Reconstruction of the null-test for the matter density perturbations

Savvas Nesseris\textsuperscript{1} \textsuperscript{*}, Domenico Sapone\textsuperscript{1,2} \textsuperscript{†}, and Juan García-Bellido\textsuperscript{2} \textsuperscript{‡}

\textsuperscript{1}Departamento de Física Teórica and Instituto de Física Teórica, Universidad Autónoma de Madrid IFT-UAM/CSIC, 28049 Cantoblanco, Madrid, Spain
\textsuperscript{2}Departamento de Física, Universidad de Chile, Blanco Encalada 2008, Santiago, Chile

We systematically study the null test for the growth rate data first presented in \cite{1} and we generalize it for modified gravity models. We reconstruct it using various combinations of data sets, such as the $f\sigma$ and $H(z)$ or Type Ia supernovae (SnIa) data. We perform the reconstruction in two different ways, either by directly binning the data or by fitting various dark energy models. We also examine how well the null test can be reconstructed by future data by creating mock catalogs based on the cosmological constant model, a model with strong dark energy perturbations, the $f(R)$ and $f(G)$ models, and the large void LTB model that exhibit different evolution of the matter perturbations. We find that with future data similar to an LSST-like survey, the null test will be able to successfully discriminate between these different cases at the 5$\sigma$ level.

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\textbf{I. INTRODUCTION}

The late-time accelerated expansion of the Universe has forced cosmologists to revise our understanding of the Universe either by introducing a new component in the Universe called dark energy \cite{2} or by modifying directly the laws of gravity \cite{3}. Within the framework of Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, we can account for a phase of accelerated expansion by simply introducing a cosmological constant ($\Lambda$), even though this model gives rise to severe coincidence and fine-tuning problems, observations are still consistent with a dark energy component that has the same characteristics of the cosmological constant \cite{4,5}.

Unfortunately, these experiments are not able to give us information neither on the variation on time of a dark energy component nor on the clustering properties of such dark energy component. Moreover, recent observations have not a sufficient sensitivity to be able to distinguish between a dark energy component or a modified gravity model, even though the two classes of models can be arbitrarily alike, see \cite{6}. It is still important to be able to discard some of the model that manifest a different behavior.

Furthermore, future experiments have been planned to collect a large amount of data with high accuracy and it would be interesting to find tests that are able to confirm our assumptions. Consistency checks are usually model independent tests able to determine if the assumptions made are violated. In this paper we make use of the consistency check found in \cite{1}. In the latter, we introduced a new null test specifically for the growth of matter perturbations and, as far as we know, this is the first null test that accounts for perturbations on the matter fluids. The evolution of the matter density contrast is governed by the evolution of the Hubble parameter and by the evolution of all the other clustering components \cite{7,10}. Hence it is a complementary probe for the dark energy because, while many different dark energy models give the same expansion history they usually differ at perturbation level (depending on the intrinsic characteristics of the fluid itself) and they will affect the evolution of the matter density contrast.

Moreover, as it is well known, modified gravity models can also be re-interpreted as effective dark energy models with their own effective perturbed quantities and consequently the growth of matter density will be influenced by these effective perturbations, see \cite{8}. Hence, it was necessary to find a null test that accounted for the growth of matter density fields.

Finally, the paper is organized as follows: in Section II we report the main equations for the growth of matter density contrast; in Section III we review the derivation of the null test and generalize it to include modified gravity models, in Section IV we construct the null test and discuss its implications. Finally, in Section VII we reconstruct the null test with a variety of data and in Section VII we summarize our results.

\textbf{II. EVOLUTION OF MATTER DENSITY CONTRAST}

The growth of matter in the Universe under the assumption of homogeneity and isotropy is governed by the second order differential equation:

\begin{equation}
\delta''(a) + \left( \frac{3}{a} + \frac{H'(a)}{H(a)} \right) \delta'(a) - \frac{3}{2} \frac{\Omega_m G_{\text{eff}}(a)/G N}{a^5 H(a)^2 / H_0^2} \delta(a) = 0,
\end{equation}

where $a$ is the scale factor and $H(a)$ is the Hubble parameter.
where \( H(a) \) is the Hubble parameter, \( \Omega_m \) is the matter density contrast today and \( H_0 \) is the Hubble constant and where we introduced the effective Newtonian constant \( G_{\text{eff}}(a) \) which accounts for dark energy perturbations or for a variety of modified gravity models, see Refs. [11–14].

Under the assumption that the Universe is currently dominated by a dark energy component with a constant equation of state and negligible dark energy perturbations, i.e. \( G_{\text{eff}}(a)/G_N = 1 \), then Equation (1) can be easily solved analytically. The differential equation (1) has in general two solutions that correspond to two different physical modes, a decaying and a growing one, that in a matter dominated Universe in GR behave as \( \delta = a^{-3/2} \) and as \( \delta = a \) respectively. Since we are only interested in the latter, we demand that at early times \( a_i \ll 1 \), usually during matter domination, the initial conditions have to be chosen as \( \delta(a_i) \simeq a_i \) and \( \delta'(a_i) \simeq 1 \). When \( G_{\text{eff}}(a)/G_N = 1 \) we get GR as a subcase, while in general for modified gravity theories, the term \( G_{\text{eff}} \) can be scale-and time-dependent.

For a flat GR model with a constant dark energy equation of state \( w \), the exact solution of Eq. (1) for the growing mode is given by [15][17]

\[
\delta(a) = a \, 2F_1 \left[ -\frac{1}{3w}, \frac{1}{2} - \frac{1}{2w}; 1 - \frac{5}{6w}; a^{-3w}(1 - \Omega_m^{-1}) \right]
\]

for \( H(a)^2/H_0^2 = \Omega_m a^{-3} + (1 - \Omega_m)a^{-3(1+w)} \),

(2)

where \( 2F_1(a; b; c; z) \) is a hypergeometric function, see Ref. [15] for more details. In more general cases, for instance admitting that the dark energy equation of state parameter is a function of time, it is impossible to find a closed form analytical solution for Eq. (1), but in Ref. [21] it was shown that the growth rate \( f(a) \equiv \frac{dn\delta}{dn/a} \) can be approximated as

\[
f(a) = \Omega_m(a) \gamma(a) \quad (3)
\]

\[
\Omega_m(a) \equiv \frac{\Omega_m}{H(a)^2/H_0^2} \quad (4)
\]

\[
\gamma(a) = \frac{\ln f(a)}{\ln \Omega_m(a)} \simeq \frac{3(1 - w)}{5 - 6w} + \cdots \quad (5)
\]

a more general expression for the growth index can be found in [15]. We should note that the approximation for \( \gamma \) is valid at first order for a dark energy model with a constant \( w \), while for LCDM (\( w = -1 \)) we have \( \gamma = \frac{6}{11} \simeq 0.545 \). Furthermore, it is easy to convert Eq. (1) into an equation for the growth rate \( f(a) \equiv \frac{dn\delta}{dn/a} \), which can be found to be:

\[
f'(a) + \left( \frac{2}{a} + \frac{H'(a)}{H(a)} \right) f(a) + \frac{1}{a} f(a)^2 - \frac{3\Omega_m G_{\text{eff}}(a)/G_N}{2a^4 H(a)^2/H_0^2} = 0 \quad (6)
\]

with initial conditions \( f(a_0) = 1 \) for \( a_0 \ll 1 \) (usually \( a_0 \simeq 10^{-3} \)).

In later sections we will use the previous equations to construct a null test for the growth rate of matter density perturbations. The null test is a function of redshift \( z \) that however has to be constant for all \( z \) under some assumptions, eg that GR is valid or homogeneity holds (since Eq. (1) was evaluated under the assumption of homogeneity and isotropy). Any deviation from the expected result then indicates the failure of one or more of the assumptions. Typical examples in cosmology include the \( \Omega_K(z) \) test of Clarkson et al. [19] or the \( Om \) statistic of Shaferle et al., see Refs. [20], [54]. Also, early examples of null tests for the growth data were shown in Refs. [23] and [57]. However, the former suffers from the problem that we need to know \( \delta(z) \) at \( z \to \infty \), while the latter, known as the \( \theta \) test requires some mild assumptions for \( G_{\text{eff}} \). On the contrary, we will show that our new null test does not suffer from any of these problems.

In this paper we expand our work from Ref. [1], creating a null test that can be used also for more sophisticated cosmological models; we also reconstruct our new null test with a variety of both real and mock data. The last are created using different cosmologies in order to test the validity and the accuracy of our test.

### III. LAGRANGIAN FORMULATION

In this section we will review the derivation of the null test using the Lagrangian formulation and we will expand it for modified gravity theories. This can be done by again constructing a Lagrangian for Eq. (1) and with the help of Noether’s theorem we can find an associated conserved quantity. If we assume that the Lagrangian can be written as

\[
\mathcal{L} = \mathcal{L}(a, \delta(a), \delta'(a)) \quad (7)
\]

then the Euler-Lagrange equations become:

\[
\frac{\partial \mathcal{L}}{\partial \delta} - \frac{d}{da} \frac{\partial \mathcal{L}}{\partial \delta'} = 0. \quad (8)
\]

We can assume a Lagrangian of the form

\[
\mathcal{L} = T - V
\]

\[
T = \frac{1}{2} f_1(a, H(a)) \delta'(a)^2
\]

\[
V = \frac{1}{2} f_2(a, H(a)) \delta(a)^2
\]

where the second and third terms are the ‘kinetic’ and ‘potential’ terms respectively and \( f_1 \) and \( f_2 \) are two functions that need to be found. Substituting the last equations in the Euler-Lagrange Eq. (8), we get

\[
\delta''(a) + \left( \frac{\partial a f_1(a, H)}{f_1(a, H)} + \frac{H'(a)\partial H f_1(a, H)}{f_1(a, H)} \right) \delta'(a)
\]

\[
+ \frac{f_2(a, H)}{f_1(a, H)} \delta(a) = 0. \quad (9)
\]

Comparing Eq. (9) with Eq. (1) we immediately find that

\[
f_1(a, H(a)) = a^3 H(a)/H_0
\]

\[
f_2(a, H(a)) = -\frac{3\Omega_m G_{\text{eff}}(a)/G_N}{2a^3 H(a)/H_0}.
\]
Then the Lagrangian $\mathcal{L}$ and the ‘Hamiltonian’ $H$ of the system become:

$$\mathcal{L} = T - V = \frac{1}{2} a^3 H(a)/H_0 \delta'(a)^2 + 3 \Omega_m G_{\text{eff}}(a)/G_N \delta(a)^2$$

$$H = T + V = \frac{1}{2} a^3 H(a)/H_0 \delta'(a)^2 - 3 \Omega_m G_{\text{eff}}(a)/G_N \delta(a)^2.$$

(10)

Unfortunately, since the Hamiltonian $H$ explicitly depends on ‘time’, i.e. the scale factor, then the energy of the system is not conserved.

Now that we have obtained the Lagrangian for the system, we can use Noether’s theorem to find a conserved quantity that will be later translated into the null test. So, if we have an infinitesimal transformation $X$ with a generator

$$X = \alpha(\delta) \frac{\partial}{\partial \delta} + \frac{da(\delta)}{da} \frac{\partial}{\partial a}$$

$$\frac{da(\delta)}{da} = \partial \alpha, \quad (12)$$

such that

$$L_X \mathcal{L} = 0, \quad (14)$$

then

$$\Sigma = \alpha(a) \frac{\partial \mathcal{L}}{\partial \delta}.$$  

(15)

is a constant of ‘motion’ for the Lagrangian of Eq. (10), see Ref. 22 for an application in Scalar-Tensor cosmology and more details. From Eq. (15) we get that

$$\Sigma = a^3 H(a)/H_0 \alpha(\delta) \delta'(a),$$

(16)

while from Eq. (14) we get

$$\alpha'(a) a^3 H(a)/H_0 \delta'(a) + 3 \Omega_m G_{\text{eff}}(a)/G_N \delta(a) \alpha(a) = 0. \quad (17)$$

The latter can be solved in favor of $\alpha(a)$ to give

$$\alpha(a) = c e^{- \int a \frac{3 \Omega_m G_{\text{eff}}(a)/G_N \delta(a)}{2 a^2 H(a)/H_0} dx} \quad (18)$$

where $c$ is an integration constant and $a_0$ can be chosen to be either 0 or 1. Then the constraint becomes

$$\Sigma = a^3 H(a)/H_0 \delta'(a) e^{- \int a \frac{3 \Omega_m G_{\text{eff}}(a)/G_N \delta(a)}{2 a^2 H(a)/H_0} dx} \quad (19)$$

where we have redefined $\Sigma$ to absorb $c$. Choosing appropriately $a_0$ can lead to convenient values for $\Sigma$, for example for $a_0 = 1$ then it is easy to see that $\Sigma = \delta'(1)$ and for $a_0 < 1$ then $\Sigma \sim \left(\Omega_m a_0^3\right)^{1/2}$, while in general we have $\Sigma = a_0^3 H(a_0) \delta'(a_0)$. We have checked numerically the validity of Eq. (19) for several different cosmologies and values of the parameters.

Eq. (19) can also be written in terms of the growth rate $f(a) \equiv \frac{da}{d\ln a}$. As a consequence, the growth factor can be found to be $\delta(a) = \delta(a_0) e^{f(a_0) \frac{1}{f}(x) dx}$ and Eq. (19) can be rewritten as

$$\Sigma/\delta(a_0) = a^2 H(a) f(a) e^{\int a_0 \frac{f(x)}{f(a_0)} \frac{3 \Omega_m G_{\text{eff}}(a)/G_N}{2 a^2 H(a)/H_0} dx}. \quad (20)$$

Taking into account that $\Sigma = a_0^3 H(a_0) \delta'(a_0)$, we get that the LHS of the previous equation can be re-expressed as

$$\Sigma/\delta(a_0) = a_0^3 H(a_0) \delta'(a_0)/\delta(a_0) = a_0^2 H(a_0) f(a_0). \quad (21)$$

so that Eq. (20) becomes:

$$\frac{a^2 H(a) f(a)}{a_0^2 H(a_0) f(a_0)} e^{\int a_0 \frac{f(x)}{f(a_0)} \frac{3 \Omega_m G_{\text{eff}}(a)/G_N}{2 a^2 H(a)/H_0} dx} = 1. \quad (22)$$

Taking the derivative of Eq. (22) with respect to the scale factor $a$, we obtain Eq. (16). This means that Eq. (22) is a first integral of “motion” of Eq. (6).

However observations can measure directly only $f \sigma_8(a) \equiv f(a) \sigma_8(a)$, where $\sigma_8(a) = \sigma_8(a = 1) = (\delta(a_0))/\delta(a_0)$ and they are not able to give directly $\delta(a)$, hence we need to transform Eq. (19) to be able to test it directly with observations. Taking into account that

$$f \sigma_8(a) \equiv f(a) \sigma_8(a) = \xi a \delta'(a), \quad (23)$$

where $\xi \equiv \frac{\sigma_8(a = 1)}{\sigma_8(a = 1)}$, we have that

$$\delta(a) = \delta(a_0) + \int a_0^a f \sigma_8(x) \frac{dx}{\xi x}. \quad (24)$$

Then, Eq. (19) can be written as

$$\frac{a^2 H(a) f \sigma_8(a)}{a_0^2 H(a_0) f \sigma_8(a_0)} e^{- \frac{2 \Omega_m f(a_0)}{G_N} \int a_0^a \frac{f \sigma_8(x)}{x^2 H(x)/H_0^2 f \sigma_8(x)} dx} = 1. \quad (25)$$

It is clear that the expressions of Eqs. (22)-(25) have to be constant for all redshifts $z$, so in the next Section we will use them to construct a null test. Any deviation from unity will imply the presence of new physics or systematics in the data.

IV. THE NULL TEST

In this Section we will use Eqs. (22)-(25) and assume $G_{\text{eff}}/G_N = 1$ to construct a new null test for the growth rate. Since this equation only holds for GR with the FLRW metric, deviations point to either new physics or
systems in the data. We have explicitly tested in the case of \( w = \text{const.} \), where the analytical solution is known, that Eqs. (22) - (25) are valid at all redshifts.

In order to create our null test, we implement Eqs. (22) - (25). We now have two equivalent forms of the null test:

\[
\mathcal{O}(z) = \frac{a^2 H(a) f(a)}{a^2 H(a_0) f(a_0)} e^{\int_{a_0}^a \left( f(x) - \frac{3m a^2}{2x^2 H(x)^2} \right) dx} \quad (26)
\]

\[
\mathcal{O}(z) = \frac{a^2 H(a) f_\sigma(a)}{a^2 H(a_0) f_\sigma(a_0)} \times \exp \left[ -\frac{4\Omega_m}{5} \int_{a_0}^a \frac{\sigma_{8(x=1)}}{\sigma_8(x)} \left( \frac{f_{\sigma(x)}}{f_\sigma(x_0)} - 1 \right) dx \right] \quad (27)
\]

Both forms are totally equivalent: Eq. (26) is expressed in terms of the growth rate \( f(a) \) which is not a direct measurable quantity but it makes the expression for the null test much simpler and it will useful (as it will be clear later on) for testing directly specific models; Eq. (27) is written in terms of direct measurable quantities and it will be extremely useful to test the data. It is clear that in both cases we should have \( \mathcal{O}(z) = 1 \) at all redshifts, and any deviation from unity could be due to several reasons:

- Detection of modified gravity and non-constant \( G_{\text{eff}} \).
- New physics or a presence of shear or strong dark energy perturbations.
- Deviation from the FLRW metric and homogeneity.
- Tension between \( H(z) \) (obtained directly or derived) and \( f_\sigma(a) \) data.

In the next Section we will test the above expression with the help of mocks based on different models.

V. COSMOLOGICAL MODELS

In this section we schematically report the different cosmologies used to create mocks catalogs.

A. \( w \)CDM model

If the Universe is filled by a dark energy component, with constant equation of state parameter \( w \), then the Hubble equation can be written as:

\[
H(a)^2/H_0^2 = \Omega_m a^{-3} + (1 - \Omega_m) a^{-3(1+w)} \quad (28)
\]

If the dark energy component is not a cosmological constant, i.e. if \( w \neq -1 \), then dark energy is able to cluster. The scale at which this dark energy component can cluster depends on the intrinsic characteristic of the fluid itself, namely: pressure perturbations \( \delta p \) which is related to the sound speed and anisotropic stress \( \sigma \) which is usually related to the viscosity of the fluid (see [7][10]).

Since we are interested in examining the effect of the \( f_\sigma \) data on the null test, we choose an \( f(R) \) model that is exactly \( \Lambda \)CDM at the background level, but is significantly different at the perturbations level. This way, we can disentangle the effects of the modified gravity, \( G_{\text{eff}}(a) \) from the background acceleration. One such degenerate model was studied in Ref. [23], where the \( f(R) \) action was found to be:

\[
S = \frac{1}{8\pi G_N} \int d^4x \sqrt{-g} \left( f(R)/2 + S_m \right), \quad (30)
\]

where \( S_m \) is the action term for the matter fields and

\[
f(R) = R - 2\Lambda + \alpha H^2_0 \left( \frac{\Lambda}{R - 3\Lambda} \right) b \quad 2F_1 \left( b, \frac{3}{2}; 3b, \frac{13}{6}; \frac{\Lambda}{R - 3\Lambda} \right), \quad (31)
\]

where \( b = \frac{1}{12} (-7 + \sqrt{13}) \) and the parameter \( \alpha \) is dimensionless and determines how strong the effects of the modified gravity are. We should note that with this Lagrangian we can recover GR at early times \( (a \ll 1) \), i.e. \( G_{\text{eff}}/G_N \sim 1 \) or \( f(R) \sim 1 \) and that it passes all criteria for viability of \( f(R) \) models, as shown in Ref. [24]. For this model we have by construction

\[
H(a)^2 = H_0^2 \left( \Omega_m a^{-3} + 1 - \Omega_m \right), \quad (32)
\]

while Newton’s constant is [12]:

\[
G_{\text{eff}}/G_N = \frac{1 + 4\xi^2 m}{F + 3\xi m}, \quad (33)
\]

\[
m = \frac{F_R}{F}, \quad (34)
\]

\[
F = f_R = \frac{\partial f}{\partial R}. \quad (35)
\]
In this case our default parameters for the mocks are: \((\Omega_m, \sigma_8) = (0.3, 0.8)\) and we also considered the two different cases \(\alpha = (0.002, 0.2)\).

C. Gauss-Bonnet model

Another interesting case are the \(f(G)\) models, where \(G\) is the Gauss-Bonnet term \(G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}\). Again, we are primarily interested in the effects of the modification of gravity, so we will use the \(f(G)\) degenerate model of Ref. [24], that is exactly \(\Lambda \text{CDM}\) at the background level. Then the action is given by [24]:

\[
S = \frac{1}{8\pi G_N} \int d^4x \sqrt{-g} (R/2 + f(G)) + S_m, 
\]

where

\[
f(G) = -3H_0^2(1 - \Omega_m) + \alpha H_0^2G \int \frac{a(G)H(G)/H_0}{G^2} dG.
\]

In the last equation the first term corresponds to the cosmological constant, we have neglected a term that was just proportional to \(G\) as it does not contribute in the field equations. The cosmological perturbations of the \(f(G)\) models were studied in Ref. [25], where it was shown that the growth factor for the matter perturbations \(\delta_m\) satisfies the evolution equation (using the subhorizon approximation \(k \gg aH\)):

\[
\ddot{\delta}_m + C_1(k,a)\dot{\delta}_m + C_2(k,a)\delta_m \approx 0, 
\]

where the functions \(C_1(k,a)\) and \(C_2(k,a)\) where first derived in [25] and are given in Appendix B of Ref. [24] for completeness. In the GR limit Eq. (38) reduces to

\[
\ddot{\delta}_m + 2H\dot{\delta}_m - \frac{3}{2} \Omega_m a^{-3} \delta_m = 0
\]

so comparing these two expressions we can define an effective Newton’s constant:

\[
G_{\text{eff}}(k,a)/G_N = \frac{C_2(k,a)}{-\frac{3}{2} \Omega_m a^{-3}},
\]

which is valid only under the subhorizon approximation \(k \gg aH\).

Even though these models suffer from instabilities in the matter density perturbations during the matter era as shown in [25], we still use them to make mocks since they exhibit rich phenomenology due to the presence of the second term containing \(C_1(k,a)\) in Eq. (38). This makes them ideal candidates for our null test, as \(C_1(k,a)\) cannot be described by a single \(G_{\text{eff}}\) term, thus will produce deviations from unity. In this case our default parameters for the mocks are: \((\Omega_m, \sigma_8) = (0.3, 0.02, 0.8)\).

D. LTB model

An alternative to \(\Lambda\) for explaining the current acceleration are inhomogeneous Universe models in which the effective acceleration is caused by our special position as observers inside a huge underdense region of space. One of the simplest models to study the effect of such large inhomogeneities is the spherically symmetric Lemaître-Tolman-Bondi model [20–28]. In this large void model, the metric is given by

\[
ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Psi)\gamma_{ij}dx^i dx^j, 
\]

where \(\Psi = \Phi + \alpha H(r,t)\) and \(\epsilon = (H_T - H_L)/(2H_T + H_L)\) is small, \(1\) as observations seem to confirm [30, 31]. We can use the ADM formalism and express our perturbed LTB metric as

\[
ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Psi)\gamma_{ij}dx^i dx^j, 
\]

where \(\gamma_{ij} = \text{diag}(X^2(r,t), A^2(r,t), A^2(r,t)\sin^2\theta)\). Within this formalism, the growing mode of the density contrast is given by [31]

\[
\delta(r,t) = \frac{\dot{A}(r,t)}{r} \mathcal{F}_1 \left[ 1, 2, \frac{7}{2}; u \right],
\]

where \(u = k(r)A(r,t)/F(r)\) and \(F(r) = H_0^2(r)\Omega_M(r)r^3\) specifies the local matter density today.

We can now calculate the growth rate of density perturbations, noting that here the matter density parameter \(\Omega_M(r)\) is a function of redshift via both time \(t\) and the radial coordinate \(r\). In LTB models, this is in principle an arbitrary function which must be chosen appropriately in each case. In the case of the constrained GBH model

\[1\] We now have two different expansion rates, \(H_T(r,t) = \dot{A}/A\) and \(H_L(r,t) = \dot{A}/A'\), corresponding to the transverse and along the line of sight expansion rates, respectively.
The parameters are given by
\[ \Omega_M(r) = 1 + (\Omega_M^0 - 1) \frac{1 - \tanh[(r - r_0)/2\Delta r]}{1 + \tanh[r_0/\Delta r]} \]  \hspace{1cm} (44)
\[ H_0(r) = H_0 \left[ \frac{1}{1 - \Omega_M(r)} - \frac{\Omega_M(r)}{(1 - \Omega_M(r))^{3/2}} \times \right. 
\left. \arcsinh \sqrt{\frac{1 - \Omega_M(r)}{\Omega_M(r)}} \right], \]  \hspace{1cm} (45)

with
\[ r_0 = 3.0 \text{ Gpc}, \ \Delta r = r_0, \ \Omega_M^0 = 0.71, \ \Omega_M^0 = 0.19, \]  \hspace{1cm} (46)

where these values have been chosen to best fit the supernovae and BAO data \cite{30, 34}. Within this model, the growth function, i.e. the logarithmic derivative of the density contrast, is given by \cite{15}
\[ f(z) = \Omega_M^{1/2}(z) \left[ P_{-5/2}^{m} \left( \frac{m^{-1/2}(z)}{P_{-5/2}^{1/2}} \right) \right], \]  \hspace{1cm} (47)

where \( P_m^m(u) \) are the associated Legendre polynomials and \( \Omega_m(z) \) is the fraction of matter density to critical density, as a function of redshift.\(^2\) This function (47) is identical to the instantaneous growth function of matter density in an open Universe, where the local matter density \( \Omega_M \) is given by \( \Omega_m(z) \) at that redshift. This is a good approximation only in LTB models with small cosmic shear, see Ref. \cite{35}.

\(^2\) The matter density in LTB model is given by \( \rho(r,t) = \rho'(r,t)A'(r,t)A'(r,t) \). Note that this is different from \( \Omega_M(r) = F'(r)/A'(r,t)A'(r,t)H_0^2(r), \) which gives the mass radial function today, see Ref. \cite{32}.
we are forced to discard the last 9 data points for \( H(z) \) (as we want to avoid having too wide bins). Because the number of data for both catalogs is quite small the choice of the bins is quite restricted. We decided to opt for two different binning: first we chose 4 and then 3 bins, both equally spaced and we evaluate the observables at the mean redshift of the bins.

It is important to notice that, in order for the consistency check \( \mathcal{O} \) to hold, we need to evaluate quantities at the same redshift. We show the results in Fig. 1. As can be seen from the figure, the number of bins affects the results; in the 4-bins case the null test \( \mathcal{O} \) is far from unity implying that the actual data do not give a \( \Lambda \)CDM scenario as at redshift \( \sim 0.5 \) the reference cosmology is at almost 3\( \sigma \)'s away. However, in the 3-bins case the data predict a \( \Lambda \)CDM scenario already at 1\( \sigma \). The reason why we have such different results is due to the number of data points we are considering. At the moment, we have few data especially for the growth factor and also not uniformly distributed, leaving some bins with only two points and making the binning technique not fully reliable.

### B. Mock catalogs

As mentioned before, we also use mock catalogs based on different cosmologies to test \( \mathcal{O}(z) \) for two main reasons: first, to evaluate how much the errors on the null test will be with future experiments; second, to examine the validity and the generality of the null test \( \mathcal{O}(z) \).

We used different cosmologies to evaluate the mock catalogs: 1) \( \Lambda \)CDM with \( w = -1 \) to recover the \( \Lambda \)CDM limit and another set of data using \( w = -0.8 \) which allows perturbations in the dark energy sector; 2) \( f(R) \) model with \( (\Omega_m, \sigma_8) = (0.3, 0.8) \) and we also considered the two different cases \( \alpha = (0.002, 0.2) \); 3) \( f(G) \) model with \( (\Omega_m, \alpha, \sigma_8) = (0.3, 0.02, 0.8) \); and 4) \( LTB \) model with \( (r_0, \Delta r, r_0, h_0, \Omega_M^0) = (3.0\text{Gpc}, 3.0\text{Gpc}, 0.71, 0.19) \). The details of the models can be found in Section IV. We created two different catalogs for each cosmology: the Hubble parameter and the \( f\sigma_8(z) \). Since we are more interested in testing the consistency check \( \mathcal{O}(z) \) rather than worrying about systematics in the data, we evaluated the Hubble and growth parameters uniformly distributed in the range \( z \in [0, 2] \) divided into 20 equally spaced bins of step \( \Delta z = 0.1 \): the \( H(z) \) and the \( f\sigma_8(z) \) were estimated as its theoretical value plus a gaussian error (that can be negative or positive) and constant errors of 0.2 and 0.006, respectively; the values of the errors were obtained using the Fisher matrix approach and having in mind a setup similar to Euclid-like and LSST-like surveys [54, 55], i.e. evaluating the sensitivity that future survey will have to measure the Hubble parameter and the growth of matter.

#### C. Binning the mock catalogs

In what follows we report the results obtained by binning the data in the mock catalogs that we created using different cosmologies. We use the null test valid for the \( \Lambda \)CDM model given by Eq. (27) and we analyze the mocks. In practice we ask ourselves, if the Universe is different from \( \Lambda \)CDM, how accurately we can test it? As we are analyzing mocks created using a cosmology different from the \( \Lambda \)CDM we expect the null test to fail, i.e. to be different from unity at all the redshifts.

To analyze the mock data we decided to use two different binning: first, we used 4 bins from redshift 0 until redshift 0.8 to compare it with the results from the actual data; second, we used 10 bins using all the data, i.e. we extended our analysis up to \( z = 2.0 \). As both catalogs contain the same number of points and they are uniformly distributed, the mean redshift in each bin will be the same for each cosmology. In Tab. IV we report the values of the \( \mathcal{O}(z) \) for the different cosmologies in the 4-bins case. In the same table, next to each value of the null test, we present the confidence level, i.e. how many sigmas the values of the null test are from unity, if the value is smaller than 1 then the value of the null test \( \mathcal{O}(z) \) is within 1\( \sigma \) close to unity, if the value is larger than one, then the value corresponds to the sigmas away the null test is.

#### Table III: Null test \( \mathcal{O}(z) \) with the corresponding \( 1\sigma \) errors using actual data divided in 4 and 3 bins with the corresponding confidence level.

| z     | \( \mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)} \) | \( \sigma_s \) | \( \mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)} \) | \( \sigma_s \) |
|-------|-----------------------------------------------|---------------|-----------------------------------------------|---------------|
| 0.101 | 1.000 ± 0.076                                 | 0.30          | 1.000 ± 0.066                                 | 0.30          |
| 0.304 | 0.889 ± 0.041                                 | 2.740         | 0.974 ± 0.039                                 | 0.656         |
| 0.506 | 0.896 ± 0.038                                 | 2.744         | 0.986 ± 0.035                                 | 0.395         |
| 0.709 | 0.963 ± 0.037                                 | 0.9900        | ...                                           | ...           |

$\text{DATA}$

$4 \text{ bins}$

$3 \text{ bins}$

$z$

$\mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)}$

$\sigma_s$

$z$

$\mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)}$

$\sigma_s$
| $z$ | $\sigma_{O(z)}$ | $\sigma_{\sigma}$ | $\sigma_{\sigma}$ | $\sigma_{\sigma}$ | $\sigma_{\sigma}$ | $\sigma_{\sigma}$ | $\sigma_{\sigma}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.1 | 1.000 ± 0.010  | 0.000 ± 0.010  | 0.000 ± 0.010  | 0.000 ± 0.010  | 0.000 ± 0.010  | 0.000 ± 0.010  | 0.000 ± 0.010  |
| 0.3 | 0.994 ± 0.013  | 0.447 ± 0.013  | 2.302 ± 0.013  | 1.292 ± 0.013  | 1.924 ± 0.013  | 0.983 ± 0.012  | 1.428 ± 0.012  |
| 0.5 | 0.989 ± 0.022  | 0.498 ± 0.022  | 1.421 ± 0.022  | 0.978 ± 0.022  | 0.987 ± 0.020  | 2.154 ± 0.014  | 1.612 ± 0.085  |
| 0.7 | 0.979 ± 0.032  | 0.637 ± 0.032  | 1.203 ± 0.031  | 0.964 ± 0.032  | 1.111 ± 0.027  | 4.205 ± 0.027  | 1.941 ± 0.153  |

Table IV: Null test $O(z)$ with 1σ errors for the five cosmologies used in this work. We also show the confidence level for each test at each redshifts, values less than 1 indicate that the null test is consistent with unity at 1σ, if it is larger it corresponds to the sigmas away the null test is.

![Figure 2](image)

Figure 2: The results for the null test by binning the mock catalogs for the $H(z)$ and $f\sigma_8$ data. Left panel: the $w$CDM with dark energy perturbations (cyan circle), the $f(R)$ (red square) and the $f(G)$ (purple diamond). Right panel: the LTB model.

In Fig. 2 we show the result for four cosmologies\(^3\) and in Tab. IV we report the values found for the null test, the corresponding errors and the confidence level.

If we test the $\Lambda$CDM mock catalogs, we get a result that it consistency with 1 already at 1σ, see Tab. IV when we use a different mock catalog for instance the $w$CDM one then $O(z)$ is less than 1 at more than 2σ’s at almost any redshift, which is the results that we would expected as the growth of the matter density contrast increases because of the dark energy perturbations. Using the $f(R)$ mocks the null test gives values closer to unity indicating that it will be more difficult to differentiate the $\Lambda$CDM and the $f(R)$ model; this is due to the fact that the $f(R)$ model used in this paper has a Hubble parameter which is exactly $\Lambda$CDM and an $\alpha$ of 0.002, hence the modification to the growth $f\sigma_8(z)$ is small. When we use the mocks from $f(G)$ and LTB cosmologies, which both models give substantially different behavior of the Hubble parameter and the growth of matter density contrast, the deviation from unity of the null test $O(z)$ becomes more evident, in fact we found that the $f(G)$ can be ruled out at more than 4σ’s and the LTB at more than 9σ’s. The results up to $z = 2$ can be found in Tab. IV.

D. Model testing

An interesting alternative to binning is to fit the data, either real or mock, to the $\Lambda$CDM and $w$CDM models and then reconstruct the null test $O(z)$. In Fig. 3 we show the results of reconstructing the null test with the real data fitted by the $\Lambda$CDM (left) and $w$CDM (right) models respectively. Clearly, the null test as reconstructed with the real data seems to be compatible with unity at the 1.5σ level.

Next, we will also test how well the null test will be reconstructed with future data. For this, we also consider the different cosmologies mentioned in a previous section and fit the mocks with both the $\Lambda$CDM and $w$CDM models. The reason for this is that we want to make a direct test of the standard cosmological model with as few extra assumptions as possible. We should stress that in this case any deviation from unity implies a breakdown of either the fundamental assumptions of the standard cosmological model, i.e. homogeneity, the validity of GR etc, or that the DE models used are not a good description of the data.

In Fig. 4 we show the results for the null test for the $\Lambda$CDM (left) and $w$CDM (right) using $\Lambda$CDM mocks (first row) and the DE perturbations (second row) for the $H(z)$ and $f\sigma_8$ data. In Fig. 5 we show the results for the null test for the $\Lambda$CDM model for the mock $H(z)$ and $f\sigma_8$ data based on the $f(R)$ model for $\alpha = 0.002$ (first

\(^3\) We excluded $\Lambda$CDM for sake of space
row left) and $\alpha = 0.2$ (first row right). On the second row we show the results for the $f(G) H(z)$ and $f\sigma_8$ data for the $\Lambda$CDM model (left) and $w$CDM (right) models respectively. Finally, in Fig. 6 we show the results for the null test for the LTB $H(z)$ and $f\sigma_8$ mocks fitted with the $\Lambda$CDM (left) and $w$CDM (right).

We find that the $O(z)$ null test will be particularly successful at detecting deviations from GR at high significance ($\gtrsim 5\sigma$), especially of the $f(R)$ and $f(G)$ types (Fig. 3), but also deviations from the FRW metric (Fig. 6). This is due to the fact that these models have significantly different evolution for the matter density perturbations, which is encoded in the $G_{\text{eff}}$ and can be detected by the null test.

1. Alternative data and theories

As an extra check we also use alternative data instead of just the $H(z)$, namely the Supernovae type Ia (SnIa) to reconstruct the Hubble parameter. In particular we used the latest Union 2.1$^4$ set of 580 SnIa data of Suzuki et al. [56] that spans from redshift 0.015 up to 1.4.

The results for this reconstruction are shown in Fig. 7 for the $\Lambda$CDM and in Fig. 8 for the $w$CDM. We find that they are in excellent agreement with that of the $H(z)$ data shown earlier, thus eliminating any possibility of bias due to the use of the particular data used to reconstruct the Hubble expansion history.

Finally, we also consider $f(R)$ models in the reconstruction of the null test. Specifically, in Fig. 9 we show the results for the null test for the $f(R)$ model for the real SnIa and $f\sigma_8$ data (left) and the $H(z)$ and $f\sigma_8$ data data (right). Again the results are in good agreement, thus demonstrating that the null test is not particularly sensitive on the model used.

VII. CONCLUSIONS

In this paper we have reconstructed the null test for the growth rate data first presented in Ref. [1] by using the $H(z)$, SnIa and $f\sigma_8$ data. We performed the reconstruction in two different ways: by directly binning the data and by fitting the data to various dark energy models like the $\Lambda$CDM and $w$CDM and then calculating the null test. We find that both methods have different advantages; the former uses as few assumptions as possible while the latter directly tests the standard cosmological model.

We have also generalized the null test and extended it for modified gravity models and models with strong DE perturbations, by taking into account the $G_{\text{eff}}$ term in Eq. (1). We have explicitly checked that when this term is taken into account, then the null test is constant as expected for modified gravity models. This allows us to verify that deviations from unity in the original version of the null test presented in Ref. [1] can indeed also be attributed to modifications of gravity.

We have found that deviations from unity could be due to several reasons, either new physics including modifications of gravity and strong dark energy perturbations, or breakdowns of one of the basic assumptions of the standard cosmological model, i.e. deviation from the FLRW metric and homogeneity or finally, a possible tension between $H(z)$ (obtained directly or derived) and the $f\sigma_8$ data. In all cases due to the nature of the null tests and that they have to be constant at all redshifts, it is enough to have a statistically significant deviation at one redshift to detect one of the above reasons. A possible limitation at the moment is that the null test cannot tell us which of the above reasons would be responsible for that deviation though. However, our growth null test will be extremely useful if joined with other null tests, like the $\Omega_K(z)$ presented in [10] which is able to test the assumptions of homogeneity and isotropy of the Universe.

We also examined how well the null test can be reconstructed by future data by creating mock catalogs based on a LSST-like survey and on the $\Lambda$CDM model, a model

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$^4$ The SnIa data can be found in [56] and in [57].
Figure 4: The results for the null test for the ΛCDM (left) and wCDM (right) using ΛCDM mocks (first row) and the DE perturbations (second row) for the $H(z)$ and $f\sigma_8$ data.

with strong DE perturbations, the $f(R)$ and $f(G)$ models, and the large void LTB model that exhibit different evolution of the matter perturbations. This was done so as to examine how well our null test can be reconstructed using the data from upcoming surveys.

Our results were presented in Figs. 1-9. We found that when reconstructed with real data the null test is consistent with unity at the 2σ level, with both the binning and the model testing methods. However, when we reconstruct it with the mock data based on the specifications of a LSST-like survey and various models that go beyond the ΛCDM, i.e., the $f(R)$, $f(G)$ models and the LTB, we find that the null test can detect deviations from unity at the 5σ and also 9σ level.

Overall, the novelty of our null test is that it can directly test the fundamental assumptions of the standard cosmological model with as few assumptions as possible. Therefore, it will definitely prove to be an invaluable tool in the near future given the plethora of upcoming surveys that will produce high quality data.

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Appendix A: Null test for the binning method

In this section we report the null test in terms of redshift $z$ that we used for binning the data and its derivatives with respect to four observables to evaluate the propagated error. The null test $\mathcal{O}$ becomes:
we chose $z_0$ to be equal the first redshift available, hence all the quantities like $H(z_0)$ and $f\sigma_8(z_0)$ are the first binned values of the data. Eq. (A1) depends also on $H_0$ which is in general a complicated parameter to measure, for this reason we use instead $\Omega_{m_0}H_0^2 = 100^2\Omega_{m_0}h^2 = 10^4\omega_m$ where $\omega_m$ is a parameter given by Cosmic Microwave Background experiments and easy to measure with great accuracy. It is also important to notice that

\begin{equation}
O(z) = \frac{(1 + z_0)^2 H(z) f\sigma_8(z)}{(1 + z)^2 H(z_0) f\sigma_8(z_0)} \exp\left\{ \frac{3}{2} \Omega_{m_0}H_0^2 \int_{z_0}^{z} \frac{(1 + x)^2}{H(x)^2 f\sigma_8(x)} \left[ \frac{\delta(z_0)}{\delta(z = 0)} - \int_{z_0}^{x} \frac{f\sigma_8(y)}{1 + y} \, dy \right] \, dx \right\} \tag{A1}
\end{equation}
when $z_0$ approaches to 0 then we have that $H(z_0) \sim H_0$, however, this term should never be thought as the real Hubble constant (like the one appearing in the exponent) but it has to be considered as the value of the Hubble parameter at the lowest redshift because the only true Hubble constant, i.e. that comes directly from the theory is the one appearing in the exponent).

For sake of completeness we also write the derivatives of Eq. (A1) with respect to the four observables that will be used to evaluate the propagated error on the quantity $\mathcal{O}(z)$ and these are:
Table V: Null test $\mathcal{O}(z)$ with $1\sigma$ errors for the five cosmologies used in this work up to $z = 2.0$. We also show the confidence level for each test at each redshift, values less than 1 indicate that the null test is consistent with unity at 1$\sigma$, if it is larger it corresponds to the sigmas away the null test is.

| $z$ | $\mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)}$ | $\sigma'$ | $\mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)}$ | $\sigma'$ | $\mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)}$ | $\sigma'$ | $\mathcal{O}(z) \pm \sigma_{\mathcal{O}(z)}$ | $\sigma'$ |
|-----|----------------------------------|--------|----------------------------------|--------|----------------------------------|--------|----------------------------------|--------|
| 0.1 | 1.000 ± 0.010                    | 0      | 1.000 ± 0.010                    | 0      | 1.000 ± 0.009                    | 0      | 1.000 ± 0.010                    | 0      |
| 0.3 | 0.994 ± 0.013                    | 0.447  | 0.971 ± 0.013                    | 1.230  | 0.975 ± 0.013                    | 1.924  | 0.983 ± 0.012                    | 1.428  |
| 0.5 | 0.989 ± 0.022                    | 0.496  | 0.970 ± 0.021                    | 1.421  | 0.978 ± 0.022                    | 0.987  | 0.986 ± 0.020                    | 2.154  |
| 0.7 | 0.979 ± 0.032                    | 0.634  | 0.962 ± 0.031                    | 1.203  | 0.964 ± 0.032                    | 1.111  | 0.885 ± 0.027                    | 4.205  |
| 0.9 | 0.980 ± 0.045                    | 0.460  | 0.951 ± 0.041                    | 1.196  | 0.950 ± 0.043                    | 1.166  | 0.850 ± 0.036                    | 4.203  |
| 1.1 | 0.961 ± 0.055                    | 0.707  | 0.947 ± 0.052                    | 1.032  | 0.935 ± 0.054                    | 1.265  | 0.801 ± 0.043                    | 4.597  |
| 1.3 | 0.968 ± 0.068                    | 0.464  | 0.949 ± 0.062                    | 0.821  | 0.915 ± 0.064                    | 1.329  | 0.784 ± 0.052                    | 4.139  |
| 1.5 | 0.960 ± 0.079                    | 0.504  | 0.945 ± 0.073                    | 0.755  | 0.927 ± 0.076                    | 0.955  | 0.781 ± 0.062                    | 3.559  |
| 1.7 | 0.955 ± 0.091                    | 0.492  | 0.919 ± 0.081                    | 0.796  | 0.931 ± 0.088                    | 0.781  | 0.751 ± 0.069                    | 3.612  |
| 1.9 | 0.943 ± 0.101                    | 0.557  | 0.928 ± 0.093                    | 0.770  | 0.903 ± 0.097                    | 1.003  | 0.749 ± 0.079                    | 3.189  |

\[
\frac{\partial \log \mathcal{O}(z)}{\partial H(z)} = \frac{1}{H(z)} - 3 \times 10^4 \Omega_{m_0} h^2 \int_0^z \frac{(1+x)^2}{H(x)^3 f_s(x)} \left[ \sigma_s(z=0) - \int_0^z \frac{f_s(y)}{1+y} dy \right] dx \\
\frac{\partial \log \mathcal{O}(z)}{\partial f_s(z)} = -\frac{1}{f_s(z)} - \frac{3}{2} \times 10^4 \Omega_{m_0} h^2 \int_0^z \frac{(1+x)^2}{H(x)^2 f_s(x)} \left[ -\sigma_s(z=0) + \int_0^z \frac{f_s(y)}{1+y} dy - f_s(x) \log(1+x) \right] dx (A2)
\]

\[
\frac{\partial \log \mathcal{O}(z)}{\partial \sigma_s(z=0)} = \frac{3}{2} \times 10^4 \Omega_{m_0} h^2 \int_0^z \frac{(1+x)^2}{H(x)^2 f_s(x)} dx (A3)
\]

\[
\frac{\partial \log \mathcal{O}(z)}{\partial \Omega_{m_0} h^2} = \frac{3}{2} \times 10^4 \Omega_{m_0} h^2 \int_0^z \frac{(1+x)^2}{H(x)^2 f_s(x)} \left[ \sigma_s(z=0) - \int_0^z \frac{f_s(y)}{1+y} dy \right] dx (A4)
\]

\[
\sigma_{\mathcal{O}(z)} = \sqrt{\left(\frac{\partial \log \mathcal{O}(z)}{\partial H(z)}\right)^2 \sigma_{\mathcal{H}(z)}^2 + \left(\frac{\partial \log \mathcal{O}(z)}{\partial f_s(z)}\right)^2 \sigma_{f_s(z)}^2 + \left(\frac{\partial \log \mathcal{O}(z)}{\partial \Omega_{m_0} h^2}\right)^2 \sigma_{\Omega_{m_0} h^2}^2 + \left(\frac{\partial \log \mathcal{O}(z)}{\partial \sigma_s(z=0)}\right)^2 \sigma_{\sigma_s(z=0)}^2} (A6)
\]

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