On the soft $p$-converse to a theorem of Gross–Zagier and Kolyvagin

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Abstract
We give a proof of a soft version of the $p$-converse to a theorem of Gross–Zagier and Kolyvagin for non-CM elliptic curves with good ordinary reduction at $p > 3$ under the irreducibility assumption on the residual representation. In particular, no condition on the conductor is imposed. Combining with the known results, we obtain the equivalence

$$\text{rk}_ZE(Q) = 1, \#\Sha(E/Q) < \infty \iff \ord_{s=1} L(E, s) = 1$$

for every elliptic curve $E$ over $Q$.

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1 Statement of the main result

Let $E$ be an elliptic curve over $Q$ and $p$ be a prime. The $p^\infty$-Selmer group of $E$ is defined by

$$\text{Sel}(Q, E[p^\infty]) = \ker \left( H^1(Q, E[p^\infty]) \to \prod_v \frac{H^1(Q_v, E[p^\infty])}{E(Q_v) \otimes Q_p/Z_p} \right)$$

where $v$ runs over the places of $Q$, and it encodes the arithmetic of $E$ via the fundamental exact sequence

$$0 \to E(Q) \otimes Q_p/Z_p \to \text{Sel}(Q, E[p^\infty]) \to \Sha(E/Q)[p^\infty] \to 0.$$
Denote by $V$ the two dimensional Galois representation over $\mathbb{Q}_p$ associated to $E$. The Selmer group of $V$ is defined by $\text{Sel}(\mathbb{Q}, V) = \ker \left( H^1(\mathbb{Q}, V) \to \prod_v H^1(\mathbb{Q}_v, V) \right)$ in the same manner.

The aim of this article is to give a succinct proof of the following $p$-converse result to the theorem of Gross–Zagier and Kolyvagin on elliptic curves of analytic rank one.

**Theorem 1.1** If

1. $\text{cork}_{\mathbb{Z}_p} \text{Sel}(\mathbb{Q}, E[\mathbb{Q}_p]) = 1$,
2. Perrin–Riou’s conjecture on Kato’s zeta elements holds (Conjecture 2.2),
3. the Iwasawa main conjecture (inverting $p$) holds (Conjecture 2.5), and
4. the restriction map $\text{res}_p : \text{Sel}(\mathbb{Q}, V) \to E(\mathbb{Q}_p) \otimes \mathbb{Q}_p$ is an isomorphism,

then $\text{ord}_s = 1 L(E, s) = 1$. In particular, $\text{rk}_{\mathbb{Z}} E(\mathbb{Q}) = 1$ and $\# \text{III}(E/\mathbb{Q}) < \infty$.

Perrin–Riou’s conjecture is recently proved by Bertolini–Darmon–Venerucci [3] and Burungale–Skinner–Tian [6] independently (Theorem 2.3). Also, the Iwasawa main conjecture (inverting $p$) is proved for non-CM elliptic curves with good ordinary reduction at $p > 3$ by Kato [13], Skinner–Urban [24], and Wan [29] if $E[p]$ is an irreducible Galois representation (Theorem 2.6). As an application of these results, we obtain the following statement.

**Corollary 1.2** Let $E$ be a non-CM elliptic curve over $\mathbb{Q}$ and $p > 3$ a good ordinary prime for $E$ such that $E[p]$ is an irreducible mod $p$ Galois representation. If

1. $\text{cork}_{\mathbb{Z}_p} \text{Sel}(\mathbb{Q}, E[\mathbb{Q}_p]) = 1$, and
2. the restriction map $\text{res}_p : \text{Sel}(\mathbb{Q}, V) \to E(\mathbb{Q}_p) \otimes \mathbb{Q}_p$ is an isomorphism,

then $\text{ord}_s = 1 L(E, s) = 1$. In particular, $\text{rk}_{\mathbb{Z}} E(\mathbb{Q}) = 1$ and $\# \text{III}(E/\mathbb{Q}) < \infty$.

**Remark 1.3** (1) Assumption (res) also appears in [23, Theorem B] in a slightly different context. See also [23, Remark 2.9.1.(x)]. The $p$-converse with Assumption (res) is called the soft $p$-converse.

(2) When $p = 3$, Corollary 1.2 still holds if we further assume that there exists a prime $\ell$ exactly dividing the conductor of $E$ such that $E[p]$ is ramified at $\ell$ [24, Theorem 3.33].

(3) For a non-CM elliptic curve $E$, the density of good ordinary primes for $E$ is one [21, Corollaire 2 to Théorème 20, §8.2] and $E[p]$ is an irreducible Galois representation for a sufficiently large prime $p \gg 0$ [22, Theorem in §2.1 of Chapter IV].

Since Assumption (res) holds if we further assume $\# \text{III}(E/\mathbb{Q})[p^{\infty}] < \infty$ in Corollary 1.2, we obtain the following statement.

**Corollary 1.4** Let $E$ be a non-CM elliptic curve over $\mathbb{Q}$. If $\text{rk}_{\mathbb{Z}} E(\mathbb{Q}) = 1$ and $\# \text{III}(E/\mathbb{Q})[p^{\infty}] < \infty$ for at least one good ordinary prime $p > 3$ such that $E[p]$ is irreducible, then $\text{ord}_s = 1 L(E, s) = 1$. In particular, $\# \text{III}(E/\mathbb{Q}) < \infty$.

**Remark 1.5** We are informed that Perrin-Riou’s conjecture and Corollaries 1.2 and 1.4 also appear in [6].

In their pioneering works, Skinner [23] and W. Zhang [31] independently proved the first general results towards the $p$-converse under certain assumptions on the conductor.
of elliptic curves. In [23, Theorem A], the conductor is square-free and satisfies a certain existence condition on split/non-split reduction primes. In [31, Theorem 1.3], the conductor satisfies a certain ramification condition arising from the arithmetic of Shimura curves. Our result completely removes these assumptions on the conductor.

Together with the work of Rubin [19] and Burungale–Tian [8] on the CM case, Corollary 1.4 completes the converse to the theorem of Gross–Zagier and Kolyvagin for elliptic curves over \( \mathbb{Q} \). In other words, we have the equivalence

\[
\text{rk}_\mathbb{Z} E(\mathbb{Q}) = 1, \; \# \text{III}(E/\mathbb{Q}) < \infty \iff \ord_{s=1} L(E, s) = 1
\]

for every elliptic curve \( E \) over \( \mathbb{Q} \).

On the other hand, the \( p \)-converse without Assumption (res), which is called the strong \( p \)-converse, can be obtained from the Heegner point main conjecture and other standard ingredients as explained in [1, 11, 30]. The Heegner point main conjecture can be replaced by the Kolyvagin conjecture [31]. We avoid using the Heegner point main conjecture here since it is currently not proved in general despite significant progresses towards it. Even the formulation of the conjecture at additive reduction primes is an open problem.

The \( p \)-converse itself is studied extensively in various forms [2, 6, 8, 10, 11, 25, 26, 28, 30]. See also [7] for the survey of the recent developments.

2 Preliminaries

2.1 Kato’s zeta elements

Let \( T \) be the Tate module of an elliptic curve \( E \) and \( V = T \otimes \mathbb{Q}_p \). Denote by \( \omega_E \) the Néron differential of a global minimal Weierstrass equation for \( E \), and \( \Omega^\pm \) the real and imaginary Néron periods for \( E \), respectively. Adapting the convention of [13, §6.3 and Theorem 16.2], we have the period map \( \text{per}_E \) from the space of global differential one-forms on \( E \) to the first Betti cohomology of \( E \) over \( \mathbb{C} \) such that

\[
\text{per}_E(\omega_E) = \Omega_E^+ \cdot \gamma^+ + \Omega_E^- \cdot \gamma^-
\]

for some non-zero \( \gamma^\pm \in V_Q(-1)^c = \pm 1 \), respectively, where \( V_Q(-1) \) is the first Betti cohomology of \( E \) over \( \mathbb{Q} \) and \( c \) is the complex conjugation. We identify \( V_Q(-1) \otimes \mathbb{Q}_p \) with the first \( p \)-adic étale cohomology \( V(-1) \) of \( E \) over \( \mathbb{Q}_p \), so \( \gamma^\pm \in V(-1) \).

Let \( \gamma = \gamma^+ + \gamma^- \in V(-1) \). Following [13, Theorem 12.5.(1)], Kato’s zeta element

\[
z^p_\gamma \in \varprojlimit_n H^1(\mathbb{Q}(\zeta_{p^n}), V(-1))
\]
for \( V(-1) \) over \( \mathbb{Q}(\zeta_{p^n}) \) is obtained from \( \gamma \) where \( \lim_{\rightarrow n} \) is taken with respect to the corestriction. Kato’s zeta element \( z_{\text{Kato}} \in H^1(\mathbb{Q}, V) \) for \( V \) over \( \mathbb{Q} \) is defined by the image of \( z^{(p)}_{\gamma} \) under the composition

\[
\lim_{\rightarrow n} H^1(\mathbb{Q}(\zeta_{p^n}), V(-1)) \rightarrow \lim_{\rightarrow n} H^1(\mathbb{Q}(\zeta_{p^n}), V) \rightarrow H^1(\mathbb{Q}, V).
\]

Let \( \mathbb{Q}_\infty \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \) and \( \Lambda = \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] \) the Iwasawa algebra. Write \( H^1_{\text{Iw}}(\mathbb{Q}, V) = \lim_{\rightarrow n} H^1(\mathbb{Q}_n, V) \) where \( \mathbb{Q}_n \) is the cyclic subextension of \( \mathbb{Q} \) of order \( p^n \) in \( \mathbb{Q}_\infty \). Kato’s zeta element \( z_{\text{Kato}}^{\infty} \in H^1_{\text{Iw}}(\mathbb{Q}, V) \) for \( V \) over \( \mathbb{Q}_\infty \) is similarly defined by the image of \( z^{(p)}_{\gamma} \) in \( H^1_{\text{Iw}}(\mathbb{Q}, V) \).

**Remark 2.1** If \( p \) is odd and the image of the Galois representation \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(V) \) contains a conjugate of \( \text{SL}_2(\mathbb{Z}_p) \), then \( \gamma \) can be chosen in \( T(-1) \) and all the zeta elements lie in the Galois cohomologies with integral coefficients. In other words, \( V \) can be replaced by \( T \) in the above discussion. See [15, Theorem 5.1] and [13, Theorem 12.5.(4)].

Kato’s explicit reciprocity law [13, Theorem 12.5.(1)] says that

\[
\exp^* \circ \text{res}_p(z_{\text{Kato}}) = (1 - a_p(E) \cdot p^{-1} + p^{-1}) \cdot \frac{L(E, 1)}{\Omega_E^+} \cdot \omega_E
\]

where \( \exp^* \) is Bloch–Kato’s dual exponential map.

### 2.2 Perrin-Riou’s conjecture

We recall Perrin-Riou’s conjecture on Kato’s zeta elements [17, Conjecture in §3.3.2].

**Conjecture 2.2** (Perrin-Riou) Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and \( p \) be a prime. If \( L(E, 1) = 0 \), then there exists a global point \( P \in E(\mathbb{Q}) \) satisfying the following properties.

1. The point \( P \) has infinite order if and only if \( \text{ord}_{s=1} L(E, s) = 1 \).
2. The following equality holds in \( \mathbb{Q}_p \) up to multiplication by a non-zero rational number:

\[
\log_{\omega_E}(\text{res}_p(z_{\text{Kato}})) = \left( \log_{\omega_E}(P) \right)^2
\]

where \( \log_{\omega_E} \) is the \( p \)-adic Lie group logarithm corresponding to \( \omega_E \).

Conjecture 2.2 is recently proved by Bertolini–Darmon–Venerucci [3, Theorem A] and Burungale–Skinner–Tian [6] independently.

**Theorem 2.3** (Bertolini–Darmon–Venerucci) If \( E \) has semi-stable reduction at an odd prime \( p \), then Conjecture 2.2 is true.

**Remark 2.4** In [6], Theorem 2.3 is proved for a good reduction prime \( p \geq 5 \).
We also refer the reader to [5, 9, 27] for other approaches towards Perrin-Riou’s conjecture.

2.3 Iwasawa main conjecture

We review the Iwasawa main conjecture (inverting \( p \)) without \( p \)-adic \( L \)-functions à la Kato [15, §6], [13, Conjecture 12.10].

The \( p \)-strict Selmer group of \( E[p^\infty] \) over \( \mathbb{Q}_n \) is defined by

\[
\mathrm{Sel}_0(\mathbb{Q}_n, E[p^\infty]) = \ker(\mathrm{res}_p : \mathrm{Sel}(\mathbb{Q}_n, E[p^\infty]) \to E(\mathbb{Q}_n, p) \otimes \mathbb{Q}_p / \mathbb{Z}_p)
\]

where \( \mathbb{Q}_n, p \) is the \( p \)-adic completion of \( \mathbb{Q}_n \). Write \( \mathrm{Sel}_0(\mathbb{Q}_\infty, E[p^\infty]) = \varprojlim_n \mathrm{Sel}_0(\mathbb{Q}_n, E[p^\infty]) \) and \((-)^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p / \mathbb{Z}_p) \). The rational version of the Iwasawa main conjecture can be written as follows.

**Conjecture 2.5 (IMC)** As ideals of \( \Lambda \otimes \mathbb{Q}_p \), we have

\[
\text{char}_{\Lambda \otimes \mathbb{Q}_p} \left( \frac{H^1_{Iw}(\mathbb{Q}, V)}{z_{Kato}} \right) = \text{char}_{\Lambda \otimes \mathbb{Q}_p} \left( \mathrm{Sel}_0(\mathbb{Q}_\infty, E[p^\infty])^\vee \otimes \mathbb{Z}_p / \mathbb{Q}_p \right).
\]

**Theorem 2.6** (Kato, Skinner–Urban, Wan) If \( E \) has good ordinary reduction at \( p \geq 5 \) and \( E[p] \) is an irreducible Galois representation, then Conjecture 2.5 is true.

**Proof** See [13, Theorem 12.5 and Theorem 17.4], [24, Theorem 3.33], and [29, Theorem 4].

3 Proof of Theorem 1.1

We extract the following statement from Conjecture 2.2.

**Corollary 3.1** If Conjecture 2.2 holds, then the following statements are equivalent.

1. \( \text{ord}_{s=1} L(E, s) = 0 \).
2. The global point \( P \) in Conjecture 2.2 has infinite order.
3. \( \text{res}_p(z_{Kato}) \) is non-zero.

Although the \( L(E, 1) = 0 \) assumption is incorporated in Conjecture 2.2, it can be removed in Theorem 1.1 thanks to the following theorem.

**Theorem 3.2** (Gross–Zagier, Kolyvagin, Rubin, Kato) Let \( E \) be an elliptic curve over \( \mathbb{Q} \). If \( L(E, 1) \neq 0 \), then \( \mathrm{Sel}(\mathbb{Q}, E[p^\infty]) \) is finite.

**Proof** See [12] and [14] for the general case. See also [18, Theorem A] for the CM case and [20, Theorem 8.1] and [13, Theorem 14.2] for the non-CM case.

**Proposition 3.3** Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and \( p \) be a prime. If the Iwasawa main conjecture inverting \( p \) holds (Conjecture 2.5) and \( \text{cork}_{\mathbb{Z}_p} \mathrm{Sel}(\mathbb{Q}, E[p^\infty]) = 1 \), then the following statements are equivalent.

\[ \blacksquare \] Springer
(1) \( \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \) is finite.

(2) \( z_{\text{Kato}} \) is non-zero.

(3) \( \text{res}_p(z_{\text{Kato}}) \) is non-zero.

**Proof** Write \( f_0, f_z \in \Lambda \otimes \mathbb{Q}_p \) to be the distinguished polynomials such that

\[
(f_0) = \text{char}_{\Lambda \otimes \mathbb{Q}_p} (\text{Sel}_0(\mathbb{Q}_\infty, E[p^\infty])^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p),
(f_z) = \text{char}_{\Lambda \otimes \mathbb{Q}_p} \left( \frac{H^1_{\text{Iw}}(\mathbb{Q}, V)}{z_{\text{Kato}}^\infty} \right).
\]

as ideals of \( \Lambda \otimes \mathbb{Q}_p \), respectively. The control theorem for \( p \)-strict Selmer groups says that the restriction map

\[
\text{res} : \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \to \text{Sel}_0(\mathbb{Q}_\infty, E[p^\infty])^\Gamma
\]

has finite kernel and cokernel where \( \Gamma = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \). Thus, the finiteness of \( \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \) is equivalent to the finiteness of \( \text{Sel}_0(\mathbb{Q}_\infty, E[p^\infty])^\Gamma \). The latter is also equivalent to \( 1(f_0) \neq 0 \) where \( 1 \) is the trivial character. On the other hand, \( z_{\text{Kato}} = 0 \) if and only if \( 1(f_z) \neq 0 \) since \( H^1_{\text{Iw}}(\mathbb{Q}, V) \simeq \Lambda \otimes \mathbb{Q}_p \) [13, Theorem 12.4.(2)]. Conjecture 2.5 implies that \( 1(f_0) \neq 0 \) if and only if \( 1(f_z) \neq 0 \). Thus, the equivalence between (1) and (2) follows.

It suffices to check (2) \( \Rightarrow \) (3) since (3) \( \Rightarrow \) (2) is trivial. We now assume \( z_{\text{Kato}} \) is non-zero. Consider the exact sequence

\[
0 \longrightarrow \text{Sel}_0(\mathbb{Q}, V) \longrightarrow \text{Sel}(\mathbb{Q}, V) \longrightarrow \text{res}_p E(\mathbb{Q}_p) \otimes \mathbb{Q}_p.
\]

By the equivalence between (1) and (2), \( \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \) is finite. Since \( E[p^\infty] \simeq V/T \), there exists a natural map \( \text{Sel}_0(\mathbb{Q}, V) \to \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \) coming from the exact sequence \( 0 \to T \to V \to V/T \to 0 \). Since the kernel and the cokernel of the map lie in compact \( \mathbb{Z}_p \)-modules, the finiteness of \( \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \) is equivalent to \( \text{Sel}_0(\mathbb{Q}, V) = 0 \). Applying the same argument to usual Selmer groups, we also have the equivalence between \( \text{cork}_{\mathbb{Z}_p} \text{Sel}(\mathbb{Q}, E[p^\infty]) = 1 \) and \( \dim_{\mathbb{Q}_p} \text{Sel}(\mathbb{Q}, V) = 1 \). Thus, the restriction map \( \text{res}_p \) is an isomorphism of one-dimensional \( \mathbb{Q}_p \)-vector spaces.

By Kato’s explicit reciprocity law (1) and Theorem 3.2, we have \( z_{\text{Kato}} \in \text{Sel}(\mathbb{Q}, V) \). Since \( z_{\text{Kato}} \neq 0 \) and \( \text{res}_p(z_{\text{Kato}}) \) is also non-zero.

**Remark 3.4** Although (2) \( \Rightarrow \) (3) immediately follows from Assumption (res), we would like to use Assumption (res) minimally.

The following lemma completes the proof.

**Lemma 3.5** If \( \text{cork}_{\mathbb{Z}_p} \text{Sel}(\mathbb{Q}, E[p^\infty]) = 1 \) and \( \text{res}_p : \text{Sel}(\mathbb{Q}, V) \simeq E(\mathbb{Q}_p) \otimes \mathbb{Q}_p \), then \( \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \) is finite.

**Proof** As in the proof of Proposition 3.3, \( \text{cork}_{\mathbb{Z}_p} \text{Sel}(\mathbb{Q}, E[p^\infty]) = 1 \) if and only if \( \dim_{\mathbb{Q}_p} \text{Sel}(\mathbb{Q}, V) = 1 \). If \( \text{res}_p : \text{Sel}(\mathbb{Q}, V) \simeq E(\mathbb{Q}_p) \otimes \mathbb{Q}_p \), then \( \text{Sel}_0(\mathbb{Q}, V) = 0 \) due to (2). Thus, the finiteness of \( \text{Sel}_0(\mathbb{Q}, E[p^\infty]) \) follows. □
Theorem 1.1 follows from the combination of Corollary 3.1, Theorem 3.2, Proposition 3.3, and Lemma 3.5. The last “In particular” part follows from the work of Gross–Zagier [12] and Kolyvagin [14] with a choice of a suitable imaginary quadratic field. The existence of such an imaginary quadratic field is ensured by the work of Bump–Friedberg–Hoffstein [4] or Murty–Murty [16].

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Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

References

1. Burungale, A.A., Castella, F., Kim, C.-H.: A proof of Perrin-Riou’s Heegner point main conjecture. Algebra Number Theory 15(10), 1627–1653 (2021)
2. Burungale, A.A., Castella, F., Skinner, C., Tian, Y.: $p^\infty$-Selmer groups and rational points on CM elliptic curves. Ann. Math. Québec 46(2), 324–346 (2022)
3. Bertolini, M., Darmon, H., Venerucci, R.: Heegner points and Beilinson-Kato elements: a conjecture of Perrin-Riou. Adv. Math. 398, 108172 (2022)
4. Bump, D., Friedberg, S., Hoffstein, J.: Nonvanishing theorems for $L$-functions of modular forms and their derivatives. Invent. Math. 102(3), 543–618 (1990)
5. Büyükboduk, K., Pollack, R., Sasaki, S.: $p$-adic Gross–Zagier formula at critical slope and a conjecture of Perrin-Riou, preprint, arXiv:1811.08216
6. Burungale, A.A., Skinner, C., Tian, Y.: Elliptic curves and Beilinson–Kato elements: rank one aspects, preprint
7. Burungale, A.A., Skinner, C., Tian, Y.: The Birch and Swinnerton-Dyer conjecture: a brief survey, Nine mathematical challenges—an elucidation (Providence, RI) (A. Kechris, N. Makarov, D. Ramakrishnan, and X. Zhu, eds.), Proc. Sympos. Pure Math., vol. 104, Amer. Math. Soc., 2021, pp. 11–29
8. Burungale, A.A., Tian, Y.: $p$-converse to a theorem of Gross-Zagier, Kolyvagin and Rubin. Invent. Math. 220(1), 211–253 (2020)
9. Büyükboduk, K.: Beilinson–Kato and Beilinson–Flach elements, Coleman–Rubin–Stark classes, Heegner points and a conjecture of Perrin-Riou, Development of Iwasawa Theory – the Centennial of K. Iwasawa’s Birth (Tokyo) (Masato Kurihara, Kenichi Bannai, Tadashi Ochiai, and Takeshi Tsuji, eds.), Adv. Stud. Pure Math., vol. 86, Mathematical Society of Japan, 2020, pp. 141–193
10. Castella, F., Grossi, G., Lee, J., Skinner, C.: On the anticyclotomic Iwasawa theory of rational elliptic curves at Eisenstein primes. Invent. Math. 227, 517–580 (2022)
11. Castella, F., Wan, X.: Perrin-Riou’s main conjecture for elliptic curves at supersingular primes, preprint
12. Gross, B., Zagier, D.: Heegner points and derivatives of $L$-series. Invent. Math. 84(2), 225–320 (1986)
13. Kato, K.: $p$-adic Hodge theory and values of zeta functions of modular forms. Astérisque 295, 117–290 (2004)

Kolyvagin, V.: Euler systems, The Grothendieck Festschrift Volume II (Pierre Cartier, Luc Illusie, Nicholas M. Katz, Gerard Laumon, Yuri Manin, and Kenneth A. Ribet, eds.), Progr. Math., vol. 87, Birkhäuser Boston, 1990, pp. 435–483
15. Kurihara, M.: On the Tate Shafarevich groups over cyclotomic fields of an elliptic curve with super-singular reduction I. Invent. Math. 149, 195–224 (2002)
16. Murty, M.R., Murty, V.K.: Mean values of derivatives of modular $L$-series. Ann. Math. 133(3), 447–475 (1991)
17. Perrin-Riou, B.: Fonctions $L$ $p$-adiques d’une courbe elliptique et points rationnels. Ann. Inst. Fourier (Grenoble) 43(4), 945–995 (1993)
18. Rubin, K.: Tate-Shafarevich groups and $L$-functions of elliptic curves with complex multiplication. Invent. Math. 89, 527–559 (1987)
19. Rubin, K.: $p$-adic variants of the Birch and Swinnerton-Dyer conjecture for elliptic curves with complex multiplication, $p$-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991) (Providence, RI) (Barry Mazur and Glenn Stevens, eds.), Contemp. Math., vol. 165, American Mathematical Society, 1994, pp. 71–80
20. Karl, R.: Euler systems and modular elliptic curves, Galois representations in arithmetic algebraic geometry (Durham, 1996) (Anthony Scholl and Richard Taylor, eds.), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge University Press, 1998, pp. 351–367
21. Serre, J.-P.: Quelques applications du théorème de densité de Chebotarev. Publ. Math. Inst. Hautes Études Sci. 54, 323–401 (1981)
22. Serre, J.-P.: Abelian $\ell$-adic representations and elliptic curves, Advanced Book Classics, Addison-Wesley Publishing Company, 1989, With the collaboration of Willem Kuyk and John Labute
23. Skinner, C.: A converse to a theorem of Gross, Zagier, and Kolyvagin. Ann. Math. 191(2), 329–354 (2020)
24. Skinner, C., Urban, E.: The Iwasawa main conjectures for $GL_2$. Invent. Math. 195(1), 1–277 (2014)
25. Sweeting, N.: Kolyvagin’s conjecture and patched Euler systems in anticyclotomic Iwasawa theory, preprint, arXiv:2012.11771
26. Skinner, C., Zhang, W.: Indivisibility of Heegner points in the multiplicative case, preprint, arXiv:1407.1099
27. Venerucci, R.: Exceptional zero formulae and a conjecture of Perrin-Riou. Invent. Math. 203(3), 923–972 (2016)
28. Venerucci, R.: On the $p$-converse of the Kolyvagin-Gross-Zagier theorem. Comment. Math. Helv. 91(3), 397–444 (2016)
29. Wan, X.: The Iwasawa main conjecture for Hilbert modular forms. Forum Math. Sigma 3, e18 (2015)
30. Wan, X.: Heegner point Kolyvagin system and Iwasawa main conjecture. Acta Math. Sin. 37(1), 104–120 (2021)
31. Zhang, W.: Selmer groups and the indivisibility of Heegner points. Camb. J. Math. 2(2), 191–253 (2014)

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