On the Power and Limits of Dynamic Pricing in Combinatorial Markets

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Abstract

We study the power and limits of optimal dynamic pricing in combinatorial markets; i.e., dynamic pricing that leads to optimal social welfare. Previous work by Cohen-Addad et al. [EC’16] demonstrated the existence of optimal dynamic prices for unit-demand buyers, and showed a market with coverage valuations that admits no such prices. However, finding the frontier of markets (i.e., valuation functions) that admit optimal dynamic prices remains an open problem. In this work we establish positive and negative results that narrow the existing gap.

On the positive side, we provide tools for handling markets beyond unit-demand valuations. In particular, we characterize all optimal allocations in multi-demand markets. This characterization allows us to partition the items into equivalence classes according to the role they play in achieving optimality. Using these tools, we provide a poly-time optimal dynamic pricing algorithm for up to 3 multi-demand buyers.

On the negative side, we establish a maximal domain theorem, showing that for every non-gross substitutes valuation, there exist unit-demand valuations such that adding them yields a market that does not admit an optimal dynamic pricing. This result is reminiscent of the seminal maximal domain theorem by Gul and Stacchetti [JET’99] for Walrasian equilibrium. Yang [JET’17] discovered an error in their original proof, and established a different, incomparable version of their maximal domain theorem. En route to our maximal domain theorem for optimal dynamic pricing, we provide the first complete proof of the original theorem by Gul and Stacchetti.

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1 Introduction

We study the power and limitations of (anonymous) pricing schemes in achieving optimal social welfare in combinatorial markets. We consider combinatorial markets with $m$ heterogeneous, indivisible goods, and $n$ buyers with (publicly known) complex, idiosyncratic preferences over bundles of items. In the most general form, every buyer has a monotone valuation function $v_i : 2^m \to \mathbb{R}_{\geq 0}$, which assigns a real (non-negative) value to every bundle, and the goal is to allocate items to buyers in a way that maximizes the social welfare; i.e., the sum of buyer values.

If only computational constraints are taken into account, a welfare maximizing allocation can be efficiently computed in the full information setting whenever the buyers exhibit preferences as simple as gross-substitutes [34]. Moreover, the celebrated VCG mechanism [38, 7, 25] achieves this even in strategic settings. Real world markets, however, usually employ simpler mechanisms, such as pricing, and understanding their performance is of great interest.

To this end, we study the welfare maximization problem by means of an anonymous pricing scheme. Apart from being simple, pricing schemes are attractive since they lack an all powerful central authority. Once the prices are set, the buyers arrive and simply choose a desired set of items from the available inventory. This procedure is quite prevalent in market scenarios.

Formally, the seller sets items prices $p = (p_1, \ldots, p_m) \in \mathbb{R}_{\geq 0}^m$, buyers arrive sequentially (in an arbitrary order), and every buyer chooses a bundle $T$ (from the remaining items) that maximizes the (quasi-linear) utility: $u_i(T, p) = v_i(T) - \sum_{j \in T} p_j$, breaking ties arbitrarily.

Maximizing welfare by pricing is challenging even for relatively simple valuations and even when the seller has full knowledge of the valuation profile, due to the large degree of freedom it leaves to the buyers. A reader familiar with the fundamental notion of Walrasian equilibrium, which dates back to the 19th century [39] (also known as market/pricing/competitive equilibrium), may conclude that the problem is solved for any market that admits a Walrasian equilibrium. A Walrasian equilibrium is a pair of an allocation $S = (S_1, \ldots, S_n)$ and prices $p = (p_1, \ldots, p_m)$, such that for every buyer $i$, $S_i$ maximizes $i$’s utility given $p$. By the first welfare theorem, every Walrasian equilibrium maximizes social welfare.

Are Walrasian prices a solution to our problem? The answer, as was previously observed [10, 28], is no. Walrasian prices alone cannot resolve a market without coordinating the tie breaking. If a buyer is faced with multiple utility-maximizing bundles, it is crucial that a central authority coordinate the tie breaking in accordance with the corresponding optimal allocation. In real-world markets, however, buyers are only faced with the prices and choose a desired bundle by themselves without caring about global efficiency. [10] demonstrated that lacking a tie-breaking coordinator, Walrasian pricing can lead to an arbitrarily bad allocation. Moreover, they showed that no pricing whatsoever can achieve more than $2/3$ of the optimal social welfare in the worst case, even when restricted to unit-demand buyers (where every buyer has value for every item, and the value for a set is the maximum value of any item in the set).

In order to circumvent this state of affairs, [10] propose a more powerful pricing scheme, namely dynamic pricing, where the seller dynamically updates prices in between buyer arrivals (based on the remaining buyers and the current inventory).

Consider the following example, given by [10], for a market that has no optimal static pricing but does have an optimal dynamic pricing.

**Example 1.** The market consists of 3 unit-demand buyers Alice, Bob and Carl, and three items $a, b, c$. Alice has value 1 for $a, b$ and 0 for $c$, Bob has value 1 for $b, c$ and 0 for $a$, and Carl has value...
1 for $a, c$ and 0 for $b$. Consider the initial prices $p_a = p_b = p_c = \frac{1}{2}$. Since the market is symmetric, we can assume w.l.o.g. that Alice comes first. At these prices she takes either $a$ or $b$, and let’s assume (again w.l.o.g.) that she takes $a$. In the second round, the seller can price $b$ at $\frac{1}{4}$ and $c$ at $\frac{1}{2}$. If Bob arrives next then he takes $b$, and if Carl arrives next then he takes $c$. The remaining item can then be priced at 0 to guarantee that it is taken by the last buyer. In any case, social welfare is maximized.

For a given valuation class $C$, the existence of an optimal dynamic pricing reduces to the existence of an adequate initial price vector. If there always is an initial price vector guaranteeing that regardless of the identity of the first buyer, and regardless of how she breaks ties, the bundle she consumes can always be completed to an optimal allocation, then some optimal allocation can always be achieved, by induction. The following definition of optimal dynamic pricing follows (the formal definition appears as Definition 2.1):

**Definition.** A price vector $p$ is an *optimal dynamic pricing* (hereafter, *dynamic pricing*) if for every buyer $i$, and every utility-maximizing bundle $T$ for $i$ (given $p$), $T$ is allocated to $i$ in some optimal allocation.

Due to the first welfare theorem, Walrasian pricing can be defined in the following way, which closely resembles the definition of dynamic pricing:

**Definition.** A price vector $p$ is a *Walrasian pricing* if there is an optimal allocation $S = (S_1, \ldots, S_n)$ such that $S_i$ maximizes $i$’s utility (given $p$) for every buyer $i$.

Despite the similarity between the two definitions, they are incomparable. [10] provide an example of a market with coverage valuations that has a Walrasian pricing but not a dynamic pricing. For the converse direction, we borrow from [20] an example of a market that has no Walrasian pricing, and show that it does admit a dynamic pricing (see section 6).

What is known about the existence of these two pricing notions? For Walrasian pricing we know much. Kelso and Crawford [30] prove that every gross-substitutes market (a strict super-set of unit-demand, and a strict subset of submodular) admits a Walrasian pricing. Moreover, Gul and Stacchetti [26] show that this result is tight in the following sense:

**Theorem 1.1** (Maximal Domain Theorem for Walrasian Equilibrium [26]). Let $v_1$ be a non gross-substitutes valuation. Then, there exist unit-demand valuations $v_2, \ldots, v_\ell$ for some $\ell$ such that the valuation profile $(v_1, v_2, \ldots, v_\ell)$ does not admit a Walrasian equilibrium\(^1\).

The combination of these results completely characterizes the existence of Walrasian pricing for classes of valuations that contain unit-demand valuations.

For dynamic pricing, however, the picture is significantly less clear. [10] show that unit-demand markets always admit dynamic prices. Moreover, their result extends to gross-substitutes markets in which there is a unique optimal allocation. For markets without unique optimum, no positive result beyond unit-demand buyers is known. On the negative end, the state of affairs is even worse; we only know of the coverage valuations example mentioned above, which does not admit a dynamic pricing. The following natural question arises:

What markets (i.e., what valuation classes) can be resolved optimally using dynamic pricing schemes?

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\(^1\)Yang [40] discovered an error in the proof; details to follow.
1.1 Our Results

In this work we shrink the gap between possibility and impossibility guarantees for optimal dynamic pricing, from both ends.

Positive Results. A natural extension of unit-demand valuations is multi-demand valuations, where every buyer $i$ has a cap (a.k.a demand) $k_i$ on the number of desired items, and the value for a set is the sum of the values for the $k_i$ most valued items in the set. Our main positive result is the following:

Theorem 1.2. Every market with up to 3 multi-demand buyers admits a dynamic pricing. Moreover, the prices can be computed in polynomial (in the number of items $m$) time, using value queries.

A major component in the analysis is the following theorem that characterizes the set of optimal allocations in multi-demand markets (with any number of buyers, and any demand vector). For the sake of simplicity, we present the theorem for markets in which all items are allocated in every optimal allocation, and in which the total demand of the players equals supply (however, an analogous result holds also for the case where demand exceeds supply, see Appendix B).

Theorem (Optimal Allocation Characterization). (Informal) In a market with $n$ multi-demand buyers, an allocation $A$ is optimal if and only if the following hold:

- The number of items that each buyer receives equals that buyer’s demand.
- If item $x$ is allocated to buyer $i$, then there is an optimal allocation where $x$ is allocated to $i$.

In addition to extending the result of [10] beyond unit-demand buyers, we also contribute to the domain of unit-demand buyers. In the pricing scheme proposed by [10] buyers may have to break ties. In section 4 we propose an alternative scheme for unit-demand buyers, where tie breaking is eliminated altogether.

Theorem 1.3. Every market with unit-demand buyers admits a dynamic pricing scheme where every buyer has a unique utility-maximizing bundle at every stage of the procedure.

Negative Results. In 2017, Yang [40] discovered an error in the proof of the maximal domain theorem for Walrasian equilibrium by Gul and Stacchetti. Unfortunately, the error is not in the analysis of the construction, rather in the construction itself. Indeed, Yang showed an instance of their constructed market that does admit a Walrasian equilibrium. In the same work he proved an alternative, incomparable theorem: for every non gross-substitutes valuation there is a (single) gross-substitutes valuation for which the obtained market has no Walrasian equilibrium. Yang’s theorem is stronger, in the sense that only a single valuation is added to guarantee the non-existence of Walrasian equilibrium. On the other hand, Gul and Stacchetti’s theorem is stronger in the sense that the added valuations have the simple structure of unit-demand valuations. In section 5 we provide a complete proof of the maximal domain theorem as it was originally stated. The proof is a major component in our main negative result, a maximal domain theorem for dynamic pricing:

Theorem 1.4. Let $v_1$ be a non gross-substitutes valuation. Then, there are unit-demand valuations $v_2, \ldots, v_\ell$ for some $\ell$ such that the valuation profile $(v_1, v_2, \ldots, v_\ell)$ does not admit a dynamic pricing.

\[2\] A value query for a valuation $v$ receives a set $S$ as input, and returns $v(S)$, see Section 2.
While this result is analogous to the theorem by Gul and Stacchetti, we emphasize that the notions of dynamic pricing and Walrasian pricing are incomparable, and thus none of the maximal domain theorems directly implies the other.

1.2 Our Techniques

Techniques for Positive Results. Our starting point is the dynamic pricing scheme of [10] for unit-demand buyers. In a nutshell, their scheme computes an optimal allocation \( X = (x_1, \ldots, x_n) \) (the item \( x_i \) is allocated to buyer \( i \)) and then constructs a complete, weighted directed graph in which the vertices are the items. An edge \( x_i \rightarrow x_j \) in this graph represents a preference constraint, requiring that buyer \( i \) strongly prefer the item \( x_i \) over \( x_j \), relative to the output prices. Hereafter, we term this graph the “preference graph”.

The best we could hope for is to compute prices relative to which all edge constraints are satisfied. In this case, all buyers would strongly prefer their items over the rest, and the allocation obtained after the last buyer leaves the market would be \( X \). Consequently, this would imply an optimal static pricing scheme for unit-demand buyers, which is impossible. However, [10] prove the following two claims:

- An edge \( x_i \rightarrow x_j \) participates in a 0-weight cycle iff there is an alternative optimal allocation in which \( x_j \) is allocated to buyer \( i \).
- If 0-weight cycles are removed from the graph, then one can compute prices that satisfy the remaining edge constraints.

Thus, the scheme removes every edge that participates in a 0-weight cycle, and then computes the prices as per the second bullet above. Relative to these prices, every buyer strongly prefers her allocated item to every other item, except (perhaps) for the set of items that are allocated to her in some alternative optimal allocation. Since every buyer takes at most one favorite item (the buyers are unit-demand), this property guarantees that allocating this item to the buyer is consistent with an optimal allocation (not necessarily \( X \)), as desired.

A similar approach can be taken in the multi-demand setting as well. Indeed, the same techniques can be adapted to output prices, relative to which every buyer strongly prefers each of her allocated items to every other item, except perhaps for those that are allocated to her in some alternative optimal allocation. However, this property is not sufficient to ensure optimal welfare for multi-demand buyers, where a buyer may take more than one item. To get intuition for what may go wrong, note that the fact that item \( x \) is allocated to buyer \( i \) in some optimal allocation and also item \( y \) is allocated to \( i \) in some optimal allocation does not imply that there exists a single optimal allocation where both \( x \) and \( y \) are allocated to \( i \). This problem is demonstrated in the following running example, which generalizes the example by [10] above.

Running Example. Consider a market with 5 items \( a, b, c, d, e \) and 3 buyers, 1, 2, 3. Buyers 1 and 2 are both 2-demand, and buyer 3 is unit-demand. Buyer 1 values \( a, b, c, d \) at 1 and \( e \) at 0, Buyer 2 values \( c, d, e \) at 1 and \( a, b \) at 0, and buyer 3 values \( a, b, e \) at 1 and \( c, d \) at 0. One can verify that allocating \( a, b \) to 1, \( c, d \) to 2 and \( e \) to 3 maximizes social welfare. Moreover, both \( c \) and \( d \) are allocated to 1 in alternative optimal allocations (different ones). However, if 1 takes both \( c \) and \( d \), then the resulting social welfare can be at most 4, while \( OPT = 5 \).

This example demonstrates that removing every edge that participates in a 0-weight cycle from the preference graph might lead to a sub-optimal allocation. Our challenge is to remove enough
edges so that on the one hand, all 0-weight cycles are eliminated (this will enable us to compute the prices), and on the other hand, every deviation from \( X \) implied by the edge removals is consistent with some alternative optimal allocation.

Our first step is to gain a better structural understanding of optimal allocations in multi-demand markets. This is cast in the optimal allocation characterization theorem above, which essentially says that an allocation is optimal iff every item is allocated to a buyer that receives it in some optimal allocation. This simple characterization is quite powerful and leads to a significant simplification of the problem: For the sake of optimizing social welfare, the concrete values of the buyers for any specific item are unimportant. The important feature of any item is the set of buyers that receive it in some optimal allocation. Two items are essentially equivalent if their corresponding sets of buyers coincide. This observation allows us to group items into equivalence classes, providing a compact view of the market. For example, in markets with up to 3 multi-demand buyers, there are at most 8 (non-empty) equivalence classes, while the total number of items can be arbitrarily large.

Equipped with this new view of the market, we construct a new directed graph, termed the “item-equivalence graph”, which abstracts away the irrelevant information encoded in the preference graph. The vertices are these equivalence classes (refined after intersecting them with the bundles from the initial optimal allocation \( X \)), and there is an edge \( C \rightarrow D \) whenever the buyer to which the items in \( C \) are allocated (in the allocation \( X \)) also receives every item in \( D \) in some alternative optimal allocation. We prove that there is a correspondence between cycles in the item-equivalence graph and 0-weight cycles in the preference graph. Thus our challenge is reduced to removing enough edges from the first (and translate these removals back to the second), in a way that eliminates all cycles, but also guarantees the following: every deviation by any buyer from her prescribed bundle, implied by the edge removals, allows the other buyers to simultaneously compensate for their “stolen” items by replacing them with items from other relevant equivalence classes. The optimal allocation characterization theorem will then guarantee that the obtained allocation is indeed optimal. We devise an edge-removal method satisfying these requirements whenever the number of buyers is at most 3.

The optimal allocation characterization theorem and the item equivalence graph may be of independent interest, and may prove useful in the context of other problems related to multi-demand markets.

**Techniques for Negative Results.** The starting point for our negative results is the following simple observation: If a valuation profile admits a unique optimal allocation, then any dynamic pricing is also a Walrasian pricing. Thus, for the sake of proving our maximal domain theorem for dynamic pricing, it suffices to modify the construction given by Gul and Stacchetti in their theorem for Walrasian equilibrium in a way that preserves the correctness of the proof, but also guarantees a unique optimal allocation for the construction.

In their proof, Gul and Stacchetti characterized gross-substitutes as the class of valuations that exhibit the single improvement property:

**Theorem** ([26]). A valuation \( v \) is gross-substitutes if and only if for any bundle \( A \) and price vector \( p \) such that \( A \) is not utility-maximizing relative to \( p \), there is a bundle \( B \) such that \( |A \setminus B|, |B \setminus A| \leq 1 \) and \( u(A, p) < u(B, p) \).

This characterization is used to claim the existence of some bundle \( A \) and item prices \( p \), relative to which \( A \) is not utility maximizing for \( v_1 \) (\( v_1 \) is the assumed non gross-substitutes valuation), and any preferred bundle \( B \) satisfies \( |B \setminus A| \geq 2 \) or \( |A \setminus B| \geq 2 \). The proof then considers such a bundle
minimizing \(|A \triangle B|\) and splits to cases according to whether \(|B \setminus A| \geq 2\) or \(|A \setminus B| \geq 2\). Suitable unit-demand valuations are then constructed for each case.

In the first case (\(|B \setminus A| \geq 2\)), we could not manage to modify the construction so as to guarantee uniqueness of the optimal allocation. However, we did manage to perturb the valuations just enough so that every dynamic pricing can be converted to a Walrasian pricing, while maintaining the correctness of the construction (i.e., the construction does not admit a Walrasian pricing).

As for the second case in their proof, Yang [40] discovered a counter-example for the construction. In section 5.1, we harness a new characterization of gross-substitutes to prove that if \(v_1\) is not gross-substitutes, then one can always find \(p\) and \(A\) for which the corresponding minimizer \(B\) satisfies \(|B \setminus A| \geq 2\). Therefore, only the first case in the original proof had to be considered in the first place, and we obtain both maximal domain theorems (for Walrasian pricing and dynamic pricing) simultaneously. This new characterization is a variant of the price-based characterization by Reijnierse et al. [35] in which the item prices are guaranteed to be non-negative (in contrast to the item prices in [35]).

**Open problems.** Our results suggest questions for future research. The most obvious one is whether our positive result for 3 multi-demand buyers can be extended to any number of buyers. We hope that the tools we develop in this work, including the optimal allocation characterization theorem and the item equivalence graph, which are applicable to multi-demand markets of any size, will prove useful in this extension.

More generally, it is still open whether any market with gross substitutes valuations admits an optimal dynamic pricing.

### 1.3 Related Work

The notion of Walrasian equilibrium was defined for divisible-goods as early as the 19th century [39]. This notion was later extended to combinatorial markets, where Kelso and Crawford [30] introduced the class of gross-substitutes valuations as a class for which a natural ascending auction reaches a Walrasian equilibrium. Gul and Stacchetti later showed via their maximal domain theorem that gross-substitutes is the frontier of this existence result [26, 40]. Gross-substitutes valuations have been introduced independently in different fields, under different names, and under seemingly different definitions [11, 12, 13, 33]; see [31] for a comprehensive survey of gross-substitutes valuations. In order to circumvent the non-existence of a market equilibrium under broader valuation classes, relaxations of market equilibrium were introduced [22, 15], and behavioral biases were harnessed [3, 19].

Posted price mechanisms were shown to be useful in combinatorial markets. Feldman, Gravin and Lucier [21] showed how to compute simple “balanced” static prices in order to obtain at least half of the optimal welfare for submodular valuations, even in the case where the seller has only Bayesian knowledge about the valuations. This idea was generalized by Duetting et al. [14], and was shown to be useful even in the face of complementarities between items [6].

Cohen-Addad et al. [10] and Hsu et al. [28] were the first to demonstrate that Walrasian prices cannot even approximate the optimal welfare in the absence of a centralized tie-breaking coordinator. Cohen-Addad et al. resolved this issue by adjusting prices dynamically for unit-demand valuations. They also showed an instance of coverage valuations where a Walrasian equilibrium exists and yet dynamic prices cannot guarantee optimal welfare. On the other hand, Hsu et al. showed that under some conditions, minimal Walrasian prices guarantee near-optimal welfare for a strict subclass of
gross-substitutes valuations\textsuperscript{3}. Ezra et al. [18] and Eden et al. [17] established better guarantees via static pricing for simpler markets (identical items and binary unit-demand, respectively), in comparison to [21].

Posted price mechanisms have been shown to be useful in additional settings with different objective functions, including revenue maximization in combinatorial markets [27, 4, 5, 1], cost minimization in online scheduling [23, 16, 29], and a variety of other online resource allocation problems [8, 9, 24, 2].

2 Preliminaries

We consider a setting with a finite set of indivisible items \( M \) (with \( m := |M| \)) and a set of \( n \) buyers (or players). Every buyer has a valuation function \( v : 2^M \rightarrow \mathbb{R}_{\geq 0} \). As standard, we assume monotonicity and normalization of all valuations, i.e. \( v(S) \leq v(T) \) whenever \( S \subseteq T \), and \( v(\emptyset) = 0 \). A valuation profile of \( n \) buyers is denoted \( v = (v_1, \ldots, v_n) \) and we assume that it is known by all. An allocation is a vector \( A = (A_1, \ldots, A_n) \) of disjoint subsets of \( M \), indicating the bundles of items given to each player (not all items have to be allocated). The social welfare of an allocation \( A \) is given by \( SW(A) = \sum_{i=1}^{n} v_i(A_i) \). An optimal allocation is an allocation that achieves the maximum social welfare among all allocations.

A pricing or a price vector is a vector \( p \in \mathbb{R}_m^\geq \) indicating the price of each item. We assume a quasi-linear utility, i.e. the utility of a buyer \( i \) from a bundle \( S \subseteq M \) given prices \( p \) is \( u_i(S, p) = v_i(S) - \sum_{x \in S} p_x \). The demand correspondence of buyer \( i \) given \( p \) is the collection of utility maximizing bundles \( D_p(v) := \arg \max_{S \subseteq M} \{ u_i(S, p) \} \).

2.1 Dynamic Pricing

In the dynamic pricing problem buyers arrive to the market in an arbitrary and unknown order. Before every buyer arrival new prices are set to the items that are still available. The arriving buyer then chooses an arbitrary utility-maximizing bundle based on the current prices and available items. The goal is to set the prices so that for any arrival order and any tie breaking choices by the buyers, the obtained social welfare is optimal. We are interested in proving the guaranteed existence of an optimal dynamic pricing for any market composed entirely of buyers from a given valuation class \( C \). It can be easily shown by induction that the problem is reduced to proving the guaranteed existence of item prices \( p \) such that any utility-maximizing bundle of any buyer can be completed to an optimal allocation. In other words, we can rephrase the dynamic pricing problem as follows:

\textbf{Definition 2.1.} An \textit{optimal dynamic pricing} (hereafter, dynamic pricing) for the buyer profile \( v = (v_1, \ldots, v_n) \) is a price vector \( p \in \mathbb{R}_m^\geq \) such that for any buyer \( i \) and any \( S \in D_p(v_i) \) there is an optimal allocation \( O \) in which player \( i \) receives \( S \).

In the case that a dynamic pricing exists, we will also be interested in efficiently computing it. We assume access to the valuation function of every buyer \( i \) through a value oracle, namely, given a set \( S \subseteq M \) the oracle returns \( v_i(S) \).

\textbf{Definition 2.2.} A \textit{poly-time dynamic pricing scheme} is a \( \text{poly}(m, n) \)-time algorithm that computes a dynamic pricing given access to the valuation profile through value oracles.

\textsuperscript{3}Tran [37] recently showed that the class of matroid-based valuation is a strict subclass of gross-substitutes valuations.
2.2 Valuation Classes

This paper considers the following valuation classes:

- **Unit-Demand Valuations**: A valuation \( v : 2^M \rightarrow \mathbb{R}_{\geq 0} \) is **unit-demand** if for all \( S \subseteq M \),
  \[ v(S) = \max_{x \in S} \{ v(x) \} \].

- **Gross-Substitutes Valuations**: A valuation \( v : 2^M \rightarrow \mathbb{R}_{\geq 0} \) is **gross-substitutes** if for any two price vectors \( p, q \) such that \( p \leq q \) (point-wise), and for any \( A \in D_p(v) \) there is a bundle \( B \in D_q(v) \) such that \( A \cap \{ x \in M \mid p_x = q_x \} \subseteq B \).

- **Multi-Demand Valuations**: A valuation \( v : 2^M \rightarrow \mathbb{R}_{\geq 0} \) is **k-demand** if for all \( S \subseteq M \),
  \[ v(S) = \max_{S' \subseteq S : |S'| \leq k} \{ \sum_{x \in S'} v(x) \} \].

Note that for \( k = 1 \) we obtain a unit-demand valuation. Furthermore, every multi-demand valuation is gross-substitutes.

3 Dynamic Pricing for Multi-Demand Buyers

In this section we prove Theorem 1.2, namely we establish a \( \text{poly}(m) \) dynamic pricing scheme for \( n = 3 \) multi-demand buyers. As we shall see most of the tools we use hold for any number of buyers \( n \). For this entire section we fix a multi-demand buyer profile \( v = (v_1, \ldots, v_n) \) over the item set \( M \), where each \( v_i \) is \( k_i \)-demand. We assume w.l.o.g. that all items are essential for optimality (i.e. all items are allocated in every optimal allocation) since otherwise we can price all unnecessary items at \( \infty \) in every round to ensure that no player takes any of them (and price the rest of the items as if the unnecessary items do not exist). Note that under this assumption, each optimal allocation gives buyer \( i \) at most \( k_i \) items, for every \( i \). In particular we have \( m \leq \sum_{i=1}^n k_i \). For the sake of simplicity we further assume for the rest of this section that every optimal allocation gives each buyer \( i \) exactly \( k_i \) items, and thus \( m = \sum_{i=1}^n k_i \). The case \( m < \sum k_i \) introduces substantial technical difficulty and we defer its treatment to Appendix B. In section 3.1 we go over the tools used in our dynamic pricing scheme. In section 3.2 we present the dynamic pricing scheme for \( n = 3 \) buyers.

3.1 Tools and Previous Solutions

In section 3.1.1 we present the main combinatorial construct of our solution, namely the preference-graph, which generalizes the construct given by [10] in their solution for unit-demand buyers. In section 3.1.2 we explain the obstacles for generalizing the approach of [10] to the multi-demand setting. Finally, in sections 3.1.3 and 3.1.4 we develop the necessary machinery needed to overcome these obstacles. All the tools we develop and their properties hold for any number of buyers \( n \).

3.1.1 The Preference Graph and an Initial Pricing Attempt

Let \( O \) be an arbitrary optimal allocation. The preference graph based on \( O \) is the directed graph \( H \) whose vertices are the items in \( M \). Furthermore there is a special ‘source’ vertex denoted \( s \). For any two different players \( i, j \) and items \( x \in O_i, y \in O_j \) we have a directed edge \( e = x \rightarrow y \) with weight \( w(e) = v_i(x) - v_i(y) \). We also have a 0-weight edge \( s \rightarrow x \) for every item \( x \in M \). Since an optimal allocation can be computed in \( \text{poly}(n, m) \) time with value queries (the valuations are gross
substitutes, see [31]), it follows that the preference graph can also be computed in \(\text{poly}(n, m)\) time with value queries. When \(|O_i| = 1\) for every \(i\), the graph is exactly the one introduced by [10] in their unit-demand solution. We remark that a similar graph structure has been used by Murota in order to compute Walrasian equilibria in gross-substitutes markets ([32]). The proofs of the following two claims and corollary are deferred to Appendix A.

**Claim 3.1.** Let \(C := x_1 \to x_2 \to \cdots x_k \to x_1\) be a cycle in \(H\), where \(x_i\) is allocated to player \(i\) in \(O\) and \(x_i \neq x_j\) for every \(i \neq j\). Then the weight of the cycle is \(w(C) = \text{SW}(O) - \text{SW}(A)\) where \(A\) is the allocation obtained from \(O\) by transferring \(x_{i+1}\) to player \(i\) for every \(i\) (we identify player \(k + 1\) with player 1).

**Corollary 3.2.** Every cycle in \(H\) has non-negative weight.

Corollary 3.2 implies that the weight of the min-weight path from \(s\) to \(x\), denoted \(\delta(s, x)\), is well-defined for any item \(x\).

**Claim 3.3.** Let \(p_x := -\delta(s, x)\) for every item \(x\). Let \(i\) be some player, and let \(x, y\) be items such that \(x \in O_i, y \notin O_i\). Then:

1. \(p_x \geq 0\).
2. \(v_i(x) - p_x \geq v_i(y) - p_y\).
3. \(v_i(x) - p_x \geq 0\).

Note that the utility player \(i\) obtains from any bundle of size at most \(k_i\) is the sum of the individual utilities obtained by the individual items. That is, for any \(S = \{s_1, \ldots, s_t\} \subseteq M, t \leq k_i\) and prices \(p\) we have \(u_i(S, p) = \sum_{j=1}^{t} v_i(s_j) - p_j\). Thus, Claim 3.3 shows that setting the prices \(p_x = -\delta(s, x)\) almost achieves the requirements of dynamic pricing. The prices are non-negative as required and any player \(i\) is maximally happy by taking the designated bundle \(O_i\) upon arriving first to the market. However, since the inequalities in Claim 3.3 are not strict, the incoming player might deviate from the designated bundle. If the price of some real item is 0, then nothing prevents the incoming player from taking it, irrespective of whether she already achieved her demand through other items. Furthermore, if \(v_i(x) - p_x = v_i(y) - p_y\), then player \(i\) might take \(y\) instead of \(x\). Finally, if \(v_i(x) - p_x = 0\), then player \(i\) might choose not to take \(x\). All of these deviations might lead to a sub-optimal allocation.

To illustrate, consider a simple example of a market composed of two unit-demand buyers and two items, each of which is valued at 1 by both buyers. There are two optimal allocations here (each allocating one item to each player) inducing a social welfare of 2. It is easy to see that the resulting preference graph has only 0-weight edges, implying a price of 0 to both items. This in turn implies that every non-empty bundle is demanded by both buyers and in particular the first incoming buyer might take both items, resulting in a partial allocation that cannot be completed to an optimal one.

If we could decrease the weight of all edges by some small enough constant \(\varepsilon\), then we would get a strong inequality in parts (2) and (3) of Claim 3.3, guaranteeing that player \(i\)'s most preferable items are \(O_i\) and that she takes no less than \(k_i\) items when arriving first. The item prices could then be increased by another tiny constant so as to also satisfy part (1) with strong inequality, guaranteeing that she also takes no more than \(k_i\) items, hereby establishing that the only bundle in her demand is \(O_i\). However, decreasing edge weights can introduce cycles of negative weight in \(H\), in which case the min-weight path from \(s\) to \(x\) is not defined for any \(x\) in such a cycle!
3.1.2 Solution for Unit-Demand Valuations and its Failure to Generalize

The approach taken in [10] to solve this issue in the setting of unit-demand players (where $|O_i| = 1$ for every $i$) is to remove every edge that participates in a 0-weight cycle in $H$ (thus leaving only positive weight cycles, by Corollary 3.2) and then decrease the remaining edge weights by a small enough $\epsilon$ so that all leftover cycles still have positive weight. Removing an edge $x \rightarrow y$ for $x \in O_i$, $y \notin O_i$ cancels the preference guarantee of Claim 3.3 (part 2), leading to a possible deviation by buyer $i$ from taking $x$ to taking $y$. However, since 0-weight cycles correspond to alternative optimal allocations (see Claim 3.1 with $w(C) = 0$), then this is not a problem: if the edge $x \rightarrow y$ was removed, then there is an optimal allocation in which player $i$ receives $y$ instead of $x$. As for the edges $x \rightarrow y$ that were not removed, the $\epsilon$ decrement causes $i$ to strongly prefer $x$ over $y$. The other inequalities of Claim 3.3 would also be strict, and we are thus guaranteed that the incoming player indeed takes a one-item bundle that is part of some optimal allocation, as desired. This approach works in the unit-demand setting, but poses problems in our setting, as illustrated in the running example (presented in the introduction).

[Running Example] Figure 1 shows two 0-weight cycles in the preference graph (out of the four existing 0-weight cycles) based on the initial optimal allocation that gives buyer 1 $\{a, b\}$, buyer 2 $\{c, d\}$ and buyer 3 $\{e\}$. If all such cycles are removed, then nothing prevents 1 from taking items $c$ and $d$ instead of $a$ and $b$. At this point buyer 2 has only the item $e$ to compensate for the two “stolen” items $c$ and $d$. It follows that the partial allocation cannot be completed to an optimal one.

Item $e$ in the running example is in some sense a “bottleneck” in the preference graph, and a more sophisticated method of eliminating 0-weight cycles must be employed instead of simply removing all edges that participate in some 0-weight cycle. To be more precise, we state our informal goal:

**Goal.** Remove a set of edges from the preference graph so that no 0-weight cycles are left, and every possible deviation implied by the removed edges is consistent with some optimal allocation.

In order to gain a better understanding of the tolerable deviations from the designated bundles $O_i$, we first characterize the structure of the collection of optimal allocations.
3.1.3 Legal Allocations

**Definition 3.4.**

- An item $x \in M$ is *legal* for player $i$ if there is some optimal allocation $O = (O_1, \ldots, O_n)$ such that $x \in O_i$.
- A bundle $S \subseteq M$ is *legal* for player $i$ if $|S| = k_i$ and every $x \in S$ is legal for player $i$.
- A *legal allocation* $A = (A_1, \ldots, A_n)$ is an allocation in which $A_i$ is legal for player $i$, for every $i$.

In a legal allocation every player $i$ receives exactly $k_i$ items, each of which is allocated to her in some optimal allocation. Note that a legal bundle for buyer $i$ might not form a part of any optimal allocation (e.g., the bundle $\{c, d\}$ for buyer 1 in the running example). The following theorem provides a characterization of the collection of optimal allocations.

**Theorem 3.5.** An allocation is legal if and only if it is optimal.

Before proving Theorem 3.5, we state a corollary of it. Recall that a price vector $p$ is a dynamic pricing iff for any $i$ and $S \in D_p(v_i)$ there is an optimal allocation $O$ in which player $i$ receives $S$.

**Corollary 3.6.** A price vector $p$ is a dynamic pricing if for every player $i$ and $S \in D_p(i)$, $S$ is legal for player $i$ and there exists an allocation of the items $M \setminus S$ to the other players in which every player receives a bundle that is legal for her.

Thus, going back to our informal Goal (see section 3.1.2), Theorem 3.5 determines the deviations from the bundles $O_i$ which are tolerable. A buyer can only deviate to items that are legal for her, in a way that the leftover items can be partitioned “legally” among the rest of the buyers. We now prove Theorem 3.5.

**Proof.** Legality follows from optimality due to our assumption that every optimal allocation allocates exactly $k_i$ items to every player $i$. For the other direction we first prove the theorem in the special case of a unit-demand market, i.e. $k_i = 1$ for every player $i$, and then we show how the general case reduces to the unit-demand market case.

**Lemma 3.7.** If $k_i = 1$ for all $i$ then any legal allocation is optimal.

**Proof.** Let $L = (\{\ell_1\}, \ldots, \{\ell_n\})$ be a legal allocation, and let $O = (\{o_1\}, \ldots, \{o_n\})$ be some optimal allocation. We show that $SW(L) = SW(O)$. By definition, for every $i$ there is an optimal allocation $O^i$ in which the item $\ell_i$ is allocated to player $i$ and optimality implies that $SW(O^i) = SW(O)$. Summing over all $i$ we get

$$\sum_{i=1}^{n} SW(O) = \sum_{i=1}^{n} SW(O^i)$$  \hspace{1cm} (1)

The right side in (1) accounts for the welfare of $n$ optimal allocations, each of which is by itself a sum of $n$ terms of the form $v_j(x)$ where every item and player appears exactly once (recall the assumption that every optimal allocation allocates all items, i.e., is a perfect matching). Thus, the right side in (1) is a sum of $n^2$ terms in which every item and player appears exactly $n$ times.
Consider the bipartite graph $G = (X, Y; E)$ in which the left side $X$ is the set of items, the right side $Y$ is the set of players and there is an edge $\{x, j\}$ for each summand $v_j(x)$ appearing in the right side of (1). Then $G$ is an $n$-regular bipartite graph (possibly with multi-edges), and note that the (perfect) matching induced by the legal allocation $L$ appears in $G$ (since the term $v_i(\ell_i)$ appears in $SW(O^i)$, for every $i$). If we erase the edges of that matching from $G$ then we are left with an $(n - 1)$-regular bipartite graph. It is a well-known fact that in this case the edges of $G$ can be split into $n - 1$ perfect matchings $P^1, \ldots, P^{n-1}$. Thinking of these matchings as allocations, and together with the allocation $L$, we get
\[
\sum_{i=1}^{n} SW(O) = \sum_{i=1}^{n} SW(O^i) = SW(L) + \sum_{i=1}^{n-1} SW(P^i).
\]
Since $SW(O) \geq SW(L)$ and $SW(O) \geq SW(P^i)$ for every $i$ (by optimality of $O$), it must be the case that all weak inequalities are in fact equalities, establishing in particular that $SW(O) = SW(L)$, as desired.

We now describe the reduction from the general case to the unit-demand market case. We are given a market with $n$ agents where each agent is $k_i$-demand and $\sum_i k_i = m$. Our reduction keeps the same set of items $M$, but splits each agent $i$ to $k_i$ identical unit-demand agents, where for each copy, the value of the agent for an item $x \in M$ is simply $v_i(x)$. Clearly, the number of unit demand agents as a result of this reduction is $\sum_i k_i = m$. Let $N_i$ be the set of unit demand bidders that corresponds to agent $i$, and let $N'' = \bigcup_{i \in N} N_i$.

Given an allocation $O = (O_i)_{i \in N}$ for the original market, where $|O_i| = k_i$ for every $i$, the corresponding allocation in the unit-demand market splits the $|O_i|$ different items arbitrarily between the unit demand bidders corresponding to bidder $i$, with each bidder receiving exactly one item. Notice that the social welfare achieved by this allocation is the same as in the original allocation. Similarly, given an allocation $O' = (O'_i)_{i \in N''}$ in the unit-demand market, the corresponding allocation in the original market gives all the items allocated to agents in $N_i$ to agent $i$. Again, since $|N_i| = k_i$ the resulting allocation achieves the same welfare as the original one.

**Lemma 3.8.** An allocation is legal in the original market if and only if it is legal in the corresponding unit-demand market. An allocation is optimal in the original market if and only if it is optimal in the corresponding unit-demand market.

**Proof.** Consider an optimal allocation in the original market and recall our assumption that optimal allocations give each player $i$ exactly $k_i$ items. The corresponding allocation in the unit-demand market obtains the same value. Similarly, given an optimal allocation in the unit-demand market where each agent gets one item, the corresponding allocation in original market obtains the same value. Therefore, an allocation is optimal in the original market if and only if it is optimal in the corresponding unit-demand market.

Similar reasoning shows that given a legal allocation in the original market, the corresponding allocation in the unit-demand market is legal as well and vice versa.

To complete the proof of Theorem 3.5, take a legal allocation in the original market. According to Lemma 3.8 it is also legal in the corresponding unit-demand market. From Lemma 3.7, we get that it is optimal in the unit-demand market. Again, by Lemma 3.8, we get that the corresponding allocation in the original market is also optimal.
3.1.4 The Item-Equivalence Graph

Let $O$ be some optimal allocation and $H$ the corresponding preference graph. For every player $i$ and set of players $C \subseteq [n] \setminus \{i\}$, we denote by $B_{i,C}$ the set of items allocated to player $i$ in $O$ that can alternatively be allocated to any player in $C$ as a part of some optimal allocation (and only to the players in $C$). For example, $B_{1,\{2,3\}}$ is the set of items $x \in O_1$ such that there are optimal allocations $O',\tilde{O}$ in which $x$ is allocated to players 2, 3 (respectively), and for any other player $j \notin \{1,2,3\}$, there is no optimal allocation in which $x$ is allocated to $j$. Formally,

$$B_{i,C} := \left\{ x \in O_i \left| \forall j \in \{i\} \cup C \ x \text{ is legal for } j \right. \left. \forall j \notin \{i\} \cup C \ x \text{ is not legal for } j \right\}$$

In other words, $B_{i,C}$ is the set of items allocated to $i$ in $O$ and whose set of players to which they are legal is exactly $\{i\} \cup C$. We make a few observations:

- The sets $B_{i,C}$ form a partition of $M$ (some of these sets might be empty sets).
- Let $x \in O_i$ and $y \in O_j$ for $i \neq j$. If $x \rightarrow y$ participates in a 0-weight cycle in $H$ and $y \in B_{j,C}$, then $i \in C$.

The second observation holds since if $x \rightarrow y$ participates in a 0-weight cycle, then there is an alternative optimal allocation in which $y$ is allocated to player $i$ (see Claim 3.1 with $w(C) = 0$).

**Definition 3.9** (Item-Equivalence Graph). Given an optimal allocation $O$, its associated *item-equivalence graph* is the directed graph $B = (T,D)$ with vertices

$$T = \{B_{i,C} \mid i \in [n], C \subseteq [n] \setminus \{i\}, B_{i,C} \neq \emptyset\}$$

and directed edges

$$D = \{B_{i,C_1} \rightarrow B_{j,C_2} \mid i \in C_2\}.$$  

For example, $(B_{1,\emptyset} \rightarrow B_{2,\{1,4\}})$ and $(B_{2,\{1,5\}} \rightarrow B_{6,\{2\}})$ are edges in the item-equivalence graph (assuming that the participating sets are non-empty), whereas, for example, $(B_{1,\emptyset} \rightarrow B_{1,\{2\}})$ and $(B_{2,\{1\}} \rightarrow B_{3,\{1,4\}})$ are not. Note also that the number of vertices is polynomial in $m$.

The next claim shows that the item-equivalence graph can be computed efficiently.

**Claim 3.10.** Given an optimal allocation $O$, its associated item-equivalence graph can be computed in $\text{poly}(m,n)$ time and value queries.

**Proof.** Clearly the main problem is determining the non-empty sets $B_{i,C}$. We can efficiently determine the set $B_{i,C}$ that any item $x \in O_i$ belongs to by executing the following sub-routine: go over all players $j \in [n] \setminus \{i\}$ and compute the optimal social welfare in the residual market obtained by fixing $x$ to player $j$. Denote the result by $\text{opt}_j$ and compare $\text{opt}_j$ with the optimal social welfare in the original market, denoted by $\text{opt}$. $x$ belongs to the set $B_{i,C}$ for the set of players $C = \{j \in [n] \setminus \{i\} \mid \text{opt}_j = \text{opt}\}$. \qed

The following lemma uses Theorem 3.5 to establish a correspondence between 0-weight cycles in $H$ and cycles in $B$.

**Lemma 3.11.** Let $O$ be an optimal allocation and let $H$ and $B$ be the corresponding preference graph and item-equivalence graph, respectively. Then:
1. If \( B_{i_1, C_1} \rightarrow \cdots \rightarrow B_{i_k, C_k} \rightarrow B_{i_1, C_1} \) is a cycle in \( B \) then for any items \( x_1 \in B_{i_1, C_1}, \ldots, x_k \in B_{i_k, C_k} \), the cycle \( C = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k \rightarrow x_1 \) is a 0-weight cycle in \( H \).

2. If \( C = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k \rightarrow x_1 \) is a 0-weight cycle in \( H \), and \( x_\ell \in B_{i_\ell, C_\ell} \) for every \( 1 \leq \ell \leq k \) then \( C' := B_{i_1, C_1} \rightarrow \cdots \rightarrow B_{i_k, C_k} \rightarrow B_{i_1, C_1} \) is a cycle in \( B \).

**Proof.** Let \( C = B_{i_1, C_1} \rightarrow \cdots \rightarrow B_{i_k, C_k} \rightarrow B_{i_1, C_1} \) be a cycle in \( B \), let \( x_1 \in B_{i_1, C_1}, \ldots, x_k \in B_{i_k, C_k} \) and note that the edges \( x_i \rightarrow x_{i+1} \) exist in \( H \) (since these are items that belong to different players). Consider the cycle \( C' = x_1 \rightarrow \cdots \rightarrow x_k \rightarrow x_1 \) in \( H \). We need to show that \( w(C') = 0 \). We can assume w.l.o.g. that all the items \( x_\ell \) are different as otherwise \( C' \) is a union of two or more cycles for which this assumption holds (these cycles are derived from corresponding sub-cycles of \( C \), and the weight of a union of 0-weight cycles is 0. By definition of \( B \) we have \( i_\ell \in C_{\ell+1} \) for every \( 1 \leq \ell \leq k \) (again we identify \( k+1 \) with 1) and so we conclude that the allocation \( A \) obtained from \( O \) by passing \( x_{\ell+1} \) to player \( i_\ell \) is a legal allocation and thus optimal (by Theorem 3.5). Since \( O \) is also optimal, we conclude by Claim 3.1 that

\[
w(C) = SW(O) - SW(A) = 0.
\]

We now prove part 2. Let \( C = x_1 \rightarrow \cdots \rightarrow x_k \rightarrow x_1 \) be a 0-weight cycle in \( H \). Again we can assume w.l.o.g. that all items are different. For all \( \ell \) let \( i_\ell \) and \( C_\ell \) be the player and set such that \( x_\ell \in B_{i_\ell, C_\ell} \). By Claim 3.1 the allocation \( A \) obtained from \( O \) by passing \( x_{\ell+1} \) to player \( i_\ell \) is optimal and thus legal. We conclude that \( i_\ell \in C_{\ell+1} \) for all \( 1 \leq \ell \leq k \) implying that the edges \( B_{i_\ell, C_\ell} \rightarrow B_{i_{\ell+1}, C_{\ell+1}} \) exist in \( B \). Therefore \( C' = B_{i_1, C_1} \rightarrow \cdots \rightarrow B_{i_k, C_k} \rightarrow B_{i_1, C_1} \) is a cycle in \( B \).

As explained in section 3.1.2, our main challenge in the dynamic pricing problem is to come up with a method to remove all 0-weight cycles from \( H \) in a way that each potential deviation of any player \( i \) from the designated bundle \( O_i \), that emanates from the edge removals, is consistent with some optimal allocation. In particular the method must overcome the “bottleneck problem” (as illustrated in Figure 1). Lemma 3.11 allows us to shift the focus from removing 0-weight cycles in \( H \) to removing cycles in \( B \) and translate these removals back to \( H \).

**Running Example** Figure 2 shows the (simple) item-equivalence graph obtained from the initial optimal allocation. Each of the items \( a, b \) is allocated to buyer 3 in some other optimal allocation (and is never allocated to buyer 2). Thus \( a, b \in B_{i_1, \{3\}} \). Similarly we have \( c, d \in B_{i_2, \{1\}} \) and \( e \in B_{i_3, \{2\}} \). Note that removing any of the edges of the item-equivalence graph makes it cycle-free. Thus, by Lemma 3.11, if we choose one of the edges \( e_1, e_2, e_3 \) and remove all edges in the preference graph corresponding to the chosen edge, then the preference graph will remain cycle-free as desired in the Goal in section 3.1.2. Now, removing the edges corresponding to \( e_1 \) could cause player 1 to take the bundle \( \{c, d\} \) instead of his designated bundle \( \{a, b\} \), and this cannot be completed to an optimal allocation. On the other hand, removing the preference graph edges that correspond to the edges \( e_2 \) and \( e_3 \) is fine. If player 2 arrives first to the market, then the removal of edge \( e_2 \) might cause her to take the item \( e \) instead of \( c \) or \( d \), and both options are consistent with some optimal allocation. Likewise if player 3 arrives first and takes \( a \) or \( b \) instead of \( e \) then this too can be completed to an optimal allocation. The important property here is that \( B_{i_3, \{2\}} \) has minimal size in the cycle, and thus removing its incoming and outgoing edges introduces tolerable potential deviations.

### 3.2 Solution for 3 Multi-Demand Buyers

We are now ready to present the dynamic pricing scheme for \( n = 3 \) multi-demand buyers.
Remark 3.12. The algorithm makes use of the item-equivalence graph. We abuse notation and instead of writing \( B_{i,j} \) (\( B_{i,j,k} \)) we write \( B_{ij} \) (\( B_{ijk} \)). Thus the vertices of the item-equivalence graph for three players are

\[
\begin{align*}
B_{1,\emptyset} & \quad B_{2,\emptyset} & \quad B_{3,\emptyset} \\
B_{12} & \quad B_{21} & \quad B_{31} \\
B_{13} & \quad B_{23} & \quad B_{32} \\
B_{123} & \quad B_{213} & \quad B_{312} \\
B_{132} & \quad B_{231} & \quad B_{321}
\end{align*}
\]

where each column corresponds to a different player.

ALGORITHM 1: Dynamic Pricing Scheme for 3 Multi-Demand Buyers.

**Input:** Multi-demand valuations \((v_1, v_2, v_3)\).

**Output:** prices \( p = (p_x)_{x \in M} \).

1. Compute some optimal allocation \( \mathbf{O} \).
2. Compute the preference graph \( H \) and the item-equivalence graph \( B \) based on \( \mathbf{O} \).
3. Mark all edges that participate in a cycle of size 2 in \( B \).
4. In each of the cycles \( B_{13} \rightarrow B_{21} \rightarrow B_{32} \rightarrow B_{13} \) and \( B_{12} \rightarrow B_{31} \rightarrow B_{23} \rightarrow B_{12} \) (if these exist) choose a set of minimal size and mark its incoming edge and outgoing edge in the cycle.
5. For every edge \( B_{i,c_1} \rightarrow B_{i,c_2} \) in \( B \) that was marked, and for every \( x \in B_{i,c_1}, y \in B_{i,c_2} \), remove the edge \( x \rightarrow y \) from \( H \). Denote the obtained graph by \( H' \).
6. Let \( \Delta > 0 \) be the difference in social welfare between the optimal and 2nd optimal allocation. Denote \( \epsilon := \frac{\Delta}{\text{poly}(m)} \) and for every edge \( e \) that was not removed (except for edges starting at the source vertex \( s \)) update its weight to \( w'(e) = w(e) - \epsilon \).
7. Compute the min-weight paths from \( s \) to every \( x \) in \( H' \), and let \( \delta(s,x) \) be its weight. For every item \( x \) set the price \( p_x = -\delta(s,x) + \epsilon \).
8. return \((p_x)_{x \in M}\)

[Running Example] Figure 3 demonstrates the graphs \( H, B \) and \( H' \) obtained in the pricing scheme when run on our example, based on the optimal allocation \( O_1 = \{a,b\}, O_2 = \{c,d\}, O_3 = \{e\} \). The edges that get marked in the item-equivalence graph are \( e_2 \) and \( e_3 \) in step 4. This translates to the removal of the outgoing edges from \( c,d \) to \( e \) and from \( e \) to \( a,b \) when transitioning from \( H \) to \( H' \) and consequently no 0-weight cycles are left. This remains true also after subtracting \( \epsilon \) from every edge that does not touch \( s \).

As stated before (Section 3.1.1 and Claim 3.10), computing \( \mathbf{O}, H \) and \( B \) can be done in polynomial time. Finding the cycles in \( B \) can also be done efficiently (\( B \) has a constant number of nodes) as well as computing min-weight paths. Thus the algorithm indeed runs in \( \text{poly}(m) \) time as desired.

**Lemma 3.13.** After step 5 every cycle in \( H' \) has strictly positive weight.
The edges $e_2$ and $e_3$ in the item-equivalence graph were marked in step 4. Consequently, the edges from $c, d$ to $e$ and from $e$ to $a, b$ were removed from the original preference graph.

Figure 3: The various graphs obtained in the execution of Algorithm 1 on our running example. The edges $e_2$ and $e_3$ in the item-equivalence graph were marked in step 4. Consequently, the edges from $c, d$ to $e$ and from $e$ to $a, b$ were removed from the original preference graph.
Lemma 3.14. For any item $x$, $p_x > 0$.
Proof. The 0-weight path $s \rightarrow x$ is some path from $s$ to $x$, and thus $\delta(s, x) \leq 0$. Thus $p_x = -\delta(s, x) + \epsilon > 0$ as desired.

Lemma 3.15. For any player $i$, $x \in O_i$ and $y \notin O_i$, if $e = x \rightarrow y \in H'$ then $u_i(x, p) > u_i(y, p)$.
Proof. By the triangle inequality we have
\[
\begin{align*}
\delta(s, x) + w'(x \rightarrow y) &\geq \delta(s, y) \\
\delta(s, x) + v_i(x) - v_i(y) - \epsilon &\geq \delta(s, y) \\
v_i(x) - (-\delta(s, x) + \epsilon) - \epsilon &\geq v_i(y) - (\delta(s, y) + \epsilon) \\
v_i(x) - p_x - \epsilon &\geq v_i(y) - p_y \\
v_i(x) - p_x &> v_i(y) - p_y
\end{align*}
\]

The claim follows.

Lemma 3.16. For any player $i$ and item $x \in O_i$ we have $v_i(x) - p_x > 0$.
Proof. Consider a min-weight path from $s$ to $x$ in $H'$, $s \rightarrow x_1 \rightarrow \cdots \rightarrow x_k = x$, and for every $1 \leq j \leq k$ let $i_j$ be the player such that $x_j \in O_{i_j}$ (note that $i_k = i$). Since every cycle in $H'$ has positive weight (Lemma 3.13) it must be the case that all the vertices $x_i$ are different (otherwise this is not a min-weight path) and $k \leq m$. We have
\[
v_{i_k}(x_k) - p_{x_k} = v_{i_k}(x_k) + \delta(s, x) - \epsilon = v_{i_k}(x_k) + \sum_{j=1}^{k-1} (v_{i_j}(x_j) - v_{i_j}(x_{j+1}) - \epsilon) - \epsilon = \sum_{j=1}^{k} v_{i_j}(x_j) - \sum_{j=1}^{k-1} v_{i_j}(x_{j+1}) - \epsilon(k - 1) - \epsilon
\]
\[ \geq \text{SW}(O) - \text{SW}(A) - \epsilon m \]

where \( A \) is the allocation obtained from \( O \) by passing the item \( x_{j+1} \) to player \( i_j \) for all \( j \), and dis-allocating \( x_1 \). Therefore, \( \sum_{j=1}^{k} v_j(x_j) - \sum_{j=1}^{k-1} v_j(x_{j+1}) = \text{SW}(O) - \text{SW}(A) \). By the assumption that every optimal allocation allocates all items, we conclude that \( A \) is a sub-optimal allocation and therefore the last term is positive as desired (note that \( \epsilon \) is sufficiently small). \qed

We are now ready to prove that the output of our dynamic pricing scheme meets the requirements of Corollary 3.6. This is cast in the following lemma:

**Lemma 3.17.** Let \( p \) be the price vector output by the dynamic pricing scheme. Then, for every player \( i \) and \( S \in D_p(i) \):

1. \( S \) is legal for player \( i \).
2. \( S \) can be completed to a legal allocation, i.e. there exists an allocation of the items \( M \setminus S \) to the other players in which every player receives a bundle that is legal for her.

**Proof.** We prove for \( i = 1 \) (the same proof applies also for \( i = 2, 3 \)). We first prove Part 1. We start by showing that every \( S \in D_p(1) \) is of size \( k_1 \). Since all item prices are positive (Lemma 3.14) and player 1 is \( k_1 \)-demand, it cannot be the case that player 1 maximizes utility with a bundle consisting of more than \( k_1 \) items. Furthermore, by Lemma 3.16 there are at least \( k_1 \) legal items from which she derives positive utility. Combining, every demanded bundle has exactly \( k_1 \) items. Now, for any two items \( x, y \) where \( x \in O_1 \) and \( y \) is not legal for player 1, the edge \( x \rightarrow y \) was not removed in the transition from \( H \) to \( H' \) (since there is no corresponding edge in the item-equivalence graph that could have been marked). Thus, player 1 strongly prefers \( x \) over \( y \) (by Lemma 3.15) and we conclude that every demanded bundle contains only legal items, as desired.

We proceed to prove part 2. Let \( S \in D_p(1) \). We refer to the items in \( S \setminus O_1 \) as the items that player 1 ‘stole’ from players 2 and 3, and to the items in \( O_1 \setminus S \) as those player 1 ‘left behind’. We need to show that players 2 and 3 can compensate for their stolen items in a ‘legal manner’, that is, by completing their leftover bundles \( O_2 \setminus S \) and \( O_3 \setminus S \) to \( k_2 \) and \( k_3 \) legal items, respectively. The first step is to determine where the stolen and left behind items are taken from. This is done in the following two claims:

**Claim 3.18.** \( (O_1 \setminus S) \subseteq (B_{12} \cup B_{13} \cup B_{123}) \).

**Proof.** Since \( O_1 = B_{1,\emptyset} \cup B_{12} \cup B_{13} \cup B_{123} \), it is enough to show that \( B_{1,\emptyset} \) is contained in any demanded bundle of Player 1. Note that \( B_{1,\emptyset} \) does not participate in any cycle in the item-equivalence graph, implying that none of its outgoing edges were marked. This in turn implies (by Lemma 3.15) that player 1 strongly prefers every item of \( B_{1,\emptyset} \) over every item \( y \notin O_1 \). Furthermore, she derives positive utility from these items (Lemma 3.16). We conclude that the items in \( B_{1,\emptyset} \) are contained in every demanded bundle, as desired. \qed

**Claim 3.19.** \( (S \setminus O_1) \subseteq (B_{21} \cup B_{213} \cup B_{31} \cup B_{312}) \).

**Proof.** The claim follows directly from the fact that \( S \) is legal for player 1. \qed
We denote

\[ a_2 := \lvert (O_1 \setminus S) \cap B_{12} \rvert \]
\[ a_3 := \lvert (O_1 \setminus S) \cap B_{13} \rvert \]
\[ a_{23} := \lvert (O_1 \setminus S) \cap B_{123} \rvert \]
\[ b_2 := \lvert (S \setminus O_1) \cap B_{21} \rvert \]
\[ b_{23} := \lvert (S \setminus O_1) \cap B_{213} \rvert \]
\[ b_3 := \lvert (S \setminus O_1) \cap B_{31} \rvert \]
\[ b_{32} := \lvert (S \setminus O_1) \cap B_{312} \rvert \]

In words, \( a_2 \) is the number of items player 1 left behind in \( B_{12} \), \( b_2 \) is the number of items she ‘stole’ from player 2 out of the items in \( B_{21} \), \( b_{32} \) is the amount she ‘stole’ from player 3 out of the items in \( B_{312} \), etc. By the previous two claims and by the fact that \( \lvert O_1 \setminus S \rvert = \lvert S \setminus O_1 \rvert \) (since \( \lvert O_1 \rvert = \lvert S \rvert \)) we have

\[ b_2 + b_{23} + b_3 + b_{32} = \lvert S \setminus O_1 \rvert = \lvert O_1 \setminus S \rvert = a_2 + a_{23} + a_3 \quad (2) \]

Consider the bipartite graph \( G \) whose left side consists of the items in \( S \setminus O_1 \) and whose right side consists of the items in \( O_1 \setminus S \), with edges \((x, y)\) whenever the stolen item \( x \) can be replaced by the leftover item \( y \) legally (e.g., if \( x \in O_2 \), then \( y \in B_{12} \cup B_{123} \)). Specifically, \( G \) is composed of a bi-clique between the stolen items from \( B_{21} \cup B_{213} \) (the stolen items of player 2) and the leftover items from \( B_{12} \cup B_{123} \) (these are the leftover items that are legal for player 2), and of another bi-clique between the stolen items of \( B_{31} \cup B_{312} \) (the stolen items of player 3) and the leftover items of \( B_{13} \cup B_{123} \) (the leftover items that are legal for player 3). If there is a perfect matching in \( G \), then every stolen item can be replaced with the item it was matched to in the perfect matching, resulting in a legal allocation, and we are done. Thus we assume that there is no perfect matching in \( G \). In this case Hall’s condition does not hold for \( G \). One can verify that this implies one of the following:

\[ b_2 + b_{23} > a_2 + a_{23} \quad \text{or} \quad b_3 + b_{32} > a_3 + a_{23} \]

Assume w.l.o.g. that \( b_2 + b_{23} > a_2 + a_{23} \). Then, by equation (2), we have \( a_3 > b_3 + b_{32} \geq 0 \). We claim that this implies \( b_{23} = 0 \). The reason is that otherwise, player 1 stole some item, denoted \( y \), from \( B_{213} \) and left behind some item, denoted \( x \), in \( B_{13} \). But this cannot be the case since this would imply (by Lemma 3.15) that the edge \( x \rightarrow y \) was removed in the transition from \( H \) to \( H' \), but the edge \( B_{13} \rightarrow B_{213} \) was never marked in the pricing scheme. Therefore \( b_{23} = 0 \) and \( b_2 > a_2 + a_{23} \geq 0 \). The combination of \( b_2 > 0 \) and \( a_3 > 0 \) implies that the edge \( B_{13} \rightarrow B_{21} \) was marked in step 4, and so one of \( B_{13}, B_{21} \) is of minimal size in the cycle \( B_{13} \rightarrow B_{21} \rightarrow B_{32} \rightarrow B_{13} \). In particular,

\[ |B_{32}| = \min \{ |B_{13}|, |B_{21}| \} \]
\[ \geq \min \{ a_3, b_2 \} \]
\[ \geq \min \{ a_3 - (b_3 + b_{32}), b_2 - (a_2 + a_{23}) \} \]
\[ = b_2 - (a_2 + a_{23}) , \]

where the equality holds by equation (2). In order to complete \( S \) to a legal allocation, player 2 compensates for his stolen \( b_2 \) items by taking the \( a_2 + a_{23} \) items player 1 left behind in \( B_{12} \cup B_{123} \) and by “stealing” \( b_2 - (a_2 + a_{23}) \) items from \( B_{32} \) (indeed there are enough items there for player 2 to steal). Player 3 now has to compensate for the items stolen from her by both players, a total of
\[(b_{32} + b_3) + (b_2 - (a_2 + a_{23})) = a_3\] items. Since player 1 left precisely this number of items in \(B_{13}\), player 3 can take them. Note that the resulting allocation is indeed legal and thus optimal.

\[\square\]

4 Dynamic Pricing for Unit-Demand Bidders Without Tie-Breaking

In [10] the authors present a poly-time dynamic pricing scheme that leads to an optimal allocation for unit-demand buyers. In their pricing scheme, buyers may have multiple bundles in their demand, but every tie-breaking rule leads to optimal welfare. In this section, we show how the scheme from [10] can be modified so that all buyers have exactly one bundle in their demand; i.e., pricing \(p\) such that for every buyer \(i\), \(|D_p(v_i)| = 1\), and the unique bundle in \(D_p(v_i)\) is legal. This property decreases the uncertainty regarding the obtained outcome.

As explained in section 3.2, we can assume w.l.o.g. that all items are necessary for optimality (i.e., are allocated in every optimal allocation), thus \(m \leq n\). Moreover, we think of each optimal allocation as if it allocates exactly \(n\) items, some of which are imaginary items (recall that for unit-demand buyers, an imaginary item is legal for a player if there is some optimal allocation in which that player receives no item; see Appendix B).

Our scheme also works by removing 0-weight cycles from the preference graph, but does so using a different approach than [10]. Recall that in a 0-weight cycle, every item is legal for the player possessing the preceding item in the cycle (this is an immediate corollary of Claim 3.1). In [10], the whole cycle is removed, potentially allowing each buyer on the cycle to take the subsequent item. Here, we take some item \(x\) on the cycle and replace its preceding item with a “copy” of item \(x\). This causes the item to be “owned” by more than one player, but this should not be concerning; the preference graph properties guarantee that the item is demanded by all players owning it, in addition to being legal for them. We show that we can repeatedly apply this item duplication procedure until no 0-weight cycles remain in the preference graph. As before, all edge weights are decreased by a sufficiently small \(\epsilon\), enabling us to set prices such that every demand correspondence contains the corresponding buyer’s associated item, and that item only.
The dynamic pricing scheme is presented below:

**ALGORITHM 2:** Dynamic Pricing Scheme for Unit-Demand Buyers Without Tie-Breaking.

| Input: Unit-demand valuations \( (v_1, \ldots, v_n) \). |
| Output: Prices \( p = (p_x)_{x \in M} \). |

1. Compute some optimal allocation \( O = (a_1, \ldots, a_n) \).
2. For every item \( a_i \), initiate a counter \( c(a_i) \leftarrow 1 \). This counter keep track of the number of players \( a_i \) is associated with.
3. Construct a graph \( H \) with a vertex \( d_i \) for every buyer \( i \) and directed weighted edges between every two vertices, as follows:
   - For every item \( a_i \), initiate the associated item \( d_i := a_i \).
   - For every \( i \neq j \), initiate the edge weight \( w(d_i \rightarrow d_j) := v_i(d_i) - v_j(d_j) \).
4. While there is a 0-weight cycle \( C \) in \( H \) in which all items are different do
   - Choose an edge \( d_i \rightarrow d_j \) in \( C \) such that \( d_j \in \arg\max_{d_k \in C} (c(d_k)) \).
   - Update \( d_i := d_j \).
   - \( c(d_i) := c(d_i) - 1 \).
   - \( c(d_j) := c(d_j) + 1 \).
5. Update edge weights as in Step 5.
6. Decrease the weight of every edge in \( H \) by a small enough \( \varepsilon > 0 \) so that the weight of every cycle remains positive, except for the edges between vertices whose associated items are the same.
7. Add a dummy vertex \( s \) to \( H \) with a 0-weighted edge \( s \rightarrow d_i \) for every \( i \).
8. For every item \( a \) set the price \( p_a \) as follows:
   - If \( c(a) = 0 \) then
     - set \( p_a = \infty \).
   - Else
     - Let \( d_i \) be a vertex with which \( a \) is associated and set \( p_a = -\delta(s, d_i) + \epsilon \).
     - /* Note that the weight of any edge that connects two vertices that are associated with the same item is 0. Thus there is no ambiguity in this step. */
9. Return \( (p_x)_{x \in M} \).

**Lemma 4.1.** The loop in step 6 terminates.

**Proof.** The sum of squares of the counters of all items strictly increases by at least 1 in every iteration. Since this sum is upper bounded by \( n^2 \), we conclude that there are at most \( n^2 \) iterations. \( \square \)

**Lemma 4.2.** For every player \( i \) and item \( x \) that is associated with \( i \) at some stage of the pricing scheme, \( x \) is legal for player \( i \).

**Proof.** We prove by induction that the claim holds for all players in every iteration of the loop (step 6). Before the first iteration, every player \( i \) is associated with the item \( a_i \) that is allocated to her in the optimal allocation \( O \), and \( a_i \) is indeed legal for player \( i \). Assume that the claim holds after the \( k \)th iteration. Let \( C = d_{i_1} \rightarrow d_{i_2} \rightarrow \cdots \rightarrow d_{i_k} \rightarrow d_{i_1} \) be the 0-weight cycle considered in the \((k + 1)\)th iteration. By the induction hypothesis, for every \( 1 \leq j \leq \ell \) the item \( d_{i_j} \) is allocated to buyer \( i_j \) in some optimal allocation \( O^j \). Consider the multi-edge bipartite graph \( G \) whose left side consists of all buyers and whose right side consists of all items, and the edges are exactly all the edges of the matchings \( \{O^j\}_{1 \leq j \leq \ell} \). Since every \( O^j \) is an optimal allocation, and we assume that all allocations are perfect matchings, then \( G \) is an \( \ell \)-regular graph. The sum of all the weights of the edges in \( G \) is \( \sum_{j=1}^{\ell} SW(O^j) = \ell \cdot OPT \). Furthermore, \( G \) contains the (partial) matching \( m = \{i_j - d_{i_j}\}_{1 \leq j \leq \ell} \). Since
We start by explaining how the lemma is derived from the above three conditions. If \(d\) is a 0-weight cycle, then, by Claim 3.1, \(SW(m) = SW(m')\) where \(m'\) is the matching obtained from \(m\) by re-matching the item \(d_{ij}\) to player \(i_{j-1}\) for all \(j\). We conclude that if we replace the matching \(m\) with \(m'\), and call the new bipartite graph \(G'\), then \(G'\) is still an \(\ell\)-regular graph with \(SW(G) = SW(G')\). Similar to the proof of Theorem 3.5, we conclude that \(G'\) is a union of \(\ell\) different perfect matchings \(O_j\), \(1 \leq j \leq \ell\), and since \(O_j\) is optimal for every \(j\), then \(SW(O_j) \geq SW'(O_j)\) for every \(j\). From here we get \(SW(O_j) = SW'(O_j)\) for every \(j\). This concludes the proof since the union of the (optimal) matchings \(O_j\) contains all the edges \(i_{j-1} - d_{ij}\), implying that \(d_{ij}\) is legal for player \(i_{j-1}\), and after the \((k + 1)\)-th iteration exactly one of the edges of \(m\) is replaced by one of the edges of \(m'\).

**Corollary 4.3.** For every player \(i\), the item associated with player \(i\) at the end of the pricing scheme is legal for her.

**Lemma 4.4.** At the end of step 6 every cycle has strictly positive weight.

*Proof.* Using a very similar reasoning as in the proof of Lemma 4.2, it follows that in every iteration of step 6, every cycle in which all items are different has non-negative weight. Combining this with the loop condition, we conclude that at the end of Step 6 every cycle in which all items are different has strictly positive weight.

Now consider a cycle \(C = d_{i_1} \rightarrow d_{i_2} \rightarrow \cdots \rightarrow d_{i_k} \rightarrow d_{i_1}\) in which some item appears more than once. Assume w.l.o.g. that \(d_{i_1}\) and \(d_{i_k}\) represent the same item for some \(1 < k \leq \ell\). Clearly, the weight of \(C\) is the sum of the weights of the two cycles \(C_1 = d_{i_1} \rightarrow \cdots d_{i_{k-1}} \rightarrow d_{i_1}\) and \(C_2 = d_{i_k} \rightarrow \cdots d_{i_{k-1}} \rightarrow d_{i_k}\). Observe that both \(C_1\) and \(C_2\) contain less item repetitions than \(C\). We apply this process recursively on \(C_1\) and \(C_2\) until there are no repetitions, and conclude that the weight of \(C\) equals the sum of the weights of the obtained cycles, each of which is one in which all items are different. The lemma follows.

We conclude with the following Lemma.

**Lemma 4.5.** Let \(d_i\) be the final item associated with player \(i\) in the pricing scheme, and let \(p\) be the output price vector. Then, \(D_p(v_i)\) contains the unique bundle \(\{d_i\}\) if \(d_i\) is a real item, and \(\emptyset\) otherwise.

*Proof.* It is convenient to think of the price of an imaginary item as always being 0. Under this convention we prove:

1. \(p_x > 0\) for any real item \(x\).

2. If \(d_i\) is real then \(v_i(d_i) - p_{d_i} > 0\).

3. For any real item \(x \neq d_i\) we have \(v_i(d_i) - p_{d_i} > v_i(x) - p_x\).

We start by explaining how the lemma is derived from the above three conditions. If \(d_i\) is imaginary, then condition 3 shows that player \(i\) derives negative utility from every real item, implying that \(D_p(v_i) = \{\emptyset\}\) as required. If, on the other hand, \(d_i\) is real, then conditions 2,3 show that player \(i\) strongly prefers \(d_i\) over every other (real) item and over getting nothing. Together with condition 1 and the fact that player \(i\) is unit-demand we obtain \(D_p(v_i)\) contains only \(\{d_i\}\), as required.

We now prove the three conditions:
• **Condition 1:** If $p_x = \infty$, then the condition clearly holds. Otherwise, $x$ appears in the (final) preference graph and $p_x > 0$ as in the proof of Lemma 3.14.

• **Condition 2:** Let $s \rightarrow d_{i_1} \rightarrow \cdots \rightarrow d_{i_k} = d_i$ be a min-weight path to $d_i$. If $d_i$ and $d_{i_1}$ represent the same item, then in particular $p_{d_i} = p_{d_{i_1}} = \epsilon$ and the condition holds whenever $\epsilon > 0$ is small enough since $v_i(d_i) > 0$ ($d_i$ is legal for $i$ and recall our assumption that all items are necessary for optimality). Otherwise, the weight of the edge $e = d_{i_k} \rightarrow d_{i_1}$ is $w(e) = v_i(d_i) - v_i(d_{i_1}) - \epsilon$. The path together with the edge $e$ closes a cycle which has positive weight (Lemma 4.4). The weight of this cycle is

$$\delta(s, d_i) + w(e) = \delta(s, d_i) + v_i(d_i) - v_i(d_{i_1}) - \epsilon = v_i(d_i) - p_{d_i} - v_i(d_{i_1}) > 0.$$  

The claim follows.

• **Condition 3:** Let $x \neq d_i$ be some real item. If $p_x = \infty$ then the condition clearly holds. Otherwise there is some buyer $j$ with $d_j = x$, and by the triangle inequality we have

$$\delta(s, d_i) + w(d_i \rightarrow d_j) \geq \delta(s, d_j).$$

Substituting $w(d_i \rightarrow d_j)$ by $v_i(d_i) - v_i(d_j) - \epsilon$ gives us

$$v_i(d_i) - (-\delta(s, d_i) + \epsilon) > v_i(d_j) - (-\delta(s, d_j) + \epsilon) = v_i(d_j) - p_{d_j}.$$

If $d_i$ is a real item, then this is exactly condition 3. Otherwise, $d_i$ is imaginary and we get

$$v_i(d_i) - p_{d_i} = 0 > v_i(d_i) - (-\delta(s, d_i) + \epsilon) > v_i(d_j) - p_{d_j},$$

as required (recall that $-\delta(s, d_i) \geq 0$).

\[\square\]

## 5 Maximal Domain Theorems for Walrasian Equilibrium and Dynamic Pricing

In this section we prove Theorems 1.1 and 1.4, namely the maximal domain theorems for Walrasian equilibrium and dynamic pricing, respectively. In section 5.1 we state and prove a variant of the price based gross-substitutes characterization by Reijnierse et al. [35]. As a direct corollary we obtain the following theorem, which will be needed to prove the maximal domain theorems (for a discussion about the differences between this version and the original one, see Section 1.2).

**Theorem 5.1.** Let $v$ be a non gross-substitutes valuation. Then there are bundles $A, B \subseteq M$ and a price vector $p$ such that:

1. $|B \setminus A| = 2$.
2. $|A \setminus B| \leq 1$.
3. $B \in \arg \min_C : u(C, p) > u(A, p)(|C \Delta A|)$.  

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As explained in the introduction, the original proof of Theorem 1.1 takes a non gross-substitutes valuation \( v_1 \) and uses the single-improvement property to claim the existence of some bundle \( A \) and item prices \( p \), relative to which \( A \) is not utility maximizing for \( v_1 \), and any preferred bundle \( B \) satisfies \( |B \setminus A| \geq 2 \) or \( |A \setminus B| \geq 2 \). The proof then considers such a bundle \( B \) minimizing \( |A \triangle B| \), and splits to cases according to whether \( |B \setminus A| \geq 2 \) or \( |A \setminus B| \geq 2 \).

Recall that Yang \([40]\) discovered an error in the construction of the second case (\( |A \setminus B| \geq 2 \)). Theorem 5.1 allows us to focus only on the first case. Since the original proof of that case is correct, combining it with Theorem 5.1 provides a full and complete proof of Theorem 1.1.

In section 5.2 we prove Theorem 1.4. The market we construct is a perturbed version of the original construction by Gul and Stacchetti, that satisfies the following two claims:

- If the market admits a dynamic pricing then it admits a Walrasian equilibrium.
- The market does not admit a Walrasian equilibrium.

The second claim constitutes a complete (and slightly altered compared to the original) proof of Theorem 1.1, and the combination of the two claims yields Theorem 1.4.

### 5.1 A Characterization of Gross-Substitutes

In their paper, Reijnierse et al. prove the following:

**Theorem 5.2** ([35]). A valuation \( v \) is gross-substitutes if and only if the following two conditions hold:

- For every pair of different items \( x, y \) and bundle \( S \subseteq M \setminus \{x, y\} \), we have
  \[
  v(S \cup \{x\}) + v(S \cup \{y\}) \geq v(S) + v(S \cup \{x, y\})
  \]  
  (SM)

- For every triplet of different items \( x, y, z \) and bundle \( S \subseteq M \setminus \{x, y, z\} \), we have
  \[
  v(S \cup \{x\}) + v(S \cup \{y, z\}) \leq \max\{v(S \cup \{y\}) + v(S \cup \{x, z\}), v(S \cup \{z\}) + v(S \cup \{x, y\})\}
  \]  
  (RGP)

The first condition is the well-known submodularity condition. The conditions (SM) and (RGP) have analogous “price” counterparts:

**Lemma 5.3** ([35]). A valuation \( v \) satisfies (SM) and (RGP) if and only if it satisfies the following two conditions:

- There are no vector \( p \in \mathbb{R}^{|M|} \) (possibly with negative entries), two different items \( x, y \) and a bundle \( S \subseteq M \setminus \{x, y\} \) for which
  \[
  D_p(v) = \{S, S \cup \{x, y\}\}
  \]  
  (P-SM)

- There are no vector \( p \in \mathbb{R}^{|M|} \) (possibly with negative entries), three different items \( x, y, z \) and a bundle \( S \subseteq M \setminus \{x, y, z\} \) for which
  \[
  D_p(v) = \{S \cup \{x\}, S \cup \{y, z\}\}
  \]  
  (P-RGP)
Theorem 5.5. A valuation \( v \) assume 1, and decrease the price of \( p_x \) and \( p_y \) by a small enough amount so that \( S \cup \{ x, y \} \) becomes the unique utility-maximizing bundle, and \( S \) becomes the only 2nd best bundle. It would appear that taking the vector \( \mathbf{p} \) together with \( A := S \) and \( B = S \cup \{ x, y \} \) proves Theorem 5.1. However, the prices obtained from Lemma 5.3 can be negative (and indeed are in the known construction) and therefore are unsuitable. The same problem arises when assuming that (P-RGP) is violated.

The following is a different version of Lemma 5.3 with non-negative prices.

**Lemma 5.4.** A valuation \( v \) satisfies (SM) and (RGP) if and only if it satisfies the following two conditions:

- There are no nonnegative price vector \( \mathbf{p} \), two different items \( x, y \) and a bundle \( S \subseteq M \setminus \{ x, y \} \) for which \( p_x, p_y > 0 \) and

\[
\{ S, S \cup \{ x, y \} \} \subseteq D_{\mathbf{p}}(v) \subseteq \{ T \mid T \subseteq S \} \cup \{ T \cup \{ x, y \} \mid T \subseteq S \} \quad \text{(NP-SM)}
\]

- There are no nonnegative price vector \( \mathbf{p} \), three different items \( x, y, z \) and a bundle \( S \subseteq M \setminus \{ x, y, z \} \) for which \( p_x, p_y, p_z > 0 \) and

\[
\{ S \cup \{ x \}, S \cup \{ y, z \} \} \subseteq D_{\mathbf{p}}(v) \subseteq \{ T \cup \{ x \} \mid T \subseteq S \} \cup \{ T \cup \{ y, z \} \mid T \subseteq S \} \quad \text{(NP-RGP)}
\]

The combination of Theorem 5.2 and Lemma 5.4 imply:

**Theorem 5.5.** A valuation \( v \) is gross-substitutes if and only if it satisfies (NP-SM) and (NP-RGP).

The proof of Lemma 5.4, which to a large extent is adapted from [31] and [36], is deferred to Appendix C. We now show how Theorem 5.5 implies Theorem 5.1.

**Proof of Theorem 5.1.** Let \( v \) be valuation that is not gross substitutes. By Theorem 5.5, there is a nonnegative price vector \( \mathbf{p} \) and a bundle \( S \) such that one of the following holds:

1. There are items \( x, y \notin S \) for which \( p_x, p_y > 0 \) and

\[
\{ S, S \cup \{ x, y \} \} \subseteq D_{\mathbf{p}}(v) \subseteq \{ T \mid T \subseteq S \} \cup \{ T \cup \{ x, y \} \mid T \subseteq S \}
\]

2. There are items \( x, y, z \notin S \) for which \( p_x, p_y, p_z > 0 \) and

\[
\{ S \cup \{ x \}, S \cup \{ y, z \} \} \subseteq D_{\mathbf{p}}(v) \subseteq \{ T \cup \{ x \} \mid T \subseteq S \} \cup \{ T \cup \{ y, z \} \mid T \subseteq S \}
\]

Assume 1, and decrease the price of \( x \) and \( y \) by a small enough \( \epsilon > 0 \) so that the new prices are still nonnegative and \( S \) derives the 2nd highest utility under the new prices. Observe that all utility maximizing bundles under the updated prices contain \( x, y \), and \( S \cup \{ x, y \} \) is such a bundle. Thus, if we choose \( A := S \) and \( B := S \cup \{ x, y \} \), then \( A, B \) satisfy \( \|B \setminus A\| = 2 \), \( \|A \setminus B\| = 0 \), and every other utility-maximizing bundle \( C \) satisfies \( \|C \triangle A\| \geq \|B \triangle A\| \), as required. Likewise, if bullet 2 holds, then we can take \( A := S \cup \{ x \} \), \( B = S \cup \{ y, z \} \) together with the price vector \( \mathbf{p} \) after \( p_y \) and \( p_z \) have been decreased by a small enough amount.
5.2 Proofs of the Maximal Domain Theorems

In this section we prove Theorems 1.1 and 1.4.

Proof of Theorems 1.1 and 1.4. Assume that \( v_1 \) is not gross-substitutes. Thus Theorem 5.1 implies the existence of bundles \( A, B \) and a price vector \( p \) for which \( |B \setminus A| = 2, |A \setminus B| \leq 1, \) and \( B \in \text{arg min}_C : u_1(C, p) > u_1(A, p) \). Denote \( B \setminus A = \{b_1, b_2\} \), and if \( |A \setminus B| = 1 \) then we denote \( A \setminus B = \{a\} \). We now introduce our collection of unit demand buyers. The first, denoted \( v_2 \), values each \( b \in B \setminus A \) at \( p_b + v_1(M) + 1 + \varepsilon'_b \) and values every other item at 0. Moreover, if \( A \setminus B \) is not empty, then we have a buyer \( v_a \) that values \( a \) at \( p_a + \varepsilon_a \) and values every other item at 0. Similarly, we have a buyer \( v_b \) for each item \( b \in B \setminus A \) that values \( b \) at \( p_b + \varepsilon_b \) and values every other item at 0. The values \( \varepsilon_a, \varepsilon_b, \varepsilon'_b \) are defined later. Finally, we have a buyer \( v_c \) for each \( c \in M \setminus (A \cup B) \) that values \( c \) at \( v_1(M) + 1 \) and values every other item at 0. Our goal is to set the numbers \( \varepsilon_a, \varepsilon_b, \varepsilon'_b \) such that the following two requirements are satisfied: if the market admits a dynamic pricing then it admits a Walrasian equilibrium, and the market does not admit a Walrasian equilibrium. The combination of the two requirements clearly implies both theorems. To this end, consider the collection \( \mathcal{A} \) of all allocations that satisfy the following properties:

- \( a \) is allocated to one of \( \{v_1, v_a\} \).
- Each item \( b \in B \setminus A \) is allocated to one of \( \{v_1, v_2, v_b\} \).
- Each item \( c \notin A \cup B \) is allocated to \( v_c \).
- Buyer \( v_2 \) takes exactly one item out of \( B \setminus A \).
- The items in \( A \cap B \) are all allocated to \( v_1 \).

We would like to set the numbers \( \varepsilon_a, \varepsilon_b, \varepsilon'_b \) such that no two allocations in \( \mathcal{A} \) have the same social welfare. When do two allocations \( \mathbf{O}^1, \mathbf{O}^2 \in \mathcal{A} \) have the same social welfare? Consider the following table that specifies the difference between \( \mathbf{O}^1 \) and \( \mathbf{O}^2 \):

|       | \( \mathbf{O}^1 \) | \( \mathbf{O}^2 \) |
|-------|-------------------|-------------------|
| 1     | \( C \)           | \( D \)           |
| 2     | \( e \)           | \( f \)           |
| \( a \) | \( G \)           | \( H \)           |
| \( b \in B \setminus A \) | \( I \)           | \( J \)           |

\( C \) and \( D \) are the bundles allocated to buyer 1 in \( \mathbf{O}^1 \) and \( \mathbf{O}^2 \), respectively. \( e \) and \( f \) are the items allocated to buyer 2 in \( \mathbf{O}^1 \) and \( \mathbf{O}^2 \), respectively. \( G \in \{a\} \) equals \( \{a\} \) if \( a \) is allocated to buyer \( v_a \) in \( \mathbf{O}^1 \) and otherwise \( G = \emptyset \). \( I \subseteq B \setminus A \) is the set of items \( b \in B \setminus A \) that are allocated to buyer \( v_b \) in \( \mathbf{O}^1 \). \( H, J \) are defined similarly. Allocations \( \mathbf{O}^1 \) and \( \mathbf{O}^2 \) have the same social welfare exactly when

\[
\begin{align*}
\nu_1(C) + (p_e + v_1(M) + 1 + \varepsilon'_e) + \sum_{a \in G} (p_a + \varepsilon_a) + \sum_{b \in I} (p_b + \varepsilon_b) + \sum_{c \in M \setminus (A \cup B)} (v_1(M) + 1) = \\
\nu_1(D) + (p_f + v_1(M) + 1 + \varepsilon'_f) + \sum_{a \in H} (p_a + \varepsilon_a) + \sum_{b \in J} (p_b + \varepsilon_b) + \sum_{c \in M \setminus (A \cup B)} (v_1(M) + 1)
\end{align*}
\]

which in turn, by rearranging, occurs exactly when

\[
\varepsilon'_e - \varepsilon'_f + \sum_{a \in G \setminus H} \varepsilon_a - \sum_{a \in H \setminus G} \varepsilon_a + \sum_{b \in I \setminus J} \varepsilon_b - \sum_{b \in J \setminus I} \varepsilon_b =
\]

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To achieve unique welfare for each allocation \( O \in \mathcal{A} \) we must set \( \varepsilon_a, \varepsilon_b, \varepsilon'_b \) so that equation (3) never holds whenever \( O^1 \neq O^2 \). The bottom expression in (3) is a function of \( C, D, e, f, G, H, I, J \) and all of \( \varepsilon_a, \varepsilon_b, \varepsilon'_b \) are in the top expression. If we set these values so that the top expression never evaluates to 0, but also small enough so that its absolute value is always smaller than the smallest possible non-zero absolute value of the bottom expression, then equality never holds, as desired. To this end denote the bottom expression of equation (3) by \( d_{C,D,e,f,G,H,I,J} \), and define \( \delta \) to be the minimal positive absolute value of \( d_{C,D,e,f,G,H,I,J} \) among all possible choices of \( C, D, e, f, G, H, I, J \). If \( |d_{C,D,e,f,G,H,I,J}| = 0 \) for all possible choices, then we set \( \delta = 1 \). We also define

\[
\varepsilon := \min \left\{ \frac{\delta}{2}, \frac{u_1(B,p) - u_1(A,p)}{4} \right\}.
\]

We now finally define the numbers \( \varepsilon_a, \varepsilon_b, \varepsilon'_b \) and complete the construction. We set:

\[
\begin{align*}
\varepsilon_{b_1} &= \varepsilon / 2^1 \\
\varepsilon_{b_2} &= \varepsilon / 2^2 \\
\varepsilon_a &= \varepsilon / 2^3 \\
\varepsilon'_{b_1} &= \varepsilon / 2^4 \\
\varepsilon'_{b_2} &= \varepsilon / 2^5
\end{align*}
\]

We claim that whenever \( O^1 \) and \( O^2 \) are different allocations, the top expression of equation (3) has positive absolute value that is smaller than \( \delta \). To see this, note that each of \( \varepsilon_a, \varepsilon_{b_1}, \varepsilon_{b_2}, \varepsilon'_{b_1}, \varepsilon'_{b_2} \) appears at most once in the top expression of equation (3), and at least one appears whenever \( O^1 \) and \( O^2 \) are not the same allocation. Take the number with the smallest power of 2 in the denominator and assume w.l.o.g. that it is preceded with a minus sign. Then, even if the rest of the numbers appear with a plus sign, the entire expression still evaluates to a negative value strictly between -\( \delta \) and 0. By definition of \( \delta \) the equation cannot hold in this case and we obtain the desired uniqueness. We have proved:

**Claim 5.6.** For every two different allocations \( O^1, O^2 \in \mathcal{A} \) we have \( SW(O^1) \neq SW(O^2) \).

**Corollary 5.7.** Each item \( x \in M \setminus (A \cap B) \) is allocated to the same player in every optimal allocation.

**Proof.** Let \( O \) be an optimal allocation. The following hold with respect to \( O \):

- \( a \) is allocated to one of \( \{v_1, v_a\} \) (otherwise the welfare can be increased by reallocating \( a \) to \( v_a \)).

- Each item \( b \in B \setminus A \) is allocated to one of \( \{v_1, v_2, v_b\} \) (otherwise the welfare can be increased by reallocating \( b \) to \( v_b \)).

- Each item \( c \in M \setminus (A \cup B) \) is allocated to \( v_c \) (similarly).

- \( v_2 \) takes exactly one item out of \( M \setminus (A \cap B) \), and this item is in \( B \setminus A \) (similarly).
Therefore, if we begin with the allocation $O$ and reallocate all the items in $A \cap B$ to $v_1$ then the resulting allocation is in $\mathcal{A}$. Furthermore, since the unit demand players value each item in $A \cap B$ at 0 and $v_1$ is monotone, we conclude that this modification does not incur a loss in welfare, implying that optimality is preserved. The claim follows since there is at most one optimal allocation in $\mathcal{A}$ (by Claim 5.6) and the modification does not reallocate any item in $M \setminus (A \cap B)$.

Corollary 5.7 can be rephrased as follows:

**Corollary 5.8.** There is some partition of $M \setminus (A \cap B)$, denoted $\{S_1, S_2\} \cup \{S_x\}_{x \in M \setminus (A \cap B)}$ such that in every optimal allocation the bundle received by player $v_i$ contains $S_i$ and perhaps also some subset of $A \cap B$.

We are now ready to prove:

**Lemma 5.9.** If the market admits a dynamic pricing, then it admits a Walrasian equilibrium.

**Proof.** Let $q$ be a dynamic pricing for the market. Recall that for any player $v_i$ and any $S \in D_q(v_i)$ there is some optimal allocation in which $v_i$ receives the bundle $S$. Thus, by Corollary 5.8, $S_i \in S$ for any $S \in D_q(v_i)$. Furthermore, for any player $v_i \neq v_1$ the items in $A \cap B$ do not add anything to the utility, implying that $S_i \in D_q(v_i)$. Moreover, even if we update $q$ so that all items in $A \cap B$ are priced at 0, and denote the new price vector by $q'$, then we would still have $S_i \in D_q(v_i)$. Note also that this update can only make player $v_1$ want $A \cap B$ more than before. Thus $S_1 \cup (A \cap B) \in D_{q'}(v_1)$.

We have thus shown that the allocation $(S_1 \cup (A \cap B), S_2, (S_x)_{x \in M \setminus (A \cap B)})$ together with the prices $q'$ constitute a Walrasian equilibrium, as desired.

It is left to prove that the market does not admit a Walrasian equilibrium.

**Lemma 5.10.** The market composed of the buyers $\{v_1, v_2\} \cup \{v_x\}_{x \in M \setminus (A \cap B)}$ does not admit a Walrasian equilibrium.

We remark that Lemma 5.10 proves Theorem 1.1 and that the proof is mainly adapted from the original proof of Theorem 1.1.

**Proof.** Assume towards contradiction that the allocation $Y = (Y_1, (Y_x)_{x \in M \setminus (A \cap B)})$ together with the price vector $t$ is a Walrasian equilibrium. Let $X = (X_1, (X_x)_{x \in M \setminus (A \cap B)})$ be the allocation obtained from $Y$ by reallocating all of $A \cap B$ to $v_1$. Define the price vector $q$ as follows:

$$
\begin{align*}
q_a &= p_a + \varepsilon_a \\
q_x &= 0 & x \in A \cap B \\
q_x &= t_x & x \in M \setminus A
\end{align*}
$$

The same arguments as in the original proof of Theorem 1.1 show that:

- $X, q$ is a Walrasian equilibrium,
- $A \cap B \subseteq X_1 \subseteq A \cup B$
- $X_2 = \{b_i\}$ for one of $i = 1, 2$, implying that $B \setminus X_1 \neq \emptyset$. Assume w.l.o.g. that $X_2 = \{b_2\}$.

The last two bullets the following:

- $A \setminus X_1$ equals one of $\{a\}, \emptyset$
\( X_1 \setminus A \) equals one of \( \{b_1\}, \emptyset \) and in any case, since \(|B \setminus A| = 2\), we have \(|A \cap X_1| < |A \cap B|\). By the minimality of \( B \) we have \( u_1(X_1, p) \leq u_1(A, p) < u_1(B, p) \). Assume that \( X_1 \setminus A = \{b_1\} \). Consider the difference \( u_1(A, p) - u_1(X_1, p) \geq 0 \). By how much does the difference change when modifying the prices from \( p \) to \( q \)? The items in \( A \cap X_1 \) do not contribute to the change (the prices of these items cancel out when evaluating the difference). \( A \setminus X_1 \) contributes no less than \(-\varepsilon_a\). Moreover, \( b_1 \notin X_{b_1} \), implying \( q_{b_1} \geq p_{b_1} + \varepsilon_{b_1} \) (otherwise \( v_{b_1} \) would prefer having \( b_1 \)). We conclude that \( X_1 \setminus A \) contributes at least \( \varepsilon_{b_1} \) to the difference change. But by definition, each \( b \in B \setminus A \) satisfies \( \varepsilon_b - \varepsilon_a > 0 \) and in particular the difference is strictly larger at the prices \( q \) compared to \( p \). Thus we have

\[
u_1(X_1, q) < u_1(A, q)\]

which is a contradiction since \((X, q)\) is a Walrasian equilibrium (implying in particular that \( X_1 \) is a favorite bundle for \( v_1 \) with respect to \( q \)). Now assume that \( X_1 \setminus A = \emptyset \). Denote \( d := u_1(B, p) - u_1(A, p) > 0 \). When passing from \( p \) to \( q \), the total price of of \( A \setminus X_1 \) increased by at most \( \varepsilon_a \leq \varepsilon \leq d/4 \) (recall the definition of \( \varepsilon \)). Together with \( u_1(X_1, p) \leq u_1(A, p) \) we have

\[
u_1(X_1, q) \leq u_1(A, q) + d/4. \tag{4}\]

Now, since \( X_1 \subseteq A \) and \( X_2 = \{b_2\} \), we must have \( b_1 \in X_{b_1} \), implying

\[
q_{b_1} \leq p_{b_1} + \varepsilon_{b_1} < p_{b_1} + \frac{d}{4}
\]

where the second inequality holds by definition of \( \varepsilon_{b_1} \), and the first holds since otherwise \( v_{b_1} \) would rather not have \( b_1 \). Moreover, since buyer \( v_2 \) (weakly) prefers \( b_2 \) over \( b_1 \), then we have

\[
p_{b_1} + v_1(M) + 1 + \varepsilon'_{b_1} - q_{b_1} \leq p_{b_2} + v_1(M) + 1 + \varepsilon'_{b_2} - q_{b_2} \quad \Leftrightarrow \quad q_{b_2} \leq p_{b_2} + q_{b_1} - p_{b_1} + \varepsilon'_{b_2} - \varepsilon'_{b_1}
\]

and since \( q_{b_1} \leq p_{b_1} + \varepsilon_{b_1} \), we also have

\[
q_{b_2} \leq p_{b_2} + \varepsilon_{b_1} + \varepsilon'_{b_2} - \varepsilon'_{b_1} \leq p_{b} + \frac{d}{4}
\]

We have shown that \( q_b \leq p_b + \frac{d}{4} \) for every \( b \in B \setminus A \), and conclude that when passing from \( p \) to \( q \), the total price of \( B \setminus A \) increased by at most \( d/2 \). Since the total price change of \( A \setminus B \) did not decrease then we have

\[
u_1(B, q) - u_1(A, q) \geq u_1(B, p) - u_1(A, p) - d/2 \tag{5}\]

Combining (4) and (5) we get

\[
u_1(B, q) - u_1(X_1, q) \geq \frac{d}{4} + u_1(B, p) - u_1(A, p) - d/2 - u_1(A, q) - d/4
\]

\[
= d - \frac{3}{4}d > 0
\]

and again this is a contradiction since \((X, q)\) is a Walrasian equilibrium.
6 A Budget-Additive Market with Dynamic Pricing and without Walrasian Equilibrium

Consider the example given by Feldman and Lucier [20], with 4 players $c_i, d_i$ for $i \in \{1, 2\}$, and 7 items $a_i, b_i, \alpha_i$ for $i \in \{1, 2\}$ and $\beta$. Each of the buyers is budget-additive with budget 2, meaning that the value for a bundle $S$ is $v(S) = \min\{2, \sum_{x \in S} v(\{x\})\}$. For $i \in \{1, 2\}$, $c_i$ has value 1 for $a_i, b_i, \alpha_i$ and value 0 for the rest of the items, and $d_i$ has value 2 for $\beta$, value 1 for $a_i, b_i$ and value 0 for the rest. It can be easily verified that $OPT = 7$ in this market. Furthermore, [20] showed that the market does not admit a Walrasian equilibrium. However, we claim that it does admit an optimal dynamic pricing. To see this, consider the first round prices

$$p_{\alpha_1}^1 = p_{\alpha_2}^1 = p_{\beta}^1 = \epsilon$$
$$p_{a_1}^1 = p_{a_2}^1 = 2\epsilon$$
$$p_{b_1}^1 = p_{b_2}^1 = 3\epsilon$$

We split to cases according to the first incoming buyer, and we can assume w.l.o.g. that it is either $c_1$ or $d_1$ (the other cases are symmetric):

1. Buyer $c_1$ arrives first. Under the above prices he takes $\alpha_1$ and $a_1$. At this point, we set the following prices for all subsequent rounds:

$$p_{\beta}^2 = p_{b_1}^2 = \epsilon$$
$$p_{a_2}^2 = p_{a_2}^2 = 2\epsilon$$
$$p_{b_2}^2 = 3\epsilon$$

The earlier of $d_1, d_2$ to arrive takes $\beta$. If $d_1$ arrives later he takes $b_1$ and $c_2$ takes $a_2$ and $\alpha_2$. If $d_2$ arrives later and before $c_2$ he takes $a_2, b_2$ and $c_2$ takes $\alpha_2$. If $d_2$ arrives last, $c_2$ takes $a_2$ and $\alpha_2$ and $d_2$ takes $b_2$. $OPT$ is achieved in any alternative.

2. Buyer $d_1$ arrives first. In this case, the initial prices are used throughout. Buyer $d_1$ takes $\beta$. $c_1$ takes $a_1, \alpha_1$ regardless of his place in line. If $d_2$ arrives before $c_2$, he takes $a_2, b_2$ and $c_2$ takes $\alpha_2$. Otherwise, $c_2$ takes $a_2, \alpha_2$ and $d_2$ takes $b_2$. Again, $OPT$ is achieved.

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A Preference Graph Properties

A.1 Proof of Claim 3.1

Proof. The weight of the cycle is

\[ w(C) = \sum_{i=1}^{k} w(x_i \rightarrow x_{i+1}) \]

\[ = \sum_{i=1}^{k} v_i(x_i) - v_i(x_{i+1}) \]
Let \( y \) be an arbitrary item. If \( y \not\in \{x_1, \ldots, x_k\} \) then it is allocated to the same player in \( O \) and in \( A \), ergo contributing 0 to the difference \( \text{SW}(O) - \text{SW}(A) \). If, on the other hand, \( y = x_i \), then it contributes \( v_i(x_i) - v_{i-1}(x_i) \) to the difference (since it is allocated to player \( i \) in \( O \) and to player \( i-1 \) in \( A \)). We have established that

\[
\text{SW}(O) - \text{SW}(A) = \sum_{i=1}^{k} v_i(x_i) - v_{i-1}(x_i) = \sum_{i=1}^{k} v_i(x_i) - v_i(x_{i+1}) = w(C)
\]

\( \square \)

### A.2 Proof of Corollary 3.2

**Proof.** If all vertices in the cycle are different then this is immediate by Claim 3.1 and the fact that \( O \) is optimal. If some vertices are repeated then the cycle is a union of two or more cycles with no vertex repetition (whose weights are non-negative). The claim holds since the weight of the cycle equals the sum of the weights of the repetition-free sub-cycles. \( \square \)

### A.3 Proof of Claim 3.3

**Proof.** Part 1 holds since the edge \( s \to x \) is a particular path from \( s \) to \( x \) and its weight is 0 (and the weight of a min-weight path from \( s \) to \( x \) can only be smaller). Part 2 holds by the triangle inequality:

\[
\begin{align*}
&\delta(s, x) + w(x \to y) \geq \delta(s, y) \\
&\delta(s, x) + v_i(x) - v_i(y) \geq \delta(s, y) \\
&v_i(x) - p_x \geq v_i(y) - p_y
\end{align*}
\]

To show part 3, consider a min-weight path \( s \to x_1 \to \cdots \to x_k = x \) from \( s \) to \( x \). Let’s assume first that \( x_1 \notin O_i \). Then the edge \( x \to x_1 \) exists and the weight of the cycle obtained by combining the edge with the path is:

\[
\delta(s, x) + w(x \to x_1) = -p_x + v_i(x) - v_i(x_1) \geq 0
\]

where the inequality holds by Corollary 3.2, and the result follows (recall that valuations are normalized and monotone implying \( v_i(x_1) \geq 0 \)). We now assume that \( x_1 \in O_i \). If \( k = 1 \) (i.e., the path is simply the edge \( s \to x \) then \( p_x = 0 \) and the result follows. Otherwise, \( x_2 \notin O_i \) and the edge \( x \to x_2 \) does exist. The weight of the cycle \( x_2 \to \cdots \to x \to x_2 \) is

\[
0 \leq \delta(x_2, x) + w(x \to x_2) = \delta(x_2, x) + v_i(x) - v_i(x_2) = (v_i(x_1) - v_i(x_2)) + \delta(x_2, x) + (v_i(x) - v_i(x_1)) = \delta(s, x) + v_i(x) - v_i(x_1) = -p_x + v_i(x) - v_i(x_1)
\]

and again the result follows. \( \square \)
B Dynamic Pricing for 3 Multi-Demand Buyers: the Case where Demand Exceeds Supply

In this section, we generalize the result of Section 3.2 for the case where \( m \leq \sum_i k_i \). A natural approach would be to introduce imaginary items with value 0 to all players, and apply the same techniques as in Section 3.2 to the obtained market. This approach ultimately succeeds, but introduces non-trivial challenges along the way which should be handled carefully. In particular, establishing the equivalent of Lemma 3.16 for the generalized setting (Lemma B.5) requires new ideas and more subtle arguments.

We fix a buyer profile \( \mathbf{v} = (v_1, \ldots, v_n) \) over the item set \( M \), where each \( v_i \) is \( k_i \)-demand. As explained in section 3 we can assume w.l.o.g. that all items are essential for optimality (i.e. each item is allocated in every optimal allocation), implying that every optimal allocation hands at most \( k_i \) items to player \( i \), for every \( i \). In section 3 we made the simplifying assumption that each optimal allocation hands exactly \( k_i \) items to player \( i \) for every \( i \). This was necessary for the proof of Theorem 3.5, which was crucial for establishing the correctness of the dynamic pricing scheme. In general though, the number of items might be smaller than \( \sum_i k_i \), in which case not all players exhaust their demand in every optimal allocation. In order to simulate this condition and present a dynamic pricing scheme that follows the same ideas of the scheme in the simplified setting, we introduce to the market \( \sum_{i=1}^n k_i - m \) “imaginary items”, valued at 0 by all players, and for every original optimal allocation in which a player \( i \) receives less than \( k_i \) items, we think of it as if the amount of received items is exactly \( k_i \), where some of the items can be imaginary. Furthermore, it is convenient to think of the price of an imaginary item as always being 0. We formalize these ideas in the following:

**Definition B.1.**

1. The **augmented market** of a buyer profile \( \mathbf{v} = (v_1, \ldots, v_n) \) is the buyer profile \( \mathbf{v}' = (v'_1, \ldots, v'_n) \) defined on the item set \( M' = M \cup \{d_1, \ldots, d_{\sum_{i=1}^n k_i - m}\} \), where for every \( i \), \( v'_i \) is \( k_i \)-demand with

\[
v'_i(x) = \begin{cases} v_i(x) & x \in M \\ 0 & x \in \{d_1, \ldots, d_{\sum_{i=1}^n k_i - m}\} \end{cases}
\]

The items \( d_1, \ldots, d_{\sum_{i=1}^n k_i - m} \) are called **imaginary items**.

2. An **augmented optimal allocation** is any original optimal allocation augmented with the additional imaginary items such that every player \( i \) receives exactly \( k_i \) items. Formally, an allocation \( O' \) is an augmented optimal allocation if every player \( i \) is allocated exactly \( k_i \) items, and there is an optimal allocation \( O \) such that for every original item \( x \in M \) we have that \( x \in O_i \) iff \( x \in O'_i \).

3. For any price vector \( \mathbf{p} \) on \( M \), its **augmented price vector** \( \mathbf{p}' \) is the price vector on \( M' \) where

\[
p'_x = \begin{cases} p_x & x \in M \\ 0 & \text{otherwise} \end{cases}
\]

**Remark B.2.**
• Since \( v'_i \) and \( v_i \) coincide on the set of real items \( M \), we abuse notation and use \( v_i \) when referring to \( v'_i \) (and similarly for \( u_i \) and \( u'_i \)).

• Every augmented optimal allocation in the augmented market is essentially an optimal allocation in the original market, padded with the appropriate number of imaginary items, to match demand.

• Since imaginary items always have value and price of 0, it follows that for any player \( i \), bundle \( S \subseteq M \) such that \(|S| \leq k_i \) and price vector \( p \) on \( M \) we have \( S \in D_p(i) \iff S \cup \{d_1, \ldots, d_{k_i-|S|}\} \in D_{p'}(i) \).

In our dynamic pricing scheme we adjust the tools used in the simplified setting to accommodate settings where demand exceeds supply. In particular, we still use the preference graph, except that its vertex set corresponds to all items, including imaginary ones, and its edges are defined with respect to some augmented optimal allocation \( O \). Analogous reasoning gives us the following claim. Recall that \( \delta(s, x) \) is the min-weight path from \( s \) to \( x \) in the preference graph \( H \).

**Claim B.3.** Consider the (non-negative) prices \( p_x = -\delta(s, x) \) for every real item \( x \), and the augmented prices \( p'_i \). Let \( i, x, y \) be such that \( x \in O_i, y \notin O_i \), and both \( x, y \) can be either real or imaginary. Then player \( i \) weakly prefers \( x \) over \( y \), i.e.

\[
v_i(x) - p'_x \geq v_i(y) - p'_y
\]

An item \( x \in M' \) (real or imaginary) is called legal for player \( i \) if there is some augmented optimal allocation \( O' = (O'_1, \ldots, O'_k) \) such that \( x \in O'_i \). Note that an imaginary item is legal for some player \( i \) if there is an optimal allocation in the original market in which player \( i \) receives strictly less than \( k_i \) items. Legal bundles and allocations are defined as in the main exposition. Theorem 3.5 then directly translates to our setting as follows. An allocation in the augmented market is legal iff it is augmented optimal, implying the following lemma.

**Lemma B.4.** A price vector \( p \) is a dynamic pricing for the original market if for every player \( i \) and \( S \in D_p(i) \):

1. The augmented set \( S' = S \cup \{d_1, \ldots, d_{\min(k_i-|S|, k_i-m)}\} \) is legal for player \( i \) in the augmented market.

2. There exists an allocation of the items \( M' \setminus S' \) to the other players in which every player receives a bundle that is legal for her.

The item-equivalence graph is also defined analogously. An imaginary item \( x \) is in the set \( B_{i,C} \) iff there is an optimal allocation in the original market in which player \( i \) receives strictly less than \( k_i \) items. Furthermore, for any two imaginary items \( x \in B_{i,C_1}, y \in B_{j,C_2} \) for \( i \neq j \) we have \( \{i\} \cup C_1 = \{j\} \cup C_2 \) (all imaginary items are legal for the same set of players since they are valued the same by all players). Lemma 3.11 carries over to our setting as well. Equipped with the modified tools, the dynamic pricing scheme is defined analogously to the main exposition, only that prices are set only for the real items (but based on the preference graph that includes the imaginary items). The only part of the analysis that does not carry over directly from the main exposition is the proof of Lemma 3.16 (ensuring a strictly positive utility from every item \( x \in O_\i \)). This lemma was crucial to argue that each demanded set of player \( i \) contains exactly \( k_i \) items. In our setting it is required...
in order to argue that each such demanded set contains at least \(|O_i \cap M|\) items (i.e. the amount of real items in \(O_i\)), implying in particular that for each \(S \in D_p(i)\),

\[|S'| = |S \cup \{d_1, \ldots, d_{\min(k_i-|S|, \Sigma_{k_i-m})}\}| = k_i^s,
\]

which is needed for the proof of Part 1 of Lemma B.4 (the analog of Lemma 3.17, whose proof also directly carries over to our setting).

We next explain why the proof of Lemma 3.16 does not carry over to augmented markets. In the proof, we argued that if \(x\) is imaginary, in which case \(x\) must be the case that all the vertices \(x\) are different (otherwise this is not a min-weight path). We split to cases:

1. The item \(x_1\) is real. In this case, the proof is identical to that of Lemma 3.16.

2. The edge \(x \rightarrow x_1\) exists (i.e., \(x\) and \(x_1\) do not belong to the same buyer and the edge was not removed in the transition from \(H\) to \(H'\)). In this case the cycle \(C\) obtained by connecting \(x\) to \(x_1\) exists in \(H'\) and its weight is positive (Lemma 3.13). Thus we have

\[
0 < w(C) = \delta(s, x) + w(x \rightarrow x_1) = \delta(s, x) + v_i(x) - v_i(x_1) - \epsilon \\
\leq v_i(x) - (-\delta(s, x) + \epsilon) = v_i(x) - p_x
\]

as desired.

3. One of the edges \(x_d \rightarrow x_{d+1}\) of the path does not correspond to an edge in the item-equivalence graph. I.e., if \(x_d \in B_{i_d,C_d}, x_{d+1} \in B_{i_{d+1}, C_{d+1}}\), then \(i_d \notin C_{d+1}\). In this case there is no augmented optimal allocation in which \(x_{d+1}\) is allocated to player \(i_d\) and thus the allocation \(A\) obtained from \(O\) by passing the item \(x_{j+1}\) to player \(i_j\) for all \(1 \leq j \leq k - 1\) and dis-allocating \(x_1\) is not optimal. Here again, the proof follows from the same reasoning as the proof of Lemma 3.16.

In the remaining cases we assume that the edge \(x \rightarrow x_1\) does not exist, \(x_1\) is imaginary and that all path edges correspond to item-equivalence graph edges that were not marked in the transition from \(H\) to \(H'\). In particular, all inner vertices of the path must belong to 2-index vertices of the item-equivalence graph (i.e. vertices of the form \(B_{i,j}\)), since all outgoing edges from 3-index vertices in the item-equivalence graph were marked in step 3, and 1-index sets have no incoming edges.

4. \(x_1\) belongs to a 3-indexed set of the item-equivalence graph. Thus it cannot be an inner vertex and we have \(x_1 = x_k = x\). However this is a contradiction since \(x\) is a real item and \(x_1\) is imaginary.

Lemma B.5. For any player \(i\) and real item \(x \in O_i \cap M\), \(v_i(x) - p_x > 0\).

Proof. Consider a min-weight path from \(s\) to \(x\) in \(H'\), \(s \rightarrow x_1 \rightarrow \cdots \rightarrow x_k = x\) (recall that \(H'\) is the modified preference graph), and suppose that \(x_j \in O_{i_j}\) for all \(j\) (with \(i_k = i\)). Since every cycle in \(H'\) has positive weight (Lemma 3.13) it must be the case that all the vertices \(x_i\) are different (otherwise this is not a min-weight path). We split to cases:

1. The item \(x_1\) is real. In this case, the proof is identical to that of Lemma 3.16.

2. The edge \(x \rightarrow x_1\) exists (i.e., \(x\) and \(x_1\) do not belong to the same buyer and the edge was not removed in the transition from \(H\) to \(H'\)). In this case the cycle \(C\) obtained by connecting \(x\) to \(x_1\) exists in \(H'\) and its weight is positive (Lemma 3.13). Thus we have

\[
0 < w(C) = \delta(s, x) + w(x \rightarrow x_1) \\
= \delta(s, x) + v_i(x) - v_i(x_1) - \epsilon \\
\leq v_i(x) - (-\delta(s, x) + \epsilon) = v_i(x) - p_x
\]

as desired.
Remark: All cases up to now did not make any assumption on the structure of \( O \). In the remaining cases we use the following terminology: given a cycle in the item-equivalence graph in which all vertices are different, “applying the cycle” means choosing an arbitrary item from each vertex in the cycle, followed by returning the augmented optimal allocation obtained by reallocating each of the items to the player possessing the preceding item in the cycle.

5. \( x_1 \) belongs to a 1-indexed set. W.l.o.g. \( x_1 \in B_{1,\emptyset} \). If \( x_k \not\in O_1 \), then the edge \( x_k \rightarrow x_1 \) exists in \( H' \) and this is handled in case 1. Thus we assume that \( x_k \) belongs to player 1. \( x_2 \not\in B_{213} \cup B_{312} \) as otherwise \( x_2 \) is an inner vertex that belongs to a 3-indexed set, contradicting our assumption. Thus \( x_2 \in B_{21} \) or \( x_2 \in B_{31} \). Assume \( x_2 \in B_{21} \). \( x_3 \not\in B_{123} \cup B_{12} \) (corresponding edges were marked in Step 3) and also \( x_3 \not\in B_{312} \) (cannot be inner vertex and cannot be final vertex since it does not belong to player 1). The remaining possibility is that \( x_3 \in B_{32} \). In this case we cannot have \( x_4 \in B_{13} \) (one of the edges \( B_{21} \rightarrow B_{32} \) was marked in Step 4). Similarly we cannot have \( x_4 \in B_{23} \cup B_{213} \) (corresponding edges were marked in Step 3). We conclude that \( x_4 = x = x_k \in B_{123} \), and the (item-equivalence graph) path is

\[ B_{1,\emptyset} \rightarrow B_{21} \rightarrow B_{32} \rightarrow B_{123} \]

In the analogous case where \( x_2 \in B_{31} \), the resulting path is

\[ B_{1,\emptyset} \rightarrow B_{31} \rightarrow B_{23} \rightarrow B_{123} \]

We now show that there is an alternative augmented optimal allocation in which both paths do not exist in the item-equivalence graph (i.e., one of the sets in each path is empty). We can then update the algorithm by adding a pre-processing step in which the base augmented optimal allocation is updated to the new one if it so happens that the imaginary items belong to \( B_{1,\emptyset} \). In the new allocation the current case is vacuous. To this end, note that the cycles \( C_1 = B_{21} \rightarrow B_{32} \rightarrow B_{123} \rightarrow B_{21} \) and \( C_2 = B_{31} \rightarrow B_{23} \rightarrow B_{123} \rightarrow B_{31} \) are cycles in the item-equivalence graph. Consider the following procedure:

While one of the cycles \( C_1, C_2 \) exists in the item-equivalence graph (i.e. the corresponding vertices are non-empty sets), “apply” one of them.

Note that each application of \( C_1 \) decreases \(|B_{21}|, |B_{32}|, |B_{123}| \) by 1 (and increases \(|B_{23}|, |B_{312}|, |B_{12}| \) by 1). Each application of \( C_2 \) decreases \(|B_{31}|, |B_{23}|, |B_{123}| \) by 1 (and increases \(|B_{32}|, |B_{213}|, |B_{13}| \) by 1). In particular, the sum

\[ |B_{21}| + |B_{123}| + |B_{31}| \]

strictly decreases after each iteration. We conclude that the procedure must end and none of these paths exists in the obtained augmented optimal allocation.

6. \( x_1 \) is imaginary and belongs to a 2-indexed set of the item-equivalence graph. W.l.o.g. \( x_1 \in B_{32} \). It cannot be the case that \( x_2 \not\in B_{23} \cup B_{213} \) (the corresponding edges were marked in Step 3). If \( x_2 \in B_{123} \), then \( x_2 = x \) but the edge \( x_2 \rightarrow x_1 \) was not removed in the transition to \( H' \), and this case was covered in part 2. The remaining possibility is that \( x_2 \in B_{13} \). Note that the edge \( x_2 \rightarrow x_1 \) exists in \( H' \) and thus \( x_2 \neq x \). It cannot be the case that \( x_3 \in B_{21} \) (one of the edges \( B_{32} \rightarrow B_{13}, B_{13} \rightarrow B_{21} \) was marked in Step 4). \( x_3 \not\in B_{312} \cup B_{31} \) (the corresponding
implying
\[ x_3 = x \in B_{213} \] and the corresponding item-equivalence graph path is
\[ B_{32} \rightarrow B_{13} \rightarrow B_{213} \]
(indeed, the edge \( x_3 \rightarrow x_1 \) does not exist in \( H' \) since the edge \( B_{213} \rightarrow B_{32} \) was marked in Step 3). Furthermore, every imaginary item can belong either to \( B_{32} \) or \( B_{23} \). In the analogous case where \( x_1 \in B_{23} \), the resulting item-equivalence graph path is
\[ B_{23} \rightarrow B_{12} \rightarrow B_{312} \]
As in the previous case we will show that there is a “cycle application” procedure that results in an alternative augmented optimal allocation in which none of these paths exists. Denote the cycles
\[ C_1 = B_{32} \rightarrow B_{13} \rightarrow B_{213} \rightarrow B_{32} \]
\[ C_2 = B_{23} \rightarrow B_{12} \rightarrow B_{312} \rightarrow B_{23} \]
and our assumption is that \( C_1 \) exists in the bottleneck graph. The procedure goes as follows:

- While \(|B_{13}|, |B_{213}|, |B_{12}|, |B_{312}| \geq 1 \) (all inequalities hold)
  - Apply \( C_1 \), then apply \( C_2 \)
- If \(|B_{12}| = 0 \) or \(|B_{312}| = 0 \), apply \( C_1 \) \( \min\{|B_{13}|, |B_{213}|, |B_{32}|\} \) times and terminate.
- Otherwise (i.e., \(|B_{13}| = 0 \) or \(|B_{213}| = 0 \)), apply \( C_2 \) \( \min\{|B_{12}|, |B_{312}|, |B_{23}|\} \) times and terminate.

Each application of \( C_1 \) decreases \(|B_{13}|, |B_{213}|, |B_{32}| \) by 1, and increases \(|B_{123}|, |B_{23}|, |B_{31}| \) by 1. Each application of \( C_2 \) decreases \(|B_{12}|, |B_{312}|, |B_{23}| \) by 1 and increases \(|B_{123}|, |B_{32}|, |B_{21}| \) by 1. Therefore, in total, each iteration of the loop decreases each of \(|B_{13}|, |B_{213}|, |B_{12}|, |B_{312}| \) by 1 and the loop in the process ends, with either \( C_1 \) or \( C_2 \) non-existent. The final step takes care of eliminating the other cycle (note that applying the leftover cycle does not resurrect its counterpart cycle).

\[ \square \]

## C Proof of Lemma 5.4

**Proof.** We first show that \((SM)\) is equivalent to \((NP-SM)\), and then we show that under the assumption that \((SM)\) holds, \((RGP)\) is equivalent to \((NP-RGP)\). The combination implies the lemma.

Assume that \((NP-SM)\) does not hold, i.e., there are corresponding non-negative price vector \( p \), items \( x, y \) and a bundle \( S \). Then,
\[
S, S \cup \{x, y\} \in D_p(v) \\
S \cup \{x\}, S \cup \{y\} \notin D_p(v)
\]
implies
\[
v(S) + v(S \cup \{x, y\}) - 2 \cdot \sum_{d \in S} p_d - p_x - p_y > v(S \cup \{x\}) + v(S \cup \{y\}) - 2 \cdot \sum_{d \in S} p_d - p_x - p_y.
\]

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We conclude that (SM) is violated. For the converse direction, assume (SM) is violated; i.e.,
\[ v(S) + v(S \cup \{x, y\}) = v(S \cup \{x\}) + v(S \cup \{y\}) + \delta \]
for some \( \delta > 0 \). We define the prices \( p \) as follows. Set \( p_d = \infty \) for any \( d \in S \cup \{x, y\} \) to guarantee that no item outside of \( S \cup \{x, y\} \) is demanded. Set \( p_d = 0 \) for any \( d \in S \). Finally, let \( \epsilon := \delta/2 \) and set
\[ p_x := v(x|S \cup \{y\}) - \epsilon \]
\[ p_y := v(y|S \cup \{x\}) - \epsilon \]
First we show that \( p_x, p_y \) are positive:
\[ p_x := v(S \cup \{x, y\}) - v(S \cup \{y\}) - \frac{1}{2}(v(S) + v(S \cup \{x, y\}) - v(S \cup \{x\}) - v(S \cup \{y\})) \]
\[ = \frac{1}{2}(v(S \cup \{x, y\}) - v(S \cup \{y\}) + v(S \cup \{x\}) - v(S)) \]
\[ \geq \frac{1}{2}(v(S \cup \{x, y\}) - v(S \cup \{y\})) \]
\[ \geq \frac{1}{2}\delta \]
\[ > 0, \]
where the first and second inequalities hold since \( v(S \cup \{x\}) - v(S) \geq 0 \) (\( v \) is monotone). \( p_y > 0 \) can be shown similarly. The definitions of \( p_x \) and \( p_y \) directly imply
\[ u(S \cup \{x, y\}) > u(S \cup \{x\}), u(S \cup \{y\}) \]
and it is also immediate that
\[ p_x + p_y = v(S \cup \{x, y\}) - v(S) \]
implying
\[ u(S \cup \{x, y\}) = u(S). \]
To summarize,
\[ u(S), u(S \cup \{x, y\}) > u(S \cup \{x\}), u(S \cup \{y\}) \quad (6) \]
Finally, for any \( T \subseteq S \) and \( E \subseteq \{x,y\} \) we have (recall that \( p_d = 0 \) for all \( d \in S \) and that \( v \) is monotone):
\[ u(T \cup E) = v(T \cup E) - \sum_{d \in E} p_d \]
\[ \leq v(S \cup E) - \sum_{d \in E} p_d \]
\[ = u(S \cup E) \]
and this together with the inequalities (6) imply that \( S, S \cup \{x, y\} \) are demanded and any other demanded bundle must be of the form \( T \) or \( T \cup \{x,y\} \) for \( T \subseteq S \). Thus (NP-SM) is violated, as required.

We proceed to prove that under the assumption that \( v \) satisfies (SM), Conditions (RGP) and (NP-RGP) are equivalent.
Assume that (NP-RGP) does not hold, i.e., there are corresponding non-negative price vector $p$, items $x,y,z$ and a bundle $S$. Then

\[
S \cup \{x\}, S \cup \{y, z\} \in D_p(v)
\]

\[
S \cup \{x, z\}, S \cup \{x, y\}, S \cup \{y\}, S \cup \{z\} \notin D_p(v)
\]

In particular we have

\[
v(S \cup \{x\}) - p_x + v(S \cup \{y, z\}) - p_y - p_z > v(S \cup \{y\}) - p_y + v(S \cup \{x, z\}) - p_x - p_z
\]

\[
v(S \cup \{x\}) - p_x + v(S \cup \{y, z\}) - p_y - p_z > v(S \cup \{z\}) - p_z + v(S \cup \{x, y\}) - p_x - p_y
\]

implying

\[
v(S \cup \{x\}) + v(S \cup \{y, z\}) > v(S \cup \{y\}) + v(S \cup \{x, z\})
\]

(7)

\[
v(S \cup \{x\}) + v(S \cup \{y, z\}) > v(S \cup \{z\}) + v(S \cup \{x, y\})
\]

(8)

and thus (RGP) is violated as required.

For the converse direction, assume that (7) and (8) hold. We define the prices $p$ as follows. Set $p_d = \infty$ for any $d \notin S \cup \{x, y, z\}$ to guarantee that no item outside of $S \cup \{x, y, z\}$ is demanded. By rearranging the assumed inequalities we get

\[
v(z \mid S \cup \{y\}) > v(z \mid S \cup \{x\}) \geq 0
\]

\[
v(y \mid S \cup \{z\}) > v(y \mid S \cup \{x\}) \geq 0
\]

Therefore, by setting $p_y = v(y \mid S \cup \{y\}) - \epsilon$ and $p_z = v(z \mid S \cup \{y\}) - \epsilon$ for a sufficiently small $\epsilon$, $p_y$ and $p_z$ are positive and

\[
u(S \cup \{y, z\}) > u(S \cup \{y\}), u(S \cup \{z\}) > u(S),
\]

(9)

where the first inequality is a direct implication of the definition of $p_y$ and $p_z$ and the second inequality is implied by the first since $v$ is submodular. Next we define $p_x$ so that the utility from $S \cup \{x\}$ equals that of $S \cup \{y, z\}$. Specifically:

\[
p_x = v(S \cup \{x\}) - v(S \cup \{y, z\}) + p_y + p_z
\]

\[=
v(S \cup \{x\}) - v(S \cup \{y, z\}) + v(S \cup \{y, z\}) - v(S \cup \{z\}) - \epsilon + v(S \cup \{y, z\}) - v(S \cup \{y\}) - \epsilon
\]

\[= v(S \cup \{x\}) + v(S \cup \{y, z\}) - v(S \cup \{z\}) - v(S \cup \{y\}) - 2\epsilon
\]

\[> v(S \cup \{y\}) + v(S \cup \{x, z\}) - v(S \cup \{z\}) - v(S \cup \{y\}) - 2\epsilon
\]

\[> v(S \cup \{z\}) - v(S \cup \{z\}) - 2\epsilon
\]

\[> 0,
\]

where the first inequality holds by (7), the second holds by monotonicity and the third holds by (8) and for a small enough $\epsilon$. Finally, set $p_d = 0$ for every $d \in S$. Observe that indeed all prices are nonnegative as required. Next we show that

\[
u(S \cup \{x\}) > u(S \cup \{x, y\})
\]

(10)

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\[ u(S \cup \{x\}) > u(S \cup \{x, z\}) \]  
(11)

Adding the two inequalities together and applying the submodularity of \( v \) gives
\[ u(S \cup \{x\}) > u(S \cup \{x, y, z\}) \]  
(12)

We show that (10) holds ((11) is analogous). This amounts to showing that the marginal contribution of \( y \) to the bundle \( S \cup \{x\} \) is negative:
\[
v(y|S \cup \{x\}) - p_y = v(S \cup \{x, y\}) - v(S \cup \{x\}) - p_y
= v(S \cup \{x, y\}) - v(S \cup \{x\}) - (v(S \cup \{y, z\}) - v(S \cup \{z\}) - \epsilon)
= -(v(S \cup \{y, z\}) + v(S \cup \{x\}) - v(S \cup \{x, y\}) - v(S \cup \{z\}) + \epsilon)
< 0
\]
where the inequality holds by (8) for a sufficiently small \( \epsilon > 0 \). To summarize, the combination of (9), (10), (11), (12) together with \( u(S \cup \{x\}) = u(S \cup \{y, z\}) \) establishes that each of
\[
u(S \cup \{x\}),
\]
\[
u(S \cup \{y, z\})
\]
is strictly greater than each of
\[
u(S),
\]
\[
u(S \cup \{y\}),
\]
\[
u(S \cup \{z\}),
\]
\[
u(S \cup \{x, z\}),
\]
\[
u(S \cup \{x, y\}),
\]
\[
u(S \cup \{x, y, z\}).
\]

Now, for any \( T \subseteq S \) and any \( E \subseteq \{x, y, z\} \) we have (recall that \( p_d = 0 \) for all \( d \in S \))
\[
u(T \cup E) = v(T \cup E) - \sum_{d \in E} p_d
\leq v(S \cup E) - \sum_{d \in E} p_d
= u(S \cup E)
\]

Therefore, demanded bundles can only be of the form \( T \cup \{x\} \) or \( T \cup \{y, z\} \) for \( T \subseteq S \), and \( S \cup \{x\}, S \cup \{y, z\} \) are demanded, implying that (NP-RGP) is violated. This concludes the proof. \( \square \)