One-loop functional renormalization group study for the dimensional reduction and its breakdown in the long-range random field O(N) spin model near lower critical dimension

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We consider the random-field O(N) spin model with long-range exchange interactions which decay with distance r between spins as $r^{-d-\sigma}$ and/or random fields which correlate with distance r as $r^{-d+\rho}$, and reexamine the critical phenomena near the lower critical dimension by use of the perturbative functional renormalization group. We compute the analytic fixed points in the one-loop beta functions, and study their stability. We find that the analytic fixed point which governs the phase transition in the system with the long-range correlations of random fields can be destabilized by the nonanalytic perturbation in both cases that the exchange interactions between spins are short-ranged and long-ranged. For the system with the long-range exchange interactions and uncorrelated random fields, we clarify the boundary between critical behaviors in systems with long-range and short-range exchange interactions. We show that the $d \to d - \sigma$ dimensional reduction holds at the leading order of the $d - 2\sigma$ expansion and for $N > 2(4 + 3\sqrt{3}) \approx 18.3923 \cdots$. For the system with the long-range exchange interactions and the long-range correlated random fields, we show that the $d \to d - \sigma - \rho$ dimensional reduction does not hold within the present framework, as far as $N$ is finite.

I. INTRODUCTION

The random-field O(N) spin model is the model in which nonrandom exchange interactions between spins are ferromagnetic and external magnetic fields are random. To clarify the critical phenomena in this model is one of the fundamental problems in the disordered spin system, and there are a lot of intensive studies on this\textsuperscript{1–4}. The $d \to d - \theta$ dimensional reduction gives an important clue to clarify the nature of this model. The $d \to d - \theta$ dimensional reduction means that the effect of random fields reduces the spatial dimension by $\theta$; namely, the critical phenomena in $d$-dimensional random-field system is equivalent to that in the $(d-\theta)$-dimensional corresponding pure system. Here $\theta$ denotes the exponent describing that the flow of the renormalized temperature goes to zero under the renormalization-group iteration. If the $d \to d - \theta$ dimensional-reduction prediction is correct, all critical exponents in the $d$-dimensional random-field system should be the same as those in the corresponding pure system in $\theta$ dimensions less.

In the spin system with the short-range ferromagnetic exchange interactions and the uncorrelated random fields (SR), the $d \to d - 2$ dimensional reduction and its breakdown are one of the central issues. This conjecture was obtained by the perturbation theory\textsuperscript{5–7} and the supersymmetry argument\textsuperscript{8}. Rigorous proofs have shown that the $d \to d - 2$ dimensional-reduction prediction is incorrect below four dimensions in the case of the random-field Ising model ($N = 1$ case)\textsuperscript{9,10}. The $d \to d - 2$ dimensional reduction and its breakdown for the random-field O(N) spin model above four dimensions have been intensively studied. Fisher studied the critical phenomena in $4 + \epsilon$ dimensions by use of the O(N) nonlinear-sigma model.\textsuperscript{11} He showed that all possible higher-rank random anisotropies which are all relevant operators are generated by the perturbative functional renormalization-group iteration of the O(N) nonlinear-sigma model with only the random field term. Then, he treated the O(N) nonlinear-sigma model including the random-field and all the random-anisotropy terms, and derived the one-loop beta function in $4 + \epsilon$ dimensions. He showed that there is no singly unstable fixed point corresponding to the $d \to d - 2$ dimensional reduction at $O(\epsilon)$, and concluded that the $d \to d - 2$ dimensional-reduction prediction is incorrect near four dimensions. The one-loop beta function obtained by Fisher and the two-loop beta function extended by Le Doussal and Wiese\textsuperscript{11} and Tissier and Tarjus\textsuperscript{13} have been examined carefully.\textsuperscript{12,14} The singly unstable fixed point corresponding to the $d \to d - 2$ dimensional reduction exists for $N > 18 - (49/5)\epsilon$, although it has the weak nonanalyticity. However, it is unstable with respect to the perturbation with nonanalyticity for $N < 2(4 + 3\sqrt{3}) - (3\sqrt{3})/2\epsilon$. Thus, the $d \to d - 2$ dimensional reduction holds for $N > 2(4 + 3\sqrt{3}) - (3\sqrt{3})/2\epsilon$, and the critical exponents of the connected and the disconnected correlation functions $\eta$ and $\tilde{\eta}$ satisfy $\tilde{\eta} = \eta$. Whereas, in $N < 2(4 + 3\sqrt{3}) - (3\sqrt{3})/2\epsilon$, the critical phenomena is governed by the fixed point with the nonanalyticity, and thus the $d \to d - 2$ dimensional reduction is broken. Moreover, a complete theoretical explanation of the $d \to d - 2$ dimensional reduction and its breakdown has been provided through the nonperturbative functional renormalization group\textsuperscript{16–19}.

In the case where the ferromagnetic exchange interactions are short-ranged, and the random fields are correlated over the distance $r$ as $r^{-d+\rho}$ (LRF), the $d \to d - 2 - \rho$ dimensional reduction and its breakdown are still under debate. Kardar, McClain and Taylor per-
formed the renormalization-group calculation near the upper critical dimension \( d_u = 6 + \rho \), and concluded that the \( d \to d - 2 - \rho \) dimensional reduction is broken at \( O(\epsilon^2) \) in \( \epsilon = d_u - d \). Bray pointed out an error in Kardar, McClain and Taylor’s result but their conclusion still holds. Chang and Abrahams carried out the one-loop renormalization-group calculation for the \( O(N) \) nonlinear-sigma model near the lower critical dimension \( d_l = 4 + \rho \), and showed that the \( d \to d - 2 - \rho \) dimensional reduction is broken at \( O(\epsilon) \) in \( \epsilon = d_l - d \). Fedorenko and Küehne examined the one-loop beta functions of the \( O(N) \) nonlinear-sigma model including not only the uncorrelated and the long-range correlated random fields but also all the uncorrelated and the long-range correlated random anisotropies which are missed in the work by Chang and Abrahams. They showed that the correlation length exponent \( \nu \) and the phase diagram obtained by Chang and Abrahams are incorrect, and the exponents \( \eta \) and \( \bar{\eta} \) are correct only in a region controlled by the singly unstable fixed point with the weaker nonanalyticity.

In the case where the long-range ferromagnetic exchange interactions decay with distance \( r \) between spins as \( r^{-\sigma} \) and random fields are uncorrelated (LRE), the \( d \to d - \sigma \) dimensional reduction and its breakdown are still under debate. Young performed the renormalization-group calculation near the upper critical dimension \( d_u = 3\sigma \), and concluded that the \( d \to d - \sigma \) dimensional reduction is broken at \( O(\epsilon^2) \) in \( \epsilon = d_u - d \). Bray pointed out an error in Young’s result but the conclusion still holds. Chang and Abrahams carried out the one-loop renormalization-group calculation for the \( O(N) \) nonlinear-sigma model near the lower critical dimension \( d_l = 2\sigma \), and showed that the \( d \to d - \sigma \) dimensional reduction holds at \( O(\epsilon) \) in \( \epsilon = d_l - d \). However, the \( O(N) \) nonlinear-sigma model studied by Chang and Abrahams does not contain an infinite number of relevant operators which should be included in the model. Recently, Balog, Tarrus and Tissier studied the critical phenomena of one-dimensional random-field Ising model with the long-range exchange interactions and uncorrelated random fields by use of the nonperturbative renormalization-group analysis. And then, we calculate the critical exponents \( \eta, \bar{\eta} \) and \( \nu \) evaluated at each of four analytic fixed points, and discuss the critical properties of the system for each universality class. We find that the destabilization of the analytic fixed point controlling the critical behavior in the system with the long-range correlations of random fields can be caused by the perturbation with nonanalyticity in both cases that the exchange interactions between spins are short-ranged and long-ranged. In the system with LRE, we clarify the boundary between critical behaviors in systems with long-range and short-range exchange interactions on the basis of the relation between \( \eta \) and \( \bar{\eta} \). And we reconsider the validity of the \( d \to d - \sigma \) dimensional reduction by putting \( \rho = 0 \). In the system with LRF, we show that \( d \to d - \sigma - \rho \) dimensional reduction breaks down within the present framework, as far as \( N \) is finite.

The organization of this paper is as follows. In Sec. II we study the systems with the short-range exchange interactions, namely, the SR and LRF cases. On the basis of the argument on the stability of the analytic fixed point by Baczyk, Tarrus, Tissier and Balog, we perform the one-loop functional renormalization group analysis. We show that the analytic fixed point which governs the phase transition in the system with LRF can be destabilized by the perturbation with nonanalyticity. In Sec. III we study the systems with the long-range exchange interactions, namely, the LRE and LREF cases. We treat the one-loop beta functions, and carefully analyze the properties of the analytic fixed points and their stability. It is shown that the analytic fixed point controlling the critical behavior in the system with LRE becomes unstable against the perturbation with nonanalyticity for \( N < 2(4 + 3\sqrt{3}) \approx 18.3923 \cdots \), which is the same as the case of the system with SR. Also, we show that the destabilization of the analytic fixed point which governs the phase transition in the system with LREF can occur due to the nonanalytic perturbation. As a result, we obtain a certain region in the plane of the parameters
Thus, the critical exponent \( \eta \) evaluated at the analytic fixed point which controls the critical behavior in the system with LRE. Our results agree with those by Sak and Blöte that the effect of the long-range exchange interactions is relevant for \( \sigma < 2 - \eta_M \), where \( \eta_M \) denotes the critical exponent of the connected correlation function in the corresponding short-range system. And we show that the \( d \to d - \sigma \) dimensional reduction holds at the leading order of the \( d - 2\sigma \) expansion and for \( N > 2(4 + 3\sqrt{3}) \approx 18.3923 \cdots \). In Sec. VII we calculate the critical exponents \( \eta, \bar{\eta} \) and \( \nu \) evaluated at the analytic fixed point which controls the critical behavior in the system with LREF. We find that the value of the critical exponent \( \nu \) does not coincide with that predicted by the \( d \to d - \rho - \sigma \) dimensional reduction for finite \( N \). Thus, the \( d \to d - \rho - \sigma \) dimensional reduction does not hold within the present framework, as far as \( N \) is finite. Sec. VII summarizes our results.

II. CRITICAL PHENOMENA AT ZERO TEMPERATURE OF LONG-RANGE CORRELATED RANDOM FIELD \( O(N) \) SPIN MODEL WITH SHORT-RANGE EXCHANGE INTERACTIONS IN \( 4 + \epsilon \) DIMENSIONS

This section is intended as a reexamination of the critical phenomena at zero temperature of the long-range correlated random field \( O(N) \) spin model with short-range exchange interactions in \( 4 + \epsilon \) dimensions. We discuss the nature of analytic fixed points and their stability. And we calculate the critical exponents evaluated at the analytic fixed point which controls the critical behavior in the system with SR and with LRF.

A. Model

Let us consider an \( N \)-component vector spin system where an \( N \)-component vector spin \( S(x) \) with a fixed-length constraint \( S(x)^2 = 1 \) couples to a random field. In order to carry out the average over the random field, we use the replica method. The critical phenomena of the long-range correlated random field \( O(N) \) spin model with the short-range exchange interactions near lower critical dimension is described by the \( O(N) \) nonlinear-sigma model of the following replica partition function \( Z \) and effective action \( \beta H_{\text{rep}} \)

\[
Z = \int \prod_{\alpha=1}^n D S^\alpha \delta(S^\alpha(x)^2 - 1) e^{-\beta H_{\text{rep}}},
\]

\[
\beta H_{\text{rep}} = \frac{a^{2-d}}{2T} \int_x^n \sum_{\alpha=1}^n S^\alpha(x) \cdot (-\partial^2) S^\alpha(x)
- \frac{a^{-d}}{2T^2} \sum_{\alpha,\beta}^n R_1(S^\alpha(x) \cdot S^\beta(x))
- \frac{a^{-d-\rho}}{2T^2} \int_{x,x'} \sum_{\alpha,\beta}^n g(x-x') R_2(S^\alpha(x) \cdot S^\beta(x')),
\]

where \( a \) is the ultraviolet cutoff, and \( \int_x := \int d^d x \). The replica indices denoted by Greek indices take values \( \alpha, \beta, \ldots = 1, \ldots, n \). The first term in the action \( (1) \) is the kinetic term which corresponds to the short-range exchange interactions between spins. The parameter \( T \) is the dimensionless temperature. The function \( R_i(S^\alpha \cdot S^\beta) \) \( (i = 1, 2) \) represents the random field and all the random anisotropies, and is given by

\[
R_i(S^\alpha \cdot S^\beta) = \sum_{r=1}^{\infty} \Delta_{i,r}(S^\alpha \cdot S^\beta)^r.
\]

Here, \( \Delta_{i,r} \) denotes the strength of the random field and the \( r \)-th rank random anisotropy \( (r = 1 \) is the random field, and \( r = 2 \) is the random second-rank anisotropy). The subscript \( i = 1 \) corresponds to the uncorrelated random fields and random anisotropies, and the subscript \( i = 2 \) corresponds to the long-range correlated random fields and random anisotropies with \( g(x-x') \sim |x-x'|^{-d+\rho} \). The lower critical dimension of this model is \( d_l = 4 + \rho \). In the present study, we consider the case of \( 0 \leq \rho < \epsilon \).

B. One-loop beta functions and the zero-temperature fixed points

To perform the renormalization group transformation, we put each replicated vector spin \( S^\alpha(x) \) as a combination of a slow field \( n_0^\alpha(x) \) of the unit length and fast fields \( \varphi_i^\alpha(x) \), \( i = 1, \ldots, N - 1 \) such that

\[
S^\alpha(x) = n_0^\alpha(x) \sqrt{1 - \varphi^\alpha(x)} + \varphi^\alpha(x)
\sim n_0^\alpha(x) - \frac{1}{2} (\varphi^\alpha(x)^2) n_0^\alpha(x) + \varphi^\alpha(x),
\]

\[
\varphi^\alpha(x) = \sum_{i=1}^{N-1} \varphi_i^\alpha(x) e_i^\alpha(x),
\]

where the unit vectors \( e_i^\alpha(x) \) are perpendicular to each other and also to the vector \( n_0^\alpha(x) \). Integrating out the fast fields \( \varphi_i^\alpha(x) \), and calculating the new replicated action \( \beta H'_{\text{rep}} \) up to the second order of the perturbation expansion, we get the one-loop beta functions for \( T, R_1 \) and \( R_2 \), which have been obtained by Fedorenko and Kühnel. The one-loop beta function for the temperature \( T \) is

\[
\partial_T T = -(d - 2)T + (N - 2)T(T + R_1(1) + R_2(1)),
\]
where \( \partial_t \) denotes a derivative with respect to \( t = \log l \) with \( l \) being the length-scale parameter which increases toward the infrared direction. Here we have rescaled \( T, R_1 \) and \( R_2 \) by \( 2/(4\pi^{3/2}\Gamma(d/2)) \). We find that \( T = 0 \) is the fixed point, at which the parameter \( T \) is irrelevant for \( d > 2 \). The one-loop beta functions at \( T = 0 \) for \( R_1 \) and \( R_2 \) are

\[
\begin{aligned}
\partial_t R_1(z) & = -\epsilon R_1(z) + 2(N - 2)(R_1'(1) + R_2'(1))R_1(z) \\
& - (N - 1)z(R_1'(1) + R_2'(1))R_1'(z) \\
& + (1 - z^2)(R_1'(1) + R_2'(1))R_1''(z) \\
& + \frac{1}{2}(N - 2 + z^2)(R_1'(1) + R_2'(1))^2 \\
& - z(1 - z^2)(R_1'(1) + R_2'(1))(R_1''(z) + R_2''(z)) \\
& + \frac{1}{2}(1 - z^2)^2(R_1'(1) + R_2'(1))^2, \\
\partial_t R_2(\zeta) & = -\epsilon R_2(\zeta) + 2(N - 2)(R_1'(1) + R_2'(1))R_2(\zeta) \\
& - (N - 1)\zeta(R_1'(1) + R_2'(1))R_2'(\zeta) \\
& + (1 - \zeta^2)(R_1'(1) + R_2'(1))R_2''(\zeta) \\
& + \frac{1}{2}(1 - \zeta^2)^2(R_1'(1) + R_2'(1))^2,
\end{aligned}
\tag{6}
\]

where \( z = n_0^\alpha(x) \cdot n_0^\beta(x), \zeta = n_0^\alpha(x) \cdot n_0^\beta(x') \). Here we have put \( d = 4 + \epsilon \). Practically, the beta functions for the first and second derivatives of \( R_1 \) and \( R_2 \) play a central role in the critical phenomena at zero temperature near the lower critical dimension. The beta functions at \( T = 0 \) for \( R_1', R_1'', R_2' \) and \( R_2'' \) in \( d = 4 + \epsilon \) are

\[
\begin{aligned}
\partial_t R_1'(z) & = -\epsilon R_1'(z) + (N - 3)(R_1'(1) + R_2'(1))R_1'(z) \\
& - (N - 1)z(R_1'(1) + R_2'(1))R_1''(z) \\
& + (1 - z^2)(R_1'(1) + R_2'(1))R_1'''(z) \\
& + \frac{1}{2}(N - 4 + 4z^2)(R_1'(1) + R_2'(1))^2 \\
& - z(1 - z^2)(R_1'(1) + R_2'(1))(R_1''(z) + R_2''(z)) \\
& - 3z(1 - z^2)(R_1'(1) + R_2'(1))^2 \\
& + (1 - z^2)^2(R_1'(1) + R_2'(1))^2(R_1''(z) + R_2''(z)), \\
\partial_t R_1''(z) & = -\epsilon R_1''(z) - 4(R_1'(1) + R_2'(1))R_1'''(z) \\
& - (N + 3)z(R_1'(1) + R_2'(1))R_1''''(z) \\
& + (1 - z^2)(R_1'(1) + R_2'(1))^2R_1''''(z) \\
& + 10z(R_1'(1) + R_2'(1))^2(R_1''(z) + R_2''(z)) \\
& + (N - 4 + 7z^2)(R_1'(1) + R_2'(1))^2 \\
& - z(1 - z^2)(R_1'(1) + R_2'(1))(R_1''''(z) + R_2''''(z)) \\
& + (N - 6 + 13z^2)(R_1'(1) + R_2'(1))^2 \\
& - 11z(1 - z^2)(R_1'(1) + R_2'(1))(R_1''''(z) + R_2''''(z)) \\
& + (1 - z^2)^2(R_1'(1) + R_2'(1))(R_1''''(z) + R_2''''(z)) \\
& + (1 - z^2)^2(R_1'(1) + R_2'(1))^2,
\end{aligned}
\tag{8}
\]

The properties of the fixed point solution \( (R_1'(z)^*, R_2'(z)^*) \) are determined under the condition that \( |R_1'(z)| \) and \( |R_2'(\zeta)| \) remain finite during the renormalization group flows. We discuss the properties of the fixed point solution \( (R_1'(z)^*, R_2'(\zeta)^*) \). Eq. (10) is linear in the function \( R_2'(\zeta) \), which can be solved analytically. Solving the fixed point equation \( \partial_t R_2'(\zeta)^* = 0 \), we can find that the fixed point solution \( R_2'(\zeta)^* \) is analytic on \( \zeta \). Next, we assume that the functions \( R_1'(z) \) and \( R_2'(\zeta) \) take the following form:

\[
\begin{aligned}
R_1'(z) & = R_1'(1) - R_1''(1)(1 - z) + \cdots \\
& + a_1(1 - z)^{\alpha + \epsilon} + \cdots, \\
R_2'(\zeta) & = R_2'(1) - R_2''(1)(1 - \zeta) \\
& + \frac{R_2''''(1)}{2}(1 - \zeta)^2 + \cdots,
\end{aligned}
\tag{12}
\]

where \( \alpha > 0 \). To keep \( |R_1'(z)| \) and \( |R_2'(\zeta)| \) finite, the following condition on the function \( \zeta \) is required:

\[
\alpha = \frac{1}{2} \quad \text{or} \quad \alpha \geq 1.
\tag{14}
\]

Thus, the fixed point function \( R_1(z)^* \) also has the same behavior of \( (1 - z)^{\alpha} \) with \( \alpha^* = 1/2 \) or \( \alpha^* \geq 1 \). Only in the case of \( \alpha^* = 1/2 \), \( R_1^*(z) \) diverges. We use the term “cuspy” on a function with \((1 - z)^{1/2}\) and “cuspless” if the first and the second derivatives of a function are finite.

**C. Stability of fixed points and critical exponents** \( \eta, \bar{\eta} \) and \( \nu \)

The critical exponents \( \eta \) and \( \bar{\eta} \) of the connected and disconnected correlation functions are expressed by use of \( R_1'(1)^* \) and \( R_2'(1)^* \) which are the values of \( R_1'(1) \) and \( R_2'(1) \) at the fixed point:

\[
\begin{aligned}
\eta & = R_1'(1)^* + R_2'(1)^*, \\
\bar{\eta} & = (N - 1)(R_1'(1)^* + R_2'(1)^*) - \epsilon.
\end{aligned}
\tag{15}
\]

The critical exponent \( \nu \) of the correlation length is given by the inverse of the maximal eigenvalue of the scaling matrix at the fixed point. Then, we find the fixed points by solving \( \partial_t R_1'(1) = 0, \partial_t R_1'(1)^* = 0, \partial_t R_2'(1) = 0 \) and \( \partial_t R_2'(1)^* = 0 \), study their stability, and calculate the critical exponents \( \eta, \bar{\eta} \) and \( \nu \) in the following.
The fixed points are
\[(R'_1(1)^*, R'_2(1)^*, R'_{1+}(1)^*, R'_2(1)^* )
= \left( \frac{\epsilon}{N-2}, 0, \frac{\epsilon[N-8+\sqrt{(N-2)(N-18)}}{2(N+7)(N-2)}, 0 \right), \] (17)

\[(R'_1(1)^*, R'_2(1)^*, R'_{1+}(1)^*, R'_2(1)^*)
= \left( \frac{\epsilon}{N-2}, 0, \frac{\epsilon[N-8-\sqrt{(N-2)(N-18)}}{2(N+7)(N-2)}, 0 \right), \] (18)

Here, we have introduced the reduced variable \(\hat{\epsilon}\):
\[\hat{\epsilon} = \frac{\epsilon}{\epsilon - \rho}. \] (21)

The stability of the cuspless fixed points with respect to the cuspless perturbation can be investigated by calculating eigenvalues of the 4 \(\times\) 4 scaling matrix whose elements are the first derivatives of the beta functions \(\partial_{\lambda} R'_1(1), \partial_{\lambda} R'_2(1), \partial_{\lambda} R'_{1+}(1), \partial_{\lambda} R'_2(1)\) at the cuspless fixed points.

The cuspless fixed points (17) and (18) exist for \(N \geq 18\). The eigenvalues \(\lambda_1, \ldots, \lambda_4\) of the scaling matrix at the cuspless fixed points (17) and (18) are given by
\[
\lambda_1 = \epsilon, \] (22)
\[
\lambda_2 = (\epsilon - \rho) \frac{(N-3)\hat{\epsilon} - (N-2)}{N-2}, \] (23)
\[
\lambda_3^\pm = \pm \sqrt{N-18 \over N-2}, \] (24)
\[
\lambda_4 = - (\epsilon - \rho + \frac{4\epsilon}{N-2}). \] (25)

Thus, the cuspless fixed point (17) is multiply unstable. If \(1 < \hat{\epsilon} < (N-2)/(N-3)\), namely \(\lambda_2 < 0\), the cuspless fixed point (18) is singly unstable. Due to \(R'_2(1)^* = 0\), the long-range correlations of random fields and random anisotropies are irrelevant, and thus the cuspless fixed point (18) governs the phase transition in the system with SR. The critical exponents \(\eta_{SR}\) of the connected correlation function and \(\tilde{\eta}_{SR}\) of the disconnected correlation function at the cuspless fixed point (18) are
\[
\eta_{SR} = \frac{\epsilon}{N-2}, \] (26)
\[
\tilde{\eta}_{SR} = \frac{\epsilon}{N-2}. \] (27)

And the critical exponent \(\nu_{SR}\) which characterizes the divergence of the correlation length in the vicinity of transition is
\[
\nu_{SR} = \frac{1}{\epsilon}. \] (28)

Whereas, in \(N \leq 18\), the cuspless fixed points (17) and (18) merge and annihilate, and thus the beta functions have no cuspless fixed point of \(O(\epsilon)\).

The cuspless fixed points (19) and (20) exist for \(N > 3\) and
\[
\hat{\epsilon} \geq \hat{\epsilon}_{\text{cuspless}} = \frac{6 + 2\sqrt{N+7}}{N-3}. \] (29)

The eigenvalues \(\lambda_1, \ldots, \lambda_4\) of the scaling matrix at the
fixed points (19) and (20) are given by

\[
\begin{align*}
\lambda_1 &= (\epsilon - \rho) \frac{N - 2}{N - 3} - \frac{\epsilon}{2} + \frac{\epsilon}{2} \sqrt{1 + \frac{4[N - 2 - \epsilon(N - 3)]}{\epsilon^2(N - 3)^2}}, \\
\lambda_2 &= (\epsilon - \rho) \frac{N - 2}{N - 3} - \frac{\epsilon}{2} - \frac{\epsilon}{2} \sqrt{1 + \frac{4[N - 2 - \epsilon(N - 3)]}{\epsilon^2(N - 3)^2}}, \\
\lambda_3^+ &= \pm \epsilon \sqrt{\left\{ (N - 3)\epsilon - 6 \right\}^2 - 4(N + 7)/(N - 3)}, \\
\lambda_4 &= -(\epsilon - \rho) \frac{N + 1}{N - 3}.
\end{align*}
\]

(30)

(31)

(32)

(33)

Thus, the cuspless fixed point (19) is multiply unstable. If \( \epsilon > (N - 2)/(N - 3) \), namely \( \lambda_3 < 0 \), the cuspless fixed point (20) is singly unstable. Due to \( R_2^*(1)^* \), the effect of the long-range correlation of random fields and random anisotropies appears, and then the cuspless fixed point (20) governs the phase transition in the system with LRF. Thus, the critical exponents \( \eta_{\text{LRF}} \) and \( \bar{\eta}_{\text{LRF}} \) at the cuspless fixed point (20) are

\[
\eta_{\text{LRF}} = \frac{\epsilon - \rho}{N - 3},
\]

(34)

\[
\bar{\eta}_{\text{LRF}} = \frac{2\epsilon - (N - 1)\rho}{N - 3}.
\]

(35)

These exponents satisfy the Schwartz-Soffer inequality \( \bar{\eta}_{\text{LRF}} \leq 2\eta_{\text{LRF}} \) and saturate the generalized Schwartz-Soffer inequality \( \bar{\eta}_{\text{LRF}} \leq 2\eta_{\text{LRF}} - \rho \). And the inverse of the exponent \( \nu_{\text{LRF}} \) is

\[
\nu_{\text{LRF}}^{-1} = \frac{(N - 2)\epsilon - \rho}{N - 3} \left[ 1 - \frac{(N - 3)\epsilon}{2(N - 2)} \right] + \frac{(N - 3)\epsilon}{2(N - 2)} \sqrt{1 + \frac{4[N - 2 - \epsilon(N - 3)]}{\epsilon^2(N - 3)^2}}.
\]

(36)

Whereas, in \( \epsilon \leq (6 + 2\sqrt{N + 7})/(N - 3) \), the cuspless fixed points (19) and (20) merge and annihilate, and thus the beta functions have no cuspless fixed point of \( O(\epsilon) \).

As Tissier and Tarjus (TT) and co-workers argued in Refs.\(^{11,14,17}\), the cuspless fixed points (18) and (20) have weaker nonanalyticities \( (1 - z)^\alpha^* \) with a noninteger \( \alpha^* \geq 1 \). The weaker nonanalyticity is called “sub-cusp”. We refer to the cuspless fixed points (18) and (20) as “SR TT FP” and “LRF TT FP”, respectively. The weaker nonanalyticity does not alter the flow equations for \( R_1^*(1) \) and \( R_2^*(1) \). The power \( \alpha^* \) is obtained as follows. Calculating the flow of \( a_l \) in Eq. (12), we have

\[
\partial_t a_l = a_l A_{\alpha + 1}(R_1^*(1)^*, R_2^*(1)^*, R_3^*(1)^*, R_2^*(1)^*),
\]

\[
A_{\alpha + 1}(R_1^*(1)^*, R_2^*(1)^*, R_3^*(1)^*, R_2^*(1)^*) = 2[R_1^*(1)^* + R_2^*(1)^* + 3(R_1^*(1)^* + R_2^*(1)^*)]\alpha^2 - [(N - 5)(R_1^*(1)^* + R_2^*(1)^*)]
\]

\[
-(N + 7)(R_1^*(1)^* + R_2^*(1)^*)]\alpha
\]

\[
+(N + 1)(R_1^*(1)^* + R_2^*(1)^* + R_1^*(1)^* + R_2^*(1)^*) - \epsilon.
\]

(37)

(38)

The power \( \alpha^* \) is determined from

\[
A_{\alpha^* + 1}(R_1^*(1)^*, R_2^*(1)^*, R_3^*(1)^*, R_2^*(1)^*) = 0.
\]

(39)

Substituting the SR TT FP (18) and the LRF TT FP (20) into the above equation, we have explicit expressions for \( \alpha^* \), respectively. Here, we treat only the LRF case (see Ref.\(^{14}\) for the SR case). From Eqs. (38) and (39), we obtain the following quadratic equation for \( \alpha^* \):

\[
\left( 2 + \frac{3}{N + 7} \right) ((N - 3)\epsilon - 6)
\]

\[
-\sqrt{\{(N - 3)\epsilon - 6 \}^2 - 4(N + 7)} \right) \alpha^*^2
\]

\[
-(N - 5) + \frac{1}{2} \left( (N - 3)\epsilon - 6 \right).
\]

\[
-\sqrt{\{(N - 3)\epsilon - 6 \}^2 - 4(N + 7)} \right) \alpha^*
\]

\[
+\frac{N + 1}{2(N + 7)} \left( (N - 3)\epsilon - 6 \right)
\]

\[
-\sqrt{\{(N - 3)\epsilon - 6 \}^2 - 4(N + 7)} = 0.
\]

(40)

Solving the above quadratic equation, we obtain the solution \( \alpha^* \) as a function of \( N \) and \( \epsilon \). It goes to \( N/2 + O(1) \) at large \( N \). The graphs of \( \alpha^* \) as \( \epsilon \) are depicted in Fig. 11.

We proceed to investigate the stability of the cuspless fixed points with respect to the cuspy perturbation, following the work by Baczky, Tarjus, Tissier and Balog.\(^{15}\) The eigenvalue \( \lambda \) relating to the cuspy deformation from the cuspless fixed points is

\[
\lambda = \Lambda_{3/2}(R_1^*(1)^*, R_2^*(1)^*, R_3^*(1)^*, R_2^*(1)^*)
\]

(41)

Substituting the SR TT FP (18) and the LRF TT FP (20) into the above equation, we have explicit expressions for \( \lambda \), respectively. The eigenvalues \( \lambda_{\text{SR}} \) for the system
with SR and $\lambda_{\text{LRF}}$ for the system with LRF as follows:

$$
\lambda_{\text{SR}} = -\frac{\epsilon}{4(N+7)} \left[3(N+4)\sqrt{\frac{N-18}{N-2}} - N + 8\right],
$$

$$
\lambda_{\text{LRF}} = -\frac{\epsilon - \rho}{4(N-3)(N+7)} \times \left[3(N+4)\sqrt{\{(N-3)\epsilon - 6\}^2 - 4(N+7)} + (N+16)(N-3)\epsilon - 6\right] - 2(N+7)(N-8).
$$

In the case of the system with SR, we find that, below $N = N_{\text{cusp}} = 2(4 + 3\sqrt{3}) \approx 18.3923\ldots$, the eigenvalue $\lambda_{\text{SR}}$ takes a positive value. Thus, the cuspy perturbation becomes relevant, which the SR TT FP [18] is multiply unstable with respect to the cuspy perturbation for $N < N_{\text{cusp}}$. Whereas it remains singly unstable with respect to the perturbation with and without the cuspy behavior for $N > N_{\text{cusp}}$. As shown in Ref.17, there exists a single unstable cuspy fixed point below $N = N_{\text{cusp}}$. As $N$ decreases from sufficiently large $N$, the fixed point which governs the phase transition in the system continuously changes from the SR TT FP to the singly unstable cuspy fixed point at $N = N_{\text{cusp}}$ before $N$ reaches $N = 18$. Accordingly, the values of the critical exponents $\gamma_{\text{SR}}$ and $\gamma_{\text{LRF}}$ deviate from the dimensional-reduction results [20] and [27] below $N = N_{\text{cusp}}$.

In the case of the system with LRF, the eigenvalue $\lambda_{\text{LRF}}$ takes a positive value below

$$
\hat{\epsilon} = \epsilon_{\text{cusp}} = \frac{3(N+4)\sqrt{N^2 - 8N + 48} - (N+4)(N-20)}{4(N-2)(N-3)}.
$$

Since $\epsilon_{\text{cusp}} > \epsilon_{\text{cuspless}}$ for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$, the LRF TT FP [20] is destabilized by the cuspy perturbation for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$ and $\hat{\epsilon} < \epsilon_{\text{cusp}}$. Even in this case, a singly unstable cuspy fixed point which governs the phase transition in the system is considered to exist for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$ and $\hat{\epsilon} < \epsilon_{\text{cusp}}$.

Finally, we calculate the eigenfunction which belongs to the eigenvalue [41]. Solving the eigenvalue equation, we obtain two solutions. One takes the form of $(1 - z)^{n-(\hat{\alpha})}$ with $\alpha_-(\hat{\lambda}) = 1/2$ when $z \to 1$, and the other takes the form of $(1 - z)^{n-\hat{\alpha}}$. Both solutions individually diverge in $z = -1$. The physical eigenfunction is represented as a linear combination of two solutions of the eigenvalue equation, in which the coefficients should be chosen to eliminate the singularities at $z = -1$. The power $\alpha_+(\lambda)$ of the function $(1 - z)^{n-\hat{\alpha}}$ can be obtained by imposing

$$
\Lambda_{\alpha_+ + 1}(R_1^1(\lambda), R_2^1(\lambda), R_3^1(\lambda), R_4^1(\lambda)) = \Lambda_{3/2}(R_1^1(\lambda), R_2^1(\lambda), R_3^1(\lambda), R_4^1(\lambda)).
$$

In the case of the system with SR, substituting the SR TT FP [18] into Eq. (45), we have

$$
\alpha_+(\lambda_{\text{SR}}) = \frac{1}{4}(N - 10 + \sqrt{(N - 2)(N - 18)}).
$$

For $N \geq 18$, $\alpha_+(\lambda_{\text{SR}})$ takes $\alpha_+(\lambda_{\text{SR}}) \geq 2$.

In the case of the system with LRF, substituting the LRF TT FP [20] into Eq. (45), we have

$$
\alpha_+(\lambda_{\text{LRF}}) = \frac{(N - 14)(N - 3)\epsilon + (N - 2)(N - 6)}{2[3(N-3)\epsilon + N - 2] + \sqrt{[(N-3)\epsilon - 6]^2 - 4(N+7)}}.
$$

The power $\alpha_+(\lambda_{\text{LRF}})$ takes $\alpha_+(\lambda_{\text{LRF}}) \geq 1 + \sqrt{3}$ for $N \geq \frac{N_{\text{cusp}}}{\epsilon_{\text{cusp}}}$ and $\hat{\epsilon} \geq \frac{N_{\text{cusp}} - 2}{N_{\text{cusp}} - 3}$, and $\alpha_+(\lambda_{\text{LRF}}) \geq 1$ for $4(1 + \sqrt{7}) \leq N < \frac{N_{\text{cusp}}}{\epsilon_{\text{cusp}}}$ and $\hat{\epsilon} \geq \epsilon_{\text{cuspless}}$. However, we should note that, for $N < 4(1 + \sqrt{7})$ and in the region of

$$
\epsilon_{\text{cuspless}} \leq \hat{\epsilon} < \frac{(N - 2)(N^2 + 32)}{(N - 3)(N - 8)(N + 16)},
$$

$\alpha_+(\lambda_{\text{LRF}}) < 1$, which is in contradiction with the condition [14]. Thus, for $N < 4(1 + \sqrt{7})$ and in the region [45], the cuspy deformation from the LRF TT FP [20] is unphysical. Then, the destabilization of the LRF TT FP [20] by the cuspy perturbation does not occur for $\hat{\epsilon} > 2(3 + 19\sqrt{7})/111$.

The regions where the various fixed points are singly unstable are depicted in Fig.2. Outside the areas where the SR TT and the LRF TT FPs are singly unstable, the cuspy fixed point is considered to control the critical behavior in the system. Particularly, in the region of $1 \leq \hat{\epsilon} < 2(3 + 19\sqrt{7})/111 \approx 1.32$, the destabilization of the SR TT and the LRF TT FPs by the cuspy perturbation is caused at $N_{\text{cusp}}$ for the SR TT FP, and at $\hat{\epsilon}_{\text{cusp}}$ for the LRF TT FP, respectively.
We start from the $O(N)$ nonlinear-sigma model with the replica effective action
\[ \beta H_{\text{rep}} = \frac{a^{\sigma-d}}{2T} \int_x \sum_{\alpha=1}^N S^\alpha(x) \cdot (-\partial^2)^{\sigma/2} S^\alpha(x) \]
\[ - \frac{a^{-d-\rho}}{2T^2} \int_x \sum_{\alpha,\beta} R_1(S^\alpha(x) \cdot S^\beta(x)) \]
\[ - \frac{a^{-d-\rho}}{2T^2} \int_{x,x'} \sum_{\alpha,\beta} g(x-x')R_2(S^\alpha(x) \cdot S^\beta(x')). \]

The first term in the action (49) is the kinetic term which corresponds to the long-range exchange interactions between spins. The operator $(-\partial^2)^{\sigma/2}$ denotes the fractional Laplacian in the Euclidean space. In the present study, we consider the case of $\sigma < 2$. The parameter $T$ denotes the dimensionless temperature. The function $R_i(z)$ $(i=1,2)$ represents the random field and all the random anisotropies, which is defined by Eq.(2). The lower critical dimension of this model is $d_l = 2\sigma + \rho$.

### B. One-loop beta functions

To carry out the renormalization group transformation, it is convenient to use the momentum representation. The fractional Laplacian $(-\partial^2)^{\sigma/2}$ is written by its Fourier transformation:
\[ (-\partial^2)^{\sigma/2} f(x) = \int_k k^\sigma \tilde{f}(k) e^{ikx}, \]  
(50)
where $kx = k^{(1)}x^{(1)} + \cdots + k^{(d)}x^{(d)}$, $k^\sigma = (k^{(1)})^2 + \cdots + k^{(d)^2}^{\sigma/2}$, and $\int_k \equiv \int \frac{d^dk}{(2\pi)^d}$. The correlation of the random fields $g(x-x')$ is written as
\[ g(x-x') \sim \frac{1}{|x-x'|^{d-\rho}} = \int_k k^{-\rho} e^{ik(x-x')}, \]  
(51)
in the momentum representation. The $N$-component replicated vector spin $S^\alpha(x)$ of the magnetization (3) is rewritten in the momentum representation as follows:

\[ S^\alpha(x) \simeq n^\alpha_0(x) - \frac{1}{2} (\varphi^\alpha(x)^2) n^\alpha_0(x) + \varphi^\alpha(x) \]
\[ = \int_k \tilde{n}^\alpha_0(k) e^{ikx} - \frac{1}{2} \sum_{k_1,\cdots,k_4} \left( \sum_{i,j} \varphi^\alpha_i(k_1) \bar{\varphi}^\alpha_j(k_2) \varphi^\alpha_i(k_3) \bar{\varphi}^\alpha_j(k_4) \cdot \tilde{e}^\alpha_i(k_4) \right) \tilde{n}^\alpha_0(k) e^{i(k_1+\cdots+k_4+k)x} \]
\[ + \int_{k_1,k_2} \left( \sum_{i=1}^{N-1} \varphi^\alpha_i(k_1) \tilde{e}^\alpha_i(k_2) \right) e^{i(k_1+k_2)x}. \]  
(52)
We integrate out the fast fields $\varphi_n^c(k)$, and calculate the new replicated action $\beta H^\text{rep}_T$ up to the second order of the perturbation expansion. After rewriting $\beta H^\text{rep}$ in the coordinate representation again, we can then obtain the one-loop beta functions for $T$, $R_1$ and $R_2$. The one-loop beta function for the temperature $T$ is

$$
\partial_t T = -(d - \sigma)T + (N - 1)T(T + R_1'(1) + R_2'(1)).
$$

(53)

Here we have rescaled $T$, $R_1$ and $R_2$ by $2/(4\pi)^{d/2} \Gamma(d/2)$. For $d > \sigma$, we find that $T = 0$ is the fixed point, at which the parameter $T$ is irrelevant.

The one-loop beta functions at $T = 0$ for $R_1$ and $R_2$ are

$$
\partial_t R_1(z) = -\varepsilon R_1(1) + 2(N - 1)(R_1'(1) + R_2'(1))R_1(z) - (N - 1)z(R_1'(1) + R_2'(1))R_1(z) + (1 - z^2)(R_1'(1) + R_2'(1))R_1''(z)
$$

$$
\partial_t R_2(z) = -\varepsilon R_2(1) + 2(N - 1)(R_1'(1) + R_2'(1))R_2(z) - (N - 1)z(R_1'(1) + R_2'(1))R_2(z) + (1 - z^2)(R_1'(1) + R_2'(1))R_2''(z).
$$

(54)

(55)

Here we have put $d = 2\sigma + \rho + \epsilon$. To study the fixed points and their stability, we consider the beta functions for their derivative. Differentiating Eqs. (54) and (55) with respect to $z$ and $\zeta$ respectively, we obtain the one-loop beta functions for $R_1'(z)$, $R_1''(z)$, $R_2'(\zeta)$ and $R_2''(\zeta)$:

$$
\partial_t R_1'(z) = -\varepsilon R_1'(1) + (N - 1)(R_1'(1) + R_2'(1))R_1'(z) - (N + 1)z(R_1'(1) + R_2'(1))R_1''(z) + (1 - z^2)(R_1'(1) + R_2'(1))R_1'''(z)
$$

$$
\partial_t R_1''(z) = -\varepsilon R_1''(1) - 2(R_1'(1) + R_2'(1))R_1'''(z) - (N + 1)z(R_1'(1) + R_2'(1))R_1''''(z) + (1 - z^2)(R_1'(1) + R_2'(1))R_1''''''(z)
$$

$$
\partial_t R_2'(\zeta) = -\varepsilon R_2'(1) - 2(R_1'(1) + R_2'(1))R_2'(\zeta) - (N + 1)z(R_1'(1) + R_2'(1))R_2''(\zeta) + (1 - \zeta^2)(R_1'(1) + R_2'(1))R_2'''(\zeta).
$$

(56)

We discuss the properties of the fixed point solution $(R_1'(z)^*, R_2'(\zeta)^*)$. First, we investigate the fixed point solution $R_2'(\zeta)^*$ for Eq. (53). Since Eq. (53) is linear in the function $R_2'(\zeta)$, the fixed point equation $\partial_t R_2'(\zeta)^* = 0$ can be solved analytically. The fixed point equation $\partial_t R_2'(\zeta)^* = 0$ takes the form

$$
(1 - \zeta^2)(R_1'(1)^* + R_2'(1)^*)R_2''(\zeta)^* - (N + 1)\zeta(R_1'(1)^* + R_2'(1)^*)R_2'''(\zeta)^* + [(N - 1)(R_1'(1)^*) + R_2'(1)^*) - \varepsilon R_2'(1)^* = 0.
$$

(59)

The solutions of this equation have regular singular points at $\zeta = \pm 1$ for the interval $-1 \leq \zeta \leq 1$. Under the condition of $|R_2'(\zeta)^*| < \infty$ on the interval $-1 \leq \zeta \leq 1$, the solutions of Eq. (60) can be expressed in terms of the Gaussian hypergeometric function:

$$
R_2'(\zeta)^* = \begin{cases} 
C_2 F_1(x_1, x_2; y; 1 - \zeta)/2 & \text{around } \zeta = 1 \\
C'_2 F_1(x_1, x_2; y; 1 + \zeta)/2 & \text{around } \zeta = -1
\end{cases}
$$

(61)

where $C$ and $C'$ are constants fulfilling the condition $|R_2'(\zeta)^*| < \infty$. Here, the generalized hypergeometric function is defined by the following series expansion:

$$
m F_n(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n; z) = \sum_{k=0}^{\infty} \frac{(x_1)_k (x_2)_k \cdots (x_m)_k z^k}{(y_1)_k (y_2)_k \cdots (y_n)_k} k!
$$

(62)

$$
(x)_k = \Gamma(x + k)/\Gamma(x).
$$

(63)
And, $x_1$, $x_2$ and $y$ are

$$x_1, x_2 = \frac{1}{2} \left[ N \pm \sqrt{N^2 + 4 \left\{ N - 1 - \frac{\epsilon}{R_1'(1)^* + R_2'(1)^*} \right\}} \right],$$

$$y = \frac{N + 1}{2}. \quad (64)$$

Thus, the fixed point solution $R_2'(\zeta)^*$ is an analytic function on $\zeta$. Next, we examine the renormalization group flow of $R_1'(z)$. We assume that the functions $R_1'(z)$ and $R_2'(\zeta)$ take the forms given by Eqs. (12) and (13) with $\alpha > 0$. To keep $|R_1'(1)|$ and $|R_2'(1)|$ finite, the following condition on the function (12) is required;

$$\alpha = \frac{1}{2} \quad \text{or} \quad \alpha \geq 1. \quad (66)$$

The fixed point solution $R_1'(z)^*$ also has the same singularity. Only in the case of $\alpha = 1/2$, $R''(1)^*$ diverges.

C. Stability of cuspless fixed points

The critical exponents $\eta$ and $\overline{\eta}$ are expressed by use of $R_1'(1)^*$ and $R_2'(1)^*$:

$$\eta = 2 - \sigma,$$

$$\overline{\eta} = (N - 1)(R_1'(1)^* + R_2'(1)^*) - (2\sigma + \rho + \epsilon - 4). \quad (67)$$

The critical exponent $\nu$ is determined from the inverse of the maximum eigenvalue of the $4 \times 4$ scaling matrix at the fixed point. Then, we find the fixed points by solving $\partial_t R_1'(1)^* = 0$, $\partial_t R_2'(1)^* = 0$, $\partial_t R_2'(2)^* = 0$ and $\partial_t R_2'^*(1)^* = 0$, and study their stability.

The cuspless fixed points are

$$(R_1'(1)^*, \overline{R_2'(1)^*}, R_1'^*(2)^*, \overline{R_2'^*(2)^*})$$

$$= \left( \frac{\epsilon + \rho}{N}, 0, \frac{\epsilon + \rho + \epsilon - 2\sqrt{(N - 2)(N - 18)} + 8}{2N(N + 7)}, 0 \right). \quad (69)$$

$$(R_1'^*(1)^*, R_2'^*(1)^*, R_1'^*(2)^*, \overline{R_2'^*(2)^*})$$

$$= \left( \frac{\epsilon + \rho}{N}, 0, \frac{\epsilon + \rho + \epsilon - 2\sqrt{(N - 2)(N - 18)} - 8}{2N(N + 7)}, 0 \right). \quad (70)$$

$$(R_1'(1)^*, \overline{R_2'(1)^*}, R_1'^*(2)^*, \overline{R_2'^*(2)^*})$$

$$= \left( \frac{\epsilon^2}{(N - 1)^2 \rho}, \frac{\epsilon^2[(N - 1)\tilde{\epsilon} - N]}{(N - 1)^2 \rho}, \frac{\epsilon[(N - 1)\tilde{\epsilon} - 8 + \sqrt{([N - 1]\tilde{\epsilon}^2 - 4)N^2 + 7}]}{2(N + 7)(N - 1)}, 0 \right). \quad (71)$$

Here, we have introduced the reduced variable $\tilde{\epsilon}$:

$$\tilde{\epsilon} = \frac{\epsilon + \rho}{\epsilon}. \quad (73)$$

The stability of the cuspless fixed points with respect to the cuspless perturbation can be investigated by calculating eigenvalues of the $4 \times 4$ scaling matrix whose elements are the first derivatives of the beta functions $\partial_t R_1'(1)$, $\partial_t R_1''(1)$, $\partial_t R_2'(1)$ and $\partial_t R_2''(1)$ at the cuspless fixed points.

The cuspless fixed points (69) and (70) exist for $N \geq 18$. The eigenvalues $\lambda_1, \ldots, \lambda_4$ of the scaling matrix at the cuspless fixed points (69) and (70) are given by

$$\lambda_1 = \epsilon + \rho,$$

$$\lambda_2 = -\epsilon \left( 1 - \frac{N - 1}{N} \tilde{\epsilon} \right),$$

$$\lambda_3^\pm = \pm (\epsilon + \rho) \sqrt{\frac{N - 2(N - 18)}{N}},$$

$$\lambda_4 = -\epsilon \left( 1 + \frac{2}{N} \tilde{\epsilon} \right). \quad (74)$$

Thus, the fixed point (69) is multiply unstable. If $1 \leq \tilde{\epsilon} < N/(N - 1)$ or $0 \leq \rho < \epsilon/(N - 1)$, the fixed point (70) is simply unstable. Due to $R_2'(1)^* = 0$, the long-range correlations of random fields and random anisotropies are irrelevant, and thus the fixed point (70) governs the phase transition in the system with LRE. Whereas, in $N \leq 18$, the cuspless fixed points (69) and (70) merge and annihilate, and thus the beta functions have no cuspless fixed point of $O(\epsilon)$.

The cuspless fixed points (71) and (72) exist for $N > 1$ and

$$\tilde{\epsilon} \geq \tilde{\epsilon}_{\text{cuspless}} = \frac{8 + 2\sqrt{N + 7}}{N - 1}. \quad (78)$$

The eigenvalues $\lambda_1, \ldots, \lambda_4$ of the scaling matrix at the cuspless fixed points (71) and (72) are given by

$$\lambda_1 = \epsilon \left[ \frac{N}{N - 1} - \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2} \sqrt{\frac{1 + 4[N - \tilde{\epsilon}(N - 1)]}{\tilde{\epsilon}^2(N - 1)^2}} \right],$$

$$\lambda_2 = \epsilon \left[ \frac{N}{N - 1} - \frac{\tilde{\epsilon}}{2} - \frac{\tilde{\epsilon}}{2} \sqrt{\frac{1 + 4[N - \tilde{\epsilon}(N - 1)]}{\tilde{\epsilon}^2(N - 1)^2}} \right],$$

$$\lambda_3^\pm = \pm (\epsilon + \rho) \sqrt{\frac{1 - 4[N - 9 + 4\tilde{\epsilon}(N - 1)]}{\tilde{\epsilon}^2(N - 1)^2}},$$

$$\lambda_4 = -\epsilon \left( N + 1 \frac{N}{N - 1} \tilde{\epsilon} \right). \quad (80)$$
Thus, the fixed point \((71)\) is multiply unstable. If \(\varepsilon > N/(N-1)\) or \(\rho > \varepsilon/(N-1)\), the fixed point \((72)\) is singly unstable. Due to \(R^*_a(1) > 0\), the effect of the long-range correlation of random fields and random anisotropies appears, and then the fixed point \((72)\) governs the phase transition in system with LREF.

The cuspless fixed points \((70)\) and \((72)\) have subcuspy singularities \((1 - z)\alpha^*\) with a noninteger \(\alpha^* \geq 1\). Then, we call the singly unstable fixed points \((70)\) and \((72)\) as the “LRE TT FP” and the “LREF TT FP” respectively. The power \(\alpha^*\) is obtained as follows. Calculating the flow of \(a_t\) in Eq. \((12)\), we have

\[
\frac{\partial a_t}{\partial t} = a_0\Lambda_{\alpha+1}(R^*_1(1)^*, R^*_2(1)^*, R^*_1(1)^*, R^*_2(1)^*), \quad (83)
\]

\[
\Lambda_{\alpha+1}(R^*_1(1)^*, R^*_2(1)^*, R^*_1(1)^*, R^*_2(1)^*) = 2[R^*_1(1)^* + R^*_2(1)^* + 3(R^*_1(1)^* + R^*_2(1)^*)]\alpha^2
\]

\[
-[(N-5)(R^*_1(1)^* + R^*_2(1)^*)] \alpha^2
\]

\[
-(N+7)(R^*_1(1)^* + R^*_2(1)^*)] \alpha^2
\]

\[
+(N+1)(R^*_1(1)^* + R^*_2(1)^* + R^*_1(1)^* + R^*_2(1)^*)
\]

\[
-(\varepsilon + \rho).
\]

The power \(\alpha^*\) is determined from

\[
\Lambda_{\alpha^*+1}(R^*_1(1)^*, R^*_2(1)^*, R^*_1(1)^*, R^*_2(1)^*) = 0. \quad (85)
\]

Substituting the LRE TT FP \((70)\) and the LREF TT FP \((72)\) into the above equation, we have explicit expressions for \(\alpha^*\) respectively. Firstly, substituting the LRE TT FP \((70)\) into Eq. \((85)\), we obtain the following quadratic equation for \(\alpha^*\):

\[
\left(2 + \frac{3}{N+7}[N - 8 - \sqrt{(N-2)(N-18)}] \right) \alpha^*^2
\]

\[
-\left(N - 5 - \frac{1}{2} \left[ N - 8 - \sqrt{(N-2)(N-18)} \right] \right) \alpha^*
\]

\[
+1 + \frac{N+1}{2(N+7)} [N - 8 - \sqrt{(N-2)(N-18)}] = 0. \quad (86)
\]

Solving the above equation, we have the solution \(\alpha^*\) as a function of \(N\). It goes to \(N/2 + O(1)\) at large \(N\). The solution is the same as that of the system with SR. The graph of \(\alpha^* = \alpha^*(N)\) is depicted in Fig.\(3\) and \(4\).

Next, substituting the LREF TT FP \((72)\) into Eq. \((85)\), we obtain the following quadratic equation for \(\alpha^*\):

\[
\left(2 + \frac{3}{N+7}[(N-1) \varepsilon - 8] - \sqrt{\{(N-1) \varepsilon - 8\}^2 - 4(N+7)} \right) \alpha^*^2
\]

\[
-\left(N - 5 + \frac{1}{2} [(N-1) \varepsilon - 8] \right) \alpha^*
\]

\[
-\left[(N-1) \varepsilon - 8\right] \sqrt{\{(N-1) \varepsilon - 8\}^2 - 4(N+7)} \right) \alpha^*
\]

\[
+N + 1 - (N-1) \varepsilon + \frac{N+1}{2(N+7)} [(N-1) \varepsilon - 8
\]

\[
-\sqrt{\{(N-1) \varepsilon - 8\}^2 - 4(N+7)}] = 0. \quad (87)
\]

Solving the above equation, we obtain the solution \(\alpha^*\) as a function of \(N\) and \(\varepsilon\). It goes to \(N/2 + O(1)\) at large \(N\). The graphs of \(\alpha^* = \alpha(N, \varepsilon)\) for some values of \(\varepsilon\) are depicted in Fig.\(4\).

We investigate the stability of the TT fixed points with respect to the cuspy perturbation can be investigated by calculating the eigenvalue \(\lambda\) relating to the cuspy deformation from the singly unstable fixed point, which is given by

\[
\lambda = \Lambda_{3/2}(R^*_1(1)^*, R^*_2(1)^*, R^*_1(1)^*, R^*_2(1)^*). \quad (88)
\]

Substituting the LRE TT FP \((70)\) and the LREF TT FP \((72)\) into the above equation, we obtain explicit expressions for \(\lambda\) respectively. The eigenvalues \(\lambda_{\text{LRE}}\) for the system with LRE and \(\lambda_{\text{LREF}}\) for the system with LREF...
are as follows:

\[
\lambda_{\text{LRE}} = -\frac{(\epsilon + \rho)(N - 2)}{4N(N + 2)} \times \left[3(N + 4)\sqrt{\frac{N - 18}{N - 2}} - \frac{N - 8}{N - 2} - N + 8\right].
\]  

(89)

\[
\lambda_{\text{LREF}} = -\frac{1}{4(N - 1)(N + 7)} \times \left[3(N + 4)\sqrt{[(N - 1)\hat{\epsilon} - 8]^2 - 4(N + 7)} - 2(N + 7)(N - 8) + (N + 16)[(N - 1)\hat{\epsilon} - 8]\right].
\]  

(90)

In the case of the system with LRE, we find that, below \(N = N_{\text{cusp}} = 2(4 + 3\sqrt{3}) \approx 18.3923 \ldots\), the eigenvalue \(\lambda_{\text{LRE}}\) takes a positive value. Thus, the cuspy perturbation becomes relevant, which the LRE TT FP (70) is multiply unstable with respect to the cuspy perturbation for \(N < N_{\text{cusp}}\). Whereas it remains singly stable with respect to the perturbation with and without the cuspy behavior for \(N > N_{\text{cusp}}\).

In the case of the system with LREF, the eigenvalue \(\lambda_{\text{LREF}}\) takes a positive value below

\[
\hat{\epsilon} = \hat{\epsilon}_{\text{cusp}} = \frac{3(N + 4)\sqrt{N^2 - 8N + 48} - (N^2 - 24N - 64)}{4(N - 1)(N - 2)}.
\]  

(91)

Since \(\hat{\epsilon}_{\text{cusp}} \geq \hat{\epsilon}_{\text{cuspless}}\) for \(4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})\), the LREF TT FP (72) is destabilized by the cuspy perturbation for \(4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})\) and \(\hat{\epsilon} < \hat{\epsilon}_{\text{cusp}}\). Even in this case, a singly unstable cuspy fixed point which governs the phase transition in the system is considered to exist for \(4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})\) and \(\hat{\epsilon} < \hat{\epsilon}_{\text{cusp}}\).

Finally, we calculate the eigenfunction which belongs to the eigenvalue (88). Solving the eigenvalue equation, we obtain two solutions. One takes the form of \((1 - z)^{\alpha_-(\lambda)}\) with \(\alpha_-(\lambda) = 1/2\) when \(z \to 1\), and the other takes the form of \((1 - z)^{\alpha_+(\lambda)}\). Both solutions individually diverge in \(z = -1\). The physical eigenfunction is represented as a linear combination of two solutions of the eigenvalue equation, in which the coefficients should be chosen to eliminate the singularities at \(z = -1\). The power \(\alpha_+(\lambda)\) of the function \((1 - z)^{\alpha_+(\lambda)}\) can be obtained by imposing

\[
\Lambda_{\alpha_+1}(R_1(1)^*, R_2(1)^*, R_1'(1)^*, R_2'(1)^*) = \Lambda_{3/2}(R_1(1)^*, R_2(1)^*, R_1'(1)^*, R_2'(1)^*). \quad (92)
\]

In the case of the system with LREF, substituting the LRE TT FP (70) into Eq. (92), we have

\[
\alpha_+(\lambda_{\text{LREF}}) = \frac{1}{4}(N - 10 + \sqrt{(N - 2)(N - 18)}). \quad (93)
\]

For \(N \geq 18\), \(\alpha_+(\lambda_{\text{LREF}})\) takes \(\alpha_+(\lambda_{\text{LREF}}) \geq 2\).
IV. CRITICAL PHENOMENA IN THE SYSTEM WITH LRE IN $2\sigma + \rho + \epsilon$ DIMENSIONS

In this section, we study the critical phenomena controlled by the LRE TT FP (70). We calculate the critical exponents $\eta$, $\tilde{\eta}$ and $\nu$ at $O(\epsilon)$, and clarify the boundary of the parameter $\sigma$ between the critical behaviors in systems with the long-range and the short-range exchange interactions. And we put $\rho = 0$, and investigate the $d \rightarrow d - \sigma$ dimensional reduction.

Substituting the LRE TT FP (70) into Eqs. (67) and (68), we obtain the critical exponents $\eta_{\text{LRE}}$ and $\tilde{\eta}_{\text{LRE}}$ at the LRE TT fixed point (72):

$$\eta_{\text{LRE}} = 2 - \sigma,$$

$$\tilde{\eta}_{\text{LRE}} = 4 - 2\sigma - \frac{\epsilon + \rho}{N}.$$  (96)

These exponents satisfy $\eta_{\text{LRE}} \geq (4 - d)/2$, $\tilde{\eta}_{\text{LRE}} \geq 4 - d$ and the Schwartz-Soffer inequality $\tilde{\eta}_{\text{LRE}} \leq 2\eta_{\text{LRE}}$. In the large $N$ limit, the relation between $\eta_{\text{LRE}}$ and $\tilde{\eta}_{\text{LRE}}$ satisfies $\tilde{\eta}_{\text{LRE}} = 2\eta_{\text{LRE}}$, which is identical to the result of the previous study for the critical properties of the random field spherical model by Vojta and Schreiber.57 For finite $N$ but $N > N_{\text{cusp}}$, the relation between $\eta_{\text{LRE}}$ and $\tilde{\eta}_{\text{LRE}}$ satisfies $2\eta_{\text{LRE}} - \tilde{\eta}_{\text{LRE}} = (\epsilon + \rho)/N$ for $\epsilon = d - 2\sigma - \rho$. Our result is consistent with the result of 1/N expansion study by Bray.58 He showed $2\eta_{\text{LRE}} - \tilde{\eta}_{\text{LRE}} = \epsilon/N$ for $\epsilon = d - 2\sigma$ by the use of the 1/N expansion. Thus, the relation $2\eta_{\text{LRE}} - \tilde{\eta}_{\text{LRE}} = (d - 2\sigma)/N$ holds in the region where the scaling behavior in the system is controlled by the LRE TT fixed point.

The relation between $\eta_{\text{LRE}}$ and $\tilde{\eta}_{\text{LRE}}$ is classified on the basis of the value of $\sigma$ as follows:

1. $\sigma < 2 - \frac{\epsilon + \rho}{N}$ : $\tilde{\eta}_{\text{LRE}} > \eta_{\text{LRE}}$,  
2. $\sigma = 2 - \frac{\epsilon + \rho}{N}$ : $\tilde{\eta}_{\text{LRE}} = \eta_{\text{LRE}}$,  
3. $\sigma > 2 - \frac{\epsilon + \rho}{N}$ : $\tilde{\eta}_{\text{LRE}} < \eta_{\text{LRE}}$,  

for $N > N_{\text{cusp}}$. Since $\eta_{\text{LRE}} \leq \tilde{\eta}_{\text{LRE}} \leq 2\eta_{\text{LRE}}$, the case 3 is unphysical. Thus, the critical value $\sigma = \sigma_{\text{LS}}$ which separates between the long-range and the short-range exchange regimes of the theory is

$$\sigma_{\text{LS}} = 2 - \frac{\epsilon + \rho}{N}.$$  (101)

Here, we comment on the critical value $\sigma_{\text{LS}}$. If $\sigma > 2 - (\epsilon + \rho)/2$, the spatial dimension in the present system is above four. Then, we put $d = 2\sigma + \rho + \epsilon = 4 + \epsilon'$ ($0 < \epsilon' \ll 1$). The critical value $\sigma_{\text{LS}}$ is rewritten in terms of $\epsilon'$ as follows:

$$\sigma_{\text{LS}} = 2 - \frac{\epsilon'}{N - 2}.$$  (102)

Since the exponent $\eta$ of the random field $O(N)$ spin model with SR in $4 + \epsilon'$ dimensions is $\eta = \eta_{\text{SR}} = \epsilon'/(N - 2)$ at $O(\epsilon')$ and for $N > N_{\text{cusp}}$, our result confirms that the critical value $\sigma_{\text{LS}}$ which separates between the long-range and the short-range exchange regimes of the theory is

$$\sigma_{\text{LS}} = 2 - \eta_{\text{SR}}.$$  (103)

We turn to compute the exponent $\nu_{\text{LRE}}$ of the correlation length. The critical exponent $\nu_{\text{LRE}}$ is determined from the inverse of the maximal eigenvalue given by Eq.(11). Thus, we obtain the inverse of the critical exponent $\nu_{\text{LRE}}^{-1}$ as

$$\nu_{\text{LRE}}^{-1} = \epsilon + \rho.$$  (104)

If we put $\rho = 0$, the spatial dimension in the present system becomes $d = 2\sigma + \epsilon$, and then $\nu_{\text{LRE}}^{-1}$ is

$$\nu_{\text{LRE}}^{-1} = \epsilon,$$  (105)

which is in agreement with that in the pure long-range system in $\sigma$ dimensions less. Therefore, the $d \rightarrow d - \sigma$ dimensional reduction holds at $O(\epsilon)$ and for $N > N_{\text{cusp}}$.

V. CRITICAL PHENOMENA IN SYSTEM WITH LREF IN $2\sigma + \rho + \epsilon$ DIMENSIONS

In this section, we study the critical phenomena controlled by the LREF TT fixed point (72). In the same way as the previous section, we determine the power $\alpha_*$ characterizing the weaker nonanalyticity of the LREF TT fixed point (72), examine the stability of the LREF TT fixed point (72) with respect to the cuspy perturbation, and calculate the critical exponents $\eta$, $\tilde{\eta}$ and $\nu$. And we investigate the $d \rightarrow d - \sigma - \rho$ dimensional reduction and the $d \rightarrow d - 2$ dimensional reduction.

Substituting the LREF TT FP (72) into Eqs. (67) and (68), we obtain the critical exponents $\eta_{\text{LREF}}$ and $\tilde{\eta}_{\text{LREF}}$ at LREF TT fixed point (72):

$$\eta_{\text{LREF}} = 2 - \sigma,$$  (106)

$$\tilde{\eta}_{\text{LREF}} = 4 - 2\sigma - \rho.$$  (107)

These exponents satisfy the Schwartz-Soffer inequality $\tilde{\eta}_{\text{LREF}} \leq 2\eta_{\text{LREF}}$, and saturate the generalized Schwartz-Soffer inequality $\tilde{\eta}_{\text{LREF}} \leq 2\eta_{\text{LREF}} - \rho$. And the inverse of the critical exponent $\nu_{\text{LREF}}$ is

$$\nu_{\text{LREF}}^{-1} = \left[ \frac{N}{N - 1} - \frac{\epsilon}{2} + \frac{\epsilon'}{2} \right] + \frac{4(N - \epsilon(N - 1))}{\epsilon^2(N - 1)^2 - \epsilon'}.$$  (108)

Since $\lim_{N \rightarrow \infty} \nu_{\text{LREF}}^{-1} = \epsilon$, in the large $N$ limit, the exponent $\nu_{\text{LREF}}$ agrees with that of the pure long-range system in $\sigma + \rho$ dimensions less. However, as long as $N$ is finite, $\nu_{\text{LREF}}^{-1} \neq \epsilon$. Thus, the $d \rightarrow d - \sigma - \rho$ dimensional reduction is broken for finite $N$. Hence, the $d \rightarrow d - 2$ dimensional reduction in the case of $\rho = 2 - \sigma$ is also broken for finite $N$, although the exponents $\eta_{\text{LREF}}$ and $\tilde{\eta}_{\text{LREF}}$ satisfy $\tilde{\eta}_{\text{LREF}} = \eta_{\text{LREF}} = 2 - \sigma$ for $N > 1$ and $\sigma < 2 - \epsilon/(N - 1)$. The graphs of $(\epsilon \nu_{\text{LREF}})^{-1}$ for
some values of $N$ are depicted in Fig. 6. It shows that $(\epsilon_{\mathrm{LRE}})^{\frac{1}{-}}$ tends to draw to 1 as the value of the parameter $\sigma$ decreases. Then, one expects that $(\epsilon_{\mathrm{LRE}})^{\frac{1}{-}}$ reaches 1 if the value of the parameter $\sigma$ decreases even further. However, it is impossible to study within the present framework, since the nontrivial fixed point of $O(\epsilon)$ disappears.

VI. SUMMARY

In this paper, we have reexamined the critical phenomena of the long-range random field $O(N)$ spin model near the lower critical dimension by using the $O(N)$ nonlinear-sigma model with the random fields and all possible higher-rank random anisotropies. By the use of the perturbative functional renormalization group, we have investigated the stability of the analytic fixed points in the one-loop beta functions. Also, we have calculated the critical exponents $\eta$, $\bar{\eta}$, and $\nu$ evaluated at the analytic fixed point controlling the critical behavior in the system.

We have shown that the analytic fixed point controlling the critical behavior in the system with the long-range exchange interactions and the uncorrelated random fields, there exists the once-unstable fixed point corresponding to the $d \to d - \sigma$ dimensional reduction for $N > N_{\text{cusp}} = 2(4 + 3\sqrt{3}) \simeq 18.3923 \ldots$. Although it has the subcusp, the weaker nonanalyticity does not change the value of the fixed point. Then, the critical exponents $\eta_{\mathrm{LRE}}$ and $\bar{\eta}_{\mathrm{LRE}}$ are $\eta_{\mathrm{LRE}} = 2 - \sigma$ and $\bar{\eta}_{\mathrm{LRE}} = 4 - 2\sigma - (d - 2\sigma)/N$ respectively, and satisfy the relation $2\eta_{\mathrm{LRE}} - \bar{\eta}_{\mathrm{LRE}} = (d - 2\sigma)/N$. And, the inverse of the exponent $\nu_{\mathrm{LRE}}$ takes $\nu_{\mathrm{LRE}}^{-1} = \epsilon$ at $O(\epsilon)$ in $\epsilon = d - 2\sigma$. Thus, the $d \to d - \sigma$ dimensional reduction holds at the leading order of the $d - 2\sigma$ expansion and for $N > N_{\text{cusp}}$. For $N < N_{\text{cusp}}$, the nonanalyticity occurring by the appearance of the linear cusp breaks down the $d \to d - \sigma$ dimensional reduction. This is considered to violate the simple relation between the exponents. Thus, one expects that the critical scaling behavior in the spin system with the long-range exchange interactions and the uncorrelated random fields is described by three independent exponents. And we have showed that, on the basis of the condition $\eta \leq \bar{\eta} \leq 2\eta$, the critical value $\sigma_{LS}$ of the boundary between critical behaviors in systems with long-range and short-range exchange interactions is $\sigma_{LS} = 2 - \eta_{SR}$, where $\eta_{SR}$ denotes the critical exponent of the connected correlation function in the corresponding short-range system.

We have studied the critical phenomena in the spin system with the long-range exchange interactions and the long-range correlations of the random fields. We have investigated the $d \to d - \sigma - \rho$ dimensional reduction and the $d \to d - 2$ dimensional reduction. Although the critical exponents $\eta_{\mathrm{LRE}}$ and $\bar{\eta}_{\mathrm{LRE}}$ satisfy $\eta_{\mathrm{LRE}} = 2 - \sigma$ and $2\eta_{\mathrm{LRE}} - \bar{\eta}_{\mathrm{LRE}} = \rho$, the $d \to d - \sigma - \rho$ dimensional reduction does not holds within the present analysis, as long as $N$ is finite; the exponent $\nu_{\mathrm{LRE}}$ does not coincide with that of the pure long-range system in $\sigma + \rho$ dimensions less. Thus, the $d \to d - 2$ dimensional reduction in the case of $\rho = 2 - \sigma$ is also broken for finite $N$. The result does not contradict that in our previous study for the three-dimensional long-range random field Ising model. Since our present study by the use of the perturbative renormalization group has been restricted to $\epsilon + \rho = \epsilon + 2 - \sigma \sim O(\epsilon)$, only the breakdown of the $d \to d - 2$ dimensional reduction has been observed. Then, to study the $d \to d - 2$ dimensional reduction and its breakdown in the $(\sigma + 2 + \epsilon)$-dimensional long-range random field $O(N)$ spin model, the non-perturbative analysis are needed.

Finally, we comment on the validity of the $d \to d - \sigma$ dimensional reduction in the system with the long-range exchange interactions and the uncorrelated random fields near the lower critical dimension and for $N > N_{\text{cusp}}$. As shown in the previous works by Young and Bray, the value of $\nu_{\mathrm{LRE}}^{-1}$ coincides with that of the pure long-range system in $\sigma$ dimensions less at the leading order in $\epsilon = d_{u} - d$ near the upper critical dimension $d_{u} = 3\sigma$. However, it fails at $O(\epsilon^2)$. Thus, although we have showed that the $d \to d - \sigma$ dimensional reduction holds at the leading order in $\epsilon = d - d_{l}$ near the lower critical dimension $d_{l} = 2\sigma$ and for $N > N_{\text{cusp}}$ in the present work, there is room for doubt whether it holds beyond one loop, even if $N > N_{\text{cusp}}$. Further studies by using the higher-loop calculation should shed light on this problem.

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