A Gibbsian Random Tree with Nearest Neighbour Interaction

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Abstract
We revisit the random tree model with nearest-neighbour interaction as described in Dunlop and Mardin (J Stat Phys 189:38, 2022), enhancing growth. When the underlying free Bienaymé–Galton–Watson (BGW) model is sub-critical, the (non-Markov) model with interaction exhibits a phase transition between sub- and super-critical regimes. In the critical regime, using tools from dynamical systems, we show that the partition function of the model approaches a limit at rate $n^{-1}$ in the generation number $n$. In the critical regime with almost sure extinction, we also prove that the mean number of external nodes in the tree at generation $n$ decays like $n^{-2}$. Finally, we give a spin representation of the random tree, opening the way to tools from the theory of Gibbs states, including FKG inequalities. We extend the construction in Dunlop and Mardin (J Stat Phys 189:38, 2022), when the law of the branching mechanism of the free BGW process has unbounded support.

KeywordsGalton–Watson · Gibbs random tree · Criticality · Fixed point · Correlation inequalities · FKG · Extinction

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1 Introduction

We extend the random tree process with nearest-neighbour interaction as described in [5] to the case of entire branching mechanisms, not necessarily polynomial. For this process favouring growth, the current law of a tree is Gibbsian, namely the one of the free BGW tree tilted by a Boltzmann weight function at inverse temperature $\beta = \log b > 0$ involving the energy of past nearest neighbour productive individuals (Sect. 2). We show that this (non-Markov) process with interaction exhibits a phase transition between sub- and super-critical regimes only when the underlying free BGW process is sub-critical. The transition of its eventual extinction probability is discontinuous at some $b = b_c > 1$. The computation of $b_c$ requires the one of the fixed points of the underlying transformation as a two-dimensional dynamical system giving the evolution of the partition function (Sects. 3 and 4). In Sects. 5 and 6, focus is on the approach to the critical point obtained when the two fixed points of the subcritical regime merge. Using the fact that the critical fixed point is partially hyperbolic in the sense of Takens, we show that the partition function of the model approaches a limit at rate $n^{-1}$ in the generation number $n$ (Sect. 5). In the critical regime with almost sure extinction, we also prove that the mean number of external nodes in the tree at generation $n$ decays like $n^{-2}$ (Sect. 6). These results confirm and extend the initial guesses discussed in [5]. Section 7 illustrates heuristically our results on a simple particular case amenable to a one-dimensional problem. Finally, in Sect. 8, we give a spin representation of the random tree, opening the way to tools from the theory of Gibbs states, including FKG inequalities. We extend the construction in [5] when the law of the branching mechanism of the free BGW process has unbounded support. Correlation inequalities (Griffiths, GKS, FKG, Lebowitz,...) are important tools in the mathematical analysis of statistical mechanical models as well as in the rigorous studies of quantum field theoretical models [1]. The FKG (for Fortuin, Kasteleyn and Ginibre) inequality is surely the most prominent one. Since it is essentially a result of the kind: if two summable functions/random variables are monotonous (i.e. both non-decreasing or non-increasing) then they are positively correlated, in other words their covariance is positive, it has already made its place in introductory books on probability theory, [2]. We prove the FKG inequality for our model which involves a nearest-neighbour pair interaction energy.

2 Model

Recall $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+ = \{1, 2, \ldots\}$. For $n \in \mathbb{N}$, let $T_n$ denote the set of planar rooted trees $\omega$ where the distance to the root, also called height, is bounded by $n + 1$. By convention the links between neighbouring connected nodes have length one. At distance $n + 1$ from the root, one finds the offspring of the nodes which lie at distance $n$ from the root, if any. These offspring are considered immature, carrying no information other than their number, and do not belong to the set of nodes $\{i \in \omega\}$. They will be called external nodes. The distance from a node $i$ to the root is denoted $|i|$. For any tree $\omega \in T_n$, for any node $i \in \omega$, implying $|i| \leq n$, let $X_i = X_i(\omega) \in \mathbb{N}$ denote the number of offspring of $i$, namely the number of neighbouring nodes away from the root. The root is node 0. Nodes are labeled à la Neveu [9]:

$$X_0, X_1, \ldots X_{X_0}, X_{11}, \ldots X_{1X_1}, X_{21}, \ldots X_{2X_2}, \ldots$$ (1)
Except for the root, the distance of a node to the root equals its number of digits in Neveu notation. We denote $a(i)$ the parent (first ancestor) of $i$, namely the neighbouring node towards the origin.

A non-interacting probability measure on $T_n$ is defined as follows. As it is a measure on a discrete set, it is enough to specify the probability of an atom, a tree $\omega$. Let $(p_k)_{k=0}^{\infty}$ be a partition of unity, $p_k \geq 0$ and $\sum_k p_k = 1$. For definiteness assume $p_0 > 0$ and $p_0 + p_1 < 1$. The non-interacting probability of a tree $\omega \in T_n$ will be

$$P_{GW}^{}(\omega) = \prod_{i \in \omega} p_{X_i(\omega)}. \tag{2}$$

The superscript $GW$ is for Galton–Watson. The normalization of the probability is easily verified by induction over $n$. Then, given a pair interaction energy $N \times N \ni (X, Y) \mapsto \varphi(X, Y) \in \mathbb{R} \tag{3}$ and a boundary condition $X_{a(0)} = x \in \mathbb{Z}_+$ specifying the offspring of a virtual ancestor for the origin, we define a Hamiltonian with first ancestor interaction,

$$H_{n}^{x}(\omega) = \sum_{i \in \omega} \varphi(X_{a(i)}, X_i) \tag{4}$$

and an interacting probability measure on $T_n$,

$$P_{n}^{x}(\omega) = \left( \mathbb{E}_{n}^{x} \right)^{-1} P_{GW}^{x}(\omega)e^{-\beta H_{n}^{x}(\omega)}, \tag{5}$$

$$\mathbb{E}_{n}^{x} = \sum_{\omega \in T_n} P_{GW}^{x}(\omega)e^{-\beta H_{n}^{x}(\omega)} \tag{6}$$

where $\beta \geq 0$ is the inverse temperature. Accordingly, for any observable $f(\omega)$,

$$\mathbb{E}_{n}^{x} f = \sum_{\omega \in T_n} P_{n}^{x}(\omega) f(\omega). \tag{7}$$

Observe that, even for the BGW measure in the subcritical case, the measure is supported by an infinite number of trees.

From (5) and (6), there is no one-step transition probability for $P_{n}^{x}(\omega)$: the process with interaction is not given by a Markov transition kernel. Given the nearest neighbour character of the interaction, it may however be described as a Markov field. The spin representation in Sect. 8 could be used to establish the Markov field property. Indeed the hard core (indicator) factor in (61) making a BGW tree from a spin configuration is a nearest neighbour interaction.

Our main example is

$$\varphi(X, Y) = -1_{X \geq 2}1_{Y \geq 2} \tag{8}$$

where the indicator function $1_A$ takes value one if event $A$ occurs and zero otherwise. This choice of interaction is motivated by an analogy with the Ising model.

Our motivation is complementary to that of the original BGW model and subsequent applications. The BGW model, besides being a simple genealogy model, has been used to describe the growth, possibly limited by extinction, of various networks, notably social networks and spread of epidemics. We assume that a node, once created, remains alive, so that the set of nodes in a BGW model can only grow or freeze. It is then natural to consider the possibility of interaction between neighbouring nodes on the BGW tree, and the Gibbs formalism is well suited for this purpose. It may be particularly relevant as the second stage
of a phenomenon where the first stage is the birth and growth of a network à la BGW until saturation at some height or time \( n \) followed by a second stage aiming at equilibrium with the chosen interaction. A dynamics for the second stage would obey the detailed balance condition with respect to the Gibbs measure \([5]\). The time scales of the two stages may be different, forbidding merging the two.

Tilting a reference measure for trees by a Gibbs factor is not new: this problem was addressed in \([11]\) with the counting measure on Cayley trees for reference measure and an interaction equal to the number of leaves (vertices of coordination number one). The interaction \((4)\) is a two-body potential. A possible one-body potential could be the total number of nodes, obtained by adding a constant to \( \phi \) in \((4)\), or the total number of leaves as in \([11]\).

3 A Dynamical System

Here we assume \((8)\) and \( n \geq 0 \). For a tree \( \omega \in T_n \), let \( N_n \) be the number of external nodes,

\[
N_n = \sum_{i \in \omega, |i| = n} X_{i_1 \ldots i_n}. \tag{9}
\]

Let \( N_n = L_n + Q_n \) where \( L_n \) denotes the number of external nodes whose parent has one offspring, and \( Q_n \) denotes the number of external nodes whose parent has two or more offspring. For \( u, v > 0 \) let

\[
\Xi^n_x(u, v) = \sum_{\omega \in T_n} \mu^{GW}(\omega) e^{-\beta H_n^x(\omega)} u^{L_n} v^{Q_n} \tag{10}
\]

with \( H_n^x(\omega) \) as \((4)-(8)\), implying

\[
\Xi^n_x(u, v) = \Xi^{2}_n(u, v) \quad \forall \ x \geq 2.
\]

The partition function \((6)\) is \( \Xi^n_x(1, 1) \). Let \( b = e^\beta \geq 1 \) and

\[
\Xi_0(u, v) = \begin{pmatrix} \Xi^1_0(u, v) \\ \Xi^2_0(u, v) \end{pmatrix} = \begin{pmatrix} p_0 + p_1 u + R(v) \\ p_0 + p_1 u + b R(v) \end{pmatrix} \equiv F(u, v), \tag{11}
\]

defining \( F \), where we assume

\[
R(v) := \sum_{k=2}^{\infty} p_k v^k < \infty \quad \forall \ v \tag{12}
\]

so that \( F(\cdot) \) maps \([1, \infty) \times [1, \infty)\) into itself and more precisely into \( D = \{(u, v) : 1 \leq u \leq v \leq bu\} \). Then for \( n \geq 1 \), summing over the choice of external nodes at distance \( n + 1 \) (equivalently over the \( X_i \)'s with \( |i| = n \)) yields, for \( x = 1, 2 \),

\[
\Xi^n_x(u, v) = \Xi^{x-1}_n(F(u, v)) = \cdots = \Xi^1_0(F^{(n)}(u, v)), \tag{13}
\]

Equivalently, starting from the root,

\[
\Xi_n = \begin{pmatrix} \Xi^1_n \\ \Xi^2_n \end{pmatrix} = \begin{pmatrix} p_0 + p_1 \Xi^{n-1}_n + R(\Xi^{2}_{n-1}) \\ p_0 + p_1 \Xi^{n-1}_n + b R(\Xi^{2}_{n-1}) \end{pmatrix} = F(\Xi^{1}_{n-1}, \Xi^{2}_{n-1}) = F(\Xi_{n-1}) = \cdots = F^{(n)}(\Xi_0) = F^{(n+1)}(\Xi_{-1}) = F^{(n+1)}(u, v) \tag{14}
\]

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where the arguments \((u, v)\) have been omitted for legibility except at the end. We have not defined the model with \(n = -1\) but the initial condition at \(n = 0\) of the recurrence, equation (11), can be pushed to \(n = -1\), with \(\Xi_{-1}(u, v) = (u, v)\).

If for some value of \(b\) the orbit of \((1, 1)\) under iteration of \(F\) goes to infinity as \(n \to \infty\), then from (5), (6) and (13), for any constant, the probability that the total number of nodes is less than the constant goes to zero as \(n \to \infty\). We say that the model is super-critical.

Otherwise the model is sub-critical or critical. We define \(T_\infty\) as the set of finite planar rooted trees. If \(\omega \in T_p\) then \(H_x^n(\omega) = H_x^p(\omega)\), whatever \(n \geq p\). For any \(\omega \in T_\infty\), for \(n > |\omega|\), the probability to observe \(\omega\) in \(T_n\), given by (5), obeys

\[
P_x^n(\omega) = \frac{\mathbb{P}^{GW}(\omega)e^{-\beta H_x^n(\omega)}}{\sum_{\omega' \in T_n} \mathbb{P}^{GW}(\omega')e^{-\beta H_x^n(\omega')}} \to \frac{\mathbb{P}^{GW}(\omega)e^{-\beta H_x^\infty(\omega)}}{\sum_{\omega' \in T_\infty} \mathbb{P}^{GW}(\omega')e^{-\beta H_x^\infty(\omega')}} = P_x^\infty(\omega) > 0.
\]

(15)

Indeed, the limit denominator in (15) is finite in the subcritical or critical regimes.

If the underlying BGW is super-critical, the interacting model is also super-critical, as can be shown by correlation inequalities, Sect. 8.

Our main motivation is criticality with interaction. We therefore assume that the underlying BGW model with branching mechanism \(p_0 + p_1 v + R(v)\) is sub-critical:

\[
\bar{k} := \sum_{k=0}^{\infty} k p_k = p_1 + R'(1) < 1.
\]

(16)

As a consequence of Theorem 1 below, we obtain the following proposition:

**Proposition 1** Suppose the underlying free BGW process is subcritical or critical. Then, the eventual extinction probability of the Gibbs tilted process

\[
\rho = \lim_{n \to \infty} P_x^n (N_n = 0)
\]

exists

with,

\[
\rho = \begin{cases} 
1 & \text{if } b \leq b_c \\
0 & \text{if } b > b_c > 1 
\end{cases}
\]

(17)

**Proof** Eventual extinction is defined by the event in which the number of external nodes \(N_n\) goes to zero as \(n\) tends to infinity. In the super-critical regime, the probability of eventual extinction is nil, due to the divergence of the partition function, see Theorem 1 below. In the sub-critical and critical regimes, the probability of eventual extinction is one due to the convergence of the partition function, see Theorem 1. There exists thus \(b_c > 1\) such that the probability \(\rho\) of eventual extinction is one for \(b \leq b_c\) and zero for \(b > b_c\), a discontinuous transition. The computation of \(b_c := b_c((p_k)_k)\), is given as the solution of (21) and (22) together with \(v = v_c, u = u_c\) below.

This differs from the continuous transition occurring in the free BGW model at \(k = \bar{k} = 1\), as defined in (16).

The potential (8) favors proliferation. Would we change its sign, a ’dual’ phase transition for the Gibbs model would require a supercritical free BGW model. We do not consider that case, being concerned here by an interaction model acting as a selection of paths mechanism reinforcing growth of the free process.


4 Fixed Point

A partial order on \( D \) is defined by

\[
(u, v) \geq (u', v') \text{ if and only if } u \geq u' \text{ and } v \geq v'
\]

(18)

whereby \( F \) is partially monotone:

\[
(u, v) \geq (u', v') \Rightarrow F(u, v) \geq F(u', v').
\]

(19)

Denote \((u_n, v_n) = F^{(n)}(1, 1)\), starting from \((u_0, v_0) = (1, 1)\). We have \((u_1, v_1) \geq (1, 1)\) and by induction \((u_{n+1}, v_{n+1}) \geq (u_n, v_n) \forall n\). If there is a fixed point \((u_c, v_c)\) in \(\{u \geq 1, v \geq 1\}\) then \(F\) being partially monotone implies

\[
(1, 1) \leq (u_n, v_n) \leq (u_{n+1}, v_{n+1}) \leq (u_c, v_c)
\]

(20)

implying convergence. A fixed point \((u, v)\) of \(F\) is given by

\[
u = \frac{p_0 + R(v)}{1 - p_1}, \quad v = \frac{p_0}{1 - p_1} + \frac{(1 - p_1)b + p_1}{1 - p_1} R(v).
\]

(21)

As \(R(v)\) is strictly convex for \(v \geq 0\), there are 0, 1 or 2 real fixed points in \(\{v \geq 1\}\). If there is more than one real fixed point, consider two fixed points \((u_f, v_f)\) and \((u'_f, v'_f)\), with \((u_f, v_f) < (u'_f, v'_f)\), then they are partially ordered because \(R\) is non-decreasing. The sequence converges to the smaller because \((u_0, v_0) < (u_f, v_f)\) and \(F\) is partially monotone. If there is just one fixed point in \(\{u \geq 1, v \geq 1\}\), the sequence converges to it. If there is no real fixed point in \(\{v \geq 1\}\), the sequence diverges to infinity.

The critical regime is the borderline between 0 fixed point and two fixed points (saddle node bifurcation), where the bisector line \(v\) is tangent to the convex curve given by the right-hand-side of (21), so that at a critical fixed point \((u_c, v_c)\), in addition to (21) we have

\[
1 = \frac{(1 - p_1)b + p_1}{1 - p_1} R'(v)
\]

(22)

and \((b_c, v_c)\) will be solution of the pair (21) and (22) with \((b, v)\) as unknowns. From (22) we extract

\[
b = \frac{1}{R'(v)} - \frac{p_1}{1 - p_1}
\]

whereby (21) becomes

\[
v = \frac{p_0}{1 - p_1} + \frac{R(v)}{R'(v)} = f(v).
\]

(23)

By virtue of (16) and \(R(1) = 1 - p_0 - p_1\), we have \(f(1) > 1\). On the other hand

\[
R'(v) = \sum_{k=2}^{\infty} kp_k v^{k-1} \geq 2 \frac{R(v)}{v}
\]

(24)

so that \(f(v) < v\) for \(v\) large and \(v = f(v)\) has solutions with \(v \geq 1\). Hence \((b_c, v_c)\), and \(u_c\) given by (21).

The fixed point equation can be solved by a series expansion in the spirit of Lagrange [4]: let \(z\) be defined as a function of \(w\) in terms of a parameter \(\alpha\) by \(z = w + \alpha \phi(z)\). Lagrange’s
inversion theorem, also called a Lagrange expansion, states that any function $F$ of $z$ can be expressed as a power series in $\alpha$ which converges for sufficiently small $\alpha$ and has the form

$$F(z) = F(w) + \sum_{n \geq 1} \frac{\alpha^n}{n!} \partial^n_{w} \left[ \phi(w)^n F'(w) \right].$$

In our context, we need to solve the fixed point equations:

(i) $p_0 + p_1 u + R(v) = u$

(ii) $p_0 + p_1 u + bR(v) = v$

Substituting $u$ as a function of $v$ in (i) into (ii) and setting $p_1 = 1 - p_1$, $b = p_1 + p_1 b$, $z = p_1 v$ and $R(z) = R(z/p_1)$, (ii) becomes ($R$ being an entire function),

$$z = p_0 + bR(z).$$

So, with $u = (p_0 + R(z))/p_1 = F(z)$ and $F'(p_0) = R'(p_0)/p_1$

$$\bar{p}_1 v = z = p_0 + \sum_{n \geq 1} \frac{\bar{b}^n}{n!} \partial^n_{p_0} \left[ R(p_0)^n \right]$$

$$u = F(z) = F(p_0) + \sum_{n \geq 1} \frac{\bar{b}^n}{n!} \partial^n_{p_0} \left[ R(p_0)^n F'(p_0) \right]$$

$$\bar{p}_1 u = p_0 + R(p_0) + \sum_{n \geq 1} \frac{\bar{b}^n}{n!} \partial^n_{p_0} \left[ R(p_0)^n R'(p_0) \right]$$

which yields a Taylor expansion of the first fixed point $(u, v)$ in (i), (ii) (i.e. the nearest to $(1, 1)$) in terms of the control parameter $\bar{b}$ when $b < b_c$. These series expansions of the critical point are convergent when $b \leq b_c$. When $b = b_c$, the two fixed points merge. This occurs when $\bar{b}_c := p_1 + \bar{p}_1 b_c$, characterized by

$$\bar{b}_c R'(v_c) = 1$$

which yields an implicit expression of $b_c$.

By a second Lagrange inversion formula ($R'(0) = 0$), we have ([4], page 159)

$$v_c = \sum_{n \geq 1} \frac{\bar{b}_c^n}{n!} \partial^n_{v} \left[ \left( \frac{v}{R'(v)} \right)^n \right] \bigg|_{v=0} = \sum_{n \geq 1} \frac{\bar{b}_c^n}{n} \left[ v^{n-1} \right] \left( \frac{v}{R'(v)} \right)^n$$

as a Taylor expansion of $v_c$ in terms of powers of $\bar{b}_c^{-1}$ The Taylor expansion of $u_c$ follows from $p_0 + p_1 u + R(v) = u$, so with

$$u_c = \frac{1}{\bar{p}_1} (p_0 + R(v_c)),$$

where $R(v_c)$ is obtained from the expansion of $v_c$ by Fa à di Bruno formula [10].

From now on, $b = b_c((p_k)_k)$, solution of (21) and (22) together with $v = v_c, u = u_c$. We have

$$R'(v_c) = \frac{1 - p_1}{1 + (b - 1)(1 - p_1)} \equiv R'_c$$

which implies $R'(1) < 1 - p_1$: the underlying BGW model must be sub-critical for a critical point to exist.
5 Critical Regime: Approach to Fixed Point

The tangent map near \((u_c, v_c)\),

\[
DF = \begin{pmatrix} p_1 & R'(v) \\ p_1 b R'(v) \end{pmatrix},
\]

then has eigenvalues 1 and

\[
\lambda_2 = (b - 1) p_1 R'_c = \frac{(b - 1)(1 - p_1)p_1}{(b - 1)(1 - p_1) + 1}, \quad 0 < \lambda_2 < p_1 < 1
\]

(30)
The eigenvector of the eigenvalue \(\lambda_2\) is proportional to the vector

\[
\begin{pmatrix} -R'_c \\ \frac{p_1}{p_1 (b - 1)(b - 1)} \\ -1 \end{pmatrix}.
\]

From the expression of \(b = b_c\) and, from (28), \(\frac{1}{b - 1} > R'_c > 0\), thus we conclude that the components of this vector have opposite signs. This corresponds to a partially hyperbolic fixed point, as considered by Takens [12], with \(s = 1, c = 1, u = 0\) in Takens’ notation, and one can check that the Sternberg non-resonance conditions are satisfied. It follows that the center manifold [7, 8] is regular. We first translate the fixed point to the origin,

\[
\begin{align*}
  u &= u_c + x \\
  v &= v_c + y
\end{align*}
\]

so that the map now reads

\[
G(x, y) = \begin{pmatrix} p_1 x + R'_c y + R''_c y^2/2 + O(y^3) \\ p_1 x + b R'_c y + b R''_c y^2/2 + O(y^3) \end{pmatrix}.
\]

(31)

where \(R''_c = R''(v_c)\). We look for the center manifold in the form of a graph

\[
x = f(y) = a y + c y^2 + O(y^3).
\]

(32)

By definition, if \(M\) is in the center manifold then \(G(M)\) is also in the center manifold. Therefore

\[
G((f(y), y)) = (f(y'), y')
\]

for some

\[
y' = g(y) = g_1 y + g_2 y^2 + O(y^3).
\]

(33)

Expanding to second order the relation

\[
G((f(y), y)) = (f(g(y)), g(y)),
\]

(34)

we find

\[
\begin{align*}
  a &= \frac{1}{(b - 1)(1 - p_1) + 1} = \frac{R'_c}{1 - p_1}, \\
  c &= \frac{R''_c}{2} \frac{(b - 1)p_1}{(b - 1)(1 - p_1)^2 + 1}, \\
  g_1 &= 1.
\end{align*}
\]
\[ g_2 = p_1c + \frac{R''c}{2}b = \frac{R''c}{2}[(b - 1)(1 - p_1) + 1]^2 > 0. \]  

(35)

In order to find the asymptotics of

\[ y_{n+1} = g(y_n) \]

with \( y_1 < 0 \) small, we set

\[ z_n = \frac{1}{y_n}, \]

so that (33) becomes

\[ z_{n+1} = z_n - g_2 + \frac{g_2^2}{z_n} + O\left(\frac{1}{z_n^2}\right). \]  

(36)

with \( z_n \rightarrow -\infty \) as \( n \rightarrow +\infty \). Therefore for all \( g_2 > \epsilon > 0 \) there exists \( n_1 > 1 \) so that for all \( n > n_1 \),

\[ \left| \frac{g_2^2}{z_n} + O\left(\frac{1}{z_n^2}\right) \right| \leq \epsilon. \]

Hence \( \forall n > n_1 \)

\[ z_{n+1} \leq z_n - g_2 + \epsilon, \]

\[ z_n \leq z_{n_1} - (n - n_1)(g_2 - \epsilon). \]

Inserting into (36) yields, \( \forall n > n_1 \),

\[ z_{n+1} = z_n - g_2 + \frac{O(1)}{z_{n_1} - (n - n_1)(g_2 - \epsilon)}, \]

\[ z_n = -n\frac{g_2}{g_2} + O(\log n), \]

\[ y_n = -\frac{1}{n\frac{g_2}{g_2}} + O\left(\frac{\log n}{n^2}\right). \]

From the Theorem in section 1 of [12] there exists a diffeomorphism \( \Phi \) from a neighbourhood \( V_0 \) of \( (u_c, v_c) \) to a neighbourhood of the origin in \( \mathbb{R}^2 \) such that \( F = \Phi^{-1} \circ N \circ \Phi \), where \( N \) is the map

\[ N\left(\frac{\xi}{\eta}\right) = \left(\frac{\phi(\xi)}{\lambda(\xi) \eta}\right). \]

and \( \phi \) and \( \lambda \) are regular functions satisfying \( \phi(0) = 0, \phi'(0) = 1 \), and \( \lambda(0) = \lambda_2 \).

Since \( \lim_{n \rightarrow \infty} (u_n, v_n) = (u_c, v_c) \) there exists \( n_0 > 0 \) such that for an \( n > n_0 \) we have \( (u_n, v_n) \in V_0 \). For \( n > n_0 \) we define \( (\xi_n, \eta_n) \) by

\[ \left(\frac{\xi_n}{\eta_n}\right) = \Phi\left(\frac{u_n}{v_n}\right). \]

Since \( \Phi \) is continuous we have

\[ \lim_{n \rightarrow \infty} (\xi_n, \eta_n) = (0, 0). \]

We also have

\[ \xi_{n+1} = \phi(\xi_n) \]
Fig. 1 Center manifold \((W^c)\), stable manifold \((W^s)\) of the fixed point \((u_c, v_c)\) and an orbit for a critical \(b\). The picture is centred at the fixed point. The parameters are \(p_0 = 0.3, p_1 = 0.69, p_2 = 0.01\) and 

\[
\eta_{n+1} = \lambda(\xi_n) \eta_n.
\]

Since \((\xi_n)\) tends to zero when \(n\) tends to infinity, we can find \(n_1 \geq n_0\) such that for any \(n > n_1\) we have

\[
|\lambda(\xi_n)| < \frac{1 + |\lambda_2|}{2} < 1.
\]

Therefore \((\eta_n)\) converges to zero exponentially fast.

The regular curve \(s \to \Phi^{-1}(0, s)\) defined for \(|s|\) small enough is a “fast” manifold. The orbit of any initial condition on this manifold is contained in it and converges exponentially fast to \((u_c, v_c)\). This curve passes through \((u_c, v_c)\) (for \(s = 0\)) and is tangent at this point to the eigendirection of \(DF(u_c, v_c)\) corresponding to the eigenvalue \(\lambda_2\). As observed previously, this eigendirection does not intersect the negative cone \(C^-\) of the \((u, v)\) plane with apex \((u_c, v_c)\). Therefore, there exists a ball \(V\) centered in \((u_c, v_c)\), contained in \(V_0\) such that the intersection of the “fast” curve with \(V \cap C^-\) reduces to the origin. See Fig. 1.

Since \((\xi_n, \eta_n)\) tends to the origin when \(n\) tends to infinity, we can find \(n_2 \geq n_1\) such that for any \(n > n_2\) we have \((\xi_n, \eta_n) \in \Phi(V)\). From the above argument we have \(\xi_n \neq 0\).

We now define for \(n > n_2\) a new sequence of points \((\tilde{u}_n, \tilde{v}_n)\) by

\[
\begin{pmatrix}
\tilde{u}_n \\
\tilde{v}_n
\end{pmatrix}
= \Phi^{-1}
\begin{pmatrix}
\xi_n \\
0
\end{pmatrix}.
\]

Since \(\phi(\xi_n) = \xi_{n+1}\) we have (with \(b = b_c\))

\[
F(\tilde{u}_n, \tilde{v}_n) = (\tilde{u}_{n+1}, \tilde{v}_{n+1}).
\]

This sequence of points is on the center manifold (and differs from \((u_c, v_c)\)) since \(\xi_n \neq 0\), and therefore we have from a previous result

\[
\tilde{v}_n = v_c - \frac{1}{ng_2} + \mathcal{O}\left(\frac{\log n}{n^2}\right).
\]
We also have (since $\tilde{u}_n - u_c = f(\tilde{v}_n - v_c)$)
\[ \tilde{u}_n = u_c - \frac{a}{n g_2} + O \left( \frac{\log n}{n^2} \right) . \]

Since $\Phi^{-1}$ is locally Lipschitz near the origin we have for $n$ large enough
\[ |\tilde{u}_n - u_n| + |\tilde{v}_n - v_n| \leq O(1) |\eta_n| \]
which converges to zero exponentially fast as we have seen before. Summarizing the above results, we formulate the following theorem:

**Theorem 1** Consider a sub-critical Bienaymé–Galton–Watson Markov chain given by (2) with $p_0 + p_1 < 1$ and $\sum_{k=0}^{\infty} k p_k < 1$. For $n \in \mathbb{N}$ and $b = \exp(\beta) > 1$ and $x = 1$ or 2, consider the measure on planar rooted trees of height bounded by $n+1$ given by (4)–(8). Then the partition function (6) is an increasing function of $n$, and there exists $b = b_c((p_k)_k) > 1$ such that $\Xi_n^x$ converges to a finite limit $\Xi_{\infty}^x$ as $n \to \infty$ if and only if $b \leq b_c$. Moreover, if $b = b_c$, then
\[ \Xi_n^1 - \Xi_{\infty}^1 = -\frac{1}{n g_2} + O \left( \frac{\log n}{n^2} \right), \quad \Xi_n^2 - \Xi_{\infty}^2 = -\frac{a}{n g_2} + O \left( \frac{\log n}{n^2} \right), \quad (37) \]
where $a$ and $g_2$ are given by (35).

### 6 Critical Regime: Number of External Nodes

From (10),
\[ \left( \Xi_n^1 L_n, \Xi_n^2 L_n \right) = \left( \Xi_n^1, \Xi_n^2 \right)_{u=v=1}, \quad \left( \Xi_n^1 Q_n, \Xi_n^2 Q_n \right) = \left( \Xi_n^1, \Xi_n^2 \right)_{u=v=1} \quad (38) \]

In view of Theorem 1, it suffices to study, using (14) and the chain rule, the tangent map
\[ D \Xi_n(u, v)|_{u=v=1} = DF^{(n+1)}(u, v)|_{u=v=1} = DF(u_n, v_n)DF(u_{n-1}, v_{n-1}) \ldots DF(u_1, v_1)DF(1, 1) . \quad (39) \]

In the sequel we sometimes use the simplified notation
\[ DF^{(k)}(u, v) \equiv DF_{u,v}^k \]
for the differential of the $k$-th iterate of $F$ evaluated at $(u, v)$.

**Theorem 2** There exists a constant $C > 1$ such that for $b = b_c$
\[ C^{-1} \leq \liminf_{n \to \infty} n^2 \begin{pmatrix} 1 & 0 \end{pmatrix} DF^1_{1,1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \leq \limsup_{n \to \infty} n^2 \begin{pmatrix} 1 & 0 \end{pmatrix} DF^n_{1,1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \leq C. \]

The same result holds for the other component and the vector $(0, 1)$.

The proof of this theorem requires the following two lemmas. Let $\mathcal{C}$ denote the positive cone of $\mathbb{R}^2$. A partial order of $\mathbb{R}^2$ is associated to this cone by $\vec{v} \geq \vec{w}$ if $\vec{v} - \vec{w} \in \mathcal{C}$ (see for example [3], chapter XVI, Sect. 1).
Lemma 1. Let $u_0 > 0$ and $v_0 > 0$. Let $\vec{u}$ and $\vec{v}$ be two vectors in the interior of $\mathcal{C}$. Let $\Lambda > 1$ be such that

$$\Lambda^{-1} \vec{u} \leq \vec{w} \leq \Lambda \vec{v}.$$  

(such a finite $\Lambda$ always exists). Then for $b = b_c$ and for any integer $k$

$$\Lambda^{-1} \mathcal{D}F^k_{u_0, v_0} \vec{u} \leq \mathcal{D}F^k_{u_0, v_0} \vec{w} \leq \Lambda \mathcal{D}F^k_{u_0, v_0} \vec{v},$$

In particular

$$\Lambda^{-1} \leq \frac{(1, 0) \mathcal{D}F^k_{u_0, v_0} \vec{w}}{(1, 0) \mathcal{D}F^k_{u_0, v_0} \vec{v}} \leq \Lambda,$$

and similarly for the other component.

Proof The first result follows from the fact that if $u > 0$, $v > 0$ and $b = b_c$, the entries of the matrix $\mathcal{D}F_{u, v}$ are positive. The second result follows from the definition of the order. \qed

For $b = b_c$ we can diagonalize the matrix $\mathcal{D}F(u_c, v_c)$:

$$S \mathcal{D}F(u_c, v_c) S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} \frac{R_c'}{1-p_1} & \frac{-R_c'}{p_1(b-1)R_c' - 1} \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & \frac{R_c'}{p_1(b-1)R_c' - 1} \\ 1 & \frac{R_c'}{1-p_1} \end{pmatrix}$$

$$|S^{-1}| = \frac{R_c'}{1-p_1} \frac{1-p_1 (b-1) R_c'}{p_1 (b-1) R_c' - 1}$$  \hspace{1cm} (40)

We then use the matrix $S$ for an approximate diagonalization of $\mathcal{D}F(u, v)$ near the fixed point $(u_c, v_c)$:

$$A(v) = S \mathcal{D}F(u, v) S^{-1}, \quad A(v_c) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$  \hspace{1cm} (41)

The variable $u$ has been omitted from $A(\cdot)$ because the matrix $\mathcal{D}F(u, v)$ actually does not depend upon $u$. We obviously have for any $n_0 > 0$, $b = b_c$, and $(u_{n_0}, v_{n_0}) = F^{n_0}(1, 1)$

$$\mathcal{D}F(u_n, v_n) \mathcal{D}F(u_{n-1}, v_{n-1}) \ldots \mathcal{D}F(u_{n_0}, v_{n_0}) = S^{-1} A(v_n) A(v_{n-1}) \ldots A(v_{n_0}) S.$$  \hspace{1cm} (42)

Hence the asymptotic behaviour of $\mathcal{D}F^n$ will follow from the asymptotic behaviour of the product of the $A$ matrices. Note also that $A(v_n)$ converges to $A(v_c)$.

In a neighbourhood of $v_c$ we can write

$$A(v) = \begin{pmatrix} 1 + \alpha(v) & \beta(v) \\ \gamma(v) & \lambda_2 + \delta(v) \end{pmatrix}$$  \hspace{1cm} (43)

with

$$\alpha(v) = \alpha'(v_c)(v - v_c) + \mathcal{O}((v - v_c)^2)$$  \hspace{1cm} (44)
and similarly for $\beta$, $\gamma$ and $\delta$. These can be computed at leading order from

$$DF(u, v) = \left( p_1 R_c' + (v - v_c) R_c'' \right) + \mathcal{O}((v - v_c)^2).$$

For any $\theta \in \mathbb{R}$ such that $\theta \neq -\frac{1 + \alpha(v)}{\beta(v)}$ we have

$$A(v) \left( \frac{1}{\theta} \right) = \rho(v, \theta) \left( \frac{1}{\tilde{\theta}(v, \theta)} \right)$$

with

$$\rho(v, \theta) = 1 + \alpha(v) + \beta(v) \theta$$

$$\tilde{\theta}(v, \theta) = \frac{\gamma(v) + (\lambda_2 + \delta(v)) \theta}{1 + \alpha(v) + \beta(v) \theta}.$$

**Lemma 2** There exists $n_0 > 1$ large enough such that the sequence $(\theta_j)_{j \geq n_0}$ defined recursively by $\theta_{n_0} = 0$ and

$$\theta_{j+1} = \frac{\gamma(v_j) + (\lambda_2 + \delta(v_j)) \theta_j}{(1 + \alpha(v_j)) + \beta(v_j) \theta_j}$$

satisfies

$$\theta_j = -\frac{\alpha'(v_c)}{j (1 - \lambda_2)} + o(j^{-1}).$$

**Proof** We take $n_0$ large enough such that for any $n > n_0$ the numbers $\alpha(v_n)$, $\beta(v_n)$, $\gamma(v_n)$ and $\delta(v_n)$ are small enough. We have already the asymptotics of $v_j - v_c$ as (37). The asymptotics of $\theta_j$ follows from

$$\theta_{j+1} = \frac{\gamma(v_j) + (\lambda_2 + \delta(v_j)) \theta_j}{(1 + \alpha(v_j)) + \beta(v_j) \theta_j} = \frac{\alpha'(v_c)}{s_2 j} + \lambda_2 \theta_j + \mathcal{O} \left( \frac{\log j}{j^2} \right).$$

yielding

$$\theta_j = -\frac{\alpha'(v_c)}{j (1 - \lambda_2)} + o(j^{-1}).$$

**Lemma 3** Let $(\theta_j)$ be the sequence defined in Lemma 2. Then for some $C > 1$

$$C^{-1} n^{-\alpha'(v_c)/s_2} < \prod_{j=n_0}^{n} \rho(v_j, \theta_j) < C n^{-\alpha'(v_c)/s_2}.$$

**Proof** Since $(\theta_n)$ tends to zero by Lemma 2 the dominant factor is

$$\prod_{j=n_0}^{n} (1 + \alpha(v_j)).$$

The result follows from the asymptotic behavior of $\Xi_n^2$ in Theorem 1. \qed
The (1, 1) - entry of $S^{-1} A(v) S$ is found to be
\[
1 + \frac{1}{|S^{-1}|} \tilde{R}''_c (v - v_c) \frac{1}{p_1 [(b - 1) \tilde{R}'_c - 1]}
\]
so that
\[
\alpha' (v_c) = \frac{1}{|S^{-1}|} \frac{1}{p_1 [(b - 1) \tilde{R}'_c - 1]}
\]
\[
= \frac{\tilde{R}''_c}{p_1 + (1 - p_1) [p_1 + b (1 - p_1)]}
\]
\[
= 2g_2.
\]

**Proof of Theorem 2** Consider for example
\[
DF_{1,1}^n \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
\]
the other case is similar. For $n > n_0$ we have by the chain rule
\[
DF_{1,1}^n \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = DF_{1,1}^{n-n_0, v_0} DF_{1,1}^{n_0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]
Since the entries of the matrix $DF_{1,1}^{n_0}$ are all strictly positive, the vector
\[
DF_{1,1}^{n_0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
\]
belongs to the interior of $\mathcal{C}$. Let
\[
\tilde{v} = S^{-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \left( \begin{array}{c} \frac{\tilde{R}'_c}{1 - p_1} \\ 1 \end{array} \right)
\]
which is also in the interior of $\mathcal{C}$. We apply Lemma 1 with $k = n - n_0$ and
\[
\tilde{w} = DF_{1,1}^{n_0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]
We obtain that there exists $\Lambda > 1$ (which depends on $n_0$) such that for any $n > n_0$
\[
\Lambda^{-1} \leq \frac{(1, 0) DF_{1,1}^n \left( \begin{array}{c} 1 \\ 0 \end{array} \right)}{(1, 0) DF_{1,1}^{n-n_0, v_0} \tilde{v}} \leq \Lambda,
\]
and similarly for the other component.
The result follows using (42) and Lemma 3.

The four components are related as follows:
\[
E_1^1 L_n = \frac{1}{u_{n+1}} (1, 0) S^{-1} A(v_n) A(v_{n-1}) \ldots A(1) S \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
\]
\[
E_2^2 L_n = \frac{1}{v_{n+1}} (0, 1) S^{-1} A(v_n) A(v_{n-1}) \ldots A(1) S \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
\]
\[
E_1^1 Q_n = \frac{1}{u_{n+1}} (1, 0) S^{-1} A(v_n) A(v_{n-1}) \ldots A(1) S \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\]
\[ \mathbb{E}_n^2 Q_n = \frac{1}{v_{n+1}} (0, 1) S^{-1} A(v_n) A(v_{n-1}) \ldots A(1) S \left( \frac{0}{1} \right) \] (49)

\[ \mathbb{E}_n^1 N_n = \frac{1}{u_{n+1}} (1, 0) S^{-1} A(v_n) A(v_{n-1}) \ldots A(1) S \left( \frac{1}{1} \right) \leq \mathbb{E}_n^2 N_n = \frac{1}{v_{n+1}} (0, 1) S^{-1} A(v_n) A(v_{n-1}) \ldots A(1) S \left( \frac{1}{1} \right) . \] (50)

Of course all are positive, and the intermediate inequality is a consequence of the FKG inequality (see Theorem 3 below).

**Corollary 1** Assume the hypotheses of Theorem 1, and let \( b = b_c \). Let \( N_n \) be the number of external nodes. Then for some \( C > 1 \), for all \( n \geq 1 \),

\[ C^{-1} < n^2 \mathbb{E}_n^x N_n < C. \] (51)

**7 A Simple Case: \( p_1 = 0 \)**

When \( p_1 = 0 \), the interaction (4) with (8) can be given a simpler interpretation. Let us call active nodes the nodes \( i \in \omega \in T_n \) such that \( X_i > 0 \), and denote \( A_n(\omega) \) their number. When \( p_1 = 0 \) we have

\[ -H_n^2(\omega) = A_n(\omega), \quad -H_n^1(\omega) = \tilde{A}_n(\omega) - 1_{X_0 > 0}. \] (52)

Denote \( \tilde{N}_n(\omega) \) the number of nodes \( i \in \omega \). For \( \omega' \in T_{n+1} \), let \( \omega(\omega') \in T_n \) be obtained from \( \omega' \) by removing the nodes with \( |i| = n + 1 \). Then

\[ \tilde{N}_{n+1}(\omega') = \tilde{N}_n(\omega) + Q_n(\omega). \] (53)

When \( p_1 = 0 \), the dynamical system in Sects. 5 and 6 collapses to a simpler one dimensional problem, for which the heuristics is easy. We now have

\[ g(y) = y + g_2 y^2 + O(y^3), \quad g_2 = b R_c''/2 \]

The ansatz \( y_n \simeq -\gamma n^{-\alpha} \) yields

\[ \alpha = 1, \quad \gamma = 1/g_2, \quad y_n \simeq -2/(b R_c'' n) . \]

The mean number of external nodes in generation \( n \), last generation, equals the average of \( Q_n \), given by (38),

\[ \mathbb{E}_n^x Q_n = \frac{1}{\mathbb{E}_n^x} \frac{\partial \Xi_n^x}{\partial v} \bigg|_{u=v=1} \] (54)

with (14),

\[ \Xi_n = \left( \begin{array}{c} p_0 + R(\Xi_{n-1}^2) \\ p_0 + b R(\Xi_{n-1}^2) \end{array} \right) \] (55)

so that

\[ \frac{\partial \Xi_n}{\partial v} \bigg|_{u=v=1} = R' (\Xi_{n-1}^2) \frac{\partial \Xi_{n-1}^2}{\partial v} \bigg|_{u=v=1} \left( \begin{array}{c} 1 \\ b \end{array} \right) \] (56)

\[ \frac{\partial \Xi_n^2}{\partial v} \bigg|_{u=v=1} = b R'(v_n) \frac{\partial \Xi_{n-1}^2}{\partial v} \bigg|_{u=v=1} \simeq \left( b R'_c + b R''_c y_n \right) \frac{\partial \Xi_{n-1}^2}{\partial v} \bigg|_{u=v=1} \] (57)
Using \(bR'_c = 1\), inserting \(y_n\) and iterating yields
\[
\frac{\partial \Xi_n^x}{\partial v}
\bigg|_{u=v=1}
\sim \prod_{m=1}^{n} (1 - 2/m) \sim \exp(-2 \log n) \sim n^{-2}
\]
where \(x = 1\) or \(2\) and the proportionality differs by a factor \(b\) according to \(x\).

8 Spin Model

A spin representation of a random tree gives some access to tools from the theory of Gibbs states of spin models. Here we extend the representation from the bounded offspring distribution \([5]\) to the unbounded case. Let \(n \geq 0\). The set of possible sites or nodes \(i\) is
\[
\Lambda_n = \bigoplus_{n' = 0}^{n} (\mathbb{Z}_+)^{n'}, \quad (\mathbb{Z}_+)^0 = \{0\}.
\]
The origin is \(i = 0\). The other sites or nodes have a non-random Neveu label \(i = i_1 i_2 \ldots i_n\) for some \(n' \leq n\), each \(i_{n''}\) is the rank of the ancestor of \(i\) in generation \(n''\), for each \(1 \leq n'' \leq n'\). The rank of a node is denoted \(r(i)\). We extend the random variable \(X_i(\omega)\) to the whole of \(\Lambda_n\) by assigning the value \(X_i(\omega) = -1\) to all the possible “phantom” nodes not in the tree \(\omega\). The set of configurations \(\chi = (X_i)_{i \in \Lambda_n}\), where \(X_i \in \{-1\} \cup \mathbb{N}\), is
\[
\Omega_n = \{[-1] \cup \mathbb{N}\}^{\Lambda_n}
\]
It is endowed with partial order: \(\chi \geq \chi'\) if and only if \(X_i \geq X_i'\) for all \(i\). A function \(F : \Omega_n \rightarrow \mathbb{R}\) is termed non-decreasing if and only if \(\chi \geq \chi' \Rightarrow F(\chi) \geq F(\chi')\).

**Proposition 2** Let \(n \geq 1\). By convention, let \(p_{-1} = 1\). Fix a boundary condition \(X_{a(0)} \in \mathbb{Z}_+\).

For \(\chi = (X_i)_{i \in \Lambda_n} \in \Omega_n\) let
\[
\mu^{GW}(\chi) = \prod_{i \in \Lambda_n} p_{X_i}
\bigg(1_{X_{a(i)} \geq r(i)} 1_{X_i \geq 0} + 1_{X_{a(i)} < r(i)} 1_{X_i < 0}\bigg).
\]
Let \(\omega\) denote a tree sampled from a BGW chain at time \(n\). Recall
\[
P^{GW}(\omega) = \prod_{i \in \omega} p_{X_i(\omega)}.
\]
Then there is a bijection \(\chi \leftrightarrow \omega\) between the support of \(\mu^{GW}\) and the set of BGW trees at time \(n\), and \(\mu^{GW} \sim P^{GW}\):
\[
\chi \mapsto \omega(\chi), \quad P^{GW}(\omega(\chi)) = \mu^{GW}(\chi) \quad \omega \mapsto \chi(\omega), \quad \mu^{GW}(\chi(\omega)) = P^{GW}(\omega)
\]

**Proof** This proposition extends to unbounded offspring distributions a similar result for bounded distributions \([5]\). The proof is essentially the same. Note that \(r(i) > X_{a(i)} \Rightarrow X_i = -1\). Therefore \(\mu\) almost surely all but a finite number of \(X_i\) take value \(-1\).

Proposition 2 allows to prove correlation inequalities \([6]\). In particular:

**Theorem 3** (FKG inequality) Let \(n \geq 0\) and \(x \in \mathbb{Z}_+\). Assume \((4)\) and, for all \(X, Y, X', Y' \in \{-1\} \cup \mathbb{N}\)
\[
\varphi(X, Y) + \varphi(X', Y') \geq \varphi(X \wedge X', Y \wedge Y') + \varphi(X \vee X', Y \vee Y').
\]
Then for any non-decreasing functions $F, G : \Omega_n \rightarrow \mathbb{R}$

$$\mathbb{E}_n^x (FG) - (\mathbb{E}_n^x F)(\mathbb{E}_n^x G) \geq 0.$$ \hspace{1cm} (65)

**Remark 1** An example obeying (64) is

$$\varphi(X, Y) = -1_{X \geq k} 1_{Y \geq l}, \hspace{0.5cm} k, l \in \mathbb{N}$$ \hspace{1cm} (66)

**Remark 2** For $\beta = 0$, Theorem 3 is a statement about the usual BGW Markov chain. A trivial example goes as follows: let $p_0 + p_2 = 1$ and $n = 1$. The set of possible nodes in Neveu notation is $\{0, 1, 2\}$. There are 5 states of respective probabilities $(p_0, p_2, p_0, p_2, p_0, p_2^2, p_0^2)$ and

$$\mathbb{E} X_1 X_2 = p_0 + 4p_2^3 \geq (\mathbb{E} X_1)(\mathbb{E} X_2) = (-p_0 + 2p_2^2)^2,$$

or also

$$\mathbb{E} X_1 1_{X_1 \geq 0} X_2 1_{X_2 \geq 0} = 4p_2^3 \geq (\mathbb{E} X_1 1_{X_1 \geq 0})(\mathbb{E} X_2 1_{X_2 \geq 0}) = 4p_2^4.$$

In the sub-critical or critical regimes with interaction, the resulting monotonicity in $n$ could be used to prove convergence as $n \nearrow \infty$ of all moments of positive increasing functions of the $X_i$’s, similarly to Theorem 1 in [5], assuming suitable bounds on these moments.

**Proof** Here we prove inequality (50), $\mathbb{E}_n^1 N_n \leq \mathbb{E}_n^2 N_n$. We have

$$H_n^2(\omega) = H_n^1(\omega) - 1_{X_0 \geq 2}(\omega)$$ \hspace{1cm} (67)

Consider for $\lambda \in [0, 1]$

$$H_n^{2,\lambda}(\omega) = H_n^1(\omega) - \lambda 1_{X_0 \geq 2}(\omega)$$ \hspace{1cm} (68)

so that

$$H_n^{2,0}(\omega) = H_n^1(\omega), \hspace{0.5cm} H_n^{2,1}(\omega) = H_n^2(\omega)$$ \hspace{1cm} (69)

Let us rewrite (9) as

$$N_n = \sum_{|i|=n} X_{i_1...i_n} 1_{X_i > 0}$$ \hspace{1cm} (70)

Then by the FKG inequality

$$\frac{d}{d\lambda} \mathbb{E}_n^{2,\lambda} N_n = \mathbb{E}_n^{2,\lambda} N_n 1_{X_0 \geq 2} - 2 \mathbb{E}_n^{2,\lambda} N_n \mathbb{E}_n^{2,\lambda} 1_{X_0 \geq 2} \geq 0$$ \hspace{1cm} (71)

and

$$\mathbb{E}_n^2 N_n - \mathbb{E}_n^1 N_n = \int_0^1 d\lambda \frac{d}{d\lambda} \mathbb{E}_n^{2,\lambda} N_n \geq 0$$ \hspace{1cm} (72)

□

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**Declarations**

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References

1. Battle, G.A., Rosen, L.: The FKG inequality for the Yukawa 2 quantum field theory. J. Stat. Phys. 22, 123–192 (1980)
2. Berger, Q., Caravenna, F., Dai Pra, P.: Introduction aux Probabilités, Dunod, (2021)
3. Birkhoff, G.: Lattice theory. Am. Math. Soc. (1967)
4. Comtet, L.: Analyse Combinatoire, Tome 1. Presses Universitaires de France, Paris (1970)
5. Dunlop, F., Mardin, A.: Galton-Watson trees with first ancestor interaction. J. Stat. Phys. 189, 38 (2022)
6. Friedli, S., Velenik, Y.: Statistical Mechanics of Lattice Systems: a Concrete Mathematical Introduction. Cambridge University Press, Cambridge (2017)
7. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York (1983)
8. Kuznetsov, Y.: Elements of Applied Bifurcation Theory. Springer, New York (1998)
9. Neveu, J.: Arbres et processus de Galton–Watson. Ann. Inst. Henri Poincaré Probab. Stat. 22, 199–207 (1986)
10. Stanley, R.P.: Enumerative Combinatorics, vol. 2. CUP, Cambridge (1999)
11. Steele, J.M.: Gibbs’measures on combinatorial objects and the central limit theorem for an exponential family. Probab. Eng. Inform. Sci. 1, 47–59 (1987)
12. Takens, F.: Partially hyperbolic fixed points. Topology 10, 133–147 (1971)

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