REINFORCING A PHILOSOPHY: LITTLEWOOD–PALEY
THEORY FOR THE MOMENT CURVE OVER GENERIC LOCAL
FIELDS
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Abstract. Using the Girard–Newton formulae, I give a simple proof of isotropic
square function estimates for extension operators along the moment curve in
generic local fields. Using Bezout’s Theorem and the Implicit Function The-
orem, I give an alternate, sharper proof for real or complex non-degenerate,
polynomial curves.

1. Introduction

The purpose of this paper is to expose two simple arguments for square function
estimates of the moment curve by utilizing the number theory paradigm:

Special subvarieties dictate the analysis.

This paradigm will be investigated in the context of proving a square function
estimate for the moment curve over generic local fields. My hope is to provide
some context to recent works in harmonic analysis and relate them to common,
elementary ideas in analytic number theory. Stating the results requires a brief
set-up taken from [1].

In this paper, let $K$ denote a (one-dimensional) local field; its non-trivial topology
is determined by the metric associated to a fixed absolute value $| \cdot |_K : K \to \mathbb{R}_{>0}$. On
$\mathbb{R}$, this will be the usual absolute value. On $\mathbb{C}$, we choose our absolute value so that
$|x + iy|_\mathbb{C} = \max\{|x|, |y|\}$ where the latter is the usual absolute value on $\mathbb{R}$. In the
results below, $K$ will be fixed; so, I usually suppress the dependence on $K$ in future
notations. For $n \in \mathbb{N}$, extend the metric to $K^n$ by defining $|x| = \max\{|x_1|, \ldots, |x_n|\}$
for $x = (x_1, \ldots, x_n) \in K^n$. Always assume that the dimension $n$ is at least two.

When $K$ is a non-Archimedean local field, let $\mathfrak{o} := \{x : |x| \leq 1\}$ denote its ring
of integers and $\mathfrak{p} := \{x : |x| < 1\}$ denote its maximal ideal. The image of $| \cdot |_K$ is isomorphic to $\mathbb{Z}$ as an abelian group and $K$ comes with a uniformizing element,
say $\pi$, generating this group. When $K = \mathbb{R}$, let $\mathfrak{o} = [0, 1]$, and when $K = \mathbb{C}$, let
$\mathfrak{o} := \{x + y\sqrt{-1} : x, y \in [0, 1]\}$.

For a non-Archimedean local field $K$, define its scales as $\mathcal{R}(K) := \{|\pi|^s : s \in \mathbb{Z}_{\geq 0}\}$. For a scale $\delta \in \mathcal{R}(K)$, let $\mathcal{P}_\delta(K)$ denote the partition of $\mathfrak{o}$ into closed (and
also open) balls of radius $\delta$; that is, $\mathcal{P}_\delta := \{(i + \mathfrak{p}^s)_{i \in \mathbb{Z}/\mathfrak{p}^s} \}$ for some $s \in \mathbb{N}$. Over
$\mathbb{R}$, define its scales as $\mathcal{R}(\mathbb{R}) := \{|R^{-1} : R \in \mathbb{N}\}$, and for each $\delta$ in $\mathcal{R}(\mathbb{R})$, define
the partition $\mathcal{P}_\delta(\mathbb{R}) := \{(i\delta, (j + 1)\delta) : i = 0, \ldots, \delta^{-1} - 1\}$. Over $\mathbb{C}$, define its
scales as $\mathcal{R}(\mathbb{C}) := \{|R^{-1} : R \in \mathbb{N}\}$, and for each $\delta$ in $\mathcal{R}(\mathbb{C})$, define the partition
$\mathcal{P}_\delta(\mathbb{C}) := \{(j\delta, (j + 1)\delta) + i[k\delta, (k + 1)\delta) : j, k = 0, \ldots, \delta^{-1} - 1\}$.

For non-Archimedean fields, we fix a non-trivial additive character $\psi(\cdot)$ such that
$\psi(0) = 1$ and $\psi(1/\pi) \neq 1$. On $\mathbb{R}$, $\psi(t) := e^{-2\pi i t}$ denotes the usual character, and on
$\mathbb{C}$, $\psi(z) := e^{-2\pi i \text{Tr}_{\mathbb{R}/\mathbb{C}}(z)/2} = e^{-2\pi i t}$ for $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.
(This is the usual character when interpreting $\mathbb{C}$ as $\mathbb{R}^2$ in the usual way. And over $\mathbb{R}$ and $\mathbb{C}$, $\pi$ denotes the standard mathematical constant.) On each local field, the
Haar measure $d\xi$ on $K$ is normalized so that the measure of $\mathfrak{o}$ is 1. For a function
$f$, write its Fourier transform as $\check{f}$.

Throughout, assume that the functions $f$ are compactly supported, measurable
functions. For a fixed local field $K$, a measurable set $I$ in $K$ and a curve $\gamma : I \to K^n$,
define the extension operator along $\gamma$ over $I$ as

$$E_I f(x) := \int_I f(\xi) c(\gamma(\xi) \cdot x) \, d\xi$$

for points $x \in \mathbb{R}^n$.

For $\delta \in \mathcal{R}(K)$, define the square function along $\gamma$ at scale $\delta$ as

$$S_\delta f(x) := \left( \sum_{j \in \mathcal{P}_\delta} |E_j f(x)|^2 \right)^{1/2}.$$  

Suppose that $w : K^n \to K$ is a reasonable function. For each $c \in K^n$ and each $R \in K \setminus \{0\}$, define

$$w(x) := w\left(\frac{x-c}{R}\right) \quad \text{for} \quad x \in K^n.$$  

I will also write this as $w_B$ when $B$ is the associated box with center $c$ and side-lengths $|R|$. Finally, define the constant

$$H_\gamma(w) := \sup_{\delta \in \mathcal{R}(K)} \sup_{c \in K^n} \sup_{f \neq 0} \|E_\delta f\|_{L^{2n}(w_{\delta}^{-1})} / \| S_\delta f\|_{L^{2n}(w_{\delta}^{-1})}.$$  

Suppose that $w : \mathbb{R} \to \mathbb{R}$ is a Schwartz function which is non-negative, at least 1 on the unit interval $[0, 1]$ and for which $\hat{w}$ is supported on $[-1, 1]$ and non-negative. We define the $n$-dimensional version of $w$ as $W(x) := w(x_1) \cdots w(x_n)$ for $x \in \mathbb{R}^n$. Then $W$ is at least 1 on the cube $[0, 1]^n$ while $\hat{W}$ is supported on the cube $[-1, 1]^n$. On $\mathbb{C}$, define our function $W(x + iy)$ to be $W(x) W(y)$ for $x, y \in \mathbb{R}^n$. Finally, on a non-Archimedean field $K$, we define $W := 1_{\{|x| \leq 1\}}$. Then $W$ is supported on and is 1 on the cube $\sigma^n$ while $\hat{W}$ is supported, and also 1, on the cube $\sigma^n$.

For a non-Archimedean local field $K$, define the constants $C_K := 1$. For $K = \mathbb{R}$, define the constant $C_\mathbb{R} := 7$. For $K = \mathbb{C}$, define the constant $C_\mathbb{C} := 7^2$. We can finally state the theorems.

**Theorem 1.** Let $K$ be a local field and $\gamma(T) := (T, T^2, \ldots, T^n)$. Assume that the characteristic of $K$ is either 0 or greater than $n$. We have the inequality $H_\gamma(W) \leq (C_K n)^{1/2}$. In other words, for each scale $\delta \in \mathcal{R}(K)$ and $c \in K^n$, we have the inequality

$$\|E_\delta f\|_{L^{2n}(W_{\delta}^{-1})} \leq (C_K n)^{1/2} \| S_\delta f\|_{L^{2n}(W_{\delta}^{-1})}. \tag{1}$$

The bound (1) is close to sharp. Indeed, Stirling’s Approximation $n! \sim \sqrt{2\pi n} (n/e)^n$ and the following lower bound implies that (1) is asymptotically off by a constant.

**Theorem 2.** Fix $n \geq 2$. For each local field $K$ and polynomial curve $\gamma \in K[T]^n$,

$$H_\gamma(W) \geq (n!)^{1/2n}. \tag{2}$$

The problem of finding sharp constants and extremizers for square function estimates appears interesting. Inspired by [19, 12], I give two sharper, more general versions of Theorem 1 for Archimedean fields using calculus and classical algebraic geometry. Define $\eta_K$ to be 1 if $K$ is $\mathbb{R}$ and 2 if $K$ is $\mathbb{C}$. For a curve $\gamma : \sigma \to \mathbb{R}^n$, define its Lipschitz norm

$$\ell(\gamma) = \sup_{i=1, \ldots, n} \sup_{t \neq s} \frac{|\gamma_i(t) - \gamma_i(s)|}{|t-s|}.$$  

Observe that $\ell(\gamma)$ is finite when $\gamma$ is polynomial and $\sigma$ is compact.

**Theorem 3.** Let $K$ be $\mathbb{R}$ or $\mathbb{C}$ and $\gamma \in K[T]^n$ be a non-degenerate polynomial curve. We have the inequality $H_\gamma(W) \leq 2[\ell(\gamma)] + 1)^{1/2} \left( \prod_{i=1}^n \deg(\gamma_i) \right)^{1/2n}. $
When $\gamma$ is the moment curve, the proof of Theorem 3 demonstrates that we have ‘strong diagonal behavior’. For more general non-degenerate curves, any off-diagonal behavior is constrained by Bezout’s theorem. These give the special subvarieties under-pinning square function estimates.

Over $\mathbb{R}$, there are much sharper bounds for fewnomials.

**Theorem 4.** Let $\gamma$ be a non-degenerate, polynomial curve in $\mathbb{R}^n$ such that the total number of monomials appearing in $\gamma$ is $M$. We have the inequality

$$H_{\gamma}(W) \leq (2|f(\gamma)| + 1)^{1/2} \left(2^{M(M-1)/2}(n + 1)^M \right)^{1/2n}.$$ 

With only a minor modification, the proofs of Theorem 3 and 4 generalize to non-degenerate Pfaff curves in $\mathbb{R}^n$. The statement is more technical than its proof, so I refer the reader to [9, 10] to extract what is needed.

In the form described above, [4] proved Theorem 1 for real non-degenerate curves; their result is far more general at the cost of an inexplicit constant. A new proof for Theorem 1 when $n = 2$ was given in [1]. Subsequently, [6] proved the bound $H_{\gamma}(W)^{2n} \leq n!$ for non-Archimedean fields of sufficiently large characteristic. Combined with Theorem 2, we see that $H_{\gamma}(W) = (n!)^{1/2n}$ for the moment curve over non-Archimedean local fields of characteristic 0 or characteristic exceeding $n$.

The arguments proving Theorems 1 and 2 are closely related to the starting point for Vinogradov’s celebrated mean value theorems. While these arguments should be well-known, they seem to be known to only a few. After consulting with experts, I believe the proofs of Theorems 3 and 4 appear to be new. I now give a concise overview of some antecedents to these results.

Historically, the main interest in such square function estimates arises from exponential sum estimates in analytic number theory; specifically, Vinogradov’s mean value theorems and Waring’s problem where the main choice of functions for input into (1) were (after a rescaling) $f := \sum_{i=1}^{N} \delta_{i/N}$ as $N$ tends to infinity. For these functions, the first result is due Vinogradov; see [8, Vinogradov’s ‘pigeon-hole lemma’ on page 12] and [7, Chapter 4, Section 2]. Linnik subsequently developed a version of Vinogradov’s methods for local fields; see [11, pages 71-72].

In harmonic analysis, related results appeared later. A Cantor–Lebesgue theorem for the circle appeared in [2], followed by a $L^2 \rightarrow L^2$-discrete restriction estimate for the circle in [20]. For more general functions, [4] attributes a version of Theorem 1 to [3] when $n = 2$ and $K = \mathbb{R}$ while [6] attributes a version of this theorem to [14, 15] when $n = 3$ and $n \geq 4$ respectively; each version is for square functions related to Bochner–Riesz operators instead of the square functions herein.

Uniting all of these works is the aforementioned paradigm from number theory. My use of ‘analysis’ there does not refer to any mathematical field, but to quantitative considerations of problems related to certain equations. This paradigm has long played a salient role in number theory and is prominent in analytic number theory through Manin’s conjecture and the circle method. In harmonic analysis, this paradigm is understated despite its long-standing presence therein.

Let me take a moment to describe how this paradigm arises in the aforementioned works. In Vinogradov’s pigeon-hole lemma and [11, ‘Linnik’s Lemma’ on pages 71-72], this paradigm arises by demonstrating diagonal behavior for certain Vinogradov systems with an added transversality assumption; over $\mathbb{R}$ transversality can be encoded as distinct 1-separated integers (note that Vinogradov’s methods easily generalize to 1-separated real points), and over the $p$-adics, transversality is usually encoded $1/p$ separation between the $p$-adic intervals. In [4, 1, 6] and this paper, this paradigm appears in the diagonal behavior of isotropic boxes covering a neighborhood of the underlying variety. In [2, 20] this paradigm appears in the geometry of two intersecting circles. In [3, 14, 15], this paradigm appears in the
almost orthogonality of off-diagonal Kakeya type information. In all cases, this information is drawn out through a reduction using orthogonality and multilinearity considerations.

**Outline of the paper.** In Section 3, I reduce the proof of Theorem 1 to bounds on the number of special subvarieties via a standard use of orthogonality and the Cauchy–Schwarz inequality. This reduction is encoded in Lemma 5. In Section 4, I prove Proposition 6 which, in combination with Lemma 5, yields Theorem 1. At the end of this section, I prove the lower bound Theorem 2. In Section 5, I prove Theorems 3 and 4. In Section 6, I discuss the connection to Superorthogonality.

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2. Scratch

3. Reduction to special subvarieties

Fix a local field $K$ and a curve $\gamma : \mathbb{A}^1 \to K^n$. For each scale $\delta \in \mathcal{R}(K)$, each $\epsilon > 0$ and each $n$-tuple $I$ in $\mathcal{P}_\delta^n$, define $S(\delta, I; \epsilon)$ to be the set of $n$-tuples $J \in \mathcal{P}_\delta^n$ such that

$$| \sum_{i=1}^n (\gamma(t_i) - \gamma(s_i)) | \leq \epsilon$$

(3)

for some $s \in I$ and some $t \in J$. In other words,

$$S(\delta, I; \epsilon) := \{ J \in \mathcal{P}_\delta^n : \text{there exist points } s \in I, t \in J \text{ satisfying (3)} \}.$$

Define $S_\gamma := \max_{\delta \in \mathcal{R}} \max_{I \in \mathcal{P}_\delta^n} |S(\delta, I; \delta^n)|$.

**Lemma 5.** Let $K$ be a local field, $n \geq 2$ and $\gamma \in K[T]^n$ be a curve in $K^n$. Then

$$\| H_\gamma(W) - (S_\gamma)^{1/2n} \|_{L^{2n}(\mathbb{A}^1_k)} \leq (S_\gamma)^{1/2n}.$$

(4)

**Proof.** Fix $n \geq 2$. Fix a local field $K$. Fix a scale $\delta \in \mathcal{R}(K)$. Let $B$ be a box of sidelengths $\delta^{-n}$ in $K^n$. Without loss of generality, it suffices to take $B$ centered at the origin. Write $E_\delta = \sum_{I \in \mathcal{P}_\delta} E_I$ by the linearity of integration. Fubini’s theorem implies that

$$|E_\delta f|^2_{L^{2n}(\mathbb{A}^1_k)} = \sum_{I \in \mathcal{P}_\delta} \sum_{J \in \mathcal{P}_\delta} \int_{K^n} (\prod_{i=1}^n E_{I,f}(x)) (\prod_{j=1}^n E_{J,f}(x)) W_B(x) \, dx.$$ 

Fourier inversion and the properties of $W_B$ imply that

$$\|E_\delta f\|^2_{L^{2n}(\mathbb{A}^1_k)} = \sum_{I \in \mathcal{P}_\delta} \sum_{J \in S(\delta, I, \delta^n)} \int (\prod_{i=1}^n E_{I,f}(x)) (\prod_{j=1}^n E_{J,f}(x)) W_B.$$ 

For more details of this orthogonality, see [4, Section 6].

Applying Fubini’s theorem once more as well as the Cauchy–Schwarz inequality on the sum over $I \in \mathcal{P}_\delta^n$, we deduce that

$$\|E_\delta f\|^2_{L^{2n}(\mathbb{A}^1_k)} = \int \sum_{I \in \mathcal{P}_\delta} (\prod_{i=1}^n E_{I,f}) (\sum_{J \in S(\delta, I, \delta^n)} \prod_{j=1}^n E_{J,f}) W_B \leq \left( \int (\sum_{I \in \mathcal{P}_\delta} |\prod_{i=1}^n E_{I,f}|^2)^{1/2} \left( \sum_{I \in \mathcal{P}_\delta} \sum_{J \in S(\delta, I, \delta^n)} \prod_{j=1}^n |E_{J,f}|^2 \right)^{1/2} W_B. $$
Applying the Cauchy–Schwarz inequality on the inner sum over $J \in S(\delta, I; \delta^n)$ and the triangle inequality on the second outer sum over $I \in \mathcal{P}_\delta^n$, we deduce that
\[
\|E_{\delta f}\|_{L^{2n}(W_n)}^2 \leq \int \left( S_\delta f \right)^n \left( \sum_{I \in \mathcal{P}_\delta^n} \sum_{J \in S(\delta, I; \delta^n)} \left| \prod_{j=1}^n E_{J_j} f_j^2 \right| \right)^{1/2} \, W_B
\]
\[
\leq S_\gamma^n \int \left( \sum_{I \in \mathcal{P}_\delta^n} \sum_{J \in S(\delta, I; \delta^n)} \left| \prod_{j=1}^n E_{J_j} f_j^2 \right| \right)^{1/2} \, W_B = S_\gamma \|S_{\delta f}\|_{L^{2n}(W_n)}^2.
\]

By the symmetry of the inequalities (3): $J \in S(\delta, I; \delta^n)$ if and only if $I \in S(\delta, J; \delta^n)$. Use this symmetry to invert the double sum and apply the triangle inequality on the innermost summand over $I \in S(\delta, J; \delta^n)$ to deduce that
\[
\|E_{\delta f}\|_{L^{2n}(W_n)}^2 \leq S_\gamma^n \int \left( \sum_{I \in \mathcal{P}_\delta^n} \sum_{J \in S(\delta, I; \delta^n)} \left| \prod_{j=1}^n E_{J_j} f_j^2 \right| \right)^{1/2} \, W_B
\]
\[
\leq S_\gamma \int \left( \sum_{I \in \mathcal{P}_\delta^n} \left| \prod_{j=1}^n E_{J_j} f_j^2 \right| \right)^{1/2} \, W_B = S_\gamma \|S_{\delta f}\|_{L^{2n}(W_n)}^2.
\]

Taking $2n^{th}$-roots on both sides of the inequality completes the proof. \hfill \square

4. Diagonal behavior of the Vinogradov system

Our main estimate uniformly bounds the size of $S(\delta, I; \delta^n)$ which implies that $S_\gamma$ is finite. Via Lemma 5, our particular bound for the cardinality of $S(\delta, I; \delta^n)$ immediately implies Theorem 1; I leave this implication to the reader.

**Proposition 6.** Let $\gamma := (T, T^2, \ldots, T^n)$ be the moment curve in a fixed local field $K$ of characteristic 0 or greater than $n$. We have the bound
\[
S_\gamma \leq (C_K \cdot n)^n.
\] (5)

See [4, Proposition 1.3], [1, Proposition 3.1] and [6, Proposition 1.2] for analogous propositions. Compare with classical versions of this proposition such as its original form due to Vinogradov exposed in [8, Vinogradov’s ‘pigeon-hole lemma’ on page 12], [18, pages 37–44], [7, Chapter 4, Section 2] and [16, Lemma 6.3 in Chapter VI on pages 121-122]. Linnik developed a $p$-adic version exposed in [17, pages 58–60] and [11, pages 71-72]; these use an added transversality assumption between the intervals which we crucially do not have at our disposal.

Let us take a moment to describe the idea underlying Proposition 6. Suppose that $s$ and $t$ are points in $K^n$. If $t$ is a permutation of $s$, then this pair of points is a solution to (3) with $\epsilon = 0$. It transpires that the converse is true. To see this converse, define the elementary symmetric polynomials $\sigma_j(X_1, \ldots, X_n)$ to be $\sum_{i_1 < i_2 < \cdots < i_j} X_{i_1} \cdots X_{i_j}$ for $j \in \mathbb{N}$ and recall the Girard–Newton equations:
\[
(-1)^{j-1} j \sigma_j(X_1, \ldots, X_n) = \sum_{i=0}^{j-1} (-1)^i (X_1^{j-i} + \cdots + X_n^{j-i}) \sigma_i(X_1, \ldots, X_n).
\] (6)

The Girard–Newton equations imply that for each coordinate $t_j$, there exists an $s_i = t_j$. The special subvarieties discussed previously are these ‘almost diagonal’ varieties given by $t_j = s_i$ for each $j$ and some $i$. Fixing a point $s$, the number of such points $t$ (and equivalently special subvarieties) is at most $n^n$. The critical insight underlying Proposition 6 is that a fattened version of this argument continues to hold for $S(\delta, I; \delta^n)$.

Going further, the Girard–Newton equations (6) and the Fundamental Theorem of Algebra imply that permutations are the only solutions. Obtaining a fattened version of this ‘strong diagonal property’ was an essential feature in [6]; the strong
diagonal property took the form: if $J \in S(\delta, I, \delta^n)$, then $J$ is a permutation of $I$. We will see this strong diagonal behavior again in Proposition 7 below.

**Proof of Proposition 6.** Fix $n \geq 2$. First, I will prove the proposition for non-Archimedean local fields of characteristic 0 or of characteristic greater than $n$. For these cases, the constant $C_K$ appearing in statement of the proposition is 1. At the end of the proof, I will indicate the necessary changes when $K$ is $\mathbb{R}$ or $\mathbb{C}$.

Fix a non-Archimedean local field $K$ of characteristic 0 or of characteristic greater than $n$. Fix a scale $\delta \in \mathcal{R}(K)$ and an $n$-tuple of intervals $I$ in $P^\delta_n$. It suffices to show the bound $|S(\delta, I; \delta^n)| \leq n^n$. For $s \in K^n$, define the univariate polynomial

$$G(s; X) := \prod_{i=1}^n (X - s_i)$$

where $X$ is the variable. If $s \in I, J \in S(\delta, I; \delta^n)$ and $t \in J$, then the Girard–Newton equations (6) imply that (3) holds, for $\epsilon = \delta^n$, with the elementary symmetric polynomials in place of the power symmetric polynomials. In turn, this implies

$$|G(t; x) - G(s; x)| \leq \delta^n \quad \text{for all } x \in \sigma.$$  \hfill (7)

Taking $x = t_j$, we find that $|G(s; t_j)| \leq \delta^n$ for each $j = 1, \ldots, n$. The pigeonhole principle implies that there exists an $i = 1, \ldots, n$ such that $|s_i - t_j| \leq \delta$. Since the local field $K$ is non-Archimedean, $t_j \in I_i$ where $I$ is written as $(I_1, \ldots, I_n)$, and as a result, $J_j = I_i$. It is now obvious that there are at most $n$ choices for each of the $n$ coordinates. Therefore, there are at most $n^n$ choices overall.

Now assume that $K$ is $\mathbb{R}$ or $\mathbb{C}$. The first difference in the proof is that the transfer from power symmetric polynomials to elementary symmetric polynomials induces a loss of a factor at most $2n^2$ in (7). To see this, observe that (6) implies

$$|j \sigma_j(s_1, \ldots, s_n) - j \sigma_j(t_1, \ldots, t_n)| \leq 2n \sum_{i=0}^{j-1} \left| \sigma_i(s_1, \ldots, s_n) - \sigma_i(t_1, \ldots, t_n) \right|.$$  \hfill (8)

This implies that for each $j = 1, \ldots, n$, we have the bound

$$|\sigma_j(s_1, \ldots, s_n) - \sigma_j(t_1, \ldots, t_n)| \leq 2n \delta^n.$$  \hfill (9)

Consequently, for all $n \geq 2$, we have

$$|G(t; x) - G(s; x)| \leq 2n^2 \delta^n \leq (3\delta)^n \quad \text{for all } x \in \sigma.$$  \hfill (10)

The second difference arises from the necessity to account for possible neighbors of intervals arising from the above inequality. This loses a factor of $(3 \cdot 2 + 1)^n = 7^n$ or $(3 \cdot 2 + 1)^{2n} = 49^n$ in the estimate for $|S(\delta, I; \delta^n)|$ for $\mathbb{R}$ and $\mathbb{C}$ respectively. The remaining details of the proof for Archimedean fields are left to the reader. \hfill \Box

There is a small analytic improvement for Archimedean fields. For $n \geq 7$, we can improve $2n^2 < 3^n$ to $2n^2 < 2^n$. Therefore, when $n \geq 7$, we have a factor of $(2 \cdot 2 + 1)^n = 7^n$ or $(2 \cdot 2 + 1)^{2n} = 49^n$ in the above estimate for $|S(\delta, I; \delta^n)|$ for $\mathbb{R}$ and $\mathbb{C}$ respectively. For any $\rho \in (1, 2)$, we have $2n^2 < \rho^n$ for sufficiently large $n$ and similar improvements can be made. Asymptotically as the dimension tends to infinity, this improves the factors $C_K$ towards 3 and $C_C$ towards $3^2$.

**Warning.** A common mistake is the following: For an $n$-tuple of intervals $I \in P^\delta_n$, there are exactly $n!$ permutations of $I$. This is not true because there are fewer than $n!$ permutations when the same interval may appears more than once in the $n$-tuple. The effect of this is that *one does not deduce* for each function $f$ the equality

$$\|E_\sigma f \|_{L^{2n}(W_n)} = (n!)^{1/2n} \|S_\delta f \|_{L^{2n}(W_n)}$$

in non-Archimedean fields; instead, only the upper bound (that is, $\leq$) is deduced.
Despite this, that upper bound is sharp. This brings us to the proof of Theorem 2.

Proof of Theorem 2. By mollifying the functions \( f := \sum_{i=1}^{N} \delta_{i/N} \) for appropriate, large \( N \in \mathbb{N} \), a lower bound for \( H \), relates to counting solutions to the system of equations \( \sum_{i=1}^{n} (\gamma(t)_{i} - \gamma(s)_{i}) = 0 \) where \( s_{1}, t_{1}, \ldots, s_{n}, t_{n} \in \{1/N, 2/N, \ldots, N/N\} \).

To be precise, \( \|E_{\varphi}f\|_{L^{2n}(T_{\varphi}W_{\varphi-n})}^{2n} \) counts the number of such solutions which is \( n!N^{n} + O(N^{n-1}) \). One see this by fixing one set of variables, say \( s \), and observing that any permutation \( t \) of \( s \) is also a solution. Additionally, there are strictly less than \( n!N^{n} \) solutions by the reasoning in Remark 4. Meanwhile, \( \|S_{N-1}f\|_{L^{2n}(T_{\varphi}W_{\varphi})}^{2n} \) is simply the diagonal contribution \( N^{n} \). (I have omitted a factor of \( \int W_{\varphi} \) which appears in calculating both \( L^{2n} \)-norms.) Taking the limit as \( N \) goes to infinity, we see that \( H_{\gamma} \geq (n!)^{1/2n} \). \( \square \)

5. Special subvarieties underlying Theorems 3 and 4

In this section, I indicate how to prove Theorems 3 and 4. The essential point is that, for non-degenerate curves in Archimedean fields, the Implicit Function Theorem allows us to upgrade uniform bounds on counting estimates to fat estimates as in Proposition 6. For the uniform bounds on counting estimates, we use Bezout’s theorem and improvements to it.

Using Lemma 5, the following proposition immediately implies Theorem 3.

Proposition 7. Let \( \gamma := (\gamma_{1}, \ldots, \gamma_{n}) \) be a non-degenerate, polynomial curve in \( K = \mathbb{R} \) or \( \mathbb{C} \). We have the bound

\[
S_{\gamma} \leq (2[\ell(\gamma)] + 1)^{n\eta K} \prod_{i=1}^{n} \deg(\gamma_{i}).
\]

Proof of Proposition 7. Let \( K \) be \( \mathbb{R} \) or \( \mathbb{C} \). Fix \( n \in \mathbb{N} \) to be two or more, fix \( \delta \in \mathcal{R}(K) \) and fix \( I \in \mathcal{P}_{\delta}^{n} \). Suppose for the moment that \( I \) is comprised of \( n \) distinct intervals. The Wronskian \( \det(\gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)) \) does not vanish for all \( t \in a \) Consequently, the determinant

\[
|\det(\gamma'(x_{1}), \gamma'(x_{2}), \ldots, \gamma'(x_{n}))| \geq \prod_{1 \leq i < j \leq n} |x_{i} - x_{j}|
\]

is non-zero for all \( x \in I \); it can become arbitrarily small if two intervals of \( I \) are adjacent, but this is not an issue for us.

Suppose that \( S(\delta, I, \delta^{n}) \) is not empty so that there exists \( x \in I \) and \( y \in o^{n} \) and \( h \in K^{n} \) satisfying (3) with \( \epsilon = \delta^{n} \) and

\[
\sum_{i=1}^{n} (\gamma_{j}(x_{i}) - \gamma_{j}(y_{i})) = h_{j} \quad \text{for} \quad j = 1, \ldots, n.
\]

Thus, \( |h_{j}| \leq \delta^{n} \) for \( j = 1, \ldots, n \). By a small perturbation, we may assume the strict inequality \( |h_{j}| < \delta^{n} \) for \( j = 1, \ldots, n \). By Bezout’s theorem there are at most \( \prod_{i=1}^{n} \deg(\gamma_{i}) \) possibilities for \( y \). Using the non-singularity estimate (9), the Implicit Function Theorem implies that for any such \( y \), there exists an open set \( U_{y} \) in \( K^{n} \) upon which (3) holds for some \( \epsilon > 0 \). The side-lengths of \( U_{y} \) are \( \leq \ell(\gamma) \delta \). Since \( \mathbb{R} \) and \( \mathbb{C} \) are connected, this forces \( J \) to be one of the \( \leq \prod_{i=1}^{n} \deg(\gamma_{i}) \) possible intervals containing \( y \) or one of each such interval’s \( \ell(\gamma) \delta \) neighbors to the left or right (and up or down when \( K = \mathbb{C} \)) for a total of at most \( (2[\ell(\gamma)] + 1)^{n\eta K} \prod_{i=1}^{n} \deg(\gamma_{i}) \) possibilities.

1 Here and below, I am considering \( \gamma \) as a column vector.

2 Technically, Bezout’s theorem gives that there are at most \( \prod_{i=1}^{n} \deg(\gamma_{i}) \) non-singular components. There are no singular components since the curve is non-degenerate. Since there are \( n \) equations, this means the components are 0-dimensional. In other words, they are points.
The cases where $J$ is not comprised of distinct intervals is proved similarly. The key difference here is to use the fact that a subsystem of the curve has non-vanishing Wronskian on $\mathbb{R}$. This is sufficient to apply Bezout’s theorem to deduce that there are at most $\prod \deg(\gamma_i)$ possibilities for each point $y \in J$. Once again, the Implicit Function Theorem and Lipschitz bound forces these and their neighbors to persist at the fattened level.

**Remark 1.** Although Bezout’s theorem and the Implicit Function Theorem are true over local fields, the above argument is restricted to Archimedean fields $\mathbb{R}$ and $\mathbb{C}$ for two reasons: The first reason is that, in non-Archimedean fields, I do not know if non-vanishing of the Wronskian (that is, non-degeneracy of the curve) implies non-singularity of the curve like in (9). This first reason is not an issue for the moment curve. The second reason is that, for Archimedean fields, I use analytic continuation when applying the Inverse Function Theorem to stop the open sets $U_y$ from jumping around as $y$ varies. This relies on the connectedness of $\mathbb{R}$ and $\mathbb{C}$, but non-Archimedean fields are totally disconnected. The second reason is an obstruction for this method to apply to the moment curve. This obstruction is overcome in [6] by making delightful use of how polynomial roots cluster.

Using Lemma 5, the following proposition immediately implies Theorem 4.

**Proposition 8.** Let $\gamma$ be a non-degenerate, polynomial curve in $\mathbb{R}^n$ such that the total number of monomials appearing in $\gamma$ is $M$. 

$$S_{\gamma} \leq (2\ell(\gamma) + 1)^n 2^{M(M-1)/2}(n+1)^M.$$  

(10)

The proof of the Proposition 8 is almost identical to the proof of Proposition 7, but utilizes improvements to Bezout’s theorem in $\mathbb{R}$ from [9, 10] which says that, over $\mathbb{R}$, one may improve the upper bound in Bezout’s theorem to $2^{M(M-1)/2}(n+1)^M$. I leave remainder of the proof of Proposition 8 to the reader.

**6. On Superorthogonality**

In this section I assume that the reader is familiar with [5], and I rephrase some terminology from therein. Suppose that $(f_{j_1}, \ldots, f_{j_{2n}})$ is a tuple of functions from a finite collection of functions $\{f_j\}_{j \in J}$. Call a $2n$-tuple of functions $(f_{j_1}, \ldots, f_{j_{2n}})$ “$2n$-superorthogonal” if

$$\int f_{j_1} f_{j_2} \cdots f_{j_{2n-1}} f_{j_{2n}} = 0.$$  

(11)

Define the ensuing ‘Types’ as subcollections of $J^{2n}$ satisfying the following properties:

- **Type I**: If $(j_1, j_3, \ldots, j_{2n-1})$ is not a permutation of $(j_2, j_4, \ldots, j_{2n})$, then $(f_{j_1}, \ldots, f_{j_{2n}})$ is $2n$-superorthogonal.
- **Type I**: If some value $j_k$ appears an odd number of times in the $2n$-tuple $(j_1, \ldots, j_{2n})$, then $(f_{j_1}, \ldots, f_{j_{2n}})$ is $2n$-superorthogonal.
- **Type II**: If some value $j_k$ appears exactly once in the $2n$-tuple $(j_1, \ldots, j_{2n})$, then $(f_{j_1}, \ldots, f_{j_{2n}})$ is $2n$-superorthogonal.
- **Type III**: If some value $j_k$ appears exactly once and is strictly greater than all other values appearing in the $2n$-tuple $(j_1, \ldots, j_{2n})$, then $(f_{j_1}, \ldots, f_{j_{2n}})$ is $2n$-superorthogonal.
- **Type IV**: If all values of the $2n$-tuple $(j_1, \ldots, j_{2n})$ are distinct, then $(f_{j_1}, \ldots, f_{j_{2n}})$ is $2n$-superorthogonal.

Each successive Type implies the next in the sense that if a subcollection is Type $I^*$, then it is also Type $I$, and so on.

For this section, take $\gamma := (T, T^2, \ldots, T^n)$ to be the moment curve. Since the moment curve is non-degenerate and has degree $n$, the proof of Proposition 7 reveals
that $S(I, \delta^n)$ is the set of permutations of $I$ along with their neighbors. This
gives ‘Type I* almost superorthogonality’ as described in [13, Section 6] and [5, Section 6].
The ‘almost’ refers to the neighbors.

In contrast, the proof of Theorem 1 allows for tuples which are not Type I*,
I, II, or III. To be precise, let us consider superorthogonality for 6-tuples of
an extension operator for the moment curve $(T, T^2, T^3)$. Let $f$ be a function and
define the functions $f_j := E_j f$ where $I_j$ is an interval in a fixed, sufficiently fine
partition. Let us regard $j$ simply as integers in, say, $[1, N]$ for some large $N$. In
the proof of Proposition 6 (and consequently in Theorem 1), I could have $j_1 = 2$
and $j_2 = j_3 = j_4 = j_5 = j_6 = 1$ potentially give

$$\int f_{j_1} f_{j_2} f_{j_3} f_{j_4} f_{j_5} f_{j_6} \neq 0.$$ 

In other words, I do not prove that (11) holds for the 6-tuple $(2, 1, 1, 1, 1, 1)$. Type I*
fails because $(2, 1, 1)$ is not a permutation of $(1, 1, 1)$. Type I fails because 1
appears an odd number of times (as does 2). Type II fails because 2 appears as an
index precisely once. Type III fails because 2 is the unique largest index.

Curiously, the proof of Proposition 6 crucially uses the property that the tuple
$(j_2, \ldots, j_n)$ is a subset of the tuple $(j_1, \ldots, j_{2n-1})$ when interpreting both tuples
as sets (this means ignoring multiplicity and order of values arising in the tuples).
Obviously, this satisfies Type IV superorthogonality. As a consequence, one may
bypass Lemma 5 and reformulate Proposition 6 to use Theorem 1 of [5] to deduce,
when the field is Archimedean and of an appropriate characteristic, that $H_\gamma(W) \leq 2^{1/2}(2n! - 1)^{1/2}$.

Finally, the underlying symmetry of (7) implies Type II superorthogonality
which gives a better constant. When $n = 2$ or 3, one easily refines the bounds
on $H_\gamma(W)$ to $2!$ and $3!$ respectively. Unfortunately, the best bound deducible from
this method exceeds $n!$ when $n \geq 4$.

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