A Closed Bianchi I Universe in String Theory

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Abstract

A special Bianchi I universe is constructed in 4D string theory. Geometrically it represents a 3D anti-de-Sitter space crossed with a flat direction whereas in terms of an associated conformal field theory it is an extremal case of a gauged WZNW theory with target the coset $SU(1,1) \times R^2/R$. Some of its properties are discussed.

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Recently investigations of string theory have converged on the study of nontrivial backgrounds, in hope that new information on the structure of quantum gravity may be obtained. A method which has proven particularly helpful in discovering novel solutions is the construction of exact conformal field theories on the world-sheet via gauging the Wess-Zumino-Witten-Novikov (WZWN) sigma model defined on a group of appropriate topology. Since Witten’s demonstration\(^1\) that such approach, in case of the \(SL(2, R)\) group on a \(2D\) space-time, yields black hole solutions, a plethora of new configurations have been obtained. These solutions turn out to admit both black hole (or black string)\(^{1-6}\) and cosmological interpretations\(^{7-11}\).

In this letter I will consider a new conformal field theory on the world sheet, which can be understood as a diagonal Bianchi I cosmology. It turns out to be a closed, anisotropic universe, which recollapses after existing only a finite time, much like the recently constructed example of Nappi and Witten\(^{10}\). They have found a closed recollapsing universe constructed as a conformal field theory on the coset \(SU(1, 1) \times SU(2)/R^2\) and investigated its singularity structure. However, this solution is free of curvature singularities, as its global structure is that of a \(S^1 \times R^2\) anti-de-Sitter space-time crossed with a flat direction \(R\). It is plagued with singularities in its causal structure, as closed timelike loops exist in it. It represents an extremal case of the gauged WZWN model on the coset \(SU(1, 1) \times R^2/R\) and some internal conformal theory, where gauging is performed with taking the limit of the coupling of one of the free bosons to \(\infty\) and keeping the other decoupled. This in effect decouples the \(SU(1, 1)\) group, too. Such extremal limit corresponds to the one discussed by Horne and Horowitz\(^4\) in relation to extremal black strings in \(3D\). A peculiarity of the solution is the constant dilaton, whose evolution is suppressed by a cancelation between the Kalb-Ramond charge and the effective cosmological constant, resulting from the central charge deficit.

The action defining the effective field theory on the target space is, in the world sheet
\[ S = \int d^4x \sqrt{G} e^{-\sqrt{2}k\Phi} \left( \frac{1}{2\kappa^2} R - H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \partial_{\mu} \Phi \partial^{\mu} \Phi + \Lambda \right) \]  

(1)

where \( G_{\mu\nu} \) and \( H_{\mu\nu\lambda} = \partial_{[\lambda} B_{\mu\nu]} \) and \( \Phi \) are the world sheet target metric, the Kalb-Ramond antisymmetric field strength and the dilaton. Braces denote antisymmetrization over enclosed indices. Here the cosmological constant has been included to represent the central charge deficit \( \Lambda = \frac{2}{3} \delta c_T = \frac{2}{3} (c_T - 4) \geq 0 \). It arises as the difference of the internal theory central charge and the total central charge for a conformally invariant theory \( c_{tot} = 26 \). In the case I will consider, when the sigma model

\[
S_\sigma = \frac{1}{\pi} \int d^2 \sigma (G_{\mu\nu} + \sqrt{\frac{2}{3}} B_{\mu\nu}) \partial_+ X^\mu \partial_- X^\nu
\]

(2)

is defined on the coset \( SU(1,1) \times R^2/R \), the central charge of the target for level \( k \) is \( c_T = (3k/k - 2) + 2 - 1 \), where \( 2 \) and \( -1 \) correspond to the two free bosons and gauging, respectively. The \( \sqrt{2/3} \) in Eq. (2) is introduced following the normalization adopted in (1). The central charge deficit, by the formulae above, will be given by

\[
\delta c_T = \frac{6}{k - 2} \simeq \frac{6}{k}
\]

(3)

in the semiclassical limit \( k \to \infty \), where the theory is most reliable.

I will first consider the effective field theory as described by the action (1) and search for a specific class of solutions featuring toroidal symmetry and constant dilaton. Such solutions were noticed in [4] as a special class of extremal black strings in 3D. So, for the background metric of the Bianchi I type,

\[
ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dx^2 + e^{2\mu} dy^2 + e^{2\eta} dz^2
\]

(4)

where the metric functions depend only on time, it is easy to derive beta function equations from (1). The axion equations \( \partial_{[\lambda} H_{\nu\sigma]} = 0 \) and \( \partial_{\mu} \left( \sqrt{G} \exp(-\sqrt{2}\Phi) H^{\mu\lambda\sigma} \right) = 0 \) are solved by (the topological charge term)

\[
B = 2Y \ dx \wedge dy
\]

\[
\dot{Y} = -\frac{Q}{2} \exp(\sqrt{2}\Phi + \nu + \lambda + \mu - \eta)
\]

(5)
where the overdot denotes the time derivative. With the choice of gauge \( \nu = \lambda + \mu \) and after enforcing \( \Phi = \Phi_0 = \text{const} \) (which implies \( \eta = \text{const} \) and by rescaling the \( z \) coordinate it can be set equal to zero), they reduce to

\[
\Lambda = \frac{Q^2}{3} e^{2\sqrt{2}\Phi_0} \\
\dot{\mu} = \ddot{\lambda} = -\Lambda e^{2(\lambda+\mu)} \\
2\dot{\lambda}\dot{\mu} = -\Lambda e^{2(\lambda+\mu)}
\]

which are very easy to solve. The solution is,

\[
ds^2 = -\frac{\xi^2}{2\Lambda} \frac{dt^2}{\cosh^2 \xi t} + \frac{e^{2\lambda_0 + \xi t} dx^2 + e^{2\mu_0 - \xi t} dy^2}{\cosh \xi t} + dz^2
\]

\[
B = \left( B_0 - \sqrt{\frac{3}{4\Lambda}} \xi \tanh \xi t \right) dx \wedge dy
\]

with integration constants related by \( Q^2 = 3\Lambda \exp(-2\sqrt{2}\Phi_0) \) and \( \exp(\lambda_0 + \mu_0) = \xi/\sqrt{2\Lambda} \).

After a coordinate transformation \( \tau = \tanh \xi t \), \( x \to \exp(-\lambda_0)x \) \( y \to \exp(-\mu_0)y \), which imposes \( -1 \leq \tau \leq 1 \), the solution becomes

\[
ds^2 = -\frac{1}{2\Lambda} \frac{d\tau^2}{1 - \tau^2} + (1 + \tau) dx^2 + (1 - \tau) dy^2 + dz^2
\]

\[
B = \left( B_0 - \sqrt{\frac{3}{2}} \tau \right) dx \wedge dy
\]

This solution apparently can be extended to describe one cosmological sector \( -1 \leq \tau \leq 1 \) and two “static” string-like branches \( \tau \leq -1 \) and \( \tau \geq 1 \). This is the case discussed in Ref. [4-6] where the group was \( SL(2, R) \). However, a future oriented observer inside the cosmological patch, starting from anywhere with \( |\tau| < 1 \) can not enter the future “static” branch, which is clear since \( \tau = 1 \) is a well-defined horizon. Therefore, this observer is confined within the cosmological sector throughout its life, and it is in this way how he/she perceives the solution. I will hence concentrate on the cosmological sector. An additional coordinate transformation \( \tau = \sin \theta \) recasts the metric to

\[
ds^2 = -\frac{1}{2\Lambda} d\theta^2 + (1 + \sin \theta) dx^2 + (1 - \sin \theta) dy^2 + dz^2
\]
where $\theta \in (-\pi/2, \pi/2)$ by the constraints on $\tau$ (I choose the principal branch to define $\theta$). This form of the metric suggests another possibility to extend the solution. First, one should note that a universe described by the metric above undergoes expansion from zero volume (at $\theta = -\pi/2$) through maximum size (at $\theta = 0$) back to collapse (at $\theta = \pi/2$). A simple calculation shows that $R_{3\mu\nu\lambda} = 0$ and that the restriction to a 3D slice $z = \text{const}$ is a maximally symmetric anti-de-Sitter space-time with $R_{\mu\nu\lambda\sigma} = -(\Lambda/2)(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$. Hence curvature singularities do not appear. However, if the time coordinate is compactified to a circle $S^1$, time-like loops appear, and singularities plague causal structure of the space-time. This apparent violation of the cosmic censorship conjecture, appearing through the (hidden) singular behavior and reduction of the metric to three dimensions at $\theta = \pm\pi/2$ is remedied by the fact that the whole universe collapses then, as volume vanishes, just as discussed by Nappi and Witten\textsuperscript{10}. Yet, at the first sight one might think that it is possible to extend the solution past $\theta = \pm\pi/2$ and obtain a solution representing an oscillating cosmos, perhaps even avoiding the closed time-like lines.

This dilemma is answered properly by the WZWN construction. The resolution of this question is a demonstration of the power of the WZWN constructions to resolve ambiguities such as the one above. It firmly states that by the construction, the time coordinate must live on a circle.

The WZWN approach starts with a choice of a group manifold $G$ on which sigma model on level $k$ is

$$S_\sigma = \frac{k}{4\pi} \int d^2 \sigma Tr \left( g^{-1} \partial_+ g^{-1} \partial_- g \right) - \frac{k}{12\pi} \Gamma(g)$$

$$\Gamma(g) = \int_M d^3 \zeta Tr \left( g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right)$$

The action is gauged with introduction of gauge fields which couple to an anomaly-free subgroup of the invariance group $G \times G$ of the sigma model. It is well known that the only anomaly-free subgroups are vector, axial vector and chiral subgroups. I will choose
the axial gauging of the theory, starting with the group manifold \( SU(1, 1) \times R^2 \) and then factor out the one-parameter axial invariance subgroup of the target \( SU(1, 1) \) mixed with translations along one of the free bosons. This choice is similar to that of Refs. \[4,5\]. The only difference is the global structure of the group \( SU(1, 1) \) here versus theirs \( SL(2, R) \).

Since the group \( SU(1, 1) \) is locally given by

\[
g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}
\]

(11)

the explicit form of the chosen ungauged sigma model is

\[
S_\sigma = -\frac{k}{4\pi} \int d^2 \sigma \left( \partial_+ ud\partial_+ v + \partial_- ud\partial_+ v + \partial_+ a\partial_- b + \partial_- a\partial_+ b \right) + \frac{k}{2\pi} \int d^2 \sigma \ln u \left( \partial_+ a\partial_- b - \partial_- a\partial_+ b \right) + \frac{1}{\pi} \int d^2 \sigma \sum_j \left( \partial_+ x_j \partial_- x_j \right)
\]

(12)

The gauge transformations are

\[
\delta a = 2\epsilon a \quad \delta b = 2\epsilon b \quad \delta u = \delta v = 0
\]

\[
\delta x_1 = 2\epsilon c \quad \delta x_2 = 0 \quad \delta A_j = -\partial_j \epsilon
\]

(13)

and the gauged sigma model (9) is

\[
S_\sigma(g, A) = S_\sigma(g) + \frac{k}{2\pi} \int d^2 \sigma A_+ \left( b\partial_- a - a\partial_- b - u\partial_- v + v\partial_- u + \frac{4c}{k} \partial_- x_1 \right) + \frac{k}{2\pi} \int d^2 \sigma A_- \left( b\partial_+ a - a\partial_+ b - v\partial_+ u + u\partial_+ v + \frac{4c}{k} \partial_+ x_1 \right) + \frac{k}{2\pi} \int d^2 \sigma 4A_+ A_- \left( 1 + \frac{2c^2}{k} - uv \right)
\]

(14)

There is no \( A_\pm \) dependent contributions from second term from (10) in (14) since the gauge group is anomaly-free. The next step is to integrate out the gauge fields, fix the gauge of the group (by choosing \( a = \pm b \) to eliminate the unpleasant logarithm from (12), and picking the sign so that still \( \det g = 1 \) ) rescale \( x_1 \to (2c/\sqrt{k}) x_1 \) and take the limit \( c \to \infty \) which effectively decouples the \( SU(1, 1) \) part from the gauge fields. In the end, the resulting action can be rewritten as

\[
S_{\sigma \text{ eff}} = -\frac{k}{8\pi} \int d^2 \sigma \frac{v^2 \partial_+ ud\partial_+ v + u^2 \partial_- v\partial_+ v + (2 - uv)(\partial_+ ud\partial_- v + \partial_- ud\partial_+ v)}{(1 - uv)}
\]

\[
+ \frac{1}{\pi} \int d^2 \sigma \left( 2(1 - uv)\partial_+ x_1 \partial_- x_1 + \partial_+ x_2 \partial_- x_2 \right)
\]

\[
+ \frac{\sqrt{k}}{2\pi} \int d^2 \sigma \left( (u\partial_- v - v\partial_- u)\partial_+ x_1 + (v\partial_+ u - u\partial_+ v)\partial_- x_1 \right)
\]

(15)
A transformation of coordinates covering the cosmological sector of the model is, with 
\[ \theta \in (-\pi/2, \pi/2) , \]
\[ u = i e^{\frac{2i}{\sqrt{2}}y \sqrt{1 - \sin \theta}} \quad v = -i e^{\frac{2i}{\sqrt{2}}y \sqrt{1 - \sin \theta}} \]  
(16)
which differs from the one employed in Ref. [4,5] by where it is valid in the parameter space, and thence lends to the above cosmological interpretation of the solution. Relabeling \( x_1 = x, \ x_2 = z \) and comparing with the sigma model action (2) yields the answer for the metric and the axion:

\[ G_{\mu\nu} = \begin{pmatrix} -\frac{k}{8} & 0 & 0 & 0 \\ 0 & 1 + \sin \theta & 0 & 0 \\ 0 & 0 & 1 - \sin \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] 
\[ B_{\mu\nu} = \begin{pmatrix} 0 & 0 & \sqrt{\frac{3}{2}}(1 - \sin \theta) & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2}(1 - \sin \theta) & 0 \\ 0 & -\frac{\sqrt{3}}{2}(1 - \sin \theta) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]  
(17)

The dilaton can be found either from a careful computation of the Jacobian determinant arising from integrating out the gauge fields, or from the associated effective action. This has already been done in the ansatz preceding Eq. (6), where it was a constant. Indeed, inspection of the Jacobian matrix before the limit \( c \to \infty \) is taken shows that it is \( \propto 1/(1 + (2c^2/k) - uv) = (k/2c^2)/(1 + (k/2c^2)(1 - uv)) \) (see Ref. [3]). As \( c \to \infty \) the non-constant terms decouple and do not contribute to the dilaton. The final answer for the metric indeed is Eq. (9), after using \( \Lambda = 4/k \). As a result of this construction, it is clear that the time \( \theta \) lives on a circle, since \( \theta' = \theta + n\pi \) represent the same group element of \( SU(1,1) \). Hence global structure of the manifold is \( S^1 \times R^3 \) and continuation of \( \theta \) beyond the interval \( (-\pi/2, \pi/2) \) inevitably leads to the appearance of time-like loops and singularities in the causal structure of the space-time. Notice that the point \( \theta = -\pi/2 \) ( \( \tau = -1 \) ) where the gauge fixing breaks down corresponds to the “Big Bang” in this model, and as it was argued before, may not be a reliable point of the theory. In
other words, it is expected\textsuperscript{12}) that a different (topological) field theory should describe the model in the vicinity of \( \theta = -\pi/2 \) and account for effects of quantum gravity. This is underlined even more with the above cosmological interpretation.

In summary, I have considered a gauged WZWN model on the coset \( SU(1,1) \times R^2/R \) and have demonstrated that a sector of the conformal theory admits a cosmological interpretation. The universe described by it is a 3D anti-de-Sitter manifold crossed with a flat direction. It evolves as a closed universe, starting from a “Big Bang” and reaching a “Big Crunch” after a finite comoving time. The “Big Crunch” is in some sense welcome in the model since it prevents appearance of causal singularities which would exist if time is to be extended beyond the moments of creation and annihilation.

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