DISPERSE ESTIMATES FOR SCHRÖDINGER OPERATORS: A SURVEY

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1. INTRODUCTION

The purpose of this note is to give a survey of some recent work on dispersive estimates for the Schrödinger flow

\[ e^{itH} P_c, \quad H = -\Delta + V \quad \text{on} \quad \mathbb{R}^d, \quad d \geq 1 \]

where \( P_c \) is the projection onto the continuous spectrum of \( H \). \( V \) is a real-valued potential that is assumed to satisfy some decay condition at infinity. This decay is typically expressed in terms of the point-wise decay \( |V(x)| \leq C \langle x \rangle^{-\beta} \), for all \( x \in \mathbb{R}^d \) and some \( \beta > 0 \). Throughout this paper, \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \). Occasionally, we will use an integrability condition \( V \in L^p(\mathbb{R}^d) \) (or a weighted variant thereof) instead of a point-wise condition. These decay conditions will also be such that \( H \) is asymptotically complete, i.e.,

\[ L^2(\mathbb{R}^d) = L^2_{p.p.}(\mathbb{R}^d) \oplus L^2_{a.c.}(\mathbb{R}^d) \]

where the spaces on the right-hand side refer to the span of all eigenfunctions, and the absolutely continuous subspace, respectively.

The dispersive estimate for (1) which we will be most concerned with is of the form

\[ \sup_{t \neq 0} |t|^d \| e^{itP_c f} \|_{L^\infty} \leq C \| f \|_1 \quad \text{for all} \quad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d). \]

Interpolating with the \( L^2 \) bound \( \| e^{itH} P_c f \|_2 \leq C \| f \|_2 \) leads to

\[ \sup_{t \neq 0} |t|^{\frac{d}{2} - \frac{1}{p'}} \| e^{itH} P_c f \|_{L^{p'}_{\text{p}'}(\mathbb{R}^d)} \leq C \| f \|_p \quad \text{for all} \quad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d). \]

where \( 1 \leq p \leq 2 \). It is well-known that via a \( T^*T \) argument \( \textbf{4} \) gives rise to the class of Strichartz estimates

\[ \| e^{itH} P_c f \|_{L^q_t(L^p_x)(\mathbb{R}^d)} \leq C \| f \|_2, \quad \text{for all} \quad \frac{2}{q} + \frac{d}{p} = \frac{d}{2}, \quad 2 < q \leq \infty. \]

The endpoint \( q = 2 \) is not captured by this approach, see Keel and Tao \( \textbf{53} \).

In heuristic terms, for the free problem \( V = 0 \) the rate of decay \( |t|^{-\frac{d}{2}} \) in \( \textbf{2} \) follows from \( L^2 \)-conservation and the classical Newton law \( \ddot{x} = 0 \) which leads to the trajectories \( x(t) = vt + x_0 \). Mathematically, \( \textbf{2} \) follows from the explicit solution

\[ (e^{-it\Delta}f)(x) = C_d t^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\frac{|x-y|^2}{4t}} f(y) \, dy. \]

For general \( V \neq 0 \) no explicit solutions are available, and one needs to proceed differently.

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If $V$ is small and $d \geq 3$, then one can proceed perturbatively. We will give examples of such arguments in Section 2. A purely perturbative approach cannot work in the presence of bound states of $H$ since those need to be removed. In other words, in the presence of bound states the nature of the spectral measure and/or resolvents of $H$ becomes essential. Since it is well-known that bound states can arise for arbitrarily small potentials in dimensions $d = 1, 2$, see Theorem XIII.11 in Reed and Simon [62], we conclude that a perturbative approach will necessarily fail in those dimensions. On the other hand, if $d = 3$ and $V$ satisfies the Rollnik condition
\[ \|V\|_{\text{Roll}}^2 := \int_{\mathbb{R}^6} \frac{|V(x)||V(y)|}{|x-y|^2} \, dx dy < \infty, \]
then Kato [52] showed that $-\Delta + V$ is unitarily equivalent with $-\Delta$ provided $4\pi\|V\|_{\text{Roll}} < 1$. Similar conditions are known for unitary equivalence if $d \geq 4$.

Dispersive estimates for large $V$ and $d = 3$ were established by Rauch [61] and Jensen, Kato [48]. In contrast to [2], these authors measured the decay on weighted $L^2(\mathbb{R}^3)$, i.e., they proved that
\[ \left\| e^{itH} P_c w f \right\|_2 \leq C |t|^{-\frac{5}{2}} \|f\|_2 \]
with $w(x) = e^{-\rho(x)}$ with some $\rho > 0$ and $V$ exponentially decaying (Rauch) or $w(x) = (x)^{-\sigma}$ for some $\sigma > 0$ and $V$ decaying at a power rate (Jensen, Kato). In addition, they needed to assume that the resolvent of $H$ has the property that
\[ \limsup_{\lambda \to 0} \|w(H - (\lambda \pm i0)^{-1}) w\|_{L^2 \to L^2} < \infty. \]

This condition is usually referred to as zero energy being neither an eigenvalue nor a resonance. While it is clear what it means for zero to be an eigenvalue of $H$, the notion of a resonance depends on the norms relative to which the resolvent is required to remain bounded at zero energy, see [6]. In the context of $L^2$ with power weights, which are most commonly used, one says that there is a resonance at zero if there exists a distributional solution $f$ of $Hf = 0$ with the property that $f \notin L^2(\mathbb{R}^3)$ but such that $(x)^{-\sigma} f \in L^2(\mathbb{R}^3)$ for all $\sigma > \frac{1}{2}$. With this definition the following holds: [6] is valid for $w(x) = (x)^{-\frac{1}{2} - \varepsilon}$ for any $\varepsilon > 0$ iff zero is neither an eigenvalue nor a resonance. The proof proceeds via the Fredholm alternative and the mapping properties of $(-\Delta + (\lambda + i0))^{-1}$ on weighted $L^2(\mathbb{R}^3)$ spaces, see Section 2. The notion of a resonance arises also in other dimensions, and we will discuss the cases $d = 1, 2$ in the corresponding sections below. If $|V(x)| \leq C (x)^{-2-\varepsilon}$ with $\varepsilon > 0$ arbitrary, and $d \geq 5$, then $H$ cannot have any resonances at zero energy. This is due to the fact that under these assumptions $(-\Delta)^{-1} V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Rauch and Jensen, Kato went beyond [53] by showing that if zero is an eigenvalue and/or a resonance, then [5] fails. In fact, they observed that if zero is a resonance but not an eigenvalue, then
\[ C^{-1} < \sup_{\|f\|_2 = 1} \sup_{t \geq 1} \left| t \right|^{-\frac{5}{2}} \left\| e^{itH} P_c f \right\|_2 < C < \infty. \]

Furthermore, this loss of decay can occur also if zero is an eigenvalue even though $P_c$ is understood to project away the corresponding eigenfunctions. They obtained these results as corollaries of asymptotic expansions of $e^{itH}$ as $t \to \infty$ on weighted $L^2$ spaces.

These asymptotic expansions are basically obtained as the Fourier transforms of asymptotic expansions of the resolvents (or rather, the imaginary part of the resolvents) around zero energy. In odd dimensions the latter are of the form, with $\Im z > 0$,
\[ (-\Delta + V - z^2)^{-1} = z^{-2} A_{-2} + z^{-1} A_{-1} + A_0 + z A_1 + O(z^2) \quad \text{as} \quad z \to 0 \]
where the $O$-term is understood in the operator norm on a suitable weighted $L^2$-space. These expansions can of course be continued to higher order $z^m$, with the degree of the weights in $L^2$
needed to control the error $O(z^m)$ increasing with $m$. In addition, the decay of $V$ needs to increase with $m$ as well. The operator $-A_{-2}$ is the orthogonal projection onto the eigenspace of $H$, and $A_{-1}$ is a finite rank operator related to both the eigenspace and the resonance functions. In odd dimensions, the free resolvent $(-\Delta + z^2)^{-1}$ is analytic for all $z \neq 0$ (and if $d \geq 3$ for all $z \in \mathbb{C}$), whereas in even dimensions the Riemann surface of the free resolvent is that of the logarithm. In practical terms, this means that $\epsilon$ needs to include (inverse) powers of $\log z$ in even dimensions.

In [26] and [17], Jensen derived analogous expansions for the resolvent around zero energy (and thus for the evolution as $t \to \infty$) in dimensions $d \geq 4$. Resolvent expansion at thresholds for the cases $d = 1$ and $d = 2$ were treated by Bollé, Gesztesy, Wilk [27], and Bollé, Gesztesy, Danneels [3], [4]. However, their approach requires separate treatment of the cases $\int V \, dx = 0$ and $\int V \, dx \neq 0$. Moreover, for $d = 2$ only the latter case was worked out. A unified approach to resolvent expansions was recently found by Jensen and Nenciu in [49]. Their method can be applied to all dimensions, but in [19] the authors only present $d = 1, 2$ in detail, because for those cases novel results are obtained by their method. The method developed by Jensen and Nenciu was applied by Erdogan and the author for $d = 3$, see [28], [29], and by the author for $d = 2$, see [68]. A very general treatment of resolvent expansions as in [17] and of local $L^2$ decay estimates can be found in Murata’s paper [54]. It is general in the sense that Murata states expansions in all dimensions, and covers the case of elliptic operators as well. However, his method is partially implicit in the sense that the coefficients of the singular powers in $\epsilon$ depend on operators that are solutions of certain equations, but those equations are not solved explicitly.

The first authors to address [4] were Journeé, Soffer, and Sogge [51]. Under suitable decay and regularity conditions on $V$, and under the assumption that zero is neither an eigenvalue nor a resonance they proved [2] for $d \geq 3$. In addition, they conjectured that [41] should hold for all $V$ such that $|V(x)| \leq C(x)^{-2-\epsilon}$ with arbitrary $\epsilon > 0$ and for which $-\Delta + V$ has neither an eigenvalue nor a resonance at zero energy.

The decay rate $|x|^{2-\epsilon}$, which corresponds to $L^2(R^d)$ integrability, plays a special role in dispersive estimates in particular, and the spectral theory of $-\Delta + V$ in general. On the one hand, potentials that decay more slowly than $|x|^{-2}$ at infinity can lead to operators with infinitely many negative bound states. On the other hand, in [34] and [33] Burq, Planchon, Stalker, and Tahvildar-Zadeh obtain Strichartz estimates for

$$i\partial_t u + \Delta u - \frac{a}{|x|^2} u = 0,$$

provided $a > -(d-2)^2/4$ and $d \geq 2$, and they show that this condition is also necessary. Furthermore, for the case of the wave equation, it is known that point-wise decay estimates fail in the attractive case $a < 0$, see the work of Planchon, Stalker, and Tahvildar-Zadeh.

For $d = 3$ the assumptions on $V$ in [51] are $|V(x)| \leq C(x)^{-7-\epsilon}$, $V \in L^1(R^3)$, and some small amount of differentiability of $V$. These requirements were subsequently relaxed by Yajima [83], [84], and [85], who proved much more, namely the $L^p$ boundedness of the wave operators for $1 \leq p \leq \infty$. A different approach, which lead to even weaker conditions on $V$ was found by Rodnianski and the author [63] (for small $V$), as well as by Goldberg and the author [35] (for large $V$). In addition, the aforementioned conjecture from [51] is proved in [63] (for large $V$).

Finally, Goldberg [34] proved that [2] — and not just [41] — holds for all $V$ for which $|V(x)| \leq C(x)^{-2-\epsilon}$ with arbitrary $\epsilon > 0$ and for which $-\Delta + V$ has neither an eigenvalue nor a resonance at zero energy. In fact, he only required a suitable $L^p$ condition, see Section [1]. In contrast, trying to adapt [35] to higher dimensions has lead Goldberg and Viscan [87] to show that for $d \geq 4$, [2] fails unless $V$ has some amount of regularity, i.e., decay alone is insufficient for [2] to hold if $d \geq 4$. 

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More precisely, they exhibit potentials \( V \in C_{\text{comp}}^{\frac{d-1}{2}}(\mathbb{R}^d) \) for which the dispersive \( L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \) decay with power \( t^{-\frac{d}{4}} \) fails.

The first results for \( d = 1 \) are due to Weder \cite{Weder1, Weder2, Weder3}, see also Artbazar and Yajima \cite{Artbazar}. These authors make use of the following explicit expression for the resolvent. If \( \Im z > 0 \), then
\[
(-\partial_z^2 + V - z^2)^{-1}(x,y) = \frac{f_+(x,z)f_-(y,z)}{W(z)} \quad \text{if } x > y
\]

and symmetrically if \( x < y \). Here \( f_{\pm} \) are the Jost solutions defined as solutions of
\[
-f_\pm''(\cdot, z) + V f_\pm(\cdot, z) = z^2 f_\pm(\cdot, z)
\]

with the asymptotics
\[
f_+(x,z) \sim e^{izx} \quad \text{as } x \to \infty
\]
\[
f_-(x,z) \sim e^{-izx} \quad \text{as } x \to -\infty,
\]

and \( W(z) = W[f_+(\cdot, z), f_-(\cdot, z)] \) is their Wronskian. These Jost solutions are known to exist and have boundary values as \( \Im z \to 0^+ \) as long as \( V \in L^1(\mathbb{R}) \) (in particular, this proves that the spectrum of \( H \) is purely a.c. on \((0, \infty)\) for such \( V \)). In order for these boundary values \( f_{\pm}(\cdot, \lambda) \) to be continuous at \( \lambda = 0 \) one needs to require that \( \langle x \rangle V(x) \in L^1(\mathbb{R}) \). In that case we say that zero energy is a resonance iff \( W(0) = 0 \). Note that the free case \( V = 0 \) has a resonance at zero energy, since then \( f_{\pm}(\cdot, 0) = 1 \). This condition is equivalent to the existence of a bounded solution \( f \) of \( H f = 0 \) (in particular, zero cannot be an eigenvalue).

Using some standard properties of the Jost solutions, see \cite{Goldberg}, Goldberg and the author proved that
\[
\|e^{itH}P_c f\|_{L^\infty(\mathbb{R})} \leq C|t|^{-\frac{d}{4}}\|f\|_{L^1(\mathbb{R})}
\]
provided \( \langle x \rangle V(x) \in L^1(\mathbb{R}) \) and provided zero is not a resonance. Note that in terms of pointwise decay, this is in agreement with the \( \langle x \rangle^{-2} \) threshold mentioned above. If zero is a resonance, then the same estimate holds for all \( V \) such that \( \langle x \rangle^2 V(x) \in L^1(\mathbb{R}) \). In Section \ref{sec:res} below, we present a variant of \eqref{eq:improved} with faster decay that seems to be new. It states that under sufficient decay on \( V \) and provided zero is not a resonance,
\[
\|\langle x \rangle^{-1}e^{itH}P_c f\|_{L^\infty(\mathbb{R})} \leq C t^{-\frac{d}{4}}\|\langle x \rangle f\|_{L^1(\mathbb{R})}
\]
for all \( t > 0 \). This estimate was motivated by the work of Murata \cite{Murata} and Buslaev and Perelman \cite{BuslaevPerelman} where such improved decay was obtained on \( L^2(\mathbb{R}) \) and with weights of the form \( \langle x \rangle^{3.5+\epsilon} \). It combines dispersive decay and the rate of propagation for \( H \). However, to the best of the author’s knowledge, \cite{Sjolin} has not appeared before and we therefore include a complete proof in Section \ref{sec:proof}

A version of \eqref{eq:improved} for the evolution of linearized nonlinear Schrödinger equations was crucial to the recent work \cite{Krieger} by Krieger and the author on stable manifolds for all supercritical NLS in one dimension.

Generally speaking, there is a very important difference between the one-dimensional dispersive bounds and those in other dimensions that have been proved so far, namely with regard to the constants. Indeed, in the one-dimensional case these constants exhibit an explicit dependence on the potential via the Jost solutions, which are solutions to a Volterra integral equation. On the other hand, in higher dimensions one resorts to a Fredholm alternative argument in order to invert the operator \( H - (\lambda^2 \pm i0) \). This indirect argument is traditionally used to prove the so-called limiting absorption principle for the resolvent, see Agmon \cite{Agmon} and \cite{Weder1} below. Any constructive proof of such an estimate for the perturbed resolvent would be most interesting, as it would allow for quantitative constants in dispersive estimates. Such a result was achieved by Rodnianski and
Tao, see [66] as well as their article in this volume. More generally, their work deals with dispersive estimates for the Schrödinger operator on $\mathbb{R}^n$ (or other manifolds) with variable metrics and is thus closely related to the subject matter of this article. Unfortunately, it is outside the scope of this review to discuss this exciting field of research. For example, see Bourgain [8], Doi [26], Burq, Gerard, Tzvetkov [12] (as well as other papers by these authors), Hassell, Tao, Wunsch [39], [40], Smith, Sogge [71], and Staffilani, Tataru [72].

For the wave equation with a potential, dispersive estimates have also been developed in recent years, see Cuccagna [20], Georgiev and Visciglia [31], Pierfelice [58], Planchon, Stalker, and Tahvildar-Zadeh [59], [60], d’Ancona and Pierfelice [22], as well as Stalker and Tahvildar-Zadeh [73]. There is some overlap with the results here, in particular with respect to certain bounds on the resolvent, but we will restrict ourselves to the Schrödinger equation. For Klein-Gordon, see Weder’s work [78].

Much of the work in this paper has been motivated by nonlinear problems (see e.g. Bourgain’s book [11], in particular pages 17–27). In recent years there has been much interest in the asymptotic stability of standing waves of the focussing NLS

$$i\partial_t \psi + \Delta \psi + f(|\psi|^2)\psi = 0. \quad (10)$$

A “standing wave” here refers to a solution of the form $\psi(t, x) = e^{i\alpha^2 t}\phi(x)$ where $\alpha \neq 0$ and

$$\alpha^2 \phi - \Delta \phi = f(\phi^2)\phi, \quad (11)$$
or any solution obtained from this one by applying the symmetries of the NLS, namely Galilei, scaling, and modulation (if the nonlinearity is critical, then there is one more symmetry by the name of pseudoconformal). Most work has been devoted to the standing wave generated by the ground state, i.e., a positive, decaying, solution of (11). In fact, such a solution must be radial and decay exponentially. Linearizing (10) around a standing wave yields a system of Schrödinger equations with non-selfadjoint matrix operator

$$\mathcal{H} = \begin{bmatrix} -\Delta + \alpha^2 - U & -W \\ W & \Delta - \alpha^2 + U \end{bmatrix} \quad (12)$$

and exponentially decaying, real-valued potentials $U$, $W$. In order to address the question of asymptotic stability of standing waves, one needs to study the spectrum of $\mathcal{H}$, as well as prove dispersive estimates for $e^{it\mathcal{H}}$ restricted to the stable subspace (which is defined as the range of a suitable Riesz projection). In the following sections we will mostly report on work on the scalar case rather than the system case. However, most of what is being said can be generalized to systems, see e.g. [21], [61], [29], [69], [55]. Although it may seem that the exponential decay of the potential in (12) may simplify matters greatly, this turns out not to be the case. In fact, the method from the paper [35], which is concerned with weakening the decay assumptions on $V$ in the scalar, three-dimensional case, has lead to the resolution of some open questions about matrix operators as in (12), see [29], [69], [55].

2. Dimensions three and higher

We start with a perturbative argument for small $V$ that can be considered as a sketch of the method from [51]. As above, let $H = -\Delta + V$ and suppose $d \geq 3$. Define

$$M_0 = \sup_{\|f\|_{L^2}} \sup_{0 \leq t \leq 1} \frac{\|e^{itH_0} f\|_{2+\infty}}{\|f\|_{2+\infty}} \quad M(T) = \sup_{0 \leq t \leq T} \sup_{\|f\|_{1+2}} \frac{\|e^{itH} f\|_{2+\infty}}{\|f\|_{2+\infty}}. \quad (13)$$

Here

$$\|f\|_{1+2} = \|f\|_{L^1 \cap L^2}, \quad \|f\|_{2+\infty} = \inf_{f_1 + f_2 = f} (\|f_1\|_2 + \|f_2\|_\infty). \quad (14)$$
Then the Duhamel formula
\[ e^{itH} = e^{itH_0} + i \int_0^t e^{i(t-s)H_0} V e^{isH} \, ds \]
implies that
\[ M(T) \leq M_0 + \langle T \rangle^{\frac{d}{2}} \int_0^T M_0(t-s)^{-\frac{d}{2}} \|V\|_{1\cap\infty} M(T) \langle s \rangle^ {-\frac{d}{2}} \, ds \leq M_0 + C \|V\|_{1\cap\infty} M_0 M(T). \]
Consequently, as long as
\[ C \|V\|_{1\cap\infty} M_0 \leq \frac{1}{2}, \]
we obtain the bound
\[ \sup_{T \geq 0} M(T) \leq 2M_0. \]
Note first that such an argument necessarily fails if \( d = 1, 2 \) due to the non-integrability of \( t^{\frac{-d}{2}} \) at infinity. Moreover, the are spectral reasons for this failure which we outlined in the introduction. Second, we would like to point out that it equally applies to time-dependent potentials provided the evolution \( e^{itH} \) is replaced with the propagator of the associated Schrödinger equation. The inclusion of the space \( L^2 \) allows us to deal with the singularity of \( t^{\frac{-d}{2}} \) at \( t = 0 \) which arises in the \( L^1 \to L^\infty \) estimate. In order to avoid it, Journé, Soffer, and Sogge use the bound
\[ \|e^{-itH_0} V e^{itH_0}\|_{p \to p} \leq \|\hat{V}\|_1, \]
which holds uniformly in \( 1 \leq p \leq \infty \). This explains the origin of the condition \( \hat{V} \in L^1 \) in their paper.

The main difficulty in [51] is of course the fact that \( V \) is large. Let us first present an unpublished argument of Ginibre [32] in dimensions \( d \geq 3 \) that allows passing from the weighted (or local) decay (5) to global decay, albeit in the form of a \( L^1 \cap L^2 \to L^2 + L^\infty \) estimate rather than the one in (2). Applying the Duhamel formula twice, we obtain
\[ e^{itH} P_c = e^{itH_0} P_c + i \int_0^t e^{i(t-s)H_0} V P_c e^{isH_0} \, ds \]
(13)
As long as \( V \) decays sufficiently rapidly so as to absorb the weights \( w \), i.e., such that
\[ \|w^{-1}V\|_{L^1 \cap L^\infty} < \infty, \]
we can combine the \( L^1 \cap L^2 \to L^2 + L^\infty \) bound
\[ \|e^{itH_0}\|_{2+\infty} \leq C \langle t \rangle^ {-\frac{d}{2}} \|f\|_{1\cap2} \]
with (13) as above to conclude from (13) that (14) also holds for \( H \). Here we also used that \( P_c : L^1 \cap L^\infty \to L^1 \cap L^\infty \) which holds provided all eigenfunctions of \( H \) with negative eigenvalue belong to \( L^1 \cap L^\infty \) (recall that zero is assumed not to be an eigenvalue). That property, however, follows from Agmon’s exponential decay bound [2] and Sobolev imbedding provided \( V \) also has some small of regularity.

This argument, however, does not shed much light on the question of \( L^1 \to L^\infty \) bounds (without assuming more regularity on \( V \)). The inclusion of \( L^2 \) is undesirable for a number of reasons, the main one being nonlinear applications. We therefore proceed differently, and first recall the small-potential argument from [63] in \( d = 3 \). For certain standard details we refer the reader to [63].
The starting point is the standard fact

\[ e^{itH}P_{ac} = \int_0^\infty e^{it\lambda}E_{ac}(d\lambda), \]

where \( E_{ac} \) is the absolutely continuous part of the spectral resolution. Its density is given by

\[ \frac{dE_{ac}(\lambda)}{d\lambda} = \frac{1}{2\pi i}[(H - (\lambda + i0))^{-1} - (H - (\lambda - i0))^{-1}] \]
on \( \lambda > 0 \). As already mentioned before, Kato’s theorem \[52\] insures that \( E_{ac} = E \) provided

\[ \|V\|_{\tilde{K}}^2 := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2} \, dx \, dy < (4\pi)^2. \]

Let \( R_V(z) = (-\Delta + V - z)^{-1} \) and \( R_0(z) = (-\Delta - z)^{-1} \). Then with \( V \) as in \[16\], for all \( f, g \in L^2(\mathbb{R}^3) \) and \( \varepsilon \geq 0 \) one has the Born series expansion

\[ \langle R_V(\lambda \pm i\varepsilon)f, g \rangle - \langle R_0(\lambda \pm i\varepsilon)f, g \rangle = \sum_{\ell=1}^\infty (-1)^\ell \langle R_0(\lambda \pm i\varepsilon)(VR_0(\lambda \pm i\varepsilon))^\ell f, g \rangle \]

It is well-known that the resolvent \( R_0(z) \) for \( \Im z \geq 0 \) has the kernel

\[ R_0(z)(x,y) = \frac{\exp(i\sqrt{z}|x-y|)}{4\pi|x-y|} \]

with \( \Im(\sqrt{z}) \geq 0 \). Then there is the following simple lemma that is basically an instance of stationary phase. For the proof we refer the reader to \[63\].

**Lemma 2.1.** Let \( \psi \) be a smooth, even bump function with \( \psi(\lambda) = 1 \) for \(-1 \leq \lambda \leq 1 \) and \( \text{supp}(\psi) \subset [-2,2] \). Then for all \( t \geq 1 \) and any real \( a \),

\[ \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda} \sin(a\sqrt{\lambda}) \psi(\frac{\sqrt{\lambda}}{L}) \, d\lambda \right| \leq Ct^{-\frac{3}{2}}|a| \]

where \( C \) only depends on \( \psi \).

In addition to \[16\], we will assume that

\[ \|V\|_k := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} \, dy < 4\pi \]

In \[63\] this norm was introduced by the name of **global Kato norm** (it is closely related to the well-known Kato norm, see Aizenman and Simon \[3, 70\]). The following lemma explains to some extent why condition \[20\] is needed. Iterated integrals as in \[21\] will appear in a series expansion of the spectral resolution of \( H = -\Delta + V \). For the sake of completeness, and in order to show how these global Kato norms arise, we reproduce the simple proof from \[63\].

**Lemma 2.2.** For any positive integer \( k \) and \( V \) as above,

\[ \sup_{x_0, x_{k+1} \in \mathbb{R}^3} \int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=0}^k |x_j - x_j + 1|} \sum_{\ell=0}^k |x_\ell - x_{\ell+1}| \, dx_1 \ldots dx_k \leq (k+1)\|V\|_k^k. \]

**Proof.** Define the operator \( A \) by the formula

\[ Af(x) = \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} f(y) \, dy. \]
Observe that the assumption \(20\) on the potential \(V\) implies that \(A : L^\infty \to L^\infty\) and \(\|A\|_{L^\infty \to L^\infty} \leq c_0\) where we have set \(c_0 := \|V\|_K\) for convenience. Denote by \(\langle,\rangle\) the standard \(L^2\) pairing. In this notation the estimate \(21\) is equivalent to proving that the operators \(B_k\) defined as

\[
B_k f = \sum_{m=0}^{k} \langle f, A^{k-m} 1 \rangle A^m 1
\]

are bounded as operators from \(L^1 \to L^\infty\) with the bound

\[
\|B_k\|_{L^1 \to L^\infty} \leq (k + 1)c_0^k.
\]

For arbitrary \(f \in L^1\) one has

\[
\|B_k f\|_{L^\infty} \leq \sum_{m=0}^{k} \|A^{k-m}\|_{L^\infty \to L^\infty} \|A^m 1\|_{L^\infty} \|f\|_{L^1}
\]

\[
\leq \sum_{m=0}^{k} c_0^k \|f\|_{L^1} \leq (k + 1)c_0^k \|f\|_{L^1},
\]

as claimed. \(\square\)

We are now in a position to prove the small \(V\) result from \([63]\). In \([58]\), Perfelice obtained an analogous result for the wave equation.

**Theorem 2.3.** With \(H = -\Delta + V\) and \(V\) satisfying the conditions \(16\) and \(20\), one has the bound

\[
\|e^{itH}\|_{L^1 \to L^\infty} \leq Ct^{-\frac{3}{2}}
\]

in three dimensions.

**Proof.** Fix a real potential \(V\) as above, as well as any \(L \geq 1\), and real \(f, g \in C_0^\infty(\mathbb{R}^3)\). Then applying \(17\), \(18\), Lemma 2.1 and Lemma 2.2 in this order, we obtain

\[
sup_{L \geq 1} \left| \langle e^{itH} \psi(\sqrt{H}/L)f, g \rangle \right|
\]

\[
\leq sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda} \psi(\sqrt{\lambda}/L) \langle E'(\lambda)f, g \rangle d\lambda \right|
\]

\[
= sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda} \psi(\sqrt{\lambda}/L) \sum_{k=0}^{\infty} \langle R_0(\lambda + i0)(VR_0(\lambda + i0))^k f, g \rangle d\lambda \right|
\]
To proceed, we now use the explicit form of the free resolvent. This yields

\[
\leq \sum_{k=0}^{\infty} \int_{\mathbb{R}^d} |f(x_0)||g(x_{k+1})| \int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^{k} |V(x_j)|}{\prod_{j=0}^{k} 4\pi |x_j - x_{j+1}|}.
\]

(22)

\[
\leq Ct^{-\frac{3}{2}} \sum_{k=0}^{\infty} \int_{\mathbb{R}^d} |f(x_0)||g(x_{k+1})| \int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^{k} |V(x_j)|}{(4\pi)^{k+1} \prod_{j=0}^{k} |x_j - x_{j+1}|} \sum_{\ell=0}^{k} |x_\ell - x_{\ell+1}| d(x_1, \ldots, x_k) dx_0 dx_{k+1}
\]

\leq Ct^{-\frac{3}{2}} \|f\|_1 \|g\|_1,
\]

since \(|V|_{C^k} < 4\pi\). In order to pass to (22) one uses the explicit representation of the kernel of \(R_0(\lambda + i0)\), see (13), which leads to a \(k\)-fold integral. Next, one interchanges the order of integration in this iterated integral.

The next step is to remove the smallness assumption on \(V\). This was done in [35] for potentials decaying like \(|V(x)| \leq C(x)^{-\beta}\) with \(\beta > 3\). The proof required splitting the energies into the regions \([0, \infty)\) (the “large energies”) and \([0, \lambda_0]\) (the “small energies”) where \(\lambda_0 > 0\) is small. In the regime of large energies, one expands the resolvent \(R_V\) into a finite Born series

\[
R_V(\lambda^2 \pm i0) = \sum_{\ell=0}^{2m+1} R_0(\lambda^2 \pm i0)(-VR_0(\lambda^2 \pm i0))^\ell
\]

(23)

\[
+ R_0(\lambda^2 \pm i0)(VR_0(\lambda^2 \pm i0))^m VR_V(\lambda^2 \pm i0)V(0)(\lambda^2 \pm i0)V^m R_0(\lambda^2 \pm i0)
\]

where \(m\) is any positive integer. All but the last term (which involves \(R_V\)) is treated by the same argument from [33] that we sketched previously. To bound the contribution of the final term in (23), let \(R_0^\pm(\lambda^2) := R_0(\lambda^2 \pm i0)\). Moreover, set

\[
G_{\pm,x}(\lambda^2)(x_1) := e^{\pm i\lambda|x|} R_0(\lambda^2 \pm i0)(x_1, x) = \frac{e^{\pm i\lambda(|x_1| - |x|)}}{4\pi|x_1 - x|}.
\]

Similar kernels appear already in Yajima’s work [86] (see his high energy section). Hence, we are led to proving that

\[
\left| \int_{0}^{\infty} e^{it\lambda^2} e^{\pm i\lambda(|x| + |y|)} (1 - \chi(\lambda/\lambda_0))\lambda VR^\pm_V(\lambda^2)V(R_0^\pm(\lambda^2)V)^m G_{\pm,y}(\lambda^2), (R_0^\pm(\lambda^2)V)^m G^*_{\pm,x}(\lambda^2) \right| d\lambda \leq |t|^{-\frac{3}{2}}
\]

(24)

uniformly in \(x, y \in \mathbb{R}^d\). Here \(\chi\) is a bump function which is equal to one on a neighborhood of the origin. The estimate (24) is proved by means of stationary phase and the limiting absorption principle. The latter refers to estimates of the form, with \(\lambda > 0\) and \(\sigma > \frac{1}{2}\),

\[
\|R_0(\lambda^2 \pm i0)f\|_{L^{2,-\sigma}} \leq C(\lambda) \|f\|_{L^{2,\sigma}},
\]

(25)

where \(L^{2,\sigma} = \langle x\rangle^{-\sigma} L^{2}\), see [1]. Similar estimates also hold for the derivatives of \(R_0\) in \(\lambda\). Moreover, \(C(\lambda)\) decays power-like with \(\lambda \to \infty\). By means of the resolvent identity and arguments of Agmon and Kato analogous estimates hold for \(R_V(\lambda^2 \pm i0)\) (this essentially amounts to the absence of
imbedded eigenvalues in the continuous spectrum). These properties insure that the integrand in \[24\], viz.

\[ a_{x,y}(\lambda) := (1 - \chi(\lambda/\lambda_0))\langle V R_0^\pm(\lambda^2) VR_0^\pm(\lambda^2) V^{m} G_{\pm,y}(\lambda^2), (R_0^\pm(\lambda^2) V)^m G^*_{\pm,x}(\lambda^2) \rangle \]

decays at least as fast as \( \lambda^{-2} \) (provided \( m \) is large) and is twice differentiable, say. Moreover, due to the presence of the functions \( G_{\pm,y} \) and \( G_{\pm,x} \) at the edges, one checks that if the critical point \( \lambda_1 = \frac{|x|+|y|}{2t} \) of the phase falls into the support of this integrand, which requires \( \lambda_1 \geq \lambda_0 \), then the entire integrand is bounded by

\[ t^{-\frac{1}{2}} |a_{x,y}(\lambda_1)| \leq C t^{-\frac{1}{2}} \langle(x)\langle(y)\rangle^{-1} \leq C t^{-\frac{3}{2}}, \]

as desired.

In the low-energy regime \( \lambda \in [0,\lambda_0] \), one writes

\[ \langle e^{it\chi(\sqrt{H}/\lambda_0)} P_{a.c.} f,g \rangle = \int_{0}^{\infty} e^{it\lambda^2} \chi(\lambda/\lambda_0) \left( [R_{\lambda^2} + i0] - R_{\lambda^2} - i0 \right) f,g \frac{d\lambda}{\pi i} \]

and proceeds via the resolvent identity

\[ R_{\lambda^2}(\lambda^2) = R_{0}^\pm(\lambda^2) - R_{0}^\pm(\lambda^2) V (I + R_{0}^\pm(\lambda^2) V)^{-1} R_{0}^\pm(\lambda^2). \]

Expanding \( R_{0}^\pm(\lambda^2) \) around zero, the invertibility of \( I + R_{0}^\pm(\lambda^2) V \) reduces to the invertibility of \( S_0 := I + R_{0}^\pm(0) V \).

However, the latter is equivalent to zero energy being neither an eigenvalue nor a resonance. Writing \( R_{0}^\pm(\lambda^2) = R_{0}(0) + B^\pm(\lambda) \), we conclude that

\[ [I + R_{0}^\pm(\lambda^2) V]^{-1} = S_{0}^{-1}[I + B^\pm(\lambda) V S_{0}^{-1}]^{-1} =: S_{0}^{-1} \tilde{B}^\pm(\lambda). \]

Some elementary calculations based on the explicit form of the kernel of \( R_0 \) and the decay of \( V \) then reduce the \( t^{-\frac{3}{2}} \) dispersive decay to the finiteness of

\[ \int_{-\infty}^{\infty} \| \chi_{0}(\tilde{\tilde{B}}^+) (u) \|_{HS(-1,-1)} du \quad \text{and} \quad \int_{-\infty}^{\infty} \| \chi_{0}(\tilde{\tilde{B}}^+) (u) \|_{HS(-1,-1)} du \]

where the norm is that of the Hilbert-Schmidt operators from \( L^{-1-\varepsilon}(\mathbb{R}^3) \to L^{-2-\varepsilon}(\mathbb{R}^3) \). Expanding into a Neumann series

\[ \tilde{B}^+(\lambda) = [I + B^+(\lambda) V S_{0}^{-1}]^{-1} = \sum_{n=0}^{\infty} ( - B^+(\lambda) V S_{0}^{-1} )^{n} \]

and making careful use of the explicit kernel

\[ B^\pm(\lambda)(x,y) = \frac{e^{i\lambda|x-y|} - 1}{4\pi|x-y|} \]

finishes the proof, see \[35\].

This argument was extended in various directions. First Yajima \[87\] and independently, Erdogan and the author \[28\], have adapted it to the case of zero energy being an eigenvalue and/or a resonance. The difference is of course that in this case \( S_0 \) as in \[24\] is no longer invertible and \( (I + R_{0}^\pm(\lambda^2) V)^{-1} \) involves singular powers of \( \lambda \). Yajima uses the expansion from \[18\] for this purpose, whereas \[28\] use the method from \[49\]. The latter is based on the symmetric resolvent identity and is therefore entirely situated in \( L^2 \) rather than weighted \( L^2 \). The following theorem is from \[28\]. Yajima proves the same, but assuming less decay on \( V \).
Theorem 2.4. Assume that \( V \) satisfies \( |V(x)| \leq C|x|^{-\beta} \) with \( \beta > 10 \) and assume that there is a resonance at energy zero but that zero is not an eigenvalue. Then there is a time dependent rank one operator \( F_t \) such that

\[
\left\| e^{itH} P_{ac} - t^{-1/2} F_t \right\|_{L^1 \rightarrow L^\infty} \leq C t^{-3/2},
\]

for all \( t > 0 \) and \( F_t \) satisfies

\[
\sup_t \| F_t \|_{L^1 \rightarrow L^\infty} < \infty, \quad \lim \sup_{t \rightarrow \infty} \| F_t \|_{L^1 \rightarrow L^\infty} > 0.
\]

A similar result holds also in the presence of eigenvalues, but in general \( F_t \) is no longer of rank one.

The paper [29] extends these methods further, namely to the case of systems of the type that arise from linearizing NLS around a ground state standing wave.

In another direction, Goldberg has improved on the method from [35] in several aspects. In [33], he proves that \( V \in L^{2(1+\epsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \) suffices for the dispersive estimate (assuming of course that zero is neither an eigenvalue nor a resonance). This amount of integrability is analogous to the \( \beta > 3 \) point-wise decay from [34]. Goldberg’s result requires a substitute for (25) on \( L^p(\mathbb{R}^3) \) spaces, rather than weighted \( L^2 \) spaces. Such a substitute exists, and is known to be related to the Stein-Tomas theorem in Fourier analysis, see [74]. It was first obtained for the free resolvent by Kenig, Ruiz, and Sogge [54], and extended to perturbed resolvents by Goldberg and the author [36], Stein-Tomas theorem in Fourier analysis, see [74]. It was first obtained for the free resolvent by Kenig, Ruiz, and Sogge [54], and extended to perturbed resolvents by Goldberg and the author [36], as well as Ionescu and the author [44]. For example, in \( \mathbb{R}^3 \) the bound from [54] takes the form

\[
\| R_0(\lambda^2 + i\epsilon)f \|_{L^4(\mathbb{R}^3)} \leq C \lambda^{-\frac{1}{2}} \| f \|_{L^4(\mathbb{R}^3)},
\]

and in [36] it is proved that

\[
\sup_{0<\epsilon<1, \lambda \geq \lambda_0} \left\| (\Delta + V - (\lambda^2 + i\epsilon))^{-1} \right\|_{L^4 \rightarrow L^4} \leq C(\lambda_0, V) \lambda^{-\frac{1}{2}}.
\]

for all real-valued \( V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3), p > \frac{3}{2} \) and every \( \lambda_0 > 0 \). This of course requires absence of imbedded bound states in the continuous spectrum, which was proved for the same class of \( V \) by Ionescu and Jerison [44]. A very different approach from the one in [36] to estimates of the form (27) was found in [45], which is related to [67]. [45] applies to all dimensions \( d \geq 2 \) and quite general perturbations (including magnetic ones) of \( -\Delta \), but it also does not rely on [44]. In fact, as in Agmon’s classical paper [1] it is shown that the imbedded eigenvalues form a discrete set outside of which a bound as in (27) holds (albeit on somewhat different spaces). Moreover, this is obtained under the assumption that \( V \in L^p(\mathbb{R}^d) \) for some \( \frac{d}{2} \leq p \leq \frac{d+1}{2} \). The upper limit of \( \frac{d+1}{2} \) is natural in some ways, since Ionescu and Jerison have found a smooth, real-valued potential in \( L^p(\mathbb{R}^d) \) for all \( p > \frac{d+1}{2} \) which has an imbedded eigenvalue. The lower limit of \( d/2 \) is the usual one for self-adjointness purposes.

Returning to dispersive estimates, Goldberg [34] proved that even \( V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \) with \( p < \frac{3}{2} < q \) suffices for a dispersive estimate with the usual restriction on zero energy. Note that this is nearly critical with respect to the natural scaling of the Schrödinger equation in \( \mathbb{R}^3 \). One of his main observations for the low energy argument was that for such \( V \) (and assuming that zero is neither an eigenvalue nor a resonance)

\[
\sup_{\lambda \in \mathbb{R}} \left\| (I + VR_0^+(\lambda^2))^{-1} \right\|_{L^1 \rightarrow L^1} < \infty,
\]

see [34] for further details. As far as high energies are concerned, Goldberg noticed that the Born series estimate from [63] can be improved so that the \( k \)-th term is bounded by \( (\lambda_1^{-\frac{k}{2}} \| V \|)^k \).
with \( \|V\| = \max(\|V\|_p, \|V\|_q) \) as opposed to \((\|V\|_K/4\pi)^k\). Choosing \(\lambda_1\) large this guarantees a convergent series.

3. THE ONE-DIMENSIONAL CASE

We will not repeat the discussion of the one-dimensional theorems from the introduction where the results from [79], [77], [41], or [35] were described. Rather, we would like to focus on a novel estimate that exploits the absence of a resonance by means of weights and obtains a better rate of decay. It was motivated by the work of Murata [56] and Buslaev, Perelman [15] on improved local estimate that exploits the absence of a resonance by means of weights and obtains a better rate of decay in the absence of resonances in dimension one. Note that the weight \( \langle x \rangle \) is optimal in the sense that it cannot be replaced with \( \langle x \rangle^\tau, \tau < 1 \).

**Theorem 3.1.** Suppose \( V \) is real-valued and \( \|\langle x \rangle^4 V\|_1 < \infty \). Let \( H = -\frac{d^2}{dx^2} + V \) have the property that zero energy is not a resonance. Then

\[
\|\langle x \rangle^{-1} e^{iH} P_{ac} f\|_\infty \leq C t^{-\frac{3}{2}} \|\langle x \rangle f\|_1
\]

for all \( t > 0 \).

**Proof.** Let \( \lambda_0 = \|\langle x \rangle V\|_2 \) and suppose \( \chi \) is a smooth cut-off such that \( \chi(\lambda) = 0 \) for \( \lambda \leq \lambda_0 \) and \( \chi(\lambda) = 1 \) for \( \lambda \geq 2\lambda_0 \). Recall that

\[
R_0(\lambda \pm i0)(x) = \frac{\pm i}{2\sqrt{\lambda}} e^{\pm i|x|\sqrt{\lambda}}.
\]

Hence,

\[
|\langle R_0(\lambda + i0)(VR_0(\lambda + i0))^n f, g \rangle| \leq (2\sqrt{\lambda})^{-n-1} \|V\|_1^n \|f\|_1 \|g\|_1,
\]

and the Born series

\[
(28) \quad R_V(\lambda \pm i0) = \sum_{n=0}^{\infty} R_0(\lambda \pm i0)(-VR_0(\lambda \pm i0))^n
\]

converges in the operator norm \( L^1(\mathbb{R}) \to L^{\infty}(\mathbb{R}) \) provided \( \lambda > \lambda_0 \). The absolutely continuous part of the spectral measure is given by

\[
\langle E_{a.c.}(d\lambda)f, g \rangle = \langle \frac{1}{2\pi i} [R_V(\lambda + i0) - R_V(\lambda - i0)] f, g \rangle d\lambda.
\]

Therefore, integrating by parts once yields

\[
(29) \quad \langle e^{itH} \chi(H) f, g \rangle = -(4\pi t)^{-1} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{d}{d\lambda} \left[ \chi(\lambda^2) (R_0(\lambda^2 + i0)(VR_0(\lambda^2 + i0))^n f, g) \right] d\lambda
\]

where we have first changed variables \( \lambda \to \lambda^2 \). Summation and integration may be exchanged because the Born series converges absolutely in the \( L^1(d\lambda) \) norm, and the domain of integration is extended to \( \mathbb{R} \) via the identity \( R_0(\lambda^2 - i0) = R_0((-\lambda)^2 + i0) \). The kernel of \( R_0(\lambda^2+i0)(VR_0(\lambda^2+i0))^n \) is given explicitly by the formula

\[
R_0(\lambda^2 + i0)(VR_0(\lambda^2 + i0))^n(x, y) = \frac{1}{(2\lambda)^{n+1}} \int_{\mathbb{R}^n} \prod_{j=1}^{n} V(x_j) e^{i\lambda(|x-x_1|+|y-x_n|+\sum_{k=2}^{n} |x_k-x_{k-1}|)} dx
\]
with \( dx = dx_1 \ldots dx_n \). Hence, in view of the derivative in (29),

\[
|\langle e^{itH} \chi(H) f, g \rangle | \
\leq C |t|^{-1} \sum_{n=0}^{\infty} (2\sqrt{\lambda_0})^{-n-1} \sup_{\alpha \in \mathbb{R}} \left| \int_{-\infty}^{\infty} e^{i(\lambda^2 + \alpha \lambda)} \chi(\lambda^2) \lambda^{-n-1} \lambda_0^{(n+1)/2} d\lambda \right| \|\langle x \rangle V \|_1 \|\langle x \rangle f\|_1 \|\langle x \rangle g\|_1 \\
+ C |t|^{-1} \sum_{n=0}^{\infty} (2\sqrt{\lambda_0})^{-n-1} \sup_{\alpha \in \mathbb{R}} \left| \int_{-\infty}^{\infty} e^{i(\lambda^2 + \alpha \lambda)} \chi'(\lambda^2) \lambda^{-n-1} \lambda_0^{(n+1)/2} d\lambda \right| \|V\|_1 \|f\|_1 \|g\|_1
\]

(30)

\[
\leq C(V) |t|^{-\frac{1}{2}} \|\langle x \rangle f\|_1 \|\langle x \rangle g\|_1 .
\]

We used the dispersive bound for the one-dimensional Schrödinger equation to pass to (30), observing in particular that

\[
\sup_{n \geq 0} \left\| \chi(\lambda^2) \lambda^{-n-1} \lambda_0^{(n+1)/2} \right\| < \infty
\]

where the norm refers to the total variation norm of measures.

It remains to consider small energies, i.e., those \( \lambda \) for which \( \chi(\lambda^2) \neq 1 \). In this case, we let \( f_j(\cdot, \lambda) \) for \( j = 1, 2 \) be the Jost solutions. They satisfy

\[
\left( -\frac{d^2}{dx^2} + V - \lambda^2 \right) f_j(x, \lambda) = 0, \ f_1(x, \lambda) \sim e^{ix\lambda} \ \text{as} \ x \to \infty, \ f_2(x, \lambda) \sim e^{-ix\lambda} \ \text{as} \ x \to -\infty
\]

for any \( \lambda \in \mathbb{R} \). Furthermore, if \( \lambda \neq 0 \), then

(31)

\[
f_1(\cdot, \lambda) = \frac{R_1(\lambda)}{T(\lambda)} f_2(\cdot, \lambda) + \frac{1}{T(\lambda)} f_2(\cdot, \lambda), \ f_2(\cdot, \lambda) = \frac{R_2(\lambda)}{T(\lambda)} f_1(\cdot, \lambda) + \frac{1}{T(\lambda)} f_1(\cdot, \lambda),
\]

where \( T(\lambda) = \frac{2i\lambda}{W(\lambda)} \) with \( W(\lambda) = W[f_1(\cdot, \lambda), f_2(\cdot, \lambda)] \) and

\[
R_1(\lambda) = -\frac{T(\lambda)}{2i\lambda} W[f_1(\cdot, \lambda), f_2(\cdot, \lambda)], \ R_2(\lambda) = \frac{T(\lambda)}{2i\lambda} W[f_1(\cdot, \lambda), f_2(\cdot, \lambda)].
\]

Then the jump condition of the resolvent \( R_V \) across the spectrum takes the form

\[
\left( R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0) \right)(x, y) = \frac{|T(\lambda)|^2}{-2i\lambda} (f_1(x, \lambda) f_1(y, \lambda) + f_2(x, \lambda) f_2(y, \lambda))
\]

with \( \lambda \geq 0 \). Let us denote the distorted Fourier basis by

\[
ee(x, \lambda) = \frac{1}{\sqrt{2\pi}} \left\{ \begin{array}{ll}
T(\lambda) f_1(\cdot, \lambda) & \text{if} \ \lambda \geq 0 \\
T(-\lambda) f_2(x, \lambda) & \text{if} \ \lambda < 0
\end{array} \right.
\]

see Weder’s papers [29] and [77] for more details on this basis. Then the evolution \( e^{itH} (1 - \chi(H)) P_{a.c.} \) can be written as

(32)

\[
\langle e^{itH} (1 - \chi(H)) P_{a.c.} \phi, \psi \rangle = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_{0}^{\infty} 2\lambda e^{i\lambda^2} (1 - \chi(\lambda^2)) \left( R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0) \right)(x, y) d\lambda \phi(x) \psi(y) dx dy
\]

\[
= \int_{-\infty}^{\infty} e^{it\lambda^2} (1 - \chi(\lambda^2)) \langle \psi, e^\cdot, \lambda \rangle (e^\cdot, \lambda, \phi) d\lambda.
\]
Our assumption that zero energy is not a resonance implies that $T(\lambda) = \alpha \lambda + o(\lambda)$ where $\alpha \neq 0$. In particular, $T(0) = 0$ and $R_1(0) = R_2(0) = -1$. Integrating by parts in (32) therefore yields

$$\langle e^{itH}(1 - \chi(H))P_{a.c.}, \psi \rangle$$

(33)

$$= -\frac{1}{4\pi it} \int_0^\infty e^{it\lambda^2} \partial_\lambda \left[(1 - \chi(\lambda^2))(T(\lambda))^2 \lambda^{-1} \langle \psi, f_1(\cdot, \lambda) \rangle \langle f_1(\cdot, \lambda), \phi \rangle \right] d\lambda$$

$$- \frac{1}{4\pi it} \int_{-\infty}^0 e^{it\lambda^2} \partial_\lambda \left[(1 - \chi(\lambda^2))(T(\lambda))^2 \lambda^{-1} \langle \psi, f_2(\cdot, -\lambda) \rangle \langle f_2(\cdot, -\lambda), \phi \rangle \right] d\lambda.$$

By symmetry, it will suffice to treat the integral involving $f_1(\cdot, \lambda)$. We distinguish three cases, depending on where the derivative $\partial_\lambda$ falls. We start with the integral

(34)

$$\int_0^\infty e^{it\lambda^2} \omega(\lambda)f_1(x, \lambda)f_1(y, -\lambda) \, d\lambda,$$

where we have set $\omega(\lambda) = \partial_\lambda[(1 - \chi(\lambda^2))(T(\lambda))^2\lambda^{-1}]$. By the preceding, $\omega$ is a smooth function with compact support in $[0, \infty)$. As usual, we will estimate (34) by means of a Fourier transform in $\lambda$. Since we are working on a half-line, this will actually be a cosine transform. Let $\tilde{\omega}$ be another cut-off function satisfying $\tilde{\omega} = \omega$. Then

(35)

$$\left| \int_0^\infty e^{it\lambda^2} \omega(\lambda)f_1(x, \lambda)f_1(y, -\lambda) \, d\lambda \right| \leq C|t|^{-\frac{3}{2}} \| \omega f_1(x, \cdot) \|_1 \| \tilde{\omega} f_1(y, \cdot) \|_1.$$

It remains to estimate

(36)

$$[\omega f_1(x, \cdot)]^\vee(u) := \int_0^\infty \cos(u\lambda) \omega(\lambda) f_1(x, \lambda) \, d\lambda$$

in $L^1$ relative to $u$. The second $L^1$-norm in (35) is treated the same way. We need to consider the cases $x \geq 0$ and $x \leq 0$ separately. In the former case,

(37)

$$[\omega f_1(x, \cdot)]^\vee(u) := \int_0^\infty \cos(u\lambda) e^{ix\lambda} \omega(\lambda) e^{-ix\lambda} f_1(x, \lambda) \, d\lambda$$

$$= \frac{1}{2} \int_0^\infty e^{i(x+u)\lambda} \omega(\lambda) e^{-ix\lambda} f_1(x, \lambda) \, d\lambda + \frac{1}{2} \int_0^\infty e^{i(x-u)\lambda} \omega(\lambda) e^{-ix\lambda} f_1(x, \lambda) \, d\lambda.$$

If $||u| - |x|| \leq |x|$, then we simply estimate

$$||[\omega f_1(x, \cdot)]^\vee(u)|| \leq C.$$

On the other hand, if $||u| - |x|| > |x|$, then we integrate by parts in (37):

(38)

$$[\omega f_1(x, \cdot)]^\vee(u) = -\frac{1}{2i(x+u)} \omega(0)f_1(x, 0) - \frac{1}{2i(x-u)} \omega(0)f_1(x, 0)$$

$$- \frac{1}{2i(x+u)} \int_0^\infty e^{i(x+u)\lambda} \partial_\lambda \left[ \omega(\lambda) e^{-ix\lambda} f_1(x, \lambda) \right] \, d\lambda$$

$$- \frac{1}{2i(x-u)} \int_0^\infty e^{i(x-u)\lambda} \partial_\lambda \left[ \omega(\lambda) e^{-ix\lambda} f_1(x, \lambda) \right] \, d\lambda.$$

Since

$$\sup_{x \geq 0, \lambda} |\partial_\lambda^j [\omega(\lambda) e^{-ix\lambda} f_1(x, \lambda)]| \leq C(V),$$

for $j = 0, 1, 2$, it follows that

$$||[\omega f_1(x, \cdot)]^\vee(u)|| \leq C \frac{|x|}{|x^2 - u^2|} + C(u + x)^{-2} + C(u - x)^{-2}.$$
The conclusion is that

\[ \int_{\mathbb{R}} |[\omega f_1(x, \cdot)]^\prime (u)| \, du \leq C(x). \]

To deal with \( x \leq 0 \), we use (39). Thus,

\[ [\omega f_1(x, \cdot)]^\prime (u) = \int_{0}^{\infty} \cos(u\lambda)\omega(\lambda) \left( \frac{R_1(\lambda)}{T(\lambda)} + \frac{1}{T(\lambda)} \right) f_2(x, \lambda) \, d\lambda \]

\[ + \int_{0}^{\infty} \cos(u\lambda)\omega(\lambda) \left( \frac{1}{T(\lambda)} f_2(x, \lambda) - f_2(x, -\lambda) \right) \, d\lambda. \]

Set \( \omega_1 = \omega(\lambda) \left( \frac{R_1(\lambda)}{T(\lambda)} + \frac{1}{T(\lambda)} \right) \). Then (40) can be written as

\[ \int_{0}^{\infty} \cos(u\lambda)\omega(\lambda) \left( \frac{R_1(\lambda)}{T(\lambda)} + \frac{1}{T(\lambda)} \right) f_2(x, \lambda) \, d\lambda = \int_{0}^{\infty} \cos(u\lambda)e^{-ix\lambda}\omega_1(\lambda)e^{ix\lambda} f_2(x, \lambda) \, d\lambda. \]

Hence it can be treated by the same arguments as (36) with \( x \geq 0 \). Indeed, simply use that

\[ \sup_{x \leq 0, \lambda} |\partial_\lambda [\omega_1(\lambda)e^{ix\lambda} f_2(x, \lambda)]| \leq C(V). \]

On the other hand, (41) is the same as (with \( \partial_2 \) being the partial derivative with respect to the second variable of \( f_2 \))

\[ \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda)\omega(\lambda) \left( \frac{\lambda}{T(\lambda)} \right) \partial_2 f_2(x, \lambda\sigma) \, d\lambda d\sigma \]

\[ = \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda)e^{-ix\lambda}\omega_2(\lambda) \partial_2 [e^{ix\lambda} f_2(x, \lambda\sigma)] \, d\lambda d\sigma \]

\[ - ix \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda)e^{-ix\lambda}\omega_2(\lambda) e^{ix\lambda} f_2(x, \lambda\sigma) \, d\lambda d\sigma, \]

where we have set \( \omega_2(\lambda) = \omega(\lambda) \left( \frac{\lambda}{T(\lambda)} \right) \) (a smooth, compactly supported function in \([0, \infty)\)). We will focus on the second integral (43), since the first one (42) is similar. We will integrate by parts in \( \lambda \), but only on the set \(|\sigma x \pm u| \geq 1\). Then

\[ - ix \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda)e^{-ix\lambda}\omega_2(\lambda) e^{ix\lambda} f_2(x, \lambda\sigma) \, d\lambda \chi_{|\sigma x \pm u| \geq 1} \, d\sigma \]

\[ = \int_{-1}^{1} \frac{x}{2(-\sigma x + u)} \omega_2(0) f_2(x, 0) \chi_{|\sigma x \pm u| \geq 1} \, d\sigma + \int_{-1}^{1} \frac{x}{2(\sigma x - u)} \omega_2(0) f_2(x, 0) \chi_{|\sigma x \pm u| \geq 1} \, d\sigma \]

\[ + \int_{-1}^{1} \frac{x}{2(-\sigma x + u)} \int_{0}^{\infty} e^{i(-\sigma x + u)\lambda} \partial_\lambda \left[ \omega_2(\lambda) e^{ix\lambda} f_2(x, \lambda) \right] \, d\lambda \chi_{|\sigma x \pm u| \geq 1} \, d\sigma \]

\[ + \int_{-1}^{1} \frac{x}{2(\sigma x - u)} \int_{0}^{\infty} e^{i(-\sigma x - u)\lambda} \partial_\lambda \left[ \omega_2(\lambda) e^{ix\lambda} f_2(x, \lambda) \right] \, d\lambda \chi_{|\sigma x \pm u| \geq 1} \, d\sigma. \]

The first two integrals here (which are due to the boundary \( \lambda = 0 \)) contribute

\[ \int_{-1}^{1} \frac{x}{2(-\sigma x + u)} \omega_2(0) f_2(x, 0) \chi_{|\sigma x \pm u| \geq 1} \, d\sigma + \int_{-1}^{1} \frac{x}{2(\sigma x - u)} \omega_2(0) f_2(x, 0) \chi_{|\sigma x \pm u| \geq 1} \, d\sigma = 0, \]
where we performed a change of variables $\sigma \mapsto -\sigma$ in the second one. Integrating by parts one more time in (44) and (45) with respect to $\lambda$ implies

\[
\int_{-\infty}^{\infty} \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda)e^{-i\lambda x}\omega_2(\lambda)e^{i\lambda x}\sigma f_2(x, \lambda\sigma) d\lambda d\sigma \left| x \right| du \leq C \int_{-\infty}^{\infty} \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda) \omega_2(\lambda)e^{i\lambda x}\sigma f_2(x, \lambda\sigma) d\lambda d\sigma \left| x \right| du \leq C \left| x \right|.
\]

Finally, the cases $|\sigma x + u| \leq 1$ and $|\sigma x - u| \leq 1$ each contribute at most $C|\sigma|$ to the $u$-integral. Hence

\[
\int_{-\infty}^{\infty} \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda)e^{-i\lambda x}\omega_2(\lambda)e^{i\lambda x}\sigma f_2(x, \lambda\sigma) d\lambda d\sigma \left| x \right| du \leq C \left| x \right|.
\]

Since (42) can be treated the same way (in fact, the bound is $O(1)$), we obtain

\[
\int_{-\infty}^{\infty} \int_{-1}^{1} \int_{0}^{\infty} \cos(u\lambda)\omega(\lambda)\frac{\lambda}{T(\lambda)} \partial_2 f_2(x, \lambda\sigma) d\lambda d\sigma \left| x \right| du \leq C \left| x \right|.
\]

In view of (39), (40), and (41),

\[
\left\| [\omega f_1(x, \cdot)]' \right\|_1 \leq C \left| x \right| \quad \forall \, x \in \mathbb{R},
\]

which in turn implies that

\[
\left| \int_{0}^{\infty} e^{it\lambda^2} \omega(\lambda)f_1(x, \lambda)f_1(y, -\lambda) d\lambda \right| \leq C \left| t \right|^{-\frac{1}{2}} \left| x \right| \left| y \right|
\]

for all $x, y \in \mathbb{R}$, see (44). This is the desired estimate on (33), but only for the case when $\partial_\lambda$ falls on the factors not involving $f_1$. We now consider the case when $\partial_\lambda$ falls on $f_1(x, \lambda)$. The integral in which $\partial_\lambda$ falls on $f_1(y, -\lambda)$ is analogous. Hence, we need to estimate

\[
\int_{0}^{\infty} e^{it\lambda^2}(1 - \chi(\lambda^2))[T(\lambda)]^{2\lambda^{-1}} \partial_\lambda f_1(x, \lambda) f_1(y, -\lambda) d\lambda
\]

\[
= \int_{0}^{\infty} e^{it\lambda^2}(1 - \chi(\lambda^2))T(-\lambda)\lambda^{-1} \partial_\lambda[T(\lambda)f_1(x, \lambda)] f_1(y, -\lambda) d\lambda
\]

\[
+ \int_{0}^{\infty} e^{it\lambda^2}(1 - \chi(\lambda^2))T(-\lambda)T'(\lambda)\lambda^{-1} f_1(x, \lambda) f_1(y, -\lambda) d\lambda.
\]

The integral in (48) is of the same form as that in (34). It therefore suffices to control (47). Let $\omega_3(\lambda) = (1 - \chi(\lambda^2))T(-\lambda)\lambda^{-1}$. By the same reductions as before, we need to show that

\[
\left\| [\omega_3 \partial_\lambda[T(\lambda)f_1(x, \cdot)]]' \right\|_1 \leq C \left| x \right| \quad \forall \, x \in \mathbb{R}.
\]

Thus consider

\[
\int_{0}^{\infty} \cos(u\lambda)\omega_3(\lambda)\partial_\lambda[T(\lambda)f_1(x, \lambda)] d\lambda = ix \int_{0}^{\infty} \cos(u\lambda)e^{ix\lambda}\omega_3(\lambda)T(\lambda)e^{-ix\lambda} f_1(x, \lambda) d\lambda
\]

\[
+ \int_{0}^{\infty} \cos(u\lambda)e^{ix\lambda}\omega_3(\lambda)\partial_\lambda[T(\lambda)e^{-ix\lambda} f_1(x, \lambda)] d\lambda.
\]
If $x \geq 0$, integrating by parts leads to

$$
ix \int_0^\infty \cos(u\lambda)e^{ix\lambda}\omega_3(\lambda)T(\lambda)e^{-ix\lambda}f_1(x, \lambda) \ d\lambda
$$

$$
= -\frac{ix}{2i(x+u)} \int_0^\infty e^{i(x+u)\lambda} \partial_\lambda \left[ \omega_3(\lambda)T(\lambda)e^{-ix\lambda}f_1(x, \lambda) \right] \ d\lambda
$$

$$
- \frac{ix}{2i(x-u)} \int_0^\infty e^{i(x-u)\lambda} \partial_\lambda \left[ \omega_3(\lambda)T(\lambda)e^{-ix\lambda}f_1(x, \lambda) \right] \ d\lambda
$$
as well as

$$
\int_0^\infty \cos(u\lambda)e^{ix\lambda}\omega_3(\lambda)\partial_\lambda[T(\lambda)e^{-ix\lambda}f_1(x, \lambda)]d\lambda
$$

$$
= -\left. \frac{1}{2i(x+u)} \omega_3(0)\partial_\lambda[T(\lambda)e^{-ix\lambda}f_1(x, \lambda)] \right|_{\lambda=0} - \frac{1}{2i(x-u)} \omega_3(0)\partial_\lambda[T(\lambda)e^{-ix\lambda}f_1(x, \lambda)] \right|_{\lambda=0}
$$

$$
- \frac{1}{2i(x+u)} \int_0^\infty e^{i(x+u)\lambda} \partial_\lambda \left[ \omega_3(\lambda)\partial_\lambda[T(\lambda)e^{-ix\lambda}f_1(x, \lambda)] \right] \ d\lambda
$$

$$
- \frac{1}{2i(x-u)} \int_0^\infty e^{i(x-u)\lambda} \partial_\lambda \left[ \omega_3(\lambda)\partial_\lambda[T(\lambda)e^{-ix\lambda}f_1(x, \lambda)] \right] \ d\lambda.
$$

Integrating by parts one more time in (49) implies

$$
\left| ix \int_0^\infty \cos(u\lambda)e^{ix\lambda}\omega_3(\lambda)T(\lambda)e^{-ix\lambda}f_1(x, \lambda) \ d\lambda \right|
$$

$$
\leq C|x|(1 + |x-u|)^{-2} + C|x|(1 + |x+u|)^{-2}
$$

uniformly in $x \geq 0$, whereas (50) is treated the same way as (38). One needs to use here that

$$
\sup_{x \geq 0, \lambda} \left| \partial_\lambda^j[\omega_3(\lambda)e^{-ix\lambda}f_1(x, \lambda)] \right| \leq C(V),
$$

for $j = 0, 1, 2, 3$ which follows from $\|\langle x \rangle^4V\|_1 < \infty$. Consequently, we have proved that

$$
\int_\mathbb{R} \left| \int_0^\infty \cos(u\lambda)\omega_3(\lambda)\partial_\lambda[T(\lambda)f_1(x, \lambda)]d\lambda \right| du \leq C\langle x \rangle
$$

uniformly in $x \geq 0$.

Next, we deal with the case $x \leq 0$. In view of (31),

$$
T(\lambda)f_1(\cdot, \lambda) = R_1(\lambda)f_2(\cdot, \lambda) + f_2(\cdot, -\lambda).
$$

This implies that

$$
\int_0^\infty \cos(u\lambda)\omega_3(\lambda)\partial_\lambda[T(\lambda)f_1(x, \lambda)]d\lambda
$$

$$
= \int_0^\infty \cos(u\lambda)e^{-ix\lambda}\omega_3(\lambda)\partial_\lambda[R_1(\lambda)e^{ix\lambda}f_2(x, \lambda)]d\lambda - ix \int_0^\infty \cos(u\lambda)e^{-ix\lambda}\omega_3(\lambda)R_1(\lambda)e^{ix\lambda}f_2(x, \lambda)d\lambda
$$

$$
+ \int_0^\infty \cos(u\lambda)e^{-ix\lambda}\omega_3(\lambda)\partial_\lambda[e^{ix\lambda}f_2(x, -\lambda)]d\lambda + ix \int_0^\infty \cos(u\lambda)e^{ix\lambda}\omega_3(\lambda)e^{-ix\lambda}f_2(x, -\lambda)d\lambda.
$$

The two integrals which are not preceded by factors of $ix$ are treated just as in (50). The only difference here is that the estimates are uniform in $x \leq 0$ rather than $x \geq 0$. On the other hand,
the integrals preceded by $ix$ need to be integrated by parts in $\lambda$. It is important to check that the boundary terms at $\lambda = 0$ do not contribute to this case. Indeed, these boundary terms are

$$\begin{align*}
\frac{x}{2(u-x)}\omega_3(0)R_1(0)f_2(x,0) & - \frac{x}{2(u+x)}\omega_3(0)R_1(0)f_2(x,0) \\
- \frac{x}{2(u+x)}\omega_3(0)f_2(x,0) & - \frac{x}{2(x-u)}\omega_3(0)f_2(x,0) = 0,
\end{align*}$$

since $R_1(0) = -1$. Hence, integrating by parts leads to an expression similar to $49$. The conclusion is that $\{51\}$ satisfies

$$\int_\mathbb{R} \left| \int_0^{\infty} \cos(u\lambda)\omega_3(\lambda)\partial_\lambda [T(\lambda)f_1(x,\lambda)] \, d\lambda \right| \, du \leq C(x)$$

uniformly in $x \leq 0$, and we are done. □

In $55$ the same bound is proved for non-selfadjoint systems of the type that arise by linearizing NLS around a ground state standing wave. It is crucial for proving the existence of stable manifolds for all super-critical NLS in one dimension.

In dimension one, there is some recent work of Cai [17] on dispersion for Hill’s operator. More precisely, let $H = -\frac{d^2}{dx^2} + q$ where $q$ is periodic and such that its spectrum has precisely one gap. It is well-known that such $q$ are characterized in terms of Weierstrass elliptic functions. As part of his Caltech Ph.D. thesis, Cai showed that for this $H$ one always has

$$\|e^{itH}f\|_\infty \leq Ct^{-\frac{1}{2}}\|f\|_1, \quad t \geq 1$$

and that generically in the potential one can replace $\frac{1}{2}$ with $\frac{1}{3}$.

4. THE TWO-DIMENSIONAL CASE

The following two-dimensional dispersive estimate was obtained in $68$.

**Theorem 4.1.** Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function such that $|V(x)| \leq C(1 + |x|)^{-\beta}$, $\beta > 3$. Assume in addition that zero is a regular point of the spectrum of $H = -\Delta + V$. Then

$$\|e^{itH}P_{ac}(H)f\|_\infty \leq C|t|^{-1}\|f\|_1$$

for all $f \in L^1(\mathbb{R}^2)$.

The definition of zero being a regular point amounts to the following, see Jensen, Nenciu [49]: Let $V \not\equiv 0$ and set $U = \text{sign}V$, $v = |V|^\frac{1}{2}$. Let $P_v$ be the orthogonal projection onto $v$ and set $Q = I - P_v$. Finally, let

$$(G_0f)(x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| f(y) \, dy.$$

Then zero is regular iff $Q(U + vG_0v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.

Jensen and Nenciu study $\ker[Q(U + vG_0v)Q]$ on $QL^2(\mathbb{R}^2)$. It can be completely described in terms of solutions $\Psi$ of $H\Psi = 0$. In particular, its dimension is at most three plus the dimension of the zero energy eigenspace, see Theorem 6.2 and Lemma 6.4 in [49]. The extra three dimensions here are called resonances. Hence, the requirement that zero is a regular point is the analogue of the usual condition that zero is neither an eigenvalue nor a resonance of $H$. An equivalent characterization of a regular point was given in [6], albeit under the additional assumption that $\int_{\mathbb{R}^2} V(x) \, dx \neq 0$.

As far as the spectral properties of $H$ are concerned, we note that under the hypotheses of Theorem [14] the spectrum of $H$ on $[0, \infty)$ is purely absolutely continuous, and that the spectrum is pure point on $(-\infty, 0)$ with at most finitely many eigenvalues of finite multiplicities. The latter
follows for example from Stoiciu [75], who obtained Birman-Schwinger type bounds in the case of two dimensions.

Theorem 4.1 appears to be the first \( L^1 \to L^\infty \) bound with \(|t|^{-1}\) decay in \( \mathbb{R}^2 \). Yajima [86] and Jensen, Yajima [50] proved the \( L^p(\mathbb{R}^2) \) boundedness of the wave operators under stronger decay assumptions on \( V(x) \), but only for \( 1 < p < \infty \). Hence their result does not imply Theorem 4.1.

Local \( L^2 \) decay was studied by Murata [56], but he does not consider \( L^1 \to L^\infty \) estimates.

The main challenge in two dimensions is of course the low energy part. This is due to the fact that the free resolvent \( R_0^\pm(\lambda^2) = (-\Delta - (\lambda^2 \pm i0))^{-1} \) has the kernel \((H_0^\pm)\) being the Hankel functions

\[
R_0^\pm(\lambda^2)(x,y) = \pm \frac{i}{2} H_0^\pm(\lambda|x-y|),
\]

which is singular at energy zero (which, just as in dimension one, expresses the fact that the free problem has a resonance at zero). It is a consequence of the asymptotic expansion of Hankel functions that for all \( \lambda > 0 \),

\[
R_0^\pm(\lambda^2) = \left[ \pm \frac{i}{4} - \frac{1}{2\pi^2} \lambda \log(\lambda/2) \right] P_0 + G_0 + E_0^\pm(\lambda).
\]

Here \( P_0 := \int_{\mathbb{R}^2} f(x) \, dx, G_0 f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x-y| f(y) \, dy \), and the error \( E_0^\pm(\lambda) \) has the property that

\[
\| \sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}} |E_0^\pm(\lambda)(\cdot,\cdot)| \| + \| \sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E_0^\pm(\lambda)(\cdot,\cdot)| \| \lesssim 1
\]

with respect to the Hilbert-Schmidt norm in \( \mathcal{B}(L^{2,s}(\mathbb{R}^2), L^{2,-s}(\mathbb{R}^2)) \) with \( s > \frac{3}{2} \). These error estimates may seem artificial, but they allow for the least amount of decay on \( V \). The following lemma from [68] contains the expansion of the perturbed resolvent around energy zero needed in the proof of Theorem 4.1. It displays an important idea from [49], namely to re-sum infinite series of powers of \( \log \lambda \) into one function \( h_\pm(\lambda) \). This feature is crucial for our purposes. Given \( V \neq 0 \), set \( U = \text{sign} \, V, v = |V|^\frac{1}{2} \). Let \( P_v \) be the orthogonal projection onto \( v \) and set \( Q = I - P_v \). Finally, let \( D_0 = (Q(U + vG_0v)Q)^{-1} \) on \( QL^2(\mathbb{R}^2) \)

Lemma 4.2. Suppose that zero is a regular point of the spectrum of \( H = -\Delta + V \). Then for some sufficiently small \( \lambda_1 > 0 \), the operators \( M^\pm(\lambda) := U + v R_0^\pm(\lambda^2)v \) are invertible for all \( 0 < \lambda < \lambda_1 \) as bounded operators on \( L^2(\mathbb{R}^2) \), and one has the expansion

\[
M^\pm(\lambda)^{-1} = h_\pm(\lambda)^{-1} - S + QD_0 Q + E^\pm(\lambda),
\]

where \( h_\pm(\lambda) = a \log \lambda + z, a \) is real, \( z \) complex, \( a \neq 0 \), \( 3z \neq 0 \), and \( h_\pm(\lambda) = \overline{h_\pm(\lambda)} \). Moreover, \( S \) is of finite rank and has a real-valued kernel, and \( E^\pm(\lambda) \) is a Hilbert-Schmidt operator that satisfies the bound

\[
\| \sup_{0<\lambda<\lambda_1} \lambda^{-\frac{1}{2}} |E^\pm(\lambda)(\cdot,\cdot)| \|_{HS} + \| \sup_{0<\lambda<\lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)(\cdot,\cdot)| \|_{HS} \lesssim 1
\]

where the norm refers to the Hilbert-Schmidt norm on \( L^2(\mathbb{R}^2) \). Finally, let \( R_V^\pm(\lambda^2) = (-\Delta + V - (\lambda^2 \pm i0))^{-1} \). Then

\[
R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2).
\]

This is to be understood as an identity between operators \( L^{2,\frac{1}{2}+\epsilon}(\mathbb{R}^2) \to L^{2,-\frac{1}{2}-\epsilon}(\mathbb{R}^2) \) for some sufficiently small \( \epsilon > 0 \).

The low energy part of the proof of Theorem 4.1 is based on a careful estimation of the contribution of each of the terms in (54) to \( R_V \) in (53) by means of the method of stationary phase, see [68].
Murata [50] discovered that under the assumptions of Theorem 4.1
\[
\|we^{itH}P_{ac}(H)wf\|_2 \leq C|t|^{-1}(\log t)^{-2}\|f\|_2
\]
provided \(w(x) = \langle x \rangle^{-\sigma}\) with some sufficiently large \(\sigma > 0\). In other words, he obtained improved local \(L^2\) decay provided zero energy is regular. Needless to say, such improved decay is impossible for the \(L^1 \to L^\infty\) bound, but a weighted \(L^1 \to L^\infty\) estimate as in Theorem 3.1 with the improved \(|t|^{-1}(\log t)^{-2}\) decay is quite possibly true but currently unknown. Due to the integrability of this decay at infinity, such a bound would be useful for the study of nonlinear asymptotic stability of (multi) solitons in dimension two.

5. Time-dependent potentials

It seems unreasonable to expect a general theory of dispersion for the Schrödinger equation
\[(57)\]
\[i\partial_t \psi + \triangle \psi + V(t, \cdot)\psi = 0\]
for time-dependent potentials \(V(t, \cdot)\). While the \(L^2\) norm is preserved for real-valued \(V\), it is well-known that in contrast to time-independent \(V\) higher \(H^s\) norms can grow in this case, see e.g. Bourgain [9], [10], and Erdogan, Killip, Schlag [27].

The classical work of Davies [24], Howland [41], [42], [43], and Yajima [82], deals with scattering for real-valued trigonometric potentials that are small at infinity (rather than vanishing).

Dispersive estimates were obtained by Rodnianski and the author [63] for small but not necessarily decaying time-dependent potentials in \(\mathbb{R}^3\), whereas the case of decaying \(V\) and dimensions \(\geq 2\) was studied by Naibo, Stepanov [57], and d’Ancona, Pierfelice, Visciglia [23]. In particular, the result from [63] insures that in \(\mathbb{R}^3\) and for small \(\varepsilon\)
\[i\partial_t \psi + \triangle \psi + \varepsilon F(t)V(x)\psi = 0\]
has the usual \(t^{-\frac{3}{2}}\) dispersive \(L^1 \to L^\infty\) decay for any real-valued trigonometric polynomial \(F(t)\) (or more generally, any quasi-periodic analytic function \(F(t)\)) and \(V\) satisfying \(\|V\|_\mathcal{K} < \infty\), see [20].

Another much studied case is that of time-periodic \(V\), see [24], [43], and [82]. Suppose \(T > 0\) is the smallest period of \(V\). Then the theory of (57) reduces to that of the Floquet operator \(U = U(T, 0)\). The Floquet operator can exhibit bound states and the question arises as to the
existence and ranges of the wave operators (the so called completeness problem) as well as the structure of the discrete spectrum. These issues are addressed in the aforementioned references.

More recently, in [30], Galtbayar, Jensen, and Yajima show that on the orthogonal complement of the bound states of the Floquet operator the solutions decay locally in $L^2(\mathbb{R}^3)$. In addition, O. Costin, R. Costin, Lebowitz, and Rohlenko [19], [18], have made a very detailed analysis of some special models with time-periodic potentials. More precisely, they have found and applied a criterion that ensures scattering of the wave function. On the level of the Floquet operator this means that there is no discrete spectrum. It would be interesting to obtain dispersive estimates for these cases.

Another well-studied class of time-dependent potentials are the so-called charge transfer models. These are Hamiltonians of the form

$$ H(t) = -\Delta + \sum_{j=1}^{m} V_j(\cdot - v_j t) $$

where $\{v_j\}_{j=1}^{m}$ are distinct velocities and $V_j$ are well-localized potentials. They admit localized states that travel with each of these potentials and asymptotically behave like the sum of bound states of each of the “channel Hamiltonians”

$$ H(t) = -\Delta + V_j(\cdot - v_j t) . $$

Those are of course Galilei transformed bound states of the corresponding stationary Hamiltonians. Yajima [51] and Graf [38] proved that these Hamiltonians are asymptotically complete, i.e., that as $t \to \infty$ each state decomposes into a sum of wave functions associated with each of the channels, including the free channel.

Rodnianski, Soffer, and the author obtained dispersive estimates for these models in the spaces $L^1 \cap L^2 \to L^2 \cap L^\infty$. Later, Cai [16] as part of his Caltech thesis removed $L^2$ from these bounds. Such estimates were needed in order to prove asymptotic stability of $N$-soliton solutions, see [65].

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