Jost solutions and quantum conserved quantities of an integrable derivative nonlinear Schrödinger model

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Abstract

We study differential and integral relations for the quantum Jost solutions associated with an integrable derivative nonlinear Schrödinger (DNLS) model. By using commutation relations between such Jost solutions and the basic field operators of DNLS model, we explicitly construct first few quantum conserved quantities of this system including its Hamiltonian. It turns out that this quantum Hamiltonian has a new kind of coupling constant which is quite different from the classical one. This modified coupling constant plays a crucial role in our comparison between the results of algebraic and coordinate Bethe ansatz for the case of DNLS model. We also find out the range of modified coupling constant for which the quantum $N$-soliton state of DNLS model has a positive binding energy.

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1 Introduction

Conserved quantities associated with quantum integrable models in low dimensions have recently found interesting applications in many topics of physics like exact calculations of transport properties in mesoscopic electronic devices and distribution of energy level spacing in quantum chaotic systems [1,2]. In the framework of quantum inverse scattering method (QISM), one can formally generate such conserved quantities by expanding the trace of monodromy matrix in a power series of spectral parameter [3-6]. The Lax operator associated with monodromy matrix satisfies quantum Yang-Baxter equation (QYBE). As a result, all of these quantum conserved quantities commute among themselves. Thus, for constructing a quantum integrable field model or spin chain, it is natural to start with a suitable quantum Lax operator which satisfies QYBE and find out corresponding conserved quantities including the Hamiltonian.

However, explicit construction of these conserved quantities in terms of basic quantum field or spin operators often turns out to be a challenging task which has inspired the application of several ingenious techniques. For example, in the case of one dimensional quantum integrable spin chains like Heisenberg model, supersymmetric t-J model and Hubbard model, one can explicitly construct the conserved quantities in a recursive way by using appropriate ‘ladder operators’ [7-10]. While dealing with 1 + 1-dimensional classically integrable field theoretical systems like nonlinear Schrödinger (NLS) model, it is again possible to explicitly construct the conserved quantities in a recursive way by solving corresponding Riccati equations. However this recursive method of finding the conserved quantities does not usually work for quantum integrable field models, where the presence of normal ordering might lead to non-uniformness in the asymptotic expansion of monodromy matrix in powers of spectral parameter. As a result, it may not be possible to obtain all quantum conserved quantities simply as normal ordered versions of the corresponding classical conserved quantities [11-13]. Fortunately, however, this problem does not occur for the case of some lower conserved quantities of quantum NLS model, which are generated by first few terms in the asymptotic expansion of monodromy matrix [14]. Consequently, conserved quantities associated with number of particles, momentum as well as Hamiltonian of the quantum NLS model can be obtained just as normal ordered versions of the corresponding classical conserved quantities. The Hamiltonian of quantum integrable Sine-Gordon model can also be obtained in a similar way from the corresponding classical Hamiltonian [3,5].

Even though the Hamiltonians of quantum integrable field models usually coincide with the normal ordered versions of the corresponding classical Hamiltonians, there is no guarantee that this thumb rule will always be obeyed. The main purpose of the present article is to construct the quantum Hamiltonian of a derivative nonlinear Schrödinger
(DNLS) model through the corresponding Lax operator and explore how this quantum Hamiltonian is related to its classical counterpart. In this context it may be noted that there exist two variants of classically integrable DNLS model in $1 + 1$-dimension [15,16], which have found applications in physical systems like circularly polarized nonlinear Alfven waves in a plasma [17,18]. However, only one among these variants of DNLS model is known to be associated with an ultralocal Poisson Bracket (PB) structure which is very suitable for quantization through QISM [19,20]. The equation of motion for such classical DNLS model is given by [16]

\[ i \partial_t \psi(x,t) + \partial_{xx} \psi(x,t) - 4i \xi \psi^*(x,t) \psi(x,t) \partial_x \psi(x,t) = 0 , \quad (1.1) \]

where $\partial_t \equiv \frac{\partial}{\partial t}$, $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_{xx} \equiv \frac{\partial^2}{\partial x^2}$ and $\xi$ is a real parameter representing the strength of the nonlinear interaction term. The Lax operator related to this DNLS model may be written in the form [19,21]

\[ U(x, \lambda) = i \left( \frac{\xi \psi^*(x) \psi(x) - \lambda^2/4}{\lambda \psi(x)} - \frac{\xi \lambda \psi^*(x) \psi(x) + \lambda^2/4}{\lambda^2} \right) , \quad (1.2) \]

where $\lambda$ denotes the spectral parameter and $\psi(x), \psi^*(x)$ represent field variables at some fixed time (which is suppressed here and all along in the following). By solving the Riccati equation associated with Lax operator (1.2), one can explicitly construct the conserved quantities for this DNLS model in a recursive way. The first few among such infinite number of classical conserved quantities, representing the mass, momentum and Hamiltonian of the DNLS system respectively, are given by [19]

\[ N = \int_{-\infty}^{+\infty} \psi^*(x) \psi(x) \, dx , \quad P = -i \int_{-\infty}^{+\infty} \psi^*(x) \partial_x \psi(x) \, dx , \quad (1.3a, b) \]

\[ H = \int_{-\infty}^{+\infty} \left\{ -\psi^*(x) \partial_{xx} \psi(x) + i \xi \psi^2(x) \partial_x \psi^2(x) \right\} \, dx . \quad (1.3c) \]

The field variables appearing in the Lax operator (1.2) obey the following equal time PB structure: $\{ \psi(x), \psi(y) \} = \{ \psi^*(x), \psi^*(y) \} = 0$, $\{ \psi(x), \psi^*(y) \} = -i \delta(x - y)$. With the help of this ultralocal PB structure, it can be shown that the Lax operator (1.2) satisfies classical Yang-Baxter equation. As a result, infinite number of conserved quantities associated with DNLS model (1.1) yield vanishing PB relations among themselves [19]. This fact establishes the classical integrability of DNLS model (1.1) in the Liouville sense.

It is remarkable that the integrability property of the above mentioned classical DNLS model can be preserved even after quantization. In this quantized version of DNLS model, the basic field operators satisfy equal time commutation relations given by

\[ [\psi(x), \psi(y)] = [\psi^\dagger(x), \psi^\dagger(y)] = 0 , \quad [\psi(x), \psi^\dagger(y)] = \hbar \delta(x - y) , \quad (1.4) \]

$h$ being the Planck’s constant. The corresponding vacuum state is defined through the relation: $\psi(x)|0\rangle = 0$. The most natural way of constructing such quantum integrable
DNLS model, possessing infinite number of mutually commuting conserved quantities, is to first find out the quantum analogue of classical Lax operator (1.2) which would satisfy the QYBE. However, it can be easily shown that QYBE is not satisfied if the normal ordered version of classical Lax operator (1.2) is directly chosen as the quantum Lax operator of DNLS model. The correct form of this quantum Lax operator, satisfying QYBE in continuum, is given by [21]

\[ U_q(x, \lambda) = i \left\{ \begin{array}{cc} f \psi^\dagger(x)\psi(x) - \lambda^2/4 & \xi \lambda \psi^\dagger(x) \\ \lambda \psi(x) & -g \psi^\dagger(x)\psi(x) + \lambda^2/4 \end{array} \right\}, \tag{1.5} \]

where \( f = \frac{\xi e^{-i\alpha/2}}{\cos \alpha/2} \), \( g = \frac{\xi e^{i\alpha/2}}{\cos \alpha/2} \) and \( \alpha \) a real parameter \( -\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} \) which is uniquely determined through the relation

\[ \sin \alpha = -\bar{h} \xi. \tag{1.6} \]

Thus the quantum Lax operator (1.5) depends not only on the parameter \( \xi \), but also on the Planck’s constant \( \hbar \). It is clear from eqn.(1.6) that, for any fixed value of \( \xi \), \( \alpha \rightarrow 0 \) limit is essentially equivalent to \( \hbar \rightarrow 0 \) limit. Since \( f \rightarrow \xi \) and \( g \rightarrow \xi \) at \( \alpha \rightarrow 0 \) limit, the quantum Lax operator (1.5) reproduces the classical Lax operator (1.2) at \( \hbar \rightarrow 0 \) limit. With the help of Lax operator (1.5) or its lattice version [19,20], one can easily construct the monodromy matrix of quantum DNLS model in continuum. Quantum conserved quantities can be defined formally through the diagonal elements of this monodromy matrix, by expanding them in the power series of spectral parameter. By applying algebraic Bethe ansatz to such formally defined quantum conserved quantities, one can derive their exact eigenfunctions and eigenvalues for scattering as well as bound soliton states [19,21]. Moreover, one can also construct the reflection operators for the DNLS model satisfying the Zamolodchikov-Faddeev algebra and find out the \( S \)-matrix for the two body scattering [21].

In spite of these studies on quantum DNLS model, the problem of explicitly constructing its conserved quantities in terms of basic field operators like \( \psi(x) \) and \( \psi^\dagger(x) \) has not been addressed so far. In particular it is not known whether, in analogy with the quantum NLS model and sine-Gordon model, the Hamiltonian of quantum DNLS model can also be obtained as the normal ordered version of the corresponding classical Hamiltonian (1.3c). The explicit form of such quantum Hamiltonian would clearly play a central role in interpreting various properties of this field model in the language of associated quantum mechanical many-particle system. In this context it should be observed that, if the normal ordered version of classical Hamiltonian (1.3c) is projected on an \( N \)-particle Hilbert space, that would yield an \( N \)-particle bosonic system interacting through the derivative \( \delta \)-function potential [22,23], where \( \xi \) represents the strength of the interaction. Equation (1.6) however imposes a restriction on the value of this coupling constant as
$|\xi| \leq \frac{1}{h}$. Thus it is evident that, if the normal ordered version of the classical Hamiltonian (1.3c) represents the quantum Hamiltonian of DNLS model, the corresponding $N$-particle bosonic system cannot be solved through QISM for $|\xi| > \frac{1}{h}$. On the other hand, it is known that this $N$-particle bosonic system with derivative $\delta$-function interaction can be solved exactly for any value of its coupling constant through the coordinate Bethe ansatz [22-24]. Thus one faces a rather curious limitation about the applicability of algebraic Bethe ansatz to the case of quantum DNLS model.

It is clear that, some direct method of finding the explicit form of quantum Hamiltonian associated with the Lax operator (1.5) of DNLS model may help us to resolve the above mentioned problem. In this context, we recall a work by Case [11] where first few conserved quantities of the quantum NLS model are explicitly constructed and their spectra are also derived in the following way. At first, Jost solutions associated with the Lax operator of quantum NLS system are considered. The scattering data, i.e. elements of monodromy matrix, are identified with the Wronskians corresponding to these Jost solutions. Subsequently it is proposed that the commutators between quantum conserved quantities of the NLS model and Wronskians obey the so called ‘fundamental relation’. This relation can generate the spectra of all quantum conserved quantities in an algebraic way. The explicit form of the first few quantum conserved quantities of NLS model are obtained from the requirement of satisfying this fundamental relation.

The above mentioned way of constructing quantum conserved quantities and finding their spectra is clearly different from the usual algebraic Bethe ansatz in QISM. However, in complete analogy with QISM, finding an appropriate quantum Lax operator is the starting point of Case’s approach. So this approach gives us valuable insight about the explicit form of quantum conserved quantities which can be obtained from the trace of monodromy matrix in QISM. In this article we shall study quantum DNLS model through this approach which is complimentary to QISM. In Section 2, we briefly recapitulate the construction of quantum Lax operator of DNLS model through a variant of QISM which is directly applicable to field theoretical systems and also discuss how the related conserved quantities can be diagonalised through algebraic Bethe ansatz [21]. In Section 3 we use the quantum Lax operator and monodromy matrix, obtained through QISM, for defining the Jost solutions of DNLS model. It is surprisingly found that, in contrast to the case of NLS model, differential equations satisfied by Jost solutions associated with boundary conditions at $x \to \infty$ and $x \to -\infty$ do not coincide with each other. Using the Wronskians and some other bilinear functions of these Jost solutions, in Section 4 we propose the ‘fundamental relation’ for the DNLS model and derive the spectra for all conserved quantities which would satisfy this relation. Here we also discuss how the conserved quantities satisfying the above relation are related to the conserved quantities which are formally defined in the framework of QISM. In Section 5, we discuss about
the necessary tools for finding out the explicit form of conserved quantities satisfying
the fundamental relation. In particular, we derive the commutation relations between
the Wronskians and basic field operators of the system. In Section 6, we construct the
explicit form of first few conserved quantities of the quantum DNLS model including
its Hamiltonian. Interestingly, it is found that the interaction part of this quantum
Hamiltonian has a new kind of coupling constant which is quite different from the classical
one. Here we also derive the condition on this coupling constant for which the quantum
N-soliton state of DNLS model has a positive binding energy. Section 7 is the concluding
section.

2 Application of QISM to DNLS model

As mentioned earlier, the monodromy matrix plays a key role in formally generating
the quantum conserved quantities of DNLS model and in diagonalising those conserved
quantities through QISM. With the help of Lax operator (1.5), one can define the quantum
monodromy matrix of DNLS model on a finite interval as

\[ T_{x_2}^{x_1}(\lambda) = : \mathcal{P} \exp \int_{x_1}^{x_2} U_q(x, \lambda) dx : , \quad (2.1) \]

where \( \mathcal{P} \) denotes the path ordering and the symbol :: denotes the normal ordering of
operators. It is evident that this monodromy matrix satisfies differential equations of the
form

\[ \frac{\partial}{\partial x_2} T_{x_2}^{x_1}(\lambda) = : U_q(x_2, \lambda) T_{x_2}^{x_1}(\lambda) : , \quad \frac{\partial}{\partial x_1} T_{x_1}^{x_2}(\lambda) = - : T_{x_1}^{x_2}(\lambda) U_q(x_1, \lambda) : . \quad (2.2a, b) \]

By using these differential equations and canonical commutation relations (1.4), it can be
shown that the direct product of two such quantum monodromy matrices satisfies QYBE
given by [21]

\[ R(\lambda, \mu) T_{x_2}^{x_1}(\lambda) \otimes T_{x_2}^{x_1}(\mu) = T_{x_2}^{x_1}(\mu) \otimes T_{x_1}^{x_2}(\lambda) R(\lambda, \mu) . \quad (2.3) \]

Here \( R(\lambda, \mu) \) is a \((4 \times 4)\) matrix with c-number elements like

\[ R(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s(\lambda, \mu) & t(\lambda, \mu) & 0 \\
0 & t(\lambda, \mu) & s(\lambda, \mu) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (2.4) \]

with \( t(\lambda, \mu) = \frac{\lambda^2 - \mu^2}{\lambda^2 q - \mu^2 q^{-1}} \), \( s(\lambda, \mu) = \frac{q-q^{-1}}{\lambda^2 q - \mu^2 q^{-1}} \lambda \mu \) and \( q = e^{-i\alpha} \). It is mentioned earlier that
the real parameter \( \alpha \), which is present both in Lax operator (1.5) and \( R \)-matrix (2.4), is
fixed through the relation (1.6). Consequently, QISM is applicable for quantum DNLS model when the parameter $\xi$ satisfies a restriction given by $|\xi| \leq \frac{1}{\hbar}$.

Next, by using the expression of $T_{x_1}^{x_2}(\lambda)$ in (2.1), we define the quantum monodromy matrix on an infinite interval limit as

$$ T(\lambda) = \lim_{x_2 \to +\infty} e(-x_2, \lambda) T_{x_1}^{x_2}(\lambda) e(x_1, \lambda) = T_+(x, \lambda) T_-(x, \lambda), \quad (2.5) $$

where $e(x, \lambda) = e^{-\frac{\alpha^2 x^2}{4} \sigma_3}$ and

$$ T_+(x, \lambda) = \lim_{x_2 \to +\infty} e(-x_2, \lambda) T_{x}^{x_2}(\lambda), \quad T_-(x, \lambda) = \lim_{x_1 \to -\infty} T_{x_1}^{x}(\lambda) e(x_1, \lambda). \quad (2.6a,b) $$

Taking into account that the quantum Lax operator (1.5) obeys certain symmetry properties [21] and assuming $\lambda$ to be a real parameter, one can express the quantum monodromy matrix (2.5) in a symmetric form given by

$$ T(\lambda) = \begin{pmatrix} \, A(\lambda) & -\xi B^\dagger(\lambda) \, \\ B(\lambda) & A^\dagger(\lambda) \end{pmatrix}, \quad (2.7) $$

and find that these operator valued elements satisfy relations like $A(-\lambda) = A(\lambda)$, $B(-\lambda) = -B(\lambda)$. Moreover, it is easy to show that these elements act on the vacuum state as: $A(\lambda)|0\rangle = |0\rangle$, $B(\lambda)|0\rangle = 0$. With the help of eqns.(2.3) and (2.5), one may now obtain QYBE for the quantum monodromy matrix on an infinite interval as [21]

$$ R(\lambda, \mu) C_+(\lambda, \mu) T(\lambda) \otimes T(\mu) C_-(\lambda, \mu) = C_+(\mu, \lambda) T(\mu) \otimes T(\lambda) C_-(\mu, \lambda) R(\lambda, \mu), \quad (2.8) $$

where

$$ C_+(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \rho_+(\lambda, \mu) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_-(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \rho_-(\lambda, \mu) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.9) $$

and

$$ \rho_\pm(\lambda, \mu) = \mp \frac{2i\hbar \xi \lambda \mu}{\lambda^2 - \mu^2} + 2\pi i \hbar \xi \lambda \mu (\lambda^2 - \mu^2) = \mp \frac{2i\hbar \xi \lambda \mu}{\lambda^2 - \mu^2} \mp i\epsilon. $$

By inserting the explicit expressions for $R(\lambda, \mu)$ (2.4), $C_\pm(\lambda, \mu)$ (2.9) and $T(\lambda)$ (2.7) to QYBE (2.8) and comparing its matrix elements from both sides, we finally obtain

$$ [A(\lambda), A(\mu)] = 0, \quad [A(\lambda), A^\dagger(\mu)] = 0, \quad [B(\lambda), B(\mu)] = 0, \quad (2.10a,b,c) $$

$$ A(\lambda) B^\dagger(\mu) = \frac{\mu^2 q - \lambda^2 q^{-1}}{\mu^2 - \lambda^2 - i\epsilon} B^\dagger(\mu) A(\lambda), \quad (2.10d) $$

$$ B(\mu) A(\lambda) = \frac{\mu^2 q - \lambda^2 q^{-1}}{\mu^2 - \lambda^2 - i\epsilon} A(\lambda) B(\mu), \quad (2.10e) $$

$$ B(\mu) B^\dagger(\lambda) = \tau(\lambda, \mu) B^\dagger(\lambda) B(\mu) + 4\pi \hbar \lambda \mu \delta(\lambda^2 - \mu^2) A^\dagger(\lambda) A(\lambda), \quad (2.10f) $$
where \( \tau(\lambda, \mu) = \left[ 1 + \frac{\hbar^2 \xi^2 \lambda^2 \mu^2}{(\lambda^2 - \mu^2)^2} - \frac{4\hbar^2 \xi^2 \lambda^2 \mu^2}{(\lambda^2 - \mu^2 - i\epsilon)(\lambda^2 - \mu^2 + i\epsilon)} \right]. \)

Due to eqn.(2.10a) it follows that all operator valued coefficients occurring in the expansion of \( \ln A(\lambda) \) in powers of \( \lambda \) must commute among themselves. Consequently, \( \ln A(\lambda) \) may be treated as the generator of conserved quantities for the quantum integrable DNLS model. For the purpose of diagonalising these quantum conserved quantities, we first notice that the commutation relation (2.10f) contains product of singular functions

\[
(\lambda^2 - \mu^2 - i\epsilon)^{-1}(\lambda^2 - \mu^2 + i\epsilon)^{-1},
\]

which does not make sense at the limit \( \lambda \to \mu \). As a result, actions of operators \( B^\dagger(\lambda), B(\mu) \) are not well defined on the Hilbert space [4,25] and generate states which are not normalised on the \( \delta \)-function. However, it is well known that, one can avoid this type of problem in the case of NLS model by considering the quantum analogue of classical reflection operators [3,26]. So, for the case of DNLS model also we consider a reflection operator given by

\[
R^\dagger(\lambda) = B^\dagger(\lambda)(A^\dagger(\lambda))^{-1}
\] (2.11)

and its adjoint \( R(\lambda) \). By using eqns.(2.10a-f), we find that such reflection operators satisfy

well defined commutation relations like [21]

\[
R^\dagger(\lambda)R^\dagger(\mu) = S^{-1}(\lambda, \mu) R^\dagger(\mu)R^\dagger(\lambda),
\]

\[
R(\lambda)R(\mu) = S^{-1}(\lambda, \mu) R(\mu)R(\lambda),
\]

\[
R^\dagger(\lambda)R(\mu) = S(\lambda, \mu) R(\mu)R^\dagger(\lambda) + 4\pi\hbar \lambda^2 \delta(\lambda^2 - \mu^2),
\] (2.12)

where

\[
S(\lambda, \mu) = \frac{\lambda^2 q - \mu^2 q^{-1}}{\lambda^2 q^{-1} - \mu^2 q}.
\] (2.13)

It is evident that these commutation relations are encoded in a form of Zamolodchikov-Faddeev algebra [3,27] and \( S(\lambda, \mu) \) (2.13) represents the nontrivial \( S \)-matrix element of two-body scattering between the related quasi-particles. It is easy to check that this \( S(\lambda, \mu) \) satisfies the relations

\[
S^{-1}(\lambda, \mu) = S(\mu, \lambda) = S^*(\lambda, \mu),
\] (2.14)

and remains nonsingular at the limit \( \lambda \to \mu \). As a result, the action of operators like \( R^\dagger(\lambda) \) on the vacuum would produce well defined states which can be normalised on the \( \delta \)-function.

The commutation relation between \( A(\lambda) \) and \( R^\dagger(\mu) \) may be derived by using eqns.(2.10b) and (2.10d) as

\[
A(\lambda)R^\dagger(\mu) = \frac{\mu^2 q - \lambda^2 q^{-1}}{\mu^2 - \lambda^2 - i\epsilon} R^\dagger(\mu)A(\lambda).
\] (2.15)
By applying the above commutation relation and also using $A(\lambda)|0\rangle = |0\rangle$, it can be shown that

$$A(\lambda) |\mu_1, \mu_2, \cdots, \mu_N\rangle = \prod_{r=1}^{N} \left( \frac{\mu_r^2 q - \lambda^2 q^{-1}}{\mu_r^2 - \lambda^2 - i\epsilon} \right) |\mu_1, \mu_2, \cdots, \mu_N\rangle,$$  (2.16)

where $|\mu_1, \mu_2, \cdots, \mu_N\rangle \equiv R^\dagger(\mu_1)R^\dagger(\mu_2)\cdots R^\dagger(\mu_N)|0\rangle$ and $\mu_j$s are all distinct real or pure imaginary numbers. Thus the states $|\mu_1, \mu_2, \cdots, \mu_N\rangle$ diagonalise the generator of conserved quantities for the quantum DNLS model. However, by using eqn.(2.16), one finds that the eigenvalues corresponding to different expansion coefficients of $\ln A(\lambda)$ would be complex quantities in general. To make the eigenvalues real, we define another operator $\hat{A}(\lambda)$ through the relation: $\hat{A}(\lambda) \equiv A(\lambda e^{-\frac{i\alpha}{2}})$ and expand $\ln \hat{A}(\lambda)$ as

$$\ln \hat{A}(\lambda) = \sum_{n=0}^{\infty} \frac{iC_n}{\lambda^{2n}}.$$  (2.17)

With the help of eqns.(2.16) and (2.17), one can easily find out the real eigenvalues associated with all $C_n$s:

$$C_0|\mu_1, \mu_2, \cdots, \mu_N\rangle = \alpha N |\mu_1, \mu_2, \cdots, \mu_N\rangle,$$  (2.18a)

$$C_n|\mu_1, \mu_2, \cdots, \mu_N\rangle = \frac{2}{n} \sin(\alpha n) \left\{ \sum_{j=1}^{N} \mu_j^{2n} \right\} |\mu_1, \mu_2, \cdots, \mu_N\rangle,$$  (2.18b)

where $n \geq 1$. Till now it is assumed that $\mu_j$s are some real or pure imaginary parameters, for which $|\mu_1, \mu_2, \cdots, \mu_N\rangle$ represents a scattering state. We can also construct the quantum soliton states or bound states for DNLS model by choosing complex values of $\mu_j$ given by [19,21]

$$\mu_j = \mu \exp \left[ i\alpha \left( \frac{N+1}{2} - j \right) \right],$$  (2.19)

where $\mu$ is a real or pure imaginary parameter and $j \in [1, 2, \cdots N]$. Similar to the case of scattering states, one can find out the real eigenvalues corresponding to all $C_n$s for these quantum soliton states of DNLS model.

Thus, by applying QISM, it is possible to obtain the exact eigenvalues as well as eigenstates for the quantum conserved quantities of DNLS model which are defined formally through the expansion (2.17). However, the important problem of expressing these conserved quantities through basic field operators like $\psi(x)$ and $\psi^\dagger(x)$ has not been explored so far. In analogy with the classical case, $C_0$, $C_1$ and $C_2$ should be related to the number operator, momentum operator and the Hamiltonian of the quantum DNLS model respectively. So it should be particularly interesting to find out the explicit form of these first three conserved quantities. To this end, we shall study quantum Jost solutions of the DNLS model.
3 Jost Solutions of quantum DNLS model

It may be recalled that the differential equations satisfied by the Jost solutions of quantum NLS model are defined through the corresponding Lax operator [11]. As a result all Jost solutions of NLS model, defined through boundary conditions at \( x \to +\infty \) or \( x \to -\infty \), satisfy exactly the same form of coupled differential equations. At present, however, we shall not directly use the Lax operator (1.5) for obtaining the differential equations associated with Jost solutions of quantum DNLS model. Instead, we shall identify appropriate elements of the matrices \( T_+(x, \lambda) \) (2.6a) and \( T_-(x, \lambda) \) (2.6b) as Jost solutions corresponding to boundary conditions at \( x \to +\infty \) and \( x \to -\infty \) respectively. The differential equations satisfied by \( T_-(x, \lambda) \) will give us in a natural way the differential equations for Jost solutions corresponding to boundary conditions at \( x \to +\infty \) and \( x \to -\infty \) respectively. It will turn out that, contrary to the case of NLS model, quantum Jost solutions of DNLS model associated with boundary conditions at \( x \to +\infty \) and \( x \to -\infty \) satisfy different types of coupled differential equations. Due to eqn.(2.5), the elements of monodromy matrix (2.7) can be expressed as Wronskians of such Jost solutions.

To proceed in the above mentioned way, let us express \( T_-(x, \lambda) \) (2.6b) in elementwise form as

\[
T_-(x, \lambda) = \begin{pmatrix} \phi_1(x, \lambda) & \bar{\phi}_1(x, \lambda) \\ \phi_2(x, \lambda) & \bar{\phi}_2(x, \lambda) \end{pmatrix},
\]

where \( \phi(x, \lambda) \equiv \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix} \) and \( \bar{\phi}(x, \lambda) \equiv \begin{pmatrix} \bar{\phi}_1(x, \lambda) \\ \bar{\phi}_2(x, \lambda) \end{pmatrix} \) are two Jost solutions corresponding to boundary conditions at \( x \to -\infty \). Due to eqn.(2.2a), \( T_-(x, \lambda) \) satisfies a differential equation given by

\[
\partial_x T_-(x, \lambda) = : U_q(x, \lambda) T_-(x, \lambda) :.
\]

Substituting the explicit form of \( T_-(x, \lambda) \) (3.1) to (3.2), we find that the components of \( \phi(x, \lambda) \) and \( \bar{\phi}(x, \lambda) \) satisfy exactly the same form of coupled differential equations given by

\[
\partial_x \rho_1(x, \lambda) = -\frac{i\lambda^2}{4}\rho_1(x, \lambda) + i f \psi^\dagger(x) \rho_1(x, \lambda) \psi(x) + i \xi \psi^\dagger(x) \rho_2(x, \lambda),
\]

\[
\partial_x \rho_2(x, \lambda) = \frac{i\lambda^2}{4}\rho_2(x, \lambda) - ig \psi^\dagger(x) \rho_2(x, \lambda) \psi(x) + i \lambda \rho_1(x, \lambda) \psi(x),
\]

where \( \begin{pmatrix} \rho_1(x, \lambda) \\ \rho_2(x, \lambda) \end{pmatrix} \) may be chosen either as \( \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix} \) or as \( \begin{pmatrix} \bar{\phi}_1(x, \lambda) \\ \bar{\phi}_2(x, \lambda) \end{pmatrix} \). Thus \( \rho(x, \lambda) \equiv \begin{pmatrix} \rho_1(x, \lambda) \\ \rho_2(x, \lambda) \end{pmatrix} \) represents the general form of Jost solutions defined through boundary conditions at \( x \to -\infty \). Next, by taking the \( x \to -\infty \) limit of \( T_-(x, \lambda) \) (2.6b), we obtain

\[
T_-(x, \lambda) \xrightarrow{x \to -\infty} e^{-\frac{i\lambda x}{4} \sigma_3}.
\]
Substituting the matrix form of $\mathcal{T}_-(x, \lambda)$ (3.1) to the relation (3.4), we obtain the boundary conditions associated with Jost solutions $\phi(x, \lambda)$ and $\bar{\phi}(x, \lambda)$ as

$$
\begin{pmatrix}
\rho_1(x, \lambda) \\
\rho_2(x, \lambda)
\end{pmatrix}
\to
\begin{pmatrix}
\rho_1^0 e^{-\frac{i \lambda^2 x}{4}} \\
\rho_2^0 e^{-\frac{i \lambda^2 x}{4}}
\end{pmatrix},
$$

(3.5)

where $\rho_1^0 = 1, \rho_2^0 = 0$ for $\rho(x, \lambda) = \phi(x, \lambda)$ and $\rho_1^0 = 0, \rho_2^0 = 1$ for $\rho(x, \lambda) = \bar{\phi}(x, \lambda)$. Using the boundary conditions (3.5), we can convert the differential equations (3.3) to their integral forms as

$$
\begin{align*}
\rho_1(x, \lambda) &= \rho_1^0 e^{-\frac{i \lambda^2 x}{4}} + i \int_{-\infty}^{x} dz e^{\frac{i \lambda^2}{4}(z-x)} \left\{ f \psi^\dagger(z) \rho_1(z, \lambda) \psi(z) + \xi \lambda \psi^\dagger(z) \rho_2(z, \lambda) \right\}, \\
\rho_2(x, \lambda) &= \rho_2^0 e^{\frac{i \lambda^2 x}{4}} + i \int_{-\infty}^{x} dz e^{\frac{i \lambda^2}{4}(x-z)} \left\{ -g \psi^\dagger(z) \rho_2(z, \lambda) \psi(z) + \lambda \rho_1(z, \lambda) \psi(z) \right\}.
\end{align*}
$$

(3.6a, b)

With the help of these integral relations it is easy to show that, for the case of real $\lambda$, the components of Jost solutions $\phi(x, \lambda)$ and $\bar{\phi}(x, \lambda)$ are related as

$$
\bar{\phi}_1(x, \lambda) = -\xi \phi_2^\dagger(x, \lambda), \quad \bar{\phi}_2(x, \lambda) = \phi_1^\dagger(x, \lambda).
$$

(3.7)

Next we try to find out the differential equations for the Jost solutions corresponding to boundary conditions at $x \to +\infty$. To this end, we express $\mathcal{T}_+(x, \lambda)$ (2.6a) in elementwise form as

$$
\mathcal{T}_+(x, \lambda) = \begin{pmatrix}
\chi_2(x, \lambda) & -\chi_1(x, \lambda) \\
\bar{\chi}_2(x, \lambda) & -\bar{\chi}_1(x, \lambda)
\end{pmatrix},
$$

(3.8)

where $\chi(x, \lambda) \equiv \begin{pmatrix} \chi_1(x, \lambda) \\ \chi_2(x, \lambda) \end{pmatrix}$ and $\bar{\chi}(x, \lambda) \equiv \begin{pmatrix} \bar{\chi}_1(x, \lambda) \\ \bar{\chi}_2(x, \lambda) \end{pmatrix}$ represent two Jost solutions corresponding to boundary conditions at $x \to +\infty$. Due to relation (2.2b), $\mathcal{T}_+(x, \lambda)$ satisfies a differential equation given by

$$
\partial_x \mathcal{T}_+(x, \lambda) = -: \mathcal{T}_+(x, \lambda) \mathcal{U}_q(x, \lambda) :.
$$

(3.9)

Substituting the elementwise form of $\mathcal{T}_+(x, \lambda)$ (3.8) to (3.9), it is easy to see that the components of $\chi(x, \lambda)$ and $\bar{\chi}(x, \lambda)$ satisfy exactly the same form of coupled differential equations given by

$$
\begin{align*}
\partial_x \chi_1(x, \lambda) &= -\frac{i \lambda^2}{4} \chi_1(x, \lambda) + ig \psi^\dagger(x) \chi_1(x, \lambda) \psi(x) + i \xi \lambda \psi^\dagger(x) \chi_2(x, \lambda), \\
\partial_x \chi_2(x, \lambda) &= \frac{i \lambda^2}{4} \chi_2(x, \lambda) - if \psi^\dagger(x) \chi_2(x, \lambda) \psi(x) + i \lambda \chi_1(x, \lambda) \psi(x),
\end{align*}
$$

(3.10)
where \( \begin{pmatrix} \tau_1(x, \lambda) \\ \tau_2(x, \lambda) \end{pmatrix} \) may be chosen as either \( \begin{pmatrix} \chi_1(x, \lambda) \\ \chi_2(x, \lambda) \end{pmatrix} \) or \( \begin{pmatrix} \bar{\chi}_1(x, \lambda) \\ \bar{\chi}_2(x, \lambda) \end{pmatrix} \). Thus \( \tau(x, \lambda) \equiv \begin{pmatrix} \tau_1(x, \lambda) \\ \tau_2(x, \lambda) \end{pmatrix} \) represents the general form of Jost solutions defined through boundary conditions at \( x \to \infty \) and \( x \to -\infty \). Next, by taking the \( x \to +\infty \) limit of \( T_+(x, \lambda) \) (2.6a), we obtain

\[
T_+(x, \lambda) \xrightarrow{x \to +\infty} e^{\frac{i\lambda^2}{4} x} \sigma_3. \tag{3.11}
\]

Substituting the explicit form of \( T_+(x, \lambda) \) (3.8) to the above relation, it is easy to find out the boundary conditions associated with Jost solutions \( \chi(x, \lambda) \) and \( \bar{\chi}(x, \lambda) \) as

\[
\begin{pmatrix} \tau_1(x, \lambda) \\ \tau_2(x, \lambda) \end{pmatrix} \xrightarrow{x \to +\infty} \begin{pmatrix} \tau_1^0 e^{-\frac{i\lambda^2}{4} x} \\ \tau_2^0 e^{\frac{i\lambda^2}{4} x} \end{pmatrix}, \tag{3.12}
\]

where \( \tau_1^0 = 0, \tau_2^0 = 1 \) for \( \tau(x, \lambda) = \chi(x, \lambda) \) and \( \tau_1^0 = -1, \tau_2^0 = 0 \) for \( \tau(x, \lambda) = \bar{\chi}(x, \lambda) \). Using the boundary conditions (3.12), we can convert the differential equations (3.10) to their integral forms as

\[
\begin{align*}
\tau_1(x, \lambda) &= \tau_1^0 e^{-\frac{i\lambda^2}{4} x} - i \int_x^{\infty} dz e^{\frac{i\lambda^2}{4} (z-x)} \left\{ g \psi(z) \tau_1(z, \lambda) \psi(z) + \xi \lambda \psi(z) \tau_2(z, \lambda) \right\}, \\
\tau_2(x, \lambda) &= \tau_2^0 e^{\frac{i\lambda^2}{4} x} - i \int_x^{\infty} dz e^{\frac{i\lambda^2}{4} (x-z)} \left\{ -f \psi(z) \tau_2(z, \lambda) \psi(z) + \lambda \tau_1(z, \lambda) \psi(z) \right\}.
\end{align*} \tag{3.13a,b}
\]

By using these integral relations it is easy to show that, for the case of real \( \lambda \), the components of Jost solutions \( \chi(x, \lambda) \) and \( \bar{\chi}(x, \lambda) \) are related as

\[
\bar{\chi}_1(x, \lambda) = -\chi_2^\dagger(x, \lambda), \quad \bar{\chi}_2(x, \lambda) = \frac{1}{\xi} \chi_1^\dagger(x, \lambda). \tag{3.14}
\]

Comparing eqns.(3.10) and (3.3), we notice that quantum Jost solutions of DNLS model, associated with boundary conditions at \( x \to +\infty \) and \( x \to -\infty \), satisfy two different sets of coupled differential equations. These two sets of differential equations are related to each other through an interchange of \( f \) and \( g \). However, since both \( f \) and \( g \) coincide with the coupling constant \( \xi \) at \( \hbar \to 0 \) limit, eqns.(3.3) and (3.10) have an identical form at this classical limit. It may also be observed that, due to vanishing boundary condition on the basic field variables, eqns. (3.3) and (3.10) have the same asymptotic form at \( |x| \to \infty \) limit.

Now we want to express the elements of quantum monodromy matrix (2.7) in terms of Jost solutions as obtained above. To this end, we substitute the elementwise form of \( T_-(x, \lambda) \) (3.1) and \( T_+(x, \lambda) \) (3.8) to eqn.(2.5) and compare it with (2.7). In this way, we obtain

\[
A(\lambda) = \chi_2(x, \lambda) \phi_1(x, \lambda) - \chi_1(x, \lambda) \phi_2(x, \lambda),
\]

12
\[ A^\dagger(\lambda) = \bar{\chi}_2(x, \lambda)\bar{\phi}_1(x, \lambda) - \bar{\chi}_1(x, \lambda)\bar{\phi}_2(x, \lambda), \]
\[ B(\lambda) = \chi_2(x, \lambda)\phi_1(x, \lambda) - \chi_1(x, \lambda)\phi_2(x, \lambda), \]
\[ B^\dagger(\lambda) = -\frac{1}{\xi}\chi_2(x, \lambda)\bar{\phi}_1(x, \lambda) + \frac{1}{\xi}\chi_1(x, \lambda)\bar{\phi}_2(x, \lambda). \quad (3.15a, b, c, d) \]

Since the l.h.s. of eqns. (3.15a-d) do not depend at all on the variable \( x \), the r.h.s. of these equations should also be independent of this variable (in spite of its explicit appearance). By taking \( x \to +\infty \) or \( x \to -\infty \) limit in the r.h.s. of eqns. (3.15a-d) and using boundary conditions (3.5) or (3.12) respectively, we obtain

\[ A(\lambda) = \lim_{x \to -\infty} e^{-\frac{i\lambda^2}{4}x} \chi_2(x, \lambda) = \lim_{x \to +\infty} e^{\frac{i\lambda^2}{4}x} \phi_1(x, \lambda), \quad (3.16a) \]
\[ A^\dagger(\lambda) = -\lim_{x \to -\infty} e^{\frac{i\lambda^2}{4}x} \bar{\chi}_1(x, \lambda) = \lim_{x \to +\infty} e^{-\frac{i\lambda^2}{4}x} \bar{\phi}_2(x, \lambda), \quad (3.16b) \]
\[ B(\lambda) = \lim_{x \to -\infty} e^{-\frac{i\lambda^2}{4}x} \chi_2(x, \lambda) = \lim_{x \to +\infty} e^{-\frac{i\lambda^2}{4}x} \phi_2(x, \lambda), \quad (3.16c) \]
\[ B^\dagger(\lambda) = \frac{1}{\xi} \lim_{x \to -\infty} e^{\frac{i\lambda^2}{4}x} \chi_1(x, \lambda) = -\frac{1}{\xi} \lim_{x \to +\infty} e^{\frac{i\lambda^2}{4}x} \bar{\phi}_1(x, \lambda). \quad (3.16d) \]

Next, let us define the quantum Wronskian associated with the general form of Jost solutions \( \tau(x, \lambda) \) and \( \rho(x, \lambda) \) as

\[ \Lambda_{\rho, \tau}(x, \lambda) = \tau_2(x, \lambda)\rho_1(x, \lambda) - \tau_1(x, \lambda)\rho_2(x, \lambda). \quad (3.17) \]

Comparing eqns. (3.15) and (3.17) for all possible choice of \( \tau(x, \lambda) \) and \( \rho(x, \lambda) \), we find that

\[ A(\lambda) = \Lambda_{\phi, \chi}(x, \lambda), \quad B(\lambda) = \Lambda_{\bar{\phi}, \bar{\chi}}(x, \lambda), \]
\[ A^\dagger(\lambda) = \Lambda_{\bar{\phi}, \bar{\chi}}(x, \lambda), \quad B^\dagger(\lambda) = -\frac{1}{\xi} \Lambda_{\phi, \chi}(x, \lambda). \quad (3.18) \]

Thus the quantum Wronskian (3.17) represents all elements of the monodromy matrix (2.7) in a general form.

Since the elements of monodromy matrix (2.7) do not depend on the variable \( x \), the quantum Wronskian (3.17) must also be independent of this variable. However, we should be able to demonstrate this fact in a direct way by showing that \( \Lambda_{\rho, \tau}(x, \lambda) \) has a vanishing derivative with respect to the variable \( x \). To this end, we consider a general type of quantum integrable field model whose Jost solutions \( \rho(x, \lambda) \) and \( \tau(x, \lambda) \) satisfy differential equations given by

\[ \partial_x \rho(x, \lambda) = :L^-(x, \lambda)\rho(x, \lambda) :, \quad \partial_x \tau(x, \lambda) = :L^+(x, \lambda)\tau(x, \lambda) :, \quad (3.19) \]

\( L^\pm(x, \lambda) \) being some \((2 \times 2)\)-matrices with elements \( L^\pm_{ij}(x, \lambda) \). As before, the quantum Wronskian associated with this general case may be defined through eqn. (3.17). For the
sake of convenience, let us ignore at present the effect of normal ordering in eqn.(3.19) and treat all quantum variables as commuting classical variables. In this way it can be easily shown that, the derivative of Wronskian (3.17) with respect to the variable \( x \) will vanish if the elements of \( L^+(x, \lambda) \) and \( L^-(x, \lambda) \) are related as

\[
L^+_{ij}(x, \lambda) = L^-_{ij}(x, \lambda),
\]

where \( i \neq j \). Thus it follows that, \( L^+(x, \lambda) \) would coincide with \( L^-(x, \lambda) \) when it satisfies the traceless condition. Since the Lax operators of quantum NLS model and almost all other integrable systems satisfy this traceless condition, \( L^+(x, \lambda) \) and \( L^-(x, \lambda) \) coincide for these cases. However, the quantum Lax operator (1.5) of DNLS model does not satisfy this condition. Consequently, the corresponding \( L^+(x, \lambda) \) and \( L^-(x, \lambda) \) matrices should not coincide with each other. Expressing eqns.(3.3) and (3.10) in matrix form, we find that \( L^-(x, \lambda) \) matrix of DNLS model is same as \( U_q(x, \lambda) \) (1.5) and \( L^+(x, \lambda) \) may be obtained from \( U_q(x, \lambda) \) by interchanging \( f \) and \( g \). Since these matrices satisfy the relation (3.20), we may conclude that the Wronskian (3.17) of DNLS model has a vanishing derivative with respect to the variable \( x \). A more rigorous proof about the coordinate independence of this Wronskian, taking into account the noncommutative nature of quantum operators, will be given in Sec.5 of this article.

4 Spectrum Generating Algebra for DNLS Model

By following the approach of Ref.11, here we shall propose the ‘fundamental relation’ for the DNLS model and explore its connection with the spectrum generating algebra. In analogy with the quantum Wronskian (3.17), let us define another operator associated with the Jost solutions of DNLS model as

\[
\Gamma_{\rho,\tau}(x, \lambda) = \tau_2(x, \lambda)\rho_1(x, \lambda) + \tau_1(x, \lambda)\rho_2(x, \lambda).
\]

This \( \Gamma_{\rho,\tau}(x, \lambda) \) and quantum Wronskian (3.17) are two basic ingredients which are needed for defining the fundamental relation of DNLS model. Now we propose that, the quantum conserved quantities \( (I_n) \) of DNLS model would annihilate the vacuum state and obey the fundamental relation given by

\[
\left[ I_n, \Lambda_{\rho,\tau}(\lambda) \right] = \frac{\hbar\lambda^{2n}}{2^{n+1}} \int_{-\infty}^{+\infty} \partial_y \Gamma_{\rho,\tau}(y, \lambda) \, dy
\]

\[
= \frac{\hbar\lambda^{2n}}{2^{n+1}} \left\{ \Gamma_{\rho,\tau}(+\infty, \lambda) - \Gamma_{\rho,\tau}(-\infty, \lambda) \right\},
\]

where \( \Lambda_{\rho,\tau}(\lambda) \) is the generating algebraic invariant of DNLS model.
where \( n \) is any nonnegative integer. Since \( \Lambda_{\rho,\tau}(x,\lambda) \) \((3.17)\) does not depend on the coordinate \( x \), we have suppressed this variable in the l.h.s. of above relation.

Next, we shall discuss how the fundamental relation (4.2) leads to the spectrum generating algebra for all quantum conserved quantities of DNLS model. To this end, it is needed to find out the \( x \to \pm \infty \) limit of \( \Gamma_{\rho,\tau}(x,\lambda) \). For all possible choices of \( \rho \) and \( \tau \), \( \Gamma_{\rho,\tau}(x,\lambda) \) \((4.1)\) may be explicitly written as

\[
\begin{align*}
\Gamma_{\phi,\chi}(x,\lambda) &= \chi_2(x,\lambda)\phi_1(x,\lambda) + \chi_1(x,\lambda)\phi_2(x,\lambda), \\
\Gamma_{\bar{\phi},\bar{\chi}}(x,\lambda) &= \bar{\chi}_2(x,\lambda)\bar{\phi}_1(x,\lambda) + \bar{\chi}_1(x,\lambda)\bar{\phi}_2(x,\lambda), \\
\Gamma_{\phi,\bar{\chi}}(x,\lambda) &= \chi_2(x,\lambda)\bar{\phi}_1(x,\lambda) + \chi_1(x,\lambda)\bar{\phi}_2(x,\lambda), \\
\Gamma_{\bar{\phi},\chi}(x,\lambda) &= \bar{\chi}_2(x,\lambda)\phi_1(x,\lambda) + \bar{\chi}_1(x,\lambda)\phi_2(x,\lambda).
\end{align*}
\]

Substituting the asymptotic forms of Jost solutions \((3.5), (3.12)\) to the \( x \to \pm \infty \) limits of relations \((4.3a-d)\) and subsequently using \((3.16a-d)\), we find that

\[
\begin{align*}
\Gamma_{\phi,\chi}(\pm \infty, \lambda) &= A(\lambda), \\
\Gamma_{\bar{\phi},\bar{\chi}}(\pm \infty, \lambda) &= -A^\dagger(\lambda), \\
\Gamma_{\phi,\bar{\chi}}(\pm \infty, \lambda) &= \mp B(\lambda), \\
\Gamma_{\bar{\phi},\chi}(\pm \infty, \lambda) &= \mp \xi B^\dagger(\lambda).
\end{align*}
\]

Inserting \((4.4)\) to the fundamental relation \((4.2)\) and also using \((3.18)\), we get

\[
\begin{align*}
\left[ I_n, A(\lambda) \right] &= 0, \\
\left[ I_n, A^\dagger(\lambda) \right] &= 0, \\
\left[ I_n, B(\lambda) \right] &= -\hbar\lambda^{2n}B(\lambda), \\
\left[ I_n, B^\dagger(\lambda) \right] &= \hbar\lambda^{2n}B^\dagger(\lambda).
\end{align*}
\]

With the help of eqns.\((4.5b,d)\), we can find out the commutation relation between the quantum conserved quantities and reflection operators \((2.11)\) as

\[
\left[ I_n, R^\dagger(\lambda) \right] = \frac{\hbar\lambda^{2n}}{2n} R^\dagger(\lambda).
\]

By using the above commutation relation and assuming that \( I_n \)'s annihilate the vacuum state, it is easy to show that these conserved quantities satisfy eigenvalue equations given by

\[
I_n \left\{ \mu_1, \mu_2, \ldots, \mu_N \right\} = \left( \frac{\hbar}{2n} \sum_{j=1}^{N} \mu_j^{2n} \right) \left\{ \mu_1, \mu_2, \ldots, \mu_N \right\},
\]

where \( \left\{ \mu_1, \mu_2, \ldots, \mu_N \right\} \equiv R^\dagger(\mu_1)R^\dagger(\mu_2) \cdots R^\dagger(\mu_N)|0\). Consequently, the commutation relation \((4.6)\) may be treated as the spectrum generating algebra for the quantum conserved quantities of DNLS model.

It should be noted that, eigenstates of \( I_n \) are same as Bethe states which we have already used in the framework of QISM to diagonalise the quantum conserved quantities
appearing in the expansion (2.17). Thus, it is natural to expect a connection between these $I_n$'s and the conserved quantities which are formally defined through the expansion (2.17). For establishing this connection, let us assume that the Bethe states $|\mu_1, \mu_2, \cdots, \mu_N\rangle$ represent a complete set of states in the corresponding Hilbert space. Thus two operators would coincide if they can be simultaneously diagonalised through these complete set of states and their eigenvalues always match with each other. Comparing eqns.(4.7) with (2.18a,b), it is easy to find that

$$C_0 = \frac{\alpha}{\hbar} I_0, \quad C_n = \frac{2^{n+1}}{n\hbar} \sin(\alpha n) I_n.$$

(4.8)

Substituting (4.8) to (2.17), we obtain the expansion of $\ln \hat{A}(\lambda)$ in terms of $I_n$'s as

$$\ln \hat{A}(\lambda) = \frac{i\alpha}{\hbar} I_0 + \frac{i}{\hbar} \sum_{n=1}^{\infty} \frac{2^{n+1}}{n\lambda^{2n}} \sin(\alpha n) I_n.$$

(4.9)

We can also define conserved quantities for DNLS model through reflection operators as

$$I'_n = \frac{1}{2^{n+1}\pi} \int_0^{\infty} \mu^{2n-1} R^\dagger(\mu) R(\mu) d\mu.$$

(4.10)

By using the commutation relations between reflection operators (2.12), which are derived in the framework of QISM, we obtain

$$[I'_n, I'_m] = 0, \quad [I'_n, R^\dagger(\lambda)] = \frac{\hbar \lambda^{2n}}{2^n} R^\dagger(\lambda).$$

(4.11a,b)

With the help of (4.11b), one can easily show that $|\mu_1, \mu_2, \cdots, \mu_N\rangle$ are eigenfunctions of $I'_n$ with exactly the same eigenvalues as found in the case of $I_n$ and conclude that $I_n = I'_n$. Consequently, equation (4.10) yields an expression of $I_n$ through the reflection operators of DNLS model.

Finally, let us investigate whether the fundamental relation may also lead to the spectrum of a general quantum integrable field model whose Jost solutions satisfy the relations (3.19). For this purpose, we assume that $\mathcal{L}^\pm(x, \lambda)$ matrices have the following asymptotic form at $|x| \to \infty$ limit:

$$\mathcal{L}^\pm(x, \lambda) \longrightarrow i \begin{pmatrix} l(\lambda) & 0 \\ 0 & -l(\lambda) \end{pmatrix},$$

(4.12)

where $l(\lambda)$ is a function of the spectral parameter. Due to these asymptotic forms of $\mathcal{L}^\pm(x, \lambda)$, the corresponding Jost solutions can be defined through boundary conditions given by

$$\rho(x, \lambda) \xrightarrow{x \to -\infty} \begin{pmatrix} \rho_1^0 e^{il(\lambda)x} \\ \rho_2^{-i\lambda}_0 e^{-i(-\lambda)x} \end{pmatrix}, \quad \tau(x, \lambda) \xrightarrow{x \to +\infty} \begin{pmatrix} \tau_1^0 e^{i\lambda x} \\ \tau_2^0 e^{-i\lambda x} \end{pmatrix}.$$  

(4.13)
Similar to the case of DNLS model, here we choose \( \rho_1^0 = 1 \), \( \rho_0^0 = 0 \) when \( \rho(x, \lambda) \equiv \phi(x, \lambda) \), \( \rho_1^0 = 0 \), \( \rho_0^0 = 1 \) when \( \rho(x, \lambda) \equiv \bar{\phi}(x, \lambda) \), \( \tau_1^0 = 0 \), \( \tau_0^0 = 1 \) when \( \tau(x, \lambda) \equiv \chi(x, \lambda) \) and \( \tau_1^0 = -1 \), \( \tau_0^0 = 0 \) when \( \tau(x, \lambda) \equiv \bar{\chi}(x, \lambda) \). The quantum Wronskian and \( \Gamma_{\rho,\tau}(x,\lambda) \) operator associated with these Jost solutions are defined through eqns. (3.17) and (4.1) respectively. By treating quantum operators as commuting classical variables and using the condition (3.20), we have already shown that \( \Lambda_{\rho,\tau}(x,\lambda) \) is independent of the variable \( x \). Here we assume that this Wronskian would remain independent of \( x \), even if the noncommuting nature of quantum operators are taken into account. Now we propose that hermitian conserved quantities \( (\mathcal{I}_n) \) associated with this general integrable field model satisfy fundamental relation of the form

\[
[\mathcal{I}_n, \Lambda_{\rho,\tau}(\lambda)] = q_n(\lambda) \left\{ \Gamma_{\rho,\tau}(+\infty, \lambda) - \Gamma_{\rho,\tau}(-\infty, \lambda) \right\},
\]

where \( n \) is any nonnegative integer and \( q_n(\lambda) \) is some real function of \( \lambda \) whose explicit form depends on the system concerned. Taking \( x \to \pm\infty \) limits of \( \Lambda_{\rho,\tau}(x,\lambda) \) (3.17) and using (4.13), we find that

\[
\Lambda_{\phi,\chi}(\lambda) = \lim_{x \to -\infty} e^{il(\lambda)x} \chi_2(x, \lambda) = \lim_{x \to +\infty} e^{-il(\lambda)x} \phi_1(x, \lambda),
\]

\[
\Lambda_{\bar{\phi},\chi}(\lambda) = -\lim_{x \to -\infty} e^{il(\lambda)x} \chi_1(x, \lambda) = \lim_{x \to +\infty} e^{-il(\lambda)x} \bar{\phi}_1(x, \lambda).
\]

Similarly, by taking \( x \to \pm\infty \) limits of \( \Gamma_{\rho,\tau}(x,\lambda) \) (4.1) and comparing them with (4.15), it is easy to show that

\[
\Gamma_{\phi,\chi}(\pm\infty, \lambda) = \Lambda_{\phi,\chi}(\lambda), \quad \Gamma_{\bar{\phi},\chi}(\pm\infty, \lambda) = \pm \Lambda_{\bar{\phi},\chi}(\lambda).
\]

Inserting (4.16) to (4.14), we find that

\[
[\mathcal{I}_n, \Lambda_{\phi,\chi}(\lambda)] = 0, \quad [\mathcal{I}_n, \Lambda_{\bar{\phi},\chi}(\lambda)] = 2q_n(\lambda) \Lambda_{\bar{\phi},\chi}(\lambda).
\]

By using (4.17b) and assuming that \( \mathcal{I}_n \)'s annihilate the vacuum state, we obtain the spectra for these conserved quantities as

\[
\mathcal{I}_n |\mu_1, \mu_2, \ldots, \mu_N\rangle = \left( 2 \sum_{i=1}^{N} q_n(\mu_i) \right) |\mu_1, \mu_2, \ldots, \mu_N\rangle,
\]

where \( |\mu_1, \mu_2, \ldots, \mu_N\rangle \equiv \Lambda_{\phi,\chi}(\mu_1)\Lambda_{\phi,\chi}(\mu_2) \cdots \Lambda_{\phi,\chi}(\mu_N)|0\rangle \).

Thus, the fundamental relation (4.14) is powerful enough to generate the spectra of conserved quantities for a class of quantum integrable field models associated with Lax equations (3.19). In the rest of this article, however, we shall restrict our attention only to quantum DNLS model and try to explicitly construct first few quantum conserved quantities which would satisfy the corresponding fundamental relation (4.2). Necessary tools for such construction will be discussed in the next section.
5 Commutation relations between the quantum Wronskian and basic field operators

Since the quantum Wronskian (3.17) of DNLS model is expressed as a bilinear function of Jost solutions, at first we consider the commutation relations between these Jost solutions and basic field operators of the system. In analogy with the case of NLS model [11], one may take the arguments of Jost solutions and field operators at exactly the same space point and try to evaluate their commutation relations (e.g., commutators of the form $[\rho_i(x, \lambda), \psi(x)]$). By using the integral relations (3.6) and canonical commutation relations (1.4), it can be easily checked that the commutators $[\rho_i(x, \lambda), \psi(x)]$ lead to indeterminant integrals of the form $\int_{-\infty}^{\infty} \delta(x - z) F(z) dz$, where $F(z)$ is some function of $z$. Such indeterminant integrals, which also appear in the case of NLS model, may be expressed in the form $\int_{-\infty}^{\infty} \delta(x - z) F(z) dz = \frac{1}{2} F(x)$ [11]. However, as will be explained shortly, the above mentioned convention of fixing indeterminant integrals would lead to the violation of Jacobi identity in the case of DNLS model. So, instead of trying to calculate commutators of the form $[\rho_i(x, \lambda), \psi(x)]$, at present we shall study commutators like $[\rho_i(y, \lambda), \psi(x)]$ in the limit $y \rightarrow x$.

To begin with, let us consider the commutators $[\rho_i(y, \lambda), \psi(x)]$ and $[\rho_i(y, \lambda), \psi^\dagger(x)]$ in the region $y < x$. For this case, all fields $\psi(z)$, $\psi^\dagger(z)$ appearing in the integral relations (3.6a,b) would commute with $\psi(x)$, $\psi^\dagger(x)$. Consequently, we obtain $[\rho_i(y, \lambda), \psi(x)] = [\rho_i(y, \lambda), \psi^\dagger(x)] = 0$ in the region $y < x$. The $y \rightarrow x$ limit of these commutation relations may be expressed in the form

$$[\rho_i(x''', \lambda), \psi(x)] = [\rho_i(x''', \lambda), \psi^\dagger(x)] = 0,$$

(5.1)

where the notation $\rho_i(x''', \lambda) \equiv \lim_{\epsilon \rightarrow 0^+} \rho_i(x - \epsilon, \lambda)$ is introduced and $\epsilon \rightarrow 0^+$ limit is taken after evaluating all commutators.

Next, we consider the commutators $[\rho_i(y, \lambda), \psi(x)]$ and $[\rho_i(y, \lambda), \psi^\dagger(x)]$ in the region $y > x$. For this case, however, eqns.(3.6a,b) lead to rather complicated integral relations which are difficult to solve in a closed form for arbitrary values of $x$ and $y$. So, for the sake of convenience, we shall try to evaluate such commutators only at the limit $y \rightarrow x$. In analogy with the previous case, we introduce a notation given by $\rho_i(x', \lambda) \equiv \lim_{\epsilon \rightarrow 0^+} \rho_i(x + \epsilon; \lambda)$. We are interested in calculating commutators like $[\rho_i(x', \lambda), \psi(x)] \equiv \lim_{\epsilon \rightarrow 0^+} [\rho_i(x + \epsilon, \lambda), \psi(x)]$, where $\epsilon \rightarrow 0^+$ limit should be taken at the final stage after evaluating all commutation relations. By using integral relations (3.6a,b) and canonical commutation relations (1.4) we obtain

$$[\rho_i(x', \lambda), \psi(x)] = -i\hbar f \rho_1(x', \lambda) \psi(x) - i\hbar \xi \lambda \rho_2(x', \lambda),$$

(5.2a)

$$[\rho_i(x', \lambda), \psi^\dagger(x)] = i\hbar f \psi^\dagger(x) \rho_1(x', \lambda),$$

(5.2b)
\[
\begin{align*}
\left[ \rho_2(x', \lambda), \psi(x) \right] &= i\hbar \rho_2(x', \lambda) \psi(x), \\
\left[ \rho_2(x', \lambda), \psi^\dagger(x) \right] &= -i\hbar \psi^\dagger(x) \rho_2(x', \lambda) + i\hbar \lambda \rho_1(x', \lambda).
\end{align*}
\]

The details of derivation for one of the above commutation relations is given in Appendix A. It is clear from the relations (5.1) and (5.2) that the commutators \([\rho_i(y, \lambda), \psi(x)]\) and \([\rho_i(y, \lambda), \psi^\dagger(x)]\) are discontinuous at the point \(y = x\). By repeatedly applying the commutation relations (5.2), we easily obtain

\[
\begin{align*}
\left[ \rho_1(x', \lambda), \psi^2(x) \right] &= h f(h f - 2i) \rho_1(x', \lambda) \psi^2(x) \\
&\quad - i\hbar \xi \lambda \left\{ 2 + i\hbar(f - g) \right\} \rho_2(x', \lambda) \psi(x), \\
\left[ \rho_1(x', \lambda), \psi^{i2}(x) \right] &= h g(2i - h f) \psi^{i2}(x) \rho_1(x', \lambda), \\
\left[ \rho_2(x', \lambda), \psi^2(x) \right] &= h g(2i + h g) \rho_2(x', \lambda) \psi^2(x), \\
\left[ \rho_2(x', \lambda), \psi^{i2}(x) \right] &= -h g(2i + h g) \psi^{i2}(x) \rho_2(x', \lambda) \\
&\quad + i\hbar \lambda \left\{ 2 + i\hbar(f - g) \right\} \psi^\dagger(x) \rho_1(x', \lambda).
\end{align*}
\]

We would like to make a comment at this point. Since the integral relations of \(\rho_1(x - \epsilon, \lambda)\) and \(\rho_1(x + \epsilon, \lambda)\) coincide with each other at the limit \(\epsilon \to 0^+\), one may say that the operators \(\rho_i(x', \lambda)\) and \(\rho_i(x'', \lambda)\) are same in the ‘weak sense’. However, we have already observed that the commutators \([\rho_i(y, \lambda), \psi(x)]\) and \([\rho_i(y, \lambda), \psi^\dagger(x)]\) are discontinuous at the point \(y = x\). As a result, operators of the form \(\Delta_i(x, \lambda) \equiv \rho_i(x'', \lambda) - \rho_i(x', \lambda)\) yield nontrivial commutation relations with \(\psi(x)\) and \(\psi^\dagger(x)\). Thus, borrowing a terminology from the theory of constrained Hamiltonian systems [28], we may say that the operators \(\rho_i(x', \lambda)\) and \(\rho_i(x'', \lambda)\) differ from each other in the ‘strong sense’. While deriving commutation relations like (5.2) in Appendix A, we have neglected some operators which become trivial in the weak sense at \(\epsilon \to 0\) limit. This procedure does not affect the validity of relations (5.2) in the weak sense. However, it is reasonable to ask whether the relations (5.2) are also valid in the strong sense. To investigate this point, one may try to evaluate commutators like \([\rho_i(x', \lambda), \psi^2(x)]\) and \([\rho_i(x', \lambda), \psi^{i2}(x)]\) from the first principles. This can be achieved with the help of integral relations (3.6a,b) and canonical commutation relations (1.4), by evaluating at first the commutators \([\rho_i(z, \lambda), \psi(y)\psi(x)]\) and \([\rho_i(z, \lambda), \psi^\dagger(y)\psi^\dagger(x)]\) in the region \(z > y > x\) and taking \(y, z \to x\) limit at the final stage. One can verify that such a procedure will exactly reproduce the relations (5.3), which are obtained through repeated applications of the commutation relations (5.2). This fact suggests that the commutation relations (5.2) are valid not only in the weak sense, but also in the strong sense.

Next, we try to evaluate commutation relations between basic field operators and Jost solutions defined through boundary conditions at \(x \to +\infty\). At first, we consider
the commutators \([\tau_i(y, \lambda), \psi(x)]\) and \([\tau_i(y, \lambda), \psi^\dagger(x)]\) in the region \(y > x\). For this case, all fields \(\psi(z)\), \(\psi^\dagger(z)\) contained in the integral relations (3.13a,b) would commute with \(\psi(x), \psi^\dagger(x)\). As a result, we get trivial relations like \([\tau_i(y, \lambda), \psi(x)] = [\tau_i(y, \lambda), \psi^\dagger(x)] = 0\) in the region \(y > x\). The \(y \to x\) limit of these commutation relations may be expressed in the form

\[
[\tau_i(x', \lambda), \psi(x)] = [\tau_i(x', \lambda), \psi^\dagger(x)] = 0 ,
\tag{5.4}
\]

where \(\tau_i(x', \lambda) \equiv \lim_{\epsilon \to 0+} \tau_i(x + \epsilon, \lambda)\). Next, we consider the commutators \([\tau_i(y, \lambda), \psi(x)]\) and \([\tau_i(y, \lambda), \psi^\dagger(x)]\) in the region \(y < x\). However, it is difficult to find out these commutators in a closed form for arbitrary values of \(x\) and \(y\). So, we shall evaluate such commutators only at the limit \(y \to x\). By using integral relations (3.13a,b) and canonical commutation relations (1.4), we obtain

\[
\begin{align*}
[\tau_1(x'', \lambda), \psi(x)] &= i\hbar g \tau_1(x'', \lambda)\psi(x) + i\hbar \xi \lambda \tau_2(x'', \lambda), \tag{5.5a} \\
[\tau_1(x'', \lambda), \psi^\dagger(x)] &= -i\hbar g \psi^\dagger(x)\tau_1(x'', \lambda), \tag{5.5b} \\
[\tau_2(x'', \lambda), \psi(x)] &= -i\hbar f \tau_2(x'', \lambda)\psi(x), \tag{5.5c} \\
[\tau_2(x'', \lambda), \psi^\dagger(x)] &= i\hbar f \psi^\dagger(x)\tau_2(x'', \lambda) - i\hbar \lambda \tau_1(x'', \lambda), \tag{5.5d}
\end{align*}
\]

where \(\tau_i(x'', \lambda) \equiv \lim_{\epsilon \to 0+} \tau_i(x - \epsilon, \lambda)\). It is clear from the relations (5.4) and (5.5) that the commutators \([\tau_i(y, \lambda), \psi(x)]\) and \([\tau_i(y, \lambda), \psi^\dagger(x)]\) are discontinuous at the point \(y = x\). By repeatedly applying the commutation relations (5.5), we also get

\[
\begin{align*}
[\tau_1(x'', \lambda), \psi^2(x)] &= \hbar g(2i + \hbar g) \tau_1(x'', \lambda)\psi^2(x) \\
&+ i\hbar \xi \lambda \left\{ 2 + i\hbar(f - g) \right\} \tau_2(x'', \lambda)\psi(x) , \tag{5.6a} \\
[\tau_1(x'', \lambda), \psi^\dagger^2(x)] &= -\hbar g(2i + \hbar g)\psi^\dagger^2(x) \tau_1(x'', \lambda) , \tag{5.6b} \\
[\tau_2(x'', \lambda), \psi^2(x)] &= \hbar f(hf - 2i) \tau_2(x'', \lambda)\psi^2(x) , \tag{5.6c} \\
[\tau_2(x'', \lambda), \psi^\dagger^2(x)] &= \hbar f(2i - hf) \psi^\dagger^2(x)\tau_2(x'', \lambda) \\
&- i\hbar \lambda \left\{ 2 + i\hbar(f - g) \right\} \psi^\dagger(x)\tau_1(x'', \lambda) . \tag{5.6d}
\end{align*}
\]

Till now we have derived all possible commutation relations between Jost solutions and basic field operators, which will be needed for our calculation of quantum conserved quantities. Next, we consider commutation relations between two Jost solutions associated with different boundary conditions, i.e. commutators of the type \([\rho_i(y, \lambda), \tau_j(x, \lambda)]\) at the limit \(y \to x\). By using the integral relations (3.6), (3.13) and canonical commutation relations (1.4), it can be shown that \([\rho_i(x', \lambda), \tau_j(x, \lambda)] = [\rho_i(x'', \lambda), \tau_j(x, \lambda)] = 0\). Thus, unlike the previous cases, the commutator \([\rho_i(y, \lambda), \tau_j(x, \lambda)]\) is continuous at the limit
y \to x$. Consequently, by following the method of extension \cite{4}, one may define the commutator $[\rho_i(x, \lambda), \tau_j(x, \lambda)]$ either as $[\rho_i(x', \lambda), \tau_j(x, \lambda)]$ or as $[\rho_i(x'', \lambda), \tau_j(x, \lambda)]$. For both of these cases, one obtains the trivial result given by

$$[\rho_i(x, \lambda), \tau_j(x, \lambda)] = 0. \tag{5.7}$$

Thus it is evident that, we can freely interchange the ordering of $\rho_i(x, \lambda)$ and $\tau_j(x, \lambda)$ in the expressions of quantum Wronskian (3.17) and $\Gamma_{\rho,\tau}(x, \lambda)$ operator (4.1).

Next, we want to calculate the derivatives for bilinears of Jost solutions, i.e. quantities like $\partial_x (\rho_i(x, \lambda)\tau_j(x, \lambda))$. By using eqns.(3.3) and (3.10), it is easy to see that such a derivative is given by the sum of few terms, each of which is a product of Jost solutions and basic field operators with arguments corresponding to exactly the same space point. It is a standard practice \cite{4,11} to express these terms in a form so that the operator $\psi^\dagger(x)$ ($\psi(x)$) is always placed at the extreme left (right), while the ordering of the remaining factors remains completely unchanged. For example, if the term $\rho_i(x, \lambda)\psi(x)\psi^\dagger(x)\tau_j(x, \lambda)$ appears in a differential equation, it should be transformed to $\psi^\dagger(x)\rho_i(x, \lambda)\tau_j(x, \lambda)\psi(x)$. For the purpose of expressing all terms in the above mentioned fashion, it is needed to use the commutation relations between basic fields and Jost solutions associated with exactly same space point, i.e. commutators of the form $[\rho_i(x, \lambda), \psi(x)]$, $[\rho_i(x, \lambda), \psi^\dagger(x)]$, $[\tau_i(x, \lambda), \psi(x)]$ and $[\tau_i(x, \lambda), \psi^\dagger(x)]$. We have commented earlier that, evaluation of these commutators through integral relations (3.6) and (3.13) would lead to indeterminant integrals like $\int_{-\infty}^{x} \delta(x-z)F(z)dz$, where $F(z)$ is some function of $z$. Similar to the case of NLS model \cite{11}, one may now try to fix these indeterminant integrals through a convention given by $\int_{-\infty}^{x} \delta(x-z)F(z)dz = \frac{1}{2}F(x)$. It can be easily checked that the above mentioned way of fixing indeterminant integrals and calculating $[\rho_i(x, \lambda), \psi(x)]$, $[\rho_i(x, \lambda), \psi^\dagger(x)]$ is essentially same as defining these commutators as

$$[\rho_i(x, \lambda), \psi(x)] \equiv \frac{1}{2}[\rho_i(x', \lambda) + \rho_i(x'', \lambda), \psi(x)],$$

$$[\rho_i(x, \lambda), \psi^\dagger(x)] \equiv \frac{1}{2}[\rho_i(x', \lambda) + \rho_i(x'', \lambda), \psi^\dagger(x)]. \tag{5.8}$$

evaluating them through the relations (5.1) and (5.2) and substituting the argument $x$ in place of $x'$ and $x''$ at the final stage. Similarly, one can calculate $[\tau_i(x, \lambda), \psi(x)]$ and $[\tau_i(x, \lambda), \psi^\dagger(x)]$, by defining them exactly like (5.8) and using the relations (5.4) and (5.5). Explicit results for all of these commutation relations are given in Appendix B. However we find in Appendix B that, unlike the case of NLS model, these commutation relations violate the Jacobi identity. Consequently, for the case of present DNLS model, it is not meaningful to define commutation relations between Jost solutions and field operators with arguments at exactly same space point through the prescription (5.8).

The above mentioned problem, which arises in the computation of $\partial_x (\rho_i(x, \lambda)\tau_j(x, \lambda))$, can be bypassed through the method of extension \cite{4}. According to this method, the
ith a small amount $\delta$ and $\delta \to 0$ limit is taken after evaluating all relevant commutation relations. The final result obtained in this way must be independent of the sign of $\delta$. By applying this method of extension, and using differential equations (3.3),(3.10) as well as commutators (5.1), (5.2), (5.4), (5.5), we obtain

$$
\begin{align*}
\partial_x \left( \rho_1(x, \lambda) \tau_1(x, \lambda) \right) &= i\lambda \left\{ \xi \psi^\dagger(x) \rho_2(x, \lambda) \tau_2(x, \lambda) + \rho_1(x, \lambda) \tau_1(x, \lambda) \psi(x) \right\}, \\
\partial_x \left( \rho_2(x, \lambda) \tau_1(x, \lambda) \right) &= i\lambda \left\{ \xi \psi^\dagger(x) \rho_2(x, \lambda) \tau_2(x, \lambda) + \rho_1(x, \lambda) \tau_1(x, \lambda) \psi(x) \right\}, \\
\partial_x \left( \rho_1(x, \lambda) \tau_1(x, \lambda) \right) &= -\frac{i\lambda^2}{2} \rho_1(x, \lambda) \tau_1(x, \lambda) + i(f + g) \psi^\dagger(x) \rho_1(x, \lambda) \tau_1(x, \lambda) \psi(x) \\
&\quad + i\xi \lambda \psi^\dagger(x) \Gamma_{\rho, \tau}(x, \lambda), \\
\partial_x \left( \rho_2(x, \lambda) \tau_2(x, \lambda) \right) &= \frac{i\lambda^2}{2} \rho_2(x, \lambda) \tau_2(x, \lambda) - i(f + g) \psi^\dagger(x) \rho_2(x, \lambda) \tau_2(x, \lambda) \psi(x) \\
&\quad + i\lambda \Gamma_{\rho, \tau}(x, \lambda) \psi(x).
\end{align*}
$$

Details of derivation for one of the above differential equations is given in Appendix C. Using eqns.(3.17) and (5.9a,b), we find that

$$
\partial_x \Lambda_{\rho, \tau}(x, \lambda) = 0. 
$$

Thus we are able to explicitly show that, the quantum Wronskian (3.17) remains independent of the variable $x$ even if the noncommutative nature of related operators are taken into account. Taking advantage of this fact, we often use the notation $\Lambda_{\rho, \tau}(\lambda)$, instead of $\Lambda_{\rho, \tau}(x, \lambda)$, to denote the quantum Wronskian. We are also interested in computing the derivative of $\Gamma_{\rho, \tau}(x, \lambda)$ operator (4.1), since it appears in the r.h.s. of the fundamental relation (4.2). With the help of eqns.(5.9a,b), we easily obtain

$$
\partial_x \Gamma_{\rho, \tau}(x, \lambda) = 2i\lambda \left( \xi \psi^\dagger(x) \rho_2(x, \lambda) \tau_2(x, \lambda) + \rho_1(x, \lambda) \tau_1(x, \lambda) \psi(x) \right). 
$$

By using eqns.(5.11) and (5.9c,d), one can further show that

$$
\frac{\lambda}{4} \partial_x \Gamma_{\rho, \tau}(x, \lambda) - \frac{(f + g)}{2\lambda} \psi^\dagger(x) \partial_x \Gamma_{\rho, \tau}(x, \lambda) \psi(x) = \Theta_{\rho, \tau}(x, \lambda), 
$$

where

$$
\Theta_{\rho, \tau}(x, \lambda) = \xi \psi^\dagger(x) \partial_x \left( \rho_2(x, \lambda) \tau_2(x, \lambda) \right) - \partial_x \left( \rho_1(x, \lambda) \tau_1(x, \lambda) \right) \psi(x). 
$$

Finally, we try to find out commutation relations between the quantum Wronskian and basic field operators. Since $\Lambda_{\rho, \tau}(y, \lambda)$ is shown to be independent of $y$, commutators like $[\Lambda_{\rho, \tau}(y, \lambda), \psi(x)]$ and $[\Lambda_{\rho, \tau}(y, \lambda), \psi^\dagger(x)]$ should not depend on the choice of argument $y$. For the case of NLS model, such commutators are calculated for the choice $y = x$ [11]. However, we have already seen in Appendix B that this choice leads to the violation of

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Jacobi identity for the case of DNLS model. So, instead of choosing \( y = x \), at present we shall calculate the commutators \([\Lambda_{\rho,\tau}(y,\lambda), \psi(x)]\) and \([\Lambda_{\rho,\tau}(y,\lambda), \psi^\dagger(x)]\) at the limit \( y \to x \). For this purpose, we introduce quantities like \( \Lambda_{\rho,\tau}(x',\lambda) \equiv \lim_{\epsilon \to 0^+} \Lambda_{\rho,\tau}(x + \epsilon,\lambda) \) and \( \Lambda_{\rho,\tau}(x'',\lambda) \equiv \lim_{\epsilon \to 0^+} \Lambda_{\rho,\tau}(x - \epsilon,\lambda) \). Using eqns. (5.1), (5.2), (5.4) and (5.5), we find that the commutators \([\Lambda_{\rho,\tau}(x',\lambda), \psi(x)]\) and \([\Lambda_{\rho,\tau}(x'',\lambda), \psi(x)]\) yield the same result which may be expressed as

\[
[\Lambda_{\rho,\tau}(\lambda), \psi(x)] = -i\hbar f \rho_1(x,\lambda) \tau_2(x,\lambda) \psi(x) - i\hbar g \rho_2(x,\lambda) \tau_1(x,\lambda) \psi(x) - i\hbar \xi \lambda \rho_2(x,\lambda) \tau_2(x,\lambda) .
\] (5.14a)

In Appendix D we present the details for deriving the above relation. Similarly, the commutators \([\Lambda_{\rho,\tau}(x',\lambda), \psi^\dagger(x)]\) and \([\Lambda_{\rho,\tau}(x'',\lambda), \psi^\dagger(x)]\) yield the same result given by

\[
[\Lambda_{\rho,\tau}(\lambda), \psi^\dagger(x)] = i\hbar f \psi^\dagger(x) \rho_1(x,\lambda) \tau_2(x,\lambda) + i\hbar g \psi^\dagger(x) \rho_2(x,\lambda) \tau_1(x,\lambda) - i\hbar \lambda \rho_1(x,\lambda) \tau_1(x,\lambda) .
\] (5.14b)

Using eqns. (5.14a,b) and (5.9a-d), one can also find out the derivatives of \([\Lambda_{\rho,\tau}(\lambda), \psi(x)]\) and \([\Lambda_{\rho,\tau}(\lambda), \psi^\dagger(x)]\) as

\[
\partial_x [\Lambda_{\rho,\tau}(\lambda), \psi(x)] = \hbar \lambda (f + g) \rho_1(x,\lambda) \tau_1(x,\lambda) \psi^2(x) \\
+ \frac{\hbar \xi \lambda^3}{2} \rho_2(x,\lambda) \tau_2(x,\lambda) + \hbar \xi \lambda^2 \Gamma_{\rho,\tau}(x,\lambda) \psi(x) \\
- i\hbar \left( f \rho_1(x,\lambda) \tau_2(x,\lambda) + g \rho_2(x,\lambda) \tau_1(x,\lambda) \right) \partial_x \psi(x) ,
\] (5.15a)

and

\[
\partial_x [\Lambda_{\rho,\tau}(\lambda), \psi^\dagger(x)] = -\hbar \lambda (f + g) \psi^\dagger(x) \rho_1(x,\lambda) \tau_2(x,\lambda) \\
- \frac{\hbar \lambda^3}{2} \rho_1(x,\lambda) \tau_1(x,\lambda) + \hbar \xi \lambda^2 \psi^\dagger(x) \Gamma_{\rho,\tau}(x,\lambda) \\
+ i\hbar \partial_x \psi^\dagger(x) \left( f \rho_1(x,\lambda) \tau_2(x,\lambda) + g \rho_2(x,\lambda) \tau_1(x,\lambda) \right) .
\] (5.15b)

We are further interested in evaluating commutation relations between \( \Lambda_{\rho,\tau}(y,\lambda) \) and the square of basic field operators. Proceeding as before, it is shown in Appendix D that the commutators \([\Lambda_{\rho,\tau}(x',\lambda), \psi^2(x)]\) and \([\Lambda_{\rho,\tau}(x'',\lambda), \psi^2(x)]\) yield the same result given by

\[
[\Lambda_{\rho,\tau}(\lambda), \psi^2(x)] = \hbar f (hf - 2i) \rho_1(x,\lambda) \tau_2(x,\lambda) \psi^2(x) \\
- \hbar g (2i + hg) \rho_2(x,\lambda) \tau_1(x,\lambda) \psi^2(x) \\
- i\hbar \xi \lambda \left( 2 + i\hbar (f - g) \right) \rho_2(x,\lambda) \tau_2(x,\lambda) \psi(x) .
\] (5.16a)

Similarly, the commutators \([\Lambda_{\rho,\tau}(x',\lambda), \psi^\dagger(x)]\) and \([\Lambda_{\rho,\tau}(x'',\lambda), \psi^\dagger(x)]\) yield

\[
[\Lambda_{\rho,\tau}(\lambda), \psi^\dagger(x)] = -\hbar f (hf - 2i) \psi^\dagger(x) \rho_1(x,\lambda) \tau_2(x,\lambda) \\
+ \hbar g (2i + hg) \psi^\dagger(x) \rho_2(x,\lambda) \tau_1(x,\lambda) \\
- i\hbar \lambda \left( 2 + i\hbar (f - g) \right) \psi^\dagger(x) \rho_1(x,\lambda) \tau_1(x,\lambda) .
\] (5.16b)
All of these relations will be extensively used in our calculation of quantum conserved quantities for the DNLS model.

6 Explicit Construction of the Quantum Hamiltonian and its spectrum

Here we try to find out the explicit form of the first few quantum conserved quantities of DNLS model, which would satisfy the fundamental relation (4.2). Analogous to the classical case (1.3a), we take the first quantum conserved quantity to be

\[ I_0 = \int_{-\infty}^{+\infty} \psi^\dagger(x)\psi(x) dx. \] (6.1)

Using (5.14a,b), we find that

\[ [\Lambda_{\rho,\tau}(\lambda), I_0] = \int_{-\infty}^{+\infty} \left\{ \Lambda_{\rho,\tau}(\lambda), \psi^\dagger(x) \psi(x) + \psi^\dagger(x) \Lambda_{\rho,\tau}(\lambda) \psi(x) \right\} dx = -i\hbar \int_{-\infty}^{+\infty} \left\{ \rho_1 \tau_1 \psi(x) + \xi \psi^\dagger(x) \rho_2 \tau_2 \right\} dx. \] (6.2)

Note that, in the above relation and in the rest of this section, we omit the arguments of Jost solutions \( \rho_i(x, \lambda) \) and \( \tau_i(x, \lambda) \) for the sake of convenience. With the help of (5.11), equation (6.2) can be simplified as

\[ [\Lambda_{\rho,\tau}(\lambda), I_0] = -\frac{\hbar}{2} \int_{-\infty}^{+\infty} \partial_x \Gamma_{\rho,\tau}(x, \lambda) dx = -\frac{\hbar}{2} [\Gamma_{\rho,\tau}(+\infty, \lambda) - \Gamma_{\rho,\tau}(-\infty, \lambda)]. \] (6.3)

So one concludes that for \( n = 0 \), the fundamental relation (4.2) is satisfied by \( I_0 \).

By imitating its classical counterpart (1.3b), the second quantum conserved quantity may be taken as

\[ I_1 = -i \int_{-\infty}^{+\infty} \psi^\dagger(x) \partial_x \psi(x) dx. \] (6.4)

Neglecting some integrals of total derivatives which lead to vanishing surface terms, one can write the commutation relation between \( \Lambda_{\rho,\tau}(\lambda) \) and \( I_1 \) (6.4) as

\[ [\Lambda_{\rho,\tau}(\lambda), I_1] = i \int_{-\infty}^{+\infty} \left\{ \partial_x \left[ \Lambda_{\rho,\tau}(\lambda), \psi^\dagger(x) \right] \psi(x) - \psi^\dagger(x) \partial_x \left[ \Lambda_{\rho,\tau}(\lambda), \psi(x) \right] \right\} dx. \]

Applying further (5.15a,b) and neglecting some integrals of total derivatives, we find that

\[ [\Lambda_{\rho,\tau}(\lambda), I_1] = \hbar \int_{-\infty}^{+\infty} \left\{ -\frac{i\lambda^3}{2} \left( \xi \psi^\dagger(x) \rho_2 \tau_2 + \rho_1 \tau_1 \psi(x) \right) \right. \\
+ \psi^\dagger(x) \left( f \partial_x \left( \rho_1 \tau_2 \right) + g \partial_x \left( \rho_2 \tau_1 \right) \right) \psi(x) \\
- i\lambda(f + g) \psi^\dagger(x) \left( \xi \psi^\dagger(x) \rho_2 \tau_2 + \rho_1 \tau_1 \psi(x) \right) \psi(x) \right\}. \]
Using (5.9a,b) and (5.11) to simplify the r.h.s. of above relation, we readily obtain
\[
\left[ \Lambda_{\rho,\tau}(\lambda), I_1 \right] = -\frac{i\hbar \lambda^2}{4} \int_{-\infty}^{+\infty} \frac{\partial \Gamma_{\rho,\tau}(x,\lambda)}{\partial x} dx = -\frac{\hbar \lambda^2}{4} \left[ \Gamma_{\rho,\tau}(+\infty,\lambda) - \Gamma_{\rho,\tau}(-\infty,\lambda) \right]. \tag{6.5}
\]
Thus \( I_1 \) satisfies the fundamental relation (4.2) for \( n = 1 \).

Finally, we try to calculate the quantum Hamiltonian of DNLS model. In analogy with its classical counterpart (1.3c), we propose that this quantum Hamiltonian can be written in the form
\[
I_2 = I_2^{(1)} + i\xi_q I_2^{(2)}, \tag{6.6}
\]
where
\[
I_2^{(1)} = -\int_{-\infty}^{+\infty} \psi^\dagger(x) \partial_x \psi(x) dx, \quad I_2^{(2)} = \int_{-\infty}^{+\infty} \psi^{12}(x) \partial_x \psi^2(x) dx, \tag{6.7a,b}
\]
and \( \xi_q \) is some yet undetermined coupling constant. Neglecting some integrals of total derivatives which lead to vanishing surface terms, one can write the commutation relation between \( \Lambda_{\rho,\tau}(\lambda) \) and \( I_2^{(1)} \) (6.7a) as
\[
\left[ \Lambda_{\rho,\tau}(\lambda), I_2^{(1)} \right] = \int_{-\infty}^{+\infty} \left\{ \partial_x \left[ \Lambda_{\rho,\tau}(\lambda), \psi^\dagger(x) \right] \partial_x \psi(x) + \partial_x \psi^\dagger(x) \partial_x \left[ \Lambda_{\rho,\tau}(\lambda), \psi(x) \right] \right\} dx.
\]
Using (5.15a,b) to evaluate the commutators appearing in the r.h.s of above relation and neglecting again integrals of some total derivatives, we obtain
\[
\left[ \Lambda_{\rho,\tau}(\lambda), I_2^{(1)} \right] = \hbar \int_{-\infty}^{+\infty} \left[ -\xi \lambda^2 \psi^\dagger(x) \partial_x \Gamma_{\rho,\tau}(x,\lambda) \psi(x) + \lambda(f + g) \psi^\dagger(x) \Theta_{\rho,\tau}(x,\lambda) \psi(x) \right.
\]
\[
- \frac{\lambda^3}{2} \Theta_{\rho,\tau}(x,\lambda) + 2\lambda(f + g) \psi^\dagger(x) \left( \xi \partial_x \psi^\dagger(x) \rho_2 \tau_2 - \rho_1 \tau_1 \partial_x \psi(x) \right) \psi(x) \big] dx, \tag{6.8}
\]
where \( \Theta_{\rho,\tau}(x,\lambda) \) is given by (5.13). Using the identity (5.12) and substituting explicit values of \( f \) and \( g \) (i.e., \( f = \xi e^{-i\alpha/2}/(\cos \alpha/2) \), \( g = \xi e^{i\alpha/2}/(\cos \alpha/2) \)), equation (6.8) can be written in the form
\[
\left[ \Lambda_{\rho,\tau}(\lambda), I_2^{(1)} \right] = \hbar \int_{-\infty}^{+\infty} \left[ -\frac{\lambda^4}{8} \partial_x \Gamma_{\rho,\tau}(x,\lambda) - 2\xi^2 \psi^{12}(x) \partial_x \Gamma_{\rho,\tau}(x,\lambda) \psi^2(x) \right.
\]
\[
+ 4\lambda \xi \psi^\dagger(x) \left( \xi \partial_x \psi^\dagger(x) \rho_2 \tau_2 - \rho_1 \tau_1 \partial_x \psi(x) \right) \psi(x) \big] dx. \tag{6.9}
\]
Next, we consider the commutation relation between \( \Lambda_{\rho,\tau}(\lambda) \) and \( I_2^{(2)} \) (6.7b). Neglecting the integral of a total derivative, we can write this commutator as
\[
\left[ \Lambda_{\rho,\tau}(\lambda), I_2^{(2)} \right] = \int_{-\infty}^{+\infty} \left\{ \left[ \Lambda_{\rho,\tau}(\lambda), \psi^{12}(x) \right] \partial_x \psi^2(x) - \partial_x \psi^{12}(x) \left[ \Lambda_{\rho,\tau}(\lambda), \psi^2(x) \right] \right\} dx.
\]
Applying (5.16a,b), neglecting again integrals of some total derivatives, and also using
relations like
\[ \partial_x (\rho_1 \tau_2) = \partial_x (\rho_2 \tau_1) = \frac{1}{2} \partial_x \Gamma_{\rho,\tau}(x, \lambda), \]
the above equation can be brought in the form
\[
\left[ \Lambda_{\rho,\tau}(\lambda), I_2^{(2)} \right] = \frac{\hbar}{2} \int_{-\infty}^{+\infty} \left[ \left( h f^2 - h g^2 - 2if - 2ig \right) \psi^\dagger(x) \partial_x \Gamma_{\rho,\tau}(x, \lambda) \psi^2(x) \right.
\]
\[ + 4i\lambda(2 + \hbar(f - g)) \psi^\dagger(x) \left( \xi \partial_x \psi^\dagger(x) \rho_2 \tau_2 - \rho_1 \tau_1 \partial_x \psi(x) \right) \psi(x) \] \[ \left. \right] dx. \number{6.10} \]

Using eqns.(6.9), (6.10) (with explicit values of \( f, g \)) and (1.6), we find that the quantum
Hamiltonian (6.6) would satisfy the fundamental relation given by
\[ \left[ \Lambda_{\rho,\tau}(\lambda), I_2 \right] = -\frac{\hbar \lambda^4}{8} \int_{-\infty}^{+\infty} \partial_x \Gamma_{\rho,\tau}(x, \lambda) dx = -\frac{\hbar \lambda^4}{8} \left[ \Gamma_{\rho,\tau}(+\infty, \lambda) - \Gamma_{\rho,\tau}(-\infty, \lambda) \right], \number{6.11} \]
provided the parameter \( \xi_q \) is chosen as
\[ \xi_q = \frac{\xi}{\sqrt{1 - \hbar^2 \xi^2}}. \number{6.12} \]

By substituting (6.12) in (6.6), we get an explicit expression for the quantum Hamiltonian
of DNLS model as
\[ I_2 = \int_{-\infty}^{+\infty} \left\{ -\psi^\dagger(x) \partial_x \psi(x) + \frac{i\xi}{\sqrt{1 - \hbar^2 \xi^2}} \psi^\dagger(x) \partial_x \psi^2(x) \right\} dx. \number{6.13} \]

Thus, it is established that the above quantum Hamiltonian satisfies the fundamental
relation (4.2) for \( n = 2 \). Even though this quantum Hamiltonian (6.13) is not manifestly
Hermitian, we can easily make it Hermitian by adding some integrals of total derivatives
which lead to vanishing surface terms. Comparing (6.13) with (1.3c) we surprisingly find
that, due to quantum effect, the coupling constant of the system is modified. Conse-
fquently, unlike most other integrable systems, the quantum Hamiltonian of DNLS model
can not be obtained from its classical counterpart by simply applying the normal ordering
prescription. It is interesting to note that, eqn.(6.12) is somewhat similar to the relation
between rest mass and dynamical mass of a relativistic particle given by:
\[ m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \]
where \( m_0, m \) and \( v/c \) play the role of \( \xi, \xi_q \) and \( \hbar \xi \) respectively. The \( v/c \to 0 \) limit is like
\( \hbar \to 0 \) limit (for a fixed \( \xi \)) in our case. Just as the dynamical mass of a relativistic particle
coincide with its rest mass in the nonrelativistic limit, the quantum coupling constant \( \xi_q \)
(6.12) coincides with the bare coupling constant \( \xi \) at \( \hbar \to 0 \) limit. On the other hand,
the \( v/c \to 1 \) limit is analogous to \( |\xi| \to \frac{1}{\hbar} \) limit in our case. Just as the dynamical mass
of a particle goes to infinity at ultrarelativistic limit, \( \xi_q \) (6.12) goes to infinity at \( |\xi| \to \frac{1}{\hbar} \)
limit. Consequently, even though QYBE restricts the value of \( \xi \) as \( |\xi| \leq \frac{1}{\hbar} \), there exists
no such restriction on the value of corresponding quantum coupling constant $\xi_q$ (6.12). Thus the apparent limitation about the applicability of QISM in solving quantum DNLS Hamiltonian for the full range of its coupling constant is resolved in a very nice way.

It is evident that $I_0$ (6.1) and $I_1$ (6.4) represent the number operator and momentum operator respectively for the quantum DNLS system. Substituting $n = 0, 1$ and $2$ in equation (4.7), one can explicitly write down the eigenvalue relations for $I_0, I_1$ and $I_2$ as

\begin{align*}
I_0 |\mu_1, \mu_2, \cdots, \mu_N\rangle &= \hbar N |\mu_1, \mu_2, \cdots, \mu_N\rangle, \quad \text{(6.14a)} \\
I_1 |\mu_1, \mu_2, \cdots, \mu_N\rangle &= \left(\frac{\hbar}{2} \sum_{j=1}^{N} \mu_j^2\right) |\mu_1, \mu_2, \cdots, \mu_N\rangle, \quad \text{(6.14b)} \\
I_2 |\mu_1, \mu_2, \cdots, \mu_N\rangle &= \left(\frac{\hbar}{4} \sum_{j=1}^{N} \mu_j^4\right) |\mu_1, \mu_2, \cdots, \mu_N\rangle. \quad \text{(6.14c)}
\end{align*}

Let us now compare these eigenvalue relations with those obtained through the technique of coordinate Bethe ansatz. Projecting the bosonic Hamiltonian (6.13) on an $N$-particle Hilbert space [22], we get

$$
\mathcal{H}_N = -\hbar \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2i\hbar^2 \xi_q \sum_{l<m} \delta(x_l - x_m) \left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_m}\right). \quad \text{(6.15)}
$$

The eigenvalues for this Hamiltonian with derivative $\delta$-function interaction and corresponding momentum operator can be derived through the method of coordinate Bethe ansatz [22,23]. It is easy to check that such eigenvalues completely match with our result in eqns.(6.14b,c) when we identify the momentum parameters ($k_j$) of coordinate Bethe ansatz with the spectral parameters ($\mu_j$) of present approach through the relation: $k_j \equiv \frac{\mu_j^2}{2}$. The eigenfunctions of the Hamiltonian (6.15) can also be constructed through coordinate Bethe ansatz. If, for the simplest $N = 2$ case, such eigenfunction is chosen in $x_1 < x_2$ region as $f(x_1, x_2) = e^{i(k_1 x_1 + k_2 x_2)}$, then its form in $x_1 > x_2$ region would be given by [22,24]

$$
f(x_1, x_2) = A(k_1, k_2)e^{i(k_1 x_1 + k_2 x_2)} + B(k_1, k_2)e^{i(k_2 x_1 + k_1 x_2)}
$$

where $A(k_1, k_2) = \frac{k_1 - k_2 + i\hbar \xi_q (k_1 + k_2)}{k_1 - k_2}$ and $B(k_1, k_2) = 1 - A(k_1, k_2)$ are the so called ‘matching coefficients’. With the help of these matching coefficients, one can easily find out the $S$-matrix for two-body scattering as [24]

$$
S(k_1, k_2) = A(k_1, k_2)A(k_2, k_1)^{-1} = \frac{k_1 - k_2 + i\hbar \xi_q (k_1 + k_2)}{k_1 - k_2 - i\hbar \xi_q (k_1 + k_2)}. \quad \text{(6.16)}
$$

Using eqns.(6.12) and (1.6), we can express $\xi_q$ as: $\xi_q = -\frac{1}{\hbar} \tan \alpha$. Putting this form of $\xi_q$ in eqn.(6.16), and identifying momentum parameters with spectral parameters through relations like $k_1 \equiv \frac{\alpha^2}{2}$, $k_2 \equiv \frac{\beta^2}{2}$, we find that this $S$-matrix (6.16) exactly matches with our earlier result (2.13) which is derived in the framework of QISM. The fact that the
‘renormalized’ coupling constant $\xi_q$ appears in the projected DNLS Hamiltonian (6.15), instead of its classical counterpart $\xi$, plays a crucial role in this comparison between the results of coordinate and algebraic Bethe ansatz.

It is also interesting to compare the results of coordinate and algebraic Bethe ansatz for the soliton sector of quantum DNLS model. By applying QISM it is found that, the distribution of complex spectral parameters for such quantum $N$-soliton state is given by the relation (2.19). Taking the square of both sides of this relation and substituting $k_j$ in place of $\mu_j^2/2$, we obtain

$$k_j = \frac{\mu_j^2}{2} \exp \left[ i \alpha (N + 1 - 2j) \right], \quad (6.17)$$

where $j \in [1, 2, \cdots N]$. This equation coincides with the momentum distribution in coordinate Bethe ansatz corresponding to the quantum $N$-soliton states of DNLS Hamiltonian (6.13) [22,23]. Again, the fact that the modified coupling constant appears in the Hamiltonian (6.13) allows us to exactly match the results of coordinate and algebraic Bethe ansatz.

By using the eigenvalue relations (6.14b,c), we can also calculate the binding energy for the above mentioned quantum $N$-soliton states. Substituting the values of complex $\mu_j$ (2.19) to (6.14b), we obtain the momentum eigenvalue corresponding to these $N$-soliton states as

$$P = \frac{\hbar \mu^2}{2} \sum_{j=1}^{N} e^{i\alpha(N+1-2j)} = \frac{\hbar \mu^2 \sin \alpha N}{2 \sin \alpha}. \quad (6.18)$$

Similarly, by substituting $\mu_j$ (2.19) to (6.14c), we obtain the energy eigenvalue corresponding to these states as

$$E = \frac{\hbar \mu^4}{4} \sum_{j=1}^{N} e^{2i\alpha(N+1-2j)} = \frac{\hbar \mu^4 \sin(2\alpha N)}{4 \sin(2\alpha)}. \quad (6.19)$$

To calculate binding energy, we assume that the momentum $P$ (6.18) of the $N$-soliton state is equally distributed among $N$ number of single-particle scattering states. The real (pure imaginary) spectral parameter associated with each of these single-particle states is denoted by $\mu_0$. With the help of eqns.(6.14b) and (6.18), we obtain

$$\mu_0^2 = \frac{\mu^2 \sin(\alpha N)}{N \sin \alpha}. \quad (6.20)$$

Using eqn.(6.14c), one can easily calculate the total energy for $N$ number of such single-particle scattering states as

$$E' = \frac{\hbar N}{4} \mu_0^4 = \frac{\hbar \mu^4 \sin^2 \alpha N}{4N \sin^2 \alpha}. \quad (6.21)$$
Subtracting $E$ (6.19) from $E'$ (6.21), we obtain the binding energy of quantum $N$-soliton state as

$$E_B = E' - E = \frac{\hbar \mu^A \sin \alpha N}{4 \sin \alpha} \left\{ \frac{\sin \alpha N}{N \sin \alpha} - \frac{\cos \alpha N}{\cos \alpha} \right\}. \quad (6.22)$$

Substituting $N = 2$ to the above relation, we obtain $E_B = \frac{\hbar \mu^A}{2} \sin^2 \alpha$. Thus we get $E_B > 0$ for any nonzero value of $\alpha$. For $N = 3$, (6.22) takes the form $E_B = \frac{2\hbar \mu^A}{3} \sin^2 \alpha (3 - 4 \sin^2 \alpha)$. Here we get $E_B > 0$ only if $|\alpha| < \frac{\pi}{3}$. Applying the method of induction, we find that the condition $E_B > 0$ is in fact valid within the range $|\alpha| < \frac{\pi}{N}$ for all values of $N$ [29]. Thus to obtain quantum $N$-soliton states with positive binding energy, the coupling constant of DNLS model should be restricted within the region $|\xi_q| < \frac{\hbar}{\mu} \tan \left( \frac{\pi}{N} \right)$.

## 7 Concluding Remarks

In analogy with the ‘fundamental relation’ of NLS model [11], in this article we propose the fundamental relation (4.2) for DNLS model. This fundamental relation plays a key role in our construction of quantum conserved quantities of DNLS model and their spectra. However, from the technical point of view, our construction of quantum conserved quantities is much more complicated than the case of NLS model due to the following reasons. Quantum Jost solutions and their commutation relations with basic field operators are extensively used to obtain the quantum conserved quantities of DNLS model. It turns out that, in contrast to the case of NLS model, differential equations satisfied by these Jost solutions corresponding to boundary conditions at $x \to \infty$ and $x \to -\infty$ do not coincide with each other. This salient feature of DNLS model is connected with the fact that its quantum Lax operator (1.5) has a nonvanishing trace. We also find that, unlike the case of NLS model, the commutation relation between Jost solutions of DNLS model and basic field operators with arguments at exactly the same space point lead to the violation of Jacobi identity. So we are compelled to use commutation relations between Jost solutions and basic field operators with slightly shifted arguments in our calculation of quantum conserved quantities.

Proceeding in the above mentioned way, we are able to explicitly construct the quantum Hamiltonian and few other conserved quantities of DNLS model through basic field operators of this system. Surprisingly we find that, unlike the cases of most other integrable systems, this quantum Hamiltonian (6.13) can not be obtained as normal ordered version of the corresponding classical Hamiltonian (1.3c). This is due to the fact that a new kind of coupling constant ($\xi_q$), quite different from the classical one ($\xi$), appears in the quantum Hamiltonian of the DNLS model. Thus we obtain the explicit form of the quantum Hamiltonian of DNLS model, which has been defined earlier in the framework of QISM in a formal way. Interestingly, the relation (6.12) between $\xi$ and $\xi_q$ is rather
similar to the relation between rest mass and dynamical mass of a relativistic particle. Just as the dynamical mass of a relativistic particle coincides with its rest mass in the nonrelativistic limit, $\xi_q$ coincides with $\xi$ at $\hbar \to 0$ limit. In the ultrarelativistic limit, the dynamical mass of a particle tends towards infinity. In a similar way, $\xi_q$ can take arbitrary large value at $|\xi| \to \frac{1}{\hbar}$ limit. Consequently, we can apply QISM to the quantum DNLS model for the full range of its coupling constant, even though QYBE restricts the value of $\xi$ as $|\xi| \leq \frac{1}{\hbar}$. Due to the presence of modified coupling constant in the quantum Hamiltonian (6.13), we are also able to consistently match various results of algebraic and coordinate Bethe ansatz in the case of DNLS model. The $S$-matrix for two particle scattering and the distribution of single-particle momentum for quantum $N$-soliton states are two such examples where the results of algebraic and coordinate Bethe ansatz match with each other. We also calculate the binding energy for the quantum $N$-soliton state of DNLS model and find out the range of coupling constant for which this binding energy has a positive value.

As a future study, it might be interesting to find out the higher quantum conserved quantities of DNLS model by using its fundamental relation and investigate whether the coupling constants appearing in such higher conserved quantities also differ from their classical counterparts. It is well known that, higher quantum conserved quantities of NLS model can not be expressed in normal ordered form as the integral of a one-dimensional density [12,13]. A similar situation may also arise for the case of higher quantum conserved quantities of the DNLS model. It should be noted that, the present approach of using fundamental relation for the construction of quantum conserved quantities and their spectra might be applicable to many other integrable systems. In this article, we have already discussed the possibility of such construction for a class of quantum integrable field models associated with $2 \times 2$ Lax equations (3.19). It should also be interesting to study fundamental relations for the case of discrete quantum integrable models like Heisenberg spin-$\frac{1}{2}$ chain, supersymmetric t-J model, Hubbard model etc. and explore how these fundamental relations lead to the construction of corresponding conserved quantities along with their spectra.
Appendix A

Here we give a detailed derivation of the commutation relation (5.2a). At first, we shall evaluate the commutator \([\rho_1(x + \epsilon, \lambda), \psi(x)]\) for the case \(\epsilon > 0\) and take \(\epsilon \to 0\) limit at the final stage. Using the integral relation of \(\rho_1(x, \lambda)\) (3.6a) and canonical commutation relations (1.4), we find that

\[
[\rho_1(x + \epsilon, \lambda), \psi(x)] = if \int_{-\infty}^{x+\epsilon} dz \, e^{\frac{iA^2}{4}(z-x-\epsilon)} \left[\psi(\bar{z}), \psi(x)\right] \rho_1(z, \lambda) \psi(z)
+ if \int_{-\infty}^{x+\epsilon} dz \, e^{\frac{iA^2}{4}(z-x-\epsilon)} \left[\psi(\bar{z}), \psi(x)\right] \rho_1(z, \lambda) \psi(z)
+ i\xi \lambda \int_{-\infty}^{x+\epsilon} dz \, e^{\frac{iA^2}{4}(z-x-\epsilon)} \left[\psi(\bar{z}), \psi(x)\right] \rho_2(z, \lambda)
+ i\xi \lambda \int_{-\infty}^{x+\epsilon} dz \, e^{\frac{iA^2}{4}(z-x-\epsilon)} \left[\psi(\bar{z}), \psi(x)\right] \rho_2(z, \lambda)
\]

\[= -ihfe^{\frac{iA^2}{4}(-\epsilon)} \rho_1(x, \lambda) \psi(x) - ih\xi \lambda e^{\frac{iA^2}{4}(-\epsilon)} \rho_2(x, \lambda) + \Omega + \Omega'. \quad (A1)\]

where

\[
\Omega = if \int_{x}^{x+\epsilon} dz \, e^{\frac{iA^2}{4}(z-x-\epsilon)} \left[\psi(\bar{z}), \psi(x)\right] \rho_1(z, \lambda) \psi(z)
+ i\xi \lambda \int_{x}^{x+\epsilon} dz \, e^{\frac{iA^2}{4}(z-x-\epsilon)} \left[\psi(\bar{z}), \psi(x)\right] \rho_2(z, \lambda). \quad (A2)\]

The lower limits of integrals appearing in the r.h.s. of eqn.(A2) are fixed by using the fact that the commutator \([\rho_1(z, \lambda), \psi(x)]\) becomes trivial for the case \(z < x\). Next, we rewrite eqn.(A1) as

\[
[\rho_1(x + \epsilon, \lambda), \psi(x)] = -ihfe^{\frac{iA^2}{4}(-\epsilon)} \rho_1(x + \epsilon, \lambda) \psi(x) - ih\xi \lambda e^{\frac{iA^2}{4}(-\epsilon)} \rho_2(x + \epsilon, \lambda) + \Omega + \Omega', \quad (A3)\]

where

\[
\Omega' = -ihfe^{\frac{iA^2}{4}(-\epsilon)} \left[\rho_1(x, \lambda) - \rho_1(x + \epsilon, \lambda)\right] \psi(x) - ih\xi \lambda e^{\frac{iA^2}{4}(-\epsilon)} \left[\rho_2(x, \lambda) - \rho_2(x + \epsilon, \lambda)\right].
\]

It is clear that the above expression of \(\Omega'\) vanishes at \(\epsilon \to 0\) limit. Let us now assume that commutators like \([\rho_i(z, \lambda), \psi(x)]\) do not produce any singular term at the limit \(z \to x\). Due to this assumption, the operator \(\Omega\) (A2) would also vanish at \(\epsilon \to 0\) limit. Consequently, by taking \(\epsilon \to 0\) limit of (A3), we obtain the commutation relation (5.2a). Other commutation relations appearing in (5.2) can also be derived in a similar fashion. It should be noted that, the forms of finally derived equations (5.2) justify in a self-consistent way our assumption about the absence of singular terms in commutators like \([\rho_i(z, \lambda), \psi(x)]\) at \(z \to x\) limit.
Appendix B

Here we derive the commutation relations between Jost solutions and field operators associated with the same space point through the prescription (5.8) and show that these commutation relations violate the Jacobi identity. Inserting the commutators (5.1) and (5.2) to the expression (5.8), and substituting the arguments $x'$ and $x''$ by $x$ at the final stage, we find that

$$
\left[ \rho_1(x, \lambda), \psi(x) \right] = -\frac{i\hbar}{2} \rho_1(x, \lambda)\psi(x) - \frac{i\hbar\xi\lambda}{2} \rho_2(x, \lambda), \quad (B.1)
$$

$$
\left[ \rho_1(x, \lambda), \psi^\dagger(x) \right] = \frac{i\hbar}{2} \psi^\dagger(x)\rho_1(x, \lambda), \quad (B.2)
$$

$$
\left[ \rho_2(x, \lambda), \psi(x) \right] = \frac{i\hbar}{2} \rho_2(x, \lambda)\psi(x), \quad (B.3)
$$

$$
\left[ \rho_2(x, \lambda), \psi^\dagger(x) \right] = -\frac{i\hbar}{2} \psi^\dagger(x)\rho_2(x, \lambda) + \frac{i\hbar\lambda}{2} \rho_1(x, \lambda). \quad (B.4)
$$

Similarly, one can calculate the commutators $[\tau_i(x, \lambda), \psi(x)]$ and $[\tau_i(x, \lambda), \psi^\dagger(x)]$, by defining them exactly like (5.8) and using the relations (5.4) as well as (5.5). In this way, we obtain

$$
\left[ \tau_1(x, \lambda), \psi(x) \right] = \frac{i\hbar}{2} \tau_1(x, \lambda)\psi(x) + \frac{i\hbar\xi\lambda}{2} \tau_2(x, \lambda), \quad (B.2.1)
$$

$$
\left[ \tau_1(x, \lambda), \psi^\dagger(x) \right] = -\frac{i\hbar}{2} \psi^\dagger(x)\tau_1(x, \lambda), \quad (B.2.2)
$$

$$
\left[ \tau_2(x, \lambda), \psi(x) \right] = -\frac{i\hbar}{2} \tau_2(x, \lambda)\psi(x), \quad (B.2.3)
$$

$$
\left[ \tau_2(x, \lambda), \psi^\dagger(x) \right] = \frac{i\hbar}{2} \psi^\dagger(x)\tau_2(x, \lambda) - \frac{i\hbar\lambda}{2} \tau_1(x, \lambda). \quad (B.2.4)
$$

Due to eqn.(5.7), Jost solutions $\rho_i(x, \lambda)$ and $\tau_j(x, \lambda)$ commute with each other.

By successively using the commutators (B.1.1) and (B.2.3), we find that

$$
\left[ \tau_2(x, \lambda), [\rho_1(x, \lambda), \psi(x)] \right] = -\frac{\hbar^2f^2}{4} \rho_1(x, \lambda)\tau_2(x, \lambda)\psi(x). \quad (B.3)
$$

Next, by applying the commutators (B.2.3), (B.1.1) and (5.7), we obtain

$$
\left[ \rho_1(x, \lambda), \psi(x), \tau_2(x, \lambda) \right] = \frac{\hbar^2f^2}{4} \rho_1(x, \lambda)\tau_2(x, \lambda)\psi(x) + \frac{\hbar^2f\xi\lambda}{4} \rho_2(x, \lambda)\tau_2(x, \lambda). \quad (B.4)
$$

Finally, by using eqns.(B.3), (B.4) and (5.7), it is easy to check that

$$
\left[ \tau_2(x, \lambda), [\rho_1(x, \lambda), \psi(x)] \right] + \left[ \rho_1(x, \lambda), [\psi(x), \tau_2(x, \lambda)] \right] + \left[ \psi(x), [\tau_2(x, \lambda), \rho_1(x, \lambda)] \right] = \frac{\hbar^2f\xi\lambda}{4} \rho_2(x, \lambda)\tau_2(x, \lambda). \quad (B.5)
$$

Thus it is evident that the set of commutation relations (B1), (B2) and (5.7) violate the Jacobi identity.
Appendix C

For deriving the relation (5.9a) through the method of extension, we shift the argument of $\rho_1(x, \lambda)$ by a very small amount $\delta$ and find out $\partial_x \left( \rho_1(x + \delta, \lambda) \tau_2(x, \lambda) \right)$ for both positive and negative $\delta$. For both cases, $\delta \to 0$ limit will be taken at the final stage. It will be shown that the final result is independent of the sign of $\delta$.

Let us first take a positive $\delta$. Using eqns.(3.3) and (3.10) we get

$$\partial_x \left( \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) \right) = \partial_x \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) + \rho_1(\tilde{x}, \lambda) \partial_x \tau_2(x, \lambda)$$

$$= \left\{ i f \psi^\dagger(\tilde{x}) \rho_1(\tilde{x}, \lambda) \psi(\tilde{x}) + i \xi \lambda \psi^\dagger(\tilde{x}) \rho_2(\tilde{x}, \lambda) \right\} \tau_2(x, \lambda)$$

$$+ \rho_1(\tilde{x}, \lambda) \left\{ - i f \psi^\dagger(x) \tau_2(x, \lambda) \psi(x) + i \lambda \tau_1(x, \lambda) \psi(x) \right\}, \quad (C1)$$

where $\tilde{x} \equiv x + \delta$. The r.h.s. of (C1) should be written in a way such that an operator like $\psi^\dagger(x)$ ($\psi(x)$) is always placed at the extreme left (right) of each term. The $\delta \to 0$ limit should be taken after rewriting the r.h.s of (C1) in the above mentioned way with the help of commutators $[\psi(\tilde{x}), \tau_2(x, \lambda)]$ and $[\rho_1(\tilde{x}, \lambda), \psi^\dagger(x)]$. Thus we have to ultimately use the $\delta \to 0$ limit of commutators $[\psi(\tilde{x}), \tau_2(x, \lambda)]$ and $[\rho_1(\tilde{x}, \lambda), \psi^\dagger(x)]$, which are given by eqns.(5.5c) and (5.2b) respectively. By using these equations and dropping terms which vanish at $\delta \to 0$ limit, we obtain

$$\partial_x \left( \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) \right)$$

$$= i f (1 + i hf) \left[ \psi^\dagger(\tilde{x}) \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) \psi(\tilde{x}) - \psi^\dagger(x) \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) \psi(x) \right]$$

$$+ i \xi \lambda \psi^\dagger(\tilde{x}) \rho_2(\tilde{x}, \lambda) \tau_2(x, \lambda) + i \lambda \rho_1(\tilde{x}, \lambda) \tau_1(x, \lambda) \psi(x)$$

$$= i \xi \lambda \psi^\dagger(x) \rho_2(x, \lambda) \tau_2(x, \lambda) + i \lambda \rho_1(x, \lambda) \tau_1(x, \lambda) \psi(x). \quad (C2)$$

Next, we consider the case $\delta < 0$. In this case also we obtain a relation of the form (C1), where $\tilde{x} \equiv x + \delta$. Again we want to rewrite the r.h.s. of (C1) in a way such that an operator like $\psi^\dagger(x)$ ($\psi(x)$) is always placed at the extreme left (right) of each term. However it is already found in Sec.5 that, $[\psi(\tilde{x}), \tau_2(x, \lambda)] = [\rho_1(\tilde{x}, \lambda), \psi^\dagger(x)] = 0$ for any negative value of $\delta$. By using these commutation relations and dropping terms which vanish at $\delta \to 0$ limit, we obtain

$$\partial_x \left( \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) \right) = i f \left[ \psi^\dagger(\tilde{x}) \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) \psi(\tilde{x}) - \psi^\dagger(x) \rho_1(\tilde{x}, \lambda) \tau_2(x, \lambda) \psi(x) \right]$$

$$+ i \xi \lambda \psi^\dagger(\tilde{x}) \rho_2(\tilde{x}, \lambda) \tau_2(x, \lambda) + i \lambda \rho_1(\tilde{x}, \lambda) \tau_1(x, \lambda) \psi(x)$$

$$= i \xi \lambda \psi^\dagger(x) \rho_2(x, \lambda) \tau_2(x, \lambda) + i \lambda \rho_1(x, \lambda) \tau_1(x, \lambda) \psi(x). \quad (C3)$$

Comparing the r.h.s. of (C2) and (C3), we find that $\partial_x \left( \rho_1(x, \lambda) \tau_2(x, \lambda) \right)$ is given by eqn.(5.9a) in a regularisation independent way.
Appendix D

For the purpose of deriving the relation (5.14a), we write down $\Lambda(x', \lambda)$ and $\Lambda(x'', \lambda)$ explicitly as

\[
\begin{align*}
\Lambda(x', \lambda) &= \rho_1(x', \lambda)\tau_2(x', \lambda) - \rho_2(x', \lambda)\tau_1(x', \lambda), \\
\Lambda(x'', \lambda) &= \rho_1(x'', \lambda)\tau_2(x'', \lambda) - \rho_2(x'', \lambda)\tau_1(x'', \lambda).
\end{align*}
\]

(D1) (D2)

Using (D1), (5.2a), (5.2c) and (5.4), we find that

\[
\begin{align*}
[\Lambda(x', \lambda), \psi(x)] &= \left[\rho_1(x', \lambda), \psi(x)\right]\tau_2(x', \lambda) - \left[\rho_2(x', \lambda), \psi(x)\right]\tau_1(x', \lambda) \\
&= -ihf\rho_1(x', \lambda)\tau_2(x', \lambda)\psi(x) - ihg\rho_2(x', \lambda)\tau_1(x', \lambda)\psi(x) \\
&= -ih\bar{\xi}\rho_2(x', \lambda)\tau_2(x', \lambda)\psi(x).
\end{align*}
\]

(D3)

Similarly, using (D2), (5.1), (5.5a) and (5.5c), we get

\[
\begin{align*}
\begin{align*}
[\Lambda(x'', \lambda), \psi(x)] &= \rho_1(x'', \lambda)\tau_2(x'', \lambda)\psi(x) - \rho_2(x'', \lambda)\tau_1(x'', \lambda)\psi(x) \\
&= -ihf\rho_1(x'', \lambda)\tau_2(x'', \lambda)\psi(x) - ihg\rho_2(x'', \lambda)\tau_1(x'', \lambda)\psi(x) \\
&= -ih\bar{\xi}\rho_2(x'', \lambda)\tau_2(x'', \lambda)\psi(x).
\end{align*}
\end{align*}
\]

(D4)

Comparing (D3) and (D4), we find that \([\Lambda(x', \lambda), \psi(x)]\) and \([\Lambda(x'', \lambda), \psi(x)]\) lead to the same result (in the weak sense) given by eqn.(5.14a).

Next, we want to derive the relation (5.16a). Using eqns.(D1), (5.3a), (5.3c) and (5.4), we obtain

\[
\begin{align*}
[\Lambda(x', \lambda), \psi^2(x)] &= \left[\rho_1(x', \lambda), \psi^2(x)\right]\tau_2(x', \lambda) - \left[\rho_2(x', \lambda), \psi^2(x)\right]\tau_1(x', \lambda) \\
&= hf(hf - 2i)\rho_1(x', \lambda)\tau_2(x', \lambda)\psi^2(x) \\
&= -hg(2i + hg)\rho_2(x', \lambda)\tau_1(x', \lambda)\psi^2(x) \\
&= -ih\bar{\xi}(2 + ih(f - g))\rho_2(x', \lambda)\tau_2(x', \lambda)\psi(x).
\end{align*}
\]

(D5)

Similarly, using (D2), (5.1), (5.6a) and (5.6c), we get

\[
\begin{align*}
[\Lambda(x'', \lambda), \psi^2(x)] &= \rho_1(x'', \lambda)\tau_2(x'', \lambda)\psi^2(x) - \rho_2(x'', \lambda)\tau_1(x'', \lambda)\psi^2(x) \\
&= hf(hf - 2i)\rho_1(x'', \lambda)\tau_2(x'', \lambda)\psi^2(x) \\
&= -hg(2i + hg)\rho_2(x'', \lambda)\tau_1(x'', \lambda)\psi^2(x) \\
&= -ih\bar{\xi}(2 + ih(f - g))\rho_2(x'', \lambda)\tau_2(x'', \lambda)\psi(x).
\end{align*}
\]

(D6)

Comparing (D5) and (D6), again we find that \([\Lambda(x', \lambda), \psi^2(x)]\) and \([\Lambda(x'', \lambda), \psi^2(x)]\) lead to the same result given by eqn.(5.16a).
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