Modes of failure in disordered solids

Subhadeep Roy\textsuperscript{1,2}, Soumyajyoti Biswas\textsuperscript{3} and Purusattam Ray\textsuperscript{1}

\textsuperscript{1} The Institute of Mathematical Sciences, Taramani, Chennai-600113, India.
\textsuperscript{2} Earthquake Research Institute, University of Tokyo, 1-1-1 Yayoi, Bunkyo, 113-0032 Tokyo, Japan.
\textsuperscript{3} Max Planck Institute for Dynamics and Self-Organization, Am Fussberg 17, Göttingen, Germany.

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The two principal ingredients determining the failure modes of disordered solids are the strength of heterogeneity and the length scale of the region affected in the solid following a local failure. While the latter facilitates damage nucleation, the former leads to diffused damage – the two extreme nature of the failure modes. In this study, using the random fiber bundle model as a prototype for disorder solids, we classify every failure modes that are the results of interplay between these two effects. We obtain scaling criteria for the different modes and propose a general phase diagram that provides a framework for understanding previous theoretical and experimental attempts of interpolation between these modes. As the fiber bundle model is a long standing model for interpreting various features of stressed disordered solids, the general phase diagram can serve as a guiding principle in anticipating the responses of disordered solids in general.

Response of a disordered solid subjected to stress provide a vital route in predicting imminent breakdown in those systems. Understanding such responses is a major goal for myriads of situations starting from micro-fracture to earthquakes\textsuperscript{1}. The apparent independence of the effect of the structural details in the static and dynamic responses of the disordered solids, for example roughness of a fractured front, avalanche size distributions etc., fueled decades of efforts in modeling these phenomena using simple, generic and minimal ingredients\textsuperscript{2}. The focus of these studies is on the understanding of the mechanical stability of the systems, precursor to catastrophic failure and also to explore the possibility of universality in the sense of critical phenomena. However, while there can be scale free behavior of response functions indicating criticality, there can also be nucleation driven abrupt failures. Therefore, such association of fracture with critical phenomena is not straightforward (see e.g.\textsuperscript{2,3,4}).

It is, however, known that the two main factors that determine such modes of failures are the strength of disorder and the range of interaction within the solid in terms of stress transfer. The aim of this work is to classify all the phases arising out of the interplay of these two effects and to arrive at criteria in distinguishing such phases, thereby providing a framework for understanding all the modes of failure using a simple model for the disordered solids.

Presence of heterogeneity increases the precursory signals prior to failure\textsuperscript{5}. The strain energy is dissipated within a short range of crack propagation in heterogeneous solids, as opposed to those lacking heterogeneity. Strong heterogeneity, therefore, compelling the system transit from a brittle like to a quasi-brittle like failure mode\textsuperscript{5}. Such a transition in porous media was observed in Ref.\textsuperscript{7}, while the disorder (porosity) spanned two decades in magnitude. The apparent contradiction of scale free size distribution for acoustic emission and subsequent damage nucleation was also observed in Ref.\textsuperscript{5}. There have been many other experiments and simulations describing the effect of increased disorder on roughness\textsuperscript{6}, pattern formation in spring networks\textsuperscript{10}, damage nucleation and percolation in random fuse models\textsuperscript{11,12} etc. As for the range of stress redistribution, linear elastic fracture mechanics predict a \(1/r^2\) type load redistribution around an Inglis crack\textsuperscript{13,14}. However this form is not always guaranteed and can change due to finite width of the sample\textsuperscript{15}, correlation in disorder\textsuperscript{16}, size of agglomerate\textsuperscript{17} and so on. In this work we attempt to characterize the formation of spatial and temporal correlation arising out of the interplay of the stress redistribution, which enhances damage nucleation, and the presence of disorder, which leads to diffused damage\textsuperscript{18,20}.

In this Letter, we report a phase diagram in the stress redistribution range and strength of disorder that captures all failure modes arising out of the interplay between these two. We consider the fiber bundle model\textsuperscript{2,21}, which has been widely used as a generic model for fracture in disordered system over many years. With the help of the phase diagram, we can now identify all its modes of failure, classify previous attempts to interpolate between some of those modes and most importantly arrive at scaling prescriptions in categorizing and predicting such failure modes. The scaling prescriptions differ from their equilibrium, and often intuitive, counterparts (say, in Ising model), making them interesting also from the point of view of critical phenomena.

Among the many modeling approaches that attempt to capture the statistics of failure of disordered solids, fiber bundle model is arguably the simplest. Introduced in the textile engineering\textsuperscript{22}, it has been proven very useful in reproducing behaviors near failure\textsuperscript{2}. It is a set of elements arranged in a lattice, each having a finite failure threshold drawn randomly from a distribution. On application of load, the elements — fibers — fail irreversibly...
and redistribute their load in a pre-defined neighborhood. The avalanche statistics and also the roughness of fracture propagation front arising out of its intermittent dynamics, compares favorably with experiments \[23\].

Here we simulate the failure in fiber bundle model in one and two dimensions – the one dimensional case is an idealized but the simplest one, while the two dimensional case is more realistic and has been used to model failure in fibrous materials (e.g. fiber reinforced composites) for many years \[24\], \[25\]. We choose the failure thresholds of the fibers from a distribution of the form \(p(x) \sim 1/x\) within a range \([10^{-\beta} : 10^\beta]\). For high values of \(\beta\), the distribution becomes very broad, making the system a highly disordered one. Physically, this implies varying strength of impurities in the system, that can significantly influence the overall critical strength of the system. Following the failure of a fiber, the load on the failed fiber is redistributed uniformly up to a distance \(R\). In one dimension, this is simply \(R\) surviving neighboring fibers on either side of the failed one. In two-dimensions we search along positive and negative \(x\) and \(y\) axes and go up to a distance \(x_+\), \(x_-\), \(y_+\), and \(y_-\). until \(R\) surviving neighbors are found along each direction. We then redistribute the load within the rectangular region \((x_+, y_+), (x_-, y_+), (x_+, y_-), (x_-, y_-)\) (assuming the origin at the failed fiber). Of course, there can be other choices, for example a circular region of radius \(R\). While that could work well for higher values of \(R\), but for smaller values there could be situations where there were no surviving fibers within that region. Moreover, such details are unlikely to affect the scaling behavior, which is also evident from the fact that our prediction matches well with power-law load redistribution studied in Ref. \[24\].

With changes in these two parameters (\(\beta\) and \(R\)) we get the different failures modes of the model. We will first describe the phase diagram to explain the different modes. Subsequently we will discuss the methods of drawing the boundaries and relate them to previous numerical and experimental attempts of interpolations.

**Phase diagram:** Intuitively, we expect a nucleating failure for low values of \(R\) and \(\beta\). This resembles brittle failures of perfectly crystalline structures. The failure thresholds of each part of the system are almost same, therefore an initial failure and subsequent load concentration around it (due to low \(R\) values) compels the subsequent damages to be near that initial damage and it will continue to grow. Thus small \(R\) and \(\beta\) imply high spatial correlation in damage. This damage nucleation can be prevented by either redistributing the load of a failed fiber to a relatively large distance, or by increasing the disorder such that the nearby fiber can have high failure threshold which compels distant fibers to fail first.

On the other hand, higher the number of fibers breaking due to stress redistribution, higher is the temporal correlation (we will present quantitative measures later). The temporal correlation in damage, i.e. avalanches, also behave similarly with \(R\) and \(\beta\). Small \(R\) and \(\beta\) imply higher correlation. The difference is that the temporal correlation does not vanish at the same values of \(R\) and \(\beta\), as the spatial correlation. The phase diagram (Fig. 1), therefore, has regions where temporal correlation exists without spatial correlation, hence giving interesting phases for the model.

Below we first describe each of the phases depicted in Fig. 1 and then describe the quantitative measures for drawing the boundaries between the phases.

**B: Brittle nucleating:** As soon as the weakest fiber is broken, the entire system collapses starting from damage nucleation happening next to the failed fiber. This is a brittle like failure (like in ceramics, say) and have both temporal and spatial correlations. The avalanche is a catastrophic failure here, with size \(\sim L\). **D: Brittle percolating:** The system here also collapses following the breaking of the weakest fiber, but as \(R\) is large enough, the subsequent damage is spatially uncorrelated i.e. multiple damage nucleation zones are formed. **A: Quasi-brittle nucleating:** In this region, the system fails after multiple stable states, hence the nature of failure is quasi-brittle. In this region, an apparent random failure eventually forms a spatially correlated failure i.e. the system begins with a scale free avalanche distribution, but for larger systems the final failure is nucleation driven (see reference \[24\], \[28\] for electrical analogue). **C: Quasi-brittle percolating:** This is the region where the \(R\) and \(\beta\) combination is such that although the spatially correlation has vanished, the temporal correlation exists. This is the region with scale free size distribution (exponent \(\sim 5/2\)) of the avalanches. **E: High disorder limit:** In this region, neither the spatial correlation nor the temporal correlation exists. As can be seen, this region appears even for very low \(R\) values, given the disorder distribution is broad enough (high \(\beta\)).

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**FIG. 1.** The figure shows all the regions on \(R - \beta\) plane for 1d and 2d bundle. \(B\) and \(D\) are brittle regions and show abrupt failure. \(A\) and \(C\) show quasi-brittle response. Only difference is, in region \(A\) and \(B\) rupture process is spatially correlated. In region \(E\) (spatially uncorrelated) and \(F\) (spatially correlated), the failure process is mainly dominated by stress increment.
The response of the model, to external stress, is perfectly linear in region B and D. While in other regions, a nonlinear behavior is observed in stress v/s strain curve (see Fig.8 of supplementary material (SM)).

**Phase boundaries:** The various phases described above are separated by phase boundaries drawn on specific criteria. We will describe those now. But before proceeding, it is useful to mention here that a general way to determine spatial correlation is to monitor the cluster density with fraction of broken fibers. Fig. 2 shows the variation of cluster density $n_p$ (number of cluster divided by system size) with fraction of broken bonds $1-U$, at different $R$ and $\beta$ values, for both one and two dimension. In one dimension, the number of clusters of broken fibers is simply the number of side by side broken and unbroken fibers present. $U$ is the fraction of surviving fibers at any time, then for complete random failure, the number of side by side broken and unbroken fiber will be $U(1-U)$ (normalized by system size). Any deviation of $n_p$ from this function would indicate spatial correlation. A quantitative measure for such departure is the area under this function would indicate spatial correlation. A quantitative measure for such departure is the area under this function would indicate spatial correlation. A quantitative measure for such departure is the area under this function would indicate spatial correlation. A quantitative measure for such departure is the area under this function would indicate spatial correlation.

$\int_0^1 U(1-U)dU = 1/6$. At low $R$ and $\beta$, the area under the curve deviates from $A_{1d}$ (See Fig.1 of SM). For two dimensions the situation is qualitatively similar. But the general shape of the curve for random failure is not known. However, there are many numerical studies in terms of random site percolation (see Ref. and references therein) that looks at density of patches under random occupations (see Fig.2).

**A-C boundary:** The deviation of the $n_p$ v/s $1-U$ curves from the random case determines this boundary. This gives a crossover scale $R_c$, which scales with the system size as $L^{2/3}$ in one-dimension. In two dimension the scaling changes to $R_c \sim L^\beta$, with $b = 0.85 \pm 0.01$ (see Fig.2 of SM). One interesting implication of the scaling is, when the load sharing is a power law, the effective range of the load redistribution can be shown to be (see SM) $R_{eff} \sim L^{3-\gamma}$, where $\gamma$ is the power of the load redistribution process. At crossover, therefore, we have $\gamma_c = 3 - b$. But $b < 1$, giving $\gamma_c > 2 (= 2.15)$. A direct simulation of the power law load sharing also verifies this, where the moment statistics of the cluster statistics were studied (see Fig.3 of SM). This now confirms with much improved precision an earlier result [26] where $\gamma_c > 2$ was obtained.

**B-D boundary:** The nature of this crossover line is the same as before and is drawn by monitoring the cluster density. The crossover length scale $R_c$ now scales non-universally with $L$, as $R_c \sim L^\xi$ with $\xi = \zeta(\beta)$ [31].

**F-E boundary:** Finally, this boundary is also drawn from the same measure and the crossover scale here is independent of the system size.

**D-C boundary:** This class of boundaries separate brittle to quasi-brittle transitions. Particularly, in the brittle region, the breaking of the weakest fiber will cause the breakdown of the entire system. Hence, by measuring the fraction of surviving fibers in the last stable configuration before breakdown ($U_c$), we track the transition from brittle (with $U_c = 1$) to quasi-brittle (with $U_c < 1$) region. A phase transition occurs only across this line [32, 33], with no system size dependence of the transition line.

**B-A boundary:** This boundary is also drawn with the same criterion as before. There is, however, a system size dependence of the line and it gets shifted to higher $\beta$ value with increasing system size in a inverse logarithmic manner (see Ref. as well as Fig.4 in SM), for a particular stress release range.

**C-E boundary:** This boundary separates the completely uncorrelated phase from temporally correlated quasi-brittle region C. We evaluate it in two different ways and the results match for the two cases: (i) We measure the scaling of the average avalanche size $\langle s \rangle \sim L^{\xi}$ (see Fig.3). In the quasi-brittle region, $\xi$ is a (decreasing) function of $\beta$ and eventually $\langle s \rangle$ becomes independent of $L$ in the percolating region E. The $\beta$ value at which $\langle s \rangle$ becomes system size independent gives the boundary between C and E, because system size independence signifies complete removal of correlation in the system. (ii) A second way to approach the problem is to measure the number of stress increment $N_s$ and the number of times $N_r$ stress were redistributed during the entire time of survival of the system. When $N_s$ outruns $N_r$, i.e. more fibers break due to stress increment (without spatial or temporal correlations) than due to stress redistributions, the uncorrelated region E is obtained. We find that the disorder strength $\beta_2$ when this happens scales with the system size as: $\beta_2(L) = \beta_2(\infty) + L^{-\alpha}$, with $\alpha = 1/2$ (see Fig.4). Also, the system size scaling of $(N_s - N_r)$ is given
by

\( (N_a - N_r) = L^\alpha \Phi[(\beta_2 - \beta_2(L))L^{\gamma}] \) \hspace{1cm} (1)

with \( \alpha = \gamma = 1/2 \). The \( \beta_2 \) obtained in this way matches with the boundary obtained from the scaling of the average avalanche size before. Hence we conclude that \( \beta_2 \) is the range of disorder beyond which the system becomes completely uncorrelated (region E). The scaling is also observed in two dimension, with exponents \( \alpha_{2d} = \gamma_{2d} = 1 \) (see Fig.9 in SM).

Across this boundary the temporal correlation vanishes but the spatial correlation still exists.

At a certain stress release range \( R, \beta_2 \) is not being observed to change much while we alter the system size (see Fig.7 in SM). Also we have found an upper value of disorder \( \beta^* \) beyond which \( \langle s \rangle \) remains constant at 1 independent of both disorder and system size effect and the fibers break through external stress increment only without a single redistribution (See Fig.5 and Fig.6 in SM).

Finally, using the criteria outlined above, we arrive at the quantitative phase diagram for fiber bundle model in one and two dimensions (see Fig.4). Almost all the studies in fiber bundle model fall in some point of this phase diagram. The most studied region being the region \( C \), which is also historically the earliest. Subsequently region \( A \) was studied, which is qualitatively different from region \( C \) in the sense that we no longer observe scale free avalanche statistics here. We provide a scaling criterion to separate these two regions.

There have been many studies over the years in interpolating between various phases of the fiber bundle model described above. Among these, most efforts were concentrated in interpolating between regions \( A \) and \( C \), because this region gives the critical interaction range below which the eventual failure will be nucleation dominated, a much debated topic in fracture [4]. The crossover from \( A \) to \( C \) was also accessed in Ref. [35] by tuning the elastic modulus of the bottom plate of the model, which in turn controls the range of interaction. In Ref. [32], the authors moved from region \( D \) to \( C \) in the mean-field limit. The value of \( \beta \) was exactly calculated in the mean-field limit [33].

Many experimental observations, like brittle (region \( B \) & \( D \)) to quasi-brittle/ductile (region \( A \) & \( C \)) transition [4], scale free size distribution for acoustic emission [8], subsequent damage nucleation [5], etc., can also be explained by this phase diagram. Such properties are characteristic of region \( A \), where the so called ‘finite size criticality’ [28] is observed, i.e. the system starts off giving scale free avalanches, but the final failure is nucleation driven. Unlike random resistor network [27], where nucleation always dominates in the final failure mode, in the fiber bundle model phase diagram, there exists a temporally correlated failure mode that sustains in the thermodynamics limit.

In conclusion, we have obtained a phase diagram for failure of disordered solids using random fiber bundle model. We describe five distinct modes of failure with varying disorder (\( \beta \)) and stress release range (\( R \)). Disorder effects the abruptness of the failure process while the stress release range influences the correlation between successive rupturing of fibers. Interplay of these two affects leads to spatial and/or temporal correlation or random failures. The resulting phase diagram gives a frame-
work for understanding previous theoretical and experimental attempts to interpolate between these different failure modes.

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[1] B. K. Chakrabarti and L. G. Benguigui, Statistical Physics of Fracture and Breakdown in Disordered Systems (Oxford University Press, Oxford, 1997); M. Sahimi, Heterogeneous Materials II: Nonlinear and Breakdown Properties (Springer-Verlag, New York, 2003); S. Biswas, P. Ray, B. K. Chakrabarti, Statistical physics of fracture, breakdown and earthquake (Wiley-VCH, July 2015).

[2] S. Pradhan, A. Hansen, B. K. Chakrabarti, Rev. Mod. Phys 82, 499 (2010).

[3] Y. Moreno, J. B. Gómez, A. F. Pacheco, Phys. Rev. Lett. 85, 2865 (2000).

[4] S. Zapperi, P. Ray, H. E. Stanley, A. Vespignani, Phys. Rev. Lett. 78, 1408 (1997).

[5] J. Vasseur, F. B. Wadsworth, Y. Lavallée, A. F. Bell, I. G. Main, D. B. Dingwell, Sci. Rep. 5, 13259 (2015).

[6] T.-F. Wong, P. Baud, J. Struct. Geol. 44, 25 (2012).

[7] R. Li, K. Sieradzki, Phys. Rev. Lett. 68, 1168 (1992).

[8] A. Guarino, A. Garcimartin, S. Ciliberto, Phys. Rev. B 6, 13 (1998).

[9] V. V. Silberschmidt, Int. J. Fract. 140, 73 (2006).

[10] I. Malakhovsky, M. A. J. Michels, Phys. Rev. B 76, 144201 (2007).

[11] P. K. V. V. Nukala, S. immovni, S. Zapperi, J. Stat. Mech 2004 P08001 (2004).

[12] B. Kahng, G. G. Batrouni, S. Redner, L. de Arcangelis, H. J. Herrmann, Phys. Rev. B 37, 7625 (1988).

[13] J. Schmittbuhl, S. Roux, J.-P. Vilotte, K. J. Mäkäy, Phys. Rev. Lett. 74, 1787 (1995).

[14] S. Ramanathan, D. Értas, D. S. Fisher, Phys. Rev. Lett. 79, 873 (1997).

[15] Z. C. Xia, J. W. Hutchinsion, J. Mech. Phys. Solids 48, 1107 (2000).

[16] C. K. Peng, S. Havlin, M. Schwartz, H. E. Stanley, Phys. Rev. A 44, R2239 (1991).

[17] K. Kendall, N. McN Alford, W. J. Clegg, J. D. Birchall, Nature 339, 130 (1989).

[18] W. A. Curtin, J. Mech. Phys. Solids 41, 217 (1993).

[19] D. Amitrano, J. Grasso, D. Hantz, Geophys. Res. Lett. 26, 2109 (1999).

[20] D. De Tommasi, G. Puglisi, G. Saccomandi, Phys. Rev. Lett. 100, 085502 (2008).

[21] A. Hansen, P. C. Henner & S. Pradhan, The Fiber Bundle Model: Modeling Failure in Materials Wiley VCH Berlin (2015).

[22] F. T. Peirce, J. Text. Inst. 17, T355 (1926).

[23] D. Bonamy, J. Phys. D 42, 214014 (2009).

[24] S. L. Phoenix, Fibre Sci. and Tech. 7, 15 (1974).

[25] S. L. Phoenix, Int. J. Engrg. Sci. 13, 287 (1975).

[26] R. C. Hidalgo, Y. Moreno, F. Kun, H. J. Herrmann, Phys. Rev. E 65, 046148 (2002).

[27] A. A. Moreira, C. L. N. Oliveira, A. Hansen, N. A. M.Araujo, H. J. Herrmann, and J. S. Andrade, Jr., Phys. Rev. Lett. 109, 255701 (2012).

[28] A. Shekhawat, S. Zapperi, and J. P. Sethna, Phys. Rev. Lett. 110, 185505 (2013).

[29] R. M. Ziff, S. R. Finch, V. S. Adamchik, Phys. Rev. Lett. 79, 3447 (1997).

[30] S. Biswas, S. Roy, P. Ray, Phys. Rev. E 91, 050105(R) (2015).

[31] S. Roy, S. Biswas, P. Ray, arXiv:1510.00687 (2016).

[32] S. Roy, P. Ray, EPL 112, 26004 (2015).

[33] C. Roy, S. Kundu, S. S. Manna, Phys. Rev. E 91, 032103 (2015).

[34] Subhadeep Roy, arXiv:1707.02422 (2017).

[35] A. Stormo, K. S. Gjerden, A. Hansen, Phys. Rev. E 86, 025101(R) (2012).