Efimov spectrum in bosonic systems with increasing number of particles

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It is well-known that three-boson systems show the Efimov effect when the two-body scattering length $a$ is large with respect to the range of the two-body interaction. This effect is a manifestation of a discrete scaling invariance (DSI). In this work we study DSI in the $N$-body system by analysing the spectrum of $N$ identical bosons obtained with a pairwise gaussian interaction close to the unitary limit. We consider different universal ratios such as $E_N^0/E_3^0$ and $E_N^1/E_3^0$, with $E_N^0$ being the energy of the ground $(i = 0)$ and first-excited $(i = 1)$ state of the system, for $N \leq 16$. We discuss the extension of the Efimov radial law, derived by Efimov for $N = 3$, to general $N$.

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I. INTRODUCTION

In the unitary limit, when the two-body scattering length $a$ of two identical bosons goes to $\pm \infty$, the three-boson spectrum has an infinite set of bound states, $E_3^N$, approaching zero in a geometrical progression. This is called Efimov effect and is related to a discrete scaling invariance (DSI) in the system of three identical bosons with total angular momentum $L = 0$. Explicitly, the ratio $E_3^{n+1}/E_3^n$ is equal to $e^{-2\pi/s_0} \approx 1/515.03$ with $s_0$ an universal number. As the absolute value of $a$ takes finite values, the three-body highest bound states disappear either into the atom-dimer continuum ($a > 0$) or in the three-atom continuum ($a < 0$). In recent years, the spectrum of the three-boson system has been extensively studied in the $(1/a, \kappa)$ plane, where $\kappa^2 = mE/h^2$. When one boson is added to the three-boson system, the resulting four-body system at the unitary limit has two bound states, one deep ($E_4^0$) and one shallow ($E_4^1$) with the ratios $E_4^0/E_3^0 \approx 4.6$ and $E_4^1/E_3^0 \approx 1.001$, having a universal character. This particular form of the spectrum has been recently studied up to six bosons.

In the present work we investigate the spectrum of $N \leq 16$ bosons. We compute different universal ratios, $E_N^0/E_3^0$ and $E_N^1/E_3^0$, close to the unitary limit to study the consequence of the three-boson DSI in the $N$-body system. In particular we extract some relations for $E_N^0 \to 0$.

II. $N$-BOSON ENERGY SPECTRUM

Following Ref. [3], we describe the $N$-boson system using a two-body gaussian (TBG) potential

$$V(r) = V_0 e^{-r^2/\ell^2},$$

and we solve the Schrödinger equation with mass parameter $\hbar^2/m = 43.281307$ (a$_0$)$^2$K. With $r_0 = 10$ a$_0$ and $V_0 = -1.2343566$K the model reproduces the binding energy and the scattering length of two helium atoms described by a widely used He-He interaction, the LM2M2 potential [4]. Varying the strength $V_0$ of the potential the $(a^{-1}, \kappa)$ plane can be explored.

To solve the Schrödinger equation for $N$ bosons we use the Hyperspherical Harmonic (HH) method in the version proposed in Ref. [5]. This method reproduces the values given in Ref. [6] up to $N = 6$ and it gives improved convergence with increasing $N$ as discussed in [7]. Here we extend the HH calculations up to $N = 16$. A first presentation of our results is given in Fig. [1]. We show the $N$-boson spectrum in the form of the energy wave number $\kappa_N = \text{sign}(E_N)|[E_N/(\hbar^2/m)]^{1/2}|$ given in terms of the inverse value of the two-body scattering length $a$, both expressed in units of the van der Waals length $\ell = 10.2$ a.u., calculated for the LM2M2 potential [1]. The ground state wave numbers are given by solid lines starting from $N = 3$ (black solid line) up to $N = 16$ (gray solid line). The excited state wave numbers are given by the same colored dashed lines and in most cases this state is bound and close to the $N - 1$ ground state. Starting at $N = 7$ the second excited state is shown by the colored dotted lines. In the

Fig. 1. We show the $N$-boson spectrum in the form of the energy wave number $\kappa_N = \text{sign}(E_N)|[E_N/(\hbar^2/m)]^{1/2}|$ given in terms of the inverse value of the two-body scattering length $a$, both expressed in units of the van der Waals length $\ell = 10.2$ a.u., calculated for the LM2M2 potential [1]. The ground state wave numbers are given by solid lines starting from $N = 3$ (black solid line) up to $N = 16$ (gray solid line). The excited state wave numbers are given by the same colored dashed lines and in most cases this state is bound and close to the $N - 1$ ground state. Starting at $N = 7$ the second excited state is shown by the colored dotted lines. In the
explored region, this state is bound for large positive values of \( a \) for \( N \geq 10 \). As \( a \) becomes negative this state becomes progressively unbound and at values of \( \xi \approx -95^\circ \) (see below) it disappears for all the studied \( N \). Defining the polar coordinates
\[
H_N^2 = 1/a^2 + E_N/(\hbar^2/m) \tag{2}
\]
\[
\xi = \arctan(E_N/(\hbar^2/ma^2)) \tag{3}
\]
which in the \( N = 3 \) case reduce to the polar coordinates introduced by Efimov, we plot in Fig. 1 the lines at constant values of \( \xi \) and circles at constant values of \( H \) for eye guidance. The DSI manifests itself via constant values between energy ratios at fixed values of the angle \( \xi \). In order to study DSI in the \( N \)-boson spectrum, we show the ratios between the ground state polar coordinate \( H_N^0 \) and the three-body ground state coordinate \( H_3^0 \) in Fig. 2. A clear trend of constant behavior can be observed as \( \xi \) becomes negative (\( \xi < -\pi/2 \)).

The radial law, derived by V. Efimov for three-boson systems \([8]\), gives the energy spectrum as a function of \( a \). From our results (see also Ref. \([9]\)) we propose the extension of the radial law to general values of \( N \), in which the limit of the \( N \)-boson ground state energies, is given by the equation
\[
E_N^0 + \frac{h^2}{ma^2} = \exp[\Delta(\xi)/s_0] \frac{h^2(\kappa_N^0)^2}{m} \tag{4}
\]
Here \( \kappa_N^0 \) is the wave number corresponding to the energy at the resonant limit and \( \Delta(\xi) \) an universal function. The parametrization of the universal function \( \Delta(\xi) \) has been determined using effective field theory and it can be found in Ref. \([1]\). Fixing the angle \( \xi \), the values of \( a \) in the above equation are different for each \( N \). Accordingly experimental studies can be done in systems in which the scattering length can be modified as in trapped atoms using Fesbach resonances (see for example Ref. \([10]\)).

A second remark deduced from our results regards the behavior of the energy curves for \( N = 3 - 8 \) approaching the \( \xi = -\pi \) axis as shown in Fig. 3. The quantity \( H_N^0 = \kappa_N \cos \xi \) is almost constant as the clusters reach the \( N \)-body continuum and, as a consequence, the energy wave number follows a circle line. Moreover, on the axis, the difference \( |\ell/a_N^-| - |\ell/a_N^-| \) is almost constant in all cases (a similar observation has been made in Ref. \([11]\), see also Ref. \([12]\)). Applying Eq. (4) to \( \xi = -\pi \) we obtain
\[
\kappa_N^0 a_N^- = \exp[-\Delta(-\pi)/s_0] \approx -1.56(5) \tag{5}
\]
where we have used the prediction of Ref. \([1]\). The above equation is a generalization of the \( N = 3 \) case for a general \( N \). The observed constant difference between the \( a_N \) together with Eq. (5) can be used to propose the following universal linear behavior
\[
\frac{\kappa_N^0}{\kappa_3^0} = 1 + \gamma (N - 3) \tag{6}
\]
with \( \gamma = (\kappa_N^0/\kappa_3^0) - 1 \) an universal constant. The ratio \( \kappa_N^0/\kappa_3^0 \) can be extracted, for example, from Ref. \([2]\) resulting in \( \kappa_N^0/\kappa_3^0 = 1 + 1.147(N - 3) \). The square of this relation gives a quadratic dependence of the energy with \( N \) at the unitary limit which has been already observed in Ref. \([13]\) though with different constants.

III. CONCLUSIONS

In the present work, we have shown the spectrum of a system of \( N \) identical bosons obtained with a fixed-range gaussian potential. Varying its strength, we have explored an extended range of values of the two-body scattering length including the unitary limit, \( a \to \infty \). In particular, we have explored the \( \xi = -\pi \) axis where the \( N \)-body clusters disappear into the \( N \)-body continuum and we have computed various ratios at fixed values of the angle \( \xi \) in order to study DSI. The main results of this analysis is the extension of the Efimov radial law to general \( N \), given in Eq. (4), and the linear relation with \( N \) between \( \kappa_N^0 \) and the three-body parameter \( \kappa_3^0 \). Further studies of this subject are in progress.

Acknowledgments

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FIG. 1: (Color on line) The $N = 3 - 16$ ground state (solid lines) and excited state (dashed line) energy wave numbers (in units of the van der Waals length $\ell$) as a function of $\ell/a$. For $N \geq 7$ the second excited state is also shown (dotted line). The states organized progressively starting with the $N = 3$ (black line on the top) up to the $N = 16$ (grey line on the bottom). For the sake of illustration, lines at constant values of $\xi$ and circles at constant values of $H$ are displayed.

FIG. 2: Ratios of the $N$-boson ground state polar coordinate $H^0_N$ with respect to the three-boson coordinate $H^0_3$ as a function of the angle $\xi$. 

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FIG. 3: The energy wave number, in units of $\ell$ close to threshold for $N = 8$ (red line) up to $N = 3$ (black line)

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