High Energy Theorems at Large-\(N\)

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ABSTRACT

Sum rules for products of two, three and four QCD currents are derived using chiral symmetry at infinite momentum in the large-\(N\) limit. These exact relations among meson decay constants, axialvector couplings and masses determine the asymptotic behavior of an infinite number of QCD correlators. The familiar spectral function sum rules for products of two QCD currents are among the relations derived. With this precise knowledge of asymptotic behavior, an infinite number of large-\(N\) QCD correlators can be constructed using dispersion relations. A detailed derivation is given of the exact large-\(N\) pion vector form factor and forward pion-pion scattering amplitudes.

PACS: 11.30.Rd; 12.38.Aw; 11.55.Jy; 11.30.Er
1. Introduction

In the large-$N$ limit all QCD mesonic correlation functions are sums of tree graphs [1,2]. The machinery of dispersion relations, which relates correlators at different energies, is transparent in this context since all dispersion relations are saturated by single-particle states. Of course dispersion relations require knowledge of asymptotic behavior and traditionally this knowledge has been assumed or taken from phenomenologically inspired models like Regge pole theory. What is lacking is a fundamental principle based in QCD which determines the asymptotic behavior of correlation functions. The dominance of single-particle states at large-$N$ has a related advantage: this limit is particularly suited to a symmetry point of view since it is easy to place single-particle states in representations of a symmetry group and extract consequences for matrix elements. In fact, there is a deep relation between asymptotic behavior and symmetries. For instance, requiring asymptotic behavior consistent with unitarity implies a spin-flavor algebra which constrains the spectrum of the large-$N$ baryons [3]. Another classic example is string theory, where both the spectrum of states and the asymptotic behavior of S-matrix elements (which are QCD-like) are determined by symmetries on the string world sheet. In this paper I show that in large-$N$ QCD, chiral symmetry determines the asymptotic behavior of an infinite number of mesonic correlation functions. Armed with this asymptotic information, exact large-$N$ correlators can be constructed using dispersion relations in a facile way.

There are well known QCD predictions of asymptotic behavior for products of two currents with nontrivial chiral quantum numbers [4]. These asymptotic constraints — known as spectral function sum rules (SFSR’s)— are naturally derived in the context of the operator product expansion (OPE) [5]. The fundamental ingredient of their derivation is one of the remarkable properties of the OPE: coefficient functions in the OPE are insensitive to vacuum properties and therefore transform with respect to the full unbroken global symmetry group of the underlying theory, regardless of whether this symmetry is spontaneously broken in the low-energy theory [5]. Hence the SFSR’s exist because the full global chiral symmetry group of QCD can be used for classification purposes in the region of large Euclidean momenta. One might then wonder whether the SFSR’s can be derived directly using chiral symmetry without recourse to the OPE. In full QCD a direct symmetry approach is hindered by complicated analyticities arising from various continuum contributions to the two-point functions. On the other hand, in the large-$N$ limit, the two-point functions are determined by the exchange of an infinite number of single-particle states. I will show that the SFSR’s at large-$N$ are purely consequences of chiral symmetry and are easily derived directly from the chiral representation theory and/or the chiral algebra without recourse to the OPE. Relevant in this respect is a special class
of Lorentz frames in which a component of momentum is taken to be very large compared to typical hadronic scales (e.g. the infinite momentum frame).

The purpose of this work is not to find an amusing way of reproducing the SFSR’s. Rather it is to see if the direct symmetry approach leads to other sum rules which have been overlooked or simply obscured by the complexities of the OPE. I will derive an infinite number of (new) sum rules in the large-$N$ limit. These sum rules rely on precisely the same theoretical assumptions as the SFSR’s and in fact depend on the SFSR’s for consistency. Just as the SFSR’s determine the asymptotic behavior of two-point functions which in turn can be constructed using dispersion relations, all of the new chiral sum rules determine the asymptotic behavior of specific three- or four-point functions in large-$N$ QCD. Dispersion relations can then be used to construct the exact correlators. I will consider several examples in detail.

In section 2 I review some basic relevant facts about large-$N$ QCD. The technology of the operator product expansion (OPE) is then used to review the derivation of the spectral function sum rules. In section 3 I derive the SFSR’s and an infinite number of new sum rules directly using chiral symmetry and its representation theory in the infinite momentum frame. In section 4 I identify some of these sum rules with asymptotic constraints and explicitly construct exact large-$N$ correlators using dispersion relations. I conclude in section 5. In an appendix I derive a set of sum rules directly from the chiral algebra.

2. Spectral Function Sum Rules at Large-$N$

This section reviews known material which is essential for what follows. In this paper I consider QCD with two massless flavors. This theory has an $SU(2)_L \times SU(2)_R$ chiral symmetry. In the large-$N$ limit, assuming confinement, this symmetry breaks spontaneously to the $SU(2)_V$ isospin subgroup [6]. (Strictly speaking, large-$N$ QCD has a $U(2)_L \times U(2)_R$ chiral symmetry. In this paper the additional Goldstone mode and its associated $U(1)$ charge are ignored.) In the large-$N$ limit the mesons have the most general quantum numbers of the quark bilinear $\bar{Q}\Gamma Q$ where $\Gamma$ is some arbitrary spin structure. This means that all mesons have zero or unit isospin. The corresponding chiral transformation properties of the mesons will be discussed below. The $SU(2) \times SU(2)$ algebra is expressed via the commutation relations

$$[Q^5_a, Q^5_b] = i\epsilon_{abc}T_c \quad [T_a, Q^5_b] = i\epsilon_{abc}Q^5_c \quad [T_a, T_b] = i\epsilon_{abc}T_c,$$

(1)

where $T^a$ are $SU(2)_V$ generators normalized so that $T^a = \tau_a/2$ and $Q^5_a$ are the $SU(2)_A$ generators that are broken by the vacuum, at rest. Note that $V = L + R$ and $A = L - R$. The conserved QCD currents associated with these charges are:
\[ A_\mu^a = \bar{Q} \gamma_\mu \gamma_5 T^a Q \]
\[ V_\mu^a = \bar{Q} \gamma_\mu T^a Q. \] (2)

These currents satisfy the commutation relations
\[ [Q_5^a, V_\mu^b] = i \epsilon^{abc} A_\mu^c \]
\[ [Q_5^a, A_\mu^b] = i \epsilon^{abc} V_\mu^c. \] (3)

Therefore, \( V_\mu^a \) and \( A_\mu^a \) form a complete (six dimensional) chiral multiplet; they transform as \((3, 1) \oplus (1, 3)\) with respect to \( SU(2) \times SU(2) \).

In the low-energy theory, the axialvector current \( A_\mu^a \) has a nonvanishing matrix element between the vacuum and the Goldstone boson (pion) states:
\[ \langle 0 | A_\mu^a | \pi^b \rangle = \delta^{ab} F_\pi p_\mu. \] (4)

Both conserved currents have nonvanishing matrix elements between the vacuum and vector meson states:
\[ \langle 0 | A_\mu^a | A_\nu^b (\lambda) \rangle = \delta^{ab} F_A M_A \epsilon^{(\lambda)}_\mu \]
\[ \langle 0 | V_\mu^a | V_\nu^b (\lambda) \rangle = \delta^{ab} F_V M_V \epsilon^{(\lambda)}_\mu \] (5)

where \( \epsilon^{(\lambda)}_\mu \) is the vector meson polarization vector and \( \lambda \) is a helicity label.

Consider the time-ordered product of currents:
\[ \Pi_{LR}^{\mu \nu}(q) \delta_{ab} = 4i \int d^4 x e^{iqx} \langle 0 | T L_\mu^a(x) R_\nu^b(0) | 0 \rangle, \] (6)

where the \( SU(2)_{L,R} \) QCD currents \( L \) and \( R \) transform as \((3, 1) \oplus (1, 3)\), respectively, with respect to \( SU(2) \times SU(2) \). It follows that \( \Pi_{LR} \) transforms as \((3, 3)\). Lorentz invariance and current conservation allow the decomposition
\[ \Pi_{LR}^{\mu \nu}(q) = (q^\mu q^\nu - g^{\mu \nu} q^2) \Pi_{LR}(Q^2), \] (7)

where \( Q^2 = -q^2 \). Because the function \( \Pi_{LR}(Q^2) \) carries nontrivial chiral quantum numbers, it vanishes to all orders in QCD perturbation theory. Hence, asymptotic freedom implies \( \Pi_{LR}(Q^2) \to 0 \) as \( Q^2 \to \infty \) and \( \Pi_{LR} \) satisfies the unsubtracted dispersion relation:
\[ \Pi_{LR}(Q^2) = \frac{1}{\pi} \int \frac{dt \text{Im} \Pi_{LR}(t)}{t + Q^2}. \] (8)

In the OPE we have the expansion,
\[ \Pi_{LR}(Q^2) \to \sum_{n=1}^{\infty} \frac{\langle \mathcal{O} \rangle^{d=2n}_{(3,3)}}{Q^{2n}} \] (9)
where the coefficients $\langle O \rangle_{(3,3)}^{d=2n}$ of mass dimension $d = 2n$ transform as $(3, 3)$ with respect to $SU(2) \times SU(2)$. This is the case regardless of whether $SU(2) \times SU(2)$ is spontaneously broken in the low-energy theory. The coefficient functions always transform with respect to the full unbroken symmetry group of the underlying theory [5].

In the large-$N$ limit $\Pi_{LR}$ is given by a sum of single-particle states [7]. Using Eq. (4)-Eq. (7) one obtains

$$-\frac{1}{\pi} \text{Im} \, \Pi_{LR}(t) = F_\pi^2 \delta(t) + \sum_A F_A^2 \delta(t - M_A^2) - \sum_V F_V^2 \delta(t - M_V^2). \quad (10)$$

Evaluating the dispersion relation then gives

$$-Q^2 \Pi_{LR}(Q^2) = F_\pi^2 + \sum_A \frac{Q^2 F_A^2}{Q^2 + M_A^2} - \sum_V \frac{Q^2 F_V^2}{Q^2 + M_V^2}, \quad (11)$$

which can be expanded in inverse powers of $Q^2$:

$$\Pi_{LR}(Q^2) \rightarrow \left\{ \sum_V F_V^2 - \sum_A F_A^2 - F_\pi^2 \right\} \frac{1}{Q^2} + \sum_{m=1}^\infty \left\{ \sum_V F_V^2 M_V^{2m} - \sum_A F_A^2 M_A^{2m} \right\} \frac{(-1)^m}{Q^{2m+2}}. \quad (12)$$

The coefficients in this expansion can be matched to the OPE, giving

$$\langle O \rangle_{(3,3)}^{d=2} = \sum_V F_V^2 - \sum_A F_A^2 - F_\pi^2 \quad (13a)$$

$$\langle O \rangle_{(3,3)}^{d=2n} = (-1)^{n-1} \left\{ \sum_V F_V^2 M_V^{2n-2} - \sum_A F_A^2 M_A^{2n-2} \right\} \quad n > 1. \quad (13b)$$

There are no QCD gauge invariant local operators with $d = 2$. The only $d = 4$ operator is $F^2$ where $F$ is the gluon field strength tensor. However, this operator is a chiral singlet. Hence the first two terms in the OPE must vanish and we have the two spectral function sum rules:

$$\langle O \rangle_{(3,3)}^{d=2} = 0 \quad (14a)$$

$$\langle O \rangle_{(3,3)}^{d=4} = 0 \quad (14b)$$

which are referred to in the text as $SFSR1$ and $SFSR2$, respectively. These sum rules are illustrated diagrammatically in Fig. 1. Of course the sums span an infinite number of states in the large-$N$ limit. The exact large-$N$ two-point function may then be written as

1 Some interesting consequences of these sum rules when truncated to a small number of states are discussed in Ref. [7].
\[ \sum \text{Fig. 1: Spectral function sum rules for products of two QCD currents.} \]

The top diagram is \textit{SFSR1} and the bottom diagram is \textit{SFSR2}. The cross represents a mass-squared insertion.

\[ \Pi_{LR}(q^2) = \sum_V \frac{F_V^2 M_V^4}{q^4 (M_V^2 - q^2)} - \sum_A \frac{F_A^2 M_A^4}{q^4 (M_A^2 - q^2)}, \tag{15} \]

which manifests the correct asymptotic behavior.

Since narrow single-particle states are all that survive in the large-\(N\) limit, one may wonder whether it is possible to obtain the SFSR’s directly using \(SU(2) \times SU(2)\) symmetry, with no reference to the OPE. One might expect that the pion together with the infinite number of vector and axialvector states in some sense fill out a representation of the full unbroken \(SU(2) \times SU(2)\) group. We will see that this is the case. I reiterate that the main motivation of the symmetry approach is to find sum rules for other \(n\)-point functions in a simple way. For instance, consider adding an external pion line to one of the diagrams in Fig. 1 so that a vector current turns into an axial current (see Fig. 4). Intuitively is would seem that there should be a sum rule analogous to the SFSR’s for such a graph. This is in fact the case, as we will see.

3. Chiral Symmetry at Infinite Momentum

3.1 The Infinite Momentum Frame

The fact that OPE coefficients do not feel vacuum properties enables the derivation of the SFSR’s by making use of the full unbroken chiral symmetry group \([5]\). In order to obtain the SFSR’s using symmetry arguments it is necessary to work directly with the matrix elements of the currents between the states and the vacuum, rather than with the correlation function \(\Pi_{LR}\). But clearly the manner in which the meson decay constants are defined through the matrix elements of Eq. (4) and Eq. (5) implies an asymmetry between states of different spin. The symmetric appearance of the decay constants in the SFSR’s (for instance, \textit{SFSR1} is invariant with respect to \(F_A \leftrightarrow F_\pi\) for each \(A\)) is a consequence of taking \(Q^2 \rightarrow \infty\) in the OPE. In studying matrix elements it is therefore convenient to work in a Lorentz frame in which the pions and the vector mesons appear the same. This is easy to achieve. At rest we define the vector meson polarization vectors:
Figure 2: Collinear kinematics of the infinite momentum frame where $p_3 \to \infty$. Conservation of helicity, $\lambda$, follows from invariance with respect to rotations about the 3-axis. Parity takes A to B. A rotation by $\pi$ about the 2-axis restores the original configuration with the sign of the spin (helicity) reversed.

\[ \epsilon_\mu^{(+)} = (0, 1, 0, 0) \quad \epsilon_\mu^{(-)} = (0, 0, 1, 0) \quad \epsilon_\mu^{(0)} = (0, 0, 0, 1). \quad (16) \]

Consider boosting all particles along the 3-axis (or the observer along the negative 3-axis) to $p_\mu = (p_0, 0, 0, p_3)$. We then have $\epsilon_\mu^{(\pm)}$ unchanged and

\[ \epsilon_\mu^{(0)} = \frac{p_3}{M}, \frac{0}{M}, \frac{0}{M}, \frac{p_0}{M}. \quad (17) \]

Now as we take $p_3 \to \infty$,

\[ \epsilon_\mu^{(0)} = \frac{p_\mu}{M} + O\left(\frac{M}{p_3}\right) \quad (18) \]

and we have

\[ \langle 0 | A_\mu^a | \pi^b \rangle = \delta^{ab} F_{\pi} p_\mu \]
\[ \langle 0 | A_\mu^a | A^b \rangle^{(0)} = \delta^{ab} F_{A} p_\mu \]
\[ \langle 0 | V_\mu^a | V^b \rangle^{(0)} = \delta^{ab} F_{V} p_\mu. \quad (19) \]

Therefore, as the momentum is taken large compared to the mass scales in the problem, the $\lambda = 0$ vector mesons act like Goldstone bosons. This is familiar from the Goldstone boson equivalence theorem in electroweak physics [8]. The kinematical conditions described above are known as the infinite momentum frame. One advantage of this frame is that boost invariance is preserved along the 3-direction.

We can use boost invariance to define the amplitudes:
\[ \langle 0 | A^a_0 - A^a_3 | \alpha \rangle \equiv (p_0 - p_3) \langle 0 | A^a | \alpha \rangle \]
\[ \langle 0 | V^a_0 - V^a_3 | \alpha \rangle \equiv (p_0 - p_3) \langle 0 | V^a | \alpha \rangle , \]

where the matrix elements of \( V^a \) and \( A^a \) are constants and \( \alpha \) represents a physical meson state. It then follows from Eq. (19) and Eq. (20) that

\[ \langle 0 | A_a | \pi_b \rangle = \delta_{ab} F_\pi \]
\[ \langle 0 | A_a | A_b \rangle^{(0)} = \delta_{ab} F_A \]
\[ \langle 0 | V_a | V_b \rangle^{(0)} = \delta_{ab} F_V . \]  

To reiterate, the matrix elements of the pions and the \( \lambda = 0 \) components of the vector mesons look the same in the infinite momentum frame. This is ideal when using symmetry arguments to relate matrix elements.

3.2 Selection Rules at Infinite Momentum

We have the following selection rules at infinite momentum\(^2\):

- Invariance with respect to rotations about the 3-axis implies helicity conservation. We can look at sectors of states labelled by helicity. This explains why the \( \lambda = 0 \) components of the vector mesons are on par with the pions at infinite momentum; all \( \lambda = 0 \) states look like scalars. We will be interested in the helicity conserving pion transition matrix elements between physical meson states \( \beta \) and \( \alpha \) which are related to the axial charges in the infinite momentum frame by:

\[ M_a(p' \lambda' \beta, p \lambda \alpha) = (F_\pi)^{-1} (m^2_\alpha - m^2_\beta)[[\lambda \lambda'] \langle \beta | Q^5_a | \alpha \rangle^{(\lambda \lambda')} \delta_{\lambda' \lambda} . \]  

- The axial charges annihilate the vacuum in the broken phase:

\[ Q^5_5 | 0 \rangle = 0 . \]  

This means that the chiral algebra is useful for classifying hadron states\(^3\). We will discuss the manner in which symmetry breaking effects appear in this frame below. Since meson states are quark bilinears in the large-\( N \) limit and quarks transform as \( (2, 1) \) and \( (1, 2) \), mesons transform as combinations of \( (2, 2) \), \( (3, 1) \), \( (1, 3) \) and \( (1, 1) \) irreducible representations of \( SU(2) \times SU(2) \). Charge conjugation leaves \( (2, 2) \) and \( (1, 1) \) unchanged

\( \footnote{2\ This formalism is worked out in great detail in Ref. 9, Ref. 10 and Ref. 11.} \)

\( \footnote{3\ An easy way to see this is to express the QCD axial charges in light-front coordinates. Then it is clear that \( Q^5_5 \) does not carry vacuum quantum numbers [12] and so annihilates the vacuum.} \)
and interchanges \((1, 3)\) and \((3, 1)\). Physical meson states have definite charge conjugation, \(C\), and isospin, \(I\), and therefore are linear combinations of the isovectors \(|(2, 2)_a\rangle\), \(|(1, 3)_a\rangle - |(3, 1)_a\rangle|\sqrt{2} \equiv |V_a\rangle\) and \(|(1, 3)_a\rangle + |(3, 1)_a\rangle|\sqrt{2} \equiv |A_a\rangle\) and the isoscalars \(|(2, 2)_a\rangle\) and \(|(1, 1)_a\rangle\). Roman subscripts are isospin indices. Only \(|V_a\rangle\) changes sign under charge conjugation. The action of the generators on the states of definite chirality can be obtained using tensor analysis:

\[
i(A_a|Q^5_b|V_c) = i \epsilon_{abc} \delta_{ij} i((2, 2)_a|Q^5_b|(2, 2)_c)_j = i \delta_{ab} \delta_{ij} \]

\[
i((2, 2)_a|T_b|A_c)_j = i(A_a|T_b|A_c)_j = i(V_a|T_b|V_c)_j = i \epsilon_{abc} \delta_{ij}.
\]

- Invariance with respect to a combined space inversion and rotation through \(\pi\) about an axis perpendicular to the 3-axis implies:

\[
^{(\lambda)}\langle \alpha|Q^5_a|\beta \rangle^{(\lambda)} = ^{(-\lambda)}\langle \alpha|Q^5_a|\beta \rangle^{(-\lambda)} P_a P\beta (-1)^{J\beta - J_\alpha + 1}
\]

where \(P\) and \(J\) are parity and spin, respectively. For \(\lambda = 0\) this implies the selection rule:

\[
\eta_\beta = -\eta_\alpha,
\]

else \(\langle \alpha|Q^5_a|\beta \rangle = 0\), where \(\eta \equiv P(-1)^J\) is normality. From Eq. (22) it then follows that only \((\lambda = 0)\) states of opposite normality communicate by pion exchange.

- Conservation of \(G\)-parity implies the selection rule

\[
G_\beta = -G_\alpha
\]

else \(\langle \alpha|Q^5_a|\beta \rangle = 0\). The product of normality and \(G\)-parity is a symmetry. Hence it follows that physical meson states fall into representations labelled by \(G\eta\).

A geometric picture of several of the selection rules is given in Fig. 2.

3.3 The Representation Theory

Using the infinite momentum frame selection rules it is straightforward to identify the most general pion \(SU(2) \times SU(2)\) representation in the large-\(N\) limit. Since the pion is a Lorentz scalar all states in its chiral representation have \(\lambda = 0\). This representation can contain meson states of any spin. Of course all mesons have \(\lambda = 0\) components. Too, the pion has \(G\eta = +1\) as do all states in its representation. Since the pion has \(I = 1\) and \(C = +1\) it is a linear combination of \(|(2, 2)_a\rangle\) and \(|A_a\rangle\) states. The other physical states which are linear combinations of \(|(2, 2)_a\rangle\) and \(|A_a\rangle\) states necessarily have \(\eta = -1\) and
Table 1: Members of the pion $SU(2) \times SU(2)$ representation at infinite momentum. The $\lambda = 0$ components of meson states of arbitrary spin with allowed quantum numbers can participate in this representation.

| $I$ | $G$ | $J$ | $P$ | $C$ | $\eta$ | $G\eta$ |
|-----|-----|-----|-----|-----|--------|--------|
| $\pi$ | 1 | − | 0 | − | + | − | + |
| $P$ | 1 | − | even | − | + | − | + |
| $A$ | 1 | − | odd | + | + | − | + |
| $V$ | 1 | + | odd | − | − | + | + |
| $S$ | 0 | + | even | + | + | + | + |

$G = -1$ but can have $P = -1$, $J =$even or $P = +1$, $J =$odd. The former are labelled $P$ and the latter $A$. The physical states with $\eta = +1$ and $G = +1$ can have $P = -1$, $J =$odd or $P = +1$, $J =$even. Bose statistics requires that the former have $I = 1$ (and thus be linear combinations of $|V_a\rangle$ states with $C = -1$) and the latter have $I = 0$ (and thus be linear combinations of $|(2, 2)_0\rangle$ and $|(1, 1)\rangle$ states with $C = +1$). These states are labelled $V$ and $S$, respectively. Allowed states are listed in Table 1 together with their quantum numbers. Thus, the most general chiral representation involving the pion and these states is:

$$|\pi_a\rangle = \sum_{i=1}^{m} u_{1i} |A_{a}\rangle_i + \sum_{i=m+1}^{n+m} u_{1i} |(2, 2)_{a}\rangle_i$$  \hspace{1cm} (28a)$$

$$|A_{a}\rangle_{2(0)} = \sum_{i=1}^{m} u_{2i} |A_{a}\rangle_i + \sum_{i=m+1}^{n+m} u_{2i} |(2, 2)_{a}\rangle_i$$

$$\vdots$$

$$|A_{a}\rangle_{l(0)} = \sum_{i=1}^{m} u_{li} |A_{a}\rangle_i + \sum_{i=m+1}^{n+m} u_{li} |(2, 2)_{a}\rangle_i$$  \hspace{1cm} (28b)$$

$$|P_{a}\rangle_{l+1(0)} = \sum_{i=1}^{m} u_{(l+1)i} |A_{a}\rangle_i + \sum_{i=m+1}^{n+m} u_{(l+1)i} |(2, 2)_{a}\rangle_i$$

$$\vdots$$

$$|P_{a}\rangle_{n+m(0)} = \sum_{i=1}^{m} u_{(n+m)i} |A_{a}\rangle_i + \sum_{i=m+1}^{n+m} u_{(n+m)i} |(2, 2)_{a}\rangle_i$$  \hspace{1cm} (28c)$$
\[ |V_a \rangle_i^{(0)} = \sum_{j=1}^{m} w_{ij} |V_a \rangle_j \]
\[ \vdots \]
\[ |V_a \rangle_m^{(0)} = \sum_{j=1}^{m} w_{mj} |V_a \rangle_j \]
\[ |S \rangle_1^{(0)} = \sum_{i=m+1}^{n+m} v_{(m+1)i} |(2, 2)_0 \rangle_i + \sum_{i=m+n+1}^{m+n+o} v_{(m+1)i} |(1, 1) \rangle_i \]
\[ \vdots \]
\[ |S \rangle_{n+o}^{(0)} = \sum_{i=m+1}^{m+n} v_{(m+n+o)i} |(2, 2)_0 \rangle_i + \sum_{i=m+n+1}^{m+n+o} v_{(m+n+o)i} |(1, 1) \rangle_i \]

where \( \hat{u}, \hat{w} \) and \( \hat{v} \) are real mixing matrices, subject to the orthonormality relations

\[ \sum_{k=1}^{n+m} u_{ik} u_{jk} = \sum_{k=1}^{m} w_{ik} w_{jk} = \sum_{k=m+1}^{m+n+o} v_{ik} v_{jk} = \delta_{ij}. \]  

The dimension of the most general representation is \( \text{Dim} = 2(2n + 3m) + o \) where \( n \) is the number of \( (2, 2) \) representations, \( m \) is the number of \( (1, 3) \oplus (3, 1) \) representations, and \( o \) is the number of \( (1, 1) \) representations. We work with \( \text{Dim} = \text{finite} \) but it should be understood that strictly speaking \( \text{Dim} = \infty \) in large-\( N \) QCD.

It is convenient to group the pion, the axialvector states and the pseudoscalar states into the \( n + m \)-component vector \( \Pi_i = (\pi, A_2...A_l, P_{l+1}...P_{n+m}) \). We can then express the pion multiplet in the compact form:

\[ |\Pi_a \rangle_i = \sum_{j=1}^{m} u_{ij} |A_a \rangle_j + \sum_{j=m+1}^{n+m} u_{ij} |(2, 2)_a \rangle_j \]
\[ i = 1 \ldots (m + n) \]  
\[ |V_a \rangle_i = \sum_{j=1}^{m} w_{ij} |V_a \rangle_j \]
\[ i = 1 \ldots m \]  
\[ |S \rangle_i = \sum_{j=m+1}^{n+m} v_{ij} |(2, 2)_0 \rangle_j + \sum_{j=n+m+1}^{m+n+o} v_{ij} |(1, 1) \rangle_j \]
\[ i = 1 \ldots (n + o). \]

The helicity label has been suppressed for ease of notation. It will be reintroduced below. The pion and vector meson decay constants generalized from Eq. (21) are given by

\[ \langle 0 | A_a | \Pi_b \rangle_i = \delta_{ab} F_{\Pi_i} \]  
\[ \langle 0 | V_a | V_b \rangle_i = \delta_{ab} F_{V_i} \]
and we now define the matrix elements of definite chirality:

\[ \langle 0| V_a | V_b \rangle_j = \langle 0| A_a | A_b \rangle_j \equiv \delta_{ab} F_j \]  
(32a)

\[ \langle 0| V_a | A_b \rangle_j = \langle 0| A_a | V_b \rangle_j = 0. \]  
(32b)

Now using Eq. (30), Eq. (31) and Eq. (32) gives

\[ F_{\Pi_i} = \sum_{j=1}^{m} u_{ij} F_j \]  
(33a)

\[ F_{V_i} = \sum_{j=1}^{m} w_{ij} F_j. \]  
(33b)

Of course these decay constants are nonvanishing only for pion and vector meson states while the sums over states span all spins according to Table 1. The axialvector couplings are defined by

\[ i \langle \Pi_b | Q^5_a | S \rangle_j = -i \delta_{ab} G_{S_{j \Pi_i}} / F_\pi \]  
(34a)

\[ i \langle \Pi_b | Q^5_a | V_c \rangle_j = -i \epsilon_{abc} G_{V_{j \Pi_i}} / F_\pi. \]  
(34b)

From Eq. (24), Eq. (30) and Eq. (34) it follows that

\[ G_{S_{j \Pi_i}} / F_\pi = -\sum_{k=m+1}^{m+n} u_{ik} v_{jk} \]  
(35a)

\[ G_{V_{j \Pi_i}} / F_\pi = \sum_{k=1}^{m} u_{ik} w_{jk}. \]  
(35b)

Note that the axial couplings are completely determined by the mixing matrices. It is now easy to find relations that are independent of mixing matrices by contracting Eq. (33) and Eq. (35).

### 3.4 Chiral Symmetry Constraints

There is one relation involving decay constants only:

\[ \sum_{i=1}^{m+n} F_{\Pi_i} F_{\Pi_i} = \sum_{i=1}^{m} F_{V_i} F_{V_i}. \]  
(36)

There are three relations which contract one decay constant with one axial coupling:
\[
\sum_{i=1}^{m} F_{V_i} G_{V_i \Pi_k} = F_\pi F_{\Pi_k} \quad (37a)
\]
\[
\sum_{j=1}^{m+n} F_{\Pi_j} G_{V_i \Pi_j} = F_\pi F_{V_i} \quad (37b)
\]
\[
\sum_{j=1}^{m+n} F_{\Pi_j} G_{S_i \Pi_j} = 0. \quad (37c)
\]

There are three relations which contract two axial couplings:

\[
\sum_{j=m+1}^{m+n+o} G_{S_j \Pi_i} G_{S_j \Pi_k} + \sum_{j=1}^{m} G_{V_j \Pi_i} G_{V_j \Pi_k} = F^2_\pi \delta_{ik} \quad (38a)
\]
\[
\sum_{i=1}^{m+n} G_{V_j \Pi_i} G_{V_k \Pi_i} = F^2_\pi \delta_{jk} \quad (38b)
\]
\[
\sum_{i=1}^{m+n} G_{S_j \Pi_i} G_{V_k \Pi_i} = 0. \quad (38c)
\]

These are the basic sum rules. (In an appendix these sum rules are derived directly from the chiral algebra.) They can be further contracted with axial couplings and decay constants to give other sum rules. In compact notation the complete set of sum rules which follow from Eq. (36)-Eq. (38) is:

*Two-point functions (SFSR1):*

\[
\sum_{V} F_{V}^2 - \sum_{A} F_{A}^2 = F_\pi^2 \quad (39)
\]

*Three-point functions (one external current):*

\[
\sum_{V} F_{V} G_{V \pi} = F_\pi^2 \quad (40a)
\]
\[
\sum_{V} F_{V} G_{V A_i}^{(0)} = F_\pi F_{A_i} \quad (40b)
\]
\[
\sum_{V} F_{V} G_{V P_i}^{(0)} = 0 \quad (40c)
\]
\[
\sum_{A} F_{A} G_{V_i A_i}^{(0)} + F_\pi G_{V_i \pi} = F_\pi F_{V_i} \quad (40d)
\]
\[
\sum_{A} F_{A} G_{S_i A_i}^{(0)} + F_\pi G_{S_i \pi} = 0 \quad (40e)
\]
Figure 3: Sum rules for three-point functions with one external current (Eq. (40)). The uppermost sum rule constrains the pion vector form factor.

**Three-point functions (two external currents):**

\[ \sum_{V,A} F_V G_{VA}^{(0)} F_A = F_\pi \sum_{A} F_A^2 \]  \hspace{1cm} (41)

**Four-point functions (no external currents):**

\[ \sum_{s} G_{s\pi}^2 + \sum_{v} G_{v\pi}^2 = F_\pi^2 \]  \hspace{1cm} (42a)

\[ \sum_{s} G_{s\pi} G_{s A_i}^{(0)} + \sum_{v} G_{v\pi} G_{v A_i}^{(0)} = 0 \]  \hspace{1cm} (42b)

\[ \sum_{s} G_{s\pi} G_{s P_i}^{(0)} + \sum_{v} G_{v\pi} G_{v P_i}^{(0)} = 0 \]  \hspace{1cm} (42c)

\[ \sum_{s} G_{s P_i}^{(0)} G_{s A_k}^{(0)} + \sum_{v} G_{v P_i}^{(0)} G_{v A_k}^{(0)} = 0 \]  \hspace{1cm} (42d)

\[ \sum_{s} G_{s A_i}^{(0)} G_{s A_k}^{(0)} + \sum_{v} G_{v A_i}^{(0)} G_{v A_k}^{(0)} = F_\pi^2 \delta_{ik} \]  \hspace{1cm} (42e)

\[ \sum_{s} G_{s P_i}^{(0)} G_{s P_k}^{(0)} + \sum_{v} G_{v P_i}^{(0)} G_{v P_k}^{(0)} = F_\pi^2 \delta_{ik} \]  \hspace{1cm} (42f)
Figure 4: Sum rule for three-point functions with two external currents (Eq. (41)).

\[
G_{V_i\pi}G_{V_k\pi} + \sum_A G_{V_iA}(0)G_{V_kA}(0) + \sum_P G_{V_iP}G_{V_kP} = F_\pi^2 \delta_{ik}
\] (42g)

\[
G_{S_i\pi}G_{V_k\pi} + \sum_A G_{S_iA}(0)G_{V_kA}(0) + \sum_P G_{S_iP}G_{V_kP} = 0
\] (42h)

Four-point functions (one external current):

\[
\sum_{A,S} F_A G_{SA}(0)G_{S\pi} + \sum_{A,V} F_A G_{VA}(0)G_{V\pi} = 0
\] (43a)

\[
\sum_{A,S} F_A G_{SA}(0)G_{SA_i} + \sum_{A,V} F_A G_{VA}(0)G_{VA_i} = F_\pi^2 F_{A_i}
\] (43b)

\[
\sum_{A,S} F_A G_{SA}(0)G_{SP_i} + \sum_{A,V} F_A G_{VA}(0)G_{VP_i} = 0
\] (43c)

\[
\sum_V F_V G_{V\pi}G_{V_i\pi} + \sum_{V,A} F_V G_{VA}(0)G_{V_iA} + \sum_{V,P} F_V G_{VP}(0)G_{V_iP} = F_\pi^2 F_{V_i}
\] (43d)

\[
\sum_V F_V G_{V\pi}G_{S_{i\pi}} + \sum_{V,A} F_V G_{VA}(0)G_{S_{iA}} + \sum_{V,P} F_V G_{VP}(0)G_{S_{iP}} = 0
\] (43e)

Four-point functions (two external currents):

\[
\sum_{A,S,A'} F_A G_{SA}(0)G_{SA'}F_{A'} + \sum_{A,V,A'} F_A G_{VA}(0)G_{VA'}F_{A'} = F_\pi^2 \sum_A F_A^2
\] (44a)

\[
\sum_{V,V'} F_V G_{V\pi}G_{V'\pi}F_{V'} + \sum_{V,A,V'} F_V G_{VA}(0)G_{VA'}F_{V'} + \sum_{V,P,V'} F_V G_{VP}(0)G_{VP'}F_{V'} = F_\pi^2 \sum_V F_V^2.
\] (44b)

We have reintroduced the helicity label to denote those couplings which generally have nonzero helicity components which are not constrained by the pion representation. The first sum rule is the first spectral function sum rule, \textit{SFSR1}. Here we see that there are an infinite number of additional sum rules which are at precisely the same level of theoretical rigor. The new sum rules are illustrated diagrammatically in Fig. 3 (Eq. (40)), Fig. 4 (Eq. (41)), Fig. 5 (Eq. (42)), Fig. 6 (Eq. (43)) and Fig. 7 (Eq. (44)).
3.5 Symmetry Breaking Constraints

Given that the chiral charges annihilate the vacuum at infinite momentum, one may wonder how symmetry breaking manifests itself. Of course it must be present in order to split states within chiral multiplets. At infinite momentum spontaneous symmetry breaking implies

\[ [Q^5_a, \hat{H}^\infty_{QCD}] \neq 0 \] (45)

where \( \hat{H}^\infty_{QCD} \) is the QCD Hamiltonian at infinite momentum. This is simply the statement —familiar to practitioners of light-front field theory— that at infinite momentum, the effects of spontaneous symmetry breaking appear as explicit breaking terms in the Hamiltonian. At infinite momentum the Hamiltonian can be formally expanded as:

\[ \hat{H}^\infty_{QCD} = \sqrt{\hat{P}^2 + \hat{M}^2} = |\hat{P}| + \frac{\hat{M}^2}{2|\hat{P}|} + O(|\hat{P}|^{-3}). \] (46)

Therefore, spontaneous chiral symmetry breaking implies

\[ [Q^5_a, \hat{M}^2] \neq 0 \] (47)

where \( \hat{M}^2 \) is the hadronic mass-squared matrix. The general solution to Eq. (47) is

\[ \hat{M}^2 = \hat{M}^2_{(1,1)} + \sum_{\mathcal{R}} \hat{M}^2_{\mathcal{R}} \] (48)
Figure 6: Sum rules for four-point functions with one external current (Eq. (43)). Dotted lines are pions.

where \((1,1)\) is of course the singlet representation and \(\mathcal{R}\) is a nontrivial \(SU(2) \times SU(2)\) representation. There is no sense in which \(\hat{M}^2_{\mathcal{R}}\) is small. From the allowed representations for the states it is straightforward to show that —without loss of generality— the most general symmetry breaking mass-squared matrix can be written as

\[
\sum_{\mathcal{R}} \hat{M}^2_{\mathcal{R}} = \hat{M}^2_{(2,2)} + \hat{M}^2_{(3,3)},
\]

(49)

The matrix elements of \(\hat{M}^2_{(1,1)}\) between the states of definite chirality are defined as

\[
i \langle \mathbf{V}_a | \hat{M}^2_{(1,1)} | \mathbf{V}_b \rangle_j = i \langle \mathbf{A}_a | \hat{M}^2_{(1,1)} | \mathbf{A}_b \rangle_j \equiv \delta_{ab} m^2_{ij} \quad (50a)
\]

\[
i \langle (2,2)_a | \hat{M}^2_{(1,1)} | (2,2)_b \rangle_j \equiv \delta_{\alpha\beta} n^2_{ij} \quad (50b)
\]

\[
i \langle (1,1) | \hat{M}^2_{(1,1)} | (1,1) \rangle_j \equiv \eta^2_{ij} \quad (50c)
\]

The matrix elements of \(\hat{M}^2_{(2,2)}\) are defined as

\[
i \langle \mathbf{A}_a | \hat{M}^2_{(2,2)} | (2,2)_b \rangle_j = i \langle (2,2)_a | \hat{M}^2_{(2,2)} | \mathbf{A}_b \rangle_j \equiv \delta_{ab} \bar{n}^2_{ij} \quad (51a)
\]
The matrix elements of $\hat{M}^2$ from which we obtain using Eq. (30), Eq. (50), Eq. (51), and Eq. (52):

$$\sum_{A, A'} \hat{M}^2_{AA'} \cdots \hat{M}^2_{AA'} = \sum_{A} \hat{M}^2_{AA}. \quad \sum_{V, V'} \hat{M}^2_{VV'} \cdots \hat{M}^2_{VV'} = \sum_{V} \hat{M}^2_{VV}.$$

**Figure 7**: Sum rules for four-point functions with two external currents (Eq. (44)). Dotted lines are pions.

$$i\langle (1, 1)|\hat{M}^2_{(2,2)}|(2, 2)_{0}\rangle = i\langle (2, 2)_{0}|\hat{M}^2_{(2,2)}|(1, 1)\rangle \equiv \delta_{ij}^2. \quad (51b)$$

The matrix elements of $\hat{M}^2_{(3,3)}$ are defined as

$$-i\langle V_a|\hat{M}^2_{(3,3)}|V_b\rangle_j = i\langle A_a|\hat{M}^2_{(3,3)}|A_b\rangle_j \equiv \delta_{ab}\bar{m}_{ij}. \quad (52)$$

The physical masses of the pion multiplet states are

$$i\langle \Pi_a|\hat{M}^2|\Pi_b\rangle_j = \delta_{ij}\delta_{ab}M_{\Pi_i}^2 \quad (53a)$$

$$i\langle V_a|\hat{M}^2|V_b\rangle_j = \delta_{ij}\delta_{ab}M_{V_i}^2 \quad (53b)$$

$$i\langle S|\hat{M}^2|S\rangle_j = \delta_{ij}M_{S_i}^2, \quad (53c)$$

from which we obtain using Eq. (30), Eq. (50), Eq. (51), and Eq. (52):

$$\delta_{ij}M_{\Pi_i}^2 = \sum_{k=1}^{m} \sum_{l=1}^{m} u_{ik}u_{jl}\{m_{kl}^2 + \bar{m}_{kl}^2\} + \sum_{k=m+1}^{m+n} \sum_{l=m+1}^{m+n} u_{ik}u_{jl}\bar{n}_{kl}^2 + 2\sum_{k=1}^{m} \sum_{l=m+1}^{m+n} u_{ik}u_{jl}\bar{n}_{kl}^2 \quad (54a)$$

$$\delta_{ij}M_{V_i}^2 = \sum_{k=1}^{m} \sum_{l=1}^{m} w_{ik}w_{jl}\{m_{kl}^2 - \bar{m}_{kl}^2\} \quad (54b)$$

$$\delta_{ij}M_{S_i}^2 = \sum_{k=m+1}^{m+n} \sum_{l=m+1}^{m+n} v_{ik}v_{jl}\bar{n}_{kl}^2 + \sum_{k=m+1}^{m+n} \sum_{l=m+1}^{m+n} v_{ik}v_{jl}\bar{\sigma}_{kl}^2 + 2\sum_{k=m+1}^{m+n} \sum_{l=m+1}^{m+n} v_{ik}v_{jl}\bar{\sigma}_{kl}^2. \quad (54c)$$

Contracting with the decay constants of Eq. (33) it is then easy to obtain the sum rule:

$$\sum_{i=1}^{m} \sum_{j=1}^{m} F_{V_i} F_{V_j} \delta_{ij}M_{V_i}^2 - \sum_{i=1}^{m} \sum_{j=1}^{m+n} F_{\Pi_i} F_{\Pi_j} \delta_{ij}M_{\Pi_i}^2 = -2\sum_{k=1}^{m} \sum_{l=1}^{m} F_{k} F_{l}\bar{m}_{kl}^2. \quad (55)$$

We also have the relation
\[ \sum_{i=1}^{n+m} \sum_{j=1}^{m} F_{\Pi_i} F_{\Pi_j i} \langle \Pi_a | \hat{M}_{(3,3)}^2 | \Pi_b \rangle_j = - \sum_{i=1}^{m} \sum_{j=1}^{m} F_{V_i} F_{V_j i} \langle V_a | \hat{M}_{(3,3)}^2 | V_b \rangle_j = \delta_{ab} \sum_{k=1}^{m} \sum_{l=1}^{m} F_k F_l \tilde{m}_{kl}^2. \]  

(56)

In compact notation, combining Eq. (55) and Eq. (56), we have

\[ \sum \Pi V F_{\Pi}^2 M_{\Pi}^2 - \sum A A' F_{A}^2 M_{A}^2 = -2 \sum \Pi \Pi' F_{\Pi} F_{\Pi'} \langle \Pi | \hat{M}_{(3,3)}^2 | \Pi' \rangle. \]  

(57)

Comparing with the OPE in Eq. (13b) gives

\[ \langle O \rangle_{d=4}^{(2,2)} = 2 \sum \Pi \Pi' F_{\Pi} F_{\Pi'} \langle \Pi | \hat{M}_{(3,3)}^2 | \Pi' \rangle. \]  

(58)

Of course this OPE coefficient vanishes in QCD. Therefore \( \hat{M}_{(3,3)}^2 = 0 \) and

\[ \hat{M}^2 = \hat{M}_{(1,1)}^2 + \hat{M}_{(2,2)}^2 \]  

(59)

in large-\( N \) QCD. This chiral decomposition of the mass-squared matrix therefore implies SFSR2,

\[ \sum \Pi V F_{\Pi}^2 M_{\Pi}^2 - \sum A A' F_{A}^2 M_{A}^2 = 0. \]  

(60)

It is interesting that one can use the OPE to constrain the form of the meson mass-squared matrix. This result justifies the assumption of Coleman and Witten that the order parameter of chiral symmetry breaking transforms purely as \( (2, 2) \) in the large-\( N \) limit [6].

It is now straightforward to find new relations that are independent of mixing matrices by contracting Eq. (35) with Eq. (54):

\[ \sum_{j=m+1}^{m+n+o} G_{S_j \Pi_i} G_{S_j \Pi_k} (M_{\Pi_i}^2 + M_{\Pi_k}^2 - 2 M_{S_j}^2) - \sum_{j=1}^{m} G_{V_j \Pi_i} G_{V_j \Pi_k} (M_{\Pi_i}^2 + M_{\Pi_k}^2 - 2 M_{V_j}^2) = 0 \]  

(61a)

\[ \sum_{k=1}^{m+n} G_{V_j \Pi_k} G_{V_j \Pi_k} (M_{V_j}^2 + M_{V_j}^2 - 2 M_{\Pi_k}^2) = 0 \]  

(61b)

\[ \sum_{k=1}^{m+n} G_{S_i \Pi_k} G_{V_j \Pi_k} (M_{V_j}^2 - M_{S_i}^2) = 0. \]  

(61c)
The last sum rule is trivially satisfied via Eq. (38c). (In an appendix these sum rules are derived directly from the chiral algebra.) It is also straightforward to relate the axial couplings and masses to the symmetry breaking parameters. For instance, one finds

$$i \langle \Pi_a | \hat{M}_2^2 | \Pi_b \rangle_j F_\pi^2 = \delta_{ab} \left[ \sum_{l=1}^{m} G_{V,l} G_{V,l} (M_{\Pi_i}^2 + M_{\Pi_j}^2 - 2M_{\Pi_i}^2) \right], \quad (62)$$

a result which we will return to below.

In compact notation the complete set of mass-squared sum rules which follow from Eq. (57) and Eq. (61) are:

**Two-point functions (SFSR2):**

$$\sum_V F_V^2 M_V^2 - \sum_A F_A^2 M_A^2 = 0 \quad (63)$$

**Four-point functions (no external currents):**

$$\sum_S G_S^2 M_S^2 - \sum_V G_{V,S}^2 M_V^2 = 0 \quad (64a)$$

$$\sum_S G_{S,A_i}^2 (M_{A_i}^2 - 2M_S^2) - \sum_V G_{V,S} G_{V,A_i} (M_{A_i}^2 - 2M_V^2) = 0 \quad (64b)$$

$$\sum_S G_{S,P_i}^2 (M_{P_i}^2 - 2M_S^2) - \sum_V G_{V,S} G_{V,P_i} (M_{P_i}^2 - 2M_V^2) = 0 \quad (64c)$$

**Figure 8:** Sum rules for four-point functions with no external currents (Eq. (64)). Dotted lines are pions. Crosses represent mass-squared insertions.
\[
\sum_{A,S} A_a S_s - \sum_{A,V} A_a V_v = 0
\]

\[
\sum_{A,S} A_a S_s - \sum_{A,V} A_a V_v = 0
\]

\[
\sum_{A,S} A_a S_s - \sum_{A,V} A_a V_v = 0
\]

\[
\sum_{V} V_v + \sum_{V,A} V_{vA} = 0
\]

\[\textbf{Figure 9:} \text{ Sum rules for four-point functions with one external current (Eq. (65)). Dotted lines are pions. Crosses represent mass-squared insertions.}\]

\[
\sum_{S} G_{SA}^{(0)} G_{SA}^{(0)} (M_{A_i}^2 + M_{A_k}^2 - 2M_S^2) - \sum_{V} G_{VA}^{(0)} G_{VA}^{(0)} (M_{A_i}^2 + M_{A_k}^2 - 2M_V^2) = 0 \quad (64d) \\
\sum_{S} G_{SP}^{(0)} G_{SP}^{(0)} (M_{P_i}^2 + M_{P_k}^2 - 2M_S^2) - \sum_{V} G_{VP}^{(0)} G_{VP}^{(0)} (M_{P_i}^2 + M_{P_k}^2 - 2M_V^2) = 0 \quad (64e) \\
\sum_{S} G_{SP}^{(0)} G_{SA}^{(0)} (M_{A_i}^2 + M_{A_k}^2 - 2M_S^2) - \sum_{A} G_{VA}^{(0)} G_{VA}^{(0)} (M_{A_i}^2 + M_{A_k}^2 - 2M_A^2) = 0 \quad (64f) \\
G_{V_i\pi} G_{V_k\pi} (M_{V_i}^2 + M_{V_k}^2) + \sum_{A} G_{V_iA}^{(0)} G_{V_kA}^{(0)} (M_{V_i}^2 + M_{V_k}^2 - 2M_A^2) \\
+ \sum_{P} G_{V_iP}^{(0)} G_{V_kP}^{(0)} (M_{V_i}^2 + M_{V_k}^2 - 2M_P^2) = 0 \quad (64g) \\
\]

\[\textbf{Four-point functions (one external current):}\]

\[
\sum_{S,A} G_{SA}^{(0)} F_A (M_A^2 - 2M_S^2) - \sum_{V,A} G_{VA}^{(0)} F_A (M_A^2 - 2M_V^2) = 0 \quad (65a) \\
\sum_{S,A} G_{SA}^{(0)} F_A (M_{A_i}^2 + M_A^2 - 2M_S^2) - \sum_{V,A} G_{VA}^{(0)} F_A (M_{A_i}^2 + M_A^2 - 2M_V^2) = 0 \quad (65b) \\
\sum_{S,A} G_{SP}^{(0)} F_A (M_{P_i}^2 + M_A^2 - 2M_S^2) - \sum_{V,A} G_{VP}^{(0)} F_A (M_{P_i}^2 + M_A^2 - 2M_V^2) = 0 \quad (65c) \\
\sum_{V} G_{V\pi} F_V (M_{V_i}^2 + M_V^2) + \sum_{A,V} G_{V\pi} F_V (M_{V_i}^2 + M_V^2 - 2M_A^2) \\
\]
Figure 10: Sum rules for four-point functions with two external currents (Eq. (66)). Dotted lines are pions. Crosses represent mass-squared insertions.

\[ \sum_{A,S,A'} F_A G_{A,A'} G_{S,S'} F_{A'} (M_A^2 + M_{A'}^2 - 2M_S^2) - \sum_{A,V,A'} F_A G_{V,A} G_{V,A'} F_{A'} (M_A^2 + M_{A'}^2 - 2M_V^2) = 0 \]

\[ \sum_{V,V'} F_V G_{V,V'} F_{V'} (M_V^2 + M_{V'}^2) + \sum_{V,A,V'} F_V G_{V,A} G_{V',A'} F_{V'} (M_V^2 + M_{V'}^2 - 2M_A^2) + \sum_{V,P,V'} F_V G_{V,P} G_{V',P'} F_{V'} (M_V^2 + M_{V'}^2 - 2M_P^2) = 0 \] (66b)

Notice that the first sum rule is the second spectral function sum rule, \(SFSR2\). Here again we see that there are an infinite number of additional sum rules which are at precisely the same level of theoretical rigor. The new sum rules are illustrated diagramatically in Fig. 8 (Eq. (64)), Fig. 9 (Eq. (65)) and Fig. 10 (Eq. (66)). (Many additional sum rules can be constructed by contracting the meson masses, decay constants and axial couplings.)

4. Large-\(N\) Asymptotics

Of the chiral sum rules derived, there are only five which involve decay constants and axial couplings with only \(\lambda = 0\) components. They are:

\[ \sum_V F_V^2 - \sum_A F_A^2 = F_\pi^2 \] (67a)

\[ \sum_V F_V M_V^2 - \sum_A F_A^2 M_A^2 = 0 \] (67b)
\[
\sum_{V} F_{V} G_{V \pi} = F_{\pi}^{2} \quad (67c)
\]
\[
\sum_{S} G_{S \pi}^{2} + \sum_{V} G_{V \pi}^{2} = F_{\pi}^{2} \quad (67d)
\]
\[
\sum_{S} G_{S \pi}^{2} M_{S}^{2} - \sum_{V} G_{V \pi}^{2} M_{V}^{2} = 0. \quad (67e)
\]

We have seen that the SFSR’s, Eq. (67a) and Eq. (67b), determine the asymptotic behavior of the two-point function \( \Pi_{LR} \) and in turn allow the construction of the full correlation function in the large-\( N \) limit. In this section, the correlators relevant to the other three sum rules will be identified and constructed using dispersion relations in precisely the same manner.

4.1 The Pion Vector Form Factor

Consider first a familiar product of three currents. The vector form factor of the pion is defined by

\[
\langle \pi^{a} p | V_{\mu}^{b} | \pi^{c} p \rangle = \epsilon^{abc} (p_{\mu} + p_{\mu}) F_{V} (q^{2}). \quad (68)
\]

Charge conservation implies \( F_{V} (0) = 1 \). The pion form factor is an analytic function in the complex-\( t \) plane with a cut extending along the real axis from the origin (the \( t \)-channel \( \pi \pi \) threshold in the chiral limit). Therefore, \( F_{V} (t) \) satisfies a dispersion relation. Say the asymptotic behavior of \( F_{V} (t) \) is given by:

\[
F_{V} (t) \xrightarrow{t \to \infty} t^{m}, \quad (69)
\]

where the integer \( m \) determines the number of subtractions. It is possible to show in QCD that \( m \leq 0 \) [13]. Hence there is at most one subtraction. With one subtraction at the origin and making use of \( F_{V} (0) = 1 \) we have:

\[
F_{V} (t) = 1 + \frac{t}{\pi} \int \frac{Im F_{V} (t') dt'}{t' (t' - t - i\epsilon)}. \quad (70)
\]

In the large-\( N \) limit, the form factor is determined by an infinite number of meson exchanges between the vector current and the \( \pi \pi \) channel. Therefore, the absorptive part may be written as the formal sum over states:

\[
Im F_{V} (t) = \sum_{V} \frac{F_{V} G_{V \pi} t \pi}{F_{\pi}^{2}} \delta (t - M_{V}^{2}) \quad (71)
\]

which gives

\[
F_{V} (t) = 1 + \sum_{V} \frac{F_{V} G_{V \pi}}{F_{\pi}^{2}} \frac{t}{(M_{V}^{2} - t)}. \quad (72)
\]
It is convenient to expand this form factor in inverse powers of $t$:

\[
F_V(t) \xrightarrow{t \to \infty} \frac{1}{F_\pi^2} \left\{ F_\pi^2 - \sum_V F_V G_{V\pi} \right\} - \frac{1}{F_\pi^2} \sum_{n=1}^{\infty} \left\{ \sum_V F_V G_{V\pi} M_V^{2n} \right\} \frac{1}{t^n}.
\] (73)

But we have derived the exact relation

\[
\sum_V F_V G_{V\pi} = F_\pi^2.
\] (74)

One can further verify directly from Eq. (33), Eq. (35) and Eq. (54b) that generally there is no inverse power, $n$, of $t$ for which

\[
\sum_V F_V G_{V\pi} M_V^{2n}
\] (75)

vanishes. Hence $m = -1$ and the asymptotic behavior of the pion form factor is determined by chiral symmetry in the large-$N$ limit. The pion form factor satisfies an unsubtracted dispersion relation and is given by:

\[
F_V(t) = \sum_V \frac{F_V G_{V\pi}}{F_\pi^2} \frac{M_V^2}{(M_V^2 - t)}.
\] (76)

subject to the normalization condition $F_V(0) = 1$. We emphasize that this is the exact form factor in the large-$N$ limit\(^4\).

4.2 Pion-pion Scattering

Next we consider correlation functions which are the time-ordered product of four currents. We will focus on the forward $\pi - \pi$ scattering amplitudes in a basis of definite $t$-channel isospin as a function of the crossing-odd variable $\nu = s - u$. The crossing-odd amplitude is pure $I_t = 1$ and the crossing-even amplitude contains $I_t = 0$ and $I_t = 2$. The analytic structure of the $\pi - \pi$ scattering amplitudes is well known\(^{[15]}\). Leading order in chiral perturbation theory\(^ {[16]}\) determines the small-$\nu$ behavior at $t = 0$:

\[
T_1^1(\nu, 0) = \frac{\nu}{F_\pi^2},
\] (77a)

\[
T_1^0(\nu, 0) = T_1^2(\nu, 0) = 0.
\] (77b)

Say the asymptotic behavior is given by

\(^4\) This form factor was studied in the context of the large-$N$ approximation in Ref. 14. There the asymptotic constraint, Eq. (74), was assumed rather than derived.
\[ T^1_t(\nu, 0) \rightarrow \nu^{2m_1 + 1} \]  
\[ T^2_t(\nu, 0) \rightarrow \nu^{2m_2} \]  
\[ T^0_t(\nu, 0) \rightarrow \nu^{2m_0} \]  

where the integers \( m_i \) determine the number of subtractions. The Froissart bound implies the upper bound \( m_i \leq 0 \) [17].

We will consider first the \( I_t = 1 \) amplitude, which is crossing-odd. Saturating the Froissart bound (assuming the most severe asymptotic behavior) we make one subtraction at \( \nu_0 = 0 \) which yields

\[
\frac{T^1_t(\nu, 0)}{\nu} = \lim_{\nu_0 \to 0} \frac{T^1_t(\nu_0, 0)}{\nu_0} + \frac{2\nu^2}{\pi} \int_0^\infty d\nu' \frac{\text{Im } T^1_t(\nu', 0)}{\nu'^2(\nu'^2 - \nu^2)}. \tag{79}
\]

Using Eq. (77a) and going to a basis of s-channel isospin gives

\[
T^1_t(\nu, 0) = \frac{\nu}{F^2_\pi} + \frac{\nu^3}{F^4_\pi} \int_0^\infty d\nu' \left[ \frac{2}{3} \text{Im } T^0_s(\nu', 0) + \text{Im } T^1_s(\nu', 0) - \frac{2}{3} \text{Im } T^2_s(\nu', 0) \right] \frac{\nu'^2(\nu'^2 - \nu^2)}{\nu'^2(\nu'^2 - \nu^2)}. \tag{80}
\]

In the large-\( N \) limit we can write the absorptive parts as formal sums over single-particle states:

\[
\text{Im } T^0_s(\nu, 0) = \sum_v \frac{4\pi M_v^4 G_{v\pi}^2}{F^4_\pi} \delta(\nu - 2M_v^2) \tag{81a}
\]
\[
\text{Im } T^0_s(\nu, 0) = \sum_s \frac{6\pi M_s^4 G_{s\pi}^2}{F^4_\pi} \delta(\nu - 2M_s^2). \tag{81b}
\]

Of course \( \text{Im } T^2_s \) vanishes since there are no \( I = 2 \) mesons in the large-\( N \) limit [2]. It immediately follows that

\[
T^1_t(\nu, 0) = \frac{\nu}{F^2_\pi} + \frac{\nu^3}{F^4_\pi} \left[ \sum_v \frac{G_{v\pi}^2}{4M_v^4 - \nu^2} + \sum_s \frac{G_{s\pi}^2}{4M_s^4 - \nu^2} \right]. \tag{82}
\]

Expanding in inverse powers of \( \nu \) gives:

\[
T^1_t(\nu, 0) \rightarrow \frac{1}{F^4_\pi} \left\{ \sum_v \frac{G_{v\pi}^2}{4M_v^4 - \nu^2} - \sum_s \frac{G_{s\pi}^2}{4M_s^4 - \nu^2} \right\} \nu - \frac{1}{F^4_\pi} \sum_{n=1}^\infty \left( \sum_v G_{v\pi}^2 (4M_v^4)^n + \sum_s G_{s\pi}^2 (4M_s^4)^n \right) \frac{1}{\nu^{2n-1}}. \tag{83}
\]
But we have derived the exact relation

\[
\sum V_{V\pi}^2 + \sum S_{S\pi}^2 = F_\pi^2. \tag{84}
\]

Moreover, the coefficients of all inverse powers of \(\nu\) are positive definite. Hence it is clear that \(m_1 = -1\). Therefore, chiral symmetry determines that \(T^1_t(\nu, 0)\) satisfies an unsubtracted dispersion relation which yields

\[
T^1_t(\nu, 0) = \frac{4\nu}{F_\pi^4} \left[ \sum V_{V\pi}^2 M_V^4 + \sum S_{S\pi}^2 M_S^4 \right] \tag{85}
\]

subject to the normalization condition \(\lim_{\nu_0 \to 0} T^1_t(\nu_0, 0)/\nu_0 = 1/F_\pi^2\). This condition (Eq. (84)) is sometimes expressed via the optical theorem as

\[
\frac{1}{F_\pi^2} = \frac{1}{\pi} \int \frac{d\nu}{\nu} \left[ \frac{1}{3} \sigma_\pi^0(\nu) + \frac{1}{2} \sigma_\pi^1(\nu) - \frac{4}{5} \sigma_\pi^2(\nu) \right] \tag{86}
\]

which is the Adler-Weisberger sum rule for \(\pi - \pi\) scattering\(^5\). Next we turn to the \(I_t = 2\) amplitude. The \(I_t = 0\) and \(I_t = 2\) amplitudes are crossing even. With one subtraction at \(\nu_0 = 0\) we have

\[
T^2_t(\nu, 0) = \lim_{\nu_0 \to 0} T^2_t(\nu_0, 0) + \frac{4\nu^2}{F_\pi^4} \int_0^\infty d\nu' \frac{\text{Im} T^2_s(\nu', 0)}{\nu' (\nu'^2 - \nu^2)}. \tag{87}
\]

Using Eq. (77b) yields

\[
T^2_t(\nu, 0) = \frac{\nu^2}{\pi} \int_0^\infty d\nu' \frac{\text{Im} T^0_s(\nu', 0) - \text{Im} T^1_s(\nu', 0) + \frac{1}{2} \text{Im} T^2_s(\nu', 0)}{\nu' (\nu'^2 - \nu^2)}, \tag{88}
\]

in a basis of s-channel isospin. Saturating with the sum over states gives

\[
T^2_t(\nu, 0) = -\frac{2\nu^2}{F_\pi^4} \left[ \sum V_{V\pi}^2 M_V^2 - \sum S_{S\pi}^2 M_S^2 \right]. \tag{89}
\]

Expanding in inverse powers of \(\nu\) yields:

\[
T^2_t(\nu, 0) \xrightarrow{\nu \to \infty} \frac{2}{F_\pi^4} \left\{ \sum V_{V\pi}^2 M_V^2 - \sum S_{S\pi}^2 M_S^2 \right\} + \frac{1}{F_\pi^4} \sum_{n=1}^\infty \left\{ \sum V_{V\pi}^2 (2M_V^2)^{2n+1} - \sum S_{S\pi}^2 (2M_S^2)^{2n+1} \right\} \frac{1}{\nu^{2n}}. \tag{90}
\]

But we have derived the exact relation

\(^5\) In effect, the sum rules of Eq. (42) comprise the complete set of Adler-Weisberger sum rules for \(\pi\)-meson scattering.
\[ \sum_{V} G_{V \pi V}^2 M_V^2 = \sum_{S} G_{S \pi S}^2 M_S^2. \]  

(91)

One can further verify directly from Eq. (35) and Eq. (54) that generally there is no power \( n \) for which

\[ \sum_{V} G_{V \pi V}^2 M_V^{4n+2} = \sum_{S} G_{S \pi S}^2 M_S^{4n+2}. \]  

(92)

It is clear that \( m_2 = -1 \) and chiral symmetry determines that \( T^0_l(\nu, 0) \) satisfies an unsubtracted dispersion relation which yields

\[ T^0_l(\nu, 0) = -\frac{8}{F_{\pi}^4} \left[ \sum_{V} \frac{G_{V \pi V}^2 M_V^6}{4 M_V^4 - \nu^2} - \sum_{S} \frac{G_{S \pi S}^2 M_S^6}{4 M_S^4 - \nu^2} \right]. \]  

(93)

subject to the normalization condition \( \lim_{\nu_0 \to 0} T^0_l(\nu_0, 0) = 0 \). This condition (Eq. (91)) is sometimes expressed via the optical theorem as the superconvergent sum rule:

\[ 0 = \int d\nu \left[ \frac{1}{3} \sigma_0^s(\nu) - \frac{1}{2} \sigma_1^s(\nu) + \frac{1}{6} \sigma_2^s(\nu) \right]. \]  

(94)

Next we turn to the \( I_t = 0 \) amplitude. With one subtraction at \( \nu_0 = 0 \) we have

\[ T^0_l(\nu, 0) = \lim_{\nu_0 \to 0} T^0_l(\nu_0, 0) + \frac{2\nu^2}{\pi} \int_{0}^{\infty} \frac{d\nu' \text{Im } T^0_l(\nu', 0)}{\nu'(\nu'^2 - \nu^2)}. \]  

(95)

Again using Eq. (77b) gives

\[ T^0_l(\nu, 0) = \frac{\nu^2}{\pi} \int_{0}^{\infty} \frac{d\nu' \left[ \frac{2}{3} \text{Im } T^0_s(\nu', 0) + 2 \text{Im } T^1_s(\nu', 0) + \frac{10}{3} \text{Im } T^2_s(\nu', 0) \right]}{\nu'(\nu'^2 - \nu^2)}. \]  

(96)

in a basis of s-channel isospin. Saturating with the sum over states gives

\[ T^0_l(\nu, 0) = \frac{2\nu^2}{F_{\pi}^4} \left[ 2 \sum_{V} \frac{G_{V \pi V}^2 M_V^2}{4 M_V^4 - \nu^2} + \sum_{S} \frac{G_{S \pi S}^2 M_S^2}{4 M_S^4 - \nu^2} \right]. \]  

(97)

Expanding in inverse powers of \( \nu \) yields:

\[ T^0_l(\nu, 0) \underset{\nu \to \infty}{\longrightarrow} -\frac{2}{F_{\pi}^4} \left\{ 2 \sum_{V} G_{V \pi V}^2 M_V^2 + \sum_{S} G_{S \pi S}^2 M_S^2 \right\} \]

\[ -\frac{1}{F_{\pi}^4} \sum_{n=1}^{\infty} \left\{ 2 \sum_{V} G_{V \pi V}^2 (2M_V^2)^{2n+1} + \sum_{S} G_{S \pi S}^2 (2M_S^2)^{2n+1} \right\} \frac{1}{\nu^{2n}}. \]  

(98)

Clearly chiral symmetry does not constrain the asymptotic behavior and so \( m_0 = 0 \) (maximal strength consistent with the Froissart bound). However, chiral symmetry gives an
interesting interpretation to the leading coefficient of the $I_t = 0$ amplitude in the $1/\nu$ expansion. Using Eq. (62), Eq. (91) and Eq. (98) gives the remarkable equation

$$T_0^0(\nu, 0) \xrightarrow{\nu \to \infty} \frac{1}{F_\pi^2} \langle \pi_a | \hat{M}_{(2,2)}^2 | \pi_b \rangle \, \delta_{ab} + O(\nu^{-2}).$$

(99)

Since the $I_t = 1$ and $I_t = 2$ amplitudes fall off at large $\nu$, it follows that the asymptotic behavior of the total cross-section is determined by the symmetry breaking part of the mass-squared matrix\(^6\). This result was known long ago [9,18]. Here it is seen to be exact in the large-$N$ limit.

### 4.3 Summary

We have used the relations,

1. \[ \sum_V F_V G_{V\pi} = F_{\pi}^2 \] \hspace{1cm} (100a)
2. \[ \sum_s G_{s\pi}^2 + \sum_V G_{V\pi}^2 = F_{\pi}^2 \] \hspace{1cm} (100b)
3. \[ \sum_V G_{V\pi}^2 M_V^2 = \sum_s G_{s\pi}^2 M_S^2 \] \hspace{1cm} (100c)

which were derived from the pion chiral representation (and the chiral algebra in an appendix) to determine the asymptotic behavior of the pion vector form factor and the forward $\pi - \pi$ scattering amplitudes, respectively:

1. \[ F_V(t) \xrightarrow{t \to \infty} t^{-1} \] \hspace{1cm} (101a)
2. \[ T_1^1(\nu, 0) \xrightarrow{\nu \to \infty} \nu^{-1} \] \hspace{1cm} (101b)
3. \[ T_2^2(\nu, 0) \xrightarrow{\nu \to \infty} \nu^{-2} \] \hspace{1cm} (101c)
4. \[ T_0^0(\nu, 0) \xrightarrow{\nu \to \infty} \nu^0. \] \hspace{1cm} (101d)

Using dispersion theory the precise large-$N$ form factor and forward scattering amplitudes were constructed:

1. \[ F_V(t) = \sum_V \frac{F_V G_{V\pi}}{F_{\pi}^2} \frac{M_V^2}{(M_V^2 - t)}; \hspace{1cm} F_V(0) = 1 \] \hspace{1cm} (102a)
2. \[ T_1^1(\nu, 0) = \frac{4
u}{F_{\pi}^4} \left[ \sum_V G_{V\pi} M_V^4 + \sum_s G_{s\pi} M_S^4 \right]; \hspace{1cm} \lim_{\nu_0 \to 0} \frac{T_1^1(\nu_0, 0)}{\nu_0} = \frac{1}{F_{\pi}^2} \] \hspace{1cm} (102b)

---

\(^{6}\) Attempts to construct $\hat{M}_{(2,2)}^2$ explicitly in QCD are discussed in the fascinating papers of Ref. 18 (see also Ref. 19).
\[ T_i^2(\nu, 0) = -\frac{8}{F^4_\pi} \left[ \sum_v G_{v\tau}^2 M_v^6 \frac{\nu}{4M_v^4 - \nu^2} - \sum_s G_{s\pi}^2 M_s^6 \frac{\nu}{4M_s^4 - \nu^2} \right]; \quad \lim_{\nu_0 \to 0} T_i^2(\nu_0, 0) = 0. \quad (102c) \]

\[ T_i^0(\nu, 0) = \frac{2\nu^2}{F^4_\pi} \left[ 2 \sum_v G_{v\tau}^2 M_v^2 \frac{\nu}{4M_v^4 - \nu^2} + \sum_s G_{s\pi}^2 M_s^2 \frac{\nu}{4M_s^4 - \nu^2} \right]. \quad (102d) \]

In similar fashion one can derive the asymptotic behavior of and construct other form factors and forward scattering amplitudes from the remaining sum rules. In order to make contact between the results of this paper and phenomenology, the infinite space of meson states must be truncated to a few low-lying states. The constraints implied by finite dimensional saturation schemes on the low-energy constants of chiral perturbation theory are discussed elsewhere. A phenomenological study of the form factors and forward scattering amplitudes in turn requires going beyond the zero-width form of the large-\(N\) correlators which necessitates subsuming higher order effects in the \(1/N\) expansion. Considerations similar to those made here lead to sum rules involving other nonconserved currents (e.g. scalar and pseudoscalar currents). Chiral symmetry constraints at infinite momentum for the ground state large-\(N\) baryons have been studied in Ref. 21 and Ref. 22. Analogs of the SFSR’s for heavy mesons have been studied in the Large-\(N\) limit in Ref. 23.

5. Conclusion

The remarkable fact that operator product expansion coefficients do not feel vacuum properties implies that the full global symmetry of an underlying theory can have profound algebraic relevance in the low-energy theory even if the full global symmetry is spontaneously broken. The spectral function sum rules are classic examples of this fact in QCD. Traditionally these sum rules are derived using the OPE. In this paper I rederived the SFSR’s using chiral symmetry directly, both from the representation theory and from the chiral algebra (see appendix). The advantage of the symmetry method is that all consequences of chiral symmetry can be derived at once for arbitrary \(n\)-point functions. Hence using this method I derived an infinite number of new sum rules for 3- and 4-point functions in the large-\(N\) limit. These sum rules are at precisely the same level of rigor as the SFSR’s. The sum rules constrain the asymptotic behavior of correlation functions and thereby allow the derivation of exact large-\(N\) correlators using dispersion relations. It is somewhat surprising that there are fundamental constraints on the asymptotic behavior of scattering amplitudes which go beyond what is implied by unitarity (via the Froissart bound). Of particular interest is the interplay between the asymptotic behavior of the total cross-section and the symmetry breaking part of the low-energy Hamiltonian or mass-squared matrix.

Generally the order parameter of chiral symmetry breaking in large-\(N\) QCD transforms as a sum of \((2, 2)\) and \((3, 3)\) representations. However, I found that the \((3, 3)\) part of the
The low-energy Hamiltonian is proportional to an OPE coefficient which vanishes in QCD. This constitutes a proof that the order parameter of chiral symmetry breaking in large-$N$ QCD transforms purely as $(2, 2)$, a result which is consistent with what is assumed in the Coleman-Witten proof of chiral symmetry breaking at large-$N$ [6].

The results of this paper demonstrate the fundamental importance of the infinite momentum frame in exploiting chiral symmetry constraints in QCD. Given the deep Euclidean nature of the OPE it is not surprising that the infinite momentum frame is relevant in extracting consequences of the full chiral group in the low-energy theory. It would be interesting to verify some of the new sum rules for 3- and 4-point functions using the OPE method of Ref. 5.

**Acknowledgements**

This work was supported by the U.S. Department of Energy grant DE-FG02-93ER-40762 (at Maryland) and DE-FG03-97ER-41014 (at Washington). I thank Markus Luty for valuable conversations.

**Appendix**

In this appendix I give an algebraic derivation of the basic sum rules, Eq. (36) -Eq. (38) and Eq. (61). Consider the commutators involving the chiral charges and the vector and axialvector currents:

\[
[Q_5^a, V_\mu^b] = i\epsilon^{abc} A_\mu^c \tag{103a}
\]

\[
[Q_5^a, A_\mu^b] = i\epsilon^{abc} V_\mu^c \tag{103b}
\]

\[
[Q_5^a, Q_5^b] = i\epsilon^{abc} T_\mu^c. \tag{103c}
\]

We can take matrix elements of these commutators between physical states at infinite momentum. Consider, for instance,

\[
\langle 0| [Q_5^a, V_\mu^b]|\Pi^c\rangle_i = i\epsilon^{abc} \langle 0| A_\mu^c|\Pi^c\rangle_i. \tag{104}
\]

Using $Q_5^a|0\rangle = 0$ and inserting a complete set of states gives

\[
-\sum_{j=1}^{m} \langle 0| V_\mu^b |V_j^f\rangle_{j,j} \langle V_j^f| Q_5^a |\Pi^c\rangle_i = i\epsilon^{abc} \langle 0| A_\mu^c|\Pi^c\rangle_i. \tag{105}
\]

Here we have also used helicity conservation. Using Eq. (19) and Eq. (34) we then have

\[
-\sum_{j=1}^{m} (\delta^{bf} F_{V_j p_\mu})(i\epsilon^{ace} G_{V_j p_\Pi}/F_\pi) = i\epsilon^{abc} (\delta^{ce} F_{\Pi p_\mu}) \tag{106}
\]
from which immediately follows
\[ \sum_{j=1}^m F_{V_j} G_{V_j\Pi_i} = F_\pi F_{\Pi_k}. \]  
(107)

Similarly, we have
\[ \langle 0|[Q_5^a, A_\mu]|V^c\rangle_i = i\epsilon^{abc}\langle 0|V_\mu^c|V^c\rangle_i, \]  
(108a)
\[ \langle 0|[Q_5^a, A_\mu]|S_i\rangle_i = 0 \]  
(108b)

which lead to
\[ \sum_{j=1}^{m+n} F_{\Pi_j} G_{V_i\Pi_j} = F_\pi F_{V_i} \]  
(109a)
\[ \sum_{j=1}^{m+n} F_{\Pi_j} G_{S_i\Pi_j} = 0. \]  
(109b)

Now contracting Eq. (107) with \( F_{\Pi_k} \) and Eq. (109a) with \( F_{V_i} \) implies
\[ \sum_{i=1}^{m+n} F_{\Pi_i} F_{V_i} = \sum_{i=1}^m F_{V_i} F_{V_i}. \]  
(110)

Similarly we can take matrix elements of the commutator, Eq. (103c). Consider, for instance,
\[ j\langle V_e|[Q_5^a, Q_5^b]|V_d\rangle_i = i\epsilon_{abc} j\langle V_e|T_c|V_d\rangle_i. \]  
(111)

Inserting a complete set of states gives
\[ \sum_{k=1}^{m+n} j\langle V_e|Q_5^a\Pi_l\rangle_k k\langle \Pi_l|Q_5^b|V_d\rangle_i - (a \leftrightarrow b) = i\epsilon_{abc} j\langle V_e|T_c|V_d\rangle_i, \]  
(112)
and using Eq. (24) and Eq. (34) we obtain
\[ \sum_{k=1}^{m+n} (i\epsilon_{alc} G_{V_j\Pi_k}/F_\pi)(i\epsilon_{lbd} G_{V_k\Pi_k}/F_\pi) - (a \leftrightarrow b) = i\epsilon_{abc}(i\delta_{ij}\epsilon_{ecd}). \]  
(113)

Finally a bit of algebra gives
\[ \sum_{i=1}^{m+n} G_{V_j\Pi_i} G_{V_k\Pi_i} = F^2_\pi \delta_{jk}. \]  
(114)
Similarly, we have

\[ j\langle \Pi_c|[Q^5_a, Q^5_b]|\Pi_d\rangle_i = i\epsilon_{abc} j\langle \Pi_c|T_c|\Pi_d\rangle_i \]  
\tag{115a} \]

\[ j\langle S|[Q^5_a, Q^5_b]|V_d\rangle_i = 0 \]  
\tag{115b} \]

which lead to

\[ \sum_{j=m+1}^{m+n+o} G_{S_jn_i}G_{S_jn_k} + \sum_{j=1}^{m} G_{V_jn_i}G_{V_jn_k} = F^2\pi\delta_{ik} \]  
\tag{116a} \]

\[ \sum_{i=1}^{m+n} G_{S_jn_i}G_{V_kn_i} = 0. \]  
\tag{116b} \]

Hence we have reproduced Eq. (36)-Eq. (38) from Eq. (103).

Taking $\hat{M}^2_{(2,2)}$ to transform as the zeroth component of the $(2, 2)$ representation implies

\[ [Q^5_a, \hat{M}^2_{(2,2)a}] = i\hat{M}^2_{(2,2)a} \]  
\tag{117} \]

and

\[ [Q^5_a, \hat{M}^2_{(2,2)b}] = -i\delta_{ab}\hat{M}^2_{(2,2)a}. \]  
\tag{118} \]

Since

\[ [Q^5_a, \hat{M}^2_{(1,1)}] = 0 \]  
\tag{119} \]

we can write

\[ [Q^5_a, [Q^5_b, \hat{M}^2]] = \delta_{ab}\hat{M}^2_{(2,2)}. \]  
\tag{120} \]

Taking matrix elements of this commutation relation with the physical states of the pion multiplet implies nontrivial sum rules. From

\[ j\langle \Pi_c|[Q^5_a, [Q^5_b, \hat{M}^2]]|\Pi_d\rangle_i \propto \delta_{ab} \]  
\tag{121a} \]

\[ j\langle V_c|[Q^5_a, [Q^5_b, \hat{M}^2]]|V_d\rangle_i = 0 \]  
\tag{121b} \]

\[ j\langle S|[Q^5_a, [Q^5_b, \hat{M}^2]]|S\rangle_i \propto \delta_{ab} \]  
\tag{121c} \]

\[ j\langle S|[Q^5_a, [Q^5_b, \hat{M}^2]]|V_d\rangle_i = 0 \]  
\tag{121d} \]

follow
\[
\sum_{j=m+1}^{m+n+o} G_{S_{j \Pi_i}} G_{S_{j \Pi_k}} (M_{\Pi_i}^2 + M_{\Pi_k}^2 - 2M_{S_j}^2) \\
- \sum_{j=1}^{m} G_{V_{j \Pi_i}} G_{V_{j \Pi_k}} (M_{\Pi_i}^2 + M_{\Pi_k}^2 - 2M_{V_j}^2) = 0 \tag{122a}
\]
\[
\sum_{k=1}^{m+n} G_{V_{i \Pi_k}} G_{V_{i \Pi_k}} (M_{V_i}^2 - M_{V_j}^2) = 0 \tag{122b}
\]
\[
\sum_{k=1}^{m+n} G_{S_{i \Pi_k}} G_{V_{j \Pi_k}} (M_{V_j}^2 - M_{S_i}^2) = 0. \tag{122c}
\]

which are the sum rules of Eq. (61).

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