Weylian reduction theory for self-similar models

Makoto Yoshikawa*

55-201 Daido-cho, Kitashirakawa, Sakyo-ku, Kyoto 606-8274, Japan
*E-mail: yoshikawa.m.th@gmail.com

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A geometrical method of formulating self-similar models in general relativity or in other gravitational theories is presented. The method consists of two techniques: (1) a Kaluza–Klein-like dimensional reduction technique for self-similar spacetimes, and (2) a systematic method of describing tensor fields on a self-similar spacetime in terms of fields on the reduced space. It is shown that the reduced space is a Weyl–Dirac conformal manifold and a self-similar model is formulated as a conformally covariant differential equation system.

1. Introduction

Self-similar models in general relativity have been widely studied in recent decades. They are attractive objects of study, mainly because they often play important roles in developing understanding of the dynamical features of general relativity [1–4], but a more practical reason is that they are relatively easy to study. Specifically, since self-similarity is a continuous symmetry, it (together with other continuous symmetries imposed on the system, such as spherical symmetry) reduces the number of coordinates on which unknown variables depend and, thus, reduces the basic equations to differential equations on a lower dimensional space. We term this process the dimensional reduction of the equation system. For example, the basic equations for a spherically symmetric self-similar model can be dimensionally reduced to an ordinary differential equation system, which is much easier to analyze than spherically symmetric models without self-similarity.

The dimensional reduction process described above is usually performed by selecting a coordinate system that is suitably adapted to self-similarity. However, there are a number of criteria for choosing a “preferable” coordinate system adapted to self-similarity, which are dependent on the purpose of the particular work, the physical and mathematical properties of the individual model, and personal preferences. Consequently, diverse formulations have been developed for each major self-similar model. This is not an ideal situation for researchers studying self-similar models, because formulations based on different coordinate systems are often related in a non-trivial way, and this causes considerable difficulty when results obtained from different formulations must be compared. See, e.g., Refs. [5,6].

This situation motivates us to take another approach, i.e., to develop a geometrical dimensional reduction technique that enables us to formulate self-similar models in a completely coordinate-free way. In this paper, we achieve this by introducing ideas from the Kaluza–Klein (KK) theory. In the simplest KK theory on $n + 1$ dimensions, one assumes homogeneity in the direction of the
single extra dimension and dimensional reduction with respect to the homogeneity is then performed in order to obtain a “reduced theory” on the \( n \)-dimensional orbit manifold. This process, called KK reduction, is accompanied by a clear geometrical perspective based on the fiber bundle theory, which allows us to link the homogeneous symmetry along the extra dimension with the internal gauge symmetry of the reduced theory (see, e.g., Ref. [7]). We modify the KK reduction procedure so that the homogeneity along the extra dimension is replaced by self-similarity, but without affecting the geometrical perspective.

The dimensional reduction technique introduced in this paper can be partly viewed as a variant of Scherk–Schwarz’s generalized dimensional reduction [8], in the sense that it allows some controlled dependence along the extra dimension. In fact, several studies have applied this approach to self-similarity for the purpose of obtaining a particular kind of massive supergravity theory on ten dimensions from the 11D supergravity [9,10]. Our purpose in this paper is to provide a general technique applicable to the formulation of various self-similar models in a wide variety of gravitational theories, so we focus on the reduction procedure itself and the underlying mathematical structure without assuming any particular theory. Another difference is the mathematical framework employed in describing the reduced geometry; note that, for our purposes, the Weyl geometric framework is preferable to the Riemann geometric framework employed in the previous works.

The organization of the paper is as follows. Section 2 is a preliminary section, in which a brief review of Weyl geometry [11,12] is given. We also introduce the concept of the Weyl–Dirac (WD) manifold, which plays the central role in subsequent sections. In Sect. 3, we focus on homothety, the self-similarity of the metric structure of a spacetime. We conduct the KK-like reduction of \((n+1)\)-dimensional spacetimes admitting homothety and show that they can be identified with \(n\)-dimensional WD manifolds. This identification provides us with the fundamental geometrical framework necessary for the dimensional reduction theory to be developed. In Sect. 4, we develop a tool for describing various tensor fields on \((n+1)\)-dimensional self-similar spacetimes, along with algebraic and differential operations acting on them, in terms of the Weyl theoretical framework on the \(n\)-dimensions. We focus primarily on self-similar classes, the classes of tensor fields subject to self-similarity conditions, because, in self-similar models, not only the metric field but also other fundamental fields should be subject to suitable self-similarity conditions so that the dimensional reduction can be performed. Section 5 outlines and comments on the mechanism through which our mathematical framework formulates self-similar models, while Sect. 6 gives concluding remarks.

We end this section by defining our notation. In this paper, we work mostly within the smoothness class of \(C^\infty\) and use the terms “smooth” and “differentiable” to indicate this class. This is for the sake of simplicity only, and generalizations to wider classes would be straightforward. We employ abstract index notation [13], as well as index-free notation to denote a tensor field (or a tensor at a point), which is suitable for developing a geometrical (coordinate-free) dimensional reduction theory without abandoning the convenience of index notation. A tensor field (or a tensor) in the index-free notation is given in boldface, and we write \(t \sim t^{p_1 \cdots p_i \cdots q_j} \) (or \(t^{p_1 \cdots p_i} \sim t^{q_1 \cdots q_j} \sim t \) to indicate that \(t^{p_1 \cdots p_i} \) and \(t^{q_1 \cdots q_j} \) represent the same tensor field but in different notations. For a differentiable manifold \(\mathcal{M}\) and two collections of abstract indices \((p_1 \cdots p_i),(q_1 \cdots q_j)\), we denote by \(\mathcal{T}_{q_1 \cdots q_j}^{p_1 \cdots p_i}(\mathcal{M})\) the collection of smooth tensor fields on \(\mathcal{M}\) of valence \((i, j)\), labeled by the abstract indices \(p_1 \cdots p_i\) and \(q_1 \cdots q_j\). For example, \(\mathcal{T}^p(\mathcal{M})\) and \(\mathcal{T}_q(\mathcal{M})\) represent, respectively, the space of smooth tangent vector fields and the space of smooth one-forms on \(\mathcal{M}\), while \(\mathcal{T}(\mathcal{M})\) is the space of smooth functions on \(\mathcal{M}\), which will also be written in the conventional notation \(C^\infty(\mathcal{M})\).
2. Weyl geometry

Weyl geometry was introduced in 1918 by H. Weyl in his attempt to unify gravitation and electromagnetism; it is now considered to be the prototype of subsequent gauge theories [11,12,14,15]. In Weyl geometry, we study conformal manifolds with an additional structure; an affine connection subject to a compatibility condition, which will be given below. Recall that a conformal manifold is a pair \((\mathcal{M}_c, \mathcal{C})\) of a differentiable manifold \(\mathcal{M}_c\) and a conformal class \(\mathcal{C}\) on it, and a conformal class on \(\mathcal{M}_c\) is a collection of pseudo-Riemannian metrics on the manifold of the form \(\mathcal{C} = \{\gamma\} = \{e^{2\rho} \gamma \mid \rho \in C^\infty(\mathcal{M}_c)\}\). We say that \(\mathcal{C} = \{\gamma\}\) is of Euclidean (resp. Lorentzian) signature if the metric signature of \(\gamma\) is \((+, +, \ldots, +)\) (resp. \((- , +, \ldots, +)\)). Although one does not focus on a specific metric in \(\mathcal{C}\) in principle, it is often useful in actual calculations to select a “working metric” \(\gamma \sim \gamma_{pq}\) from \(\mathcal{C}\), called a gauge. Changing from one working metric to another is called a gauge transformation or a Weyl transformation, and can be written in the form

\[
\gamma_{pq} \rightarrow \gamma'_{pq} = e^{2\rho}\gamma_{pq},
\]

for some \(\rho \in C^\infty(\mathcal{M}_c)\).

A Weyl connection \(\mathcal{D}_p\) on \((\mathcal{M}_c, \mathcal{C})\) is defined as a torsion-free affine connection on \(\mathcal{M}_c\) compatible with \(\mathcal{C}\) in the following sense: for any \(\gamma_{pq} \sim \gamma \in \mathcal{C}\), there is \(W_p \in \mathcal{T}_p(\mathcal{M}_c)\) such that

\[
\mathcal{D}_p \gamma_{qr} = -2W_p \gamma_{qr}.
\]

One can safely replace “any \(\gamma \in \mathcal{C}\)” in the statement with “some \(\gamma \in \mathcal{C}\)”, because, if \(\gamma_{pq}\) satisfies (2), then \(e^{2\rho}\gamma_{pq}\) satisfies (2) with \(W_p\) replaced by \(W_p - (d\rho)_p\). A Weyl connection on a conformal manifold is not unique due to the freedom of choosing \(W_p\).

A Weyl structure on \((\mathcal{M}_c, \mathcal{C})\) is defined as a map \(A_p : \mathcal{C} \rightarrow \mathcal{T}_p(\mathcal{M}_c)\) such that the image \(A_p(\gamma)\) transforms as an Abelian gauge field under the Weyl transformation (1), i.e.,

\[
A_p(e^{2\rho}\gamma) = A_p(\gamma) - (d\rho)_p, \quad (\gamma \in \mathcal{C}, \rho \in C^\infty(\mathcal{M}_c)).
\]

A Weyl manifold is defined as a conformal manifold equipped with a Weyl structure, and will be denoted by \((\mathcal{M}_c, \mathcal{C}, A_p)\).

Note that defining a Weyl structure on a conformal manifold is equivalent to giving a Weyl connection on the same space [16]. Indeed, given any Weyl connection \(\mathcal{D}_p\) on \((\mathcal{M}_c, \mathcal{C})\), we have a unique Weyl structure, \(A_p : \gamma \mapsto W_p\), specified by condition (2). Conversely, any Weyl structure \(A_p\) on \((\mathcal{M}_c, \mathcal{C})\) uniquely specifies a Weyl connection such that (2) holds for \(W_p = A_p(\gamma)\) for any \(\gamma \in \mathcal{C}\), which is given by

\[
\mathcal{D}_p V^q = \nabla_p^{(\gamma)} V^q + C_p^q r(\gamma) V^r, \quad (V^q \in \mathcal{T}^q(\mathcal{M}_c)),
\]

\[
\text{with} \quad C_p^q r, (\gamma) = A_p(\gamma) \delta^q_r + A_r(\gamma) \delta^q_p - \gamma_{pr} \gamma^{qs} A_s(\gamma),
\]

and where \(\nabla_p^{(\gamma)}\) represents the Levi–Civita connection with respect to \(\gamma\). Note that \(\gamma^{pq}\) is the inverse of \(\gamma_{pq}\); \(\gamma^{pr} \gamma_{rq} = \gamma_{qr} \gamma^{rp} = \delta^p_q\).

There are several intrinsic curvature tensors on \((\mathcal{M}_c, \mathcal{C}, A_p)\). First, we have the distance curvature \(\mathcal{F}_{pq}\), where

\[
\mathcal{F}_{pq} \equiv (dA(\gamma))_{pq} = \mathcal{D}_p A_q(\gamma) - \mathcal{D}_q A_p(\gamma).
\]

We also have the curvature tensor \(\mathcal{R}^P_{pqrs}\) with respect to the Weyl connection, i.e.,

\[
\mathcal{R}^P_{pqrs} V^q = [\mathcal{D}_r, \mathcal{D}_s] V^p, \quad (V^p \in \mathcal{T}^p(\mathcal{M}_c)).
\]
This satisfies the usual identities
\[ \mathcal{R}^p_{
abla qrs} = -\mathcal{R}^p_{qsr}, \] (7)
\[ \mathcal{R}^p_{\nabla qrs} = 0, \quad \text{(1st Bianchi identity)}, \] (8)
\[ \mathcal{R}_{\nabla t} \mathcal{R}^p_{qrs} = 0, \quad \text{(2nd Bianchi identity)}, \] (9)
where the brackets on the indices indicate skew-symmetrization. In contrast to the Riemann curvature tensor, the skew-symmetricity with respect to the first two indices does not always hold. Indeed, it is not difficult to verify that the skew-symmetric part \( \omega^p_{\nabla qrs} \equiv \gamma^p_{\nabla t} \mathcal{R}^t_{qrs} \), called the direction curvature, can be written as \([12,16]\)

\[ \omega^p_{\nabla qrs} = \mathcal{R}^p_{qrs} - \delta^p_q \mathcal{F}_{rs}. \] (10)

We contract \( \omega^p_{\nabla qrs} \) to obtain the Weylian counterpart of the Ricci tensor, so that

\[ \omega_{pq} = \omega^r_{prq}. \] (11)

Unlike the Ricci tensor, \( \omega_{pq} \) is not symmetric, and

\[ \omega_{[pq]} = \left( \frac{\dim \mathcal{M}_c}{2} - 1 \right) \mathcal{F}_{pq}, \] (12)

while the contracted Bianchi identity is analogous to the Riemannian counterpart, with

\[ \gamma^p_{pq} \mathcal{F}_{pq} - \frac{1}{2} \gamma^p_{pq} \mathcal{F}_{qr} \omega_{pq} = 0. \] (13)

Let us introduce some additional concepts that are helpful in working with conformal or Weyl manifolds. A Weyl covariant tensor field on \( (\mathcal{M}_c, \mathcal{C}) \) is any map \( T^{p_1\cdots p_i}_{q_1\cdots q_j} : \mathcal{C} \to \mathcal{T}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \) satisfying

\[ T^{p_1\cdots p_i}_{q_1\cdots q_j}(e^{2\rho} \mathcal{Y}) = e^{\omega \rho} T^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{Y}), \quad (\forall \mathcal{Y} \in \mathcal{C}, \ \forall \rho \in C^\infty(\mathcal{M}_c)), \] (14)

where \( \omega \) is a real number called the Weyl weight (or weight for short) of \( T^{p_1\cdots p_i}_{q_1\cdots q_j} \). If \( \omega = 0 \), it is simply called a Weyl invariant tensor field. We denote by \( \omega \mathcal{M}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c, \mathcal{C}) \) (or \( \omega \mathcal{M}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \) for short) the collection of Weyl covariant tensor fields on \( (\mathcal{M}_c, \mathcal{C}) \) of Weyl weight \( \omega \) taking values in \( \mathcal{T}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \). We can naturally identify \( \mathcal{T}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \) with the collection of Weyl invariant tensor fields \( \mathcal{B}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \) by identifying each \( t^{p_1\cdots p_i}_{q_1\cdots q_j} \in \mathcal{T}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \) with the constant map \( \mathcal{Y} \mapsto t^{p_1\cdots p_i}_{q_1\cdots q_j} \).

Each \( \omega \mathcal{M}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \) forms an \( \mathbb{R} \)-vector space with respect to addition and scalar multiplication, defined by

\[ (T^{p_1\cdots p_i}_{q_1\cdots q_j} + S^{p_1\cdots p_i}_{q_1\cdots q_j})(\mathcal{Y}) = T^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{Y}) + S^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{Y}), \] (15)
\[ (a T^{p_1\cdots p_i}_{q_1\cdots q_j})(\mathcal{Y}) = a T^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{Y}). \] (16)

Other tensor algebra operations for Weyl covariant tensor fields, i.e., tensor product, contraction, and index permutation, are defined similarly. For example, the tensor product is defined by

\[ (T^{p_1\cdots p_i}_{q_1\cdots q_j} S^{r_1\cdots r_k}_{s_1\cdots s_l})(\mathcal{Y}) = T^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{Y}) S^{r_1\cdots r_k}_{s_1\cdots s_l}(\mathcal{Y}), \] (17)

and this gives an \( \mathbb{R} \)-bilinear map:

\[ \text{Tensor Product:} \quad \omega \mathcal{M}^{p_1\cdots p_i}_{q_1\cdots q_j}(\mathcal{M}_c) \times \omega \mathcal{M}^{r_1\cdots r_k}_{s_1\cdots s_l}(\mathcal{M}_c) \to \omega + \omega \mathcal{M}^{p_1\cdots p_i r_1\cdots r_k}_{q_1\cdots q_j s_1\cdots s_l}(\mathcal{M}_c). \] (18)
The tensor product for the Weyl covariant scalar fields \((i = j = k = l = 0)\) is simply called the multiplication. The contraction and index permutations provide the following \(\mathbb{R}\)-linear maps:

\[
N - \text{Contraction: } \omega_{\frac{p_1}{p_2} \cdots \frac{p_{ij}}{p_{q_1 \cdots q_j}}}(\mathcal{M}_c) \rightarrow \omega_{\frac{p_1}{p_2} \cdots \frac{p_{ij}}{p_{q_1 \cdots q_j}}}(\mathcal{M}_c),
\]

Index Permutations:

\[
\omega_{\frac{p_1}{p_2} \cdots \frac{p_{ij}}{p_{q_1 \cdots q_j}}}(\mathcal{M}_c) \rightarrow \omega_{\frac{p_1}{p_2} \cdots \frac{p_{ij}}{p_{q_1 \cdots q_j}}}(\mathcal{M}_c).
\]

All these operations coincide with the usual forms when they act on Weyl invariant tensor fields.

The Weyl covariant metric \(q_{pq}: \mathcal{C} \rightarrow \mathcal{T}_{pq}(\mathcal{M}_c)\) on \((\mathcal{M}_c, \mathcal{C})\) is defined by \(q_{pq}(\gamma) = \gamma_{pq}\), and is an element of \(\mathfrak{M}_{pq}(\mathcal{M}_c)\). The contravariant counterpart \(q^{pq}\) of \(q_{pq}\) is given by \(q^{pq}(\gamma) = \gamma^{pq}\), and belongs to \(\mathfrak{M}^{pq}(\mathcal{M}_c)\). The tensor product and contraction operations allow us to define the index raising and lowering operations using \(q^{pq}\) and \(q_{pq}\), which change the Weyl weight by \(-2\) and \(+2\), respectively.

It is sometimes useful to consider the \(\mathbb{R}\)-graded vector space

\[
\mathfrak{M}^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c) = \bigoplus_{\omega \in \mathbb{R}} \omega_{\frac{p_1}{p_2} \cdots \frac{p_{ij}}{p_{q_1 \cdots q_j}}}(\mathcal{M}_c),
\]

for each valence type \(\frac{p_1}{p_2} \cdots \frac{p_{ij}}{p_{q_1 \cdots q_j}}\). An element of these spaces will also be referred to as a Weyl covariant tensor field; it is homogeneous if it has a definite Weyl weight, and inhomogeneous otherwise. We note that not only homogeneous elements, but also inhomogeneous elements of \(\mathfrak{M}^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c)\), can be naturally regarded as maps from \(\mathcal{C}\) to \(\mathcal{T}_{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c)\). All algebraic operations above can be extended straightforwardly to act on these spaces. Evidently, the space of Weyl covariant scalar fields \(R_{\mathfrak{M}}(\mathcal{M}_c) \equiv \mathfrak{M}(\mathcal{M}_c) = \bigoplus_{\omega \in \mathbb{R}} \omega \mathfrak{M}(\mathcal{M}_c)\) forms a commutative \(\mathbb{R}\)-graded ring with respect to the addition and multiplication, and every other space \(\mathfrak{M}^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c)\) is an \(\mathbb{R}\)-graded \(R_{\mathfrak{M}}(\mathcal{M}_c)\)-module.

The Weyl covariant derivative of \(T^{p_1 \cdots p_i}_{q_1 \cdots q_j} \in \mathfrak{M}^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c)\) is defined by

\[
(D_r T^{p_1 \cdots p_i}_{q_1 \cdots q_j})(\gamma) = D_r (T^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\gamma)) + A_r (\omega T^{p_1 \cdots p_i}_{q_1 \cdots q_j})(\gamma).
\]

Here, \(\omega\) is the weight operator, i.e., the linear operator on \(\mathfrak{M}^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c)\) that multiplies each homogeneous summand by its Weyl weight. Clearly, this coincides with the Weyl connection operator \(D_r\) when it acts on Weyl invariant tensor fields. It is also easy to see that \(D_r\) preserves the Weyl covariance and the \(\mathbb{R}\)-grading (Weyl weight), so it gives a graded \(\mathbb{R}\)-linear map from each Weyl covariant class \(\mathfrak{M}^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c)\) to \(\mathfrak{M}^{p_1 \cdots p_i}_{q_1 \cdots q_j}(\mathcal{M}_c)\). Moreover, it satisfies other standard properties of covariant derivatives: the Leibniz rule and commutativity with contractions and index permutations. Thus, \(D_r\) is a natural extension of \(\mathcal{D}_i\) to act on Weyl covariant classes. Another important property is that it annihilates the Weyl covariant metric, such that

\[
D_r q_{qr} = 0, \quad D_r q^{qr} = 0.
\]

This implies that the Weyl covariant derivative also commutes with the operations of index raising and lowering.

Finally, we define WD manifolds.\(^\dagger\) A WD manifold is a quad \((\mathcal{M}_c, \mathcal{C}, A_p, \phi)\) where the former three elements constitute a Weyl manifold \((\mathcal{M}_c, \mathcal{C}, A_p)\), and the latter \(\phi\) is a Weyl covariant scalar field on \((\mathcal{M}_c, \mathcal{C}, A_p)\) of Weyl weight \(+1\). We assume that \(\phi\) is strictly positive-valued, i.e., \(\phi(\gamma)\) is

\(^\dagger\) \(p_1 \cdots p_i \cdots p_i\) represents \(p_1 \cdots p_{i-1} p_{i+1} \cdots p_i\), and this represents \(p_{i+1} \cdots p_i\) when \(i = 1\) and \(p_{i+1} \cdots p_{i-1}\) when \(i = 0\). Similar abbreviations will be employed throughout this paper.

\(^\dagger\) Named after the Weyl–Dirac theory, which was developed by P. A. M. Dirac while trying to revive Weyl’s theory as a physically viable approach by appending a Weyl covariant scalar field [17].
a strictly positive-valued function on $\mathcal{M}$ for any (or, equivalently, some) $\gamma \in \mathcal{C}$. With the additional scalar field, we have a preferred gauge $\gamma^E \in \mathcal{C}$, called the Einstein gauge, specified by the condition $\phi^i(\gamma^E) = 1$. This implies that $(\mathcal{M}, \mathcal{C}, A_p, \phi)$ also has the aspect of a pseudo-Riemannian manifold $(\mathcal{M}, \gamma_{pq})$ equipped with a one-form $A_p^E \equiv A_p(\gamma^E)$. Due to the redundant structure, $(\mathcal{M}, \mathcal{C}, A_p, \phi)$ has another “natural” affine connection other than the Weyl connection, namely, the Levi–Civita connection $\gamma^E_p$ in the Einstein gauge. More generally, when other “natural” gauge conditions can be placed using $\phi$ or other available fields, we have more “natural” affine connections.

3. Dimensional reduction of simple homothetic spacetimes

Self-similarity of a spacetime $(\mathcal{M}, g_{ab})$ is characterized by the existence of a homothety (or a proper homothety, to be precise) acting upon it [18], i.e., a smooth one-parameter transformation group $h : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, $(\tau, p) \mapsto h_\tau(p)$ satisfying

$$h_\tau^* g_{ab} = e^{2\tau} g_{ab}, \quad (\forall \tau \in \mathbb{R}),$$

(24)

where $h_\tau^*$ denotes the pullback by $h_\tau$. The generating vector field of a homothety is called a homothetic vector field. Note that $\xi \sim \xi^a \in \mathfrak{T}^a(\mathcal{M})$ is a homothetic vector field on $(\mathcal{M}, g_{ab})$ iff it is complete and satisfies

$$\mathcal{L}_\xi g_{ab} = 2g_{ab}.$$  

(25)

Throughout this paper, the homothetic vector field generating $h$ will be denoted by $\xi_h$ (or $\xi^a_h$, $\xi^b_h$, ···).

The main object of study in this paper is a triple $(\mathcal{M}, g_{ab}, h)$ of a differentiable manifold $\mathcal{M}$, a smooth metric $g_{ab}$ on $\mathcal{M}$ of Lorentzian signature, and a proper homothety $h$ acting on $(\mathcal{M}, g_{ab})$. We say that the triple $(\mathcal{M}, g_{ab}, h)$ is a simple homothetic spacetime if $g_{ab}h^{a}_h h^{b}_h$ has constant sign on $\mathcal{M}$ with no zero points. Hereafter, $(\mathcal{M}, g_{ab}, h)$ is always a simple homothetic spacetime of dimension $n + 1$ ($n \geq 1$), $\mathcal{M}_h = \mathcal{M}/h$ is the orbit space of $h$, and $\pi$ is the natural projection from $\mathcal{M}$ onto $\mathcal{M}_h$. We employ lower-case Latin letters $a, b, c, \cdots$ as abstract indices associated with the (co)tangent spaces of $\mathcal{M}$, which are raised and lowered with $g^{ab}$ and $g_{ab}$ as usual.

3.1. Basic structure of a simple homothetic spacetime

For a simple homothetic spacetime $(\mathcal{M}, g_{ab}, h)$, let $\psi$ be the function on $\mathcal{M}$ given by

$$\psi = \sqrt{|g_{ab} \xi^a_h \xi^b_h|},$$  

(26)

Then, $\psi$ is a smooth strictly positive-valued function on $\mathcal{M}$ satisfying

$$h_\tau^* \psi = e^{\tau} \psi, \quad (\forall \tau \in \mathbb{R}),$$  

(27)

or, equivalently,

$$\partial_\xi_h \psi = \psi.$$  

(28)

This implies that, along each orbit of $h$, $\psi$ is a monotonically increasing function onto $(0, \infty)$, so each orbit intersects with the level set $\mathcal{N}_E \equiv \psi^{-1}(1)$ exactly once. Thus, the projection $\pi$ restricted to $\mathcal{N}_E$,

$$\pi|_{\mathcal{N}_E} : \mathcal{N}_E \to \mathcal{M}_h,$$  

(29)

is a bijection. Note that $d\psi$ is non-zero at any point of $\mathcal{M}$ because of (28) and the strict positivity of $\psi$, so the level set $\mathcal{N}_E$ is an $n$-dimensional closed submanifold of $\mathcal{M}$. Consequently, one can
introduce the structure of an $n$-dimensional differentiable manifold on $\mathcal{M}_h$ by requiring the map (29) to be a diffeomorphism.\(^3\)

We define a reference surface of $(\mathcal{M}, g_{ab}, h)$ as a smooth map $\mathcal{I}: \mathcal{M}_h \to \mathcal{M}$ such that

$$\pi \circ \mathcal{I} = \text{id}_{\mathcal{M}_h}. \quad (30)$$

Since the inverse of the diffeomorphism (29) gives a smooth embedding $\mathcal{I}_E : \mathcal{M}_h \to \mathcal{M}$ satisfying (30), $(\mathcal{M}, g_{ab}, h)$ has at least one reference surface. Actually, it has as many reference surfaces as elements of $C^\infty(\mathcal{M}_h)$, as will be shown later. Any reference surface $\mathcal{I}$ is an embedding, because condition (30) implies that $\mathcal{I}$ is a homeomorphism into $\mathcal{M}$ and that the pushforward $(\mathcal{I}_E)_x : T_x \mathcal{M}_h \to T_{\mathcal{I}(x)} \mathcal{M}$ is injective for any $x \in \mathcal{M}_h$. Thus, the image $\mathcal{I}(\mathcal{M}_h)$ is always an $n$-dimensional submanifold of $\mathcal{M}$. Whenever there is no danger of confusion, we will also refer to the submanifold $\mathcal{I}(\mathcal{M}_h)$ as a reference surface and denote it by $\mathcal{I}$.

Given a reference surface $\mathcal{I}$ of $(\mathcal{M}, g_{ab}, h)$, we have a unique function $\eta_{\mathcal{I}} \in C^\infty(\mathcal{M})$ characterized by the conditions

$$\mathcal{I}^* \eta_{\mathcal{I}} = 0, \quad (31)$$

$$h^* \eta_{\mathcal{I}} = \eta_{\mathcal{I}} + \tau, \quad (\forall \tau \in \mathbb{R}). \quad (32)$$

Actually, $\eta_{\mathcal{I}}$ is explicitly given by

$$\eta_{\mathcal{I}} = \log \psi - (\mathcal{I} \circ \pi)^* \log \psi. \quad (33)$$

We call $\eta_{\mathcal{I}}$ the homothetic scale function with respect to $\mathcal{I}$. Let $F_{\mathcal{I}} : \mathcal{M} \to \mathcal{M}_h \times \mathbb{R}$ be defined by

$$F_{\mathcal{I}}(p) = (\pi(p), \eta_{\mathcal{I}}(p)).$$

Then, $F_{\mathcal{I}}$ is a diffeomorphism, because it has the smooth inverse map $F_{\mathcal{I}}^{-1}(x, \tau) = h_x \circ \mathcal{I}(x)$. This shows that a simple homothetic spacetime $(\mathcal{M}, g_{ab}, h)$ always has the direct product structure $\mathcal{M} \cong \mathcal{M}_h \times \mathbb{R}$.

Let $\mathcal{R}$ be the collection of all reference surfaces of $(\mathcal{M}, g_{ab}, h)$. For $\mathcal{I} \in \mathcal{R}$ and $\rho \in C^\infty(\mathcal{M}_h)$, we define $h_{\rho} \mathcal{I} : \mathcal{M}_h \to \mathcal{M}$ by

$$(h_{\rho} \mathcal{I})(x) = h(\rho(x), \mathcal{I}(x)). \quad (34)$$

It is easy to verify that $h_{\rho} \mathcal{I}$ is in $\mathcal{R}$. Thus, each $\mathcal{I} \in \mathcal{R}$ defines a map

$$C^\infty(\mathcal{M}_h) \to \mathcal{R}, \quad \rho \mapsto h_{\rho} \mathcal{I}. \quad (35)$$

This map is bijective. Indeed, it is clearly injective, and any $\mathcal{I}_1 \in \mathcal{R}$ can be written as $\mathcal{I}_1 = h_{\rho_1} \mathcal{I}$ with $\rho_1 = \mathcal{I}_1^* \eta_{\mathcal{I}}$. It is also easy to see that the homothetic scale function with respect to $h_{\rho} \mathcal{I}$ is related to that with respect to $\mathcal{I}$ by

$$\eta_{h_{\rho} \mathcal{I}} = \eta_{\mathcal{I}} - \pi^* \rho. \quad (36)$$

---

\(^3\) The topological structure on $\mathcal{M}_h$ given in this way is actually equal to the quotient topology. Note that this does not hold if $h$ is replaced by a one-parameter group of isometries (e.g., the irrational translations on the 2D torus).
3.2. WD manifolds as dimensionally reduced simple homothetic spacetimes

For any reference surface \( \mathscr{S} \) of \( (\mathcal{M}, g_{ab}, \mathfrak{h}) \), the conformally rescaled metric \( e^{-2\eta_{\mathscr{S}}} g_{ab} \) is invariant under the actions of \( \mathfrak{h}^\mu \) because, by (24) and (32), any change of \( e^{-2\eta_{\mathscr{S}}} \) cancels any change of \( g_{ab} \). Thus, \( \mathfrak{h} \) serves as a 1D isometry group for the conformally rescaled spacetime \( (\mathcal{M}, e^{-2\eta_{\mathscr{S}}} g_{ab}) \) whose orbit space is an \( n \)-dimensional manifold \( \mathcal{M}_h \), so the usual KK argument can be applied. Specifically, for any point \( x \) of the orbit manifold \( \mathcal{M}_h \) and for a local coordinate system \( (X^1, \ldots, X^n) \) on \( \mathcal{M}_h \) around \( x \), the pulled-back coordinate functions \( X_\mu^o \equiv \pi^*X^\mu (\mu = 1, \ldots, n) \) together with \( \eta_{\mathscr{S}} \) form a local coordinate system on \( \mathcal{M} \) around the orbit \( \pi^{-1}(x) \). (We call a local coordinate system \( (X^1_o, \ldots, X^n_o, \eta_{\mathscr{S}}) \) obtained in this way a homothetic local coordinate system, in accordance with Ref. [1].) In terms of the local coordinate system, \( \xi_h \) can be written as

\[
\xi_h = \left( \frac{\partial}{\partial \eta_{\mathscr{S}}} \right)_{X^1_o, \ldots, X^n_o},
\]

and the conformally rescaled metric \( e^{-2\eta_{\mathscr{S}}} g \) can be written in the KK form

\[
e^{-2\eta_{\mathscr{S}}} g = \sum_{\mu, \nu=1}^n q_{\mu\nu}(X_o) dX_\mu o dX_\nu o + \varepsilon \phi_{\mathscr{S}}(X_o)^2 \left( d\eta_{\mathscr{S}} - \sum_{\mu=1}^n A^o_{\mu}(X_o) dX_\mu o \right)^2,
\]

where \( \phi_{\mathscr{S}}, A^o_{\mu}, \) and \( q_{\mu\nu}\) are \( n \)-variable functions independent of \( \eta_{\mathscr{S}} \), they can be considered as local coordinate representations of a scalar field, a one-form, and a symmetric tensor field, respectively, on \( \mathcal{M}_h \). Let us adopt upper-case Latin letters \( A, B, C, \ldots \) as abstract indices associated with the (co)tangent spaces of \( \mathcal{M}_h \), and denote these fields on \( \mathcal{M}_h \) by \( \phi(\mathscr{S}), A_\lambda(\mathscr{S}), \) and \( q_{AB}(\mathscr{S}) \), respectively, where “(\( \mathscr{S} \))” is attached in order to indicate that they are dependent on the choice of \( \mathscr{S} \in \mathfrak{S} \). Specifically, they are locally given by

\[
\phi(\mathscr{S}) = \phi_{\mathscr{S}}(X),
\]

\[
A_\lambda(\mathscr{S}) = \sum_\mu A^o_{\mu}(X)(dX^\mu)_\lambda,
\]

\[
q_{AB}(\mathscr{S}) = \sum_{\mu, \nu} q_{\mu\nu}(X)(dX^\mu)_A(dX^\nu)_B,
\]

where \( X \) abbreviates \( (X^1, \ldots, X^n) \). Then, the metric can be written in a coordinate-free form as

\[
g_{ab} = e^{2\eta_{\mathscr{S}}} \left\{ \pi^*(q(\mathscr{S}) + \varepsilon \pi^*(\phi_{\mathscr{S}})^2 (d\eta_{\mathscr{S}} - \pi^*(A(\mathscr{S})))^2 \right\}_{ab}.
\]

To elucidate the geometrical aspects of the fields (39)–(41), it is useful to introduce the following fields on \( \mathcal{M} \):

\[
\xi_{a}^h \equiv \varepsilon \psi^{-2} g_{ab} \xi_{b}^h,
\]

\[
p^a_{ab} \equiv g_{ab} - \varepsilon \psi^2 \xi_{a}^h \xi_{b}^h.
\]

Note that \( \xi_{a}^h \) is normalized so that \( \xi_{a}^h \xi_{a}^h = 1 \) (take care not to confuse \( \xi_{a}^h \) with \( g_{ab} \xi_{b}^h \)), and \( p^a_{ab} \) is defined so that \( p^a_{ab} = \delta^a_b - \xi_{a}^h \xi_{b}^h \) is the orthogonal projection tensor to the orthogonal complement
to $\xi^a_h$. By (26) and (37)–(41), we have
\begin{align}
\psi &= e^{\eta} \pi^* \phi(\mathcal{S}), \\
\xi^a_h &= (d\eta \cdot \pi^* A(\mathcal{S}))^a, \\
p_{ab} &= (e^{2\eta} \pi^* q(\mathcal{S}))_{ab},
\end{align}
where the sign of $\phi$ is chosen so that $\phi(\mathcal{S}) > 0$. By applying $\mathcal{S}^*$ to these and using (30) and (31), we arrive at the following geometrical expressions for $\phi(\mathcal{S}), A_\lambda(\mathcal{S}),$ and $q_{AB}(\mathcal{S})$:
\begin{align}
\phi(\mathcal{S}) &= \mathcal{S}^* \psi, \\
A_\lambda(\mathcal{S}) &= -(\mathcal{S}^* \xi^b_\lambda), \\
q_{AB}(\mathcal{S}) &= (\mathcal{S}^* p)_{AB}.
\end{align}

To see the dependence of these fields on the choice of $\mathcal{S} \in \mathcal{R}$, let us apply (45)–(47) to two arbitrary reference surfaces, $\mathcal{S}$ and $h_\rho \mathcal{S}$. Using (36) and the injectivity of $\pi^*$, we obtain
\begin{align}
\phi(h_\rho \mathcal{S}) &= e^\rho \phi(\mathcal{S}), \\
A_\lambda(h_\rho \mathcal{S}) &= A_\lambda(\mathcal{S}) - (d\rho)_\lambda, \\
q_{AB}(h_\rho \mathcal{S}) &= e^{2\rho} q_{AB}(\mathcal{S}).
\end{align}

These should be contrasted with the corresponding result in KK theory, in which $A_\lambda$ follows the same transformation law as above, while $\phi$ and $q_{AB}$ are invariant [7]. Needless to say, this difference comes from the existence of the factor $e^{2\eta} \mathcal{S}$ in (42).

One can easily see that the symmetric tensor field $q(\mathcal{S}) \sim q_{AB}(\mathcal{S})$ gives a smooth metric on $\mathcal{M}_h$ of Lorentzian signature (when $s = +$) or Euclidean signature (when $s = -$). Thus we deduce from (53) and the bijectivity of (35) that
\begin{equation}
\mathcal{C}_h \equiv \{ q(\mathcal{S}) \mid \mathcal{S} \in \mathcal{R} \}
\end{equation}
is a conformal class on $\mathcal{M}_h$ of Lorentzian or Euclidean signature, and can be identified with $\mathcal{R}$ by the map $\mathcal{R} \ni \mathcal{S} \mapsto q(\mathcal{S}) \in \mathcal{C}_h$. With this identification, $A_\lambda$ and $\phi$ are maps from $\mathcal{C}_h$ to $\mathcal{S}_\lambda(\mathcal{M}_h)$ and $C^\infty(\mathcal{M}_h)$, respectively. Furthermore, by (51) and (52), $A_\lambda$ is a Weyl structure on the conformal manifold $(\mathcal{M}_h, \mathcal{C}_h)$, and $\phi$ is a Weyl covariant scalar field on $(\mathcal{M}_h, \mathcal{C}_h)$ of Weyl weight 1, which is obviously strictly positive-valued. Consequently, the quad $(\mathcal{M}_h, \mathcal{C}_h, A_\lambda, \phi)$ is a WD manifold of dimension $n$. We term this process of obtaining a WD manifold $(\mathcal{M}_h, \mathcal{C}_h, A_\lambda, \phi)$ from a simple homothetic spacetime $(\mathcal{M}, g_{ab}, h)$ the dimensional reduction of $(\mathcal{M}, g_{ab}, h)$, and refer to $(\mathcal{M}_h, \mathcal{C}_h, A_\lambda, \phi)$ as the reduced WD manifold. Choosing a gauge $\gamma \in \mathcal{C}_h$ in the reduced WD manifold corresponds to choosing a reference surface $\mathcal{S} \in \mathcal{R}$, and a Weyl transformation $\gamma \mapsto e^{2\rho} \gamma$ in the reduced WD manifold corresponds to the change of reference surface from $\mathcal{S}$ to $h_\rho \mathcal{S}$. In particular, the Einstein gauge $\gamma^E$ corresponds to the reference surface $\mathcal{S}_E$ represented by the level set $N_E$.

The reduced WD manifold $(\mathcal{M}_h, \mathcal{C}_h, A_\lambda, \phi)$ has the full geometrical information of the original $(\mathcal{M}, g_{ab}, h)$. Indeed, if we are given $(\mathcal{M}_h, \mathcal{C}_h, A_\lambda, \phi)$, then, as seen in the previous subsection, we know that the original $\mathcal{M}$ has the direct product structure $\mathcal{M} \approx \mathcal{M}_h \times \mathcal{R}$ on which $h$ acts as $h_{\tau}(p, \eta) = (p, \eta + \tau)$. Hence, we can reconstruct $g_{ab}$ from the Weyl–Dirac structure by applying (42) in a suitable gauge $\gamma = q(\mathcal{S}) \in \mathcal{C}_h$, where $s = +$ (resp. $s = -$) when the conformal structure $\mathcal{C}_h$ is of Lorentzian (resp. Euclidean) signature. Actually, this correspondence between simple homothetic manifolds and WD manifolds is one-to-one in the following sense.
Theorem 3.1. The dimensional reduction procedure provides one-to-one correspondence, up to equivalence, between (a) the class of \((n + 1)\)-dimensional simple homothetic spacetimes, and (b) the class of \(n\)-dimensional WD manifolds with a conformal structure of Lorentzian or Euclidean signature. Here, two simple homothetic spacetimes \((\mathcal{M}, g_{ab}, \eta)\) and \((\mathcal{M}', g'_{ab}, \eta')\) are defined to be equivalent if

\[
\exists \Phi : \mathcal{M} \to \mathcal{M}' \text{ (diffeomorphism) s.t. } \Phi^* g'_{ab} = g_{ab} \text{ and } \Phi \circ \eta = \eta' \circ \Phi \quad (\forall \tau \in \mathbb{R}),
\]

and two WD manifolds \((\mathcal{N}, \mathcal{C}, A_\lambda, \phi)\) and \((\mathcal{N}', \mathcal{C}', A'_\lambda, \phi')\) are defined to be equivalent if

\[
\exists \varphi : \mathcal{N} \to \mathcal{N}' \text{ (diffeomorphism) s.t.}
\]

\[
\forall \gamma' \in \mathcal{C}' \left[ \varphi^* \gamma' \in \mathcal{C} \text{ and } A_\lambda(\varphi^* \gamma') = \varphi^* A'_\lambda(\gamma') \text{ and } \phi(\varphi^* \gamma') = \varphi^* \phi'(\gamma') \right].
\]

The proof is straightforward and is therefore omitted.

3.3. Correspondence of connections

In KK theory, it is well known that a link exists between the Levi–Civita connections on the full spacetime and on the reduced spacetime, via the concept of “horizontal lift” \([7]\). In this section, we will follow the same lines to see the correspondence given by Theorem 3.1 from the perspective of connections.

At each point \(p\) of \((\mathcal{M}, g_{ab}, \eta)\), the horizontal tangent space \(H_p\) is defined as the orthogonal complement to \(\xi_h\) in the tangent space \(T_p\mathcal{M}\). As was previously mentioned, the orthogonal projection from \(T_p\mathcal{M}\) to \(H_p\) is given by \(w^a \mapsto p^a_b w^b\). A vector field \(v\) on \(\mathcal{M}\) is defined as horizontal if \(v_p \in H_p\) at any point \(p \in \mathcal{M}\). Let us denote by \(\pi_p \mathcal{M}\) the restriction of the pushforward \((\pi_\ast)_p : T_p\mathcal{M} \to T_{\pi(p)}\mathcal{M}_b\) to \(H_p\). Then \(\sigma_p : H_p \to T_{\pi(p)}\mathcal{M}_b\) is a linear isomorphism, since \((\pi_\ast)_p\) is a surjection from \(T_p\mathcal{M}\) to \(T_{\pi(p)}\mathcal{M}_b\) that annihilates \(\xi_h\). The horizontal lift of a vector field \(V \sim V^\lambda\) on \(\mathcal{M}_b\), labeled \(\uparrow V\) or \(\uparrow V^a, \uparrow V^b, \ldots\), is defined as the vector field on \(\mathcal{M}\) given by

\[
\uparrow V_p = \sigma_p^{-1}(V_{\pi(p)}), \quad (p \in \mathcal{M}).
\]

If we take a local coordinate system \(X = (X^1, \ldots, X^n)\) on \(\mathcal{M}_b\) and write \(V = \sum_{\mu} V^\mu(X) \frac{\partial}{\partial X^\mu}\), the horizontal lift \(\uparrow V\) can be written in terms of the homothetic local coordinate system \((X^1_\eta, \ldots, X^n_\eta, \eta_\lambda)\), where

\[
\uparrow V = \sum_{\mu=1}^n V^\mu(X_\eta) \left( \frac{\partial}{\partial X^\mu_\eta} + A_\mu^\lambda(X_\eta) \frac{\partial}{\partial \eta_\lambda} \right).
\]

Thus, the horizontal lift of a smooth vector field is also smooth. The following properties can be readily verified:

\[
[\xi_h, \uparrow V] = 0,
\]

\[
\uparrow (fV) = \pi^* f \uparrow V,
\]

\[
\partial_\lambda \uparrow V = \pi^*(\partial_\lambda f),
\]

\[
\partial_\lambda \eta_\lambda = \pi^*(V^\lambda A_\lambda(\lambda)),
\]

where \(f\) is any smooth function on \(\mathcal{M}_b\).

Our claim is stated in the following theorem.
Theorem 3.2. The Weyl connection $\mathcal{D}_\mathcal{A}$ on the reduced WD manifold $(\mathcal{M}_h, \mathcal{C}_h, A_\mathcal{A}, \phi)$ can be completely characterized by the condition

$$\forall U^A, V^A \in \mathcal{T}^h(\mathcal{M}_h), \quad (U^C \mathcal{D}_C V)^a = p_a^b U^c \nabla_c U^b. \quad (61)$$

Here, $\nabla_a$ is the Levi–Civita connection on $(\mathcal{M}, g_{ab})$.

To prove this theorem, we first show the following lemma.

Lemma 3.3. For an affine connection $D_a$ on $\mathcal{M}$ and a reference surface $\mathcal{S}$ of $(\mathcal{M}, g_{ab}, h)$, there is a unique affine connection $\tilde{D}_\mathcal{S}$ on $\mathcal{M}_h$ such that, for any smooth vector fields $U, V$ on $\mathcal{M}_h$,

$$(\tilde{D}_\mathcal{S} U) \big|_x = (\pi_a)_{\mathcal{S}(x)} (D_a U) \big|_{\mathcal{S}(x)}. \quad (\forall x \in \mathcal{M}_h) \quad (62)$$

If, moreover, $D_a$ is torsion-free, then $\tilde{D}_\mathcal{S}$ is also torsion-free.

Proof. It is evident that $\tilde{D}_\mathcal{S} V$ given by (62) is smooth and the map $(U, V) \mapsto \tilde{D}_\mathcal{S} V$ is $\mathbb{R}$-bilinear. For $f \in C^\infty(\mathcal{M}_h)$, it follows from (58) and (59) that

$$(\tilde{D}_\mathcal{S} U V)_x = (\pi_a)_{\mathcal{S}(x)} (f(x) (D_a U V)_{\mathcal{S}(x)}) = (f \tilde{D}_\mathcal{S} U V)_x,$$

$$(\tilde{D}_\mathcal{S} f V)_x = (\pi_a)_{\mathcal{S}(x)} (\pi^* f (D_a U V) + (\partial_a \pi^* f) \uparrow V) = (f \tilde{D}_\mathcal{S} U V + (\partial_a f) V)_x.$$

Thus, $\tilde{D}_\mathcal{S}$ is an affine connection on $\mathcal{M}_h$.

Let $\Theta^\mathcal{S}$ be the torsion tensor for $\tilde{D}_\mathcal{S}$; $\Theta^\mathcal{S} (U, V) = \tilde{D}_\mathcal{S} U V - \tilde{D}_\mathcal{S} U V - [U, V]$. One can easily see, from (56) or (59), that $(\pi_a)_{\pi^* (U^1)} = [U, V]_{\pi^* (p)}$ holds at any $p \in \mathcal{S}$. Thus we have

$$(\Theta^\mathcal{S} (U, V))_x = (\pi_a)_{\mathcal{S}(x)} ((D_a U V - D_a V U)_{\mathcal{S}(x)}), \quad (\forall x \in \mathcal{M}_h).$$

If $D_a$ is torsion-free, the r.h.s. vanishes for any $U, V$, which means that $\Theta^\mathcal{S} = 0$.

Proof of Theorem 3.2. By Lemma 3.3, for an arbitrarily chosen $\mathcal{S}$, we have a unique torsion-free affine connection $\tilde{\nabla}_\mathcal{S}$ on $\mathcal{M}_h$ such that

$$(\tilde{\nabla}_\mathcal{S} U V)^a = (\pi_a)_{\mathcal{S}(x)} (D_a U V)_{\mathcal{S}(x)} \quad (\forall x \in \mathcal{M}_h), \quad (63)$$

for any smooth vector fields $U \sim U^A$ and $V \sim V^B$ on $\mathcal{M}_h$. By the dimensional reduction procedure described in Sect. 3.2 and the definition of the horizontal lift, we deduce that

$$\gamma_{AB} U^A V^B = \mathcal{S}^* (g_{ab} U^a V^b),$$

$$(\mathcal{S}^* U^a)_{\mathcal{S}} U^c = (\mathcal{S}^* (U^a - \pi^* (U^A A_\mathcal{A} (\mathcal{S}))) \xi_b)_{\mathcal{S}(x)}, \quad (\forall x \in \mathcal{M}_h), \quad (64)$$

where $\gamma_{AB}$ is the gauge corresponding to $\mathcal{S}$. Using these together with (57), we compute

$$U^C (\tilde{\nabla}_\mathcal{S} \gamma_{AB}) V^R W^B = \mathcal{S}^* (\partial_1 U (g_{ab} U^a V^b) - g_{ab} \nabla_U U^a \cdot W^b - g_{ab} U^a \nabla_U W^b)$$

$$- U^A A_\mathcal{A} (\gamma) \mathcal{S}^* (\partial_1 h (g_{ab} U^a W^b)), \quad (65)$$

$$- 2U^C A_C (\gamma) \gamma_{AB} V^R W^B.$$ 

As $U^A, V^A,$ and $W^A \in \mathcal{T}^h(\mathcal{M}_h)$ are arbitrary, we obtain $\tilde{\nabla}_\mathcal{S} q_{BC} = -2 A_\mathcal{A} (\gamma) q_{BC}$. This proves that $\tilde{\nabla}_\mathcal{S}$ is equal to the Weyl connection $\mathcal{D}_\mathcal{A}$ on $(\mathcal{M}_h, \mathcal{C}_h, A_\mathcal{A}, \phi)$, irrespective of the choice of $\mathcal{S}$.

Recall that a WD manifold has other “natural” affine connections than the Weyl connection. The above theorem asserts, nevertheless, that the Levi–Civita connection on the $(n + 1)$-dimensions singles out the Weyl connection $\mathcal{D}_\mathcal{A}$. Hence, the Weyl connection is preferable among others when one wants to investigate Riemannian geometric structure on the $(n + 1)$-dimensions via the reduced geometry.
4. Self-similar tensor field calculus

4.1. Modules of self-similar tensor fields

In a self-similar model, a suitable self-similarity condition adapted to the homothety, \( h \), of the background spacetime \( \mathcal{M}, g_{ab} \) is imposed on every fundamental field of the theory. For a tensor field \( t^{a_1 \cdots a_r}_{b_1 \cdots b_s} \), the self-similarity condition usually takes the form

\[
\mathcal{h}^* t^{a_1 \cdots a_r}_{b_1 \cdots b_s} = e^{\omega \tau} t^{a_1 \cdots a_r}_{b_1 \cdots b_s}, \quad (\forall \tau \in \mathbb{R}),
\]

or, equivalently,

\[
\mathcal{L}_\xi t^{a_1 \cdots a_r}_{b_1 \cdots b_s} = \omega t^{a_1 \cdots a_r}_{b_1 \cdots b_s}.
\]

Here, \( \omega \) is a real number called the self-similarity weight (or weight for short) of \( t^{a_1 \cdots a_r}_{b_1 \cdots b_s} \). The condition (66) implies that the tensor field \( t^{a_1 \cdots a_r}_{b_1 \cdots b_s} \), which should transform as \( t^{a_1 \cdots a_r}_{b_1 \cdots b_s} \rightarrow r^{\omega \tau} t^{a_1 \cdots a_r}_{b_1 \cdots b_s} \) under the scale transformation \( g_{ab} \rightarrow r^2 g_{ab} \), so \( \omega \) is defined such that the physical quantity represented by \( t^{a_1 \cdots a_r}_{b_1 \cdots b_s} \) has the dimension of (length)\(^\omega \) in suitable units. We call a tensor field \( t^{a_1 \cdots a_r}_{b_1 \cdots b_s} \) subject to (66) a self-similar tensor field on \( \mathcal{M}, g_{ab}, h \) of (self-similarity) weight \( \omega \). For example, the metric tensor field \( g_{ab} \) is a self-similar tensor field of weight 2, and its contravariant counterpart \( g^{ab} \) is a self-similar tensor field of weight \(-2\). A self-similar tensor field of weight 0 is also said to be \( h \)-invariant. The homothetic vector field \( \xi^a \) and the one-form \( \xi^a \) given by (43) are examples of \( h \)-invariant tensor fields.

Given a valence type \( a_1 \cdots a_r \) and \( \omega \in \mathbb{R} \), we denote by \( \omega \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}, g_{ab}, h) \) (or \( \omega \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) for short) the collection of smooth self-similar tensor fields on \( \mathcal{M}, g_{ab}, h \) of weight \( \omega \) with abstract indices \( a_1 \cdots a_r \). Obviously, \( \omega \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) is an \( \mathbb{R} \)-subspace of \( \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \). Since any finite number of non-zero self-similar tensor fields with the same indices \( a_1 \cdots a_r \) but with different weights are \( \mathbb{R} \)-linearly independent, the \( \mathbb{R} \)-subspace \( \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) of \( \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) spanned by \( \bigcup_{\omega \in \mathbb{R}} \omega \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) is indeed the direct sum

\[
\mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) = \bigoplus_{\omega \in \mathbb{R}} \omega \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}).
\]

In other words, \( \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) is \( \mathbb{R} \)-graded by the self-similarity weight \( \omega \). An element of \( \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) is also referred to as a self-similar tensor field on \( \mathcal{M}, g_{ab}, h \); it is said to be homogeneous if it has a definite weight and inhomogeneous otherwise.

The tensor product, contraction, and index permutation operations provide the maps:

\[
\text{Tensor Product: } \mathcal{S}_{b_1 \cdots b_s} (\mathcal{M}) \times \mathcal{S}_{d_1 \cdots d_r} (\mathcal{M}) \rightarrow \mathcal{S}_{b_1 \cdots b_s d_1 \cdots d_r} (\mathcal{M}), \quad (69)
\]

\[
\text{ (i) } - \text{ Contraction: } \mathcal{S}^{a_1 \cdots a_r}_{b_1} (\mathcal{M}) \rightarrow \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}), \quad (70)
\]

\[
\text{Index Permutations: } \mathcal{S}_{b_1 \cdots b_s} (\mathcal{M}) \rightarrow \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}). \quad (71)
\]

Moreover, the following facts can be easily verified:

(1) The space of self-similar scalar fields \( R(\mathcal{M}) \equiv \mathcal{S}(\mathcal{M}) \) forms a commutative \( \mathbb{R} \)-graded ring;

(2) Each \( \mathcal{S}^{a_1 \cdots a_r}_{b_1 \cdots b_s} (\mathcal{M}) \) is an \( \mathbb{R} \)-graded module over the coefficient ring \( R(\mathcal{M}) \);

(3) The tensor product, contraction, and index permutations are graded \( R(\mathcal{M}) \)-(bi)linear operations.

Note that these are analogous to the properties of Weyl covariant classes described in Sect. 2.
We define a tensor field \( t \sim t^{a_1 \cdots a_r} b_1 \cdots b_s \) on \( \mathcal{M} \) to be horizontal if any possible contraction of \( t \) with \( \xi^a \) or \( \xi^a \) vanishes, i.e.,

\[
t^{a_1 \cdots a_r} b_1 \cdots b_s \delta_{a_i}^b = 0, \quad (1 \leq i \leq r) \quad \text{and} \quad t^{a_1 \cdots a_r} b_1 \cdots b_j \delta^b_j = 0, \quad (1 \leq j \leq s).
\]

Clearly, this is a generalization of the horizontality for vector fields described in Sect. 3.3. The collection of all horizontal elements of \( \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \) will be denoted by \( \mathcal{H}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \) or \( \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \) for short. We have \( \mathcal{H}(\mathcal{M}) = \mathcal{S}(\mathcal{M}) = \mathcal{R}_{\mathcal{E}}(\mathcal{M}) \) for scalar fields \( (r = s = 0) \), while \( \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \subseteq \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \) for other fields \( (r + s > 0) \). It is easy to see that each \( \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \) is a graded \( \mathcal{R}_{\mathcal{E}}(\mathcal{M}) \)-submodule of \( \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \), so it can be decomposed as

\[
\mathcal{H}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) = \bigoplus_{\omega \in \mathbb{R}} \omega \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}),
\]

where \( \omega \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) = \mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}) \cap \omega \mathcal{H}_{b_1 \cdots b_s}^{a_1 \cdots a_r}(\mathcal{M}). \)

The tensor product, contraction, and index permutation operations preserve horizontality, so they also provide maps (69)–(71) with \( \mathcal{S}_{\mathcal{E}}(\mathcal{M}) \) replaced by \( \mathcal{H}(\mathcal{M}) \).

Let us see some examples. Various tensor fields on \( \mathcal{M} \) that appeared in the preceding sections belong to horizontal self-similar classes or self-similar classes. For example,

\[
\begin{align*}
\xi^a & \in 0\mathcal{S}^a(\mathcal{M}), \quad \xi^b \in 0\mathcal{S}_a(\mathcal{M}), \quad \psi \in 1\mathcal{S}(\mathcal{M}) = 1\mathcal{H}(\mathcal{M}), \\
g_{ab} & \in 2\mathcal{S}_{ab}(\mathcal{M}), \quad \delta^a \in 0\mathcal{S}_a^a(\mathcal{M}), \quad g_{ab} \in 2\mathcal{S}_{ab}(\mathcal{M}), \\
p_{ab} & \in 2\mathcal{S}_{ab}(\mathcal{M}), \quad p^a \in 0\mathcal{S}_a(\mathcal{M}), \quad p_{ab} \in 2\mathcal{S}_{ab}(\mathcal{M}), \\
1^{V^a} & \in 0\mathcal{S}^a(\mathcal{M}), \quad (V^a \in \mathcal{S}(\mathcal{M}_h)), \\
(\pi^*W)_{a_1 \cdots a_s} & \in 0\mathcal{S}_{a_1 \cdots a_s}(\mathcal{M}), \quad (W_{\dot{a}_1 \cdots \dot{a}_s} \in \mathcal{S}(\mathcal{M}_h)).
\end{align*}
\]

### 4.2. Dimensional reduction of self-similar tensor fields

We now go on to discuss dimensional reduction of self-similar tensor fields on \( (\mathcal{M}, g_{ab}, \mathfrak{h}) \) using a step-by-step process from the simplest case of self-similar scalar fields to more general cases. In what follows in this section, we represent by \( \mathcal{H}_\gamma \) the reference surface of \( (\mathcal{M}, g_{ab}, \mathfrak{h}) \) corresponding to \( \gamma \in \mathcal{E}_\mathfrak{h} \) in the sense described in Sect. 3.2, and by \( \eta_\gamma \) the homothetic scale function with respect to \( \mathcal{H}_\gamma \). Then, (36) yields the Weyl transformation law for \( \eta_\gamma \), where

\[
\eta_{e^2\gamma} = \eta_\gamma - \pi^*\mathcal{P}.
\]

(a) Self-similar scalar fields. For \( f \in \mathcal{R}_{\mathcal{E}}(\mathcal{M}) \), let \( \Phi_f \) be defined by

\[
\Phi_f : \mathcal{E}_\mathfrak{h} \rightarrow C^\infty(\mathcal{M}_h), \quad \gamma \mapsto \mathcal{H}_\gamma f.
\]

We claim that \( \Phi_f \) is an element of \( \mathcal{R}_{\mathcal{M}}(\mathcal{M}_h) \), the \( \mathbb{R} \)-graded ring of Weyl covariant scalar fields on the reduced WD manifold. Indeed, if \( f \) is a homogeneous element of weight \( \omega \), then, for any \( \gamma \in \mathcal{E}_\mathfrak{h} \), \( e^{-\omega\gamma} f \) is constant along each orbit of \( \mathfrak{h} \) and, hence, is equal to \( \pi^*\mathcal{H}_\gamma(\gamma) f \). So we have

\[
f = e^{\omega\gamma} \pi^*\Phi_f(\gamma).
\]
It is clear from the above argument that the map

\[ R_\mathcal{I}(\mathcal{M}) \to R_\mathfrak{M}(\mathcal{M}_h), \quad f \mapsto \Phi_f, \tag{83} \]

preserves the \( \mathbb{R} \)-grading. Moreover, this also preserves ring operations, since the pullback map \( \mathcal{I}_* \) commutes with addition and multiplication. Therefore, (83) is a graded ring homomorphism.

Let us show that (83) is invertible. For \( F \in R_\mathfrak{M}(\mathcal{M}_h) \), we define the lift \( \uparrow F \) by

\[ \uparrow F = \sum_j e^{\omega_j \gamma} \pi^* F_j(\gamma), \tag{84} \]

where \( F_j \) represents the homogeneous summand of \( F \) of weight \( \omega_j \). The Weyl transformation law (80) guarantees that the r.h.s. is independent of the choice of \( \gamma \in \mathfrak{C}_h \). Obviously, \( \uparrow F \) belongs to \( R_\mathcal{I}(\mathcal{M}) \), so we have a well defined map

\[ R_\mathfrak{M}(\mathcal{M}_h) \to R_\mathcal{I}(\mathcal{M}), \quad F \mapsto \uparrow F. \tag{85} \]

It is clear by construction that \( \Phi_{\uparrow F} = F \) holds for any \( F \in R_\mathfrak{M}(\mathcal{M}_h) \). From (82) and the linearity of (83), \( \uparrow (\Phi_f) = f \) also holds for any \( f \in R_\mathcal{I}(\mathcal{M}) \). Thus, (85) is the inverse of (83), and the claim is proved.

To summarize, we have proved:

**Theorem 4.1.** The lifting map (85) is a graded ring isomorphism.

Consequently, it is reasonable to define the *reduced field* of a self-similar scalar field \( f \) as the unique Weyl covariant scalar field \( F \in R_\mathfrak{M}(\mathcal{M}_h) \) such that \( f = \uparrow F \), which is equal to the \( \Phi_f \) given by (81).

In addition to the ring operations, more scalar field operations can be incorporated in our framework. The following two operations are elementary and would be of importance in actual applications.

i) **Substitution into a function** \( \varphi \mapsto h(\varphi) \)

Here, \( h \) is any smooth real function on \( \mathbb{R} \). This operation acts on scalar fields of weight 0. Specifically, it gives a map \( 0^\mathfrak{G}(\mathcal{M}) \to 0^\mathfrak{G}(\mathcal{M}) \) on the \((n+1)\)-dimensional side, and \( 0^\mathfrak{W}(\mathcal{M}_h) \to 0^\mathfrak{W}(\mathcal{M}_h) \) on the \( n \)-dimensional side. The lifting map simply commutes with the operation

\[ h(\uparrow \sigma) = \uparrow (h(\sigma)), \quad (\sigma \in 0^\mathfrak{W}(\mathcal{M}_h)). \tag{86} \]

Similarly, we can also consider the substitution operation into a multi-variable function \( h(x_1, \ldots, x_N) \).

ii) **Power operation** \( \varphi \mapsto \varphi^\kappa (\kappa \in \mathbb{R}) \)

This might seem to be a special case of the substitution operation given above, but the difference is that it can act on any homogeneous scalar field. Indeed, this gives a map \( \omega^\mathfrak{G}(\mathcal{M}) \to \kappa \omega^\mathfrak{G}(\mathcal{M}) \) on the \((n+1)\)-dimensional side, and \( \omega^\mathfrak{W}(\mathcal{M}_h) \to \kappa \omega^\mathfrak{W}(\mathcal{M}_h) \) on the \( n \)-dimensional side for each \( \omega \in \mathbb{R} \). When \( \kappa < 0 \), scalar fields having zero points should be avoided. The power operation also commutes with the lifting map

\[ (\uparrow \sigma)^\kappa = \uparrow (\sigma^\kappa), \quad (\sigma \in \omega^\mathfrak{W}(\mathcal{M}_h)). \tag{87} \]
(b) Horizontal self-similar tensor fields. For $p \in \mathcal{M}$, let $H_p$ be the horizontal tangent space at $p$ and $H^*_p$ be the dual space to $H_p$, which can be naturally identified with the annihilator subspace of $(\xi_p)_p$ in the cotangent space $T^*_p \mathcal{M}$. For given abstract indices $a_1^{(1)} \ldots a_r^{(r)}$, we denote by $(H_p)_{b_1^{(1)} \ldots b_s^{(s)}}$ the tensor product of $r$ copies of $H_p$ labeled by $a_1, \ldots, a_r$, and $s$ copies of $H^*_p$ labeled by $b_1, \ldots, b_s$. Then, a self-similar tensor field $t^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}}$ belongs to the horizontal self-similar class $s_t^{a_1^{(1)} \ldots a_r^{(r)}}(\mathcal{M})$ iff its value at each $p \in \mathcal{M}$ is in $(H_p)_{b_1^{(1)} \ldots b_s^{(s)}}$. Let $\sigma_p : H_p \rightarrow T(\mathcal{M})_b$ be the bijection given in Sect. 3.3. The tensor product of $r$ copies of the inverse map $\sigma_p^{-1}$ and $s$ copies of the transpose map $t^\sigma_p$ gives a linear isomorphism

$$\sigma_p^{-1}(a_1) \cdots \sigma_p^{-1}(a_r) \sigma_p(b_1) \cdots \sigma_p(b_s) : [T(\mathcal{M})_b]^{A_1 \ldots A_r} \rightarrow [H_p]^{a_1 \ldots a_r},$$

where $[T(\mathcal{M})_b]^{A_1 \ldots A_r}$ represents the tensor product of $r$ copies of $T(\mathcal{M})_b$ labeled by $A_1, \ldots, A_r$, and $s$ copies of $T(\mathcal{M})^*_b$ labeled by $B_1, \ldots, B_s$. When $r = s = 0$, this is the identity map on $\mathbb{R}$. For $T^{A_1 \ldots A_r}_{b_1^{(1)} \ldots b_s^{(s)}} \in \mathbb{M}^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}}(\mathcal{M}_b)$, the horizontal lift of $T^{A_1 \ldots A_r}_{b_1^{(1)} \ldots b_s^{(s)}}$, labeled $\uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}}$, is the tensor field on $\mathcal{M}$, which is defined as follows. If $T^{A_1 \ldots A_r}_{b_1^{(1)} \ldots b_s^{(s)}}$ is a homogeneous element of weight $\omega$, then $\uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}}$ is given by

$$\uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}} = e^{\omega \pi(p)} (\sigma_p^{-1}(a_1) \cdots \sigma_p^{-1}(a_r) \sigma_p(b_1) \cdots \sigma_p(b_s) (T^{A_1 \ldots A_r}_{b_1^{(1)} \ldots b_s^{(s)}}(\mathcal{M})))_{p(\mathcal{M})},$$

which is independent of the choice of $\gamma \in \mathcal{C}_b$ due to (80); if $T^{A_1 \ldots A_r}_{b_1^{(1)} \ldots b_s^{(s)}}$ is inhomogeneous, then $\uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}}$ is the sum of the horizontal lift of each homogeneous summand. This generalizes the definition of horizontal lift for vector fields given in Sect. 3.3, since (89) reduces to (55) when $r = 1, s = 0$, and $\omega = 0$. Since the transpose of $\sigma_p$ coincides with the transpose of $(\pi_s)_p$, for $W_B \in \mathbb{T}_B(\mathcal{M}_b) = 0 \mathbb{M}_B(\mathcal{M}_b)$, we have

$$\uparrow W_B = (\pi^* W)_b.$$

More generally, for $T_{B_1 \ldots B_s} \in \mathbb{M}_{B_1 \ldots B_s}(\mathcal{M}_b)$, we have

$$\uparrow T_{B_1 \ldots B_s} = \sum_j e^{\omega_j \gamma} (\pi^* T_j(\gamma))_{b_1 \ldots b_s},$$

where $T_j_{B_1 \ldots B_s}$ is the homogeneous summand of $T_{B_1 \ldots B_s}$ of weight $\omega_j$. This reduces to (84) when $s = 0$, so the above definition of horizontal lift is also consistent with that for scalar fields.

We proceed to establish basic properties of horizontal lifting for Weyl covariant tensor fields.

Theorem 4.2. The operation of horizontal lifting gives a graded $\mathbb{R}$-linear isomorphism

$$\mathbb{M}^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}}(\mathcal{M}_b) \rightarrow s_{t}^{a_1^{(1)} \ldots a_r^{(r)}}(\mathcal{M}), \quad T^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}} \mapsto \uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}},$$

and commutes with the tensor algebra operations as follows:

- **Tensor Product:** For $T^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}} \in \mathbb{M}^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}}(\mathcal{M}_b)$, $S^{C_1 \ldots C_t}_{D_1^{(1)} \ldots D_t^{(t)}} \in \mathbb{M}^{C_1 \ldots C_t}_{D_1^{(1)} \ldots D_t^{(t)}}(\mathcal{M}_b)$,

$$\uparrow (T S)^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)} c_1^{(1)} \ldots c_t^{(t)} d_1^{(1)} \ldots d_t^{(t)}} = \uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}} \uparrow S^{c_1^{(1)} \ldots c_t^{(t)} d_1^{(1)} \ldots d_t^{(t)}}.$$

- **(i)**-Contraction:** For $T^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}} \in \mathbb{M}^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}}(\mathcal{M}_b)$,

$$\uparrow (C^i T)^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}}_{i j \ldots r} = \uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}} b_i^{(i)},$$

where $C^i T)^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}} \equiv T^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}} d^{B_j}_{A_j}.$

- **Index Permutations:** For $T^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}} \in \mathbb{M}^{A_1 \ldots A_r}_{B_1^{(1)} \ldots B_s^{(s)}}(\mathcal{M}_b)$ and permutations $\sigma, \tau$.

$$\uparrow (P^\sigma_T)^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}} = \uparrow T^{a_1^{(1)} \ldots a_r^{(r)} b_1^{(1)} \ldots b_s^{(s)}} b_{\sigma(1)} \ldots b_{\sigma(r)} \ldots b_{\tau(1)} \ldots b_{\tau(s)}.$$

Note that $\delta^b_A$ and $\delta^B_A$ represent the abstract Kronecker deltas on $\mathcal{M}$ and on $\mathcal{M}_b$, respectively.
Proof To prove that (92) is well defined, it suffices to verify that \( \uparrow T^{a_1 \cdots a_r} b_1 \cdots b_s \) belongs to \( \mathcal{S}_{b_1 \cdots b_s}^ {a_1 \cdots a_r} (\mathcal{M}) \) for any \( \mathcal{T}^{\lambda_1 \cdots \lambda_r} b_1 \cdots b_s \in \mathcal{M}^{\lambda_1 \cdots \lambda_r} (\mathcal{M}) \). Using a partition of unity, \( \mathcal{T}^{\lambda_1 \cdots \lambda_r} b_1 \cdots b_s \) can be written as a locally finite sum

\[
\mathcal{T}^{\lambda_1 \cdots \lambda_r} b_1 \cdots b_s = \sum_{\lambda} F_{\lambda} V_{\lambda,1} \cdots V_{\lambda,r} W_{\lambda,1} \cdots W_{\lambda,s},
\]

where \( F_{\lambda} \in R_{\mathcal{C}_{\mathcal{M}}} (\mathcal{M}) \); \( V_{\lambda,1}, \ldots, V_{\lambda,r} \in \mathcal{T}^\lambda (\mathcal{M}) \); and \( W_{\lambda,1}, \ldots, W_{\lambda,s} \in \mathcal{T}_{\lambda} (\mathcal{M}) \) for each \( \lambda \). It follows from the definition of horizontal lift and (84), (55), and (90) that

\[
\uparrow T^{a_1 \cdots a_r} b_1 \cdots b_s = \sum_{\lambda} \uparrow F_{\lambda} \uparrow V_{\lambda,1} \cdots \uparrow V_{\lambda,r} (\pi^* W_{\lambda,1})_{b_1} \cdots (\pi^* W_{\lambda,s})_{b_s}.
\]

For each \( \lambda \), we have \( \uparrow V_{\lambda,1}, \ldots, \uparrow V_{\lambda,r} \in 0 \mathcal{S}^\lambda (\mathcal{M}) \); \( (\pi^* W_{\lambda,1})_{b_1}, \ldots, (\pi^* W_{\lambda,s})_{b_s} \in 0 \mathcal{S}_{b_1 \cdots b_s}^\lambda (\mathcal{M}) \); and \( \uparrow F_{\lambda} \in R_{\mathcal{C}_{\mathcal{M}}} (\mathcal{M}) \) from (78), (79), and Theorem 4.1, respectively. Thus, \( \uparrow T^{a_1 \cdots a_r} b_1 \cdots b_s \) is in \( \mathcal{S}_{b_1 \cdots b_s}^ {a_1 \cdots a_r} (\mathcal{M}) \) and the map (92) is well defined. It is clear from the above argument that (92) is \( \mathbb{R} \)-linear and preserves \( \mathbb{R} \)-grading. The properties (93)–(95) follow readily from the definition of horizontal lift and the exponential law.

To complete the proof, we show that (92) is invertible. For \( t \sim t^{a_1 \cdots a_r} b_1 \cdots b_s \in \mathcal{S}_{b_1 \cdots b_s}^ {a_1 \cdots a_r} (\mathcal{M}) \), let \( \Phi_{t^{a_1 \cdots a_r} b_1 \cdots b_s} : \mathcal{C}_{\mathcal{B}} \rightarrow \mathcal{T}_{b_1 \cdots b_s}^{a_1 \cdots a_r} (\mathcal{M}) \) be given by

\[
(\Phi_{t^{a_1 \cdots a_r} b_1 \cdots b_s} (\gamma))_\lambda = (\sigma_{\mathcal{J}^\lambda} (x))_\lambda^{-1} (\sigma_{\mathcal{J}^\lambda} (x))_{a_1} \cdots (\sigma_{\mathcal{J}^\lambda} (x))_{a_r} (\sigma_{\mathcal{J}^\lambda} (x))_{b_1} \cdots (\sigma_{\mathcal{J}^\lambda} (x))_{b_s} (t^{a_1 \cdots a_r} b_1 \cdots b_s)_{\mathcal{J}^\lambda} (x).
\]

For arbitrary \( W_{A_1}, \ldots, W_{A_r} \in \mathcal{T}^\lambda (\mathcal{M}) \) and \( V_{A_1}^1, \ldots, V_{A_s}^s \in \mathcal{T}_{\lambda} (\mathcal{M}) \), we have

\[
\Phi_{t^{a_1 \cdots a_r} b_1 \cdots b_s} (\gamma) W_{A_1}^1 \cdots W_{A_r}^r V_{A_1}^1 \cdots V_{A_s}^s = \mathcal{J}^\lambda (t^{a_1 \cdots a_r} b_1 \cdots b_s (\pi^* W_{A_1}^1)_{a_1} \cdots (\pi^* W_{A_s}^s)_{b_s} V_{A_1}^1 \cdots V_{A_s}^s).
\]

As the r.h.s. is smooth, \( \Phi_{t^{a_1 \cdots a_r} b_1 \cdots b_s} \) is actually a map into \( \mathcal{T}_{b_1 \cdots b_s}^{a_1 \cdots a_r} (\mathcal{M}) \). It also follows from (99) and the argument in (a) that the map

\[
\mathcal{C}_{\mathcal{B}} \rightarrow C^\infty (\mathcal{M}), \quad \gamma \mapsto \Phi_{t^{a_1 \cdots a_r} b_1 \cdots b_s} (\gamma) W_{A_1}^1 \cdots W_{A_r}^r V_{A_1}^1 \cdots V_{A_s}^s,
\]

gives an element of \( R_{\mathcal{C}_{\mathcal{M}}} (\mathcal{M}) \), which implies that \( \Phi_{t^{a_1 \cdots a_r} b_1 \cdots b_s} \) is in \( \mathcal{M}_{b_1 \cdots b_s}^{a_1 \cdots a_r} (\mathcal{M}) \). Thus, we have a map

\[
\mathcal{S}_{b_1 \cdots b_s}^{a_1 \cdots a_r} (\mathcal{M}) \rightarrow \mathcal{M}_{b_1 \cdots b_s}^{a_1 \cdots a_r} (\mathcal{M}), \quad t^{a_1 \cdots a_r} b_1 \cdots b_s \mapsto \Phi_{t^{a_1 \cdots a_r} b_1 \cdots b_s}.
\]

It is now straightforward to confirm that this is the inverse map of (92).

This theorem reduces to Theorem 4.1 when \( r = s = 0 \). If we identify the coefficient rings \( R_{\mathcal{C}_{\mathcal{M}}} (\mathcal{M}) \) and \( R_{\mathcal{C}_{\mathcal{M}}} (\mathcal{M}) \) by Theorem 4.1, (92) is an isomorphism in the category of graded \( R_{\mathcal{C}_{\mathcal{M}}} (\mathcal{M}) \)-modules.

As in the scalar field case, we define the reduced field of a horizontal self-similar tensor field \( t^{a_1 \cdots a_r} b_1 \cdots b_s \) as the unique \( \mathcal{T}^{\lambda_1 \cdots \lambda_r} b_1 \cdots b_s \in \mathcal{M}^{\lambda_1 \cdots \lambda_r} (\mathcal{M}) \) such that \( t^{a_1 \cdots a_r} b_1 \cdots b_s = \uparrow T^{a_1 \cdots a_r} b_1 \cdots b_s \).

Let us see some examples. From (45) and (47), we find

\[
p_{ab} = \uparrow q_{ab},
\]

\[
\psi = \uparrow \phi.
\]

Since \( p_a^b \) and \( p_a^b \) are horizontal self-similar tensor fields that can be completely characterized by the properties \( p_a^b v^b = v^a \) (\( \forall v^a \in 0 \mathcal{S}^a (\mathcal{M}) \)) and \( p_{ac} p^b_c = p_a^b \), respectively, we deduce from

\[\text{16/24}\]
Theorem 4.2 that

\begin{align}
\rho^a_b &= \uparrow a^a_b, \quad (102) \\
p^{ab} &= \uparrow q^{ab}. \quad (103)
\end{align}

For a horizontal tensor field, one can use \( p^{ab} \) and \( p_{ab} \), instead of \( g^{ab} \) and \( g_{ab} \), to raise and lower the indices. So (100) and (103) imply the following corollary.

**Corollary 4.3.** The operation of horizontal lifting (92) commutes with the index raising and lowering operations.

**(c) General self-similar tensor fields.** From (102) we have

\[ \delta^a_b = \uparrow \delta^a_b + \xi^a_b \xi^b_a. \quad (104) \]

For \( t^{a_1 \cdots a_r b_1 \cdots b_s} \in \bar{\mathcal{G}}^{a_1 \cdots a_r b_1 \cdots b_s} (\mathcal{M}) \), we write \( t^{a_1 \cdots a_r b_1 \cdots b_s} = \delta^{a_1} \cdots \delta^{a_r} \delta^{b_1} \cdots \delta^{b_s} \phi^{c_1 \cdots c_r} a_{d_1 \cdots d_s} \), substitute (104) into each \( \delta^a_b \), and then expand. This yields \( t^{a_1 \cdots a_r b_1 \cdots b_s} \) expressed as a sum of tensor products of horizontal self-similar tensor fields and copies of \( \xi^a_b \) and \( \xi^b_a \). By applying Theorem 4.2 to each of the horizontal components, we conclude that \( t^{a_1 \cdots a_r b_1 \cdots b_s} \) can be uniquely written in the form

\[ t^{a_1 \cdots a_r b_1 \cdots b_s} = \uparrow T(0) t^{a_1 \cdots a_r b_1 \cdots b_s} + \sum_{j=1}^r \xi^a_b \uparrow T(1) t^{a_1 \cdots a_j \cdots a_r b_1 \cdots b_s} + \sum_{k=1}^s \xi^b_a \uparrow T(1) t^{a_1 \cdots a_r b_1 \cdots b_k \cdots b_s} + \sum_{1 \leq j < k \leq r} \xi^a_b \xi^b_a \uparrow T(2) t^{a_1 \cdots a_j \cdots a_k \cdots a_r b_1 \cdots b_s} + \sum_{j=1}^r \sum_{k=1}^s \xi^a_b \xi^b_a \uparrow T(2) t^{a_1 \cdots a_j \cdots a_k \cdots a_r b_1 \cdots b_s} + \sum_{1 \leq j < k \leq s} \xi^b_a \xi^a_b \uparrow T(2) t^{a_1 \cdots a_r b_1 \cdots b_j \cdots b_k \cdots b_s} + \cdots + \xi^a_b \cdots \xi^a_b \xi^b_a \cdots \xi^b_a \uparrow T(r+s) t^{a_1 \cdots a_r b_1 \cdots b_s}.
\]

We call this the **horizontal decomposition** of \( t^{a_1 \cdots a_r b_1 \cdots b_s} \). In this way, any \( t^{a_1 \cdots a_r b_1 \cdots b_s} \in \bar{\mathcal{G}}^{a_1 \cdots a_r b_1 \cdots b_s} (\mathcal{M}) \) can be reduced to a series of the Weyl covariant tensor fields \( (T(0) t^{a_1 \cdots a_r b_1 \cdots b_s}, T(1) t^{a_1 \cdots a_r b_1 \cdots b_s}, T(k) t^{a_1 \cdots a_r b_1 \cdots b_s}) \) on the reduced WD manifold. We term this process the **dimensional reduction** of \( t^{a_1 \cdots a_r b_1 \cdots b_s} \), and the resultant Weyl covariant tensor fields \( T(0) t^{a_1 \cdots a_r b_1 \cdots b_s}, T(1) t^{a_1 \cdots a_r b_1 \cdots b_s}, T(k) t^{a_1 \cdots a_r b_1 \cdots b_s} \) are referred to as the **reduced horizontal components**. Note that, for a homogeneous self-similar tensor field of self-similarity weight \( \omega \), all the reduced horizontal components are also homogeneous and of Weyl weight \( \omega \), since \( \xi^a_b \) and \( \xi^b_a \) are \( h \)-invariant.

Along with the reduced horizontal components, the algebraic expression (105) is also important for the following reasons: (1) it gives us information on how to reconstruct the original self-similar tensor field from the reduced horizontal components, and (2) it can be used, with the aid of Theorem 4.1 and Theorem 4.2, to determine how operations on self-similar tensor fields can be translated into operations on the reduced horizontal components. Thus, when working with reduced horizontal components, it is often convenient to have them embedded in the horizontal decomposition (105), rather than to deal with each of them separately.

Let us see some examples. The horizontal decompositions of the metric tensors \( g_{ab} \) and \( g^{ab} \) are

\[ g_{ab} = \uparrow q_{ab} + \xi^b_b \xi^b_a \phi(\phi^2), \quad (106) \]
\[ g^{ab} = \uparrow q^{ab} + \xi^a_a \xi^b_b \phi^{-2}. \quad (107) \]
while the volume form \((dV)^{(a_1 a_2 \cdots a_{n+1})}\) on \((\mathcal{M}, g_{\alpha\beta})\) is a self-similar tensor field of weight \(n + 1\), which can be horizontally decomposed as

\[
(dV)^{(a_1 a_2 \cdots a_{n+1})} = \pm (n + 1) \xi^h_{\alpha} ((\phi dV)^{(a_2 \cdots a_{n+1})}).
\]  

(108)

Here, \((dV)^{(\alpha_1 \cdots \alpha_n)}\) is the Weyl covariant tensor field of weight \(n\) such that, for each \(\gamma \in \mathfrak{C}_h\), \((dV)^{(\alpha_1 \cdots \alpha_n)}(\gamma)\) is the volume form on \((\mathcal{M}_h, g_{\alpha\beta})\). The sign on the r.h.s. of (108) is determined from the orientations of \(\mathcal{M}\) and \(\mathcal{M}_h\). We have another example of horizontal decomposition (104), and also other trivial examples (100)–(103).

The dimensional reduction theory for general self-similar tensor fields given above is sufficient for application to self-similar models. However, as in the preceding cases, it is also possible to describe the orientations of \(M\) and the reduced WD manifold. This can be achieved by introducing new abstract variables \(\Xi_h \sim \Xi^h, \Xi^h, \ldots\), which can be “lifted” to give \(\uparrow \Xi_h = \xi^a_h\) and \(\uparrow \Xi^h = \xi_{ab}\), so that one can consider the modules generated by tensor products of Weyl covariant tensor fields and copies of these variables. This approach would give us a formally elegant description of the principle of dimensional reduction, but we will not elaborate on it here, since such an abstraction does not seem to provide any advantages in actual use.

4.3. Calculating covariant derivatives

The horizontal decomposition (105) implies that, to calculate the Levi–Civita covariant derivative of a self-similar tensor field, we need the horizontal decompositions of the following two types of objects:

1. \(\nabla_a \xi^b_h\) and \(\nabla_a \xi^{h}_{\alpha}\);
2. Covariant derivatives of horizontally lifted Weyl covariant tensor fields.

It is useful to introduce the following quantities:

\[
\alpha_\lambda \equiv \phi^{-1}D_\lambda \phi,
\]  

(109)

\[
\Psi_{\alpha\beta} \equiv q_{\alpha\beta} - \frac{1}{2} \phi^2 F_{\alpha\beta},
\]  

(110)

which are Weyl covariant tensor fields of weight 0 and 2, respectively. Note that (109) is the Weyl structure \(A_\lambda\) in the Einstein gauge, presented here in manifestly Weyl invariant form. It is not difficult to verify the following:

\[
\nabla_a \uparrow F = \uparrow (DF)_a + \xi^b_a \downarrow (\omega F),
\]  

(111)

\[
\xi^b_a \xi^a_b \xi^b_h = 1,
\]  

(112)

\[
g_{ab} \xi^a_h \nabla_b \uparrow V^b = \uparrow (\phi^2 \alpha_\lambda V^\lambda),
\]  

(113)

\[
g_{ab} \xi^a_h \nabla_b \uparrow U^b = - \uparrow (q_{\alpha\beta} U^\lambda U^\beta),
\]  

(114)

\[
g_{ab} \xi^a_h [ \uparrow U, \uparrow V^b ] = \uparrow (\phi^2 F_{\alpha\beta} U^\lambda V^\beta),
\]  

(115)

where \(F \in R_{\mathfrak{g}\mathfrak{g}}(\mathcal{M}_h)\) and \(U^\lambda, V^\lambda \in \mathfrak{T}^\lambda(\mathcal{M}_h)\) are arbitrary. The last two identities immediately yield

\[
g_{ab} \xi^a_h \nabla_b \uparrow V^b = - \uparrow (\Psi_{\alpha\beta} U^\lambda V^\beta).
\]  

(116)

Hence, we obtain the following theorem.
Theorem 4.4. $\nabla_a \xi_b^a$ and $\nabla_a \xi_b^a$ are self-similar tensor fields of weight 0, and are horizontally decomposed as

$$\nabla_a \xi_b^a = \uparrow \Psi_a^b - s \xi_a^b (\phi^2 \alpha)^b + \xi_b^h \uparrow \alpha_a + \xi_a^h \xi_b^a, \quad (117)$$

$$\nabla_a \xi_b^a = s \uparrow (\phi^{-2} \Psi)_{ab} - \xi_b^a \uparrow \alpha_b - \xi_b^a \uparrow \alpha_a - \xi_b^a \xi_b^a, \quad (118)$$

For $V^A \in \mathfrak{X}(\mathcal{M}_8)$ and $W_A \in \mathfrak{X}(\mathcal{M}_8)$, we use Theorem 3.2 and the above formulæ to obtain the following derivative formulæ for $\uparrow V^a$ and $\uparrow W_a$:

$$\nabla_a \uparrow V^b = \uparrow (DV)_a^b + \xi_b^h \uparrow (V \cdot \Psi_a^b) - \uparrow \phi^2 \Psi_a^b + \xi_a^h \uparrow \alpha_a \xi_b^a,$$

$$\nabla_a \uparrow W_b = \uparrow (DW)_{ab} - \xi_a^h \uparrow (\Psi \cdot W)_{ab} - \uparrow \phi^2 \Psi_a^b + \xi_a^h \uparrow \alpha_a \xi_b^a,$$

where $(V \cdot \Psi)^A = V^B \Psi_B^A$, $(\Psi \cdot V)^A = \Psi_A^B V^B$, and $(\Psi \cdot W)^A = \Psi_A^B W_B$. An arbitrary Weyl covariant tensor field $T^{A_1 \ldots A_r}_{B_1 \ldots B_s}$ can be written in the form (96), so we apply (111), (119), and (120) to establish the following.

Theorem 4.5. For any $T^{A_1 \ldots A_r}_{B_1 \ldots B_s} \in \mathfrak{M}^{A_1 \ldots A_r}_{B_1 \ldots B_s}(\mathcal{M})$, $\nabla_c \uparrow T^{a_1 \ldots ar}_{b_1 \ldots bs}$ is a self-similar tensor field that is horizontally decomposed as

$$\nabla_c \uparrow (T)^{a_1 \ldots ar}_{b_1 \ldots bs} = \uparrow (DT)_c^{a_1 \ldots ar}_{b_1 \ldots bs} + \xi_b^h \uparrow (\Omega_T + \Psi \ast T)^{a_1 \ldots ar}_{b_1 \ldots bs}$$

$$- \sum_{j=1}^r \xi_b^h \uparrow (s \phi^{-2} \Psi_j (T))^{a_1 \ldots ar}_{b_1 \ldots bs} - \sum_{k=1}^s \xi_b^h \uparrow (\Psi_s T)^{a_1 \ldots ar}_{b_1 \ldots bs}$$

$$+ \sum_{j=1}^r \xi_b^h \uparrow (\alpha_j (T))^{a_1 \ldots ar}_{b_1 \ldots bs} + \sum_{k=1}^s \xi_b^h \uparrow (s \phi^2 \alpha_T)^{a_1 \ldots ar}_{b_1 \ldots bs}.$$ 

Here,

$$(\Psi \ast T)^{A_1 \ldots A_r}_{B_1 \ldots B_s} \equiv \sum_{k=1}^s \Psi_{B_k}^C T^{A_1 \ldots A_r}_{B_1 \ldots B_s} \Psi_{C}^A - \sum_{k=1}^s \Psi_{B_k}^C T^{A_1 \ldots A_r}_{B_1 \ldots B_s} \Psi_{C}^A,$$ 

$$\Psi^{(j)} (T)^{A_1 \ldots A_r}_{B_1 \ldots B_s} \equiv \Psi_{C}^A T^{A_1 \ldots A_r}_{B_1 \ldots B_s}, \quad (j = 1, \ldots, r),$$

$$\alpha_j^A (T)^{A_1 \ldots A_r}_{B_1 \ldots B_s} \equiv \alpha_j^A T^{A_1 \ldots A_r}_{B_1 \ldots B_s}, \quad (j = 1, \ldots, r),$$

$$\Psi_{s}^A (T)^{A_1 \ldots A_r}_{B_1 \ldots B_s} \equiv \Psi_{s}^A T^{A_1 \ldots A_r}_{B_1 \ldots B_s}, \quad (s = 1, \ldots, s).$$

By applying Theorem 4.4 and Theorem 4.5 to the horizontal decomposition (105), we can obtain the horizontal decomposition formula for the covariant derivative of any self-similar tensor field. A direct consequence of the result is that the Levi–Civita covariant derivative operator gives a graded $\mathbb{R}$-linear map $\nabla_c : \mathfrak{G}^{a_1 \ldots ar}_{b_1 \ldots bs}(\mathcal{M}) \to \mathfrak{G}^{a_1 \ldots ar}_{c b_1 \ldots bs}(\mathcal{M})$ for each valence type $a_1 \ldots ar$.

4.4. Curvature tensors

The derivative formulæ obtained above can be used to show that the curvature tensors on $(\mathcal{M}, g_{ab}, h)$ are self-similar tensor fields, and to obtain their horizontal decompositions, which are given below.
Riemann tensor:  The Riemann tensor $R^a_{bcd}$ on $(\mathcal{M}, g_{ab}, \mathfrak{h})$ is a self-similar tensor field of weight 0. We evaluate $R^a_{bcd}v^b = [\nabla_c, \nabla_d]v^b$ to obtain the horizontal decomposition

$$R^{ab}_{cd} = \uparrow R^a_{(0)cd} + 2\varepsilon^{[a}_{b} \uparrow R^b_{(1)cd} + 2s\varepsilon^{b}_{[a} \phi^2 R^b_{(1)d]d} - 4\varepsilon^{[a}_{b} \uparrow R^b_{(2)d]d},$$

(125)

where the reduced horizontal components are given by

$$R^a_{(0)BCD} = \mathcal{W}^a_{BCD} + \frac{S}{2}\phi^2 (\mathcal{F}^a_{[C} \mathcal{F}^B_{D]} - \mathcal{F}^a_{B} \mathcal{F}^C_{D}) + \mathcal{D}^a_{C[D} \mathcal{F}_{B]} - q_{[C} \mathcal{F}^a_{D]}B,
$$

(126)

$$R^a_{(1)BCD} = \mathcal{D}_{[C} \mathcal{F}^a_{D]} + \phi^{-1} \mathcal{D}_{[C} \phi \cdot \mathcal{F}^a_{D]} - \phi^{-1} \mathcal{D}^a_{B} \phi \cdot \mathcal{F}^C_{D} - 2s\phi^{-3} q_{[C} \mathcal{D}^a_{D]} \phi,$$

(127)

$$R^a_{(2)BD} = \frac{S}{4} \phi^2 \mathcal{F}^a_{BC} \mathcal{F}^C_{D} - \phi^{-1} \mathcal{D}^a_{(B} \mathcal{D}^a_{D)} \phi,$$

(128)

Here, the parentheses on the indices indicate symmetrization.

Ricci tensor:  The Ricci tensor $Ric_{ab} \equiv R^{c}_{c=ab}$ is also a self-similar tensor field of weight 0. The horizontal decomposition is given by

$$Ric_{ab} = \uparrow Ric_{(0)ab} + 2\varepsilon^{b}_{(a} \uparrow Ric_{(1)b]a} + \varepsilon^{b}_{(a} \varepsilon^{h}_{(b)} \uparrow Ric_{(2)},$$

(129)

where the reduced horizontal components are

$$Ric_{(0)AB} = \mathcal{W}^a_{AB} - \frac{S}{2}\phi^2 \mathcal{F}^a_{AC} \mathcal{F}^B_{C} - \phi^{-1} \mathcal{D}^a_{(A} D^B \phi - s(n - 1)\phi^{-2} q_{AB},$$

(130)

$$Ric_{(1)A} = -\frac{S}{2} \phi^{-1} D^B (\phi^3 \mathcal{F}^a_{A} B) + (n - 1)\phi^{-1} \mathcal{D}^a_{A} \phi,$$

(131)

$$Ric_{(2)} = \frac{1}{4} \phi^4 \mathcal{F}^a_{AB} \mathcal{F}^B_{A} - s\phi D^2 \phi,$$

(132)

Here, $D^2 \equiv q^{AB} \mathcal{D}^A \mathcal{D}^B$.

Ricci scalar:  The Ricci scalar $R \equiv g^{ab} Ric_{ab}$ is a self-similar scalar field of weight $-2$, and can be written as

$$R = \uparrow (q_{AB} \mathcal{W}^a_{AB} - \frac{S}{4} \mathcal{F}^{a}_{AB} \mathcal{F}^{a}_{AB} - 2\phi^{-2} D^2 \phi - s n(n - 1)\phi^{-2}).$$

(133)

4.5. Dimensional reduction of non-self-similar tensor fields

To end this section, we discuss an approach to tensor fields that are not assumed to belong to self-similar classes. Briefly, the idea used here is a generalization of the KK expansion theory. We will show below that it can be incorporated into the framework constructed in this section, but with a slight extension.

We state that a tensor field $t^{a_1 \cdots a_r}_{b_1 \cdots b_s}$ on $(\mathcal{M}, g_{ab}, \mathfrak{h})$ has weak (self-similarity) weight $\omega$ if, for any $\mathfrak{h}$-invariant 1-forms $w^a_{(1)}, \ldots, w^a_{(r)}$ and $\mathfrak{h}$-invariant vector fields $v^a_{(1)}, \ldots, v^a_{(s)}$ on $\mathcal{M}$,

$$e^{-\omega \tau} (t^{a_1 \cdots a_r} w^a_{(1)} \cdots w^a_{(r)} v^a_{(1)} \cdots v^a_{(s)}) \mathfrak{h}_{1(p)},$$

is Fourier transformable in a suitable sense (ordinary, $L^2$, distributional, or so on, depending on the given circumstances) with respect to the parameter $\tau$ along each orbit of $\mathfrak{h}$. For such $t^{a_1 \cdots a_r}_{b_1 \cdots b_s}$,
we have a family of complex tensor fields \( \hat{t}_{(\omega+ik)}^{a_1\cdots ar} b_1\cdots b_r \) \((k \in \mathbb{R})\) on \( \mathcal{M} \) characterized by the condition

\[
\forall w_1^{a(1)} \ldots w_r^{a(r)} \in 0\mathcal{G}_a(\mathcal{M}), \quad \forall t^{a_1} \ldots v_{(s)}^{a_s} \in 0\mathcal{G}^a(\mathcal{M}), \quad \forall p \in \mathcal{M},
\]

\[
\left( \hat{t}_{(\omega+ik)}^{a_1\cdots ar} b_1\cdots b_r w_{a_1}^{(1)} \ldots w_{a_r}^{(r)} v_{(1)}^{b_1} \ldots v_{(s)}^{b_s} \right)_p
\]

\[
= \int_{-\infty}^{+\infty} \frac{d \tau}{2\pi i} e^{-(\omega+ik)\tau} \left( \hat{t}_{(\omega)}^{a_1\cdots ar} b_1\cdots b_r w_{a_1}^{(1)} \ldots w_{a_r}^{(r)} v_{(1)}^{b_1} \ldots v_{(s)}^{b_s} \right)_{b_r(p)}. \quad (134)
\]

It is easy to verify that each of the complex tensor fields \( \hat{t}_{(\omega+ik)}^{a_1\cdots ar} b_1\cdots b_r \) \((k \in \mathbb{R})\) satisfies the self-similarity condition (66), but with the complex self-similarity weight \( \omega + ik \). Using this and the Fourier inversion formula, we obtain

\[
t^{a_1\cdots ar} b_1\cdots b_r = \int_{-\infty}^{+\infty} d\lambda \hat{t}_{(\lambda)}^{a_1\cdots ar} b_1\cdots b_r. \quad (135)
\]

It is straightforward to generalize the entire argument in Sect. 4.2 to complex self-similar tensor fields of complex self-similarity weights so that it can be applied to \( \hat{t}_{(\lambda)}^{a_1\cdots ar} b_1\cdots b_r \). Then, we obtain the “generalized” horizontal decomposition

\[
t^{a_1\cdots ar} b_1\cdots b_r = \int_{-\infty}^{+\infty} d\lambda \hat{T}_{(\lambda,0)}^{a_1\cdots ar} b_1\cdots b_r + \sum_{j=1}^{r} \mathbb{S}^{a_j} \int_{-\infty}^{+\infty} d\lambda \hat{T}_{(\lambda,1)}^{a_1\cdots ar \setminus a_j} b_1\cdots b_r
\]

\[
+ \sum_{k=1}^{s} \mathbb{S}_{b_k} \int_{-\infty}^{+\infty} d\lambda \hat{T}_{(\lambda,k)}^{a_1\cdots ar} b_1\cdots b_k \cdots. \quad (136)
\]

Here, the reduced horizontal components \( \hat{T}_{(\lambda,0)}^{a_1\cdots ar} b_1\cdots b_r, \hat{T}_{(\lambda,1)}^{a_1\cdots ar \setminus a_j} b_1\cdots b_r, \hat{T}_{(\lambda,k)}^{a_1\cdots ar} b_1\cdots b_k \cdots \) are now complex Weyl covariant tensor fields of complex Weyl weights. The algebraic and differential operations on \( t^{a_1\cdots ar} b_1\cdots b_r \) can be translated to those on the reduced horizontal components using the same technique described in Sects. 4.2 and 4.3 for self-similar tensor fields. We can also consider tensor fields that are “inhomogeneous” with respect to the weak weight, similar to the inhomogeneous self-similar tensor fields discussed in the preceding subsections.

A tensor field \( t^{a_1\cdots ar} b_1\cdots b_r \) does not have more than one weak self-similarity weight in most instances. However, if \( t^{a_1\cdots ar} b_1\cdots b_r \) is supported on a domain given by \( \eta_i > c_0 \) (resp. \( \eta_i < c_0 \)) for some \( c_0 \in \mathbb{R} \), then there exists \( \omega_0 \in \mathbb{R} \) such that \( \omega \) can be any value on \( (\omega_0, +\infty) \) (resp. \( (-\infty, \omega_0) \)). In this case, one can associate \( \hat{t}_{(\lambda)}^{a_1\cdots ar} b_1\cdots b_r \) with the Laplace transform of \( t^{a_1\cdots ar} b_1\cdots b_r \) along the orbit of homothety, so it is holomorphic with respect to \( \lambda \).

## 5. Formulation of self-similar models

### 5.1. Dimensional reduction of basic equations

Let us consider a theory defined on an \((n+1)\)-dimensional spacetime \((\mathcal{M}, g_{ab})\) whose fundamental degrees of freedom are tensor fields \( g \sim g_{ab}, t_1 \sim t_1^{a\cdots b\cdots}, \ldots, t_m \sim t_m^{a\cdots b\cdots} \) on \( \mathcal{M} \). We assume that the basic equations for the theory can be written as tensorial equations, such that

\[
E_{(0)} = 0, \quad E_{(1)} = 0, \ldots, E_{(m)} = 0,
\]

where \( E_{(0)} \sim E_{(0)}^{ab}, E_{(1)} \sim E_{(1)}^{a\cdots b\cdots}, \ldots, E_{(m)} \sim E_{(m)}^{a\cdots b\cdots} \) can be written in terms of the fundamental fields \( g, t_1, \ldots, t_m \) and the curvature tensors via the operations described in the preceding
subsections. We say that the equation system is *scale covariant* if there are constants \( \omega_1, \ldots, \omega_m, \Omega_0, \Omega_1, \ldots, \Omega_m \) such that

\[
E_{(j)}[r^2 g, r^{\omega_1} t_1, \ldots, r^{\omega_m} t_m] = r^{\Omega_j} E_{(j)}[g, t_1, \ldots, t_m], \quad (j = 0, 1, \ldots, m)
\]

(138) holds for any \( r \in \mathbb{R} \) and any configuration of \( g, t_1, \ldots, t_m \). Obviously, if the equation system is scale covariant, the solution space to the equation system admits the scaling symmetry, i.e., it is invariant under the scale transformation \( (g, t_1, \ldots, t_m) \to (r^2 g, r^{\omega_1} t_1, \ldots, r^{\omega_m} t_m) \).

To obtain a self-similar model from the theory, we assume a simple homothety \( h \) on the spacetime \((\mathcal{M}, g_{ab})\), with \( t_1, \ldots, t_m \) subject to the self-similarity condition (66) with suitable self-similarity weights. Then \( E_{(0)}, E_{(1)}, \ldots, E_{(m)} \) are also self-similar tensor fields on \((\mathcal{M}, g_{ab}), \) so (137) hold if every homogeneous summand of every horizontal component vanishes for each \( E_{(0)}, E_{(1)}, \ldots, E_{(m)} \). In this way, the equation system on \((n + 1)\)-dimensions for the tensor fields \( g, t_1, \ldots, t_m \) is reduced to equations on the \( n \)-dimensions for the reduced horizontal components of \( g, t_1, \ldots, t_m \), which are written in a Weyl covariant way.

If equation system (137) is scale covariant, it is reasonable to choose the self-similarity weights of \( t_1, \ldots, t_m \) to be equal to \( \omega_1, \ldots, \omega_m \) in condition (138). Then, the self-similar tensor fields \( E_{(0)}, E_{(1)}, \ldots, E_{(m)} \) are homogeneous of weight \( \Omega_0, \Omega_1, \ldots, \Omega_m \), respectively. This implies that the reduced equation system consists of as many equations as the original equation system (137) and, hence, can be expected to have exactly the required number of equations to determine the unknown functions. In this sense, the scale covariance (138) is a sufficient condition for the theory to give a valid self-similar model. If, on the contrary, there is an inhomogeneous \( E_{(j)} \), the reduced equation system is overdetermined unless there is degeneracy among the homogeneous summands.

Models with fields that are not subject to self-similarity can be formulated in the same way using the complexified framework introduced in Sect. 4.5. In principle, it is applicable to any model provided one can set a suitable homothetic background, but, practically, it would be most suitable for linear perturbative models.

5.2. Variational formulation of self-similar models

It is natural to wonder whether a self-similar model can be formulated within a variational principle, as in KK theory, by imposing self-similarity directly on the action integral. Unfortunately, the answer is negative, as will be explained below.

Let us consider an action integral

\[
I[g, t_1, \ldots, t_m] = \int_{\mathcal{M}} \mathcal{L} \, dV,
\]

(139)

where the Lagrangian \( \mathcal{L} \) can be written in terms of the fundamental fields \( g \sim g_{ab}, t_1 \sim t_1^{a_{1...}b_{...}}, \ldots, t_m \sim t_m^{a_{1...}b_{...}} \) and the curvature tensors via the algebraic and differential operations. For simplicity, we assume that the action is scale covariant, i.e., there are \( \omega_1, \ldots, \omega_m, \Omega \in \mathbb{R} \) such that

\[
I[r^2 g, r^{\omega_1} t_1, \ldots, r^{\omega_m} t_m] = r^{\Omega} I[g, t_1, \ldots, t_m], \quad (\forall r \in \mathbb{R}).
\]

(140)

Then, the equations of motion for the action are also scale covariant, so it gives a valid self-similar model. Under the self-similarity conditions, \( \mathcal{L} \, dV \) is a self-similar \((n + 1)\)-form on \((\mathcal{M}, g_{ab}, h)\) of weight \( \Omega \), so we can use (108) and (46) to horizontally decompose it as

\[
\mathcal{L} \, dV = \pm d\eta \wedge L(\phi \, dV),
\]

(141)

where \( L \in \Omega_{n-1}(\mathcal{M}), \) with \( L = \mathcal{L} \) and \( \eta \) the homothetic scale function. Hence, we can write the action as a multiple integral over \( \mathcal{M} \approx \mathcal{M}_h \times \mathbb{R} \). However, the self-similarity of the integrand
implies that the integral with respect to $\eta$ over $\mathbb{R}$ diverges unless $L = 0$. Furthermore, unlike in KK theory, a homothetic spacetime cannot be periodically identified with respect to $\eta$ to avoid the divergence. Thus, the standard KK technique cannot be applied to convert the action (139) into an integral over $\mathcal{M}_h$.

One may still expect, by comparing expression (141) with the KK counterpart, that the integral of $\phi L d\nu$ over $\mathcal{M}_h$ would give the correct reduced action. Unfortunately, this also does not work in general, because a Weyl covariant Lagrangian density does not always give Weyl covariant equations of motion, unless it is Weyl invariant. To see this in detail, let us write the variation of $L d\nu$ as

$$\delta(L d\nu) = (E + \nabla_c B^c) d\nu.$$  \hfill (142)

Then, $E = E(0)_{ab} \delta g^{ab} + E(1)_{a\cdots b\cdots \delta t(1)} + \cdots + E(m)_{a\cdots b\cdots \delta t(m)} = 0$ for any variations gives the equations of motion (137). Under the self-similarity conditions, $E$ and $B^c$ are self-similar fields of weight $\Omega - n - 1$, and can be horizontally decomposed as

$$E = \epsilon,$$

$$B^c = (\beta_H^c) + \xi^c \beta^c||,$$  \hfill (143)

where $\epsilon, \beta^c||, \beta_H^c$ can be written in terms of the reduced horizontal components of $g, t(1), \ldots, t(m)$ and their variations. Clearly, the condition that $\epsilon = 0$ for any variations means that the correct reduced equations for the self-similar model are obtained. On the other hand, we use Theorem 4.4, Theorem 4.5, and (4) to obtain

$$\delta(\phi L d\nu) = \left\{ \phi \epsilon + \Omega \phi \left( \beta^c + A_c \beta_H^c \right) \right\} d\nu + \text{(total derivatives)}.\hfill (145)$$

This implies that, if the original action $I$ is scale invariant, i.e., $\Omega = 0$, the “reduced action”

$$\int_{\mathcal{M}_h} \phi L d\nu$$

gives the correct equations of motion. Otherwise, the equations of motion are accompanied by the non-Weyl-covariant extra terms $\Omega \phi (\beta^c + A_c \beta_H^c)$, unless $\beta^c||$ and $\beta_H^c$ vanish identically. Note that most self-similar models studied to date are obtained from scale covariant actions with $\Omega \neq 0$, so the action approach described here cannot be applied. Indeed, for instance, the Einstein–Hilbert action is a scale covariant action with $\Omega = n - 1$, and one can easily see that the extra terms identically vanish only when the spacetime dimension $n + 1$ is equal to 2. In short, the primary cause of difficulty in the action approach is the fact that the scaling symmetry is an on-shell symmetry, not an action symmetry, in most theories employed in studies of self-similar models.

6. Conclusion

It has been shown that an $(n + 1)$-dimensional simple homothetic spacetime can be dimensionally reduced to a $n$-dimensional WD manifold in a manner analogous to the KK dimensional reduction. In addition, we have developed a systematic means of describing self-similar tensor fields on a homothetic spacetime in terms of Weyl covariant tensor fields on the reduced WD manifold via horizontal decomposition. This also enables us to describe various algebraic operations and Levi–Civita covariant derivatives acting on self-similar tensor fields in terms of algebraic operations, along with Weyl covariant derivative operations acting on the reduced Weyl covariant tensor fields. This method is applicable to the formulation of various self-similar models, which can be conducted by reducing the basic equations to Weyl covariant equations on the reduced WD manifold, while the variational formulation analogous to that in the KK theory is not applicable in general. Unlike conventional formulations of self-similar models, our method is completely geometrical, and does not require a specific coordinate system.
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