The Effect of Flow on the Resonance Absorption of Slow MHD Waves in Magnetic Flux Tubes

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Abstract

In this paper, we study kink and sausage oscillations in the presence of longitudinal background flow. We study the resonance absorption of kink and sausage modes in the slow continuum, under magnetic pore conditions, in the presence of flow. We determine the dispersion relation, and solve it numerically to find the frequencies and damping rates of the slow kink and sausage surface modes. We also obtain an analytical solution for the damping rate of the slow surface mode in the long wavelength limit. We show that in the presence of plasma flow, resonance absorption can result in strong damping for forward waves, and can be considered as an efficient mechanism to justify the extremely rapid damping of slow surface sausage waves observed in magnetic pores. Moreover, the plasma flow reduces the efficiency of resonance absorption to damp backward waves. With respect to pore conditions, resonance instability is avoided in our model.

\textbf{Unified Astronomy Thesaurus concepts: Solar physics (1476); Solar photosphere (1518); Solar atmosphere (1477)}

1. Introduction

The mechanism for the heating of the solar corona (and the coronae of stars in general) is not yet fully understood. Several non-thermal mechanisms have been proposed to explain this phenomenon, and the problem of justifying this phenomenon remains. Clearly, the heating must be tied to the magnetic field, as it is obvious that the heated areas have a non-potential magnetic field. Plasma is bounded by magnetic field lines, and can form many types of visible structure. One of these is the propagation of magnetohydrodynamic (MHD) waves and their damping. Resonance absorption was first proposed as the damping mechanism of MHD waves by Ionson (1978). With the launch of space satellites, the interest of theoretical physicists in studying waves in the solar atmosphere, in particular the use of resonance absorption, increased. Nakariakov et al. (1999) reported transverse oscillations in coronal loops with high damping rate. Ruderman & Roberts (2002) expressed the idea that the observed period of oscillation and its damping time can be used to determine the transverse density distribution in a coronal magnetic loop. This method was later used by many researchers (e.g., Goossens et al. 2002, 2008, 2012; Arregui et al. 2007; McEwan et al. 2008; Wang et al. 2009; Wang 2011; Moreels & Van Doorsselaere 2013; Soler et al. 2014; Moreels et al. 2015a, 2015b; Ebrahimi & Karami 2016; Wang 2016; Raes et al. 2017).

Since the source of the high-temperature energy of the corona originates from the convection zone below the surface of the Sun, it is important to study the dynamics of MHD waves in the photosphere and chromosphere (e.g., Jess et al. 2015; Jess & Verth 2016). In the photosphere, in addition to Alfvén resonance, energy transfer by slow resonance absorption is of particular importance. Yu et al. (2017a) showed that slow resonance absorption can affect the damping of waves in the photosphere. They also found that the resonance damping of the fast surface kink mode is much stronger than that of the slow surface kink mode. Yu et al. (2017b) considered a linear profile for density and pressure in the transitional layers. They showed that in cases where damping via the Alfvén continuum is weak, the resonance absorption in the slow continuum may be an effective mechanism for damping sausage and kink slow surface modes. Sadeghi & Karami (2019) investigated resonance absorption in the presence of a weak magnetic twist in the photosphere’s condition. They concluded that a magnetic twist could affect more intense damping. In this paper, we study the effect of flow on slow sausage and kink MHD waves, as observed by the Dunn Solar Telescope in Grant et al. (2015).

Observations by Brekke et al. (1997) Winebarger et al. (2001, 2002), Teriaca et al. (2004), Doyle et al. (2006), Ofman & Wang (2008), and Tian et al. (2008, 2009) showed that plasma flows in magnetic flux tubes are present everywhere in the solar atmosphere. Soler et al. (2011) reported that the flow velocities are usually less than 10% of the plasma Alfvén speed. Grant et al. (2015) investigated wave damping observed in upwardly propagating sausage mode oscillations, contained within a magnetic pore. They showed that the waves propagate through only 0.25 of its wavelength before damping, whereas theoretically, one would expect the wave to survive for the distance of a few wavelengths. They also showed that the average upflow speed in the photosphere is about one-third Alfvén speed. In addition, higher speeds, up to about 1.15 Alfvén speed, have been observed by Grant et al. (2015). MHD oscillations of flowing plasma have been investigated by a number of researchers, e.g., Goossens et al. (1992), and Bahari & Shahhosaini (2020). Joarder et al. (1997) investigated the resonance instability of MHD waves in the presence of plasma flow. They showed that if the plasma velocity is greater than a certain value, it will cause instability. Soler et al. (2011) studied the damping length of resonantly kink waves in static flux tubes, both analytically and numerically. They showed that flow affects the wavelength and the damping length due to resonance absorption. Bahari (2018) considered propagating kink MHD waves in the presence of magnetic twist and plasma flow. He showed that the damping of the waves depends on the direction of plasma flow, and the wavenumber of the wave. Bahari et al. (2020) studied the propagation and instability of kink waves in a twisted magnetic tube in the presence of flow. They showed that, for particular values of flow speed in coronal
continuously. In the cylindrical coordinate, the magnetic equilibrium and stationary quantities are constant, and a consists of interior and exterior regions, in which the magnetic field, in the direction of the tube axis. The model presented. We consider a model similar to that introduced by Yu et al. (2017b), but with plasma flow included. In Section 2, the model, and the basic equations of motion governing the MHD waves are presented. We find the dispersion relation in the case of no inhomogeneous layer in Section 3. In Section 4, we obtain the dispersion relation in the presence of the inhomogeneous layer, using the connection formula for the slow continuum. In Section 5, the results of our numerical calculations are discussed. Finally, we conclude the paper in Section 6.

2. Equations of Motion and Model

The linear perturbations of homogeneous flowing magnetized plasma are governed by the following equations, as given by Kadomtsev (1966):

\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{B} = - \nabla \rho - \frac{1}{\mu_0} \left( \delta \mathbf{B} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \delta \mathbf{B}) \right),
\]

\[
\delta \mathbf{B} = - \nabla \times (\mathbf{B} \times \xi),
\]

\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = - \nabla p + \rho \mathbf{g},
\]

\[
\frac{\partial}{\partial t} \delta \mathbf{p} = \rho \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial}{\partial t} \mathbf{v} \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{v} - \frac{1}{\mu_0} \left( \delta \mathbf{B} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \delta \mathbf{B}) \right),
\]

\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{E} = 0,
\]

\[
\frac{\partial}{\partial t} \delta \mathbf{E} = \mathbf{E} \cdot \nabla \mathbf{v} + \frac{\partial}{\partial t} \mathbf{v} \times \mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{v}) - \frac{1}{\mu_0} \left( \delta \mathbf{B} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \delta \mathbf{B}) \right).
\]

where \(\rho\) and \(\mathbf{B}\) are background quantities, representing density, kinetic pressure, plasma velocity, and magnetic field, respectively; \(\xi\) is the Lagrangian displacement vector, and \(\delta \rho\) and \(\delta \mathbf{B}\) are the Eulerian perturbations of the pressure and magnetic field, respectively. Here, \(\gamma\) is the ratio of specific heats (taken to be 5/3 in this work), and \(\mu_0\) is the permeability of free space.

We consider a flux tube model with a unidirectional magnetic field, in the direction of the tube axis. The model consists of interior and exterior regions, in which the equilibrium and stationary quantities are constant, and a transitional layer, in which the background quantities vary continuously. In the cylindrical coordinate, the magnetic field is

\[
\mathbf{B} = (0, 0, B_z(r)).
\]

Plasma pressure and magnetic field must be satisfied in the hydrostatic equilibrium equation:

\[
\frac{d}{dr} \left( p + \frac{B_z^2}{2\mu_0} \right) = 0.
\]

Here, the background plasma density and magnetic field are assumed to be the same as those considered by Sadeghi & Karami (2019)

\[
\rho(r) = \begin{cases} 
\rho_i, & r \leq r_i, \\
\rho_i + (\rho_e - \rho_i) \left( \frac{r - r_i}{r_e - r_i} \right), & r_i < r < r_e, \\
\rho_e, & r \geq r_e,
\end{cases}
\]

where \(r_i = R - l/2\), and \(r_e = R + l/2\). Here, \(R\) and \(l\) are the tube radius and the thickness of the inhomogeneous layer, respectively, and

\[
B_z^2(r) = \begin{cases} 
B_z^2, & r \leq r_i, \\
B_z^2 + \left( B_{zi}^2 - B_z^2 \right) \left( \frac{r - r_i}{r_e - r_i} \right), & r_i < r < r_e, \\
B_{ze}^2, & r \geq r_e.
\end{cases}
\]

In these equations \(\rho_i\) and \(\rho_e\) are the constant densities of the interior and exterior regions of the flux tube, respectively; \(B_{zi}\) and \(B_{ze}\) are the interior and exterior constant longitudinal magnetic fields, respectively. For the magnetic pore conditions examined in this work, the density inside the tube is slightly larger than the density outside of the tube, but the magnetic field inside the tube is much stronger than the magnetic field outside the tube. Putting Equation (5) into the magnetohydrostatic Equation (3), we obtain the background gas pressure as follows:

\[
p(r) = \begin{cases} 
\rho_i, & r \leq r_i, \\
\rho_i + \left( \rho_e - \rho_i \right) \left( \frac{r - r_i}{r_e - r_i} \right), & r_i < r < r_e, \\
\rho_e, & r \geq r_e,
\end{cases}
\]

where \(\rho_i\) is an arbitrary constant. The plasma flow is considered to be in the direction of the magnetic field lines, and occurs in the various regions as follows:

\[
v_z(r) = \begin{cases} 
v_{zi}, & r \leq r_i, \\
v_{zi} + \left( v_{ze} - v_{zi} \right) \left( \frac{r - r_i}{r_e - r_i} \right), & r_i < r < r_e, \\
v_{ze}, & r \geq r_e,
\end{cases}
\]

where \(v_{zi}\) and \(v_{ze}\) are the constant flow of the interior and exterior regions of the flux tube, respectively. In addition, we define the following quantities:

\[
v_{A(i,e)}^2 \equiv \frac{B_{z(i,e)}^2}{\rho_{0(i,e)}},
\]

\[
v_{v(i,e)}^2 \equiv \frac{\gamma (\rho_{i,e})^\gamma}{\rho_{0(i,e)}},
\]

\[
v_{c(i,e)}^2 \equiv \frac{v_{A(i,e)}^2 v_{A(i,e)}^2}{v_{v(i,e)}^2 + v_{A(i,e)}^2},
\]

where \(v_{A(i,e)}\), \(v_{v(i,e)}\), and \(v_{c(i,e)}\) are the interior/exterior Alfvén, sound, and cusp velocities, respectively.

Since the hydrostatic equilibrium is only a function of \(r\), all the perturbed quantities, including \(\xi\) and \(\delta \mathbf{P}_T\), can be Fourier analyzed as

\[
(\xi, \delta \mathbf{P}_T) \propto e^{(m \phi + k_c z - \omega t)}.
\]

Here, \(\omega\) is the oscillation frequency, \(m\) is the azimuthal wavenumber for which only integer values are allowed, and \(k_c\)
is the longitudinal wavenumber in the $z$ direction. We study both forward and backward waves, propagating in the positive and negative $z$ directions, respectively. For both waves, the longitudinal wavenumber is restricted to positive values; the oscillation frequency is positive for forward waves, and negative for backward waves. The perturbed quantity, $\delta P_T = \delta p + B \cdot \delta B / \rho_0$, is the Eulerian perturbation of total (gas and magnetic) pressure. Putting Equation (12) into (1a)–(c), we obtain two coupled first order differential equations:

$$\mathcal{D} \frac{d(r \xi)}{dr} = -r C_2 \delta P_T,$$

$$\mathcal{D} \frac{d \delta P_T}{dr} = C_3 \xi.$$  \hspace{1cm} (13a, 13b)

The above equations were derived for the first time by Hain & Lust (1958), and later by Goedbloed (1971), Appert et al. (1974), and Sakurai et al. (1991). Here, the multiplicative factors are defined as

$$\mathcal{D} \equiv \rho (v_z^2 + v_\perp^2) (\Omega^2 - \omega_\perp^2) (\Omega^2 - \omega_c^2),$$

$$C_2 \equiv \Omega^4 - \left( k_z^2 + \frac{m^2}{r^2} \right) (v_z^2 + v_\perp^2) (\Omega^2 - \omega_c^2),$$

$$C_3 \equiv \rho \mathcal{D} (\Omega^2 - \omega_\perp^2),$$

in which

$$f_B \equiv k_z B_z,$$

$$\omega_\perp^2 \equiv \frac{f_B^2}{\rho_0},$$

$$\omega_c^2 \equiv \left( \frac{v_z^2}{v_\perp^2 + v_\perp^2} \right) \omega_\perp^2.$$

Here $\Omega = \omega - \omega_f$ is the Doppler-shifted frequency, where $\omega_f = \omega_z v_z$ is the flow frequency, $\omega_\perp = \omega_z v_\perp$ is the Alfvén frequency, and $\omega_c = k_z v_c$ is the cusp frequency. Also $v_\perp = B_c / \sqrt{\rho_0 \rho}$ is the Alfvén speed, $v_c = \sqrt{\gamma p / \rho}$ is the sound speed, and $v_\perp = \sqrt{\gamma p / \rho_0}$ is the cusp speed.

Following Edwin & Roberts (1983), by combining Equations (13a) and (13b), one can eliminate radial displacement, and obtain a second order ordinary differential equation for $\delta P_T$:

$$\frac{d^2 \delta P_T}{dr^2} + \frac{1}{r} \frac{d \delta P_T}{dr} - \left( k_z^2 + \frac{m^2}{r^2} \right) \delta P_T = 0,$$  \hspace{1cm} (15)

where

$$k_z^2 \equiv \frac{(\omega_\perp^2 - \Omega^2)(\omega_\perp^2 - \Omega^2)}{(\omega_\perp^2 + v_\perp^2)(\omega_c^2 - \Omega^2)}.$$  \hspace{1cm} (16)

The solutions of Equation (15) in the interior ($r \leq r_i$) and exterior ($r \geq r_e$) regions are given by

$$\delta P_{T_i}(r) = A_i I_0(k_z r),$$

$$\delta P_{T_e}(r) = A_e K_0(k_z r),$$  \hspace{1cm} (17a, 17b)

where $A_i$ and $A_e$ are constants. Here, $I_0$ and $K_0$ are the modified Bessel functions of the first and second kind, respectively. Replacing the solutions 17(a) and (b) into Equation 13(b), the radial displacement can be determined as

$$\xi_{r_i}(r) = \frac{A_i}{\rho_i (\Omega^2 - \omega_\perp^2)} I_0'(k_z r),$$

$$\xi_{r_e}(r) = \frac{A_e}{\rho_e (\Omega^2 - \omega_\perp^2)} K_0'(k_z r),$$  \hspace{1cm} (18a, 18b)

in which the prime denotes the differentiation of the function with respect to its argument. These solutions are used in the following two sections to determine the dispersion relation of the tube oscillations.

3. Dispersion Relation for the Case of No Inhomogeneous Layer

In this section we consider a flux tube without the inhomogeneous layer, and obtain the dispersion relation of the oscillations. For this purpose, the solutions obtained for $\xi_r$ and $\delta P_T$ in the previous section, inside and outside the tube (i.e., Equations 17(a)–18(b)), must be placed within the following boundary conditions:

$$\xi_{r_i} |_{r=R} = \xi_{r_e} |_{r=R},$$

$$\delta P_{Ti} |_{r=R} = \delta P_{Te} |_{r=R},$$  \hspace{1cm} (19a, 19b)

where $R$ is the tube radius. The dispersion relation can then be determined, after some algebra, as

$$\rho_i (\Omega_i^2 - \omega_\perp^2) - \frac{k_z}{k_{re}} \rho_e (\Omega_e^2 - \omega_\perp^2) Q_m = 0,$$  \hspace{1cm} (20)

where

$$Q_m = \frac{I_0'(k_z R) K_0(k_{re} R)}{I_0(k_z R) K_0'(k_{re} R)}.$$

For the case of no flow ($\Omega_i = \Omega_e = \omega$), the dispersion relation reduces to the result obtained by Edwin & Roberts (1983), and Yu et al. (2017b). We have solved the dispersion relation (20) numerically, and have determined the phase speed, $\Omega_i/\omega_\perp$, of the slow surface sausage ($m = 0$) and kink ($m = 1$) modes versus $k_z R$ for various values of the flow parameter $v_{zi}/v_{si}$, for which the results are displayed in Figure 1. Panels (a) and (b) refer to forward sausage and kink modes, and panels (c) and (d) refer to backward sausage and kink modes, respectively. The figure shows that (i) for a given value of $k_z R$, forward waves when the flow speed increases, the Doppler-shifted phase speed decreases and for backward waves the magnitude of the phase speed increases. (ii) For a given flow speed, $v_{zi}/v_{si}$, as $k_z R$ increases, the Doppler-shifted phase speed for forward decreases, and the magnitude of the Doppler-shifted phase speed for backward increases. (iii) For $k_z R \ll 1$, for both forward and backward waves, $\Omega_i/\omega_\perp$ goes to $\omega_{zi}/\omega_{si}$. (iv) These results show that for specific values of the flow speed, the Doppler-shifted phase speed is between the internal and external values of the cusp speed of the flux tube. (vi) For the case of no flow, the result of Yu et al. (2017b) is recovered.

4. Dispersion Relation in the Presence of Inhomogeneous Layer and Resonance Absorption

In this section we consider a flux tube with an inhomogeneous boundary layer. According to Equations (4)–(6), the
density, magnetic field, and pressure change continuously from the inside to the outside of the tube, so in this case the Doppler-shifted frequency \( \Omega \) of the waves may be equal to the cusp \( \omega_c \) or Alfvén \( \omega_A \) frequency. According to Yu et al. (2017b), under photosphere conditions, the oscillation frequency will be equal to the cusp frequency at a point in the boundary layer which causes a singularity in the equations of motion. This phenomenon is known as cusp resonance absorption.

Sakurai et al. (1991) showed that the solutions inside and outside the tube can be connected, using the connection formula

\[
[\xi_r] \equiv \xi_{rc}(r_c) - \xi_{ri}(r_i) = -i\pi \frac{\text{Sign } \Omega}{|\Delta_c|} \frac{\mu \omega_c^2}{rB^2\omega_A^2} \delta P_{Tc},
\]

\[
[\delta P_r] \equiv \delta P_{rc}(r_c) - \delta P_{ri}(r_i) = 0,
\]

where \( [\xi_r] \) and \( [\delta P_r] \) represent the jumps for the radial component of Lagrangian displacement, and the total pressure perturbation across the inhomogeneous (resonance) boundary, which connects the solutions inside and outside of the flux tube. The subscript \( c \) in \( \Delta_c \) shows that this quantity must be calculated at the cusp resonance point \( r_c \), where we determine its location subsequently. We obtain the dispersion relation in the presence of flow by substituting the solutions 17(a)–18(b) into the connection formula 21(a) and (b). The result is as follows:

\[
\rho_i(\Omega_i^2 - \omega_{ci}^2) - \rho_c(\Omega_c^2 - \omega_{ce}^2) \frac{k_{ni}}{k_{re}} Q_m + i\pi \frac{\text{Sign } \Omega k_c^2}{|\Delta_c| \rho_c} \times \left( \frac{v_i^2}{v_i^2 + v_A^2} \right) \rho_i \rho_c (\Omega_i^2 - \omega_{ci}^2)(\Omega_c^2 - \omega_{ce}^2) \frac{G_m}{k_{re}} = 0,
\]
where \( G_m = \frac{\kappa_\alpha(k_\alpha, r_c)}{\kappa_\alpha(k_\alpha, r_i)} \). It is clear that in the absence of plasma flow, this equation reduces to the dispersion relation obtained by Yu et al. (2017b).

To display the background quantities in the boundary layer, we define the variable \( \delta = \frac{\rho}{\rho_i} \), which varies from 0 to 1 in the boundary layer. Using Equations (4)–(6), one can write the quantities \( v_i = \sqrt{\rho_i/\rho} \) \( v_A = |B_i|/\sqrt{\mu_0\rho} \) in the inhomogeneous boundary layer as functions of \( \delta \):

\[
v_i^2 = v_A^2 \left[ 1 + \delta (\chi v_{sci}^2 - 1) \right] / \left[ 1 + \delta (\chi - 1) \right],
\]

(23)

\[
v_A^2 = v_A^2 \left[ 1 + \delta (\chi v_{sci}^2 - 1) \right] / \left[ 1 + \delta (\chi - 1) \right],
\]

(24)

and the cusp speed, \( v_c = \frac{v_{sci}}{v_i + v_A^2/v_c^2} \), in the inhomogeneous layer \((r_i < r < r_c)\) becomes

\[
v_c^2 = \frac{v_s^2 v_A^2 [1 + \delta (\chi v_{sci}^2 - 1)][1 + \delta (\chi v_{sci}^2 - 1)]}{[1 + \delta (\chi - 1)][v_s^2 (1 + \delta (\chi v_{sci}^2 - 1)) + v_A^2 (1 + \delta (\chi v_{sci}^2 - 1))]}
\]

(25)

in which \( \chi \equiv \rho_0/\rho_i \), \( v_{sci} \equiv v_{sci}/v_{sci} \), and \( v_{sci} \equiv v_A/v_{sci} \). Using Equations (23)–(25), we plot the sound, Alfvén, and cusp speeds under magnetic pore conditions in Figure 2. The figure shows that in order for the surface sausage and kink modes to resonantly damp in the slow continuum, we should have \( v_c < v_{sci} \). In addition, the body sausage and kink modes can resonantly damp in the slow continuum under the condition \( v_c < v_{sci} < v_{sci} \). Here, \( v_{sci} \) is the maximum value of the cusp speed in the transition layer.

Note that, according to Yu et al. (2017b), the position of the cusp resonance point, \( r_c \), is obtained by setting \( \Omega^2 = \omega_c^2 \mid_{r=r_c} = \kappa_\alpha v_c^2 \mid_{r=r_c} \). Consequently, the resulting equation, in terms of the variable \( \delta_c = \delta \mid_{r=r_c} = \frac{r_c - r_i}{r_c - r_i} \), yields the following second order equation:

\[
A \delta_c^2 + B \delta_c + C = 0,
\]

(26)

where \( A, B, \) and \( C \) are similar to the constants defined in Equations (55)–(57) in Yu et al. (2017b). The solutions for \( \delta_c \) are

\[
\delta_{c1} = -\frac{B}{2A} + \sqrt{\frac{B^2 - 4AC}{2A}}, \quad (27)
\]

\[
\delta_{c2} = -\frac{B}{2A} - \sqrt{\frac{B^2 - 4AC}{2A}}. \quad (28)
\]

For the slow surface sausage and kink modes, the oscillation frequency \( \Omega/\omega_{AI} \) is below \( v_{sci} \), which means that only \( \delta_{c2} \) satisfies this condition in Yu et al. (2017b).

Next, we calculate the parameter \( \Delta_c \), which appeared in the dispersion relation (22). To this end, using Equation (25) and \( \omega_c^2(r_c) = k_c^2 v_c^2 \mid_{r=r_c} \), we obtain

\[
\Delta_c \equiv \left[ \frac{d}{dr}(\Omega^2 - \omega_c^2) \right]_{r=r_c} = -2\left(\omega - \omega_f\right) \frac{d\omega_f}{dr} + \omega_f \frac{d\omega_c}{dr} \mid_{r=r_c} = -2\left(\omega - \omega_f(r_c)\right) (\omega_f - \omega_f) \mid_{l} = -\frac{\omega_c^2(r_c)}{l}
\]

\[
\times \left\{ \frac{\chi v_{sci}^2 - 1}{1 + \delta (\chi v_{sci}^2 - 1)} \right\} - \frac{\chi v_{sci}^2 - 1}{1 + \delta (\chi v_{sci}^2 - 1)} + \frac{\chi v_{sci}^2 - 1}{1 + \delta (\chi v_{sci}^2 - 1)} - \frac{v_c^2 (v_{sci}^2 - 1)}{v_s^2 [1 + \delta (\chi v_{sci}^2 - 1)] + v_A^2 [1 + \delta (\chi v_{sci}^2 - 1)]} \right\} \mid_{r=r_c}, \quad (29)
\]

\[
= \Delta_c (r_c) = \frac{d}{dr}(\Omega^2 - \omega_c^2) \mid_{r=r_c}. \quad (29)
\]
4.1. Weak Damping Limit—Slow Continuum

In this subsection we study the dispersion relation (22) in the weak damping limit. We first rewrite the dispersion relation as

\[ D_{AR} + iD_{AI} = 0, \]  

where \( D_{AR} \) and \( D_{AI} \) are the real and imaginary parts of Equation (22), respectively, given by

\[ D_{AR} = \rho_{c}(\Omega_e^2 - \omega_{\Delta_e}^2) - \rho_{e}(\Omega_e^2 - \omega_{\Delta_e}^2) \frac{k_{ri}}{k_{re}} Q_m, \]  

\[ D_{AI} = \frac{\pi \rho_{c} \rho_{e} k_{s}^2}{k_{re}} \text{Sign} \Omega \left| \left( \frac{v_{ii}^2}{v_{ki}^2} + \frac{v_{ie}^2}{v_{ke}^2} \right) \right|^2 \times (\Omega_e^2 - \omega_{\Delta_e}^2)(\Omega_e^2 - \omega_{\Delta_e}^2) G_m. \]  

Note that in Equations (31) and (32) we have the complex frequency \( \omega = \omega_i + i\gamma_m \). As in Goossens et al. (1992) in the limit of weak damping, i.e., \( \gamma_m \ll \omega_i \), the imaginary part \( \gamma_m \) is given by

\[ \gamma_m = -D_{AI}(\omega) \frac{\partial D_{AR}}{\partial \omega} \bigg|_{\gamma_m}^{-1}. \]  

We now simplify Equation (33) to obtain the damping rate of surface sausage modes in the weak damping limit, i.e., \( \gamma_m \ll \omega_i \). To this end, we first calculate \( \frac{\partial D_{AR}}{\partial \omega} \) from Equation (31) as follows:

\[ \frac{\partial D_{AR}}{\partial \omega} = 2\rho_{c} \Omega_i - 2\rho_{e} \Omega_e \frac{k_{ri}}{k_{re}} Q_m - \rho_{e} (\Omega_e - \omega_{\Delta_e}^2) \]

\[ \times \left( \frac{1}{k_{re} \partial \omega} - \frac{k_{ri}}{k_{re} \partial \omega} \right) Q_m - \rho_{e} (\Omega_e^2 - \omega_{\Delta_e}^2) \]

\[ = \frac{k_{ri} \partial Q_m}{k_{re} \partial \omega}. \]  

From Equation (16), one can obtain

\[ \frac{dk_{ri}}{d\omega} = -\Omega_i^2 (\Omega_i^2 - 2\omega_{\Delta_i}^2) \]

\[ = \frac{-\Omega_i^2 (\Omega_i^2 - 2\omega_{\Delta_i}^2)}{(v_{ii}^2 + v_{ki}^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^2 k_{ri}}, \]  

\[ \frac{dk_{re}}{d\omega} = -\Omega_i^2 (\Omega_i^2 - 2\omega_{\Delta_i}^2) \]

\[ = \frac{-\Omega_i^2 (\Omega_i^2 - 2\omega_{\Delta_i}^2)}{(v_{ie}^2 + v_{ke}^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^2 k_{re}}. \]  

With the help of Equations (35) and (36), the following expression for \( \frac{dQ_m}{d\omega} \) is determined:

\[ \frac{dQ_m}{d\omega} = x_{Pm} \frac{\Omega_i^2 (\Omega_i^2 - 2\omega_{\Delta_i}^2)}{(\omega_{ai}^2 - \Omega_i^2)(\omega_{ai}^2 - \Omega_e^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^2} + y_{Sm} \frac{\Omega_i^3 (\Omega_i^2 - 2\omega_{\Delta_i}^2)}{(\omega_{ei}^2 - \Omega_i^2)(\omega_{ei}^2 - \Omega_e^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^3}, \]  

with the definitions

\[ P_m = \left( \frac{\mathcal{I}_m'(x)}{\mathcal{I}_m(x)} - \frac{\mathcal{I}_m'(y)}{\mathcal{I}_m(y)} \right) \text{K}_m(y), \]

\[ S_m = \left( \frac{1 - \text{K}_m'(y)^2}{\text{K}_m'(y)^2} \right) \frac{\mathcal{I}_m'(x)}{\mathcal{I}_m(x)}. \]  

in which \( x = k_{ri} \) and \( y = k_{re} c \). Substituting Equation (37) into Equation (34) yields

\[ \frac{\partial D_{AR}}{\partial \omega} = 2\rho_{c} \Omega_i - 2\rho_{e} \Omega_e \frac{k_{ri}}{k_{re}} Q_m - \rho_{e} (\Omega_e^2 - \omega_{\Delta_e}^2) \frac{k_{ri}}{k_{re}} \]

\[ \times \left( \frac{(Q_m + x_{Pm})(\Omega_i^2 - 2\omega_{\Delta_i}^2)\Omega_i^3}{(\omega_{ai}^2 - \Omega_i^2)(\omega_{ai}^2 - \Omega_e^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^2} \right) \]

\[ \times \left( \frac{(Q_m - y_{Sm})(\Omega_i^2 - 2\omega_{\Delta_i}^2)\Omega_i^3}{(\omega_{ei}^2 - \Omega_i^2)(\omega_{ei}^2 - \Omega_e^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^3} \right). \]  

Finally, substituting Equations (32) and (39) into Equation (33), one can obtain \( \gamma_m \) in the limit of weak damping for the surface modes in the slow continuum:

\[ \gamma_m \bigg|_{\omega = \omega_i} = -\frac{\pi \rho_{c} \rho_{e} k_{s}^2}{k_{re}} \text{Sign} \Omega \left| \left( \frac{v_{ii}^2}{v_{ki}^2} + \frac{v_{ie}^2}{v_{ke}^2} \right) (\Omega_i^2 - \omega_{\Delta_i}^2)(\Omega_e^2 - \omega_{\Delta_e}^2) G_m \right| \]

\[ = \frac{2 (\Omega_i - \chi_{\Delta_e} k_{re} Q_m) - \chi T_m}{\Omega_i - \chi_T}, \]  

where

\[ T_m = (\Omega_e^2 - \omega_{\Delta_e}^2) \frac{k_{ri}}{k_{re}} \left( \frac{(Q_m + x_{Pm})(\Omega_i^2 - 2\omega_{\Delta_i}^2)\Omega_i^3}{(\omega_{ai}^2 - \Omega_i^2)(\omega_{ai}^2 - \Omega_e^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^2} \right) \]

\[ \times \left( \frac{(Q_m - y_{Sm})(\Omega_i^2 - 2\omega_{\Delta_i}^2)\Omega_i^3}{(\omega_{ei}^2 - \Omega_i^2)(\omega_{ei}^2 - \Omega_e^2)(\Omega_i^2 - \omega_{\Delta_i}^2)^3} \right). \]  

Equation (40) can be further simplified in the long wavelength limit, as shown in the following subsection.

4.2. Weak Damping in Long Wavelength Limit—Slow Continuum

In the limit \( k_{s} R \ll 1 \) i.e., \( k_{s} R \ll 1 \), and \( k_{s} R \ll 1 \), we can obtain a more simplified expression for \( \gamma_m \) using the asymptotic expansion of \( Q_m, G_m, P_m \), and \( S_m \). For the sausage \( (m = 0) \) mode in the slow continuum, we obtain (see Appendix A)

\[ \gamma_{0e} = 2\pi^3 \text{Sign} \Omega \left[ \frac{(-1)^{\frac{\theta_0 - \theta_1}{2}}}{|\Delta_e| R} \frac{\omega_{ai}^2(\Omega_i^2 - \omega_{\Delta_i}^2)^3}{3\omega_{ei}^6(\omega_{ai}^2 - \Omega_e^2)(\omega_{ei}^2 - \Omega_e^2)(\omega_{ei}^2 - \omega_{\Delta_i}^2)} \right] \]

\[ \times (k_{s} R)^4 \ln^3(k_{s} R). \]  

Note that the resonance damping depends on the slope of the flow speed, \( \left( \frac{v_i - v_e}{l} \right) \). If the value of the first term in Equation (29) becomes larger, then the value of \( \Delta_e \) will be smaller, and the damping is more intense. For the kink \( (m = 1) \) mode in the slow continuum, we obtain (see Appendix B)

\[ \gamma_{1e} = -\pi^3 \text{Sign} \Omega \left[ \frac{(-1)^{\frac{\theta_0 - \theta_1}{2}}}{8|\Delta_e| R} \frac{\omega_{ai}^2(\omega_{ai}^2 - \omega_{\Delta_i}^2)^2}{\omega_{ai}^2(\omega_{ai}^2 - \omega_{\Delta_i}^2)^2} \right] (k_{s} R)^4. \]  

In Equations (42) and (43), the value of \( \theta \) is 0 and 1 for forward and backward waves, respectively. Under magnetic pore conditions \( (v_{\Delta_e} = 0) \) the expression obtained for \( \gamma_{0e} \) and \( \gamma_{1e} \)
reduces to

\[
\gamma_0 = \frac{2\pi \chi^2 \text{Sign} \Omega}{|\Delta_zR|} \left[ \frac{(-1)^i \omega^2 \omega_i^2 \Omega^6}{3 \omega_{iA}^4 \omega_i^4 + 8 \omega_{iA}^8 \omega_i^2 \Omega^2 \ln(kzR)} \right] \\
\times (kzR)^4 (\ln(kzR))^{3/2} \\
\gamma_{1c} = -\frac{\pi \chi^2 \text{Sign} \Omega}{8|\Delta_zR|} \left[ \frac{(-1)^i \omega^4 \omega_i^4 \Omega^2}{6 \omega_{iA}^4 \omega_i^4 - 8 \omega_{iA}^8 \omega_i^2 \Omega^2} \right] (kzR)^4. 
\]

(44)

(45)

In the absence of flow (\(v_i = v_e = 0\)), \(\Omega = \omega_i\), such that

\[
\gamma_0 = \frac{2\pi \chi^2 \text{Sign} \Omega}{|\Delta_zR|} \left[ \frac{(-1)^i \omega^2 \omega_i^2 \Omega^6}{3 \omega_{iA}^4 \omega_i^4 + 8 \omega_{iA}^8 \omega_i^2 \Omega^2 \ln(kzR)} \right] \\
\times (kzR)^4 (\ln(kzR))^{3/2} \\
\gamma_{1c} = -\frac{\pi \chi^2 \text{Sign} \Omega}{8|\Delta_zR|} \left[ \frac{(-1)^i \omega^4 \omega_i^4 \Omega^2}{6 \omega_{iA}^4 \omega_i^4 - 8 \omega_{iA}^8 \omega_i^2 \Omega^2} \right] (kzR)^4. 
\]

(46)

(47)

where these relations are the same as those of Equations (79) in Sadeghi & Karami (2019), and (38) in Yu et al. (2017b), respectively.

Note that in our model, resonance instability occurs when \(\gamma_0\) (or \(\gamma_{1c}\)) are positive. In Equation (42), we have \(\Omega^2 > \omega_i^2\), under magnetic pore conditions, together with \(\ln(kzR) < 0\). Based on Equation (42) for both forward (\(\theta = 0\), Sign\(\Omega > 0\)) and backward (\(\theta = 1\), Sign\(\Omega < 0\)) waves, \(\gamma_0 < 0\) is always the case, and resonance instability does not occur. In addition, for the kink modes from Equation (43), the same reason as mentioned above, the stability is preserved (i.e., \(\gamma_{1c} < 0\)). Moreover, for small and moderate wavelengths, our numerical results also confirm that resonance instability is avoided in our model.

5. Numerical Results

In this section we solve the dispersion relation in Equation (22) numerically, to obtain the frequencies and damping rates of the slow surface sausage and kink modes, and we compare the analytical results, as given in Equation (40), with the numerical results. Under magnetic pore conditions, following Grant et al. (2015), we set the model parameters as \(v_{Ai} = 12 \text{ km s}^{-1}\), \(v_{ce} = 0 \text{ km s}^{-1}\) (i.e., \(B_z = 0\)), \(v_s = 7 \text{ km s}^{-1}\), \(v_e = 11.5 \text{ km s}^{-1}\), \(v_{ci} = 6.0464 \text{ km s}^{-1}\) (\(\omega_0.8638 \omega_{vi}\)), and \(v_{se} = 0 \text{ km s}^{-1}\). We have assumed the flow outside the tube to be zero (\(v_{ce} = 0 \text{ km s}^{-1}\)). Note that the dispersion relations, i.e., Equations (20) and (22), are symmetric under the exchange \((\omega, v_i)\), with \((-\omega, -v_i)\). Therefore, it is sufficient to consider only the positive values of flow velocity with both positive and negative values of oscillation frequency, i.e., forward and backward waves in the presence of upward plasma flow. Our numerical results are shown in Figures 3–11.

Figures 3 and 4 represent the dependence of the phase speed (or normalized frequency) \(v/v_i \equiv \omega_i/\omega_{vis}\), the Doppler-shifted phase speed \(\Omega/\omega_{vis}\), and the damping rate \(-\gamma_0/\omega_r\) \((\gamma_0/\omega_r)\) of the slow surface sausage modes for forward and backward waves on \(kzR\), for various values of flow parameter, and various thicknesses of the inhomogeneous layer, \(I/R = 0.1, 0.2\). The left panels of these figures demonstrate that for forward waves, and various values of flow parameter \(v_{ci}/v_i = (10^{-5}, 0.2, 0.4, 0.6, 0.8)\): (i) The value of the phase speed \(v/v_i\) increases with an increase in the flow parameter \(v_{ci}/v_i\). (ii) The minimum value of the Doppler-shifted phase speed decreases with increased flow speed. (iii) As flow speed increases, the maximum value of \(-\gamma_0/\omega_r\) also increases. For small values of flow parameter, as the flow speed increases, the maximum value of \(-\gamma_0/\omega_r\) occurs at smaller \(kzR\), whereas for larger values of flow parameter, as the flow speed increases, the maximum value of \(-\gamma_0/\omega_r\) occurs at larger \(kzR\). (iv) The dashed-line curves in these figures represent the analytical results of the damping rate \(-\gamma_0/\omega_r\), as evaluated via Equation (40). These curves show that for weak damping (i.e., \(\gamma_0 \ll \omega_r\)), and in the long wavelength limit (i.e., \(kzR \ll 1\), the oscillation frequency is not affected by the presence of the transitional layer. This is also confirmed by our numerical results. (v) For a given \(I/R\), the minimum value of the damping time to period ratio, \(\tau_D/T = \omega_r/(2\pi)|\gamma_0|\) decreases with increasing \(v_{ci}/v_i\). For instance, in the case where \(I/R = 0.1\), and \(kzR = 1\), the value of \(\tau_D/T\) for \(v_{ci}/v_i = 0.8\) changes by \(-95\%\) less than in the case where there is no flow. As such, the dependence of the damping rate (time) and the plasma flow is of interest. Several researchers have obtained similar results for the sausage modes in photospheric conditions. Yu et al. showed that for \(I/R = 0.1\), the minimum value of the damping time to period ratio is \(\tau_D/T = 14.11\). Yu et al. (2017b), and Sadeghi & Karami (2019) showed that for \(I/R = 0.1\), the minimum value of the damping time to period ratio is \(\tau_D/T = 10.2\) for twist parameter \(B_{zi}/B_z = 0.3\), while our results show that the minimum value of the damping time to period ratio for high upflow is much lower. For instance, for the flow parameter \(v_{ci}/v_i = 0.8\), the minimum value of the damping time to period ratio is 0.66. (vii) For \(kzR \rightarrow 0\), we see that the damping rate goes to zero for finite values of flow parameter; this is in agreement with the analytical relation of Equation (44).

In the right panels of Figures 3 and 4 we plot the phase speed (or normalized frequency) \(v/v_i \equiv \omega_i/\omega_{vis}\), the normalized Doppler-shifted frequency \(\Omega/\omega_{vis}\), and the damping rate \(-\gamma_0/\omega_r\) of the slow surface sausage modes for backward wave, versus \(kzR\) for various flow parameters, \(v_{ci}/v_i = (10^{-5}, 0.1, 0.2, 0.3)\), and various thicknesses of the inhomogeneous layer, \(I/R = (0.1, 0.2)\). The figures show that (i) the magnitude of the phase speed decreases with increasing flow. (ii) The magnitude of the Doppler-shifted phase speed increases with increasing flow. (iii) As the flow speed increases, the maximum value of \(-\gamma_0/\omega_r\) decreases, shifting to larger \(kzR\). (vi) For a given \(I/R\), the minimum value of \(\tau_D/T\) increases with increasing \(v_{ci}/v_i\). For instance, for the case where \(I/R = 0.1\), and \(kzR = 1\), the value of \(\tau_D/T\) for \(v_{ci}/v_i = 0.3\) changes by \(-238\%\) more than the case where there is no flow. Due to the fact that at high flow parameters for backward waves, the Doppler-shifted frequencies are outside of the slow continuum, it is plotted up to the flow parameter, 0.3.

Figures 5 and 6 show the dependence of the phase speed (or normalized frequency) \(v/v_i \equiv \omega_i/\omega_{vis}\), the Doppler-shifted phase \(\Omega/\omega_{vis}\) and the damping rate \(-\gamma_0/\omega_r\) \((\gamma_0/\omega_r)\) with respect to slow surface sausage modes, for forward and backward waves on the inhomogeneous layer \((l/R)\), for various flow parameters and \(kzR = (0.5, 2)\). The left panels of Figures 5 and 6 show that for forward waves and various values of flow parameter \(v_{ci}/v_i = (10^{-5}, 0.2, 0.4, 0.6, 0.8)\): (i) the frequency increases with increasing flow, \(v_{ci}/v_i\). (ii) With increasing \(I/R\) for \(kzR \ll 1\), the Doppler-shifted frequency increases, whereas for \(kzR \gg 1\), the Doppler-shifted frequency reaches a maximum value, before approaching \(v_{ci}/v_i\). (iii) For \(kzR \ll 1\), the Doppler-shifted frequency decreases when the flow increases; for \(kzR \gg 1\), when the Doppler-shifted frequency reaches above...
Figure 3. The left panels represent the forward sausage waves, and the diagrams in (a), (b), and (c) represent the phase speed \(v/v_{Ai}\equiv \omega_r/\omega_{si}\), the Doppler-shifted phase speed \(\Omega/\omega_{si}\), and the damping rate \(-\gamma_0/c_i\) as functions of \(k_z R\), for various values of the flow parameter. The right panels are the same as the left panels, but for backward sausage waves. With respect to the damping rate, the dashed curves represent the analytical solutions determined via Equation (40). The dashed curves in the other diagrams show the results obtained in the case of no boundary layer i.e., Equation (20). The other parameters of the tube are \(l/R = 0.1\), \(v_{Ai} = 12\text{ km s}^{-1}\), \(v_{ Ae} = 0\text{ km s}^{-1}\) (i.e., \(B_{ Ae} = 0\)), \(v_{se} = 7\text{ km s}^{-1}\), and \(v_{ce} = 11.5\text{ km s}^{-1}\).
Figure 4. Same as Figure 3, but for $l/R = 0.2$. 

(a) Forward Sausage Waves
(b) Forward Sausage Waves
(c) Forward Sausage Waves
(d) Backward Sausage Waves
(e) Backward Sausage Waves
(f) Backward Sausage Waves
Figure 5. The left panels refer to the forward sausage waves, and the diagrams in (a), (b), and (c) represent the phase speed \( v/v_s \equiv \omega_r/\omega_s \), the Doppler-shifted phase speed \( \Omega/\omega_s \), and the damping rate \( -\gamma_0/\omega_s \) as functions of \( l/R \), for various values of flow parameter. The right panels are the same as the left panels, but for backward sausage waves. For all panels, we have assumed that \( k_R = 0.5 \); other parameters are as given in Figure 3.
Figure 6. As Figure 5, but for \( k_{oR} = 2 \).
Figure 7. The left panels refer to forward kink waves, and the diagrams in (a), (b), and (c) represent the phase speed $v/v_{si} \equiv \omega_r/\omega_{si}$, the Doppler-shifted phase speed $\Omega/\omega_{si}$, and the damping rate $-\gamma_1 c/\omega_r$, respectively, as functions of $k_z R$, for various values of flow parameter. The right panels are the same as the left panels, but for backward kink waves. With respect to the damping rate, the dashed curves represent the analytical solutions determined via Equation (40). The dashed curves in the other diagrams show the results obtained for the case with no boundary layer i.e., Equation (20). For all panels, we have assumed $l/R = 0.1$; the other parameters are the same as those given in Figure 3.
Figure 8. As Figure 7, but for $I/R = 0.2$
The left panels refer to forward kink waves, and the diagrams in (a), (b), and (c) represent the phase speed \( v/v_{si} \equiv \omega_r/\omega_{si} \), the Doppler-shifted phase speed \( \Omega/\omega_{si} \), and the damping rate \( -\gamma_1/\omega_r \) as functions of \( l/R \), for various values of flow parameter. The right panels are the same as the left panels, but for backward kink waves. For all panels, we have assumed \( k_z R = 0.5 \); the other parameters are as given in Figure 3.

**Figure 9.** The left panels refer to forward kink waves, and the diagrams in (a), (b), and (c) represent the phase speed \( v/v_{si} \equiv \omega_r/\omega_{si} \), the Doppler-shifted phase speed \( \Omega/\omega_{si} \), and the damping rate \( -\gamma_1/\omega_r \) as functions of \( l/R \), for various values of flow parameter. The right panels are the same as the left panels, but for backward kink waves. For all panels, we have assumed \( k_z R = 0.5 \); the other parameters are as given in Figure 3.
Figure 10. As Figure 9, but for $k_{\text{eff}} = 2$. 
v_{ci}/v_{si}$, it decreases with increasing flow, and goes to the value of $v_{ci}/v_{si}$. (iv) For a given $kR$, the value of the damping rate increases, and the value of the damping time to period ratio decreases with increasing flow. For example, for $kR = 2$, the minimum value of $\tau_D/T$ for $v_{ci}/v_{si} = 0.8$ decreases $\sim 93\%$ with respect to the case where there is no flow.

The right panels of Figures 5 and 6 show the dependence of the phase speed (or normalized frequency) $\omega/\omega_{si}$, the Doppler-shifted phase speed $\Omega/\omega_{si}$, and the damping rate $\gamma_{ci}/\omega_{si}$ of the slow surface sausage modes for backward waves on the inhomogeneous layer (l/R), for various flow parameters $v_{ci}/v_{si} = (10^{-5}, 0.1, 0.2, 0.3)$, where $kR = (0.5, 4)$. The figures show that (i) the magnitude of the phase speed increases with increasing flow. (ii) With increasing $l/R$ for $kR \ll 1$, the Doppler-shifted frequency decreases, but for $kR \gg 1$ the Doppler-shifted frequency reaches a minimum value, and then approaches $-v_{ci}/v_{si}$. (iii) For $kR \ll 1$, the magnitude of the Doppler-shifted frequency increases when the flow increases. For $kR \gg 1$, the Doppler-shifted frequency decreases with increasing flow, and then goes to the value of $-v_{ci}/v_{si}$. (iv) For a given $kR$, the values of the damping rate decrease, and the values of the damping time to period ratio increase with increasing flow. For example, for $kR = 2$, the minimum value of $\tau_D/T$ for $v_{ci}/v_{si} = 0.3$ increases $\sim 278\%$ as compared with the case where there is no flow.

We plot the results for kink waves in Figures 7 and 8. As is the case for the sausage modes in Figures 3 and 4 for the forward wave (the left panels), as flow speed increases, the maximum value of $-\gamma_{ci}/\omega_{ci}$ also increases. For small values of the flow parameter, as the flow speed increases, the maximum value of $-\gamma_{ci}/\omega_{ci}$ shifts to smaller $kR$, whereas for large values of flow parameter, as the flow speed increases, the maximum value of $-\gamma_{ci}/\omega_{ci}$ shifts to larger $kR$. For a given $l/R$, the minimum value of $\tau_D/T$ decreases with increasing $v_{ci}/v_{si}$. For instance, for the case where $l/R = 0.2$, the minimum value of $\tau_D/T$ for $v_{ci}/v_{si} = 0.8$ changes by $\sim 57\%$ less than for the case where there is no flow. Yu et al. (2017b) showed that for $l/R = 0.2$, the minimum value of $\tau_D/T$ is about 18.8, but our result gives a value of about 2.8. Soler et al. (2009) have obtained a value of about 1000 for $l/R = 0.2$. In addition, for the backward wave (the right panels) the maximum value of $\gamma_{ci}/\omega_{ci}$ decreases, and its position moves to larger $kR$ when $v_{ci}/v_{si}$ increases.

Figures 9 and 10 are similar to Figures 5 and 6, but for kink modes. The results show that the effects of flow on the slow resonance absorption of sausage and kink modes are almost the same. The effect of slow resonance in the presence of flow on wave damping is significant under photospheric conditions.

It should be noted that for the case with no flow, our results are similar to the results of Yu et al. (2017b). When the flow is very small (i.e., $v_{ci}/v_{si} = 10^{-5}$), the results overlap with the no-flow case. Figure 11 shows the minimum value of damping time to period ratio ($\tau_D/T$) for the forward wave of the slow surface sausage (solid line) and kink (dashed line) modes, versus upflow velocity ($v_{ci}/v_{si}$). This figure shows that when the upflow velocity increases, the minimum value of damping time to period ratio can be considerably reduced. For instance, for the upflow velocity value $v_{ci}/v_{si} = 0.87$, the damping time to period ratio of the surface sausage mode will reach about $3.0$. This confirms that resonant absorption in the presence of flow can be considered as an effective mechanism to justify the rapid damping of the slow surface sausage mode observed by Grant et al. (2015). Note that, for all of the results given in Figure 11, the longitudinal wavenumber is in the observational range, i.e., $kR \ll 5$. In this range, the minimum number of oscillations increases slightly for flow speeds larger than $v_{ci}/v_{si} = 0.87$.

6. Conclusions

In this paper we have studied the effect of the flow parameter on frequencies and damping rates in the slow continuum of slow sausage and kink waves in magnetic flux tubes, under solar photospheric (or magnetic pore) conditions. We considered a straight cylindrical flux tube with three regions inside the tube, in the annulus region, and outside the tube, in which the linear density, squared magnetic field (linear pressure), and linear flow profiles are considered in the annulus region or transitional layer. We solved the dispersion relation numerically, and obtained the phase speed (or normalized frequency) $\omega/\omega_{si}$, the normalized Doppler-shifted frequency, the damping rate $\gamma_{mc}/\omega_{mc}$, and the damping time to period ratio $\tau_D/T$ of the slow surface sausage and kink modes for forward and backward waves under photospheric (magnetic pore) conditions. Our results show that:

1. For forward waves, the frequency and the damping rate increase when the flow parameter increases, whereas for backward waves, the frequency and the damping rate decrease when the flow parameter increases.
2. For forward waves, the damping time to period ratio decreases when the flow parameter increases, but for backward waves, the damping time to period ratio increases when the flow parameter increases.
3. For a given $l/R$, the Doppler-shifted frequency approaches $\Omega/\omega_{ci} \rightarrow v_{ci}/v_{si}$ for forward waves, and approaches $\Omega/\omega_{ci} \rightarrow -v_{ci}/v_{si}$ for backward waves, and $\gamma_{mc}/\omega_{ci} \rightarrow 0$ for both forward and backward waves, in both the long- and short-wavelength limit.
4. For a given $kR$, the maximum value of $\gamma_{mc}/\omega_{ci}$ (minimum value of $\tau_D/T$) increases (decreases) for forward waves, and decreases (increases) for backward waves.
5. For the case where $l/R = 0.1$, the minimum value of $\tau_D/T$ for $v_{ji}/v_{si} = 0.6$, for example, changes $\sim 89\%$ less for forward sausage waves, and for backward sausage waves, the minimum value of $\tau_D/T$ for $v_{ji}/v_{si} = 0.3$, changes $\sim 204\%$ more with respect to the case where there is no flow. In addition, the kink mode changes $\sim 83\%$ less for forward waves, and $\sim 272\%$ more for backward waves with respect to the case where there is no flow. Based on these results, it can be said that flow has a significant effect on the resonance absorption of the slow surface sausage and kink modes in magnetic flux tubes under magnetic conditions.

6. For the case where $l/R = 0.1$, and $v_{ji}/v_{si} = 0.87$, the damping time to period ratio of the surface sausage mode can achieve $\tau_D/T = 0.3$. For comparison, based on a static tube (no flow) where $l/R = 0.1$, Yu et al. (2017b) obtained $\tau_D/T = 14.11$. This confirms that the resonance absorption in the presence of plasma flow justifies the extremely rapid damping of the slow surface sausage mode observed by Grant et al. (2015). It should be noted that more recently, Geeraerts & Van Doorsselaere (2000) studied the effect of electrical resistivity on the damping of slow surface sausage modes. They showed that for the magnetic Reynolds number $10^4$, $kR = 0.7$, and $l/R = 0.01$, the damping to period ratio is about $\tau_D/T \sim 0.25$. This shows that electrical resistivity can play an important role in wave damping, and can greatly reduce the number of oscillations of slow surface sausage modes in photospheric magnetic pores. Another alternative mechanism that may be studied is to consider the effects of both magnetic twist and plasma flow. It is possible that the presence of these two effects may cause more intense damping, and might therefore be able to verify these observations. However, we will postpone this investigation for a future work.

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**Appendix A**

*Weak Damping in the Long Wavelength Limit for the Sausage Mode*

For the sausage mode $m = 0$, we have

$$Q_0 = \frac{I_0^2(x)K_0(y)}{I_0(x)K_0'(y)} = -\frac{I_0(x)K_0'(y)}{I_0(x)K_0(y)} \approx \frac{xy(\ln(y/2) + \gamma_e)}{2},$$

(A1)

$$G_0 = \frac{K_0(y)}{K_0'(y)} = \frac{K_0(y)}{-K_1(y)} \approx -\frac{\ln(y/2) - \gamma_e}{1/y},$$

(A2)

$$P_0 = \left(\frac{I_0^2(x)}{I_0(x)} - \frac{I_0^2(x)}{I_0(x)}\right) \frac{K_0(y)}{K_0'(y)} \approx y \left(\frac{1}{2} - \frac{3y^2}{16}\right),$$

(A3)

$$S_0 = \left(1 - \frac{K_0'(y)K_0(y)}{K_0(y)^2}\right) \frac{I_0'(x)}{I_0(x)} \approx \frac{x}{2} (1 + \ln(y/2) + \gamma_e).$$

(A4)

Inserting Equations (A1)–(A4) into Equation (41) yields

$$T_0 = (\Omega_e^2 - \omega_A^2) \frac{k_R}{k_{re}} \left(\frac{xy}{\ln(y/2) + \gamma_e} \left(1 - \frac{2y^2}{16}\right)(\Omega_e^2 - 2\omega_A^2)\Omega_e^2}{(\omega_A^2 - \Omega_e^2)(\Omega_e^2 - \omega_A^2)(\Omega_e^2 - \omega_{ce}^2)} + \frac{xy}{\ln(y/2) + \gamma_e} \left(\frac{2(\omega_{ce}^2 - \Omega_e^2)(\omega_A^2 - \Omega_e^2)(\Omega_e^2 - \omega_{ce}^2)}{2(\omega_{ce}^2 - \Omega_e^2)(\omega_A^2 - \Omega_e^2)(\Omega_e^2 - \omega_{ce}^2)}\right)\right),$$

(A5)

where $\ln(y/2) + \gamma_e = \ln(y)$. In the limit $k_R \ll 1$ ($\Omega_e \approx \omega_{ce}$), the above relation becomes singular. To avoid singularity, we need to evaluate the quantity $\alpha$. To this end, following Yu et al. (2017b), we first replace $\Omega_e^2 = \omega_A^2 - \alpha$ into Equation (16) and obtain

$$k_A^2 \approx \frac{\omega_A^2 (\omega_A^2 - \omega_{ce}^2)(\Omega_e^2 - \omega_A^2)}{\alpha (\omega_A^2 - \omega_{ce}^2)} = \frac{k_R^2}{\alpha},$$

(A6)

where we have used the definition $\omega_A^2 = \frac{\omega_{se}^2}{\omega_{ce}^2}$ in obtaining the second equality of the above relation. In the next, the dispersion relation (20) in long wavelength limit ($k_R \ll 1$) reads

$$\rho_s(\Omega_e^2 - \omega_A^2) - k_R^2 R^2 \ln(k_R),$$

(A7)

Now, replacing $k_R^2$ from Equation (A6) into (7), the quantity $\alpha$ can be obtained as follows:

$$\alpha = \frac{\chi \omega_A^3}{2 \omega_A^2} (\Omega_e^2 - \omega_A^2) k_R^2 R^2 \ln(k_R),$$

(A8)

where $\Omega_e = \omega_A + k_s (v_{si} - v_{se})$ and

$$k_A^2 = -\frac{2\omega_A^2 \omega_{ce}^2}{\chi \omega_A^3 (\Omega_e^2 - \omega_A^2) R^2 \ln(k_R)},$$

(A9)

Replacing Equations (A8) and (A9) into Equation (A10), we obtain

$$T_0 = \left(-\frac{4\omega_{ce}^6}{\chi^2 \omega_A^6 \omega_{ce}^2 (\Omega_e^2 - \omega_A^2) k_R^2 R^2} - \frac{3\omega_{ce}^8 \omega_A^2 \ln(y)}{2 \chi^3 \omega_A^4 (\Omega_e^2 - \omega_A^2) k_R^2 R^2 \ln(k_R)} - \frac{\omega_A^4 \omega_{ce}^2 (\Omega_e^2 - 2\omega_A^2) \Omega_e^2}{\chi \omega_A^2 (\omega_{ce}^2 - \Omega_e^2) (\Omega_e^2 - \omega_A^2) \ln(k_R)}\right).$$

(A11)

where, finally, we obtain

$$T_0 = \left(-\frac{3\omega_{ce}^8 \omega_A^2 + 8\omega_A^6 \omega_{ce}^2 (\Omega_e^2 - \omega_A^2) \ln(k_R)}{2 \chi^3 \omega_A^4 (\Omega_e^2 - \omega_A^2) k_R^2 R^2 \ln(k_R)},$$

(A12)
Substituting Equation (A12) in (40) we have
\[
\gamma_0 = \frac{\pi \rho c H \text{Sign} \Omega}{k \rho R} \left( \frac{y}{x} \right)^2 \left( \omega_{ci}^2 - \omega_{Ai}^2 \right) (\Omega_x^2 - \omega_{Ac}^2) k_x R
\]
and after some algebra we arrive at
\[
\gamma_0 = \frac{2\pi \chi \text{Sign} \Omega}{|\Delta| R} \left\{ \frac{\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)^2}{3\omega_{ci}^4 \omega_{Ai}^4 + 8\chi^2 \omega_{Ai}^8 \omega_{ci}^6 (\Omega_x^2 - \omega_{Ac}^2) \ln(kR)} \right\} \times (kR)^4 \ln(kR),
\]
(A13)

\[
\gamma_0 = \frac{2\pi \chi \text{Sign} \Omega}{|\Delta| R} \times \left\{ \frac{\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)^3}{3\omega_{ci}^4 \omega_{Ai}^4 + 8\chi^2 \omega_{Ai}^8 \omega_{ci}^6 (\Omega_x^2 - \omega_{Ac}^2) \ln(kR)} \right\} \times (kR)^4 \ln(kR).
\]
(A14)

**Appendix B**

**Weak Damping in the Long Wavelength Limit for the Kink Mode**

For the kink mode \( m = 1 \), we have
\[
Q_1 = \frac{I''(x)K_1(x) - I_1(x)K'(x)}{I_1(x)K_1'(x)} = -\frac{K_1(y)I_1(x) + I_2(x)}{K_0(y) + K_2(y)} \approx -\left( \frac{y}{x} + \frac{y}{4} \right).
\]
(B1)

\[
G_1 = \frac{K_1(y)}{K_1'(y)} = -\frac{2K_1(y)}{K_0(y) + K_2(y)} \approx \frac{1}{y} + \frac{1}{4} + \frac{1}{2} \ln(y/2 + \gamma_c) = -y,
\]
(B2)

\[
P_1 = \left( \frac{I''(x)}{I_1(x)} - \frac{I'(x)}{I_1(x)} \right) K_1(y) \approx -y \left( \frac{1}{4} - \frac{1}{x^2} \right).
\]
(B3)

\[
S_1 = \left( 1 - \frac{K''(y)}{K_1'(y)} \right) \frac{I''(x)}{I_1(x)} \approx -\frac{1 + (1 + 3 \ln(y))^2}{x}.
\]
(B4)

Inserting Equations (B1)–(B4) into (41) yields
\[
T_1 = (\Omega_x^2 - \omega_{Ac}^2) \frac{k_x}{k_R} \left( -x \left( \frac{1}{4} - \frac{1}{x^2} \right) \right) (\Omega_x^2 - \omega_{ci}^2) (\Omega_x^2 - 2\omega_{ci}^2) (2\omega_{ci}^2 - \Omega_x^2)
\]
\[
\times \left( \frac{(\omega_{ci}^2 - \Omega_x^2)(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)}{(\omega_{ci}^2 - \Omega_x^2)(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)} \right) \left( \frac{\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)^3}{3\omega_{ci}^4 \omega_{Ai}^4 + 8\chi^2 \omega_{Ai}^8 \omega_{ci}^6 (\Omega_x^2 - \omega_{Ac}^2) \ln(kR)} \right) \times (kR)^4 \ln(kR),
\]
(B5)

For \( \Omega_x^2 = \omega_{ci}^2 - \alpha \), we obtain
\[
T_1 = (\Omega_x^2 - \omega_{Ac}^2) \left( \frac{\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)^3}{3\omega_{ci}^4 \omega_{Ai}^4 + 8\chi^2 \omega_{Ai}^8 \omega_{ci}^6 (\Omega_x^2 - \omega_{Ac}^2) \ln(kR)} \right) \times \left( \frac{\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)}{(\omega_{ci}^2 - \Omega_x^2)(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)} \right) \times (kR)^4 \ln(kR),
\]
(B6)

The dispersion relation (20) in long wavelength limit \( (kR \ll 1) \) for \( m = 1 \) reads
\[
\rho_1 (\omega_{ci}^2 - \omega_{Al}^2) + \rho_2 (\Omega_x^2 - \omega_{Ac}^2) \left( 1 + \frac{\gamma_c}{4} \right) = 0,
\]
(B7)

then, replacing \( k_1^2 \) from Equation (A6) into (A7), the quantity \( \alpha \) can be obtained as follows:
\[
\alpha = \frac{\chi}{4} \frac{\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)}{(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)R^2},
\]
(B8)

\[
k_1^2 = \frac{4 \omega_{ci}^2 \omega_{Ai}^2 - \chi \omega_{ci}^2 (\Omega_x^2 - \omega_{Ac}^2)}{\chi \omega_{ci}^2 (\Omega_x^2 - \omega_{Ac}^2)R^2},
\]
(B9)

and by putting Equations (B8) and (B9) in (B6), and keeping only the sentence proportional to the sentence \( \frac{1}{k^2R^4} \) we obtain
\[
T_1 = \frac{8\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)}{(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)} \left( \frac{8\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)}{(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)} - 1 \right)^2.
\]
(B10)

In the following, with the help of Equation (40), we get
\[
\gamma_{1c} = -\frac{\pi \chi \text{Sign} \Omega}{k \rho R} \left| \frac{y^2}{x^2 + y^2} \right| \left( \omega_{ci}^2 - \omega_{Al}^2 \right) (\Omega_x^2 - \omega_{Ac}^2) k_x R
\]
\[
\times \left( \frac{\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)}{(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)} - 1 \right)^2 \left( \frac{8\omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2)}{(\omega_{ci}^2 - \Omega_x^2)(\Omega_x^2 - \omega_{Ac}^2)} - 1 \right)^2.
\]
(B11)

this can be simplified as
\[
\gamma_{1c} = -\frac{\pi \chi \text{Sign} \Omega}{8|\Delta| R} \frac{1}{\omega_{Al}^2 (\omega_{ci}^2 - \omega_{Al}^2)} (\Omega_x^2 - \omega_{Ac}^2) \left( \omega_{ci}^2 \omega_{Ai}^2 (\Omega_x^2 - \omega_{Ac}^2) - (kR)^4 \right).
\]
(B12)

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