Comparison study of image segmentation techniques by
curvature-driven flow of graphs

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Abstract
In this paper, we deal with image segmentation by a curvature-driven flow of graphs. We focus on images, where the segmented objects can be represented by 1D graphs of functions. Compared to the usual direct approach, such a description benefits from certain advantages. Three methods to image segmentation are discussed and applied in terms of the graph formulation. Then, all methods are compared in the qualitative computational study.

Keywords image segmentation, curvature flow of graphs, gradient flow

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1. Introduction
In the field of digital image processing, retrieving particular shapes (segments) in images and their extraction in a form suitable for further analysis is an attractive task, and many different approaches have been developed (e.g., graph cut methods or statistical methods \cite{1,2}). In this paper, we discuss partial differential equation-based methods for image segmentation. The segmented object is represented by a planar curve given as a stationary solution of geometrical evolution equation which schematically reads as

\[
\text{normal velocity} = \text{curvature} + \text{force}.
\] (1)

From the analytical point of view, problem (1) can be treated either by indirect methods for implicit interface tracking - i.e., the level set method or the phase-field method, where the curve is defined as a particular level set of a higher-dimensional field. Such approaches usually lead to a computationally-demanding task with a large number of degrees of freedom. Moreover, indirect methods are exclusively designed for closed curve dynamics. The significant advantage of methods based on the level set approach or phase-field approach is the ability to capturing the complex topological changes of the curve, such as several curves being merged into a single one, or splitting of one curve to several ones. For the application of these methods within the context of image segmentation, we refer the reader to, e.g. \cite{3-5}.

On the other hand a direct approach to the problem (1) provides a simple and straightforward framework for capturing the curve dynamics directly since a moving polygon represents the curve. The most usual direct approach is the parametrization of the curve and reformulation of the motion law (1) in terms of the system of parabolic degenerate partial differential equations for the curve parametrization \cite{6,7}. Such an approach is fast and straightforward and also suitable for open curve dynamics \cite{8}. The usual drawback of this approach is that it cannot capture the topological changes (merging or splitting) on its own \cite{9}. Also, regularization of the motion law and redistribution of the discretization points along the curve in the tangential direction may be necessary for many applications in image processing or material science \cite{10}.

In this paper, we deal with the special case of segmentation of a class of objects whose edges can be represented as graphs of one-dimensional function. Our approach is based on the graph formulation of the problem (1). For details on this approach, see \cite{11}. In this paper, we employ the direct graph formulation resulting in one parabolic PDE. The advantages of our approach are its simplicity and also no additional requirements for regularization or tangential redistribution. For segmentation, we propose the reformulation of three different methods for the case of curvature-driven flow of 1D graphs - minimal radius approach \cite{12}, gradient flow method \cite{13}, and stepwise method \cite{14}. Our goal is to apply these methods to the segmentation of a single object in a grayscale image. For simplicity, we suppose the object has the following properties: a graph of a 1D function can represent a single object in a grayscale image and its contours. Under such assumptions, we can easily determine the segmentation curve (graph of a function) by a simple numerical approximation scheme of the respective motion laws without employing additional regularization, tangential redistribution of discretization points, or external algorithms for topological changes. Then, a comparison of these methods is demonstrated on testing images.
2. Method for image segmentation

This section summarizes three model equations for the evolution of the image segmentation curve represented by a function graph and originating in the general geometric evolution equation (1).

Let us consider an open planar curve \( \Gamma \) with fixed endpoints. The curve \( \Gamma \) is represented by a graph of a function \( f : [a,b] \times [0,T_{\text{max}}] \rightarrow \mathbb{R} \), i.e., \( \Gamma = \text{graph}(f) = \{(x,f(x,t)) : x \in [a,b], t \in [0,T_{\text{max}}]\} \). Then, basic geometric quantities of our interest are the subject of straightforward calculations. The unit tangential and normal vectors to \( \Gamma \) and the curvature \( \kappa \) of \( \Gamma \) are given as the following

\[
\mathbf{t} = \frac{(1,f_x(x,t))}{\sqrt{1 + f_x^2(x,t)}}, \quad \mathbf{n} = \mathbf{t}^\perp, \quad \kappa = \frac{-f_{xx}(x,t)}{\left(1 + f_x^2(x,t)\right)^{3/2}}. \tag{2}
\]

Here, \( f_x \) and \( f_{xx} \) denote the first and second derivatives of \( f \) with respect to \( x \), respectively. The definition of the unit normal vector \( \mathbf{n} \) and the orientation of the curve \( \Gamma \) corresponds to the convention \( \det(\mathbf{t}, \mathbf{n}) = 1 \). Then, the normal velocity \( v \) to the curve \( \Gamma \) is given as

\[
v = \frac{f_t(x,t)}{\sqrt{1 + f_x^2(x,t)}}. \tag{3}
\]

In the segmentation study presented in this paper, we consider grayscale images represented by the image intensity function \( I = I(x), I(x) \in \{0,1,\ldots, 255\} \). Here, the variable \( x \) describes the coordinates of particular pixels. The range of \( I \) describes the color intensity of each pixel. Thus \( I \) is a piecewise constant function. Values \( I = 0 \) and \( I = 255 \) correspond to the pure black and white color, respectively. The other values express the shades of a gray color. We suppose that the image's background corresponds to the black color (\( I = 0 \)), while the segmented object is more related to the white color (bigger values of \( I \)).

In what follows, we briefly introduce three different methods for image segmentation originating in flow (1) adapted for the curvature flow of graphs.

Minimal radius method. This approach to image segmentation was discussed by Pauš et al. in [12]. The geometric evolution equation for the segmentation curve is proposed as the following

\[
v = -\kappa + F. \tag{3}
\]

Considering the graph description of the segmentation curve \( \Gamma \), flow (3) reads

\[
f_t = \frac{f_{xx}}{1 + f_x^2} + \sqrt{1 + f_x^2} F. \tag{4}
\]

The external forcing term reads as \( F = F_{\text{min}} + (F_{\text{max}} - F_{\text{min}})I/255 \), where two free parameters \( F_{\text{max}} > 0 \) and \( F_{\text{min}} < 0 \) correspond to the black color (background) to the white color (segmented object), respectively. It is known from the dynamics of the closed curves that factor \( 1/|F| \) corresponds to the minimal radius, which a shrinking circle can attain. Thus, small values of \( F \) are responsible for the big radius. It is difficult for the segmentation curve to penetrate through narrow gaps or capture sharp edges in such a choice.

Consequently, the bigger values of \( F \) are set, the finer and more precise shapes can be captured. The natural difficulty of this model is, how to set these two parameters \( F_{\text{min}} \) and \( F_{\text{max}} \) properly. In [12], several computational experiments with different choices of model parameters (including tangential redistribution) were presented, providing a basic understanding of the proper choice of \( F_{\text{min}} \) and \( F_{\text{max}} \). The experience from computational experiments shows that this method can be successfully applied in high contrast images.

Gradient flow method for the evolution of closed curves (including various applications) is extensively discussed in, e.g., [13] and references therein. Let \( \gamma(x) > 0 \) be a sufficiently smooth and inhomogeneous energy density along the curve \( \Gamma \). Then the energy of the curve \( \Gamma \) is \( E(\Gamma) = \int_{\Gamma} \gamma ds \), and its gradient flow is given by the geometric evolution equation

\[
v = -\gamma(x) \kappa - \nabla \gamma(x) \cdot \mathbf{n}. \tag{5}
\]

The idea of image segmentation by gradient flow of curves is discussed in, e.g., [13]. Given an image intensity function \( I(x) \in [0,1] \), we define the energy density function as \( \gamma(x) = f(|\nabla I(x)|) \), where the auxiliary function \( f \) serves as the smooth edge detector and \( I = I/255 \) is the image intensity function scaled between 0 and 1. Since the edges of the image correspond to parts where the image intensity gradient \( |\nabla I(x)| \) is large, the value of the edge detector \( f(|\nabla I(x)|) \) is expected to be low. Such expectation can be realized by choice of \( f \) as, e.g., \( f(s) = 1/(1 + s^2) \). Consequently, since the flow (5) minimizes the energy \( E(\Gamma) \), the curve \( \Gamma \) in the image moves towards the edges, where \( |\nabla I(x)| \) is large and attains the stationary shape.

Compared to the minimal radius method, which behaves well in high contrast images, computational experiments show that the Gradient flow method often requires a pre-processing of the image, usually in the form of initial smoothing by a Gaussian filter.

The Gradient flow method belongs to the family of so-called active contour models, which in some cases exhibit undesirable behavior during the curve evolution, as it has been reported in, e.g., [7, 15]. The typical problem in the gradient flow method is that the segmentation curve cannot penetrate concavities, as it is depicted in Fig. 1. This unintended behavior is caused by the very small image intensity in the background (i.e., very small driving force) and the distance of the initial condition from the object's edge. As the graph is evolved according to flow (5), it tends to straighten and eventually comes to a state with almost zero curvature. If this happens far from the edges, there is no way to catch the desired edge. If the segmentation curve \( \Gamma \) is a graph of a function, we propose a simple solution based on considering additional local force term \( p(\kappa_1, I(x), |\nabla I(x)|) \) acting on particular points of the curve \( \Gamma \) provided the defined conditions are fulfilled. If the conditions for \( p \) are well formulated for the image, it pushes the curve through critical regions, such as large regions with very small image intensity gradient or noisy parts of the image.

In the graph description of segmentation curve \( \Gamma \), the
of the flame front, i.e., located in the burning region. In the graph modification, this means the point \( \mathbf{x} \) is enclosed by the flame front, which corresponds to the position of point \( \mathbf{x} \) on the curve \( \Gamma \) lies below the edge of the segmented region. If \( I(\mathbf{x}) > I^* \), then the point \( \mathbf{x} \) is outside the burning region, which corresponds to the position of point \( \mathbf{x} \) of \( \Gamma \) above the segmented edge. The corresponding graph evolution equation for flow (5) with the pushing force is

\[
f_t = \gamma(\mathbf{x}) \left( \frac{f_{\mathbf{x}}}{1 + f_x^2} - \sqrt{1 + f_x^2} (\nabla I(\mathbf{x}) \cdot \mathbf{n}) + p(\kappa, \mathbf{I}, |\nabla \mathbf{I}|) \right).
\]

(6)

**Stepwise method** was originally proposed by Uegata et al. in [14] and originated in the minimal radius method. The original purpose of this method was to segment the flame/smoldering fronts in images from combustion experiments, where the minimal radius method was reported to produce unsatisfactory results [14], mainly because of the contrast and smoothness of the flame/smoldering interface. In this method, the normal velocity for the segmentation curve is given as

\[
v = \delta(\mathbf{x}) \kappa + G(\mathbf{x}),
\]

\[
\delta(\mathbf{x}) = (2J(\mathbf{x}) - 1)^2, \quad J(\mathbf{x}) = \min \left\{ \frac{I(\mathbf{x})}{2I^*} \right\},
\]

(7)

\[
G(\mathbf{x}) = (2J(\mathbf{x}) - 1)(G_0(2J(\mathbf{x}) - 1) + 1).
\]

In this model, \( G_0 \) is a positive parameter, and the quantity \( I^* \) serves as a threshold. If \( I(\mathbf{x}) > I^* \), then the point \( \mathbf{x} \) is enclosed by the flame front, i.e., located in the burning region. In the graph modification, this means the point \( \mathbf{x} \) on the curve \( \Gamma \) lies below the edge of the segmented image. If \( I(\mathbf{x}) < I^* \), then \( \mathbf{x} \) is outside the burning region, which corresponds to the position of point \( \mathbf{x} \) of \( \Gamma \) above the segmented edge. The corresponding graph formulation for the stepwise method reads as

\[
f_t = \delta(\mathbf{x}) \left( \frac{f_{\mathbf{x}}}{1 + f_x^2} + \sqrt{1 + f_x^2} G(\mathbf{x}) \right).
\]

(8)

3. **Computational examples**

We present the comparative study of all three methods discussed above. In our numerical computations, equations (4), (6), and (8) were discretized by second-order finite differences in space, and the time integration was done by the explicit Euler method. Such choice is based on our approach to the solution, where the segmentation curve is represented by a graph. This significantly simplifies resulting evolution equations. When more sophisticated approach is used (as, e.g., in [10], where tangential effects are taken into account), we recommend to consider higher order Runge-Kutta method or semi-implicit method. In every case, the segmentation curve \( \Gamma \) was discretized by \( N = 400 \) points, and the time increment for the explicit Euler method was set as \( \Delta t = 1/(4N^2) \).

As the first example, we verified all three methods on a testing image with a zig-zag shape white region - see Fig. 3. The source photo is a courtesy of Unsplash database (https://unsplash.com) and was used in agreement with the Unsplash license. Compared to the previous testing image, the background is not uniformly black, and also a noise is present, which makes the segmentation procedure more complicated. In the **Minimal radius method**, choices \( F_{\text{min}} = -150 \) and \( F_{\text{max}} = 100 \) provided acceptable results. The **Stepwise method** with the choice of parameters \( I^* = 100, G_0 = 1 \) was able to segment the mountain profile; however, this method could not correctly capture fine details on the top of the mountain. The **Gradient flow method** produced very good results with most details, but the strong artificial image pushing and Gaussian smoothing before computation were necessary to overcome the background’s image noise and intensity gradation. The artificial pushing term was set as \( p = -80 \) in the case when \( |\nabla I(\mathbf{x})| \leq 10 \).

4. **Conclusions**

We presented the direct formulation of the curvature-driven flow of graphs in the 1D case and applied this approach to a special case of image segmentation, where the contours of an image can be represented as 1D graphs of functions. We discussed three methods available in the literature for closed curves for the segmentation techniques and reformulated them to the graph description. Compared to the direct approach, the significant advantage of our formulation is that it does not require any regularization of governing equations or additional sta-
bilization by means of tangential redistribution of discretization points, as it is usual in the parametric description of curves. We compared all three approaches on two examples. In the first example of an artificial image with smooth intensity gradation, all three methods could segment the image precisely provided the model parameters are chosen correctly. In the second example, a real image of Mount Fuji was used. Our computations suggest that the performance of all three methods strongly depends on the choice of model parameters. We recommend the Gradient flow method with the correctly chosen artificial pushing. This choice can consistently produce the best qualitative results. However, the optimal choice of model parameters in the artificial pushing term remains an open problem. If pushing is too weak, the segmentation curve might not penetrate the gaps. If pushing is too strong, unnecessary oscillations can arise. As these problems do not require a long computational time, estimation of suitable parameters can be based on experience and few consequent computations. As another way to approach the optimal set of parameters in our segmentation problem, we can consider to employ the statistical analysis and machine learning-based estimation. This will be the subject of future research.

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