Current presentation for the super-Yangian double $DY(gl(m|n))$ and Bethe vectors

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Abstract. Bethe vectors are found for quantum integrable models associated with the supersymmetric Yangians $Y(gl(m|n))$ in terms of the current generators of the Yangian double $DY(gl(m|n))$. The method of projections onto intersections of different types of Borel subalgebras of this infinite-dimensional algebra is used to construct the Bethe vectors. Calculation of these projections makes it possible to express the supersymmetric Bethe vectors in terms of the matrix elements of the universal monodromy matrix. Two different presentations for the Bethe vectors are obtained by using two different but isomorphic current realizations of the Yangian double $DY(gl(m|n))$. These Bethe vectors are also shown to obey certain recursion relations which prove their equivalence.

Bibliography: 30 titles.

Keywords: Bethe vector, current algebra, monodromy matrix, Gauss decomposition, projection.

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1. Introduction

The calculation of form factors and correlation functions in quantum integrable models is one of the most important problems in the area of exactly solvable models in statistical physics and low-dimensional quantum mechanics. A lot of results were obtained in this direction starting from the earliest years in the development of the Quantum Inverse Scattering Method (QISM) [1], [2]. For models connected with various deformations of the affine algebra $\widehat{\mathfrak{gl}}(2)$ one of the most important results is a determinant presentation for the particular case of the scalar product in which one of vectors is an eigenvector of the transfer matrix [3]. This result lets us go directly to the problem of calculating the correlation functions [4] of the local operators in integrable models (see the survey [5] and the references there).

One of the most important notions of the QISM is a Bethe vector. In $\widehat{\mathfrak{gl}}(2)$-based models the Bethe vector is a monomial in the upper-right element of the monodromy matrix (the creation operator) applied to the pseudo-vacuum vector. It depends on a set of complex variables called Bethe parameters. The distinguishing feature of these vectors is that they become eigenvectors of the transfer matrix if the Bethe parameters satisfy a special system of equations (the Bethe equations). In this case we call them on-shell Bethe vectors. Otherwise, if the Bethe parameters are generic complex numbers, then the corresponding vectors are called off-shell Bethe vectors, or simply Bethe vectors. In this paper we deal with the universal monodromy matrix. This means that it depends only on the underlying algebra generators. We refer to the corresponding Bethe vectors as universal Bethe vectors.

The main purpose of this paper is to study Bethe vectors in the Yangian double $\text{DY}(\mathfrak{gl}(m|n))$. Our first goal is to obtain explicit formulae for them. The second
goal is to derive formulae for the action of the monodromy matrix entries on the
off-shell Bethe vectors. Achieving these two goals enables us to pose the prob-
lem of calculating the scalar products of Bethe vectors, which in turn is necessary
for studying the form factors and correlation functions in integrable models with
underlying $\mathfrak{gl}(m|n)$ supersymmetry.

For models connected with higher-rank symmetries, the QISM is based on the
so-called nested Bethe ansatz, which was elaborated in the pioneering papers [6]–[8].
There a recursive procedure was developed for constructing Bethe vectors corre-
sp ending to the algebra $\widehat{\mathfrak{gl}}(N)$ from the known Bethe vectors of the algebra $\widehat{\mathfrak{gl}}(N−1)$.
Formally, this method enables us to obtain explicit formulae for Bethe vectors in
terms of certain polynomials in the creation operators (upper triangular entries of
the monodromy matrix) acting on the pseudo-vacuum vector. However, the pro-
cedure is quite involved, and therefore no explicit representations were obtained in
the early works mentioned above, with the exception of graphical representations
found by Reshetikhin in [9] for models with the algebra $\mathfrak{gl}(3)$. The use of this dia-
gram technique yielded a formula for the scalar products of off-shell Bethe vectors
in terms of sums over partitions of the sets of Bethe parameters (a sum formula).

In [10] and [11] the Bethe vectors for the integrable models associated with
deformed algebras $\widehat{\mathfrak{gl}}(N)$ were obtained as the traces of products of the monodromy
matrices, $R$-matrices, and certain projections. These results were generalized to
supersymmetric algebras in [12]. This approach makes it possible in some cases
to calculate the norms of the nested Bethe vectors, but not their scalar products.

An alternative approach to the construction of Bethe vectors was proposed
in [13]. This method explores the relation between two different realizations of
the quantized Hopf algebra $U_q(\widehat{\mathfrak{gl}}(N))$ associated with the affine algebra $\widehat{\mathfrak{gl}}(N)$, the
first in terms of the universal monodromy matrix $T(z)$ and the RTT commutation
relations, and the second in terms of the total currents, which are defined by the
Gauss decomposition of the monodromy matrix $T(z)$ [24]. Further, it was shown
in [14] that the two different types of formulae for the universal off-shell Bethe
vectors (constructed from the monodromy matrix) are related to the two different
current realizations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(N))$ and their associated
projections.

Moreover, the approach using the current generators of the deformed current
algebras makes it possible to calculate the action of the monodromy matrix elements
on the universal Bethe vectors. These action formulae turned out to be very useful
for calculating form factors in the different quantum integrable models connected
with rational and trigonometric deformations of the affine algebra $\widehat{\mathfrak{gl}}(3)$ [15]–[17].
Recently, similar results were obtained for the models with the superalgebras $\widehat{\mathfrak{gl}}(1|2)$
and $\widehat{\mathfrak{gl}}(2|1)$ in [18] and [19]. In these works the explicit formulae for the Bethe
vectors and the action formulae in [20] and [21] were used in an essential way.

In the present paper we use the approach of [13]. In this framework the universal
off-shell Bethe vector is defined as a projection of a product of total currents applied
to the pseudo-vacuum vector. We defer the detailed definition to §3, because it
requires the introduction of many new concepts and new notation. For the same
reason, we postpone a description of our main results to §3.5. Here we would like to
mention only that we construct explicit formulae for the universal Bethe vectors in
terms of the current generators of the Yangian double $DY(gl(m|n))$ for two different Gauss decompositions of the universal monodromy matrix and two different current realizations of this algebra. These different Gauss decompositions correspond to the embeddings of $DY(gl(m-1|n))$ or $DY(gl(m|n-1))$ in $DY(gl(m|n))$. On the level of the RTT realization these embeddings are either in the lower-right corner or in the upper-left corner of the universal monodromy matrix. Using the first or the second type of these embeddings, we obtain two different representations for the Bethe vectors, which we denote by $\mathbb{B}(\mathring{t})$ and $\mathbb{B}(\overline{t})$, respectively, where $\mathring{t}$ is a set of Bethe parameters (3.11). We prove that these two representations are equivalent, that is, $\mathbb{B}(\mathring{t}) = \mathbb{B}(\overline{t})$.

The paper is organized as follows. In §2 we introduce the necessary notation used for calculations in graded vector spaces, as well as the RTT and current realizations of the algebra $DY(gl(m|n))$. In §3 we define universal Bethe vectors using the notion of projections onto intersections of different types of Borel subalgebras. As already mentioned, §3.5 contains the main results obtained in this paper. §4 contains calculations of the action of the monodromy matrix elements on Bethe vectors in the generic case of $DY(gl(m|n))$. It is proved there, using these action formulae, that the vectors we have constructed become on-shell Bethe vectors if the supersymmetric Bethe equations for the Bethe parameters are satisfied. In §5 we calculate the projections of a product of currents and present explicit formulæ for the off-shell Bethe vectors as sums over partitions of the Bethe parameters. In Appendix A we introduce the notion of composed currents and study the relation between them and the Gauss coordinates of the universal monodromy matrix. Appendix B describes important properties of the projections. Appendix C shows how the Izergin and Cauchy determinants arise in the course of resolving the hierarchical relations in the determination of explicit formulæ for the off-shell Bethe vectors.

2. Universal monodromy matrix

In this paper we adopt the following approach. We do not consider any specific supersymmetric exactly solvable models defined by a particular monodromy matrix $T(z)$ satisfying the standard RTT relation. Instead, we treat a $T$-operator (2.3) as the universal monodromy matrix whose matrix elements are the generating series of the full set of generators of the Yangian double $DY(gl(m|n))$ acting in a generic representation space of this algebra, which is a rational deformation of the affine algebra $\widehat{gl}(m|n)$. These representations are not specified, except for the requirement that left and right pseudo-vacuum vectors exist, which ensures the applicability of the algebraic Bethe ansatz. To construct Bethe vectors we will use only the one $T$-operator $T^+(z)$ from the dual pair $\{T^+(z), T^-(z)\}$ which generates the whole algebra $DY(gl(m|n))$. The eigenvalues $\lambda_i(z)$ of the diagonal matrix elements on the pseudo-vacuum vectors (see (2.12) and (2.13)) are free functional parameters which can be set equal to zero if necessary.

We first give a definition of $\mathbb{Z}_2$-graded linear spaces and their multiplication rules, and we describe matrices acting in these spaces.

2.1. $\mathbb{Z}_2$-graded linear spaces and notation. Let $\mathbb{C}^{m|n}$ be a $\mathbb{Z}_2$-graded linear space with a basis $e_i$, $i = 1, \ldots, m + n$, where the vectors $\{e_1, e_2, \ldots, e_m\}$ are even
and the vectors \( \{ e_{m+1}, e_{m+2}, \ldots, e_{m+n} \} \) are odd. The \( \mathbb{Z}_2 \)-grading of the indices is as follows:

\[
[i] = 0 \quad \text{for } i = 1, 2, \ldots, m \quad \text{and} \quad [i] = 1 \quad \text{for } i = m + 1, m + 2, \ldots, m + n. \tag{2.1}
\]

Let \( E_{ij} \in \text{End}(\mathbb{C}^{m|n}) \) be the matrix with the only non-zero entry equal to 1 at the intersection of the \( i \)th row and \( j \)th column.

The basis vectors \( e_i \) and the matrices \( E_{ij} \) have the following grading:

\[
[e_i] = [i] \quad \text{and} \quad [E_{ij}] = [i] + [j] \mod 2.
\]

The tensor product is also graded according to the rule

\[
(E_{ij} \otimes E_{kl}) \cdot (E_{pq} \otimes E_{rs}) = (-)^{(k_l) + (l_p) + (p_q) + (q_r)} E_{ij} E_{pq} \otimes E_{kl} E_{rs}.
\]

Let \( P \) be the graded permutation operator acting in the tensor product \( \mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n} \) as follows:

\[
P = \sum_{a,b=1}^{m+n} (-)^{[b]} E_{ab} \otimes E_{ba}.
\]

Let

\[
g(u, v) = \frac{c}{u - v}
\]

be a rational function of the spectral parameters \( u \) and \( v \) and let \( c \) be a deformation parameter. By rescaling the spectral parameters it is always possible to set \( c = 1 \), but we will keep it for later convenience.

We define \( R(u, v) \in \text{End}(\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n}) \) as a rational supersymmetric R-matrix associated with the vector representation of \( \mathfrak{gl}(m|n) \),

\[
R(u, v) = I \otimes I + g(u, v) P, \tag{2.2}
\]

where we have introduced the identity matrix in \( \mathbb{C}^{m|n} \) by

\[
I = \sum_{i=1}^{m+n} E_{ii}.
\]

### 2.2. Commutation relations for the universal monodromy matrix.

The superalgebra \( DY(\mathfrak{gl}(m|n)) \) is a graded associative algebra with unit \( I \) and is generated by the modes \( T_{i,j}^{(\ell)} \), \( \ell \in \mathbb{Z} \), \( 1 \leq i, j \leq N + 1 \), of the T-operators

\[
T^\pm(u) = I \otimes 1 + \sum_{\ell \geq 0} \sum_{i,j=1}^{N+1} E_{ij} \otimes T_{i,j}^{(\ell)} u^{-\ell-1}, \tag{2.3}
\]

where \( \ell \geq 0 \) (respectively, \( \ell < 0 \)) refers to the + index (respectively, the - index) in \( T^\pm(u) \) and \( N = m + n - 1 \) is the number of simple roots of the superalgebra \( \mathfrak{gl}(m|n) \). The monodromy matrix elements \( T_{i,j}^{\pm}(u) \) are subject to the relations

\[
R(u, v) \cdot (T^\mu(u) \otimes I) \cdot (I \otimes T^\nu(v)) = (I \otimes T^\nu(v)) \cdot (T^\mu(u) \otimes I) \cdot R(u, v), \tag{2.4}
\]

where \( \mu, \nu \in \{ \pm \} \).
where $\mu, \nu = \pm$. For the monodromy matrix\(^1\) $T(u)$ to be globally even, we fix the grading of the monodromy matrix elements as follows:

$$[T_{i,j}(u)] = [i] + [j] \mod 2.$$  

The tensor product of matrices and algebra generators is also graded, that is,

$$(E_{ij} \otimes T_{i,j}(u)) \cdot (E_{kl} \otimes T_{k,l}(v)) = (-)^{([i]+[j])([k]+[l])}E_{ij}E_{kl} \otimes T_{i,j}(u)T_{k,l}(v).$$

The subalgebras formed by the modes $T_{i,j}^{(\ell)}$ (for $\ell \geq 0$ and for $\ell < 0$) of the $T$-operators $T^\pm(u)$ are the standard Borel subalgebras $U(b^\pm) \subset DY(gl(m|n))$. These Borel subalgebras are Hopf subalgebras of $DY(gl(m|n))$. Their coalgebraic structure is given by the graded coproduct

$$\Delta(T^\pm_{i,j}(u)) = \sum_{k=1}^{n+m} (-)^{([i]+[k])([k]+[j])}T^\pm_{k,j}(u) \otimes T^\pm_{i,k}(u).$$  

(2.5)

By the commutation relations (2.4) the universal transfer matrix $t(u)$, defined as the supertrace

$$t(u) = \text{str}(T^+(u)) \equiv \sum_{i=1}^{n+m} (-)^{[i]} T^+_{i,i}(u)$$  

(2.6)

of the universal monodromy matrix $T^+(u)$, commutes for arbitrary values of the spectral parameters:

$$[t(u), t(v)] = 0.$$  

Thus, it can be regarded as a generating function for the commuting integrals of motion in the corresponding supersymmetric quantum integrable model.

All the commutation relations (2.4) can be rewritten in the form

$$[T^\mu_{i,j}(u), T^\nu_{k,l}(v)] \equiv T^\mu_{i,j}(u)T^\nu_{k,l}(v) - (-)^{([i]+[j])([k]+[l])}T^\nu_{k,l}(v)T^\mu_{i,j}(u)$$

$$= (-)^{[i]} \cdot T^\nu_{k,l}(v)g(u,v)(T^\nu_{k,l}(v)T^\mu_{i,j}(u) - T^\nu_{i,l}(u)T^\mu_{i,j}(v)),  

(2.7)$$

where $\mu, \nu = \pm$. Renaming in (2.7) the indices and the spectral parameters by $i \leftrightarrow k$, $j \leftrightarrow l$, and $u \leftrightarrow v$, we obtain the equivalent relation

$$[T^\mu_{i,j}(u), T^\nu_{k,l}(v)] = T^\mu_{i,j}(u)T^\nu_{k,l}(v) - (-)^{([i]+[j])([k]+[l])}T^\nu_{k,l}(v)T^\mu_{i,j}(u)$$

$$= (-)^{[j]} \cdot T^\nu_{i,l}(u)g(u,v)(T^\nu_{i,l}(u)T^\mu_{i,j}(v) - T^\mu_{i,j}(v)T^\nu_{i,l}(u)).$$  

(2.8)

Note that, according to the commutation relations (2.7) and (2.8), the odd matrix elements of the monodromy matrix do not commute, in contrast to the even ones:

$$T^\mu_{i,j}(u)T^\nu_{i,j}(v) = \frac{h[i]}{h[j]}(v,u) T^\nu_{i,j}(v)T^\mu_{i,j}(u).$$  

(2.9)

\(^1\)We use the notation $T(u)$ to denote either $T^+(u)$ or $T^-(u)$ when both matrices share the same properties.
Here and below we use the graded rational functions

\[ f[i](u, v) = 1 + g[i](u, v) = 1 + \frac{c[i]}{u - v} = \frac{u - v + c[i]}{u - v}, \quad h[i](u, v) = \frac{f[i](u, v)}{g[i](u, v)} \]

and

\[ c[i] = (-)^{[i]} c. \]

Below we also use the notation

\[ \epsilon_{i,j} = 1 - \delta_{i,j}, \]

where \( \delta_{i,j} \) is the Kronecker symbol.

### 2.3. Morphism of \( DY(\mathfrak{gl}(m|n)) \), singular vectors, and Gauss decompositions.

Since the R-matrix (2.2) and the universal monodromy matrix (2.3) are globally even, one can easily check that the map

\[ \Psi: T_{i,j}^\pm(u) \rightarrow (-)^{[i][j] + 1} T_{j,i}^\mp(u) \]  

is an antimorphism of \( DY(\mathfrak{gl}(m|n)) \) which is a super- (or equivalently, graded) transposition compatible with the notion of super-trace. This map satisfies

\[ \Psi(A \cdot B) = (-)^{[A][B]} \Psi(B) \cdot \Psi(A) \]

for arbitrary elements \( A, B \in DY(\mathfrak{gl}(m|n)) \) and will be used to relate right and left states, or equivalently, Bethe vectors and the dual ones.

Let \( |0\rangle \) and \( \langle 0| \) be vectors satisfying the conditions

\[ T_{i,j}^\pm(u)|0\rangle = 0, \quad i > j, \quad T_{i,i}^\pm(u)|0\rangle = \lambda_i^\pm(u)|0\rangle, \quad i = 1, \ldots, N + 1, \]

\[ \langle 0|T_{i,j}^\pm(u) = 0, \quad i < j, \quad \langle 0|T_{i,i}^\pm(u) = \lambda_i^\pm(u)|0\rangle, \quad i = 1, \ldots, N + 1, \]

where in (2.12) the monodromy matrix elements are acting to the right, while in (2.13) they are acting to the left. Such vectors, if they exist, are called singular vectors. If the pseudo-vacuum vectors \( |0\rangle \) and \( \langle 0| \) belong to the finite-dimensional representations of the Yangian double \( DY(\mathfrak{gl}(m|n)) \), then the functions \( \lambda_i^\pm(u) \) are coinciding rational functions of the spectral parameter [22] expanded in the different domains: the function \( \lambda_i^+(u) \) is a series with respect to \( u^{-1} \) and the same function \( \lambda_i^-(u) \) is a series with respect to \( u \). In what follows we will use the same notation \( \lambda_i(u) \) for the functions \( \lambda_i^\pm(u) \).

For the T-operators fixed by the relations (2.4) we have two possibilities for introducing the Gauss coordinates. The first possibility is to introduce \( F_{j,i}^\pm(u) \),

\[ F_{j,i}^\pm(u) = \left. \frac{d^{[i][j] + 1} \Psi}{d u^{[i][j] + 1}} \right|_{u \to 0} \]

\[ \Psi(A \cdot B) = (-)^{[A][B]} \Psi(B) \cdot \Psi(A) \]

for arbitrary elements \( A, B \in DY(\mathfrak{gl}(m|n)) \) and will be used to relate right and left states, or equivalently, Bethe vectors and the dual ones.

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\[ F_{j,i}^\pm(u) = \left. \frac{d^{[i][j] + 1} \Psi}{d u^{[i][j] + 1}} \right|_{u \to 0} \]
E_{i,j}^\pm(u), 1 \leq i < j \leq N + 1, and k_\ell^\pm(u), \ell = 1, \ldots, N + 1, such that

\begin{equation}
T_{i,j}^\pm(u) = F_{j,i}^\pm(u)k_i^\pm(u) + \sum_{1 \leq \ell < i} F_{j,\ell}^\pm(u)k_\ell^\pm(u)E_{\ell,i}^\pm(u), \tag{2.14}
\end{equation}

\begin{equation}
T_{i,i}^\pm(u) = k_i^\pm(u) + \sum_{1 \leq \ell < i} F_{i,\ell}^\pm(u)k_\ell^\pm(u)E_{\ell,i}^\pm(u), \tag{2.15}
\end{equation}

\begin{equation}
T_{j,i}^\pm(u) = k_i^\pm(u)E_{i,j}^\pm(u) + \sum_{1 \leq \ell < i} F_{i,\ell}^\pm(u)k_\ell^\pm(u)E_{\ell,j}^\pm(u). \tag{2.16}
\end{equation}

In the second case we introduce \( \hat{F}_{j,i}^\pm(u), \hat{E}_{i,j}^\pm(u), 1 \leq i < j \leq N + 1, and \hat{k}_\ell^\pm(u), \ell = 1, \ldots, N + 1, such that

\begin{equation}
T_{i,j}^\pm(u) = \hat{F}_{j,i}^\pm(u)\hat{k}_j^\pm(u) + \sum_{j < \ell \leq N + 1} (-)^{(l+[i])((j+[j])\hat{F}_{\ell,i}^\pm(u)\hat{k}_\ell^\pm(u)\hat{E}_{\ell,j}^\pm(u), \tag{2.17}
\end{equation}

\begin{equation}
T_{j,j}^\pm(u) = \hat{k}_j^\pm(u) + \sum_{j < \ell \leq N + 1} (-)^{(l+[j])\hat{F}_{\ell,j}^\pm(u)\hat{k}_\ell^\pm(u)\hat{E}_{\ell,j}^\pm(u), \tag{2.18}
\end{equation}

\begin{equation}
T_{j,i}^\pm(u) = \hat{k}_j^\pm(u)\hat{E}_{i,j}^\pm(u) + \sum_{j < \ell \leq N + 1} (-)^{(l+[i])((j+[j])\hat{F}_{\ell,j}^\pm(u)\hat{k}_\ell^\pm(u)\hat{E}_{\ell,j}^\pm(u). \tag{2.19}
\end{equation}

One can verify that the antimorphism (2.10) and the Gauss decomposition (2.14)–(2.16) imply the following formulae for the Gauss coordinates:

\[ \Psi(F_{j,i}^\pm(u)) = (-)^{|i||j|^1}E_{i,j}^\mp(u), \quad \Psi(E_{i,j}^\pm(u)) = (-)^{|j||i|^1}F_{j,i}^\mp(u), \]
\[ \Psi(k_\ell^\pm(u)) = k_\ell^\mp(u). \tag{2.20} \]

Similarly,

\[ \Psi(\hat{F}_{j,i}^\pm(u)) = (-)^{|i||j|^1}\hat{E}_{i,j}^\mp(u), \quad \Psi(\hat{E}_{i,j}^\pm(u)) = (-)^{|j||i|^1}\hat{F}_{j,i}^\mp(u), \]
\[ \Psi(\hat{k}_\ell^\pm(u)) = \hat{k}_\ell^\mp(u). \]

The Gauss decomposition formulae also imply that

\[ E_{i,j}^\pm(u)|0\rangle = \hat{E}_{i,j}^\pm(u)|0\rangle = 0, \quad i < j, \quad k_\ell^\pm(u)|0\rangle = \hat{k}_\ell^\pm(u)|0\rangle = \lambda_\ell^\pm(u)|0\rangle; \]
\[ \langle 0|F_{j,i}^\pm(u) = \langle 0|\hat{F}_{j,i}^\pm(u) = 0, \quad i < j, \quad \langle 0|E_{i,j}^\pm(u) = \langle 0|\hat{E}_{i,j}^\pm(u) = \langle 0|\hat{E}_{j,i}^\pm(u) = \lambda_\ell^\pm(u)|0\rangle. \]

2.4. Current realizations of \( \text{DY}(\text{gl}(m|n)) \). Let

\[ F_i(u) = F_{i+1,i}(u) - F_{i+1,i}(u) \quad \text{and} \quad E_i(u) = E_{i,i+1}(u) - E_{i,i+1}(u) \]

be total currents [23]. Note that according to (2.20) we have

\[ \Psi(F_i(u)) = (-)^{|i||i|^1}E_i(u) = -E_i(u), \]
\[ \Psi(E_i(u)) = (-)^{|i+1||i|^1}F_i(u) = (-)^{\delta_{i,m}}F_i(u), \quad i = 1, \ldots, N. \tag{2.21} \]

This proves that the graded transposition is an idempotent of order 4 and its square counts the number of odd elements modulo 2.
Using straightforward calculations [24], [25] and the Gauss decomposition (2.14)–(2.16), we can obtain the following non-trivial commutation relations in the form
\[
E_k^\pm(u)F_i(v)F_j^\pm(u)^{-1} = f_{ij}(v, u)F_i(v),
\]
(2.22)
\[
k_{i+1}(u)F_i(v)F_{i+1}(u)^{-1} = f_{i+1}(u, v)F_i(v),
\]
\[
k_{i+1}(u)^{-1}E_i(v)F_i^\pm(u) = f_{ij}(v, u)E_i(v),
\]
(2.23)
\[
k_{i+1}(u)^{-1}E_i(v)F_{i+1}(u) = f_{i+1}(u, v)E_i(v),
\]
where \(\delta(u, v)\) is the rational \(\delta\)-function given by (2.32). These calculations also lead to the Serre relations. For the simple root currents \(F_i(u)\), \(i = 1, \ldots, N\), they have the form
\[
\text{Sym}_{u_1, u_2} \left((u_2 - u_1)\delta_{i,m} - c_{i+1}[F_i(u_1)F_i(u_2)F_{i+1}(v)]
- 2F_i(u_1)F_{i+1}(v)F_i(u_2) + F_{i+1}(v)F_i(u_1)F_i(u_2)\right) = 0,
\]
(2.29)
\[
\text{Sym}_{u_1, u_2} \left((u_1 - u_2)\delta_{i,m} + c_{i}[F_i(u_1)F_i(u_2)F_{i-1}(v)]
- 2F_i(u_1)F_{i-1}(v)F_i(u_2) + F_{i-1}(v)F_i(u_1)F_i(u_2)\right) = 0,
\]
(2.30)
\[
\text{Sym}_{u_1, u_2} \left((u_1 - u_2 + c)[F_m(u_1)F_m(u_2)F_{m-1}(v)F_{m+1}(v)]
- 2F_m(u_1)F_{m-1}(v)F_m(u_2)F_{m+1}(v)
+ 2c \cdot F_{m-1}(v)F_m(u_1)F_m(u_2)F_{m+1}(v)
+ (u_2 - u_1 + c)[F_{m-1}(v)F_{m+1}(v)F_m(u_1)F_m(u_2)
- 2F_{m-1}(v)F_m(u_1)F_{m+1}(v)F_m(u_2)]\right) = 0.
\]
(2.31)
Analogous formulae for the currents \(E_i(u)\), \(i = 1, \ldots, N\), can be obtained by applying the antimorphism \(\Psi\) to these relations. This amounts to replacing \(F_i(u)\) by \(E_i(u)\) and \(c\) by \(-c\) in (2.29)–(2.31).

The rational, or equivalently, additive \(\delta\)-function used in (2.28) can be represented as a difference of two series:
\[
\delta(u, v) = \delta(v, u) = \frac{1}{(u - v)} - \frac{1}{(u - v)} = \sum_{n \in \mathbb{Z}} \frac{v^n}{u^{n+1}},
\]
(2.32)
where
\[
\frac{1}{(u - v)} = \frac{1}{u} \sum_{k \geq 0} \left(\frac{v}{u}\right)^k \quad \text{and} \quad \frac{1}{(u - v)} = -\frac{1}{v} \sum_{k \geq 0} \left(\frac{u}{v}\right)^k.
\]
(2.33)
Here the symbol $>$ in the rational function $\frac{1}{(u-v)_+}$ means that $|u| > |v|$ and this rational function should be represented as the first series in (2.33). In turn, the symbol $<$ in the rational function $\frac{1}{(u-v)_-}$ means that $|u| < |v|$ and this rational function should be represented as the second series in (2.33). Below we will also use the notation $\frac{1}{(u-v)_{\lessgtr}}$ to stress that one can use either of the two series expansions in (2.33) for the rational function $\frac{1}{u-v}$.

It is known [14] that another current realization of the Yangian double $\mathcal{DY}(\mathfrak{gl}(m|n))$ can be obtained using a different Gauss decomposition of the monodromy matrix, as in (2.17)–(2.19). The commutation relations between the Cartan currents $\widehat{k}_i^\pm(u)$ and the simple root total currents $\widehat{F}_i(u)$ and $\widehat{E}_i(u)$ given by

$$\widehat{F}_i(u) = \widehat{F}_{i+1,i}(u) - \widehat{F}_{i+1,i}(u), \quad \widehat{E}_i(u) = \widehat{E}_{i,i+1}(u) - \widehat{E}_{i,i+1}(u)$$

are gathered below:

$$\widehat{k}_i^\pm(u) \widehat{F}_i(v) \widehat{k}_i^\pm(u)^{-1} = f_{[i]}(v, u) \widehat{F}_i(v),$$

$$\widehat{k}_{i+1}^\pm(u) \widehat{F}_i(v) \widehat{k}_{i+1}^\pm(u)^{-1} = f_{[i+1]}(u, v) \widehat{F}_i(v),$$

$$\widehat{k}_i^\pm(u)^{-1} \widehat{E}_i(v) \widehat{k}_i^\pm(u) = f_{[i]}(v, u) \widehat{E}_i(v),$$

$$\widehat{k}_{i+1}^\pm(u)^{-1} \widehat{E}_i(v) \widehat{k}_{i+1}^\pm(u) = f_{[i+1]}(u, v) \widehat{E}_i(v),$$

$$((u-v)\epsilon_{i,m} + c_{[i]}) \widehat{F}_i(u) \widehat{F}_i(v) = ((u-v)\epsilon_{i,m} - c_{[i]}) \widehat{F}_i(v) \widehat{F}_i(u),$$

$$((u-v)\epsilon_{i,m} - c_{[i]}) \widehat{E}_i(u) \widehat{E}_i(v) = ((u-v)\epsilon_{i,m} + c_{[i]}) \widehat{E}_i(v) \widehat{E}_i(u),$$

$$(u-v-c_{[i+1]}) \widehat{F}_i(u) \widehat{F}_{i+1}(v) = (u-v) \widehat{F}_{i+1}(v) \widehat{F}_i(u),$$

$$(u-v) \widehat{E}_i(u) \widehat{E}_{i+1}(v) = (u-v-c_{[i+1]}) \widehat{E}_{i+1}(v) \widehat{E}_i(u),$$

$$[\widehat{E}_i(u), \widehat{F}_j(v)] = \widehat{E}_i(u) \widehat{F}_j(v) - (-)^{(l[i]+[i+1])(l[j]+[j+1])} \widehat{F}_j(v) \widehat{E}_i(u)$$

$$= \delta_{i,j} c_{[i+1]} \delta(u, v)(\widehat{k}_i^+(u) \cdot \widehat{k}_{i+1}^+(u)^{-1} - \widehat{k}_i^-(v) \cdot \widehat{k}_{i+1}^-(v)^{-1}).$$

The Serre relations for the simple root currents $\widehat{E}_i(u)$, $i = 1, \ldots, N$, now have the form

$$\text{Sym}_{u_1, u_2}(((u_2-u_1)\delta_{i,m} - c_{[i+1]}) (\widehat{E}_i(u_1) \widehat{E}_i(u_2) \widehat{E}_{i+1}(v)$$

$$- 2 \widehat{E}_i(u_1) \widehat{E}_{i+1}(v) \widehat{E}_i(u_2) + \widehat{E}_{i+1}(v) \widehat{E}_i(u_1) \widehat{E}_i(u_2)) = 0,$$

$$\text{Sym}_{u_1, u_2}(((u_1-u_2)\delta_{i,m} + c_{[i]}) (\widehat{E}_i(u_1) \widehat{E}_i(u_2) \widehat{E}_{i-1}(v)$$

$$- 2 \widehat{E}_i(u_1) \widehat{E}_{i-1}(v) \widehat{E}_i(u_2) + \widehat{E}_{i-1}(v) \widehat{E}_i(u_1) \widehat{E}_i(u_2)) = 0,$$


\[ \text{Sym}_{u_1, u_2} ((u_1 - u_2 + c) [\hat{E}_m(u_1)\hat{E}_m(u_2)\hat{E}_{m-1}(v_1)\hat{E}_{m+1}(v_2) \\
- 2\hat{E}_m(u_1)\hat{E}_{m-1}(v_1)\hat{E}_m(u_2)\hat{E}_{m+1}(v_2)] \\
+ 2c\hat{E}_{m-1}(v_1)\hat{E}_m(u_1)\hat{E}_m(u_2)\hat{E}_{m+1}(v_2) \\
+ (u_2 - u_1 + c) [\hat{E}_{m-1}(v_1)\hat{E}_{m+1}(v_2)\hat{E}_m(u_1)\hat{E}_m(u_2) \\
- 2\hat{E}_{m-1}(v_1)\hat{E}_m(u_1)\hat{E}_{m+1}(v_2)\hat{E}_m(u_2)] = 0. \] (2.44)

Thanks to the antimorphism \(\Psi\), there are analogous relations for the currents \(\hat{F}_i(u)\), \(i = 1, \ldots, N\), with the replacements \(\hat{E}_i(u) \rightarrow \hat{F}_i(u)\) and \(c \rightarrow -c\) in the formulae (2.42)–(2.44). The action of the antimorphism (2.10) on the currents \(\hat{F}_i(u), \hat{E}_i(u)\), and \(\hat{k}_\ell(u)\) is given by the same formulae as in (2.21).

Note that in the commutation relations (2.24), (2.25), (2.37), and (2.38) one can replace \(c_{ij}\) by \(c_{i+1j}\). Indeed, \(c_{ij} = c_{i+1j}\) when \(i \neq m\), while for \(i = m\) the factor \((u - v)\epsilon_{i,m}\) vanishes, and thus it does not matter whether we use \(c_{ij}\) or \(c_{i+1j}\).

### 3. Universal Bethe vectors

It follows from the commutation relations (2.4) that the subalgebras \(U^\pm\) generated by the modes of the T-operators \(T^{(n)}_{ij}\) form two Borel subalgebras of \(DY(\mathfrak{gl}(m|n))\). Moreover, by (2.5) they are Hopf subalgebras. We call \(U^\pm\) the standard Borel subalgebras of the Yangian double \(DY(\mathfrak{gl}(m|n))\).

As we already mentioned, the universal Bethe vectors are constructed from the matrix elements of one universal monodromy matrix \(T^{\pm}_{ij}\). These operators belong to the standard ‘positive’ Borel subalgebra \(U^+\). The goal of this section is to express the universal Bethe vectors in terms of the current generators of the Yangian double \(DY(\mathfrak{gl}(m|n))\), using the approach developed in [13], [14], and [26].

In this paper we consider formulae for the Bethe vectors compatible with two different ways of embedding an algebra of smaller rank in an algebra of larger rank. Namely, from the explicit formulae for the right Bethe vectors \(\mathbb{B}(\vec{i})\) (see (5.17)) one can conclude that the Bethe vector \(\mathbb{B}(\vec{i})\) is obtained by resolving the hierarchical relations based on the embedding of the Yangian double \(DY(\mathfrak{gl}(m - 1|n))\) in the larger algebra \(DY(\mathfrak{gl}(m|n))\). Similarly, it follows from (5.25) that the Bethe vector \(\mathbb{B}(\vec{i})\) is obtained by resolving the hierarchical relations based on the embedding of the Yangian double \(DY(\mathfrak{gl}(m|n - 1))\) in the larger algebra \(DY(\mathfrak{gl}(m|n))\). To express the Bethe vectors \(\mathbb{B}(\vec{i})\) and \(\hat{\mathbb{B}}(\vec{i})\) in terms of the current generators we will use two different types of Gauss decompositions of the monodromy matrix elements and the corresponding current generators [14].

The general theory of the relation between Bethe vectors and currents was developed in the paper [26] and then applied in [13] and [14] to the construction of the hierarchical Bethe vectors for quantum integrable models associated with the quantum affine algebra \(U_q(\mathfrak{gl}(N))\). The main tool used in those papers was the language of projections onto intersections of Borel subalgebras of different type.
To describe the Bethe vectors $\mathcal{B}(\bar{t})$ and $\mathcal{C}(\bar{t})$ we will use the current Borel subalgebras associated with the Gauss decomposition (2.14)–(2.16) and the antimorphism (2.10). For the Bethe vectors $\mathcal{B}(\bar{t})$ and $\mathcal{C}(\bar{t})$ we will use the same antimorphism and the current Borel subalgebras associated with the second Gauss decomposition (2.17)–(2.19).

### 3.1. Notation and conventions.

We will denote sets of variables by bars over letters: $\overline{u}$, $\overline{v}$, and so on. To simplify formulae below, we use a shortened notation for products of functions depending on one or two variables. Namely, whenever we indicate that a function $\lambda_j$ depends on a set of variables, the notation $\lambda_j(\overline{u})$ stands for the product of the functions $\lambda_j(u_\ell)$ over the set $\overline{u}$. Similarly, the notation $f_{[i]}(\overline{u}, \overline{v})$ (or $g_{[i]}(\overline{u}, \overline{v})$, or $h_{[i]}(\overline{u}, \overline{v})$) denotes the double product of these functions over the corresponding sets. For example,

$$\lambda_j(\overline{u}) = \prod_{u_\ell \in \overline{u}} \lambda_j(u_\ell) \quad \text{and} \quad f_{[i]}(\overline{u}, \overline{v}) = \prod_{u_\ell \in \overline{u}, \, v_\ell' \in \overline{v}} f_{[i]}(u_\ell, v_\ell').$$

Moreover, we use the same convention when considering products of commuting operators. For example,

$$T_{i,j}(\overline{u}) = \prod_\ell T_{i,j}(u_\ell) \quad \text{for } [i] + [j] = 0 \text{ mod } 2.$$

We also introduce several rational functions which will appear in the text below. First, for any function $x(u_1, u_2)$ we set

$$\Delta_x(\overline{u}) = \prod_{1 \leq \ell < \ell' \leq a} x(u_{\ell'}, u_\ell) \quad \text{and} \quad \Delta'_x(\overline{u}) = \prod_{1 \leq \ell < \ell' \leq a} x(u_\ell, u_{\ell'}),$$

where $a = \#\overline{u}$.

Second, for arbitrary sets of parameters $\overline{u}$ and $\overline{v}$ we define

$$\gamma_i(\overline{u}) = \frac{\Delta_{f_{[i]}}(\overline{u})}{\Delta_{h}(\overline{u})^{\delta_{i,m}}} \quad \text{and} \quad \gamma_i(\overline{u}, \overline{v}) = \frac{f_{[i]}(\overline{u}, \overline{v})}{h(\overline{u}, \overline{v})^{\delta_{i,m}}}.$$

The first function coincides with $\Delta_{f_{[i]}}(\overline{u})$ for $i \neq m$ and with $\Delta_g(\overline{u})$ for $i = m$. The second function coincides with $f_{[i]}(\overline{u}, \overline{v})$ for $i \neq m$ and with $g(\overline{u}, \overline{v})$ for $i = m$.

Similarly, we define

$$\gamma_{i}(\overline{u}) = \frac{\Delta_{f_{[i+1]}}(\overline{u})}{\Delta_{h}(\overline{u})^{\delta_{i,m}}} \quad \text{and} \quad \gamma_{i}(\overline{u}, \overline{v}) = \frac{f_{[i+1]}(\overline{u}, \overline{v})}{h(\overline{u}, \overline{v})^{\delta_{i,m}}}.$$ 

For $i \neq m$,

$$\gamma_{i}(\overline{u}) = \Delta_{f_{[i+1]}}(\overline{u}) \quad \text{and} \quad \gamma_{i}(\overline{u}, \overline{v}) = f_{[i+1]}(\overline{u}, \overline{v}),$$

while for $i = m$,

$$\gamma_{m}(\overline{u}) = \Delta'_g(\overline{u}) \quad \text{and} \quad \gamma_{m}(\overline{u}, \overline{v}) = g(\overline{u}, \overline{v}).$$

Note that the function $\gamma_{m}(\overline{u})$ differs from $\gamma_{m}(\overline{u})$ by the factor $(-)^{\#\overline{u}(\#\overline{u}-1)/2}$. Similarly,

$$\gamma_{m}(\overline{u}, \overline{v}) = (-)^{\#\overline{u}(\#\overline{u}-1)/2} \gamma_{m}(\overline{u}, \overline{v}).$$

Also, note that $\gamma_{i}(\overline{u}) = \gamma_{i}(\overline{u})$ and $\gamma_{i}(\overline{u}, \overline{v}) = \gamma_{i}(\overline{u}, \overline{v})$ for $i \neq m$. 


3.2. Deformed symmetrization. For any formal series $G(\bar{\ell})$ depending on the set of variables $\bar{\ell}$ (see (3.11) below) we define the deformed symmetrization (or $c$-symmetrization) to be the sum

$$\overline{\text{Sym}}_c G(\bar{\ell}) = \sum_{\sigma \in S_r} \prod_{s=1}^{N} \prod_{\ell < \ell'}^{\sigma^+(\ell) > \sigma^+(\ell')} \frac{(t^s_{\sigma^+(\ell')}) - t^s_{\sigma^+(\ell)}}{(t^s_{\sigma^+(\ell')}) - t^s_{\sigma^+(\ell)}} \epsilon_{s,m} + c[s] \bar{G}(\sigma \bar{\ell}), \quad (3.3)$$

where $S_r = S_{r_1} \times \cdots \times S_{r_N}$ is the direct product of the groups $S_{r_s}$ of permutations of the integers $1, \ldots, r_s$, $s = 1, \ldots, N$, and $\sigma \bar{\ell}$ is the corresponding permuted set of Bethe parameters (3.11). By the arguments at the end of §2.4, the formula for the deformed symmetrization can easily be written as

$$\overline{\text{Sym}}_c G(\bar{\ell}) = \sum_{\sigma \in S_r} \prod_{s=1}^{N} \prod_{\ell < \ell'}^{\sigma^+(\ell) > \sigma^+(\ell')} \frac{(t^s_{\sigma^+(\ell')}) - t^s_{\sigma^+(\ell)}}{(t^s_{\sigma^+(\ell')}) - t^s_{\sigma^+(\ell)}} \epsilon_{s,m} + c[s] \bar{G}(\sigma \bar{\ell}). \quad (3.4)$$

In what follows we will use either (3.3) or (3.4), depending on the situation.

We say that a series $Q(\bar{\ell})$ is $c$-symmetric if

$$\overline{\text{Sym}}_c Q(\bar{\ell}) = \left( \prod_{s=1}^{N} r_s! \right) Q(\bar{\ell}).$$

Note that for $s = m$ the product over $\ell$ and $\ell'$ is equal to $(-)^{P(\sigma^m)}$, where $P(\sigma^m)$ is the parity of the permutation $\sigma^m$, and the sum over all permutations $\sigma^m$ is nothing but the antisymmetrization over the set $\bar{\ell}^m$.

3.3. The Bethe vector $\mathbb{B}(\bar{\ell})$ and the dual Bethe vector $\mathbb{C}(\bar{\ell})$. We first explain the relation between the Bethe vector $\mathbb{B}(\bar{\ell})$ and the current presentation (2.22)-(2.28).

Let $U_F \subset DY(gl(m|n))$ be the $DY(gl(m|n))$ subalgebra generated by the modes of the simple root currents $F_i^{(\ell)}, i = 1, \ldots, N, \ell \in \mathbb{Z}$, and by the modes of the ‘positive’ Cartan currents $k_j^{(\ell)}, j = 1, \ldots, N+1, \ell' \geq 0$. In the framework of the quantum double construction, the subalgebra $U_E \subset DY(gl(m|n))$ dual to $U_F$ is generated by the modes of the simple root currents $E_i^{(\ell)}, i = 1, \ldots, N, \ell \in \mathbb{Z}$, and by the modes of the ‘negative’ Cartan currents $k_j^{(\ell')}, j = 1, \ldots, N+1, \ell' < 0$.

We call the subalgebras $U_F$ and $U_E$ current Borel subalgebras. They are Hopf subalgebras of $DY(gl(m|n))$ with respect to the so-called Drinfeld coproduct

$$\Delta^{(D)}(F_i(z)) = F_i(z) \otimes 1 + k_{i+1}^{+}(z)(k_{i}^{+}(z))^{-1} \otimes F_i(z),$$

$$\Delta^{(D)}(k_j^{\pm}(z)) = k_j^{\pm}(z) \otimes k_j^{\mp}(z),$$

$$\Delta^{(D)}(E_i(z)) = 1 \otimes E_i(z) + E_i(z) \otimes k_{i+1}^{-}(z)(k_{i}^{-}(z))^{-1}, \quad (3.5)$$

which obviously differs from the coproduct given by (2.5).

5Recall that $N = m + n - 1$ is the number of simple roots of the superalgebra $gl(m|n)$.
In order to express the Bethe vectors $\mathcal{B}(\vec{t})$ in terms of the current generators, we need only the one current Borel subalgebra $U_F$ and its coalgebraic properties given by the first two equalities in (3.5). Consider the following intersections of this current Borel subalgebra with the standard Borel subalgebras $U_{\pm}$:

$$U_F^- = U_F \cap U^- \quad \text{and} \quad U_F^+ = U_F \cap U^+.$$  \hspace{1cm} (3.6)

Each of these intersections is a subalgebra of $DY(gl(m|n))$ [26], and they are coideals with respect to the coproduct (3.5):

$$\Delta^{(D)}(U_F^+) = U_F^+ \otimes U_F \quad \text{and} \quad \Delta^{(D)}(U_F^-) = U_F \otimes U_F^-.$$  \hspace{1cm} (3.7)

To see this we introduce the expansion of the following combination of Cartan currents:

$$k_{i+1}^+(z)(k_i^+(z))^{-1} = 1 + \sum_{\ell > 0} k_{i}^{(\ell)} z^{-\ell - 1}.$$  \hspace{1cm} (3.8)

The properties (3.7) become obvious in view of (3.8).

According to the Cartan–Weyl construction of the Yangian double we have to find a global ordering on the generators of this algebra. There are two different choices for this ordering. We choose the ordering such that elements in the subalgebra $U_F^-$ precede elements of the subalgebra $U_F^+$ [26], [27]. We say that an arbitrary element $F \in U_F$ is ordered if it is represented in the form

$$F = F_- \cdot F_+,$$

where $F_\pm \in U_F^\pm$.

According to the general theory [26] one can define the projections of any ordered elements of the subalgebra $U_F$ on the subalgebras (3.6) using the formulae

$$P_f^+(F_- \cdot F_+) = \varepsilon(F_-) F_+, \quad P_f^-(F_- \cdot F_+) = F_\varepsilon(F_+), \quad F_\pm \in U_F^\pm,$$  \hspace{1cm} (3.9)

where the counit map $\varepsilon : U_F \to \mathbb{C}$ is defined by the rules

$$\varepsilon(F_i^{(\ell)}) = 0, \quad \varepsilon(1) = 1, \quad \varepsilon(k_j^{(\ell)}) = 0.$$  \hspace{1cm} (3.10)

Let $\overline{U}_F$ be the completion of $U_F$, which is formed by infinite sums of monomials that are ordered products of the form

$$A_{i_1}^{(\ell_1)} \cdots A_{i_a}^{(\ell_a)}, \quad \ell_1 \leq \cdots \leq \ell_a,$$

where $A_{i_\ell}$ is either $F_{i_\ell}^{(\ell)}$ or $k_{i_\ell}^{(\ell)}$. It can be proved [26] that

1) the action of the projections (3.9) extends to the algebra $\overline{U}_F$;
2) for any $F \in \overline{U}_F$ with $\Delta^{(D)}(F) = F' \otimes F''$ we have

$$F = P_f^-(F') \cdot P_f^+(F'').$$  \hspace{1cm} (3.10)
The formula (3.10) is an important tool for calculating the universal Bethe vectors. It allows us to present an arbitrary product of currents in the ordered form using simple formulae for the Drinfeld current coproducts.

Now we can define the universal Bethe vector. Let

$$\bar{t} = \{t_1^1, \ldots, t_{r_1}^1; t_1^2, \ldots, t_{r_2}^2; \ldots; t_1^N, \ldots, t_{r_N}^N\}$$  \hspace{1cm} (3.11)

be a set of parameters. The superscript labels the different types of Bethe parameters and refers to the simple root numbering, and the subscript counts the number of parameters of a given type. There are \(r_\ell\) Bethe parameters of type \(\ell = 1, \ldots, N\).

Let 

$$\leftarrow A_a \rightarrow A_a$$

respectively, 

$$\leftarrow A_a \rightarrow A_a$$

denote the ordered product of non-commuting operators \(A_a\) such that \(A_\ell\) is on the right (respectively, on the left) of \(A_{\ell'}\) for \(\ell' \geq \ell\):

$$\prod_{j \geq a \geq i} A_a = A_j A_{j-1} \cdots A_{i+1} A_i \quad \text{and} \quad \prod_{i \leq a \leq j} A_a = A_i A_{i+1} \cdots A_{j-1} A_j.$$  \hspace{1cm} (3.12)

We define an ordered product of total currents,

$$\mathcal{F}(\bar{t}) = \prod_{1 \leq a \leq N} \left( \prod_{1 \leq \ell \leq r_a} F_a(t_\ell^a) \right),$$  \hspace{1cm} (3.13)

which is a formal series with respect to the ratios \(t_b^a/t_c^a\) (\(b > c\)) and \(t_i^a/t_j^a\) (\(i > j\)) and takes values in the completion \(\overline{U}_F\) (see [26]). The product (3.12) has poles for some values of the ratios \(t_b^a/t_c^a\) and \(t_i^a/t_j^a\). The operator-valued coefficients at these poles take values in the completion \(\overline{U}_F\) and can be identified with composed root currents (see Appendix A). Note also that in view of the commutation relations between currents, the product (3.12) as well as its projections are \(c\)-symmetric.

Let us introduce the normalized product of currents

$$F(\bar{t}) = \prod_{1 \leq a \leq N} \frac{\prod_{\ell=1}^N \gamma_\ell(t_\ell^a)}{\prod_{\ell=1}^{N-1} f(\ell+1)_{(\ell+1, \ell)}} \mathcal{F}(\bar{t}),$$  \hspace{1cm} (3.14)

where \(\gamma_\ell\) is given by (3.1). Then the universal off-shell Bethe vector \(\mathbb{B}(\bar{t})\) is defined as the action of the projection on this normalized product, applied to the singular vector \(|0\rangle\):

$$\mathbb{B}(\bar{t}) = P_f^+ (F(\bar{t})) \prod_{s=1}^N \lambda_s(\bar{t}^s)|0\rangle.$$  \hspace{1cm} (3.15)

Note that in view of the commutation relations (2.24) and (2.26) between currents the normalized product of currents (3.13) is symmetric with respect to permutations of Bethe parameters of the same type.

The normalization of the universal off-shell Bethe vector is chosen so that it removes all zeros and poles originating from products of currents. For example, according to the commutation relations (2.24), the products of currents \(F_\ell(\bar{t}^\ell)\) have poles when \(t_j^\ell - t_i^\ell + c_{[\ell]} = 0\) for \(j > i\) and \(\ell \neq m\), and zeros for all \(\ell\) when \(t_j^\ell - t_i^\ell = 0\). The potential singularities are compensated by the rational
functions in the numerator of the prefactor in (3.13). On the other hand, the products of currents $\mathcal{F}_t(\bar{t}^t)\mathcal{F}_{t+1}(\bar{t}^{t+1})$ have poles when $t_j^{t+1} - t_i^t = 0$ and zeros when $t_j^{t+1} - t_i^t + c_{[t+1]} = 0$ for all $i, j$. These possible singularities are compensated by the product of the rational functions $f_{[t+1]}(\bar{t}^{t+1}, \bar{t}^t)^{-1}$ in the denominator of the prefactor in (3.13).

Our strategy is to calculate first the projection in (3.14) and then to rewrite the result of this calculation as some polynomial in the monodromy matrix elements. This will be done in §5. Then we define the dual Bethe vector $\mathcal{C}(\bar{t})$ by the formula

$$\mathcal{C}(\bar{t}) = \Psi(\mathcal{B}(\bar{t})),$$  \hspace{1cm} (3.15)

where the antilinear transformation (2.10) is extended from the algebra to vectors of the representation of this algebra using the relations $\Psi(|0\rangle) = |0\rangle$ and $\Psi(|\langle0\rangle|) = |\langle0\rangle|)$.

Alternatively, the formula for the dual Bethe vector can be found via the projection method and another choice of the current Borel subalgebra, the Drinfeld coproduct, and the associated projections from the ordered product of currents

$$\mathcal{E}(\bar{t}) = \prod_{N \geq a \geq 1} \left( \prod_{r_a \geq t_1} E_a(t_a^t) \right).$$

We do not perform these calculations in this paper.

### 3.4. The Bethe vector $\hat{\mathcal{B}}(\bar{t})$ and the dual Bethe vector $\hat{\mathcal{C}}(\bar{t})$.

For the Bethe vector $\hat{\mathcal{B}}(\bar{t})$ and the dual Bethe vector $\hat{\mathcal{C}}(\bar{t})$ one has to explore the second current realization (2.35)–(2.41) of the Yangian double $DY(\mathfrak{gl}(m|n))$ given by the currents $\hat{F}_i(z)$, $\hat{E}_i(z)$, and $\hat{k}_j^\pm(z)$, which are related to the monodromy matrix elements through the Gauss decomposition (2.17)–(2.19) and the Frenkel–Ding formulae (2.34).

As in the previous subsections, to describe the Bethe vector $\hat{\mathcal{B}}(\bar{t})$ we define a Borel subalgebra $\hat{U}_F$ such that the ‘positive’ Cartan currents $\hat{k}_j^+(z)$ are in $\hat{U}_F$ and have the coalgebraic properties

$$\hat{\Delta}^{(D)}(\hat{F}_i(z)) = 1 \otimes \hat{F}_i(z) + \hat{F}_i(z) \otimes \hat{k}_i^+(z)(\hat{k}_{i+1}^+(z))^{-1},$$

$$\hat{\Delta}^{(D)}(\hat{k}_j^+(z)) = \hat{k}_j^+(z) \otimes \hat{k}_j^+(z).$$  \hspace{1cm} (3.16)

We again consider the intersections of this current Borel subalgebra with the standard Borel subalgebras $\hat{U}_\pm$,

$$\hat{U}^-_F = \hat{U}_F \cap \hat{U}^- \quad \text{and} \quad \hat{U}^+_F = \hat{U}_F \cap \hat{U}^+, \hspace{1cm} (3.17)$$

and check the coideal properties of these intersections,

$$\hat{\Delta}^{(D)}(\hat{U}^+_F) = \hat{U}_F \otimes \hat{U}^+_F \quad \text{and} \quad \hat{\Delta}^{(D)}(\hat{U}^-_F) = \hat{U}^-_F \otimes \hat{U}_F$$

with respect to the coproduct (3.16).

Using the same cycling ordering for the Cartan–Weyl generators of $\hat{U}_F$ as we used for ordering elements in $U_F$, we say that an arbitrary element $\hat{\mathcal{F}} \in \hat{U}_F$ is ordered if

$$\hat{\mathcal{F}} = \hat{\mathcal{F}}_- \cdot \hat{\mathcal{F}}_+,$$

where $\hat{\mathcal{F}}_\pm \in \hat{U}_F^\pm$. 

Again, according to the general theory formulated in [26] one can define the projections of any ordered elements of the subalgebras $\hat{U}_F$ and $\hat{U}_E$ on the subalgebras (3.17) by using the formulae

$$\hat{P}_f^+(\hat{F}_- \cdot \hat{F}_+) = \hat{\varepsilon}(\hat{F}_-) \hat{F}_+,$$

$$\hat{P}_f^-(\hat{F}_- \cdot \hat{F}_+) = \hat{\varepsilon}(\hat{F}_+) \hat{F}_-$$

(3.18)

where the counit map $\hat{\varepsilon}: \text{DY}(\mathfrak{gl}(m|n)) \rightarrow \mathbb{C}$ is defined by the rules

$$\hat{\varepsilon}(\hat{F}_i^{(\ell)}) = 0 \quad \text{and} \quad \hat{\varepsilon}(\hat{k}_j^{(\ell)}) = 0,$$

and $\hat{F}_i^{(\ell)}$ and $\hat{k}_j^{(\ell)}$ are modes of the currents $\hat{F}_i(z)$ and $\hat{k}_j^+(z)$ in the second current realization of the Yangian double $\text{DY}(\mathfrak{gl}(m|n))$.

Defining the completion $\hat{U}_F$, we can verify [26] that:

1) the action of the projections (3.18) extends to the algebras $\hat{U}_F$;
2) for any $\hat{F} \in \hat{U}_F$ with $\hat{\Delta}(\hat{F}) = \hat{F} \otimes \hat{F}$ we have

$$\hat{F} = \hat{P}_f^{-}(\hat{F}^\prime) \cdot \hat{P}_f^{+}(\hat{F}^\prime).$$

(3.19)

For the set (3.11) of Bethe parameters we consider the normalized ordered product of currents

$$\hat{F}(\tilde{t}) = \frac{\prod_{\ell=1}^{N} \hat{\gamma}_{\ell}(\tilde{t})}{\prod_{\ell=1}^{N-1} \hat{f}_{[\ell+1]}(\tilde{t}, \hat{t})} \hat{F}(\tilde{t}),$$

(3.20)

where

$$\hat{F}(\tilde{t}) = \prod_{N \geq a \geq 1} \left( \prod_{r_a \geq \ell \geq 1} \hat{P}_{a}(t_a^{(\ell)}) \right).$$

(3.21)

The universal off-shell Bethe vectors associated with the second current realization of the Yangian double $\text{DY}(\mathfrak{gl}(m|n))$ are defined in terms of the action of the above projections on the singular vector $|0\rangle$ as follows:

$$\hat{B}(\tilde{t}) = \hat{P}_f^+(\hat{F}(\tilde{t})) \prod_{s=1}^{N} \lambda_{s+1}(\tilde{t}^s)|0\rangle.$$

(3.22)

The normalization of this universal off-shell Bethe vector is again chosen in such a way as to remove all zeros and poles arising from products of currents.

The dual Bethe vector $\hat{C}(\tilde{t})$ is defined using the antimorphism (2.10):

$$\hat{C}(\tilde{t}) = \Psi(\hat{B}(\tilde{t})).$$

(3.23)

3.5. Main results. In this paper we verify the following.

- The two different ways of constructing the Bethe vectors lead in the end to the same result, that is,

$$\hat{B}(\tilde{t}) = \hat{B}(\tilde{t}) \quad \text{and} \quad \hat{C}(\tilde{t}) = \hat{C}(\tilde{t}).$$

(3.24)

In §4 we will prove this statement for the Bethe vectors $\hat{B}(\tilde{t})$ and $\hat{B}(\tilde{t})$ only. The proof for the dual vectors $\hat{C}(\tilde{t})$ and $\hat{C}(\tilde{t})$ follows from application of the antimorphism $\Psi$ to the first equality in (3.24).
Bethe vectors become on-shell, or equivalently, become eigenvectors of the supersymmetric transfer matrix $t(z)$ (2.6) with the eigenvalue (4.78), if the Bethe equations (4.75) for the parameters (3.11) are satisfied.

Explicit formulae for the Bethe vectors in terms of the monodromy matrix elements are given by (5.17) and (5.25). Explicit formulae for the dual vectors can be obtained using the antimorphism (2.10).

The coproduct properties for the Bethe vectors are given in the relations (4.8) and (4.9). They express the coproduct of a Bethe vector in terms of Bethe vectors belonging to the two copies of $DY'(gl(m|n))$ arising under application of the coproduct.

4. Formulae for the action of the monodromy matrix elements

The goal of the present section is to prove that the Bethe vectors $B(\bar{t})$ and $bB(\bar{t})$ coincide. After obtaining formulae for the universal off-shell Bethe vectors in terms of elements of the monodromy matrix (see §5), we will see that a direct proof of the equality (3.24) is a rather complicated combinatorial problem. Instead, we will prove it by checking that both of these vectors satisfy the same recurrence relations with respect to the action of the upper triangular and diagonal monodromy matrix elements on these vectors. To check this statement it is not necessary to get explicit formulae for the universal off-shell Bethe vectors in terms of the monodromy matrix. Before starting this analysis, we show that the Bethe vectors $B(\bar{t})$ and $bB(\bar{t})$ have the same coproduct properties that follow from the coproduct (2.5) for the monodromy matrix.

4.1. Coproduct properties of the Bethe vectors. Calculating the coproduct of the product of the currents $F_i(t)$ using the first formula in (3.5), we get that the Drinfeld coproduct of the ordered product of simple root currents $\mathcal{F}(\bar{t})$ is

$$
\Delta^{(D)}(\mathcal{F}(\bar{t})) = \sum_{0 \leq s_1 \leq r_1} \cdots \sum_{0 \leq s_N \leq r_N} \prod_{\ell=1}^{N} \frac{1}{s_{\ell}! (r_{\ell} - s_{\ell})!} \times \text{Sym}_{\bar{t}} \left( \prod_{s=1}^{N} \prod_{\ell=s+1}^{r_{s}} k_{s+1}^+ (t_{s}^+ t_{s}^-)^{-1} \otimes \mathcal{F}(\bar{t}^{''}) \right),
$$

(4.1)

where the sets $\bar{t}'$ and $\bar{t}''$ are

$$
\bar{t}' = \{ t_1^1, \ldots, t_1^{s_1}; t_1^2; \ldots; t_1^{r_1}; \ldots, t_N^1, \ldots, t_N^{s_N} \},
$$

$$
\bar{t}'' = \{ t_1^{s_1+1}; \ldots; t_1^{r_1}; t_2^1; \ldots; t_2^{r_2}; \ldots; t_N^{s_N+1}; \ldots, t_N^{r_N} \},
$$

and $Z_{\bar{t}}$ is the rational function

$$
Z_{\bar{t}} = \prod_{a=1}^{N-1} \prod_{s_a < \ell \leq r_a} \frac{t_a^a - t_{\ell}^{a+1} - c_{[a+1]}}{t_{\ell}^{a+1} - t_{\ell}^a} = \prod_{a=1}^{N-1} \prod_{s_a < \ell \leq r_a} f_{[a+1]}(t_{\ell}^{a+1}, t_{\ell}^a).
$$

The formula (4.1) enables us to obtain the coalgebraic properties of the normalized product of currents (3.13) with respect to the Drinfeld coproduct. Indeed,
the $c$-symmetrization can be transformed into the usual symmetrization over the set $\{\bar{t}^s\}$ due to the property

$$
\gamma_s(\bar{t}^s) \overline{\text{Sym}_{\bar{t}^s}}(G(\bar{t}^s)) = \text{Sym}_{\bar{t}^s}(\gamma_s(\bar{t}^s)G(\bar{t}^s)).
$$

Then the symmetrization can be replaced by the sum over partitions and subsequent symmetrization over each subset:

$$
\text{Sym}_{\bar{t}^s}(\cdot) = \sum_{\bar{t}^s = \{\bar{t}^1, \ldots, \bar{t}^N\}} \text{Sym}_{\bar{t}^1} \text{Sym}_{\bar{t}^2} \ldots \text{Sym}_{\bar{t}^N}(\cdot).
$$

Here the summation is over the partitions of the set $\{\bar{t}^s\}$ into two disjoint subsets $\{\bar{t}^s_1\}$ and $\{\bar{t}^s_2\}$ with cardinalities $\#\bar{t}^s_1 + \#\bar{t}^s_2 = \#\bar{t}^s$, where

$$
\bar{t} = \{\bar{t}^1, \ldots, \bar{t}^N\} \Rightarrow \bar{t}_1 \cup \bar{t}_2
$$

and

$$
\bar{t}_1 = \{\bar{t}^1_1, \ldots, \bar{t}^N_1\}, \quad \bar{t}_2 = \{\bar{t}^1_2, \ldots, \bar{t}^N_2\}.
$$

Using (4.2) and (4.3) and the fact that the normalized product of currents $F(\bar{t})$ is symmetric with respect to permutations in each set of Bethe parameters $\bar{t}_1^s$, $s = 1, \ldots, N$, we can transform (4.1) into a sum over the partitions given by (4.4) and (4.5):

$$
\Delta^{(D)}(F(\bar{t})) = \sum_{\text{part}} \frac{\prod_{s=1}^N \gamma_s(\bar{t}^s_1, \bar{t}^s_2)}{\prod_{s=1}^{N-1} f_{s+1}(\bar{t}^{s+1}_1, \bar{t}^s_1)} F(\bar{t}_1) \prod_{s=1}^N k^+_s(\bar{t}^s_1) k^+_s(\bar{t}^s_1)^{-1} \otimes F(\bar{t}_2). \tag{4.6}
$$

With the help of the Drinfeld coproduct (3.16) for the second current realization of $DY(gl(m|n))$ we can show that the coproduct of the normalized product of currents (3.20) is given by

$$
\Delta^{(D)}(\widehat{F}(\bar{t})) = \sum_{\text{part}} (-)^{\#\bar{t}^s_1 \cdot \#\bar{t}^s_2} \prod_{s=1}^N \gamma_s(\bar{t}^s_1, \bar{t}^s_2) \widehat{F}(\bar{t}_1) \otimes \widehat{F}(\bar{t}_2) \prod_{s=1}^N \hat{k}_s^+ (\bar{t}^s_1) \hat{k}_s^+ (\bar{t}^s_2)^{-1}, \tag{4.7}
$$

where the summation is over the disjoint subsets defined by (4.4) and (4.5).

We can use the formulae (4.6) and (4.7) to establish the coproduct properties of the universal Bethe vectors (3.14) and (3.22). It was proved in [26] that for any elements $\mathcal{F} \in \overline{U}_F$ and $\mathcal{F} \in \overline{U}_F$ the following equations hold:

$$
\Delta(P_f^+(\mathcal{F})) \equiv (P_f^+ \otimes P_f^+)(\Delta^{(D)}(\mathcal{F})) \mod U_F^+ \otimes J,
$$

$$
\Delta(\widehat{P}_f^+(\mathcal{F})) \equiv (\widehat{P}_f^+ \otimes \widehat{P}_f^+)(\widehat{\Delta}^{(D)}(\mathcal{F})) \mod \widehat{U}_F^+ \otimes \widehat{J},
$$

where $J$ and $\widehat{J}$ are ideals in the corresponding subalgebras which annihilate the singular vector $|0\rangle$. A proper definition of these ideals is given in the beginning of the next subsection. Using these equalities and the formulae (4.6) and (4.7), we get that

$$
\mathbb{B}(\bar{t}) = \sum_{\text{part}} \frac{\prod_{s=1}^N \gamma_s(\bar{t}^s_1, \bar{t}^s_2)}{\prod_{s=1}^{N-1} f_{s+1}(\bar{t}^{s+1}_1, \bar{t}^s_1)} \mathbb{B}^{(1)}(\bar{t}_1) \prod_{s=1}^N \lambda^{(1)}_s(\bar{t}^s_1) \mathbb{B}^{(2)}(\bar{t}_2) \prod_{s=1}^N \lambda^{(2)}_s(\bar{t}^s_2). \tag{4.8}
$$
and
\[
\mathbb{B}(\vec{t}) = \sum_{\text{part}} \frac{(-)^{\#\vec{t}^n \cdot \#\vec{t}^n}}{\prod_{s=1}^{N-1} f_{s+1}(\vec{t}_{II}^s, \vec{t}_1^s)} \mathbb{B}^{(1)}(\vec{t}_1) \prod_{s=1}^N \lambda_s^{(1)}(\vec{t}_2^s) \otimes \mathbb{B}^{(2)}(\vec{t}_{II}) \prod_{s=1}^N \lambda_s^{(2)}(\vec{t}_1^s).
\]

Taking (3.2) into account, we conclude that the universal Bethe vectors \( \mathbb{B}(\vec{t}) \) and \( \mathbb{B}(\vec{t}) \) have the same coproduct properties, which indicates that they may coincide. Below we will show that they satisfy the same recurrence relations, thereby proving that they do coincide.

The coproduct formulae (4.1) and (4.6) are very powerful tools for calculating the projection of a product of currents. Indeed, using the fundamental property in (3.10) of the projections \( P_f^\pm \), we get from (4.1) that

\[
\mathcal{F}(\vec{t}) = \sum_{0 \leq s_1 \leq r_1} \cdots \sum_{0 \leq s_N \leq r_N} \prod_{\ell=1}^N \frac{1}{s_{\ell}! (r_{\ell} - s_{\ell})!} \text{Sym}_t \left( Z_t(\vec{t}) P_f^- (\mathcal{F}(\vec{t}')) P_f^+ (\mathcal{F}(\vec{t}'')) \right),
\]

and from (4.6) that
\[
\mathcal{F}(\vec{t}) = \sum_{\vec{t}' \Rightarrow \{\vec{t}_1, \vec{t}_{II}\}} \prod_{s=1}^N \gamma_s(\vec{t}_1, \vec{t}_1') \prod_{s=1}^{N-1} f_{s+1}(\vec{t}_{II}^s, \vec{t}_1^s) P_f^- (\mathcal{F}(\vec{t}_1)) \cdot P_f^+ (\mathcal{F}(\vec{t}_{II})).
\]

This equality and the analogous equality for the product of currents \( \widehat{F}_t(t) \) will be used in §5 to solve the hierarchical relations for the nested Bethe vectors and to obtain explicit formulae for them in terms of the monodromy matrix elements. This will be achieved by an explicit calculation of the projection of the corresponding products of currents, which reduces to a calculation presented in Appendix C.

4.2. Ideals of the Yangian double and presentations of the projections.

To calculate the action of monodromy matrix elements on Bethe vectors, we have to formulate an important auxiliary statement about the action of monodromy matrix elements \( T_{i,j}^+ (z) \) on ‘negative’ projections of composed currents \( P_f^- (F_{k,l}(w)) \) and \( \widehat{P}_f^- (F_{k,l}(w)) \) modulo certain ideals. This can be proved in the same way as used in [27] for the quantum affine algebra \( U_q(\widehat{\mathfrak{g}}(N)) \), and therefore we just sketch it below.

Let \( U_F^\pm \) and \( U_E^\pm \) be the intersections of the standard Borel subalgebras \( U^\pm \) and the current Borel subalgebras \( U_F \) and \( U_E \) used in §3.3. Let \( I \subset DY(\mathfrak{gl}(m|n)) \) be the ideal constructed from the elements of the form \( \mathcal{F} \cdot \mathcal{E} \) such that \( \mathcal{F} \in U_F \), \( \mathcal{E} \in U_E \), and \( \varepsilon(\mathcal{F}) = 0 \). Here and below, \( \varepsilon \) is the counit in the Hopf algebra \( DY(\mathfrak{gl}(m|n)) \). It is clear from the definition (3.9) of the projection \( P_f^+ \) that the whole ideal \( I \) is annihilated by it: \( P_f^+(I) = 0 \). Let \( K \subset DY(\mathfrak{gl}(m|n)) \) be the ideal generated by the elements which contain any combination of the ‘negative’ Cartan currents \( k_i^- (w) \). By the commutation relations in \( DY(\mathfrak{gl}(m|n)) \), \( K \) is indeed an ideal because the ‘negative’ Cartan currents cannot be annihilated by any of the commutation relations in \( DY(\mathfrak{gl}(m|n)) \). Let \( J \subset DY(\mathfrak{gl}(m|n)) \) be the ideal generated by the elements of the form \( \mathcal{F} \cdot \mathcal{E}_+ \) such that \( \mathcal{E}_+ \in U_E^+, \mathcal{F} \in U_F^+, \) and \( \varepsilon(\mathcal{E}_+) = 0 \). By
the definition of this ideal, any element in $J$ annihilates the right vacuum vector: $J|0\rangle = 0$. Below we will use the symbols $\sim_I$, $\sim_K$, and $\sim_J$ to denote equalities in the Yangian double $DY(gl(m|n))$ modulo terms from the corresponding ideals $I$, $K$, and $J$. Similarly, starting from the current Borel subalgebras $\hat{U}_F$ and $\hat{U}_E$, we define the ideals $\hat{I}$, $\hat{K}$, and $\hat{J}$ and the equivalence relations $\sim_{\hat{I}}$, $\sim_{\hat{K}}$, and $\sim_{\hat{J}}$.

Since the off-shell Bethe vectors defined in (3.14) and (3.22) obviously do not belong to the ideals $I$ and $K$ nor the ideals $\hat{I}$ and $\hat{K}$, we can compute the action of the monodromy matrix elements on the Bethe vectors modulo these ideals. Moreover, since the ideals $J$ and $\hat{J}$ annihilate the vacuum vector $|0\rangle$, we can also skip the terms from these ideals when calculating the action of the monodromy matrix on the projections of currents.

Using the commutation relations (2.7) and (2.8) between the Gauss coordinates of the 'positive' and 'negative' monodromy matrices, as well as the relations (A.32) and (A.36) between the 'negative' projections of composed currents and the Gauss coordinates, we can prove the following.

**Proposition 4.1** (see [27]).

\begin{align*}
T_{i,j}^+(z) \cdot P^-_j(F_{k,l}(w)) &\sim_{I,K} -\phi_k c_{[l,k]} \delta_{j,l} g(z,w) T_{i,k}^+(z), & (4.12) \\
T_{i,j}^+(z) \cdot \hat{P}^-_j(\hat{F}_{k,l}(w)) &\sim_{\hat{I},\hat{K}} -\hat{\phi}_l c_{[l,k]} \delta_{i,l} g(z,w) T_{i,j}^+(z), & (4.13)
\end{align*}

where $c_{[l,k]}$ is given by (A.30), and\(^6\)

\begin{align*}
\phi_k &= (-)^{[i]+[j]}[k]+[i][j] & \text{for } k > j, \\
\hat{\phi}_l &= (-)^{1+[i]} & \text{for } l < i. & (4.14)
\end{align*}

**Remark 4.1.** One can extend the values of the indices $k$ and $l$ in (4.14) to the values $k = j$ and $l = i$:

\begin{align*}
\phi_k &= 1 & \text{for } k = j & \text{and} & \hat{\phi}_l &= 1 & \text{for } l = i
\end{align*}

(this extension will be justified later; see Proposition 4.6).

**Sketch of the proof of Proposition 4.1.** The appearance of the Kronecker symbols $\delta_{j,l}$ and $\delta_{i,k}$ in (4.12) and (4.13), respectively, was proved in [27]. Let us give arguments which fix the rest of the terms on the right-hand side of (4.12) and (4.13), including the phases (4.14). To do this we consider the equations (4.12) and (4.13) applied to a right singular vector.

It is clear from the Gauss decompositions (2.14) that

\[ F_{k,l}^-(w)|0\rangle = T_{l,k}^-(w)T_{l,l}^-(w)^{-1}|0\rangle. \]

Then the equation (2.8) can be interpreted as

\begin{align*}
T_{i,j}^+(z)T_{l,k}^-(w)T_{l,l}^-(w)^{-1} &\sim_{I,K} (-)^{[k][i]+[j][j]} g(z,w) T_{i,k}^+(z)T_{l,j}^-(w)T_{l,l}^-(w)^{-1}, & (4.15)
\end{align*}

\(\text{The asymmetry in the symbols } \phi_k \text{ and } \hat{\phi}_l \text{ is related to the asymmetry in the different Gauss decompositions.}\)
and due to the Kronecker symbol $\delta_{j,l}$ on the right-hand side of (4.12) the ‘negative’ monodromy matrix elements on the right-hand side of (4.15) cancel each other. Taking (A.32) into account, we get that $\phi_k = (-)^{[k][l]+[i]i[1]}$.

Similarly, it follows from the Gauss decomposition (2.17) that

$$\hat{F}_{i,l}(w)|0\rangle = T_{i,k}^{-}(w) T_{k,k}^{-1}(w)|0\rangle.$$  

Then the equation (2.7) can be interpreted as

$$T_{i,j}^{+}(z) T_{l,k}^{-}(w) T_{k,k}^{-1}(w) \sim \hat{T}_{i,k}^{-}(-1)^{[i][l]} g(z,w) T_{i,j}^{+}(z) T_{l,k}^{-}(w) T_{k,k}^{-1}(w),$$  

and due to the Kronecker symbol $\delta_{i,k}$ on the right-hand side of (4.13) the ‘negative’ monodromy matrix elements on the right-hand side of (4.16) disappear, leading to $\hat{\phi}_l = (-)^{1+[i]}$. □

We conclude this subsection by formulating the following proposition.

**Proposition 4.2.** The off-shell Bethe vectors given by (3.14) and (3.22) satisfy the same recurrence relations following from the action by the upper triangular monodromy matrix elements $T_{i,j}(z)$, $i \leq j$, on these vectors. This implies that the Bethe vectors coincide:

$$\mathbb{B}(\hat{t}) = \overline{\mathbb{B}}(\hat{t}).$$

The proof of this proposition will be given in the next two subsections, §§ 4.3 and 4.4.

### 4.3. Auxiliary presentations for the projections

To calculate the action of the upper triangular and diagonal monodromy matrix elements on the Bethe vectors (3.14) and (3.22), we have to obtain a special presentation for the projections of the products of simple root total currents. A systematic way to get such a presentation is based on techniques elaborated in [28]. Below we use the results contained in that paper, adapting them to the case under consideration.

**Proposition 4.3.** The following identities hold for $i < j$:

$$P_f^{-}(F_i(t^i) \cdots F_j(t^j)) = \sum_{\ell=0}^{i-j} c_{i,\ell-1}^{-1} \prod_{s=i}^{i+\ell-1} g_{|s+\ell|}(t^{s+1}, t^s) P_f^{-}(F_{i+\ell, j}(t^{i+\ell})) \times P_f^{-}(F_{i+\ell+1}(t^{i+\ell+1}) \cdots F_j(t^j)), \quad (4.17)$$

$$\hat{P}_f^{-}(\hat{F}_j(t^j) \cdots \hat{F}_i(t^i)) = \sum_{\ell=0}^{j-i} c_{j,\ell-1}^{-1} \prod_{s=j-\ell}^{j+1} g_{|s+\ell|}(t^{s+1}, t^s) \hat{P}_f^{-}(\hat{F}_{j+\ell-1}(t^{j-\ell-1}) \cdots \hat{F}_i(t^i)) \times \hat{P}_f^{-}(\hat{F}_{j+\ell}(t^{j+\ell}) \cdots \hat{F}_i(t^i)). \quad (4.18)$$

**Proof.** The two equalities can be proved similarly, using the definitions of the projections. Therefore, we give a detailed proof only for (4.17). We start from the definition

$$P_f^{-}(F_i(t^i) \cdots F_j(t^j)) = P_f^{-}(F_i(t^i)) \cdot P_f^{-}(F_{i+1}(t^{i+1}) \cdots F_j(t^j))$$

$$+ P_f^{-}(F^{(i)}_i(t^i) F_{i+1}(t^{i+1}) F_{i+2}(t^{i+2}) \cdots F_j(t^j)). \quad (4.19)$$

The proof of this proposition will be given in the next two subsections, §§ 4.3 and 4.4.
Using the definition
\[ F_i^{(+)}(t^i) = \int dw \frac{F_i(w)}{t^i - w} \]
and the commutation relation
\[ F_i(u)F_{i+1}(v) = \frac{u - v - c_{i+1}}{u - v} F_{i+1}(v)F_i(u) - \delta(u, v)F_{i+2,i}(v), \]
which is a particular case of the definition of the composed current (A.1) or (A.9), we get that
\[ F_i^{(+)}(t^i)F_{i+1}(t^{i+1}) = f_{i+1}(t^{i+1}, t^i)F_{i+1}(t^{i+1})F_i^{(+)}(t^i; t^{i+1}) + c_{i+1}^{-1} g_{i+1}(t^{i+1}, t^i)F_{i+2,i}(t^{i+1}), \]
where \( F_i^{(+)}(t^i; t^{i+1}) = F_i^{(+)}(t^i) - \frac{c_{i+1}}{t^{i+1} - t^i + c_{i+1}} F_i^{(+)}(t^{i+1}) \). Because of the commutativity of the current \( F_i(t) \) with \( F_{i+2}(t^{i+2}) \cdots F_j(t^j) \), the first term in (4.21) vanishes under the ‘negative’ projection in the second term of (4.19). On the other hand, by the second relation in (A.13),
\[ F_{i+2,i}(t^{i+1}) = -\mathcal{J}_{F_i^{(0)}}(F_{i+1}(t^{i+1})) + c_{i+1} F_{i+1}(t^{i+1})F_i^{(+)}(t^{i+1}), \]
where the operators \( \mathcal{J}_{F_i^{(0)}}(\cdot) \) are called screening operators and are defined by (B.1). The second term on the right-hand side of (4.22) also vanishes under the ‘negative’ projection in the second line of (4.19). Thus, (4.19) turns into
\[ P_f^-(F_i(t^i) \cdots F_j(t^j)) = P_f^-(F_i(t^i)) \cdot P_f^-(F_{i+1}(t^{i+1}) \cdots F_j(t^j)) - c_{i+1}^{-1} g_{i+1}(t^{i+1}, t^i) \mathcal{J}_{F_i^{(0)}}(P_f^-(F_{i+1}(t^{i+1}) \cdots F_j(t^j))). \]

In the second line of (4.23) we obtain the ‘negative’ projection of the product of currents \( F_{i+1}(t^{i+1}) \cdots F_j(t^j) \). Therefore, we can use this equality recursively to get in the first step that
\[ P_f^-(F_i(t^i) \cdots F_j(t^j)) = P_f^-(F_i(t^i)) \cdot P_f^-(F_{i+1}(t^{i+1}) \cdots F_j(t^j)) + c_{i+2}^{-1} g_{i+1}(t^{i+1}, t^i) P_f^-(F_{i+2,i}(t^{i+1}))P_f^-(F_{i+2}(t^{i+2}) \cdots F_j(t^j)) + c_{i+3}^{-1} g_{i+1}(t^{i+1}, t^i) g_{i+2}(t^{i+2}, t^{i+1}) \times \mathcal{J}_{F_i^{(0)}}(\mathcal{J}_{F_{i+1}^{(0)}}(P_f^-(F_{i+2}(t^{i+2}) \cdots F_j(t^j)))). \]

where we have again used (4.22) and the commutativity of the screening operators and the projections (see Appendix B). Continuing this recursion process, we prove (4.17). The equality (4.18) can be proved similarly starting from the commutation relations
\[ \hat{F}_{i+1}(u)\hat{F}_i(v) = \frac{u - v + c_{i+1}}{(u - v)} \hat{F}_i(v)\hat{F}_{i+1}(u) + \delta(u, v)\hat{F}_{i+2,i}(v) \]
and using the first equality in (A.17). \( \square \)
For each simple root index \( i = 1, \ldots, N \) we introduce the following notation for ordered products of currents:

\[
\mathcal{F}_i(t^i) = F_i(t^i_1) \cdots F_i(t^i_{\lambda_i}) \quad \text{and} \quad \widetilde{\mathcal{F}}_i(t^i) = \widetilde{F}_i(t^i_1) \cdots \widetilde{F}_i(t^i_{\lambda_i}).
\]

Using the normal ordering relation (3.10) (in the form (4.10)) and (4.17), we can prove the following statement.

**Proposition 4.4.** The equality

\[
P^+_f (\mathcal{F}_1(t^1) \cdots \mathcal{F}_N(t^N)) = P^+_f (\mathcal{F}_1(t^1) \cdots \mathcal{F}_{N-1}(t^{N-1})) \cdot \mathcal{F}_N(t^N)
\]

\[- \sum_{\ell=1}^{N} \mathbb{C}_{\ell} \text{Sym}_{t^1, \ldots, t^N} \left[ G_{\ell} (t^{\ell-1}, \ldots, t^N) G^{-1} \right] P^{-}_f \left( F_{N+1, \ell} (t^{\ell}_1) \right)
\]

\[\times P^+_f (\mathcal{F}_1(t^1) \cdots \mathcal{F}_{\ell-1}(t^{\ell-1}) \mathcal{F}_\ell(t^\ell_1) \cdots \mathcal{F}_{N-1}(t^{N-1}_1)) \cdot \mathcal{F}_N(t^N_1) \] + \mathbb{W} \quad (4.24)

holds, where for \( 1 \leq \ell \leq N \) the rational functions

\[G_{\ell}(t^{\ell-1}, \ldots, t^N) = f[\ell] (t^{\ell}_1, t^{\ell-1}) \prod_{s=\ell}^{N-1} g_{[s+1]} (t^{s+1}_1, t^s_1) f_{[s+1]} (t^{s+1}_1, t^s_1) \quad (4.25)\]

appear along with the combinatorial factors

\[\mathbb{C}_{\ell} = \prod_{s=\ell}^{N} \frac{1}{(r_s - 1)!} \quad (4.26)\]

In (4.24), \( \mathbb{W} \) denotes terms having the structure \( P^-_f (F_{j_1, i_1} (w_1)) P^-_f (F_{j_2, i_2} (w_2)) \mathcal{F} \) with \( j_1 \geq j_2 \) for some element \( \mathcal{F} \in \mathcal{U}_F \).

In (4.24) we used the shortened notation

\[\overline{t}^\ell_1 = \{t^\ell_1, \ldots, t^\ell_{i-1}, t^\ell_{i+1}, \ldots, t^\ell_{\lambda_\ell}\}, \quad i = 1, \ldots, \lambda_\ell,\]

where the Bethe parameter \( t^\ell_i \) is omitted from the set \( \overline{t}^\ell \), \( \ell = 1, \ldots, N \).

By (4.12), the action of any monodromy matrix element \( T^+_{i,j} (z) \) on the terms \( \mathbb{W} \) belongs to the ideal \( I \), except for the terms proportional to \( \delta_{j_1, i_1} \delta_{j_2, i_2} \). These terms are irrelevant in view of the condition \( j_1 \geq j_2 > i_2 \).

**Proof.** It was proved in [28] that the projection

\[P^+_f (\mathcal{F}_1(t^1) \cdots \mathcal{F}_N(t^N))\]

can be represented in the form\(^7\)

\[P^+_f (\mathcal{F}_1 \cdots \mathcal{F}_N) = \sum \mathcal{P} (P^-_f (F_{N+1, \ell})) \cdot P^+_f (\mathcal{F}_1 \cdots \mathcal{F}_{N-1}) \cdot \mathcal{F}_N \quad (4.27)\]

where \( \mathcal{P} (P^-_f (F_{N+1, \ell})) \) is a certain polynomial with rational coefficients in the ‘negative’ projections of the composed currents \( F_{N+1, \ell}, \ell = 1, \ldots, N \), and the \( \mathcal{F}_\ell \)

\(^7\)In fact, this was proved in [28] for the case of the currents \( \widetilde{F}_\ell \), but it can easily be repeated for the currents \( F_\ell \), leading to (4.27).
are the products of currents corresponding to the simple roots \( \ell \). For brevity we did not write the arguments of the currents in (4.27).

It was shown in [28] that only ‘negative’ projections of currents \( P_f^{-} (F_{N+1, \ell}(t)) \) appear on the right-hand side of (4.27). The other ‘negative’ projections of currents \( P_f^{-} (F_{\ell', \ell}(t)) \) with \( N \geq \ell' > \ell \) do not appear. The main reason for such a phenomenon is the factorization of projections of products of currents. We will demonstrate this phenomenon below in the simplest non-trivial case of \( N = 2 \), using the normal ordering relation (4.10).

Moreover, by (4.12) it is enough to keep in (4.27) only the first-order polynomials in the ‘negative’ projections of composed currents. Indeed, after the action of the monodromy matrix element \( T_{i,j}^{+}(z) \) on a product of two ‘negative’ projections of composed currents \( P_f^{-} (F_{N+1, \ell_1}(t)) \cdot P_f^{-} (F_{N+1, \ell_2}(t)) \), the terms which are not in the ideals \( I \) and \( K \) are proportional to \( \delta_{j,\ell_1} \delta_{N+1,\ell_2} \), and they vanish because \( \ell_2 < N+1 \).

Let us show how relations of the type (4.27) arise in the simple case of \( m = 2 \) and \( n = 1 \). We rename the sets of parameters as \( \bar{t}_1 \equiv \bar{u} \) and \( \bar{t}_2 \equiv \bar{v} \) with cardinalities \( \# \bar{u} = a \) and \( \# \bar{v} = b \) to simplify the formulae below. In this case the formula (4.10) can be rewritten as

\[
P_f^+ (F_1(u_1) \cdots F_1(u_a) \cdot F_2(v_1) \cdots F_2(v_b)) = F_1(u_1) \cdots F_1(u_a) \cdot F_2(v_1) \cdots F_2(v_b)
\]

\[
- \text{Sym}_{\bar{u}} \frac{1}{(a-1)!} P_f^{-} (F_1(u_1)) P_f^+ (F_1(u_2) \cdots F_1(u_a) \cdot F_2(v_1) \cdots F_2(v_b))
\]

\[
- \text{Sym}_{\bar{u}} \frac{f(v_1, \bar{u})}{(b-1)!} P_f^{-} (F_2(v_1)) P_f^+ (F_1(u_1) \cdots F_1(u_a) \cdot F_2(v_2) \cdots F_2(v_b))
\]

\[
- \text{Sym}_{\bar{u}, \bar{v}} \frac{f(v_1, \bar{u}_1)}{(a-1)! (b-1)!} P_f^{-} (F_1(u_1) F_2(v_1))
\]

\[
\times P_f^+ (F_1(u_2) \cdots F_1(u_a) \cdot F_2(v_2) \cdots F_2(v_b)) + \mathbb{W}. \tag{4.28}
\]

We keep the double symmetrized term in (4.28) because it is the source of the ‘negative’ projection of composed currents \( P_f^{-} (F_{3,1}(v)) \) (see (4.29) below), while the quadratic terms from \( P_f^{-} (F_1(u_1) F_2(v_1)) \) disappear in the next step of the recursion.

Applying (4.28) recursively, we can replace the ‘positive’ projections by the corresponding products of total currents. Using the equality

\[
P_f^{-} (F_1(u) F_2(v)) = P_f^{-} (F_1(u)) P_f^{-} (F_2(v)) + c^{-1} g(v, u) P_f^{-} (F_{3,1}(v)), \tag{4.29}
\]

which is a direct consequence of (4.20), we obtain instead of (4.28) the equality of formal series (recall that \( F_{i+1,i}(t) \equiv F_i(t) \))

\[
P_f^+ (F_1(u_1) \cdots F_1(u_a) \cdot F_2(v_1) \cdots F_2(v_b)) = F_1(u_1) \cdots F_1(u_a) \cdot F_2(v_1) \cdots F_2(v_b)
\]

\[
- \text{Sym}_{\bar{u}} \frac{1}{(a-1)!} P_f^{-} (F_{2,1}(u_1)) F_1(u_2) \cdots F_1(u_a) \cdot F_2(v_1) \cdots F_2(v_b)
\]

\[
- \text{Sym}_{\bar{u}} \frac{f(v_1, \bar{u})}{(b-1)!} P_f^{-} (F_{3,2}(v_1)) F_1(u_1) \cdots F_1(u_a) \cdot F_2(v_2) \cdots F_2(v_b)
\]

\[
- \text{Sym}_{\bar{u}, \bar{v}} \frac{c^{-1} g(v_1, u_1) f(v_1, \bar{u}_1)}{(a-1)! (b-1)!}
\]

\[
\times P_f^{-} (F_{3,1}(v_1)) F_1(u_2) \cdots F_1(u_a) \cdot F_2(v_2) \cdots F_2(v_b) + \mathbb{W},
\]
where the terms denoted by $\mathcal{W}$ again belong to the ideal $I$ after the action of any monodromy matrix element. Finally, using the normal ordering rule (4.10) for the product of currents $T_1$, we can replace these products by their ‘positive’ projections to obtain
\[
P_f^+ (F_1(u_1) \cdots F_1(u_a) \cdot F_2(v_1) \cdots F_2(v_b))
= P_f^+ (F_1(u_1) \cdots F_1(u_a)) \cdot F_2(v_1) \cdots F_2(v_b)
- \sum_{\nu} \text{Sym}_{\nu} \frac{f(v_1, \nu)}{(b - 1)!} P_f^- (F_{3,2}(v_1)) P_f^+ (F_1(u_1) \cdots F_1(u_a)) \cdot F_2(v_2) \cdots F_2(v_b)
- \sum_{\nu} \text{Sym}_{\nu, \nu} \frac{c^{-1} (v_1, u_1) f(v_1, \nu_1)}{(a - 1)! (b - 1)!} P_f^- (F_{3,1}(v_1)) \times P_f^+ (F_1(u_2) \cdots F_1(u_a)) \cdot F_2(v_2) \cdots F_2(v_b) + \mathcal{W}.
\] (4.30)

We see that the terms containing the ‘negative’ projection of a current $P_f^- (F_{2,1}(u_1))$ disappear from the final formula (4.30).

Now we prove the statement of Proposition 4.4 in the general case, using the normal ordering relation (4.10). Taking into account the arguments above, we write
\[
P_f^+ (T_1(\bar{t}^1) \cdots T_{N-1}(\bar{t}^{N-1}) T_N(\bar{t}^N)) = T_1(\bar{t}^1) \cdots T_{N-1}(\bar{t}^{N-1}) T_N(\bar{t}^N)
- \sum_{\ell=1}^N \mathbb{C} \text{Sym}_{\bar{t}^1, \ldots, \bar{t}^N} \left[ f_{[t]}(t_1^\ell, \bar{t}^{\ell-1}) \prod_{s=\ell}^{N-1} f_{[t_s+1]}(t_1^{s+1}, \bar{t}^s) P_f^- (F_\ell(t_1^\ell) \cdots F_N(t_1^N)) \times P_f^+ (T_1(\bar{t}^1) \cdots T_{\ell-1}(\bar{t}^{\ell-1}) \cdot T_\ell(\bar{t}_1^\ell) \cdots T_N(\bar{t}_1^N)) \right] + \mathcal{W},
\] (4.31)

where we keep only the terms containing $P_f^- (F_\ell(t_1^\ell) \cdots F_N(t_1^N))$ as the source of the ‘negative’ projection of a composed current $P_f^- (F_{N+1,\ell}(t_1^N))$, and $\mathcal{W}$ denotes terms which give elements of the ideal $I$ after the action of any monodromy matrix element $T_{i,j}^+ (z)$. Using (4.17), we can replace (4.31) by
\[
P_f^+ (T_1(\bar{t}^1) \cdots T_{N-1}(\bar{t}^{N-1}) T_N(\bar{t}^N)) = T_1(\bar{t}^1) \cdots T_{N-1}(\bar{t}^{N-1}) T_N(\bar{t}^N)
- \sum_{\ell=1}^N \mathbb{C} \text{Sym}_{\bar{t}^1, \ldots, \bar{t}^N} \left[ G_\ell(\bar{t}^{\ell-1}, \ldots, \bar{t}^N) c_{[t, N+1]}^{-1} P_f^- (F_{N+1,\ell}(t_1^N)) \times P_f^+ (T_1(\bar{t}^1) \cdots T_{\ell-1}(\bar{t}^{\ell-1}) \cdot T_\ell(\bar{t}_1^\ell) \cdots T_N(\bar{t}_1^N)) \right] + \mathcal{W}.
\] (4.32)

Now we can use a result from [28] asserting that only ‘negative’ projections of composed currents $P_f^- (F_{N+1,\ell}(t_1^N))$, $\ell = 1, \ldots, N$, appear on the right-hand side of (4.27). This allows us to replace the first term $T_1(\bar{t}^1) \cdots T_{N-1}(\bar{t}^{N-1}) T_N(\bar{t}^N)$ on the right-hand side of (4.32) by
\[
P_f^+ (T_1(\bar{t}^1) \cdots T_{N-1}(\bar{t}^{N-1}) T_N(\bar{t}^N)).
\]

Similarly, the ‘positive’ projections of products of composed currents
\[
P_f^+ (T_1(\bar{t}^1) \cdots T_{\ell-1}(\bar{t}^{\ell-1}) \cdot T_\ell(\bar{t}_1^\ell) \cdots T_N(\bar{t}_1^N))
\]
under the sum sign in (4.32) can be replaced by

$$P_f^+ (\mathcal{F}_1(t^1) \cdots \mathcal{F}_{\ell-1}(t^{\ell-1}) \cdot \mathcal{F}_\ell(t^\ell) \cdots \mathcal{F}_{N-1}(t^{N-1}) \cdot \mathcal{F}_N(t^N)),$$

and this replacement changes only the structure of the elements in \( W \). This finishes the proof of Proposition 4.4. □

Similarly, using (3.19) and (4.18), we can prove the following statement.

**Proposition 4.5.** The equality

$$\hat{P}_f^+ (\mathcal{F}_N(t^N) \cdots \mathcal{F}_1(t^1)) = P_f^+ (\mathcal{F}_N(t^N) \cdots \mathcal{F}_2(t^2)) \cdot \mathcal{F}_1(t^1)$$

$$- \sum_{\ell=1}^{N} \text{Sym}_{t^\ell, \ldots, t^1} [\mathcal{C}_\ell \hat{G}_\ell(t^1, \ldots, t^{\ell+1}) c_{[1,\ell+1]}^{-1} \hat{P}_f^- (\mathcal{F}_{\ell+1,1}(t^1_{\ell}))$$

$$\times \hat{P}_f^+ (\mathcal{F}_N(t^N) \cdots \mathcal{F}_{\ell+1}(t^{\ell+1}) \mathcal{F}_\ell(t^\ell_{\ell}) \cdots \mathcal{F}_2(t^2_{\ell}) \cdot \mathcal{F}_1(t^1_{\ell})) + \hat{W} \quad (4.33)$$

holds, where for \( 1 \leq \ell \leq N \) the rational functions

$$\hat{G}_\ell(t^1, \ldots, t^{\ell+1}) = f_{[\ell+1]}(t^{\ell+1}, t^{\ell}_{\ell}) \prod_{s=1}^{\ell-1} g_{[s+1]}(t^{s+1}_{s+1}, t^s_{s}) f_{[s+1]}(t^{s+1}_{s+1}, t^s_{s}) \quad (4.34)$$

appear along with the combinatorial factors

$$\mathcal{C}_\ell = \prod_{s=1}^{\ell} \frac{1}{(r_s - 1)!}. \quad (4.35)$$

The symbol \( \hat{W} \) denotes terms with the structure \( P_f^- (\hat{F}_{j_1,1}(w_1)) P_f^- (\hat{F}_{j_2,1}(w_2)) \).

Again, the action of any monodromy matrix element \( T_{i,j}^+(z) \) on \( \hat{W} \) belongs to the ideal \( \hat{I} \) in view of (4.13). The terms not belonging to this ideal are proportional to \( \delta_{i,j}, \delta_{1,j_2} \), and they vanish due to the condition \( 1 < j_2 \).

**4.4. Action of the monodromy matrix element \( T_{i,j}^+(z) \).** Let us apply the monodromy matrix element \( T_{i,j}^+(z) \) from the left to (4.24) and (4.33). As one can easily verify, the structure of the action formulae differs significantly in the cases \( i \leq j \) and \( i > j \).

The action of the monodromy matrix elements \( T_{i,j}(z) \) for \( i < j \) leads to recursion relations which relate Bethe vectors depending on fewer Bethe parameters to Bethe vectors depending on more of these parameters. If we prove that the action formulae for \( i < j \) are the same for \( \mathcal{B}(t) \) and \( \hat{\mathcal{B}}(\bar{t}) \), then this will mean that these vectors satisfy the same recurrence relations, and thus \( \mathcal{B}(t) \) and \( \hat{\mathcal{B}}(\bar{t}) \) coincide.

The action formulae for the diagonal monodromy matrix elements \( T_{i,i}(z) \) lead to the Bethe equations. They prove that the Bethe vectors become eigenvectors of the transfer matrix if the Bethe equations are satisfied.

Finally, the action formulae for the monodromy matrix elements \( T_{i,j}(z) \) with \( i > j \) are necessary for calculating the scalar products of Bethe vectors. This last problem is beyond the scope of the present paper, and we will consider the general action
formule in this case in a separate publication. From now on, we restrict ourselves to the action of the monodromy matrix elements $T_{i,j}(z)$ with $i \leq j$.

We introduce the shortened notation

$$\mathcal{F}_\ell \equiv \mathcal{F}(i^\ell), \quad \mathcal{F}'_\ell \equiv \mathcal{F}(i^\ell_1), \quad \text{and} \quad \mathcal{F}''_\ell \equiv \mathcal{F}(i^\ell_{rec})$$

and the analogous notation

$$\mathcal{F}_\ell \equiv \mathcal{F}(i^\ell), \quad \mathcal{F}'_\ell \equiv \mathcal{F}(i^\ell_1) \quad \text{and} \quad \mathcal{F}''_\ell \equiv \mathcal{F}(i^\ell_{rec}),$$

where $i^\ell_1 = \{i^\ell\} \setminus \{i^\ell_1\}$ and $i^\ell_{rec} = \{i^\ell\} \setminus \{i^\ell_{rec}\}$ are the sets of Bethe parameters of the same type with either the first or the last element omitted.

For $1 \leq \ell \leq N$ we introduce the two sets of rational functions

$$\mathcal{G}_\ell^p(\bar{t}^{\ell-1}, \ldots, \bar{t}^q) = f_{[q]}(t^\ell_1, \bar{t}^{\ell-1})$$

$$\times \prod_{s=\ell}^{q-1} g_{[s+1]}(t^{s+1}_1, t^s_1)f_{[s+1]}(t^{s+1}_1, \bar{t}^s_1), \quad \ell \leq q \leq N,$$

$$\mathcal{G}_\ell^p(\bar{t}^{p_1}, \ldots, \bar{t}^{p+1}) = f_{[p+1]}(\bar{t}^{p+1}, t^\ell_{rec})$$

$$\times \prod_{s=p}^{\ell-1} g_{[s+1]}(t^{s+1}_{rec}, t^s_{rec})f_{[s+1]}(t^{s+1}_{rec}, \bar{t}^s_{rec}), \quad 1 \leq p \leq \ell.$$  \hfill (4.36)

The rational functions in (4.25) and (4.34) are particular cases of the functions in (4.36):

$$\mathcal{G}_\ell(\bar{t}) \equiv \mathcal{G}^N_\ell(\bar{t}) \quad \text{and} \quad \mathcal{G}(\bar{t}) \equiv \mathcal{G}^1_\ell(\bar{t}).$$

For $q = j + 1, \ldots, N + 1$ and $p = 1, \ldots, i - 1$ we also define the rational functions

$$\mathcal{Z}^q_j(z; \bar{t}) = g(z, t^{q-1}_1)\mathcal{G}^{q-1}_j(\bar{t}) \quad \text{and} \quad \mathcal{Z}^p_i(z; \bar{t}) = g(z, t^p_{rec})\mathcal{G}^{p-1}_i(\bar{t}).$$  \hfill (4.37)

We extend these definitions to $q = j$ and $p = i$ by setting $\mathcal{Z}^j_j(z; \bar{t}) = \mathcal{Z}^i_i(z; \bar{t}) \equiv 1$. Finally, let

$$C^{\ell'}_{\ell} = \prod_{s=\ell}^{\ell' - 1} \frac{1}{(r_s - 1)!}.$$ \hfill (4.38)

Then the combinatorial factors given by (4.26) and (4.35) are

$$C_\ell \equiv C^N_\ell \quad \text{and} \quad \widehat{C}_\ell \equiv C^1_\ell.$$

**Proposition 4.6.** The following equivalence relations hold:

$$T^+_{i,j}(z) \cdot P^+_f(\mathcal{F}_1 \cdots \mathcal{F}_N) \sim_{I,K} \sum_{q=j}^{N+1} \text{Sym}_{\bar{t}^{q-1}, \bar{t}^{q-1}} \left[ \phi_q C^{q-1}_j \mathcal{Z}^q_j(z; \bar{t}) \right]$$

$$\times T^+_{i,q}(z) \cdot \mathcal{F}_1 \cdots \mathcal{F}_{j-1} \mathcal{F}'_{j} \cdots \mathcal{F}_{q-1} \cdot \mathcal{F}_q \cdots \mathcal{F}_N,$$  \hfill (4.39)

$$T^+_p(z) \cdot \mathcal{F}_N \cdots \widehat{\mathcal{F}_1} \sim_{I,R} \sum_{p=1}^i \text{Sym}_{\bar{t}^{p-1}, \bar{t}^{p-1}} \left[ \phi_p C^p_{i-1} \mathcal{Z}^p_i(z; \bar{t}) \right]$$

$$\times T^+_p(z) \cdot \mathcal{F}_N \cdots \widehat{\mathcal{F}_1} \widehat{\mathcal{F}}'_{i-1} \cdots \widehat{\mathcal{F}}'_p \cdot \mathcal{F}_p \cdots \widehat{\mathcal{F}_1},$$ \hfill (4.39)
where the sign factors $\phi_q$ for $q = j+1, \ldots, N+1$ and $\hat{\phi}_p$ for $p = 1, \ldots, i-1$ are given by (4.14), and $\hat{\phi}_i = \hat{\phi}_{\bar{i}} \equiv 1$.

Proof. We begin the proof with the relation (4.38). Assume that $j = N + 1$. Then by (4.12), under the action of $T_{i,N+1}^+(z)$ the sum over $\ell$ on the right-hand side of (4.24) and also the terms $\mathbb{W}$ give elements of the ideal $I$. As a result,

$$T_{i,N+1}^+(z) \cdot P_f^+ (\mathcal{F}_1 (\bar{t}^1) \cdots \mathcal{F}_N (\bar{t}^N))$$

$$\sim_{I,K} T_{i,N+1}^+(z) \cdot P_f^+ (\mathcal{F}_1 (\bar{t}^1) \cdots \mathcal{F}_{N-1} (\bar{t}^{N-1})) \cdot \mathcal{F}_N (\bar{t}^N).$$

(4.40)

Again using (4.24) for the projection $P_f^+ (\mathcal{F}_1 (\bar{t}^1) \cdots \mathcal{F}_{N-1} (\bar{t}^{N-1}))$, we can continue this process and get that

$$T_{i,N+1}^+(z) \cdot P_f^+ (\mathcal{F}_1 (\bar{t}^1) \cdots \mathcal{F}_N (\bar{t}^N)) \sim_{I,K} T_{i,N+1}^+(z) \cdot \mathcal{F}_1 (\bar{t}^1) \cdots \mathcal{F}_N (\bar{t}^N).$$

(4.41)

Assume now that $j \leq N$. Then by (4.12), besides the first term as in (4.40) there will be a contribution of the term corresponding to $\ell = j$ in the sum on the right-hand side of (4.24), so that

$$T_{i,j}^+(z) \cdot P_f^+ (\mathcal{F}_1 \cdots \mathcal{F}_N) \sim_{I,K} T_{i,j}^+(z) \cdot P_f^+ (\mathcal{F}_1 \cdots \mathcal{F}_{N-1}) \cdot \mathcal{F}_N$$

$$+ \operatorname{Sym}_{\bar{t}^j, \ldots, \bar{t}^N} [\phi_{N+1} g(z, t_1^N) \mathcal{C}_{\bar{t}^j} \mathcal{G}_{\bar{t}^j} (\bar{t}) \times T_{i,N+1}^+(z) \cdot \mathcal{F}_1 \cdots \mathcal{F}_{j-1} \mathcal{F}_j^j \cdots \mathcal{F}_{N-1}^j \cdot \mathcal{F}_N].$$

(4.42)

In view of (4.41) and (4.12) we can omit the projection operator $P_f^+$ applied to the product of currents $\mathcal{F}_1 \cdots \mathcal{F}_{j-1} \mathcal{F}_j^j \cdots \mathcal{F}_{N-1}^j$.

We leave the second term on the right-hand side of (4.42) as it is and consider the first term. In this term we have the projection $P_f^+ (\mathcal{F}_1 \cdots \mathcal{F}_{N-1})$ and we can again use the presentation (4.24) for a product of currents in the smaller-rank algebra $\mathfrak{gl}(m|n-1)$. As before, the only contribution comes from the regular term and the one term with $\ell = j$ in the sum over $\ell$. We obtain

$$T_{i,j}^+(z) \cdot P_f^+ (\mathcal{F}_1 \cdots \mathcal{F}_N) \sim_{I,K} T_{i,j}^+(z) \cdot P_f^+ (\mathcal{F}_1 \cdots \mathcal{F}_{N-2}) \cdot \mathcal{F}_{N-1} \mathcal{F}_N$$

$$+ \operatorname{Sym}_{\bar{t}^j, \ldots, \bar{t}^{N-1}} [\phi_{N+1} g(z, t_1^{N-1}) \mathcal{C}_{\bar{t}^j} \mathcal{G}_{\bar{t}^j} (\bar{t}) \times T_{i,N+1}^+(z) \cdot \mathcal{F}_1 \cdots \mathcal{F}_{j-1} \mathcal{F}_j^j \cdots \mathcal{F}_{N-1}^j \cdot \mathcal{F}_N]$$

$$+ \operatorname{Sym}_{\bar{t}^j, \ldots, \bar{t}^{N-1}} [\phi_{N+1} g(z, t_1^{N-1}) \mathcal{C}_{\bar{t}^j} \mathcal{G}_{\bar{t}^j} (\bar{t}) \mathcal{T}_{i,N+1}^+(z) \cdot \mathcal{F}_1 \cdots \mathcal{F}_{j-1} \mathcal{F}_j^j \cdots \mathcal{F}_{N-1}^j \cdot \mathcal{F}_N].$$

Continuing this process, we conclude that the action of the monodromy matrix element $T_{i,j}^+(z)$ on the projection $P_f^+ (\mathcal{F}_1 \cdots \mathcal{F}_N)$ modulo elements in the ideals $I$ and $K$ is given by

$$T_{i,j}^+(z) \cdot P_f^+ (\mathcal{F}_1 \cdots \mathcal{F}_N) \sim_{I,K} T_{i,j}^+(z) \cdot \mathcal{F}_1 \cdots \mathcal{F}_N$$

$$+ \sum_{q = j+1}^{N+1} \operatorname{Sym}_{\bar{t}^j, \ldots, \bar{t}^{q-1}} [\phi_{q} g(z, t_1^{q-1}) \mathcal{C}_{\bar{t}^j} \mathcal{G}_{\bar{t}^j} (\bar{t}) \times T_{i,q}^+(z) \cdot \mathcal{F}_1 \cdots \mathcal{F}_{j-1} \mathcal{F}_j^j \cdots \mathcal{F}_{q-1} \mathcal{F}_q \cdots \mathcal{F}_N].$$

(4.43)
Using the relation (4.33) and arguments similar to those above, we get that

\[
T_{i,j}^+(z) \cdot \hat{P}_f^+ (\hat{\mathcal{F}}_N \cdots \hat{\mathcal{F}}_1) \sim \hat{T}_{i,j}^R \cdot \hat{T}_{i,j}^+(z) \cdot \hat{\mathcal{F}}_N \cdots \hat{\mathcal{F}}_1 \\
+ \sum_{p=1}^{i-1} \text{Sym}_{e_p, \ldots, e_{i-1}} [\phi_p g(z, t_{r_p}) \hat{C}^p_{i-1} \hat{G}^p_{i-1} (t)] \\
\times T_{p,j}^+(z) \cdot \hat{\mathcal{F}}_N \cdots \hat{\mathcal{F}}_i \hat{\mathcal{F}}''_{i-1} \cdots \hat{\mathcal{F}}''_1 \cdots \hat{\mathcal{F}}_p \cdots \hat{\mathcal{F}}_1. \tag{4.44}
\]

With the notation (4.37) the formulae (4.43) and (4.44) are equivalent to the assertion of Proposition 4.6. \(\square\)

The next step is to use the explicit representations for the monodromy matrix element \(T_{i,j}^+(z)\) in terms of the Gauss coordinates (2.14),

\[
T_{i,q}^+(z) = \sum_{1 \leq p \leq i} F^+_{q,p}(z) k^+_p(z) E^+_{p,i}(z), \tag{4.45}
\]

and in terms of the ‘hatted’ Gauss coordinates (2.17),

\[
T_{p,j}^+(z) = \sum_{j \leq q \leq N+1} (-)^{([q]+[p])([q]+[j])} \hat{F}^+_{q,p}(z) \hat{k}^+_q(z) \hat{E}^+_{j,q}(z), \tag{4.46}
\]

where we have formally set \(F^+_{i,i}(z) = \hat{F}^+_{j,j}(z) = E^+_{i,i}(z) = \hat{E}^+_{j,j}(z) \equiv 1\). These representations allow us to move the Gauss coordinates \(E^+_{p,i}(z)\) and \(E^+_{j,q}(z)\) through the corresponding products of currents.

As we will demonstrate below, for \(i \leq j\) these permutations transform the product of currents in (4.38) into the product

\[
\mathcal{F}_1 \cdots \mathcal{F}_{p-1} \cdot \mathcal{F}_p'' \cdots \mathcal{F}_{i-1} \cdot \mathcal{F}_i \cdots \mathcal{F}_{j-1} \cdot \mathcal{F}_j' \cdots \mathcal{F}'_{q-1} \cdot \mathcal{F}_q' \cdots \mathcal{F}_N, \tag{4.47}
\]

and the product of currents in (4.39) into the product

\[
\hat{\mathcal{F}}_N \cdots \hat{\mathcal{F}}_q' \cdots \hat{\mathcal{F}}'_{i-1} \cdot \hat{\mathcal{F}}_i \cdots \hat{\mathcal{F}}_{j-1} \cdots \hat{\mathcal{F}}_j' \cdots \hat{\mathcal{F}}''_{i-1} \cdots \hat{\mathcal{F}}''_p \cdots \hat{\mathcal{F}}_{p-1} \cdots \hat{\mathcal{F}}_1 \tag{4.48}
\]

for \(p = 1, \ldots, i\) and \(q = j, \ldots, N + 1\).

According to (A.29) and (A.35), the Gauss coordinates \(F^+_{q,p}(z)\) and \(\hat{F}^+_{q,p}(z)\) can be replaced by the total composed currents \(F_{q,p}(z)\) and \(\hat{F}_{q,p}(z)\) modulo terms in the ideal \(I\). Then by (A.5) and (A.7) the products of currents

\[
\mathcal{F}_p'' \cdots \mathcal{F}_{i-1}'' \cdot \mathcal{F}_i \cdots \mathcal{F}_{j-1}'' \cdot \mathcal{F}_j' \cdots \mathcal{F}'_{q-1}
\]

and

\[
\hat{\mathcal{F}}_{q-1}'' \cdots \hat{\mathcal{F}}_j'' \cdots \hat{\mathcal{F}}_i \cdots \hat{\mathcal{F}}_{i-1}'' \cdots \hat{\mathcal{F}}''_p \cdot \hat{\mathcal{F}}_{p-1} \cdots \hat{\mathcal{F}}_1 \tag{4.49}
\]

in (4.47) and (4.48) will be extended by the corresponding simple root currents depending on the auxiliary parameter \(z\).

This observation shows that the action of the monodromy matrix element \(T_{i,j}^+(z)\) on the projections of currents \(P^+_f(\mathcal{F}_1 \cdots \mathcal{F}_N)\) and \(\hat{P}^+_f(\hat{\mathcal{F}}_N \cdots \hat{\mathcal{F}}_1)\) have a similar
structure. This is the first sign that the recursion relations for the Bethe vectors (3.14) and (3.22) coincide.

Let us be more precise. In view of (A.37) the Gauss coordinate $E_{p+1}^+(z)$ commutes with all the products of currents $\mathcal{F}_q \cdots \mathcal{F}_{j-1}$ except $\mathcal{F}_p \cdots \mathcal{F}_{i-1}$. This is because by (A.37) the Gauss coordinate $E_{p+1}^+(z)$ is constructed from modes of the currents $E_p(z), E_{p+1}(z), \ldots, E_{i-1}(z)$. From the commutation relations (2.28) for the simple root total currents we obtain the commutation relations of the simple root Gauss coordinates,

$$[E_{i,i+1}^+(v), F_{i+1,i}^+(u)] = \frac{c_{i+1}}{v-u} (k_{i+1}^+(v)k_i^+(v)^{-1} - k_{i+1}^+(u)k_i^+(u)^{-1}),$$

$$[E_{i,i+1}^-(v), F_{i+1,i}^-(u)] = \frac{c_{i+1}}{v-u} (k_{i+1}^+(v)k_i^+(v)^{-1} - k_{i+1}^-(u)k_i^-(u)^{-1}),$$

which also follow from (2.8). From this we conclude that

$$[E_{p,p+1}^+(z), F_p(t)] \sim K g_{[p+1]}(t, z)\psi_p^+(t),$$

where

$$\psi_p^+(t) = k_{p+1}^+(t)k_p^+(t)^{-1}.$$  

We recall that $[\cdot, \cdot]$ is the graded commutator defined in (2.28). Using this commutation relation, the commutation relations of the Cartan currents with the total currents $F_p(t)$, and the definition of deformed symmetrization (3.3), we have

$$[E_{p,p+1}^+(z), \mathcal{F}_p(\bar{t}^p)] \sim K \frac{(-)^{(r_p-1)}\delta_{p,m}}{(r_p-1)!} \text{ASym}_{\bar{t}^p} [g_{[p+1]}(t_{r_p}^p, z)\mathcal{F}_p(\bar{t}^p)|\psi_p^+(t_{r_p}^p)|]. \quad (4.50)$$

Let us explain the appearance of the phase factor $(-)^{(r_m-1)}$ in this formula for $p = m$. Using the definition of the graded commutator in (2.28), the commutativity $\psi_m^+(t)F_m(t') = F_m(t')\psi_m^+(t)$, and the anticommutativity of the currents $F_m(t)$, we conclude that

$$[E_{m,m+1}^+(z), \mathcal{F}_m(\bar{t}^m)] \sim K \frac{(-)^{r_m-1}}{(r_m-1)!} \text{ASym}_{\bar{t}^m} (g(z, t_{r_m}^m)F_m(t_{1}^m) \cdots F_m(t_{r_m}^m)),$$

where the symbol $\text{ASym}_{\bar{t}^m} (\cdot)$ stands for antisymmetrization over the set of variables $\bar{t}^m$. It coincides with the deformed symmetrization $\text{Sym}_{\bar{t}^m} (\cdot)$ (see (3.3)) over the same set.

Within the product of screening operators $\mathcal{J}_{E_{i-1}}^{(0)} \cdots \mathcal{J}_{E_{p+1}}^{(0)}$ in the formula (A.37) for the Gauss coordinate $E_{p+1}^+(z)$, only the screening operator $\mathcal{J}_{E_{p+1}}^{(0)}$ does not commute with the Cartan current $k_{p+1}^+(t_{r_p}^p)$:

$$\mathcal{J}_{E_{p+1}}^{(0)} (k_{p+1}^+(t_{r_p}^p)) = -c_{[p+1]}k_{p+1}^+(t_{r_p}^p)E_{p+1,p+2}^+(t_{r_p}^p),$$
which can be obtained from the commutation relation (2.23). Again using (A.37), we find that
\[
[E_{p,i}^+(z), \mathcal{F}_p(p)] \sim K \frac{(-)^{(r_p-1)\delta_{p,m}}}{(r_p-1)!} \text{Sym}_{i,p}[g_{[p+1]}(t_p^p, z), \mathcal{F}_p(p_p^p)\psi_p^+(t_p^p)E_{p+1,i}^+(t_p^p)].
\]

In view of the result
\[
E_{p,i}^+(z) \cdot \mathcal{F}_{p+1} \cdots \mathcal{F}_{i-1} \sim J 0
\]
we can represent (4.51) as an action of the Gauss coordinate $E_{p,i}^+(z)$ on the product of currents $\mathcal{F}_p \cdots \mathcal{F}_{i-1}$ modulo elements in the ideals $K$ and $J$
\[
E_{p,i}^+(z) \cdot \mathcal{F}_p(p) \cdots \mathcal{F}_{i-1}(\bar{t}) \sim_K J \frac{(-)^{(r_p-1)\delta_{p,m}}}{(r_p-1)!} \text{Sym}_{i,p}[g_{[p+1]}(t_p^p, z), \mathcal{F}_p(p_p^p)\psi_p^+(t_p^p)
\times E_{p+1,i}^+(t_p^p) \cdot \mathcal{F}_{p+1}(\bar{t}^{p+1}) \cdots \mathcal{F}_{i}(\bar{t})].
\]
In the last line of (4.52) we can use (4.51) again, and by repeating the calculations finally get that
\[
E_{p,i}^+(z) \cdot \mathcal{F}_p(p) \cdots \mathcal{F}_{i-1}(\bar{t}^{i-1}) \cdot \mathcal{F}_i(\bar{t}) \sim_K J \epsilon_p \prod_{s=p}^{i-1} (-)^{(r_s-1)\delta_{s,m}}
\times \left[ C_{p+1}^{q-1} \widehat{Z}_q^q(z; \bar{t}) \mathcal{F}_p(p_p^p) \cdots \mathcal{F}_{i-1}(\bar{t}^{i-1}) \mathcal{F}_i(\bar{t}) \right.
\times \left. \prod_{s=p}^{i-1} k_s^{+(t_s^p)} k_s^{+(t_s^1)} \right],
\]
where $\epsilon_p$ is the sign factor
\[
\epsilon_i = 1 \quad \text{and} \quad \epsilon_p = (-)^{1+[i]} \quad \text{for} \quad p = 1, 2, \ldots, i-1.
\]
We recall that the rational function $\widehat{Z}_q^q(z; \bar{t})$ is defined by (4.36) and (4.37).

Similarly, taking into account that the Gauss coordinate $\widehat{E}_{j,q}^+(z)$ does not commute only with the product of currents $\mathcal{F}_{q-1}(\bar{t}^{q-1}) \cdots \mathcal{F}_j(\bar{t})$ in the product (4.49), we find that
\[
\sim_{\bar{K}, \bar{J}} \widehat{e}_q \prod_{s=j}^{q-1} (-)^{(r_s-1)\delta_{s,m}} \text{Sym}_{\bar{t}^{q-1}, \bar{t}^{q-1}} \left[ C_{j+1}^{q-1} \widehat{Z}_j^q(z; \bar{t}) \mathcal{F}_{j-1}(\bar{t}^{j-1}) \cdot \mathcal{F}_j(\bar{t}_j) \right.
\times \left. \mathcal{F}_j(\bar{t}) \mathcal{F}_{j-1}(\bar{t}^{j-1}) \prod_{s=j}^{q-1} k_s^{+(t_s^1)} k_s^{+(t_s^1)} \right],
\]
where $\widehat{e}_q$ is the sign factor
\[
\widehat{e}_j = 1 \quad \text{and} \quad \widehat{e}_q = (-)^{([j]+[j])}(-)^{[j]} \quad \text{for} \quad q = j+1, j+2, \ldots, N,
\]
and the rational function $\widehat{Z}_j^q(z; \bar{t})$ is defined by (4.36) and (4.37).
The Gauss coordinates $F_{q,p}^+(z)$ and $\widehat{F}_{q,p}^+(z)$ in (4.45) and (4.46) can be replaced by the products of the corresponding currents (see the formulae (A.5), (A.29) and (A.7), (A.34), respectively):

\[
F_{q,p}^+(z) \sim \prod_{s=p}^{q-2} f_{[s+1]}(z_{s+1}, z_s)^{-1} F_p(z_p) \cdots F_{q-1}(z_{q-1}) \bigg|_{z_p=\ldots=z_{q-1}=z},
\]

\[
\widehat{F}_{q,p}^+(z) \sim \prod_{s=p}^{q-2} f_{[s+1]}(z_{s+1}, z_s)^{-1} \widehat{F}_p(z_p) \cdots \widehat{F}_{q-1}(z_{q-1}) \bigg|_{z_p=\ldots=z_{q-1}=z},
\]

where we have changed the order in the products of currents and have introduced an auxiliary set $\mathcal{Z} = \{z_p, \ldots, z_{q-1}\}$, which in the end should all be set equal to the parameter $z$.

Combining (4.38), the Gauss decomposition (4.45), the action (4.53) of the Gauss coordinates $E_{p,n}^+(z)$, and the formula (4.57), we can obtain the action formulae of the monodromy matrix elements $T_{i,j}^+(z)$ on the unnormalized Bethe vector

\[
\mathcal{B}(\mathcal{I}) = P_j^+ (\mathcal{F}(\mathcal{I})) \prod_{\ell=1}^{N} \lambda_{\ell}(\mathcal{I}^{\ell})|0\rangle,
\]

where the ordered product of simple root currents $\mathcal{F}(\mathcal{I})$ is given by (3.12). We have

\[
T_{i,j}^+(z) \cdot \mathcal{B}(\mathcal{I}) = \sum_{p=1}^{N+1} \sum_{q=j}^{N+1} \phi_{q} \epsilon_p C_{p}^{q-1} C_{j}^{q-1} \prod_{s=p}^{q-1} (-)^{(r_s-1)} \delta_{s,m} \times \text{Sym}_{\mathcal{I}_p, \ldots, \mathcal{I}_r, \ldots, \mathcal{I}_{q-1}} \left[ \frac{\mathcal{Z}_p^\ell(z; \mathcal{I}_p, \ldots, \mathcal{I}_q)}{\mathcal{X}(\mathcal{Z}; \mathcal{I}_p, \ldots, \mathcal{I}_q)} \right] \times \mathcal{B}(\mathcal{I}_1, \ldots, \mathcal{I}_p, \{z_p, \mathcal{I}_r\}, \ldots, \{z_{i-1}, \mathcal{I}_{r_{i-1}}\}, \{z_i, \mathcal{I}_i\}, \ldots, \{z_{j-1}, \mathcal{I}_{j-1}\}, \{z_j, \mathcal{I}_j\}, \ldots, \{z_{q-1}, \mathcal{I}_{q-1}\}, \mathcal{I}_q, \ldots, \mathcal{I}_N) \times \frac{\lambda_{p+1}(\mathcal{I}_r) \cdots \lambda_i(\mathcal{I}_{r_{i-1}}) \lambda_j(\mathcal{I}_1) \cdots \lambda_{q-1}(\mathcal{I}_{r_{q-1}})}{\lambda_p(z_p) \cdots \lambda_{q-1}(z_{q-1})} \bigg|_{z_p=\ldots=z_{q-1}=z},
\]

where we have introduced yet another rational function $\mathcal{X}(\mathcal{Z}, \mathcal{I}_p, \ldots, \mathcal{I}_q)$ depending on the auxiliary set $\mathcal{Z}$ and the Bethe parameters:

\[
\mathcal{X}(\mathcal{Z}; \mathcal{I}_p, \ldots, \mathcal{I}_q) = \prod_{s=p}^{q-1} f_{[s+1]}(z_{s+1}, \{z_s, \mathcal{I}_s\}) \times \prod_{s=i}^{j-1} f_{[s+1]}(z_{s+1}, \{z_s, \mathcal{I}_s\}) \prod_{s=j}^{q-2} f_{[s+1]}(z_{s+1}, \{z_s, \mathcal{I}_s\}) f_{[p]}(\mathcal{I}_r, z_p)^{-1}.
\]

Similarly, using (4.39), the Gauss decomposition (4.46), the action (4.55) of the Gauss coordinate $\widehat{F}_{j,q}^+(z)$, and the formula (4.58), we can calculate the action
the definitions of these factors in (4.14), (4.54), and (4.56), we observe that

seems to differ from

\[ \phi_q = (-)^{|q|}[j]+(|q|+|j|)[i] \]  

seems to differ from

\[ \tilde{\epsilon}_q = (-)^{|q|}[j]+(|q|+|j|)[p]. \]  

However, this is not true, because of the restrictions on \( p, i, j, \) and \( q \). If the parities of the indices \([p]\) and \([i]\) coincide, then the factors (4.63) and (4.64) also coincide. Now consider the case where the parities of \([p]\) and \([i]\) are different. Recall that \( p \leq i \). By the definition of the grading (see (2.1)), this means that \([p] = 0\) and \([i] = 1\). But in this subsection we consider the action of diagonal and upper triangular monodromy matrix elements \( T_{i,j}^+(z) \) on Bethe vectors. This means that there is the restriction \( p \leq i \leq j \leq q \), so that if \([p] \neq [i]\), then \([j] = [q] = 1\) and both factors in (4.63) and (4.64) are equal to \(-1\). Below we will denote these phase factors as

\[ \phi_q \epsilon_p = \tilde{\phi}_p \tilde{\epsilon}_q = \varphi_{p,q}. \]
We can now restore the normalizations of the Bethe vectors (3.14) and (3.22) and observe that the actions of the diagonal and upper triangular monodromy matrix elements on these Bethe vectors lead to the same recurrence relations. This means that the Bethe vectors given by (3.14) and (3.22) coincide.

We start our restoration of the normalization with the Bethe vectors (3.14) using (4.59). Note that the deformed symmetrization in the action formula (4.61) turns into the usual symmetrization in (4.66) in view of the property (4.2). Using the explicit expressions for the rational functions (4.25), (4.34), and (4.60), we get that

\[
T_{i,j}^{+}(z) \cdot \mathbb{B}(\bar{t}) = \sum_{p=1}^{i} \sum_{q=1}^{N+1} \varphi_{p,q} C_{p}^{i-1} C_{q}^{i-1} \times \text{Sym}_{t_{p},...,t_{i-1},\bar{t}_{i-1},...,\bar{t}_{q-1}}[\mathbb{D}(\bar{t}) \Lambda(z, \bar{t}) \Lambda(z; \bar{t}) \mathbb{B}(\{z, \bar{t}\})],
\]

(4.66)

where the sign factor \( \varphi_{p,q} \) is given by (4.65), and the Bethe vector \( \mathbb{B}(\{z, \bar{t}\}) \) on the right-hand side of this equality depends on the following set of parameters:

\[
\{z, \bar{t}\} = \{\bar{t}, \ldots, \bar{t}^{p-1}, \{z, \bar{t}_{p}\}, \ldots, \{z, \bar{t}_{r_{i-1}}\}, \{z, \bar{t}\}, \ldots, \{z, \bar{t}^{q-1}\}, \{z, \bar{t}_{1}^{\prime}\}, \ldots, \{z, \bar{t}_{1}^{N-1}\}, \bar{t}_{1}^{\prime}, \ldots, \bar{t}_{N}\}.
\]

The rational function \( \mathbb{D}(\bar{t}) \) is given by the product

\[
\mathbb{D}(\bar{t}) = \prod_{s=p}^{i-1} f_{[s]}(t_{s}^{p}, \bar{t}_{s}^{p}) \prod_{s=j}^{q-1} f_{[s]}(\bar{t}_{s}^{1}, t_{s}^{1}) \prod_{s=j}^{q-1} h(t_{s}^{1}, \bar{t}_{s}^{1}) \delta_{s,m} \prod_{s=j}^{q-1} h(t_{s}^{1}, \bar{t}_{s}^{1}) \delta_{s,m}.
\]

(4.67)

The form of the other two rational functions \( \Upsilon(z, \bar{t}) \) and \( \Lambda(z; \bar{t}) \) strongly depends on the values of \( p \) and \( q \). For \( p < i \) and \( q > j \)

\[
\Upsilon(z, \bar{t}) = f_{[i]}(z, \bar{t}^{p-1}) f_{[j]}(\bar{t}^{q}, z) \prod_{s=i}^{i-1} h(\bar{t}_{s}^{p}, z) \delta_{s,m} \prod_{s=j}^{j-1} h(\bar{t}^{q}, z) \delta_{s,m} \prod_{s=j}^{q-1} h(\bar{t}^{q}, \bar{t}^{q}) \delta_{s,m}
\times g(z, t_{s}^{p}) \prod_{s=p}^{i-2} f_{[s]}(t_{s}^{p}, t_{s}^{p}) g(z, \bar{t}_{s}^{q}) \prod_{s=j}^{q-2} f_{[s]}(\bar{t}_{s}^{1}, \bar{t}_{s}^{1}) \prod_{s=j}^{q-1} g_{[s]}(t_{s}^{p}, t_{s}^{p}) g_{[s]}(\bar{t}_{s}^{q}, \bar{t}_{s}^{q}) \prod_{s=j}^{q-1} g_{[s]}(\bar{t}_{s}^{1}, \bar{t}_{s}^{1})
\]

\[
\Lambda(z; \bar{t}) = \lambda_{p+1}(t_{p}^{p}) \cdots \lambda_{i}(t_{r_{i-1}}^{p-1}) \lambda_{j}(t_{1}^{1}) \cdots \lambda_{q-1}(t_{1}^{q-1})
\]

\[
= \frac{\lambda_{p+1}(z) \cdots \lambda_{q-1}(z)}{\lambda_{p+1}(z) \cdots \lambda_{q-1}(z)},
\]

for \( p = i \) and \( q > j \)

\[
\Upsilon(z, \bar{t}) = f_{[i]}(z, \bar{t}^{p-1}) f_{[j]}(\bar{t}^{q}, z) \prod_{s=i}^{j-1} h(\bar{t}^{q}, z) \delta_{s,m}
\times g(z, t_{s}^{p}) \prod_{s=p}^{i-2} f_{[s]}(t_{s}^{p}, t_{s}^{p}) \prod_{s=j}^{q-1} g_{[s]}(t_{s}^{p}, t_{s}^{p}) \prod_{s=j}^{q-1} g_{[s]}(\bar{t}_{s}^{1}, \bar{t}_{s}^{1})
\]

\[
\Lambda(z; \bar{t}) = \frac{\lambda_{j}(t_{1}^{1}) \cdots \lambda_{q-1}(t_{1}^{q-1})}{\lambda_{i+1}(z) \cdots \lambda_{q-1}(z)},
\]

\[
= \frac{\lambda_{j}(t_{1}^{1}) \cdots \lambda_{q-1}(t_{1}^{q-1})}{\theta_{1+i, q-1}},
\]
for $p < i$ and $q = j$

$$
\Psi(z, \bar{t}) = f_p(z, \bar{t}^{n-1}) f_j(\bar{t}^j, z) \prod_{s=p}^{i-1} h(\bar{t}^s, z) \delta_{s,m} \times \prod_{s=i}^{j-1} h(\bar{t}^s, z) \delta_{s,m} \frac{g(z, t_p^s) \prod_{s=p}^{i-2} g_{s+1}(t_{s+1}^s, t_{s+1}^s)}{\prod_{s=p}^{i-2} f_{s+1}(t_{s+1}^s, t^s)} ,
$$

$$
\Lambda(z; \bar{t}) = \frac{\lambda_{p+1}(t_p^s) \cdots \lambda_i(t_{i-1}^s)}{(\lambda_{p+1}(z) \cdots \lambda_j(z))^\theta_{i+1,j-1}} ,
$$

and finally, for $p = i$ and $q = j$

$$
\Psi(z, \bar{t}) = f_i(z, \bar{t}^{i-1}) f_j(\bar{t}^j, z) \prod_{s=i}^{j-1} h(\bar{t}^s, z) \delta_{s,m} ,
$$

$$
\Lambda(z; \bar{t}) = \frac{\lambda_i(z) \delta_{ij}}{(\lambda_{i+1}(z) \cdots \lambda_j(z))^\theta_{i+1,j-1}} ,
$$

where $\theta_{i,j}$ is the Heaviside step function

$$
\theta_{i,j} = \begin{cases} 
1, & i \leq j, \\
0, & i > j. 
\end{cases}
$$

Now we restore the normalization of the Bethe vectors (3.22) using (4.61). Again using the explicit expressions for the rational functions (4.25), (4.34), and (4.62), we obtain the action formula

$$
T_{i,j}^+ (z) \cdot \mathbb{B}(\bar{t}) = \sum_{p=1}^{i} \sum_{q=j}^{N+1} \varphi_{p,q} \mathbb{C}_p^{i-1} \mathbb{C}_j^{q-1} \times \text{Sym}_{\bar{t}_p, \ldots, \bar{t}^{i-1}, \bar{t}_j, \ldots, \bar{t}^{q-1}} [\mathbb{D}(\bar{t}) \Psi(z, \bar{t}) \Lambda(z; \bar{t}) \mathbb{B}([z, \bar{t}])], \tag{4.68}
$$

where the only difference from the action formula (4.66) is that the function $\mathbb{D}(\bar{t})$ is replaced by

$$
\mathbb{D}(\bar{t}) = \prod_{s=p}^{i-1} f_{s+1}(t_{s+1}^s, t_s^s) \prod_{s=j}^{q-1} \frac{f_{s+1}(\bar{t}_1^s, t_1^s)}{h(t_{s+1}^s, t_s^s)^{\delta_{s,m}}} \prod_{s=j}^{q-1} \frac{f_{s+1}((\bar{t}_1^s, t_1^s)\delta_{s,m}}{h(t_{s+1}^s, t_s^s)^{\delta_{s,m}}} . \tag{4.69}
$$

Comparing the action formulae (4.66) and (4.68), we can prove Proposition 4.2 if we prove that the functions $\mathbb{D}(\bar{t})$ and $\mathbb{D}(\bar{t})$ actually coincide. First of all, we recall that for $s \neq m$ the rational functions $f_{s+1}(u, v)$ and $f_{s+1}(u, v)$ in the definitions of the functions (4.67) and (4.69) coincide. A difference is possible only in the case when $s = m$, since by definition

$$
\frac{f_{[m]}(u, v)}{u - v} = \frac{u - v + c}{u - v} \quad \text{and} \quad \frac{f_{[m+1]}(u, v)}{u - v} = \frac{u - v - c}{u - v} .
$$

Assume first that $m \notin \{p, \ldots, i - 1\}$ and $m \notin \{j, \ldots, q - 1\}$. Then the functions (4.67) and (4.69) coincide. If $m \in \{p, \ldots, i - 1\}$, then both the factors
in the functions $\mathbb{D}(\vec{t})$ and $\widehat{\mathbb{D}}(\vec{t})$ that depend on the Bethe parameters $\vec{t}^m$ are equal to $g(\vec{t}^m, \vec{t}^m)$. Similarly, if $m \in \{j, \ldots, q - 1\}$, then these factors are equal to $g(\vec{t}^1, \vec{t}^1)$. This means that in the Yangian double $DY(gl(m|n))$ the Bethe vectors constructed using the first current realization (3.14) coincide with the Bethe vectors constructed using the second current realization (3.22).

This concludes the proof of the main statement formulated in Proposition 4.2.

4.5. Actions of the diagonal elements and the Bethe equations. In this subsection we consider the action of the universal transfer matrix $t(z)$ in (2.6) on Bethe vectors. For this we must find the action of the diagonal monodromy matrix elements. Hence we should set $i = j$ on the right-hand side of the action formula (4.66). Since the action formulae (4.66) and (4.68) are equivalent, we use the first of them. We have

$$t(z) \cdot \mathbb{B}(\vec{t}) = \sum_{i=1}^{N+1} \sum_{p=1}^{N+1} (-)^i \varphi_{p,q} C_{p+1}^{i+1} C_i^{q-1} \times \text{Sym}_{p_1, \ldots, p_{q-1}} \left[ \mathbb{D}(\vec{t}) \mathbb{Y}(z, \vec{t}) \Lambda(z; \vec{t}) \mathbb{B}(\{z, \vec{t}\}'), \right] \tag{4.70}$$

where

$$\{z, \vec{t}\}' = \{\vec{t}^1, \ldots, \vec{t}^{p-1}, \{z, \vec{t}^p\}, \ldots, \{z, \vec{t}^{1-1}\}, \{z, \vec{t}^1\}, \ldots, \{z, \vec{t}^{1-1}\}, \vec{t}^q, \ldots, \vec{t}^N\}$$

and, we recall, $N = m + n - 1$.

Among all the terms on the right-hand side of (4.70) there are the so-called 'wanted' terms corresponding to $p = q = i$. One can easily see that their sum is equal to

$$\sum_{i=1}^{N+1} (-)^i \lambda_i(z) f_{[i]}(z, \vec{t}^{i-1}) f_{[i]}(\vec{t}^i, z) \mathbb{B}(\vec{t}).$$

Let us compare the terms in (4.70) coming from the actions of the monodromy matrix elements $T_{i,i}(z)$ and $T_{i+1,i+1}(z)$. In both cases they correspond to the terms in the sums over $p$ and $q$ on the right-hand side of (4.66) for $p = i$ and $q = i + 1$. For the action of the matrix element $(-)^{[i]} T_{i,i}(z)$ these terms are

$$\frac{1}{(r_i - 1)!} \text{Sym}_{\vec{t}^i} \left[ \frac{\lambda_i(t^i_{r_i})}{f_{[i]}(t^i_{r_i}, t^i_{r_i})} \frac{f_{[i]}(\vec{t}^i, t^i_{r_i})}{h(\vec{t}^i, t^i_{r_i})} \mathbb{B}(\vec{t}^1, \ldots, \vec{t}^{i-1}, \{z, \vec{t}^i\}, \vec{t}^{i+1}, \ldots, \vec{t}^N) \times g(z, t^i_{r_i}) f_{[i]}(z, \vec{t}^{i-1}) f_{[i]}(\vec{t}^{i+1}, z) h(\vec{t}^i, z) \delta_{i,m} \right]. \tag{4.71}$$

For the action of the matrix element $(-)^{[i+1]} T_{i+1,i+1}(z)$ the analogous terms are

$$\frac{(-)^{1+(r_i-1)} \delta_{i,m}}{(r_i - 1)!} \text{Sym}_{\vec{t}^i} \left[ \frac{\lambda_{i+1}(t^i_{r_i})}{f_{[i]}(t^i_{r_i}, t^i_{r_i})} \frac{f_{[i]}(\vec{t}^i, t^i_{r_i})}{h(\vec{t}^i, t^i_{r_i})} \delta_{i,m} \times \mathbb{B}(\vec{t}^1, \ldots, \vec{t}^{i-1}, \{z, \vec{t}^i\}, \vec{t}^{i+1}, \ldots, \vec{t}^N) \times g(z, t^i_{r_i}) f_{[i]}(z, \vec{t}^{i-1}) f_{[i]}(\vec{t}^{i+1}, z) h(\vec{t}^i, z) \delta_{i,m} \right]. \tag{4.72}$$
The symmetrizations in (4.71) and (4.72) can be replaced by summations over \( \ell = 1, \ldots, r_i \):

\[
\sum_{\ell=1}^{r_i} \frac{\lambda_i(t_i^\ell)}{f_{[i]}(t_i^\ell, t_i'^\ell)} \frac{f_{[i]}(t_i^\ell, t_i'^\ell)}{h(t_i^\ell, t_i'^\ell)^{\delta_{i,m}}} \mathbb{B}(t_i^1, \ldots, t_i^{i-1}, \{z, \bar{t}_i^\ell\}, t_i^{i+1}, \ldots, t_i^N) \\
\times g(z, t_i^\ell) f_{[i]}(z, t_i^{i-1}) f_{[i+1]}(t_i^{i+1}, z) h(t_i^\ell, z)^{\delta_{i,m}}
\]

(4.73)

and

\[
-(r_i-1)^{\delta_{i,m}} \sum_{\ell=1}^{r_i} \frac{\lambda_{i+1}(t_i^\ell)}{\lambda_i(t_i^\ell)} \left[ \frac{f_{[i]}(t_i^\ell, t_i'^\ell)}{h(t_i^\ell, t_i'^\ell)^{\delta_{i,m}}} \frac{h(t_i^\ell, t_i'^\ell)}{f_{[i]}(t_i^\ell, t_i'^\ell)} \right] \mathbb{B}(t_i^1, \ldots, t_i^{i-1}, \{z, \bar{t}_i^\ell\}, t_i^{i+1}, \ldots, t_i^N)

\times g(z, t_i^\ell) f_{[i]}(z, t_i^{i-1}) f_{[i+1]}(t_i^{i+1}, z) h(t_i^\ell, z)^{\delta_{i,m}}
\]

(4.74)

If the set of Bethe parameters \( \bar{t} \) satisfies the system of equations

\[
\lambda_{i+1}(t_i^\ell) = (-)^{\delta_{i,m}} \frac{f_{[i]}(t_i^\ell, t_i'^\ell)}{h(t_i^\ell, t_i'^\ell)^{\delta_{i,m}}} \frac{h(t_i^\ell, t_i'^\ell)}{f_{[i]}(t_i^\ell, t_i'^\ell)} \mathbb{B}(t_i^1, \ldots, t_i^{i-1}, \{z, \bar{t}_i^\ell\}, t_i^{i+1}, \ldots, t_i^N)
\]

(4.75)

then the terms in (4.73) and (4.74) cancel each other. If \( i \neq m \), then the equations (4.75) become the standard Bethe equations analogous to those arising in the algebra \( \mathfrak{gl}(N+1) \):

\[
\lambda_{i+1}(t_i^\ell) = \frac{f_{[i]}(t_i^\ell, t_i'^\ell)}{f_{[i]}(t_i^\ell, t_i'^\ell)} \mathbb{B}(t_i^1, \ldots, t_i^{i-1}, \{z, \bar{t}_i^\ell\}, t_i^{i+1}, \ldots, t_i^N)
\]

(4.76)

For \( i = m \) the Bethe equations (4.75) simplify to

\[
\lambda_{m+1}(t_m^\ell) = \frac{f(t_m^\ell, \bar{t}_m^{m-1})}{f(t_m^\ell, \bar{t}_m^{m+1})}
\]

(4.77)

This simplified form of the Bethe equations is typical for the models of free fermions, but one should remember that in the case under consideration the parameters \( t_m^\ell \) are coupled through the equations (4.76) with \( i = m \pm 1 \).

If the Bethe equations are satisfied, then the Bethe vector becomes an eigenvector of the transfer matrix (2.6):

\[
t(z) \cdot \mathbb{B}(\bar{t}) = \tau(z; \bar{t}) \mathbb{B}(\bar{t})
\]

with the eigenvalue

\[
\tau(z; \bar{t}) = \sum_{i=1}^{N+1} (-)^{[i]} \lambda_i(z) f_{[i]}(z, \bar{t}_i^{i-1}) f_{[i]}(\bar{t}_i^1, z).
\]

(4.78)

In this case we call \( \mathbb{B}(\bar{t}) \) an on-shell Bethe vector. Note that the Bethe equations (4.76) and (4.77) can be regarded as the condition of absence of poles of the eigenvalue (4.78) at the points \( z = t_i^\ell \).
Let us verify that all the remaining ‘unwanted’ terms in the action of the transfer matrix (2.6) on the on-shell Bethe vector vanish. To do this we calculate the general coefficient of the Bethe vector
\[ \mathcal{B}(\vec{t}^1, \ldots, \vec{t}^{i-1}, \{z, \vec{t}^i_b\}, \ldots, \{z, \vec{t}^{i+a}_b\}, \vec{t}^{i+a+1}, \ldots, \vec{t}^N) \] (4.79)
for fixed \( i \) and \( a > 0 \) in the sum over the index \( \ell_b \) of the Bethe parameters \( t^b_{\ell_1} \) for \( b = i, \ldots, i + a \). These sums arise from the symmetrizations in (4.70). One can see that a vector with Bethe parameters as in (4.79) can arise only from the actions of the diagonal monodromy matrix elements \( T^{+}_{b,b}(z) \) with \( b = i, \ldots, i + a + 1 \). To get such a vector one must take the term with \( p = i \) and \( q = i + a + 1 \) in the sums over \( p \) and \( q \) in (4.66). Recalling the definition of the phase factor \( \varphi_{i,i+a+1} \) in (4.65) for each \( b = i, \ldots, i + a + 1 \) and denoting it by \( \varphi_{i,i+a+1}(b) \), we find that
\[ (-)^{b}[\varphi_{i,i+a+1}(b)] = \begin{cases} 1 & \text{for } b = i, \\ (-)^{1+b} & \text{for } b = i + 1, \ldots, i + a, \\ -1 & \text{for } b = i + a + 1. \end{cases} \]
Substituting the explicit Bethe equation in the function \( \Lambda(z; \vec{t}) \), we get that the coefficient of the Bethe vector (4.79) on the right-hand side of the action formula (4.70) is proportional to the expression
\[ g(z, t^i_{\ell_1})^{-1} - \sum_{b=i+1}^{i+a} g(t^b_{\ell_b}, t^{b-1}_{\ell_{b-1}})^{-1} - g(z, t^{i+a}_{\ell_a})^{-1}, \]
which obviously vanishes. We note that the same trivial identity was used in [27] (see the unnumbered formula on p. 29 of that paper) to prove that a universal off-shell Bethe vector becomes on-shell if the Bethe equations are satisfied.

5. Explicit formulae for the universal Bethe vectors

5.1. Hierarchical relations for the Bethe vectors \( \mathcal{B}(\vec{t}) \). By calculating the ‘positive’ projection in the formula (3.14) for the Bethe vector \( \mathcal{B}(\vec{t}) \), we can obtain a hierarchical recurrence relation which connects the Bethe vectors constructed for the Yangian double \( DY(\mathfrak{gl}(m|n)) \) with the Bethe vectors for \( DY(\mathfrak{gl}(m-1|n)) \). Let us separate the product of currents \( \mathcal{F}_1(\vec{t}^1) = F_1(t^1_1) \cdots F_1(t^1_{\ell_1}) \) from the product of the other currents \( \mathcal{F}_2(\vec{t}^2 \cdots \mathcal{F}_N(\vec{t}^N)) \) and apply the normal ordering rule (4.11) to the latter product. It is obvious from this rule that in order to obtain the desired hierarchical relations for the Bethe vectors (see (5.3) below) it is sufficient to calculate the projection
\[ P^+_{f^1} (\mathcal{F}_1(\vec{t}^1) \cdot P^-_{f^2} (\mathcal{F}_2(\vec{t}^2) \cdots \mathcal{F}_N(\vec{t}^N))). \] (5.1)

Using the property \( P^-_{f^1} (\mathcal{F} \cdot P^+_{f^2} (\mathcal{F}')) = 0 \) for arbitrary elements \( \mathcal{F}, \mathcal{F}' \in \mathcal{U}_F \) such that \( \varepsilon(\mathcal{F}') = 0 \), we reduce the problem to the calculation of the projections
\[ P^+_{f^1} (\mathcal{F}_1(\vec{t}^1) \cdot P^-_{f^2} (\mathcal{F}_2(\vec{t}^2) \cdot P^-_{f^3} (\mathcal{F}_3(\vec{t}^3) \cdots P^-_{f^N} (\mathcal{F}_N(\vec{t}^N))))). \] (5.2)

\(^{8}\)The case \( a = 0 \) was considered above to obtain the Bethe equations.
This calculation is given in Appendix C, where it is shown to provide an answer in the form of a sum over partitions of the sets $\tilde{\ell}^1$ and $\tilde{\ell}'_1$, $\ell = 2, \ldots, N$, of the Bethe parameters in the expression (5.2).

To obtain the hierarchical relations for the Bethe vectors in the framework of this approach, we use the formula (4.11) to rewrite the Bethe vector (3.14) as a sum over partitions of the sets of Bethe parameters

\[ \tilde{\ell}' = \{\tilde{\ell}^2, \ldots, \tilde{\ell}^N\} \Rightarrow \tilde{\ell}'_1 \cup \tilde{\ell}'_{II}, \]

where

\[ \tilde{\ell}'_1 = \{\tilde{\ell}^2_1, \ldots, \tilde{\ell}^N_1\} \quad \text{and} \quad \tilde{\ell}'_{II} = \{\tilde{\ell}^2_{II}, \ldots, \tilde{\ell}^N_{II}\}. \]

The primed set of Bethe parameters $\tilde{\ell}'$ differs from the full set $\tilde{\ell}$ of these parameters (3.11) by excluding the Bethe parameters of the first type $\tilde{\ell}^1$. It follows from (4.11) and the properties of the projections that

\[ \mathbb{B}^{(m|n)}(\tilde{\ell}) = \sum_{\tilde{\ell}' \Rightarrow \tilde{\ell}'_1 \cup \tilde{\ell}'_{II}} \frac{\gamma_1(\tilde{\ell}^1)}{f_2(\tilde{\ell}^2_1, \tilde{\ell}^1)} P^+_f(F_{2,1}(\tilde{\ell}^1_1) \cdots F_{2,1}(\tilde{\ell}^1_{r_1}) P^{-}_f(F(\tilde{\ell}^1_1))) k_1^+ (\tilde{\ell}^1) \]

\[ \times \frac{1}{f_2(\tilde{\ell}^2_{II}, \tilde{\ell}^1)} \frac{\prod_{s=2}^{N} \gamma_s(\tilde{\ell}^s_{II}, \tilde{\ell}^s_1)}{\prod_{s=2}^{N-1} f_{[s+1]}(\tilde{\ell}^s_{II+1}, \tilde{\ell}^s_1)} \mathbb{B}^{(m-1|n)}(\tilde{\ell}'_{II}) \prod_{s=2}^{N} \lambda_s(\tilde{\ell}^s_1), \quad (5.3) \]

where we have identified $\tilde{\ell}^1_1$ with $\tilde{\ell}^1$ and used the fact that the Cartan currents $k_1^+(z)$ commute with all the currents $F_s(t')$, $s = 2, \ldots, N$. Let

\[ \mathcal{X}(\tilde{\ell}) = \frac{\gamma_1(\tilde{\ell}^1)}{f_2(\tilde{\ell}^2, \tilde{\ell}^1)} P^+_f(F_{2,1}(\tilde{\ell}^1_1) \cdots F_{2,1}(\tilde{\ell}^1_{r_1}) P^{-}_f(F(\tilde{\ell}^1_1))) k_1^+ (\tilde{\ell}^1), \quad (5.4) \]

where $\tilde{\ell}' = \{\tilde{\ell}^1, \ldots, \tilde{\ell}^N\}$. Then the expression on the first line of the right-hand side of (5.3) is equal to

\[ \mathcal{X}(\tilde{\ell}^1, \tilde{\ell}'_1). \quad (5.5) \]

To calculate the ‘positive’ projection of the product of currents

\[ F_{2,1}(\tilde{\ell}^1_1) \cdots F_{2,1}(\tilde{\ell}^1_{r_1}) \]

and the ‘negative’ projection $P^{-}_f(F(\tilde{\ell}^1_1))$ in (5.4), we use the formulae (C.23) and (C.25) for different $i$, starting from larger $i$ to smaller $i$. We use the first formula (C.23) for $i = m + 1, \ldots, N$, going from $i = N$ to $i = m + 1$, and the second formula (C.25) for $i = 2, \ldots, m$, going from $i = m$ to $i = 2$.

The results in Appendix C show that the sets $\tilde{\ell}^\ell$ will always be further divided into subsets. To describe this, for each subset $\tilde{\ell}^\ell$, $\ell = 1, \ldots, N$, we introduce the subdivision

\[ \tilde{\ell}^\ell \Rightarrow \{\tilde{\ell}^\ell_\ell, \tilde{\ell}^\ell_{\ell+1}, \ldots, \tilde{\ell}^\ell_N\} \quad (5.6) \]

such that the following constraints hold for the cardinalities of the subsets:

\[ \#\tilde{\ell}^\ell_q = \#\tilde{\ell}^\ell_{q'} \quad \text{for all } \ell \neq \ell' \quad \text{and} \quad q = \max(\ell, \ell'), \ldots, N. \quad (5.7) \]

In (5.6) and (5.7) the superscripts of the subsets, as usual, describe the type of the Bethe parameters, while the subscripts count the subsets in the subdivision (5.6).
One should not confuse this notation with the notation $\tilde{t}_i^t = \tilde{t}^t \setminus \{t_i^t\}$ used in the previous section §4.

Moreover, to get a non-trivial result in the calculation of the ‘positive’ projection in (5.4), we have to impose the following restrictions on the cardinalities of the subsets $\tilde{t}^s$:

$$\#\tilde{t}^1 > \#\tilde{t}^2 > \cdots > \#\tilde{t}^N \geq 0.$$ 

Appendix C shows how the Izergin determinant $K_{[i]}(\overline{y}|\overline{x})$ [29] (see (C.22)) arises in the calculation of the projections. It also shows how the result of these calculations can be rewritten in the form of sums over partitions of the sets of Bethe parameters into subsets. Let

$$K_0(\overline{y}|\overline{x}) = K_{[i]}(\overline{y}|\overline{x}) \quad \text{for } i = 1, \ldots, m,$$

$$K_1(\overline{y}|\overline{x}) = K_{[i]}(\overline{y}|\overline{x}) = K_0(\overline{x}|\overline{y}) \quad \text{for } i = m + 1, \ldots, N.$$ 

It is also convenient to introduce, for any sets $\overline{y}$ and $\overline{x}$ of the same cardinality, the following product of rational functions:

$$C(\overline{y}|\overline{x}) = g(\overline{y}, \overline{x})h(\overline{x}, \overline{x}). \quad (5.8)$$

We consider in greater detail the calculation of the projections, using the results obtained in Appendix C. The subdivision (5.6) can be represented using the table

$\begin{array}{cccccccc}
\tilde{t}_1^1 & \cup & \tilde{t}_2^1 & \cup & \cdots & \cup & \tilde{t}_{m-1}^1 & \cup & \tilde{t}_m^1 \\
\tilde{t}_2^1 & \cup & \cdots & \cup & \tilde{t}_{m-1}^1 & \cup & \tilde{t}_m^1 & \cup & \tilde{t}_{m+1}^1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\tilde{t}_{m-1}^m & \cup & \tilde{t}_{m-1}^{m-1} & \cup & \cdots & \cup & \tilde{t}_N^{m-1} & \cup & \tilde{t}_N^m \\
\tilde{t}_m^m & \cup & \tilde{t}_{m+1}^m & \cup & \cdots & \cup & \tilde{t}_N^m & \cup & \tilde{t}_N^{m+1} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\tilde{t}_N^{N-1} & \cup & \tilde{t}_N^N
\end{array}$

(5.9)

where the cardinalities of all the subsets in the same column are equal.

For any set $\overline{w}$ of cardinality $\#\overline{w} = d$ we introduce the following ordered product of composed (or simple, when $j = i + 1$) currents:

$$\mathcal{F}_{j,i}(\overline{w}) = F_{j,i}(w_1) \cdot F_{j,i}(w_2) \cdots F_{j,i}(w_d). \quad (5.10)$$

Then the ‘negative’ projection $P_f^-(F(\tilde{t}^t))$ in the definition of the element (5.4) can be written in the form

$$\begin{align*}
\prod_{s=2}^{N} \frac{\gamma_s(\tilde{t}^s)}{f_{[s+1]}(\tilde{t}_s)} & P_f^-(\overline{\mathcal{F}}_{3,2}(\tilde{t}^2)P_f^-(\cdots P_f^-(\mathcal{F}_{N,N-1}(\tilde{t}^{N-1}) \\
& \times P_f^-(\mathcal{F}_{N+1,N}(\tilde{t}^N))) \cdots ),
\end{align*}$$
and in the first step we have to calculate
\[
\frac{\gamma_{N-1}(\bar{t}^N-1)\gamma_N(\bar{t}^N)}{f[\bar{N}](\bar{t}^N, t^N-1)} P_f^-(\mathcal{F}_{N,N-1}(\bar{t}^N-1) \cdot P_f^-(\mathcal{F}_{N+1,N}(\bar{t}^N))), \quad (5.11)
\]
using either (C.23) or (C.25), depending on the relation between \(m\) and \(N\).

If \(m < N\), and hence \([N] = 1\), then we have to use (C.23) in order to obtain for the element (5.11) a sum over the partitions \(t^N-1 \Rightarrow \{\bar{t}^N_{N-1} \cup t^N_N\}\) such that \(#\bar{t}^N_{N-1} = #\bar{t}^N_N\) (see the next-to-last line in the table above), where we identify the sets \(\bar{t}^N_N \equiv t^N_N\):

\[
(\mathcal{C})^{-#\bar{t}^N_N} \gamma_N(\bar{t}^N-1) \sum_{\bar{t}^N-1 \Rightarrow \{\bar{t}^N_{N-1} \cup t^N_N\}} \frac{f_1(\bar{t}^N_{N-1}, t^N_N) K_1(t^N_N | \bar{t}^N_{N-1})}{f_1(t^N_N, \bar{t}^N_{N-1}) f_1(\bar{t}^N_N, t^N_N)}
\times P_f^-(\mathcal{F}_{N+1,N-1}(t^N_N) \cdot \mathcal{F}_{N,N-1}(\bar{t}^N_{N-1})).
\]

On the other hand, if \(m = N\), and hence \([N] = 0\) (this case corresponds to the algebra \(\mathfrak{gl}(m|1)\)), then we have to use (C.25) in order to obtain the element (5.11) again as a sum over the same partitions of \(\bar{t}^N-1\):

\[
c^{-#\bar{t}^N_N} \gamma_N(\bar{t}^N-1) \sum_{\bar{t}^N-1 \Rightarrow \{\bar{t}^N_{N-1} \cup t^N_N\}} \frac{f_0(\bar{t}^N_{N-1}, t^N_N) C(t^N_N | \bar{t}^N_{N-1})}{f_0(t^N_N, \bar{t}^N_{N-1}) f_0(\bar{t}^N_N, t^N_N)}
\times \Delta_h(t^N_N)^{-1} P_f^-(\mathcal{F}_{N+1,N-1}(t^N_N) \cdot \mathcal{F}_{N,N-1}(\bar{t}^N_{N-1})).
\]

The next step is to calculate the projections
\[
\frac{\gamma_{N-2}(\bar{t}^N-2)\gamma_{N-1}(\bar{t}^N-1)}{f[\bar{N}-1](\bar{t}^N-1, \bar{t}^N-2)} P_f^-(\mathcal{F}_{N-1,N-2}(t^N-2) \cdot P_f^-(\mathcal{F}_{N+1,N-1}(t^N_N) \cdot \mathcal{F}_{N,N-1}(\bar{t}^N_{N-1}))),
\]
using (C.23) for \(m < N - 1\) and (C.25) for \(m = N - 1\). Continuing the calculation of the element (5.4) using first (C.23) and then (C.25), we eventually get that

\[
\mathcal{F}(\bar{t}) = \sum_{t^N = \{\bar{t}^N_N, t^N_{N+1}, \ldots, t^N_N\}} \prod_{\ell=1}^{N-1} \prod_{\ell+1 \leq q \leq N} f_{\ell+1}(t^N_{\bar{q}+1}, t^N_{\bar{q}+1})^{-1}
\times \prod_{\ell=1}^{N} \prod_{\ell \leq q < q' \leq N} \frac{f_{[\ell]}(t^N_{\bar{q}'}, t^N_{\bar{q}'})}{f_{[\ell]}(t^N_{\bar{q}}, t^N_{\bar{q}})} \cdot \delta_{\ell,m} \quad \prod_{q=2}^{m-1} \prod_{\ell=2}^{q} K_{[\ell]}(t^N_{\bar{q}} | t^N_{\bar{q}'-1})
\times \prod_{q=m+1}^{N} \prod_{\ell=m+1}^{q} K_{[\ell]}(t^N_{\bar{q}} | t^N_{\bar{q}'-1}) \prod_{q=m}^{N} \prod_{\ell=2}^{m} C(t^N_{\bar{q}} | t^N_{\bar{q}'-1}) \prod_{q=m}^{N} \Delta_h(t^N_{\bar{q}'-1}^{-1})
\times \gamma_1(\bar{t}^N) P_f^+(\mathcal{F}_{N+1,N}(\bar{t}^N) \cdot \mathcal{F}_{m+1,N}(t^N_m) \cdot \mathcal{F}_{m+1,m+1}(t^N_{m+1}) \cdot \mathcal{F}_{m+1,m+1}(t^N_{m+1}) \cdots \mathcal{F}_{2,1}(t^N_1)) k^+_1(\bar{t}^N), \quad (5.12)
\]

The projection in the last line in (5.12) can be calculated by the method in [13]. Being multiplied from the right by the product of the Cartan currents \(k^+_1(\bar{t}^N)\),
it can be expressed in terms of an ordered product of monodromy matrix elements $T_{1,\ell}(t)$, $\ell = 2, \ldots, N + 1$. This shows that the hierarchical relations which we have resolved by calculating the projections in (5.3) are compatible with the embedding of $DY(\mathfrak{gl}(m - 1|n))$ in $DY(\mathfrak{gl}(m|n))$.

Finally, the element (5.4) is given as a multiple sum over partitions:

$$
\mathcal{X}(\bar{t}) = \sum_{\bar{t}' \Rightarrow \{\bar{t}_{\ell}^{1}, \ldots, \bar{t}_{\ell}^{N}\}} \prod_{\ell=1}^{N-1} \prod_{\ell' = 1, \ldots, N} f_{[\ell+1]}(\bar{t}_{q'}^{\ell+1}, \bar{t}_{\ell}^{\ell})^{-1}
\times \prod_{\ell=1}^{N} \prod_{\ell' = q, q' \leq N} \frac{f_{[\ell]}(\bar{t}_{q'}^{\ell}, \bar{t}_{\ell}^{\ell})}{h_{[\ell]}(\bar{t}_{q}^{\ell}, \bar{t}_{\ell}^{\ell})^{q, q'}^{\ell}} \prod_{q=2}^{m-1} \prod_{\ell=2}^{q} K_{[\ell]}(\bar{t}_{q}^{\ell}, \bar{t}_{q}^{\ell-1})
\times \prod_{q=m+1}^{N} \prod_{\ell = m+1}^{q} K_{[\ell]}(\bar{t}_{q}^{\ell}, \bar{t}_{q}^{\ell-1}) \prod_{q=m}^{N} \prod_{\ell=2}^{m} C(\bar{t}_{q}^{\ell}, \bar{t}_{q}^{\ell-1})
\times T_{1, N+1}(\bar{t}_{N}) T_{1, N}(\bar{t}_{N-1}) \cdots T_{1, m+1}(\bar{t}_{m}) T_{1, m}(\bar{t}_{m-1}) \cdots T_{1, 2}(\bar{t}_{1}).
$$

(5.13)

Here we have used the notation

$$
T_{i,j}(\bar{w}) = \Delta_{h}(\bar{w})^{-1} T_{i,j}(w_{1}) T_{i,j}(w_{2}) \cdots T_{i,j}(w_{d-1}) T_{i,j}(w_{d})
$$

(14)

for any set $\bar{w}$ of cardinality $\# \bar{w} = d$ and for $|i| + |j| = 1$. It is obvious that by the commutation relation (2.9) this product of odd matrix elements is symmetric with respect to permutations of the parameters $w_{i}$.

5.2. The Bethe vectors $B(\bar{t})$. We substitute the expression (5.13) with the subsets in (5.5) into the hierarchical relation (5.3), and then we repeat the same procedure for the Bethe vector $B^{(m - 1|n)}(\bar{t}_{\Pi}')$ in the second line of (5.3). In the end we will obtain an explicit expression for the Bethe vector $B^{(m|n)}(\bar{t})$ as a sum over multiple partitions of the set of Bethe parameters. Each term of this sum is a rational coefficient multiplied by symmetric products of monodromy matrix elements. To describe this expression it is necessary to introduce a more convenient indexing of the multiple partitions.

For all $\ell = 1, \ldots, N$ we introduce the partition of the sets of Bethe parameters

$$
\bar{t}_{\ell} = \bigcup_{q=1}^{\ell} \bigcup_{q' = \ell}^{N} \bar{t}_{q, q'},
$$

(15)

indexed by pairs of positive integers $q, q'$ such that

$$
1 \leq q \leq \ell \leq q' \leq N.
$$

We also introduce ordering rules $<$ and $\preceq$ for these pairs according to the following convention:

$$
q, q' < p, p' \text{ if } q < p, \forall q', p' \text{ or } q = p, q' < p',
$$

(16)
and

\[ q, q' \preceq p, p' \quad \text{if} \quad q < p, \quad \forall q', p', \quad \text{or} \quad q = p, \quad q' < p', \quad \text{or} \quad q = p, \quad q' = p'. \]

Using this notation and combining (5.3) with (5.13), we obtain for the Bethe vector the expression

\[ B(\vec{t}) = B(\vec{t})|0\rangle, \]

where the pre-Bethe vector \( B(\vec{t}) \) is given by a sum over the partitions (5.15):

\[
B(\vec{t}) = \sum_{\text{part}} \prod_{q,q'<p,p'} \prod_{\ell=1}^{N-1} f_{[\ell+1]}(\tilde{t}^\ell_{p,p'}, \tilde{t}^\ell_{q,q'})^{-1} \prod_{\ell=1}^{N} g(\tilde{t}^m_{p,p'}, \tilde{t}^m_{q,q'}) \prod_{\ell \neq m} f_{[\ell]}(\tilde{t}^\ell_{p,p'}, \tilde{t}^\ell_{q,q'})
\]

\[
\times \prod_{q=1}^{m-1} \prod_{q'=q+1}^{m-1} \prod_{\ell=q+1}^{m-1} K_\ell(\tilde{t}^\ell_{q,q'} | \tilde{t}^\ell_{q,q'} - 1) \prod_{q=1}^{m-1} \prod_{q'=q+1}^{m-1} \prod_{\ell=q+1}^{m-1} C(\tilde{t}^\ell_{q,q'} | \tilde{t}^\ell_{q,q'} - 1)
\]

\[
\times \prod_{q=1}^{m-1} \prod_{q'=m+1}^{q'} \prod_{\ell=m+1}^{q'} K_\ell(\tilde{t}^q_{q,q'} | \tilde{t}^q_{q,q'} - 1) \prod_{q=1}^{m-1} \prod_{q'=m+1}^{q'} \prod_{\ell=m+1}^{q'} K_\ell(\tilde{t}^\ell_{q,q'} | \tilde{t}^\ell_{q,q'} - 1)
\]

\[
\times \prod_{1 \leq q \leq m} \left( \prod_{N+1 \geq q' > q + 1} \prod_{m \geq q'} T_{q,q'}(\tilde{t}^{q}_{q,q'} - 1) \prod_{N+1 \geq q' > q + 1} \prod_{m \geq q'} T_{q,q'}(\tilde{t}^{q}_{q,q'} - 1) \right) \prod_{N+1 \geq q' > q + 1} \prod_{m \geq q'} T_{\ell,\ell}(\tilde{t}^{\ell}_{q,q'}). \tag{5.17}
\]

The partitions of the Bethe parameters can be pictured as an ordered table which is the following union of diagrams analogous to (5.9):

\[
\bigcup_{\ell=1, \ldots, N} \tilde{t}^\ell_{\ell, \ell} \cup \tilde{t}^\ell_{\ell, \ell+1} \cup \cdots \cup \tilde{t}^\ell_{\ell, N-1} \cup \tilde{t}^\ell_{\ell, N} \\
\cup \tilde{t}^{\ell+1}_{\ell, \ell+1} \cup \tilde{t}^{\ell+1}_{\ell, \ell+1} \cup \cdots \cup \tilde{t}^{\ell+1}_{\ell, N-1} \cup \tilde{t}^{\ell+1}_{\ell, N} \\
\cup \tilde{t}^{\ell N-1}_{\ell, \ell-N} \cup \tilde{t}^{\ell N-1}_{\ell, \ell-N} \\
\cup \tilde{t}^{\ell N}_{\ell, \ell-N} 
\tag{5.18}
\]

The ordering means that if \( \ell' < \ell \), then the diagram corresponding to \( \ell' \) in (5.18) is on the left of the diagram corresponding to \( \ell \). The ordering rules (5.16) mean literally that if \( q, q' \prec p, p' \), then the subset \( \tilde{t}^\ell_{q,q'} \) in the \( \ell \)th row is located to the left of the subset \( \tilde{t}^\ell_{p,p'} \) in the \( \ell \)th row of the diagram. All the subsets in the same column have the same cardinality. The subsets which describe a partition of Bethe parameters of the same type are in the same row of the diagram (see examples of such tables in (5.19), (5.21), and (5.23)).

**Example 5.1.** Let us look at the formula (5.17) in some particular cases of small \( m \) and \( n \).
The case $m = 2$ and $n = 1$. In this case $N = m + n - 1 = 2$ and the partitions of the sets $\ell^1$ and $\ell^2$ can be pictured by the following union of two diagrams:

\[
\begin{align*}
\ell^1 & : \quad \{\ell^1_{1,1} \cup \ell^1_{1,2}\} \\
\ell^2 & : \quad \{\ell^2_{1,2} \cup \ell^2_{2,2}\}
\end{align*}
\]

(5.19)

In this case the formula (5.17) simplifies:

\[
\mathcal{B}^{(2)(1)}(\ell^1,\ell^2) = \sum_{\text{part}} f(\ell^2,\ell^1)^{-1} f(\ell^1_{1,2},\ell^1_{1,1}) g(\ell^2_{1,2},\ell^2_{1,2}) C(\ell^2_{1,2}|\ell^1_{1,2}) \\
\times \mathbb{T}_{1,3}(\ell^1_{1,2}) \mathbb{T}_{1,2}(\ell^1_{1,1}) \mathbb{T}_{2,3}(\ell^2_{1,2}) \mathbb{T}_{2,2}(\ell^2_{1,2}).
\]

(5.20)

After the identifications $\ell^1_{1,1} \equiv \bar{\nu}_1$, $\ell^1_{1,2} \equiv \bar{\nu}_1$, $\ell^2_{1,2} \equiv \bar{\nu}_1$, and $\ell^2_{2,2} \equiv \bar{\nu}_1$, we recover from (5.20) the expression (5.32) for the Bethe vector.

The case $m = 2$ and $n = 2$. The partitions (5.15) can be described by the following table:

\[
\begin{align*}
\ell^1 & : \quad \{\ell^1_{1,1} \cup \ell^1_{1,2} \cup \ell^1_{1,3}\} \\
\ell^2 & : \quad \{\ell^2_{1,2} \cup \ell^2_{1,3} \cup \ell^2_{2,2} \cup \ell^2_{2,3}\} \\
\ell^3 & : \quad \{\ell^3_{1,3} \cup \ell^3_{2,3} \cup \ell^3_{3,3}\}
\end{align*}
\]

(5.21)

It corresponds to the union of three diagrams of the form (5.9). With this notation, (5.17) takes the form

\[
\mathcal{B}^{(2)(2)}(\ell^1,\ell^2,\ell^3) = \sum_{\text{part}} f_0(\ell^1_{1,2},\{\ell^1_{1,1} \cup \ell^1_{1,2}\})^{-1} f_0(\{\ell^2_{1,3} \cup \ell^2_{2,2} \cup \ell^2_{2,3}\},\ell^1)^{-1} \\
\times f_1(\ell^3_{1,3},\{\ell^2_{1,2} \cup \ell^2_{1,3}\})^{-1} f_1(\{\ell^2_{2,3} \cup \ell^3_{3,3}\},\ell^2)^{-1} \\
\times f_0(\ell^1_{1,3},\{\ell^1_{1,2} \cup \ell^1_{1,1}\}) f_0(\ell^1_{1,2},\ell^1_{1,1}) \\
\times g(\ell^2_{1,3},\{\ell^2_{2,2} \cup \ell^3_{1,3} \cup \ell^2_{2,2}\}) g(\ell^2_{2,2},\{\ell^2_{1,3} \cup \ell^2_{1,2}\}) g(\ell^2_{1,3},\ell^2_{1,2}) \\
\times f_1(\ell^3_{1,3},\{\ell^3_{1,3} \cup \ell^3_{2,3}\}) f_1(\ell^3_{1,3},\ell^3_{1,3}) \\
\times C(\ell^1_{1,2},\ell^1_{1,2}) C(\ell^2_{1,3},\ell^2_{1,3}) K_1(\ell^3_{1,3},\ell^3_{1,3}) K_1(\ell^3_{1,3},\ell^3_{2,3}) \\
\times \mathbb{T}_{1,4}(\ell^1_{1,3}) \mathbb{T}_{1,3}(\ell^1_{1,2}) \mathbb{T}_{1,2}(\ell^1_{1,1}) \mathbb{T}_{2,4}(\ell^2_{2,3}) \mathbb{T}_{2,3}(\ell^2_{2,3}) \mathbb{T}_{3,4}(\ell^3_{3,3}) \\
\times \mathbb{T}_{2,2}(\ell^2_{1,2}) \mathbb{T}_{2,2}(\ell^2_{1,3}) \mathbb{T}_{3,3}(\ell^3_{1,3}) \mathbb{T}_{3,3}(\ell^3_{2,3}).
\]

(5.22)

There is a rule for constructing a pre-Bethe vector from any given table of partitions (5.15). We demonstrate this rule for the diagram (5.21), considering each line in (5.22) and explaining all the factors in this formula with the help of (5.21).

- For a given subset $i^*_t$ in the $t$th row of the diagram (5.21) the first and second lines in (5.22) (which correspond to the values $\ell = 2, 3$) are products of the reciprocal functions $f[\ell](i^*_t, i^*_{k,l})^{-1}$, where the subset $i^*_{k,l}$ is either above or on the left of the starting subset $i^*_t$.

- The third, fourth, and fifth lines in (5.22) correspond to certain products formed for each row of the diagram in accordance with the following rule. For the rows corresponding to $i^\ell$ with $\ell < m$ (respectively, with $\ell = m$ or with $\ell > m$) we form products of the functions $f_0(\bar{x}, \bar{y})$ (respectively, $g(\bar{x}, \bar{y})$, or $f_1(\bar{x}, \bar{y})$). In these
products the subset $\overline{x}$ is to the right of the subset $\overline{y}$ in each row of the diagram (5.21).

- The sixth line in (5.22) is a product of Cauchy determinants or Izergin determinants for neighbouring pairs of subsets $(\overline{t}_{i,j}^k, \overline{t}_{i,j}^{k-1})$ belonging to the same column of the diagram corresponding to some $\ell$.

For $\ell = 1, \ldots, m - 1$ and any pair $(\overline{t}_{i,j}^k, \overline{t}_{i,j}^{k-1}) \equiv (\overline{x}, \overline{y})$, we use:
  - the Izergin determinant $K_0(\overline{x}|\overline{y})$ if $\ell + 1 \leq k \leq j \leq m - 1$;
  - the normalized Cauchy determinant $C(\overline{x}|\overline{y})$ (5.8) if $\ell + 1 \leq k \leq m \leq j \leq N$;
  - the Izergin determinant $K_1(\overline{x}|\overline{y})$ if $m + 1 \leq k \leq j \leq N$.

For $\ell = m, \ldots, N - 1$ and any pair $(\overline{t}_{i,j}^k, \overline{t}_{i,j}^{k-1}) \equiv (\overline{x}, \overline{y})$, we use the Izergin determinant $K_1(\overline{x}|\overline{y})$ if $\ell + 1 \leq k \leq j \leq N$.

Note that the asymmetry between the cases $\ell < m$ and $\ell \geq m$ is due to the hierarchical relation (5.3), which is based on the series of inclusions $\mathfrak{gl}(m|n) \supset \mathfrak{gl}(m - 1|n) \supset \cdots \supset \mathfrak{gl}(1|n) \supset \mathfrak{gl}(n)$.

In our example of the diagram (5.21) there are four such pairs

$$(\overline{t}_{1,2}^1, \overline{t}_{1,2}^1), \quad (\overline{t}_{1,3}^3, \overline{t}_{1,3}^1), \quad (\overline{t}_{1,3}^4, \overline{t}_{1,3}^2), \quad \text{and} \quad (\overline{t}_{2,3}^4, \overline{t}_{2,3}^2).$$

There are no $K_0(\overline{x}|\overline{y})$ determinants in this example, but they can appear for higher $m$. For instance, they appear in the Bethe vector for the algebra $\mathfrak{gl}(3|2)$ and are constructed for the pair of subsets $(\overline{t}_{1,2}^4, \overline{t}_{1,2}^1)$ using the diagram in (5.23).

- The seventh line is an ordered product of monodromy matrix elements $T_{i,j}$ with $i < j$ and depends on the subsets $\overline{t}_{i,j}^{k-1}$. It is the usual product for even matrix elements (that is, when $[i] + [j] = 0 \bmod 2$) and the normalized product (5.14) otherwise. The order of the factors in the product is from top to bottom for the lines and from right to left within a line, as becomes clear upon comparing the seventh line in (5.22) and the diagram (5.21).

- The last line in (5.22) is the product of the diagonal matrix elements depending on the remaining subsets of Bethe parameters which were not used in the previous line. The index of a diagonal matrix element $T_{i,i}$ coincides with the number of the line in the diagram. The order in this product is irrelevant, because the diagonal elements commute when the pre-Bethe vector (5.22) acts on the pseudo-vacuum vector $|0\rangle$.

**The case $m = 3$ and $n = 2$.** The Bethe vectors in this case can be constructed by the rules described above on the basis of the following table of partitions of the Bethe parameters $\overline{t}^1, \overline{t}^2, \overline{t}^3$, and $\overline{t}^4$:

$\overline{t}^1 : \overline{t}_{1,1}^i \cup \overline{t}_{1,2}^i \cup \overline{t}_{1,3}^i \cup \overline{t}_{1,4}^i$

$\overline{t}^2 : \overline{t}_{1,2}^i \cup \overline{t}_{1,3}^i \cup \overline{t}_{1,4}^i \cup \overline{t}_{2,2}^i \cup \overline{t}_{2,3}^i \cup \overline{t}_{2,4}^i$

$\overline{t}^3 : \overline{t}_{1,3}^i \cup \overline{t}_{1,4}^i \cup \overline{t}_{2,3}^i \cup \overline{t}_{2,4}^i \cup \overline{t}_{3,2}^i \cup \overline{t}_{3,3}^i \cup \overline{t}_{3,4}^i$

$\overline{t}^4 : \overline{t}_{1,4}^i \cup \overline{t}_{2,4}^i \cup \overline{t}_{3,4}^i \cup \overline{t}_{4,4}^i$ (5.23)

**5.3. The Bethe vectors $\widehat{H}(\overline{t})$.** In a completely analogous way one can obtain for the Bethe vectors (3.22) defined by means of the second current realization of the Yangian double $DY(\mathfrak{gl}(m|n))$ hierarchical relations which are compatible with the embedding of $DY(\mathfrak{gl}(m|n - 1))$ in $DY(\mathfrak{gl}(m|n))$. Another possibility for obtaining these hierarchical relations is to apply a special map to (5.3) and (5.13). This
morphism was discussed in [30]. It maps the Bethe vectors $\mathbb{B}(\bar{t})$ of $DY(\mathfrak{gl}(m|n))$ to the Bethe vectors $\mathbb{B}(\bar{t})$ of $DY(\mathfrak{gl}(n|m))$ (see (5.26) and the discussion that follows). Thus, using this map and the exchange $m \leftrightarrow n$, we can obtain an explicit hierarchical relation for the Bethe vector $\mathbb{B}(\bar{t})$. We do not give it here, but we give an analogue of (5.17) for $\mathbb{B}(\bar{t})$.

Again, for all $\ell = 1, \ldots, N$ we introduce a partition of the sets of Bethe parameters analogous to (5.15):

$$\tilde{t}^\ell = \bigcup_{q=1}^{\ell} \bigcup_{q'=\ell}^{N} \tilde{t}^\ell_{q',q},$$

indexing by pairs of positive integers $q, q'$ with

$$1 \leq q \leq \ell \leq q' \leq N.$$ 

We also introduce the ordering rules $\succ$ and $\succcurlyeq$ for these pairs according to the following conventions:

$$p', p \succ q', q \text{ if } p' > q', \forall p, q \text{ or } p' = q', p > q,$$

and

$$p, p' \succ q, q' \text{ if } p' > q', \forall p, q, \text{ or } p' = q', p > q, \text{ or } p' = q', p = q.$$ 

In this notation we have for the Bethe vector the expression

$$\mathbb{B}(\bar{t}) = \mathbb{B}(\bar{t})|0\rangle,$$

where the pre-Bethe vector $\mathbb{B}(\bar{t})$ is given by the sum over the partitions (5.24),

$$\mathbb{B}(\bar{t}) = \sum_{\text{part}} \prod_{p', p \succ q', q}^{N-1} \prod_{\ell=1}^{N} \mathcal{f}_{[\ell+1]}(\tilde{t}^\ell_{p',p}, \tilde{t}^\ell_{q',q})^{-1} \times \prod_{p', p \succ q', q} g(\tilde{t}^m_{q',q}, \tilde{t}^m_{p',p}) \prod_{\ell=1}^{N} \mathcal{f}_{[\ell+1]}(\tilde{t}^\ell_{p',p}, \tilde{t}^\ell_{q',q})$$

$$\times \prod_{q'=m+2}^{N} \prod_{q=m+1}^{q'-1} \prod_{\ell=m+1}^{q'-1} K[\ell](\tilde{t}^\ell_{q',q}, \tilde{t}^\ell_{q',q}) \prod_{q'=m+1}^{N} \prod_{q=1}^{m-1} \prod_{\ell=m+1}^{q-1} K[\ell](\tilde{t}^\ell_{q',q}, \tilde{t}^\ell_{q',q})$$

$$\times \prod_{q'=m+1}^{N} \prod_{q=1}^{m-1} \prod_{\ell=q}^{q'-1} K[\ell](\tilde{t}^\ell_{q',q}, \tilde{t}^\ell_{q',q}) \prod_{q'=m+1}^{N} \prod_{q=1}^{m-1} \prod_{\ell=q}^{q'-1} K[\ell](\tilde{t}^\ell_{q',q}, \tilde{t}^\ell_{q',q})$$

$$\times \prod_{\ell=1}^{N} \prod_{\ell=q}^{q'-1} \prod_{m=q+1}^{\ell} T_{q,q'+1}(\tilde{t}^\ell_{q',q}) \prod_{\ell=q}^{q'-1} \prod_{m=q+1}^{\ell} T_{q,q'+1}(\tilde{t}^\ell_{q',q}),$$

(5.25)
and where in contrast to (5.8) we normalize the Cauchy determinant \( \tilde{C}(\bar{\gamma}|\bar{x}) \) as follows:

\[
\tilde{C}(\bar{\gamma}|\bar{x}) = g(\bar{x}, \bar{\gamma}) h(\bar{\gamma}, \bar{\gamma}) = C(\bar{x}|\bar{\gamma}).
\]

The partitions of the Bethe parameters used in (5.25) also can be pictured using an ordered union of diagrams analogous to (5.9):

\[
\bigcup_{\ell=N,...,1} \bar{t}_\ell^0 \cup \bar{t}_\ell^1 \cup ... \cup \bar{t}_\ell^{\ell-1} \cup \bar{t}_{\ell-1}^\ell \cup ... \cup \bar{t}_{\ell-2}^\ell \cup \bar{t}_{\ell-1}^\ell
\]

The ordering here is opposite to the one used in the table (5.18). This means that a triangle for a smaller \( \ell \) in (5.18) is to the right of a triangle for a larger \( \ell \). All the subsets in a given column again have the same cardinality. The subsets which describe partitions of the Bethe parameters of the same type are in the same row of the table (see examples of such tables in (5.27) and (5.29)).

We note that the two realizations (5.17) and (5.25) are related by the morphism \( \varphi \) defined in [30] by

\[
\varphi: \begin{cases} 
DY(\mathfrak{gl}(m|n)) \to DY(\mathfrak{gl}(n|m)), \\ T_{i,j}(x) \mapsto (-1)^{|i||j|+|j|+1} \bar{T}_{j',i'}(x), & k' = m + n + 1 - k. 
\end{cases}
\]

Indeed, starting from the pre-Bethe vector \( \mathcal{B}(\bar{t}) \in DY(\mathfrak{gl}(m|n)) \) and applying \( \varphi \) to it, we get the pre-Bethe vector \((-1)^{\#\bar{t} - \#\mathcal{B}(\bar{s})} \bar{\mathcal{B}}(\bar{s}) \in DY(\mathfrak{gl}(n|m))\), where the set \( \bar{t} \) is divided into subsets \( \bar{t}_{i,j}^\ell \) satisfying (5.15), while the set \( \bar{s} \) is divided into subsets \( \bar{s}_{i,j}^\ell \) satisfying (5.24). The relation between these partitions is given by \( \bar{t}_{i,j}^\ell = \bar{s}_{i-j-1,j}^{\ell-1} \), where \( k' = m + n + 1 - k \) for any \( k \). In particular, \( \varphi(\mathcal{B}(\bar{t})) = (-1)^{\#\bar{t} - \#\mathcal{B}(\bar{s})} \bar{\mathcal{B}}(\bar{s}) \) when \( m = n \), as can be checked in the example \( m = n = 2 \) described by (5.22) and (5.30).

**Example 5.2.** For \( m = 2 \) and \( n = 1 \) the partition (5.24) can be pictured using the table

\[
\begin{array}{c}
\bar{t}^2 : \\
\bar{t}_{2,2}^3 \\
\bar{t}_{2,1}^2 \\
\bar{t}_{1,1}^1 \\
\bar{t}^1 :
\end{array}
\]

and the formula (5.25) reduces to

\[
\hat{\mathcal{B}}^{(2|1)}(\bar{t}^1, \bar{t}^2) = \sum_{\text{part}} f(\bar{t}^2, \bar{t}^1)^{-1} f(\bar{t}_{2,1}^1, \bar{t}_{1,1}^1) g(\bar{t}_{2,2}^3, \bar{t}_{2,1}^2, \bar{t}_{1,1}^1) K_0(\bar{t}_{2,1}^2|\bar{t}_{2,1}^1)
\times T_{1,3}(\bar{t}_{2,1}^2) T_{2,3}(\bar{t}_{2,2}^3) T_{1,2}(\bar{t}_{1,1}^1) T_{2,2}(\bar{t}_{2,1}^1),
\]

which implies (5.33) (see below) after the identifications \( \bar{t}_{2,1}^1 \equiv \bar{u}_I, \bar{t}_{2,1}^2 \equiv \bar{u}_I, \bar{t}_{2,2}^3 \equiv \bar{v}_II, \) and \( \bar{t}_{2,1}^2 \equiv \bar{v}_I, \)
In the case $m = 2$ and $n = 2$ the partitions (5.24) can be described using the following union of diagrams:

\[
\bar{t}^3 : \bar{t}^3_{3,3} \cup \bar{t}^3_{3,2} \cup \bar{t}^3_{3,1} \\
\bar{t}^2 : \bar{t}^2_{3,2} \cup \bar{t}^2_{3,1} \cup \bar{t}^2_{2,2} \cup \bar{t}^2_{2,1} \\
\bar{t}^1 : \bar{t}^1_{3,1} \cup \bar{t}^1_{2,1} \cup \bar{t}^1_{1,1}
\]

(5.29)

According to this table, the formula (5.25) takes the form

\[
\hat{B}^{(2|2)}(\bar{t}^1, \bar{t}^2, \bar{t}^3) = \sum_{\text{part}} f_0(\bar{t}^2, {\bar{t}^3_{2,1} \cup \bar{t}^3_{1,1}})^{-1} f_0(\{\bar{t}^3_{3,2} \cup \bar{t}^3_{2,1}\}, \bar{t}^3_{3,1})^{-1}
\]

\[
\times f_1(\bar{t}^3_{3,1}, \{\bar{t}^3_{3,2} \cup \bar{t}^3_{2,1}\})^{-1} f_1(\{\bar{t}^3_{3,2} \cup \bar{t}^3_{2,1}\}, \bar{t}^2)^{-1}
\]

\[
\times f_0(\bar{t}^3_{1,1}, \{\bar{t}^3_{2,1} \cup \bar{t}^3_{1,1}\}) f_0(\bar{t}^1_{2,1}, \bar{t}^1_{1,1})
\]

\[
\times g(\{\bar{t}^3_{3,2} \cup \bar{t}^3_{2,2} \cup \bar{t}^3_{2,1}\}, \bar{t}^3_{3,2}) g(\{\bar{t}^3_{2,2} \cup \bar{t}^3_{2,1}\}, \bar{t}^3_{2,1}) g(\bar{t}^3_{2,1}, \bar{t}^2_{2,2})
\]

\[
\times f_1(\{\bar{t}^3_{3,2} \cup \bar{t}^3_{2,2}\}, \bar{t}^3_{2,2}) f_1(\bar{t}^3_{3,2}, \bar{t}^3_{2,3})
\]

\[
\times K_0(\bar{t}^3_{2,1}|\bar{t}^3_{2,1}) K_0(\bar{t}^3_{2,2}|\bar{t}^3_{2,1}) \hat{C}(\bar{t}^3_{3,1}|\bar{t}^3_{2,1}) \hat{C}(\bar{t}^3_{3,2}|\bar{t}^3_{2,2})
\]

\[
\times T_{1,4}(\bar{t}^3_{1,3}) T_{2,4}(\bar{t}^3_{2,3}) T_{3,4}(\bar{t}^3_{3,3}) T_{1,3}(\bar{t}^2_{2,1}) T_{2,3}(\bar{t}^2_{2,2}) T_{1,2}(\bar{t}^1_{2,1})
\]

\[
\times T_{2,2}(\bar{t}^1_{3,1}) T_{2,2}(\bar{t}^1_{2,1}) T_{3,3}(\bar{t}^2_{3,2}) T_{3,3}(\bar{t}^2_{3,1}).
\]

Comparing (5.30) and the diagram (5.29), we can formulate the rules for associating with a partition diagram an explicit formula for the Bethe vector in a way similar to that in the previous subsection. We leave this as an exercise for the interested reader.

5.4. Dual Bethe vectors and examples for \( \text{DY} (\mathfrak{gl}(2|1)) \). In order to obtain explicit expressions for the dual Bethe vectors \( \mathcal{C}(\bar{t}) \) and \( \hat{C}(\bar{t}) \) we have to exploit the definition and the properties of the antismorphism (2.10), (2.11). It is clear that for even operators \( \Psi(T_{i,j}(\bar{u})) = T_{j,i}(\bar{u}) \). Consider an odd monodromy matrix element \( T_{i,j}(u) \) for \( i < j \). This means that \( [i] = 0 \) and \( [j] = 1 \), and it follows from the commutation relations (2.9) that for any set \( \bar{u} \) with cardinality \#\( \bar{u} = a \) the product \( T_{i,j}(\bar{u}) \) given by (5.14) is symmetric with respect to permutations of the parameters \( u_i \).

For an odd monodromy matrix element \( T_{i,j}(u) \) with \( i > j \) and the set \( \bar{u} \) we define the product

\[
T_{i,j}(\bar{u}) = \Delta_h(\bar{u})^{-1} T_{i,j}(u_1) T_{i,j}(u_2) \cdots T_{i,j}(u_{a-1}) T_{i,j}(u_a),
\]

which is also symmetric with respect to permutations in the set \( \bar{u} \) due to the commutation relations (2.9).

Let us apply the antismorphism (2.10) to the product \( T_{i,j}(\bar{u}) \) with \( i < j \). Using the property (2.11), we get for \( i < j \) that

\[
\Psi(T_{i,j}(\bar{u})) = \Delta_h(\bar{u}) \Psi(T_{i,j}(u_1)) \Psi(T_{i,j}(u_2)) \cdots \Psi(T_{i,j}(u_{a-1})) \Psi(T_{i,j}(u_a))
\]

\[
= (-)^{a(a-1)/2} \Delta_h(\bar{u}) \Psi(T_{i,j}(u_1)) \Psi(T_{i,j}(u_2)) \cdots \Psi(T_{i,j}(u_{a-1})) \Psi(T_{i,j}(u_a))
\]

\[
= (-)^{a(a-1)/2} \Psi(T_{j,i}(\bar{u})).
\]
Similarly, for $i < j$ we can calculate that

$$
\Psi(T_{j,i}(\overline{u})) = (-)^{a(a+1)/2}\mathbb{T}_{i,j}(\overline{u})
$$

(5.31)

by taking into account that in this case

$$
\Psi(T_{j,i}(u)) = (-)^{|i|(|i|+1)}T_{i,j}(u) = -T_{i,j}(u).
$$

The relation (5.31) shows that for any $i$ and $j$ such that $|i| + |j| = 1$

$$
\Psi(\Psi(T_{i,j}(\overline{u}))) = (-)^a T_{i,j}(\overline{u}),
$$

and the antimorphism $\Psi$ is an idempotent of fourth order.

Thus, we have described the action of $\Psi$ on symmetric products of even and odd operators. Applying this action to the pre-Bethe vectors $\mathbb{B}(\vec{\ell})$ in (5.17) and $\widehat{\mathbb{B}}(\vec{\ell})$ in (5.25), we obtain explicit expressions for the dual pre-Bethe vectors $\mathbb{C}(\vec{\ell})$ and $\widehat{\mathbb{C}}(\vec{\ell})$, respectively. Up to a common sign factor they are still given by (5.17) and (5.25) with the opposite order of operator products and the replacement $T_{i,j} \rightarrow T_{j,i}$. Let us give explicit formulae for the particular case of the (dual) Bethe vectors $\mathbb{B}(\vec{\ell})$, $\widehat{\mathbb{B}}(\vec{\ell})$, $\mathbb{C}(\vec{\ell})$, and $\widehat{\mathbb{C}}(\vec{\ell})$ defined by (5.20) and (5.28) and connected with the Yangian double $DY(\text{gl}(2|1))$. In this case we have two sets of Bethe parameters $\vec{\ell}^\ell$ with cardinalities $\#\vec{\ell}^\ell = r_\ell$, $\ell = 1, 2$, which we rename as $\vec{\ell}^1 \equiv \vec{u}$ and $\vec{\ell}^2 \equiv \vec{v}$ with cardinalities $r_1 = a$ and $r_2 = b$. The formulae (3.14), (3.15), (3.22), and (3.23) for these Bethe vectors take the form

$$
\mathbb{B}_{a,b}(\overline{u}, \overline{v}) = f(\overline{v}, \overline{u})^{-1}\sum g(\overline{v}_1, \overline{u}_1)f(\overline{u}_1, \overline{u}_{11})g(\overline{u}_{11}, \overline{v}_1)h(\overline{u}_1, \overline{u}_1)
$$

(5.32)

$$
\times T_{1,3}(\overline{u}_1)T_{1,2}(\overline{u}_{11})T_{2,3}(\overline{v}_{11})\lambda_2(\overline{v}_1)|0\rangle,
$$

$$
\widehat{\mathbb{B}}_{a,b}(\overline{u}, \overline{v}) = f(\overline{v}, \overline{u})^{-1}\sum K_p(\overline{v}_1|\overline{u}_1)f(\overline{u}_1, \overline{u}_{11})g(\overline{u}_{11}, \overline{v}_1)
$$

(5.33)

$$
\times T_{1,3}(\overline{v}_1)T_{2,3}(\overline{u}_{11})T_{1,2}(\overline{u}_{11})\lambda_2(\overline{u}_1)|0\rangle,
$$

$$
\mathbb{C}_{a,b}(\overline{u}, \overline{v}) = (-)^{b(b-1)/2}f(\overline{v}, \overline{u})^{-1}\sum g(\overline{v}_1, \overline{u}_1)f(\overline{u}_1, \overline{u}_{11})g(\overline{u}_{11}, \overline{v}_1)h(\overline{u}_1, \overline{u}_1)
$$

(5.34)

$$
\times |0\rangle\lambda_2(\overline{v}_1)T_{3,2}(\overline{v}_{11})\cdot T_{2,1}(\overline{u}_{11})\cdot T_{3,1}(\overline{u}_1),
$$

$$
\widehat{\mathbb{C}}_{a,b}(\overline{u}, \overline{v}) = (-)^{b(b-1)/2}f(\overline{v}, \overline{u})^{-1}\sum K_p(\overline{v}_1|\overline{u}_1)f(\overline{u}_1, \overline{u}_{11})g(\overline{u}_{11}, \overline{v}_1)
$$

(5.35)

$$
\times |0\rangle\lambda_2(\overline{u}_1)T_{2,1}(\overline{u}_{11})T_{3,2}(\overline{v}_{11})T_{3,1}(\overline{v}_1),
$$

where the sums run over partitions of the sets $\overline{u} \Rightarrow \{\overline{u}_1, \overline{u}_{11}\}$ and $\overline{v} \Rightarrow \{\overline{v}_1, \overline{v}_{11}\}$ such that $\#\overline{u}_1 = \#\overline{v}_1 = p \leq \min(a, b)$.

The formulae (5.32)–(5.35) were already used in the series of papers [18]–[20] to calculate the form factors of the monodromy matrix elements in the supersymmetric quantum integrable models associated with the super-Yangian $Y(\text{gl}(2|1))$.

Appendix A. Composed currents and Gauss coordinates

In the completed algebras $\mathcal{U}_F$, $\mathcal{U}_E$, $\widehat{\mathcal{U}}_F$, and $\widehat{\mathcal{U}}_E$ a product of total currents has some specific analytical properties. This means that if one performs the normal ordering of the current generators in these products, then one can see the
pole structure of this product, which is encoded in the commutation relations of the total currents. This normal ordering procedure demonstrates that the products $F_i(u)F_{i+1}(v)$, $E_{i+1}(v)E_i(u)$, $\widehat{F}_{i+1}(v)\widehat{F}_i(u)$, and $\widehat{E}_i(u)\widehat{E}_{i+1}(v)$ have simple poles at $u = v$. We define the composed currents $F_{j,i}(u)$, $E_{i,j}(u)$, $\widehat{F}_{j,i}(u)$, and $\widehat{E}_{i,j}(u)$ for $1 \leq i < j \leq m + n$ inductively as residues:

$$F_{j,i}(v) = \text{res}_{u=v} F_{a,i}(v)F_{j,a}(u) = -\text{res}_{u=v} F_{a,i}(u)F_{j,a}(v), \quad (A.1)$$

$$E_{i,j}(v) = \text{res}_{u=v} E_{a,j}(u)E_{i,a}(u) = -\text{res}_{u=v} E_{a,j}(v)E_{i,a}(u), \quad (A.2)$$

$$\widehat{F}_{j,i}(v) = \text{res}_{u=v} \widehat{F}_{j,a}(u)\widehat{F}_{a,i}(u) = -\text{res}_{u=v} \widehat{F}_{j,a}(v)\widehat{F}_{a,i}(u), \quad (A.3)$$

$$\widehat{E}_{i,j}(v) = \text{res}_{u=v} \widehat{E}_{i,a}(v)\widehat{E}_{a,j}(u) = -\text{res}_{u=v} \widehat{E}_{i,a}(u)\widehat{E}_{a,j}(v), \quad (A.4)$$

where $i < a < j$ and we have denoted the simple root currents as follows: $F_i(u) \equiv F_{i+1,i}(u)$, $E_i(u) \equiv E_{i,i+1}(u)$, $\widehat{F}_i(u) \equiv \widehat{F}_{i+1,i}(u)$, and $\widehat{E}_i(u) \equiv \widehat{E}_{i,i+1}(u)$.

Calculating the residues in (A.1)–(A.4) with the help of the commutation relations (2.26), (2.27), (2.39), and (2.40), respectively, we obtain

$$F_{j,i}(v) = c_{[i+1]} \cdots c_{[j-1]} F_{j,j-1}(v)F_{j-1,j-2}(v) \cdots F_{i+1,i}(v), \quad (A.5)$$

$$E_{i,j}(v) = c_{[i+1]} \cdots c_{[j-1]} E_{i,i+1}(v)E_{i+1,i+2}(v) \cdots E_{j-1,j}(v), \quad (A.6)$$

$$\widehat{F}_{j,i}(v) = c_{[i+1]} \cdots c_{[j-1]} \widehat{F}_{i+1,i}(v)\widehat{F}_{i+2,i+1}(v) \cdots \widehat{F}_{j-1,j}(v), \quad (A.7)$$

$$\widehat{E}_{i,j}(v) = c_{[i+1]} \cdots c_{[j-1]} \widehat{E}_{j-1,j}(v)\widehat{E}_{j-2,j-1}(v) \cdots \widehat{E}_{i+1,i}(v). \quad (A.8)$$

Let us prove one of these formulae, namely, (A.5). Consider (A.1) for $j = i + 2$ and $a = i + 1$. Since we know that the product $F_{i+1,i}(v)F_{i+2,i+1}(u)$ has a simple pole at $u = v$, we can calculate the residue in (A.1) as follows:

$$F_{i+2,i}(v) = \text{res}_{u=v} F_{i+1,i}(v)F_{i+2,i+1}(u) = (u - v)F_{i+1,i}(v)F_{i+2,i+1}(u)\bigg|_{u=v}$$

$$= (u - v + c_{[i+1]})F_{i+2,i+1}(u)F_{i+1,i}(v)\bigg|_{u=v} = c_{[i+1]}F_{i+2,i+1}(v)F_{i+1,i}(v).$$

Here we have used the commutation relation (2.26) in passing from the first line to the second line. Now we perform the analogous calculation in the case of the current $F_{i+3,i}(v)$, using the simple root current $F_{i+3,i+2}(u)$ and the composed current $F_{i+2,i}(v)$ that we just calculated. By the commutativity of $F_{i+3,i+2}(u)$ and $F_{i+1,i}(v)$ we get that

$$F_{i+3,i}(v) = c_{[i+1]}c_{[i+2]}F_{i+3,i+2}(v)F_{i+2,i+1}(v)F_{i+1,i}(v).$$

Iterating the calculation, we get the formula (A.5). The proof of the formulæ (A.6)–(A.8) is completely analogous.

The composed currents are important in calculating the universal Bethe vectors using the formulæ (3.14) and (3.22). In this section we show that the projections of composed currents discussed in §4 coincide with the Gauss coordinates of the universal monodromy matrix (2.14)–(2.16) and (2.17)–(2.19) up to some unessential prefactors. To do this we rewrite the defining formulæ for the composed currents in integral form.
Both equations in (A.1) can be expressed in terms of contour integrals:

\[
F_{j,i}(v) = -\oint_{C_0} du F_{a,i}(v) F_{j,a}(u) + \oint_{C_\infty} du \frac{u - v + c_{[a]} F_{j,a}(u) F_{a,i}(v)}{(u - v)^{<}} = -\oint_{C_\infty} du F_{a,i}(u) F_{j,a}(v) + \oint_{C_0} du \frac{u - v - c_{[a]} F_{j,a}(v) F_{a,i}(u)}{(u - v)^{>}},
\]

where \(C_0\) and \(C_\infty\) are small closed contours around the points 0 and \(\infty\) on the complex \(u\)-plane. The rational functions \(1/(u - v)^{<}\) are defined by the series in (2.33).

For any formal series \(G(u) = \sum_{\ell \in \mathbb{Z}} G^{(\ell)} u^{-\ell-1}\) we define \(G^{(\pm)}(u)\) by

\[
G^{(\pm)}(u) = \pm \sum_{\ell \geq 0} G^{(\ell)} u^{-\ell-1}.
\]

It is obvious that the half-currents \(F^{(\pm)}\) and \(E^{(\pm)}\) coincide with the corresponding projections of currents only for the simple root currents \(F_i(u)\) and \(E_i(u)\). For the composed currents this is not the case, but nevertheless one can prove that

\[
P_f^+ (F^{(\pm)}_{j,i}(u) \cdot \mathcal{F}) = 0, \quad P_f^- (\mathcal{F} \cdot F^{(\pm)}_{j,i}(u)) = 0,
\]

\[
P_e^+ (\mathcal{E} \cdot E^{(\pm)}_{i,j}(u)) = 0, \quad P_e^- (E^{(\pm)}_{j,i}(u) \cdot \mathcal{E}) = 0
\]

for any elements \(\mathcal{F} \in \overline{U}_F\) and \(\mathcal{E} \in \overline{U}_E\). Similar properties can be formulated for the projections \(P_f^\pm\) and \(P_e^\pm\).

Using the notation (A.10) and calculating the formal contour integrals in (A.9) as

\[
\oint_{C_0} du G(u) = \oint_{C_\infty} du G(u) = G^{(0)},
\]

we obtain the following expressions for the composed currents \(F_{j,i}(v)\):

\[
F_{j,i}(v) = [F^{(0)}_{j,a}, F_{a,i}(v)] - c_{[a]} F^{-}_{a,i}(v) F_{a,i}(v) = [F_{j,a}(v), F^{(0)}_{a,i}] + c_{[a]} F_{j,a}(v) F^{(+)_{a,i}}(v).
\]

For the composed currents \(E_{i,j}(v)\) defined by (A.2) we have

\[
E_{i,j}(v) = -\oint_{C_0} du E_{a,j}(u) E_{i,a}(v) + \oint_{C_\infty} du \frac{u - v + c_{[a]} E_{i,a}(v) E_{a,j}(u)}{(u - v)^{<}} = -\oint_{C_\infty} du E_{a,j}(v) E_{i,a}(u) + \oint_{C_0} du \frac{u - v - c_{[a]} E_{i,a}(u) E_{a,j}(v)}{(u - v)^{>}},
\]

or by using (A.12) we get for these composed currents that

\[
E_{i,j}(v) = [E^{(0)}_{i,a}, E^{(0)}_{a,j}] - c_{[a]} E_{i,a}(v) E^{-}_{a,j}(v) = [E^{(0)}_{i,a}, E_{a,j}(v)] + c_{[a]} E^{(+)_{i,a}}(v) E_{a,j}(v).
\]
Similarly, for the currents $\hat{F}_{j,i}(v)$ defined by (A.3) we have
\[
\hat{F}_{j,i}(v) = \oint_{C_\infty} du \hat{F}_{j,a}(u)\hat{F}_{a,i}(v) - \oint_{C_0} du \frac{u - v + c[a]}{(u - v)} \hat{F}_{a,i}(v)\hat{F}_{j,a}(u)
\]
\[
= \oint_{C_0} du \hat{F}_{j,a}(v)\hat{F}_{a,i}(u) - \oint_{C_{\infty}} du \frac{u - v - c[a]}{(u - v)} \hat{F}_{a,i}(u)\hat{F}_{j,a}(v),
\]
(A.16)
or after calculating these formal contour integrals we get that
\[
\hat{F}_{j,i}(v) = [\hat{F}_{j,a}^{(0)}, \hat{F}_{a,i}(v)] + c[a] \hat{F}_{a,i}(v)\hat{F}_{j,a}^{(+)}(v)
\]
\[
= [\hat{F}_{j,a}(v), \hat{F}_{a,i}^{(0)}] - c[a] \hat{F}_{a,i}^{(-)}(v)\hat{F}_{j,a}(v).
\]
(A.17)

Finally, for the composed currents $\hat{E}_{i,j}(v)$ defined by (A.4) we can calculate
\[
\hat{E}_{i,j}(v) = \oint_{C_\infty} du \hat{E}_{i,a}(v)E_{a,j}(u) - \oint_{C_0} du \frac{u - v + c[a]}{(u - v)} \hat{E}_{a,j}(u)\hat{E}_{i,a}(v)
\]
\[
= \oint_{C_0} du \hat{E}_{i,a}(u)\hat{E}_{a,j}(v) - \oint_{C_{\infty}} du \frac{u - v - c[a]}{(u - v)} \hat{E}_{a,j}(v)\hat{E}_{i,a}(u),
\]
(A.18)
or, equivalently,
\[
\hat{E}_{i,j}(v) = [\hat{E}_{i,a}(v), \hat{E}_{a,j}^{(0)}] + c[a] \hat{E}_{a,j}^{(+)}(v)\hat{E}_{i,a}(v)
\]
\[
= [\hat{E}_{i,a}^{(0)}, \hat{E}_{a,j}(v)] - c[a] \hat{E}_{a,j}^{(-)}(v)\hat{E}_{i,a}(v).
\]
(A.19)

**Projections of composed currents.** The formulae (A.13), (A.15), (A.17), and (A.19) are very useful for calculating the projections of composed currents. Indeed, let us take $a = j - 1$ in the first line of (A.13) and apply the ‘positive’ projection $P_f^+$ defined by (3.9) to both sides of this equality. Similarly, we can consider the second line in (A.13) for $a = i + 1$ and apply the ‘negative’ projection $P_f^-$ to this equality. Using the properties of the projections (A.11), we have
\[
P_f^+(F_{j,i}(v)) = [F_{j,j-1}^{(0)}, P_f^+(F_{j-1,i}(v))],
\]
\[
P_f^-(F_{j,i}(v)) = [P_f^-(F_{j,i+1}(v)), F_{i+1,i}^{(0)}],
\]
(A.20)
where we have used the commutativity of the projections with the adjoint action of the zero modes of the simple root currents, which will be proved in Appendix B. Then the equations (A.20) can easily be iterated to obtain
\[
P_f^+(F_{j,i}(v)) = \mathcal{J}_{F_{j-1}^{(0)} F_{j-2}^{(0)} \cdots F_{i+1}^{(0)}}(F_{i+1,i}^{(0)}),
\]
\[
P_f^-(F_{j,i}(v)) = (-)^{j-i} \mathcal{J}_{F_{i+1}^{(0)} F_{i+2}^{(0)} \cdots F_{j-2}^{(0)}}(F_{j,j-1}^{(0)}),
\]
(A.21)
where we have used the relation between the projections of the simple root currents and the Gauss coordinates: $P_f^\pm(F_{i+1,i}(v)) = \pm F_{i+1,i}^{\mp}(v)$.

In a quite similar way we can get from (A.15), (A.17), and (A.19) that
\[
P_e^+(E_{i,j}(v)) = (-)^{j-i} \mathcal{J}_{E_{i-1}^{(0)} E_{i-2}^{(0)} \cdots E_{i+1}^{(0)}}(E_{i+1,i}^{(0)}),
\]
\[
P_e^-(E_{i,j}(v)) = - \mathcal{J}_{E_{i+1}^{(0)} E_{i+2}^{(0)} \cdots E_{j-2}^{(0)}}(E_{j-j-1}^{(0)}),
\]
(A.22)
The math expressions and text from the page are as follows:

\[ \hat{P}_f^- (\hat{F}_{j,i} (v)) = - \mathcal{F}_{F_{j-1,0}} \mathcal{F}_{F_{j-2,0}} \cdots \mathcal{F}_{F_{i+1,0}} (\hat{F}_{i+1,i} (v)), \]  
\[ \hat{P}_f^+ (\hat{F}_{j,i} (v)) = (-)^{j-i-1} \mathcal{F}_{F_{j-1,0}} \mathcal{F}_{F_{j-2,0}} \cdots \mathcal{F}_{F_{i+1,0}} (\hat{F}_{j,j-1} (v)), \]  
\[ \hat{P}_e^- (\hat{E}_{j,i} (v)) = (-)^{j-i} \mathcal{F}_{E_{j-1,0}} \mathcal{F}_{E_{j-2,0}} \cdots \mathcal{F}_{E_{i+1,0}} (\hat{E}_{i,i+1} (v)), \]  
\[ \hat{P}_e^+ (\hat{E}_{i,j} (v)) = \mathcal{F}_{E_{i-1,0}} \mathcal{F}_{E_{i-2,0}} \cdots \mathcal{F}_{E_{i+1,0}} (\hat{E}_{i,j-1} (v)). \]

In the rest of this section, we are going to show that the ‘positive’ projections of composed currents given by the first lines in (A.21) and (A.22) and the second lines in (A.23) and (A.24) coincide with the Gauss coordinates of the universal monodromy operator \( T_{i,j}^+ (v) \). To do this, we consider the relation (2.8) for \( i \to i, j \to j - 1, k \to j - 1, l \to j, \) and \( i < j - 1 \):

\[ [T_{i,j-1}^+(u), T_{j,j-1}^+(v)] = \frac{c[j-1]}{u - v} (T_{i,j}^+(u) T_{j-1,j-1}^+(v) - T_{i,j}^+(v) T_{j-1,j-1}^+(u)). \]

To obtain (A.25) from (2.7), we take into account that

\[ (-)^{[[i]+[j-1]]+[i][j]} = 1 \]

for any \( i \) and \( j \) satisfying \( i < j - 1 \), and the sign factor \( (-)^{[j][[i]+[j-1]]+[i][j]} \) is equal to \( (-)^{[j-1]} \).

One can easily see from the Gauss decomposition and the mode expansions of the Gauss coordinates (2.3) that the zero modes of the monodromy matrix elements coincide with the zero modes of the corresponding currents:

\[ \text{res}_{v \to \infty} v T_{i,i+1}^+(v) = (T_{i,i+1}^+(0)) = (F_{i,i+1}^+(0)) = (\hat{F}_{i,i+1}^+(0)) = F_i^+(0) = \hat{F}_i^+(0), \]

\[ \text{res}_{v \to \infty} v T_{i+1,i}^+(v) = (T_{i+1,i}^+(0)) = (E_{i+1,i}^+(0)) = (\hat{E}_{i,i+1}^+(0)) = E_i^+(0) = \hat{E}_i^+(0). \]

We multiply (A.25) by \( v \) and let \( v \to \infty \). By (A.26), this relation becomes

\[ c[j-1] T_{i,j}^+(u) = \mathcal{F}_{F_{j-1}^+(0)} T_{i,j-1}^+(u), \]

or, equivalently,

\[ c[j-1] (F_{j,i}^+(u) k_i^+(u) + \cdots) = \mathcal{F}_{F_{j-1}^+(0)} (F_{j-1,i}^+(u) k_i^+(u) + \cdots), \]

where the dots denote the terms given by the Gauss decomposition (2.14). One can use weight arguments to prove that the contribution of these terms vanishes, and by the commutativity of the Cartan current \( k_i^+(u) \) with the zero mode \( F_i^+(0) \) for \( i < j - 1 \), we get from (A.28) that

\[ c[j-1] F_{j,i}^+(u) = \mathcal{F}_{F_{j-1}^+(0)} F_{j-1,i}^+(u). \]

Iterating this relation for ‘positive’ Gauss coordinates, we obtain

\[ c[i,j] F_{j,i}^+(u) = \mathcal{F}_{F_{j-1}^+(0)} \cdots \mathcal{F}_{F_i^+(0)} (F_{i+1,i}^+(u)) = P^+_f (F_{j,i} (u)). \]
in accordance with the first line in (A.21), where we use the notation
\[ c_{[i,j]} = c_{[i+1]} c_{[i+2]} \cdots c_{[j-2]} c_{[j-1]} \]  
(A.30)
In particular, we set \( c_{[i,i+1]} = 1 \).

The formula (A.29) describes the connection between the ‘positive’ projection of composed currents and the ‘positive’ Gauss coordinates. The connection between the ‘negative’ projection of composed currents and the ‘negative’ Gauss coordinates is more complicated. To find it we apply the ‘negative’ projection to the first equality in (A.13) for \( a = j - 1 \) to obtain
\[ P_f^-(F_{j,i}(u)) = (\mathcal{S}_{F_{j-1}^{(0)}} - c_{[j-1]}F_{j,j-1}^-(u))P_f^-(F_{j-1,i}(u)), \]  
(A.31)
where we have used the equality \( F_{j,j-1}^-(u) = F_{j,j-1}^-(u) \) between ‘negative’ half-currents and ‘negative’ Gauss coordinates. Iterating (A.31), we obtain for the ‘negative’ projection an expression which uses only the zero-mode screening operators and the ‘negative’ Gauss coordinates:
\[ P_f^-(F_{j,i}(u)) = - (\mathcal{S}_{F_{j-1}^{(0)}} - c_{[j-1]}F_{j,j-1}^-(u))(\mathcal{S}_{F_{j-2}^{(0)}} - c_{[j-2]}F_{j-1,j-2}^-(u)) \times \cdots \\
\times (\mathcal{S}_{F_{i+1}^{(0)}} - c_{[i+1]}F_{i+2,i+1}^-(u))F_{i+1,i}^-(u), \]
where in the last step we have used the relation
\[ P_f^-(F_{i+1,i}(u)) = -F_{i+1,i}^-(u). \]
Multiplying out the parentheses in the equality above, we finally get that
\[ P_f^-(F_{j,i}(u)) = -c_{[i,j]} (F_{j,i}^-(u) + \sum_{\ell=1}^{j-i-1} (-\ell) \sum_{j > i_2 > \cdots > i_1 > i} F_{j,i_1}^-(u) \cdots F_{i_2,i_1}^-(u) F_{i_1,i}^-(u)). \]  
(A.32)
This expression is very useful for calculating the action of the monodromy matrix elements on Bethe vectors.

On the other hand, we can establish a connection between the projection of the composed current given by the second line in (A.23), and the Gauss coordinate defined by the relation (2.17). To do this we consider (2.7) for \( i \rightarrow j, j = k \rightarrow i + 1, l \rightarrow j, i < j - 1 \), which reduces to
\[ [T^+_{i,i+1}(u), T^+_{i+1,j}(v)] = \frac{c_{[i+1]}}{u-v} (T^+_{i+1,i+1}(v)T^+_{i,j}(u) - T^+_{i,i+1}(u)T^+_{i,j}(v)). \]  
(A.33)
As before, the factor \((-)^{([i]+[i+1]+[j])} \) is equal to 1 for any \( i \) and \( j \) satisfying \( i < j - 1 \), and the sign factor \((-)^{[i]+[i+1]+[j]} \) is equal to \((-)^{[i+1]} \). Multiplying the equality (A.33) by \( u \) and letting \( u \rightarrow \infty \), we obtain from (2.17) a relation between the Gauss coordinates:
\[ c_{[i+1]} F^+_{j,i}(v) = -\mathcal{S}_{F_i^{(0)}}(\hat{F}^+_{j,i+1}(v)). \]  
(A.34)
Iterating this equality, we find that
\[ c_{[i,j]} F^+_{j,i}(u) = (-)^{j-i-1} \mathcal{S}_{F_i^{(0)}} \cdots \mathcal{S}_{F_{j-2}^{(0)}}(\hat{F}^+_{j,j-1}(u)) = \hat{P}_f^+(\hat{F}_{j,i}(v)). \]  
(A.35)
For the relation between the ‘negative’ projection of composed currents and the ‘negative’ Gauss coordinates we have

\[
\hat{P}^-_j(\hat{F}_{j,i}(u)) = -c_{[i,j]}\left(\hat{F}^-_{j,i}(u) + \sum_{\ell=1}^{j-i-1} (-\ell) \sum_{j>i_\ell \ldots > i_1} \hat{F}^-_{i_\ell,i}(u) \hat{F}^-_{i_{\ell-1},i_\ell}(u) \cdots \hat{F}^-_{i_1,i}(u) \right).
\]

(A.36)

Again starting from (2.8) for \(i \rightarrow i + 1, j \rightarrow i, k \rightarrow j, l \rightarrow i + 1, \) and \(i < j - 1,\) we obtain a connection between the Gauss coordinate \(\hat{P}^+_e(\hat{E}_{i,j}(v))\) and the projection of the composed current \(\hat{P}^+_e(\hat{E}_{i,j}(v))\) by using analogous arguments and the Gauss decomposition (2.19):

\[
c_{[i,j]}\hat{E}^+_i,j(v) = \mathcal{F}^{(0)}_{i,j} \cdots \mathcal{F}^{(0)}_{j-2} \hat{E}^+_{j-1,j}(v) = \hat{P}^+_e(\hat{E}_{i,j}(v)).
\]

Finally, from the relation (2.7) for \(i \rightarrow j - 1, j \rightarrow i, k \rightarrow j, l \rightarrow j - 1, \) \(i < j - 1\) and (2.16) we get that

\[
c_{[i,j]}\hat{E}^+_i,j(u) = (-)^{j-i-1} \mathcal{F}^{(0)}_{j-1} \cdots \mathcal{F}^{(0)}_{i+1} \hat{E}^+_{i+1}(u) = P^+_e(E_{i,j}(v)). \quad (A.37)
\]

Summarizing the above considerations, we conclude that the ‘positive’ projections of composed currents coincide with the corresponding Gauss coordinates of the universal monodromy operator. The formulae for the connection for the ‘negative’ projections of composed currents are a bit more complicated, and one can also obtain formulae similar to (A.32) and (A.36) for the other two types of composed currents \(E_{i,j}(u)\) and \(\hat{E}_{i,j}(u)\).

\[
\text{Appendix B. Commutativity of the projections and the screening operators}
\]

The adjoint actions by the zero modes of simple root currents \(F^{(0)}_i, E^{(0)}_i\) and \(\hat{F}^{(0)}_i, \hat{E}^{(0)}_i\) play an important role. For any elements \(F \in U_F, E \in U_E, \hat{F} \in \hat{U}_F,\) and \(\hat{E} \in \hat{U}_E\) we introduce the screening operators

\[
\mathcal{F}^{(0)}_i(F) \equiv [F^{(0)}_i, F], \quad \mathcal{F}^{(0)}_i(E) \equiv [E^{(0)}_i, E],
\]

\[
\mathcal{F}^{(0)}_i(\hat{F}) \equiv [\hat{F}^{(0)}_i, \hat{F}], \quad \mathcal{F}^{(0)}_i(\hat{E}) \equiv [\hat{E}^{(0)}_i, \hat{E}]. \quad (B.1)
\]

One can check that the intersections of standard and current Borel subalgebras are all stable under the corresponding action of the screening operators.

Let us check, for example, that the subalgebras \(U_F^\pm\) defined by (3.6) are invariant under the adjoint action of the screening operators \(\mathcal{F}^{(0)}_i\) for \(i = 1, \ldots, N.\) It follows from (3.10) that any element \(F \in U_F\) can be represented in the normal ordered form \(F = \sum_\ell F^{(-)}_\ell \otimes F^{(+)}_\ell,\) where \(F^{(\pm)}_\ell \in U_F^\pm\) by definition. Then

\[
\mathcal{F}^{(0)}_i(F) = \sum_\ell \mathcal{F}^{(0)}_i(F^{(-)}_\ell) \cdot F^{(+)}_\ell + \sum_\ell F^{(-)}_\ell \cdot \mathcal{F}^{(0)}_i(F^{(+)}_\ell),
\]
and by the definition (3.9) of the projection $P_f^+$ we have

$$P_f^+ (\mathcal{F}_{\ell}^{(0)} (\mathcal{F})) = \sum_{\ell} \varepsilon (\mathcal{F}_{\ell}^{(0)} (\mathcal{F}_{\ell}^{(-)})) \cdot \mathcal{F}_{\ell}^{(+)} + \sum_{\ell} \varepsilon (\mathcal{F}_{\ell}^{(-)}) \cdot \mathcal{F}_{\ell}^{(0)} (\mathcal{F}_{\ell}^{(+)}). \tag{B.2}$$

The first sum on the right-hand side of (B.2) vanishes because $\mathcal{F}_{\ell}^{(0)} (\mathcal{F}_{\ell}^{(-)}) \in U_F^-$ if $\varepsilon (\mathcal{F}_{\ell}^{(-)}) = 0$. It also vanishes if $\varepsilon (\mathcal{F}_{\ell}^{(-)}) = 1$ in view of the definition of the screening operators and the commutation relations

$$\mathcal{F}_{\ell}^{(0)} (k_i^-(u)) = c_{[i]} F_i^-(u) k_i^-(u) \quad \text{and} \quad \mathcal{F}_{\ell}^{(0)} (k_{i+1}^- (u)) = -c_{[i+1]} F_i^-(u) k_{i+1}^- (u),$$

which easily follow from (2.22). Since $\varepsilon (\mathcal{F}_{\ell}^{(-)}) \in \mathbb{C}$, the equality (B.2) can be rewritten in the form

$$P_f^+ (\mathcal{F}_{\ell}^{(0)} (\mathcal{F})) = \mathcal{F}_{\ell}^{(0)} \left( \sum_{\ell} \varepsilon (\mathcal{F}_{\ell}^{(-)}) \cdot (\mathcal{F}_{\ell}^{(+)} \right) = \mathcal{F}_{\ell}^{(0)} (P_f^+ (\mathcal{F})),$$

which proves the assertion. The commutativity of the projections and the other relevant screening operators can be proved similarly.

**Appendix C. Calculation of the projection**

Let $\bar{v}$ be a set of variables with cardinality $\# \bar{v} = b$. Consider a product of composed currents (A.5)

$$F_{j_1,i}(v_1) \cdot F_{j_2,i}(v_2) \cdots F_{j_{b−1},i}(v_{b−1}) \cdot F_{j_b,i}(v_b), \tag{C.1}$$

with the following restrictions on the indices of the composed currents:

$$j_1 \geq j_2 \geq \cdots \geq j_{b−1} \geq j_b \geq i + 1. \tag{C.2}$$

In previous papers on the method of projections these products were called *strings*.

For any $\ell, \ell' = 1, \ldots, N$ with $\ell \leq \ell'$ denote by $U_{\ell, \ell'}$ the subalgebra of $U_F$ generated by the modes of the currents $F_\ell (t), F_{\ell+1} (t), \ldots, F_{\ell'} (t)$. Then $U_{\ell, \ell'}^\varepsilon = U_{\ell, \ell'} \cap \ker \varepsilon$ is the corresponding augmentation ideal.

**Proposition C.1.** The commutation relations between composed currents imply the equality

$$F_{i,i−1}(u_1) \cdots F_{i,i−1}(u_a) \cdot P_f^- \left( F_{j_1,i}(v_1) \cdot F_{j_2,i}(v_2) \cdots F_{j_{b−1},i}(v_{b−1}) \cdot F_{j_b,i}(v_b) \right)$$

$$= \frac{c_{[i]}^{−b}}{(a−b)!} \text{Sym}_{\pi} \left[ \prod_{\ell=1}^{b} g_{[i]}(v_{\ell}, u_{\ell}) \prod_{1 \leq \ell < \ell' \leq b} f_{[i]}(u_{\ell}, u_{\ell'}) f_{[i]}(v_{\ell'}, u_{\ell}) \prod_{\ell=1}^{b} \prod_{\ell' = b+1}^{a} f(u_{\ell}, u_{\ell'}) \right.$$

$$\times F_{j_1,i−1}(u_1) \cdot F_{j_2,i−1}(u_2) \cdots F_{j_{b−1},i−1}(u_{b−1}) \cdot F_{j_b,i−1}(u_b) \cdot F_{i,i−1}(u_{b+1}) \cdots F_{i,i−1}(u_a) \bigg] \mod P_f^- (U_{i,j−1}^\varepsilon \cdot U_{i−1,j−1}^\varepsilon). \tag{C.3}$$
\textbf{Proof.} In what follows, equality of elements \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) in the subalgebra \( \overline{U}_F \) modulo elements of the form \( P_f(U_{i,j-1}^c) \cdot U_{i-1,j-1} \) will be denoted by \( \mathcal{A}_1 \sim_{i,j} \mathcal{A}_2 \).

Let us prove (C.3) step by step. First of all, we observe that the ‘negative’ projection of the product of composed currents (C.1) with the restrictions (C.2) can be factorized \([13], [14]:\)

\[
P_f^-(F_{j_1,i_1}(v_1) \cdot F_{j_2,i_2}(v_2) \cdots F_{j_{b-1},i_{b-1}}(v_{b-1}) \cdot F_{j_{b},i_{b}}(v_{b}))
= P_f^-(F_{j_1,i_1}(v_1; v_2, \ldots, v_{b})) \cdot P_f^-(F_{j_2,i_2}(v_2; v_3, \ldots, v_{b})) \cdots P_f^-(F_{j_{b},i_{b}}(v_{b})),
\]

where \( F_{j,i}(v_1; v_2, \ldots, v_{b}) \) is the linear combination

\[
F_{j,i}(v_1; v_2, \ldots, v_{b}) = F_{j,i}(v_1) - \sum_{\ell=2}^{b} h_{[i]}(v_\ell, v_{i})^{-1} \prod_{\ell' = 2, \ell' \neq \ell}^{b} f_{[i]}(v_{\ell'}, v_{i}) F_{j,i}(v_\ell)
\]

\[\text{(C.4)}\]

of composed currents of the same type. Next, we observe that due to the first relation for the composed currents in (A.13) we have

\[
P_f^-(F_{j,i}(v)) + F_{j,i}^{(-)}(v) \sim_{i,j} \mathcal{A}_{F_{j-1,i}^{(0)}} P_f^-(F_{j-1,i}(v)) + F_{j-1,i}^{(-)}(v).
\]

Iterating this relation, we find that

\[
P_f^-(F_{j,i}(v)) + F_{j,i}^{(-)}(v) \sim_{i,j} \mathcal{A}_{F_{j-1,i}^{(0)}} \cdots \mathcal{A}_{F_{i+1,i}^{(0)}} P_f^-(F_{i+1,i}(v)) + F_{i+1,i}^{(-)}(v),
\]

and since \( P_f^-(F_{i+1,i}(v)) + F_{i+1,i}^{(-)}(v) = 0, \) we arrive at the relation

\[
P_f^-(F_{j,i}(v)) \sim_{i,j} -F_{j,i}^{(-)}(v).
\]

This means that

\[
P_f^-(F_{j_1,i_1}(v_1) \cdot F_{j_2,i_2}(v_2) \cdots F_{j_{b-1},i_{b-1}}(v_{b-1}) \cdot F_{j_{b},i_{b}}(v_{b}))
\sim_{i,j} (-)^b F_{j_1,i_1}^{(-)}(v_1; v_2, \ldots, v_{b}) \cdot F_{j_2,i_2}^{(-)}(v_2; v_3, \ldots, v_{b}) \cdots F_{j_{b},i_{b}}^{(-)}(v_{b}).
\]

\[\text{(C.5)}\]

Hence, by calculating the projection (5.2) one can move the terms of the form \( P_f^-(U_{i+1,j-1}^c) \) to the left through the product of currents \( F_1(u) \cdots F_{i-1}(u), \) where they disappear under the action of the ‘positive’ projection \( P_f^+ \). This fact allows us to replace the product of currents and the ‘negative’ projection on the left-hand side of (C.3) by the product

\[
(-)^b F_{i-1}^{(-)}(u_1) \cdots F_{i,i-1}^{(-)}(u_a) \cdot F_{j_1,i_1}^{(-)}(v_1; v_2, \ldots, v_{b}) \cdot F_{j_2,i_2}^{(-)}(v_2; v_3, \ldots, v_{b}) \cdots F_{j_{b},i_{b}}^{(-)}(v_{b}).
\]

The commutation relations between the product of the currents \( F_{i,i-1}(u) \) and the ‘negative’ half-currents \( F_{j,i}^{(-)}(v) \) can be calculated with the help of the relation

\[
F_{i,i-1}(u) F_{j,i}^{(-)}(v) = f_{[i]}(v,u)(F_{j,i}^{(-)}(v) - h_{[i]}(v,u)^{-1} F_{j,i}^{(-)}(u)) F_{i,i-1}(u)
\]

\[+ c_{[i]}^{-1} g_{[i]}(u,v) F_{j,i}(u).
\]

\[\text{(C.6)}\]
The latter equality is a consequence of the commutation relations
\[ F_{i,i-1}(u)F_{j,i}(v) = f_{[i]}(v,u) \ F_{j,i}(v)F_{i,i-1}(u) - \delta(u,v)F_{j,i-1}(u) \]
between simple root currents and composed currents and the definition of the ‘negative’ half-current
\[ F_{j,i}^{(-)}(v) = -\sum_{p<0} F_{j,i}^{(p)} u^{-p-1}. \]

Using the commutation relations (C.6), we get that
\[
F_{i,i-1}(u_{1}) \cdots F_{i,i-1}(u_{a}) \cdot F_{j,i}^{(-)}(v) \\
= f_{[i]}(v, \overline{v}) F_{j,i}^{(-)}(v; u_{1}, \ldots, u_{a}) \cdot F_{i,i-1}(u_{1}) \cdots F_{i,i-1}(u_{a}) \\
+ \sum_{q=1}^{a} c_{[i]}^{-1} g_{[i]}(u_{q}, v) \prod_{q'=q+1}^{a} \frac{(u_{q} - u_{q'}) c_{i,m+1} + c_{[i]}}{(u_{q} - u_{q'}) c_{i,m+1} - c_{[i]}} \\
\times F_{i,i-1}(u_{1}) \cdots F_{i,i-1}(u_{q-1}) \cdot F_{i,i-1}(u_{q+1}) \cdots F_{i,i-1}(u_{a}) \cdot F_{j,i-1}(u_{q}), \quad (C.7)
\]
where
\[
\tilde{F}_{j,i}^{(-)}(v; u_{1}, \ldots, u_{a}) = F_{j,i}^{(-)}(v) - \sum_{\ell=1}^{a} h_{[i]}(v, u_{\ell})^{-1} \prod_{q=1}^{a} \frac{f_{[i]}(u_{\ell}, u_{q})}{f_{[i]}(v, u_{q})} F_{j,i}^{(-)}(u_{\ell}). \quad (C.8)
\]

The linear combination of ‘negative’ half-currents (C.8) in the first term on the right-hand side of (C.7) commutes with all the products of currents
\[ F_{i-2}(u), \ldots, F_{1}(u). \]
Therefore, this term eventually disappears under the action of the ‘positive’ projection in (5.2). To transform the sum over \( q \) on the right-hand side of (C.7), we move the composed current to the right using for \( i \neq m + 1 \) the commutation relation
\[ F_{j,i-1}(u_{2})F_{i,i-1}(u_{1}) = f_{[i]}(u_{1}, u_{2})^{-1}F_{i,i-1}(u_{1})F_{j,i-1}(u_{2}) \quad (C.9) \]
and for \( i = m + 1 \) the commutation relation
\[ F_{j,m}(u_{2})F_{m+1,m}(u_{1}) = -f_{[m+1]}(u_{2}, u_{1})^{-1}F_{m+1,m}(u_{1})F_{j,m}(u_{2}) \]
or, what is the same,
\[ F_{j,m}(u_{2})F_{m+1,m}(u_{1}) = -f(u_{1}, u_{2})^{-1}F_{m+1,m}(u_{1})F_{j,m}(u_{2}). \quad (C.10) \]
Here we have used the fact that \([m + 1] = 1\) and \( f_{1}(u_{2}, u_{1}) = f(u_{1}, u_{2})\). The two cases \( i \neq m + 1 \) and \( i = m + 1 \) can be combined into one formula, and by the definition (3.3) of the deformed symmetrization the sum in (C.7) can be written as
\[
F_{i,i-1}(u_{1}) \cdots F_{i,i-1}(u_{a}) \cdot F_{j,i}^{(-)}(v) \\
\sim_{i,j} \frac{c_{[i]}^{-1}}{(a - 1)!} \overline{\text{Sym}}_{a} (g_{[i]}(u_{a}, v) F_{i,i-1}(u_{1}) \cdots F_{i,i-1}(u_{a-1}) \cdot F_{j,i-1}(u_{a})), \quad (C.11)
\]
or, equivalently,

\[
F_{i,i-1}(u_1) \cdots F_{i,i-1}(u_a) \cdot F_{j,i}^{(-)}(v) \\
\sim_{i,j} \frac{c_{[i]}^{-1}}{(a - 1)!} \bar{\text{Sym}}_{\pi} \left( g_{[i]}(u_1, v) f_{[i]}(u_1, \overline{\nu}_1) F_{j,i-1}(u_1) \cdot F_{i,i-1}(u_2) \cdots F_{i,i-1}(u_a) \right).
\]

(C.12)

Here we have to use the commutation relations (C.9) and (C.10) in order to obtain (C.12) from (C.11).

By using the definition of the linear combination of half-currents (C.4) and the summation formula

\[
g_{[i]}(u, v_1) f_{[i]}(\overline{\nu}_1, u) = g_{[i]}(u, v_1) f_{[i]}(\overline{\nu}_1, v_1) + \sum_{\ell = 2}^{b} g_{[i]}(u, v_\ell) g_{[i]}(v_1, v_\ell) \prod_{\ell' = 2, \ell' \neq \ell}^{b} f_{[i]}(v_\ell', v_\ell)
\]

we can now rewrite the equality (C.12) as

\[
F_{i,i-1}(u_1) \cdots F_{i,i-1}(u_a) \cdot F_{j,i}^{(-)}(v_1, v_2, \ldots, v_b) \sim_{i,j} \frac{c_{[i]}^{-1}}{(a - 1)!} \\
\times \bar{\text{Sym}}_{\pi} \left( g_{[i]}(u_1, v_1) f_{[i]}(u_1, \overline{\nu}_1) \frac{f_{[i]}(\overline{\nu}_1, u_1)}{f_{[i]}(\overline{\nu}_1, v_1)} F_{j,i-1}(u_1) \cdot F_{i,i-1}(u_2) \cdots F_{i,i-1}(u_a) \right).
\]

We can use this result for calculating the commutation of the product of currents \( F_{i,i-1}(u_1) \cdots F_{i,i-1}(u_a) \) with the ‘negative’ projection (C.5) modulo terms which vanish under the action of the ‘positive’ projection in (5.1). The result gives us the proof of the relation (C.3). Note that the deformed symmetrization \( \bar{\text{Sym}}_{\pi} \) over the set \( \overline{\nu} \) becomes the usual antisymmetrization over this set for \( i = m + 1 \).

We stress the meaning of (C.3). Moving the ‘negative’ projection of the string (C.1) through the product of currents \( F_{i,i-1}(u_1) \cdots F_{i,i-1}(u_a) \), we obtain linear combinations of analogous strings

\[
F_{j_1,i-1}(u_1) \cdot F_{j_2,i-1}(u_2) \cdots F_{j_a,i-1}(u_a)
\]

(C.13)

modulo terms which are irrelevant for calculation of the ‘positive’ projection in the definition of the Bethe vector (3.14), and with the restrictions

\[
j_1 \geq j_2 \geq \cdots \geq j_{a-1} \geq j_a \geq i,
\]

(C.14)

so that the first \( b \) indices \( j_\ell, \ell = 1, \ldots, b \), in the string (C.13) coincide with the corresponding indices in the string (C.1), and the remaining indices are equal to \( i \): \( j_{b+1} = \cdots = j_a = i \).

This linear combination is given by the deformed symmetrization over the set \( \overline{\nu} \), which can be reduced to a sum over partitions of this set. We describe these partitions.

Let \( p_1 \) be the number of equal indices \( j_\ell \) starting from \( j_1 \). Then let \( p_2 \) be the number of equal indices \( j_\ell \) starting from \( j_{p_1+1} \), and so on. Assume that the whole
set of indices $j_\ell$ is divided into $s$ subsets of identical indices with cardinalities $p_l$, $l = 1, \ldots, s$, and all $p_l > 0$. The integer $s$ counts the number of groups of composed currents of the same type in the string (C.1). It is clear that $1 \leq s \leq b$, including the cases when all the currents are the same ($s = 1$) or all currents are different ($s = b$). The restriction on the indices in the product of composed currents (C.1) induces a natural decomposition

$$\overline{v} = \{v_1, v_2, \ldots, v_{b-1}, v_b\} \Rightarrow \{\overline{v}^1, \ldots, \overline{v}^s\} \quad (C.15)$$

of the set $\overline{v}$ into $s$ disjoint subsets with cardinalities $\#\overline{v}^q = p_q$, $q = 1, \ldots, s$. Here we had to use a superscript to count these subsets, and this superscript should not be confused with the index which characterizes the type of Bethe parameters.

Assume that $a > b$. Let us decompose the set $\overline{v}$ into $s + 1$ disjoint subsets

$$\overline{v} = \{u_1, u_2, \ldots, u_{a-1}, u_a\} \Rightarrow \{\overline{v}^1, \ldots, \overline{v}^s, \overline{v}^{s+1}\} \quad (C.16)$$

such that

$$\#\overline{v}^q = p_q > 0 \quad \text{and} \quad \#\overline{v}^{s+1} = a - b.$$ 

The last subset $\overline{v}^{s+1}$ can be empty for the terms with $a = b$ in (C.3). According to the definition of the sizes of the subsets $\overline{v}^q$, $q = 1, \ldots, s$, we have

$$j_1 = \cdots = j_{p_1} > j_{p_1+1} = \cdots = j_{p_2} > \cdots > j_{p_{s-1}+1} = \cdots = j_{p_s} > i.$$ 

Let

$$j_{p_{\ell-1}+1} = \cdots = j_{p_\ell} = j'_\ell$$

for $\ell = 1, \ldots, s$. Using the definition of the ordered product of composed or simple currents of the same type given by (5.10) and dividing the initial set of variables $\overline{v}$ in (C.15) into the subsets $\overline{v}^q$, $q = 1, \ldots, s$, we can transform the string (C.1) as follows:

$$F_{j_1, i}(v_1) \cdot F_{j_2, i}(v_2) \cdots F_{j_{b-1}, i}(v_{b-1}) \cdot F_{j_b, i}(v_b) \rightarrow \mathcal{F}_{j'_1, i}(\overline{v}_1) \cdot \mathcal{F}_{j'_2, i}(\overline{v}_2) \cdots \mathcal{F}_{j'_{s-1}, i}(\overline{v}^{s-1}) \cdot \mathcal{F}_{j'_s, i}(\overline{v}^s). \quad (C.17)$$

Denote the ordered product of currents on the right-hand side of (C.17) by

$$\mathcal{F}_{\overline{v}, i}(\overline{v}) = \mathcal{F}_{j'_1, i}(\overline{v}_1) \cdot \mathcal{F}_{j'_2, i}(\overline{v}_2) \cdots \mathcal{F}_{j'_{s-1}, i}(\overline{v}^{s-1}) \cdot \mathcal{F}_{j'_s, i}(\overline{v}^s),$$

where $\overline{v}' = \{j'_1, \ldots, j'_s\}$ and $j'_1 > j'_2 > \cdots > j'_s > i$.

Similarly, after dividing the set $\overline{v}$ into the subsets (C.16), we transform the string (C.13) into

$$\mathcal{F}_{\overline{v}, i-1}(\overline{v}) = \mathcal{F}_{j'_1, i-1}(\overline{v}^1) \cdot \mathcal{F}_{j'_2, i-1}(\overline{v}^2) \cdots \mathcal{F}_{j'_{s-1}, i-1}(\overline{v}^{s}) \cdot \mathcal{F}_{i, i-1}(\overline{v}^{s+1}), \quad (C.18)$$

where $\overline{v}' = \{j'_1, \ldots, j'_s, i\}$.

In order to rewrite the sum over permutations of the elements of the set $\overline{v}$ on the right-hand side of (C.3), we multiply both sides of (C.18) by the rational function $\Delta_{f_{[i]}(\overline{v})} \Delta_{h_{[i]}(\overline{v})}\delta_{i,m+1}$. Then using the fact that for any formal series $G(\overline{v})$ the
deformed symmetrization (or antisymmetrization in the case when \( i = m + 1 \)) can be transformed into the usual symmetrization over \( \vec{u} \), that is, using

\[
\frac{\Delta f_{[i]}(\vec{u})}{\Delta h_{[i]}(\vec{u})^{\delta_i,m+1}} \text{Sym}_{\vec{u}}(G(\vec{u})) = \text{Sym}_{\vec{u}}\left( \frac{\Delta f_{[i]}(\vec{u})}{\Delta h_{[i]}(\vec{u})^{\delta_i,m+1}} G(\vec{u}) \right),
\]

we can replace it by the sum over partitions (C.16) and by symmetrizations over the subsets in the partition:

\[
\text{Sym}_{\vec{u}}(\cdot) = \sum_{\vec{u}\Rightarrow\{\vec{u}^1,\ldots,\vec{u}^s,\vec{u}^{s+1}\}} \text{Sym}_{\vec{u}^1} \text{Sym}_{\vec{u}^2} \cdots \text{Sym}_{\vec{u}^s} \text{Sym}_{\vec{u}^{s+1}}(\cdot).
\]

Below we use the fact that after multiplication of both sides of (C.3) by the rational function \( \Delta f_{[i]}(\vec{u})\Delta h_{[i]}(\vec{u})^{-\delta_i,m+1} \), we can sum over the symmetrizations in all the disjoint subsets \( \vec{u}^q \), \( q = 1, \ldots, s + 1 \), on the right-hand side of (C.3).

For any composed current \( F_{j,i}(u), j > i \), we introduce its parity \( \mu_{i,j} \) defined by

\[
\mu_{i,j} = [i] + [j] = \begin{cases} 1, & i \leq m \leq j - 1, \\ 0, & i > m \text{ or } m > j - 1. \end{cases}
\]

We refer to composed currents with parity 1 as odd and to those with parity 0 as even. Using the commutation relations for simple root currents, one can check that the commutation relations between even composed currents are the same as for even simple root currents, while odd composed currents anticommute:

\[
\begin{align*}
(u - v - c_{[i]} F_{j,i}(u)) F_{j,i}(v) &= (u - v + c_{[i]} F_{j,i}(v)) F_{j,i}(u) \quad \text{for } \mu_{i,j} = 0, \\
F_{j,i}(u) F_{j,i}(v) &= -F_{j,i}(v) F_{j,i}(u) \quad \text{for } \mu_{i,j} = 1. \tag{C.19}
\end{align*}
\]

If \( m + 1 < i \leq N \), then it is clear from the restrictions (C.2) and (C.14) that only even currents (simple and composed alike) appear in both sides of (C.3). Otherwise, for \( i = m + 1 \) all the currents (again, simple and composed alike) on the right-hand side of (C.3) are odd. But if \( 1 < i \leq m \), then there are both odd and even currents on the right-hand side of (C.3), and according to the structure of the initial string (C.1) all the odd currents are to the left of all the even currents. In this case there are \( s' \) \( (1 \leq s' < s) \) factors in the string which are products of the same odd currents. In view of the commutation relations (C.19) for composed currents, the symmetrizations over the subsets \( \vec{u}^q \) with \( q = 1, \ldots, s' \) and over those with \( q = s' + 1, \ldots, s + 1 \) will be implemented differently. For \( m + 1 \leq i \leq N \) the symmetrizations over all the subsets \( \vec{u}^q \) for \( q = 1, \ldots, s + 1 \) are the same. The number \( s' \) can be calculated as follows:

\[
s' = \sum_{\ell=1}^{s} \mu_{i,j'_\ell}, \tag{C.20}
\]
We first consider the case \( m + 1 \leq i \leq N \). Multiplying both sides of (C.3) by the function \( \gamma_{i-1}(\uppi) \), we get that

\[
\gamma_{i-1}(\uppi) \mathcal{F}_{i,i-1}(\uppi) \cdot P_f^{-1}(\mathcal{F}_{j,i}(\uppi)) \sim_{i,j_1} \frac{c_{[i]}^{b}}{\Delta f_{[i]}(\uppi)} \sum_{\uppi} f_{[i]}(\uppi^q, \uppi^{q'}) \\
\times \prod_{q<q'}^{s} f_{[i]}(\uppi^q, \uppi^q') \gamma_{i-1}(\uppi) \mathcal{F}_{j,i-1}(\uppi)
\]

\[
\times \prod_{q=1}^{s} \text{Sym}_{\uppi} \left[ \Delta_{f_{[i]}}'(\uppi^q) \prod_{\ell} g_{[i]}(v_{\ell}, u_{\ell}) \prod_{\ell < \ell'} f_{[i]}(v_{\ell'}, u_{\ell}) \right] \quad \text{(C.21)}
\]

where we have used the fact that the product of the function \( \gamma_{i-1}(\uppi) \) and the string (C.18) is symmetric with respect to permutations within each subset \( \uppi^q \). In particular, this symmetry allows us to get rid of symmetrization over the subset \( \uppi^{s+1} \) and cancel the combinatorial factor \( (a - b)!^{-1} \) in (C.3). Note that if \( i = m + 1 \), then all the currents in the product \( \mathcal{F}_{j,m}(\uppi) \) become odd, and the symmetry with respect to permutations of the variables in each subset \( \uppi^q \) is ensured by the function \( \gamma_{m}(\uppi) = \Delta g_{[m]}(\uppi) \).

The remaining symmetrization over each subset \( \uppi^q, q = 1, \ldots, s \), is the well-known Izergin determinant \([29]\) defined for two sets \( \uppi \) and \( \uppi \) with the same cardinality \( \#\uppi = \#\uppi = p \) as follows:

\[
K_{[i]}(\uppi | \uppi) = \text{Sym}_{\uppi} \left[ \Delta_{f_{[i]}}(\uppi) \prod_{k=1}^{p} g_{[i]}(y_k, x_k) \prod_{k < k'} f_{[i]}(y_{k'}, x_{k'}) \right] \\
= \Delta g_{[i]}(\uppi) \Delta g_{[i]}'(\uppi) h_{[i]}(\uppi, \uppi) \det \left[ g_{[i]}(y_k, x_{k'}) \right] \quad \text{(C.22)}
\]

Thus, we conclude that if the index \( i \) belongs to the interval \( m + 1 \leq i \leq N \), then (C.3) can be rewritten as a sum over partitions of \( \uppi \) which is determined by the string \( \mathcal{F}_{j,i}(\uppi) \):

\[
\gamma_{i-1}(\uppi) \mathcal{F}_{i,i-1}(\uppi) \cdot P_f^{-1}(\mathcal{F}_{j,i}(\uppi)) \\
\sim_{i,j_1} \frac{c_{[i]}^{b}}{\Delta f_{[i]}(\uppi)} \sum_{\uppi} f_{[i]}(\uppi^q, \uppi^{q'}) \prod_{q<q'} f_{[i]}(\uppi^q, \uppi^q') \\
\times \prod_{q=1}^{s} K_{[i]}(\uppi^q | \uppi^q) \gamma_{i-1}(\uppi) \mathcal{F}_{j,i-1}(\uppi) \quad \text{(C.23)}
\]

Consider now the case when \( 1 < i \leq m \). As mentioned above, in this case the product of currents \( \mathcal{F}_{j,i-1}(\uppi) \) contains both odd and even composed currents. Therefore, to perform symmetrization over the subsets \( \uppi^q \) we have to use different approaches for odd and even currents.

Let \( s', 1 \leq s' \leq s \), be the number of products of the same odd currents on the right-hand side of (C.3), which is given by (C.20). Then the symmetrization over the subsets \( \uppi^q \) for \( s' < q \leq s + 1 \) in (C.21) is exactly the same as described above.
It leads to the appearance of Izergin determinants depending on the corresponding sets of variables. Since variables in the subsets $\bar{\pi}^q$ for $1 \leq q \leq s'$ become arguments of odd anticommuting currents, the relation (3.3) takes the following form after multiplication by the function in (3.1):

$$
\gamma_{i-1}(\bar{\pi}) \mathcal{F}_{i-1}(\bar{\pi}) \cdot P_f \left( \mathcal{F}_{i-1}(\bar{\pi}) \right)
$$

$$
\sim_{i,j_1} \frac{c_{[i]}^{b}}{\Delta f_{[i]}(\bar{\pi})} \sum_{\bar{\pi} = \{\bar{\pi}^1, \ldots, \bar{\pi}^{s+1}\}} \prod_{q < q'} \prod_{q < q'} f_{[i]}(\bar{\pi}, \bar{\pi}'') \prod_{q = s'+1} K_{[i]}(\bar{\pi}'', \bar{\pi}'')
$$

$$
\times \prod_{q=1}^{s'} \text{Sym}_{\pi^q} \Delta'_{g_{[i]}(\bar{\pi})} \prod_{\ell < \ell'} g_{[i]}(v_{\ell}, u_{\ell}) \prod_{\ell < \ell'} f_{[i]}(v_{\ell'}, u_{\ell'})
$$

$$
\times \gamma_{i-1}(\bar{\pi}) \prod_{q=1}^{s'} \Delta'_{h_{[i]}(\bar{\pi})} \mathcal{F}_{i-1}(\bar{\pi}),
$$

(C.24)

where we have used the factorization $\Delta'_{f_{[i]}(\bar{\pi})} = \Delta'_{g_{[i]}(\bar{\pi})} \Delta'_{h_{[i]}(\bar{\pi})}$.

The fact that the products of odd currents on the right-hand side of (C.24) can be taken out from under the sign for symmetrization over the subsets $\pi^q$, $q = 1, \ldots, s'$, follows from the observation that for $1 < i \leq m$ the function $\gamma_{i-1}(\bar{\pi}) = \Delta f_{[i-1]}(\bar{\pi})$ contains the factors $\Delta h_{[i-1]}(\bar{\pi})$ and $\Delta g_{[i-1]}(\bar{\pi})$. The first factor together with the function $\Delta h_{[i]}(\bar{\pi})$ gives a function that is symmetric with respect to the variables in the subset $\bar{\pi}$:

$$
\Delta h_{[i-1]}(\bar{\pi}) \Delta'_{h_{[i]}(\bar{\pi})} = \Delta h_{[i]}(\bar{\pi}) \Delta'_{h_{[i]}(\bar{\pi})} = h_{[i]}(\bar{\pi}, \bar{\pi}) \quad \text{for} \quad 1 < i \leq m,
$$

while the second factor $\Delta g_{[i-1]}(\bar{\pi})$ makes symmetric the product of the odd currents depending on the variables in $\bar{\pi}^q$.

We denote the normalized symmetrization in the third line of (C.24) by $\mathcal{C}_{[i]}(\bar{\pi}|\bar{\pi})$:

$$
\mathcal{C}_{[i]}(\bar{\pi}|\bar{\pi}) = \Delta'_{h_{[i]}(\bar{\pi})} \text{Sym}_{\pi} \left[ \Delta'_{g_{[i]}(\bar{\pi})} \prod_{\ell < \ell'} g_{[i]}(v_{\ell}, u_{\ell}) \prod_{\ell < \ell'} f_{[i]}(v_{\ell'}, u_{\ell'}) \right]_{v_{\ell}, v_{\ell'} \in \bar{\pi}, u_{\ell}, u_{\ell'} \in \bar{\pi}}
$$

This function is proportional to the Cauchy determinant, as follows from the chain of equalities

$$
\mathcal{C}_{[i]}(\bar{\pi}|\bar{\pi}) = \Delta'_{h_{[i]}(\bar{\pi})} \Delta'_{h_{[i]}(\bar{\pi})} \text{ASym}_{\pi} \left[ \prod_{\ell < \ell'} g_{[i]}(v_{\ell}, u_{\ell}) \prod_{\ell < \ell'} f_{[i]}(v_{\ell'}, u_{\ell'}) \right]_{v_{\ell}, v_{\ell'} \in \bar{\pi}, u_{\ell}, u_{\ell'} \in \bar{\pi}}
$$

$$
= \Delta'_{f_{[i]}(\bar{\pi})} \Delta f_{[i]}(\bar{\pi}) \text{ASym}_{\pi} \left[ \prod_{\ell} g_{[i]}(v_{\ell}, u_{\ell}) \right]_{v_{\ell} \in \bar{\pi}, u_{\ell} \in \bar{\pi}}
$$

$$
= \Delta'_{f_{[i]}(\bar{\pi})} \Delta f_{[i]}(\bar{\pi}) \text{ASym}_{\pi} \left[ g_{[i]}(\bar{\pi}, \bar{\pi}) = \Delta'_{h_{[i]}(\bar{\pi})} \Delta h_{[i]}(\bar{\pi}) g_{[i]}(\bar{\pi}, \bar{\pi}),
$$

where the symbol $\text{ASym}_{\pi}$ means antisymmetrization with respect to the set $\bar{\pi}$. 
Thus, for $1 < i \leq m$ the relation (C.3) can be represented as the following sum over partitions:

$$\gamma_{i-1}(\overline{u}) F_{i,i-1}(\overline{u}) \cdot P_f^{-}(\mathcal{F}_{y,i}(\overline{v}))$$

$$\sim_{i,j} \frac{c_{i,j}}{\Delta f_{i,j}(\overline{v})} \sum_{\pi \Rightarrow \{\pi^1, ..., \pi^s, \pi^{s+1}\}} \prod_{q < q'}^{s+1} f_{i,j}(\overline{u}^{q'}, \overline{u}^{q}) \prod_{q < q'}^{s} f_{i,j}(\overline{u}^{q'}, \overline{u}^{q})$$

$$\times \prod_{q=1}^{s'} C_{i,j}(\overline{u}^{q} | \overline{u}^{q'}) \prod_{q=s'+1}^{s} K_{i,j}(\overline{u}^{q} | \overline{u}^{q'}) \gamma_{i-1}(\overline{v}) F_{y,i-1}(\overline{v}),$$

where $s'$ is given by (C.20).

Now we apply (C.23) and (C.25) to the calculation of the projection (5.2) and thereby obtain the recursion relation for the Bethe vectors (3.14).

We should add to (C.23) and (C.25) the rule for ordering the subsets $\pi^{q}$. As we indicated in the definition of the string (C.18), the subsets with smaller indices occur in more complicated composed currents to the left in (C.18).

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