ON ESTIMATES OF TRANSITION DENSITY FOR SUBORDINATE BROWNIAN MOTIONS WITH GAUSSIAN COMPONENTS IN $C^{1,1}$-OPEN SETS

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Abstract. We consider a subordinate Brownian motion $X$ with Gaussian components when the scaling order of purely discontinuous part is between 0 and 2 including 2. In this paper we establish sharp two-sided bounds for transition density of $X$ in $\mathbb{R}^d$ and $C^{1,1}$-open sets. As a corollary, we obtain a sharp Green function estimates.

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1. Introduction

One of the most important notions in probability theory and analysis is the heat kernel. The transition density $p(t, x, y)$ of a Markov process $X$ is the heat kernel of infinitesimal generator $\mathcal{L}$ of $X$ (also called the fundamental solution of $\partial_t u = \mathcal{L} u$), whose explicit form does not exist usually. Thus obtaining sharp estimates of heat kernel $p(t, x, y)$ is a fundamental problem in both fields. Recently, for a large class of purely discontinuous Markov processes, the sharp heat kernel estimates were obtained in [8, 9, 3, 4]. A common property of all purely discontinuous Markov processes considered so far in the estimates of the heat kernel was that the scaling order was always strictly between 0 and 2. In [19], Ante Mimica succeeded in obtaining sharp heat kernel estimates for purely discontinuous subordinate Brownian motions when the scaling order is between 0 and 2 including 2. For heat kernel estimates of processes with diffusion parts, mixture of Brownian motion and stable process was considered in [21] and diffusion process with jumps was considered in [10].

For any open subset $D \subset \mathbb{R}^d$, let $X^D$ be a subprocess of $X$ killed upon leaving $D$ and $p_D(t, x, y)$ be a transition density of $X^D$. An infinitesimal generator $\mathcal{L}|_D$ of $X^D$ is the infinitesimal generator $\mathcal{L}$ with zero exterior condition. $p_D(t, x, y)$ is also called the Dirichlet heat kernel for $\mathcal{L}|_D$ since it is the fundamental solution to exterior Dirichlet problem with respect to $\mathcal{L}|_D$. There are many results for Dirichlet heat kernel estimates in open subsets.

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of \( \mathbb{R}^d \) (see [13, 5, 6, 7, 11, 12]). In [13], second-named author, jointly with Zhen-Qing Chen and Renming Song, obtains sharp two-sided estimates for the Dirichlet heat kernels in \( C^{1,1} \)-open sets of a large class of subordinate Brownian motions with Gaussian components. Very recently, In [16] second-named author, jointly with Ante Mimica, establish sharp two-sided estimates for the Dirichlet heat kernels in \( C^{1,1} \)-open sets of subordinate Brownian motions without Gaussian components whose scaling order is not necessarily strictly below 2.

In this paper, we continue the journey on investigating the sharp two-sided estimates of heat kernels both in the whole space and \( C^{1,1} \)-open sets. Here, we consider subordinate Brownian motions with Gaussian components when the scaling order of purely discontinuous part is between 0 and 2 including 2. Such processes were not considered in [21, 10, 13, 16].

Let us describe the results of the paper in more detail. We start with a description of the setup of this paper.

Let \( S = (S_t)_{t \geq 0} \) be a subordinator (increasing 1-dimensional Lévy process) whose Laplace transform of \( S_t \) is of the form

\[
E e^{-\lambda S_t} = e^{-t\psi(\lambda)}, \quad \lambda > 0,
\]

where \( \psi \) is called the Laplace exponent of \( S \). Without loss of generality, we assume the drift of \( \psi \) is equal to 1 so that \( \psi \) has the expression \( \psi(\lambda) = \lambda + \phi(\lambda) \) with \( \phi(\lambda) := \int_{(0, \infty)} (1 - e^{-\lambda t})\mu(dt) \).

(1.1)

Here, \( \mu \) is a Lévy measure of \( S \) satisfying \( \int_{(0, \infty)} (1 \wedge t)\mu(dt) < \infty \).

Let \( X = (X_t)_{t \geq 0} \) be a subordinate Brownian motion with subordinator \( S = (S_t)_{t \geq 0} \), where \( W = (W_t)_{t \geq 0} \) is a Brownian motion independent of \( S \). Then \( X \) is rotationally invariant Lévy process whose characteristic function is \( \psi(|\xi|^2) = |\xi|^2 + \phi(|\xi|^2) \). One can view \( X \) as an independent sum of a Brownian motion and purely discontinuous subordinate Brownian motion i.e., \( X_t = B_t + Y_t \) where \( B \) is a Brownian motion and \( Y \) is a subordinate Brownian motion, independent of \( B \), with subordinator \( T \) whose Laplace exponent of \( T \) is \( \phi \). If the scaling order of \( \phi \) is 2, one can say that the process \( X \) is very close to Brownian motion. (See Corollary 1.4)

The Lévy density (jumping kernel) \( J \) of \( X \) is given by

\[
J(x) = j(|x|) = \int_0^{\infty} (4\pi t)^{-d/2} e^{-|x|^2/4t}\mu(dt).
\]

The function \( J(x) \) determines a Lévy system for \( X \): for any non-negative measurable function \( f \) on \( \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \) with \( f(s, y, y) = 0 \) for all \( y \in \mathbb{R}^d \), any stopping time \( T \) (with respect to the filtration of \( X \)) and any \( x \in \mathbb{R}^d \),

\[
E_x \left[ \sum_{s \leq T} f(s, X_{s-}, X_s) \right] = E_x \left[ \int_0^T \left( \int_{\mathbb{R}^d} f(s, X_s, y)J(X_s - y)dy \right) ds \right]. \quad (1.2)
\]

We first introduce the following scaling conditions for a function \( f : (0, \infty) \to (0, \infty) \).
Definition 1.1. Suppose $f$ is a function from $(0, \infty)$ to $(0, \infty)$.

(1) We say that $f$ satisfies $L_a(\gamma, C_L)$ if there exist $a \geq 0$, $\gamma > 0$, and $C_L \in (0, 1]$ such that
\[ \frac{f(\lambda x)}{f(\lambda)} \geq C_L x^\gamma \quad \text{for all } \lambda > a \text{ and } x \geq 1, \]

(2) We say that $f$ satisfies $U_a(\delta, C_U)$ if there exist $a \geq 0$, $\delta > 0$, and $C_U \in [1, \infty)$ such that
\[ \frac{f(\lambda x)}{f(\lambda)} \leq C_U x^\delta \quad \text{for all } \lambda > a \text{ and } x \geq 1. \]

Remark 1.2. According to [16, Remark 2.2], if we assume in addition $f$ is increasing, then the following holds.

(1) If $f$ satisfies $L_b(\gamma, C_L)$ with $b > 0$ then $f$ satisfies $L_a(\gamma, (\frac{a}{b})^\gamma C_L)$ for all $a \in (0, b]$:
\[ \frac{f(\lambda x)}{f(\lambda)} \geq \left( \frac{a}{b} \right)^\gamma C_L x^\gamma, \quad x \geq 1, \lambda \geq a. \]

(2) If $f$ satisfies $U_b(\delta, C_U)$ with $b > 0$ then $f$ satisfies $U_a(\delta, \frac{f(b)}{f(a)} C_U)$ for all $a \in (0, b]$:
\[ \frac{f(\lambda x)}{f(\lambda)} \leq \frac{f(b)}{f(a)} C_U x^\delta, \quad x \geq 1, \lambda \geq a. \]

Throughout this paper we denote $p^{(2)}(t, x)$ the transition density of $B$ (and $W$). i.e.,
\[ p^{(2)}(t, x) = (4\pi t)^{-d/2} \exp(-\frac{|x|^2}{4t}). \]

We assume that $\mu(0, \infty) = \infty$ and denote $q(t, x)$ the transition density of $Y$ and $p(t, x)$ the transition density of $X$. $q(t, x)$ and $p(t, x)$ are of the forms
\[ p(t, x) = \int_{(0, \infty)} (4\pi s)^{-d/2} e^{-\frac{|x|^2}{4s}} \mathbb{P}(S_t \in ds), \quad q(t, x) = \int_{(0, \infty)} (4\pi s)^{-d/2} e^{-\frac{|x|^2}{4s}} \mathbb{P}(T_t \in ds) \]
for $x, y \in \mathbb{R}^d$ and $t > 0$. These imply that for all $t > 0$, $p(t, x) \leq p(t, y)$ and $q(t, x) \leq q(t, y)$ if $|x| \geq |y|$.

Throughout this paper the constants $r_0, R_0, \lambda_0, A_0$, and $C_i, i = 1, 2, \ldots$ will be fixed. While, we use $c_1, c_2, \ldots$ to denote generic constants, whose exact values are not important and the labeling of the constants $c_1, c_2, \ldots$ starts anew in the statement of each result and its proof. For $a, b \in \mathbb{R}$ we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Notation $f(x) \simeq g(x), x \in I$ means that there exists constants $c_1, c_2 > 0$ such that $c_1 f(x) \leq g(x) \leq c_2 g(x)$ for $x \in I$.

The following is the first main result of this paper. Throughout this paper, $S = (S_t)_{t \geq 0}$ is a subordinator whose Laplace exponent $\psi$ is $\lambda + \phi(\lambda)$ and we denote $H(\lambda) := \phi(\lambda) - \lambda \phi'(\lambda)$.

Theorem 1.3. Let $X = (X_t)_{t \geq 0}$ be a subordinate Brownian motion whose characteristic exponent is $\psi(|\xi|^2) = |\xi|^2 + \phi(|\xi|^2)$.

(1) Suppose $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$. For every $T, M > 0$, there exist positive constants $c_1, c_2, c_3, c_4$, and $c_5$ such that for all $0 < t \leq T$ and $|x| \leq M/2$,
\[ c_1^{-1} \left( t^{-d/2} \wedge (p^{(2)}(t, c_2 x) + q(t, c_3 x)) \right) \leq p(t, x) \leq c_1 \left( t^{-d/2} \wedge (p^{(2)}(t, c_4 x) + q(t, c_5 x)) \right). \]
are the second main results in this paper. Thus there exist positive constants \( c_0, c_7, c_8, c_9, \) and \( c_{10} \) such that for all \( t > 0 \) and \( x \in \mathbb{R}^d \),
\[
\begin{aligned}
c_0^{-1} \left( (t^{-d/2} \wedge \phi^{-1}(t^{-1})^{d/2}) \wedge (p^{(2)}(t, c_7 x) + q(t, c_8 x)) \right) &\leq p(t, x) \leq c_0 \left( (t^{-d/2} \wedge \phi^{-1}(t^{-1})^{d/2}) \wedge (p^{(2)}(t, c_9 x) + q(t, c_{10} x)) \right).
\end{aligned}
\]

As an application, we obtain sharp two-sided estimate for Green function of transient subordinate Brownian motion \( X = (X_t)_{t \geq 0} \) \((d \geq 3)\). If \( X \) is transient, then the following Green function is well-defined and finite.
\[
G(x, y) = G(x - y) = \int_0^\infty p(t, x - y) dt, \quad x, y \in \mathbb{R}^d, x \neq y.
\]

**Corollary 1.4.** Let \( d \geq 3 \). Suppose \( H \) satisfies \( L_0(\gamma, C_L) \) and \( U_0(\delta, C_U) \) with \( \delta < 2 \).
Then
\[
G(x) \asymp |x|^{-d} (|x|^2 \wedge \phi(|x|^{-2})^{-1}), \quad x \in \mathbb{R}^d.
\]

Consider \( \phi \) is given in Example 1.8 (2) below. Then
\[
|x|^{-d} \phi(|x|^{-2})^{-1} \asymp \begin{cases}
|x|^{-d+2} \log \frac{1}{|x|} & |x| < \frac{1}{2} \\
|x|^{-d+2} & |x| \geq \frac{1}{2}.
\end{cases}
\]
Thus \( G(x) \asymp G^{(2)}(x), \quad x \in \mathbb{R}^d \) where \( G^{(2)}(x) = c |x|^{-d+2} \) is the Green function of the Brownian motion. This shows that how close this process is to the Brownian motion and Green function estimates may not detect the difference between our \( X \) and the Brownian motion.

Let \( D \subset \mathbb{R}^d \) \((\text{when } d \geq 2)\) be a \( C^{1,1} \) open set with \( C^{1,1} \) characteristics \((R_0, \Lambda_0)\), that is, there exists a localization radius \( R_0 > 0 \) and a constant \( \Lambda_0 > 0 \) such that for every \( z \in \partial D \) there exist a \( C^{1,1} \)-function \( \varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R} \) satisfying \( \varphi(0) = 0, \nabla \varphi(0) = (0, ..., 0) \), \( ||\nabla \varphi||_\infty \leq \Lambda_0, |\nabla \varphi(x) - \nabla \varphi(w)| \leq \Lambda_0 |x - w| \) and an orthonormal coordinate system \( CS_z \) of \( z = (z_1, \cdots, z_{d-1}, z_d) := (\bar{z}, \bar{z}_d) \) with origin at \( z \) such that \( D \cap B(z, R_0) = \{ y = (\bar{y}, y_d) \in B(0, R_0) \cap CS_z : y_d > \varphi(\bar{y}) \} \). The pair \((R_0, \Lambda_0)\) will be called the \( C^{1,1} \) characteristics of the open set \( D \). By a \( C^{1,1} \) open set in \( \mathbb{R} \) with a characteristic \( R_0 > 0 \), we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least \( R_0 \) and the infimum of the distances between these intervals is at least \( R_0 \).

Throughout this paper we denote \( p_D(t, x, y) \) the transition density of \( X^D \). The following are the second main results in this paper.

**Theorem 1.5.** Let \( X = (X_t)_{t \geq 0} \) be a subordinate Brownian motion whose characteristic exponent is \( \psi(|\xi|^2) = |\xi|^2 + \phi(|\xi|^2) \). Suppose \( H \) satisfies \( L_a(\gamma, C_L) \) and \( U_a(\delta, C_U) \) with \( \delta < 2 \) for some \( a > 0 \) and \( D \) is a bounded \( C^{1,1} \) open set in \( \mathbb{R}^d \) with characteristics \((R_0, \Lambda_0)\). Then for every \( T > 0 \) there exist positive constants \( c_1, c_2, c_3, a_U, a_L \) such that
1. For any \((t, x, y) \in (0, T] \times D \times D\), we have
\[
\begin{aligned}
p_D(t, x, y) &\leq c_1 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \\
&\times \left( t^{-d/2} \wedge (p^{(2)}(t, c_2(x - y)) + \frac{tH(|x - y|^2)}{|x - y|^d} + \phi^{-1}(t^{-1})^{d/2}e^{-a_U|x-y|^2\phi^{-1}(t^{-1})}} \right).
\end{aligned}
\]
(2) For any \((t, x, y) \in (0, T] \times D \times D\), we have
\[
p_D(t, x, y) \geq c_1^{-1} \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) 
\times \left( t^{-d/2} \wedge (p^{(2)}(t, c_3(x - y)) + \frac{tH(|x - y|^2)}{|x - y|^d} + \phi^{-1}(t^{-1})d/2 e^{-a_L|x-y|^2\phi^{-1}(t^{-1})}) \right).
\]
(1.5)

(3) For any \((t, x, y) \in [T, \infty) \times D \times D\), we have
\[
p_D(t, x, y) \approx e^{-\lambda_1 t} \delta_D(x) \delta_D(y),
\]
where \(-\lambda_1 < 0\) is the largest eigenvalue of the generator of \(X^D\).

We say that the path distance in a domain (connected open set) \(U\) is comparable to the Euclidean distance with characteristic \(\lambda_0\) if for every \(x\) and \(y\) in \(U\) there is a rectifiable curve \(l\) in \(U\) which connects \(x\) to \(y\) such that the length of \(l\) is less than or equal to \(\lambda_0|x - y|\). Clearly, such a property holds for all bounded \(C^{1,1}\) domains, \(C^{1,1}\) domains with compact complements, and domain consisting of all the points above the graph of a bounded globally \(C^{1,1}\) function.

**Theorem 1.6.** Let \(X = (X_t)_{t \geq 0}\) be a subordinate Brownian motion whose characteristic exponent is \(\psi(|\xi|^2) = |\xi|^2 + \phi(|\xi|^2)\). Suppose \(H\) satisfies \(L_0(\gamma, C_L)\) and \(U_0(\delta, C_U)\) with \(\delta < 2\) and \(D\) is an unbounded \(C^{1,1}\) open set in \(\mathbb{R}^d\) with characteristics \((R_0, \Lambda_0)\). Then for every \(T > 0\) there exists \(c_1, c_2, c_3, a_U, a_L\) such that
(1) For any \((t, x, y) \in (0, T] \times D \times D\), we have
\[
p_D(t, x, y) \leq c_1 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) 
\times \left( t^{-d/2} \wedge (p^{(2)}(t, c_2(x - y)) + \frac{tH(|x - y|^2)}{|x - y|^d} + \phi^{-1}(t^{-1})d/2 e^{-a_U|x-y|^2\phi^{-1}(t^{-1})}) \right).
\]
(2) If the path distance in each connected component of \(D\) is comparable to the Euclidean distance with characteristic \(\lambda_0\), then for any \((t, x, y) \in (0, T] \times D \times D\), we have
\[
p_D(t, x, y) \geq c_1^{-1} \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) 
\times \left( t^{-d/2} \wedge (p^{(2)}(t, c_3(x - y)) + \frac{tH(|x - y|^2)}{|x - y|^d} + \phi^{-1}(t^{-1})d/2 e^{-a_L|x-y|^2\phi^{-1}(t^{-1})}) \right).
\]

Define \(G_D(x, y) = \int_0^\infty p_D(t, x, y)dt\), Green function of \(X^D\). The following is Green function estimate of \(X^D\).

**Corollary 1.7.** Suppose \(H\) satisfies \(L_a(\gamma, C_L)\) and \(U_a(\delta, C_U)\) with \(\delta < 2\) for some \(a > 0\) and \(D\) is a bounded \(C^{1,1}\) open set in \(\mathbb{R}^d\) with characteristics \((R_0, \Lambda_0)\). Then
\[
G_D(x, y) \approx g_D(x, y), \quad x, y \in D,
\]
where

\[
g_D(x, y) := \begin{cases} 
\frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } d \geq 3, \\
\log \left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } d = 2, \\
\left(\delta_D(x)\delta_D(y)\right)^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|} & \text{when } d = 1.
\end{cases}
\] (1.6)

Denote by \(G_D^{(2)}(x, y)\) the Green function of Brownian motion in \(D\). It is known (see [15]) that \(G_D^{(2)} \asymp g_D(x, y)\) when \(x\) and \(y\) are in the same component of \(D\), and \(G_D^{(2)}(x, y) = 0\) otherwise. Thus when \(D\) is a bounded \(C^{1,1}\) connected open subset of \(\mathbb{R}^d\), \(G_D(x, y) \asymp G_D^{(2)}(x, y)\), while our heat kernel estimates (Theorem 1.5) are different from heat kernel estimates of Brownian motion in \(D\).

These are examples where the scaling order of \(\phi\) is not strictly between 0 and 2.

**Example 1.8.** (1) Let \(\phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})}\), where \(\beta \in (0, 2)\). Then

\[\phi^{-1}(\lambda) \asymp \begin{cases} 
\lambda^{2/\beta} & 0 < \lambda < 2 \\
\lambda \log \lambda & \lambda \geq 2
\end{cases}
\]

\[H(\lambda) \asymp \begin{cases} 
\lambda^{-\beta/2} & 0 < \lambda < 2 \\
\lambda (\log \lambda)^2 & \lambda \geq 2
\end{cases}
\]

Hence, \(H\) satisfies \(L_0(\gamma, C_L)\) and \(U_0(\delta, C_U)\) with some \(\gamma, C_L, C_U\) and \(\delta < 2\).

(2) Let \(\phi(\lambda) = \frac{\lambda}{\log(1+\lambda)} - 1\). Then

\[\phi^{-1}(\lambda) \asymp \begin{cases} 
\lambda & 0 < \lambda < 2 \\
\lambda \log \lambda & \lambda \geq 2
\end{cases}
\]

\[H(\lambda) \asymp \begin{cases} 
\lambda^{2/\beta} & 0 < \lambda < 2 \\
\lambda (\log \lambda)^2 & \lambda \geq 2
\end{cases}
\]

Hence, \(H\) satisfies \(L_0(\gamma, C_L)\) and \(U_2(\delta, C_U)\) with some \(\gamma, C_L, C_U\) and \(\delta < 2\).

Suppose \(D\) is a bounded \(C^{1,1}\) open set with \(\text{diam}(D) < 1/2\) and \(\psi(\lambda) = \lambda + \phi(\lambda)\), where \(\phi\) is the one in above two cases. Then for \(t < 1/2\), there exist positive constants \(c_1, c_2, c_3, a_U,\) and \(a_L\) such that

\[
p_D(t, x, y) \leq c_1 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right)
\times \left(t^{-d/2} \wedge \left(p^{(2)}(t, c_2(x-y)) + \frac{t}{|x-y|^{d+2}(\log \frac{1}{|x-y|})^2} + t^{-d/2} \left(\log \frac{1}{t}\right)^{d/2} e^{-a_U \frac{|x-y|^2}{x-y}} \log \right)\right),
\]

and

\[
p_D(t, x, y) \geq c_1^{-1} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right)
\times \left(t^{-d/2} \wedge \left(p^{(2)}(t, c_3(x-y)) + \frac{t}{|x-y|^{d+2}(\log \frac{1}{|x-y|})^2} + t^{-d/2} \left(\log \frac{1}{t}\right)^{d/2} e^{-a_L \frac{|x-y|^2}{x-y}} \log \right)\right).
\]
2. Heat kernel estimates in $\mathbb{R}^d$

Throughout this paper, $X = (X_t)_{t \geq 0}$ is a subordinate Brownian motion whose characteristic exponent is $\psi(|\xi|^2) = |\xi|^2 + \phi(|\xi|^2)$. In this section we obtain estimates of transition density of the subordinate Brownian motion $X$. The following are heat kernel estimates for $q(t, x)$, which is transition density of $Y$. Recall that $Y$ is a subordinate Brownian motion with subordinator $T$ whose Laplace exponent of $T$ is $\phi$ and $H(\lambda) = \phi(\lambda) - \lambda \phi'(\lambda)$.

**Theorem 2.1** ([19] [16]). (i) If $\phi$ satisfies $L_a(\gamma, C_L)$ for some $a > 0$, then for every $T > 0$ there exist $C_1 = C_1(T) > 1$ and $a_U > 0$ such that for all $t \leq T$ and $x \in \mathbb{R}^d$,

$$q(t, x) \leq C_1 \left( \phi^{-1}(t^{-1})^{d/2} \wedge (t|x|^{-d}H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2}e^{-a_U|z|^2\phi^{-1}(t^{-1})}) \right),$$

(2.1)

and

$$q(t, x) \geq C_1^{-1}\phi^{-1}(t^{-1})^{d/2}, \quad \text{if} \quad t\phi(|x|^{-2}) \geq 1.$$  

(2.2)

Consequently, the Lévy density (jumping kernel) $J$ satisfies

$$J(x) = \lim_{t \to 0} q(t, x)/t \leq C_1|x|^{-d}H(|x|^{-2}).$$

Furthermore, if $a = 0$, then (2.1) and (2.2) hold for every $t > 0$ and $x \in \mathbb{R}^d$.

(ii) If $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$, then for every $T, M > 0$ there exist $C_1 = C_1(a, \gamma, C_L, \delta, C_U, T, M) > 0$ and $a_L > 0$ such that for all $t \leq T$ and $|x| < M$,

$$q(t, x) \geq C_1^{-1} \left( \phi^{-1}(t^{-1})^{d/2} \wedge (t|x|^{-d}H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2}e^{-a_L|z|^2\phi^{-1}(t^{-1})}) \right).$$

(2.3)

Consequently, the Lévy density $J$ satisfies

$$J(x) = \lim_{t \to 0} q(t, x)/t \asymp |x|^{-d}H(|x|^{-2}), \quad |x| < M.$$  

(2.4)

Furthermore, if $a = 0$, then (2.3) and (2.4) hold for all $t > 0$ and $x \in \mathbb{R}^d$.

We will use following formula of transition density $p(t, x)$ of $X$, which is given by

$$p(t, x) = \int_{\mathbb{R}^d} p^{(2)}(t, x - y)q(t, y)dy.$$ 

2.1. Upper bounds. In this subsection we will prove the upper bounds for the transition density. First, we observe the following simple upper bound of $p(t, x)$. (See [21].)

**Lemma 2.2.** For every $(t, x) \in (0, \infty) \times \mathbb{R}^d$,

$$p(t, x) \leq \exp(|x|^2/(4t))p^{(2)}(t, x).$$

In particular if $t \geq |x|^2$, we have $p(t, x) \leq e^{1/4}p^{(2)}(t, x).$
Proof. Since \( p^{(2)}(t, x - y) \leq p^{(2)}(t, x) \exp(|x|^2/(4t)) \), we have
\[
p(t, x) = \int_{\mathbb{R}^d} p^{(2)}(t, x - y)q(t, y)dy
\leq p^{(2)}(t, x) \exp(|x|^2/(4t)) \int_{\mathbb{R}^d} q(t, y)dy = \exp(|x|^2/(4t))p^{(2)}(t, x).
\]

The next two lemmas will be used several times in this paper.

Lemma 2.3 ([19] Lemma 2.1(a)]. For any \( \lambda > 0 \) and \( x \geq 1 \),
\[\phi(\lambda x) \leq x\phi(\lambda) \quad \text{and} \quad H(\lambda x) \leq x^2H(\lambda).\]

Lemma 2.4 ([19] Lemma 2.1(b)]. For \( a \geq 0 \) if \( H \) satisfies \( L_a(\gamma, C_L) \) (resp. \( U_a(\delta, C_U)\)), then \( \phi \) satisfies \( L_a(\gamma, C_L) \) (resp. \( U_a(\delta \wedge 1, C_U)\)).

Since the proofs are basically same, we provide the proof for the case \( a > 0 \) in the next two results.

Lemma 2.5. Suppose \( \phi \) satisfies \( L_a(\gamma, C_L) \) for some \( a > 0 \) (\( L_0(\gamma, C_L) \), respectively). For \( T > 0 \), there exists a positive constant \( c \) such that for \( 0 < t \leq T \) (\( t > 0 \), respectively) and \( t\phi(|x|^{-2}) \geq 1 \),
\[p(t, x) \leq cq(t, x).\]

Proof. Since \( \phi \) satisfies \( L_a(\gamma, C_L) \), for all \( y \in \mathbb{R}^d \) and \( 0 < t \leq T \), \( q(t, y) \leq c_1\phi^{-1}(t^{-1})^{d/2} \) by Theorem 2.1(i). By [22], for all \( 0 < t \leq T \) and \( t\phi(|x|^{-2}) \geq 1 \), we have that \( q(t, x) \geq C_1^{-1}\phi^{-1}(t^{-1})^{d/2} \). Hence,
\[
p(t, x) = \int_{\mathbb{R}^d} p^{(2)}(t, x - y)q(t, y)dy
\leq c_1\phi^{-1}(t^{-1})^{d/2} \int_{\mathbb{R}^d} p^{(2)}(t, x - y)dy \leq c_1C_1q(t, x).
\]

Let \( \tilde{p}^{(2)}(t, x) := (4\pi t)^{-d/2} \exp\{-|x|^2/(16t)\} \).

Lemma 2.6. Suppose \( H \) satisfies \( L_a(\gamma, C_L) \) and \( U_a(\delta, C_U) \) for some \( a > 0 \) (\( L_0(\gamma, C_L) \) and \( U_0(\delta, C_U) \), respectively) with \( \delta < 2 \). For \( T, M > 0 \) there exist positive constants \( c \) and \( c_0 < 1 \) such that for all \( 0 < t \leq T \) and \( x \in \mathbb{R}^d \) satisfying \( |x| < M \) (\( t > 0 \) and \( x \in \mathbb{R}^d \), respectively) and \( t\phi(|x|^{-2}) \leq 1 \),
\[p(t, x) \leq c \max(\tilde{p}^{(2)}(t, x), q(t, c_0x)).\]
Proof. We divide the integral

\[ p(t, x) = \int_{\mathbb{R}^d} p^{(2)}(t, x - y)q(t, y)dy \]

\[ = \int_{|y-x|>|x|/2} p^{(2)}(t, x - y)q(t, y)dy + \int_{|y-x|\leq|x|/2} p^{(2)}(t, x - y)q(t, y)dy \]

\[ =: I_1 + I_2. \]

(i) For \(|y - x| > |x|/2\), \(\exp\{-|x - y|^2/(4t)\} \leq \exp\{-|x|^2/(16t)\}\). Therefore,

\[ I_1 \leq (4\pi t)^{-d/2} \exp(-|x|^2/(16t)) \int_{|y-x|>|x|/2} q(t, y)dy \leq \tilde{p}^{(2)}(t, x). \]

(ii) For \(|y - x| \leq |x|/2\), we have \(|x|/2 \leq |y| \leq 3|x|/2\). Therefore, using (2.1), Lemma 2.3, and (2.3), we have

\[ I_2 \leq c_1 \int_{|y-x|\leq|x|/2} p^{(2)}(t, x - y)(t|y|^{-d}H(|y|^2) + \phi^{-1}(t^{-1})^{d/2}e^{-a_U|y|^2\phi^{-1}(t^{-1})})dy \]

\[ \leq c_1 \int_{|y-x|\leq|x|/2} p^{(2)}(t, x - y)((2)^d|x|^{-d}H(2|x|^2) + \phi^{-1}(t^{-1})^{d/2}e^{-a_U|y|^2\phi^{-1}(t^{-1})})dy \]

\[ \leq c_2 \left( \int_{|y-x|\leq|x|/2} p^{(2)}(t, x - y)dy \right) \left( t|c_0|x|^{-d}H(|c_0x|^2) + \phi^{-1}(t^{-1})^{d/2}e^{-a_U|c_0x|^2\phi^{-1}(t^{-1})} \right) \]

\[ \leq c_3 q(t, c_0x), \]

where \(c_0 = a_{U, 2}^{1/2}/2a_L^{1/2} < 1\). \(\square\)

Remark 2.7. In the proof of Lemma 2.6, we just used that \(\phi\) satisfies \(L_a(\gamma, C_L)\) until the last inequality. Thus if \(\phi\) satisfies \(L_a(\gamma, C_L)\) for some \(a > 0\) \((L_0(\gamma, C_L), \text{respectively})\), then for \(T > 0\) there exists a positive constant \(c\) such that for all \(0 < t \leq T\) \((t > 0, \text{respectively})\) and \(x \in \mathbb{R}^d\) satisfying \(t\phi(|x|^2) \leq 1\),

\[ p(t, x) \leq c\max(\tilde{p}^{(2)}(t, x), t|x|^{-d}H(|x|^2) + \phi^{-1}(t^{-1})^{d/2}e^{-(a_U/4)|x|^2\phi^{-1}(t^{-1})}). \]

Consequently, combining Lemma 2.2 and Lemma 2.3 we can obtain upper bounds for \(p(t, x)\):

If \(\phi\) satisfies \(L_a(\gamma, C_L)\) for some \(a > 0\), then for \(0 < t \leq T\) and \(x \in \mathbb{R}^d\)

\[ p(t, x) \leq c_1 \left( t^{-d/2} \wedge (t^{-d/2}e^{-|x|^2/(c_2t)} + t|x|^{-d}H(|x|^2) + \phi^{-1}(t^{-1})^{d/2}e^{-c_3|x|^2\phi^{-1}(t^{-1})}) \right), \]

and if \(\phi\) satisfies \(L_0(\gamma, C_L)\) then for \(t > 0\) and \(x \in \mathbb{R}^d\)

\[ p(t, x) \leq c_4 \left( t^{-d/2} \wedge \phi^{-1}(t^{-1})^{d/2} \right) \wedge (t^{-d/2}e^{-|x|^2/(c_5t)} + \frac{tH(|x|^2)}{|x|^d} + \phi^{-1}(t^{-1})^{d/2}e^{-a_6|x|^2\phi^{-1}(t^{-1})}). \]
2.2. Lower bounds. In this subsection we will prove the lower bounds for the transition density. As the subsection 2.1, we provide the proof for the case $a > 0$ only.

Let $\hat{p}^{(2)} := (4\pi t)^{-d/2} \exp(-|x|^2/t)$.

**Lemma 2.8.** Suppose $\phi$ satisfies $L_a(\gamma, C_L)$ for some $a > 0$ ($L_0(\gamma, C_L)$, respectively). For $T > 0$ there exists a positive constant $c$ such that for $0 < t \leq T$ ($t > 0$, respectively) and $x \in \mathbb{R}^d$ satisfying $t\phi(|x|^{-2}) \leq 1$,

$$p(t, x) \geq c\hat{p}^{(2)}(t, x).$$

**Proof.** If $|y| \leq \phi^{-1}(t^{-1})^{-1/2}$, then $|y| \leq \phi^{-1}(t^{-1})^{-1/2} \leq |x|$. Therefore $|y - x| \leq 2|x|$ and hence $\exp(-|x - y|^2/(4t)) \geq \exp(-|x|^2/t)$. By (2.2), $q(t, y) \geq C_1^{-1}\phi^{-1}(t^{-1})^{d/2}$ for $0 < t \leq T$. Thus,

$$p(t, x) \geq \int_{B(0, \phi^{-1}(t^{-1})^{-1/2})} \hat{p}^{(2)}(t, x - y)q(t, y)dy$$

$$\geq c_1(4\pi t)^{-d/2} \exp(-|x|^2/t)\phi^{-1}(t^{-1})^{d/2}(\phi^{-1}(t^{-1})^{-1/2})^d = c_2\hat{p}^{(2)}(t, x).$$

$\square$

**Lemma 2.9.** Suppose $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$ ($L_0(\gamma, C_L)$ and $U_0(\delta, C_U)$, respectively). For $T, M > 0$ there exist positive constants $c$ and $\hat{c}_0 > 1$ such that for all $0 < t \leq T$, $x \in \mathbb{R}^d$ satisfying $|x| < \frac{2}{3}M$ ($t > 0$ and $x \in \mathbb{R}^d$, respectively) and every $y \in B(x, |x|/2)$, it holds that

$$q(t, y) \geq cq(t, \hat{c}_0 x).$$

**Proof.** Note that $y \in B(x, |x|/2)$ implies $|y| < \frac{3}{4}|x| < M$. We consider four cases separately.

**Case (1):** When $t\phi(|x|^{-2}) \geq 1$ and $t\phi(|y|^{-2}) \geq 1$. Using (2.2),

$$q(t, x) \leq C_1\phi^{-1}(t^{-1})^{d/2} \leq C_2^2 q(t, y).$$

**Case (2):** When $t\phi(|x|^{-2}) \geq 1$ and $t\phi(|y|^{-2}) \leq 1$. Using (2.3), $|y| < \frac{3}{4}|x|$, $|x|^2\phi^{-1}(t^{-1}) \leq 1$, and (2.1), we have

$$q(t, y) \geq C_1^{-1} t |y|^{-d}H(|y|^{-2}) + \phi^{-1}(t^{-1})^{d/2} e^{-a_L|y|^2\phi^{-1}(t^{-1})} \geq C_1^{-1} \phi^{-1}(t^{-1})^{d/2} e^{-a_L|y|^2\phi^{-1}(t^{-1})}$$

$$\geq C_1^{-1} \phi^{-1}(t^{-1})^{d/2} e^{-\frac{a_L}{2} |y|^2 \phi^{-1}(t^{-1})} \geq C_1^{-1} \phi^{-1}(t^{-1})^{d/2} e^{-\frac{a_L}{2} |x|^2 \phi^{-1}(t^{-1})} \geq C_1^{-1} \phi^{-1}(t^{-1})^{d/2} \geq c_1 q(t, x).$$

**Case (3):** When $t\phi(|x|^{-2}) \leq 1$ and $t\phi(|y|^{-2}) \geq 1$. Using (2.2), $tH(|x|^{-2}) \leq t\phi(|x|^{-2}) \leq 1$, $|x|^{-d} \leq \phi^{-1}(t^{-1})^{d/2}$, and (2.1), we have

$$q(t, y) \geq C_1^{-1} \phi^{-1}(t^{-1})^{d/2} \geq c_3(tH(|x|^{-2})|x|^{-d} + \phi^{-1}(t^{-1})^{d/2} e^{-a_L|y|^2\phi^{-1}(t^{-1})}) \geq c_4 q(t, x).$$
Case (4): When \( t\phi(|x|^2) \leq 1 \) and \( t\phi(|y|^2) \leq 1 \). Using (2.8), \(|y| < 3|x|/2\), and (2.11), we have

\[
q(t, y) \geq C_1^{-1}(t|y|^{-d}H(|y|^{-2}) + \phi^{-1}(t^{-1})^{d/2}e^{-a_L|y|^2\phi^{-1}(t^{-1})})
\]

\[
\geq C_1^{-1}(t(2/3)|x|^{-d}H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2}e^{-a_L|x|^2\phi^{-1}(t^{-1})})
\]

\[
\geq c_5(t|\tilde{c}_0|^{-d}H(|\tilde{c}_0|^{-2}) + \phi^{-1}(t^{-1})^{d/2}e^{-a_U|\tilde{c}_0|^2\phi^{-1}(t^{-1})})
\]

\[
\geq c_6q(t, \tilde{c}_0x),
\]

where \( \tilde{c}_0 = \frac{3}{2}(a_L/a_U)^{1/2} > 1 \). Since \( q(t, x) \geq q(t, \tilde{c}_0 x) \) by (1.3), we conclude that \( q(t, y) \geq cq(t, \tilde{c}_0 x) \) for some \( c > 0 \).

- **Lemma 2.10.** Suppose \( H \) satisfies \( L_\alpha(\gamma, C_L) \) and \( U_\alpha(\delta, C_U) \) with \( \delta < 2 \) for some \( a > 0 \) \((L_0(\gamma, C_L) \) and \( U_0(\delta, C_U) \), respectively). For \( T, M > 0 \) there exist constants \( c \) and \( \tilde{c}_0 > 1 \) such that for all \( 0 < t \leq T \) and \( x \in \mathbb{R}^d \) satisfying \(|x| < \frac{2}{3}M \) \((t > 0 \) and \( x \in \mathbb{R}^d \), respectively) and \( t \leq |x|^2 \),

\[
p(t, x) \geq cq(t, \tilde{c}_0 x).
\]

**Proof.** By Lemma 2.9 \( q(t, y) \geq c_1q(t, \tilde{c}_0 x) \) for every \( y \in B(x, |x|/2) \). Using this and change of variable, we have

\[
p(t, x) = q(t, \tilde{c}_0 x) \int_{\mathbb{R}^d} \frac{q(t, y)}{q(t, \tilde{c}_0 x)} p^{(2)}(t, x-y) dy
\]

\[
\geq c_1q(t, \tilde{c}_0 x) \int_{B(x, |x|/2)} p^{(2)}(t, x-y) dy
\]

\[
= c_1q(t, \tilde{c}_0 x) \int_{B(0, t^{-1/2}|x|)} p^{(2)}(1, u) du
\]

\[
\geq c_1 \left( \int_{B(0, 1/2)} p^{(2)}(1, u) du \right) q(t, \tilde{c}_0 x) = c_2q(t, \tilde{c}_0 x).
\]

- **Lemma 2.11.** Suppose \( H \) satisfies \( L_\alpha(\gamma, C_L) \) and \( U_\alpha(\delta, C_U) \) with \( \delta < 2 \) for some \( a > 0 \) \((L_0(\gamma, C_L) \) and \( U_0(\delta, C_U) \), respectively). For \( T, M > 0 \) there exists a constant \( c \) such that for all \( 1 \leq t \leq T \) and \( x \in \mathbb{R}^d \) satisfying \(|x| < M/2 \) \((t \geq 1 \) and \( x \in \mathbb{R}^d \), respectively) and \( t\phi(|x|^2) \geq 1 \),

\[
p(t, x) \geq cq(t, x).
\]

**Proof.** Assume \( t\phi(|x|^2) \geq 1 \) and let \( b = M\phi^{-1}(T^{-1})^{1/2}/2 \). Note that we have \( q(t, x) \leq C_1\phi^{-1}(t^{-1})^{d/2} \) by (2.11).

If \(|y - x| \leq b\phi^{-1}(t^{-1})^{-1/2} \), then \(|y| \leq |y - x| + |x| \leq (b + 1)\phi^{-1}(t^{-1})^{-1/2} \) and \(|y| \leq |x - y| + |x| \leq b\phi^{-1}(t^{-1})^{-1/2} + |x| \leq M \). Thus by (2.3), we have

\[
q(t, y) \geq C_1^{-1}\phi^{-1}(t^{-1})^{d/2}e^{-a_L|y|^2\phi^{-1}(t^{-1})} \geq c_1\phi^{-1}(t^{-1})^{d/2} \text{ for } |y - x| \leq b\phi^{-1}(t^{-1})^{-1/2}.
\]
Therefore, using the above inequality and change of variable

\[ p(t, x) \geq \int_{|x-y| \leq b\phi^{-1}(t^{-1})^{-1/2}} p^{(2)}(t, x-y)q(t, y) dy \]

\[ \geq c_2\phi^{-1}(t^{-1})^{d/2} \int_{|x-y| \leq b\phi^{-1}(t^{-1})^{-1/2}} p^{(2)}(t, x-y) dy \]

\[ = c_2\phi^{-1}(t^{-1})^{d/2} \int_{|u| \leq bt^{-1/2}\phi^{-1}(t^{-1})^{-1/2}} p^{(2)}(1, u) du \]

\[ \geq c_2\phi^{-1}(t^{-1})^{d/2} \int_{|u| \leq b\phi^{-1}(1)^{-1/2}} p^{(2)}(1, u) du \]

\[ \geq c_3\phi^{-1}(t^{-1})^{d/2} \]

\[ \geq c_4q(t, x). \]

In the third inequality, we use \( \frac{\phi^{-1}(1)}{\phi^{-1}(t^{-1})} \geq t \) which follows from Lemma 2.3 (See also [19, Lemma 3.1(i)]).

\[ \square \]

**Lemma 2.12.** Suppose \( \phi \) satisfies \( L_a(\gamma, C_L) \) for some \( a \geq 0 \). For \( T \geq 1 \) there exists a positive constant \( c \) such that for all \( t \leq T \) and \( t\phi(|x|^2) \geq 1 \),

\[ p(t, x) \geq cp^{(2)}(t, x). \]

**Proof.** We may assume that \( a < \phi^{-1}(t^{-1}) \) by Remark 1.2. By the condition \( L_a(\gamma, C_L) \) on \( \phi \) (See also [19, Lemma 3.1(ii)]) and Lemma 2.3, we have for \( t \leq T \),

\[ C_L^{-1/\gamma}T^{1/\gamma}\phi^{-1}(t^{-1}) \geq \phi^{-1}(Tt^{-1}) \geq Tt^{-1}\phi^{-1}(1). \]  

(2.5)

If \( |y| \leq \phi^{-1}(t^{-1})^{-1/2} \), then \( q(t, y) \geq C_1^{-1}\phi^{-1}(t^{-1})^{d/2} \) by (2.2). Also \( |x-y| \leq |x|+|y| \leq 2\phi^{-1}(t^{-1})^{-1/2} \) and (2.5) imply

\[ \exp(-\frac{|x-y|^2}{4t}) \geq \exp(-\frac{4\phi^{-1}(t^{-1})^{-1}}{4t}) \geq \exp(-C_L^{-1/\gamma}\phi^{-1}(1)^{-1}T^{1/\gamma-1}). \]

Therefore for \( t \leq T \) and \( t\phi(|x|^2) \geq 1 \),

\[ p(t, x) \geq \int_{|y| \leq \phi^{-1}(t^{-1})^{-1/2}} p^{(2)}(t, x-y)q(t, y) dy \]

\[ \geq c_1\phi^{-1}(t^{-1})^{d/2} \int_{|y| \leq \phi^{-1}(t^{-1})^{-1/2}} p^{(2)}(t, x-y) dy \]

\[ \geq c_1\phi^{-1}(t^{-1})^{d/2}(4\pi t)^{-d/2}\int_{|y| \leq \phi^{-1}(t^{-1})^{-1/2}} dy \]

\[ \geq c_2(4\pi t)^{-d/2} \geq c_2p^{(2)}(t, x). \]

\[ \square \]

**Proof of Theorem 1.3.** Combining Lemma 2.4, Lemma 2.2, Lemma 2.5, and Lemma 2.6, we get the upper bound of \( p(t, x) \). Using Lemma 2.4, Lemma 2.8, Lemma 2.10, Lemma 2.11, and Lemma 2.12, we get the lower bound of \( p(t, x) \). See Figure 1.

\[ \square \]
3. Dirichlet heat kernel estimates in $C^{1,1}$-open sets

Recall that $p_D(t,x,y)$ is the transition density of $X^D$. In this section we obtain the sharp estimates of $p_D(t,x,y)$ in $C^{1,1}$-open sets.

3.1. Lower bounds. In this subsection we derive the lower bound estimate on $p_D(t,x,y)$ when $D$ is a $C^{1,1}$-open set. When $D$ is unbounded, we assume that the path distance in each connected component of $D$ is comparable to the Euclidean distance. Since the proofs are almost identical, we will provide a proof when $D$ is a bounded $C^{1,1}$-open set. We will use some relation between killed subordinate Brownian motions and subordinate killed Brownian motions.

Let $T_t$ be a subordinator whose Laplace exponent $\phi$ is given by (1.1). Then $t + T_t$ is a subordinator which has the same law as $S_t$. So $\{X_t; t \geq 0\}$ starting from $x$ has the same distribution as $\{B_{t+T_t}; t \geq 0\}$ starting from $x$. Suppose that $U$ is an open subset of $\mathbb{R}^d$. We denote by $B^U$ the part process of $B$ killed upon leaving $U$. The process $\{Z_t^U; t \geq 0\}$ defined by $Z_t^U = B_{t+T_t}^U$ is called a subordinate killed Brownian motion in $U$. Let $q_U(t,x,y)$ be the transition density of $Z^U$. Denote by $\zeta^{Z^U}$ the lifetime of $Z^U$. Clearly, $Z_t^U = B_{t+T_t}$ for every $t \in [0, \zeta^{Z^U})$. Therefore we have

$$p_U(t,z,w) \geq q_U(t,z,w) \quad \text{for} \quad (t,z,w) \in (0,\infty) \times U \times U.$$ 

In the next proposition we will use [19, Proposition 2.4]. Note that there is a typo in [19, Proposition 2.4]. $\alpha\phi^{-1}(\beta^{-1})$ in the display there should be $\alpha\phi^{-1}(\beta^{-1})$. 

Figure 1. Regions of heat kernel estimates for $p(t,x)$. Dotted line corresponds to $t = |x|^2$ and full line corresponds to $t\phi(|x|^{-2}) = 1$. Note that heat kernel estimate is different to [10, Theorem 1.4], when $t\phi(|x|^{-2}) \leq 1$ and $|x| > 1$. $p^{(2)}(t,c_1 x)$ additionally appear in our case.
Proposition 3.1. Suppose that $D$ is a $C^{1,1}$-open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda_0)$. If $D$ is bounded, we assume that $\phi$ satisfies $L_a(\gamma, C_L)$ for some $a > 0$. If $D$ is unbounded, we assume that $\phi$ satisfies $L_0(\gamma, C_L)$ and the path distance in each connected component of $D$ is comparable to the Euclidean distance with characteristic $\lambda_0$. For any $T > 0$ there exist positive constants $c_1 = c_1(R_0, \Lambda_0, \lambda_0, T, \phi)$ and $c_2 = (R_0, \Lambda_0, \lambda_0)$ such that for all $t \in (0, T]$ and $x, y$ in the same connected component of $D$,

$$p_D(t, x, y) \geq c_1 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \phi^{-1}(t^{-1})^{d/2} e^{-c_2|x-y|^2 \phi^{-1}(t^{-1})}.$$

Proof. Suppose that $x$ and $y$ are in the same component of $D$ and $\rho \in (0, 1)$ is the constant in [19, Proposition 2.4] for $\alpha = 2$ and $\beta = 1$. Without loss of generality, let $T \geq 1$ and $a$ satisfies $T < \rho \phi(a)^{-1}$ by Remark [12]. Let $\tilde{p}_D(t, z, w)$ be the transition density of $B^D$ (killed Brownian motion) and $q_D(t, x, y)$ be the transition density of $B^D_{S_1}$ (subordinate killed Brownian motion). By [14, Theorem 3.3] (see also [22, Theorem 1.2]) (where the comparability condition on the path distance in each component of $D$ with the Euclidean distance is used if $D$ is unbounded), there exists positive constants $c_3 = c_3(R_0, \Lambda_0, \lambda_0, T, \phi)$ and $c_4 = c_4(R_0, \Lambda_0, \lambda_0)$ such that for any $(s, z, w) \in (0, \phi^{-1}(\rho T^{-1})^{-1}] \times D \times D$,

$$\tilde{p}_D(s, z, w) \geq c_3 \left(1 \wedge \frac{\delta_D(z)}{\sqrt{s}}\right) \left(1 \wedge \frac{\delta_D(w)}{\sqrt{s}}\right) s^{-d/2} e^{-c_4|z-w|^2/2}.$$

(Although not explicitly mentioned in [14], a careful examination of the proofs in [14] reveals that the constants $c_3$ and $c_4$ in the above lower bound estimate can be chosen to depend only on $(R_0, \Lambda_0, \lambda_0, T, \phi)$ and $(R_0, \Lambda_0, \lambda_0)$, respectively.)

We have that for $0 < t \leq T$,

$$p_D(t, x, y) \geq q_D(t, x, y) = \int_{(0, \infty)} \tilde{p}_D(s, x, y) \mathbb{P}(S_t \in ds)$$

$$\geq c_3 \int_{[2^{-1} \phi^{-1}(t^{-1})^{-1}, \phi^{-1}(\rho t^{-1})^{-1}]} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{s}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{s}}\right) s^{-d/2} e^{-c_4|x-y|^2/s} \mathbb{P}(S_t \in ds)$$

$$\geq c_3 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{\phi^{-1}(\rho t^{-1})^{-1}}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{\phi^{-1}(\rho t^{-1})^{-1}}}\right) \phi^{-1}(\rho t^{-1})^{d/2} e^{-2c_4|x-y|^2 \phi^{-1}(t^{-1})}$$

$$\times \mathbb{P}(2^{-1} \phi^{-1}(t^{-1})^{-1} \leq S_t \leq \phi^{-1}(\rho t^{-1})^{-1}).$$

Since $0 < t < \rho \phi(a)^{-1}$, using the condition $L_a(\gamma, C_L)$ on $\phi$ (also see [19, Lemma 3.1(ii)])], we have

$$\phi^{-1}(\rho t^{-1}) = \phi^{-1}(t^{-1}) \phi^{-1}(t^{-1})^{-1} \geq C_L^{1/\gamma} \rho^{1/\gamma} \phi^{-1}(t^{-1}).$$

Using this and [25], we have

$$\phi^{-1}(\rho t^{-1}) \geq C_L^{2/\gamma} \rho^{1/\gamma} T^{1-1/\gamma} \phi^{-1}(1) t^{-1}. $$
Using the last two displays and [19, Proposition 2.4] we get
\[
p_D(t, x, y) \geq c_5 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \phi^{-1}(\rho t^{-1}) \frac{d/2}{e^{-2c_4|x-y|^2\phi^{-1}(t^{-1})} \tau},
\]
where \(\tau\) is the constant in [19, Proposition 2.4] for \(\alpha = 2\) and \(\beta = 1\). \(\square\)

**Lemma 3.2.** For any positive constants \(a, b\) and \(T\), there exists \(c > 0\) such that for all \(z \in \mathbb{R}^d\) and \(0 < t \leq T\),
\[
\inf_{y \in B(z, at^{1/2})} \mathbb{P}^y(\tau_{B(z, at^{1/2})} > bt) \geq c
\]

**Proof.** See [13, Lemma 2.3]. \(\square\)

Although the proof of the following Lemma is similar to that of Lemma 2.4 of [13], we give the proof again to make the paper self-contained.

**Lemma 3.3.** Suppose \(H\) satisfies \(L_a(\gamma, C_L)\) and \(U_a(\delta, C_U)\) with \(\delta < 2\) for some \(a > 0\) \((L_0(\gamma, C_L)\) and \(U_0(\delta, C_U)\), respectively). Then for every \(T > 0, M > 0\) and \(b > 0\) there exists \(c > 0\) such that we have that for all \(t \in (0, T]\) and \(u, v \in \mathbb{R}^d\) satisfying \(|u - v| \leq M/2\) \((u, v \in \mathbb{R}^d, \) respectively)
\[
p_E(t, u, v) \geq c(t^{-d/2} \wedge t|u - v|^{-d} H(|u - v|^{-2}))
\]
where \(E := B(u, bt^{1/2}) \cup B(v, bt^{1/2})\).

**Proof.** We fix \(b > 0\) and \(u, v \in \mathbb{R}^d\) satisfying \(|u - v| \leq M/2\), and let \(r_t := bt^{1/2}\). If \(|u - v| \leq r_t/2\), by [13, Lemma 2.1] (with \(\sqrt{\lambda} = r_t\) and \(D = B(0, 1)\)),
\[
p_E(t, u, v) \geq \inf_{|z| < r_t/2} p_{B(0, r_t)}(t, 0, z) = \inf_{|z| < r_t/2} p_{B(0, r_t)}(r_t^2(t/r_t^2), 0, z)
\]
\[
\geq c_1 t^{-d/2} \left(1 \wedge \frac{r_t}{\sqrt{t}}\right) \left(1 \wedge \frac{r_t}{2\sqrt{t}}\right) e^{-c_4 r_t^2/t} \geq c_3 t^{-d/2}.
\]
If $|u - v| \geq r_t/2$, since the distance between $B(u, r_t/8)$ and $B(v, r_t/8)$ is at least $r_t/4$, we have by the strong Markov property and the Lévy system of Lemma 3.4. Suppose the Gaussian component and another off-diagonal estimate component.

Proof. By Lemma 2.3 there exist $c_1, c_2, R$ and $T$ so that $t^{-d/2}e^{-r^2/(c_1t)} + \phi^{-1}(t^{-1})^{d/2}e^{-c_2r^2\phi^{-1}(t^{-1})} \leq c_3 r^{-d} H(r^{-2})$ for every $r \geq R$ and $t \in (0, T]$.

Thus using (2.4) and Lemma 2.3 we have

$$p_E(t, u, v) \geq c_5 t j(2|u - v|) \geq c_6 t 2^{-d} |u - v|^{-d} H(|u - v|^{-2}/4) \geq 2^{-d-4} c_6 t |u - v|^{-d} H(|u - v|^{-2}).$$

The next lemma say that if $x$ and $y$ are far away, the jumping kernel component dominates the Gaussian component and another off-diagonal estimate component.

Lemma 3.4. Suppose $\phi$ satisfies $L_a(\gamma, C_L)$ for some $a \geq 0$. For any given positive constants $c_1, c_2, R$ and $T$, there is a positive constant $c_3 = c_3(R, T, c_1, c_2)$ so that $t^{-d/2}e^{-r^2/(c_1t)} + \phi^{-1}(t^{-1})^{d/2}e^{-c_2r^2\phi^{-1}(t^{-1})} \leq c_3 r^{-d} H(r^{-2})$ for every $r \geq R$ and $t \in (0, T]$.

Proof. By Lemma 2.3 there exist $c_4 > 0$ and $c_5 > 0$ such that

$$r^{-d} H(r^{-2}) \geq r^{-d-4} \geq c_4 e^{-c_5 r} \text{ for every } r > 1.$$

For $r > 1 \vee (2c_1 c_5 T) \vee \frac{2c_4}{c_2} \phi^{-1}(T^{-1})^{-1}$ and $t \in (0, T]$, we have following inequalities

$$r^2/(2c_1t) > c_5 r, \quad c_2 r^2 \phi^{-1}(t^{-1})/2 > c_5 r.$$
\[ t^{-d/2-1} e^{-r^2/(2c_1 t)} \leq t^{-d/2-1} e^{-1/(2c_1 t)} \leq \sup_{0 < s \leq T} s^{-d/2-1} e^{-1/(2c_1 s)} =: c_6 < \infty, \]

and

\[
\phi^{-1}(t^{-1})^{d/2} t^{-1} e^{-c_2 r^2 \phi^{-1}(t^{-1})/2} \leq \sup_{0 < s \leq T} \phi^{-1}(s^{-1})^{d/2} s^{-1} e^{-c_2 \phi^{-1}(s^{-1})/2} \leq T^{1/\gamma - 1/\gamma} C_L^{1/\gamma - 1} \sup_{0 < s \leq T} \phi^{-1}(s^{-1})^{d/2+1} e^{-c_2 \phi^{-1}(s^{-1})/2} =: c_7 < \infty.
\]

In the last inequality we have used (2.5). (Without loss of generality, we can assume that \( a \leq \phi^{-1}(T^{-1}). \) Therefore when \( r > 1 \lor (2c_1 c_5 t) \lor \frac{2cs}{c_2} \phi^{-1}(T^{-1})^{-1} \) and \( t \in (0, T], \) we have

\[ t^{-d/2} e^{-r^2/(c_1 t)} \leq c_6 t e^{-r^2/(2c_1 t)} \leq c_6 t e^{-c_7 r} \leq (c_6/c_4) tr^{-d} H(r^{-2}) \]

and

\[ \phi^{-1}(t^{-1})^{d/2} e^{-c_2 r^2 \phi^{-1}(t^{-1})/2} \leq c_7 t e^{-c_2 r^2 \phi^{-1}(t^{-1})/2} \leq c_7 t e^{-c_7 r} \leq (c_7/c_4) tr^{-d} H(r^{-2}). \]

When \( R \leq r \leq 1 \lor (2c_1 c_5 T) \lor \frac{2cs}{c_2} \phi^{-1}(T^{-1})^{-1} \) and \( t \in (0, T], \) clearly

\[ t^{-d/2} e^{-r^2/(c_1 t)} \leq t \left( \sup_{s \leq T} s^{-d/2-1} e^{-R^2/(c_1 s)} \right) \leq c_8 tr^{-d} H(r^{-2}) \]

and

\[ \phi^{-1}(t^{-1})^{d/2} e^{-c_2 r^2 \phi^{-1}(t^{-1})/2} \leq t \left( \sup_{s \leq T} \phi^{-1}(s^{-1})^{d/2} s^{-1} e^{-c_2 R^2 \phi^{-1}(s^{-1})} \right) \leq c_9 tr^{-d} H(r^{-2}). \]

\[ \square \]

**Proof of Theorem 1.5 (2) and Theorem 1.6 (2).** Since two proofs are almost identical, we just prove Theorem 1.5 (2). First note that the distance between two distinct connected components of \( D \) is at least \( R_0. \) Since \( D \) is a \( C^{1,1} \) open set, it satisfies the uniform interior ball condition with radius \( r_0 = r_0(R_0, \Lambda_0) \in (0, R_0] \): there exists \( r_0 = r_0(R_0, \Lambda_0) \in (0, R_0] \) such that for any \( x \in D \) with \( \delta_D(x) < r_0, \) there are \( z_x \in \partial D \) so that \( |x - z_x| = \delta_D(x) \) and that \( B(x_0, r_0) \subset D \) for \( x_0 = z_x + r_0(x - z_x)/|x - z_x|. \) Set \( T_0 = (r_0/4)^2. \) Using such uniform interior ball condition, by considering the cases \( \delta_D(x) < r_0 \) and \( \delta_D(x) > r_0, \) there exists \( L = L(r_0) > 1 \) such that, for all \( t \in (0, T_0] \) and \( x, y \in D, \) we can choose \( \xi^t_x \in D \cap B(x, L \sqrt{t}) \) and \( \xi^t_y \in D \cap B(y, L \sqrt{t}) \) so that \( B(\xi^t_x, 2 \sqrt{t}) \) and \( B(\xi^t_y, 2 \sqrt{t}) \) are subsets of the connected components of \( D \) that contains \( x \) and \( y, \) respectively.

We first consider the case \( t \in (0, T_0]. \) Note that by the semigroup property,

\[ p_D(t, x, y) \geq \int_{B(\xi^t_x, \sqrt{t})} \int_{B(\xi^t_y, \sqrt{t})} p_D(t/3, x, u)p_D(t/3, u, v)p_D(t/3, v, y)du dv. \] \hspace{1cm} (3.1)

For \( u \in B(\xi^t_x, \sqrt{t}), \) we have

\[ \delta_D(u) \geq \sqrt{t} \quad \text{and} \quad |x - u| \leq |x - \xi^t_x| + |\xi^t_x - u| \leq L \sqrt{t} + \sqrt{t} = (L + 1) \sqrt{t}. \]
Thus by Lemma 2.1, for \( t \in (0, T_0] \),
\[
\int_{B(\xi^t, \sqrt{t})} p_D(t/3, x, u) du \geq c_3 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \int_{B(\xi^t, \sqrt{t})} \left( 1 \wedge \frac{\delta_D(u)}{\sqrt{t}} \right) t^{-d/2} e^{-c_4|x-u|^2/t} du \geq c_3 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) t^{-d/2} c e^{-c_4|\xi^t-x|^2/t} \geq c_5 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right).
\]

Similarly, for \( t \in (0, T_0] \) and \( v \in B(\xi^t, \sqrt{t}) \),
\[
\int_{B(\xi^t, \sqrt{t})} p_D(t/3, y, v) dv \geq c_5 \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right).
\]

Using (3.1), Proposition 3.3, symmetry and (3.2)–(3.3), we have
\[
p_D(t, x, y)
\geq c_6 \int_{B(\xi^t, \sqrt{t})} \int_{B(\xi^t, \sqrt{t})} p_D(t/3, x, u)p_{B(\xi^t, \sqrt{t})}(t/3, u, v)p_D(t/3, v, y) du dv
\geq c_6 \int_{B(\xi^t, \sqrt{t})} \int_{B(\xi^t, \sqrt{t})} p_D(t/3, x, u) \left( t^{-d/2} \wedge \left( \frac{tH(|u-v|^2)}{|u-v|^d} \right) \right) p_D(t/3, v, y) du dv
\geq c_6 \left( \inf_{(u,v) \in B(\xi^t, \sqrt{t}) \times B(\xi^t, \sqrt{t})} \left( t^{-d/2} \wedge \left( \frac{tH(|u-v|^2)}{|u-v|^d} \right) \right) \right) \times \int_{B(\xi^t, \sqrt{t})} \int_{B(\xi^t, \sqrt{t})} p_D(t/3, x, u) p_D(t/3, v, y) du dv
\geq c_6 c_5^2 \left( \inf_{(u,v) \in B(\xi^t, \sqrt{t}) \times B(\xi^t, \sqrt{t})} \left( t^{-d/2} \wedge \left( \frac{tH(|u-v|^2)}{|u-v|^d} \right) \right) \right) \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right).
\]

Suppose that \( |x - y| \geq \sqrt{t}/8 \) and \( t \in (0, T_0] \). Then we have that for \( (u, v) \in B(\xi^t, \sqrt{t}) \times B(\xi^t, \sqrt{t}) \),
\[
|u - v| \leq |u - \xi^t_x| + |\xi^t_x - x| + |x - y| + |y - \xi^t_y| + |\xi^t_y - v| \\
\leq 2(1 + L) \sqrt{t} + |x - y| \leq (16(1 + L)|x - y|).
\]
Thus using Lemma 2.3 we have
\[
\inf_{(u,v) \in B(\xi^t, \sqrt{t}) \times B(\xi^t, \sqrt{t})} \left( t^{-d/2} \wedge \left( \frac{tH(|u-v|^2)}{|u-v|^d} \right) \right) \geq c_7 \left( t^{-d/2} \wedge \left( \frac{tH(|x-y|^2)}{|x-y|^d} \right) \right).
\]

Therefore, for \( |x - y| \geq \sqrt{t}/8 \) and \( t \in (0, T_0] \)
\[
p_D(t, x, y) \geq c_8 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left( t^{-d/2} \wedge \left( \frac{tH(|x-y|^2)}{|x-y|^d} \right) \right).
\]

Using the inequality (3.5), we will obtain the sharp lower bound estimates by considering the following three cases.

Case (1): Suppose that \( |x - y| \geq \sqrt{t}/8 \), \( t \in (0, T_0] \), and \( x \) and \( y \) are contained in same
connected component of $D$. Combining with (3.4), Proposition 3.1 and [13, Lemma 2.1], we conclude that

$$p_D(t, x, y) \geq c_9 \left( \frac{1 \wedge \delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \times \left( t^{-d/2} \right) \left( t^{-d/2} e^{-\frac{|x-y|^2}{c_{11}t}} + \phi^{-1}(t^{-1})d/2 e^{-c_{10}|x-y|^2\phi^{-1}(t^{-1})} \right).$$

(3.6)

Case (2): Suppose that $|x - y| \geq \sqrt{t}/8$, $t \in (0, T_0]$, and $x$ and $y$ are contained in two distinct connected components of $D$. By (3.5) and Lemma 3.4, we have the same conclusion in (3.6).

Case (3): Suppose that $|x - y| < \sqrt{t}/8$ and $t \in (0, T_0]$. In this case $x$ and $y$ are in the same connected component. For $(u, v) \in B(\xi^t_x, \sqrt{t}) \times B(\xi^t_y, \sqrt{t})$, $|u - v| \leq 2(1 + L)\sqrt{t} + |x - y| \leq (2(1 + L) + 8^{-1})\sqrt{t}$. Thus by [13, Lemma 2.1], we have that for every $(u, v) \in B(\xi^t_x, \sqrt{t}) \times B(\xi^t_y, \sqrt{t})$,

$$p_D(t/3, u, v) \geq c_{12} \left( 1 \wedge \frac{\delta_D(u)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(v)}{\sqrt{t}} \right) t^{-d/2} e^{-c_{13}|u-v|^2/t} \geq c_{14}t^{-d/2}.$$  

Therefore by (3.1)–(3.3), for $t \leq T_0$,

$$p_D(t, x, y) \geq c_{14}t^{2}(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) t^{-d/2} \geq c_{14}t^2 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left( t^{-d/2} e^{-\frac{|x-y|^2}{c_{11}t}} + \phi^{-1}(t^{-1})d/2 e^{-c_{16}|x-y|^2\phi^{-1}(t^{-1})} \right) \times \left( \frac{1 \wedge \delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right).$$

(3.7)

Combining the above three cases, we get (1.5) for $t \in (0, T_0]$. When $T > T_0$ and $t \in (T_0, T]$, observe that $T_0/3 \leq t - 2T_0/3 \leq T - 2T_0/3 \leq (T/T_0 - 2/3)T_0$, that is, $t - 2T_0/3$ is comparable to $T_0/3$ with some universal constants that depend only on $T$ and $T_0$. Using the inequality

$$p_D(t, x, y) \geq \int_{B(\xi^t_x, \sqrt{T_0})} \int_{B(\xi^t_y, \sqrt{T_0})} p_D(T_0/3, x, u)p_D(t - 2T_0/3, u, v)p_D(T_0/3, v, y)dudv \geq \int_{B(\xi^t_x, \sqrt{T_0})} \int_{B(\xi^t_y, \sqrt{T_0})} p_D(T_0/3, x, u)p_B(u, \sqrt{T_0/2})p_B(v, \sqrt{T_0/2})(t - 2T_0/3, u, v)p_D(T_0/3, v, y)dudv$$

instead of (3.1) and following the argument in (3.4) and (3.5) we have

$$p_D(t, x, y) \geq c_{17}(1 \wedge \frac{\delta_D(x)}{\sqrt{T_0}}) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{T_0}} \right) \left( T_0^{-d/2} \wedge H(t, x, y)^{-2} \right).$$
Consider the cases \(|x - y| \geq \sqrt{T_0}/8\) and \(|x - y| < \sqrt{T_0}/8\) separately and follow the above three cases. Then since \(\frac{\partial p}{\partial t} \leq T_0 < t\) for \(t \in [T_0, T]\), we can obtain (1.5) for \(t \in [T_0, T]\) and hence for \(t \in (0, T]\). \(\Box\)

3.2. **Upper bounds.** In this subsection we derive the upper bound estimate on \(p_D(t, x, y)\) when \(D\) is a \(C^{1,1}\)-open set (not necessarily bounded). We use the following lemma in [13, Lemma 3.1].

**Lemma 3.5 ([13, Lemma 3.1]).** Suppose that \(U_1, U_3, E\) are open subsets of \(\mathbb{R}^d\) with \(U_1, U_3 \subset E\) and \(\text{dist}(U_1, U_3) > 0\). Let \(U_2 := E \setminus (U_1 \cup U_3)\). If \(x \in U_1\) and \(y \in U_3\), then for every \(t > 0\),

\[
p_E(t, x, y) \leq \mathbb{P}_x(X_{\tau_{U_1}} \in U_2) \left( \sup_{s \leq t} p_E(s, z, y) \right) + \int_0^t \mathbb{P}_x(\tau_{U_1} > s) \mathbb{P}_y(\tau_E > t - s) ds \left( \sup_{u \in U_1, z \in U_3} J(u, z) \right) \leq \mathbb{P}_x(X_{\tau_{U_1}} \in U_2) \left( \sup_{s \leq t} p(s, z, y) \right) + (t \wedge \mathbb{E}_x[\tau_{U_1}]) \left( \sup_{u \in U_1, z \in U_3} J(u, z) \right). \tag{3.8}
\]

Note that by Remark 2.7 or Theorem 1.3 there exist positive constants \(c, a_U, C_2\) such that

\[
p(t, x, y) \leq c \left( t^{-d/2} \wedge \left( t^{-d/2} e^{-\frac{|x-y|^2}{2a_U^2}} + \frac{tH(|x-y|^{-2})}{|x-y|^d} + \phi^{-1}(t^{-1})d/2 e^{-a_U|x-y|^2}\phi^{-1}(t^{-1}) \right) \right). \tag{3.10}
\]

The boundary Harnack principle for subordinate Brownian motions with Gaussian components was proved in [18] for any \(C^{1,1}\)-open set, see [18, Theorem 1.2]. In [18, Theorem 1.2], it is assumed that \(\phi\) is a complete Bernstein function and that the Lévy density \(\mu\) of \(S\) satisfies growth condition near zero, i.e., for any \(K > 0\), there exists \(c = c(K) > 1\) such that \(\mu(r) \leq cr(2r)\).

Note that in the proof of [18, Theorem 1.2], as a consequence of the growth condition of Lévy density of \(S\) and assumption that \(\phi\) is a complete Bernstein function, in fact, the following conditions of Lévy density \(j\) of \(X\) are actually used (see [18, (2.7), (2.8)]); for any \(K > 0\), there exists \(c_1 = c_1(K) > 1\) such that

\[
j(r) \leq c_1 j(2r), \quad \text{for} \quad r \in (0, K), \tag{3.11}
\]

and there exists \(c_2 > 1\) such that

\[
j(r) \leq c_2 j(r + 1), \quad \text{for} \quad r > 1. \tag{3.12}
\]

If, instead of the assumption that \(\phi\) is a complete Bernstein function and Lévy density \(\mu\) satisfies growth condition near zero, we assume that \(H\) satisfies \(L_0(\gamma, C_L)\) and \(U_0(\delta, C_U)\) with \(\delta < 2\), then by (2.4), (3.11) and (3.12) hold. Thus the boundary Harnack principle still hold. But if we assume that \(H\) satisfies \(L_0(\gamma, C_L)\) and \(U_0(\delta, C_U)\) with \(\delta < 2\) for some \(a > 0\), then (3.12) may not holds. Nonetheless, if we only consider harmonic functions not only vanishing continuously on \(D^c \cap B(Q, r)\), \(Q \in \partial D\), but also zero on \(D^c\), then we don’t need the condition (3.12). Thus we have the following modified theorem.
Theorem 3.6. Let $D$ is a $C^{1,1}$-open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda_0)$. If $D$ is bounded, then we assume that $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$. If $D$ is unbounded, then we assume that $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$. Then there exists a positive constant $c = c(d, \Lambda_0, R_0)$ such that for $r \in (0, R_0]$, $Q \in \partial D$ and any nonnegative function $f$ in $\mathbb{R}^d$ which is harmonic in $D \cap B(Q, r)$ with respect to $X$, zero on $D^c$ and vanishes continuously on $D^c \cap B(Q, r)$, we have

$$\frac{f(x)}{\delta_D(x)} \leq c \frac{f(y)}{\delta_D(y)} \quad \text{for every } x, y \in D \cap B(Q, r/2). \quad (3.13)$$

Proof. Since we have explained before the statement of the theorem why theorem holds for the unbounded case, we will just prove the theorem when $D$ is bounded and $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$.

Let $\mathbb{R}_+^d = \{ x = (x_1, ..., x_{d-1}, x_d) : (x, x_d) \in \mathbb{R}^d : x_d > 0 \}$, $V$ is the potential measure of the ladder height process of $X^d_t$, where $X^d_t$ is d-th component of $X_t$, and $w(x) = V((x_d)^+)$. We first show that \[18\] Proposition 3.3 holds under our assumptions, i.e., we claim that for any positive constants $r_0$ and $M$, we have

$$\sup_{x \in \mathbb{R}^d: 0 < x_d < M} \int_{B(x, r_0) \cap \mathbb{R}_+^d} w(y) j(|x - y|) dy < \infty, \quad (3.14)$$

Once we have (3.14), then for $f$ which is harmonic in $D \cap B(Q, r)$ with respect to $X$, zero on $D^c$ and vanishes continuously on $D^c \cap B(Q, r)$ where $r \in (0, R_0]$ and $Q \in \partial D$, we can follow the proofs of \[18\] Theorem 5.3 and Theorem 1.2] line by line without using (3.12).

By \[1\] Theorem 5.7, page 79] and \[17\] Lemma 2.1, $V$ is absolutely continuous and has a continuous and strictly positive density $v$ such that $v(0+) = 1$. It is also well known that $V$ is subadditive, i.e., $V(s+t) \leq V(s) + V(t)$, $s, t \in \mathbb{R}$ (See \[1\] page 74],) and $V(\infty) = \infty$. Without loss of generality we assume that $\bar{x} = 0$. Note that for $0 < x_d < M$ and $y \in B(x, r_0)^c$,

$$w(y) = V((y_d)^+) \leq V(|y|) \leq V(M + |x - y|) \leq V(M) + V(|x - y|). \quad (3.15)$$

Let $L(r) = \int_r^\infty r^{d-1} j(r) dr$, then by \[2\] (2.23)], $L(r) \leq c_1/V(r)^2$. Using (3.15), the integration by parts and \[2\] (2.23)] twice, we have

$$\sup_{x \in \mathbb{R}^d: 0 < x_d < M} \int_{B(x, r_0)^c \cap \mathbb{R}_+^d} w(y) j(|x - y|) dy$$

$$\leq \sup_{x \in \mathbb{R}^d: 0 < x_d < M} \int_{B(x, r_0)^c} (V(M) + V(|x - y|) j(|x - y|) dy$$

$$\leq c_2 \int_{r_0}^\infty (V(M) + V(r)) r^{d-1} j(r) dr = c_2 L(r_0)(V(M) + V(r_0)) + c_2 \int_{r_0}^\infty V'(r) L(r) dr$$

$$\leq c_2 L(r_0)(V(M) + V(r_0)) + c_3 \int_{r_0}^\infty \frac{V'(r)}{V(r)^2} dr < \infty.$$

We have proved (3.14).
For the remainder of this section, we follow proofs of [13, Proposition 3.2 and Theorem 1.1(i)]. First note that for $C^{1,1}$-open set $D \subset \mathbb{R}^d$ with characteristics $(R_0, \Lambda_0)$, there exists $r_0 = r_0(R_0, \Lambda_0) \in (0, R_0]$ such that $D$ satisfies the uniform interior and uniform exterior ball conditions with radius $r_0$. We will use such $r_0 > 0$ in the proof of the next proposition and Theorem [13] (1).

**Proposition 3.7.** Let $D$ is a $C^{1,1}$-open set in $\mathbb{R}^d$ with characteristics $(R_0, \Lambda_0)$. If $D$ is bounded, then we assume that $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$. If $D$ is unbounded, then we assume that $H$ satisfies $L_0(\gamma, C_L)$ and $U_0(\delta, C_U)$ with $\delta < 2$. For every $T > 0$, there exists $c > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t, x, y) \leq c \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \times \left( t^{-d/2} \wedge \left( t^{-d/2} e^{-\frac{|x-y|^2}{4|\alpha|^2}} + tH\left( \frac{|x-y|^2}{|x-y|^2} \right) + \phi^{-1}(t^{-1}) \frac{d}{2} e^{-\frac{a_U}{4}|x-y|^2}\phi^{-1}(t^{-1}) \right) \right),$$

(3.16)

where the constants $C_2$ and $a_U$ are from (3.10).

**Proof.** We prove the proposition for the case that $D$ is bounded, $H$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$ only, because the proof of the other case is almost identical.

Fix $T > 0$ and $t \in (0, T]$. Let $x, y \in D$. We just consider the case $\delta_D(x) < r_0\sqrt{t}/(16\sqrt{T}) \leq r_0/16$, if not, we can directly obtain (3.16) by (3.10). Choose $x_0 \in \partial D$ and $x_1 \in D$ such that $\delta_D(x) = |x - x_0|$ and $x_1 = x_0 + \frac{r_0\sqrt{T}}{16\sqrt{T}}n(x_0)$, respectively, where $n(x_0) = (x - x_0)/|x_0 - x|$ be the unit inward normal of $D$ at the boundary point $x_0$. Define

$$U_1 := B(x_0, r_0\sqrt{T}/(8\sqrt{T})) \cap D.$$ 

Since (3.14) holds, by [13, Lemma 4.3]

$$\mathbb{E}_x[\tau_{U_1}] \leq c_1 \sqrt{t}\delta_D(x).$$

(3.17)

Using Theorem 3.6 and $\delta_D(x_1) = \frac{r_0\sqrt{T}}{16\sqrt{T}}$, we have

$$\mathbb{P}_x(X_{\tau_U} \in D \setminus U_1) \leq c_2 \mathbb{P}_x(X_{\tau_U} \in D \setminus U_1) \frac{\delta_D(x)}{\delta_D(x_1)} \leq c_2 \frac{16\sqrt{T}\delta_D(x)}{r_0\sqrt{t}} \leq c_3 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right).$$

(3.18)

Thus by (3.18) and (3.17) we have

$$\mathbb{P}_x(\tau_D > t/2) \leq \mathbb{P}_x(\tau_U > t/2) + \mathbb{P}_x(X_{\tau_U} \in D \setminus U_1) \leq \left( 1 \wedge \left( \frac{2}{t} \mathbb{E}_x[\tau_{U_1}] \right) \right) + \mathbb{P}_x(X_{\tau_U} \in D \setminus U_1) \leq c_4 \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right).$$

(3.19)

Now we estimate $p_D(t, x, y)$ considering two cases separately. Let $c_5 := (d/2) \vee ((dC_{L_{\gamma}}^{1/\gamma} T^{1/\gamma-1}\phi^{-1}(1)^{-1})/(2C_2a_U)) \vee (r_0^2/(4C_2T))$ where $a_U$ and $C_2$ are the constants in (3.10).
Case (1): $|x - y| \leq 2(C_2c_5)^{1/2}\sqrt{t}$. By the semigroup property,

$$p_D(t, x, y) = \int_D p_D(t/2, x, z)p_D(t/2, y, z)dz \leq \left( \sup_{z,w \in D} p(t/2, z, w) \right) \int_D p_D(t/2, x, z)dz.$$ 

Using Theorem 1.3 and (3.19) in the above display, we have

$$p_D(t, x, y) \leq c_6(t/2)^{-d/2}e^{c_{10}t} (\tau_D > t/2) \leq c_6c_2^{d/2}t^{-d/2} \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right).$$

Since $|x - y|^2/(4C_2t) \leq c_5$, we have

$$p_D(t, x, y) \leq c_6c_2^{d/2}e^{c_5t} e^{-|x-y|^2/(4C_2t)} \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right). \quad (3.20)$$

Case (2): $|x - y| \geq 2(C_2c_5)^{1/2}\sqrt{t}$. Define

$$U_3 := \{ z \in D : |z - x| > |x - y|/2 \} \quad \text{and} \quad U_2 := D \setminus (U_1 \cup U_3). \quad (3.21)$$

For $z \in U_2$,

$$\frac{3}{2} |x - y| \geq |x - y| + |x - z| \geq |z - y| \geq |x - y| - |x - z| \geq \frac{|x - y|}{2}. \quad (3.22)$$

By our choice of $c_5$ and (2.5) (we can assume that $a \leq \phi^{-1}(T^{-1})$ by Remark 1.2), we have

$$t \leq |x - y|^2/(2dC_2) \quad \text{and} \quad \phi^{-1}(t^{-1}) \leq C_L^{-1/2}T^{-1/2}t_0^{-1} \leq \frac{a}{2d} \frac{|x - y|^2}{|x| |y|}. \quad (3.23)$$

Thus by (3.10) and (3.22),

$$\sup_{s \leq t, z \in U_2} p(s, z, y) \leq \left( s^{-d/2}e^{-2|x-z|^2/4C_2t} + H(|x - z|^{-2}) \right) \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right).$$

For the last inequality, we argue as follows: by the proof of [19, Corollary 1.3]

$$2^{d+1} t \frac{H(|x - y|^{-2})}{|x - y|^{d}} + \phi^{-1}(t^{-1}) d/2 e^{-\frac{a}{2d}|x-y|^2 \phi^{-1}(t^{-1})} \leq c_9 t \phi(|x - y|^{-2}) \frac{\delta_D(x)}{|x - y|^{d}}. \quad (3.24)$$

On the other hand, by Lemma 2.3

$$t \frac{\phi(|x - y|^{-2})}{|x - y|^{d}} \leq \frac{T \phi((4C_2c_5)^{-1}t^{-1})}{|x - y|^{d}} \leq \frac{c_{10} t^{-d/2}}{|x - y|^{d}}.$$
Therefore
\[ 2^{d+4}t \frac{H(|x-y|^2)}{|x-y|^d} + \phi^{-1}(t^{-1})^{d/2}e^{-\frac{a_U}{4}|x-y|^2\phi^{-1}(t^{-1})} \leq c_{11}t^{-d/2}. \] (3.25)

For \( u \in U_1 \) and \( z \in U_3 \), since \( |z-x|/2 > |x-y|/4 \geq \frac{r_0\sqrt{t}}{4\sqrt{T}} \),
\[ |u-z| \geq |z-x| - |x-x_0| - |x_0-u| \geq |z-x| - \frac{r_0\sqrt{t}}{4\sqrt{T}} \geq \frac{|z-x|}{2} > \frac{|x-y|}{4} \geq \frac{r_0\sqrt{t}}{4\sqrt{T}}. \]

Thus \( \text{dist}(U_1, U_3) > 0 \) and by (2.4) and Lemma 2.3 we have
\[ \sup_{u \in U_1, z \in U_3} J(u, z) \leq \sup_{|u-z| \geq \frac{|x-y|}{2}} \frac{C_1H(|u-z|^2)}{|u-z|^d} \leq \frac{C_14dH(16|x-y|^2)}{|x-y|^d} \leq \frac{C_14d+4H(|x-y|^2)}{|x-y|^d}. \] (3.26)

By the same argument in (3.18), we can apply Theorem 3.6 to get
\[ \mathbb{P}_x(X_{\tau_{U_1}} \in U_2) \leq c_{12}\mathbb{P}_x(X_{\tau_{U_1}} \in U_2) \frac{\delta_D(x)}{\delta_D(x_1)} \leq c_{13} \frac{\delta_D(x)}{\sqrt{t}}. \] (3.27)

Applying (3.17), (3.23), (3.26) and (3.27) in the inequality (3.9), we obtain
\[
p_D(t, x, y) 
\leq c_{14} \frac{\delta_D(x)}{\sqrt{t}} \left( t^{-d/2} \wedge \left( t^{-d/2}e^{-\frac{a_U}{4}|x-y|^2} + t\frac{H(|x-y|^2)}{|x-y|^d} + \phi^{-1}(t^{-1})^{d/2}e^{-\frac{a_U}{4}|x-y|^2\phi^{-1}(t^{-1})} \right) \right) 
+ c_{15} \sqrt{t}\delta_D(x) \frac{H(|x-y|^2)}{|x-y|^d} 
\leq c_{16} \frac{\delta_D(x)}{\sqrt{t}} \left( t^{-d/2} \wedge \left( t^{-d/2}e^{-\frac{a_U}{4}|x-y|^2} + t\frac{H(|x-y|^2)}{|x-y|^d} + \phi^{-1}(t^{-1})^{d/2}e^{-\frac{a_U}{4}|x-y|^2\phi^{-1}(t^{-1})} \right) \right). 
\]

In the last line, (3.25) is used. Combining above two cases, we have completed the proof of the proposition. \(\square\)

**Proof of Theorem 1.5 (1) and Theorem 1.6 (1).** We only prove Theorem 1.5 (1), because both proofs are almost identical. Using (3.8) and (3.16) instead of (3.9) and (3.10) respectively, we will follow the proof of Proposition 3.7.

Fix \( T > 0 \). Let \( t \in (0, T] \) and \( x, y \in D \). By Proposition 3.7, Theorem 1.3 and symmetry, we only need to prove Theorem 1.5 (1) when \( \delta_D(x) \wedge \delta_D(y) < r_0\sqrt{t}/(16\sqrt{T}) \leq r_0/16 \). Thus we assume that \( \delta_D(x) \wedge \delta_D(y) < r_0\sqrt{t}/(16\sqrt{T}) \leq r_0/16 \). Define \( x_0, x_1 \) and \( U_1 \) in the same way as in the proof of Proposition 3.7 and let \( c_1 := ((d+1)/2)\wedge((dC_L^{-1/7}T^{1/7}-1)^{-1}/(C_2a_U)) \wedge (r_0^2/(16C_2T)) \) where \( a_U \) and \( C_2 \) are constants in (3.16). Now we estimate \( p_D(t, x, y) \) by considering the following two cases separately.

**Case (1):** \( |x-y| \leq 4(C_2c_1)^{1/2}\sqrt{t} \). By the semigroup property and symmetry,
\[
p_D(t, x, y) = \int_D p_D(t/2, x, z)p_D(t/2, z, y)dz \leq \left( \sup_{z \in D} p_D(t/2, y, z) \right) \int_D p_D(t/2, x, z)dz. 
\]
Using Proposition 3.7 and (3.19) in the above inequality, we have

\[ p_D(t, x, y) \leq c_2 t^{-d/2} \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \mathbb{P}_x(\tau_D > t/2) \]
\[ \leq c_2 t^{-d/2} \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \]
\[ \leq c_2 e^{c_1 t^{-d/2}} e^{-|x-y|^2/(16C_2 t)} \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right). \]

(3.28)

**Case (2):** \( |x - y| \geq 4(C_2c_1)^{1/2}\sqrt{t}. \) Define \( U_2 \) and \( U_3 \) in the same way as in (3.21). Then by the same way (3.22) and (3.26) hold.

By our choice of \( c_1 \), we have \( t \leq |x - y|^2 / (8(d + 1)C_2) \). Using this and the fact that \( s \to s^{-(d+1)/2}e^{-\beta/s} \) is increasing on the interval \((0, 2\beta/(d + 1)]\), we get for \( s \leq t \),

\[ s^{-(d+1)/2}e^{-\frac{|x-y|^2}{16c_2^2t}} \leq t^{-(d+1)/2}e^{-\frac{|x-y|^2}{16c_2^2t}}. \]

Thus by (3.16) and (3.22),

\[ \sup_{s \leq t, z \in U_2} p_D(s, z, y) \]
\[ \leq c_3 \sup_{s \leq t, z \in U_2} \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{s}} \right) \]
\[ \times \left( s^{-d/2} \wedge \left(s^{-d/2}e^{-\frac{|x-y|^2}{4c_2^2s}} + \frac{sH(|z - y|^{-2})}{|z - y|^d} + \phi^{-1}(s^{-1})d/2 e^{-\frac{\gamma}{16}|z-y|^2\phi^{-1}(s^{-1})} \right) \right) \]
\[ \leq c_3 \delta_D(y) \sup_{s \leq t, |z - y| \geq \frac{|x-y|}{2}} \left( s^{-(d+1)/2}e^{-\frac{|x-y|^2}{4c_2^2s}} + \frac{\sqrt{s}H(|z - y|^{-2})}{|z - y|^d} + \phi^{-1}(s^{-1})d/2 e^{-\frac{\gamma}{16}|z-y|^2\phi^{-1}(s^{-1})} \right) \]
\[ \leq c_4 \delta_D(y) \left( \sup_{s \leq t} s^{-(d+1)/2}e^{-\frac{|x-y|^2}{16c_2^2s}} + \frac{2d\sqrt{t}H(4|x-y|^{-2})}{|x - y|^d} + s^{-1/2}\phi^{-1}(s^{-1})d/2 e^{-\frac{\gamma}{16}|x-y|^2\phi^{-1}(s^{-1})} \right) \]
\[ \leq c_5 \frac{\delta_D(y)}{\sqrt{t}} \left( t^{-d/2}e^{-\frac{|x-y|^2}{16c_2^2t}} + \frac{tH(|x - y|^{-2})}{|x - y|^d} \right) \]
\[ + c_4 \delta_D(y) \left( \sup_{s \leq t} s^{-1/2}e^{-\frac{\gamma}{16}|x-y|^2\phi^{-1}(s^{-1})} \right) \left( \sup_{s \leq t} \phi^{-1}(s^{-1})d/2 e^{-\frac{\gamma}{16}|x-y|^2\phi^{-1}(s^{-1})} \right). \]

(3.29)

Now we find upper bounds of \( s^{-1/2}e^{-\frac{\gamma}{16}|x-y|^2\phi^{-1}(s^{-1})} \) and \( \phi^{-1}(s^{-1})d/2 e^{-\frac{\gamma}{16}|x-y|^2\phi^{-1}(s^{-1})} \) for \( s \leq t. \) By our choice of \( c_1 \) and (2.5), we have

\[ t \leq \frac{a_U}{16d} C_L^{1/\gamma} T^{1-1/\gamma} \phi^{-1}(1)|x - y|^2 \quad \text{and} \quad \phi^{-1}(t^{-1})^{-1} \leq C_L^{1/\gamma} T^{1-1/\gamma} t^{\phi^{-1}(1)^{-1}} \leq \frac{a_U}{16d} |x - y|^2. \]
Using this and the fact that $s \to s^{-d/2}e^{-\beta s}$ is increasing on the interval $(0, 2\beta/d]$, we get
\[
\delta_D(y) \left( \sup_{s \leq t} s^{-1/2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(s^{-1})} \right) \left( \sup_{s \leq t} \phi^{-1}(s^{-1}) s^{-1/2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(s^{-1})} \right) \\
\leq \delta_D(y) \left( \sup_{s \leq t} s^{-1/2} e^{-\frac{aU}{2d} |x-y|^2 C_L^{1/2} T^{1-1/\gamma} \phi^{-1}(1) s^{-1}} \right) \left( \sup_{s \leq t} \phi^{-1}(s^{-1}) s^{-1/2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(s^{-1})} \right) \\
\leq \delta_D(y) t^{-1/2} e^{-\frac{aU}{2d} C_L^{1/2} T^{1-1/\gamma} \phi^{-1}(1)} \phi^{-1}(t^{-1}) d^{-2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(t^{-1})} \\
= \frac{\delta_D(y)}{\sqrt{t}} e^{-\frac{aU}{2d} C_L^{1/2} T^{1-1/\gamma} \phi^{-1}(1)} \phi^{-1}(t^{-1}) d^{-2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(t^{-1})}. \tag{3.30}
\]

Therefore combine (3.29) with (3.30) we get
\[
\sup_{s \leq t, z \in U_2} p_D(s, z, y) \\
\leq c_0 \frac{\delta_D(y)}{\sqrt{t}} \left( t^{-d/2} e^{-\frac{|x-y|^2}{16c_2^2 t}} + tH(|x-y|^2) \right) + \phi^{-1}(t^{-1}) d^{-2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(t^{-1})} \\
\leq c_1 \frac{\delta_D(y)}{\sqrt{t}} \left( t^{-d/2} e^{-\frac{|x-y|^2}{16c_2^2 t}} + tH(|x-y|^2) \right) + \phi^{-1}(t^{-1}) d^{-2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(t^{-1})}.
\tag{3.31}
\]

In the last inequality we have used similar argument as the one leading (3.25). On the other hand by (3.19) we have
\[
\int_0^t \mathbb{P}_x(\tau_{U_1} > s) \mathbb{P}_y(\tau_D > t-s)ds \leq \int_0^t \mathbb{P}_x(\tau_D > s) \mathbb{P}_y(\tau_D > t-s)ds \\
\leq c_8 \int_0^t \frac{\delta_D(x)}{\sqrt{s}} \frac{\delta_D(y)}{\sqrt{t-s}} ds = c_9 \delta_D(x) \delta_D(y) \int_0^1 \frac{1}{\sqrt{r(1-r)}} dr = c_9 \delta_D(x) \delta_D(y). \tag{3.32}
\]

Using (3.26), (3.27), (3.31) and (3.32) in the inequality (3.8), we conclude that
\[
p_D(t, x, y) \\
\leq c_{10} \frac{\delta_D(x) \delta_D(y)}{t} \left( t^{-d/2} e^{-\frac{|x-y|^2}{16c_2^2 t}} + tH(|x-y|^2) \right) + \phi^{-1}(t^{-1}) d^{-2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(t^{-1})} \\
+ c_{11} \frac{\delta_D(x) \delta_D(y)}{t} tH(|x-y|^2) \frac{1}{|x-y|^d} \\
\leq c_{12} \frac{\delta_D(x) \delta_D(y)}{t} \left( t^{-d/2} e^{-\frac{|x-y|^2}{16c_2^2 t}} + tH(|x-y|^2) \right) \frac{1}{|x-y|^d} + \phi^{-1}(t^{-1}) d^{-2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(t^{-1})} \\
= c_{12} \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \\
\times \left( t^{-d/2} e^{-\frac{|x-y|^2}{16c_2^2 t}} + tH(|x-y|^2) \right) \frac{1}{|x-y|^d} + \phi^{-1}(t^{-1}) d^{-2} e^{-\frac{aU}{2d} |x-y|^2 \phi^{-1}(t^{-1})}.
\]

In the second inequality above, we also used similar argument as the one leading (3.25). This combined with (3.28) above completes the proof. \qed

**Proof of Theorem 1.5 (3).** The proof is same as [13] Theorem 1.1(iii), (iv)]. We should
consider Theorem 1.5(1) and Theorem 1.5(2) instead of [13, Theorem 1.1(ii)] and [13, Theorem 2.6], respectively. We omit the proof.

4. Green function estimates

In this section we give the proof of Corollaries 1.4 and 1.7.

**Proof of Corollary 1.4.** Note that by Lemma 2.3, if \(|x| \leq 1, \phi(1) \phi(|x|^{-2})^{-1} \geq |x|^2\), and if \(|x| > 1, \phi(1) \phi(|x|^{-2})^{-1} < |x|^2\). We split the integral

\[
G(x) = \int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} p(t, x) dt + \int_{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2}^{\infty} p(t, x) dt.
\]

By Remark 2.7 and (3.24),

\[
\int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} p(t, x) dt
\]

\[
\leq c_1 \int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} \left( t^{-d/2} e^{-c_2 |x|^2/t} + t |x|^{-d} H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2} e^{-c_3 |x|^2 \phi^{-1}(t^{-1})} \right) dt
\]

\[
\leq c_4 \left( \int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} t^{-d/2} e^{-c_2 |x|^2/t} dt + \int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} t |x|^{-d} \phi(|x|^{-2}) dt \right)
\]

\[
\leq c_5 \left( \int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} t^{-d/2} (|x|^2/t)^{-d/2} dt + |x|^{-d} \phi(|x|^{-2}) \int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} t dt \right)
\]

\[
\leq c_6 \left( |x|^{-d} \phi(|x|^{-2})^{-1} \wedge |x|^2 \right) + |x|^{-d} \phi(|x|^{-2})^{-1} \wedge |x|^2 \phi(|x|^{-2})
\]

\[
\leq c_7 \left( |x|^{-d+2} \wedge |x|^{-d} \phi(|x|^{-2})^{-1} \right).
\]

For \(|x| \leq 1\), using Lemma 2.2,

\[
\int_0^{\infty} p(t, x) dt = \int_0^{|x|^2} p(t, x) dt \leq c_8 \int_0^{|x|^2} t^{-d/2} dt = \frac{2c_8}{d-2} |x|^{-d+2}.
\]

For \(|x| > 1\), using Lemma 2.3 and change of variables,

\[
\int_0^{\phi(1) \phi(|x|^{-2})^{-1} \wedge |x|^2} p(t, x) dt = \int_0^{\infty} p(t, x) dt \leq c_9 \int_0^{\phi(1) \phi(|x|^{-2})^{-1}} \phi^{-1}(t^{-1})^{d/2} dt
\]

\[
= c_9 \int_0^{c_{10} |x|^{-2}} s^{d/2} \left( -\frac{1}{\phi(s)} \right)' ds = c_9 \int_0^{c_{10} |x|^{-2}} s^{d/2-1} \frac{y \phi'(s)}{\phi(s)^2} ds
\]

\[
\leq c_9 \int_0^{c_{10} |x|^{-2}} s^{d/2-1} \frac{1}{\phi(s)} ds,
\]

in the last inequality we use \(\lambda \phi'(\lambda) \leq \phi(\lambda)\) because \(\phi\) is represented by (1.1). Since \(d \geq 3\), we have \(\frac{d}{2} - 2 > -1\). Hence

\[
\int_0^{c_{10} |x|^{-2}} s^{d/2-1} \frac{1}{\phi(s)} ds \leq \frac{c_{11}}{\phi(|x|^{-2})} \int_0^{c_{10} |x|^{-2}} s^{d/2-1} \frac{|x|^{-2}}{s} ds = c_{12} |x|^{-d} \phi(|x|^{-2})^{-1}.
\]
On the other hand, for $|x| > 1$, by Lemma 2.11 and the condition of $L_0(\gamma, C_L)$ on $\phi$, we have

$$G(x) \geq \int_{|x|^{-2} - 1}^{\infty} p(t, x) dt$$

$$\geq c_{13} \int_{|x|^{-2} - 1}^{2|\phi(|x|^{-2})^{-1}} \phi^{-1}(t^{-1})^{d/2} dt \geq c_{13} \phi^{-1} \left( \frac{\phi(|x|^{-2})}{2} \right)^{d/2} \frac{1}{\phi(|x|^{-2})} \geq c_{13} \left( C_L / 2 \right)^{d/(2\gamma)}.$$  

When $|x| \leq 1$, by Lemma 2.11 and Lemma 2.12, we have

$$G(x) \geq \int_{|x|^{-2} - 1}^{2|\phi(|x|^{-2})^{-1}} p(t, x) dt \geq c_{14} \int_{|x|^{-2} - 1}^{2|\phi(|x|^{-2})^{-1}} t^{-d/2} \wedge \phi^{-1}(t^{-1})^{d/2} dt \geq c_{15} \int_{|x|^{-2} - 1}^{2|\phi(|x|^{-2})^{-1}} t^{-d/2} dt \geq c_{15} \left( \frac{|x|^{-d}}{2^{d/2}} \right)^2.$$  

Third inequality holds because for $t \leq 2$, $t^{-d/2} \leq c_\phi^{-1}(t^{-1})^{d/2}$ for some $c_{16} > 0$. Hence we conclude that

$$G(x) \asymp |x|^{-d+2} \wedge |x|^{-d} \phi(|x|^{-2})^{-1}.$$  

\[ \square \]

**Proof of Corollary 1.7.** Recall that $g_D(x, y)$ is defined in Corollary 1.7. Let $T := \text{diam}(D)^2$. Then we have (see the proof of [13, Corollary 1.3])

$$\int_0^T \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) p^{(2)}(t, c(x - y)) dt + \int_T^\infty e^{-\lambda t} \delta_D(x) \delta_D(y) dt \asymp g_D(x, y).$$

By Theorem 1.5 (2) and (3), we can easily obtain $G_D(x, y) \geq c_1 g_D(x, y)$.

Next we consider the upper bound for $G_D(x, y)$. By Theorem 1.5 (1), for the bounded $C^{1,1}$-open set $D,$

$$G_D(x, y) = \int_0^\infty p_D(t, x, y) dt$$

$$\leq c_1 \int_0^T \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right)$$

$$\times \left( t^{-d/2} \wedge \left( p^{(2)}(t, c_2(x - y)) + tH(\frac{|x - y|^{-2}}{|x - y|^d}) + \phi^{-1}(t^{-1})^{d/2} e^{-a_\gamma|x - y|^2 \phi^{-1}(t^{-1})} \right) \right)$$

$$+ c_1 \int_T^\infty e^{-\lambda t} \delta_D(x) \delta_D(y) dt$$

$$\leq c_1 \int_0^T \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right)$$

$$\times \left( p^{(2)}(t, c_2(x - y)) + \left( t^{-d/2} \wedge \left( tH(\frac{|x - y|^{-2}}{|x - y|^d}) + \phi^{-1}(t^{-1})^{d/2} e^{-a_\gamma|x - y|^2 \phi^{-1}(t^{-1})} \right) \right) \right)$$

$$+ c_1 \int_T^\infty e^{-\lambda t} \delta_D(x) \delta_D(y) dt$$

$$\leq c_2 \left( g_D(x, y) + \int_0^T \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left( t^{-d/2} \wedge \frac{t}{|x - y|^{d+2}} \right) dt \right).$$
For the last inequality, we use \( tH(|x-y|^2) \), boundedness of \( D \) and Lemma 2.3:

\[
\frac{tH(|x-y|^2)}{|x-y|^d} + \frac{\phi^{-1}(t^{-1})d^2}{|x-y|^d} e^{-a_U|x-y|^2\phi^{-1}(t^{-1})} \leq \frac{c_3 t\phi(|x-y|^2)}{|x-y|^{d+2}} \leq \frac{c_4 t}{|x-y|^{d+2}}.
\]

To complete the proof, it suffices to show that

\[
\int_0^T \left( 1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left( t^{-d/2} \wedge \frac{t}{|x-y|^{d+2}} \right) \leq c_5 g_D(x, y),
\]

which is \([13, (4.1)]\). Thus the remaining proof is same as the part of proof starting on the page 135 in \([13, Corollary 1.3]\). So we omit it.

\[\square\]

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