GREAT SPHERE FOLIATIONS AND MANIFOLDS
WITH CURVATURE BOUNDED ABOVE

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Abstract. The survey is devoted to Toponogov’s conjecture, that if a complete simply connected Riemannian manifold with sectional curvature ≤ 4 and injectivity radius ≥ π/2 has extremal diameter π/2, then it is isometric to CROSS. In Section 1 the relations of problem with geodesic foliations of a round sphere are considered, but the proof of conjecture on this way is not complete. In Section 2 the proof based on recent results and methods for topology and volume of Blaschke manifolds is given.

1. Great circle foliations

The interest to fibrations of the \( n \)-sphere by great \( \nu \)-spheres is propelled by the Blaschke problem and by extremal theorems in Riemannian geometry.

On a round sphere a foliation by geodesics is the same thing as a great circle fibration. The most simple of them – Hopf fibration with fibers \( \{S^1\} \), can be given as a collection of intersections of \( S^{2n-1} \) with all holomorphic 2-planes \( \{\sigma = x \wedge Jx\} \), where \( J \) is a complex structure – a linear operator in \( \mathbb{R}^{2n} \), given for some orthonormal basis \( \{e_i\} \) by the rule

\[
Je_{2i-1} = e_{2i}, \quad Je_{2i} = -e_{2i-1}, \quad (1 \leq i \leq n).
\]

Let \( F_0(S^{2n-1}) \) denote a space of all Hopf fibrations of sphere \( S^{2n-1} \), for \( n = 2 \) see Chapter 1. Each fiber spans corresponding two-plane through the origin in \( \mathbb{R}^{2n} \) and hence determines a point in the Grassmann manifold \( G(2, 2n) \).

2 Definition ([GW] for \( n = 2 \)). The skew-Hopf fibration is given by intersections of \( S^{2n-1} \) with all holomorphic 2-planes \( \{\sigma = x \wedge Jx\} \), where \( J \) is an almost complex structure – a linear operator in \( \mathbb{R}^{2n} \), given for some affine basis \( \{e_i\} \) by the rule (1).

By other words, the skew-Hopf fibration is obtained from the Hopf fibration by applying a nondegenerate linear transformation of \( \mathbb{R}^{2n} \) and then projection the images of fibers back to the \( S^{2n-1} \).

Let \( F_1(S^{2n-1}) \) denote a space of all skew-Hopf fibrations of sphere \( S^{2n-1} \). Let \( F(S^{2n-1}) \) denote a space of all oriented great circle fibrations of sphere \( S^{2n-1} \).

For simplicity we consider below geodesic foliations on three-sphere \( (n > 2 \) see [Rov 2]). The space \( F_0(S^3) \) is 2-dimensional and homeomorphic to a pair of

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disjoint two-spheres. The 8-dimensional space $\mathcal{F}_1(S^3)$ is a disjoint union of two copies of $S^2 \times \mathbb{R}^6$. Both these spaces are homogeneous [GW]:

$$\mathcal{F}_0(S^3) = O(4)/U(2), \quad \mathcal{F}_1(S^3) = GL(4,\mathbb{R})/GL(2,\mathbb{C}).$$ (3)

The space $\mathcal{F}(S^3)$ is infinite dimensional. Let $V$ be a Hopf unit vector field and $D^2$ a small ball transverse to $V$ at point $p$. As was shown in [GW], a small $C^1$-perturbations of $V$ on $D^2$, which are identity on the neighbourhood of boundary $\partial D^2$, lead to different great circle foliations of $S^3$.

For example, $G(2,4) \equiv S^2 \times S^2$ [GW], and a Hopf fibration $h \in \mathcal{F}_0(S^3)$ can be recognized by the fact that its orbit space $M_h$ appears inside the Grassmanian as $\{\text{point}\} \times S^2$ or $S^2 \times \{\text{point}\}$.

For two 2-planes $P$ and $Q$ in $\mathbb{R}^4$ the smallest and the largest angles $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$ from interval $[0,\frac{\pi}{2}]$, that any line in $P$ makes with the plane $Q$, are called the principal angles between certain planes. The same angles result upon interchanging the roles of $P$ and $Q$. The relative position of $P$ and $Q$ in $\mathbb{R}^4$ is completely determined by these principal angles in sense of rigid motion of $\mathbb{R}^4$.

One can always choose an orthonormal basis $e_1, e_2, e_3, e_4$ for $\mathbb{R}^4$ so that $e_1, e_2$ is an orthonormal basis for $P$ and $\cos \alpha_{\text{min}} e_1 + \sin \alpha_{\text{min}} e_3$, $\cos \alpha_{\text{max}} e_2 + \sin \alpha_{\text{max}} e_4$ is an orthonormal basis for $Q$ [GW]. Any two equidistant great circles from $S^3$ (in particular two leaves of Hopf fibration) determine two 2-planes from $\mathbb{R}^4$ with equal principal angles.

Not all submanifolds of $G(2,4)$ can appear in the role of $M_f$ for $f \in \mathcal{F}(S^3)$.

4 Theorem [GW]. A submanifold in $G(2,4) \equiv S^2 \times S^2$ corresponds to a fibration $f \in \mathcal{F}(S^3)$ if and only if it is a graph of a distance decreasing map $\tilde{f}$ from either $S^2$ factor to the other. Fibration $f$ is differentiable if and only if the corresponding map $\tilde{f}$ is differentiable with $|d\tilde{f}| \leq 1$.

Hence, the space $\mathcal{F}(S^3)$ deformation retracts to the space $\mathcal{F}_0(S^3)$.

The catalogue of great circle fibrations of the three-sphere (in Theorem 4) is one of first nontrivial examples in which one has a clear overview of all possible geodesic foliations of a fixed Riemannian manifold. There are plenty of nondifferentiable great circle fibrations of $S^3$. There also exist discontinuous fillings of $S^3$ by great circles: one can fill the closed solid torus $x_1^2 + x_2^2 \geq x_3^2 + x_4^2$ on $S^3$ as for Hopf original fibration and then fill the remaining open solid torus as for Hopf fibration with reversed screw sense [GW].

5 Corollary [GW]. Each fibration $f \in \mathcal{F}(S^3)$ contains a pair of orthogonal fibers.

6 Theorem [Gag]. Each skew-Hopf fibration $f \in \mathcal{F}_1(S^3)$ corresponds to a distance decreasing map $f : S^2 \to S^2$ with convex image in a semi-sphere, which can be decomposed as: a) orthogonal projection of $S^2$ to a plane through the center of the sphere; followed by b) a distance decreasing linear map from one 2-plane to another; and finally c) inverse projection onto $S^2$.

Thus the space $\mathcal{F}_1(S^3)$ is stratified with respect to values of rank $r$ of linear map from point b): each fibration with $r = 2$ has exactly one pair of antipodes, in degenerate case $r = 1$ each fibration has 1-parameter family (Hopf torus) of pairs of antipodes, case $r = 0$ is the same as Hopf fibrations.

By Corollary 5 for each $f \in \mathcal{F}(S^3)$ there exists orthonormal special basis $\{e_i\}$ in $\mathbb{R}^4$ such, that the sphere $S^3$ intersects with 2-planes $e_1 \wedge e_2$ and $e_3 \wedge e_4$ by
pair of orthogonal fibers, (which is unique when \( r = 2 \)). It is not difficult to see, that in special basis for each vector \( \lambda = (1, 0, \lambda_3, \lambda_4) \) there exists a unique vector \( h = (0, 1, h_3, h_4) \) such, that the intersection of the plane \( \lambda \wedge h \) with sphere \( S^3 \) is a fiber of \( f \). Hence, we may correspond to foliation \( f \in F(S^3) \) a diffeomorphism of 2–plane

\[
\varphi = (\varphi_1, \varphi_2) : \mathbb{R}^2 \to \mathbb{R}^2, \quad h_3 = \varphi_1(\lambda_3, \lambda_4), \quad h_4 = \varphi_2(\lambda_3, \lambda_4). \quad (7)
\]

If \( f \in F_0(S^3) \), then the corresponding \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear orthogonal map.

**8 Lemma [Rov 1]**. The space \( F_1(S^3) \) is characterized in \( F(S^3) \) by the property, that \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear operator and without real eigenvalues.

**Proof.** Let \( f \in F_1(S^3) \), and an almost complex structure \( J \) in \( \mathbb{R}^4 \) for special basis is given by block matrix:

\[
J = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}, \quad A^2 = B^2 = -E, \quad (9)
\]

moreover, the matrices \( A, B \) have no real eigenvalues. It is easy to see, that \( \varphi \) is a linear operator on \( \mathbb{Bbb} \mathbb{R}^2 \) given by the following matrix without real eigenvalues

\[
F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a = \frac{a_{33} - a_{11}}{a_{21}}, \quad b = \frac{a_{34}}{a_{21}}, \quad c = \frac{a_{43}}{a_{21}}, \quad d = \frac{a_{44} - a_{11}}{a_{21}}. \quad (10)
\]

Conversely, let \( \varphi \) be a linear transformation of \( \mathbb{R}^2 \) with matrix \( F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) without real eigenvalues, i.e. the discriminant of characteristic quadratic equation is negative \( D = (a - d)^2 + 4bc < 0 \). Then the matrix of form (9) with coefficients

\[
a_{11} = -a_{22} = -\frac{a + d}{\sqrt{-D}}, \quad a_{33} = -a_{44} = \frac{a - d}{\sqrt{-D}}
\]

\[
a_{12} = \frac{2(bc - ad)}{\sqrt{-D}}, \quad a_{21} = \frac{2}{\sqrt{-D}}, \quad a_{34} = \frac{2b}{\sqrt{-D}}, \quad a_{43} = \frac{2c}{\sqrt{-D}} \quad (11)
\]

defines an almost complex structure in \( \mathbb{R}^4 \). It is easy to see, that the for induced foliation \( \tilde{f} \in F_1(S^3) \) the corresponding operator \( \tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2 \) has the following matrix \( F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

We study the skew-Hopf fibrations in relations with interesting space \( F_R(S^{2n-1}) \) of analytic geodesic foliations of sphere \( S^{2n-1} \), see [Top 1-3] for more general case.

**12 Definition [Top 1]**. The subspace \( F_R(S^{2n-1}) \subset F(S^{2n-1}) \) consists of fibrations, for which there exists a tensor (multi-linear function) \( R : \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4 \) with properties:

- \( R \) curvature symmetries

\[
R(x, y, z, w) = R(z, w, x, y) = -R(x, y, w, z),
\]

\[
R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0
\]

\[
R(x, y, z, w) = R(z, w, x, y) = -R(x, y, w, z),
\]

\[
R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0
\]
for almost each unit vector $x \in \mathbb{R}^{2n}$ there exists unique $2$–plane $\sigma \ni x$, with condition

$$R(x, y, x, y) = 1, \quad (y \in \sigma, \ y \perp x, \ |y| = 1),$$

(14)\footnote{Remark.} The analogous fact is true for $\mathbb{R}^n$ with complex structure $J$, standard metric and corresponding Hopf fibration have the properties $R_1 - R_3$.

13 Theorem [Rov 1]. \( \mathcal{F}_R(S^3) = \mathcal{F}_1(S^3). \)

Proof. For each $x \in \mathcal{F}(S^3)$ there exists special orthonormal basis such, that $2$–planes $e_1 \wedge e_2$ and $e_3 \wedge e_4$ intersect with sphere $S^3$ by fibers. Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a diffeomorphism, which by Lemma 8 corresponds to fibration $f$.

1. We shall state firstly the inclusion $\mathcal{F}_R(S^3) \subset \mathcal{F}_1(S^3)$.

Let $f \in \mathcal{F}_R(S^3)$. For a special basis in view of $R_1 - R_3$ we have

$$R_{1214} = \delta_{24}, \quad R_{2124} = \delta_{14}, \quad R_{3444} = \delta_{14}, \quad R_{3444} = \delta_{13},$$

where $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ are components of tensor $R$. Let us calculate the sectional curvature of $R$:

$$K(\lambda, h) = \frac{R(\lambda, h, \lambda, h)}{\lambda^2 h^2 - (\lambda, h)^2} = \frac{Q}{\lambda^2 h^2 - (\lambda, h)^2} + 1.$$ 

The quadratic form $Q$ from variables $\lambda_3, \lambda_4, h_3, h_4$ is given by formula

$$Q = \sum_{i,k}(R_{2i2k} - \delta_{ik})\lambda_i \lambda_k + \sum_{j,p}(R_{1j1p} - \delta_{jp})h_j h_p + 2 \sum_{i,p}(R_{12i4} + R_{24i1})\lambda_i h_j. \quad (15)$$

The $2$–parameter family of vectors $(\lambda_3, \lambda_4, h_3, h_4)$, which correspond to fibers of $f$, in view of $R_2 - R_3$ lies in kernel of form $Q$. In view of $R_3$ the subspace $\ker Q$ is $2$–dimensional, and hence the functions $h_3(\lambda_3, \lambda_4), h_4(\lambda_3, \lambda_4)$ of variables $\lambda_3, \lambda_4$ are linear. Since the fibers do not intersect in $S^3$, then linear operator $\varphi$ has no real eigenvalues. By Lemma 8 $f \in \mathcal{F}_1(S^3)$.

2. We shall now state the inverse inclusion $\mathcal{F}_1(S^3) \subset \mathcal{F}_R(S^3)$.

Let $f \in \mathcal{F}_1(S^3)$. Denote by $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix of linear operator $\varphi$, which corresponds to $f$ in special ortho-basis, see Lemma 8. In view multi-linearity and symmetries of tensor $R$ it is sufficient to define only some of components $\{R_{ijkl}\}$. It is easy to see, that properties $R_1 - R_3$ for tensor $R$ and $f$ are true, when we assume for arbitrary $\gamma < 0, \beta < 0$

$$R_{1214} = \delta_{24}, \quad R_{2124} = \delta_{14}, \quad R_{3444} = \delta_{14}, \quad R_{1313} = 1 = \gamma,$$

$$R_{1414} - 1 = \beta, \quad R_{2424} - 1 = b^2 \gamma + d^2 \beta, \quad R_{1323} = a \gamma, \quad R_{2324} = ab \gamma + cd \beta,$$

$$R_{1234} = \frac{1}{3}(-c \beta + b \gamma), \quad R_{2334} - 1 = a^2 \gamma + c^2 \beta, \quad R_{1424} = d \beta, \quad R_{1314} = 0,$$

$$R_{2431} = \frac{1}{3}(2b \gamma + c \beta), \quad R_{2341} = -\frac{1}{3}(2c \beta + b \gamma). \quad (16)$$

Thus $f \in \mathcal{F}_R(S^3)$.
A complete even-dimensional simply connected Riemannian manifold $M$ with sectional curvature $0 < K_M \leq 4$ has injectivity radius $\text{inj}(M) \geq \frac{\pi}{2}$ and, hence, the diameter $\text{diam}(M) \geq \frac{\pi}{2}$. For complete odd-dimensional simply connected $M$ the same inequalities for $\text{inj}(M)$ and $\text{diam}(M)$ are true under more strong curvature restriction $1 \leq K_M \leq 4$. The optimal value for the curvature pinching constant $\delta$ in the last case is unknown [AM], but for $\delta < \frac{1}{9}$ the proposition is wrong in view of Berge’s example [Ber 2].

17 Remark. M. Berge considered a family of Riemannian metrics $g_s$, $(0 < s \leq 1)$ on the odd-dimensional spheres $S^{2n+1}$, which are defined by shrinking the standard metric in the direction of the Hopf circles in such a way that their lengths with respect to $g_s$ become $2\pi s$. The range of the sectional curvature of such a metric $g_s$ is in the interval $[s^2, 4 - 3s^2]$. Clearly, $\pi s < \pi / \sqrt{4 - 3s^2}$ for $s^2 < \frac{1}{3}$. This means that for any $\delta \in (0, \frac{1}{9})$ there exists a M. Berger metric $g_s$ whose sectional curvature $K$ is $\delta$-pinched and whose injectivity radius is strictly less than $\pi / \sqrt{\max K}$. From considering the curvature along horizontal geodesics, we obtain that for any $0 < s < 1$ the conjugate radius of $g_s$ is strictly greater than $\pi / \sqrt{\max K}$.

Note, that $\text{inj}(M)$ is always not more then $\text{diam}(M)$. The manifolds with $\text{inj}(M) = \text{diam}(M)$ are exactly Blaschke manifolds [Bes].

In situations when the extremal value for curvature, diameter or volume of manifold is considered (under others given conditions), one often obtains, that manifold is isometric to a model space from a finite list. The bright example of such extremal theorems is the following

18 Theorem (minimal diameter) [Ber 1]. Let $M$ be a complete, connected, simply connected Riemannian manifold with sectional curvature $1 \leq K_M \leq 4$ and diameter $\frac{\pi}{2}$. Then $M$ is isometric to CROSS: round sphere of curvature $4$ or projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$ with its canonical metric.

M. Berger used direct geometric arguments to see the curvature tensor in enough detail to prove that such a manifold must be locally symmetric and hence (since simply connected) symmetric space. Appealing to the classification of these finished the proof. J. Cheeger and D. Ebin [CE] also give a geometricaly more direct proof for this result. H. Gluck and co-authors [GWZ] reproved constructively this theorem, used Berger’s geometric arguments to show that the exponential map from the tangent cut locus to the cut locus is a fibration of a round sphere by parallel great spheres, and hence a Hopf fibration. Then they see how this fibration encodes the curvature tensor and use this to display an isometry between $M$ and a round sphere or projective space. Latter Berger’s theorem was generalized in some directions:

1) stability results: there exists a constant $\delta_n < \frac{1}{4}$ such that any $n$-dimensional, complete, simply connected Riemannian manifold $M^n$ with $\delta_n \leq K_M \leq 1$ is homeomorphic to CROSS [Ber 3], for compact odd-dimensional manifold such $\delta$ is universal and less than $\frac{1}{4}(1 + 10^{-6})^{-2}$ [AM].

2) upper curvature bound is replaced by corresponding lower bound for the diameter or radius: a complete, simply connected Riemannian manifold $M^n$ with $K_M \geq 1$ and radius $\text{rad}(M) \geq \frac{\pi}{2}$ is either homeomorphic to the sphere or the universal covering $\tilde{M}$ is isometric to CROSS [Wil].

Analogous result with diameter is obtained by [GG3], where the only case of $\mathbb{C}aP^2$ is unknown. Recall that the radius $\text{rad}(M)$ of a compact connected Rie-
mannian manifold is defined as the infimum of the function $p \to \text{rad}_p(M) =: \max_{q \in M} \text{dist}(p,q)$. Clearly, $\text{inj}(M) \leq \text{rad}(M) \leq \text{diam}(M)$. One of key point in these results is the studying of Riemannian foliations on the round sphere. For 1- and 3-dimensional leaves they are always Hopf fibrations $[GG]$, a partial classification of Riemannian foliations on $S^{15}$ with 7-dimensional leaves is obtained by $[Wil],[Lu]$.

Thus it is natural to investigate $V^m(-\infty,4)$ – a complete simply connected Riemannian manifold, whose sectional curvature $K_V \leq 4$ and injectivity radius $\text{inj}(V) \geq \frac{\pi}{2}$. Since this manifold has lower estimate for diameter, the case of extremal value $\frac{\pi}{2}$ of diameter is especially interesting. Note that inequality $\text{inj}(V) \geq \frac{\pi}{2}$ is equivalent to condition, that the perimeter of every nondegenerate geodesic biangle in $V^m(-\infty,4)$ is not less than $\pi$.

19 Theorem $[Top 2,3]$. The manifold $M = V^{2n+1}(-\infty,4)$ with diameter $\frac{\pi}{2}$ is isometric to sphere of curvature 4. The manifold $M = V^{2n}(-\infty,4)$ with extremal diameter $\frac{\pi}{2}$ is isometric to sphere of curvature 4, either geodesics are grouped into families (as for projective spaces):

- $F_1$) for every point $p \in M$ and any vector $\lambda \in T_pM$ there exists a-dimensional ($a = 2, 4, 8$ and if $a = 8$, then $\dim M = 16$) subspace $d(\lambda) \subset T_pM$ such, that all geodesics $\gamma \subset M$, $(\gamma(0) = p, \gamma'(0) \in d(\lambda))$ form a totally geodesic submanifold $F(p,\lambda)$, which is isometric to round sphere $S^a(4)$;
- $F_2$) for all nonzero vectors $\lambda_1, \lambda_2 \in T_pM$ the submanifolds $F(p,\lambda_1), F(p,\lambda_2)$ coincide, either their intersection consists only of one point $p$.

The key point of Theorem 19 is the following result:

20 Theorem $[Top 2]$. If a Riemannian manifold $V^n(-\infty,4)$ has a closed geodesic $\gamma$ with length $\pi$ and index $a - 1$, then there is a-dimensional totally geodesic submanifold containing $\gamma$, which is isometric to $a$-dimensional sphere of curvature 4.

We outline the idea of the proof of Theorem 20.

21 Lemma $[Top 2]$. Under the conditions and with the notations of Theorem 20, any two points $P$ and $Q$ of $\gamma$ whose mutual distance on $\gamma$ is $\frac{\pi}{2}$ are conjugate of multiplicity $a - 1$.

The proof of Lemma 21 uses the condition $\text{inj}(V) \geq \frac{\pi}{2}$ and well-known Lemma of the calculus of variations.

From Lemma 21 we obtain by an easy induction

22 Lemma $[Top 2]$. Under the conditions and with the notations of Theorem 20, there exists an arc $\sigma$ of length $> \frac{\pi}{2}$ on the geodesic $\gamma$ and a $(a - 1)$-parameter family of parallel vector fields $\nu$ along $\sigma$ such that the Riemann curvature for the two-dimensional directions in $\nu$ along $\gamma$ is equal to 4.

Using Lemma 22 for all the fields $\nu$ we can construct a sequence of triangles $\Delta_n(\nu)$ whose perimeter is strictly less then $\pi$ and which converges to $\gamma$. It follows from condition $\text{inj}(V) \geq \frac{\pi}{2}$ that in every triangle $\Delta_n(\nu)$ we can span a cone $K_n(\nu)$ obtained as the set of the shortest lines between the vertices of $\Delta_n(\nu)$ and the opposite edges. For these cones the following Lemma holds.
23 Lemma [Top 2]. The Gauss curvature in points of \( K_n(\nu) \) does not exceed 4.

Lemma 23 follows from Synge’s Theorem.

24 Lemma [Top 2]. The area of the cone \( K_n(\nu) \) is not greater than the area of the triangle \( \Delta_n^L(\nu) \) on the sphere of curvature 4 whose sides have the length of the corresponding sides of \( \Delta_n(\nu) \).

Lemma 24 follows from a Theorem of A.D.Aleksandrov, see [Top 2]. From Lemmas 23 and 24 we obtain an upper bound of the integral curvature of \( K_n(\nu) \). On the other hand, the Gauss-Bonnet Theorem for the integral curvature of \( K_n(\nu) \) can be used to obtain a lower bound for the angle sum of \( K_n(\nu) \). A comparison of these bounds shows that the Gauss curvature of \( K_n(\nu) \) is everywhere almost equal to 4 and the area of \( K_n(\nu) \) is almost equal to \( \pi \).

Passing to the limit for \( n \to \infty \), we see that there exists a \( (a-1) \)-parameter family of surfaces \( \{F\} \) that are isometric to the 2-dimensional hemisphere of curvature 4 and whose boundary is \( \gamma \).

It is now easily shown that the union of all surfaces of that family is a \( a \)-dimensional surface \( F_a \) which is isometric to the \( a \)-dimensional sphere. From the previous results and \( inj(V) \geq \frac{\pi}{2} \) it follows easily that \( F_a \) is a totally geodesic surface in \( M \).

The proof of Theorem 19 follows similar reasoning, analog to the preceding reduction, only we need the certain topological results and in particular a Theorem of W.Brouder.

V. Toponogov [Top 1-3] conjectured, that a manifold \( V^{2n}(-\infty, 4) \) with extremal diameter \( diam(V) = \frac{\pi}{2} \) is isometric to CROSS.

Note, that the tangent \( a \)-planes to submanifolds \( \{F(p, \lambda)\} \) (in Theorem 19) induce \( (a-1) \)-dimensional great sphere foliation of the round sphere \( S_p \) in the tangent space \( T_pM \). In case \( K_M > 0 \) for almost each point \( p \in M \) such foliation \( f_p \) is related with function of sectional curvature in \( T_pM \) by the following way (compare with 12) [Top 1]:

\( R_2 \) for almost each vector \( x \in T_pM \) there exists unique \( a \)-dimensional subspace \( V \ni x \) with condition

\[ K(x, y) = 4, \ (y \in V), \]

\( R_3 \) if \( a \)-dimensional subspace \( V \) contains a fiber of \( f_p \), then

\[ K(x, y) = 4, \ (x, y \in V). \]

The natural strategy to prove, that Riemannian manifold \( M \) with properties \( R_1, R_2 \) is isometric to CROSS, (and, hence, to prove Toponogov’s conjecture) is to deduce firstly, that induced great sphere foliations in tangent spheres \( \{S_p\}, \ (p \in M) \) are Hopf fibrations. The last claim was conjectured in [Lemma 6, Top 1] for foliation with properties \( R_2, R_3 \) and curvature symmetries \( R_4 \). From the consideration at one point of \( M \) (see results of 1) one can only obtain, that such foliations on tangent spheres \( \{S_p\}, \ (p \in M) \), are skew-Hopf fibrations (the proof of [Lemma 6, Top 1] is wrong). By other words, manifold in Theorem 19, when \( a = 2 \), admits an almost complex structure \( J : TM \to TM \) with identity

\[ (\nabla_x J)x = 0, \ (x \in TM), \]

and with constant holomorphic curvature (totally geodesic submanifolds \( F(p, \lambda) \) are \( J \)-invariant). Note, that Hermitian manifold \( (M, J) \) with above identity is
called *nearly Kahlerian*. We don’t know is it possible to prove locally, that \( J \) is Hermitian.

Latter the Toponogov’s conjecture was proved on another way [RovT]: by using of global integral geometrical methods by M. Berge and J. Kazdan and recent topological results for manifolds with closed geodesics.

Manifolds \( M \) with properties \( F_1, F_2 \) are the particular case of Blaschke manifolds. We shall give a short survey.

### 25. Manifolds with closed geodesics.

A compact Riemannian manifold \( M \) is called a \( C_π \)-**manifold**, if all its geodesics are closed of equal length \( π \). This class includes (A.Allamigeon and F.Warner) **Blaschke manifolds**, for which all cut loci \( \text{Cut}(p) \subset T_pM, (p \in M) \) are round spheres of constant radius and dimension. The examples are CROSS: a sphere or projective spaces over a classical fields.

If \( M \) is a simply connected \( C_π \)-manifold, then it is homotopically equivalent to CROSS (R.Bott and H.Samelson).

For Blaschke manifold the exponential map \( \exp_p : T_pM \to C(p) \), restricted on sphere \( S_p \) with radius \( d(p,C(p)) \), defines a *great sphere foliation*, for CROSS this foliation is Hopf fibration. Since every great sphere foliation of \( S^N \) is homeomorphic to Hopf fibration [Sat], (partial cases in works by H.Gluck, F.Warner, C.Yang), then a simply connected Blaschke manifold is homeomorphic to its model CROSS.

The well-known **Blaschke conjecture**, that any Blaschke manifold is isometric to its model CROSS, had been proved for spherical case by following scheme [Bes]:

1) using integral geometry in the space of geodesics one shows that \( \text{vol}(M^N) \geq \text{vol}(S^N, \text{can}) \) with equality of volumes if and only if \( M \) is isometric to \( (S^N, \text{can}) \),

2) on the other hand, one uses topological arguments to show that the **Weinstein integer** \( i(M) = \text{vol}(M^N) / \text{vol}(S^N, \text{can}) \), which has a description by cogomology of the space of oriented geodesics of given \( C_π \)-manifold, is actually one.

The evident analog of Blaschke conjecture for \( C_π \)-manifolds is wrong already when model CROSS is \( S^2 \), but unknown for non-spherical case.

The step 2) is related with **weak Blaschke conjecture** by C.T.Yang, that all Blaschke manifolds have right volumes. It was proven in [Yan] for complex projective space \( CP^n \) and for \( C_π \)-manifolds homeomorphic to model CROSS in [Rez 1,2].

In view of facts in 25 the missing link for proving Toponogov’s conjecture is the follows

### 26 Theorem [RovT].

If a Riemannian manifold \( M, (\dim M = an) \) has the properties \( F_1, F_2 \), then its volume is not less then volume of \( KP^n(4), (\dim \mathbb{K} = a) \) and equality holds if and only if \( M \) is isometric to \( KP^n(4) \).

**Proof.** We use the scheme [App. D, Bes] with modification by splitting of Jacobi equation and volume measure along family \( \{F(p, \lambda)\} \). The same proof (for Blaschke manifold with taut geodesics) is given in [Heb].

Let \( \mu \) be a volume measure on \( M \), the whole measure \( V(M) = \int_M d\mu \) of manifold is called *volume*. The volumes of projective spaces \( KP^n(4), (\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{C}_0) \) with standard metrics and diameter \( \frac{\pi}{2} \) and volume of sphere \( S^{an-1}_1 \) of radius 1 will be denoted by \( V(KP^n) \) and \( V(S^{an-1}) \), their numerical values are known. The total space of fibration \( \pi : UM \to M \) of unit spheres tangent to \( M \) is endowed by canonical metric and measure \( \mu_1 = \sigma \otimes \mu \), where \( \sigma \) is standard volume measure.
on $S^{an-1}$. Thus $V(UM) = V(S^{an-1})V(M)$. The unit geodesical vector field $Z$ on $UM$ is defined, – the projections of the integral curves of $Z$ are geodesics in $M$, the induced dynamical system $\xi$ on $UM$ is called geodesical flow. Note, that the measure $\mu_1$ is invariant under geodesical flow $\xi$ on manifold $UM$ [Bes].

Let $\gamma_u$, $(u = (p, \lambda) \in UM)$ be a unit speed geodesic with initial values $\gamma_u(0) = p$, $\gamma_u'(0) = \lambda$ and $\xi$ be a evolution map of geodesical flow, i.e. $\pi(\xi^t(u)) = \gamma_u'(t)$.

Let $f(u, t)$, where $u \in UM$, $t \in \mathbb{R}_+$, denotes a volume form on $M$ in polar coordinates, i.e. at the point $\exp(\lambda \bar{t})$ it is true $d\mu = f(u, t)d\sigma \otimes dt$. This function $f(u, t)$ may be calculated with the help of Jacobi fields by the following way. Let $\{\lambda_i\}$, $(2 \leq i \leq an)$ be orthonormal basis of subspace $\lambda^+$ (orthogonal complement to $\lambda$) and $\{Y_i\}$, $(2 \leq i \leq an)$ – Jacobi fields along $\gamma_u$ with initial values $Y_i(0) = 0$, $Y_i'(0) = \lambda_i$. Then for all $t$ it is true: $f(u, t) = |\det Y_2(t) \wedge \cdots \wedge Y_{an}(t)|$ [Bes]. In our case the vectors $\lambda_2, \ldots, \lambda_a$ are chosen tangent to $d(\lambda)$, and vectors $\lambda_{a+1}, \ldots, \lambda_{an}$ – orthogonal to $d(\lambda)$. Along geodesic $\gamma_u$ the curvature transformation $R(\gamma_u, \gamma_u), \gamma_u$ has two invariant subspaces: the tangent space to the totally geodesic constant curvature submanifold $F(p, \lambda)$ containing $\gamma_u$, on which it is a multiplication by 4, and the space orthogonal to this.

Since the Jacobi fields $Y_2(t), \ldots, Y_a(t)$ are tangent to totally geodesic submanifold $F(p, \lambda)$ with constant curvature 4, they are given by known formula:

$$Y_i(t) = \left(\frac{1}{2} \sin 2t\right)\bar{\lambda}_i, \ (2 \leq i \leq a),$$

where $\bar{\lambda}_i \in \lambda_i$ is parallel vector field along $\gamma_u$. In view of $F_2$ the vector fields $Y_{a+1}(t), \ldots, Y_{an}(t)$ are non-zero for $0 < t < \pi$, they span the normal bundle to $F(p, \lambda)$ along geodesic $\gamma_u$. Thus,

$$f(u, t) = f_1(u, t)\left|\frac{1}{2} \sin 2t\right|^{a-1},$$

where function $f_1(u, t) = |\det Y_{a+1}(t) \wedge \cdots \wedge Y_{an}(t)|$ is positive for $0 < t < \pi$. We shall denote

$$\varphi(u, t) = f_1(u, t)\left|\frac{1}{2} \sin 2t\right|^{a-1},$$

i.e.

$$f(u, t) = \varphi(u, t)\left|\frac{1}{2} \sin 2t\right|^{a-1}.$$  

In particular, for model CROSS $KP^n(4)$ it is true $\varphi(u, t) = \sin t$.

27 Lemma. $\pi V^2(M) = \int_{UM} (\int_0^\pi (\int_0^{\pi-x} f(\xi^z(u), t)dt)dx) d\mu_1$.

Proof of Lemma 27. Since diameter and injectivity radius of $M$ are $\bar{\pi}$, then $\forall p \in M$ and a ball $B(p, \frac{\bar{\pi}}{2})$ it is true:

$$V(M) = V(B(p, \frac{\bar{\pi}}{2})) = \int_{U_p M} \left(\int_0^{\pi/2} f(u, t)dt\right)d\sigma.$$

With the help of equality $f(-u, t) = f(u, \pi - t)$, $(\forall u \in UM)$ [App. D, Bes], we obtain:

$$\int_{U_p M} \left(\int_0^{\pi/2} f(-u, t)dt\right)d\sigma = \int_{U_p M} \left(\int_0^{\pi/2} f(u, \pi - t)dt\right)d\sigma = \int_{U_p M} \left(\int_0^\pi f(u, t)dt\right)d\sigma.$$

(28)
Thus the integration on interval \([0, \pi]\) gives us the double volume

\[
2V(M) = \int_{U_\rho M} \left( \int_0^\pi f(u, t)dt \right) d\sigma. \tag{29}
\]

We integrate (29) over \(M\)

\[
2V^2(M) = \int_M \left( \int_{U_\rho M} \left( \int_0^\pi f(u, t)dt \right) d\sigma \right) dp = \int_{U_M} \left( \int_0^\pi f(u, t)dt \right) d\mu_1
\]

and then integrate the last equality over interval \([0, \pi]\) with using the invariance of measure \(\mu_1\) under geodesic flow

\[
2\pi V^2(M) = \int_{U_M} \left( \int_0^\pi \int_0^\pi f(\xi^x(u), t)dx \right) dt \mu_1. \tag{30}
\]

Since \(\xi^x(-u) = -\xi^{x-x}(u), (u \in UM, 0 \leq x \leq \pi)\) and the map \(u \to -u\) is diffeomorphism of manifold \(UM\) which preserve measure \(\mu_1\), then in view of \(f(-u, t) = f(u, \pi - t)\) it is true

\[
\int_{U_M} \left( \int_0^\pi \int_0^\pi f(\xi^x(u), \pi-t)dt \right) dx dt \mu_1 = \int_{U_M} \left( \int_0^\pi \int_{\pi-x}^\pi f(\xi^x(u), t) dt \right) dx dt \mu_1 = \]

\[
\int_{U_M} \left( \int_0^\pi \int_0^{\pi-x} f(\xi^x(u), \pi-t) dt \right) dx dt \mu_1 = \int_{U_M} \left( \int_0^{\pi-x} \int_0^\pi f(\xi^x(u), t) dt \right) dx dt \mu_1.
\]

Thus the integral (30) may be broken onto two equal parts. \(\Box\)

**31 Lemma.** For any \(u \in UM\) it is true

\[
J(u) = \int_0^\pi \left( \int_0^{\pi-x} f(\xi^x(u), t) dt \right) dx \geq \pi \frac{V(KP^n)}{V(S^{an-1})}
\]

with equality for only case of \(K(\gamma'(x), y) \equiv 1, (y \perp d(\gamma'(x)), 0 \leq x \leq \pi)\).

**Proof of Lemma 31.** We shall use the Holder inequality of order \(p = an - a\)

\[
\int g_1 g_2 \leq \left( \int g_1^p \right)^{\frac{1}{p}} \left( \int g_2^q \right)^{\frac{1}{q}},
\]

where equality holds if and only if \(g_1^p = \rho g_2^q\), \((\rho = \text{const})\). In our case the functions will be

\[
g_1(t) = \varphi(t) \frac{1}{2} \sin 2t |\sin(t)|^{an-a}, \quad g_2(t) = (\sin t)^{an-a-1} \frac{1}{2} \sin 2t |\sin(t)|^{\frac{(an-a-1)(a-1)}{an-1}}
\]

and the condition \(g_1 = \rho g_2^\frac{1}{p}\) takes a form: \(\varphi = \rho \sin\). Thus

\[
J(u) = \int_0^\pi \left( \int_0^\pi \varphi^{an-a}(\xi^x(u), x-y) \frac{1}{2} \sin 2(y-x)(a-1) dy \right) dx \geq
\]

\[
\int_0^\pi \left( \int_0^{\pi-x} \varphi^a(\xi^x(u), x-y) \frac{1}{2} \sin 2(y-x) dy \right) dx
\]

\[
\int_0^\pi \left( \int_0^{\pi-x} \varphi^{an-a}(\xi^x(u), x-y) \frac{1}{2} \sin 2(y-x)dy \right) dx \geq
\]

\[
\int_0^\pi \left( \int_0^{\pi-x} \varphi^{an-a}(\xi^x(u), x-y) \frac{1}{2} \sin 2(y-x)dy \right) dx
\]
From above it follows, that our manifold $M$ is true the inequality (proof is the same, as in [App. D, Bes])

$$\int_0^\pi \int_0^x \varphi(\xi^x(u), y - x) \sin^{a-n-1}(y - x) \left| \frac{1}{2} \sin 2(y - x) \right|^{a-1} dy dx \frac{an-a}{a}.$$ (32)

Since the submanifolds $\{F(p, \lambda)\}$ are totally geodesic, then for function $\varphi(u, x)$ it is true the inequality (proof is the same, as in [App. D, Bes])

$$\varphi(\xi^x(u), z) \geq \varphi(u, x) \varphi(u, x + z) \int_x^{x+z} dt \frac{\varphi^2(u, t)}{\varphi^2(u, t)}.$$ 

With the help of Kazdan’s inequality [App. E, Bes] with weight function

$$\rho(y - x) = \sin(y - x)^{an-a} \frac{1}{2} \sin 2(y - x)|^{a-1}$$

we shall estimate the numerator in (32)

$$\int_0^\pi \left( \int_x^\pi \varphi(\xi^x(u), y - x) \sin(y - x)^{an-a} \left| \frac{1}{2} \sin 2(y - x) \right|^{a-1} dy dx \right) \geq$$

$$\int_0^\pi \left( \int_x^\pi \frac{\varphi(\xi^x(u), y - x) \varphi(u, y) \varphi^2(u, y)}{\varphi^2(u, t)} \sin(y - x)^{an-a} \left| \frac{1}{2} \sin 2(y - x) \right|^{a-1} dtdy dx \right) \geq$$

$$\int_0^\pi \left( \int_x^\pi \sin(y - x)^{an-a} \left| \frac{1}{2} \sin 2(y - x) \right|^{a-1} dy dx \right) = \beta(a, n).$$

Thus

$$J(u) \geq \beta(a, n) = \int_0^\pi \left( \int_x^\pi \sin(y - x)^{an-a} \left| \frac{1}{2} \sin 2(y - x) \right|^{a-1} dy dx \right).$$

The equality holds for only case of (see [App. D, Bes])

$$K(\gamma'_u(x), y) = 1, \ (0 \leq x \leq \pi, \ y \perp d(\gamma'_u(x))).$$

We shall show below, that $\beta(a, n) = \pi V(KP^n)_{V(S^{an-1})}$. \qed

Continue the proof of Theorem 26. From Lemma 27 and Lemma 31 it follows

$$\pi V^2(M) \geq \int_{UM} \beta(a, n) d\mu_1 = V(UM) \beta(a, n).$$

Since $V(UM) = V(M)V(S^{an-1})$, then $\pi V^2(M) \geq \beta(a, n)V(M)V(S^{an-1})$, i.e.

$$V(M) \geq \frac{1}{\pi} \beta(a, n)V(S^{an-1}).$$

The equality holds for only case of $M$ being with constant sectional curvature

$$K(\lambda, y) = 1, \ (\lambda \in U_p M, \ y \perp d(\lambda), \ p \in M).$$

From above it follows, that our manifold $M$ has positive $\frac{1}{\pi}$—pinched sectional curvature and by Theorem 18 $M$ is isometric to $KP^n(4)$. If we repeat the above for model CROSS $KP^n(4)$, then obtain the numeric value of $\beta(a, n)$:

$$V(KP^n) = \frac{1}{\pi} \beta(a, n)V(S^{an-1}) \Rightarrow \beta(a, n) = \frac{V(KP^n)}{\pi V(S^{an-1})}.$$ \qed

From the above statement and facts about Blaschke manifolds it follows
33 Theorem [RovT]. Riemannian manifold $M$, $(\dim M = an)$ with the properties $F_1, F_2$ is isometric to CROSS.

From Theorem 33 and Theorem 19 it follows the confirmation of Toponogov’s conjecture:

34 Theorem [RovT]. Riemannian manifold $V^m(-\infty, 4)$ with extremal diameter $\frac{\pi}{2}$ is isometric to CROSS.

35 Corollary (diameter rigidity). A complete, connected, simply connected Riemannian manifold $M^{2n}$ with sectional curvature $0 < K_M < 4$ and diameter $\text{diam}(M) = \frac{\pi}{2}$ is isometric to CROSS.

Theorem 34 and Corollary 35 generalize the Theorem 18 by M. Berge.

Theorem 34 has many corollaries. Below is one of them.

Projective planes $KP^2(4), (K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{C}a)$ with standard metrics are at the same time Riemannian manifolds with dimensions $2a = 2, 4, 8, 16$. Totally geodesic spheres $\{KP^1(4) = S^a(4)\}$ (with dimension $a$ and constant curvature 4) play the role of straight lines in $KP^2(4)$, and axioms of projective geometry are true:

$P_1$: for all two different points there exists exactly one straight line, which connects them,

$P_2$: every two different straight lines have intersection at exactly one point.

Thus we obtain differential geometrical test of projective planes over $\mathbb{C}, \mathbb{H}, \mathbb{C}a$.

36 Corollary. Assume, that $M^{2a}$, $(a > 1)$ be a complete Riemannian manifold with the conditions $P_1, P_2$ and straight lines are totally geodesic submanifolds isometric to Euclidean sphere $S^a(4)$. Then $a = 2, 4, 8$ and $M^{2a}$ is isometric to projective plane $KP^2(4), (K = \mathbb{C}, \mathbb{H}, \mathbb{C}a)$ with standard metric.
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