Lattice Algorithms for Multivariate Approximation in Periodic Spaces with General Weight Parameters

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Abstract

This paper provides the theoretical foundation for the construction of lattice algorithms for multivariate $L_2$ approximation in the worst case setting, for functions in a periodic space with general weight parameters. Our construction leads to an error bound that achieves the best possible rate of convergence for lattice algorithms. This work is motivated by PDE applications in which bounds on the norm of the functions to be approximated require special forms of weight parameters (so-called POD weights or SPOD weights), as opposed to the simple product weights covered by the existing literature. Our result can be applied to other lattice-based approximation algorithms, including kernel methods or splines.

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1 Introduction

This paper provides a theoretical foundation for the construction of lattice algorithms for multivariate $L_2$ approximation in the worst case setting, for functions in a periodic space with general weight parameters. Our construction leads to an error bound that achieves the best possible rate of convergence for lattice algorithms. We will provide a background in the Introduction, assuming little prior knowledge from the reader, and highlight our new contribution together with our motivation for this work. Section 2 provides the mathematical formulation of the problem and reviews known results, while Section 3 proves the main theorem.

Lattice rules have been developed since the late 1950s as cubature rules for multivariate periodic integrands characterized by absolutely convergent Fourier series. In recent years lattice rules have also been successfully used for non-periodic integrands (by way of random shifts or tent transformation). Lattice rules represent a branch of the family of quasi-Monte Carlo (QMC) methods. The other significant branch of QMC methods encompasses digital nets and sequences. Reference books and surveys include [36, 46, 17, 18, 8, 33, 12, 32, 11, 34, 41]. Our interest lies in the situations where the dimensionality, $d$, or the number of variables, is very large, say, in the hundreds or thousands. Much of the research focus in the last two decades has been on the concept of strong tractability [38, 39, 40]: loosely speaking, it means seeking error bounds that are independent of dimension $d$ (or, in the case of polynomial tractability, with error bounds that grow only polynomially as $d$ increases).

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By now it is well known that these desired dimension-independent error bounds hold for carefully chosen lattice rules and suitably defined weighted function space for the integrands [11]. The first studied setting involves the so-called product weights [49, 50], where one weight parameter \( \gamma_j > 0 \) is associated with each coordinate direction \( x_j \) to describe the significance of the integrand \( f(x) = f(x_1, \ldots, x_d) \) in the direction \( x_j \). The typical condition to ensure dimension independence is that the sequence \( \{\gamma_j\} \) decays fast enough to ensure that the series is summable, i.e., \( \sum_{j=1}^{\infty} \gamma_j < \infty \). Faster rates of decay of the weights \( \gamma_j \) then enable faster rates of convergence in the dimension-independent error bounds, provided that the integrands are sufficiently smooth and the cubature rules are capable of benefiting from the higher smoothness.

The component-by-component (CBC) construction of lattice rules that can achieve dimension-independent cubature error bounds in weighted spaces is another milestone in the past 20 years [47, 11]. These constructions are proved to achieve the optimal convergence rates [23], while fast CBC algorithms (based on the fast Fourier transform) allow these constructions to easily reach tens of thousands of dimensions with millions of points [42, 41]. For example, in the periodic Hilbert space setting where the squared Fourier coefficients decay at the rate of \( \alpha > 1 \) (corresponding roughly to \( \alpha/2 \) available mixed derivatives), the optimal convergence rate is \( O(n^{-\alpha/2+\delta}) \), \( \delta > 0 \), where the implied constant is independent of \( d \) provided that \( \sum_{j \geq 1} \gamma_j^{1/\alpha} < \infty \) for the case of product weights, while the cost for a fast CBC construction with \( n \) points up to dimension \( d \) is \( O(d n \log(n)) \) operations.

Lattice rules have also been analyzed in the context of multivariate approximation. We refer to the resulting algorithms as lattice algorithms; they can be described as follows. For a function with an absolutely convergent Fourier series, we approximate this function by first truncating the series expansion to a finite index set, and then approximating the remaining Fourier coefficients (which are integrals of the function against each basis function) by lattice truncating the series expansion to a finite index set, and then approximating the remaining.

Two main strategies have been employed in the literature: one strategy is to construct lattice algorithms to directly minimize the error bound [29, 30]; the other strategy is to construct lattice algorithms which exactly reconstruct the function on a given finite index set (the latter are called reconstruction lattices) [20, 21, 45, 1, 24]. Both strategies can make use of CBC constructions. It is also possible to combine both strategies in one CBC construction. Also related are spline algorithms or kernel methods [52, 53] and collocation [35, 48] using lattice points.

Though product weights are easy to work with, they may not be the appropriate model to describe the dimension structure of the target function, as we now explain. In accordance with the concepts of effective dimension [2] and multivariate decomposition [28], every function in \( d \) dimensions can be written (in more than one way) as a sum of \( 2^d \) terms, \( f = \sum_{\mathbf{u} \subseteq \{1, \ldots, d\}} f_{\mathbf{u}} \), where each term \( f_{\mathbf{u}} \) depends only on a subset \( \mathbf{u} \) of the \( d \) variables, namely, \( x_j \) for \( j \in \mathbf{u} \). By using an appropriate orthogonal decomposition for the function space, the terms \( f_{\mathbf{u}} \) are mutually orthogonal. We can moderate the importance of each term \( f_{\mathbf{u}} \) by using a weight parameter \( \gamma_{\mathbf{u}} \) for each subset \( \mathbf{u} \) in the function space definition. These weights \( \gamma_{\mathbf{u}} \) are called general weights [13, 11]. Product weights are then the special case of general weights in which \( \gamma_{\mathbf{u}} = \prod_{j \in \mathbf{u}} \gamma_j \), that is, the weight associated with the group of variables indexed by the subset \( \mathbf{u} \) is obtained by taking the product of the weights \( \gamma_j \) corresponding to the variables \( x_j \) with \( j \in \mathbf{u} \). The full generality of general weights allows more flexibility in modeling the functions, but comes at an exponential cost in \( d \) for the CBC construction. So compromises have been made by researchers by imposing further structure on the weights, including order dependent weights where \( \gamma_{\mathbf{u}} \) depends only on the cardinality of the set \( \mathbf{u} \), and finite order weights where \( \gamma_{\mathbf{u}} \) is zero for all \( \mathbf{u} \) with cardinality greater than a prescribed number. More interestingly, recent works on PDEs with random coefficients
have led to the invention of a new form of weights called *POD weights – product and order dependent weights*, which combines the features of product weights and order dependent weights, and even to *SPOD weights – smoothness-driven product and order dependent weights*, which involves an inner structure depending on the smoothness property of the function space.

A theoretical justification for the CBC construction of lattice rules for integration under the general weights setting has been known for some time [13], while fast CBC algorithms for POD weights and SPOD weights have only been developed in recent times, driven by the need in PDE applications [27, 10]. The basic model involves an elliptic PDE with a random coefficient [3, 27, 25] which is parameterized by a sequence of stochastic variables (our integration variables) and the goal is to compute the expected value (an integral with respect to the large number or even an infinite number of stochastic variables) of a linear functional $G(\cdot)$ of the PDE solution $u$ with respect to the spatial variables (with spatial dimension 1, 2, or 3). To be able to apply the known integration error bounds for lattice rules, a key step in the analysis is to estimate the norm of the integrand $f = G(u)$. This requires us to “differentiate the PDE” [3, 25], to obtain the regularity of $G(u)$ with respect to the stochastic variables. Estimates of the norm and our desire for dimension-independent error bounds together lead to the choice of POD weights or SPOD weights for the function space, and in turn create the need to construct lattice rules appropriate to these weights.

**New contribution**

Motivated by the strong desire to obtain higher order moments or other statistics of the quantities of interest rather than just the expected value, we seek in future work to apply lattice algorithms directly to the PDE solution $u$ at all spatial points as a function of the stochastic variables. However, all presently available theory on lattice algorithms for approximation has been for the unweighted setting or just with product weights. So to proceed we must

- provide a theoretical justification in the periodic setting for the CBC construction of lattice algorithms for approximation with general weights; and
- develop the fast CBC algorithms for the construction of lattice algorithms with special structure of weights, especially POD weights and SPOD weights.

This paper will address the first point, while a companion paper [6] will address the second point. Both papers involve novel elements and significant new results that cannot be obtained by trivial generalizations of existing results.

Specifically, in the periodic Hilbert space setting where the squared Fourier coefficients decay at the rate of $\alpha > 1$, the optimal convergence rate for integration is $O(n^{-\alpha/2+\delta})$, $\delta > 0$, as mentioned earlier, see [50]. The optimal algorithm for $L_2$ approximation in this setting based on the class of *arbitrary linear information* (implying that all Fourier coefficients can be obtained exactly) can achieve the same convergence rate $O(n^{-\alpha/2+\delta})$, $\delta > 0$, see [37]. However, if we restrict to the class of *standard information* where only function values are available, then it has been an open problem whether the same rate can be achieved with no dependence of the error bound on the dimension $d$. A general (non-constructive) result in [31] yields the convergence rate $O(n^{-(\alpha/2)(1/(1+1/\alpha)+\delta)})$, $\delta > 0$, which is nearly optimal for large $\alpha$ but loses a factor of nearly $1/2$ in the rate when $\alpha$ is small. A very recent manuscript [22] appears to have solved this open problem.

For algorithms that use function values at lattice points, it is proved that the best possible convergence rate is $O(n^{-\alpha/4+\delta})$, $\delta > 0$; see [1] for a lower bound which proved the unavoidable
gap in the convergence rates between integration and approximation. We prove in this paper
that a generating vector for a lattice algorithm and general weights can be obtained by a CBC
construction to achieve this best possible error bound
\[ O(n^{-\alpha/4+\delta}), \quad \delta > 0, \]
with the implied constant independent of \( d \), provided that the general weights satisfy the condi-
tion \( \sum_{\mathbf{u} \subset \mathbb{N}, |\mathbf{u}| < \infty} \gamma_\mathbf{u} \left[ 2\zeta(\alpha \lambda)\right] |\mathbf{u}| < \infty \), where \( \lambda = 1/(\alpha - 4\delta) \). Here the summation is
over all finite subsets of positive integers \( \mathbb{N} := \{1, 2, \ldots\} \), \(|\mathbf{u}|\) denotes the cardinality of the set \( \mathbf{u} \), and \( \zeta(x) := \sum_{h=1}^{\infty} h^{-x} \) denotes the Riemann zeta function.

At this point the result can only be said to be semi-constructive, in that a CBC construction
with fully general weights has a prohibitively high computational cost. In our companion paper
[6] we develop fast CBC algorithms for weights with special structure, including so-called POD
weights and SPOD weights.

Though the best possible convergence rate for algorithms based on lattice points cannot match
the rate of a general optimal algorithm in this setting (i.e., \( O(n^{-\alpha/4+\delta}) \) versus \( O(n^{-\alpha/2+\delta}) \),
\( \delta > 0 \), see above), lattice-based algorithms have a number of advantages including simplicity and
efficiency in applications, making them still attractive and competitive.

2 Problem formulation and review of known results

2.1 Lattice rules and lattice algorithms

We consider one-periodic real-valued \( L_2 \) functions defined on \([0,1]^d\) with absolutely convergent
Fourier series
\[ f(x) = \sum_{h \in \mathbb{Z}^d} \hat{f}_h e^{2\pi i h \cdot x}, \quad \text{with} \quad \hat{f}_h := \int_{[0,1]^d} f(x) e^{-2\pi i h \cdot x} \, dx, \]
where \( \hat{f}_h \) are the Fourier coefficients and \( h \cdot x = h_1 x_1 + \cdots + h_d x_d \) denotes the usual dot product.

A (rank-1) lattice rule [46] with \( n \) points and generating vector \( \mathbf{z} \in \{1, \ldots, n-1\}^d \) approxi-
mates the integral of \( f \) by
\[ I(f) := \int_{[0,1]^d} f(x) \, dx \approx Q(f) := \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k \mathbf{z}}{n}\right), \]
where the braces around a vector indicate that we take the fractional part of each component in
the vector. Using the character property
\[ \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i k h \cdot \mathbf{z}/n} = \begin{cases} 1 & \text{if } h \cdot \mathbf{z} \equiv_n 0, \\ 0 & \text{if } h \cdot \mathbf{z} \not\equiv_n 0, \end{cases} \]
it is easy to show that the integration error is
\[ Q(f) - I(f) = \sum_{h \in \mathbb{Z}^d \setminus \{0\} \atop h \equiv 0} \hat{f}_h, \quad (2.1) \]
where \( \equiv_n \) denotes congruence modulo \( n \).

A lattice algorithm for multivariate approximation [29] with \( n \) points and generating vector
\( \mathbf{z} \in \{1, \ldots, n-1\}^d \), together with an index set \( \mathcal{A}_d \subset \mathbb{Z}^d \), approximates the function \( f \) by first
When measured in the worst case setting, the worst case integration error is

\[ e_{\mathrm{wor-int}}(z) := \sup_{f \in H_d, \|f\|_d \leq 1} |I(f) - Q(f)| = \left( \sum_{h \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|r(h)\|} \right)^{1/2}. \]

The initial integration error is \( e_{\mathrm{wor-int}} := \sup_{f \in H_d, \|f\|_d \leq 1} |I(f)| = 1 \). It is proved in [13] that for general weights \( \gamma_u \), if \( n \) is prime, a generating vector \( z \) can be obtained by a CBC construction to achieve the integration error bound

\[ |I(f) - Q(f)| \leq \left( \frac{1}{n - 1} \sum_{u \in \mathbb{N}, |u| < \infty} \gamma_u^\lambda \left[2\zeta(\alpha\lambda)\right]^{\gamma_u} \right)^{1/(2\lambda)} \|f\|_d \quad \text{for all } \lambda \in \left(\frac{1}{n}, 1\right]. \]
The result generalizes to non-prime $n$, with $n - 1$ replaced by the Euler totient function $\varphi(n) := \{1 \leq z \leq n : \gcd(z, n) = 1\}$. Fast CBC algorithms for integration with product weights, order dependent weights, POD weights and SPOD weights have been developed in [42, 4, 26, 19].

### 2.4 Approximation

For the approximation problem we can follow [29, 30] to define the index set $A_d$ with some parameter $M > 0$ by

$$A_d(M) := \{ h \in \mathbb{Z}^d : r(h) \leq M \}, \quad (2.5)$$

with the difference being that here we have general weights determining the values of $r(h)$ in (2.4) while [29, 30] considered product weights. We can then bound the first sum in the $L_2$ approximation error (2.3) by

$$\sum_{h \in A_d(M)} |\hat{h}|^2 = \sum_{h \in A_d(M)} |\hat{f}_h|^2 r(h) \frac{1}{r(h)} \leq \| f \|_d^2 \frac{1}{M},$$

since $r(h) > M$ for $h \notin A_d(M)$. The second sum in (2.3) contains the integration error of the function $g_h(x) := f(x) e^{-2\pi i h \cdot x}$ so from (2.1) we obtain

$$|\hat{f}_h - \hat{f}_h| = |I(g_h) - Q(g_h)|^2 = \left| \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} (\hat{g}_h, e^{2\pi i \ell \cdot z})^2 \right| = \left| \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{r(h + \ell)} \right|^2$$

leading to

$$\sum_{h \in A_d(M)} |\hat{f}_h - \hat{f}_h|^2 \leq \| f \|_d^2 E_d(z), \quad \text{with} \quad E_d(z) := \sum_{h \in A_d(M)} \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{r(h + \ell)}.$$

Combining these bounds yields the worst case $L_2$ approximation error bound

$$e_{n,d,M}^{\text{wor-app}}(z) := \sup_{f \in H_d, \| f \|_{d \leq 1}} \| f - A(f) \|_{L_2} \leq \left( \frac{1}{M} + E_d(z) \right)^{1/2}. \quad (2.6)$$

More precisely, it was proved in [29] that $e_{n,d,M}^{\text{wor-app}}(z) \leq 1/M + \varphi(T_z)$, where $\varphi(T_z)$ denotes the spectral radius of some matrix $T_z$ depending on the generating vector $z$. The $1/M$ term arose from the truncation to the finite index set, while the spectral radius arose from the cubature approximations of the remaining coefficients. Though the elements of the matrix $T_z$ were known explicitly, there was no simple expression for the spectral radius and therefore it was upper bounded by its trace, leading to the quantity $E_d(z)$. The initial approximation error is given by

$$e_{0,d}^{\text{wor-app}} := \sup_{f \in H_d, \| f \|_{d \leq 1}} \| f \|_{L_2} = \max_{u \in \{1:d\}^d} \gamma_u^{1/2}.$$
It is worth noting that \( n \) needs to be large enough in relation to \( M \). For example, we must have \( (n - 1)^{\alpha}/\gamma_{(1)} > M \), since otherwise \( h^* := (n - 1, 0, \ldots, 0) \) belongs to \( A_d(M) \) because \( r(h^*) = (n - 1)^{\alpha}/\gamma_{(1)} \), as a result of which the sum over \( h \) and \( \ell \) in \( E_d(z) \) contains a pair \((h^*, \ell^*) \) with \( \ell^* := (n - 0, \ldots, 0) \) which contributes the value of \( \gamma_{(1)} \) in \( E_d(z) \), leading to the sum not converging to zero as \( n \to \infty \). This is ensured below by the condition \( n \geq \kappa M^{1/\alpha} \).

For product weights it is proved in [29, 30] that for \( n \) prime a generating vector \( z \) can be obtained by a CBC construction to achieve

\[
E_d(z) \leq \left( \frac{1}{\mu^{\lambda}} \right) \frac{|A_d(M)|}{n - 1} \prod_{j=1}^{d} \left( \left( 1 + 2(1 + \mu^{\lambda}) \right) \zeta(\alpha \lambda) \gamma_{j}^{\alpha} \right)^{1/\lambda} \quad \text{for all } \lambda \in (\frac{1}{\alpha}, 1],
\]

and for all \( \mu \in (0, (1 - 1/\kappa)^{\alpha}] \) where \( \kappa > 1 \) is such that \( n \geq \kappa M^{1/\alpha} \). Combining this with the bounds on the cardinality of the index set [29, 30]

\[
\gamma_{(1)}^{1/\lambda} \leq |A_d(M)| \leq M^{q} \prod_{j=1}^{d} \left( 1 + 2\zeta(\alpha q) \gamma_{j}^{q} \right) \quad \text{for all } q > \frac{1}{\alpha},
\]

we balance the two terms in (2.6) by taking

\[
\frac{1}{M} = \frac{M^{q/\lambda}}{n^{1/\lambda}} \quad \text{and } \quad q = \lambda \in (\frac{1}{\alpha}, 1],
\]

to obtain the convergence rate \( e_{n,d,M}^{wor-app}(z) = O(n^{-1/(4\lambda)}) = O(n^{-\alpha/(4+4)}), \) \( \delta > 0, \) with \( \lambda = 1/(\alpha - 4\delta) \), where the implied constant is independent of \( d \) provided that \( \sum_{j=1}^{\gamma_{j}^{1}} < \infty \). The cost of the fast CBC algorithm based on \( E_d(z) \) with product weights is \( O(|A_d(M)| d n \log(n)) \) operations.

Alternatively, since \( r(h) \leq M \) for \( h \in A_d(M) \), we can bound \( E_d(z) \) by

\[
E_d(z) \leq \sum_{h \in A_d(M)} \sum_{\ell \in \mathbb{Z}^{d} \setminus \{0\}} \frac{1}{r(h + \ell)} \leq M S_d(z), \quad \text{with } \quad S_d(z) := \sum_{h \in \mathbb{Z}^{d}} \sum_{\ell \in \mathbb{Z}^{d} \setminus \{0\}} \frac{1}{r(h + \ell)}.
\]  

A variant of the quantity \( S_d(z) \) first appeared in the context of a Lattice-Nyström method for Fredholm integral equations of the second kind [9]. (Note, however, that in [9] the quantity was defined as the square root of the double sum.) The advantage of working with \( S_d(z) \) instead of \( E_d(z) \) is that there is no dependence on the index set \( A_d(M) \), thus the error analysis is simpler and the construction cost is lower.

For product weights it is proved in [9] that for \( n \) prime a generating vector \( z \) can be obtained by a CBC construction to achieve

\[
S_d(z) \leq \left( \frac{1}{\mu^{\lambda}} \right) \frac{1}{n} \prod_{j=1}^{d} \left( \left( 1 + 2(1 + \mu^{\lambda})^{1/2} \right) \zeta(\alpha \lambda) \gamma_{j}^{\alpha} \right)^{1/\lambda} \quad \text{for all } \lambda \in (\frac{1}{\alpha}, 1],
\]

and for all \( \mu \in (0, 2^{-3\alpha}] \). Now we balance the two terms in (2.6) by taking

\[
\frac{1}{M} = \frac{M}{n^{1/\lambda}} \quad \text{and } \quad \lambda \in (\frac{1}{\alpha}, 1],
\]
to obtain again the convergence rate $e_{\text{wor-app}}(z) = O(n^{-1/(4\lambda)}) = O(n^{-\alpha/4+\delta})$, $\delta > 0$, with
\[ \lambda = 1/(\alpha - 4\delta), \]
where the implied constant is independent of $d$ provided that $\sum_{j\geq 1} \gamma_j^4 < \infty$.

The cost of the fast CBC algorithm based on $S_d(z)$ with product weights is only $O(dn \log(n))$ operations.

The goal of this paper is to obtain an analogous error bound for $S_d(z)$ with general weights. This is our Theorem 3.5 below.

Note that one can of course apply lattices that are designed for integration with general weights directly for function approximation, but as shown in [29] this will lead to a worse convergence rate compared to designing lattice algorithms specifically for the purpose of approximation.

2.5 Other related results

In this paper we consider $L_2$ approximation in the worst case setting. Instead of measuring the error in the $L_2$ norm, one can also consider other $L_p$ norms, including the $L_\infty$ norm [1]. Instead of the worst case setting, one can also consider the average case setting where the function space is equipped with a Gaussian probability measure with a prescribed mean and covariance function [30]. As already mentioned earlier, instead of attempting to reduce the error criterion directly, one can look for reconstruction lattices which exactly reproduce functions whose Fourier series are solely supported on a finite index set [20, 21, 45]. Instead of the periodic setting, one can also consider the related cosine space or Chebyshev space of nonperiodic functions [48, 7, 44, 24]. One can also consider lattice algorithms in the context of discrete least square approximation [24].

Also related are spline algorithms or kernel methods [51, 52, 53] and collocation [35, 48] based on lattice points. In a reproducing kernel Hilbert space with a “shift-invariant” kernel (as we have in the periodic setting here), the structure of the lattice points allows the required linear system to be solved in $O(n \log(n))$ operations. Since splines have the smallest worst case $L_2$ approximation error among all algorithms that make use of the same sample points (see for example [53]), the lattice generating vectors from this paper can be used in a spline algorithm and the worst case error bound from this paper will carry over as an immediate upper bound with no further multiplying constant. The advantage of a spline over the lattice algorithm (2.2) is that there is no presence of the index set $A_d$ so is extremely efficient in practice.

The best possible rate of convergence for lattice algorithms for approximation is proved recently in [1] to be only half of the optimal rate of convergence for lattice rules for integration (i.e., $O(n^{-\alpha/4+\delta})$ versa $O(n^{-\alpha/2+\delta})$, $\delta > 0$). This is a negative point for lattice algorithms, since there are other approximation algorithms such as Smolyak algorithms or sparse grids which do not suffer from this loss of convergence rate. However, as discussed in [1], lattice algorithms have their advantages in terms of simplicity in construction and point generation, and stability and efficiency in application, making them still attractive and competitive despite the reduced convergence rate.

3 New results on approximation with general weights

3.1 Size of the index set

Although the index set $A_d(M)$ does not appear in the expression for $S_d(z)$ in (2.7), it does impact the lattice algorithm $A(f)$ defined in (2.2). Here we provide a bound on its cardinality.
Lemma 3.1. For all \( d \geq 1, M > 0 \) and \( q > 1/\alpha \) we have

\[ |A_d(M)| \leq M^q \sum_{u \subseteq \{1:d\}} [2\zeta(\alpha q)]^{|u|} |\gamma_u|^q. \]

Proof. We can write

\[
|A_d(M)| = \sum_{h \in A_d(M)} 1 = \sum_{h \in \mathbb{Z}^d, r(h) \leq M} 1 = \sum_{u \subseteq \{1:d\}} \sum_{h \in \mathbb{Z}^d, \text{supp}(h) \subseteq u} 1 \]
\[
= \sum_{u \subseteq \{1:d\}} \sum_{h_u \in (\mathbb{Z} \setminus \{0\})^{|u|}} 1 = \sum_{u \subseteq \{1:d\}} \sum_{h_u \in B_u(\gamma_u M)} 1 = \sum_{u \subseteq \{1:d\}} |B_u(\gamma_u M)|,
\]

where we introduced the auxiliary set (treating \( \gamma_u \) as part of the argument \( m \))

\[ B_u(m) := \left\{ h_u \in (\mathbb{Z} \setminus \{0\})^{|u|} : \prod_{j \in u} |h_j|^\alpha \leq m \right\}, \quad u \subseteq \{1:d\}. \]

The result holds if we can show that, for all \( u \subseteq \{1:d\}, M > 0 \) and \( q > 1/\alpha \),

\[ |B_u(m)| \leq [2\zeta(\alpha q)]^{|u|} m^q. \]

Since the coordinates are equivalent, it suffices to consider the set

\[ \tilde{B}_s(m) := \left\{ h \in (\mathbb{Z} \setminus \{0\})^s : \prod_{j=1}^s |h_j|^\alpha \leq m \right\}, \]

and show that, for all \( s \geq 0, m > 0 \) and \( q > 1/\alpha \),

\[ |\tilde{B}_s(m)| \leq [2\zeta(\alpha q)]^s m^q. \quad (3.1) \]

We have \(|\tilde{B}_0(m)| = 0\) and \(|\tilde{B}_1(m)| = 2[m^{1/\alpha}]\), so (3.1) holds trivially for \( s = 0, 1 \). For \( s \geq 2 \) and assuming that (3.1) holds with \( s \) replaced by \( s - 1 \), we have

\[
|\tilde{B}_s(m)| = \sum_{h \in (\mathbb{Z} \setminus \{0\})^s, \prod_{j=1}^s |h_j|^\alpha \leq m} 1 = \sum_{h_s \in (\mathbb{Z} \setminus \{0\})^s} \sum_{h_{s-1} \in (\mathbb{Z} \setminus \{0\})^{s-1}, \prod_{j=1}^{s-1} |h_j|^\alpha \leq \frac{m}{|h_s|^\alpha}} 1
\]
\[
= \sum_{h_s \in (\mathbb{Z} \setminus \{0\})^s} |\tilde{B}_{s-1}(\frac{m}{|h_s|^\alpha})| \leq \sum_{h_s \in (\mathbb{Z} \setminus \{0\})^s} [2\zeta(\alpha q)]^{s-1}(\frac{m}{|h_s|^\alpha})^q = [2\zeta(\alpha q)]^s m^q.
\]

Hence (3.1) holds for all \( s \geq 0 \). This completes the proof. \( \Box \)

3.2 Dimension-wise decomposition of the error criterion

With product weights, the error criterion \( S_d(z) \) in (2.7) can be expressed recursively as

\[ S_d(z) = (1 + 2\zeta(2\alpha)\gamma_d^2) S_{d-1}(z_1, \ldots, z_{d-1}) + \theta_d(z), \]

where \( \theta_d \) is an expression which captures all the contribution of the new component \( z_d \). The precise formula for \( \theta_d \) in the case of product weights does not matter here. The main point is
that this recursion provided the inductive step to prove the bound (2.8) for the CBC construction of \( z \) based on minimizing \( S_d(z_1, \ldots, z_s) \) for each \( s = 1, 2, \ldots, d \). Moreover, this means that the result for product weights is extensible in \( d \).

The situation with general weights is quite different: the recursion turns out to be rather complicated because “future” weights get tangled up! To enable us to describe this complication, we introduce a temporary notation, \( S_d(z) = S_d(z; \left\{ \gamma_u \right\}_{u \subseteq \{1:d\}}) \), to show its explicit dependence on the weights \( \left\{ \gamma_u \right\} \). We show in the proof of the following lemma that, with respect to any input sequence \( \left\{ \beta_u \right\} \) (“replaceable” in every function call), we have

\[
S_d(z; \left\{ \beta_u \right\}_{u \subseteq \{1:d\}}) = S_{d-1}(z_1, \ldots, z_{d-1}; \left\{ \beta_u \right\}_{u \subseteq \{1:d-1\}}) + 2\zeta(2\alpha) S_{d-1}(z_1, \ldots, z_{d-1}; \left\{ \beta_{u \cup \{d\}} \right\}_{u \subseteq \{1:d-1\}}) + \theta_d(z; \left\{ \beta_u \right\}_{u \subseteq \{1:d\}}),
\]

(3.2)

with \( \theta_d \) defined as in (3.5) below. That is, the error criterion in \( d \) dimensions with input sequence \( \left\{ \beta_u \right\} \) depends on the error criterion in \( d - 1 \) dimensions with input sequence \( \left\{ \beta_u \right\}, \) as well as on the error criterion in \( d - 1 \) dimensions in which each parameter \( \beta_u \) in the input sequence is “replaced” by a corresponding “future” parameter \( \beta_{u \cup \{d\}} \).

This dependence on “future” weights means that a CBC construction which minimizes \( S_d(z_1, \ldots, z_s) \) one dimension at a time cannot work here, because there is no way to establish a valid induction argument! To overcome this difficulty, we need to fix a priori a value of \( d \) to be the target final dimension, and then use the recursion (3.2) to decompose \( S_d(z; \left\{ \gamma_u \right\}_{u \subseteq \{1:d\}}) \) all the way down to the first dimension, to yield a dimension-wise decomposition of the error criterion as shown below. A similar strategy has previously been used in [43, 14].

**Lemma 3.2.** Let \( d \geq 1 \) be fixed and a sequence of weights \( \left\{ \gamma_u \right\}_{u \subseteq \{1:d\}} \) be given. We can write

\[
S_d(z) = \sum_{s=1}^{d} T_{d,s}(z_1, \ldots, z_s),
\]

(3.3)

where, for each \( s = 1, 2, \ldots, d, \)

\[
T_{d,s}(z_1, \ldots, z_s) := \sum_{w \subseteq \{s+1:d\}} [2\zeta(2\alpha)]^{\left| w \right|} \theta_s(z_1, \ldots, z_s; \left\{ \gamma_u \right\}_{u \subseteq \{1:s\}}),
\]

(3.4)

\[
\theta_s(z_1, \ldots, z_s; \left\{ \beta_u \right\}_{u \subseteq \{1:s\}}) := \sum_{h \in \mathbb{Z}^s} \sum_{\ell \in \mathbb{Z}^s, \ell \neq 0} \frac{\beta_{\text{supp}(h)}}{r'(h)} \frac{\beta_{\text{supp}(h+\ell)}}{r'(h + \ell)}
\]

(3.5)

with \( r'(h) := \prod_{j \in \text{supp}(h)} |h_j|^\alpha \).

**Proof.** We remark that the quantity \( r'(h) \) is essentially \( r(h) \) without the weight parameter. We generalize the definition in (2.7): for each \( s = 1, 2, \ldots \) and any input sequence \( \left\{ \beta_u \right\}_{u \subseteq \{1:s\}} \), we define

\[
S_s(z_1, \ldots, z_s; \left\{ \beta_u \right\}_{u \subseteq \{1:s\}}) := \sum_{h \in \mathbb{Z}^s} \sum_{\ell \in \mathbb{Z}^s \setminus \{0\}} \frac{\beta_{\text{supp}(h)}}{r'(h)} \frac{\beta_{\text{supp}(h+\ell)}}{r'(h + \ell)}
\]
so that \(S_d(z_1, \ldots, z_d; \{\gamma_u\}_{u \subseteq \{1:d\}})\) agrees with (2.7). We define additionally \(S_0 := 0\). By separating the cases (i) \(h_s = \ell_s = 0\), (ii) \(h_s \neq 0\) and \(\ell_s = 0\), (iii) \(\ell_s \neq 0\), we obtain

\[
S_s(z_1, \ldots, z_s; \{\beta_u\}_{u \subseteq \{1:s\}}) = \sum_{h \in \mathbb{Z}^{s-1}} \frac{\beta_{\supp(h)} \beta_{\supp(h + \ell)}}{r'(h) r'(h + \ell)}
\]

\[+ \sum_{h_s \in \mathbb{Z}^s \setminus \{0\}} \frac{1}{h_s} \sum_{h \in \mathbb{Z}^{s-1}} \frac{\beta_{\supp(h) \cup \{s\}} \beta_{\supp(h + \ell) \cup \{s\}}}{r'(h) r'(h + \ell)} + \theta_s(z_1, \ldots, z_s; \{\beta_u\}_{u \subseteq \{1:s\}})
\]

where \(\theta_s\) is as defined in (3.5). This proves (3.2) by taking \(s = d\).

Abbreviating temporarily the above recursion by

\[
S_s(\{\beta_u\}_u) = S_{s-1}(\{\beta_u\}_u) + c S_{s-1}(\{\beta_{u \cup \{s\}}\}_u) + \theta_s(\{\beta_u\}_u),
\]

with \(c := 2\zeta(2\alpha)\), we can write

\[
S_d(\{\gamma_u\}_u) = S_{d-1}(\{\gamma_u\}_u) + c S_{d-1}(\{\gamma_{u \cup \{d\}}\}_u) + \theta_d(\{\gamma_u\}_u)
\]

\[= S_{d-2}(\{\gamma_u\}_u) + c S_{d-2}(\{\gamma_{u \cup \{d-1\}}\}_u) + \theta_{d-1}(\{\gamma_u\}_u)
\]

\[+ c S_{d-2}(\{\gamma_{u \cup \{d\}}\}_u) + c^2 S_{d-2}(\{\gamma_{u \cup \{d-1,d\}}\}_u) + c \theta_{d-1}(\{\gamma_{u \cup \{d\}}\}_u)
\]

\[+ \theta_d(\{\gamma_u\}_u)
\]

\[= S_{d-3}(\{\gamma_u\}_u) + c S_{d-3}(\{\gamma_{u \cup \{d-2\}}\}_u) + \theta_{d-2}(\{\gamma_u\}_u)
\]

\[+ c S_{d-3}(\{\gamma_{u \cup \{d-1\}}\}_u) + c^2 S_{d-3}(\{\gamma_{u \cup \{d-2,d-1\}}\}_u) + c \theta_{d-2}(\{\gamma_{u \cup \{d\}}\}_u)
\]

\[+ \theta_{d-1}(\{\gamma_u\}_u)
\]

\[+ c S_{d-3}(\{\gamma_{u \cup \{d\}}\}_u) + c^2 S_{d-3}(\{\gamma_{u \cup \{d-1,d\}}\}_u) + c^3 S_{d-3}(\{\gamma_{u \cup \{d-2,d-1\}}\}_u) + c^2 \theta_{d-2}(\{\gamma_{u \cup \{d-1,d\}}\}_u)
\]

\[+ c \theta_{d-1}(\{\gamma_{u \cup \{d\}}\}_u)
\]

\[+ \theta_d(\{\gamma_u\}_u).
\]

Continuing this way to decompose the terms until we reach \(S_0 = 0\), we eventually end up with the expression in the lemma.

\[\square\]

### 3.3 Component-by-component construction

Algorithm 3.3 below outlines a CBC construction for the generating vector \(z\). Lemma 3.4 provides the essential averaging argument needed in the proof of Theorem 3.5. The main result, Theorem 3.6, is that we achieve the best possible convergence rate for lattice algorithms as proven in [1].

**Algorithm 3.3.** Given \(n \geq 2\), a fixed \(d \geq 1\), and a sequence of weights \(\{\gamma_u\}_{u \subseteq \{1:d\}}\), the generating vector \(z^* = (z_1^*, \ldots, z_n^*)\) is constructed as follows: for each \(s = 1, \ldots, d\), with \(z_1^*, \ldots, z_{s-1}^*\) fixed, choose \(z_s \in \{1, \ldots, n - 1\}\) to minimize \(T_{d,s}(z_1^*, \ldots, z_{s-1}^*, z_s)\) given by (3.4).
Lemma 3.4. Let \( n \) be prime. For any \( s \geq 1 \), any input sequence \( \{ \beta_u \}_{u \subseteq \{1:s\}} \), all values of \( z_1, \ldots, z_{s-1} \), and all \( \lambda \in (\frac{1}{\lambda}, 1] \), we have

\[
\frac{1}{n-1} \sum_{z_s=1}^{n-1} \left[ \Theta_s(z_1, \ldots, z_{s-1}, z_s; \{ \beta_u \}_{u \subseteq \{1:s\}}) \right]^\lambda \\
\leq \frac{\tau}{n} \left( \sum_{z_s \in \{1:s\}} \beta_u^\lambda [2\zeta(\alpha \lambda)]^{|w|} \right) \left( \sum_{u \subseteq \{1:s\}} \beta_u^\lambda [2\zeta(\alpha \lambda)]^{|w|} \right),
\]

(3.6)

where \( \tau := \max(6, 2.5 + 2^{2\alpha \lambda + 1}) \).

Proof. We have from the formula (3.5) and Jensen’s inequality \( \sum_k a_k \leq (\sum_k a_k^\lambda)^{1/\lambda} \) for all \( a_k \geq 0 \) that

\[
\text{Avg} := \frac{1}{n-1} \sum_{z_s=1}^{n-1} \left[ \Theta_s(z_1, \ldots, z_{s-1}, z_s; \{ \beta_u \}_{u \subseteq \{1:s\}}) \right]^\lambda \\
\leq \frac{1}{n-1} \sum_{z_s=1}^{n-1} \sum_{t_1, \ldots, t_{s-1}} \sum_{e \in \mathbb{Z}^s, \ e \neq 0} \left( \frac{\beta_{\text{supp}(h)}^\lambda \beta_{\text{supp}(h + \ell)}^\lambda}{r^\lambda(h) r^\lambda(h + \ell)} \right) \\
= \frac{1}{n-1} \sum_{h \in \mathbb{Z}^s} \sum_{e \in \mathbb{Z}^s, \ e \neq 0} \left( \frac{\beta_{\text{supp}(h)}^\lambda \beta_{\text{supp}(h + \ell)}^\lambda}{r^\lambda(h) r^\lambda(h + \ell)} \right) \\
= \sum_{h \in \mathbb{Z}^s} \sum_{e \in \mathbb{Z}^s, \ e \neq 0} \left( \frac{\beta_{\text{supp}(h)}^\lambda \beta_{\text{supp}(h + \ell)}^\lambda}{r^\lambda(h) r^\lambda(h + \ell)} \right),
\]

(3.7)

where we separated the terms depending on whether or not \( \ell_s \) is a multiple of \( n \). In particular, we used the fact that for \( n \) prime and \( \ell_s \not\equiv n \), the product \( \ell_s z_s \) covers each number from 1 to \( n-1 \) in some order as \( z_s \) runs from 1 to \( n-1 \).

Next we obtain an upper bound by dropping the conditions on the dot product in both terms in (3.7) (thus dropping all dependence on \( z_1, \ldots, z_{s-1} \)), and define

\[
G(h_s, \ell_s) := \sum_{h \in \mathbb{Z}^{s-1}} \sum_{e \in \mathbb{Z}^{s-1}} \left( \frac{\beta_{\text{supp}(h, h_s)}^\lambda \beta_{\text{supp}(h, h_s) + (\ell, \ell_s)}^\lambda}{r^\lambda(h, h_s) r^\lambda((h, h_s) + (\ell, \ell_s))} \right)^\lambda,
\]

so that

\[
\text{Avg} \leq \frac{1}{n-1} \sum_{h, \ell \in \mathbb{Z}^s, \ e \neq 0} G(h_s, \ell_s) + \sum_{h, \ell \in \mathbb{Z}^s, \ e \neq 0} G(h_s, \ell_s) \\
\leq \frac{1}{n-1} \sum_{h, \ell \in \mathbb{Z}^s, \ e \neq 0} G(h_s, \ell_s) + \sum_{h, \ell \in \mathbb{Z}^s, \ e \neq 0} G(h_s, \ell_s) + \sum_{h, \ell \in \mathbb{Z}^s, \ e \neq 0} G(h_s, \ell_s),
\]

where we further upper bounded by dropping the condition \( \ell_s \not\equiv n \) in the first term and then splitting the remaining term into the cases \( h_s \equiv 0 \) and \( h_s \not\equiv 0 \).
Thus and similarly, we have

\[ G(h_s, \ell_s) = \left( \sum_{h \in \mathbb{Z}^{-1}} \frac{\beta^\lambda_{\text{supp}(h, h_s)}}{r'(h, h_s)^\lambda} \right) \left( \sum_{q \in \mathbb{Z}^{-1}} \frac{\beta^\lambda_{\text{supp}(q, h_s + \ell_s)}}{r'(q, h_s + \ell_s)^\lambda} \right) \]

\[ = \begin{cases} \frac{1}{|h_s|^{\lambda \alpha}} |P| & \text{if } h_s = 0 \text{ and } \ell_s \neq 0, \\ \frac{1}{|h_s|^{\lambda \alpha}} Q & \text{if } h_s \neq 0 \text{ and } \ell_s = -h_s, \\ \frac{1}{|h_s|^{\lambda \alpha}} \frac{1}{|h_s + \ell_s|^{\lambda \alpha}} Q^2 & \text{if } h_s \neq 0 \text{ and } \ell_s \neq -h_s, \end{cases} \]

with the abbreviations

\[ P := \sum_{u \in \{1, s-1\}} \beta^\lambda_u [2 \zeta(\alpha \lambda)]^{\lceil u \rceil} \quad \text{and} \quad Q := \sum_{u \in \{1, s-1\}} \beta^\lambda_{u+\{s\}} [2 \zeta(\alpha \lambda)]^{\lceil u \rceil}. \]

Thus

\[ W_1 := \sum_{h \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}\setminus\{0\}} G(h, \ell) \]

\[ = \sum_{\ell \in \mathbb{Z}\setminus\{0\}} P \frac{\Omega}{|\ell|^{\lambda \alpha}} + \sum_{h \in \mathbb{Z}\setminus\{0\}} \frac{\Omega P}{|h|^{\lambda \alpha}} + \sum_{h \in \mathbb{Z}\setminus\{0\}} \sum_{\ell \in \mathbb{Z}\setminus\{0, h\}} \frac{\Omega^2}{|h|^{\lambda \alpha} |h + \ell|^{\lambda \alpha}} \]

\[ \leq 2 \zeta(\alpha \lambda) P \frac{\Omega}{n^{\lambda \alpha}} + 2 \zeta(\alpha \lambda) Q P + \sum_{h \in \mathbb{Z}\setminus\{0\}} \sum_{q \in \mathbb{Z}\setminus\{0\}} \frac{\Omega^2}{|h|^{\lambda \alpha} |q|^{\lambda \alpha}} \]

\[ = 2 [2 \zeta(\alpha \lambda)] P \frac{\Omega}{n^{\lambda \alpha}} + [2 \zeta(\alpha \lambda)]^2 Q^2, \]

and similarly

\[ W_2 := \sum_{h \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}\setminus\{0\}} \sum_{h \neq 0} G(h, \ell) \]

\[ = \sum_{\ell \in \mathbb{Z}\setminus\{0\}} P \frac{\Omega}{|\ell n|^{\lambda \alpha}} + \sum_{h \in \mathbb{Z}\setminus\{0\}} \frac{\Omega P}{|hn|^{\lambda \alpha}} + \sum_{h \in \mathbb{Z}\setminus\{0\}} \sum_{\ell \in \mathbb{Z}\setminus\{0, h\}} \frac{\Omega^2}{|hn|^{\lambda \alpha} |(h + \ell)n|^{\lambda \alpha}} \]

\[ \leq 2 \zeta(\alpha \lambda) P \frac{\Omega}{n^{\lambda \alpha}} + 2 \zeta(\alpha \lambda) Q P + \sum_{h \in \mathbb{Z}\setminus\{0\}} \sum_{q \in \mathbb{Z}\setminus\{0\}} \frac{\Omega^2}{|hn|^{\lambda \alpha} |qn|^{\lambda \alpha}} \]

\[ \leq 2 [2 \zeta(\alpha \lambda)] P \frac{\Omega}{n^{\lambda \alpha}} + [2 \zeta(\alpha \lambda)]^2 Q^2. \]

Moreover, we have

\[ W_3 := \sum_{h \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}\setminus\{0\}} G(h, \ell) = \sum_{h \in \mathbb{Z}\setminus\{0\}} \sum_{\ell \neq 0} \sum_{h \neq 0} \frac{\Omega^2}{|h|^{\lambda \alpha} |h + \ell n|^{\lambda \alpha}} \]

\[ = \Omega^2 \sum_{h \in \mathbb{Z}\setminus\{0\}} \left( \frac{1}{|h|^{\lambda \alpha}} \sum_{\ell \in \mathbb{Z}} \frac{1}{|h + \ell n|^{\lambda \alpha}} - \frac{1}{|h|^{2 \lambda \alpha}} \right). \]
where we separated out the case $\ell = 0$. Writing $h = pn + k$ with $k$ being the remainder modulo $n$, we obtain

$$W_3 = Q^2 \sum_{k = -\lfloor (n-1)/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \sum_{p \in \mathbb{Z}} \left( \frac{1}{|pn + k|^{\alpha \lambda}} \sum_{\ell \in \mathbb{Z}} \frac{1}{|pn + k + \ell n|^{\alpha \lambda}} - \frac{1}{|pn + k|^{2\alpha \lambda}} \right)$$

$$= Q^2 \sum_{k = -\lfloor (n-1)/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \sum_{p \in \mathbb{Z}} \left( \frac{1}{|pn + k|^{\alpha \lambda}} \sum_{q \in \mathbb{Z}} \frac{1}{|qn + k|^{\alpha \lambda}} - \frac{1}{|pn + k|^{2\alpha \lambda}} \right)$$

$$= Q^2 \sum_{k = -\lfloor (n-1)/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \left( \left( \sum_{p \in \mathbb{Z}} \frac{1}{|pn + k|^{\alpha \lambda}} \right)^2 - \sum_{p \in \mathbb{Z}} \frac{1}{|pn + k|^{2\alpha \lambda}} \right)$$

$$\leq Q^2 \sum_{k = -\lfloor (n-1)/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \left( \left( \frac{1}{|k|^{\alpha \lambda}} + \sum_{p \in \mathbb{Z} \setminus \{0\}} \frac{1}{|pn|^{\alpha \lambda}|1 + k/(pn)|^{\alpha \lambda}} \right)^2 - \frac{1}{|k|^{2\alpha \lambda}} \right).$$

Now for $|k| \leq \lfloor (n-1)/2 \rfloor$ and $|p| \geq 1$, we have $|1 + k/(pn)| \geq 1/2$, and so

$$W_3 \leq Q^2 \sum_{k = -\lfloor (n-1)/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \left( \left( \frac{1}{|k|^{\alpha \lambda}} + \sum_{p \in \mathbb{Z} \setminus \{0\}} \frac{1}{|pn|^{\alpha \lambda}|1/2\lambda|^{\alpha \lambda}} \right)^2 - \frac{1}{|k|^{2\alpha \lambda}} \right)$$

$$= Q^2 \sum_{k = -\lfloor (n-1)/2 \rfloor}^{\lfloor (n-1)/2 \rfloor} \left( \frac{2^{\alpha \lambda + 1} \zeta(\alpha \lambda)}{n^{\alpha \lambda}} + \left( \frac{2^{\alpha \lambda + 1} \zeta(\alpha \lambda)}{n^{\alpha \lambda}} \right)^2 \right)$$

$$\leq Q^2 \left( 4\zeta(\alpha \lambda) \frac{2^{\alpha \lambda + 1} \zeta(\alpha \lambda)}{n^{\alpha \lambda}} + (n - 1) \left( \frac{2^{\alpha \lambda + 1} \zeta(\alpha \lambda)}{n^{\alpha \lambda}} \right)^2 \right) \leq \frac{2^{2\alpha \lambda + 1} \zeta(\alpha \lambda)^2 Q^2}{n^{\alpha \lambda}},$$

where we used $\alpha \lambda > 1$.

Combining the bounds on $W_1, W_2, W_3$, and using $1/(n - 1) \leq 2/n$ and $1/n^{\alpha \lambda} \leq 1/n$ and $1/n^{2\alpha \lambda} \leq 1/(2n)$, we obtain

$$\text{Avg} \leq \frac{2^2 [2\zeta(\alpha \lambda)]^2 P Q + 2 [2\zeta(\alpha \lambda)]^2 Q^2}{n} + \frac{2 [2\zeta(\alpha \lambda)]^2 Q^2 + 1/2 [2\zeta(\alpha \lambda)]^2 Q^2}{n} + \frac{2^{2\alpha \lambda + 1} \zeta(\alpha \lambda)^2 Q^2}{n}$$

$$= \frac{6 [2\zeta(\alpha \lambda)]^2 P Q + \tau_0 [2\zeta(\alpha \lambda)]^2 Q^2}{n}, \quad \tau_0 := 2.5 + 2^{2\alpha \lambda + 1}. $$

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Writing $\tau := \max(6, \tau_0)$, we have

$$Avg \leq \frac{\tau}{n} \left( \sum_{u \subseteq \{1, s\}} \beta^\lambda_{u,s} [2\zeta(\alpha \lambda)]^{[u]} \right) \times \left( \sum_{u \subseteq \{1, s\}} \beta^\lambda_u [2\zeta(\alpha \lambda)]^{[u]} \right)$$

$$= \frac{\tau}{n} \left( \sum_{s \subseteq u \subseteq \{1, s\}} \beta^\lambda_u [2\zeta(\alpha \lambda)]^{[u]} \right) \times \left( \sum_{s \subseteq u \subseteq \{1, s\}} \beta^\lambda_u [2\zeta(\alpha \lambda)]^{[u]} \right)$$

$$= \frac{\tau}{n} \left( \sum_{s \subseteq u \subseteq \{1, s\}} \beta^\lambda_u [2\zeta(\alpha \lambda)]^{[u]} \right) \times \left( \sum_{s \subseteq u \subseteq \{1, s\}} \beta^\lambda_u [2\zeta(\alpha \lambda)]^{[u]} \right).$$

This completes the proof. \qed

**Theorem 3.5.** Let $n$ be prime. For fixed $d \geq 1$ and a given sequence of weights $\{\gamma_u\}_{u \subseteq \{1, d\}}$, a generating vector $z$ obtained from the CBC construction following Algorithm 3.3 satisfies for all $\lambda \in (\frac{1}{n}, 1)$,

$$S_d(z) \leq \left( \frac{\tau}{n} \left( \sum_{\emptyset \neq u \subseteq \{1, d\}} |u| \gamma^\lambda_u [2\zeta(\alpha \lambda)]^{[u]} \right) \left( \sum_{u \subseteq \{1, d\}} \gamma^\lambda_u [2\zeta(\alpha \lambda)]^{[u]} \right) \right)^{1/\lambda}, \quad (3.8)$$

where $\tau := \max(6, 2.5+2^{\alpha+1})$. Furthermore, if the weights are such that there exists a constant $\xi \geq 1$ (which may depend on $\lambda$) such that

$$\gamma^\lambda_{u,w,m} \leq \xi \frac{\gamma^\lambda_u}{[2\zeta(\alpha \lambda)]^{[w]}} \quad \text{for all} \quad u \subseteq \{1 : s\}, \quad w \subseteq \{s+1 : d\}, \quad s \geq 1, \quad d \geq 1, \quad (3.9)$$

then (3.8) holds with $\tau$ replaced by $\tau \xi$ and with the $|u|$ factor inside the first sum replaced by 1.

**Proof.** Let $z^* = (z^*_1, \ldots, z^*_d)$ denote the generating vector obtained from Algorithm 3.3. We have from (3.3) that

$$S_d(z^*) = \sum_{s=1}^{d} T_{d,s}(z^*_1, \ldots, z^*_s).$$

For each $s = 1, \ldots, d$, the component $z^*_s$ is chosen to minimize the quantity $T_{d,s}(z^*_1, \ldots, z^*_s, z_s)$ over all $z_s \in \{1, \ldots, n-1\}$. Since the minimum must be smaller than or equal to the average, for all $\lambda \in (\frac{1}{n}, 1)$ we have

$$[T_{d,s}(z^*_1, \ldots, z^*_s)]^\lambda \leq \frac{1}{n-1} \sum_{z_s=1}^{n-1} [T_{d,s}(z^*_1, \ldots, z^*_s, z_s)]^\lambda$$

$$= \frac{1}{n-1} \sum_{z_s=1}^{n-1} \left( \sum_{w \subseteq \{s+1 : d\}} [2\zeta(2\alpha)]^{[w]} \theta_{\alpha}(z^*_1, \ldots, z^*_s, z_s; \{\gamma_{u,w,m}\}_{u \subseteq \{1, s\}}) \right)^\lambda$$

$$\leq \frac{1}{n-1} \sum_{z_s=1}^{n-1} \sum_{w \subseteq \{s+1 : d\}} [2\zeta(2\alpha)]^{\lambda[w]} \left[ \theta_{\alpha}(z^*_1, \ldots, z^*_s, z_s; \{\gamma_{u,w,m}\}_{u \subseteq \{1, s\}}) \right]^\lambda$$

$$= \sum_{w \subseteq \{s+1 : d\}} [2\zeta(2\alpha)]^{\lambda[w]} \left( \frac{1}{n-1} \sum_{z_s=1}^{n-1} \left[ \theta_{\alpha}(z^*_1, \ldots, z^*_s, z_s; \{\gamma_{u,w,m}\}_{u \subseteq \{1, s\}}) \right]^\lambda \right),$$

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where we used Jensen’s inequality.

Now for every $w \subseteq \{s + 1 : d\}$, we apply Lemma 3.4 with $z_1 = z_1^*$, ..., $z_s = z_s^*$ and input sequence $\beta_u = \gamma_u$ for each $u \subseteq \{1 : s\}$. In other words, Lemma 3.4 is applied $2^{d-s}$ times, each time with a different input sequence depending on $w$. Using $[2\zeta(2\alpha)]^\lambda \leq [2\zeta(\alpha)]^2$ and (3.6), we obtain

$$[T_d, s(z^*)]^\lambda \leq \sum_{w \subseteq \{s + 1 : d\}} \frac{\tau}{n} \left( \sum_{u \subseteq \{1 : s\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right) \left( \sum_{u \subseteq \{1 : s\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right)$$

$$= \frac{\tau}{n} \sum_{w \subseteq \{s + 1 : d\}} \left( \sum_{s \subseteq u \subseteq \{1 : s\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right) \left( \sum_{u \subseteq \{1 : s\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right)$$

$$\leq \frac{\tau}{n} \left( \sum_{w \subseteq \{s + 1 : d\}} \max_{s \subseteq u \subseteq \{1 : s\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right) \left( \sum_{u \subseteq \{1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right)$$

$$= \frac{\tau}{n} \left( \sum_{s = 1}^{d} \max_{w \subseteq \{s + 1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right) \left( \sum_{u \subseteq \{1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right)^{1/\lambda}.$$

This leads to

$$S_d(z^*) \leq \frac{\sum_{s = 1}^{d} \left( \frac{\tau}{n} \sum_{w \subseteq \{s + 1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right) \left( \sum_{u \subseteq \{1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right)^{1/\lambda}}{1/\lambda}.$$

(3.10)

We remark at this point that, unlike a typical induction proof for a CBC construction, there is no induction in this proof.

We consider two ways to proceed. The first way is to replace the maximum in (3.10) by the sum, which yields

$$S_d(z^*) \leq \frac{\sum_{s = 1}^{d} \sum_{w \subseteq \{1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \left( \sum_{u \subseteq \{1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right)^{1/\lambda}}{1/\lambda}.$$

The second way is to apply the assumption (3.9) in (3.10), so that the maximum drops out to yield

$$S_d(z^*) \leq \frac{\sum_{s = 1}^{d} \sum_{w \subseteq \{1 : s\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \left( \sum_{u \subseteq \{1 : d\}} \gamma_u [2\zeta(\alpha)]^{\lvert u \rvert} \right)^{1/\lambda}}{1/\lambda},$$

which does not contain the factor $\lvert u \rvert$ inside the first sum.□
We summarize the main conclusion of this paper in the following theorem, which states that we achieve the best possible convergence rate for lattice algorithms as shown in [1].

**Theorem 3.6.** Given $d \geq 1$, $\alpha > 1$ and weights $\{\gamma_u\}_{u \subseteq \mathbb{N}}$, let $n$ be prime and $M > 0$. The lattice algorithm (2.2), with index set (2.5) and generating vector $z$ obtained from the CBC construction following Algorithm 3.3, satisfies for all $\lambda \in \left(\frac{1}{\alpha}, 1\right]$,

$$e_{n,d,M}^{\text{wor-app}}(z) \leq \left(\frac{1}{M} + M S_d(z)\right)^{1/2} \leq \left(\frac{1}{M} + M \left(\frac{\tau}{n} \left(\sum_{u \subseteq \{1:d\}} |u| \gamma_u^{\lambda} [2\zeta(\alpha \lambda)] |u|\right) \left(\sum_{u \subseteq \{1:d\}} \gamma_u^{\lambda} [2\zeta(\alpha \lambda)] |u|\right)^{1/\lambda}\right)^{1/2},$$

where $\tau = \max(6, 2.5 + 2^{2\alpha \lambda} + 1)$.

Taking $M = n^{1/(2\lambda)}$, we obtain a simplified upper bound

$$e_{n,d,M}^{\text{wor-app}}(z) \leq \sqrt{2} n^{1/(4\lambda)} \left(\sum_{u \subseteq \{1:d\}} \max(|u|, 1) \gamma_u^{\lambda} [2\zeta(\alpha \lambda)] |u|\right)^{1/\lambda}. $$

**Hence**

$$e_{n,d,M}^{\text{wor-app}}(z) = O(n^{-\alpha/4 + \delta}), \quad \delta > 0,$$

where the implied constant is independent of $d$ provided that

$$\sum_{u \subseteq \mathbb{N}, |u| < \infty} \max(|u|, 1) \gamma_u^{\lambda} [2\zeta(\alpha \lambda)] |u| < \infty.$$

**If** the weights satisfy (3.9) for some $\xi \geq 1$ **then** the $|u|$ factor inside the sums can be replaced by 1 as long as $\tau$ is replaced by $\tau \xi$.

**Proof.** The theorem is a consequence of combining (2.6), (2.7), (3.8) and then balancing the terms by choosing $M$ in relation to $n$ according to (2.9) and then taking $\lambda = 1/(\alpha - 4\delta)$.

As a closing remark we note that $\max(|u|, 1) \leq (e^{1/e}) |u| = (1.4446 \cdots)^{|u|}$. This means that the constant is independent of $d$ if

$$\sum_{u \subseteq \mathbb{N}, |u| < \infty} \gamma_u^{\lambda} [2e^{1/e} \zeta(\alpha \lambda)] |u| < \infty.$$

This condition is slightly more demanding, but easier on the eyes, and suggests that the factor of $|u|$ which popped up in the estimates is not really worse than some of the other estimates which were already made on the way.

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