Origin of the anomalous Hall effect in two-band chiral superconductors

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We consider the origin of the anomalous Hall effect in a general model of a clean two-band chiral superconductor. Within the Kubo formalism we derive an analytic expression for the high-frequency ac Hall conductivity valid close to the critical temperature. This expression involves two distinct gauge-invariant time-reversal-odd bilinear (TROB) functions involving the pairing potential and its Hermitian conjugate. We argue that the existence of at least one of these TROBs generically implies a nonzero ac Hall conductivity. The TROBs allow us to clarify the roles of intra- and interband pairing, and provide a straightforward criterion for a superconducting state to exhibit the anomalous Hall effect. We briefly exemplify our results with model calculations for a chiral $p$-wave pairing state in strontium ruthenate and a chiral $d$-wave pairing state on the honeycomb lattice.

I. INTRODUCTION

Chiral superconductivity is an exotic pairing state characterized by the spontaneous breaking of time-reversal symmetry and a handed winding of the gap phase around the Fermi surface [1]. Chiral pairing states have been proposed for a number of superconductors, most notably Sr$_2$RuO$_4$ [2], UPt$_3$ [3, 4], URu$_2$Si$_2$ [5], and twisted bilayer graphene [6], although a definitive interpretation of the experimental evidence remains elusive. The observation of the polar Kerr effect, which is closely related to the anomalous Hall effect (AHE) [7, 8], is a key signature of time-reversal symmetry breaking (TRSB) and a handed winding of the gap phase around the Fermi surface [1]. Chiral pairing states have been observed in various candidate superconductors, such as Sr$_2$RuO$_4$ [9], UPt$_3$ [10], URu$_2$Si$_2$ [11], Bi/Ni bilayers [12], PrO$_4$Si$_{12}$ [13], and UTe$_2$ [14], providing strong evidence for the presence of chiral pairing states in these materials.

The origin of the polar Kerr effect in a chiral superconductor has been the subject of much debate, as the breaking of time-reversal symmetry by the pairing potential is not sufficient to explain the presence of the AHE. Specifically, in chiral superconductors time-reversal symmetry is broken in the relative momentum coordinate of the Cooper pair, while it is the center-of-mass coordinate which couples to the external electric field in the AHE [1, 15]. Thus, the AHE is vanishing in single-band superconductors except in the presence of impurities that break the translational symmetry which necessitates that these coordinates are independent [16–20]. On the other hand, the relative and center-of-mass coordinates are coupled in multiband superconductors [15], so mechanisms intrinsic to the clean superconductor can contribute to the AHE. Such an intrinsic contribution has been theoretically demonstrated in a large number of models [15, 21–28]. Nevertheless, it is unclear what conditions the pairing potential in a multiband model should satisfy for the existence of the AHE.

Within a simple model of chiral $d$-wave pairing on the honeycomb lattice, Ref. [27] identified a TRSB bilinear combination of the pairing potential and its Hermitian conjugate as critical to the appearance of the AHE. Dubbed the “time-reversal-odd bilinear” (TROB), this quantity is explicitly gauge-invariant and breaks time-reversal symmetry by construction. Although the simplicity of the model studied in Ref. [27] made the wider relevance of the TROB uncertain, the authors speculated that a nonzero TROB is crucial for the existence of an intrinsic AHE in a multiband chiral superconductor. In this manuscript we demonstrate that this is indeed the case by analytically calculating the ac Hall conductivity for a generic model of a two-band superconductor. Our work establishes general conditions on the form of the pairing potential required for the existence of a TROB.

We begin in Sec. II by introducing a general model of a two-band system where the normal state Hamiltonian includes all terms allowed by inversion and time-reversal symmetry. Using the Kubo formalism, in Sec. III we analytically determine the leading contribution to the ac Hall conductivity in the limit of high frequencies and a small gap, revealing that the Hall conductivity depends upon two distinct TROBs. We evaluate these TROBs for arbitrary even- and odd-parity pairing states in Sec. III A, which reveals the general importance of interband pairing, but suggests that purely-intraband pairing may also play a role. Although our analysis is performed in an asymptotic limit, we argue in Sec. III B that our conclusions should hold more generally. Our analysis is made concrete in Sec. IV by considering two model systems: chiral $p$-wave pairing in Sr$_2$RuO$_4$ [15] and $d$-wave superconductivity on the honeycomb lattice [27]. Finally, we conclude in Sec. V with an outlook for future work.

II. MODEL

We consider a general superconducting system in which the electronic states are described by their momentum, spin, and an additional discrete degree of freedom which we denote as the “orbital”. In a model with two
orbitals the single-particle Hamiltonian is written in the Bogoliubov de Gennes (BdG) formalism as
\[
\mathcal{H} = \frac{1}{2} \sum_k \Psi_k^\dagger \left( \begin{array}{cc} H_0(k) & \Delta(k) \\ \Delta^\dagger(k) & -H_0^T(-k) \end{array} \right) \Psi_k,
\]
where
\[
\Psi_k = \left( \begin{array}{c} \hat{a}_{k} \\ \hat{a}_{-k}^\dagger \end{array} \right)
\]
with
\[
\hat{a}_k^T = (\hat{a}_{k,1,\uparrow}, \hat{a}_{k,1,\downarrow}, \hat{a}_{k,2,\uparrow}, \hat{a}_{k,2,\downarrow}),
\]
and \(\hat{a}_{k,p,\sigma}\) the annihilation operator for an electron with momentum \(k\) and spin \(\sigma\) in the orbital \(p\). Both \(H_0\) and \(\Delta\) are \(4 \times 4\) matrices in the combined orbital-spin space, and transform under inversion \(I\) and time-reversal \(\Theta\) as
\[
I : H_0(k) = U_1^\dagger H_0(-k) U_1 = H_0(k),
\]
\[
I : \Delta(k) = U_1^\dagger \Delta(-k) U_1^T = \pm \Delta(k),
\]
\[
\Theta : H_0(k) = U_T^\dagger H_0^T(-k) U_T = H_0(k),
\]
\[
\Theta : \Delta(k) = U_T^\dagger \Delta^\dagger(-k) U_T^T \neq \Delta(k),
\]
where \(U_1\) and \(U_T\) are unitary matrices. Whereas the normal state Hamiltonian \(H_0\) is assumed to be symmetric under both spatial inversion and time-reversal, the pairing potential \(\Delta(k)\) is assumed to have definite parity but breaks time-reversal symmetry. Although the derivation in Section III proceeds identically for mixed-parity pairing states, we ignore this case here since it typically requires broken inversion symmetry in the normal state.

The normal state Hamiltonian matrix can be decomposed as
\[
H_0(k) = \sum_{\alpha,\beta=0}^3 h_{\alpha\beta}(k) \eta_{\alpha} \otimes \sigma_{\beta},
\]
where \(\eta_{\alpha} (\sigma_{\beta})\) are Pauli matrices encoding the orbital (spin) degree of freedom. Hermiticity requires the \(h_{\alpha\beta}(k)\) coefficients to be real-valued, while the presence of inversion and time-reversal permit just six \((\alpha, \beta)\) pairs in Eq. (8). These correspond to five mutually anticommuting matrices \(\gamma_{1...5}\), along with the identity matrix \(1_4 = \eta_0 \otimes \sigma_0\). The particular form of the matrices \(\gamma_i\) depends on the system under consideration. For example, a system with two orbitals of the same parity has \(U_1 = 1_4\) and \(U_T = \eta_0 \otimes i\sigma_2\), and the allowed terms in Eq. (8) are
\[
(\alpha, \beta) = (0,0), (1,0), (3,0), (2,1), (2,2), (2,3).
\]

For a system with orbitals exchanged by inversion symmetry such that \(U_1 = \eta_1 \otimes \sigma_0\) and \(U_T = \eta_0 \otimes i\sigma_2\), the allowed terms are
\[
(\alpha, \beta) = (0,0), (2,0), (3,0), (1,1), (1,2), (1,3).
\]

| \(U_1^{(0)}\) | \(U_T^{(0)}\) | \(U_1^{(0)}\) | \(U_T^{(0)}\) |
|---|---|---|---|
| \(\eta_0\)| \(\eta_0\)| \(\eta_0\)| \(\eta_0\)|
| \(\eta_1\)| \(\eta_0\)| \(\eta_0\)| \(\eta_0\)|
| \(\eta_2\)| \(\eta_0\)| \(\eta_0\)| \(\eta_0\)|
| \(\eta_3\)| \(\eta_0\)| \(\eta_0\)| \(\eta_0\)|

Table I. The \((\alpha, \beta)\) terms permitted in Eq. (8) for systems with each possible form of inversion and time-reversal, enumerated by the orbital parts of \(U_1\) and \(U_T\). In each case the allowed terms correspond to the identity along with five mutually anticommuting matrices. \(U_1^{(0)} = \eta_2\) is not allowed as it leads to \(U_1 U_T^T = +1\), which does not correspond to a spin-half system. We exclude cases where inversion does not commute with time-reversal (i.e. \(U_1 U_T \neq U_T U_1\)) and where the inversion operator is not its own inverse (i.e. \(U_1 U_1 \neq I\)).

All other possibilities are summarized in Table I. Although some of these may appear atypical, they can each be related to one of the cases noted above via a canonical transformation.

It is convenient to re-write the normal state Hamiltonian as
\[
\tilde{H}_0(k) = h_{00}(k) 1_4 + \tilde{h}(k) \cdot \gamma,
\]
where \(h_{00}(k)\) is an even function of momentum, \(\gamma = (\gamma_1, \ldots, \gamma_5)\) is a vector of the gamma matrices, and \(\tilde{h}(k) = (h_{\alpha_1\beta_1}, \ldots, h_{\alpha_5\beta_5})\) is a vector of the corresponding coefficients. For future reference we define the “flattened” normal state Hamiltonian
\[
\tilde{H}_0(k) = \frac{H_0(k) - h_{00}(k) 1_4}{|h(k)|} = \tilde{h}(k) \cdot \gamma,
\]
where \(\tilde{h} = h/|h|\). This is traceless and satisfies the eponymous property
\[
\tilde{H}_0(k) [k, \pm, s] = \pm [k, \pm, s],
\]
where \([k, \pm, s]\) are eigenstates of \(\tilde{H}_0\) with the good quantum numbers momentum \(k\), band \(\pm\), and pseudospin \(s\). The existence of a degenerate pseudospin index is guaranteed by the inversion and time-reversal symmetries [29, 30].

Analogously to Eq. (8), the pairing potential \(\Delta\) can be decomposed in terms of orbital and spin Pauli matrices, but its form is not so restricted because inversion and time-reversal symmetries can be broken in the superconducting state. We only enforce that the potential satisfies fermionic antisymmetry, requiring that \(\Delta(k) = -\Delta^\dagger(-k)\). For convenience, we define the transformed pairing potential
\[
\tilde{\Delta}(k) = \Delta(k) U_T^T,
\]
which transforms analogously to $H_0$ under point symmetry operations, and has the useful property that $\Delta$ and $\Delta^\dagger$ are time-reversed counterparts. Although the particular form of $\Delta(k)$ is set by the details of the system, we note that for an even-parity pairing state it will be a linear combination of the six $\eta_\alpha \otimes \sigma_\beta$ matrices which are allowed to appear in the normal state Hamiltonian; the potential for an odd-parity pairing state will involve the other ten matrices.

**III. HALL CONDUCTIVITY AND TIME-REVERSAL-ODD BILINEARS**

The frequency-dependent Hall conductivity is defined as

$$\sigma_H(\omega) = \frac{i}{2\omega} \lim_{\omega \to i0^+} \left[ \pi_{xy}(i\omega_n) - \pi_{yx}(i\omega_n) \right],$$

(15)

where $\pi_{ab}$ is the current-current correlation function [31]

$$\pi_{ab}(i\omega_n) = -\frac{1}{N} \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau J_a(\tau) J_b(0) \rangle,$$

(16)

$N$ is the number of lattice points in the crystal, $\tau$ is the imaginary time, $J_a$ is the $a^{th}$ component of the current operator

$$J_a = e \sum_k \Psi_k^\dagger v^a_k \Psi_k,$$

(17)

and the velocity matrix $v^a_k$ is given by

$$v^a_k = \begin{pmatrix} v^{a,n} & 0 \\ 0 & v^{a,h} \end{pmatrix} = \begin{pmatrix} \frac{\partial H_0(k)}{\partial k_n} & 0 \\ 0 & \frac{\partial H_0(k)}{\partial k_h} \end{pmatrix}.$$

(18)

In the linear response regime $\pi_{ab}(i\omega_n)$ is evaluated at the one-loop level, yielding

$$\pi_{ab}(i\omega_n) = \frac{e^2}{2N\beta} \sum_{k,m} \text{Tr} \left\{ v^a_k \delta_{\omega_n+i\nu_m} v^b_k \delta_{k} \right\},$$

(19)

where $\delta_{\omega_n+i\nu_m}$ is the Matsubara Green’s function corresponding to Eq. (1). Note that in Eq. (19) $\omega_n = 2n\pi/\beta$

is a Bose Matsubara frequency, and $\nu_m = (2m + 1)\pi/\beta$ is a Fermi Matsubara frequency. For convenience we set $\hbar = 1$.

While Eq. (19) can be evaluated for an arbitrary pairing potential, the resulting analytic expression is generally very complicated and offers only limited insight. Progress can be made by considering the high-frequency, small-gap limit, obtained by neglecting terms in Eq. (19) of lower order than $\omega^{-2}$ and higher order than $|\Delta|^2$. This choice is justified since experiments are often performed in the high-frequency regime, and the gap is small compared to other relevant energy scales; the small-gap approximation is expected to be particularly accurate close to the critical temperature. Performing a diagrammatic expansion of Eq. (19), an approximate expression for the intrinsic anomalous Hall conductivity in this limit is given by [27]

$$\sigma_H(\omega) \approx \frac{i e^2}{2N\omega^2\beta} \sum_{k,m} \text{Tr} \left\{ [\nu \wedge v] G_0 H_\Delta G_0 H_\Delta G_0 \right\},$$

(20)

where $[a \wedge b] = a^\dagger b^\nu - a^\nu b^\dagger$, and explicit momentum and frequency indices have been dropped for convenience. $H_\Delta$ is the pairing part of the BdG Hamiltonian,

$$H_{\Delta,k} = \begin{pmatrix} 0 & \Delta(k) \\ \Delta^\dagger(k) & 0 \end{pmatrix},$$

(21)

and $G_0 = G_{0,k,i\nu_m}$ is the normal state Matsubara Green’s function in the Nambu representation

$$G_{0,k,i\nu_m} = \begin{pmatrix} G_{0,k,i\nu_m}^a & 0 \\ 0 & G_{0,k,i\nu_m}^h \end{pmatrix}.$$

(22)

With these expressions, the trace in Eq. (20) is split into two terms. By enforcing that $H_0$ is symmetric under both inversion and time-reversal, and noting that time-reversal commutes with inversion (i.e. $U_I U_T = U_T U_I^\dagger$, as time-reversal is antiunitary), it can be shown that $[v^{a,n} \wedge v^{a,h}] = -U_I^T [v^{a,n} \wedge v^{a,h}] U_T^\dagger$ and $U_I^T [v^{a,n} \wedge v^{a,h}] U_T = -[v^{a,n} \wedge v^{a,h}]$, from which we obtain

$$\sigma_H(\omega) \approx \frac{i e^2}{2N\omega^2\beta} \sum_{k,m} \left\{ \text{Tr} \left\{ [\nu \wedge v] G_{0,k,i\nu_m}^a \Delta U_T G_{0}^h U_T^\dagger \Delta^\dagger G_{0}^a \right\} \right\} + \text{Tr} \left\{ [\nu \wedge v] G_{0,k,i\nu_m}^a U_T^\dagger G_{0}^h U_T \Delta^\dagger G_{0}^a U_T \Delta G_{0}^h U_T \right\}$$

(23)

in the high-frequency, small-gap limit. The electron-like Green’s functions can be written in terms of projection operators onto each normal state energy band:

$$G_{0,k,i\nu_m}^a = \sum_{\pm} \frac{P_{k,\pm}}{i\nu_m - E_{k,\pm}}.$$
The projection operators are defined in terms of Eq. (12) as 
\[ \mathcal{P}_{k,\pm} = (1 \pm \hat{H}_0(k))/2, \]
and therefore
\[ \mathcal{G}_{0,k,ivm}^{e} = \frac{1 + \hat{H}_0(k)}{2(i\nu_m - E_{k,+})} + \frac{1 - \hat{H}_0(k)}{2(i\nu_m - E_{k,-})} \]
\[ = a_{k,vm} 1 + b_{k,vm} \hat{H}_0(k), \] (25)
where
\[ a_{k,vm} = \frac{1}{2} \left[ (i\nu_m - E_{k,+})^{-1} + (i\nu_m - E_{k,-})^{-1} \right], \] (26)
\[ b_{k,vm} = \frac{1}{2} \left[ (i\nu_m - E_{k,+})^{-1} - (i\nu_m - E_{k,-})^{-1} \right]. \] (27)

Analogously, the hole-like Green’s function has the form
\[ \mathcal{G}_{0,k,ivm}^{h} = c_{k,vm} 1 + d_{k,vm} \hat{H}_0^T(k), \] (28)
where 
\[ c_{k,vm} = -a_{k,-vm} \quad \text{and} \quad d_{k,vm} = -b_{k,-vm}. \]
Further, because \( \hat{H}_0 \) is symmetric under time-reversal,
\[ U_T^{\dagger} \mathcal{G}_{0,k,ivm}^{h} U_T = c_{k,vm} 1 + d_{k,vm} \hat{H}_0(k). \] (29)
Substituting Eq. (25) and Eq. (29) into Eq. (23), and using the relationship between \( a_{k,vm} \) (\( b_{k,vm} \)) and \( c_{k,vm} \) (\( d_{k,vm} \)), we obtain after some manipulation
\[ \sigma_{H}(\omega) = \frac{ie^2}{2Nu^2\beta} \sum_{k,vm} \left[ a^2 c \text{ Tr}\{ [v^e \wedge v^e] TROB_1 \} + b^2 c \text{ Tr}\{ [\hat{H}_0[v^e \wedge v^e] \hat{H}_0] \text{ TROB}_1 \} + abc \text{ Tr}\{ [\hat{H}_0, [v^e \wedge v^e]] \text{ TROB}_1 \} \right. \]
\[ + a^2 d \text{ Tr}\{ [v^e \wedge v^e] \text{ TROB}_2 \} + b^2 d \text{ Tr}\{ [\hat{H}_0[v^e \wedge v^e] \hat{H}_0 \text{ TROB}_2 \} + abd \text{ Tr}\{ [\hat{H}_0, [v^e \wedge v^e]] \text{ TROB}_2 \}, \] (30)
where momentum and frequency indices are suppressed for clarity. The key feature of Eq. (30) is the introduction of two distinct time-reversal-odd bilinears (TROBs), which are defined in terms of the pairing potential:
\[ \text{TROB}_1 = \hat{\Delta} \hat{\Delta}^\dagger - \hat{\Delta}^\dagger \hat{\Delta}, \] (31)
\[ \text{TROB}_2 = \hat{\Delta} \hat{H}_0 \hat{\Delta}^\dagger - \hat{\Delta}^\dagger \hat{H}_0 \hat{\Delta}. \] (32)
TROB$_1$ was initially introduced in [27]. It can be verified that both TROBs are the difference of a gauge-invariant quantity and its time-reversed value, making them odd under time-reversal by construction. The significance of the TROBs is that they translate the TRSB in the particle-particle channel to the observable particle-hole channel.

This direct dependence of the anomalous Hall conductivity on TROBs is the core result of this paper. We observe that the presence of at least one non-vanishing TROB is a necessary condition for a non-zero Hall conductivity in the high-frequency, small-gap limit. Although this is not a sufficient condition, fine-tuning is required for Eq. (30) to be vanishing if either TROB is non-zero. Importantly, although the TROBs are only nonzero for a TRSB pairing potential, a vanishing TROB does not imply that the pairing potential is time-reversal symmetric. As such, the TROBs significantly constrain which superconducting states can give rise to a non-zero anomalous Hall conductivity. Both TROBs are straightforward to calculate, and hence provide an easy mechanism for identifying candidate Hamiltonians worth further examination.

Another important aspect of Eq. (30) is the presence of the wedge product of velocity matrices \([v^e \wedge v^e]\). Since this product is vanishing in a single-band system, it implies that a multiband model is necessary for an anomalous Hall conductivity.

A. Interband vs intraband pairing

Interband pairing has been identified by a number of authors as another necessary condition for an AHE [15, 27]. This is notable because intraband pairing generally guarantees the robustness of the superconducting instability, while intraband pairing competes, and is therefore detrimental to the formation of a superconducting state [32]. The connection between, and compatibility of, these claims and our result is of interest. Here we show that the TROBs directly evidence the key role played by interband pairing, while also demonstrating that purely intraband pairing can generate a nonzero Hall conductivity.

1. Even-parity pairing

The general form of an even-parity pairing potential in the band-pseudospin basis is
\[ \hat{\Delta}_e = \frac{1}{2} (\psi_+ + \psi_-) b_0 \otimes s_0 + \frac{1}{2} (\psi_+ - \psi_-) b_3 \otimes s_0 \]
\[ + \psi_e b_1 \otimes s_0 + \mathbf{d}_e \cdot (b_2 \otimes s), \] (33)
where the first line describes intraband pseudospin-singlet pairing with potential \( \psi_{\pm} \) in band \( \pm \), while \( \psi_e \) and \( \mathbf{d}_e \) are the interband pseudospin-singlet and -triplet potentials. The Pauli matrices \( b_i \) and \( s_p \) encode the band and pseudospin degrees of freedom, respectively, and all
pairing potentials are even functions of the momentum. Because the pseudospin and band indices are even under inversion, the inversion operator is trivial in this representation, and so the matrices appearing in Eq. (33) are essentially the same as in the case of Eq. (9). The two TROBs evaluate as

\[
\begin{align*}
\text{TROB}_1 &= 2i\mathbf{d}_c \times \mathbf{d}_c^* \cdot (b_0 \otimes s) - 4i\text{Im}\{\psi_+ \mathbf{d}_c^*\} \cdot (b_3 \otimes s) \\
&\quad - 2\text{Im}\{(\psi_+ - \psi_-)\mathbf{d}_c^*\} b_2 \otimes s_0 \\
&\quad + 2\text{Im}\{(\psi_+ - \psi_-)\mathbf{d}_c^*\} \cdot (b_1 \otimes s), \\
\text{TROB}_2 &= -2i\mathbf{d}_c \times \mathbf{d}_c^* \cdot (b_3 \otimes s) + 4i\text{Im}\{\psi_- \mathbf{d}_c^*\} \cdot (b_0 \otimes s) \\
&\quad - 2\text{Im}\{(\psi_+ + \psi_-)\mathbf{d}_c^*\} b_2 \otimes s_0 \\
&\quad + 2\text{Im}\{(\psi_+ + \psi_-)\mathbf{d}_c^*\} \cdot (b_1 \otimes s).
\end{align*}
\]

We observe that the pairing potential must have interband components for either TROB to be nonzero, which is generically the case if the pairing potential has a nontrivial matrix structure when expressed in the orbital-spin basis. Note that the velocity operators are not generally diagonal in the band-pseudospin picture, and so the off-diagonal components of the TROBs are still relevant to the existence of the Hall conductivity. We note that the diagonal blocks of TROB$_1$ also play a crucial role in the inflation of point or line nodes into Bogoliubov Fermi surfaces [33].

2. Odd-parity pairing

Adopting the same band-pseudospin basis as above, a general odd-parity pairing potential has the form

\[
\Delta_o = \frac{1}{2} (\mathbf{d}_+ + \mathbf{d}_-) \cdot (b_0 \otimes s) + \frac{1}{2} (\mathbf{d}_+ - \mathbf{d}_-) \cdot (b_3 \otimes s) \\
+ \psi_o b_2 \otimes s_0 + \mathbf{d}_o \cdot (b_1 \otimes s),
\]

where in addition to the intraband triplet potentials $\mathbf{d}_\pm$ there are also interband singlet $\psi_o$ and triplet $\mathbf{d}_o$ potentials. Here all potentials are odd in momentum. Evaluating the TROBs we obtain

\[
\begin{align*}
\text{TROB}_1 &= i\mathbf{d}_+ \times \mathbf{d}_+^* \cdot (b_0 + b_3) \otimes s \\
&\quad + i\mathbf{d}_- \times \mathbf{d}_-^* \cdot (b_0 - b_3) \otimes s \\
&\quad + 2i\mathbf{d}_o \times \mathbf{d}_o^* \cdot (b_0 \otimes s) + 4i\text{Im}\{\psi_o \mathbf{d}_o^*\} \cdot (b_3 \otimes s) \\
&\quad + 2\text{Im}\{\mathbf{d}_o \cdot (\mathbf{d}_+ - \mathbf{d}_-)^*\} b_2 \otimes s_0 \\
&\quad - 2\text{Im}\{\psi_o \mathbf{d}_o \cdot (\mathbf{d}_+ - \mathbf{d}_-)^*\} \cdot (b_1 \otimes s), \\
\text{TROB}_2 &= i\mathbf{d}_+ \times \mathbf{d}_+^* \cdot (b_0 + b_3) \otimes s \\
&\quad - i\mathbf{d}_- \times \mathbf{d}_-^* \cdot (b_0 - b_3) \otimes s \\
&\quad + 2i\mathbf{d}_o \times \mathbf{d}_o^* \cdot (b_3 \otimes s) - 4i\text{Im}\{\psi_o \mathbf{d}_o^*\} \cdot (b_0 \otimes s) \\
&\quad + 2\text{Im}\{\mathbf{d}_o \cdot (\mathbf{d}_+ - \mathbf{d}_-)^*\} b_2 \otimes s_0 \\
&\quad - 2\text{Im}\{\psi_o \mathbf{d}_o \cdot (\mathbf{d}_+ - \mathbf{d}_-)^*\} \cdot (b_1 \otimes s) \\
&\quad - 2\text{Im}\{\psi_o \mathbf{d}_o \cdot (\mathbf{d}_+ + \mathbf{d}_-)^*\} \cdot (b_1 \otimes s).
\end{align*}
\]

In contrast to the even-parity case, we observe that a purely-intraband potential with $\mathbf{d}_\pm \times \mathbf{d}_\pm^* \neq 0$ can give rise to a nonzero TROB. Although such nonunitary pairing states are generically realized in TRSB single-band systems, the presence of the wedge product of velocity matrices in Eq. (30) is nevertheless only nonzero in a multiband system, reiterating the necessity of multiple bands for an anomalous Hall conductivity.

B. Away from the high-frequency, small-gap limit

Although Eq. (30) only rigorously applies in the high-frequency, small-gap limit, the conclusion that a nonzero TROB implies a nonzero Hall conductivity is generically valid, since any correction terms are unlikely to exactly cancel the leading-order contribution Eq. (30). In considering the converse statement, we start by noting that the high-frequency limit can be rigorously defined in terms of the commutator of the current operators

\[
\sigma_H(\omega) \approx \frac{i}{N \omega^2} \langle [J_x, J_y] \rangle.
\]

Note that this expression does not require the small-gap restriction. We can identify the sum in Eq. (30) as the small-gap approximation of the expectation value in Eq. (39). This expectation value also appears in a sum rule for the imaginary part of the Hall conductivity

\[
\int_{-\infty}^{\infty} \omega \text{Im}\{\sigma_H(\omega)\} \, d\omega = -\frac{i\pi}{N} \langle [J_x, J_y] \rangle.
\]

The expectation value in Eq. (40) is unlikely to be zero unless the Hall conductivity itself is vanishing at all frequencies. Further, for the expectation value to be nonzero when both TROBs are vanishing implies that the leading contributions to Eq. (39) and Eq. (40) must be fourth-order or higher in the pairing potential; for these higher-order contributions to be nonzero but the second-order contribution to vanish places extremely stringent conditions on both the pairing potential and the normal state Hamiltonian, which would not generically be satisfied. We thus consider it likely that vanishing TROBs imply a vanishing Hall conductivity in a clean system at arbitrary frequency and temperature.

IV. EXAMPLE CALCULATIONS

To exemplify the role of the TROBs, here we present calculations for two model systems.

A. Strontium Ruthenate

Strontium ruthenate (Sr$_2$RuO$_4$) is a key candidate material for chiral superconductivity, with both muon spin relaxation [34] and polar Kerr experiments [8, 9] providing strong evidence of TRSB in its superconducting state. The theory of the anomalous Hall effect in Sr$_2$RuO$_4$
has been considered by several authors [15, 21, 22], all working on the assumption that it is a chiral p-wave superconductor [35–37]. Although recent experiments have thrown significant doubt on this proposition [38, 39], we nevertheless adopt this picture in order to make contact with the existing literature. We follow Ref. [15] and adopt a minimal two-dimensional model of Sr$_2$RuO$_4$ involving the ruthenium $d_{xz}$ and $d_{yz}$ orbitals. Since the orbitals transform trivially under inversion and time-reversal, Eq. (9) enumerates the terms permitted in $H_0$. We take the following tight-binding model:

$$h_{00} = -t_1(\cos k_x + \cos k_y) - \mu, \quad h_{10} = 2t_3 \sin k_x \sin k_y, \quad h_{30} = -t_2(\cos k_x - \cos k_y), \quad$$

and isotropic spin orbit coupling $h_{33} = \lambda$; the (2, 1) and (2, 2) terms do not appear in this two-dimensional model because $h_{21}$ and $h_{22}$ are odd under reflection about the $k_z = 0$ mirror plane. Setting $t_1 = 1$, $\mu = 1$, $t_2 = 0.8$, $t_3 = 0.1$, and $\lambda = 0.25$, this model qualitatively reproduces the observed $\alpha$ and $\beta$ Fermi surfaces [40–42].

We focus on pairing states belonging to $E_u$ irreducible representations of the $D_{6h}$ point group, which being two dimensional, naturally occur in TRSB combinations. A general $E_u$ pairing state is given by $\Delta = \Delta_{03}\eta_0 \otimes \sigma_3 + \Delta_{33}\eta_3 \otimes \sigma_3 + \Delta_{20}\eta_2 \otimes \sigma_0 + \Delta_{33}\eta_3 \otimes \sigma_3$, where each $\Delta_{\alpha\beta}$ is an odd chiral function of momentum. The TROBs evaluate as

$$\text{TROB}_1 = 4 \left[ \text{Im}(\tilde{\Delta}_{13}\tilde{\Delta}_{31})\eta_2 \otimes \sigma_0 + \text{Im}(\tilde{\Delta}_{33}\tilde{\Delta}_{20})\eta_1 \otimes \sigma_3 + \text{Im}(\tilde{\Delta}_{20}\tilde{\Delta}_{13})\eta_3 \otimes \sigma_3 \right],$$

$$\text{TROB}_2 = 4 \left[ (\tilde{h}_{10} \text{Im}(\tilde{\Delta}_{20}\tilde{\Delta}_{33}) + \tilde{h}_{30} \text{Im}(\tilde{\Delta}_{13}\tilde{\Delta}_{20}) + \tilde{h}_{23} \text{Im}(\tilde{\Delta}_{33}\tilde{\Delta}_{13}) )\eta_0 \otimes \sigma_3 + (\tilde{h}_{30} \text{Im}(\tilde{\Delta}_{03}\tilde{\Delta}_{23}) + \tilde{h}_{23} \text{Im}(\tilde{\Delta}_{33}\tilde{\Delta}_{03}) )\eta_1 \otimes \sigma_3 + (\tilde{h}_{10} \text{Im}(\tilde{\Delta}_{03}\tilde{\Delta}_{13}) + \tilde{h}_{23} \text{Im}(\tilde{\Delta}_{33}\tilde{\Delta}_{03}) )\eta_3 \otimes \sigma_3 \right].$$

In general, both TROBs are non-zero, although they can each individually vanish depending on the details of a given model. Further, each term in Eq. (41) and Eq. (42) involves two pairing channels: although chiral pairing in a single channel is sufficient to break time-reversal symmetry, pairing in at least two channels is required for the appearance of an anomalous Hall conductivity.

As a specific model for further examination we take the intraorbital spin-triplet pairing state considered in Ref. [15]

$$\tilde{\Delta} = \tilde{\Delta}_{03}\eta_0 \otimes \sigma_3 + \tilde{\Delta}_{33}\eta_3 \otimes \sigma_3,$$

with the $p$-wave form factors $\tilde{\Delta}_{03} = \tilde{\Delta}_{03}(\sin k_x + i \sin k_y)$ and $\tilde{\Delta}_{33} = \tilde{\Delta}_{33}(\sin k_x - i \sin k_y)$. Only TROB$_2$ is nonzero for this system, evaluating to

$$\text{TROB}_2 = 4 \text{Im}(\tilde{\Delta}_{03}\tilde{\Delta}_{33}^*) (\tilde{h}_{10}\eta_2 \otimes \sigma_0 - \tilde{h}_{23}\eta_1 \otimes \sigma_3).$$

Microscopically, TROB$_2$ can be interpreted as an orbital- or spin-angular-momentum-polarized interorbital hoping term.

To study the temperature-dependence of the TROB, we determine the pairing potential amplitudes $\tilde{\Delta}_{0,01}$ and $\tilde{\Delta}_{0,31}$ self-consistently. To this end, we introduce a phenomenological pairing interaction $V_{\nu\nu'}$, which scatters a Cooper pair from channel $\nu'$ to channel $\nu$. The interaction strengths were taken to be $V_{03,03} = -0.2$, $V_{33,33} = -0.265$, and $V_{03,33} = V_{33,03} = 0.03$. These values were chosen such that $\tilde{\Delta}_{0,33}$ exhibits a nonmonotonic temperature dependence, as shown in Fig. 1(a). The existence of TROB$_2$ in this state is confirmed by the thermal expectation value of the dimensionless quantity $\text{TROB}_2 = \text{TROB}_2/|\tilde{\Delta}_{03}\tilde{\Delta}_{0,33}|$, which is shown in Fig. 1(b). The nonmonotonic temperature dependence of the TROB is particularly helpful when comparing the approximate form of the Hall conductivity Eq. (30) with the full calculation, which is shown in Fig. 1(c). We additionally include the prediction of the high-frequency limit Eq. (39). The two approximate results are in excellent qualitative agreement with the full calculation, and also in reasonable quantitative agreement; the quantitative agreement improves at higher values of the frequency.

Although this underscores the fact that the small-gap approximation captures the leading contribution to the Hall conductivity, we note that this approximation appears to break down as the temperature goes to zero.

### B. The honeycomb lattice

An example of a TRSB even-parity superconductor with a nonzero anomalous Hall conductivity is provided by the nearest-neighbor chiral $d$-wave pairing state on the honeycomb lattice [43, 44]. The anomalous Hall conductivity of this system was analyzed in detail in [27], which included the introduction of the TROB concept. In the notation used here, the honeycomb lattice model is written as

$$H_0 = h_{00}\eta_0 \otimes \sigma_0 + h_{10}\eta_1 \otimes \sigma_0 + h_{20}\eta_2 \otimes \sigma_0,$$

where $\eta_\mu$ encodes the sublattice degree of freedom. The allowed terms are given by $h_{00} = \mu, \quad h_{10} = \eta_1 \otimes \sigma_0$.
\[ \Delta = \tilde{\Delta}_{10} \eta_1 \otimes \sigma_0 + \tilde{\Delta}_{20} \eta_2 \otimes \sigma_0, \]  

(46)

with the amplitudes \( \tilde{\Delta}_{10} = \Delta_0 (\cos k_x - \cos \frac{\sqrt{3}}{2} k_y - i \sqrt{3} \sin \frac{k_x}{2} \sin \frac{\sqrt{3} k_y}{2} ) \) and \( \tilde{\Delta}_{20} = -\Delta_0 (\sin k_x + \sin \frac{\sqrt{3}}{2} k_y + i \sqrt{3} \cos \frac{k_x}{2} \sin \frac{\sqrt{3} k_y}{2} ) \). Only TROB1 is nonzero for this model, taking the value

\[ \text{TROB}_1 = 4 \text{Im} \{ \tilde{\Delta}_{10} \tilde{\Delta}^*_{20} \} \eta_3 \otimes \sigma_0. \]  

(47)

The form of TROB1 is equivalent to the loop current term in Haldane’s model of the anomalous quantum Hall effect, and the anomalous Hall conductivity of the model is directly related to this quantity [27]. The absence of TROB2 is an artifact of the simplicity of the model: by including the symmetry-allowed Kane-Mele spin-orbit coupling [45] \( h_{33} \eta_3 \otimes \sigma_3 = \lambda \sin \frac{\sqrt{3}}{2} k_y (\cos \frac{\sqrt{3}}{2} k_x - \cos \frac{\sqrt{3}}{2} k_y ) \eta_3 \otimes \sigma_3 \) in the normal state Hamiltonian, we find

\[ \text{TROB}_2 = 4 \hat{h}_{33} \text{Im} \{ \tilde{\Delta}_{10} \tilde{\Delta}^*_{20} \} \eta_0 \otimes \sigma_3, \]  

(48)

while TROB1 is left unchanged.

The TROB results for the honeycomb lattice are readily understood in terms of the band-pseudospin picture. In the absence of spin-orbit coupling, the pseudospin can be chosen as identical to the real spin, and so the interband pairing is purely singlet. Since the intraband singlet potentials have opposite sign, it follows that only TROB1 is nonzero. The addition of the spin-orbit coupling introduces interband triplet pairing, and the second term in Eq. (35) for TROB2 is therefore nonzero.

\[ \prod_{j=1}^n \left[ H_0 (k) - E_{k,j} \right] = 0 \]  

(49)

where \( E_{k,j} \) are the \( n \) distinct doubly-degenerate eigenvalues of the normal state. This implies that we can define

- Motivated by the observation of the polar Kerr effect in chiral-superconductor candidate materials, in this manuscript we have considered the origin of the anomalous Hall conductivity in a general model of a two-band superconductor. Our analysis reveals a key role for two distinct gauge-invariant time-reversal-odd bilinear functions (TROBs) of the pairing potential: if at least one of these TROBs is present, we may generically expect that the anomalous Hall conductivity is nonzero. Although our result is rigorously valid in the high-frequency, small-gap limit, we argued that it applies more generally. The TROBs strongly constrain the form of the pairing potential which generates an anomalous Hall conductivity. Explicitly calculating the TROBs in a pseudospin-band basis, we found that interband pairing generically gives rise to an anomalous Hall conductivity, but also demonstrated that purely-intraband nonunitary triplet pairing can make a contribution. Our general conclusions are illustrated with two specific examples of model chiral \( p \)-wave and \( d \)-wave superconductors.

For the purposes of analytical clarity, our study has been restricted to a model with a twofold orbital degree of freedom. It nevertheless appears straightforward to generalize our analysis to systems with more orbital degrees of freedom. For example, more realistic models of the low-energy electronic structure of Sr\(_2\)RuO\(_4\) typically involve at least three bands [21, 22, 24, 28, 46]. We speculate that in a system with \( n \) orbital degrees of freedom there will be \( n \) distinct TROBs; this follows from the observation that the normal state Hamiltonian \( H_0 (k) \) of such a system obeys the characteristic polynomial
\( n \) independent TROBs

\[
\text{TROB}_j = \Delta \tilde{H}_0^{-1} \tilde{\Delta} - \tilde{\Delta} \tilde{H}_0^{-1} \Delta,
\]

(50)

where \( \tilde{H}_0 \) is the traceless part of \( H_0 \). Due to the characteristic polynomial Eq. (49), TROBs defined in terms of powers of \( H_0 \) higher than \( n - 1 \) can be written in terms of \( \text{TROB}_{j=1,\ldots,n} \). We leave a detailed analysis of this situation to future work.

Further generalization of our results should consider going beyond the small-gap limit, to explore the relevance of TROBs at all energy scales. Another direction for future work is to extend our argument to noncentrosymmetric systems, which could be relevant to time-reversal symmetry-breaking candidate superconductors LaNiC\(_2\) [47] and twisted bilayer graphene [6].

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