THE WEIGHTED BERGMAN SPACES AND PSEUDOREFLECTION GROUPS

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Abstract. We consider a bounded domain $\Omega \subseteq \mathbb{C}^d$ which is a $G$-space for a finite pseudoreflection group $G$. For each one-dimensional representation of the group $G$, the relative invariant subspace of the weighted Bergman space on $\Omega$ is isometrically isomorphic to a weighted Bergman space on the quotient domain $\Omega/G$. Consequently, formulae involving the weighted Bergman kernels and projections of $\Omega$ and $\Omega/G$ are established. As a result, a transformation rule for the weighted Bergman kernels under a proper holomorphic mapping with $G$ as its group of deck transformations is obtained in terms of the character of the sign representation of $G$. Explicit expressions for the weighted Bergman kernels of several quotient domains (of the form $\Omega/G$) have been deduced to demonstrate the merit of the described formulae.

1. Introduction

Suppose that $\Omega \subseteq \mathbb{C}^d$ is a bounded domain which is a $G$-space for an action of a finite group $G$. In this article, we focus on the following question:

- For any $G$-space $\Omega$, how the weighted Bergman kernels and weighted Bergman projections of $\Omega$ and $\Omega/G$ are related?

The quotient $\Omega/G$ can be given the structure of a complex analytic space which is biholomorphically equivalent to some domain in $\mathbb{C}^d$ whenever $G$ is a finite pseudoreflection group [9, Subsection 3.1.1] [31, Proposition 1]. Henceforth we confine our attention to a finite pseudoreflection group $G$ to establish formulae between the weighted Bergman kernels of the domain $\Omega$ and the quotient domain $\Omega/G$ via the one-dimensional representations of $G$. This in turn invokes identities involving associated weighted Bergman projections of $\Omega$ and $\Omega/G$. For the sign representation of $G$, the transformation formula for the weighted Bergman kernels and the weighted Bergman projections generalize Bell’s transformation rule for the Bergman kernels under proper holomorphic maps in [11, p. 687, Theorem 1] to the weighted Bergman kernels and the Bergman projection formula in [10, p. 167, Theorem 1] to weighted Bergman projections, respectively. As a result, we are able to overcome the limitation of removing the critical points of the proper holomorphic map in the transformation rule for the Bergman kernel. However, our approach is entirely different from that of Bell. We largely use various tools from invariant theory of finite pseudoreflection groups and an analytic version of well known Chevalley-Shephard-Todd theorem obtained in [14] by the author and her collaborators. We start by recalling notions involving finite pseudoreflection groups which are necessary to state the main results.

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A pseudoreflection on $\mathbb{C}^d$ is a linear homomorphism $\sigma : \mathbb{C}^d \to \mathbb{C}^d$ such that $\sigma$ has finite order in $GL(d, \mathbb{C})$ and the rank of $(\text{id} - \sigma)$ is 1. A group generated by pseudoreflections is called a pseudoreflection group. For example, any finite cyclic group, the symmetric group $\mathfrak{S}_d$ on $d$ symbols, the dihedral groups are pseudoreflection groups [31]. A pseudoreflection group $G$ acts on $\mathbb{C}^d$ by $\sigma \cdot z = \sigma^{-1}z$ for $\sigma \in G$ and $z \in \mathbb{C}^d$ and the group action extends to the set of all complex-valued functions on $\mathbb{C}^d$ by

$$\sigma(f)(z) = f(\sigma^{-1} \cdot z), \text{ for } \sigma \in G \text{ and } z \in \mathbb{C}^d. \quad (1.1)$$

We call $f$ a $G$-invariant function if $\sigma(f) = f$ for all $\sigma \in G$. If $\Omega$ is a $G$-space under the above action then the quotient $\Omega/G$ is biholomorphically equivalent to the domain $\theta(\Omega)$, where $\theta : \mathbb{C}^d \to \mathbb{C}^d$ is a basic polynomial map associated to the finite pseudoreflection group $G$ [14, 39]. So we work with the domain $\theta(\Omega)$ instead of $\Omega/G$. Special cases of such quotients have been studied in many instances. For example, $D^d/\mathfrak{S}_d$ (the cartesian product of $d$ copies of the unit disc $D$ in the complex plane) is realized as the symmetrized polydisc (denoted by $\mathbb{G}_d$) in [15] and in [13], Bender et al. realized a monomial polyhedron as a quotient domain $\Omega/G$ for $\Omega \subseteq D^d$ and a finite abelian group $G$. The symmetrized polydisc has been studied extensively in the last two decades in the context of function theory [1, 25, 32], operator theory [4, 5, 16, 22] and geometry [2, 3, 6]. Recently, in [20] the symmetrized polydisc and in [13] the monomial polyhedron were considered in connection with $L^p$-regularity of the Bergman projection.

For an analytic Hilbert module $\mathcal{H}$ on $\Omega$, the relative invariant subspace associated to a one-dimensional representation $\varrho$ of $G$, is defined by

$$R^G_\varrho(\mathcal{H}) = \{f \in \mathcal{H} : \sigma(f) = \chi_\varrho(\sigma)f \text{ for all } \sigma \in G\},$$

where $\chi_\varrho$ denotes the character of the representation $\varrho$. We show that the subspace $R^G_\varrho(\mathcal{H})$ is a reproducing kernel Hilbert space. The elements of $R^G_\varrho(\mathcal{H})$ are said to be $\varrho$-invariant. There exists a polynomial (unique up to a constant multiple), say $\ell_\varrho$, which forms a basis of the ring of $\varrho$-invariant polynomials as a free module over the ring of $G$-invariant polynomials in $d$ variables [36, p. 139, Theorem 3.1]. A characterization of the relative invariant subspace is that every $f \in R^G_\varrho(\mathcal{H})$ is divisible by the polynomial $\ell_\varrho$ and the quotient is in the ring of $G$-invariant holomorphic functions on $\Omega$ (cf. Lemma 2.8). The polynomial $\ell_\varrho$ plays an important role in our discussion. An explicit expression for $\ell_\varrho$ (unique up to a constant multiple) has been obtained from the representation $\varrho$ [36, p. 139, Theorem 3.1] (cf. Lemma 2.5).

We now briefly describe the results we have proved in this article.

1.1. On the weighted Bergman kernels. Given a continuous weight function $\omega : \Omega \to (0, \infty)$, $L^2_\omega(\Omega)$ denotes the Hilbert space of Lebesgue measurable functions (equivalence classes of functions) on $\Omega$ which are square integrable with respect to the measure $\omega(z)dV(z)$, where $dV$ is the normalized Lebesgue measure on $\Omega$. The weighted Bergman space $K^2_\omega(\Omega)$ is the closed subspace consisting of holomorphic functions in $L^2_\omega(\Omega)$. For $\omega \equiv 1$, $K^2_\omega(\Omega)$ reduces to the Bergman space $K^2(\Omega)$. We consider a weight function of the form $\omega = \bar{\omega} \circ \theta$ for a continuous map $\bar{\omega} : \theta(\Omega) \to (0, \infty)$. For each one-dimensional representation $\varrho$ of $G$, we set $\omega_\varrho(\theta(z)) = |J_\varrho(z)|^2 \bar{\omega}(\theta(z))$, where $J_\varrho$ is the determinant of the complex jacobian matrix of the basic polynomial map $\theta$. The following theorem is the main result of this article.
Theorem 1.1. Let $G$ be a finite pseudoreflection group and a bounded domain $\Omega \subseteq \mathbb{C}^d$ be a $G$-space. Then for each one-dimensional representation $\varrho$ of $G$, the relative invariant subspace $R_{\varrho}^2(\mathbb{A}_o(\varrho))$ is isometrically isomorphic to the weighted Bergman space $\mathbb{A}_o^2(\varrho, \Omega)$, where $\varrho$ is a basic polynomial map associated to the group $G$.

Moreover, the reproducing kernel $\mathcal{B}_{\varrho}$ of the weighted Bergman space $\mathbb{A}_o^2(\varrho, \Omega)$ is given by

$$\mathcal{B}_{\varrho}(\theta(z), \theta(w)) = \frac{1}{\ell_o(z)\ell_o(w)} \sum_{\sigma \in G} \chi_{\varrho}(\sigma^{-1}) \mathcal{B}_{\varrho}(\sigma^{-1} \cdot z, w) \quad z, w \in \Omega,$$

where $\mathcal{B}_{\varrho}$ is the reproducing kernel of the weighted Bergman space $\mathbb{A}_o^2(\varrho, \Omega)$, $\chi_{\varrho}$ denotes the character of the representation $\varrho$ and $\ell_o$ is as described in Lemma 2.5.

We provide a number of applications of Theorem 1.1 in Section 5 to determine explicit formulae for the weighted Bergman kernels of various quotient domains. For example, the weighted Bergman space $\mathbb{A}_o^2(\mathbb{D}^d)$ with the $G_d$-invariant weight function $\omega(z) = \prod_{i=1}^{d}(1-|z_i|^2)^{-\lambda}$, $\lambda > 1$, has two relative invariant subspaces $\mathbb{A}_o^2_{\text{ant}}(\mathbb{D}^d) = \{ f \in \mathbb{A}_o^2(\mathbb{D}^d) : \sigma(f) = \text{sign}(\sigma)f \text{ for } \sigma \in G_d \}$ and $\mathbb{A}_o^2_{\text{sym}}(\mathbb{D}^d) = \{ f \in \mathbb{A}_o^2(\mathbb{D}^d) : \sigma(f) = f \text{ for } \sigma \in G_d \}$ associated to only one-dimensional representations of $G_d$. The subspace $\mathbb{A}_o^2_{\text{ant}}(\mathbb{D}^d)$ is isometrically isomorphic to some weighted Bergman space on $G_d$ with the reproducing kernel

$$\mathcal{B}_{\text{ant}}(s(z), s(w)) = \frac{\det \left( (1-z_i\overline{w}_j)^{-\lambda} \right)_{i,j=1}^d}{\prod_{i<j}(z_i-z_j)(\overline{w}_i-\overline{w}_j)}, \quad z, w \in \mathbb{D}^d,$$

and $\mathbb{A}_o^2_{\text{sym}}(\mathbb{D}^d)$ is isometrically isomorphic to another weighted Bergman space on $G_d$ whose reproducing kernel is given by

$$\mathcal{B}_{\text{sym}}(s(z), s(w)) = \text{perm} \left( (1-z_i\overline{w}_j)^{-\lambda} \right)_{i,j=1}^d, \quad z, w \in \mathbb{D}^d,$$

where $\text{perm} A$ denotes the permanent of the matrix $A$ (cf. Proposition 5.3 and Proposition 5.4).

1. The case of abelian groups. If $G$ is a finite abelian group, each irreducible representation of $G$ is one-dimensional. In this case, the weighted Bergman kernel $\mathbb{A}_o^2(\varrho, \Omega)$ is isometrically isomorphic to an orthogonal direct sum of weighted Bergman spaces on $\varrho(\Omega)$, that is,

$$\mathbb{A}_o^2(\varrho, \Omega) \cong \bigoplus_{\varrho \in \hat{G}} \mathbb{A}_o^2(\varrho, \Omega),$$

where $\hat{G}$ is the set of equivalence classes of irreducible representations of $G$. The reproducing kernel of each weighted Bergman space $\mathbb{A}_o^2(\varrho, \Omega)$ can be obtained from Equation (1.2). Therefore, we have the following identity involving weighted Bergman kernels:

$$\mathcal{B}_{\varrho}(z, w) = \frac{1}{|G|} \sum_{\varrho \in \hat{G}} \ell_{\varrho}(z)\mathcal{B}_{\varrho}(\theta(z), \theta(w)) \overline{\ell_{\varrho}(w)}.$$

Recently, in [33] Nagel and Pramanik made an analogous observation when a basic polynomial map of some finite abelian group is given by a monomial type mapping.

2. The case of sign representation. The one-dimensional representation, $\text{sgn} : G \rightarrow \mathbb{C}^*$ is defined by

$$\text{sgn}(\sigma) = (\det(\sigma))^{-1},$$

(1.3)
see [36]. In the proceedings, we refer this representation as sign representation. For the sign representation, Equation (1.2) reduces to the following identity which is worth mentioning:

\[
B_z(\theta(z), \theta(w)) = \frac{1}{J_\theta(z)J_\theta(w)} \sum_{\sigma \in G} \det(\sigma) B_\omega(\sigma^{-1} \cdot z, w) \quad \text{for } z, w \in \Omega. \tag{1.4}
\]

For \( \omega \equiv 1 \), this identity emerges as a very convenient tool to determine explicit formula for the Bergman kernel of \( \Omega/G \) in terms of the Bergman kernel of \( \Omega \). The Bergman kernel plays a crucial role in complex analysis and complex geometry. For instance, an explicit expression for the Bergman kernel of a domain is essential for understanding the boundary behaviour of geodesics arising from the Bergman metric of the domain and for characterizing Lu Qi-keng domains, see [17, 18, 26, 37]. We demonstrate the efficiency of Equation (1.4) by obtaining explicit formulae for the Bergman kernels for several domains, namely, symmetrized polydisc, monomial polyhedron, a subclass of complex ellipsoids and \( D^2/D_{2k} \) for the dihedral group \( D_{2k} \).

3. Weighted Bergman kernel and proper holomorphic maps. We consider a proper holomorphic map \( f : \Omega \to \tilde{\Omega} \) with \( G \) as the deck transformation group. Such a map \( f : \Omega \to \tilde{\Omega} \) is called proper holomorphic map factored by automorphisms \( G \) in [9, 23], see also [14, p. 7]. A transformation rule for the weighted Bergman kernels under a proper holomorphic map with \( G \) as the deck transformation group is established as following:

\[
B_z(f(z), f(w)) = \frac{1}{J_f(z)J_f(w)} \sum_{\sigma \in G} \det(\sigma) B_\omega(\sigma^{-1} \cdot z, w) \quad \text{for } z, w \in \Omega, \tag{1.5}
\]

where the weight is of the form \( \omega = \tilde{\omega} \circ f \), \( B_\omega \) and \( B_{\tilde{\omega}} \) are the reproducing kernels of \( A_\omega^2(\Omega) \) and \( A_{\tilde{\omega}}^2(\tilde{\Omega}) \), respectively. As expected, for \( \omega \equiv 1 \), Equation (1.5) overlaps with the transformation rule described by Steven Bell in [11, p. 687, Theorem 1]. We emphasize that Equation (1.5) works for the critical points of \( f \) as well.

4. Bergman kernels for Rudin’s domains. A family of quotient domains of the form \( B_d/G \) is described in [34], where \( B_d \) denotes the open unit ball with respect to the \( \ell^2 \)-norm on \( \mathbb{C}^d \) and the group \( G \) is a conjugate to a finite pseudoreflection group. Following [12, p. 427], we refer such domains as Rudin’s domains. The domain \( \Omega \subset \mathbb{C}^d \) is a Rudin’s domain if and only if there exists a proper holomorphic map \( F : B_d \to \Omega \). Then the Bergman kernel \( B_\Omega \) of a Rudin’s domain \( \Omega \) is given by the following formula:

\[
B_\Omega(F(z), F(w)) = \frac{1}{J_F(z)J_F(w)} \sum_{\sigma \in G} J_{\sigma^{-1}}(z) \left( 1 - \langle \sigma^{-1} \cdot z, w \rangle \right)^{d+1}, \quad z, w \in B_d,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{C}^d \).

1.2. On the weighted Bergman projections. The weighted Bergman projection \( P_\Omega^\omega : L_\omega^2(\Omega) \to A_\omega^2(\Omega) \) is defined by

\[
(P_\Omega^\omega \phi)(z) = \langle \phi, B_\omega(\cdot, z) \rangle = \int_\Omega \phi(w) B_\omega(z, w) \omega(w) dV(w), \quad \phi \in L_\omega^2(\Omega),
\]

where \( B_\omega \) is the reproducing kernel of the weighted Bergman space \( A_\omega^2(\Omega) \). For a holomorphic function \( \phi \) in \( L_\omega^2(\Omega) \), we have \( (P_\Omega^\omega \phi)(z) = \langle \phi, B_\omega(\cdot, z) \rangle = \phi(z) \).
Theorem 1.2. For each one-dimensional representation $\rho$ of $G$, the weighted Bergman projections $P_{\rho(\Omega)}^\omega$ and $P_{\Omega}^\omega$ are related by the following formula:

$$P_{\rho(\Omega)}^\omega(\ell_\phi (\phi \circ \theta)) = \ell_\phi (P_{\rho(\Omega)}^\omega (\phi) \circ \theta), \quad \phi \in L^2_{\omega_{\rho}^2}(\rho(\Omega)), \quad (1.6)$$

where $P_{\rho(\Omega)}^\omega : L^2_{\omega_{\rho}^2}(\rho(\Omega)) \to \mathcal{H}_{\omega_{\rho}^2}(\rho(\Omega))$ is the weighted Bergman projection.

That is, we get $k$ formulae involving weighted Bergman projections, where $k$ is the number of one-dimensional representations in $G$. For a fixed one-dimensional representation $\rho \in \hat{G}$, Equation (1.6) is true for the weight tuple $(\omega, \omega_\rho)$ whenever $\omega_\rho(\theta(z)) = |\ell_\phi(z)|^2 |\ell_{\rho}(z)|^2 \omega(z)$ for $z \in \Omega$. For the sign representation of $G$, Equation (1.6) generalizes Bell’s formula for Bergman projections [11, Equation 2.2, p. 686] to the weighted Bergman projections for the weight tuples $(\omega, \tilde{\omega})$, where $\omega = \tilde{\omega} \circ \theta$.

2. Pseu-doreflection groups and analytic Hilbert modules

We begin by recalling a number of useful definitions and standard results about pseudoreflection groups.

2.1. Chevalley-Shephard-Todd Theorem. Let $G$ be a finite pseudoreflection group. Recall that $G$ acts on the set of functions on $\mathbb{C}^d$ by $\sigma(f)(z) = f(\sigma^{-1} \cdot z)$. A function is said to be $G$-invariant if $\sigma(f)(z) = f, \text{ for all } \sigma \in G$. The ring of all complex polynomials in $d$ variables is denoted by $\mathbb{C}[z_1, \ldots, z_d]$. Moreover, the set of all $G$-invariant polynomials, denoted by $\mathbb{C}[z_1, \ldots, z_d]^G$, forms a subring and coincides with the relative invariant subspace $P^G_{\text{trivial}}(\mathbb{C}[z_1, \ldots, z_d])$ associated to the trivial representation of $G$. Chevalley, Shephard and Todd characterize finite pseudoreflection groups in the following theorem.

Theorem (CST Theorem). [19, p. 112, Theorem 3] The invariant ring $\mathbb{C}[z_1, \ldots, z_d]^G$ is equal to $\mathbb{C}[\theta_1, \ldots, \theta_d]$, where $\theta_i$’s are algebraically independent homogeneous polynomials if and only if $G$ is a finite pseudoreflection group.

The collection of homogeneous polynomials $\{\theta_i\}_{i=1}^d$ is called a homogeneous system of parameters (hsop) or basic polynomials associated to the pseudoreflection group $G$. Although a hsop is not unique but the degrees of $\theta_i$’s are unique for $G$ up to order. The map $\theta : \mathbb{C}^d \to \mathbb{C}^d$, defined by

$$\theta(z) = (\theta_1(z), \ldots, \theta_d(z)), \quad z \in \mathbb{C}^d \quad (2.1)$$

is called a basic polynomial map associated to the group $G$. The following proposition shows that a pseudoreflection group $G$ always induces a canonical polynomial proper mapping.

Proposition 2.1. [14, p. 16, Proposition 5.4] Let $\Omega \subseteq \mathbb{C}^d$ be a $G$-invariant domain. Then

(i) $\theta(\Omega)$ is a domain, and 
(ii) $\theta : \Omega \to \theta(\Omega)$ is a proper holomorphic map with $G$ as the group of deck transformations.

Moreover, if $\theta'$ is another basic polynomial map associated to the group $G$, then the domain $\theta'(\Omega)$ is biholomorphically equivalent to $\theta(\Omega)$ [14, p. 12]. The following proposition illustrates a significance of a basic polynomial map.
Proposition 2.2. Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic map with the finite pseudoreflection group $G$ as the group of deck transformations. Then there exists a unique biholomorphic map $\hat{f} : \hat{\Omega}(\Omega_1) \to \Omega_2$ such that $f = \hat{f} \circ \theta$, where $\theta : \Omega_1 \to \hat{\Omega}(\Omega_1)$ is a basic polynomial map associated to the pseudoreflection group $G$.

We use the analytic version of Chevalley-Shephard-Todd theorem to prove this proposition. Let $\mathcal{O}(\Omega)$ denote the ring of all holomorphic functions on $\Omega$.

Theorem 2.3 (Analytic CST Theorem). [14, p. 12] Let $G$ be a finite pseudoreflection group and $\Omega \subseteq \mathbb{C}^d$ be a $G$-invariant domain. For any $G$-invariant holomorphic function $f$ on $\Omega$, there exists a unique $\hat{f} \in \mathcal{O}(\theta(\Omega))$ such that $f = \hat{f} \circ \theta$.

A function $f : \Omega_1 \to \Omega_2$ is said to be $G$-invariant if $f(\sigma^{-1} \cdot z) = f(z)$ for all $\sigma \in G, z \in \Omega_1$. Clearly, if $f = (f_1, \ldots, f_d) : \Omega_1 \to \Omega_2$ is a $G$-invariant holomorphic map, then each $f_i$ is a $G$-invariant holomorphic map on $\Omega_1$. By analytic CST theorem, we get $f_i = \hat{f}_i \circ \theta$, where $\hat{f}_i \in \mathcal{O}(\theta(\Omega_1))$. That is, $f = \hat{f} \circ \theta$, where the function $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_d) : \theta(\Omega_1) \to \Omega_2$ is uniquely determined.

Proof of Proposition 2.2. Since the group of deck transformations of $f$ is $G$, $f$ is $G$-invariant. So there exists a unique holomorphic map $\hat{f}$ such that $f = \hat{f} \circ \theta$. In order to show that $\hat{f} : \theta(\Omega_1) \to \Omega_2$ is biholomorphic, it is enough to prove that the map is bijective. Then from [35, p. 303, Theorem 15.1.8], it follows that $\hat{f}$ is biholomorphic.

Being a proper holomorphic map, $\hat{f}$ is surjective. Since $f = \hat{f} \circ \theta$, the map $\hat{f}$ is also surjective. Arguing by contradiction, suppose that $\hat{f}$ is not injective, then there exist two distinct $z_1, z_2 \in \theta(\Omega_1)$ such that $\hat{f}(z_1) = \hat{f}(z_2)$. Suppose that for $w_i \in \Omega_1$, we have $\theta(w_i) = z_i$, for $i = 1, 2$. That implies

$$f(w_1) = (\hat{f} \circ \theta)(w_1) = \hat{f}(z_1) = \hat{f}(z_2) = (\hat{f} \circ \theta)(w_2) = f(w_2).$$

Since $G$ is the group of deck transformations of $f$, there is an element $\sigma \in G$ such that $\sigma(w_1) = w_2$. Consequently, $G$-invariance of $\theta$ implies that $\theta(\sigma(w_1)) = \theta(w_1)$, that is, $\theta(w_2) = \theta(w_1)$, that is, $z_1 = z_2$, which is a contradiction. $\blacksquare$

Moreover, the complex jacobian $J_{\theta} = \det \left( \frac{\partial \theta_j}{\partial z_i} \right)_{i,j=1}^d$ has an association with the sign representation of $G$. A detailed discussion regarding one-dimensional representations of pseudoreflection groups is provided which consequently establishes the association.

2.2. one-dimensional representations of pseudoreflection groups.

Definition 2.4. A hyperplane $H$ in $\mathbb{C}^d$ is called reflecting if there exists a pseudoreflection in $G$ acting trivially on $H$.

For a pseudoreflection $\sigma \in G$, define $H_{\sigma} := \ker(\id - \sigma)$. By definition, the subspace $H_\sigma$ has dimension $d - 1$. Clearly, $\sigma$ fixes the hyperplane $H_\sigma$ pointwise. Hence each $H_\sigma$ is a reflecting hyperplane. By definition, $H_\sigma$ is the zero set of a non-zero homogeneous linear polynomial $L_\sigma$ on $\mathbb{C}^d$, determined up to a non-zero constant multiplier, that is, $H_\sigma = \{ z \in \mathbb{C}^d : L_\sigma(z) = 0 \}$. Moreover, the elements of $G$ acting trivially on a reflecting hyperplane forms a cyclic subgroup of $G$.

Let $H_1, \ldots, H_t$ denote the distinct reflecting hyperplanes associated to the group $G$ and the corresponding cyclic subgroups are $G_1, \ldots, G_t$, respectively. Suppose $G_i = \langle a_i \rangle$ and the order of each $a_i$ is $m_i$ for $i = 1, \ldots, t$. For every one-dimensional representation
of \( G \), there exists a unique \( t \)-tuple of non-negative integers \((c_1, \ldots, c_t)\), where \( c_i \)'s are the least non-negative integers that satisfy the following:

\[
\varrho(a_i) = (\det(a_i))^{c_i}, \quad i = 1, \ldots, t. \tag{2.2}
\]

The \( t \)-tuple \((c_1, \ldots, c_t)\) solely depends on the representation \( \varrho \). The character of the one-dimensional representation \( \varrho, \chi_\varrho : G \to \mathbb{C}^* \) coincides with the representation \( \varrho \). The set of polynomials relative to the representation \( \varrho \) is given by

\[
R^G_\varrho(\mathbb{C}[z_1, \ldots, z_d]) = \{ f \in \mathbb{C}[z_1, \ldots, z_d] : \sigma(f) = \chi_\varrho(\sigma)f, \text{ for all } \sigma \in G \}.
\]

The elements of the subspace \( R^G_\varrho(\mathbb{C}[z_1, \ldots, z_d]) \) are said to be \( \varrho \)-invariant polynomials.

Stanley proves a typical property of the elements of \( R^G_\varrho(\mathbb{C}[z_1, \ldots, z_d]) \) in [36, p. 139, Theorem 3.1].

**Lemma 2.5.** [36, p. 139, Theorem 3.1] Suppose that the linear polynomial \( \ell_i \) is a defining function of \( H_i \) for \( i = 1, \ldots, t \). The homogeneous polynomial \( \ell_\varrho = \prod_{i=1}^t \ell_i^{c_i} \) is a generator of the module \( R^G_\varrho(\mathbb{C}[z_1, \ldots, z_d]) \) over the ring \( \mathbb{C}[z_1, \ldots, z_d]^G \), where \( c_i \)'s are unique non-negative integers as described in Equation (2.2).

In particular, the sign representation of a finite pseudoreflection group \( G \), described in Equation (1.3), is given by \( \text{sgn}(a_i) = (\det(a_i))^{m_i-1}, \quad i = 1, \ldots, t \), [36, p. 139, Remark (1)] and it has the following property.

**Corollary 2.6.** [38, p. 616, Lemma] Let \( H_1, \ldots, H_t \) denote the distinct reflecting hyperplanes associated to the group \( G \) and let \( m_1, \ldots, m_t \) be the orders of the corresponding cyclic subgroups \( G_1, \ldots, G_t \), respectively. Suppose that the linear polynomial \( \ell_i \) is a defining function of \( H_i \) for \( i = 1, \ldots, t \). Then for a non-zero constant \( c \),

\[
J_\varrho(z) = c \prod_{i=1}^t \ell_i^{m_i-1}(z) = \ell_{\text{sgn}}(z).
\]

Consequently, \( J_\varrho \) is a basis of the module \( R^G_{\text{sgn}}(\mathbb{C}[z_1, \ldots, z_d]) \) over the ring \( \mathbb{C}[z_1, \ldots, z_d]^G \).

Clearly, the character of the sign representation is given by \( \chi_{\text{sgn}} : G \to \mathbb{C}^* \)

\[
\chi_{\text{sgn}}(\sigma) = (\det(\sigma))^{-1}, \quad \sigma \in G. \tag{2.3}
\]

Now, for any one-dimensional representation \( \varrho \) of \( G \), we generalize the notion of \( \varrho \)-invariance for analytic Hilbert modules.

**Definition 2.7.** We recall the following definitions from [21] and [24].

1. A Hilbert space \( \mathcal{H} \) is said to be a **Hilbert module** over an algebra \( \mathcal{A} \) if the map \((f, h) \mapsto T_f(h) := f \cdot h, f \in \mathcal{A}, h \in \mathcal{H} \) defines an algebra homomorphism \( f \mapsto T_f \) of \( \mathcal{A} \) into \( \mathcal{L}(\mathcal{H}) \).

2. A Hilbert module \( \mathcal{H} \) over \( \mathbb{C}[z_1, \ldots, z_d] \) is said to be an **analytic Hilbert module** if
   (a) the Hilbert space \( \mathcal{H} \) consists of holomorphic functions on some bounded domain \( \Omega \subseteq \mathbb{C}^d \),
   (b) \( \mathbb{C}[z_1, \ldots, z_d] \subseteq \mathcal{H} \) is dense in \( \mathcal{H} \) and
   (c) \( \mathcal{H} \) possesses a reproducing kernel on \( \Omega \).

The module action in an analytic Hilbert module is given by pointwise multiplication, that is, \( m_p(h)(z) = p(z)h(z), h \in \mathcal{H}, z \in \Omega \).

For example, a weighted Bergman space on some bounded domain \( \Omega \subseteq \mathbb{C}^d \) is an analytic Hilbert module over \( \mathbb{C}[z_1, \ldots, z_d] \) on \( \Omega \).
Let \( \mathcal{H} \) be an analytic Hilbert module consisting holomorphic functions on \( \Omega \), where \( \Omega \) is a \( G \)-invariant domain. We also assume that the reproducing kernel \( K \) of \( \mathcal{H} \) is \( G \)-invariant, that is,

\[
K(\sigma \cdot z, \sigma \cdot w) = K(z, w) \text{ for all } \sigma \in G.
\]

For a one-dimensional representation \( \rho \) of \( G \), recall that the relative invariant subspace of \( \mathcal{H} \), defined by

\[
\mathcal{R}_\rho^G(\mathcal{H}) = \{ f \in \mathcal{H} : \sigma(f) = \chi_\rho(\sigma)f \text{ for all } \sigma \in G \}.
\]

The elements of the subspace \( \mathcal{R}_\rho^G(\mathcal{H}) \) are said to be \( \rho \)-invariant. A result analogous to Lemma 2.5 is proved for \( \mathcal{R}_\rho^G(\mathcal{H}) \) in the next lemma.

**Lemma 2.8.** Let \( \mathcal{H} \subseteq \mathcal{O}(\Omega) \) be an analytic Hilbert module and \( f \in \mathcal{R}_\rho^G(\mathcal{H}) \). Then \( \ell_\phi \) divides \( f \) and the quotient \( \frac{f}{\ell_\phi} \) is a \( G \)-invariant holomorphic function on \( \Omega \), where \( \ell_\phi \) is as in Lemma 2.5.

**Proof.** For a fixed \( 1 \leq i \leq t \), suppose that the reflecting hyperplane \( H_i \) is fixed by the cyclic subgroup \( G_i \) and the pseudoreflection \( a_i \) generates \( G_i \). Using a linear change of coordinates in \( \Omega \), we consider a new coordinate system \( y_1 = \ell_i, y_2 = x_2, \ldots, y_d = x_d \). In this new coordinate system, we have \( a_i = \text{diag}(\omega_i, 1, \ldots, 1) \), where \( \omega_i \) is a primitive \( m_i \)-th root of unity. Then,

\[
\begin{align*}
    f(a_i^{-1} \cdot (y_1, \ldots, y_d)) &= \chi_\rho(a_i)f(y_1, \ldots, y_d) \\
    f(\omega_i y_1, y_2, \ldots, y_d) &= (\det(a_i))^{c_i} f(y_1, \ldots, y_d) \\
    &= \omega_i^{c_i} f(y_1, \ldots, y_d).
\end{align*}
\]

Therefore, \( f(y_1, \ldots, y_d) \) is divisible by \( y_i^{c_i} \). Now changing the coordinates, we conclude that \( f(x_1, \ldots, x_d) \) is divisible by \( \ell_i^{c_i} \).

If \( z \notin H_i \cap \Omega \), then \( \ell_i(z) \neq 0 \) and this is true in a neighbourhood of \( z \). On the other hand, if \( H_i \cap \Omega \) is non-empty and \( z \in H_i \cap \Omega \), then it follows from Weierstrass Division Theorem [28, p. 11] that \( \ell_i \) divides \( f \) in a neighbourhood of \( z \) since \( \ell_i \) is irreducible. As the holomorphic functions which are obtained locally, patch to give a global holomorphic function, since both \( f \) and \( \ell_i \) are defined on all of \( \Omega \), it follows that \( \ell_i \) divides \( f \) in \( \mathcal{O}(\Omega) \). Repetitive application of this argument gives us that the quotient function \( \frac{f}{\ell_\phi} \) is in \( \mathcal{O}(\Omega) \). These steps can be repeated for all distinct hyperplanes associated to the group \( G \). Thus one gets that \( \ell_\phi \) divides \( f \) and the quotient function is holomorphic. Moreover, \( \ell_\phi \) is \( \rho \)-invariant from [36, p. 138, Theorem 2.3]. The quotient of a \( \rho \)-invariant function by a \( \rho \)-invariant function is clearly \( G \)-invariant. Hence \( \frac{f}{\ell_\phi} \) is \( G \)-invariant.

**Remark 2.9.** Let \( f \in \mathcal{R}_\rho^G(\mathcal{H}) \). Combining Theorem 2.3 and Lemma 2.8, we write \( \frac{f}{\ell_\phi} = \widehat{f} \circ \theta \) for a unique \( \widehat{f} \in \mathcal{O}(\theta(\Omega)) \). That is, \( f = (\widehat{f} \circ \theta)\ell_\phi \). In other words, \( \mathcal{R}_\rho^G(\mathcal{H}) \subseteq \{ (\widehat{f} \circ \theta)\ell_\phi \mid \widehat{f} \in \mathcal{O}(\theta(\Omega)) \} \).

### 2.3. Orthogonal projections

Each one-dimensional representation \( \rho \) of \( G \) induces the unique linear operator \( \mathbb{P}_\rho \colon \mathcal{H} \to \mathcal{H} \) given by

\[
\mathbb{P}_\rho \phi = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\rho(\sigma^{-1}) \phi \circ \sigma^{-1}, \quad \phi \in \mathcal{H},
\]

where \( \chi_\rho \) is the character of the representation \( \rho \) and \( |G| \) denotes the order of the group \( G \).
Lemma 2.10. Let $G$ be a finite pseudoreflection group and $\Omega$ be a $G$-invariant domain in $\mathbb{C}^d$. If $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ is an analytic Hilbert module with the $G$-invariant reproducing kernel, the operator $P_\phi : \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto the subspace $R^G_\phi(\mathcal{H})$.

Proof. The reproducing kernel of the analytic Hilbert module $\mathcal{H}$ is $G$-invariant. So $P_\phi$ is an orthogonal projection on $\mathcal{H}$ by [14, p. 15]. It remains to show that $P_\phi \mathcal{H} = R^G_\phi(\mathcal{H})$. For any $\tau \in G$, $\phi \in P_\phi \mathcal{H}$,

$$
\tau(\phi) = \tau(P_\phi \phi) = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\phi(\sigma^{-1}) \phi \circ \sigma^{-1} \tau^{-1}
$$

$$
= \frac{1}{|G|} \sum_{\eta \in G} \chi_\phi(\eta^{-1} \tau) \phi \circ \eta^{-1} \text{ (taking } \eta = \tau \sigma) \n$$

$$
= \chi_\phi(\tau) \phi.
$$

Hence $\phi$ is in $R^G_\phi(\mathcal{H})$.

To prove $R^G_\phi(\mathcal{H}) \subseteq P_\phi \mathcal{H}$, consider $\phi \in R^G_\phi(\mathcal{H})$. Then we have

$$
P_\phi(\phi) = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\phi(\sigma^{-1}) \phi \circ \sigma^{-1} = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\phi(\sigma^{-1}) \chi_\phi(\sigma) \phi
$$

$$
= \frac{1}{|G|} \sum_{\sigma \in G} \phi = \phi.
$$

This completes the proof. \hfill \blacksquare

The group action in Equation (1.1) is known as the left regular representation. The left regular representation of $G$ admits the isotypic decomposition $\oplus_{\phi \in \hat{G}} n_\phi \phi$, where the positive integer $n_\phi = \chi_\phi(id)$ is the multiplicity of $\phi$ in the left regular representation of $G$ and $\hat{G}$ is the set of equivalence classes of irreducible representations of $G$. The linear map $P_\phi$ is the orthogonal projection onto the isotypic component associated to the irreducible representation $\phi$ in the decomposition of the left regular representation on $\mathcal{H}$.

Theorem 2.11. [14] An analytic Hilbert module $\mathcal{H}$ on a $G$-space $\Omega$ with $G$-invariant kernel admits an orthogonal decomposition $\mathcal{H} = \oplus_{\phi \in \hat{G}} P_\phi \mathcal{H}$ and each $P_\phi \mathcal{H}$ is isometrically isomorphic to a reproducing kernel Hilbert space $\mathcal{H}_\phi \subseteq \mathcal{O}(\theta(\Omega)) \otimes \mathbb{C}^{n_\phi}$, where $n_\phi = \chi_\phi(id)$.

3. On the weighted Bergman modules

In this section, we prove Theorem 1.1 and Theorem 1.2. We start by recalling weighted Bergman spaces. Let $\Omega$ be a bounded domain in $\mathbb{C}^d$. For a continuous weight function $\omega : \Omega \to (0, \infty)$, the weighted Bergman space $\mathcal{A}^2_\omega(\Omega)$ is defined by

$$
\mathcal{A}^2_\omega(\Omega) = \{ f \in \mathcal{O}(\Omega) : \int_{\Omega} |f(z)|^2 \omega(z) dV(z) < \infty \},
$$

where $dV$ is the normalized Lebesgue measure on $\Omega$. Clearly, the weighted Bergman space $\mathcal{A}^2_\omega(\Omega)$ is an analytic Hilbert module.

Let $G$ be a finite pseudoreflection group and $\Omega$ be a $G$-space. We consider a weight function of the form $\omega = \bar{\omega} \circ \theta$ for a continuous map $\bar{\omega} : \theta(\Omega) \to (0, \infty)$, where $\theta$ is a basic polynomial map associated to the group $G$. Let $\mathcal{U}_d$ denote the group of unitary operators on $\mathbb{C}^d$. Since any finite group $G$ generated by pseudoreflections on $\mathbb{C}^d$ is a subgroup of $\mathcal{U}_d$, it can be shown that the reproducing kernel of the weighted Bergman
space $A^2_\omega(\Omega)$ is $G$-invariant as the weight function $\omega$ is $G$-invariant. Specializing Lemma 2.10 to the weighted Bergman module $A^2_\omega(\Omega)$, we obtain the following result.

**Lemma 3.1.** Let $G$ be a finite pseudoreflection group and $\Omega \subseteq \mathbb{C}^d$ be a $G$-invariant domain. Then $P_\theta : A^2_\omega(\Omega) \to A^2_\omega(\Omega)$ is an orthogonal projection and $P_\theta(A^2_\omega(\Omega)) = R^G_\theta(A^2_\omega(\Omega))$. 

For each one-dimensional representation $\varrho \in \hat{G}$, note that the function $|\ell_\varrho(\omega)|^2/|f_\varrho(\omega)|^2$ is a positive-valued continuous function on $\Omega \setminus \bigcup_{i=1}^{t} H_i$, where the hyperplanes $H_i$'s are as in Corollary 2.6. Since $\ell_\varrho$ is $\varrho$-invariant, there exists a unique positive integer, say $m_\varrho$, such that $\ell_\varrho^{m_\varrho}$ is $G$-invariant. Then $\ell_\varrho^{m_\varrho}(z) = f_\varrho(\theta(z))$ for a unique polynomial $f_\varrho$. The function $\omega_\varrho : \theta(\Omega) \setminus \theta(\bigcup_{i=1}^{t} H_i) \to (0, \infty)$ defined by

$$\omega_\varrho(u) = \frac{|f_\varrho(u)|^{2/m_\varrho}}{|f_{sgn}(u)|^{2/m_2}} \omega(u), \quad u \in \theta(\Omega) \setminus \theta(\bigcup_{i=1}^{t} H_i)$$

is continuous. With each one-dimensional representation $\varrho \in \hat{G}$, we associate the linear map $\Gamma_\varrho : A^2_\omega(\theta(\Omega)) \to A^2_\omega(\Omega)$ defined by

$$\Gamma_\varrho \psi = \frac{1}{\sqrt{|G|}} (\psi \circ \theta) \ell_\varrho.$$

Note that

$$\|\Gamma_\varrho \psi\|^2 = \frac{1}{|G|} \int_{\Omega} |\psi \circ \theta(z)|^2 |\ell_\varrho(z)|^2 \omega(z) dV(z)$$

$$= \frac{1}{|G|} \int_{\Omega \setminus \bigcup_{i=1}^{t} H_i} |\psi \circ \theta(z)|^2 \frac{|f_\varrho(\theta(z))|^{2/m_\varrho}}{|f_{sgn}(\theta(z))|^{2/m_2}} \omega(z) dV(z)$$

$$= \int_{\theta(\Omega) \setminus \theta(\bigcup_{i=1}^{t} H_i)} |\psi(z)|^2 \omega_\varrho(z) dV(z) = \|\psi\|^2,$$

the last equality follows since the set $\theta(\Omega) \setminus \theta(\bigcup_{i=1}^{t} H_i)$ is of Lebesgue measure 0. So $\Gamma_\varrho$ is an isometry. Therefore, the range of $\Gamma_\varrho$ is closed and $\Gamma_\varrho(A^2_\omega(\theta(\Omega)))$ is a reproducing kernel Hilbert space. Moreover, we show in the next lemma that it coincides with the relative invariant subspace associated to the representation $\varrho$.

**Lemma 3.2.** Let $G$ be a finite pseudoreflection group and $\Omega$ be a $G$-invariant domain in $\mathbb{C}^d$. Suppose that $\theta : \Omega \to \theta(\Omega)$ is a basic polynomial map associated to the group $G$. Then $\Gamma_\varrho : A^2_\omega(\theta(\Omega)) \to R^G_\varrho(A^2_\omega(\Omega))$ is unitary.

**Proof.** Since $\Gamma_\varrho$ is an isometry, it is enough to prove that $\Gamma_\varrho(A^2_\omega(\theta(\Omega))) = R^G_\varrho(A^2_\omega(\Omega))$. Suppose that $h \in A^2_\omega(\theta(\Omega))$. Then for every $\sigma \in G$,

$$\sigma(\Gamma_\varrho h) = \frac{1}{\sqrt{|G|}} \sigma((h \circ \theta) \ell_\varrho) = \frac{1}{\sqrt{|G|}} \sigma(\ell_\varrho) \sigma(h \circ \theta)$$

$$= \frac{1}{\sqrt{|G|}} \chi_\varrho(\sigma) \ell_\varrho(h \circ \theta) = \chi_\varrho(\sigma) \Gamma_\varrho h.$$

Hence $\Gamma_\varrho h$ is a $\varrho$-invariant element.
On the other hand, consider \( h \in R^G_\varphi(A^2_\omega(\Omega)) \). By Remark 2.9, we write \( h = \ell_\varphi(\hat{h} \circ \theta) \), for \( \hat{h} \in \mathcal{O}(\theta(\Omega)) \). Clearly,
\[
\|h\|^2 = \int_\Omega |h(z)|^2 \omega(z)dV(z) = \int_\Omega |\ell_\varphi(z)|^2 |\hat{h} \circ \theta(z)|^2 \omega(z)dV(z) = |G| \int_{\theta(\Omega)} |\hat{h}(z)|^2 \omega(z)dV(z).
\]
This shows that \( \hat{h} \in A^2_\omega(\theta(\Omega)) \) and \( \Gamma_\varphi(\sqrt{|G|} \hat{h}) = h \). Therefore, \( h \in \Gamma_\varphi(A^2_\omega(\theta(\Omega))) \).

**Remark 3.3.** From Remark 2.9, we have \( R^G_\varphi(A^2_\omega(\Omega)) \subseteq \{ \ell_\varphi(\hat{f} \circ \theta) \mid \hat{f} \in \mathcal{O}(\theta(\Omega)) \} \).

Whereas Lemma 3.2 asserts that
\[
R^G_\varphi(A^2_\omega(\Omega)) = \{ \ell_\varphi(\hat{f} \circ \theta) \mid \hat{f} \in A^2_\omega(\theta(\Omega)) \}.
\]
We refer \( \ell_\varphi \) by generating polynomial of \( R^G_\varphi(A^2_\omega(\Omega)) \).

Now we have all the ingredients to prove Theorem 1.1.

**Proof of Theorem 1.1.** The first part follows from Lemma 3.2.

In Lemma 2.10, we show that \( \mathbb{P}_\varphi \) is the orthogonal projection onto the subspace \( R^G_\varphi(A^2_\omega(\Omega)) = \Gamma_\varphi(A^2_\omega(\theta(\Omega))) \). So the reproducing kernel \( B_\varphi \) of \( \Gamma_\varphi(A^2_\omega(\theta(\Omega))) \) is given by \( B_\varphi(z,w) = \langle \mathbb{P}_\varphi(B_\omega), (B_\omega)_z \rangle \). Then
\[
B_\varphi(z,w) = (\mathbb{P}_\varphi(B_\omega)_w)(z) = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\varphi(\sigma^{-1}) B_\omega(\sigma^{-1} \cdot z, w). \tag{3.2}
\]
An analogous calculation as in [27, p. 6] yields the following formula for the reproducing kernel \( B_\varphi \) of \( \Gamma_\varphi(A^2_\omega(\theta(\Omega))) \):
\[
B_\varphi(z,w) = \frac{1}{|G|} \ell_\varphi(z)B_\omega(\theta(z), \theta(w))\overline{\ell_\varphi(w)} \text{ for } z,w \in \Omega, \tag{3.3}
\]
where \( B_\omega \) is the kernel of \( A^2_\omega(\theta(\Omega)) \). Combining Equation (3.2) and Equation (3.3), we have the desired result.

**Remark 3.4.** For a fixed \( w \in \Omega \), as a function of \( z \), we write \( B_\varphi(z,w) = (\mathbb{P}_\varphi(B_\omega)_w)(z) = \ell_\varphi(z)(B_\omega)_w \circ \theta(z) \) for some \( (B_\omega)_w \in \mathcal{O}(\theta(\Omega)) \). Similarly, \( B_\varphi(z,w) = (\mathbb{P}_\varphi(B_\omega)_z)(w) = \overline{\ell_\varphi(w)}((B_\omega)_z \circ \theta(w)) \) for some \( (B_\omega)_z \in \mathcal{O}(\theta(\Omega)) \), when \( z \) is fixed but arbitrary. Since the variables \( z \) and \( w \) are independent of each other, \( B_\varphi(z,w) \) divisible by \( \ell_\varphi(z)\overline{\ell_\varphi(w)} \) for every \( z,w \in \Omega \). Therefore, the right side of Equation (1.2) is well-defined, even if \( z \) or \( w \) belongs to \( N_\varphi = \{ z \in \Omega : \ell_\varphi(z) = 0 \} \).

Additionally, if \( G \) is abelian, we have the following result.

**Proposition 3.5.** Suppose that \( G \) is an abelian group, then
\[
A^2_\omega(\Omega) \equiv \bigoplus_{\varphi \in G} A^2_\omega(\theta(\Omega)),
\]
and
\[
B_\omega(z,w) = \frac{1}{|G|} \sum_{\varphi \in G} \ell_\varphi(z)B_\omega(\theta(z), \theta(w))\overline{\ell_\varphi(w)}, \ z,w \in \Omega,
\]
where \( B_\omega \) and \( B_\omega \) are the reproducing kernels of \( A_\omega(\Omega) \) and \( A^2_\omega(\theta(\Omega)) \), respectively, and the polynomial \( \ell_\varphi \) is as described in Lemma 2.5 for \( \varphi \in \hat{G} \).
Proof. Since $G$ is a finite abelian group, each irreducible representation of $G$ is one-dimensional. The isotypic decomposition of the left regular representation of $G$ on $A^2_\omega(\Omega)$ yields the orthogonal decomposition $A^2_\omega(\Omega) = \bigoplus_{\rho \in \hat{G}_\omega} P_{\rho} A^2_\omega(\Omega)$ (cf. Theorem 2.11). Moreover, $P_{\rho}$ is the orthogonal projection onto the subspace $R^G_{\rho}(A^2_\omega(\Omega))$ (cf. Lemma 3.1) which is isometrically isomorphic to $A^2_\omega(\theta(\Omega))$ (cf. Lemma 3.2). Thus, the first part follows.

For every fixed $w \in \Omega$, the function $B_\omega(\cdot, w) \in A^2_\omega(\Omega)$ and it admits the decomposition $B_\omega(z, w) = \sum_{\rho \in \hat{G}} P_{\rho} B_\omega(z, w)$. Then Equation (3.3) proves the second part.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. First note that for a fixed $w \in \Omega$,

$$
(G_\rho(B_{\omega})(\theta(w))(z) = \frac{1}{\sqrt{|G|}} \langle \ell_\rho(z) B_{\omega}(\theta(z), \theta(w)) = \frac{1}{\sqrt{|G|}} \frac{1}{\ell_\rho(w)} \sum_{\sigma \in G} \langle \chi_\rho(\sigma)(\theta(w) \cdot z, w). \rangle
$$

Moreover, the $G_\rho$ map can be extended to $L^2_\omega(\theta(\Omega))$ isometrically. Therefore, for a fixed but arbitrary $w \in \Omega$ we have the following: $\phi \in L^2_\omega(\theta(\Omega))$,

$$
(P^\omega_{\theta(w)}(\phi)(\theta(w)) = \langle \phi, (B_{\omega})(\theta(w)) = \langle \Gamma \phi, \Gamma (B_{\omega})(\theta(w)) = \sum_{\sigma \in G} \chi_\rho(\sigma^{-1})(B_{\omega}) \sigma \cdot w, w \rangle
$$

Hence the result follows.

Sign representation. For the sign representation of $G$, we observe that the relative invariant subspace $R^G_{\text{sgn}}(A^2_\omega(\Omega))$ is isometrically isomorphic to the Bergman space $A^2_\omega(\theta(\Omega))$, where $\omega = \tilde{\omega} \circ \theta$. In particular, for the weight function $\omega \equiv 1$, we have that the Bergman space on $\theta(\Omega)$ is isometrically isomorphic to a subspace $R^G_{\text{sgn}}(A^2(\Omega))$ of $A^2(\Omega)$. Subsequently, we get the following results.

We recall from Equation (2.3) that the character of the sign representation is given by $\chi_{\text{sgn}}(\sigma^{-1}) = \text{det}(\sigma)$ for $\sigma \in G$. Also from Corollary 2.6, we have that $\ell_{\text{sgn}} = J_\theta$. Therefore, the following corollary is an immediate consequence of Theorem 1.1.

Corollary 3.6. The weighted Bergman kernel $B_\omega$ of $A^2_\omega(\theta(\Omega))$ is given by the following formula:

$$
B_\omega(z, \theta(w)) = \frac{1}{J_\theta(z) J_\theta(w)} \sum_{\sigma \in G} \det(\sigma) B_\omega(\sigma^{-1} \cdot z, w) \quad \text{for } z, w \in \Omega,
$$

(3.4)
where $B_\omega$ is the reproducing kernel of $A^2_\omega(\Omega)$ and $J_\theta$ is the determinant of the complex jacobian matrix of the basic polynomial map $\theta$.

Now we state a formula involving the weighted Bergman projections $P^\omega_{\Omega_1} : L^2_\omega(\Omega) \to A^2_\omega(\Omega)$ and $P^\omega_{\theta(\Omega)} : L^2_\omega(\theta(\Omega)) \to A^2_\omega(\theta(\Omega))$ which follows immediately from Theorem 4.1. This is a generalization of [10, p. 167, Theorem 1] to the weighted Bergman projections. However, the choice of proper holomorphic map in [10, p. 167, Theorem 1] is restricted here to a basic polynomial map associated to some finite pseudoreflection group.

**Corollary 3.7.** The weighted Bergman projections $P^\omega_{\Omega_1}$ and $P^\omega_{\theta(\Omega)}$ are related to

$$P^\omega_{\Omega_1}(J_\theta(\phi \circ \theta)) = J_\theta((P^\omega_{\theta(\Omega)}(\phi) \circ \theta), \phi \in L^2_\omega(\theta(\Omega)),$$

where $\omega = \tilde{\omega} \circ \theta$.

4. **Proper Holomorphic maps and Bergman Kernels**

In this section, we prove a transformation formula for weighted Bergman kernels under a proper holomorphic map whose group of deck transformations is either a finite pseudoreflection of group or a conjugate to a finite pseudoreflection group.

Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic map with the finite pseudoreflection group $G$ as the group of deck transformations and $\omega : \Omega_1 \to (0, \infty)$ is a continuous function of the form $\omega = \tilde{\omega} \circ f$ for a continuous function $\tilde{\omega} : \Omega_2 \to (0, \infty)$. A transformation rule for the weighted Bergman kernels of $A^2_\omega(\Omega_1)$ and $A^2_\omega(\Omega_2)$ under the proper holomorphic map $f$ is established in next theorem.

**Theorem 4.1.** The reproducing kernels $B_\omega$ of $A^2_\omega(\Omega_1)$ and $B_\omega$ of $A^2_\omega(\Omega_2)$ transform according to

$$B_\omega(f(z), f(w)) = \frac{1}{J_f(z)J_f(w)} \sum_{\sigma \in G} \det(\sigma) B_\omega(\sigma^{-1} \cdot z, w) \quad \text{for } z, w \in \Omega_1.$$

**Proof.** It follows from Proposition 2.2 that there exists a unique biholomorphic map $\tilde{f} : \theta(\Omega_1) \to \Omega_2$ such that $f = \tilde{f} \circ \theta$. We write $\omega' = \tilde{\omega} \circ \tilde{f}$ and thus $\omega = \omega' \circ \theta$.

Under the biholomorphic map $\tilde{f} : \theta(\Omega_1) \to \Omega_2$, the weighted Bergman kernels $B_\omega'$ of $A^2_\omega'(\theta(\Omega_1))$ and $B_\omega'$ of $A^2_\omega(\Omega_2)$ are related as:

$$B_\omega(\tilde{f}(\theta(z)), \tilde{f}(\theta(w))) = \frac{1}{J_{\tilde{f}}(\theta(z))J_{\tilde{f}}(\theta(w))} B_{\omega'}(\theta(z), \theta(w)), \quad (4.1)$$

see [29]. Hence we get the result combining Equation (4.1) and Corollary 3.6.

In particular, for $\omega \equiv 1$ we get the following the transformation formula for the Bergman kernels of $\Omega_1$ and $\Omega_2$ which overlaps with the Bell’s transformation formula described in [11, p. 687, Theorem 1]. We emphasize that our transformation formula works for the critical points of $f$ as well.

**Corollary 4.2.** Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic map with a finite pseudoreflection group $G$ as the group of deck transformations. Then

$$B_2(f(z), f(w)) = \frac{1}{J_f(z)J_f(w)} \sum_{\sigma \in G} \det(\sigma) B_1(\sigma^{-1} \cdot z, w), \quad z, w \in \Omega_1, \quad (4.2)$$

where $B_1$ and $B_2$ denote the Bergman kernels of $\Omega_1$ and $\Omega_2$, respectively.
The Bergman kernel of a domain is rational if it is a rational function of the coordinates. If the Bergman kernel $B_1$ and the proper map $f$ are rational, then from Equation (4.2) it is clear that $B_2$ is also rational.

**Corollary 4.3.** Suppose that the Bergman kernel of $\Omega_1$ is rational. If there exists a proper holomorphic map $f : \Omega_1 \to \Omega_2$ which is rational and the group of deck transformations of $f$ is a finite pseudoreflection group $G$, then the Bergman kernel of $\Omega_2$ is also rational.

### 4.1. Groups Conjugate to Pseudoreflection groups.
Suppose that $\Omega_1$ and $\Omega_2$ are two domains in $\mathbb{C}^d$ and $G \subseteq \text{Aut}(\Omega_1)$ is a finite pseudoreflection group. The group $\tilde{G} = \Psi^{-1}G\Psi$ (4.3)
is said to be a conjugate to the pseudoreflection group $G$ by an automorphism $\Psi \in \text{Aut}(\Omega_1)$. Let $F : \Omega_1 \to \Omega_2$ be a proper holomorphic map with the group of deck transformations $\tilde{G}$. Equivalently,

$$F^{-1}F(z) = \bigcup_{\sigma \in \tilde{G}} \{\sigma(z)\} \quad \text{for } z \in \Omega_1.$$ (4.4)

The proper holomorphic map $F$ satisfying Equation (4.4) is referred as factored by automorphisms $\tilde{G}$ in [9, 23]. We obtain a characterization for such proper holomorphic maps in the following proposition.

**Proposition 4.4.** Suppose that $F : \Omega_1 \to \Omega_2$ is a proper holomorphic map between two bounded domains in $\mathbb{C}^d$ and $G \subseteq \text{Aut}(\Omega_1)$ is a conjugate to a pseudoreflection group $G$ by the automorphism $\Psi \in \text{Aut}(\Omega_1)$. Then $F$ is factored by $\tilde{G}$ if and only if $F = \Phi \circ \theta \circ \Psi$, where $\theta$ is a basic polynomial map associated to the group $G$ and $\Phi$ is a biholomorphic map from $\theta(\Omega_1)$ to $\Omega_2$.

**Proof.** Suppose that the proper holomorphic map $F$ is factored by the group $\Psi^{-1}G\Psi$. Then

$$F^{-1}F(z) = \bigcup_{\sigma \in G} \{(\Psi^{-1} \circ \sigma \circ \Psi)(z)\} \quad \text{for all } z \in \Omega_1.$$ Consider the map $f = F \circ \Psi^{-1}$. For every $z \in \Omega_1$, $f^{-1}f(z) = \bigcup_{\sigma \in G} \{\sigma(z)\}$, that is, the proper holomorphic map $f : \Omega_1 \to \Omega_2$ is factored by the finite pseudoreflection group $G$. From Proposition 2.2, we get that $f = \Phi \circ \theta$, where $\Phi$ is a biholomorphic map from $\theta(\Omega_1)$ to $\Omega_2$ and $\theta$ is a basic polynomial map associated to the group $G$. Therefore, $F$ can be written in the desired way.

Conversely, assume that we can express $F = \Phi \circ \theta \circ \Psi$, where $\Phi$ is a biholomorphic map from $\theta(\Omega_1)$ to $\Omega_2$, $\theta$ is a basic polynomial map associated to $G$ and $\Psi \in \text{Aut}(\Omega_1)$. Note that $\Phi \circ \theta$ is factored by the group $G$. Since $\Phi \circ \theta(z) = F \circ \Psi^{-1}(z)$ for all $z \in \Omega_1$, the result follows.

**Theorem 4.5.** Suppose that $\Omega_i$, for $i = 1, 2$ are two bounded domains in $\mathbb{C}^d$ and $F : \Omega_1 \to \Omega_2$ is a proper holomorphic map which is factored by $\tilde{G} \subseteq \text{Aut}(\Omega_1)$, where $\tilde{G}$ is as in Equation (4.3). Then $B_2$ can be expressed in terms of $B_1$ by the following formula:

$$B_2(F(z), F(w)) = \frac{1}{J_F(z)J_F(w)} \sum_{\sigma \in G} J_{\Psi_\sigma}(z)B_1(\Psi_\sigma(z), w),$$ (4.5)

where $B_1$ is the Bergman kernel of the domain $\Omega_1$, $\Psi \in \text{Aut}(\Omega_1)$ and $\Psi_\sigma = \Psi^{-1} \circ \sigma^{-1} \circ \Psi$ for $\sigma \in G$. 


Proof. The Bergman kernel $B_1$ transforms under the automorphism $\Psi \in \text{Aut}(\Omega_1)$ following [30, p. 419, Proposition 12.1.10]:

$$B_1(\Psi(z), \Psi(w)) = \frac{1}{J_{\psi}(z) J_{\psi}(w)} B_1(z, w) \text{ for } z, w \in \Omega_1.$$  

Therefore, for a fixed $w \in \Omega_1$,

$$\begin{align*}
(B_1)_{\Psi(w)}(z) &= \frac{1}{J_{\psi}(w) J_{\psi}(\Psi^{-1}(z))} (B_1)_w(\Psi^{-1}(z)) \\
(B_1)_{\Psi(w)}(\sigma^{-1} \cdot z) &= \frac{1}{J_{\psi}(w) J_{\psi}(\Psi^{-1}(\sigma^{-1} \cdot z))} (B_1)_w(\Psi^{-1}(\sigma^{-1} \cdot z)) \\
(B_1)_{\Psi(w)}(\sigma^{-1} \cdot \Psi(z)) &= \frac{1}{J_{\psi}(w) J_{\psi}(\Psi^{-1} \circ \sigma^{-1} \circ \Psi(z))} (B_1)_w(\Psi^{-1} \circ \sigma^{-1} \circ \Psi(z)).
\end{align*}$$

Let $\Psi_{\sigma} := \Psi^{-1} \circ \sigma^{-1} \circ \Psi$. Then $\Psi \circ \Psi_{\sigma} = \sigma^{-1} \circ \Psi$. Application of the chain rule on the both sides yields $J_{\psi}(\Psi(z)) J_{\Psi_{\sigma}}(z) = J_{\sigma^{-1}}(\Psi(z)) J_{\Psi}(z)$. Hence we have

$$B_1(\sigma^{-1} \cdot \Psi(z), \Psi(w)) = \frac{J_{\Psi_{\sigma}}(z)}{J_{\Psi}(w) J_{\Psi}(z) J_{\Psi_{\sigma}}(\Psi(z))} B_1(\Psi_{\sigma}(z), w). \quad (4.6)$$

Suppose that $B_\theta$ denotes the Bergman kernel of the domain $\theta(\Omega_1)$. From Equation (4.2), we have

$$B_\theta(\theta(\Psi(z)), \theta(\Psi(w))) = \frac{1}{J_{\theta}(\Psi(z)) J_{\theta}(\Psi(w))} \sum_{\sigma \in G} \det(\sigma) B_1(\sigma^{-1} \cdot \Psi(z), \Psi(w)),$$

for $z, w \in \Omega_1$. Equation (4.6) implies that

$$\begin{align*}
B_\theta(\theta(\Psi(z)), \theta(\Psi(w))) &= \frac{1}{J_{\theta}(\Psi(z)) J_{\theta}(\Psi(w))} \sum_{\sigma \in G} \det(\sigma) \frac{J_{\Psi_{\sigma}}(z)}{J_{\Psi}(w) J_{\Psi}(z) J_{\Psi_{\sigma}}(\Psi(z))} B_1(\Psi_{\sigma}(z), w) \\
&= \frac{1}{J_{\theta \cdot \Psi}(z) J_{\theta \cdot \Psi}(w)} \sum_{\sigma \in G} \det(\sigma) \frac{J_{\Psi_{\sigma}}(z)}{J_{\Psi}(w) J_{\Psi}(z) J_{\Psi_{\sigma}}(\Psi(z))} B_1(\Psi_{\sigma}(z), w).
\end{align*}$$

Since $F = \Phi \circ \theta \circ \Psi$, where $\Phi : \theta(\Omega_1) \to \Omega_2$ is a biholomorphism, we have the following:

$$\begin{align*}
B_2(F(z), F(w)) &= \frac{1}{J_{\Psi}(z) J_{\Psi}(w)} \sum_{\sigma \in G} \det(\sigma) \frac{J_{\Psi_{\sigma}}(z)}{J_{\Psi}(w) J_{\Psi}(z) J_{\Psi_{\sigma}}(\Psi(z))} B_1(\Psi_{\sigma}(z), w).
\end{align*}$$

Note that for every $z \in \Omega_1$, $J_{\sigma^{-1}}(\Psi(z)) = \det(\sigma)$, so $\frac{\det(\sigma)}{J_{\sigma^{-1}}(\Psi(z))} = 1$. This completes the proof. 

\[\blacksquare\]

Remark 4.6. The map $\iota : \Omega_1 \to \Omega_1$ given by $\iota(z) = z$, is indeed an automorphism of the domain $\Omega_1$. A trivial observation is that Equation (4.5) coincides with Equation (4.2) for $\Psi = \iota$. 
5. Applications

The Bergman kernel on the domain $\Omega \subseteq \mathbb{C}^d$ is denoted by $B_\Omega$. We fix this notation for the rest of our discussion. Let $\mathbb{B}_d$ be the unit ball with respect to the $\ell^2$-norm induced by the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^d$. The Bergman kernel of $\mathbb{B}_d$ is given by [7, p. 172]

$$B_{\mathbb{B}_d}(z, w) = (1 - \langle z, w \rangle)^{-(d+1)}. \quad (5.1)$$

Let $\mathbb{D}^d = \{z \in \mathbb{C}^d : |z_1|, \ldots, |z_d| < 1\}$, the unit ball with respect to $\ell^\infty$-norm, be the polydisc in $\mathbb{C}^d$. The Bergman kernel of $\mathbb{D}^d$ is given by

$$B_{\mathbb{D}^d}(z, w) = \prod_{j=1}^d (1 - z_j w_j)^{-2}. \quad (5.2)$$

In this section, we obtain formulae for the weighted Bergman kernels of several domains which are biholomorphically equivalent to some quotient domains of the form $\Omega/G$, where $\Omega = \mathbb{B}_d$ or $\mathbb{D}^d$ and $G$ is a finite pseudoreflection group. This demonstrates an useful application of Theorem 1.1 and Theorem 4.5.

Suppose that for a finite pseudoreflection group $G$, a basic polynomial map associated to the group $G$ is denoted by $\theta$. Clearly, $J_\theta$ is again a polynomial. If $\mathbb{B}_d$ (or $\mathbb{D}^d$) is $G$-invariant, then the kernel function $B_{\theta(B_d)}(z, w)$ (or $B_{\theta(D^d)}(z, w)$) is rational from Corollary 4.3. In this section, we consider domains (except Rudin’s domains) which can be realized as $\theta(\mathbb{D}^d)$ for a basic polynomial map $\theta$ associated to some finite pseudoreflection group $G$. This provides classes of domains with rational Bergman kernels.

5.1. Rudin’s domain. Rudin characterizes proper holomorphic mappings from $\mathbb{B}_d$ onto a domain $\Omega \subseteq \mathbb{C}^d, d > 1$, in [34, p. 704, Theorem 1.6], see also [8, p. 506]. The result is stated as:

**Theorem.** [34, p. 704, Theorem 1.6] Suppose that $F : \mathbb{B}_d \to \Omega$ is a proper holomorphic mapping from the open unit ball $\mathbb{B}_d$ in $\mathbb{C}^d (d > 1)$ onto a domain $\Omega$ in $\mathbb{C}^d$ with multiplicity $m > 1$. Then there exists a unique finite pseudoreflection group $G$ of order $m$ such that

$$F = \Phi \circ \theta \circ \Psi,$$

where $\Psi$ is an automorphism of $\mathbb{B}_d$, $\theta$ is a basic polynomial mapping associated to $G$ and $\Phi : \theta(\mathbb{B}_d) \to \Omega$ is biholomorphic.

In other words, any proper holomorphic mapping from the open unit ball $\mathbb{B}_d$ onto the domain $\Omega$ is factored by automorphisms $G$, where $G$ is a conjugate to the finite pseudoreflection group $G$ by the automorphism $\Psi \in \text{Aut}(\mathbb{B}_d)$. Such a domain $\Omega$ is referred as Rudin’s domain in [12, p. 427]. Here, we include formulae for the Bergman kernels of Rudin’s domains to exhibit a direct application of Theorem 4.5. Moreover, we show that the Bergman kernels of a subclass of Rudin’s domains are rational.

The group of unitary operators on $\mathbb{C}^d$, $U_d$, leaves the open unit ball $\mathbb{B}_d$ invariant. Since any pseudoreflection group $G$ acting on $\mathbb{C}^d$ is a subgroup of $U_d$, $\mathbb{B}_d$ is $G$-invariant. Now we use Theorem 4.5 to get the following result.

**Theorem 5.1.** Suppose that $F : \mathbb{B}_d \to \Omega$ is a proper holomorphic mapping. Then the Bergman kernel $B_\Omega$ is given by the following formula:

$$B_{\Omega}(F(z), F(w)) = \frac{1}{J_F(z) J_F(w)} \sum_{\sigma \in G} \frac{J_{\Psi_\sigma}(z)}{(1 - \langle \Psi_\sigma(z), w \rangle)^{d+1}}, \quad (5.4)$$
where $\Psi \in \operatorname{Aut}(\mathbb{B}_d)$ and $\Psi_\sigma = \Psi^{-1} \sigma^{-1} \Psi$ for $\sigma \in G$.

The automorphisms of the open unit ball $\mathbb{B}_d$ are given by certain rational functions [35, p. 28, Theorem 2.2.5]. So the functions $\Psi_\sigma(z)$ and $J_{\Psi_\sigma}(z)$ are rational in $z$. A proper holomorphic map from $\mathbb{B}_d$ to a bounded circular domain (containing the origin) in $\mathbb{C}^d$ is always rational [12, p. 429, Theorem 3]. Then the next corollary follows from Equation (5.4).

**Corollary 5.2.** [12, p. 427] Suppose that the circular domain $\Omega \subset \mathbb{C}^d$ contains the origin and there exists a proper holomorphic map from the open unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ onto $\Omega$. Then the Bergman kernel of $\Omega$ is rational.

5.2. Symmetrized Polydisc. The permutation group on $d$ symbols is denoted by $S_d$. The group $S_d$ acts on $\mathbb{C}^d$ by permuting its coordinates, that is,

$$\sigma \cdot (z_1, \ldots, z_d) = (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(d)})$$

for $\sigma \in S_d$ and $(z_1, \ldots, z_d) \in \mathbb{C}^d$.

Clearly, the open unit polydisc $\mathbb{D}_d$ is invariant under the action of the group $S_d$.

Let $s_k$ denote the elementary symmetric polynomials of degree $k$ in $d$ variables, for $k = 1, \ldots, d$. The symmetrization map $s := (s_1, \ldots, s_d) : \mathbb{C}^d \to \mathbb{C}^d$ is a basic polynomial map associated to the pseudoreflection group $S_d$. The domain $\mathbb{G}_d := s(\mathbb{D}_d)$ is known as the symmetrized polydisc.

The weight function $\omega(z) = \prod_{j=1}^{d}(1 - |z_j|^2)^{1/2}$ is $S_d$-invariant, so there exists continuous function $\tilde{\omega} : \mathbb{G}_d \to (0, \infty)$ such that $\omega = \tilde{\omega} \circ s$. For $\lambda > 1$, the reproducing kernel of the weighted Bergman space $\mathbb{A}^2_\omega(\mathbb{D}_d)$ is $E^1_d(z, w) = \prod_{j=1}^{d}(1 - z_j w_j)^{-\lambda}$.

The symmetric group $S_d$ has only two one-dimensional representations in $\mathbb{G}_d$. Those are the sign representation and the trivial representation of $S_d$.

For the sign representation. The relative invariant subspace subspace $R_{\text{sgn}}(\mathbb{A}^2_\omega(\mathbb{D}_d))$ is isometrically isomorphic to the Bergman space $\mathbb{A}^2_\tilde{\omega}(\mathbb{G}_d)$ (cf. Theorem 1.1). Therefore, the weighted Bergman kernel $B_{\mathbb{G}_d}$ of $\mathbb{A}^2_\tilde{\omega}(\mathbb{G}_d)$ can be derived in terms of the weighted Bergman kernel of $E^1_d(z, w)$. From Equation (1.4), the weighted Bergman kernel $B_{\mathbb{G}_d}$ is given by the formula:

$$B_{\mathbb{G}_d}(s(z), s(w)) = \frac{1}{J_s(z)J_s(w)} \sum_{\sigma \in S_d} \det(\sigma) B^1_d(z, \sigma^{-1} \cdot z, w).$$

Note that $\det(\sigma) = \text{sgn}(\sigma^{-1})$, for $\sigma \in S_d$. Therefore,

$$B_{\mathbb{G}_d}(s(z), s(w)) = \frac{1}{J_s(z)J_s(w)} \sum_{\sigma \in S_d} \text{sgn}(\sigma^{-1}) B^1_d(z, \sigma \cdot w) = \frac{1}{J_s(z)J_s(w)} \sum_{\sigma \in S_d} \text{sgn}(\sigma^{-1}) \prod_{i=1}^{d}(1 - z_i \bar{w}_{\sigma^{-1}(i)})^{-\lambda} = \frac{1}{J_s(z)J_s(w)} \det((1 - z_i \bar{w}_j)^{-\lambda})^{d}_{i,j=1}.$$

Note that $J_s(z) = \prod_{i<j}(z_i - z_j)$ [25, p. 370, Lemma 10].
Proposition 5.3. The weighted Bergman kernel of $\mathbb{A}_w^2(G_d)$ is given by

$$
B_w(z, w) = \frac{\det \left( (1 - z_i \bar{w}_j)^{-\lambda} \right)_{i,j=1}^d}{\prod_{i<j} (z_i - z_j)(\bar{w}_i - \bar{w}_j)}, \quad z, w \in \mathbb{D}^d.
$$

In particular, for $\lambda = 2$ the weight function $\omega \equiv 1$ (consequently, $\tilde{\omega} \equiv 1$) and the Bergman kernel of the symmetrized polydisc $G_d$ can be deduced from the above result which is

$$
B_{G_d}(z, w) = \frac{\det \left( (1 - z_i \bar{w}_j)^{-2} \right)_{i,j=1}^d}{\prod_{i<j} (z_i - z_j)(\bar{w}_i - \bar{w}_j)}.
$$

This expression of the Bergman kernel of the symmetrized polydisc was obtained in [25, p. 369, Proposition 9] using Bell’s transformation rule on the set of regular values of the symmetrization map $s$. A different approach is followed to derive this formula in [32, p. 2306, Theorem 2.3].

For the trivial representation. The trivial representation of $\mathcal{S}_d$ is given by $\text{tr} : \mathcal{S}_d \to \mathbb{C}^*$ such that $\text{tr}(\sigma) = 1$ for all $\sigma \in \mathcal{S}_d$. The generating polynomial $\ell_{tr}$ is a constant polynomial, so we can choose $\ell_{tr} \equiv 1$. Then from Theorem 1.1, the relative invariant subspace $R_{tr}^{\mathbb{D}^d}(\mathbb{A}_w^2(\mathbb{D}^d))$ is isometrically isomorphic to the Bergman space $\mathbb{A}_w^2(G_d)$, where the weight function is given by

$$
\bar{\omega}_1(s(z)) = \frac{1}{\prod_{i<j}|z_i - z_j|^2} \tilde{\omega}(s(z)). \quad (5.5)
$$

An explicit expression for the reproducing kernel $B_{\bar{\omega}_1}$ of the weighted Bergman space $\mathbb{A}_{\bar{\omega}_1}^2(G_d)$ is derived using Equation (1.2):

$$
B_{\bar{\omega}_1}(z, w) = \sum_{\sigma \in \mathcal{S}_d} B_{\omega}^{(\ell_{tr})}(\sigma^{-1} \cdot z, w)
$$

$$
= \sum_{\sigma \in \mathcal{S}_d} \prod_{i=1}^d (1 - z_i \bar{w}_{\sigma^{-1}(i)})^{-\lambda}
$$

$$
= \text{perm} \left( (1 - z_i \bar{w}_j)^{-\lambda} \right)_{i,j=1}^d,
$$

where perm$A$ denotes the permanent of the matrix $A$.

Proposition 5.4. Let $\bar{\omega}_1 : G_d \setminus s(N) \to (0, \infty)$ be the continuous function defined as in Equation (5.5), where $N = \{z \in \mathbb{D}^d : z_i = z_j \text{ for at least two } i, j, i \neq j\}$. The reproducing kernel $B_{\bar{\omega}_1}$ of the weighted Bergman space $\mathbb{A}_{\bar{\omega}_1}^2(G_d)$ is given by

$$
B_{\bar{\omega}_1}(z, w) = \text{perm} \left( (1 - z_i \bar{w}_j)^{-\lambda} \right)_{i,j=1}^d, \quad z, w \in \mathbb{D}^d,
$$

where perm$A$ denotes the permanent of the matrix $A$.

5.3. The domain $\mathbb{D}^2/D_{2k}$. Let $D_{2k} = \langle \delta, \sigma : \delta^k = \sigma^2 = 1, \sigma \delta \delta^{-1} = \delta^{-1} \rangle$ be the dihedral group of order $2k$. We define its action on $\mathbb{C}^2$ via the faithful representation $\pi$ defined by

$$
\pi : D_{2k} \to GL(2, \mathbb{C}) : \delta \mapsto \begin{bmatrix} \omega_k & 0 \\ 0 & \omega_k^{-1} \end{bmatrix}, \sigma \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
$$

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where \( \omega_k \) denotes a primitive \( k \)-th root of unity. We write the matrix representation of the group action with respect to the standard basis of \( \mathbb{C}^2 \). The polynomial map \( \phi(z_1, z_2) = (z_1^k + z_2^k, z_1z_2) \) for \((z_1, z_2) \in \mathbb{C}^2 \), is a basic polynomial map associated to the pseudoreflection group \( D_{2k} \). The open unit bidisc \( \mathbb{D}^2 \) is invariant under this action. The restriction map \( \phi: \mathbb{D}^2 \to \phi(\mathbb{D}^2) := D_{2k} \) is a proper holomorphic map of multiplicity \( 2k \) and is factored by automorphisms \( D_{2k} \). Clearly, \( J_\phi(z_1, z_2) = k(z_1^k - z_2^k) \).

The number of one-dimensional representations of the dihedral group \( D_{2k} \) in \( \hat{D}_{2k} \) is \( 2 \) if \( k \) is odd and \( 4 \) if \( k \) is even. Clearly, for every \( k \in \mathbb{N} \) the trivial representation of \( D_{2k} \) and the sign representation of \( D_{2k} \) are in \( \hat{D}_{2k} \).

For the sign representation. Since the relative invariant subspace subspace \( R_{\text{tr}}^{D_{2k}}(\mathbb{A}^2(\mathbb{D}^2)) \) is isometrically isomorphic to the Bergman space \( \mathbb{A}^2(D_{2k}) \), we derive the Bergman kernel of the domain \( D_{2k} \) using Corollary 3.6. Note that
\[
\det(\sigma^j \delta^i) = \begin{cases} 
1 & \text{if } i = 0, \\
-1 & \text{if } i = 1,
\end{cases}
\]
for \( j = 0, \ldots, k - 1 \). Recalling Equation (5.2), \( B_{D^2}(z, w) = \frac{1}{((1 - z_1 w_1)(1 - z_2 w_2))^2} \), we conclude form Equation (3.4) that:
\[
B_{D_{2k}}(\phi(z_1, z_2), \phi(w_1, w_2)) = \frac{1}{k^2(z_1^k - z_2^k)(w_1^k - w_2^k)} \sum_{i=1}^{k} (B_{D^2}(\delta^i \cdot z, w) - B_{D^2}(\sigma \delta^i \cdot z, w))
\]
\[
= \frac{1}{k^2(z_1^k - z_2^k)(w_1^k - w_2^k)} \times \sum_{i=1}^{k} \left( \frac{1}{(1 - \omega_k z_1 w_1)^2(1 - \omega_k^{-1} z_2 w_2)^2} - \frac{1}{(1 - \omega_k z_2 w_2)^2(1 - \omega_k^{-1} z_1 w_1)^2} \right).
\]
After a tedious but straightforward calculation, we state the following proposition.

**Proposition 5.5.** The Bergman kernel of \( D_{2k} \) is given by the following formula:
\[
B_{D_{2k}}(\phi(z_1, z_2), \phi(w_1, w_2)) = \frac{(z_1 - z_2)}{k^2(z_1^k - z_2^k)(w_1^k - w_2^k)} \times \sum_{i=1}^{k} \left( \frac{2(1 + z_1 z_2 w_1 w_2) - (z_1 + z_2)(\omega_k^i w_1 + \omega_k^{-i} w_2)(\omega_k^i w_1 - \omega_k^{-i} w_2)}{(\omega_k w_1^k + \omega_k^{-i} w_2)(z_1 X_1 + z_2 X_2) + z_1 z_2(\omega_k^i w_1 + \omega_k^{-i} w_2)^2} \right)^2,
\]
where \( \omega_k \) is a primitive \( k \)-th root of unity and \( X_\ell = 1 + z_1^2 w_1 w_2, \) for \( \ell = 1, 2 \).

For the trivial representation. The trivial representation of \( D_{2k} \) is given by \( \text{tr} : D_{2k} \to \mathbb{C}^* \) such that \( \text{tr}(\sigma) = 1 \) for all \( \sigma \in D_{2k} \). The generating polynomial \( \ell_{tr} \) can be taken as \( \ell_{tr} \equiv 1 \). From Theorem 1.1, we get that the relative invariant subspace \( R_{\text{tr}}^{D_{2k}}(\mathbb{A}^2(\mathbb{D}^2)) \) is isometrically isomorphic to the Bergman space \( \mathbb{A}^2(D_{2k}) \), where the weight function is given by
\[
\omega(\phi(z)) = \frac{1}{k(z_1^k - z_2^k)}, \quad (5.6)
\]
Hence we have the following from Equation (1.2):
Proposition 5.6. Let \( \omega : \mathcal{D}_{2k} \setminus \phi(N) \to (0, \infty) \) be the continuous function defined as in Equation (5.6), where \( N = \{ z \in \mathbb{D}^2 : z_1 = \omega k z_2, \omega_k \text{ is a } k\text{-th root of unity} \} \). The reproducing kernel \( \mathcal{B}_{\mathcal{D}_{2k}}^\omega \) of the weighted Bergman space \( \mathcal{A}_{\mathcal{D}_{2k}}^\omega \) is given by

\[
\mathcal{B}_{\mathcal{D}_{2k}}^\omega(\phi(z_1, z_2), \phi(w_1, w_2)) = \sum_{i=1}^{k} \frac{1}{(1 - \omega_k^i z_1 \bar{w}_1)^2 (1 - \omega_k^{i-1} z_2 \bar{w}_2)^2} + \frac{1}{(1 - \omega_k^i z_2 \bar{w}_1)^2 (1 - \omega_k^{i-1} z_1 \bar{w}_2)^2}, \quad z, w \in \mathbb{D}^2.
\]

For two additional one-dimensional representations while \( k = 2k' \). While \( k \) is even, \( \mathcal{D}_{2k} \) has two more one-dimensional representations. We refer those by \( g_1 \) and \( g_2 \), where

\[
g_1(\delta) = -1 \quad \text{and} \quad g_1(\tau) = 1 \text{ for } \tau \in \langle \delta^2, \sigma \rangle, \\
g_2(\delta) = -1 \quad \text{and} \quad g_2(\tau) = 1 \text{ for } \tau \in \langle \delta^2, \delta \sigma \rangle.
\]

The generating polynomials associated to \( g_1 \) and \( g_2 \) are given by \( \ell_{g_1}(z) = z_1^{k'} + z_2^{k'} \) and \( \ell_{g_2}(z) = z_1^{k'} - z_2^{k'} \), respectively. Let the weight functions \( \tilde{\omega}_1 : \mathcal{D}_{2k} \setminus \phi(N_1) \to (0, \infty) \) and \( \tilde{\omega}_2 : \mathcal{D}_{2k} \setminus \phi(N_2) \to (0, \infty) \) be defined as the following:

\[
\tilde{\omega}_1(\phi(z)) = \frac{1}{k(z_1^{k'} - z_2^{k'})}, \quad \text{and} \quad \tilde{\omega}_2(\phi(z)) = \frac{1}{k(z_1^{k'} + z_2^{k'})},
\]

where \( N_1 = \{ z \in \mathbb{D}^2 : z_1 = \omega k z_2, \omega_k \text{ is a } k'\text{-th root of unity} \} \) and \( N_2 = \{ z \in \mathbb{D}^2 : z_1^{k'} + z_2^{k'} = 0 \} \). Now a similar approach as above (using Theorem 1.1) will lead to explicit expressions for the reproducing kernels of the weighted Bergman spaces \( \mathcal{A}_{\mathcal{D}_{2k}}^{\tilde{\omega}_1} \) and \( \mathcal{A}_{\mathcal{D}_{2k}}^{\tilde{\omega}_2} \).

5.4. Monomial Polyhedron. For \( d \geq 2 \), a \( d \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Q}^d \) of rational numbers and a \( d \)-tuple of complex numbers \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \), we denote \( z^\alpha := \prod_{k=1}^{d} z_k^{\alpha_k} \). Consider a matrix \( B \in M_d(\mathbb{Q}) \). We enumerate the row vectors of \( B \) by \( \mathcal{F} = \{ b^1, \ldots, b^d \} \), where \( b^k = (b_1^k, \ldots, b_d^k) \). The monomial polyhedron associated to \( B \) is defined by

\[
\mathcal{U} = \{ z \in \mathbb{C}^d : |z^{b^k}| < 1 \text{ for all } 1 \leq k \leq d \},
\]

unless for some \( 1 \leq k, j \leq d \), the quantity \( z_j^{b_j^k} \) is not defined due to the division of zero, see [13, Equation 1.1]. Without loss of generality, we assume that \( B \in M_d(\mathbb{Z}) \), \( \det(B) > 0 \) and \( B^{-1} \geq 0 \) [13, Equation 3.3].

Set \( A = \text{adj } B \). The rows of \( A \) are enumerated by \( \{ a^1, \ldots, a^d \} \). We borrow the proper holomorphic map \( \Phi_A : \mathbb{D}^d_{L(B)} \to \mathcal{U} \) defined by

\[
\Phi_A(z) = (z^{a^1}, \ldots, z^{a^d}), \quad \text{for } z \in \mathbb{D}^d_{L(B)}
\]

from [13, Theorem 3.12], where \( \mathbb{D}^d_{L(B)} \) is the product of some copies of the unit disc with some copies of the punctured unit disc. The proper map \( \Phi_A \) is of quotient type with pseudoreflection group \( G \) and \( G \) is isomorphic to the direct product of cyclic groups \( \prod_{i=1}^{d} \mathbb{Z}/\delta_i \mathbb{Z} \), where each \( \delta_i \in \mathbb{Z} \) is coming from the Smith Normal form of the matrix \( A \), that says, \( A = PDQ \), where \( P, Q \in GL_d(\mathbb{Z}) \) and \( D = \text{diag}(\delta_1, \ldots, \delta_d) \in M_d(\mathbb{Z}) \). Let \( \varphi : \prod_{i=1}^{d} \mathbb{Z}/\delta_i \mathbb{Z} \to G \) be a group isomorphism. Set \( S_G = \{ n = (n_1, \ldots, n_d) \in \mathbb{N}^d : 1 \leq\)
\( n_i \leq |\delta_i| \). We write

\[
G = \{ \sigma_n : \psi(\prod_{i=1}^{d} \omega_{n_i}^{\delta_i}) = \sigma_n \text{ for } n = (n_1, \ldots, n_d) \in S_G \},
\]

where \( \omega_{n_i}^{\delta_i} \) is a primitive \( |\delta_i| \)-th root of unity for \( i = 1, \ldots, d \).

**Proposition 5.7.** The Bergman kernel \( B_\mathcal{U} \) of the monomial polyhedron \( \mathcal{U} \) is given by:

\[
B_\mathcal{U}(\Phi_A(z), \Phi_A(w)) = \frac{1}{(\det A)^2} \prod_{i=1}^{d} z_i^{\alpha_i} \sum_{n \in S_G} \prod_{i=1}^{d} \omega_{n_i}^{\delta_i} B_{D^d}(\sigma_n^{-1} \cdot z, w), \tag{5.7}
\]

where \( B_{D^d} \) is the Bergman kernel of \( D^d \).

**Proof.** We use Corollary 4.2 to get the following:

\[
B_\mathcal{U}(\Phi_A(z), \Phi_A(w)) = \frac{1}{J_{\Phi_A}(z)} \sum_{\sigma_n \in G} \det(\sigma_n) B_{D^d}(\sigma_n^{-1} \cdot z, w),
\]

where the Bergman kernel of \( D^d \) is denoted by \( B_{D^d} \). Note that \( B_{D^d}(z, w) = B_{D^d}(z, w) \), whenever \( z, w \in D^d \). Since the character remains unchanged under group isomorphism, the representation \( \mu \) of \( G \), as described in Equation (2.3), gives \( \chi_\mu(\sigma_n^{-1}) = \det(\sigma_n) = \prod_{i=1}^{d} \omega_{n_i}^{\delta_i} \) for every \( n \in S_G \). Moreover, from [13, Lemma 3.8]

\[
J_{\Phi_A}(z) = \det A \cdot \prod_{i=1}^{d} \frac{z_i^{\alpha_i}}{z_i}
\]

Thus the result follows. \( \square \)

In [13, Proposition 3.22], Chakrabarti et al. proved rationality of the Bergman kernel of the monomial polyhedron using Bell’s transformation rule for the Bergman kernels under a proper holomorphic mapping. Since \( J_{\Phi_A}(\cdot) \) is rational, we conclude the following from Equation (5.7).

**Corollary 5.8.** The Bergman kernel of monomial polyhedron \( \mathcal{U} \) is rational.

### 5.4.1. Fat Hartogs Triangle.

Let \( \gamma \) be a positive integer. For the matrix \( B = \begin{pmatrix} \gamma & -1 \\ 0 & 1 \end{pmatrix} \), the domain \( \mathcal{U} \) can be written as \( \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1 \} \). This turns out to be a subclass of ‘fat Hartogs triangle’ [33, p. 4533]. We denote such domains by \( \Omega_\gamma \). The map \( \Phi_A : D^2_{L(B)} \to \Omega_\gamma \), given by \( (z_1, z_2) \mapsto (z_1^\gamma, z_2) \), is a proper holomorphic map which is factored by a group isomorphic to \( \mathbb{Z}_\gamma \). From Equation (5.7), we write the Bergman kernel for the domain \( \Omega_\gamma \) as follows:

\[
B_{\Omega_\gamma}(\Phi_A(z), \Phi_A(w)) = \frac{1}{\gamma^2(z_2 w_2)^\gamma} \sum_{\sigma \in \mathbb{Z}_\gamma} \det(\sigma) B_{D^2}(\sigma^{-1} \cdot z, w)
\]

\[
= \frac{1}{\gamma^2(z_2 w_2)^\gamma} \sum_{k=1}^{\gamma} \omega_\gamma^k B_{D^2}(\omega_\gamma^k z_1, \omega_\gamma^{-k} z_2, (w_1, w_2))
\]

\[
= \frac{1}{\gamma^2(z_2 w_2)^\gamma} \sum_{k=1}^{\gamma} \omega_\gamma^k \frac{1}{(1 - \omega_\gamma^k z_1 w_1)^2(1 - \omega_\gamma^{-k} z_2 w_2)^2},
\]

where \( \omega_\gamma \) is a primitive \( \gamma \)-th root of unity.
5.5. Complex Ellipsoid. A subclass of complex ellipsoids is given by
\[ \Omega_{p,q} := \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2p} + |z_2|^{2q} < 1 \} \]
for \( p = \frac{1}{q}, q = \frac{1}{p} \in \mathbb{N} \). The holomorphic map \( \Phi : \mathbb{B}_2 \to \Omega_{p,q} \) defined by \( (z_1, z_2) \mapsto (z_1^p, z_2^q) \) is proper. The map \( \Phi \) is factored by automorphisms \( G \subseteq \text{Aut}(\mathbb{B}_2) \), where \( G \) is isomorphic to the pseudoreflection group \( \mathbb{Z}_p \times \mathbb{Z}_q \). We use Corollary 4.2 to determine a formula for the Bergman kernel of \( \Omega_{p,q} \), denoted by \( \mathcal{B}_{\Omega_{p,q}} \). Therefore, the kernel \( \mathcal{B}_{\Omega_{p,q}} \) is given by
\[
\mathcal{B}_{\Omega_{p,q}}(\Phi(z_1, z_2), \Phi(w_1, w_2)) = \mathcal{B}_2(\omega_1, \omega_2) = \frac{1}{(pq)^2(z_1 w_1)^{p-1}(z_2 w_2)^{q-1}} \sum_{\alpha_1=1}^{p} \sum_{\alpha_2=1}^{q} \omega_p^{\alpha_1} \omega_q^{\alpha_2} \mathcal{B}_2(\omega)^{\alpha_1}(z_1, w_1) \omega^{\alpha_2}(z_2, w_2) \]
where \( (z_1, z_2), (w_1, w_2) \in \mathbb{B}_2 \), and \( \omega_k \) denotes a primitive \( k \)-th root of unity.

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