THE LOOP-STABLE HOMOTOPY CATEGORY OF ALGEBRAS

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Abstract. Let $\ell$ be a commutative ring with unit. Garkusha constructed a functor from the category of $\ell$-algebras into a triangulated category $D$, that is a universal excisive and homotopy invariant homology theory. Later on, he provided different descriptions of $D$, as an application of his motivic homotopy theory of algebras. Using these, it can be shown that $D$ is triangulated equivalent to a category, denote it by $\mathcal{R}$, whose objects are pairs $(A, m)$ with $A$ an $\ell$-algebra and $m$ an integer, and whose Hom-sets can be described in terms of homotopy classes of morphisms. All these computations, however, require a heavy machinery of homotopy theory. In this paper, we give a more explicit construction of the triangulated category $\mathcal{R}$ and prove its universal property, avoiding the homotopy-theoretic methods and using instead the ones developed by Cortiñas-Thom for defining $kk$-theory. Moreover, we give a new description of the composition law in $\mathcal{R}$, mimicking the one in the suspension-stable homotopy category of bornological algebras defined by Cuntz-Meyer-Rosenberg. We also prove that the triangulated structure in $\mathcal{R}$ can be defined using either extension or mapping path triangles.

1. Introduction

Throughout this text, $\ell$ is a commutative ring with unit and $\text{Alg}_\ell$ stands for the category of associative and not necessarily unital $\ell$-algebras and $\ell$-algebra homomorphisms.

Algebraic $kk$-theory is a bivariant homology theory on $\text{Alg}_\ell$, defined by Cortiñas-Thom in [1] as a completely algebraic analogue of Kasparov’s $KK$-theory. It consists of a triangulated category $\text{kk}$ endowed with a functor $j : \text{Alg}_\ell \to \text{kk}$ that is excisive (E), polynomial homotopy invariant (H) and $M_\infty$-stable (M). The excision property (E) means that every short exact sequence in $\text{Alg}_\ell$ that splits as a sequence of $\ell$-modules gives rise to a distinguished triangle upon applying $j$. The property (M) is a kind of matrix-stability. The functor $j$ is moreover universal with the above properties: any other functor from $\text{Alg}_\ell$ into a triangulated category satisfying (E), (H) and (M) factors uniquely through $j$. A remarkable property of $kk$-theory is that it recovers Weibel’s homotopy $K$-theory [1] Theorem 8.2.1]. The work of Cortiñas-Thom was motivated by the work of Cuntz on the bivariant $K$-theory of locally convex algebras [2]. In particular, the Hom-sets and the composition law in $kk$ were defined explicitly in terms of polynomial homotopy classes of $\ell$-algebra homomorphisms.

With completely different methods, Garkusha constructed in [4] various universal bivariant homology theories of algebras. Starting with an admissable category of $\ell$-algebras $\mathcal{R}$ and a class $\mathcal{F}$ of fibrations on it, he inverted certain weak equivalences to obtain a triangulated category $D(\mathcal{R}, \mathcal{F})$. The latter is endowed with a functor $\mathcal{R} \to D(\mathcal{R}, \mathcal{F})$ that is $\mathcal{R}$-excisive, polynomial homotopy invariant and universal with these properties [4] Theorem 2.6]. At this stage, Garkusha did not consider any matrix-stability, motivated by the fact that some interesting admissible categories —such as the one of commutative algebras—

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are not closed under matrices. He showed, moreover, that different matrix-stabilities can be added later on. In particular, he gave a new construction of $kk$-theory, taking $\mathfrak{R}$ to be the class $\mathfrak{R}_{\text{spl}}$ of $\ell$-split surjections and adding $M_n$-stability.

This paper is concerned with the universal excisive and homotopy invariant homology theory of $\ell$-algebras $-D(\text{Alg}_\ell, \mathfrak{R}_{\text{spl}})$ in the notation of [4]. Garkusha provided different descriptions of $D(\text{Alg}_\ell, \mathfrak{R}_{\text{spl}})$ as an application of his motivic homotopy theory of algebras; we proceed to briefly recall some details of this. Garkusha showed in [5] that the universal homology theories defined in [4] are represented by spectra. In particular, for every pair of algebras $(A, B)$, he explicitly constructed an $\Omega$-spectrum $\mathcal{B}(A, B)$ such that

$$\text{Hom}_{D(\text{Alg}_\ell, \mathfrak{R}_{\text{spl}})}(A, \Omega^n B) \cong \pi_n \mathcal{B}(A, B)$$

for all $n \in \mathbb{Z}$ [5] Comparison Theorem B]; see also [6] Theorem 4.4 and Corollary 4.3]. The homotopy groups of $\mathcal{B}(A, B)$ can be computed as follows; see [5] Corollary 7.1 and [11] Theorem 4.11:

$$\pi_n \mathcal{B}(A, B) \cong \underset{\text{colim}}{\text{colim}} [J^v A, B^{\mathbb{Z}_{>0}}]$$

Here, the square brackets stand for homotopy classes, $JA$ is the kernel of the universal extension of $A$ (see Section 2.20), and $B^{\mathbb{Z}_v}$ is the ind-algebra of polynomial functions on the $N$-dimensional cube that vanish at the boundary of the cube (see Example 2.6 and Section 2.10). It is easily deduced from the above that $D(\text{Alg}_\ell, \mathfrak{R}_{\text{spl}})$ is equivalent to a category, denote it by $\mathfrak{R}$, whose objects are pairs $((A, m))_{A \in \text{Alg}_\ell, m \in \mathbb{Z}}$ and whose morphisms are defined by:

$$\text{Hom}_{\mathfrak{R}}((A, m), (B, n)) = \underset{\text{colim}}{\text{colim}} [J^{m+n} A, B^{\mathbb{Z}_{>0}}]$$

(1)

We call $\mathfrak{R}$ the loop-stable homotopy category of algebras. We emphasize that the cited computations in [5] and [6] all require a heavy machinery of homotopy theory. In this paper, we avoid this kind of methods and use instead the ones developed by Cortiñas-Thom in [11] to give a more explicit construction of $\mathfrak{R}$ and prove its universal property. In particular, we provide a new explicit formula for the composition law in $\mathfrak{R}$ in terms of (1). The main results of the paper can be summarized as follows.

**Theorem 1.1** (Theorem 4.8 Proposition 9.12 and Theorem 10.12). Let $\mathfrak{R}$ be the category whose objects are pairs $(A, m)$ with $A$ an $\ell$-algebra and $m$ an integer, whose morphisms are defined by (1) and whose composition law is induced by the operation $\ast$ of Definition 3.4. Then $\mathfrak{R}$ admits a triangulated structure such that the obvious functor $j : \mathfrak{R} \to \text{Alg}_\ell$, $j(A) = (A, 0)$, is a universal excisive and homotopy invariant homology theory. Moreover, this triangulated structure can be defined using either extension or mapping path triangles.

If we regard $kk$-theory as the algebraic counterpart of Kasparov’s $K$-theory, then we should think of $\mathfrak{R}$ as the algebraic counterpart of the category $\Sigma \text{Ho}$ defined by Cuntz-Meyer-Rosenberg in the context of bornological algebras [3] Section 6.3]. Notice that the Hom-sets in $\mathfrak{R}$ (1) and the Hom-sets in $\Sigma \text{Ho}$ are defined in a similar fashion. However, two technical issues arise when we want to mimic in $\mathfrak{R}$ the definition of the composition law in $\Sigma \text{Ho}$. The first one is how to make sense of a sign that appears when permuting coordinates. This is easily handled, since $B^{\mathbb{Z}_v}$ is a group object in the category of ind-algebras if $\ell \geq 1$ and this group is abelian if $\ell \geq 2$ [11] Theorem 3.10]. The second difficulty is that $(B^{\mathbb{Z}_v})^{\mathbb{Z}_v} \neq B^{\mathbb{Z}_{v \cdot n}}$ in $\text{Alg}_\ell$ due to the failure of the exponential law [4]. Remark 3.1.4]. To deal with this, we use certain morphisms $\mu_{M,N} : (B^{\mathbb{Z}_v})^{\mathbb{Z}_v} \to B^{\mathbb{Z}_{v \cdot n}}$ that induce isomorphisms in any excisive and homotopy invariant homology theory [11] Section 3.1]. Modulo these technicalities, we can define a composition law in $\mathfrak{R}$ essentially as in
natural transformation

The ind-category.

2.1.3. \( f \) 

path triangles (Proposition 9.12). In section 10 we prove that the functor

The symbol \( \otimes \) indicates tensor product over \( \mathbb{Z} \). If \( C \) is a category and \( X, Y \) are two of its objects, we may write \( C(X, Y) \) instead of \( \text{Hom}_C(X, Y) \).

2. Preliminaries

We regard simplicial \( \ell \)-algebras as simplicial sets via the forgetful functor \( \text{Alg}_\ell \to \text{Set} \). The symbol \( \otimes \) indicates tensor product over \( \mathbb{Z} \). If \( C \) is a category and \( X, Y \) are two of its objects, we may write \( C(X, Y) \) instead of \( \text{Hom}_C(X, Y) \).

2.1. Directed diagrams. Let \( C \) be a category. A directed diagram in \( C \) is a functor \( X : I \to C \), where \( I \) is a filtering poset; we often write \( (X, I) \) or \( X_* \) for such a functor. We shall consider different notions of morphisms between directed diagrams:

2.1.1. Fixing the filtering poset. Let \( I \) be a filtering poset. We write \( C^I \) for the category of functors \( X : I \to C \) with natural transformations as morphisms.

2.1.2. Varying the filtering poset. We write \( C^< \) for the category whose objects are the directed diagrams in \( C \) and whose morphisms are defined as follows. Let \( (X, I) \) and \( (Y, J) \) be two directed diagrams. A morphism from \( (X, I) \) to \( (Y, J) \) consists of a pair \( (f, \theta) \) where \( \theta : I \to J \) is a functor and \( f : X \to Y \circ \theta \) is a natural transformation.

For fixed \( I \), there is a functor \( \text{id}_I : C^I \to C^< \) that is the identity on objects and sends a natural transformation \( f \) to the morphism \( (f, \text{id}_I) \).

2.1.3. The ind-category. We write \( C^{\text{ind}} \) for the category whose objects are the directed diagrams in \( C \) and whose hom-sets are defined by:

\[
\text{Hom}_{C^{\text{ind}}}((X, I), (Y, J)) := \lim_{\substack{\rightarrow \atop i \in I}} \text{colim}_{j \in J} \text{Hom}_C(X_i, Y_j)
\]

There is a functor \( C^< \to C^{\text{ind}} \) that acts as the identity on objects and that sends a morphism \( (f, \theta) : (X, I) \to (Y, J) \) to the morphism:

\[
\{f_i : X_i \to Y_{\theta(i)}\}_{i \in I} \in \lim_{\substack{\rightarrow \atop i \in I}} \text{colim}_{j \in J} \text{Hom}_C(X_i, Y_j)
\]

Lemma 2.2. Let \( I \) be a filtering poset and let \( C \) be a category. Then there exist functors \( b_I : (C^I)^< \to C^< \) and \( c_I : (C^I)^{\text{ind}} \to C^{\text{ind}} \) such that, for every filtering poset \( J \), the following
diagram commutes:

\[
\begin{array}{ccc}
(\mathcal{C}^I)^I & \xrightarrow{\alpha_j} & (\mathcal{C}^I)^I \\
\cong & & \cong \\
\mathcal{C}I & \xrightarrow{b_I} & \mathcal{C}I
\end{array}
\] (3)

**Proof.** If \( X : J \to \mathcal{C}^I \) is a directed diagram in \( \mathcal{C}^I \), define both \( b_I(X) \) and \( c_I(X) \) to be the functor \( I \times J \to \mathcal{C} \) obtained by the exponential law. If \( (f, \theta) : (X, J) \to (Y, K) \) is a morphism in \( (\mathcal{C}^I)^I \), then \( f : b_I(X) \to b_I(Y) \circ (\text{id}_J \times \theta) \) is a natural transformation of functors \( I \times J \to \mathcal{C} \); this defines \( b_I \) on morphisms. Let us define \( c_I \) on morphisms. If \( X : J \to \mathcal{C}^I \) and \( Y : K \to \mathcal{C}^I \) are directed diagrams in \( \mathcal{C}^I \), we should define a function:

\[
\text{Hom}_{\mathcal{C}^I}(X, Y) \longrightarrow \text{Hom}_{\mathcal{G}^\text{ind}}(c_I(X), c_I(Y)) = \lim_{(i,j) \in I \times J} \text{Hom}_{\mathcal{G}^\text{ind}}(X_I(i), c_I(Y))
\]

This is equivalent to defining compatible functions:

\[
\pi_{(i,j)} : \text{Hom}_{\mathcal{G}^\text{ind}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{G}^\text{ind}}(X_I(i), c_I(Y))
\]

Fix \( (i, j) \in I \times J \). We let \( \pi_{(i,j)} \) be the following composite, that we proceed to explain:

\[
\text{Hom}_{\mathcal{G}^\text{ind}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{G}^\text{ind}}(X_I(i), Y_j(i)) \longrightarrow \text{Hom}_{\mathcal{G}^\text{ind}}(X_I(i), c_I(Y))
\]

The function on the left is induced by the functor \( \mathcal{G}^I \to \mathcal{G} \) that evaluates a diagram at the object \( i \in I \). The function in the middle is precomposition with the obvious morphism \( X_I(i) \to Y_j(i) \) in \( \mathcal{G}^\text{ind} \). The function on the right is composition with the obvious morphism \( Y_j(i) \to c_I(Y) \) in \( \mathcal{G}^\text{ind} \). It is straightforward but tedious to verify that the definitions above determine functors \( b_I \) and \( c_I \) making the diagram (3) commute. \( \square \)

### 2.3. Simplicial enrichment of algebras

Let \( \mathcal{S} \) be the category of simplicial sets. We briefly recall the simplicial enrichment of \( \text{Alg}_\ell \), introduced by Cortiñas-Thom in [1] Section 3. Let \( p \geq 0 \) and let \( \mathbb{Z}^p \) := \( \mathbb{Z} \{[0, \ldots, 1] \} / (1 - \sum t_i) \). We shall think of \( \mathbb{Z}^p \) as the ring of polynomial functions on the \( p \)-dimensional simplex. A non-decreasing function \( \varphi : \{p\} \to \{q\} \) induces a ring homomorphism \( \mathbb{Z}^p \to \mathbb{Z}^q \) by the formula:

\[
t_i \mapsto \sum_{\varphi(j) = i} t_j
\]

This defines a simplicial ring \( \mathbb{Z}^A \). For \( B \in \text{Alg}_\ell \), put \( B^A := \mathbb{Z}^A \otimes B \); this is a simplicial \( \ell \)-algebra. For \( X \in \mathcal{S} \), put \( B^X := \text{Hom}_{\mathcal{S}}(X, B^A) \). It is easily verified that \( B^X \) is an \( \ell \)-algebra with the pointwise operations; we shall think of it as the \( \ell \)-algebra of polynomial functions on \( X \) with coefficients in \( B \).

**Remark 2.4.** In general \( (B^X)^Y \neq B^{X \times Y} \) — this already fails when \( X \) and \( Y \) are standard simplices; see [1] Remark 3.1.4).

Let \( A \) and \( B \) be two \( \ell \)-algebras. The simplicial mapping space from \( A \) to \( B \) is the simplicial set \( \text{Hom}_{\text{Alg}_\ell}(A, B^A) \). For \( A, B \in \text{Alg}_\ell \) and \( X \in \mathcal{S} \) we have an adjunction isomorphism:

\[
\text{Hom}_{\mathcal{S}}(X, \text{Hom}_{\text{Alg}_\ell}(A, B^A)) \cong \text{Hom}_{\text{Alg}_\ell}(A, B^X)
\]
2.5. Simplicial pairs. A simplicial pair is a pair \((K, L)\) where \(K\) is a simplicial set and \(L \subseteq K\) is a simplicial subset. A morphism of pairs from \((K', L')\) to \((K, L)\) is a morphism of simplicial sets \(f : K' \to K\) such that \(f(L') \subseteq L\). A pair \((K, L)\) is finite if \(K\) is a finite simplicial set.

**Example 2.6.** Let \(I := \Delta^1\) and \(\partial I := \{0, 1\} \subseteq I\). For \(n \geq 1\), let \(I^n := I \times \cdots \times I\) be the \(n\)-fold direct product and let \(\partial I^n\) be the boundary of \(I^n\):

\[
\partial I^n := [(\partial I) \times I \times \cdots \times I] \cup [I \times (\partial I) \times \cdots \times I] \cup \cdots \cup [I \times \cdots \times I \times (\partial I)]
\]

Put \(I^0 := \Delta^0\) and \(\partial I^0 := \emptyset\). We will write \(\mathcal{E}_n\) for the simplicial pair \((I^n, \partial I^n)\).

**Example 2.7.** We shall think of a simplicial set \(K\) as a simplicial set. It equals the cartesian product when restricted to simplicial sets.

**Definition 2.8.** Let \((K, L)\) and \((K', L')\) be simplicial pairs. The box product of \((K, L)\) with \((K', L')\) is defined as the simplicial pair:

\[
(K, L) \boxtimes (K', L') := (K \times K', (K \times L') \cup (L \times K'))
\]

It is easily verified that \(\boxtimes\) is a symmetric monoidal operation with unit \(\Delta^0\) on (finite) simplicial pairs. It equals the cartesian product when restricted to simplicial sets.

**Example 2.9.** For \(m, n \geq 0\), we have \(\mathcal{E}_m \boxtimes \mathcal{E}_n = \mathcal{E}_{m+n}\).

In the rest of the paper we will only consider finite simplicial pairs, omitting the word ‘finite’ from now on.

2.10. Functions vanishing on a subset. Let \(sd : \mathcal{S} \to \mathcal{S}\) be the subdivision functor; see [8, Section III.4]. We have a natural transformation \(\gamma : sd \to \text{id}_\mathcal{S}\) called the last vertex map. For a simplicial pair \((K, L)\), an \(\ell\)-algebra \(B\) and \(r \geq 0\), put:

\[
B^{(K, L)}_r := \ker\left( B^{\partial K} \longrightarrow B^{\partial L} \right)
\]

The last vertex map induces morphisms \(B^{(K, L)}_r \to B^{(K, L)}_{r+1}\) and we usually regard \(B^{(K, L)}_\ast\) as a directed diagram in \(\text{Alg}_\ell\):

\[
B^{(K, L)}_\ast : B^{(K, L)}_0 \longrightarrow B^{(K, L)}_1 \longrightarrow B^{(K, L)}_2 \longrightarrow \cdots
\]

**Example 2.11.** \(B^{(\mathcal{S})}_\ast = B^{\mathcal{A}_0}_\ast\) is the constant \(\mathbb{Z}_{\geq 0}\)-diagram \(B\).

**Lemma 2.12** ([11 Lemma 2.10]; cf. [1, Proposition 3.1.3]). Let \((K, L)\) be a (finite) simplicial pair, let \(B\) be an \(\ell\)-algebra and let \(r \geq 0\). Then \(\mathbb{Z}^{(K, L)}_\ast\) is a free abelian group and there is a natural \(\ell\)-algebra isomorphism \(B \otimes \mathbb{Z}^{(K, L)}_\ast \cong B^{(K, L)}_\ast\).

2.13. Multiplication morphisms. Let \((K, L)\) and \((K', L')\) be simplicial pairs, let \(B \in \text{Alg}_\ell\) and let \(r, s \geq 0\). Recall from [11, Section 3.1] that the multiplication in the simplicial commutative ring \(\mathbb{Z}^\ast\) induces an \(\ell\)-algebra homomorphism

\[
\mu^\ast_B : (B^{(K, L)}_r)^{(K', L')} \longrightarrow B^{(K, L) \boxtimes (K', L')}_{r+s}
\]

such that the following statements hold:

(1) The morphisms \(\mu^\ast_B\) are covariant in \(B\) and contravariant in \((K, L)\) and in \((K', L')\).
2. Define $\theta : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by $\theta(r, s) = r + s$. Then, for varying $r$ and $s$, the morphisms (3) assemble into a morphism in $\text{Alg}_\ell$:

$$\left(\mu_B^{(K, L), (K', L')} : (B_r^{(K, L)})_s \to B_r^{(K', L')}\right) \in \text{Alg}_\ell$$

(3) The morphisms (4) are associative in the obvious way.

(4) The composite

$$B_r^{(K, L)} \cong (B_r^{(K, L)})_s \xleftarrow{\mu_r^{K,L,S^0}} B_r^{(K, L) \sqcup S^0} \cong B_r^{(K, L)}$$

is the transition morphism of $B_r^{(K, L)}$. An analogous statement holds for $\mu_B^{0, (K, L)}$.

To alleviate notation when there is no risk of confusion, we often write $\mu^{(K, L), (K', L')}$ or even just $\mu$ instead of $\mu^{(K, L), (K', L')}$. We also write $\mu^{n, m}$ instead of $\mu^{n, n}$.

2.14. Polynomial homotopy. Two morphisms $f_0, f_1 : A \to B$ in $\text{Alg}_\ell$ are elementary homotopic if there exists an $\ell$-algebra homomorphism $f : A \to B[i]$ such that $\text{ev}_0 \circ f = f_1$ and $\text{ev}_1 \circ f = f_0$. Here, $\text{ev}_i$ stands for the evaluation at $i$. Elementary homotopy is reflexive and symmetric, but it is not transitive. We let $\sim$ be its transitive closure and we call $f_0$ and $f_1$ homotopic if $f_0 \sim f_1$. It is easily shown that $f_0 \sim f_1$ if there exist $r \in \mathbb{N}$ and $f : A \to B_r^{K}$ such that the following diagrams commute:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B_r^{K} \\
\downarrow{f_0} & & \downarrow{d_r} \\
B & \cong & B_r^{K'}
\end{array}
$$

Homotopy is compatible with composition, and we have a category $[\text{Alg}_\ell]$ whose objects are $\ell$-algebras and whose morphisms are polynomial homotopy classes of $\ell$-algebra homomorphisms [7, Lemma 1.1]. There is an obvious functor $\text{Alg}_\ell \to [\text{Alg}_\ell]$. Now let $A_\ast$ and $B_\ast$ be two directed diagrams in $\text{Alg}_\ell$ and let $f, g : A_\ast \to B_\ast$ be two morphisms in $[\text{Alg}_\ell]^{\text{ind}}$. Following [11, Definition 3.1.1], we say that $f$ and $g$ are homotopic if they become equal upon applying the functor $[\text{Alg}_\ell]^{\text{ind}} \to [\text{Alg}_\ell]^{\text{ind}}$. We will write:

$$[A_\ast, B_\ast] := \text{Hom}_{[\text{Alg}_\ell]^{\text{ind}}}((A, I), (B, J))$$

It turns out that the homotopy groups of the simplicial mapping space between algebras can be described in terms of polynomial homotopy classes of morphisms:

**Theorem 2.15** ([11, Theorem 3.10]). For any pair of $\ell$-algebras $A$ and $B$ and any $n \geq 0$, there is a natural bijection:

$$\pi_n \text{Hom}_{\text{Alg}_\ell}(A, B)^\wedge \cong [A, B_r^{\wedge}]$$

(5)

**Remark 2.16.** Let $A$ and $B$ be two $\ell$-algebras and let $n \geq 1$. Endow the set $[A, B_r^{\wedge}]$ with the group structure for which (5) is a group isomorphism. This group structure is abelian if $n \geq 2$. Moreover, if $f : A \to A'$ and $g : B \to B'$ are morphisms in $[\text{Alg}_\ell]$, then the following functions are group homomorphisms:

$$f^* : [A', B_r^{\wedge}] \to [A, B_r^{\wedge}]$$

$$g_* : [A, B_r^{\wedge}] \to [A, (B')_r^{\wedge}]$$
Example 2.17. Let \( \omega \) be the automorphism of \( B^\otimes \) defined by \( \omega(t_0) = t_1 \) and \( \omega(t_1) = t_0 \); then \( \omega \) induces an automorphism of \( B^\otimes \). It is easily shown that if \( g : A \to B^\otimes \) is an \( \ell \)-algebra homomorphism and \( [g] \) represents its class in \([A, B^\otimes]\), then \([\omega \circ g] = [g]^{-1}\) in \([A, B^\otimes]\); see [11] Example 3.12 for details.

Example 2.18. Fix \( r \geq 1 \). Note that there is an automorphism of \( sd' I \) exchanging both endpoints of \( I \). We have a pushout square

\[
\begin{array}{ccc}
\Delta^0 & \xrightarrow{d'} & sd'I \\
\downarrow{d'} & & \downarrow{sd'I} \\
sd'I & \xrightarrow{sd' I} & sd'r+1 I
\end{array}
\]

that shows that \( sd^{r+1} I \) can be obtained by concatenating two copies of \( sd'I \); the endpoint \( I \) in the first copy gets identified with the 0 in the second one. Since \( B^\otimes : \text{Set} \to \text{Alg}_\ell \) preserves products, it follows that \( B^{sd'r+1} \cong B^{sd'I} \times B^{sd'I} \). Thus, two morphisms \( f, g : A \to B^{sd'I} \) such that \( d_0 f = d_1 g \) determine a morphism \( f \circ g : A \to B^{sd'r+1} \), that we call the concatenation of \( f \) and \( g \). In particular, we can always concatenate \( f, g : A \to B^\otimes \) to obtain \( f \circ g : A \to B^\otimes r+1 \). It is not difficult to show that the group structure in \([A, B^\otimes]\) is induced by concatenation; we will not use this fact, however.

Example 2.19. Let \( m, n \geq 1 \) and let \( c : I^m \times I^n \to I^m \times I^n \) be the commutativity isomorphism. Then \( c \) induces an isomorphism \( c^* : B_n^\otimes \to B_m^\otimes \), that in turn induces a bijection \( c^* : [A, B_n^\otimes] \to [A, B_m^\otimes] \). We claim that the latter is multiplication by \((-1)^{mn}\). Indeed, this follows from Theorem 2.15 and the well known fact that permuting two coordinates in \( \Omega^k \) induces multiplication by \((-1)^{mn}\) upon taking \( \pi_0 \).

2.20. Extensions and classifying maps. An extension is a short exact sequence

\[
\varepsilon : A \longrightarrow B \longrightarrow C
\]

of \( \ell \)-algebras that splits as a sequence of \( \ell \)-modules. A morphism of extensions is a morphism of diagrams in \( \text{Alg}_\ell \). We often write \((\varepsilon, s)\) to indicate that we consider \( \varepsilon \) together with a particular \( \ell \)-linear splitting \( s \). A strong morphism of extensions \((\varepsilon', s') \to (\varepsilon, s)\) is a morphism of extensions \((a, b, c) : \varepsilon' \to \varepsilon\) that is compatible with the specified splittings; i.e. such that the following square of modules commutes:

\[
\begin{array}{ccc}
B' & \xrightarrow{c} & C' \\
\downarrow{b} & & \downarrow{c} \\
B & \xrightarrow{a} & C
\end{array}
\]

Remark 2.21. If \( \varepsilon \) is an extension and \( G \) is a ring, then \( \varepsilon \otimes G \) is an extension. In particular, if \((K, L)\) is a simplicial pair, then \( \varepsilon_{\tau(K, L)} \) is an extension by Lemma 2.12.

Let \( \text{Mod}_\ell \) be the category of \( \ell \)-modules. The forgetful functor \( F : \text{Alg}_\ell \to \text{Mod}_\ell \) has a right adjoint \( \overline{T} \); put \( T := \overline{T} \circ F \). For an \( \ell \)-algebra \( A \), let \( \eta_A : TA \to A \) be the counit of this adjunction and put \( JA := \ker \eta_A \). The universal extension of \( A \) is the sequence

\[
\mathcal{U}_A : \xrightarrow{\eta_A} TA \xrightarrow{\eta_A} A ,
\]

split by the unit map \( \sigma_A : FA \to F\overline{T}(FA) \). We will always use \( \sigma_A \) as a splitting for \( \mathcal{U}_A \).
Proposition 2.22 (cf. [1] Proposition 4.4.1). Let (6) be an extension with splitting s and let \( f : D \to C \) be an \( \ell \)-algebra homomorphism. Then there exists a unique strong morphism of extensions \( \mathcal{U}_D \to (\mathcal{E}, s) \) extending \( f \):

\[
\begin{array}{ccc}
\mathcal{U}_D & \xrightarrow{\xi} & JD \\
\downarrow & & \downarrow \\
(\mathcal{E}, s) & \xrightarrow{f} & C
\end{array}
\]

The morphism \( \xi \) is called the classifying map of \( f \) with respect to \( (\mathcal{E}, s) \). When \( D = C \) and \( f = \text{id}_C \) we call \( \xi \) the classifying map of \( (\mathcal{E}, s) \).

Proposition 2.23 (cf. [1] Proposition 4.4.1). In the hypothesis of Proposition 2.22 the homotopy class of the classifying map \( \xi \) does not depend upon the splitting \( s \).

Thus, we may refer to the classifying map of (6) as a homotopy class \( JC \to A \) without specifying a splitting for (6).

Proposition 2.24 (cf. [1] Proposition 4.4.2). Let \( \mathcal{E}_i : A_i \to B_i \to C_i \) be an extension with classifying map \( \xi_i \) \((i = 1, 2)\). Let \( (a, b, c) : \mathcal{E}_1 \to \mathcal{E}_2 \) be a morphism of extensions. Then the following square of algebras commutes up to homotopy:

\[
\begin{array}{ccc}
JC_1 & \xrightarrow{\xi_1} & JC_2 \\
\downarrow & & \downarrow \\
A_1 & \xrightarrow{a} & A_2
\end{array}
\]

Moreover, if we consider specific splittings for the \( \mathcal{E}_i \) and \((a, b, c)\) is a strong morphism of extensions then the square above commutes on the nose.

Lemma 2.25 (cf. [1] Corollary 4.4.4). The functor \( J : \text{Alg}_\ell \to \text{Alg}_\ell \) preserves homotopy, and thus defines a functor \( J : [\text{Alg}_\ell] \to [\text{Alg}_\ell] \).

2.26. Path extensions. For \( B \in \text{Alg}_\ell \) and \( n \geq 0 \), put:

\[
P(n, B)_\bullet := B_{\mathbb{Z}_{\geq 0}[I^{(1)}]}
\]

We will write \((PB)_\bullet\) instead of \(P(0, B)_\bullet\). The inclusions

\[
\mathbb{S}_{n+1} = \mathbb{S}_n \sqcup \mathbb{S}_1 \supseteq \mathbb{S}_n \sqcup (I, \{1\}) \supseteq \mathbb{S}_n \sqcup \{0\} \equiv \mathbb{S}_n
\]

induce a sequence of \( \mathbb{Z}_{\geq 0}\)-diagrams:

\[
\begin{array}{ccc}
\mathcal{P}_{n, B} : B_{\mathbb{S}_{n+1}} & \xrightarrow{\rho} & P(n, B)_\bullet \\
\downarrow & & \downarrow \\
& B_{\mathbb{S}_{n+1}} & \xrightarrow{\rho} & P(n, B)_\bullet
\end{array}
\]

We claim that (7) is an extension. Exactness at \( P(n, B)_\bullet \) holds because the functor \( B^{\text{op}} : \mathbb{S} \to \text{Alg}_\ell\) preserves pushouts and we have:

\[
\partial I^{n+1} = [(\partial I^n \times I) \cup (I^n \times \{1\})] \cup (I^n \times \{0\})
\]

Exactness at \( B_{\mathbb{S}_{n+1}} \) follows from the fact that both \( B_{\mathbb{S}_{n+1}} \) and \( P(n, B)_\bullet \) are subalgebras of \( B^{\text{op}}(I^{n+1}) \). Consider the element \( t_0 \in \mathbb{S}^{A^1} \); \( t_0 \) is actually in \( \mathbb{S}_{(I^{(1)})} \) since \( d_0(t_0) = 0 \). It is easily verified that the composite

\[
B_{\mathbb{S}^{(I^{(1)})}} \xrightarrow{\rho(t_0)} B_{\mathbb{S}^{(I^{(1)})}} \cong (B_{\mathbb{S}^{(I^{(1)})}})_0 \xrightarrow{\mu} B_{\mathbb{S}^{(I^{(1)})}}
\]

is a splitting for (7); we will always use this as a splitting for (7).
Example 2.27. By naturality of $\mu$, there is a strong morphism of extensions:

$$\begin{array}{ccc}
\mathcal{P}_{m,B} & \xrightarrow{\mu} & \mathcal{P}(m,B) \\
B^{\overline{m}+1} & \xrightarrow{\mu} & P(n+m,B_{r+r}) \\
\mathcal{P}_{n+m,B} & \xrightarrow{\mu} & B^{\overline{m}+n+1}
\end{array}$$

Example 2.28. For $n = 0$, the extension (7) takes the form:

$$\begin{array}{c}
\mathcal{P}_{0,B} : B^{\overline{1}+1} \longrightarrow (PB), \\
\longmapsto B^{\overline{0}+0} \cong B
\end{array} \tag{8}$$

This is the loop extension of [11, Section 4.5] and we will write $\lambda_B$ for its classifying map.

Example 2.29. Define $\overline{P}(n,B)_* := B^{\bullet,\{1\}\sqcup \Xi_n}$. The inclusions

$$\Xi_{1+n} = \Xi_1 \sqcup \Xi_n \supseteq (I,\{1\}) \sqcup \Xi_n \supseteq [0] \sqcup \Xi_n$$

induce a sequence that turns out to be an extension:

$$\begin{array}{ccc}
\mathcal{P}_{n,B} : B^{\overline{1}+n} & \longrightarrow & \overline{P}(n,B) \\
\longmapsto B^{\overline{0}+n}
\end{array} \tag{9}$$

The commutativity isomorphism $c : I^n \times I \to I \times I^n$ induces an isomorphism of extensions $c^* : \mathcal{P}_{n,B} \to \mathcal{P}_{n,B}$.

Example 2.30. It will be useful to have a more explicit description of (8). We have:

$$\begin{cases}
B^{\overline{0}} = B[t_0]/(1-t_0) & \cong B \\
B^{\overline{1}} = B[t_0, t_1]/(1-t_0 - t_1) & \cong B[t], \quad t_1 \leftrightarrow t.
\end{cases} \tag{10}$$

Under these identifications, the face morphisms $B^{\overline{1}} \to B^{\overline{0}}$ coincide with the evaluations $ev_i : B[t] \to B$ for $i = 0, 1$. We have:

$$(PB)_0 = \ker \left( B^{\overline{1}} \xrightarrow{d_0} B^{\overline{0}} \right) \cong \ker \left( B[t] \xrightarrow{ev_1} B \right) = (t-1)B[t]$$

$$(PB)_1 = \ker \left( (PB)_0 \xrightarrow{d_1} B^{\overline{0}} \right) \cong \ker \left( (t-1)B[t] \xrightarrow{ev_0} B \right) = (t^2 - t)B[t]$$

Hence, the extension (8) is isomorphic to:

$$\begin{array}{c}
(r^2 - r)B[t] \xrightarrow{\text{incl}} (t-1)B[t] \xrightarrow{ev_0} B
\end{array} \tag{11}$$

The section in (8) identifies with the morphism $B \to (t-1)B[t]$, $b \mapsto b(1-t)$.

We now want a description of (8) once a subdivision has been made. Recall that $sd\Delta^1$ fits into the following pushout:

$$\begin{array}{ccc}
\Delta^0 & \xrightarrow{d^0} & \Delta^1 \\
\downarrow & & \downarrow \\
\Delta^1 & \xrightarrow{d^0} & sd\Delta^1
\end{array}$$

Since the functor $B^\bullet : \Delta_{op} \to \text{Alg}_\ell$ preserves limits, we have:

$$B^{sd\Delta^1} \cong B[t]_{ev_1} \times_{ev_1} B[t] = \{(p,q) \in B[t] \times B[t] : p(1) = q(1)\}$$

The endpoints of $sd\Delta^1$ are the images of the coface maps $d^1 : \Delta^0 \to \Delta^1$ whose codomains are each of the two copies of $\Delta^1$ contained in $sd\Delta^1$. We get:

$$(PB)_1 = \ker \left( B^{sd\Delta^1} \xrightarrow{d^1} B^{sd\Delta^1} \right) \cong B[t]_{ev_1} \times_{ev_1} tB[t]$$
\[ B_1^{\bar{\mathcal{S}}} = \ker \left( B^{d\partial^{\Delta}_1} \rightarrow B^{d(d\partial^{\Delta}_1)} \right) \cong tB[t] \times_{\ev_1} tB[t] \]

In this description of \((PB)_1\) a choice has been made, since the two endpoints of \(sd\Delta^1\) are indistinguishable. The extension \(\mathcal{S}\) is isomorphic to:

\[ tB[t] \times_{\ev_1} tB[t] \xrightarrow{\text{incl}} B[t] \times_{\ev_1} tB[t] \xrightarrow{\ev_0 \circ \pr_1} B \]  \hspace{1cm} (12)

Here \(\pr_1 : B[t] \times_{\ev_1} tB[t] \rightarrow B[t]\) is the projection into the first factor. The section in \(\mathcal{S}\) identifies with the morphism:

\[ B \rightarrow B[t] \times_{\ev_1} tB[t], \quad b \mapsto (b(1-t), 0). \]

The last vertex map induces a strong morphism of extensions from (11) to (12); this morphism has the following components:

\[ (t^2 - t)B[t] \rightarrow tB[t] \times_{\ev_1} tB[t], \quad p \mapsto (p, 0); \]
\[ (t - 1)B[t] \rightarrow B[t] \times_{\ev_1} tB[t], \quad p \mapsto (p, 0). \]

**Lemma 2.31.** Let \(B \in \Alg_r, n \geq 1\) and \(r \geq 0\). Then \(P(n, B)_r\) is contractible.

**Proof.** Let \(\vartheta : I \times I \rightarrow I\) be the unique morphism of simplicial sets that satisfies:

\[ \begin{array}{c}
\begin{array}{c}
(0, 0) \\
(0, 1), (1, 0), (1, 1)
\end{array}
\end{array} \xrightarrow{\theta} \begin{array}{c}
\begin{array}{c}
\rightarrow 0 \\
\rightarrow 1
\end{array}
\end{array} \]

It is easily verified that \(\vartheta\) induces a morphism of pairs \(\vartheta : (I, \{1\}) \Box I \rightarrow (I, \{1\})\). The latter, in turn, induces an \(\ell\)-algebra homomorphism:

\[ f := (\bar{\mathcal{S}} \Box \vartheta)^* : P(n, B)_r = B_r^{\mathcal{S}, \Box(I, \{1\})} \longrightarrow B_r^{\bar{\mathcal{S}}, \Box(I, \{1\}) \Box I} \]

The coface maps \(d^i : \Delta^0 \rightarrow I\) induce \(\ell\)-algebra homomorphisms:

\[ \delta_i := (\bar{\mathcal{S}} \Box \vartheta(I, \{1\}) \Box d^i)^* : B_r^{\mathcal{S}, \Box(I, \{1\}) \Box I} \longrightarrow B_r^{\bar{\mathcal{S}}, \Box(I, \{1\}) \Box \Delta^i} \cong P(n, B)_r \]

Note that \(\delta_0 \circ f = 0\) and \(\delta_1 \circ f = \id_{P(n, B)_r}\). By \([5\text{ Hauptlemma (2)}], \delta_0 \circ f \text{ and } \delta_1 \circ f\) represent the same class in \([P(n, B)_r, P(n, B)_r]\). Indeed, both morphisms represent the same class in \([P(n, B)_r, B_{0}^{\partial (I-1)}]\). For the polynomial homotopies constructed in op. cit. preserve our boundary conditions. \(\square\)

2.32. **Exchanging loop functors.** Let \(B \in \Alg_r\) and let \(m, n \geq 0\). We proceed to define, by induction on \(n\), a natural transformation:

\[ \kappa_B^{m,n} : J^n(B_r^{\bar{\mathcal{S}}}) \longrightarrow (J^n B_r)^{\bar{\mathcal{S}}} \]

Let \(k_B^m : J(B_r^{\bar{\mathcal{S}}}) \rightarrow (J B_r)^{\bar{\mathcal{S}}}\) be the classifying map of the extension \((\mathcal{S}_B)^{\bar{\mathcal{S}}}\). For \(n \geq 1\), define inductively \(k_B^{m+1,n}\) as the composite:

\[ J^{n+1}(B_r^{\bar{\mathcal{S}}}) \xrightarrow{J(k_B^{m,n})} J(J^n B_r)^{\bar{\mathcal{S}}} \xrightarrow{\kappa_B^{m,n}} (J^{n+1} B_r)^{\bar{\mathcal{S}}} \]

Define \(J^n\) as the identity functor of \(\Alg_r\) and let \(k_B^{0,m}\) be the identity of \(B_r^{\bar{\mathcal{S}}}\). When there is no risk of confusion, we may write \(k_B^{m,n}\) instead of \(k_B^{m,n}\) to alleviate notation. The next result follows from an easy induction on \(n = p + q\).
**Lemma 2.33.** Let $p, q, r \geq 0$ and let $B$ be an $\ell$-algebra. Then $k^{p+q,m}_B = k^{q,m}_{B^p} \circ J^q(k^{p,m}_B)$.

That is, the following diagram in $\text{Alg}_\ell$ commutes:

\[
\begin{array}{ccc}
J^{p+q}(B_{r}^{\Xi_{n}}) & \xrightarrow{\delta^{p+q}} & (J^{p+q}B)_{r}^{\Xi_{n}} \\
\downarrow J^q(J^{p}B)_{r}^{\Xi_{n}} & & \downarrow (J^{p}B)_{r}^{\Xi_{n}} \\
J^q(J^{p}B)_{r}^{\Xi_{n}} & \xrightarrow{\delta^{p}} & (J^{p}B)_{r}^{\Xi_{n}}
\end{array}
\]

This lemma is explained as follows. We have $J^{p+q} = J^q \circ J^p$. Thus, in order to exchange $J^{p+q}$ and (?) $\Xi_{n}$, we may first exchange $J^p$ and (?) $\Xi_{n}$ and then $J^q$ and (?) $\Xi_{n}$.

**Remark 2.34.** Replacing $\Xi_{n}$ with any simplicial pair $(K, L)$, we can define morphisms $J^n(B^{[K, L]}) \to (J^nB)^{[K, L]}$ and prove Lemma 2.33 in this setting.

The following result is an analog of Lemma 2.33. Its statement is, however, more complicated since $(B^{\Xi_{n}})_{n} \neq B^{\Xi_{n,p}}$.

**Lemma 2.35.** Let $p, q, r, s \geq 0$ and let $B$ be an $\ell$-algebra. Then the following diagram in $\text{Alg}_\ell$ commutes:

\[
\begin{array}{cccc}
J^n\left((B_{r}^{\Xi_{s}})_{s}^{\Xi_{r}}\right) & \xrightarrow{k^{s}_{n}} & (J^nB_{r})_{s}^{\Xi_{r}} & \xrightarrow{(k^{s}_{n})^{\Xi_{r}}} & (J^nB)_{r}^{\Xi_{s}} \\
\downarrow & & \downarrow & & \downarrow \\
J^n\left(B_{r+s}^{\Xi_{r+s}}\right) & \xrightarrow{k^{s}_{n}} & (J^nB)_{r+s}^{\Xi_{r+s}}
\end{array}
\]

**Proof.** We proceed by induction on $n$. The case $n = 1$ follows from Proposition 2.24 applied to the following strong morphism of extensions:

\[
\begin{array}{cccc}
(U_{B_{r+s}}^{\Xi_{s}})_{s}^{\Xi_{r}} & \xrightarrow{(U_{B_{r+s}}^{\Xi_{s}})_{s}^{\Xi_{r}}} & (J(B_{r+s})^{\Xi_{s}})_{s}^{\Xi_{r}} & \xrightarrow{(J(B_{r+s})^{\Xi_{s}})_{s}^{\Xi_{r}}} & (J(B_{r+s})^{\Xi_{s}})_{s}^{\Xi_{r}} \\
\downarrow & & \downarrow & & \downarrow \\
(U_{B_{r+s}}^{\Xi_{s}})_{s}^{\Xi_{r}} & \xrightarrow{(U_{B_{r+s}}^{\Xi_{s}})_{s}^{\Xi_{r}}} & (J(B_{r+s})^{\Xi_{s}})_{s}^{\Xi_{r}} & \xrightarrow{(J(B_{r+s})^{\Xi_{s}})_{s}^{\Xi_{r}}} & (J(B_{r+s})^{\Xi_{s}})_{s}^{\Xi_{r}} \\
\downarrow & & \downarrow & & \downarrow \\
(UB_{r+s})_{r+s}^{\Xi_{r+s}} & \xrightarrow{(UB_{r+s})_{r+s}^{\Xi_{r+s}}} & (JB)_{r+s}^{\Xi_{r+s}} & \xrightarrow{(JB)_{r+s}^{\Xi_{r+s}}} & (JB)_{r+s}^{\Xi_{r+s}} \\
\downarrow & & \downarrow & & \downarrow \\
(UB_{r+s})_{r+s}^{\Xi_{r+s}} & \xrightarrow{(UB_{r+s})_{r+s}^{\Xi_{r+s}}} & (JB)_{r+s}^{\Xi_{r+s}} & \xrightarrow{(JB)_{r+s}^{\Xi_{r+s}}} & (JB)_{r+s}^{\Xi_{r+s}}
\end{array}
\]

Now suppose that the diagram commutes for $n$; we will show it also commutes for $n + 1$. The following diagram commutes by inductive hypothesis and naturality of $\kappa^{n,1}_{r}$; we omit...
we still denote $F$ composing this with the functor $g$ are two homotopic $F$ and let $F$ be a functor. Then $F$ induces $F^{\text{ind}} : \text{Alg}_{\ell}^{\text{ind}} \to (\text{Alg}_{\ell})^{\text{ind}}$; composing this with the functor $c_I$ of Lemma 2.2, we get a functor $\text{Alg}_{\ell}^{\text{ind}} \to \text{Alg}_{\ell}^{\text{ind}}$ that we still denote $F$. This happens, for example, in the following cases:

(i) $I = \emptyset$ and $F = J : \text{Alg}_{\ell} \to \text{Alg}_{\ell}$;
(ii) $I = \emptyset$ and $F = (\cdot)^X : \text{Alg}_{\ell} \to \text{Alg}_{\ell}$ for any $X \in \mathbb{S}$;
(iii) $I = \mathbb{Z}_{\geq 0}$ and $F = (?)^{(K,L)} : \text{Alg}_{\ell} \to \text{Alg}_{\ell}^{\mathbb{Z}_{\geq 0}}$ for any simplicial pair $(K,L)$;
(iv) $I$ any poset and $F = ? @ C_* : \text{Alg}_{\ell} \to \text{Alg}_{\ell}^1$, with $C_* \in (\text{Alg}_{\ell})^I$.

In all these examples, $F$ has the aditional property of being homotopy invariant: if $f$ and $g$ are two homotopic $\ell$-algebra homomorphisms then, for all $i \in I$, $F(f_i)$ and $F(g_i)$, are homotopic. Because of this, $F$ induces a functor $F : [\text{Alg}_{\ell}] \to [\text{Alg}_{\ell}]^I$ and thus a functor $F : [\text{Alg}_{\ell}]^{\text{ind}} \to [\text{Alg}_{\ell}]^{\text{ind}}$; here we are using Lemma 2.2 once more. It is easily seen that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Alg}_{\ell}^{\text{ind}} & \xrightarrow{F} & \text{Alg}_{\ell}^{\text{ind}} \\
\downarrow & & \downarrow \\
[\text{Alg}_{\ell}]^{\text{ind}} & \xrightarrow{F} & [\text{Alg}_{\ell}]^{\text{ind}}
\end{array}
\]

Thus, we can apply functors \((i)\)-(iv) to objects and morphisms in $[\text{Alg}_{\ell}]^{\text{ind}}$ and, if we start at $\text{Alg}_{\ell}^{\text{ind}}$, we may apply these functors and take homotopy classes in any order we like.
Lemma 2.37. The construction above determines a natural transformation $(\nu^\circ \theta)_{\mathcal{C}}$ as endofunctors of $[\text{Alg}_\ell]^\text{ind}$; we would like to consider $\mu^m_{\ell}$ as a natural transformation between them. We now proceed to explain how certain morphisms from $F : \text{Alg}_\ell \to \text{Alg}_J^\ell$ to $G : \text{Alg}_\ell \to \text{Alg}_J^\ell$ induce natural transformations between the associated endofunctors of $[\text{Alg}_\ell]^\text{ind}$—here, $I$ and $J$ may be different filtering posets, and $F$ and $G$ are homotopy invariant functors.

Let $F : \text{Alg}_\ell \to \text{Alg}_J^\ell$ and $G : \text{Alg}_\ell \to \text{Alg}_J^\ell$ be two homotopy invariant functors. Consider a pair $(\nu, \theta)$ where $\theta : I \to J$ is a functor and $\nu : F \to \theta^*G$ is a natural transformation of functors $\text{Alg}_\ell \to \text{Alg}_J^\ell$. This means that:

(a) For each $A \in \text{Alg}_\ell$ we have $\nu_A : F(A) \to G(A) \circ \theta \in \text{Alg}_J^\ell$;
(b) For each morphism $f : A \to A'$ in $\text{Alg}_\ell$, the following diagram in $\text{Alg}_J^\ell$ commutes:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\nu_A} & G(A) \circ \theta \\
F(f) \downarrow & & \downarrow \theta \circ G(f) \\
F(A') & \xrightarrow{\nu_{A'}} & G(A') \circ \theta
\end{array}
$$

Let $(C, K) \in [\text{Alg}_\ell]^\text{ind}$. We will define $\nu_{C,i} \in [F(C_i), G(C_i)\ast]$. For each pair $(i, k) \in I \times K$, let $(\nu_{C_i})_{(i,k)}$ be the class of the morphism $(\nu_{C_i}) : F(C_k) \to G(C_k)\ast(i)$ in $[F(C_k), G(C_i)\ast]$. It is easily verified that the $(\nu_{C_i})_{(i,k)}$ are compatible and assemble into a morphism:

$$
\nu_{C_i} = \{(\nu_{C_i})_{(i,k)}\} \in \lim_{(i,k)} [F(C_k), G(C_i)\ast] = [F(C_i)\ast, G(C_i)\ast]
$$

**Lemma 2.37.** The construction above determines a natural transformation $\nu : F \to G$ of functors $[\text{Alg}_\ell]^\text{ind} \to [\text{Alg}_J^\ell]^\text{ind}$. That is, for every morphism $f \in [C_i, D_i]$, the following diagram in $[\text{Alg}_\ell]^\text{ind}$ commutes:

$$
\begin{array}{ccc}
F(C_i)\ast & \xrightarrow{\nu_{C,i}} & G(C_i)\ast \\
F(f) \downarrow & & \downarrow G(f) \\
F(D_i)\ast & \xrightarrow{\nu_{D,i}} & G(D_i)\ast
\end{array}
$$

**Proof.** It is a straightforward verification. \(\square\)

**Example 2.38.** Regard $\kappa^m_{\ell} : (\ell^m)! : (\ell^m)! \to (\ell^m)!$ as a natural transformation between (homotopy invariant) functors $\text{Alg}_\ell \to \text{Alg}_\ell \omega_{\omega}$. By Lemma 2.37 we can also regard $\kappa^m_{\ell}$ as a natural transformation:

$$
\begin{array}{ccc}
(\ell^m)! & \xrightarrow{\kappa^m_{\ell}} & (\ell^m)! \\
\ast \downarrow & & \ast \downarrow \\
[\text{Alg}^\ell]^\text{ind} & \xrightarrow{[\text{Alg}^\ell]^\text{ind}} & [\text{Alg}^\ell]^\text{ind}
\end{array}
$$

**Example 2.39.** Consider the (homotopy invariant) functors $F : \text{Alg}_\ell \to \text{Alg}_\ell \omega_{\omega} \times \omega_{\omega}$, $F(B) = (B^\omega_{\omega})\mathcal{E}$, and $G : \text{Alg}_\ell \to \text{Alg}_\ell \omega_{\omega} \times \omega_{\omega} ; G(B) = (B^\omega_{\omega})\mathcal{E}$. Define $\theta : \omega_{\omega} \times \omega_{\omega} \to \omega_{\omega}$ by $\theta(r, s) = r + s$. Then $\theta$ is a functor and $\mu^m_{\ell} : F \to \theta^*G$ is a natural transformation. By Lemma 2.37 the pair $(\mu^m_{\ell}, \theta)$ induces a natural transformation:

$$
\begin{array}{ccc}
(\ell^m)! \mathcal{E} & \xrightarrow{(\mu^m_{\ell})_{\mathcal{E}}} & (\ell^m)! \mathcal{E} \\
\ast \downarrow & & \ast \downarrow \\
[\text{Alg}_\ell]^\text{ind} & \xrightarrow{[\text{Alg}_\ell]^\text{ind}} & [\text{Alg}_\ell]^\text{ind}
\end{array}
$$

**Remark 2.40.** We have just seen we can regard $J$ and $(?)\mathcal{E}$ as endofunctors of $[\text{Alg}_\ell]^\text{ind}$. Moreover, we can consider $\kappa^m_{\ell}$ and $\mu^m_{\ell}$ as natural transformations between endofunctors of $[\text{Alg}_\ell]^\text{ind}$. In the sequel, we shall do this without further mention.
2.41. **Group structure.** Let \( n \geq 1 \). For \( A, B \in \text{Alg}_{\ell} \), the set \([A, B^\infty_{\text{alg}}]\) has a natural group structure, that is abelian if \( n \geq 2 \); see Remark 2.16. We claim that this assertion remains true if we replace \( A \) and \( B \) by arbitrary ind-objects in \([\text{Alg}_{\ell}]\). Indeed, for \((A, I), (B, J) \in [\text{Alg}_{\ell}]^{\text{ind}}\), we have:

\[
[A, (B^\infty_{\text{alg}})] \cong \lim_{I \to J} [A_i, (B^\infty_{\text{alg}})]
\]  

Since limits and filtered colimits of groups are computed respectively as limits and filtered colimits of sets, the right hand side of (13) has a natural group structure, that is abelian if \( n \geq 2 \). This can be summarized as follows.

**Lemma 2.42.** Let \( B_* \in [\text{Alg}_{\ell}]^{\text{ind}} \) and let \( n \geq 1 \). Then \((B^\infty_{\text{alg}})_{\text{set}}\) is a group object in \([\text{Alg}_{\ell}]^{\text{ind}}\), which is abelian if \( n \geq 2 \). Moreover, a morphism \( g \in [B_*, B_*] \) induces a morphism of group objects \( g_* \in [(B^\infty_{\text{alg}}), (B^\infty_{\text{alg}})] \).

We conclude the section with a proposition that relates the group structure in \([A_*, (B^\infty_{\text{alg}})]\) with the loop functors \( J \) and \(?\). We postpone its proof to the Appendix A.

**Proposition 2.43.** Let \( A, B \in \text{Alg}_{\ell} \), let \( K \) be a filtering poset, let \( C_* \in (\text{Alg}_{\ell})^K \) and let \( m, n \geq 1 \). Then the following composite functions are group homomorphisms:

\[
\begin{align*}
(i) & \quad [A, B^\infty_{\text{alg}}] \xrightarrow{J^n} [J^n A, J^n (B^\infty_{\text{alg}})] \\
(ii) & \quad [A, B^\infty_{\text{alg}}] \xrightarrow{\gamma_0 \gamma_*} [A \otimes C_*, B^\infty_{\text{alg}} \otimes C_*] \cong [A \otimes C_*, (B \otimes C_*)^\infty_{\text{alg}}] \\
(iii) & \quad [A, (B^\infty_{\text{alg}})] \xrightarrow{(\mu^{m*}_n)} [A, B^\infty_{\text{alg}}] \\
(iv) & \quad [A, B^\infty_{\text{alg}}] \xrightarrow{(?)^m} [A_{\text{alg}}^\infty, (B^\infty_{\text{alg}})] \xrightarrow{(\mu^{m*}_n)} [A_{\text{alg}}^\infty, B^\infty_{\text{alg}}] \\
\end{align*}
\]

In (ii) the bijection on the right is induced by the obvious isomorphism of \( K \times \mathbb{Z}_{\geq 0}\)-diagrams \( B^\infty_{\text{alg}} \otimes C_* \cong (B \otimes C_*)^\infty_{\text{alg}}\).

### 3. The loop-stable homotopy category

Let \( f : A \to B^\infty_{\text{alg}} \) be an \( \ell\)-algebra homomorphism. By Proposition 2.22 there exists a unique strong morphism of extensions \( \mathcal{Y}_A \to \mathcal{P}_{n,B} \) extending \( f \):

\[
\begin{array}{cccccc}
\mathcal{Y}_A & \xrightarrow{f} & \mathcal{P}_{n,B} \\
\downarrow \mathcal{Y}_A & & \downarrow \mathcal{P}_{n,B} \\
\mathcal{P}_{B^\infty_{\text{alg}}} & \xrightarrow{B^\infty_{\text{alg}}} & \mathcal{P}_{B^\infty_{\text{alg}}} \\
\end{array}
\]

We will write \( \Lambda^n(f) \) for the classifying map of \( f \) with respect to \( \mathcal{P}_{n,B} \).

**Remark 3.1.** We have \( \Lambda^n(f) = \Lambda^n(\text{id}_{B^\infty_{\text{alg}}}) \circ J(f) \). Indeed, this follows from the uniqueness statement in Proposition 2.22 and the fact that the following diagram exhibits a strong morphism of extensions \( \mathcal{Y}_A \to \mathcal{P}_{n,B} \) that extends \( f \):

\[
\begin{array}{cccccc}
\mathcal{Y}_A & \xrightarrow{f} & \mathcal{P}_{B^\infty_{\text{alg}}} \\
\downarrow \mathcal{Y}_A & & \downarrow \mathcal{P}_{B^\infty_{\text{alg}}} \\
\mathcal{P}_{B^\infty_{\text{alg}}} & \xrightarrow{B^\infty_{\text{alg}}} & \mathcal{P}_{B^\infty_{\text{alg}}} \\
\end{array}
\]
Remark 3.2. We have \( \Lambda^n(f) = \mu_B^{1,n} \circ f^{B^2_\infty} \circ \lambda_A \). Indeed, this follows from the uniqueness statement in Proposition \ref{2.22} and the fact that the following diagram exhibits a strong morphism of extensions \( \mathcal{U}_A \to \mathcal{P}_{n,B} \) that extends \( f \):

\[
\begin{array}{ccccccc}
\mathcal{U}_A & \rightarrow & JA & \rightarrow & TA & \rightarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{0,A} & \rightarrow & A_0 \times \mathcal{U}_0 & \rightarrow & P(0,A) & \rightarrow & A \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{0,B} & \rightarrow & (B^2_\infty) & \rightarrow & P(0,B^2_\infty) & \rightarrow & B^2_\infty \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{n,B} & \rightarrow & B^{2n+1} & \rightarrow & P(n,B) & \rightarrow & B^2_\infty \\
\end{array}
\]

If \( f, g : A \to B^2_\infty \) are homotopic morphisms, then \( \Lambda^n(f) \) and \( \Lambda^n(g) \) are homotopic too. Thus, we can regard \( \Lambda^n \) as a function \( \Lambda^n_{A,B} : [A, B^2_\infty] \to [JA, B^2_\infty] \). For ind-algebras \((A, I)\) and \((B, J)\), define \( \Lambda^n_{A, B} : [A_*, (B_*)_{\infty}] \to [J(A_*), (B_*)_{\infty}] \) as the function:

\[
\lim \colim \Lambda^n_{A, B} : \lim \colim [A_i, (B_i)_{\infty}] \rightarrow \lim \colim [J(A_i), (B_i)_{\infty}]
\]

**Lemma 3.3.** Let \( A_*, B_* \in \text{Alg}_{\mathcal{I}}^{\text{ind}} \) and let \( n \geq 1 \). Then the functions

\[ \Lambda^n_{A_*, B_*} : [A_*, (B_*)_{\infty}] \rightarrow [J(A_*), (B_*)_{\infty}] \]

are group homomorphisms.

**Proof.** It suffices to prove the case \( A_* = A \in \text{Alg}_{\mathcal{I}} \) and \( B_* = B \in \text{Alg}_{\mathcal{I}} \). Consider the following strong morphism of extensions \( \mathcal{U}_{B^2_\infty} \to \mathcal{P}_{n,B} \) extending the identity:

\[
\begin{array}{ccccccc}
\mathcal{U}_{B^2_\infty} & \rightarrow & J(B^2_\infty) & \rightarrow & T(B^2_\infty) & \rightarrow & B^2_\infty \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
(\mathcal{U}_B)_{\infty} & \rightarrow & (JB)_\infty & \rightarrow & (TB)_\infty & \rightarrow & B^2_\infty \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
(\mathcal{P}_{0,B})_{\infty} & \rightarrow & (B^2_0)_\infty & \rightarrow & (PB)^2_\infty & \rightarrow & B^2_\infty \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{n,B} & \rightarrow & B^{2n+1} & \rightarrow & P(n,B) & \rightarrow & B^2_\infty \\
\end{array}
\]

By the uniqueness statement in Proposition \ref{2.22} we have:

\[ \Lambda^n(\text{id}_{B^2_\infty}) = c^* \circ \mu_B^{1,n} \circ (\lambda_B)^{2n} \circ k_B^{1,n} \quad (14) \]

Then, by Remark 3.1 \( \Lambda^n_{A,B} \) equals the composite:

\[
\begin{array}{cccccc}
[A, B^2_\infty] & \xrightarrow{\gamma_B^{1,n} \circ \text{JT}(?)} & [JA, (JB)_\infty] & \xrightarrow{(\lambda_B)_*} & [JA, (B^2_0)_\infty] & \xrightarrow{(\nu_B)_*} & [JA, B^{2n+1}_\infty] & \xrightarrow{c^*} & [JA, B^{\infty+1}_\infty] \\
\end{array}
\]
Let $A$ into the algebraic setting. The following two lemmas are straightforward verifications.

Lemma 4.2. Let $f, g : (A, m) \to (B, n)$ be the following composite in $[\text{Alg}_r]^{\text{ind}}$:

$$J^{N_1+N_2} A \xrightarrow{J^{N_1}(f)} J^{N_1}(B \times_{N_2} \cdots) \xrightarrow{J^{N_1}(g)} J^{N_1}(B) \xrightarrow{g^{N_1}} (C \times_{N_1} N_2) \xrightarrow{\mu^{N_1}} C^{N_1+N_2}$$

Then define $\langle g \circ f \rangle := \langle g \star f \rangle \in \mathcal{R}((A, m), (C, k))$. We will show in Lemma 4.7 that $\langle g \star f \rangle$ does not depend upon the choice of the representatives $f$ and $g$. Then, in Theorem 4.8 we will prove that the composition just defined makes $\mathcal{R}$ into a category.

4. WELL-DEFINEDNESS OF THE COMPOSITION

We closely follow [3] Section 6.3], making appropriate changes to translate the proof into the algebraic setting. The following two lemmas are straightforward verifications.

Lemma 4.1. Let $A_*, B_* \in [\text{Alg}_r]^{\text{ind}}$. Then:

(i) If $f \in [A_*, A_*]$, then $\Lambda^n(g \circ f) = \Lambda^n(g) \circ J(f) \in [J(A_*), (B_* \times_{N_1} N_2)]$.

(ii) If $h \in [B_*, A_*]$, then $\Lambda^n(h \times g) = h^{N_1} \circ \mu^{N_1} \in [J(A_*), (B_* \times_{N_1} N_2)]$.

Lemma 4.2. Let $A_*, B_* \in [\text{Alg}_r]^{\text{ind}}$ and let $f \in [A_*, (B_* \times_{N_1} N_2)]$. Then:

$$\Lambda^{n+m} (\mu^{n,m}_B \circ f) = \mu^{n,m+1}_B \circ \Lambda^m(f) \in [J(A_*), (B_* \times_{N_1} N_2)]$$

Lemma 4.3 (cf. [3] Lemma 6.30). Let $B \in \text{Alg}_r$. Then the following diagram in $[\text{Alg}_r]^{\text{ind}}$ commutes:

$$
\begin{array}{ccc}
J^2 B & \xrightarrow{(\lambda_{JB})^{-1}} & J(B \times_{N_1} N_2) \\
\downarrow & & \downarrow \kappa_{B_1}^{N_1} \\
(JB)_N & \xrightarrow{\kappa_{B_1}^{N_1}} & (J(B \times_{N_1} N_2))_{N_1}
\end{array}
$$

(15)

Here, $(\lambda_{JB})^{-1}$ is the inverse of $\lambda_{JB}$ in the group $[J^2 B, (JB)_{N_1}]$.

Proof. Recall from Example 2.30 that, for $A \in \text{Alg}_r$, the extension $\mathcal{O}_{0,A}$ is isomorphic to:

$$
\begin{array}{ccc}
(t^2 - t) A[t] & \xrightarrow{\text{inc}} & (t - 1) A[t] & \xrightarrow{ev_0} A & \quad (r = 0) \\
t A[t] & \xrightarrow{ev} & A[t] & \xrightarrow{ev_0 \circ \text{ev}_r} A & \quad (r = 1)
\end{array}
$$

In the rest of the proof we will make these identifications without further mention.

Define

$$I := \ker \left( t(TB)[t] \xrightarrow{ev_1} TB \xrightarrow{\eta_B} B \right)$$

The result now follows from Proposition 2.43 (i) and (iii) and Example 2.19. □

Definition 3.4 (cf. [3] Section 6.3). We proceed to define a category $\mathcal{R}$, that we will call the loop-stable homotopy category. The objects of $\mathcal{R}$ are the pairs $(A, m)$ where $A$ is an object of $\text{Alg}_r$ and $m \in \mathbb{Z}$. For two objects $(A, m)$ and $(B, n)$, put:

$$\mathcal{R}((A, m), (B, n)) := \text{colim} \{ J^{m+n} A, B \mathbb{N}_{n+m} \}
$$

Here, the colimit is taken over the morphisms $\Lambda^{n+m}$ of Lemma 3.3 and $n$ runs over the integers such that both $m + n \geq 0$ and $n + v \geq 0$. We often write $\langle f \rangle$ for the element of $\mathcal{R}((A, m), (B, n))$ represented by $f \in [J^{m+n} A, B \mathbb{N}_{n+m}]$. The composition in $\mathcal{R}$ is defined as follows. Represent elements of $\mathcal{R}((A, m), (B, n))$ and $\mathcal{R}((B, n), (C, k))$ by $f \in [J^{m+n} A, B \mathbb{N}_{n+m}]$ and $g \in [J^{m+n} B, C \mathbb{N}_{n+m}]$ respectively. To simplify notation, write:

$$N_1 := m + v, \quad N_2 := n + v, \quad N_3 := n + w, \quad \text{and} \quad N_4 := k + w.$$
and let $s : B \to t(TB)[t]$ be given by $s(b) = \sigma_B(b)t$ —recall that $\sigma_B$ is the section of the universal extension $\mathcal{U}_B$. We have an extension:

$$(\mathcal{E}', s) : I \xrightarrow{\text{incl}} t(TB)[t] \xrightarrow{\eta_B \circ \text{ev}_1} B$$

Now put

$$E := \{(p, q) \in t(TB)[t] \times t(JB)[t] : p(1) = q(1)\}$$

and let $s' : I \to E$ be given by $s'(p) = (p, p(1)t)$ —here we use that $\text{ev}_1 : I \to TB$ factors through $JB$. It is easily seen that we have an extension:

$$(\mathcal{E}', s') : (t^2 - t)(JB)[t] \xrightarrow{(0, \text{incl})} E \xrightarrow{\text{pr}_1} I$$

Let $\chi : JB \to I$ be the classifying map of $(\mathcal{E}', s)$. The following diagram exhibits a strong morphism of extensions $\mathcal{U}_B \to \mathcal{U}_B$ extending id$B$:

$$
\begin{array}{ccc}
\mathcal{U}_B & \xrightarrow{\chi} & JB \\
\downarrow \mathcal{E}' & & \downarrow \mathcal{E}' \\
\mathcal{U}_B & \xrightarrow{\text{incl}} & J(B) \\
\downarrow \mathcal{E}'' & & \downarrow \mathcal{E}'' \\
\mathcal{P}_{0,JB} & \xrightarrow{\text{incl}} & J(B) \\
\downarrow \omega & & \downarrow \omega \\
(t^2 - t)(JB)[t] & \xrightarrow{\text{incl}} & (t(JB)[t] \\
\downarrow \mathcal{J}'' & & \downarrow \mathcal{J}'' \\
\mathcal{P}_{0,JB} & \xrightarrow{\mathcal{J}''} & (t - 1)(JB)[t] \\
\downarrow \omega & & \downarrow \omega \\
(t^2 - t)(JB)[t] & \xrightarrow{\text{incl}} & JB \\
\end{array}
$$

It follows that $\text{ev}_1 \circ \chi = \text{id}_{JB}$. Now let $\omega$ be the automorphism of $(JB)[t]$ given by $\omega(t) = 1 - t$ and consider the following strong morphism of extensions $\mathcal{U}_{JB} \to \mathcal{P}_{0,JB}$ extending id$_{\mathcal{P}_{0,JB}}$:

$$
\begin{array}{ccc}
\mathcal{U}_{JB} & \xrightarrow{\omega} & J^2B \\
\downarrow \mathcal{J}'' & & \downarrow \mathcal{J}'' \\
(JB)^{\mathcal{J}_1} & \xrightarrow{\text{incl}} & J^2B \\
\downarrow \omega & & \downarrow \omega \\
(JB)^{\mathcal{J}_1} & \xrightarrow{(0, \text{incl})} & (t^2 - t)(JB)[t] \\
\downarrow (\text{incl}_0) & & \downarrow (\text{incl}_0) \\
\mathcal{U}_{JB} & \xrightarrow{\text{incl}} & J^2B \\
\downarrow \mathcal{J}'' & & \downarrow \mathcal{J}'' \\
(JB)^{\mathcal{J}_1} & \xrightarrow{\text{incl}} & J^2B \\
\downarrow \omega & & \downarrow \omega \\
(JB)^{\mathcal{J}_1} & \xrightarrow{(0, \text{incl})} & (t^2 - t)(JB)[t] \\
\end{array}
$$

Since $\omega^{-1} = \omega$, we get that the classifying map of $\chi$ with respect to $(\mathcal{E}'', s')$ equals the composite:

$$J^2B \xrightarrow{\lambda_B} (t^2 - t)(JB)[t] \xrightarrow{\omega} (t^2 - t)(JB)[t]$$

Define $\theta : E \to (TB)^{\mathcal{J}_1} = t(TB)[t] \times_{ev_1} t(TB)[t]$ by $\theta(p, q) = (q, p)$ and consider the following morphisms of extensions:

$$
\begin{array}{ccc}
\mathcal{U}_{JB} & \xrightarrow{\omega} & J^2B \\
\downarrow \mathcal{J}'' & & \downarrow \mathcal{J}'' \\
(JB)^{\mathcal{J}_1} & \xrightarrow{\text{incl}_0} & J^2B \\
\downarrow \omega & & \downarrow \omega \\
(JB)^{\mathcal{J}_1} & \xrightarrow{(0, \text{incl})} & (t^2 - t)(JB)[t] \\
\downarrow (\text{incl}_0) & & \downarrow (\text{incl}_0) \\
\mathcal{U}_{JB} & \xrightarrow{\text{incl}} & J^2B \\
\downarrow \mathcal{J}'' & & \downarrow \mathcal{J}'' \\
(JB)^{\mathcal{J}_1} & \xrightarrow{\text{incl}_0} & J^2B \\
\downarrow \omega & & \downarrow \omega \\
(JB)^{\mathcal{J}_1} & \xrightarrow{(0, \text{incl})} & (t^2 - t)(JB)[t] \\
\end{array}
$$

(16)
Note that the morphism $\mathcal{E}' \to (\mathcal{U}_B)^{\Xi_1}$ is not strong. Put $\psi := (0, \eta_B) \circ \chi : JB \to B_1^{\Xi_1}$. By Proposition \[\ref{prop:2.24}\] applied to \[\ref{eq:16}\], the following diagram commutes in $[\text{Alg}_\ell]$:

\[
\begin{array}{ccc}
F^2 B & \xrightarrow{J(\psi)} & J(B_1^{\Xi_1}) \\
\downarrow \text{id} & & \downarrow \gamma^1 \\
F^2 B & \xrightarrow{\lambda_B} (JB)^{\Xi_1}_0 & \xrightarrow{\omega} (JB)^{\Xi_1}_0 & \xrightarrow{\gamma} (JB)^{\Xi_1}_1
\end{array}
\]

Here $\gamma^* = (\text{incl}, 0) : (i^2 - t)(JB)[t] \to (t(JB)[t]_{ev_0} \times_{ev_1} t(JB)[t]$ is the morphism induced by the last vertex map; see Example \[\ref{ex:2.30}\]. The proof will be finished if we show that $J(\psi)$ equals $J(\lambda_B)$ in $[\text{Alg}_\ell]$. By Lemma \[\ref{lem:2.25}\] it suffices to show that $\psi$ equals $\lambda_B$ in $[\text{Alg}_\ell]$. Define $\beta : t(TB)[t] \to (PB)_1$ by $\beta(p) = (\eta_B(p)(1 - t), 0)$. The following diagram exhibits a strong morphism of extensions $\mathcal{U}_B \to \mathcal{P}_{0,B}$ extending $\text{id}_B$:

\[
\begin{array}{ccc}
\mathcal{U}_B & \xrightarrow{J} & TB & \xrightarrow{B_1^{\Xi_1}} & B \\
(\mathcal{E}, s) & & \downarrow \text{incl} & & \downarrow \text{id} \\
\mathcal{P}_{0,B} & \xrightarrow{t(TB)[t]} & B
\end{array}
\]

It follows that $\lambda_B$ equals the composite $JB \xrightarrow{\chi} I \xrightarrow{\beta} B_1^{\Xi_1}$, which is easily seen to be homotopic to $\psi$.

\[\square\]

**Lemma 4.4.** Let $B \in \text{Alg}_\ell$ and let $v_n = (-1)^n$. Then the following diagram in $[\text{Alg}_\ell]^{\text{ind}}$ commutes:

\[
\begin{array}{ccc}
F^{n+1} B & \xrightarrow{F^{n+1} (1_B)} & F^n (B_1^{\Xi_1}) \\
(\lambda_B)^{F^{n+1}} & \downarrow \gamma^{n+1} & \downarrow \gamma^n \\
(\lambda_B)^{F^n} & \xrightarrow{\gamma^n} & (F^n B_1^{\Xi_1})
\end{array}
\]

**Proof.** We prove the result by induction on $n$. The case $n = 1$ is Lemma \[\ref{lem:4.3}\]. Suppose that the result holds for $n \geq 1$. We have:

\[
\begin{align*}
\kappa_B^{n+1,1} \circ F^{n+1} (1_B) &= \kappa_B^{1,1} \circ J^{n+1} (1_B) (\lambda_B) \\
&= \kappa_B^{1,1} \circ J (k_B^{n+1} \circ F^n (1_B)) \\
&= \kappa_B^{1,1} \circ J ((\lambda_B) F^n) \\
&= \kappa_B^{1,1} \circ J (k_B^{n+1} \circ F^n (1_B)) \\
&= \kappa_B^{1,1} \circ J (k_B^{n+1} \circ F^n (1_B)) \\
&= [\lambda_B^{F^n}]^{-1} F^n = (\lambda_B^{F^n})^{-1} (F^n (1_B))^{F^{n+1}} (\text{by the case } n = 1)
\end{align*}
\]

Then the result holds for $n + 1$.  

\[\square\]
Lemma 4.5 (cf. [3, Lemma 6.29]). Let $B \in \text{Alg}_\ell$ and let $n \geq 0$. Then the following diagram in $\text{Alg}_\ell^{\text{ind}}$ commutes:

\[
\begin{array}{c}
J(B_n^\text{alg}) \\
\downarrow (-1)^n \Lambda^m(\text{id}_{B_n}) \\
B_n^{\text{alg}+1} \\
\downarrow \mu^1_n \\\n(J_B)_n^{\text{alg}+1}
\end{array}
\]

Proof. We have to prove the equality of two morphisms in $\text{Alg}_\ell^{\text{ind}}$: since $[J(B_n^\text{alg}), B_n^{\text{alg}+1}] = \lim_r [J(B_r^\text{alg}), B_r^{\text{alg}+1}]$, it will be enough to show that both morphisms are equal when projected to $[J(B_r^\text{alg}), B_r^{\text{alg}+1}]$, for every $r$. The result now follows from [14]; the appearance of the sign $(-1)^n$ is explained in Example 2.19. \hfill \square

Lemma 4.6. Let $B \in \text{Alg}_\ell$ and let $m, n \geq 0$. Then, we have:

\[
(-1)^n \Lambda^m(\kappa_B^{n,m}) = \kappa_B^{n,m+1} \circ J^n \Lambda^m(\text{id}_{B_n}) \in [J^{n+1}(B_n^\text{alg}), (J^n B)_n^{\text{alg}+1}]
\]

Proof. On one hand, the diagram below commutes in $\text{Alg}_\ell^{\text{ind}}$ by Lemmas 2.35 and 4.4.

On the other hand, by Remark 3.2 we have:

\[
\kappa_B^{n,m+1} \circ J^n(\mu_B^{n,m}) = \kappa_B^{n,m+1} \circ J^n \Lambda^m(\text{id}_{B_n})
\]

Note that we use Proposition 2.43(iii) to handle the sign $(-1)^n$ in the latter equation. \hfill \square

Lemma 4.7 (cf. [3, Lemma 6.32]). Let $f \in [J^{N_1}A, B_n^\text{alg}]$ and $g \in [J^{N_2}B, C_n^\text{alg}]$. Then:

\[
\Lambda^{N_1}(g) \star f = \Lambda^{N_1+N_2}(g \star f) = \Lambda^{N_1}(f) \in [J^{N_1+N_2+1}A, C_n^\text{alg}]
\]

Proof. First, we have:

\[
(-1)^{N_2} \Lambda^{N_1+N_2}(g \star f) = \Lambda^{N_1+N_2}(\mu_C^{N_1,N_2} \circ \kappa_B^{N_1,N_2} \circ J^{N_1}(f))
\]

\[
= \Lambda^{N_1+N_2}(\mu_C^{N_1,N_2} \circ \kappa_B^{N_1,N_2}) \circ J^{N_1+1}(f)
\]

\[
= \mu_C^{N_1,N_2+1} \circ \Lambda^{N_1}(g \circ \kappa_B^{N_1,N_2}) \circ J^{N_1+1}(f)
\]

Here the equalities follow from the definition of $g \star f$, Lemma 4.1(i) Lemma 4.2 and Lemma 4.1 (iii) in that order. We are also using Proposition 2.43(iv) and Lemma 3.3 to handle the sign $(-1)^{N_2}$. 


Secondly, we have:

\[-(-)^{N_N} \Lambda_{N_i}(g) \ast f = \]

\[= (-1)^{N_N} \mu_G \circ (g^{\oplus 2})^2 \circ \Lambda_{N_i} \circ J^{N_i+1}(f) \]

\[= (-1)^{N_N} \mu_G \circ (g^{\oplus 2})^2 \circ \Lambda_{N_i} \circ J^{N_i+1}(f) \]

\[= (-1)^{N_N} \mu_G \circ (g^{\oplus 2})^2 \circ \Lambda_{N_i} \circ J^{N_i+1}(f) \]

Here the equalities follow from the definition of $\ast$ and Remark \[\ref{5.2}\] the functoriality of $\langle ? \rangle^2$, the associativity of $\mu$ and the naturality of $\mu$ —in that order.

Finally, we have:

\[-(-)^{N_N} \Lambda_{N_i}(f) = \]

\[= (-1)^{N_N} \mu_G \circ (g^{\oplus 2})^2 \circ \Lambda_{N_i} \circ J^{N_i+1}(f) \]

\[= (-1)^{N_N} \mu_G \circ (g^{\oplus 2})^2 \circ \Lambda_{N_i} \circ J^{N_i+1}(f) \]

\[= (-1)^{N_N} \mu_G \circ (g^{\oplus 2})^2 \circ \Lambda_{N_i} \circ J^{N_i+1}(f) \]

Here the equalities follow from the definition of $\ast$, Remark \[\ref{5.1}\] and the functoriality of $J^{N_N}$ —in that order.

Thus, $\Lambda_{N_N} + \Lambda_{N_i}(g \ast f) = g \ast \Lambda_{N_i}(f)$ by Lemma \[\ref{4.6}\] and to prove that $\Lambda_{N_N} + \Lambda_{N_i}(g \ast f) = \Lambda_{N_i}(g) \ast f$ it is enough to show that:

\[\Lambda_{N_N} \left( \Lambda_{N_i}(f) \right) = (-1)^{N_N} \mu_G \circ (J^{N_i+1}, \Lambda_{N_i}) \circ J^{N_i+1} \circ \Lambda_{N_i} \circ \Lambda_{N_i}(f) \]

We have:

\[\Lambda_{N_N} \left( \Lambda_{N_i} \right) = \Lambda_{N_N} \left( \Lambda_{N_i} \right) \circ J^{N_i+1} \circ \Lambda_{N_i} \circ \Lambda_{N_i}(f) \]

\[\Lambda_{N_N} \left( \Lambda_{N_i} \right) \circ J^{N_i+1} \circ \Lambda_{N_i} \circ \Lambda_{N_i}(f) \]

\[\Lambda_{N_N} \left( \Lambda_{N_i} \right) \circ J^{N_i+1} \circ \Lambda_{N_i} \circ \Lambda_{N_i}(f) \]

The first equality is trivial and the others follow from Lemma \[\ref{4.1}\], Lemma \[\ref{4.5}\] and Lemma \[\ref{2.33}\] —in that order. 

\[\square\]

**Theorem 4.8.** The composition described in Definition 3.4 is well-defined and makes $\mathcal{S}$ into a category.

**Proof.** The composition is well-defined by Lemma \[\ref{4.7}\]. The associativity is a straightforward but lengthy verification. 

\[\square\]

**Example 4.9.** There is a functor $j : \text{Alg}_\mathcal{G} \to \mathcal{S}$ defined by $A \mapsto (A, 0)$ on objects, that sends a morphism $f : A \to B$ to its class in $\mathcal{S}((A, 0), (B, 0))$. This functor clearly factors through $\text{Alg}_\mathcal{G} \to [\text{Alg}_\mathcal{G}]$. We often write $A$ and $f$ instead of $j(A)$ and $j(f)$.

5. **Additivity**

The hom-sets in $\mathcal{S}$ are abelian groups; indeed, they are defined as the filtered colimit of a diagram of abelian groups. Next we show that the composition is bilinear.

**Lemma 5.1.** The composition in $\mathcal{S}$ is bilinear:
Proof. Let $g \in [J^{n+v}B, C^\otimes_{n+v}]$ represent an element $\beta \in \mathcal{S}(B, n, (C, k))$. Let us show that $\beta_1 : \mathcal{S}(A, m, (B, n)) \to \mathcal{S}(A, m, (C, k))$ is a group homomorphism. Represent elements $\alpha, \alpha' \in \mathcal{S}(A, m, (B, n))$ by $f, f' \in [J^{n+v}A, B^\otimes_{n+v}]$ —we may assume that $n + v \geq 2$ by choosing $v$ large enough. To alleviate notation, write

$$N_1 := m + v, \quad N_2 := n + v, \quad N_3 := n + w \quad \text{and} \quad N_4 := k + w.$$ 

By definition of the composition in $\mathcal{S}$, the following diagram of sets commutes:

$$[J^{N_1}A, B^\otimes_{n+v}] \xrightarrow{g \circ f} [J^{N_1+N_2}A, C^\otimes_{n+2N_3}]$$

Here, the vertical arrows are structural morphisms into the colimits —hence they are group homomorphisms. Since $\beta_1(\alpha + \alpha') = (g \star (f + f'))$, to prove that $\beta_1(\alpha + \alpha') = \beta_1(\alpha) + \beta_1(\alpha')$ it suffices to show that $g \star ?$ is a group homomorphism. But $g \star ?$ can be factored as the following composite of group homomorphisms, as we proceed to explain:

$$[J^{N_1}A, B^\otimes_{n+v}] \longrightarrow [J^{N_1+N_2}A, (J^{N_1+N_2}B)^\otimes_{n+2N_3}] \longrightarrow [J^{N_1+N_2}A, (C^\otimes_{n+2N_3})^\otimes_{n+2N_3}] \longrightarrow [J^{N_1+N_2}A, C^\otimes_{n+2N_3}]$$

The morphism on the left is the one in Proposition 2.43(ii). The morphism in the middle is $(-1)^{N_1+N_2}g_\ast$. The morphism on the right is the one in Proposition 2.43(1).

Now let $f \in [J^{n+w}B, C^\otimes_{n+w}]$ represent an element $\alpha \in \mathcal{S}(A, m, (B, n))$. Let us show that $\alpha' : \mathcal{S}(B, n, (C, k)) \to \mathcal{S}(A, m, (C, k))$ is a group homomorphism. Represent elements $\beta, \beta' \in \mathcal{S}(B, n, (C, k))$ by $g, g' \in [J^{n+w}B, C^\otimes_{n+w}]$ —we may assume that $k + w \geq 2$ by choosing $w$ large enough. As before, write

$$N_1 := m + v, \quad N_2 := n + v, \quad N_3 := n + w \quad \text{and} \quad N_4 := k + w.$$ 

By definition of the composition in $\mathcal{S}$, the following diagram of sets commutes:

$$[J^{N_1}B, C^\otimes_{n+w}] \xrightarrow{g \circ f} [J^{N_1+N_2}A, C^\otimes_{n+2N_3}]$$

Since $\alpha'(\beta + \beta') = ((g + g') \star f)$, to prove that $\alpha'(\beta + \beta') = \alpha'(\beta) + \alpha'(\beta')$ it suffices to show that $? \star f$ is a group homomorphism. But $? \star f$ and the following composite of group homomorphisms differ only by the sign $(-1)^{N_2N_3}$:

$$[J^{N_1}B, C^\otimes_{n+w}] \xrightarrow{\text{Proposition 2.43(iv)}} [J^{N_3}B, C^\otimes_{n+w}] \xrightarrow{(g^{N_1} \circ f^{N_2} \circ f)^*} [J^{N_1+N_2}A, C^\otimes_{n+2N_3}]$$

Proposition 5.2. The category $\mathcal{S}$ is additive.

Proof. By Lemma 5.11 it suffices to show that $\mathcal{S}$ has finite products.
Let $B, C \in \text{Alg}_\ell$ and let $n \in \mathbb{Z}$. Let us first show that $(B \times C, n)$ is a product of $(B, n)$ and $(C, n)$ in $\mathcal{R}$. For any $(A, m) \in \mathcal{R}$, we have:

$$\mathcal{R}((A, m), (B \times C, n)) = \text{colim}_{v, r} [J^{m+r}_A, (B \times C)_r^\sim]$$

$$\cong \text{colim}_{v, r} [J^{m+r}_A, B_r^\sim \times C_r^\sim]$$

$$\cong \text{colim}_{v, r} \left\{ [J^{m+r}_A, B_r^\sim] \times [J^{m+r}_A, C_r^\sim] \right\}$$

$$\cong \text{colim}_{v, r} [J^{m+r}_A, B_r^\sim] \times \text{colim}_{v, r} [J^{m+r}_A, C_r^\sim]$$

$$= \mathcal{R}((A, m), (B, n)) \times \mathcal{R}((A, m), (C, n))$$

Here we use that the functors $(?)^\sim_r : \text{Alg}_\ell \rightarrow \text{Alg}_\ell$ and $\text{Alg}_\ell \rightarrow [\text{Alg}_\ell]$ commute with finite products, and that filtered colimits of sets commute with finite products.

To prove that $\mathcal{R}$ has finite products, we reduce to the special case above. We will show in Lemma 7.1 that, for any $(B, n) \in \mathcal{R}$ and any $p \geq 1$, we have an isomorphism $(B, n) \cong (J^p B, n - p)$. Using this, any pair of objects of $\mathcal{R}$ can be replaced by a new pair of objects—each of them isomorphic to one of the original ones—with equal second coordinate. The proof of Lemma 7.1 relies only on Lemma 4.3 and the definition of $\star$. □

6. Excision

In this section we closely follow [1] Section 6.3. Let $f : A \rightarrow B$ be a morphism in $\text{Alg}_\ell$. The mapping path $(P_f)_*$ is the $\mathbb{Z}_{\geq 0}$-diagram in $\text{Alg}_\ell$ defined by the pullbacks:

$$\begin{array}{ccc}
(P_f)_r & \xrightarrow{\pi_r} & A \\
\downarrow & & \downarrow f \\
(PB)_r & \xrightarrow{d_1} & B
\end{array}$$

(17)

Note that $\pi_r$ is a split surjection in $\text{Mod}_\ell$, since so is $d_1$. Define $\iota_f$ as the unique morphism that makes the following diagram commute:

$$\begin{array}{ccc}
B_r^\sim & \xrightarrow{\iota_f} & (P_f)_r \\
\downarrow \text{inc} & & \downarrow \pi_r \\
(PB)_r & \xrightarrow{d_1} & B
\end{array}$$

(18)

**Lemma 6.1** (cf. [1] Lemma 6.3.1). Let $f : A \rightarrow B$ be a morphism in $\text{Alg}_\ell$ and let $C \in \text{Alg}_\ell$. Then the following sequence is exact:

$$\text{colim}_s \mathcal{R}(C, (P_f)_s) \xrightarrow{(\pi_f)_s} \mathcal{R}(C, A) \xrightarrow{f_*} \mathcal{R}(C, B)$$

(19)

**Proof.** Let $s \geq 0$ and note that $\mathcal{R}(C, (PB)_s) = 0$ since $(PB)_s$ is contractible. Then the following composite is zero, because it factors through $\mathcal{R}(C, (PB)_s)$:

$$\mathcal{R}(C, (P_f)_s) \xrightarrow{(\pi_f)_s} \mathcal{R}(C, A) \xrightarrow{f_*} \mathcal{R}(C, B)$$

This shows that the composite in (19) is zero.
Now let \( g : J^mC \rightarrow A_s^{\tilde{\alpha}} \) be a morphism in \( \text{Alg}_\ell \) that represents an element \( \alpha \in \mathcal{R}(C,A) \) such that \( f_s(\alpha) = 0 \in \mathcal{R}(C,B) \). Increasing \( m \) and \( s \) if necessary, we may assume that the following composite is nullhomotopic:

\[
J^mC \xrightarrow{g} A_s^{\tilde{\alpha}} \xrightarrow{f_s^{2m}} B_s^{\tilde{\alpha}}
\]

This implies the existence of a commutative square of algebras:

\[
\begin{array}{ccc}
J^mC & \xrightarrow{g} & A_s^{\tilde{\alpha}} \\
\downarrow & & \downarrow \phi_s^{2m} \\
(B_s^{\tilde{\alpha}})_r^{(I,1)} & \xrightarrow{d_1} & B_s^{\tilde{\alpha}}
\end{array}
\]

(20)

Since \( (?)_r^{\tilde{\alpha}} \) : \( \text{Alg}_\ell \rightarrow \text{Alg}_\ell \) commutes with finite limits, we get the following pullback upon applying this functor to (17):

\[
\begin{array}{ccc}
(P_f)_s^{\tilde{\alpha}} & \xrightarrow{\pi_f^{2m}} & A_s^{\tilde{\alpha}} \\
\downarrow & & \downarrow \phi_s^{2m} \\
(B_s^{\tilde{\alpha}})_r^{(I,1)} & \xrightarrow{d_1} & B_s^{\tilde{\alpha}}
\end{array}
\]

Note that \((B_s^{\tilde{\alpha}})_r^{(I,1)} \cong (B_s^{\tilde{\alpha}})_r^{(I,1)}\). Then the commutativity of (20) determines a morphism \( J^mC \rightarrow ((P_f)_s^{\tilde{\alpha}})_r^{(I,1)} \), that in turn gives an element \( \beta \in \mathcal{R}(C,(P_f)_r) \) mapping to \( \alpha \).

**Definition 6.2.** Let \( f : A \rightarrow B \) be a morphism in \( \text{Alg}_\ell \). We call \( f \) a \( \mathcal{R} \)-equivalence if it becomes invertible upon applying \( j : \text{Alg}_\ell \rightarrow \mathcal{R} \).

**Lemma 6.3** (c.f. [1] Lemma 6.3.2). Let \( f \) be a morphism in \( \text{Alg}_\ell \) that is a split surjection in \( \text{Mod}_\ell \). Then the natural inclusions \( \ker f \rightarrow (P_f)_r \) are \( \mathcal{R} \)-equivalences for all \( r \).

**Proof.** The proof is like that of [1] Lemma 6.3.2].

Let \( f : A \rightarrow B \) be a morphism in \( \text{Alg}_\ell \) that is a split surjection in \( \text{Mod}_\ell \). As explained in the discussion following [1] Lemma 6.3.2], Lemma 6.3 implies that the morphisms \((P_f)_r \rightarrow (P_f)_{r+1}\) are \( \mathcal{R} \)-equivalences for all \( r \geq 0 \). Indeed, this follows from the ‘two out of three’ property of \( \mathcal{R} \)-equivalences. Combining this fact with Lemma 6.1 we get:

**Corollary 6.4.** Let \( f : A \rightarrow B \) be a morphism in \( \text{Alg}_\ell \) that is a split surjection in \( \text{Mod}_\ell \) and let \( C \in \text{Alg}_\ell \). The following sequence is exact:

\[
\mathcal{R}(C,(P_f)_0) \xrightarrow{(\pi_f)_0} \mathcal{R}(C,A) \xrightarrow{f_*} \mathcal{R}(C,B)
\]

**Corollary 6.5** (Corollary 6.3.3]). Let \( f : A \rightarrow B \) be a morphism in \( \text{Alg}_\ell \). Recall the definitions of \( \pi_f \) and \( \iota_f \) from (17) and (18) respectively. Let \( \phi_f : B_0^{\tilde{\pi}^1} \rightarrow (P_{\pi_f})_0 \) be the unique morphism that makes the following diagram commute:

```
B_0^{\tilde{\pi}^1} \xrightarrow{\phi_f} (P_{\pi_f})_0 \xrightarrow{\pi_f} (P_f)_0 \xrightarrow{\iota_f} (PA)_0 \xrightarrow{d_1} A
```
Lemma 6.7. Let \( \mathcal{S} \) be a \( \mathbb{K} \)-equivalence.

Proof. The morphism \( \pi_f \) is a split surjection in \( \text{Mod}_A \). The result then follows from Lemma 6.3 if we show that \( \tau_f : B_{f}^2 \rightarrow (P_f)_0 \) is a kernel of \( \pi_f \), and the latter is easily verified. \( \square \)

Corollary 6.6 (II Corollary 6.3.4). Let \( D \in \text{Alg}_A \). Then the following diagram in \( \text{Alg}_A \):

The proof of [1, Corollary 6.3.4] carries over verbatim in this setting. \( \square \)

Lemma 6.7. Let \( f : A \rightarrow B \) be a morphism in \( \text{Alg}_A \) and let \( \phi_f \) be the morphism defined in Corollary 6.5. Then the following diagram in [\( \text{Alg}_A \)] commutes:

\[
\begin{array}{ccc}
A^2_0 & \xrightarrow{(f^2)_0} & B^2_0 \\
\downarrow \text{id} & & \downarrow \text{id} \\
A^2_0 & \xrightarrow{\pi_f} & (P_f)_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
(P_f)_0 & \xrightarrow{\tau_f} & A \\
\downarrow \phi_f & & \downarrow \text{id} \\
(P_f)_0 & \xrightarrow{\pi_f} & A \\
\end{array}
\]

Proof. The map \( \phi_f : B^2_0 \rightarrow (P_f)_0 \) is given by \( (p(t),a,q(t)) \mapsto (p(0),0) \). The map \( \tau_f : B^2_0 \rightarrow (P_f)_0 \) is given by \( (p(t),0) \mapsto (p(t),0) \). We also have:

\[
\begin{align*}
PB &= (t-1)B[t] \\
B^2_0 &= (t^2-t)B[t] \\
P_f &= \{ (p(t),a) \in PB \times A : p(0) = f(a) \}
\end{align*}
\]

The map \( \phi_f : P_f \rightarrow A \) is given by \( (p(t),a) \mapsto a \). The map \( \tau_f : B^2_0 \rightarrow P_f \) is given by \( (p(t),a) \mapsto (p(t),a) \). We also have:

\[
\begin{align*}
PA &= (t-1)A[t] \\
A^2_0 &= (t^2-t)A[t] \\
P_{\tau_f} &= \{ (p(t),a,q(t)) \in PB \times A \times PA : p(0) = f(a), a = q(0) \} \\
&= \{ (p(t),q(t)) \in PB \times PA : p(0) = f(q(0)) \}
\end{align*}
\]

The map \( \phi_f : B^2_0 \rightarrow P_{\tau_f} \) is given by \( (p(t),0) \mapsto (p(t),0) \). The map \( \tau_f : A^2_0 \rightarrow P_{\tau_f} \) is given by \( (0,q(t)) \mapsto (0,q(t)) \). To prove the result, it suffices to show that the following morphisms \( A^2_0 \rightarrow P_{\tau_f} \) are homotopic:

\[
\begin{align*}
\phi_f \circ (f^2)_0^{-1} : q(t) &\mapsto (f(q(1-t)),0) \\
\tau_f : q(t) &\mapsto (0,q(t))
\end{align*}
\]

We have:

\[
P_{\tau_f}[u] = \{ (p(t,u),q(t,u)) \in (t-1)B[t,u] \times (t-1)A[t,u] : p(0,u) = f(q(0,u)) \}
\]

Let \( H : A^2_0 \rightarrow P_{\tau_f}[u] \) be the homotopy defined by:

\[
H(q(t)) = (f(q((1-t)u)),q(1-(1-t)(1-u)))
\]

Then \( H \) is an elementary homotopy between the morphisms in (21). \( \square \)
Theorem 6.8 ([1] Theorem 6.3.6). Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an extension in Alg. Then, for any $D \in \text{Alg}$, the following sequence is exact:

$$\mathcal{R}(D, B^0_0) \xrightarrow{(g^2)_*} \mathcal{R}(D, C^0_0) \xrightarrow{\partial} \mathcal{R}(D, A) \xrightarrow{f_*} \mathcal{R}(D, B) \xrightarrow{g_*} \mathcal{R}(D, C)$$

Here, the morphism $\partial$ is the composite:

$$\mathcal{R}(D, C^0_0) \xrightarrow{(\pi_g)_*} \mathcal{R}(D, (P_g)_0) \xrightarrow{\pi_*} \mathcal{R}(D, A)$$

Proof. Both $g$ and $\pi_g$ are split surjections in Mod; then, the following sequence is exact by Corollary [6.4]:

$$\mathcal{R}(D, (P_{\pi_g})_0) \xrightarrow{(\pi_g)_*} \mathcal{R}(D, (P_g)_0) \xrightarrow{(\pi_g)_*} \mathcal{R}(D, B) \xrightarrow{g_*} \mathcal{R}(D, C) \quad (22)$$

We have a commutative diagram:

![Diagram](image)

The morphism $A \to (P_g)_0$ is a $\mathcal{R}$-equivalence by Lemma [6.3]. Thus, we can replace $(\pi_g)_*$ in (22) by the composite

$$\mathcal{R}(D, (P_g)_0) \xrightarrow{=} \mathcal{R}(D, A) \xrightarrow{f_*} \mathcal{R}(D, B)$$

and we get an exact sequence:

$$\mathcal{R}(D, (P_{\pi_g})_0) \xrightarrow{=} \mathcal{R}(D, A) \xrightarrow{f_*} \mathcal{R}(D, B) \xrightarrow{g_*} \mathcal{R}(D, C) \quad (23)$$

By Corollary [6.5] we can identify $(\phi_g)_* : \mathcal{R}(D, C^0_0) \xrightarrow{=} \mathcal{R}(D, (P_{\pi_g})_0)$ and (23) becomes:

$$\mathcal{R}(D, C^0_0) \xrightarrow{(\pi_{\pi_g})_*} \mathcal{R}(D, A) \xrightarrow{f_*} \mathcal{R}(D, B) \xrightarrow{g_*} \mathcal{R}(D, C) \quad (24)$$

It is easily verified that $\partial$ is the leftmost morphism in (24); indeed, this follows from the equality $\pi_{\pi_g} \circ \phi_g = \iota_g$.

By Lemma [6.7] we have a commutative diagram as follows:

![Diagram](image)

Notice that $\ker \partial = \ker (\iota_g) : \mathcal{R}(D, C^0_0) \to \mathcal{R}(D, (P_g)_0)$. Thus, to finish the proof, it suffices to show that the top row in (25) is exact. Since $(\phi_g)_*$ and $(\phi_{\pi_g})_*$ are isomorphisms
by Corollary 6.5, the top row in (25) is exact if and only if the bottom one is. But $\pi_{x_j}$ is a split surjection in $\text{Mod}_r$, and so the bottom row in (25) is exact by Corollary 6.4. \hfill \Box

7. The translation functor

Define a functor $L : \mathcal{R} \to \mathcal{R}$ as follows. For $(A, m) \in \mathcal{R}$, put $L(A, m) := (A, m + 1)$. Let $L$ act as the identity on morphisms. It is clear that $L$ is an automorphism of $\mathcal{R}$.

Recall the definition of $\lambda_B : JB \to B_0^{\geq 1}$ from Example 2.28. For $m \in \mathbb{Z}$ we can consider:

$$\langle \lambda_B \rangle \in \mathcal{R}((JB, m), (B, 1 + m))$$

$$\langle \text{id}_{JB} \rangle \in \mathcal{R}((B, 1 + m), (JB, m))$$

These two morphisms are mutually inverses in $\mathcal{R}$, as we prove below. From now on, each time we identify $(B, 1 + m) \cong (JB, m)$ it will be through these isomorphisms. Using this identification $n$ times we get an isomorphism $(B, n + m) \cong (J^n B, m)$. It is easily verified that the latter is represented by $\text{id}_{J^n B}$.

**Lemma 7.1.** The morphisms in (26) are mutually inverses.

**Proof.** We have that $\langle \text{id}_{JB} \rangle \circ \langle \lambda_B \rangle = \langle \text{id}_{JB} \star \lambda_B \rangle$, where $\text{id}_{JB} \star \lambda_B$ equals the following composite in $[\text{Alg}_x]^{\text{ind}}$:

$$J(JB) \xrightarrow{J(\lambda_B)} J(B_0^{\geq 1}) \xrightarrow{\text{id}_{B_0^{\geq 1}}} (JB)_0^{\geq 1}$$

By Lemma 4.3, $\text{id}_{JB} \star \lambda_B = \lambda_{JB}$ and thus $\langle \text{id}_{JB} \rangle \circ \langle \lambda_B \rangle = \langle \lambda_{JB} \rangle = \text{id}_{(JB,m)}$.

We have that $\langle \lambda_B \rangle \circ \langle \text{id}_{JB} \rangle = \langle \lambda_B \star \text{id}_{JB} \rangle$. It is easily verified that $\lambda_B \star \text{id}_{JB} = \lambda_B$ and thus $\langle \lambda_B \rangle \circ \langle \text{id}_{JB} \rangle = \langle \lambda_B \rangle \circ \langle \text{id}_{B(1+m)} \rangle$. \hfill \Box

We can also consider:

$$\langle \lambda_B \rangle \in \mathcal{R}((B, 1 + m), (B_0^{\geq 1}, m))$$

$$\langle \text{id}_{B_0^{\geq 1}} \rangle \in \mathcal{R}((B_0^{\geq 1}, m), (B, 1 + m))$$

We will show below that these morphisms are mutually inverses. Each time we identify $(B, 1 + m) \cong (B_0^{\geq 1}, m)$ it will be through these isomorphisms.

**Lemma 7.2 (11 Lemma 6.3.10).** The morphism $\lambda_B : JB \to B_0^{\geq 1}$ is a $\mathcal{R}$-equivalence.

**Proof.** The proof of [11 Lemma 6.3.10] works verbatim, using Theorem 6.8. \hfill \Box

**Lemma 7.3.** The morphisms in (27) are mutually inverses.

**Proof.** We claim that $\langle \lambda_B \rangle$ is an isomorphism. To see this, notice that it equals the following composite:

$$(B, 1 + m) \xrightarrow{\text{id}_{JB}} (JB, m) \xrightarrow{L^n(\lambda_B)} (B_0^{\geq 1}, m)$$

Here, $\langle \text{id}_{JB} \rangle$ is an isomorphism by Lemma 7.1 and $L^n(\lambda_B)$ is an isomorphism by Lemma 7.2. This proves the claim. To prove the lemma it suffices to show that $\langle \text{id}_{B_0^{\geq 1}} \rangle \circ \langle \lambda_B \rangle = \text{id}_{(B,1+m)}$. This follows immediately from the definitions. \hfill \Box

**Remark 7.4.** By Lemma 7.3 there is a natural isomorphism $j \circ (?)_0^{\geq 1} \cong L \circ j$ of functors $\text{Alg}_x \to \mathcal{R}$. Hence, a morphism $f$ in $\text{Alg}_x$ is a $\mathcal{R}$-equivalence if and only if $f_0^{\geq 1}$ is.

**Lemma 7.5.** The morphisms $B_r^{\geq a} \to B_{r+1}^{\geq a}$ are $\mathcal{R}$-equivalences.
Proof. We proceed by induction on \( n \). The result holds for \( n = 0 \) as in this case \( B_f^{\geq 0} \to B_f^{\geq 0} \) is the identity of \( B \). For the inductive step, consider the morphism of extensions:

\[
\begin{array}{ccc}
B_f^{\geq 0} & \xrightarrow{ \mu^m } & P(m, B)_{r+1} \\
\downarrow & & \downarrow \\
B_f^{\geq 0} & \xrightarrow{ \mu^m } & B_f^{\geq 0}
\end{array}
\]

(28)

The middle vertical morphism is a \( \mathfrak{R} \)-equivalence since both its source and target are contractible by Lemma 2.31. The right vertical morphism is a \( \mathfrak{R} \)-equivalence by induction hypothesis. Then the left vertical morphism is a \( \mathfrak{R} \)-equivalence too by Theorem 6.8.

Lemma 7.6. The morphisms \( \mu_B^{m,n} : (B_f^{\geq n})^\gamma \to B_f^{\geq n+1} \) are \( \mathfrak{R} \)-equivalences for all \( m \) and \( n \).

Proof. Let us start with the case \( n = 1 \). Consider the following morphism of extensions:

\[
\begin{array}{ccc}
(B_f^{\geq n})^\gamma & \xrightarrow{ \mu^m } & P(B_f^{\geq n}), \\
\mu^{m+1} & & \mu^{m+1} \\
B_f^{\geq m+1} & \xrightarrow{ (\gamma^*)^r } & B_f^{\geq m+1}
\end{array}
\]

(29)

The result follows from Theorem 6.8 since \( P(B_f^{\geq n}) \) and \( P(m, B)_{r+1} \) are contractible by Lemma 2.31 and \( (\gamma^*)^r \) is a \( \mathfrak{R} \)-equivalence by Lemma 7.5.

We will prove the general case by induction on \( n \). Suppose that \( \mu_B^{m,n} \) is a \( \mathfrak{R} \)-equivalence for all \( m \). The following square commutes by the associativity of \( \mu \):

\[
\begin{array}{ccc}
\left((B_f^{\geq n})^\gamma\right)^\gamma & \xrightarrow{ \mu^m } & \left(B_f^{\geq n}\right)^\gamma, \\
\mu^m & & \mu^m \\
B_f^{\geq m+1} & \xrightarrow{ \mu^m } & B_f^{\geq m+1}
\end{array}
\]

The horizontal morphisms are \( \mathfrak{R} \)-equivalences by the case \( n = 1 \) and the left vertical morphism is a \( \mathfrak{R} \)-equivalence since \( \mu_B^{m,n+1} \) is; see Remark 7.4. Then \( \mu_B^{m,n+1} \) is a \( \mathfrak{R} \)-equivalence. □

Lemma 7.7. The identity of \( B_f^{\geq n} \) induces an isomorphism \( \langle \text{id}_{B_f^{\geq n}} \rangle \in \mathfrak{R}((B_f^{\geq n}, m), (B, n + m)). \)

Proof. By Lemma 7.5 we may assume that \( r = 0 \). We will proceed by induction on \( n \). The case \( n = 1 \) holds by Lemma 7.3. Suppose now that the result holds for \( n \geq 1 \). It is easily verified that the following diagram in \( \mathfrak{R} \) commutes:

\[
\begin{array}{ccc}
(B_0^{\geq n}, m) & \xrightarrow{ L^n(\mu_B^{n,1}) } & (B_0^{\geq n}, 1 + m) \\
\downarrow \text{id}_{B_f^{n+1}} & & \downarrow \text{id}_{B_f^{n+1}} \\
(B_0^{\geq n}, m) & \xrightarrow{ (\text{id}_{B_f^{n+1}}) } & (B, n + 1 + m)
\end{array}
\]

The top horizontal morphism is an isomorphism by Lemma 7.3, the right vertical one is an isomorphism by induction hypothesis, and \( L^n(\mu_B^{n,1}) \) is an isomorphism by Lemma 7.6. □
Remark 7.8. By Lemma [7.7] we have natural isomorphisms \( j \circ (\bar{g}_{0}^n) \cong L^n \circ j \) and \( (\bar{g}_{0}^n) \cong L^{n+1} \circ j \). Indeed, it is easily seen that the following squares commute for every morphism \( f : A \to B \) in \( \text{Alg}_f \):

\[
\begin{array}{c}
\begin{array}{c}
 (A_0^n, 0) \\
 \downarrow \text{id}_{A_0^n} \\
 (A, n)
\end{array}
\quad \quad \quad \\
\begin{array}{c}
 ((A_0^n, 0), 0) \\
 \downarrow \text{id}_{(A_0^n, 0)} \\
 (A, n+1)
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
 (B_0^n, 0) \\
 \downarrow \text{id}_{B_0^n} \\
 (B, n)
\end{array}
\quad \quad \quad \\
\begin{array}{c}
 ((B_0^n, 0), 0) \\
 \downarrow \text{id}_{(B_0^n, 0)} \\
 (B, n+1)
\end{array}
\end{array}
\end{array}
\]

Lemma 7.9. Let \( \alpha \in \mathcal{R}(A, m), (B, n) \) be represented by \( f : J^{m+n}A \to B_{r}^{\bar{g}_{0}^{n+1}} \). Then the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{c}
 (A, m) \\
 \downarrow \text{id}_{(A, m)} \\
 (J^{m+n}A, -u)
\end{array}
\quad \quad \quad \\
\begin{array}{c}
 (B, n) \\
 \downarrow \text{id}_{(B, n)} \\
 (B_{r}^{\bar{g}_{0}^{n+1}}, -u)
\end{array}
\end{array}
\]

Proof. It is a straightforward computation. □

8. Long exact sequences associated to extensions

Lemma 8.1. Let \( \varepsilon : A \xrightarrow{\bar{g}} B \xrightarrow{\bar{f}} C \) be an extension. Then, for every \( l \)-algebra \( D \), there is a natural long exact sequence of abelian groups:

\[
\cdots \to \mathcal{R}(D, (A, n)) \xrightarrow{f} \mathcal{R}(D, (B, n)) \xrightarrow{g} \mathcal{R}(D, (C, n)) \xrightarrow{\bar{g}} \mathcal{R}(D, (A, n-1)) \to \cdots
\]

Proof. By Theorem [6.8] applied to the extension \( \varepsilon_0^{\bar{g}_{0}^n} \) we have an exact sequence:

\[
\mathcal{R}(D, (B_0^{\bar{g}_{0}^{n+1}})) \xrightarrow{g} \mathcal{R}(D, (C_0^{\bar{g}_{0}^{n+1}})) \xrightarrow{\bar{g}} \mathcal{R}(D, (A_0^{\bar{g}_{0}^n})) \xrightarrow{f} \mathcal{R}(D, (B_0^{\bar{g}_{0}^n})) \xrightarrow{g} \mathcal{R}(D, (C_0^{\bar{g}_{0}^n}))
\]

Under the identifications described in Remark [7.8] the latter becomes:

\[
\mathcal{R}(D, (B, n+1)) \xrightarrow{g} \mathcal{R}(D, (C, n+1)) \xrightarrow{\bar{g}} \mathcal{R}(D, (A, n)) \xrightarrow{f} \mathcal{R}(D, (B, n)) \xrightarrow{g} \mathcal{R}(D, (C, n))
\]

For varying \( n \geq 0 \), these sequences assemble into an exact sequence, infinite to the left, ending in \( \mathcal{R}(D, (C, 0)) \). It remains to show how to extend this sequence to the right. Upon applying \( \mathcal{R}(D_0^{\bar{g}_{0}^n}, ?) \) to \( \varepsilon \), we get an exact sequence:

\[
\mathcal{R}(D_0^{\bar{g}_{0}^n}, (B, 1)) \xrightarrow{g} \mathcal{R}(D_0^{\bar{g}_{0}^n}, (C, 1)) \xrightarrow{\bar{g}} \mathcal{R}(D_0^{\bar{g}_{0}^n}, A) \xrightarrow{f} \mathcal{R}(D_0^{\bar{g}_{0}^n}, B) \xrightarrow{g} \mathcal{R}(D_0^{\bar{g}_{0}^n}, C)
\]

After identifying \( \mathcal{R}(D_0^{\bar{g}_{0}^n}, ?) \equiv \mathcal{R}((D, n), ?) \equiv \mathcal{R}(D, L^{-n}(?)) \), this sequence becomes:

\[
\mathcal{R}(D, (B, 1-n)) \xrightarrow{g} \mathcal{R}(D, (C, 1-n)) \xrightarrow{\bar{g}} \mathcal{R}(D, (A, -n)) \xrightarrow{f} \mathcal{R}(D, (B, -n)) \xrightarrow{g} \mathcal{R}(D, (C, -n))
\]

Now glue these for varying \( n \geq 0 \) to extend the exact sequence to the right. □

Lemma 8.2. Let \( D \in \text{Alg}_f \). For every pullback square in \( \text{Alg}_f \)

\[
\begin{array}{c}
\begin{array}{c}
 B' \\
 \downarrow \\
 B
\end{array}
\quad \quad \quad \\
\begin{array}{c}
 C' \\
 \downarrow \\
 C
\end{array}
\end{array}
\]

\[
\begin{array}{c}
 B' \xrightarrow{g} C' \\
 \downarrow \\
 B \xrightarrow{g} C
\end{array}
\]
where \( g \) is a split surjection in \( \text{Mod}_r \), there is a long exact Mayer-Vietoris sequence:

\[
\cdots \rightarrow \mathcal{R}(D, (B', n)) \rightarrow \mathcal{R}(D, (B, n)) \oplus \mathcal{R}(D, (C', n)) \rightarrow \mathcal{R}(D, (C, n)) \rightarrow \mathcal{R}(D, (B', n-1)) \rightarrow \cdots
\]

**Proof.** It follows from Lemma 8.1 and the argument explained in [3, Theorem 2.41]. \( \square \)

**Corollary 8.3.** Let \( f \) be any morphism in \( \text{Alg}_r \). Then the morphisms \( (P_f)_r \rightarrow (P_f)_{r+1} \) are \( \mathcal{R} \)-equivalences for all \( r \).

**Proof.** Pulling back the path extension \( \mathcal{P}_{A,B} \) along \( f : A \rightarrow B \), we get a long exact Mayer-Vietoris sequence that takes the following form, since \((PB)_r \) is contractible:

\[
\cdots \rightarrow \mathcal{R}(D, (A, 1)) \rightarrow \mathcal{R}(D, (B, 1)) \rightarrow \mathcal{R}(D, (P_f)_r)) \rightarrow \mathcal{R}(D, (A)) \rightarrow \mathcal{R}(D, (B)) \rightarrow \cdots
\]

As this sequence is natural in \( r \), the result follows from the five lemma and Yoneda. \( \square \)

### 9. Triangulated structure

We will use the definition of triangulated category given in [9]. As explained in [1, Section 6.5], the definition of triangulated category is self dual, and we will actually prove that \( \mathcal{R}^{op} \) is a triangulated category, since this is more natural in our context. Recall the definitions of \( \pi_f \) and \( t_f \) from [17] and [18].

**Definition 9.1.** We call mapping path triangle to a diagram in \( \mathcal{R} \) of the form

\[
\Delta_{f,n} : \quad L(B, n) \xrightarrow{\partial_{f,n}} ((P_f)_0, n) \xrightarrow{L^*(\pi_f)} (A, n) \xrightarrow{L^*(f)} (B, n),
\]

where \( f : A \rightarrow B \) is a morphism in \( \text{Alg}_r \), \( n \in \mathbb{Z} \) and \( \partial_{f,n} \) equals the composite:

\[
(B, n + 1) \xrightarrow{(\text{id}, \text{id})} (B^0_{n+1}, n) \xrightarrow{(-1)^{n+1}L^*(f)} ((P_f)_0, n)
\]

A distinguished triangle in \( \mathcal{R} \) is a triangle isomorphic (as a triangle) to some \( \Delta_{f,n} \).

We are ready to verify that \( \mathcal{R} \) satisfies the axioms of a triangulated category with the translation functor \( L \) and the distinguished triangles defined above.

**Axiom 9.2 (TR0).** Any triangle which is isomorphic to a distinguished triangle is itself distinguished. For any \( B \in \text{Alg}_r \) and any \( n \in \mathbb{Z} \), the following triangle is distinguished:

\[
L(B, n) \xrightarrow{0} (B, n) \xrightarrow{\text{id}} (B, n)
\]

**Proof.** It follows from the fact that the mapping path \( (P_{\text{id}})_0 \equiv (PB)_0 \) is contractible. \( \square \)

**Axiom 9.3 (TR1).** Every morphism \( \alpha \) in \( \mathcal{R} \) fits into a distinguished triangle of the form:

\[
\Delta(Y) \xrightarrow{Z} X \xrightarrow{\alpha} Y
\]

**Proof.** By Lemma [7,9] we can assume that \( X = (C, k) \), \( Y = (B, k) \) and \( \alpha = L^k(f) \) with \( f : C \rightarrow D \) a morphism in \( \text{Alg}_r \). In this case \( \alpha \) fits into the mapping path triangle \( \Delta_{f,k} \). \( \square \)

**Definition 9.4.** Consider a triangle \( \Delta \) in \( \mathcal{R} \):

\[
\alpha : \quad L(Z) \xrightarrow{\alpha} X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z \quad (30)
\]

Define the rotated triangle \( R(\Delta) \) by:

\[
R(\Delta) : \quad L(Y) \xrightarrow{-L^Y} L(Z) \xrightarrow{-\alpha} X \xrightarrow{-\beta} Y
\]
Remark 9.5. As explained in [4] Definition 6.51, we have an isomorphism:

\[ R(\Delta) \cong (L(Y) \xrightarrow{\gamma} L(Z) \xrightarrow{\beta} X \xrightarrow{\alpha} Y) \]

Axiom 9.6 (TR2). A triangle \( \triangle \) is distinguished if and only if \( R(\Delta) \) is.

Proof. Let us first show that if \( \triangle \) is distinguished, then \( R(\Delta) \) is distinguished as well. It suffices to prove that the rotation of a mapping path triangle is distinguished. Let \( f : A \to B \) be a morphism in Alg\( _\ell \) and consider the following mapping path triangles:

\[
\begin{align*}
\triangle_{f,n} : \quad L(B, n) &\xrightarrow{\partial_{f,n}} ((P_f)_0, n) \xrightarrow{L^*(\pi_1)} (A, n) \xrightarrow{L^*(f)} (B, n) \\
\triangle_{\pi_1,n} : \quad L(A, n) &\xrightarrow{\partial_{\pi_1,n}} ((P_{\pi_1})_0, n) \xrightarrow{L^*(\pi_1)} ((P_f)_0, n) \xrightarrow{L^*(f)} (A, n)
\end{align*}
\]

Let \( \epsilon : L(B, n) \to ((P_{\pi_1})_0, n) \) be the following composite, where \( \phi_f \) is the morphism defined in Corollary 6.5,

\[ (B, n + 1) \xrightarrow{(\phi_f)^*} (B_0, n) \xrightarrow{(-1)^n L^*(\phi_f)} ((P_{\pi_1})_0, n) \]

Notice that \( \epsilon \) is an isomorphism by Corollary 6.5. It follows from Lemma 6.7 that we have an isomorphism \( R(\Delta_{f,n}) \cong \triangle_{\pi_1,n} \) as follows:

\[
\begin{align*}
R(\Delta_{f,n}) &\xrightarrow{\epsilon} L(A, n) \xrightarrow{\partial_{\pi_1,n}} (P_{\pi_1})_0, n) \xrightarrow{L^*(\pi_1)} ((P_f)_0, n) \xrightarrow{L^*(f)} (A, n) \\
&\xrightarrow{\text{id}} L(A, n) \xrightarrow{\partial_{f,n}} ((P_f)_0, n) \xrightarrow{L^*(f)} (A, n)
\end{align*}
\]

This shows that the rotation of a mapping path triangle is distinguished.

We still have to prove that if \( R(\Delta) \) is distinguished, then \( \Delta \) is distinguished. We claim that if \( R^3(\Delta) \) is distinguished, then \( \triangle \) is distinguished; suppose for a moment that this claim is proved. If \( R(\Delta) \) is distinguished then \( R^3(\Delta) \) is distinguished —because \( R \) preserves distinguished triangles— and so \( \triangle \) is distinguished —by the claim. Thus, the proof will be finished if we prove the claim. Let \( \Delta \) be the triangle in (30). Then:

\[ R^3(\Delta) \cong (L^2(Z) \xrightarrow{-L^2} L(X) \xrightarrow{L^\phi} L(Y) \xrightarrow{L^\gamma} L(Z)) \]

Suppose that \( R^3(\Delta) \) is distinguished. Then there exists a morphism \( f : A \to B \) in Alg\( _\ell \) that fits into an isomorphism of triangles as follows:

\[
\begin{align*}
L^2(Z) &\xrightarrow{-L^2} L(X) \xrightarrow{L^\phi} L(Y) \xrightarrow{L^\gamma} L(Z) \\
&\xrightarrow{\text{id}} L(A, n) \xrightarrow{\partial_{f,n}} ((P_f)_0, n) \xrightarrow{L^*(f)} (A, n) \xrightarrow{L^*(f)} (B, n)
\end{align*}
\]

Upon applying \( L^{-1} \) to (31) we get a commutative diagram as follows:

\[
\begin{align*}
L(Z) &\xrightarrow{-a} X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z \\
&\xrightarrow{\text{id}} (B, n) \xrightarrow{\partial_{f,n-1}} ((P_f)_0, n - 1) \xrightarrow{L^*(f)(n)} (A, n - 1) \xrightarrow{L^*(f)} (B, n - 1)
\end{align*}
\]
Thus, the vertical morphisms in the latter diagram assemble into an isomorphism of triangles $\triangle \cong \triangle_{f,n-1}$. Then $\triangle$ is distinguished. 

\begin{lemma}
Let $f : A \to B$ be a morphism in $\text{Alg}_\ell$, let $k \in \mathbb{Z}$ and let $n \geq 0$. Then there is a morphism of triangles:

\[
\begin{array}{cccc}
\triangle_{f,k+n} & L(B, k + n) & \rightarrow (P_{f})_{0}, k + n & \rightarrow (A, k + n) & \rightarrow (B, k + n) \\
= & \Downarrow (\text{id}_{v_{k+n}}) & = & \Downarrow (\text{id}_{v_{k+n}}) & = \\
\triangle_{J(f),k} & L(J^{k}B, k) & \rightarrow ((P_{\text{id}})_{0}, k) & \rightarrow (J^{k}A, k) & \rightarrow (J^{k}B, k)
\end{array}
\]

\begin{proof}
It is enough to construct the morphism for $n = 1$ and then consider the composite:

\[
\triangle_{f,k+1} \rightarrow \triangle_{J(f),k+1} \rightarrow \triangle_{J^{2}(f),k+2} \rightarrow \cdots \rightarrow \triangle_{J^{k}(f),k}
\]

Let $c : (P_{f})_{0} \rightarrow (P_{J(f)})_{0}$ be the morphism defined by the following diagram in $\text{Alg}_\ell$:

\[
\begin{array}{cccc}
J(P_{f})_{0} & \rightarrow & P(JB)_{0} & \rightarrow JBP_{0} \\
J(\pi_{1}) & \rightarrow & \,
(\text{id}_{c}) & \downarrow
\end{array}
\]

\[
\begin{array}{cccc}
J(P_{f})_{0} & \rightarrow & (P_{J(f)})_{0} & \rightarrow JBP_{0} \\
\pi_{J(f)} & \rightarrow & J(\text{id}) & \downarrow
\end{array}
\]

\[
\begin{array}{cccc}
J(P_{f})_{0} & \rightarrow & (P_{J(f)})_{0} & \rightarrow JBP_{0} \\
\pi_{J(f)} & \rightarrow & J(\text{id}) & \downarrow
\end{array}
\]

\[
\begin{array}{cccc}
J(P_{f})_{0} & \rightarrow & (P_{J(f)})_{0} & \rightarrow JBP_{0} \\
\pi_{J(f)} & \rightarrow & J(\text{id}) & \downarrow
\end{array}
\]

It is easily verified that the following diagram commutes, where the unlabelled vertical morphisms are induced by the natural isomorphism $j \circ J \cong L \circ j : \text{Alg}_\ell \rightarrow \mathcal{R}$:

\[
\begin{array}{cccc}
(B_{0}^{\varepsilon_{1}}, k + 1) & \rightarrow (P_{f})_{0}, k + 1 & \rightarrow (A, k + 1) & \rightarrow (B, k + 1) \\
\rightarrow & \Downarrow (\text{id}_{v_{k+n}}) & = & \Downarrow (\text{id}_{v_{k+n}}) & = \\
(JB_{0}^{\varepsilon_{1}}, k) & \rightarrow (J^{k}B_{0}^{\varepsilon_{1}}, k) & \rightarrow (JA, k) & \rightarrow (JB, k) \\
\Downarrow (\sigma_{k}^{+}) & \Downarrow (\text{id}_{\sigma_{k}^{+}}) & \Downarrow (\text{id}_{\sigma_{k}^{+}}) & \Downarrow (\text{id}_{\sigma_{k}^{+}}) & \Downarrow (\text{id}_{\sigma_{k}^{+}})
\end{array}
\]

Another straightforward computation shows that the following square also commutes:

\[
\begin{array}{cccc}
(B, k + 2) & \rightarrow (B_{0}^{\varepsilon_{1}}, k + 1) \\
\rightarrow & \Downarrow (\text{id}_{v_{k+n}}) & = & \Downarrow (\text{id}_{v_{k+n}}) & = \\
(JB, k + 1) & \rightarrow (JB_{0}^{\varepsilon_{1}}, k) \\
\Downarrow (\sigma_{k}^{+}) & \Downarrow (\text{id}_{\sigma_{k}^{+}}) & \Downarrow (\text{id}_{\sigma_{k}^{+}}) & \Downarrow (\text{id}_{\sigma_{k}^{+}})
\end{array}
\]

To get the desired morphism of triangles $\triangle_{f,k+1} \rightarrow \triangle_{J(f),k}$, join both diagrams above. 
\end{proof}
\end{lemma}
Lemma 9.8. Let $f : A \rightarrow B$ be a morphism in $\text{Alg}_r$, let $k \in \mathbb{Z}$ and let $n \geq 0$. Then there is an isomorphism of triangles:

\[
\begin{array}{ccccccc}
\Delta_{f^{2n+k}} & L(B_r^{\sim_k}, k) & ((P_{f^{2n+k}})_0, k) & (A_r^{\sim_k}, k) & (B_r^{\sim_k}, k) \\
\cong & \cong & \cong & \cong & \cong \\
\Delta_{f^{n+k}} & L(B, n + k) & ((P_f)_0, n + k) & (A, n + k) & (B, n + k)
\end{array}
\]

Proof. It is easily verified that there is an isomorphism $((P_{f^0})_0)^{\sim_k} \cong (P_{f^{2n}})_0$. We have a commutative diagram as follows, where the vertical morphisms from the second row to the first one are induced by the natural isomorphism $j \circ (\cdot)^{\sim_k} \cong L^n \circ j : \text{Alg}_r \rightarrow \mathbb{R}$:

\[
\begin{array}{ccccccc}
(B_0^{\sim_k}, n + k) & L^{n+k}(\cdot) & ((P_{f^0})_0, n + k) & L^{n+k}(\cdot) & (A, n + k) & L^{n+k}(\cdot) & (B, n + k) \\
\cong & \cong & \cong & \cong & \cong \\
((B_0^{\sim_k})^{\sim_k}, k) & L^1(\cdot) & (((P_{f^0})_0)^{\sim_k}, k) & L^1(\cdot) & (A_k^{\sim_k}, k) & L^1(\cdot) & (B_k^{\sim_k}, k) \\
\cong & \cong & \cong & \cong & \cong & \cong & \cong \\
((B_0^{\sim_k})^{\sim_k}, k) & L^1(\cdot) & (((P_{f^0})_0)^{\sim_k}, k) & L^1(\cdot) & (A_k^{\sim_k}, k) & L^1(\cdot) & (B_k^{\sim_k}, k)
\end{array}
\]

A straightforward computation shows that the following square commutes:

\[
\begin{array}{ccccccc}
(B, n + k + 1) & (\text{id}_{B_0^{\sim_k}}) & (B_0^{\sim_k}, n + k) \\
\cong & \cong & \cong \\
(B_k^{\sim_k}, k + 1) & (-1)^n(\text{id}_{\cdot}) & ((B_0^{\sim_k})^{\sim_k}, k)
\end{array}
\]

To get the desired isomorphism of triangles $\Delta_{f^{2n+k}} \cong \Delta_{f^{n+k}}$, join both diagrams above. \(\square\)

Axiom 9.9 (TR3). For every diagram of solid arrows as follows, in which the rows are distinguished triangles, there exists a dotted arrow that makes the whole diagram commute.

\[
\begin{array}{ccccccc}
L(Z') & X' & Y' & Z' \\
& \downarrow & \downarrow & \downarrow \\
L(Z) & X & Y & Z
\end{array}
\]

(32)

Proof. We follow [3], Axiom 6.53, making appropriate changes. Let us begin with a special case. Consider a commutative square in $[\text{Alg}_r]$:

\[
\begin{array}{cccc}
A' & f' & B' \\
\downarrow & \downarrow & \downarrow \\
A & f & B
\end{array}
\]

\[
A' \xrightarrow{f'} B' \\
\downarrow \quad \quad \quad \downarrow \\
A \xrightarrow{f} B
\]
Suppose that (32) takes the following form, where the rows are mapping path triangles:

\[
\begin{array}{cccccc}
L(B', n) & \xrightarrow{\partial_{n+}} & ((P_f)_0, n) & \xrightarrow{L^r(\pi)} & (A', n) & \xrightarrow{L^r(f)} & (B', n) \\
\downarrow{\partial_{n+}} & & \downarrow{L^r(\pi)} & & \downarrow{L^r} & & \\
L(B, n) & & ((P_f)_0, n) & & (A, n) & & (B, n)
\end{array}
\]

(33)

We want to show that a dotted arrow exists in this case. Let \( H : A' \to B^0 \) be a homotopy such that \( d_1 \circ H = f \circ a \) and \( d_0 \circ H = b \circ f' \); we may assume \( r \geq 1 \). We have:

\[
(P_f)_r = \{(x, y) \in A' \times (PB')_r \mid f'(x) = d_1(y)\}
\]

\[
(P_f)_{r+1} = \{(x, y) \in A \times (PB)_{r+1} \mid f(x) = d_1(y)\}
\]

Define a morphism \( c : (P_f)_r \to (P_f)_{r+1} \) by the formula:

\[
c(x, y) = (a(x), H(x) \bullet P(b)(y))
\]

Here, \( \bullet \) stands for concatenation of paths; see Example 2.18. Note that this concatenation makes sense since \( d_0(H(x)) = b(f'(x)) = b(d_1(y)) = d_1(P(b)(y)) \). Moreover, \( c(x, y) \) is indeed an element of \((P_f)_{r+1}\) since we have:

\[
d_1(H(x) \bullet P(b)(y)) = d_1(H(x)) = f(a(x))
\]

Let \( \chi : ((P_f)_0, n) \to ((P_f)_0, n) \) be the composite:

\[
\begin{array}{cccccc}
((P_f)_0, n) & \xrightarrow{\pi^r} & ((P_f)_r, n) & \xrightarrow{L^r(c)} & ((P_f)_{r+1}, n) & \xrightarrow{\pi^r} & ((P_f)_0, n)
\end{array}
\]

We claim that taking the dotted arrow in (33) equal to \( \chi \) makes the whole diagram commute. It is easily verified that the following square commutes in \( \text{Alg}_I \), and this implies that the middle square in (33) commutes:

\[
\begin{array}{ccc}
(P_f)_r & \xrightarrow{\pi^r} & A' \\
\downarrow{c} & & \downarrow{a} \\
(P_f)_{r+1} & \xrightarrow{\pi^r} & A
\end{array}
\]

Another straightforward computation shows that the following diagram commutes in \([\text{Alg}_I]\), and this implies that the left square in (33) commutes:

\[
\begin{array}{cccc}
(B'_{r+1}) & \xrightarrow{\pi^r} & (P_f)_r \\
\downarrow{b^2} & & \downarrow{c} \\
B'_{r+1} & & (P_f)_{r+1}
\end{array}
\]

This finishes the proof of the axiom in this special case.

In the general case, we may suppose that both triangles are mapping path triangles, so that (32) equals the following diagram, for some morphisms \( f : A \to B \) and \( f' : A' \to B' \):

\[
\begin{array}{cccccc}
\Delta f \cdot \Delta x & L(B', k') & \xrightarrow{((P_f)_0, k')} & (A', k') & \xrightarrow{(B', k')} \\
& & \downarrow{a} & & \downarrow{\beta} \\
\Delta f \cdot \Delta k & L(B, k) & \xrightarrow{((P_f)_0, k)} & (A, k) & \xrightarrow{(B, k)}
\end{array}
\]

(34)
We may choose \( l \) and \( r \) large enough so that \( \alpha \) is represented by 
\[
\aligned
J^{k+i}A & \to A^{'2k+i}, \\
J^{k+i}B & \to B^{'2k+i},
\endaligned
\]
and the following square in \([\mathrm{Alg}_\ell]\) commutes:

\[
\begin{array}{ccc}
J^{k+i}A & \to & J^{k+i}B \\
\downarrow a & & \downarrow b \\
A^{'2k+i} & \to & B^{'2k+i}
\end{array}
\]

By the special case we have already proven, we can extend \( a \) and \( b \) to a morphism of triangles 
\[
\delta_{f,k} \to \delta_{f^{'2k+i},-l}
\]

\( \Delta f, \delta_{f,k} \to \Delta f^{'2k+i}, \delta_{f,k} \) is a morphism of triangles that extends the diagram of solid arrows in (34). □

**Axiom 9.10 (TR4).** Let \( \alpha : X \to X' \) and \( \pi' : X' \to Y \) be composable morphisms in \( \mathcal{K} \) and put \( \pi := \pi' \circ \alpha \). Then there exist commutative diagrams as follow, where the rows and columns of the diagram on the left are distinguished triangles.

\[
\begin{array}{cccc}
L^2Y & \xrightarrow{L\pi'} & LX' & \xrightarrow{L\pi} & LY \\
\downarrow & \cdot & \downarrow & \cdot & \downarrow \\
0 & \xrightarrow{\phi} & Z'' & \xrightarrow{\phi} & 0 \\
\downarrow & \cdot & \downarrow & \cdot & \downarrow \\
LY & \xrightarrow{\pi} & X & \xrightarrow{\pi} & Y \\
\downarrow & \cdot & \downarrow & \cdot & \downarrow \\
LY & \xrightarrow{\pi'} & X' & \xrightarrow{\pi'} & Y
\end{array}
\]

**Proof.** Let \( \alpha : X \to X' \) and \( \pi' : X' \to Y \) be composable morphisms and consider the following diagram in \( \mathcal{K} \):

\[
\begin{array}{cccc}
X & \xrightarrow{\alpha} & X' & \xrightarrow{\pi'} & Y
\end{array}
\]

We say that (35) satisfies (TR4) if the axiom holds for this particular pair of morphisms. The following remarks are straightforward:

(i) If two diagrams like (35) are isomorphic, then one satisfies (TR4) if and only if the other does.

(ii) Any diagram like (35) is isomorphic to the diagram below, for some \( n \in \mathbb{Z} \) and some morphisms \( a : A \to B \) and \( b : B \to C \) in \( \mathrm{Alg}_\ell \):

\[
\begin{array}{cccc}
(A, n) & \xrightarrow{L^{(a)}} & (B, n) & \xrightarrow{L^{(b)}} & (C, n)
\end{array}
\]

(Apply Lemma 7.9 twice, once for \( \alpha \) and once for \( \pi' \).)

(iii) The diagram (35) satisfies (TR4) if and only if the following diagram does:

\[
\begin{array}{cccc}
LY & \xrightarrow{L\pi'} & LX' & \xrightarrow{L\pi} & LY
\end{array}
\]

By (i), (ii) and (iii), we may assume that (35) is of the form

\[
\begin{array}{cccc}
(A, 0) & \xrightarrow{a} & (B, 0) & \xrightarrow{b} & (C, 0)
\end{array}
\]
where \( a : A \to B \) and \( b : B \to C \) are morphisms in \( \text{Alg}_c \). We will proceed as explained in [1, Axiom 6.5.7] but we will provide some more details. Put \( c := b \circ a : A \to C \). We will use the identifications in Lemma [6,7] so that we have, for example:

\[
(P\text{C})_0 = (t - 1)C[t]
\]

\[
C^\xi_1 = (t^2 - t)C[t]
\]

\[
(P_b)_0 = \{(p(t), y) \in C[t] \times B : p(0) = b(y) \text{ and } p(1) = 0 \}
\]

\[
(P_c)_0 = \{(q(t), z) \in C[t] \times A : q(0) = c(z) \text{ and } q(1) = 0 \}
\]

Recall from (17) and (18) the definitions of \( \pi_b : (P_b)_0 \to C \) and \( \iota_b : C^\xi_1 \to (P_b)_0 \). For example, the morphism \( \pi_b : (P_b)_0 \to B \) is defined by \( \pi_b(p(t), y) = y \). The morphism \( \eta : (P_c)_0 \to (P_b)_0, \eta(q(t), z) = (q(t), a(z)) \), makes the following diagram in \( \text{Alg}_c \) commute:

\[
\begin{array}{ccc}
C^\xi_1 & \xrightarrow{\iota_b} & (P_b)_0 \\
\downarrow{\text{id}} & & \downarrow{\eta} \\
C^\xi_1 & \xrightarrow{\iota} & (P_c)_0 \\
\end{array}
\]

By functoriality of the mapping path construction, there is a morphism \( \theta \) making the following diagram commute:

\[
\begin{array}{ccc}
0 & \xrightarrow{(\pi_b, \theta)} & B^\xi_1 \\
\downarrow{\iota_b} & & \downarrow{\iota} \\
(P_b)_0 & \xrightarrow{\theta} & (P_c)_0 \\
\end{array}
\]

(36)

We claim that \( \theta \) is a \( \mathcal{R} \)-equivalence; indeed, it is a split surjection with contractible kernel, as we proceed to explain. We have:

\[
(P(P_b)_0)_0 = \left\{ (p(t), s, y(s)) \in C[t, s] \times B[s] : \begin{array}{l}
p(0, s) = b(y(s), \quad p(1, s) = 0, \\
p(t, 1) = 0 \text{ and } y(1) = 0
\end{array} \right\}
\]

\[
(P_c)_0 = \{ (p(t), y(s), q(t), z) \in (P(P_b)_0)_0 \times (P_c)_0 : (p(t, 0), y(0)) = (q(t), a(z)) \}
\]

In the description of \( (P_c)_0 \) above, \( (p(t, s), y(s), q(t), z) \) satisfies \( q(t) = p(t, 0) \) so that we can get rid of \( q \) as long as we keep \( p \). Hence, we have:

\[
(P_c)_0 = \left\{ (p(t, s), y(s), z) \in C[t, s] \times B[s] \times A : \begin{array}{l}
p(0, s) = b(y(s), \quad p(1, s) = 0, \\
p(t, 1) = 0, \quad y(1) = 0, \quad p(0, 0) = c(z) \quad \text{and } y(0) = a(z)
\end{array} \right\}
\]
It is easily seen that, using the latter description of \((P_\eta)_0\), the morphism \(\theta : (P_\eta)_0 \to (P_\eta)_0\) is given by \(\theta(p(t, s), y(s), z) = (y(t), z)\). We have:

\[
\ker \theta = \{(p(t, s), 0, 0) \in (P_\eta)_0 \}
\]

\[\cong \{p(t, s) \in C[t, s] : p(0, s) = 0, p(1, s) = 0 \text{ and } p(t, 1) = 0\}\]

It is easily verified that \(\ker \theta\) is contractible. Moreover, \(\theta\) is a split surjection with section:

\[(P_\eta)_0 \ni (y(t), z) \mapsto (b(y)(1 - (1 - s)(1 - t)), y(s), z) \in (P_\eta)_0\]

Upon applying \(j\) to \((36)\) and identifying \((B_0, 0) \cong (B, 1)\), we get the following diagram in \(\mathcal{R}\) whose rows and columns are mapping path triangles; the diagram clearly commutes, except maybe for the squares \(*\) and \(\#\):

\[
\begin{array}{c}
(C, 2) \xrightarrow{-\partial_{0,0}} ((P_\eta)_0, 1) \xrightarrow{L_1(\partial_{0,0})} (B, 1) \xrightarrow{L_1(\theta)} (C, 1) \\
0 \xrightarrow{-\partial_{0,0}} ((P_\eta)_0, 0) \xrightarrow{id} ((P_\eta)_0, 0) \xrightarrow{id} 0 \\
(C, 1) \xrightarrow{\pi_0} ((P_\eta)_0, 0) \xrightarrow{\pi_0} (A, 0) \xrightarrow{c} (C, 0) \\
(C, 1) \xrightarrow{\eta} ((P_\eta)_0, 0) \xrightarrow{\pi_0} (B, 0) \xrightarrow{b} (C, 0) \\
(C, 1) \xrightarrow{\eta} ((P_\eta)_0, 0) \xrightarrow{\pi_0} (B, 0) \xrightarrow{b} (C, 0)
\end{array}
\]

The composite \(c \circ \pi_0 : (P_\eta)_0 \to C\) is easily seen to be nullhomotopic, so that the square \(*\) commutes. The composite

\[
(C_0, \tilde{\gamma}), ((P_\eta)_0, \tilde{\gamma}) \xrightarrow{(\kappa \circ \pi_0)} (P_\eta)_0
\]

is easily seen to factor through \(\ker \theta\), which is contractible; this implies that the square \(*\) commutes too.

We still have to show that the following diagram commutes:

\[
\begin{array}{c}
(B, 1) \xrightarrow{L_1(b)} (C, 1) \\
(B, 1) \xrightarrow{\theta^{-1}(\alpha_{0,0})} ((P_\eta)_0, 0) \xrightarrow{-\partial_{0,0}} ((P_\eta)_0, 0)
\end{array}
\]

It is easily seen that the commutativity of the square above is implied by the commutativity of the following diagram in \([\text{Alg}_C]\):

\[
\begin{array}{c}
B_0, \tilde{\gamma} \xrightarrow{b^{\tilde{\gamma}}} C_0, \tilde{\gamma} \\
\phi \downarrow \quad \downarrow \phi \\
(P_\eta)_0 \xrightarrow{\pi_0} (P_\eta)_0
\end{array}
\]

(37)

Here the morphism \(\xi\) is given by \(\xi(y(t), z) = (b(y)(t), z)\). The square in \((37)\) commutes on the nose. The triangle in \((37)\) commutes in \([\text{Alg}_C]\), as we proceed to explain. Consider the
following elementary homotopies $H_1, H_2 : (P_\eta)_0 \to (P_\ell)_0[u]$:

$$H_1(p(t, s), y(s), z) = (p(tu, t), z)$$

$$H_2(p(t, s), y(s), z) = (p(t, tu), z)$$

Then $ev_{u=0} \circ H_1 = \xi \circ \theta$, $ev_{u=1} \circ H_1 = ev_{u=1} \circ H_2$ and $ev_{u=0} \circ H_2 = \pi_\eta$, showing that $\xi \circ \theta = \pi_\eta$ in $\text{Alg}_\ell$. This finishes the proof of (TR4).

We showed that $\mathcal{R}$ is a triangulated category with the distinguished triangles being those triangles isomorphic to mapping path triangles. As in the topological setting [3 Section 6.6], the distinguished triangles could also be defined using extension triangles; we proceed to give the details of this.

**Definition 9.11.** Let $\xi : A \xrightarrow{\xi} B \xrightarrow{\xi} C$ be an extension in $\text{Alg}_\ell$ with classifying map $\xi : JC \to A$. Let $n \in \mathbb{Z}$ and let $\partial_{\xi,n}$ be the composite:

$$(C, n + 1) \xrightarrow{(\text{id},n)} (JC, n) \xrightarrow{(-1)^nL^n(\xi)} (A, n)$$

We call **extension triangle** to a diagram in $\mathcal{R}$ of the form:

$$\Delta_{\xi,n} : L(C, n) \xrightarrow{\partial_{\xi,n}} (A, n) \xrightarrow{L^n(f)} (B, n) \xrightarrow{L^n(g)} (C, n)$$

**Proposition 9.12 (3 Section 6.6).** A triangle in $\mathcal{R}$ is distinguished if and only if it is isomorphic to an extension triangle.

**Proof.** Let us show first that every mapping path triangle is isomorphic to an extension triangle. Let $g : B \to C$ be any morphism in $\text{Alg}_\ell$. Consider the mapping cylinder:

$$Z_g := \{(p, b) \in C[t] \times B : p(0) = g(b)\}$$

Using the identifications in Lemma 6.7 we have:

$$(P_g)_0 := \{(p, b) \in (t-1)C[t] \times B : p(0) = g(b)\}$$

It is easily verified that the following diagram is an extension in $\text{Alg}_\ell$:

$$\Delta_{Z_g,0} : (B, 1) \xrightarrow{\partial_{Z_g,0}} ((P_g)_0, 0) \xrightarrow{\text{id}} (Z_g, 0) \xrightarrow{\epsilon} (C, 0)$$

Let $\text{pr} : Z_g \to B$ be the natural projection; $\text{pr}$ is easily seen to be a homotopy equivalence inverse to $b \mapsto (g(b), b)$. We claim that there is an isomorphism of triangles as follows:

$$\Delta_{Z_g,0} : (B, 1) \xrightarrow{\partial_{Z_g,0}} ((P_g)_0, 0) \xrightarrow{\text{id}} (Z_g, 0) \xrightarrow{\epsilon} (C, 0)$$

The middle and right squares clearly commute but we still have to show that $\partial_{Z_g,0} = \partial_{\xi,0}$. Let $\omega : C_{-n}^\Xi \to C_{-n}^\Xi$ be the automorphism defined by $\omega(p(t)) = p(1 - t)$. Consider the following morphism of extensions, where the vertical map in the middle is defined by $p(t) \mapsto (p(1 - t), 0)$:

$$\mathcal{R}_{0,C} \xrightarrow{C_{0}^\Xi} (PC)_0 \xrightarrow{C} C$$

$$\mathcal{P}_{0,C} \xrightarrow{t_{\text{sc}}^\Xi} (P_g)_0 \xrightarrow{\text{id}} Z_g \xrightarrow{\text{id}} C$$
By Proposition 2.24 the classifying map of $\mathcal{Z}_g$ equals $\tau_g \circ \omega \circ \lambda_C$; this is easily seen to imply that $\partial_{\mathcal{Z}_g,0} = \partial_{\mathcal{Z}_g,0}$.

Let us now show that every extension triangle is isomorphic to a mapping path triangle. Let $\mathcal{E} : A \to B \to C$ be an extension in $\text{Alg}_C$. Let $h : A \to (P_g)_0$ be the natural morphism, that is a $\mathcal{K}$-equivalence by Lemma 6.3. We claim that there is an isomorphism of triangles:

$$\Delta_{\mathcal{E},0} : (C, 1) \xrightarrow{\partial_{\mathcal{E},0}} (A, 0) \xrightarrow{f} (B, 0) \xrightarrow{g} (C, 0)$$

The middle and right squares clearly commute but we still have to show that $(h \circ \partial_{\mathcal{E},0}) = \partial_{\mathcal{E},0}$. Above we proved that the classifying map of $\mathcal{Z}_g$ equals $\tau_g \circ \omega \circ \lambda_C$ in $[\text{Alg}_C]^\bullet$. By Proposition 2.24, the classifying map of $\mathcal{Z}_g : (C, 1) \to ((P_g)_0, 0) \to (B, 0) \to (C, 0)$. Let $\xi : JC \to A$ be the classifying map of $\mathcal{E}$ and consider the following morphism of extensions, where the middle vertical map is $b \mapsto (g(b), b)$:

$$\mathcal{E} : A \xrightarrow{f} B \xrightarrow{g} C$$

By Proposition 2.24 the classifying map of $\mathcal{Z}_g$ equals $h \circ \xi$ in $[\text{Alg}_C]^\bullet$. Then $\tau_g \circ \omega \circ \lambda_C = h \circ \xi$ in $[\text{Alg}_C]^\bullet$, and this is easily seen to imply that $j(h) \circ \partial_{\mathcal{E},0} = \partial_{\mathcal{E},0}$. □

10. Universal property

Recall from [1] Section 6.6] the definition of an excisive homology theory with values in a triangulated category.

**Definition 10.1.** Let $(\mathcal{T}, L)$ be a triangulated category. An excisive homology theory with values in $\mathcal{T}$ consists of the following data:

(i) a functor $X : \text{Alg}_C \to \mathcal{T}$;
(ii) a morphism $\delta_{\mathcal{E}} \in \text{Hom}_\mathcal{T}(LX(C), X(A))$ for every extension $\mathcal{E}$ in $\text{Alg}_C$:

$$\mathcal{E} : A \xrightarrow{f} B \xrightarrow{g} C$$

These morphisms $\delta_{\mathcal{E}}$ are subject to the following conditions:

(a) For every extension (38), the following triangle is distinguished:

$$\Delta_{\mathcal{E}} : LX(C) \xrightarrow{\delta_{\mathcal{E}}} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

(b) The triangles $\Delta_{\mathcal{E}}$ are natural with respect to morphisms of extensions.

**Example 10.2.** Let $\mathcal{E} : A \to B \to C$ be an extension in $\text{Alg}_C$. Recall from Proposition 9.12 that we have a distinguished triangle in $\mathcal{R}$:

$$\Delta_{\mathcal{E},0} : (C, 1) \xrightarrow{\partial_{\mathcal{E},0}} (A, 0) \xrightarrow{j(f)} (B, 0) \xrightarrow{j(g)} (C, 0)$$

Moreover, it follows from Proposition 2.24 that $\partial_{\mathcal{E},0}$ is natural with respect to morphisms of extensions. Then the functor $j : \text{Alg}_C \to \mathcal{R}$ together with the morphisms $\partial_{\mathcal{E},0}$ is an excisive homology theory.
A graded category is a pair $\mathcal{A}$, $L$ where $\mathcal{A}$ is an additive category and $L$ is an automorphism of $\mathcal{A}$. It $(\mathcal{A}, L)$ is a graded category and $X$ is an object of $\mathcal{A}$, we will often write $(X, n)$ instead of $L^n(X)$. A graded functor $F : (\mathcal{A}, L) \rightarrow (\mathcal{A'}, L')$ is an additive functor $F : \mathcal{A} \rightarrow \mathcal{A'}$ such that $F \circ L = L' \circ F$. Let $F, G : (\mathcal{A}, L) \rightarrow (\mathcal{A'}, L')$ be graded functors. A graded natural transformation $\nu : F \rightarrow G$ is a natural transformation $\nu$ such that $L'(\nu_X) = \nu_{L(X)} : L'F(X) \rightarrow L'G(X)$ for all $X \in \mathcal{A}$.

Example 10.3. A triangulated category is a graded category.

Example 10.4. Let $(\mathcal{T}, L)$ be a triangulated category. Put $\mathcal{A} := \mathcal{T}^I$ where $I = \{0 \rightarrow 1\}$ is the interval category; then $\mathcal{A}$ is an additive category. It is easily seen that $L$ induces a translation functor in $\mathcal{A}$ that makes $\mathcal{A}$ into a graded category.

Example 10.5. Let GrAb be the category whose objects are $\mathbb{Z}$-graded abelian groups and whose morphisms graded morphisms of degree zero. Then $A = \text{GrAb}$ is a graded category with the translation functor $L$ defined by $L(M)_n = M_{n+1}$, $n \in \mathbb{Z}$, $M \in \text{GrAb}$.

Definition 10.6. Let $(\mathcal{A}, L)$ be a graded category. A $\delta$-functor with values in $\mathcal{A}$ consists of the following data:

(i) a functor $X : \text{Alg}_\ell \rightarrow \mathcal{A}$ that preserves finite products;

(ii) a morphism $\delta_\mathcal{E} \in \text{Hom}_\mathcal{A}(LX(C), X(A))$ for every extension $\mathcal{E}$ in $\text{Alg}_\ell$:

\[
\begin{array}{c}
\xymatrix{ A \ar[r] & B \ar[r] & C }
\end{array}
\]

These morphisms $\delta_\mathcal{E}$ are subject to the following conditions:

(a) $\delta_\mathcal{E} : LX(C) \rightarrow X(A)$ is an isomorphism if $X(B) = 0$;

(b) The morphisms $\delta_\mathcal{E}$ are natural with respect to morphisms of extensions.

Example 10.7. An excisive homology theory $X : \text{Alg}_\ell \rightarrow \mathcal{T}$ is a $\delta$-functor.

Example 10.8. Let $X, Y : \text{Alg}_\ell \rightarrow \mathcal{T}$ be excisive homology theories and let $\nu : X \rightarrow Y$ be a natural transformation such that, for every extension $\mathcal{E}$, the following diagram commutes:

\[
\begin{array}{c}
\xymatrix{ LX(C) \ar[r]^{\delta^X_\mathcal{E}} & X(A) \ar[d]^{\nu_A} }
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{ LY(C) \ar[r]^{\delta^Y_\mathcal{E}} & Y(A) }
\end{array}
\]

Let $\mathcal{A} = \mathcal{T}^I$ be the graded category of Example 10.4. Then the natural transformation $\nu$ induces a $\delta$-functor $\text{Alg}_\ell \rightarrow \mathcal{A}$; that is, we still denote $\nu$. Explicitly, the functor $\nu : \text{Alg}_\ell \rightarrow \mathcal{A}$ is defined as follows:

\[
A \mapsto (\nu_A : X(A) \rightarrow Y(A))
\]

\[
f \in \text{Hom}_{\text{Alg}_\ell}(A, B) \mapsto (X(f), Y(f)) \in \text{Hom}_{\mathcal{A}}(\nu_A, \nu_B)
\]

For an extension $\mathcal{E}$, the morphism $\delta_\mathcal{E} \in \text{Hom}_{\mathcal{A}}(L(\nu_C), \nu_A)$ is defined by:

\[
\delta_\mathcal{E} := (\delta^X_\mathcal{E}, \delta^Y_\mathcal{E}) \in \text{Hom}_{\mathcal{A}}(L(\nu_C), \nu_A)
\]

We will show that $j : \text{Alg}_\ell \rightarrow \mathcal{R}$ is the universal excisive and homotopy invariant homology theory, in the sense of [21, Section 6.6]. In order to deal with natural transformations, we will work in the slightly more general setting of $\delta$-functors. From now on, fix a homotopy invariant $\delta$-functor $X$ with values in a graded category $(\mathcal{A}, L)$. A morphism in $\text{Alg}_\ell$ will be called an $X$-equivalence if it becomes invertible upon applying $X$. For example, the following morphisms are $X$-equivalences:
1) The morphisms $B^\mathbb{Z}_r \rightarrow B^\mathbb{Z}_{r+1}$ for any $B \in \text{Alg}_\ell$, $n \in \mathbb{Z}_{\geq 0}$ and $r \geq 0$. This follows by induction on $n$. For the inductive step, apply $X$ to the morphism of extensions (28) and use that $P(n, B)$ is contractible.

2) The morphisms $\mu_{m,n} : (B^\mathbb{Z}_r)^{\otimes n} \rightarrow B^\mathbb{Z}_{r+s}$ for $B \in \text{Alg}_\ell$ and $m, n, r, s \geq 0$. We will only use this fact in the special case $n = 1$ and $r = s = 0$. Proceed by induction on $m$. For the inductive step, apply $X$ to the morphism of extensions (29) and then use that the middle terms are contractible.

Since $X$ is homotopy invariant, $X$ induces a functor $[\text{Alg}_\ell] \rightarrow \mathcal{A}$ that we still denote $X$. Let $[\text{Alg}_\ell]^{\text{ind}}$ be the full subcategory of $[\text{Alg}_\ell]^{\text{ind}}$ whose objects are ind-objects $(A, I)$ such that:

(i) $I$ has an initial object $i_0$;

(ii) all the transition morphisms $A_i \rightarrow A_j$ are $X$-equivalences.

Note that, for any $B \in \text{Alg}_\ell$, the ind-object $B^\mathbb{Z}_0$ is in $[\text{Alg}_\ell]^{\text{ind}}$.

It is easily verified that $[\text{Alg}_\ell]^{\text{ind}}$ has finite products and that the functor $[\text{Alg}_\ell] \rightarrow [\text{Alg}_\ell]$ commutes with finite products. Then $[\text{Alg}_\ell]^{\text{ind}}$ has finite products too; explicitly, the product of $(A, I)$ and $(B, J)$ is the object $(A \times B, I \times J)$ with the obvious projections. Since $X$ is a $\delta$-functor, we have natural isomorphisms:

$$X(A_i \times B_j) \cong X(A_i) \oplus X(B_j)$$

Using this, it is easily seen that the product of two objects of $[\text{Alg}_\ell]^{\text{ind}}$ is again an object of $[\text{Alg}_\ell]^{\text{ind}}$. This shows that $[\text{Alg}_\ell]^{\text{ind}}$ has finite products.

Let $(A, I), (B, J) \in [\text{Alg}_\ell]^{\text{ind}}$. A morphism $f \in [A, B]$ is a collection $\{f_i\}_{i \in I}$ of homotopy classes of morphisms $f_i : A_i \rightarrow B_{i(0)}$ subject to certain compatibility relations.

**Lemma 10.9.** There is a functor $\tilde{X} : [\text{Alg}_\ell]^{\text{ind}} \rightarrow \mathcal{A}$ such that $\tilde{X}(A, I) = X(A_0)$ and such that $\tilde{X}(f)$ is the composite

$$X(A_0) \xrightarrow{X(f_0)} X(B_{i(0)}) \xrightarrow{\cong} X(B_0)$$

for any $f \in [A, B]$. Moreover, $\tilde{X}$ preserves finite products.

**Proof.** It is easily verified that $\tilde{X}$ is indeed a well-defined functor. The fact that $\tilde{X}$ preserves finite products follows from the fact that $X : [\text{Alg}_\ell] \rightarrow \mathcal{A}$ does. \qed

If $B \in \text{Alg}_\ell$ and $n \geq 1$, then $B^\mathbb{Z}_0$ is a group object in $[\text{Alg}_\ell]^{\text{ind}}$; see Lemma 24.2. It follows from Lemma 10.9 that $X(B^\mathbb{Z}_0)$ has a group object structure induced by that of $B^\mathbb{Z}_0$. Since $\mathcal{A}$ is an additive category, every object of $\mathcal{A}$ is naturally an abelian group object. Thus, $X(B^\mathbb{Z}_0)$ has two group object structures: the one coming from $B^\mathbb{Z}_0$ and the other from being an object of $\mathcal{A}$. By the Eckmann-Hilton argument, both group structures coincide and the function

$$\tilde{X} : [A, B^\mathbb{Z}_0] \rightarrow \text{Hom}_\mathcal{A}(X(A_0), X(B^\mathbb{Z}_0))$$

is a group homomorphism for every $(A, I)$ in $[\text{Alg}_\ell]^{\text{ind}}$.

Let $A \in \text{Alg}_\ell$ and let $\mathcal{A}_A$ be the universal extension of $A$. Since $TA$ is contractible, there is an isomorphism:

$$\delta_{\mathcal{A}_A} : (X(A), 1) \xrightarrow{\cong} (X(TA), 0)$$

Put $i^{\mathcal{A}}_A := \delta_{\mathcal{A}_A}$ and define inductively $i^{\mathcal{A}, n+1}_A$ as the composite:

$$(X(A), n + 1) \xrightarrow{Li^{\mathcal{A}}_A} (X(J^nA), 1) \xrightarrow{i^{\mathcal{A}}_A} (X(J^{n+1}A), 0)$$
Let $i_{A}^{0}$ be the identity of $(X(A), 0)$. It is easily verified by induction on $n = p + q$ that the following equality holds for $p, q \geq 0$:

$$i_{A}^{p+q} = i_{p,A}^{q} \circ L^{q}(i_{A}^{p})$$

The morphisms $i_{A}^{p}$ assemble into a natural isomorphism $L^{p} \circ X \cong X \circ J^{p}(?) : \text{Alg}_{\mathcal{E}} \to \mathcal{A}$. Let $A \in \text{Alg}_{\mathcal{E}}$ and let $\mathcal{P}_{0,A}$ be the path extension of $A$. Since $\mathcal{P}(A)_{0}$ is contractible, there is an isomorphism:

$$\delta \varphi_{A} : (X(A), 1) \xrightarrow{\sim} (X(A_{0}^{\mathcal{E}}), 0)$$

Put $i^{3,1}_{A} := \delta \varphi_{A}$ and define inductively $i^{n}_{A}$ as the composite:

$$\begin{array}{ccc}
(X(A), n) & \xrightarrow{L(i^{n}_{A})} & (X(A_{0}^{\mathcal{E}}), 1) \\
& \xrightarrow{i^{1,n}_{A}} & (X((A_{0}^{\mathcal{E}}))_{0}^{\mathcal{E}}), 0) \\
& \xrightarrow{i^{n}_{A}} & (X(A_{0}^{\mathcal{E}})), 0)
\end{array}$$

Let $i^{n}_{A}$ be the identity of $(X(A), 0)$. It is easily verified by induction on $n = p + q$ that the following equality holds for $p, q \geq 0$:

$$i^{p+q}_{A} = X(\mu^p_{A}) \circ i_{A}^{q} \circ L^{p}(i_{A}^{q})$$

The morphisms $i^{p,n}_{A}$ assemble into a natural isomorphism $L^{n} \circ X \cong X \circ (\gamma)_{0}^{n} : \text{Alg}_{\mathcal{E}} \to \mathcal{A}$.

**Lemma 10.10.** Let $N_{2}, N_{3} \geq 0$ and let $B \in \text{Alg}_{\mathcal{E}}$. Then the following diagram in $\mathcal{A}$ commutes up to the sign $(-1)^{N_{2}N_{3}}$:

$$\begin{array}{ccc}
(X(B), N_{2} + N_{3}) & \xrightarrow{L^{N_{2}}(i^{N_{2}}_{B})} & (X(B_{0}^{\mathcal{E}})), N_{3}) \\
& \xrightarrow{i^{N_{2},N_{3}}_{B}} & (X((J^{N_{2}}(B_{0}^{\mathcal{E}}))), 0)
\end{array}$$

**Proof.** If $N_{2} = 0$ or $N_{3} = 0$ there is nothing to prove. Upon applying the functor $\tilde{X}$ to the diagram (15) we get that the following diagram in $\mathcal{A}$ commutes up to the sign $-1$; the case $N_{2} = N_{3} = 1$ follows easily from this:

$$\begin{array}{ccc}
X(J^{2}B) & \xrightarrow{X(J^{2}B)} & X(J(B_{0}^{\mathcal{E}})) \\
& \xrightarrow{X(J^{2}B)} & X(J(B_{0}^{\mathcal{E}}))
\end{array}$$

Once we know the case $N_{2} = N_{3} = 1$, the case $N_{3} = 1$ with arbitrary $N_{2}$ follows by an easy induction on $N_{2}$. Once we know the result for $N_{2} = 1$ and arbitrary $N_{3}$, the general case follows by induction on $N_{3}$. $\square$

**Theorem 10.11.** Let $(\mathcal{A}, L)$ be a graded category and let $X : \text{Alg}_{\mathcal{E}} \to \mathcal{A}$ be a homotopy invariant $\delta$-functor. Then there exists a unique graded functor $\tilde{X} : \mathcal{R} \to \mathcal{A}$ such that $\tilde{X} \delta_{\mathcal{E},0} = \delta \varphi$ for every extension $\mathcal{E}$ and such that the following diagram commutes:

$$\begin{array}{ccc}
\text{Alg}_{\mathcal{E}} & \xrightarrow{j} & \mathcal{R} \\
X \downarrow & \tilde{X} \downarrow & \mathcal{A}
\end{array}$$
Proof. Define $\tilde{X}$ on objects by $\tilde{X}(A, m) := (X(A), m)$. To define $\tilde{X}$ on morphisms we must define, for every pair of objects of $\mathcal{C}$, a group homomorphism:

$$
\tilde{X}_{(A,m),(B,n)} : \mathcal{C}((A, m), (B, n)) \rightarrow \text{Hom}_{\mathcal{C}}((X(A), m), (X(B), n)) \tag{40}
$$

Let $\tilde{X}^v$ be the dotted composite:

$$
\begin{array}{ccc}
[J^{m+v}A, B^\varnothing_{m+v}] & \xrightarrow{\tilde{X}} & \text{Hom}_{\mathcal{C}}((X(J^{m+v}A), 0), (X(B^\varnothing_{m+v}), 0)) \\
\downarrow & & \downarrow \cong \circ \gamma_{J^{m+v}} \\
\tilde{X}^v & & \text{Hom}_{\mathcal{C}}((X(A), m + v), (X(B), n + v)) \\
\downarrow & & \downarrow \cong \circ L^v \\
\tilde{X} & & \text{Hom}_{\mathcal{C}}((X(A), m), (X(B), n))
\end{array}
$$

The function $\tilde{X}^v$ is a group homomorphism by the discussion following Lemma 10.19. Moreover, it is easily verified that this diagram commutes:

$$
\begin{array}{ccc}
[J^{m+v}A, B^\varnothing_{m+v}] & \xrightarrow{\tilde{X}^v} & \text{Hom}_{\mathcal{C}}((X(A), m), (X(B), n)) \\
\downarrow \alpha^{m+v} & & \downarrow \alpha^{m+v} \\
[J^{m+1+v}A, B^\varnothing_{m+1+v}] & \xrightarrow{\tilde{X}^{m+1+v}} & \text{Hom}_{\mathcal{C}}((X(A), m + 1 + v), (X(B), n + 1 + v))
\end{array}
$$

Thus, the morphisms $\tilde{X}^v$ induce the desired group homomorphism $\tilde{X}_{(A,m),(B,n)}$ in (40), this defines $\tilde{X}$ on morphisms. It is straightforward but tedious to verify that the definitions above indeed give rise to an additive functor $\tilde{X} : \mathcal{C} \rightarrow \mathcal{A}$.

When verifying that $\tilde{X}$ preserves composition, Lemma 10.10 is needed to show that the signs in Definition 3.4 work out. We clearly have $X = \tilde{X} \circ j$ and $L \circ \tilde{X} = \tilde{X} \circ L$.

Let us now show that $\tilde{X}^v(\partial_{\mathcal{C},0}) = \delta_{\mathcal{C}}$ for every extension $\mathcal{E}$ in $\text{Alg}_{\mathcal{C}}$. Consider an extension as follows, with classifying map $\xi : JC \rightarrow A$:

$$
\mathcal{E} : A \xrightarrow{f} B \xrightarrow{g} C
$$

Recall from Definition 9.11 that $\partial_{\mathcal{E},0}$ equals the composite:

$$
(C, 1) \xrightarrow{(d_{J\mathcal{C}}, 0)} (JC, 0) \xrightarrow{\xi} (A, 0)
$$

Upon applying $\tilde{X}$ we get:

$$
(X(C), 1) \xrightarrow{\tilde{X}^v_{JC}} (X(JC), 0) \xrightarrow{\tilde{X}(\xi)} (X(A), 0)
$$

By naturality of $\delta$, we have:

$$
\tilde{X}(\partial_{\mathcal{E},0}) = X(\xi) \circ \tilde{j}^v_{JC} = X(\xi) \circ \delta_{\mathcal{C}} = \delta_{\mathcal{E}}
$$

It remains to check the uniqueness of $\tilde{X}$. Let $\tilde{X} : \mathcal{C} \rightarrow \mathcal{A}$ be any graded functor with the properties described in the statement of this theorem. Let $\alpha \in \mathcal{C}((A, m), (B, n))$ be represented by $f : J^{m+v}A \rightarrow B^\varnothing_{m+v}$ and let $\gamma' : B^\varnothing_{m+v} \rightarrow B^\varnothing_{m+v}$ be the morphism induced by
the iterated last vertex map. By Lemma 7.9, the following diagram in $K$ commutes:

\[
\begin{array}{c}
(A, m) \xrightarrow{\alpha} (B, n) \\
\downarrow \quad \downarrow \\
(J^m + v A, -v) \xrightarrow{L^-(f)} (B_f^v, -v) \xrightarrow{L^-(y')} (B_0^v, -v)
\end{array}
\]

Upon applying $\overline{X}$ we get the following commutative diagram in $\mathcal{A}$:

\[
\begin{array}{c}
(X(A), m) \xrightarrow{X(\alpha)} (X(B), n) \\
\downarrow \quad \downarrow \\
(X(J^m + v A), -v) \xrightarrow{L^-(i^{(\text{ext})})} (X(B_f^v), -v) \xrightarrow{L^+(y)} (X(B_0^v), -v)
\end{array}
\]

It follows that $\overline{X}$ is the functor defined above. □

As a corollary we get:

**Theorem 10.12** ([5] Comparison Theorem B). Let $(\mathcal{T}, L)$ be a triangulated category and let $X : \text{Alg}_L \to \mathcal{T}$ be an excisive and homotopy invariant homology theory. Then there exists a unique triangulated functor $\overline{X} : \mathcal{R} \to \mathcal{T}$ such that $\overline{X}(\delta_E) = \delta_E$ for every extension $E$, and such that the following diagram commutes:

\[
\begin{array}{c}
\text{Alg}_L \xrightarrow{j} \mathcal{R} \\
\downarrow \quad \downarrow \\
X \quad \overline{X}
\end{array}
\]

Proof. By Theorem 10.11 there exists a unique graded functor $\overline{X}$ making the diagram commute and such that $\overline{X}(\delta_E, 0) = \delta_E$ for every extension $\mathcal{E}$ in $\text{Alg}_L$. It remains to check that $\overline{X}$ sends distinguished triangles in $\mathcal{R}$ to distinguished triangles in $\mathcal{T}$. By Proposition 9.12 it suffices to show that $\overline{X}$ sends extension triangles $\Delta_{\mathcal{E}, 0}$ to distinguished triangles in $\mathcal{T}$, but this follows immediately from the fact that $\overline{X}(\delta_E, 0) = \delta_E$. □

Remark 10.13. One way to summarize Theorem 10.12 is to say that $j : \text{Alg}_L \to \mathcal{R}$ is the universal excisive and homotopy invariant homology theory with values in a triangulated category. As explained in the introduction, such a theory was already constructed by Garkusha [4] Theorem 2.6 (2)] using completely different methods. Both constructions are, of course, equivalent since they satisfy the same universal property.

**Theorem 10.14.** Let $F : \text{Alg}_L \to \text{Alg}_L$ be a functor satisfying the following properties:

1) $F$ preserves homotopic morphisms;
2) $F$ preserves extensions.

Then there exists a unique triangulated functor $\overline{F}$ making the following diagram commute:

\[
\begin{array}{c}
\text{Alg}_L \xrightarrow{j} \mathcal{R} \\
\downarrow \quad \downarrow \\
F \quad \overline{F}
\end{array}
\]
Let $F_1, F_2 : \text{Alg}_\ell \to \text{Alg}_\ell$ be two functors with the properties above and let $\eta : F_1 \to F_2$ be a natural transformation. Then there exists a unique (graded) natural transformation $\nu : \bar{F}_1 \to \bar{F}_2$ such that $\bar{\eta}(A) = j(\eta_A)$ for all $A \in \text{Alg}_\ell$.

**Proof.** The existence and uniqueness of $\bar{F}$ follow from Theorem 10.12 once we notice that $j \circ F : \text{Alg}_\ell \to \mathcal{R}$ is an excisive and homotopy invariant homology theory. Let us show now the existence of $\bar{\eta}$. For every $A \in \text{Alg}_\ell$ put:

$$\nu_A := j(\eta_A) \in \mathcal{R}(F_1(A), F_2(A))$$

The $\nu_A$ assemble into a natural transformation $\nu : j \circ F_1 \to j \circ F_2 : \text{Alg}_\ell \to \mathcal{R}$. Let $\mathcal{A} := \mathcal{R}(I)$ where $I$ is the interval category. Recall from Example 10.8 that $\nu$ induces a homotopy invariant $\delta$-functor $\text{Alg}_\ell \to \mathcal{A}$ if we show that the diagram (39) commutes. Let $\mathcal{E} : A \longrightarrow B \longrightarrow C$

be an extension in $\text{Alg}_\ell$. Then we have a morphism of extensions in $\text{Alg}_\ell$:

$$
\begin{array}{c}
F_1(A) \xrightarrow{F_1(f)} F_1(B) \xrightarrow{F_1(g)} F_1(C) \\
\downarrow \nu_A \quad \quad \quad \downarrow \nu_A \\
F_2(A) \xrightarrow{F_2(f)} F_2(B) \xrightarrow{F_2(g)} F_2(C)
\end{array}
$$

Since $j$ sends extensions to triangles in a natural way, the following diagram in $\mathcal{R}$ commutes:

$$
\begin{array}{c}
(F_1(C), 1) \xrightarrow{\delta^\nu f_1} (F_1(A), 0) \\
\downarrow L(\nu A) \quad \quad \quad \downarrow \nu_A \\
(F_2(C), 1) \xrightarrow{\delta^\nu f_2} (F_2(A), 0)
\end{array}
$$

Thus, $\nu$ induces a homotopy invariant $\delta$-functor $\text{Alg}_\ell \to \mathcal{A}$, which in turn induces a graded functor $\bar{\nu} : \mathcal{R} \to \mathcal{A}$ by Theorem 10.11. It is easily verified that this graded functor $\bar{\nu}$ corresponds to the desired natural transformation $\bar{\eta} : \bar{F}_1 \to \bar{F}_2$.

**Remark 10.15.** Let $F, F', F'' : \text{Alg}_\ell \to \text{Alg}_\ell$ be functors satisfying the hypothesis of Theorem 10.14 and let $\eta : F \to F'$ and $\eta' : F' \to F''$ be natural transformations. Then $\eta'' \circ \eta = \eta' \circ \bar{\eta}$.

**APPENDIX A.**

In this appendix we prove Proposition 10.13.

**Proposition A.1.** Let $A, B \in \text{Alg}_\ell$, let $K$ be a filtering poset, let $C_* \in (\text{Alg}_{Z0})^K$ and let $m, n \geq 1$. Then the following composite functions are group homomorphisms:

(i) $[A, B_1] \xrightarrow{J^m} [J^m A, J^m(B_1)] \xrightarrow{J^m f} [J^m A, (J^m B_1)]$

(ii) $[A, B_1] \xrightarrow{\gamma \otimes C_*} [A \otimes C_*, B_1 \otimes C_*] \cong [A \otimes C_*, (B \otimes C_*)]$.

(iii) $[A, (B_1)_{B_2}] \otimes C_* \xrightarrow{J^m f} [A \otimes C_* , (B \otimes C_*)]$.

(iv) $[A, B_1] \xrightarrow{J^m f} [A \otimes C_*, (B_1 \otimes C_*)]$.

In (ii) the bijection on the right is induced by the obvious isomorphism of $K \times \mathbb{Z}_{\geq 0}$-diagrams $B_1 \otimes C_* \cong (B \otimes C_*) B_2$. 


We can easily reduce to the case $C_\mu$. The inductive step is straightforward once we notice that $\Phi$ of simplicial sets $\psi$. It is straightforward to show that this defines a morphism. The latter is in turn natural in $p$. This dotted function is natural in $p$. Proof. Write $\Phi^{p,m}_{A,B}$ for the composite function in (11). Let us prove it is a group homomorphism. We proceed by induction on $n$. In the case $n = 1$, we claim that there is a morphism of simplicial sets $\varphi : \text{Hom}_{\text{Alg}}(A, B^\Delta) \to \text{Hom}_{\text{Alg}}(JA, (JB)^\Delta)$ inducing $\Phi^{1,m}_{A,B}$ under the identifications of Theorem 2.15. Indeed, for $f \in \text{Hom}_{\text{Alg}}(A, B^\Delta)$, let $\varphi(f) \in \text{Hom}_{\text{Alg}}(JA, (JB)^\Delta)$ be the classifying map of $f$ with respect to $(\emptyset B)^\Delta$. It is easily verified that the following diagram commutes, proving the case $n = 1$.

$$
\begin{array}{ccc}
(\Omega^m \text{Ex}^\infty \text{Hom}_{\text{Alg}}(A, B^\Delta))_0 & \cong & \text{Hom}_{\text{Alg}}(A, \hat{B}^\infty) \\
\downarrow \quad \text{Ex}^{\infty} \varphi & & \downarrow \quad \mu_m^n \circ \iota_r \\
(\Omega^m \text{Ex}^\infty \text{Hom}_{\text{Alg}}(JA, (JB)^\Delta))_0 & \cong & \text{Hom}_{\text{Alg}}(A, (JB)^\infty)
\end{array}
$$

The inductive step is straightforward once we notice that $\Phi^{p+1,m}_{A,B} = \Phi^{p,m}_{pA,pB} \circ \Phi^{p,m}_{A,B}$. Write $\Psi_C$ for the composite function in (11). Let us prove it is a group homomorphism. We can easily reduce to the case $C = C \in \text{Alg}_S$. Again, we claim that there is a morphism of simplicial sets $\psi : \text{Hom}_{\text{Alg}}(A, B^\Delta) \to \text{Hom}_{\text{Alg}}(A \otimes C, (B \otimes C)^\Delta)$ inducing $\Psi_C$ upon taking $\pi_m$ and making the identifications of Theorem 2.15. For $p \geq 0$, let $\psi_p$ be the composite:

$$
\text{Hom}_{\text{Alg}}(A, B^{\Delta'}) \xrightarrow{\psi_p} \text{Hom}_{\text{Alg}}(A \otimes C, B^{\Delta'} \otimes C) \cong \text{Hom}_{\text{Alg}}(A \otimes C, (B \otimes C)^{\Delta'})
$$

It is straightforward to show that this defines a morphism $\psi$ with the desired properties.

Let us now prove that the function $(\mu^m_{B,r})_r$ in (11) is a group homomorphism. Note that

$$
\text{Hom}_{\text{Alg}}(A, B^\Delta) \otimes \text{Hom}_{\text{Alg}}(A \otimes C, B^{\Delta'} \otimes C) \cong \text{Hom}_{\text{Alg}}(A \otimes C, (B \otimes C)^{\Delta'})
$$

and write $\iota_r : [A, (B^\infty_r)^\infty] \to [A, (B^\infty_r)^\infty]$ for the natural morphism into the colimit. It suffices to show that $(\mu^m_{B,r}) \circ \iota_r$ is a group homomorphism for all $r$. For fixed $r$, $s$ and $p$, consider the dotted function that makes the following diagram commute:

$$
\begin{array}{ccc}
(\text{Ex}^\infty \text{Hom}_{\text{Alg}}(A, (B^\infty_r)^\Delta))_p & \cong & \text{Hom}_{\text{Alg}}(A, (B^\infty_r)^\Delta) \\
\downarrow \quad \text{Ex}^{\infty} \varphi & & \downarrow \quad (\mu^m_{B,r})_r \circ \iota_r \\
(\Omega^m \text{Ex}^\infty \text{Hom}_{\text{Alg}}(A, B^\Delta))_p & \cong & \text{Hom}_{\text{Alg}}(A, B^\infty_r)^{\Delta'}
\end{array}
$$

This dotted function is natural in $p$ and thus induces a morphism of simplicial sets:

$$
\text{Ex}^\infty \text{Hom}_{\text{Alg}}(A, (B^\infty_r)^\Delta) \longrightarrow \Omega^m \text{Ex}^{\infty} \text{Hom}_{\text{Alg}}(A, B^\Delta)
$$

The latter is in turn natural in $s$ and thus induces a morphism $\Theta_r$ upon taking colimit:

$$
\Theta_r : \text{Ex}^\infty \text{Hom}_{\text{Alg}}(A, (B^\infty_r)^\Delta) \longrightarrow \Omega^m \text{Ex}^{\infty} \text{Hom}_{\text{Alg}}(A, B^\Delta)
$$

We claim that $(\mu^m_{B,r})_r \circ \iota_r$ equals the group homomorphism

$$
\pi_n \Theta_r : \pi_n \text{Ex}^\infty \text{Hom}(A, (B^\infty_r)^\Delta) \longrightarrow \pi_n \Omega^m \text{Ex}^{\infty} \text{Hom}(A, B^\Delta) \cong \pi_{m+n} \text{Ex}^\infty \text{Hom}(A, B^\Delta)
$$
upon making the identifications of Theorem 2.15. Indeed, this claim follows from the commutativity of the diagram below, which ultimately reduces to the naturality of \( \mu \).

\[
\begin{array}{ccc}
\text{Hom}_{\text{Alg}}(A, (B_{\infty}^{\mathbb{Z}})_x) & \xrightarrow{(\mu^{\mathbb{Z}}_{B_{\infty}})_x} & \text{Hom}_{\text{Alg}}(A, B_{\infty}^{x_{\infty}}) \\
\cong & & \cong \\
\left(\Omega^m\Omega^\infty\text{Hom}_{\text{Alg}}(A, B^{\lambda})\right)_0 & \xrightarrow{\Omega^m\Omega^\infty\text{Hom}_{\text{Alg}}(A, B^{\lambda})_0} & \left(\Omega^m\Omega^\infty\text{Hom}_{\text{Alg}}(A, B^{\lambda})\right)_0
\end{array}
\]

It remains to show that the composite function in (iv) is a group homomorphism. The functors \( ? \otimes \mathbb{Z} \) and \( ? \) are naturally isomorphic. Then, by Proposition A.1 (ii) applied to \( C_* = \mathbb{Z} \), we have a group homomorphism:

\[
\Psi_{x_{\infty}^x} : [A, B_{\infty}^{x_{\infty}}] \rightarrow [A^{x_{\infty}}, (B_{\infty}^{x_{\infty}})^{x_{\infty}}]
\]

Let \( c : I^m \times I^m \xrightarrow{\cong} I^m \times I^m \) be the commutativity isomorphism. It is easily verified that the following diagram commutes:

\[
\begin{array}{ccc}
[A, B_{\infty}^{x_{\infty}}] & \xrightarrow{(?)_{x_{\infty}}} & [A^{x_{\infty}}, (B_{\infty}^{x_{\infty}})^{x_{\infty}}] & \xrightarrow{(\mu^{\mathbb{Z}}_{B_{\infty}})_x} & [A^{x_{\infty}}, B_{\infty}^{x_{\infty}}] \\
\downarrow & & & & \downarrow c' \\
[A^{x_{\infty}}, (B_{\infty}^{x_{\infty}})^{x_{\infty}}] & \xrightarrow{(\mu^{\mathbb{Z}}_{B_{\infty}})_x} & [A^{x_{\infty}}, B_{\infty}^{x_{\infty}}]
\end{array}
\]

The function \( (\mu^{\mathbb{Z}}_{B_{\infty}})_x \) is a group homomorphism by Proposition A.1 (iii) and \( c' \) is multiplication by \((-1)^{m_0}\). The result follows.

\[\square\]

[1] Guillermo Cortiñas and Andreas Thom, Bivariant algebraic K-theory, J. Reine Angew. Math. 610 (2007), 71–123, DOI 10.1515/CRELLLE.2007.068. MR2359851 (2008i:19003)

[2] Joachim Cuntz, Bivariant K-theory and the Weyl algebra, K-Theory 35 (2005), no. 1-2, 93–137, DOI 10.1007/s10977-005-3464-0. MR2240217

[3] Joachim Cuntz, Ralf Meyer, and Jonathan M. Rosenberg, Topological and bivariant K-theory, Oberwolfach Seminars, vol. 36, Birkhäuser Verlag, Basel, 2007. MR2340673 (2008j:19001)

[4] Grigory Garkusha, Universal bivariant algebraic K-theories, J. Homotopy Relat. Struct. 8 (2013), no. 1, 67–116, DOI 10.1007/s40062-012-0013-4. MR3031594

[5] Grigory Garkusha and Daniel G. Quillen, Bivariant algebraic K-theory, I, Doc. Math. 19 (2014), 1207–1269, MR3291646

[6] Grigory Garkusha and Daniel G. Quillen, Bivariant algebraic K-theory, II, Ann. K-Theory 1 (2016), no. 3, 275–316, DOI 10.2140/akt.2016.1.275. MR3529093

[7] S. M. Gersten, Homotopy theory of rings, J. Algebra 19 (1971), 396–415, MR0291253

[8] Paul G. Goerss and John F. Jardine, Simplicial homotopy theory, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR1711612 (2001d:55012)

[9] Amnon Neeman, Triangulated categories, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR1812507

[10] Daniel G. Quillen, Homotopical algebra, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967. MR0223432

[11] E. Rodríguez Cirone, The homotopy groups of the simplicial mapping space between algebras, available at https://arxiv.org/abs/1803.06887

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