Further Results on the Convergence of the Pavon–Ferrante Algorithm for Spectral Estimation

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Abstract

In this paper, we provide a detailed analysis of the global convergence properties of an extensively studied and extremely effective fixed-point algorithm for the Kullback–Leibler approximation of spectral densities, proposed by Pavon and Ferrante in [1]. Our main result states that the algorithm globally converges to one of its fixed points.

Index Terms

Approximation of spectral densities, spectral estimation, generalized moment problems, Kullback–Leibler divergence, fixed-point iteration, convergence analysis.

I. INTRODUCTION

In recent years, the problem of approximating—in an optimal sense—a spectral density with another one satisfying some given constraints has received considerable attention in the control and signal processing community. In [2], Georgiou and Lindquist considered the following formulation of the above-mentioned approximation problem: Find the optimal Kullback–Leibler approximation of a spectral density given

(i) an a-priori estimate of the spectrum describing a zero-mean second-order stationary process, and

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(ii) asymptotic state-covariance data that are typically inconsistent with the a-priori estimate.

These data are obtained by feeding the above process to a measurement device consisting of a bank of rational filters.

According to this formulation, the approximation problem turns into a (convex) optimization problem with integral constraints which falls into the celebrated category of (generalized) moment problems. During the past century, the latter class of problems has played a crucial role in many areas of the engineering and mathematical sciences, see e.g. [3], [4] and references therein. In particular, some noteworthy (generalized) moment problems, which are close relatives of the Georgiou–Lindquist approximation problem, are the covariance extension problem [5, Ch. 12.5], [6], [7], [8], THREE-like spectral estimation [9], [10], [11], [12], the classical Nevanlinna–Pick interpolation problem [13], [14], [15], and its generalization, the augmented Basic Interpolation Problem (aBIP) [16], [17]. Moreover, variations of the latter classes of problems have generated a huge stream of literature in recent years, see for instance [18], [19], [20], [21], [22], [23], [24], [25], [26]. Among the large number of applications emerging from the class of (generalized) moment problems it is worth mentioning, besides spectral estimation, those related to system modeling/identification and optimal $H_\infty$ control [27], [28], [29].

In [2], the optimization problem was approached by resorting to the dual problem which is finite-dimensional but, in general, it does not admit a closed-form solution. Also, standard gradient-based minimization techniques for the numerical solution of the dual problem have proved to be computationally demanding and severely ill-conditioned. In order to tackle these issues, in [1], Pavon and Ferrante proposed an alternative iterative method for the solution of the Georgiou–Lindquist approximation problem. The Pavon–Ferrante algorithm is a surprisingly simple and efficient nonlinear fixed-point iteration in the set of positive semi-definite unit trace matrices. Furthermore, the algorithm exhibits very attractive and robust properties from a numerical viewpoint, since it can be implemented via the solution of an algebraic Riccati equation and a Lyapunov equation [30]. On the other hand, despite the huge amount of numerical evidences, proving the convergence of the latter algorithm to a prescribed set of fixed points—which provide the solution of the approximation problem—has revealed to be an highly non-trivial challenge [31], [30]. In particular, in [30] it has been shown that the Pavon–Ferrante algorithm is locally convergent to the aforesaid set of fixed points, through a rather tortuous yet enlightening proof. Nonetheless, a proof of global convergence of the algorithm, though conjectured and supported by a large number of numerical simulations, has so far been elusive.
The present paper addresses this problem. Specifically, we consider a cost functional arising from the formulation of the dual problem and we show that the latter is decreasing along the trajectories generated by the Pavon–Ferrante algorithm. This provides an answer to a conjecture raised in [30, Sec. V] and leads to the main contribution of the paper, namely, a proof of global convergence of the Pavon–Ferrante algorithm towards its set of fixed points.

Paper structure. The paper is organized as follows. In Section II, we introduce the Georgiou–Lindquist spectral approximation problem and its solution via the Pavon–Ferrante algorithm. In Section III, we establish some auxiliary results. Section IV contains the proof of global convergence to the set of fixed points of the Pavon–Ferrante algorithm. Finally, Section V collects some concluding remarks.

Notation. We let \( \mathbb{Z}, \mathbb{C}, \mathbb{R}, \mathbb{R}_{>0}, \mathbb{R}_{\geq 0}, \) and \( \mathbb{T} \) denote the set of integer, complex, real, positive real, non-negative real numbers, and the unit circle in the complex plane, respectively. We denote by \( \mathbb{C}^{n \times m} \) the set of \( n \times m \) matrices with complex entries. Given \( A \in \mathbb{C}^{n \times m} \), \( A^* \) will denote the Hermitian transpose of \( A \). We write \( A \succeq 0 \) \( (A > 0) \) to mean that \( A = A^* \in \mathbb{C}^{n \times n} \) is positive semi-definite (positive definite, respectively). For \( A \succeq 0 \), \( A^{1/2} \) will denote the principal matrix square root of \( A \), i.e., the unique positive semi-definite Hermitian matrix whose square is \( A \). We endow the space \( \mathbb{C}^n \) with the standard inner product \( \langle x, y \rangle = x^* y \) and norm \( \|x\|^2 := \langle x, x \rangle \), for \( x, y \in \mathbb{C}^n \), and the space of Hermitian matrices of dimension \( n \times n \), denoted by \( \mathbb{H}_n \), with the trace inner product \( \langle X, Y \rangle := \text{tr}(XY^*) \) and Frobenius norm \( \|X\|_F^2 := \text{tr}(XX^*) \), for \( X, Y \in \mathbb{H}_n \), where \( \text{tr}(\cdot) \) denotes the trace operator. Moreover, we denote by \( \mathcal{S}_n \) the convex set of \( n \times n \) positive semi-definite unit trace matrices. For a matrix-valued function \( G(z) \) in the complex variable \( z \), \( G^*(z) \) will denote the analytic continuation of the function that for \( z \in \mathbb{T} \) equals the Hermitian transpose of \( G(z) \). Finally, \( \mathcal{C}(\mathbb{T}) \) will denote the set of continuous functions on \( \mathbb{T} \), and \( \mathcal{C}_+(\mathbb{T}) \) the set of continuous functions on \( \mathbb{T} \) which take (strictly) positive values on the same region, i.e., the space of continuous coercive discrete-time spectral density functions.

II. Problem statement

In this section, we first review the spectral density approximation problem treated in [2]. Then, we discuss its solution via the algorithm proposed in [1].
A. The Georgiou–Lindquist approximation problem

Let \( \{ y(t), t \in \mathbb{Z} \} \) be a zero-mean purely nondeterministic second-order stationary process and assume that an a-priori estimate \( \Psi(e^{j\theta}) \in \mathcal{C}_+(\mathbb{T}) \) of the spectral density of \( y(t) \) is given. Consider the rational matrix transfer function

\[
G(z) = (zI - A)^{-1}B, \quad A \in \mathbb{C}^{n \times n}, \quad B \in \mathbb{C}^{n \times 1},
\]

of the discrete-time system

\[
x(t + 1) = Ax(t) + By(t), \quad t \in \mathbb{Z},
\]

where \( A \) is Schur stable, i.e., all the eigenvalues of \( A \) are strictly inside \( \mathbb{T} \), and the pair \((A, B)\) is reachable. Note that the \( n \)-dimensional process \( x(t) := [x_1(t), x_2(t), \ldots, x_n(t)]^\top \) coincides with the output of a bank of \( n \) filters \( G(z) := [g_1(z), g_2(z), \ldots, g_n(z)]^\top \) fed by \( y(t) \).

Suppose we know the steady-state covariance of the process \( x(t) \), which we denote by \( \Sigma > 0 \). Given the estimate \( \Psi(e^{j\theta}) \) and the steady-state covariance \( \Sigma \), the task is to estimate the spectral density of the process \( y(t) \). To this end, we need to find the spectral density \( \hat{\Phi}(e^{j\theta}) \in \mathcal{C}_+(\mathbb{T}) \) which is the “closest possible”, in a suitable sense, to the a-priori estimate \( \Psi \) among all spectra \( \Phi(e^{j\theta}) \in \mathcal{C}_+(\mathbb{T}) \) satisfying the constraint

\[
\int_{-\pi}^{\pi} G(e^{j\theta})\Phi(e^{j\theta})G^*(e^{j\theta}) \frac{d\theta}{2\pi} = \int G_\theta \Phi_\theta G^*_\theta = \Sigma.
\]

(In order to lighten the notation, throughout the paper we let \( G_\theta := G(e^{j\theta}), \Phi_\theta := \Phi(e^{j\theta}), \Psi_\theta := \Psi(e^{j\theta}) \), and for integrals we use the above shorthand, where the integration takes place on the unit circle w.r.t. the normalized Lebesgue measure.) By using the Kullback–Leibler divergence [32] as “measure of closeness” between spectral densities, namely,

\[
\mathcal{S}(\Phi_\theta \| \Psi_\theta) := \int \Psi_\theta \log \frac{\Psi_\theta}{\Phi_\theta},
\]

the problem can be formally stated as follows.

**Problem 1** (Georgiou–Lindquist approximation problem [2]). Let \( \Psi_\theta \in \mathcal{C}_+(\mathbb{T}) \) and \( \Sigma \in \mathbb{H}_n \), \( \Sigma > 0 \). Find \( \hat{\Phi}_\theta \in \mathcal{C}_+(\mathbb{T}) \) that solves

\[
\min_{\Phi_\theta \in \mathcal{K}} \mathcal{S}(\Phi_\theta \| \Psi_\theta),
\]

\(^1\)On the problem of estimating covariance matrices from measurements obtained by linear filtering see also [22], [23].
where

\[ \mathcal{K} := \left\{ \Phi_\theta \in C_+(\mathbb{T}) : \int G_\theta \Phi_\theta G_\theta^* = \Sigma \right\}. \quad (1) \]

The variational analysis outlined in [2] (see also [1], [30] where some additional details are spelled out and [10], [33] for the existence part) leads to the following result.

**Theorem 1.** The set \( \mathcal{K} \) as defined in (1) is non-empty if and only if there exists \( H \in \mathbb{C}^{1 \times n} \) s.t.

\[ \Sigma - A\Sigma A^* = BH + H^* B^*. \quad (2) \]

Moreover, assuming that the above condition is fulfilled, there exists \( \hat{\Lambda} \in \mathbb{H}_n \) such that

\[ G_\theta^* \hat{\Lambda} G_\theta > 0, \quad \forall \theta \in [-\pi, \pi), \quad (3) \]

\[ \int G_\theta \frac{\Psi_\theta}{G_\theta^* \hat{\Lambda} G_\theta} G_\theta^* = \Sigma. \quad (4) \]

For any such \( \hat{\Lambda} \),

\[ \hat{\Phi}_\theta := \frac{\Psi_\theta}{G_\theta^* \hat{\Lambda} G_\theta} \quad (5) \]

is the unique solution of Problem 1.

Supposing the feasibility condition (2) satisfied, in view of the above theorem, Problem 1 can be reduced to the problem of finding \( \Lambda \in \mathbb{H}_n \) satisfying conditions (3)-(4).

In [2], Georgiou and Lindquist exploited duality theory to arrive at the equivalent convex optimization problem

\[ \min_{\Lambda \in \mathcal{L}} \mathcal{J}(\Lambda), \quad (6) \]

where

\[ \mathcal{J} : \mathcal{L} \rightarrow \mathbb{R} \]

\[ \Lambda \mapsto \text{tr}(\Lambda \Sigma) - \int \Psi_\theta \log G_\theta^* \Lambda G_\theta, \quad (7) \]

and

\[ \mathcal{L} := \{ \Lambda \in \mathbb{H}_n : G_\theta^* \Lambda G_\theta > 0, \forall \theta \in [-\pi, \pi) \}. \]
In [2, Thm. 5] it has been established that the above dual problem admits a unique solution \( \hat{\Lambda} \) on \( \mathcal{L}(\Gamma) := \mathcal{L} \cap \text{Range} \Gamma \), where \( \Gamma \) is the linear operator defined by

\[
\Gamma: \mathcal{C}(T) \rightarrow \mathbb{H}_p,
\Phi_\theta \longmapsto \int G_\theta \Phi_\theta G_\theta^*,
\tag{8}
\]

and such a solution satisfies (4), so that \( \hat{\Lambda} \) returns the optimal estimate \( \hat{\Phi}_\theta \) via (5). After a suitable parametrization of \( \mathcal{L}(\Gamma) \), problem (6) can be numerically solved using Newton-like minimization methods [2, Sec. VII]. However, as mentioned in the introduction, these techniques are affected by several numerical issues related to unboundedness of the gradient of \( J(\cdot) \) around the neighborhood of the boundary and high computational burden due to a large number of backstepping iterations [2, Sec. VII], [1], [31], [30].

B. The Pavon–Ferrante algorithm

An alternative, numerically robust algorithm for the solution of Problem 1 has been proposed by Pavon and Ferrante in [1] and further investigated in [31], [30]. Before presenting the algorithm, we introduce some simplifications in the formulation of Problem 1; namely, we suppose

(i) the a-priori estimate \( \Psi_\theta \) to be such that \( \int \Psi_\theta = 1 \), and

(ii) the steady-state covariance \( \Sigma \) to be normalized to identity, i.e., \( \Sigma = I \).

These assumptions can be made without any loss of generality, as explained in [30, Remark 2.3]. Furthermore, notice that, with these simplifications in place, the cost functional \( J(\cdot) \) in (7) becomes

\[
J(\Lambda) = \text{tr}(\Lambda) - \int \Psi_\theta \log G_\theta^* \Lambda G_\theta.
\tag{9}
\]

The Pavon–Ferrante algorithm is a fixed-point iteration of the form

\[
\Lambda_{k+1} = \Theta(\Lambda_k) := \int \Lambda_k^{1/2} G_\theta \left( \frac{\Psi_\theta}{G_\theta^* \Lambda_k G_\theta} \right) G_\theta^* \Lambda_k^{1/2},
\tag{10}
\]

\(^2\)It is worth observing however that Problem (6) has in general infinitely many solutions on \( \mathcal{L} \).
for \( k \in \mathbb{Z}, \, k \geq 0 \), where the initialization is taken to be a positive definite trace-one matrix \( \Lambda_0 > 0 \). Iteration (10) features several remarkable properties. Firstly, it preserves unit trace and positivity. Furthermore, by introducing the sets

\[
\mathcal{M} := \{ \Lambda \in \mathbb{S}_n : G^*_\theta \Lambda G_\theta > 0, \ \forall \theta \in [-\pi, \pi) \},
\]

\[
\mathcal{M}_+ := \{ \Lambda \in \mathcal{M} : \Lambda > 0 \} \subset \mathcal{M},
\]

it has been shown in [1, Thm. 4.1] that \( \Theta(\cdot) \) maps elements of \( \mathcal{M} (\mathcal{M}_+) \) into elements of \( \mathcal{M} (\mathcal{M}_+, \text{respectively}) \).

Secondly, and most importantly, if iteration (10) converges to a positive definite fixed point of \( \Theta(\cdot) \), say \( \hat{\Lambda} > 0 \), then \( G^*_\theta \hat{\Lambda} G_\theta > 0, \ \forall \theta \in [-\pi, \pi) \), and, by multiplying Eq. (10) on both sides by \( \hat{\Lambda}^{-1/2} \),

\[
\int G_\theta \frac{\Psi_\theta}{G^*_\theta \hat{\Lambda} G_\theta} G^*_\theta = I,
\]

so that conditions (3)-(4) are satisfied. As a consequence, if the feasibility condition in (2) is satisfied by \( \Sigma = I \), such a \( \hat{\Lambda} \) yields the solution of Problem 1 via (5). Importantly, such a fixed point always exists. In fact, let \( \mathcal{S} \) denote the space of \( \Lambda \in \mathbb{H}_n \) satisfying (3)-(4), in [30, Thm. 3.2] it has been shown that the set of positive definite fixed points of iteration (10) defines a non-empty open convex set \( \mathcal{P} \) of the space \( \mathcal{S} \). To conclude, we remark that the positive definite ones are not the only fixed points of iteration (10) which provide the solution to Problem 1. Indeed, in the closure of \( \mathcal{P} \) there exist singular elements which still satisfy (3)-(4) and, thus, solve Problem 1 via (5), see [30, Sec. II-B]. In general, however, singular fixed points are not guaranteed to satisfy conditions (3)-(4) and, therefore, to solve Problem 1 via (5).

### III. Preliminary Results

In this section, we collect some auxiliary results which will be used in the proof of the main theorems presented in the next section. The first result is a consequence of Jensen’s inequality [32, Thm. 2.6.2].

**Lemma 1.** Let \( X \subseteq \mathbb{R} \). Consider an integrable function \( f: X \to \mathbb{R}_{>0} \) and an integrable function \( w: X \to \mathbb{R}_{>0} \) satisfying \( \int_X w(x) \, dx = 1 \), then

\[
\log \int_X w(x)f(x) \, dx \geq \int_X w(x) \log f(x) \, dx,
\]

and the equality is attained if and only if \( f(x) \) is constant a.e. on \( X \).
Another ancillary lemma is stated and proved below.

**Lemma 2.** Let $X \subseteq \mathbb{R}$. Let $w: X \rightarrow \mathbb{R}$ and $f: X \times X \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions with $f(\cdot, \cdot)$ symmetric, i.e. $f(x, y) = f(y, x)$ for all $x, y \in X$. Then it holds

$$
\int_X \int_X w(x)^2 f(x, y) \, dx \, dy \geq \int_X \int_X w(x) w(y) f(x, y) \, dx \, dy.
$$

**(13)**

**Proof.** Thanks to the symmetry of $f(\cdot, \cdot)$, we have

$$
\int_X \int_X w(x)^2 f(x, y) \, dx \, dy = \int_X \int_X \left( \frac{w(x)^2 + w(y)^2}{2} \right) f(x, y) \, dx \, dy.
$$

Now, since $w(x)^2 + w(y)^2 \geq 2w(x)w(y)$, due to the fact that $(w(x) - w(y))^2 \geq 0$, for all $x, y \in X$, the claim follows.

Next we focus the attention on the function $J(\cdot)$, as defined in Eq. (9). This function will play a key role in the convergence analysis presented in Section IV.

**Lemma 3.** $J(\cdot)$ is a continuous and bounded function on $\mathcal{S}_n$.

**Proof.** We first note that, in view of the stability of $A$ and reachability of the pair $(A, B)$, for all $\Lambda \in \mathcal{S}_n$, $G^*_\theta \Lambda G_\theta$ is a nonzero rational spectral density analytic on (an open annulus containing) $\mathbb{T}$. This in turn implies that $\log G^*_\theta \Lambda G_\theta$ is integrable on $\mathbb{T}$, see e.g. [34, p. 64]. Since $\Psi_\theta$ is bounded on $\mathbb{T}$, $\Psi_\theta \log G^*_\theta \Lambda G_\theta$ is again integrable on $\mathbb{T}$. This in turn implies that $J(\cdot)$ is bounded on $\mathcal{S}_n$. Now let $\bar{\Lambda} \in \mathcal{S}_n$ and consider any sequence $\{\Lambda_k\}_{k \geq 0}$ in $\mathcal{S}_n$ such that $\lim_{k \rightarrow \infty} \Lambda_k = \bar{\Lambda}$. The corresponding sequence $\{G^*_\theta \Lambda_k G_\theta\}_{k \geq 0}$ is composed of nonzero rational spectral densities analytic on $\mathbb{T}$ and such that $\lim_{k \rightarrow \infty} G^*_\theta \Lambda_k G_\theta = G^*_\theta \bar{\Lambda} G_\theta$ uniformly on $\mathbb{T}$, where the limit $G^*_\theta \bar{\Lambda} G_\theta$ is a spectral density as before. Hence, from [35, Cor. 4.6], it follows that the sequence $\{\log G^*_\theta \Lambda_k G_\theta\}_{k \geq 0}$ is uniformly integrable on $\mathbb{T}$. Eventually, since $\Psi_\theta$ is bounded on $\mathbb{T}$, $\{\Psi_\theta \log G^*_\theta \Lambda_k G_\theta\}_{k \geq 0}$ is again uniformly integrable, so that by Vitali’s convergence theorem [36, p. 133] it holds

$$
\lim_{k \rightarrow \infty} J(\Lambda_k) = 1 - \lim_{k \rightarrow \infty} \int \Psi_\theta \log G^*_\theta \Lambda_k G_\theta
$$

$$
= 1 - \int \lim_{k \rightarrow \infty} \Psi_\theta \log G^*_\theta \Lambda_k G_\theta
$$

$$
= 1 - \int \Psi_\theta \log G^*_\theta \bar{\Lambda} G_\theta = J(\bar{\Lambda}),
$$

which proves continuity.
Consider the orthogonal complement (w.r.t. the trace inner product in $\mathbb{H}_n$) of Range $\Gamma$, where the linear operator $\Gamma$ has been defined in (8). This quantity has been shown in [18, Sec. IV-A] to be given by

$$(\text{Range } \Gamma)^\perp = \{ X \in \mathbb{H}_n : G^{*}_\theta X G_\theta = 0, \ \forall \theta \in [-\pi, \pi] \}. \quad (14)$$

Given $\Lambda \in \mathfrak{S}_n$ and any non-zero matrix $\Lambda^\perp \in (\text{Range } \Gamma)^\perp$ such that $\Lambda + \Lambda^\perp \geq 0$, we observe that $\mathcal{J}(\Lambda) = \mathcal{J}(\Lambda + \Lambda^\perp)$ and $\Lambda + \Lambda^\perp \in \mathfrak{S}_n$, since every element in $(\text{Range } \Gamma)^\perp$ is traceless [30, Sec. II].

At this point, we analyze the behavior of $\mathcal{J}(\cdot)$ in the region of the boundary of $\mathfrak{S}_n$ defined by

$$\mathcal{N} := \{ \Lambda \in \mathfrak{S}_n : \exists \bar{\theta} \in [-\pi, \pi) \text{ s.t. } G^{*}_\bar{\theta} \Lambda G_\bar{\theta} = 0 \}. \quad (15)$$

The following lemma provides a useful result in this regard.

**Lemma 4.** Suppose $\Psi_\theta \in C_+(\mathbb{T})$. For all $\bar{\Lambda} \in \mathcal{N}$, the (right-sided) directional derivative of $\mathcal{J}(\cdot)$ at $\bar{\Lambda}$ along any direction $\Delta \bar{\Lambda} \in \mathbb{H}_n$ such that $\bar{\Lambda} + \Delta \bar{\Lambda} \in \mathcal{M}$ takes the value $-\infty$.

**Proof.** Let $\bar{\Lambda} \in \mathcal{N}$. First, we note that $\bar{\Lambda} + \varepsilon \Delta \bar{\Lambda} \in \mathcal{M}$ for all $\Delta \bar{\Lambda} \in \mathbb{H}_n$ such that $\bar{\Lambda} + \Delta \bar{\Lambda} \in \mathcal{M}$, and for all $\varepsilon \in (0, 1]$. The (right-sided) directional derivative of $\mathcal{J}(\cdot)$ at $\bar{\Lambda}$ in the direction $\Delta \bar{\Lambda}$ is given by

$$\nabla \mathcal{J}(\bar{\Lambda}; \bar{\Lambda} + \Delta \bar{\Lambda}) := \lim_{\varepsilon \to 0^+} \frac{\mathcal{J}(\bar{\Lambda} + \varepsilon \Delta \bar{\Lambda}) - \mathcal{J}(\bar{\Lambda})}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \left( \frac{1}{\varepsilon} \text{tr}(\bar{\Lambda} + \varepsilon \Delta \bar{\Lambda}) - \frac{1}{\varepsilon} \text{tr}(\bar{\Lambda}) - \frac{1}{\varepsilon} \int \Psi_\theta \log \frac{G^{*}_\theta (\bar{\Lambda} + \varepsilon \Delta \bar{\Lambda}) G_\theta}{G^{*}_\theta \Lambda G_\theta} \right)$$

$$= - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int \Psi_\theta \log \left( 1 + \frac{G^{*}_\theta \Delta \bar{\Lambda} G_\theta}{G^{*}_\theta \Lambda G_\theta} \right)$$

$$= - \int \Psi_\theta \frac{G^{*}_\theta \Delta \bar{\Lambda} G_\theta}{G^{*}_\theta \Lambda G_\theta},$$

where we exploited the fact that $\text{tr}(\Delta \bar{\Lambda}) = 0$, and, in the last step, the Taylor expansion of $\log(1 + x)$. Eventually, since (i) $\bar{\Lambda} \in \mathcal{N}$, and (ii) $\Delta \bar{\Lambda}$ is such that $\bar{\Lambda} + \Delta \bar{\Lambda} \in \mathcal{M}$, there exists (at least) a frequency $\bar{\theta} \in [-\pi, \pi)$ such that $G^{*}_\bar{\theta} \Lambda G_\bar{\theta} = 0$ and $G^{*}_\bar{\theta} \Delta \bar{\Lambda} G_\bar{\theta} \neq 0$. Since $\Psi_\theta > 0$ for all $\theta \in [-\pi, \pi)$, this in turn implies $\int \Psi_\theta \frac{G^{*}_\bar{\theta} \Delta \bar{\Lambda} G_\bar{\theta}}{G^{*}_\bar{\theta} \Lambda G_\bar{\theta}} = \infty$, which yields the thesis. $\square$

**Remark 1.** Lemma 4 provides a characterization of the elements of $\mathcal{N}$ in terms of directional derivatives of $\mathcal{J}(\cdot)$ along directions belonging to the subset of $\mathfrak{S}_n$ given by $\mathcal{M}$. Notice that this result typically does not characterize all directional derivatives of $\mathcal{J}(\cdot)$ evaluated at elements in
Along directions pointing to \( \mathcal{N} \). However, for a particular subset of \( \mathcal{N} \) this is indeed the case. Let \( x \in \mathbb{C}^n, \|x\| = 1 \), and let \( P_x := xx^* \) denote the orthogonal projection onto the subspace spanned by \( x \). Consider the following subset of \( \mathcal{N} \)

\[ \mathcal{N}_0 := \{ P_x \in \mathcal{S}_n : \exists \theta \in [-\pi, \pi), \bar{x} \in \mathbb{C}^n, \|\bar{x}\| = 1, \text{s.t.} \]

(i) \( \langle \bar{x}, G_{\theta} \rangle = 0 \), and

(ii) \( \langle x, G_{\theta} \rangle \neq 0, \forall x \in \mathbb{C}^n, \|x\| = 1, x \neq \bar{x} \}. \] (16)

By exploiting the same argument of the proof of Lemma 4, it follows that the directional derivative of \( J(\cdot) \) evaluated at \( \bar{\Lambda} \in \mathcal{N}_0 \) along any direction \( \Delta \bar{\Lambda} \in \mathbb{H}_n \setminus \{0\} \) such that \( \bar{\Lambda} + \Delta \bar{\Lambda} \in \mathcal{S}_n \) is unbounded below. Moreover, it is worth noticing that, for the particular case \( n = 2 \), it holds \( \mathcal{N}_0 \equiv \mathcal{N} \).

To conclude this section, we recall the discrete-time version of LaSalle’s invariance principle, whose proof can be found in [37, Prop. 2.6].

**Proposition 1** (Discrete-time LaSalle’s invariance principle). Consider a discrete-time system

\[ x(t + 1) = f(x(t)), \quad x(0) \in \mathcal{X}, \quad t \geq 0, \]

where \( f : \mathcal{X} \rightarrow \mathcal{X} \) is continuous and \( \mathcal{X} \) is an invariant and compact set. Suppose \( V(\cdot) \) is a continuous function of \( x \in \mathcal{X} \), bounded below and satisfying

\[ \Delta V(x) := V(f(x)) - V(x) \leq 0, \quad \forall x \in \mathcal{X}, \]

that is \( V(x) \) is non-increasing along (forward) trajectories of the dynamics. Then any trajectory converges to the largest invariant subset \( \mathcal{I} \) contained in \( \mathcal{E} := \{ x \in \mathcal{X} : \Delta V(x) = 0 \} \).

**IV. GLOBAL CONVERGENCE ANALYSIS**

In this section, we present the main results of this note. The first result (Theorem 2) states that the cost function (9) is always non-increasing along the trajectories of (10). This result provides a positive answer to a conjecture raised in the conclusive part of [30].

**Theorem 2.** For every \( \Lambda \in \mathcal{S}_n \) it holds

\[ \Delta J(\Lambda) := J(\Theta(\Lambda)) - J(\Lambda) \leq 0, \] (17)
where $\mathcal{J}(\cdot)$ has been defined in (9). Moreover $\Delta \mathcal{J}(\Lambda) = 0$ if and only if $\Theta(\Lambda) = \Lambda + \Lambda^\perp$, with $\Lambda^\perp \in (\text{Range } \Gamma)^\perp$, where $\Gamma$ is the linear operator defined in Eq. (8).

**Proof.** By plugging the expression of $\Theta(\cdot)$ into (17), we get

$$\Delta \mathcal{J}(\Lambda) = \mathcal{J}(\Theta(\Lambda)) - \mathcal{J}(\Lambda) \leq 0 \iff \int \Psi_\theta \log G_\theta^* \Theta(\Lambda) G_\theta - \int \Psi_\theta \log G_\theta^* \Lambda G_\theta \geq 0 \iff \int \Psi_\theta \log f_\theta \geq 0,$$

(18)

where $f_\theta := \frac{G_\theta^* \Theta(\Lambda) G_\theta}{G_\theta^* \Lambda G_\theta}$ is well-defined and strictly positive on $\mathcal{T}$, since $\Theta(\Lambda)$ has the same rank and kernel of $\Lambda \in \mathcal{G}_n$, cf. [30, Prop. 2.1].

Firstly, we notice that the following inequality holds

$$\int \Psi_\theta \log f_\theta = -2 \int \Psi_\theta \log f_\theta^{1/2} \geq -2 \log \int \Psi_\theta f_\theta^{1/2},$$

(19)

which is a consequence of Lemma 1.

Secondly, by defining $\Pi_\theta := \frac{\Lambda^{1/4} G_\theta G_\theta^* \Lambda^{1/4}}{G_\theta^* \Lambda G_\theta}$, we have the following chain of equations

$$1 = \int \Psi_\theta f_\theta^{1/2} \Psi_\theta = \int \int f_\theta^{1/2} \Psi_\theta \Psi_\omega \langle \Pi_\theta, \Pi_\omega \rangle \quad (20)$$

$$\geq \int \int f_\theta^{1/2} \Psi_\theta \Psi_\omega \langle \Pi_\theta, \Pi_\omega \rangle \quad (21)$$

$$= \left( \int \Psi_\theta f_\theta^{1/2} \Pi_\theta, \int \Psi_\omega f_\omega^{1/2} \Pi_\omega \right)$$

$$= \left\| \Lambda^{1/2} \right\|_F^2 \left\| \int \Psi_\theta f_\theta^{1/2} \Pi_\theta \right\|_F^2 \quad (22)$$

$$\geq \left\| \Lambda^{1/2} \right\|_F^2 \left( \int \Psi_\theta f_\theta^{1/2} \Pi_\theta \right)^2 \quad (23)$$

$$= \left( \int \Psi_\theta f_\theta^{1/2} \right)^2 \quad (24)$$

where

- Eq. (20) follows by noticing that $f_\theta = \int \Psi_\omega \langle \Pi_\theta, \Pi_\omega \rangle$,
- Eq. (21) uses Lemma 2 applied to the symmetric function $\Psi_\theta \Psi_\omega \langle \Pi_\theta, \Pi_\omega \rangle$,
- Eq. (22) exploits the fact that $\left\| \Lambda^{1/2} \right\|_F^2 = \text{tr}(\Lambda) = 1$,
- Eq. (23) follows from Cauchy–Schwarz inequality, and
- Eq. (24) uses the fact that $\langle \Lambda^{1/2}, \Pi_\theta \rangle = 1$.
Eventually, a combination of the two sets of inequalities (19) and (20)-(24) yields $\int \Psi_\theta \log f_\theta \geq 0$ which, in turn, implies $\Delta \mathbb{J}(\Lambda) \leq 0$, in view of equivalence (18).

Now it remains to prove that we attain equality in (17) if only if $\Lambda \in S_n$ is such that $\Theta(\Lambda) = \Lambda + \Lambda^\perp$ with $\Lambda^\perp \in (\text{Range } \Gamma)^\perp$. In view of the definition of $(\text{Range } \Gamma)^\perp$ given in Eq. (14), the “if” part becomes straightforward. Indeed, if $\Theta(\Lambda) = \Lambda + \Lambda^\perp$, we have

$$
\Delta \mathbb{J}(\Lambda) = \int \Psi_\theta \log \frac{G^*_\theta \Theta(\Lambda) G_\theta}{G^*_\theta \Lambda G_\theta} = \int \Psi_\theta \log \frac{G^*_\theta (\Lambda + \Lambda^\perp) G_\theta}{G^*_\theta \Lambda G_\theta} = 0.
$$

So it remains to prove the “only if” part, i.e., if equality in (17) is attained for $\Lambda$ then $\Theta(\Lambda) = \Lambda + \Lambda^\perp$ with $\Lambda^\perp \in (\text{Range } \Gamma)^\perp$. To this end, we notice that a necessary condition for (17) to hold with equality is to have (19) satisfied with equality. By Lemma 1, this implies that the function $f_\theta$ is constant for every $\theta \in [-\pi, \pi)$, namely

$$
f_\theta = \frac{G^*_\theta \Theta(\Lambda) G_\theta}{G^*_\theta \Lambda G_\theta} = \kappa, \quad \forall \theta \in [-\pi, \pi),
$$

where $\kappa > 0$ is a real constant. Now equality in (17) is attained (if and) only if $\kappa = 1$, and therefore we have that

$$
G^*_\theta \Theta(\Lambda) G_\theta = G^*_\theta \Lambda G_\theta, \quad \forall \theta \in [-\pi, \pi).
$$

From the latter equation and by definition of $(\text{Range } \Gamma)^\perp$ in (14), it follows that $\Theta(\Lambda) = \Lambda + \Lambda^\perp$, $\Lambda^\perp \in (\text{Range } \Gamma)^\perp$. This completes the proof.

The following theorem is based on the previous one and states that iteration (10) always converges to the set of fixed points of $\Theta(\cdot)$.

**Theorem 3.** The trajectories generated by iteration (10) converge for all $\Lambda_0 \in S_n$ to elements belonging to $\mathcal{F} := \{ \Lambda \in S_n : \Theta(\Lambda) = \Lambda \}$.

**Proof.** The proof consists of an application of the discrete-time version of LaSalle’s invariance principle (Proposition 1). The natural candidate Lyapunov function $V(\cdot)$ of Proposition 1 is given in this case by $\mathbb{J}(\Lambda)$ which is continuous and bounded for every $\Lambda \in S_n$ (Lemma 3), and, by virtue of Theorem 2, non-increasing along the (forward) trajectories of the dynamics (10). Hence, by LaSalle’s invariance principle, we have that the (forward) trajectories generated by iteration (10) converges to the largest invariant set $\mathcal{I}$ contained in

$$
\mathcal{E} := \{ \Lambda \in S_n : \Delta \mathbb{J}(\Lambda) = 0 \}.
$$
Therefore, it remains to show that the trajectories in $I$ consist of fixed points of $\Theta(\cdot)$ only, that is, $I \equiv \mathcal{F}$. To this end, by Theorem 2, we know that the elements $\Lambda \in \mathcal{E}$ satisfy the condition
\[
\Theta(\Lambda) = \Lambda + \Lambda^\perp,
\] (25)
with $\Lambda^\perp \in (\text{Range } \Gamma)^\perp$. In view of the latter constraint on the dynamics $(10)$ and the definition of $(\text{Range } \Gamma)^\perp$ in $(14)$, it follows that any trajectory belonging to $\mathcal{E}$ must obey to the recurrence relation
\[
\Lambda_{k+1} = \Lambda^{1/2}_k M \Lambda^{1/2}_k, \quad \Lambda_0 \in \mathcal{E}, \; k \geq 0,
\] (26)
where
\[
M := \int_{-\pi}^{\pi} \Psi_\theta \frac{G_\theta G_\theta^*}{G_\theta^* A_0 G_\theta},
\]
depends on the initial condition $\Lambda_0$ only, in view of Eq. (25). Now, since (26) must generate unit trace trajectories starting from any $\Lambda_0 \in \mathcal{S}_n$, we have $\text{tr}(\Lambda_0) = \text{tr}(\Lambda_1) = \text{tr}(\Lambda_2) = 1$. By exploiting the cyclic property and the linearity of the trace, this in turn implies that
\[
\begin{align*}
\text{tr}(\Lambda_0) - 2\text{tr}(\Lambda_1) + \text{tr}(\Lambda_2) &= \text{tr}(\Lambda_0) - 2\text{tr}(\Lambda_0^{1/2} M \Lambda_0^{1/2}) + \text{tr}(M \Lambda_0^{1/2} M \Lambda_0^{1/2}) \\
&= \text{tr}(\Lambda_0) - \text{tr}(\Lambda_0^{3/4} M \Lambda_0^{1/4}) - \text{tr}(\Lambda_0^{1/4} M \Lambda_0^{3/4}) + \text{tr}(\Lambda_0^{1/4} M \Lambda_0^{1/2} M \Lambda_0^{1/4}) \\
&= \text{tr}(\Lambda_0^{1/4} (I - M) \Lambda_0^{1/4})^2 = \left\|\Lambda_0^{1/4} (I - M) \Lambda_0^{1/4}\right\|_F^2 = 0.
\end{align*}
\]
The previous equation is satisfied if and only if $\Lambda_0^{1/4} (I - M) \Lambda_0^{1/4} = 0$, or, equivalently, if and only if
\[
\Lambda_0 = \Lambda_0^{1/2} M \Lambda_0^{1/2}.
\]
From the previous equation and Eq. (26) it readily follows that $\Lambda_0$ must be a fixed point of $\Theta(\cdot)$. This ends the proof.

As a final result, we characterize a whole family of fixed points of $\Theta(\cdot)$ that are not asymptotically stable. The following result provides a partial answer to another conjecture of [30, Sec. V] claiming that orthogonal rank-one projections which do not belong to the closure of the set of positive definite fixed points $\mathcal{P}$ are unstable equilibrium points of $\Theta(\cdot)$.

**Proposition 2.** The set $\mathcal{N}_0$ defined in Eq. (16) consists of fixed points that are not asymptotically stable for the dynamics $(10)$.

**Proof.** Let $\hat{\Lambda} \in \mathcal{N}_0$. Notice that $\hat{\Lambda}$ is a rank-one orthogonal projection so that $\hat{\Lambda}$ is a fixed point of $\Theta(\cdot)$ by [1, Prop. 4.3]. Now observe that, in view of Lemma 4 and Remark 1, all the (right-sided)
directional derivatives at $\mathbb{J}(\bar{\Lambda})$ along directions pointing to $\mathcal{S}_n$ take the value $-\infty$. This implies that in a sufficiently small neighbourhood $U$ of $\bar{\Lambda}$, it holds $\mathbb{J}(\bar{\Lambda}) > \mathbb{J}(\Lambda)$, $\forall \Lambda \in U \cap \mathcal{S}_n$, $\Lambda \neq \bar{\Lambda}$. In light of this, the claim follows from the fact that $\mathbb{J}(\cdot)$ is non-increasing along trajectories of the dynamics (10) (Theorem 2).

V. CONCLUDING REMARKS

In this paper, we analyzed the global convergence properties of an extensively studied fixed-point algorithm for the Kullback–Leibler approximation of spectral densities introduced by Pavon and Ferrante in [1]. Our main result states that the Pavon–Ferrante algorithm globally converges to one of its fixed points.

A question which remains unanswered in the paper concerns global convergence of the Pavon–Ferrante algorithm to a fixed point leading to the solution of the Georgiou–Lindquist spectral approximation problem, and, in particular, to the set $\mathcal{P}$ of positive definite fixed points. A possible approach to guarantee convergence to a positive definite fixed point is to modify the Pavon–Ferrante iteration by adding at each step a suitable “correction” term belonging to $(\text{Range } \Gamma)^\perp$ which prevents the iteration to approach the boundary of $\mathcal{S}_n$. Notice in particular that, in view of Theorem 2, the presence of such terms does not affect the decreasing behavior of $\mathbb{J}(\cdot)$ along the trajectories of the iteration, and, consequently, the convergence argument used in the proof of Theorem 3. This aspect will be the subject of future investigation.

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REFERENCES

[1] M. Pavon and A. Ferrante, “On the Georgiou–Lindquist approach to constrained Kullback–Leibler approximation of spectral densities,” *IEEE Trans. Autom. Control*, vol. 51, no. 4, pp. 639–644, 2006.

[2] T. T. Georgiou and A. Lindquist, “Kullback–Leibler approximation of spectral density functions,” *IEEE Trans. Inf. Theory*, vol. 49, no. 11, pp. 2910–2917, 2003.

[3] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*. Edinburgh: Oliver & Boyd, 1965.

[4] C. I. Byrnes and A. Lindquist, “Important moments in systems and control,” *SIAM J. Control Optim.*, vol. 47, no. 5, pp. 2458–2469, 2008.

[5] A. Lindquist and G. Picci, *Linear Stochastic Systems*, ser. Series in Contemporary Mathematics. Springer-Verlag Berlin Heidelberg, 2015.
[6] T. T. Georgiou, “Realization of power spectra from partial covariance sequences,” *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 35, no. 4, pp. 438–449, 1987.

[7] C. I. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Matveev, “A complete parameterization of all positive rational extensions of a covariance sequence,” *IEEE Trans. Autom. Control*, vol. 40, no. 11, pp. 1841–1857, 1995.

[8] C. I. Byrnes, S. V. Gusev, and A. Lindquist, “A convex optimization approach to the rational covariance extension problem,” *SIAM J. Control Optim.*, vol. 37, no. 1, pp. 211–229, 1998.

[9] C. I. Byrnes, T. T. Georgiou, and A. Lindquist, “A new approach to spectral estimation: A tunable high-resolution spectral estimator,” *IEEE Trans. Signal Process.*, vol. 48, no. 11, pp. 3189–3205, 2000.

[10] T. T. Georgiou, “Spectral estimation via selective harmonic amplification,” *IEEE Trans. Autom. Control*, vol. 46, no. 1, pp. 29–42, 2001.

[11] F. Ramponi, A. Ferrante, and M. Pavon, “A globally convergent matricial algorithm for multivariate spectral estimation,” *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2376–2388, 2009.

[12] ———, “On the well-posedness of multivariate spectrum approximation and convergence of high-resolution spectral estimators,” *Systems & Control Lett.*, vol. 59, no. 3, pp. 167–172, 2010.

[13] C. I. Byrnes, T. T. Georgiou, and A. Lindquist, “A generalized entropy criterion for Nevanlinna–Pick interpolation with degree constraint,” *IEEE Trans. Autom. Control*, vol. 46, no. 6, pp. 822–839, 2001.

[14] A. Blomqvist, A. Lindquist, and R. Nagamune, “Matrix-valued Nevanlinna–Pick interpolation with complexity constraint: An optimization approach,” *IEEE Trans. Autom. Control*, vol. 48, no. 12, pp. 2172–2190, 2003.

[15] C. I. Byrnes, T. T. Georgiou, A. Lindquist, and A. Megretski, “Generalized interpolation in $H^{\infty}$ with a complexity constraint,” *Trans. Am. Math. Soc.*, vol. 358, no. 3, pp. 965–987, 2006.

[16] H. Dym, “A basic interpolation problem,” *Holomorphic spaces*, vol. 33, pp. 381–423, 1998.

[17] V. Bolotnikov and H. Dym, *On boundary interpolation for matrix valued Schur functions*. American Mathematical Soc., 2006, vol. 181, no. 856.

[18] A. Ferrante, M. Pavon, and F. Ramponi, “Hellinger versus Kullback–Leibler multivariable spectrum approximation,” *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 954–967, 2008.

[19] T. T. Georgiou and A. Lindquist, “A convex optimization approach to ARMA modeling,” *IEEE Trans. Autom. Control*, vol. 53, no. 5, pp. 1108–1119, 2008.

[20] F. P. Carli, A. Ferrante, M. Pavon, and G. Picci, “A maximum entropy solution of the covariance extension problem for reciprocal processes,” *IEEE Trans. Autom. Control*, vol. 56, no. 9, pp. 1999–2012, 2011.

[21] A. Ferrante, C. Masiero, and M. Pavon, “Time and spectral domain relative entropy: A new approach to multivariate spectral estimation,” *IEEE Trans. Autom. Control*, vol. 57, no. 10, pp. 2561–2575, 2012.

[22] M. Zorzi and A. Ferrante, “On the estimation of structured covariance matrices,” *Automatica*, vol. 48, no. 9, pp. 2145–2151, 2012.

[23] A. Ferrante, M. Pavon, and M. Zorzi, “A maximum entropy enhancement for a family of high-resolution spectral estimators,” *IEEE Trans. Autom. Control*, vol. 57, no. 2, pp. 318–329, 2012.

[24] M. Zorzi, “A new family of high-resolution multivariate spectral estimators,” *IEEE Trans. Autom. Control*, vol. 59, no. 4, pp. 892–904, 2014.

[25] ———, “Multivariate spectral estimation based on the concept of optimal prediction,” *IEEE Trans. Autom. Control*, vol. 60, no. 6, pp. 1647–1652, 2015.

[26] T. T. Georgiou and A. Lindquist, “Likelihood analysis of power spectra and generalized moment problems,” *IEEE Trans. Autom. Control (To appear)*, 2017.
[27] J. D. Stefanovski, “$\mathcal{H}_\infty$ problem with nonstrict inequality and all solutions: Interpolation approach,” *SIAM J. Control Optim.*, vol. 53, no. 4, pp. 1734–1767, 2015.

[28] ——, “A reformulation of augmented basic interpolation problem and an application to $\mathcal{H}_\infty$ control,” *Linear Algebra Appl.*, vol. 485, pp. 103–123, 2015.

[29] ——, “New interpolation solution and application in system modeling and optimal control with prescribed distance to instability,” *Int. J. Robust Nonlin.*, vol. 26, no. 11, pp. 2455–2477, 2016.

[30] A. Ferrante, F. Ramponi, and F. Ticozzi, “On the convergence of an efficient algorithm for Kullback–Leibler approximation of spectral densities,” *IEEE Trans. Autom. Control*, vol. 56, no. 3, pp. 506–515, 2011.

[31] A. Ferrante, M. Pavon, and F. Ramponi, “Further results on the Byrnes–Georgiou–Lindquist generalized moment problem,” in *Modeling, Estimation and Control*. Springer Berlin Heidelberg, 2007, pp. 73–83.

[32] T. M. Cover and J. A. Thomas, *Elements of information theory*. John Wiley & Sons, 2012.

[33] T. T. Georgiou, “The structure of state covariances and its relation to the power spectrum of the input,” *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1056–1066, 2002.

[34] Yu. A. Rozanov, *Stationary random processes*. San Francisco: Holden-Day, 1967.

[35] H. I. Nurdin, “Spectral factorization of a class of matrix-valued spectral densities,” *SIAM J. Control Optim.*, vol. 45, no. 5, pp. 1801–1821, 2006.

[36] W. Rudin, *Real and complex analysis*. New York: McGraw-Hill, Inc., 1987.

[37] J. P. LaSalle, *The Stability and Control of Discrete Processes*, ser. Applied Mathematical Sciences. New York: Springer Verlag, 1986, vol. 62.