On position operator spectral measure
for deformed oscillator in the case
of indeterminate Hamburger moment problem

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Abstract

The spectral measure of the position (momentum) operator $X$ for $q$-deformed oscillator is calculated in the case of the indeterminate Hamburger moment problem. The exposition is given for concrete choice of generators for $q$-oscillator algebra, although developed technique apply for every other cases with indeterminate moment problem. The Stieltjes transformation $m(z)$ of spectral measure is expressed in terms of the entries of Jacobi matrix $X$ only. The direct connection between values of parameters labeling the spectral measures and related selfadjoint extensions of $X$ is established.

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1 Introduction

The connection of the quantum harmonic oscillator with the Hermite polynomials $H_n(x)$ is well-known from the early days of quantum theory. The coordinate operator $X = \frac{1}{\sqrt{2}} (b + b^\dagger)$, being the Jacobi matrix in the number representation

$$N|n\rangle = b^\dagger b|n\rangle = n|n\rangle,$$  

$n = 0, 1, 2, \ldots,$

has the eigenvectors, components of which in coordinate representation are proportional to $H_n(x)$.

The discovery of the quantum groups gave rise to connection of quantum algebras to variety of $q$-special functions. In particular, it was pointed out that deformed oscillator algebra $A_q$ [1-6] is related to $q$-Hermite polynomials (see e.g. [7, 8, 9]). The possibility of different choice of $q$-oscillator generators $a(\lambda), a^\dagger(\lambda)$ results in [10-13] different behaviour.
of the corresponding Jacobi matrix entries

\[(X(q, \lambda))_{i,j} = b_i \delta_{i,j-1} + b_{i-1} \delta_{i,j+1}\]

and different q-Hermite polynomials \(H_n(x; q, \lambda)\).

The spectral measure of a selfadjoint extension of the symmetric operator \(X(q, \lambda)\), and of the orthogonal q-polynomials is related to the classical Hamburger moment problem \([14]-[18]\). This moment problem can be determined (the measure is unique) or indeterminate (a family of measures) according to the behaviour of the Jacobi matrix entries \([14, 15]\). It was pointed out in \([19]\) that even for the quantum harmonic oscillator the Hamburger moment problem for the Jacobi matrices \(J^{(k)} = b^k + (b^*)^k\) (appearing in the description of higher power squeezed states) is indeterminate for \(k > 2\). As far as we know, for all the choices of q-oscillator generators \([20, 21]\), when the corresponding q-Hermite polynomials and the spectral measures where known from the q-analysis \([22, 23, 24]\) the moment problems are determined.

The calculation of the spectral measure of the position (momentum) operator \(X(q, \lambda)\) in the case of the indeterminate Hamburger moment problem is the aim of this paper. Although the developed construction applies to each choice of generators of the q-oscillator algebra \(A_q\), the particular expressions and proofs are given for the q-oscillator \([4, 5]\) with nonzero entries of the Jacobi matrix equal to \(b_n = \sqrt{n+1}\), where "symmetric" basic number \(a\) being defined as [a] = \(\frac{q^a - q^{-a}}{q - q^{-1}}\). Let us note that [a] is exponentially growing for \(q > 1\), as well as for \(0 < q < 1\) in contrast with the q-number \([a; q] = \frac{1 - q^a}{1 - q}\) usual for q-analysis \([24]\) and basic hypergeometric functions \([22, 23]\).

Following to the general theory of the moment problem and Jacobi matrices \([14, 15, 16]\) we express the Stieltjes transformation \(m(z)\) of the spectral measure in terms of the entries \(b_n\) of Jacobi matrix only. We also establish the direct connection between the values of parameters labeling the spectral measures and associated selfadjoint extensions of position operator \(X\), which was missing in the general considerations.

The paper is organized as follows. The relations between harmonic oscillator and Hermite polynomials, main formulas of the moment problem and definitions of the q-deformed oscillator algebra \(A_q\) and q-Hermite polynomials \([1]\) are briefly reviewed in Sec.2. In this section the Hamburger power moment problem for deformed oscillator of considered type is also formulated. This moment problem is indeterminate one. In this case extremal spectral measure is concentrated \([14, 15, 16]\) on the set of zeros of entire function expressed in terms of the related orthogonal polynomials. Let us note that in the cases considered, for example, in the papers \([17, 25, 26]\) where such measures, also for indeterminate Hamburger moment problem, are constructed the founded measures are expressed in terms of the standard for q-analysis symbol \([\alpha; q]\) and represent the q-analogues of some classical special functions. The entries of
the Jacobi matrix for the \( q \)-oscillator generators under concideration are expressed in terms of the symmetric \( q \)-number \([n]\) or its one parameter generalization \([n; q, \lambda] = q^{\lambda n}[n]\). The spectral measures we are interested in, are expressed as infinite series with this \( q \)-numbers. These series were not study yet in the \( q \)-analysis and they could be of interest by themselves. Our main considerations and computations are given in Sec.3. In this section we first compute the value \( m(i) \) of the Stieltjes transformation \( m(z) \) of the spectral measure at the point \( i \), and after that we found the value of \( m(z) \) at arbitrary complex value \( z \). Then in the Sec.4 we give the construction of the support of spectral measure \( \sigma_{\varphi_0} \). In Sec.5 concrete examples of the spectral measures \( \sigma_0 \) and \( \sigma_\pi \) are given.

2 Background material on \( q \)-oscillator, classical moment problem and \( q \)-Hermite polynomials

2.1 Spectral theory of Jacobi matrices, classical moment problem and \( q \)-orthogonal polynomials

We recall some of the results on the spectral theory of Jacobi matrices and their relation with orthogonal polynomials\(^2\).

Let operator \( X \) acting on the standard orthonormal basis \( \{e_n \mid n \in \mathbb{Z}_+\} \) of \( l^2(\mathbb{Z}_+) \) by

\[
X e_n = b_n e_{n+1} + a_n e_n + b_{n-1} e_{n-1}, \quad a_n \geq 0, \ b_n \in \mathbb{R}.
\]  

(2.1)

Then \( X \) can be represented (in \( l^2(\mathbb{Z}_+) \)) by the infinite dimensional three-diagonal matrix

\[
X = \begin{pmatrix}
a_0 & b_0 & 0 & 0 & \cdots & \cdots \\
0 & a_1 & b_1 & 0 & \cdots & \cdots \\
0 & 0 & a_2 & b_2 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix},
\]  

(2.2)

which is known as a Jacobi matrix.

With such Jacobi matrix one can associate polynomials \( P_n(x) \) of degree \( n \) in \( x \) by the recurrence relation

\[
b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x) = x P_n(x),
\]  

(2.3)

\(^2\)For more information we refer to the books by Berezanskii \([15]\) [Ch. VII, §1], Ahiezer \([14]\) and Shohat and Tamarkin \([16]\).
with "initial conditions"

\[ P_0(x) = 1, \quad P_{-1}(x) = 0. \]  \hfill (2.4)

In the following we assume that \( a_n, b_n \in \mathbb{R} \) and \( a_n = 0, b_n > 0 \). Under this condition all polynomials \( P_n(x) \) have real coefficients and \( P_n(-x) = (-1)^n P_n(x) \).

Note that the recurrence relation (2.3) has two linearly independent solutions. The polynomials \( P_n(x) \) solving the recurrence relation (2.3) and fulfilling the initial conditions (2.4) are called the polynomials of the first kind. One has \( \deg P_n(x) = n \).

The independent set of solutions of the relation (2.3) consists of polynomials \( Q_n(x) \) fulfilling the initial conditions

\[ Q_0(x) = 0, \quad Q_1(x) = \frac{1}{b_0}, \]  \hfill (2.5)

and we have \( \deg Q_n(x) = n - 1 \). Such polynomials are called the polynomials of the second kind for the Jacobi matrix \( X \) (2.2).

These polynomials of the first and second kind are related through

\[ P_{n-1}(x)Q_n(x) - P_n(x)Q_{n-1}(x) = \frac{1}{b_{n-1}}, \quad (n = 1, 2, 3, \ldots). \]  \hfill (2.6)

With the Jacobi matrix \( X \) (2.2) one can associate the so-called Hamburger (power) moment problem [14]. For given number sequence \( s_n \) find the measure \( \sigma \) on the line such that

\[ s_n = \int_{-\infty}^{\infty} t^n d\sigma(t), \quad n = 0, 1, 2, \ldots; \quad s_n = \alpha_0, \quad t^n = \sum_{i=0}^{n} \alpha_i P_i(t; q). \]  \hfill (2.7)

If such measure is defined uniquely then the moment problem is determined one. One says that the moment problem is indeterminate if one has infinite family of measures which solves the relation (2.7) for given sequence of moments \( s_n \). There is one to one correspondence between \( X \) and \( s_n \) [14, 15] :

\[ s_n = (e_0, X^n e_0). \]  \hfill (2.8)

The matrix (2.2) defines the operator \( X \) which is symmetric, and its deficiency indices are \( (0, 0) \) (which mean that its closure \( \overline{X} \) is a selfadjoint operator) or \( (1, 1) \) (in this case \( \overline{X} \) has many selfadjoint extensions).

Note that the deficiency indices are \( (0, 0) \), and thus \( \overline{X} \) is selfadjoint in \( \ell^2(\mathbb{Z}_+) \), if and only if the related orthonormal polynomials \( P_n(x) \), correspond to a determined moment problem. By Favard’s theorem, for a selfadjoint operator \( \overline{X} \) there exists a unique positive measure \( m \) on the real line such that the polynomials \( P_n(x) \) are orthonormal

\[ \int_{\mathbb{R}} P_n(x)P_m(x) dm(x) = \delta_{n,m}. \]  \hfill (2.9)
The measure is obtained by $m(B) = \langle E(B)e_0, e_0 \rangle$, where $B \subseteq \mathbb{R}$ is a Borel set, and $E$ denotes the spectral decomposition of the selfadjoint operator $\mathbf{X}$. The point spectrum corresponds to the discrete mass points in $dm$. The spectral decomposition $E$ of $\mathbf{X}$ is related to the orthogonality measure by

$$\langle E(B)e_n, e_m \rangle = \int_B P_n(x)P_m(x) \, dm(x),$$

where $B \subseteq \mathbb{R}$ is a Borel set.

If, in general case $a_n \neq 0$, $|a_n| \leq C$, $n = 0, 1, 2, \ldots$, and there exist $N$, such that the inequality $b_{n-1}b_{n+1} \leq b_n^2$ is hold for all $n > N$ then in the case when

$$\sum_{n=0}^{\infty} \frac{1}{b_n} < \infty, \quad (2.10)$$

operator $\mathbf{X}$ is not selfadjoint operator, and has infinitely many selfadjoint extensions.

The operator $\mathbf{X}$ has deficiency indices $(0, 0)$ iff the series $\sum_{n=0}^{\infty} |P_n(z)|^2$ is divergent for all $z \in \mathbb{C}$, $\text{Im} \ z \neq 0$. If the operator $\mathbf{X}$ has deficiency indices $(1, 1)$ then this series converges for all $z \in \mathbb{C}$, $\text{Im} \ z \neq 0$. In the case when $\mathbf{X}$ has deficiency indices $(1, 1)$ the deficiency subspaces $N_\varphi$ are all one-dimensional ones and spanned by the vectors $\sum_{n=0}^{\infty} P_n(z) |n>.

The spectral properties of selfadjoint extensions of operator $\mathbf{X}$ are intimately linked with the properties of measures which solves the related moment problem. Let a parameter $\varphi_0$ label selfadjoint extensions $\mathbf{X}_{\varphi_0}$ of the operator $\mathbf{X}$ in the case of indeterminate moment problem. Then the spectral measure $\sigma_{\varphi_0}$ gives the “extremal” solution of the Hamburger moment problem (2.7) related to the selfadjoint extension $\mathbf{X}_{\varphi_0}$.

It is known [27], that for complex number $\omega$

$$\omega = c_\infty(i) - ie^{-i\varphi_0}r_\infty(i) \quad (2.11)$$

one has

$$\omega = \int_{-\infty}^{\infty} \frac{\sigma_{\varphi_0}(d\lambda)}{\lambda - i}. \quad (2.12)$$

The center $c_\infty(i)$ and radius $r_\infty(i)$ of the limit Weyl - Hamburger circle are given by the formulas

$$c_\infty(i) = \frac{i}{2} - \sum_{k=0}^{\infty} Q_k(i; q) P_k(-i; q) \quad \sum_{k=0}^{\infty} \left| P_k(i; q) \right|^2, \quad r_\infty(i) = \left( 2 \sum_{k=0}^{\infty} \left| P_k(i; q) \right|^2 \right)^{-1}. \quad (2.13)$$
The Stieltjes transformation
\[ m(z) = \int_{-\infty}^{\infty} \frac{\sigma_{\varphi_0}(d\lambda)}{\lambda - z} \]  
(2.14)
of the measure \( \sigma_{\varphi_0} \) is connected with its value \( m(i) \) at the point \( i \) by the relation
\[ m(z) = \frac{E_0(z, i)m(i) + E_1(z, i)}{D_0(z, i)m(i) + D_1(z, i)}. \]  
(2.15)
Here \( E_i = \lim_{n \to \infty} E_i^{(n)} \), \( D_i = \lim_{n \to \infty} D_i^{(n)} \), and
\[
\begin{align*}
E_0^{(n-1)}(z, i) &= \sqrt{n} \{ Q_n(z; q)P_{n-1}(i; q) - P_n(i; q)Q_{n-1}(z; q) \}, \\
E_1^{(n-1)}(z, i) &= \sqrt{n} \{ Q_n(z; q)Q_{n-1}(i; q) - Q_n(i; q)Q_{n-1}(z; q) \}, \\
D_0^{(n-1)}(z, i) &= \sqrt{n} \{ P_n(i; q)P_{n-1}(z; q) - P_n(i; q)P_{n-1}(z; q) \}, \\
D_1^{(n-1)}(z, i) &= \sqrt{n} \{ Q_n(i; q)P_{n-1}(z; q) - Q_n(i; q)P_{n-1}(z; q) \}.
\end{align*}
\]  
(2.16)
From given \( m(z) \) one can reconstruct the spectral measure \( \sigma_{\varphi_0} \) using the inverse Stieltjes transformation
\[ \sigma_{\varphi_0}(\Delta) = \lim_{\tau \to 0} \int_{\Delta} \psi(\gamma; \tau)d\gamma; \quad z = \gamma + i\tau, \quad \psi(\gamma; \tau) = \frac{m(z) - m(\bar{z})}{2\pi i}. \]  
(2.17)

### 2.2 Harmonic oscillator

It is well-known that the quantum-mechanical operators of position \( X_b \) and canonically conjugate momentum \( P_b \) are selfadjoint operators with real line as continuous spectrum. These operators are unbounded and on the dense domain in the Hilbert space \( \mathcal{F} \) they fulfill the famous commutation relations
\[ [X_b, P_b] = i\hbar I, \]  
(2.18)
ultimately connected with the Heisenberg uncertainty relation.

These operators related with creation \( b^\dagger \) and annihilation \( b \) operators by the formulae
\[ X_b = \frac{1}{\sqrt{2}} (b^\dagger + b), \quad P_b = \frac{1}{i\sqrt{2}} (b^\dagger - b). \]  
(2.19)
Operators \( b^\dagger, b \) are also unbounded, densely defined in \( \mathcal{F} \) and adjoint to each other \( ((b^\dagger) = b) \). Consider the realization \( \mathcal{F} = \ell^2(\mathbb{Z}_+) \) in which the selfadjoint number operator \( N_b = b^\dagger b \) \( ((N_b)^\dagger = N_b) \) is diagonal. There exist the so-called vacuum state \( |0\rangle \) defined as unique state
annihilated by \( b, b|0\rangle = 0 \). Other basis vectors are obtained by action of the creation operator \( b^\dagger \)
\[
|n\rangle = \frac{1}{\sqrt{n!}} \left( b^\dagger \right)^n |0\rangle, \quad n = 0, 1, 2, \ldots
\] (2.20)

On these basis states the operators \( b^\dagger, b \) and \( N_b \) acts according to
\[
N_b |n\rangle = n |n\rangle ;
\]
\[
b^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \quad n \geq 0;
\]
\[
b |n\rangle = \sqrt{n} |n-1\rangle , \quad n \geq 1, \quad b |0\rangle = 0.
\] (2.21)

This realization is known as the Fock (or number) representation. According to the famous von Neumann theorem this representation is unique irreducible representation up to unitary equivalence.

The operator \( X_b \) in this realization is represented by infinite dimensional Jacobi matrix
\[
X_b = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & b_0 & 0 & 0 & \cdots \cdots \\
0 & b_0 & b_1 & 0 & \cdots \cdots \\
0 & b_0 & b_1 & b_2 & \cdots \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix},
\] (2.22)

with \( b_n = \sqrt{n + 1} \). Obviously operator \( X_b \) is symmetric and due to
\[
\sum_{k=0}^{\infty} \frac{1}{b_k} = \infty,
\] (2.23)
the operator \( X_b \) is selfadjoint. It is wellknown that operators \( X_b = \frac{1}{\sqrt{2}} (b^\dagger + b) \) and \( P_b = \frac{1}{i\sqrt{2}} (b^\dagger - b) \) are related by the unitary transformation \( b^\dagger \rightarrow ib \). These operators (and their closures) are simultaneously selfadjoint or not selfadjoint. So, in what follows, we restrict consideration to the case of position operator and its \( q \)-analogues.

In the theory of Lie groups and quantum mechanics, special functions appear as particular matrix elements (overlap coefficients) of appropriate operators in corresponding representations (realizations): examples are exponential functions, as coherent states in the Bargmann-Fock (holomorphic) representation of \( \mathcal{H}_0 \)
\[
\exp(\pi z) = \langle w|z \rangle, \quad |z\rangle = e^{zb^\dagger}|0\rangle, \quad b|z\rangle = z|z\rangle,
\] (2.24)
and Hermite polynomials, as eigenvectors of the operator \( N \), in the coordinate representation,

\[
H_n(x) \sim \langle n|x \rangle, \quad (b + b^\dagger)|x\rangle = \sqrt{2} x|x\rangle.
\]

To find the generalized eigenvectors for position operator \( X_b \)

\[
X_b |x\rangle = x |x\rangle \tag{2.25}
\]

we expand the state \(|x\rangle \) in Fock basis \(|n\rangle \)

\[
|x\rangle = \sum_{n=0}^{\infty} P_n(x) |n\rangle \tag{2.26}
\]

substitute (2.20) and (2.21) into (2.25). Then from (2.21) we obtain the following recurrence relation

\[
\sqrt{2} x P_n(x) = \sqrt{n} P_{n-1}(x) + \sqrt{n+1} P_{n+1}(x), \quad n \geq 0, \tag{2.27}
\]

for coefficients \( P_n(x) \) of (2.26) with initial conditions

\[
P_0(x) = 1, \quad P_{-1}(x) = 0. \tag{2.28}
\]

This means that

\[
P_n(x) = \frac{1}{\sqrt{2^n n!}} H_n(x), \tag{2.29}
\]

where \( H_n(x) \) is the usual Hermite polynomials

\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right), \tag{2.30}
\]

with recurrence relation

\[
x H_n(x) = n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x). \tag{2.31}
\]

The basis states of the Fock representation, i.e. eigenvectors for the number operator \( N = b^\dagger b \) are represented by

\[
|n_b\rangle = \left[ n! 2^n \sqrt{\pi} \right]^{-1/2} H_n(x) e^{-\frac{x^2}{2}} \tag{2.32}
\]

The Hermite polynomials are orthogonal polynomials

\[
\int_{-\infty}^{\infty} H_m(x) H_n(x) d\sigma(x) = \delta_{mn} d_n^2, \tag{2.33}
\]
with measure
\[ d\sigma(x) = e^{-x^2} \, dx \] (2.34)
and normalization
\[ d_2^n = 2^n \, n! \, \sqrt{\pi} \] (2.35)

The classical Hamburger moment problem connected with Jacobi matrix \( X_b \) (2.22) is determined ones and its unique solution is given by (2.34). This means that the operator \( X_b \) (and \( P_b \)) has zero deficiency indices \((0, 0)\) and thus selfadjoint.

### 2.3 Position operator for deformed oscillator and \( q \)-Hermite polynomials

In view of its connection with quantum groups and algebras the deformed oscillator became the rather popular subject in the last years (see e.g. [10, 29]). Here we briefly mention only main definitions and some of the properties of \( q \)-oscillator needed in the following.

Let us remark that for the \( q \)-deformed oscillator the situation became significantly richer and interesting that in the case of the usual harmonic oscillator recalled above. Indeed in this case together with \( q \)-analogue of the Fock representation there is plenty of inequivalent representations. Moreover, the related Hamburger power moment problem can be indeterminate thus admitting different spectral measures. From the point of view of the functional analysis this means that position operator \( X \) for the \( q \)-oscillator may have deficiency indices \((1, 1)\) and not only \((0, 0)\). In the first case the operator \( X \) has many different selfadjoint extensions.

The deformed oscillator algebra, \( \mathcal{A}_q \), is generated by three elements \( a, a^\dagger, N \) with defining relations\(^3\)

\[ aa^\dagger - qa^\dagger a = 1, \] (2.36)
\[ [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \] (2.37)

Note that the generator \( N \) is considered here as an independent element, and we restrict ourselves to the case of positive real \( q \in (0, \infty) \). The algebra \( \mathcal{A}_q \) has a central element [30],

\[ \zeta = q^{-N}([N; q] - a^\dagger a), \] (2.38)

where

\[ [N; q] := (1 - q^N)/(1 - q) \] (2.39)

is the standard basic number of \( q \)-analysis [23].

\(^3\)The relations (2.36) appears in [3]. In [1] this relations are studied in connection with the generalization of the Veneziano amplitude, by substitution of the \( q \)-\( \Gamma \)-function instead of the standard \( \Gamma \)-function.
In the original papers, the irreducible representation of $A_q$ with the vacuum state $|0\rangle$ ($a|0\rangle = 0$) was considered. The oscillator-type representation space $H_0$, in the basis of eigenvectors of the operator $N$, is

$$H_0 = \{ |n\rangle; \quad n = 0, 1, 2, \ldots; \quad a|0\rangle = 0, \quad |n\rangle = ([n;q]!!)^{-1/2}(a^\dagger)^n|0\rangle \}. \quad (2.40)$$

Due to the existence of a non-trivial central element $\zeta$, in addition to $H_0$, the algebra $A_q$ has a set of inequivalent irreducible representations ($0 < q < 1$) in the spaces $H_\gamma (\gamma \geq \gamma_c = (1-q)^{-1})$ parametrized by the value of the central element $\zeta = -\gamma [30]$, with the spectrum of $N$, the set of all integers $\mathbb{Z}$.

Considering $A_q$ as an associative algebra, any invertible transformation of the generators is admissible; in particular, there are some natural sets of the generators: $[3, 10]$, $AA^\dagger - q^{1/2}A^\dagger A = q^{-N/2}$, $[N, A] = -A$, $[N, A^\dagger] = A^\dagger$, (2.41)

related to the quantum algebra $sl_q(2)$ via the Schwinger realization $[4, 5]$, and the following set related to the $sl_q(2)$ algebra by a contraction procedure with fixed $q$ $[30]$,

$$[\alpha, \alpha^\dagger] = q^{-N}, \quad [N, \alpha] = -\alpha, \quad [N, \alpha^\dagger] = \alpha^\dagger. \quad (2.42)$$

The equivalence of these generators is given by the equalities $a = q^{N/2} A = q^{N/4} A$ $[3, 10]$, with an obvious one-parameter generators generalization, namely,

$$a(\lambda) = q^{-\frac{\lambda N}{2}} a, \quad a^\dagger(\lambda) = a^\dagger q^{-\frac{\lambda N}{2}}. \quad (2.43)$$

This leads to the commutation relations (still one degree of freedom)

$$a(\lambda)a^\dagger(\lambda) - q^{1-\lambda}a^\dagger(\lambda)a(\lambda) = q^{-\lambda N}. \quad (2.44)$$

One more formal parameter $\nu \in \mathbb{R}$ can be added by a shift $N \rightarrow N + \nu$. The corresponding set of $A_q$ generators is denoted by $W_{p, r}(q)$ $[13]$. As a consequence of (2.44), namely,

$$a(\lambda)(a^\dagger(\lambda))^m = (pa^\dagger(\lambda))^m a(\lambda) + (pa^\dagger(\lambda))^{m-1}r^N[m; \frac{r}{p}], \quad (2.45)$$

the normalized basis vectors of $H_0$ in terms of $a^\dagger(\lambda)$ are given by

$$|n\rangle = ([n;q,\lambda]!!)^{-1/2}(a^\dagger)^n|0\rangle$$

with the factorials defined as

$$[n;q,\lambda]! = \prod_{k=1}^n [k; q, \lambda], \quad [m; q, \lambda] = q^{\lambda(1-m)}[m; q]. \quad (2.46)$$

10
The classical moment problem refers also to $q$-Hermite polynomials: the latter are nothing but polynomials of the first kind \[14, 15\] for a Jacobi matrix $\mathcal{J}$ which is constructed as a “generalized coordinate" from the $q$-oscillator creation and annihilation operators \[9\],

$$
\mathcal{J}(\lambda) = a(\lambda) + a^\dagger(\lambda), \quad \mathcal{J}(\lambda) |x\rangle_\lambda = 2x |x\rangle_\lambda.
$$

Due to \[2.47\], these $q$-Hermite polynomials satisfy the following three-term recurrence relation:

$$
b_n(\lambda)H_{n-1}(x; q, \lambda) + b_{n+1}(\lambda)H_{n+1}(x; q, \lambda) = x H_n(x; q, \lambda).
$$

The measure entering into the $q$-Hermite polynomials $H_n(x; q, \lambda)$ orthogonality relations is connected with the solution of the Hamburger moment problem: this measure is known explicitly for some cases (see e.g.\[14, 15\]). This connection of the moment problem with Jacobi matrices gives rise to a generalized deformation of the oscillator identifying the matrix $\lambda k \delta_{n+1,k}, \lambda > 0$ with an annihilation operator $a$. Then one gets the Wigner commutation relation $[a, a^\dagger] = F(N)$ with $F(n) = b_n^2 - b_n^2$ and its central element $\zeta = (c^2(N) - a^\dagger a) + \text{const}$ (see also \[31, 32, 33\]). The $q$-special functions related to the other irreducible representations $H_\gamma$ of $\mathcal{A}_q$ are discussed in \[11\].

Let us recall that there exists several variants $q$-generalizations of Hermite polynomials (see for example \[34\]-\[39\]) but in all this definitions, based on the standard $q$-analysis, always nonsymmetrical basic number $[a; q]$ is used. Thus such $q$-Hermite polynomials are little to do with symmetrical $q$-oscillator. As far as we know our article \[9\] is the only work in which the attempt to define $q$-Hermite polynomials related to the symmetrical $q$-oscillator is initiated. In \[9\] we take as defining property of the $q$-Hermite polynomials the three point recurrence relation

$$
xH_n^q(x) = [n]H_{n-1}^q(x) + H_{n+1}^q(x), \quad H_0^q(x) = 1,
$$

and more physical condition that, as in the usual case, the generating function

$$
g(z, x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n^q(x)
$$

is the eigenfunction of coordinate operator $X = a^\dagger + a$ in $q$-holomorphic representation

$$
X = a^\dagger + a = z + qD_z, \quad X g(z, x) = x g(z, x),
$$

where $qD_z$ is symmetrical difference $q$-derivation defined by

$$
qD_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}.
$$
This gives the difference equation
\[ qD_z g(z, x) = (x - z)g(z, x). \] (2.54)

If we represent \( g(z, x) \) in the form
\[ g(z, x) = 1 + g_1(z, x) + g_2(z, x) + \ldots, \] (2.55)
we obtain the sequence of equations
\[ qD_z g_n(z, x) = (x - z)g_{n-1}(z, x), \quad g_0(z, x) = 1. \] (2.56)

Using modification on the case of symmetric basic number of the Jackson’s \( q \)-integral, one easily solves this equations.

As result one obtains
\[
\begin{align*}
g_0(z, x) &= 1, \\
g_1(z, x) &= xz - \frac{z^2}{[2]} - \frac{z^2}{[1][2]}, \\
g_2(z, x) &= x^2z^2 - \frac{xz^3}{[2][3]} ([1] + [2]) + \frac{z^4}{[4][3][1]}; \\
g_3(z, x) &= x^3z^3 - \frac{x^2z^4}{[3][4]} ([1] + [2] + [3]) + \frac{z^5}{[5][3][1]} \bigl([3][1] + [4]([1] + [2])\bigr) - \frac{z^6}{[6][5][3][1]}; \\
g_4(z, x) &= x^4z^4 - \frac{x^3z^5}{[4][5]} ([1] + [2] + [3] + [4]) + \frac{x^2z^6}{[6][7]} \bigl([3][1] + [4]([1] + [2]) + [5][1] + [2] + [3]\bigr) - \frac{xz^7}{[7][8]} \bigl([5][3][1] + [6][3][1] + [4]([1][2])\bigr) + \frac{z^6}{[6][7][5][3][1]}. \end{align*}
\] (2.57)

Now if we substitute (2.57) into (2.55) and join the terms with equal degree we obtain
\[
\begin{align*}
H_0^3(x) &= 1, \\
H_1^3(x) &= x, \\
H_2^3(x) &= x^2 - [1], \\
H_3^3(x) &= x^3 - x([1] + [2]), \\
H_4^3(x) &= x^4 - x^2([1] + [2] + [3]) + [3][1], \\
H_5^3(x) &= x^5 - x^3([1] + [2] + [3] + [4]) + x([3][1] + [4]([1] + [2])) + \frac{z^6}{[6][7][5][3][1]}. \end{align*}
\] (2.58)
It is not hard to check that such defined \( q \)-Hermite polynomials fulfil recurrent relation (2.50) which can be used to obtain concrete form of all other polynomials. General expression of \( q \)-Hermite polynomial looks as

\[
H^q_n(x) = x^n + \sum_{k=1}^{\epsilon(n/2)} (-1)^k x^{n-2k} \left( \sum_{m_k=2k-1}^{n-1} [m_k] \sum_{m_{k-1}=2k-3}^{m_k-2} [m_{k-1}] \sum_{m_{k-2}=2k-5}^{m_{k-2}-2} [m_{k-2}] \cdots \sum_{m_1=1}^{m_2-2} [m_1] \right).
\]

(2.59)

Here \( \epsilon(x) = \text{Ent}(x) \) denotes the integer part of \( x \).

2.4 The Hamburger power moment problem for symmetricaly deformed oscillator

Below we consider the deformed oscillator algebra \( \mathcal{A}_q \), \( q \in \mathbb{R} \) with fixed generators \( a^\pm, N \) fulfilling the commutation rules

\[
a^-a^+ - qa^+a^- = q^{-N}, \quad [N, a^\pm] = \pm a^\pm,
\]

(2.60)

and usual hermiticity conditions \( (a^\pm)^\dagger = a^\mp, N^\dagger = N \).

In the Fock representation action of the generators are given by

\[
N |n\rangle = n |n\rangle; \quad a^+ |n\rangle = \sqrt{n+1} |n+1\rangle, \quad n \geq 0;
\]

\[
a^- |n\rangle = \sqrt{n} |n-1\rangle, \quad n \geq 1, \quad a^- |0\rangle = 0;
\]

(2.61)

where

\[
|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle
\]

and we use the notation

\[
[\alpha] \equiv [\alpha]_q = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}.
\]

(2.62)

According to these relations in the Fock space with given above basis \( \{|n\rangle\}_{n=0}^\infty \) the position operator \( X := a^+ + a^- \) is described by Jacobi matrix \( \{2.22\} \) in which now \( b_k = \sqrt{k+1}, \quad k = 0, 1, 2, \ldots \).

From eigenvalue equation \( X |x\rangle = x |x\rangle \), \( |x\rangle = \sum_{n=0}^\infty P_n(x; q) |n\rangle \), one obtains following recurrent relations for \( q \)-Hermitian polynomials \( P_n(x; q) \)

\[
\sqrt{n} P_{n-1}(x; q) + \sqrt{n+1} P_{n+1}(x; q) = xP_n(x; q), \quad n \geq 1
\]

\[
P_1(x; q) = xP_0(x; q), \quad P_0(x; q) = 1.
\]

(2.63)
These q-Hermitian polynomials $P_n(x; q)$ are polynomials of the 1-st kind for Jacobi matrix $X$. Note that polynomials $Q_n(x; q)$ of the 2-nd kind for the same Jacobi matrix fulfil the same recurrence relations (2.63) but with other initial conditions

$$Q_1(x; q) = 1, \quad Q_0(x; q) = 0.$$  

The wellknown connection [14] between polynomials of the 1-st and 2-nd kind looks in our case as

$$P_{n-1}(x; q)Q_n(x; q) - P_n(x; q)Q_{n-1}(x; q) = \frac{1}{\sqrt{n}} \quad \text{(2.64)}$$

Because for $q > 0$, $q \neq 1$ we have

$$\sqrt{n}\sqrt{n+2} \leq [n+1], \quad n \geq 0, \quad \sum_{n=0}^{\infty} [n+1]^{-\frac{3}{2}} < \infty$$

then, according to ([15], thm 1.5) the closure $\overline{X}$ of the position operator $X$ is only closed but not selfadjoint operator and has deficiency indices $(1.1)$. This means that coordinate operator $X$ has family of selfadjoint extensions.

The related Hamburger moment problem is indeterminate one, and there exist family of measures $\sigma$ fulfilling relations (2.7). The spectral measure $\sigma_{\varphi_0}$ gives the ”extremal” solution of the Hamburger moment problem (2.7) related to the selfadjoint extension $X_{\varphi_0}$.

**Remark.** The Stieltjes transformation $m(z)$ of the spectral measure $\sigma_{\varphi_0}$ one can obtain by the relation [14]

$$m(z) = \frac{A(z)t - C(z)}{B(z)t - D(z)} \quad \text{(2.65)}$$

where the elements of Nevanlinna matrix

$$\begin{pmatrix} A(z) & C(z) \\ B(z) & D(z) \end{pmatrix} \quad \text{(2.66)}$$

are connected with $E_i^{(n)}$ and $D_i^{(n)}$ by the relations

$$\begin{align*}
A(z) &= E_1(z; 0) \quad B(z) = -D_1(z; 0) \\
C(z) &= E_0(z; 0) \quad D(z) = -D_0(z; 0)
\end{align*} \quad \text{(2.67)}$$

Unfortunately, there is no way of finding at once the value $t$ in (2.63) which correspond to the considered selfadjoint extension of $X$, labelled by $\varphi_0$. To obtain the Stieltjes transformation $m(z)$ of the spectral measure $\sigma_{\varphi_0}$ we need to use the more complicated relation (2.13).

As it was mentioned above the inverse Stieltjes transformation [28, 13] allows to reconstruct the spectral measure $\sigma_{\varphi_0}$ from given $m(z)$ according to formula (2.17).
3 Stieltjes transformation $m(z)$ of the spectral measure

3.1 Computation of $m(i)$

In this section we will get the explicit form of the Stieltjes transformation $m(i)$ of the spectral measure $\sigma_{\varphi_0}$ in term of the elements of Jacobi matrix $X$. In this connection we need to consider some non-standard $q$-series, which are possibly of a special interest by themselves. The polynomials $P_n$ and $Q_n$ can be represented in the form

$$P_n(x; q) = \sum_{m=0}^{\infty} (-1)^m \frac{\alpha_{2m-1, n-1} x^{n-2m}}{[n]!}$$  \hspace{1cm} (3.1)

$$\alpha_{-1, n-1} \equiv 1; \quad \alpha_{2m-1, n-1} = \sum_{k_1=2m-1}^{n-1} [k_1] \sum_{k_2=2m-3}^{k_1-2} [k_2] \cdots \sum_{k_m=2}^{k_{m-1}-2} [k_m], \quad m \geq 1 \hspace{1cm} (3.2)$$

$$Q_{n+1}(x; q) = \sum_{m=0}^{\infty} (-1)^m \frac{\beta_{2m, n} x^{n-2m}}{[n+1]!}$$  \hspace{1cm} (3.3)

$$\beta_{0, n} \equiv 1; \quad \beta_{2m, n} = \sum_{k_1=2m-1}^{n} [k_1] \sum_{k_2=2m-2}^{k_1-2} [k_2] \cdots \sum_{k_m=2}^{k_{m-1}-2} [k_m], \quad m \geq 1, \hspace{1cm} (3.4)$$

recall that $\epsilon(x) = \operatorname{Ent}(x)$ denotes the integer part of $x$. We remark that the relations (3.1)-(3.4) give us the non-standard representation for the polynomials $P_n$ and $Q_n$. So we need to obtain further some properties of the coefficients $\alpha_{m,n}$ and $\beta_{m,n}$.

From recurrence relations for $P_n$ and $Q_n$ (2.63) it follows that the coefficients $\alpha_{m,n}$ and $\beta_{m,n}$ fulfills the relations

$$\alpha_{2m-1, n} = [n] \alpha_{2m-3, n-2} + \alpha_{2m-1, n-1}; \hspace{1cm} (3.5)$$

$$\beta_{2m, n} = [n] \beta_{2m-2, n-2} + \beta_{2m, n-1}. \hspace{1cm} (3.6)$$

From the relations (2.11), (2.12) and (2.14) we have

$$m(i) = \int_{-\infty}^{\infty} \frac{\sigma_{\varphi_0}(d\lambda)}{\lambda - i} = c_\infty(i) - i e^{-i\varphi_0} r_\infty(i). \hspace{1cm} (3.7)$$

Thus to compute $m(i)$ we must compute the center $c_\infty(i)$ and the radius $r_\infty(i)$ of Weyl - Hamburger circle at a point $z = i$. 

15
Using the formula (5) from ([14], ch.1, §2) and (3.1) we have
\[
\sum_{k=0}^{n-1} |P_k(i; q)|^2 = \frac{\sqrt{n}}{2^n} (P_n(i; q) P_{n-1}(i; q) - P_{n-1}(i; q) P_n(i; q))
\]
\[
= \frac{1}{[n-1]!} \left( \sum_{\alpha=0}^{r(n)} \alpha_{2m-1,n-1} \right) \left( \sum_{\alpha=0}^{r(n-1)} \alpha_{2m-1,n-2} \right)
\]
(3.8)

Let us introduce the auxiliary functions
\[
\Psi(q) = \lim_{s \to \infty} \Psi_s(q), \quad \Psi_s(q) = 1 + \sum_{k=1}^{s} \frac{[2k-1]!!}{[2k]!!}
\]
(3.9)
\[
\Phi(q) = \lim_{s \to \infty} \Phi_s(q), \quad \Phi_s(q) = 1 + \sum_{k=1}^{s} \frac{[2k]!!}{[2k+1]!!}
\]
(3.10)

Here
\[
[2n]!! = [2n] [2n-2] \cdots [2], \quad [2n-1]!! = [2n-1] [2n-3] \cdots [1].
\]
It is not difficult to check that for \( q > 0, q \neq 1 \) the functions \( \Psi(q) \) (3.9) and \( \Phi(q) \) (3.10) are well defined as convergent q-series. In fact it is follow from the inequality \([2s] \geq ([2s-1][2s+1])^{\frac{1}{2}}\) that
\[
\frac{[2k-1]!!}{[2k]!!} \leq \frac{[1]^{\frac{1}{2}}}{[2k+1]^{\frac{1}{2}}}
\]
(3.11)

In view of \( \sum_{n=0}^{\infty} [n+1]^{-\frac{1}{2}} < \infty \) for \( q > 0, q \neq 1 \) we have that \( \Psi(q) \) is well defined. The same is true of \( \Phi(q) \).

It is convenient to take out the leading terms \( \alpha_{2(n+1)-1,n-1} \) and \( \alpha_{2(n+1)-2,n-2} \) from the sums in the first and the second parenthessises in the right hand side of the expression (3.8). This gives us the common factor equal to

1) \( \alpha_{2p-1,2p-1} = [2p]!! \psi_p(q), \quad \text{if} \quad n = 2p + 1 \)
2) \( \alpha_{2p-1,2p-2} = [2p-1]!! \psi_{p-1}(q), \quad \text{if} \quad n = 2p \)

(3.12)

where we take into account that from (3.2) and \( \Phi(q) \) (3.10) we have
\[
\alpha_{2p-1,2p} = [2p]!! \psi_p(q), \quad \alpha_{2p-1,2p-1} = [2p-1]!!.
\]
(3.13)

This allows us rewrite (3.8) in the form

1) \( \sum_{k=0}^{2p} |P_k(i; q)|^2 = \psi_p(q) \sum_{m=0}^{p} \frac{\alpha_{2m-1,2p}}{\alpha_{2p-1,2p}} \sum_{m=0}^{p} \frac{\alpha_{2m-1,2p-1}}{\alpha_{2p-1,2p-1}} \)
2) \( \sum_{k=0}^{2p-1} |P_k(i; q)|^2 = \psi_{p-1}(q) \sum_{m=0}^{p} \frac{\alpha_{2m-1,2p-1}}{\alpha_{2p-1,2p-1}} \sum_{m=0}^{p-1} \frac{\alpha_{2m-1,2p-2}}{\alpha_{2p-1,2p-2}} \)

(3.14)

16
With help of the recurrent relations (3.3) one obtain

\[
(A_\alpha^{(1)})_p := \sum_{m=0}^{p} \frac{\alpha_{2m-1,2p-1}}{\alpha_{2p-1,2p-1}} = 1 + \sum_{k=1}^{p} \frac{1}{[2k-1]!!} \sum_{m=0}^{k-1} \alpha_{2m-1,2k-2},
\]

\[
(A_\alpha^{(2)})_p := \sum_{m=0}^{p} \frac{\alpha_{2m-1,2p}}{\alpha_{2p-1,2p}} = 1 + \frac{1}{\Psi(p)} \sum_{k=1}^{p} \frac{1}{[2k]!!} \sum_{m=0}^{k-1} \alpha_{2m-1,2k-1};
\]  

(3.15)

It is clear, that \((A_\alpha^{(i)})_p, i = 1, 2,\) are the positive increasing quantities. So exist the limits \(A_\alpha^{(i)} := \lim_{p \to \infty} (A_\alpha^{(i)})_p, i = 1, 2.\) It follows from (3.14) and the inequalities

\[
\sum_{k=0}^{\infty} |P_k(i,q)|^2 < \infty, \quad \Psi(q) < \infty, \quad (q > 0, q \neq 1),
\]

(3.16)

that the expressions

\[
A_\alpha^{(1)} = 1 + \sum_{k=1}^{\infty} \frac{1}{[2k-1]!!} \sum_{m=0}^{k-1} \alpha_{2m-1,2k-2},
\]

(3.17)

\[
A_\alpha^{(2)} = 1 + \frac{1}{\Psi(q)} \sum_{k=1}^{\infty} \frac{1}{[2k]!!} \sum_{m=0}^{k-1} \alpha_{2m-1,2k-1},
\]

(3.18)

are well defined for \(q > 0, q \neq 1.\)

Finally from the relations (2.13), (3.14) - (3.18) we obtain the following expression for the radius of limit Weyl - Hamburger circle

\[
r_\infty(i) = (2A_\alpha^{(1)} \Psi(q) A_\alpha^{(2)})^{-1}.
\]

(3.19)

Now we go to consideration of the center \(c_\infty(i) = \lim_{n \to \infty} c_n(i),\) where (see [14])

\[
c_n(i) = -\frac{D_1^{(n-1)}(-i;i)}{2i \sum_{k=0}^{\infty} |P_k(i)|^2}
\]

(3.20)

For terms of the numerator of this expression

\[
D_1^{(n-1)}(-i;i) = A(n) + B(n)
\]

(3.21)

in view of the relations (2.16), (3.1) and (3.3) one obtain

\[
A(n) = -\sqrt{n} Q_{n-1}(i) P_n(-i) = \frac{1}{[n-1]!} \left( \sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \beta_{2m,n-2} \right) \left( \sum_{m=0}^{n-1} \alpha_{2m-1,n-1} \right)
\]

17
$B(n) = \sqrt{n}Q_n(i)P_{n-1}(i) = \frac{1}{[n-1]!} \left( \sum_{m=0}^{\epsilon(n-1)} \beta_{2m,n-1} \right) \left( \sum_{m=0}^{\epsilon(n-1)} \alpha_{2m-1,n-2} \right)$

From (3.4) and (3.10) one obtain also

$$\beta_{2p,2p+1} = [2p+1]!! \Phi(q), \quad \beta_{2p,2p} = [2p]!!$$

Using (3.13) and (3.22) and repeat the calculations used above in consideration of the expression (3.19) we obtain

$$\lim_{p \to \infty} A(2p) = \lim_{p \to \infty} B(2p + 1) = A_\alpha^{(1)}A_\beta^{(1)};$$

$$\lim_{p \to \infty} A(2p + 1) = \lim_{p \to \infty} B(2p) = \Psi(q)\Phi(q)A_\alpha^{(2)}A_\beta^{(2)},$$

where together with (3.17) and (3.18) we use the notation

$$A_\beta^{(1)} = 1 + \sum_{k=1}^{\infty} \frac{1}{[2k]!!} \sum_{m=0}^{k-1} \beta_{2m,2k-1}$$

$$A_\beta^{(2)} = 1 + \frac{1}{\Phi(q)} \sum_{k=2}^{\infty} \frac{1}{[2k-1]!!} \sum_{m=0}^{k-2} \beta_{2m,2k-2}$$

It is not difficult to check that $A_\beta^{(1)}$ and $A_\beta^{(2)}$ are well defined for $q > 0, \ q \neq 1$.

Finally from the relations (3.19)-(3.21) and (3.23) we obtain

$$C_\infty(i) = \frac{i}{2} \left( A_\alpha^{(1)}A_\beta^{(1)} + \Psi(q)A_\alpha^{(2)}\Phi(q)A_\beta^{(2)} \right)$$

From (3.7), (3.19) and (3.26) we have

$$m(i) = \frac{-\sin \varphi_0}{A_\alpha^{(1)}\Psi(q)A_\alpha^{(2)}} + i \frac{-\cos \varphi_0 + A_\alpha^{(1)}A_\beta^{(2)} + \Psi(q)A_\alpha^{(2)}\Phi(q)A_\beta^{(2)}}{A_\alpha^{(1)}\Psi(q)A_\alpha^{(2)}}$$

Remark 1. Note that from (3.1) and (3.3) it follows that

$$Q_{n-1}(i; q)P_n(i; q) = (-1)^nQ_{n-1}(i; q)P_n(-i; q) = (-1)^{n-1}A(n) \frac{[n]}{\frac{1}{2}}$$

$$Q_n(i; q)P_{n-1}(i; q) = (-1)^{n-1}Q_n(i; q)P_{n-1}(-i; q) = (-1)^{n-1}B(n) \frac{[n]}{\frac{1}{2}}$$

Then from (2.64), (3.23) and (3.28) we have

$$W = -\Psi(q)A_\alpha^{(2)}\Phi(q)A_\beta^{(2)} + A_\alpha^{(1)}A_\beta^{(1)} = 1$$
Remark 2. We emphasize that all results of the present section are hold for every Jacobi matrix (2.22) with restrictions

\[ b_n b_{n+2} \leq (b_{n+1})^2, \quad n \geq 0; \quad \sum_{n=0}^{\infty} (b_n)^{-\frac{1}{2}} < \infty \]

which provide the associated moment problem to be indeterminate.

3.2 Computation of the \( m(z) \)

For computation of the \( m(z) \) (2.15) we must find the functions (2.16). To this end we represent the complex variable \( z \) in the polar form

\[ s = q^\kappa e^{i\varphi}, \quad q > 0, \quad q \neq 1, \quad \kappa \in \mathbb{R}, \quad 0 \leq \varphi \leq 2\pi \] (3.30)

For computation the functions (2.16) we use the same reasoning as in the sect.1 for obtaining the formula (3.19). For doing so we must introduce together with auxiliary quantities \( A_{\varepsilon}^{(i)} (\varepsilon = \alpha, \beta; \ i = 1, 2) \) also the following expressions

\[ A_{\alpha, \kappa}^{(1)}(\varphi) := 1 + \sum_{k=1}^{\infty} \frac{1}{[2k-1]!!} \sum_{m=0}^{k-1} (-1)^{k-m} (q^\kappa e^{i\varphi})^{2(k-m)} \alpha_{2m-1, 2k-2}; \] (3.31)

\[ A_{\alpha, \kappa}^{(2)}(\varphi) := 1 + \frac{1}{\Psi(q)} \sum_{k=1}^{\infty} \frac{1}{[2k]!!} \sum_{m=0}^{k-1} (-1)^{k-m} (q^\kappa e^{i\varphi})^{2(k-m)} \alpha_{2m-1, 2k-1}; \] (3.32)

\[ A_{\beta, \kappa}^{(1)}(\varphi) := 1 + \sum_{k=1}^{\infty} \frac{1}{[2k]!!} \sum_{m=0}^{k-1} (-1)^{k-m} (q^\kappa e^{i\varphi})^{2(k-m)} \beta_{2m, 2k-1}; \] (3.33)

\[ A_{\beta, \kappa}^{(2)}(\varphi) := 1 + \frac{1}{\Phi(q)} \sum_{k=2}^{\infty} \frac{1}{[2k-1]!!} \sum_{m=0}^{k-2} (-1)^{k-m-1} (q^\kappa e^{i\varphi})^{2(k-m-1)} \beta_{2m, 2k-2}; \] (3.34)

Note that \( A_{\varepsilon, \kappa=0}^{(i)}(\varphi = \frac{\pi}{2}) = A_{\varepsilon}^{(i)} \). In terms of the above notations we have

\[ E_0(z; i) = ie^{i\varphi} q^\kappa \Psi(q) A_{\alpha}^{(2)}(\varphi) A_{\beta, \kappa}^{(2)}(\varphi) + A_{\alpha}^{(1)} A_{\beta, \kappa}^{(1)}(\varphi); \] (3.35)

\[ E_1(z; i) = -i A_{\beta, \kappa}^{(1)}(\varphi) \Phi(q) A_{\beta}^{(2)} + e^{i\varphi} q^\kappa A_{\beta}^{(1)} \Phi(q) A_{\beta, \kappa}^{(2)}(\varphi); \] (3.36)

\[ D_1(z; i) = ie^{i\varphi} q^\kappa \Psi(q) A_{\alpha, \kappa}^{(2)}(\varphi) \Phi(q) A_{\beta}^{(2)} + A_{\beta}^{(1)} A_{\alpha, \kappa}^{(1)}(\varphi); \] (3.37)

\[ D_0(z; i) = i \Psi(q) A_{\alpha}^{(2)} A_{\alpha, \kappa}^{(1)}(\varphi) - e^{i\varphi} q^\kappa A_{\alpha}^{(1)} \Psi(q) A_{\alpha, \kappa}^{(2)}(\varphi); \] (3.38)
Let us denote
\[ A^{(i)}_{\varepsilon,\kappa}(\varphi) = A^{(i)R}_{\varepsilon,\kappa}(\varphi) + i A^{(i)I}_{\varepsilon,\kappa}(\varphi) \] (3.39)

Using the relations (3.29) and (3.39) from (2.67), (3.27) and (3.35) - (3.38) one obtains
\[ m(z) = \frac{N_{m(z)}}{D_{m(z)}} \]

where
\[ N_{m(z)} = \left\{ -\frac{1}{2} N_1 - \frac{\sin \varphi_0}{2} N_2 + \frac{\cos \varphi_0}{2} N_3 \right\} - i \left\{ -\frac{1}{2} N_4 + \frac{\sin \varphi_0}{2} N_5 + \frac{\cos \varphi_0}{2} N_6 \right\} \]
(3.40)

\[ D_{m(z)} = \left\{ -\frac{1}{2} D_1 - \frac{\sin \varphi_0}{2} D_2 + \frac{\cos \varphi_0}{2} D_3 \right\} - i \left\{ -\frac{1}{2} D_4 + \frac{\sin \varphi_0}{2} D_5 + \frac{\cos \varphi_0}{2} D_6 \right\} \]
(3.41)
where we use the following notation

From (3.40) and (3.41) and their analogues for $$m$$

Thus for reconstruction of the spectral measure

The formulas analogous to (3.40) and (3.41) for $$D$$

$$D_3 = \frac{A_{\alpha,\kappa}^{(1)R}(\varphi)}{A_\alpha^{(1)}} - q^\kappa \cos \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)I}(\varphi)}{\Psi(q)A_\alpha^{(2)}} - q^\kappa \sin \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)R}(\varphi)}{\Psi(q)A_\alpha^{(2)}}$$

$$D_4 = \frac{A_{\alpha,\kappa}^{(1)}(\varphi)}{A_\alpha^{(1)}} - q^\kappa \cos \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)R}(\varphi)}{\Psi(q)A_\alpha^{(2)}} + q^\kappa \sin \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)I}(\varphi)}{\Psi(q)A_\alpha^{(2)}}$$

$$D_5 = \frac{A_{\alpha,\kappa}^{(1)R}(\varphi)}{A_\alpha^{(1)}} - q^\kappa \cos \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)}(\varphi)}{\Psi(q)A_\alpha^{(2)}} - q^\kappa \sin \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)R}(\varphi)}{\Psi(q)A_\alpha^{(2)}}$$

$$D_6 = -\frac{A_{\alpha,\kappa}^{(1)}I(\varphi)}{A_\alpha^{(1)}} - q^\kappa \cos \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)R}(\varphi)}{\Psi(q)A_\alpha^{(2)}} + q^\kappa \sin \varphi \frac{\Psi(q)A_{\alpha,\kappa}^{(2)I}(\varphi)}{\Psi(q)A_\alpha^{(2)}}$$

The formulas analogous to (3.40) and (3.41) for $$m(\overline{\tau})$$ can be obtained from (3.40) and (3.41) by replacement $$\varphi \to -\varphi$$ with help of the relations

$$A_{\varepsilon,\kappa}^{(i)I}(-\varphi) = -A_{\varepsilon,\kappa}^{(i)I}(\varphi), \quad i = 1, 2; \quad \varepsilon = \alpha, \beta. \quad (3.42)$$

4 Support of the spectral measure $$\sigma_{\varphi_0}$$

From (3.40) and (3.41) and their analogues for $$m(\overline{\tau})$$ we check that $$\text{Re}(m(z) - m(\overline{\tau})) = 0$$. Thus for reconstruction of the spectral measure $$\sigma_{\varphi_0}$$ from its Stieltjes transformation $$m(z)$$ according to the relation (2.63) we must find $$\text{Im}(m(z) - m(\overline{\tau}))$$. We have

$$\text{Im}(m(z) - m(\overline{\tau})) =$$

$$\frac{q^\kappa}{2} \left( \text{Im} B_{\alpha,\beta,\kappa}^{(+)\alpha} + \cos \varphi \varphi_0 \text{Im} B_{\alpha,\beta,\kappa}^{(-)\alpha} \right) + \sin \varphi \varphi_0 \text{Im} A_{\alpha,\beta,\kappa}^{\alpha} \quad (4.1)$$

where we use the following notation

$$A_{\alpha,\kappa}^{(\pm)} := \left( \frac{A_{\alpha,\kappa}^{(1)}(\varphi)}{A_\alpha^{(1)}} \right)^2 \pm q^\kappa \left( \frac{\Psi(q)A_{\alpha,\kappa}^{(2)}(\varphi)}{\Psi(q)A_\alpha^{(2)}} \right)^2 \quad (4.2)$$

$$A_{\alpha,\beta,\kappa} := \frac{A_{\alpha,\kappa}^{(1)}(\varphi)A_{\beta,\kappa}^{(1)}(\varphi) + q^\kappa \Psi(q)A_{\beta,\kappa}^{(2)}(\varphi)\Phi(q)A_{\beta,\kappa}^{(2)}(\varphi)}{A_\alpha^{(1)}\Psi(q)A_\alpha^{(2)}} \quad (4.3)$$

$$B_{\alpha,\beta,\kappa}^{(\pm)} := e^{i\varphi} \left[ \frac{A_{\alpha,\kappa}^{(1)}(\varphi)\Phi(q)A_{\beta,\kappa}^{(2)}(\varphi)}{\left( A_\alpha^{(1)} \right)^2} \pm \frac{A_{\beta,\kappa}^{(1)}(\varphi)\Psi(q)A_{\alpha,\kappa}^{(2)}(\varphi)}{\left( \Psi(q)A_\alpha^{(2)} \right)^2} \right] \quad (4.4)$$
\[
B_\alpha := e^{i\varphi} \frac{A_{\alpha,\kappa}^{(1)}(\varphi) \Psi(q) A_{\alpha,\kappa}^{(2)}(\varphi)}{A_{\alpha}^{(1)}\Psi(q)A_{\alpha}^{(2)}}
\]  

Recall that according to (2.65)

\[
\psi(\gamma; \tau) = \frac{m(z) - m(\zeta)}{2\pi i} = \frac{N_\psi}{D_\psi}
\]  

In the limit \(\varphi \to 0\) or \(\varphi \to \pi\) we have

\[
\lim_{\varphi \to \frac{\pi}{2}} N_\psi = 0
\]  

\[
\lim_{\varphi \to \frac{\pi}{2}} D_\psi = 2\pi \left( A_c^\psi \cos \frac{\varphi_0}{2} + x A_s^\psi \sin \frac{\varphi_0}{2} \right)^2 \equiv \Lambda_{\varphi_0}(x),
\]

where

\[
A_c^\psi = \frac{A_{\alpha,\kappa}^{(1)}(\frac{\pi}{2} \pm \frac{\pi}{2})}{A_{\alpha}^{(1)}}, \quad A_s^\psi = \frac{\Psi(q)A_{\alpha,\kappa}^{(2)}(\frac{\pi}{2} \mp \frac{\pi}{2})}{\Psi(q)A_{\alpha}^{(2)}},
\]

and

\[
x = q^\kappa e^{i(\frac{\pi}{2} \pm \frac{\pi}{2})}.
\]

It is wellknown ([13] ch.VII, sect.1) that in nondefinite case spectrum of \(X_{\varphi_0}\) is discreet, real and can have accumulation points only in infinity. The support of the spectral measure \(\sigma_{\varphi_0}\) coincides with the spectrum of \(X_{\varphi_0}\), and, as follows from (4.7) and (4.8), it coincides also with set \(\Pi_{\varphi_0}\) of zeros for function \(\Lambda_{\varphi_0}(x)\). Note that the function \(\Lambda_{\varphi_0}(x)\) is even, and to find the set

\[
\Pi_{\varphi_0} = \{\pm x_k\}_{k=1}^{M} \leq \infty, \quad 0 < x_1 < x_2 < \ldots < x_M,
\]

we must solve the equation

\[
A_c^\psi(0) \cos \frac{\varphi_0}{2} + x A_s^\psi(0) \sin \frac{\varphi_0}{2} = 0,
\]

where

\[
A_c^\psi(0) = \frac{A_{\alpha,\kappa}^{(1)}(0)}{A_{\alpha}^{(1)}}, \quad A_s^\psi(0) = \frac{\Psi(q)A_{\alpha,\kappa}^{(2)}(0)}{\Psi(q)A_{\alpha}^{(2)}}.
\]

As follows from (3.31) and (3.32), we have (with \(x = q^\kappa\))

\[
A_{\alpha,\kappa}^{(1)}(0) = 1 + \sum_{k=1}^{\infty} \frac{1}{[2k-1]!!} \sum_{m=0}^{k-1} (-1)^{k-m} x^{2(k-m)} a_{2m-1,2k-2} = 
\]

\[
P_0(x) + \sum_{k=1}^{\infty} (-1)^k \sqrt{\frac{[2k-2]!!}{[2k-1]!!}} x^k P_{2k-1}(x)
\]

22
\[
\Psi(q)A_{\alpha,\kappa}^{(2)}(0) = \Psi(q) + \sum_{k=1}^{\infty} \frac{1}{[2k]!!} \sum_{m=0}^{k-1} (-1)^{k-m} x^{2(k-m)} \alpha_{2m-1,2k-1}
\]
\[
P_0(x) + \sum_{k=1}^{\infty} (-1)^k \sqrt{\frac{[2k-2]!!}{[2k-1]!!}} x P_{2k-1}(x) = (-1)^n \sqrt{\frac{[2n]!!}{[2n-1]!!}} P_{2n}(x)
\]
(4.13)

Note that with help of the recurrent relation (2.63) we can rewrite the \(n\)-th partial sum for the series in (4.12) as
\[
P_0(x) + \sum_{k=1}^{n} (-1)^k \sqrt{\frac{[2k-2]!!}{[2k-1]!!}} x P_{2k-1}(x) = (-1)^n \sqrt{\frac{[2n]!!}{[2n-1]!!}} P_{2n}(x)
\]
(4.14)

In what follows we for simplicity we not consider the general case of arbitrary self adjoint extension \(X_{\varphi_0}\) of coordinate operator \(X\) and restrict ourself to the concrete cases \(X_0 (\varphi_0 = 0)\) and \(X_\pi (\varphi_0 = \pi)\). In the case \(\varphi_0 = 0\) the support of the spectral measure, that is the set \(\Pi_0 = \{\pm x_k\}_{k=1}^{M<\infty}\) of zeros must be finded from the equation
\[
P_0(x) + \sum_{k=1}^{\infty} (-1)^k \sqrt{\frac{[2k-2]!!}{[2k-1]!!}} x P_{2k-1}(x) = 0
\]
(4.15)

As follows from relations (2.65), (1.1), (1.4) and (1.2), the function \(\psi(\gamma; \tau)\) for the case \(\varphi_0 = 0\) is given by
\[
\psi(\gamma; \tau) = \frac{q^\kappa}{\pi} \text{Im} \left( B_{\alpha,\beta,\kappa}^{(-)} + B_{\alpha,\beta,\kappa}^{(+)} \right) A_{\alpha,\kappa}^{(-)} + A_{\alpha,\kappa}^{(+)}
\]
(4.16)

Analogously for the case \(\varphi_0 = \pi\) the set of zeros \(\Pi_\pi = \{\pm x_k\}_{k=1}^{M<\infty}\) consist from the roots of the equation
\[
x(P_0(x) + \sum_{k=1}^{\infty} (-1)^k \sqrt{\frac{[2k-1]!!}{[2k]!!}} P_{2k}(x)) = 0,
\]
(4.17)

and function \(\psi(\gamma; \tau)\) is given by
\[
\psi(\gamma; \tau) = \frac{q^\kappa}{\pi} \text{Im} \left( B_{\alpha,\beta,\kappa}^{(+)} - B_{\alpha,\beta,\kappa}^{(-)} \right) A_{\alpha,\kappa}^{(+)} - A_{\alpha,\kappa}^{(-)}
\]
(4.18)

5 Construction of the spectral measures \(\sigma_0\) and \(\sigma_\pi\).

5.1 The case of the measure \(\sigma_0 (\varphi_0 = 0)\)

From (4.17), it follows that \(\sigma_0(\Delta) = 0\) if interval \(\Delta\) does not contain points \(x_k \in \Pi_0\) and in the case when \(\Delta\) contains only the single point \(x_k\), we have
\[
\sigma_0(\Delta) \equiv \sigma_0(x_k) > 0.
\]
(5.1)
According to (2.65) and (5.1), for $x_k \in \Pi_0$ (we take for definiteness $x_k > 0$)

$$\sigma_0(x_k) = \lim_{\delta \to 0} \lim_{\tau \to 0} \int_{x_k-\delta}^{x_k+\delta} \psi(\gamma; \tau) d\gamma$$ \hspace{1cm} (5.2)

For computation of the right hand side of the relation (5.2) we decompose function $\psi(\gamma; \tau)$ in the vicinity of the point $z = \gamma + i\tau = (x_k, 0)$. Using (4.2), (4.4) and (4.16) we obtain

$$\psi(\gamma; \tau) = \frac{q^k}{\pi} \frac{\text{Im} \left( e^{i\varphi} A^{(1)}_{\alpha, \kappa}(\varphi) \Phi(q) A^{(2)}_{\beta, \kappa}(\varphi) \right)}{A^{(1)}_{\alpha, \kappa}} \hspace{1cm} (5.3)$$

Using the relations

$$q^k e^{i\varphi} = \gamma + i\tau \iff q^k = \frac{\gamma + i\tau}{\sqrt{\gamma^2 + \tau^2}}, \quad e^{i\varphi} = \frac{\gamma + i\tau}{\sqrt{\gamma^2 + \tau^2}}$$

and taking into account the notation (3.39) we rewrite (5.3) in the form

$$\psi(\gamma; \tau) = \frac{1}{\pi} \frac{\gamma G_{\gamma} + \tau G_{\tau}}{A^{(1)}_{\alpha, \kappa} + A^{(1)}_{\alpha, \kappa}}$$ \hspace{1cm} (5.4)

where

$$G_{\gamma} = A^{(1)}_{\alpha, \kappa} \Phi(q) A^{(2)}_{\beta, \kappa} - A^{(1)}_{\alpha, \kappa} \Phi(q) A^{(2)}_{\beta, \kappa}$$

$$G_{\tau} = A^{(1)}_{\alpha, \kappa} \Phi(q) A^{(2)}_{\beta, \kappa} + A^{(1)}_{\alpha, \kappa} \Phi(q) A^{(2)}_{\beta, \kappa}$$

Using the relations

$$\text{Re} \left( \gamma + i\tau \right)^{2s} = \gamma^{2s} - C^2_{2s} \gamma^2 \tau^{2s-2} + \ldots + (-1)^{s-1} \tau^{2s}$$

$$\text{Im} \left( \gamma + i\tau \right)^{2s} = \gamma \tau \{ C^1_{2s} \gamma^{2s-2} - C^3_{2s} \gamma^{2s-4} \tau^2 + \ldots + (-1)^{s-1} C^2_{2s-1} \tau^{2s-2} \}$$ \hspace{1cm} (5.5)

and formulas (3.31), (3.34) and (3.39) we obtain

$$A^{(1)}_{\alpha, \kappa}(\gamma, \tau) = 1 + \sum_{k=1}^{\infty} \frac{1}{(2k-1)!} \sum_{m=0}^{k-1} \frac{(-1)^{k-m}}{m!} \left\{ C^1_{2(k-m)} \gamma^{2(k-m)-1} - C^3_{2(k-m)} \gamma^{2(k-m)-2} \right\} \alpha_{2m-1,2k-2};$$ \hspace{1cm} (5.6)

$$A^{(1)}_{\alpha, \kappa}(\gamma, \tau) = \gamma \tau \sum_{k=1}^{\infty} \frac{1}{(2k-1)!} \sum_{m=0}^{k-1} \frac{(-1)^{k-m}}{m!} \left\{ C^1_{2(k-m)} \gamma^{2(k-m)-1} - C^3_{2(k-m)} \gamma^{2(k-m)-2} \right\} \alpha_{2m-1,2k-2}$$ \hspace{1cm} (5.7)
\[ \Phi(q)A^{(2)}_{\beta,\kappa}(\gamma, \tau) = \Phi(q) + \sum_{k=2}^{\infty} \frac{1}{(2k-1)!!} \sum_{m=0}^{k-2} (-1)^{k-m-1} \left\{ \gamma^{2(k-m-1)} - C_{2(k-m-1)}^2 \gamma^{2(k-m-2)} \tau^2 + \ldots + (-1)^{k-m-1} \gamma^{2(k-m-1)} \right\} \beta_{2m,2k-2} \] (5.8)

\[ \Phi(q)A^{(2)}_{\beta,\kappa}(\gamma, \tau) = \tau \sum_{k=1}^{\infty} \frac{1}{(2k-1)!!} \sum_{m=0}^{k-2} (-1)^{k-m-1} \left\{ C_{2(k-m-1)}^1 \gamma^{2(k-m-1)} - C_{2(k-m-1)}^3 \gamma^{2(k-m-2)} \tau^2 + \ldots + (-1)^{k-m-2} C_{2(k-m-1)} \gamma \right\} \beta_{2m,2k-2} \] (5.9)

Let \( \tilde{\gamma} := \gamma - x_k \) and \( f_1(\tilde{\gamma}; \tau) = \)

\[ f_1(\tilde{\gamma}; \tau) = A^{(1)}_{\alpha,\kappa}(\tilde{\gamma} + x_k; \tau) \quad \varphi_1(\tilde{\gamma}; \tau) = \Phi(q)A^{(2)}_{\beta,\kappa}(\tilde{\gamma} + x_k; \tau) \]

\[ f_2(\tilde{\gamma}; \tau) = A^{(1)}_{\alpha,\kappa}(\tilde{\gamma} + x_k, \tau) \quad \varphi_2(\tilde{\gamma}; \tau) = \Phi(q)A^{(2)}_{\beta,\kappa}(\tilde{\gamma} + x_k; \tau) \]

If we expand these functions in Taylor series about point \( M(0, 0) \) and save leading terms we obtain

\[ f_1(\tilde{\gamma}; \tau) = \tilde{\gamma} c_1(\tilde{\gamma}) + \tau^2 c_2(\tilde{\gamma}) + \tau^4 c_3(\tilde{\gamma}) + \ldots \] (5.10)

where

\[ c_1(\tilde{\gamma}) = \sum_{k=1}^{\infty} \frac{\partial^k f_1(0; 0)}{k!} \tilde{\gamma}^{k-1} \quad c_1(0) = \frac{\partial f_1}{\partial \tilde{\gamma}}(0; 0) \]

\[ c_2(\tilde{\gamma}) = \sum_{k=0}^{\infty} \frac{\partial^k f_1(0; 0)}{(k + 2)!} \tilde{\gamma}^{k-1} \quad c_2(0) = \frac{\partial^2 f_1}{\partial \tilde{\gamma}^2}(0; 0) \frac{1}{2!} \] (5.11)

\[ c_3(\tilde{\gamma}; \tau) = \sum_{s=2}^{\infty} \tau^{2(s-2)} \sum_{k=0}^{\infty} \frac{\partial^k f_1(0; 0)}{\partial \tilde{\gamma}^k \partial \tau^{2s}} \tilde{\gamma}^k \quad c_3(0; 0) = \frac{\partial^4 f_1}{\partial \tau^2}(0; 0) \frac{1}{4!} \]

\[ f_2(\tilde{\gamma}; \tau) = \tau c_4(\tilde{\gamma}) + \tau^3 c_5(\tilde{\gamma}) + \ldots \] (5.12)

\[ c_4(\tilde{\gamma}) = \sum_{k=0}^{\infty} \frac{\partial^k+1 f_2(0; 0)}{(k + 1)!} \tilde{\gamma}^k \quad c_4(0) = \frac{\partial f_2}{\partial \tilde{\gamma}}(0; 0) \]

\[ c_5(\tilde{\gamma}; \tau) = \sum_{s=1}^{\infty} \tau^{2(s-1)} \sum_{k=0}^{\infty} \frac{\partial^k+2 f_2(0; 0)}{\partial \tilde{\gamma}^k \partial \tau^{2s+1}} \tilde{\gamma}^k \quad c_5(0; 0) = \frac{\partial^3 f_2}{\partial \tau^3}(0; 0) \frac{1}{3!} \] (5.13)

\[ \varphi_1(\tilde{\gamma}; \tau) = c_6(\tilde{\gamma}) + \tau^2 c_7(\tilde{\gamma}) + \tau^4 c_8(\tilde{\gamma}) + \ldots \] (5.14)

\[ c_6(\tilde{\gamma}) = \sum_{k=0}^{\infty} \frac{\partial^k \varphi_1(0; 0)}{(k + 1)!} \tilde{\gamma}^k \quad c_6(0) = \varphi_1(0; 0) \]

\[ c_7(\tilde{\gamma}) = \sum_{k=0}^{\infty} \frac{\partial^k+2 \varphi_1(0; 0)}{(k + 2)!} \tilde{\gamma}^k \quad c_7(0) = \frac{\partial^2 \varphi_1}{\partial \tau^2}(0; 0) \frac{1}{2!} \] (5.15)

\[ c_8(\tilde{\gamma}; \tau) = \sum_{s=2}^{\infty} \tau^{2(s-2)} \sum_{k=0}^{\infty} \frac{\partial^k+3 \varphi_1(0; 0)}{\partial \tilde{\gamma}^k \partial \tau^{2s+1}} \tilde{\gamma}^k \quad c_8(0) = \frac{\partial^4 \varphi_1}{\partial \tau^4}(0; 0) \frac{1}{4!} \]
Using the formulas (5.10) - (5.17) we can rewrite quantities $c_9(\gamma)$ and $c_{10}(\gamma; \tau)$ in the form

$$c_9(\gamma) = \frac{\tau^3 c_{10}(\gamma; \tau)}{\varphi^2(0; 0)}$$

(5.16)

$$c_{10}(\gamma; \tau) = \sum_{s=1}^{\infty} \tau^{2(s-1)} \sum_{k=0}^{s-1} \partial^{k+2s+1} \varphi^2(0; 0) \frac{\tau^k}{(k + 2s + 1)!}$$

Using (5.4), we can rewrite (5.2) for $x_k > 0$ in the form

$$\sigma_0(x_k) = \frac{1}{\pi} \lim_{\delta \to 0} \lim_{\tau \to 0} \int_{-\delta}^{\delta} \frac{B_1(\gamma; \tau)}{B_2(\gamma; \tau)} d\gamma$$

(5.18)

Using the formulas (5.10) - (5.17) we can rewrite quantities $B_1(\gamma; \tau)$ and $B_2(\gamma; \tau)$ entering (5.18) in the form

$$B_1(\gamma; \tau) = \tau (b_1(\gamma) + \gamma b_2(\gamma)) + \tau^3 b_3(\gamma; \tau)$$

(5.19)

$$B_2(\gamma; \tau) = \gamma c_1^2(\gamma) + \tau^2 b_4(\gamma) + \tau^4 b_5(\gamma; \tau)$$

(5.20)

where

$$b_1(\gamma) = -c_4(\gamma)c_6(\gamma)$$

$$b_2(\gamma) = (\gamma + x_k) (c_1(\gamma)c_9(\gamma) - c_4(\gamma)c_6(\gamma)) + c_1(\gamma)c_9(\gamma)$$

(5.21)

$$b_4(\gamma) = 2\gamma c_1^2(\gamma) c_2(\gamma) + c_4^2(\gamma), \quad b_4(0) = c_4^2(0).$$

In what follows we do not use the expressions for $b_3(\gamma; \tau)$ and $b_5(\gamma; \tau)$ and note only that

$$b_3(0; \tau) = x_k \left\{ (c_2(0) + \tau^2 c_3(0))(c_9(0) + \tau^2 c_{10}(0)) - (c_7(0) + \tau^2 c_8(0))(c_4(0) + \tau^2 c_5(0)) \right\} + (c_2(0) + \tau^2 c_3(0))(c_6(0) + \tau^2 c_7(0) + \tau^4 c_8(0)) + (c_4(0) + \tau^2 c_5(0))(c_8(0) + \tau^2 c_{10}(0))$$

(5.22)

$$b_5(0; 0) = c_2^2(0) + 2c_4(0)c_5(0)$$

(5.23)

From (5.11), definition of $f_1$ and (5.9) we have

$$c_1(0) = \sum_{l=1}^{\infty} \frac{1}{[2l - 1]!!} \sum_{m=0}^{l-1} (-1)^{l-m} 2 (l - m) x_k^{2(l-m)-1} a_{2m-1,2l-2} =$$

$$\frac{d}{dx} \left( \frac{P_0(x)}{x} + \sum_{l=1}^{\infty} (-1)^l \sqrt{\frac{[2l - 2]!!}{[2l - 1]!!}} x P_{2l-1}(x) \right) (x_k)$$

(5.24)
Analogously from (5.15) and (5.8) we have

\[ c_6(0) = \Phi(q) + \sum_{l=2}^{\infty} \frac{1}{[2l-1]!!} \sum_{m=0}^{l-2} (-1)^{l-m-1} x_k^{2(l-m-1)} \beta_{2m,2l-2} = \]

\[ Q_1(x_k) + \sum_{l=2}^{\infty} (-1)^{l-1} \sqrt{\frac{[2l-2]!!}{[2l-1]!!}} Q_{2l-1}(x_k) \]

Thus from (5.18) and (5.19) we obtain

\[ \sigma_0(x_k) = \frac{1}{\pi} \lim_{\delta \to 0} \lim_{\tau \to 0} \tau \left\{ \int_{-\delta}^{\delta} \frac{x_k b_1(\tilde{\gamma})}{B_2(\tilde{\gamma}; \tau)} d\tilde{\gamma} + \int_{-\delta}^{\delta} \frac{\tilde{\gamma} b_2(\tilde{\gamma})}{B_2(\tilde{\gamma}; \tau)} d\tilde{\gamma} + \tau^2 \int_{-\delta}^{\delta} \frac{b_3(\tilde{\gamma})}{B_2(\tilde{\gamma}; \tau)} d\tilde{\gamma} \right\} \]

Using (5.24), (5.22) and relations

\[ B_2(\tilde{\gamma}; \tau) = \frac{1}{\alpha(\tilde{\gamma}; \tau)} \left( 1 - \tau \frac{b_5(\tilde{\gamma}; \tau)}{\alpha(\tilde{\gamma}; \tau)} + \ldots \right) \]

\[ c_4(0) = c_1(0) + 2x_k \sum_{l=1}^{\infty} \frac{[2l-2]!!}{[2l-1]!!} \Psi_{l-1}(q) \]

we see that in the limit \( \delta \to 0, \tau \to 0 \)

\[ \tau \int_{-\delta}^{\delta} \frac{b_3(\tilde{\gamma})}{B_2(\tilde{\gamma}; \tau)} d\tilde{\gamma} \sim b_3(0; \tau) \int_{-\delta}^{\delta} \frac{d\tilde{\gamma}}{c_1(0) \tilde{\gamma}^2 + \tau^2 c_4^2(0)} = \frac{2b_3(0; \tau)}{c_1(0)c_4(0)} \arctan \frac{\delta c_1(0)}{\tau c_4(0)}. \]

So in the limit \( \tau \to 0 \) the 3-rd term in the sum in (5.26) disappears. The same is truth in the case when \( c_1(0) = 0 \).

Quite similarly when \( \delta \to 0, \tau \to 0 \)

\[ \tau \int_{-\delta}^{\delta} \frac{\tilde{\gamma} b_2(\tilde{\gamma})}{B_2(\tilde{\gamma}; \tau)} d\tilde{\gamma} \sim \tau b_2(0) \int_{-\delta}^{\delta} \frac{d\tilde{\gamma}}{c_1^2(0) \tilde{\gamma}^2 + \tau^2 c_4^2(0)} = 0 \]

(5.29)

Note, that (5.29) still hold also in the case when \( c_1(0) = 0 \).

Finally, in the limit \( \delta \to 0, \tau \to 0 \)

\[ \tau \int_{-\delta}^{\delta} \frac{x_k b_1(\tilde{\gamma})}{B_2(\tilde{\gamma}; \tau)} d\tilde{\gamma} \sim -\frac{2x_k c_6(0)}{c_1(0)} \arctan \frac{\delta c_1(0)}{\tau c_4(0)} \]

(5.30)
From (5.26), (5.28) - (5.30) we have

$$\sigma_0(x_k) = \begin{cases} 
0 & \text{if } c_1(0) = 0, \\
-x_k \left( \frac{c_6(0)}{c_1(0)} \right) & \text{if } c_1(0) \neq 0.
\end{cases}$$

(5.31)

According to (5.24), (5.25) and (5.31) we have $\sigma_0(x_k) = 0$ if

$$\frac{d}{dx} \left( A^{(1)}_{a, \kappa}(0) \right) (x_k) = \begin{cases} 
0 & \text{otherwise}\n\end{cases}$$

otherwise

$$\sigma_0(x_k) = -x_k \frac{Q_1(x_k) + \sum_{l=2}^{\infty} (-1)^{l-1} \sqrt{\frac{[2l-2]!!}{[2l-1]!!}} Q_{2l-1}(x_k)}{\frac{d}{dx} \left( P_0(x) + \sum_{l=1}^{\infty} (-1)^l \sqrt{\frac{[2l-2]!!}{[2l-1]!!}} x P_{2l-1}(x) \right) (x_k)}$$

(5.32)

One can to check the positiveness of the measure $\sigma_0$ [14].

5.2 The case of the measure $\sigma\pi \ (\varphi_0 = \pi)$.

Quite similarly one can consider the case when $\varphi_0 = \pi$. We omit related calculations and give only final expression for the measure. Namely, $\sigma_\pi(x_k) = 0$ if

$$\frac{d}{dx} \left( \Psi(q) A^{(2)}_{a, \kappa}(0) \right) (x_k) = \begin{cases} 
0 & \text{otherwise}\n\end{cases}$$

otherwise

$$\sigma_\pi(x_k) = \frac{1}{x_k} \sum_{l=1}^{\infty} \frac{1}{[2l]!!} \sum_{m=0}^{l-1} (-1)^{l+m} x_k 2^{l-m} (l-m-1) \beta_{2m,2l-1}$$

(5.33)

In view of (3.1)-(3.4) one can to rewrite (5.33) as

$$\sigma_\pi(x_k) = \frac{1}{x_k} \frac{Q_1(x_k) + \sum_{l=1}^{\infty} (-1)^l \sqrt{\frac{[2l-1]!!}{[2l]!!}} x_k Q_{2l}(x_k)}{\frac{d}{dx} \left( P_0(x) + \sum_{l=1}^{\infty} (-1)^l \sqrt{\frac{[2l-1]!!}{[2l]!!}} P_{2l}(x) \right) (x_k)}$$

(5.34)
6 Conclusion

As follows from (2.65), (2.67), (4.15), (4.17) and general results from [14] (ch.3 §4) measure $\sigma_0$ which correspond to Stieltjes transformation $m_0(z)$ from (2.65) is equal to $\sigma_\pi$ which correspond to the selfadjoint extension $\mathcal{X}$ with $\phi_0 = \pi$. At the same time measure $\sigma_\infty$ which correspond to $m_\infty(z)$ connected with the selfadjoint extension $\mathcal{X}$ with $\phi_0 = 0$.

It is not too hard to find general connection between values of the parameter $t = t_0$ and the related selfadjoint extensions $\mathcal{X}$ labeled by $\phi_0$ which looks as

$$t_0 = -\text{ctg}\frac{\phi_0}{2} \frac{\psi A^{(2)}_\alpha}{A^{(1)}_\alpha}.$$  \hspace{1cm} (6.35)

It will be interesting to obtain the interpretation of the entire function

$$F_t(z; q) = [A^{(1)}_{\alpha,\kappa}(0)](z) t - [\psi(q) A^{(2)}_{\alpha,\kappa}(0)](z),$$  \hspace{1cm} (6.36)

which roots are supports of the spectral measure $\sigma_t$, in terms of a one or another standard $q$-special function.

We would like to stress once more that all results reported in this work are hold for each Jacobi matrix (2.22) which entries fulfill the restrictions

$$[b_n] [b_{n+2}] \leq [b_{n+1}]^2, \quad n \geq 0$$

$$\sum_{n=0}^{\infty} [b_n]^{-\frac{1}{2}} < \infty$$

which provide the associated moment problem to be indeterminate.

This allows to expect that our results may be useful also in other problems in which indeterminate moment problem arised.
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