Theory of zones on Zeeman manifolds
A new approach to the infinities of QED

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Abstract

The Zeeman-Hamilton operator of free charged particles are identified with the Laplacians of certain Riemannian manifolds, called Zeeman manifolds. The quantum Hilbert space, $\mathcal{H}$, decomposes into subspaces (Zeeman zones) which are invariant under the action both of the Zeeman operator and the natural Heisenberg group representation. Thus a well defined particle theory and zonal geometry can be developed on each zone separately. The most surprising result is that quantities those divergent on the global setting appear to be finite on the zonal setting. Even the zonal Feynman integral is well defined.

This zonal interpretation of particles has fundamental effect both on the physical and mathematical view of these objects. The points are non-existing on a zone, for instance. One should introduce the concept of zonal point-spread, defined by certain wave functions. As a result, the zonal particles are not point- but point-spread-objects. The theory developed below includes explicit computation of objects such as the waves defining the point-spreads, the zonal Wiener-Kac and Dirac-Feynman flows (which then define the corresponding measures on the path-spaces), and the corresponding zonal Feynman-Kac formulas. It will be tested against several well known effect such as the Aharanov-Bohm effect and Lamb shift. There is also explained why these extended charged particles do not blow up.

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1 Introduction

The confusing infinities (divergent integrals), stubbornly present in calculations since the very beginning of quantum electrodynamics, are controlled by renormalization today. This pertubative tool provides the desired finite quantities by differences of infinities. The legacy of difficulties came from concepts such as point mass and point charge of classical electron theory, which provided the first warning that a point electron will have infinite electromagnetic self-mass; the mass $e^2/(6\pi ac^2)$ for a surface distribution of charge with radius $a$ blows up for $a \to 0$. In quantum field theory the Hamiltonian of the field is proportional to this electromagnetic self-mass. This is why this infinity launched one of the deepest crises in the history of physics.

Mathematically speaking, the infinities mostly due to the infinite trace of kernels such as the Wiener-Kac kernel $e^{-tH}$ (providing the fundamental solution of the heat equation), or, Dirac-Feynman kernel $e^{-tHi}$ (providing the fundamental solution of the Schrödinger equation). As a result, they assign infinite measures to physical objects such as self-mass, self-charge, e.t.c.. Also the Feynman measure, which is analogous to the well defined Wiener-Kac measure on the path-spaces, requires renormalization.

This paper gives a new non-perturbative approach to this problem. In the first step the quantum Hilbert space $\mathcal{H}$ (on which the quantum Hamilton operator $H$ is acting) is decomposed into the direct sum of $H$-invariant subspaces called Zeeman zones. Then, all the actions such as the heat- or Feynman-flow are considered on these invariant subspaces separately. It turns out that both the Wiener-Kac and Dirac-Feynman kernels are of the trace class on each zone, furthermore, both define the corresponding zonal measures on the path-spaces rigorously.

Three types of Hamilton operators are considered in this paper. (1) The classical Zeeman operator $H_Z$ corresponding to free charged particles with exterior (orbiting) spin. This spin is exhibited by the angular momentum operator associated with the magnetic dipole moment. (2) The Pauli operator, which provides also inner spin to the orbiting spin. (3) The relativistic Dirac operator. These Hamiltonians are the most important ones in quantum physics. They were primarily introduced for explaining the Zeeman effect.
There are several new features also in the manner by which these operators are introduced in this paper. The original historic model for establishing the Zeeman operator was a charged particle orbiting in a constant magnetic field which is perpendicular to the plane where the particle is orbiting. The Hamilton function of this system can be found by the Maxwell equations. In the hypothesis of particles having orbiting spin one assumes that the Hamiltonian of charged particles are exactly of this form. This hypothesis, so to speak, adds a little magnet to the particle. Assumption \( E = 0 \), imposed for the constant electromagnetic field, fixes an inertial system in which the orbiting particle is considered. In fact, in all other systems the field appears with a non-vanishing electric field \( E \). By this observation, all considerations developed here become relativistic. Even the Pauli operator, operating on spinors having two components, will be identified with the Dirac operator acting on four component spinors. This is why they are called Pauli-Dirac (PD) operators.

An other new interpretation is that the Zeeman Hamiltonian operator, corresponding to the above Hamiltonian function, is identified with the Laplacian of a non-compact Riemannian manifold. In case of a single particle orbiting in the \((x, y)\)-plane this manifold is nothing but the Heisenberg group, parametrized by \((x, y, t)\), whose center, parametrized by \(t\), is periodic established by factorization with a lattice. The Riemannian metric, \(g\), is the natural left-invariant metric on this manifold. Despite the positive definite metric, the considerations are still relativistic, due to the above observation.

One can generalize this idea to higher dimensional Heisenberg groups. This model corresponds to more particles orbiting in the same inertial system determined by the same constant magnetic field. Much more interesting is the generalization to two-step nilpotent Lie groups whose center is factorized by a co-compact lattice. This model corresponds to more particles being in different constant magnetic fields, thus each of them determines an individual inertial system with individual time. This mathematical model matches Dirac’s multi-time theory. All these Riemannian manifolds are called Zeeman manifolds.

It is well known that quantum physics is lacking of a comprehensive mathematical model which is able to explain the big diversity of phenomenons experienced in quantum theory. (The Schrödinger equation itself is not enough for explaining many of these experiences.) The above Riemannian manifolds
partially furnish this missing piece, serving as a relativistic space-time background for the theory presented here. It should be pointed out that the fixed inertial system appearing in the model resolves the concern of Pauli, brought up against Dirac’s relativistic electron theory. Indeed, beside the hole theory, Pauli strongly criticized also the probabilistic argument developed on the relativistic space-time. According to Pauli, probabilistic amplitudes do not have any meaning on the Minkowski space-time, since they must be defined on the space. Note that the above depicted space-time concept really offers a resolution for this concern. In fact, the probabilistic theory will be developed on the space determined by the fixed inertial system. The constant magnetic field, so to speak, serves as a bridge for descending from the Minkowski 4-space to the 3-space, needed for the probabilistic interpretations.

It should be pointed out that the mathematical model depicted so far does not go beyond the Schrödinger equation, therefore, it is not enough for explaining the effects which are not in the scope of the classical theory. The lacking piece what should be furnished yet is the spectral Zeeman zone decomposition, \( \mathcal{H} = \bigoplus_{a=0}^{\infty} \mathcal{H}(a) \), of the quantum Hilbert space. On the complex \( z = (x, y) \)-plane the total quantum Hilbert space, \( \mathcal{H} \), is the \( L^2 \)-Hilbert space of complex valued functions defined on \( \mathbb{C} \), where the inner product is defined by a Gaussian density \( e^{-\lambda z \overline{z}} \). This Hilbert space is spanned by the polynomials written in terms of \( z \) and \( \overline{z} \).

The zones are described in many different levels in this paper. They can be defined by Gram-Schmidt orthogonalization such that the starting zone, \( \mathcal{H}^{(0)} \) is the holomorphic zone spanned by the holomorphic polynomials. Then \( \mathcal{H}^{(1)} \) is defined by applying the Gram-Schmidt orthogonalization to the function space \( \mathcal{G}^{(1)} \) spanned by functions of the form \( z^a h(z) \), where \( h(z) \) is an arbitrary holomorphic polynomial. The procedure is continued by successive orthogonalization of spaces \( \mathcal{G}^{(a)} \) spanned by functions of the form \( \overline{z}^a h(z) \).

There is shown in the paper that the zones on the complex plane are the irreducible invariant subspaces of the natural Heisenberg group representation, thus each zone is suitable for establishing its own physics. In higher dimensional cases, describing multiple of particles, the irreducibility fails, however, in this case a zone further decomposes into subspaces by which exclusion principles and particles such as Bosonic and Fermionic ones can be defined.

Yet an other description explores the fact that the zones are invariant under
the action of the Hamilton operator and, by using explicit spectrum computations, the zones are introduced by the spectrum of the magnetic dipole moment operator. This technique actually uses polarization. A zone corresponds to a magnetic state of the particles.

All the important spectral theoretical objects on Zeeman manifolds are explicitly established. They include the spectrum, the zonal projection operators, the zonal Wiener-Kac and Dirac-Feynman kernels, and the zonal partition functions. Also the zonal Feynman-Kac type formulas, for both the Wiener-Kac and Feynman measure, are established in a novel form. One of the spectaculars of this theory is that all the quantities which appear as infinities regarding the total space $\mathcal{H}$ become finite ones on the zonal setting. Particularly, also the zonal Feynman measures are well defined which can be explicitly computed.

The consideration of particles as zonal objects has a deep effect also on the mathematical approach to the geometry formed within a zone. The most important impact is that there do not exist points in the original sense on a zone. By using the total Hilbert space $\mathcal{H}$ the points, $X$, can be identified with the Dirac delta distribution $\delta_X(Y) = \sum \phi_i(X)\delta(Y)$, where $\{\phi_i\}$ is an orthonormal basis on $\mathcal{H}$. Regarding a zone, $\mathcal{H}^{(a)}$, this distribution can be defined by those functions, $\phi_i^{(a)}$, which are in the zone. In other words, on a zone the points can be introduced by the kernel function belonging to the projection regarding the zone. On the holomorphic zone, for instance, this kernel is the well known Bergman kernel $\delta_x^{(0)}(y) = q e^{x^2+y^2-1/2(x^2+y^2)}$. It is fascinating to see that one gets the kernel of projection onto another zone, $\mathcal{H}^{(a)}$, just by multiplying the Bergman kernel by the Laguerre polynomial $L_n(|x-y|^2)$. Thus the zonal points are point-spreads which are explicitly described by the above wave functions.

In physics the idea of matter-wave was developed by de Broglie, whose theory got its final form in the Schrödinger equation. However, the Schrödinger theory still needs the point and other relating concepts such as point-mass and point-charge (see Weiskopf’s explanation in the next section), which then lead to the confusing infinities in quantum field theory. Mathematics did not follow de Broglie’s idea and it still enforced the point concept, from which the de Broglie theory departed. The contradiction between the two approach had been lifted by the duality principle, asserting that matter manifests itself sometime as wave while in other cases as point particle.
The zonal theory, called point spread geometry, establishes de Broglie’s idea also on mathematical level. The point spreads on a zone are considered to be the most compressed mathematical objects which still spread out over the whole space. They are the most compressed states possible for a zonal particle. These physical objects can be also in other states described by the zonal wave functions. The question remains if a charge spread is stable if it spreads as a point spread. It is pointed out that these charged point-spreads are not stable (they blow up), however, the zonal Dirac-Feynman flow corresponding to the zonal Schrödinger equation moves these spreads to stable, so called, solid charge spreads. The duality principle is overtaken also to the zonal theory. The zonal objects manifests themselves sometimes by the zonal wave functions and sometimes as solid zonal spreads.

The theory is worked out also for the Pauli-Dirac operator (anomalous zones) and it is tested against several effects such as Aharanov-Bohm effect and Lamb shift. Regarding the Lamb-shift, it should be emphasized that the theory developed so far concerns free particles, meaning that the Hamilton operator does not contain potential functions $V$ attributed to other sources such as nucleus. In case of Lamb-shift, regarding the non-existence of doublets in the spectrum of an electron in a hydrogen atom, there is such $V$ considered which is the Coulomb potential of the nucleus.

The problem arising about this potential is that the zones are not invariant with respect to multiplication with this radial function. Therefore, this operator does not fit the zonal theory at all. This operator, by expressing non-local interactions, contradicts also relativity. Einstein was able to eliminate the similar problem arising about Newton’s gravitation law by the theory of general relativity. Despite this intimidating analogy this problem will be considered in two different ways.

The first one defines the action of the zonal Coulomb potential operator, $V^{(a)}$, on a zonal function $f^{(a)}$ by projecting $Vf^{(a)}$ back to $H^{(a)}$. It turns out that also this zonal operator is an integral operator whose smooth kernel, $V^{(a)}(x, y)$, spreads out (non-trivially) onto the whole space. Thus the zonal Coulomb fields express local interactions. The theory will be tested against the Lamb shift in a hydrogen atom where the nucleus has zonal Coulomb potential. There are no doublets in the spectrum in this case.

In the second way the zonal theory is extended onto curved manifolds. Thus
the potential \( V \) is inbuilt into a curved metric, like in Einstein’s general relativity.

The following review meticulously goes back through the details of the zonal theory which is just sketchily described above. Most of the mathematical formulas will be only stated with no proofs. The rather long mathematical details are in \([Sz5, Sz6]\).

## 2 Grand review of zonal theory

(A) Brief history of renormalization.

The confusing infinities represented by divergent integrals are still present since the very beginning of quantum electrodynamics. The difficulties originate from concepts such as point mass and point charge of classical electron theory, which provided the first warning that a point electron will have infinite electromagnetic self-mass; the mass \( e^2/6\pi ac^2 \) for a surface distribution of charge with radius \( a \) blows up for \( a \to 0 \). In his classic paper V. F. Weisskopf (1939) explains the trouble caused by this divergence as follows: “Quantum kinematics shows that the radius of the electron must be assumed to be zero. It is easily proved that the product of the charge densities at two different points, \( \rho(\mathbf{r} - \xi/2) \times \rho(\mathbf{r} + \xi/2) \), is a delta-function \( e^2\delta(\xi) \). In other words: if one electron alone is present, the probability of finding a charge density simultaneously at two different points is zero for every finite distance between the points. Thus the energy of the electrostatic field is infinite as

\[
W_{st} = \lim_{a \to 0} e^2/a.
\]

This infinity had given rise to the greatest confusion at the earliest time of field quantization theory (cf. the pioneering work of Heisenberg and Pauli, published in (1929)), where the above integral represents the Hamiltonian, \( H_{field} = H_f \), of the Coulomb field. Then this crisis further escalated without finding any technique by which the incessantly growing number of infinities in the theory could have been handled. (A fascinating account on the development of quantum electrodynamics from its beginning up-to the 50’s can be found in *Silvan S. Schweber; QED and the men who made it: Dyson, Feynman, Schwinger, and Tomonaga; Princeton University Press* (1994). The most important original papers concerning this topic are reprinted in
“Selected Papers on Quantum Electrodynamics”, edited by Julian Schwinger, Dover Publications, 1958).

The breakthrough on the problem grew out of discussions at the Theoretical Physics Conference on Shelter Island, June 2 to 4; 1947, devoted for finding a satisfactory theoretical explanation for the so called Lamb shift measured between the two second level, 2s and 2p (resp. 2s_{1/2} and 2p_{1/2} in the Dirac theory), of the hydrogen atom. According to the Schrödinger (resp. Dirac) equation, this second level is degenerated and the two levels occur at the same energy. Yet, Lamb and Retherford (1947) found that there is indeed a small separation.

The infinities experienced in QED confused all earlier attempts to calculate this difference. Then; at the Shelter Conference; Kramers, Schwinger and Weisskopf, and Oppenheimer had suggested that the possible explanation might be the shift of the energy levels caused by the interaction of the electron with the radiation field. Though also this shift comes out infinite in all theories, the most strongly divergent term can be identified with an electromagnetic mass which must exist for a bound as well as for a free electron. Therefore, this mass is already included in the observed mass of the electron and one must subtract from the theoretical expression the corresponding expression for a free electron of the same average kinetic energy. In this computation the desired finite quantity is produced by the difference of two infinities.

For a clear exposition we need to explain this computation technique in a more mathematical way. In the simple situation when the infinity is due to the infinity of the trace of a Green kernel, \( K(H_0) \), derived from the Hamiltonian \( H_0 \) of a free particle, the divergent term is usually removed by perturbation. In this scheme the particle is imagined to be placed in a field described, for instance, by a potential \( V \). Therefore, the Hamiltonian of the bounded particle is \( H = H_0 + V \). Usually the trace of \( K(H) \) is still infinite, however, for appropriate \( V \)'s the difference \( \text{Tr}(K(H) - K(H_0)) \) is a finite quantity. By this technique one can produce, for instance, from a non-trace class heat kernel, \( e^{-tH_0} \), a relative heat kernel, \( e^{-tH} - e^{-tH_0} \), of the trace class. This latter kernel allows to introduce important object such as relative zeta- and eta-functions, which could have not been defined regarding the original kernel.
As it is well known, well defined zeta- and eta-functions describe the spectra of the Hamilton operators. What do then the relative zeta- and eta-functions describe? The answer to this question is this: By adding $V$ to the Hamiltonian, the free particle becomes a bounded one. This change causes shifts in the spectral lines. The relative functions describe, the shifts exhibited between the two spectra. For instance, the multiplicity of each spectrum-element of a free Zeeman electron is infinity, while this multiplicity drops to 2 for the bounded electron in a hydrogen atom. The relative objects describe these drastic changes (splitting) of the spectral lines.

The Lamb-Retherford experiment demonstrated that even the doublets do not exist in the spectrum of hydrogen atom. The earlier computations retained only the Coulomb potential which is the main term in the interaction. To explain the Lamb-shift, i. e. the non-degeneracy of the energy levels in the hydrogen atom, one had to consider additional interactions between the electron, the proton and the quantized electromagnetic field. The additional terms in the total Hamiltonian, which exhibit the energies due to these interactions, are called radiative correction.

Dropping of the divergent term from a mathematical expression is called renormalization, or, regularization. The idea of renormalization gained ground also in mathematics, used for spectral investigations on non-compact manifolds. This type of investigations started out with [OPS], which was apparently motivated by [A] and [Po], written in physics.

Regarding the Lamb shift the first non-relativistic computations using renormalization were completed by Bethe by the end of the Shelton Island Conference. Then this idea was further developed by Dyson, Feynman, Schwinger, Tomonaga, and others and the calculations came out in excellent agreement with the observed value measured in the Lamb-Retherford experiment. Yet, this scheme for getting rid of infinities is thought to be a theory which is imperfect on several counts. First, it is only perturbative, second, infinities occur even if they can be isolated and hidden $\nabla$.

In this article the problem of infinities is approached from a completely different angle. Here a natural spectral decomposition of the quantum Hilbert space will be considered and the desired finite quantities will appear on these sectors (zones). This idea has fundamental effect also on the mathematics describing the geometry of sectors. For instance, one can not think of points
on a sector. Indeed, they are substituted by a point-spread concept, defined by the wave functions \( \delta_X^{(s)}(Y) \) which are introduced by projecting the Dirac \( \delta_X(Y) \)-functionals onto the sector \( s \). The elimination of the point concept, which has been the fang of quantum theory since the very beginning, is a key idea in this article. It terminates the infinities on the sector-level. It terminates even Weisskopf’s above described argument, since his \( \delta_\xi \) becomes \( \delta_\xi^{(s)} \), which describes indeed a positive probability for the zonal particle being at two different points. In building up this new theory one should rethink many of the fundamental problems arising in the theory of finite many particles.

(B) Zeeman and Pauli operators.

(B1) Classical Hamiltonians. The Hamilton function, \( H_F \), and the corresponding quantum operator, \( H_Q \), of a charged particle in an electromagnetic field given by the vector potential \( a \) and scalar potential \( \phi \) are:

\[
H_F = \frac{1}{2\mu} |p - \frac{e}{c} a|^2 + V + e\phi, \tag{1}
\]

\[
H_Q = -\frac{\hbar^2}{2\mu} \Delta - \frac{\hbar e}{2\mu c^2} (a \cdot \nabla + \nabla \cdot a) + \frac{e^2}{2\mu c^2} a^2 + e\phi + V, \tag{2}
\]

where \( e \) is the charge, \( \mu \) is the mass, and \( V \) is the potential energy originated from other sources such as nucleus.

One of the most important version of this operator is the Zeeman operator representing a charged particle orbiting about the origin of the \((x, y)\)-plane in a constant magnetic field \( B = B\partial_z \) perpendicular to the plane. This field is described by the vector potential \( a_{(x, y)} = (B/2)(-y, x) \). There is also assumed that both \( V \) and \( \phi \) vanish, meaning that the particle is free and only the constant magnetic field is present.

The latter assumption can be interpreted such that the free particle is considered in that unique inertial system where \( E = 0 \) and \( B \neq 0 \) holds. Note that in the other inertial systems the constant electromagnetic field appears with a non-vanishing constant electric field \( E \neq 0 \). Because of this fixed inertial system, all the considerations developed below can be linked to the relativistic notion.

It should be emphasized again that the Zeeman zone decomposition is established for free particles. Bounded particles in potential fields \( V \) will be
considered later, after analyzing the transitions and shifts the potential fields produce among the zones and spectral lines. Also the Lamb-shift can be considered only after this analysis.

So-far the constant magnetic field is externally added to the particle. The hypothesis of orbiting spin in the classical theory means that the Hamilton operator of the charged particle is supposed to be the same as of the particle orbiting in a constant magnetic field. This hypothesis, so to speak, internally adds a little magnet to the particle. The orbiting spin is exhibited by the angular momentum operator discussed below.

An other important remark is that the Zeeman operator will be considered only on the \((x,y)\)-plane, thus it has the form

\[
H_Z = -\frac{\hbar^2}{2\mu} \Delta - \frac{\hbar eB}{2\mu c} D_z \vec{\cdot} + \frac{e^2 B^2}{8\mu c^2} (x^2 + y^2),
\]

where \(D_z \vec{\cdot} = x\partial_y - y\partial_x\). In order to point out the finer differences between the classical theory and the one presented here, we should make further remarks about these Hamiltonians.

**Operator** \(H_{ZF} \). The Zeeman Hamiltonian does not involve the Hamiltonian \(H_f\) of the constant magnetic field, discussed above. In classical field theory this term is introduced by renormalization. In the point-particle theory this field energy is meaningless anyway. In the point-spread theory, however, the field energy of the “little magnet inside of the point-spread” does have a real meaning, providing an additional term to the Zeeman Hamiltonian. This combined particle-field Hamiltonian will be denoted by \(H_{ZF} = H_Z + H_f\).

Operator \(H_Q\) leaves on the 3-dimensional space, while \(H_Z\) is defined over \(\mathbb{R}^2\). The 3- and 2-dimensional versions differ from each other just by the operator \(-\left(\hbar^2/2\mu\right)\partial^2_z\), thus the spectral computations with respect to these two cases can be easily compared. Also the 2-dimensional version (3) is well known. It has been intensely investigated, since its first appearance in [AC], in connection with the *Aharonov-Bohm (AB) effect* [AB]. This AB-phenomena got a lot of attention in the near past and will be considered also in this article. A brief account on this problem is as follows.

**The AB effect** produces relative phase shift between two electron beams enclosing a magnetic flux even if they do not touch the magnetic
field (cf. the thought experiment performed with the AB-solenoid in [AB]).

This predicted effect has no explanation in the classical mechanics and it
contradicts even relativity. Indeed, according to this theory, all fields must
interact only locally and since the electrons do not reach the regions where the
fields are, the effect can not be the production of the fields themselves. Yet,
this effect was clearly demonstrated by the Tonomura experiments [Ton1],
[Ton2].

In their paper Aharonov and Bohm explained the predicted effect by the
“significance of electromagnetic potentials in the quantum theory”. In the
classical electrodynamics these potentials, which are unique only upto gauge
transformations, are just convenient mathematical tools which do not have
any physical meaning. The only physical objects are the fields by which the
fundamental equations of motions can always be expressed. Nevertheless,
these potentials are needed to obtain a classical canonical formalism.

In quantum mechanics, however, the equations of motion of a particle are
replaced by the Schrödinger equation, which is obtained from a canonical
formalism. Thus this equation does involve the potentials, resulting that “in
quantum theory an electron (for example) can be influenced by the potentials
even though all the field regions are excluded from it”. Then it is pointed
out, in [AB], that this effect depend only on the gauge-invariant quantity
\( \oint \mathbf{a} \cdot d\mathbf{x} = \int \mathbf{H} \cdot ds \), so that in reality they can be expressed in terms of the
fields inside the circuit. Yet, because of the relativistic notions the effect
can not be interpreted as due to the fields themselves. There is no other
way out from this controversy but to retain a local theory by regarding the
potential \( \mathbf{a} \) as a physical variable. “This means that we must be able to
define the physical difference between two quantum states which differ only
by gauge transformation. It will be shown in a future paper that in a system
containing undefined number of charged particles (i. e., a superposition
of states of different total charge), a new Hermitian operator, essentially an
angle variable, can be introduced which may give a meaning to the gauge.”

In the announced second article (1964) the AB-effect is linked to super-
conductivity and super-fluidity, where the above angle variable was used to
explain these super-phenomenas. Yet, a convincing mathematical model ex-
plaining the AB effect is still missing from the literature sofar.

The zonal theory offers a quite different explanation to this problem. Indeed,
a zonal electron is considered as an electron-spread, reaching every region in finite distance. Actually, exactly the zones contribute the desired physical meaning to the potential. The strong attachment of the zones to the potential is revealed, for instance, by the fact that the zones can be defined by the quantization of the magnetic dipole moment. Also note that the zones are not invariant with respect to the gauge transformations. These facts clearly demonstrate that the potential reveals itself by the zones and the point-spreads defined by the zones.

Beyond this new explanation of the AB effect, the point-spread concept offers explanations also for number of other problems. Its most important impact is certainly the cancellation of infinities from the theory of finite many particles.

(B4) Quantum numbers. Term involving $D_z\cdot$ in operator $H_Q$ is called angular momentum operator. In what follows, it is denoted by $L$. It represents the orbiting spin of the particle. Note that the rest part of the Hamilton operator is a harmonic oscillator operator, $O$, which commutes with $L$.

By the standard method, the eigenfunctions of $O$ are sought in the form $f(r^2)\varphi_l$, where the $f$ is a radial function and $\varphi_l$ is a homogeneous harmonic polynomial of order $l$. For any fixed $\varphi_l$ the $O$ induces a differential operator acting on $f$. This operator depends just on $l$ and the eigenfunctions of this radial operator appear in the form $f_k(r^2) = u_k(r^2)e^{-\lambda r^2/2}$, where the $u_k$ is a $k^{th}$-order Laguerre polynomial and the parameter $\lambda$ is described below. Thus the multiplicity of the eigenvalue for fixed $k$ and $l$ is the dimension of the space of the $l^{th}$-order spherical harmonics $H^{(l)}$. The integers $k + l$ resp. $l$ are called principal resp. azimuthal quantum numbers.

Now let also the angular momentum operator be acting. Since it commutes with the Euclidean Laplacian, the space $H^{(l)}$ is invariant under its action. It is well known that the $iD_z\cdot$ has the eigenvalues $\{-l, -l + 1, \ldots, l - 1, l\}$, which are called magnetic quantum numbers, denoted by $m$. Thus adding $L$ to the $O$ causes shifts in the energies emerging on the same level with respect to the operator $O$. This shift is called Zeeman effect whose existence was demonstrated by the Stern-Gerlach experiment. This experiment approved the hypothesis of magnetic dipole moment (orbiting spin, little magnet) existing in charged particles such as electrons.

(B5) Pauli operator. Although the Schrödinger wave equation gives ex-
cellent agreement with experiment in predicting the spectral lines, yet small discrepancies have been found which can be removed by assuming, besides the orbiting spin, also the existence of an intrinsic angular momentum, $\mathbf{S}$, of magnitude $-\hbar/2\mu c$. It is well known the many efforts made to establish an adequate spin concept [10]. The ultimate solution of this problem is due to Pauli, in the non-relativistic case, and to Dirac in the relativistic case. In Pauli’s theory the intrinsic angular momentum operator acts on spinor fields having two components, while Dirac’s theory applies to 4-component spinor fields. In Pauli’s theory the constant magnetic field $B\partial_z$ makes the following contribution to the Hamilton operator:

$$H_S = -\frac{\hbar}{2\mu c} \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}. \quad (4)$$

The Hamilton operator $H_P = H_Z + H_S$, where the $H_Z$ acts on the 2-component spinor fields as a scalar operator, is called Pauli Hamiltonian. A particle corresponding to $H_Z$ resp. $H_P$ is called Zeeman- resp. Pauli-particle.

(C) Mathematical modeling; Zeeman manifolds

Zeeman operator identified with a Riemannian Laplacian. Most surprising feature of the Zeeman operator $H_Z$ is that it can be pinned down as the Laplacian on a Riemannian manifold. As far as the author knows, this interpretation has not been recognized in the literature so far. We use this Riemannian manifold as the fundamental mathematical model describing the space-time structure on the quantum level. Although the metric is positive definite, this model fulfills the relativistic criteria. By the first version of these manifolds single Zeeman-, or, Pauli-particles are modeled.

This fundamental Zeeman manifold is a Riemannian circle bundle over $\mathbb{R}^2$, defined by factorizing the center of the 3-dimensional Heisenberg group endowed with a left-invariant metric.

The Lie algebra $\mathfrak{n} = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ (where $\mathbb{R}$ is the center) of the 3D-Heisenberg group can be described in terms of the natural complex structure $J$, acting on $\mathbb{R}^2$, and the natural inner product $\langle \cdot, \cdot \rangle$, defined on $\mathfrak{n} = \mathbb{R}^3$, by the formula $\langle [X,Y], Z \rangle = \langle tJ(X), Y \rangle$, where the 3-vectors $X, Y$ and $Z$ are in $\mathbb{R}^2$ and $\mathbb{R}$ respectively, furthermore, $t$ is the coordinate of $Z$ in $\mathbb{R}$. The map $Z \to tJ = J_Z$ associates skew endomorphisms acting on $\mathbb{R}^2$ to the elements,
They satisfy the relation $J_Z^2 = -|Z|^2 \text{id}$. Thus the metric Lie algebra is completely determined by the system

$$\{n = v \oplus z, \langle \cdot, \cdot \rangle, J_Z\}, \quad (5)$$

where $v = \mathbb{R}^2$ and $z$ are called X- and Z-space respectively. With higher dimensional X- and Z-spaces this system defines the Heisenberg type Lie algebras introduced by Kaplan [Ka]. If the Clifford condition $J_Z^2 = -|Z|^2 \text{id}$ is dropped for the skew endomorphisms, the above system defines a most general 2-step nilpotent Lie algebra. The considerations will be extended to these general cases, however, the discussion proceeds with the fundamental 3-dimensional case.

Note that there are two options, $J$ or $J' = -J$, for choosing a complex structure on $\mathbb{R}^2$. The two Lie algebras, $n$ and $n'$ are isometrically isomorphic by the map $(X, Z) \rightarrow (X, Z' = -Z)$.

The Lie group defined by this Lie algebra is denoted by $N$, furthermore, the left-invariant extension of the inner product $\langle \cdot, \cdot \rangle$, defined on the tangent space $T_{(0,0)}(N) = n$ at the origin, is denoted by $g$. The exponential map is one-to-one whose inverse identifies the group $N$ with its Lie algebra $n$. Thus also the group lives on the same linear space $(X, Z)$. Then the group multiplication is given by:

$$(X, Z)(X^*, Z^*) = (X + X^*, Z + Z^* + \frac{1}{2}[X, X^*]). \quad (6)$$

On the linear coordinate systems $\{x^1, x^2, t\}$, defined by the natural basis $\{E_i, E_t, e_t\}$, the left-invariant extensions of the vectors $E_i, e_t$ are of the form

$$X_i = \partial_i + \frac{1}{2}\langle [X, E_i], e_t \rangle \partial_t = \partial_i + \frac{1}{2}\langle J(X), E_i \rangle \partial_t ; \quad Z = \partial_t, \quad (7)$$

where $\partial_i = \partial/\partial x^i, \partial_t = \partial/\partial t$. Then for the Laplacian, $\Delta$, acting on functions we have:

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\partial^2_{tt} + \partial_t D\bullet, \quad (8)$$

where $\Delta_X$ is the Euclidean Laplacian on the X-space and $D\bullet$ means differentiation (directional derivative) with respect to the vector field

$$D : X \rightarrow J(X) \quad (9)$$
tangent to the X-space.

The above Laplacian is not the desired Zeeman operator yet. This surprising interpretation can be established on center-periodic Heisenberg groups defined by an L-periodic lattice \( \Gamma_L = \{ \gamma L | \gamma \in \mathbb{Z} \} \) on the center. Since the \( \Gamma \) is a discrete subgroup of isometries, one can consider the factor manifold \( \Gamma \backslash N \) with the factor metric. The factor manifold is a principal circle bundle over the base space \( v \) such that the circles \( C_X = \pi^{-1}(X) \) over the points \( X \in v \) are of constant length \( L \). Then the projection \( \pi : \Gamma \backslash N \rightarrow v \) projects the inner product from the horizontal subspace (defined by the orthogonal complement to the circles) to the Euclidean inner product \( \langle \cdot, \cdot \rangle \) on the X-space.

By using the Fourier-Weierstrass decomposition

\[
L^2(\Gamma \backslash N) = \bigoplus_{\gamma} \text{FW}^{(\gamma)}(\gamma), \tag{10}
\]

where \( \text{FW}^{(\gamma)} \) consists of functions of the form

\[
\phi^{(\gamma)}(X, Z) = \varphi(X)e^{i2\pi t/L}, \tag{11}
\]

the Laplacian can be established in the following particular form.

By (8), the function spaces \( \text{FW}^{\gamma} \) are invariant under the action of the Laplacian. More precisely we have:

\[
\Delta \phi^{(\gamma)} = (\square^{(\lambda)} \varphi)e^{i2\pi t/L}, \quad \text{where} \quad \square^{(\lambda)} = \Delta_X + 2iD_{\lambda} \bullet -4\lambda^2 \left( 1 + \frac{1}{4}|X|^2 \right), \quad \lambda = \frac{\pi \gamma}{L}. \tag{12}
\]

and \( D_{\lambda} \bullet = \lambda D \bullet \) means directional derivative along the the vector field \( X \rightarrow \lambda J(X) = J_\lambda(X) \). If \( \lambda < 0 \), the \( J \) and \( \lambda \) are exchanged for \(-J\) and \(-\lambda\) respectively. Thus one can assume that \( \lambda > 0 \).

Apart from the constant term \(-4\lambda^2\), operator \( \square^{(\lambda)} \) is nothing but the Zeeman operator \( H_{Zf} \) described in (3). The surplus constant term will be identified by the field-energy of the constant magnetic field in the charge spread, thus the above operator is, actually, \( H_{Zf} \) described earlier. The precise description of identification of the Zeeman operator (3) with the Laplacian \( \square_{\lambda} \) acting on the invariant subspace \( \text{FW}^{\gamma} \) is as follows. The macroscopic Zeeman operator
is defined by $\hbar = \mu = 1$. Then $H_{Zf} = -(1/2)\Box(\lambda)$, where $\lambda = -eB/2c$. Note that particles with negative charge correspond to the cases $\gamma > 0$, i.e., they are attached to $J$, while particles with positive charges are attached to $-J$.

On the quantum (microscopic) level, the periodicity $L$ and the parameter $\lambda$ are exchanged for $L_\hbar = \hbar L$ and $\lambda_\hbar = \lambda/\hbar$ respectively. This process means nothing but scaling of the periodicity by $\hbar$. Then we have $H_{Zf} = -(h^2/2\mu)\Box(\lambda_\hbar)$. By scaling also the Euclidean metric on the X-space by $\hbar/\sqrt{\mu}$, one has $H_{Zf} = -(1/2)\Box(\lambda_\hbar)$. In the following we proceed with the macroscopic operator, however, the previous formulas allow an easy transfer from the macroscopic level to the microscopic one.

By a straightforward generalization, described later, these operators can be introduced on higher dimensional Heisenberg groups defined by a complex structure, $J$, on an even dimensional Euclidean space $\mathbb{R}^k$. Let $(z_1, \ldots, z_{k/2})$, where $z_i = q_i + ip_i$ and $\partial p_i = J(\partial q_i)$, be a complex coordinate system regarding $J$. Then this system identifies $\mathbb{R}^k$ with $\mathbb{C}^{k/2}$. The circle bundle, defined by factorization of the center $\mathbb{R}$, determines quantum operators depending just on one parameter $\lambda$. In the most general situation the center of a two-step nilpotent group is factorized by a lattice, resulting a torus bundle over the X-space. Then one arrives to operators depending on different parameters $\lambda_i > 0$ which are defined for each complex coordinate plane $z_i$. As it will be explained later, the above operators correspond to the Hamiltonians of a system where $k/2$ number of charged particles are circulating in a constant magnetic field. When there is only one $\lambda$ involved, the particles are considered to be identical upto the sign of the charge.

(D) Introducing the zones

(D1) Quantum Hilbert space $\mathcal{H}$. The quantum operator is acting on smooth functions primarily. This action extends to the $L^2$-Hilbert space $\mathcal{H} = L^2_\mathbb{C}(\mathbb{R}^k)$ of complex valued functions endowed with the inner product $\langle f, g \rangle = \int f\overline{g}dX$ by the Friedrics extension. This Hilbert space can be identified with the weighted Hilbert space $L^2_{C,\eta_{\lambda_i}}(\mathbb{C}^{k/2})$, endowed with the Gaussian density $\eta_{\lambda_i}$, in the following way.

First the density is defined. For $\lambda_i = 1$, $\forall i$, it has the simple form $\eta = e^{-\sum z_i\overline{z}_i}$, while in general it is $\eta_{\lambda_i} = e^{-\sum \lambda_i z_i\overline{z}_i}$. The Hilbert space $L^2_{C,\eta_{\lambda_i}}(\mathbb{C}^{k/2})$ is spanned by the polynomials written in terms of holomorphic and antiholomorphic
coordinates. The above mentioned identification of this Hilbert space with the standard Hilbert space $L^2_{C}$ endowed with the standard Euclidean density $\eta = 1$ is established by the map

$$L^2_{C\eta,\lambda_i} \rightarrow L^2_{\mathbb{C}}, \quad \psi \rightarrow \psi e^{-\frac{1}{2} \sum \lambda_i z_i \overline{z}_i}. \quad (14)$$

(D2) **Zones defined by Heisenberg group representations.** The Zeeman zone decomposition of the Hilbert space $\mathcal{H} = L^2_{C\eta,\lambda_i}$ can be introduced in two entirely different ways. In the first way these subspaces are defined by the semi-irreducible invariant subspaces of the natural reducible complex Heisenberg algebra representation. The other definition explores that these subspaces are invariant under the action of the quantum operator. The zones are introduced by a spectral decomposition such that the spectrum of the Hamilton operator is explicitly computed and, then, the eigen-functions are sorted into different classes corresponding to the distinct magnetic states defined by the quantization of the magnetic dipole moment.

The Lie bracket on the complex Heisenberg algebra is the restriction of the Poisson bracket to the set $\{z_i, \overline{z}_i, c\}$ of holomorphic resp. antiholomorphic coordinates and constant $c$. The real Heisenberg algebra is hidden in this complex algebra which can be recovered by certain formulas established in [Sz4]. The representation of this complex Heisenberg algebra is introduced by

$$\rho_C(z_i)(\psi) = (-\partial_{\overline{z}_i} + \lambda_i z_i)\psi, \quad \rho_C(\overline{z}_i)(\psi) = \partial_{z_i}\psi. \quad (15)$$

This representation satisfies the well known Heisenberg relations, however, it is not unitary on the whole complex algebra. It becomes unitary by restricting it to the real sub-algebra.

The most remarkable feature of this complex representation is that the $\rho_C$ is a reducible representation on the whole Hilbert space $\mathcal{H} = L^2_{C\eta,\lambda_i}$. Indeed, the holomorphic subspace $\mathcal{H}^{(0)}$ spanned by the holomorphic polynomials $z_1^{a_1} \ldots z_k^{a_k}$ is obviously an irreducible invariant subspace. In the literature usually this irreducible representation is called complex (Fock) representation and no thorough investigation of the whole reducible representation has been implemented yet.

By the first definition, the zones are introduced by the semi-irreducible invariant subspaces of this reducible representation. This decomposition can
be established by the standard Gram-Schmidt orthogonalization such that the Gram-Schmidt process is applied to the series $G^{(a)}$, $a = 0, 1, \ldots$, of subspaces, where $G^{(a)}$ is spanned by the subspaces $\mathcal{Z}_1^{a_1} \ldots \mathcal{Z}_k^{a_k} \mathcal{H}^{(0)}$ satisfying $a_1 + \cdots + a_k = a$. Clearly $\mathcal{H}^{(0)} = G^{(0)}$ holds. The next subspace, $\mathcal{H}^{(1)}$, is defined as the orthogonal complement of $\mathcal{H}^{(0)}$ in $G^{(0)} \oplus G^{(1)}$, e. t. c., the higher order subspaces are defined inductively.

Representation $\rho$ is irreducible on $\mathcal{H}^{(0)}$ and it is irreducible on the higher order zones if and only if $k = \dim(\mathcal{V}) = 2$. In the higher dimensional cases the zones of higher order are reducible. For instance, the zone-functions defined by a fixed $\mathcal{Z}_i$ form an irreducible invariant subspace in the higher order zones. This is an explanation for the meaning of the name semi-irreducible.

It should be noted that this reducibility in the higher dimensions has a much deeper meaning. In fact, the complex dimension $k/2$ is interpreted as number of particles and by considering subspaces defined by eigenfunctions satisfying certain symmetry properties with respect to coordinate exchanges, one can define Bosonic or Fermionic particles and one can introduce important concepts such as exclusion principles. More about this important topic can be found later.

(D3) Zones established by spectral decomposition (polarization).
The other technique by which the Zeeman zones are established is based on explicit computation of the spectrum and eigen functions. Next only systems involving one parameter $\lambda$ are considered. If $\lambda = 0$, the spectrum is the continuous spectrum of the Euclidean Laplacian $\Delta_X$, thus we suppose $\lambda \neq 0$. In this case the spectrum is discrete whose computation is carried out by the following ideas.

First note that the Hamilton operator $H_Z$ is the sum of the harmonic oscillator operator $\mathbf{O} = \Delta_X - \pi^2 \lambda^2 |X|^2$ and the quantum angular momentum operator $\mathbf{L}_\lambda = 2\pi i D_\lambda \cdot$. These operators commute with each other. There is a well known technique, using Hermite polynomials, by which the eigenfunctions and eigen-values of operator $\mathbf{O}$ are explicitly determined. Then one generates a function space by letting the operators $\mathbf{L}_\lambda^n, n = 1, 2, \ldots$, successively act on an eigenfunction determined in the first step. It still consists of eigenfunctions of $\mathbf{O}$ but, additionally, it is invariant also under the action of $\mathbf{L}_\lambda$. The latter operator has the distinct eigenvalues $(2p - l)\lambda p, p = 0, \ldots, l$. In the last step the generated function space is decomposed according to these
eigenvalues, in order to have the eigenfunctions of the complete operator. These eigenfunctions appear in the form $h^{(p,l-p)}(X) e^{-\frac{1}{2}|X|^2}$, where the $l^\text{th}$-order polynomial $h$ is determined by its leading term $z_{i_1} \cdots z_{i_p} \bar{z}_{j_1} \cdots \bar{z}_{j_{l-p}}$. Indexes $p$ resp. $v = l - p$ are nothing but the number (degree) of holomorphic resp. antiholomorphic coordinates involved into the leading term of the sought polynomial. The eigensubspace spanned by these functions is denoted by $H^{(p,v)}$, where $p$ and $v$ are the so called holomorphic and antiholomorphic indexes.

In order to describe these eigenfunctions on the complex plane write the complex numbers $z$ in the polar form $z = re^{i\alpha}$. Then the above constructed $h^{(p,v)}$ has a uniquely determined term $e^{i(p-v)\alpha}$ which is multiplied by a radial function $f(r)$. I. e., $h^{(p,v)}(r,\alpha) = f(r)e^{i(p-v)\alpha}$. The functions of this form are called polarized functions. The method of polarization straightforwardly extends to the higher dimensional cases. Note that polarized eigenfunctions appear in the form $F(r)H^{(\tilde{p},\tilde{l}-\tilde{p})}$, where $H^{(\tilde{p},\tilde{l}-\tilde{p})}$ is an $\tilde{l}^\text{th}$-order homogeneous harmonic polynomial (spherical harmonics). Thus it is in the eigen-subspace of magnetic dipole moment operator $D \cdot$ belonging to the eigenvalue (magnetic quantum number) $m = \tilde{p} - (\tilde{l} - \tilde{p}) = 2\tilde{p} - \tilde{l}$. In this case $\tilde{p}$ has the same meaning as $p$ has in the first case, but it is counted only with respect to the polynomial $H$. Also $\tilde{l}$ concerns strictly this polynomial. These are very important differences between the quantum numbers $p,l,m$ and $\tilde{p},\tilde{l},m$. In standard spectrum-computation of electrons the eigenfunctions are sought exactly in the second form. The precise calculations reveal that function $F(r)$ appears in the form

$$F(r) = u_{(\lambda,n,\tilde{l},m)}(\lambda r^2)e^{-\frac{1}{2}\lambda r^2},$$

(16)

where $u_{(\lambda,n,\tilde{l},m)}(t)$ is an appropriate Laguerre polynomial of the $n^{\text{th}}$-other. The eigenvalue belonging to such an eigenfunction is

$$\mu_{(\lambda,n,\tilde{l},m)} = -((4n + 4\tilde{p} + k)\lambda + 4k\lambda^2).$$

(17)

Thus the relations between the two set of quantum numbers are: $l = 2n + \tilde{l}$, $p = \tilde{p} + n$, and $m = p - (l - p) = \tilde{p} - (\tilde{l} - \tilde{p})$.

Let it be pointed out again that the quantum numbers are defined in the classical electron theory by the indexes $n, \tilde{p}$ and $\tilde{l}$, derived from the spherical
harmonics technique. If the electron is in the quantum state described by
the \( \tilde{l} \)-th order homogeneous harmonic polynomials, then the eigenvalues of the
dipole moment operator are \( m = 2\tilde{p} - \tilde{l} = p - \tilde{\nu} = l - 2\tilde{\nu} \), where \( p = 0, \ldots, l \).
Number \( m, \tilde{l} \) and \( n + \tilde{l} \) are called magnetic, azimuthal, and principal quantum numbers respectively. By the above relations these classical quantum
numbers can be expressed by the ones derived from the first representation of
eigen-functions and vice versa.

The explicit form

\[
(2p + (k/2))\lambda + 2k\lambda^2
\]  

(18)

of the spectrum of \( H_{ZI} = -(1/2)\Box_{\lambda} \) shows that the elements depend only on \( p \) and they are independent of \( \nu \). If \( -J_{\lambda_1,\ldots,\lambda_s}^2 \) has \( s \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_s \), then the actual Laplace-eigenvalues depend on the corresponding \( p_i \)'s and they are independent of the \( \nu_i \)'s. It follows that each spectrum element has infinite multiplicity. The spectral Zeeman zone decomposition is
defined such that a zone consists of functions having the same antiholomorphic indexes \( (\nu_1, \ldots, \nu_s) \) (which are called also Zeeman zone indexes (ZZI)).

On these subspaces (zones) the spectrum is not just discrete but each eigenvalue has finite multiplicity. One can prove that these spectrally defined zones are equal to those defined earlier, i. e., they are the semi-irreducible invariant subspaces with respect to the natural complex Heisenberg algebra representations \( \text{Sz5} \).

If \( k = 2 \), the multiplicity of eigenvalues is 1 on each zone and any 2 zones are
isospectral. If \( k > 2 \), the elements of the spectra on two distinct zones are the same, however, the multiplicities are different. One can see this by observing
that the multiplicity of an eigenvalue depends both on the holomorphic and
antiholomorphic indexes (cf. Splitting Theorem 2.3 in \( \text{Sz5} \)).

The higher multiplicities of eigenvalues on the zones of higher dimensional
complex spaces \( \mathbb{C}^{k/2} \) allow to introduce important concepts of multi-particles theory. In case of \( k = 2 \) one can choose \( J \) or \( -J \) for the constructions. If one of them is attached to a particle with positive charge, the other is attached to a particle with the same amount of negative charge.

In case of two particles, i. e. \( k = 4 \) and \( \{z_1, z_2\} \) is a complex coordinate system on the X-space, there are two types of eigenfunctions in a zone. One of them is symmetric while the other is skew symmetric with respect to the
coordinate-exchange $z_1 \leftrightarrow z_2$. Pauli used such skew eigenfunctions to establish his exclusion principle. According to this idea, particles described by these functions can not simultaneously be in the same quantum state (dislike each other). They are called Fermions in the literature, which are characterized by the property that the probability for being in the same quantum state is zero. The explanation for this phenomena is that the particles have charges of the same sign, therefore, a repulsive force is acting between them. In the symmetric case the particles have opposite charges and attract each other. This explains that such particles can simultaneously be in the same quantum states. This property is labeled by the name *Bosonic* in the literature. Both the Fermionic and Bosonic sub-zones are invariant under the action of the Hamilton operator and the multiplicity of each eigenvalue is one on them. Thus each zone bears the possibility to assign charges with the same or opposite sign to the particles. Thus a zone can be endowed with the Pauli or anti-Pauli (Bosonic) exclusion principle, by restricting it onto the corresponding sub-zone.

The options for choosing signs for the charges get more and more variegated by increasing the number, $k/2$, of the particles. In these general cases the invariant sub-zones correspond to dividing the complex coordinates into two classes, say $\{z_1, \ldots, z_r\}$ and $\{z_{r+1}, \ldots, z_{k/2}\}$. The idea is that both groups correspond to particles having same-sign-charges, however, the sign of the charge with respect to the first group is opposite to the sign in the second group. I. e., the skew-symmetric (Fermionic) property is valid within both groups and they are symmetric regarding exterior coordinate-exchanges between the two classes. Note that these sub-zones are not supposed to be invariant with respect to the action of the Heisenberg group representation. This is clearly exhibited on the irreducible holomorphic zone and is present also in the classical theory where the Heisenberg group is acting always irreducibly.

By summing up, the usage of reducible Heisenberg group representation is one of the most important distinguishing feature of the theory presented here. Actually, this reducibility is unavoidable. In fact, for a fixed azimuthal quantum number, the distinct magnetic quantum numbers are assigned to distinct Zeeman zones. I. e., the zones correspond to the quantization of the magnetic dipole moment and particles in different zones are considered to be in distinct independent magnetic quantum state. This observation
clearly shows that there is no way to avoid the reducible Heisenberg algebra representation, in other words, this physical system can not be completely described in the framework of an irreducible Heisenberg group representation. By neglecting Zeeman zones one neglects existing magnetic quantum states. Due to the Stone-Neumann theorem, asserting that the infinite dimensional unitary representations of the Heisenberg group are uniquely determined up to multiplications with complex unities, quantum theory has always been preferring irreducible representations.

(E) Fundamentals of point-spread geometry.

(E1) Zonal point spreads. The point-spreads on a zone $\mathcal{H}^{(a)}$ are defined by the operator $P^{(a)}$ projecting the total Hilbert space $\mathcal{H}$ onto the zone. It turns out that these operators are integral operators with a smooth Hermitian integral kernel $\delta^{(a)}_Z(W)$. It can be defined by restricting the Dirac $\delta$-kernel, defined by an orthonormal basis $\{\varphi_i\}$ of $\mathcal{H}$ by $\delta_Z(W) = \sum_i \varphi_i(Z)\varphi_i(W)$, onto the zones. This restriction can be established by considering an orthonormal basis $\{\varphi^{(a)}_i\}_{i=1}^\infty$ on $\mathcal{H}^{(a)}$. Then the sought projection-kernel is $\delta^{(a)}_Z(W) = \sum \varphi^{(a)}_i(Z)\varphi^{(a)}_i(W)$. The sum $\sum_a \delta^{(a)}_Z(W)$ all of these zonal Dirac $\delta$’s is the total $\delta_Z(W)$. However, this is just a formal definition of the sought zonal kernels. Fortunately enough, all of them can be explicitly computed.

First note that the kernel regarding the holomorphic zone is the well known Bergman kernel. In case of a single parameter $\lambda$, it has the following well known form:

$$\delta^{(0)}_{\lambda Z}(W) = (\lambda^{k/2}/\pi^{k/2})e^{\lambda(Z\overline{W}-(1/2)(|Z|^2+|W|^2))}.$$  \hfill (19)

One of the new features of this article is that these projection kernels are explicitly computed not just for the holomorphic zone but for all zones. According to these computations, projection $P^{(a)}_{\lambda}$ is of the form

$$P^{(a)}_{\lambda}(f)_Z = \int \frac{\lambda^{k/2}}{\pi^{k/2}}L^{(k/2)-1}_a(\lambda|Z-W|^2)e^{\lambda Z\overline{W}}f(W)e^{-\lambda|W|^2}dW,$$  \hfill (20)

where $L^{(k/2)-1}_a(t)$ is the corresponding Laguerre polynomial of $a^{th}$-order. (Formula $Z \cdot \overline{W} = \langle Z,W \rangle + i\langle Z,J(W) \rangle$ is often used in establishing the formulas mentioned below.)
According to (20), a general projection kernel differs from the Bergman kernel just by a multiplicative Laguerre polynomial. More precisely we have:

$$
\delta^{(a)}(W) = \frac{\lambda^{k/2}}{\pi^{k/2}} L_a^{((k/2)-1)}(\lambda|Z - W|^2) e^{\lambda(Z \cdot W - \frac{\lambda}{2}|Z|^2 + |W|^2)}.
$$

(21)

These zonal kernels can be interpreted such that a point particle concentrated at a point $Z$ appears on the zone as an object which spreads around $Z$ as a wave-package with wave-function described by the above kernel explicitly.

The wave-package interpretation of physical objects started out with the de Broglie theory. This concept was finalized in the Schrödinger equation. The mathematical formalism did not follow this development, however, and the Schrödinger theory is built up on such mathematical background which does not exclude the existence of the controversial point objects. On the contrary, Weisskopf’s above argument points out that an electron must be considered as a point-object in the Schrödinger theory as well. An other demonstration for the presence of point particles in classical theory is the duality principle, stating that objects manifest themselves sometime as waves and sometime as point particles. The bridge between the two visualizations is built up in Born’s probabilistic theory, where the probability for a particle, attached to a wave $\xi$, can be found on a domain $D$ is measured by $\int_D \xi \overline{\xi}$.

Although the points are ostracized from the zonal theory, the point-spreads still bear some reminiscence of the point-particles. For instance, they are the most compressed wave-packages and all the other wave-functions in the zone can be expressed as a unique superposition of the point-spreads. If $\xi$ is a zone-function, the above integral measures the probability that the center of a pointspread is on the domain $D$. This interpretation restores, in some extend, the duality principle in the zonal theory.

Function $\delta^{(a)}(Z)\overline{\delta^{(a)}(Z)}$ is called the density of the spread around $Z$. By this reason, function $\delta^{(a)}(Z)$ is called spread-amplitude. Both the spread-amplitude and spread-density generate well defined measures on the path-space consisting of continuous curves connecting two arbitrary points. Both measures can be constructed by the method applied in constructing the Wiener measure.

The point-spread concept bears some remote reminiscence of Heisenberg’s suggestion (1938) for the existence of a fundamental length $L$, analogously to $\hbar$, such that field theory was valid only for distances larger than $L$ and...
so divergent integrals would be cut off at that distance. This idea has never became an effective theory, however. Other distant relatives of the point-spread concept are the smeared operators, i.e. those suitably averaged over small regions of space-time, considered by Bohr and Rosenfeld in quantum field theory.

**E2) Global Wiener-Kac and Dirac-Feynman kernels.** The kernels in the title are introduced in the following unified form:

\[ e^{-\sigma t H_0} = d_{\sigma}(t, X, Y) = \sum e^{-t\sigma \mu_i \psi_i(X) \overline{\psi_i(Y)}}, \]  

(22)

where \( \{\mu_i > 0\} \) is the discrete spectrum of the Zeeman operator \( H_Z \) and functions in the orthonormal basis \( \{\psi_i\} \) are eigenfunctions of \( H_Z \). Furthermore, the parameters \( \sigma = 1 \) and \( \sigma = i \) correspond to the Wiener-Kac resp. Dirac-Feynman kernels. Both kernels satisfy the condition

\[ \lim_{t \to +0} d_{\sigma}(t, X, Y) = \delta_{X}(Y), \]  

(23)

where \( \delta \) is the complex Dirac-delta kernel. The kernels attached to operator \( H_{ZF} = -(1/2) \Box_{(\lambda)} \) are multiples of (22) by \( e^{2\sigma \lambda^2 t} \).

Despite the infinite multiplicities, these infinite function series converge to

\[ d_1(t, X, Y) = \left( \frac{\lambda}{2\pi \sinh(\lambda t)} \right)^{k/2} e^{-\frac{\lambda}{2} \coth(\lambda t)|X-Y|^2 + i\langle X, J(Y) \rangle}, \]  

\[ d_1(t, X, Y) = \left( \frac{\lambda}{2\pi \sin(\lambda t)} \right)^{k/2} e^{i\{\frac{\lambda}{2} \cot(\lambda t)|X-Y|^2 - \langle X, J(Y) \rangle\}}, \]  

(24)

(25)

There are numerous differences between these two kernels. They provide the fundamental solutions for the heat- and the Schrödinger equation respectively. The first one defines the well established *Wiener-Kac measure* on the continuous path-spaces while this construction technique can not be carried over to the DF-kernel in order to construct the *Feynman measure*. In fact, one arrives (by this technique) to the well known divergent integrals, meaning that the approximating measures in the construction do not extend to a continuous complex measure on the path-space. It is known from the history of QED that Kac, who tried to understand Feynman, was able to introduce a well defined measure on the path-spaces only by the kernel \( d_1 \), which in the Euclidean case (\( \lambda = 0 \)) is nothing but the Wiener measure by which the
Brownian motion is introduced. Later, in the Feynman-Kac formulas, the Radon-Nikodym derivative of the more general Wiener-Kac measure with respect to the Euclidean Wiener measure was determined. This well-defined measure is a very important tool in quantum theory even today. Despite the controversies, also the Feynman measure is still a very powerful intuitive tool in QED. It should be also mentioned that none of these kernels is of the trace class, therefore, objects such as zeta functions, eta functions, e. t. c. can be defined only by regularizations.

(E3) Zonal Wiener-Kac and Dirac-Feynman kernels. Since the global flows leave the zones invariant, the zonal WK- and DF-kernels can be defined by restricting the global kernels onto the zones. The particular beauty of this theory is that all these objects can explicitly be computed. Unlike the global ones, both zonal kernels are of the trace class, having well defined partition functions. In case of a single parameter $\lambda$, the zonal WK-kernels and the corresponding partition functions are of the form

$$d_1^{(a)}(t, X, Z) = \left(\frac{\lambda e^{-\lambda t}}{\pi}\right)^{\frac{k}{2}} \cdot L_a^{(\frac{k}{2}-1)}(\lambda|X - Z|^2)e^{\lambda(-\frac{1}{2}|X|^2+\frac{1}{2}|Z|^2)+e^{-2\lambda t}(X, Z+1J(Z))}, \quad (26)$$

while the corresponding zonal DF-kernels and partition functions are:

$$d_1^{(a)}(t, X, Z) = \left(\frac{\lambda e^{-\lambda t}}{\pi}\right)^{\frac{k}{2}} \cdot L_a^{(\frac{k}{2}-1)}(\lambda|X - Z|^2)e^{\lambda(-\frac{1}{2}|X|^2+\frac{1}{2}|Z|^2)+e^{-2\lambda t}(X, Z+1J(Z))} = e^{-\frac{k\lambda t}{2}}e^{\lambda(\lambda(-\lambda t)-1)}(X, Z+1J(Z)) \delta_\lambda^{(a)}(X, Z), \quad (27)$$

$$Z_1^{(a)}(t) = Tr d_1^{(a)}(t) = \left(a + (k/2) - 1\right) a e^{-\lambda t} \frac{e^{-\lambda t}}{(1 - e^{-2\lambda t})^{\frac{k}{2}}}, \quad (28)$$

By these formulas the zonal zeta and eta functions can be introduced with no regularization. These formulas are beyond the scope of this review.

(F) Linking to the blackbody radiation; Solid zonal particles

(F1) Blackbody radiation and specific heat of solids. The above partition functions allow to link the evolution of the point-spreads driven
by the zonal WK- resp. DF-flow to the blackbody radiation in equilibrium. Quantum theory grew out from this historic problem, concerning the amount of energy $U(\nu)d\nu$ radiated by the blackbody in the frequency range between $\nu$ and $\nu + d\nu$. In equilibrium the rate at which the walls emit this frequency is balanced by the rate they absorb this frequency. Experiments show that the $U(\nu)$ depends only on the temperature $T$, and not on the material of which the walls are made.

The old theory predicted that this radiation yields the Rayleigh-Jeans law: $U(\nu)d\nu \sim \kappa T \nu^2 d\nu$ which contradicted the empirical curve. The empirical curve is fairly good described by the Wien law: $U(\nu)d\nu \sim \nu^3 e^{-h\nu/\kappa T}$, where $\kappa$ is Boltzmann’s constant.

The controversy arising between theory and experiment was resolved by Planck by the hypothesis that the energy attached to frequency $\nu$ is restricted to the integral multiple of the basic unit $\hbar$, i.e. $E_n = n\hbar\nu$, where $n$ is any positive integer number. Furthermore, the probability that the wall emits-absorbs an energy-quanta $E_n$ is $W(n) \sim e^{-E_n/\kappa T} = e^{-n\hbar\nu/\kappa T}$. Thus by normalization we have: $W(n) = e^{-n\hbar\nu/\kappa T}(1 - e^{-n\hbar\nu/\kappa T})$. Then a simple calculation yield that the mean energy is $\tilde{E} = \sum E_n W(n) = \hbar \nu e^{-\hbar\nu/\kappa T}/(1 - e^{-\hbar\nu/\kappa T})$, from which the Planck distribution $U(\nu) \sim \hbar \nu^3 e^{-\hbar\nu/\kappa T}/(1 - e^{-\hbar\nu/\kappa T})$ follows. The latter distribution is the exact form of the Wien law. Let it also be mentioned that the Rayleigh-Jeans law is derived by assuming the equipartition of the energy.

Einstein proposed to adopt Planck’s idea to solids in order to explain the experimental curve describing the specific heat, defined by the change in energy with temperature, of materials. This curve approaches zero at absolute zero, and rises asymptotically to $\kappa$ per atom at high temperatures. In contrast to the blackbody radiation, where all possible frequencies can occur, the materials have only one frequency, which is the characteristic frequency of the substance. Then $\partial_T \tilde{E} = (h\nu)^2 e^{-\hbar\nu/\kappa T}/\kappa T^2 (1 - e^{-\hbar\nu/\kappa T})^2$, which formula clearly fits the curve of specific heat.

(F2) WK-flow linked to blackbody radiation of solids. In case of the WK-kernel the microscopic heat flow formula can be linked to the blackbody radiation of solids by the substitutions $\nu = 1, T = 1/t, \kappa = 2\pi \mu / \lambda$, where $\lambda = |e|B/2c$. The possible integral values for the quantized energy are $n = 2p + (k/2)$, where the $p$ can take any integral value between 0 and $\infty$.
For \( k = 2 \) the multiplicity of eigenvalues on a zone is 1 and the probability \( W_1(n) \) is defined by \( W_1(n) \sim e^{-E_n/\kappa T} = e^{-\hbar \lambda/2\pi \mu} \), thus one gets the exact probability by dividing this by \( Z_1^{(a)}(\hbar t/2\pi \mu) \). These quantities are independent from the zone-index \( a \). If \( k > 2 \), then multiplicities occurs and these formulas should be defined accordingly. These quantities depend on the zone index via the combinatorial factor in (27). Since these general formulas can be easily recovered by multiplying with the combinatorial factor and the appropriate exponentiation with \( k/2 \), we proceed with the case \( k = 2 \). A simple calculation yields

\[
\sum_n E_n e^{-E_n \lambda/2\pi \mu} = -\frac{2\pi \mu}{\lambda} \partial_t Z_1^{(a)}(\hbar t/2\pi \mu) =
\]

thus the average energy is

\[
\tilde{E}_1(T) = h + 2h \frac{e^{-2h \lambda/2\pi \mu}}{1 - e^{-2h \lambda/2\pi \mu}},
\]

(30)

Note that the second term is the average energy in the blackbody radiation of solids with the characteristic frequency \( \nu = 2 \). Thus Einstein’s specific heat formula is

\[
\partial_T \tilde{E}_1(T) = (2h)^2 e^{-2h/\kappa T}/\kappa T^2 (1 - e^{-2h/\kappa T})^2.
\]

(31)

Also note that at \( t = 0 \), which corresponds to \( T = \infty \), the WK-flow starts out with the equipartition of energy, i.e., with the Rayleigh-Jeans law what is turned to the Wien law for all \( t > 0 \), or, \( T < \infty \). At the start all the quantities \( Z_1, E_1, \partial_T E_1 \) are infinities.

(F3) Stable (solid) charge spreads. The same computations lead to completely different interpretations in case of the DF-kernel. The analogously defined complex measure \( W_1 \) is interpreted as probability amplitude for the energy emission-absorption in the blackbody radiation in equilibrium. Then the average energy and specific heat amplitudes are

\[
\tilde{E}_1(T) = h + 2h \frac{e^{-2h \lambda/\kappa T}}{1 - e^{-2h \lambda/\kappa T}} , \quad \partial_T \tilde{E}_1(T) = (2h)^2 \frac{e^{-2h \lambda/\kappa T}}{\kappa T^2 (1 - e^{-2h \lambda/\kappa T})^2}.
\]

(33)
In the following these formulas are analyzed in terms of \( t \). The DF-specific heat does not bear the same physical meaning attributed to the WK-specific heat. In fact, the limit of \( |\partial_t \tilde{E}_i| \) is \( \kappa \), both at 0 and \( \infty \). On the other hand, an interesting interpretation can be given for the other functions.

Both \( Z_i(t) \) and \( \tilde{E}_i(t) \) are \( L \)-periodic functions, where \( L = \kappa/\hbar \). At the lattice points \( nL/2 \), where \( n \) can be arbitrary integer, they have limit at the infinity, furthermore, the density functions \( |Z_i|^2 \) and \( |\tilde{E}_i|^2 \) attain their minimum exactly at points \( (n + (1/2))L/2 \) of the periodicity intervals. Thus starting from the left endpoint \( P_n \) on an interval the infinite average energy drops to its minimum at the quarter point \( P_{n+1/4} \), then it grows up to the infinity at the right endpoint \( P_{n+1} \). This phenomena is exhibited also by the function \( d_i(t, X, X) = (\lambda e^{-\lambda t}/\pi) e^{\lambda(e^{-2\lambda t} - 1)}|X|^2 \) pointwise, since \( |d_i(t, X, X)|^2 \) attains maximum, for any fixed \( X \), exactly at the end- and mid-points of the periodicity intervals and the minimum is taken at the quarter points \( P_{n+1/4} \) and \( P_{n+3/4} \).

The tension amplitude and density at \( X \) are defined by

\[
\tau^{(a)}(X) = \partial_t d_i^{(a)}(t, X, X) \quad \text{and} \quad |\tau^{(a)}(X)|^2
\]

respectively. Thus there is no tension in the minimum state, while it is infinity at the maximum places. This and the above considerations suggest that the uniquely determined spreads

\[
d_i^{(a)}(n + (1/4))L, X, Z) = -i e^{-2\lambda XZ} \delta_X^{(a)}(X, Z)
\]

\[
= -\frac{i\hbar}{\pi} L_a(\lambda|X - Z|^2) e^{-\lambda((1/2)|X+Z|^2 + i\langle X, J(Z) \rangle)}
\]

\[
d_i^{(a)}(n + (3/4))L, X, Z) = i e^{-2\lambda XZ} \delta_X^{(a)}(X, Z)
\]

\[
= \frac{i\hbar}{\pi} L_a(\lambda|X - Z|^2) e^{-\lambda((1/2)|X+Z|^2 + i\langle X, J(Z) \rangle)}
\]

at the quarter points should be considered as the stable charge spreads. If the particle is in a higher average energy state then, due to the tensions, it drops down to a stable minimum state.

Also note that the Dirac delta spreads represent the Rayleigh-Jeans law and the DW-flow moves the spread down to the minimal state, which represents
the Wien law. The introduction of stable zonal charge spreads seems to terminate the de Broglie-Schrödinger’s waves from the theory. The contradiction can be resolved, however, by the duality principle, asserting that charged materials behave sometimes as waves and sometimes as particles being in the stable charge-spread state.

This interpretation provide a plain explanation not just for the problem why stable charge spreads do not blow up but it predicts also the spontaneous emission of accelerated electrons. It is well known that such electrons radiate even when no light is incident. The explanation for this phenomena is as follows. The accelerations displace the electron spread from the stable minimal state and it moves back by radiating the energy excess gained by the acceleration.

(G) Zonal WK- and Feynman-measures.

(G1) Construction of WK- and Feynman-measures. The existence of zonal WK-measures on path-spaces follow from the well defined global WK-measure. Nevertheless, it is rather striking that the zonal DF-kernels are not just of the trace class but they well define, regarding any zone, a complex measure \( dw_{T_{xy}}^{(a)}(\omega) \) on the space \( P^T_{xy} \) of continuous paths \( \omega: [0, T] \to \mathbb{R}^k \) connecting points \( x \) and \( y \). (This path-space is topologized with the topology of uniform convergence. By the one-point compactification \( M = \mathbb{R}^k \cup \infty \) the \( P_{xy}(M) \) becomes a compact topological space.) Well defined measures, \( dv_{xy}^{T(a)}(\omega) \) and \( d(\nu_{xy}^{T(a)}) \), can be constructed also by the zonal spread amplitudes \( \delta^{(a)} \) and densities \( \delta^{(a)} \delta^{(a)} \) respectively.

All these measures are constructed by the same ideas the Wiener measure, regarding the Euclidean heat kernel \( E(t, p, q) \), was established. This construction starts out with the elementary fact that the Borel \( \sigma \)-algebra (generated by the open sets) on \( P^T_{xy}(M) \) can also be generated by the fibred sets \( \rho_t^{-1}(B) \subset P^T_{xy}(M) \), where \( t = 0 < t_1 < \cdots < t_n < T \) is a fixed subdivision, \( \rho_t: P^T_{xy}(M) \to M^n = M \times \cdots \times M \) is the evaluation map defined by \( \rho_t(x) = (x(t_1), \ldots, x(t_n)) \), and \( B \) is a Borel subset of \( M^n \). Measure \( w_{E_{xy}} \) on a fibred set \( \rho_t^{-1}(B) \) is defined by

\[
 w_{E_{xy}}(\rho_t^{-1}(B)) = \int_B E(t_1, x, m_1)E(t_2 - t_1, m_1, m_2) \ldots E(T - t_n, m_n, y)dm_1 \ldots dm_n. \tag{37}
\]
By classical results, such as Riesz’ theorem (concerning the measure representation of bounded linear functionals on the Banach space of continuous functions defined on a compact metrizable space where the Banach norm is defined by $\sup |f|$) and the Stone-Weierstrass theorem (asserting that the curves $\rho_t^{-1}(x_1, \ldots, x_n)$ corresponding to $T_n^1 < \cdots < T_n^{n-1}$ are dense in $\mathcal{P}_xy(M)$), this construction determines a complex countably additive regular Borel measure $w^T_{Exy}$ on $\mathcal{P}_xy(M)$, satisfying

$$w^T_{Exy}(\mathcal{P}_xy(M)) = E(T, x, y). \quad (38)$$

The same idea works out for any of the kernels mentioned above. The construction is completed by proving the uniform boundedness of the approximating measures. This is established by a particular integral formula in [Sz5]. Although these measures are constructed on the total space $\mathcal{P}_xy(M)$, the zonal measures are concentrated on curves which can be described by functions constituting the zones. We do not consider these technical details in this paper.

(G2) Zonal Feynman-Kac type formulas, Feynman’s only stopwatch. It should also be pointed out that the zonal Euclidean heat- or DF-kernels are not defined because the zones are not invariant with respect to the action of the Euclidean Laplacian $\Delta_X$. Therefore, the original Feynman-Kac formulas can not be formally carry over to the zones. This is why $d\nu^T_{xy}(\omega)$ and $d(\nu^T_{xy})^{T(a)}$ are introduced. In the zonal Feynman-Kac type formulas the Radon-Nikodym derivative of $d\nu^T_{axy}$ with respect to the latter measure are explicitly computed. The Radon-Nikodym derivative of the zonal Feynman measure with respect to the zonal Wiener-Kac measure is also established. These zonal FK-type formulas have the following explicit form:

$$d_i^{(a)}(T, x, y) = \int_{\mathcal{P}_xy} e^{-\frac{kT}{2} - 2 \int_0^T |\omega(\tau)|^2 d\tau} d\nu^{T(a)}_{xy}(\omega), \quad (39)$$

$$d_i^{(a)}(T, x, y) = \int_{\mathcal{P}_xy} e^{-\frac{kT}{2} - 2 \int_0^T |\omega(\tau)|^2 d\tau} d\nu^{T(a)}_{xy}(\omega). \quad (40)$$

The Radon-Nikodym derivative of $d\nu^T_{xy}$ resp. the zonal Feynman measure with respect to the zonal WK-measure are:
\[ d\nu^{T(a)}_{xy}(\omega) = e^{kT} + 2 \int_0^T |\omega(\tau)|^2 d\tau d\nu^{T(a)}_{1xy}(\omega), \]  \hspace{1cm} (41)

\[ dw^{T(a)}_{xy}(\omega) = e^{(kT + 2)\int_0^T |\omega(\tau)|^2 d\tau} (1 - i) dw^{T(a)}_{1xy}(\omega), \]  \hspace{1cm} (42)

which establish the most direct connection between the three measures. The above formulas have the very same form with respect to any kernel \( e^{-\sigma H_x} \), where the \( \sigma \) is arbitrary unit complex number. One should just substitute \( \sigma \) for \( i \), however, the last \( i \) in the first equation should be left alone, obviously.

The zonal Feynman measure of the whole set of curves connecting \( x \) and \( y \) is

\[ w^{T(a)}_{xy}(\mathcal{P}^T_{xy}(M)) = d^{T(a)}_{i}(T, x, y). \]  \hspace{1cm} (43)

By an intuitive interpretation of Feynman, where he referred to the global kernel and measure, the motion of a particle (electron, photon, e. t. c.) from \( x \) to \( y \) is timed by a “stopwatch” whose hand starts rapidly turning when leaving \( x \) and it is stopped when the particle arrives to \( y \). The “hand” of the stopwatch is considered as a complex unit number and this timing is performed along each continuous curve connecting \( x \) and \( y \). Suppose that the particle moves from \( x \) to \( y \) during the same outer time \( T \), which is not the time measured by the stopwatch. Speaking hypothetically, the particle is moving along each of the paths \( \omega : [0, T] \to \mathbb{R}^k, \omega(0) = x, \omega(T) = y \) connecting \( x \) and \( y \). Thus, by the finite arrow (hand of the stopwatch), each curve is represented by a unit complex number at \( y \). On the other hand, there is also a complex measure defined on the path-space, by which these final arrows are integrated, producing a single final arrow (complex number). This unique complex number is then \( d^{T}_{i}(T, x, y) \) which is called probability amplitude. The positive real number \( d^{T}_{i} \) defines the probability density at \( y \).

For constructing the measure on the path-space, Feynman used the global kernel \( d_{i} \) in the same way how \( d_{i} \) is used for constructing the well defined Wiener-Kac measures. In case of the Feynman measures, however, the approximating measures diverge and they do not extend into a continuous complex measure defined on the Borel sets of the path-space. That is why the above intuitive (yet very beautiful) idea is considered to be mathematically imperfect.
On the zonal setting, however, this idea works out perfectly and both the zonal Feynman measure and the zonal Feynman stopwatch can be explicitly determined. The zonal probability density is defined by

$$\rho_{xy}^{T(a)} = \pi^{k/2} d_i^{(a)}(T, x, y) d_i^{(a)}(T, x, y),$$  \hfill (44)

meaning that, for a Borel set $B$, the integral $\int_B \rho_{z,y}^{(a)} db$ measures the probability that the point spread about $x$ can be caught, at the time $T$, among the point spreads whose centers are on the set $B$. The probability regarding the whole space is always 1, regardless the time $T$. Feynman’s stopwatch, the turning unit complex number, is explicitly determined by the zonal Radon-Nikodym derivative (density)

$$\frac{d\omega_{xy}^{T(a)}}{d\nu_{xy}^{T(a)}}(\omega) = e^{-\frac{kT^2}{2} - 2 \int_0^T |\omega(\tau)|^2 d\tau}^i,$$  \hfill (45)

which is the integrand in (40). The fascinating thing is that Feynman has only one stopwatch for all of the zones. The zone-depending object is the measure $d\nu_{xy}^{T(a)}$ by which the arrows at $y$ are integrated. Also the densities in the zonal Radon-Nikodym derivatives (39), (41), and (42) are independent from the zones.

The very same statement can be established for probabilities defined by a normalized zonal wave function $\psi^{(a)}(t, X)$ satisfying the Schrödinger equation. Such a function is the uniquely determined extension of the initial function $\psi^{(a)}(0, X)$. The extension is defined by the convolution formula $\psi^{(a)}(t, X) = d_i^{(a)}(t, X, Z) \ast_Z \psi^{(a)}(0, Z)$. The probability concerning the density function $\psi^{(a)}\bar{\psi}^{(a)}$ is interpreted as the likely-hood that the zonal object described by the wave function can be caught on a Borel set $B$ at the time $T$. Like the first one, also this probability satisfies the conservation law. All these statements are particular exhibitions of the theorem asserting that the Feynman-Dirac zonal flows define a unitary semigroup, $U_t^{(a)}$, on each zone. Thus one has a unitary semigroup, $U_t = \oplus_a U_t^{(a)}$, on the whole $L^2$ function space.
As there is explained earlier, the probabilistic theory is the bridge connecting the de Broglie-Schrödinger waves with the particles defined by the stable zonal charge spreads.

**(H) Zeeman manifolds with higher dimensional centers.**

The mathematical model for interpreting the Zeeman operator as the Laplacian on a Riemannian manifold has been, so-far, a Riemannian circle bundle, defined by factorizing the centers on Heisenberg groups which is endowed with a left invariant metrics. This idea works out also on metric two-step nilpotent Lie groups, which are rudimentary described in section (C), whose center \( z \) is factorized by a lattice \( \Gamma_Z \). This center is considered as an abstract higher dimensional space such that an element \( Z \in z \) is identified with the endomorphism \( J_Z : v \rightarrow v \) and its natural inner product is defined by \( \langle Z_1, Z_2 \rangle = -Tr(J_{Z_1} \circ J_{Z_2}) \). Formulas (5)-(13) in (C) apply also to these general cases, just the Laplacian (13) appears in a slightly different form. Upto isomorphism, the Lie algebra of such a group is uniquely determined by a linear space, \( J_z \), of skew endomorphisms acting on the Euclidean space \( v \). Two 2-step nilpotent groups are isometrically isomorphic if and only if the corresponding endomorphism spaces are conjugate.

The rather large class of Riemannian torus bundles introduced in this way are called also Zeeman manifold. Below also particular Zeeman manifolds are introduced. It is remarkable that for the so called Clifford-Zeeman manifolds even classification can be implemented, which may be used for classifying the charged particles investigated in this theory.

The Laplacian on the Riemannian group \( (N_J_z, g) \), defined by the endomorphism space \( J_z \), has the explicit form:

\[
\Delta = \Delta_X + \Delta_Z + \frac{1}{4} \sum_{\alpha, \beta=1}^r \langle J_\alpha(X), J_\beta(X) \rangle \partial^2_{\alpha\beta} + \sum_{\alpha=1}^r \partial_\alpha D_\alpha \bullet, \tag{46}
\]

which leaves the function spaces \( FW^{(\gamma)} \) spanned by the functions of the form \( \Psi^{(\gamma)}(X, Z) = \psi(X) e^{2\pi i (Z_\gamma, Z)} = \psi(X) e^{2\pi i (Z_\gamma, Z)} \), for all lattice points \( Z_\gamma \in \Gamma_Z \) (resp. \( Z_\gamma \in \pi\Gamma_Z \)), invariant. Its action on such a function space can be described in the form \( \Delta(\Psi^{(\gamma)})(X, Z) = \Box^{(\gamma)}(\psi)(X) e^{2\pi i (Z_\gamma, Z)} \), where operator
□(γ), acting on $L^2(v)$, is of the form

$$\Box(\gamma) = \Delta X + 2\pi i D(\gamma) \bullet -4\pi^2(|Z_\gamma|^2 + \frac{1}{4}|J_{Z_\gamma}(X)|^2) \tag{47}$$

$$= \Delta X + 2iD_{Z_\gamma} \bullet -4(|Z_\gamma|^2 + \frac{1}{4}|J_{Z_\gamma}(X)|^2).$$

Thus the Zeeman operator appears on the invariant subspaces defined by the Fourier-Weierstrass decomposition. The spectral investigations on these manifolds are reduced to investigate this operator on each Fourier-Weierstrass subspace separately.

The particles represented by these Riemannian torus bundles are called Zee-man molecules. A physical interpretation of factorization by the lattice $\Gamma_Z = \{Z_\gamma\}$ is that there is a quantization considered also on the space of torque-axes $Z$ of the magnetic dipole moment.

There are special Z-molecules, defined by particular endomorphism spaces, which are particularly interesting. The Heisenberg-type or Cliffordian endomorphism spaces are attached to Clifford modules (representations of Clifford algebras). They are characterized by the property $J^2_Z = -|Z|^2 id$, for all $Z \in \mathbf{z}$. [Ka]. The corresponding molecules are called Clifford-Zeeman molecules. The well known classification of Clifford modules provides classification also for the Clifford endomorphism spaces and molecules. A brief account on this classification theorem is as follows.

If $r = \text{dim}(J_Z) \not\equiv 3(\text{mod}4)$, then there exist (up to equivalence) exactly one irreducible $H$-type endomorphism space acting on a $\mathbb{R}^{n_r}$, where the dimension $n_r$, depending on $r$, is described below. This endomorphism space is denoted by $J_r^{(1)}$. If $r = 3(\text{mod}4)$, then there exist (up to equivalence) exactly two non-equivalent irreducible $H$-type endomorphism spaces acting on $\mathbb{R}^{n_r}$ which are denoted by $J_r^{(1,0)}$ and $J_r^{(0,1)}$ respectively. They are connected by the relation $J_r^{(1,0)} \simeq -J_r^{(0,1)}$.

The values $n_r$ corresponding to $r = 8p, 8p + 1, \ldots, 8p + 7$ are

$$n_r = 2^{4p}, 2^{4p+1}, 2^{4p+2}, 2^{4p+2}, 2^{4p+3}, 2^{4p+3}, 2^{4p+3}, 2^{4p+3}. \tag{48}$$
The reducible Cliffordian endomorphism spaces can be built up by these irreducible ones. They are denoted by \( J_r^{(a)} \) resp. \( J_r^{(a,b)} \). The corresponding Lie algebras are denoted by \( h_r^{(a)} \) resp. \( h_r^{(a,b)} \). In the latter case the X-space is defined by the \((a + b)\)-times product \( \mathbb{R}^{nr} \times \cdots \times \mathbb{R}^{nr} \) such that on the last \( b \) component the action of a \( J_Z \) is defined by \( J_Z^{(0,1)} \) and on the first \( a \) components this action is defined by \( J_Z^{(1,0)} \). In the first case this process should be applied only on the corresponding \( a \)-times product.

In a Clifford endomorphism space each endomorphism anticommutes with all perpendicular endomorphisms. In other words, all endomorphisms are anticommutators. A more general concept can be introduced by the *anticommutative endomorphism spaces* where all endomorphisms are anticommutators. They can be built up, in a non-trivial way, by Clifford endomorphism spaces. Roughly speaking, a CZ-molecule is the compound of irreducible molecules of the same type while an *anticommutative Z-molecule* is the compound of CZ-molecules of different types in general.

Originally, the metric groups \((N_J, g)\) were used, in many different ways, for constructing isospectral Riemannian metrics with different local geometries. The author’s results regarding such constructions are published in [Sz1, Sz2, Sz3, Sz4] which contain also detailed history about this topic. These examples include isospectral pairs of metrics on ball\( \times \)torus-, sphere\( \times \)torus-, ball-, and sphere-type manifolds. Among these examples the most striking are those constructed both on sphere- and sphere\( \times \)torus-type manifolds. One of the metrics in the isospectral pair is homogeneous while the other one is not even locally homogeneous. These isospectrality constructions are implemented such that on some of the irreducible subspaces \( \mathbb{R}^{nr} \) the endomorphism spaces \( J_r^{(1,0)} \) (resp. \( J_r^{(0,1)} \)) are switched to \( J_r^{(0,1)} \) (resp. \( J_r^{(1,0)} \)). It turns out that the Riemannian space, resulted by this switching, has a completely different local geometry, yet, the considered domains in the original and the new Riemann spaces are isospectral. Endomorphism spaces \( J_r^{(1,0)} \) and \( J_r^{(0,1)} \) are considered to be representing irreducible CZ-particles having opposite charges. Thus the isospectrality theorem can be physically interpreted as follows:

*By charging some of the irreducible CZ-particles in a CZ-molecule by the same amount of the opposite charge the spectra of the considered domains remain the same, however, the local geometry is drastically changed in general.*

Most of these isospectrality statements are established by constructing in-
tertwining operators, while some are proved by explicit computations of the spectrum. These computations are different from the one developed for the Zeeman zones. They are, rather, the relative of techniques applied in physics for computing the spectra of charged particles in a Coulomb potential field. If such potential is present the eigenfunctions can be sought just in the form $F(r)H^{(p, l-p)}$, described in section (D3). There is also explained that the Zeeman zones can not be constructed by these eigenfunctions.

(I) The Pauli-Dirac operators.

(I1) Introducing the PaDi-operator. In this review the Pauli-Dirac operator is considered only on the plane, $\mathbb{C}$, of complex numbers $z = (z_1, z_2)$. In higher dimensions this operator is the sum of operators defined on the complex coordinate planes. These operators are square roots of the Hamilton operators, introduced by means of matrices

\[
\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_1, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Matrices $\sigma_1$ and $\sigma_2$, which are built up by the well known Pauli spin matrices, are called canonically conjugate spin matrices. They satisfy the commutativity relations

\[
\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Such pairs can be defined for any pair $\{u_1, u_2\}$ of perpendicular unit complex numbers, which define the coefficients before the Pauli matrices. The above matrices correspond to $u_1 = (1+i)/\sqrt{2}$, $u_2 = \overline{u}_1$. Any of such pairs $(\sigma_1, \sigma_2)$ is appropriate to establish a PaDi-operator. Note that $u_2$ is not the conjugate of $u_1$ in general; this is true just for the above matrices. I. e., canonically conjugate spin matrices are generated by perpendicular and not conjugate complex numbers.

The PaDi-operator is defined by
\[ \mathcal{PD} = \frac{1}{\sqrt{2}} \sum_{j=1}^{2} \sigma_j (i \partial z_j - a^j) + 2\sigma_0 \lambda = \left( \begin{array}{c}
\frac{2\lambda}{\sqrt{2}} \\
\frac{-1+i\sqrt{2}}{\sqrt{2}} (2\partial_z - \lambda z) \\
\frac{1-i\sqrt{2}}{\sqrt{2}} (2\partial_z - \lambda z) \\
-2\lambda
\end{array} \right). \]

This PaDi-operator is attached to the Hamiltonian \( H_{Zf} \). The corresponding operator attached to \( H_Z \) is defined by omitting the second term, \( \lambda \sigma_0 \), in (51). They are distinguished by the denotations \( \mathcal{PD}_{Zf} \) and \( \mathcal{PD}_Z \).

The PaDi-operator acts on \( \mathbb{C}^2 \)-valued functions, called 2-component spinor fields, which are written in the form \( \phi = (\varphi_1, \varphi_2) \). The inner product of spinor fields \( \phi \) and \( \gamma \) is defined by

\[ \langle \phi, \gamma \rangle = \int_{\mathbb{R}^2} \sum_i \varphi_i(X) \overline{\gamma_i(X)} dX. \]

The corresponding \( L^2 \) spinor Hilbert space is denoted by \( \mathcal{S} \).

In order to compute the squared operators, we express \( \mathcal{PD} \) in a more explicit form. Since the vector potential is of the form \( a = \lambda (-z_2, z_1) \), thus

\[ \mathcal{PD}_Z(\phi) = (\mathcal{D}_1(\varphi_2), \mathcal{D}_2(\varphi_1)), \quad \text{where} \]

\[ \mathcal{D}_1 = \frac{1+i}{\sqrt{2}} (2\partial_z - \lambda z) \quad \text{and} \quad \mathcal{D}_2 = \frac{-1+i}{\sqrt{2}} (2\partial_z + \lambda \overline{z}). \]

Though the component operators act only on smooth functions, by Friedriech extension, their action extends to the function space \( L^2_{\mathbb{C}} \). By (14), this space is isomorphic to the weighted Hilbert space \( \mathcal{H} = L^2_{\mathbb{C}, \eta_{\lambda}} \), defined by the Gaussian density \( \eta_{\lambda} = e^{-\lambda |X|^2} \). This isomorphism defines an isomorphism also between \( \mathcal{S}_{\eta} \) and \( \mathcal{S} \).

On the weighted Hilbert space operators (54) appear in the form
\[ \mathcal{D}_1 = \sqrt{2}(1 + i)(\partial_z - \lambda z) \quad \mathcal{D}_2 = \sqrt{2}(-1 + i)\partial_z, \quad \text{(55)} \]

therefore the map

\[ \rho_C(z) = \frac{1}{\sqrt{2}(-1 + i)} \mathcal{D}_2, \quad \rho_C(\bar{z}) = \frac{-1}{\sqrt{2}(1 + i)} \mathcal{D}_1 \quad \text{(56)} \]

is nothing but the natural complex unitary Heisenberg algebra representation, described in (15). Thus, by \( \mathcal{D}^*_1 = \mathcal{D}_2, \mathcal{D}^*_2 = \mathcal{D}_1 \), we have

\[ \mathcal{P}\mathcal{D}_Z^2 = \frac{1}{2}(\mathcal{D}_1\mathcal{D}_2, \mathcal{D}_2\mathcal{D}_1) = H_Z - \lambda\sigma_0. \quad \text{(57)} \]

This computation shows that the appearance of \( \lambda\sigma_0 \) in the squared operator is due to Heisenberg’s commutation relations. We also have \( \mathcal{P}\mathcal{D}_Z^2 f = -\frac{1}{\mathcal{P}^2}\Box - \lambda\sigma_0 \), where the last term can be explained by the same argument. Thus the square of the PaDi-operator is exactly the classical Pauli operator.

(I2) The relativistic property of Pauli- and PaDi-operators. The latter operator is characterized as the non-relativistic spin operator, however, next we show that in the situation given in this paper both the Pauli- and the PaDi-operator are relativistic. This fact is proved below by pointing out the exact matching of the PaDi-operator with the original Dirac operator acting on 4-spinors.

The complete Dirac operator, which involves also the potential \( V \) due to the nucleus, is defined on a coordinate system \((t, x_1, x_2, x_3)\) on the 4-space by

\[ -\frac{\hbar}{i\epsilon} \frac{\partial}{\partial t} - eV - \sum_{r=1}^{3} \alpha_r \left( \frac{\hbar}{i} \frac{\partial}{\partial x_r} + \frac{e}{\epsilon} a^r \right) - \alpha_0 mc = -\frac{\hbar}{i\epsilon} \frac{\partial}{\partial t} - H_D, \quad \text{(58)} \]
where $H_D$ is the Dirac-Hamilton operator, furthermore,

$$
\alpha_1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \alpha_2 = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
\end{pmatrix}, \quad (59)
$$

$$
\alpha_3 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix}, \quad \alpha_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix},
$$

The charged particle is considered in that unique inertial system where the constant electromagnetic field defining the orbiting spin has vanishing electric field and constant magnetic field determined by the vector potential \( a = \lambda(-x_2, x_1, 0) \). Thus the relativistic 4-potential is \( (0, a^1, a^2, 0) \). In this case the \( H_D \) can be restricted onto the \((x_1, x_2)\)-plane, meaning that the system is completely described by such 4-spinors which are defined on the plane and for which both the first- (corresponding to \( t \)) and the fourth-component (corresponding to \( x_3 \)) vanish. Thus they are, actually, 2-spinors defined by the second and third components. From the \( 4 \times 4 \) spin matrices only the \( 2 \times 2 \)-matrices in the middle should be retained, since the rest part define only trivial operations. Note that these middle matrices are exactly the \( \sigma \)-matrices defined with respect to the perpendicular unit complex numbers \( u_1 = 1 \) and \( u_2 = -i \). Therefore, this restricted \( H_D \) is nothing but the Pauli-Dirac operator \( \mathcal{P}D \).

The above Dirac equation clearly suggest the form of the PaDi-operator for bounded particles as well for free particles \((V = 0)\). In case of \( H_Z \) the rest mass \( m \) is neglected, while for \( H_{Zf} \) the rest-mass is defined by \( m = \lambda/c \).

This identification of the PaDi-operator with the Dirac operator proves the relativistic property of the PaDi-operator immediately. A general Pauli operator or PaDi-operator can not be derived from the Dirac operator in this simple way and they are indeed non-relativistic operators which can be derived from the Dirac operator just by non-relativistic limit.

Besides of finding the relativistic version of the Schrödinger equation, Dirac’s main interest was to find a linear equation which derives positive probability
densities. His main concern was that the second order relativistic Klein-Gordon equation defined negative probabilities. It is noteworthy that Pauli was very critical of this probabilistic argument of Dirac. According to him, concept such as “the probability of a particle to be at \( x \) in space” is meaningless for relativistic particles, thus it is meaningless to interpret the wave \( \xi(x) \) as probability amplitude. He regarded the Dirac equation, as well as the Klein-Gordon equation, as the field equation rather than as the equation of probability amplitude as Dirac preferred (cf. the famous Pauli-Weisskopf article where the authors pushed for the resurrection of the Klein-Gordon field by quantizing the KG-equation as well as the Maxwell equations).

Pauli’s concern about the relativistic probability amplitude is solved in this article by identifying the Dirac operator with the PaDi-operator. This operator lives on the space determined by the unique inertial system where \( E = 0 \) and \( B \neq 0 \) hold. Thus also the probability amplitude is defined rather on the space then on the space-time.

Also the hole-theory is completely avoidable by assuming the existence of such particles only which are observable in the mathematical model depicted here (observation practically means performing the Stern Gerlach experiment). Particles of negative energy have never been observed by this measuring so far. Even transition from positive to negative energy state is meaningless, since the new particle is not observable by the same observation even in the case when the new particle exist. For instance, to observe a negative charge reduces the problem to choosing a complex structure \( J \). Positive charges are observable only on models defined by \( -J \). Thus the result of a transition from a state to an other one of opposite charge is not observable by the model defined by \( J \). Yet, this argument does not exclude the existence of positrons. On the contrary, these objects are most definitely predicted by this mathematical model.

**Spectra of PaDi-operators; anomalous Zeeman zones.** On \( S \), both the eigenfunctions and eigenvalues of \( \mathcal{PD} \) can be explicitly determined. These computations should be carried out first for \( \mathcal{PD}^2 \). By representing the \( C^2 \)-valued functions in the form \( \phi = (\phi_1, \phi_2) \), the eigenfields of \( -\Box_\gamma - 2\lambda\sigma_0 \) with eigenvalue \( \mu \) appear in the form \( \phi_1 = (\varphi, 0) \) or \( \phi_2 = (0, \varphi) \), where \( \varphi \) is an eigenfunction of \( -\Box_\gamma \) with eigenvalue, say, \( \nu \). Then the eigenvalue corresponding to \( \phi_j \) is \( \mu_j = \nu + 2(-1)^j\lambda \). The explicit eigenvalues, \( -\lambda(4p + k + 4\lambda k) \), of operator \( \Box_\gamma \) are known from (18). Thus also the spectrum of
$\mathcal{PD}^2$ is explicitly determined.

Then the eigenvalue problem regarding the PaDi-operator $\mathcal{PD}_{Zf}$ can be easily completed. In fact, all the eigenvalues $\mu_j$ above are strictly positive. Furthermore, for the fields

$$\psi_{j+} = Q_j(\phi_j + \frac{1}{\sqrt{\mu_j}}\mathcal{PD}(\phi_j)), \quad \psi_{j-} = Q_j(\phi_j - \frac{1}{\sqrt{\mu_j}}\mathcal{PD}(\phi_j))$$

we have

$$\mathcal{PD}(\psi_{j+}) = \sqrt{\mu_j}\psi_{j+}, \quad \mathcal{PD}(\psi_{j-}) = -\sqrt{\mu_j}\psi_{j-}. \quad (61)$$

(In these formulas the technical constant $Q_j$ is defined such that for a function satisfying $||\varphi|| = 1$ also $||\psi_{j\pm}|| = 1$ must be satisfied. Accordingly $Q_j = 1/((1 - 2(-1)^j|Z_\gamma|)^2 + 1)^{1/2}$.) Thus (60) provides the eigenspinors of $\mathcal{PD}_{Zf}$ with the explicit eigenvalues described in (61).

The lowest eigenvalue, $2\lambda + 4\lambda^2$, belongs to the eigenfunctions $\varphi_0^{(a)} = \bar{\psi}^{(a)}$ on a zone $\mathcal{H}^{(a)}$. For $\mathcal{PD}_Z^2$ we have $\mu_{10} = 0$. This zero eigenvalue of the classical Pauli operator is a current interest in the literature. Note that, due to the additional constant $4\lambda^2$, this eigenvalue is non-zero for $\mathcal{PD}_{Zf}^2$. On microscopic level this additional term involves $\hbar^2$, thus it is negligible for weak magnetic fields. It is not negligible, however, for strong magnetic fields.

Due to $\mu_{10} = 0$, the eigenfunctions of $\mathcal{PD}_Z$ are well defined by (60) for all but this zero eigenvalue. To cover this missing case, the $\psi_1^{(a)}$ is defined by $\phi_1$ and $\psi_{10}^{(a)} = 0$. The constant $Q = 1/\sqrt{2}$ is independent of $j$. Hereby, the eigenvalues and eigenfunctions both of $\mathcal{PD}_Z$ and $\mathcal{PD}_{Zf}$ are explicitly determined.

According to (60), there are two types of eigenspinors, $\psi_1$ and $\psi_2$, depending on the position the generating function $\varphi$ is placed. The spinor spaces spanned by these eigenspinors are denoted by $S_1$ and $S_2$ respectively. The fields in these spaces represent the states of particles’ position and momentum respectively. The first question one should consider if there is an overlapping between these two spaces? It turns out that $S_2$ is completely contained in $S_1$, furthermore, $S = S_1$. Actually, these relations turn out to be true in a more puzzling way: The eigenstates regarding the position and momentum are described by the very same eigenspinors. One can use these relations to establish the **uncertainty principle**: The spinor fields can not be used at the
same time to describe both the position and momentum eigenstates. In other words, if one has a complete information about one type of states, there is no information about the other type of states.

The anomalous zones $S^{(a)}_{\pm}$, $S^{(a)}_{1\pm}$, and $S^{(a)}_{2\pm}$ are spanned by the appropriate eigenspinors derived from the zones $H^{(a)}$ according to the following formulas

$$S^{(a)} = H^{(a)} \times H^{(a)} = S^{(a)}_+ \oplus S^{(a)}_- = S^{(a)}_1 \oplus S^{(a)}_1,$$  \hfill (62)

$$S^{(a)}_2 = H^{(a)}_{\mu_1 > 0} \times H^{(a)} = S^{(a)}_2 \oplus S^{(a)}_2 \subset S^{(a)}_1.$$  \hfill (63)

The whole spinor space is the direct sum of the anomalous zones, i.e.,

$$S = S_1 = \oplus_{a=0}^{\infty} S^{(a)} = \oplus_{a=0}^{\infty} S^{(a)}_1, \quad S_2 = \oplus_{a=0}^{\infty} S^{(a)}_2 \subset S_1.$$  \hfill (64)

(I4) Anomalous kernels. The anomalous zones can be similarly analyzed than the normal ones. This theory includes explicit establishing of an anomalous kernels regarding projections, heat flows and PaDi-flows. In this review only the projections onto the anomalous zone $S^{(a)}_j$ is described. This $\mathbb{C}^2 \otimes \mathbb{C}^2$-valued kernel $Q^{(a)}_j(X, Y)$ has the following component functions

$$Q^{(a)}_{(1)11}(X, Y) = \frac{\lambda}{2\pi} (L_a(\lambda|X - Y|^2) e^{\lambda XY} + (\lambda XY)^a) e^{-\frac{1}{\lambda}(|X|^2 + |Y|^2)},$$  \hfill (65)

$$Q^{(a)}_{(1)22}(X, Y) = \frac{\lambda}{2\pi} L_a(\lambda|X - Y|^2) e^{-\frac{1}{\lambda}(|X|^2 + |Y|^2)},$$  \hfill (66)

$$Q^{(a)}_{(1)12}(X, Y) = Q^{(a)}_{(1)21}(X, Y) = 0,$$  \hfill (67)

$$Q^{(a)}_{(2)11}(X, Y) = \frac{\lambda}{2\pi} (L_a(\lambda|X - Y|^2) e^{\lambda XY} - (\lambda XY)^a) e^{-\frac{1}{\lambda}(|X|^2 + |Y|^2)},$$  \hfill (68)

$$Q^{(a)}_{(2)22}(X, Y) = \frac{\lambda}{2\pi} L_a(\lambda|X - Y|^2) e^{-\frac{1}{\lambda}(|X|^2 + |Y|^2)},$$  \hfill (69)

$$Q^{(a)}_{(2)12}(X, Y) = Q^{(a)}_{(2)21}(X, Y) = 0.$$  \hfill (70)

By this kernel the concept of spinning point spreads can be introduced. More complicated formulas describe the projections $Q^{(a)}_j$ onto the subzones $S^{(a)}_{j\pm}$. Further complications arise when the anomalous heat- and PaDi-flows are described. Yet these computations are manageable which can be used for establishing a well defined anomalous zonal Feynman measure along with explicit stopwatch spinors.
(J) Zonal Coulomb law, Lamb shift.

The main reason for the Zeeman zones are established only for free particles ($V = 0$) is that they are not invariant with respect to multiplications with non-holomorphic functions such as the radial Coulomb potential $V = Q/r$. In subsequent papers the theory of zones will be extended to bounded particles in two different ways. In one of them the quantum Coulomb operator $V$ (multiplication with $V$) is modified such that it leaves the zones invariant. A natural modified operator is defined by projecting $V(H^{(a)})$ back to $H^{(a)}$. Then this zonal Coulomb operator, $V^{(a)}$, is an integral operator with a smooth kernel which can be explicitly computed. Note that the zonal Coulomb forth acts locally, thus the non-relativistic nature of the Coulomb-forth action is terminated. Other problems caused by the original Coulomb law are terminated too. For instance, there are eigenvalues with multiplicities 2 (doublets) in the spectrum of the Zeeman-Coulomb operator $H_Z + V$. The existence of these doublets are argued in the Lamb-Retherford experiment (Lamb shift). It is fascinating to see that this multiplicity drops down to one with respect to each eigenvalue in the global spectrum of the hydrogen atom having the zonal Coulomb potentials $V^{(a)}$. In other words, the real cause of the doublets in the spectrum of the hydrogen atom is the non-relativistic Coulomb law.

There is another way to build in the Coulomb law into the zonal theory, namely, by building in the potential $V$ into the metric of a curved Riemannian manifold. This idea is borrowed from Einstein’s general relativity, obviously. There is a natural generalization of 2-step nilpotent Lie groups, leading to the concept of 2-step nilpotent-type Riemannian manifolds, which can be used to carry out this idea.

A 2-step nilpotent Lie algebra is defined by an endomorphism space, $E$, consisting of skew endomorphisms acting on a Euclidean space $v$. In the place of the Euclidean space $v$, consider a Riemannian manifold $(M, g)$ endowed by a smooth field, $E(p)$, of skew endomorphism spaces. If one assumes that the $E(p)$ is spanned by auto-parallel complex structures, $J_Z(X)$, where $Z \in z$, the system $\{M, g, E(p) = J_Z(p)\}$ is called Kähler complex. Such systems were investigated in the literature just for dimensionality’s $dim(z) = 1, 3$ which correspond to Kähler resp. hyperkähler manifolds.

Principal bundle $(M, M \times z, z)$ corresponds to $(v, v \times z, z)$ in case of nilpotent
groups. The most important objects on this bundle are the *gauge connections* \( \omega \) which, by definition, satisfy the structure equation \( d\omega = \Omega = \Omega^\alpha e_\alpha \), where the curvature form \( \Omega \) is defined by means of the endomorphisms \( J e_\alpha \). The gauge metrics \( g_\omega \) on the gauge-bundle is defined by the horizontal lift of metric \( g \) on \( M \) to the horizontal subspace.

Up to gauge transforms, this metric is unique, defining a 2-step nilpotent type Riemannian manifold. The generalized Zeeman manifolds are defined by factorizing \( z \). The Zeeman manifolds defined by Kähler complexes form an important subclass. Particularly interesting are the Bergman-Zee manifolds constructed on the bounded domains of \( \mathbb{C}^n = \mathbb{R}^{2n} \). Zonal analysis along with several spectral investigations is the next step in developing this theory. The Laplacian on these Riemannian manifolds appears as a generalized Zeeman operator with a “built in electric potential”. Thus the generalized zonal theory takes the contributions attributed to electric potentials into account.

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