The Algebra of Strand Splitting. II.
A Presentation for the Braid Group on One Strand

MATTHEW G. BRIN

May 28, 2004

1. Introduction

In [3], we give descriptions of a braided version $BV$ of Thompson’s group $V$ as well as a group $\hat{BV}$ that contains $BV$ as a subgroup and that is somewhat easier to work with. The paper [3] contains both geometric and algebraic descriptions of these two groups and shows that for each group the two descriptions are of isomorphic groups. An infinite presentation for $\hat{BV}$ is also given in [3]. The current paper computes a finite presentation for $\hat{BV}$ as well as two infinite and two finite presentations for $BV$.

There is a very close relationship (similar to that between an Artin group and its corresponding Coxeter group) between $BV$ and $V$ as well as between $\hat{BV}$ and a corresponding group $\hat{V}$. The relationship is close enough that, with no extra work, our calculations also yield finite and infinite presentations of $V$ and finite presentations of $\hat{V}$. Infinite presentations of $\hat{V}$ are given in [3].

The similarity of the relationships to that of an Artin group to its corresponding Coxeter group is expressed by the fact that presentations for $V$ and $\hat{V}$ can be obtained from the presentations for $BV$ and $\hat{BV}$ by simply declaring the squares of certain generators to be the identity. We give multiple presentations of the groups $V$ and $BV$ since one set more closely resembles existing presentations of Thompson’s groups (see, for example, [5]), while the other set emphasizes the Artin/Coxeter relation since the presentations (finite and infinite) for $V$ are gotten from the corresponding presentations for $BV$ by declaring the squares of all the generators to be the identity. See Corollary 4.15 and the last paragraph of Theorem 2.

The finite presentation for $V$ given in [5] has fewer relations than the finite presentation that we obtain for $V$, but the presentation in [5] is harder to relate to the presentation of $BV$.

We work entirely with the algebraic structure of the groups above, and we refer the reader to [3] for the geometric descriptions that reveal the structures as braid groups. The title refers to the fact that $\hat{BV}$ can be viewed as a braid group on countably many strands that are allowed to split and recombine as long as the order of splits and recombinations is remembered. The subgroup $BV$ corresponds to the subgroup in which all splitting and recombining is confined to the first strand and in which all braiding is confined to the strands obtained from splitting the first strand. Thus $BV$ can be thought of as the “braid group on one strand” where splitting and joining is allowed.

In the next section, we list what we need from [3] and also give needed facts about another of Thompson’s groups.

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1AMS Classification (2000): primary 20F05, secondary 20F36, 20E99, 20B07
2. INTRODUCING THE GROUPS

2.1. The groups $\hat{V}$ and $\hat{BV}$. The following presentations come from Theorem 1 of [3] where $\Lambda = \{\lambda_0, \lambda_1, \ldots\}$ and $\Sigma = \{\sigma_0, \sigma_1, \ldots\}$.

(1) \[ \hat{V} = \langle \Lambda \cup \Sigma \mid \lambda_q \lambda_m = \lambda_m \lambda_{q+1}, \quad m < q, \sigma_m^2 = 1, \quad m \geq 0, \sigma_m \sigma_n = \sigma_n \sigma_m, \quad |m - n| \geq 2, \sigma_m \sigma_{m+1} \sigma_m = \sigma_{m+1} \sigma_m \sigma_m, \quad m \geq 0, \sigma_q \lambda_m = \lambda_m \sigma_{q+1}, \quad m < q, \sigma_m \lambda_m = \lambda_{m+1} \sigma_m \sigma_m, \quad m \geq 0, \sigma_m \lambda_{m+1} = \lambda_m \sigma_m \sigma_m, \quad m \geq 0, \sigma_q \lambda_m = \lambda_m \sigma_q, \quad m > q + 1 \rangle. \]

Some of the relations are redundant. The relations $\sigma_m \lambda_{m+1} = \lambda_m \sigma_{m+1} \sigma_m$ follow from the relations $\sigma_m \lambda_m = \lambda_{m+1} \sigma_m \sigma_{m+1}$ in $\hat{V}$ by bringing each $\sigma_m$ and $\sigma_{m+1}$ to the other side of the equality. Similarly, the relations $\sigma_m' \lambda_{m+1} = \lambda_m \sigma_{m+1} \sigma_m'$ follow from the relations $\sigma_m' \lambda_m = \lambda_{m+1} \sigma_m \sigma_{m+1}$ in $\hat{BV}$. Also, the exponents $\epsilon$ can be eliminated from several of the relations in $\hat{BV}$ because of the group setting. The relations are listed to be used and not to give minimal presentations.

The submonoid of $\hat{V}$ generated by the positive powers of the generators in $\Lambda$ is isomorphic to the submonoid of $\hat{BV}$ generated by the positive powers of generators from $\Lambda$ and we will use $F$ to denote both of these submonoids.

The submonoid of $\hat{V}$ generated by the elements of $\Sigma$ and their inverses is isomorphic to the infinite symmetric group and will be denoted by $S_\infty$. The submonoid of $\hat{BV}$ generated by the elements of $\Sigma$ and their inverses is isomorphic to the braid group on infinitely many strands and will be denoted by $B_\infty$. We will often have need to refer separately to the relations of $B_\infty$, which are

(3) \[ \sigma_m \sigma_n = \sigma_n \sigma_m, \quad |m - n| \geq 2, \]

(4) \[ \sigma_m \sigma_{m+1} \sigma_m = \sigma_{m+1} \sigma_m \sigma_{m+1}, \quad m \geq 0. \]

We regard $S_\infty$ as the set of finite permutations of $N$, and the generator $\sigma_i$ corresponds to the transposition that switches $i$ and $i + 1$. The relations of $S_\infty$ are \[ and \[ as well as

(5) \[ \sigma_m^2 = 1, \quad m \geq 0. \]
The subgroup of $S_\infty$ or $B_\infty$ generated by $\{\sigma_0, \sigma_1, \ldots, \sigma_{n-2}\}$ will be denoted, respectively, by $S_n$ or $B_n$. That is, $S_n$ permutes the first $n$ elements $\{0, 1, \ldots, n-1\}$ of $\mathbb{N}$ and $B_n$ is the braid group on $n$ strands.

The submonoid of $\hat{V}$ generated by $\mathcal{F} \cup S_\infty$ has the structure of a Zappa-Szép product $\mathcal{F} \bowtie S_\infty$ and the submonoid of $\hat{B}V$ generated by $\mathcal{F} \cup B_\infty$ has the structure of a Zappa-Szép product $\mathcal{F} \bowtie B_\infty$. The Zappa-Szép product is a generalization of the semidirect product in that neither factor is required to be normal in the result.

The details of the Zappa-Szép product are discussed in [3] and in more detail in [2] and need not concern us here beyond the following facts.

(Z1) The Zappa-Szép product $A \bowtie B$ of two monoids $A$ and $B$ is defined on the set $A \times B$ and sending all $a \in A$ to $(a, 1)$ and all $b \in B$ to $(1, b)$ are isomorphic embeddings of $A$ and $B$ into $A \bowtie B$.

(Z2) The multiplication on $A \times B$ gives the equality $(a, 1)(1, b) = (a, b)$ so it makes sense to write $ab$ for the pair $(a, b)$ in $A \times B$. The reader should note that it is not always the case that $(1, b)(a, 1) = (a, b)$.

(Z3) Every element in $A \bowtie B$ is uniquely expressible as a product $ab$ with $a \in A$ and $b \in B$.

The group $\hat{V}$ is a group of right fractions of $\mathcal{F} \bowtie S_\infty$ and $\hat{B}V$ is a group of right fractions of $\mathcal{F} \bowtie B_\infty$. What we need to know about groups of right fractions is the following where $M$ is a monoid and $G$ is its group of right fractions. See Section 2.3 of [3] for more details.

(R1) There is an isomorphic embedding $i : M \to G$.

(R2) For every $g \in G$ there are $p$ and $n$ in $M$ so that $g = (i(p))(i(n))^{-1}$.

2.2. The monoid $\mathcal{F}$ and the group $F$. The only relations in $\hat{V}$ and $\hat{B}V$ that apply to the submonoids called $\mathcal{F}$ are those in the first line of the presentations [1] and [2]. These relations have remarkable properties and we need the following standard facts about these relations as applied to $\mathcal{F}$ and the associated Thompson group $F$ with presentation

$$(6) \quad F = \langle \Lambda \mid \lambda_q \lambda_m = \lambda_m \lambda_{q+1}, \quad m < q \rangle$$

The second fact below quotes Proposition 3.3 of [3] while the first quotes that proposition and Proposition 6.1 of [3]. The third is Lemma 2.1.5 of [4] or Theorem 4.3 of [5].

(F1) The monoid $\mathcal{F}$ has presentation given by [6] regarded as a monoid presentation.

(F2) Each element of $\mathcal{F}$ can be written uniquely as a word $\lambda_{i_0} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$ for which $i_0 \leq i_1 \leq \cdots \leq i_k$.

(F3) Every proper quotient of the group $F$ is abelian.

2.3. The groups $V$ and $BV$. Section 7 of [3] identifies $V$ and $BV$ as consisting of those elements of $\hat{V}$ and $\hat{B}V$, respectively, of the form $(F\beta)(G\gamma)^{-1}$ with $F$ and $G$ from $\mathcal{F}$ and $\beta$ and $\gamma$ from $S_\infty$ or $B_\infty$, as appropriate, that satisfy the following properties for some integer $n \geq 0$.

(S1) The lengths of $F$ and $G$ as words in the generators from $\Lambda$ are both equal to $n$.

(S2) Both $F$ and $G$ can be written as words in the generators from $\Lambda$ in the form $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}$ so that $i_j < j$ for all $j$ with $1 \leq j \leq n$. 


(S3) Both $\beta$ and $\gamma$ can be expressed as words in elements of $\{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$.

In (S1), the lengths are well defined because of (F1) and the fact that the relations in (6) preserve length. Note that (S3) says that $\beta$ and $\gamma$ lie in $S_{n+1}$ or $B_{n+1}$ as appropriate.

3. Finite presentations for $\hat{V}$ and $\hat{BV}$

The infinite presentations given in (16) and (17) fit nicely into a machine due to Thompson that takes certain structures and reduces them from infinite to finite in (6) preserve length. Note that (S3) shows that $\beta$ and $\gamma$ lie in $S_{n+1}$ or $B_{n+1}$ as appropriate.

Using the finite generating sets just given, and using $\lambda_i = \lambda_0^{1-i}\lambda_1\lambda_0^{i-1}$ and $\sigma_i = \lambda_0^{1-i}\sigma_1\lambda_0^{i-1}$ for $i > 1$. This shows that the generating sets can be reduced to $\{\lambda_0, \lambda_1, \sigma_0, \sigma_1\}$ for both $\hat{V}$ and $\hat{BV}$. This is not minimal, but we will ignore that for now.

The relations break into two types: those that involve only one variable to express all the subscripts involved (such as the “braid relation” $\sigma_m\sigma_{m+1}\sigma_m = \sigma_{m+1}\sigma_m\sigma_{m+1}$) and those that involve two variables to express all the subscripts involved (such as $\sigma_m\sigma_n = \sigma_m\sigma_n$ whenever $|m - n| \geq 2$).

A relation of the first type in which the minimum subscript is at least 1 can be conjugated on both sides by $\lambda_0^i$ with $i > 0$ to uniformly elevate all of the subscripts in the relation by $i$. Since all subscripts are non-negative, only two relations from each such family are needed to generate all the relations in that family: one in which the subscript 0 appears, and one in which the minimum subscript is 1. For example, all of the braid relations $\sigma_m\sigma_{m+1}\sigma_m = \sigma_{m+1}\sigma_m\sigma_{m+1}$ follow from $\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1$ and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

Relations of the second type can have their appearance altered by rewriting the larger of the two subscripts. For example, $\sigma_q\lambda_m = \lambda_m\sigma_q$ for $m > q + 1$ can be written as $\sigma_q\lambda_{q+k} = \lambda_{q+k}\sigma_q$ for $k > 1$. Each value of $k$ gives a family of relations determined by the lower subscript. Again, conjugating by powers of $\lambda_0$ allow us to generate all relations for a fixed value of $k$ from two relations: one in which the lower subscript is 0, and one in which the lower subscript is 1. The task is to get all values of $k$ from a finite number of relations.

All of the relations of the second type are statements about the result of a conjugation: $X^{-1}YX = Z$. A commutativity statement comes under this description as well. All can be arranged so that the symbol with the lower subscript can play the role of conjugator, represented by $X$. If the conjugator $X$ is some $\sigma_q$ and $k \geq 3$, then conjugating the relation by $\lambda_i^{q+2}$ with $i > 0$ will elevate the value of $k$ by $i$ while keeping the value of $q$ untouched. This requires knowing the relations $\sigma_q\lambda_{q+2} = \lambda_{q+2}\lambda_q$ for all $q$ and knowing inductively the result of conjugating $\lambda_{i+j}$ and/or $\sigma_{i+j}$ by $\lambda_i$ for values of $j$ satisfying $j < k$. 
If the conjugator \(X\) is some \(\lambda_m\) and \(k \geq 3\), then the generator in the role of \(Y\) can be replaced by a conjugate using a lower subscript. For example,
\[
\lambda_m^{-1} \sigma_{m+k} \lambda_m = \lambda_m^{-1} (\lambda_m^{-1} \sigma_{m+k-1} \lambda_{m+1}) \lambda_m.
\]
Under an inductive hypothesis based on \(k\) and the homomorphic behavior of conjugation, the right expression is equal to \(\lambda_m^{-1} \sigma_{m+k} \lambda_m + 2\) which equals \(\sigma_{m+k+1}\) appealing again to the inductive hypothesis.

Using these arguments and the comments following \([2]\) about redundant relations, the reader can verify that the presentations above can be reduced to the following. In the presentations below, we have exploited the fact that one of the relations we are left with for \(\hat{V}\) and \(\tilde{BV}\) is \(\sigma_0 \lambda_0 = \lambda_1 \sigma_0 \sigma_0\). This allows us to write \(\lambda_1 = \sigma_0 \lambda_0 \sigma_0^{-1} \sigma_0^{-1}\).

**Lemma 3.1.** The groups \(\hat{V}\) and \(\tilde{BV}\) are presented as groups by the following in which \(\sigma_i = \lambda_i^{1-i} \sigma_1 \lambda_i^{-1}\) and \(\lambda_i = \lambda_i^{1-i} \lambda_1 \lambda_i^{-1}\) for \(i > 1\), as well as \(\lambda_1 = \sigma_0 \lambda_0 \sigma_0^{-1} \sigma_0^{-1}\) are definitions:
\[
\hat{V} = \langle \lambda_0, \sigma_0, \sigma_1 | \lambda_1^{-1} \lambda_2 \lambda_1 = \lambda_3, \quad \lambda_1^{-1} \lambda_3 \lambda_1 = \lambda_4, \\
\sigma_0^2 = 1, \quad \sigma_1^2 = 1, \\
[\sigma_0, \sigma_2] = [\sigma_0, \sigma_3] = 1, \quad [\sigma_3, \sigma_4] = [\sigma_1, \sigma_1] = 1, \\
\sigma_0 \sigma_1 \sigma_0 = \sigma_1 \sigma_0 \sigma_1, \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \\
\lambda_1^{-1} \sigma_2 \lambda_1 = \sigma_3, \quad \lambda_1^{-1} \sigma_3 \lambda_1 = \sigma_4, \\
\sigma_0 \lambda_0 = \lambda_1 \sigma_0 \sigma_1, \quad \sigma_1 \lambda_1 = \lambda_2 \sigma_1 \sigma_2, \\
[\sigma_0, \sigma_2] = [\sigma_0, \sigma_3] = 1, \quad [\sigma_1, \lambda_3] = [\sigma_1, \lambda_4] = 1, \rangle
\]

\[
\tilde{BV} = \langle \lambda_0, \sigma_0, \sigma_1 | \lambda_1^{-1} \lambda_2 \lambda_1 = \lambda_3, \quad \lambda_1^{-1} \lambda_3 \lambda_1 = \lambda_4, \\
[\sigma_0, \sigma_2] = [\sigma_0, \sigma_3] = 1, \quad [\sigma_3, \sigma_4] = [\sigma_1, \sigma_1] = 1, \\
\sigma_0 \sigma_1 \sigma_0 = \sigma_1 \sigma_0 \sigma_1, \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \\
\lambda_1^{-1} \sigma_2 \lambda_1 = \sigma_3, \quad \lambda_1^{-1} \sigma_3 \lambda_1 = \sigma_4, \\
\sigma_0 \lambda_0 = \lambda_1 \sigma_0 \sigma_1, \quad \sigma_1 \lambda_1 = \lambda_2 \sigma_1 \sigma_2, \\
\sigma_0^{-1} \lambda_0 = \lambda_1 \sigma_0^{-1} \sigma_0^{-1}, \quad \sigma_1^{-1} \lambda_1 = \lambda_2 \sigma_1^{-1} \sigma_1^{-1}, \\
[\sigma_0, \sigma_2] = [\sigma_0, \sigma_3] = 1, \quad [\sigma_1, \lambda_3] = [\sigma_1, \lambda_4] = 1, \rangle
\]

### 4. Infinite presentations for \(V\) and \(BV\)

#### 4.1. Generators

Generators are easy to find.

**Lemma 4.1.** The union of the three infinite sets
\[
\{v_n = \lambda_0^{n+1} \lambda_1 \lambda_0^{-n-2} \mid n \in \mathbb{N}\}, \\
\{\pi_n = \lambda_0^{n+2} \sigma_1 \lambda_0^{-n-2} \mid n \in \mathbb{N}\}, \\
\{\overline{\pi}_n = \lambda_0^{n+1} \sigma_0 \lambda_0^{-n-1} \mid n \in \mathbb{N}\}
\]

generates \(V\) and \(BV\).

**Proof.** Assume that we can show that any \(v = W \beta \lambda_0^{-n}\) is a product of the claimed generators when \(W\) has length \(n\) and satisfies \([2]\) and \(\beta\) is in \(S_{n+1}\) or \(B_{n+1}\). Then
any \( (F\beta)(G\gamma)^{-1} \) satisfying \([S1]–[S3]\) can be written as \( (F\beta \lambda_0^n)(G\gamma \lambda_0^{-n})^{-1} \) and will be a product of the claimed generators. Thus we concentrate on a word such as \( u \).

We first assume that \( \beta = 1 \) and argue inductively on the length of \( W \) that \( W\lambda_0^n \) is product of the generators in \( \{v_0, v_1, \ldots\} \). If \( W \) has length \( n \) and satisfies \([S2]\), then \( W\lambda_i \) satisfies \([S2]\) if and only if \( 0 \leq i \leq n \). Now if \( W\lambda_0^n \) is a product of the \( v_j \), then so is \( W\lambda_0\lambda_0^{-n-1} \), so we may as well assume \( 1 \leq i \leq n \) which gives \( 1-i \leq 0 \).

This leads to the calculation

\[
W\lambda_0^{-n}v_{n-i} = W\lambda_0^{-n}\lambda_0^{n-i+1}\lambda_1\lambda_0^{i-n-2} = W\lambda_0^{i-1}\lambda_1\lambda_0^{i-n-2} = W\lambda_1\lambda_0^{n-1}
\]

which finishes the induction.

For \( \beta \) non-trivial, we will have our result inductively if \( W\beta\sigma_i\lambda_0^{-n} = ug \) for some generator \( g \) whenever \( 0 \leq i < n \). Note that there is nothing to show unless \( n \geq 1 \).

For \( i = 0 \), we have \( W\beta\lambda_0^{-n}\pi_{n-1} = W\beta\sigma_0\lambda_0^{-n} \). For \( 0 < i < n \), we have

\[
W\beta\lambda_0^{-n}\pi_{n-i-1} = W\beta\lambda_0^{-n}\lambda_0^{n-i+1}\sigma_1\lambda_0^{-n+i-1} = W\beta\lambda_0^{i-1}\sigma_1\lambda_0^{-n+i-1} = W\beta\sigma_i\lambda_0^{-n}.
\]

This completes the proof. \( \square \)

### 4.2. Relations

The groups \( V \) and \( BV \) sit inside groups with a known set of relations. It is not hard to find relations for the smaller groups. The problem is finding enough relations among them to get a presentation. We start with lists of relations that are demonstrably true.

**Lemma 4.2.** In the following list, the relations \([4]\) through \([10]\) hold in \( BV \) and all the relations below hold in \( V \).

(7) \( v_q v_m = v_m v_{q+1} \), \( m < q \),
(8) \( \pi_q v_m = v_m \pi_{q+1} \), \( m < q \),
(9) \( \pi_m v_{m+1} = v_{m+1} \pi_m \pi_{m+1} \), \( m \geq 0 \), \( \epsilon = \pm 1 \),
(10) \( \pi_q v_m = v_m \pi_q \), \( m > q + 1 \),
(11) \( \pi_q v_m = v_m \pi_q \), \( m < q \),
(12) \( \pi_m v_{m+1} = \pi_m \pi_{m+1} \), \( m \geq 0 \), \( \epsilon = \pm 1 \),
(13) \( \pi_m \pi_m = \pi_m \pi_m \), \( |m - q| \geq 2 \),
(14) \( \pi_{m+1} \pi_m = \pi_{m+1} \pi_m \pi_{m+1} \), \( m \geq 0 \),
(15) \( \pi_{q+1} v_m = \pi_{q+1} \pi_{q+1} \), \( q \geq m + 2 \),
(16) \( \pi_m v_{m+1} = \pi_m v_{m+1} \pi_{m+1} \), \( m \geq 0 \),
(17) \( \pi_m = 1 \), \( m \geq 0 \),
(18) \( \pi_m = 1 \), \( m \geq 0 \).

**Proof.** It is straightforward to verify the claimed relations from the definitions in Lemma 4.1 and the relations that hold in \( \hat{V} \) and \( \hat{BV} \) as given in \( [4] \) and \( [2] \). We
verify (7) as an example. We assume \( m < q \) and calculate
\[
v_q v_m = \lambda_0^{q+1} \lambda_1 \lambda_0^{-q-2} \lambda_0^{m+1} \lambda_1 \lambda_0^{-m-2}
\]
\[
= \lambda_0^{q+1} \lambda_1 \lambda_0^{-m-q-1} \lambda_1 \lambda_0^{-m-2}
\]
\[
= \lambda_0^{q+1} \lambda_1 \lambda_0^{-m-q-2} \lambda_1 \lambda_0^{-q-3}.
\]
\[
v_m v_{q+1} = \lambda_0^{m+1} \lambda_1 \lambda_0^{-m-2} \lambda_0^{q+2} \lambda_1 \lambda_0^{-q-3}
\]
\[
= \lambda_0^{m+1} \lambda_1 \lambda_0^{-m-q} \lambda_1 \lambda_0^{-q-3}
\]
\[
= \lambda_0^{q+1} \lambda_1 \lambda_0^{-m-q-2} \lambda_1 \lambda_0^{-q-3}.
\]

\[\square\]

The relations above are highly redundant. In fact, the family of generators is also overly large since the relations (12) can be used to define each \( \pi_n \) as \( \pi_n v_n \pi^{-1}_{n+1} \). However, the large supply of relations will make them easy to work with.

### 4.3. A subgroup isomorphic to \( F \).

Understanding words only in the \( v_i \) will be important.

**Lemma 4.3.** The subgroup of \( V \) or \( BV \) generated by \( \{v_i \mid i \in \mathbb{N}\} \) is isomorphic to the group \( F \) with presentation (2).

**Proof.** By comparing (7) with (6), we see that the subgroup in question is a quotient of \( F \). By (3), \( F \) has no proper non-abelian quotients, so we need only show \( v_2 = v_0^{-1} v_1 v_0 \neq v_1 \). But if \( v_1 = v_2 \), then
\[
v_2 v_1^{-1} = (\lambda_0^3 \lambda_1 \lambda_0^{-4})(\lambda_0^3 \lambda_1^{-1} \lambda_0^{-2}) = \lambda_0^3 \lambda_1 \lambda_0^{-1} \lambda_0^{-3}
\]
is trivial and so we would have \( \lambda_1 = \lambda_2 \). But the normal form given in (2) forbids this equality. \[\square\]

### 4.4. Calculations from the relations for \( BV \).

From this point we derive consequences of the relations in Lemma 4.2. In order to insure that the work we do applies equally to \( V \) and \( BV \), we will avoid making any use at all of the relations (6) and (8) until we loudly announce that we are ready to resume using them. Given two words \( w \) and \( w' \) in the generators of Lemma 4.2, we will say \( w \sim w' \) to mean that \( w \) can be converted to \( w' \) by use of the relations (4) through (10) of Lemma 4.2.

There is a function from words in the \( \pi_i \) and their inverses into the group \( S_\infty \) of finitely supported permutations of \( \mathbb{N} \). In fact, it is a homomorphism from the subgroup of \( BV \) generated by the \( \pi_i \), but we do not know this now, and have no need to know it. The function is gotten by taking \( \pi_i \) to the transposition that interchanges \( i \) and \( i+1 \). Under this assignment, any word \( w \) in the \( \pi_i \) and their inverses gives a permutation that we continue to call \( w \). This is used in the next lemma.

**Lemma 4.4.** If \( W \) is the set of words in the \( \pi_i \) and their inverses, and if \( V \) is the set \( \{v_0, v_1, \ldots\} \), then there are functions \( (w, v) \mapsto w' \) from \( W \times V \to W \) and \( (v, w) \mapsto w \cdot v \) from \( V \times W \to W \) so that for \( w \in W \) and \( v_m \in V \), we have \( w v_m \sim v_j w^{(m)} \) where \( j = w(m) \) and \( v_m^{-1} w \sim (v_m \cdot w)^{-1} \) where \( k = w^{-1}(m) \).
Further, if \( w \) is a word in \( \{\pi_i, \pi_i^{-1} \mid 0 \leq i \leq k\} \), then \( v_m \cdot w \) and \( w^{v_m} \) are words in \( \{\pi_i, \pi_i^{-1} \mid 0 \leq i \leq k+1\} \) and \( w(m) = m \) if \( m > k + 1 \) and \( w(m) \leq k + 1 \) if \( m \leq k + 1 \).

**Proof.** From (6), we get \( v_{m+1}^{-1} \pi_m \sim \pi_m^{-1} \pi_{m+1}^{-1} v_m^{-1} \) which on inversion gives

\[
(19) \qquad \pi_m^{-1} v_m \sim v_m \pi_m^{-1} \pi_{m+1}^{-1}.
\]

This combines with (3), (4) and (10) and an induction argument to give \( u v_m \sim v_j u^{v_m} \). Inverting (12), conjugating (3) and (10) by \( v_m \) and combining with \( v_{m+1}^{-1} \pi_m \sim \pi_m^{-1} \pi_{m+1}^{-1} v_m^{-1} \) and an induction argument gives \( v_{m+1}^{-1} w \sim (v_m \cdot w)^{-1} \). The last paragraph of the statement comes from the forms of (3), (4), (10) and (11) and the fact that under the hypotheses of that paragraph, \( w \) maps to a permutation in \( S_\infty \) that fixes all \( j \in \mathbb{N} \) with \( j > k + 1 \).

The interaction of \( u v_m \) and \( \pi_q \) are partly covered by (11) and (12). The next lemma covers the rest.

**Lemma 4.5.** The following hold for all \( k > 0 \):

\[
\pi_m v_m \sim (v_m v_{m+1} \cdots v_m k-2 v_{m+k-1}) \pi_m^{-1} (\pi_m^{-1} \pi_{m+k-1} \cdots \pi_m),
\]

\[
v_{m+1}^{-1} \pi_m \sim (\pi_m^{-1} \pi_{m+1} \cdots \pi_{m+k}) v_{m+1} (v_{m+1} \cdots v_{m+k-2})^{-1}.
\]

**Proof.** From the inverted form of (12), we get \( \pi_m^{-1} \sim v_m \pi_{m+1}^{-1} \pi_m^{-1} \), and inductively, get

\[
\pi_m v_{m+k} \sim (v_m v_{m+1} \cdots v_{m+k-2} v_{m+k-1}) \pi_{m+k} (\pi_{m+k-1} \cdots \pi_m).
\]

where we have used (10), (13) and (14). The other formula is a similar exercise starting with the uninverted form of (12).

**Lemma 4.6.** Let \( W \) be a word in the generators of Lemma 4.4. Then \( W \sim LMR \) where \( L \) is a word in \( \{v_i \mid i \in \mathbb{N}\} \), \( R \) is a word in \( \{v_i^{-1} \mid i \in \mathbb{N}\} \) and \( M \) is a word in \( \{\pi_i, \pi_i^{-1} \mid i \in \mathbb{N}\} \).

**Proof.** We start by assuming that \( x \) is represented by a word in \( \{v_i, v_i^{-1} \mid i \in \mathbb{N}\} \). From (7), we get \( v_m^{-1} v^q \sim v^{q-1} v_m^{-1} \) and \( v^q v_m \sim v_m v_{q+1}^{-1} \) whenever \( m < q \). This allows negative powers with low subscripts to pass to the right and positive powers with low subscripts to pass to the left, each at the expense of raising the already higher subscript. When there is a possible ambiguity (negative power wants to pass to the right of a positive power), then one of the subscripts will be lower and the ambiguity is resolved, or we have \( v_i^{-1} v_i \) which is replaced by 1. Thus we can achieve the promised form with \( M = 1 \).

Next we assume that \( x \) is represented by a word in positive and negative powers of the generators of Lemma 4.4 in which there are no appearances of any \( \pi_i \) or \( \pi_i^{-1} \). Now Lemma 4.4 allows us to pass negative powers of the \( v_i \) from left to right and positive powers of \( v_i \) from right to left over appearances of the \( \pi_i \). This comes at a
cost of possibly increasing the number of appearances of the $\pi_i^j$ (if (9) is used), but the number of $v_i^j$ cannot go up. An easy induction on a complexity derived from the word representing $x$ completes the argument. This is essentially the argument that would go to show that a rewriting system derived from the relevant relations is terminating.

Now we assume that $x$ is represented by an arbitrary word in the positive and negative powers of the generators of Lemma 4.1. We will use Lemma 4.5 to pass appearances of the $v_i^j$ over appearances of the $\pi_j^k$. Unfortunately, the number of appearances of the $v_i^j$ might go up, but the number of $\pi_j^k$ will not.

Consider the rightmost appearance of some $\pi_j^k$ in the word representing $x$. From the previous paragraphs, we can assume that all appearances of positive powers of the $v_i$ that appear somewhere to the right of this $\pi_j^k$ are immediately to its right. Now an application of (11), (12) or the first relation in Lemma 4.5 reduces by one the number of positive powers of $v_i$ to the right of this (altered) rightmost appearance of $\pi_j^k$ in the word. If Lemma 4.6 is used, then extra appearances of the $\pi_i$ are introduced, but these can be gotten “out of the way” by the previous paragraphs. Eventually, there will be no appearances of positive powers of the $v_i$ to the right of the rightmost appearance of an $\pi_j^k$ in the word. An induction on a complexity derived from the word shows that all positive powers of the $v_i$ can be moved to the extreme left of the word. Now negative powers can be moved to the right using the second relation from Lemma 4.5 and altered forms of (11) and (12).

We now concentrate on the subword $M$ obtained from Lemma 4.6. It is a word in \{\(\pi_i, \pi_i^{-1}\pi_i, \pi_i^{-1}\) | \(i \in \mathbb{N}\)}. We can map $M$ to a word in the generators of $BV$ using the definitions in Lemma 4.1. Under the mapping, each letter in $M$ becomes some $\sigma_i^j$ conjugated by a power of $\lambda_0$. We will be happiest when all the conjugations are by the same power of $\lambda_0$. The appearances of $\pi_i$ in $M$ give us more flexibility in achieving this because of the following observation in which $k \geq 0$:

\[
\pi_i = \lambda_0^{i+2} \sigma_1 \lambda_0^{-i-2} = \lambda_0^{i+2} \lambda_0^{-k} \sigma_1 \lambda_0^{-i-2} = \lambda_0^{i+k+2} \sigma_{k+1} \lambda_0^{-(i+k)-2}.
\]

However, the $\pi_i$ are more complicated. This motivates the next definitions.

Because of the definition $\pi_n = \lambda_0^{n+1} \sigma_0 \lambda_0^{-n-1}$, we define the height of $\pi_n$ to be the subset \{\(n+1\)\} \(\subseteq\) \(\mathbb{N}\). We define the height of $\pi_n$ to be the subset \{\(j \mid j \geq n+2\)\} \(\subseteq\) \(\mathbb{N}\). If $M$ is a word in \{\(\pi_i, \pi_i^{-1}\pi_i, \pi_i^{-1}\) | \(i \in \mathbb{N}\)\}, then we define the height of $M$ to be the intersection of the heights of all the letters in $M$. Note that it can easily be that the height of $M$ is empty. One of our goals is the next lemma.

**Lemma 4.7.** Let $W$ be a word in the generators of Lemma 4.1. Then $W \sim LMR$ as specified in Lemma 4.6 so that the height of $M$ is not empty.

We need some technical information about the behavior of height. This will help in the proof and application of Lemma 4.7. In the following lemmas, a monosyllable is a word in \{\(\pi_i, \pi_i^{-1}\pi_i, \pi_i^{-1}\) | \(i \in \mathbb{N}\)\} with exactly one appearance of $\pi_i^{\pm 1}$.

**Lemma 4.8.** Let $M$ be a monosyllable of height \{\(h\)\} and let \(0 \leq n < h\). Then the following are true where $M'$ is some monosyllable of height \{\(h+1\)\} and \(0 \leq j < h\).
The different appearances of $M'$ and $j$ are not to be taken as representing the same values.

(a) $M \sim M'v_j^{-1}$,
(b) $M \sim v_jM'$,
(c) $Mv_m \sim M'$ or $Mv_m \sim v_jM'$,
(d) $v_m^{-1}M \sim M'$ or $v_m^{-1}M \sim M'v_j^{-1}$.

Proof. The word $M$ has the form $\Sigma_1\overrightarrow{\pi}_{\gamma}^{\pi_0}_{h-1}\Sigma_2$ where $\gamma \in \{-1, 1\}$ and $\Sigma_1$ and $\Sigma_2$ are words in $\{\pi_i, \pi_i^{-1} \mid 0 \leq i \leq h - 2\}$.

To prove (a), we get $\overrightarrow{\pi}_{\gamma}^{\pi_0}_{h-1}\pi_{\gamma}^{\pi_0}_{h-1}\overrightarrow{\gamma}_{h-1}\pi_{\gamma}^{\pi_0}_{h-1}$ from (4.3), and from Lemma 4.10 we get $v_1^{-1}\Sigma_2 \sim \Sigma_2v_j$ where $0 \leq j < h$ and $\Sigma_2$ is a word in $\{\pi_i, \pi_i^{-1} \mid 0 \leq i \leq h - 1\}$. Now $M \sim \Sigma_1\pi_0\overrightarrow{\gamma}_{h-1}\Sigma_2v_j$. The proof of (b) is similar.

We consider (c). By Lemma 4.3 $\Sigma_2v_m \sim v_1\Sigma_2$ where $0 \leq k < h$ and $\Sigma_2'$ is a word in $\{\pi_i, \pi_i^{-1} \mid 0 \leq i \leq h - 1\}$. If $k = h - 1$, then $\overrightarrow{\pi}_{\gamma}^{\pi_0}_{h-1}\pi_{\gamma}^{\pi_0}_{h-1}\overrightarrow{\pi}_{h-1}\pi_{\gamma}^{\pi_0}_{h-1}$ and we have $Mv_m \sim \Sigma_1\pi_0\overrightarrow{\pi}_{h-1}\pi_{\gamma}^{\pi_0}_{h-1}\Sigma_2$. If $k < h - 1$, then $\overrightarrow{\pi}_{\gamma}^{\pi_0}_{h-1}\pi_{\gamma}^{\pi_0}_{h-1}\overrightarrow{\pi}_{h-1}\pi_{\gamma}^{\pi_0}_{h-1}$ and, by Lemma 4.3 $\Sigma_2v_k \sim v_1\Sigma_2$ where $0 \leq j < h$ and $\Sigma_1$ is a word in $\{\pi_i, \pi_i^{-1} \mid 0 \leq i \leq h - 1\}$. Now $Mv_m \sim v_1\Sigma_2\overrightarrow{\pi}_{\gamma}^{\pi_0}_{h-1}\Sigma_2$. The proof of (b) is similar. □

**Lemma 4.9.** Let $W = M_1M_2\cdots M_t$ where each $M_i$ is a monosyllable of height $\{h_i\}$ and $h_1 \leq h_2 \leq \cdots \leq h_t$. Then $W \sim M_1'v_1M_2'v_2 \cdots M_t'v_t$ or $W \sim M_1'M_2' \cdots M_t'$ where $0 \leq j < h_t$ and each $M_i'$ is a monosyllable of height $\{h_i + 1\}$.

Proof. This is a repetitive application of Lemma 4.8. We start by replacing $M_1$ by $M_1'v_1^{-1}$ using Lemma 4.8(a), and looking at $v_1^{-1}M_2$. This either gives $M_2'$ or $M_2'v_2^{-1}$ by Lemma 4.8(d). In the first case, we continue by applying Lemma 4.8(a) to $M_3$, and in the second case, we continue by applying Lemma 4.8(d) to $v_2^{-1}M_3$. Eventually, we finish. □

**Lemma 4.10.** Let $M$ be a word in $\{\pi_i, \pi_i^{-1}, \pi_1, \pi_1^{-1} \mid i \in \mathbb{N}\}$ with a non-empty height that contains $h \in \mathbb{N}$. Then there are $i < h$, $j < h$, $\epsilon \in \{0, 1\}$ and $\delta \in \{0, 1\}$ so that $M \sim v_iM'$ and $M \sim M''v_j^{-\delta}$ so that $M'$ and $M''$ are words in $\{\pi_i, \pi_1^{-1}, \pi_1, \pi_1^{-1} \mid i \in \mathbb{N}\}$ each with non-empty height containing $h + 1$.

Proof. If $M$ contains no appearances of any $\overrightarrow{\pi}_{\gamma}^{\pi_0}_{i}$, then the height of $M$ is an infinite subset of $\mathbb{N}$ and we can let $\epsilon$ and $\delta$ be 0 and 0 and let $M' = M'' = M$.

If there are appearances of the $\overrightarrow{\pi}_{\gamma}^{\pi_0}_{i}$ in $M$, then $M$ is a concatenation of monosyllables of constant height. We get the form $M''v_j^{-\delta}$ by a direct application of Lemma 4.8(d) and we get the form $v_iM'$ by applying Lemma 4.8(d) to $M''v_j^{-\delta}$ and then inverting the result. □

**Proof of Lemma 4.7.** Let $W \sim LMR$ as given by Lemma 4.6. We may assume that $M$ has at least one appearance of some $\overrightarrow{\pi}_{i}^{\pm 1}$. Thus $M$ is a concatenation $M_1M_2\cdots M_t$ of monosyllables. Let $\{h_i\}$ be the height of the monosyllable $M_i$. If we alter $M$ by the use of the Lemmas 4.8 and 4.9 then we continue to refer to the monosyllables as $M_i$ and the heights as $\{h_i\}$. If we apply Lemma 4.9 repeatedly to $M_t$ we can raise the height of $M_t$ possibly at the expense of introducing extra terms into the subword $R$. Thus we can assume that $h_t \geq h_i$ for all $i < t$. Now we apply Lemma 4.9 to $M_{t-1}M_t$ repeatedly so that we can assume $h_{t-1} \geq h_i$ for $i < t - 1$ and $h_{t-1} \leq h_t$. Eventually, we get $h_1 \leq h_2 \leq \cdots \leq h_t$. □
Now we invert the word $LMR$ which still has the form given by Lemma 4.10 and $M$ is again a concatenation of monosyllables $M_1M_2\cdots M_t$ with associated heights satisfying $h_1 \geq h_2 \geq \cdots \geq h_t$. We repeat the above process, but now we can obtain $h_{t-1} = h_t$ after applying Lemma 4.10 to $M_t$, and we get $h_{t-2} = h_{t-1} = h_t$ after applying Lemma 4.10 to $M_{t-1}M_t$, and so forth. Eventually, we get equal heights for all the monosyllables which completes the proof. \( \square \)

We now consider all of $LMR$ as obtained from Lemma 4.10 and its mapping to a word in the generators of $\overline{BV}$. From Lemma 4.7 we can assume that $M$ maps to a word of the form $\lambda_0^iw_{\lambda_0^h}$ where $w$ is a word in $\{\sigma_i, \sigma_i^{-1} \mid i \in \mathbb{N}\}$. Since $L$ is a word in $\{v_i \mid i \in \mathbb{N}\}$, we can map $L$ to a word in the form

\[(20) \quad \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}\lambda_0^{-k}.\]

That one value of $k$ correctly appears in two places in \(20\) follows easily from the definitions of the $v_i$ and from the relations in \(2\). We are not concerned with the uniqueness of $k$ and will say the height of $L$ is no more than $k$ if $L$ maps to a word in the form of \(20\). Since $R^{-1}$ is a word in $\{v_i \mid i \in \mathbb{N}\}$, we can say the height of $R$ is no more than $k$ if the height of $R^{-1}$ is no more than $k$. Our next goal is the following improvement on Lemmas 4.6 and 4.7.

Lemma 4.11. Let $W$ be a word in the generators of Lemma 4.1. Then there is a $k$ and $W \sim LMR$ as specified in Lemma 4.7 so that the height of $M$ contains $k$ and the heights of $L$ and $R$ are no more than $k$.

We need one more lemma about the behavior of height.

Lemma 4.12. Let $L$ be a word in $\{v_i \mid i \in \mathbb{N}\}$ of height no more than $k$. Then $Lv_m$ has height no more than $k+1$ if $m \leq k-1$ and has height no more than $m+2$ if $m \geq k-1$.

Proof. We have $L = \Lambda\lambda_0^{-k}$ where $\Lambda$ is a word in $\{\lambda_i \mid i \in \mathbb{N}\}$. Let $m = k-1+j$ so that $v_m = \lambda_0^k+\lambda_1\lambda_0^{-k-j-1}$. Now $Lv_m = \lambda_0^k\lambda_1\lambda_0^{-k-j-1}$. If $j \geq 0$, then $Lv_m$ has height no more than $k+j+1 = m+2$ and if $j \leq 0$, then $Lv_m = \Lambda\lambda_{j+1}\lambda_0^{-k-1}$ and has height no more than $k+1$. \( \square \)

Proof of Lemma 4.11. Let $W \sim LMR$ as given by Lemma 4.7. Let $L$ have height no more than $k_1$ and let $R$ have height no more than $k_2$. If $M$ has no appearances of any $\pi_i^{\pm 1}$, then the height of $M$ is an infinite set and there is nothing to prove. Thus assume that $M$ is a concatenation of monosyllables and has height $\{h\}$.

Assume first that $k_1 \leq h$ and $k_2 > h$. Use Lemma 4.10 to replace $M$ by $v_m^\epsilon M'$ where $m < h$, where $\epsilon \in \{0, 1\}$ and where $M'$ has height $\{h+1\}$. If $\epsilon = 0$, then the height of $Lv_m^\epsilon$ is no more than $k_1$ which is less than $h + 1$. Otherwise from Lemma 4.12, the height of $Lv_m^\epsilon$ is no more than $k_1+1$ or $m+2$, both of which are no more than $h+1$. Thus repeated uses of Lemma 4.10 give us the desired result. If $k_1 > h$ and $k_2 \leq h$, then we use Lemma 4.11 repeatedly replacing $M$ by $M'v_m^\delta$ with $\delta \in \{0, 1\}$.

Now assume $k_1 > h$ and $k_2 > h$. Now repeated uses of Lemma 4.10 replacing $M$ by $v_m^\epsilon M'$ will achieve $h \geq k_2$ at the expense of raising $k_1$. Now we are in one of the cases of the previous paragraph. \( \square \)
4.5. Infinite presentations. We now are ready to resume using the relations (17) and (18) when dealing with \( V \). In the following \( \sim \) will use only relations (9) through (16) in the case of \( BV \) and will use (17) and (18) in addition in the case of \( V \). The difference in the application will be clear.

**Proposition 4.13.** Let \( W \) be a word in the generators of Lemma 4.1. If \( W \) represents the trivial element of either \( V \) or \( BV \), then \( W \sim \phi \) where \( \phi \) is the empty word.

**Proof.** We may assume that \( W \) has the form \( LMR \) as given by Lemma 4.11.

We first argue that \( W \sim \phi \) follows if we can show \( M \sim \phi \). If \( M \sim \phi \), then \( W \sim LR \) which is a word in \( \{ v_i, v_i^{-1} \mid i \in \mathbb{N} \} \). By Lemma 4.3, the subgroup generated by \( \{ v_i, v_i^{-1} \mid i \in \mathbb{N} \} \) is isomorphic to the group \( F \) whose relations are precisely the relations in (7). Thus if \( LR \) is the identity, then \( W \sim \phi \).

It remains to show that \( M \sim \phi \).

We know that \( M \) has a height that includes some \( h \) and that \( L \) and \( R \) have height no more than \( h \). Thus the mapping of \( W \) into the generators of \( \hat{V} \) or \( BV \) takes the form \( \Lambda_1 \lambda_0^{-s} \lambda_0^h \lambda_0^{-s} \lambda_0 \Lambda_2 \) where \( \Lambda_1 \) and \( \Lambda_2^{-1} \) are words in \( \{ \lambda_i \mid i \in \mathbb{N} \} \), \( w \) is a word in \( \{ \sigma_i \mid i \in \mathbb{N} \} \) and \( s \leq h \) and \( t \leq h \) both hold. Thus the image of \( W \) is really of the form \( \Lambda_1 w \Lambda_2 \) where \( w \) is the same as above and \( \Lambda_1 \) and \( \Lambda_2^{-1} \) are words in \( \{ \lambda_i \mid i \in \mathbb{N} \} \).

We can say more about the letters in \( w \). The mapping of the elements of \( M \) into \( W \) took each letter of \( M \) to some \( \sigma_i \) conjugated by \( \lambda_0^h \). In particular, each \( \pi_i^{j+1} \) in \( M \) is of the form \( \pi_i^{j+1} \) with \( \epsilon \in \{ -1, 1 \} \). This maps to \( \lambda_0^h \sigma_i \lambda_0^{-h} \). Each \( \pi_i \) must have \( i \leq h - 2 \) and maps to \( \lambda_0^i \sigma_i \lambda_0^{-i} \). However as noted before Lemma 4.7 it can also be mapped to \( \lambda_0^{i-k+2} \sigma_i \lambda_0^{-i-k-2} \). Setting \( (i + k) + 2 = h \) gives \( 1 + k = (h - 1) - i \).

The previous paragraph gives a simple formula for getting the word \( w \) from \( M \). We have that

\[
M = X_{i_1}^r X_{i_2}^r \cdots X_{i_r}^r
\]

where each \( X \) is either the symbol \( \pi \) or the symbol \( \pi \). The resulting word \( w \) is

\[
\sigma_{j_1}^r \sigma_{j_2}^r \cdots \sigma_{j_r}^r
\]

where each \( j_k = (h - 1) - i_k \).

We first argue that it suffices to show that (22) reduces to the trivial word modulo the relations in \( \hat{V} \) or \( BV \) that relate only to the \( \sigma_i \); specifically the relations on (9) and (10) in either case and (11), in addition, in the case of \( V \).

Assume a sequence of applications of (3), (11) and perhaps (13) reduce (22) to the empty word. The relation (13) reads \( \sigma_m \sigma_n = \sigma_n \sigma_m \) if \( |m - n| \geq 2 \). However, \( \sigma_m \) comes from \( X_p \) and \( \sigma_n \) comes from \( X_q \) and with \( p = (h - 1) - m \) and \( q = (h - 1) - n \). But \( |p - q| = |m - n| \geq 2 \) and the relations of Lemma 4.2 imply that \( X_p X_q = X_q X_p \) so the application of (3) to (22) can be mirrored by a corresponding application of either (13) or (15) to (21). After the corresponding applications, the new version of (21) maps to the new version of (22) under the mapping given by the paragraphs above. An identical argument says that an application of (11) to (22) can be mirrored by a corresponding application of either (13) or (15) to (21). Finally, applications of (15) can be mirrored by applications of either (17) or (18). Note that applications
of (17) or (18) will only be needed if (15) is needed. However, (5) only applies to \( \hat{V} \) and so (17) or (18) will only be needed if we are dealing with \( V \).

Thus the reduction of (22) to the empty word gives instructions to achieve a reduction of (21) to the trivial word that uses appropriate relations in each case.

Now we must argue that (22) can be trivialized by applications of (3), (4) and perhaps (5).

We are assuming the word \( \Lambda_1'w\Lambda_2' \) represents the identity in \( \hat{V} \) or \( \hat{BV} \), that \( w \) is the word (22) above, and that \( \Lambda_1' \) and \( (\Lambda_2')^{-1} \) are words in \( \{\lambda_i \mid i \in \mathbb{N}\} \). Thus \( \Lambda_1'w = (\Lambda_2')^{-1} \) as elements of \( \hat{V} \) or \( \hat{BV} \). Since \( \hat{V} \) and \( \hat{BV} \) are the groups of fractions of the monoid \( \mathcal{F} \bowtie S_\infty \) and \( \mathcal{F} \bowtie B_\infty \), respectively, the embedding property (R1) of groups of fractions says that \( \Lambda_1'w = (\Lambda_2')^{-1} \) as elements of \( \mathcal{F} \bowtie S_\infty \) or \( \mathcal{F} \bowtie B_\infty \). From the uniqueness of representation in Zappa-Szép products (Z3), this says that \( w = 1 \) as an element of \( S_\infty \) or \( B_\infty \). Since the relations (3) and (4) suffice to present \( B_\infty \) and the relations (3), (4) and (5) suffice to present \( S_\infty \), we are done with the proof. \( \square \)

We are now ready to give an infinite presentation of \( V \) and \( BV \).

**Theorem 1.** The groups \( V \) and \( BV \) are presented by the generators of Lemma 4.2 and the relations (7) through (16) of Lemma 4.2 in the case of \( BV \) and the relations (7) through (18) of Lemma 4.2 in the case of \( V \).

**Corollary 4.14.** The group \( BV \) is presented by the generators \( v_n \) and \( \pi_n \) for \( n \in \mathbb{N} \) and by the relations

\[
\begin{align*}
v_q v_m &= v_m v_{q+1}, & m < q, \\
\pi_m v_m &= v_{m+1} \pi_m \pi_{m+1}, & m \geq 0, \epsilon = \pm 1, \\
\pi_q v_m &= v_m \pi_q, & m > q + 1, \\
\pi_q v_m &= v_m \pi_{q+1}, & m < q, \\
\pi_m &= \pi_{m+1} v_m^{-1} \pi_m, & m \geq 0, \\
\pi_q \pi_m &= \pi_m \pi_q, & |m - q| \geq 2, \\
\pi_m \pi_{m+1} \pi_m &= \pi_{m+1} \pi_m \pi_{m+1}, & m \geq 0, \\
\pi_q \pi_m &= \pi_m \pi_q, & q \geq m + 2, \\
\pi_m \pi_{m+1} \pi_m &= \pi_{m+1} \pi_m \pi_{m+1}, & m \geq 0
\end{align*}
\]

in which \( \pi_n = \pi_n v_n \pi_n^{-1} \) is taken as a definition for each \( n \in \mathbb{N} \).

The group \( V \) is presented by the generators \( v_n \) and \( \pi_n \) for \( n \in \mathbb{N} \) and by the relations with definitions listed above for \( BV \) with the addition of the relations

\[
\pi_m^2 = \pi_m = 1, \quad m \geq 0.
\]

**Proof.** The definitions for the \( \pi_n \) are obtained from the relations (12) with \( \epsilon = 1 \). Now the relations (8) follow from the definitions of the \( \pi_n \) and from (7) and (11). The fifth line of relations above is obtained from (12) with \( \epsilon = -1 \). \( \square \)

It is not clear if there is any way to conclude the relations \( \pi_m^2 = 1 \) from the others in the case of \( V \).
4.6. A “Coxeter group/Artin group” pair. The choice of generators for the
presentations in Corollary 4.14 was made mostly to mirror the traditional pre-
sentations of Thompson’s groups. Instead, we could have turned the definition
π_0 = π_{n+1}n^{-1}n into v_n = π_{n-1}nπ_{n+1} to get the following.

Corollary 4.15. The group BV is presented by the generators π_n and π_{n+1} for
n ∈ N and by the relations in Corollary 4.14 in which v_n = π_{n-1}nπ_{n+1} is taken
as a definition for each n ∈ N.

The group V is presented by the generators π_n and π_{n+1} for n ∈ N and by the
relations with definitions listed above for BV with the addition of the relations
π_m = π_{m+1} = 1, m ≥ 0.

5. Finite presentations for V and BV

We can now extract finite presentations from Corollary 4.14.

Theorem 2. The group BV has the following presentation. The equalities v_i = v_0^{1-i}v_1v_{i-1} and π_i = π_0^{1-i}π_1v_i for all i ≥ 2, and π_i = π_1v_iπ_{i-1} for all i ∈ N are
taken as definitions.

BV = ⟨v_0, v_1, π_0, π_1 | v_2v_1 = v_1v_3, v_3v_1 = v_1v_4, π_2v_1 = v_1π_3, π_3v_1 = v_1π_4,
π_0v_0 = v_1π_0π_1, π_1v_1 = v_2π_1, π_0^{-1}v_0 = v_1π_0^{-1}π_1^{-1}, π_1^{-1}v_1 = v_2π_1^{-1}π_2^{-1},
π_0v_2 = v_2π_0, π_0v_3 = v_3π_0, π_1v_3 = v_3π_1, π_1v_4 = v_4π_1,
π_2v_0 = v_0π_2, π_0π_3 = π_3π_0, π_1π_3 = π_3π_1, π_1π_4 = π_4π_1,
π_0π_1π_0 = π_1π_0π_1, π_1π_2π_1 = π_2π_1π_2, π_2π_0 = π_0π_2, π_3π_0 = π_0π_3,
π_3π_1 = π_1π_3, π_4π_1 = π_1π_4, π_0π_1π_0 = π_1π_0π_1, π_1π_2π_1 = π_2π_1π_2⟩.

A presentation for the group V is obtained from that for BV by adding the
relations π_0^2 = π_1^2 = π_0^3 = π_1^3 = 1.

The generating set for the presentations of both V and BV can be replaced by
{π_0, π_1, π_0, π_1} if the definitions above are replaced by the equalities π_i = v_0^{1-i}π_1v_0^{-1}
and π_i = v_0^{-i}π_1v_0^{1-i} for all i ≥ 2, and v_i = v_0^{-i}π_1π_{i+1} for all i ∈ N.

Proof. The first presentations for BV and V are extracted from the presentations of Corollary 4.14 using the arguments from Section 3 in the same way that the
presentations of Lemma 4.1 are extracted from those of (1) and (2). Each family of
relations in Corollary 4.14 gives rise to 2 or four relations in the above list. The pre-
sentations of the last paragraph can be shown equivalent via Tietze transformations
to the first presentations of the statement.

Note that several relations from Theorem 2 become redundant in the case of V
since the π_m and π_{m+1} become their own inverses.
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Department of Mathematical Sciences
State University of New York at Binghamton
Binghamton, NY 13902-6000
USA
email: matt@math.binghamton.edu