Gauge transformations in non-perturbative chiral gauge theories

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Abstract

We reconsider gauge-transformation properties in chiral gauge theories on the lattice observing all pertinent information and show that these properties are actually determined in a general way for any gauge group and for any value of the index. In our investigations we also clarify several related issues.

1 Introduction

Gauge invariance of the chiral determinant on the lattice has been considered a major problem [1]. Particular developments aiming at gauge invariance have been presented in Ref. [2] and have been the basis of various further works. Motivated by our observation that actually more information is available on the transformation properties of the bases involved, we here reinvestigate the subject. We also extend the considerations beyond the vacuum sector admitting zero modes and any value of the index.

We first show how the expressions for the chiral determinant in Ref. [2], which are based on gauge variations, can be evaluated further and find that with the covariance requirement for the current introduced there everything is fixed without any further assumptions. It thus becomes obvious that the developments in Ref. [2] are not relevant for the question of gauge invariance, which we discuss in detail.

In our more general analysis we then reveal that not allowing arbitrary switching to different equivalence classes of pairs of bases is the general principle which also implies covariance of the current mentioned above. To admit zero modes and any value of the index we consider appropriate forms of fermionic correlation functions and investigate their behavior under finite gauge transformations, finding gauge covariance up to constant phase factors.

In Section 2 we collect general relations. In Section 3 we consider variations of the effective action and the special case of Ref. [2]. In Section 4 we use finite transformations to analyze general correlation functions. Section 5 contains our conclusions.
2 General relations

2.1 Basic quantities

The chiral projections $P_+$ and $P_-$ are subject to

$$P_+ D = D P_-,$$  \hspace{1cm} (2.1)

where $D$ is the Dirac operator. They can be expressed as

$$P_- = \frac{1}{2}(1 - \gamma_5 G), \quad P_+ = \frac{1}{2}(1 + \bar{G} \gamma_5),$$  \hspace{1cm} (2.2)

which because of $P_+^\dagger = P_- = P_-^2$ and $\bar{P}_+^\dagger = \bar{P}_+ = \bar{P}_+^2$ implies $G^{-1} = G^\dagger = \gamma_5 G \gamma_5$ and $\bar{G}^{-1} = \bar{G}^\dagger = \gamma_5 \bar{G} \gamma_5$. Requiring $D$ to be $\gamma_5$-Hermitian and normal and $G$ and $\bar{G}$ to be functions of $D$ we get $\bar{N} - N = I$ for the numbers of anti-Weyl and Weyl degrees of freedom $\bar{N} = \text{Tr} \bar{P}_+$ and $N = \text{Tr} P_-$ and the index $I$ of $D$. For more details on the operator properties we refer to the recent analysis in Ref. [3].

Integrating out the Grassmann variables basic fermionic correlation functions for the Weyl degrees of freedom are given by [4]

$$\langle \psi_{\sigma_{r+1}} \ldots \psi_{\sigma_N} \bar{\psi}_{\bar{\sigma}_{r+1}} \ldots \bar{\psi}_{\bar{\sigma}_N} \rangle_f = \frac{1}{r!} \sum_{\sigma_1 \ldots \sigma_r} \sum_{\bar{\sigma}_1 \ldots \bar{\sigma}_r} \bar{\Upsilon}^{\sigma_1 \ldots \sigma_N} \Upsilon_{\bar{\sigma}_1 \ldots \bar{\sigma}_N} D_{\sigma_1 \sigma_1} \ldots D_{\sigma_r \sigma_r}$$  \hspace{1cm} (2.3)

with the alternating multilinear forms

$$\Upsilon_{\sigma_1 \ldots \sigma_N} = \sum_{i_1, \ldots, i_N = 1}^N \epsilon_{i_1, \ldots, i_N} u_{\sigma_1 i_1} \ldots u_{\sigma_N i_N}, \quad \bar{\Upsilon}_{\bar{\sigma}_1 \ldots \bar{\sigma}_N} = \sum_{j_1, \ldots, j_N = 1}^N \epsilon_{j_1, \ldots, j_N} \bar{u}_{\bar{\sigma}_1 j_1} \ldots \bar{u}_{\bar{\sigma}_N j_N}.$$  \hspace{1cm} (2.4)

The bases $\bar{u}_{\bar{\sigma}j}$ and $u_{\sigma i}$ in (2.4) satisfy

$$P_- = uu^\dagger, \quad u^\dagger u = \mathbb{1}_w, \quad \bar{P}_+ = \bar{u}\bar{u}^\dagger, \quad \bar{u}^\dagger \bar{u} = \mathbb{1}_{\bar{w}}.$$  \hspace{1cm} (2.5)

General fermionic functions are linear combinations of the basic ones (2.3).

2.2 Subsets of bases

By (2.5) the bases are only fixed up to unitary transformations, $u^{[S]} = u S$, $\bar{u}^{[\bar{S}]} = \bar{u} \bar{S}$, under which the forms (2.4) get multiplied by factors $\det_w S$ and $\det_{\bar{w}} \bar{S}$, respectively, and therefore the correlation functions (2.3) by a factor

$$\det_w S \cdot \det_{\bar{w}} \bar{S}^\dagger = e^{i\vartheta}.$$  \hspace{1cm} (2.6)

\(^1\)To compare with vector theory we can consider its formulation analogous to (2.3) with $\text{Tr} \mathbb{1}$ instead of $\bar{N}$ and $N$ and see that because of $\bar{u} = u$ we there get $\det S \cdot \det S^\dagger = 1$ instead of (2.6).
Therefore, in order that full correlation functions remain invariant, we have to require

\[ \vartheta = \text{const}, \quad (2.7) \]

i.e. that the phase \( \vartheta \) is independent of the gauge field. While without condition (2.7) all bases related to a chiral projection are connected by unitary transformations, with it the total set of pairs of bases \( u \) and \( \bar{u} \) is decomposed into inequivalent subsets, beyond which legitimate transformations do not connect. This has the important consequence that for the formulation of the theory one has to restrict to one of such subsets.

Different ones of the indicated subsets, which obviously are equivalence classes, are related by pairs of basis transformations for which \( \vartheta \) in (2.6) depends on the gauge field. The phase factor \( e^{i\vartheta(U)} \) then determines how the results of the theory differ for the respective classes. In view of such differences one has to decide which class is appropriate for the description of physics, for which there is, however, so far no criterion.

### 2.3 Gauge transformations

Since \( G \) and \( \bar{G} \) are functions of \( D \) the gauge-transformation behavior \( D' = TD\bar{T}^\dagger \) is inherited by them and then also by the chiral projections, which thus satisfy \( P'_- = TP_-\bar{T}^\dagger \) and \( \bar{P}'_+ = \bar{T}\bar{P}_+\bar{T}^\dagger \) in accordance with (2.7). In addition to the case where none of the chiral projections commutes with \( T \) the case where one of them is constant and thus commutes is of interest (examples of which are the particular choices in Ref. [3] and in Ref. [2], respectively).

Considering \( [T, P_-] \neq 0 \) we note that given a solution \( u \) of the conditions (2.5), then \( Tu \) is a solution of the transformed conditions (2.5). To account for the fact that \( u \) and \( u' \) are only fixed up to unitary transformations we introduce the unitary transformation \( S \) getting \( u' = TuS \) for all solutions of the transformed conditions. Analogous considerations apply to \( [T, \bar{P}_+] \neq 0 \). In the case where \( [T, P_-] \neq 0 \) and \( [T, \bar{P}_+] \neq 0 \) we thus have the general relations

\[ u' = TuS, \quad \bar{u}' = T\bar{u}\bar{S}. \quad (2.8) \]

For the phase \( \Theta \) in

\[ \det_w S \cdot \det_w S^\dagger = e^{i\Theta(T)} \quad (2.9) \]

using (2.6) with (2.7) we then immediately get

\[ \Theta(1) = \text{const}, \quad (2.10) \]

i.e. independence of the gauge field at least for \( T = 1 \).

In the case where \( [T, P_-] \neq 0 \) and \( \bar{P}_+ = \text{const} \) the equivalence class of pairs of bases always contains constant \( \bar{u}_c \). This follows since given a pair \( u, \bar{u} \) the basis \( \bar{u} \) is generally related to \( \bar{u}_c \) by a unitary transformation \( \bar{u} = \bar{u}_c S_c \). Then transforming \( u \) as \( u = u_c S_c \), where the unitary \( S_c \) is subject to \( \det_w S_c \cdot \det_w S_c^\dagger = \text{const} \), according to (2.6) with (2.7) the
pair \( u_e, \tilde{u}_c \) is in the same equivalence class as the pair \( u, \tilde{u} \). Analogously for a transformed pair \( u', \tilde{u}' \) we get the equivalent one \( u'_e, \tilde{u}_c \). Instead of (2.8) we then have

\[
u'_e = T u_e \tilde{S}, \quad \tilde{u}_c = \text{const},
\]

(2.11)

with unitary \( \tilde{S} \), and instead of (2.9) obtain

\[
\det w \tilde{S} = e^{i\tilde{\Theta}(T)}.
\]

(2.12)

Using (2.6) with (2.7) we thus get the analogon to (2.10),

\[
\tilde{\Theta}(1) = \text{const},
\]

(2.13)

i.e. again independence of the gauge field at least for \( T = 1 \).

3 Variational approach

3.1 General relations

We define general gauge-field variations of a function \( \phi(U) \) by

\[
\delta \phi(U) = \left. \frac{d \phi(U(t))}{dt} \right|_{t=0}, \quad U_\mu(t) = e^{t B^{\text{left}}/B^{\text{right}}}_\mu U_\mu e^{-t B^{\text{left}}/B^{\text{right}}}_\mu,
\]

(3.1)

where \( (U_\mu)_{n'n''} = U_{\mu n} \delta^4_{n'n''+\bar{\mu}} \) and \( (B^{\text{left/right}}_{\mu})_{n'n''} = B^{\text{left/right}}_{\mu n} \delta^4_{n'n''} \). The special case of gauge transformations then is straightforwardly described by

\[
B^{\text{left}}_{\mu} = B^{\text{right}}_{\mu} = B.
\]

(3.2)

In the case of gauge transformations we can use the definition (3.1) and the finite transformation relations to obtain the related variations explicitly. For operators with \( \mathcal{O}(t) = T(t) \mathcal{O} T^\dagger(t) \) and \( T(t) = e^{tB} \) this gives

\[
\delta^\mathcal{O} \mathcal{O} = [B, \mathcal{O}].
\]

(3.3)

In the case \([T, \mathcal{P}_-] \neq 0, [T, \tilde{\mathcal{P}}_+] \neq 0\) according to (2.8) we have for the bases \( u(t) = T(t) u \mathcal{S}(t), \tilde{u}(t) = T(t) \tilde{u} \mathcal{S}(t) \) and obtain

\[
\delta^\mathcal{O} u = B u + u \mathcal{S}^\dagger \delta^\mathcal{O} \mathcal{S}, \quad \delta^\mathcal{O} \tilde{u} = B \tilde{u} + \tilde{u} \tilde{\mathcal{S}}^\dagger \delta^\mathcal{O} \tilde{\mathcal{S}}.
\]

(3.4)

In the case \([T, \mathcal{P}_-] \neq 0, \tilde{\mathcal{P}}_+ = \text{const}\) according to (2.11) we get

\[
\delta^\mathcal{O} u_e = B u_e + u_e \tilde{\mathcal{S}}^\dagger \delta^\mathcal{O} \tilde{\mathcal{S}}, \quad \delta^\mathcal{O} \tilde{u}_c = 0.
\]

(3.5)
3.2 Effective action

Requiring absence of zero modes of $D$ (and thus also restricting to the vacuum sector) the effective action can be considered, for the variation of which one gets

$$\delta \ln \det_{\bar{w}w}(\bar{u}^\dagger Du) = \text{Tr}(P_- D^{-1} \delta D) + \text{Tr}(\delta u u^\dagger) - \text{Tr}(\delta \bar{u} \bar{u}^\dagger),$$

(3.6)

in which due to (3.3)

$$\text{Tr}(P_- D^{-1} \delta G D) = \text{Tr}(\bar{B} \bar{P} +) - \text{Tr}(BP_-).$$

(3.7)

In the case $[\mathcal{T}, P_-] \neq 0, [\mathcal{T}, \bar{P}+] \neq 0$ we obtain with (3.4)

$$\text{Tr}(\delta \bar{G} \bar{u} \bar{u}^\dagger) = \text{Tr}(B \bar{P} +) + \text{Tr}(\bar{S}^\dagger \delta^c \bar{S}),$$

(3.8)

and therefore

$$\delta^c \ln \det_{\bar{w}w}(\bar{u}^\dagger Du) = \text{Tr}(\bar{S}^\dagger \delta^c \bar{S}) - \text{Tr}(\bar{S}^\dagger \delta^c \bar{S}).$$

(3.9)

(3.10)

In the case $[\mathcal{T}, P_-] \neq 0, \bar{P} + = \text{const}$ we get with (3.5)

$$\text{Tr}(\delta^c u_e u_e^\dagger) = \text{Tr}(\bar{S}^\dagger \delta^c \bar{S}) + \text{Tr}(BP_-),$$

(3.11)

and remembering that $u, \bar{u}$ and $u_e, \bar{u}_e$ are in the same equivalence class

$$\delta^c \ln \det_{\bar{w}w}(\bar{u}^\dagger Du) = \delta^c \ln \det_{\bar{w}w}(\bar{u}_e^\dagger Dw_e) = \text{Tr}(\bar{S}^\dagger \delta^c \bar{S}) + \text{Tr}(B \bar{P} +),$$

(3.12)

where now $\text{Tr}(BP_+)$ is constant.

3.3 Special case of Lüscher

Lüscher [2] considers the variation of the effective action imposing the Ginsparg-Wilson relation $\{\gamma_5, D\} = D \gamma_5 D$ and using chiral projections which correspond to the choice $\bar{G} = 1$ and $G = 1 - D$ in (2.2). He assumes $\bar{P} +$ to be represented by constant bases so that he is effectively starting from the pair $u_e, \bar{u}_e$ of our formulation.

An important point in Lüscher’s work is the definition of a current $j_{\mu\nu}$ by

$$\text{Tr}(\delta u_e u_e^\dagger) = -i \sum_{\mu, \nu} \text{tr}_g(\eta_{\mu\nu} \bar{j}_{\mu\nu}), \quad \delta U_{\mu\nu} = \eta_{\mu\nu} U_{\mu\nu},$$

(3.13)

(3.14)

which he requires to transform gauge-covariantly.
His generator is given by 

$$\eta_{\mu n} = B_{\mu n}^\text{left} - U_{\mu n} B_{\mu}^\text{right} U_{\mu n}^\dagger$$

in terms of our left and right generators. We get explicitly

$$j_{\mu n} = i(U_{\mu n} \rho_{\mu n} + \rho_{\mu n}^\dagger U_{\mu n}^\dagger), \quad \rho_{\mu n,\alpha'} = \sum_{j,\sigma} u_{j\sigma}^\dagger \frac{\partial u_{j\sigma}}{\partial U_{\mu n,\alpha'\alpha}}.$$ (3.15)

The requirement of gauge-covariance

$$j_{\mu n}' = e^{B_{n+\mu}} j_{\mu n} e^{-B_{n+\mu}}$$

because of

$$U_{\mu n}' = e^{B_{n+\mu}} U_{\mu n} e^{-B_{n+\mu}}$$

implies that one must have

$$\rho_{\mu n}' = e^{B_n} \rho_{\mu n} e^{-B_n}.$$ (3.16)

which with (2.11) leads to the condition

$$\sum_{j,k} \tilde{S}_{kj} \frac{\partial \tilde{S}_{jk}}{\partial U_{\mu n,\alpha'\alpha}} = 0.$$ (3.17)

Using (3.17) it follows that

$$\text{Tr}_w (\tilde{S}_\dagger \delta \tilde{S}) = 0.$$ (3.18)

Because of 

$$\text{Tr}_w (\tilde{S}_\dagger \delta \tilde{S}) = \delta \ln \det w \tilde{S}$$

it is seen that (3.18) requires 

$$\det w \tilde{S}$$

to be independent of the gauge field. With (2.12) we thus obtain

$$\tilde{\Theta}(T) = \text{const},$$ (3.19)

i.e. that (2.13) extends to all $T$.

Since (3.18) implies 

$$\text{Tr}_w (\tilde{S}_\dagger \delta \tilde{S}) = 0$$

and because of the particular form 

$$\tilde{P}_+ = \frac{1}{2} (1 + \gamma_5) \mathbb{1}$$

we now get from (3.13)

$$\delta^2 \ln \det w (\tilde{u}_\dagger D \tilde{u}) = \frac{1}{2} \text{Tr} \, B,$$ (3.20)

i.e. a definite result following without any further assumptions.

Because 

$$\exp \left( \frac{1}{2} \text{Tr} \, B \right)$$

in (3.20) depends only on the gauge transformation but not on the gauge field the chiral determinant is gauge invariant up to a constant (gauge-field independent) phase factor, which is 

$$\exp (i \tilde{\Theta} + \frac{1}{2} \text{Tr} \, B)$$

as will be confirmed by the general analysis in Section 4.3.

### 3.4 Both chiral projections non-commuting

In the case $[T, P_-] \neq 0$, $[T, \tilde{P}_+] \neq 0$, starting analogously from 

$$\text{Tr} (\delta u u_\dagger) - \text{Tr} (\delta \tilde{u} \tilde{u}_\dagger)$$

as in (3.14) from 

$$\text{Tr} (\delta u e u_\dagger),$$

one straightforwardly arrives at the analogon of (3.18),

$$\text{Tr}_w (\tilde{S}_\dagger \delta \tilde{S}) - \text{Tr}_w (\tilde{S}_\dagger \delta \tilde{S}) = 0.$$ (3.21)

This with (2.9) leads to

$$\Theta(T) = \text{const},$$ (3.22)

generalizing (2.10) to all $T$. With (3.21) we get from (3.10)

$$\delta^2 \ln \det w (\tilde{u}_\dagger D \tilde{u}) = 0,$$ (3.23)

i.e. again a definite result following without any further assumptions. The chiral determinant thus is gauge invariant up to a constant phase factor, i.e. up to the factor $\exp (i \tilde{\Theta})$ as will be confirmed by the general analysis in Section 4.2.
3.5 Discussion

The definite result (3.20) shows that the developments presented in Ref. [2] are not relevant for the question of gauge invariance. It reveals that whatever the gauge-field dependences of the bases might be their gauge-transformation properties are such that gauge variations of the effective action in the special case there are equal to $\frac{1}{2} \text{Tr} B$. (Furthermore, also the aim $\delta \ln \det \bar{w} (\bar{u}^\dagger D u) = 0$ in Ref. [2] disagrees with the precise result (3.20)).

In Ref. [2] a main argument was that without the anomaly cancelation condition one would be unable to cancel the anomaly term. However, as has become explicit here this is not true. Indeed, in the case considered there the basis term being of form $\text{Tr}(B P_{-})$ just compensates the respective contribution in the anomaly term (3.7) so that one gets cancelation up to the irrelevant quantity $\text{Tr}(B \bar{P}_{+}) = \frac{1}{2} \text{Tr} B$. In the case where both chiral projections do not commute with $T$ the contribution $\text{Tr}(B P_{-}) - \text{Tr}(B \bar{P}_{+})$ of the bases even fully compensates the anomaly term.

Thus in detail the considerations of topological fields (anyway only feasible in the Abelian case) and the substantial additional assumptions in Ref. [2] turn out to be irrelevant for gauge invariance.

It is to be emphasized in the present context that here as well as in Ref. [2] and in the related discussion [1] one is concerned with the formulation on the finite lattice and the non-perturbative description. Thus one cannot a priori expect to find the same situation as in continuum perturbation theory.3

4 General analysis

4.1 Equivalence-class requirements

So far the conditions (3.19) and (3.22) have emerged as consequences of the covariance requirement for Lüscher’s current and have been seen to prevent the addition of completely arbitrary terms to the gauge variation of the effective action. We now turn to the general principle from which these conditions follow.

In the case where $[T, P_{-}] \neq 0$ and $[T, \bar{P}_{+}] \neq 0$ from (2.8) with (2.9),

$$u' = T u S, \quad \bar{u}' = T \bar{u} \bar{S}, \quad \det_w S \cdot \det_{\bar{w}} \bar{S}^\dagger = e^{i\Theta},$$

(4.1)

and in the case where $[T, P_{-}] \neq 0$ and $\bar{P}_{+} = \text{const}$ from and (2.11) with (2.12),

$$u'_c = T u_c \bar{S}, \quad \bar{u}_c = \text{const}, \quad \det_w \bar{S} = e^{i\tilde{\Theta}},$$

(4.2)

it is seen that admitting gauge-field dependence of $\Theta$ and of $\tilde{\Theta}$, respectively, means to allow switching to arbitrary inequivalent subsets of pairs of bases. Such combinations of gauge

2For an observation with respect to general gauge-field dependences see the end of Section 4.1.

3In the limit of a perturbation expansion the compensating basis terms vanish so that one gets the usual setting of continuum perturbation theory where the anomaly cancelation condition is needed [4].
transformations with transformations to arbitrary different equivalence classes of pairs of bases would obviously introduce severe ambiguities. Thus to avoid these ambiguities by requiring (3.22),

\[ \Theta = \text{const}, \]  

(4.3)

and (3.19),

\[ \tilde{\Theta} = \text{const}, \]  

(4.4)

respectively, turns out to be appropriate, which also accounts for the fact that to describe physics one must restrict to one of the equivalence classes.

The important observation here is that given an equivalence class of pairs of bases in this way the equivalence class after the transformation remains uniquely determined, which is possible because in the case of gauge transformations we have the explicit relations (2.8) and (2.11), respectively. In contrast to this, considering general gauge-field dependences, given an equivalence class for one set of fields there is so far no criterion determining the equivalence class after a general change of the fields.

4.2 Non-commuting chiral projections

Considering the transformation of correlation functions in the case where \([\mathcal{T}, P_-] \neq 0\) and \([\mathcal{T}, \bar{P}_+] \neq 0\) using (4.1) we obtain

\[ \langle \psi'_{\sigma_1} \ldots \psi'_{\sigma_R} \bar{\psi}'_{\bar{\sigma}_1} \ldots \bar{\psi}'_{\bar{\sigma}_R} \rangle = \exp(i\Theta) \sum_{\sigma_1, \ldots, \sigma_R} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_R} \mathcal{T}_{\sigma_1 \bar{\sigma}_1} \ldots \mathcal{T}_{\sigma_R \bar{\sigma}_R} \langle \psi_{\sigma_1} \ldots \psi_{\sigma_R} \bar{\psi}_{\bar{\sigma}_1} \ldots \bar{\psi}_{\bar{\sigma}_R} \rangle \mathcal{T}_{\sigma_1 \bar{\sigma}_1}^\dagger \ldots \mathcal{T}_{\sigma_R \bar{\sigma}_R}^\dagger. \]  

(4.5)

Due to (4.3) the correlation functions thus turn out to transform gauge-covariantly up to a constant phase factor, i.e. up to the factor \(e^{i\Theta}\).

4.3 One constant chiral projection

In the case where \([\mathcal{T}, P_-] \neq 0\) and \(\bar{P}_+ = \text{const}\) we can rewrite \(\bar{u}_c\) as

\[ \bar{u}_c = \mathcal{T} \bar{u}_c S_T \]  

(4.6)

where \(S_T\) because of \([\mathcal{T}, \bar{P}_+] = 0\) is unitary. Using this and (4.2) we get for the transformation of the correlation functions again the form (4.5) but with \(\Theta\) being replaced by \(\tilde{\Theta} + \theta_T\) where \(\theta_T\) is given by

\[ e^{i\theta_T} = \det_{\omega} S_T^\dagger = \det_{\omega} (\bar{u}_c^\dagger \mathcal{T} \bar{u}_c). \]  

(4.7)

Since (4.7) does not depend on in the gauge field and because \(\tilde{\Theta}\) according to (4.4) is also constant the correlation functions thus are again seen to show gauge-covariant behavior up to a constant phase factor, i.e. up to the factor \(e^{i(\tilde{\Theta} + \theta_T)}\).

To calculate \(i\theta_T\) we note that with \([\mathcal{T}, \bar{P}_+] = 0\) and \(\mathcal{T} = e^B\) we get \(\bar{u}_c^\dagger \mathcal{T} \bar{u}_c = \bar{u}_c^\dagger e^{B\mathcal{T}_+} \bar{u}_c\) and the eigenequations \(B\bar{P}_+ \bar{u}_j^d = \omega_j \bar{u}_j^d\) and \(\bar{P}_+ \bar{u}_j^d = \bar{u}_j^d\). With this we obtain \(\det_{\omega} (\bar{u}_c^\dagger e^{B\mathcal{T}_+} \bar{u}_c) = \prod_j e^{\omega_j} = \exp(\text{Tr}(B\bar{P}_+))\), so that we find \(i\theta_T = \text{Tr}(B\bar{P}_+)\). For \(\bar{P}_+ = \frac{1}{2}(1 + \gamma_5)\mathbb{1}\\) we then have \(i\theta_T = \frac{1}{2} \text{Tr} B\).
5 Conclusions

We have given an unambiguous derivation of the gauge-transformation properties in chiral
gauge theories on the finite lattices observing that there are more informations on the bases
available which must not be ignored.

We have first considered the subject in terms of variations of the effective action In this
context we have shown that satisfying the covariance requirement for Lüscher’s current
the gauge variation leads to a definite field-independent quantity without any further assumptions. This means that the developments presented in Ref. [2] are irrelevant for
the question of gauge invariance.

In detail it has become explicit that on the lattice the anomaly term is canceled without
imposing a respective condition. Thus the considerations of topological fields and the
substantial additional assumptions in Ref. [2] have turned out to be not relevant for
gauge invariance.

In our more general analysis we then have pointed out that not allowing to combine
gauge transformations with arbitrary switching to different equivalence classes of pairs of
bases is the general principle which also implies covariance for Lüscher’s current.

In order to extend the considerations beyond the vacuum sector we have investigated
the behavior of correlation functions also in the presence of zero modes and for any value
of the index using finite gauge transformations. We have found that fermionic correlation
functions transform gauge-covariantly up to constant phase factors.

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