Interaction effects on PT-symmetry breaking transition in atomic gases

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Non-Hermitian systems having parity-time (PT) symmetry can undergo a transition, spontaneously breaking the symmetry. Ultracold atomic gases provide an ideal platform to study interaction effects on the transition. We consider a model system of N bosons of two components confined in a tight trap. Radio frequency and laser fields are coupled to the bosons such that the single particle Non-Hermitian Hamiltonian \( h_{PT} = -i\Gamma\sigma_z + J\sigma_x \), which has PT-symmetry, can be simulated in a passive way. We show that when interatomic interactions are tuned to maintain the symmetry, the PT-symmetry breaking transition is affected only by the SU(2) variant part of the interactions parameterized by \( \delta g \). We find that the transition point \( \Gamma_{\text{tr}} \) decreases as \( |\delta g| > N \) increases; in the large \( |\delta g| \) limit, \( \Gamma_{\text{tr}} \) scales as \( \sim |\delta g|^{-(N-1)} \). We also give signatures of the PT-symmetric and the symmetry breaking phases for the interacting bosons in experiment.

Study of non-Hermitian systems is constantly enriching our knowledge derived from Hermitian ones \cite{11, 12}. Of particular interest are a class of non-Hermitian systems having the parity-time (PT) symmetry \cite{10, 11}. A representative model of the class is a two-level system whose Hamiltonian is of the form \( h_{PT} = -i\Gamma\sigma_z + J\sigma_x \); under the combined transformation of complex conjugate and the swap \( |\uparrow\rangle \leftrightarrow |\downarrow\rangle \), \( h_{PT} \) is invariant \cite{13}. Parameter tuning across the critical point \( \Gamma_{\text{tr}} = J \) gives rise to the transition of the two-level system from the PT-symmetric phase to the symmetry breaking phase where exponentially growing or decaying modes set in. PT-symmetry breaking transition has been widely investigated in electromagnetic \cite{4, 5, 14, 15}, and mechanical systems \cite{16}. The transition is the cornerstone of exceptional properties regarding light propagation \cite{19, 21}, lasing \cite{22, 23} and topological energy transfer \cite{25, 26}.

Recently PT-symmetry breaking transition was successfully demonstrated in a gas of two component noninteracting \(^9\text{Li}\) atoms in a passive way \cite{27}; in the experiment, the application of a radio-frequency field and a laser inducing loss in one component of the atoms leads to, apart from kinetic energy, the single particle Hamiltonian \( h = -i\Gamma + h_{PT} \), as the term \(-i\Gamma\) gives rise to an overall decay. This approach circumvents the difficulty of realizing an atom gain in quantum simulation of PT-symmetric non-Hermitian Hamiltonians in atomic gases \cite{3}. On the other hand, Feshbach resonance enables unprecedented control of interactions in ultracold atomic gases \cite{28}, and deterministic preparation is achievable for a sample of variable \( N \) atoms \cite{29, 32}. These capabilities make ultracold atoms an ideal platform to probe interaction effects on PT-symmetry breaking transition \cite{33}.

In this work, we consider \( N \) interacting two component bosons confined in a tight harmonic trap such that their spatial wave-function is frozen to be the ground harmonic state. The bosons are subject to the radio frequency field and the laser as in Ref. \cite{27}. Feshbach resonance is used to tune the interaction Hamiltonian of the \( N \) bosons to maintain the PT-symmetry. We find that in this interacting system, PT-symmetry breaking transition depends on only the SU(2) variant part of the interactions parameterized by \( \delta g \). The transition point \( \Gamma_{\text{tr}} \) decreases as \( |\delta g| > N \) increases. In the large \( |\delta g| \) limit, \( \Gamma_{\text{tr}} \) is suppressed as \( \sim |\delta g|^{-(N-1)} \). Finally we show how the modification on the transition by the interactions can be detected experimentally.

Figure (1) gives a schematic of the system that we consider. Bosonic atoms with two internal states denoted by \( |\uparrow\rangle \) and \( |\downarrow\rangle \) are confined in a harmonic trap potential \( V(r) = \frac{1}{2}m\omega_0^2r^2 \), where \( m \) is the atomic mass. For simplicity, we assume the confinement being so tight, i.e., \( \omega_0 \) is much larger than any other energy scales to be considered, that the spatial wave-function of the bosons is frozen to be the single particle ground state \( \phi_0(r) \) of the harmonic trap. A radio-frequency field of frequency equal to the internal energy difference \( E_\uparrow - E_\downarrow \) is used to flip the atoms between the two internal states \( |\uparrow\rangle \) and \( |\downarrow\rangle \) with Rabi frequency \( J \). An additional laser is coupled to the atoms in state \( |\uparrow\rangle \) and results in a loss rate \( 4\Gamma \) of the atom number in the state. We take \( h = 1 \) throughout.

In the absence of interatomic interactions, the Hamiltonian for each bosons spanned by \( |\uparrow\rangle \) and \( |\downarrow\rangle \) is non-Hermitian and is given by \( h = -i\Gamma + h_{PT} \), apart from the ground harmonic state energy \cite{27}. The interatomic interaction Hamiltonian of the bosons is given by

\[
H_{\text{int}} = \sum_{j \neq k} \left[ \frac{g_{\uparrow\uparrow}\sigma_z^{(j)} + 1}{2} \sigma_z^{(k)} + 1 + \frac{g_{\downarrow\downarrow}\sigma_z^{(j)} - 1}{2} \sigma_z^{(k)} - 1 \right],
\]

where \( g_{\sigma\sigma'} = (4\pi a_{\sigma\sigma'}/m) \int dr |\phi_0(r)|^4 \) and \( a_{\sigma\sigma'} \) is the s-wave scattering length \cite{34, 35}, and \( \sigma^{(j)} \) are the Pauli
matrices for the \( j \)th boson. The interaction Hamiltonian is \( \mathcal{PT} \)-symmetric only if \( g_{\uparrow\downarrow} = g_{\downarrow\uparrow} \); in atomic gases, this condition can be experimentally fulfilled by the technique of Feshbach resonance [28]. We focus on this situation \( g \equiv g_{\uparrow\downarrow} \) in the following discussion.

Therefore, the total Hamiltonian of \( N \) interacting bosons is given by

\[
H = H_{\mathcal{PT}} + C
\]

\[
H_{\mathcal{PT}} = -2i\Gamma S_z + 2JS_x + \delta g (S_x)^2
\]

\[
C = -i\mathcal{N}T + \frac{g}{2} N(N-1) - \frac{\delta g}{4} N^2,
\]

where \( S = \sum_{j=1}^{N} \sigma^{(j)} \), and \( \delta g \equiv g - g_{\uparrow\downarrow} \). Compared with the noninteracting case, the \( \mathcal{PT} \)-symmetry breaking transition of the interacting boson system is now determined by \( H_{\mathcal{PT}} \). Since there are two internal states \( |\uparrow\rangle \) and \( |\downarrow\rangle \) to accommodate \( N \) bosons, the dimension of the Hilbert space shall be \( N + 1 \). It is easy to assure oneself that such a space is spanned by the streched states \( |N/2,m\rangle \) with \( m = -N/2, -N/2 + 1, \ldots, N/2 - 1, N/2 \) where \( S^z|N/2,m\rangle = N(N+2)/4|N/2,m\rangle \) and \( S_z|N/2,m\rangle = m|N/2,m\rangle \). In this space, the matrix element of \( H_{\mathcal{PT}} \) becomes

\[
(H_{\mathcal{PT}})_{mn} \equiv \langle N/2,m|H_{\mathcal{PT}}|N/2,m'\rangle
= (-2i\Gamma m + \delta g m^2)\delta_{m,m'}
+ J \sqrt{(N/2)(N/2+1) - m^2} \delta_{m,m'+1}
+ J \sqrt{(N/2)(N/2+1) - m^2} \delta_{m,m'-1}.
\]

The transition occurs when some eigenvalues of \( H_{\mathcal{PT}} \) coalesce and turn complex afterwards. Note that when \( \delta g = 0 \), the interactions drop out of \( H_{\mathcal{PT}} \); this is the situation that the scattering lengths \( a_{\sigma\sigma'} \) become all the same and the interatomic interactions are SU(2) invariant. This dependence of the transition on the interactions is because if the interatomic interactions are SU(2) invariant, i.e., \([S,H_{\text{int}}] = 0\), the noninteracting \( \mathcal{PT} \)-symmetric non-Hermitian Hamiltonian \(-2i\Gamma S_z + 2JS_x \) and \( H_{\text{int}} \) are commutative and can be diagonalised simultaneously.

The above non-Hermitian Hamiltonian formalism is related to the Lindblad equation describing the bosons subject to pure loss in the following way. In terms of the field operator \( b_\sigma (b_\sigma^\dagger) \) which annihilates (creates) a boson of internal state \( \sigma \) in the ground state of the harmonic trap, the Lindbald equation for the density matrix \( \rho \) of the bosons is given by [30]

\[
\frac{d\rho}{dt} = -i[H_{\sigma\sigma}\rho] - 2\Gamma(b_\uparrow^\dagger b_\downarrow \rho + \rho b_\downarrow b_\uparrow^\dagger) + 4\Gamma b_\downarrow \rho b_\uparrow^\dagger,
\]

where \( H_{\sigma\sigma} \) is the Hermitian Hamiltonian of the bosons in the absence of the external lossy laser coupling. Given that in experiment the initial density matrix \( \rho(0) \) shall be always block diagonalized in the number of bosons, i.e., \( \rho_{\alpha\beta}(t = 0) \) is zero unless \( \alpha = \beta \), where \( \rho_{\alpha\beta}(t) \equiv P_\alpha \rho(t) P_\beta \) and \( P_\alpha \) is the \( \alpha \) boson subspace projection operator, so is \( \rho(t) \). If the last term \( 4\Gamma b_\downarrow \rho b_\uparrow^\dagger \) in Eq. (6) were not there, the time dependent density matrix would be given by \( \rho(t) = U(t)\rho(0)U^\dagger(t) \) with \( U(t) = e^{-iHt} \) and \( H = H_s - i2\Gamma b_\downarrow b_\uparrow^\dagger \); since the projection of \( H \) in the \( N \) boson subspace is just the non-Hermitian Hamiltonian \( H \) in Eq. (2), i.e., \( H = P_N H P_N \), the properties of \( H \) would determine the time evolution of \( \rho(t) \).

To access the importance of the term \( 4\Gamma b_\downarrow \rho b_\uparrow^\dagger \) in Eq. (6) to a typical experiment starting with \( N \) particles, we note that initially only \( \rho_{NN}(t = 0) \) is nonzero. Since the pure loss can only cause the particle number to decrease, for all the following time, \( \rho_{\alpha\beta}(t) = 0 \) if \( \alpha > N \) or \( \beta > N \). Thus, from Eq. (6) one can first obtain \( \rho_{NN}(t) = U(t)\rho_{NN}(0)U^\dagger(t) \); the term \( 4\Gamma b_\downarrow \rho b_\uparrow^\dagger \) has no effects on \( \rho_{NN}(t) \) since the projection of the term involves only \( \rho_{NN+1,N+1}(t) \) which is identically zero. Note that \( H \) commutes with the total particle number. From hereon, one can show \( \rho_{N-1,N-1}(t) = f_0^t d\tau U(t-\tau)[4\Gamma b_\downarrow \rho_{NN}(\tau) b_\uparrow^\dagger U^\dagger(t-\tau) \rho_{NN}(\tau) b_\downarrow b_\uparrow^\dagger U(t-\tau)] \). Likewise, one can solve all the rest \( \rho_{\alpha\alpha}(t) \) for \( \alpha < N \) in a cascade; the non-diagonal parts are always zero, i.e., \( \rho_{\alpha\beta} = 0 \) for \( \alpha \neq \beta \). The above argument justifies one to study the time evolution of the purely lossy system by analyzing the non-Hermitian Hamiltonian \( H \) from Eq. (2). Of course, calculations of observables should resort to the density matrix \( \rho(t) \). This justification shall also apply to other similar purely lossy systems.

We start with analyzing the non-Hermitian Hamiltonian \( H \) of two interacting bosons. For \( N = 2 \), the Hamiltonian \( H_{\mathcal{PT}} \) in the basis \( \{|\uparrow\rangle \langle \uparrow|, \{|\uparrow\rangle \langle \downarrow| \}

\]

\]

FIG. 1: Schematic of the system. Bosons are spatially confined in the single particle ground state of a tight harmonic potential. A resonant radio-frequency field couples two internal states of each bosons, \( |\uparrow\rangle \) and \( |\downarrow\rangle \), with Rabi frequency \( J \). An additional laser couples the internal state \( |\uparrow\rangle \) to an another excited state \( |e\rangle \) and results in a number loss in state \( |\uparrow\rangle \) of rate \( 4\Gamma \).
The fit to the numerically calculated points gives \( \Gamma_{tr}/J \sim |\delta g/J|^{-(N-1)} \), agreeing with the argument given in the text.

The corresponding characteristic polynomial is

\[
f(\lambda) = -\lambda^3 + 2\lambda^2 \delta g + [4(J^2 - \Gamma^2) - \delta^2] \lambda - 4J^2 \delta g,
\]

whose zeros are the eigenvalues of \( H_{PT} \). Since all the coefficients of the cubic polynomial \( f(\lambda) \) are real, one of the three zeros of \( f(\lambda) \) is real definite. When \( \Gamma \) is increased from zero, the rest two zeros of \( f(\lambda) \), which are also real in the first place, coalesce at the \( \mathcal{PT} \)-symmetry transition point and become complex afterwards. This coalescence occurs when the discriminant of \( f(\lambda) \)

\[
\Delta(J, \Gamma, \delta g) = 16(16J^2 - \Gamma^2)^3 + (J^4 - 20J^2 \Gamma^2 - 8\Gamma^4)\delta g^2 - \Gamma^2 \delta g^4
\]

is zero. When \( \delta g = 0 \), \( \Delta(J, \Gamma_{tr}, 0) = 0 \) retrieves the known transition point \( \Gamma_{tr}/J = 1 \). For nonzero \( \delta g \), Fig. (2) shows that \( \Gamma_{tr}/J \) is suppressed more and more as \( |\delta g/J| \) increases. By Eq. (3), it is manifest from that \( \Gamma_{tr}/J \) is even in \( \delta g/J \), and one finds \( (\Gamma_{tr}/J)^2 \approx 1 - 3|\delta g/J|^2/2^3/3^3 \) for \( |\delta g/J| \ll 1 \), and \( \Gamma_{tr}/J \approx |\delta g/J|^{-1} \) for \( |\delta g/J| \gg 1 \).

The suppression of \( \Gamma_{tr}/J \) in the limit \( |\delta g/J| \gg 1 \) is readily understood by inspecting the \( \mathcal{PT} \)-symmetric Hamiltonian, Eq. (4). In such a limit, we recast \( H_{PT} = H_{PT,L} + V \) with

\[
H_{PT,L} = \begin{bmatrix} \delta g & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta g \end{bmatrix},
\]

\[
V = \begin{bmatrix} -2\Gamma & \sqrt{2}J & 0 \\ \sqrt{2}J & 0 & \sqrt{2}J \\ 0 & \sqrt{2}J & 2\Gamma \end{bmatrix}. \tag{10}
\]

To the leading order, \( H_{PT,L} \) yields right away that the two states \( |\uparrow\rangle |\uparrow\rangle \) and \( |\downarrow\rangle |\downarrow\rangle \) are degenerate and share the same eigenvalue \( \delta g \), and the eigenvalue of the third state \( (|\uparrow\rangle |\downarrow\rangle + |\downarrow\rangle |\uparrow\rangle)/\sqrt{2} \) is zero, well separated apart from \( \delta g \). Since the \( \mathcal{PT} \)-symmetry transition is expected to happen at the point where eigenvalues of \( H_{PT} \) coalesce, to find the effects of \( \Gamma \) and \( J \) on the two degenerate eigenvalues originally equal to \( \delta g \), we use \( V \) to carry out a perturbation calculation to derive the effective Hamiltonian in the subspace spanned by the states \( |\uparrow\rangle |\uparrow\rangle \) and \( |\downarrow\rangle |\downarrow\rangle \); we find that to second order of \( V \) the effective Hamiltonian is given by

\[
H_{eff} = \delta g + 2J^2/\delta g + \begin{bmatrix} -2\Gamma & 2J^2/\delta g \\ 2J^2/\delta g & 2\Gamma \end{bmatrix}. \tag{11}
\]

This effective Hamiltonian yields \( \Gamma_{tr}/J \approx |J/\delta g|^{-1} \), the same as from requiring \( \Delta(J, \Gamma_{tr}, \delta g) = 0 \).

For \( N > 2 \), we numerically diagonalise \( H_{PT} \) and find that the \( \mathcal{PT} \) transition is always due to the coalescence of a pair of eigenvalues of \( H_{PT} \). Figure (2) shows that the critical value \( \Gamma_{tr}/J \) is also symmetric in \( \delta g/J \), which is because, under the transformation \( \delta g \to -\delta g \) and \( S \to -S \), we have \( H_{PT} \to -H_{PT} \). We find that for fixed \( N \), \( \Gamma_{tr}/J \) decreases as \( |\delta g/J| \) increases, while for fixed \( \delta g/J \), as \( N \) increases, \( \Gamma_{tr}/J \) is more and more suppressed.

Figure (3) shows that in the large \( |\delta g/J| \) limit, \( \Gamma_{tr}/J \sim |\delta g/J|^{-(N-1)} \). This asymptotic behavior can be understood by an analysis similar to the one given above for \( N = 2 \). For arbitrary \( N \), in the large \( |\delta g| \) limit,
we separate $H_{PT}$ from Eq. (3) as $H_{PT} = H_{PT,L} + V$ with $H_{PT,L} = \delta g (S_z)^2$ and $V = -2t (S_x + 2JS_z)$. In such a limit, the leading order Hamiltonian $H_{PT,L}$ gives rise to a pair of degenerate eigenvalues in each subspace spanned by $|N/2, m\rangle$ and $|N/2, -m\rangle$; the eigenvalues $m^2 \delta g$ are all well separated from each other. To determine the transition, we use $V$ to derive the effective Hamiltonian $H_{eff}$ in the each two dimensional subspace. It is easy to convince oneself that to the lowest order of $J$, the diagonal elements are $\langle N/2, \pm m | H_{eff} | N/2, \pm m \rangle = m^2 \delta g \mp 2im \Gamma$, and the off-diagonal elements are generated at order of $V^2|m|$, resulting in $\langle N/2, \mp m | H_{eff} | N/2, \mp m \rangle \sim J^2 |m| / \delta g^2 |m|^{-1}$. Thus, by diagonalizing the $2 \times 2$ matrix of $H_{eff}$, we find the transition point $\Gamma_{tr}/J \sim |\delta g/J|^{-2|m|^{-1}}$ for each two dimensional subspaces. Given that the maximum value of $|m|$ equals $N/2$, overall, the $N$-body system enters into the symmetry breaking phase first at $\Gamma_{tr}/J \sim |\delta g/J|^{-4(N-1)}$.

The relation between the non-Hermitian Hamiltonian formalism and the Lindblad equation for our system given above indicates that the signatures of the $\mathcal{PT}$-symmetric and symmetry breaking phases governed by $H_{PT}$ in Eq. (3) can be detected experimentally in the following way. Let one prepare the experiment initially with $N$ bosons $|\alpha\rangle$ such that $\rho_{NN}(t) = e^{2N\Gamma t} \rho_{NN}(0)e^{-iH_{PT}t}$, the quantity $e^{2N\Gamma t} \rho_{NN}(t)$ shall have qualitatively different time dependent behaviors in the symmetric and symmetry breaking phases. For example, by the high accuracy atom number detection achieved experimentally [32], one can measure the rescaled probability of finding $N$ bosons $P(N,t) = e^{2N\Gamma t} \rho_{NN}(t)$ with $P(N,t) = \text{Tr}\rho_{NN}(t)$. In contrast, the total number of atoms was measured to distinguish the two phases for the noninteracting $^6$Li atoms [27]. In our interacting case, the total number of atoms ceases to be a good observable for the purpose since the observable depends on not only $\rho_{NN}(t)$ but also $\rho_{\alpha\alpha}(t)$ for $\alpha < N$ whose dynamics is not determined by a single Hamiltonian $H$ for $\alpha$ bosons. Figure 4 plots the rescaled probability $P(2,t)$ in an experiment starting with $N = 2$ bosons and $\delta g/J = 1$ for various values of $\Gamma/J$. Note that for $N = 2$ and $\delta g/J = 1$, $\Gamma_{tr}/J \approx 0.538$. Figure 4 shows that $P(2,t)$ is bounded in the $\mathcal{PT}$-symmetric phase, and grows exponentially in the symmetry breaking one. Due to the interactions, the point $\Gamma/J = 3/4$ is already in the symmetry breaking phase while its value is still smaller than the critical value $\Gamma_{tr}/J = 1$ for the noninteracting case.

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