Schwarzschild radius from Monte Carlo calculation of the Wilson loop in supersymmetric matrix quantum mechanics

Masanori Hanada, Akitsugu Miwa, Jun Nishimura, and Shingo Takeuchi

1 Department of Particle Physics, Weizmann Institute of Science, Rehovot 76100, Israel
2 Institute of Physics, University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8902, Japan
3 Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India
4 High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan
5 Department of Particle and Nuclear Physics, School of High Energy Accelerator Science, Graduate University for Advanced Studies (SOKENDAI), Tsukuba 305-0801, Japan
6 Asia Pacific Center for Theoretical Physics (APCTP), Pohang, Gyeongbuk 790-784, Korea

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Introduction.— String/gauge duality, which originated from the AdS/CFT correspondence [1], has been investigated intensively over the past decade. Remarkable developments have been achieved including generalization to various cases, confirmation by explicit calculations, and applications to various branches of physics such as hadron physics and condensed matter physics.

From the viewpoint of string theory, the duality enables us to study quantum aspects of gravity including its non-perturbative effects from the gauge theory side, which is more tractable. In this regard it is important to understand how gauge theory captures the information of space-time geometry [2]. Based on the duality at finite temperature [3], one can show that a temporal Wilson loop operator in gauge theory is related directly to the Schwarzschild radius [4], which is a fundamental quantity that characterizes the dual black hole geometry. (See also Refs. [5] for related works.) In this Letter we confirm this prediction by first-principle calculations on the gauge theory side. We calculate the Wilson loop on the gauge theory side in the strongly coupled regime by performing Monte Carlo simulation of supersymmetric matrix quantum mechanics with 16 supercharges. The results reproduce the expected power-law behavior up to a constant shift, which is explainable as $\alpha'$ corrections on the gravity side.

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In the string/gauge duality it is important to understand how the space-time geometry is encoded in gauge theory observables. We address this issue in the case of the D0-brane system at finite temperature $T$. Based on the duality, the temporal Wilson loop operator $W$ in gauge theory is expected to contain the information of the Schwarzschild radius $R_{\text{Sch}}$ of the dual black hole geometry as $\log\langle W \rangle = R_{\text{Sch}}/(2\pi\alpha' T)$. This translates to the power-law behavior $\log\langle W \rangle = 1.89 \cdot (T/\lambda^{1/3})^{-3/5}$, where $\lambda$ is the 't Hooft coupling constant. We calculate the Wilson loop on the gauge theory side using a lattice approach [11].

Here we apply this method to the calculation of the Wilson loop operator, and demonstrate that one can extract the Schwarzschild radius of the dual black hole geometry from it. See Ref. [12] for earlier discussions on a similar issue in the same model using other observables and other calculation techniques.

Wilson loop in the dual string theory.— Let us review the calculation of the Wilson loop based on the string/gauge duality [6] for general D-branes. In addition to a stack of $N$ D-branes, which are placed on top of each other creating a curved background geometry, we consider a single probe D-brane, which is placed far away from them in parallel. Since D-branes are objects which a fundamental string can end on, we may consider such a string stretched between the probe D-brane and one of the $N$ D-branes. The amplitude for the string propagating along a certain loop $C$ on the D-brane can be calculated in two different ways.

First in the worldvolume theory of the $N$ D-branes, the process is viewed as a heavy test particle in the fundamental representation of the $U(N)$ group propagating along the loop. The amplitude is therefore given by

$$A = \langle W(C) \rangle e^{-MT},$$

where $W(C)$ represents the Wilson loop associated with
the target space with the metric $g$. Here we also restrict ourselves to small bosonic part is given by the Polyakov action

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\hat{h}} \left( \hat{h}^{ab} \partial_a x^M \partial_b x^N + \alpha' \phi R_{(2)} \right).$$

Here $x^M$ represents the embedding of the worldsheet into the target space with the metric $g_{MN}$ ($M, N = 1, \ldots, 10$), while $R_{(2)}$ represents the two-dimensional scalar curvature defined for the worldsheet metric $h_{ab}$ ($a, b = 1, 2$). The effective string coupling is given as $e^\phi$ in terms of the dilaton field $\phi$. We have omitted a term in $S_P$ depending on the NS-NS $B$-field since the background we are considering does not have non-zero $B$ field.

In what follows, we will be mostly interested in the parameter region, in which the string coupling is small that we only have to consider the disk amplitude in $W_{(2)}$. We also restrict ourselves to small $\alpha'$, which corresponds to a large string tension, so that the path integral $W_{(2)}$ is dominated by the saddle-point configuration. In this parameter region, one can use the classical solution to the supergravity as the background. According to the dictionary of the string/gauge duality, the parameter region corresponds to taking the planar large-$N$ limit with large 't Hooft coupling constant on the gauge theory side.

Equating (1) and (2), we obtain a formula which relates the Wilson loop in the strongly coupled gauge theory to the string amplitude on the classical background geometric. The explicit check of this formula has been discussed in [13, 14, 15]. The obtained result indeed agrees with the prediction from the gravity side. The agreement can be understood also from the scale invariance of the worldsheet theory near the D3-branes [16].

The D0-brane case.— From now on, let us restrict ourselves to the D0-brane case. The gauge theory side is described by the supersymmetric MQM

$$S_{SQM} = \frac{\lambda}{N} \int_0^\beta dt \text{tr} \left\{ \frac{1}{2} D_t X_i \right\}^2 - \frac{1}{4} |X_i, X_j|^2$$

$$+ \frac{1}{2} \psi_\alpha D_t \psi_\alpha - \frac{1}{4} \psi_\alpha (\gamma_i)_{\alpha\beta} [X_i, \psi_\beta] \right\},$$

where $D_t = \partial_t - i [A(t), \cdot\cdot\cdot]$, $\lambda$ represents the mass of the test particle on the gauge theory side is given by the position of the probe D0-brane. Following the proposal [6], we consider a string D-brane and the probe D-brane.

The effective string coupling is given as $\beta = 1/N$ introduced to make the quantity finite in the planar large-$N$ limit. We have used the same normalization on the right hand side of (2).

The gravity dual of the supersymmetric MQM is given by the near-horizon geometry of the (Euclidean) near-extremal black 0-brane solution in type IIA supergravity. In particular, the metric is given by

$$ds^2 = \frac{U^{7/2} f(U)}{\sqrt{d_0 \lambda}} dt^2 + \frac{\sqrt{d_0 \lambda}}{U^{7/2} f(U)} dU^2 + \frac{\sqrt{d_0 \lambda}}{U^{3/2}} d\Omega_8^2,$$

where $f(U) = 1 - U_0^7/U^7$ and $d_0 \equiv 2^7 \pi^{9/2} \Gamma(7/2)$. The Schwarzschild radius and the inverse Hawking temperature are given by

$$R_{Sch} = \alpha' U_0, \quad \beta = \frac{4}{7} \pi \sqrt{d_0 \lambda} U_0^{-5/2}.$$
both sides of \( \boxed{3} \). This follows also from the prescription proposed in Ref. [17] based on T-duality. Thus we obtain

\[
\log(\mathcal{W}(\mathcal{C})) = \frac{\beta U_0}{2\pi} = \frac{\beta R_{\text{Sch}}}{2\pi \alpha'} = 1.89 \left( \frac{T}{\lambda^{1/2}} \right)^{-3/5}, \tag{9}
\]

where we have used \( \boxed{7} \).

The range of validity.— Let us recall the range of validity for the supergravity description \( \boxed{10} \). By changing the target-space coordinates as \( U = U_0 u^{2/5} \) and \( t = \sqrt{rac{4}{5}} u T - \frac{1}{5} / \pi \), the metric \( \boxed{6} \) and the effective string coupling \( e^\phi \) become

\[
\frac{ds^2}{\alpha'} = (d_0^{1/3} K)^{3/5} \left[ \frac{4}{25} \left( \tilde{f}(u) du^2 + \frac{du^2}{f(u)} \right) + d\Omega_5^2 \right], \tag{10}
\]

\[
e^\phi = \frac{(2\pi)^2}{N} \left( d_0^{-1/3} K \right)^{21/10}, \quad K = \frac{7\lambda^{1/3}}{4\pi u T}, \tag{11}
\]

where \( \tilde{f}(u) \equiv u^2(1 - u^{-14/5}) \). From \( \boxed{10} \), one finds that the geometry asymptotes at large \( u \) to a geometry which is conformally equivalent to \( \text{AdS}_5 \times S^5 \) \( \boxed{13} \), and that the typical length scale of the geometry is given by \( \rho \equiv (u T / \lambda^{1/3})^{-3/10} \alpha'^{1/2}. \) This scale should be much larger than the string length \( \alpha'^{1/2} \) for the \( \alpha' \) corrections to the supergravity action to be negligible. Hence, \( u T / \lambda^{1/3} \ll 1 \). In this case, the first term in \( \boxed{6} \), which is proportional to \( \rho^2 \), becomes large, and the semi-classical treatment for the string amplitude \( \boxed{2} \) is also justified. Note, however, that we have introduced \( U_\infty \). Assuming that we only need to require \( U_\infty / U_0 \) to be large (but finite), we may assume \( u \) to be finite as well. Then we obtain the condition \( T / \lambda^{1/3} \ll 1 \).

We also need to require the effective string coupling \( e^\phi \) to be small. From \( \boxed{11} \), we obtain \( N^{-10/21} \ll T / \lambda^{1/3} \) noting that \( u \geq 1 \) in our finite temperature set-up.

\( \alpha' \) corrections.— Let us discuss possible subleading terms in \( \boxed{6} \) due to \( \alpha' \) corrections on the gravity side. There are three effects one should consider: (I) the coupling with the background \( \phi \) field represented by the second term in \( \boxed{6} \). (II) \( \alpha' \) corrections to the background fields that appear in the action \( \boxed{3} \), and (III) the quantum fluctuation of the string worldsheet including fermionic degrees of freedom in evaluating \( \boxed{2} \). In order to discuss the next-leading order terms, we can treat each of these effects separately.

The effect (I) yields a constant term and a logarithmic term with respect to \( T / \lambda^{1/3} \) in \( \boxed{9} \) as one can see from \( \boxed{3} \) and \( \boxed{11} \). The constant term includes \( \log N \), but this is canceled by the prefactor \( 1 / N \) in \( \boxed{2} \) as it should. The effect (II) can be neglected at this order since \( \alpha' \) corrections to the type IIA supergravity action starts only at the \( \alpha'' \) order \( \boxed{19} \). The effect (III) yields a constant term to \( \boxed{9} \). This effect is discussed also in the case of D3-branes \( \boxed{20} \). In fact a logarithmic term can appear from it as well due to the insertion of the ghost zero mode \( \boxed{14} \).

Monte Carlo simulation.— We perform Monte Carlo simulation of the model \( \boxed{4} \) and calculate the temporal Wilson loop \( \boxed{5} \) to check the prediction \( \boxed{9} \). We use the Fourier-mode simulation method \( \boxed{12} \), in which we take the static diagonal gauge \( A(t) = \frac{1}{n} \text{diag}(\alpha_1, \ldots, \alpha_N) \) with \( -\pi < \alpha_a \leq \pi \), and introduce a cutoff \( \Lambda \) on the Fourier modes as \( X_i(t) = \sum_{n=-\Lambda}^{\Lambda} X_{i,n} e^{i n t} \), where \( \omega = \frac{2\pi}{n T} \).

Supersymmetry at \( T = 0 \), which is broken only due to finite \( \Lambda \), is shown to recover rapidly as \( \Lambda \to \infty \) in a simpler model \( \boxed{1} \). The effective ’t Hooft coupling constant is given by \( \lambda_{\text{eff}} = \lambda / T^3 \). In actual simulation we set \( \lambda = 1 \) without loss of generality, so that high/low \( T \) corresponds to weak/strong coupling, respectively.

Integration over the fermionic matrices yields a Pfaffian \( \text{PfM} \), which is complex in general. According to the standard reweighting method, one uses \( |\text{PfM}| \) to generate configurations, and includes the effect of the phase when one calculates the expectation values. In fact \( \text{PfM} \) is almost real positive at sufficiently high \( T \), but the fluctuation of the phase becomes larger as \( T \) decreases, which causes the so-called sign problem. It turned out, however, that the results of the reweighting method in the temperature regime where the sign problem is not so severe are actually in good agreement with what we obtain by simply neglecting the phase. We interpret this as an effect of the large-\( N \) limit, in which the fluctuations of single trace observables vanish. For the same reason, it is expected that \( \log(\langle W \rangle) \) agrees with \( \log(\langle |W| \rangle) \) in the large-\( N \) limit. We therefore calculate the latter in an ensemble generated with \( |\text{PfM}| \). Complete justification of these simplifications is left for future investigations.

We evaluate \( \boxed{5} \) as a limit \( W = \lim_{\nu \to -\infty} W_{\nu} \), where

\[
W_{\nu} = \frac{1}{N} \text{tr} \prod_{k=0}^{\nu-1} \left[ 1 + \frac{\beta}{\nu} \{ tA + \bar{n} X_i(t_k) \} \right] \tag{12}
\]

with \( t_k = \frac{\beta}{\nu} \). The matrices \( X_i(t_k) \) are obtained as the inverse Fourier transform of the configurations generated by our simulation. Using the asymptotic behavior \( W_{\nu} \simeq W + \frac{\langle|W|\rangle}{\nu} \) at large \( \nu \), we can make a reliable extrapolation to \( \nu = \infty \). As the unit vector \( \bar{n} \), we have used the ones in all 9 directions with plus or minus sign in front, and averaged over them to increase statistics.

In Fig. \( \boxed{1} \) we plot \( \langle |W| \rangle \) against \( T^{-3/5} \). As \( T \) decreases (to the right on the horizontal axis), the data show a clear linear growth with a slope consistent with the value 1.89 predicted in \( \boxed{9} \). In fact we can fit our data to \( \langle |W| \rangle = 1.897 T^{-3/5} - C \), where \( C = 4.95 \) for \( N = 4 \) and \( C = 4.58 \) for \( N = 6 \). The data points for \( N = 8 \) are very close to those for \( N = 6 \). Note that the constant term and the logarithmic term predicted from the gravity side are difficult to distinguish numerically. We therefore consider that the value of \( C \) extracted above actually represents the sum of the two terms at the temperature regime investigated.
FIG. 1: The plot of $\langle \log |W| \rangle$ for $\lambda = 1$ against $T^{-3/5}$. The cutoff $\Lambda$ is chosen as follows: $\Lambda = 12$ for $N = 4$; $\Lambda = 0.6/T$ for $N = 6, 8$; $\Lambda = 4$ for $N = 14$; $\Lambda = 6$ for $N = 17$. The dashed line represents the results of the high-temperature expansion up to the next-leading order with extrapolations to $N = \infty$, which are obtained by applying the method in Ref. [10]. The solid line and the dotted line represent fits for $N = 6$ and $N = 4$ respectively, to straight lines with the slope 1.89 predicted from the gravity side at the leading order.

Summary.— We have presented the first Monte Carlo calculations of the Wilson loop in a supersymmetric gauge theory at strong coupling. Up to subleading terms anticipated from the analysis on the gravity side, our results are in precise agreement with the prediction from the dual supergravity. This is a new and highly nontrivial evidence for the string/gauge duality. It would be nice to obtain the subleading terms explicitly from the gravity side, which will provide a nontrivial check of the duality including $\alpha'$ corrections. It is also interesting to extend this work to $N = 4$ SYM on $\mathbb{R} \times S^3$, which is possible by using the equivalence [21] in the planar limit between the SYM and a mass-deformed MQM around a multi-fuzzy-sphere background. The equivalence is confirmed by explicit calculations at weak coupling [22].

The fact that we were able to see the Schwarzschild radius of the dual black hole geometry by simulating large-$N$ matrices gives us strong support and a firm ground for using matrix model simulations to study quantum gravity. [23]. Note that the gauge theory description is valid also at small $\lambda$ and small $N$, where the dual supergravity description is no longer valid. Of particular interest is to study the parameter region corresponding to M-theory.

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