THE RIEMANN PROBLEM FOR THE STOCHASTICALLY PERTURBED NON-VISCOUS BURGERS EQUATION AND THE PRESSURELESS GAS DYNAMICS MODEL

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Abstract. Proceeding from the method of stochastic perturbation of a Langevin system associated with the non-viscous Burgers equation we construct a solution to the Riemann problem for the non-interacting particles and sticky particles systems. We analyze the difference in the behavior of discontinuous solutions for these two models and relations between them.

Let us consider the Cauchy problem for the non-viscous Burgers equation:

$$\frac{\partial u}{\partial t} + (u, \nabla) u = 0, \quad u(x,0) = u_0(x),$$  \hspace{1cm} (1)

where $u(x,t) = (u_1, ..., u_n)$ is a vector-function $\mathbb{R}^{n+1} \to \mathbb{R}^n$. It is well known that on the smooth solutions of the Burgers equation this equation is equivalent to the system of ODEs

$$\dot{x}(t) = u(t,x(t)), \quad \dot{u}(t,x(t)) = 0$$  \hspace{1cm} (2)

for the characteristics $x = x(t)$. We associate with (2) the following system of stochastic differential equations:

$$dX_k(t) = U_k(t)dt + \sigma d(W_k)_t,$$

$$dU_k(t) = 0, \quad k = 1, ..., n,$$

$$X(0) = x, \quad U(0) = u,$$  \hspace{1cm} (3)

where $(X(t),U(t))$ runs in the phase space $\mathbb{R}^n \times \mathbb{R}^n$, $\sigma > 0$ is a constant and $(W_k)_t = (W_k)_t$, $k = 1, ..., n$, is the n-dimensional Brownian motion.

Let us introduce a function

$$\hat{u}(t,x) = \frac{\int_{\mathbb{R}^n} uP(t,x,u)du}{\int_{\mathbb{R}^n} P(t,x,u)du},$$  \hspace{1cm} (4)

where $P(t,x,u)$ is the probability density in position and velocity space. This value has a sense of the conditional expectation of $U$ for fixed position $X$ \[^{[1]}\]. If we choose

$$P_0(x,u) = \delta(u - u_0(x))f_0(x) = \prod_{k=1}^{n} \delta(u_k - (u_0(x))_k)f_0(x),$$  \hspace{1cm} (5)

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where \( f_0(x) \) is an arbitrary sufficiently regular nonnegative function such that \( \int_{\mathbb{R}^n} f_0(x)dx = 1 \), then \( \hat{u}(0, x) = u_0(x) \). Certain properties of \( \hat{u}(t, x) \) was studied in [2] and [3].

The density \( P = P(t, x, u) \) obeys the Fokker-Planck equation

\[
\frac{\partial P}{\partial t} = \left[ -\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k} + \sum_{k=1}^{n} \frac{1}{2} \sigma_k^2 \frac{\partial^2}{\partial x_k^2} \right] P, \tag{6}
\]

subject to initial data (5).

We apply the Fourier transform in the variables \( x \) and \( u \) to (6) and (5) and obtain the Cauchy problem for \( \hat{P} = \hat{P}(t, \lambda, \xi) \):

\[
\frac{\partial \hat{P}}{\partial t} = -\frac{1}{2} \lambda^2 |\lambda|^2 \hat{P} + \lambda \frac{\partial \hat{P}}{\partial \xi}, \tag{7}
\]

\[
\hat{P}(0, \lambda, \xi) = \int_{\mathbb{R}^n} e^{-i(x, \lambda)} e^{-i(u_0(s))} f_0(s)ds, \tag{8}
\]

which solution is given by the following formula:

\[
\hat{P}(t, \lambda, \xi) = \hat{P}(0, \lambda, \xi + \lambda t) e^{-\frac{1}{2} \lambda^2 |\lambda|^2 t}. \tag{9}
\]

The inverse Fourier transform allows to find the density \( P(t, x, u) \):

\[
P(t, x, u) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x, \lambda)} e^{i(u_0(s))} \hat{P} d\lambda d\xi = \frac{1}{(\sqrt{2\pi} t)^n} \int_{\mathbb{R}^n} \delta(u - u_0(s)) f_0(s) e^{-\frac{|u_0(s) + x - u|^2}{2\sigma^2 t}} ds. \tag{10}
\]

Then we substitute \( P(t, x, u) \) in (3) and get the following expression for \( \hat{u}(t, x) \):

\[
\hat{u}(t, x) = \int_{\mathbb{R}^n} \frac{u_0(s) f_0(s) e^{-\frac{|u_0(s) + x - u|^2}{2\sigma^2 t}} ds}{\int_{\mathbb{R}^n} f_0(s) e^{-\frac{|u_0(s) + x - u|^2}{2\sigma^2 t}} ds}. \tag{11}
\]

It should be noted that integrals in (11) are defined also for a wider class of \( f_0(x) \) then the probability density of the particle positions in the space at the initial moment of time. If the integral \( \int_{\mathbb{R}^n} f_0(x)dx \) diverges (for example, for \( f_0(x) = \text{const} \)), we consider the domain \([-L, L]^n\), where \( L > 0 \) and use another definition of \( \hat{u}(t, x) \):

\[
\hat{u}(t, x) = \lim_{L \to +\infty} \frac{\int_{[-L, L]^n} u_0(s) f_0(s) e^{-\frac{|u_0(s) + x - u|^2}{2\sigma^2 t}} ds}{\int_{[-L, L]^n} f_0(s) e^{-\frac{|u_0(s) + x - u|^2}{2\sigma^2 t}} ds}. \tag{12}
\]

Evidently, this definition coincides with (11) for \( f_0(x) \in L_1(\mathbb{R}^n) \).

The following property of \( \hat{u}(t, x) \) holds:

**Proposition 1.** Let \( u_0(x) \) and \( f_0(x) \) be bounded functions of class \( C^1 \) and the solution to the respective Cauchy problem (1) keeps smoothness for \( t < t_* \leq +\infty \). Then \( \hat{u}(t, x) \) tends to solution of problem (7) as \( \sigma \to 0 \) for any fixed \((t, x) \in \mathbb{R}^{n+1}, t < t_* \).
Proof. Let us denote \( J(u_0(x)) \) the Jacobian matrix of the map \( x \mapsto u_0(x) \). As it was shown in \([4]\) (Theorem 1), if \( J(u_0(x)) \) has at least one eigenvalue negative for a certain point \( x \in \mathbb{R}^n \), then the classical solution to (1) fails to exist beyond a positive time \( t_* \). Otherwise, \( t_* = \infty \). The matrix \( C(t, x) = (I + tJ(u_0(x))) \), where \( I \) is the identity matrix, fails to be invertible for \( t = t_* \).

The formula (11) (or (12)) implies

\[
\lim_{\sigma \to 0} \hat{u}(t, x) = \frac{\int_{\mathbb{R}^n} u_0(s) f_0(s) \lim_{\sigma \to 0} \frac{1}{(2\pi\sigma)^n} e^{-\frac{|u_0(x)+s-x|^2}{2\sigma^2}} ds}{\int_{\mathbb{R}^n} f_0(s) \lim_{\sigma \to 0} \frac{1}{(2\pi\sigma)^n} e^{-\frac{|u_0(x)+s-x|^2}{2\sigma^2}} ds}.
\]

If \( p(t, x, s) = u_0(s)t + s - x \), we can use locally the implicit function theorem and find \( s = s(t, x, p) \). Moreover, \( dp = \det(C(t, s)) ds \). Therefore,

\[
\lim_{\sigma \to 0} \hat{u}(t, x) = \frac{\int_{\mathbb{R}^n} u_0(s(p)) f_0(s(p)) \det(C(t, s(p)))^{-1} \delta(p) \, dp}{\int_{\mathbb{R}^n} f_0(s(p)) \det(C(t, s(p)))^{-1} \delta(p) \, dp} = u_0(s_0(t, x)),
\]

where we denote by \( s_0(t, x) = s(t, x, 0) \) the vector-function which obeys the following vectorial equation:

\[
u_0(s_0(t, x))t + s_0(t, x) - x = 0. \tag{13}\]

Let us show that \( u(t, x) = u_0(s_0(t, x)) \) satisfies the Burgers equation, that is

\[
\sum_{j=1}^n \partial_j(u_0,i)(s_0,j)_t + \sum_{j,k=1}^n u_{0,j} \partial_k(u_0,i)s_0,kx_j = 0, \quad i = 1, \ldots, n, \tag{14}\]

and \( u_0(s_0(0, x)) = u_0(x) \). Here we denote by \( u_{0,i} \) and \( s_0,i \) the \( i \)-th components of vectors \( u_0 \) and \( s_0 \), respectively.

We differentiate (13) with respect to \( t \) and \( x_j \) to get the matrix equations:

\[
\sum_{j=1}^n C_{ij}(s_0,j)_t + u_{0,i} = 0, \quad i = 1, \ldots, n,
\]

and

\[
\sum_{k=1}^n C_{ik}(s_0,k)x_j + \delta_{ij} = 0, \quad i, j = 1, \ldots, n,
\]

where \( \delta_{ij} \) is the Kronecker symbol. The equations imply

\[
(s_0,j)_t = -\sum_{i=1}^n (C^{-1})_{ij} u_{0,i}, \quad (s_0,k)x_j = -(C^{-1})_{jk}, \quad j = 1, \ldots, n. \tag{15}\]

It remains now only to substitute (15) into (14).

Further, (13) implies \( u_0(s_0(0, x)) = u_0(x) \). \( \square \)

It is important to note that \( s_0(t, x) \) is unique for all \( t \) for which the solution to the Burgers equation \( u(t, x) \) is smooth.
Let us denote \( \rho(t, x) = \int f_0(s) e^{-\|s - u_0(t, x)\|^2 / 2\sigma^2} ds. \) From (10) we have
\[
\rho(t, x) = \frac{1}{(\sqrt{2\pi\sigma^2})^n} \int f_0(s) e^{-\|s - u_0(t, x)\|^2 / 2\sigma^2} ds.
\]
(16)

It can be readily checked that in the one-dimensional case the functions \( \rho(t, x) \) and \( \hat{u}(t, x) \) solve the following system:
\[
\partial_t \rho + \partial_x (\rho \hat{u}) = \frac{1}{2} \sigma^2 \partial_{xx} \rho,
\]
(17)
\[
\partial_t (\rho \hat{u}) + \partial_x (\rho \hat{u}^2) = -\int \frac{(u - \hat{u})^2}{2} \partial_x P \, du + \frac{1}{2} \sigma^2 \partial_{xx} (\rho \hat{u}).
\]
(18)

The equation (17) follows from the Fokker-Planck equation (6) directly. To prove (17) we note that definitions of \( \hat{u}(t, x) \) and \( \rho(t, x) \) imply
\[
\partial_t (\rho \hat{u}) = \partial_t \int uP(t, x, u) du = \int uP'(t, x, u) du
\]
\[
= -\int u^2 P_x'(t, x, u) du + \frac{1}{2} \sigma^2 \int uP_{xx}'(t, x, u) du
\]
\[
= -\int u^2 P_x'(t, x, u) du + \frac{1}{2} \sigma^2 \partial_{xx} \rho \hat{u}.
\]
Further,
\[
\partial_x (\rho \hat{u}^2) = \hat{u}^2 \partial_x \rho + 2\rho \hat{u} \partial_x \hat{u}
\]
\[
= 2\hat{u} \partial_x \rho - \hat{u}^2 \partial_x \rho = \int 2\hat{u} uP_x'(t, x, u) du - \int \hat{u}^2 P_x'(t, x, u) du.
\]

A certain analog of system (17), (18) was obtained in [5].

Let us denote \( f(t, x) = \lim_{\sigma \to 0} \rho(t, x) \). Taking into account Proposition 1 and the Fokker-Plank equations, as a limit \( \sigma \to 0 \) for smooth \( f(t, x) \) and \( u(t, x) \) we obtain the system of pressureless gas dynamic (e.g. [6]) in any space dimensions:
\[
\partial_t f + \text{div}_x (fu) = 0,
\]
(19)
\[
\partial_t (fu) + \nabla (fu \otimes u) = 0.
\]

However the formula (11) has sense also for discontinuous initial data \( (f_0(x), u_0(x)) \). For the sake of simplicity we dwell on the one-dimensional case and consider the following initial data:
\[
f_0(x) = f_1 + f_2 \theta(x - x_0),
\]
(20)
\[
u_0(x) = u_1 + u_2 \theta(x - x_0),
\]
(21)
where \( \theta(x - x_0) \) is the Heaviside function. Without loss of generality we assume \( x_0 = 0 \).
Definition 1. We call the couple of functions \((f(t, x), u(t, x))\) the generalized solution to the problem \(14, 20, 21\) in the sense of free particles, if for almost all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n\)

\[
f(t, x) = \lim_{\varepsilon \to 0} \lim_{\sigma \to 0} \rho^\varepsilon(t, x), \quad u(t, x) = \lim_{\varepsilon \to 0} \lim_{\sigma \to 0} \tilde{u}^\varepsilon(t, x),
\]

where \((\rho^\varepsilon(t, x), \tilde{u}^\varepsilon(t, x))\) satisfy the system \(17, 18\) with initial data \((f^\varepsilon_0(x), u^\varepsilon_0(x))\) from the class \(C^1(\mathbb{R}^n)\) such that \(\lim_{\varepsilon \to 0} f^\varepsilon_0(x) = f_0(x), \lim_{\varepsilon \to 0} u^\varepsilon_0(x) = u_0(x)\) for almost all fixed \(x \in \mathbb{R}^n\).

Proposition 2. The solution \((f(t, x), u(t, x))\) to the problem \(14, 20, 21\) in the sense of free particles does not depend of the choice of the couple \((f^\varepsilon_0(x), u^\varepsilon_0(x))\) in \(C^1(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)\).

Proof. Let us choose two couples of smoothed functions \((f^\varepsilon_{01}(x), u^\varepsilon_{01}(x))\) and \((f^\varepsilon_{02}(x), u^\varepsilon_{02}(x))\) such that

\[
\lim_{\varepsilon \to 0} f^\varepsilon_{01}(x) = \lim_{\varepsilon \to 0} f^\varepsilon_{02}(x) = f_0(x), \quad \lim_{\varepsilon \to 0} u^\varepsilon_{01}(x) = \lim_{\varepsilon \to 0} u^\varepsilon_{02}(x) = u_0(x)
\]

for any fixed \(x \in \mathbb{R}^n\). Then the couple

\[
(f_0^\varepsilon(x), u_0^\varepsilon(x)) = (f^\varepsilon_{01}(x) - f^\varepsilon_{02}(x), u^\varepsilon_{01}(x) - u^\varepsilon_{02}(x)) \in C^1(\mathbb{R}) \cap C_b(\mathbb{R}^n)
\]

can be considered as initial data for the problem \(19\). To prove the proposition we have to show that the respective solution is identically zero.

Indeed, from \(16\) we have

\[
f(t, x) = \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^n} f_0^\varepsilon(s) \delta(s - s_0(t, x)) ds \right) = \lim_{\varepsilon \to 0} (f^\varepsilon_{01}(s_0(t, x)) - f^\varepsilon_{02}(s_0(t, x))) = 0.
\]

Here \(s_0(t, x)\) is a solution of \(13\) as before. Analogously proceeding from \(11\), we prove that \(u(t, x) \equiv 0\). □

Our purpose is to find relations between a stable solution to the Riemann problem of system that can be obtained as a limit \(\sigma \to 0\) from \(17, 18\) (as we will show below it is not necessarily looks like \(19\)) and the couple \((f(t, x), u(t, x))\).

The Riemann problem in 1D case. According to Definition 1 we must consider the smoothed initial data instead of \(20\) and \(21\). As follows from Proposition 1 we can choose any couple of smoothed initial data we want. It will be convenient to consider the piecewise linear approximation of initial data of the form

\[
f_0^\varepsilon(x) = \begin{cases} 
  f_1, & x \leq -\varepsilon, \\
  \frac{f_2}{2\varepsilon}x + f_1 + \frac{f_2}{2}, & -\varepsilon < x < \varepsilon, \\
  f_1 + f_2, & x \geq \varepsilon,
\end{cases}
\]

\[
u_0^\varepsilon(x) = \begin{cases} 
  u_1, & x \leq -\varepsilon, \\
  \frac{u_2}{2\varepsilon}x + u_1 + \frac{u_2}{2}, & -\varepsilon < x < \varepsilon, \\
  u_1 + u_2, & x \geq \varepsilon,
\end{cases}
\]

where \(f_1, f_2, u_1\) and \(u_2\) are constants.

Note that these functions can be pointwisely approximated by functions from the class \(C^1(\mathbb{R}^n)\).
From (10) we can find the density $\rho^\varepsilon(t, x)$ corresponding to the smoothed initial data $(f_0^\varepsilon(x), u_0^\varepsilon(x))$:

$$\rho^\varepsilon(t, x) = f_1 \Phi \left( \frac{C^\varepsilon}{\sigma \sqrt{t}} \right) + (f_1 + f_2) \Phi \left( -\frac{C^\varepsilon}{\sigma \sqrt{t}} \right) + I_1^\varepsilon, \quad (24)$$

where $\Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{s^2}{2}} ds$ is the Gauss function, $C^\varepsilon_- = u_1 t - x - \varepsilon$, $C^\varepsilon_+ = (u_1 + u_2)t - x + \varepsilon$, and

$$I_1^\varepsilon = F^\varepsilon(t, x) \left[ \Phi \left( \frac{C^\varepsilon_-}{\sigma \sqrt{t}} \right) - \Phi \left( \frac{C^\varepsilon_+}{\sigma \sqrt{t}} \right) \right] + O \left( \sigma e^{-\frac{1}{4\varepsilon}} \right), \quad (25)$$

The expression for $I_1^\varepsilon$ can be written out, however, we are interested only in behavior of $I_1^\varepsilon$ as $\sigma \to 0$. It can be calculated that $\lim_{\sigma \to 0} F^\varepsilon(t, x) = 0$.

To find $\hat{u}(t, x)$ we compute the numerator in formula (11):

$$\frac{1}{\sqrt{2\pi t\sigma}} \int_{\mathbb{R}} f_0^\varepsilon(s) f_0^\varepsilon(s) e^{-\frac{(s - s - 2\varepsilon)^2}{2\sigma^2}} ds$$

$$= u_1 \rho^\varepsilon(t, x) + u_2 (f_1 + f_2) \Phi \left( -\frac{C^\varepsilon_+}{\sigma \sqrt{t}} \right) + I_2^\varepsilon, \quad (26)$$

where

$$I_2^\varepsilon = N^\varepsilon(t, x) \left[ \Phi \left( \frac{C^\varepsilon_-}{\sigma \sqrt{t}} \right) - \Phi \left( \frac{C^\varepsilon_+}{\sigma \sqrt{t}} \right) \right] + O \left( \sigma^2 + \sigma e^{-\frac{1}{4\varepsilon}} \right), \quad (27)$$

where $N^\varepsilon(t, x) = \left( \frac{u_2}{u_2 t + 2\varepsilon} (x - (u_1 + u_2)t + \frac{u_2}{2}) \right) F^\varepsilon(t, x)$ and $\lim_{\varepsilon \to 0} N^\varepsilon(t, x) = 0$.

Thus, we have the following result:

$$\hat{\hat{u}}^\varepsilon(t, x) = u_1 + \frac{u_2 (f_1 + f_2) \Phi \left( -\frac{C^\varepsilon_+}{\sigma \sqrt{t}} \right) + I_2^\varepsilon}{f_1 \Phi \left( \frac{C^\varepsilon_-}{\sigma \sqrt{t}} \right) + (f_1 + f_2) \Phi \left( -\frac{C^\varepsilon_+}{\sigma \sqrt{t}} \right) + I_1^\varepsilon}, \quad (28)$$

where $I_1^\varepsilon$ and $I_2^\varepsilon$ are given by in (24) and (25), respectively. Note that $\frac{C^\varepsilon_-}{\sigma} \to \pm\infty$ as $\sigma \to 0$.

Now we can find the generalized solution to the Riemann problem as

$$f(t, x) = \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \rho^\varepsilon(t, x) \right), \quad u(t, x) = \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \hat{\hat{u}}^\varepsilon(t, x) \right).$$

Let us introduce the points $\hat{x}_1^\varepsilon = u_1 t - \varepsilon$ and $\hat{x}_2^\varepsilon = (u_1 + u_2)t + \varepsilon$. Their velocities are $u_1$ and $u_1 + u_2$, respectively.

We consider two cases:

1. $u_2 > 0$ (velocity of the point $\hat{x}_2^\varepsilon$ is higher than velocity of the point $\hat{x}_1^\varepsilon$). We can find $f^\varepsilon(t, x) = \lim_{\sigma \to 0} \rho^\varepsilon(t, x)$ from (24). Let us note that this formula contains $F^\varepsilon(t, x)$. It is easy to see that

$$\lim_{\varepsilon \to 0} \hat{x}_1^\varepsilon = x - u_1 t, \quad \lim_{\varepsilon \to 0} \hat{x}_2^\varepsilon = x - (u_1 + u_2)t.$$
and \( \lim_{\varepsilon \to 0} F^\varepsilon(t, x) = 0 \). Thus,

\[
f(t, x) = \lim_{\varepsilon \to 0} f^\varepsilon(t, x) = \begin{cases} 
  f_1, & x < u_1 t, \\
  \frac{f_1 + f_2}{2}, & x = u_1 t, \\
  0, & u_1 t < x < (u_1 + u_2)t, \\
  \frac{f_1 + f_2}{2}, & x = (u_1 + u_2)t, \\
  f_1 + f_2, & x > (u_1 + u_2)t.
\end{cases}
\]

Further, from (27) we find the solution of the gas dynamic system with smooth initial data \( u^\varepsilon(t, x) = \lim_{\varepsilon \to 0} \tilde{u}_\varepsilon(t, x) \) as follows:

\[
u^\varepsilon(t, x) = \begin{cases} 
  u_1, & x < \tilde{x}_1^\varepsilon, \\
  u_1 + \frac{N(\varepsilon)}{F(\varepsilon)}, & \tilde{x}_1^\varepsilon \leq x \leq \tilde{x}_2^\varepsilon, \\
  u_1 + u_2, & x > \tilde{x}_2^\varepsilon.
\end{cases}
\]

It can be shown that

\[
\lim_{\varepsilon \to 0} \frac{N(\varepsilon)}{F(\varepsilon)} = \lim_{\varepsilon \to 0} \left( \frac{u_2}{2} + \frac{u_2}{u_2 t + 2\varepsilon} (x - (u_1 + \frac{u_2}{2})t) \right) = \frac{x}{t} - u_1.
\]

Thus, we get the following solution:

\[
u(t, x) = \lim_{\varepsilon \to 0} u^\varepsilon(t, x) = \begin{cases} 
  u_1, & x < u_1 t, \\
  \frac{x}{t}, & u_1 t \leq x \leq (u_1 + u_2)t, \\
  u_1 + u_2, & x > (u_1 + u_2)t.
\end{cases}
\]

We can see that the velocity includes the rarefaction wave. It is well known stable solution to the Burgers equation with initial data (21).

It is interesting to note that if we compute the limit in \( \varepsilon \) first we get the solution \( u(t, x) = u_1 + u_2 \theta(x - (u_1 + \frac{u_2}{2})t) \), which is unstable with respect to small perturbations.

2. \( u_2 < 0 \) (the velocity of \( \tilde{x}_2^\varepsilon \) is higher than the velocity of \( \tilde{x}_1^\varepsilon \)). From (24) and (27) we find as before:

\[
f(t, x) = \begin{cases} 
  f_1, & x < (u_1 + u_2)t, \\
  \frac{2f_1 + f_2}{3}, & x = (u_1 + u_2)t, \\
  \frac{2f_1 + f_2}{3}, & (u_1 + u_2)t < x < u_1 t, \\
  \frac{3f_1 + 2f_2}{2}, & x = u_1 t, \\
  f_1 + f_2, & x > u_1 t,
\end{cases}
\]

\[
u(t, x) = \begin{cases} 
  u_1, & x < (u_1 + u_2)t, \\
  u_1 + \frac{f_1 + f_2}{2f_1 + f_2} u_2, & (u_1 + u_2)t \leq x \leq u_1 t, \\
  u_1 + u_2, & x > u_1 t.
\end{cases}
\]

Remark We can consider in this framework the singular Riemann problem with initial density \( f_0(x) = f_1 + f_2 \theta(x - x_0) + f_3 \delta(x) \).
The Hugoniot conditions and the spurious pressure. As we have been proved, if \( f \) and \( u \) are smooth, they solve the pressureless gas dynamics system. Now we ask the question which system satisfy the solution of this system with jumps in the sense of free particles.

The system of conservation laws (19) implies two Hugoniot conditions that should be held on the jumps of the solution \( u \). This signifies the solution satisfies the system in the sense of integral identities. If we denote by \( D \) the velocity of the jump and \( [h(y)] = h(y + 0) - h(y - 0) \) the value of the jump, then the continuity equation and the momentum conservation give \([f]D = [fu]\) and \([fu]D = [fu^2]\), respectively.

In the case \( u_2 > 0 \) the velocity is continuous, therefore the Hugoniot conditions hold trivially.

We should check these conditions for the jumps in the case \( u_2 < 0 \). An easy computation shows that the first one is satisfied, however, the second one does not hold. To understand the reason let us estimate the integral term in (18) on the generalized solutions in the case \( u_2 < 0 \):

\[
\int \left( u - \hat{u}(t, x) \right)^2 P_x(t, x, u) \, du =
\]

\[
= \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} f_0(s)(u_0(s) - \hat{u}(t, x))^2 \left( e^{-\frac{(u_0(s)+\hat{u}(t, x)-x)^2}{2\sigma^2}} \right)_x \, ds =
\]

\[
= -\frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} f_0(s)(u_0(s) - u_{FP}(t, x))^2 \left( e^{-\frac{(u_0(s)+\hat{u}(t, x))^2}{2\sigma^2}} \right)_s \, ds +
\]

\[
-2(u_{FP}(t, x) - \hat{u}(t, x)) \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} f_0(s)(u_0(s) - u_{FP}(t, x)) \left( e^{-\frac{(u_0(s)+\hat{u}(t, x))^2}{2\sigma^2}} \right)_s \, ds =
\]

\[
= I_1 + I_2 + I_3.
\]

The integrals \( I_2 \) and \( I_3 \) tend to zero as \( \sigma \to 0 \) due to properties of the Riemann data since \( \hat{u}(t, x) \to u_{FP}(t, x) \) for almost all \( x \in \mathbb{R} \). Let us estimate \( I_1 \).

\[
I_1 = -\frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} f_1(u_1 - \hat{u}(t, x))^2 \left( e^{-\frac{(u_1+\hat{u}(t, x))^2}{2\sigma^2}} \right)_s \, ds -
\]

\[
\frac{1}{\sqrt{2\pi\sigma}} \int_{-u_2}^{0} (f_1 + f_2) ((u_1 + u_2) - \hat{u}(t, x))^2 \left( e^{-\frac{(u_1+u_2+\hat{u}(t, x))^2}{2\sigma^2}} \right)_s \, ds =
\]

\[
= -\frac{1}{\sqrt{2\pi\sigma}} \frac{f_1(f_1 + f_2)^2u_2^2}{(2f_1 + f_2)^2} \left( e^{-\frac{(u_1+\hat{u}(t, x))^2}{2\sigma^2}} - e^{-\frac{(u_1+u_2+\hat{u}(t, x))^2}{2\sigma^2}} \right) -
\]

\[
-\frac{1}{\sqrt{2\pi\sigma}} \frac{f_2^2(f_1 + f_2)^2}{(2f_1 + f_2)^2} \left( e^{-\frac{(u_1+\hat{u}(t, x))^2}{2\sigma^2}} - e^{-\frac{(u_1+u_2+\hat{u}(t, x))^2}{2\sigma^2}} \right).
\]
Thus,

$$I_1 = \frac{f_1(f_1 + f_2)u_2^2}{(2f_1 + f_2)}(\delta(x - (u_1 + u_2)t) - \delta(x - u_1 t)), \quad \sigma \to 0.$$  

Thus, the integral term corresponds to a spurious pressure between the jumps $x = (u_1 + u_2)t$ and $x = u_1 t$, namely,

$$p = \frac{f_1(f_1 + f_2)u_2^2}{(2f_1 + f_2)}(\theta(x - (u_1 + u_2)t) - \theta(x - u_1 t)).$$  

(29)

The Hugoniot condition $[fu]D = [fu^2 + p]$ is satisfied with this kind of pressure.

Thus, we get the following theorem.

**Theorem 1.** The generalized solution to the Riemann problem [20, 21] for the pressureless gas dynamics system in the sense of free particles (according to Definition 1) in the case of a discontinuous velocity ($u_2 < 0$) solves in fact the gas dynamics system with a pressure defined by [22].

**Sticky particles model vs non-interacting particles model.** In our model the particles are allowed to go through the discontinuity as one particle does not feel the others. However, if we are in the frame of the sticky particles model we should assume that the particles meeting one other stick together on the jump.

The non-interacting particles model and the sticky particles model are equivalent for smooth velocities, however, if the velocity has a jump, the solutions behavior differs drastically. Nevertheless, we can study the solution to the Riemann problem in the case of $u_2 < 0$ for the sticky particles model, too, basing on the solution obtained in the present paper. Indeed, the jump position $x(t)$ is a point between $x_1(t) = (u_1 + u_2)t$ and $x_2(t) = u_1 t$. The mass $m(t)$ accumulates in the jump due to the impenetrability of the discontinuity with the velocity

$$m(t) = (x(t) - (u_1 + u_2)t)((2f_1 + f_2) - f_1) + (u_1 t - x(t))((2f_1 + f_2) - (f_1 + f_2))$$

$$= -((u_1 + u_2)(f_1 + f_2) - u_1 f_1)t + x(t)f_2 = -[uf]t + [fx](t),$$

where $[ ]$ stands for a jump value. Further, if we change heuristically the overlapped mass between $x_1$ and $x_2$ to the mass concentrated at a point, then from the condition of equality of momenta in the both cases we can find the velocity of the point singularity:

$$(u_1 + u_2)((2f_1 + f_2) - f_1)(x(t) - (u_1 + u_2)t) + u_1 f_1 (u_1 t - x(t))$$

$$= -[u^2f]t + [uf]x(t) = m(t)\dot{x}(t).$$

Thus, to find the position of the point singularity we get the equation

$$([f]x(t) - [uf]t) \dot{x}(t) = [uf]x(t) - [u^2f]t,$$  

(30)

subject to initial data $x(0) = 0$. The respective solution is

$$x(t) = \frac{1}{[f]} \left([uf] \pm \sqrt{[uf]^2 - [f][u^2f]} \right) t,$$  

(31)

where the sign should be chosen from the condition $x_1(t) < x(t) < x_2(t)$. It can be readily shown that the latter condition is satisfied either for plus or minus in the formula (31). The condition coincides with the Lax stability condition $u_1 < \dot{x}(t) < u_1 + u_2$. 
The formulas describing the amplitude of the delta-function in the density component and the singularity position obtained earlier in [8],[9],[10] give the same result.

It is worth mentioning that the spurious pressure (29) does not arise in the sticky particles model.

**Back to the Burgers equation.** Now we are able to get the solution to the Cauchy problem (1), (21). Let us assume that \( f_0(x) = \text{const} \), that is \( f_2 = 0 \). Then for the case \( u_2 < 0 \) from (30) we obtain the well known formula for a position of the jump:

\[
x(t) = \left[ \frac{u^2}{2|u|} \right] t = \frac{u_+ + u_-}{2}.
\]

If \( u_2 > 0 \), the solution is continuous and it is given by the formula (28).

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