Reiterative $m_n$-Distributional Chaos of Type $s$ in Fréchet Spaces

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Abstract
The main aim of this paper is to consider various notions of (dense) $m_n$-distributional chaos of type $s$ and (dense) reiterative $m_n$-distributional chaos of type $s$ for general sequences of linear not necessarily continuous operators in Fréchet spaces. Here, $(m_n)$ is an increasing sequence in $[1, \infty)$ satisfying $\lim \inf_{n \to \infty} \frac{m_n}{n} > 0$ and $s$ could be $0, 1, 2, 2+, 2\frac{1}{2}, 3, 1+, 2-, 2_{Bd}, 2_{Bd}+$. We investigate $m_n$-distributionally chaotic properties and reiteratively $m_n$-distributionally chaotic properties of some special classes of operators like weighted forward shift operators and weighted backward shift operators in Fréchet sequence spaces, considering also continuous analogues of introduced notions and some applications to abstract partial differential equations.

Keywords $m_n$-distributional chaos of type $s$ · Reiterative $m_n$-distributional chaos of type $s$ · $\lambda$-distributional chaos of type $s$ · Reiterative $\lambda$-distributional chaos of type $s$ · Fréchet spaces

Mathematics Subject Classification 47A06 · 47A16

1 Introduction and Preliminaries

Assume that $X$ is a Fréchet space. A linear operator $T$ on $X$ is said to be hypercyclic iff there exists an element $x \in D_\infty(T) \equiv \bigcap_{n \in \mathbb{N}} D(T^n)$ whose orbit $\text{Orb}(x, T) \equiv \{T^n x : n \in \mathbb{N}_0\}$ is dense in $X$; $T$ is said to be topologically transitive, resp. topologically mixing, iff for every pair of open non-empty subsets $U$, $V$ of $X$, there exists $n_0 \in \mathbb{N}$
such that $T^{n_0}(U) \cap V \neq \emptyset$, resp. there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n_0$, $T^n(U) \cap V \neq \emptyset$. A linear operator $T$ on $X$ is said to be chaotic iff it is topologically transitive and the set of periodic points of $T$, defined by $\{x \in D_{\infty}(T) : (\exists n \in \mathbb{N})(T^n x = x)\}$, is dense in $X$. For more details about topological dynamics of linear operators in Fréchet spaces, we refer the reader to the monographs by Bayart and Matheron [2] and Grosse-Erdmann and Peris [16].

A strong motivational factor for genesis of this paper presents the fact that the structural results established in the foundational paper by Bernardes Jr et al. [6] have not yet been seriously elucidated and completely reexamined for sequences of linear continuous operators in Fréchet spaces (cf. also the article by Conejero et al. [11], as well as [3, 31, 32]). In our recent joint research study with A. Bonilla [10], we have analyzed reiterative distributional chaos in Banach spaces and observed for the first time how the techniques developed in [6] can be successfully applied in the analysis of dense Li-Yorke chaos. In this paper, which is partly conceptualized as a certain addendum to the papers [6] and [10, 11], the construction of distributionally irregular vectors developed in the proof of [6, Theorem 15] is essentially applied in the analysis of dense reiterative $m_n$-distributional chaos of type $s$ in Fréchet spaces. Besides that, the paper contains a great number of other novelties, particularly those concerned with the deeper analysis of distributionally chaotic linear continuous operators $T$ on Banach space $X$ for which the condition

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \|T^j x\| = +\infty \quad \text{for all} \quad x \in X \setminus \{0\} \quad (1.1)$$

holds true (the existence of such an operator, acting on the space $X := l^p(\mathbb{N})$ for some $p \in [1, \infty)$ or $X := c_0(\mathbb{N})$, has been recently proved in [7, Theorem 25]). We revisit the Godefroy–Schapiro criterion and its continuous analogue, the Desch–Schappacher–Webb criterion, in our new framework.

Concerning motivation, mention should be also made of the recent research study by Xiong et al. [36], where $\lambda$-distributionally chaotic continuous mappings between compact metric spaces have been analyzed ($\lambda \in (0, 1]$). Following this approach, we further specify linear distributional chaos in Fréchet spaces by using the concepts of lower and upper (Banach) $m_n$-densities and, in particular, the lower and upper (Banach) $\lambda$-densities. Speaking-matter-of-factly, we analyze various notions of (dense) $m_n$-distributional chaos of type $s$ and (dense) reiterative $m_n$-distributional chaos of type $s$ for general sequences of linear operators in Fréchet spaces, where $(m_n)$ is an increasing sequence in $[1, \infty)$ satisfying $\lim \inf_{n \to \infty} \frac{m_n}{n} > 0$ and $s$ symbolically takes one of the values $0, 1, 2, 2+, 2\frac{1}{2}, 3, 1+, 2-, 2_Bd, 2_Bd+$; before we go any further, we want to say that any $m_n$-distributionally chaotic sequence and, in particular, any $\lambda$-distributionally chaotic sequence, needs to be distributionally chaotic, i.e., 1-distributionally chaotic, as well as that the converse statement is not true in general.

The organization and main ideas of this paper can be briefly summarized as follows. After collecting some necessary preliminaries about Fréchet spaces we are working with (separability or infinite-dimensionality is not assumed a priori), we
provide basic definitions and properties of lower and upper (Banach) $m_n$-densities in Sect. 1.1. In the second section of paper, we fix notions and introduce several different types of (reiterative) distributional chaos. In particular, we analyze distributional chaos of type $s \in \{1, 2, 2Bd+, 2\frac{1}{4}, 3\}$, reiterative distributional chaos of type $s \in \{0, 1, 1+, 2, 2+, 2Bd, 2Bd+\}$ and Li-Yorke chaos; in Sect. 2.1, we provide several observations, results and open problems for orbits of single operators in Fréchet spaces. (We feel it is our duty to say that it might be much better to summarize the implications between different notions of chaos studied in the paper in some figures, as it has been done in our joint paper with Prof. A. Bonilla [10]; despite this, we have followed another path here because we really work with a great deal of different notions of chaos in this paper.) In this subsection, we prove several results regarding the existence of distributionally chaotic operators which are not $m_n$-distributionally chaotic for certain types of sequences $(m_n)$. In particular, the following is shown: Suppose that $X := c_0(\mathbb{N})$ or $X := l^p(\mathbb{N})$ for some $p \in [1, \infty)$.

(a) There exists a weighted forward shift operator $T \in L(X)$ which is $\lambda$-distributionally chaotic for any number $\lambda \in (0, 1]$ and which additionally satisfies (1.1).

(b) For each number $\lambda \in (0, 1]$, there exists a weighted forward shift operator $T \in L(X)$ satisfying (1.1), which is $\lambda$-distributionally chaotic and not $\lambda'$-distributionally chaotic for any $\lambda' \in (0, \lambda)$.

(c) For each of the numbers $a > 0$ and $b \in (0, 1)$, there exists a weighted forward shift operator $T \in L(X)$ satisfying (1.1), which is $(2^{an^b})$-distributionally chaotic.

Moreover, in the statements (a)–(c), the corresponding weight $(\omega_j)_{j \in \mathbb{N}}$ can be chosen to consist of sufficiently large blocks of 2’s and sufficiently large blocks of (1/2)’s. In Proposition 2.15, we reconsider the notion of (dense) Li-Yorke chaos following an idea from [10]. The main purpose of Sect. 3 is to investigate associated notions of reiterative $m_n$-distributionally irregular vectors of type $s$ and reiterative $m_n$-distributionally irregular manifolds of type $s$; in a series of results presented in Sect. 3.1, we reconsider and slightly improve [6, Proposition 7-Proposition 9, Theorem 12] and [11, Theorem 3.7, Corollary 3.12] for $m_n$-distributional chaos and $\lambda$-distributional chaos. In contrast to orbits of linear continuous operators, the situation is much more complicated with general sequences because (reiterative) distributional chaos of type $s$ and Li-Yorke chaos can occur even in finite-dimensional spaces. The main aim of Sect. 4 is to analyze dense $m_n$-distributional chaos of type $s$, dense reiterative $m_n$-distributional chaos of type $s$ and dense Li-Yorke chaos for general sequences of linear operators in Fréchet spaces; as in our recent research study [11], this section aims to show that the results about various types of dense (reiterative) distributional chaos of type $s$ and dense Li-Yorke chaos can be formulated in this general setting. We extend the well-known result by Luo and Hou [28, Theorem 3.5], examining dense $m_n$-distributionally chaotic properties of the unilateral weighted backward shift operator in $l^p(\mathbb{N})$ and $c_0(\mathbb{N})$ spaces, where $1 < p < \infty$ and the corresponding weight is given by $\omega_j := 2j/2j - 1$ for $j \in \mathbb{N}$. (In contrast to the case $p = 1$ considered in [28], we obtain completely different results in case $p > 1$.) In Sect. 4.1, we introduce continuous analogues of the lower $m_n$-densities and, after clarifying some results for the families of linear not necessarily continuous operators defined on the nonnegative real axis, we briefly explain how we

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can provide certain applications in the qualitative analysis of \( f \)-distributionally chaotic and \( \lambda \)-distributionally chaotic solutions of the abstract (fractional) partial differential equations in Fréchet spaces; here, \( f : [0, \infty) \to [1, \infty) \) is an increasing mapping satisfying \( \lim \inf_{t \to +\infty} \frac{f(t)}{t} > 0 \). Section 5 is reserved for giving final observations and open problems that we have not been able to solve (see also Problems 2.17 and 2.18 proposed earlier, in Sect. 2.1; the analysis of reiterative \( m_n \)-distributional chaos of type \( s \) for composition operators is not carried out here). Before explaining the notation used in the paper, the author would like to express his sincere gratitude to Prof. A. Bonilla, J. A. Conejero, M. Murillo-Arcila and X. Wu for many stimulating discussions during the preparation of manuscript.

Henceforth, we assume that \( X \) is a Fréchet space over the field \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \) and that the topology of \( X \) is induced by the fundamental system \( (p_n)_{n \in \mathbb{N}} \) of increasing seminorms. The translation invariant metric \( d : X \times X \to [0, \infty) \), defined by

\[
d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}, \quad x, y \in X,
\]

enjoys the following properties: \( d(x + u, y + v) \leq d(x, y) + d(u, v) \), \( x, y, u, v \in X \); \( d(cx, cy) \leq (|c| + 1)d(x, y) \), \( c \in \mathbb{K}, x, y \in X \), and \( d(\alpha x, \beta x) \geq \frac{|\alpha - \beta|}{1 + |\alpha - \beta|} d(0, x), \quad x \in X, \quad \alpha, \beta \in \mathbb{K} \). By \( Y \), we denote possibly another Fréchet space over the same field of scalars as \( X \); the topology of \( Y \) is induced by the fundamental system \( (p_n^Y)_{n \in \mathbb{N}} \) of increasing seminorms. Define the translation invariant metric \( d_Y : Y \times Y \to [0, \infty) \) by replacing \( p_n(\cdot) \) with \( p_n^Y(\cdot) \) in (1.2). If \( (X, \| \cdot \|) \) or \( (Y, \| \cdot \|_Y) \) is a Banach space, then we assume that the distance of two elements \( x, y \in X \), \( y \in Y \) is given by \( d(x, y) := \|x - y\| \) \( (d_Y(x, y) := \|x - y\|_Y) \).

Keeping in mind this terminological change, our structural results clarified in Fréchet spaces continue to hold in the case that \( X \) or \( Y \) is a Banach space. If \( A \subseteq \mathbb{N} \), we denote its complement by \( A^c \); for any \( s \in \mathbb{R} \), we set \([s] := \sup\{l \in \mathbb{Z} : s \leq l\}\) and \([s] := \inf\{l \in \mathbb{Z} : s \leq l\}\). Let us recall that an infinite subset \( A \) of \( \mathbb{N} \) is called syndetic iff its difference set, defined as usually, is bounded from above.

For a linear operator \( A \) on \( X \), by \( D(A), R(A) \) and \( \sigma_p(A) \) we denote its domain, range and point spectrum, respectively. Suppose that \( C \in L(X) \) is injective. Set \( p_n^C(x) := p_n(C^{-1}x), n \in \mathbb{N}, x \in R(C) \). Then \( p_n^C(\cdot) \) is a seminorm on \( R(C) \) and the calibration \( (p_n^C)_{n \in \mathbb{N}} \) induces a Fréchet locally convex topology on \( R(C) \); we denote this space simply by \( [R(C)] \). Notice that \( [R(C)] \) is a Banach space provided that \( X \) is. By \( I \), we denote the identity operator on \( X \). (In this paper, we analyze only single-valued linear operators; for various extensions in multi-valued setting and related results obtained recently, see [25–27] and our forthcoming monograph [20].)

### 1.1 Lower and Upper Densities

Suppose that \( A \subseteq \mathbb{N} \). As it is well known, the lower density of \( A \), denoted by \( d(A) \), is defined by

\[
d(A) := \lim \inf_{n \to \infty} \frac{|A \cap [n, \infty)|}{n+1},
\]

where \( [n, \infty) := \{k \in \mathbb{N} : \kappa \geq n\} \) and \( \kappa \) is the upper density of \( A \), denoted by \( \kappa(A) \), is defined by

\[
\kappa(A) := \lim \sup_{n \to \infty} \frac{|A \cap [n, \infty)|}{n+1}.
\]
\[ d(A) := \lim_{n \to \infty} \inf \frac{|A \cap [1, n]|}{n}, \]

and the upper density of \( A \), denoted by \( \overline{d}(A) \), is defined by
\[ \overline{d}(A) := \lim_{n \to \infty} \sup \frac{|A \cap [1, n]|}{n}. \]

Further on, the lower Banach density of \( A \), denoted by \( \underline{Bd}(A) \), is defined by
\[ \underline{Bd}(A) := \lim_{s \to +\infty} \lim_{n \to \infty} \inf \frac{|A \cap [n+1, n+s]|}{s} \]

and the (upper) Banach density of \( A \), denoted by \( \overline{Bd}(A) \), is defined by
\[ \overline{Bd}(A) := \lim_{s \to +\infty} \lim_{n \to \infty} \sup \frac{|A \cap [n+1, n+s]|}{s}. \]

Then
\[ 0 \leq \underline{Bd}(A) \leq d(A) \leq \overline{d}(A) \leq \overline{Bd}(A) \leq 1, \]
\[ d(A) + \overline{d}(A) = 1 \text{ and } \underline{Bd}(A) + \overline{Bd}(A) = 1. \]

The following notions of lower and upper densities for a subset \( A \subseteq \mathbb{N} \) have been recently analyzed in [23]:

**Definition 1.1** Suppose that \((m_n)\) is an increasing sequence in \([1, \infty)\) and \(q \in [1, \infty)\). Then:

(i) The lower \((m_n)\)-density of \( A \), denoted by \( d_{m_n}(A) \), is defined by
\[ d_{m_n}(A) := \lim_{n \to \infty} \inf \frac{|A \cap [1, m_n]|}{m_n}. \]

(ii) The upper \((m_n)\)-density of \( A \), denoted by \( \overline{d}_{m_n}(A) \), is defined by
\[ \overline{d}_{m_n}(A) := \lim_{n \to \infty} \sup \frac{|A \cap [1, m_n]|}{m_n}. \]

(iii) The lower \(q\)-density of \( A \), denoted by \( d_q(A) \), is defined by
\[ d_q(A) := \lim_{n \to \infty} \inf \frac{|A \cap [1, n^q]|}{n^q}. \]

(iv) The upper \(q\)-density of \( A \), denoted by \( \overline{d}_q(A) \), is defined by
\[ \overline{d}_q(A) := \lim_{n \to \infty} \sup \frac{|A \cap [1, n^q]|}{n^q}. \]
We will use the following simple result:

**Lemma 1.2** Let $q \geq 1$ and $A = \{n_1, n_2, \ldots, n_k, \ldots\}$, where $(n_k)$ is a strictly increasing sequence of positive integers. Then $d_q (A) = \liminf_{k \to \infty} \frac{k}{n^q_k}$ and $d_q (A) > 0$ iff there exists a finite constant $L > 0$ such that $n_k \leq L k^q$, $k \in \mathbb{N}$.

In our further work, the following notion from [23] will be crucially important:

**Definition 1.3** Let $(m_n)$ be an increasing sequence in $[1, \infty)$, $q \in [1, \infty)$ and $A \subseteq \mathbb{N}$. Then we define:

(i) The lower $l; (m_n)$-Banach density of $A$, denoted shortly by $Bd_{l;m_n} (A)$, by

$$Bd_{l;m_n} (A) := \liminf_{s \to +\infty} \liminf_{n \to \infty} \frac{|A \cap [n + 1, n + m_s]|}{s}.$$ 

(ii) The lower $u; (m_n)$-Banach density of $A$, denoted shortly by $Bd_{u;m_n} (A)$, by

$$Bd_{u;m_n} (A) := \limsup_{s \to +\infty} \liminf_{n \to \infty} \frac{|A \cap [n + 1, n + m_s]|}{s}.$$ 

(iii) The lower $l; q$-Banach density of $A$, denoted shortly by $Bd_{l;q} (A)$, by

$$Bd_{l;q} (A) := \liminf_{s \to +\infty} \liminf_{n \to \infty} \frac{|A \cap [n + 1, n + s^q]|}{s}.$$ 

(iv) The lower $u; q$-Banach density of $A$, denoted shortly by $Bd_{u;q} (A)$, by

$$Bd_{u;q} (A) := \limsup_{s \to +\infty} \liminf_{n \to \infty} \frac{|A \cap [n + 1, n + s^q]|}{s}.$$ 

The above notion is not completely explored even in the case that $m_n = n^q$ for some $q > 1$. For example, we have operated with $\liminf_{n \to \infty}$ in all the above expressions but not with $\inf_{n \in \mathbb{N}}$, which will cause some troubles in the proof of Theorem 4.10 below. (These notions are no longer equivalent in general case $m_n \neq n$.) In order for the proof of Theorem 4.16 to work, we need to slightly modify the notion introduced in [23] by operating with $\limsup_{n \to \infty}$ in place of $\limsup_{n \to \infty}$:

**Definition 1.4** Let $(m_n)$ be an increasing sequence in $[1, \infty)$, $q \in [1, \infty)$ and $A \subseteq \mathbb{N}$. Then we define:

(i) The (upper) $l; (m_n)$-Banach density of $A$, denoted shortly by $Bd_{l;m_n} (A)$, by

$$Bd_{l;m_n} (A) := \liminf_{s \to +\infty} \sup_{n \in \mathbb{N}} \frac{|A \cap [n + 1, n + m_s]|}{s}.$$ 

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(ii) The (upper) $u: (m_n)$-Banach density of $A$, denoted shortly by $\overline{Bd}_{u,m_n}(A)$, by
\[
\overline{Bd}_{u,m_n}(A) := \limsup_{s \to +\infty} \sup_{n \in \mathbb{N}} \left| A \cap [n + 1, n + m_n] \right| / s.
\]

(iii) The $l : q$-Banach density of $A$, denoted shortly by $\overline{Bd}_{l,q}(A)$, by
\[
\overline{Bd}_{l,q}(A) := \liminf_{s \to +\infty} \inf_{n \in \mathbb{N}} \left| A \cap [n + 1, n + s^q] \right| / s.
\]

(iv) The $u : q$-Banach density of $A$, denoted shortly by $\overline{Bd}_{u,q}(A)$, as follows
\[
\overline{Bd}_{u,q}(A) := \limsup_{s \to +\infty} \sup_{n \in \mathbb{N}} \left| A \cap [n + 1, n + s^q] \right| / s.
\]

The analysis of the above densities is completely without scope of this paper, and we only want to mention that $\overline{Bd}_{l,q}(A)$ can be strictly greater than the quantity
\[
\overline{Bd}_{l,q}(A) := \liminf_{s \to +\infty} \limsup_{n \to +\infty} \frac{|A \cap [n+1,n+s^q]|}{s},
\]
analyzed in [23], provided that $q > 1$. For example, if $A := \{n^2 : n \in \mathbb{N}\}$ and $q = 2$, then it can be easily seen that $\overline{Bd}_{l,q}(A) \geq 1$ as well that $\overline{Bd}_{l,q}(A) = 0$ since for each $q > 0$ and $s \in \mathbb{N}$ one has $\sup_{n \to +\infty} \frac{|A \cap [n+1,n+s^q]|}{s} \leq 1/s$.

We will use the following simple lemma, as well (cf. [23,27]):

**Lemma 1.5** Suppose that $A \subseteq \mathbb{N}$.

(i) Let $\liminf_{n \to +\infty} m_n / n > 0$. Then $\overline{Bd}_{l,m_n}(A) = 0$ iff $\overline{Bd}_{u,m_n}(A) = 0$ iff $A$ is finite or $A$ is infinite non-syndetic.

(ii) Let $\liminf_{n \to +\infty} m_n / n > 0$. Then $d_{m_n}(A) > 0$ provided that $A$ is syndetic.

2 Reiterative $m_n$-Distributional Chaos of Type $s$

Denote by $R$ the class consisting of all increasing sequences $(m_n)$ of positive reals satisfying $\liminf_{n \to +\infty} m_n / n > 0$, i.e., there exists a finite constant $L > 0$ such that $n \leq Lm_n$, $n \in \mathbb{N}$. If this is the case, then for each positive constant $\alpha > 0$ we have that $(\alpha m_n) \in R$. Unless stated otherwise, we assume that $(m_n) \in R$ henceforth.

In this section, it is assumed that, for every $j \in \mathbb{N}$, $T_j : D(T_j) \subseteq X \to Y$ is a linear operator, $T : D(T) \subseteq X \to X$ is a linear operator and $\tilde{X}$ is a non-empty subset of $X$. Let a number $\delta > 0$ and two elements $x, y \in \bigcap_{j \in \mathbb{N}} D(T_j) \cap \tilde{X}$ be given. We set
\[
F_{x,y,m_n}(\delta) := d_{m_n} \left( \left\{ j \in \mathbb{N} : d_Y(T_j x, T_j y) < \delta \right\} \right),
\]
\[
G_{x,y,m_n}(\delta) := d_{m_n} \left( \left\{ j \in \mathbb{N} : d_Y(T_j x, T_j y) \geq \delta \right\} \right).
\]
\[ H_{x,y,m_n}(\delta) := \bar{d}_{m_n}\left(\{ j \in \mathbb{N} : d_y(T_jx, T_jy) < \delta \}\right), \]

and

\[ I_{x,y,m_n}(\delta) := \bar{d}_{m_n}\left(\{ j \in \mathbb{N} : d_y(T_jx, T_jy) \geq \delta \}\right). \]

If a pair \((x, y)\) satisfies

\begin{enumerate}[(DC1)]
  \item \(G_{x,y,m_n} \equiv 0\) and \(F_{x,y,m_n}(\sigma) = 0\) for some \(\sigma > 0\), or
  \item \(G_{x,y,m_n} \equiv 0\) and \(I_{x,y,m_n}(\sigma) > 0\) for some \(\sigma > 0\), or
  \item \(G_{x,y,m_n} \equiv 0\) and \(I_{x,y,m_n}(\sigma) > 0\) for some \(\sigma > 0\), or
\end{enumerate}

(DC3) there exist real numbers \(a > 0\) and \(b > 0\) and \(c > 0\) such that \(F_{x,y,m_n}(\delta) < c < H_{x,y,m_n}(\delta)\) for all \(0 < \delta < b\), or

(DC3) there exist real numbers \(a > 0\), \(b > 0\) and \(c > 0\) such that \(F_{x,y,m_n}(\delta) < c < H_{x,y,m_n}(\delta)\) for all \(0 < \delta < b\), then \((x, y)\) is called a \(\tilde{X}_{m_n}\)-distributionally chaotic pair of type \(s \in \{1, 2, 2\frac{1}{2}, 3\}\) for \((T_j)_{j \in \mathbb{N}}\). The sequence \((T_j)_{j \in \mathbb{N}}\) is said to be \(\tilde{X}_{m_n}\)-distributionally chaotic of type \(s\) (\(m_n\)-distributionally chaotic of type \(s\), if \(\tilde{X} = X\) if there exists an uncountable set \(S \subseteq X\) such that every pair \((x, y)\) of distinct points in \(S\) is \(\tilde{X}_{m_n}\)-distributionally chaotic pair of type \(s\) for \((T_j)_{j \in \mathbb{N}}\); the sequence \((T_j)_{j \in \mathbb{N}}\) is said to be \(\tilde{X}_{m_n}\)-distributionally chaotic (\(m_n\)-distributionally chaotic, if \(\tilde{X} = X\) if there exist an uncountable set \(S \subseteq X\) and a number \(\sigma > 0\) such that for every pair \((x, y)\) of distinct points in \(S\) we have \(G_{x,y,m_n} \equiv 0\) and \(F_{x,y,m_n}(\sigma) = 0\). Furthermore, if \(S\) can be chosen to be dense in \(\tilde{X}\), then we say that \((T_j)_{j \in \mathbb{N}}\) is densely \(\tilde{X}_{m_n}\)-distributionally chaotic of type \(s\) (densely \(m_n\)-distributionally chaotic of type \(s\), if \(\tilde{X} = X\); densely \(\tilde{X}_{m_n}\)-distributionally chaotic (densely \(m_n\)-distributionally chaotic, if \(\tilde{X} = X\).

A linear operator \(T : D(T) \subseteq X \rightarrow X\) is said to be (densely) \(\tilde{X}_{m_n}\)-distributionally chaotic (of type \(s\)) (densely) \(m_n\)-distributionally chaotic (of type \(s\), if \(\tilde{X} = X\) if the sequence \((T_j \equiv T^j)_{j \in \mathbb{N}}\) is. In this case, \(S\) is called a \(\tilde{X}_{m_n}\)-distributionally scrambled set (of type \(s\)) for the sequence \((T_j)_{j \in \mathbb{N}}\) (the operator \(T\).

As mentioned on [7, p. 798], the notion of (dense) distributional chaos of type 1 and the notion of (dense) distributional chaos coincide for operators on Fréchet spaces; after establishing Theorem 3.12, we will see that the same statement holds for the notion of (dense) \(m_n\)-distributional chaos of type 1 and the notion of (dense) \(m_n\)-distributional chaos. On the other hand, the situation is completely different for general sequences of linear continuous operators:

**Example 2.1** It is clear that there exist two infinite sets \(A, B \subseteq \mathbb{N}\) such that \(\bar{d}(A) = \bar{d}(B) = 1\) and \(\mathbb{N} = A \cup B\). Set \(X := \mathbb{K}, T_j := 2I (j \in A)\) and \(T_j := 0 (j \in B)\). Using the fact that the set \(S - S\) cannot be bounded away from zero if \(S\) is an uncountable subset of \(\mathbb{K}\), we can simply show that the sequence \((T_j)\) is distributionally chaotic of type 1 and not distributionally chaotic.

Before proceeding further, we would like to note that the \(\lambda\)-distributional chaos of sequence \((T_j)_{j \in \mathbb{N}}\) for some \(\lambda \in (0, 1)\) implies \(\lambda'\)-distributional chaos of \((T_j)_{j \in \mathbb{N}}\) for all numbers \(\lambda' \in [\lambda, 1)\). The converse statement is not true in general, as we will see later.
Next, for a given number \( \delta > 0 \), we set
\[
BF_{x,y,m_n}(\delta) := Bd_{I,m_n}\left( \{ j \in \mathbb{N} : d_Y(T_j x, T_j y) < \delta \} \right),
\]
\[
BG_{x,y,m_n}(\delta) := Bd_{I,m_n}\left( \{ j \in \mathbb{N} : d_Y(T_j x, T_j y) \geq \delta \} \right),
\]
and
\[
BI_{x,y,m_n}(\delta) := \overline{Bd_{I,m_n}}\left( \{ j \in \mathbb{N} : d_Y(T_j x, T_j y) \geq \delta \} \right).
\]

**Definition 2.2** We say that the sequence \( (T_j)_{j \in \mathbb{N}} \) is \( \tilde{T}_{n,m} \)-reiteratively distributionally chaotic of type 1 iff there exists an uncountable set \( S \subseteq \bigcap_{j \in \mathbb{N}} D(T_j) \bigcap \tilde{X} \) such that for every pair \((x, y) \in S \times S\) of distinct points \( BF_{x,y,m_n}(\sigma) = 0 \) for some \( \sigma > 0 \) and there exist \( c \in (0, \liminf_{n \to \infty} \frac{m_n}{n}) \) and \( r > 0 \) such that \( G_{x,y,m_n}(\delta) \leq c \) for all \( 0 < \delta < r \). If \( G_{x,y,m_n} \equiv 0 \), we say that \( (T_j)_{j \in \mathbb{N}} \) is \( \tilde{T}_{n,m} \)-reiteratively distributionally chaotic of type 1+.

We say that \( (T_j)_{j \in \mathbb{N}} \) is \( \tilde{T}_{n,m} \)-reiteratively distributionally chaotic of type 2 iff there exists an uncountable set \( S \subseteq \bigcap_{j \in \mathbb{N}} D(T_j) \bigcap \tilde{X} \) such that for every pair \((x, y) \in S \times S\) of distinct points we have \( BG_{x,y,m_n} \equiv 0 \) and \( I_{x,y,m_n}(\sigma) > 0 \) for some \( \sigma > 0 \). Finally, we say that \( (T_j)_{j \in \mathbb{N}} \) is \( \tilde{T}_{n,m} \)-reiteratively distributionally chaotic of type 2+ if there exists an uncountable set \( S \subseteq \bigcap_{j \in \mathbb{N}} D(T_j) \bigcap \tilde{X} \) such that for every pair \((x, y) \in S \times S\) of distinct points we have \( (BG_{x,y,m_n} \equiv 0) \) \( G_{x,y,m_n} \equiv 0 \) and \( BI_{x,y,m_n}(\sigma) > 0 \) for some \( \sigma > 0 \).

In the case that the number \( \sigma > 0 \) does not depend on the choice of pair \((x, y) \in S \times S\), then we say that \( (T_j)_{j \in \mathbb{N}} \) is \( \tilde{T}_{n,m} \)-reiteratively distributionally chaotic of type 2+ or \( \tilde{T}_{n,m} \)-reiteratively distributionally chaotic of type 2BD+.

A series of elaborate and very plain counterexamples shows that the conclusions established in our previous research study [10] are no longer true for general sequences of linear continuous operators, even on finite-dimensional spaces (for more details about this problematic, see [24]). This also holds for \( \tilde{T}_{n,m} \)-reiterative distributional chaos of types 0, 1+, and 2−, which are introduced as follows:

**Definition 2.3** We say that the sequence \( (T_j)_{j \in \mathbb{N}} \) is \( \tilde{T}_{n,m} \)-reiteratively distributionally chaotic of type 0, resp. 1+ [2−], iff there exist an uncountable set \( S \subseteq \bigcap_{j \in \mathbb{N}} D(T_j) \bigcap \tilde{X} \) and \( \sigma > 0 \) such that for each pair \( x, y \in S \) of distinct points we have \( BF_{x,y,m_n}(\sigma) = 0 \) and \( BG_{x,y,m_n} \equiv 0 \), resp. \( BF_{x,y,m_n}(\sigma) = 0 \) and \( G_{x,y,m_n} \equiv 0 \) \( [F_{x,y,m_n}(\sigma) = 0] \) and \( BG_{x,y,m_n} \equiv 0 \).

The notions of reiteratively \( \tilde{T}_{n,m} \)-distributionally scrambled set of type \( s \) and dense \( \tilde{T}_{n,m} \)-reiterative distributional chaos of type \( s \), where \( s \in \{ 1, 1^+, 1^-, 2, 2^+, 2BD, 2BD+, 0, 2^- \} \) are introduced as above.

It is worth noting that the notions introduced in Definitions 2.2 and 2.3 extend the notions introduced in [10, Definition 1.3], where we have only analyzed the uniform...
reiterative distributional chaos of type \( s \); for example, in [10], we have considered only the notion of reiterative distributional chaos of type \( 2+ \) but not of type \( 2 \). Some implications trivially can be clarified, like reiterative distributional chaos of type \( 1^+ \) implies reiterative distributional chaos of type \( 1+ \), which further implies reiterative distributional chaos of type \( 1 \).

We will accept the following agreements. If \( \tilde{X} = X \) or \( m_n \equiv n \), then we remove the prefixes “\( \tilde{X} - \)” and “\( m_n - \)” from the terms and notions. For example, dense reiterative distributional chaos of type \( 3 \) is dense reiterative \( X_n \)-distributional chaos of type \( 3 \). If \( m_n \equiv n^{1/\lambda} \) for some \( \lambda \in (0, 1] \), then, as a special case of the above definitions, we obtain the notions of (dense, reiterative) \( X_\lambda \)-distributional chaos of type \( s \), (dense, reiterative) \( \lambda \)-distributional chaos of type \( s \), (reiterative) \( X_\lambda \)-scrambled sets of type \( s \) and (reiterative) \( \lambda \)-scrambled sets of type \( s \) for operators and their sequences; for \( \lambda = 1 \), we obtain the notions of (dense, reiterative) \( X \)-distributional chaos of type \( s \), (dense, reiterative) distributional chaos of type \( s \), (reiterative) \( X \)-scrambled sets of type \( s \) and (reiterative) scrambled sets of type \( s \) for operators and their sequences, and so on and so forth.

**Remark 2.4**

(i) It is clear that, if \( (m'_n) \in \mathbb{R} \) and \( m_n \leq m'_n, \ n \in \mathbb{N} \), then \( (m'_n) \)-distributional chaos/(\( m'_n \))-distributional chaos of type \( 1 \) or \( 2 \) implies \( (m_n) \)-distributional chaos/(\( m_n \))-distributional chaos of the same type, while reiterative \( (m'_n) \)-distributional chaos of type \( 1, 1^+, 1+ \), \( 0 \) or \( 2- \) implies reiterative \( (m_n) \)-distributional chaos of the same type.

(ii) If \( (m_n) \in \mathbb{R} \), then \( (m_n) \)-distributional chaos/(\( m_n \))-distributional chaos of type \( 1 \) or \( 2 \) implies distributional chaos/distributional chaos of the same type, while reiterative \( (m_n) \)-distributional chaos of type \( 1, 1^+, 1+ \), \( 0 \) or \( 2- \) implies reiterative distributional chaos of the same type. In particular, any \( m_n \)-distributionally chaotic sequence is distributionally chaotic.

(iii) It is worth noting that the distributional chaos of type \( s \in \{1, 2, 3\} \) for backward shift operators in Köthe sequence spaces has been analyzed for the first time by Wu et al. [33], under certain assumptions other from ours.

Finally, we will use the notion of Li-Yorke chaos below. Li-Yorke chaos in Fréchet spaces has been recently investigated by Bernardes Jr et al. [8] and Kostić [22] (see also [4, 5, 9, 33–35]):

**Definition 2.5** We say that the sequence \( (T_j)_{j \in \mathbb{N}} \) is \( \tilde{X} \)-Li-Yorke chaotic iff there exists an uncountable set \( S \subseteq \bigcap_{j \in \mathbb{N}} D(T_j) \bigcap \tilde{X} \) such that for every pair \( (x, y) \in S \times S \) of distinct points, we have

\[
\liminf_{j \to \infty} d_Y(T_jx, T_jy) = 0 \quad \text{and} \quad \limsup_{j \to \infty} d_Y(T_jx, T_jy) > 0.
\]

In this case, \( S \) is called a \( \tilde{X} \)-scrambled set for \( (T_j)_{j \in \mathbb{N}} \) and each such pair \( (x, y) \) is called a \( \tilde{X} \)-Li-Yorke pair for \( (T_j)_{j \in \mathbb{N}} \). We say that \( (T_j)_{j \in \mathbb{N}} \) is densely \( \tilde{X} \)-Li-Yorke chaotic iff \( S \) can be chosen to be dense in \( \tilde{X} \).

We refer the reader to [8, 22] for the notion of Li-Yorke (semi-)irregular vectors. Any notion introduced above is accepted also for a single linear continuous operator.
Due to Proposition 1.2, if \( \lambda \in (0, 1] \) and \( A = \{n_1, n_2, \ldots, n_k, \ldots\} \), where \( (n_k) \) is a strictly increasing sequence of positive integers, then \( d_{1/\lambda}(A) = 0 \) iff for any finite constant \( L > 0 \) there exists \( k \in \mathbb{N} \) such that \( n_k > Lk^{1/\lambda} \). Therefore, it is very simple to construct two disjoint subsets \( A \) and \( B \) of \( \mathbb{N} \) such that \( \mathbb{N} = A \cup B \) and \( d_{1/\lambda}(A) = d_{1/\lambda}(B) = 0 \) for each number \( \lambda \in (0, 1] \); for example, set \( a_n := \sum_{i=1}^n 2^2 \) \((n \in \mathbb{N})\), \( A := \bigcup_{n \in \mathbb{N}} \{a_n, a_n+1\} \) and \( B := \mathbb{N} \setminus A \). After that, set \( X := \mathbb{K} \), \( T_j := j I \) \((j \in \mathbb{A})\) and \( T_j := 0 \) \((j \in B)\). Then it can be simply checked that the sequence \((T_j)_{j \in \mathbb{N}}\) is densely \( \lambda \)-distributionally chaotic for each number \( \lambda \in (0, 1] \), and that the corresponding scrambled set \( S \) can be chosen to be the whole space \( X \).

### 2.1 A Few Remarks and Open Problems for Orbits of Single Operators

We start this subsection with the observation that the property (DC2) with \( m_n \equiv n \) for \((T_j)_{j \in \mathbb{N}}\) is not equivalent to the mean Li-Yorke chaos for \((T_j)_{j \in \mathbb{N}}\), as for continuous mappings on compact metric spaces (cf. [7, 9]). Counterexamples exist even for orbits of weighted forward shift operators on \( l^p \) spaces, where \( 1 \leq p < \infty \); for more details about the mean Li-Yorke chaos, the reader may consult [9, 13] and references cited therein.

Further on, let us recall that there is no Li-Yorke chaotic operator on a Fréchet space that is compact [8], so that the situation appearing in Example 2.6 cannot occur for orbits on finite-dimensional spaces. Concerning infinite-dimensional spaces, the situation is completely different: There exists a continuous linear operator \( T \) on \( c_0(\mathbb{N}) \) or \( l^p(\mathbb{N}) \), where \( 1 \leq p < \infty \), which is \( \lambda \)-distributionally chaotic for any number \( \lambda \in (0, 1] \), not hypercyclic and which additionally satisfies some other requirements. To verify this, we will prove the following proper extension of [7, Theorem 25]; cf. also [5, 24, Remark 21] and Theorem 2.10:

**Theorem 2.7** Suppose that \( X := c_0(\mathbb{N}) \) or \( X := l^p(\mathbb{N}) \) for some \( p \in [1, \infty) \). Then there exists a continuous linear operator \( T \) on \( X \) which is \( \lambda \)-distributionally chaotic for any number \( \lambda \in (0, 1] \) and which additionally satisfies (1.1) as well as \( \lim_{j \to \infty} T^j x = 0 \) for some \( x \in X \) iff \( x = 0 \).

**Proof** Without loss of generality, we may assume that \( X = l^2(\mathbb{N}) = l^2 \). Consider a weighted forward shift \( T = F_\omega : l^2 \to l^2 \), defined by \( F_\omega(x_1, x_2, \ldots) \mapsto (0, \omega_1 x_1, \omega_2 x_2, \ldots) \), where the sequence of weights \( \omega = (\omega_k)_{k \in \mathbb{N}} \) consists of sufficiently large blocks of 2’s of lengths \( b_1, b_2, \ldots, \) and sufficiently large blocks of \((1/2)’s \) of lengths \( a_1, a_2, \ldots, \). More precisely, let \( b_n := 2^{2(n-1)^2} \) and \( a_n := 2^{2(n)^2} \) \((n \in \mathbb{N})\). Let a number \( \lambda \in (0, 1) \) be fixed. To see that \( T \) is \( \lambda \)-distributionally chaotic, it suffices to show that for each \( n \in \mathbb{N} \) and we have \( d_{1/\lambda}(\{j \in \mathbb{N} : \|T^j e_1\| < 2^n\}) = 0 \) and \( d_{1/\lambda}(\{j \in \mathbb{N} : \|T^j e_1\| > 2^{-n}\}) = 0 \), where \( e_1 = (1, 0, 0, \ldots) \). Toward this end,
observe that there exist a finite subset \( A \subseteq \mathbb{N} \) and a number \( n_0 \in \mathbb{N} \) such that \( \{ j \in \mathbb{N} : \| T_j e_1 \| \leq 2^n \} \subseteq A \bigcup \{ 2(k_1 + 1) + 2k_2 + \cdots + 2k_s - n, 2(a_1 + a_2 + \cdots + a_s) + n \} = B \). Let \( L > 0 \) be arbitrarily chosen. Then there exists sufficiently large \( s \in \mathbb{N} \) such that, with \( j = (2n + 1)(s - 1) + \sum_{w=1}^{s-1}(s - w)(a_w - b_w) \) and \( n_j = 2 \sum_{w=1}^{s} b_w - n \), we have \( n_j > \lambda^{1/L} \). Therefore, \( d_{1/L}(B) = 0 \) due to Proposition 1.2. The second equation can be proved analogously, so that \( T \) is \( \lambda \)-distributionally chaotic. Furthermore, it is clear that \( T \) cannot be hypercyclic as well as that the condition \( \lim_{j \to \infty} T_j x = 0 \) for some \( x \in l^2 \) implies \( x = 0 \). To complete the proof, it suffices to show that (1.1) holds with \( x = e_1 \). To estimate the term \( \frac{1}{N} \sum_{j=1}^{N} \| T_j e_1 \| \), we will consider separately two possible cases:

1. There exists \( n \in \mathbb{N} \) such that \( N \in [b_n + \sum_{s=1}^{n-1} (a_s + b_s), \sum_{s=1}^{n} (a_s + b_s)] \). Then we have

\[
\frac{1}{N} \sum_{j=1}^{N} \| T_j e_1 \| = \frac{1}{N} \sum_{j=1}^{N} \omega_1 \omega_2 \cdots \omega_j \geq \frac{1}{2na_n} \sum_{j=0}^{N} \omega_1 \omega_2 \cdots \omega_j \\
\geq \frac{1}{2na_n} \sum_{j=0}^{2^n - 1} 2^j = \frac{2^{b_n - a_n - 1} - 1}{2na_n} \to +\infty, \quad n \to +\infty.
\]

2. There exists \( n \in \mathbb{N} \) such that \( N \in [\sum_{s=1}^{n-1} (a_s + b_s), b_n + \sum_{s=1}^{n-1} (a_s + b_s)] \). Then we have

\[
\frac{1}{N} \sum_{j=1}^{N} \| T_j e_1 \| = \frac{1}{N} \sum_{j=1}^{N} \omega_1 \omega_2 \cdots \omega_j \geq \frac{1}{2nb_n} \sum_{j=0}^{N} \omega_1 \omega_2 \cdots \omega_j \\
\geq \frac{1}{2nb_n} \sum_{j=0}^{2^{b_n - a_n - 2} - 1} 2^j = \frac{2^{a_n - 1} - 1}{2nb_n} \to +\infty, \quad n \to +\infty.
\]

The proof of the theorem is thereby complete.

\[\square\]

**Remark 2.8** Our construction is much more easier and transparent than the construction employed in [7, Theorem 25] for case \( \lambda = 1 \). Arguing as in [7, Remark 26] and the proof of Theorem 2.7, we can prove the existence of an invertible continuous linear operator \( T \) on \( X := c_0(\mathbb{Z}) \) or \( X := l^p(\mathbb{Z}) \) for some \( p \in [1, \infty) \), satisfying the all required properties from Theorem 2.7.

Let \( \lambda \in (0, 1] \). Recall that the class of dynamical systems on the plane \( \mathbb{R}^2 \), named winding systems, have been introduced in [36, Section 5] in order to show that the \( \lambda \)-distributional chaos of an operator \( T \) on a compact metric space does not imply \( \lambda' \)-distributional chaos of \( T \) for any number \( \lambda' \in (0, \lambda) \). Before solving in the affirmative the corresponding problem for orbits of single-valued linear operators in Banach spaces satisfying equation (1.1), we would like to show (cf. also Corollary 4.5 and Example 4.6 below) that there exist a continuous linear operator \( T \) on \( X := c_0(\mathbb{N}) \) and a sequence...
\((P_j(z))_{j \in \mathbb{N}}\) of nonzero real polynomials such that the sequence \((T_j = P_j(T))_{j \in \mathbb{N}}\) in \(L(X)\) satisfies that there exists a dense linear submanifold \(X_0\) of \(X\) with \(\lim_{j \to \infty} T_j x = 0\), \(x \in X_0\), as well as that \((T_j)_{j \in \mathbb{N}}\) is \(\lambda\)-distributionally chaotic and not \(\lambda'\)-distributional chaotic for any \(\lambda' \in (0, \lambda)\):

**Example 2.9** Suppose that \(0 < \lambda' < \lambda\), \(X := c_0(\mathbb{N})\), \(a_n := \lfloor n^{2/\lambda} \ln n \rfloor\) and \(b_n := \lfloor n^{2/\lambda} \ln n \rfloor + n (n \in \mathbb{N})\). Set \(S := \bigcup_{n \geq 1} [b_{n-1}, a_n)\). Then it can be easily seen that

\[
\inf_{n \to \infty} \frac{1 + 2 + \cdots + n}{a_\lambda^n} = 0
\]

and

\[
\inf_{n \to +\infty} \frac{1 + 2 + \cdots + (n - 1)}{b_{\lambda'}^n} = +\infty.
\]

Set \(T(x_1, x_2, x_3, \ldots) := (2x_2, 2x_3, 2x_4, \ldots)\) for all \((x_n)_{n \in \mathbb{N}}\), as well as \(T_j := T^j\), if \(j \in S\), and \(T_j := 2^{-j} T\), if \(j \in S^c\). Since the finite linear combinations of the vectors \(e_n\) from the standard basis of \(c_0\) form a dense submanifold satisfying the prescribed assumption \(\lim_{j \to \infty} T_j x = 0\), \(x \in X_0\) and the vector \((\frac{1}{n})_{n \in \mathbb{N}}\) is \(\lambda\)-distributionally unbounded for \((T_j)_{j \in \mathbb{N}}\), Corollary 4.2 implies that \((T_j)_{j \in \mathbb{N}}\) is densely \(\lambda\)-distributionally chaotic. On the other hand, \((T_j)_{j \in \mathbb{N}}\) cannot be \(\lambda'\)-distributionally chaotic since for each \(x \in X\) and \(\sigma > 0\) we have the existence of a finite set \(D \subseteq \mathbb{N}\) such that \(S^c \setminus D \subseteq \{ j \in \mathbb{N} : \|T_j x\| < \sigma \}\), and therefore, \(\inf_{n \to \infty} (\{ j \in \mathbb{N} : \|T_j x\| < \sigma \}) = +\infty\).

We continue by stating the following existence-type result closely related with Theorem 2.7 and Example 2.9:

**Theorem 2.10** Suppose that \(X := c_0(\mathbb{N})\) or \(X := l^p(\mathbb{N})\) for some \(p \in [1, \infty)\). Then for each number \(\lambda, \lambda' \in (0, 1)\) there exists a continuous linear operator \(T\) on \(X\) satisfying (1.1), \(\lim_{j \to \infty} T_j x = 0\) for some \(x \in X\) iff \(x = 0\), which is \(\lambda\)-distributionally chaotic and not \(\lambda'\)-distributionally chaotic for any \(\lambda' \in (0, \lambda)\).

**Proof** Let \(\lambda, \lambda' \in (0, 1)\) and \(\lambda' \in (0, \lambda)\). As above, we may assume without loss of generality that \(X = l^2\). The construction of a weighted forward shift \(T \equiv F_\omega : l^2 \to l^2\), defined by \(F_\omega(x_1, x_2, \ldots) \mapsto (0, \omega_1 x_1, \omega_2 x_2, \ldots)\), where the sequence of weights \(\omega = (\omega_k)_{k \in \mathbb{N}}\) consists of sufficiently large blocks of 2’s of lengths \(b_1, b_2, \ldots\), and sufficiently large blocks of \((1/2)\)'s of lengths \(a_1, a_2, \ldots\) now goes as follows. Set

\[
a_0 := 0, \quad a_n := \frac{1}{2} \left[ \lfloor (n + 1)^{4/\lambda} \ln(n + 1) \rfloor - \lfloor n^{4/\lambda} \ln n \rfloor \right], \quad n \in \mathbb{N}
\]

and

\[
b_n := a_{n-1} + \frac{1}{2} (3n^2 - 3n + 1), \quad n \in \mathbb{N}.
\]
Then

\[ 2(b_1 + \cdots + b_n) = \lfloor n^{4/\lambda} \ln n \rfloor + n^3, \quad n \in \mathbb{N} \]

and

\[ 2(a_1 + \cdots + a_n) = \lfloor (n + 1)^{4/\lambda} \ln (n + 1) \rfloor, \quad n \in \mathbb{N}. \]

Arguing as in the previous theorem, we get that the existence of an integer \( n \in \mathbb{N} \) such that \( N \in [b_n + \sum_{s=1}^{n-1} (a_s + b_s), \sum_{s=1}^{n} (a_s + b_s)] \) implies

\[ \frac{1}{N} \sum_{j=1}^{N} \| T^j e_1 \| \geq \frac{2^{b_n-a_n-1} - 1}{2na_n}, \]

as well as that the existence of an integer \( n \in \mathbb{N} \) such that \( N \in [\sum_{s=1}^{n-1} (a_s + b_s), b_n + \sum_{s=1}^{n-1} (a_s + b_s)] \) implies

\[ \frac{1}{N} \sum_{j=1}^{N} \| T^j e_1 \| \geq \frac{1}{2nb_n} \sum_{j=0}^{b_n-1-a_n-2} 2^j \geq \frac{2^{b_n-1-a_n-2} - 1}{2nb_n}. \]

Since \( \lim_{j \to \infty} T^j x = 0 \) for some \( x \in X \) iff \( x = 0 \) and

\[ \lim_{n \to +\infty} \frac{2^{b_n-a_n-1}}{2n \min(a_n, b_n-1)} = +\infty, \]

the foregoing arguments show that it is sufficient to show that the following holds for each integer \( k \in \mathbb{Z} \):

1. \( d_{1/\lambda'}(\{ j \in \mathbb{N} : \| T^j e_1 \| > 2^k \}) = +\infty \),
2. \( d_{1/\lambda}(\{ j \in \mathbb{N} : \| T^j e_1 \| > 2^k \}) = 0 \) and \( d_{1/\lambda'}(\{ j \in \mathbb{N} : \| T^j e_1 \| < 2^k \}) = 0 \).

But, \([1.-2.]\) can be verified to be true on the basis of our consideration from Example 2.9.

\[ \square \]

**Remark 2.11** Arguing as in [7, Remark 26] and the proof of above theorem, we can prove the existence of an invertible continuous linear operator \( T \) on \( X := c_0(\mathbb{Z}) \) or \( X := l^p(\mathbb{Z}) \) for some \( p \in [1, \infty) \), satisfying the all required properties from Theorem 2.10.

Now, we state the following useful proposition:

**Proposition 2.12** Suppose that \( T \) is a Banach space, \( T \in L(X) \) and (1.1) holds. Then we have:

(i) Let \((m_n) \in \mathbb{R} \) and there exist \( n_0 \in \mathbb{N} \) such that \( m_n \geq n \| T \|^n, n \in \mathbb{N}, n \geq n_0 \). Then \( T \) cannot be \( m_n \)-distributionally chaotic (of type 1) of types 2, 2\(_{BD} \) or reiteratively \( m_n \)-distributionally chaotic of types 1+, 1\(_+\).
(ii) \( \text{Let for each } c > 0 \text{ there exist } n_0 \in \mathbb{N} \text{ such that } (m_n) \in \mathbb{R} \text{ and } m_n \geq n \|T\|^c n, \ n \in \mathbb{N}, \ n \geq n_0. \text{ Then } T \text{ cannot be reiteratively } m_n\text{-distributionally chaotic of type 1.} \)

**Proof** We will prove only (i), for \( m_n\text{-distributional chaos. Assume to the contrary that } T \text{ is } m_n\text{-distributionally chaotic and (1.1) holds. Then } \|T\| > 1 \text{ and it is not difficult to see that there exist a nonzero vector } x \in X \text{ and a strictly increasing sequence } (n_k) \text{ of positive integers satisfying that there exists } k_0 \in \mathbb{N} \text{ such that } k_0 \geq n_0 \text{ and the interval } [1, m_{n_k}] \text{ contains at least } m_{n_k} - n_k \text{ integers } j \in \mathbb{N} \text{ for which } \|T^j x\| \leq 1 (k \geq k_0). \text{ For any other integer } j \in [1, m_{n_k}], \text{ it is almost trivial to show that } \|T^j x\| \leq \|T\|^{n_k}, \text{ so that } \frac{1}{m_{n_k}} \sum_{i=1}^{m_{n_k}} \|T^i x\| \leq \frac{1}{m_{n_k}} \left[ m_{n_k} - n_k + n_k \right] \|T\|^{n_k} \leq 2, \text{ for } k \geq k_0. \text{ This clearly contradicts (1.1), so that } T \text{ is not } m_n\text{-distributionally chaotic.} \)

As already shown in Theorem 2.10, the notions of \( \lambda\)-distributional chaos and \( \lambda'\)-distributional chaos do not coincide for orbits of linear continuous operators on Banach spaces \( (\lambda, \ \lambda' \in (0, 1], \ \lambda \neq \lambda'). \) Using Proposition 2.12, it is almost trivial to show that the notions being \( \lambda\)-distributionally chaotic for each number \( \lambda \in (0, 1] \) and being \( m_n\)-distributionally chaotic for each sequence \( (m_n) \in \mathbb{R} \) do not coincide for orbits of linear continuous operators on Banach spaces, as well:

**Example 2.13** Consider the weighted forward shift \( T := F_{\omega} \colon l^2 \rightarrow l^2 \) from Theorem 2.7. Then we know that \( T \) is \( \lambda\)-distributionally chaotic for each number \( \lambda \in (0, 1] \) and (1.1) holds. Let \( (m_n) \in \mathbb{R} \) and \( m_n \geq n \|T\|^n, \ n \in \mathbb{N}. \) Due to Proposition 2.12, \( T \) cannot be \( (m_n)\)-distributionally chaotic (of type 1) of types 2, 2_{\text{Bd}} or reiteratively distributionally chaotic of types 1+, 1^+.

Keeping in mind Theorem 2.7 and Proposition 2.12, it is quite natural to ask whether there exists a weighted forward shift operator \( T = F_{\omega} \) satisfying (1.1) and the additional property that \( T \) is \( m_n\)-distributionally chaotic for some sequence \( (m_n) \in \mathbb{R} \) growing ultrapolynomially at plus infinity. As the next theorem shows, the answer is affirmative:

**Theorem 2.14** Suppose that \( X := c_0(\mathbb{N}) \) or \( X := l^p(\mathbb{N}) \) for some \( p \in [1, \infty). \) Then for each of the numbers \( a > 0 \) and \( b \in (0, 1), \) there exists a weighted forward shift operator \( T \) on \( X \) satisfying (1.1), \( \lim_{j \rightarrow \infty} T^j x = 0 \) for some \( x \in X \) iff \( x = 0, \) which is \( 2^{anb} \)-distributionally chaotic.

**Proof** Without loss of generality, we may assume that \( a = 1 \) and \( X = l^2. \) We construct a weighted forward shift \( T = F_{\omega} \colon l^2 \rightarrow l^2, \) defined by \( F_{\omega}(x_1, x_2, \ldots) \mapsto (0, \omega_1 x_1, \omega_2 x_2, \ldots), \) where the sequence of weights \( \omega = (\omega_k)_{k \in \mathbb{N}} \) consists of sufficiently large blocks of 2's of lengths \( b_1, \ b_2, \ldots, \) and sufficiently large blocks of \((1/2)'s \) of lengths \( a_1, \ a_2, \ldots \) as done below. First of all, let us recall that for each number
\( n \in \mathbb{Z} \) there exist two finite subsets \( A, A' \subseteq \mathbb{N} \) and two numbers \( n_0, n'_0 \in \mathbb{N} \) such that
\[
\{ j \in \mathbb{N} : |T^j e_1| \leq 2^n \} = A \bigcup \bigcup_{s \geq n_0} [2(b_1 + b_2 + \cdots + b_s) - n, 2(a_1 + a_2 + \cdots + a_s) + n]
\]
and
\[
\{ j \in \mathbb{N} : |T^j e_1| \geq 2^n \} = A' \cup \bigcup_{s \geq n'_0} [2(a_1 + a_2 + \cdots + a_s) + n, 2(b_1 + b_2 + \cdots + b_s + 1) - n].
\]
Keeping in mind this fact as well as the final part of proof of Theorem 2.7, the points \([1, 2]\), it suffices to construct two strictly increasing sequences \( (a_n) \) and \( (b_n) \) of natural numbers satisfying that there exists an integer \( n_1 \in \mathbb{N} \) such that the following holds:

1. \( b_n < a_n < b_{n+1} < a_{n+1} < \cdots \) for \( n \geq n_1 \);
2. \( b_{n+1} \geq 2a_n \) for \( n \geq n_1 \);
3. \( \lim_{n \to \infty} \frac{2^{b_n}}{a_n} = +\infty \);
4. \( \lim_{k \to \infty} \frac{\log_2 2^{b_{4^k + 1}}}{b_1 + \cdots + b_{4^k + 1}} = 0 \);
5. \( \lim_{k \to \infty} \frac{\log_2 2^{a_{4^k + 1}}}{b_1 + \cdots + b_{4^k + 1}} = 0 \).

We define the sequence \( (b_n) \) inductively by \( b_1 := 2 \) and \( b_{n+1} := \lfloor n^{-2} 2^{b_n/2} \rfloor \) for \( n \geq 1 \). Then (iii) holds and, since \( b \in (0, 1) \), it is checked at once that for each finite number \( \sigma > 0 \) we have

\[
\lim_{n \to \infty} \frac{n^\sigma b_n}{(\log_2 2^{b_n + 1})^{1/\sigma}} = 0. \tag{2.1}
\]

Next, we define the sequence \( (a_n) \) in the following way: If there exists \( k \in \mathbb{N} \) such that \( n = 4^k + 1 \), then we set

\[
a_n := \left\lfloor 2^{k(b_1 + \cdots + b_{4^k + 1})} - 1 \right\rfloor;
\]
otherwise, we set \( a_n := b_n + 1 \). Then

\[
\frac{b_1 + \cdots + b_{4^k + 1}}{(\log_2 2^{a_{4^k + 1}})^{1/\sigma}} = 1/k, \quad k \in \mathbb{N}
\]

and (v) trivially holds. Furthermore, the requirements (i)–(ii) can be also trivially verified due to the condition \( b \in (0, 1) \). To prove (iii), observe that there exist sufficiently large finite numbers \( k_0 \in \mathbb{N} \) and \( d > 0 \) such that for all integers \( k \geq k_0 \) one has:

\[
\frac{a_1 + \cdots + a_{4^k + k}}{(\log_2 2^{b_{4^k + 1}})^{1/\sigma}} \leq \frac{k_0 d + (4^k + k)(b_{4^k + 1})}{(\log_2 2^{b_{4^k + 1}})^{1/\sigma}}.
\]

Using this estimate, the requirement (iii) follows by applying (2.1). This completes the proof of theorem. \( \square \)

We continue by stating the following extension of [10, Proposition 1.1], where we observe that the \( (m_n) \)-reiterative distributional chaos of type 0 for orbits of linear continuous operators on Banach spaces is equivalent to the Li-Yorke chaos. Furthermore, we reexamine the same question for operators on Fréchet spaces and obtain only some partial results in this direction:

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Proposition 2.15 Suppose that $T \in L(X)$ and $(m_n) \in \mathbb{R}$. Consider the following statements:

(a) $T$ is (densely) Li-Yorke chaotic.
(b) $T$ is (densely) $m_n$-reiteratively distributionally chaotic of type 0.
(c) $T$ is (densely) reiteratively distributionally chaotic of type 0.

Then we have the following:

(i) If $X$ is a Banach space, then the statements (a), (b) and (c) are mutually equivalent.
(ii) If $X$ is a Fréchet space, then we have $(b) \Rightarrow (c) \Rightarrow (a)$. Moreover, the validity of (a) implies that there exist an uncountable set $S \subseteq X$, a positive integer $m \in \mathbb{N}$ and a strictly increasing sequence $(j_k)$ in $\mathbb{N}$ such that $\lim_{j \to \infty} p_m(T^{j}x) = +\infty$ as well as that for each number $s > 0$ there exist a finite number $F(s) > 0$ and an integer $k_0 = k_0(s) \in \mathbb{N}$ such that for each pair $y, z \in S$ of distinct points we have $BG_{y, z, m_n} \equiv 0$ and

$$\left\{ j \in \mathbb{N} : d(T^jy, T^jz) < F(s) \right\} \cap \left[ j_k - [m_s], j_k \right] = \emptyset \text{ for all } k \geq k_0. \quad (2.2)$$

**Proof** The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are trivial and for the proof of (i) we only need to show that (a) implies (b). Toward this end, let us recall that $T$ is Li-Yorke chaotic iff $T$ admits an irregular vector $x \in X$ for $T$, i.e., the vector $x \in X$ such that the sequence $(T^jx)$ is unbounded and has a subsequence converging to zero. In the Banach space setting, this means that there exist two strictly increasing sequences $(l_k)$ in $\mathbb{N}$ and $(j_k)$ in $\mathbb{N}$ such that $\lim_{k \to \infty} \|T^{l_k}x\| = 0$ and $\lim_{k \to \infty} \|T^{j_k}x\| = +\infty$. Let $s > 0$ be fixed, and let $\sigma > 0$ be arbitrarily chosen. Set $S := \text{span}\{x\}$. By definitions of $Bd_{l_k, m_n}(-)$ and $m_n$-reiterational distributional chaos of type 0, it suffices to prove that

$$\liminf_{n \to \infty} \frac{\left\{ j \in \mathbb{N} : \|T^{j}x\| > \sigma \right\} \cap [n + 1, n + m_s]}{s} = 0 \quad (2.3)$$

and

$$\liminf_{n \to \infty} \frac{\left\{ j \in \mathbb{N} : \|T^{j}x\| < \sigma \right\} \cap [n + 1, n + m_s]}{s} = 0. \quad (2.4)$$

It is clear that there exist two strictly increasing sequences of positive integers $(l'_k)$ and $(j'_k)$ with unbounded differences such that $\|T^{l'_k}x\| < \sigma (2 + \|T\|)^{-k^2 - m_s - 1}/2$ and $\|T^{j'_k}x\| > 2\sigma (2 + \|T\|)^{k^2 + m_s + 1}$ for all $k \in \mathbb{N}$. An elementary line of reasoning shows that the sets $\{ j \in \mathbb{N} : \|T^{j}x\| > \sigma \} \cap [l'_k, l'_k + [m_s]]$ and $\{ j \in \mathbb{N} : \|T^{j}x\| < \sigma \} \cap [j'_k - [m_s], j'_k]$ are empty, finishing the proofs of (2.3)–(2.4) and (i). In the Fréchet space setting, we have that there exist an integer $m \in \mathbb{N}$ and a strictly increasing sequence $(j_k)$ in $\mathbb{N}$ such that $\lim_{k \to \infty} p_m(T^{j_k}x) = +\infty$. Set, as above, $S := \text{span}\{x\}$ and fix numbers $s > 0$, $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}$ ($\alpha \neq \beta$). Due to the continuity of $T$, we can find two strictly increasing sequences $(c_j)$ and $(r_j)$ of positive integers such that $pr_j(Ty) \leq c_j pr_{j+1}(y)$ for all $y \in X$ and $j \in \mathbb{N}$. Define
\[ F(s) := \sum_{a=[m_s]+m}^{\infty} 2^{-1-r_a+1}. \]

It remains to be proved that \( BG_{\alpha x, \beta x, m_a} \equiv 0 \) and (2.2) holds with \( y = \alpha x \) and \( z = \beta x \). For the first equality, choose \( \epsilon > 0 \) arbitrarily. Suppose that \( \lim_{k \to \infty} T^{l_k} x = 0 \) and \( j \in [l_k, l_k + [m_s]] \) for some \( k \in \mathbb{N} \). It is clear that there exists an integer \( a \in \mathbb{N} \) such that \( \sum_{l \geq r_a} 2^{-l} \leq \epsilon / 2 \). Then we have

\[
d(T^j \alpha x, T^j \beta x) \leq \sum_{l=1}^{r_a} \frac{1}{2^l} \left( \frac{p_l(T^j(\alpha - \beta)x)}{1 + p_l(T^j(\alpha - \beta)x)} \right) + \sum_{l > r_a} \frac{1}{2^l} \left( \frac{p_l(T^j(\alpha - \beta)x)}{1 + p_l(T^j(\alpha - \beta)x)} \right)
\]

so that there exists a sufficiently large number \( k_0' = k_0'(s) \in \mathbb{N} \) such that for each integer \( k \in \mathbb{N} \) with \( k \geq k_0' \) and for each integer \( j \in [l_k, l_k + [m_s]] \) we have \( d(T^j \alpha x, T^j \beta x) \leq \epsilon \). Hence, \( BG_{\alpha x, \beta x, m_a} \equiv 0 \). Suppose now that \( k \in \mathbb{N}, j \in \mathbb{N} \cap [j_k - [m_s], j_k] \) and \( a \geq [m_s] + m \). Then it is easy to see that \( p_{r_a}(T^j x) \geq [c_{a-1}c_{a-2} \cdots c_{a-(j_k-j)}]^{-1}p_{r_a-(j_k-j)}(T^{j_k} x) \), so that

\[
d(\alpha T^j x, \beta T^j x) \geq \sum_{a=[m_s]+m}^{\infty} 2^{-r_a+1} \frac{\alpha - \beta}{1 + \alpha - \beta} \frac{|\alpha - \beta|p_{r_a}(T^j x)}{p_{r_a}(T^j x)}
\]

since \( \lim_{k \to \infty} p_{m}(T^{j_k} x) = +\infty \), the above implies the existence of a sufficiently large number \( k_0 = k_0(s) \in \mathbb{N} \) such that for each \( k \in \mathbb{N} \) with \( k \geq k_0, j \in \mathbb{N} \cap [j_k - [m_s], j_k] \)
and $a \geq \lceil m_x \rceil + m$, we have $d(\alpha T^j x, \beta T^j x) \geq F(s)$. This gives (2.2) and completes the proof of proposition. \qed

**Remark 2.16** By the proof of above proposition, we have the following equivalence relations in Banach spaces:

(i) An element $x \in X$ is Li-Yorke unbounded for $T$ (i.e., $\inf_{j \in \mathbb{N}} \| T^j x \| = +\infty$) iff for each sequence $(m_n) \in \mathbb{R}$ there exists an infinite set $B \subseteq \mathbb{N}$ such that $\lim_{j \in B} \| T^j x \| = +\infty$ and $Bd_{\ell^1;m_n} (B^c) = 0$.

(ii) An element $x \in X$ is Li-Yorke near to zero for $T$ (i.e., $\sup_{j \in \mathbb{N}} \| T^j x \| = 0$) iff for each sequence $(m_n) \in \mathbb{R}$ there exists an infinite set $B \subseteq \mathbb{N}$ such that $\lim_{j \in B} \| T^j x \| = 0$ and $Bd_{\ell^1;m_n} (B^c) = 0$.

For operators on Banach spaces, it is well known that the chaos DC2 is equivalent to chaos DC1 (see [7, Theorem 2]). This basically follows from an application of [6, Proposition 8], and the first problem that we want to announce is a question whether the condition (i)' in the formulation of this proposition implies the condition (i) for operators in Fréchet spaces:

**Problem 2.17** Let $X$ be a Fréchet space and let $T \in L(X)$. Consider the following conditions:

(i)': There exist $\epsilon > 0$, a sequence $(y_k) \in X$ and an increasing sequence $(N_k) \in \mathbb{N}$ such that $\lim_{k \to \infty} y_k = 0$ and

$$\text{card} \left( \{ 1 \leq j \leq N_k : d(T^j y_k, 0) > \epsilon \} \right) \geq \epsilon N_k, \quad k \in \mathbb{N}. $$

(i): There exist $\epsilon > 0$, a sequence $(y_k) \in X$ and an increasing sequence $(N_k) \in \mathbb{N}$ such that $\lim_{k \to \infty} y_k = 0$ and

$$\lim_{k \to \infty} \frac{1}{N_k} \text{card} \left( \{ 1 \leq j \leq N_k : d(T^j y_k, 0) > \epsilon \} \right) = 1. $$

Is it true that (i)' implies (i)?

Now, we would like to raise the following issues closely linked with Problem 2.17:

**Problem 2.18** Let $X$ be a Fréchet space and let $T \in L(X)$. Is it true that:

(i) $T$ satisfies (DC2) iff $T$ satisfies (DC1)?

(ii) $T$ is reiteratively distributionally chaotic of type 2 $-$ iff $T$ is reiteratively distributionally chaotic of type 2?

To the best knowledge of the author, it is still unknown whether the answers to Problems 2.17 and 2.18 are affirmative for certain classes of shift operators in Fréchet sequence spaces and certain classes of composition operators in Fréchet function spaces, and whether a counterexample of such an operator not fulfilling the equivalence relation (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) of Proposition 2.15 really exists.
3 Reiterative $m_n$-Distributionally Irregular Vectors of Type $s$ and Reiterative $m_n$-Distributionally Irregular Manifolds of Type $s$

In this section, we investigate various notions of (reiterative) $m_n$-distributionally irregular vectors of type $s$ and (reiterative) $m_n$-distributionally irregular manifolds of type $s$. We start by introducing the following definition:

**Definition 3.1** Suppose that for each $j \in \mathbb{N}$, $T_j : D(T_j) \subseteq X \to Y$ is a linear operator. Then we say that:

(i) $x$ is (reiteratively) $m_n$-distributionally near to $0$ for $(T_j)_{j \in \mathbb{N}}$ iff there is $A \subseteq \mathbb{N}$ such that $(Bd_{I,m_n}(A^c) = 0) \downarrow m_n(A^c) = 0$ and $\lim_{j \in A, j \to \infty} T_j x = 0$;

(ii) $x$ is (reiteratively) $m_n$-distributionally $m$-unbounded for $(T_j)_{j \in \mathbb{N}}$ iff there is $B \subseteq \mathbb{N}$ such that $(Bd_{I,m_n}(B^c) = 0) \downarrow m_n(B^c) = 0$ and $\lim_{j \in B, j \to \infty} p_{f_j}(T_j x) = \infty$; $x$ is said to be (reiteratively) $m_n$-distributionally unbounded for $(T_j)_{j \in \mathbb{N}}$ if there exists $q \in \mathbb{N}$ such that $x$ is (reiteratively) $m_n$-distributionally $q$-unbounded for $(T_j)_{j \in \mathbb{N}}$ (if $Y$ is a Banach space, this simply means that $\lim_{j \in B, j \to \infty} \|T_j x\|_Y = \infty$).

In the following two definitions, we introduce separately the notions of $m_n$-distributionally irregular vectors of type $s \in \{1, 2, 2^1, 3, 2Bd\}$ and reiteratively $m_n$-distributionally irregular vectors of type $s \in \{0, 1, 1+, 2, 2Bd, 2-\}$:

**Definition 3.2** Suppose that for each $j \in \mathbb{N}$, $T_j : D(T_j) \subseteq X \to Y$ is a linear operator. Then we say that:

(i) $x$ is an $m_n$-distributionally irregular vector for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is $m_n$-distributionally near to $0$ for $(T_j)_{j \in \mathbb{N}}$ and $x$ is $m_n$-distributionally unbounded for $(T_j)_{j \in \mathbb{N}}$;

(ii) $x$ is an $m_n$-distributionally irregular vector of type $1$ for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is $m_n$-distributionally near to $0$ for $(T_j)_{j \in \mathbb{N}}$ and $F_{x,0,m_n}(\sigma) = 0$ for some $\sigma > 0$;

(iii) $x$ is an $m_n$-distributionally irregular vector of type $2$ for $(T_j)_{j \in \mathbb{N}}$ iff there exists a finite number $\sigma > 0$ such that $G_{x,0,m_n} \equiv 0$ and $I_{x,0,m_n}(\sigma) > 0$;

(iv) $x$ is an $m_n$-distributionally irregular vector of type $2^1$ for $(T_j)_{j \in \mathbb{N}}$ iff there exist real numbers $c > 0$ and $r > 0$ such that $F_{x,0,m_n}(\delta) < c < H_{x,0,m_n}(\delta)$ for all $0 < \delta < r$;

(v) $x$ is an $m_n$-distributionally irregular vector of type $3$ for $(T_j)_{j \in \mathbb{N}}$ iff there exist real numbers $b > a > 0$ and $c > 0$ such that $F_{x,0,m_n}(\delta) < c < H_{x,0,m_n}(\delta)$ for all $a \leq \delta \leq b$;

(vi) $x$ is an $m_n$-distributionally irregular vector of type $2Bd$ for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is $m_n$-distributionally near to zero and there exists a finite number $\sigma > 0$ such that $B_{I_{x,0,m_n}(\sigma)} > 0$.

**Definition 3.3** Suppose that for each $j \in \mathbb{N}$, $T_j : D(T_j) \subseteq X \to Y$ is a linear operator. Then we say that:

(i) $x$ is a reiteratively $m_n$-distributionally irregular vector of type $0$ for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is reiteratively $m_n$-distributionally near to $0$ for $(T_j)_{j \in \mathbb{N}}$ and $x$ is reiteratively $m_n$-distributionally unbounded for $(T_j)_{j \in \mathbb{N}}$;

(ii) $x$ is a reiteratively $m_n$-distributionally irregular vector of type $1$ for $(T_j)_{j \in \mathbb{N}}$ iff there exist two finite numbers $c \in (0, \lim_{n \to \infty} \frac{m_n}{n})$ and $r > 0$ such that...
Reiterative $m_n$-Distributional Chaos of Type $s$ in Fréchet Spaces

that $G_{x,0,m_n}(\delta) \leq c$ for $0 < \delta < r$, and $x$ is reiteratively $m_n$-distributionally unbounded for $(T_j)_{j \in \mathbb{N}}$;

(iii) $x$ is a reiteratively $m_n$-distributionally irregular vector of type $1^+$ for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is $m_n$-distributionally near to zero and $x$ is reiteratively $m_n$-distributionally unbounded for $(T_j)_{j \in \mathbb{N}}$;

(iv) $x$ is a reiteratively $m_n$-distributionally irregular vector of type $2$ for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is reiteratively $m_n$-distributionally near to zero and there exists $\sigma > 0$ such that $I_{x,0,m_n}(\sigma) > 0$;

(v) $x$ is a reiteratively $m_n$-distributionally irregular vector of type $2_{Bd}$ for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is reiteratively $m_n$-distributionally near to zero and there exists a finite number $\sigma > 0$ such that $B I_{x,0,m_n}(\sigma) > 0$;

(vi) $x$ is a reiteratively $m_n$-distributionally irregular vector of type $2_{-}$ for $(T_j)_{j \in \mathbb{N}}$ iff $x$ is reiteratively $m_n$-distributionally near to zero and $x$ is $m_n$-distributionally unbounded for $(T_j)_{j \in \mathbb{N}}$.

We will employ the following notion, as well:

**Definition 3.4** Suppose that for each $j \in \mathbb{N}, T_j : D(T_j) \subseteq X \to Y$ is a linear operator. Let $\{0\} \neq X' \subseteq \tilde{X}$ be a linear manifold and let $s \in \{1, 2, 2^\frac{1}{2}, 3, 2_{Bd}\}$. Then we say that:

(i) $X'$ is (dense, if $X'$ is dense in $\tilde{X}$) $\tilde{X}_{m_n}$-distributionally irregular manifold (of type $s$) for $(T_j)_{j \in \mathbb{N}}$ iff any element $x \in (X' \cap \bigcap_{j=1}^{\infty} D(T_j)) \setminus \{0\}$ is an $m_n$-distributionally irregular vector (of type $s$) for $(T_j)_{j \in \mathbb{N}}$;

(ii) $X'$ is a (dense, if $X'$ is dense in $\tilde{X}$) uniformly $\tilde{X}_{m_n}$-distributionally irregular manifold for $(T_j)_{j \in \mathbb{N}}$ if, in addition to the above, there exists $m \in \mathbb{N}$ such that any vector $x \in (X' \cap \bigcap_{j=1}^{\infty} D(T_j)) \setminus \{0\}$ is $m_n$-distributionally $m$-unbounded;

(iii) $X'$ is (dense, if $X'$ is dense in $\tilde{X}$) $\tilde{X}_{m_n}$-distributionally irregular manifold of type $2_{Bd}$ for $(T_j)_{j \in \mathbb{N}}$ iff $X'$ is (dense, if $X'$ is dense in $\tilde{X}$) $\tilde{X}_{m_n}$-distributionally irregular manifold of type $2_{Bd}$ for $(T_j)_{j \in \mathbb{N}}$ and the number $\sigma > 0$ in Definition 3.2(iii) is independent of choice of element $x \in (X' \cap \bigcap_{j=1}^{\infty} D(T_j)) \setminus \{0\}$.

**Definition 3.5** Suppose that for each $j \in \mathbb{N}, T_j : D(T_j) \subseteq X \to Y$ is a linear operator. Let $\{0\} \neq X' \subseteq \tilde{X}$ be a linear manifold and let $s \in \{0, 1, 1^+, 1^+, 2, 2_{Bd}, 2_{-}\}$. Then we say that:

(i) $X'$ is (dense, if $X'$ is dense in $\tilde{X}$) reiteratively $\tilde{X}_{m_n}$-distributionally irregular manifold of type $s$ for $(T_j)_{j \in \mathbb{N}}$ iff any element $x \in (X' \cap \bigcap_{j=1}^{\infty} D(T_j)) \setminus \{0\}$ is a reiteratively $m_n$-distributionally irregular vector of type $s$ for $(T_j)_{j \in \mathbb{N}}$;

(ii) If $s \in \{0, 1, 1^+\}$, then we say that $X'$ is a (dense, if $X'$ is dense in $\tilde{X}$) uniformly $\tilde{X}_{m_n}$-reiteratively distributionally irregular manifold of type $s$ for $(T_j)_{j \in \mathbb{N}}$ if, in addition to the above, there exists $m \in \mathbb{N}$ such that any vector $x \in (X' \cap \bigcap_{j=1}^{\infty} D(T_j)) \setminus \{0\}$ is reiteratively $m_n$-distributionally $m$-unbounded;

(iii) If $s = 2_{-}$, then we say that $X'$ is a (dense, if $X'$ is dense in $\tilde{X}$) uniformly $\tilde{X}_{m_n}$-reiteratively distributionally irregular manifold of type $s$ for $(T_j)_{j \in \mathbb{N}}$ if, in addition to the above, there exists $m \in \mathbb{N}$ such that any vector $x \in (X' \cap \bigcap_{j=1}^{\infty} D(T_j)) \setminus \{0\}$ is $m_n$-distributionally $m$-unbounded;

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(iv) If \( s \in \{ 2^+, 2Bd+ \} \), then we say that \( X' \) is (dense, if \( X' \) is dense in \( \tilde{X} \)) reiteratively \( \tilde{X}_{m_n} \)-distributionally irregular manifold of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \) iff any element \( x \in (X' \cap \bigcap_{j=1}^{\infty} D(T_j)) \backslash \{ 0 \} \) is a reiteratively \( m_n \)-distributionally irregular vector of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \) and the number \( \sigma > 0 \) in Definition 3.3(iv)–(v) is independent of \( x \).

All the above notions are accepted for a linear operator \( T : D(T) \subseteq X \to X \) by using the sequence \( (T_j \equiv T^j)_{j \in \mathbb{N}} \) for definitions. Further on, we will accept similar agreements as before. If \( \tilde{X} = X \) or \( m_n = n \), then we remove the prefixes “\( \tilde{X} \)-” and “\( m_n \)-” from the terms and notions. For example, dense uniformly reiteratively distributionally irregular manifold of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \) is dense uniformly \( X_n \)-reiteratively distributionally irregular manifold of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \). If \( m_n = n^{1/\lambda} \) for some \( \lambda \in (0, 1] \), then, as a special case of the above definitions, we obtain the notions of (reiteratively) \( \lambda \)-distributionally near to zero vectors, (reiteratively) \( \lambda \)-distributionally \((m\text{-})\)-unbounded vectors and (reiteratively) \( \lambda \)-distributionally irregular vectors of type \( s \) for sequence \( (T_j)_{j \in \mathbb{N}} \) (operator \( T \)); a similar terminology is used for manifolds. In the case that \( \lambda = 1 \), then we remove the prefix “\( \lambda \)-” from the terms and notions.

The following statements hold:

A. Using the elementary properties of metric, it can be simply verified that \( X' \) is a \( \tilde{X}_{m_n} \)-scrambled set for \( (T_j)_{j \in \mathbb{N}} \) whenever \( X' \) is a uniformly reiteratively \( \tilde{X}_{m_n} \)-distributionally irregular manifold for \( (T_j)_{j \in \mathbb{N}} \). Furthermore, let \( s \in \{ 1, 2^1, 3, 2Bd, 2Bd+ \} \); then \( X' \) is a \( \tilde{X}_{m_n} \)-scrambled set of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \) whenever \( X' \) is an \( \tilde{X}_{m_n} \)-distributionally irregular manifold of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \). On the other hand, we can simply verify that if \( 0 \neq x \in \tilde{X} \) is an \( m_n \)-distributionally irregular vector/\( m_n \)-distributionally irregular vector of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \), then \( X' \equiv \text{span}\{x\} \) is a uniformly \( \tilde{X}_{m_n} \)-distributionally irregular manifold for \( (T_j)_{j \in \mathbb{N}} /\tilde{X}_{m_n} \)-distributionally irregular manifold of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \).

If \( s = 3 \), then we need to additionally impose that \( Y \) is a Banach space.

B. Let \( s \in \{ 0, 1, 1^+, 2^- \} \). Using the elementary properties of metric, it can be simply verified that \( X' \) is a reiteratively \( m_n \)-scrambled set of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \) whenever \( X' \) is a uniformly reiteratively \( m_n \)-distributionally irregular manifold of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \). Furthermore, \( X' \) is a reiteratively \( m_n \)-scrambled set of type \( 2 \) (\( 2Bd \)) for \( (T_j)_{j \in \mathbb{N}} \) whenever \( X' \) is a reiteratively \( m_n \)-distributionally irregular manifold of type \( 2/2^+ \) (\( 2Bd/2Bd^+ \)) for \( (T_j)_{j \in \mathbb{N}} \). On the other hand, we can simply verify that if \( 0 \neq x \in \tilde{X} \) is a reiteratively \( m_n \)-distributionally irregular vector of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \), then \( X' \equiv \text{span}\{x\} \) is a (uniformly, if \( s \in \{ 2, 2Bd \} \)) reiteratively \( \tilde{X}_{m_n} \)-distributionally irregular manifold of type \( s \) for \( (T_j)_{j \in \mathbb{N}} \).

### 3.1 Structural Results for \( m_n \)-Distributionally Irregular Vectors

A fairly complete analysis of (reiteratively) \( m_n \)-distributionally irregular vectors of type \( s \) and corresponding (reiteratively) \( m_n \)-distributionally irregular manifolds of type \( s \) is far from being easy and trivial. In this subsection, we shall primarily focus our attention on the notions of \( m_n \)-distributional chaos, reiterative \( m_n \)-distributional
chaos of types $1$, $1^+$, and explain how the results from [6, Section 2] can be slightly extended for $m_n$-distributional chaos (cf. also [20]).

We start by stating the following extension of [6, Proposition 7] and the equivalence relations (i) ⇔ (ii) ⇔ (iii) of [6, Proposition 8] (concerning this proposition, we feel duty bound to say that the equivalence with (i)' and (ii)' for orbits of a single operator on Banach space is not attainable for $m_n$-distributional chaos, as far as we can see):

**Proposition 3.6** Let $(m_n) \in \mathbb{R}$, $m \in \mathbb{N}$ and $(T_j)_{j \in \mathbb{N}}$ be a sequence in $L(X, Y)$.

(i) The following assertions are equivalent:

(a) Suppose that there exist a number $\epsilon > 0$, a zero sequence $(y_k)$ in $X$ and a strictly increasing sequence $(N_k)$ in $\mathbb{N}$ such that

$$m_{N_k} - \left\{ j \in \mathbb{N} : p^Y_m(T_j y_k) > \epsilon \right\} \cap [1, m_{N_k}] \leq \frac{N_k}{k}, \quad k \in \mathbb{N}. \quad (3.1)$$

(b) The set of $m_n$-distributionally $m$-unbounded vectors for $(T_j)_{j \in \mathbb{N}}$ is non-empty.

(c) The set of $m_n$-distributionally $m$-unbounded vectors for $(T_j)_{j \in \mathbb{N}}$ is residual in $X$.

(ii) The following assertions are equivalent:

(a)' Suppose that there exist a number $\epsilon > 0$, a zero sequence $(y_k)$ in $X$ and a strictly increasing sequence $(N_k)$ in $\mathbb{N}$ such that

$$m_{N_k} - \left\{ j \in \mathbb{N} : d_Y(T_j y_k, 0) > \epsilon \right\} \cap [1, m_{N_k}] \leq \frac{N_k}{k}, \quad k \in \mathbb{N}. \quad (3.2)$$

(b)' The set of $m_n$-distributionally unbounded vectors for $(T_j)_{j \in \mathbb{N}}$ is non-empty.

(c)' The set of $m_n$-distributionally unbounded vectors for $(T_j)_{j \in \mathbb{N}}$ is residual in $X$.

**Proof** We will only outline the main details for showing the implication (a) ⇒ (c) in (i) since the use of arguments contained in the proof of [6, Proposition 7] is possible with appropriate modifications described as follows. For each $k \in \mathbb{N}$, we set

$$M_k := \left\{ x \in X : (\exists n \in \mathbb{N}) \, \text{s.t.} \, m_n - \left\{ j \in \mathbb{N} : p^Y_m(T_j x) > k \right\} \cap [1, m_n] \leq \frac{n}{k} \right\}.$$ 

Then it is very plain to show that for each $k \in \mathbb{N}$ the set $M_k$ is open as well as that the set $\bigcap_{k \in \mathbb{N}} M_k$ is consisted solely of $m_n$-distributionally $m$-unbounded vectors for $(T_j)_{j \in \mathbb{N}}$. It remains to be proved that for each $k \in \mathbb{N}$ the set $M_k$ is dense. To see this, we can repeat almost literally the arguments contained in the proof of above-mentioned Proposition 7, with the same terminology used and the sets

$$A := \left\{ 1 \leq j \leq m_n : p^Y_m(T_j u) > \epsilon \right\},$$

$$B_s := \left\{ 1 \leq j \leq m_n : p^Y_m(T_j u_s) \leq k \right\}, \quad s = 0, 1, \ldots, 2k(1 + m_n) - 1,$$
where
\[ u_s := x + \frac{\delta su}{2k(1 + m_n)c}, \quad s = 0, 1, \ldots, 2k(1 + m_n) - 1, \]
and \( n \) is chosen so that
\[ m_n - \left| \{ j \in \mathbb{N} : p^Y_m(T_j y_k) > \epsilon \} \cap [1, m_n] \right| \leq m_n - \frac{n}{2k}; \]
cf. (3.1).

\[ \square \]

Corollary 3.7 Let \( \lambda \in (0, 1] \), \( m \in \mathbb{N} \) and \( (T_j)_{j \in \mathbb{N}} \) be a sequence in \( L(X, Y) \).

(i) The following assertions are equivalent:

(a) Suppose that there exist a number \( \epsilon > 0 \), a zero sequence \( (y_k) \) in \( X \) and a strictly increasing sequence \( (N_k) \) in \( \mathbb{N} \) such that
\[ N_k^{1/\lambda} - \left| \{ j \in \mathbb{N} : p^Y_m(T_j y_k) > k \} \cap [1, N_k^{1/\lambda}] \right| \leq \frac{N_k}{k}, \quad k \in \mathbb{N}. \]

(b) The set of \( \lambda \)-distributionally \( m \)-unbounded vectors for \( (T_j)_{j \in \mathbb{N}} \) is non-empty.

(c) The set of \( \lambda \)-distributionally \( m \)-unbounded vectors for \( (T_j)_{j \in \mathbb{N}} \) is residual in \( X \).

(ii) The following assertions are equivalent:

(a) Suppose that there exist a number \( \epsilon > 0 \), a zero sequence \( (y_k) \) in \( X \) and a strictly increasing sequence \( (N_k) \) in \( \mathbb{N} \) such that
\[ N_k^{1/\lambda} - \left| \{ j \in \mathbb{N} : d_Y(T_j y_k, 0) > \epsilon \} \cap [1, N_k^{1/\lambda}] \right| \leq \frac{N_k}{k}, \quad k \in \mathbb{N}. \]  

(b) The set of \( \lambda \)-distributionally unbounded vectors for \( (T_j)_{j \in \mathbb{N}} \) is non-empty.

(c) The set of \( \lambda \)-distributionally unbounded vectors for \( (T_j)_{j \in \mathbb{N}} \) is residual in \( X \).

Considering the sets
\[ M_{k,m} := \left\{ x \in X : (\exists n \in \mathbb{N}) \text{ s.t. } m_n - \left| \{ j \in \mathbb{N} : p^Y_m(T_j x) < 1/k \} \cap [1, m_n] \right| \leq \frac{n}{k} \right\}, \]
we can similarly prove the following extension of [6, Proposition 9]:

Proposition 3.8 Suppose that \( (T_j)_{j \in \mathbb{N}} \) is a sequence in \( L(X, Y) \). If the set of those vectors \( x \in X \) for which there exists a set \( B \subseteq \mathbb{N} \) such that \( d_{m_n}(B^c) = 0 \) and \( \lim_{j \in B} T_j x = 0 \) is dense in \( X \), then the set of \( m_n \)-distributionally near to zero vectors for \( (T_j)_{j \in \mathbb{N}} \) is residual in \( X \).

Corollary 3.9 Suppose that \( \lambda \in (0, 1] \) and \( (T_j)_{j \in \mathbb{N}} \) is a sequence in \( L(X, Y) \). If the set of those vectors \( x \in X \) for which there exists a set \( B \subseteq \mathbb{N} \) such that \( d_{1/\lambda}(B^c) = 0 \) and \( \lim_{j \in B} T_j x = 0 \) is dense in \( X \), then the set of \( \lambda \)-distributionally near to zero vectors for \( (T_j)_{j \in \mathbb{N}} \) is residual in \( X \).
Keeping in mind Propositions 3.6 and 3.8, we can state the following extensions of the first parts in [11, Theorem 3.7, Corollary 3.12] for \(m_n\)-distributional chaos:

**Proposition 3.10** Let \((m_n) \in \mathbb{R}\) and \((T_j)_{j \in \mathbb{N}}\) be a sequence in \(L(X, Y)\). Suppose that the set consisting of those vectors \(x \in X\) for which there exists a set \(B \subseteq \mathbb{N}\) such that 
\[
d_{m_n}(B^c) = 0 \quad \text{and} \quad \lim_{j \in B} T_j x = 0
\]
is dense in \(X\), as well as that there exist a number \(\varepsilon > 0\), a zero sequence \((y_k)\) in \(X\) and a strictly increasing sequence \((N_k)\) in \(\mathbb{N}\) such that (3.2) holds. Then the set of \(m_n\)-distributionally irregular vectors for \((T_j)_{j \in \mathbb{N}}\) is residual in \(X\).

**Corollary 3.11** Let \(\lambda \in (0, 1]\) and \((T_j)_{j \in \mathbb{N}}\) be a sequence in \(L(X, Y)\). Suppose that the set consisting of those vectors \(x \in X\) for which there exists a set \(B \subseteq \mathbb{N}\) such that 
\[
d_{1/\lambda}(B^c) = 0 \quad \text{and} \quad \lim_{j \in B} T_j x = 0
\]
is dense in \(X\), as well as that there exist a number \(\varepsilon > 0\), a zero sequence \((y_k)\) in \(X\) and a strictly increasing sequence \((N_k)\) in \(\mathbb{N}\) such that (3.3) holds. Then the set of \(\lambda\)-distributionally irregular vectors for \((T_j)_{j \in \mathbb{N}}\) is residual in \(X\).

If \(T \in L(X)\), then we are in a position to extend the assertion of [6, Theorem 12] for \(m_n\)-distributional chaos in a rather technical way. For these purposes, we introduce the \(m_n\)-Distributionally Chaotic Criterion and \(\lambda\)-Distributionally Chaotic Criterion in the following ways:

- (DCC\(_{mn}\)) There exist a number \(\varepsilon > 0\), a set \(B \subseteq \mathbb{N}\), two sequences \((x_k)\) and \((y_k)\) in \(X\) as well as a strictly increasing sequence \((N_k)\) of natural numbers such that 
\[
d_{m_n}(B^c) = 0, \quad \lim_{n \in B} T^n x_k = 0, \quad y_k \in \text{span}\{x_n : n \in \mathbb{N}\}, \quad \lim_{k \to \infty} y_k = 0\]
and (3.2) holds with \(T_j \equiv T^j\);

- (DCC\(_\lambda\)) There exist a number \(\varepsilon > 0\), a set \(B \subseteq \mathbb{N}\), two sequences \((x_k)\) and \((y_k)\) in \(X\) as well as a strictly increasing sequence \((N_k)\) of natural numbers such that 
\[
d_{1/\lambda}(B^c) = 0, \quad \lim_{n \in B} T^n x_k = 0, \quad y_k \in \text{span}\{x_n : n \in \mathbb{N}\}, \quad \lim_{k \to \infty} y_k = 0\]
and (3.3) holds with \(T_j \equiv T^j\).

Then we have:

**Theorem 3.12** Suppose that \(T \in L(X)\) and \((m_n) \in \mathbb{R}\). Then the following assertions are equivalent:

(i) \(T\) satisfies (DCC\(_{mn}\)).

(ii) There is an \(m_n\)-distributionally irregular vector for \(T\).

(iii) \(T\) is \(m_n\)-distributionally chaotic.

(iv) There is an \(m_n\)-distributionally chaotic pair of type 1 for \(T\).

(v) \(T\) is \(m_n\)-distributionally chaotic of type 1.

(vi) There is an \(m_n\)-distributionally irregular vector of type 1 for \(T\).

**Proof** The equivalence of (i), (ii), (iii) and (iv) has been already proved. The implication (ii) \(\Rightarrow\) (vi) is trivial, the implication (vi) \(\Rightarrow\) (v) follows from the last statement in [A.], while the implication (v) \(\Rightarrow\) (iv) follows directly from definition. This completes the proof.

**Corollary 3.13** Suppose that \(T \in L(X)\) and \(\lambda \in (0, 1]\). Then the following assertions are equivalent:
(i) \( T \) satisfies (DCC\(_\lambda\)).
(ii) There is a \(\lambda\)-distributionally irregular vector for \( T \).
(iii) \( T \) is \(\lambda\)-distributionally chaotic.
(iv) There is a \(\lambda\)-distributionally chaotic pair for \( T \).
(v) \( T \) is \(\lambda\)-distributionally chaotic of type 1.
(vi) There is a \(\lambda\)-distributionally irregular vector of type 1 for \( T \).

Concerning reiterative \( m_n \)-distributional chaos of types 1 and \( 1^+ \), we will first state and prove the following two lemmas:

Lemma 3.14 Suppose that for each \( j \in \mathbb{N} \), \( T_j : D(T_j) \subseteq X \to Y \) is a linear operator, and \( x \in \bigcap_{j \in \mathbb{N}} D(T_j) \). Then there exists a finite number \( r > 0 \) such that \( G_{x,0,m_n}(\delta) \leq c \) for \( 0 < \delta < r \) iff there exist a finite number \( d \in (0, \liminf_{n \to \infty} \frac{m_n}{n}) \) and an infinite set \( A \subseteq \mathbb{N} \) such that \( \underline{d}_{m_n}(A^c) = d \) and \( \lim_{j \in A} T_j x = 0 \).

**Proof** Suppose first that there exists a finite number \( r > 0 \) such that \( G_{x,0,m_n}(\delta) \leq c \) for \( 0 < \delta < r \). Let \( k \in \mathbb{N} \) and \( k > 1/r \). Then there exists a positive integer \( n_k \in \mathbb{N} \) such that \( \delta y(T_j x, 0) < 1/k \). Let \( A_k \) denote the collection of such numbers and let \( A := \bigcup_{k \in \mathbb{N}, k > 1/r} A_k \). Then it can be easily seen that \( \underline{d}_{m_n}(A^c) = c \) and \( \lim_{j \in A} T_j x = 0 \). For the converse statement, we can simply prove that the existence of a finite number \( d \in (0, \liminf_{n \to \infty} \frac{m_n}{n}) \) and an infinite set \( A \subseteq \mathbb{N} \) such that \( \underline{d}_{m_n}(A^c) = d \) and \( \lim_{j \in A} T_j x = 0 \) implies \( G_{x,0,m_n}(\delta) \leq c \equiv (d + \liminf_{n \to \infty} \frac{m_n}{n})/2 \) for \( 0 < \delta < r \). \( \square \)

Lemma 3.15 Suppose that for each \( j \in \mathbb{N} \) we have \( T_j \in L(X, Y) \). Denote by \( \mathcal{X} \) the set consisting of all vectors \( x \in X \) for which there exists an infinite set \( A \subseteq \mathbb{N} \) such that \( \underline{d}_{m_n}(A^c) = d \) and \( \lim_{j \in A} T_j x = 0 \). Then \( \mathcal{X} \) is residual if it is dense.

**Proof** For \( k, \ m \in \mathbb{N} \), set
\[
M_{k,m} := \left\{ x \in X : (\exists n \in \mathbb{N}) \ m_n - \left| \left\{ 1 \leq j \leq m_n : p_m^Y(T_j x) < k^{-1} \right\} \right| \geq n(c + k^{-1}) \right\}.
\]
Clearly, \( M_{k,m} \) is open and dense (since \( M_{k,m} \supseteq \mathcal{X} \)), so the set \( X_1 = \bigcap_{k,m} M_{k,m} \) is residual and contains \( \mathcal{X} \). \( \square \)

Keeping in mind the above lemmas, Remark 2.16 and the proof of [10, Theorem 2.3], we can deduce the following result:

**Theorem 3.16** Suppose \( X \) is a Banach space and \( T \in L(X) \). Then we have the following:

(i) \( T \) is reiteratively \( m_n \)-distributionally chaotic of type 1 iff there exists a reiteratively \( m_n \)-distributionally irregular vector \( x \) of type 1.
(ii) \( T \) is reiteratively \( m_n \)-distributionally chaotic of type \( 1^+ \) iff there exists a reiteratively \( m_n \)-distributionally irregular vector \( x \) of type \( 1^+ \).
Proof We will prove only (i) because the part (ii) can be deduced analogously. Due to [B.], the existence of a reiteratively $m_n$-distributionally irregular vector $x$ of type 1 implies that $T$ is reiteratively $m_n$-distributionally chaotic of type 1. On the other hand, Lemma 3.14 and the consideration from Remark 2.16 together imply that a vector $x \in X$ is reiteratively $m_n$-distributionally irregular vector of type 1 for $T$ iff $x$ is a Li-Yorke irregular vector $x$ for $T$ and there exists an infinite set $A \subseteq \mathbb{N}$ such that $d_{m_n}(A^c) = d < \liminf_{n \to \infty} \frac{m_n}{n}$ and $\lim_{j \in A} T_j x = 0$. Suppose now that $T$ is reiteratively $m_n$-distributional chaotic of type 1. Then there exists a pair $(x, y)$ of distinct points such that $B_{x, y, m_n}(\sigma) = 0$ for some $\sigma > 0$ and there exist $c \in (0, \liminf_{n \to \infty} \frac{m_n}{n})$ and $r > 0$ such that $G_{x, y, m_n}(\delta) \leq c$ for all $0 < \delta < r$. Let $u = x - y$ and consider

$$Y_1 = \overline{\text{span}(\text{Orb}(u, T))},$$

which is an infinite-dimensional closed $T$-invariant subspace of $X$. Consider the operator $S \in B(Y_1)$ obtained by restricting $T$ to $Y_1$.

Then $S$ is reiteratively $m_n$-distributionally chaotic of type 1 in $Y_1$ because $\text{span}\{u\}$ is a corresponding reiteratively $m_n$-distributionally chaotic scrambled set of type 1. Thus, by [8, Corollary 5], $S$ has a residual set of points on $Y_1$ with orbit unbounded. Moreover, by Lemma 3.15, $S$ has a residual set of points $z$ on $Y_1$ for which there exists an infinite set $A \subseteq \mathbb{N}$ such that $d_{m_n}(A^c) = d < \liminf_{n \to \infty} \frac{m_n}{n}$ and $\lim_{j \in A} T_j z = 0$. This yields that $S$ has a residual set of reiteratively $m_n$-distributionally irregular vectors of type 1, so that $T$ has a reiteratively $m_n$-distributionally irregular vector of type 1.

4 Dense Reiterative $m_n$-Distributional Chaos

In this section, we will see that the method proposed in the proof of [6, Theorem 15] provides a safe and sound way for the examination of dense reiterative $m_n$-distributional chaos of type $s$ in Fréchet spaces. The first structural result of ours, which in combination with Theorem 3.12, provides an extension of [6, Theorem 15], and the second part of [11, Theorem 3.7] reads as follows:

**Theorem 4.1** Suppose that $X$ is separable, $(m_n) \in \mathbb{R}$, $(T_j)_{j \in \mathbb{N}}$ is a sequence in $L(X, Y)$, $X_0$ is a dense linear subspace of $X$, as well as:

(i) $\lim_{j \to \infty} T_j x = 0$, $x \in X_0$,
(ii) there exists an $m_n$-distributionally unbounded vector $y \in X$ for $(T_j)_{j \in \mathbb{N}}$.

Then $(T_j)_{j \in \mathbb{N}}$ is densely $m_n$-distributionally chaotic, and moreover, the scrambled set $S$ can be chosen to be a dense uniformly $m_n$-distributionally irregular submanifold of $X$.

Proof We will only outline the main details. Without loss of generality, we may assume that

$$p_Y^m(T_j x) \leq p_{j+m}(x) \text{ for all } x \in X \text{ and } j, m \in \mathbb{N} \quad (4.1)$$
as well as that a set $B \subseteq \mathbb{N}$ satisfies $d_{m_n}(B^c) = 0$ and \( \lim_{n \to \infty, n \in B} p_Y^1(T_n y) = +\infty \). Using the equality $d_{m_n}(B^c) = 0$, we can construct a sequence $(x_k)_{k \in \mathbb{N}}$ in $X_0$ and a strictly increasing sequence $(j_k)_{k \in \mathbb{N}}$ of positive integers such that, for every $k \in \mathbb{N}$, one has: $p_k(x_k) \leq 1$,

$$\left| \{1 \leq j \leq m_{j_k} : p_Y^1(T_j x_k) \geq k2^j \} \right| \geq m_{j_k} - \frac{j_k}{k}$$

and

$$\left| \{1 \leq j \leq m_{j_k} : p_Y^k(T_j x_s) < 1/k \} \right| \geq m_{j_k} - \frac{j_k}{k}, \quad s = 1, \ldots, k - 1.$$

Take any strictly increasing sequence $(r_q)_{q \in \mathbb{N}}$ in $\mathbb{N}\setminus\{1\}$ such that

$$r_{q+1} \geq 1 + r_q + m_{j_{r_q+1}} \text{ for all } q \in \mathbb{N}. \quad (4.2)$$

Let $\alpha \in \{0, 1\}^\mathbb{N}$ be a sequence defined by $\alpha_n = 1$ iff $n = r_q$ for some $q \in \mathbb{N}$. Further on, let $\beta \in \{0, 1\}^\mathbb{N}$ contain an infinite number of 1’s and let $\beta_q \leq \alpha_q$ for all $q \in \mathbb{N}$. If $\beta_k = 1$ for some $k \in \mathbb{N}$ and $x_{\beta} = \sum_{q=1}^{\infty} \beta_{r_q}x_{r_q}/2^q$, then for each $j \in [1, m_{j_k}]$ such that $p_Y^1(T_j x_{\beta}) \geq k2^k$ and $p_Y^k(T_j x_s) < 1/r_k$ for $s < r_k$, we have: $1 + j \leq 1 + m_{j_k} \leq 1 + m_{j_{r_k-1}} \leq r_q$ for $q > k$:

$$p_Y^1(T_j x_{\beta}) \geq r_k - \sum_{q<k} p_Y^1(T_j x_{r_q})/2^{r_q} - \sum_{q\geq k} p_Y^1(T_j x_{r_q})/2^{r_q}$$

$$\geq r_k - \sum_{q<k} p_Y^1(T_j x_{r_q})/2^{r_q} - \sum_{q\geq k} p_Y^{1+j}(x_{r_q})/2^{r_q}$$

$$\geq r_k - \sum_{q<k} \frac{1}{2^{r_q}r_q} - \sum_{q\geq k} \frac{1}{2^{r_q}} \geq r_k - 1,$$

which implies that for each of the fixed distinct numbers $\alpha_1$, $\alpha_2 \in \mathbb{N}$ there exists a positive integer $k_0(\alpha_1, \alpha_2)$ such that for each $k \geq k_0(\alpha_1, \alpha_2)$ one has:

$$d_Y(T_k \alpha_1 x_{\beta}, T_k \alpha_1 x_{\beta}) \geq 2^{-1} \frac{|\alpha_1 - \alpha_2|(r_k - 1)}{1 + |\alpha_1 - \alpha_2|(r_k - 1)} \geq 4^{-1};$$

hence,

$$\lim_{k \to \infty} \frac{\left| \{1 \leq j \leq m_{j_k} : d_Y(T_k \alpha_1 x_{\beta}, T_k \alpha_2 x_{\beta}) < 4^{-1} \} \right|}{j_k} = 0. \quad (4.3)$$

Furthermore, if $j \in [1, m_{j_{k+1}}]$ and $p_Y^{r_{k+1}}(T_j x_s) < 1/(r_k + 1)$ for $s < r_k + 1$, then we have $1 + r_k + j \leq 1 + r_k + m_{j_{k+1}} \leq 1 + r_{q-1} + m_{j_{r_{q-1}+1}} \leq r_q$ due to (4.2), and therefore,
\[ p_{r_k+1}^Y(T_j x_\beta) \leq \sum_{q \leq k} \frac{p_{r_k+1}^Y(T_j x_{r_q})}{2^q} + \sum_{q > k} \frac{p_{r_k+1}^Y(T_j x_{r_q})}{2^{r_q}} \]
\[ \leq \sum_{q \leq k} \frac{1}{2^q (r_k + 1)} + \sum_{q > k} \frac{p_{1+j+r_k}(x_{r_q})}{2^{r_q}} \]
\[ \leq \frac{1}{2(r_k + 1)} + \sum_{q > k} \frac{1}{2^q} \leq \frac{1}{r_k + 1}, \]

which clearly implies
\[ d_Y(T_j x_\beta, 0) = \sum_{q=1}^{r_k+1} \frac{1}{2^q} \frac{p_q(T_j x_\beta)}{1 + p_q(T_j x_\beta)} + \sum_{q=r_k+1}^{\infty} \frac{1}{2^q} \frac{p_q(T_j x_\beta)}{1 + p_q(T_j x_\beta)} \leq \frac{1}{r_k + 2} + \frac{1}{2^{r_k}}. \]

(4.4)

This yields that for each \( \epsilon > 0 \) and for each pair of distinct numbers \( \alpha_1, \alpha_2 \in \mathbb{K} \) we have:
\[ \lim_{k \to \infty} \left| \left\{ 1 \leq j \leq m_{j_k} : d_Y(T_k \alpha_1 x_\beta, T_k \alpha_2 x_\beta) \geq \epsilon \right\} \right|_{j_{r_k}} = 0. \]

(4.5)

By (4.3)–(4.5), we get that the sequence \( (T_j)_{j \in \mathbb{N}} \) is \( m_n \)-distributionally chaotic, with \( S = \text{span}\{x_\beta\} \) as a 1/4-scrambled set. The final statement of theorem now follows similarly as in the proof of [6, Theorem 15].

□

The following corollaries are immediate:

**Corollary 4.2** Suppose that \( X \) is separable, \( \lambda \in (0, 1] \), \( (T_j)_{j \in \mathbb{N}} \) is a sequence in \( L(X, Y) \), \( X_0 \) is a dense linear subspace of \( X \), as well as:

(i) \( \lim_{j \to \infty} T_j x = 0, x \in X_0 \),

(ii) there exists a \( \lambda \)-distributionally unbounded vector \( y \in X \) for \( (T_j)_{j \in \mathbb{N}} \).

Then \( (T_j)_{j \in \mathbb{N}} \) is densely \( \lambda \)-distributionally chaotic, and moreover, the scrambled set \( S \) can be chosen to be a dense uniformly \( \lambda \)-distributionally irregular submanifold of \( X \).

**Corollary 4.3** Suppose that \( X \) is separable, \( (T_j)_{j \in \mathbb{N}} \) is a sequence in \( L(X, Y) \), \( X_0 \) is a dense linear subspace of \( X \), as well as:

(i) \( \lim_{j \to \infty} T_j x = 0, x \in X_0 \),

(ii) there exist \( y \in X \) and \( m \in \mathbb{N} \) such that \( \lim_{j \to \infty} p_m^Y(T_j y) = +\infty \).

Then \( (T_j)_{j \in \mathbb{N}} \) is densely \( m_n \)-distributionally chaotic for each sequence \( (m_n) \in \mathbb{R} \), and moreover, the corresponding scrambled set \( S \) can be chosen to be a dense uniformly \( m_n \)-distributionally irregular submanifold of \( X \).
Remark 4.4 It is worth noting that Corollary 4.2 provides a generalization of [6, Theorem 16] for sequences of operators, where the case $\lambda = 1$ has been considered. On the other hand, Corollary 4.3 provides a generalization of [6, Corollary 17] for sequences of operators; in this statement, the authors have shown that the Godefroy–Schapiro criterion (see [14] and [16, Theorem 3.1]) implies dense distributional chaos.

Suppose now that $T_j : D(T_j) \subseteq X \to X$ is a linear mapping, $C \in L(X)$ is an injective mapping with dense range, as well as

$$R(C) \subseteq D(T_j) \text{ and } T_j C \in L(X) \text{ for all } j \in \mathbb{N}. \quad (4.6)$$

Then (4.6) implies that, for every $j \in \mathbb{N}$, the mapping $T'_j : R(C) \to X$ defined by $T'_j(Cx) := T_j Cx$, $x \in X$, $j \in \mathbb{N}$ is an element of the space $L([R(C)], X)$. By Theorem 4.1, we immediately obtain the following corollary.

**Corollary 4.5** Let $(m_n) \in \mathbb{R}$, let $\lambda \in (0, 1]$, and let the above conditions hold.

(i) Suppose that $X$ is separable, $X_0$ is a dense linear subspace of $X$, as well as:

(a) $\lim_{j \to \infty} T_j Cx = 0$, $x \in X_0$,
(b) there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq \mathbb{N}$ such that $d_{m}(B^c) = 0$ and $\lim_{j \to \infty, j \in B} p_m(T_j Cx) = \infty$, resp. $\lim_{j \to \infty, j \in B} \|T_j Cx\| = \infty$ if $X$ is a Banach space.

Then the sequence $(T_j)_{j \in \mathbb{N}}$ is densely $m_n$-distributionally chaotic for each sequence $(m_n) \in \mathbb{R}$, and moreover, the scrambled set $S$ can be chosen to be a dense uniformly $m_n$-distributionally irregular submanifold of $X$.

(ii) Suppose that $X$ is separable, $X_0$ is a dense linear subspace of $X$, as well as:

(a) $\lim_{j \to \infty} T_j Cx = 0$, $x \in X_0$,
(b) there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq \mathbb{N}$ such that $d_{1/\lambda}(B^c) = 0$ and $\lim_{j \to \infty, j \in B} p_m(T_j Cx) = \infty$, resp. $\lim_{j \to \infty, j \in B} \|T_j Cx\| = \infty$ if $X$ is a Banach space.

Then the sequence $(T_j)_{j \in \mathbb{N}}$ is densely $\lambda$-distributionally chaotic, and moreover, the scrambled set $S$ can be chosen to be a dense uniformly $\lambda$-distributionally irregular submanifold of $X$.

(iii) Suppose that $X$ is separable, $X_0$ is a dense linear subspace of $X$, as well as:

(a) $\lim_{j \to \infty} T_j Cx = 0$, $x \in X_0$,
(b) there exist $x \in X$ and $m \in \mathbb{N}$ such that $\lim_{j \to \infty} p_m(T_j Cx) = +\infty$.

Then the sequence $(T_j)_{j \in \mathbb{N}}$ is densely $m'_n$-distributionally chaotic for each sequence $(m'_n) \in \mathbb{R}$, and moreover, the corresponding scrambled set $S$ can be chosen to be a uniformly $m'_n$-distributionally irregular submanifold of $X$.

In [11, Example 3.8–Example 3.10, Corollary 3.12], we have considered only orbits of linear operators; it is clear that Corollary 4.5(ii) provides an extension of the second part in [11, Corollary 3.12], where the case $\lambda = 1$ has been analyzed, as well as that Corollary 4.5(iii) can be used to further improve the conclusions obtained in [11,
Example 4.6 (see [11, Example 3.9] and references cited therein) Denote by \( F \) and \( F^{-1} \) the Fourier transform on the real line and its inverse transform, respectively. Assume that \( X := L^2(\mathbb{R}), \) \( c > b/2 > 0, \) \( \Omega := \{ \lambda \in \mathbb{C} : \Re \lambda < c - b/2 \} \) and \( \mathcal{A}_c u := u'' + 2buxu' + cu \) is the bounded perturbation of the one-dimensional Ornstein–Uhlenbeck operator acting with domain \( D(\mathcal{A}_c) := \{ u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R}) \}. \) Then it is well known that \( \mathcal{A}_c \) generates a strongly continuous semigroup, \( \Omega \subseteq \sigma_p(\mathcal{A}_c), \) and for any open connected subset \( \Omega' \) of \( \Omega \) which admits a cluster point in \( \Omega, \) one has \( E = \text{span}\{g_i(\lambda) : \lambda \in \Omega', i = 1, 2, \} \), where \( g_1 : \Omega \to X \) and \( g_2 : \Omega \to X \) are defined by \( g_1(\lambda) := F^{-1}(e^{-\frac{\lambda^2}{2}}|\xi|^{-\frac{b+c}{2}})(\cdot), \lambda \in \Omega \) and \( g_2(\lambda) := F^{-1}(e^{-\frac{\lambda^2}{2}}|\xi|^{-\frac{b+c}{2}})(\cdot), \lambda \in \Omega. \) Assume that \( (P_j(z))_{j \in \mathbb{N}} \) is a sequence of nonzero complex polynomials such that there exists an open connected subset \( \Omega' \) of \( \Omega \) such that \( \lim_{j \to \infty} P_j(\lambda) = 0, \lambda \in \Omega' \) as well as that there exists a number \( \lambda \in \Omega \) such that \( |P_j(\lambda)| > 1. \) Due to Corollary 4.5(ii), we get that the sequence of operators \( (P_j(\mathcal{A}_c))_{j \in \mathbb{N}} \) is densely \( m_n \)-distributionally chaotic for each sequence \( (m_n) \in \mathbb{R}. \)

Assume, for the time being, that \( X \) is a Fréchet sequence space in which \((e_n)_{n \in \mathbb{N}}\) is a basis and \((\omega_n)_{n \in \mathbb{N}}\) is a sequence of positive weights; for more details, see [16, Section 4.1]. Consider the unilateral weighted backward shift \( T_\omega : D(T_\omega) \subseteq X \to X, \) given by

\[
T_\omega \langle x_n \rangle_{n \in \mathbb{N}} := \langle w_n x_{n+1} \rangle_{n \in \mathbb{N}}, \quad \langle x_n \rangle_{n \in \mathbb{N}} \in X, \tag{4.7}
\]

about which we assume that it is not necessarily continuous. Albeit it is without scope of this paper to consider \( m_n \)-distributionally chaotic properties of unbounded bilateral weighted shift operators (see [24], where we have recently initiated the study of disjoint distributionally chaotic properties of such operators), we will state here only one result regarding this question, which can be deduced with the help of Corollary 4.5(iii) and the proof of [24, Theorem 4.11]:

**Theorem 4.7** Let \( X := l^p \) for some \( 1 \leq p < \infty \) or \( X := c_0, \) and let \( (m_n) \in \mathbb{R}. \) Suppose that there exists a bounded sequence \((a_n)_{n \in \mathbb{N}}\) of positive reals such that for each \( k \in \mathbb{N} \) we have

\[
B_k := \sup_{n \in \mathbb{N}} \left[ a_{k+n} \prod_{i=n}^{k+n-1} \omega_i \right] < \infty.
\]

Then the following holds:

(i) If \( p \neq 2 \) and \( X = l^p \) or \( X = c_0, \) as well as \( \sum_{k=1}^{\infty} \frac{1}{B_k} < \infty, \)
or

\[(\text{ii}) \quad X = l^2 \text{ and } \sum_{k=1}^{\infty} \frac{1}{B_k} < \infty,\]

then the operator \(T_\omega\) is densely \(m_n\)-distributionally chaotic.

An illustrative example of application can be simply given:

**Example 4.8** Let \(X := l^p\) for some \(p \in [1, \infty)\) or \(X := c_0\), and let \(\omega_n := n^j\) for some \(j > 0\). Then we can apply Theorem 4.7 in order to see that the operator \(T_\omega\) is densely \(m_n\)-distributionally chaotic for any \((m_n) \in \mathbb{R}\).

The result of Theorem 4.7 states that the weighted backward shift operator is densely \(m_n\)-distributionally chaotic for any sequence \((m_n) \in \mathbb{R}\). We want only to add that it could be natural to ask whether the conditions given in this theorem can be weakened to some conditions depending directly on the sequence \((m_n)\).

In the following example, we construct \(m_n\)-distributionally unbounded vectors directly (cf. also Example 4.17 below):

**Example 4.9** Let us recall that L. Luo and B. Hou have considered the case \(X := l^1(\mathbb{N})\) and \((\omega_n)_{n \in \mathbb{N}} := (\frac{2n}{2n-1})_{n \in \mathbb{N}}\), showing that the corresponding operator \(T_\omega\) is topologically mixing, absolutely Cesàro bounded and therefore not distributionally chaotic ([28]). The analysis has been recently continued in [7], where it has been shown that the operator \(T_\omega\) cannot be distributionally chaotic of type 3. Consider now the following cases:

1. \(X := l^p(\mathbb{N})\) for some \(p \in (1, \infty)\). Due to Stirling’s formula, we have that \(\beta(n) := \prod_{j=1}^{n} \omega_i \sim \sqrt{n!} n \to +\infty\). Applying [28, Proposition 3.1], we get that the operator \(T_\omega\) is topologically mixing iff \(p > 1\) as well as that \(T_\omega\) is chaotic iff \(p > 2\). Now, we will prove that for each \(p > 1\) and each sequence \((m_n) \in \mathbb{R}\), the operator \(T_\omega\) is densely \(m_n\)-distributionally chaotic. Let \(\epsilon \in (0, 1)\) be arbitrarily chosen. Then it is clear that \((x_n)_{n \in \mathbb{N}} := (n^{-(1+\epsilon)/p})_{n \in \mathbb{N}} \in X\) as well that there exists a positive finite constant \(c > 0\) such that

\[\omega_n \omega_{n+1} \ldots \omega_{n+j} = \beta(n+j)/\beta(n-1) \geq c \sqrt{1 + \frac{j}{n}} \quad \text{for all } n, j \in \mathbb{N}. \quad (4.8)\]

We will prove that the vector \(x = (x_n)_{n \in \mathbb{N}}\) satisfies \(\lim_{j \to \infty} \|T_\omega^j x\| = +\infty\), so that the final conclusion follows by applying Theorem 5.3. Using (4.8) and the well-known result regarding the estimates of partial sums defining the Riemann zeta function, we get that

\[\|T_\omega^j x\| = \left(\sum_{n=1}^{\infty} |\omega_n \omega_{n+1} \ldots \omega_{n+j} x_{n+j+1}|^p\right)^{1/p} \geq c \left(\sum_{n=1}^{\infty} \left(1 + \frac{j}{n}\right)^{p/2} |x_{n+j+1}|^p\right)^{1/p} \geq c \sqrt{j} \left(\sum_{n=1}^{\infty} n^{-\frac{p}{2}} |x_{n+j+1}|^p\right)^{1/p}\]
there exists a reiteratively $m_n$-distributionally unbounded vector $x$.

(ii) On the other hand, Lemma 1.5(i) implies that $T_{\omega}$ is densely $m_n$-distributionally chaotic for any sequence $(m_n) \in \mathbb{R}$. Due to [16, Theorem 4.8], the operator $T_{\omega}$ is both topologically mixing and chaotic.

We are turning back to the case in which $X$ is a general Fréchet space under our consideration, by stating the following result:

**Theorem 4.10** Suppose that $X$ is separable, $(T_j)_{j \in \mathbb{N}}$ is a sequence in $L(X, Y)$, $X_0$ is a dense linear subspace of $X$, as well as:

(i) $\lim_{j \to \infty} T_j x = 0$, $x \in X_0$.

(ii) there exists a reiteratively $m_n$-distributionally unbounded vector $x \in X$ for $(T_j)_{j \in \mathbb{N}}$.

Then there exists a dense uniformly reiteratively $m_n$-distributionally irregular manifold of type $1^+$ for $(T_j)_{j \in \mathbb{N}}$, and particularly, $(T_j)_{j \in \mathbb{N}}$ is densely reiteratively $m_n$-distributionally chaotic of type $1^+$.

**Proof** We will only outline the main details. Without loss of generality, we may assume that (4.1) holds as well as that, due to (b), there exists a set $B \subseteq \mathbb{N}$ such that $Bd_{T_{:\omega}}(B^c) = 0$ and $\lim_{j \in B} p_Y^1(T_j y) = +\infty$. Using this, we can construct a sequence $(x_k)_{k \in \mathbb{N}}$ in $X_0$, a strictly increasing sequence $(j_k)_{k \in \mathbb{N}}$ of positive integers and a sequence $(B_k)_{k \in \mathbb{N}}$ of subsets consisted of positive integers such that $\sup B_k < \inf B_{k+1}$ for all $k \in \mathbb{N}$, as well as that, for every $j, k \in \mathbb{N}$, one has: $B_k \subseteq [j_k + 1, j_k + m_k] \cap B$, $|B_k| \geq m_k - (s_k/k)$, $p_k(x_k) \leq 1$, $p_Y^1(T_j x_k) \geq k 2^j$ for $j \in B_k$, $p_Y^1(T_j x_s) \leq 1/(k+1)$ for all $s \in \mathbb{N}_{k-1}$ and $j \in \mathbb{N}$ with $j \in B_k$, as well as:

$$\frac{|\{1 \leq j \leq m_k : p_Y^1(T_j x_s) < 1/k\}|}{j_k} \geq 1 - k^{-2}, \quad s \in \mathbb{N}_{k-1}.$$

In particular, for each $k \in \mathbb{N}$ we have:

$$p_Y^1(T_j x_s) \leq 1/(k+1), \quad s \in \mathbb{N}_{k-1}, \quad j \in B_k. \quad (4.9)$$

Set $B' := \bigcup_{k \in \mathbb{N}} B_k$. Since $Bd_{T_{:\omega}}(B^c) = 0$, it is clear that $\liminf_{s \to \infty} \inf_{n \in \mathbb{N}} [B' \cap [n+1, n+m_s]] = 0$, which simply implies that $\liminf_{s \to \infty} \inf_{n \in \mathbb{N}} [B' \cap [n+1, n+m_s]] = 0$. On the other hand, Lemma 1.5(i) implies that $Bd_{T_{:\omega}}((B')^c) = 0$, provided that $(B')^c$ is finite of infinite non-syndetic. If $(B')^c$ is syndetic, then it is clear that there exists a finite constant $d > 0$ such that $Bd_{T_{:\omega}}((B')^c) \leq \liminf_{s \to \infty} (m_s/s) \leq d \liminf_{s \to \infty} \inf_{n \in \mathbb{N}} [B' \cap [n+1, n+m_s]] = 0$; summa summarum, we have $Bd_{T_{:\omega}}((B')^c) = 0$. Take now any strictly increasing sequence
(r_q)_{q \in \mathbb{N}} \text{ of positive integers such that } r_{q+1} \geq 1 + r_q + j_{r_q+1} \text{ for all } q \in \mathbb{N}. \text{ Let } 
abla \in \{0, 1\}^\mathbb{N} \text{ be a sequence defined by } \alpha_q = 1 \text{ iff } n = r_q \text{ for some } q \in \mathbb{N}. \text{ Further on, let } \eta \in \{0, 1\}^\mathbb{N} \text{ contain an infinite number of } 1's \text{ and let } \beta_q \leq \alpha_q \text{ for all } q \in \mathbb{N}. \text{ If } \beta_{r_k} = 1 \text{ for some } k \in \mathbb{N} \text{ and } x_\beta = \sum_{q=1}^\infty \beta_q x_{r_q} / 2^{q}, \text{ then for each } j \in B_{j_{r_k}} \text{ we have (see (4.9) and observe that for such values of } j \text{ we have } 1 + q \leq 1 + j_{r_k} \leq r_k + 1 \leq r_q \text{ for } q > k):

\begin{align*}
p_Y^1(T_j x_\beta) & \geq r_k - \sum_{q<k} p_Y^1(T_j x_{r_q}) / 2^{r_q} - \sum_{q>k} p_Y^1(T_j x_{r_q}) / 2^{r_q} \\
& \geq r_k - \sum_{q<k} p_Y^1(T_j x_{r_q}) / 2^{r_q} - \sum_{q>k} p_{1+j}(x_{r_q}) / 2^{r_q} \\
& \geq r_k - \sum_{q<k} 2^{r_q}(1 + r_q) / 2^{r_q} - \sum_{q>k} 1 / 2^{r_q} \geq r_k - 2.
\end{align*}

Hence, \( \lim_{j \in B'} p_Y^1(T_j x_\beta) = \infty \) and the vector \( x_\beta \) is reiteratively \( m_n \)-distributionally 1-unbounded. Arguing in the same way as in the proof of Theorem 4.1, we get that \( x_\beta \) is \( m_n \)-distributionally near to zero. The conclusion of theorem now simply follows by copying the final part of proof of \([6, \text{Theorem } 15]\).

For the orbits of a linear continuous operator \( T \in L(X) \), the condition (ii) in the formulation of Theorem 4.10 is equivalent to its unboundedness; therefore, Theorem 4.10 provides an extension of \([8, \text{Corollary } 21]\). We can also state the following corollary:

**Corollary 4.11** Suppose \( T \in L(X) \), \( X \) is a separable Banach space, \( X_0 \) is a dense linear subspace of \( X \) and \( \lim_{j \to \infty} T^j x = 0, x \in X_0 \). Then \( T \) is densely Li-Yorke chaotic iff \( T \) is densely reiteratively \( m_n \)-distributionally chaotic of type 0 or 1 +.

**Proof** All non-trivial that we need to show is that the dense Li-Yorke chaos for \( T \) implies dense reiterative \( m_n \)-distributional chaos of type 1 + for \( T \). Assume that \( T \) has an unbounded orbit. Arguing as in the first part of proof of Proposition 2.15, we get that \( T \) admits a reiteratively \( m_n \)-distributionally unbounded orbit. Applying Theorem 4.10, we get that there exists a dense uniformly reiteratively \( m_n \)-distributionally irregular manifold of type 1 + for \( T \).

**Example 4.12** It is worth noting that Corollary 4.11 can be applied in the analysis of multiplication operators and their adjoints in Hilbert spaces. For example, the Li-Yorke chaos of an adjoint multiplication operator \( M_\phi^* \) considered in \([8, \text{Theorem } 26(ii)]\) is equivalent not only to its hypercyclicity but also to the reiterative distributional chaos of type \( s \) for any \( s \in \{0, 1, 1^+\} \); this follows from Corollary 4.11 and the arguments contained in the proof of \([14, \text{Theorem } 4.5]\).

Arguing as in the proof of \([6, \text{Theorem } 15]\) and Theorem 4.10 above, we can similarly deduce the following extension of last mentioned result for general sequences of linear continuous operators (the extension is proper even for orbits of linear continuous operators):
Theorem 4.13  Suppose that $X$ is separable, $(m_n) \in \mathbb{R}$, $(T_j)_{j \in \mathbb{N}}$ is a sequence in $L(X, Y)$, $X_0$ is a dense linear subspace of $X$, as well as:

(i) $\lim_{j \to \infty} T_j x = 0$, $x \in X_0$.
(ii) there exists a vector $y \in X$ such that the sequence $(T_j y)_{j \in \mathbb{N}}$ is unbounded.

Then there exist a number $m \in \mathbb{N}$ and a dense submanifold $W$ of $X$ consisting of those vectors $x$ for which the sequence $(p_m(T_j x))_{j \in \mathbb{N}}$ is unbounded and which are $m_n$-distributionally near to zero for $(T_j)_{j \in \mathbb{N}}$. In particular, $(T_j)_{j \in \mathbb{N}}$ is densely Li-Yorke chaotic.

Remark 4.14  (i) It is worth noting that Theorems 4.1 and 4.13 provide extensions of [10, Theorem 3.1, Corollary 3.2], where the orbits of an operator in Banach space have been considered.
(ii) A slight generalization of Theorem 4.13 for disjoint Li-Yorke chaotic operators has been recently established and proved in [26].

Now, we will state and prove the following result, which is closely linked with Theorems 4.1 and 4.13:

Theorem 4.15  Suppose that $X$ is separable, $(T_j)_{j \in \mathbb{N}}$ is a sequence in $L(X, Y)$, $X_0$ is a dense linear subspace of $X$, as well as:

(i) $\lim_{j \to \infty} T_j x = 0$, $x \in X_0$.
(ii) there exist $m \in \mathbb{N}$, $c > 0$, $x \in X$ and set $B \subseteq \mathbb{N}$ such that $\overline{d}_{m_n}(B) = c$ and $\lim_{j \in B} p_m(T_j x) = +\infty$.

Then there exists a dense $m_n$-distributionally irregular manifold of type 2 for $(T_j)_{j \in \mathbb{N}}$, and particularly, $(T_j)_{j \in \mathbb{N}}$ is densely $m_n$-distributionally chaotic of type 2.

Proof  In order to see that Eq. (4.1) can be again used with $m = 1$, it suffices to observe the following:

1. Suppose that $Y$ is a pure Fréchet space. Then we can always construct a fundamental system $(p_m^Y(\cdot))_{m \in \mathbb{N}}$ of increasing seminorms on the space $X$, inducing the same topology on $X$, so that $p_m^Y(T_j x) \leq p_{j+m}^Y(x)$, $x \in X$, $j$, $m \in \mathbb{N}$.

2. If $(Y, \| \cdot \|_Y)$ is a Banach space, then we can renorm $Y$ by using the fundamental system $(p_m^Y(\cdot) \equiv m \cdot \|\cdot\|_m \in \mathbb{N})$, turning $Y$ into a linearly and topologically homeomorphic Fréchet space for which the induced metric $d_Y : Y \times Y \to [0, 1]$ satisfies $d_Y(x, y) \leq \|x - y\| \sum_{m=0}^{\infty} m^{-m}$ for all $x$, $y \in Y$ as well as the implication: $\|x\| > k^{-1}$ for some $x \in X$ and $k \in \mathbb{N}$ $\Rightarrow$ $d_Y(x, 0) \geq 1/2(k+1)$.

In our new situation, we can construct a sequence $(x_k)_{k \in \mathbb{N}}$ in $X_0$ and a strictly increasing sequence $(j_k)_{k \in \mathbb{N}}$ of positive integers such that, for every $k \in \mathbb{N}$, one has: $p_k(x_k) \leq 1$,

$$\left| \left\{ 1 \leq j \leq m_{j_k} : p_Y^Y(T_j x_k) > 2^k \right\} \right| \geq c j_k \left( 1 - k^{-2} \right)$$

and

$$\left| \left\{ 1 \leq j \leq m_{j_k} : p_Y^Y(T_j x_k) < 1/k \right\} \right| \geq m_{j_k} - \frac{c j_k}{2 k}, \quad s \in \mathbb{N}_{k-1}.$$
The remaining part of proof can be deduced by repeating verbatim the arguments used in the proofs of Theorem 4.1 and [6, Theorem 15].

We can also clarify the following statement regarding the existence of dense $m_n$-distributionally irregular manifolds of type 2$_{B^d}$:

**Theorem 4.16** Suppose that $X$ is separable, $(T_j)_{j \in \mathbb{N}}$ is a sequence in $L(X, Y)$, $X_0$ is a dense linear subspace of $X$, as well as:

(i) $\lim_{j \to \infty} T_j x = 0$, $x \in X_0$.
(ii) there exist $m \in \mathbb{N}$, $c > 0$, $x \in X$ and set $B \subseteq \mathbb{N}$ such that $Bd_{l:m_n}(B) = c$ and $\lim_{j \in B} p^Y_m(T_j x) = +\infty$.

Then there exists a dense $m_n$-distributionally irregular manifold of type 2$_{B^d}$ for $(T_j)_{j \in \mathbb{N}}$, and particularly, $(T_j)_{j \in \mathbb{N}}$ is densely $m_n$-distributionally chaotic of type 2$_{B^d}$.

**Proof** Without loss of generality, we may assume that (4.1) holds and $m = 1$. Since $Bd_{l:m_n}(B) = c$, we can find two strictly increasing sequences $(sk)_{k \in \mathbb{N}}$ and $(jk)_{k \in \mathbb{N}}$ of positive integers such that for each $k \in \mathbb{N}$ we have that the set $B_k := B \cap [jk + 1, jk + ms_k]$ contains at least $sk(c - k^{-1})$ integers. Set $B' := \bigcup_{k \in \mathbb{N}} B_k$. Then

$$\frac{1}{sk} \sup_{n \in \mathbb{N}} |B' \cap [n + 1, n + ms_k]| \geq c(1 - k^{-1}), \quad k \in \mathbb{N},$$

which implies

$$Bd_{l:m_n}(B') = \lim_{s \to \infty} \inf_{s} \frac{\sup_{n \in \mathbb{N}} |B' \cap [n + 1, n + ms]|}{s} \geq c.$$ 

Now, we can construct a sequence $(x_k)_{k \in \mathbb{N}}$ in $X_0$ such that, for every $k \in \mathbb{N}$, one has: $p_k(x_k) \leq 1$, $p^Y_l(T_j x_k) \geq 2k^2$ for $j \in B_k$, $p^Y_l(T_j x_s) \leq 1/(k + 1)$ for all $s \in \mathbb{N}_{k-1}$ and $j \in \mathbb{N}$ with $j \in B_k$, as well as:

$$\frac{\{1 \leq j \leq mk : p^Y_l(T_j x_s) < 1/k\}}{jk} \geq 1 - k^{-2}, \quad s \in \mathbb{N}_{k-1}.$$

The final conclusion follows similarly as in the proof of Theorem 4.1.

We can simply formulate corollaries of Theorems 4.10, 4.15 and 4.16 for dense $\lambda$-reiterative distributional chaos of type $1^+$, dense $\lambda$-distributional chaos of type 2 and dense $\lambda$-distributional chaos of type $2_{B^d}$, respectively, as well as corollaries of Theorems 4.10, 4.13, 4.15 and 4.16 for dense $(\lambda -) m_n$-reiterative distributional chaos of type $1^+$, certain types of dense Li-Yorke chaos, dense $(\lambda -) m_n$-distributional chaos of type 2 and dense $(\lambda -) m_n$-distributional chaos of type $2_{B^d}$, respectively, for linear unbounded operators (see Corollaries 4.2 and 4.5). Concerning the possible applications of Theorems 4.1, 4.10, 4.13, 4.15 and 4.16 and these corollaries to unbounded linear operators, we would like to note that the additional use of regularizing operator $C \in L(X)$ does not take a right effect in the investigation of dense Li-Yorke chaos, in...
contrast to the notion of dense distributional chaos (cf. the second part of [11, Corollary 3.12]). On the other hand, there exists a great number of possible applications of the above-mentioned results to the sequences of bounded linear operators in Banach spaces, and we will present here only one illustrative:

**Example 4.17** Let $X := l^1$, $B \subseteq \mathbb{N}$ and $\overline{B}d(B) = 1$. Further on, let for each $j \in \mathbb{N}$ we have that $(\omega_j^1)_{n \in \mathbb{N}}$ is a bounded sequence of positive reals such that $\omega_j^1 > j^3$ for all $j \in B$, and let for each $j \in \mathbb{N}$ we have that $(a_j^1)_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive integers such that the sequence $(a_j^1)_{j \in \mathbb{N}}$ is strictly increasing, as well. Set

$T_j \langle x_n \rangle_{n \in \mathbb{N}} := \langle \omega_j^1 x_{a_j^1}, \omega_j^2 x_{a_j^2}, \ldots \rangle, \langle x_n \rangle_{n \in \mathbb{N}} \in X.$

Define $x_n := 0$ if $n \notin \bigcup_{j \in B} a_j^1$ and $x_n := j^{-2}$ if $n = a_j^1$ for some $j \in B$. Then it can be easily seen that the vector $(x_n)_{n \in \mathbb{N}}$ is reiteratively distributionally unbounded of type $1^+$ for $(T_j)_{j \in \mathbb{N}}$. By Theorem 4.1, it readily follows that the sequence $(T_j)_{j \in \mathbb{N}}$ is reiteratively distributionally chaotic of type $1^+$.

### 4.1 An Application to Abstract Partial Differential Equations

It is almost straightforward and rather technical to transfer all results proved in this section by now for operator families defined on the nonnegative real axis. For the sake of brevity and better exposition, we will consider here only continuous analogues of Theorem 4.1 and Corollaries 4.2 and 4.3.

Let $T(t) : D(T(t)) \subseteq X \rightarrow Y$ be a linear possibly not continuous mapping ($t \geq 0$). By $Z(T)$, we denote the set of all $x \in X$ such that $x \in D(T(t))$ for all $t \geq 0$ as well as that the mapping $t \mapsto T(t)x$, $t \geq 0$ is continuous. Denote by $m(\cdot)$ the Lebesgue measure on $[0, \infty)$ and by $F$ the class consisting of all increasing mappings $f : [0, \infty) \rightarrow [1, \infty)$ satisfying that $\lim \inf_{t \rightarrow +\infty} \frac{f(t)}{t} > 0$.

We will use the following continuous counterpart of Definition 1.1:

**Definition 4.18** [23] Let $A \subseteq [0, \infty)$, let $f \in F$, and let $q \in [1, \infty)$. Then:

(i) The lower $f$-density of $A$, denoted by $d_f(A)$, is defined through:

$$d_f(A) := \liminf_{t \rightarrow \infty} \frac{m(A \cap [0, f(t)])}{t}.$$ 

(ii) The lower $qc$-density of $A$, denoted by $d_{qc}(A)$, is defined through:

$$d_{qc}(A) := \liminf_{t \rightarrow \infty} \frac{m(A \cap [0, t^q])}{t}.$$ 

We introduce the notion of $\tilde{X}_f$-distributional chaos as follows:

**Definition 4.19** Suppose that $\tilde{X}$ is a non-empty subset of $X$, $T(t) : D(T(t)) \subseteq X \rightarrow Y$ is a linear possibly not continuous mapping ($t \geq 0$) and $f \in F$. If there exist an
uncountable set $S \subseteq Z(T) \cap \tilde{X}$ and $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x, y \in S$ of distinct points, we have that

$$d_f \left( \{t \geq 0 : d_Y(T(t)x, T(t)y) < \sigma \} \right) = 0,$$

and

$$d_f \left( \{t \geq 0 : d_Y(T(t)x, T(t)y) \geq \epsilon \} \right) = 0,$$

then we say that $(T(t))_{t \geq 0}$ is $\tilde{X}_f$ distributionally chaotic ($f$-distributionally chaotic, if $\tilde{X} = X$). Furthermore, we say that $(T(t))_{t \geq 0}$ is densely $\tilde{X}_f$-distributionally chaotic iff $S$ can be chosen to be dense in $\tilde{X}$. The set $S$ is said to be $\sigma_{\tilde{X}_f}$-scrambled set ($\sigma_f$-scrambled set in the case that $\tilde{X} = X$) of $(T(t))_{t \geq 0}$.

If $q \geq 1$ and $f(t) := 1 + t^q$ ($t \geq 0$), then we particularly obtain the notions of (dense) $\tilde{X}_q$-distributional chaos, (dense) $q$-distributional chaos, $\sigma_{\tilde{X}_q}$-scrambled set and $\sigma_q$-scrambled set for $(T(t))_{t \geq 0}$.

The basic result for applications is the following counterpart of Theorem 4.1, whose proof is very similar to that of the aforementioned theorem and therefore omitted (cf. also the proof of [11, Theorem 4.1]):

**Theorem 4.20** Suppose that $X$ is separable, $f \in F$ and $m_n := \lceil f(n) \rceil$, $n \in \mathbb{N}$. Suppose, further, that $(T(t))_{t \geq 0} \subseteq L(X, Y)$ is strongly continuous, $X_0$ is a dense linear subspace of $X$, as well as:

(i) $\lim_{t \to \infty} T(t)x = 0$, $x \in X_0$,

(ii) there exist a vector $y \in X$, a set $B \subseteq [0, \infty)$ and a number $m \in \mathbb{N}$ such that

$$\lim_{t \to \infty} \frac{f(t) - |B \cap [1, f(t)]|}{t} = 0$$

and

$$\lim_{t \to \infty, t \in B} p^m_Y(T(t)y) = +\infty. \quad (4.10)$$

Then $(T(t))_{t \geq 0}$ is densely $f$-distributionally chaotic and the corresponding scrambled set $S$ can be chosen to be a dense uniformly $f$-distributionally irregular submanifold of $X$. Furthermore, for each fixed number $t_0 > 0$, we have that the operator $T(t_0)$ is densely $m_n$-distributionally chaotic, and moreover, the corresponding scrambled set $S_{t_0}$ can be chosen to be a dense uniformly $m_n$-distributionally irregular submanifold of $X$.

We also attach the following obvious counterparts of Corollaries 4.2 and 4.3:

**Corollary 4.21** Suppose that $X$ is separable, $\lambda \in (0, 1]$, $(T(t))_{t \geq 0} \subseteq L(X, Y)$ is strongly continuous, $X_0$ is a dense linear subspace of $X$, as well as:
Reiterative $m_n$-Distributional Chaos of Type $s$ in Fréchet Spaces

(i) $\lim_{t \to \infty} T(t)x = 0$, $x \in X_0$,
(ii) there exist a vector $y \in X$, a set $B \subseteq [0, \infty)$ and a number $m \in \mathbb{N}$ such that

$$\liminf_{n \to \infty} \frac{t^{1/\lambda} - |B \cap [1, t^{1/\lambda}]|}{t} = 0$$

and (4.10) holds.

Then $(T(t))_{t \geq 0}$ is densely $\lambda$-distributionally chaotic and the corresponding scrambled set $S$ can be chosen to be a dense uniformly $\lambda$-distributionally irregular submanifold of $X$. Furthermore, for each fixed number $t_0 > 0$, we have that the operator $T(t_0)$ is densely $m_n$-distributionally chaotic, and moreover, the corresponding scrambled set $S_{t_0}$ can be chosen to be a dense uniformly $m_n$-distributionally irregular submanifold of $X$; here, $m_n := \lceil n^q \rceil$ for all $n \in \mathbb{N}$.

**Corollary 4.22** Suppose that $X$ is separable, $(T(t))_{t \geq 0} \subseteq L(X, Y)$ is strongly continuous, $f \in F$, $X_0$ is a dense linear subspace of $X$, as well as:

(i) $\lim_{t \to \infty} T(t)x = 0$, $x \in X_0$.
(ii) there exist a vector $y \in X$ and a number $m \in \mathbb{N}$ such that (4.10) holds with $B = \mathbb{N}$.

Then $(T(t))_{t \geq 0}$ is densely $f$-distributionally chaotic and the corresponding scrambled set $S$ can be chosen to be a dense uniformly $f$-distributionally irregular submanifold of $X$. Furthermore, for each fixed number $t_0 > 0$, we have that the operator $T(t_0)$ is densely $m_n$-distributionally chaotic, and moreover, the corresponding scrambled set $S_{t_0}$ can be chosen to be a dense uniformly $m_n$-distributionally irregular submanifold of $X$; here, $m_n := \lfloor f(n) \rfloor$ for all $n \in \mathbb{N}$.

We continue with the observation that we have not used any semigroup property of operator family $(T(t))_{t \geq 0}$ under our consideration so that the results of this subsection are applicable in the qualitative analysis of solutions for certain classes of the abstract (multi-term) fractional differential equations with Caputo derivatives. Here, we will present only one example of possible application of this type; for more details on how we can incorporate Theorem 4.20 and Corollaries 4.21 and 4.22 in the qualitative analysis of solutions to abstract fractional PDEs, the reader may consult [19, Section 3.3]:

**Example 4.23** (cf. [17,21] for the notion) Suppose that $X$ is a symmetric space of non-compact type and rank one, $p > 2$, the parabolic domain $P_p$ and the positive real number $c_p$ possess the same meaning as in [17]. Suppose, further, that $\Delta_{X,p}^{\frac{1}{2}}$ denotes the corresponding Laplace–Beltrami operator and $P(z) = \sum_{j=0}^{n} a_j z^j$, $z \in \mathbb{C}$ is a non-constant complex polynomial with $a_n > 0$. Consider the abstract fractional Cauchy problem:

$$D_2^{2a} u(t) + cu(t) = -e^{it} P(\Delta_{X,p}^{\frac{1}{2}}) D_1^{2a} u(t), \quad t \geq 0,$$

$$u^{(k)}(0) = u_k, \quad k = 0, \ldots, \lceil 2a \rceil - 1,$$
where $0 < a < 2$, $c > 0$ and $|\theta| < \min\left(\frac{\pi}{2} - n \arctan \frac{|p-2|}{2\sqrt{p-1}}, \frac{\pi}{2} - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - \frac{\pi}{2}a\right)$. Then $-e^{i\theta} P(\Delta_{X,p})$ generates an exponentially bounded, analytic resolvent propagation family $((R_\theta, p, 0(t))_{t \geq 0}, \ldots, (R_\theta, p, [2a]-1(t))_{t \geq 0})$ of certain angle; see [19] for the notion. Applying Corollary 4.22 and the analysis from [21, Example 2.8], we can show that the condition

$$-e^{i\theta} P\left(\text{int}(P_p)\right) \cap \left\{ (it)^a + c(it)^{-a} : t \in \mathbb{R}\setminus\{0\} \right\} \neq \emptyset$$

implies that $(R_\theta, p, 0(t))_{t \geq 0}$ is densely $f$-distributionally chaotic and for each $t_0 > 0$ the operator $T(t_0)$ is densely $m_n$-distributionally chaotic, where $m_n := \lceil f(n) \rceil$ for all $n \in \mathbb{N}$ ($f \in F$).

## 5 Conclusions, Final Remarks and Open Problems

In this section, we provide several observations and remarks about results obtained so far and ask some questions. In the considerations of backward shift operators, we will always assume that $X$ is a Fréchet sequence space in which $(e_n)_{n \in \mathbb{N}}$ is basis and $(\omega_n)_{n \in \mathbb{N}}$ is a sequence of positive weights; furthermore, we will always assume that the unilateral weighted backward shift $T_{\omega}$, given by (4.7), is a continuous linear operator on $X$. Recall that the finite linear combinations of vectors from the basic $(e_n)_{n \in \mathbb{N}}$ form a dense submanifold of $X$.

First of all, we would like to ask the following questions:

**Problem 5.1** Suppose $(m_n) \in \mathbb{R}$ and $T_{\omega}$ is distributionally chaotic. Is it true that $T_{\omega}$ is $m_n$-distributionally chaotic?

**Problem 5.2** Let $(m_n) \in \mathbb{R}$, let $T \in L(X)$ satisfy that there exists a dense submanifold $X_0$ of $X$ such that $\lim_{n \to \infty} T^n x = 0$ for all $x \in X_0$, and let $T$ be distributionally chaotic. Is it true that $T$ is $m_n$-distributionally chaotic?

Concerning these problems, irrelevant of the fact whether the answers to them are affirmative or not, we would like to note that combining Theorems 3.12 and 4.1 immediately yields the following extension of [6, Theorem 25]:

**Theorem 5.3** Suppose that $(m_n) \in \mathbb{R}$ and $T \in L(X)$ satisfies that there exists a dense submanifold $X_0$ of $X$ such that $\lim_{n \to \infty} T^n x = 0$ for all $x \in X_0$. Then the following assertions are equivalent:

(i) $T$ is $m_n$-distributionally chaotic.

(ii) $T$ is densely $m_n$-distributionally chaotic (of type 1).

(iii) There exists an $m_n$-distributionally unbounded (irregular) vector for $T$.

(iv) There exists a dense uniformly $m_n$-distributionally irregular submanifold for $T$.

The following generalization of [6, Theorem 26, Corollary 27] can be proved as for distributional chaos (similarly we can reconsider the statements [6, Theorem 29-Theorem 30, Corollary 31-Corollary 32] for $m_n$-distributional chaos in Fréchet.
sequence spaces in which \((e_n)_{n \in \mathbb{Z}}\) is a basis; with the exception of [6, Problem 23], we obtain further extensions of all other statements established in [6, Section 3] for sequences of operators):

**Theorem 5.4**

(i) Suppose that \((m_n) \in \mathbb{R}\) and the operator \(T\) is given by (4.7) with the weight \(w_n \equiv 1\) \((n \in \mathbb{N})\). Let there exist a subset \(S\) of natural numbers such that the series \(\sum_{n \in S} e_n\) converges in \(X\), and \(d_{m_n}(S^c) = 0\). Then the operator \(T\) is densely \(m_n\)-distributionally chaotic.

(ii) Suppose that \((m_n) \in \mathbb{R}\) and the operator \(T\) satisfies that there exists a subset \(S\) of natural numbers such that the series \(\sum_{n \in S} (\prod_{i=1}^{n} \omega_i)^{-1} e_n\) converges in \(X\), and \(d_{m_n}(S^c) = 0\). Then the operator \(T\) is densely \(m_n\)-distributionally chaotic.

**Corollary 5.5**

(i) Suppose that \(\lambda \in (0, 1]\) and the operator \(T\) is given by (4.7) with the weight \(w_n \equiv 1\) \((n \in \mathbb{N})\). Let there exist a subset \(S\) of natural numbers such that the series \(\sum_{n \in S} e_n\) converges in \(X\), and \(d_{1/\lambda}(S^c) = 0\). Then the operator \(T\) is densely \(\lambda\)-distributionally chaotic.

(ii) Suppose that \(\lambda \in (0, 1]\) and the operator \(T\) satisfies that there exists a subset \(S\) of natural numbers such that the series \(\sum_{n \in S} (\prod_{i=1}^{n} \omega_i)^{-1} e_n\) converges in \(X\), and \(d_{1/\lambda}(S^c) = 0\). Then the operator \(T\) is densely \(\lambda\)-distributionally chaotic.

Now, we would like to ask the following:

**Problem 5.6** Suppose that \(X := l^p(\mathbb{N})\) for some \(p \in [1, \infty)\) or \(X := c_0(\mathbb{N})\), and \(\lambda \in (0, 1]\). Is it true that there exists a densely \(\lambda\)-distributionally chaotic backward shift operator \(T\) which is \(\lambda\)-distributionally chaotic and not \(\lambda'\)-distributionally chaotic for any number \(\lambda' \in (0, \lambda)\)?

Regarding distributionally chaotic backward shift operators, mention should be also made of papers [30,31,34,37]. For the sake of brevity, we will not reconsider the related problematic for \(m_n\)-distributional chaos here.

Concerning applications to the abstract partial differential equations of first order whose solutions are governed by strongly continuous semigroups, it is clear that our results from Sect. 4.1 can be employed at any place where the Desch–Schappacher–Webb criterion [12] is employed (see, e.g., [1,11] and references cited therein); concerning applications to the abstract ill-posed partial differential equations of first order whose solutions are governed by fractionally integrated \(C\)-semigroups, our results can be used to the equations considered in [11] and [18, Subsection 3.1.4]. But, if we are in a position, for example, in which the requirements of the Desch–Schappacher–Webb criterion hold for a strongly continuous semigroup \((T(t))_{t \geq 0}\), then we always have the existence of a dense linear subspace \(X_0\) of \(X\) satisfying \(\lim_{t \to \infty} T(t)x = 0\) for all \(x \in X_0\), so that it is quite natural to ask the following:

**Problem 5.7** Suppose that \((T(t))_{t \geq 0}\) is a distributionally chaotic, strongly continuous semigroup on \(X\) and there exists a dense linear subspace \(X_0\) of \(X\) satisfying \(\lim_{t \to \infty} T(t)x = 0\) for all \(x \in X_0\). Is it true that \((T(t))_{t \geq 0}\) is \(f\)-distributionally chaotic for all \(f \in F\)?

It seems very plausible that [11, Theorem 4.2] admits a reformulation for \(m_n\)-distributional chaos, so that a positive solution to Problem 5.2 immediately answers
Problem 5.7 in the affirmative. (Observe, however, that it is not clear how one can reconsider the above-mentioned theorem for fractional solution operator families.)

Suppose finally that $1 \leq p < \infty$. We refer the reader to [12, Definition 4.3] for the notions of an admissible weight function $\rho : [0, \infty) \to (0, \infty)$ and the Banach spaces $L^p_{\rho}([0, \infty), \mathbb{K}), C_{\rho, p}([0, \infty), \mathbb{K})$. We close the paper by proposing the following continuous counterpart of Problem 5.6:

**Problem 5.8** Suppose that $\lambda \in (0, 1]$. Can we find an admissible weight function $\rho(\cdot)$ and a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X := L^p_{\rho}([0, \infty), \mathbb{K})$ or $X := C_{\rho, p}([0, \infty), \mathbb{K})$, which is $\lambda$-distributionally chaotic and not $\lambda'$-distributionally chaotic for any number $\lambda' \in (0, \lambda)$?

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