OSCILLATING SEQUENCES OF HIGHER ORDERS AND TOPOLOGICAL SYSTEMS OF QUASI-DISCRETE SPECTRUM

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Abstract. Fully oscillating sequences are orthogonal to all topological dynamical systems of quasi-discrete spectrum in the sense of Hahn-Parry. This orthogonality concerns not only simple but also multiple ergodic means. It is stronger than that required by Sarnak’s conjecture.

1. Main Statements

In this note, we introduce the notion of oscillating sequences of higher orders and point out its relation to topological dynamical systems of quasi-discrete spectrum in the sense of Hahn-Parry [16]. This relationship is an orthogonality between oscillating sequences of all orders and systems of quasi-discrete spectrum. This orthogonality is stronger than that required by Sarnak’s conjecture (see [27, 28] for Sarnak’s conjecture). Actually it is proved that multiple ergodic averages along polynomial times and weighted by a fully oscillating sequence tend to zero everywhere. The oscillation may be reduced to some finite order if the system has a quasi-discrete spectrum of finite order.

A sequence of complex numbers $c = (c_n)_{n \geq 0}$ is said to be oscillating of order $d$ ($d \geq 1$) if for any real polynomial $P \in \mathbb{R}_d[z]$ of degree less than or equal to $d$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i P(n)} = 0.$$  

Then we write $(c_n) \in \text{OSC}_d$. The set OSC$_d$ is decreasing as $d$ increases to infinity. If $(c_n)$ is oscillating of all orders $d \geq 1$, we say that it is fully oscillating. Then we write $(c_n) \in \text{OSC}_\infty$. We say that $(c_n)$ is oscillating of exact order $d$ if it is oscillating of order $d$ but not oscillating of order $d + 1$. Then we write $(c_n) \in \text{OSC}^*_d$. Recall that the oscillation of order one means nothing but that the Fourier-Bohr spectrum of the sequence is empty. Therefore the oscillations of higher orders are improvements of the fact that the Fourier-Bohr spectrum is empty. The notion of oscillation of order 1 appeared in [11].

Let us make a precise statement. In the following, $d^\ell q$ denotes the degree of a polynomial $q$.

Theorem 1. Let $\ell \geq 1$ be an integer. Assume that

(i) $q_1, \ldots, q_\ell \in \mathbb{Q}[z]$ are $\ell$ polynomials taking values in $\mathbb{N}$ with $\Delta := \max(d^\ell q_1, \ldots, d^\ell q_\ell)$;

(ii) $(X, T)$ is a dynamical system of quasi-discrete spectrum of order $d_T$ ($1 \leq d_T \leq \infty$);

(iii) $(c_n)$ is an oscillating sequence of order $d_T \Delta$ such that $\sum_{n=1}^{N} |c_n| = O(N)$.

Then for any continuous function $F \in C(X^\ell)$ and any point $x \in X$, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n F(T^{q_1(n)}x, \ldots, T^{q_\ell(n)}x) = 0.$$  

We need recall the notion of quasi-discrete spectrum, due to Hahn-Parry [16] which is based on Abramov’s theory on measure-theoretic dynamics [1] using the concept of quasi-eigenfunction due to Halmos and von Neumann [17]. In the present note, not as in [16], we don’t assume that our dynamical systems are totally minimal, but transitive.

By a dynamical system we mean a pair $(X, T)$ where $X$ is a compact metric space and $T: X \to X$ is a continuous map. Assume that $(X, T)$ is transitive, i.e. the orbit $O(x) :=$
\{T^n x : n \geq 0\} of some x \in X is dense in X. Let C(X) be the Banach algebra of continuous complex valued functions on X and let

\[ G(X) := \{ f \in C(X) : \forall x \in X, |f(x)| = 1 \} \]

which is a group under multiplication. All groups appearing below are subgroups of G(X).

The transformation T induces an isometry \( V_T \) of the algebra C(X), defined by \( V_T f(x) = f(Tx) \). We say that \( f \in C(X) \), \( f \neq 0 \), is an eigenfunction if there is a complex number \( \lambda \in \mathbb{C} \) for which

\[ f \circ T = \lambda f. \]

The number \( \lambda \) is called an eigenvalue. Let \( H_1 \) be the group of all eigenvalues. The eigenfunctions corresponding to the eigenvalue 1 are called invariant functions. The transitivity of T implies that invariant functions are constant functions, and \( H_1 \subset K \) where K is the group \( \{ z \in \mathbb{C} : |z| = 1 \} \) under multiplication, and eigenfunctions have constant modulus. We define \( G_1 \) to be the group of all eigenfunctions \( f \) belonging to \( G(X) \). We see that \( H_1 \subset G_0 \subset G_1 \), where we make the convention \( G_0 := H_1 \).

Now we define \( H_n \) and \( G_n \) \((n \geq 2)\) by induction. Assume \( H_1 \subset H_2 \subset \cdots \subset H_n \) and \( G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n \) have already been defined such that \( H_{i+1} \subset G_i \) for \( 0 \leq i < n \). Let \( G_{n+1} \) be the set of all \( f_{n+1} \in G(X) \) such that there is a \( g_n \in G_n \) with

\[ f_{n+1} \circ T = g_n f_{n+1}. \]

Let \( H_{n+1} \) be the set of all \( g_n \in G_n \) for which there is an \( f_{n+1} \in G_{n+1} \) verifying (1.4). Denote

\[ G := \bigcup_{n=1}^{\infty} G_n, \quad H := \bigcup_{n=1}^{\infty} H_n. \]

The elements of \( H \) are called quasi-eigenvalues and those of \( G \) are called quasi-eigenfunctions.

A dynamical system \((X, T)\) is said to have quasi-discrete spectrum if the algebra generated by the quasi-eigenfunctions is dense in \( C(X) \), or equivalently the linear span of quasi-eigenvalues is dense in \( C(X) \) because \( G \) is a multiplicative group. By using Stone-Weierstrass theorem, this is equivalent to saying that quasi-eigenfunctions separate points of \( X \). If, furthermore, \( G_d = G_{d+1} \) and \( \Delta \) is the least such integer, we say that \((X, T)\) has quasi-discrete spectrum of order \( \Delta \).

It was proved in [19] that all minimal affine transformations of any connected compact abelian group have quasi-discrete spectrum. This, together with Theorem 3 implies immediately the following result, the conclusion of which is stronger than Theorem 1.1 in [23], where the multiple ergodic means (some times called non-conventional ergodic averages) are not concerned.

**Corollary 1.** Let \( X \) be a connected compact abelian group and \( T \) be a minimal affine transformation on \( X \). Then \((X, T)\) has quasi-discrete spectrum of some order \( \Delta \) and the conclusion \((1.2)\) in Theorem 1 holds where

1. \( q_1, \cdots, q_{\ell} \in \mathbb{Q}[x] \) are \( \ell \) \((\geq 1)\) polynomials taking values in \( \mathbb{N} \) with \( \Delta := \max_{1 \leq j \leq \ell} d^0 q_j \);
2. \( (c_n) \) is an oscillating sequence of order \( \Delta \) such that \( \sum_{n=1}^{\infty} |c_n| = O(N) \).

The polynomial dynamics on the ring \( \mathbb{Z}_p \) of \( p \)-adic integers have been well investigated (see [2] [12]). Basing on the minimal decomposition theorem in [12] and on Theorem 4 we can prove the following result.

**Corollary 2.** Assume \( X = \mathbb{Z}_p \) and \( T \) is a polynomial in \( \mathbb{Z}_p[z] \). Then the conclusion \((1.2)\) in Theorem 3 holds where

1. \( q_1, \cdots, q_{\ell} \in \mathbb{Q}[x] \) are \( l \) \((\geq 1)\) polynomials taking values in \( \mathbb{N} \) with \( \Delta := \max_{1 \leq j \leq l} d^0 q_j \);
2. \( (c_n) \) is an oscillating sequence of order \( \Delta \) such that \( \sum_{n=1}^{\infty} |c_n| = O(N) \).

The same conclusion holds for dynamics defined by rational functions of good reduction for which a minimal decomposition is established in [3].

Recall that Sarnak’s conjecture concerns the orthogonality between the Möbius function \( \mu \) and a topological system \((X, T)\) with zero entropy:

\[ \forall f \in C(X), \forall x \in X, \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(n) f(T^n x) = 0. \]
Sarnak’s conjecture are proved for some special cases, for example, nilsequences \([13]\), horocycle flows \([5]\), the Thue-Morse sequence \([20]\) etc. Using our Theorem 4 and the proof of Theorem 4.1 in \([8]\), we can get the following result.

**Corollary 3.** Let \((Y, S)\) be a unique ergodic topological system with quasi-discrete spectrum, having \(\tau\) as the invariant measure. Let \((X, T)\) be a unique ergodic topological system having \(\sigma\) as the invariant measure. Suppose there exists a topological factor \(\pi: X \to Y\), which is also an isomorphism between the two measure-theoretic dynamical systems \((X, T, \sigma)\) and \((Y, S, \tau)\). Then Sarnak’s conjecture holds for \((X, T)\), even the Möbius function in \((1.5)\) can be replaced by any fully oscillating sequence or oscillating sequence of finite order (depending on the order of the quasi-discrete spectrum of \((Y, S)\)).

Here are some example of oscillating sequences:

- Let \(\alpha\) be an irrational number and \(d \geq 1\). The sequence \((e^{2\pi in^{d+1}\alpha})_{n \geq 0}\) is oscillating of exact order \(d\), because \(n^{d+1}\alpha + P(n)\) is uniformly distributed modulo for any \(P \in \mathbb{R}_d[x]\).
- The Möbius function \((\mu(n))_{n \geq 0}\) is fully oscillating \([3]\) \([15]\) (see also \([24]\)).
- The random sequences of \(-1\) or \(1\) (with respect to the symmetric Bernoulli probability) are fully oscillating almost surely. This is a special case of subnormal sequences.

Let us recall the notion of subnormality from probability theory which was introduced by Kahane \([21]\). A real random variable \(\xi\) is said to be subnormal if

\[\mathbb{E}e^{\lambda \xi} \leq e^{\lambda^2/2}, \quad \forall \lambda \in \mathbb{R}.\]

By *subnormal sequence* we mean a sequence of independent subnormal variables. An example of subnormal sequences is an i.i.d sequence of symmetric Bernoulli variables.

**Theorem 2.** A subnormal sequence is fully oscillating almost surely.

### 2. Proofs

**Proof of Theorem 1.** First observe that the Banach algebra \(C(X^d)\) of continuous functions defined on the product space \(X^d\) possesses a dense subalgebra \(\mathcal{L}_1\) where \(\mathcal{L}_1\) is the subspace of linear combinations of \(g_1 \otimes \cdots \otimes g_\ell\) with \(g_1, \cdots, g_\ell \in C(X)\), called tensor product of \(g_1, \cdots, g_\ell\), which are defined by

\[g_1 \otimes \cdots \otimes g_\ell(x_1, \cdots, x_\ell) = g_1(x_1)g_2(x_2)\cdots g_\ell(x_\ell).\]

The density is a consequence of Stone-Weierstrass theorem. Since quasi-eigenfunctions are dense in \(C(X)\) by (ii), the algebra \(C(X^d)\) possesses another dense subspace \(\mathcal{L}_2\), which consists of linear combinations of tensor products of quasi-eigenfunctions. Notice that \(\mathcal{L}_2\) is actually a subalgebra, because the quasi-eigenfunctions form a group.

On the other hand, since \(\sum_n^{N-1} |c_n| = O(N)\), there exists a constant \(D > 0\) such that

\[\forall F \in C(X^d), \quad \left\| \frac{1}{N} \sum_{n=0}^{N-1} c_n F(T^{\eta_1(n)}g_1, \cdots, T^{\eta_\ell(n)}g_\ell) \right\|_\infty \leq D \|F\|_\infty.\]

So, using an approximation argument, we have only to prove \((1.2)\) for \(F = g_1 \otimes \cdots \otimes g_\ell\), where \(g_1, \cdots, g_\ell\) are quasi-eigenfunctions.

Let \(f\) be a quasi-eigenfunction of order \(k\) (\(\leq d\)). Then there exist \(f_0 \in G_0, f_1 \in G_1, \cdots, f_{k-1} \in G_{k-1}, f_k = f\) such that

\[f_k \circ T = f_{k-1}f_k, \cdots, f_2 \circ T = f_1f_2, f_1 \circ T = f_0f_1.\]

Then, for each integer \(n \geq 0\), we have

\[(2.1) \quad f(T^n x) = f_0(x)\binom{k}{n} f_1(x)\binom{k-1}{n-1} \cdots f_{k-1}(x)\binom{1}{n-k} f_k(x)\binom{k}{n}.

(Remark that \(\binom{n}{k} = 0\) for \(0 \leq n < k\)). This can be easily proved by induction on \(n\) using Pascal’s formula. Fix \(x \in X\). Let \(\theta_j \in [0, 1]\) be the number such that \(f_j(x) = e^{2\pi i \theta_j}, 0 \leq j \leq k\). Then we can write the equality \((2.1)\) as

\[(2.2) \quad \forall n \geq 0, \quad f(T^n x) = e^{2\pi i Q(n)}\]

where

\[Q(z) = \sum_{j=0}^{k} \theta_j \binom{z}{k-j} \in \mathbb{R}[z].\]
Now assume $F = g_1 \otimes \cdots \otimes g_{\ell}$, where $g_j$ is a quasi-eigenfunction of order $k_j$ ($1 \leq j \leq \ell$). Applying (2.2) to each $g_j$, we get a polynomial $Q_j \in \mathbb{R}_k[z]$ such that
\[ g_j(T^n x) = e^{2\pi i Q_j(n)}. \]
Therefore
\[ F(T^{q_1(n)} x, \ldots, T^{q_{\ell}(n)} x) = g_1(T^{q_1(n)} x) \cdots g_{\ell}(T^{q_{\ell}(n)} x) = e^{2\pi i P(n)} \]
where
\[ P = \sum_{j=1}^\ell Q_j \circ q_j \in \mathbb{R}_d \Delta[z] \]
(recall that $\Delta = \max(d^\ell q_1, \ldots, d^\ell q_{\ell})$). Thus
\[
\frac{1}{N} \sum_{n=0}^{N-1} c_n F(T^{q_1(n)}, \ldots, T^{q_{\ell}(n)}) = \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i P(n)}
\]
which tends to zero, by (iii).

If $(X, T)$ is not of quasi-discrete spectrum, the tensor products of quasi-eigenfunctions still span a subalgebra of $C(X^\ell)$. Its closure is a closed subspace of $C(X^\ell)$, that we denote by $C_T(X^\ell)$. The conclusion of Theorem 1 remains true for all $F$ in $C_T(X^\ell)$.

**Proof of Corollary 2.** By the main theorem in [12], $Z_p = M \cup B$ (disjoint union) where $M$ is a disjoint union of at most countably many minimal sets $M_i$ and every point in $B$ is mean attracted to one of minimal sets in $M$ (see [11] for the definition of mean attraction), and each subsystem $T$: $M_i \to M_i$ is either a finite cycle or is conjugate to an adding machine. So, $T$: $M_i \to M_i$ has discrete spectrum, i.e. quasi-discrete spectrum of order 1. Thus (1.2) holds for all $x \in M_i$, by Theorem 1. For $x \in B$, we can prove (1.2) as in [11] (see the end of the proof of Theorem 1 in [11]).

The same result in Corollary 2 holds for $p$-adic dynamics defined by rational functions with good reduction, because a minimal decomposition is also proved for such dynamics in [9]. Recall that any rational function $\phi \in \mathbb{Q}_p[z]$ having good reduction is 1-Lipschitz continuous on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$:
\[ \rho(\phi(P_1), \phi(P_2)) \leq \rho(P_1, P_2) \quad \text{for all } P_1, P_2 \in \mathbb{P}^1(\mathbb{Q}_p) \]
where $\rho(\cdot, \cdot)$ is the spherical metric on $\mathbb{P}^1(\mathbb{Q}_p)$ (see [29, p.59]).

**Proof of Theorem 2.** Consider a subnormal sequence $(\xi_n)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a finite sequence of trigonometric polynomials of $s$ variables $f_n(t_1, \cdots, t_s)$ of order less than or equal to $N$. Then consider the random trigonometric polynomial
\[ P(t) = \sum_n \xi_n f_n(t). \]
There exists an absolute constant $C > 0$ such that
\[
\mathbb{P} \left( \|P\|_\infty \leq C \sqrt{s \sum \|f_n\|_\infty^2 \log N} \right) \leq \frac{1}{N^2}.
\]
We refer this inequality as Littlewood-Salem-Kahane inequality ([21] p. 70, Theorem 3. See [13] for a different proof and a generalization). Basing on this inequality, we can prove the conclusion as follows.

We have to prove that for almost all $\omega \in \Omega$ and for all polynomials $P \in \mathbb{R}[z]$, we have
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \xi_n(\omega) e^{2\pi i P(n)} = 0.
\]
Let $d = d^p P$ be the degree of $P$ and $P(x) = \sum_{j=0}^{d} t_j x^j$ with $(t_0, t_1, \cdots, t_d) \in \mathbb{R}^{d+1}$. Then
\[ e^{2\pi i P(n)} = e^{2\pi i (t_0 + t_1 n + \cdots + t_d n^d)} =: f_n(t_0, t_1, \cdots, t_d). \]
We regard $f_n$ as a trigonometric polynomial of $d + 1$ variables, of degree $1 + n + \cdots + n^d \leq (d + 1)N^d$. Thus
\[
\sup_{P \in \mathbb{R}_d[z]} \left| \sum_{n=0}^{N-1} \xi_n(\omega) e^{2\pi i P(n)} \right| = \max_{t \in \mathbb{T}^{d+1}} \left| \sum_{n=0}^{N-1} \xi_n(\omega) f_n(t) \right|.
\]
Observe that \(\|f_n\|_\infty = 1\). By Littlewood-Salem-Kahane inequality (2.3), we have
\[
\mathbb{P} \left( \sup_{P \in \mathbb{R}[z]} \left| \sum_{n=0}^{N-1} \xi_n(\omega)e^{2\pi P(n)} \right| \geq C(d + 1)\sqrt{(d + 1)N \log N} \right) \leq \frac{1}{N^2}.
\]
Then, by Borel-Cantelli lemma, for almost all \(\omega\),
\[
\sup_{P \in \mathbb{R}[z]} \left| \sum_{n=0}^{N-1} \xi_n(\omega)e^{2\pi P(n)} \right| = O(\sqrt{N \log N}).
\]
Since \(d\)'s are countable, this implies (2.4).

We have actually proved that almost surely
\[
\frac{1}{N} \sum_{n=0}^{N-1} \xi_n(\omega)e^{2\pi P(n)} = O_{d, \omega}(\sqrt{\frac{\log N}{N}}),
\]
where the constant involved in \(O\) is uniform for all polynomials of degree less than some fixed \(d\). In other words, it depends only on the degree \(d\) and on \(\omega\).

3. Some remarks

1. Hahn and Parry [16] restricted their study of quasi-discrete systems to minimal homeomorphisms and they made the assumptions that the system is totally minimal and has no non-trivial eigenvalues of finite order. For our concern, it is not necessary to make these assumptions.

Assume the space \(X\) is connected. Then the minimality of \(T\) implies automatically the total minimality. Under the extra assumption that there is no non-trivial eigenvalues of finite order, a minimal homeomorphism of quasi-discrete spectrum must be uniquely ergodic and totally ergodic (Theorem 1 and Theorem 2 in [16]), and two minimal homeomorphisms of quasi-discrete spectrum are conjugate iff their systems of quasi-eigenvalues are equivalent (Theorem 3 in [16]). It is easy to see that the total minimality implies no eigenvalues of finite order. Then \((X, T)\) has no non-trivial eigenvalues of finite order if \(X\) is connected and \(T\) is minimal.

We don't assume that \(X\) is connected. This allows us to include interesting dynamics on disconnected spaces. Here is an illustrative example. Let us consider an affine system \(Tx = ax + b\) on the ring \(\mathbb{Z}_p\) of \(p\)-adic integers, with \(a, b \in \mathbb{Z}_p\). It is well known that \(T\) is minimal iff \(a = 1 \mod p\) and \(b \neq 0 \mod p\) (see [2] [10]). Under the assumption \(a = 1 \mod p\) and \(b \neq 0 \mod p\), the dynamics \(T\) is minimal and uniquely ergodic and it has discrete spectrum [10]. But \(T\) is not totally minimal and not totally ergodic either (because \(T^m\) is not minimal when \(m = 0 \mod p\)), and all the eigenvalues of \(T\) are of finite order!

2. It was also proved by Hahn and Parry [16] that under the hypothesis that there is no non-trivial eigenvalues of finite order, a minimal homeomorphism of quasi-discrete spectrum can be represented as an affine transformation of a compact abelian group. Theorem 2 in [18] states that if there exists a totally minimal affine transformation on a compact abelian group, the group must be connected. Hoare and Parry proved in [19] that all minimal affine transformations of any connected compact abelian group have quasi-discrete spectrum, to which Theorem 1 applies (Corollary 1).

3. Dynamics on disconnected spaces like the symbolic space \(\{0, 1, \cdots, m - 1\} (m \geq 2)\) are worthy of study. But such dynamics were excluded in the works [16] [18] [19]. The dynamics of polynomial dynamics on \(\mathbb{Z}_p\) have been well investigated (see [2] [12]). If such a polynomial is minimal, then it is conjugate to the adding machine \(x \mapsto x + 1\) and therefore it has not only quasi-discrete spectrum but discrete spectrum and the eigenvalues are all of finite order. Polynomials in \(\mathbb{Z}_p[z]\) and rational functions in \(\mathbb{Q}_p[z]\) of good reductions share a minimal decomposition [12][8]. Therefore they locally behave like an adding machine (in general adding machine on \(\prod_{k=1}^{\infty} \mathbb{Z}/m_k\mathbb{Z}\) for some sequence of integers \((m_k)\)).

4. Under the assumption that \((X, T)\) is a uniquely ergodic model of some totally ergodic measure-theoretic system \((Y, \sigma, S)\) with quasi-discrete spectrum in the sense of Abramov [1], Abdalaoui, Lemańczyk and de la Rue proved (Theorem 1 in [3]) that for any \(f \in C(X)\) and
any $x \in X$ we have
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \nu_n f(T^n x) = 0 \]
where $\nu : \mathbb{N} \to C$ is a multiplicative function with $|\nu(n)| \leq 1$ and $\sum_{n=0}^{N} \nu(n) = o(N)$.
Actually the condition of quasi-discrete spectrum in the sense of Abramov can be replaced by the weaker condition of Asymptotic Orthogonal Powers (Theorem 2 in [3]): for any $f, g \in L^2(\sigma)$ with $\int f \sigma = \int g \sigma = 0$, we have
\[ \lim_{p,q \to \infty} \sup_{\kappa \in J^s(S^p, S^q)} \left| \int_{Y \times Y} f \otimes g \, d\kappa \right| = 0 \]
where $p, q$ are different primes and $J^s(S^p, S^q)$ is the set of ergodic joinings of $S^p$ and $S^q$.
In this setting, the total ergodicity plays an important role in the arguments by joining and the proof is based on Bourgain–Sarnak–Ziegler orthogonality criterion [3]. Theorem 1 in [3] and Corollary 3 in the present note have some overlaps, but no one can implies the other.

5. Huang, Wang and Zhang [20] proved that Möbius sequence is orthogonal to all topological models of an ergodic system with discrete spectrum, answering affirmatively a question raised by Downarowicz and Glasner [7]. The result recovers and generalizes some results obtained in [11] and [30]. Actually they found a sufficient condition for one point observable to be orthogonal to the Möbius sequence. The proof uses the estimate on short averages of non-pretentitious multiplicative functions due to Matomäki, Radziwill and Tao [25].

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