Duality for compact group actions on operator algebras and applications: irreducible inclusions and Galois correspondence

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ABSTRACT. We consider compact group actions on C*- and W*- algebras. We prove results that relate the duality property of the action (as defined in the Introduction) with other relevant properties of the system such as the relative commutant of the fixed point algebras being trivial (called the irreducibility of the inclusion) and also to the Galois correspondence between invariant C*-subalgebras containing the fixed point algebra and the class of closed normal subgroups of the compact group.

1 Introduction and auxiliary results

A C*-dynamical system (respectively a W*-dynamical system) is a triple $(A, G, \alpha)$ where $A$ is a C*-algebra (respectively a W*-algebra), $G$ a locally compact group and $\alpha$ an action of $G$ on $A$, that is, for each $g \in G$, $\alpha_g$ is an automorphism of $A$ such that the mapping $g \rightarrow \varphi(\alpha_g(a))$ is continuous for every $a \in A$ and every $\varphi$ in the dual $A^*$ of the C*-algebra $A$ (respectively $\varphi$ in the predual $A_*$ of the W*-algebra $A$). In the rest of this paper, we will assume that $G$ is a compact group.

In [15], Takesaki proved the following

Theorem A ([15], Theorem 1). Let $(A, G, \alpha)$ be a W*-dynamical system with $G$ compact. Suppose that the commutant $S$ of $\{\alpha_g : g \in G\}$ in the group $Aut(A)$ of all automorphisms of $A$ acts ergodically on $A$. Then, if $\beta \in Aut(A)$
leaves $A^\alpha$ pointwise invariant and commutes with $S$, it is of the form $\beta = \alpha_g$ for some $g \in G$.

He also proved a result about a Galois correspondence between $G$ and $S$ invariant subalgebras of $A$ containing $A^\alpha$ and the set of normal subgroups of $G$ [15, Theorem 3], due to Kishimoto [8, Theorem 7]. We will discuss this result and one of its $C^*$-dynamical systems analogs in Section 3.

Notice that if $(A, G, \alpha)$ is a $W^*$-dynamical system with $G$ compact such that $(A^\alpha)' \cap A = CI$, so if the inclusion $A^\alpha \subset A$ is an irreducible inclusion of factors, then the automorphism group $S = \{adu : u \in A^\alpha \text{ unitary}\}$ acts ergodically on $A$ and commutes with $\alpha$. Therefore, a consequence of Theorem A is the following:

**Corollary B** [15, Corollary 2]. If the relative commutant $(A^\alpha)' \cap A$ of $A^\alpha$ in $A$ reduces to scalars, then every automorphism $\beta$ of $A$ that leaves $A^\alpha$ pointwise invariant is of the form $\beta = \alpha_g$ for some $g \in G$.

The conclusion of Theorem A and Corollary B will be referred in the next sections as the duality property of the action $\alpha$.

In Section 2, we will prove a converse of Corollary B for $W^*$-dynamical systems and study the corresponding problem for $C^*$-dynamical systems.

For $C^*$-dynamical systems there are several notions of ergodicity. We mention some of them that will be used in this paper.

Let $S$ be a group of automorphisms of a $C^*$ algebra $A$

a) $S$ is called minimal, [9], if the only non zero -invariant hereditary subalgebra of $A^\alpha$ is $A$.

b) $S$ is said to be topologically transitive, [9], if for every $B_1, B_2$ non zero $S$-invariant hereditary subalgebras of $A$, we have $B_1B_2 \neq \{0\}$. In [2] it is noticed that $S$ is topologically transitive if and only if, for every $x, y$ non zero elements of $A$ there exists $s \in S$ such that $xs(y) \neq 0$.

Clearly a)$\Rightarrow$b) but not the other way around. The above two notions generalize the classical concepts for commutative $C^*$-dynamical systems.

In [2] the authors introduced the notion of strong topological transitivity:

c) $S$ is said to be strongly topologically transitive if for every finite set $\{x_i, y_i\}_{i=1}^n$ of non zero elements of $A$ such that $\sum x_i \otimes y_i \neq 0$, then there exists $s \in S$ such that $\sum x_is(y_i) \neq 0$. The authors proved a version of Theorem A for $C^*$-dynamical systems under the strong topological transitivity assumption.

Clearly, c)$\Rightarrow$b) in general, but it is not known whether b)$\Rightarrow$c) in general. Also we do not know whether a)$\Rightarrow$c) in general.

In order to set the framework for $C^*$-dynamical systems, notice that if $(A, G, \alpha)$ is a $W^*$-dynamical system with $G$ compact and $\alpha$ faithful such that $(A^\alpha)' \cap A = CI$, then every $\alpha_g, g \neq e$ is properly outer, i.e. it is not implemented by a unitary element $u \in A$ (otherwise, $u \in (A^\alpha)' \cap A = CI$, contradiction). For $C^*$-algebras, the proper outerness of automorphisms is defined [10], [3], using
the algebra of local multipliers, \(M_{\text{loc}}(A)\), (as denoted in [1]), or equivalently, the algebra of essential multipliers, \(M^\infty(A)\), first defined by Pedersen in [10]. This algebra is defined as the inductive limit of the multiplier algebras \(M(I)\) where \(I\) is an essential ideal of \(A\) (an ideal is said to be essential if its annihilator in \(A\) equals \(\{0\}\)). In this definition, it is assumed, as proven in [11, Proposition 3. 12. 8.], that if \(I_1 \subset I_2\) are essential ideals then there is a unit preserving inclusion \(M(I_2) \subset M(I_1)\).

In the next lemma, if \(B\) is a C*-algebra, \(B^{**}\) stands for the second dual of \(B\). For each automorphism \(\alpha\) of \(B\), we denote by \(\alpha^{**}\) the second dual of \(\alpha\).

**1.1. Lemma** Let \((B, G, \alpha)\) be a C*-dynamical system and \(M(B)\) the multiplier algebra of \(B\). Then,

i) For every \(x \in M(B)\) and \(\varphi \in B^*\) the mapping \(g \to \varphi(\alpha_g^{**}(x))\) is continuous.

ii) For every \(x \in M(B)\) there exists a unique element \(P^\alpha(x) \in M(B)^\alpha\) such that

\[
\varphi(P^\alpha(x)) = \int \varphi(\alpha_g^{**}(x))dg.
\]

for every \(\varphi \in B^*

iii) The mapping \(P^\alpha : M(B) \to M(B)^\alpha\) is a (norm one) conditional expectation continuous in the \(B^*\) topology of \(M(B)\).

iv) If \(\rho\) is a faithful representation of \(B\) then the normal extension \(\rho^*\) of \(\rho\) to \(B^{**}\) is faithful on \(M(B)\), the function \(g \to \varphi(\rho^*(\alpha_g^{**}(x)))\) is continuous for every \(x \in M(B)\) and \(\varphi \in (\rho^*(B^{**}))_*\), where \((\rho^*(B^{**}))_*\) denotes the predual of \(\rho^*(B^{**})\) and

\[
\varphi(\rho^*(P^\alpha(x))) = \int \varphi(\rho^*(\alpha_g^{**}(x)))dg
\]

for every \(x \in M(B)\) and \(\varphi \in (\rho^*(B^{**}))_*\).

**Proof.** 1) Let \(\tilde{B}\) be C*-algebra generated by \(B\) and the unit of \(B^{**}\) and \(\tilde{B}_{sa}\) the self adjoint part of \(\tilde{B}\). Then, by [Ped. 3.12.9.] we have

\[
M(B)_{sa} = (\tilde{B}_{sa})^m \cap (\tilde{B}_{sa})_m
\]

where \((\tilde{B}_{sa})^m\) (respectively \((\tilde{B}_{sa})_m\)) denotes the strong limits in \(B^{**}\) of all increasing nets in \(\tilde{B}_{sa}\) (respectively the limits of all decreasing nets in \(\tilde{B}_{sa}\). Let \(x \in M(B)_{sa}\). Then there exists an increasing net \(\{a_\nu = a_{\nu} + c_\nu I\}_\nu\) and a decreasing net \(\{b_{\mu} = b_{\mu} + d_{\mu} I\}_\mu\), both in \(\tilde{B}_{sa}\) that converge strongly to \(x\) in \(B^{**}\). Clearly, for every \(\nu\) and \(\mu\) and for every \(\varphi \in B^*\), the mappings \(g \to \varphi(\alpha_{a_\nu}^{**}(a_\nu)) = \varphi(\alpha_{a_\nu}(a_{\nu})) + c_\nu \varphi(I)\) and \(g \to \varphi(\alpha_{b_{\mu}}^{**}(b_{\mu})) = \varphi(\alpha_{b_{\mu}}(b_{\mu})) + d_{\mu} \varphi(I)\) are continuous and \(\lim_\nu \varphi(\alpha_{a_\nu}^{**}(a_\nu)) = \lim_\mu \varphi(\alpha_{b_{\mu}}^{**}(b_{\mu})) = \varphi(\alpha_{a_{\nu}}^{**}(x))\). Therefore, if \(\varphi \in B^*\) is a positive functional, the mapping \(g \to \varphi(\alpha_g^{**}(x))\) is both lower and upper semi continuous and therefore continuous. Since this is true for positive functionals, it follows that it holds for every \(\varphi \in B^*\).
ii) In [5, Theorem 1.1.] it is proven that both dual pairs of Banach spaces $(M(B), B^*)$ and $(B^*, M(B))$ possess the Krein property and therefore the corresponding (Pettis) integral can be performed. They also proved that the $B^*$-topology and the strict topology of $M(B)$ are consistent.

iii) This follows from i) and ii).

iv) The fact that $\rho'$ is faithful on $M(B)$ follows from [11, Corollary 3.12.5.]. To prove the continuity of the mapping and the subsequent equality, notice that if $\varphi \in (\rho''(B^{**}))^*$, then the composition $\varphi \circ \rho'' \in B^*$ and we can apply i).

Now, let $(B, G, \alpha)$ be a C*-dynamical system with $G$ compact and $B$ a prime C*-algebra (i.e. a C*-algebra, $B$ such that every ideal of $B$ is essential). In [3, the proof of implication 6$\Rightarrow$7 and Proposition 3.3.] it is proven that every ideal of $B$ contains an $\alpha$-invariant ideal and therefore, $M_{loc}(B)$ is the inductive limit of the multiplier algebras $M(I)$ where $I$ is a non zero $\alpha$-invariant ideal of $B$. In what follows we will assume that $B$ has a faithful factorial representation. It is known that this is the case if, in particular, $B$ is a separable prime C*-algebra.

We can assume that $B \subset B(H)$ for some Hilbert space, $H$, and $M = B''$ is a factor. If this is the case, then for every ideal $I \subset B$ we have $I'' = M$. Therefore the predual $M_*$ of $M$ is canonically embedded in the dual $I^*$ of $I$. Also, according to [10, Corollary 3.12.5], the multiplier algebra $M(I)$ of $I$ can be canonically embedded in $M$ for every ideal $I \subset B$ and, therefore, $M_{loc}(B)$ can be identified with the norm closure

\[
\bigcup \{M(I) : I \subset B \text{ is an } \alpha\text{-invariant ideal}\} \subset M.
\]

Since $B''$ is a factor, and therefore for every non zero ideal $I \subset B$, $I'' = B''$, if $(B'')^*$ denotes the predual of $B''$, it follows that $(B'')^* \subset I^*$ for every non zero ideal $I$ of $B$. Since by Lemma 1.1. iv) the (faithful) representation restricted to $I$ can be extended to a normal representation of $I''$ which is faithful on $M(I)$, for every non zero ideal $I \subset B$ and $x \in M(I) \subset B''$, we can identify $\alpha(x) \in I''$ with an element $\alpha(x) \in B''$. Using these facts and Lemma 1.1.

1.2. Lemma For every $x \in M_{loc}(B)$ there exists a unique $P^\alpha(x) \in M_{loc}(B)$ such that the mapping $g \mapsto \varphi(\alpha(x))$ is continuous for every $\varphi \in (B^*)^*$ and

\[
\varphi(P^\alpha(x)) = \int \varphi(\alpha(x))dg
\]

for all $\varphi \in (B^*)^*$ and $P^\alpha$ is a conditional expectation of $M_{loc}(B)$ onto the fixed point algebra $(M_{loc}(B))^\alpha$.

Proof. This follows from Lemma 1.1. iv) and the fact that if $I \subset B$ is an $\alpha$-invariant ideal, we have $M(I)^\alpha = M(I^\alpha)$. This latter equality follows from the fact that every approximate identity of $I^\alpha$ is also an approximate identity of $I$. ■
1.3. Remark  Let $B \subset B(H)$ be a $C^*$-algebra such that $B^*$ is a factor, so $M_{loc}(B) \subset B^*$. If $x$ is such that $BxB = \{0\}$, then $x = 0$. Indeed, if this is the case, we see immediately that $B^*xB^* = \{0\}$, so $x = 0$.

For simplicity, in the next section we will write $\alpha_g(x)$ instead of $\alpha^*_g(x)$ for $x \in M_{loc}(B)$. Also, the element $P^\alpha(x), x \in M_{loc}(B)$, from Lemma 1.2. will be denoted, simply

$$P^\alpha(x) = \int \alpha_g(x)dg$$

If $G$ is a compact group, the set of equivalence classes of unitary irreducible representations of $G$ is denoted by $\hat{G}$ and called dual object (or dual space) of the group $G$. For each $\pi \in \hat{G}$ we denote also by $\pi$ a representative of that class. If $\pi$ is an irreducible unitary representation of $G$, and $d_\pi$ the dimension of the corresponding Hilbert space, $H_\pi$, $\chi_\pi(g) = d_\pi^{-1} \sum_{i=1}^{d_\pi} \pi_i(g)$ denotes its character.

Now let $(\mathcal{A},G,\alpha)$ be a $\mathcal{W}^*$- or a $\mathcal{C}^*$-dynamical system.

1.4. Notations  i) For each $\pi$ denote

$$A^\pi_1(\pi) = \left\{ \int \chi_\pi(g)\alpha_g(a)dg : a \in \mathcal{A} \right\} \subset \mathcal{A}$$

the corresponding spectral subspace of $\mathcal{A}$.

ii) $A^\pi_2(\pi) = \{[a_{ij}] \in \mathcal{A} \otimes \mathcal{B}(H_\pi) : (\alpha_g \otimes \iota)([a_{ij}]) = [a_{ij}] (1 \otimes \pi_g), g \in G \}$.

In [12, JFA] it is noticed that each entry, $a_{ij}$, is the element of $A^\pi_1(\pi)$ such that $\alpha_g(a_{ij}) = \sum_1^{d_\pi} \pi_j(g)a_{i1}$.

iii) $\hat{\pi}$ denotes the conjugate representation of $\pi$. Clearly, $A^\pi_1(\hat{\pi}) = A^\pi_1(\pi)^*$. iv) The Arveson spectrum of the action is defined as $sp(\alpha) = \{ \pi \in \hat{G} : A^\pi_1(\pi) \neq \{0\} \}$.

v) The linear span of $\{mn : m,n \in A^\pi_2(\pi)\}$ is denoted by $A^\pi_2(\pi)A^\pi_2(\pi)$ and the linear span of $\{mn^* : m,n \in A^\pi_2(\pi)\}$ is denoted $A^\pi_2(\pi)A^\pi_2(\pi)^*$. It is straightforward to check that $A^\pi_2(\pi)A^\pi_2(\pi)^*$ is a two sided ideal of the fixed point algebra $(\mathcal{A} \otimes \mathcal{B}(H_\pi))^{\alpha_{ad}}$ and $A^\pi_2(\pi)^*A^\pi_2(\pi)$ an ideal of $(\mathcal{A} \otimes \mathcal{B}(H_\pi))^{\alpha_{ad}}$ for all $\pi \in \hat{G}$, the action $\alpha$ is called saturated.

2 Duality for compact group actions versus irreducible inclusions.

In this section we will state and prove a converse of Corollary B for $\mathcal{W}^*$- and $\mathcal{C}^*$-dynamical systems and also prove the Corollary B for some $\mathcal{C}^*$-dynamical systems. The following example shows that a verbatim converse of Corollary B is false.
2.1. Example Let \( K \) be a finite dimensional Hilbert space of dimension larger than 1 and \( G \) the compact group of all unitary operators in \( A = B(K) \). If \( \alpha_g(x) = gxg^{-1}, g \in G, x \in A \), then \( A^\alpha = \mathbb{C}I \), so \((A^\alpha)^\prime \cap A = A \neq \mathbb{C}I \), but the action has the duality property.

In the above example, clearly, both \( A^\alpha \) and \( A \) are factors so even if we add the condition that \( A^\alpha \) and \( A \) are factors to the duality property the conclusion that \((A^\alpha)^\prime \cap A = \mathbb{C}I \) is false for non abelian compact groups. We mention that if \( G \) is a compact abelian group then the duality property plus the assumption that \( A^\alpha \) and \( A \) are factors imply that the inclusion \( A^\alpha \subset A \) is irreducible, i.e. \((A^\alpha)^\prime \cap A = \mathbb{C}I \). This fact was proven in [3, Theorem]. Thus we will be concerned with compact non abelian groups \( G \). We start with the following.

2.2. Lemma Let \((A, G, \alpha)\) be a C*-dynamical system (respectively a W*-dynamical system) with \( G \) compact. If \((A \otimes B(H_\pi))^\alpha \otimes \text{ad}\pi\) are prime C*-algebras (respectively W* factors) for all \( \pi \in \text{sp}(\alpha) \), then \( \text{sp}(\alpha) \) is closed under tensor products. If in addition, \( \alpha \) is faithful (that is \( \alpha_g \neq \iota \) if \( g \neq e \)), then \( \text{sp}(\alpha) = \hat{G} \).

Proof. Recall that, by definition, \( \text{sp}(\alpha) \) is said to be closed under tensor products if every irreducible component of the tensor product \( \pi_1 \otimes \pi_2, \pi_1, \pi_2 \in \text{sp}(\alpha) \) is in \( \text{sp}(\alpha) \). Since \((A \otimes B(H_\pi))^\alpha \otimes \text{ad}\pi\) are prime (respectively factors), it follows that \( A^\pi_1(\pi_1)^* A^\pi_2(\pi_2), \pi \in \text{sp}(\alpha) \), are essential ideals of \((A \otimes B(H_\pi))^\alpha \otimes \text{ad}\pi\) (respectively weakly dense ideals in the case of W*-dynamical systems). Thus, \( \text{sp}(\alpha) \) coincides with the spectrum defined in [6, Definition 3.1. (1)] for compact quantum groups. Applying [6, Lemma 5.5] to the particular case of groups, it follows that \( \text{sp}(\alpha) \) is closed under tensor products. Since \( \text{sp}(\alpha) \) is also closed under conjugation by the definition of the spectrum (since \( A_1(\pi) = A_1(\pi)^* \)), we can apply [7, Theorem 28.9.] and derive that the annihilator \( \text{sp}(\alpha)_\perp \) of \( \text{sp}(\alpha) \) is a closed normal subgroup \( G_0 \) of \( G \) and the annihilator \( G_0^\alpha \) of \( G_0 \) in \( \hat{G} \) equals \( \text{sp}(\alpha) \). It follows that, if \( \alpha \) is faithful, \( G_0 = \{e\} \), so \( \text{sp}(\alpha) = \hat{G} \). \( \square \)

As we mentioned in Section 1, in this Section 2, if \((A, G, \alpha)\) is a C*-dynamical system with \( G \) compact, we will assume that \( A \) has a faithful factorial representation, or, equivalently, as we stated before, that \( A \subset B(H) \) for some Hilbert space \( H \) and \( A^\prime \) is a factor, so, in particular, \( A \) is a prime C*-algebra. This assumption will allow us to use the identification of \( M_{\text{loc}}(A) \) with a norm closed subalgebra of \( A^\prime \) and use Lemma 1.2. for \( B = A \). Notice that if \((A, G, \alpha)\) is a W*-dynamical system, we do not need the algebra \( M_{\text{loc}}(A) \) and thus, in Corollary 2.4. below we will assume that \( A \) is a factor. We will also assume that the action \( \alpha \) is faithful.

The next result is our converse of Corollary B for C*-dynamical systems. Our version of Corollary B itself will be given in Theorem 2.5.

2.3. Theorem Let \((A, G, \alpha)\) be a C*-dynamical system with \( G \) compact as above. If

i) \((A \otimes B(H_\pi))^\alpha \otimes \text{ad}\pi, \pi \in \text{sp}(\alpha) \) are prime C*-algebras and
ii) If \( \beta \) is an automorphism of \( M_{\text{loc}}(A) \) which leaves \( A^\alpha \) pointwise invariant then \( \beta = \alpha_g \) for some \( g \in G \), then \( sp(\alpha) = \hat{G} \) and \( (A^\alpha)' \cap M_{\text{loc}}(A) = CI \).

**Proof.** The equality \( sp(\alpha) = \hat{G} \) follows from the above Lemma 2.2. We will prove next the second part of the conclusion. Let \( u \in (A^\alpha)' \cap M_{\text{loc}}(A) \) be a unitary element. Then \( \beta(a) = adu(a) = uau^* \) is an automorphism of \( M_{\text{loc}}(A) \) that leaves \( A^\alpha \) pointwise invariant. By the hypothesis ii) there exists \( g_0 \in G \) such that \( \beta = \alpha_{g_0} \). Let \( \pi \in \hat{G} \) and \( m \in A_0^\alpha(\pi), m \neq 0 \). Then, by the definition of \( A_0^\alpha(\pi) \) we have \( (\alpha_g \odot \iota)(m) = m(I \otimes \pi_g) \) for every \( g \in G \), in particular, for \( g = g_0 \). Since \( \alpha_{g_0} = \beta = adu \) it follows that

\[
(u \otimes 1)m = m(u \otimes \pi_{g_0}).
\]

(1)

Clearly, \( u \otimes \pi_{g_0} \in M_{\text{loc}}(A) \otimes B(H_\pi) = M_{\text{loc}}(A \otimes B(H_\pi)) \). This latter equality follows from the fact that every ideal of \( A \otimes B(H_\pi) \) is of the form \( I \otimes B(H_\pi) \) where \( I \) is an ideal of \( A \). Let \( n \in A_0^\alpha(\pi) \). By multiplying (1) on the right by \( n^* \) we get

\[
(u \otimes 1)mn^* = m(u \otimes \pi_{g_0})n^*.
\]

(2)

By applying \( \alpha_g \odot \iota, g \in G \) to equality (2), since \( mn^* \in A^\alpha \otimes B(H) \) and then using the definition of \( A_0^\alpha(\pi) \) in the right hand side of (2), we get

\[
(\alpha_g(u \otimes 1)mn^* = m((\alpha_g \odot ad\pi_g)(u \otimes \pi_{g_0}))n^*.
\]

(3)

By integrating both sides of (3), we get

\[
(P^\alpha(u \otimes I)mn^* = mP^\alpha \odot ad\pi(u \otimes \pi_{g_0})n^*.
\]

(4)

Since \( u \in (A^\alpha)' \cap M_{\text{loc}}(A) \) and this latter algebra is \( \alpha \)-invariant we have that \( P^\alpha(u) \in (A^\alpha)' \cap M_{\text{loc}}(A)^\alpha \). On the other hand, according to [3, Proposition 3.3.], \( M_{\text{loc}}(A)^\alpha \subset M_{\text{loc}}(A^\alpha) \), so \( P^\alpha(u) \in (A^\alpha)' \cap M_{\text{loc}}(A^\alpha) \). Since by hypothesis \( A^\alpha \) is a prime \( \mathbb{C}^* \)-algebra, from [3, Proposition 3.1.] it follows that \( P^\alpha(u) = \lambda I \) for some scalar \( \lambda \). Therefore

\[
\lambda(I \otimes I)mn^* = mP^\alpha \odot ad\pi(u \otimes \pi_{g_0})n^*.
\]

(4')

Let \( c, d \in A_0^\alpha(\pi) \). By multiplying (4') on the right by \( c \) and on the left by \( d^* \) we have

\[
\lambda(I \otimes I)d^* mn^*c = d^* mP^\alpha \odot ad\pi(u \otimes \pi_{g_0})n^*c.
\]

(5)

We noticed in Section 1 that \( J_\pi = A_0^\alpha(\pi)A_0^\alpha(\pi) \) is an ideal of \( (A \otimes B(H_\pi))^\alpha \odot ad\pi \). Therefore, from (5) it follows that

\[
J_\pi(\lambda(I \otimes I) - P^\alpha \odot ad\pi(u \otimes \pi_{g_0}))(J_\pi = \{0\}.
\]

(6)

So, if we denote \( x = \lambda(I \otimes I) - P^\alpha \odot ad\pi(u \otimes \pi_{g_0}) \in (M_{\text{loc}}(A \otimes B(H_\pi))^\alpha \odot ad\pi \) we have

\[
J_\pi x J_\pi = \{0\}.
\]

(6')
and since by [3, Proposition 3.3.], \( M_{loc}(A \otimes B(H_\pi))^{\alpha \otimes ad}\pi \subset M_{loc}((A \otimes B(H_\pi))^{\alpha \otimes ad}\pi) \), and \((A \otimes B(H_\pi))^{\alpha \otimes ad}\pi \) is a prime C*-algebra by hypothesis i), it follows that \( J_\pi \) is an essential ideal of \((A \otimes B(H_\pi))^{\alpha \otimes ad}\pi \), so \( M_{loc}((A \otimes B(H_\pi))^{\alpha \otimes ad}\pi = M_{loc}(J_\pi) \). By taking \( B = J_\pi \) in Remark 1.3., it follows that \( x = 0 \). Therefore

\[
\lambda(I \otimes I) = \int (\alpha_g(u) \otimes \pi_{g_{0,0}})dg.
\]

(7)

Let us calculate the sum of diagonal elements in (7):

\[
\lambda d_\pi = \sum_{l,k,j} \pi_{lk}(g)\pi_{kj}(g_0)\pi_{jl}(g^{-1})\alpha_g(u)dg =
\]

\[
= \sum_{k,j} \pi_{kj}(g_0) \int \sum_l \pi_{jl}(g^{-1})\pi_{lk}(g)\alpha_g(u)dg =
\]

\[
\sum_{k,j} \pi_{kj}(g_0) \int \delta_{kj}\alpha_g(u)dg
\]

where \( \delta_{kj} \) is the Kronecker symbol. Therefore

\[
\lambda d_\pi = tr(\pi_{g_0})P^\alpha(u) = \lambda tr(\pi_{g_0})
\]

(8)

Hence, if \( \lambda \neq 0 \) then \( tr(\pi_{g_0}) = d_\pi \) for all \( \pi \in \hat{G} \). It follows that \( \pi_{g_0} = I \) for every \( \pi \in \hat{G} \). Since \( \hat{G} \) separates the points of \( G \) we get \( g_0 = e \), the neutral element of \( G \). Therefore, \( \beta = adu = u \), so \( u \in A' \cap M_{loc}(A) \). Since \( A \) is prime by hypothesis, from [3, Proposition 3.1.] it follows that \( u \) is a scalar multiple of the identity. Now, let \( a \in (A^\alpha)' \cap M_{loc}(A) \) be a positive element of norm 1. Then, there exists a unitary element, \( u \in (A^\alpha)' \cap M_{loc}(A) \) such that \( a = \frac{1}{2}(u + u^*) \). Since \( P^\alpha \) is faithful, it follows that \( P^\alpha(a) \neq 0 \) and therefore, \( P^\alpha(u) \neq 0 \). According to the previous arguments, \( u \) and therefore, \( a \) is a scalar multiple of the identity and the proof is completed. \( \blacksquare \)

The following is a consequence of the proof of the above Proposition.

2.4. Corollary Let \((A, G, \alpha)\) be a W*-dynamical system with \( G \) compact. If

i) \((A \otimes B(H_\pi))^{\alpha \otimes ad}\pi, \pi \in sp(\alpha)\) and \( A \) are factors and

ii) If \( \beta \) is an automorphism of \( A \) which leaves \( A^\alpha \) pointwise invariant then \( \beta = \alpha_g \) for some \( g \in G \),

then \( sp(\alpha) = \hat{G} \) and \((A^\alpha)' \cap A = \mathbb{C}I\).

Proof. If \((A \otimes B(H_\pi))^{\alpha \otimes ad}\pi, \pi \in sp(\alpha)\) are factors, then \( A_2^\alpha(\pi)^*A_2^\alpha(\pi), \pi \in sp(\alpha)\), are weakly dense ideals of \((A \otimes B(H_\pi))^{\alpha \otimes ad}\pi\) and the rest of the proof of Theorem 2.3. carries over to the case of W*-dynamical systems. \( \blacksquare \)

Next we will prove that hypothesis of the above corollary is equivalent with the irreducibility of the inclusion \( A^\alpha \subset A \) for W*-dynamical systems.
2.5. Theorem Let \((A, G, \alpha)\) be a \(W^*\)-dynamical system with \(G\) compact and \(\alpha\) faithful. The following statements, i) and ii), are equivalent

i) \((A \otimes B(H_\pi))^{\alpha \otimes ad\pi}, \pi \in sp(\alpha)\) and \(A\) are factors.

ii) If \(\beta\) is an automorphism of \(A\) which leaves \(A^\alpha\) pointwise invariant then \(\beta = \alpha_g\) for some \(g \in G\).

Under each of the conditions i) and ii) we have \(sp(\alpha) = \widehat{G}\).

Proof. The implication i)\(\Rightarrow\)ii) and the fact that i) implies the equality \(sp(\alpha) = \widehat{G}\) follows from the above Corollary. Suppose ii) holds. Then, according to [16, Proposition 6.2.], applied to the particular case of compact groups, the relative commutant of \(A\) in the crossed product \(A \times_\alpha G\) equals \(CI\). In particular, this means that \(A\) and \(A \times_\alpha G\) are factors. Applying [12 Cor. 3.12.] to the case of \(W^*\)-dynamical systems (as noticed at the end of the paper), it follows that \(sp(\alpha) = \widehat{G}\) and \((A \otimes B(H_\pi))^{\alpha \otimes ad\pi}, \pi \in \widehat{G}\) are factors, so i1) holds. The condition i2) follows from Corollary B of Takesaki, stated in Section 1.

The next result is concerned with \(C^*\)-dynamical systems. To prove the converse of Theorem 2.3., i.e. a version of Corollary B for \(C^*\)-dynamical systems, we need a condition that is stronger than the existence of a faithful factorial representation of \(A\). Namely, we will assume that there exists a faithful irreducible representation \(\rho\) of \(A\) such that \(\rho(A^\alpha)' \cap \rho(A)^\alpha = CI\). Clearly such a representation is factorial and, since \(M_{loc}(A) \subset \rho(A)^\alpha\), it implies that \((A^\alpha)' \cap M_{loc}(A) = CI\). Examples of such systems are given in [3] for compact abelian groups \(G\) and a stronger condition than this, namely the existence of a faithful irreducible representation \(\rho\) of \(A\) such that the restriction of \(\rho\) to \(A^\alpha\) is also irreducible, for non abelian compact groups in [4]. So, we will assume that \(A \subset B(H)\) for some Hilbert space \(H\) and \((A^\alpha)' \cap A^\alpha = CI\).

2.6. Proposition Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system with \(G\) compact and \(\alpha\) faithful. Suppose that \(A \subset B(H)\) for some Hilbert space \(H\) and \((A^\alpha)' \cap A^\alpha = CI\). Then,

i) \((A \otimes B(H_\pi))^{\alpha \otimes ad\pi}, \pi \in \widehat{G}\) are prime and

ii) If \(\beta\) is an automorphism of \(M_{loc}(A)\) which leaves \(A^\alpha\) pointwise invariant then \(\beta = \alpha_g\) for some \(g \in G\).

Proof. Applying [3, Corollary 5.4.], it follows that the hypothesis implies that the unitary group of \(A^\alpha\) (with unit adjoined if necessary) acts strongly topologically transitively on \(A\) (strong topological transitivity as defined in [2, Introduction]). Since obviously the action of the unitary group of \(A^\alpha\) commutes with \(\alpha\) we can apply [2, Corollary 2.2.] to obtain the equality \(sp(\alpha) = \widehat{G}\).

Since \((A^\alpha)' \cap A^\alpha = CI\) and obviously \((A^\alpha)^\alpha \subset A^\alpha\) it follows that \((A^\alpha)^\alpha\) is a factor. From the discussion in Section 1 it follows that \(M_{loc}(A^\alpha) \subset (A^\alpha)^\alpha\), so, in particular, \((A^\alpha)' \cap M_{loc}(A^\alpha) = CI\). From [3, Proposition 3.1.] it follows that \(A^\alpha\) is a prime \(C^*\)-algebra. Since \(\alpha\) commutes with a strongly topologically transitive...
action and \( A^\alpha \) is prime, we can apply [13, Proposition 5.4.] and we get that the crossed product \( A \times_\alpha G \) is also a prime C*-algebra. From [12, Corollary 3.12.] it follows that \( (A \otimes B(H_\pi))^{\alpha \otimes \text{ade}} \), \( \pi \in \hat{G} \) are prime so i1) is proven. i2) follows from the updated version of [2, Theorem 2.1.] given in [3, proof of the implication 12\( \Rightarrow \)13].

3 Duality for compact actions and Galois correspondence

If \((A, G, \alpha)\) is a C*-dynamical system with \( G \) compact, and \( B \) is an \( \alpha \)-invariant C*- subalgebra of \( A \) such that \( A^\alpha \subset B \subset A \), in [14] it is defined a closed normal subgroup of \( G \)

\[
G^B = \{ g \in G : \alpha_g(b) = b, b \in B \}
\]

and a subalgebra of \( A \)

\[
A^{G^B} = \{ a \in A : \alpha_g(a) = a, g \in G^B \}
\]

In [14, Lemma 13 and Corollary 9] we proved that if there exists a subgroup \( S \subset \text{Aut}(A) \) which acts minimally on \( A \), that commutes with \( \alpha \), then \( A^{G^0} = B \). We have also shown that this is false if \( S \) acts strongly topologically transitively on \( A \). In what follows we will complete that result as follows. Suppose as above that there exists a subgroup \( S \subset \text{Aut}(A) \) which acts minimally on \( A \), that commutes with \( \alpha \). If \( G_0 \subset G \) is a closed normal subgroup of \( G \), let

\[
A^{G_0} = \{ a \in A : \alpha_g(a) = a, g \in G_0 \} .
\]

and

\[
G^{A^{G_0}} = \{ g \in G : \alpha_g(a) = a, a \in A^{G_0} \} .
\]

In Proposition 3.2. below, we will prove that \( G^B = G_0 \) where \( B = A^{G_0} \).

This will follow from the following

3.1. Theorem Let \((A, G, \alpha)\) be a C*-dynamical system with \( G \) compact and \( \alpha \) faithful. Suppose that there exists a subgroup \( S \subset \text{Aut}(A) \) which acts minimally on \( A \), that commutes with \( \alpha \). Suppose as above that there exists a subgroup \( S \subset \text{Aut}(A) \) which acts minimally on \( A \), and commutes with \( \alpha \). Then, if \( \beta \) is an automorphism of \( A \) which commutes with \( S \) and leaves \( A^\alpha \) pointwise invariant, there exists \( g \in G \) such that \( \beta = \alpha_g \).

Proof. Applying [14, Lemma 13], it follows that, under the hypothesis of the theorem the action \( \alpha \) is saturated, i.e. \( A_3^\alpha(\pi)^* A_4^\alpha(\pi) \) is dense in \((A \otimes B(H_\pi))^{\alpha \otimes \text{ade}} \) for all \( \pi \in \hat{G} \). Since \( S \subset \text{Aut}(A) \) acts minimally on \( A \), it is in particular topologically transitive and therefore, by [9, Proposition 2.14.] the
only fixed points of the extension of $S$ to the multiplier algebra $M(A)$ are scalars multiples of the identity. In [9], [13] we have referred to this property as weak ergodicity of $S$. The conclusion of the theorem follows from [13, Theorem 3.3].

3.2. Proposition Let $(A, G, \alpha)$ be a $C^*$-dynamical system with $G$ compact and $\alpha$ faithful. Suppose that there exists a subgroup $S \subset \text{Aut}(A)$ which acts minimally on $A$, that commutes with $\alpha$. Then, if $G_0$ is a closed normal subgroup of $G$, $B = A^{G_0}$ is $S$- and $\alpha$-invariant, $A^\alpha \subset B$ and $G^B = G_0$.

Proof. Clearly $A^\alpha \subset B = A^{G_0} \subset A$ and, since $S$ commutes with $\alpha$, $B$ is $S$-invariant. Since $G_0$ is a normal subgroup, $B$ is also $\alpha$-invariant. From the definitions of $B$ and $G^B$ we get that $G_0 \subset G^B$. Now consider the $C^*$-dynamical system $(A, G_0, \alpha|_{G_0})$ and let $g \in G^B$. Then $\alpha_g$ leaves $A^{G_0} = B$ pointwise invariant and commutes with the minimal action $S$. Applying Theorem 3.1., it follows that there exists $g_0 \in G_0$ such that $\alpha_g = \alpha_{g_0}$. Since $\alpha$ is faithful, $g = g_0 \in G_0$, so $G^B = G_0$.

The next result completes and improves the results in [14].

3.3. Theorem Let $(A, G, \alpha)$ be a $C^*$-dynamical system with $G$ compact and $\alpha$ faithful. Suppose that there exists a subgroup $S \subset \text{Aut}(A)$ which acts minimally on $A$, that commutes with $\alpha$. Then the assignments $B \rightarrow G^B$ and $G_0 \rightarrow A^{G_0}$ are inverse to each other maps between the set of all $S$- and $\alpha$-invariant $C^*$ subalgebras of $A$ that contain $A^\alpha$ and the set of all closed normal subgroups of $G$.

Proof. From [14, Lemma 13 and Corollary 9] it follows that if $B$ is $S$- and $\alpha$-invariant and $A^\alpha \subset B$, then $A^{G_0} = B$. From Proposition 3.2. above it follows that if $G_0$ is a closed normal subgroup of $G$, then $G^{A^{G_0}} = G_0$ and the proof is completed.

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