CROSSED MODULES AND SYMMETRIC COHOMOLOGY OF GROUPS

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Abstract. This paper links the third symmetric cohomology (introduced by Staic [9] and Zarelua [11]) to crossed modules with certain properties. The equivalent result in the language of 2-groups states that an extension of 2-groups corresponds to an element of $HS^3$ iff it possesses a section which preserves inverses in the 2-categorical sense. This ties in with Staic’s (and Zarelua’s) result regarding $HS^2$ and abelian extensions of groups.

1. Introduction

Let $G$ be a group and $M$ be a $G$-module. Symmetric cohomology $HS^\ast(G, M)$ was introduced by M. Staic [8] as a variant of classical group cohomology. A. Zarelua’s prior definition [11] of exterior cohomology $H^\ast\lambda(G, M)$ is very closely related to this, as shown in [6]. Namely, we proved in [6] that the symmetric cohomology has a functorial decomposition

$$HS^\ast(G, M) = H^\ast\lambda(G, M) \oplus H^\ast\delta(G, M),$$

where $H^3\lambda(G, M) = 0$ when $0 \leq i \leq 4$ or $M$ has no elements of order two.

There are natural homomorphisms $\alpha^n : HS^n(G, M) \to H^n(G, M)$ (and $\beta^n : H^n\lambda(G, M) \to H^n(G, M)$), which are isomorphisms for $n = 0, 1$ and a monomorphism for $n = 2$. This was shown by Staic in [9]. According to our results in [6], the map $\alpha^n$ is an isomorphism for $n = 2$ if $G$ has no elements of order two. In this case, $\alpha^3$ is a monomorphism. More generally, $\alpha^n$ is an isomorphism if $G$ is torsion free.

Classical $H^2(G, M)$ classifies extensions of $G$ by $M$. Staic showed in [9] that $HS^2(G, M)$ classifies a subclass of extensions of $G$ by $M$, namely those that have a section preserving inversion.

It is also well-known that $H^3(G, M)$ classifies the so-called crossed extensions of $G$ by $M$ [4], or equivalently $H^3(G, M)$ classifies 2-groups $\Gamma$, with $\pi_0(\Gamma) = G$ and $\pi_1(\Gamma) = M$.

The main result of this paper is that for groups $G$ with no elements of order 2 the group $HS^3(G, M)$ classifies crossed extensions of $G$ by $M$ with a certain condition on the section.

There has been interest in the mathematical community in exploring symmetric cohomology. Several papers have already been published on this topic, e.g. [2], [3], [7], [10].

2. Preliminaries

2.1. Preliminaries on symmetric cohomology of groups. Let $G$ be a group and $M$ be a $G$-module. Recall that the group cohomology $H^\ast(G, M)$ is defined as the cohomology of the cochain complex $C^\ast(G, M)$, where the group of $n$-cochains of $G$ with coefficients in $M$ is the set of functions from $G^n$ to $M$: $C^n(G, M) =$
\(\{\phi : G^n \to M\}\) and coboundary map \(d^n : C^n(G, M) \to C^{n+1}(G, M)\) is defined by
\[
d^n(\phi)(g_0, g_1, \ldots, g_n) = g_0 \cdot \phi(g_1, \ldots, g_n)
+ \sum_{i=1}^{n}(-1)^i\phi(g_0, \ldots, g_{i-2}, g_i, g_{i-1}, g_{i+1}, \ldots, g_n)
+ (-1)^{n+1}\phi(g_0, \ldots, g_{n-1}).
\]

A cochain \(\phi\) is called \textit{normalised} if \(\phi(g_1, \ldots, g_n) = 0\) whenever some \(g_j = 1\). The collection of normalised cochains is denoted by \(C^*_N(G, M)\) and the classical normalisation theorem claims that the inclusion \(C^*_N(G, M) \to C^*(G, M)\) induces an isomorphism on cohomology.

In \([8]\) Staic introduced a subcomplex \(CS^*(G, M) \subset C^*(G, M)\), whose homology is known as the symmetric cohomology of \(G\) with coefficients in \(M\) and is denoted by \(HS^*(G, M)\). The definition is based on an action of \(\Sigma_{n+1}\) on \(C^n(G, M)\) (for every \(n\)) compatible with the differential. In order to define this action, it is enough to define how the transpositions \(\tau_i = (i, i+1), 1 \leq i \leq n\) act. For \(\phi \in C^n(G, M)\) one defines:
\[
(\tau_i\phi)(g_1, g_2, g_3, \ldots, g_n) = \begin{cases} 
-g_i\phi(g_1, g_2, g_3, \ldots, g_n), & \text{if } i = 1, \\
-\phi(g_1, \ldots, g_{i-2}, g_i^{-1}g_{i-1}g_i, g_i, g_{i+1}, \ldots, g_n), & 1 < i < n, \\
-\phi(g_1, g_2, g_3, \ldots, g_{n-1}g_n, g_n^{-1}), & \text{if } i = n.
\end{cases}
\]

Denote by \(CS^n(G, M)\) the subgroup of the invariants of this action. That is, \(CS^n(G, M) = C^n(G, M)\Sigma_{n+1}\). Staic proved that \(CS^*(G, M)\) is a subcomplex of \(C^*(G, M)\) \([8]\), \([9]\) and hence the groups \(HS^*(G, M)\) are well-defined. There is an obvious natural transformation
\[
\alpha^n : HS^n(G, M) \to H^n(G, M), \quad n \geq 0.
\]

According to \([8]\), \([9]\), \(\alpha^n\) is an isomorphism if \(n = 0, 1\) and is a monomorphism for \(n = 2\). For extensive study of the homomorphism \(\alpha^n\) for \(n \geq 2\) we refer to \([6]\).

Denote by \(CS^*_N(G, M)\) the intersection \(CS^*(G, M) \cap C^*_N(G, M)\). Unlike to the classical cohomology the inclusion \(CS_N^*(G, M) \to CS^*(G, M)\) does not always induces an isomorphism on cohomology. The groups \(H^n_N(G, M) = H^*(CS_N^*(G, M))\) are isomorphic to the so called \textit{exterior cohomology of groups} introduced by Zarelua in \([11]\). According to \([6]\) the canonical map
\[
\gamma_n : H^n_N(G, M) \to HS^n(G, M)
\]
induced by the inclusion \(CS_N^*(G, M) \to CS^*(G, M)\), is an isomorphism if \(n \leq 4\), or \(M\) has no elements of order two.

2.2. \textbf{Symmetric extensions and} \(HS^2\). It is a classical fact, that \(H^2(G, M)\) classifies the extension of \(G\) by \(M\) and \(H^3(G, M)\) classifies the crossed extensions of \(G\) by \(M\). One can ask what objects classify the symmetric cohomology groups \(HS^2(G, M)\) and \(HS^3(G, M)\). The answer to this question in the dimension two was given in \([6]\). The aim of this work is to prove a similar result in the dimension three.

Let \(G\) be a group and \(M\) be a \(G\)-module. Recall that an extension of \(G\) by \(M\) is a short exact sequence of groups
\[
0 \to M \xrightarrow{i} K \xrightarrow{p} G \to 0
\]
such that for any \(k \in K\) and \(m \in M\), one has \(ki(m)k^{-1} = i(p(k)m)\). An \(s\)-section to this extension is a map \(s : G \to K\) such that \(p \circ s(x) = x\) for all \(x \in G\). Let \(\text{Ext}_{Gr}(G, M)\) be the category whose objects are extensions of \(G\) and \(M\) and...
morphism are commutative diagrams

\[
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow{id} & & \downarrow{id} \\
0 & \rightarrow & K \\
\end{array}
\quad
\begin{array}{ccc}
& \varepsilon & \rightarrow & p \\
\downarrow{\phi} & & \downarrow{\phi'} \\
& K' & \rightarrow & G \\
\end{array}
\quad
\begin{array}{ccc}
& \rightarrow & 0 \\
\end{array}
\]

The set of connected components of the category \textbf{Exgr}(G, M) is denoted by \textbf{Exgr}(G, M). It is well-known that there exists a natural map \textbf{Exgr}(G, M) \rightarrow H^2(G, M), which is a bijection. To construct this map, one needs to choose an s-section \(s\) and then define \(f \in C^2(G, M)\) by

\[
s(x)s(y) = i(f(x, y))s(xy).
\]

One checks that \(f\) is a 2-cocycle and its class in \(H^2\) is independent of the chosen s-section \(s\).

Let \(0 \rightarrow M \xrightarrow{\iota} K \xrightarrow{p} G \rightarrow 0\) be an extension and \(s\) be a s-section. Then \(s\) is called symmetric if \(s(x^{-1}) = s(x)^{-1}\) holds for all \(x \in G\). An extension is called symmetric if it possesses a symmetric s-section. The symmetric extensions form a full subcategory \textbf{ExS}(G, M) of the category \textbf{Exgr}(G, M). The set of connected components of \textbf{ExS}(G, M) is denoted by \textbf{ExS}(G, M). The main result of [9] claims that the restriction of the bijection \textbf{Exgr}(G, M) \rightarrow H^2(G, M) on \textbf{ExS}(G, M) yields a bijection \textbf{ExS}(G, M) \rightarrow HS^2(G, M).

2.3. Crossed Modules. Recall the classical relationship between third cohomology of groups and crossed modules. A crossed module is a group homomorphism \(\partial : T \rightarrow R\) together with an action of \(R\) on \(T\) satisfying:

\[
\partial(\ell(t)) = r\partial(t)r^{-1} \text{ and } \partial s = ts^{-1}, \quad r \in R, t, s \in T.
\]

It follows from the definition that the image \(Im(\partial)\) is a normal subgroup of \(R\), and the kernel \(\text{Ker}(\partial)\) is in the center of \(T\). Moreover the action of \(R\) on \(T\) induces an action of \(G\) on \(\text{Ker}(\partial)\), where \(G = \text{Coker } \partial\).

A morphism from a crossed module \(\partial : T \rightarrow R\) to a crossed module \(\partial' : T' \rightarrow R'\) is a pair of group homomorphisms \((\phi : T \rightarrow T', \psi : R \rightarrow R')\) such that

\[
\psi \circ \partial = \partial' \circ \phi, \quad \phi(\ell(t)) = \psi(r)\phi(t), r \in R, t \in T.
\]

For a group \(G\) and for a \(G\)-module \(M\) one denotes by \textbf{Xext}(G, M) the category of exact sequences

\[
0 \rightarrow M \rightarrow T \xrightarrow{\partial} R \rightarrow G \rightarrow 0,
\]

where \(\partial : T \rightarrow R\) is a crossed module and the action of \(G\) on \(M\) induced from the crossed module structure coincides with the prescribed one. The morphisms in \textbf{Xext}(G, M) are commutative diagrams

\[
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow{id} & & \downarrow{id} \\
0 & \rightarrow & T \\
\downarrow{\phi} & & \downarrow{\phi'} \\
0 & \rightarrow & R \\
\downarrow{\psi} & & \downarrow{id} \\
0 & \rightarrow & G \\
\end{array}
\quad
\begin{array}{ccc}
& \rightarrow & 0 \\
\end{array}
\]

where \((\phi, f)\) is a morphism of crossed modules \((T, G, \partial) \rightarrow (T', G', \partial')\). We let \textbf{Xext}(G, M) be the class of the connected components of the category \textbf{Xext}(G, M). Objects of the category \textbf{Xext}(G, M) are called crossed extensions of \(G\) by \(M\).

It is a classical fact (see for example [4]) that there is a canonical bijection

\[
\xi : \textbf{Xext}(G, M) \rightarrow H^3(G, M).
\]

The map \(\xi\) has the following description. Let \(0 \rightarrow M \rightarrow T \xrightarrow{\partial} R \xrightarrow{p} G \rightarrow 0\) be a crossed extension. An s-section of it is a pair of maps \((s : G \rightarrow R, \sigma : G \times G \rightarrow T)\)
for which the following hold
\[ ps(x) = x, \quad s(x)s(y) = \partial(\sigma(x,y))s(xy), \quad x, y \in G. \]

An \( s \)-section is called \textit{normalised} if \( s(1) = 1 \) and \( \sigma(1,x) = 1 = \sigma(x,1) \) for all \( x \in G \).

It is clear that every crossed extension has a normalised \( s \)-section. Any (normalised) \( s \)-section \((s, \sigma)\) gives rise to a (normalised) 3-cocycle \( f \in Z^3(G, M) \) defined by
\[
(2.3.1) \quad f(x, y, z) = s(x)\sigma(y, z)\sigma(x, yz)\sigma(xy, z)^{-1}\sigma(x, y)^{-1}
\]

To make dependence of \( f \) on \( \sigma \) and \( s \) we sometimes write \( f_\sigma \) or even \( f_{s, \sigma} \) instead of \( f \). Then the map \( \xi \) assigns the class of \( f \) in \( H^3(G, M) \) to the class of \( 0 \to M \to T \overset{\partial}{\to} R \overset{p}{\to} G \to 0 \) in \( \text{Ext}(G, M) \).

3. A CHARACTERISATION OF SYMMETRIC COCYCLES

In this section we prove the following auxiliary results, which will be used in the next section.

\textbf{Lemma 1.} i) If \( \phi \in CS^N_3(G, M) \), \( n \geq 2 \) then \( \phi(g_1, \cdots, g_n) = 0 \), whenever \( g_{i+1} = g_i^{-1} \) for some \( 1 \leq i \leq n - 1 \).

\textit{Proof.} i) By definition we have \((\tau_i(\phi) + \phi)(g_1, \cdots, g_n) = 0\) for any \( g_1, \cdots, g_n \in G \). If \( g_{i+1} = g_i^{-1} \) for some \( i \), then \( \tau_i(\phi)(g_1, \cdots, g_n) = 0 \) by the normalisation condition. Hence \( \phi(g_1, \cdots, g_i, g_i^{-1}, \cdots, g_n) = 0 \). □

The converse in general is not true, however we have the following important fact.

\textbf{Lemma 2.} If \( \phi \in CS^N_3(G, M) \) is a cocycle, \( n \geq 2 \), then \( \phi \in CS^N_3(G, M) \) iff \( \phi(g_1, \cdots, g_n) = 0 \), whenever \( g_{i+1} = g_i^{-1} \) for some \( 1 \leq i \leq n - 1 \).

\textit{Proof.} Thanks to Lemma 1 we need to prove that \( \tau_i(\phi) + \phi = 0 \), if \( \phi(g_1, \cdots, g_n) = 0 \), whenever \( g_{k+1} = g_k^{-1} \) for some \( 1 \leq k \leq n - 1 \).

By assumption
\[
x_1\phi(x_2, \cdots, x_{n+1}) + \sum_{k=1}^{n} (-1)^k \phi(x_1, \cdots, x_kx_{k+1}, \cdots, x_{n+1})
+ (-1)^{n+1} \phi(x_1, \cdots, x_{n}) = 0.
\]

for any \( x_1, \cdots, x_{n+1} \in G \). First we take
\[
x_k = \begin{cases} 
    g_1, & k = 1, \\
    g_1^{-1}, & k = 2, \\
    g_1g_2, & k = 3, \\
    g_{k-1}, & k \geq 4.
\end{cases}
\]
to obtain
\[
(\tau_1(\phi) + \phi)(g_1, \cdots, g_n) = 0.
\]

Next, fix \( 1 < i < n \) and put
\[
x_k = \begin{cases} 
    g_k, & k \leq i, \\
    g_i^{-1}, & k = i + 1, \\
    g_ig_{i+1}, & k = i + 2, \\
    g_{k-1}, & k \geq i + 3.
\end{cases}
\]
to obtain
\[
((-1)^i\tau_1(\phi) + (-1)^{i+2}(\phi)(g_1, \cdots, g_n) = 0.
\]

Thus \( \tau_i(\phi) + \phi = 0 \), \( 1 < i < n \).
Finally, we take
\[ x_k = \begin{cases} g_k, & k \leq n, \\ g_{n-k}^{-1}, & k = n+1. \end{cases} \]
to obtain
\[ ((-1)^{n-1} \tau_n \phi + (-1)^{n+1} \phi)(g_1, \ldots, g_n) = 0. \]
Thus \( \tau_n \phi + \phi = 0 \) and Lemma follows. \( \square \)

In particular a cocycle \( \phi \in C^3_N(G, M) \) is symmetric (and hence defines a class in \( HS^3(G, M) = H^3_N(G, M) \) iff
\[ \phi(x, x^{-1}, y) = \phi(x, y, y^{-1}) \]
for all \( x, y \in G \).

Next, Lemma helps us to distinguish boundary elements in \( CS^3_N(G, M) \).

**Lemma 3.** Suppose \( \phi(x, y, z) \) is a (normalised) symmetric cocycle. Also suppose that it is a coboundary: so there exists a \( g(x, y) \in C^2_N(G, M) \) such that
\[ \phi(x, y, z) = xg(y, z) - g(xy, z) + g(x, yz) - g(x, y). \]
Then \( g \) is symmetric iff \( g(x, x^{-1}) = 0 \) for all \( x \in G \).

**Proof.** Recall that \( g \) is symmetric iff
\[ g(x, y) = -xg(x^{-1}, xy) = -g(xy, y^{-1}). \]
If these conditions hold, we can take \( y = x^{-1} \) to obtain
\[ g(x, x^{-1}) = -g(1, x) = 0, \]
because \( g \) is normalised. Conversely, assume \( g(x, x^{-1}) = 0 \) for all \( x \in G \). Since \( \phi \) is symmetric we have
\[ 0 = \phi(x, y, y^{-1}) = xg(y, y^{-1}) - g(xy, y^{-1}) + g(x, 1) - g(x, y) \]
and
\[ 0 = \phi(x, x^{-1}, z) = xg(x^{-1}, z) - g(1, z) + g(x, x^{-1}z) + g(x, x^{-1}). \]
Since \( g \) is normalized we have \( g(x, 1) = g(1, z) = 0 \). By assumption, we also have \( g(y, y^{-1}) = 0 = g(x, x^{-1}) \). Hence,
\[ g(x, y) + g(xy, y^{-1}) = 0 \]
and \( g(x, x^{-1}z) + g(x, x^{-1}z) = 0 \).
Replacing \( z \) by \( xy \) in the last equality, one gets symmetric conditions on \( g \). \( \square \)

**4. Third symmetric cohomology and crossed modules**

We start with proving the following result, which links symmetric cocycles and crossed modules. Our notations are the same as at the end of Section 2.3.

**Proposition 4.** The class of \( 0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0 \) in \( \text{Xext}(G, M) \) lies in the image of the composite map
\[ HS^3(G, M) \xrightarrow{\alpha^3} H^3(G, M) \xrightarrow{\xi^{-1}} \text{Xext}(G, M) \]
iff the crossed extension \( 0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0 \) has a normalised \( s \)-section \((s, \sigma)\) for which the following two identities hold
\[ s(x)\sigma(x^{-1}, y)\sigma(x, x^{-1}y) = \sigma(x, x^{-1}), \]
\[ \sigma(x, y)\sigma(xy, y^{-1}) = s(x)\sigma(y, y^{-1}). \]
Theorem 8. Let $s$ be a normalised $s$-section of a crossed extension

$$0 \to M \to T \xrightarrow{p} R \xrightarrow{\partial} G \to 0$$

By Proposition 4 the corresponding 3-cocycle is symmetric if

$$s(x)\sigma(x^{-1}, y)\sigma(x, x^{-1}y)\sigma(1, y)^{-1}\sigma(x, x^{-1})^{-1} = 1$$

and

$$s(x)\sigma(y, y^{-1})\sigma(x, 1)\sigma(xy, y^{-1})^{-1}\sigma(x, y)^{-1} = 1$$

Since $\sigma(1, -) = 1 = \sigma(-, 1)$, we obtain

$$s(x)\sigma(x^{-1}, y)\sigma(x, x^{-1}y) = \sigma(x, x^{-1})$$

and

$$s(x)\sigma(y, y^{-1}) = \sigma(x, y)\sigma(xy, y^{-1})$$

and we are done. \hfill \Box

Definition 5. A normalised $s$-section $(s, \sigma)$ of a crossed extension

$$0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$$

is called weakly symmetric if the following identities hold

i) $s(x^{-1}) = s(x)^{-1}$,

ii) $\sigma(x, x^{-1}) = 1$, $x, y \in G$.

We have the following easy fact.

Lemma 6. Let $G$ be a group which has no elements of order two. Then any crossed extension

$$0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$$

has a weakly symmetric $s$-section.

Proof. In this case $G \setminus \{1\}$ is a disjoint union of two element subsets of the form

$\{x, x^{-1}\}$, $x \neq 1$. Let us choose a representative in each class. If $x$ is a representative, we set $s(x)$ to be an element in $p^{-1}(x)$. We then extend $s$ to whole $G$ by $s(1) = 1$ and $s(x^{-1}) = s(x)^{-1}$, where $x$ is a representative. We see that for $y = x^{-1}$, one has $s(x)s(y)s(xy)^{-1} = 1$. Thus one can choose $\sigma$ with property $\sigma(x, x^{-1}) = 1$ and lemma follows. \hfill \Box

Definition 7. A weakly symmetric $s$-section $(s, \sigma)$ of a crossed extension

$$0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$$

is called symmetric if the following identities hold

i) $\sigma(x, y) \cdot s(x)\sigma(x^{-1}, xy) = 1$, $x, y \in G$.

ii) $\sigma(x, y) \cdot \sigma(xy, y^{-1}) = 1$, $x, y \in G$.

A crossed extension

$$0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$$

is called symmetric if it has a symmetric $s$-section.

Symmetric crossed extensions of $G$ by $M$ form a subset $\text{XextS}(G, M)$ of the set of $\text{Xext}(G, M)$.

Theorem 8. Let $G$ be a group which has no elements of order two, then there is a natural bijection

$$HS^3(G, M) \cong \text{XextS}(G, M).$$

Proof. By Lemma 6 any crossed extension has a weakly symmetric $s$-section $(s, \sigma)$. By Proposition 4 the corresponding 3-cocycle is symmetric if

$$s(x)\sigma(x^{-1}, y)\sigma(x, x^{-1}y) = 1,$$

$$\sigma(x, y)\sigma(xy, y^{-1}) = 1.$$
Now, if we replace $x$ by $x^{-1}$ in the first identity and then act by $s(x)$, we see that these conditions are exactly ones in the Definition 7. Hence by Proposition 4, the image of the composite map

$$HS^3(G, M) \xrightarrow{\alpha^3} H^3(G, M) \cong \text{Xext}(G, M)$$

is exactly $\text{XextS}(G, M)$. On the other hand, since $G$ has no elements of order two, the map $\alpha^3$ is injective. This follows from the part ii) of Corollary 4.4 [6], because $HS^3 = H^3_3$, thanks to Theorem 3.9 [6]. It follows that the induced map $HS^3(G, M) \to \text{Xext}(G, M)$ is a bijection. □

5. Interpretation in terms of 2-groups

For us (strict) 2-groups are group objects in the category of small categories. Thus it is a category $C$ (in fact a groupoid) equipped with a bifunctor $\cdot : C \times C \to C$, $(a, b) \mapsto a \cdot b$ which is strictly associative and satisfies group object axioms. These objects are also known under the name categorical groups and 1-cat-groups see, [5].

Recall the relationship between crossed modules and 2-groups [5]. Let $\partial : T \to R$ be a crossed module. It defines a 2-group $\text{Cat}_{\to R}$. Objects of $\text{Cat}_{\to R}$ are elements of $R$. A morphism from $r \in R$ to $r' \in R$ is an element $t \in T$ such that

$$r' = \partial(t) r,$$

In this situation we use the notation $r \overset{t}{\to} r'$. The composite of arrows $r \overset{t}{\to} r' \overset{t'}{\to} r''$ is $r \overset{t' t}{\to} r''$. It is clear that $r \overset{1}{\to} r$ is the identity arrow $\text{Id}_r$ in the category $\text{Cat}_{\to R}$.

Any morphism in $\text{Cat}_{\to R}$ is an isomorphism. The inverse of $r \overset{t}{\to} r'$ is $r' \overset{t^{-1}}{\to} r$. As usual we set $M = \text{Ker}(\partial)$. Observe that any $m \in M$ defines an endomorphism $r \overset{m}{\to} r$ of $r \in R$ and conversely, any endomorphism of $r$ has this form.

The bifunctor

$$\text{Cat}_{\to R} \times \text{Cat}_{\to R} \to \text{Cat}_{\to R}$$

given on objects by the multiplication rule in the group $R$, while on morphisms it is given by

$$(r \overset{t}{\to} z) \cdot (x \overset{z}{\to} y) = r x \overset{t(z s)}{\to} z y, \quad r, x, y, z \in R, s, t \in T.$$

In particular, we have

$$(x \overset{z}{\to} y) \cdot \text{Id}_z = x z \overset{t}{\to} y z,$$

$$\text{Id}_r \cdot (x \overset{z}{\to} y) = r x \overset{t r}{\to} r y.$$

It is well-known that any 2-group is isomorphic to the 2-group of the form $\text{Cat}_{\to R}$. For a uniquely defined (up to isomorphism) crossed module $\partial : T \to R$, see for example [5].

In particular crossed modules gives rise to monoidal categories. So, we can consider monoidal functors. We recall the corresponding definition. Let $C$ and $D$ be 2-groups. An s-functor $F : C \to D$ is a pair $(F, \xi)$, where $F : C \to D$ is a functor and $\xi$ is a natural transformation from the composite functor $C \times C \to C$ to the composite functor $C \times \text{Cat}_{\to R} \to \text{Cat}_{\to R}$. Thus for any objects $x$ and $y$ of $C$ we have a morphism $\xi_{x, y} : F(x \cdot y) \to F(x) \cdot F(y)$, which is natural in $x$ and $y$.

In what follows, we will assume that $(F, \xi)$ is normalised, meaning that $F(1) = 1$ and $\xi_x = \text{Id}$ if $x = 1$ or $y = 1$. Thus for any object $x$ we have a morphism $\xi(x, x^{-1}) : 1 \to F(x) \cdot F(x^{-1})$, which will play an important role later.
An $s$-functor is \textit{monoidal} if for any objects $x, y, z$ of the category $\mathcal{C}$ the diagram
\[
\begin{array}{ccc}
F(x \cdot y \cdot z) & \xrightarrow{\xi_{x,y,z}} & F(x \cdot y) \cdot F(z) \\
\downarrow \xi_{x,y,z} & & \downarrow \xi_{x,y} \cdot \text{id}_{F(z)} \\
F(x) \cdot F(y \cdot z) & \xrightarrow{\text{id}_{F(z)} \cdot \xi_{y,z}} & F(x) \cdot F(y) \cdot F(z)
\end{array}
\]
commutes.

Let $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ be a crossed extension. In this situation we have two $2$-groups $\mathcal{C}_T \to R$ and $\mathcal{C}_G$. The second one is $G$ considered as a discrete category (equivalently, the $2$-group, corresponding to the crossed module $1 \to G$).

The homomorphism $p$ yields the strict monoidal functor $\mathcal{C}_T \to \mathcal{C}_G$, which is still denoted by $p$.

One can consider sections of $p$. We will ask different level of compatibility of sections with monoidal structures. The weakest condition to ask to such a section is to be a functor. Since $\mathcal{C}_A$ is a discrete category, we see that such a section of the functor $p$ is nothing but a set section of the map $p : R \to G$. Next, is to ask to the functor $\mathcal{C}_G \to \mathcal{C}_T \to R$ to be an $s$-functor. Call them $s$-sections of $p$. One easily, observes that there is a one-to-one correspondence between $s$-sections $(F, \xi)$ of the functor $p$ and $s$-sections of a crossed extension $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$. In fact, if $(s, \sigma)$ is an $s$-section, then $(F, \xi)$ is an $s$-functor $\mathcal{C}_G \to \mathcal{C}_T \to R$, for which $F \circ p = \text{id}_{\mathcal{C}_G}$. Here the functor $F$ and natural transformation $\xi$ are defined as follows. Since $\mathcal{C}_G$ is a discrete category, the functor $F$ is uniquely determined by the rule:

\[ F(g) = s(g), \quad g \in G. \]

Next, the natural transformation $\xi$ is uniquely determined by the family of morphisms

\[ \xi_{g,h} = \left( s(gh) \xrightarrow{\sigma(g,h)} s(g)s(h) \right), \quad g, h \in G. \]

Even stronger assumption is to ask to the pair $(F, \xi)$ to be a monoidal functor. As the following well-known result shows this condition is a $2$-dimensional analogue of a splitting in a short exact sequence.

\textbf{Proposition 9.} The class of a crossed extension

\[ 0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0 \]

is zero iff the strict monoidal functor $p : \mathcal{C}_T \to R \to \mathcal{C}_G$ has a section $\mathcal{C}_G \to \mathcal{C}_T \to R$, which is monoidal. That is there exists an $s$-section $(s, \sigma)$ of $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$, for which the corresponding $s$-functor $(F, \xi)$ is monoidal.

Since we did not find appropriate reference we give the proof.

\textbf{Proof.} Let $(s, \sigma)$ be an $s$-section of $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$. Then the diagram in the definition of the monoidal functor for $(F, \xi)$ has the form

\[
\begin{array}{ccc}
s(x \cdot y \cdot z) & \xrightarrow{\sigma(x,y,z)} & s(x \cdot y) \cdot s(z) \\
\downarrow \sigma(x,y,z) & & \downarrow \sigma(x,y) \\
s(x) \cdot s(y \cdot z) & \xrightarrow{s(x) \sigma(y,z)} & s(x) \cdot s(y) \cdot s(z)
\end{array}
\]

Thus commutativity of this diagram is equivalent to the vanishing of the $3$-cocycle $f$ in (2.3.1). So, if part is done. Conversely, if $f_\sigma$ defined in (2.3.1) is a coboundary and

\[ f_\sigma(x, y, z) = xk(y, z) - k(xy, z) + k(x, yz) - k(x, y) \]
for a function \( k : G^2 \to M \), then \((s, \tau)\) is also an \( s\)-section for which \( f_\tau = 0 \). Here \( \tau(g, h) = \sigma(g, h) - k(g, h) \).

We now introduce another conditions to an \( s\)-functor \((F, \xi)\), which are weaker than the monoidal functor.

**Definition 10.** An \( s\)-functor \((F, \xi)\) is called symmetric, if

\[
\begin{align*}
F(x) & \xrightarrow{\xi(x, y^{-1})} F(xy) \cdot F(y^{-1}) \\
\downarrow \quad \downarrow & \quad \downarrow \quad \downarrow \\
F(x) \cdot F(y) \cdot F(y^{-1}) & \implies F(x) \cdot F(y) \cdot F(y^{-1})
\end{align*}
\]

and

\[
\begin{align*}
F(y) & \xrightarrow{\xi(x, x^{-1}y)} F(x) \cdot F(x^{-1}y) \\
\downarrow \quad \downarrow & \quad \downarrow \quad \downarrow \\
F(x) \cdot F(x^{-1}) \cdot F(y) & \implies F(x) \cdot F(x^{-1}) \cdot F(y)
\end{align*}
\]

Then we have the following obvious facts:

- Any monoidal functor is symmetric \( s\)-functor.
- The \( s\)-functor corresponding to an \( s\)-section \((s, \sigma)\) is symmetric, if \((s, \sigma)\) satisfies the condition listed in Proposition 4.

Thus 2-groups, for which the corresponding class in \( H^3(G, M) \) lies in the image of \( HS^3(G, M) \) can be characterized, as those, for which there exists a symmetric \( s\)-functor \( Ca_G \to Ca_T \to R \), which is a section of \( p : Ca_T \to Ca_G \).

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