Combining Trajectory Data with Energy Functions for Improved Region of Attraction Estimation

Abstract—The increasing penetration of inverter based sources (IBS) has been raising a lot of concerns for the power system operators around the world. The high level of complexity of this kind of generators make model-based stability analysis a difficult task for operators. This work presents a method to estimate the Region of Attraction (ROA) based on synchrophasor measurements, through trajectory polynomial fitting using Bernstein polynomials, of a converse Lyapunov function of the system and its derivative. The method is derived from the expansion of a previous ROA, which guarantees an inner approximation and is not dependent on the chosen approximation method, in this case the Bernstein polynomials. The mathematical derivation is described and applied to the Single Machine Infinite Bus (SMIB) system.

Index Terms—WAMS, Region of Attraction, Bernstein polynomials, Lyapunov Stability.

I. INTRODUCTION

Stability analysis is of utmost importance for the secure planning and operation of modern power systems. Of particular interest is the Transient Angular Stability of a system, defined as the ability of the system to maintain rotor angle synchronism following a disturbance and its subsequent angle excursion [1]. Given a Stable Equilibrium Point (SEP), the Region of Attraction (ROA), defined as the set of initial conditions for which the system tends to the SEP, provides a metric of the strength of angular stability of the system. Moreover, knowledge of the ROA can be used to provide protection parameters and limits of operation that maintain the stability and safety of the system.

For general nonlinear systems there does not exist an analytical expression for the ROA. In the absence of an analytical expression, there is a need for methods that can compute approximations of the ROA. Classically, methods that approximate the ROA rely on precise system models [2], [3]. However, the increasing penetration IBS has resulted in more complex machine models and more dynamic operating points. For such complex systems it has become increasingly intractable to use classical model-based methods to accurately approximate the ROA [4]. Fortunately, the use of synchrophasors has provided new sets of system data with minimal model dependency. Some works have already explored the use of synchrophasors for ROA estimation in the literature.

In [5], data driven ROA estimation is realized by application of energy function analysis, using PMU measurements, to monitor tie-lines dynamic power flows. The method is applied to the WECC power system collected data. Authors from [6] propose a novel qualitative method to determine transient stability of power systems. Global phase portraits (GPPs) including the singularity at infinity show the beginning and ending points of the portraits, and indicate limits of an attractor’s basin. In [7] the stability analysis of power systems is performed constructing Lyapunov functions (LF), making use of semidefinite programming and sum of squares recent advances. Since power systems have a nonlinear trigonometric representation, an algebraic reconfiguration method is applied to represent power systems dynamics as polynomial differential algebraic equations, required by sum of squares tools. This method requires knowledge of the complete vector field. The authors from [8] and [9] propose the use of measurement data to make model-based stability analysis, there’s is no guarantee of inner approximation of the ROA.

The works of [4], [6] make analytical approaches that improves over classical energy function methods. Most of importantly, there is no appropriate discussion of the use of the derivative of the LF in measurement-based methods for the analysis of the power system stability, nor an analytical discussion of an inner approximation of the ROA using trajectory data.

The main contribution of this paper, presented in Theorem 3 shows how the existence of two functions, V1 and V2, can provide a certifiable inner approximation of the ROA of a given system. Specifically, if V2 is a LF and V1 (that may not necessarily be a LF) is decreasing along the solution map inside a “donut”-shaped region, \( \{x \in \mathbb{R}^n : \gamma_1 \leq V_1(x) \leq \gamma_2\} \), we show that it is possible to construct an improved ROA estimation, as compared with the ROA approximation yielded alone by the LF, V2.

II. NOTATION

We denote the ball of radius \( \eta > 0 \) centered at \( x \in \mathbb{R}^n \) by \( B_\eta(x) := \{y \in \mathbb{R}^n : ||x - y||_2 < \eta\} \), where \( || \cdot ||_2 \) is the euclidean norm. Let \( C(\Omega, \Theta) \) be the set of continuous functions with domain \( \Omega \subset \mathbb{R}^n \) and image \( \Theta \subset \mathbb{R}^m \).

For \( \alpha \in \mathbb{N}^n \) we denote the partial derivative \( \partial^\alpha f(x) := \)
\( \Pi^n \frac{\partial^n f}{\partial x^n}(x) \) where by convention if \( \alpha = [0, \ldots, 0]^T \) we denote 
\( D^\alpha f(x) := f(x) \). We denote the set of \( \iota \) th continuously differentiable functions by 
\( C^\iota(\Omega, \Theta) := \{ f \in C(\Omega, \Theta) : D^\alpha f \in C(\Omega, \Theta) \} \) for all \( \alpha \in \mathbb{N}^n \) such that \( \sum_{i=1}^n \alpha_j \leq \iota \).

For \( V \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}) \) we denote \( \nabla V := \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right)^T \).

We denote the space of polynomials \( p : \Omega \to \Theta \) by \( \mathcal{P}(\Omega, \Theta) \) and polynomials with degree at most \( d \in \mathbb{N} \) by \( \mathcal{P}_d(\Omega, \Theta) \).

### III. Stability of Ordinary Differential Equations

Consider a dynamical system, represented by a nonlinear ordinary differential equation (ODE) of the form

\[
\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \in [0, \infty)
\]

(1)

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is the vector field and \( x_0 \in \mathbb{R}^n \) is the initial condition. WLOG throughout this paper we will consider the initial condition to the origin.

We look for a neighborhood of the origin. We say a set \( U \) contains a neighborhood of the origin. 2) \( ROA \)

expression for their associated solution maps. In the absence of \( ROA \)

of polynomials with degree at most \( d \in \mathbb{N} \) by \( \mathcal{P}_d(\Omega, \Theta) \).

### Definition 1.

Consider an ODE (1) defined by some vector field \( f \). We say a set \( U \subset \mathbb{R}^n \) is asymptotically stable if 1) \( U \) contains a neighborhood of the origin. 2) \( U \) is invariant. That is for all \( x \in U \) we have that \( \phi_f(x, t) \in U \) for all \( t \geq 3 \) For all \( x \in U \) we have that \( \lim_{t \to \infty} \| \phi_f(x, t) \| = 0 \).

Furthermore, we say the set \( U \) is exponentially stable if there exists \( \mu, \delta > 0 \) such \( \| \phi_f(x, t) \| < \mu e^{-\delta t} \) for all \( t \geq 0 \).

A system given by an ODE (1) is said to be locally asymptotically stable if it has a non-empty asymptotically stable set. The system is said to be locally exponentially stable if it has a non-empty asymptotically stable set.

For a solution map \( \phi_f \) associated with some ODE (1) the main aim of this paper is to estimate the set:

\[
ROA_f := \{ x \in \mathbb{R}^n : \lim_{t \to \infty} \| \phi_f(x, t) \| = 0 \}
\]

(2)

called the Region of Attraction (ROA). Intuitively the ROA is the set of all initial conditions that result in the solution map converging to the origin and is hence the union of all asymptotically stable sets of the ODE.

There is no universal method for exactly solving nonlinear ODEs, hence there is no general way of obtaining an analytical expression for their associated solution maps. In the absence of a precise analytical solution it is challenging to directly compute \( ROA_f \) using Eq. (2). Thus, over the years, arguably the most commonly used method to estimate \( ROA_f \) is Lyapunov’s second method that indirectly estimates \( ROA_f \) using Lyapunov Functions (LFs); functions that are globally non-negative that decrease along the solution map. The following theorem shows how the sublevel set of a LF can approximate \( ROA_f \).

### Theorem 1 (LaSalle’s Invariance Principle [10]). Consider an ODE (1) defined by some vector field \( f \in C^1(\mathbb{R}^n, \mathbb{R}^n) \).

Suppose there exits \( V \in C^1(\mathbb{R}^n, \mathbb{R}) \) and a set \( \Omega \subset \mathbb{R}^n \) with \( 0 \in \Omega \) such that

\[
V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \in \Omega \setminus \{0\},
\]

\[
\nabla V(x)^T f(x) \leq 0 \text{ for all } x \in \Omega,
\]

\[
\phi_f(x, t) \in \{ x \in \Omega : V(x) \leq 0 \}
\]

for all \( t \geq 0 \) iff \( x = 0 \).

If \( \gamma > 0 \) is such that \( \{ x \in \mathbb{R}^n : V(x) \leq \gamma \} \subseteq ROA_f \).

Theorem 1 shows that for a given ODE, if we can find a LF, then we can construct an inner-approximate of the ROA of the ODE. However, this theorem does not show that there must necessarily exists a LF for a given ODE or that the ROA of the ODE can be exactly characterized by an LF.

In [11] it was shown that for locally asymptotically stable ODEs there exists a LF of the form,

\[
W(x) := \int_0^\infty \alpha(||\phi_f(x, t)||^2)\, dt,
\]

(3)

where \( \alpha : [0, \infty) \to [0, \infty) \) is some monotonically increasing function with \( \alpha(0) = 0 \). The converse LF given in Eq. (3) is called the maximal LF because it has the property that its \( n \)-sublevel set is equal to \( ROA_f \), that is \( \{ x \in \mathbb{R}^n : W(x) < \alpha \} = ROA_f \).

Unfortunately, the maximal LF, given in Eq. (3), is unbounded outside of \( ROA_f \) and therefore is not continuous over compact sets that contain points outside of \( ROA_f \). This makes approximating maximal LFs challenging. Fortunately, it has been shown in [12] that for locally exponentially stable ODEs of Form (1), there exists a bounded and continuous LF of the form,

\[
V^*_{\lambda, \beta}(x) := \begin{cases} 1 - \exp \left( -\lambda \int_0^\infty ||\phi_f(x, t)||^2 \, dt \right) & \text{if } x \in ROA_f \\ 1, & \text{otherwise} \end{cases}
\]

(4)

where \( \lambda > 0 \) and \( \beta \in \mathbb{N} \). Moreover, for sufficiently large \( \lambda \) and \( \beta \), this converse LF, \( V^*_{\lambda, \beta} \), is Lipschitz continuous and hence differentiable almost everywhere (by Rademacher’s theorem).

Furthermore, the \( 1 \)-sublevel set of this converse LF yields the ROA of the given ODE, that is \( \{ x \in \mathbb{R}^n : V^*_{\lambda, \beta}(x) < 1 \} = ROA_f \).

### IV. Fitting Bernstein Polynomials to Converse Lyapunov Functions

Given an ODE of Form (1), in this section we show that by using an ODE solver to generate trajectory data, Eq. (4) can be used to construct input-output data of a converse LF. By fitting a function to this data we can approximate this converse LF. This converse LF has the property that its \( 1 \)-sublevel set
is equal to the ROA of the ODE. Thus by examining the 1-sublevel sets of our approximation we can hope to construct an estimation of the ROA of the ODE.

Specifically, we fit polynomial functions to the generated input/output data of the converse LF. Although there are many ways to fit polynomials to data (each having their relative advantages and disadvantages), we have chosen a method based on Bernstein approximations. As we will next see, Bernstein’s method for fitting polynomials to data is an optimization free approach that is guaranteed to converge uniformly.

A. Bernstein Approximation of Smooth Functions

We now provide a brief description of how Bernstein polynomials can approximate smooth functions. For a more in-depth overview of the field we refer to [13]. Now, recalling from Section II that we \( \mathcal{P}_d(\mathbb{R}^n, \mathbb{R}) \) as the set of d-degree polynomials we next define the Bernstein operator that maps onto \( \mathcal{P}_d(\mathbb{R}^n, \mathbb{R}) \).

**Definition 2.** We denote the degree \( d \in \mathbb{N} \) Bernstein operator by \( B_d : C(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathcal{P}_{d \times n}(\mathbb{R}^n, \mathbb{R}) \) and for \( V \in C(\mathbb{R}^n, \mathbb{R}) \) we define \( B_dV \in \mathcal{P}_{d \times n}(\mathbb{R}^n, \mathbb{R}) \) by

\[
B_dV(x) = \sum_{i=1}^n \sum_{k_i=1}^d \frac{d!}{k_1! \cdots k_n!} \prod_{j=1}^n \left( \frac{d}{k_j} \right) x^{k_j} (1 - x_j)^{d-k_j}.
\]

We note that, given a function \( V \in C(\mathbb{R}^n, \mathbb{R}) \), we can calculate the polynomial \( B_dV \) through application of Eq. (4) with only knowledge of the values of \( V \) at uniformly gridded points in \([0,1]^n\). Thus, in order to calculate \( B_dV \) it is not necessary to have an analytic expression of \( V \). We next recall that \( B_dV \rightarrow V \) uniformly as \( d \rightarrow \infty \). Moreover, although the Bernstein approximation in Eq. (5) only involves the value of the 0'th differential order of \( V \), if \( V \) is differentiable, then it follows that the derivative of \( B_dV \) will also converge to the derivative of \( V \). This is a particularly useful when it comes to approximating converse LFs because we would also like our approximation to be a LF itself. Thus to make our approximation, \( B_dV \), have the property that it decreases along solution trajectories, \( \dot{P}(x) < 0 \), we ensure the derivative of \( P \) also approximates the derivative of the converse LF, \( \dot{V}^*(x) < 0 \).

**Theorem 2** (Multivariate uniform approximation by Bernstein polynomials, see Theorem 4 in [13]). Given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \) suppose \( D^nV \in C(\mathbb{R}^n, \mathbb{R}) \) then it follows

\[
\lim_{d \rightarrow \infty} \sup_{x \in [0,1]^n} |D^nB_dV(x) - D^nV(x)| = 0.
\]

Theorem 2 shows that Eq. (5) can be used to approximate functions over \([0,1]^n\). Note, using the same methodology, we may also approximate a function, \( V \), over some set \([a,b]^n\) where \( a < b \). In order to do this we first apply a linear coordinate change mapping \([a,b]^n\) to \([0,1]^n\), defining \( \tilde{V}(x) := V\left(\frac{x_1-a_1}{b_1-a_1}, \ldots, \frac{x_n-a_n}{b_n-a_n}\right)\). We then apply Eq. (5) to approximate \( \tilde{V}(x) \) over \([0,1]^n\), yielding \( B_d\tilde{V} \) such that \( \lim_{d \rightarrow \infty} \sup_{x \in [0,1]^n} |D^nB_d\tilde{V}(x) - D^n\tilde{V}(x)| = 0 \) (by Theorem 2). Finally, we again change the coordinates, mapping \([0,1]^n\) back to \([a,b]^n\), defining \( \tilde{J}(x) := B_d\tilde{V}\left((b_1-a_1)x_1 + a_1, \ldots, (b_n-a_n)x_n + a_n\right) \). It then follows that

\[
\lim_{d \rightarrow \infty} \sup_{x \in [a,b]^n} |D^nJ(x) - D^nV(x)| = 0.
\]

B. Generating Data from Converse Lyapunov Functions

Converse LFs can be approximated by polynomials using Eq. (5). However, in order to apply Eq. (5) we must know the value of the converse LF at uniformly gridded points in \([a,b]^n\) (note approximation over \([a,b]^n\) rather than \([0,1]^n\) can be achieved through linear coordinate transformations, see Eq. (7)).

Given a set of initial conditions, \( \{x_i\}_{1 \leq i \leq N} \subset [a,b]^n \), and a terminal trajectory time \( T > 0 \), it is possible to generate trajectory data \( D_{i,j} := \phi_f(x_i, (j-1)\Delta t) \), where \( \Delta t > 0 \) is some small time-step and \( 1 \leq j \leq \frac{T+1}{\Delta t} \). This can be achieved using any ODE solver, for instance, Matlab’s ODE45. Of course, in order to use an ODE solver this does require complete knowledge of the vector field, \( f \). In the case where the model of the system is unknown it may still be possible to generate the required trajectory data from experimental data.

Now, given \( \lambda > 0 \) and \( \beta \in \mathbb{N} \) trajectory data, \( D \in \mathbb{R}^{N \times (K+1)} \), for sufficiently large \( K \in \mathbb{N} \), we can approximate the value of the corresponding converse LF given in Eq. (4) in the following way

\[
V^*_\lambda,\beta(x_0) \approx 1 - e^{-\lambda W(x_0)},
\]

where \( W(x_0) \approx \sum_{j=1}^{K+1} D_{i,j}^{2\beta} \Delta t \).

V. IMPROVING ROA ESTIMATION WITH APPROXIMATED CONVERSE LYAPUNOV FUNCTIONS

Given an ODE defined by some vector field \( f \), we have shown that through applications of Eqs. (5) and (8) that it is possible to numerically construct Bernstein polynomial approximation, \( B_dV^*_\lambda,\beta \) for some \( d \in \mathbb{N} \lambda > 0 \) and \( \beta \in \mathbb{N} \), of the converse LF, \( V^*_\lambda,\beta \), given in Eq. (4). Moreover, assuming that \( D^nV^*_\lambda,\beta \) is continuous, where \( \alpha \in \mathbb{N}^n \), Theorem 2 can be used to show that \( \lim_{d \rightarrow \infty} D^nB_dV^*_\lambda,\beta \rightarrow D^nV^*_\lambda,\beta \) uniformly in \([a,b]^n\) (note approximation over \([a,b]^n\) rather than \([0,1]^n\) can be achieved through linear coordinate transformations, see Eq. (7)).

Ideally, for sufficiently large \( d \in \mathbb{N} \lambda > 0 \) and \( \beta \in \mathbb{N} \), our approximation, \( B_dV^*_\lambda,\beta \), will also be a LF over some set containing the origin; that is \( B_dV^*_\lambda,\beta(0) = 0 \), \( B_dV^*_\lambda,\beta(x) \geq 0 \) for all \( x \in \Omega \) where \( \Omega \) is some small time-step and \( 1 \leq j \leq \frac{T+1}{\Delta t} \).
set of $B_d V^*_{\lambda,\beta}$ yields an inner approximation of the ROA of the given ODE.

Unfortunately, despite the fact $B_d V^*_{\lambda,\beta}$ tends to $V^*_{\lambda,\beta}$ as $d \to \infty$ and $V^*_{\lambda,\beta}$ is a LF, $B_d V^*_{\lambda,\beta}$ is not necessarily a LF. To see this we first note that, for a given $x_0 \in [a, b]^n$ if $\nabla V^*_{\lambda,\beta}(x_0) T f(x_0) < 0$, then $\frac{\partial}{\partial t} V^*_{\lambda,\beta}(x_0) \leq a > 0$, if $T > \frac{\partial}{\partial t}$. This follows by Theorem 2 that there exists $D \in N$ such that $|\nabla B_d V^*_{\lambda,\beta}(x_0) - \nabla V^*_{\lambda,\beta}(x_0)||_2 < \frac{\partial}{\partial t} V^*_{\lambda,\beta}(x_0)||_2$ for all $D > D$. Thus, using the Cauchy-Swarz inequality we have that for all $D > D$,

$$\nabla B_d V^*_{\lambda,\beta}(x_0) T f(x_0) \leq ||\nabla B_d V^*_{\lambda,\beta}(x_0)| - \nabla V^*_{\lambda,\beta}(x_0)||_2 $$

Eq. (9) shows that for sufficiently large $d$, whenever $V^*_{\lambda,\beta}$ is strictly decreasing along the solution map we also have that $B_d V^*_{\lambda,\beta}$ is strictly decreasing along the solution map. However, at the origin $V^*_{\lambda,\beta}$ is not strictly decreasing along the solution map, that is $\nabla V^*_{\lambda,\beta}(f(0)) = 0$. Because of this fact and the fact $B_d V^*_{\lambda,\beta}$ is continuous, in general for some finite $D \in N$ our approximation will be such that that $\nabla B_d V^*_{\lambda,\beta}(x) T f(x) = 0$ for all $x$ in some small neighborhood of the origin. Thus, in general, for finite $D \in N$, it follows that we will have $B_d V^*_{\lambda,\beta}(x) T f(x) = 0$ for all $x$ in some “donut shaped” region, $\{y \in R^n : \gamma_1 \leq B_d V^*_{\lambda,\beta}(x) \leq \gamma_2\}$ for some $\gamma_1 < \gamma_2$, as opposed to a sublevel set $\{y \in R^n : B_d V(x) \leq \gamma_2\}$.

Although, $B_d V^*_{\lambda,\beta}$ is not a LF, and therefore cannot certify the origin is asymptotically stable, we will next show, in Prop. 1 that functions strictly decreasing along the solution map inside some “donut shaped” region can still be used to certify that the solution map must enter some ball of radius $\eta > 0$ around the origin, $B_\eta(0)$.

**Proposition 1.** Consider an ODE (11) defined by some vector field $d f = C^1 (R^n, R^n)$. Suppose $V \in C^1 (R^n, R)$, $\eta > 0$. $\gamma_1 < \gamma_2$ and $a > 0$ are such that

- $\{y \in R^n : V(y) \leq \gamma_1\} \subset B_\eta(0) \subset \{y \in R^n : V(y) \leq \gamma_2\}$
- For all $x \in \{y \in R^n : \gamma_1 \leq V(y) \leq \gamma_2\}$ we have that $\nabla V(x) T f(x) < -a < 0$.

Then it follows that for any $x \in \{y \in R^n : V(y) \leq \gamma_2\}$ there exists $T > 0$ such that $\phi_f(x, T) \in B_\eta(0)$.

Moreover, $\{y \in R^n : V(y) \leq \gamma_2\}$ is an invariant set. That is for all $x \in \{y \in R^n : V(y) \leq \gamma_2\}$ we have that $\phi_f(x, t) \in \{y \in R^n : V(y) \leq \gamma_2\}$ for all $t \geq 0$.

**Proof.** We first show that $\{y \in R^n : V(y) \leq \gamma_2\}$ is an invariant set. For $x \in \{y \in R^n : \gamma_1 \leq V(y) \leq \gamma_2\}$ let us denote the first time the solution map leaves the set $\{y \in R^n : \gamma_1 \leq V(y) \leq \gamma_2\}$ by $T_x := \inf\{T : V(\phi_f(x, T)) > \gamma_2 \text{ or } V(\phi_f(x, T)) < \gamma_1\}$.

Now, by Eq. (10) it follows that

$$\frac{d}{dt} V(\phi_f(x, t)) = \nabla V(\phi_f(x, t)) f(\phi_f(x, t)) < -a$$

for all $x \in \{y \in R^n : \gamma_1 \leq V(y) \leq \gamma_2\}$ and $t \in [0, T_x]$. (13)

By integrating Eq. (13) over $[0, t]$ it follows that,

$$V(\phi_f(x, t)) < V(x) - at \leq \gamma_2 - at < \gamma_2$$

for all $x \in \{y \in R^n : \gamma_1 \leq V(y) \leq \gamma_2\}$ and $t \in [0, T_x]$. (14)

For contradiction suppose that $\{y \in R^n : V(y) \leq \gamma_2\}$ is not an invariant set. Then there exists $x \in \{y \in R^n : V(y) \leq \gamma_2\}$ and $0 < T < \infty$ such that $\phi_f(x, T) \notin \{y \in R^n : V(y) \leq \gamma_2\}$, implying $V(\phi_f(x, T)) > \gamma_2$.

Let us denote the function $g(t) := V(\phi_f(x, t))$. It follows that $g$ is continuous since the composition of two continuous functions is continuous and $V$ and $\phi_f(x, \cdot)$ are both continuous functions.

WLOG we assume $x$ and $T$ are such that $g(t) = V(\phi_f(x, t)) > \gamma_1$ for all $t \in [0, T]$. Note that we can assume this WLOG since if there exists $s \in (0, T]$ such that $g(s) = V(\phi_f(x, s)) \leq \gamma_1$ then by the Intermediate Value Theorem (using the fact $g$ is continuous) there must exist $c \in [s, T]$ such that $g(c) = \frac{\gamma_2}{\gamma_2 - \gamma_1} > \gamma_2$. In this case we relabel $x$ as $\phi_f(x, T)$ and $T$ as $T - s$ (using the fact $\phi_f(x, s, t) = \phi_f(x, t + s)$. Note that we may relabel $x$ and $T$ as many times required until there does not exist any $s \in (0, T)$ such that $g(s) = V(\phi_f(z, s)) \leq \gamma_1$.

Now $\gamma_1 < g(0) = V(x) \leq \gamma_2$ and $g(T) = V(\phi_f(x, T)) > \gamma_2$. By the Intermediate Value Theorem (using the fact $g$ is continuous) there must exist a $c \in [0, T]$ such that $g(c) = V(\phi_f(x, c)) = \gamma_2$. Clearly $c < T_x$ (recalling $T_x$ is defined in Eq. (12)). Thus applying Eq. (14) at $t = c$ we get

$$\gamma_2 = V(\phi_f(x, c)) < \gamma_2,$$

providing a contradiction. Hence, $\{y \in R^n : V(y) \leq \gamma_2\}$ is an invariant set.

We next show that for $x \in \{y \in R^n : V(y) \leq \gamma_2\}$ there exists $T > 0$ such that

$$\phi_f(x, T) \in B_\eta(0).$$

(15)

Note that since $\{y \in R^n : V(y) \leq \gamma_1\} \subset B_\eta(0)$ it follows that if $x \in \{y \in R^n : V(y) \leq \gamma_2\}$ then $x \in \{y \in R^n : \gamma_1 \leq V(y) \leq \gamma_2\}$, thus Eq. (14) can be applied for such $x$. Also note that since $\{y \in R^n : V(y) \leq \gamma_2\}$ is an invariant set it follows that $T_x$, given in Eq. (12), can be simplified to

$$T_x := \inf\{T : V(\phi_f(x, T)) < \gamma_1\}.$$ (16)

Now, if $T_x < \infty$ then $V(\phi_f(x, T_x)) \leq \gamma_1$. This implies $\phi_f(x, T_x) \in \{y : V(y) \leq \gamma_1\} \subset B_\eta$, showing Eq. (15). On the other hand if we suppose for contradiction that $T_x$ is unbounded then $V(\phi_f(x, t)) \geq \gamma_1$ for all $t \geq 0$ and Eq. (14)
holds for all $t \geq 0$. In particular if we select $t = \frac{2a-2\gamma}{2a} > 0$ (noting $\gamma_1 < \gamma_2$) in Eq. (13) we get that
\[
\gamma_1 \leq V(\phi_f(x,t)) \leq \gamma_2 - at = \frac{\gamma_2 + \gamma_1}{2} < \gamma_1,
\]
providing a contradiction showing that $T_x < \infty$ and hence showing Eq. (15) must hold. \hfill \square

In Prop. [1] we have proposed conditions, based on some function we denote here as $V_1$, that certify that the solution map of a given ODE must enter some ball, $B_\eta(0)$. Recall that $V_1$ can be found by approximating a converse LF by Bernstein polynomials.

We next show that if there exists a LF, $V_2$, that certifies that $B_\eta(0)$ is an asymptotically stable set, then $V_1$ and $V_2$ can be used together to provide an inner approximation of the ROA of the given ODE.

**Theorem 3.** Consider an ODE (1) defined by some vector field $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Suppose there exists $V_1, V_2 \in C^1(\mathbb{R}^n, \mathbb{R})$ such that for some $\eta > 0$, $\gamma_1 < \gamma_2$ and $a > 0$, $V_1$ satisfies
\[
\{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_1 \} \subseteq B_\eta(0) \cap \{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_2 \},
\]
and for all $x \in \{ y \in \mathbb{R}^n : \gamma_1 \leq V_1(y) \leq \gamma_2 \}$ we have that
\[
\nabla V_1(x)^T f(x) < -a < 0.
\]
Moreover, for some $\gamma_0 > 0$ and $\Omega \subseteq \mathbb{R}^n$, such that $0 \in \Omega$, suppose $V_2$ satisfies
\[
\begin{align*}
V_2(0) & = 0 \text{ and } V_2(x) > 0 \text{ for all } x \in \Omega \setminus \{0\}, \\
\nabla V_2(x)^T f(x) & < 0 \text{ for all } x \in \Omega, \\
\phi_f(x,t) & \in \{ x \in \Omega : \nabla V_1(x)^T f(x) = 0 \} \text{ for all } t \geq 0 \text{ iff } x = 0, \\
B_\eta(0) & \subseteq \{ y \in \mathbb{R}^n : V_2(y) \leq \gamma_3 \} \subseteq \Omega.
\end{align*}
\]
Then $\{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_2 \} \cup \{ y \in \mathbb{R}^n : V_2(y) \leq \gamma_3 \} \subseteq \text{ROA}_f$.

**Proof.** By Theorem [1] it follows that $\{ y \in \mathbb{R}^n : V_2(y) \leq \gamma_3 \} \subseteq \text{ROA}_f$.

If $x \in \{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_2 \}$ then by Prop. [1] there exists $T > 0$ such that
\[
z := \phi_f(x,T) \in B_\eta(0) \subseteq \{ y \in \mathbb{R}^n : V_2(y) \leq \gamma_3 \}.
\]
Since $\{ y \in \mathbb{R}^n : V_2(y) \leq \gamma_3 \} \subseteq \text{ROA}_f$ it follows that
\[
\lim_{t \to \infty} ||\phi_f(z,t)|| = 0.
\]
Therefore, using the semi-group properties of solution maps we have that
\[
\lim_{t \to \infty} ||\phi_f(x,t) - t - T|| = \lim_{t \to \infty} ||\phi_f(z,t) - s|| = 0,
\]
implying $x \in \text{ROA}_f$. Because the same argument can be used for any $x \in \{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_2 \}$ it follows that $\{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_2 \} \subseteq \text{ROA}_f$.

VI. THE ENERGY FUNCTION OF THE SMIB SYSTEM

Theorem [3] shows how two functions, $V_1$ and $V_2$, can be used to produce an inner approximation of the ROA of a given ODE. Recall from Section [I] that $V_1$ may be found by fitting Bernstein polynomials to converse LFs. On the other hand, $V_2$ may be found by linearizing the system and then computing a quadratic LF for the linear systems using Linear Matrix Inequalities (LMIs), see [14]. Alternatively, in the case that the vector field is polynomial, $V_2$ may be computed using Sum-of-Square (SOS) programming, see [15].

In this paper we take an alternative approach to computing $V_2$. For many systems it is possible to derive an energy function. The value of this function provides the total energy of the system for each element of the state space. As we will next see, energy functions are LFs over some neighborhood of the equilibrium point and hence can be used as $V_2$ in Theorem 3.

Unlike linearization or SOS based methods, finding energy functions does not involve solving an optimization problem. Rather, it involves careful consideration of the physics of the particular system in order to derive the potential and kinetic energies for each element of the state space of the system. We next give a brief overview of the derivation of the energy function of the Single Machine Infinite Bus (SMIB) system; for a more in-depth analysis see [16].

Consider the SMIB system defined by the following ODE:
\[
\dot{\delta}(t) = \omega(t)
\]
\[
2H\dot{\omega}(t) = P_m - \frac{E'EB}{X_{eq}} \sin(\delta(t)) - D\omega(t),
\]
where $P_m, E', E_B, X_{eq}, D \in \mathbb{R}$ are known constants. The equilibrium point of this system occurs at $[\delta_{eq}]^T \in \mathbb{R}^2$, where $\delta_{eq} := \sin^{-1}\left(\frac{P_m X_{eq}}{E'EB}\right)$. To be consistent with Section [III] we make the coordinate change $\delta = \delta - \delta_{eq}$, mapping the equilibrium point $[\delta_{eq}]^T$ to the origin. Now, denoting the state as $x = [\delta, \omega]$, the system given in Eq. (18) can be written as an ODE of Form (1) with the following vector field,
\[
f_{SMB}(x) := \left(1/2H\right)\left(P_m - \frac{E'EB}{X_{eq}} \sin(x_1 + \delta_{eq}) - Dx_2\right).
\]
It was shown in [16] that the potential and kinetic energies of the SMIB system are given by,
\[
V_{pe}(x) := -P_m x_1 + \frac{E'V}{X_{eq}} \cos(\delta_{eq}) - \cos(x_1 + \delta_{eq})
\]
\[
V_{ke}(x) := Hx_2^2.
\]
The potential and kinetic energies can then be combined together to give the energy function,
\[
V_E(x) := V_{pe}(x) + V_{ke}(x)
\]
\[
= -P_m x_1 + \frac{E'V}{X_{eq}} \cos(\delta_{eq}) - \cos(x_1 + \delta_{eq}) + Hx_2^2.
\]
The energy function given in Eq. (19) can be used as a candidate LF. Clearly $V(0) = 0$ and it is shown in [16] that
\( V(x) > 0 \) about some punctured neighborhood of the origin (excluding the point \( 0 \in \mathbb{R}^n \)). Moreover,

\[
\nabla V(x)^T f_{\text{SMIB}}(x) = [-P_m + \frac{E'V}{X_{eq}} \sin(x_1 + \delta_{ep}), 2Hz_2]^T f_{\text{SMIB}}(x)
\]

\[
= x_2 \left( -P_m + \frac{E'V}{X_{eq}} \sin(x_1 + \delta_{ep}) \right) + x_2 \left( P_m - \frac{E'EB}{X_{eq}} \sin(x_1 + \delta_{ep}) - \tilde{D}x_2 \right)
\]

\[
= -\tilde{D}x_2^2 \leq 0 \text{ for all } x \in \mathbb{R}^n.
\]

It was shown in [16] that \( \phi_{T_{\text{SMIB}}}(x,t) \in \{ x \in \mathbb{R}^n : x_2 = 0 \} \) if \( x = 0 \). Thus, by Theorem 1 the largest inner-approximation of the ROA of the ODE (18) that \( V_E \) can yield is given by \( \{ x \in \mathbb{R}^n : V_E(x) \leq \gamma^* \} \subseteq \text{ROA}_{T_{\text{SMIB}}} \)

where \( \gamma^* \in \arg \sup_{\gamma \in \mathbb{R}} \gamma \) \( \in \mathbb{R} \)

such that \( \{ x \in \mathbb{R}^n : V_E(x) \leq \gamma \} \subseteq \Omega_E \),

and \( \Omega_E := \{ 0 \} \cup \{ (x \in \mathbb{R}^n : V_E(x) > 0) \cap \{ x \in \mathbb{R}^n : \nabla V_E(x)^T f_{\text{SMIB}}(x) \leq 0 \} \}. \)

We have graphically solved \( \text{Opt. (20)} \), finding \( \gamma^* = 5.722 \), and plotted the boundary of \( \{ x \in \mathbb{R}^n : V_E(x) \leq \gamma^* \} \) as the blue line in Fig. 1.

VII. NUMERICAL EXAMPLE: ESTIMATING THE ROA OF THE SMIB SYSTEM

Consider the SMIB system given in Eq. (18) with model constants: \( P_m, E', E_B, X_{eq}, D \in \mathbb{R} \), \( \omega = 1pu \), \( M = 2H = 0.0212s^2/\text{rad} \), \( X_{eq} = 0.28pu \), \( P_m = 1pu \), \( E_B = 1pu \), \( E' = 1.21pu \), \( D = 1.03 \).

We approximate \( \text{ROA}_{T_{\text{SMIB}}} \) by finding functions \( V_1 \) and \( V_2 \) in Theorem 3. To find \( V_1 \) we fit a Bernstein polynomial to the convex LF, \( V^*_{X,\beta} \) (given in Eq. (4)), for \( \lambda = 10 \) and \( \beta = 1 \) over the set \([-0.75\pi, 0.75\pi] \times [-55, 55] \) using Eqs. (5) and (7) for \( d = 25 \). Application of Eq. (5) for \( d = 25 \) requires knowledge of the value of \( V^*_{X,\beta} \) at 625 grid points that is found from trajectory data using Eq. (8). Note, the Bernstein fitting uses all grid points in the rectangle, unstable or not. Finally, we use \( V_E \), given in Eq. (19), as \( V_2 \). We have plotted the boundaries of the sets \( \{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_1 \} \) and \( \{ y \in \mathbb{R}^n : V_2(y) \leq \gamma_2 \} \) in Fig. 1 as the blue and black lines respectively, where \( \gamma_1 = 0.55 \) and \( \gamma_2 = 5.722 \) (found by solving Opt. (20)) are such that Theorem 3 shows \( \{ y \in \mathbb{R}^n : V_1(y) \leq \gamma_1 \} \cup \{ y \in \mathbb{R}^n : V_2(y) \leq \gamma_2 \} \subseteq \text{ROA}_{T_{\text{SMIB}}} \), providing an inner approximation of \( \text{ROA}_{T_{\text{SMIB}}} \). Moreover, we have plotted the boundary of the set \( \{ y \in \mathbb{R}^n : \nabla V_1(y)^T f_{\text{SMIB}}(y) \leq 0 \} \) as the dotted black line. Showing \( \nabla V_1(y)^T f_{\text{SMIB}}(y) \) may not be negative around a neighborhood of the origin as expected from Sec. V.

To demonstrate the accuracy of our approximation of \( \text{ROA}_{T_{\text{SMIB}}} \) we have carried out extensive Monte Carlo simulations of \( \phi_{T_{\text{SMIB}}} \) for over 10,000 initial conditions to estimate the stable and unstable regions, colored green and red respectively in Fig. 1. Although this Monte Carlo method can estimate \( \text{ROA}_{T_{\text{SMIB}}} \) well (in two dimensions) it does not account for simulation error or provide a LF, hence it cannot certify whether the green region is truly an inner approximation of \( \text{ROA}_{T_{\text{SMIB}}} \), unlike the method proposed in this paper.

Using both \( V_1 \) and \( V_2 \) we have improved the inner approximation of \( \text{ROA}_{T_{\text{SMIB}}} \) as compared to the approximation yielded by the energy function, \( V_2 \), alone. Our approximation provides an almost analytical solution for the boundary of \( \text{ROA}_{T_{\text{SMIB}}} \) at the top right quadrant, an important region where the system is most likely to operate in a practical situation.

VIII. CONCLUSION

This work proposes a novel methodology for improving the ROA of transient angular stability, using Bernstein polynomial estimation of converse Lyapunov function applied to PMU data. The method is derived and applied to the SMIB system, providing expanded ROA. The method is capable of providing an inner approximation of the ROA which is based on the estimation of both the converse Lyapunov Function and its derivative. The method is able to improve the ROA provided by the traditional energy function method, particularly at the typical region of operation of the machine. Additionally, the method is not limited to the Bernstein polynomial approximation, so estimators that can handle better high dimensions can be explored, which is a future work the authors intend to pursue. Long Version.

REFERENCES

[1] P. Kundur, N. J. Balu, and M. G. Lauby, Power system stability and control. McGraw-hill New York, 1994, vol. 7.

[2] Y.-H. Moon, B.-K. Choi, and T.-H. Roh, “Estimating the domain of attraction for power systems via a group of damping-reflected energy functions,” Automatica, vol. 36, no. 3, pp. 419-425, 2000.
[3] L. Jin, H. Liu, R. Kumar, J. D. Mc Calley, N. Elia, and V. Ajjarapu, “Power system transient stability design using reachability based stability-region computation,” in Proceedings of the 37th Annual North American Power Symposium, 2005. IEEE, 2005, pp. 338–343.

[4] Z. Shuai, C. Shen, X. Liu, Z. Li, and S. John Shen, “Transient angle stability of virtual synchronous generators using lyapunov’s direct method,” IEEE Transactions on Smart Grid, vol. 10, no. 4, pp. 4648–4661, July 2019.

[5] J. H. Chow, A. Chakrabortty, M. Arcak, B. Bhargava, and A. Salazar, “Synchronized phasor data based energy function analysis of dominant power transfer paths in large power systems,” IEEE Transactions on Power Systems, vol. 22, no. 2, pp. 727–734, May 2007.

[6] M. Ma, W. Jie, Z. Wang, and M. W. Khan, “Global geometric structure of the transient stability regions of power systems,” IEEE Transactions on Power Systems, pp. 1–1, 2019.

[7] M. Anghel, F. Milano, and A. Papachristodoulou, “Algorithmic construction of lyapunov functions for power system stability analysis,” IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 60, no. 9, pp. 2533–2546, 2013.

[8] C. Zhai and H. D. Nguyen, “Estimating the region of attraction for power systems using gaussian process and converse lyapunov function,” IEEE Transactions on Control Systems Technology, 2021.

[9] B. K. Colbert and M. M. Peet, “Using trajectory measurements to estimate the region of attraction of nonlinear systems,” in 2018 IEEE Conference on Decision and Control (CDC). IEEE, 2018, pp. 2341–2347.

[10] H. K. Khalil, Nonlinear systems. Prentice hall, 2002.

[11] A. Vannelli and M. Vidyasagar, “Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems,” Automatica, vol. 21, no. 1, pp. 69–80, 1985.

[12] M. Jones and M. M. Peet, “Converse lyapunov functions and converging inner approximations to maximal regions of attraction of nonlinear systems,” arXiv preprint arXiv:2103.12825, 2021.

[13] A. Y. Veretennikov and E. Veretennikova, “On partial derivatives of multivariate bernstein polynomials,” Siberian Advances in Mathematics, vol. 26, no. 4, pp. 294–305, 2016.

[14] I. O. Gomes, E. S. Tognetti, R. C. Oliveira, and P. L. Peres, “Local stability analysis and estimation of domains of attraction for nonlinear systems via takagi-sugeno fuzzy modeling,” in 2019 IEEE 58th Conference on Decision and Control (CDC). IEEE, 2019, pp. 4823–4828.

[15] M. M. Peet, “Exponentially stable nonlinear systems have polynomial lyapunov functions on bounded regions,” IEEE Transactions on Automatic Control, vol. 54, no. 5, pp. 979–987, 2009.

[16] J. Chow and J. J. Sanchez-Gasca, Power System Modeling, Computation, and Control. JohnWiley & Sons Ltd, 2020.