dirichlet growth condition as a first step to partial regularity. We globally assume the constants $\mu_1, \mu_c, \mu_2 > 0$ to be fixed, in fact most constants will depend on those.

Morrey’s Dirichlet growth criterion would imply local Hölder continuity of $R$ once we know that $\rho^{1-3/p}\|D\rho\|_{L^p(B_\rho(x_0))}$ is bounded by some $c\rho^\mu$ for all $B_\rho(x_0) \subset \Omega$. The following Lemma establishes a discrete version of this which allows the same conclusion.

**Lemma 4.1 (discrete Morrey condition)** We fix $\mu \in (0,1)$. Then there exists constants $\varepsilon_0 > 0$, $\theta \in (0,1)$, and $\rho_0 \in (0,1)$ such that the following holds for every minimizer $(\varphi, R) \in W^{1,2}(B^3, \mathbb{R}^3) \times W^{1,p}(B^3, SO(3))$ of $J$ subject to its boundary data. For every ball $B_{\rho_0}(x_0) \subset B^3$ with $|x_0| \leq \frac{1}{2}$ and any $\rho \in (0, \rho_0)$, the condition

$$\rho^{2\mu} \leq \rho^{-1}\|D\varphi\|_{L^2(B_\rho(x_0))}^2 + \rho^{p-3}\|D\rho\|_{L^p(B_\rho(x_0))}^p \leq \varepsilon_0$$

imply

$$(\theta \rho)^{-1}\|D\varphi\|_{L^2(B_{\theta \rho_0}(x_0))}^2 + (\theta \rho)^{p-3}\|D\rho\|_{L^p(B_{\theta \rho_0}(x_0))}^p$$

$$\leq \theta^{2\mu}(\|D\varphi\|_{L^2(B_{\rho_0}(x_0))}^2 + \rho^{p-3}\|D\rho\|_{L^p(B_{\rho_0}(x_0))}^p).$$

**Proof.** Assume that the assertion does not hold. Then there are balls $B_{\rho_i}(x_i)$ with $|x_i| \leq \frac{1}{2}$ and $\rho_i \searrow 0$ such that

$$\gamma_i := \rho_i^{-1/2}\|D\varphi\|_{L^2(B_{\rho_i}(x_i))} \searrow 0,$$

$$\delta_i := \rho_i^{1-3/p}\|D\rho\|_{L^p(B_{\rho_i}(x_i))} \searrow 0,$$

and $\rho_i^{2\mu} \leq \gamma_i^2 + \varepsilon^p$, but

$$(\theta \rho_i)^{-1}\|D\varphi\|_{L^2(B_{\theta \rho_i}(x_i))}^2 + (\theta \rho_i)^{p-3}\|D\rho\|_{L^p(B_{\theta \rho_i}(x_i))}^p$$

$$> \theta^{2\mu}(\|D\varphi\|_{L^2(B_{\rho_i}(x_i))}^2 + \rho_i^{p-3}\|D\rho\|_{L^p(B_{\rho_i}(x_i))}^p).$$

Now we do a suitable rescaling and define $(\varphi_i, R_i) \in W^{1,2}(B^3, \mathbb{R}^3) \times W^{1,p}(B^3, N_i)$ by

$$\varphi_i(x) := \gamma_i^{-1}(\varphi(x_i + \rho_i x) - \bar{\varphi}(i)),$$

$$R_i(x) := \delta_i^{-1}(R(x_i + \rho_i x) - \bar{R}(i)),$$

where here $\bar{\varphi}(i)$ and $\bar{R}(i)$ are the mean values

$$\bar{\varphi}(i) := \int_{B_{\rho_i}(x_i)} \varphi \, dx, \quad \bar{R}(i) := \int_{B_{\rho_i}(x_i)} R \, dx,$$

and $N_i$ is the rescaled and shifted target manifold $N_i := \delta_i^{-1}(N - \bar{R}(i))$. Then $(\varphi_i, R_i)$ minimizes the rescaled functional

$$J_i(\tilde{\varphi}, \tilde{R}) := \frac{\gamma_i^2}{\gamma_i^2 + \delta_i^p} \int_{B^3} |P((\bar{R}(i) + \delta_i \tilde{R})^t D\tilde{\varphi} - \rho_i \gamma_i^{-1} I)|^2 \, dx + \frac{\delta_i^p}{\gamma_i^2 + \delta_i^p} \int_{B^3} |D\tilde{R}|^p \, dx.$$
Here the denominator is chosen such that, after passing to a subsequence, we may assume $\frac{\gamma_i^2}{\gamma_i^2 + \delta_i^2} \to \sigma$ and $\frac{\delta_i}{\gamma_i^2 + \delta_i^2} \to 1 - \sigma$ for some $\sigma \in [0,1]$. Note also that $\frac{\gamma_i^2}{\gamma_i^2 + \delta_i^2}(\rho_i \gamma_i^{-1})^2 \to 0$ since $\rho_i^2/(\gamma_i^2 + \delta_i^2) \to 0$ and $\mu < 1$. And the sequence $\overline{R}(i)$ is certainly bounded, since it cannot leave the convex hull of $SO(3)$ in $\mathbb{R}^{3\times3}$. We even have

$$\text{dist}(\overline{R}(i), SO(3)) \leq c \rho_i^{-3/p} \| R - \overline{R}(i) \|_{L^p(B_{\rho_i}(x_i))} \leq c \rho_i^{1-3/p} \| DR \|_{L^p(B_{\rho_i}(x_i))} = c \delta_i \to 0,$$

and therefore we can assume $\overline{R}(i) \to T$ for some $T \in SO(3)$.

We now consider the corresponding subsequence of our $J_i$-minimizers $(\varphi_i, R_i)$ and hope that it converges (in a suitable sense) to a minimizer of the limit functional

$$J_\infty(\tilde{\varphi}, \tilde{R}) := \sigma \int_{B^3} |P(T^i D\tilde{\varphi})|^2 \, dx + (1 - \sigma) \int_{B^3} |D\tilde{R}|^p \, dx.$$ 

This may be not quite true, but almost, since it holds away from $\partial B^3$. This is stated and proved in Lemma 4.2 below. From that we know that on every compact subset of $B^3$, $(\varphi_i, R_i)$ converges in $W^{1,2} \times W^{1,p}$-norm to a minimizer $(\varphi_\infty, R_\infty)$ of $J_\infty$. Note that the $N_i$ converge locally in Hausdorff distance to some 3-dimensional subspace of $\mathbb{R}^{3\times3}$ which we denote by $N_\infty$. Actually, $N_\infty$ is the limit of the affine tangent spaces $T_{\overline{R}(i)} + T_{\overline{R}(i)} N_i$, and $0 \in N_\infty$ because all $R_i$ have mean value 0.

In preparation of Lemma 4.2, we note that we have weak (sub-)convergence $(\varphi_i, R_i) \rightharpoonup (\varphi_\infty, R_\infty)$. This holds because $\| D\varphi_i \|_{L^2(B^3)} = 1$ and $\| DR_i \|_{L^p(B^3)} = 1$ by construction, and both $\varphi_i$ and $R_i$ have mean values 0.

Up to passing to another subsequence, we then have

$$\overline{R}(i) + \delta_i R_i \to T \quad (3)$$

pointwise almost everywhere, for the constant matrix $T$ found above. We see this by combining the pointwise convergences $\overline{R}(i) \to T$, $R_i \to R_\infty$ and $\delta_i \to 0$ in

$$\overline{R}(i) + \delta_i R_i = \overline{R}(i) + \delta_i R + \delta_i (R_i - R) \to \lim_{i \to \infty} \overline{R}(i) = T.$$

This explains why we consider $J_\infty$ the right limit functional.

Having applied Lemma 4.2, we return to our proof of Lemma 4.1, and we have to distinguish three cases. In what follows, $J_{i,r}$ means the same functional as $J_i$, but with integration over $B_r$ instead of $B^3$.

The first case is $\sigma \in (0, 1)$. Then $(\varphi_\infty, R_\infty)$ can only minimize $J_{\infty, r}$ if $\varphi_\infty$ is a minimizer of $\int_{B_r} |P(T^i D\tilde{\varphi})|^2 \, dx$ and $R_\infty$ is a minimizer of $\int_{B_r} |D\tilde{R}|^p \, dx$. Then $\varphi_\infty$ solves an elliptic system with constant coefficients, and hence is Hölder continuous. And $R_\infty$ (now with values in a vector space, contrary to $R$) solves the $p$-Laplace system considered by Uhlenbeck [Uh] who also proved Hölder continuity. More precisely, we have the regularity estimates

$$\int_{B_{\theta}} |D\varphi_\infty|^2 \, dx \leq c \theta^{2\nu} \theta \int_{B_{\theta}} |D\varphi_\infty|^2 \, dx,$$

$$\int_{B_{\theta}} |DR_\infty|^p \, dx \leq c \theta^{\nu} \theta^{3-p} \int_{B_{\theta}} |DR_\infty|^p \, dx$$

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for all $\nu \in (0, 1)$, $\theta < \frac{1}{2}$, with $c$ depending only on $\nu$ (and $P$, which is considered fixed). The first of the estimates is a well-known standard estimate. Note that the equation $\varphi_\infty$ solves depends on $T$, but the constant $c$ does not, since it only depends on the ellipticity constant of the operator, i.e. only on $P$. The second estimate follows from Uhlenbeck's Hölder regularity result (saying that the $p$-harmonic $R_\infty$ is in $C^{0,\mu}$ for every $\mu \in (0, 1)$, since it is even in $C^{1,\alpha}$), using [Lu, Lemma 2(a)].

We now choose $\theta$ small enough in order to reach the desired contradiction. By norm convergence on $B_\theta$ and weak convergence on $B_1 = B^3$, we find, for $\nu := \frac{\mu+1}{2}$,

$$\lim_{i \to \infty} \frac{1}{\gamma_i^2 + \delta_i^p} ((\theta \rho_i)^{-1} \| D\varphi_{\theta i} \|_{L^2(B_{\theta \rho_i}(x_i))}^2 + (\theta \rho_i)^{p-3} \| DR_{\rho_i} \|_{L^p(B_{\theta \rho_i}(x_i))}^p)$$

$$= \lim_{i \to \infty} \frac{1}{\gamma_i^2 + \delta_i^p} (\theta^{-1} \gamma_i^2 \| D\varphi_i \|_{L^2(B_\theta)}^2 + \theta^{p-3} \delta_i^p \| DR_{\rho_i} \|_{L^p(B_\theta)}^p)$$

$$= \sigma \theta^{-1} \| D\varphi_\infty \|_{L^2(B_\theta)}^2 + (1 - \sigma) \theta^{p-3} \| DR_\infty \|_{L^p(B_\theta)}^p$$

$$\leq c \theta^{2\nu} (\sigma \| D\varphi_\infty \|_{L^2(B_1)}^2 + (1 - \sigma) \| DR_\infty \|_{L^p(B_1)}^p)$$

$$\leq c \theta^{2\nu}$$

(4)

if $\theta$ is taken small enough to have $c \theta^{1-\mu} \leq 1$. This contradicts our original assumption in the first case.

The second case is $\sigma = 0$, in which case we have a minimizer $R_\infty$ of $\int_{B_r} |D\tilde{R}|^p dx$, but the information on $\varphi_\infty$ has been lost in the limit. However, we still have the estimate (4), since the $\| D\varphi_\infty \|_{L^2(B_\theta)}$-term we now cannot estimate has the coefficient 0, anyway. The third case, $\sigma = 1$, uses the same arguments with $R$ taking the role of $\varphi$.

This proves the Lemma, up to Lemma 4.2 below.

The compactness lemma is proven almost exactly as in Luckhaus' paper, we formulate a sketchy proof only for the reader’s convenience.

**Lemma 4.2 (compactness)** In the proof of Lemma 4.1, the weakly convergent sequence $(\varphi_i, R_i)$ of $J_1$-minimizers converges even in $W^{1,2} \times W^{1,p}$ on every compact subset of $B^3$, and the limit $(\varphi_\infty, R_\infty)$ minimizes the limit functional $J_\infty$.

**Sketch of proof.** Let $\psi \in W^{1,2}(B^3, \mathbb{R}^3)$ and $Q \in W^{1,p}(B^3, N^\infty)$ be given such that $\psi = \varphi_\infty$ and $Q = R_\infty$ on some neighborhood of $\partial B^3$. Then (...) there exists $r$ close to 1 such that $\psi = \varphi_\infty$ and $Q = R_\infty$ almost everywhere on $\partial B_r$, and there exist $Q_i \in W^{1,p}(B_r, N_i)$ such that $Q_i \to Q$ in $W^{1,p}(B^3, \mathbb{R}^{3 \times 3})$,

$$\| D\psi \|_{L^2(\partial B_r)} + \sup_i (\| D\varphi_i \|_{L^2(\partial B_r)} + \| DQ_i \|_{L^p(\partial B_r)} + \| DR_{\rho_i} \|_{L^p(\partial B_r)}) =: K < \infty$$

and

$$\lim_{i \to \infty} \| Q_i - R_i \|_{L^p(\partial B_r)} = 0.$$ (6)

Now the “Luckhaus Lemma” (Lemma 1 from [Lu]) provides us with two sequences of functions $\zeta_i \in W^{1,2}(B^3, \mathbb{R}^3)$ and $P_i \in W^{1,p}(B^3, \mathbb{R}^{3 \times 3})$ as well as number sequences
These equations together with the assumptions give that the $\zeta_i$ are bounded in $W^{1,2}(B^3, \mathbb{R}^3)$ and the $P_i$ are bounded in $W^{1,p}(B^3, \mathbb{R}^{3 \times 3})$.

Denote by $J_{i,r}$ the variant of $J_i$ where integration is over $B_r$ instead of $B^3$. We apply the pointwise convergence in (3) together with the dominated convergence theorem in the first step, and weak lower semicontinuity in the second to infer

\[
J_{i,r}(\varphi_i, R_i) = \lim_{i \to \infty} \frac{\gamma_i^2}{\gamma_i^2 + \delta_i^p} \int_{B_r} |P((\mathcal{R}(i) + \delta_i R_i)^t D\varphi_i - \rho_i \gamma_i^{-1} I)|^2 \, dx \\
+ \lim_{i \to \infty} \frac{\delta_i^p}{\gamma_i^2 + \delta_i^p} \int_{B_r} |DR_i|^p \, dx.
\]

\[
\leq \liminf_{i,j \to \infty} \frac{\gamma_i^2}{\gamma_i^2 + \delta_i^p} \int_{B_r} |P((\mathcal{R}(i) + \delta_i R_i)^t D\varphi_j - \rho_i \gamma_i^{-1} I)|^2 \, dx \\
+ \liminf_{i,j \to \infty} \frac{\delta_i^p}{\gamma_i^2 + \delta_i^p} \int_{B_r} |DR_j|^p \, dx.
\]

\[
\leq \liminf_{i \to \infty} J_{i,r}(\varphi_i, R_i).
\]

(11)

Now we use that $(\varphi_i, R_i)$ is a $J_r$-minimizer and has the same boundary values on $\partial B_r$ as $(\zeta_i, \pi_i \circ P_i)$, where here $\pi_i$ is the nearest point retraction onto $N_i$. Since the $N_i$ are magnifications of $SO(3)$, the Lipschitz constants of $\pi_i$ (restricted to tubes of width 1, say, around $N_i$) are bounded independently on $i$. And $\pi_i \circ P_i$ differs from $P_i$ only on the annuli $B_r \setminus B_{(1-\lambda_i)r}$ where the $p$-energy of $P_i$ approaches 0. Hence the $p$-energies of $\pi_i \circ P_i$ and $P_i$ differ only by $o(1)$, and the same applies for the $p$-energies of $P_i$ and $Q_i$, as well as the $2$-energies of $\psi$ (independent from $i$) and $\zeta_i$. For example, we have

\[
\frac{\gamma_i^2}{\gamma_i^2 + \delta_i^p} \int_{B_r} \left( |P((\mathcal{R}(i) + \delta_i R_i)^t D\varphi_i - \rho_i \gamma_i^{-1} I)|^2 \\
- |P((\mathcal{R}(i) + \delta_i \pi_i \circ P_i)^t D\varphi_j - \rho_i \gamma_i^{-1} I)|^2 \right) dx
\]

\[
\leq c \frac{\gamma_i^2}{\gamma_i^2 + \delta_i^p} \int_{B_r \setminus B_{(1-\lambda_i)r}} (|D\varphi_i|^2 + \rho_i^2 \gamma_i^{-2}) dx
\]

\[
= c \frac{\rho_i^2}{\gamma_i^2 + \delta_i^p} \int_{B_{\rho_i(x_0)} \setminus B_{(1-\lambda_i)\rho_i(x_0)}} (|D\varphi|^2 + 1) dx
\]

\[
\leq c \int_{B_{\rho_i(x_0)} \setminus B_{(1-\lambda_i)\rho_i(x_0)}} (|D\varphi|^2 + 1) dx
\]

\[
\to 0.
\]
Together with similar estimates, we find $J_{i,r}(\varphi_i, R_i) = J_{i,r}(\zeta_i, \pi_i \circ P_i) + o(1)$. We combine this with the minimality of $(\varphi_i, R_i)$ to continue estimate (11) in

$$J_{\infty,r}(\varphi_\infty, R_\infty) \leq \liminf_{i \to \infty} J_{i,r}(\varphi_i, R_i) \leq \liminf_{i \to \infty} J_{i,r}(\zeta_i, \pi_i \circ P_i) \leq \liminf_{i \to \infty} J_{i,r}(\psi, Q_i) = J_{\infty,r}(\psi, Q),$$

(12)

where in the last step we have used $Q_i \to Q$ in $W^{1,p}$ and a similar reasoning as in (3) which shows $T(\zeta_i) + \delta_i Q_i \to T$ pointwise almost everywhere. This proves that $(\varphi_\infty, R_\infty)$ is $J_{\infty,r}$-minimizing with respect to its boundary values on $\partial B_r$. Since $r < 1$ can be chosen arbitrarily close to 1, we have that $(\varphi_\infty, R_\infty)$ is $J_{\infty,r}$-minimizing on every compact subset of $B_3$.

It is allowed to chose $(\psi, Q) = (\varphi_\infty, R_\infty)$ in (12), and this gives

$$\lim_{i \to \infty} J_{i,r}(\varphi_i, R_i) = J_{\infty,r}(\varphi_\infty, R_\infty),$$

which by strict convexity is easily seen to imply convergence in $W^{1,2} \times W^{1,p}$ on $B_r$. This completes the proof of Lemma 4.2.

It is now relatively standard to proceed from Lemma 4.1 to the following partial Hölder regularity statement.

**Proposition 4.3 (\(\varepsilon_0\)-regularity and partial Hölder continuity)** Assume $2 \leq p \leq 3$, $\mu \in (0, \frac{1}{p})$, and the assumptions made above. Then every $J$-minimizer $(\varphi, R) \in W^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is locally in $C^{1,\mu} \times C^{0,\mu}$ on $\Omega \setminus \Sigma$, where

$$\Sigma := \{x_0 \in \Omega : \liminf_{\rho \to 0} \rho^{-1} \|D\varphi\|_{L^2(B_{\rho/2}(x_0))}^2 + \rho^{3-p} \|DR\|_{L^p(B_{\rho/2}(x_0))}^p \geq \varepsilon_0\}$$

for some sufficiently small $\varepsilon_0 > 0$.

The set $\Sigma$ is relatively closed in $\Omega$, and $\mathcal{H}^1(\Sigma) = 0$.

**Proof.** The Hölder continuity of $R$ on $\Omega \setminus \Sigma$ is more or less exactly Luckhaus’ argument which is as follows. For the moment, let $\mu \in (0,1)$. For every $x_0 \in \Omega \setminus \Sigma$, there is some $s > 0$ such that

$$s^{-1} \|D\varphi\|_{L^2(B_s(x_0))}^2 + s^{p-3} \|DR\|_{L^p(B_s(x_0))}^p \leq 2\varepsilon_0. \quad (13)$$

The key is to prove the energy estimate

$$r^{-1} \|D\varphi\|_{L^2(B_r(x))}^2 + r^{p-3} \|DR\|_{L^p(B_r(x))}^p \leq C\varepsilon_0 \left(\frac{r}{s}\right)^{2\mu} \quad (14)$$

for every $x \in B_{s/2}(x_0)$ and every $r \in (0, \frac{s}{2})$. Once we have that, the Hölder continuity of $R$ on $B_{s/2}(x_0)$ follows using Morrey’s Dirichlet growth criterion.
To prove (14), we use Lemma 4.1, the \( \varepsilon_0 \) of which we denote by \( \tilde{\varepsilon}_0 \) here. Let \( s_0 := \min \{ \frac{s}{2}, \theta^{(p-3)/2\mu-1}\tilde{\varepsilon}_0^{1/2\mu} \} \). By (13), we have
\[
\|D\varphi\|_{L^2(B_{s_0}(x))}^2 + s_0^{p-3}\|DR\|_{L^p(B_{s_0}(x))}^p \leq C_0\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0
\]
for every \( x \in B_{s/2}(x_0) \) if we have chosen \( \varepsilon_0 \) accordingly. We abbreviate \( \Phi(r) := r^{-1}\|D\varphi\|_{L^2(B_r(x))}^2 + r^{p-3}\|DR\|_{L^p(B_r(x))}^p \). By induction, we prove the claim \( \Phi(\theta k s_0) \leq \theta^{2k\mu}\tilde{\varepsilon}_0 \) for all \( k \in \mathbb{N} \). For this clearly holds for \( k = 0 \), and let us assume that \( \Phi(\theta^{k-1}s_0) \leq \theta^{2(k-1)\mu}\tilde{\varepsilon}_0 \) has already been proved. Then either \( (\theta^{k-1}s_0)^{2\mu} < \Phi(\theta^{k-1}s_0) \), and Lemma 4.1 implies
\[
\Phi(\theta^k s_0) \leq \theta^{2\mu}\Phi(\theta^{k-1}s_0) \leq \theta^{2k\mu}\tilde{\varepsilon}_0,
\]
or \( (\theta^{k-1}s_0)^{2\mu} \geq \Phi(\theta^{k-1}s_0) \), and hence
\[
\Phi(\theta^k s_0) \leq \theta^{-3}\Phi(\theta^{k-1}s_0) \leq \theta^{-3}(\theta^{k-1}s_0)^{2\mu} = \theta^{2k\mu}\theta^{-3-2\mu}s_0^{2\mu} \leq \theta^{2k\mu}\tilde{\varepsilon}_0.
\]
Note that the smallness condition of Lemma 4.1 is fulfilled in every step. We have thus proven \( \Phi(\theta^k s_0) \leq \theta^{2k\mu}\tilde{\varepsilon}_0 \) for all \( k \), and this clearly implies (14), and hence the asserted Hölder continuity. More precisely, we have proven \( \varphi \in C^{0,\mu}_\text{loc} \) and \( R \in C^{0,\mu/2}_\text{loc} \) away from \( \Sigma \), and we will improve the statement about \( \varphi \) in a moment.

The dimension estimate for \( \Sigma \) then is a classical result, e.g. using [GM, Proposition 9.21]. What remains to be proven is the Hölder continuity of \( D\varphi \). Note that the Euler-Lagrange equation (1) for \( \varphi \) is a linear elliptic equation with coefficients and right-hand side depending on \( R \). Once we know Hölder continuity of \( R \), which we do, away from \( \Sigma \), we are in the realm of classical Schauder estimates which give us Hölder continuity of \( \varphi \) and even \( D\varphi \) whenever \( R \) is Hölder continuous. A version of that fact that fits our need precisely is [GM, Theorem 5.19] which reads as follows. Let \( u \in W^{1,2}_\text{loc}(\Omega,\mathbb{R}^m) \) be a solution to
\[
\partial_\alpha(A^{\alpha\beta}_{ij}\partial_\beta w^i) = -\partial_\alpha F^\alpha_i,
\]
with \( A^{\alpha\beta}_{ij} \in C^{0,\sigma}_\text{loc}(\Omega) \) satisfying the Legendre-Hadamard condition, for some \( \sigma \in (0,1) \). If \( F^\alpha_i \in C^{0,\sigma}_\text{loc}(\Omega) \), then we have \( Du \in C^{0,\sigma}_\text{loc}(\Omega) \).

This applies to \( \varphi \) and all \( \sigma < \frac{2}{p} \), and our theorem is proven.

**Remark.** The case \( p > 3 \) is much simpler, since then \( W^{1,p} \) already embeds into some Hölder space. We have full Hölder regularity of \( R \) in \( C^{0,p/3-1}_\text{loc}(\Omega) \) then, hence \( \varphi \in C^{1,p/3-1}_\text{loc}(\Omega) \), and the Hölder exponents can be improved by arguments from this section. We leave the details to the interested reader.

## 5 Dimension reduction for the singular set

We try to follow part (b) of Luckhaus’ theorem, where additional assumptions allow to prove that the singular set has smaller Hausdorff dimension than what can be estimated by the arguments of the previous section. Two things prevent us from applying Luckhaus’ theorem directly. Again, our integrand that has two summands
of different homogeneity is not allowed in Luckhaus’ assumptions, and his arguments have to be modified accordingly. Moreover, for the dimension reduction Luckhaus has to assume that the unknown functions take their values in a compact Riemannian manifold, which is not the case for our \( \varphi \). However, the case of \( \varphi \) being allowed to take values in all of \( \mathbb{R}^3 \) is sufficiently easy to be included in Luckhaus’ reasoning with only minor modifications.

In order to control in \( W^{1,2} \cap W^{1,p} \) some blowup sequence \((\varphi_t, R_t)\) we are going to use. we need a **monotonicity formula**. Such formulae have played a central role in the regularity theory for many functionals. The first monotonicity formula for harmonic maps has surfaced in the physics literature [89]. Our monotonicity formula controls \( r^{p-3} \int_{B_r(x_0)} |D\varphi|^2 \, dx \) and \( r^{p-3} \int_{B_r(x_0)} |DR|^p \, dx \) for minimizers \((\varphi, R)\) of \( J \). Note that \( r^{-1} \int_{B_r(x_0)} |D\varphi|^2 \, dx \) would be more natural, due to scaling invariance. But since we cannot deal with both integrands separately, we are forced to use the common factor \( r^{p-3} \) for both.

**Lemma 5.1 (monotonicity formula)** Assume \( p \in [2,3) \) and that \((\varphi, R) \in W^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))\) minimizes \( J \). Then, for every \( B_s(x_0) \subset \Omega \) and every \( r \in (0,s) \), we have

\[
(s^{p-3} + 1) \int_{B_s(x_0)} (|P(R^t D\varphi)|^2 + |DR|^p) \, dx - r^{p-3} \int_{B_r(x_0)} (|P(R^t D\varphi)|^2 + |DR|^p) \, dx
\geq \int_{B_s \setminus B_r} |x|^p (|DR|^{p-2} |\partial_{rad} R|^2 + Q(\varphi, R)) \, dx - cr^{p-1/2} (1 + \|D\varphi\|_{L^2(\Omega)}),
\]

where here

\[
Q(\varphi, R) := |P(R^t D\varphi)|^2 - |P(R^t (D\varphi - \frac{x-x_0}{|x-x_0|} \otimes \partial_{rad} \varphi))|^2 \geq 0.
\]

If \( p = 3 \), the formula holds without the first “+ 1”.

**Proof.** Let \( t \in (0,s) \). We abbreviate \( B_t := B_t(x_0) \) and assume \( x_0 = 0 \) to shorten notation. We compare \((\varphi, R)\) with \((\varphi_t, R_t) : B_s \rightarrow \mathbb{R}^3 \times SO(3)\) defined by

\[
\varphi_t(x) := \begin{cases} 
\varphi \left( \frac{t}{|x|} x \right) & \text{if } 0 < |x| < t, \\
\varphi(x) & \text{if } t \leq |x| < s,
\end{cases}
\]

\[
R_t(x) := \begin{cases} 
R \left( \frac{t}{|x|} x \right) & \text{if } 0 < |x| < t, \\
R(x) & \text{if } t \leq |x| < s.
\end{cases}
\]

By \( \partial_{rad} \) we denote the radial derivative in the direction of \( \frac{x}{|x|} \). Using the fact that \( |DR - \frac{x}{|x|} \otimes \partial_{rad} R|^2 = |DR|^2 - |\partial_{rad} R|^2 \), we calculate

\[
\frac{3-p}{t} \int_{B_t} |DR|^p = \frac{m-p}{t} \int_0^t \int_{\partial B_r} \left( |DR \left( \frac{t}{\tau} x \right)|^2 - \left| \partial_{rad} R \left( \frac{t}{\tau} x \right) \right|^2 \right)^{p/2} dH^2 \, d\tau
\leq \frac{3-p}{t} \int_0^t \int_{\partial B_r} |DR \left( \frac{t}{\tau} x \right)|^{p-2} \left( \left| DR \left( \frac{t}{\tau} x \right) \right|^2 - \left| \partial_{rad} R \left( \frac{t}{\tau} x \right) \right|^2 \right) dH^2(x) \, d\tau
\leq \frac{3-p}{t} \int_0^t \int_{\partial B_r} |DR|^{p-2} \left( |DR|^2 - |\partial_{rad} R|^2 \right) dH^2 \, d\tau
= \int_{\partial B_t} |DR|^{p-2} \left( |DR|^2 - |\partial_{rad} R|^2 \right) dH^2
= \frac{d}{dt} \int_{B_t} |DR|^p \, dx - \int_{\partial B_t} |DR|^{p-2} |\partial_{rad} R|^2 \, dH^2.
\]
The same way, but using \(|P(R'(D\varphi - \frac{\varepsilon}{\varepsilon} \otimes \partial_{rad} \varphi}))|^2 = |P(R'(D\varphi)|^2 - Q(\varphi, R)\) this time, we also have

\[
\frac{3 - p}{t} \int_{B_t} |P(R'_t D\varphi_t)|^2 \, dx \leq \frac{1}{t} \int_{B_t} |P(R'_t D\varphi_t)|^2 \, dx \\
\leq \frac{d}{dt} \int_{B_t} |P(R'_t D\varphi)|^2 \, dx - \int_{\partial B_t} Q(\varphi, R) \, dH^2. \quad (15)
\]

We use the fact that \((\varphi, R)\) minimizes \(J\) on \(B_t\) and coincides with \((\varphi_t, R_t)\) on \(\partial B_t\). This implies

\[
\int_{B_t} |P(R'(D\varphi)|^2 \, dx + \int_{B_t} |DR|^p \, dx \\
\leq \int_{B_t} |P(R'(D\varphi - I)|^2 \, dx + \int_{B_t} |P(I)|^2 \, dx \\
+ c\left( \int_{B_t} |P(R'(D\varphi - I)|^2 \, dx \right) \left. \int_{B_t} |P(I)|^2 \, dx \right)^{1/2} + \int_{B_t} |DR|^p \, dx \\
\leq \int_{B_t} |P(R'(D\varphi - I)|^2 \, dx + \int_{B_t} |DR|^p \, dx + c(t^3 + c\frac{3}{2})\|D\varphi\|_{L^2(\Omega)} \\
\leq \int_{B_t} |P(R'_t D\varphi_t)|^2 \, dx + \int_{B_t} |DR_t|^p \, dx + c(t^3 + c\frac{3}{2})\|D\varphi\|_{L^2(\Omega)} \\
\leq (1 + \varepsilon t) \int_{B_t} |P(R'_t D\varphi)|^2 \, dx + c\varepsilon^{-1} t^2 + \int_{B_t} |DR_t|^p \, dx \\
+ c(t^3 + c\frac{3}{2})\|D\varphi\|_{L^2(\Omega)}, \quad (16)
\]

where we have chosen \(\varepsilon > 0\) small enough and used (15) in the last step. Now we combine the inequalities and get

\[
\frac{d}{dt} \left( t^{p-3} \int_{B_t} (|P(R'(D\varphi)|^2 + |DR|^p) \, dx \right) \\
= t^{p-3} \left( \frac{d}{dt} \int_{B_t} (|P(R'(D\varphi)|^2 + |DR|^p) \, dx \right) - \frac{3 - p}{t} \int_{B_t} (|P(R'(D\varphi)|^2 + |DR|^p) \, dx \\
\geq t^{p-3} \left( \frac{d}{dt} \int_{B_t} (|P(R'(D\varphi)|^2 + |DR|^p) \, dx \right) - \frac{3 - p}{t} \int_{B_t} (|P(R'(D\varphi)|^2 + |DR|^p) \, dx \\
- t^{p-2} \frac{d}{dt} \int_{B_t} |P(R'(D\varphi)|^2 \, dx - ct^{p-3/2}(1 + \|D\varphi\|_{L^2(\Omega)}) \\
\geq t^{p-3} \int_{\partial B_t} (|DR|^p - |\partial_{rad} R|^2 + Q(\varphi, R)) \, dH^2 \\
- t^{p-3} \int_{B_t} |P(R'(D\varphi)|^2 \, dx - ct^{p-3/2}(1 + \|D\varphi\|_{L^2(\Omega)})
\]
Integrating from \( r \) to \( s \), we infer

\[
\begin{align*}
    s^{p-3} \int_{B_s} (|P(R_i^tD\varphi)|^2 + |DR_i|^p) \, dx - r^{p-3} \int_{B_r} (|P(R_i^tD\varphi)|^2 + |DR_i|^p) \, dx \\
    \geq \int_{B_r \setminus B_s} |x|^{p-3} (|DR_i|^{p-2} |\partial_{rad} R_i|^2 + Q(\varphi, R_i)) \, dx - \int_{B_r \setminus B_s} |P(R_i^tD\varphi)|^2 \, dx \\
    \quad - cs^{p-1/2}(1 + \|D\varphi\|_{L^2(\Omega)})
\end{align*}
\]

for \( 0 < r < s < \text{dist}(x_0, \partial\Omega) \). This proves the lemma.

The following proposition summarizes what we can infer from the monotonicity formula via a blow-up argument. We define the singular set \( \text{Sing}(\varphi, R) \) as the set of points in \( \Omega \) at which \( (\varphi, R) \) fails to be locally in \( C^{1,\mu} \times C^{0,\mu} \) for any \( \mu \in (0, 1) \).

**Proposition 5.2 (partial regularity and \( p \)-minimizing tangent maps)**

Assume \( p \in [2, 3] \) and that \( (\varphi, R) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3)) \) is a minimizer of \( J \) on \( \Omega \). Then the following statements hold.

(i) The singular set \( \text{Sing}(\varphi, R) \) is discrete in \( \Omega \). If \( p = 3 \), it is even empty.

(ii) If \( p \in (2, 3) \) and \( \text{Sing}(\varphi, R) \) is not empty, then there is at least one “\( p \)-minimizing tangent map” to \( SO(3) \), by which we mean a nonconstant continuous map \( R_\infty : B^3 \setminus \{0\} \to SO(3) \) which is radially constant and minimizes \( E^p(\tilde{R}) := \int_{B^3} |\tilde{D}\tilde{R}|^p \, dx \) among all \( W^{1,p} \)-maps to \( SO(3) \) with the same boundary values.

(iii) The same statement holds if \( p = 2 \) and \( \mu_1 = \mu_c = \mu_2 \).

The notion of \( p \)-minimizing tangent maps has been central in the regularity for \( p \)-harmonic maps. Like here, they are the obstacles to full regularity of \( p \)-harmonic maps and hence play an important role in [Lu] and in many other results on \( p \)-harmonic maps. And, of course, \( p \)-minimizing tangent maps are weakly \( p \)-harmonic maps themselves.

**Proof.** By Proposition 4.3, the singular set is a subset of the set \( \Sigma \) where “enough \( p \)-energy of \( R \) concentrates”. In what follows, we assume that \( x_0 \in \Sigma \). We denote rescaled versions of \( \varphi \) and \( R \) around \( x_0 \) by

\[
\varphi_i(x) := \rho_i^{p/2-1}(\varphi(x_0 + \rho_ix) - \bar{\varphi}(i)), \quad R_i(x) := R(x_0 + \rho_ix),
\]

where \( (\rho_i)_{i \in \mathbb{N}} \) is any strictly decreasing sequence with \( \rho_1 \) sufficiently small and \( \rho_i \searrow 0 \). Then \( (\varphi_i, R_i) \) minimizes

\[
J_i(\tilde{\varphi}, \tilde{R}) := \int_{B^3} (|P(\tilde{R}_i^tD\tilde{\varphi} - \rho_i^{p/2}I)|^2 + |D\tilde{R}_i|^p) \, dx,
\]

and this explains the scaling chosen, because the monotonicity formula from Lemma 5.1 implies that \( J_i(\varphi_i, R_i) \) stays bounded as \( i \to \infty \).

Assuming \( \rho_i \searrow 0 \), the formal limit functional is

\[
J_\infty(\tilde{\varphi}, \tilde{R}) := \int_{B^3} (|P(\tilde{R}^tD\tilde{\varphi})|^2 + |D\tilde{R}|^p) \, dx.
\]
Note that this time all \( R_i \) map to \( SO(3) \) rather than to rescaled and shifted copies of it. The bound on \( J_i(\varphi_i, R_i) \) and the fact that \( \varphi_i \) has mean value 0 imply that \( (\varphi_i, R_i) \) is bounded in \( W^{1,2}(B^3, \mathbb{R}^3) \times W^{1,\infty}(B^3, SO(3)) \). After passing to a subsequence, we can therefore assume that \( (\varphi_i, R_i) \rightharpoonup (\varphi_\infty, R_\infty) \) weakly in \( W^{1,2} \times W^{1,\infty} \) for some \((\varphi_\infty, R_\infty) \in W^{1,2}(B^3, \mathbb{R}^3) \times W^{1,\infty}(B^3, SO(3)) \).

By the same reasoning as in Lemma 4.2 (which we do not work out since it is sufficiently close to \([\text{Lu}]\)), we have convergence \((\varphi_i, R_i) \rightarrow (\varphi_\infty, R_\infty)\) in \( W^{1,2} \times W^{1,\infty} \), and \((\varphi_\infty, R_\infty)\) minimizes \( J_\infty \) with respect to its boundary values.

Inserting \( s = \rho_i, r = \rho_j \) into Lemma 5.1 and rescaling, we have

\[
(1 + \rho_i^{3-p}) \int_{B_1} (|P(R^t D\varphi_i)|^2 + |DR_i|^p) \, dx - \int_{B_1} (|P(R^t D\varphi_j)|^2 + |DR_j|^p) \, dx \\
\geq \int_{B_1 \setminus B_{\rho_j/\rho_i}} |x|^{-3} (|DR_i|^{-2} |\partial_{rad} R_i|^2 + Q(\varphi_i, R_i)) \, dx - c\rho_j^{-1/2}
\]

for \( 2 \leq p < 3 \), and the same with \( (1 + \rho_i^{3-p}) \) replaced by 1 for \( p = 3 \). Letting \( j \rightarrow \infty \) and then \( i \rightarrow \infty \), the norm convergence gives

\[
\int_{B^3} |x|^{-3} (|DR_\infty|^{-2} |\partial_{rad} R_\infty|^2 + Q(\varphi_\infty, R_\infty)) \, dx = 0.
\]

Hence \( \partial_{rad} R_\infty \equiv 0 \), and \( Q(\varphi_\infty, R_\infty) \equiv 0 \), which also implies \( \partial_{rad} \varphi_\infty \equiv 0 \). This means that both \( \varphi_\infty \) and \( R_\infty \) are radially constant. And by the definition of \( \Sigma \) and the norm convergence, \((\varphi_\infty, R_\infty)\) is not constant.

Our first aim is to prove that the singular set of \((\varphi, R)\) is discrete. The strategy for that is taken from \([\text{Lu}]\), but goes back to “Federer’s dimension reduction argument” from \([\text{Fe}]\). This is clearly a local question that we can answer by considering only a neighborhood of some point in \( \Omega \). Therefore we restrict to \( \Omega = B^3 \) and define the “\( \varepsilon_0 \)-singular set” of any \((\varphi, R)\) by

\[
S(\varphi, R; \varepsilon_0) := \left\{ x_0 \in B^3 : |x_0| < \frac{1}{2}, \right. \\
\left. r^{-1} \|D\varphi\|^2_{L^2(B_r(x_0))} + r^{-3} \|DR\|^p_{L^p(B_r(x_0))} \geq \varepsilon_0 \text{ for } 0 < r < 1 - |x_0| \right\}.
\]

By Proposition 4.3, it is sufficient to show that \( S(\varphi, R; \varepsilon_0) \) is discrete for \( \varepsilon_0 > 0 \) chosen sufficiently small.

Suppose the contrary, then there is an \( a \in S(\varphi, R; \varepsilon_0) \) that is the limit of a sequence \( \{a_i\} \in \mathbb{N} \) in \( S(\varphi, R; \varepsilon_0) \) \( \setminus \{a\} \). Then define rescaled mappings \((\varphi_i, R_i)\) as above (with \( x_0 \) replaced by \( a \)), for some \( \rho_i \searrow 0 \) that the sequence converges to a radially constant minimizer \((\varphi_\infty, R_\infty)\). Since the integrals transform naturally under the scaling involved in the definition of \( \varphi_i \) and \( R_i \), we find that \(|x_0| \leq \frac{1}{2} \) and \( a + \rho_i^{-1} x_0 \in S(\varphi, R; \varepsilon_0) \) imply \( x_0 \in S(\varphi_i, R_i; \varepsilon_0) \).

And we can also assume that we have chosen the \( \rho_i \) in such a way that the sequence \( \rho_i (a_i - a) \) has an accumulation point \( x_a \) with \( |x_a| = \frac{1}{2} \). Then we can verify that \( x_a \in S(\varphi_\infty, R_\infty; \varepsilon_0) \). But \( x_a \neq 0 \), and \((\varphi_\infty, R_\infty)\) is radially constant, and nonconstant because of the definition of \( S(\varphi, R, \varepsilon) \) and the norm convergence. Therefore, we have
\[ R x_a \subseteq S(\varphi_\infty, R_\infty; \varepsilon_0), \] but the \( \varepsilon_0 \)-singular set of a \( W^{1,p} \)-mapping for \( p \geq 2 \) in three dimensions always has vanishing one-dimensional Hausdorff measure. This is a contradiction, and we have proved Assertion (i) that the \( \varepsilon_0 \)-singular set of \( (\varphi, R) \), and hence also the singular set, is discrete. And as the \( \varepsilon_0 \)-singular set of a \( W^{1,3} \)-mapping always vanishes in three dimension, the singular set is empty in case \( p = 3 \).

Now let us assume \( p \in (2, 3) \) and prove (ii). To this end, we remark that \( (\varphi_\infty, R_\infty) \) minimizes \( J_\infty \), a functional very similar to \( J \) except for being less inhomogeneous for the lack of the term involving \( I \). In particular, a monotonicity formula for minimizers of \( J_\infty \) is proven exactly as for those of \( J \), the proof being actually slightly shorter since there is one term less involved. That monotonicity formula allows us to blow up \( (\varphi_\infty, R_\infty) \) again around 0, finding that, for every \( i \in \mathbb{N} \), the pair

\[ \varphi_{\infty,i}(x) := \rho_{i}^{p/2-1} \varphi_\infty(\rho_i x), \quad R_{\infty,i}(x) := R_\infty(\rho_i x) \]

minimizes \( J_\infty \) with respect to its boundary values. The compactness argument employed above gives that the weak limit again minimizes \( J_\infty \), but by the radial homogeneity of \( \varphi_\infty \) and \( R_\infty \), we have

\[ (\varphi_{\infty,i}, R_{\infty,i}) \to (0, R_\infty), \]

This means that \( (0, R_\infty) \) is a \( J_\infty \)-minimizer, which of course implies that \( R_\infty \) minimizes \( \int_{B^3} |D \tilde{R}|^p \, dx \) and hence is a weakly \( p \)-harmonic map. By its properties already known, including \( R_\infty \) being nonconstant by repetition of the argument above, it is a \( p \)-minimizing tangent map, and we have proved (ii).

The argument we just performed breaks down for \( p = 2 \), since this time we have actually taken advantage of the inhomogeneity of the functional. We do not know whether the same statement can be proven for \( p = 2 \), except for the special case when \( \mu_1 = \mu_c = \mu_2 \). In that case, the functional \( J_\infty \) equals

\[ J_\infty(\tilde{\varphi}, \tilde{R}) = \int_{B^3} (\mu_1 |D \tilde{\varphi}|^2 + |D \tilde{R}|^p) \, dx, \]

in which \( \tilde{\varphi} \) and \( \tilde{R} \) are “decoupled”. Which means that \( (\varphi_\infty, R_\infty) \) minimizes \( J_\infty \) iff \( \varphi_\infty \) minimizes \( \|D \tilde{\varphi}\|_{L^2(B^3)}^2 \) and \( R_\infty \) minimizes \( \|D \tilde{R}\|^p_{L^p(B^3)} \). This allows the same conclusion as in (ii) and hence proves (iii), which completes the proof of Proposition 5.2.

\[ \square \]

6 Nonexistence of \( p \)-harmonic minimizing tangent maps

We have seen in Proposition 5.2 that, in order to exclude point singularities, we need to know the non-existence of \( p \)-minimizing tangent maps (now abbreviated as \( p \)-mtm) \( B^3 \setminus \{0\} \to SO(3) \). The first thing we claim is that it suffices to exclude \( p \)-mtm to \( S^3 \) instead.
Lemma 6.1 (lift of p-mtm to $S^3$) Assume $p > 1$, and that there is a p-mtm $R : B^3 \setminus \{0\} \to SO(3)$. Then there also is a p-mtm $u : B^3 \setminus \{0\} \to S^3$.

Proof. This is due to the fact that $S^3$ is the universal cover of $SO(3)$. A locally isometric (up to a constant factor) 2-to-1 covering map $\pi : S^3 \to SO(3)$ is given by

$$\pi(w, x, y, z) := \begin{pmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2zw & 2xz + 2yw \\ 2xy + 2zw & 1 - 2x^2 - 2z^2 & 2yz - 2wx \\ 2xz - 2yw & 2yz + 2wx & 1 - 2x^2 - 2y^2 \end{pmatrix}.$$ 

Since $B^3 \setminus \{0\}$ is simply connected, there is a lift $u : B^3 \setminus \{0\} \to S^3$ of our given map $R$, satisfying $\pi \circ u = R$. Certainly, $u$ cannot be constant if $R$ is not. And $u$ minimizes $E^p$ with respect to its own boundary values on $S^2$. Assume it does not, then we had $v : B^3 \to S^3$ with $v = u$ on $S^2$ which has $E^p(v) < E^p(u)$. But then, since $\pi$ is a constant times an isometry, $E^p(\pi \circ v) < E^p(\pi \circ u) = E^p(R)$ with $\pi \circ v = R$ on $S^2$, which means $R$ would not be minimizing. This is a contradiction, hence $u$ minimizes $E^p$. It is also radially constant. Summarizing, we have found that $u$ is a p-mtm. \qed

Remark. The proof gives some interpretation of the singular example for the Cosserat model given in Section 2. The mapping $R$ there is just the equator map $(\frac{x}{|x|}, 0) : B^3 \to S^3$ projected to $SO(3)$ using $\pi$.

The Lemma means that we know Hölder continuity of minimizers in our Cosserat model once we can exclude the existence of p-mtm $B^3 \setminus \{0\} \to S^3$. But this has been done for $p = 2$.

Proposition 6.2 (a complete regularity case for $p = 2$) Assume $p = 2$, $\mu_1 = \mu_\infty = \mu_2$, and that $(\varphi, R) \in W^{1,2}(\Omega, \mathbb{R}^3 \times SO(3))$ is a minimizer of $J$ on $\Omega$. Then $(\varphi, R) \in C^{1,\mu}_{\text{loc}}(\Omega, \mathbb{R}^3) \times C^{0,\mu}_{\text{loc}}(\Omega, SO(3))$ for every $\mu \in (0, 1)$.

Proof. This is a combination of Proposition 4.3, Proposition 5.2 and the nonexistence of 2-mtm $B^3 \setminus \{0\} \to S^3$. The latter has been proven by Schoen and Uhlenbeck in [SU2, Theorem 2.7]. \qed

In order to apply part (ii) of Proposition 5.2, we would like to know about nonexistence of p-mtm $B^3 \setminus \{0\} \to S^3$. Unfortunately, the proof of [SU2] does not seem to give a hint on how to handle that, because it uses the fact that harmonic 2-spheres are very closely related to minimal immersions, an argument that does not carry over to p-harmonic maps for any $p \neq 2$. Okayasu [Ok] has published a modification of the [SU2]-argument for the target $S^3$ (and others), based on a so-called improved Kato inequality. He proves that, for harmonic maps $S^{k-1} \to N$, the trivial pointwise inequality $|D|Du| \leq |\nabla Du|$ can be improved to $|D|Du|^2 \leq \frac{\kappa_2}{\kappa_1} |\nabla Du|^2$, and the constant is optimal.

If we had an improved Kato-type inequality for p-harmonic maps, we could try to improve regularity theorems from Xin and Yang [XY] or Nakauchi [Na] the way Okayasu improved [SU2]. Unfortunately, no optimal Kato inequality for p-harmonic
maps seems to be available. There is, however, an optimal one for \( p \)-harmonic functions (i.e., maps to \( \mathbb{R} \)) that can give some orientation. It has been proven recently by Chang, Chen, and Wei [CCW, Lemma 5.4] and reads \(|\nabla Du|^2 \geq (1 + \tilde{\kappa})|D|Du||^2\), where here \( \tilde{\kappa} := \min\{\left(\frac{p-1}{m}\right)^2, 1\} \), and \( m \) is the domain dimension. We do not see how the proof could carry over to the \( S^3 \)-valued case, but we do get some improvement of Kato’s inequality, which is probably not optimal, but for \( p \searrow 2 \) reproduces Okayasu’s result.

**Lemma 6.3 (improved Kato inequality for \( p \)-harmonic maps)** Let \( p > 1 \), and let \( M \) and \( N \) be smooth complete Riemannian manifolds, \( m := \dim M \). Fix some parameter \( \varepsilon \in (0, 1] \), let

\[
\kappa := \frac{m - 1 + (\frac{1}{\varepsilon} - 1)(p - 2)^2}{m - \varepsilon} \quad \text{(which is } > \frac{1}{2}\text{)},
\]

and assume that \( u \in C^{1,\alpha}(M, N) \) is a \( p \)-harmonic mapping (which is automatically \( C^2 \) away from the points with \( Du(x) = 0 \)). Then at any \( x \in M \) with \( Du(x) \neq 0 \), we have

\[
|D|Du||^2 \leq \kappa |\nabla Du|^2,
\]

where \( \nabla \) denotes the Levi-Civita connection of \( N \).

**Proof.** By Nash’s embedding theorem, \( N \) (or some compact portion of it around \( u(x) \)) is embedded isometrically into some \( \mathbb{R}^n \). We write \( \nabla_i \) for partial derivatives with respect to the Levi-Civita connection of \( N \). The \( p \)-harmonic map equation can be written as

\[
\sum_i \nabla_i (|Du|^{p-2} \partial_i u) = 0,
\]

which is equivalent to

\[
\sum_i \nabla_i \partial_i u = \frac{p - 2}{|Du|^2} \sum_{i,j} \langle \nabla_i \partial_j u, \partial_j u \rangle \partial_i u.
\]

This implies, abbreviating \( \sum_i \nabla_i \partial_i := \tau \), known as the tension field,

\[
|\tau(u)|^2 \leq (p - 2)^2 |\nabla Du|^2.
\]

We now fix \( x \in M \) and some index \( \alpha \in \{1, \ldots, n\} \) and find an ONB \( \{b_1, \ldots, b_n\} \) of \( T_x M \) (depending on \( \alpha \)) such that \( \partial_1 u^\alpha = |Du^\alpha|b_1 \), which implies \( \partial_1 u^\alpha(x) = |Du^\alpha(x)| \) and \( \partial_j u^\alpha(x) = 0 \) for \( j \neq 1 \). This idea is from [CCW]. By using appropriate coordinates, we can also assume \( \nabla_i \partial_j u^\alpha(x) = \nabla_j \partial_i u^\alpha(x) \) for all \( i, j \).

\[
\sum_{i,j=1}^m (\nabla_i \partial_j u^\alpha)^2 \geq (\nabla_1 \partial_1 u)^2 + 2 \sum_{h=2}^m (\nabla_h \partial_1 u^\alpha)^2 + \sum_{h=2}^m (\nabla_h \partial_h u^\alpha)^2
\]

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Absorbing the last term into the left-hand side proves the lemma. 

\[
\begin{align*}
\geq & \ (\nabla_1 \partial_1 u)^2 + 2 \sum_{h=2}^{m} (\nabla_h \partial_1 u^\alpha)^2 + \frac{1}{m-1} \left( \sum_{h=2}^{m} \nabla_h \partial_h u^\alpha \right)^2 \\
= & \ (\nabla_1 \partial_1 u)^2 + 2 \sum_{h=2}^{m} (\nabla_h \partial_1 u^\alpha)^2 + \frac{1}{m-1} (\tau(u)^\alpha - \nabla_1 \partial_1 u^\alpha)^2 \\
\geq & \ \frac{m-\varepsilon}{m-1} (\nabla_1 \partial_1 u^\alpha)^2 + 2 \sum_{h=2}^{m} (\nabla_h \partial_1 u^\alpha)^2 - \frac{\varepsilon-1-1}{m-1} (\tau(u)^\alpha)^2.
\end{align*}
\]

In the last line, we have applied Young’s inequality. Using \(2 \geq \frac{m-\varepsilon}{m-1}\) and \((\nabla_h \partial_1 u^\alpha)^2 = (\partial_h|Du^\alpha|)^2\), the last estimate becomes

\[
\sum_{i,j=1}^{m} (\nabla_i \partial_j u^\alpha)^2 \geq \frac{m-\varepsilon}{m-1} |D|Du^\alpha| |Du^\alpha| - \frac{\varepsilon-1-1}{m-1} (\tau(u)^\alpha)^2. \tag{18}
\]

We have

\[
\begin{align*}
|D|Du| &= \left| D \sqrt{\sum_{\alpha} |Du^\alpha|^2} \right|^2 = \left( \sum_{\alpha} |D|Du^\alpha| |Du^\alpha| \right)^2 \\
& \leq \sum_{\alpha} |D|Du^\alpha| \sum_{\alpha} |Du^\alpha|^2 = \sum_{\alpha} |D|Du^\alpha|^2
\end{align*}
\]

and can therefore sum over \(\alpha\) in (18). Using also (17), we get

\[
|\nabla Du| \geq \frac{m-\varepsilon}{m-1} |D|Du| - \frac{\varepsilon-1-1}{m-1} |\tau(u)|^2 \\
\geq \frac{m-\varepsilon}{m-1} |D|Du| - \frac{\varepsilon-1-1}{m-1} (p-2)^2 |\nabla Du|^2.
\]

Absorbing the last term into the left-hand side proves the lemma.

**Remark.** If \(p-2\) is moderately large, there is no choice for \(\varepsilon > 0\) that makes \(\kappa < 1\). Therefore, Lemma 6.3 must be seen as a tool only for \(p\) close to 2. If \(p > 2\) is close to 2, the optimal choice of \(\varepsilon\), the one that makes \(\kappa\) smallest, is

\[
\varepsilon := \frac{1}{m-1-(p-2)^2} \left( (p-2)^2 + \sqrt{(p-2)^4 + m(m-1-(p-2)^2)(p-2)^2} \right).
\]

For \(p \ll 2\), we have \(\varepsilon \sim \sqrt{\frac{m}{m-1}} (p-2)\). \(\square\)

Now we use a slight modification of the arguments in [XY] or [Na] — the latter is slightly easier to cite for our purposes. We assume that \(u : B^k \setminus \{0\} \rightarrow S^n\) is a minimizing \(p\)-harmonic tangent map. Then we use [Na, Lemma 1], which is derived from the stability inequality,

\[
\int_{S^{k-1}} |Du|^{p-2} |D|Du| dx \geq \frac{n-p}{n+p-2} \int_{S^{k-1}} |Du|^{p+2} dx - \frac{(k-p)^2}{4} \int_{S^{k-1}} |Du|^p dx. \tag{19}
\]

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The other ingredient is [Na, Lemma 2], which we cite in a modified version. We use Nakauchi’s derivation of the lemma, but do not use $|\nabla Du| \geq |D|Du|$, and find
\[
\int_{S^{k-1}} |Du|^{p-2}(|\nabla Du|^2 + (p - 2)|D|Du|^2) \, dx
\leq \frac{k - 2}{k - 1} \int_{S^{k-1}} |Du|^{p+2} \, dx - (k - 2) \int_{S^{k-1}} |Du|^p \, dx.
\]

Using the improved Kato inequality from Lemma 6.3, we find
\[
\left( \frac{k - 1 - \varepsilon}{k - 2 + (\varepsilon^{-1} - 1)(p - 2)^2} + p - 2 \right) \int_{S^{k-1}} |Du|^{p-2}|D|Du|^2 \, dx
\leq \frac{k - 2}{k - 1} \int_{S^{k-1}} |Du|^{p+2} \, dx - (k - 2) \int_{S^{k-1}} |Du|^p \, dx.
\] (20)

Comparing (20) with (19), we could find slight improvements of the results of [XY] and [Na], but we do not bother to write them down in general form. Instead, we concentrate on the case we need to exclude point singularities in the Cosserat model, and for that we need only consider the special case $k = n = 3$. Combining (19) and (20), we then have
\[
\left( \frac{3 - p}{1 + p} \left( \frac{2 - \varepsilon}{1 + (\varepsilon^{-1} - 1)(p - 2)^2} + p - 2 \right) - \frac{1}{2} \right) \int_{S^2} |Du|^{p+2} \, dx
\leq \left( \frac{(3 - p)^2}{4} \left( \frac{2 - \varepsilon}{1 + (\varepsilon^{-1} - 1)(p - 2)^2} + p - 2 \right) - 1 \right) \int_{S^2} |Du|^p \, dx.
\]

If the coefficient on the left-hand side is $> 0$ while the one on the right-hand side is $\leq 0$, then $u$ must be constant, hence there exist no $p$-mtm. (This is the key idea from [SU2].)

Using the optimal $\varepsilon$ from the previous remark, this condition is easily verified numerically for $2 \leq p \leq \frac{32}{15}$, where the upper bound is not quite optimal and set to a fraction only for simplicity — we do not expect our method of proof to be optimal, anyway.

This means that for $2 \leq p \leq \frac{32}{15}$, there are no nonconstant $p$-minimizing tangent maps $B^3 \setminus \{0\} \to S^3$. Combining that with Proposition 4.3, Proposition 5.2 (ii), and Lemma 6.1, we have proven the following result.

**Proposition 6.4 (Hölder regularity for $p > 2$ close to 2)** Assume $p \in (2, \frac{32}{15}]$ and that $(\varphi, R) \in W^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3))$ is a minimizer of $J$ on $\Omega$. Then $(\varphi, R) \in C^{1,\mu}(\Omega, \mathbb{R}^3) \times C^{0,\mu}(\Omega, SO(3))$ for every $\mu \in (0, \frac{2}{p})$.

**Remark.** From the Propositions 6.2 and 6.4, we read off that the singular example from Section 2 cannot minimize $J$ if $p \in [2, \frac{32}{17}]$. [\square]
7 External forces and moments

Now we return to the case where the functions \( f \) and \( M \) do not vanish. That is, we again consider the functional

\[
J(\varphi, R) := \int_{\Omega} \left( |P(R^t D\varphi - I)|^2 + |DR|^p + \varphi \cdot f + R \cdot M \right) dx,
\]

where we have omitted \(- \int x \cdot f\) which only gives an additive constant and is therefore irrelevant for minimizing.

Then we have to care for some more lower order terms which do not really affect the reasoning of the previous sections. First of all, the monotonicity allows for force and moment terms in the natural Lebesgue spaces the functional allows.

**Lemma 7.1 (more general monotonicity formula)** Additionally to the assumptions of Lemma 5.1, let functions \( f \in L^2(\Omega, \mathbb{R}^3) \) and \( M \in L^q(\Omega, \mathbb{R}^{3 \times 3}) \) be given, where here \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \((\varphi, R)\) be a minimizer of the functional \( J \) now involving the corresponding force and moment potentials. Then the monotonicity formula from Lemma 5.1 still holds after it has been modified by an additional term

\[
-cs^{p-1}(\|D\varphi\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} + \|DR\|_{L^p(\Omega)} \|M\|_{L^q(\Omega)})
\]

on the right-hand side.

**Proof.** The change to be made in the proof of Lemma 5.1 is in the estimate (16) where \((\varphi, R)\) is compared with \((\varphi_t, R_t)\). Here we have to add a term \( \int_{B_t} ((\varphi_t - \varphi) \cdot f + (R_t - R) \cdot M \) to the right-hand side, which in the sequel is estimated according to

\[
\int_{B_t} ((\varphi_t - \varphi) \cdot f + (R_t - R) \cdot M ) dx
\]

\[
\leq \|\varphi_t - \varphi\|_{L^2(B_t)} \|f\|_{L^2(\Omega)} + \|R_t - R\|_{L^p(B_t)} \|M\|_{L^q(B_t)}
\]

\[
\leq ct(\|D\varphi\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} + \|DR\|_{L^p(\Omega)} \|M\|_{L^q(\Omega))}.
\]

This term is easily carried through the remaining estimates in the proof of the monotonicity formula.

For the regularity theory, we restrict to Hölder continuous data, which should be good enough for most applications. In comparison to the case without exterior forces, we have to compromise about the Hölder exponents.

**Proposition 7.2 (regularity with exterior forces and moments)** The statements of the Propositions 4.3, 5.2, 6.2, and 6.4 continue to hold in the case of nonvanishing \( f \) and \( M \) if we assume \( f \in C^0,\mu(\overline{\Omega}, \mathbb{R}^3) \) and \( M \in C^0(\overline{\Omega}, \mathbb{R}^{3 \times 3}) \). However, if \( f \neq 0 \), we have to restrict the Hölder exponent \( \mu \) to \((0, \frac{1}{2p})\) instead of \((0, \frac{p}{2})\).

**Sketch of proof.** We have to check the arguments of the proofs wherever we have used the minimality of \((\varphi, R)\) or the Euler-Lagrange equations. As a matter of fact,
this does not affect our reasoning too much, since the proofs use blowup procedures, and the new potential terms are scaled away in the blowup processes. Hence the limit functionals are the same as in the case where \( f \) and \( M \) vanish.

To see how that works, let us first have a look into the proof of Lemma 4.1. The functional that \( (\varphi_i, R_i) \) minimizes now is a modified version of the \( J_i \) given there, namely

\[
J_i(\tilde{\varphi}, \tilde{R}) := \frac{\gamma_i^2}{\gamma_i^2 + \delta_i^p} \int_{B^3} |P((R_i(i) + \delta_i \tilde{R})^4 D\tilde{\varphi} - \rho_i \gamma_i^{-1} I)|^2 \, dx + \frac{\delta_i^p}{\gamma_i^2 + \delta_i^p} \int_{B^3} |D\tilde{R}|^p \, dx \\
+ \frac{\rho_i^2}{\gamma_i^2 + \delta_i^p} \int_{B^3} (\varphi_i + \gamma_i \tilde{\varphi}(x) \cdot f(x_i + \rho_i x)) \, dx \\
+ \frac{\rho_i^2}{\gamma_i^2 + \delta_i^p} \int_{B^3} (\tilde{R}(i) + \delta_i \tilde{R}(x)) \cdot M(x_i + \rho_i x) \, dx.
\]

Remember \( \rho_i^{2\mu} \leq \gamma_i^2 + \delta_i^p \), and \( \mu < 1 \), hence the coefficient \( \frac{\rho_i^2}{\gamma_i^2 + \delta_i^p} \) of the last two integrals vanishes in the limit \( i \to \infty \). And \( (\varphi_i, R_i) \) minimizes the original \( J_\infty \) without any additional terms once we can prove that the last two integrals in \( J_i(\varphi_i, R_i) \) are bounded uniformly in \( i \). But they are, because first of all, we have

\[
\left| \int_{B^3} (\tilde{R}(i) + \delta_i \tilde{R}(x)) \cdot M(x_i + \rho_i x) \, dx \right| = \left| \rho_i^{-3} \int_{B_{\rho_i}(x_i)} R \cdot M \, dx \right| \leq c \| M \|_{C^0(\Omega)}.
\]

For the other integral, things are slightly more involved, and we need \( \mu < \frac{1}{4} \) in Lemma 4.1, which corresponds to \( \mu < \frac{1}{2p} \) in Proposition 4.3. If \( \mu < \frac{1}{4} \), we have \( \frac{\rho_i^{1/2}}{\gamma_i^2 + \delta_i^p} \to 0 \), and it is sufficient to bound \( \rho_i^{3/2} \) times the integral. We have

\[
\rho_i^{3/2} \left| \int_{B^3} (\varphi_i + \gamma_i \tilde{\varphi}(x) \cdot f(x_i + \rho_i x)) \, dx \right| \\
= \rho_i^{-3/2} \left| \int_{B_{\rho_i}(x_i)} \varphi \cdot f \, dx \right| \leq c \| \varphi \|_{L^2(B_{\rho_i}(x_i))} \| f \|_{C^0(\Omega)} \leq c \| \varphi \|_{L^2(\Omega)} \| f \|_{C^0(\Omega)}.
\]

Note that the estimate depends on \( \| \varphi \|_{L^2(\Omega)} \) for our fixed minimizer \( (\varphi, R) \). But it is used in a term that vanishes in the limit, anyway, and we can check the proof of Lemma 4.1 to find that the constants in its statement still do not depend on \( \| \varphi \|_{L^2(\Omega)} \).

In the blow-up performed in the proof of Proposition 5.2, there is a similar reasoning. Here, two integrals have to be added to the functional \( J_i \) that correspond to \( \rho_i^{3-p} \int_{B_i(x_0)} \varphi \cdot f \, dx \) and \( \rho_i^{3-p} \int_{B_i(x_0)} R \cdot M \, dx \), both of which vanish as \( i \to \infty \), even if \( p = 3 \).

There is one more change necessary in the proof of Proposition 4.3, where we apply Schauder theory in order to get Hölder continuity of \( D\varphi \). The equation considered for this now reads \( \text{div}(R^2(D\varphi - I)) = f \), with \( f \in C^{0,\mu} \) instead of 0. But this additional right-hand side in \( C^{0,\mu} \) does not affect the Schauder estimates, which are known to hold also for \( f_i - \partial_\alpha F_i^\alpha \) instead of just \( -\partial_\alpha F_i^\alpha \) if the \( f_i \) and \( F_i^\alpha \) are in \( C^{0,\mu} \).

Apart from that, there are no serious changes in the proofs, which can therefore be adapted to prove Proposition 7.2. 

\[ \square \]
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