\textbf{\textit{η}-RICCI SOLITONS ON KENMOTSU MANIFOLD WITH GENERALIZED SYMMETRIC METRIC CONNECTION}

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\textbf{Abstract.} The objective of the present paper is to study the \textit{η}-Ricci solitons on Kenmotsu manifold with generalized symmetric metric connection of type \((\alpha, \beta)\). There are discussed Ricci and \textit{η}-Ricci solitons with generalized symmetric metric connection of type \((\alpha, \beta)\) satisfying the conditions \(\bar{R} \bar{S} = 0, \bar{S} \bar{R} = 0, W_2 \bar{S} = 0, \text{ and } \bar{S} W_2 = 0\). Finally, we construct an example of Kenmotsu manifold with generalized symmetric metric connection of type \((\alpha, \beta)\) admitting \textit{η}-Ricci solitons.

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1. Introduction

A linear connection \(\nabla\) is said to be generalized symmetric connection if its torsion tensor \(T\) is of the form

\begin{equation}
T(X, Y) = \alpha\{u(Y)X - u(X)Y\} + \beta\{u(Y)\varphi X - u(X)\varphi Y\},
\end{equation}

for any vector fields \(X, Y\) on a manifold, where \(\alpha\) and \(\beta\) are smooth functions. \(\varphi\) is a tensor of type \((1, 1)\) and \(u\) is a \(1\)-form associated with a non-vanishing smooth non-null unit vector field \(\xi\). Moreover, the connection \(\nabla\) is said to be a generalized symmetric metric connection if there is a Riemannian metric \(g\) in \(M\) such that \(\nabla g = 0\), otherwise it is non-metric.

In the equation (1.1), if \(\alpha = 0\) \((\beta = 0)\), then the generalized symmetric connection is called \(\beta\)-quarter-symmetric connection \((\alpha\)-semi-symmetric connection\), respectively. Moreover, if we choose \((\alpha, \beta) = (1, 0)\) and \((\alpha, \beta) = (0, 1)\), then the generalized symmetric connection is reduced to a semi-symmetric connection and quarter-symmetric connection, respectively. Hence a generalized symmetric connections can be viewed as a generalization of semi-symmetric connection and quarter-symmetric connection. This two connection are important for both the geometry study and applications to physics. In [12], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors (see [1], [9], [10], [24], [26]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by S.Golab [11]. In [23], Sharifuddin and Hussain defined a semi-symmetric metric connection in an almost contact manifold, by setting

\begin{equation}
T(X, Y) = \eta(Y)X - \eta(X)Y.
\end{equation}
In [13], [25] and [19] the authors studied the semi-symmetric metric connection and semi-symmetric non-metric connection in a Kenmotsu manifold, respectively.

In the present paper, we defined new connection for Kenmotsu manifold, generalized symmetric metric connection. This connection is the generalized form of semi-symmetric metric connection and quarter-symmetric metric connection.

On the other hand, a Ricci soliton is a natural generalization of an Einstein metric. In 1982, R. S. Hamilton [14] said that the Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow:

\[
\frac{\partial g}{\partial t} = -2\text{Ric}(g).
\]

**Definition 1.1.** A Ricci soliton \((g, V, \lambda)\) on a Riemannian manifold is defined by

\[
\mathcal{L}_V g + 2S + 2\lambda = 0,
\]

where \(S\) is the Ricci tensor, \(\mathcal{L}_V\) is the Lie derivative along the vector field \(V\) on \(M\) and \(\lambda\) is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as \(\lambda < 0, \lambda = 0\) and \(\lambda > 0\), respectively.

In 1925, H. Levy [16] in Theorem: 4, proved that a second order parallel symmetric non-singular tensor in real space forms is proportional the metric tensor. Later, R. Sharma [22] initiated the study of Ricci solitons in contact Riemannian geometry. After that, Tripathi [28], Nagaraja et. al. [17] and others like C. S. Bagewadi et. al. [4] extensively studied Ricci solitons in almost contact metric manifolds. In 2009, J. T. Cho and M. Kimura [6] introduced the notion of \(\eta\)-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting \(\eta\)-Ricci solitons. \(\eta\)-Ricci solitons in almost paracontact metric manifolds have been studied by A. M. Blaga et. al. [2]. A. M. Blaga and various others authors also have been studied \(\eta\)-Ricci solitons in manifolds with different structures (see [3], [20]). It is natural and interesting to study \(\eta\)-Ricci solitons in almost contact metric manifolds with this new connection.

Therefore, motivated by the above studies, in this paper we study the \(\eta\)-Ricci solitons in a Kenmotsu manifold with respect to a generalized symmetric metric connection. We shall consider \(\eta\)-Ricci solitons in the almost contact geometry, precisely, on an Kenmotsu manifold with generalized symmetric metric connection which satisfies certain curvature properties: \(\bar{R}.\bar{S} = 0, \bar{S}.\bar{R} = 0, \bar{W}^2 = 0\) and \(\bar{S}.\bar{W} = 0\) respectively.

2. Preliminaries

A differentiable manifold of dimension \(n = 2m + 1\) is called almost contact metric manifold [5], if it admit a \((1,1)\) tensor field \(\phi\), a contravariant vector field \(\xi\), a 1-form \(\eta\) and Riemannian metric \(g\) which satify

\[
\phi\xi = 0,
\]

\[
\eta(\phi X) = 0
\]

\[
\eta(\xi) = 1,
\]

\[
\phi^2(X) = -X + \eta(X)\xi,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

\[
g(X, \xi) = \eta(X),
\]

In [13], [25] and [19] the authors studied the semi-symmetric metric connection and semi-symmetric non-metric connection in a Kenmotsu manifold, respectively.
for all vector fields $X$, $Y$ on $M$. If we write $g(X, φY) = Φ(X,Y)$, then the tensor field $φ$ is a anti-symmetric $(0,2)$ tensor field $[5]$. If an almost contact metric manifold satisfies

\begin{equation}
(∇_X φ)Y = g(φX,Y)ξ - η(Y)φX,
\end{equation}
\begin{equation}
∇_X ξ = X - η(X)ξ,
\end{equation}
then $M$ is called a Kenmotsu manifold, where $∇$ is the Levi-Civita connection of $g$ $[18]$.

In Kenmotsu manifolds the following relations hold $[18]$:

\begin{equation}
(∇_X η)Y = g(φX, φY)
\end{equation}
\begin{equation}
g(R(X,Y)Z, ξ) = η(R(X,Y)Z) = g(X,Z)η(Y) - g(Y,Z)η(X),
\end{equation}
\begin{equation}
R(ξ, X)Y = η(Y)X - g(X,Y)ξ,
\end{equation}
\begin{equation}
R(X,Y)ξ = η(X)Y - η(Y)X,
\end{equation}
\begin{equation}
R(ξ, X)ξ = X - η(X)ξ,
\end{equation}
\begin{equation}
S(X, ξ) = -(n-1)η(X),
\end{equation}
\begin{equation}
S(φX, φY) = S(X,Y) + (n-1)η(X)η(Y)
\end{equation}
for any vector fields $X$, $Y$ and $Z$, where $R$ and $S$ are the the curvature and Ricci tensors of $M$, respectively.

A Kenmotsu manifold $M$ is said to be generalized $η$ Einstein if its Ricci tensor $S$ is of the form

\begin{equation}
S(X,Y) = ag(X,Y) + bη(X)η(Y) + cg(φX,Y),
\end{equation}
for any $X, Y ∈ \Gamma(TM)$, where $a$, $b$ and $c$ are scalar functions such that $b \neq 0$ and $c \neq 0$. If $c = 0$ then $M$ is called $η$ Einstein manifold.

### 3. Generalized Symmetric Metric Connection in a Kenmotsu Manifold

Let $\overline{∇}$ be a linear connection and $∇$ be a Levi-Civita connection of an almost contact metric manifold $M$ such that

\begin{equation}
\overline{∇}_XY = ∇_XY + H(X,Y),
\end{equation}
for any vector field $X$ and $Y$. Where $H$ is a tensor of type $(1,2)$. For $\overline{∇}$ to be a generalized symmetric metric connection of $∇$, we have

\begin{equation}
H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)],
\end{equation}
where $T$ is the torsion tensor of $\overline{∇}$ and

\begin{equation}
g(T'(X,Y), Z) = g(T(Z,X), Y).
\end{equation}
From $[14]$ and $[3.3]$ we get

\begin{equation}
T'(X,Y) = α\{η(X)Y - g(X,Y)ξ\} + β\{-η(X)φY - g(φX,Y)ξ\}.
\end{equation}
Using $[1.1]$, $[3.2]$ and $[3.4]$ we obtain

\begin{equation}
H(X,Y) = α\{η(Y)X - g(X,Y)ξ\} + β\{-η(X)φY\}.
\end{equation}
Corollary 3.1. For a Kenmotsu manifold, generalized symmetric metric connection $\nabla$ is given by

$$\nabla_X Y = \nabla_X Y + \alpha \{\eta(Y)X - g(X,Y)\xi\} - \beta \eta(X)\phi Y.$$  

(3.6)

If we choose $(\alpha, \beta) = (1,0)$ and $(\alpha, \beta) = (0,1)$, generalized metric connection is reduced a semi-symmetric metric connection and quarter-symmetric metric connection as follows:

$$\nabla_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi,$$  

(3.7)

$$\nabla_X Y = \nabla_X Y - \eta(X)\phi Y.$$  

(3.8)

From (3.6), we have the following proposition

**Proposition 3.2.** Let $M$ be a Kenmotsu manifold with generalized metric connection. We have the following relations:

$$\nabla_X \phi Y = (\alpha + 1)\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$  

(3.9)

$$\nabla_X \xi = (\alpha + 1)\{X - \eta(X)\xi\},$$  

(3.10)

$$\nabla_X \eta Y = (\alpha + 1)\{g(X, Y) - \eta(Y)\eta(X)\},$$  

(3.11)

for any $X, Y, Z \in \Gamma(TM)$.

4. **Curvature Tensor on Kenmotsu Manifold with Generalized Symmetric Metric Connection**

Let $M$ be an $n$-dimensional Kenmotsu manifold. The curvature tensor $\bar{R}$ of the generalized metric connection $\nabla$ on $M$ is defined by

$$\bar{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$  

(4.1)

Using proposition 3.2 from (3.6) and (4.1), we have

$$\bar{R}(X, Y)Z = R(X, Y)Z + \{(-\alpha^2 - 2\alpha)g(Y, Z) + (\alpha^2 + \alpha)\eta(Y)\eta(Z)\}X$$

$$+ \{(\alpha^2 + 2\alpha)g(X, Z) + (-\alpha^2 - \alpha)\eta(X)\eta(Z)\}Y$$

$$+ \{(\alpha^2 + \alpha)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]$$

$$+ (\beta + \alpha\beta)[g(X, \phi Z)\eta(Y) - g(Y, \phi X)\eta(X)]\}\xi$$

$$+ (\beta + \alpha\beta)\eta(Y)\eta(Z)\phi X - (\beta + \alpha\beta)\eta(X)\eta(Z)\phi Y$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

is the curvature tensor with respect to the Levi-Civita connection $\nabla$. Using (4.2) and the first Bianchi identity we have

$$\bar{R}(X, Y)Z = \tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y$$

$$= 2(\beta + \alpha\beta)\{\eta(X)g(\phi Y, Z) + \eta(Y)g(X, \phi Z) + \eta(Z)g(Y, \phi X)\}.$$  

(4.4)

Hence we have the following proposition

**Proposition 4.1.** Let $M$ be an $n$-dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. If $(\alpha, \beta) = (-1, \beta)$ or $(\alpha, \beta) = (\alpha, 0)$ then the first Bianchi identity of the generalized symmetric metric connection $\nabla$ on $M$ is provided.
Using (2.10), (2.11), (2.12), (2.13) and (2.2) we give the following proposition:

**Proposition 4.2.** Let $M$ be an $n-$ dimensional Kenmotsu manifold with generalized symmetric metric connection of type $(\alpha, \beta)$. Then we have the following equations:

\[
(4.5) \quad \bar{R}(X, Y)\xi = (\alpha + 1)\{\eta(X)Y - \eta(Y)X + \beta[\eta(Y)\phi X - \eta(X)\phi Y]\}
\]

\[
(4.6) \quad \bar{R}(\xi, X)Y = (\alpha + 1)\{\eta(Y)X - g(X, Y)\xi + \beta[\eta(Y)\phi X - g(X, \phi Y)\xi]\},
\]

\[
(4.7) \quad \bar{R}(\xi, Y)\xi = (\alpha + 1)\{Y - \eta(Y)\xi - \beta\phi Y\},
\]

\[
(4.8) \quad \eta(\bar{R}(X, Y)Z = (\alpha + 1)\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z) + \beta[\eta(Y)g(X, \phi Z) - \eta(X)g(Y, \phi Z)]\}
\]

for any $X, Y, Z \in \Gamma(TM)$.

**Example 4.3.** We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let $E_1, E_2, E_3$ be a linearly independent global frame on $M$ given by

\[
(4.9) \quad E_1 = x \frac{\partial}{\partial z}, \quad E_2 = x \frac{\partial}{\partial y}, \quad E_3 = -x \frac{\partial}{\partial x}.
\]

Let $g$ be the Riemannian metric defined by

\[
g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,
\]

Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_3)$, for any $U \in TM$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_2, \phi E_2 = -E_1$ and $\phi E_3 = 0$. Then, using the linearity of $\phi$ and $g$ we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U) E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U) \eta(W)$ for any $U, W \in TM$. Thus for $E_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

Let $\nabla$ be the Levi-Civita connection with respect to the Riemannian metric $g$. Then we have

\[
(4.10) \quad [E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2,
\]

Using Koszul formula for the Riemannian metric $g$, we can easily calculate

\[
(4.11) \quad \nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1,
\]

\[
\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = 0,
\]

\[
\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.
\]

From the above relations, it can be easily seen that

\[
(\nabla_X \phi) Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad \nabla_X \xi = X - \eta(X)\xi, \quad \nabla_X \eta = X - \eta(X)\xi,
\]

for all $E_3 = \xi$. Thus the manifold $M$ is a Kenmotsu manifold with the structure $(\phi, \xi, \eta, g)$, for $\xi = E_3$. Hence the manifold $M$ under consideration is a Kenmotsu manifold of dimension three.
5. Ricci and \(\eta\)-Ricci solitons on \((M, \phi, \xi, \eta, g, \)\)

Let \((M, \phi, \xi, \eta, g,\)\) be an almost contact metric manifold. Consider the equation

\[(5.1) \quad \mathcal{L}_\xi g + 2\bar{S} + 2\lambda g + 2\mu \eta \otimes \eta = 0,\]

where \(\mathcal{L}_\xi\) is the Lie derivative operator along the vector field \(\xi\), \(\bar{S}\) is the Ricci curvature tensor field with respect to the generalized symmetric metric connection of the metric \(g\), and \(\lambda\) and \(\mu\) are real constants. Writing \(\mathcal{L}_\xi\) in terms of the generalized symmetric metric connection \(\nabla\), we obtain:

\[(5.2) \quad 2\bar{S}(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y) - 2\mu \eta(X) \eta(Y),\]

for any \(X,Y \in \chi(M)\).

The data \((g, \xi, \lambda, \mu)\) which satisfy the equation \[(5.1)\] is said to be an \(\eta\)-Ricci soliton on \(M\) \[10\]. In particular if \(\mu = 0\) then \((g, \xi, \lambda)\) is called Ricci soliton \[6\] and it is called shrinking, steady or expanding, according as \(\lambda\) is negative, zero or positive respectively \[6\].

Here is an example of \(\eta\)-Ricci soliton on Kenmotsu manifold with generalized symmetric metric connection.

**Example 5.1.** Let \(M(\phi, \xi, \eta, g)\) be the Kenmotsu manifold considered in example 4.3.

Let \(\nabla\) be a generalized symmetric metric connection, we obtain: Using the above relations, we can calculate the non-vanishing components of the curvature tensor as follows:

\[(5.3) \quad R(E_1, E_2)E_1 = E_2, \quad R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_1 = E_3 \quad R(E_1, E_3)E_3 = -E_2\]

From the equations \[(5.3)\] we can easily calculate the non-vanishing components of the ricci tensor as follows:

\[(5.4) \quad S(E_1, E_1) = -2, \quad S(E_2, E_2) = -2, \quad S(E_3, E_3) = -2\]

Now, we can make similar calculations for generalized metric connection. Using \[(3.6)\] in the above equations, we get

\[(5.5) \quad \nabla_{E_1} E_1 = -(1 + \alpha)E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = (1 + \alpha)E_1, \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -(1 + \alpha)E_3, \quad \nabla_{E_2} E_3 = \alpha E_2, \quad \nabla_{E_3} E_1 = -\beta E_2, \quad \nabla_{E_3} E_2 = \beta E_1, \quad \nabla_{E_3} E_3 = 0.\]

From \[(5.5)\], we can calculate the non-vanishing components of curvature tensor with respect to generalized metric connection as follows:

\[(5.6) \quad \nabla(R(E_1, E_2)E_1) = (1 + \alpha)^2 E_2, \quad \nabla(R(E_1, E_2)E_2) = -(1 + \alpha)^2 E_1, \quad \nabla(R(E_1, E_3)E_1) = (1 + \alpha)E_3, \quad \nabla(R(E_1, E_3)E_3) = (1 + \alpha)(\beta E_2 - E_1), \quad \nabla(R(E_2, E_3)E_2) = -(1 + \alpha)(-\beta E_1 + E_2), \quad \nabla(R(E_3, E_2)E_1) = -(1 + \alpha)\beta E_3, \quad \nabla(R(E_3, E_1)E_2) = (1 + \alpha)\beta E_3.\]
From \( (5.6) \), the non-vanishing components of the ricci tensor as follows:
\[
\overline{\mathbf{S}}(E_1, E_1) = -(1 + \alpha)(2 + \alpha), \quad \overline{\mathbf{S}}(E_2, E_2) = -(1 + \alpha)(2 + \alpha), \quad \overline{\mathbf{S}}(E_3, E_3) = -2(1 + \alpha).
\]

From \((5.2)\) and \((5.5)\) we get
\[
2(1 + \alpha)[g(e_i, e_i) - \eta(e_i)\eta(e_i)] + 2\beta h(e_i, e_i) + 2\gamma g(e_i, e_i) + 2\mu \eta(e_i)\eta(e_i) = 0
\]
for all \(i \in \{1, 2, 3\}\), and we have \(\lambda = (1 + \alpha)^2\) (i.e. \(\lambda > 0\)) and \(\mu = 1 - \alpha^2\), the data \((g, \xi, \lambda, \mu)\) is an \(\eta\)-Ricci soliton on \((\mathcal{M}, \phi, \xi, \eta, g)\). If \(\alpha = -1\) which is steady and if \(\alpha \neq -1\) which is expanding.

\[\text{6. Parallel symmetric second order tensors and } \eta\text{-Ricci solitons in Kenmotsu manifolds}\]

An important geometrical object in studying Ricci solitons is well known to be a symmetric \((0, 2)\)-tensor field which is parallel with respect to the generalized symmetric metric connection.

Now, let fix \(h\) a symmetric tensor field of \((0, 2)\)-type which we suppose to be parallel with respect to generalized symmetric metric connection \(\nabla\) that is \(\nabla h = 0\).

Applying Ricci identity \[7\]
\[
\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; Z, W) = 0,
\]
we obtain the relation
\[
h(\overline{R}(X, Y)Z, W) + h(Z, \overline{R}(X, Y)W) = 0.
\]
Replacing \(Z = W = \xi\) in \((6.2)\) and by using \((4.5)\) and by the symmetry of \(h\) follows
\[
h(\overline{R}(X, Y)\xi, \xi) = 0 \text{ for any } X, Y \in \chi(\mathcal{M}) \text{ and}
\]
\[
(\alpha + 1)\eta h(Y, \xi) - (\alpha + 1)\eta(Y)h(X, \xi) + (\alpha + 1)\eta(Y)h(\phi X, \xi) - (\alpha + 1)\eta(Y)h(\xi, \phi X) - \beta h(\phi X, \xi) - \beta h(\phi Y, \xi) = 0
\]
Putting \(X = \xi\) in \((6.3)\) and by the virtue of \((2.4)\), we obtain
\[
2(\alpha + 1)[h(Y, \xi) - \eta(Y)h(\xi, \xi)] - 2\beta h(\phi Y, \xi) = 0.
\]
or
\[
2(\alpha + 1)[h(Y, \xi) - \eta(Y)h(\xi, \xi)] - 2\beta h(\phi Y, \xi) = 0.
\]
Suppose \((\alpha + 1) \neq 0, \beta = 0\) it results
\[
h(Y, \xi) = \eta(Y)h(Y, \xi) = 0,
\]
for any \(Y \in \chi(\mathcal{M})\), equivalent to
\[
h(Y, \xi) - g(Y, \xi)h(\xi, \xi) = 0,
\]
for any \(Y \in \chi(\mathcal{M})\). Differentiating the equation \((6.7)\) covariantly with respect to the vector field \(X \in \chi(\mathcal{M})\), we obtain
\[
h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) = h(\xi, \xi)[g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi)]
\]
Using \((4.5)\) in \((6.8)\), we obtain
\[
h(X, Y) = h(\xi, \xi)g(X, Y).
\]
for any \(X, Y \in \chi(M)\). The above equation gives the conclusion:

**Theorem 6.1.** Let \((M, \phi, \xi, \eta, g, \cdot\)
be a Kenmotsu manifold with generalized symmetric metric connection also with non-vanishing \(\xi\)-sectional curvature and endowed with a tensor field of type \((0,2)\) which is symmetric and \(\phi\)-skew-symmetric. If \(h\) is parallel with respect to \(\tilde{\nabla}\), then it is a constant multiple of the metric tensor \(g\).

On a Kenmotsu manifold with generalized symmetric metric connection using equation (3.10) and \(\mathcal{L}_\xi g = 2(g - \eta \otimes \eta)\), the equation (5.2) becomes:

\[
\tilde{S}(X, Y) = -(\lambda + \alpha + 1)g(X, Y) + (\alpha + 1 - \mu)\eta(X)\eta(Y).
\]

In particular, \(X = \xi\), we obtain

\[
\tilde{S}(X, \xi) = -(\lambda + \mu)\eta(X).
\]

In this case, the Ricci operator \(\tilde{Q}\) defined by \(g(\tilde{Q}X, Y) = \tilde{S}(X, Y)\) has the expression

\[
\tilde{Q}X = -(\lambda + \alpha + 1)X + (\alpha + 1 - \mu)\eta(X)\eta(X)\xi.
\]

Remark that on a Kenmotsu manifold with generalized symmetric metric connection, the existence of an \(\eta\)-Ricci soliton implies that the characteristic vector field \(\xi\) is an eigenvector of Ricci operator corresponding to the eigenvalue \(- (\lambda + \mu)\).

Now we shall apply the previous results on \(\eta\)-Ricci solitons.

**Theorem 6.2.** Let \((M, \phi, \xi, \eta, g)\) be a Kenmotsu manifold with generalized symmetric metric connection. Assume that the symmetric \((0,2)\)-tensor filed \(h = \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta\) is parallel with respect to the generalized symmetric metric connection associated to \(g\). Then \((g, \xi, -\frac{1}{2}h(\xi, \xi), \mu)\) yields an \(\eta\)-Ricci soliton.

**Proof.** Now, we can calculate

\[
h(\xi, \xi) = \mathcal{L}_\xi g(\xi, \xi) + 2\tilde{S}(\xi, \xi) + 2\mu\eta(\xi)\eta(\xi) = -2\lambda,
\]

so \(\lambda = -\frac{1}{2}h(\xi, \xi)\). From (6.10) we conclude that \(h(X, Y) = -2\lambda g(X, Y)\), for any \(X, Y \in \chi(M)\). Therefore \(\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta = -2\lambda g\). \(\Box\)

For \(\mu = 0\) follows \(\mathcal{L}_\xi g + 2S - S(\xi, \xi)g = 0\) and this gives

**Corollary 6.3.** On a Kenmotsu manifold \((M, \phi, \xi, \eta, g)\) with generalized symmetric metric connection with property that the symmetric \((0,2)\)-tensor field \(h = \mathcal{L}_\xi g + 2S\) is parallel with respect to generalized symmetric metric connection associated to \(g\), the relation (5.1), for \(\mu = 0\), defines a Ricci soliton.

Conversely, we shall study the consequences of the existence of \(\eta\)-Ricci solitons on a Kenmotsu manifold with generalized symmetric metric connection. From (6.10) we give the conclusion:

**Theorem 6.4.** If equation (6.10) define an \(\eta\)-Ricci soliton on a Kenmotsu manifold \((M, \phi, \xi, \eta, g)\) with generalized symmetric metric connection, then \((M, g)\) is quasi-Einstein.

Recall that the manifold is called quasi-Einstein \([8]\) if the Ricci curvature tensor field \(S\) is a linear combination (with real scalars \(\lambda\) and \(\mu\) respectively, with \(\mu \neq 0\)) of \(g\) and the tensor product of a non-zero 1-from \(\eta\) satisfying \(\eta = g(X, \xi)\), for \(\xi\) a unit vector field and respectively, Einstein \([8]\) if \(S\) is collinear with \(g\).
Theorem 6.5. If \((\phi, \xi, \eta, g)\) is a Kenmotsu structure with generalized symmetric metric connection on \(M\) and \((4.9)\) defines an \(\eta\)-Ricci soliton on \(M\), then

\begin{enumerate}
  \item \(Q \circ \phi = \phi \circ Q\)
  \item \(Q\) and \(S\) are parallel along \(\xi\).
\end{enumerate}

Proof. The first statement follows from a direct computation and for the second one, note that

\begin{align*}
(\bar{\nabla}_\xi Q)X &= \bar{\nabla}_\xi QX - Q(\bar{\nabla}_\xi X), \\
(\bar{\nabla}_\xi S)(X, Y) &= \xi(S(X, Y)) - S(\bar{\nabla}_\xi X, Y) - S(X, \bar{\nabla}_\xi Y).
\end{align*}

Replacing \(Q\) and \(S\) from (6.12) and (6.11) we get the conclusion. \(\square\)

A particular case arise when the manifold is \(\phi\)-Ricci symmetric, which means that \(\phi^2 \circ \nabla Q = 0\), that fact stated in the next theorem.

Theorem 6.6. Let \((M, \phi, \xi, \eta, g)\) be a Kenmotsu manifold with generalized symmetric metric connection. If \(M\) is \(\phi\)-Ricci symmetric and \((4.9)\) defines an \(\eta\)-Ricci soliton on \(M\), then \(\mu = 1\) and \((M, g)\) is Einstein manifold [8].

Proof. Replacing \(Q\) from (6.12) in (6.14) and applying \(\phi^2\) we obtain

\begin{align*}
(\alpha + 1 - \mu)\eta(Y)[X - \eta(X)\xi] &= 0,
\end{align*}

for any \(X, Y \in \chi(M)\). Follows \(\mu = \alpha + 1\) and \(S = -(\lambda + \alpha + 1)g\). \(\square\)

Remark 6.7. In particular, the existence of an \(\eta\)-Ricci soliton on a Kenmotsu manifold with generalized symmetric metric connection which is Ricci symmetric (i.e. \(\bar{\nabla}S = 0\)) implies that \(M\) is Einstein manifold. The class of Ricci symmetric manifold represents an extension of class of Einstein manifold to which belong also the locally symmetric manifold (i.e. satisfying \(\bar{\nabla}R = 0\)). The condition \(\bar{\nabla}S = 0\) implies \(\bar{\nabla}S = 0\) and the manifolds satisfying this condition are called Ricci semi-symmetric [7].

In what follows we shall consider \(\eta\)-Ricci solitons requiring for the curvature to satisfy \(\bar{R}(\xi, X, S) = 0\), \(\bar{S}(\bar{R}(\xi, X)) = 0\), \(W_2(\xi, X, S) = 0\) and \(\bar{S}(X, W_2(\xi, X)) = 0\) respectively, where the \(W_2\)-curvature tensor field is the curvature tensor introduced by G. P. Pokhariyal and R. S. Mishra in [21].

(6.17) \[W_2(X, Y)Z = R(X, Y)Z + \frac{1}{\dim M - 1}[g(X, Z)QY - g(Y, Z)QX].\]

7. \(\eta\)-Ricci solitons on a Kenmotsu manifold with generalized symmetric metric connection satisfying \(\bar{R}(\xi, X, S) = 0\)

Now we consider a Kenmotsu manifold with with generalized symmetric metric connection \(\bar{\nabla}\) satisfying the condition

\begin{equation}
(7.1) \quad \bar{S}(\bar{R}(\xi, X)Y, Z) + \bar{S}(Y, \bar{R}(\xi, X)Z) = 0,
\end{equation}

for any \(X, Y \in \chi(M)\).

Replacing the expression of \(\bar{S}\) from (6.10) and from the symmetries of \(\bar{R}\) we get

\begin{equation}
(7.2) \quad (\alpha + 1)(\alpha + 1 - \mu)[\eta(Y)g(X, Z) + \eta(Z)g(X, Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0,
\end{equation}

respectively, where the \(W_2\)-curvature tensor field is the curvature tensor introduced by G. P. Pokhariyal and R. S. Mishra in [21].
for any $X, Y \in \chi(M)$.

For $Z = \xi$ we have

$$\tag{8.2} (\alpha + 1)(\alpha + 1 - \mu)g(\phi X, \phi Y) = 0,$$

for any $X, Y \in \chi(M)$.

Hence we can state the following theorem:

**Theorem 7.1.** If a Kenmotsu manifold with a generalized symmetric metric connection $\nabla$, $(g, \xi, \lambda, \mu)$ is an $\eta$-Ricci soliton on $M$ and satisfies $\bar{\nabla}(\xi, X), \bar{\nabla} = 0$, then the manifold is an $\eta$-Einstein manifold.

For $\mu = 0$, we deduce:

**Corollary 7.2.** On a Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{\nabla}(\xi, X), \bar{\nabla} = 0$, there is no $\eta$-Ricci soliton with the potential vector field $\xi$.

8. $\eta$-Ricci solitons on Kenmotsu manifold with generalized symmetric metric connection satisfying $\bar{\nabla} \bar{R}(\xi, X) = 0$

In this section we consider Kenmotsu manifold with a a generalized symmetric metric connection $\bar{S}$ satisfying the condition

$$\tag{8.1} \bar{S}(X, \bar{R}(Y, Z)W)\xi - \bar{S}(\xi, \bar{R}(Y, Z)W)X + \bar{S}(X, Y)\bar{R}(\xi, Z)W - \bar{S}(\xi, Y)\bar{R}(X, Z)W + \bar{S}(X, Z)\bar{R}(Y, \xi)W - \bar{S}(\xi, Z)\bar{R}(Y, X)W + \bar{S}(X, W)\bar{R}(Y, Z)\xi - \bar{S}(\xi, W)\bar{R}(Y, Z)X = 0$$

for any $X, Y, Z, W \in \chi(M)$.

Taking the inner product with $\xi$, the equation (8.1) becomes

$$\tag{8.2} \bar{S}(X, \bar{R}(Y, Z)W) - \bar{S}(\xi, \bar{R}(Y, Z)W)\eta(X) + \bar{S}(X, Y)\eta(\bar{R}(\xi, Z)W) - \bar{S}(\xi, Y)\eta(\bar{R}(X, Z)W) + \bar{S}(X, Z)\eta(\bar{R}(Y, \xi)W) - \bar{S}(\xi, Z)\eta(\bar{R}(Y, X)W) + \bar{S}(X, W)\eta(\bar{R}(Y, Z)\xi) - \bar{S}(\xi, W)\eta(\bar{R}(Y, Z)X) = 0$$

for any $X, Y, Z, W \in \chi(M)$.

For $W = \xi$, using equation (4.5), (4.6), (4.8) and (6.10) in (8.2), we get

$$\tag{8.3} (\alpha + 1)(2\lambda + \mu + \alpha + 1)[g(\phi X, \phi Y)\eta(Z) - g(\phi X, Z)\eta(Y) + \beta g(\phi X, Y)\eta(Z) - g(\phi X, Z)\eta(Y)]$$

for any $X, Y, Z, W \in \chi(M)$.

Hence we can state the following theorem:

**Theorem 8.1.** If $(M, \phi, \xi, \eta, g)$ is a Kenmotsu manifold with a generalized symmetric metric connection, $(g, \xi, \lambda, \mu)$ is an $\eta$-Ricci soliton on $M$ and satisfies $\bar{S} \bar{R}(\xi, X) = 0$. Then

$$\tag{8.4} (\alpha + 1)(2\lambda + \mu + \alpha + 1) = 0.$$

For $\mu = 0$ follows $\lambda = -\frac{\alpha + 1}{2}(\alpha \neq -1)$, therefore, we have the following corollary:

**Corollary 8.2.** On a Kenmotsu manifold with a generalized symmetric metric connection, satisfying $\bar{S} \bar{R}(\xi, X) = 0$, the Ricci soliton defined by (8.1), $\mu = 0$ is either shrinking or expanding.
9. $\eta$-Ricci soliton on $(\varepsilon)$-Kenmotsu manifold with a semi-symmetric metric connection satisfying $\bar{W}_2(\xi, X)\bar{S} = 0$

The condition that must be satisfied by $\bar{S}$ is

\[(9.1)\quad \bar{S}(\bar{W}_2(\xi, X)Y, Z) + \bar{S}(Y, \bar{W}_2(\xi, X)Z) = 0,\]

for any $X, Y, Z \in \chi(M)$.

For $X = \xi$, using (4.5), (4.6), (4.8), (6.10) and (6.17) in (9.1), we get

\[(9.2)\quad (\alpha + 1 - \mu)(-2\mu - 2\lambda + (4\alpha + 4)n)\eta(Y)\eta(Z)\]

for any $X, Y, Z \in \chi(M)$. Hence, we can state the following:

**Theorem 9.1.** If $(M, \phi, \xi, \eta, g)$ is an $(2n+1)$-dimensional Kenmotsu manifold with a generalized symmetric metric connection, $(g, \xi, \lambda, \mu)$ is an $\eta$-Ricci soliton on $M$ and $\bar{W}_2(\xi, X)\bar{S} = 0$, then

\[(9.3)\quad (\alpha + 1 - \mu)(-2\mu - 2\lambda + (4\alpha + 4)n) = 0.\]

For $\mu = 0$ follows that $\lambda = \frac{(4\alpha + 4)n}{2}$, $(\alpha \neq -1)$, therefore, we have the following corollary:

**Corollary 9.2.** On a Kenmotsu manifold with a generalized symmetric metric connection, satisfying $\bar{W}_2(\xi, X)\bar{S} = 0$, the Ricci soliton defined by (9.1), $\mu = 0$ is either shrinking or expanding.

10. $\eta$-Ricci soliton on Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{S}\bar{W}_2(\xi, X) = 0$

In this section we consider an $(\varepsilon)$-Kenmotsu manifold with a semi-symmetric metric connection $\nabla$ satisfying the condition

\[(10.1)\quad \bar{S}(X, \bar{W}_2(Y, Z)V)\xi - \bar{S}(\xi, \bar{W}_2(Y, Z)V)X + \bar{S}(X, Y)\bar{W}_2(\xi, Z)V - \bar{S}(\xi, Y)\bar{W}_2(X, Z)V + \bar{S}(X, Z)\bar{W}_2(Y, \xi)V - \bar{S}(\xi, Z)\bar{W}_2(Y, X)V + \bar{S}(X, V)\bar{W}_2(Y, Z)\xi - \bar{S}(\xi, V)\bar{W}_2(Y, Z)X = 0,

for any $X, Y, Z, V \in \chi(M)$.

Taking the inner product with $\xi$, the equation (10.1) becomes

\[(10.2)\quad \bar{S}(X, \bar{W}_2(Y, Z)V) - \bar{S}(\xi, \bar{W}_2(Y, Z)V)\eta(X) + \bar{S}(X, Y)\eta(\bar{W}_2(\xi, Z)V) - \bar{S}(\xi, Y)\eta(\bar{W}_2(X, Z)V) + \bar{S}(X, Z)\eta(\bar{W}_2(Y, \xi)V) - \bar{S}(\xi, Z)\eta(\bar{W}_2(Y, X)V) + \bar{S}(X, V)\eta(\bar{W}_2(Y, Z)\xi) - \bar{S}(\xi, V)\eta(\bar{W}_2(Y, Z)X) = 0,

for any $X, Y, Z, V \in \chi(M)$.

For $X = V = \xi$, using (4.5), (4.6), (4.8), (6.10) and (6.17) in (10.2), we get

\[(10.3)\quad \{-(\alpha + 1)(2\lambda + \alpha + 1 + \mu) + \frac{(\lambda + \alpha + 1)^2 + (\lambda + \mu)^2}{2n}\}\{\eta(X)\eta(Y) - g(X, Y)\}

+ \beta(\alpha + 1)(2\lambda + \alpha + 1 + \mu)g(\phi X, Y) = 0,

for any $X, Y, Z \in \chi(M)$. Hence, we can state:
Theorem 10.1. If \((M, \phi, \xi, \eta, g)\) is a \((2n + 1)\)-dimensional Kenmotsu manifold with generalized symmetric metric connection, \((g, \xi, \lambda, \mu)\) is an \(\eta\)-Ricci soliton on \(M\) and \(\bar{S}.\bar{W}_2(\xi, X) = 0\), then

\begin{equation}
- (\alpha + 1)(2\lambda + \alpha + 1 + \mu) + \frac{(\lambda + \alpha + 1)^2 + (\lambda + \mu)^2}{2n} = 0,
\end{equation}

and

\begin{equation}
\beta(\alpha + 1)(2\lambda + \alpha + 1) = 0.
\end{equation}

For \(\mu = 0\) we get the following corollary:

Corollary 10.2. On a Kenmotsu manifold with a generalized symmetric metric connection satisfying \(\bar{S}.\bar{W}_2(\xi, X) = 0\), the Ricci soliton defined by (5.1), for \(\mu = 0\), we have the following expressions:

(i) \(- (\alpha + 1)(2\lambda + \alpha + 1 + \frac{1}{2}(\lambda + \alpha + 1)^2 + (\lambda)^2}{2n} = 0\) and \(\beta(\alpha + 1)(2\lambda + \alpha + 1) = 0\).

(ii) If \(\alpha = -1\) or \(\alpha = -2\lambda - 1\) which is steady.

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