An immersed $CR-P_0$ element for Stokes interface problems
and the optimal convergence analysis

Haifeng Ji∗ Feng Wang† Jinru Chen‡ Zhilin Li§

Abstract

This paper presents and analyzes an immersed finite element (IFE) method for solving
Stokes interface problems with a piecewise constant viscosity coefficient that has a jump across
the interface. In the method, the triangulation does not need to fit the interface and the IFE
spaces are constructed from the traditional $CR-P_0$ element with modifications near the interface
according to the interface jump conditions. We prove that the IFE basis functions are unisolvent
on arbitrary interface elements and the IFE spaces have the optimal approximation capabilities,
although the proof is challenging due to the coupling of the velocity and the pressure. The
stability and the optimal error estimates of the proposed IFE method are also derived rigorously.
The constants in the error estimates are shown to be independent of the interface location relative
to the triangulation. Numerical examples are provided to verify the theoretical results.

keyword: Stokes equations, interface, immersed finite element, unfitted mesh, two-phase flow,
error estimates

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1 Introduction

In this paper we are interested in designing and analyzing immersed finite element (IFE) methods
for solving Stokes interface problems, also known as two-phase Stokes problems. Let $\Omega \subset \mathbb{R}^2$ be a
bounded domain with a convex polygonal boundary $\partial \Omega$, and $\Gamma$ be a $C^2$-smooth interface immersed
in $\Omega$. Without loss of generality, we assume that $\Gamma$ divides $\Omega$ into two phases $\Omega^+$ and $\Omega^-$ such that
$\Gamma = \partial \Omega^-$; see Figure 1 for an illustration. The Stokes interface problem reads: given a body force
$f \in L^2(\Omega)^2$ and a piecewise constant viscosity $\mu|_{\Omega^\pm} = \mu^\pm > 0$, find a velocity $u$ and a pressure $p$
such that

$$
-\nabla \cdot (2\mu \varepsilon(u)) + \nabla p = f \quad \text{in } \Omega^+ \cup \Omega^-,
$$

$$
\nabla \cdot u = 0 \quad \text{in } \Omega,
$$

$$
[\sigma(\mu, u, p) n]_\Gamma = 0 \quad \text{on } \Gamma,
$$

$$
[u]_\Gamma = 0 \quad \text{on } \Gamma,
$$

$$
u = 0 \quad \text{on } \partial \Omega,
$$

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*School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China (hfji@njupt.edu.cn)
†Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China (fwang@njnu.edu.cn)
‡Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China (jrchen@njnu.edu.cn)
§Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA (zhilin@math.ncsu.edu)
where \( \epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \) is the strain tensor, \( \sigma(\mu, u, p) = 2\mu\epsilon(u) - pI \) is the Cauchy stress tensor, \( I \) is the identity matrix, \( n \) is the unit normal vector of the interface \( \Gamma \) pointing toward \( \Omega^+ \), and \( [v]_\Gamma \) stands for the jump of a vector function \( v \) on the interface, i.e., \( [v]_\Gamma := v^+|_\Gamma - v^-|_\Gamma \) with \( v^\pm := v|_{\Omega^\pm} \). For simplicity, the notations of the jump \( [\cdot]_\Gamma \) and the superscripts \( +, - \) are also used for scalar or matrix-valued functions. If the restriction \( (\nabla \cdot u)|_\Gamma \) makes sense, the equation (1.2) provides an additional interface jump condition

\[
[\nabla \cdot u]_\Gamma = 0 \quad \text{on } \Gamma.
\]

Figure 1: Left diagram: geometries of an interface problem; Right diagram: an unfitted mesh.

The study of the Stokes equations is motivated to solve two-phase incompressible flows, often modeled by the Navier-Stokes equations with a discontinuous density and viscosity across a sharp interface. The Stokes interface problem is a reasonable approximation if the inertia term is negligible. For interface problems, numerical methods using unfitted meshes have attracted a lot of attention because of the relative ease of handling moving interfaces or complex interfaces. Unfitted meshes are generated independent of the interface, and can have elements cut by the interface (called interface elements), which makes it challenging to design numerical methods with optimal convergence rates due to the discontinuities in the pressure and the derivatives of velocity across the interface.

In the finite element framework, generally there are two kinds of unfitted mesh methods. One type is the XFEM [12] and the cutFEM [8], also known as the Nitsche-XFEM [28] where the finite element space is defined on each individual subdomain separated by the interface and the jump conditions are enforced weakly using a variant of Nitsche’s method. The basic idea of this kind of methods is to enrich the traditional finite element space by extra degrees of freedom on interface elements to capture the discontinuities. For the Stokes interface problems, this type of methods have been developed and analyzed in [11, 10, 37, 27, 17, 9, 36, 21]. The other type of unfitted mesh methods is the immersed finite element (IFE) method [30, 33], which modifies the traditional finite element on interface elements according to interface conditions to achieve the optimal approximation capability, while keeping the degrees of freedom unchanged. For second-order elliptic interface problems, IFE methods have been studied extensively in [32, 20, 34, 15, 25]. However, for the Stokes interface problems, there are much fewer works on IFE method in the literature. One difficulty is that the jumps of velocity and pressure are coupled together and it is difficult to modify the velocity and the pressure finite element spaces separately.

Although the idea of IFE methods was proposed in 1998 [30], the first IFE method for Stokes interface problem was developed in 2015 by Adjerid, Chaabane, and Lin in [2], in which the coupling of the velocity and pressure was taken into account in constructing the IFE spaces and an immersed \( Q_1-Q_0 \) discontinuous Galerkin method was proposed. The method then was applied to the Stokes interface problems with moving interfaces in [3], and the idea was further developed with immersed \( CR-P_1 \) and rotated \( Q_1-Q_0 \) elements in [26]. We also note that recently, a Taylor-Hood IFE was
constructed by a least-squares approximation in [11]. However, to the best of our knowledge, there is no theoretical analysis even for the optimal approximation capabilities of those IFE spaces, not mentioning the stability and the convergence of those methods. One of the major obstacles hindering the analysis is that the velocity and the pressure are also coupled in IFE spaces.

In this paper we develop and analyze an IFE method based on the immersed \textit{CR}-\textit{P}_0 elements proposed in [26] for solving Stokes interface problems. Different from [26], we propose a new bilinear form by including additional integral terms defined on the edges cut by the interface (called interface edges) to ensure the inf-sup stability and the optimal convergence. We show that these terms are important to prove the optimal convergence of the IFE method. In some sense, one need these terms to get an optimal error estimate on edges, otherwise the order of convergence is suboptimal; see the counter example in [24] for the second-order elliptic interface problems.

Besides the different scheme considered in this paper (compared with [26]) we mention the following other three new contributions of this paper. The first one is about the unisolvency (i.e., the existence and uniqueness) of the IFE basis functions. We prove the unisolvency on arbitrary triangles via a new augmented approach inspired by [31]. Note that in [26] the unisolvency is only shown on isosceles right triangles by proving the invertibility of the corresponding $14 \times 14$ coefficient matrices. It seems that the proof is tedious and cannot be extended to arbitrary triangles. Furthermore, we also provide an explicit formula for the IFE basis functions, which is convenient in the implementation. The second contribution is that we prove the optimal approximation capabilities of the IFE spaces on shape regular triangulations, although it is challenging due to the coupling of the velocity and pressure. The proof is based on some novel auxiliary functions constructed on interface elements and a $\delta$-strip argument developed by Li et al. [29] for estimating errors in the region near the interface. The third contribution is the well-known inf-sup stability result and the finite element error estimates. By establishing a new trace inequality for IFE functions and investigating the relations of the coupled velocity and pressure in IFE spaces, we prove that the coupled velocity and pressure IFE spaces satisfy the inf-sup condition with a constant independent of the meshsize and the interface location relative to the mesh. The optimal error estimates of the proposed IFE method are also derived where the errors resulting from approximating curved interfaces by line segments are taken into consideration rigorously. To the best of our knowledge, this is the first paper which gives a complete theoretical analysis for IFE methods for solving Stokes interface problems.

The rest of this paper is organized follows. In section 2 we introduce some notations and assumptions. The IFE and corresponding IFE method are presented in section 3. Section 4 is devoted to study the properties of the constructed IFE including the unisolvency of the IFE basis functions and the optimal approximation capabilities of the IFE space. In section 5 the stability and the optimal error estimates are proved. Section 6 provides some numerical experiments. Conclusions are drawn in section 7.

### 2 Preliminaries and notations

Throughout the paper we adopt the standard notation $W^k_p(\Lambda)$ for Sobolev spaces on a domain $\Lambda$ with the norm $\| \cdot \|_{W^k_p(\Lambda)}$ and the seminorm $| \cdot |_{W^k_p(\Lambda)}$. Specially, $W^2_2(\Lambda)$ is denoted by $H^2(\Lambda)$ with the norm $\| \cdot \|_{H^2(\Lambda)}$ and the seminorm $| \cdot |_{H^2(\Lambda)}$. As usual $H^1_0(\Lambda) = \{ v \in H^1(\Lambda) : v = 0 \text{ on } \partial\Lambda \}$. Given a domain $\Lambda$ with $\Lambda \cap \Omega^+ \neq \emptyset$ and $\Lambda \cap \Omega^- \neq \emptyset$, we define subregions $\Lambda^\pm := \Lambda \cap \Omega^\pm$ and a broken space

$$H^k(\Lambda^+ \cup \Lambda^-) := \{ v \in L^2(\Lambda) : v|_{\Lambda^s} \in H^k(\Lambda^s), s = +, - \}$$

(2.1)
equipped with the norm \( \| \cdot \|_{H^k(\Lambda + \cup \Lambda^{-})} \) and the semi-norm \( | \cdot |_{H^k(\Lambda + \cup \Lambda^{-})} \) satisfying
\[
\| v \|_{H^k(\Lambda + \cup \Lambda^{-})} = \| v \|_{H^k(\Lambda^+)} + \| v \|_{H^k(\Lambda^-)}, \quad | v |_{H^k(\Lambda + \cup \Lambda^{-})} = | v |_{H^k(\Lambda^+)} + | v |_{H^k(\Lambda^-)}.
\]

With the usual spaces \( \mathbb{V} := H^0_0(\Omega)^2 \) and \( M := \{ q \in L^2(\Omega) : \int_{\Omega} q = 0 \} \), the weak form of the Stokes interface problem (1.1)-(1.5) reads: find \((u, p)\) in \((\mathbb{V}, M)\) such that
\[
a(u, v) + b(v, p) = \int_{\Omega} f \cdot v \quad \forall v \in \mathbb{V},
\]
\[
b(u, q) = 0 \quad \forall q \in M,
\]
where
\[
a(u, v) := \int_{\Omega} 2\mu \varepsilon(u) : \varepsilon(v), \quad b(v, q) := -\int_{\Omega} q \nabla \cdot v.
\]

It is well-known that the problem \((2.2)\) is well-posed, that is, there exists a unique solution \((u, p)\) in \((\mathbb{V}, M)\) to the weak form \((2.2)\). For the convergence analysis we assume that the solution has a higher regularity in each sub-domain, i.e., \((u, p) \in \mathcal{H}_2 \mathcal{H}_1 \cap (\mathbb{V}, M)\), where
\[
\mathcal{H}_2 \mathcal{H}_1 := \{(v, q) : v \in H^2(\Omega^+ \cup \Omega^-)^2, q \in H^1(\Omega^+ \cup \Omega^-), [\sigma(\mu, v, q)n]_\Gamma = 0, [v]_\Gamma = 0, [\nabla \cdot v]_\Gamma = 0\}.
\]

In order to solve the problem \((2.2)\), we consider a family of triangulations \(\{T_h\}_{h>0}\) of \(\Omega\), generated independently of the interface \(\Gamma\). For each element \(T \in T_h\), let \(h_T\) denote its diameter, and define the meshsize of the triangulation \(T_h\) by \(h = \max_{T \in T_h} h_T\). We assume that \(T_h\) is shape regular, i.e., for every \(T\), there exists \(\varrho > 0\) such that \(h_T \leq \varrho r_T\) where \(r_T\) is the diameter of the largest circle inscribed in \(T\). Denote \(\mathcal{E}^h_\text{i}\) and \(\mathcal{E}^h_\text{b}\) as the sets of interior and boundary edges, respectively. The set of all edges of the triangulation then is \(\mathcal{E}^h = \mathcal{E}^h_\text{i} \cup \mathcal{E}^h_\text{b}\). Since the interface \(\Gamma\) is \(C^2\)-smooth, we can always refine the mesh near the interface to satisfy the following assumption.

**Assumption 2.1.** The interface \(\Gamma\) does not intersect the boundary of any element \(T \in T_h\) at more than two points. The interface \(\Gamma\) does not intersect the closure \(\overline{\mathcal{T}}\) for any \(e \in \mathcal{E}^h\) at more than one point.

We adopt the convention that the elements \(T \in T_h\) and edges \(e \in \mathcal{E}^h\) are open sets. The sets of interface elements and interface edges are then defined by
\[
T^\Gamma_h := \{ T \in \mathcal{T}_h : T \cap \Gamma \neq \emptyset \} \quad \text{and} \quad \mathcal{E}^\Gamma_h := \{ e \in \mathcal{E}^h : e \cap \Gamma \neq \emptyset \}.
\]

The sets of non-interface elements and non-interface edges are \(T^\text{non}_h = T_h \setminus T^\Gamma_h\) and \(\mathcal{E}^\text{non}_h = \mathcal{E}_h \setminus \mathcal{E}^\Gamma_h\).

On an edge \(e = \text{int}(\partial T_1 \cap \partial T_2)\) with \(T_1, T_2 \in T_h\), let \(n_e\) be the unit normal vector of \(e\) pointing toward \(T_2\). For a piecewise smooth function \(v\), we define the jump across the edge \(e\) by \([v]_e := v|_{T_1} - v|_{T_2}\) and the average by \(\{v\}_e := \frac{1}{2}(v|_{T_1} + v|_{T_2})\). If \(e \in \mathcal{E}^h_\text{i}\), then \(n_e\) is the unit normal vector of \(e\) pointing toward the outside of \(\Omega\), and define \([v]_e := v\) and \(\{v\}_e := v\). On a region \(\Lambda\), for any \(v^+ \in L^1(\Lambda)\) and \(v^- \in L^1(\Lambda)\), we also need the following notation
\[
[v^\pm](x) := v^+(x) - v^-(x) \quad \forall x \in \Lambda.
\]
For vector or matrix-valued functions, the notations \([.]_e\), \(\{\cdot\}_e\) and \(\mathcal{[\cdot]}\) are defined analogously. Note that the difference between \([\mathcal{[\cdot]}](x)\) and \([\mathcal{[\cdot]}](x)\) is the range of \(x\).

We approximate the interface \(\Gamma\) by \(\Gamma_h\), which is composed of all the line segments connecting the intersection points of the triangulation and the interface. The approximated interface \(\Gamma_h\) divides \(\Omega\)
into two disjoint sub-domains $\Omega_h^+ \text{ and } \Omega_h^-$ such that $\Gamma_h = \partial \Omega_h^-$. On each interface element $T \in T_h^x$, the discrete interface $\Gamma_h$ divides $T$ into two sub-elements:

$$T_h^+ := T \cap \Omega_h^+ \quad \text{and} \quad T_h^- := T \cap \Omega_h^-.$$ 

For simplicity of notation, we denote

$$\Gamma_T := \Gamma \cap T \quad \text{and} \quad \Gamma_{h,T} := \Gamma_h \cap T.$$ 

Let $\mathbf{n}_h(x)$ be the unit normal vector of $\Gamma_h$ pointing toward $\Omega_h^+$; see Figure 2 in the proof of Lemma 4.6 for an illustration. The unit tangent vectors of $\Gamma_h$ and $\Gamma$ are obtained by a $90^\circ$ clockwise rotation of $\mathbf{n}_h$ and $\mathbf{n}$, i.e., $\mathbf{t}_h(x) = R_{-\pi/2}\mathbf{n}_h(x)$ and $\mathbf{t}(x) = R_{-\pi/2}\mathbf{n}(x)$ with a rotation matrix

$$R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$ 

At the end of this section, we recall the notation $v^\pm := v|_{\Omega^\pm}$ for a function $v$ defined on the whole domain $\Omega$. Again the notation of the superscripts $s = +$ and $-$ may be different in the continuous and discrete cases due to some mismatched regions from the line segment approximation. We also use $q^\pm$ to represent $q|_{T_h^\pm}$ if no confusion can arise. Furthermore, if the function $q^s$, $s = +$ or $-$, is a polynomial, then the polynomial $q^s$ is viewed as defined on the whole element $T$, unless otherwise specified. The superscripts are used for vector or matrix-valued functions similarly.

### 3 The immersed CR-$P_0$ finite element method

#### 3.1 The IFE space

Let $P_h(T)$ be the set of all polynomials of degree less than or equal to $k$ on each $T \in T_h$. On a non-interface element $T \in T_h^{\text{non}}$, we use the standard CR-$P_0$ shape function spaces, i.e.,

$$\left( V_h(T), M_h(T) \right) = \left( P_1(T)^2, P_0(T) \right).$$

For every $T \in T_h$, the local degrees of freedom are chosen as

$$N_{i,T}(v,q) := \frac{1}{|e_i|} \int_{e_i} v_1, \quad N_{3+i,T}(v,q) := \frac{1}{|e_i|} \int_{e_i} v_2, \quad i = 1, 2, 3, \quad N_T(v,q) := \frac{1}{|T|} \int_T q,$$  \hspace{1cm} (3.2)

where $e_i \in \mathcal{E}_h$, $i = 1, 2, 3$ are edges of $T$, and $v_1$ and $v_2$ are two components of $v$, i.e., $v = (v_1,v_2)^T$.

On an interface element $T \in T_h^x$, the shape function spaces $(V_h(T), M_h(T))$ do not have the optimal approximation capabilities due to the interface jumps $\{1.3\}, \{1.4\}$ and $\{1.6\}$. The shape function spaces need to be modified according to these interface jump conditions. Given $v^\pm \in P_1(T)^2$ and $q^\pm \in P_0(T)$, we define the following discrete interface jump conditions

$$[\sigma(\mu^\pm, v^\pm, q^\pm)\mathbf{n}_h] = \mathbf{0},$$ \hspace{1cm} (3.3)

$$[v^\pm]|_{\Gamma_{h,T}} = \mathbf{0} \quad \text{(or, equivalently, } [v^\pm](\mathbf{x}_T) = \mathbf{0}, \quad [\nabla v^\pm \mathbf{t}_h] = \mathbf{0}),$$ \hspace{1cm} (3.4)

$$[\nabla \cdot v^\pm] = 0,$$ \hspace{1cm} (3.5)

where $\mathbf{x}_T$ is a point on $\Gamma_{h,T} \cap \Gamma_T$. The immersed CR-$P_0$ shape function space is then defined by

$$\begin{align*}
\mathbf{V}M_h^{IFE}(T) := \lbrace (v,q) : v|_{T_h^\pm} = v^\pm|_{T_h^\pm}, \quad v^\pm \in P_1(T)^2, \quad q|_{T_h^\pm} = q^\pm|_{T_h^\pm}, \quad q^\pm \in P_0(T), \\
(v^\pm, q^\pm) \text{ satisfying conditions } (3.3)-(3.5) \rbrace. 
\end{align*}$$  \hspace{1cm} (3.6)
Remark 3.1. Note that \( v^\pm \) and \( q^\pm \) have fourteen parameters. It is easy to check that (3.3) provides two constraints, (3.4) provides four constraints, and (3.5) provides one constraint. Intuitively, we can expect that the functions \( v^\pm \) and \( q^\pm \) satisfying conditions (3.3)-(3.5) are uniquely determined by the degrees of freedom \( N_{i,T} \), \( i = 1, \ldots, 7 \) defined in (3.3). We will prove that the unisolvence holds on arbitrary triangles in subsection 4.1.

Remark 3.2. The velocity and pressure belonging to \( VM_h^{IFE}(T) \) are coupled due to the discrete interface jump condition (3.3). In other words, if \((v_1, q_1)\) and \((v_2, q_2)\) belong to \( VM_h^{IFE}(T) \), then \((v_1, q_1 + q_2)\) and \((v_1 + v_2, q_1)\) may not belong to \( VM_h^{IFE}(T) \), instead we only have \((v_1 + v_2, q_1 + q_2) \in VM_h^{IFE}(T) \).

The global IFE space is defined by

\[
VM_h^{IFE} = \left\{ (v, q) : v|_T \in V_h(T), \ q|_T \in M_h(T) \ \forall T \in \mathcal{T}_h^{non}, \right. \\
\left. (v|_T, q|_T) \in VM_h^{IFE}(T) \ \forall T \in \mathcal{T}_h^T, \ \int_e [v]_e = 0 \ \forall e \in \mathcal{E}_h^b \right\},
\]

in which the velocity and pressure are coupled. We also define a subspace of \( VM_h^{IFE} \) to take into account the boundary condition of velocity and the constraint of pressure

\[
VM_{h,0}^{IFE} = \left\{ (v, q) : (v, q) \in VM_h^{IFE}, \ \int_e v = 0 \ \forall e \in \mathcal{E}_h^b, \ \int_\Omega q = 0 \right\}.
\]

3.2 The IFE method

To make the method easy for implementation, we define an approximate coefficient \( \mu(x) \) by

\[
\mu_h(x) = \begin{cases} \mu^+ & \text{in } \Omega_h^+ \\ \mu^- & \text{in } \Omega_h^-. \end{cases}
\]

In other words, the viscosity is adjusted in the mismatched small area due to the line segment approximation. The immersed \( CR-P_0 \) finite element method for the Stokes interface problem (1.1)-(1.3) reads: find \((u_h, p_h) \in VM_h^{IFE} \) such that

\[
A_h(u_h, p_h; v_h, q_h) = \int_\Omega f \cdot v_h \quad \forall (v_h, q_h) \in VM_{h,0}^{IFE},
\]

where the bilinear form is defined as, for all \((u_h, p_h)\) and \((v_h, q_h)\) in \( VM_h^{IFE} \),

\[
A_h(u_h, p_h; v_h, q_h) := a_h(u_h, v_h) + b_h(v_h, p_h) - b_h(u_h, q_h) + J_h(p_h, q_h),
\]

\[
a_h(u_h, v_h) := \sum_{T \in \mathcal{T}_h} \int_T 2 \mu_h \epsilon(u_h) : \epsilon(v_h) + \sum_{e \in \mathcal{E}_h^T} \frac{1}{|e|} \int_e [u_h]_e \cdot [v_h]_e,
\]

\[
- \sum_{e \in \mathcal{E}_h^T} \int_e \left\{ 2 \mu_h \epsilon(u_h) n_e \cdot [v_h]_e + \delta \{ 2 \mu_h \epsilon(v_h) n_e \cdot [u_h]_e \} + \sum_{e \in \mathcal{E}_h^T} \eta \int_e [n_e]_e \cdot [v_h]_e, \right. \\
\left. - \sum_{T \in \mathcal{T}_h} \int_T q_h \nabla \cdot v_h + \sum_{e \in \mathcal{E}_h^T} \int_e \{ q_h \} \cdot [v_h]_e, 
\]

\[
J_h(p_h, q_h) := \sum_{e \in \mathcal{E}_h^T} |e| \int_e [p_h]_e [q_h]_e,
\]

where \( \delta = \pm 1 \) and \( \eta \geq 0 \). When the parameter \( \delta = 1 \), the bilinear form \( a_h(\cdot, \cdot) \) is symmetric and the penalty \( \eta \) should be larger enough to ensure the coercivity. When \( \delta = -1 \), the bilinear form
$a_h(\cdot, \cdot)$ is non-symmetric. In general, we can choose an arbitrary $\eta \geq 0$ to ensure the coercivity; see Lemma 5.3 in section 5.

We briefly discuss the roles of different terms in the method. The second term of $a_h(\cdot, \cdot)$ is added to control the rigid body rotations so that the Korn inequality holds for Crouzeix-Raviart finite element spaces. The integral $\int_e \{2\mu h H(u) n_e\} e \cdot [v_h] e$ in the third term of $a_h(\cdot, \cdot)$ appears to make the method consistent on interface edges; and correspondingly the integral $\int_e \{2\mu h H(v) n_e\} e \cdot [u_h] e$ and the fourth term are added to make the bilinear form $a_h(\cdot, \cdot)$ coercive. We emphasize that, different from the traditional CR-P2 finite element method, these integral terms on interface edges cannot be neglected and are important to ensure the optimal convergence of the IFE method. The reason is similar to that of the nonconforming IFE methods for second-order elliptic interface problems [24]. The second term in $b_h(\cdot, \cdot)$ is needed also for the consistency on interface edges and the penalty term $J_h(\cdot, \cdot)$ controlling the jumps of the pressure is added to make the inf-sup condition satisfied.

4 Properties of the IFE

In this section, we discuss some properties of the proposed IFE. To begin with, we make some preparations. Denote $\text{dist}(x, \Gamma)$ as the distance between a point $x$ and the interface $\Gamma$, and $U(\Gamma, \delta) = \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma) < \delta\}$ as the neighborhood of $\Gamma$ of thickness $\delta$. Define the meshsize of $T_h^\Gamma$ by

$$h_\Gamma := \max_{T \in T_h^\Gamma} h_T. \quad (4.1)$$

It is obvious that $h_\Gamma \leq h$ and $\bigcup_{T \in T_h^\Gamma} T \subset U(\Gamma, h_\Gamma)$. We also use the signed distance function near the interface

$$\rho(x) = \begin{cases} \text{dist}(x, \Gamma) & \text{if } x \in \Omega^+ \cap U(\Gamma, \delta_0), \\ -\text{dist}(x, \Gamma) & \text{if } x \in \Omega^- \cap U(\Gamma, \delta_0). \end{cases}$$

**Assumption 4.1.** There exists a constant $\delta_0 > 0$ such that the signed distance function $\rho(x)$ is well-defined in $U(\Gamma, \delta_0)$ and $\rho(x) \in C^2(U(\Gamma, \delta_0))$. We also assume that $h_\Gamma < \delta_0$ so that $T \subset U(\Gamma, \delta_0)$ for all interface elements $T \in T_h^\Gamma$.

The assumption is reasonable since the interface $\Gamma$ is $C^2$-smooth. Using the signed distance function $\rho(x)$, we can evaluate the unit normal and tangent vectors of the interface as $n(x) = \nabla \rho$ and $t(x) = R_{-\pi/2} \nabla \rho$, which are well-defined in the region $U(\Gamma, \delta_0)$. We note that the functions $n_h(x)$ and $t_h(x)$ can also be viewed as piecewise constant vectors defined on interface elements. Since $\Gamma$ is $C^2$-smooth, by Rolle’s Theorem, there exists at least one point $x^* \in \Gamma \cap T$ such that $n(x^*) = n_h(x^*)$. Since $\rho(x) \in C^2(U(\Gamma, \delta_0))$, we have $n(x) \in (C^1(\overline{T}))^2$. Using the Taylor expansion at $x^*$, we further have

$$\|n - n_h\|_{L^\infty(T)} \leq Ch_T, \quad \|t - t_h\|_{L^\infty(T)} = \|R_{-\pi/2}(n - n_h)\|_{L^\infty(T)} \leq Ch_T, \quad \forall T \in T_h^\Gamma. \quad (4.2)$$

For any interface element $T \in T_h^\Gamma$, we define the region between the mismatched interfaces $\Gamma$ and $\Gamma_h$ as

$$T^\Delta := (T^- \cap T_h^+) \cup (T^+ \cap T_h^-) \quad \forall T \in T_h^\Gamma. \quad (4.3)$$

Since $\Gamma$ is $C^2$-smooth, there exists a constant $C$ depending only on the curvature of $\Gamma$ such that

$$T^\Delta \subset U(\Gamma, Ch_T^2) \quad \forall T \in T_h^\Gamma. \quad (4.4)$$

The following lemma presents a $\delta$-strip argument that will be used for the error estimate in the region near the interface; see Lemma 2.1 in [29]
Lemma 4.2. Let $\delta > 0$ be a sufficiently small number. Then it holds for any $v \in H^1(\Omega)$ that
\[
\|v\|_{L^2(U(\Gamma, \delta))} \leq C\sqrt{\delta} \|v\|_{H^1(\Omega)}.
\]
Furthermore, if $v|_T = 0$, then there holds
\[
\|v\|_{L^2(U(\Gamma, \delta))} \leq C\delta \|\nabla v\|_{L^2(U(\Gamma, \delta))}.
\]

We also need the following well-known extension result [13].

Lemma 4.3. Suppose that $v^\pm \in H^m(\Omega^\pm)$. Then there exist extensions $v_E^\pm \in H^m(\Omega)$ such that
\[
v_E^\pm|_{\Omega^\pm} = v^\pm \text{ and } |v_E^\pm|_{H^r(\Omega)} \leq C|v^\pm|_{H^r(\Omega^\pm)}, \quad i = 0, \ldots, m, \ m = 1, 2,
\]
for a constant $C > 0$ depending only on $\Omega^\pm$.

For an element $T \in \mathcal{T}_h$ with edges $e_i$, $i = 1, 2, 3$, let $W(T) := \{v \in L^2(T) : \int_{e_i} v, i = 1, 2, 3 \text{ are well defined}\}$. We define local interpolation operators $\pi_{h,T}^{CR}$, $\pi_{h,T}^0$ and $\Pi_{h,T}$ such that, for all $v \in W(T)$ and for all $(v, q) \in (W(T)^2, L^2(T))$,
\[
\begin{align*}
\pi_{h,T}^{CR} v &\in P_1(T), \quad \int_{e_i} \pi_{h,T}^{CR} v = \int_{e_i} v, \quad i = 1, 2, 3, \\
\pi_{h,T}^0 q &\in P_0(T), \quad \int_T \pi_{h,T}^0 q = \int_T q, \quad (4.5) \\
\Pi_{h,T} (v, q) &\in (V_h(T), M_h(T)), \quad N_{i,T} (\Pi_{h,T} (v, q)) = N_{i,T} (v, q), \quad i = 1, \ldots, 7.
\end{align*}
\]
Note that these interpolation operators will be used to interpolate discontinuous functions; see, e.g., (4.21). Let $v = (v_1, v_2)^T$, then we have
\[
\Pi_{h,T} (v, q) = (\pi_{h,T}^{CR} v, \pi_{h,T}^0 q) \quad \text{with} \quad \pi_{h,T}^{CR} v := (\pi_{h,T}^{CR} v_1, \pi_{h,T}^{CR} v_2)^T. \quad (4.6)
\]

For an interface element $T \in \mathcal{T}_h^I$, define a local IFE interpolation operator $\Pi_{h,T}^{IFE} : (W(T)^2, L^2(T)) \rightarrow \text{V}M_h^{IFE}(T)$ such that
\[
N_{i,T} (\Pi_{h,T}^{IFE} (v, q)) = N_{i,T} (v, q), \quad i = 1, \ldots, 7, \quad \forall (v, q) \in (W(T)^2, L^2(T)). \quad (4.7)
\]

Now the global IFE interpolation operator $\Pi_h^{IFE} : (H^1(\Omega)^2, L^2(\Omega)) \rightarrow \text{V}M_h^{IFE}$ is defined by
\[
\forall (v, q) \in (H^1(\Omega)^2, L^2(\Omega)), \quad (\Pi_h^{IFE} (v, q))|_T = \begin{cases} 
\Pi_{h,T}^{IFE} (v, q) & \text{if } T \in \mathcal{T}_h^I, \\
\Pi_{h,T} (v, q) & \text{if } T \in \mathcal{T}_h^{non}.
\end{cases} \quad (4.8)
\]

We use $\Pi_{v,q}^{IFE}$ and $\Pi_{v,q}^{IFE} q$ to represent the velocity and pressure of $\Pi_h^{IFE} (v, q)$, i.e.,
\[
\Pi_h^{IFE} (v, q) = (\Pi_{v,q}^{IFE} v, \Pi_{v,q}^{IFE} q). \quad (4.9)
\]
Note that the subscript of $\Pi_{v,q}^{IFE}$ means that the interpolation operator may depend not only on $v$ but also on $q$ since the velocity and pressure are coupled in the IFE space; see Remark 4.10 for details.

It is well-known that the local interpolation operators $\pi_{h,T}^{CR}$, $\pi_{h,T}^0$ and $\Pi_{h,T}$ are well-defined. We can introduce the standard CR basis functions by
\[
\lambda_{i,T} \in P_1(T), \quad \frac{1}{|e_j|} \int_{e_j} \lambda_{i,T} = \delta_{ij}, \quad i, j = 1, 2, 3, \quad (4.10)
\]
and the standard CR-P₀ finite element basis functions by
\[(\phi_{i,T}, \varphi_{i,T}) \in (V_h, M_h(T)), \quad N_{j,T}(\phi_{i,T}, \varphi_{i,T}) = \delta_{i,j}, \quad \forall i, j = 1, ..., 7,\] (4.11)
where \(\delta_{ij}\) is the Kronecker function. Obviously, we have
\[\phi_{i,T} = (\lambda_{i,T}, 0)^T, \quad \phi_{i+3,T} = (0, \lambda_{i,T})^T, \quad i = 1, 2, 3, \quad \phi_{7,T} = 0,\]
\[\varphi_{i,T} = 0, \quad i = 1, ..., 6, \quad \varphi_{7,T} = 1.\] (4.12)

However, the well-definedness of the IFE interpolation operator \(\Pi_{h,h}^{I FE}\) is not obvious. We need a result that the IFE shape functions in \(V M_{h}^{I FE}(T)\) can be uniquely determined by \(N_{i,T}(v, q), i = 1, ..., 7\), which will be proved in the following subsection.

### 4.1 The unisolvence of IFE shape functions

Note that for many existing IFEs developed for other interface problems, the unisolvence of IFE shape functions with respect to the degrees of freedom relies on the mesh assumption, i.e., the no-obtuse angle condition [13, 25, 26, 28]. Recently, we showed that for second-order elliptic interface problems, if integral-values on edges are used as the degrees of freedom, then the unisolvence holds on arbitrary triangles [24]. In this paper, we are able to prove that the unisolvence also holds on arbitrary elements for the proposed immersed CR-P₀ element for Stokes interface problems as well.

Now we use a new augmented approach inspired by [31] to prove the unisolvence. Without loss of generality, we consider an interface element \(T \in T_h\) for the proof. By the definition (4.3)- (4.6), it is obvious that the space \(V M_{h}^{I FE}(T)\) is not an empty set. Given a pair of IFE functions \((v, q) \in V M_{h}^{I FE}(T)\), we define \((v^{j_0}, q^{j_0})\) such that
\[(v^{j_0}, q^{j_0}) \in (V_h(T), M_h(T)), \quad N_{i,T}(v^{j_0}, q^{j_0}) = N_{i,T}(v, q), i = 1, ..., 7.\] (4.13)

From (4.5)-(4.6), we know \((v^{j_0}, q^{j_0}) = (\pi_{h,T}^{CR} V, \pi_{h,T} q)\). Recalling the notation of superscripts \(\pm\) described at the end of section 2, we set \(v^{j_0, \pm} := (v^{j_0})^{\pm}\) and \(q^{j_0, \pm} := (q^{j_0})^{\pm}\). It is easy to check that
\[\begin{align*}
\sigma(1, v^{j_0, \pm}, q^{j_0, \pm}) n_h &= 0, \quad [v^{j_0, \pm}]_{T_h,T} = 0, \quad [\nabla \cdot v^{j_0, \pm}] = 0.
\end{align*}\] (4.14)

We also define \((v^{j_1}, q^{j_1})\) such that
\[v^{j_1, \pm} := (v^{j_1})^{\pm} \in V_h(T), \quad q^{j_1, \pm} := (q^{j_1})^{\pm} \in M_h(T), \quad N_{i,T}(v^{j_1}, q^{j_1}) = 0, i = 1, ..., 7,\]
\[\begin{align*}
\sigma(1, v^{j_1, \pm}, q^{j_1, \pm}) n_h &= n_h, \quad [v^{j_1, \pm}]_{T_h,T} = 0, \quad [\nabla \cdot v^{j_1, \pm}] = 0.
\end{align*}\] (4.15)

and \((v^{j_2}, q^{j_2})\) such that
\[v^{j_2, \pm} := (v^{j_2})^{\pm} \in V_h(T), \quad q^{j_2, \pm} := (q^{j_2})^{\pm} \in M_h(T), \quad N_{i,T}(v^{j_2}, q^{j_2}) = 0, i = 1, ..., 7,\]
\[\begin{align*}
\sigma(1, v^{j_2, \pm}, q^{j_2, \pm}) n_h &= t_h, \quad [v^{j_2, \pm}]_{T_h,T} = 0, \quad [\nabla \cdot v^{j_2, \pm}] = 0.
\end{align*}\] (4.16)

The existence and uniqueness of \(v^{j_1}\) and \(v^{j_2}\) will be proved in Lemma 4.5. Combining (4.13)-(4.16), we immediately have the following lemma.

**Lemma 4.4.** Given \((v, q) \in V M_{h}^{I FE}(T)\), if we know the augmented variable
\[\begin{align*}
[\sigma(1, v^{\pm}, q^{\pm}) n_h] = c_1 n_h + c_2 t_h,
\end{align*}\] (4.17)
then the pair of functions \((v, q)\) can be written as
\[(v, q) = (v^{j_0} + c_1 v^{j_1}, q^{j_0} + c_1 q^{j_1} + c_2 q^{j_2}).\] (4.18)
We want to find the augmented variable \((c_1, c_2)^T\) so that the original interface jump condition (3.3) is satisfied. Substituting (4.18) into (3.3), we have
\[
\|[\sigma(\mu^\pm, c_1 v^{J_1,\pm} + c_2 v^{J_2,\pm}, c_1 q^{J_1,\pm} + c_2 q^{J_2,\pm}) n_h]\| = -\|[\sigma(\mu^\pm, v^{J_0,\pm}, q^{J_0,\pm}) n_h]\|
\]
\[
= -\sigma([\mu^\pm], v^{J_0}, 0)n_h.
\]
(4.19)
To derive an equation for the augmented variable \((c_1, c_2)^T\) according to (4.19), we need the following lemma about the functions \((v^{J_1, q^{J_1}})\) and \((v^{J_2, q^{J_2}})\).

**Lemma 4.5.** The functions \((v^{J_1}, q^{J_1})\) and \((v^{J_2}, q^{J_2})\) defined in (4.15) and (4.16) are unique and can be constructed explicitly as
\[
v^{J_1} = 0, \quad q^{J_1} = z - \tilde{\pi}^0_{h,T} z, \quad v^{J_2} = (w - \tilde{\pi}^{CR}_{h,T} w) t_h, \quad q^{J_2} = 0,
\]
(4.20)
where
\[
z(x) = \begin{cases} 
  z^+ &= -1 & \text{if} \ x \in T^+_h, \\
  z^- &= 0 & \text{if} \ x \in T^-_h,
\end{cases}
\]
\[
w(x) = \begin{cases} 
  w^+ &= \text{dist}(x, \Gamma_{h,T}) & \text{if} \ x \in T^+_h, \\
  w^- &= 0 & \text{if} \ x \in T^-_h.
\end{cases}
\]
(4.21)
*Proof.* First we introduce the following identities about the interface jump conditions. If \(v^{J,s} \in V_h(T)\) and \(q^{J,s} \in M_h(T)\), \(s = +, -\) satisfy
\[
\|[\sigma(1, v^{J,\pm}, q^{J,\pm}) n_h]\| = g, \quad \|[v^{J,\pm}]_{|\Gamma_{h,T}} = 0, \quad \|[\nabla \cdot v^{J,\pm}] = 0,
\]
then the following identities hold
\[
\begin{align*}
\{\nabla(v^{J,\pm} \cdot n_h)\} & = 0, \quad \{\nabla(v^{J,\pm} \cdot t_h)\} = 0, \\
\{\nabla(v^{J,\pm} \cdot t_h)\} & = g \cdot t_h, \quad \{\nabla(v^{J,\pm} \cdot n_h)\} = 0, \quad \{q^{J,\pm}\} = -g \cdot n_h.
\end{align*}
\]
(4.23)
The second and fourth identities are direct consequences of \(\|[v^{J,\pm}]_{|\Gamma_{h,T}} = 0\). The other identities can be proved easily by decomposing \(v^{J,\pm}\) into the normal direction \(n_h\) and the tangential direction \(t_h\), i.e.,
\[
\sigma(1, v^{J,\pm}, q^{J,\pm}) n_h = \left(\frac{\partial(v^{J,\pm} \cdot n_h)}{\partial n_h} - q^{J,\pm} \right) n_h + \left(\frac{\partial(v^{J,\pm} \cdot n_h)}{\partial t_h} + \frac{\partial(v^{J,\pm} \cdot t_h)}{\partial n_h}\right) t_h,
\]
\[
\nabla \cdot v^{J,\pm} = \frac{\partial(v^{J,\pm} \cdot n_h)}{\partial n_h} + \frac{\partial(v^{J,\pm} \cdot t_h)}{\partial t_h},
\]
which can also be derived easily in a new \(n_h - t_h\) coordinate system. The detailed proof can be found in the literature; see, e.g., [22, 35].

For the function \(v^{J_1}\) defined in (4.15), we set \(g = n_h\) in (4.22), then (4.23) becomes
\[
\begin{align*}
\{\nabla(v^{J_1,\pm} \cdot n_h)\} & = 0, \quad \{\nabla(v^{J_1,\pm} \cdot t_h)\} = 0, \\
\{\nabla(v^{J_1,\pm} \cdot t_h)\} & = g \cdot t_h, \quad \{\nabla(v^{J_1,\pm} \cdot n_h)\} = 0, \quad \{q^{J_1,\pm}\} = -1,
\end{align*}
\]
which together with \(\|[v^{J_1,\pm}]_{|\Gamma_{h,T}} = 0, v^{J_1,\pm} \in V_h(T), q^{J_1,\pm} \in M_h(T)\) and \(N_{i,T}(v^{J_1, q^{J_1}}) = 0, i = 1, \ldots, 7\) implies that \(v^{J_1}\) and \(q^{J_1}\) exist uniquely and can be constructed from (4.20)-(4.21). Similarly, for the function \(v^{J_2}\) defined in (4.16), with \(g = t_h\), we obtain
\[
\begin{align*}
\{\nabla(v^{J_2,\pm} \cdot n_h)\} & = 0, \quad \{\nabla(v^{J_2,\pm} \cdot t_h)\} = 0, \\
\{\nabla(v^{J_2,\pm} \cdot t_h)\} & = g \cdot n_h, \quad \{\nabla(v^{J_2,\pm} \cdot n_h)\} = 1, \quad \{q^{J_2,\pm}\} = 0.
\end{align*}
\]
Using the fact \(\|[v^{J_2,\pm}]_{|\Gamma_{h,T}} = 0, v^{J_2,\pm} \in V_h(T), q^{J_2,\pm} \in M_h(T)\) and \(N_{i,T}(v^{J_2, q^{J_2}}) = 0, i = 1, \ldots, 7\), we have
\[
v^{J_2} \cdot n_h = 0, \quad v^{J_2} \cdot t_h = w - \tilde{\pi}^{CR}_{h,T} w, \quad q^{J_2} = 0,
\]
which completes the proof. □
Since \( v^{J_1} = 0 \) and \( q^{J_2} = 0 \) from (4.20), the equation (4.19) can be simplified as
\[
\left[ \sigma(\mu^\pm, c_2 v^{J_2, \pm}, c_1 q^{J_1, \pm}) n_h \right] = -\sigma(\mu^\pm, v^{J_0}, 0) n_h. \tag{4.24}
\]
By the fact \( v^{J_2} \cdot n_h = 0 \) from (4.20), the above equation (4.24) becomes
\[
\begin{pmatrix}
- [q^{J_1, \pm}] & 0 \\
0 & \mu^\pm \nabla (v^{J_2, \pm} \cdot t_h) \cdot n_h
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
\sigma(\mu^\pm, v^{J_0}, 0) n_h - n_h \\
\sigma(\mu^\pm, v^{J_0}, 0) n_h \cdot t_h
\end{pmatrix}. \tag{4.25}
\]
Using (4.20) again, we have \(-[q^{J_1, \pm}] = 1\) and
\[
[\mu^\pm \nabla (v^{J_2, \pm} \cdot t_h) \cdot n_h] = [\mu^\pm \nabla (w - \pi^{CR}_{h,T} w) \cdot n_h] = [\mu^\pm \nabla w \cdot n_h] - [\mu^\pm \nabla \pi^{CR}_{h,T} w \cdot n_h]
= \mu^+ - (\mu^+ - \mu^-) \nabla \pi^{CR}_{h,T} w \cdot n_h. \tag{4.26}
\]
Thus, the linear system (4.25) for the augmented variable \( (c_1, c_2)^T \) becomes
\[
\begin{pmatrix}
1 & 0 \\
0 & 1 + (\mu^- / \mu^+ - 1) \nabla \pi^{CR}_{h,T} w \cdot n_h
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
\sigma(\mu^-, v^{J_0}, 0) n_h \cdot n_h \\
\sigma(\mu^- / \mu^+ - 1, v^{J_0}, 0) n_h \cdot t_h
\end{pmatrix}. \tag{4.27}
\]

![Figure 2: Diagrams of typical interface elements.](image)

**Lemma 4.6.** Let \( T \) be an arbitrary interface triangle with an arbitrary \( \Gamma_{h,T} \), and \( w \) be a piecewise linear function defined in (4.21). Then it holds that
\[
0 \leq \nabla \pi^{CR}_{h,T} w \cdot n_h \leq 1. \tag{4.28}
\]

**Proof.** Consider \( T = \triangle A_1 A_2 A_3 \) with edges \( e_1 = A_2 A_3 \), \( e_2 = A_1 A_3 \) and \( e_3 = A_1 A_2 \). Without loss of generality, we assume that the interface \( \Gamma \) cuts \( e_1 \) and \( e_2 \) at points \( D \) and \( E \). There are two cases: Case 1: \( T^+_h = \triangle EDA_3 \) (see Figure 2(a)); Case 2: \( T^-_h = \triangle EDA_3 \) (see Figure 2(b)). In Case 1, we have from (4.21) that
\[
w(x) = \begin{cases}
    n_h \cdot \overrightarrow{Dx} & \text{if } x \in \triangle EDA_3, \\
    0 & \text{if } x \in T \setminus \triangle EDA_3.
\end{cases} \tag{4.29}
\]
In order to distinguish between these two cases, we replace the notations \( n_h \) and \( w \) by \( n'_h \) and \( w' \) in Case 2. Using the fact \( n'_h = -n_h \), we have the following result according to (4.21)
\[
w'(x) = \begin{cases}
    0 & \text{if } x \in \triangle EDA_3, \\
    - n_h \cdot \overrightarrow{Dx} & \text{if } x \in T \setminus \triangle EDA_3.
\end{cases} \tag{4.30}
\]
Comparing (4.20) with (4.30), we find \( w' = w - n_h \cdot \overrightarrow{Dx} \), which implies
\[
\nabla \pi^{CR}_{h,T} w' \cdot n'_h = \nabla \pi^{CR}_{h,T} (w - n_h \cdot \overrightarrow{Dx}) \cdot (-n_h) = 1 - \nabla \pi^{CR}_{h,T} w \cdot n_h. \tag{4.31}
\]
If the estimate (4.28) holds for Case 1, then we can conclude from (4.31) that the estimate (4.28) also holds for Case 2. Therefore, we just need to consider Case 1 whose geometric configuration is given in Figure 2(a).

The proof for Case 1 is similar to that of Lemma 3.3 in [24]. By the definitions of the interpolation operator \( \pi_{h,T}^{CR} \) in (4.5) and the basis functions \( \lambda_{i,T} \) in (4.10), we have

\[
\nabla \pi_{h,T}^{CR} w \cdot n_h = \sum_{i=1}^{3} \left( \nabla \lambda_{i,T} \cdot n_h \frac{1}{|e_i|} \int_{e_i} w \right) = \nabla \lambda_{1,T} \cdot n_h \frac{1}{|e_1|} \int_{A_1D} n_h \cdot \vec{\nabla} x + \nabla \lambda_{2,T} \cdot n_h \frac{1}{|e_2|} \int_{A_2E} n_h \cdot \vec{\nabla} x. \tag{4.32}
\]

Let \( M_2 \) be the midpoint of the edge \( e_2 \) and \( Q \) be the orthogonal projection of \( M_2 \) onto the line \( A_2A_3 \). Then, it holds

\[
\nabla \lambda_1 \cdot n_h = |M_2Q|^{-1} \overrightarrow{M_2Q} \cdot n_h = |M_2Q|^{-1} R_{\pi/2} \left( \overrightarrow{M_2Q} |M_2Q|^{-1} \right) \cdot R_{\pi/2} n_h = |M_2Q|^{-1} |A_3D|^{-1} \overrightarrow{A_3D} \cdot t_h.
\]

Note that

\[
\frac{1}{|e_1|} \int_{A_1D} n_h \cdot \vec{\nabla} x ds = \frac{1}{2} |e_1|^{-1} |A_3D| n_h \cdot \overrightarrow{DA_3}.
\]

Therefore, it follows from the above identities and the fact \( |M_2Q||e_1| = |T| \) that

\[
\nabla \lambda_{1,T} \cdot n_h \frac{1}{|e_1|} \int_{A_1D} n_h \cdot \vec{\nabla} x = \frac{1}{2} |T|^{-1} (n_h \cdot \overrightarrow{DA_3})(\overrightarrow{A_3D} \cdot t_h). \tag{4.33}
\]

Analogously, we have

\[
\nabla \lambda_{2,T} \cdot n_h \frac{1}{|e_2|} \int_{A_2E} n_h \cdot \vec{\nabla} x = \frac{1}{2} |T|^{-1} (n_h \cdot \overrightarrow{DA_3})(\overrightarrow{EA_3} \cdot t_h). \tag{4.34}
\]

Substituting (4.33) and (4.34) into (4.32) yields

\[
\nabla \pi_{h,T}^{CR} w \cdot n_h = \frac{1}{2} |T|^{-1} (n_h \cdot \overrightarrow{DA_3})(\overrightarrow{ED} \cdot t_h) = \frac{1}{2} |T|^{-1} (n_h \cdot \overrightarrow{DA_3})|ED| = \frac{|T|}{|T|} \in [0,1], \tag{4.35}
\]

which completes the proof. \( \square \)

**Lemma 4.7.** For an arbitrary interface triangle \( T \in T_h^\Gamma \), the pair of functions \( (v, q) \in VM_{h}^{1FE}(T) \) is uniquely determined by \( N_{i,T}(v, q), i = 1, \ldots, 7 \). Furthermore, we have the following explicit formula

\[
(v, q) = (v^{J_0}, q^{J_0}) + (c_2 v^{J_2}, c_1 q^{J_1}) \tag{4.36}
\]

with

\[
c_1 = \sigma(\mu^- - \mu^+), \quad c_2 = \frac{\sigma(\mu^- - \mu^+, v^{J_0}, 0) n_h \cdot n_h + n_h \cdot t_h}{1 + (\mu^- - \mu^+, v^{J_0}, 0) n_h \cdot n_h}, \tag{4.37}
\]

\[
v^{J_0} = \sum_{i=1}^{6} N_{i,T}(v, q) \phi_{i,T}, \quad q^{J_0} = N_{7,T}(v, q),
\]

where \( v^{J_2}, q^{J_1} \), \( w \) and \( \phi_{i,T} \) are defined in (4.20), (4.21) and (4.11), and \( \pi_{h,T}^{CR} \) is the standard CR interpolation defined in (4.9).
Lemma 4.11. The analysis is also independent of the interface location relative to the mesh. In other words, our method may cut meshes in an arbitrary way. In the rest of our paper, we will show that the constant in \( C \) becomes the standard CR-P\(_0\) finite element space \((V_h, M_h)\).

Remark 4.8. If \( \mu^+ = \mu^- \), then \( c_1 = c_2 = 0 \) and \((v, q) = (v^0, q^0) \in (V_h(T), M_h(T))\). Therefore, the IFE space \( V_h^{IFE} \) becomes the standard CR-P\(_0\) finite element space \((V_h, M_h)\).

Remark 4.9. If \( N_i,T(v, q) = 0 \), \( i = 1, ..., 7 \), then \((v^0, q^0) = (0, 0)\). From (4.37), we also have \( c_1 = c_2 = 0 \). Hence, we conclude \((v, q) = (0, 0)\) when \((v, q) \in V_h^{IFE}(T)\) and \( N_i,T(v, q) = 0 \), \( i = 1, ..., 7 \).

Remark 4.10. From (4.37)-(4.6), we know \( v^0 = \pi_{h,T}^{CR}v \) and \( q^0 = \pi_{h,T}^0q \). Hence, the IFE interpolations of \((v, q) \in (H^1(\Omega)^2, L^2(\Omega))\) on an interface element \( T \in T_h^I \) are

\[
(\Pi_{v,q}^{IFE} v)_T = \sigma_{h,T}^{CR}v + c_2v^0 \text{ and } (\Pi_{v,q}^{IFE} q)_T = \pi_{h,T}^0q + c_1q^0
\]

with \( c_1 \) and \( c_2 \) defined in (4.37) that are independent of the pressure \( q \). From the above identities, we find that \( \Pi_{v,q}^{IFE} v \) depends only on the velocity \( v \), not on the pressure \( q \). However, \( \Pi_{v,q}^{IFE} q \) depends both on \( v \) and \( q \).

4.2 Estimates of IFE basis functions

For each interface element \( T \in T_h^I \), similar to (4.11), we define IFE basis functions by

\[
(\phi_{i,T}^{IFE}, \varphi_{i,T}^{IFE}) \in V_h^{IFE}(T), \quad N_j,T(\phi_{i,T}^{IFE}, \varphi_{i,T}^{IFE}) = \delta_{i,j}, \quad \forall i, j = 1, ..., 7.
\]

Using Lemma 4.7, we can write these IFE basis functions \((\phi_{i,T}^{IFE}, \varphi_{i,T}^{IFE})\) explicitly as

\[
\begin{align*}
\phi_{i,T}^{IFE} &= \phi_{i,T} + \frac{\sigma(\mu^- / \mu^+ - 1, \phi_{i,T}, 0) n_h \cdot t_h}{1 + (\mu^- / \mu^+ - 1) \nabla \pi_{h,T}^{CR} w \cdot n_h} (w - \pi_{h,T}^{CR} w) t_h, \quad i = 1, ..., 6, \\
\varphi_{i,T}^{IFE} &= \sigma(\mu^- - \mu^+, \phi_{i,T}, 0) n_h \cdot n_h (z - \pi_{h,T}^0 z), \quad i = 1, ..., 6, \\
\phi_{7,T}^{IFE} &= 0, \quad \varphi_{7,T}^{IFE} = 1,
\end{align*}
\]

where \( \phi_{i,T}, i = 1, ..., 6 \) are the standard CR basis functions for the velocity (see (4.12)), and \( w \) and \( z \) are known functions defined in (4.21). Also we have \((\phi_{7,T}^{IFE}, \varphi_{7,T}^{IFE}) = (\phi_{7,T}, \varphi_{7,T})\) from (4.12). We emphasize that these explicit formulas for IFE basis functions are very useful in the implementation.

From (4.35), we highlight that the denominator in the IFE basis functions (4.40) does not tend to zero even if \(|T^+_h| \to 0\) or \(|T^-_h| \to 0\). This property is important for IFE method because the interface may cut meshes in an arbitrary way. In the rest of our paper, we will show that the constant in the analysis is also independent of the interface location relative to the mesh. In other words, our method works for the case \(|T^+_h| \to 0\) or \(|T^-_h| \to 0\).

Lemma 4.11. There exists a positive constant \( C \) depending only on \( \mu^\pm \) and the shape regularity parameter \( q \) such that, for \( m = 0, 1 \),

\[
\begin{align*}
|\phi_{i,T}^{IFE}|_{W^m_\infty(T)} &\leq Ch_T^{-m}, \quad \|\phi_{i,T}^{IFE}\|_{L^\infty(T)} \leq Ch_T^{-1}, \quad i = 1, ..., 6, \\
|\phi_{7,T}^{IFE}|_{W^m_\infty(T)} &\leq 0, \quad \|\phi_{7,T}^{IFE}\|_{L^\infty(T)} = 1.
\end{align*}
\]
4.3 Approximation capabilities of the IFE space

For clarity, we first describe the main idea of the proof of approximation capabilities of the IFE space. Our target is to estimate the following term on interface element $T \in \mathcal{T}_h^I$,

$$\|(v, q) - \Pi_h^{IFE}(v, q)\|_T^2 \leq \sum_{s=\pm} 2 \left\|(v_E^s, q_E^s) - (\Pi_{h,T}^{IFE}(v, q))^s\right\|_T^2,$$

where $\| \cdot \|_T$ is a specific norm, $v_E^s$ and $q_E^s$ are extensions of $v^s$ and $q^s$ as shown in Lemma 4.3 and the notation of superscripts $s=\pm$ is described at the end of section 2. Obviously, the functions on the right-hand side can be split as

$$(v_E^s, q_E^s) - (\Pi_{h,T}^{IFE}(v, q))^s = (v_E^s, q_E^s) - (\Pi_{h,T}(v_E^s, q_E^s))^s + (\Pi_{h,T}(v_E^s, q_E^s) - (\Pi_{h,T}^{IFE}(v, q))^s).$$

The estimate of the first term (I) is standard and the main difficulty is to estimate the second term (II). Noticing that the term (II) are piecewise polynomials on the interface element $T \in \mathcal{T}_h^I$, our idea is to decompose the term (II) by proper degrees of freedom as shown in Lemma 4.13. Then we estimate every term in the decomposition to get the desired results (see Theorem 4.11). The degrees of freedom for determining the term (II) include $N_{j,T}$, $j = 1, \ldots, 7$, and others related to the interface jumps (4.33)-(4.35), which inspire us to define the following novel auxiliary functions.

On each interface element $T \in \mathcal{T}_h^I$, we define auxiliary functions $(\Psi_{i,T}, \psi_{i,T}), i = 1, \ldots, 7$ with $\Psi_{i,T}|_{T_h^\pm} = \Psi_{i,T}^\pm$, $\psi_{i,T}|_{T_h^\pm} = \psi_{i,T}^\pm$ such that

$$(\Psi_{i,T}^\pm, \psi_{i,T}^\pm) \in (V_h(T), M_h(T)), \quad N_{j,T}(\Psi_{i,T}, \psi_{i,T}) = 0, \quad j = 1, \ldots, 7, \quad (4.43)$$

Proof. It suffices to estimate the terms on the right-hand side of (4.40). First we have the following estimates about the standard CR basis functions

$$|\lambda_{i,T}|_{W_\infty^1(T)} \leq Ch_T^{-m} \quad \text{and} \quad |\phi_{i,T}|_{W_\infty^1(T)} \leq Ch_T^{-m}, \quad m = 0, 1.$$

Using the $n_h - t_h$ coordinate system, we then have

$$|\sigma(\mu^-/\mu^+ - 1, \phi_{i,T}, 0)n_h \cdot t_h| = |(\mu^-/\mu^+ - 1)(\nabla(\phi_{i,T} \cdot n_h) \cdot t_h + \nabla(\phi_{i,T} \cdot t_h) \cdot n_h)| \leq Ch_T^{-1},$$

$$|\sigma(\mu^-/\mu^+, \phi_{i,T}, 0)n_h \cdot n_h| = |2(\mu^-/\mu^+\nabla(\phi_{i,T} \cdot t_h) \cdot n_h)| \leq Ch_T^{-1}.$$

By the definitions of $w$ and $z$ in (4.21), we also have

$$|w^+|_{W_\infty^1(T)} \leq Ch_T^{1-m}, \quad |w^-|_{W_\infty^1(T)} = 0,$$

$$|\pi_{h,T}^{CR}w|_{W_\infty^1(T)} = \left| \sum_{i=1}^3 \lambda_{i,T} \frac{1}{|e_i|} \int_{e_i} w \right|_{W_\infty^1(T)} \leq Ch_T \sum_{i=1}^3 |\lambda_{i,T}|_{W_\infty^1(T)} \leq Ch_T^{1-m}, \quad (4.42)$$

$$|z^+| = 1, \quad |z^-| = 0, \quad |\pi_{h,T}^0 z| = |T|^{-1} \left| \int_T z \right| \leq \|z\|_{L_\infty(T)} \leq 1.$$

Finally, the desired estimates (4.41) are obtained by substituting (4.38) and the above estimates into (4.40). ~\hfill \square
and

\[ [\sigma(\mu^\pm, \Psi^\pm_{i,T}, \psi^\pm_{i,T})] n_h] = 0, \quad [\Psi^\pm_{i,T}](x_T) = n_h, \quad \nabla [\Psi^\pm_{i,T}] t_h = 0, \quad [\nabla \cdot \Psi^\pm_{i,T}] = 0, \]

where \( x_T \) is the same as that in (3.3).

Lemma 4.12. On each interface element \( T \in \mathcal{T}_h^I \), these auxiliary functions \( (\Psi_{i,T}, \psi_{i,T}) \), \( i = 1, \ldots, 7 \) defined in (4.43)-(4.44) exist uniquely and satisfy, for \( m = 0, 1 \),

\[
|\Psi^\pm_{i,T}|_{W^m_{\infty}(T)} \begin{cases}
\leq Ch_T^{-m} & \text{if } i = 1, 2, \\
= 0 & \text{if } i = 3, \\
\leq Ch_T^{1-m} & \text{if } i = 4, \ldots, 7,
\end{cases}
\quad |\psi^\pm_{i,T}|_{L^\infty(T)} \begin{cases}
\leq Ch_T^{-1} & \text{if } i = 1, 2, \\
\leq C & \text{if } i = 3, \ldots, 7,
\end{cases}
\]

where the constant \( C \) depends only on \( \mu^\pm \) and the shape regularity parameter \( \rho \).

Proof. The justification of the existence and uniqueness is that the coefficient matrix is the same as that for determining the IFE shape functions in the space \( V_{M^p_{h,\text{IFE}}}(T) \) if we write a 14-by-14 linear system of equations for the fourteen parameters (see Remark 3). To derive the estimates (4.45), we need explicit expressions of these auxiliary functions. First, we define \((v_i^+, q_i^-) = (0, 0)\) and \((v_i^+, q_i^+) \in (V_h(T), M_h(T)), i = 1, \ldots, 7\) such that

\[
\begin{align*}
v_1^+(x_T) &= n_h, \quad \frac{\partial (v_1^+ \cdot n_h)}{\partial n_h} = 0, \quad \frac{\partial (v_1^+ \cdot n_h)}{\partial t_h} = 0, \quad \frac{\partial (v_1^+ \cdot t_h)}{\partial n_h} = 0, \quad \frac{\partial (v_1^+ \cdot t_h)}{\partial t_h} = 0, q_1^+ = 0, \\
v_2^+(x_T) &= t_h, \quad \frac{\partial (v_2^+ \cdot n_h)}{\partial n_h} = 0, \quad \frac{\partial (v_2^+ \cdot n_h)}{\partial t_h} = 0, \quad \frac{\partial (v_2^+ \cdot t_h)}{\partial n_h} = 0, \frac{\partial (v_2^+ \cdot t_h)}{\partial t_h} = 0, q_2^+ = 0, \\
v_3^+(x_T) &= 0, \quad \frac{\partial (v_3^+ \cdot n_h)}{\partial n_h} = 0, \quad \frac{\partial (v_3^+ \cdot n_h)}{\partial t_h} = 0, \quad \frac{\partial (v_3^+ \cdot t_h)}{\partial n_h} = 0, \frac{\partial (v_3^+ \cdot t_h)}{\partial t_h} = 0, q_3^+ = -1, \\
v_4^+(x_T) &= 0, \quad \frac{\partial (v_4^+ \cdot n_h)}{\partial n_h} = 0, \quad \frac{\partial (v_4^+ \cdot n_h)}{\partial t_h} = 0, \quad \frac{\partial (v_4^+ \cdot t_h)}{\partial n_h} = 0, \frac{\partial (v_4^+ \cdot t_h)}{\partial t_h} = 1, q_4^+ = 0, \\
v_5^+(x_T) &= 0, \quad \frac{\partial (v_5^+ \cdot n_h)}{\partial n_h} = 0, \quad \frac{\partial (v_5^+ \cdot n_h)}{\partial t_h} = 1, \quad \frac{\partial (v_5^+ \cdot t_h)}{\partial n_h} = -1, \frac{\partial (v_5^+ \cdot t_h)}{\partial t_h} = 0, q_5^+ = 0, \\
v_6^+(x_T) &= 0, \quad \frac{\partial (v_6^+ \cdot n_h)}{\partial n_h} = -1, \quad \frac{\partial (v_6^+ \cdot n_h)}{\partial t_h} = 0, \quad \frac{\partial (v_6^+ \cdot t_h)}{\partial n_h} = 0, \frac{\partial (v_6^+ \cdot t_h)}{\partial t_h} = 1, q_6^+ = -2\mu^+, \\
v_7^+(x_T) &= 0, \quad \frac{\partial (v_7^+ \cdot n_h)}{\partial n_h} = 1, \quad \frac{\partial (v_7^+ \cdot n_h)}{\partial t_h} = 0, \quad \frac{\partial (v_7^+ \cdot t_h)}{\partial n_h} = 0, \frac{\partial (v_7^+ \cdot t_h)}{\partial t_h} = 0, q_7^+ = 2\mu^+.
\end{align*}
\]

By using the \( n_h, t_h \) coordinate system, it is easy to verify that the above defined functions exist uniquely and satisfy the jump conditions (4.41). If we define

\[
(v_i, q_i) = \begin{cases} (v_i^+, q_i^+) & \text{in } T^+_h, \\
(v_i^+, q_i^-) & \text{in } T^-_h, \end{cases}
\]

(4.47)
then the auxiliary functions satisfying (4.43)-(4.44) can be obtained by
\[
(\Psi_{i,T}, \psi_{i,T}) = (v_i, q_i) - \Pi_{h,T}^{IFE} (v_i, q_i) = (v_i, q_i) - \sum_{j=1}^{7} N_j(T) (v_i, q_i) (\phi_{j,T}^{IFE}, \varphi_{j,T}^{IFE}), \quad j = 1, ..., 7.
\] (4.48)

Now we estimate the terms on the right-hand side of the above identity. From (4.46), we have
\[
|v_i^+|_{W_2^m(T)} \leq C h_T^{-m}, \quad i = 1, 2, \quad |v_3^+|_{W_2^m(T)} = 0, \quad |v_i^+|_{W_2^m(T)} \leq C h_T^{-m}, \quad i = 4, ..., 7, \quad m = 1, 2,
\]
\[
|q_i^+|_{L^\infty(T)} \leq C, \quad i = 3, 6, 7, \quad |q_i^+|_{L^\infty(T)} = 0, \quad i = 1, 2, 4, 5,
\] (4.49)
which together with (3.2) leads to
\[
|N_j(T) (v_i, q_i)| \leq C, \quad j = 1, ..., 6, \quad |N_7(T) (v_i, q_i)| = 0, \quad i = 1, 2,
\]
\[
|N_j(T) (v_i, q_i)| = 0, \quad j = 1, ..., 6, \quad |N_7(T) (v_i, q_i)| \leq C,
\]
\[
|N_j(T) (v_i, q_i)| \leq C h_T, \quad j = 1, ..., 6, \quad |N_7(T) (v_i, q_i)| = 0, \quad i = 4, 5,
\]
\[
|N_j(T) (v_i, q_i)| \leq C h_T, \quad j = 1, ..., 6, \quad |N_7(T) (v_i, q_i)| \leq C, \quad i = 6, 7.
\] (4.50)
Combining (4.48)-(4.50) and (4.41), we get the desired estimates (4.45).

**Lemma 4.13.** For any \((v, q) \in (V,M)\), let \((v_E^\pm, q_E^\pm)\) be extensions of \((v^\pm, q^\pm)\) as defined in Lemma 4.3 and let \(v_{E,1}^s\) and \(v_{E,2}^s\) be two components of \(v_E^s\), i.e., \(v_E^s = (v_{E,1}^s, v_{E,2}^s)^T\), \(s = +, -\).

For any \(T \in T_h\), let \(e_i, i = 1, 2, 3\) be its edges and we set \(e_i = e_i \cap \Omega^s\). Then it holds that
\[
\Pi_{h,T} (v_{E}^\pm, q_{E}^\pm) - (\Pi_{h,T}^{IFE} (v, q))^\pm = \sum_{i=1}^{7} (\phi_{i,T}^{IFE}, \varphi_{i,T}^{IFE})^\pm \alpha_i + \sum_{i=1}^{7} (\Psi_{i,T}, \psi_{i,T})^\pm \beta_i,
\] (4.51)
where
\[
\alpha_i = \frac{1}{|e_i|} \sum_{s = +, -} \int_{e_i^s} (\pi_{h,T}^{CR} v_{E,1}^s - v_{E,1}^s), \quad \alpha_{3+i} = \frac{1}{|e_i|} \sum_{s = +, -} \int_{e_i^s} (\pi_{h,T}^{CR} v_{E,2}^s - v_{E,2}^s), \quad i = 1, 2, 3,
\]
\[
\alpha_7 = \frac{1}{|T|} \sum_{s = +, -} \int_{T^s} (\pi_{h,T}^{CR} q_{E}^s - q_{E}^s)
\] (4.52)
and
\[
\beta_1 = [\pi_{h,T}^{CR} v_{E}^s]^T (x_{T}^s), \quad \beta_2 = [\pi_{h,T}^{CR} v_{E}^s]^T (x_{T}), \quad \beta_3 = [\sigma (\mu^s, \pi_{h,T}^{CR} v_{E}^s), \pi_{h,T}^{CR} q_{E}^s]^T (n_{h,T}), \quad \beta_4 = [\sigma (\mu^s, \pi_{h,T}^{CR} v_{E}^s), \pi_{h,T}^{CR} q_{E}^s]^T (n_{h,T}) \cdot \tau_{h,T}]
\]
\[
\beta_5 = [\nabla (\pi_{h,T}^{CR} v_{E}^s)^T] \cdot \n_{h,T}, \quad \beta_6 = [\nabla (\pi_{h,T}^{CR} v_{E}^s)^T] \cdot \n_{h,T}, \quad \beta_7 = [\nabla (\pi_{h,T}^{CR} v_{E}^s)^T]
\] (4.53)

**Proof.** For simplicity of notation, we define a pair of functions \((\Xi_h, \xi_h)\) such that
\[
(\Xi_h, \xi_h)|_{T_h^s} = (\Xi_h, \xi_h) = \Pi_{h,T} (v_{E}^s, q_{E}^s) - (\Pi_{h,T}^{IFE} (v, q))^\pm.
\] (4.54)
Define another pair of functions \((\hat{\Xi}_h, \hat{\xi}_h)\) by
\[
(\hat{\Xi}_h, \hat{\xi}_h) = \sum_{i=1}^{7} (\phi_{i,T}^{IFE}, \varphi_{i,T}^{IFE}) \alpha_i + \sum_{i=1}^{7} (\Psi_{i,T}, \psi_{i,T}) \beta_i
\] (4.55)
with
\[
\alpha_i = N_i(T) (\Xi_h, \xi_h), \quad i = 1, ..., 7
\] (4.56)
Theorem 4.14. 

Next, we prove \((\Xi_h, \xi_h) = (\hat{\Xi}_h, \hat{\xi}_h)\). Using the facts that the IFE basis functions \((\phi_{i,T}^{IE}, \psi_{i,T}^{IE})\) and the constructed functions \((\Psi_{i,T}, \psi_{i,T})\) satisfy the interface jump conditions \((3.3)-(3.5)\) and \((4.43)\), we have from \((4.55)-(4.57)\) that

\[
\begin{align*}
\| (\hat{\Xi}_h - \Xi_h)^\pm \|_T &= 0, \\
\| (\hat{\Xi}_h - \Xi_h)^\pm \|_T &= 0,
\end{align*}
\]

which implies

\[
(\hat{\Xi}_h - \Xi_h, \hat{\xi}_h - \xi_h) = \Xi M_h^{IE}(T).
\]

Similarly, from \((4.39)-(4.41)\) and \((4.55)-(4.57)\), we also have

\[
N_i,T(\hat{\Xi}_h - \Xi_h, \hat{\xi}_h - \xi_h) = 0, \quad i = 1, ..., 7.
\]

In view of Remark 4.10 we therefore conclude \((\Xi_h - \hat{\Xi}_h, \xi_h - \hat{\xi}_h) = (0, 0)\), i.e., \((\Xi, \xi) = (\hat{\Xi}, \hat{\xi})\).

Now it remains to calculate the constants \(\alpha_i\) and \(\beta_i\) in \((4.56)-(4.57)\). If we define a broken interpolation operator \(\Pi_{h,T}^{BK}\) such that

\[
(\Pi_{h,T}^{BK}(v, q))|_{T_h^\pm} = \Pi_{h,T}(v^\pm_h, q^\pm_h),
\]

then \((\Xi_h, \xi_h)\) defined in \((4.54)\) can be written as

\[
(\Xi_h, \xi_h) = \Pi_{h,T}^{BK}(v, q) - \Pi_{h,T}^{IE}(v, q).
\]

By \((1.7)\), we can calculate \(\alpha_i\) in \((4.56)\) as

\[
\alpha_i = N_i,T(\Pi_{h,T}^{BK}(v, q)) - N_i,T(\Pi_{h,T}^{IE}(v, q)) = N_i,T(\Pi_{h,T}^{BK}(v, q)) - N_i,T(v, q), \quad i = 1, ..., 7,
\]

which together with \((4.61)-(4.62)\) and \((5.2)\) leads to \((4.52)\). The results in \((4.53)\) for \(\beta_i\), \(i = 1, ..., 7\) are obtained by substituting \((4.54)\) into \((4.57)\) and using the fact that \(\Pi_{h,T}^{IE}(v, q)\) satisfies the interface jump conditions \((3.3)-(3.5)\). This completes the proof of the lemma.

Theorem 4.14. For any \((v, q) \in \overline{H_2} H_1\), there exists a positive constant \(C\) independent of \(h\) and the interface location relative to the mesh such that

\[
\begin{align*}
\sum_{T \in T_h^\pm} |v_E^\pm - (\Pi_{v,q}^{IE})^\pm|_{H^m(T)}^2 &\leq C h_{\Gamma}^{-4-2m} (||v||_{H^2(\Omega_+ \cup \Omega_-)} + ||q||_{H^1(\Omega_+ \cup \Omega_-)}^2), & m = 0, 1, \quad (4.64) \\
\sum_{T \in T_h^\pm} ||q_E^\pm - (\Pi_{v,q}^{IE})^\pm||_{L^2(T)}^2 &\leq C h_{\Gamma}^2 (||v||_{H^2(\Omega_+ \cup \Omega_-)}^2 + ||q||_{H^1(\Omega_+ \cup \Omega_-)}^2), \quad (4.65)
\end{align*}
\]

where \(h_{\Gamma}\) is defined in \((4.4)\).

Proof. On each interface element \(T \in T_h^\pm\), by the triangle inequality, we have

\[
\begin{align*}
|v_E^\pm - (\Pi_{v,q}^{IE})^\pm|_{H^m(T)}^2 &\leq |v_E^\pm - \sigma_{h,T}^{IE} v_E^\pm|_{H^m(T)}^2 + ||\sigma_{h,T}^{IE} v_E^\pm - (\Pi_{v,q}^{IE})^\pm||_{H^m(T)}^2, \\
||q_E^\pm - (\Pi_{v,q}^{IE})^\pm||_{L^2(T)}^2 &\leq ||q_E^\pm - \sigma_{h,T}^{IE} q_E^\pm||_{L^2(T)}^2 + ||\sigma_{h,T}^{IE} q_E^\pm - (\Pi_{v,q}^{IE})^\pm||_{L^2(T)}^2.
\end{align*}
\]

(4.66)
The estimates of the first terms are standard

\[
|v_E^\pm - \pi_{h,T}^{CR}v_E^\pm|_H^2 \leq C h_T^{4-2m} |v_E^\pm|_{H^2(T)}^2, \quad m = 0, 1, \tag{4.67}
\]

\[
|q_E^\pm - \pi_{h,T}^{4,q} q_E^\pm|_{L^2(T)}^2 \leq C h_T^2 |q_E^\pm|_{H^1(T)}^2.
\]

For the second term on the right-hand side of (4.66), we use (4.10), (4.9) and Lemmas 4.11 and 4.12 to get

\[
|\pi_{h,T}^{CR}v_E^\pm - (\Pi_{\gamma,q}^{FE}v)^\pm|_H^2 \leq C \left( \sum_{i=1}^{7} \alpha_i^2 \| (\phi_{i,T}^{FE})^\pm \|_H^2 + \sum_{i=1}^{7} \beta_i^2 \| \psi_{i,T}^\pm \|_H^2 \right)
\]

\[
\leq C \sum_{i=1}^{6} \alpha_i^2 h_T^{2-2m} + C \sum_{i=1}^{2} \beta_i^2 h_T^{2-2m} + C \sum_{i=4}^{7} \beta_i^2 h_T^{4-2m}, \tag{4.68}
\]

\[
|\pi_{h,T}^0 q_E^\pm - (\Pi_{\gamma,q}^{FE}q)^\pm|_{L^2(T)}^2 \leq C \left( \sum_{i=1}^{7} \alpha_i^2 \| (\varphi_{i,T}^{FE})^\pm \|_{L^2(T)}^2 + \sum_{i=1}^{7} \beta_i^2 \| \varphi_{i,T}^\pm \|_{L^2(T)}^2 \right)
\]

\[
\leq C \sum_{i=1}^{6} \alpha_i^2 + C \alpha_i^2 h_T^2 + C \sum_{i=1}^{2} \beta_i^2 + C \sum_{i=4}^{7} \beta_i^2 h_T^2,
\]

where the constants \( \alpha_i \) and \( \beta_i \) are defined in (4.52) and (4.53). Next, we estimate these constants one by one. By the Cauchy-Schwarz inequality, we have

\[
\alpha_i^2 = \frac{1}{|e_i|^2} \left| \sum_{s=+,-} \int_{e_i} (\pi_{h,T}^{CR} v_{E,1}^s - v_{E,1}^s)^2 \right| \leq C |e_i|^{-1} \sum_{s=+,-} \|\pi_{h,T}^{CR} v_{E,1}^s - v_{E,1}^s\|^2_{L^2(e_i)}, \quad i = 1, 2, 3,
\]

\[
\alpha_i^2 \leq C |e_i|^{-1} \sum_{i=+,-} \|\pi_{h,T}^{CR} v_{E,2}^s - v_{E,2}^s\|^2_{L^2(e_i)}, \quad i = 4, 5, 6,
\]

\[
\alpha_i^2 = \frac{1}{|T|^2} \left| \sum_{s=+,-} \int_{T_k} (\pi_{h,T}^0 q_{E}^s - q_{E}^s)^2 \right| \leq C h_T^{-2} \sum_{s=+,-} \|\pi_{h,T}^0 q_{E}^s - q_{E}^s\|^2_{L^2(T)}.
\]

By the standard trace inequality and the standard interpolation error estimates, it follows

\[
\alpha_i^2 \leq C |e_i|^{-1} \sum_{s=+,-} \|\pi_{h,T}^{CR} v_{E}^s - v_{E}^s\|^2_{L^2(e_i)} \leq C \sum_{s=+,-} \left( h_T^{-2} \|\pi_{h,T}^{CR} v_{E}^s - v_{E}^s\|^2_{L^2(T)} \right.
\]

\[
+ \left. \|\pi_{h,T}^{CR} v_{E}^s - v_{E}^s\|^2_{H^2(T)} \right) \leq C h_T^2 \sum_{s=+,-} \|v_{E}^s\|^2_{H^2(T)}, \quad i = 1, ..., 6, \tag{4.69}
\]

\[
\alpha_i^2 \leq C \sum_{s=+,-} \|q_{E}^s\|^2_{H^2(T)}.
\]

Since \((v,q) \in H_2 \cap H_1\), the value \(v(x_T)\) is well-defined and the identity \(\tilde{v}_{E}^\pm(x_T) = 0\) holds on the point \(x_T \in \Gamma_{h,T} \cap \Gamma_T\). Therefore, the constants \( \beta_1 \) and \( \beta_2 \) in (4.53) can be estimated as

\[
\beta_i^2 \leq \left| \left[ \pi_{h,T}^{CR} v_{E}^s(x_T) \right] \right|^2 = \left| \left[ \pi_{h,T}^{CR} v_{E}^s - \tilde{v}_{E}^\pm(x_T) \right] \right|^2 \leq \left| \left[ \pi_{h,T}^{CR} v_{E}^s - \tilde{v}_{E}^\pm(x_T) \right] \right|^2_{L^\infty(T)}
\]

\[
\leq C \sum_{s=+,-} \left| \left[ \pi_{h,T}^{CR} v_{E}^s - \tilde{v}_{E}^\pm(x_T) \right] \right|^2_{L^\infty(T)} \leq C h_T^2 \sum_{s=+,-} \|v_{E}^s\|^2_{H^2(T)}, \quad i = 1, 2, \tag{4.70}
\]

where we have used the standard interpolation error estimate in the last inequality; see Theorem 4.4.20 in [4]. To estimate \( \beta_3 \) and \( \beta_4 \), we use the following notations for simplicity

\[
\sigma_\pi^\pm := \sigma(\mu^\pm, \pi_{h,T}^{CR} v_{E}^\pm, \pi_{h,T}^{4,q} q_{E}^\pm), \quad \sigma^\pm := \sigma(\mu^\pm, v_{E}^\pm, q_{E}^\pm). \tag{4.71}
\]
Noticing $\sigma^2_{\pm} n_h$ are constant vectors, we derive from (4.53) that

$$\beta_i^2 = \| t_h T \sigma^2_{\pm} n_h \|^2_{L^2(\Omega)} \leq Ch^{-2} \| t_h T \sigma^2_{\pm} n_h \|^2_{L^2(\Omega)}$$

$$= Ch^{-2} \| t_h T \sigma^2_{\pm} n_h + (t_h T \sigma^2_{\pm} n_h) \|^2_{L^2(\Omega)}$$

$$= Ch^{-2} \| t_h T \sigma^2_{\pm} n_h + (t_h T - t_h T) \sigma^2_{\pm} n + t_h T \sigma^2_{\pm} (n_h - n) + t_h T \sigma^2_{\pm} n \|^2_{L^2(\Omega)}$$

$$\leq Ch^{-2} \| t_h T \|^2_{L^2(\Omega)} + \| t_h - t_h T \|^2_{L^2(\Omega)} \| \sigma^2_{\pm} n \|^2_{L^2(\Omega)}$$

$$+ \| n_h - n \|^2_{L^2(\Omega)} \| \sigma^2_{\pm} n \|^2_{L^2(\Omega)} + Ch^{-2} \| \sigma^2_{\pm} n \|^2_{L^2(\Omega)}.$$  
(4.72)

Analogously, we can estimate $\beta_i, \ i = 3, 5, 6, 7$ as

$$\beta_3^2 \leq C \sum_{s=\pm} \sum_{i=1}^2 \left( \| v_E^2_{H^2(T)} + |q_E^2_{H^1(T)} \| \|^2_{L^2(T)} \right) + Ch^{-2} \| \sigma^2_{\pm} (\mu^\pm, v_E^\pm, q_E^\pm) n \|^2_{L^2(T)}.$$  
(4.73)

Substituting (4.69) - (4.70), (4.72) - (4.73) into (4.68) and summing up over all interface elements, we get

$$\sum_{T \in T_h^I} \| \Pi_{h,T} v_E^\pm - \Pi_{h,F} v \|^2_{H^2(T)} \leq Ch^{-2m} \sum_{s=\pm} \left( \| v_E^2_{H^2(\Omega)} + |q_E^2_{H^1(\Omega)} \|^2_{L^2(\Omega)} \right)$$

$$+ Ch^{-2m} \left( \| \| \nabla \cdot v_E^\pm \|^2_{L^2(U(\Gamma,hT))} + \| \sigma^2_{\pm} (\mu^\pm, v_E^\pm, q_E^\pm) n \|^2_{L^2(U(\Gamma,hT))} \right)$$

$$\sum_{T \in T_h^I} \| \Pi_{h,T} q_E^\pm - \Pi_{h,F} q \|^2_{L^2(T)} \leq Ch^{-2} \sum_{s=\pm} \left( \| v_E^2_{H^2(\Omega)} + |q_E^2_{H^1(\Omega)} \|^2_{L^2(\Omega)} \right)$$

$$+ C \left( \| \| \nabla \cdot v_E^\pm \|^2_{L^2(U(\Gamma,hT))} + \| \sigma^2_{\pm} (\mu^\pm, v_E^\pm, q_E^\pm) n \|^2_{L^2(U(\Gamma,hT))} \right)$$  
(4.74)

where we have used the relation $T < U(\Gamma,hT)$ for all $T \in T_h^I$ from Assumption (4.1). Since $(v, q) \in H_2 H_1$, we know from the definition (2.3) that

$$[\sigma^2_{\pm} (\mu^\pm, v_E^\pm, q_E^\pm) n]_\Gamma = 0, \| \nabla \cdot v_E^\pm \|_\Gamma = 0, \| v_E^\pm \|_\Gamma = 0 \text{ (implying } \| \nabla v_E^\pm \|_\Gamma = 0).$$  
(4.75)

Thus, by Lemma (4.2)

$$\| \| \nabla \cdot v_E^\pm \|^2_{L^2(U(\Gamma,hT))} + \| \sigma^2_{\pm} (\mu^\pm, v_E^\pm, q_E^\pm) n \|^2_{L^2(U(\Gamma,hT))}$$

$$\| \| \nabla v_E^\pm \|^2_{L^2(U(\Gamma,hT))} \leq Ch^{-2} \sum_{s=\pm} \left( \| v_E^2_{H^2(\Omega)} + |q_E^2_{H^1(\Omega)} \|^2_{L^2(\Omega)} \right) .$$  
(4.76)
where we note that the constant $C$ also depends on the curvature of $\Gamma$. Substituting the above inequality into (4.74) and combining (4.66)-(4.67), we obtain

$$
\sum_{T \in T_h^M} |v^\pm - (\Pi_{v,q}^{IFE} \mathbf{v})|^2_{H^m(T)} \leq Ch^4 - 2m \sum_{s = \pm} \left( ||v^s||^2_{H^2(\Omega)} + ||q^s||^2_{H^1(\Omega)} \right),
$$

$$
\sum_{T \in T_h^M} |q^\pm - (\Pi_{v,q}^{IFE} q)|^2_2(T) \leq Ch^2 \sum_{s = \pm} \left( ||v^s||^2_{H^2(\Omega)} + ||q^s||^2_{H^1(\Omega)} \right),
$$

which together with the extension Lemma 4.3 leads to the desired estimates (4.64) and (4.65). □

Now we are ready to prove the optimal approximation capabilities of the IFE space $\mathbf{V} M_h^{IFE}$, where the error resulting from the mismatch of $\Gamma$ and $\Gamma_h$ is considered rigorously.

**Theorem 4.15.** For any $(v, q) \in H_2 \Omega_1$, there exists a positive constant $C$ independent of $h$ and the interface location relative to the mesh such that

$$
\sum_{T \in \mathcal{T}_h} |v - \Pi_{v,q}^{IFE} v|^2_{H^m(T)} \leq Ch^4 - 2m (||v||^2_{H^2(\Omega + \cup \Omega)} + ||q||^2_{H^1(\Omega + \cup \Omega)}), \quad m = 0, 1,
$$

$$
\sum_{T \in \mathcal{T}_h} |q - \Pi_{v,q}^{IFE} q|^2_2(T) \leq Ch^2 (||v||^2_{H^2(\Omega + \cup \Omega)} + ||q||^2_{H^1(\Omega + \cup \Omega)}).
$$

**Proof.** On each non-interface element $T \in \mathcal{T}_{h}^{non}$, it follows from (4.78)-(4.79) and (4.6) that

$$
|v - \Pi_{v,q}^{IFE} v|^2_{H^m(T)} = |v - \pi_{v,T}^{CR} v|^2_{H^m(T)} \leq Ch^4 - 2m |v|^2_{H^2(T)}, \quad m = 0, 1,
$$

$$
|q - \Pi_{v,q}^{IFE} q|^2_2(T) = ||q - \pi_{v,T}^{0} q||^2_2(T) \leq Ch^2 |q|^2_{H^1(T)}.
$$

On each interface element $T \in \mathcal{T}_{h}^{I}$, in view of the relations $T = T^+ \cup T^-$ and $T^s = (T^+ \cap T_h^+) \cup (T^s \cap T_h^s)$, $s = +, -, \pm$, we have

$$
|v - \Pi_{v,q}^{IFE} v|^2_{H^m(T)} = \sum_{s = \pm} |v^s - (\Pi_{v,q}^{IFE} v)^s|^2_{H^m(T^s \cap \mathcal{T}_h)} + |v^+ - (\Pi_{v,q}^{IFE} v)^+|^2_{H^m(T^+ \cap \mathcal{T}_h)} + |v^- - (\Pi_{v,q}^{IFE} v)^-|^2_{H^m(T^- \cap \mathcal{T}_h)}.
$$

By the triangle inequality, we further obtain

$$
|v^+ - (\Pi_{v,q}^{IFE} v)^+|^2_{H^m(T^+ \cap \mathcal{T}_h)} \leq 2|v^+ - \pi_{v,T}^{+} v|^2_{H^m(T^+ \cap \mathcal{T}_h)} + 2|v^+_E - (\Pi_{v,q}^{IFE} v)^+|^2_{H^m(T^+ \cap \mathcal{T}_h)},
$$

$$
|v^- - (\Pi_{v,q}^{IFE} v)^-|^2_{H^m(T^- \cap \mathcal{T}_h)} \leq 2|v^- - \pi_{v,T}^{-} v|^2_{H^m(T^- \cap \mathcal{T}_h)} + 2|v^-_E - (\Pi_{v,q}^{IFE} v)^-|^2_{H^m(T^- \cap \mathcal{T}_h)}.
$$

Substituting (4.81) into (4.80) and using the definition (4.3), we conclude, for all $T \in \mathcal{T}_h^I$,

$$
|v - \Pi_{v,q}^{IFE} v|^2_{H^m(T)} \leq C \sum_{s = \pm} ||v^s - (\Pi_{v,q}^{IFE} v)^s||^2_{H^m(T^s \cap \mathcal{T}_h)} + C ||v^\pm||^2_{H^m(T^\pm \cap \mathcal{T}_h)}, \quad m = 0, 1.
$$

Analogously, for all $T \in \mathcal{T}_h^I$, it holds

$$
||q - \Pi_{v,q}^{IFE} q||^2_2(T) \leq C \sum_{s = \pm} ||q^s - (\Pi_{v,q}^{IFE} q)^s||^2_2(T^s \cap \mathcal{T}_h) + C ||q^\pm||^2_2(T^\pm \cap \mathcal{T}_h).
$$

Combining (4.79), (4.82)-(4.83), (4.4) and Theorem 4.14, we arrive at

$$
\sum_{T \in \mathcal{T}_h} |v - \Pi_{v,q}^{IFE} v|^2_{H^m(T)} \leq Ch^4 - 2m (||v||^2_{H^2(\Omega + \cup \Omega)} + ||q||^2_{H^1(\Omega + \cup \Omega)} + C ||v^\pm||^2_{H^m(U(\Gamma, C h^2))}),
$$

$$
||q - \Pi_{v,q}^{IFE} q||^2_2(\Omega) \leq Ch^2 (||v||^2_{H^2(\Omega + \cup \Omega)} + ||q||^2_{H^1(\Omega + \cup \Omega)} + C ||q^\pm||^2_2(U(\Gamma, C h^2))).
$$

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Hence, \( \tilde{F} \) provides the following estimates
\[
\| \mathbf{v}^E_1 \|^2_{L^2(U(\Gamma,C\hat{h}^2))} \leq Ch^4 \| \mathbf{v}^E_1 \|^2_{H^1(U(\Gamma,C\hat{h}^2))} \leq Ch^4 \sum_{s=\pm} |\mathbf{v}^E_s|^2_{H^1(\Omega)},
\]
\[
\| \nabla \mathbf{v}^E_1 \|^2_{L^2(U(\Gamma,C\hat{h}^2))} \leq Ch^2 \| \nabla \mathbf{v}^E_1 \|^2_{H^1(\Omega)} \leq Ch^2 \sum_{s=\pm} |\mathbf{v}^E_s|^2_{H^2(\Omega)},
\]
\[
\| \mathbf{q}^E_1 \|^2_{L^2(U(\Gamma,C\hat{h}^2))} \leq Ch^2 \| \mathbf{q}^E_1 \|^2_{H^1(\Omega)} \leq Ch^2 \sum_{s=\pm} |\mathbf{q}^E_s|^2_{H^1(\Omega)},
\]
where the fact \( |\mathbf{v}^E_1|_\Gamma = 0 \) is used for proving the first inequality. Substituting the above inequalities into \((4.84)\), and using the extension Lemma \([13]\), we complete the proof of the theorem.

**Remark 4.16.** As shown in Remark \([10]\), the function \( \Pi^{IF,E}_{q} \) depends only on the velocity \( \mathbf{v} \), not on the pressure \( q \). Accordingly, we can remove the term \( \|q\|^2_{H^1(\Omega^+ \cup \Omega^-)} \) on the right-hand sides of the estimates \((4.62)\) and \((4.79)\) in Theorems \([4,14] \) and \([4,15] \). Indeed, given \( (\mathbf{v},q) \in H_2^2 H_1 \), we can construct a new function \( \tilde{q} \) such that
\[
(\mathbf{v}, \tilde{q}) \in H_2^2 H_1 \quad \text{and} \quad \|\tilde{q}\|_{H^1(\Omega^+ \cup \Omega^-)} \leq C \|\mathbf{v}\|_{H^2(\Omega^+ \cup \Omega^-)},
\]
which enables us to remove the term \( \|q\|^2_{H^1(\Omega^+ \cup \Omega^-)} \) in \((4.79)\) (similarly, in \((4.62)\)) as
\[
\sum_{T \in T_h} \| \mathbf{v} - \Pi^{IF,E}_{q,T} \mathbf{v} \|^2_{H^m(T)} = \sum_{T \in T_h} \| \mathbf{v} - \Pi^{IF,E}_{\tilde{q},T} \mathbf{v} \|^2_{H^m(T)} \leq Ch^{2m} \|\mathbf{v}\|^2_{H(\Omega^+ \cup \Omega^-)} \leq Ch^{2m} \|\mathbf{v}\|^2_{H(\Omega^+ \cup \Omega^-)}.
\]

The function \( \tilde{q} \) is constructed as follows. Define \( \tilde{q}_{|\Omega^\pm} := \tilde{q}^\pm \) with \( \tilde{q}^\pm \) satisfying
\[
\tilde{q}^+ = 0 \quad \text{and} \quad \Delta \tilde{q}^- = 0 \quad \text{in} \quad \Omega^-, \quad \tilde{q}^-|_\Gamma = -[\sigma(\mathbf{v}^\pm, \mathbf{E}^E, 0)]|_\Gamma.
\]

It is easy to verify that the condition \((4.85)\) is satisfied.

## 5 Analysis of the IFE method

For all \( (\mathbf{v},q) \in (\mathbf{V},M) + VM^{IF,E}_{h,0} \), we introduce the following mesh dependent norms
\[
\|\mathbf{v}\|^2_{1,h} := \sum_{T \in T_h} |\mathbf{v}|^2_{H^1(T)}, \quad \|\mathbf{v}\|^2_{2,h} := \sum_{T \in T_h} \|\nabla \mathbf{v}\|^2_{L^2(T)} + \sum_{e \in E_h} \frac{1}{|e|} \|\mathbf{v}|_e|^2_{L^2(e)},
\]
\[
\|\mathbf{v}\|^2_{2,h} := \|\mathbf{v}\|^2_{1,h} + \sum_{e \in E_h^T} |\mathbf{v}|_e^2 \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2}
\]
\[
\|q\|^2_{2,h} := \|q\|^2_{2,h} + \sum_{e \in E_h^T} |\mathbf{v}|_e^2 \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2} \sum_{e \in E_h^T} ^{2}
\]
\[
\|\mathbf{v},q\|^2 := \|\mathbf{v}\|^2_{1,h} + \|q\|^2_{2,h} + J_h(q,q), \quad \|\mathbf{v},q\|^2 := \|\mathbf{v}\|^2_{1,h} + \|q\|^2_{2,h} + J_h(q,q).
\]

As \( \mathbf{v}|_T \in H^1(T)^2 \) for all \( T \in T_h \) from \([3,4]\), \( \int_T |\mathbf{v}|_e = 0 \) for all \( e \in E_h \) and \( \int_{\Omega} \mathbf{v} = 0 \), the Poincaré-Friedrichs inequality for piecewise \( H^1 \) functions (see \([5]\)) and the Korn inequality for piecewise \( H^1 \) vector functions (see \([6]\)) imply
\[
\|\mathbf{v}\|^2_{L^2(\Omega)} \leq C \sum_{T \in T_h} |\mathbf{v}|^2_{H^1(T)}, \quad \sum_{T \in T_h} \|\mathbf{v}\|^2_{H^1(T)} \leq C \|\mathbf{e}(\mathbf{v})\|^2_{L^2(\Omega)} + C \sum_{e \in E_h} |\mathbf{e}|^{-1} \|\mathbf{v}|_e|^2_{L^2(e)}. \quad (5.2)
\]

Hence, \( \| \cdot \| \) and \( \| \cdot \|_* \) are indeed norms for the space \((\mathbf{V},M) + VM^{IF,E}_{h,0}\).
5.1 Boundedness and coercivity

It follows from the Cauchy-Schwarz inequality that the bilinear forms $a_h(\cdot, \cdot)$ and $b_h(\cdot, \cdot)$ are bounded, i.e.,

$$a_h(u, v) \leq ||u||_{*, h} ||v||_{*, h} \quad \text{and} \quad b_h(v, q) \leq C_b ||v||_{1, h} ||q||_{*, p}. \quad (5.3)$$

where $C_b$ is a constant independent of $h$ and the interface location relative to the mesh. Furthermore, by the definitions (3.11) and (5.1) we have the following lemma.

**Lemma 5.1.** For all $(u, p)$ and $(v, q)$ belonging to $(V, M) + VM_h^{IFE}$, it holds

$$A(u, p; v, q) \leq C_A ||(u, p)||_{*} ||(v, q)||_{*}, \quad (5.4)$$

where $C_A$ is a positive constant independent of $h$ and the interface location relative to the mesh.

To prove the coercivity of the bilinear form $a_h(\cdot, \cdot)$, we need a trace inequality for IFE functions. For all $(v_h, q_h) \in VM_h^{IFE}(T)$ on an interface element $T \in \mathcal{T}_h$, since $v_h \in H^1(T)^2$, we have the standard trace inequality

$$||v_h||_{L^2(\partial T)} \leq C(h^{-1/2} ||v_h||_{L^2(T)} + h^{1/2} ||\nabla v_h||_{L^2(T)}). \quad (5.5)$$

However, the standard trace inequality cannot be applied to $\nabla v_h$ directly since the components of $\nabla v_h$ no longer belong to $H^1(T)$. We establish the trace inequality for IFE functions in the following lemma.

**Lemma 5.2.** For any interface element $T \in \mathcal{T}_h$, there exists a positive constant $C$ independent of $h_T$ and the interface location relative to the mesh such that

$$||\nabla v_h||_{L^2(\partial T)} \leq C h^{-1/2} ||\nabla v_h||_{L^2(T)} \quad \forall (v_h, q_h) \in VM_h^{IFE}(T). \quad (5.6)$$

**Proof.** From Lemma 4.7 and the definition (4.20), we know $v_h = \pi_{h, T}^{CR} v_h + c_2(w - \pi_{h, T}^{CR} w)t_h$ with $w$ and $c_2$ defined in (4.21) and (4.37), respectively. Using $\pi_{h, T}^{CR} v_h \in P_1(T)^2$, $\pi_{h, T}^{CR} w \in P_1(T)$, $|\nabla w^+| = 1$, and (4.42), we have

$$||\nabla v_h||_{L^2(\partial T)} \leq ||\nabla \pi_{h, T}^{CR} v_h||_{L^2(\partial T)} + |c_2| (||\nabla \pi_{h, T}^{CR} w||_{L^2(\partial T)} + ||\nabla w^+||_{L^2(\partial T)})$$

$$\leq C h^{-1/2} ||\nabla \pi_{h, T}^{CR} v_h||_{L^2(T)} + C|c_2|h^{1/2}. \quad (5.7)$$

From (4.37) and (4.38) the constant $|c_2|$ can be estimated as

$$|c_2| = \frac{\sigma(\mu^-/\mu^+ - 1, \pi_{h, T}^{CR} v_h, 0) n_h \cdot t_h}{1 + (\mu^-/\mu^+ - 1)|\nabla \pi_{h, T}^{CR} w |} \leq C |\nabla \pi_{h, T}^{CR} v_h|.$$  

Combining the above inequalities, we get

$$||\nabla v_h||_{L^2(\partial T)} \leq C h^{-1/2} ||\nabla \pi_{h, T}^{CR} v_h||_{L^2(T)}. \quad (5.7)$$

Let $e_i, i = 1, 2, 3$ be edges of $T$ and $v_h = (v_{h1}, v_{h2})^T$, then $\pi_{h, T}^{CR} v_h = (\pi_{h, T}^{CR} v_{h1}, \pi_{h, T}^{CR} v_{h2})^T$. By choosing a constant $c_T = |T|^{-1} \int_T \pi_{h, T}^{CR} v_h$, we can derive

$$||\nabla \pi_{h, T}^{CR} v_{h1}||_{L^2(T)} = ||\nabla \pi_{h, T}^{CR} (v_{h1} - c_T)||_{L^2(T)} \leq \sum_{i=1}^{3} \frac{1}{|e_i|} \left| \int_{e_i} (v_{h1} - c_T) \right| \lambda_i |H^1(T)|$$

$$\leq C \sum_{i=1}^{3} h_T^{-1/2} ||v_{h1} - c_T||_{L^2(e_i)} \leq C \left( h_T^{-1} ||v_{h1} - c_T||_{L^2(T)} + ||v_{h1}||_{H^1(T)} \right)$$

$$\leq C |v_{h1}|_{H^1(T)},$$

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which together with a similar estimate for $\pi_{h,T}^{CR}v_h$ implies
\[
\|\nabla \pi_{h,T}^{CR}v_h\|_{L^2(T)} \leq C\|\nabla v_h\|_{L^2(T)}.
\] (5.8)

Substituting this result into (5.7), we complete the proof of the lemma.

□

The coercivity of $a_h(\cdot, \cdot)$ is given in the following lemma.

**Lemma 5.3.** There exists a positive constant $C_a$ independent of $h$ and the interface location relative to the mesh such that
\[
a_h(v_h, v_h) \geq C_a\|v_h\|_{H^1}^2, \quad \forall (v_h, q_h) \in V_{h,0}^{IF E}
\] (5.9)
is true for $\delta = -1$ with an arbitrary $\eta \geq 0$ and is true for $\delta = 1$ with a sufficiently large $\eta$.

**Proof.** From (5.6), the Cauchy-Schwarz inequality and the relation $|\epsilon\| \leq C|\nabla \epsilon|$, we obtain
\[
\sum_{e \in \mathcal{E}_h} \int_e \{2\mu_h \epsilon(v_h) \mathbf{n}_e \cdot [v_h] e \} \cdot [v_h] e \leq C \left( \sum_{e \in \mathcal{E}_h} |\epsilon| \|\nabla \epsilon\|_{L^2(e)} \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} |\epsilon|^{-1} \|v_h\|_{L^2(e)}^2 \right)^{1/2}
\leq C_1 \left( \sum_{T \in \mathcal{T}_h} \|v_h\|_{H^1(T)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} |\epsilon|^{-1} \|v_h\|_{L^2(e)}^2 \right)^{1/2}
\leq \frac{\varepsilon C_1}{2} \sum_{T \in \mathcal{T}_h} \|v_h\|_{H^1(T)}^2 + \frac{1}{2\varepsilon} \sum_{e \in \mathcal{E}_h} |\epsilon|^{-1} \|v_h\|_{L^2(e)}^2,
\]
where the positive constant $C_1$ is independent of $h$ and the interface location relative to the mesh. By the second inequality in (5.2), there is another constant $C_2$ independent of $h$ and the interface location relative to the mesh such that
\[
\sum_{T \in \mathcal{T}_h} \int_T 2\mu_h \epsilon(v_h) : \epsilon(v_h) + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e [v_h] e \cdot [v_h] e \geq C_2 \sum_{T \in \mathcal{T}_h} \|v_h\|_{H^1(T)}^2.
\] (5.10)

It then follows from (3.11) that, for $\delta = 1$,
\[
a_h(v_h, v_h) = (C_2 - \varepsilon C_1) \sum_{T \in \mathcal{T}_h} \|v_h\|_{H^1(T)}^2 + (\eta - \varepsilon^{-1}) \sum_{e \in \mathcal{E}_h} |\epsilon|^{-1} \|v_h\|_{L^2(e)}^2,
\]
which implies the coercivity (5.9) with $C_a = 2^{-1}C_2$ when choosing $\eta \geq \varepsilon = C_2(2C_1)^{-1}$. And for $\delta = -1$, the result (5.9) is a direct consequence of (5.10). □

## 5.2 Norm-equivalence for IFE functions

In this subsection we show that the two norms $\| \cdot \|$ and $\| \cdot \|_*$ are equivalent for the coupled IFE functions. First we need the following lemma.

**Lemma 5.4.** For all $e \in \mathcal{E}_h$, let $\mathcal{T}_e^h$ be the set of all elements in $\mathcal{T}_h$ having $e$ as an edge, then there exists a positive constant $C$ independent of $h$ and the interface location relative to the mesh such that, for all $(v_h, q_h) \in V_{h,0}^{IF E},$
\[
|\epsilon|^{-1} \|v_h\|_{L^2(e)}^2 \leq C \sum_{T \in \mathcal{T}_e^h} |v_h|_{H^1(T)}^2 \quad \forall e \in \mathcal{E}_h,
\] (5.11)
\[
|\epsilon| \|q_h\|_{e}^2 \leq C \sum_{T \in \mathcal{T}_h} \left( |v_h|_{H^1(T)}^2 + \|q_h\|_{L^2(T)}^2 \right) \quad \forall e \in \mathcal{E}_h^T.
\] (5.12)
Proof. If \( e \in \mathcal{E}_h^\text{non} \) and \( T_h^e \subset T_h^\text{non} \), the proof of (5.11) is standard. If \( e \in \mathcal{E}_h^\text{non} \) and \( T_h^e \cap T_h^\text{f} \neq \emptyset \), or \( e \in \mathcal{E}_h^\text{f} \), noticing that \( v_h|_T \in H^1(T)^2 \) for all \( T \in T_h^\text{f} \) from (3.4), we can prove (5.11) analogously; see Lemma 4.2 in [24].

Next, we prove (5.12). For an interface element \( T \in T_h^\text{f} \), from Lemma 4.7 the pressure can be written as
\[
q_h = \pi_h^0 q_h + c_1(z - \pi_h^0 z) \quad \text{with} \quad c_1 = \sigma(\mu^- - \mu^+, \pi_{h,T}^{CR} v_h, 0)n_h \cdot n_h, \tag{5.13}
\]
where \( z \) is defined in (4.21). Let \( e \) be an edge of \( T \). It follows from (4.42) and (5.8) that
\[
|e||q_h|^2 \leq C \left( |T|^{-\frac{1}{2}} \left( \int_T q_h \right)^2 + |\pi_{h,T}^{CR} v_h|^2_{H^1(T)} \right) \leq C \left( |v_h|_{H^1(T)}^2 + ||q_h||_{L^2(T)}^2 \right), \tag{5.14}
\]
which implies the estimate (5.12).

We now prove the norm-equivalence in the following lemma.

**Lemma 5.5.** There exists a positive constant \( C_0 \) independent of \( h \) and the interface location relative to the mesh such that, for all \((v_h, q_h) \in VM_{h,0}^{IF} \),
\[
||v_h||_{1,h} \leq ||v_h||_{*,h} \leq C_0 ||v_h||_{1,h} \tag{5.15}
\]
and correspondingly,
\[
||(v_h, q_h)||_{*,h} \leq C_0 ||(v_h, q_h)|| \tag{5.16}
\]

*Proof.* The result (5.15) is obtained by using (5.1), (5.6), (5.11) and the relation \(|\varepsilon(v)| \leq C|\nabla v|\). Combining (5.13), (5.12) and the definitions in (5.1), we proved (5.10).

### 5.3 The inf-sup stability

In order to prove the stability, we first need to bound the jumps of IFE pressures on interface elements by the coupled velocity.

**Lemma 5.6.** For any \( T \in T_h^\text{f} \), there exists a positive constant \( C \) independent of \( h_T \) and the interface location relative to the mesh such that
\[
h_T ||[q_h^+]|_{L^2(\Gamma_h,T)}|^2 \leq C |v_h|_{H^1(T)}^2 \quad \forall (v_h, q_h) \in VM_{h}^{IF}(T). \tag{5.17}
\]

*Proof.* Noticing that \([q^{j_1,j_2}] = -1 \) and \( q^{d_0} \) is a constant, Lemma 4.7 gives
\[
[q_h^+] = -\sigma(\mu^- - \mu^+, \pi_{h,T}^{CR} v_h, 0)n_h \cdot n_h \quad \forall x \in T.
\]
It follows from (5.3) that
\[
h_T ||[q_h^+]|_{L^2(\Gamma_h,T)}|^2 \leq Ch_T|\Gamma_h,T||\nabla \pi_{h,T}^{CR} v_h|^2 \leq C ||\nabla \pi_{h,T}^{CR} v_h||_{L^2(T)}^2 \leq C ||\nabla v_h||_{L^2(T)}^2,
\]
which completes the proof.

We also need the stability of the IFE interpolation and some interpolation error estimates under the \( H^1 \)-regularity.
Lemma 5.7. For any $v \in H^1(T)^2$, there exists a positive constant $C$ independent of $h$ and the interface location relative to the mesh such that

$$\|\Pi_{v,q}^{IF} v\|_{H^1(T)} \leq C|v|_{H^1(T)} \quad \forall T \in T_h^\Gamma,$$

$$\|v - \Pi_{v,q}^{IF} v\|_{L^2(T)} \leq C h_T \|v\|_{H^1(T)}, \quad |v - \Pi_{v,q}^{IF} v|_{H^1(T)} \leq C|v|_{H^1(T)} \quad \forall T \in T_h^\Gamma,$$

(5.18)

where $\Pi_{v,q}^{IF} v$ is independent of $q$; see Remark 4.10.

Proof. On an interface element $T \in T_h^\Gamma$, it follows from Lemma 4.7 and Remark 4.10 that

$$\Pi_{v,q}^{IF} v = \pi_{h,T}^{CR} v + c_2 (w - \pi_{h,T}^{CR} w) t_h \quad \text{with} \quad c_2 = \frac{\sigma (\mu^- / \mu^+ - 1, \pi_{h,T}^{CR} v, 0) n_h \cdot t_h}{1 + (\mu^- / \mu^+ - 1) \nabla \pi_{h,T}^{CR} w \cdot n_h},$$

where $w$ is defined in (4.21). Similar to the proof of Lemma 5.2, we have

$$|c_2| \leq C|\pi_{h,T}^{CR} v|, \quad |\nabla w| = 1, \quad w = 0, \quad |\pi_{h,T}^{CR} w|_{W^{m,q}_2(T)} \leq Ch_T^{1-m}, \quad |\pi_{h,T}^{CR} v|_{H^1(T)} \leq |v|_{H^1(T)}.$$

The result (5.18) then is obtained from

$$\|\Pi_{v,q}^{IF} v\|_{H^1(T)} \leq \|\pi_{h,T}^{CR} v\|_{H^1(T)} + |c_2| \left( \|w\|_{H^1(T)} + |\pi_{h,T}^{CR} w|_{H^1(T)} \right) \leq \|\pi_{h,T}^{CR} v\|_{H^1(T)} + Ch_T \|\pi_{h,T}^{CR} v\|_{H^1(T)} \leq C|\pi_{h,T}^{CR} v|_{H^1(T)} \leq C|v|_{H^1(T)}.$$

From the definition (4.21), it is easy to verify $\|w\|_{L^2(T)} \leq Ch_T^2$. Therefore,

$$\|v - \Pi_{v,q}^{IF} v\|_{L^2(T)} \leq \|v - \pi_{h,T}^{CR} v\|_{L^2(T)} + |c_2| \left( \|w\|_{L^2(T)} + \|\pi_{h,T}^{CR} w\|_{L^2(T)} \right) \leq Ch_T \|v\|_{H^1(T)} + Ch_T^2 \|\pi_{h,T}^{CR} v\|_{H^1(T)} \leq Ch_T \|v\|_{H^1(T)},$$

(5.20)

which proves the first estimate of (5.19). The second estimate of (5.19) can be easily obtained by (5.18) and the triangle inequality. \hfill \Box

With these preparations, we now prove the inf-sup stability of the proposed IFE method.

Lemma 5.8. There exist a positive constant $C_3$ independent of $h$ and the interface location relative to the mesh such that, for all $(v_h, q_h) \in VM_{h,0}^{IF}$,

$$C_3 \|q_h\|_{L^2(\Omega)} \leq \sup_{(\tilde{v}_h, \tilde{q}_h) \in VM_{h,0}^{IF}} \frac{b_h(\tilde{v}_h, q_h)}{\|\tilde{v}_h\|_{L^1(h)} + \left( \sum_{T \in T_h^\Gamma} |v_h|^2_{H^1(T)} \right)^{\frac{1}{2}}} + J_h^\frac{3}{2}(q_h, q_h).$$

(5.21)

Proof. Let $(v_h, q_h) \in VM_{h,0}^{IF}$. Since $q_h$ also belongs to the space $M$, there is a function $\tilde{v} \in V$ satisfying (see Lemma 11.2.3 in [3])

$$\nabla \cdot \tilde{v} = q_h \quad \text{and} \quad \|\tilde{v}\|_{H^1(\Omega)} \leq C\|q_h\|_{L^2(\Omega)}$$

with a constant $C$ only depends on $\Omega$. Applying the integration by parts, we derive

$$\|q_h\|_{L^2(\Omega)}^2 = \int_\Omega q_h \nabla \cdot \tilde{v} = \sum_{e \in T_h} \int_{\partial e} [q_h]_e \tilde{v} \cdot n_e - \sum_{T \in T_h^\Gamma} \int_{T_h^T} [q_h]^2 \tilde{v} \cdot n_h.$$

(5.22)
Since the IFE interpolation function $\Pi_{\tilde{q}}^{IFE}\tilde{v}$ is continuous on the whole element $T$ and independent of the pressure $\tilde{q}$ (see Remark 4.10), we apply the integration by parts again to get

$$
\begin{align*}
\int_{\Gamma_{h,T}} e \cdot \nabla_{T} \cdot \Pi_{\tilde{q}}^{IFE}\tilde{v} &= \sum_{T \in \mathcal{T}_h} \int_{\partial T} \tilde{q}_h \cdot \frac{\partial \Pi_{\tilde{q}}^{IFE}\tilde{v}}{\partial n^e} - \int_{\partial T} \Pi_{\tilde{q}}^{IFE}\tilde{v} \cdot \nu_e \\
&= -\sum_{e \in E_h} \int_{\partial e} \{h\}_e \{\Pi_{\tilde{q}}^{IFE}\tilde{v}\}_e \cdot n_e + \int_{\Omega} \{\Pi_{\tilde{q}}^{IFE}\tilde{v}\}_e \cdot n_e,
\end{align*}
$$

where we have used the facts that $\int_{\partial e} \{\Pi_{\tilde{q}}^{IFE}\tilde{v}\}_e \cdot n_e = 0$ for all $e \in E_h$ and $\{h\}_e$ is a constant for all $e \in E_h$. Combining (5.22) and using the facts that $[q_h]_e$ is a constant for all $e \in E_h$ and $\int_{\partial e} (\tilde{v} - \Pi_{\tilde{q}}^{IFE}\tilde{v})_e \cdot n_e = 0$ for all $e \in E_h$ with $e \in \partial T$, we further have

$$
\begin{align*}
\|q_h\|_{L^2(\Omega)} &= -b_h(\Pi_{\tilde{q}}^{IFE}\tilde{v}, q_h) + \left(b_h(\Pi_{\tilde{q}}^{IFE}\tilde{v}, q_h) + \int_{\Omega} q_h \nabla \cdot \tilde{v}\right) \\
&= -b_h(\Pi_{\tilde{q}}^{IFE}\tilde{v}, q_h) + \sum_{e \in E_h} \int_{\partial e} [q_h]_e \{\nabla - \Pi_{\tilde{q}}^{IFE}\tilde{v}\}_e \cdot n_e - \sum_{T \in \mathcal{T}_h} \int_{\partial T} [q_h]_e \{\nabla - \Pi_{\tilde{q}}^{IFE}\tilde{v}\}_e \cdot n_e,
\end{align*}
$$

(5.24)

It follows from (5.10) that

$$
\begin{align*}
|I_1| &= \frac{b_h(\Pi_{\tilde{q}}^{IFE}\tilde{v}, q_h)}{\|\Pi_{\tilde{q}}^{IFE}\tilde{v}\|_{1,h}} \|\tilde{v}\|_{1,h} \leq \left(\sup_{(\tilde{v}_h, q_h) \in V_{h}^{IFE}} \frac{b_h(\tilde{v}_h, q_h)}{\|\tilde{v}_h\|_{1,h}}\right) C|\tilde{v}|_{H^1(\Omega)} \\
&\leq C \left(\sup_{(\tilde{v}_h, q_h) \in V_{h}^{IFE}} \frac{b_h(\tilde{v}_h, q_h)}{\|\tilde{v}_h\|_{1,h}}\right) \|q_h\|_{L^2(\Omega)}.
\end{align*}
$$

(5.25)

Since $\|\Pi_{\tilde{q}}^{IFE}\tilde{v}\|_{T} \in H^1(T)$ for all $T \in \mathcal{T}_h$, we use the standard trace inequality and the interpolation estimates (5.11) to get

$$
\begin{align*}
|I_2| &\leq \left(\sum_{e \in E_h^I} |e| \|h\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \left(\sum_{e \in E_h^I} |e|^{-1} \|\tilde{v} - \Pi_{\tilde{q}}^{IFE}\tilde{v}\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\
&\leq C J_h^{\tilde{T}}(h, q_h) \left(\sum_{T \in \mathcal{T}_h^I} \frac{h_T^{-1}}{\|\tilde{v} - \Pi_{\tilde{q}}^{IFE}\tilde{v}\|_{L^2(T)}^2 + |\tilde{v} - \Pi_{\tilde{q}}^{IFE}\tilde{v}|_{H^1(T)}^2}\right)^{\frac{1}{2}} \\
&\leq C J_h^{\tilde{T}}(h, q_h) |\tilde{v}|_{H^1(\Omega)} \leq C J_h^{\tilde{T}}(h, q_h) \|q_h\|_{L^2(\Omega)}.
\end{align*}
$$

(5.26)

Similarly, by (5.17) and a well-known trace inequality on interface elements; see Lemma 3 in [18] or
Lemma 3.1 in [38], we can bound the third term by

\[
|I_3| \leq \left( \sum_{T \in T_h^0} h_T \| q_h \|_{L^2(I_h, T)}^2 \right)^{1/2} \left( \sum_{T \in T_h^0} h_T^{1/2} \left\| \overline{\nabla} - \Pi_{V_{\tilde{q}}^{h}} \overline{\nabla}\right\|_{L^2(I_h, T)}^2 \right)^{1/2}.
\]

Since

\[
\left\| \overline{\nabla} \right\|_{L^2(\Omega)} = 0 \quad \text{for all} \quad \overline{\nabla} \in V_{\tilde{q}}^{h}.
\]

Thus, we let

\[
C_3 \| q_h \|_{L^2(\Omega)}^2 \leq b_h(\overline{\nabla}^h, q_h) + \left( \sum_{T \in T_h^0} |\nabla|^2_{H^1(T)} \right)^{1/2} \| q_h \|_{L^2(\Omega)} + J_h^Q(q_h, q_h) \| q_h \|_{L^2(\Omega)}.
\]

Substituting (5.26)–(5.27) into (5.24) we conclude the proof. \(\square\)

Theorem 5.9. There exists a positive constant \(C_*\) independent of \(h\) and the interface location relative to the mesh such that

\[
C_* \| (v_h, q_h) \| \leq \sup_{(w_h, r_h) \in VM_{h, 0}^{1 FE}} \frac{A_h(v_h, q_h; w_h, r_h)}{\| (w_h, r_h) \|} \quad \forall (v_h, q_h) \in VM_{h, 0}^{1 FE}. \tag{5.28}
\]

Proof. Let \((v_h, q_h) \in VM_{h, 0}^{1 FE}\). Since \(VM_{h, 0}^{1 FE}\) is a finite-dimensional space, we assume that the supremum in (5.21) is achieved at \((\overline{\nabla}^h, q_h^h) \in VM_{h, 0}^{1 FE}\), i.e.,

\[
\sup_{(\overline{\nabla}^h, q_h^h) \in VM_{h, 0}^{1 FE}} \frac{b_h(\overline{\nabla}^h, q_h^h)}{\| \overline{\nabla}^h \|_{1, h} \| q_h^h \|_{L^2(\Omega)}} = \frac{b_h(\overline{\nabla}^h, q_h^h)}{\| q_h \|_{L^2(\Omega)}} \quad \text{with} \quad k = \frac{\| q_h \|_{L^2(\Omega)}}{\| \overline{\nabla}^h \|_{1, h}}. \tag{5.29}
\]

Here the function \(q_h^h\) is not unique and will be specified latter. Therefore, (5.21) becomes

\[
C_3 \| q_h \|_{L^2(\Omega)}^2 \leq b_h(\overline{\nabla}^h, q_h) + \left( \sum_{T \in T_h^0} |\nabla|^2_{H^1(T)} \right)^{1/2} \| q_h \|_{L^2(\Omega)} + J_h^Q(q_h, q_h) \| q_h \|_{L^2(\Omega)}. \tag{5.30}
\]

Before continuing, we discuss some properties of the coupled functions \(\overline{\nabla}^h\) and \(q_h^h\). From Lemma 4.7 we know that \(N_{T, T}(\overline{\nabla}^h, \overline{q}_h^h)\) does not affect the function \(\overline{\nabla}^h\). Thus, we let \(N_{T, T}(\overline{\nabla}^h, \overline{q}_h^h) = 0\) for all \(T \in T_h\). Obviously, \(\overline{q}_h^h|_{T} = 0\) for all \(T \in T_h^{\text{nom}}\). On an interface element \(T \in T_h^I\), it follows from (4.36)–(4.37) that

\[
\overline{q}_h^h|_{T} = (\sigma(\mu^+ - \mu^-), \pi_{h, T}^{CR} \overline{\nabla}^h, 0) \| n_h \cdot n_h \| q_{j^I}^h
\]

with \(q_{j^I}^h\) defined in (4.20). Let \(e\) be an edge of \(T\). From (5.8) we can derive

\[
\| e \|^2_{1, e} + \| \tilde{q}_h\|^2_{L^2(T)} \leq C h_T^2 \| \nabla \pi_{h, T}^{CR} \overline{\nabla}^h\|_{L^2(T)} \leq C \| \nabla \pi_{h, T}^{CR} \overline{\nabla}^h\|_{L^2(T)} \leq C L^2(\Omega)\overline{\nabla}^h_{1, h}
\]

Thus, there exists a constant \(C_*\) independent of \(h\) and the interface location relative to the mesh such that

\[
\| \overline{\nabla}^h \|_{1, h} \leq C_* \| \overline{\nabla}^h \|_{1, h} \quad \text{and} \quad J_h^Q(\overline{\nabla}^h, \overline{q}_h^h) \leq C_* \| \overline{\nabla}^h \|_{1, h}, \tag{5.31}
\]

which mean that \(\overline{q}_h^h\) can be controlled by \(\overline{\nabla}^h\) in a proper norm.

Now we estimate the first term on the right-hand side of (5.30). From (3.11), (3.3), (5.15), (5.31) and (5.31), we have

\[
b_h(\overline{\nabla}^h, q_h) = A_h(v_h, q_h; k\overline{\nabla}^h, k\overline{q}_h^h) - a_h(v_h, k\overline{\nabla}^h) + b_h(v_h, k\overline{q}_h^h) - J_h(q_h, k\overline{q}_h^h)
\]

\[
\leq A_h(v_h, q_h; k\overline{\nabla}^h, k\overline{q}_h^h) + \| v_h \|_{1, h} \| k\overline{\nabla}^h \|_{1, h} + C_h \| v_h \|_{1, h} \| k\overline{q}_h^h \|_{1, h} + J_h^Q(q_h, q_h) J_h^Q(k\overline{q}_h^h, k\overline{q}_h^h)
\]

\[
\leq A_h(v_h, q_h; k\overline{\nabla}^h, k\overline{q}_h^h) + C_h \| v_h \|_{1, h} \| k\overline{\nabla}^h \|_{1, h} + C_* \left( C_h \| v_h \|_{1, h} + J_h^Q(q_h, q_h) \right) \| k\overline{\nabla}^h \|_{1, h}.
\]
Substituting the above inequality into (5.30), and using the arithmetic-geometric mean inequality: 

$$ab \leq \frac{a^2}{C_3} + \frac{b^2}{C_3}$$

and the fact \(\|k\vec{v}_h^*\|_{1,h} = \|q_h\|_{L^2(\Omega)}\) from (5.29), we further have

\[
C_3\|q_h\|^2_{L^2(\Omega)} \leq A_h(v_h, q_h; k\vec{v}_h^*, k\vec{q}_h^*) + \frac{2C_4}{C_3}\|v_h\|^2_{1,h} + \frac{C_3}{8}\|q_h\|^2_{L^2(\Omega)}
\]

\[
+ \frac{2C_4^2}{C_3}\|v_h\|^2_{1,h} + \frac{C_3}{8}\|q_h\|^2_{L^2(\Omega)} + \frac{2C_4^2}{C_3}\|v_h\|^2_{1,h} + \frac{C_3}{8}\|q_h\|^2_{L^2(\Omega)}
\]

which leads to

\[
\frac{3C_3}{8}\|q_h\|^2_{L^2(\Omega)} \leq A_h(v_h, q_h; k\vec{v}_h^*, k\vec{q}_h^*) + \frac{2C_4^2}{C_3}\|v_h\|^2_{1,h} + \frac{2C_4^2}{C_3}\|v_h\|^2_{1,h} + \frac{2C_4^2 + 2}{C_3}\|J_h(q_h, q_h)\).
\]

On the other hand, by Lemma 5.3 and the definition (3.11), we know

\[J_h(q_h, q_h) + C_a\|v_h\|^2_{1,h} \leq A_h(v_h, q_h; v_h, q_h).
\]

Combining this with (5.32) we get

\[
C_4\|(v_h, q_h)\|^2 = C_4 \left( J_h(q_h, q_h) + \|v_h\|^2_{1,h} + \|q\|^2_{L^2(\Omega)} \right)
\]

\[
\leq A_h(v_h, q_h; v_h + \theta k\vec{v}_h^*, q_h + \theta k\vec{q}_h^*)
\]

with

\[
\theta = \min \left( \frac{C_3C_a}{2(2C_0^2 + 2C_2^2C_3^2 + 2)}, \frac{C_3}{2(2C_0^2 + 2)} \right) \text{ and } C_4 = \min \left( \frac{3C_3\theta}{8}, \frac{1}{2}, \frac{C_a}{2} \right).
\]

Since \((v_h, q_h) \in V M_{h,0}^{IFE}\) and \((k\vec{v}_h^*, k\vec{q}_h^*) \in V M_{h,0}^{IFE}\), it holds

\[(v_h + \theta k\vec{v}_h^*, q_h + \theta k\vec{q}_h^*) = (v_h, q_h) + \theta(k\vec{v}_h^*, k\vec{q}_h^*) \in V M_{h,0}^{IFE}.
\]

By (5.31) and the fact \(\|k\vec{v}_h^*\|_{1,h} = \|q_h\|_{L^2(\Omega)}\) from (5.29), we see

\[
\|k\vec{v}_h^*\|_{1,h} + \|k\vec{q}_h^*\|_{L^2(\Omega)} + J_h^2(k\vec{v}_h^*, k\vec{q}_h^*) \leq (2C_2^2 + 1)\|k\vec{v}_h^*\|_{1,h} = (2C_2^2 + 1)\|q_h\|_{L^2(\Omega)},
\]

which leads to

\[
\|(k\vec{v}_h^*, k\vec{q}_h^*)\| \leq \sqrt{3}\|(2C_2^2 + 1)\|(v_h, q_h)\|.
\]

Therefore, we have

\[
\|(v_h + \theta k\vec{v}_h^*, q_h + \theta k\vec{q}_h^*)\| \leq \|(v_h, q_h)\| + \theta\|(k\vec{v}_h^*, k\vec{q}_h^*)\| \leq \left(1 + \sqrt{3}(2C_2^2 + 1)\right)\|(v_h, q_h)\|.
\]

Combining (5.33)-(5.35) yields the desired result (5.29) with

\[
C_s = \left(1 + \sqrt{3}(2C_2^2 + 1)\right)^{-1} C_4 > 0
\]

which is independent of \(h\) and the interface location relative to the mesh.

\[\square\]

As a consequence of Theorem 5.5, the discrete problem (5.10) is well-posed; see [7] for example.
5.4 A priori error estimates

We first derive an optimal estimate for the IFE interpolation error in terms of the norm \( \| \cdot \|_* \).

**Lemma 5.10.** Suppose \((v, q) \in \tilde{H}_2 H_1\), then there exists a constant \( C \) independent of \( h \) and the interface location relative to the mesh such that

\[
\| (v, q) - \Pi_I^{IF}(v, q) \|_* \leq Ch\|v\|_{H^2(\Omega^+ \cup \Omega^-)} + \| q \|_{H^1(\Omega^+ \cup \Omega^-)}.
\]

**Proof.** Since \((\Pi_{v,q}^{IF}) T \in H^1(T)^2\) for all \( T \in \mathcal{T}_h\), we apply the standard trace inequality to get

\[
\sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \| v - \Pi_{v,q}^{IF} v \|_{L^2(e)}^2 \leq C \sum_{T \in \mathcal{T}_h} \left( h_T^{-2} \| v - \Pi_{v,q}^{IF} v \|_{L^2(T)}^2 + \| v - \Pi_{v,q}^{IF} v \|_{H^1(T)}^2 \right).
\]

Let \( e^\pm = e \cap \Omega^\pm \). The following inequality holds

\[
\| \{2\mu \epsilon(v - \Pi_{v,q}^{IF} v)n_e\}_{e} \|_{L^2(e)}^2 = \sum_{s = \pm} \| \{2\mu \epsilon(v^s_E - (\Pi_{v,q}^{IF} v)^s)n_e\}_{e} \|_{L^2(e)}^2 \leq C \sum_{s = \pm} \| \nabla(v^s_E - (\Pi_{v,q}^{IF} v)^s) \|_{L^2(e)}^2,
\]

which together with the standard trace inequality yields

\[
\sum_{e \in \mathcal{E}_h} \| \{2\mu \epsilon(v - \Pi_{v,q}^{IF} v)n_e\}_{e} \|_{L^2(e)}^2 \leq C \sum_{T \in \mathcal{T}_h} \sum_{s = \pm} \left( \| v^s_E - (\Pi_{v,q}^{IF} v)^s \|_{H^1(T)}^2 + h_T^2 |v^s_E|_{H^2(T)}^2 \right).
\]

Analogously, we have

\[
\sum_{e \in \mathcal{E}_h} \| q - \Pi_{v,q}^{IF} q \|_{L^2(e)}^2 \leq C \sum_{T \in \mathcal{T}_h} \sum_{s = \pm} \left( \| q^s - (\Pi_{v,q}^{IF} q)^s \|_{L^2(T)}^2 + h_T^2 |q^s|_{H^1(T)}^2 \right),
\]

\[
J_h(q - \Pi_{v,q}^{IF} q, q - \Pi_{v,q}^{IF} q) \leq C \sum_{T \in \mathcal{T}_h} \sum_{s = \pm} \left( \| q^s - (\Pi_{v,q}^{IF} q)^s \|_{L^2(T)}^2 + h_T^2 |q^s|_{H^1(T)}^2 \right).
\]

Combining the above estimates with the definition of the norm \( \| \cdot \|_* \) in (5.4) and using Theorems 4.11 and 4.15, we complete the proof. \( \square \)

The following lemma concerns the consistent errors.

**Lemma 5.11.** Let \((u, p)\) and \((u_h, p_h)\) be the solutions of the problems (2.2) and (3.11), respectively. Suppose \((u, p) \in \tilde{H}_2 H_1 \cap (V, M)\). Then, there exists a constant \( C \) independent of \( h \) and the interface location relative to the mesh such that, for all \((w_h, r_h) \in VM_h^{IF}\),

\[
\| A_h(u - u_h, p - p_h; w_h, r_h) \| \leq Ch \left( \| u \|_{H^2(\Omega^+ \cup \Omega^-)} + \| p \|_{H^1(\Omega^+ \cup \Omega^-)} \right) \| (w_h, r_h) \|. \tag{5.36}
\]

**Proof.** Let \((w_h, r_h) \in VM_h^{IF}\) be arbitrary and \( n_{\partial T}\) be the unit outward normal to \( \partial T\). Multiplying (1.1) by \( w_h \) and applying integration by parts, we obtain

\[
\int \Omega \cdot w_h = \sum_{T \in \mathcal{T}_h} \left( \int_T (2\mu \epsilon(u) - p\mathbb{I}) : \nabla w_h - \int_{\partial T} (2\mu \epsilon(u) - p\mathbb{I}) n_{\partial T} \cdot w_h \right),
\]

where the integral on the interface \( \Gamma\) is canceled due to the interface condition (1.3) and the fact that \( w_h |_{T} \in C^0(T)^2 \) for all interface elements \( T \in \mathcal{T}_h^\Gamma\). Since \((u, p) \in \tilde{H}_2 H_1\), we have \([2\mu \epsilon(u) - p\mathbb{I}]_{\cdot n_e} = 0\)
for all \( e \in \mathcal{E}_h \), and
\[
\int_{\Omega} f \cdot w_h = \sum_{T \in T_h} \int_T 2\mu\epsilon(u) \cdot \epsilon(w_h) - \sum_{T \in T_h} \int_T p \nabla \cdot w_h + \sum_{e \in \mathcal{E}_h} \int_e \{ p \} \epsilon[w_h \cdot n_e] + \sum_{e \in \mathcal{E}_h} \int_e \{ 2\mu\epsilon(u) n_e \} \cdot [w_h]_e. \tag{5.37}
\]
Subtracting (3.10) from (5.37) we further obtain
\[
A_h(u - u_h, p - p_h; w_h, r_h) = -\sum_{T \in T_h} \int_T 2(\mu - \mu_h)\epsilon(u) : \epsilon(w_h) - \sum_{e \in \mathcal{E}_h} \int_e \{ p \} \epsilon[w_h \cdot n_e] + \sum_{e \in \mathcal{E}_h} \int_e \{ 2\mu\epsilon(u) n_e \} \cdot [w_h]_e := I_1 + I_2 + I_3,
\]
where we have used the facts that \( \int_{\Omega} r_h \nabla \cdot u = 0 \) from (1.2), \( \mu|_e = \mu_h|_e \) for all \( e \in \mathcal{E}_h \), and \( \{ p \} = [u] = 0 \) for all \( e \in \mathcal{E}_h \) since \( (u, p) \in \widehat{H}_2H_1 \).

We use (4.14) and Lemmas 4.2 and 4.3 to bound the first term below
\[
|I_1| \leq \| 2(\mu - \mu_h)\epsilon(u) \|_{L^2(U(\Gamma, \partial\Omega^2))} \| \epsilon(w_h) \|_{L^2(U(\Gamma, \partial\Omega^2))} \leq C \| u \|_{H^1(U(\Gamma, \partial\Omega^2))} \| w_h \|_{1, h}
\]
\[
\leq C \sum_{s=\pm} \| u^s \|_{H^2(U(\Gamma, \partial\Omega^2))} \| w_h \|_{1, h} \leq Ch \sum_{s=\pm} \| u^s \|_{H^2(\Omega)} \| w_h \|_{1, h}
\]
\[
\leq Ch \sum_{s=\pm} \| u^s \|_{H^2(\Omega^\pm) \cup \Omega^\pm)} \| w_h \|_{1, h}.
\]
Let \( T_h^e \) be the set of all elements in \( T_h \) having \( e \) as an edge. If \( T_h^e \cap T_{h}^{non} \neq \emptyset \), let \( T_e \in T_h^e \cap T_{h}^{non} \). Then, we have the standard result for the nonconforming finite elements (see, e.g., [1])
\[
\left| \int_e p[w_h \cdot n_e] \right| \leq \| p - c_e \|_{L^2(e)} \| w_h \|_{L^2(e)} \leq C |p|_{H^1(T_e)} |e|^{1/2} \| w_h \|_{L^2(e)},
\]
where \( c_e \) is an arbitrary constant. If \( T_h^e \cap T_{h}^{non} = \emptyset \) (i.e., \( T_h^e \subset T_h^{non} \)), we have, for all \( T \in T_h^e \),
\[
\left| \int_e p[w_h \cdot n_e] \right| \leq \sum_{s=\pm} \left| \int_e p^s[w_h \cdot n_e] \right| \leq C \sum_{s=\pm} \| p^s \|_{H^1(T)} |e|^{1/2} \| w_h \|_{L^2(e)}.
\]
Combining the above estimates with Lemmas 4.2 and 5.4, we further get
\[
|I_2| \leq C \left( \sum_{s=\pm} \| p^s \|_{H^1(\Omega)} \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h^{non}} |e| \| [w_h]_e \|_{L^2(e)} \right)^{1/2} \leq Ch \| p \|_{H^1(\Omega^\pm \cup \Omega^\pm)} \| w_h \|_{1, h}.
\]
Analogously, we have the following estimate for the third term
\[
|I_3| \leq Ch \| u \|_{H^2(\Omega^\pm \cup \Omega^\pm)} \| w_h \|_{1, h}.
\]
This concludes the proof. \( \square \)

We now provide the error estimate for the proposed IFE method in the following theorem.

**Theorem 5.12.** Let \((u, p)\) and \((u_h, p_h)\) be the solutions of the problems (2.2) and (5.10), respectively. Suppose \((u, p) \in H^2 \cap \mathcal{V} \cap M\), then the following error estimate holds
\[
\| (u, p) - (u_h, p_h) \| \leq C h \left( \| u \|_{H^2(\Omega^\pm \cup \Omega^\pm)} + \| p \|_{H^1(\Omega^\pm \cup \Omega^\pm)} \right), \tag{5.38}
\]
with a constant \( C \) independent of \( h \) and the interface location relative to the mesh.
Proof. Using (5.16) for the equivalence of two norms, the inf-sup stability (5.22) and the continuity (5.4) of the bilinear form $A_h(\cdot, \cdot)$, we have, for all $(v_h, q_h) \in V_h^{IFE}$,

$$
\| (u_h, p_h) - (v_h, q_h) \|_s \leq C_0 \| (u_h, p_h) - (v_h, q_h) \|
$$

$$
\leq C_0 C_s^{-1} \sup_{(w_h, r_h) \in V_h^{IFE}} \frac{A_h(u_h - v_h, p_h - q_h; w_h, r_h)}{\| (w_h, r_h) \|}
$$

$$
= C_0 C_s^{-1} \sup_{(w_h, r_h) \in V_h^{IFE}} \frac{A_h(u - v_h, p - q_h; w_h, r_h) + A_h(u_h - u, p_h - p; w_h, r_h)}{\| (w_h, r_h) \|}
$$

$$
\leq C A C_0^2 C_s^{-1} \|(u - v_h, p - q_h)\|_s + C_0 C_s^{-1} \sup_{(w_h, r_h) \in V_h^{IFE}} \frac{A_h(u_h - u, p_h - p; w_h, r_h)}{\| (w_h, r_h) \|}.
$$

It follows from Lemma 5.11 and the triangle inequality that, for all $(v_h, q_h) \in V_h^{IFE}$,

$$
\| (u, p) - (u_h, p_h) \|_s \leq \| (u, p) - (v_h, q_h) \|_s + \| (u_h, p_h) - (v_h, q_h) \|_s
$$

$$
\leq C \| (u, p) - (v_h, q_h) \|_s + C h (\| u \|_{H^2(\Omega+\cup\Omega-)} + \| p \|_{H^1(\Omega+\cup\Omega-)}).
$$

Finally, the estimate (5.38) is obtained by choosing $(v_h, q_h) = \Pi_h^{IFE}(u, p)$ and Lemma 5.10.

6 Numerical experiments

In this section, we present some numerical experiments to validate the theoretical analysis. Consider $\Omega = (-1, 1) \times (-1, 1)$ as the computational domain and use uniform triangulations constructed as follows. We first partition the domain into $N \times N$ congruent rectangles, and then obtain the triangulation by cutting the rectangles along one of diagonals in the same direction. The interface is $\Gamma = \{ (x_1, x_2)^T \in \mathbb{R}^2 : x_1^2 + x_2^2 = r_0^2 \}$ with $r_0 = 0.5$ and the exact solution $(u, p)$ is given for all $x = (x_1, x_2)^T$ by

$$
u(x) = \begin{cases} \frac{r_0^2 - |x|^2}{\mu^+} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} & \text{if } |x| < r_0, \\
\frac{r_0^2 - |x|^2}{\mu^+} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} & \text{if } |x| \geq r_0. 
\end{cases}
$$

The right-hand side $f$ and the non-homogeneous Dirichlet boundary condition $u|_{\partial \Omega}$ are determined from the exact solution.

We set $\delta = -1$ and $\eta = 0$ and use a standard approach from the finite element framework to deal with the non-homogeneous Dirichlet boundary condition. The resulting systems of equations are solved by a robust sparse direct solver from the MKL PARDISO package [1]. Note that the explicit formulas (4.40) have been used to compute the IFE basis functions. We denote the errors by $\| u_h \|_{L^2} := \| u - u_h \|_{L^2(\Omega)}$, $|u_h|_{H^1} := \| u - u_h \|_{1, h}$ and $|p_h|_{L^2} := \| p - p_h \|_{L^2(\Omega)}$ and compute them experimentally on a sequence of uniform triangulations. We test the example with the viscosity coefficient ranging from small to large jumps: $\mu^+ = 5$, $\mu^- = 1$; $\mu^+ = 1$, $\mu^- = 5$; $\mu^+ = 1000$, $\mu^- = 1$; $\mu^+ = 1$, $\mu^- = 1000$. The errors and rates of convergence are listed in Tables [11]. All data indicate that the IFE method achieves the optimal convergence rates, which in turn confirms our theoretical analysis.
Table 1: Errors of the IFE method for the example with $\mu^+ = 5$, $\mu^- = 1$.

| N   | $\|e_u\|_{L^2}$ rate | $|e_u|_{H^1}$ rate | $\|e_p\|_{L^2}$ rate |
|-----|------------------------|---------------------|-----------------------|
| 8   | 1.001E-02              | 2.020E-01           | 2.476E-01             |
| 16  | 2.688E-03 1.90         | 1.065E-01 0.92      | 1.297E-01 0.93        |
| 32  | 6.821E-04 1.98         | 5.422E-02 0.97      | 6.154E-02 1.08        |
| 64  | 1.667E-04 2.03         | 2.722E-02 0.99      | 2.971E-02 1.05        |
| 128 | 4.216E-05 1.98         | 1.364E-02 1.00      | 1.459E-02 1.03        |
| 256 | 1.054E-05 2.00         | 6.826E-03 1.00      | 7.250E-03 1.01        |
| 512 | 2.642E-06 2.00         | 3.414E-03 1.00      | 3.614E-03 1.00        |

Table 2: Errors of the IFE method for the example with $\mu^+ = 1$, $\mu^- = 5$.

| N   | $\|e_u\|_{L^2}$ rate | $|e_u|_{H^1}$ rate | $\|e_p\|_{L^2}$ rate |
|-----|------------------------|---------------------|-----------------------|
| 8   | 2.497E-02              | 6.643E-01           | 2.241E-01             |
| 16  | 6.419E-03 1.96         | 3.329E-01 1.00      | 1.172E-01 0.93        |
| 32  | 1.056E-03 2.00         | 1.667E-01 1.00      | 5.427E-02 1.11        |
| 64  | 3.997E-04 2.01         | 8.335E-02 1.00      | 2.653E-02 1.03        |
| 128 | 9.972E-04 2.00         | 4.109E-02 1.00      | 1.330E-02 1.00        |
| 256 | 2.490E-05 2.00         | 2.084E-02 1.00      | 6.631E-03 1.00        |
| 512 | 6.221E-06 2.00         | 1.042E-02 1.00      | 3.310E-03 1.00        |

Table 3: Errors of the IFE method for the example with $\mu^+ = 1000$, $\mu^- = 1$.

| N   | $\|e_u\|_{L^2}$ rate | $|e_u|_{H^1}$ rate | $\|e_p\|_{L^2}$ rate |
|-----|------------------------|---------------------|-----------------------|
| 8   | 9.349E-03              | 1.228E-01           | 3.835E-01             |
| 16  | 2.906E-03 1.69         | 6.905E-02 0.83      | 3.490E-01 0.14        |
| 32  | 8.687E-04 1.74         | 3.752E-02 0.88      | 1.759E-01 0.99        |
| 64  | 1.971E-04 2.14         | 1.976E-02 0.92      | 9.581E-02 0.88        |
| 128 | 5.417E-05 1.86         | 1.100E-02 0.85      | 5.046E-02 0.93        |
| 256 | 1.402E-05 1.95         | 5.827E-03 0.92      | 1.979E-02 1.35        |
| 512 | 3.539E-06 1.99         | 2.981E-03 0.97      | 7.686E-03 1.36        |

Table 4: Errors of the IFE method for the example with $\mu^+ = 1$, $\mu^- = 1000$.

| N   | $\|e_u\|_{L^2}$ rate | $|e_u|_{H^1}$ rate | $\|e_p\|_{L^2}$ rate |
|-----|------------------------|---------------------|-----------------------|
| 8   | 2.517E-02              | 6.636E-01           | 2.275E-01             |
| 16  | 6.444E-03 1.97         | 3.329E-01 1.00      | 1.426E-01 0.67        |
| 32  | 1.618E-03 1.99         | 1.667E-01 1.00      | 9.357E-02 0.61        |
| 64  | 4.049E-04 2.00         | 8.336E-02 1.00      | 6.253E-02 0.58        |
| 128 | 1.010E-04 2.00         | 4.169E-02 1.00      | 2.371E-02 1.40        |
| 256 | 2.518E-05 2.00         | 2.084E-02 1.00      | 1.014E-02 1.23        |
| 512 | 6.263E-06 2.01         | 1.042E-02 1.00      | 4.677E-03 1.12        |
7 Conclusions

In this paper we have developed and analyzed an IFE method for Stokes interface problems with discontinuous viscosity coefficients. The IFE space is constructed by modifying the traditional $CR-P_0$ finite element space. We have shown the unisolvence of IFE basis functions and the optimal approximation capabilities of IFE space. The stability and the optimal error estimates have been derived rigorously. This paper presents the first theoretical analysis for IFE methods for Stokes interface problems. In the future we intend to study the Stokes interface problems with non-homogeneous jump conditions and construct IFE spaces for three-dimensional Stokes interface problems.

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References

[1] Intel oneAPI Math Kernel Library PARDISO Solver - Parallel Direct Sparse Solver Interface, https://software.intel.com/content/www/us/en/develop/documentation/onemkl-developer-reference-c/top/sparse-solver-routines/onemkl-pardiso-parallel-direct-sparse-solver-interface.html.

[2] S. Adjerid, N. Chaabane, and T. Lin. An immersed discontinuous finite element method for Stokes interface problems. *Comput. Methods Appl. Mech. Engrg.*, 293:170–190, 2015.

[3] S. Adjerid, N. Chaabane, T. Lin, and P. Yue. An immersed discontinuous finite element method for the Stokes problem with a moving interface. *J. Comput. Appl. Math.*, 362:540–559, 2019.

[4] S. Brenner and L. Scott. *The mathematical theory of finite element methods*. Texts in Applied Mathematics 15, Springer, Berlin, 2008.

[5] S. C. Brenner. Poincaré-Friedrichs inequalities for piecewise $H^1$ functions. *SIAM J. Numer. Anal.*, 41:306–324, 2003.

[6] S. C. Brenner. Korn’s inequalities for piecewise $H^1$ vector fields. *Math. Comp.*, 73:1067–1087, 2004.

[7] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer, Berlin, 1991.

[8] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing. CutFEM: discretizing geometry and partial differential equations. *Internat. J. Numer. Methods Engrg.*, 104:472–501, 2015.

[9] E. Cáceres, J. Guzmán, and M. Olshanskii. New stability estimates for an unfitted finite element method for two-phase Stokes problem. *SIAM J. Numer. Anal.*, 58:2165–2192, 2020.

[10] L. Cattaneo, L. Formaggia, G. F. Iori, A. Scotti, and P. Zunino. Stabilized extended finite elements for the approximation of saddle point problems with unfitted interfaces. *Calcolo*, 52:123–152, 2015.
[11] Y. Chen and X. Zhang. A $P_2 - P_1$ partially penalized immersed finite element method for Stokes interface problems. *Int. J. Numer. Anal. Model.*, 18:120–141, 2021.

[12] T.-P. Fries and T. Belytschko. The extended/generalized finite element method: an overview of the method and its applications. *Internat. J. Numer. Methods Engrg.*, 84:253–304, 2010.

[13] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[14] R. Guo and T. Lin. A group of immersed finite-element spaces for elliptic interface problems. *IMA J. Numer. Anal.*, 39:482–511, 2019.

[15] R. Guo and T. Lin. An immersed finite element method for elliptic interface problems in three dimensions. *J. Comput. Phys.*, 414:109478, 2020.

[16] R. Guo, Y. Lin, and J. Zou. Solving two dimensional $H(curl)$-elliptic interface systems with optimal convergence on unfitted meshes. *arXiv:2011.11905*, 2020.

[17] J. Guzmán and M. Olshanski. Inf-sup stability of geometrically unfitted Stokes finite elements. *Math. Comp.*, 87:2091–2112, 2018.

[18] A. Hansbo and P. Hansbo. An unfitted finite element method, based on Nitsche’s method, for elliptic interface problems. *Comput. Methods Appl. Mech. Engrg.*, 191:5537–5552, 2002.

[19] P. Hansbo, M. G. Larson, and S. Zahedi. A cut finite element method for a Stokes interface problem. *Appl. Numer. Math.*, 85:90–114, 2014.

[20] X. He, T. Lin, and Y. Lin. The convergence of the bilinear and linear immersed finite element solutions to interface problems. *Numer. Methods Partial Differential Equations*, 28:312–330, 2012.

[21] X. He, F. Song, and W. Deng. Stabilized nonconforming Nitsche’s extended finite element method for Stokes interface problems. *arXiv:1905.04844*, 2019.

[22] K. Ito and Z. Li. Interface conditions for Stokes equations with a discontinuous viscosity and surface sources. *Appl. Math. Lett.*, 19:229–234, 2006.

[23] H. Ji. An immersed Raviart-Thomas mixed finite element method for elliptic interface problems on unfitted meshes. *arXiv:2105.03227*, 2021.

[24] H. Ji, F. Wang, J. Chen, and Z. Li. Analysis of nonconforming IFE methods and a new scheme for elliptic interface problems. *arXiv:2108.03179*, 2021.

[25] H. Ji, F. Wang, J. Chen, and Z. Li. A new parameter free partially penalized immersed finite element and the optimal convergence analysis. *arXiv:2103.10025*, 2021.

[26] D. Jones and X. Zhang. A class of nonconforming immersed finite element methods for Stokes interface problems. *J. Comput. Appl. Math.*, 392:113493, 2021.

[27] M. Kirchhart, S. Gross, and A. Reusken. Analysis of an XFEM discretization for Stokes interface problems. *SIAM J. Sci. Comput.*, 38:A1019–A1043, 2016.

[28] C. Lehrenfeld and A. Reusken. Nitsche-XFEM with streamline diffusion stabilization for a two-phase mass transport problem. *SIAM J. Sci. Comput.*, 34:A2740–A2759, 2012.

[29] J. Li, J. Markus, B. Wohlmuth, and J. Zou. Optimal a priori estimates for higher order finite elements for elliptic interface problems. *Appl. Numer. Math.*, 60:19–37, 2010.
[30] Z. Li. The immersed interface method using a finite element formulation. *Appl. Numer. Math.*, 27:253–267, 1998.

[31] Z. Li, K. Ito, and M.-C. Lai. An augmented approach for Stokes equations with a discontinuous viscosity and singular forces. *Comput. & Fluids*, 36:622–635, 2007.

[32] Z. Li, T. Lin, Y. Lin, and R. Rogers. An immersed finite element space and its approximation capability. *Numer. Methods Partial Differential Equations*, 20:338–367, 2004.

[33] Z. Li, T. Lin, and X. Wu. New Cartesian grid methods for interface problems using the finite element formulation. *Numer. Math.*, 96:61–98, 2003.

[34] T. Lin, Y. Lin, and X. Zhang. Partially penalized immersed finite element methods for elliptic interface problems. *SIAM J. Numer. Anal.*, 53:1121–1144, 2015.

[35] Z. Tan, D. V. Le, K. M. Lim, and B. C. Khoo. An immersed interface method for the incompressible Navier-Stokes equations with discontinuous viscosity across the interface. *SIAM J. Sci. Comput.*, 31:1798–1819, 2009.

[36] N. Wang and J. Chen. A nonconforming Nitsche’s extended finite element method for Stokes interface problems. *J. Sci. Comput.*, 81:342–374, 2019.

[37] Q. Wang and J. Chen. A new unfitted stabilized Nitsche’s finite element method for Stokes interface problems. *Comput. Math. Appl.*, 70:820–834, 2015.

[38] H. Wu and Y. Xiao. An unfitted $hp$-interface penalty finite element method for elliptic interface problems. *J. Comput. Math.*, 37:316–339, 2019.