Genuinely entangled subspaces of maximal dimensions (and many smaller ones) cannot be constructed from orthogonal unextendible product bases

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We consider the open problem of whether there exist genuinely entangled subspaces complementary to orthogonal unextendible product bases. We solve this problem in the negative for many cardinalities of the latter in any multiparticle scenario, including the minimal ones corresponding to genuinely entangled subspaces of the maximal dimensions in all cases. The proof is elementary and only uses the necessary and sufficient condition of [Bennett et al., PRL 82, 5383 (1999)] for a set of product states to be unextendible.

I. INTRODUCTION

Unextendible product bases (UPBs), that is incomplete sets of orthogonal product vectors, such that no other product vector orthogonal to their span exists, is a very important notion of quantum information theory [1, 2]. While being significant from a purely mathematical standpoint, UPBs are also related to several problems with more practical implications. For example, they provide examples of sets of product vectors, which cannot be distinguished with local operations and classical communication (LOCC), so-called nonlocality without entanglement phenomenon [3]. Also, by the very definition, they give rise to constructions of completely unextendible subspaces (CESs) [4–6], that is subspaces void of product vectors. Importantly, mixed states supported on such subspaces, which are necessarily entangled, have positive partial transposition (PPT) across any partition [1] and thus are indistillable, or, in other words, bound entangled (BE) [7, 8].

Much effort has thus been put into finding and characterizing UPBs in different bipartite and multipartite setups (a largely incomplete list of works includes, e.g., [9–15]). However, all those attempts did not care about the type of entanglement of entangled states in the orthocomplement of a UPB. In [16] we have thus posed a problem whether it is possible to construct multipartite genuinely unextendible product bases (GUPBs), that is UPBs which are unextendible in a stronger sense – even with a biproduct vector (i.e., a vector product across a bipartition). This amounts to asking about a construction of genuinely entangled subspaces (GESs) – subspaces composed only of genuinely multipartly entangled (GME) states – from UPBs. While there has been progress in constructing GESs [17–22], with a very recent result solving the problem in full generality [23], the problem of deciding (non)existence of GUPBs has remained basically unexplored (albeit see [24]) and open.

We address it in the present communication and show that for many cardinalities of UPBs, including the minimal permissible ones, they are always extendible with biproduct vectors, i.e., GUPBs do not exist in those cases. In turn, GESs with many dimensionalities, including the maximal ones, cannot be constructed from UPBs. Our result has two important immediate implications: it rules out the possibility of constructing PPT GME BE states with certain ranks from UPBs and shows that in many cases strongly nonlocal sets of product vectors cannot be built from GUPBs (cf. [25–28]).

II. PRELIMINARIES

We provide here a terse summary of the necessary notions and results.

We consider n-partite Hilbert spaces with equal local dimensions $d$, $\mathcal{H} = (\mathbb{C}^d)^\otimes n$, and assume $n \geq 3$ and $d \geq 3$ as these cases only are relevant. Local subsystems are denoted $A_i$ and the whole system as $A$. A state $|\psi\rangle_A \in \mathcal{H}$ is called fully product, or, simply, product, if it is a product of single-party states, $|\psi\rangle_A = |\varphi\rangle_{A_1} \otimes \cdots \otimes |\xi\rangle_{A_n}$. If it is not the case, a state is said to be entangled. Some entangled states are biproduct, that is they are product across at least one of the bipartitions of the parties $S|\bar{S}$, $S \cup \bar{S} = A$, i.e., $|\psi\rangle_A = |\zeta\rangle_S \otimes |\xi\rangle_{\bar{S}}$. If an entangled state is not biproduct we say that it is genuinely multipartly entangled (GME).

Subspaces composed only of entangled vectors are called completely entangled (CES) [4–6], among those there are subspaces only with GME vectors and they are named genuinely entangled (GES) [16–19, 21–23, 29, 30]. The maximal dimension of a GES is $(d^{n-1} - 1)(d - 1)$.

An unextendible product basis (UPB) is an incomplete set of product vectors with the property that there does not exist a product vector orthogonal to all of them [1]. Obviously, a UPB defines a CES in its orthocomplement. The simplest example of a UPB is the following four-element set of vectors from $(\mathbb{C}^2)^\otimes 3$: $S = \{|0\rangle|0\rangle|0\rangle, |1\rangle|+\rangle|-, |-, |1\rangle|+\rangle|+\rangle\}$, $|\pm\rangle = |0\rangle \pm |1\rangle$. This UPB is known under the name Shifts.

In the present paper, we ask about (non)existence of UPBs, which are not extendible in a stronger sense, namely, there do
not exist a biproduct vector orthogonal to their elements ($S$ is clearly not such a set). Such UPBs must also be bipartite UPBs across any cut of the parties [16]. We propose the following

**Definition 1.** A multipartite UPB, which is a UPB for any bipartition is called a genuinely unextendible product basis (GUPB).

Clearly, the orthocomplement of the span of a GUPB is a GES.

It is known, that UPBs do not exists in systems $\mathbb{C}^2 \otimes \mathbb{C}^m$, which implies that GUPBs do not exist when at least one of the subsystems is a qubit, which is why in what follows we assume $d \geq 3$. It is also well established that UPBs do not exist with arbitrary cardinalities. The following summarizes what is currently known regarding their sizes [9, 10, 12, 31, 32].

**Fact 2.** (i) The theoretical minimum number of elements in a UPB in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ ($d_1, d_2 \geq 3$) is

(a) $d_1 + d_2$, when $d_1, d_2 \geq 4$ are even,

(b) $d_1 + d_2 - 1$, in the remaining cases.

(ii) The achievable upper bound on the cardinality of a UPB is $d_1d_2 - 4$.

By considering all bipartite cuts of an $n$-partite system with $d$-dimensional subsystems, it follows that the minimal permissible cardinality of a GUPB is $d^{n-1} + d - 1$ when $d$ is odd and $d^{n-1} + d$ for even $d$. This already implies that maximal GESs cannot be constructed from UPBs for even $d$ and arbitrary $n –$ the maximal allowed dimensions of GESs and the minimum cardinalities of UPBs simply do not match. Moreover, GESs of dimensions one, two, and three cannot be constructed in this way either. We will be further interested in eliminating cardinalities, which are theoretically possible by Fact 2.

Our proof is based on the necessary and sufficient condition for extendibility of a set of product vectors due to Bennett et al. [1, 2]. We recall it here in a bipartite case since, as noted above, this is sufficient for our purpose.

**Fact 3.** Let $B = \{|\psi_i\rangle \otimes |\psi_i\rangle\}_{i=1}^m$, $k \geq m + n - 1$, be a set of product vectors from $\mathbb{C}^m \otimes \mathbb{C}^n$. There exists a product vector orthogonal to all elements of $B$, i.e., $B$ is extendible, if and only if there exist a partition of $B$ into disjoint sets $B_1$ and $B_2$, such that $|\psi_i\rangle$’s of $B_1$ do not span $\mathbb{C}^m$ and $|\psi_i\rangle$’s of $B_2$ do not span $\mathbb{C}^n$.

**III. MAIN RESULT**

Let us now prove the main result of the present communication, which states that in any multipartite scenario there exist cardinalities, apart from those excluded by Fact 2, not possible for GUPBs, in particular the minimal ones.

**Proposition 4.** There do not exist genuinely unextendible product bases (GUPBs) with cardinalities $k$ satisfying

$$k \leq d^{n-1} + \left\lfloor \frac{d^{n-1} - 2}{n - 1} \right\rfloor. \quad (1)$$

In particular, all UPBs with cardinalities $d^{n-1} + d - 1$ and $d^{n-1} + d$ are not GUPBs. In consequence, genuinely entangled subspaces (GESs) of maximal theoretically possible with the approach dimensions cannot be constructed from GUPBs.

**Proof.** Let $B = \{|v_i\rangle_A\}_{i=1}^k$, $|v_i\rangle_A = \bigotimes_{m=1}^n |u_m^{(i)}\rangle_{A_m}$, be a set of $k$ mutually orthogonal product vectors. Since orthogonality of the vectors from $B$ stems from orthogonality of local vectors $|u_m^{(i)}\rangle$ on different sites, for any $|v_i\rangle$, by the pigeonhole principle, there exist at least

$$s := \left\lceil \frac{k - 1}{n} \right\rceil \quad (2)$$

vectors orthogonal to this vector on one of the sites. Consider for simplicity one of the vectors, say $|v_1\rangle$, let further the said orthogonal to it vectors be $B_1 = \{|v_2\rangle, \ldots, |v_s + 1\rangle\}$ and the site $- A_1$ (see Fig. 1). It follows that the vectors from $B_1$ do not span locally on $A_1$ the whole Hilbert space, i.e., $\dim \text{span} \{|u_1^{(2)}\rangle, \ldots, |u_1^{s+1}\rangle\} < d$. Now, if the remaining vectors $B_2 = \{|v_1\rangle, |v_{s+2}\rangle, \ldots, |v_k\rangle\}$ do not span locally on $A_2 \cdots A_n$, the whole Hilbert space, then, by Fact 3, there exist a biproduct vector orthogonal to all $|v_i\rangle$’s from $B$, which is given as $|u_1^{(1)}\rangle \otimes |\xi\rangle$ with an $(n - 1)$-partite vector $|\xi\rangle \perp \text{span} B_2^{A \setminus A_1}$, where $B_2^{A \setminus A_1}$ is the set of local vectors on $A \setminus A_1$ of the set $B_2$. A sufficient condition for the local rank deficiency of $B_2$ on $A_2 \cdots A_n$ is simply that the number of states is smaller than the dimension of the local Hilbert space

$$k - s = k - \left\lceil \frac{k - 1}{n} \right\rceil \leq d^{n-1} - 1 := w. \quad (3)$$

The function $f(k) = k - \left\lceil \frac{k - 1}{n} \right\rceil$ is non-decreasing in $k$ if $\left\lceil \frac{k - 1}{n} \right\rceil$ is an integer, then $f(k + 1) = f(k)$ and thus we look for the largest $k$ satisfying (3). With this aim consider the equation

$$k - s = k - \frac{k - 1}{n} \leq d^{n-1} - 1 := w. \quad (4)$$

It holds

$$\frac{k - 1}{n} \leq s \leq \frac{k - 1}{n} + \frac{n - 1}{n}. \quad (5)$$

Plugging $k$ from (4) in the above, we obtain

$$\frac{w - 1}{n - 1} \leq s \leq \frac{w - 1}{n - 1} + 1. \quad (6)$$

It follows that the optimal value is $s = \left\lfloor \frac{w - 1}{n - 1} \right\rfloor + 1$. Inserting this into (4) we obtain the value of the largest $k$ for which (3) is satisfied

$$w + \left\lfloor \frac{w - 1}{n - 1} \right\rfloor + 1, \quad (7)$$
and in turn the claimed condition
\[ k \leq d^{n-1} + \left\lfloor \frac{d^{n-1} - 2}{n-1} \right\rfloor. \] (8)

It remains to show that it is nontrivial for any pair \((n, d)\). To this purpose it suffices to prove that
\[ d^{n-1} + d \leq d^{n-1} + \left\lfloor \frac{d^{n-1} - 2}{n-1} \right\rfloor, \] (9)
that is
\[ \left\lfloor \frac{d^{n-1} - 2}{n-1} \right\rfloor \geq d. \] (10)
One can consider instead a stronger inequality, namely
\[ \frac{d^{n-1} - 2}{n-1} - \frac{n-2}{n-1} \geq d, \] (11)
which quite obviously is true.

This proves the claim that GUPBs with the minimal permissible cardinalities do not exist.

![Multiparticle UPB](image)

**FIG. 1.** A multiparticle UPB (or any set of product vectors) can be represented as a complete graph with colored edges with colors corresponding to sites on which two vectors are mutually orthogonal. While the proof of Proposition 4 does not exploit the graph representation explicitly, it is a useful aid in visualizing it easily. The figure shows a graph for an 11-element set of mutually orthogonal product vectors in the three qutrit case. It follows that vector \(v_1\) must be orthogonal to at least four vectors on one of the sites, say \(A_1\) (red edges; all the remaining irrelevant edges are pale gray). This implies that vectors \(v_2, v_3, v_4, v_5\) (shaded green area) span locally on \(A_1\) at most two-dimensional subspace. There remains only seven vectors (shaded gray area), which means that they do not span locally on \(A_2A_3\) the whole nine-dimensional Hilbert space. In turn, by Fact 3, there exists a biproduct vector orthogonal to \(v_1\)'s. Since 11 is the minimal cardinality of a GUPB in this case, our result shows that the maximal GEBS cannot be constructed in this manner. The same argument goes through for an 12-element set, while it fails for 13 elements, as there will be nine vectors in the shaded gray area.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
n & 3 & 4 & 5 & 6 \\
\hline
3 & (11, 12) & (20, 23) & (29, 36) & (42, 53) \\
4 & (29, 35) & (68, 84) & (129, 166) & (222, 287) \\
5 & (83, 100) & (260, 319) & (629, 780) & (1302, 1619) \\
\hline
\end{array}
\]

**TABLE I.** Cardinalities of GUPBs excluded by Proposition 4. The left endpoints of all intervals are minimal permissible cardinalities of GUPBs: \(d^{n-1} + d - 1\) for odd \(d\) and \(d^{n-1} + d\) for even \(d\).

Table I shows which cardinalities apart from the minimal theoretical ones are excluded by Proposition 4 for different values of \(d\) and \(n\).

If any of the vectors in the set is orthogonal to \(s + m\) vectors on one of the sites (we are not aware of a property of UPBs ruling out this possibility), then the excluded maximal dimension gets larger exactly by \(m\). This eliminates certain orthogonality graphs for GUPBs.

Notice that the proof does not even assume that the set is a UPB, it merely refers to a set of fully product orthogonal vectors. It may be interpreted as an indication of how strong of a property is biproduct extendibility. It will be interesting to see in the future if the gap for all cardinalities can be closed (if GUPBs do not exist) without referring to UPBs. Invoking properties of UPBs, on the other hand, would make things much harder.

The reasoning may be applied to heterogeneous systems, i.e., those with different local dimensions, but we omit the derivation here.

We conclude this section with a few observations. First, if GUPBs existed the well-known method of constructing new UPBs from old ones by adding flags [33] would not work in their case. More precisely, suppose we have \(d\) UPBs in \((C^d)\otimes^n\), \(B_i = \{|i\rangle|\psi_j^{(i)}\rangle\}_{j=1}^k\), then the set \(B = \{|i\rangle\psi_j^{(i)}\rangle, i = 0, 1, \ldots, m - 1, j = 1, 2, \ldots, k_i\}\) is a UPB in \((C^d)\otimes^{(n+1)}\). Clearly, this can never be true for GUPBs as there always exists a biproduct vector orthogonal to \(B\), e.g., \(|0\rangle\otimes|B_i^+\rangle\), where \(|B_i^+\rangle\) is any vector from the orthocomplement of \(B_i\). On the other hand, it was shown that tensoring would work and GUPBs could be constructed from GUPBs in this manner [24].

Moreover, nonexistence of GUPBs in systems \((C^d)\otimes^n\) implies nonexistence of GUPBs with the same number of elements but in larger \((N\text{-partite with } N > n)\) systems with local dimensions such that their products in groups equal \(d\). For example, if there is no GUPB with \(k\) elements in \(H_{9,3} := (C^9)^\otimes^3\) then there is no GUPB with cardinality \(k\) in \(H_{3,6} := (C^3)^\otimes^6 = (C^3\otimes C^3)^\otimes^3\). Unfortunately, this fact in combination with our result cannot be used to further eliminate certain cardinalities for GUPBs. This is because the largest cardinality of a GUPB forbidden by Proposition 4 in the less partite system is smaller.
than the minimal size of a GUPB in the larger system. For instance, the largest $k$ for $\mathcal{H}_{9,3}$ is 120, while the minimal size of a GUPB in $\mathcal{H}_{3,6}$ is 245.

\section*{IV. CONCLUSIONS}

We have shown that for certain cardinalities, including the minimal ones, there do not exist multipartite unextendible product bases (UPBs), which are at the same time unextendible by biproduct vectors – we called them genuinely unextendible product bases (GUPBs). This shows that a construction of genuinely entangled subspaces (GESs) with corresponding dimensions from such bases is ruled out. This partially solves the open problem posed in [16]. The proof is elementary and only uses properties of sets of product vectors, which drives us to conjecture that GUPBs do not exist at all. We hope our result will stimulate research towards (dis)proving this conjecture.

As to the implications of the result, two main come to mind: first, it implies that genuinely multiparty entangled bound entangled states with positive partial transposition of certain ranks cannot be constructed from UPBs, second, it shows that strongly nonlocal UPBs unextendible by biproducts do not exist with all cardinalities (if at all).

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