In memory of my dear friend Alex Chigogidze

ORDERING A SQUARE

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ABSTRACT. We identify a condition on $X$ that guarantees that any finite power of $X$ is homeomorphic to a subspace of a linearly ordered space.

1. Introduction

To start our discussion, let us agree on terminology. A linearly ordered space, or a LOTS, is one whose topology is generated by open intervals and open rays with respect to some linear order on $X$. A space homeomorphic to a subspace of a LOTS is called a generalized ordered space, or a GO space. It is known (see, for example, [2]), a Hausdorff space $L$ is a GO space, if its topology can be generated by a collection of convex sets (not necessarily open) with respect to some linear order on $L$. Throughout the paper, we will also refer to LOTS and GO spaces as orderable and suborderable, respectively.

Having a topology compatible with an order is a delicate property that can be destroyed by many standard operations. Under favorable conditions, however, order-topology ties demonstrate a remarkable resistance to the product operation as demonstrated by zero-dimensional separable metric spaces. A step into a non-metrizable world quickly reveals that zero-dimensionality has to be coupled with very strong properties to achieve a desired resistance of orderability to the product operation. For example, $\omega_1$ is a zero-dimensional LOTS with "cannot be better" local properties, while $\omega_1 \times \omega_1$ has a rather rigid non-orderable structure. One can see that $\omega_1^2$ is not sub-orderable by observing that it is not hereditarily normal. Another natural example is Sorgenfrey Line, which is known to be a GO-space. The square of the line is not a GO-space. One explanation is non-normality. Observe, however, that every countable power of the Sorgenfrey Line admits a continuous injection into the irrationals.

The goal of this paper is to identify a property that may serve as a characterization of spaces with (sub) orderable finite powers. In Theorem 2.4, we present a sufficient conditions that may turn out to be a necessary one. We then observe that the condition in Theorem 2.4 is a criterion in the scope of well ordered subspaces (Theorem 2.5). We do not know if Theorem 2.4 can be reversed without narrowing its scope.

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In notation and terminology, we will follow [1]. Even though we study ordered spaces, we will not need a notation for an open interval. Therefore, we reserve \((a, b)\) to denote the ordered pair. All spaces are assumed Tychonoff. We say that a subset \(A \subseteq X\) separates distinct \(x, y \in X\) if \(A\) meets \(\{x, y\}\) by exactly one element. A family \(\mathcal{A}\) of subsets of \(X\) separates distinct \(x, y \in X\) if at least one member of the family separates \(x\) from \(y\).

2. Results

To formulate our main result we would like to introduce a property that can be extracted from many arguments leading to orderability of certain spaces.

**Definition of the \(P\)-number of \(X\).** The \(P\)-number of a space \(X\) is \(|X|\) if \(X\) is discrete. Otherwise, the \(P\)-number of \(X\) is the largest cardinal number \(\tau\) such that the intersection of any fewer than \(\tau\) open sets of \(X\) is open.

Note that the \(P\)-number of \(X\) is well-defined. Indeed, if \(X\) is not discrete, then there exists an infinite set \(A\) and \(b \in \overline{A} \setminus A\). Then the family \(\{X \setminus \{a\} : a \in A\}\) consists of open sets and has non-open intersection. Thus, the \(P\)-number of \(X\) is

\[
\min\{|\mathcal{O}| : \mathcal{O} \text{ consists of open sets and has non-open intersection}\}
\]

Since the minimum is computed over a non-empty set, it exists. In particular, the \(P\)-number of any non-discrete metric space is \(\omega\). The \(P\)-number of \(\{\omega_1\} \cup \{\alpha \in \omega_1 : \alpha \text{ is isolated}\}\) is \(\omega_1\).

Note that if a space has an \(\omega_1\)-long convergent sequence and an \(\omega\)-long convergent sequence, then the square of the space is not suborderable because the product of these two sequences is not hereditarily normal. This and similar structures are eliminated if a space has a \(\tau\)-discrete basis, where \(\tau\) is the \(P\)-number of the space. We start by identifying a necessary condition for suborderability. We then show that the found property is finitely productive.

**Theorem 2.1.** Let \(X\) have a \(\tau\)-discrete basis of clopen sets, where \(\tau\) is the \(P\)-number of \(X\). Then \(X\) is a GO-space.

**Proof.** Let \(\mathcal{B} = \bigcup_{\alpha < \tau} \mathcal{B}_\alpha\) be a basis as in the hypothesis. Since each \(\mathcal{B}_\alpha\) is a discrete family of clopen sets, we may assume that \(\mathcal{B}_\alpha\) is a cover of \(X\) for each \(\alpha\). Additionally, we may assume that \(\tau\) is infinite and \(\mathcal{B}_0 = \{X\}\).

Inductively, for each \(\alpha < \tau\), we will define a relation \(\mathcal{O}_\alpha\) on \(X\) so that \(\bigcup_{\alpha < \tau} \mathcal{O}_\alpha\) will be an order \(\prec\) on \(X\) compatible with the topology of \(X\). In addition, we will define a collection \(\mathcal{P}_\alpha\) so that \(\bigcup_{\alpha < \tau} \mathcal{P}_\alpha\) will consist of \(\prec\)-convex sets and form a basis for the topology of \(X\).
Step 0. Put $O_0 = \emptyset$ and $P_0 = \{X\}$.

Assumption ($\beta < \alpha$). Assume that for each $\beta < \alpha$, where $\alpha < \tau$, we have defined $O_\beta$ and the following hold:

A1: $O_\gamma \subset O_\beta$ if $\gamma < \beta$.

A2: If $x, y$ are separated by $\bigcup_{\gamma \leq \beta} B_\gamma$, then either $(x, y)$ or $(y, x)$, but not both, is in $O_\beta$.

A3: If $(x, y), (y, z) \in O_\beta$, then $(x, z) \in O_\beta$.

A4: $(x, x) \not\in O_\beta$ for any $x \in X$.

A5: If $x, y$ are not separated by any element of $\bigcup_{\gamma \leq \beta} B_\gamma$, then the following two statements are true:

\[ \forall z \left[ (z, x) \in O_\beta \rightarrow (z, y) \in O_\beta \right] \]
\[ \forall z \left[ (x, z) \in O_\beta \rightarrow (y, z) \in O_\beta \right] \]

Note that our assumptions hold for $\beta = 0$.

Step $\alpha < \tau$. Put $F_\alpha = \bigcup_{\beta < \alpha} B_\beta$.

Definition of $P_\alpha$: $P \in P_\alpha$ if and only if $P$ is the non-empty intersection of a maximal subfamily of $F_\alpha$ with the finite intersection property.

The next three claims will be used in our post-construction argument.

Claim 1. Any $I \in P_\alpha$ is open in $X$.

Let $I \subset F_\alpha$ be such that $I = \bigcap I$. Since each $B_\beta$ is discrete, $I$ meets each $B_\beta$ for $\beta < \alpha$ by exactly one element. Therefore, $|I| \leq \alpha < \tau$. Since the $P$-number of $X$ is $\tau$, the set $I = \bigcap I$ is open, which proves the claim.

Claim 2. If $x \in B_x \in F_\alpha$, then $x \in P \subset B_x$ for some $P \in P_\alpha$.

Since each $B_\beta$ is a disjoint cover of $X$, the family $\{B_{x,\beta} : x \in B_{x,\beta} \in B_\beta, \beta < \alpha\}$ is a maximal subfamily in $F_\alpha$ with the finite intersection property and its intersection $P$ has the desired properties, which completes the claim.

The next claim is obvious and is stated without a proof.

Claim 3. $P_\alpha$ is a partition of $X$ inscribed in each $B_\beta$, $\beta < \alpha$.

Next order elements of $B_\alpha = \{B_{\alpha,\lambda} : \lambda < \tau_\alpha\}$.

Definition of $O_\alpha$: Put $O'_\alpha = \{(x, y) : x, y \in P \in P_\alpha, x \in B_{\alpha,\lambda}, y \in B_{\alpha,\gamma}, \lambda < \gamma\}$ and $O_\alpha = O'_\alpha \cup (\bigcup_{\beta < \alpha} O_\beta)$.

Let us verify A1-A5 for $\alpha$. 
A1 check: This property follows from the fact that each $O_{\beta}$, where $\beta < \alpha$, is represented in the union that defines $O_{\alpha}$.

A2 check: Fix $x, y$ separated by $\bigcup_{\beta \leq \alpha} B_{\beta}$. If $x, y$ are separated by $B \in B_{\beta}$ for some $\beta < \alpha$, then apply A2 for $\beta$ and A1 for $\alpha$.

Otherwise, by Claim 3, there exists $P \in P_{\alpha}$ that contains $\{x, y\}$. Since $B_\alpha$ is a disjoint cover of $X$ that separates $x$ and $y$, we conclude that $x \in B_{\alpha, \lambda}$ and $y \in B_{\alpha, \gamma}$ for some $\lambda \neq \gamma$. By the definition of $O'_\alpha$, either $(x, y)$ or $(y, x)$, but not both, is in $O'_\alpha$ and, therefore, in $O_\alpha$.

A3 check: Fix $(x, y), (y, z) \in O_\alpha$.

Assume that both $(x, y), (y, z) \in O_\beta$ for some $\beta < \alpha$. Then $(x, z) \in O_\beta$ by A3 for $\beta$. Then $(x, z) \in O_\alpha$ by A1 for $\alpha$.

Assume that $(x, y) \in O_\beta$ and $(y, z) \notin O_\beta$ for some $\beta < \alpha$. By A2 for $\alpha$, we conclude that $(y, z) \notin O_\beta$. Then $y$ and $z$ are not separated by $\bigcup_{\gamma \leq \beta} B_\gamma$. Then $(x, z) \in O_\beta$ by A5 for $\beta$. Therefore, $(x, z) \in O_\alpha$ by A1 for $\alpha$.

Assume that $(x, y) \notin O_\beta$ and $(y, z) \in O_\beta$ for some $\beta < \alpha$. Then the previous argument applies.

Assume that neither $(x, y)$ nor $(y, z)$ is $\bigcup_{\beta < \alpha} O_\beta$. Then there exists $P \in P_\alpha$ such that $\{x, y, z\} \subset P$. Also, there exist $\beta < \gamma < \lambda$ such that $x \in B_{\alpha, \beta}, y \in B_{\alpha, \gamma}, z \in B_{\alpha, \lambda}$. Since $\beta < \lambda$, we conclude that $(x, z) \in O'_\alpha \subset O_\alpha$.

A4 check: Fix any $x \in X$. By assumption, $(x, x) \notin \bigcup_{\beta < \alpha} O_\beta$. By definition, $(x, x) \notin O'_\alpha$. Therefore, $(x, x) \notin O_\alpha$.

A5 check: Assume that $x$ and $y$ are not separated by $\bigcup_{\gamma \leq \alpha} B_\gamma$. Then there exist $P \in P_\alpha$ such that $\{x, y\} \subset P$ and $\gamma$ such that $x, y \in B_{\alpha, \gamma} \in B_\alpha$. We will consider the case $(z, x) \in O_\alpha$.

Assume $(z, x) \in O_\beta$ for some $\beta < \alpha$. Then $(z, y) \in O_\beta$ by A5 for $\beta$. By A1 for $\alpha$, we conclude that $(z, y) \in O_\alpha$.

Assume $(z, x) \notin O_\beta$ for any $\beta < \alpha$. Then there exists $P \in P_\alpha$ such that $\{z, x, y\} \subset P$. Since $(z, x) \in O_\alpha$ and $x \in B_{\alpha, \gamma}$ there exists $\lambda < \gamma$ such that $z \in B_{\alpha, \lambda}$. Since $y \in B_{\alpha, \gamma}$ we conclude that $(z, y) \in O'_\alpha \subset O_\alpha$.

The inductive construction is complete.

Define $\prec$ by letting $x \prec y$ if and only if $(x, y) \in \bigcup_{\alpha < \tau} O_\alpha$. Let us show that the relation is a linear order. Non-reflexivity follows from A4. Transitivity follows from A3 and A1. To check comparability, fix distinct $x, y \in X$. Since $X$ is Hausdorff and $B$ is a basis for the topology of $X$, there exists $\alpha < \tau$ such that $x$ and $y$ are separated by $B_\alpha$. By A2, either $(x, y)$ or $(y, x)$ is in $O_\alpha$.

The conclusion of the theorem follows from the next two claims.
Claim 4. $\bigcup_{\alpha<\tau} P_\alpha$ forms a basis for the topology of $X$.

To prove the claim, fix $x \in X$ and $B \in B_\alpha$ containing $x$. By Claim 3, there exists $P \in P_{\alpha+1}$ such that $x \in P \in B$. By Claim 1, $P$ is open in $X$. The claim is proved.

Claim 5. Every element of $\bigcup_{\alpha<\tau} P_\alpha$ is convex with respect to $\prec$.

Fix $P \in P_\alpha$ and $x, y \in P$ with $x \prec y$. Fix any $z$ such that $x \prec z \prec y$. Since $x, y \in P \in P_\alpha$, we conclude that $x, y$ are not separated by $\bigcup_{\beta<\alpha} B_\beta$. If $x, z$ were separated by $\bigcup_{\beta<\alpha} B_\beta$ then $(x, z)$ would have been in $O_\beta$ for $\beta < \alpha$. By A5, $(y, z)$ would have been in $O_\beta$, contradicting $z \prec y$. Therefore, $x, y, z$ are not separated by $\bigcup_{\beta<\alpha} B_\beta$. By Claim 3, $x, y, z \in P$, which proves convexity of $P$. □

To state the promised necessary condition for suborderability of finite powers, we need the following lemma.

Lemma 2.2. Let $\tau$ be the $P$-number of non-discrete spaces $X$ and $Y$. Let $X$ and $Y$ have $\tau$-discrete bases of clopen sets. Then the $P$-number of $X \times Y$ is $\tau$, and $X \times Y$ has a $\tau$-discrete basis of clopen sets.

Proof. Let $\{O_\alpha : \alpha < \kappa\}$ be a collection of open sets in $X \times Y$, where $\kappa < \tau$. Assume that $S = \bigcap_{\alpha<\kappa} O_\alpha$ is not empty. Fix any $(x, y) \in S$. Then for each $\alpha < \kappa$, there exist $U_\alpha, V_\alpha$ open neighborhoods of $x$ and $y$, respectively, such that $U_\alpha \times V_\alpha \subset O_\alpha$. Since the $P$-number of $X$ and $Y$ is $\tau$, we conclude that $U = \bigcap_{\alpha<\kappa} U_\alpha$ and $V = \bigcap_{\alpha<\kappa} V_\alpha$ are open. Therefore, $(x, y) \in U \times V \subset S$. Hence $S$ is open. Therefore, the $P$-number of $X \times Y$ is greater than or equal to $\tau$. Since $X \times Y$ contains a copy of $X$, the $P$-number of $X \times Y$ is at most $\tau$. Thus, the $P$-number of $X \times Y$ is $\tau$.

Next, let $U_\alpha$ be a discrete collection of clopen subsets of $X$ such that $\bigcup_{\alpha<\tau} U_\alpha$ is a basis for the topology of $X$. Similarly, we fix a basis $\bigcup_{\alpha<\tau} V_\alpha$ for the topology of $Y$. Put $B_{\alpha \beta} = \{U \times V : U \in U_\alpha, V \in V_\beta\}$. The family $B_{\alpha \beta}$ is a discrete collection of clopen sets, since $U_\alpha$ and $V_\beta$ are. By the definition of product topology, $\bigcup_{\alpha, \beta<\tau} B_{\alpha \beta}$ is a basis for the topology of $X \times Y$. □

The statement of the next Lemma is a simple corollary to Lemma 2.2.

Lemma 2.3. Let $\tau$ be the $P$-number of a non-discrete $X$. Let $X$ have a $\tau$-discrete basis of clopen sets. Then for any positive integer $n$, the $P$-number of $X^n$ is $\tau$ and $X^n$ has a $\tau$-discrete basis of clopen sets.
Theorem 2.1 and Lemma 2.3 imply our main result of the paper, stated as follows.

**Theorem 2.4.** If the P-number of \( X \) is \( \tau \) and \( X \) has a \( \tau \)-discrete basis of clopen sets, then \( X^n \) is a generalized ordered space for any natural number \( n \).

Theorem 2.4 implies, in particular, that if \( X \) is zero-dimensional and the weight of \( X \) is equal to the P-number of \( X \), then any finite power of \( X \) is suborderable. Since the square of the Sorgenfrey Line is not suborderable, the weight-P-number equality cannot be replaced by the density-P-number equality. Also, having P-number equal to weight locally and uniformly is not sufficient either. The space of countable ordinals serves as a counterexample.

It is natural to wonder if our sufficient condition for suborderability of \( X^2 \) is a characterization. Our next result shows that it may be, at least for a sufficiently large class of spaces. For the sake of our next result only, we say that \( X \) is character-homogeneous at points of a set \( S \subset X \) if \( \chi(x, X) = \chi(y, X) \) for all \( x, y \in S \). We will use the theorem of Engelking and Lutzer that (see [2] or [3]) "A GO-space \( X \) is paracompact if and only if no close subspace of \( X \) is homeomorphic to a closed subspace of a regular uncountable cardinal".

**Theorem 2.5.** Let \( X \) be a subspace of an ordinal. Then the following conditions are equivalent:

1. \( X \) has no stationary subspaces and is character-homogeneous at all non-isolated points.
2. \( X \) has a \( \tau \)-discrete basis of clopen sets, where \( \tau \) is the P-number of \( X \).
3. \( X^n \) is suborderable for any \( n \).

**Proof.** Proof of (1) \( \Rightarrow \) (2). We will prove the implication by induction on ordinal \( \alpha \) that can host \( X \).

**Step (n is finite).** Then \( X = \{a_1, ..., a_m\} \) for some \( m \leq n \) and the P-number of \( X \) is \( m \). The family \( \bigcup_{i \leq m} B_i \), where \( B_i = \{\{x_n\} : n = 1, ..., m\} \), is an \( m \)-discrete basis of \( X \).

**Remark.** Observe that a discrete space of any cardinality has a \( \tau \)-discrete basis for any \( \tau > 0 \).

**Assumption for \( \beta < \alpha \).** Assume that for any \( X \subset \beta < \alpha \), the implication (1) \( \Rightarrow \) (2) is true.

**Inductive Step \( \alpha \).**
Case of limit \( \alpha \). Since \( X \) has no subspaces homeomorphic to a stationary subset, by the Engelking-Lutzer Theorem, \( X \) is paracompact. Therefore, \( X \) can be written as a free sum \( \oplus_{\gamma \in \Gamma} X_\gamma \), where \( X_\gamma \subset \gamma < \alpha \) for each \( \gamma \in \Gamma \). Since \( X \) is character homogeneous at all non-isolated points, there exists a cardinal \( \tau \) such that \( \chi(x, X) = \tau \) for all non-isolated \( x \in X \). Then each \( X_\gamma \) has no subspaces homeomorphic to a stationary subset and is character homogeneous at non-isolated points. The \( P \)-number of any non-discrete \( X_\gamma \) is \( \tau \). By Inductive Assumption and Remark, each \( X_\gamma \) has a \( \tau \)-discrete basis \( B_\gamma \) of clopen sets. Since \( \{X_\gamma : \gamma \in \Gamma\} \) is a discrete cover, the family \( \bigcup_{\gamma \in \Gamma} B_\gamma \) is a \( \tau \)-discrete basis of \( X \) consisting of clopen sets.

Case of isolated \( \alpha \). Let \( \alpha = \beta + 1 \). We may assume that \( \beta \in X \) and \( \beta \) is not isolated in \( X \). Otherwise, we can reduce our scenario to a smaller ordinal.

Since \( X \) has no subspaces homeomorphic to a stationary subset, we conclude that \( X \setminus \{\beta\} \) can be written as a free sum \( \oplus_{\gamma \in \Gamma} X_\gamma \), where \( X_\gamma \subset \gamma < \beta \) for each \( \gamma \in \Gamma \) and \( |\Gamma| = cf(\beta) \). Since \( \beta \) is not isolated in \( X \), we conclude that the character of \( \beta \) is \( cf(\beta) \). Then the character at all non-isolated points of \( X \) is \( cf(\beta) \). Following the argument of the limit case, each \( X_\gamma \) has a \( cf(\beta) \)-discrete basis of clopen sets \( B_\gamma \). The family 
\[
\{X \setminus (\gamma + 1) : \gamma \in \Gamma\} \cup \{B \in B_\gamma : \gamma \in \Gamma\}
\]
is \( cf(\beta) \)-discrete basis of \( X \) consisting of clopen sets.

Proof of (2)\( \Rightarrow \) (3). This implication is Theorem 2.4.

Proof of (3)\( \Rightarrow \) (1). If \( X \) had a stationary subset or two limit points of distinct characters then \( X \times X \) would not have been hereditarily normal. This statement follows from the argument of Katetov [4] that if \( X \times Y \) is hereditarily normal then either every closed subset of \( X \) is a \( G_\delta \)-set or every countable subset of \( Y \) is closed. Since every GO space is hereditarily normal, the implication is proved. \( \square \)

We would like to finish the paper with a few questions that naturally arise as a result of our discussion.

Question 2.6. Let \( X \times X \) be suborderable. Is \( X \times X \) orderable? What if \( X \) is orderable?

Question 2.7. Let \( X \times X \) be orderable (suborderable). Is \( X^n \) orderable (suborderable)?

Question 2.8. Let \( X \) have a \( \tau \)-discrete basis of clopen sets, where \( \tau \) is the \( P \)-number of \( X \). Is \( X \times X \) orderable? What if \( \tau \) is the weight of \( X \)?
Finally, the unaccomplished goals of the paper are summarized in the next two questions.

**Question 2.9.** Assume that $X \times X$ is suborderable (or orderable). Is it true that $X$ has a $\tau$-discrete basis of clopen sets, where $\tau$ is the $P$-number of $X$?

**Question 2.10.** Assume that $X \times X$ is suborderable (or orderable) space of density $\tau$. Is it true that the weight of $X$ is equal to the $P$-number of $X$?

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**References**

[1] R. Engelking, *General Topology*, PWN, Warszawa, 1977.
[2] H. Bennet and D. Lutzer, *Linearly Ordered and Generalized Ordered Spaces*, Encyclopedia of General Topology, Elsevier, 2004.
[3] D. Lutzer, *Ordered Topological Spaces*, Surveys in General Topology, G. M. Reed, Academic Press, New York (1980), 247-296.
[4] M. Katetov, *Complete Normality of Cartesian Products*, Fund. Math., 36 (1948), 271-274.

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