Quantum Genetic Algorithm With Individuals in Multiple Registers
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Abstract—Genetic algorithms (GAs) are heuristic optimization techniques inspired by Darwinian evolution, which are characterized by successfully finding robust solutions for optimization problems. While quantum resources have allowed for the acceleration of many computational tasks, it remains an open question whether they can enhance the efficacy of evolutionary algorithms, particularly GAs. Here, we propose a subroutine-based quantum GA with individuals codified in independent registers. This distinctive codification allows our proposal to depict all the fundamental elements characterizing GAs, i.e., population-based search with selection of many individuals, crossover, and mutation. Our subroutine-based construction permits us to consider several variants of the algorithm. For instance, we first analyze the performance of two different quantum cloning machines (QCMs), a key component of the crossover subroutine. Indeed, we study two paradigmatic examples, namely, the biomimetic cloning of quantum observables and the Bužek–Hillery universal QCM. We observed a faster average convergence of the former, but better-final populations of the latter. Additionally, we analyzed the effect of introducing a mutation subroutine, concluding a minor impact on the average performance. Furthermore, we introduce a quantum channel analysis to prove the exponential convergence of our algorithm and even predict its convergence ratio.

Index Terms—Genetic algorithms (GAs), optimization, quantum computing.

I. INTRODUCTION

GENETIC algorithms (GAs) are bioinspired algorithms with well-established performance in finding resilient solutions to complex optimization problems, such as problems with exponentially scaling dimensions of the search space and noisy fitness function [1], [2], [3], [4]. In these algorithms, every element of the search space can be potentially represented by an individual and the selection toward the optimal solution is performed by Darwin-like evolution. The set of individuals is known as population, their codification is commonly known as chromosomes, and their performance is ranked by a fitness function. Although there is no formal definition of GA which univocally distinguishes them from other evolutionary algorithms, there is a general consensus about the presence of the following four characteristic elements: 1) population-based search through joint evolution of a set of individuals; 2) a selection of some of them according to a potentially randomized criteria based on their performance; 3) a crossover operation to breed new individuals; and 4) a mutation operation which randomly modifies them [5].

Optimization problems are a cornerstone in real-life applications, and consequently, huge efforts have been devoted to the development of quantum algorithms for approximate optimization problems. Indeed, obtaining an exact solution to many optimization problems is NP-complete, whereas finding an approximate solution is feasible in polynomial time. Although quantum computers are not expected to provide an exponential advantage for NP-complete problems, there is a chance for exponential speedup in approximate optimization. Nonetheless, even potential improvements in a factor or exponent could be of relevance in commercial and industrial applications and it encourages us to carry on with the research in alternative heuristic quantum approaches. During the past decades, the merge of GAs and quantum computation has been a source of new heuristic optimization methods [6], [7], [8]. Induced by the nonlinear behavior of genetic operators, most of the effort has been focused on quantum inspired GAs, which integrate some concepts of quantum mechanics to engineer new varieties of classical evolutionary algorithms [7], [9], [10], [11], [12], [13], [14], [15]. On the other hand, fully quantum approaches potentially achieving quantum speed-up have only attained partial success in the inclusion of the aforementioned characteristic elements [16], [17], [18], [19]. In 2001, Rylander et al. proposed...
in [16] a quantum GA (QGA) introducing the concepts of chromosome-register and fitness-register. The individuals are encoded in the chromosome-registers which are possibly entangled with fitness-registers. Although it was claimed that quantum superposition provides an increased searching power, this conclusion has been considered unsupported due to the lack of heuristic or analytic evidence [6]. In 2006, Udrescu et al. [17] proposed an algorithm based on quantum searching and inspired by evolutionary computation, called reduced QGA. Here, individuals are represented by a binary basis so that the whole population can be encoded as a quantum superposition in a single register. This encoding is compatible with Grover’s quantum search algorithms which leads to a speed-up in the selection process [20]. However, they conclude that there is no need for elements like crossover or mutation, which raises the question of whether it should be considered a GA [6]. Afterward, Malossini et al. [18] proposed in 2008 the quantum genetic optimization algorithm, which also employed quantum searching techniques. These techniques enhance the selection procedure similarly to the previous work, but the crossover and mutation subroutines are still classically introduced. Lately, SaiToh et al. [19] extended both previous proposals by introducing an algorithm which includes crossover and mutation as quantum subroutines. However, the treatment of the population substantially differs from its role in classical GAs. Indeed, as the selection procedure is implemented by projective measurements, only one individual is selected in each generation. However, this hinders the population-based search feature, and consequently, reduces the intrinsic exploration capacity with respect to GAs.

In this article, we propose a QGA with individuals codified in independent registers. This allows for population-based search and selection, which are characterizing elements of GAs and a distinctive feature with respect to previous approaches. Our proposal is composed of modular quantum subroutines inspired by classical GAs: selection, crossover, and mutation. The intrinsic nonlinear nature of selection and crossover leads to fundamental obstacles posed by the principles of quantum information, such as no-cloning and no-deleting theorems. Therefore, we construct our proposal inspired by a primitive GA, namely, a GA, composed of elite retention and standard fixed-index crossover. Selection is performed by a reversible sorting with ancillary qubits and a posterior partial trace of the lowest-ranked individuals. The replication step of the crossover is carried out by a partial quantum cloning machine (QCM) and the combination step is accomplished by swap gates. This modular structure of the subroutines allows us to benchmark two paradigmatic quantum approximated cloning machines: the biomimetic cloning of quantum observables (BCQOs) and the Bužek–Hillery universal QCM (UQCM). Numerical analysis shows a faster average convergence of the former, but better-final populations of the latter. Finally, mutations are introduced by randomly allocated Pauli gates. We conclude that their presence has a negligible effect upon the average performance. Finally, we present a toolbox based on the spectral theory of quantum channels. We employ it to formally prove the exponential convergence of the algorithm, as well as to predict its convergence rate. Remarkably, both this prediction and the final quantum state accurately match our numerical simulations.

Section II introduces the QGA and its constituent subroutines, and is concluded by an analysis of the convergence. Section III presents the main results regarding convergence and performance of the variants.

**II. QUANTUM GENETIC ALGORITHM**

In order to take advantage of the exploratory capacity of population-based search, we encode individuals in several independent quantum registers. This allows us to select and replicate them on a population level. More specifically, let us consider \( n \) individuals composed of \( c \)-qubits each. For simplicity, we assume that \( n \) is divisible by 4 and \( c \) is even. The individuals are encoded in the quantum state of these registers, thus, the search space is the Hilbert space associated with a \( c \)-qubit quantum register, \( \mathcal{H} \). The fitness function to be maximized is naturally introduced into quantum algorithms as a problem Hamiltonian \( H_P \), that describes its inverse cost function. Therefore, the goal of the optimization procedure is to evolve the population toward quantum states with low energies, ideally the ground state. This encoding for the individuals and optimization criteria is commonplace in the quantum optimization [21], but not in previous approaches combining quantum algorithms and GAs. For instance, combinatorial optimization problems formulated as quadratic unconstrained binary optimization are encoded in Ising Hamiltonians [22] and the problem of finding the ground state of a chemical structure is naturally represented by its electronic Hamiltonian [23].

The full QGA implementation is summarized in Algorithm 1, showing the selection, crossover, and mutation subroutines comprising the algorithm. Selection is performed

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**Algorithm 1 QGA**

\[
\begin{align*}
H_P & \leftarrow \text{problem Hamiltonian} \quad \triangleright \text{with comparison oracle} \\
n & \leftarrow \text{number of registers} \quad \triangleright \text{assumed divisible by four} \\
c & \leftarrow \text{number of qubits per register} \quad \triangleright \text{assumed even}
\end{align*}
\]

Initialize with a random state

repeat
  sort registers 1 to \( n \) according to \( H_P \)
  reset registers \( n/2 \) to \( n \)
  for \( r = 1, 2, \ldots, n/2 \)
    pseudo-clone register \( r \) to register \( n/2 + r \).
  end for
  for \( i = 1, 2, \ldots, n/4 \)
    swap the last \( c/2 \) qubits of register \( n/2 + 2i - 1 \)
    with the last \( c/2 \) qubits of register \( n/2 + 2i \).
  end for
  mutate each qubit with probability \( p_m \)
until ending criteria is met \( \triangleright G \) generations
by sorting the population and then discarding the worst individuals given by $H_P$. Crossover is performed by replicating the selected individuals employing an approximate QCM [24] and then combining their features by performing qubit swap operations. Finally, mutation is performed by applying single qubit rotations at random. The discard and copy steps in these subroutines have required approximate implementations, due to fundamental limitations imposed by the no-deleting and no-cloning theorems [25], [26]. The modular flexibility of our proposal allows us to analyze four variants given by suppressing or activating the mutation, as well as the use of two different QCMs. Namely, we used the BCQO and the Bužek–Hillery UQCM. In the following sections we detail each subroutine.

Additionally, we analyze the algorithm in terms of quantum channels. This approach is based on the operator sum representation of each of the subroutines of the QGA process, i.e., a general description of a quantum state transformation [27]. In this representation, the evolution of an arbitrary density matrix $\rho$ under a quantum channel is given by $T(\rho) = \sum_k E_k \rho E_k^\dagger$, where $E_k$ are Kraus operators satisfying that $\sum_k E_k^\dagger E_k = I$ is the identity operator. This way, each iteration is also described by a quantum channel and the full QGA process corresponds to its self-composition. Consequently, we can employ the spectral theory of quantum channels to formally prove an exponential convergence of the algorithm, as well as bound its convergence rate.

### A. Quantum Selection Subroutine

The quantum selection subroutine aims to select the best individuals of the population without measuring their state. We achieve this by sorting the individuals to discard the lower half of the population. We refer to registers $r_1$ to $r_{n/2}$ and $r_{n/2+1}$ to $r_n$ as the upper-registers and lower-registers, respectively.

Sorting networks are protocols that sort the joint state of an $n$-register system [28]. We use the bubble sort algorithm, which performs efficiently for a moderate number of registers [29]. For $n$-individual populations, this algorithm is composed of $n$ layers. The odd ones sort consecutive register pairs starting from the first register and performing $n/2$ comparisons. Whereas, the even layers start from the second register and perform $n/2 - 1$ comparisons, hence, the total number of required comparisons is $n(n-1)/2$. Analogously, quantum sorting networks substitute the classical pairwise comparison with a quantum pairwise sorting operator [30]. We propose a quantum sorting network implementation tailored to our algorithm in the following.

Let us define the problem basis as the sorted $H_P$ eigenbasis $\{|u_1\rangle, \ldots, |u_{2^n}\rangle\}$, in increasing eigenvalue order, $\epsilon_k \leq \epsilon_{k+1}$, where $H_P|u_k\rangle = \epsilon_k|u_k\rangle$. In the following, we describe states in $\mathcal{H}$ with this basis. Consequently, the problem basis for the Hilbert space $\mathcal{H}^\otimes n$ of an $n$-individual population is formed by

$$|u_k\rangle = |u_{k_1}\rangle \otimes \cdots \otimes |u_{kn}\rangle$$

with $k_i \in \{1, \ldots, 2^c\}$. We refer to $k = (k_1, \ldots, k_n)$ as the population index sequence which labels the population basis state $|u_k\rangle$.

The pairwise sorting operator is defined by concatenating a comparison oracle $O_{\text{CMP}}$ with a controlled swap $C_{\text{SWAP}}$ acting on consecutive registers $\{i, i+1\}$ and ancillary qubit $a$

$$O_{\text{CMP}}[u_k]\langle u_{k'}|_{i+1}|0\rangle_a = \begin{cases} |u_k\rangle|u_{k'}\rangle_{i+1}|0\rangle_a, & \text{if } \epsilon_k \leq \epsilon_{k'} \\ |u_k\rangle|u_{k'}\rangle_{i+1}|1\rangle_a, & \text{if } \epsilon_k > \epsilon_{k'} \end{cases}$$

and

$$C_{\text{SWAP}}|x\rangle_i|y\rangle_{i+1}|c\rangle_a = \begin{cases} |x\rangle_i|y\rangle_{i+1}|c\rangle_a, & \text{if } c = 0 \\ |y\rangle_i|x\rangle_{i+1}|c\rangle_a, & \text{if } c = 1 \end{cases}$$

where $|x\rangle|y\rangle$ represents any separable pure state. Fig. 1 shows the circuit representation for a four-register quantum sorting network...
network, where $O_{\text{CMP}}$ is denoted by controlled CMP gates. For clarity, let us note that $O_{\text{CMP}}$ can indeed be implemented without explicitly knowing the eigenvalues and eigenvectors of the problem Hamiltonian. On a general case, two additional registers can be included to compute the energy in binary representation and mark the ancillary qubit with an arithmetic comparator [31], [32], [33].

To describe the output state of the sorting subroutine, we define $s(k)$ as the sorted version of the population index sequence $k$, i.e., $s(k)$ is the population index sequence obtained by a permutation of $k$ such that $s(k)_i \leq s(k)_{i+1}$. This allows us to denote by $|u_{s(k)}⟩$ to the sorted version of the population basis state $|u_k⟩$. We also define $\sigma(k)$ as the binary sequence of instructions associated with a given input and sorting network, i.e., the bit $\sigma(k)_i$ stores the output of the $i$th comparison in the sorting network which is 1 if the pair required swapping and 0 otherwise. In the quantum circuit implementation, $\sigma(k)$ corresponds to the outputs of the ancillas given population input $|u_k⟩$.

In general, given initial state $\sum_k b_k |u_k⟩$, the sorting output is a superposition of sorted populations, possibly entangled with the ancillary qubits

$$|\Psi⟩ = \sum_k b_k |u_{s(k)}⟩ |\sigma(k)⟩.$$  \hspace{1cm} (4)

After the sorting, ancillary qubits must be discarded to proceed with the algorithm, therefore, the output state is a reduced density matrix of $|\Psi⟩⟨\Psi|$,

$$\rho_{\text{sorted}} = \sum_k \sum_{k'} b_k b_{k'}^* \delta_{\sigma(k),\sigma(k')} |u_{s(k')}⟩⟨u_{s(k')}|.$$  \hspace{1cm} (5)

where $\delta_{ij}$ is the Kronecker delta. The state $\rho_{\text{sorted}}$ is a mixture of pure states, each being a quantum superposition of states with equal sorting instructions, $\sigma(k) = \sigma(k')$. Hence, the Kraus operators of this subroutine are

$$A_k = \sum_k \delta_{\sigma(k),\sigma} |u_{s(k)}⟩⟨u_{s(k')}|.$$  \hspace{1cm} (6)

where $\kappa$ represents a sorting instruction and $k$ iterates over all possible index sequences. The vector $A_k |u_{k'}⟩ = \delta_{\sigma(k),\sigma} |u_{s(k')}⟩$ if $\sigma(k') = \kappa$ or equals zero otherwise. That is, $A_k$ transforms the states into their sorted version if they are sorted by the instructions $\kappa$, so that summing over all $\kappa$ we obtain

$$\rho_{\text{sorted}} = \sum_k A_k \rho A_k^+.$$  

The proposed subroutine provides a population-based approach to sort a set of individuals with unitary-preserving rules. This allows us to implement the selection defined by the problem Hamiltonian in a quantum-compatible manner.

B. Quantum Crossover Subroutine

In this subroutine, the states of the selected individuals in the upper-registers are approximately replicated into the lower-registers by means of QCMs. It is performed in three steps: first, the lower registers are set to a reference state $\rho_0$ defined by the QCM, then each QCM acts on the joint state of registers $i$ and $(n/2) + i$, where $1 \leq i \leq (n/2)$, and, finally, consecutive lower-registers are combined swapping their second half.

We discard the state in the lower-registers and replace it with a reference state $\rho_0^{\otimes n/2}$, i.e., we reset the lower-registers, obtaining

$$\rho_{\text{selected}} = \mathcal{W}_{\text{low}}(\rho_{\text{sorted}}) \otimes \rho_0^{\otimes n/2}.$$  \hspace{1cm} (7)

Given a reference state with spectral decomposition $\rho_0 = \sum_{i=1}^d \omega_i |e_i⟩⟨e_i|$ and $d \leq 2^n$, the Kraus operators are

$$B_{j,i_1,\ldots,i_{n/2}} = \sqrt{\omega_{i_1} \cdots \omega_{i_{n/2}}} |\otimes_{2} \cdots |e_{i_1} \cdots e_{i_{n/2}}⟩⟨j|$$  \hspace{1cm} (8)

where the states $|j⟩$ form an orthogonal basis of the state space of the lower registers, with $j = 1, \ldots, 2^{n/2}$.

We denote the QCM acting on individuals $i$ and $j$ by $T_{\text{QCM}}^{ij}$, employing superindices to represent the individuals it acts on. The global replication step is described by the operation

$$\rho_{\text{replicated}} = \sum_{i=1}^{n/2} \chi_{T_{\text{QCM}}^{ij}} (\rho_{\text{selected}})$$  \hspace{1cm} (9)

where by the tensor product of the operators $T_{\text{QCM}}^{ij}$ that act on two individuals we obtain an operator that acts on the whole population. Given a representation of $T_{\text{QCM}}^{ij}$ with Kraus operators $c_{ki}^{ij}$ for some $k = 1, \ldots, K$, the Kraus operators for the global replication are

$$C_{k_1,\ldots,k_{n/2}} = \bigotimes_{i=1}^{n/2} c_{ki}^{ij}$$  \hspace{1cm} (10)

where each element of the sequence $k_1,\ldots,k_{n/2}$ takes values from 1 to $K$. Due to this constructive definition it is convenient to preserve multiple indexes instead of replacing them with a single index running from 1 to $K^{n/2}$. We detail below the particular Kraus operators for the QCMs used in our algorithm.

Finally, we swap qubits $c/2$ to $c$ in register $(n/2) + 2i − 1$, with qubits $c/2$ to $c$ in register $(n/2) + 2i$ for all $1 \leq i \leq (n/4)$, which yields

$$U_{\text{swap}} = \prod_{i=mc/4}^{(n-2)c/2} \prod_{j=m+1}^{c} S_{2i+j,2i+c+j}$$  \hspace{1cm} (11)

where $S_{ij}$ denotes a swap between qubits $i$ and $j$, and qubits are numbered increasing from the first one in $r_1$ to the last one in $r_n$. Recall that $c$ is the number of qubits per register. The population after crossover is given by

$$\rho_{\text{crossover}} = U_{\text{swap}} \left( \sum_{i=1}^{n/2} T_{\text{QCM}}^{ij} (\rho_{\text{selected}}) \right) U_{\text{swap}}^\dagger.$$  \hspace{1cm} (12)

1) Quantum Cloning Machines: Generically, let us consider a joint system $A–B$ such that $A$ is initially in an unknown state $|\psi⟩$ and $B$ in a reference state $|R⟩$. Then, a QCM is a quantum operation mapping $|\psi⟩_A|R⟩_B$ to an output state $\rho_{AB}'$ which contains two approximated copies of the input state $|\psi⟩$ [24]. The quality of the copies is measured in terms of
the single copy fidelity, which is the fidelity of $|\psi\rangle$ with the partial states of $\rho_{AB}'$ of each clone, i.e.,

$$F_j(|\psi\rangle) = \langle \psi | \rho_j' | \psi \rangle, \quad j = A, B$$

(13)

where $\rho_A' = \text{tr} [\rho_{AB}']$ and $\rho_B' = \text{tr}_A [\rho_{AB}']$.

Formally, let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the $d$-dimensional Hilbert spaces associated with systems $A$ and $B$, respectively. In general, a QCM is described as a quantum operation $T_{\text{QCM}} : \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B$. However, the action of $T_{\text{QCM}}$ on an input state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is only considered an approximated cloning operation if $\rho_{AB} = \rho_A \otimes \rho_B$, where $\rho_A \in \mathcal{H}_A$ is the arbitrary state to be copied and $\rho_B \in \mathcal{H}_B$ is the reference state to be overwritten.

Below, we analyze the performance of our algorithm employing two possible QCMs. Namely, the BCQOs [34] and the optimal symmetric UQCM introduced by Bužek and Hillery [35], [36].

a) BCQO: For a reference state $\rho_0$ on system $B$ and a quantum observable $\theta$, the cloning operator $U(\theta, \rho_0)$ is the operator satisfying

$$\langle \theta | \rho \rangle \equiv \text{tr}[\rho \otimes |\theta\rangle \langle \theta|]_{U_{\rho_0} \otimes \rho U^\dagger} = (|\theta\rangle \otimes \theta)_{U_{\rho_0} \otimes \rho U^\dagger}$$

(14)

for any state $\rho$ on system $A$. That is, the expected value of $\theta$ on the initial state of system $A$ is the same as the expected value that is eventually obtained on both systems. If $U$ satisfies (14) for the operator $\theta$, then it can be constructed to also hold for a complete set of observables commuting with $\theta$. Additionally, the unitary operator can be straightforwardly described in the $\{|i\rangle\}_{i=1}^d$ basis which diagonalizes those observables, assuming the reference state is $\rho_0 = |1\rangle \langle 1|$. Following (8) the Kraus operators for the reset are $B_j = (I \otimes |j\rangle \langle j|)^{2nc/2} \otimes \mathbb{1}$. Additionally, the BCQO can be efficiently implemented in elementary gates employing ripple-carry addition [31].

b) UQCM: The UQCM satisfies three valuable properties: it obtains symmetric single copy fidelity $F_A = F_B$, the value of this fidelity is independent of the input pure state, and that value is the optimal value while satisfying the previous properties, which equals to $\frac{1}{2} + \frac{1}{d+1}$ with $d = 1$ of the Hilbert space. It is efficiently implemented with an ancillary register and ripple-carry adder [24], [31].

Let $\mathcal{H}_A^2$ be the subspace of $\mathcal{H}_A \otimes \mathcal{H}_B$ formed by the invariant states with respect to $A-B$ system swap $S_{AB}$, and let $S_+$ be the projection operation into $\mathcal{H}_A^2$

$$S_+ \equiv \frac{1}{2} (I_A \otimes I_B + S_{AB}).$$

(15)

It requires the reference state $\rho_0 = |1\rangle \langle 1|$, hence, the Kraus operators derived from (8) are $B_{ij} = \sqrt{\frac{1}{2(nc/2)}} |j\rangle \langle j|$, where $d = 2^e$ and the states $|j\rangle$ form an orthogonal basis of the state space of the lower-registers $r = 1, \ldots, 2^{(nc/2)}$. Then, given a state to copy $\rho_A$, the UQCM is the projection of the state $\rho_A \otimes \rho_0$ into the symmetric subspace $\mathcal{H}_A^2$ with a normalization, i.e.,

$$T_{\text{UQCM}}(\rho) = \frac{2d}{d+1} (S_+ (\text{tr}_B(\rho) \otimes \rho_0)) S_+$$

(16)

where we require the partial trace on $B$ so that the operation is well defined for every possible input state $\rho$. However, the operation is only considered an approximated cloning for $\rho = \rho_A \otimes \rho_0$, which satisfy $\text{tr}_B(\rho) = \rho_A$. The Kraus operators are

$$P_{ij} = \sqrt{\frac{2}{2^e+1}} S_+ (I^r \otimes |j\rangle \langle j|)$$

(17)

where states $|j\rangle$ and $|k\rangle$ form an orthogonal basis of the $j$th register state space. The operator $I^r \otimes |j\rangle \langle k|$ acts trivially on register $i$ and projects register $j$ with $|j\rangle \langle k|$, and $S_+^{ij}$ is the projection into the swap-invariant subspace of these registers.

C. Quantum Mutation Subroutine

Mutation aims to slightly perturb some of the individuals, to facilitate the exploration of new areas in the search space. Classically, it is often implemented by bit-flip operations according to a sufficiently small mutation probability. Similarly, its quantum analog is implemented by mutation unitary gates which are applied stochastically to each individual. To express mutation in terms of Kraus operators, let us consider a generic set of mutation gates $\{U_{\mu}^M\}_{\mu=0}^M$ with disjoint probabilities $\{\mu\}_{\mu=0}^M$ defining $U_0 = I$ for the mutation case. Then, the Kraus operators are all possible gate combinations

$$D_{\mu_1, \ldots, \mu_n} = \sqrt{\mu_1 \cdots \mu_n} U_{\mu_1} \otimes \cdots \otimes U_{\mu_n}.$$  

(18)

The most suitable set of mutation gates may depend on the characteristics of the considered problem and exploring large search spaces could require elaborated approaches, such as two-qubit gates or generalized rotations. Here, we choose an ensemble of gates composed of single-qubit mutations, namely, the Pauli gates $X$, $Y$, and $Z$. More precisely, each qubit is mutated with probability $p_m$, and the mutation gate is chosen at random from $\{X, Y, Z\}$ with equal probability. Hence, the mutation gate ensemble is $U_{\mu} \in \{X, Y, Z\} \otimes c$ with $p_\mu = (p_m/3)^k \frac{1}{c} \mu^{k}$, where $k$ is the number of nonidentity gates in $U_{\mu}$.

D. Fixed Points and Convergence

Let us call $T$ the population evolution for a single generation and $\rho_0$ the initial state for the population. The operation $T$ is the composition of the subroutines described in previous subsections, namely, reset $T_R$, crossover $T_C$, swap $T_{\text{SWAP}}$, mutation $T_M$, and sorting $T_S$ quantum channels, and it can be expressed as

$$T = T_ST_M U_{\text{SWAP}} T_C T_R$$

(19)

which is also a quantum channel. Here, we placed the sorting subroutine as the last step to force that the output state is sorted, which is necessary to determine the optimal solution. Consequently, the outcome after $G$ generations of the QGA corresponds to $G$ applications of $T$ to the initial state after a sort operation, $\rho(G; \rho_0) = T_G(T_S(\rho_0))$. Fortunately, for the purpose of analyzing the asymptotic regime, we can assume without loss of generality that the initial state is already sorted, i.e., $\rho_0 = T_S(\rho_0)$. Therefore, the performance and convergence of the algorithm are given by the algebraic properties of the quantum channel in its asymptotic limit, i.e.,
convex set of fixed points and the spectral subradius. To analyze these properties, let us introduce the eigenvectors and eigenvalues of $T$

$$T(W_l) = \lambda_l W_l, \quad \text{with } |\lambda_{l+1}| \leq |\lambda_l|. \quad (20)$$

As $T$ is a completely positive trace preserving map, according to the Perron–Frobenius Theorem, the spectral radius is one [37]. The convex set spanned by the eigenvectors corresponding to the eigenvalues with absolute value one is called the convex set of fixed points. Moreover, we assume w.l.o.g. that the first $m \geq 1$ eigenvalues are equal to one and the remaining $l > m$ satisfy $|\lambda_l| < 1$, implying no oscillating fixed points. Indeed, employing that any eigenvalue with absolute value one is a root of unity, i.e., it has the form $\exp(i(2\pi k)/\rho)$ with $k$ and $p$ integer [39], we could redefine the generation of the QGA as the $p$-composition $T^p$ of the original channel when studying the asymptotic regime. In any case, this situation is not expected when employing the aforementioned subroutines, since the algorithm converges by construction.

Given a decomposition of the initial state $\rho_{\text{in}} = \sum \omega_l W_l$ in terms of the eigenvectors from (20), we define $\Lambda = \sum_{l=1}^{m} \omega_l |\lambda_l|^2 W_l$ as the projection of $\rho_{\text{in}}$ into the convex set of fixed-points. $\Lambda$ is a density matrix so the state for the $G$th generation can be straightforwardly obtained as

$$\rho(G; \rho_{\text{in}}) = T^G(\rho_{\text{in}}) = \Lambda + \sum_{l=m+1}^{m} \omega_l |\lambda_l|^2 e^{iG\arg\lambda_l} W_l = \Lambda + O(|\lambda_{m+1}|^G). \quad (21)$$

This shows that the convergence of the QGA is exponential with a rate given by the magnitude of the second greatest eigenvalue of $T$, $|\lambda_{m+1}|$, which is called spectral subradius.

For the particular case $m = 1$, the fixed point $\Lambda$ is unique and the spectral subradius is $|\lambda_2|$. Note, that this unique fixed point depends on the problem Hamiltonian via the sorting subroutine. In this case, the expected value of an observable $\theta$ after $G$ generations can be estimated by

$$\langle \theta \rangle (G; \rho_{\text{in}}) = \text{tr}[\theta \rho(G; \rho_{\text{in}})] = \theta_\infty + \Delta |\lambda_2|^G + O(|\lambda_3|^G) \quad (22)$$

where $\theta_\infty = \text{tr}[\theta \Lambda]$ and $|\lambda_2|$ only depend on the problem Hamiltonian, whereas $\Delta = 2 \text{Re}(\omega_2 e^{iG\arg\lambda_2} \text{tr}[\theta W_2])$ depends on the initial state and the generation, but its absolute value is upper bounded by a constant.

III. RESULTS

In this section, we use two methods to benchmark four QGA variants with respect to a common set of problems. More specifically, for every problem, we simulate the evolution for each variant considering a random sample of initial states taken from a uniform Haar distribution. Additionally, we analyze the spectral properties of the quantum channel corresponding to the corresponding problem and variant on the grounds of Section II-D. The four variants consist of two different QCMs, namely, BCQO and UQCM, together with the inclusion or not of the mutation subroutine. Due to computational constraints, the size of the simulated system is limited to $n = 4$ individuals with a chromosome length of $c = 2$ qubits. Hence, the population is encoded in a total of 8 qubits. These are the minimal values allowing to implement our algorithm and providing a nontrivial search space. We use a mutation probability of $\rho_m = (1/24)$, i.e., on average, one mutation in one of the 8 qubits every 3 generations. This choice is motivated by the typical suggested values for classical GAs [5]. Our benchmark consists of a random sample of 200 problem Hamiltonians. Finally, the figures of merit for this benchmark are the quantum fidelity between the best individual and the desired Hamiltonian ground state, and the corresponding convergence speed.

Simulations were performed with matrix computations in Python-NumPy. We ran each simulation for ten generations, which were sufficient to estimate the figures of merit within the asymptotic regime due to the exponential convergence, as explained in Section II-D. Additionally, we used for each problem Hamiltonian a random sample of ten initial quantum states taken from a uniform Haar distribution, which was the same for the four variants. This is intended to avoid the emergence of biases that can affect the performance of the cloning subroutine. We have numerically observed that the figures of merit converge with as few as ten sample states, thus, this is statistically sufficient for our system size and leads to a good balance between precision and computational cost of the simulation. The mathematical reason behind this, as we will see, is that the algorithm has a unique fixed point. For each case, we recorded the problem Hamiltonian, the set of initial populations, the set of final populations, and the statistics of each individual in the Hamiltonian basis.

The quantum channel analysis was performed in MATLAB. For this analysis, we have discarded the mutation subroutine, since we have observed in previous simulations a negligible effect on the performance while leading to a substantial increase in the computational cost. First, we represent the quantum channel eigenvalue equation (20) in matrix form for both BCQO and UQCM variants applied to each problem Hamiltonian. To achieve it, we employ the vectorization $\sum_k E_k \rho E_k^\dagger \rightarrow (\sum_k E_k \otimes E_k^\dagger) |\rho\rangle$. Afterward, we computed the six greatest eigenvalues and their respective eigenvectors. Then, the fixed points are the eigenvectors corresponding to eigenvalue. Remarkably, all cases and variants studied in this article show a unique fixed point. We expect that this situation is generic, but we leave a formal proof for further research.

A. Selection of the Problem Hamiltonians

Let us note that the problem Hamiltonian exclusively plays a role in the selection subroutine, which is only sensitive to the ordinal position of the eigenvalues and not to their exact value. Since the states of the problem basis defined in Section II-A are sorted in increasing order according to their energy, the effect of the Hamiltonian can be fully described by its problem basis. Given two problem Hamiltonians with bases related by unitary $U$, the respective Kraus operators $A_k$ and $A_k'$ of their sorting subroutines, defined as in (6), are linked by a basis transformation $A'_k = U \otimes A_k (U^\dagger)^\dagger$. This is shown by the linearity of Kraus operators. This way, generating random unitaries is equivalent to generating random
nondegenerate problem Hamiltonians. We define the computational Hamiltonian $H_C$ as a diagonal Hamiltonian in the computational basis $|u_k\rangle = |k\rangle$ in canonical order, choosing without loss of generality eigenvalues proportional to their corresponding indexes, $\epsilon_k \propto k$. Afterward, we sample a random $U$ from a uniform set of unitary operations, in order to generate the different problem Hamiltonians as $U H_C U^\dagger$.

B. Figures of Merit

Let us define the QGA fidelity after $G$ generations of the algorithm and for the initial state population $\rho_{in}$. The QGA fidelity is the quantum fidelity between the state of the best individual after $G$ iterations and the exact ground state of the Hamiltonian $|u_1\rangle$

$$F_{\text{QGA}}(G; \rho_{in}) = \langle u_1 | \text{tr}_{\perp} [\rho(G; \rho_{in})] | u_1 \rangle$$

where $\text{tr}_{\perp}$ is the partial trace over all the population but the best individual, and $\rho(G; \rho_{in})$ is the state of the population at the $G$th generation. Note that $F_{\text{QGA}}$ is computed after the sorting operation since the best individual is located in the first register.

The quantum fidelity is not a suitable metric for large systems, as it rapidly tends to zero for relatively small differences [40]. In fact, in the thermodynamic limit, two states are only either the same or orthogonal. Then, the expected energy of the best individual turns out to be a better figure of merit. This quantity additionally provides an adequate method for comparing this algorithm with other optimization approaches. It is also an adequate fitness function to evaluate the performance in a realistic scenario in which the exact solution of the problem is not available. Nevertheless, the expected energy of the best individual can be altered by the particular choice of the eigenvalues, thus, the QGA fidelity is a better choice for the benchmarking analysis pursued in this article.

As $|u_1\rangle$ is a pure state, $F_{\text{QGA}}$ is the expected value of an observable. Precisely, it is the probability of measuring the ground state energy. Additionally, we have empirically observed that the algorithm has generically a unique fixed point. This allows us to estimate the evolution of the QGA fidelity according to (22)

$$F_{\text{QGA}}(G; \rho_{in}) \approx F_\infty + \beta_{in} \gamma^G$$

where $F_\infty$ is the expected value of $F_{\text{QGA}}$ in the fixed point and $0 \leq \gamma < 1$ describes the convergence rate. When the fixed point is unique, both are independent from $\rho_{in}$, whereas $\beta_{in}$ is a bounded parameter depending on the initial state. In our analysis, we employ $F_\infty$ and $\gamma$ as figures of merit characterizing the quality of the final population and the convergence velocity, respectively.

C. Quantum Channel Analysis and Numerical Simulations

We have analyzed the accuracy and convergence speed of the algorithm in terms of the fidelity between the best individual and the desired ground state. These quantities are characterized by the asymptotic value $F_\infty$ and convergence-rate $\gamma$ introduced in (24). We employed two methods to estimate their value: 1) fitting the parameters in (24) from the data obtained by the simulations described in the introduction of Section III and 2) computing them from the fixed points and spectral subradius corresponding to (22). In this section, we show the agreement between both approaches.

In the first method, we fit the parameters in (24) from the data points $F_{\text{QGA}}(G; \rho_{in})$ by least-squares method. This is performed for every problem Hamiltonian and initial population. Then, $F_\infty$ and $\gamma$ values are averaged over different initial populations to estimate a single value for each problem Hamiltonian. We only included data points after four generations to avoid biases due to the initial states.

In the second method, we compute both the fixed point and the spectral subradius of the channel by diagonalizing its matrix form. This allows us to predict $F_\infty = \langle u_1 | \text{tr}_{\perp} [\Lambda] | u_1 \rangle$ and $\gamma = |\lambda_2|$. As discussed in the introduction of Section III, we have only studied the variants without the mutation subroutine with this method, since they have a negligible effect on the performance while leading to a substantial increase in the computational cost. It is important to highlight that all the channels analyzed in this article show a unique fixed point. We expect this property to be generic, i.e., for a randomly chosen problem Hamiltonian and any variant considered in this article this feature holds. However, there are corner cases in which there is a nontrivial convex set of fixed points. For instance, in [38] a fourfold degenerate example is constructed for a problem Hamiltonian diagonal in the computational basis employing the BCQO variant without mutation. Another situation in which this might happen is when the ground state of the problem Hamiltonian is degenerate. However, we leave a complete characterization of this scenario for further research.

The comparison between these two methods is shown in Fig. 2, where each point represents a different problem Hamiltonian. The vertical axis represents the $F_\infty$ and $\gamma$ obtained by fitting the simulation data, while the horizontal axis represents the same parameters obtained via quantum channel analysis. Fig. 2(a) and (b) depict the values of $F_\infty$ obtained, respectively, for BCQO and UQCM variants, both without mutation. Fig. 2(c) and (d) depict the values of $\gamma$ obtained, respectively, for BCQO and UQCM variants, again without mutation. The $R^2$ between simulation and quantum channel analysis for $F_\infty$ are 0.999 for BCQO and 0.986 for UQCM. Similarly, the $R^2$ between simulation and quantum channel analysis for $\gamma$ are 0.756 for BCQO and 0.999 for UQCM. These correlation coefficients show strong accordance between the results obtained with both methods. This establishes the quantum channel analysis as a powerful tool to formally prove exponential convergence, as well as providing mathematical techniques to show bounds for the accuracy.

D. Convergence and Performance

In Fig. 3, we show a representative evolution of the $F_{\text{QGA}}$ for each variant, illustrating the exponential convergence modeled in (24). Recall that we used identical initial
Fig. 2. Comparison of results obtained from numerical simulations and quantum channel analysis for a sample of problem Hamiltonians. (a) and (b) Represent the final fidelity $F_{\infty}$ between the best individual and the target ground state obtained, respectively, for BCQO and UQCM variants, both without mutation. (c) and (d) Represent the fidelity convergence rate $\gamma$ obtained, respectively, for BCQO and UQCM variants, both without mutation. The large linear correlation $R^2$ is consistent with both results being the same in all cases.

Fig. 3. Representative evolution of the fidelity between the best individual and the desired Hamiltonian ground state for the four different variants. We plot the evolution for the same ten initial populations considering four QGA variants applied on the same Hamiltonian. (a) Evolution for variants without mutation and (b) evolution for variants with mutation, different markers represent variants with different cloning machines, namely, dots for BCQO variants and crosses for UQCM variants. Note that UQCM variants reach higher-fidelity values, which is consistent with the statistical performance derived in Section III-D. The stochastic behavior of qubit tracing was considered to account for the statistical information in the density matrix, whereas mutation was applied considering a different randomly chosen mutation pattern for each initial population and variant. This results in oscillations of the curves in (b), but we conclude an average convergence toward a stable fidelity value in Section III-D.

Fig. 4. Final fidelity between the best individual and the desired Hamiltonian ground state, $F_{\infty}$. We applied four variants—employing two different cloning machines, BCQO or UQCM, and excluding or including mutation—to a sample of randomly generated problem Hamiltonians. The vertical position of each point is the average value from ten initial populations for each problem Hamiltonian, and the horizontal position is jittered to reduce the visual overlap between points. BCQO variants hold the same mean and standard deviation—0.72 and 0.05—independently of the presence or absence of mutation. The UQCM variant without mutation has mean and standard deviation equal to 0.91 and 0.04, whereas, for the UQCM variant with mutation, the values are 0.89 and 0.04.

populations for the four variants. Each initial population and variant is represented by a line, and all figures correspond to the same problem Hamiltonian. Cases without mutation in Fig. 3(a) showcase a better fit than Fig. 3(b) because mutation was applied as a unitary gate despite being stochastic, producing random patterns in the fidelity. Note that in the rest of the subroutines stochastic elements are integrated in the density matrix description.

According to the numerical fit, the BCQO variant without mutation shows $\gamma$ values between 0.18 and 0.88, with an
average of 0.50 and a standard deviation of 0.13. Meanwhile, the UQCM variant without mutation shows $\gamma$ values between 0.47 and 0.79, with an average of 0.60 and a standard deviation of 0.07. Additionally, we compared the variants for each problem Hamiltonian and obtained that the BCQO variant tends to lower-$\gamma$ values in 77% of cases, with a 95% confidence interval of $\pm 6$. Therefore, the BCQO variant presents a faster convergence on average.

Fig. 4 summarizes the performance of the variants in terms of the fidelity $F_{\infty}$ between their best individual and the ground state. We can note that the success probability is above 0.6 for virtually every problem Hamiltonian. Remarkably, BCQO variants hold the same mean and standard deviation – 0.72 and 0.05 – regardless of the presence or absence of mutation. The mean and standard deviation of the UQCM variant without mutation are 0.91 and 0.04, respectively, whereas those of the UQCM variant with mutation are 0.89 and 0.04, respectively. As we can see, the UQCM variant without mutation outperforms the others in the studied cases, yielding 25% higher fidelity than BCQO ones and 2% higher fidelity than the UQCM variant with mutation. Hence, replicating the individuals with the UQCM produces generally better results in terms of fidelity with the desired Hamiltonian ground state. Note, however, that this result is obtained on a set of randomly generated Hamiltonians and that BCQO variants may improve their results for problem Hamiltonians that nearly commute with the quantum observable chosen to define the BCQO.

Overall, we observe that both $\gamma$ and $F_{\infty}$ vary more with respect to the problem Hamiltonian for the BCQO variants. This is because the cloning fidelity of BCQO strongly depends on the input state and whether it is diagonal to the basis of the cloning observable, unlike UQCM. Regarding convergence rate, entanglement could play a role, as stronger entanglement in the cloning process leads to a larger loss of information in the selection process. We have experimentally observed that BCQO cloning produces stronger entanglement than UQCM, which could intuitively explain its faster convergence. This can also be observed on a quantum-channel level, where the spectral decomposition of the reset and cloning subroutines are more extreme for BCQO than for UQCM. Indeed, the eigenvalues of the former are always either 1 or 0, originating a more abrupt collapse. Note that although the convergence was generally faster in the BCQO variant, the UQCM variant obtained generally higher fidelity. Although this may suggest a conflict between fidelity and convergence speed, a detailed analysis of all the problem instances and other variants would be required to clarify it.

The employed method for the mutation subroutine does not introduce any meaningful performance improvement. It produced a marginal effect on the BCQO variant, but also a slight performance decrease in the UQCM variant. Further research is required to understand the role of mutation and the use of other approaches. However, for the relatively small search space that we have explored in our analysis, exploitation takes precedence over exploration, which reduces the importance of this subroutine.

IV. CONCLUSION

Here, we have introduced a QGA comprising the fundamental elements which characterize GAs. This was attained by codifying the individuals in nonorthogonal quantum states supported in independent registers, a distinctive feature with respect to previous approaches. Moreover, we have codified the optimization problem in a nondiagonal Hamiltonian and the cost function in the energy of the individual with respect to this Hamiltonian. The algorithm has a modular structure composed of quantum selection, crossover, and mutation subroutines. At the cost of introducing some ancillary qubits, the selection was performed as a reversible sorting algorithm with respect to the problem Hamiltonian, tracing out the lowest-ranked individuals. We carried out the replication step in the crossover via a partial QCM and the combination of half of the genome by swapping the corresponding qubits. We have benchmarked two paradigmatic quantum approximated cloning machines: BCQOs and Bužek–Hillery UQCM. We have generated a sample of 200 random problem Hamiltonians, ran the quantum algorithm for ten generations, and compared for both cloning machines the convergence ratio and fidelity of their corresponding solutions with respect to the real ground state. Then, this numerical analysis showed that the convergence speed employing the BCQO is larger than with UQCM in 77% of the cases. However, when we focus on the fidelity of their best individual with respect to the ground state, we observed that UQCM always outperforms BCQO in the studied cases with an average improvement in the fidelity of 25%. Finally, we concluded that introducing mutations, implemented by means of randomly allocated Pauli gates, had a negligible effect on the fidelity of the best individual. In fact, even though there were small changes case by case, both the mean and the standard deviation were identical with and without mutation for the BCQO and there was a slight advantage of 2% in the absence of mutations for the UQCM. Finally, we have expressed our subroutines as quantum channels, such that each generation of the algorithm, which is also a quantum channel, is a composition of them. It follows that the iteration of the algorithm corresponds to the composition of this channel with itself. Therefore, we can employ the spectral theory of quantum channels to formally prove an exponential convergence of the algorithm toward the fixed point of the channel, as well as to bound its convergence rate by its spectral subradius. Remarkably, both this prediction and the final quantum state accurately match our numerical simulations. Indeed, the correlation coefficients between the fidelities obtained by means of numerical simulations and quantum channel techniques were $R^2 = 0.999$ for BCQO and $R^2 = 0.986$ for UQCM. Similarly, the correlation coefficients between predicted and numerically obtained convergence rates were $R^2 = 0.756$ for BCQO and $R^2 = 0.999$ for UQCM. This approach can be extended to other nonunitary iteration-based quantum algorithms.

Further investigation into the role of mutations in the protocol is a possible future research. Specifically for larger systems, it could inspire the development of new mutation methods based on the structure of the problem. Additionally, a thorough analysis of the balance between convergence speed
and the fidelity with the ground state could provide valuable insights into any fundamental limitation in this approach. Future work should also involve extensive comparison with other optimization algorithms and more relevant problem cases, which has been already initiated in [41]. Additionally, a specific implementation of the subroutines in terms of fundamental gates could be developed for a particular problem case.

REFERENCES

[1] G. Hornby, A. Globus, D. S. Linden, and J. D. Lohn, “Automated antenna design with evolutionary algorithms,” in Proc. AIAA Space, 2006, pp. 1–11.
[2] J. H. Holland, Adaptation in Natural and Artificial Systems: An Introductory Analysis with Applications to Biology, Control and Artificial Intelligence. Cambridge, MA, USA: MIT Press, 1992.
[3] K. De Jong, “Genetic algorithms are NOT function optimizers,” Found. Genet. Algorithms, vol. 2, pp. 3–17, 1993.
[4] R. S. Zebulum, M. A. C. Pacheco, and M. M. B. Velasque, Evolutionary Electronics: Automatic Design of Electronic Circuits and Systems by Genetic Algorithms. Boca Raton, FL, USA: CRC Press, 2018.
[5] M. Mitchell, An Introduction to Genetic Algorithms. Cambridge, MA, USA: MIT Press, 1996.
[6] O. A. Söfge, “Prospective algorithms for quantum evolutoinary computation,” 2008, arXiv:0804.1133.
[7] U. Roy, S. Roy, and S. NayeK, “Optimization with quantum genetic algorithm,” Int. J. Comput. Appl., vol. 102, no. 16, pp. 1–7, 2014.
[8] R. Lahoz-Beltra, “Quantum genetic algorithms for computer scientists,” Computers, vol. 5, no. 4, p. 24, 2016.
[9] A. Narayanan and M. Moore, “Quantum-inspired genetic algorithms,” in Proc. IEEE Int. Conf. Evol. Comput., 1996, pp. 61–66.
[10] K. Han and J. H. Kim, “Quantum genetic algorithm and its application to combinatorial optimization problem,” in Proc. Conge. Evol. Comput., vol. 2, 2001, pp. 1422–1429.
[11] K.-H. Han and J.-H. Kim, “Quantum-inspired evolutionary algorithm for a class of combinatorial optimization,” IEEE Trans. Evol. Comput., vol. 6, no. 6, pp. 580–593, Dec. 2002.
[12] S. Yang, M. Wang, and L. Jiao, “A novel quantum evolutoinary algorithm and its application,” in Proc. Conge. Evol. Comput., vol. 1, 2004, pp. 820–826.
[13] L. Wang, F. Tang, and H. Wu, “Hybrid genetic algorithm based on quantum computing for numerical optimization and parameter estimation,” Appl. Math. Comput., vol. 171, no. 2, pp. 1141–1156, 2005.
[14] S. Yingharenthavornchai, C. Aporntrwan, and P. Chongstitvatana, “An implementation of compact genetic algorithm on a quantum computer,” in Proc. 9th Int. Conf. Comput. Sci. Softw. Eng. (ICCSSE), 2012, pp. 131–135.
[15] B. Rylander, T. Soule, J. Foster, and J. Alves-Foss, “Quantum evolutionary programming,” in Proc. Genet. Evol. Comput. Conf. (GECCO), San Francisco, CA, USA, 2001, pp. 1005–1011.
[16] M. Udrescu, L. Prodan, and M. Vladutiu, “Implementing quantum genetic algorithms: A solution based on Grover’s algorithm,” in Proc. 3rd Conf. Comput. Front., 2006, pp. 71–82.
[17] A. Malossini, E. Blanzieri, and T. Calarco, “Quantum genetic optimization,” IEEE Trans. Evol. Comput., vol. 12, no. 2, pp. 231–241, Apr. 2008.
[18] A. SaiToh, R. Rahimi, and M. Nakahara, “A quantum genetic algorithm with quantum crossover and mutation operations,” Quant. Inf. Process. vol. 13, no. 3, pp. 737–755, 2014.
[19] L. K. Grover, “A fast quantum mechanical algorithm for database search,” in Proc. 28th Annu. ACM Symp. Theory Comput., 1996, pp. 212–219.
[20] K. P. Kalinin and N. G. Berloff, “Global optimization of spin Hamiltonians with gain-dissipative systems,” Sci. Rep. vol. 8, pp. 1–9, Dec. 2018.
[21] A. Lucas, “Ising formulations of many NP problems,” Front. Phys., vol. 2, pp. 1–4, Feb. 2014.
[22] T. Jilly et al., “The variational quantum Eigensolver: A review of methods and best practices,” Phys. Rev., vol. 986, pp. 1–128, Nov. 2022.
[23] V. Scarani, S. Ibllisdir, N. Gisin, and A. Acín, “Quantum cloning,” Rev. Mod. Phys., vol. 77, pp. 1225–1256, Nov. 2005.
[24] K. A. Pati and S. L. Braunstein, “Impossible of deleting an unknown quantum state,” Nature, vol. 404, no. 6774, pp. 164–165, 2000.