TRAVELING WAVES IN A CHEMOTAXIS MODEL WITH LOGISTIC GROWTH

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(Communicated by Zhian Wang)

Abstract. Traveling wave solutions of a chemotaxis model with a reaction term are studied. We investigate the existence and non-existence of traveling wave solutions in certain ranges of parameters. Particularly for a positive rate of chemical growth, we prove the existence of a heteroclinic orbit by constructing a positively invariant set in the three dimensional space. The monotonicity of traveling waves is also analyzed in terms of chemotaxis, reaction and diffusion parameters. Finally, the traveling wave solutions are shown to be linearly unstable.

1. Introduction. In this paper, we are interested in traveling wave solutions of the following Keller-Segel type model with a reaction

\[
\begin{align*}
    u_t &= Du_{xx} - (u\chi(v)v_x)_x + \mu u(1 - u) \\
    v_t &= \alpha u + \beta v
\end{align*}
\]

for \(x \in \mathbb{R}\) and \(t \geq 0\). Here, \(u(x, t) \geq 0\) and \(v(x, t) \geq 0\) denote the density of cell population and the chemical concentration. Constant \(D > 0\) is called the cell diffusion coefficient, and \(\mu > 0\) represents the rate of logistic cell growth. The chemosensitivity \(\chi(v) \geq 0\) describes the signal detection mechanism and measures the strength of the chemical signal. It is assumed to be \(C^1\) throughout this paper. In the second equation of (1), \(\alpha < 0\) means that the cells consume the chemical, and \(\alpha > 0\) describes that the cells secrete the chemical themselves. Constant \(\beta\) corresponds to the chemical growth rate if \(\beta > 0\), degradation rate if \(\beta < 0\) and no rate if \(\beta = 0\).

The classical chemotaxis model was proposed by Keller and Segel [11] in the 1970s to describe the aggregation of cellular slime molds \textit{Dictyostelium discoideum} in response to the chemical cyclic adenosine monophosphate (cAMP). Especially chemotaxis plays a critical role to account for a coherent pattern in biology [1, 2], so the investigation of traveling wave solutions is imperative to describe propagating patterns in studying a chemotaxis model. Traveling waves of the Keller-Segel model

2010 Mathematics Subject Classification. Primary: 35B35, 35C07, 35K57, 92C17; Secondary: 35K55.

Key words and phrases. Reaction-diffusion-chemotaxis, cell growth, traveling waves, linear instability.

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have been extensively studied by many other researchers; see [3, 4, 7, 8, 9, 12, 15, 16, 17, 18, 19, 20, 22, 23, 24] and the references therein.

In particular, the existence and non-existence of traveling waves to the Keller-Segel type models with a reaction have been developed in papers by various authors. Authors in [3, 4, 8, 9, 16, 22] studied the Keller-Segel models involving a logistic source with the logarithmic function $\chi(v) = 1/v$, which is singular at $v = 0$. Authors in [17, 19, 20] considered a constant sensitivity function $\chi(v) = \chi_0 > 0$. Particularly, Salako and Shen [19, 20] obtained results for $\chi(v) = \chi_0 > 0$ when $\alpha > 0$ and $\beta < 0$ in (1). However, traveling waves of (1) with a non-constant $C^1$ sensitivity function have not been studied yet. In fact, there are non-constant $C^1$ sensitivity functions $\chi(v)$ that are commonly used such as $\chi_0/(1 + v)^2$ (receptor law) with $\chi_0 > 0$. We also refer to [7] for more examples of $C^1$ sensitivity functions. The aim of this paper is to investigate the existence and non-existence of traveling wave solutions of the system (1) with a general $C^1$ sensitivity function $\chi(v) \geq 0$.

To prove the existence and non-existence of the traveling wave solutions, we apply the dynamical systems theory. Particularly, to show the existence of traveling wave solutions of the system (1) with a $C^1$ sensitivity function $\chi(v) \geq 0$, $\alpha < 0$ and $\beta > 0$, we construct a positively invariant set in the three dimensional phase space and obtain a heteroclinic orbit connecting two steady states. This technique is inspired by [3], where the authors studied traveling wave solutions to a Keller-Segel type model with the logarithmic sensitivity $\chi(v) = 1/v$ whose singularity was removed by the Hopf-Cole transformation. In dealing with a general $C^1$ sensitivity function $\chi(v) \geq 0$, we work on the system (1) directly to build a positively invariant set. The structures of our invariant sets help us understand the relation between the cell and chemical densities, and describe the properties of the traveling wave profiles. Furthermore, by analyzing the interaction of chemotaxis, diffusion and reaction processes, we obtain conditions for the traveling wave solutions to be monotone.

Our main theorem for the existence of traveling waves is for the case $\alpha < 0$ and $\beta > 0$, which biologically indicates that the chemoattractant such as food and oxygen is consumed by cells to balance the exponential chemical growth. For simplicity, we fix $\alpha = -1$ throughout this paper.

**Theorem 1.1.** Let $D, \mu, \beta > 0$ and $\chi(v) \geq 0$ be a $C^1$ function. Then, there is a minimal speed $s^* > 0$ such that if $s \geq s^*$, the system (1) has a traveling wave solution $(u(x,t), v(x,t)) = (U(\xi), V(\xi))$, where $\xi := x - st$, satisfying

\[
\begin{cases}
0 < U(\xi) \leq \beta V(\xi) < 1 & \text{and} \quad V'(\xi) < 0 \quad \text{for any} \quad \xi \in (-\infty, \infty), \\
(U(-\infty), V(-\infty)) = (1, \frac{1}{\beta}) & \text{and} \quad (U(\infty), V(\infty)) = (0, 0),
\end{cases}
\]

and the traveling wave solution is unique up to a translation in $\xi$.

In particular, if $0 < s^* \leq \sqrt{\frac{D}{\beta}}$, for any $s \geq s^*$, $U$ is monotonically decreasing, namely, $U'(\xi) < 0$ holds for any $\xi \in (-\infty, \infty)$.

**Remark 1.** The condition of $0 < s^* \leq \sqrt{\frac{D}{\beta}}$ in Theorem 1.1 is sufficient to have a monotonically decreasing $U(\xi)$, $\xi = x - st$, for any $s \geq s^*$. It is verified by the construction of a positive invariant set for the proof of Theorem 1.1 and by Lemma 5.3.

Theorem 1.1 implies that there are infinitely many traveling wave solutions for given two far field states. The existence of infinitely many solutions is attributed to the effect of the logistic cell growth term in (1). Moreover, the logistic growth
μ > 0 prevents a chemotactic collapse and allows pattern formations in cells. It is verified that there is no traveling wave solution for μ = 0 by showing that s∗ → ∞ as μ → 0 in Theorem 2.1, where s∗ is a minimal speed. This is analogous to the result in [22].

Now we present the results on the non-existence of traveling wave solutions for α = −1.

**Theorem 1.2.** Let D, μ > 0 and χ(v) ≥ 0 be a C1 function. Then

(i) β ≤ 0 and s ≥ 0, the system (1) has no traveling wave solution of speed s.

(ii) β > 0 and s = 0, the system (1) has no traveling wave solution of speed s satisfying 0 ≤ U ≤ 1 and U(−∞) or U(∞) = 0.

In Theorems 1.1 and 1.2, we present only the case of s ≥ 0. To examine the existence traveling wave solution for s < 0, we use change of variable ˜ξ := −ξ to change into the case of −s > 0; we can obtain results analogous to Theorems 1.1 and 1.2 for s < 0.

We also study the stability of the traveling wave solutions, that is, if initial data satisfies

\[(u, v)(x, 0) = (u_0, v_0) \rightarrow (u_±, v_±) \quad \text{as} \quad x \rightarrow ±\infty \]  

and is close to the traveling waves, whether or not the solution to the Cauchy problem (1)(3) converges to the traveling waves as time evolves. By studying the essential spectrum of the linearized operator of (1), we show the linear instability of the traveling wave solutions as in [10, 14, 18]. It is confirmed that the rate of logistic cell growth μ > 0 and the exponential chemical growth β > 0 cause the linear instability, which is biologically relevant.

**Theorem 1.3.** Let χ(v) ≥ 0 be a C3 function. If (U(x−st), V(x−st)) is a traveling wave solution of the system (1) obtained in Theorem 1.1, then it is linearly unstable against perturbations in L2(R) × L2(R).

This paper is organized as follows. In Section 2, we reformulate our system (1) and state our main results for the transformed system. The proof for the non-existence of traveling wave solution of the transformed system is shown in Section 2, and the existence is proved in Sections 3-5. In Section 6, we show the linear instability of traveling wave solutions.

2. Reformulation of the problem. A smooth traveling wave solution of (1) is of the form (u, v)(x, t) = (U, V)(ξ), ξ := x − st, satisfying

\[
\begin{cases}
U(ξ) ≥ 0, \quad V(ξ) ≥ 0, & \forall ξ ∈ (−∞, ∞), \\
(U(±∞), V(±∞)) = (u_±, v_±), \\
(U'(±∞), V'(±∞)) = (0, 0),
\end{cases}
\]  

where (u_±, v_±) are far field states and s is the wave speed. Substituting (U(ξ), V(ξ)) into the PDE system (1) and using (4), we obtain the following ODE system

\[
\begin{align*}
-ss' &= DU'' - (Uχ(V)V')' + μU(1 - U) \\
-ss'' &= βV - U.
\end{align*}
\]  

To reduce the number of parameters, we choose the following scalings

\[
\tilde{u}(x, t) := u(\sqrt{D}x, t), \quad \tilde{v}(x, t) := v(\sqrt{D}x, t), \quad \tilde{χ}(\tilde{v}) := \frac{1}{D}χ(\tilde{v})
\]
in the system (1). Then we obtain the transformed system
\[
\begin{align*}
\tilde{u}_t &= \tilde{u}_{xx} - (\tilde{u}\tilde{\chi}(\tilde{v})\tilde{v}_x)_x + \mu\tilde{u}(1 - \tilde{u}) \\
\tilde{v}_t &= \beta\tilde{v} - \tilde{u}.
\end{align*}
\] (7)

Letting
\[c := \frac{s}{\sqrt{D}}, \quad \tilde{\xi} := x - ct = x - \frac{s}{\sqrt{D}}t,\] (8)
and dropping the tildes in (7), the traveling wave system (5) is also reduced to
\[
\begin{align*}
-cU' &= U'' - (U\chi(V)V')' + \mu U(1 - U) \\
-cV' &= \beta V - U,
\end{align*}
\] (9)
where \(c \geq 0\) by \(s \geq 0\) and (8).

In the following subsections, we prove the non-existence of traveling wave solutions of (1). For the proof of the existence, we restate Theorem 1.1 for the transformed system (7) and provide the proof in the next sections.

2.1. The non-existence of traveling wave solutions. In this subsection, we prove Theorem 1.2 by showing that the transformed system (7) has no traveling wave solutions. According to (8), the proof of the non-existence of traveling waves of (7) is presented by considering two cases: (i) \(\beta \leq 0\) and \(c \geq 0\) and (ii) \(\beta > 0\) and \(c = 0\).

Proof of Theorem 1.2. (i) First, let \(\beta < 0\) and \(c > 0\). We assume to the contrary that there are non-trivial traveling wave solutions \((U(\xi), V(\xi))\) of (9) satisfying (4).

Evaluating (9) at \(\xi = \pm\infty\) and using (4) yield
\[(u_+, v_+), (u_-, v_-) \in \{(0, 0), (1, \frac{1}{\beta})\}.
\]
By ruling out the case of \(v_\pm = \frac{1}{\beta} < 0\), \((U, V)\) must be a traveling pulse-pulse solution satisfying \((u_\pm, v_\pm) = (0, 0)\). From the second equation of (9), we derive that \(V' = \frac{1}{\beta}(-\beta V + U) \geq 0\) for any \(\xi \in (-\infty, \infty)\) since \(U, V \geq 0\) and \(\beta < 0\). If there is a \(\xi_0\) such that \(V'(\xi_0) > 0\), \(V\) cannot satisfy \(v_\pm = 0\), which implies that \(V' \equiv 0\) and \(V \equiv 0\). Hence \(U \equiv \beta V \equiv 0\), and it contradicts that \(U\) and \(V\) are non-trivial. Thus, (9) cannot have non-trivial traveling wave solutions. Transforming this result with (6) and (8), the proof of Theorem 1.2-(i) for \(\beta < 0\) and \(c > 0\) is completed.

Next, when \(\beta \leq 0\) and \(c = 0\), \(U = \beta V\) from (9). It is straightforward to show that there is no non-trivial and non-negative traveling wave solution.

To the end, let \(\beta = 0\) and \(c > 0\). Introducing a new variable
\[W(\xi) := cU + U' - U\chi(V)V' - cu_-\] (10)
in (9) with \(\beta = 0\), one derives from (9) the following system
\[
\begin{align*}
U' &= -cU + \frac{1}{\varepsilon}U^2\chi(V) + W \\
V' &= uU \\
W' &= \mu U(U - 1).
\end{align*}
\] (11)
To complete the proof, it is enough to show that there exists no non-trivial traveling wave solution of (11). We assume that there is a traveling wave solution \((U(\xi), V(\xi), W(\xi))\) of (11) and derive a contradiction.
The system (11) has two continuum of equilibria \((0,v_{\pm},0)\) for any \(v_{\pm} \geq 0\), and hence \(U\) and \(W\) must be pulses. Moreover, since \(U,V \geq 0\), we have \(V' = \frac{1}{c}U \geq 0\) and \(0 \leq V \leq v_{+}\).

Now we define a \(C^1\) function

\[
L(\xi) = L(U(\xi), W(\xi)) := \frac{1}{2c}U^2 + \frac{1}{2c\mu}W^2 \geq 0, \quad \xi \in (-\infty, \infty).
\]

Then, \(L(U,V,W) = 0\) holds only when \(U = W = 0\). Letting

\[
B := \{ (U,V,W) \mid \left( \max_{0 \leq V \leq v_{+}} \chi(V) \right) U \leq \frac{c^2}{4}, \quad 0 \leq V \leq v_{+}, \quad |W| \leq \frac{c}{4} \},
\]

it is satisfied that

\[
\frac{dL}{d\xi} \leq -U^2 + \frac{U^2}{c} \left( 1 - U \chi(V) + |W| \right) \leq -\frac{1}{2}U^2, \quad \forall (U,V,W) \in B. \quad (12)
\]

Also, since \(U\) and \(W\) are non-trivial pulses and \(u_{-} = w_{-} = 0\), there is a \(\xi_0\) such that

\[
\begin{align*}
0 &\leq \left( \max_{0 \leq V \leq v_{+}} \chi(V) \right) U(\xi) \leq \left( \max_{0 \leq V \leq v_{+}} \chi(V) \right) U(\xi_0) < \frac{c^2}{8}, \quad \forall \xi < \xi_0 \\
0 &\leq |W(\xi)| \leq |W(\xi_0)| < \frac{c}{8}, \quad \forall \xi < \xi_0 \\
U(\xi_0) &> 0 \text{ or } |W(\xi_0)| > 0.
\end{align*}
\]

Then, \((U,V,W) \in B\) for any \(\xi \leq \xi_0\), and we obtain from (12) the following inequality

\[
0 = L(-\infty) \geq L(\xi) \geq L(\xi_0) = \frac{1}{2c}U^2(\xi_0) + \frac{1}{2c\mu}W^2(\xi_0) > 0,
\]

which is a contradiction. Therefore (11) cannot have non-trivial traveling wave solutions connecting \((0,v_{\pm},0)\). Consequently, (1) has no non-trivial traveling wave solution, and the proof of Theorem 1.2-(i) is completed.

(ii) For \(\beta > 0\), we assume to the contrary that the system (9) has non-trivial traveling wave solution \((U,V)\) satisfying \(0 \leq U \leq 1\) and \(U(-\infty) = 0\) or \(U(\infty) = 0\). Introducing \(W\) given in (10) to (9) and using \(c = 0\) and \(V = \frac{1}{\beta}U\), one derives the following system

\[
\begin{cases}
\left( 1 - \frac{1}{\beta}U \chi(\frac{1}{\beta}U) \right) U' = W \\
W' = -\mu U(1-U).
\end{cases}
\quad (13)
\]

Evaluating (13) at \(\xi = \pm \infty\), the equilibrium points are \((u_{\pm}, w_{\pm}) = \{(0,0),(1,0)\}\), and hence we conclude that \(W\) must be a pulse. However, since \(0 \leq U \leq 1\), we have \(W' \leq 0\) and \(W \equiv 0\). Without loss of generality, let \(u_{-} = 0\). If there is no \(\xi \in (-\infty, \infty)\) such that \(U(\xi)\chi(\frac{1}{\beta}U(\xi)) = \beta\), \(U' = 0\) and \(U \equiv 0\), which is a contradiction. Otherwise, we can pick the smallest \(\xi_0 \in (-\infty, \infty)\) such that \(U(\xi_0)\chi(\frac{1}{\beta}U(\xi_0)) = \beta\) since \(\chi(\frac{1}{\beta}u) \in C^1\) for \(0 \leq u \leq 1\). However, since \(U' = 0\) must be true for any \(\xi < \xi_0\) and \(u_{-} = 0\), \(U(\xi_0) = 0\) holds by the continuity of \(U\), which is a contradiction. Thus, for \(c = 0\), there is no non-trivial solution such that \(0 \leq U \leq 1\) and \(U(-\infty) = 0\) or \(U(\infty) = 0\), and hence (9) does not have any non-trivial traveling wave solutions. Furthermore, the proof of Theorem 1.2-(ii) is completed. \(\Box\)
2.2. Existence of traveling wave solutions. In this subsection, we prove the existence of traveling wave solutions \((U, V)\) of (1), which is desired in Theorem 1.1 for \(\beta > 0\) and \(c > 0\). To prove the existence, we define a new variable \(W(\xi)\) as in (10) with \(u_- = 1\) and integrate the first equation of (9) over \((-\infty, \xi)\). Then the following first order system is obtained from (9):

\[
\begin{align*}
U' &= -cU + \frac{1}{c} U \chi(U - \beta V) + W \\
V' &= \frac{1}{\epsilon}(U - \beta V) \\
W' &= \mu U(U - 1).
\end{align*}
\tag{14}
\]

This system has two equilibrium points

\[
O := (0, 0, 0) \quad \text{and} \quad C := (1, \frac{1}{\beta}, c).
\tag{15}
\]

Now we state a theorem for the existence of traveling wave solutions of (14) connecting the equilibrium points \(O\) and \(C\). Theorem 1.1 is a consequence of the following theorem by the transformations (6) and (8).

**Theorem 2.1.** Let \(\mu, \beta > 0\), \(\chi(v) \geq 0\) be a \(C^1\) function and

\[
c^* := \begin{cases}
\max_{0 \leq v \leq \frac{1}{\beta}} \left\{ 2\sqrt{\mu}, \sqrt{\chi(\frac{1}{\beta}) - \mu}, \sqrt{\frac{\beta}{\mu}} \chi(v) \right\} & \text{if } \chi(\frac{1}{\beta}) \geq \mu, \\
\max_{0 \leq v \leq \frac{1}{\beta}} \left\{ 2\sqrt{\mu}, \sqrt{\frac{\beta}{\mu}} \chi(v) \right\} & \text{if } \chi(\frac{1}{\beta}) < \mu.
\end{cases}
\tag{16}
\]

For each \(c \geq c^* > 0\), the system (14) has a traveling wave solution \((U(\xi), V(\xi), W(\xi))\), \(\xi := x - ct\), satisfying

\[
\begin{align*}
0 < U(\xi) &\leq \beta V(\xi) < 1, \quad 0 < W(\xi) < c \quad \forall \xi \in (-\infty, \infty) \\
V'(\xi) &< 0, \quad W'(\xi) < 0 \quad \forall \xi \in (-\infty, \infty) \\
(U, V, W)(-\infty) &= (1, \frac{1}{\beta}, c), \quad (U, V, W)(\infty) = (0, 0, 0),
\end{align*}
\tag{17}
\]

and the traveling wave solution is unique up to a translation in \(\xi\).

In particular, if \(0 < c^* \leq \sqrt{\beta}\), for \(c \geq c^*\), \(U'(\xi) < 0\) for all \(\xi \in (-\infty, \infty)\).

**Remark 2.** When a minimal speed \(c^* > \sqrt{\beta}\), Theorem 2.1 does not guarantee the monotonicity of \(U\) as shown in Figure 2-(A).

**Remark 3.** In (16) of Theorem 2.1, \(c^* \geq 2\sqrt{\mu}\) is a necessary condition for \(U\) and \(V\) to be non-negative, which is to be verified by (19). For instance, when \(\chi(v) \equiv 1\) and \(\mu = \beta = 1\), the system (14) has the unique minimal speed \(c^* = 2\sqrt{\mu} = 2\) as in Fisher’s equation [13] and (14) has a unique traveling wave solution if and only if \(c \geq 2\). Furthermore, \(U\) is monotone according to Remark 4.

In Sections 3-5, we prove Theorem 2.1 by applying the dynamical systems theory in the three dimensional space. In Section 3, we analyze the linearized system of (14) at the equilibrium points \(O\) and \(C\) given in (15). In Section 4, we construct positively invariant sets based on the results from Section 3. Finally the proof of Theorem 2.1 is fulfilled in Section 5 by showing the existence of heteroclinic orbits of (14) connecting \(O\) and \(C\).
3. Proof of Theorem 2.1 - Part 1: Linearized system. To understand backward and forward asymptotic behaviors of a trajectory \((U(\xi), V(\xi), W(\xi))\) of the system (14), in this section we investigate the behavior of the trajectory in a neighborhood of the equilibrium points \(O\) and \(C\) given in (15).

At the equilibrium point \(O\), the Jacobian matrix of (14) is

\[
\begin{pmatrix}
-c & 0 & 1 \\
\frac{1}{c} & -\frac{\beta}{c} & 0 \\
-\mu & 0 & 0
\end{pmatrix}.
\] (18)

Direct calculation gives us the following eigenvalues:

\[
\lambda_{o,1} = -\frac{\beta}{c}, \quad \lambda_{o,2} = -c - \sqrt{\frac{c^2 - 4\mu}{2}}, \quad \lambda_{o,3} = -c + \sqrt{\frac{c^2 - 4\mu}{2}}.
\] (19)

Eigenvectors associated with \(\lambda_{o,1}, \lambda_{o,2}, \lambda_{o,3}\) are

\[
\tilde{v}_{o,1} = (0, 1, 0)^t, \quad \tilde{v}_{o,i} = (\beta + c\lambda_{o,i}, 1, \beta(c + \lambda_{o,i}) - \mu c)^t, \quad i = 2, 3.
\] (20)

Then we derive the following lemma.

**Lemma 3.1.** For each \(c \geq c^*\), the equilibrium point \(O\), defined in (15), is a stable node (including the degenerate stable node). The local stable manifold \(W^{s}_{loc}(O)\) of (14) is 3-dimensional.

At the equilibrium point \(C\), the Jacobian matrix of (14) is

\[
\begin{pmatrix}
-c + \frac{1}{c} \chi \left(\frac{\beta}{\beta^2}\right) & -\frac{\beta}{c} \chi \left(\frac{1}{\beta}\right) & 1 \\
\frac{1}{c} & -\frac{\beta}{c} & 0 \\
\mu & 0 & 0
\end{pmatrix}
\] (21)

and its characteristic polynomial is

\[
P(\lambda) = -\lambda^3 - \left(c + \frac{\beta}{c} - \frac{1}{c} \chi \left(\frac{1}{\beta}\right)\right)\lambda^2 - (\beta - \mu)\lambda + \frac{\beta \mu}{c}.
\] (22)

Let \(\lambda_{c,1}, \lambda_{c,2}, \lambda_{c,3}\) be roots of \(P(\lambda)\). Noticing that

\[
\lambda_{c,1}\lambda_{c,2}\lambda_{c,3} = P(0) = \frac{\beta \mu}{c} > 0,
\]

the roots \(\lambda_{c,1}, \lambda_{c,2}, \lambda_{c,3}\) of \(P(\lambda)\) must satisfy one of the following cases:

\[
\begin{cases}
\text{Re}\lambda_{c,1} \geq 0, \quad \text{Re}\lambda_{c,2} \geq 0, \quad \text{Re}\lambda_{c,3} > 0 \\
\text{Re}\lambda_{c,1} = \text{Re}\lambda_{c,2} = 0, \quad \text{Im}\lambda_{c,1} = -\text{Im}\lambda_{c,2} \neq 0, \quad \lambda_{c,3} > 0 \\
\text{Re}\lambda_{c,1} < 0, \quad \text{Re}\lambda_{c,2} < 0, \quad \text{Re}\lambda_{c,3} > 0.
\end{cases}
\] (23)

In fact, the first and second cases in (23) are ruled out. Assuming to the contrary that the first or second case is true, we derive contradiction by considering two cases as follows.

(i) **Case of** \(0 < \beta \leq \mu\). If the first or second case in (23) is true, \(\lambda_{c,1}\lambda_{c,2} + \lambda_{c,1}\lambda_{c,3} + \lambda_{c,2}\lambda_{c,3}\) must be positive. However, we deduce from (22) that \(\lambda_{c,1}\lambda_{c,2} + \lambda_{c,1}\lambda_{c,3} + \lambda_{c,2}\lambda_{c,3} = \beta - \mu \leq 0\), which is a contradiction. The first and second cases in (23) are excluded.

(ii) **Case of** \(0 < \mu < \beta\). First, we assume to the contrary that the first case in (23) is true. Namely, all of the roots of the polynomial

\[
P(-\lambda) = \lambda^3 - \left(c + \frac{\beta}{c} - \frac{1}{c} \chi \left(\frac{1}{\beta}\right)\right)\lambda^2 + (\beta - \mu)\lambda + \frac{\beta \mu}{c}
\] (24)
have negative real parts. Then, the Hurwitz algorithm [5, p. 190] with \( P(-\lambda) \) implies both
\[
e^2 < \chi\left(\frac{1}{\beta}\right) - \beta \tag{25}
\]
and
\[
-(\beta - \mu)(e^2 + \beta - \chi\left(\frac{1}{\beta}\right)) > \beta \mu. \tag{26}
\]
Then (25), \( c \geq c^* \) and (16) lead to
\[
\chi\left(\frac{1}{\beta}\right) - \mu \leq e^2 < \chi\left(\frac{1}{\beta}\right) - \beta \implies \beta < \mu,
\tag{27}
\]
which is a contradiction to \( 0 < \mu < \beta \). Thus, the first case in (23) is excluded.

Next, assuming that the second case holds in order to derive a contradiction, it follows from (22) that
\[
\lambda_{c,1} + \lambda_{c,2} + \lambda_{c,3} = \lambda_{c,3} = -\left(c + \frac{\beta}{c} - \frac{1}{c} \chi\left(\frac{1}{\beta}\right)\right) > 0,
\]
which implies (25). Using (27) again, we derive a contradiction to \( 0 < \mu < \beta \).

From the above Case (i) and Case (ii), we therefore conclude that \( P(\lambda) \) has a unique positive root for any \( c \geq c^* \). Denote the positive root of \( P(\lambda) \) by \( \lambda_c \). Then an eigenvector \( \vec{v}_c \) of \( \lambda_c \) is written as
\[
\vec{v}_c = \left(1, \frac{-1}{c} \left(\frac{\beta}{c} + \lambda_c\right)^{-1}, \frac{\mu}{\lambda_c}\right)^t. \tag{28}
\]

The following lemma is also obtained.

**Lemma 3.2.** For each \( c \geq c^* \), the equilibrium point \( C \), defined in (15), is unstable. The local unstable manifold \( W^u_{loc}(C) \) of (14) is 1-dimensional.

The description of the asymptotic behavior of \((U,V,W)\) at \( O \) and \( C \) that is obtained in Lemmas 3.1 and 3.2 is to be used in the following section.

### 4. Proof of Theorem 2.1 - Part 2: Construction of positively invariant sets \( \mathcal{P} \) and \( \mathcal{T} \).
In order to show the existence of traveling wave solutions of the system (14) connecting two equilibrium points \( O \) and \( C \) given in (15), in this section we look for a compact region where solutions of (14) stay in the region for all positive time if their initial data lies in the interior of the region.

In Subsection 4.1, we define two compact sets \( \mathcal{P} \) for \( c > \sqrt{\beta} \) and \( \mathcal{T} \) for \( 0 < c \leq \sqrt{\beta} \) as in Figure 1. In fact, the constructions of \( \mathcal{P} \) and \( \mathcal{T} \) are characterized by the local unstable manifolds of (14) at \( C \). In Subsection 4.2, the sets \( \mathcal{P} \) and \( \mathcal{T} \) are turned out to be positively invariant to the flows of (14).

#### 4.1. Positively invariant sets.
In this subsection, we define compact sets, two of whose vertices are the equilibrium points \( O \) and \( C \). For each \( c \geq c^* \), the structure of compact sets is classified by the relation of \( c \) and \( \beta \): a pentahedron \( \mathcal{P} \) for \( c > \sqrt{\beta} > 0 \) and a tetrahedron \( \mathcal{T} \) for \( 0 < c \leq \sqrt{\beta} \) (see Figure 1).
For each $c > \sqrt{\beta} > 0$ and $c \geq c^*$, as shown in Figure 1-(A), we construct a pentahedron $\mathcal{P}$ that is surrounded by the following 5 faces

$$
F_1 := \{(U,V,W) \mid 0 \leq U = \beta V \leq 1, \ k_2 U \leq W \leq c U\},
$$
(29)

$$
F_2 := \{(U,V,W) \mid 0 \leq U \leq \beta V \leq 1, \ W = (c - k_1)U + k_1 \beta V\},
$$
(30)

$$
F_3 := \{(U,V,W) \mid U = 0, \ 0 \leq V \leq \frac{1}{\beta}, \ 0 \leq W \leq k_1 \beta V\},
$$
(31)

$$
F_4 := \{(U,V,W) \mid 0 \leq U \leq \beta V \leq 1, \ W = k_2 U\},
$$
(32)

$$
F_5 := \{(U,V,W) \mid 0 \leq U \leq 1, \ V = \frac{1}{\beta}, \ k_2 U \leq W \leq (c - k_1)U + k_1\}
$$
(33)

and that has vertices $O, C, (0, \frac{1}{\beta}, k_1), (0, 0, 0), (1, \frac{1}{\beta}, k_2)$, where

$$
k_1 := c - \frac{\beta}{c} \quad \text{and} \quad k_2 := \frac{c - \sqrt{c^2 - 4\mu}}{2}.
$$
(34)

Here, $0 < k_1 < c$ and $k_2 = -\lambda_{o,3} > 0$, where $\lambda_{o,3}$ is given in (19).

For $0 < c \leq \sqrt{\beta}$ and $c \geq c^*$, as shown in Figure 1-(B), we define a tetrahedron $\mathcal{T}$ whose vertices are $O, C, (0, \frac{1}{\beta}, 0), (1, \frac{1}{\beta}, k_2)$ and whose faces are $F_1, F_2', F_4, F_5'$, where

$$
F_2' := \{(U,V,W) \mid 0 \leq U \leq \beta V \leq 1, \ W = c U\},
$$
(35)

$$
F_5' := \{(U,V,W) \mid 0 \leq U \leq 1, \ V = \frac{1}{\beta}, \ k_2 U \leq W \leq c U\}.
$$
(36)

It is noticed that $F_2'$ and $F_5'$ are the faces $F_2$ and $F_5$ when $k_1 = 0$. In other words, $\mathcal{P}$ becomes $\mathcal{T}$ when $c = \sqrt{\beta}$.

By the local unstable manifold theorem, the local unstable manifold $W^u_{loc}(C)$ of the system (14) from Lemma 3.2 is tangent to the eigenvector $\vec{v}_c$ at $C$, where $\vec{v}_c$ is defined in (28). In the remaining part of this subsection, we verify that $\vec{v}_c$ at $C$ points to the interior of $\mathcal{P}$ or $\mathcal{T}$ or lie on them.

**Lemma 4.1.** Let $c \geq c^*$. If $0 < c \leq \sqrt{\beta}$, the vector $\vec{v}_c$, given in (28), at $C$ points to the interior point of $\mathcal{T}$. Otherwise, $\vec{v}_c$ at $C$ points to the interior of $\mathcal{P}$.

**Proof.** Define a line

$$
l := \left\{(U,V,W) \mid U - 1 = (\beta + c \lambda_c)(V - \frac{1}{\beta}) = \frac{\lambda_c}{\mu}(W - c)\right\}
$$
(37)
passing through $C$ in the direction parallel to $v_c$. Since $v_c$ is an eigenvector associated with the eigenvalue $\lambda_c$ given in (28), it suffices to verify that $l$ intersects the following regions:

$$
\begin{align*}
F_6 & := \{(U, V, W) \mid 0 \leq U < \beta V < 1, \ W = 0\} \quad \text{if} \ 0 < c \leq \sqrt{\beta}, \\
F_3 \setminus \{(U, V, W) \mid W = k_1 \beta V\} & \quad \text{if} \ c > \sqrt{\beta},
\end{align*}
$$

(38)

where $F_6$ is a part of the projection of $F_4$ which is a face of $T(\subset P)$ in (32) into the $U$-$V$ plane and $F_3$ is a face of $P$ in (31).

On the other hand, since $\lambda_c$ is a unique positive root of $P(\lambda)$ in (22), we deduce

$$
0 < c^2 \leq \beta \implies \chi \left( \frac{1}{\beta} \right) - \mu \leq \beta 
$$

\begin{align*}
\iff P(\frac{\mu}{c}) &= -\frac{\mu^2}{c^2}(\mu + \beta - \chi \left( \frac{1}{\beta} \right)) \leq 0 \\
\iff 0 < \lambda_c \leq \frac{\mu}{c},
\end{align*}

(39)

where we have used $c \geq c^*$ and (16). According to (39), we shall show that $l$ intersects the region given in (38) by considering two cases as follows.

(i) Case of $0 < \lambda_c \leq \frac{\mu}{c}$. To verify $l$ intersects $F_6$, we shall show that $(0, V^*, W^*) \in l$ satisfies

$$
0 \leq U^* < \beta V^*,
$$

(40)

Indeed, by direct calculation, we arrive at

$$
U^* = 1 - \frac{c\lambda_c}{\mu} \quad \text{and} \quad V^* = -\frac{c\lambda_c}{\mu}(\beta + c\lambda_c)^{-1} + \frac{1}{\beta}.
$$

It is straightforward to show that (40) holds by using $0 < \lambda_c \leq \frac{\mu}{c}$. Thus, we conclude that $l$ intersects $F_6$.

(ii) Case of $\lambda_c > \frac{\mu}{c} > 0$. To show that $l$ intersects $F_3 \setminus \{(U, V, W) \mid W = k_1 \beta V\}$, we shall verify that $(0, V^*, W^*) \in l$ satisfies

$$
0 < V^* < \frac{1}{\beta}, \quad 0 < W^* < k_1 \beta V^*,
$$

(41)

where $k_1$ is given in (34) and $V^*$ and $W^*$ are

$$
V^* = \frac{1}{\beta} - \frac{1}{\beta + c\lambda_c} \quad \text{and} \quad W^* = c - \frac{\mu}{\lambda_c}.
$$

(42)

Using (42) and $\lambda_c > \frac{\mu}{c} > 0$, we can easily check that $0 < V^* < \frac{1}{\beta}$ and $W^* > 0$.

Now it remains to show that $W^* < k_1 \beta V^*$ in (41). Direct calculation with (42) yields

$$
W^* < k_1 \beta V^* \iff c \left( \frac{\beta^2}{c^2} + (\beta - c)\lambda_c - \frac{\beta\mu}{c} \right) < 0.
$$

(43)

Here, since $\lambda_c$ satisfies

$$
P(\lambda_c) = 0 
\iff -\lambda_c^3 - \left( c - \frac{1}{c}(\frac{1}{\beta}) \right)\lambda_c^2 = \frac{\beta}{c} \lambda_c^2 + (\beta - \mu)\lambda_c - \frac{\beta\mu}{c},
$$

(44)
where $P(\lambda)$ is given in (22), substituting (44) into the right hand side of (43) gives us
\[
W^* < k_1 \beta V^* \iff -c\lambda^2 \left( \frac{1}{c} \chi \left( \frac{1}{\beta} \right) \right) < 0. \tag{45}
\]
In fact, the right hand side of (45) can be shown by using $\frac{c}{c} + c - \frac{1}{c} \chi \left( \frac{1}{\beta} \right) \geq 0$ from $c \geq c^*$ and (16):
\[
\lambda + c - \frac{1}{c} \chi \left( \frac{1}{\beta} \right) > \frac{\mu}{c} + c - \frac{1}{c} \chi \left( \frac{1}{\beta} \right) \geq 0.
\]
Hence, $W^* < k_1 \beta V^*$ is finally verified, and the proof of (41) is completed. Thus, we conclude that $l$ intersects $F_3 \setminus \{(U, V, W) \mid W = k_1 \beta V\}$. According to (39), if $0 < c \leq \sqrt{\beta}$ implies $0 < \lambda_c \leq \frac{\mu}{c}$, Case (i) indicates that $\vec{n}_c$ at $C$ points to the interior of $T$. Similarly, for $c > \sqrt{\beta}$, noticing that $T \subset P$, we derive from Case (i) and Case (ii) that $\vec{n}_c$ at $C$ points to the interior of $P$. \hfill $\Box$

### 4.2. Proof of positive invariance of $P$ and $T$

In this subsection, we prove that the sets $P$ and $T$ are indeed positive invariant in $\mathbb{R}^3$. The main tool is to show that at any point $(U, V, W) \in \partial P \setminus \{O, C\}$ or $\partial T \setminus \{O, C\}$ it is satisfied that
\[
\vec{n} \cdot (U', V', W') < 0,
\]
where $\vec{n}$ denotes an outward normal vector at the point (see [21, p. 116] for more details).

#### 4.2.1. Case of $c > \sqrt{\beta}$

The followings (F1)-(F5) verify that the pentahedron $P$, surrounded by the 5 faces $F_1, \cdots, F_5$ in (29)-(33), is a positive invariant set.

(F1) The face $F_1$ in (29) has an outer normal vector $\vec{n}_{F_1} := (1, -\beta, 0)^t$. At any point on the face, we have
\[
\vec{n}_{F_1} \cdot (U', V', W') = -cU + W < 0 \tag{46}
\]
except for $W = cU$. Along the line segment $W = cU$ on $F_1$, a flow $(U, V, W)$ of (14) satisfies $U' = V' = 0$ and $W' = \mu U(U - 1) < 0$ unless $(U, V, W) = O, C$. Therefore, any forward trajectory $(U(\xi), V(\xi), W(\xi))$ starting at the interior of $P$ cannot exit $P$ through $F_1$.

(F2) The face $F_2$ in (30) has an outer normal vector $\vec{n}_{F_2} := (k_1 - c, -k_1 \beta, 1)^t$, where $k_1$ is given in (34). At a point on the face $F_2$, we derive
\[
\begin{align*}
\vec{n}_{F_2} \cdot (U', V', W') & = (k_1 - c) \left( \frac{1}{c} U \chi(V)(U - \beta V) - k_1 U + k_1 \beta V \right) - \frac{k_1 \beta}{c} (U - \beta V) + \mu U(U - 1) \\
& < (k_1 - c) \left( \frac{1}{c} U \chi(V)(U - \beta V) - k_1 (U - \beta V) \right) - \frac{k_1 \beta}{c} (U - \beta V) + \mu U(U - \beta V) \\
& = \frac{(U - \beta V)}{c^2} U(\mu c^2 - \beta \chi(V)) \\
& \leq 0,
\end{align*}
\]
where we have used $0 \leq V \leq \frac{1}{\beta}$, (34), $c \geq c^*$, and (16). Then it follows that $\vec{n}_{F_2} \cdot (U', V', W') < 0$ unless $(U, V, W) \in F_1 \cap F_2$. We conclude that no forward orbit starting at the interior of $P$ can exit $P$ through the faces $F_1$ and $F_2$. 


Lemma 5.1. 2.1 is fulfilled. Investigating the monotonicity of the traveling wave solutions, the proof of Theorem 2.1 and heteroclinic orbits of (14) connecting the equilibrium points have been constructed. With these sets, in this section we prove the existence of F.

Subsection 4.1, the condition cannot exit through T. Proof of Theorem 2.1 - Part 3: A heteroclinic orbit connecting wave solution C. From (47) with one has

\[ \vec{n}_{F_4} \cdot (U', V', W') = -ck_2U + \frac{k_2}{c}U\chi(V)(U - \beta V) + k_2^2U - \mu(U - 1) \]

\[ \leq U(k_2^2 - ck_2 + \mu) \]

\[ = 0. \]

In fact, \( \vec{n}_{F_4} \cdot (U', V', W') < 0 \) if \( (U, V, W) \notin \mathcal{I} \), where \( \mathcal{I} \) is given in (48). Therefore, no forward trajectory initiating at the interior of \( \mathcal{P} \) exits \( \mathcal{P} \) through \( F_4 \).

(F5) On the face \( F_3 \) in (33), \( V' = \frac{1}{c}(U - \beta V) < 0 \) unless \( U = 1 \) and \( V = \frac{1}{\beta} \). Since (1, \( \frac{1}{\beta} \), W) is in \( F_1 \cap F_5 \) for \( 0 < W < c \), no forward orbit comes out of \( \mathcal{P} \) through \( F_5 \).

From (F1) – (F5), we deduce that no forward orbits can exit through any of the faces of pentahedron \( \mathcal{P} \). Thus, \( \mathcal{P} \) is a trapping region for the system (14).

4.2.2. Case of \( 0 < c \leq \sqrt{\beta} \). In the earlier proof that \( \mathcal{P} \) is positive invariant in Subsection 4.1, the condition \( c > \sqrt{\beta} \) is used only in (F2). Also, faces \( F_1, F_4, F_5' \) of \( \mathcal{T} \) are contained in \( \mathcal{P} \). Since \( \mathcal{T} \) is surrounded by faces \( F_1, F_2', F_4, F_5' \), it is sufficient to prove that any forward orbit at the interior of \( \mathcal{T} \) cannot exit \( \mathcal{T} \) through the face \( F_2' \).

(F2') Since \( F_2' \) in (35) is the face \( F_2 \) with \( k_1 = 0 \), an outer normal vector of \( F_2' \) is

\[ \vec{n}_{F_2'} := (-c, 0, 1). \]

From (47) with \( k_1 = 0 \), one has

\[ \vec{n}_{F_2'} \cdot (U', V', W') \leq U(U - \beta V)(-\chi(V) + \mu). \]

Using \( c \leq \sqrt{\beta}, c \geq c^* \) and (16) leads to \( \chi(V) \leq \mu \) for all \( 0 \leq V \leq \frac{1}{\beta} \), which implies \( \vec{n}_{F_2'} \cdot (U', V', W') < 0 \) in (51). Thus, any forward orbit initiating at the interior of \( \mathcal{T} \) cannot exit through \( F_2' \).

5. Proof of Theorem 2.1 - Part 3: A heteroclinic orbit connecting \( O \) and \( C \). In the previous section, the positive invariant sets \( \mathcal{P} \) and \( \mathcal{T} \) of the system (14) have been constructed. With these sets, in this section we prove the existence of heteroclinic orbits of (14) connecting the equilibrium points \( O \) and \( C \). Further, by investigating the monotonicity of the traveling wave solutions, the proof of Theorem 2.1 is fulfilled.

Lemma 5.1. Let \( c \geq c^* \) and \( c > \sqrt{\beta} \). For each \( c \), there exists a unique traveling wave solution \( (U(\xi), V(\xi), W(\xi)), \xi = x - ct \), of the system (14) satisfying

(i) \( (U, V, W)(-\infty) = C \) and \( (U, V, W)(\infty) = O \),
(ii) \( (U, V, W) \in \text{int}(\mathcal{P}) \) for any \( \xi \in (-\infty, \infty) \),
(iii) \( V' < 0 \) and \( W' < 0 \) for any \( \xi \in (-\infty, \infty) \).
Figure 2. Numerical simulations of traveling wave solutions $(U(\xi), V(\xi))$ of the system (1). For (A), $D = 1, s = 2, \beta = 0.2, \mu = 1$ and $\chi(v) = \cos(10v) - \sin(20v) + 2$. For (B), $D = 1, s = 2, \beta = 1, \mu = \frac{1}{2}$ and $\chi(v) = \frac{1}{(1+v)^2}$.

Proof. From Lemma 3.2, $W_{uloc}(C) \cap \text{int}(P)$ is non-empty since the eigenvector $\vec{v}_c$ at $C$ points into the interior of $P$. Then, the solution $(U, V, W)$ starting at $(U, V, W)(\xi_0) \in W_{loc}^u(C) \cap \text{int}(P)$ for sufficiently negative $\xi_0$ must stay in $P$ as proved in Subsection 4.2; specifically, the solution must approach the equilibrium point $O$. Furthermore, we deduce from (14) and the structure of $P$ that $V', W' < 0$. Since $W_{uloc}^u(C)$ is 1-dimensional from Lemma 3.2, the heteroclinic orbit connecting $O$ and $C$ is unique up to a translation in $\xi$. 

Now we consider a case of $c \geq c^*$ and $c \leq \sqrt{\beta}$. By the definition of $T$, it is noticed that any $(U, V, W)$ staying in $T$ satisfies

$$U' = (W - cU) + \frac{1}{c} U \chi(V)(U - \beta V) < 0,$$

and consequently $U', V', W' < 0$. In a similar way to show Lemma 5.1, the following Lemma is proved.

**Lemma 5.2.** Let $c \geq c^*$ and $c \leq \sqrt{\beta}$. For each $c$, there is a unique traveling wave solution $(U(\xi), V(\xi), W(\xi))$, $\xi = x - ct$, of the system (14) satisfying

(i) $(U, V, W)(-\infty) = C$ and $(U, V, W)(\infty) = O$,
(ii) $(U, V, W) \in \text{int}(T)$ for all $\xi \in (-\infty, \infty)$,
(iii) $U', V', W' < 0$ for all $\xi \in (-\infty, \infty)$.

In fact, when $c \geq c^*$ and $c > \sqrt{\beta}$ hold, Figure 2 presents two cases: one has non-monotone $U$ as in Figure 2-(A) and the other has monotone $U$ as in Figure 2-(B). For the rest of this section, we analyze how the chemotaxis, diffusion and reaction processes affect the monotonicity.

**Lemma 5.3.** For $c \geq c^*$, let $(U(\xi), V(\xi), W(\xi))$ be obtained from Lemma 5.1. If $0 \leq \chi(v) \leq \mu$ for any $v \in (0, \frac{1}{\sqrt{\beta}})$, $U'(\xi) < 0$ for any $(\infty, \infty)$.
Proof. Noticing from (17) that 0 < U < βV < 1 and 0 ≤ χ(V) ≤ µ for any ξ ∈ (−∞, ∞), the third equation of (14) implies that

\[ W' = µU(U - 1) ≤ χ(V)U(U - 1) < χ(V)U(U - βV). \]  

(52)

It then follows from the first equation of (14) that

\[ U' + cU = \frac{1}{c} Uχ(V)(U - βV) + W > \frac{1}{c} W' + W, \]  

(53)

where we have applied (52). Multiplying (53) by \( e^{cξ} \) gives us the following estimate

\[ c(e^{cξ}U)' > (e^{cξ}W)'. \]

Integrating the above estimate over (−∞, ξ) with respect to ξ and multiplying the resultant by \( e^{-ξ} \) yield

\[ cU > W. \]  

(54)

Applying (17) and (54) to the first equation of (14), we conclude that \( U' < 0 \) for any \( ξ \in (−∞, ∞) \).

Remark 4. A typical example of sensitivity function \( χ(v) = \frac{1}{(1 + v)^2} \) with \( µ = \frac{1}{2}, β = 1 \) and \( c = c^* = 2 \) does not satisfy any conditions given in Lemmas 5.2 and 5.3; however, \( U \) is monotonically decreasing as in Figure 2-(B). In fact, \( U \) is monotonically decreasing for any \( c ≥ c^* \) if \( χ'(v) ≤ 0 \) for any \( 0 < v < \frac{1}{β} \). It can be proved by contradiction.

In the following corollary, the monotonicity of the traveling wave solution of the speed \( c \) can be also attained by studying the upper bound of the minimal speed \( c^* \) in (16).

Corollary 1. Assume that \( 0 < c^* ≤ \sqrt{β} \). Then, for any \( c ≥ c^* \), \( U \) is monotonically decreasing.

The proof of the corollary can be directly achieved by using the definition of \( c^* \) in (16) and Lemma 5.3. As a consequence of Lemmas 5.1, 5.2 and Corollary 1, the proof of Theorem 2.1 is finally completed.

6. Linear instability of the traveling wave solutions. In this section, we study the stability of the traveling wave solutions \( (U(x - st), V(x - st)) \) of the system (1) against perturbation in a certain space as claimed in Theorem 1.3. In fact, it is confirmed that the logistic cell growth \( µ > 0 \) and the chemical growth rate \( β > 0 \) cause the linear instability of traveling wave solutions, which is biologically relevant.

Consider the Cauchy problem (1)(3) in the moving coordinate \( ξ = x - st \),

\[
\begin{align*}
  u_t &= Du_{ξξ} + su_ξ - (uχ(v)v_ξ)_ξ + µu(1 - u) \\
  v_t &= sv_ξ + βv - u,
\end{align*}
\]

(55)

with the initial data

\[(u, v)(ξ, 0) = (u_0, v_0)(ξ) \rightarrow (u_±, v_±) = \begin{cases} (0, 0) & \text{as } ξ \rightarrow ∞ \\ (1, \frac{1}{β}) & \text{as } ξ \rightarrow -∞. \end{cases} \]

(56)

The traveling wave solution \( (U(ξ), V(ξ)) \) is a perturbation in the form of

\[(u, v)(ξ, 0) = (U(ξ), V(ξ)) + (ψ(ξ, 0), η(ξ, 0)),\]
where \((\psi, \eta)\) is the perturbation \((u - U, v - V)\). Then the solution to the Cauchy problem (55)-(56) is written as
\[
(u, v)(\xi, t) = (U, V)(\xi) + (\psi, \eta)(\xi, t).
\] (57)

For any \(\chi(v) \in C^1\), reformulating the problem (1) in terms of the perturbation \((\psi, \eta)\) by using (55) and (5) and linearizing the resulting problem yield
\[
\begin{pmatrix}
\psi \\
\eta
\end{pmatrix} = \mathcal{L} \begin{pmatrix}
\psi \\
\eta
\end{pmatrix},
\] (58)
where \(\mathcal{L}\) denotes
\[
\begin{pmatrix}
\left(D \frac{\partial}{\partial \xi}\right)^2 + A_1(\xi) \frac{\partial}{\partial \xi} + A_2(\xi) & -U \chi(V) \left(\frac{\partial}{\partial \xi}\right)^2 + A_3(\xi) \frac{\partial}{\partial \xi} + A_4(\xi) \\
-1 & s \frac{\partial}{\partial \xi} + \beta
\end{pmatrix}
\] (59)
with
\[
\begin{align*}
A_1(\xi) &= s - \chi(V)V', \\
A_2(\xi) &= -\chi'(V)V'^2 - \chi(V)V'' + \mu(1 - 2U), \\
A_3(\xi) &= -U'\chi(V) - 2UV'\chi'(V), \\
A_4(\xi) &= -U'V'\chi'(V) - UV'^2\chi''(V) - UVV''\chi'(V).
\end{align*}
\]
It is noticed that \(\mathcal{L} : H^2(\mathbb{R}) \times H^2(\mathbb{R}) \to X\), where \(\mathcal{L}\) is given in (59) and \(X = L^2(\mathbb{R}) \times L^2(\mathbb{R})\), is a closed operator in the space \(X\). Noticing that
\[
A_1(\pm \infty) = s, \quad A_2(\pm \infty) = \mu(1 - 2u_{\pm}) - u_{\pm} \chi(V),
\]
the asymptotic operator \(\mathcal{L}_\pm\) at \(\xi = \pm \infty\) is
\[
\mathcal{L}_\pm = \begin{pmatrix}
\left(D \frac{\partial}{\partial \xi}\right)^2 + s \frac{\partial}{\partial \xi} + \mu(1 - 2u_{\pm}) & -u_{\pm} \chi(V) \left(\frac{\partial}{\partial \xi}\right)^2 \\
-1 & c \frac{\partial}{\partial \xi} + \beta
\end{pmatrix}.
\] (60)

Let
\[
A^\pm(\tau) := \begin{pmatrix}
-D\tau^2 + s\tau i + \mu(1 - 2u_{\pm}) & u_{\pm} \chi(V) \tau^2 \\
-1 & s\tau i + \beta
\end{pmatrix}.
\] (61)

Then, by the spectral theory in [6], the boundary of essential spectrum \(\sigma_{ess}(\mathcal{L})\) is described by the curves \(S^+ \cup S^-\), where
\[
S^\pm = \{\lambda \in \mathbb{C} \mid \det(A_{\pm}(\tau) - \lambda I) = 0 \text{ for some } \tau \in \mathbb{R}\}.
\] (62)
Particularly, evaluating \((u_+, v_+) = (0, 0)\) into \(A^+(\tau)\) in (61), \(\lambda \in S^+\) satisfies
\[
\lambda = \beta + s\tau i, \quad -D\tau^2 + \mu + s\tau i.
\] (63)
Then, \(\text{Re}(\lambda) = \beta > 0\) for the first \(\lambda\) in (63) since the chemical growth rate \(\beta\) is positive. The second \(\lambda\) satisfies \(\text{Re}(\lambda) = -D\tau^2 + \mu > 0\) for some \(\tau \in \mathbb{R}\) since the rate of logistic cell growth \(\mu\) is positive.

Since \(\sigma_{ess}(\mathcal{L}) \subset \sigma(\mathcal{L})\) where \(\sigma(\mathcal{L})\) is the spectrum of \(\mathcal{L}\), we derive the following theorem:

**Theorem 6.1.** The operator \(\mathcal{L}\) in (59) satisfies
\[
\sigma(\mathcal{L}) \cap \{\lambda \in \mathbb{C} \mid \text{Re } \lambda > 0\} \neq \emptyset.
\] (64)
This theorem asserts that the traveling waves \((U(\xi), V(\xi))\) of system (1) is linearly unstable as desired in Theorem 1.3.
Acknowledgments. The authors would like to thank the referee for reading our article carefully and for your valuable comments and suggestions to improve this work.

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Received November 2018; revised January 2019.

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