Position and momentum operators on manifolds of constant curvature

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We discuss Hermitean momentum operators given by orthonormal Killing vector fields on manifolds of constant curvature. It is shown that this definition provides a bilinear decomposition of the Laplace-Beltrami operator. The commutation relations of the momentum operators are expressed in terms of the Christoffel symbols of the underlying connection. An appropriate notion of position operators is introduced and the corresponding commutation relations of position and momentum are established as a part of the standard duality relation on the manifold. The occurring "deformation" on the right-hand side of the Heisenberg commutation relations will be uncovered by the anti-exact part of the associated dual frame forms. It is shown that this approach is compatible with the most popular examples known from literature. Applications are given for $S^1$, $S^1 \times \mathbb{R}$, the Euclidean spaces $\mathbb{R}^2$, $\mathbb{R}^3$ and finally for the nontrivial case $S^3$. For manifolds of constant positive curvature it will appear that the present approach is possible for dimensions equal to 1, 3 and 7. A no-go statement is given for $S^2$ and the hyperbolic space $H^n$, $n > 1$.

I. INTRODUCTION

Recently it has been mentioned [1][2] that there is a rigorous derivation of the Extended Uncertainty Principle (EUP) based on an uncertainty relation on 3-dimensional spaces of constant curvature [3][4]. The derivation is based on the notion of geodesic balls and reflects the influence of spatial curvature on quantum mechanics on 3-dimensional spacelike hypersurfaces of space-time. The method was applied to calculate the EUP for Rindler and Friedmann horizons to obtain corrections to the Hawking temperature and Bekenstein entropy of black holes in [1]; this was further commented on in [5]. The issue of deriving a general expression for the EUP in the case of nonhomogeneous (e.g. nontrivial) effects of curvature on the EUP has been given in [2].

However, up to now no explicit definition has been given for a Hermitean momentum operator by which the standard deviation $\sigma_P$ of the uncertainty relation in [1][2][3][5] could be provided (see also [6]). Actually, one cannot even be sure whether a well-defined and Hermitean operator does exist. In the standard approach of the EUP this problem is addressed by introduction of a phenomenological ("deformed") commutation relation of position and momentum. Nevertheless, in our case of constant curvature spaces one can hope to find an adequate representation of a momentum operator and the corresponding commutation relations purely by mathematical deduction.

To address this question, we briefly repeat the main idea and the assumption concerning the momentum operator given in the original approach of [3][5]. Let $\psi$ be a free wave function on a Riemannian manifold but confined to a geodesic ball $B_r$ of radius $r$. Due to their invariance under diffeomorphisms, geodesic balls are the natural generalization of the Heisenberg slit to curved manifolds where $r$ corresponds to the slit width, thus providing a measure of position uncertainty. Confinement to this domain is ensured by imposing Dirichlet boundary conditions on the wave function. As the Laplace-Beltrami-operator $\Delta$ (representing the squared momentum operator $\hat{p}^2 = -\hbar^2 \Delta$) is Hermitean with respect to the measure $d\mu$ of the Hilbert space in question $\mathcal{H} = L^2(B_r)$, deriving the uncertainty relation basically boils down to solving the Dirichlet eigenvalue problem [3].

The mean momentum of the particle in $B_r$ is zero (see Proposition 2) and the square of the momentum standard deviation $\sigma_p$ is given by

$$\sigma^2_p = -\hbar^2 \langle \Delta \rangle$$

(1)

For a ball of geodesic radius $r$ in a simply connected 3-manifold of constant curvature $K$, the first eigenvalue of the Laplace-Beltrami operator can be explicitly computed, such that the following uncertainty relation has been obtained in [3]:

$$\sigma_p r \geq \pi \hbar \left[ 1 - \frac{K}{\pi^2 r^2} \right]^{\frac{1}{2}}$$

(2)

Although our statement concerning the mean momentum within the compact domain $B_r$, being equal to zero seems quite obvious, proof can only be given if a suitable Hermitean momentum operator does exist.

The paper is organized as follows: In the following section, we introduce Hermitean momentum operators for spaces of constant curvature and discuss their main properties. It is shown that this definition provides a bilinear decomposition of the Laplace-Beltrami operator (Proposition 3). The commutation relations of the momentum operators are discussed. In Sec. III, an appropriate notion of position operators is introduced and the corresponding commutation relations of position and momentum are established as a part of the standard duality relation on the manifold. The associated deformation will be uncovered as the anti-exact part of the dual frame forms on the
manifold. Applications are given in Sec. IV. It is shown that this approach is compatible with the most popular examples known from literature and also provides the foundation of the uncertainty relation given in [2].

II. HERMITEAN MOMENTUM OPERATORS

Let $M$ be an $n$-dimensional smooth Riemannian manifold with metric $g$, occasionally denoted by $(\cdot, \cdot)$. At every point $p \in M$, smooth manifolds admit a tangent space $T_pM$, which is an $n$-dimensional real vector space. The basic quantum mechanical operator generated by quantization $\hat{H}$ is the operator of projection of momentum on a given smooth vector field $X$ on $M$ according to

$$P_X = -i\hbar \left( \nabla_X + \frac{1}{2} \text{div} X \right),$$

where $\nabla_X$ is the Levi-Civita connection with respect to $X$. These operators act in the Hilbert space of square integrable complex functions $L^2(M, \mu)$ endowed by the inner product

$$(f, g) = \int_M d\mu f^* g, \quad f, g \in L^2(M, \mu),$$

where $\mu$ is the standard volume measure on $M$. The statement that these operators are Hermitean with respect to the inner product is mainly based on the assumption that the boundary terms vanish after partial integration. However, in what follows, we are focused on the Hilbert space $\mathcal{H}_M \subset L^2(M, \mu)$, with

$$\mathcal{H}_M = \{ f \in C^1(M) : f|_{\partial M} = \text{const.} \}.$$  

At this point one has to check whether the momentum operator $\hat{P}_X$ still remains Hermitean because the elements in $\mathcal{H}_M$ are not supposed to vanish at the boundary. One cannot define an operator such as the momentum or Hamiltonian operator on a bounded domain without specifying boundary conditions. In mathematical terms, choosing the boundary conditions amounts to choosing an appropriate domain for the operator. If we use no boundary conditions, too many functions are eigenvectors and so the spectrum of $P_X$ is the whole complex plane. If we impose Dirichlet boundary conditions, the situation is too restrictive and one cannot find an orthonormal basis. Thus, if the functions $f$ are smooth on $M$ but constant functions at the boundary, in this case finding a domain such that $P_X$ is self-adjoint is a compromise to obtain an orthonormal basis of countable spectrum. Actually, this is the reason for our choice of $\mathcal{H}_M$.

**Proposition 1.** Let $X$ be a smooth and divergenceless vector field on $M$, such that $\text{div} X = 0$. Then $\hat{P}_X$ is Hermitean on $\mathcal{H}_M$.

**Proof.** Let $f, h \in \mathcal{H}_M$. The divergence of the product $fX$ can be written as

$$\text{div}(fX) = f \text{div}(X) + \nabla_x f.$$  

On the other hand, we have the decomposition

$$h^* \nabla_x f = \langle h^* X, \text{grad} f \rangle = \text{div}(fh^* X) - f \text{div}(h^* X).$$

This equation can be integrated with respect to the volume form $d\mu$ on $M$ as follows

$$(h, \nabla_x f) = -\int_M d\mu f \text{div}(h^* X) + \int_{\partial M} d\mu \partial f h^* \langle X, \nu \rangle.$$  

Here, we have applied Stokes’ theorem, where $d\mu_\partial$ is the volume measure with respect to the non-empty boundary $\partial M$ and $\nu$ is the non-negative outward normal on $\partial M$. Now, since $f$ and $h$ are assumed to be constant at the boundary of $M$, they can be taken out of the integration in [3] and we can apply Stokes’ theorem once more such that the remaining boundary integral on the right-hand side in [3] becomes

$$\int_{\partial M} d\mu \partial (X, \nu) = \int_M d\mu \text{div} X$$

and with the assumption $\text{div} X = 0$, we can write

$$(h, \nabla_x f) = -\int_M d\mu f \text{div}(h^* X).$$

Finally, we apply (6) and get

$$(h, \nabla_x f) + (\nabla_x h, f) = -\int_M d\mu h^* f \text{div}(X)$$

The term on the right-hand side can be absorbed into each term on the left-hand side with a prefactor $1/2$. By multiplication of the equation with $-i\hbar$, and after applying definition [3], we finally obtain

$$(h, P_X f) = (P_X h, f)$$

for all $f, h \in \mathcal{H}_M$.  

Without the divergence assumption for $X$ it is not possible to get the right-hand side in [4] to be zero on the compact domain in general. This makes the divergence criterion necessary.

Now let us consider a simple but important statement concerning the mean value of the momentum operator on a compact domain:
\textbf{Proposition 2.} Let $f \in \mathcal{H}_M$, such that $f$ satisfies the Dirichlet eigenvalue problem

$$\Delta f + \lambda f = 0 \quad (13)$$
$$f \big|_{\partial M} = 0 \quad (14)$$

\begin{proof}
For every smooth function $f$ on $M$, the eigenvalue problem (13) is the same for its real part $g := \text{Re}(f)$ and its imaginary part $h := \text{Im}(f)$. Therefore, there is a constant $\alpha \in \mathbb{R}$ such that $h = \alpha g$ and $f = cg$, for some complex number $c = 1 + ia \in \mathbb{C}$. Thus, we can write

$$(P_x f, f) = c^* c (P_x g, g) = -c^* c (g, P_x g) = -(f, P_x f) \quad (16)$$

for all $f \in \mathcal{H}_M$. The minus sign in the second line is due to the imaginary unit in the definition (3). Since $X$ is divergenceless by assumption, $P_x$ is Hermitean on $\mathcal{H}_M$. This proves the statement (15), for all $f \in \mathcal{H}_M$. \hfill \square
\end{proof}

In the approach given in (6) (Appendix A), it has been mentioned that the co-linearity of the real and imaginary part of the eigenvalue problem (13) does not necessarily imply that the corresponding wave functions are real, neither that their momentum expectation values vanish. We only share the first point of this statement, that the wave functions are not necessarily real-valued. According to Proposition 2, the mean value of the momentum operator $P_x$ on a compact domain is zero if the condition $\text{div} X = 0$ of the corresponding direction $X$ is satisfied. To resolve the remaining concerns in (6), see our remark in the appendix.

For the standard deviation applied in (11), it is appropriate to provide a bilinear decomposition of the Laplace-Beltrami operator in terms of the components of the momentum operators in (4). This can be obtained by the following:

\textbf{Proposition 3.} Let $M$ be an $n$-dimensional manifold and $X_1, \ldots, X_n$ be an orthonormal basis frame of smooth vector fields on $M$ with $\text{div} X_i = 0$, $i = 1, \ldots, n$. There is a decomposition of the Laplace-Beltrami operator, such that

$$\sum_{j=1}^n P_{X_j}^2 = -\hbar^2 \Delta \quad (17)$$

\begin{proof}
The dual frame on $M$ is given by an $n$-tuple of $1$-forms $\theta^1, \ldots, \theta^n$, which are linearly independent at each point on $M$. When applied to the vector fields $X_j$ they satisfy the duality relation

$$\theta^k(X_j) = \delta^i_j \quad (18)$$

for $j, k = 1, \ldots, n$. The vector fields $X_j$ can be expressed as a linear combination of the coordinate vector fields $\partial_\alpha = \partial/\partial x^\alpha$, such that

$$X_i = \xi^\alpha_i \partial_\alpha \quad (19)$$

where each $\xi^\alpha_i$ is a function. These can be seen as the components of a matrix $\xi$. Furthermore, each covector field $\theta^k$ can be expressed as a linear combination of the corresponding covector fields $dx^i$ given by

$$\theta^k = \eta^k_\beta dx^\beta \quad (20)$$

where each $\eta^k_\beta$ is a function, which can also be seen as the components of a matrix $\eta$. The two coordinate expressions above combine to yield $\xi^\alpha_i \eta^k_\beta = \delta^k_i$; in terms of matrices, this just says that $\xi$ and $\eta$ are inverses of each other.

$$\xi \eta = 1 \quad (21)$$

By applying the relation $\eta^k_\beta = g_{ij} \xi^i_j$ to the left-hand side we obtain $\xi g \xi^T = 1$ or equivalently

$$\xi^T \xi = g^{-1} \quad (22)$$

Now we can write

$$\sum_{i,j} \nabla_{X_i}^2 f = \delta^{ij} \nabla_{X_i}(X_j f) = \delta^{ij} \left[ (\xi^k_{i,j})^* \partial_k f + \xi^i_j \xi^k_{i,j} \nabla_{\partial_k} \partial_{\alpha,f} \right] \quad (23)$$

which has been obtained by the product rule of differentiation. On the other hand, by using the basic property $\nabla g = 0$ of the Levi-Civita connection together with (22), we obtain the following general expression

$$0 = \nabla_{\partial_\alpha} g_{ij}^k = \delta^{ij} \nabla_{\partial_\alpha} (\xi^k_{i,j})$$
$$= \delta^{ij} \nabla_{\partial_\alpha} \left( \text{div} X_i \xi^k_j + \nabla_{X_i} \xi^k_j \right)$$
$$= \delta^{ij} \nabla_{X_i} \xi^k_j \quad (24)$$

Thus, we can ignore the first term on the right-hand side of (23). After applying (22) to the second term on the right-hand side of (23), we obtain

$$\sum_{i,j} \nabla_{X_i}^2 f = g^{ij} \nabla_{\partial_i} (\partial_j f) = g^{ij} \nabla_{\partial_i} \nabla_{\partial_j} f = \Delta f \quad (25)$$
for every \( f \in \mathcal{H}_M \). With definition \( \| \) of the momentum operator in physical units, we obtain the statement \( \square \).

A momentum operator which can be considered as a foundation of \( (2) \) is given by

\[ P_{X_k} = -i\hbar \nabla_{X_k} \]

\( k = 1, \ldots, n \), are called geodesic momentum operators in the direction \( X_k \) on \( M \).

With Propositions 1, 2 and 3, the standard deviation \( (1) \) of the momentum is equivalently given by

\[ \sigma_p^2 = \sum_k \left( \langle P_{X_k} \rangle - \langle P_{X_k} \rangle \right)^2 \]

The Lie bracket of two Killing fields is still a Killing field. The momentum operators \( (26) \) thus form a Lie subalgebra of vector fields on \( M \). If \( M \) is a complete manifold, this is the Lie algebra of the translation group. The explicit form of the commutation relations can be obtained by the Christoffel symbols \( \Gamma^i_{jk} \) with respect to the Killing frame defined by

\[ \nabla_{X_j} X_k = \Gamma^l_{jk} X_l \]

In addition, \( \nabla \) is a Riemannian connection which is also torsionless. In this case the commutation relations of the momenta are

\[ [P_{X_j}, P_{X_k}] = -i\hbar c^l_{jk} P_{X_l} \]

where the structure functions can be expressed by the Christoffel symbols according to

\[ c^l_{jk} = \Gamma^l_{jk} - \Gamma^l_{kj} \]

for \( j, k, l = 1, 2, 3 \). The Christoffel symbols are not independent of each other. The following two important properties have been mentioned in \( [3] \):

\[ \Gamma^i_{jk} - \Gamma^i_{ik} = 0 \quad \text{(Killing condition)} \]
\[ \Gamma^i_{jk} - \Gamma^i_{ij} = 0 \quad \text{(Skew symmetry)} \]

The divergence property of the Killing fields can be expressed by the contraction \( \Gamma^i_{ji} = 0 \).

### III. POSITION OPERATORS

In order to discuss the commutation relations of position and momentum, let us first introduce an appropriate notion of position operators \( Q^k \) on \( M \). The operators \( Q^k \) of position type functions \( q^k \) are \( Q^k = q^k \mathbb{1} \), i.e. they are the operators of multiplication by the functions \( q^k \) - what is shown by the unity operator \( \mathbb{1} \) on the right-hand side. Occasionally we write \( Q^k \) instead of \( q^k \) so that no misunderstandings can occur. From the exterior calculus point of view, the position functions \( q^k \) can be considered as differential forms of zero degree, whose derivative is related to the basis 1-forms of the dual space of \( M \). Thus, it is obvious to ask if there exist 0-forms \( Q^k \) such that \( \delta^k = dQ^k \), for \( k = 1, \ldots, n \). The answer is given by the Poincaré lemma. Especially for differential forms on star-shaped domains (e.g. geodesic balls), every closed differential form is also exact and therefore the position functions can be obtained by integration over \( \theta^k \). In exterior calculus \( [1] \) \( [10] \), this type of integration is commonly expressed by the homotopy operator \( H \) on forms defined on \( M \). Sometimes properties of \( H \) are given in Theorem 5-3.1 of \( [8] \). For our purpose it is especially relevant to know that \( H \) is a linear transformation of \( p \)-forms into \( (p-1) \)-forms, \( p = 1, \ldots, n \), with the property to be a decomposition of the identity map on forms commonly expressed as

\[ dH + H d = \mathbb{1} \]

For the dual forms \( \theta^k \), then \( \theta^k = dH\theta^k + H d\theta^k \) allows us to associate an exact form

\[ \theta^k = dH\theta^k \]

with the base form \( \theta^k \). It is thus meaningful to refer to \( \theta_e \) as the exact part of \( \theta^k \). We accordingly refer to

\[ \theta^k_a = H d\theta^k \]

as the anti-exact part of \( \theta^k \). This decomposition of a form into the sum

\[ \theta^k = \theta^k_e + \theta^k_a \]

of an exact part and an anti-exact part is unique \( [3] \), Corollary 5-6.1). For intrinsically curved manifolds this implies a non-zero 2-form

\[ d\theta^k = d\theta^k_a \]

From the physics point of view, we propose to identify the exact part of the dual frame forms with the differential of the position functions (operators) such that \( \theta^k = dQ^k \). Hence, with regard to \( (32) \) we are ready to introduce the following:

**Definition 2.** (Position operator)

The position functions \( Q^k \) with respect to the exact part of the dual frame \( \theta^k \) on a star-shaped domain \( M \) is

\[ Q^k = H\theta^k \]
This assignment is distinguished because it tells us how to provide an adequate position operator for every curved manifold with a well-defined dual frame. The interrelation to the Heisenberg commutation relations of position and momentum can now be obtained by the trivial but general identity

\[
[Q^k, X_j] = -X_j Q^k = -dQ^k (X_j) = -\theta^k_a (X_j)
\]

which is valid for every vector field \(X_j\) on \(M\). This uncovers the Heisenberg commutation in terms of the exact part of the frame decomposition \([34]\). Therefore the duality relation \([13]\) can be expressed in terms of the commutator and the anti-exact part according to

\[
\delta_j^k = -[Q^k, X_j] + \theta^k_a (X_j)
\]

After multiplication with the physical units \(i\hbar\) and some algebraic manipulations, we obtain the Heisenberg commutation relations for position and momentum on \(M\) as

\[
[Q^k, P_{x^j}] = i\hbar \left( \delta^k_j - \theta^k_a (X_j) \right)
\]

Additional terms on the right-hand side of the Heisenberg commutation relations are commonly denoted by “deformation”. In the present approach, the deformation is just the anti-exact part of the dual frame on the manifold. For locally flat manifolds this term vanishes, such that the ordinary Heisenberg commutator will be recovered in this case. For intrinsically curved manifolds the differential of \(\theta^k\) can be assessed by the first Cartan structure equation applied to the base frame \(X_i\). Its explicit form is

\[
d\theta^k(X_i, X_j) = -\theta^k([X_i, X_j])
\]

which is generally nonzero. For a symmetric connection \([25]\), the structure functions \([30]\) are zero and we obtain the dual forms \(\theta^k\) to be closed and hence exact. This is especially the case when the momentum operators do all commute with each other. In this case the deformation must be zero and we recover the ordinary Heisenberg commutation relations of position and momentum of the flat case. Otherwise, if the Riemann curvature tensor is nonzero, the deformation cannot vanish. A simplification of the anti-exact form \(\theta^k\) with help of the structure equation \(d\theta^k = -\omega^k_j \wedge \theta^j\) is not possible. In fact, the definition of \(H\) shows that \(H(\omega^k_j \wedge \theta^j)\) is not expressible in terms of \(H(\omega^k_j)\), \(H(\theta^j)\) and the operations \(\wedge, + \) or \(d\). A further discussion of this rather mathematical point is given in \([8]\), p. 183.

The scope of the concept given so far can be understood by looking at the mathematical classification of manifolds with orthonormal Killing frames. It has been shown that a Riemannian manifold having the Killing property must be locally symmetric \([8]\). Thus, each point of a connected Riemannian manifold having the Killing property has an open neighbourhood which is isometric to an open neighbourhood in a simply connected Riemannian symmetric space \(M\). Then \(M\) also has the Killing property and, moreover, has global Killing frames. In fact, a local Killing frame exists on \(M\) because of the given local isometry, and can be extended uniquely to give a global Killing frame. The extension of each Killing vector field to a global Killing vector field is possible since the symmetry implies completeness. This extension remains orthonormal since the Riemannian structure on \(M\) is subordinate to a real analytic Riemannian structure (cf. \([11]\) p. 240, \([12]\) p. 187).

A simply connected Riemannian symmetric space is said to be irreducible if it is not the product of two or more Riemannian symmetric spaces. It can then be shown that any simply connected Riemannian symmetric space is a Riemannian product of irreducible ones. Therefore, we furthermore restrict ourselves to the irreducible, simply connected Riemannian symmetric spaces. Any simply connected Riemannian symmetric space \(M\) is of one of the following three types: (i) Euclidean type: \(M\) has vanishing curvature, and is therefore isometric to a Euclidean space. (ii) Compact type: \(M\) has nonnegative (but not identically zero) sectional curvature. (iii) Non-compact type: \(M\) has nonpositive (but not identically zero) sectional curvature.

Manifolds of constant positive curvature are known \([8]\) to have the Killing property only if the dimension of \(M\) is equal to 1, 3 or 7. For the spheres \(S^1\), \(S^3\) and \(S^7\), in fact, there is a global Killing frame. The construction depends essentially on the existence of a multiplication in \(\mathbb{R}^2\) (complex numbers), \(\mathbb{R}^4\) (quaternions), and \(\mathbb{R}^8\) (Cayley numbers).

IV. APPLICATIONS

Manifolds of one dimension are locally flat, and therefore always have the Killing property. Thus, we begin with the 1-dimensional case:

**Circle**: For the circle we can take \(X_1\) to be the unit tangent vector field, say pointing in the anti-clockwise direction. More precisely consider the situation of a connected Riemannian manifold \(M \subset S^1\) embedded in \(\mathbb{R}^2\). The general solution of the Killing equation \(L X_1 g = 0\) on \(S^1\), with metric \(ds^2 = \rho^2 d\varphi^2\) is given by \(X_1 = \xi^\varphi \partial_{\varphi}\), for \(\xi^\varphi \in \mathbb{R}\). Let \(\xi^\varphi = 1/\rho\), where \(\rho\) is the constant (hyper-) radius of the circle, then we have \(X_1^2 = 1\) and the Killing trajectory is a geodesic. The dual 1-form is given by \(\theta^1 = \rho d\varphi\). According to our definition of the position operator, we obtain \(Q^1 = \rho \varphi\). The associated momentum operator is

\[
P_{\varphi} = -i\hbar \frac{1}{\rho} \frac{\partial}{\partial \varphi}.
\]
According to Proposition 1, this operator is symmetric on any compact set \( M \subset S^1 \), with \( f = \text{const.} \) on the boundary. It has been mentioned in \([13]\) that in quantum mechanics on a circle with standard commutation relation for \( \varphi \) and \( p_\varphi \), the uncertainty relation cannot be stronger than \( \sigma_\varphi \sigma_\varphi \geq 0 \), where \( \sigma_\varphi \) and \( \sigma_p \) are the standard deviation of position and momentum. Actually, this inequality is not informative at all, since a product of two nonnegative values cannot be negative. Alternatively, we refer to the uncertainty relation in \([1]\), which is not affected by the problems in defining a proper measure of position uncertainty on manifolds mentioned in \([14]\). By applying the substitution \( r = \rho \varphi \), which corresponds to the arc-length on \( S^1 \), the uncertainty principle of \([4]\) is given by

\[
\sigma_p \Delta r \geq \pi \hbar \tag{42}
\]

where \( \Delta r \) is the measure (length) of the compact domain \( M \).

**Euclidean plane:** We consider the Euclidean plane \( \mathbb{R}^2 \) endowed with Euclidean metric. Let \( x, y \) and \( r, \varphi \) denote Cartesian and polar coordinates on the plane. Then the Killing vector fields corresponding to translations and rotations are \( X_1 = \partial_x \), \( X_2 = \partial_y \) and \( X_3 = \partial_\varphi \). Their squared vector norms are \( X_1^2 = 1, X_2^2 = 1 \) and \( X_3^2 = r^2 \). The Killing vector fields \( X_1 \) and \( X_2 \) have a constant length on the whole plane. Their trajectories are straight lines, which are geodesics. The Killing trajectories corresponding to rotations \( X_3 \) are concentric circles around the origin. The length of \( X_3 \) is constant along the circles, but nonconstant on the whole plane. The corresponding Killing trajectories are circles, which are not geodesics. Thus an appropriate Killing frame is given by \( X_1 \) and \( X_2 \), which can be transformed in polar coordinates according to

\[
X_1 = \cos \varphi \partial_r - \frac{\sin \varphi}{r} \partial_\varphi \tag{43}
\]

\[
X_2 = \sin \varphi \partial_r + \frac{\cos \varphi}{r} \partial_\varphi \tag{44}
\]

The corresponding momentum operators in physical units are given by \([26]\). The cotangential space is spanned by the orthonormal dual frame

\[
\theta^1 = \cos \varphi \, dr - r \sin \varphi \, d\varphi \tag{45}
\]

\[
\theta^2 = \sin \varphi \, dr + r \cos \varphi \, d\varphi \tag{46}
\]

The corresponding position operators \( Q^1 \) and \( Q^2 \) are given by \([30]\), such that we obtain

\[
Q^1 = r \cos \varphi \tag{47}
\]

\[
Q^2 = r \sin \varphi \tag{48}
\]

As expected, these functions correspond to the ordinary position operators \( \hat{x}, \hat{y} \) in Cartesian coordinates and the anti-exact part of the dual forms is zero.

At this point it should be emphasized that the above dual frame is different from the commonly considered (simpler) frame \( \{dr, r \, d\varphi\} \), which is also orthonormal. However, this simpler frame is inadmissible because the associated tangential frame \( \{\partial_r, \frac{1}{r} \partial_\varphi\} \) is not a Killing frame, as can easily be verified by computing the divergence \( \text{div} \partial_r = 1/r \neq 0 \). If this frame were applied anyway, then the Heisenberg commutation relations would not all be satisfied and the square of the momentum operators would no longer be identical to the Laplace-Beltrami operator.

Uncertainty relations in polar coordinates can be obtained as follows: because of the rotation symmetry one can choose \( \varphi = 0 \) to consider the radial direction, such that we have \( X_1 = \partial_r \). Since \( r \) is a geodesic radius, we can apply the inequality already given in \([42]\). However, more interesting is the 2-dimensional flat circular disc (geodesic ball) of radius \( r \). The corresponding uncertainty relation has already been discussed in \([15]\) and is given by

\[
\sigma_p r \geq j_{0,1} \hbar \tag{49}
\]

where \( j_{0,1} \approx 2.40 \) is the first positive zero of the Bessel function \( J_0 \).

**Cylinder:** We consider the product space of \( \mathbb{R} \times S^1 \) endowed with the composed standard metric. Let \( z, \varphi \) denote the Cartesian component and the polar angle and \( r \) is the constant radius of the cylinder. Then there is the following (orthonormal) Killing frame

\[
X_1 = \partial_z \tag{50}
\]

\[
X_2 = \frac{1}{r} \partial_\varphi \tag{51}
\]

with dual frame

\[
\theta^1 = dz \tag{52}
\]

\[
\theta^2 = r \, d\varphi \tag{53}
\]

The position operators on the cylinder are given by

\[
Q^1 = z \tag{54}
\]

\[
Q^2 = r \varphi \tag{55}
\]

Zero curvature of the cylinder implies the Heisenberg commutation relations are satisfied without deformation.

**Euclidean space** \( \mathbb{R}^3 \): The radial momentum operator is usually obtained through canonical quantization of the classical radial momentum \([10, 17, 18, 19]\). It has the form

\[
\hat{p}_r = -i \hbar \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \tag{56}
\]

In this case, the associated tangential vector is \( X = \partial_r \), which points in the radial direction of the Euclidean space.
in spherical coordinates. The divergence of this vector field is \( \text{div}(\partial_r) = 2/r \), such that the radial expression is compatible with \( \mathfrak{g} \). The non-vanishing divergence is not a contradiction because the domain of \( \mathfrak{g} \) is assumed to be unbounded. However, since we are working with compact domains (e.g., geodesic balls), this kind of momentum operator cannot be applied. Alternatively, in standard spherical coordinates

\[
\begin{align*}
x^1 &= r \sin \theta \cos \varphi \\
x^2 &= r \sin \theta \sin \varphi \\
x^3 &= r \cos \theta
\end{align*}
\]

we introduce the following Killing frame:

\[
\begin{align*}
X_1 &= \sin \theta \cos \varphi \partial_r + \frac{1}{r} \cos \theta \cos \varphi \partial_\theta - \frac{\sin \varphi}{r \sin \theta} \partial_\varphi \\
X_2 &= \sin \theta \sin \varphi \partial_r + \frac{1}{r} \cos \theta \sin \varphi \partial_\theta + \frac{\cos \varphi}{r \sin \theta} \partial_\varphi \\
X_3 &= \cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta
\end{align*}
\]

These vector fields are orthonormal with respect to the induced metric and we have the property \( \text{div}X_i = 0 \). Obviously, the corresponding integral curves are geodesics and the position functions are given by \( q^i = x^k(r, \theta, \varphi) \). The structure functions in \( \mathfrak{g} \) and the deformation term in \( \mathfrak{g} \) are zero. The uncertainty relation of position and momentum is given by \( \mathfrak{g} \), for \( K = 0 \).

**2-sphere:** A more sophisticated example is the situation for the 2-sphere. In \( \mathfrak{g} \), it has been proposed to consider the momentum operators

\[
\begin{align*}
\hat{p}_\theta &= -i \hbar \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta \right) \\
\hat{p}_\varphi &= -i \hbar \frac{\partial}{\partial \varphi}
\end{align*}
\]

which also contain a term of non-zero divergence. For \( X_1 = \partial_\theta \) and \( X_2 = \partial_\varphi \), this also fits into the definition \( \mathfrak{g} \). The vector field \( X_1 \) is a unit vector on \( S^2 \), but it is not a Killing vector. On the other hand, \( X_2 \) is a Killing vector on \( S^2 \), but its norm is not constant. We obtain \( \text{div} \partial_\varphi = \frac{1}{2} \cot \theta \) and \( \text{div} \partial_\varphi = 0 \). Since \( \partial_\theta \) is not divergenceless but an element of the tangential space of \( S^2 \), we do not see how to prove that \( \hat{p}_\theta \) is Hermitian on a compact domain of \( S^2 \).

Alternatively, in \( \mathfrak{g} \), there is a systematic classification of several interesting forms of momentum operators on \( S^2 \), proposed by different sources in literature. It has been shown that only one of them ((31)-(33) in \( \mathfrak{g} \)) is completely compatible with the framework of Dirac’s canonical quantization approach. According to the notation in \( \mathfrak{g} \), let \( x^i, i = 1, 2, 3 \), be the Cartesian coordinates of \( \mathbb{R}^3 \) and \( \phi^1 = \theta, \phi^2 = \varphi \) the spherical degrees of freedom. With the vector fields defined by \( X_i = (\partial_r, \phi^\alpha) \partial_{\phi^\alpha}, \alpha = 1, 2, \) the corresponding divergences are given by \( \text{div}X_1 = -2 \sin \theta \cos \varphi, \text{div}X_2 = -2 \sin \theta \sin \varphi \) and \( \text{div}X_3 = -2 \cos \theta \), which is compatible with \( \mathfrak{g} \) and reproduces the form proposed in (31)-(33) of \( \mathfrak{g} \). However, both vector fields are not Killing vectors and therefore cannot describe geodesic integral curves on \( S^2 \).

At this point it should be mentioned that each single component \( L_1, L_2, L_3 \) of the ordinary textbook angular momentum operator \( \mathbf{L} \) is a Killing vector on \( S^2 \) and moreover, \( \mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2 \) in fact corresponds to the Laplace–Beltrami operator on \( S^2 \). Although this seems quite promising, these vector fields are not normalizable. Indeed, all vector fields on the 2-sphere are inappropriate for this purpose because of the hairy ball theorem of differential topology, which states that there is generally no nonvanishing continuous tangent vector field on even-dimensional n-spheres. This makes it hard to think about what kind of vector fields should be appropriate for an adequate description of momentum operators on \( S^2 \).

**Surface in \( \mathbb{R}^3 \):** Another approach given by Cartesian momentum operators for general surfaces (of non-constant curvature) embedded in three dimensions is

\[
\hat{p}_i = -i \hbar \left( \frac{\partial}{\partial \chi^i} + \frac{1}{2} \Gamma^i_{jk} \right)
\]

with \( \Gamma^i_{jk} = \Gamma^i_{jk} \) being the once-contracted Christoffel symbol \( \mathfrak{g} \). In this case, we have the vector fields \( X_i = \delta^k_i \partial_k \) and take into account the general identity \( \Gamma^i_{jk} = \partial_k (\ln \sqrt{g}) \) to reproduce \( \mathfrak{g} \), with \( \text{div}X_i = \Gamma_i \neq 0 \). Hence, the vector fields \( X_i \) are not Killing and their norm is not constant given by \( ||X_i|| = \sqrt{g_{ii}} \).

**3-sphere** This case is especially relevant for the inequality given in the introduction. We apply standard spherical coordinates of \( \mathbb{R}^4 \), given by

\[
\begin{align*}
x^1 &= R \cos \chi \\
x^2 &= R \sin \chi \cos \theta \\
x^3 &= R \sin \chi \sin \theta \cos \varphi \\
x^4 &= R \sin \chi \sin \theta \sin \varphi
\end{align*}
\]

with fixed curvature radius \( R > 0 \) and \( 0 \leq \chi, \theta \leq \pi, 0 \leq \varphi \leq 2\pi \). For the unit sphere with \( R = 1 \), the induced metric on \( S^3 \) is

\[
ds^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)
\]

We solved the corresponding Killing equation and se-
lected the following orthonormal Killing frame:

\[
X_1 = \sin \theta \cos \varphi \frac{\partial}{\partial \chi} \\
+ (\cot \chi \cos \theta \cos \varphi - \sin \varphi) \frac{\partial}{\partial \theta} \\
- (\cot \chi \csc \theta \sin \varphi + \cot \theta \cos \varphi) \frac{\partial}{\partial \varphi}
\]

(67)

\[
X_2 = \sin \theta \sin \varphi \frac{\partial}{\partial \chi} \\
+ (\cot \chi \cos \theta \sin \varphi + \cos \varphi) \frac{\partial}{\partial \theta} \\
+ (\cot \chi \csc \theta \cos \varphi - \cot \theta \sin \varphi) \frac{\partial}{\partial \varphi}
\]

(68)

\[
X_3 = \cos \theta \frac{\partial}{\partial \chi} - \cot \chi \sin \theta \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi}
\]

(69)

The corresponding dual frame is given by

\[
\theta^1 = \sin \theta \cos \varphi \, d\chi \\
+ (\sin \chi \cos \chi \cos \theta \cos \varphi - \sin^2 \chi \sin \varphi) \, d\theta \\
- (\sin \chi \cos \chi \sin \theta \sin \varphi + \sin^2 \chi \sin \theta \cos \cos \varphi) \, d\varphi
\]

(70)

\[
\theta^2 = \sin \theta \sin \varphi \, d\chi \\
+ (\sin \chi \cos \chi \cos \theta \sin \varphi + \sin^2 \chi \cos \varphi) \, d\theta \\
+ (\sin \chi \cos \chi \sin \theta \cos \varphi - \sin^2 \chi \sin \theta \sin \cos \varphi) \, d\varphi
\]

(71)

\[
\theta^3 = \cos \theta \, d\chi - \sin \chi \cos \chi \sin \theta \, d\theta + \sin^2 \chi \sin^2 \theta \, d\varphi
\]

(72)

For these frame forms the position functions \( Q^a \) can be obtained by \( 36 \). Although the corresponding integrals can all be expressed as closed form solutions, there are many trigonometric terms which make their explicit representation somewhat lengthy. Hence, we perform the computation generically for the case \( Q^3 \). According to \( 36 \), the position functions \( Q^k \) can be computed by the homotopy operator \( H \) which is explicitly given by

\[
Q^k(x) = x^i \int_0^1 \eta_i(tx) \, dt
\]

(73)

The functions \( \eta_i \) are given by the components in \( 70-72 \). Firstly, we annotate the following integral identities:

\[
\int_0^1 \cos(t\chi) \, dt = \frac{\sin \theta}{\theta}
\]

(74)

\[
\int_0^1 \sin(t\chi) \cos(t\chi) \sin(t\theta) \, dt \\
= \frac{1}{4} \left( \frac{\sin(\theta - 2\chi)}{\theta - 2\chi} - \frac{\sin(\theta + 2\chi)}{\theta + 2\chi} \right)
\]

(75)

and

\[
\int_0^1 \sin^2(t\chi) \sin^2(t\theta) \\
= \frac{1}{16} \left( \frac{\sin(2\chi + \theta)}{\chi + \theta} + \frac{\sin(2\chi - \theta)}{\chi - \theta} \right) \\
- \frac{1}{8} \left( \frac{\sin(2\chi)}{\chi} + \frac{\sin(2\theta)}{\theta} \right) + \frac{1}{4}
\]

(76)

After substitution into \( 73 \), we obtain the position operator

\[
Q^3 = \chi \frac{\sin \theta}{\theta} \\
- \theta \left( \frac{\sin(\theta - 2\chi)}{\theta - 2\chi} - \frac{\sin(\theta + 2\chi)}{\theta + 2\chi} \right) \\
+ \frac{\varphi}{16} \left( \frac{\sin(2(\chi + \theta))}{\chi + \theta} + \frac{\sin(2(\chi - \theta))}{\chi - \theta} \right) \\
- \frac{\varphi}{8} \left( \frac{\sin(2\chi)}{\chi} + \frac{\sin(2\theta)}{\theta} \right) + \frac{\varphi}{4}
\]

(77)

The exact part \( \theta^3 \) of the frame form \( \theta^3 \) is given by applying the exterior derivative according to

\[
\theta^3 = \frac{\partial Q^3}{\partial \chi} \, d\chi + \frac{\partial Q^3}{\partial \theta} \, d\theta + \frac{\partial Q^3}{\partial \varphi} \, d\varphi
\]

(78)

We hesitate to write out this expression because there are about 30 terms which might be more easily computed with an appropriate mathematical software tool like Maple or Mathematica. The other position operators \( Q^1 \) and \( Q^2 \) can be derived by the same procedure.

The exterior derivative of \( \theta^3 \) is given by the following formula:

\[
d\theta^3 = b_1^3 \, dx^1 \wedge dx^2 + b_2^3 \, dx^3 \wedge dx^1 + b_3^3 \, dx^1 \wedge dx^2
\]

(79)

with \((x^1, x^2, x^3) = (\chi, \theta, \varphi)\) and the functions

\[
b_1^3 = 2 \sin^2 \chi \sin \theta \cos \theta \\
b_2^3 = -2 \sin \chi \cos \chi \sin^2 \theta \\
b_3^3 = 2 \sin \chi \sin \theta
\]

The corresponding components of the anti-exact part \( \theta_a^3 \) can be computed by the homotopy operator applied to \( d\theta^3 \). According to the textbook formula of \( H \) on 2-forms, we obtain

\[
a_1^3(x) = x^3 \int_0^1 t b_2^3(tx) \, dt - x^2 \int_0^1 t b_3^3(tx) \, dt \\
a_2^3(x) = x^1 \int_0^1 t b_3^3(tx) \, dt - x^3 \int_0^1 t b_2^3(tx) \, dt \\
a_3^3(x) = x^2 \int_0^1 t b_1^3(tx) \, dt - x^1 \int_0^1 t b_2^3(tx) \, dt
\]

and analogously for the remaining cases \( \theta_a^1, \theta_a^2 \). Although the explicit form of the integrals is quite lengthy, every single term can be expressed by a closed form expression. Moreover, there are no coordinate singularities such that a Taylor series expansion is possible around every point. This property is especially useful for analyzing the deformation term on the right-hand side of the Heisenberg commutator in \( 39 \).

The only remaining space of positive and constant curvature for which this approach can be applied is \( S^7 \). The
corresponding Lie-algebra can be found in [8]. On the other hand, spaces of negative constant curvature imply that there are no nontrivial (nonzero) Killing fields. This makes the notion of momenta in spaces of negative curvature somewhat dubious.

V. SUMMARY

Our starting point has been the uncertainty relation [2]. Actually, this is a sort of quantum physical application of the famous Poincaré inequality, which allows one to obtain bounds on a function using bounds on its derivatives and the geometry of its domain of definition. In the quantum physics context it has been applied to establish a sharp lower bound of the momentum standard deviation for spatially compact position domains on manifolds. Originally, this inequality has been provided without an explicit representation of a momentum operator. This problem has been clarified by our discussion. The advantage of considering the momentum operator in terms of orthonormal Killing frames is twofold: on the one hand it ensures the momentum operator to be Hermitean on every compact sub-manifold (or geodesic ball) on $M$. On the other hand, the corresponding Killing trajectories are distinguished since their associated isometries are ”translations” - that is, they are geodesics of the manifold. The latter property is especially appealing because it takes into account the foundations of general relativity which is based on the notion of geodesics.

The functions of the position operator are considered as 0-forms whose exterior derivative is identical to the exact part of the dual frame forms on the manifold. By this choice the traditional Heisenberg commutation relations are recovered if the manifold is locally flat. For intrinsically curved manifolds the traditional commutation relations have to be extended by an appropriate ”deformation” term by which the unavoidable path dependency of the parallel transport is characterized. This term is identical to the anti-exact part of the dual frame forms. This concept is distinguished because the decomposition of a dual frame into the exact part and the anti-exact part is unique.

One obvious application is the Friedmann-Lemaître-Robertson-Walker-cosmology, where each fixed spacelike time-slice corresponds to a 3-dimensional space of constant curvature. If we think of flat space just as a special realization of a positively curved manifold with infinite radius of curvature, then there can only be the three remaining possibilities $S^1$, $S^3$ and $S^7$ in nature having the Killing property.

References

[1] Dabrowski M. P. and Wagner F., Extended Uncertainty Principle for Rindler and Cosmological Horizons, Eur. Phys. J. C 79 (2019) 716-723.
[2] Dabrowski M. P. and Wagner F., Asymptotic generalized extended uncertainty principle, Eur. Phys. J. C 80 (2020) 676.
[3] Schürmann T., Uncertainty principle on 3-dimensional manifolds of constant curvature, Found. Phys. 48 (2018) 716.
[4] Schürmann T. and Hoffmann I., A closer look at the uncertainty relation of position and momentum, Found. Phys. 39 (2009) 958.
[5] Schürmann T., On the uncertainty principle in Rindler and Friedmann spacetimes, Eur. Phys. J. C 80, (2020) 141.
[6] Petruzzello L. and Wagner F., Gravitationally induced uncertainty relations in curved backgrounds, arXiv:2101.06502 Quantum Physics (2021)
[7] Śniatycki J., Geometric quantization and quantum mechanics, Springer Publications, New York 1980.
[8] D’Atri J. E. and Nickerson H. K., The existence of special orthonormal frames, J. Differential Geom. 2 (1968) 393-409.
[9] Edelen D., Applied exterior calculus, Dover Publications, New York 2005.
[10] Bishop R. L. and Goldberg S. I., Tensor analysis on manifolds, Dover Publications, New York 1980.
[11] Wolf J. A., Spaces of constant curvature, McGraw-Hill, New York, 1967.
[12] Helgason S., Differential geometry and symmetric spaces, Academic Press, New York, 1962.
[13] Golovnev A. V. and Prokhorov L. V., Uncertainty relations in curved spaces, J. Phys. A: Math. Gen. 37 (2004) 2765-2775.
[14] Trifonov D. A., Position uncertainty measures on the sphere, Proceedings of the Fifth International Conference on Geometry, Integrability and Quantization 755 (Softex, Sofia, Bulgaria 2004) 211-224.
[15] Schürmann T., The uncertainty principle in terms of isoperimetric inequalities, arXiv:1606.07936 Quantum Physics (2016).
[16] Liboff R. L., Nebenzahl I., and Fleischmann H. H. On the Radial Momentum Operator, American Journal of Physics 41 (1973) 976.
[17] Liboff R. L. Introductory Quantum Mechanics, Addison-Wesley 1998.
[18] Paz G. On the connection between the radial momentum operator and the Hamiltonian in n dimensions, Eur. J. Phys. 22 (2001) 337–341.
[19] Liu Q. H. A self-adjoint decomposition of the radial momentum operator, International Journal of Geometric Methods in Modern Physics 12 (2015) 1550028.
[20] Jahangiri L. and Panahi H., Quantum mechanical treatment of a constrained particle on two dimensional sphere, Annals of Physics 375 (2016) 407-413.
[21] Liu Q. H., Tang L. H., and Xun D. M., Geometric momentum: The proper momentum for a free particle on a two-dimensional sphere, Phys. Rev. A 84 (2011) 042101.
[22] Liu Q. H., Tong C. L. and Lai M. M. Constraint-induced mean curvature dependence of Cartesian momentum operators, J. Phys. A: Math. Theor. 40 (2007) 4161.

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APPENDIX

Remark. (cf. Proposition 2.) We briefly reconsider the starting point in [6]. Let \( \psi_n, n = 1, 2, \ldots \) be the countable point-spectrum of \([13]\) and consider a general solution given by

\[
\psi = \sum_n a_n \psi_n \tag{80}
\]

with certain given complex valued coefficients \( a_n \in \mathbb{C} \). Indeed, from this expression it is hard to see that there is a real number \( \alpha \in \mathbb{R} \), such that the wave function can be decomposed according to

\[
\psi = (1 + i\alpha) \text{Re} \psi \tag{81}
\]

To unravel this mystery, we consider the decomposition of \( a_n \) into real and imaginary parts \( b_n, c_n \in \mathbb{R} \) given by

\[
a_n = b_n + ic_n \tag{82}
\]

Likewise, we introduce the decomposition of \( \psi_n \) into its real and imaginary parts \( G_n, H_n \in \mathbb{R} \) given by

\[
\psi_n = G_n + iH_n \tag{83}
\]

with \( G_n = \text{Re} \psi_n \) and \( H_n = \text{Im} \psi_n \), for \( n = 1, 2, \ldots \). In this notation, we can rewrite the wave function \( \psi \) in terms of

\[
\psi = g + ih \tag{84}
\]

with

\[
g = \sum_n (b_n G_n - c_n H_n) \tag{85}
\]

\[
h = \sum_n (c_n G_n + b_n H_n) \tag{86}
\]

Since \( \psi \) satisfies the Dirichlet eigenvalue problem, there exists an \( \alpha \in \mathbb{R} \), such that

\[
h = \alpha g. \tag{87}
\]

The corresponding interrelation between \( b_n \) and \( c_n \) can be obtained by taking into account that there also exists a countable set of real numbers \( \alpha_n \in \mathbb{R} \), such that

\[
H_n = \alpha_n G_n \tag{88}
\]

\( n = 1, 2, \ldots \). By substitution of this expression into \((85)\) and \((86)\) and subsequently equating \( g \) and \( h \) according to \((87)\), we obtain the (hidden) dependency between \( b_n \) and \( c_n \) given by

\[
c_n = \frac{\alpha - \alpha_n}{1 + \alpha \alpha_n} b_n \tag{89}
\]

A verification of \((89)\) can be obtained by re-substitution into the original expression \((84)\), such that we get

\[
\psi = (1 + i\alpha) \sum_n \frac{1 + \alpha_n^2}{1 + \alpha \alpha_n} b_n \text{Re} \psi_n
\]

\[
= (1 + i\alpha) \text{Re} \psi \tag{90}
\]

This clarifies the decomposition \((81)\) and provides the necessary assumption to apply the proof of Proposition 2.