LOOP EQUATION IN TURBULENCE

A.A. Migdal

Physics Department, Princeton University,
Jadwin Hall, Princeton, NJ 08544-1000.
E-mail: migdal@acm.princeton.edu

Abstract

The incompressible fluid dynamics is reformulated as dynamics of closed loops $C$ in coordinate space. This formulation allows to derive explicit functional equation for the generating functional $\Psi[C]$ in inertial range of spatial scales, which allows the scaling solutions. The requirement of finite energy dissipation rate leads then to the Kolmogorov index. We find an exact steady solution of the loop equation in inertial range of the loop sizes. The generating functional decreases as $\exp\left(-A^{\frac{2}{3}}\right)$ where $A = \oint_C r \wedge dr$ is the area inside the loop. The pdf for the velocity circulation $\Gamma$ is Lorentzian, with the width $\bar{\Gamma} \propto A^{\frac{2}{3}}$. 
1 Introduction

Incompressible fluid dynamics underlies the vast majority of natural phenomena. It is described by famous Navier-Stokes equation

\[ \dot{v}_\alpha = \nu \partial_\beta v_\beta - v_\beta \partial_\beta v_\alpha - \partial_\alpha p; \quad \partial_\alpha v_\alpha = 0 \]  

which is nonlinear, and therefore hard to solve. This nonlinearity makes life more interesting, though, as it leads to turbulence. Solving this equation with appropriate initial and boundary conditions we expect to obtain the chaotic behavior of velocity field.

The simplest boundary conditions correspond to infinite space with vanishing velocity at infinity. We are looking for the translation invariant probability distribution for velocity
In order to compensate for the energy dissipation, we add the usual random force to the Navier-Stokes equations, with the short wavelength support, corresponding to large scale energy pumping.

One way to attempt to describe this probability distribution by the Hopf generating functional (the angular bracket denote time averaging, or ensemble averaging over realizations of the random forces)

\[ Z[J] = \langle \exp \left( \int d^3 r J_\alpha(r)v_\alpha(r) \right) \rangle \]

which is known to satisfy linear functional differential equation

\[ \dot{Z} = H \left[ J, \frac{\delta}{\delta J} \right] Z \]

similar to the Schrödinger equation for Quantum Field Theory, and equally hard to solve. Nobody managed to go beyond the Taylor expansion in source \( J \), which corresponds to the obvious chain of equations for the equal time correlation functions of velocity field in various points in space. The same equations could be obtained directly from Navier-Stokes equations, so the Hopf equation looks useless.

In this work we argue, that one could significantly simplify the Hopf functional without loosing information about correlation functions. This simplified functional depends upon the set of 3 periodic functions of one variable

\[ C : r_\alpha = C_\alpha(\theta) ; \quad 0 < \theta < 2\pi \]

which set describes the closed loop in coordinate space. The correlation functions reduce to certain functional derivatives of our loop functional with respect to \( C(\theta) \) at vanishing loop \( C \to 0 \).

The properties of the loop functional at large loop \( C \) also have physical significance. Like the Wilson loops in Gauge Theory, they describe the statistics of large scale structures of vorticity field, which is analogous to the gauge field strength. As we argue in this paper, the Kolmogorov scaling law corresponds to the loop functional decreasing as \( \exp \left( -A^{\frac{4}{5}} \right) \), where \( A \) is the tensor Area inside the loop. This area law emerges as a self-consistent solution of our loop equation in the inertial range of loops. By Fourier transformation of the loop functional we obtain the pdf for the velocity circulation, which turns out Lorentzian.

In Appendix A we recover the expansion in inverse powers of viscosity by direct iterations of the loop equation.

In Appendix B we study the matrix formulation of the Navier-Stokes equation, which may serve as a basis of the random matrix description of turbulence.

In Appendix C we study the reduced dynamics, corresponding to the functional Fourier transform of the loop functional. We argue, that instead of 3D Navier-Stokes equations one can use the 1D equations for the Fourier loop \( P_\alpha(\theta,t) \).
In Appendix D we discuss the relation between the initial data for velocity field and the \( P \) field, and we find particular realisation for these initial data in terms of the gaussian random variables.

In Appendix E we introduce the generating functional for the scalar products \( P_\alpha(\theta)P_\alpha(\theta') \). The advantage of this functional over the original \( \Psi[C] \) functional is the smoother continuum limit.

Finally, in Appendix F we discuss the possible numerical implementations of the reduced loop dynamics.

These four last Appendixes can be skipped at first reading. They might be needed for further development of this approach.

## 2 The Loop Calculus

We suggest to use in turbulence the following version of the Hopf functional

\[
\Psi[C] = \left\langle \exp \left( \frac{i}{\nu} \oint dC_\alpha(\theta) v_\alpha (C(\theta)) \right) \right\rangle \tag{5}
\]

which we call the loop functional or the loop field. It is implied that all angular variable \( \theta \) run from 0 to \( 2\pi \) and that all the functions of this variable are \( 2\pi \) periodic.\(^1\) The viscosity \( \nu \) was inserted in denominator in exponential, as the only parameter of proper dimension. As we shall see below, it plays the role, similar to the Planck’s constant in Quantum mechanics, the turbulence corresponding to the WKB limit \( \nu \to 0. \)\(^2\)

As for the imaginary unit \( i \), there are two reasons to insert it in the exponential. First, it makes the motion compact: the phase factor goes around the unit circle, when the velocity field fluctuates. So, at large times one may expect the ergodicity, with well defined average functional bounded by 1 by absolute value. Second, with this factor of \( i \), the irreversibility of the problem is manifest. The time reversal corresponds to the complex conjugation of \( \Psi \), so that imaginary part of the asymptotic value of \( \Psi \) at \( t \to \infty \) measures the effects of dissipation.

The loop orientation reversal \( C(\theta) \to C(2\pi - \theta) \) also leads to the complex conjugation, so it is equivalent to the time reversal. This symmetry implies, that any correlator of odd/even number of velocities should be integrated odd/even number of times over the loop, and it must enter with an imaginary/real factor. Later, we shall use this property in the area law.

\(^1\)This parametrization of the loop is a matter of convention, as the loop functional is parametric invariant.

\(^2\)One could also insert any numerical parameter in exponential, but this factor could be eliminated by space- and/or time rescaling.
We shall often use the field theory notations for the loop integrals,

\[ \Psi[C] = \exp \left( \frac{i}{\hbar} \oint_C \! d\alpha v_\alpha \right) \] (6)

This loop integral can be reduced to the surface integral of vorticity field

\[ \omega_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \] (7)

by the Stokes theorem

\[ \Gamma_C[v] \equiv \oint_C \! d\alpha v_\alpha = \int_S \! d\sigma \omega_{\mu\nu}; \quad \partial S = C \] (8)

This is the well-known velocity circulation, which measures the net strength of the vortex lines, passing through the loop \( C \). Would we fix initial loop \( C \) and let it move with the flow, the loop field would be conserved by the Euler equation, so that only the viscosity effects would be responsible for its time evolution. However, this is not what we are trying to do. We take the Euler rather than Lagrange dynamics, so that the loop is fixed in space, and hence \( \Psi \) is time dependent already in the Euler equations. The difference between Euler and Navier-Stokes equations is the time irreversibility, which leads to complex average \( \Psi \) in Navier-Stokes dynamics.

It is implied that this field \( \Psi[C] \) is invariant under translations of the loop \( C(\theta) \rightarrow C(\theta) + \text{const} \). The asymptotic behavior at large time with proper random forcing reaches certain fixed point, governed by the translation- and scale invariant equations, which we derive in this paper.

The general Hopf functional (2) reduces for the loop field for the following imaginary singular source

\[ J_\alpha(r) = \frac{i}{\nu} \oint_C \! d\alpha' \delta^3(r' - r) \] (9)

The \( \Psi \) functional involves connected correlation functions of the powers of circulation at equal times.

\[ \Psi[C] = \exp \left( \sum_{n=2}^{\infty} \frac{i^{n-1}}{n! \nu^{n-1}} \langle \Gamma_C^n[v] \rangle \right) \] (10)

This expansion goes in powers of effective Reynolds number, so it diverges in turbulent region. There, the opposite WKB approximation will be used.

Let us come back to the general case of the arbitrary Reynolds number. What could be the use of such restricted Hopf functional? At first glance it seems that we lost most of information, described by the Hopf functional, as the general Hopf source \( J \) depends upon 3 variables \( x, y, z \) whereas the loop \( C \) depends of only one parameter \( \theta \). Still, this information can be recovered by taking the loops of the singular shape, such as two infinitesimal
loops $R_1, R_2$, connected by a couple of wires

The loop field in this case reduces to

$$\Psi[C] \rightarrow \frac{1}{2} \sum_{\mu\nu} R_1 \omega_{\mu\nu}(r_1) + \frac{1}{2} \sum_{\mu\nu} R_2 \omega_{\mu\nu}(r_2)$$

where

$$\Sigma_{\mu\nu}^R = \oint_R dr_{\nu} r_{\mu}$$

is the tensor area inside the loop $R$. Taking functional derivatives with respect to the shape of $R_1$ and $R_2$ prior to shrinking them to points, we can bring down the product of vorticities at $r_1$ and $r_2$. Namely, the variations yield

$$\delta \Sigma_{\mu\nu}^R = \oint_R (dr_{\nu} \delta r_{\mu} + r_{\mu} d\delta r_{\nu}) = \oint_R (dr_{\nu} \delta r_{\mu} - dr_{\mu} \delta r_{\nu})$$

where integration by parts was used in the second term.

One may introduce the area derivative $\frac{\delta}{\delta \sigma_{\mu\nu}(r)}$, which brings down the vorticity at the given point $r$ at the loop.

$$- \nu^2 \frac{\delta^2 \Psi[C]}{\delta \sigma_{\mu\nu}(r_1) \delta \sigma_{\lambda\rho}(r_2)} \rightarrow \langle \omega_{\mu\nu}(r_1) \omega_{\lambda\rho}(r_2) \rangle$$

The careful definition of these area derivatives are of paramount importance to us. The corresponding loop calculus was developed in [2] in the context of the gauge theory. Here we rephrase and further refine the definitions and relations established in that work.

The basic element of the loop calculus is what we suggest to call the spike derivative, namely the operator which adds the infinitesimal $\Lambda$ shaped spike to the loop

$$D_\alpha(\theta, \epsilon) = \int_\theta^{\theta+2\epsilon} d\phi \left( 1 - \frac{ |\theta + \epsilon - \phi|}{\epsilon} \right) \frac{\delta}{\delta C_\alpha(\phi)}$$

The finite spike operator

$$\Lambda(r, \theta, \epsilon) = \exp (r_\alpha D_\alpha(\theta, \epsilon))$$

adds the spike of the height $r$. This is the straight line from $C(\theta)$ to $C(\theta+\epsilon)+r$, followed by
another straight line from $C(\theta + \epsilon) + r$ to $C(\theta + 2\epsilon)$.

Note, that the loop remains closed, and the slopes remain finite, only the second derivatives diverge. The continuity and closure of the loop eliminates the potential part of velocity; as we shall see below, this is necessary to obtain the loop equation.

In the limit $\epsilon \to 0$ these spikes are invisible, at least for the smooth vorticity field, as one can see from the Stokes theorem (the area inside the spike goes to zero as $\epsilon$). However, taking certain derivatives prior to the limit $\epsilon \to 0$ we can obtain the finite contribution.

Let us consider the operator

$$\Pi (r, r', \theta, \epsilon) = \Lambda \left( r, \theta, \frac{1}{2} \epsilon \right) \Lambda \left( r', \theta, \epsilon \right)$$

By construction it inserts the smaller spike on top of a bigger one, in such a way, that a polygon appears.

Taking the derivatives with respect to the vertices of this polygon $r, r'$, setting $r = r' = 0$ and antisymmetrising, we find the tensor operator

$$\Omega_{\alpha\beta}(\theta, \epsilon) = -\nu D_\alpha \left( \theta, \frac{1}{2} \epsilon \right) D_\beta \left( \theta, \epsilon \right) - \{\alpha \leftrightarrow \beta\}$$

which brings down the vorticity, when applied to the loop field

$$\Omega_{\alpha\beta}(\theta, \epsilon) \Psi [C] \overset{\epsilon \to 0}{\longrightarrow} \omega_{\alpha\beta} (C(\theta)) \Psi [C]$$
The quick way to check these formulas is to use formal functional derivatives
\[
\frac{\delta \Psi [C]}{\delta C_{\alpha}(\theta)} = C'_{\beta}(\theta) \frac{\delta \Psi [C]}{\delta \sigma_{\alpha \beta}(C(\theta))}
\] (20)

Taking one more functional derivative we find the term with vorticity times first derivative of the \( \delta \) function, coming from the variation of \( C''(\theta) \)
\[
\frac{\delta^2 \Psi [C]}{\delta C_{\alpha}(\theta) \delta C_{\beta}(\theta')} = \delta' (\theta - \theta') \frac{\delta \Psi [C]}{\delta \sigma_{\alpha \beta}(C(\theta))} + C'_{\gamma}(\theta) C'_{\lambda}(\theta') \frac{\delta^2 \Psi [C]}{\delta \sigma_{\alpha \gamma}(C(\theta)) \delta \sigma_{\beta \lambda}(C(\theta'))}
\] (21)

This term is the only one, which survives the limit \( \epsilon \to 0 \) in our relation (19).

So, the area derivative can be defined from the antisymmetric tensor part of the second functional derivative as the coefficient in front of \( \delta' (\theta - \theta') \). Still, it has all the properties of the first functional derivative, as it can also be defined from the above first variation. The advantage of dealing with spikes is the control over the limit \( \epsilon \to 0 \), which might be quite singular in applications.

So far we managed to insert the vorticity at the loop \( C \) by variations of the loop field. Later we shall need the vorticity off the loop, in arbitrary point in space. This can be achieved by the following combination of the spike operators
\[
\Lambda (r, \theta, \epsilon) \Pi (r_1, r_2, \theta + \epsilon, \delta); \ \delta \ll \epsilon
\] (22)

This operator inserts the \( \Pi \) shaped little loop at the top of the bigger spike, in other words, this little loop is translated by a distance \( r \) by the big spike.

Taking derivatives, we find the operator of finite translation of the vorticity
\[
\Lambda (r, \theta, \epsilon) \Omega_{\alpha \beta}(\theta + \epsilon, \delta)
\] (23)

and the corresponding infinitesimal translation operator
\[
D_\mu (\theta, \epsilon) \Omega_{\alpha \beta}(\theta + \epsilon, \delta)
\] (24)

which inserts \( \partial_\mu \omega_{\alpha \beta}(C(\theta)) \) when applied to the loop field.

Coming back to the correlation function, we are going now to construct the operator, which would insert two vorticities separated by a distance. Let us note that the global \( \Lambda \) spike
\[
\Lambda (r, 0, \pi) = \exp \left( r_\alpha \int_0^{2 \pi} d\phi \left( 1 - \frac{|\phi - \pi|}{\pi} \right) \frac{\delta}{\delta C_\alpha(\phi)} \right)
\] (25)

when applied to a shrunk loop \( C(\phi) = 0 \) does nothing but the backtracking from 0 to \( r \)
\[\begin{array}{c}
0 \longrightarrow r
\end{array}\]

Fig. 4

This means that the operator
\[
\Omega_{\alpha \beta}(0, \delta) \Omega_{\lambda \rho}(\pi, \delta) \Lambda (r, 0, \pi)
\] (26)
when applied to the loop field for a shrunk loop yields the vorticity correlation function

\[
\Omega_{\alpha\beta}(0, \delta) \Omega_{\lambda\rho}(\pi, \delta) \Lambda(r, 0, \pi) \Psi[0] = \langle \omega_{\alpha\beta}(0) \omega_{\lambda\rho}(r) \rangle
\]

(27)

The higher correlation functions of vorticities could be constructed in a similar fashion, using the spike operators. As for the velocity, one should solve the Poisson equation

\[
\partial_\mu^2 v_\alpha(r) = \partial_\beta \omega_{\beta\alpha}(r)
\]

(28)

with the proper boundary conditions, say, \( v = 0 \) at infinity. Formally,

\[
v_\alpha(r) = \frac{1}{\partial_\mu^2} \partial_\beta \omega_{\beta\alpha}(r)
\]

(29)

This suggests the following formal definition of the velocity operator

\[
V_\alpha(\theta, \epsilon, \delta) = \frac{1}{D_\beta^2(\theta, \epsilon)} D_\beta(\theta, \epsilon) \Omega_{\beta\alpha}(\theta, \delta); \ \delta \ll \epsilon
\]

(30)

\[
V_\alpha(\theta, \epsilon, \delta) \Psi[C] \xrightarrow{\delta, \epsilon \to 0} v_\alpha(C(\theta)) \Psi[C]
\]

(31)

Another version of this formula is the following integral

\[
V_\alpha(\theta, \epsilon, \delta) = \int d^3 \rho \frac{\rho_\beta}{4\pi|\rho|^3} \Lambda(\rho, \theta, \epsilon) \Omega_{\alpha\beta}(\theta + \epsilon, \delta)
\]

(32)

where the \( \Lambda \) operator shifts the \( \Omega \) by a distance \( \rho \) off the original loop at the point \( r = C(\theta + \epsilon) \)

3 Loop Equation

Let us now derive exact equation for the loop functional. Taking the time derivative of the original definition, and using the Navier-Stokes equation we get in front of exponential

\[
\oint_C dr_\alpha \left( \nu \partial_\beta v_\alpha - v_\beta \partial_\beta v_\alpha - \partial_\alpha p \right)
\]

(33)
The term with the pressure gradient yields zero after integration over the closed loop, and the velocity gradients in the first two terms could be expressed in terms of vorticity up to irrelevant gradient terms, so that we find

\[ \oint_C \overline{d\alpha} \left( \nu \frac{\partial \omega_\gamma}{\partial \beta} - v_\beta \omega_\gamma \right) \]  

(34)

Replacing the vorticity and velocity by the operators discussed in the previous Section we find the following loop equation (in explicit notations)

\[ -\dot{\Psi}[C] = \nu^2 \oint_C \overline{d\alpha} \left( \frac{\delta}{\delta \sigma_\beta(r)} \left( D_\beta(\theta,\epsilon) \Omega_\gamma(\theta,\epsilon) + \frac{1}{\nu} \int d^3 \rho \frac{\rho_\gamma}{4\pi |\rho|^3} \Lambda (\rho, \theta, \epsilon) \Omega_\gamma(\theta + \epsilon, \delta) \Omega_\beta(\theta, \delta) \right) \right) \Psi[C] \]  

(35)

The more compact form of this equation, using the notations of \( \mathbb{H} \), reads

\[ -\nu \dot{\Psi}[C] = \mathcal{H}_C \Psi \]  

(36)

\[ \mathcal{H}_C \equiv \nu^2 \oint_C \overline{d\alpha} \left( \frac{\delta}{\delta \sigma_\beta(r)} \left( \frac{\delta}{\delta \sigma_\gamma(r')} \left( \frac{\delta^2}{\delta \sigma_\alpha(r) \delta \sigma_\beta(r)} \right) \right) \right) \]  

(37)

Now we observe that viscosity \( \nu \) appears in front of time and spatial derivatives, like the Planck constant \( \hbar \) in Quantum mechanics. Our loop hamiltonian \( \mathcal{H}_C \) is not hermitean, due to dissipation. It contains the second loop derivatives, so it represents a (nonlocal!) kinetic term in loop space.

So far, we considered so called decaying turbulence, without external energy source. The energy

\[ E = \int d^3 r \frac{1}{2} v_\alpha^2 \]  

(38)

would eventually all dissipate, so that the fluid would stop. In this case the loop wave function \( \Psi \) would asymptotically approach 1

\[ \Psi[C] \xrightarrow{t \to \infty} 1 \]  

(39)

In order to reach the steady state, we add to the right side of the Navier-Stokes equation the usual gaussian random forces \( f_\alpha(r,t) \) with the space dependent correlation function

\[ \langle f_\alpha(r,t) f_\beta(r',t') \rangle = \delta_{\alpha\beta} \delta(t-t') F(r-r') \]  

(40)

concentrated at at small wavelengths, i.e. slowly varying with \( r-r' \).

Using the identity

\[ \langle f_\alpha(r,t) \Phi[v(\cdot)] \rangle = \int d^3 r' F(r-r') \frac{\delta \Phi[v(\cdot)]}{\delta v_\alpha(r')} \]  

(41)

which is valid for arbitrary functional \( \Phi \) we find the following imaginary potential term in the loop hamiltonian

\[ \delta \mathcal{H}_C \equiv \nu \mathcal{U}[C] = \frac{\nu}{\nu} \oint_C \overline{d\alpha} \oint_C \overline{d\alpha'} \oint_C \overline{d\alpha} \int d^3 r \int d^3 r' F(r-r') \]  

(42)
Note, that orientation reversal together with complex conjugation changes the sign of the loop hamiltonian, as it should. The potential part involves two loop integrations times imaginary constant. The first term in the kinetic part has one loop integration, one loop derivative times imaginary constant. The second kinetic term has one loop integration, two loop derivatives and real constant. The left side of the loop equation has no loop integrations, no loop derivatives, but has a factor of $i$.

The relation between the potential and kinetic parts of the loop hamiltonian depends of viscosity, or, better to say, it depends upon the Reynolds number, which is the ratio of the typical circulation to viscosity. In the viscous limit, when the Reynolds number is small, the loop wave function is close to 1. The perturbation expansion in $\frac{1}{\nu}$ goes in powers of the potential, in the same way, as in Quantum mechanics. The second (nonlocal) term in kinetic part of the hamiltonian also serves as a small perturbation (it corresponds to nonlinear term in the Navier-Stokes equation). The first term of this perturbation expansion is just

$$
\Psi[C] \rightarrow 1 - \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{F}(k)}{2\nu^3 k^2} \left| \oint_C dr_k e^{i k r} \right|^2
$$

with $\tilde{F}(k)$ being the Fourier transform of $F(r)$. This term is real, as it corresponds to the two-velocity correlation. The next term comes from the triple correlation of velocity, and this term is purely imaginary, so that the dissipation shows up.

This expansion can be derived by direct iterations in the loop space as in [2], inverting the operator in the local part of the kinetic term in the hamiltonian. This expansion is discussed in Appendix A. The results agree with the straightforward iterations of the Navier-Stokes equations in powers of the random force, starting from zero velocity.

So, we have the familiar situation, like in QCD, where the perturbation theory breaks because of the infrared divergencies. For arbitrarily small force, in a large system, the region of small $k$ would yield large contribution to the terms of the perturbation expansion. Therefore, one should take the opposite WKB limit $\nu \rightarrow 0$.

In this limit, the wave function should behave as the usual WKB wave function, i.e. as an exponential

$$
\Psi[C] \rightarrow \exp \left( \frac{i S[C]}{\nu} \right)
$$

The effective loop Action $S[C]$ satisfies the loop space Hamilton-Jacobi equation

$$
\dot{S}[C] = -i U[C] + \oint_C dr_r \int d^3 r' \frac{r'_\alpha - r_\gamma}{4\pi |r - r'|^3} \frac{\delta S}{\delta \sigma_\beta_\alpha(r)} \frac{\delta S}{\delta \sigma_\beta_\gamma(r')} \delta S
$$

The imaginary part of $S[C]$ comes from imaginary potential $U[C]$, which distinguishes our theory from the reversible Quantum mechanics. The sign of $\Im S$ must be positive definite, since $|\Psi| < 1$. As for the real part of $S[C]$, it changes the sign under the loop orientation reversal $C(\theta) \rightarrow C(2\pi - \theta)$. 
At finite viscosity there would be an additional term

\[- \nu \oint_C dr_\alpha \partial_\beta \frac{\delta S[C]}{\delta \sigma_\alpha(r)} - \nu \nu \oint_C dr_\alpha \int d^3r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta^2 S[C]}{\delta \sigma_\alpha(r) \delta \sigma_\beta(r')}\]

on the right of (44). As for the term

\[- \oint_C dr_\alpha \partial_\beta (\partial_\gamma S[C]) \frac{\delta S[C]}{\delta \sigma_\alpha(r)}\]

which formally arises in the loop equation, this term vanishes, since \(\partial_\beta S[C] = 0\). This operator inserts backtracking at some point at the loop without first applying the loop derivative at this point. As it was discussed in the previous Section, such backtracking does not change the loop functional. This issue was discussed at length in [2], where the Leibnitz rule for the operator \(\partial_\alpha \delta_{\beta\gamma}\) was established

\[\partial_\alpha \frac{\delta f(g[C])}{\delta \sigma_\gamma(r)} = f'(g[C]) \partial_\alpha \frac{\delta g[C]}{\delta \sigma_\gamma(r)}\]

In other words, this operator acts as a first order derivative on the loop functional with finite area derivative (so called Stokes type functional). Then, the above term does not appear.

The Action functional \(S[C]\) describes the distribution of the large scale vorticity structures, and hence it should not depend of viscosity. In terms of the above connected correlation functions of the circulation this corresponds to the limit, when the effective Reynolds number \(\Gamma_C \nu\) goes to infinity, but the sum of the divergent series tends to the finite limit. According to the standard picture of turbulence, the large scale vorticity structures depend upon the energy pumping, rather than the energy dissipation.

This, of course implies, that both time \(t\) and the loop size\(^3\) \(|C|\) should be greater then the viscous scales

\[t \gg t_0 = \nu^{-\frac{3}{4}} \mathcal{E}^{-\frac{1}{4}}; \quad |C| \gg r_0 = \nu^{\frac{3}{4}} \mathcal{E}^{-\frac{1}{4}}\]

where \(\mathcal{E}\) is the energy dissipation rate.

It is defined from the energy balance equation

\[0 = \partial_t \left\langle \frac{1}{2} v_\alpha^2 \right\rangle = \nu \left\langle v_\alpha \partial^2 v_\alpha \right\rangle + \langle f_\alpha v_\alpha \rangle\]

which can be transformed to

\[\frac{1}{4} \nu \left\langle \omega_{\alpha\beta}^2 \right\rangle = 3F(0)\]

The left side represents the energy, dissipated at small scale due to viscosity, and the right side - the energy pumped in from the large scales due to the random forces. Their common value is \(\mathcal{E}\).

\(^3\)As a measure of the loop size one may take the square root of the minimal area inside the loop.
We see, that constant $F(r - r')$, i.e., $\tilde{F}(k) \propto \delta(k)$ is sufficient to provide the necessary energy pumping. However, such forcing does not produce vorticity, which we readily see in our equation. The contribution from this constant part to the potential in our loop equation drops out (this is a closed loop integral of total derivative). This is important, because this term would have the wrong order of magnitude in the turbulent limit - it would grow as the Reynolds number.

Dropping this term, we arrive at remarkably simple and universal functional equation

$$\dot{S}[C] = \oint_C dr_{\alpha} \int d^3 r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta S}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta S}{\delta \sigma_{\beta\gamma}(r')}$$  (51)

The stationary solution of this equation describes the steady distribution of the circulation in the strong turbulence. Note, that the stationary solutions come in pairs $\pm S$. The sign should be chosen so, that $\Im S > 0$, to provide the inequality $|\Psi| < 1$.

4 Scaling law

The ‘Hamilton-Jacobi’ equation without the potential term (51) allows the family of the scaling solutions

$$S[C] = t^{2\kappa - 1} \phi \left[ \frac{C}{t^\kappa} \right]$$  (52)

with arbitrary index $\kappa$. The scaling function satisfies the equation

$$(2\kappa - 1)\phi[C] - \kappa \oint_C dr_{\alpha} \frac{\delta \phi[C]}{\delta \sigma_{\beta\alpha}(r)} r_\beta = \oint_C dr_{\alpha} \int d^3 r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta \phi[C]}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta \phi[C]}{\delta \sigma_{\beta\gamma}(r')}$$  (53)

The left side here was computed, using the chain rule differentiation of functional.

Asymptotically, at large time, we expect the fixed point, which is the homogeneous functional

$$S_\infty[C] = |C|^2 \frac{1}{2} f \left[ \frac{C}{|C|} \right]$$  (54)

zeroing the right side of our ‘kinetic’ functional equation

$$0 = \oint_C dr_{\alpha} \int d^3 r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta S_\infty[C]}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta S_\infty[C]}{\delta \sigma_{\beta\gamma}(r')}$$  (55)

The Kolmogorov scaling [1] would correspond to

$$\kappa = \frac{3}{2}$$  (56)

in which case one can express the $S$ functional in terms of $E$

$$S[C] = \mathcal{E} t^2 \phi \left[ \frac{C}{\sqrt{\mathcal{E} t^3}} \right]$$  (57)
One can easily rephrase the Kolmogorov arguments in the loop space. The relation between the energy dissipation rate and the velocity correlator reads

$$\mathcal{E} = \langle v_\alpha(r_0)v_\beta(0)\partial_\beta v_\alpha(0) \rangle$$

(58)

where the point splitting at the viscous scale $r_0$ is introduced. Such splitting is necessary to avoid the viscosity effects; without the splitting the average would formally reduce to the total derivative and vanish.

Instead of the point splitting one may introduce the finite loop of the viscous scale $|C| \sim r_0$, and compute this correlator in presence of such loop. This reduces to the WKB estimates

$$\omega_{\alpha\beta}(r) \rightarrow \frac{\delta S[C]}{\delta \sigma_{\alpha\beta}(r)}; \quad v_\alpha(r) = \int d^3r' \frac{r'_\gamma - r_\gamma}{4\pi|r - r'|^3} \omega_{\alpha\gamma}(r')$$

(59)

Using the generic scaling law for $S$ we find

$$\omega \sim r_0^{-\frac{1}{\kappa}}; \quad v \sim r_0^{1 - \frac{1}{\kappa}}; \quad \mathcal{E} \sim r_0^{2 - \frac{2}{\kappa}}$$

(60)

We see, that the energy dissipation rate would stay finite in the limit of the vanishing viscous scale only for the Kolmogorov value of the index. This argument looks rather cheap, but I think it is basically correct. The constant value of the energy dissipation rate in the limit of vanishing viscosity arises as the quantum anomaly in the field theory, through the finite limit of the point splitting in the correspondent energy current.$^4$

There is another version of this argument, which I like better. The dynamics of Euler fluid in infinite system would not exist, for the non-Kolmogorov scaling. The extra powers of loop size would have to enter with the size $L$ of the whole system, like $\left(\frac{|C|}{L}\right)^\epsilon$. So, in the regime with finite energy pumping rate $\mathcal{E}$ the infinite Euler system can exist only for the Kolmogorov index. This must be the essence of the original Kolmogorov reasoning.$^4$

The problem is that nobody proved that such limit exists, though. Within the usual framework, based on the velocity correlation functions, one has to prove, that the infrared divergencies, caused by the sweep, all cancel for the observables. Within our framework these problems disappear, as we shall see later.

As for the correlation functions in inertial range, unfortunately those cannot be computed in the WKB approximation, since they involve the contour shrinking to a double line, with vanishing area inside. Still, most of the physics can be understood in loop terms, without these correlation functions. The large scale behavior of the loop functional reflects the statistics of the large vorticity structures, encircled by the loop.

$^4$I am grateful to A. Polyakov and E. Siggia for inspiring comments on this subject.
5 Area law

The Wilson loop in QCD decreases as exponential of the minimal area, encircled by the loop, leading to the quark confinement. What is the similar asymptotic law in turbulence? The physical mechanisms leading to the area law in QCD are absent here. Moreover, there is no guarantee, that $\Psi[C]$ always decreases with the size of the loop.

This makes it possible to look for the simple Anzatz, which was not acceptable in QCD, namely

$$S[C] = s \left( \Sigma_{\mu\nu}^C \right)$$

where

$$\Sigma_{\mu\nu}^C = \oint_C r_\mu dr_\nu$$

is the tensor area encircled by the loop $C$. The difference between this area and the scalar area is the positivity property. The scalar area vanishes only for the loop which can be contracted to a point by removal of all the backtracking. As for the tensor area, it vanishes, for example, for the 8 shaped loop, with opposite orientation of petals.

Thus, there are some large contours with vanishing tensor area, for which there would be no decrease of the $\Psi$ functional. In QCD the Wilson loops must always decrease at large distances, due to the finite mass gap. Here, the large scale correlations are known to exist, and play the central role in the turbulent flow. So, I see no reasons to reject the tensor area Anzatz.

This Anzatz in QCD not only was unphysical, it failed to reproduce the correct short-distance singularities in the loop equation. In turbulence, there are no such singularities. Instead, there are the large-distance singularities, which all should cancel in the loop equation.

It turns out, that for this Anzatz the (turbulent limit of the) loop equation is satisfied automatically, without any further restrictions. Let us verify this important property. The first area derivative yields

$$\omega_{\mu\nu}^C(r) = \frac{\delta S}{\delta \sigma_{\mu\nu}(r)} = 2 \frac{\partial s}{\partial \Sigma_{\mu\nu}^C}$$

The factor of 2 comes from the second term in the variation

$$\frac{\delta \Sigma_{\alpha\beta}^C}{\delta \sigma_{\mu\nu}(r)} = \delta_{\alpha\mu} \delta_{\beta\nu} - \delta_{\alpha\nu} \delta_{\beta\mu}$$

Note, that the right side does not depend on $r$. Moreover, you can shift $r$ aside from the base loop $C$, with proper wires inserted. The area derivative would not change, as the contribution of wires drops.

This implies, that the corresponding vorticity $\omega_{\mu\nu}^C(r)$ is space independent, it only depends upon the loop itself. The velocity can be reconstructed from vorticity up to
irrelevant potential terms

\[ v_C^C(r) = \frac{1}{2} r_\alpha \omega^C_{\alpha \beta} \]  

(65)

This can be formally obtained from the above integral representation

\[ v_C^C(r) = \int d^3 r' \frac{r_\alpha - r'_\alpha}{4\pi |r - r'|^3} \omega^C_{\alpha \beta} \]  

(66)

as a residue from the infinite sphere \( R = |r'| \rightarrow \infty \). One may insert the regularizing factor \(|r'|^{-\epsilon}\) in \( \omega \), compute the convolution integral in Fourier space and check that in the limit \( \epsilon \rightarrow 0^+ \) the above linear term arises. So, one can use the above form of the loop equation, with the analytic regularization prescription.

Now, the \( v \omega \) term in the loop equation reads

\[ \oint_C dr_\gamma v_C^C(r) \omega^C_{\beta \gamma} \propto \Sigma^C_{\alpha \beta} \omega^C_{\alpha \beta} \omega^C_{\beta \gamma} \]  

(67)

This tensor trace vanishes, because the first tensor is antisymmetric, and the product of the last two antisymmetric tensors is symmetric with respect to \( \alpha \gamma \).

So, the positive and negative terms cancel each other in our loop equation, like the "income" and "outcome" terms in the usual kinetic equation. We see, that there is an equilibrium in our loop space kinetics.

From the point of view of the notorious infrared divergencies in turbulence, the above calculation explicitly demonstrates how they cancel. By naive dimensional counting these terms were linearly divergent. The space isotropy lowered this to logarithmic divergence in (66), which reduced to finite terms at closer inspection. Then, the explicit form of these terms was such, that they all cancelled.

This cancellation originates from the angular momentum conservation in fluid mechanics. The large loop \( C \) creates the macroscopic eddy with constant vorticity \( \omega^C_{\alpha \beta} \) and linear velocity \( v^C(r) \propto r \). This is a well known static solution of the Navier-Stokes equation. The eddy is conserved due to the angular momentum conservation. The only nontrivial thing is the functional dependence of the eddy vorticity upon the shape and size of the loop \( C \). This is a function of the tensor area \( \Sigma^C_{\mu \nu} \), rather than a general functional of the loop.

Combining this Anzatz with the space isotropy and the Kolmogorov scaling law, we arrive at the turbulent area law

\[ \Psi[C] \propto \exp \left( -B \left( \frac{\mathcal{E}}{\nu^3} \left( \Sigma^C_{\alpha \beta} \right)^2 \right)^{\frac{1}{4}} \right) \]  

(68)

The universal constant \( B \) here must be real, in virtue of the loop orientation symmetry. When the orientation is reversed \( C(\theta) \rightarrow C(2\pi - \theta) \), the loop integral changes sign, but its square, which enters here, stays invariant. Therefore, the constant in front must be real. The time reversal tells the same, since both viscosity \( \nu \) and the energy dissipation rate \( \mathcal{E} \) are time-odd. Therefore, the ratio \( \mathcal{E}/\nu^3 \) is time-even, hence it must enter \( \Psi[C] \) with the real coefficient. Clearly, this coefficient \( B \) must be positive, since \( |\Psi[C]| < 1 \).
6 Discussion

So, we found an exact solution of the loop equation in the turbulent limit. It remains to be seen, whether this is the most general solution, and is it realized in turbulent flows. Meanwhile, let us discuss its general properties, and its implications to the large scale vorticity distribution.

First of all, let us address the issue of the uniqueness of this solution. Let us take the following Ansatz

\[ S[C] = f \left( \oint_C dr_x \oint_C dr'_x W(r - r') \right) \quad (69) \]

When substituted into the static loop equation (with the area derivatives computed in Appendix A), it yields the following equation for the correlation function

\[ 0 = \oint_C dr_x \oint_C dr'_x \oint_C dr''_x U_{\alpha \beta \gamma}(r, r', r'') \quad (70) \]

\[ U_{\alpha \beta \gamma}(r, r', r'') = W(r - r') \hat{V}_\beta V_i W(r' - r'') + \text{permutations} \]

\[ \hat{V}_{\mu \nu} = \delta_{\alpha \nu} \partial_\mu - \delta_{\alpha \mu} \partial_\nu \]

The derivative \( f' \) of the unknown function drops from the static equation.

This equation should hold for arbitrary loop \( C \). Using the Taylor expansion for the Stokes type functional [2], we can argue, that the coefficient function \( U \) must vanish up to the total derivatives. An equivalent statement is that the third area derivative of this functional must vanish. Using the loop calculus (see Appendix A) we find the following equation

\[ 0 = \hat{V}_\mu \hat{V}_\nu \hat{V}_\rho \Sigma_{\mu \nu \rho} U_{\alpha \beta \gamma}(r, r', r'') \quad (71) \]

which should hold for arbitrary \( r, r', r'' \). This leads to the overcomplete system of equations for \( W(r) \) in general case. However, for the special case \( W(r) = r^2 \) which corresponds to the square of the tensor area

\[ \Sigma^2_{\alpha \beta} = -\frac{1}{2} \oint C dr_x \oint C dr'_x (r - r')^2 \]

the system is satisfied as a consequence of certain symmetry. In this case we find in the loop equation

\[ 2 \oint C dr_x \oint C dr'_x (r - r')^2 \oint C dr''_x (r'_x - r''_x) \propto \Sigma^C_{\alpha \beta} \oint C dr_x \oint C dr'_x (r - r')^2 \]

(73)

The last integral is symmetric with respect to permutations of \( \alpha, \beta \), whereas the first factor \( \Sigma^C_{\alpha \beta} \) is antisymmetric, hence the sum over \( \alpha \beta \) yields zero, as we already saw above.

This solution can also be used to get the probability distribution for the velocity circulation. For this purpose the extra factor \( \gamma \) should be inserted in the definition of the loop average

\[ \Psi [C] = \left\langle \exp \left( \frac{i \gamma}{V} \oint C dr_x v_x \right) \right\rangle \]

\[ \mathrm{(74)} \]
and the Fourier transformation should be performed

\[ P_C(\Gamma) = \frac{1}{2\pi\nu} \int_{-\infty}^{+\infty} d\gamma \exp \left( -\frac{i\gamma \Gamma}{\nu} \right) \Psi[C] \]  

(75)

Now, the \( \gamma \) dependence of \( \Psi[C] \) can be found from the dimensional analysis. It enters only in combination with viscosity \( \nu \), and the other source of viscosity, the \( \nu\partial^2 \nu \) term, was dropped in the Navier-Stokes equation. So, the asymptotic solution \([38]\) should get the extra factor of \( |\gamma| \) in the exponential.

\[ \Psi[C] \propto \exp \left( -B |\gamma| \left( \frac{E}{\nu^3} \left( \Sigma^C_{\alpha\beta} \right)^2 \right)^{\frac{1}{3}} \right) \]  

(76)

Another way to get this factor is to observe, that one could scale \( \gamma \) away from original definition, by rescaling \( E \rightarrow E|\gamma|^{-3} \), since the velocity circulation scales as \( E^{\frac{4}{3}} \).

The Fourier integral yields the Lorentz distribution

\[ P_C(\Gamma) = \frac{\bar{\Gamma}}{\pi \left( \Gamma^2 + \bar{\Gamma}^2 \right)} ; \quad \bar{\Gamma} = B \left( \frac{E}{\nu^3} \left( \Sigma^C_{\alpha\beta} \right)^2 \right)^{\frac{1}{3}} \]  

(77)

Let us answer an obvious question. The Lorentz distribution is symmetric. How does it agree with the known asymmetry of velocity correlations, in particular, the Kolmogorov triple correlation? The answer is that the Kolmogorov correlation does not imply the asymmetry of \textit{vorticity} correlations.

Taking the tensor version of the \( \frac{4}{5} \) law in arbitrary dimension \( d \)

\[ \langle v_\alpha(0)v_\beta(0)v_\gamma(r) \rangle = \frac{E}{(d-1)(d+2)} \left( \delta_{\alpha\gamma}r_\beta + \delta_{\beta\gamma}r_\alpha - \frac{2}{d}\delta_{\alpha\beta}r_\gamma \right) \]  

(78)

and differentiating, we find that

\[ \langle v_\alpha(0)v_\beta(0)\omega_\gamma\lambda(r) \rangle = 0 \]  

(79)

So, the odd vorticity correlations could, in fact, be absent, in spite of the asymmetry of the velocity distribution. Besides, the area law does not apply to small loops which are involved in the definition of the vorticity correlations in terms the loop functional. Moreover, we do not see any reason to expect these correlations to be scale invariant. From our point of view these are not the asymptotic quantities, so the viscous effect could be important.

With velocity correlations it is even worse than that. The infrared divergencies may be important as well, so that the factors like \( \left( \frac{L}{r} \right)^\delta \) could appear in the higher moments of velocity distribution. The observed violations of the Kolmogorov scaling in these moments could be attributed to these infrared divergencies. However, according to our theory, the velocity circulation agrees with the Kolmogorov scaling and has smooth Lorentz distribution in the infinite system.
It would be extremely interesting to measure the velocity circulation for large loops in real or numerical experiments, and test these predictions. Maybe the long tails of the Lorentz distribution reflect the notorious intermittency? Note, that all the even moments are infinite for the Lorentz distribution, which is quite unusual, but does not contradict any known physical requirements.

In real world this would probably mean that the \( n \)-th moments of the circulation are cut off by the finite time and finite size effects, i.e. they grow as \( \min \left\{ (\mathcal{E}t^2)^n, (\mathcal{E}L^4)^{\frac{n}{3}} \right\} \). Instead of measurements of these moments, the shape of the Lorentz distribution and the area dependence of its width \( \bar{\Gamma} \) could be tested.

It was assumed in above arguments, that the loop \( C \) consist of only one connected part. Let us now consider the more general situation, with arbitrary number \( n \) of loops \( C_1, \ldots C_n \). The corresponding Anzatz would be

\[
S_n [C_1, \ldots C_n] = s_n \left( \Sigma^1, \ldots \Sigma^n \right)
\]

where \( \Sigma^i \) are tensor areas.

This function should obey the same WKB loop equations in each variable. Introducing the loop vorticities

\[
\omega_{\mu\nu}^k = 2 \frac{\partial s_n}{\partial \Sigma_{\mu\nu}^k}
\]

which are constant on each loop, we have to solve the following problem. What are the values of \( \omega_{\mu\nu}^k \) such that the single velocity field \( v_\alpha(r) \) could produce them?

We do not see any other solutions, but the trivial one, with all equal \( \omega_{\mu\nu}^k \) and linear velocity, as before. This would correspond to

\[
s_n \left( \Sigma^1, \ldots \Sigma^n \right) = s_1 (\Sigma) ; \Sigma_{\mu\nu} = \sum_{k=1}^{n} \Sigma_{\mu\nu}^k = \oint_{\psi C_k} r_\mu dr_\nu
\]

The loop equation would be satisfied like before, with \( C = \psi C_k \). This corresponds to the additivity of loops

\[
S_n [C_1, \ldots C_n] = S_1 [\psi C_k]
\]

Note, that such additivity is the opposite to the statistical independence, which would imply that

\[
S_n [C_1, \ldots C_n] = \sum S_1 [C_k]
\]

The additivity could also be understood as a statement, that any set of \( n \) loops is equivalent to a single loop for the abelian Stokes functional. Just connect these loops by wires, and note that the contribution of wires cancels. So, if the area law holds for arbitrary single loop, than it must be additive.

This assumption may not be true, though, as it often happens in the WKB approximation. There is no single asymptotical formula, but rather collections of different WKB regions, with quantum regions in between. In our case, this corresponds to the following situation.
Take the large circular loop, for which the WKB approximation holds, and try to split it into two large circles. You will have to twist the loop like the infinity symbol ∞, in which case it intersects itself. At this point, the WKB approximation might break, as the short distance velocity correlation might be important near the self-intersection point. This may explain the paradox of the vanishing tensor area for the ∞ shaped loop. From the point of view of our area law such loop is not large at all.

This cancellation of large circulations is puzzling. Apparently, the further analytical and numerical study of the loop equation is required. One could try some variational approach, by approximating the loop by a polygon, which reduces the Ψ functional to a function of the vertices of the polygon. There are some other options, such as truncation of Fourier expansion of the loop function \( C(\theta) \). I do not expect any easy success here, because of singularities of the loop derivatives involved. Still, the beauty of the loop dynamics and its apparent reduction of the dimension of the turbulence problem, raises some hopes of the analytical advances.

The main issue, in my opinion, is the Kolmogorov scaling. Usually, the indexes in the scaling solutions of the nonlinear integral equations of QFT (so called bootstrap equations), are determined from selfconsistency of the equations, rather than from some extra requirements. However, the Kolmogorov law for triple correlation function was derived from the same Navier-Stokes equation plus the scaling assumptions about velocity correlation functions, which is equivalent to the assumptions we made in our scaling Anzatz for \( S[C] \). So, it is possible, that the dynamical value of \( \kappa \), found from selfconsistency, would coincide with the Kolmogorov value.

However, we have to keep in mind the possibility of the more general phenomenon. Namely, there might be the spectrum of solutions for the critical index \( \kappa \), corresponding to various fixed points of above Hamilton-Jacobi equation in the loop space. The steady state may not exist, if these fixed points are all unstable. In this case, \( S[C] \) would go from one fixed point to another.

This would be the next level of complexity, as compared to the strange attractors, found in dynamical systems with small number of degrees of freedom. Not just the trajectory in phase space, but the whole probability distribution functional would evolve with time. When averaged over given time interval \( T \), it would reach steady state only for the scales less than \( T^{\kappa} \). The marginal scales distribution would slowly drift, and larger scales would not be in equilibrium at all. In this case the infinite system would never reach the steady state.

This situation is not described by above area laws, but rather requires more general non-steady solutions of the loop equations. In Appendices C, D, E, F we develop the general framework for studying such solutions. We decided to place these parts of our work in Appendix because this formalism is too heavy. However, the mathematically oriented reader might find it useful.
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References

[1] A.N. Kolmogorov. *The local structure of turbulence in incompressible viscous fluid for very large Reynolds’ numbers*. C. R. Acad. Sci. URSS, 30(4):301–305, 1941.

[2] A. Migdal, *Loop equations and $\frac{1}{N}$ expansion*, Physics Reports, 201, 102, (1983).

[3] D. Gross, A. Migdal, Physical Review Letters, 717, 64, (1990), and Nuclear Physics B340, 333,(1990), M. Douglas and S. Shenker, Nuclear Physics B335, 635, (1990), E. Brezin, V. Kazakov, Physical Letters, 144, 236B, (1990).
## A Loop Expansion

Let us outline the method of direct iterations of the loop equation. The full description of the method can be found in [2]. The basic idea is to use the following representation of the loop functional

\[
\Psi[C] = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \left\{ \oint_C dr_1^{\alpha_1} \cdots \oint_C dr_n^{\alpha_n} \right\}_{\text{cyclic}} W^n_{\alpha_1 \cdots \alpha_n} (r_1, \ldots r_n) \tag{85}
\]

This representation is valid for every translation invariant functional with finite area derivatives (so called Stokes type functional). The coefficient functions \( W \) can be related to these area derivatives. The normalization \( \Psi[0] = 1 \) for the shrunk loop is implied.

In general case the integration points \( r_1, \ldots r_n \) in (85) are cyclicly ordered around the loop \( C \). The coefficient functions can be assumed cyclicly symmetric without loss of generality. However, in case of fluid dynamics, we are dealing with so called abelian Stokes functional. These functionals are characterized by completely symmetric coefficient functions, in which case the ordering of points can be removed, at expense of the extra symmetry factor in denominator

\[
\Psi[C] = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \oint_C dr_1^{\alpha_1} \cdots \oint_C dr_n^{\alpha_n} W^n_{\alpha_1 \cdots \alpha_n} (r_1, \ldots r_n) \tag{86}
\]

The incompressibility conditions

\[
\partial_{\alpha_k} W^n_{\alpha_1 \cdots \alpha_n} (r_1, \ldots r_n) = 0 \tag{87}
\]

does not impose any further restrictions, because of the gauge invariance of the loop functionals. This invariance (nothing to do with the symmetry of dynamical equations!) follows from the fact, that the closed loop integral of any total derivative vanishes. So, the coefficient functions are defined modulo such derivative terms. In effect this means, that one may relax the incompressibility constraints (87), without changing the loop functional.

To avoid confusion, let us note, that the physical incompressibility constrains are not neglected. They are, in fact, present in the loop equation, where we used the integral representation for the velocity in terms of vorticity. Still, the longitudinal parts of \( W \) drop in the loop integrals.

The loop calculus for the abelian Stokes functional is especially simple. The area derivative corresponds to removal of one loop integration, and differentiation of the corresponding coefficient function

\[
\frac{\delta \Psi[C]}{\delta \sigma_{\mu\nu}(r)} = \sum_{n=1}^{\infty} \frac{1}{n!} \oint_C dr_1^{\alpha_1} \cdots \oint_C dr_n^{\alpha_n} \hat{V}_{\mu\nu} W^n_{\alpha_1 \cdots \alpha_n} (r, r_1, \ldots r_n) \tag{88}
\]

where

\[
\hat{V}_{\mu\nu} \equiv \partial_{\mu} \delta_{\nu\alpha} - \partial_{\nu} \delta_{\mu\alpha} \tag{89}
\]
In the nonabelian case, there would also be the contact terms, with $W$ at coinciding points, coming from the cyclic ordering \[2\]. In abelian case these terms are absent, since $W$ is completely symmetric.

As a next step, let us compute the local kinetic term

$$\hat{L}_v[C] \equiv \oint_C dr_\nu \partial_{\mu} \frac{\delta \Psi[C]}{\delta \sigma_{\mu\nu}(r)}$$

Using above formula for the loop derivative, we find

$$\hat{L}_v[C] = \sum_{n=1}^{\infty} \frac{1}{n!} \oint_C dr_1^\alpha \oint_C dr_1^{\alpha_1} \ldots \oint_C dr_n^{\alpha_n} \partial^2 W_{n+1}^{\alpha_1 \ldots \alpha_n} (r, r_1, \ldots, r_n)$$

The net result is the second derivative of $W$ with respect to one variable. Note, that the second term in $\hat{V}_{\mu\nu}^\alpha$ dropped, as the total derivative in the closed loop integral.

As for the nonlocal kinetic term, it involves the second area derivative off the loop, at the point $r'$, integrated over $r'$ with the corresponding Green’s function. Each area derivative involves the same operator $\hat{V}$, acting on the coefficient function. Again, the abelian Stokes functional simplifies the general framework of the loop calculus. The contribution of the wires cancels here, and the ordering does not matter, so that

$$\frac{\delta^2 \Psi[C]}{\delta \sigma_{\mu\nu}(r) \delta \sigma_{\mu'\nu'}(r')} = \sum_{n=0}^{\infty} \frac{1}{n!} \oint_C dr_1^{\alpha_1} \ldots \oint_C dr_n^{\alpha_n} \hat{V}_{\mu\nu}^\alpha \hat{V}_{\mu'\nu'}^{\alpha'} W_{n+2}^{\alpha, \alpha', \alpha_1 \ldots \alpha_n} (r, r', r_1, \ldots, r_n)$$

Using these relations, we can write the steady state loop equation as follows

Here the light dotted lines symbolize the arguments $\alpha_k, r_k$ of $W$, the big circle denotes the loop $C$, the tiny circles stand for the loop derivatives, and the pair of lines with the arrow denote the Green’s function. The sum over the tensor indexes and the loop integrations over $r_k$ are implied.

The first term is the local kinetic term, the second one is the nonlocal kinetic term, and the right side is the potential term in the loop equation. The heavy dotted line in this term stands for the correlation function $F$ of the random forces. Note that this term is an abelian Stokes functional as well.
The iterations go in the potential term, starting with $\Psi^{C} = 1$. In the next approximation, only the two loop correction $W_{\alpha_1 \alpha_2}^2(r_1, r_2)$ is present. Comparing the terms, we note, that nonlocal kinetic term reduces to the total derivatives due to the space symmetry (in the usual terms it would be $\langle v \omega \rangle$ at coinciding arguments), so we are left with the local one.

This yields the equation

$$\nu^3 \partial^2 W_{\alpha \beta}^2(r - r') = F(r - r') \delta_{\alpha \beta}$$

modulo derivative terms. The solution is trivial in Fourier space

$$W_{\alpha \beta}^2(r - r') = -\int \frac{d^3k}{(2\pi)^3} \exp(ik(r - r')) \delta_{\alpha \beta} \tilde{F}(k)$$

Note, that we did not use the transverse tensor

$$P_{\alpha \beta}(k) = \delta_{\alpha \beta} - \frac{k_\alpha k_\beta}{k^2}$$

Though such tensor is present in the physical velocity correlation, here we may use $\delta_{\alpha \beta}$ instead, as the longitudinal terms drop in the loop integral. This is analogous to the Feynman gauge in QED. The correct correlator corresponds to the Landau gauge.

The potential term generates the four point correlation $FW^2$, which agrees with the disconnected term in the $W^4$ on the left side

$$W^4_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(r_1, r_2, r_3, r_4) \rightarrow W^2_{\alpha_1 \alpha_2}(r_1 - r_2) W^2_{\alpha_3 \alpha_4}(r_3 - r_4) +$$

In the same order of the loop expansion, the three point function will show up. The corresponding terms in kinetic part must cancel among themselves, as the potential term does not contribute. The local kinetic term yields the loop integrals of $\partial^2 W^3$, whereas the nonlocal one yields $\tilde{V} W^2 V' W^2$, integrated over $d^3 r'$ with the Greens’s function $\frac{1}{4\pi |r - r'|^3}$. The equation has the structure

Now it is clear, that the solution of this equation for $W^3$ would be the same three point correlator, which one could obtain (much easier!) by direct iterations of the Navier-Stokes equation.
The purpose of this painful exercise was not to give one more method of developing the expansion in powers of the random force. We rather verified that the loop equations are capable of producing the same results, as the ordinary chain of the equations for the correlation functions.

In above arguments, it was important, that the loop functional belonged to the class of the abelian Stokes functionals. Let us check that our tensor area Anzatz

$$\Sigma^C_{\alpha\beta} = \oint_C r_\alpha dr_\beta$$

(97)

belongs to the same class. Taking the square we find

$$\left(\Sigma^C_{\alpha\beta}\right)^2 = \oint_C dr_\beta \oint_C dr'_\beta r_\alpha r'_\alpha = -\frac{1}{2} \oint_C dr_\beta \oint_C dr'_\beta (r - r')^2$$

(98)

where the last transformation follows from the fact, that only the cross term in $(r - r')^2$ yields nonzero after double loop integration.

Any expansion in terms of the square of the tensor area reduces, therefore to the superposition of multiple loop integral of the product of $(r_i - r_j)^2$, which is an example of the abelian Stokes functional. In the limit of large area, this could reduce to the fractional power. An example could be, say

$$\Psi[C] \xlongequal{\text{2}} \exp B \left( 1 - \left(1 + \frac{E \left(\Sigma^C_{\alpha\beta}\right)^2}{\nu^3}\right)^{\frac{1}{3}} \right)$$

(99)

One could explicitly verify all the properties of the abelian Stokes functional. This example is not realistic, though, as it does not have the odd terms of expansion. In the real world such terms are present at the viscous scales. According to our solution, this asymmetry disappears in inertial range of loops (which does not apply to velocity correlators at inertial range, as those correspond to shrunk loops).

**B Matrix Model**

The Navier-Stokes equation represents a very special case of nonlinear PDE. There is a well known galilean invariance

$$v_\alpha(r, t) \to v_\alpha(r - ut, t) + u_\alpha$$

(100)

which relates the magnitude of velocity field with the scales of time and space. \footnote{At the same time it tells us that the constant part of velocity if frame dependent, so that it better be eliminated, if we would like to have a smooth limit at large times. Most of notorious large scale divergencies in turbulence are due to this unphysical constant part.} Let us make this relation more explicit.

\footnote{At the same time it tells us that the constant part of velocity if frame dependent, so that it better be eliminated, if we would like to have a smooth limit at large times. Most of notorious large scale divergencies in turbulence are due to this unphysical constant part.}
First, let us introduce the vorticity field
\[ \omega_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \] (101)
and rewrite the Navier-Stokes equation as follows
\[ \dot{v}_\alpha = \nu \partial_\beta \omega_{\beta\alpha} - v_\beta \omega_{\beta\alpha} - \partial_\alpha w; \ w = p + \frac{v^2}{2} \] (102)

This \( w \) is the well known enthalpy density, to be found from the incompressibility condition \( \text{div} v = 0 \), i.e.
\[ \partial^2 w = \partial_\alpha v_\beta \omega_{\beta\alpha} \] (103)

As a next step, let us introduce ”covariant derivative” operator
\[ D_\alpha = \nu \partial_\alpha - \frac{1}{2} v_\alpha \] (104)
and observe that
\[ 2 [D_\alpha D_\beta] = \nu \omega_{\beta\alpha} \] (105)
\[ 2D_\beta [D_\alpha D_\beta] + h.c. = \nu \partial_\beta \omega_{\beta\alpha} - v_\beta \omega_{\beta\alpha} \] (106)
where \( h.c. \) stands for hermitean conjugate.

These identities allow us to write down the following dynamical equation for the covariant derivative operator
\[ \dot{D}_\alpha = D_\beta [D_\alpha D_\beta] - D_\alpha W + h.c. \] (107)

As for the incompressibility condition, it can be written as follows
\[ [D_\alpha D_\alpha^\dagger] = 0 \] (108)
The enthalpy operator \( W = \frac{w}{\nu} \) is to be determined from this condition, or, equivalently
\[ [D_\alpha [D_\alpha W]] = [D_\alpha, D_\beta [D_\alpha D_\beta]] \] (109)

We see, that the viscosity disappeared from these equations. This paradox is resolved by extra degeneracy of this dynamics: the antihermitean part of the \( D \) operator is conserved. Its value at initial time is proportional to viscosity.

The operator equations are invariant with respect to the time independent unitary transformations
\[ D_\alpha \rightarrow S^\dagger D_\alpha S; \ S^\dagger S = 1 \] (110)
and, in addition, to the time dependent unitary transformations with
\[ S(t) = \exp \left( \frac{1}{2\nu} t u_\beta \left( D_\beta - D_\beta^\dagger \right) \right) \] (111)
corresponding to the galilean transformations.

One could view the operator $D_\alpha$ as the matrix

$$\langle i | D_\alpha | j \rangle = \int d^3r \psi_i^* (r) \nu \partial_\alpha \psi_j (r) - \frac{1}{2} \psi_i^*(r) v_\alpha (r) \psi_j (r)$$

where the functions $\psi_j (r)$ are the Fourier of Tchebyshev functions depending upon the geometry of the problem.

The finite mode approximation would correspond to truncation of this infinite size matrix to finite size $N$. This is not quite the same as leaving $N$ terms in the mode expansion of velocity field. The number of independent parameters here is $O(N^2)$ rather then $O(N)$. It is not clear whether the unitary symmetry is worth paying such a high price in numerical simulations!

The matrix model of Navier-Stokes equation has some theoretical beauty and raises hopes of simple asymptotic probability distribution. The ensemble of random hermitean matrices was recently applied to the problem of Quantum Gravity [3], which led to a genuine breakthrough in the field.

Unfortunately, the model of several coupled random matrices, which is the case here, is much more complicated then the one matrix model studied in Quantum Gravity. The dynamics of the eigenvalues is coupled to the dynamics of the "angular" variables, i.e. the unitary matrices $S$ in above relations. We could not directly apply the technique of orthogonal polynomials, which was so successful in the one matrix problem.

Another technique, which proved to be successful in QCD and Quantum Gravity is the loop equations. This method, which we are discussing at length in this paper, works in field theory problems with hidden geometric meaning. The turbulence proves to be an ideal case, much simpler then QCD or Quantum Gravity.

## C The Reduced Dynamics

Let us now try to reproduce the dynamics of the loop field by a simpler Anzatz

$$\Psi[C] = \langle \exp \left( \frac{1}{\nu} \oint dC_\alpha (\theta) P_\alpha (\theta) \right) \rangle$$

The difference with original definition (5) is that our new function $P_\alpha (\theta)$ depends directly on $\theta$ rather then through the function $v_\alpha (r)$ taken at $r_\alpha = C_\alpha (\theta)$. This is the $d \to 1$ dimensional reduction we mentioned before. From the point of view of the loop functional there is no need to deal with field $v(r)$, one could take a shortcut.

Clearly, the reduced dynamics must be fitted to the Navier-Stokes dynamics of original field. With the loop calculus, developed above, we have all the necessary tools to build this reduced dynamics.
Let us assume some unknown dynamics for the $P$ field

$$\dot{P}_\alpha(\theta) = F_\alpha(\theta, [P])$$

(114)

and compare the time derivatives of original and reduced Ansatz. We find in (113) instead of (34)

$$\frac{i}{\nu} \oint dC_\alpha(\theta) F_\alpha(\theta, [P])$$

(115)

Now we observe, that $P'$ could be replaced by the functional derivative, acting on the exponential in (113) as follows

$$\frac{\delta}{\delta C_\alpha(\theta)} \leftrightarrow -i \nu P'_\alpha(\theta)$$

(116)

This means, that one could take the operators of the Section 2, expressing velocity and vorticity in terms of the spike operator, and replace the functional derivative as above. This yields the following formula for the spike derivative

$$D_\alpha(\theta, \epsilon) = -i \nu \int_0^{\theta + 2\epsilon} d\phi \left( 1 - \frac{|\theta + \epsilon - \phi|}{\epsilon} \right) P'_\alpha(\phi) = -i \nu \int_{-1}^1 d\mu \operatorname{sgn}(\mu) P_\alpha(\theta + \epsilon(1 + \mu))$$

(117)

This is the weighted discontinuity of the function $P(\theta)$, which in the naive limit $\epsilon \to 0$ would become the true discontinuity. However, the function $P(\theta)$ has in general the stronger singularities, then discontinuity, so that this limit cannot be taken yet.

Anyway, we arrive at the dynamical equation for the $P$ field

$$\dot{P}_\alpha = \nu D_\beta \Omega_{\beta\alpha} - V_\beta \Omega_{\beta\alpha}$$

(118)

where the operators $V, D, \Omega$ of the Section 2 should be regarded as the ordinary numbers, with definition (117) of $D$ in terms of $P$.

All the functional derivatives are gone! We needed them only to prove equivalence of reduced dynamics to the Navier-Stokes dynamics.

The function $P_\alpha(\theta)$ would become complex now, as the right side of the reduced dynamical equation is complex for real $P_\alpha(\theta)$. Let us discuss this puzzling issue in more detail. The origin of imaginary units was the factor of $i$ in exponential of the definition of the loop field. We had to insert this factor to make the loop field decreasing at large loops as a result of oscillations of the phase factors. Later this factor propagated to the definition of the $P$ field.

Our spike derivative $D$ is purely imaginary for real $P$, and so is our $\Omega$ operator. This makes the velocity operator $V$ real. Therefore the $D\Omega$ term in the reduced equation (118) is real for real $P$ whereas the $V\Omega$ term is purely imaginary.

This does not contradict the moments equations, as we saw before. The terms with even/odd number of velocity fields in the loop functional are real/imaginary, but the
moments are real, as they should be. The complex dynamics of $P$ simply doubles the number of independent variables.

There is one serious problem, though. Inverting the spike operator $D_\alpha$ we implicitly assumed, that it was antihermitean, and could be regularized by adding infinitesimal negative constant to $D_\alpha^2$ in denominator. This, indeed, works perturbatively, in each term of expansion in time, or that in size of the loop, as we checked. However, beyond this expansion there would be a problem of singularities, which arise when $D_\alpha^2(\theta)$ vanishes at some $\theta$.

In general, this would occur for complex $\theta$, when the imaginary and real part of $D_\alpha^2(\theta)$ simultaneously vanish. One could introduce the complex variable

$$e^{i\theta} = z; \ e^{-i\theta} = \frac{1}{z}; \ \oint d\theta = \oint \frac{dz}{iz}$$

(119)

where the contour of $z$ integration encircles the origin around the unit circle. Later, in course of time evolution, these contours must be deformed, to avoid complex roots of $D_\alpha^2(\theta)$.

D Initial Data

Let us study the relation between the initial data for the original and reduced dynamics. Let us assume, that initial field is distributed according to some translation invariant probability distribution, so that initial value of the loop field does not depend on the constant part of $C(\theta)$.

One can expand translation invariant loop field in functional Fourier transform

$$\Psi[C] = \int DQ \delta^3 \left( \oint d\phi Q(\phi) \right) W[Q] \exp \left( i \oint d\theta C_\alpha(\theta) Q_\alpha(\theta) \right)$$

(120)

which can be inverted as follows

$$\delta^3 \left( \oint d\phi Q(\phi) \right) W[Q] = \int DC \Psi[C] \exp \left( -i \oint d\theta C_\alpha(\theta) Q_\alpha(\theta) \right)$$

(121)

Let us take a closer look at these formal transformations. The functional measure for these integrations is defined according to the scalar product

$$(A, B) = \oint \frac{d\theta}{2\pi} A(\theta) B(\theta)$$

(122)

which diagonalizes in the Fourier representation

$$A(\theta) = \sum_{-\infty}^{+\infty} A_n e^{in\theta}; \ A_{-n} = A_n^*$$

(123)

$$(A, B) = \sum_{-\infty}^{+\infty} A_n B_{-n} = A_0 B_0 + \sum_{1}^{\infty} a_n' b_n' + a_n'' b_n''; \ a_n' = \sqrt{2\Re A_n}, a_n'' = \sqrt{2\Im A_n}$$

(124)
The corresponding measure is given by an infinite product of the Euclidean measures for the imaginary and real parts of each Fourier component

\[ DQ = d^3Q_0 \prod_1^\infty d^3q'_n d^3q''_n \]  

(125) 

The orthogonality of Fourier transformation could now be explicitly checked, as

\[ \int DC \exp \left( i \int d\theta C_\alpha(\theta) (A_\alpha(\theta) - B_\alpha(\theta)) \right) = \int d^3C_0 \prod_1^\infty d^3c'_n d^3c''_n \exp \left( 2\pi i \left( C_0 (A_0 - B_0) + \sum_1^\infty c'_n (a'_n - b'_n) + c''_n (a''_n - b''_n) \right) \right) = \delta^3 (A_0 - B_0) \prod_1^\infty \delta^3 (a'_n - b'_n) \delta^3 (a''_n - b''_n) \]  

(126) 

Let us now check the parametric invariance

\[ \theta \to f(\theta); \ f(2\pi) - f(0) = 2\pi; \ f'(\theta) > 0 \]  

(127) 

The functions \( C(\theta) \) and \( P(\theta) \) have zero dimension in a sense, that only their argument transforms

\[ C(\theta) \to C(f(\theta)); \ P(\theta) \to P(f(\theta)) \]  

(128) 

The functions \( Q(\theta) \) and \( P'(\theta) \) in above transformation have dimension one

\[ P'(\theta) \to f'(\theta)P'(f(\theta)); \ Q(\theta) \to f'(\theta)Q(f(\theta)) \]  

(129) 

so that the constraint on \( Q \) remains invariant

\[ \oint d\theta Q(\theta) = \oint df(\theta)Q(f(\theta)) \]  

(130) 

The invariance of the measure is easy to check for infinitesimal reparametrization

\[ f(\theta) = \theta + \epsilon(\theta); \ \epsilon(2\pi) = \epsilon(0) \]  

(131) 

which changes \( C \) and \( (C,C') \) as follows

\[ \delta C(\theta) = \epsilon(\theta)C'(\theta); \ \delta(C,C') = \oint \frac{d\theta}{2\pi} \epsilon(\theta)2C_\alpha(\theta)C'_\alpha(\theta) = -\oint \frac{d\theta}{2\pi} \epsilon'(\theta)C^2_\alpha(\theta) \]  

(132) 

The corresponding Jacobian reduces to

\[ 1 - \oint d\theta \epsilon'(\theta) = 1 \]  

(133) 

in virtue of periodicity.

This proves the parametric invariance of the functional Fourier transformations. Using these transformations we could find the probability distribution for the initial data of

\[ P_\alpha(\theta) = -\nu \int_0^\theta d\phi Q_\alpha(\phi) \]  

(134)
The simplest but still meaningful distribution of initial velocity field is the Gaussian one, with energy concentrated in the macroscopic motions. The corresponding loop field reads

$$\Psi_0[C] = \exp \left( -\frac{1}{2} \oint dC_\alpha(\theta) \oint dC_\alpha(\theta') f(C(\theta) - C(\theta')) \right)$$  \hspace{1cm} (135)

where $f(r - r')$ is the velocity correlation function

$$\langle v_\alpha(r)v_\beta(r') \rangle = \left( \delta_\alpha\beta - \partial_\alpha\partial_\beta\partial_\mu^{-2} \right) f(r - r')$$  \hspace{1cm} (136)

The potential part drops out in the closed loop integral.

The correlation function varies at macroscopic scale, which means that we could expand it in Taylor series

$$f(r - r') \rightarrow f_0 - f_1(r - r')^2 + \ldots$$  \hspace{1cm} (137)

The first term $f_0$ is proportional to initial energy density,

$$\frac{1}{2} \langle v_\alpha^2 \rangle = \frac{d-1}{2} f_0$$  \hspace{1cm} (138)

and the second one is proportional to initial energy dissipation rate

$$\mathcal{E}_0 = -\nu \langle v_\alpha \partial_\beta v_\alpha \rangle = 2d(d-1)\nu f_1$$  \hspace{1cm} (139)

where $d = 3$ is dimension of space.

The constant term in (137) as well as $r^2 + r'^2$ terms drop from the closed loop integral, so we are left with the cross term $rr'$

$$\Psi_0[C] \rightarrow \exp \left( -f_1 \oint dC_\alpha(\theta) \oint dC_\alpha(\theta')C_\beta(\theta)C_\beta(\theta') \right)$$  \hspace{1cm} (140)

This is almost Gaussian distribution: it reduces to Gaussian one by extra integration

$$\Psi_0[C] \rightarrow \text{const} \int d^3\omega \exp \left( -\omega_{\alpha\beta}^2 \right) \exp \left( 2i\sqrt{f_1} \omega_{\mu\nu} \oint dC_\mu(\theta)C_\nu(\theta) \right)$$  \hspace{1cm} (141)

The integration here goes over all $\frac{d(d-1)}{2} = 3$ independent $\alpha < \beta$ components of the antisymmetric tensor $\omega_{\alpha\beta}$. Note, that this is ordinary integration, not the functional one. The physical meaning of this $\omega$ is the random constant vorticity at initial moment.

At fixed $\omega$ the Gaussian functional integration over $C$

$$\int DC \exp \left( i \oint d\theta \left( \frac{1}{\nu} C_\beta(\theta)P_\beta'(\theta) + 2\sqrt{f_1} \omega_{\alpha\beta}C_\alpha'(\theta)C_\beta(\theta) \right) \right)$$  \hspace{1cm} (142)

can be performed explicitly, it reduces to solution of the saddle point equation

$$P_\beta'(\theta) = 4\nu\sqrt{f_1} \omega_{\alpha\beta}C_\alpha'(\theta)$$  \hspace{1cm} (143)
which is trivial for constant $\omega$

$$C_\alpha(\theta) = \frac{1}{4\nu \sqrt{f_1}} \omega^{-1}_{\alpha\beta} P_\beta(\theta) \quad (144)$$

provided the matrix $\omega$ is invertible, which is true in general provided the matrix $\omega$ is invertible, which is true in general. We find the Gaussian probability distribution for $P$ with the correlator

$$\langle P_\alpha(\theta)P_\beta(\theta') \rangle = 2\nu \sqrt{f_1} \omega_{\alpha\beta} \text{sign}(\theta' - \theta) \quad (145)$$

Note, that antisymmetry of $\omega$ compensates that of the sign function, so that this correlation function is symmetric, as it should be. However, it is antihermitean, which corresponds to purely imaginary eigenvalues. The corresponding realization of the $P$ functions is complex!

Let us study this phenomenon for the Fourier components. Differentiating the last equation with respect to $\theta$ and Fourier transforming we find

$$\langle P_{\alpha,n}P_{\beta,m} \rangle = \frac{4\nu}{m} \delta_{-nm} \sqrt{f_1} \omega_{\alpha\beta} \quad (146)$$

This cannot be realized at complex conjugate Fourier components $P_{\alpha,-n} = P_{\alpha,n}^*$ but we could take $\bar{P}_{\alpha,n} \equiv P_{\alpha,-n}$ and $P_{\alpha,n}$ as real random variables, with correlation function

$$\langle \bar{P}_{\alpha,n}P_{\beta,m} \rangle = \frac{4\nu}{m} \delta_{nm} \sqrt{f_1} \omega_{\alpha\beta}; \quad n > 0 \quad (147)$$

The trivial realization is

$$\bar{P}_{\alpha,n} = \frac{4\nu}{n} \sqrt{f_1} \omega_{\alpha\beta} P_{\beta,n} \quad (148)$$

with $P_{\beta,n}$ being Gaussian random numbers with unit dispersion.

As for the constant part $P_{\alpha,0}$ of $P_\alpha(\theta)$, it is not defined, but it drops from equations in virtue of translational invariance.

## E W-functional

The difficulties of turbulence are hidden in the loop equation, but they show up, if you try to solve it numerically. The main problem is that one cannot get rid of the cutoffs $\epsilon, \delta \to 0$ in the definitions of the spike derivatives. These cutoffs are designed to pick up the singular contributions in the angular integrals, but with finite number of modes, such as Fourier harmonics there would be no singularities. We did not find any way to truncate degrees of freedom in the $P$ equation, without violating the parametric invariance. It very well may be, that this invariance would be restored in the limit of large number of modes, but it looks that there are too much ambiguity in the finite mode approximation.

The integration over $\omega$ should be shifted towards complex plane to avoid such degeneracy.
After some attempts, we found the simpler version of the loop functional, which can be studied analytically in the turbulent region. This is the generating functional for the scalar products \( P_\alpha(\theta_1)P_\alpha(\theta_2) \)

\[
W[S] = \left\langle \exp \left( - \oint d\theta_1 \oint d\theta_2 S(\theta_1, \theta_2)P_\alpha(\theta_1)P_\alpha(\theta_2) \right) \right\rangle \tag{149}
\]

where, as before, the averaging goes over initial data for the \( P \) field.

The time derivative of this \( W \)-functional

\[
\dot{W} = -2 \left\langle \oint d\theta_1 \oint d\theta_2 S(\theta_1, \theta_2)P_\alpha(\theta_1)\dot{P}_\alpha(\theta_2) \exp \left( - \oint d\theta_1 \oint d\theta_2 S(\theta_1, \theta_2)P_\alpha(\theta_1)P_\alpha(\theta_2) \right) \right\rangle \tag{150}
\]

can be expressed in terms of functional derivatives of \( W \) by replacing

\[
P_\alpha(\phi_1)P_\alpha(\phi_2) \rightarrow \frac{\delta}{\delta S(\phi_1, \phi_2)} \tag{151}
\]

for every scalar product of \( P \) fields, which arise after expansion of the spike derivatives \((117), (18), (30)\) in the scalar product

\[
P_\alpha(\theta_1)\dot{P}_\alpha(\theta_2) = \nu P_\alpha(\theta_1)D_\beta(\theta_2)\Omega_{\beta\alpha}(\theta_2) - P_\alpha(\theta_1)V_\beta(\theta_2)\Omega_{\beta\alpha}(\theta_2) \tag{152}
\]

This equation has the structure

\[
\dot{W} = \oint d^2\theta S(\theta_1, \theta_2) \left( A_2 \left[ \frac{\delta}{\delta S} \right] W + A_3 \left[ \frac{\delta}{\delta S} \right] D^{-2}(\theta, \epsilon)W \right) \tag{153}
\]

where \( A_k [X] \) stands for the \( k \)- degree homogenous functional of the function \( X(\theta_1, \theta_2) \).

The operator \( D^{-2} \) is also the homogeneous functional of the negative degree \( k = -1 \). It can be written as follows

\[
D^{-2}(\theta, \epsilon)W[S] = \int_0^\infty d\tau W[S + \tau U] \tag{154}
\]

with

\[
U(\theta_1, \theta_2) = \epsilon^{-2} \sgn(\theta + \epsilon - \theta_1) \sgn(\theta + \epsilon - \theta_2) \tag{155}
\]

### F Possible Numerical Implementation

The above general scheme is fairly abstract and complicated. Could it lead to any practical computation method? This would depend upon the success of the discrete approximations of the singular equations of reduced dynamics.

The most obvious approximation would be the truncation of Fourier expansion at some large number \( N \). With Fourier components decreasing only as powers of \( n \) this
approximation is doubtful. In addition, such truncation violates the parametric invariance which looks dangerous.

It seems safer to approximate \( P(\theta) \) by a sum of step functions, so that it is piecewise constant. The parametric transformations vary the lengths of intervals of constant \( P(\theta) \), but leave invariant these constant values. The corresponding representation reads

\[
P_\alpha(\theta) = \sum_{l=0}^{N} (p_\alpha(l+1) - p_\alpha(l)) \Theta(\theta - \theta_l); \quad p(N+1) = p(1), \quad p(0) = 0
\]  

(156)

It is implied that \( \theta_0 = 0 < \theta_1 < \theta_2 \ldots < \theta_N < 2\pi \). By construction, the function \( P(\theta) \) takes value \( p(l) \) at the interval \( \theta_{l-1} < \theta < \theta_l \).

We could take \( \dot{P}(\theta) \) at the middle of this interval as approximation to \( \dot{p}(l) \).

\[
\dot{p}(l) \approx \dot{P}(\bar{\theta}_l); \quad \bar{\theta}_l = \frac{1}{2} (\theta_{l-1} + \theta_l)
\]  

(157)

As for the time evolution of angles \( \theta_l \), one could differentiate (156) in time and find

\[
\dot{P}_\alpha(\theta) = \sum_{l=0}^{N} (\dot{p}_\alpha(l+1) - \dot{p}_\alpha(l)) \Theta(\theta - \theta_l) - \sum_{l=0}^{N} (p_\alpha(l+1) - p_\alpha(l)) \delta(\theta - \theta_l) \dot{\theta}_l
\]  

(158)

from which one could derive the following approximation

\[
\dot{\theta}_l \approx \frac{(p_\alpha(l) - p_\alpha(l+1))}{(p_\mu(l+1) - p_\mu(l))} \int_{\theta_l}^{\theta_{l+1}} d\theta \dot{P}_\alpha(\theta)
\]  

(159)

The extra advantage of this approximation is its simplicity. All the integrals involved in the definition of the spike derivative (117) are trivial for the stepwise constant \( P(\theta) \). So, this approximation can be in principle implemented at the computer. This formidable task exceeds the scope of the present work, which we view as purely theoretical.