Horndeski and the Sirens

Charles Dalang,1 Pierre Fleury,1,2 and Lucas Lombriser1‡

1 Département de Physique Théorique, Université de Genève, 24 quai Ernest-Ansermet, 1211 Genève 4, Switzerland
2 Instituto de Física Teórica UAM-CSIC, Universidad Autonóma de Madrid, Cantoblanco, 28049 Madrid, Spain

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Abstract

Standard sirens have been proposed as probes of alternative theories of gravity, such as Horndeski models. Hitherto, all studies have been conducted on a homogeneous-isotropic cosmological background, which is unable to consistently account for realistic distributions of matter, and for inhomogeneities in the Horndeski scalar field. Yet, the latter are essential for screening mechanisms. In this article, we comprehensively analyze the propagation of Horndeski gravitational waves in an arbitrary background spacetime and scalar field. We restrict to the class of theories in which gravitational waves propagate at light speed, and work in the geometric-optics regime. We find that kinetic braiding only produces a nonphysical longitudinal mode, whereas conformal coupling affects the amplitude of the standard transverse modes but not their polarization. We confirm that any observable deviation from general relativity depends on the local value of the effective Planck mass at emission and reception of the wave. This result is interpreted as the conservation of the number of gravitons.

I. INTRODUCTION: HORNDESKI’S ODYSSEY

Tell me of a complicated theory.
Urania, tell me how it wandered and was lost when it had attempted to replace the relativity of Einstein.

This theory, which more accurately is a crew of theories raised by Horndeski, supplements general relativity (hereafter GR) with a scalar degree of freedom. From a phenomenological point of view, Horndeski theories may be considered an attempt to model dark energy, i.e. the cause of the acceleration of cosmic expansion. From a more theoretical point of view, they could represent the effective behavior of more elaborate theories of gravity beyond GR. Theories of gravity beyond GR have indeed to disguise so as not to be noticed. Screening mechanisms can indeed cancel the effect of the scalar field when the local gravitational potential, gravitational field, or matter density exceeds some threshold. In principle, this allows Horndeski theories to be indistinguishable from GR in the Solar System, while significantly differing from it on cosmological scales. See, however, Refs. [12-14] for mitigation.

Of other perilous challenges were nonetheless awaiting Horndeski, after he escaped the Solar System; he had to navigate through the vast Universal ocean, avoiding rogue waves, traitorous cosmic flows, and the mighty creatures hidden in the cosmic web. It was indeed argued in Ref. [15] that gravitational waves (hereafter GWs) could destabilize Horndeski’s scalar field in the presence of kinetic braiding. Besides, many current and near-future tests of Horndeski theories rely on the observation of the local and large-scale structure of the Universe, including galaxy clustering, weak gravitational lensing, and redshift-space distortions, but also relativistic effects and more local observations.

But of all the dangers that Horndeski had to face, the deadliest one was undoubtedly the spell cast by the circuleọ̄ ballet of two neutron stars, which condemned the best members of his crew. By constraining the speed of GWs to be equal to the speed of light, the combined observation of GW170817 and GRB 170817 thereby ruled out a significant fraction of the parameter space of Horndeski’s theories. In particular, it has put a lot of pressure on the so-called self-accelerating models, in which the cause of the acceleration of cosmic expansion may be attributed to a genuine gravitational effect, rather than some form of dark energy.

Only two members of Horndeski’s crew survived the spell, namely Conformal Coupling and Cubic Galileon; but the gods’ wrath was still upon them, and that is how Horndeski met the Sirens. Merging binary systems of compact objects, such as black holes or neutron stars, are efficient emitters of GWs. Since the measurement of their waveform directly gives access to their distance, such objects were nicknamed standard sirens. Just like standard candles, standard sirens can be used to build a Hubble diagram, i.e. a measurement of the relation between distance and redshift across the Universe. While candles and sirens produce the same Hubble diagram in GR, differences are expected in Horndeski theories, notably through conformal coupling. That is why standard sirens were recently argued to be a key probe of gravity model beyond GR, especially in the future era of the Laser Interferometer Space Antenna (LISA).

As noticed in Ref. [17], and further explored by two of the authors of the present article, if the environments of both...
the siren and her observer are screened, which is more likely, then the aforementioned difference between electromagnetic and GW Hubble diagrams may be suppressed, thereby questioning the relevance of that probe of alternative theories of gravity, such as Horndeski’s. Specifically, it was argued that the distance measured with a standard siren, $D_G$, differs from the standard electromagnetic luminosity distance $D_L$ as

$$D_G = \frac{M^2}{M_0} D_L,$$  \hspace{1cm} (1)

where $M_s$, $M_0$ respectively denote the local effective Planck mass of the source and at the observer. Thus, if any screening mechanism enforces $M_s = M_0$, then $D_G = D_L$ and the GW and electromagnetic Hubble diagram coincide. If, on the contrary, $M_s \neq M_0$, then one must also account for that effect in the trigger of supernova explosions, which may also lead to a coincidence between the two Hubble diagrams. In other words, standard-siren tests of Horndeski theories seem to lie between Scylla and Charybdis.

At this point, we stress that every rationale underlying standard-siren tests of gravity, including Eq. (1), was hitherto made for GWs propagating on an ideal Friedmann-Lemaître-Robertson-Walker (FLRW) background, which is to the real Universe what Homer’s Odyssey is to Greek history; in particular, the FLRW background cannot consistently account for spatial variations of the Horndeski scalar field. Yet, allowing for such variations is crucial, e.g. to model screening mechanisms, but also to evaluate the impact of kinetic braiding, which couples GWs to derivatives of the scalar field.

The telos of this article – besides making epic puns – is to fill this gap, by investigating the propagation of GWs in Horndeski theories, for arbitrary spacetime backgrounds, and arbitrary variations of the scalar field. After briefly presenting the Horndeski models in Sec. II, we analyze in Sec. III the propagation of GWs within the geometric optics regime; in particular, we demonstrate the general validity of Eq. (1) beyond the homogeneous and isotropic FLRW background. Finally, in Sec. IV, we show that this relation between GW and luminosity distances is related to the conservation of the number of gravitons. We summarize and conclude in Sec. V.

Throughout this article, unlike Pheidippides, Greek indices modestly run from 0 to 3; bold symbols indicate three-vectors; a comma denotes a partial derivative, and a semicolon denotes a covariant derivative associated with the Levi-Civita connection. Symmetrization of indices follows $X_{\mu\nu} \equiv \frac{1}{2}(X_{\mu\nu} + X_{\nu\mu})$. We use the convention of Misner, Thorne, Wheeler [43] for the metric signature and the Riemann tensor. Finally, we adopt units such that $c = 1 = \hbar$ except for Sec. IV.

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4 In this article, the effective Planck mass $M$ will refer to the prefactor of Ricci curvature in the action of gravitation. It shall be distinguished from the notion of effective Newton’s constant $G_{eff}$, which more commonly refers to the quantity involved in the Poisson equation. In general, $M^{-2} \neq 8\pi G_{eff}$.

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II. HORNDESKI’S MODELS OF GRAVITY

Horndeski theories [2] form a class of extensions of GR involving an extra scalar degree of freedom. When it is nonminimally coupled to the spacetime geometry, this scalar can be seen either as a new physical interaction – a fifth force – or a modification of the laws of gravity. Specifically, Horndeski’s theories are the most general local theories involving the metric tensor $g_{\mu\nu}$ of a four-dimensional spacetime Lorentzian manifold, coupled to a scalar field $\varphi$, and whose equations of motion deriving from an action principle are second-order. This section briefly introduces the main equations of these theories, and reviews some of their properties.

A. Horndeski’s action

In the so-called Jordan frame, Horndeski’s action is

$$S = S_g[\varphi, g_{\mu\nu}] + S_m[\psi, g_{\mu\nu}],$$  \hspace{1cm} (2)

where $\psi$ refers to the matter fields of particle physics, assumed to be minimally coupled to the spacetime metric $g_{\mu\nu}$ within their action $S_m$; it is not necessary to specify the expression of $S_m$ for the purpose of this article. More importantly, $S_g$ encodes the gravitational sector of the theory.

$$S_g[\varphi, g_{\mu\nu}] = \sum_{i=2}^{5} S_i[\varphi, g_{\mu\nu}] = \frac{M_P^2}{2} \int d^4 x \sqrt{-g} \sum_{i=2}^{5} L_i,$$  \hspace{1cm} (3)

where $M_P \equiv \sqrt{\frac{1}{8\pi G}}$ is the standard Planck mass, $G$ being Newton’s constant. The four Lagrangian densities $L_i$ read

$$L_2 \equiv G_2(\varphi, X),$$  \hspace{1cm} (4)

$$L_3 \equiv G_3(\varphi, X) \varphi,$$  \hspace{1cm} (5)

$$L_4 \equiv G_4(\varphi, X) R + G_4(\varphi, X) \left[ (\Box \varphi)^2 - \varphi_{\mu\nu} \varphi^{\mu\nu} \right],$$  \hspace{1cm} (6)

$$L_5 \equiv G_5(\varphi, X) E^{\mu\nu} \varphi_{\mu\nu} - \frac{1}{6} G_5(\varphi, X) \left[ (\Box \varphi)^3 - 3 \Box \varphi \varphi^{\mu\nu} \varphi_{\mu\nu} + 2 \varphi_{\mu\nu} \varphi^{\mu\nu} \varphi_{\rho\sigma} \varphi^{\rho\sigma} \right].$$  \hspace{1cm} (7)

In the above, we have introduced the following short-hand notation: $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the D’Alembert operator, $X \equiv -\frac{1}{2} \Box \varphi \varphi$, $R$ is the Ricci scalar and $E_{\mu\nu}$ the Einstein tensor

The Horndeski class encompasses all scalar-tensor theories, but also covariant Galileons [47]. Any specific model is thereby determined by the four functions $G_{2...5}(\varphi, X)$, whose functional form is mostly free, apart from a few conditions ensuring stability and causality [20, 48]. Note however that, for a given model, the functional form of the $G_{2...5}$ is not unique. One is, indeed, free to reparameterize the scalar

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5 We choose this notation instead of the more traditional $G_{\mu\nu}$ because there are already many $G_{\mu\nu}$ in these expressions.
field as $\varphi \mapsto \tilde{\varphi}$. One can also choose to reparameterize the metric through a so-called disformal transformation

$$g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = C(\varphi) g_{\mu\nu} + D(\varphi) \varphi, \mu \varphi, \nu,$$

where $C, D$ are two arbitrary functions of the scalar field. Such a transformation preserves the general form of Horndeski’s action $\mathcal{L}[\varphi, \mu], \varphi, \mu]$ [49]. It is then said that one works in a different frame. Disformal transformations do not change the physical properties of the model – see e.g. Ref. [50] for a detailed example with conformal transformations in cosmology – but they change the physical interpretation of the metric. Indeed, since $\tilde{g}_{\mu\nu}$ is, in general, nonminimally coupled to matter, it does not indicate the times and distances actually measured by an observer.

In this article, we choose to work in the Jordan frame, i.e. the parameterization such that the metric is minimally coupled to matter. This choice ensures that gravity acts on matter via $g_{\mu\nu}$ only. In particular, interferometers designed to detect GWs, such as LIGO$^7$ and Virgo$^8$ are then sensitive to waves of the metric field $g_{\mu\nu}$ only. This means that we can focus our attention on such waves without caring too much about potential scalar waves.

The quasi-simultaneous detection, in August 2017, of the GW-signal emitted by a neutron-star binary (GW170817 [24]) and the associated $\gamma$-ray burst (GRB 170817 [25,26]) proved that the propagation speed of GWs cannot differ from the speed of light by more than a part in $10^5$ [15]. Assuming that this difference is exactly zero, and excluding fine-tuned theories [51], viable Horndeski models must satisfy [27,35]

$$G_{A,X} = G_S = 0.$$  \hspace{1cm} (9)

Two remarks may be formulated about Eq. (9). First, the equality between GW speed and the speed of light has been experimentally tested for GWs of relatively small wavelengths (on the order of $10^6$ m), i.e. relatively high energies, compared to, e.g., the relevant scales of cosmology. Thus, it may be cavalier to extrapolate this result and conclude that Eq. (9) holds at low energies [52]. Second, the neutron-star merger that produced the GW and GRB signal from which Eq. (9) was deduced, happened relatively close to us (about 40 Mpc away). It is not excluded, in principle, that Eq. (9) only holds locally, but not at cosmological distances across our past lightcone. Having pointed out these possible limitations, we will nevertheless assume in this article that Eq. (9) does hold in the entire Universe, and is indeed a property of the theory.

### B. Equations of motion

This section gives the equations of motion for $\varphi, g_{\mu\nu}$ deriving from Horndeski’s action with $G_{A,X} = G_S = 0$. These will be the starting point of the present analysis.

Let us start with what will eventually be our main focus, namely the equation of motion for the tensor field $g_{\mu\nu}$. Varying Eq. (2) with respect to $g_{\mu\nu}$, formally yields

$$\sum_{i=2}^{4} E_{i\mu\nu} = M_p^{-2} T_{\mu\nu} , \quad \text{with} \quad E_{i\mu\nu} = \frac{2 M_p^{-2} \delta S_i}{\sqrt{g}} \frac{\delta g_{\mu\nu}}{\delta g_{\mu\nu}} ,$$

and where $T_{\mu\nu} \equiv (-/\sqrt{-g}) \delta S_m / \delta g_{\mu\nu}$ is the matter energy-momentum tensor. The four pieces $E_{i\mu\nu}$ of the equation of motion explicitly read

$$E_{2\mu\nu} = -\frac{1}{2} \left( G_{2,XX} \varphi, \mu \varphi, \nu + G_{2,\varphi} \varphi, \mu \varphi, \nu \right),$$

$$E_{3\mu\nu} = - G_{3,\varphi} (\varphi, \mu \varphi, \nu + \varphi, \mu \varphi, \nu),$$

$$E_{4\mu\nu} = G_4 F_{\mu\nu} - G_{4,\varphi} \Box \varphi.$$  \hspace{1cm} (10)

Recall that, in this article, $F_{\mu\nu}$ denotes the Einstein tensor. Dividing the entire equation of motion [10] by $G_4$, one sees that matter gravitates via the effective Planck mass

$$M^2 \equiv G_4 M_P^2 = \frac{G_4}{8 \pi G}.$$  \hspace{1cm} (14)

Hence, the conformal factor $G_4$ may be seen as a multiplicative correction to the Planck mass.

#### 2. Scalar field

As mentioned earlier, we chose to work in the Jordan frame so as to maximally reduce interactions between matter and the scalar field. Thus, the dynamics of $\varphi$ will be mostly irrelevant in this article. We nevertheless give its equation of motion below for completeness. Imposing that the variation of the action [9] with respect to $\varphi$ vanishes yields

$$0 = G_{2,\varphi} + G_{2,XX} \Box \varphi - 2 G_{2,XX} \varphi, \mu \varphi, \mu \varphi, \nu X, \mu + (2 G_{3,\varphi} - 2 G_{3,\varphi} X, \mu \varphi, \nu X, \mu + G_{3,\varphi} \Box \varphi) \Box \varphi - 2 G_{3,\varphi} \varphi, \mu \varphi, \nu X, \mu + G_{3,XX} \varphi, \mu \nu X, \mu \nu - G_{3,XX} \varphi, \mu \nu \varphi, \nu \nu + R^{\mu\nu} \varphi, \mu \nu X, \mu \nu + G_{4,\varphi} R .$$  \hspace{1cm} (15)

Using Eq. (10), one can substitute the curvature terms $R_{\mu\nu}, R$ of Eq. (15), with their expression in terms of $\varphi, T_{\mu\nu}$. The resulting equation being much longer than Eq. (15), but not particularly illuminating, we will not write it down explicitly. Nevertheless, it is interesting to point out that it contains terms proportional to $G_{3,XX} \varphi, \mu \nu \varphi, \nu X, \mu$ and $G_{4,\varphi} T$, where $T \equiv T_{\mu\nu}$ is the trace of the matter energy-momentum tensor. Therefore, the scalar field is directly sourced by matter if either $G_{3,XX} \neq 0$ (kinetic braiding [53]) or $G_{4,\varphi} \neq 0$ (conformal coupling), even in the Jordan frame.

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6 This concept of frame has nothing to do with its counterpart in mechanics or relativity; it is just a particular choice for the parameterization of the degrees of freedom $\varphi, g_{\mu\nu}$ of the theory.

7 [http://www.ligo.caltech.edu](http://www.ligo.caltech.edu)

8 [http://www.virgo-gw.eu/](http://www.virgo-gw.eu/)
C. Screening

The presence of an extra scalar degree of freedom in Horndeski theories can lead to a rich gravitational phenomenology: variations of the effective Planck mass, violation of the equivalence principle \[54\], anomalous propagation of light, etc. However, such phenomena are tightly constrained in the Solar System and the Milky Way. Specifically, lunar laser ranging \[55\] imposes that \(|G_{\text{eff}}/G_{\text{Planck}}| < 10^{-14} \text{ yr}^{-1}\), where \(G_{\text{eff}}\) denotes the gravitational coupling entering the effective Poisson equation. Besides, the 21-year monitoring \[55\] of the timing of the pulsar binary PSR J1713+0747, about 1.2 kpc away from us, imposes \(|G_{\text{eff}}/G_{\text{eff}}| < 10^{-12} \text{ yr}^{-1}\) there. Regarding the effect of gravity on light propagation, and in particular the Shapiro time-delay effect, radio communication with the Cassini spacecraft \[57\] in solar conjunction, in 2002, constrained the \(\gamma\) post-Newtonian parameter as \(|\gamma - 1| < 2 \times 10^{-5}\). Together with the constraint that GWs propagate at the speed of light, these should force gravitation to be extremely close to GR \[29, 30\].

Is there any way out? It is important to note that the conclusions of Refs. \[29, 30\] rely on the assumption that Horndeski theories have the same behavior in the Solar System and on cosmic scales. However, it turns out that Horndeski theories generically possess screening mechanisms, which allow the scalar field to hide in situations relevant to, e.g., the Solar System. Conformally coupled scalar fields (which include \(f(R)\) and symmetron models \[58\]) can display the so-called chameleon mechanism \[9, 59\], where \(\phi\) is effectively suppressed when the gravitational potential is large enough. Another possible screening mechanism is the Vainshtein effect, which was originally introduced in the context of massive gravity \[11\], but is also relevant for Galileon-like models. In that case, self-interactions in the kinetic (rather than potential) part of the action of \(\phi\) lead to a suppression of its effects when the matter density is large enough. Similarly, k-mouflage \[10\] operates when the gravitational acceleration is large enough.

Screening is not always a panacea. For instance, chameleon models cannot be screened in the Solar System and in the Milky Way while self-accelerating cosmic expansion (i.e. without the help of some form of dark energy) \[13\]. Besides, the Vainshtein mechanism seems to be unable to screen cosmological time variations of the effective Planck mass, at least in Galileon models \[12\]: hence, such models cannot explain the cosmological dynamics without violating the stringent constraints on \(|G_{\text{eff}}/G_{\text{eff}}|\) or on \(|\gamma - 1|\).

D. Absence of scalar waves?

In this article, \(\textit{we will not consider waves in the scalar sector, which may seem surprising at first sight. Indeed, as pointed out just after Eq. (15),}\ \(\varphi\) is effectively sourced by matter, and must then propagate waves just like the metric does. However GW sources, such as binary systems of black holes or neutron stars, are typically strong-field regions, which are very likely to be screened. Because screening suppresses the effects of the scalar field, it is natural to expect it to suppress scalar radiation as well. This intuition was confirmed both analytically \[60, 61\] and numerically \[52\] in the Vainshtein case. Besides, in the chameleon case, scalar waves should exponentially decay out of the source due to the short range of the resulting interaction.

However, even in the absence of direct scalar radiation, some may be generated by “leakage” of tensor GWs in the scalar sector. Indeed, \(\varphi\) is sourced by Ricci curvature, and conversely curvature is sourced by \(\Box \varphi\), both may thus exchange energy as a GW propagates. A comprehensive way of addressing this issue would consist in treating scalar and tensor perturbations simultaneously, and diagonalize the resulting system of propagation equations. This analysis would be significantly more involved than the one proposed here, but it may also be unnecessary.

Consider scalar and tensor waves \(\varphi_w, g_{\mu \nu}\). Assuming for simplicity that \(G_2 = X\), and keeping only the highest-derivative terms, Eqs. (10), (15) schematically read
\[
\Box \varphi_w \sim \alpha \Box h_{\mu \nu}, \quad (16)
\]
\[
\Box h_{\mu \nu} \sim \alpha \Box \varphi_w, \quad (17)
\]
with \(\alpha \sim G_2 X G_{\text{eff}} X G_{\text{eff}}\) quantifying the amplitude of the coupling — either conformal or kinetic — between the scalar field and spacetime geometry. In the Jordan frame, the effect of \(\varphi_w\) can only be observed through its backreaction on \(h_{\mu \nu}\); what we called the leakage effect is therefore of order \(\alpha^2\). Since \(\alpha\) must be reasonably smaller than unity, the impact of scalar waves is thus expected to be subdominant. This heuristic reasoning is supported by the results of Ref. \[63\], where the amplitude of scalar waves is found to be 20 orders of magnitudes smaller than the amplitude of tensor waves in the typical LIGO frequency band, and 10 orders of magnitude smaller in the LISA band.

In what follows, the scalar field may thus be viewed as a kind of stiff medium, which will guide the propagation of GWs without actually being shaken by them.

III. GRAVITATIONAL WAVES IN HORNDESKI MODELS

The present section is the core of the article. We carefully analyze the propagation of GWs in the Horndeski theories described in Sec. [11] and the notion of distance measured with standard sirens. Contrary to previous studies, we do not restrict to a homogeneous-isotropic FLRW background spacetime, and work with an arbitrary background geometry and scalar field.

A. Linearized equations of motion

Let us assume the existence of a coordinate system such that the spacetime metric reads
\[
g_{\mu \nu} = \tilde{g}_{\mu \nu} + h_{\mu \nu}, \quad |h_{\mu \nu}| \ll 1, \quad (18)
\]
where \(\tilde{g}_{\mu \nu}\) represents the background spacetime, typically generated by astrophysical and cosmological sources, while the perturbation \(h_{\mu \nu}\) stands for GWs. The equation of motion \[10\] for \(g_{\mu \nu}\) can then be formally linearized as
\[
\sum_{i=2}^{4} \mathcal{E}_{\mu \nu} = \sum_{i=2}^{4} \tilde{\mathcal{E}}_{\mu \nu} + \delta \mathcal{E}_{\mu \nu} = M_p^2 (\tilde{T}_{\mu \nu} + \delta T_{\mu \nu}), \quad (19)
\]
where $\delta E^i_{\mu\nu}, \delta T^i_{\mu\nu}$ are linear in $h_{\mu\nu}$ and its derivatives. By definition, the background metric satisfies $\sum_{i=2}^4 \epsilon^i_{\mu\nu} = M_4^2 T^i_{\mu\nu}$, and we are left with

$$ \sum_{i=2}^4 \delta E^i_{\mu\nu} - M_4^2 \delta T^i_{\mu\nu} = 0 . \quad (20) $$

The term $\delta T^i_{\mu\nu}$ comes from the fact that the metric is always coupled to matter, whose action $S_m$ contains at least $\sqrt{-g}$. Fortunately, for standard forms of matter such as perfect fluids, electromagnetic fields, and so on, $S_m$ does not involve derivatives of the metric. Hence, $\delta T^i_{\mu\nu}$ is generally what we will call a mass-like term in the following.

1. Kinetic, damping, and mass terms

The linearized equation (20) is conveniently decomposed into kinetic terms $K^i_{\mu\nu}$, which contain second-order derivatives of $h_{\mu\nu}$; damping (or amplitude) terms $\mathcal{A}^i_{\mu\nu}$, with first-order derivatives; and mass-like terms $M^i_{\mu\nu}$ without any derivative:

$$ \delta E^2_{\mu\nu} = M^2_{\mu\nu} , \quad (21) $$

$$ \delta E^3_{\mu\nu} = M^3_{\mu\nu} + \mathcal{A}^3_{\mu\nu} , \quad (22) $$

$$ \delta E^4_{\mu\nu} = M^4_{\mu\nu} + \mathcal{A}^4_{\mu\nu} + K^4_{\mu\nu} . \quad (23) $$

Let us briefly explain the origin of the relevant terms and give their expressions. First of all, the only kinetic term comes from the perturbation of the Einstein tensor; hence, it is proportional to the standard one obtained in GR,

$$ K^4_{\mu\nu} = \frac{G_4}{2} \left[ 2 \gamma^\mu_{\rho(\mu, \nu)} - \Box \gamma_{\mu\nu} - \gamma_{\rho\sigma,\mu\nu} \bar{\sigma}^\mu_{\rho\sigma} \right] , \quad (24) $$

where $\gamma_{\mu\nu}$ denotes the usual trace-reversed metric perturbation

$$ \gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \bar{\sigma}^\mu_{\mu\nu} . \quad (25) $$

with $h \equiv h^\mu_{\mu} \equiv \bar{g}_{\mu\nu} h_{\mu\nu}$. Note that, in Eq. (24) and in all the remainder of this article, indices are raised and lowered by the background metric $\bar{g}_{\mu\nu}$; semicolons refer to background covariant derivatives $\bar{\nabla}_\mu$ and $\Box = \bar{g}_{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu$.

The damping term associated with conformal coupling comes from the second covariant derivatives of $G_4$ in Eq. (13), because these feature products between $\partial G_4$ and Christoffel symbols. Its explicit expression is

$$ \mathcal{A}^4_{\mu\nu} = \frac{1}{2} G_4^\rho (h_{\rho\mu\nu} + h_{\rho\nu\mu} - h_{\mu\rho\nu}) - G_{\lambda\rho} \gamma^{\rho\nu}_{\lambda\nu} \bar{g}^\mu_{\lambda\nu} . \quad (26) $$

Similarly, the cubic-Galileon piece leads to

$$ \mathcal{A}^3_{\mu\nu} = \frac{1}{2} G_3 X \varphi^\rho \varphi^{\sigma\nu} \left[ \frac{1}{2} \varphi^4 h_{\rho\sigma,\lambda} \bar{g}_{\mu\nu} - h_{\rho\sigma,\mu\nu} \bar{g}_{\lambda\nu} \right] + \frac{1}{2} G_3 X \varphi \gamma_{\rho\sigma} \varphi_{\mu\nu} . \quad (27) $$

The mass-like terms $M^i_{\mu\nu}$, which will be negligible in this work thanks to the eikonal approximation (see Sec. III B), are given in the appendix A for completeness.

2. Harmonic gauge

Just like in GR, the linearized equation of motion (20) enjoys a gauge invariance, because of the general covariance of Horndeski’s action and of the resulting equations of motion. Recall that, under an infinitesimal coordinate transformation $x^\mu \mapsto x^\mu + \xi^\mu$, the metric perturbation changes as

$$ h_{\mu\nu} \mapsto h_{\mu\nu} + 2 \xi_{\mu\nu} , \quad (28) $$

which we will refer to as a gauge transformation.

We can take advantage of the gauge invariance to simplify the equation of motion (20). In particular, since under a gauge transformation

$$ \gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} + \Box \xi_{\mu} + \bar{R}^\rho_{\nu} \xi^\rho , \quad (29) $$

it is always possible to impose

$$ \gamma_{\mu\nu} = 0 , \quad (30) $$

because if it were not the case initially, there would always exist a gauge field $\xi_{\mu}$ capable of ensuring it. Equation (30) is known as the harmonic, or Hilbert, or De Donder gauge. It greatly simplifies the terms of interest as

$$ K^4_{\mu\nu} = - \frac{1}{2} G_4 \Box \gamma_{\mu\nu} , \quad (31) $$

$$ \mathcal{A}^4_{\mu\nu} = \frac{1}{2} G_4 (h_{\rho\mu\nu} + h_{\rho\nu\mu} - h_{\mu\rho\nu}) , \quad (32) $$

$$ \mathcal{A}^3_{\mu\nu} = \frac{1}{2} G_3 X \varphi^\rho \varphi^{\sigma\nu} \left[ \frac{1}{2} \varphi^4 h_{\rho\sigma,\lambda} \bar{g}_{\mu\nu} - h_{\rho\sigma,\mu\nu} \bar{g}_{\lambda\nu} \right] . \quad (33) $$

B. Eikonal approximation

The equation of motion for $h_{\mu\nu}$ may now be turned into a wave-propagation equation by working in the geometric-optics regime, i.e., using the eikonal approximation.

1. Nature of the approximation

In a nutshell, the eikonal approximation consists in assuming that the typical wavelength of the GW is much smaller than all the other characteristic length scales of the problem at hand. In that regime, the GW essentially behaves as a stream of

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* The decomposition is not unique, since two covariant derivatives in $K^i_{\mu\nu}$ may be swapped at the price of introducing curvature terms, which add to the mass-like terms $M^i_{\mu\nu}$. 
particles, and all the phenomena related to its actual wave nature – interference, diffraction – can be neglected. In more concrete terms, we shall introduce the ansatz
\[ h_{\mu\nu} = \frac{1}{2} H_{\mu\nu} e^{\pm i w} + \text{c.c.} \] (34)
where \( H_{\mu\nu} \) represents the complex amplitude and polarization of the GW, \( w \) its phase, and \( \text{c.c.} \) means complex conjugate. The eikonal approximation then implies that \( w \) varies much faster than:

1. the GW amplitude,
2. the background spacetime geometry, and
3. the scalar field.

If \( \omega \sim \partial w \) denotes the typical cyclic frequency of the GW, the above translates into
\[ \omega \gg H^{-1} \partial H, \quad \sqrt{|\mathcal{K}_{\mu\nu}\partial w|}, \quad \varphi^{-1} \partial \varphi. \] (35)

In practice, \( \omega^{-1} \) can be treated as a small parameter in terms of which one performs perturbative expansions; see e.g. Refs. [43,65] for further details.

For GWs falling in the LIGO maximum sensitivity, \( \omega \sim 10^2 \text{ Hz} \), the typical wavelength is on the order of \( 10^6 \text{ m} \approx 7 \times 10^{-8} \text{ AU} \), thereby making the eikonal approximation very accurate. For the expected LISA maximum sensitivity, \( \omega \sim 10^{-3} \text{ Hz} \), typical wavelengths approach the astronomical unit, making the eikonal approximation less applicable down to stellar scales. It remains, however, entirely valid for gravitational and scalar fields evolving on galactic scales, and a fortiori on cosmological scales. See Refs. [65,66] for recent attempts to model GWs beyond the geometric-optics regime.

2. Propagation equations

In the eikonal approximation, the various terms of the linearised equation of motion (20) for \( h_{\mu\nu} \) obey a simple hierarchy
\[ \mathcal{K}^4_{\mu\nu} \gg \mathcal{A}^3_{\mu\nu}, \mathcal{A}^4_{\mu\nu} \gg M^4_{\mu\nu}, \partial T_{\mu\nu}, \] (36)
which leads us to directly neglect the mass-like terms. Substituting Eqs. (31)-(33) into the resulting \( \mathcal{K}^4_{\mu\nu} + \mathcal{A}^4_{\mu\nu} = 0 \), and subtracting its trace, we obtain the rather simple
\[ G_4 \Box h_{\mu\nu} = G_4^\phi \left[ 2 h_{\rho(\mu;\nu)} - h_{\mu\nu,\rho} \right] - G_{3,\chi} \varphi^\rho \varphi^\sigma h_{\rho\sigma;\mu\nu} \varphi_{\rho\sigma} \] (37)
Using the wave-ansatz (34) for \( h_{\mu\nu} \), we have
\[ h_{\mu\nu,\rho} = \frac{1}{2} \left( H_{\mu\nu,\rho} + i k_{\rho} H_{\mu\nu} \right) e^{i w} + \text{c.c.}, \] (38)
\[ \Box h_{\mu\nu} = \left( -k^\rho k_\rho H_{\mu\nu} + i \mathcal{D} H_{\mu\nu} + \Box H_{\mu\nu} \right) e^{i w} + \text{c.c.}, \] (39)
where \( k_\mu \equiv w_\mu \) denotes the wave four-vector associated with the eikonal \( w \), and \( \mathcal{D} \) is a linear differential operator defined as
\[ \mathcal{D} \equiv 2 k^\mu \nabla_\mu + k^\mu \mu. \] (40)

Equation (37) may now be split into its real and imaginary parts. The real part is the dispersion relation,
\[ k^\mu k_\mu = 0 \] (41)
up to negligible mass-like terms (including \( \Box H_{\mu\nu} \)), which indicates that GW wavefronts propagate at the speed of light. This is not surprising since we precisely restricted to this case via Eq. (9). Note also that Eq. (41) implies the geodesic equation \( k^\mu k_{\rho;\mu} = 0 \), because \( k_\rho \) is a null gradient, so that GWs propagate along null geodesics.

The imaginary part of Eq. (37) governs the evolution of the GW amplitude, and reads
\[ G_4 \mathcal{D} H_{\mu\nu} = 2 G_4^\phi H_{\rho(\mu;\nu)} - k^\rho G_{4,\mu} H_{\rho\nu} - G_{3,\chi} \varphi^\rho \varphi^\sigma h_{\rho\sigma;\mu\nu} \varphi_{\rho\sigma}. \] (42)

In GR, the above would reduce to \( \mathcal{D} H_{\mu\nu} = 0 \), meaning that the GW energy would dilute as the area of the wavefront grows. Here, both conformal coupling \( (G_{4,\mu}\varphi) \) and kinetic braiding \( (G_{3,\chi}\varphi) \) seem to break this law. Note that the latter was absent from earlier studies, because in FLRW \( \varphi = \varphi, r_\mu = 0 \) and \( H_{\mu\nu} = H_{\mu\nu} ; \varphi_{\mu} \), so that \( \varphi^\mu H_{\mu\nu} = 0 \). We will see, however, that this new term does not lead to observable corrections.

C. Longitudinal and transverse modes

So far, the GW amplitude \( H_{\mu\nu} \) carries six degrees of freedom \((10 - 4 \text{ due to the harmonic-gauge condition})\). In GR, these can be reduced to two transverse modes using adequate gauge transformations. Things turn out to be more complicated in Horndeski theories. Namely, there will remain four longitudinal degrees of freedom in addition to the two traditional transverse ones. Although this longitudinal mode is nonphysical, kinetic braiding seems to prevent us from gauging it away.

1. Tetrad decomposition

In order to decompose \( H_{\mu\nu} \) into longitudinal and transverse components, it is useful to introduce a null tetrad \([k_\mu, n_\mu, m_\mu, n_\mu]\), where \( k_\mu \) is the wave four-vector, \( m_\mu \) is complex, and a star denotes complex conjugation. This tetrad is

10 Although \( \phi \) is a more common notation for a phase, \( w \) has been chosen so as to avoid confusions with the scalar field.
defined with respect to the background metric; all its vectors are null, and the only nonvanishing scalar products are
\[ \tilde{g}^{\mu\nu} k_\mu n_\nu = \tilde{g}^{\mu\nu} m_\mu m_\nu = 1. \] (43)
In terms of the null tetrad, the background metric reads
\[ \tilde{g}_{\mu\nu} = 2k_\mu n_\nu + 2m_\mu m_\nu. \] (44)
While \( k_\mu, n_\mu \) must be understood as being longitudinal with respect to the GW propagation, the complex vector \( m_\mu \) and its conjugate are transverse to it. Finally, as \( k_\mu \) is tangent to a null geodesic, it is parallel-transported along itself. For convenience, we decide the other vectors \( n_\mu, m_\mu \) (and hence \( m_\mu \)) to be parallel-transported as well,
\[ k_\nu n_{\mu\nu} = k_\nu m_{\mu\nu} = 0. \] (45)

Now, the harmonic gauge condition, in the eikonal approximation, reads
\[ H_{\mu\nu} k^\nu = \frac{1}{2} H k_\mu. \] (46)

In terms of the null tetrad, this can be shown to imply the following decomposition:
\[ H_{\mu\nu} = H_{\mu\nu}^{||} + H_{\mu\nu}^\perp, \] (47)
with
\[ H_{\mu\nu}^{||} = 2iH_\mu k_\nu, \] (48)
\[ H_{\mu\nu}^\perp = H_\perp m_\mu m_\nu + H_\perp m_\mu^* m_\nu^*. \] (49)

In the transverse mode \( H_{\mu\nu}^\perp \), the two complex numbers \( H_\perp, H_\perp^* \) represent the amplitudes of the left and right circular polarizations of the GW; see Appendix C for their relation with the plus and cross polarizations. The longitudinal mode \( H_{\mu\nu}^{||} \) depends on a vector field \( H_\mu \), and hence carries four degrees of freedom. In GR, this mode can be entirely removed by a gauge transformation. This operation is not possible when \( G_{\lambda\chi} \neq 0 \), as shown in Appendix C. Importantly, this longitudinal mode should not be confused with the (physical) longitudinal mode that would be carried by scalar waves.

2. The longitudinal mode is nonphysical

A good diagnostic of the physical content of a particular metric perturbation consists in computing the associated tidal forces. At linear order, the contribution of \( h_{\mu\nu} \) to the Riemann tensor reads
\[ \delta R_{\mu\nu\rho\sigma} = \frac{1}{2} \left( h_{\mu\nu,\rho\rho} - h_{\mu\nu,\rho\sigma} - h_{\nu\sigma,\mu\rho} + h_{\nu\rho,\mu\sigma} \right) \] (50)
\[ = \frac{1}{4} \left( -H_{\mu\rho} k_\rho k_\sigma + H_{\mu\rho} k_\nu k_\sigma \right. \right.
\[ + H_{\nu\sigma} k_\mu k_\rho - H_{\nu\rho} k_\mu k_\sigma \left. \right) e^{i\omega} + c.c. \] (51)

This is directly split into a contribution of the transverse mode and a contribution of the longitudinal mode, \( \delta R_{\mu\nu\rho\sigma}^{||} + \delta R_{\mu\nu\rho\sigma}^\perp \), and it is straightforward to check that
\[ \delta R_{\mu\nu\rho\sigma}^{||} = \frac{1}{2} \left( -H_{\mu\rho} k_\nu k_\sigma + H_{\mu\rho} k_\nu k_\sigma \right. \right.
\[ + H_{\nu\sigma} k_\mu k_\rho - H_{\nu\rho} k_\mu k_\sigma \left. \right) k^{\mu\nu} + c.c. \] (52)
\[ = 0. \] (53)

Therefore, the longitudinal mode \( H_{\mu\nu}^{||} \) does not produce any curvature in the geometric optics regime, and hence it does not interact with matter. Furthermore, we will see in Sec. IV A that this mode does not even carry energy-momentum in that regime. Therefore, we can safely consider it negligible, and discard it in the remainder of this section.

3. Propagation of the transverse mode

Projecting Eq. (42) successively on \( m_\mu m_\nu \) and \( m_\mu^* m_\nu^* \) yields the following propagation equation for the transverse amplitudes
\[ D \sqrt{G_\lambda H_\lambda} = 0, \] (54)
where \( H_\lambda \) denotes either \( H_\perp \) or \( H_\perp^* \) for short. In order to interpret Eq. (54), we shall first dedicate a few lines to the quantity \( k^{\mu\nu} \), which appears in the definition (40) of \( D \). For further details, we refer the curious reader to the Sec. 2.3 of Ref. [67], about the Sachs optical scalars. Consider a narrow bundle of null geodesics emerging from a GW source with a small solid angle \( \Omega_\lambda \). This solid angle subtends a small patch of the wavefront, whose area \( A \) grows as the wave propagates (see Fig. [i]). Let \( \lambda \) be the affine parameter along the bundle such that \( k^\mu = dx^\mu/d\lambda \). It can be shown that
\[ k^\mu = \frac{1}{A} \frac{dA}{d\lambda}, \] (55)
which thus represents the local expansion rate of the wavefront. An observer who would measure the area $A_0$ of the bundle at their location could then define a notion of distance

$$D \equiv \sqrt{\frac{A_0}{\Omega_0}},$$  \hspace{3cm} (56)

so that $k^\mu_{\mu} = 2d\ln D/d\lambda$. Equation (54) is then rewritten as

$$\frac{d}{d\lambda} \left[ \sqrt{G_4(\varphi)} D H_\odot \right] = 0,$$  \hspace{3cm} (57)

which shows that the GW amplitude evolves like $1/\sqrt{G_4}D$ as the wave propagates.

Quite importantly, $D$ as defined in Eq. (56) is not the usual angular-distance distance $D_A \equiv \sqrt{\Lambda_s/\Omega_0}$, because the roles of the source and the observer have been swapped. These are nevertheless related by Etherington’s reciprocity relation (68)

$$D = (1 + z) D_A,$$  \hspace{3cm} (58)

where the $(1 + z)$ factor comes from the fact that the solid angles $\Omega_s, \Omega_0$ are subject to different relativistic aberration effects. Equation (57) is the main result of this article. It shows that in four dimensions, one may wonder whether, and how, this gravitational distance $D_G$ is related to other known definitions of distance. For that purpose, it is useful to consider how $D_G$ is extracted from observations. As such, it is simply the quantity that is used to normalize the GW amplitude. In terms of the plus and cross polarizations, one has indeed (69)

$$h_+ = \frac{2M_c^2 \omega_0(\varphi_0)}{D_G} \left[ 1 + \cos^2 \iota \right] \cos w(t),$$  \hspace{3cm} (59)

$$h_\times = \frac{2M_c^2 \omega_0(t)}{D_G} \left[ 2 \cos \iota \sin w(t) \right],$$  \hspace{3cm} (60)

or, in terms of the amplitude of circular polarizations,

$$H_\odot = \frac{2M_c^2 \omega_0(\varphi_0)}{D_G} \left[ 1 + \cos^2 \iota \right]^2,$$  \hspace{3cm} (61)

where $M_c = (1 + z) M$, is the redshifted chirp mass of the binary system producing the GW. $\omega_0(t)$ is the observed GW cyclic frequency, and $\iota$ is the inclination angle formed between the line of sight and the orbital angular momentum of the binary. Since the observed frequency is related to the emitted one via $\omega_0 = \omega_c/(1 + z)$, we conclude that

$$H_\odot \propto \frac{1 + z}{D_G},$$ \hspace{3cm} (62)

where the proportionality factor only depends on the properties of the source. Therefore, combining Eq. (62) with Eq. (57), it appears that the gravitational distance reads

$$D_G = \sqrt{\frac{G_4(\varphi_0)}{G_4(\varphi_s)}} (1 + z)^2 D_A = \sqrt{\frac{G_4(\varphi_0)}{G_4(\varphi_s)}} D_h,$$  \hspace{3cm} (63)

where $\varphi_s$ and $\varphi_0$ are the values of the scalar field at emission and reception, respectively. In the last equality, we have introduced the standard electromagnetic luminosity distance, which is related to the angular distance by the distance-duality law $D_h = (1 + z)^2 D_A$ if the number of photon is conserved between the source and the observer.

We stress that although Eq. (63) has been already widely used in the literature, it had actually never been rigorously derived in the general context of a nonhomogeneous Universe and an arbitrary distribution for the scalar field. Our result thus justifies the use of Eq. (63) even in nonlinear setups, such as when the GW propagates through screened regions, as long as the eikonal approximation holds.

The importance of the fact that the gravitational distance $D_G$ only differs from the electromagnetic luminosity distance $D_L$ by the ratio of the local values of $G_4(\varphi)$ at emission and reception, was recently emphasized by Ref. [42]. This was shown to jeopardize standard-siren tests of modified gravity. On the one hand, if screening completely washes out the effect of $G_4$ in high-density regions, thereby forcing $G_4(\varphi_s) = G_4(\varphi_e)$, then $D_G = D_L$ and hence the modification of gravity remains as hidden as Odysseus’ men inside the Trojan horse. On the other hand, if $G_4(\varphi_s) \neq G_4(\varphi_e)$, then the resulting modification of the effective Planck mass also affects the explosion mechanism of type Ia supernovae. This effectively changes their observed luminosity distance in a way that potentially cancels the difference between $D_L$ and $D_G$.

More generally, we stress that in the case of theories where variations of the effective Planck mass are a universal time evolution, standard-siren tests cannot compete with the combination of lunar laser ranging, Shapiro time delay, and GW propagation speed. These lead to constraints whose precision beats LISA forecasts by four orders of magnitude. Having said that, because of Eq. (63), standard sirens are still relevant to constrain scenarios in which the Planck mass would be constant in time, but spatially inhomogeneous.

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14 This quantity can be accessed via the accurate measurement of the time evolution of the GW frequency [40].

15 The effectiveness of this cancellation depends on the exact relation between the supernova peak luminosity and the Chandrasekhar mass.
IV. PHYSICAL INTERPRETATION: GRAVITON-NUMBER CONSERVATION

In GR, the relation between luminosity distance and angular-diameter distance can be understood as a consequence of the conservation of photon number. In this section, we demonstrate that the relation (63) between \( D_0 \) and \( D_A \) can be interpreted as graviton-number conservation. This is true in GR but also in Horndeski models, where the concept of energy of a GW, and of a graviton, must be examined with care.

A. Energy-momentum of a gravitational wave

As a first step, it is instructive to determine the energy-momentum carried by a GW in Horndeski theories. In particular, we aim to determine the impact of the couplings \( G_3, G_4 \) to the scalar field. This paragraph essentially follows the logic of Ref. [64], adapting it from GR to Horndeski models.

By analogy with any form of matter, one defines the energy-momentum of a GW through its ability to gravitate, i.e. to generate spacetime curvature. More precisely, the energy-momentum tensor \( T_{\mu\nu}^{GW} \) of a GW quantities the backreaction of the metric perturbation \( h_{\mu\nu} \) on the background metric \( g_{\mu\nu} \). In order to evaluate it, one must expand the tensor equation of motion (10) at second order in \( h_{\mu\nu} \). This formally reads

\[
\sum_{i=2}^{4} \bar{E}_{\mu\nu}^i + \delta^4 E_{\mu\nu}^i + \delta^2 E_{\mu\nu}^i = M_p^{-2}(\bar{T}_{\mu\nu} + \delta^1 T_{\mu\nu} + \delta^2 T_{\mu\nu}).
\]  

(64)

where a superscript on \( \delta \) indicates the expansion order in \( h_{\mu\nu} \).

By construction, the first-order terms cancel. The second-order terms being quadratic in \( h_{\mu\nu} \) and its derivatives, they are highly oscillating about a generically non-zero mean. Hence, one can split any such term as

\[
\delta^2 E_{\mu\nu} = \langle \delta^2 E_{\mu\nu} \rangle + \delta^2 E_{\mu\nu}^{osc},
\]  

(65)

where the brackets must be understood as a spacetime average over a region that is typically much larger than the GW wavelength, but much smaller than the other relevant length scales of the problem. On the right-hand side of Eq. (65), the second term has zero average and is rapidly varying. On the contrary, the first term evolves on much larger scales, and hence is naturally thought of as a small correction to the background equation of motion:

\[
\sum_{i=2}^{4} \bar{E}_{\mu\nu}^i = M_p^{-2}(\bar{T}_{\mu\nu} + T_{\mu\nu}^{GW}).
\]  

(66)

where we defined

\[
T_{\mu\nu}^{GW} = \langle \delta^2 T_{\mu\nu} \rangle - M_p^2 \sum_{i=2}^{4} \langle \delta^2 E_{\mu\nu}^i \rangle.
\]  

(67)

This is how a GW gravitates.

As such, \( T_{\mu\nu}^{GW} \) is quadratic in \( h_{\mu\nu} \) and its derivatives. In fact, examining Eqs (11, 12, 13), it is easy to see that any quadratic term must take either of the following four forms:

\[
h^2, \partial h, (\partial h)^2, \partial^2 h.
\]  

The terms with only one derivative of \( h \) vanish on average, because they are proportional to \( \sin w \cos w \). The terms with two derivatives then completely overcome the terms with no derivatives, since

\[
(\partial h)^2, \partial^2 h \sim \omega^2 h^2.
\]  

(68)

and \( \omega \) is very large. Hence, we will only keep those in the expression of \( T_{\mu\nu}^{GW} \). It is straightforward to check that such terms can only come from \( \bar{E}_{\mu\nu}^4 \) and more precisely from the Einstein tensor. As a result,

\[
T_{\mu\nu}^{GW} = -G_4 M_p^2 \left( \delta^2 E_{\mu\nu} \right) = -M^2 \left( \delta^2 E_{\mu\nu} \right). \]  

(69)

In other words, the only difference with GR seems to be the conformal factor \( G_4 \). However, before applying standard results too quickly, we must remember that the cubic coupling \( G_3 \) generates a longitudinal mode \( h_{GW}^{0\nu} \). This prevented us, in particular, to impose the usual transverse-traceless gauge, in which the rest of the calculation is usually performed.

Fortunately, it turns out that the longitudinal mode does not change the final result for \( T_{\mu\nu}^{GW} \). Let us prove this point. In harmonic gauge, after several integrations by parts, we find

\[
\langle \delta^2 E_{\mu\nu} \rangle = -\frac{1}{4} \left( h_{\sigma\nu,\mu} h^{\sigma\nu} - \frac{1}{2} h_{\mu} h_{\nu} \right),
\]  

(70)

in which it can be explicitly checked that the longitudinal mode \( h_{GW}^{0\nu} \) does not contribute. This mode carrying neither momentum of a single graviton is related to its wavevector as \( p^\mu \approx h k^\mu \), where \( h \) is the reduced Planck constant

\[\text{[58] Does}\]
that the picture change in Horndeski theories? In particular, does
the conformal factor $G_A(\varphi)$ change the energy of a graviton?

A heuristic argument suggests a negative answer to the
above question, i.e. $p^\mu = h k^\mu$ even in the presence of $G_A(\varphi)$.
The conformal factor may be understood as a variation of
the effective Planck mass in space and time. But since $M_P$
is not involved in $p^\mu = h k^\mu$, why should its variation affect
that relation at all? In electromagnetism, an inhomogeneous
dielectric medium is essentially equivalent to vacuum with an
inhomogeneous permittivity $\varepsilon = n^2 e_0$, where $n$ is the optical
index of the medium [73]. Yet, such a modification does not
change that $p^\mu = h k^\mu$ for photons. We do not expect things to
be different in gravitation. The remainder of this subsection is an
attempt to further justify the above.

1. Effective action

Consider the classical behavior of the transverse modes in
the geometric optics regime. The left-handed and right-handed
modes $h_{+}, h_{-}$, defined as the projection over $m_{+} m_{-}$ and $m_{+} m_{-}$,
respectively, evolve independently, just like two scalar fields.
We also know that their typical wavelength is much shorter
than all the other distance scales of the problem.

Let $\mathcal{R}$ be a region of spacetime that is much larger than
the typical wavelength of the GWs under study, but smaller than
the typical scale over which $\varphi$ varies appreciably, and much
smaller than the curvature radius of the background spacetime
geometry. Thanks to the second condition, we can work in a
normal coordinate system $x^\alpha$ in which the background metric
is almost Minkowskian, $\bar{g}_{\alpha\beta} = \eta_{\alpha\beta}$ across $\mathcal{R}$.

An effective action for this system, leading to the correct
equation of motion for the transverse mode $h_{+}$, has the form
$S_{\text{eff}}[h_{+}] + S_{\text{eff}}[h_{-}]$, with

$$S_{\text{eff}}[h_{+}] = -\frac{1}{32\pi G} \int_{\mathcal{R}} d^{4} x \, G_A(\varphi) \, h_{+}^2 \partial \varphi .$$  \hspace{1cm} (72)

In Eq. (72), $h_{\pm} \equiv (1/2)H_{\pm} e^{i \omega t} + \text{c.c.}$, and, as before, $H_{\pm}$ refers to
either of the independent polarizations $H_{+}, H_{-}$. It is understood
that, when varying the action, one has to impose $\delta \varphi_{\pm} = 0$
and $\delta \partial \varphi_{\pm} = 0$ on $\partial \mathcal{R}$. It is then straightforward to check that $\delta S_{\text{eff}}/\delta h_{\pm} = 0$
implies $G_A \delta h_{\pm} + \partial \varphi_{\pm} h_{\pm} = 0$, which is equivalent to the
transverse part of Eq. (73).

At this point it is useful to define the canonical field

$$\chi = \sqrt{\frac{G_A(\varphi)}{16\pi G}} h_{\pm} ,$$  \hspace{1cm} (73)

which may be interpreted as a normalized version of the metric
perturbation in the Einstein frame. In terms of this variable,
the effective action is found to read

$$S_{\text{eff}}[\chi] = -\frac{1}{2} \int_{\mathcal{R}} d^{4} x \left[ \chi_{,\alpha} \chi^{\alpha} + m^2 \chi^2 \right] ,$$  \hspace{1cm} (74)

with the effective mass $m^2 = \Box \sqrt{G_A}/\sqrt{G_A}$. By definition, this
mass is almost constant across $\mathcal{R}$; furthermore, it is of the
same order of magnitude as the mass-like terms that we have
neglected so far thanks to the eikonal approximation. We can
thus safely neglect it as well in what follows.

2. Quantization

Since $m^2$ can be considered constant across $\mathcal{R}$, or even
neglected, the field $\chi$ is immediately quantized as $^{17}$

$$\hat{\chi}(x) = \int \frac{d^3 k}{(2\pi)^3} \sqrt{\frac{h}{2\omega(k)}} \left( \hat{a}_k e^{ik_\alpha x^\alpha} + \hat{a}^\dagger_k e^{-ik_\alpha x^\alpha} \right) .$$ \hspace{1cm} (75)

with $\omega^2(k) = k^2 + m^2 = k^2$, and $\hat{a}_k, \hat{a}^\dagger_k$ are the usual annihilation
and creation operators for $\chi$, with commutators

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = (2\pi)^3 \delta(k - k') , \hspace{1cm} \{ \hat{a}_k, \hat{a}^\dagger_{k'} \} = 0 .$$ \hspace{1cm} (76)

so as to ensure canonical quantization $^{18}$

The vacuum quantum state $|0\rangle$ is such that $\hat{a}_k |0\rangle = 0$, while
the application of $\hat{a}_k$ populates any state with a quantum of $\chi$
with wavevector $k$. Because of the proportionality [73]
between $\chi$ and $h$, these states also describe quantum states
of the original metric perturbation $h$. It turns out that they are
also quanta of energy, as shown in the next paragraph.

3. Quanta of energy-momentum

The energy-momentum tensor of one of the polarization
modes is directly obtained from its effective action as

$$T_{\alpha\beta} = \frac{G_A(\varphi)}{16\pi G} \left[ h_{\alpha}^2 \partial_\beta - \frac{1}{2} (h_{\gamma} h_{\alpha\beta}) \eta_{\alpha\beta} \right]$$

\hspace{5.4cm} (77)

$$= \chi_{,\alpha} \chi_{,\beta} - \frac{1}{2} (\chi_{,\gamma} \chi_{,\gamma}) \eta_{\alpha\beta} .$$ \hspace{1cm} (78)

Its quantum version $\hat{T}_{\alpha\beta}$, which is an operator on a Fock space,
is obtained by simply adding hats on $\chi$. The result is then
expressed in terms of the creation and annihilation operators as

$$\hat{T}_{\alpha\beta} = \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \sqrt{\frac{h}{2\omega(k)}} \left[ -\hat{a}_k \hat{a}^\dagger_{k'} e^{ik_\alpha x^\alpha} e^{ik'_\beta x'^\beta} + \hat{a}_k \hat{a}^\dagger_{k'} e^{ik_\alpha x^\alpha} e^{-ik'_\beta x'^\beta} + \hat{a}^\dagger_k \hat{a}_{k'} e^{-ik_\alpha x^\alpha} e^{ik'_\beta x'^\beta} - \hat{a}^\dagger_k \hat{a}_{k'} e^{-ik_\alpha x^\alpha} e^{-ik'_\beta x'^\beta} \right]$$

\hspace{5.4cm} (79)

Let $\Sigma$ be a spatial slice of $\mathcal{R}$ with $t = \text{const}$. The total energy
in $\Sigma$ plays the role of the Hamiltonian, and read $^{19}$

$$\hat{E} \equiv \int_{\Sigma} d^3 x \left( \hat{T}_{00} - \langle 0 | \hat{T}_{00} | 0 \rangle \right) = \int \frac{d^3 k}{(2\pi)^3} h \omega \hat{a}^\dagger_k \hat{a}_k ,$$ \hspace{1cm} (80)

where the energy of vacuum has been subtracted as usual. One-
particle states of the form $|k\rangle \equiv \hat{a}^\dagger_k |0\rangle$ are clearly eigenstates

---

$^{17}$ For consistency, the range of $k$ over which we integrate must be such that $1/k$ remains much smaller than the size of $\mathcal{R}$.

$^{18}$ Note in particular that the $\hat{A}_\alpha$ in Eq. (75) ensures that the commutation relation between $\chi$ and its conjugate momentum $\hat{\pi} = \partial_\alpha \chi$ reads

$$[\chi(t, x), \hat{\pi}(t, y)] = h \delta(x - y) .$$

$^{19}$ This result is only approximate, due to the finiteness of $\mathcal{R}$. 
of \( \hat{E} \), with eigenvalue \( h\omega \). Hence, they qualify as gravitons, according to our definition. It can be noted that they are not eigenstates of the spatial part of the momentum operator

\[
P^{\alpha \nu} \equiv \int_\Sigma d^3x \left( \hat{T}^{\alpha \nu} - \langle 0 \vert \hat{T}^{\alpha \nu} \vert 0 \rangle \right),
\]

for which terms of the form \( \hat{\partial}_k \hat{a} - \hat{a} \hat{\partial}_k \) remain. However, the expectation value of \( P^{\alpha \nu} \) on a one-particle state reads

\[
\langle k \vert P^{\alpha \nu} \vert k \rangle = \hbar k^\alpha,
\]

which we shall call a quantum of gravitational energy-momentum.

Summarizing, in any small region \( \mathcal{R} \) of spacetime, the transverse metric perturbation modes \( h_G \) and \( h_{\Sigma} \) essentially behave as canonical scalar fields. Their kinetic coupling with \( G_4(\varphi) \) effectively translates as a negligible mass \( m^2 = \Box \sqrt{G_{4\Sigma}}/\sqrt{G_4} \). Their quantization then leads to very standard results; in particular, a graviton with wavevector \( k \) carries an energy \( E = \hbar \omega \), and a linear momentum \( p^\mu = \hbar k^\mu \). Note that \( \omega^2 = k^2 + m^2 \approx k^2 \) is observer dependent. Here, it was implicitly defined with respect a normal coordinate system \( (x^\alpha) \). In general, an observer with four-velocity \( u^\mu \) would measure \( \omega = -u^\mu k_\mu \).

### C. Graviton-number conservation

Let us finally show how the classical relation (54) may be related to the conservation of graviton number. We will follow the same rationale as Ref. [67], § 1.3.2, which concerned photon-number conservation.

Consider an arbitrary observer in spacetime with four-velocity \( u^\mu \). From the expression (74) of the energy-momentum tensor \( T_{\mu \nu}^{GW} \) of a GW, we conclude that its four-momentum density with respect to the observer is

\[
P_{GW}^\mu = -u_\nu T_{\mu \nu}^{GW} = \frac{G_4 \omega}{64 \pi G} \langle |H|^2 \rangle k^\mu,
\]

where we denoted \( \langle |H|^2 \rangle \equiv \langle |H_\Sigma|^2 + |H_{\Sigma}|^2 \rangle \) for short, and \( \omega \equiv -u_\mu k^\mu \). Physically speaking, \( \rho_{GW} \equiv -u_\mu P_{GW}^\mu \) represents the energy density of the GW in the observer’s frame. Besides, the spatial projection \( \Pi_{GW}^\mu \equiv \langle u^\rho u_\gamma + \delta_\gamma^\rho \rangle P_{GW}^\mu \) on the observer’s local space represents the density of 3-momentum of the GW, as well as its energy flux density in the observer’s frame.

It is, therefore, very natural to define the graviton flux density four-vector as

\[
J^\mu \equiv \frac{P_{GW}^\mu}{\hbar \omega} = \frac{G_4}{64 \pi G \hbar} \langle |H|^2 \rangle k^\mu.
\]

Indeed, with such a definition, \( n \equiv -u_\mu J^\mu = \rho_{GW}/E_\Sigma \) clearly represents the number of gravitons per unit volume, while \( (u^\mu u_\nu + \delta_\nu^\mu)J^\nu = \Pi_{GW}^\mu / p_\Sigma \) represents the graviton flux density, i.e. the number of gravitons crossing an arbitrary surface, per unit time and per unit area. Note that, contrary to \( P_{GW}^\mu \), \( J^\mu \) is observer-independent.

The propagation equation (54) implies that \( J^\mu \) is divergence-free. Indeed, for each polarization amplitude \( H_\Sigma, H_{\Sigma} \),

\[
(G_4 |H_\Sigma|^2 k_\mu)_\mu = 0,
\]

so that finally

\[
J^\mu_\mu = 0.
\]

Equation (86) must be understood as a continuity equation. Consider indeed an arbitrary observer, and a small spatial domain \( \Sigma \) in their frame. Let \( (x^\alpha) = (t, x^i) \) be a normal coordinate system adapted to \( \Sigma \), i.e. such that \( \hat{g}_{\alpha \beta} \approx \eta_{\alpha \beta} \) across \( \Sigma \). Integrating Eq. (86) over that spatial domain then leads to

\[
\partial_t \int_\Sigma d^3x \ n = - \int_{\partial \Sigma} dA \ n \cdot J,
\]

where \( n \) is the outgoing normal vector orthogonal to the boundary \( \partial \Sigma \) of the spatial domain \( \Sigma \). In other words, the variation of the total number of gravitons in \( \Sigma \) is given by how many gravitons enter through its boundary.

### V. SUMMARY AND CONCLUSION

In this article, we have investigated the propagation of GWs in Horndeski theories of gravity. Given the strict observational constraints set by GW170817/GRB 170817, we restricted to the class of models in which GWs propagate at the speed of light. These consist of a combination of k-essence \( G_2(\varphi, X) \) conformally coupled spacetime geometry, \( G_4(\varphi)R \), and a cubic-Galileon kinetic braiding \( G_3(\varphi, X)X \varphi \). Only the last two can potentially affect the propagation of GWs.

We have found that, on the one hand, kinetic braiding introduces a nonphysical longitudinal mode \( h_{\parallel \parallel} \) with no observational consequences. Conformal coupling, on the other hand, affects the amplitude of GWs without modifying their polarization. The main consequence is that the gravitational distance \( D_G \) measured with standard sirens reads

\[
D_G = \sqrt{\frac{G_4(\varphi_0)}{G_4(\varphi)}} D_L = \frac{M_\odot}{M_s} D_L
\]

where \( D_L \) is the electromagnetic luminosity distance, and \( \varphi_0, \varphi \) are the local values of the scalar field at the emission and reception events. Equation (88) had already been derived for GWs propagating on a homogeneous-isotropic cosmological background, with a homogeneous scalar field. The increment of the present article is the rigorous proof of the validity of Eq. (88) for any background spacetime, and for any distribution for the scalar field. In particular, it holds if the GW is lensed; or if it is emitted, received, or propagates through screened regions. Furthermore, we have demonstrated that Eq. (88) corresponds to the conservation of the number of gravitons—a property that was already suspected in Ref. [39].

Albeit very general, our analysis relies on two important assumptions: (a) scalar waves can be neglected; and (b) GWs
are in the geometric-optics regime (eikonal approximation). The latter assumption is easily satisfied as long as spacetime geometry and the scalar field vary over astronomical scales, but it breaks down in the vicinity of compact objects, and for very low-frequency GWs. The former assumption is justified by (i) the suppression of scalar radiation for screened emitters, and (ii) the weak coupling between tensor and scalar perturbations, which limits the conversion of tensor waves into scalar waves. This point may nevertheless deserve further investigation, which is beyond the scope of the present article.

Standard sirens have been argued to be a key probe of variations of the effective Planck mass $M$ in Horndeski theories. However, since $D_G$ depends on local values of the scalar field at emission and reception, screening mechanisms may significantly limit the efficiency of this probe; see Ref. [42] for a more detailed discussion. As Horndeski escapes the sirens, we may recall the words of Athena at the end of the Odyssey, Horndeski, you are adaptable; you always find solutions.

**Appendix A: Mass-like terms in GW propagation**

In Sec. [III.A1] we split the linearized equation of motion for $h_{\mu
u}$ into kinetic, damping, and mass-like terms. What we called mass-like terms were all the terms of the form $M^A_{\mu
u} = M^A_{\mu\nu\rho\sigma} h^{\rho\sigma}$, where $M^A_{\mu\nu\rho\sigma}$ is a tensor depending on background quantities only. Their expressions are

\[
M^2_{\mu\nu\rho\sigma} = \frac{1}{4} \left( G_{2,XX} \varphi,\mu,\nu,\varphi + G_{2,XX} \tilde{g}_{\mu\nu} \right) \varphi,\rho,\varphi,\sigma - \frac{1}{2} G_{2,\mu\rho} \tilde{g}_{\nu\sigma}, \tag{A1}
\]

\[
M^3_{\mu\nu\rho\sigma} = \frac{1}{2} \left( G_{3,XX} \varphi,\mu,\nu,\varphi + (G_{3,XX} + \frac{1}{2} G_{2,XX} \tilde{g}_{\mu\nu}) \varphi,\rho,\varphi,\sigma + \left( G_{3,\varphi} + X G_{3,XX} - \frac{1}{2} G_{3,XX} \varphi,\rho,\varphi,\sigma \right) \tilde{g}_{\mu\nu} \right) \varphi,\rho,\varphi,\sigma - \frac{1}{2} G_{2,\mu\rho} \tilde{g}_{\nu\sigma}, \tag{A2}
\]

\[
M^4_{\mu\nu\rho\sigma} = \frac{G_4}{2} (2 \tilde{R}_{\mu\nu\rho\sigma} + 2 \tilde{R}_{\rho\mu\nu\sigma} - 2 \tilde{R}_{\nu\rho\mu\sigma} - \tilde{R}_{\mu\sigma\nu\rho} - 4 \tilde{g}_{\mu\rho} \tilde{g}_{\nu\sigma} + 2 \tilde{g}_{\mu\rho} \tilde{g}_{\nu\sigma}) - G_{4,\mu\rho} \tilde{g}_{\nu\sigma} + \Box G_4 \tilde{g}_{\mu\rho} \tilde{g}_{\nu\sigma}. \tag{A3}
\]

Note that, in Eqs. (A1), (A2), (A3), $X$ refers to the background $X = -\tilde{g}^{\mu\nu} \varphi,\mu,\nu,\varphi,\nu/2$.

**Appendix B: Null tetrad**

Let $u^\mu$ be the four-velocity of an arbitrary observer, normalized with respect to the background metric,

\[\tilde{g}_{\mu\nu} u^\mu u^\nu = -1. \tag{B1}\]

For a wave propagating along $k^\mu$, one defines the unit four-vector $d^\mu$ associated with the spatial propagation direction of the wave in the observer’s frame as

\[k^\mu = \omega (d^\mu + u^\mu), \tag{B2}\]

with $\omega = -\tilde{g}_{\mu\nu} u^\mu k^\nu$. The auxiliary null vector $n^\mu$ can then be defined as, e.g.,

\[n^\mu = \frac{1}{2\omega} (d^\mu - u^\mu). \tag{B3}\]

The piece of plane orthogonal to both $u^\mu$, $d^\mu$ can be seen as a spatial screen, orthogonal to the wave’s propagation direction. Introduce an orthonormal basis $(s^\mu_1, s^\mu_2)$ of that screen, i.e.

\[\tilde{g}_{\mu\nu} d^\mu s^\nu_1 = \tilde{g}_{\mu\nu} d^\mu s^\nu_2 = 0, \tag{B4}\]

\[\tilde{g}_{\mu\nu} s^\mu_1 s^\nu_2 = \delta_{AB}, \tag{B5}\]

where indices $A, B$ take values in $\{1, 2\}$. The last piece of the null tetrad can then be defined as

\[m^\mu = \frac{1}{\sqrt{2}} (s^\mu_1 + i s^\mu_2). \tag{B6}\]

The set $(u^\mu, d^\mu, s^\mu_1, s^\mu_2)$ forms a “traditional” tetrad, with one timelike and three spacelike vectors. The set $(k^\mu, n^\mu, m^\mu, (m^\mu)')$, besides, forms a null tetrad with the desired properties. It is straightforward to check that the background metric is expressed in terms of both bases as

\[\tilde{g}_{\mu\nu} = -u^\mu u^\nu + d^\mu d^\nu + \delta^{AB} s^A_1 s^B_2 \tag{B7}\]

\[= 2 k^\mu n_\nu + 2 m^\mu m^\nu. \tag{B8}\]

The transverse-traceless amplitude $H^+_{\mu \nu}$ of a GW can be decomposed either in terms of the traditional tetrad, which defines the usual plus (+) and cross (×) linear polarizations; or in terms of the null tetrad, which defines the left (⟲) and right (⟳) circular polarizations,

\[H^+_{\mu \nu} = H_+ (s^\mu_1 s^\nu_1 - s^\mu_2 s^\nu_2) + H_\times (s^\mu_1 s^\nu_2 + s^\mu_2 s^\nu_1) \tag{B9}\]

\[= H_\rightarrow m^\mu m^\nu + H_\leftarrow m^\mu m^\nu. \tag{B10}\]

These are, therefore, related by

\[H_\rightarrow = H_+ - i H_\times, \tag{B11}\]

\[H_\leftarrow = H_+ + i H_\times. \tag{B12}\]
Appendix C: Removing the longitudinal mode?

We have seen in Sec. III C that, in the harmonic gauge, the GW amplitude can be decomposed into a longitudinal and a transverse mode, \(H_{\mu\nu} = H^l_{\mu\nu} + H^\perp_{\mu\nu}\), where

\[
H^l_{\mu\nu} = 2H(\mu k_\nu)
\]

(C1)

involves a vector field \(H_\mu\) whose equation of motion follows from Eq. (42). Besides, in the geometric optics regime, we have seen that the longitudinal mode produces no tidal force (III C 2) and carries no energy-momentum (IVA). This mode being nonphysical, we naturally expected it to be removable by a suitable gauge transformation.

Let \(\xi_\mu\) be a gauge field generating the gauge transformation \(h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{(\mu)}\nu\). In the geometric optics regime, it is convenient to write the gauge field as a wave

\[
\xi_\mu = \frac{1}{2} \Xi_\mu e^{iw} + \text{c.c.},
\]

(C2)

where \(w\) is the same phase as the GW’s, and \(\Xi_\mu\) is a complex amplitude. Indeed, with this ansatz, the gauge transformation of the GW amplitude simply reads

\[
H_{\mu\nu} \rightarrow H_{\mu\nu} + 2\Xi_{(\mu)\nu}.
\]

(C3)

Comparing with Eq. (C1), one may immediately conclude that choosing \(\Xi_\mu = -H_\mu\) does remove the longitudinal mode. Albeit valid in GR, this choice is nevertheless forbidden if \(G_{3,\chi} \neq 0\), because \(\Xi_\mu\) and \(H_\mu\) do not satisfy the same propagation equation. Let us elaborate on that point. For the gauge field to preserve the harmonic gauge condition \(\gamma_{\mu
\nu}^{\text{prop}} = 0\), one must ensure that \(\Box \xi_\mu + \tilde{R}_\mu^\nu \xi_\nu \approx \Box \xi_\mu = 0\); that is, in the geometric optics regime

\[
k^\mu k_\mu = 0, \quad D\xi_\mu = 0,
\]

(C4)\n
(C5)

with \(k_\mu \equiv w_\mu\), and \(D = 2k^\mu \tilde{\nabla}_\mu + k^\mu k_\nu\) as in Eq. (40). The first condition is trivially satisfied, but the second one turns out to be incompatible with the propagation equation for \(H_\mu\). Indeed, contracting Eq. (42) with \(n^\mu (\delta_\nu^\rho - \frac{1}{3} n_\nu k_\rho)\), and renaming indices, we find

\[
G_4 D^\mu H_\mu = G_4^\alpha (H^\alpha_{\mu\nu} + H_\nu k_\mu) - \frac{1}{2} G_{3,\chi} \varphi^\rho \varphi^{\gamma} H_{\rho\nu} \varphi_{\mu\gamma}.
\]

(C6)

Thus, one cannot gauge this mode away without violating the harmonic gauge condition, which was necessary for all the above. We suspect that the longitudinal mode, albeit nonphysical in the geometric regime, may possess physical information beyond geometric optics, which would explain why it cannot be simply eliminated.

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