Spectral link and macroscopic non-universality in turbulent plane Couette flow

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Abstract

In fully developed turbulent plane Couette flow the classical experimental data of frictional factor $f$ versus Reynolds number $Re$ display well-known disparities which have long remained unexplained. These disparities are accompanied by previously unnoticed disparities in the wakes, and only in the wakes, of the attendant mean-velocity profiles (MVPs). To help explain these experimental data, we apply the model of the ‘spectral link’ for MVPs to turbulent plane Couette flows. The model links the dissipative range, the inertial range, and the energetic range of the standard phenomenological model of the spectrum of turbulent kinetic energy to, respectively, the buffer layer, the log layer, and the wake of the MVPs. By assessing the empirical data using the spectral model, we argue for the existence, in plane Couette flow, of multiple states of turbulence which differ from one another only at the largest lengthscales in the flow, corresponding to the energetic range, where the spectrum is subject to finite-domain effects. Thus, the multiplicity of turbulent states is entirely consistent with small-scale universality, and the experimental data on plane Couette flow pose no challenge to the phenomenological theory of turbulence. Our findings might apply more broadly to a general class of flows engendered by moving boundaries.

Keywords: turbulent plane Couette flow, friction factor, mean-velocity profile, spectral link, macroscopic non-universality
1. Introduction

The classical experiments on turbulent plane Couette flow date back to the 1950s, to the independent experiments by Reichardt (1956, 1959) and by Robertson (1959). These classical experiments focused on the friction factor $f$. A dimensionless measure of the wall shear stress $\tau_0$, the friction factor is defined as $f \equiv \tau_0/\rho u_c^2$, where $u_c \equiv u(y = b)$, that is, the mean velocity $u$ evaluated at the centerline of the flow, $y = b$, where $b$ is half the thickness of the gap between the plates.3

From the onset, the experiments revealed a conspicuous disparity in the relation between $f$ and the Reynolds number $Re \equiv u_c b/\nu$ (figure 1), where $\nu$ is the kinematic viscosity of the fluid. In consultation with Reichardt, Robertson repeated his experiments but the disparity prevailed and has never been resolved (Robertson and Johnson 1970). Later, more experiments became available, including the experiments by Telbany and Reynolds (1982), whose measurements of $f$ are in good accord with Reichardt’s, and the experiments by Kitoh et al (2005), whose measurements of $f$ agree well with Robertson’s. In the meantime, direct numerical simulations (DNS) of turbulent plane Couette flow became possible. Remarkably, the computational $f$–$Re$ relation turns out to be quite distinct from either of the experimental $f$–$Re$ relations (figure 1). At the same value of Reynolds number $Re$, the friction factors of Robertson (1959) and Kitoh et al (2005) are consistently higher than those of Reichardt (1959) and Telbany and Reynolds (1982), but consistently lower than the computational ones.

The Reichardt–Robertson disparity has always been thought of as an anomaly, the inadvertent product of a systematic error that would sooner or later be identified and factored in. The implicit notion has been that to each value of $Re$ there should correspond a unique turbulent state and, therefore, a unique value of $f$.

In view of figure 1, we argue that this notion should be abandoned. Thus, in what follows, we shall accept as a fact the experimental and computational evidence that to each value of $Re$ there might actually correspond multiple values of $f$. Furthermore, we shall assume that each one of these values of $f$ signals a specific, alternative turbulent state.

A number of questions are then brought to the fore: How are we to characterize a turbulent state? In what ways do various turbulent states differ among themselves? Can the existence of multiple turbulent states be reconciled with the phenomenological theory of turbulence?

Before we seek to answer these questions, we turn our attention to the mean-velocity profiles (MVPs) of turbulent plane Couette flow. The MVPs have been ignored in discussions of the Reichardt–Robertson disparity. And yet, the friction factor $f$ is closely related to the MVPs (Furuichi et al 2015), and the disparities in the $f$–$Re$ relation should be reflected in concomitant disparities in the MVPs, which in turn might help shed light upon the nature of the different turbulent states that prevail in turbulent plane Couette flow.

In figure 2, we plot the MVPs for the highest $Re$ available in various experiments and computations, all for plane Couette flow. The MVPs are plotted in terms of the classical dimensionless ‘wall variables’ $\bar{y}$ and $\bar{u}$, where $\bar{y} \equiv u_c y/\nu$ and $\bar{u} \equiv u/u_c$. From figure 2 we can verify that at similar values of $Re_c$ (the friction Reynolds number, an alternative form of the $Re$ defined as $Re_c \equiv u_c b/\nu$) the various MVPs are indistinguishable everywhere in the domain of the flow except for the wake. Indeed, the disparities in the $f$–$Re$ relation are

3 In plane Couette flow the fluid is confined between two mutually parallel plates which move parallel to one another at a constant relative velocity. Here, we will consider the frame of reference in which the lower plate is stationary. No pressure gradient is imposed on the fluid; thus the total shear stress is uniform across the thickness and coincides with $\tau_0$. 

reflected in the wakes of the MVPs, and only there, in the form of varying degrees of overshooting of the log law.

Even though some of the values of $Re_\tau$ that appear in figure 2 are not entirely comparable with the others (especially for the experiment of Kitoh et al. (2005) and the computations of Avsarkisov et al. (2014), where the highest available values of $Re_\tau$ are rather small), it can be clearly seen that the overshooting of the log law is largest for the experiments of Reichardt (1959) and Telbany and Reynolds (1982), second largest for the experiments of Robertson (1959), and smallest for the computations of Pirozzoli et al. (2014) (figure 2). That is, a higher friction factor consistently corresponds to a smaller overshooting in the wakes of the MVPs.

Indeed, the overshooting can be quantified by the value of the non-dimensional velocity slope at the flow centerline, which is defined as

$$R_\tau \equiv \frac{b}{u_\tau} \frac{du}{dy} \bigg|_{y=b} = \frac{d\tilde{u}}{d \ln \tilde{y}} \bigg|_{\tilde{y}=Re_\tau}. \quad (1)$$

A higher value of $R_\tau$ indicates a larger overshooting in the wakes of the MVPs. In figure 3 it is possible to confirm the conclusion from figure 2, namely that a higher friction factor consistently corresponds to a smaller overshooting in the wakes of the MVPs (if we exclude as an outlier, likely due to scatter, the only counterexample, which occurs at a very small Reynolds number).

To summarize our discussion so far: we have ascribed the disparities in the $f$-$Re$ relation to the existence of multiple turbulent states in turbulent plane Couette flow. In addition, we have verified that these disparities are reflected in the wakes of the MVPs, and only there, in
the form of varying degrees of overshooting of the log law. In what follows, we intend to harness this latter fact in order to explain, with help from a spectral model of the MVPs (Gioia et al 2010) of turbulent plane Couette flow, the way in which the multiple turbulent states differ from one another. Thus, we turn to the spectral model.

2. The model of spectral link

As a starting point of our analysis, here we outline the model of the ‘spectral link’ for MVPs in turbulent pipe flows and channel flows (Gioia et al 2010). In the next section, we adapt this spectral model to turbulent plane Couette flows. The model is grounded on the existence of a mathematical link between the MVPs and the eddy velocity distribution of the phenomenological theory of turbulence (or, alternatively, the spectrum of turbulent kinetic energy of the phenomenological theory of turbulence).

2.1. The spectrum of turbulent kinetic energy

The spectrum of turbulent kinetic energy, or the spectrum for short, is a function of the wavenumber $k$, $E(k)$, defined by the expression $\frac{\text{d}v_k^2}{\text{d}k} = E(k)$, where $v_k$ is the characteristic velocity of a turbulent eddy of wavenumber $k$. Note that $k \propto 1/s$, where $s$ is the size of the
eddy; thus the eddy velocity distribution of the phenomenological theory is but a derivative of the spectrum. Indeed, the velocity of an eddy of size $s$, $v_s$, can be computed as

$$v_s = \sqrt{\int_{1/s}^{\infty} E(k) \, dk},$$

where

$$E(k) = \frac{2}{3} (\kappa \varepsilon)^{2/3} k^{-5/3} \exp(-\beta_d \eta k)(1 + (\beta_e/bk)^{2})^{-17/6}. $$

This expression for $E(k)$ corresponds to the standard model of the spectrum. It consists of

(i) the power-law spectrum of Kolmogórov, $\frac{2}{3} (\kappa \varepsilon)^{2/3} k^{-5/3}$, valid in the inertial range, where the dynamics of the turbulent eddies is dominated by the inertial forces of the energy cascade, with vanishing interference from viscous and finite-domain effects. Here, $\kappa \varepsilon$ is a dimensionless, non-negative constant which can be theoretically estimated as $4/5$ (Gioia et al 2010, Gioia and Chakraborty 2006), $\varepsilon$ is the turbulent power per unit mass (that is, the characteristic invariant of the energy cascade), and $-5/3$ is the spectral exponent.

(ii) The dissipative-range correction that accounts for the effect of viscosity at high wavenumber (small eddies), $\exp(-\beta_d \eta k)$, where $\beta_d$ is a dimensionless, non-negative constant known as the dissipative-range parameter, and $\eta$ is the viscous lengthscale, $\eta = \nu^{3/4} \varepsilon^{-1/4}$.

(iii) The energetic-range correction that accounts for effect of finite domain at low wavenumber (large eddies), $(1 + (\beta_e/bk)^{2})^{-17/6}$, where $\beta_e$ is a dimensionless, non-negative constant known as the energetic-range parameter, and $b$ is the size of the largest.
eddies, which size is commensurate, for plane Couette flow, with the thickness of the gap between the plates.

The standard model of the spectrum has been tested successfully by comparison with extensive measurements of the velocity fluctuations carried out over a wide range of Reynolds numbers, and by a variety of probing techniques, in laboratory flows, atmospheric flows, flows with shear and flows without shear, as well as by comparison with velocity fluctuations computed in numerical simulations (Pope 2000). In spite of its relative simplicity, the standard model provides a comprehensive, quantitative and physically meaningful account of each of the features of the spectrum evinced by the empirical data.

For future reference, note that $E(k) > 0$ for all $k$. Thus, from $v_t = \sqrt{\int_{d(k)}^{\infty} E(k) \, dk}$, we conclude that $v_t$ is a monotonically increasing function of $s$. In other words, the larger the eddy, the larger its characteristic velocity.

By substituting the standard model of the spectrum in equation (2), we obtain (Gioia et al 2010)

$$v_t = (\kappa_s \varepsilon s)^{1/3} \sqrt{I},$$

where

$$I \equiv I(\eta/s, s/b) \equiv \frac{2}{3} \int_{1}^{\infty} \xi^{-5/3} \exp(-\xi \beta_d \eta/s)(1 + (\beta_s s/b)^2/\xi^2)^{-17/6} \, d\xi,$$

and $\xi \equiv \eta$. In general, for an eddy of size $s$ in the inertial range ($\eta \ll s \ll b$), $I = 1$ and $v_t = (\kappa_s \varepsilon s)^{1/3}$, which is the eddy velocity distribution in the inertial range. For an eddy size $s$ in the dissipative range ($s \ll \eta$) or the energetic range ($s \gg b$), $I < 1$ and $v_t < (\kappa_s \varepsilon s)^{1/3}$, with the implication that the eddy is slower than an imaginary eddy of the same size in the inertial range. Thus, the dissipative-range correction and the energetic-range correction may be said to depress the spectrum relative to the power-law spectrum of Kolmogórov. Indeed, if we set $\beta_d = \beta_s = 0$—that is, if there were no dissipative-range correction and no energetic-range correction (an unphysical proposition)—$v_t = (\kappa_s \varepsilon s)^{1/3}$ for all $s$, and all eddies would be inertial.

### 2.2. Turbulent shear stress and spectral link

We seek to derive an expression for the turbulent shear stress that acts on a wetted surface $W_y$ at a distance $y$ from the wall (figure 4). Here we use a method different from the original derivation of Gioia et al (2010): our derivation is grounded on the concept of control volume (CV) (Vincenti 1982). Consider a cuboidal CV whose upper face (of surface area $S$) overlaps with $W_y$ and whose lower face is at a distance $y_1$ from the wall. The length of the CV is $X$. Here, $y_1$ and $X$ are arbitrary. Eddies of size $s$ and velocity $v_t$ bring high-momentum fluid (momentum per unit mass $\sim \rho u(y + s)$) into the CV and take low-momentum fluid (momentum per unit mass $\sim \rho u(y - s)$) out of the CV. The net momentum exchanged in a time interval $\Delta t$ is

$$\Delta M \sim S v_t \Delta t \, \rho u(y + s) - S v_t \Delta t \, \rho u(y - s) \approx 2S \rho v_t \Delta t \, \frac{du(y)}{dy}.$$  

Thus, there is a turbulent force $F$ acting on CV, $F = \Delta M/\Delta t \approx 2S \rho v_t \, \frac{du(y)}{dy}$, and the turbulent shear stress $\tau_{12}$ must be the same as this turbulent force per unit area, that is $\tau_{12} = F/S \approx \rho v_t \, \frac{du(y)}{dy}$.

Now, we have seen that $v_t$ is a monotonically increasing function of $s$. Thus, $\tau_{12}$ is also a monotonically increasing function of $s$, and the production of turbulent shear stress must be
dominated by the largest eddies that straddle Wy—that is, the eddies of size $s = y$. We conclude that

$$\tau_{12} = \kappa_{\tau} \rho y \frac{du(y)}{dy}, \quad (7)$$

where $\kappa_{\tau}$ is a dimensionless proportionality constant. Next, we set $s = y$ in equation (4) to obtain, $v_{s} = (\kappa_{\tau} y) \sqrt[3]{I}$, where $\varepsilon = \tau_{12} \frac{d\varepsilon}{dy}/\rho$ is the energy equation (or, more precisely, the turbulent kinetic budget equation (Kundu and Cohen 2002)); substituting in equation (7):

$$\tau_{12} = \kappa^2 \rho \alpha^3/\gamma_{\alpha} \left( \frac{du(y)}{dy} \right)^2, \quad (8)$$

where $\kappa \equiv (\kappa_{\tau}, \kappa_{\varepsilon})^{1/4}$ and $I \equiv I(\eta/y, y/b)$.

Equation (7) links the turbulent shear stress to $v_{s}$, the eddy velocity distribution, which is a function of the spectrum. In other words, equation (7) is the ‘spectral link’. It was first derived (along with equation (8)), for turbulent pipe flow and channel flow by Gioia et al (2010), who showed that the parameter $\kappa$ in equation (7) is but the Kármán constant of the log law, which according to the model can be expressed as $\kappa \equiv (\kappa_{\tau}, \kappa_{\varepsilon})^{1/4}$. Thus, the Kármán constant is set by $\kappa_{\tau}$ and $\kappa_{\varepsilon}$, which relate to the mechanism of momentum transfer and to the spectrum, respectively. The fact that the Kármán constant depends explicitly on a parameter related to the spectrum, namely $\kappa_{\varepsilon}$, serves as a striking reminder of the fundamental spectral nature of the model.

3. Equations of the MVPs

For turbulent plane Couette flow, the total shear stress can be expressed in the form

$$\tau(y) = \tau_{v12} + \tau_{112} = \tau_{0}. \quad (9)$$

By substituting the total shear stress and viscous shear stress $\tau_{112} \equiv \rho \frac{d\tau_{112}}{dy}$ in equation (8) and rewriting the result in terms of $Re_{\tau}$ and the wall variables $\tilde{y}$ and $\tilde{u}$, we obtain the momentum equation:
where \( I \equiv I(\eta/y, \tilde{y}/Re_\nu) \).

Similarly, by taking into account that \( \eta = \nu^{3/4} \varepsilon^{-1/4} \) and \( \varepsilon = \tau_{112} \frac{du(y)}{dy}/\rho \) (the energy equation), we obtain a recast form of the energy equation:

\[
\eta/y = \left( \frac{\tilde{d}u}{\tilde{d}y} - \left( \frac{du}{dy} \right)^2 \right)^{1/4} \tilde{y}^{-1},
\]

which is coupled to the momentum equation via the dependence of \( I \) on \( \eta/y \). Equations (10) and (11) are the equations of the MVPs for turbulent plane Couette flow.

We proceed following the analysis of Gioia et al. (2010). When the flow is laminar, the turbulent shear stress vanishes \( (\tau_{112} = 0) \) and equation (10) reduces to \( d\tilde{u}/d\tilde{y} = 1 \), which is the law of laminar flows (Kundu and Cohen 2002). When the flow is turbulent, the no-slip boundary condition (Pope 2000) gives rise to a thin laminar viscous layer next to the wall \( (y < b) \). With the no-slip boundary condition, that is, \( \tilde{u} = 0 \) for \( \tilde{y} = 0 \), the viscous layer solution reads \( \tilde{u} = \tilde{y} \). We shall show next that the thickness of the viscous layer can be determined starting with equations (10) and (11).

Consider a point \( y < b, \tau_{112} > 0 \), thus \( I = I(\eta/y, 0) > 0 \). Then \( d\tilde{u}/d\tilde{y} \) can be eliminated from equations (10) and (11) to obtain

\[
\tilde{y} = \left( \frac{\eta/y}{\kappa^3} + \kappa^{4/3} b^{1/2}(\eta/y, 0) \right)^{1/2}.
\]

Now, for a given value of \( \kappa \) and a given value of \( \beta_d \), equation (12) gives \( \tilde{y} \) as a function of \( \eta/y \). As it turns out, this function has a minimum, denoted \( \tilde{y}_c \) (figure 5), with the implication that for \( \tilde{y} < \tilde{y}_c \) there is no turbulent solution (Gioia et al. 2010). Thus, for \( \tilde{y} < \tilde{y}_c \), the only possible solution is the viscous solution, \( \tilde{u} = \tilde{y} \), and \( \tilde{y}_c \) represents the thickness of the viscous layer. For \( \kappa = 0.81 \) and \( \beta_d = 7 \), we obtain \( \tilde{y}_c = 4.2 \) (figure 5). For \( \tilde{y} > \tilde{y}_c \), turbulence prevails and the turbulent MVP can be computed by integrating equation (10) with boundary condition \( \tilde{u} = \tilde{y}_c \) at \( \tilde{y} = \tilde{y}_c \). Note that the MVP is determined by all three parameters \( (\kappa, \beta_d \text{ and } \beta_e) \) whereas the thickness of the viscous layer depends only on \( \kappa \) and \( \beta_d \).

From the equations above, it is apparent that the standard model of the spectrum should in itself be sufficient to compute, with no additional assumptions, the turbulent MVP corresponding to any given Reynolds number, over the entire domain of flow. Indeed, the model yields complete MVPs with viscous layer, buffer layer, log layer, and wake (Gioia et al. 2010). Furthermore, each part of the MVP relates to a specific spectral range: the viscous layer to the dissipative range, the buffer layer to the dissipative range, the log layer to the inertial range, and the wake to the energetic range (Gioia et al. 2010).

The one-to-one correspondence between the layers of the MVPs and the ranges of the spectrum plays a crucial role in our analysis of turbulent plane Couette flow. Specifically, in section 1 we noted that the disparities in the \( f-Re \) relation are reflected in only the wakes of the MVPs. In the spectral model the amplitude of the wake is set by only one spectral parameter: the dimensionless parameter \( \beta_e \) of the energetic-range correction. In the next section we turn to how this single parameter can shed light on the disparities.

Before discussing the results, we note that here our emphasis will be on that which the spectrum may tell us about the salient qualitative features of the disparities rather than on how accurately the spectral model can reproduce, on the basis of the standard model of the spectrum, plots of the empirical data on these disparities. In particular, our interest will be on

\[
\kappa^2 l^3/42^2 \left( \frac{d\tilde{u}}{d\tilde{y}} \right)^2 + \frac{d\tilde{u}}{d\tilde{y}} = 1,
\]
qualitative, or generic, aspects of the energetic-range correction and their relation to salient characteristics of the wake, notably the relation between the generic fact that the energetic-range correction decays for small $k$, and the concomitant fact that the MVP overshoots the log law in the wake\textsuperscript{4}. We are not considering the specific form of the energetic-range correction and the exact shape of the overshooting wake. Keeping this outlook in mind, we will not try to fit the $f$-$Re$ curves and the MVPs from the spectral model (computed using the standard model) to that from empirical data. Instead we will show the computed and the empirical ones in separate yet readily-comparable plots, and focus on the capacity of the spectral model to reproduce all of the salient features of the empirical data, allowing us to trace the cause of each one of those salient features to the corresponding feature of the spectrum.

\section*{4. Analysis}

We use the spectral model of the previous section to make the theoretical predictions shown in figures 6 and 7. The different curves correspond to different values of the dimensionless parameter $\beta_e$. Note that the spectral model can reproduce all of the salient trends in the experimental and computational data (also shown in figures 1 and 2). Since the disparities in friction factor and the attendant disparities in the wakes are accounted for by a single parameter, $\beta_e$, of the energetic-range correction, we conclude that the multiple turbulent states we introduced in section 1 can only differ at the lengthscales of the energetic-range correction, the largest lengthscales of the flow. Experimental and DNS studies of plane Couette flows

\footnote{Analysis of such generic aspects of the spectrum can be used to shed light on the scaling relations for the MVPs (Gioia and Chakraborty 2017).}
(Tsukahara et al 2006, Avsarkisov et al 2014, Lee and Moser 2018) show that these largest length scales appear as counter-rotating, streamwise vortices. While, to our knowledge, there is no systematic study of how these vortices are affected by details of the flow, we posit that the different turbulent states correspond to different configurations of the vortices, which are

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Log-linear plots of the friction factor of plane Couette flow as a function of $Re$ computed from the spectral model with $\kappa = 0.81$, $\beta_d = 7$, and three different values of $\beta_e$: $\beta_e = 1$ (black line), $\beta_e = 2$ (red line), $\beta_e = 3$ (blue line).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{Log-linear plots of the MVPs of plane Couette flow at the same value of $Re_v = 1000$ computed from the spectral model with $\kappa = 0.81$, $\beta_d = 7$, and three different values of $\beta_e$: $\beta_e = 1$ (black line), $\beta_e = 2$ (red line), $\beta_e = 3$ (blue line).}
\end{figure}
affected by flow details such as finite-domain effects or the initial conditions. We submit that such a study would shed important insight into plane Couette flows.

From our results we note that, contrary to the view that multiple turbulent states might be somewhat at odds with the phenomenological theory of turbulence, which view was put forward in a discussion of multiple turbulent states in Taylor–Couette flow (Huisman et al 2014, van der Veen et al 2016), but might be thought applicable to plane Couette flow as well, the difference between any one such state and another is totally circumscribed to the largest lengthscales in the flow. This implies that, at any given Reynolds number, all turbulent states are indistinguishable at inertial and viscous lengthscales, and that small-scale universality (Kolmogórov 1941, Liao and Su 2015, Schumacher et al 2014) holds regardless of state, in accord with the phenomenological theory.

4.1. Disparities in macroscopic properties other than the MVPs

As per the spectral model, a higher value of $\beta$ indicates stronger finite-domain effects and results in a lower friction factor and a larger overshooting in the wakes of the MVPs (figures 6 and 7). From this conclusion we expect that a change in turbulent state (and the attendant change if the $f$–$Re$ relation) should have a marked effect on the turbulent velocity fluctuation profile $V(y) \equiv u'u''_y$; this is because most of the turbulent energy is carried by the largest turbulent eddies in the flow. Indeed, we can predict that a turbulent state with a lower friction factor corresponds to stronger finite-domain effects, which in turn should correspond to higher turbulent velocity fluctuations. This prediction is confirmed in figure 8, where we plot the most recent DNS and experimental data, for the highest available values of $Re$, on the $\tilde{V}(\tilde{y})$ profile, where $\tilde{V} \equiv V/u_*^2$.
The spectral model does not directly give the turbulent velocity fluctuations. However, the turbulent wall-normal velocity fluctuation $v_{\text{rms}}(y)$ may be identified as $v_y$ (the characteristic velocity of the dominant eddy at a distance $y$ from the wall): in figure 9, we consider that at a distance $y$ from the wall, the turbulent wall-normal velocity fluctuation results mostly from the turbulent eddies centering at the wetted surface $W_y$. Then, the dominant eddies contributing to the turbulent wall-normal velocity fluctuation $v_{\text{rms}}(y)$ must be of size $s = y$ as $v_y$ is a monotonically increasing function of $s$. Thus, we assume that the $v_{\text{rms}}(y) = v_y$. From equations (7) and (8), $v_y$ (or $v_{\text{rms}}(y)$) can be directly computed from the spectral model as

$$v_y = v_{\text{rms}}(y) = (\kappa^2/\nu) y^{1/3} 1/3 y^{1/3} y^{1/3} \frac{du(y)}{dy}.$$  \hspace{1cm} (13)

We now are ready to compare our theoretical results of $v_{\text{rms}}(y)$ to the corresponding experimental and computational data. Similar to the $V(y)$ profile, the $\tilde{v}_{\text{rms}}(\tilde{y})$ profile, where $\tilde{v}_{\text{rms}} \equiv v_{\text{rms}}/u_\tau$, should also differ depending on the turbulent state. What we expect is that because of weaker finite-domain effects, a higher friction factor should also correspond to a higher turbulent wall-normal velocity fluctuation, which is again confirmed by figure 10 and can be reproduced by the spectral model by varying the dimensionless parameter $\beta_\epsilon$ of the energetic-range correction (figure 11).

5. A note on the asymptotic value of the velocity slope at the flow centerline

In plane Couette flow, the value of the non-dimensional velocity slope at the flow centerline, $R_s$, has elicited considerable interest since the 1950s when the classical experiments on turbulent Couette flow were conducted by Reichardt (1956, 1959). Regarding how $R_s$ varies with $Re$ (figure 3), while most recent DNS studies report a slow decrease of $R_s$ with increasing $Re$, different trends have been reported in various experiments. The discrepancies have been attributed to the difficult measurements of small velocity differences in experiments (Pirozzoli et al 2014). Regarding whether the value of $R_s$ will approach a constant in the limit of infinite $Re$, a convincing conclusion has never been reached because of the limitation to achieve very high Reynolds number in both experiments and simulations. Thus, the value of the non-dimensional velocity slope at the flow centerline remains an open question till now. An alternative definition of the non-dimensional velocity slope at the flow centerline is

$$S_0 \equiv \frac{h}{u_c} \frac{du}{dy} \bigg|_{y=b} \equiv \frac{\tilde{d}u}{d \ln \tilde{y}} \bigg|_{\tilde{y}=Re_c} \sqrt{\tilde{J}} = R_s \sqrt{\tilde{J}}.$$  \hspace{1cm} (14)
Here, the spectral model will be employed to address the open question on the value of the non-dimensional velocity slope at the flow centerline of plane Couette flow. $R_s$ and $S_0$ are readily to be calculated from the theoretically computed MVP and friction factor. At the flow

**Figure 10.** Plots of experimental and computational data on the turbulent wall-normal velocity fluctuation profiles of turbulent plane Couette flow, $\Delta R_e = 805$ (Telbany and Reynolds 1982), DNS from Avsarkisov et al (2014) (blue) at $R_e = 550$, and Pirozzoli et al (2014) (red) at $R_e = 986$, respectively.

**Figure 11.** Plots of the turbulent wall-normal velocity fluctuation profiles of plane Couette flow at the same value of $R_e = 1000$ computed from present spectral model with $\kappa = 0.81$, $\beta_d = 7$, and three different values of $\beta_e$: $\beta_e = 1$ (black line), $\beta_e = 2$ (red line), $\beta_e = 3$ (blue line).

Here, the spectral model will be employed to address the open question on the value of the non-dimensional velocity slope at the flow centerline of plane Couette flow. $R_s$ and $S_0$ are readily to be calculated from the theoretically computed MVP and friction factor. At the flow
centerline where the viscous shear stress is negligible, equation (10) gives that \( R_s = R_0^{3/8} / \kappa \), where \( R_0 \equiv I(\eta/b, 1) \), which depends on \( Re \) via \( \eta/b \). For a high enough Reynolds number, \( \eta/b \) reduces to zero. Thus \( R_0 \equiv I(0, 1) \) is not dependent on \( Re \) any more and \( R_s \) becomes a constant value, which is determined by the energetic-range correction to the spectrum via \( I(0, 1) \). Note that the conclusion that \( R_s \) becomes a constant value at high Reynolds numbers is valid regardless of the form of the function \( I(\eta/s, s/b) \), thus regardless of the form of the energetic-range correction to the spectrum, as long as the energetic-range correction to the spectrum is a function of \( s/b \). In the standard model of the spectrum we are using here, the dimensionless parameter \( \beta_e \) of the energetic-range correction sets the constant value of \( R_s \) at high Reynolds numbers (figure 12) as \( I_0 \) becomes

\[
I(0, 1) \equiv \frac{2}{3} \int_1^{\infty} \xi^{-5/3} (1 + \beta_e^2 / \xi^2)^{-17/6} d\xi,
\]

which is a constant value only depending on \( \beta_e \). We have argued that a certain \( \beta_e \) signifies a specific turbulent state of plane Couette flow. That is to say, for each turbulent state, there is an asymptotic constant value of \( R_s \) in the limit of high \( Re \) as shown in figure 12.

In figure 13, we also plot \( S_0 \equiv R_s \sqrt{f} \) as a function \( Re \). The decreasing rate of \( S_0 \) is getting slow in high Reynolds number region as the friction factor \( f \) is decreasing slowly. At infinite Reynolds number, if the friction factor becomes zero, then \( S_0 \) will be zero.

6. Discussion and summary

We have used the spectral model of the MVPs of turbulent plane Couette flow to revisit the Reichardt–Robertson disparity, with the following conclusions.
The Reichardt–Robertson disparity is a manifestation of the existence of multiple turbulent states in turbulent plane Couette flow.

By invoking the one-to-one, or bijective property of the spectral link, whereby each successive range of the spectrum of turbulent kinetic energy relates to one, and only one, of the successive layers of the attendant MVP (the dissipative range to the buffer layer, the inertial range to the log layer, and the energetic range to the wake), we have been able to ascertain the ways in which these multiple turbulent states differ from one another, namely only at the largest lengthscales in the flow, corresponding to the energetic range of the eddy velocity distribution.

Corollary: a change in turbulent state can only alter the wake of the attendant MVP (which is the part of the MVP dominated by the eddies of the energetic range).

At smaller lengthscales, comprising the inertial and viscous portions of the spectrum of turbulent kinetic energy, the spectrum is one and the same regardless of the turbulent state, consistent with the phenomenological theory of turbulence.

Thus, the existence of multiple turbulent states does not contradict the universality of the spectrum of turbulent kinetic energy (such as this universality is understood in the phenomenological theory).

Conversely, spectral universality does not rule out the existence of multiple turbulent states.

These conclusions might apply beyond plane Couette flow to a general class of flows engendered by moving boundaries. Consider, for instance, the recent papers on multiple states in highly turbulent Taylor–Couette flows (Huisman et al 2014, van der Veen et al 2016). These studies report two branches of global torque, which is analogous to the friction factor \( f \). These branches correspond to differences in the wake of the MVPs of azimuthal velocity and differences in the size of large-scale Taylor vortices, which correspond to differences in the

Figure 13. Log-linear plots of the non-dimensional velocity slope at the flow centerline \( S_0 \) of plane Couette flow as a function of \( Re \) computed from present spectral model with \( \kappa = 0.81, \beta_d = 7, \) and three different values of \( \beta_e: \beta_e = 1 \) (black line), \( \beta_e = 2 \) (red line), \( \beta_e = 3 \) (blue line).
energetic-range corrections to the spectrum. Our analysis might provide a unified explanation for these features.

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References

Avsarkisov V, Hoyas S, Oberlack M and Garcia-Galache J P 2014 Turbulent plane Couette flow at moderately high Reynolds number J. Fluid Mech. 751 R1–10
Bech K H, Tillmark N, Alfredsson P and Anderson H I 1995 An investigation of turbulent plane Couette flow at low Reynolds numbers J. Fluid Mech. 286 291–325
Furuichi N, Terao Y, Wada Y and Tsuji Y 2015 Friction factor and mean velocity profile for pipe flow at high Reynolds numbers Phys. Fluids 27 095108
Gioia G and Chakraborty P 2006 Turbulent friction in rough pipes and the energy spectrum of the phenomenological theory Phys. Rev. Lett. 96 044502
Gioia G and Chakraborty P 2017 Spectral derivation of the classic laws of wall-bounded turbulent flows Proc. R. Soc. A 473 20170354
Gioia G, Guttenberg N, Goldenfeld N and Chakraborty P 2010 Spectral theory of the turbulent mean-velocity profile Phys. Rev. Lett. 105 184501
Huisman S G, Veen R C, Sun C and Lohse D 2014 Multiple states in highly turbulent Taylor–Couette flow Nat. Commun. 3820 5
Kitoh O, Nakabayashi K and Nishimura F 2005 Experimental study on mean velocity and turbulence characteristics of plane Couette flow: low-Reynolds-number effects and large longitudinal vortical structure J. Fluid Mech. 539 199–227
Kolmogórov A N 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers Dokl. Akad. Nauk SSSR 30 301–5
Kundu P K and Cohen I M 2002 Fluid Mechanics 2nd edn (San Diego, CA: Academic)
Lee M J and Kim J 1991 The structure of turbulence in a simulated plane Couette flow 8th Symp. on Turbulent Shear Flows, Munich, Federal Republic of Germany, Proc. vol. 1 (University Park, PA, 9–11 September, 1991) pp 531–6
Lee M and Moser R D 2018 Extreme-scale motions in turbulent plane Couette flows J. Fluid Mech. 842 128–45
Liao Z-J and Su W-D 2015 Kolmogorov’s hypotheses and global energy spectrum of turbulence Phys. Fluids 27 041701
Pirozzoli S, Bernardini M and Orlandi P 2014 Turbulence statistics in Couette flow at high Reynolds number J. Fluid Mech. 758 327–43
Pope S B 2000 Turbulent Flows (Cambridge: Cambridge University Press)
Prandtl L. 1953 Essentials of Fluid Dynamics 3rd edn (London: Blackie & Son)
Reichardt H 1956 Über die geschwindigkeitsverteilung in einer geradlinigen turbulenten Couetteströmung Z. Angew. Math. Mech. Sonderheft 36 S26
Reichardt H 1959 Gesetzmäßigkeiten Der Geradlinigen Turbulenten Couetteströmung (Göttingen: Max-Planck Institute)
Robertson J M 1959 On turbulent plane-Couette flow (Austin, TX: University of Texas) pp 169–82
Robertson J M and Johnson H F 1970 Turbulence structure in plane Couette flow Proc. A.S.C.E., J. Eng. Mech. 96 1171
Schumacher J, Scheel J D, Krasnov D, Donzis D A, Yakhot V and Sreenivasan K R 2014 Small-scale universality in fluid turbulence Proc. Natl Acad. Sci. USA 111 10961
Telbany M M M and Reynolds A J 1982 The structure of turbulent plane Couette flow Trans. A.S.M. E., J. Fluids Eng. 104 367
Tennekes H and Lumley J L 1972 A First Course in Turbulence (Cambridge, MA: MIT Press)
Tsukahara T, Kawamura H and Shigai K 2006 DNS of turbulent Couette flow with emphasis on the large-scale structure in the core region *J. Turbul.* **7** 19

van der Veen R C, Huisman S G, Dung O-Y, Tang H L, Sun C and Lohse D 2016 Exploring the phase space of multiple states in highly turbulent Taylor–Couette flow *Phys. Rev. Fluids* **1** 024401

Vincenti W G 1982 Control-volume analysis: a difference in thinking between engineering and physics *Technol. Cult.* **23** 145