PULLBACKS OF $\kappa$ CLASSES ON $\overline{M}_{0,n}$

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Abstract. The moduli space $\overline{M}_{0,n}$ carries a codimension-$d$ cycle class $\kappa_d$. We consider the subspace $K^d_n$ of $A^d(\overline{M}_{0,n},\mathbb{Q})$ spanned by pullbacks of $\kappa_d$ via forgetful maps. We find a permutation basis for $K^d_n$, and describe its annihilator under the intersection pairing in terms of $d$-dimensional boundary strata. As an application, we give a new permutation basis of the divisor class group of $\overline{M}_{0,n}$.

1. Introduction

Mumford [D.N.] introduced the tautological, codimension-$d$ class $\kappa_d$ in the cohomology/Chow group of the moduli space $\mathcal{M}_{g,n}$. This class extends to the moduli space $\overline{M}_{g,n}$ of stable curves as well as to various partial compactifications of $\mathcal{M}_{g,n}$. Ring-theoretic relations involving $\kappa$ classes have been studied by Faber, Ionel, Pandharipande, Pixton, Zagier, Zvonkine, and several others, and play a role in the study of Gromov-Witten theory and mirror symmetry (Fab99, Ion05, Pan12, PP, PPZ16; see Pan11, Pan18 for overviews).

Here, we investigate $\kappa$ classes on $\overline{M}_{0,n}$ from a linear-algebraic and representation-theoretic perspective. The symmetric group $S_n$ acts on $\overline{M}_{0,n}$, and thus acts on its cohomology and Chow groups. Given any $T \subseteq \{1, \ldots, n\}$ with $|T| \geq 3$, there is a forgetful map $\pi_T : \overline{M}_{0,n} \to \overline{M}_{0,T}$. We set $\kappa_d^T := \pi_T^*(\kappa_d)$, and consider the subspace $K^d_n \subseteq A^d(\overline{M}_{0,n},\mathbb{Q})$ spanned by $\{\kappa_d^T\}_{T \subseteq \{1, \ldots, n\}}$; this subspace is clearly $S_n$-invariant. Recall that a permutation basis of a $G$-representation is one whose elements are permuted by the action of $G$. We show:

Theorem A. (Theorem 3.11[11]) If $n \geq 4$ and $1 \leq d \leq n - 3$, then $K^d_n$ has a permutation basis given by $\{\kappa_d^T \mid |T| \geq (d + 3), |T| \equiv (d + 3) \mod 2\}$.

1.1. Does $A^d(\overline{M}_{0,n},\mathbb{Q})$ have a permutation basis? Getzler [Get95] and Bergström-Minabe [DM13] have given algorithms to compute the character of $A^d(\overline{M}_{0,n},\mathbb{Q})$ as an $S_n$-representation. It is not clear from these algorithms whether $A^d(\overline{M}_{0,n},\mathbb{Q})$ has a permutation basis. Farkas and Gibney [FG03] have given a permutation basis for $A^1(\overline{M}_{0,n},\mathbb{Q})$. Theorem 3.15 implies that $K^1_n = A^1(\overline{M}_{0,n},\mathbb{Q})$, so:

Theorem B. The set $\{\kappa_1^T \mid |T| \geq 4, |T| \text{ even}\}$ is a permutation basis of $A^1(\overline{M}_{0,n},\mathbb{Q})$.

The basis given by Theorem B is different from the one given in [FG03], which consists of certain boundary divisors and $\psi$ classes. For odd $n$, the two bases are isomorphic as $S_n$-sets, but for even $n$ they are not.

Silversmith and the author [RS20] have produced a permutation basis for $A_2(\overline{M}_{0,n},\mathbb{Q})$, using Theorem 3.15 as an ingredient. Very recent work of Castravet and Tevelev [CT20] on the derived category of $\overline{M}_{0,n}$ implies that $A^*(\overline{M}_{0,n},\mathbb{Q}) = \bigoplus_{d=0}^{n-3} A^d(\overline{M}_{0,n},\mathbb{Q})$ has a permutation basis; its elements, however, are not of pure degree. The question of whether or not $A^d(\overline{M}_{0,n},\mathbb{Q})$ has a permutation basis for all $d$ and $n$ remains open.

1.2. The dual story in $A_d(\overline{M}_{0,n},\mathbb{Q})$ and the proof of Theorem 3.15 There is an $S_n$-equivariant intersection pairing $A_d(\overline{M}_{0,n},\mathbb{Q}) \times A^d(\overline{M}_{0,n},\mathbb{Q}) \to \mathbb{Q}$. To prove Theorem A, we show:

Theorem C. (Theorem 3.11[11]) If $n \geq 4$ and $1 \leq d \leq n - 3$, we have

1. The annihilator of $K^d_n \subseteq A^d(\overline{M}_{0,n},\mathbb{Q})$ is the subspace $V_{d,n} \subseteq A_d(\overline{M}_{0,n},\mathbb{Q})$ spanned by boundary strata whose dual trees have two or more vertices with valence at least four.

2. $Q_{d,n} := A_d(\overline{M}_{0,n},\mathbb{Q})/V_{d,n}$ is the dual of $K^d_n$. 

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It is straightforward to show that $V_{d,n}$ is contained in the annihilator of $K^d_n$, but to show equality involves a complicated induction on $n$. We use the fact if $\pi$ denotes the forgetful morphism from $\mathcal{M}_{0,n+1}$ to $\mathcal{M}_{0,n}$, then:

**Theorem D.** (Theorem 3.14(v)) If $n \geq 4$ and $1 \leq d \leq n - 3$, then we have the following (dual) exact sequences:

\[
0 \to Q_{d,n} \xrightarrow{\pi^*} Q_{d+1,n+1} \xrightarrow{\pi^*} Q_{d+1,n} \to 0
\]

\[
0 \to K^d_{n+1} \xrightarrow{\pi_+^*} K^{d+1}_{n+1} \xrightarrow{\pi_+^*} K^d_n \to 0
\]

It is difficult to use induction to study $A^d(\mathcal{M}_{0,n}, \mathbb{Q})$, partly due to the fact that $A^d(\mathcal{M}_{0,n}, \mathbb{Q}) \xrightarrow{\pi^*} A^d(\mathcal{M}_{0,n+1}, \mathbb{Q}) \xrightarrow{\pi^*} A^{d+1}(\mathcal{M}_{0,n}, \mathbb{Q})$ is not exact. This failure of exactness is also responsible for the fact that the dimensions of $A^d(\mathcal{M}_{0,n}, \mathbb{Q}) = A^{n-3-d}(\mathcal{M}_{0,n}, \mathbb{Q})$ grow exponentially with $n$, whereas $\dim(K^{n-3-d}_n)$ grows as a degree-$d$ polynomial in $n$.

### 1.3. Significance for dynamics on $M_{0,n}$

**Hurwitz correspondences** are a class of multivalued dynamical systems on $M_{0,n}$. They were introduced by Koch [Koc13] in the context of Teichmüller theory and complex dynamics on $\mathbb{P}^1$, and their dynamics were studied by the author [Ram18, Ram19b, Ram19a]. A Hurwitz correspondence $\mathcal{H}$ on $M_{0,n}$ induces a linear pushforward action on $Q_{d,n}$, and the $d$-th dynamical degree of $\mathcal{H}$ (a numerical invariant of algebraic dynamical systems) is the largest eigenvalue of this action [Ram18]. Theorem 1.15 can be used to re-interpret Theorem 10.6 of [Ram18] to conclude that $\mathcal{H}$ acts on pullbacks of $\kappa$ classes, and that this action encodes important information about the dynamics of $\mathcal{H}$.

**Theorem E.** Suppose $\mathcal{H}$ is a Hurwitz correspondence on $M_{0,n}$. If $1 \leq d \leq n - 3$, then $K^d_n$ is invariant under the pullback $\mathcal{H}^* : A^d(\mathcal{M}_{0,n}, \mathbb{Q}) \to A^d(\mathcal{M}_{0,n}, \mathbb{Q})$, and the $d$-th dynamical degree of $\mathcal{H}$ is the largest eigenvalue of the action of $\mathcal{H}^*$ on $K^d_n$.

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### Notation and conventions

For $n$ a positive integer, we denote by $[n]$ the set $\{1, \ldots, n\}$. For $A$ a finite set, we denote by $\mathbb{Q}A$ the free $\mathbb{Q}$-vector space on $A$. For $V$ a vector space, we denote by $V^\vee$ its dual. For a linear map $\mu : V \to X$, we denote by $\mu^\vee$ its dual map. If $W$ is a subspace of $V$, we denote by $W^\perp$ its annihilator in $V^\vee$.

### 2. Cycle classes on $\mathcal{M}_{0,n}$

The Chow group $A_d(\mathcal{M}_{0,n})$ is a finitely generated free abelian group generated, though not freely, by the fundamental classes of $d$ dimensional boundary strata [Kee92, KM94]. We set $A_{d,n} := A_d(\mathcal{M}_{0,n}, \mathbb{Q})$, and $A^d_n := A_{n-3-d,n}$. There is an $S_n$-equivariant non-degenerate intersection pairing $A_{d,n} \times A^d_n \to \mathbb{Q}$; this identifies $A^d_n$ with $A^d_{n'}$. A stable $n$-marked tree is a tree $\sigma$ with $n$ marked legs such that every vertex has valence at least 3 (counting the legs). Boundary strata on $\mathcal{M}_{0,n}$ are in bijection with stable $n$-marked trees; if $\sigma$ is a stable $n$-marked tree we denote by $X_\sigma$ the corresponding boundary stratum on $\mathcal{M}_{0,n}$. Boundary strata are isomorphic to products of smaller moduli spaces: $X_\sigma \cong \prod X_{\sigma, \text{Valence}(\nu)}$, where the product is over vertices $\nu$ of $\sigma$. We conclude that $X_\sigma$ is positive-dimensional if and only if its dual tree has at least one vertex with valence at least four. If $\sigma$ has exactly one vertex $\nu$ with valence at least four, then $X_\sigma$ is isomorphic to $\mathcal{M}_{0,n'}$, where $n' = \text{Valence}(\nu)$, since the factors in the above product decomposition of $X_\sigma$ corresponding to vertices other than $\nu$ are all isomorphic to single-point spaces.

**Definition 2.1.** We say that a positive-dimensional boundary stratum $X_\sigma$ is **Type I** if its dual tree $\sigma$ has exactly one vertex with valence at least four; in this case we also say that $\sigma$ is a Type I stable tree. We say that a positive-dimensional boundary stratum $X_\sigma$ is **Type II** if its dual tree $\sigma$ has two or more vertices with...
valence at least four; in this case we also say that \( \sigma \) is a type II stable tree. For \( n \geq 4 \) and \( d = 1, \ldots, n - 4 \), we set \( \mathcal{V}_{d,n} \subset A_{d,n} \) to be the subspace generated by the fundamental classes of Type II boundary strata. We set \( \mathcal{Q}_{d,n} \) to be the quotient \( A_{d,n}/\mathcal{V}_{d,n} \). Note that since \( \mathcal{V}_{d,n} \) is \( S_n \)-invariant, \( \mathcal{Q}_{d,n} \) inherits an action of \( S_n \). Also note that \( \mathcal{Q}_{d,n} \) is naturally isomorphic, as an \( \mathbb{R} \)-representation, to the pullback, to \( \mathbb{R} \)-equivariant, \( S_n \)-equivariant, linear map \( \mathcal{Q}\mathcal{S}\mathcal{P}_{d,n} \rightarrow \mathcal{Q}_{d,n} \).

**Definition 2.2.** Suppose \( \sigma \) is stable \( n \)-marked tree and \( v \) a vertex on \( \sigma \). We obtain from the pair \((\sigma,v)\) a set partition \( \Pi(\sigma,v) \) of \([n]\) as follows: \( i \) and \( j \) are in the same part of \( \Pi(\sigma,v) \) if and only if the \( i \)- and \( j \)-marked legs on \( \sigma \) are on the same connected component of \( \sigma \setminus \{v\} \). Note that the number of parts of \( \Pi(\sigma,v) \) equals the valence of \( v \). If \( \sigma \) is Type I and \( v \) is its unique vertex with valence at least four, then the partition \( \Pi(\sigma,v) \) is intrinsically associated to \( \sigma \), so we denote it by \( \Pi_*(\sigma) \). In this case we have \( \dim(X_\sigma) = |\Pi_*(\sigma)| - 3 (= \text{Valence}(v) - 3) \).

**Lemma 2.3.** Suppose that \( \sigma_1 \) and \( \sigma_2 \) are two Type I stable \( n \)-marked trees, and suppose \( \Pi_*(\sigma_1) = \Pi_*(\sigma_2) \). Then \( [X_{\sigma_1}] = [X_{\sigma_2}] \in A_{d,n} \), where \( d = \dim(X_{\sigma_1}) = \dim(X_{\sigma_2}) \).

**Proof.** This follows immediately from the fact that \( \sigma_1 \) and \( \sigma_2 \) differ only in the arrangement of trivalent subtrees. See Lemma 5.2.1 of [Ram17] for a detailed proof.

For \( n \geq 1 \) and \( d \geq -3 \), we set \( \mathcal{S}\mathcal{P}_{d,n} \) to be the set of all set partitions of \([n]\) having exactly \( d + 3 \) parts. By Lemma 2.23 if \( n \geq 4 \) and \( d \geq 1 \), then there is a well-defined map \( \mathcal{S}\mathcal{P}_{d,n} \rightarrow A_{d,n} \) sending \( \Pi \) to \([X_\sigma]\), where \( \sigma \) is any Type I stable \( n \)-marked tree such that \( \Pi = \Pi_*(\sigma) \) (it is clear that such a \( \sigma \) exists). Extending by linearity and composing with the quotient map from \( A_{d,n} \) to \( \mathcal{Q}_{d,n} \), we obtain a surjective, \( S_n \)-equivariant, linear map \( \mathcal{Q}\mathcal{S}\mathcal{P}_{d,n} \rightarrow \mathcal{Q}_{d,n} \).

**Definition 2.4.** For \( n \geq 4 \) and \( 1 \leq d \leq n - 4 \), let \( \mathcal{R}_{d,n} \) be the subspace of \( \mathcal{Q}\mathcal{S}\mathcal{P}_{d,n} \) generated by elements of the form:

\[
\{P_1 \cup P_2, P_3, P_4, \ldots, P_{d+4}\} + \{P_1, P_2, P_3 \cup P_4, \ldots, P_{d+4}\} - \{P_1 \cup P_3, P_2, P_4, \ldots, P_{d+4}\} - \{P_1, P_3, P_2 \cup P_4, \ldots, P_{d+4}\},
\]

where \( \{P_1, P_2, P_3, P_4, \ldots, P_{d+4}\} \) is a set partition of \([n]\) with \( d + 4 \) parts. Note that \( \mathcal{R}_{d,n} \) is \( S_n \)-invariant.

**Lemma 2.5.** The kernel of the surjective linear map \( \mathcal{Q}\mathcal{S}\mathcal{P}_{d,n} \rightarrow \mathcal{Q}_{d,n} \) is \( \mathcal{R}_{d,n} \).

**Proof.** This follows immediately from the description of linear relations among boundary strata given in Theorem 7.3 of [KM94], or see Lemma 5.2.4 of [Ram17] for a detailed proof.

We conclude that \( \mathcal{Q}_{d,n} \) is naturally isomorphic, as an \( S_n \) representation, to \( \mathcal{Q}\mathcal{S}\mathcal{P}_{d,n}/\mathcal{R}_{d,n} \). We use this identification throughout the paper.

2.1. **Kappa classes and the intersection pairing.** For \( d = 0, \ldots, n - 3 \), \( \mathcal{M}_{0,n} \) carries a codimension \( d \) kappa class \( \kappa_d \in A^d_{n} \), see [AC96] [AC98] for the definition and properties.

**Lemma 2.6.** Let \( n \geq 4 \) and \( 1 \leq d \leq n - 3 \). Suppose \( X_\sigma \) is a dimension \( d \) boundary stratum on \( \mathcal{M}_{0,n} \). Then

\[
[X_\sigma] \cdot \kappa_d = \begin{cases} 
0 & \text{if } X_\sigma \text{ is Type II} \\
1 & \text{if } X_\sigma \text{ is Type I}
\end{cases}
\]

**Proof.** This follows by a standard computation from Equation 1.8 of [AC96] together with Lemma 1.1 (12) of [CY11].

**Definition 2.7.** For \( T \subseteq [n] \), set \( \kappa^T_\sigma \) to be the pullback, to \( \mathcal{M}_{0,n} \), of the codimension \( d \) kappa class on \( \mathcal{M}_{0,T} \), via the natural forgetful map \( \pi_T : \mathcal{M}_{0,n} \rightarrow \mathcal{M}_{0,T} \).

**Definition 2.8.** Set \( \kappa^d_\sigma \) to be the subspace of \( A^d_{n} \) spanned by the classes \( \{\kappa_T \mid T \subseteq [n]\} \). Note that \( \kappa^d_\sigma \) is \( S_n \)-invariant.

**Lemma 2.9.** Let \( n \geq 4 \) and \( 1 \leq d \leq n - 3 \). Suppose \( X_\sigma \) is a dimension \( d \) boundary stratum on \( \mathcal{M}_{0,n} \), and \( T \subseteq [n] \). Then

\[
[X_\sigma] \cdot \kappa^T_\sigma = \begin{cases} 
0 & \text{if } X_\sigma \text{ is Type II} \\
1 & \text{if } X_\sigma \text{ is Type I and } \forall P \in \Pi_*(\sigma), P \cap T \neq \emptyset \\
0 & \text{if } X_\sigma \text{ is Type I and } \exists P \in \Pi_*(\sigma) \text{ s.t. } P \cap T = \emptyset
\end{cases}
\]
Proof. By the projection formula, $[X_\sigma] \cdot \kappa^T_{d,n} = (\pi_T)_*([X_\sigma]) \cdot \kappa_d$. By [Ram18], if $X_\sigma$ is Type II, then $\pi_*([X_\sigma])$ is either zero, or the fundamental class of a Type II boundary stratum of $\overline{M}_{0,T}$. If $X_\sigma$ is Type I, then if $\exists P \in \Pi_*(\sigma)$ s.t. $P \cap T = \emptyset$, then $\pi_*([X_\sigma]) = 0$, while if $\forall P \in \Pi_*(\sigma)$, $P \cap T \neq \emptyset$, then $\pi_*([X_\sigma])$ is the fundamental class of a Type I boundary stratum of $\overline{M}_{0,T}$. Applying Lemma 2.10 we obtain the desired result.

□

**Corollary 2.10.** (1) The subspace $\mathcal{K}^d_{d,n} \subseteq \mathcal{A}^d_{d,n}$ is orthogonal, with respect to the intersection pairing, to $\mathcal{V}^1_{d,n} \subseteq \mathcal{A}^d_{d,n}$, i.e. we have $\mathcal{K}^d_{d,n} \subseteq \mathcal{V}^1_{d,n}$.

(2) The intersection pairing on $\overline{M}_{0,n}$ descends to a pairing $Q_{d,n} \times \mathcal{K}^d_{d,n} \to \mathcal{Q}$.

We will eventually show that $\mathcal{K}^d_{d,n} = \mathcal{V}^1_{d,n}$, which implies that $\mathcal{K}^d_{d,n} = \mathcal{Q}_{d,n}$.

**Definition 2.11.** Define a pairing $\langle \cdot, \cdot \rangle : \{\text{set partitions of } [n]\} \times \{\text{subsets of } [n]\} \to \mathbb{Z}$. For a set partition $\Pi$ and subset $T$, set

$$\langle \Pi, T \rangle = \begin{cases} 1 & \forall P \in \Pi, \ P \cap T \neq \emptyset \\ 0 & \exists P \in \Pi \text{ s.t. } P \cap T = \emptyset \end{cases}$$

From Lemma 2.10 we obtain a purely combinatorial description of the pairing between $Q_{d,n}$ and $\mathcal{K}^d_{d,n}$ obtained in Corollary 2.10.

**Lemma 2.12.** For any Type I $d$-dimensional boundary stratum $X_\sigma$, and for any subset $T \subseteq [n]$, we have that $[X_\sigma] \cdot \kappa^T_{d,n} = \langle \Pi_*(\sigma), T \rangle$.

2.2. **Pushing forward and pulling back via forgetful maps.** The forgetful morphism $\pi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ induces pushforward maps $\pi_* : \mathcal{A}_{d,n+1} \to \mathcal{A}_{d,n}$ and pullback maps $\pi^* : \mathcal{A}_{d,n} \to \mathcal{A}_{d+1,n+1}$. By [Ram18], $\pi_*([\mathcal{V}_{d,n+1}]) \subseteq \mathcal{V}_{d,n}$. Also, if $X_\sigma$ is a Type II boundary stratum, then $\pi_*([X_\sigma])$ is a sum of fundamental classes of Type II boundary strata of $\overline{M}_{1,n+1}$. This implies that $\pi^*([\mathcal{V}_{d,n}]) \subseteq \mathcal{V}_{d+1,n+1}$. Thus there are induced pushforward maps $\pi_* : Q_{d,n+1} \to Q_{d,n}$ and pullback maps $\pi^* : Q_{d,n} \to Q_{d+1,n+1}$.

**Lemma 2.13.** (1) The pushforward $\pi_* : Q_{d,n+1} \to Q_{d,n}$ lifts to $\tilde{\pi}_* : \mathcal{QSP}_{d,n+1} \to \mathcal{QSP}_{d,n}$ where

$$\tilde{\pi}_*([P_1, \ldots, P_{d+3}]) = \begin{cases} 0 & \exists i \text{ s.t. } P_i = \{n+1\} \\ \{P_1 \setminus \{n+1\}, \ldots, P_{d+3} \setminus \{n+1\}\} & \text{otherwise} \end{cases}$$

(2) The pullback $\pi^* : Q_{d,n} \to Q_{d+1,n+1}$ lifts to $\tilde{\pi}^* : \mathcal{QSP}_{d,n} \to \mathcal{QSP}_{d+1,n+1}$ where $\tilde{\pi}^*(\Pi) = \Pi \cup \{\{n+1\}\}$, for $\Pi \in \mathcal{SP}_{d,n}$.

**Sketch of proof.** This lemma follows from the observation that if $X_\sigma$ is a Type I $d$-dimensional boundary stratum whose dual tree $\sigma$ has exactly one vertex with valence at least four, then

$$\pi^*([X_\sigma]) = [X_\sigma] + \text{(sum of classes of Type II boundary strata)},$$

where $\sigma'$ is a Type I stable $(n+1)$-marked tree with the property that $\Pi_*(\sigma') = \Pi_*(\sigma) \cup \{\{n+1\}\}$. □

The pushforward maps $\pi_* : \mathcal{A}_{d,n+1} \to \mathcal{A}_{d,n}$ and $\pi_* : Q_{d,n+1} \to Q_{d,n}$ are easily seen to be surjective. Since $\pi$ has positive relative dimension (equal to one), $\pi_* \circ \pi^* = 0$ on $\mathcal{A}_{*,*}$, thus also on $Q_{*,*}$.

**Lemma 2.14.** For $n \geq 4$ and $k \geq 1$, the complex $Q_{d,n} \overset{\pi_*}{\to} Q_{d+1,n+1} \overset{\pi_*}{\to} Q_{d+1,n}$ is exact.

**Proof.** We use the lifts of $\pi^*$ and $\pi_*$ to $\tilde{\pi}^* : QSP_{d,n} \to QSP_{d+1,n+1}$ and $\tilde{\pi}_* : QSP_{d+1,n+1} \to QSP_{d+1,n}$ respectively. We have:

$$\text{Ker}(\tilde{\pi}_*) = \text{Im}(\tilde{\pi}^*) + \text{Span}(\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots \} - \{P_1, P_2 \cup \{n+1\}, P_3, P_4, \ldots \})$$

and:

$$\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots \} - \{P_1, P_2 \cup \{n+1\}, P_3, P_4, \ldots \}$$

$$= (\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots \} - \{P_1, \{n+1\}, P_2 \cup P_3, P_4, \ldots \})$$

$$- \{P_1, P_3, P_2 \cup \{n+1\}, P_4, \ldots \} - \{P_1 \cup P_3, P_2, \{n+1\}, P_4, \ldots \}$$

$$- \{P_1, \{n+1\}, P_2 \cup P_3, P_4, \ldots \} - \{P_1 \cup P_3, P_2, \{n+1\}, P_4, \ldots \} \in \mathcal{R}_{k+1,n+1} + \text{Im}(\tilde{\pi}^*)$$
This implies that \( \ker(\hat{\pi}_*) = \im(\hat{\pi}^*) + \mathcal{R}_{k+1,n+1} \), which in turn implies that \( \ker(\pi_*) = \im(\pi^*) \). \( \square \)

The pullback \( \pi^* : A^d_n \rightarrow A^d_{n+1}, \) which by the projection formula is dual to \( \pi_* : A_{d,n} \rightarrow A_{d,n+1} \), restricts to \( \pi^* : K^d_n \rightarrow K^d_{n+1} \), and sends \( \kappa^T_d \) on \( \mathcal{M}_{0,n} \) to \( \kappa^T_d \) on \( \mathcal{M}_{0,n+1} \). The pushforward \( \pi_* : A^{d+1}_{n+1} \rightarrow A^d_n \) is dual to \( \pi^* : A_{d+1,n+1} \rightarrow A_{d,n} \).

**Lemma 2.15.** The pushforward \( \pi_* : A^{d+1}_{n+1} \rightarrow A^d_n \) restricts to \( \pi_* : K^{d+1}_{n+1} \rightarrow K^d_n \), with

\[
\pi_*(\kappa^T_d) = \begin{cases} 0 & n + 1 \in T \\ \kappa^T_d & n + 1 \notin T \end{cases}
\]

**Proof.** By Lemma 2.10, we have that \( \forall d, n, K^d_n \subseteq V^d_{d,n} \). Since \( \pi^*(V_{d,n}) \subseteq V_{d+1,n+1} \), and since, by the projection formula, \( \pi_* \) and \( \pi^* \) are dual maps, we have that \( \pi_*(K^{d+1}_{n+1}) \subseteq V^d_{d,n} \). This means that for \( T \subseteq [n+1] \), the class \( \pi_*(\kappa^T_d) \) is determined by the functional that it defines on \( Q_{d,n} \), i.e., by the values of \( [X_\sigma] \cdot \pi_*(\kappa^T_d) \), where \( X_\sigma \) ranges over all Type I \( d \)-dimensional boundary strata on \( \mathcal{M}_{0,n} \). Given such an \( X_\sigma \), we have, by the projection formula, by the expression for \( \pi^*([X_\sigma]) \) given in Equation 3 and by applying Lemma 2.12 twice, that

\[
[X_\sigma] \cdot \pi_*(\kappa^T_d) = \pi^*([X_\sigma]) \cdot \kappa^T_d = (\Pi_*(\sigma) \cup \{n + 1\}, T)
\]

\[
= \begin{cases} 1 & \text{if } \forall P \in \Pi_*(\sigma) \cup \{n + 1\}, P \cap T \neq \emptyset \\ 0 & \text{if } \exists P \in \Pi_*(\sigma) \cup \{n + 1\} \text{ s.t. } P \cap T = \emptyset \end{cases}
\]

\[
= \begin{cases} 1 & \text{if } \forall P \in \Pi_*(\sigma), P \cap T \neq \emptyset, \text{ and } n + 1 \in T \\ 0 & \text{if } \exists P \in \Pi_*(\sigma) \text{ s.t. } P \cap T = \emptyset, \text{ or if } n + 1 \notin T \end{cases}
\]

\[
= \begin{cases} \Pi_*(\sigma), T & \text{if } n + 1 \in T \\ 0 & \text{if } n + 1 \notin T \end{cases}
\]

\[
[X_\sigma] \cdot \kappa^T_d = \begin{cases} 1 & \text{if } n + 1 \in T \\ 0 & \text{if } n + 1 \notin T \end{cases}
\]

\( \square \)

3. Main results and proofs

3.1. The set-up. Throughout Section 3, we use the identification \( Q_{d,n} = \mathbb{Q}\text{SP}_{d,n}/\mathcal{R}_{d,n} \) introduced in Section 2, we write an element of \( Q_{d,n} \) as a \( \mathbb{Q} \)-linear combination of set partitions of \( [n] \) with \( d + 3 \) parts, rather than as a \( \mathbb{Q} \)-linear combination of fundamental classes of Type I boundary strata.

**Definition 3.1.** For \( n \geq 1 \) and \( d \geq -3 \), we set \( K^d_n \) to be the set \( \{T \subseteq [n] \mid |T| \geq (d + 3), |T| \equiv (d + 3) \mod 2\} \).

**Definition 3.2.** There is a natural linear map \( \psi_{d,n} : \mathbb{Q}K^d_n \rightarrow K^d_n \) sending \( T \) to \( \kappa^T_d \).

The map \( \psi_{d,n} \), together with the intersection pairing \( Q_{d,n} \times K^d_n \rightarrow \mathbb{Q} \) induces the pairing \( Q_{d,n} \times \mathbb{Q}K^d_n \rightarrow \mathbb{Q} \), where, for \( \Pi \in \text{SP}_{d,n} \) and \( T \in K^d_n \), \( \Pi \cdot T = \langle \Pi, T \rangle \) as in Definition 2.11. Note that if \( S \) is an \( S_n \)-set, then \( QS \) is canonically and \( S_n \)-equivariantly isomorphic to its dual. Thus \( \mathbb{Q}K^d_n = (\mathbb{Q}K^d_n)\vee \).

**Definition 3.3.** For \( d \geq -1 \), the pairing \( \langle .., \rangle \) (Definition 2.11) between set partitions and subsets of \( [n] \) induces a map \( \hat{\phi}_{d,n} : \mathbb{Q}\text{SP}_{d,n} \rightarrow (\mathbb{Q}K^d_n)\vee = \mathbb{Q}K^d_n \). For \( d \geq 1 \), \( \hat{\phi}_{d,n} \) descends to a map \( \phi_{d,n} : Q_{d,n} \rightarrow \mathbb{Q}K^d_n \).

The map \( \hat{\phi}_{d,n} \) on generators \( \Pi \in \text{SP}_{d,n} \) is given explicitly by:

\[
\hat{\phi}_{d,n}(\Pi) = \sum_{T \in K^d_n} \langle \Pi, T \rangle \cdot T.
\]

**Definition 3.4.** Define maps \( \alpha : \mathbb{Q}K^d_n \rightarrow \mathbb{Q}K^{d+1}_{n+1} \) and \( \beta : \mathbb{Q}K^{d+1}_{n+1} \rightarrow \mathbb{Q}K^d_n \), where:

\[
\alpha(T) = T \cup \{n + 1\} \quad \beta(T) = \begin{cases} T & n + 1 \notin T \\ 0 & n + 1 \in T \end{cases}
\]
Lemma 3.5. The following is an exact sequence:

\[ 0 \to \mathbb{Q}K_n^d \xrightarrow{\alpha} \mathbb{Q}K_{n+1}^{d+1} \xrightarrow{\beta} \mathbb{Q}K_n^{d+1} \to 0 \]

Sketch of proof. Observe that \( \text{Im}(\alpha) = \text{Ker}(\beta) = \mathbb{Q}\{T \in K_{n+1}^{d+1} | n + 1 \in T\}. \)

Lemma 3.6. The following diagram commutes:

\[
\begin{array}{c}
\mathbb{Q}\text{SP}_{d,n} \xrightarrow{\pi^*} \mathbb{Q}\text{SP}_{d+1,n+1} \xrightarrow{\pi} \mathbb{Q}\text{SP}_{d+1,n} \\
\downarrow \phi_{d,n} \quad \quad \quad \downarrow \phi_{d+1,n+1} \quad \downarrow \phi_{d+1,n} \\
\mathbb{Q}K_n^d \xrightarrow{\alpha} \mathbb{Q}K_{n+1}^{d+1} \xrightarrow{\beta} \mathbb{Q}K_n^{d+1}
\end{array}
\]

Proof. Commutativity of the left square: Given \( \Pi \in \text{SP}_{d,n} \), we have

\[
\phi_{d+1,n+1}(\pi^*(\Pi)) = \sum_{T \in K_{n+1}^{d+1}} (\Pi \cup \{n + 1\}, T) \cdot T = \sum_{T \in K_{n+1}^{d+1}} (\Pi \cup \{n + 1\}, T) \cdot T \\
= \sum_{T' \in K_{n}^{d}} (\Pi, T') \cdot T' \cup \{n + 1\} \\
= \alpha\left( \sum_{T' \in K_{n}^{d}} (\Pi, T') \cdot T' \right) = \alpha(\phi_{d,n}(\Pi))
\]

Commutativity of the right square: Given \( \Pi' = \{P_1, \ldots, P_{d+4}\} \in \text{SP}_{d+1,n+1} \), we may assume without loss of generality that \( n + 1 \in P_1 \). There are two cases:

Case 1: \( P_1 = \{n + 1\} \). Then \( \pi_*(\Pi') = 0 \) so \( \phi_{d+1,n}(\pi_*(\Pi')) = 0 \). Note that for \( T \subset [n + 1] \), we have that if \( n + 1 \notin T \), then \( (\Pi', T) = 0 \), while if \( n + 1 \in T \), then \( \beta(T) = 0 \). This implies that

\[
\beta(\phi_{d+1,n+1}(\Pi')) = \beta\left( \sum_{T' \in K_{n+1}^{d+1}} (\Pi', T) \cdot T \right) = \beta\left( \sum_{T' \in K_{n+1}^{d+1}} (\Pi', T) \cdot T \right) = 0
\]

Case 2: \( P_1 \neq \{n + 1\} \). Then

\[
\phi_{d+1,n}(\pi_*(\Pi')) = \sum_{T' \in K_{n+1}^{d+1}} (\{P_1 \setminus \{n + 1\}, P_2, \ldots, P_{d+4}\}, T') \cdot T' \\
= \sum_{T' \in K_{n+1}^{d+1}} \underbrace{T'}_{T' \cap P_1 \setminus \{n + 1\} \neq \emptyset \atop T' \cap P_2, \ldots, T' \cap P_{d+1} \neq \emptyset} = \beta(\phi_{d+1,n+1}(\Pi'))
\]

We use the following lemma several times; its proof follows from a standard diagram chase.

Lemma 3.7. [Variant of the Four Lemma] Suppose we have a commutative diagram of vector spaces as follows

\[
\begin{array}{ccc}
\mathcal{W}_1 \xrightarrow{f_1} \mathcal{W}_2 \xrightarrow{f_2} \mathcal{W}_3 & \xrightarrow{h_3} 0 \\
\downarrow h_1 \quad \quad \quad \downarrow h_2 \quad \downarrow h_3 \\
\mathcal{X}_1 \xrightarrow{g_1} \mathcal{X}_2 \xrightarrow{g_2} \mathcal{X}_3 & \xrightarrow{h_3} 0
\end{array}
\]

Suppose further that the bottom row is exact at \( \mathcal{X}_2 \), that the top row is exact at \( \mathcal{W}_3 \), and that \( h_1 \) and \( h_3 \) are surjective. Then \( h_2 \) is surjective.
3.2. A preliminary lemma. In this section, we prove some technical results — Lemmas 3.9, 3.10 and 3.11. These are not of independent interest, but are necessary to prove Theorem 3.14 in Section 3.3. The proofs (and statements) of these three lemmas are conceptually similar to each other, as well as to those of Theorem 3.14. All four proofs use the Four Lemma to induct on \( n \). Lemma 3.9 is required in the inductive step of Lemma 3.10, which is required in the inductive step of Lemma 3.11, which in turn is required in the inductive step of Theorem 3.14. The proofs of Lemmas 3.9 and 3.10 also involve some intricate combinatorics of set partitions and subsets.

Definition 3.8. For \( n > 0 \), we set:

\[
E_n := \{ T \subseteq \{n\} | |T| \text{ even} \}; \quad O_n := \{ T \subseteq \{n\} | |T| \text{ odd} \}; \quad F_n := \{ (P_1, P_2) | P_1 \cup P_2 = [n], P_1 \cap P_2 = \emptyset, 1 \in P_1 \}
\]

Note that \( F_n \setminus \{ ([n], \emptyset) \} \) is in canonical bijection with \( SP_{-1,n} \), so \( QF_n \) is canonically isomorphic to \( Q\{([n], \emptyset)\} \oplus QSP_{-1,n} \). There are maps \( \alpha : QE_n \to QO_{n+1} \), \( \alpha : QO_n \to QE_{n+1} \), \( \beta : QE_{n+1} \to QE_n \) and \( \beta : QO_{n+1} \to QO_n \), analogous to the maps \( \alpha \) and \( \beta \) as in Definition 3.3. Define maps \( odd_n : QF_n \to QO_n \) and \( even_n : QF_n \to QE_n \), where

\[
odd_n((P_1, P_2)) = \sum_{T \subseteq P_1} (-T) + \sum_{T \subseteq P_2} (-T); \quad even_n((P_1, P_2)) = \sum_{T \subseteq P_1} (-T) + \sum_{T \subseteq P_2} (-T)
\]

Lemma 3.9. For \( n \geq 1 \), the maps \( odd_n \) and \( even_n \) are surjective.

Proof. We induct on \( n \). Base case: \( n = 1 \). We have:

\[
F_1 = \{ ([1], \emptyset) \}; \quad E_1 = \{ \emptyset \}; \quad O_1 = \{ \{1\} \}; \quad odd_1(([1], \emptyset)) = -1; \quad even_1(([1], \emptyset)) = -2\emptyset.
\]

This establishes the base case.

Inductive hypothesis: The proposition holds up to some \( n \geq 1 \).

Inductive step: Define maps \( \tilde{\pi}_n : QF_{n+1} \to QF_n \) and \( \gamma : QF_n \to QF_{n+1} \), where:

\[
\tilde{\pi}_n((P_1, P_2)) = (P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\})
\]

\[
\gamma((P_1, P_2)) = (P_1 \cup \{n+1\}, P_2 - (P_1 \cup \{n+1\}))
\]

The diagram below has exact rows; we claim it commutes.

\[
\begin{array}{cccc}
QF_n & \xrightarrow{\gamma} & QF_{n+1} & \xrightarrow{\tilde{\pi}_n} & QF_n & \longrightarrow & 0 \\
\text{even}_n & \downarrow & \text{odd}_{n+1} & \downarrow & \text{odd}_n & & \\
0 & \longrightarrow & QE_n & \xrightarrow{\alpha} & QO_{n+1} & \xrightarrow{\beta} & QO_n & \longrightarrow & 0.
\end{array}
\]

Commutativity of the left square: For \( (P_1', P_2') \in QF_n \),

\[
odd_{n+1}(\gamma(P_1', P_2')) = odd_{n+1}(P_1' \cup \{n+1\}, P_2') - odd_{n+1}(P_1', P_2' \cup \{n+1\})
\]

\[
= \sum_{T \subseteq P_1' \cup \{n+1\}} (-T) + \sum_{T \subseteq P_2'} (-T) - \left( \sum_{T \subseteq P_1'} (-T) + \sum_{T \subseteq P_2' \cup \{n+1\}} (-T) \right)
\]

\[
= \sum_{T \subseteq P_1' \cup \{n+1\}} (-T) + \sum_{n+1 \in T} (-T)
\]

\[
= \sum_{T' \subseteq P_1'} (-T' \cup \{n+1\}) + \sum_{T' \subseteq P_2'} (-T' \cup \{n+1\})
\]

\[
= \alpha(even_n(P_1', P_2')).
\]
Commutativity of the right square: For \((P_1, P_2) \in \mathbb{Q}F_{n+1}\),
\[
\text{odd}_n(\pi'_*(P_1, P_2)) = \text{odd}_n(P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\})
\]
\[
= \sum_{T \subseteq P_1 \setminus \{n+1\}} (-T) + \sum_{T \subseteq P_2 \setminus \{n+1\}} (T)
\]
\[
= \beta(\text{odd}_{n+1}(P_1, P_2)).
\]
This proves the claim. By the inductive hypothesis, \(\text{even}_n\) and \(\text{odd}_n\) are surjective, so by the Four Lemma, \(\text{odd}_{n+1}\) is surjective.

The diagram below has exact rows; we claim it commutes.

\[
\begin{array}{ccc}
\mathbb{Q}F_n & \xrightarrow{\gamma} & \mathbb{Q}F_{n+1} \\
\text{odd}_n & \downarrow & \text{even}_{n+1} \\
0 & \xrightarrow{\alpha} & \mathbb{Q}E_{n+1} \\
\end{array}
\]
\[
\begin{array}{ccc}
\mathbb{Q}F_n & \xrightarrow{\pi'_*} & \mathbb{Q}F_n \\
\text{even}_n & \downarrow & \beta \\
0 & \rightarrow & \mathbb{Q}E_n \\
\end{array}
\]

Commutativity of the left square: For \((P'_1, P'_2) \in \mathbb{Q}F_n\),
\[
\text{even}_{n+1}(\gamma(P'_1, P'_2)) = \text{even}_{n+1}(P'_1 \cup \{n+1\}, P'_2) - \text{even}_{n+1}(P'_1, P'_2 \cup \{n+1\})
\]
\[
= \sum_{T \subseteq P'_1 \cup \{n+1\}} (-T) + \sum_{T \subseteq P'_2} (T) - \left( \sum_{T \subseteq P'_1 \cup \{n+1\}} (-T) + \sum_{T \subseteq P'_2} (T) \right)
\]
\[
= \sum_{T \subseteq P'_1 \cup \{n+1\}} (-T') + \sum_{T' \subseteq P'_2 \cup \{n+1\}} (T')
\]
\[
= \alpha(\text{odd}_n(P'_1, P'_2)).
\]

Commutativity of the right square: For \((P_1, P_2) \in \mathbb{Q}F_{n+1}\),
\[
\text{even}_n(\pi'_*(P_1, P_2)) = \text{even}_n(P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\})
\]
\[
= \sum_{T \subseteq P_1 \setminus \{n+1\}} (-T) + \sum_{T \subseteq P_2 \setminus \{n+1\}} (T)
\]
\[
= \beta(\text{even}_{n+1}(P_1, P_2)).
\]
This proves the claim. Again, by the inductive hypothesis, \(\text{odd}_n\) and \(\text{even}_n\) are surjective, so by Lemma 3.7, \(\text{even}_{n+1}\) is surjective.

We only use the fact that \(\text{odd}_n\) is surjective to proceed; we use it to prove Lemma 3.10.

**Lemma 3.10.** For all \(n \geq 2\), the map \(\tilde{\phi}_{-1,n} : \mathbb{Q}SP_{-1,n} \to \mathbb{Q}K_{n-1}\) is surjective.

**Proof.** We induct on \(n\).

**Base case:** \(n = 2\). We have \(\mathbb{S}P_{-1,2} = \{\{1\}, \{2\}\}\) and \(\mathbb{K}^{-1}_2 = \{\{1\}, \{2\}\}\). We have \(\tilde{\phi}_{2,2}(\{\{1\}, \{2\}\}) = \langle\{1\}, \{2\}\rangle \cdot \{1, 2\} = \{1, 2\}\), which shows that \(\tilde{\phi}_{2,2}\) is surjective.

**Inductive hypothesis:** The lemma holds up to some \(n \geq 2\).

**Inductive step:** Define a map \(\gamma : \mathbb{Q}F_n \to \mathbb{Q}SP_{-1,n+1}\), where \(\gamma((P'_1, P'_2)) = \{P'_1 \cup \{n+1\}, P'_2\} - \{P'_1, P'_2 \cup \{n+1\}\}\). Consider the diagram
The proposition holds up to some $n$. For all $n$, Lemma 3.11 proves the claim. Since odd, is surjective, Note that the right square commutes by Lemma 3.6, we claim the left square commutes as well.

**Commutativity of the left square:** For $(P'_1, P'_2) \in \mathbb{Q}F_n$,

$$\tilde{\phi}_{-1,n+1}(\gamma(P'_1, P'_2)) = \tilde{\phi}_{-1,n+1}(\{P'_1 \cup \{n+1\}, P'_2\}) - \tilde{\phi}_{-1,n+1}(\{P'_1\}, P'_2 \cup \{n+1\})$$

$$= \sum_{T \subseteq [n+1]} (T) - \sum_{T \subseteq [n+1]} (T)$$

$$= \sum_{T \subseteq [n]} (T') - \sum_{T \subseteq [n]} (T')$$

This proves the claim. Since odd, is surjective, and by the inductive hypothesis so is $\tilde{\phi}_{-1,n+1}$. By Lemma 3.7, $\tilde{\phi}_{-1,n+1}$ is surjective.

**Lemma 3.11.** For all $n \geq 3$, the map $\tilde{\phi}_{0,n} : \mathbb{Q}SP_{0,n} \rightarrow \mathbb{Q}K^0_n$ is surjective.

**Proof.** We induct on $n$.

**Base case:** We have $SP_{0,3} = \{\{1\}, \{2\}, \{3\}\}$, $K^0_3 = \{1, 2, 3\}$, and $\tilde{\phi}_{0,3}(\{1\}, \{2\}, \{3\}) = \{1, 2, 3\}$, so $\tilde{\phi}_{0,3}$ is surjective.

**Inductive hypothesis:** The proposition holds up to some $n \geq 4$.

**Inductive step:** Consider the following diagram, which commutes by Lemma 3.6.

$$\begin{array}{ccc}
\mathbb{Q}SP_{0,n} & \xrightarrow{\tilde{\phi}_{-1,n+1}} & \mathbb{Q}SP_{0,n+1} \\
\downarrow \tilde{\phi}_{-1,n+1} & & \downarrow \tilde{\phi}_{-1,n+1} \\
0 & \xrightarrow{\tilde{\phi}_{0,n+1}} & \mathbb{Q}K^0_{n+1} \\
\downarrow \tilde{\phi}_{0,n+1} & & \downarrow \tilde{\phi}_{0,n+1} \\
0 & \xrightarrow{\tilde{\phi}_{0,n+1}} & \mathbb{Q}K^0_{n+1} \\
\end{array}$$

By the inductive hypothesis, $\tilde{\phi}_{0,n}$ is surjective. By Lemma 3.10, $\tilde{\phi}_{-1,n}$ is surjective, so by the Four Lemma, $\tilde{\phi}_{0,n+1}$ is surjective, as desired.

3.3. An inductive proof of Theorem 3.14.

**Lemma 3.12.** $\dim \mathbb{Q}A_n = \dim \mathbb{Q}K^1_n$ for all $n \geq 4$.

**Proof.** There are no Type II 1-dimensional boundary strata, so $\forall n \geq 4$, $\mathbb{V}_{1,n} = \{0\}$ and $\mathbb{Q}A_n \cong A_{1,n}$. By [FG03], $\dim A_{1,n} = 2^{n-1} - {n \choose 2} - 1$. On the other hand,

$$\dim \mathbb{Q}K^1_n = \#\{T \subseteq [n] | |T| \text{ even}, |T| \geq 4\} = 2^{n-1} - {n \choose 2} - 1.$$
Lemma 3.13. For all \( n \geq 4 \), \( \phi_{n-3,n} \) is an isomorphism.

**Proof.** For all \( n \geq 4 \), we have that \( V_{n-3,n} = \{0\} \), \( A_{n-3,n} = Q_{n-3,n} = \mathbb{Q}\{\{1\}, \ldots, \{n\}\} \), \( K_n^{-3} = \{n\} \), and \( \phi_{n-3,n} : Q_{n-3,n} \rightarrow QK_n^{-3} \) sends \( \{1\}, \ldots, \{n\} \) to \( [n] \). Thus \( \phi_{n-3,n} \) is an isomorphism. \( \square \)

**Theorem 3.14.** For \( n \geq 4 \) and \( d \) such that \( 1 \leq d \leq n-3 \), \( \phi_{d,n} : Q_{d,n} \rightarrow QK_n^d \) is an isomorphism.

**Inductive proof of Theorem 3.14.**

**Base case:** \( n = 4 \); then \( 1 \leq d \leq n-3 \) implies that \( d = 1 = n - 3 \). By Lemma 3.13, \( \phi_{1,4} \) is an isomorphism.

**Inductive hypothesis:** For some \( n \geq 4 \), and for all \( d \) with \( 1 \leq d \leq n-3 \), we have that \( \phi_{d,n} : Q_{d,n} \rightarrow QK_n^d \) is an isomorphism.

**Inductive step:** For \( 1 \leq d \leq (n-4) \), we have the following diagram, which commutes by Lemma 3.13.

\[
\begin{array}{c}
\text{Ker}(\pi^*) & \rightarrow & Q_d,n & \xrightarrow{\pi^*} & Q_{d+1,n+1} & \xrightarrow{\pi^*} & Q_{d+1,n} & \rightarrow & 0 \\
\downarrow & & \downarrow \phi_{d,n} & & \downarrow \phi_{d+1,n+1} & & \downarrow \phi_{d+1,n} & & \\
0 & \rightarrow & QK_n^d & \xrightarrow{\alpha} & QK_{n+1}^{d+1} & \xrightarrow{\beta} & QK_{n+1}^d & \rightarrow & 0
\end{array}
\]

The top row is exact by Lemma 2.11, and the bottom row is exact by Lemma 3.6. By the inductive hypothesis, \( \phi_{d,n} \) and \( \phi_{d+1,n} \) are isomorphisms. So, by the Five Lemma, \( \phi_{d+1,n+1} \) is an isomorphism. Combining the above argument with Lemma 3.13, we conclude that for \( 2 \leq d \leq (n+1) - 3 \), \( \phi_{d,n+1} \) is an isomorphism. We also have the following diagram, which commutes by Lemma 3.6.

\[
\begin{array}{c}
\QSP_{0,n} & \xrightarrow{\pi^*} & Q_{1,n+1} & \xrightarrow{\pi^*} & Q_{1,n} & \rightarrow & 0 \\
\downarrow \delta_{0,n} & & \downarrow \phi_{1,n+1} & & \downarrow \phi_{1,n} & & \\
0 & \rightarrow & QK_0^n & \xrightarrow{\alpha} & QK_{n+1}^d & \xrightarrow{\beta} & QK_n^d & \rightarrow & 0
\end{array}
\]

where the bottom row is exact and the top row is a complex, exact at \( Q_{1,n} \). By Lemma 3.11, \( \hat{\phi}_{0,n} \) is surjective, and by the inductive hypothesis, \( \phi_{1,n} \) is an isomorphism. By Lemma 3.7, \( \phi_{1,n+1} \) is surjective. But by Lemma 3.12, \( \dim Q_{1,n+1} = \dim QK_{n+1}^d \), so \( \phi_{1,n+1} \) is an isomorphism.

\( \square \)

3.4. **Theorem 3.15** and its proof.

**Theorem 3.15.** For \( n \geq 4 \) and \( d \) such that \( 1 \leq d \leq n-3 \):

(i) We have \( K_n^d = V_{d,n} \).

(ii) The pairing \( Q_{d,n} \times K_n^d \rightarrow \mathbb{Q} \) is perfect.

(iii) The set \( \{\kappa_d^T \mid |T| \geq (d+3), |T| \equiv (d+3) \mod 2\} \) is an \( S_n \)-equivariant basis for \( K_n^d \).

(iv) The \( S_n \) actions on \( Q_{d,n} \) and \( K_n^d \) are isomorphic to the permutation representation induced by the natural action of \( S_n \) on the set \( \{T \subseteq [n] \mid |T| \geq (d+3), |T| \equiv (d+3) \mod 2\} \).

(v) The following (dual) sequences are exact:

\[
\begin{align*}
0 \rightarrow & Q_{d,n} \xrightarrow{\pi^*} Q_{d+1,n+1} \xrightarrow{\pi^*} Q_{d+1,n} \rightarrow 0 \\
0 \rightarrow & K_n^{d+1} \xrightarrow{\pi^*} K_n^{d+1} \xrightarrow{\pi^*} K_n^d \rightarrow 0
\end{align*}
\]

**Proof.** Recall the map \( \psi_{d,n} : QK_n^d \rightarrow K_n^d \) given in Definition 3.2. We have compatible pairings \( Q_{d,n} \times K_n^d \rightarrow \mathbb{Q} \) and \( Q_{d,n} \times QK_n^d \rightarrow \mathbb{Q} \), inducing maps \( \eta_{d,n} : Q_{d,n} \rightarrow (K_n^d)\psi \) and \( \phi_{d,n} : Q_{d,n} \rightarrow (QK_n^d)\psi = QK_n^d \), where, \( \phi_{d,n} \) is as in Definition 3.3. These maps satisfy: \( \phi_{d,n} = (\psi_{d,n})\psi \circ \eta_{d,n} \). By Theorem 3.14, \( \phi_{d,n} \) is an isomorphism, which implies that \( \eta_{d,n} \) is injective. On the other hand, we have by Corollary 2.10 that \( K_n^d \subset V_{d,n} = (Q_{d,n})\psi \), so \( (\eta_{d,n})\psi \) is injective as well. This implies that \( \eta_{d,n} \) is an isomorphism, proving item (iii) and (i). Since \( \eta_{d,n} \) and \( \phi_{d,n} \) are both isomorphisms, we conclude that so is \( \psi_{d,n} \), proving item (iii) and thus also item (iv).

Finally, by Theorem 3.14 and Lemma 3.6, the sequence in Equation 3 is dual to the sequence in Equation 4 which is exact by Lemma 3.5. We conclude that the sequence in Equation 5 is exact. The sequence in Equation 5 is dual to the sequence in Equation 6, so the latter sequence is exact, proving item (v). \( \square \)
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