MOTIVIC CLASSES OF CLASSIFYING STACKS OF FINITE GROUPS AND UNRAMIFIED COHOMOLOGY

FEDERICO SCAVIA

Abstract. Combining work of Peyre, Colliot-Thélène and Voisin, we give the first example of a finite group $G$ such that the motivic class of its classifying stack $BG$ in Ekedahl’s Grothendieck ring of stacks over $\mathbb{C}$ is non-trivial and $BG$ has trivial unramified Brauer group.

1. Introduction

Let $G$ be a finite group, let $k$ be a field, and let $V$ be a faithful $G$-representation over $k$. We say that the classifying stack $BG$ is stably $k$-rational if the quotient $V/G$ is stably $k$-rational, that is, $V/G \times_k \mathbb{A}^n_k$ is birationally equivalent to $\mathbb{A}^m_k$ for some $m, n \geq 0$. By the no-name lemma, this definition does not depend on $V$. The question of the stable rationality of $BG$ is a variation of the following problem, first considered by E. Noether [10]: if $V$ is the regular representation of $G$ over $k$, is the field of invariants $k(V)^G$ purely transcendental over $k$? If Noether’s problem for $G$ over $k$ has an affirmative answer, then $BG$ is stably $k$-rational.

In [13] and [17], R. Swan and V. Voskresenskiı independently constructed the first example of $G$ such that $BG$ is not stably rational. In their example, $k = \mathbb{Q}$ and $G = \mathbb{Z}/47\mathbb{Z}$. Later, D. Saltman remarked that Wang’s counterexamples to the Grunwald problem imply that $B(\mathbb{Z}/8\mathbb{Z})$ is not stably rational over $\mathbb{Q}$. A complete solution to Noether’s problem for abelian groups was given by H. Lenstra [7].

The first examples over an algebraically closed field were given by Saltman in [12]. Saltman observed that the unramified Brauer group $\text{Br}_u(K/k)$ of a purely transcendental field extension $K/k$ is trivial, and then constructed a finite group $G$ and a $k$-representation $V$ of $G$ such that $\text{Br}_u(k(V)^G/k) \neq 0$. When $k = \mathbb{C}$, E. Peyre exhibited the first examples of groups $G$ such that $BG$ is not stably rational over $\mathbb{C}$, but $\text{Br}_u(\mathbb{C}(V)^G/\mathbb{C}) = 0$. These examples satisfy $H^3_u(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$; see [11, Theorem 3.1].

In this paper, we consider a motivic variant of the problem of stable rationality of $BG$. We denote by $K_0(\text{Stacks}_k)$ the Grothendieck ring of algebraic $k$-stacks, as defined by T. Ekedahl in [6]; see Section 2. By definition, every algebraic stack $X$ of finite type over $k$ and with affine stabilizers has a class $\{X\}$ in $K_0(\text{Stacks}_k)$. The multiplicative identity of $K_0(\text{Stacks}_k)$ is $1 = \{\text{Spec } k\}$.

It is an interesting problem to compute the class $\{BG\}$ in $K_0(\text{Stacks}_k)$. We have $\{BG\} = 1$ in many cases, e.g. when $G = \mu_n$, $G = S_n$, or $G$ is a finite subgroup of $\text{GL}_3$ and $k$ is algebraically closed of characteristic zero; see [5, Proposition 3.2, Theorem 4.1] and [8, Theorem 2.4]. In all examples of finite groups $G$ such that $\{BG\} = 1$, the classifying stack $BG$ is known to be stably rational.
There are also examples of finite groups $G$ for which $\{BG\} \neq 1$ in $K_0(\text{Stacks}_k)$. In [5] Corollary 5.8, Ekedahl showed that $\{B(\mathbb{Z}/47\mathbb{Z})\} \neq 1$ in $K_0(\text{Stacks}_\mathbb{C})$. Moreover, in [5] Theorem 5.1], he showed that if $\text{Br}_r(\mathbb{C}(V)^G/\mathbb{C}) \neq 0$, then $\{BG\} \neq 1$. Thus, the examples $G$ of Swan, Voskresenski˘ı and Saltman also satisfy $\{BG\} \neq 1$.

**Question 1.1.** Does there exist a finite group $G$ such that $\text{Br}_r(\mathbb{C}(V)^G/\mathbb{C}) = 0$, but $\{BG\} \neq 1$ in $K_0(\text{Stacks}_\mathbb{C})$?

To our knowledge, this question was first asked by Ekedahl, and was posed to us by A. Vistoli.

If $H^i_{nr}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$ for some $i$, then $BG$ is not stably rational; see [9] Proposition 3.4. When $i \geq 3$, it is not known whether $H^i_{nr}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$ implies $\{BG\} \neq 1$ in $K_0(\text{Stacks}_\mathbb{C})$. We prove that this is the case if $i = 3$.

**Theorem 1.2.** Let $k$ be a field of characteristic zero, let $G$ be a finite group, and let $V$ be a faithful complex representation of $G$. Assume that $H^3_{nr}(\mathbb{C}(V)^G/\mathbb{C}, \mathbb{Q}/\mathbb{Z}) \neq 0$. Then $\{BG\} \neq 1$ in $K_0(\text{Stacks}_\mathbb{C})$.

A crucial ingredient in our proof of Theorem 1.2 is a result of J.-L. Colliot-Thélène and C. Voisin [4]; see Theorem 3.1 below.

The combination of Peyre’s examples in [11] and Theorem 1.2 has the following consequence.

**Corollary 1.3.** Question [11] has an affirmative answer.

In [14] B. Totaro asked, among other things, whether the stable rationality of $BG$ over $\mathbb{C}$ is equivalent to the condition $\{BG\} = 1$ in $K_0(\text{Stacks}_\mathbb{C})$. An affirmative answer to Totaro’s question is supported by all known examples, and also by Theorem 1.2. A proof of the equivalence seems to be out of reach of current techniques.

2. The Grothendieck ring of stacks

Let $k$ be an arbitrary field. By definition, the Grothendieck ring of varieties $K_0(\text{Var}_k)$ is the abelian group generated by isomorphism classes $\{X\}$ of $k$-schemes $X$ of finite type, modulo the relations $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed subscheme $Y \subseteq X$. The multiplication in $K_0(\text{Var}_k)$ is defined on generators by $\{X\} \cdot \{Y\} := \{X \times_k Y\}$, and we have $1 = \{\text{Spec } k\}$. We set $\mathbb{L} := \{A^1_k\}$.

Following Ekedahl [6], we define the Grothendieck ring of stacks $K_0(\text{Stacks}_k)$ as the abelian group generated by isomorphism classes $\{X\}$ of algebraic stacks $X$ with affine stabilizers and of finite type over $k$, modulo the relations $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed embedding $Y \subseteq X$, and the relations $\{E\} = \{A^r_k \times_k X\}$ for every vector bundle $E \to X$ of constant rank $r$. The product is defined on generators by $\{X\} \cdot \{Y\} := \{X \times_k Y\}$, and we have $1 = \{\text{Spec } k\}$. By [6] Theorem 1.2, the canonical ring homomorphism $K_0(\text{Var}_k) \to K_0(\text{Stacks}_k)$ induces an isomorphism $K_0(\text{Stacks}_k) \cong K_0(\text{Var}_k)[\{\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} : n \geq 1\}]$.

The following was observed by Ekedahl in [6] p. 14.

**Lemma 2.1.** Let $k$ be a field of characteristic zero. Then, as an abelian group, $K_0(\text{Var}_k)[\mathbb{L}^{-1}]$ may be presented as the abelian group generated by formal fractions of the form $\{X\}/\mathbb{L}^m$, where $X$ is a smooth projective variety and $m \geq 0$, modulo the following relations:
Theorem 3.1. □

The dimension filtration $\text{Fil}^n K_0(\text{Var}_k)[L^{-1}]$ of $K_0(\text{Var}_k)[L^{-1}]$ is defined as follows: for every $n \in \mathbb{Z}$, $\text{Fil}^n K_0(\text{Var}_k)[L^{-1}]$ is the subgroup generated by the elements $\{X\}/L^n$, where $X$ is a $k$-variety and $\dim(X) - m \leq n$. When char $k = 0$, using resolution of singularities, we see that $\text{Fil}^n K_0(\text{Var}_k)[L^{-1}]$ is generated by elements of the form $\{X\}/L^m$, where $X$ is a smooth projective $k$-variety and $\dim(X) - m \leq n$; see [6, Lemma 3.1].

We denote by $\hat{K}_0(\text{Var}_k)$ the completion of $K_0(\text{Var}_k)$ with respect to the dimension filtration. For every $n, n' \in \mathbb{Z}$, we have

$$\text{Fil}^n K_0(\text{Var}_k)[L^{-1}] \cdot \text{Fil}^{n'} K_0(\text{Var}_k)[L^{-1}] \subseteq \text{Fil}^{n+n'} K_0(\text{Var}_k)[L^{-1}].$$

It follows that the multiplication on $K_0(\text{Var}_k)$ extends to $\hat{K}_0(\text{Var}_k)$, making it into a commutative ring with identity.

For every $n \geq 1$, we have $(1 - L^n) \sum_{i \geq 0} L^{ni} = 1$ in $\hat{K}_0(\text{Var}_k)$. Therefore, we have canonical ring homomorphisms

$$K_0(\text{Var}_k) \to \hat{K}_0(\text{Var}_k) \to \hat{K}_0(\text{Var}_k).$$

3. The integral Hodge Question and unramified cohomology

Let $X$ be a smooth projective variety over $\mathbb{C}$, and let $d := \dim(X)$. For every integer $i$, we write $CH^i(X)$ for the group of algebraic cycles of codimension $i$ on $X$ modulo rational equivalence, and we set $CH_i(X) := CH^{d-i}(X)$. We have the cycle class maps

$$c_i^X : CH^i(X) \to H^{2i}(X(\mathbb{C}), \mathbb{Z}).$$

By convention, we set $CH^i(X) = 0$ and $H^{2i}(X(\mathbb{C}), \mathbb{Z}) = 0$ when $i < 0$ and $i > d$.

A cohomology class $\alpha \in H^{2i}(X(\mathbb{C}), \mathbb{Z})$ is called an integral Hodge class if its image in $H^{2i}(X(\mathbb{C}), \mathbb{C})$ is of type $(i, i)$ with respect to the Hodge decomposition of $H^{2i}(X(\mathbb{C}), \mathbb{C})$. We denote by $\text{Hdg}^{2i}(X, \mathbb{Z})$ the subgroup of integral Hodge classes of $H^{2i}(X(\mathbb{C}), \mathbb{Z})$. We have an inclusion $\text{Im}(c_i^X) \subseteq \text{Hdg}^{2i}(X, \mathbb{Z})$. We set

$$Z^{2i}(X) := \text{Hdg}^{2i}(X, \mathbb{Z})/\text{Im}(c_i^X), \quad Z_{2i}(X) := Z^{2d-2i}(X).$$

For every integer $i$, the abelian group $Z^{2i}(X)$ is finitely generated. The Hodge Conjecture for cycles of codimension $i$ on $X$ predicts that $Z^{2i}(X)$ is finite. The integral Hodge Question for cycles of codimension $i$ on $X$ asks whether $Z^{2i}(X)$ is zero. By the Lefschetz Theorem on $(1, 1)$-classes, the integral Hodge Question has an affirmative answer when $i = 1$. When $i = 2$, the integral Hodge Question has a negative answer in general, as shown by examples of M. Atiyah and F. Hirzebruch [1].
Theorem 3.1 (Colliot-Thélène, Voisin). Let \( X \) be a smooth projective variety over \( \mathbb{C} \), of dimension \( d \). Assume that there exists a smooth closed subvariety \( S \subseteq X \) of dimension \( \leq 2 \), such that the pushforward map \( CH_0(S) \to CH_0(X) \) is surjective. Then we have an isomorphism of finite groups
\[
H^3_m(X, \mathbb{Q}/\mathbb{Z}) \cong Z^4(X).
\]

Proof. See [4, Théorème 1.1]. \( \square \)

Remark 3.2. It is well known that the assumptions of Theorem 3.1 are satisfied when \( X \) is unirational. We have learned the following argument from J.-L. Colliot-Thélène.

If \( X \) is a smooth projective unirational variety over \( \mathbb{C} \), then there exist a dense open subset \( U \subseteq X \) and a surjective morphism \( \varphi : V \to U \), where \( V \) is an open subset of some affine space. If \( p_1, p_2 \in U(\mathbb{C}) \), we may find \( q_1, q_2 \in V(\mathbb{C}) \) such that \( \varphi(q_i) = p_i \) for \( i = 1, 2 \). There is a line connecting \( q_1 \) and \( q_2 \), hence, since \( X \) is complete, we find a morphism \( \mathbb{P}^1 \to X \) whose image contains \( p_1 \) and \( p_2 \). It follows that any two zero-cycles of degree 1 in \( U \) are rationally equivalent.

Now, if \( p \in X(\mathbb{C}) \), a moving lemma shows that \( p \) is rationally equivalent to a zero-cycle whose support is contained in \( U \); see [3, Complément, p. 599]. We conclude that the degree map \( \deg : CH_0(X) \to \mathbb{Z} \) is an isomorphism. Thus, the hypotheses of Theorem 3.1 are satisfied, with \( S \) a closed point of \( X \).

4. PROOF OF THEOREM 1.2

We denote by \( K_0(\text{Ab}) \) the group generated by isomorphism classes \([A]\) of finitely generated abelian groups \( A \), modulo the relations \([A \oplus B] = [A] + [B]\). As an abelian group, \( K_0(\text{Ab}) \) is freely generated by \([\mathbb{Z}]\) and \([\mathbb{Z}/p^n\mathbb{Z}]\), where \( p \) ranges among prime numbers and \( n \geq 1 \); see [6, Proposition 3.3(i)].

Proposition 4.1. Let \( i \) be an integer.

(a) There exists a group homomorphism
\[
Z_{2i} : K_0(\text{Var}_\mathbb{C})[L^{-1}] \to K_0(\text{Ab}),
\]

given by sending \( \{X\}/L^m \mapsto [Z_{2i+2m}(X)] \) for every smooth projective variety \( X \) over \( \mathbb{C} \) and every \( m \geq 0 \).

(b) The homomorphism \( Z_{2i} \) is continuous with respect to the filtration topology on \( K_0(\text{Var}_\mathbb{C})[L^{-1}] \) and the discrete topology of \( K_0(\text{Ab}) \). It thus extends uniquely to a group homomorphism
\[
\tilde{Z}_{2i} : K_0(\text{Var}_\mathbb{C}) \to K_0(\text{Ab}).
\]

Proof. (a) To show that \( Z_{2i} : K_0(\text{Var}_\mathbb{C})[L^{-1}] \to K_0(\text{Ab}) \) is well-defined, we verify that the association \( \{X\}/L^m \mapsto [Z_{2i+2m}(X)] \) respects the relations of Lemma 2.1. It is clear that (i) is satisfied.

Let \( m \geq 0 \), let \( Y \subseteq X \) be a closed embedding of smooth projective complex varieties, let \( \tilde{X} \to X \) be the blow-up of \( X \) at \( Y \), and let \( E \) be the exceptional divisor of the blow-up. Denote by \( d \) the dimension of \( X \), and by \( r \) the codimension of \( Y \) in \( X \). We want to show that
\[
[Z_{2i+2m}(X)] - [Z_{2i+2m}(Y)] = [Z_{2i+2m}(\tilde{X})] - [Z_{2i+2m}(E)].
\]
in \( K_0(\text{Ab}) \). Letting \( j = d - i - m \), we see that (4.2) is equivalent to:
\[
[Z^{2j}(X)] - [Z^{2j-2r}(Y)] = [Z^{2j}(\tilde{X})] - [Z^{2j-2r}(E)].
\]
By [16, Theorem 9.27], we have a group isomorphism
\[ \varphi^j : \oplus_{0 \leq h \leq r-2} CH^{j-1-h}(Y) \oplus CH^j(X) \xrightarrow{\sim} CH^j(\tilde{X}). \]
By [15, Theorem 7.31], we have an isomorphism of Hodge structures
\[ \oplus_{0 \leq h \leq r-2} H^{2j-2-2h}(Y,\mathbb{C},\mathbb{Z}) \oplus H^{2j}(X,\mathbb{C},\mathbb{Z}) \xrightarrow{\sim} H^{2j}(\tilde{X}(\mathbb{C}),\mathbb{Z}), \]
where the Hodge structure on \( H^{2j-2-2h}(Y,\mathbb{C},\mathbb{Z}) \) is shifted by \( (h+1,h+1) \), and so has weight \( 2j \). In particular, we have an isomorphism of groups
\[ \psi^j : \oplus_{0 \leq h \leq r-2} \text{Hdg}^{2j-2-2h}(Y,\mathbb{Z}) \oplus \text{Hdg}^{2j}(X,\mathbb{Z}) \xrightarrow{\sim} \text{Hdg}^{2j}(\tilde{X},\mathbb{Z}). \]
Comparing the explicit description of these isomorphisms, as given in the references, we see that \( \varphi^j \) and \( \psi^j \) are compatible with the cycle class maps. In other words, we have a commutative square
\[ \begin{array}{ccc}
\oplus_{0 \leq h \leq r-2} CH^{j-1-h}(Y) \oplus CH^j(X) & \xrightarrow{\varphi^j} & CH^j(\tilde{X}) \\
\downarrow \oplus_h cl^j_h & & \downarrow cl^j_h \\
\oplus_{0 \leq h \leq r-2} \text{Hdg}^{2j-2-2h}(Y,\mathbb{Z}) \oplus \text{Hdg}^{2j}(X,\mathbb{Z}) & \xrightarrow{\psi^j} & \text{Hdg}^{2j}(\tilde{X},\mathbb{Z}).
\end{array} \]
We deduce that
\[ (4.4) \quad Z^{2j}(\tilde{X}) \cong \oplus_{0 \leq h \leq r-2} Z^{2j-2-2h}(Y) \oplus Z^{2j}(X). \]
The morphism \( E \to Y \) identifies \( E \) with the projectivization of the normal bundle of \( Y \) inside \( X \). By [16, Theorem 9.25] and [15, Lemma 7.32], the pullback along \( E \to Y \) induces a commutative diagram
\[ \begin{array}{ccc}
\oplus_{0 \leq h \leq r-1} CH^{j-1-h}(Y) & \xrightarrow{\sim} & CH^{j-1}(E) \\
\downarrow \oplus_h cl^j_h & & \downarrow cl^{j-1}_E \\
\oplus_{0 \leq h \leq r-1} \text{Hdg}^{2j-2-2h}(Y,\mathbb{Z}) & \xrightarrow{\sim} & \text{Hdg}^{2j-2}(E,\mathbb{Z}),
\end{array} \]
where the horizontal arrows are isomorphisms. We deduce that
\[ (4.5) \quad Z^{2j-2}(E) \cong \oplus_{0 \leq h \leq r-1} Z^{2j-2-2h}(Y). \]
Now (4.3) follows from (4.4) and (4.4). Therefore, \( Z_{2i} \) respects all relations of type (ii).
It remains to show that \( Z_{2i} \) is compatible with relations of type (iii). Let \( X \) be a smooth projective variety of dimension \( d \), and let \( m \geq 0 \) be an integer. We must show that
\[ [Z_{2i+2m+2}(X \times_{\mathbb{C}} \mathbb{P}_d^1)] - [Z_{2i+2m+2}(X)] = [Z_{2i+2m}(X)]. \]
Setting \( j = d - i - m \), the claim becomes
\[ [Z^{2j}(X \times_{\mathbb{C}} \mathbb{P}_d^1)] - [Z^{2j-2}(X)] = [Z^{2j}(X)]. \]
\[ \text{Note that the formula of [16, Theorem 9.25] contains a typographical error: } CH_{1-r+1+k}(-) \text{ should be } CH_{1-r+1+k}(-). \]
Applying [15, Theorem 9.25] and [15, Lemma 7.32] to the trivial projective bundle $X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \to X$, we obtain a commutative square

$$
\begin{align*}
CH^j(X) \oplus CH^{j-1}(X) & \xrightarrow{\sim} CH^j(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}) \\
Hdg^{2j}(X, \mathbb{Z}) \oplus Hdg^{2j-2}(X, \mathbb{Z}) & \xrightarrow{\sim} Hdg^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}, \mathbb{Z}).
\end{align*}
$$

Thus

$$Z^{2j}(X \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}) \cong Z^{2j}(X) \oplus Z^{2j-2}(X),$$

which implies (4.10). It follows that $Z_{2i}$ respects relations of type (iii) as well, hence $Z_{2i}$ is a well-defined group homomorphism.

(b) Let $X$ be a smooth projective variety of dimension $d$, and let $m \geq d - i$. Then $2i + 2m \geq 2d$, and so

$$Z_{2i}([X]/L^m) = [Z_{2i+2m}(X)] = 0.$$  

This means that $Z_{2i}$ sends $\text{Fil}^i K_0(\text{Var}_\mathbb{C}[L^{-1}])$ to zero. Therefore, if we endow $K_0(\text{Var}_\mathbb{C}[L^{-1}])$ with the dimension filtration topology and $K_0(\text{Ab})$ with the discrete topology, the homomorphism $Z_{2i}$ is continuous. It follows that $Z_{2i}$ extends uniquely to a homomorphism $\tilde{Z}_{2i} : \tilde{K}_0(\text{Var}_\mathbb{C}) \to K_0(\text{Ab})$. 

We also denote by $Z_{2i}$ the composition

$$K_0(\text{Stacks}_\mathbb{C}) \to \tilde{K}_0(\text{Var}_\mathbb{C}) \xrightarrow{\tilde{Z}_{2i}} K_0(\text{Ab}).$$

**Proposition 4.7.** Let $G$ be a finite group, let $V$ be a faithful $G$-representation over $\mathbb{C}$, and let $X$ be a smooth projective variety over $\mathbb{C}$ that is birational to $V/G$. Then $Z_{2i}([BG]) = 0$ for every $i \geq -1$, and

$$Z_{-4}([BG]) = [H^3_{\text{nr}}(\mathbb{C}(V)G/\mathbb{C}, \mathbb{Q}/\mathbb{Z})].$$

**Proof.** Let $V$ be a faithful complex $G$-representation of dimension $d \geq 1$, and let $U \subseteq V$ be the non-empty open subscheme where $G$ acts freely. By [15, Theorem 3.4], we may write

$$\{BG\} \mathbb{L}^d = \{U/G\} + \sum_j m_j \{BH_j\} \mathbb{L}^{a_j}$$

in $K_0(\text{Stacks}_\mathbb{C})$, where the $H_j$ are distinct proper subgroups of $G$, $m_j \in \mathbb{Z}$ and $a_j \leq d - 1$.

Using resolution of singularities, we may write

$$\{U/G\} = \{X\} + \sum_q n_q \{X_q\}$$

in $K_0(\text{Var}_\mathbb{C})$, where $X$ and the $X_q$ are smooth projective varieties over $\mathbb{C}$, $X$ is birationally equivalent to $U/G$, $\dim(X_q) \leq d - 1$, and $n_q \in \mathbb{Z}$ for every $q$. We substitute (4.9) into (4.8) and divide by $\mathbb{L}^d$:

$$\{BG\} = \{X\} \mathbb{L}^{-d} + \sum_q n_q \{X_q\} \mathbb{L}^{-d} + \sum_j m_j \{BH_j\} \mathbb{L}^{a_j - d}.$$  

We apply $Z_{2i}$:

$$Z_{2i}([BG]) = [Z_{2i+2d}(X)] + \sum_q n_q[Z_{2i+2d}(X_q)] + \sum_j m_j Z_{2i+2d-2a_j}([BH_j]).$$
If $G$ is trivial, then $BG \cong \text{Spec } \mathbb{C}$ and there is nothing to prove. Assume now that $G$ is non-trivial, and that the conclusion of the proposition holds for all $i \in \mathbb{Z}$ and all proper subgroups of $G$.

By the Lefschetz Theorem on $(1,1)$-classes, we have $Z_{2d-2}(X) = Z^2(X) = 0$. Therefore, if $i \geq -1$ every term on the right hand side of (4.10) is zero. This shows that $Z^i(\{BG\}) = 0$ for all $i \geq -1$.

If $i = -2$, another application of the Lefschetz Theorem on $(1,1)$-classes shows that the right hand side of (4.10) reduces to $[Z_{2d-4}(X)] = [Z^4(X)]$. Since $X$ is birationally equivalent to $V/G$, by Theorem 3.1 we have:

$$Z^4(X) \cong H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \cong H^3_{nr}(C(V)^G/C, \mathbb{Q}/\mathbb{Z}).$$

**Proof of Theorem 1.2.** By a standard limit argument, we may assume that $k$ is finitely generated over $\mathbb{Q}$. Fix an embedding $k \hookrightarrow \mathbb{C}$. By assumption, we have $H^3_{nr}(C(V)^G/C, \mathbb{Q}/\mathbb{Z}) \neq 0$, hence by Proposition 4.7 we obtain $Z_{-4}(\{BG\}) \neq 0$. On the other hand, it is clear that $Z_{-4}(\{\text{Spec } \mathbb{C}\}) = 0$. We conclude that $\{BG\} \neq 1$ in $K_0(\text{Stacks}_\mathbb{C})$, hence $\{BG\} \neq 1$ in $K_0(\text{Stacks}_k)$.

**Acknowledgements**

I thank Jean-Louis Colliot-Thélène for making me aware of the results of [4], and for a conversation which eventually led me to Theorem 1.2. I thank Angelo Vistoli for posing Question 1.1 to me, and my advisor Zinovy Reichstein for helpful suggestions on the exposition.

**References**

[1] M. F. Atiyah and F. Hirzebruch. Analytic cycles on complex manifolds. *Topology*, 1:25–45, 1962.

[2] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. *Compositio Mathematica*, 140(4):1011–1032, 2004.

[3] Jean-Louis Colliot-Thélène. Un théorème de finitude pour le groupe de Chow des zéro-cycles d’un groupe algébrique linéaire sur un corps $p$-adique. *Invent. Math.*, 159(3):589–606, 2005.

[4] Jean-Louis Colliot-Thélène and Claire Voisin. Cohomologie non ramifiée et conjecture de Hodge entière. *Duke Math. J.*, 161(5):735–801, 2012.

[5] Torsten Ekedahl. A geometric invariant of a finite group. *arXiv preprint arXiv:0903.3148*, 2009.

[6] Torsten Ekedahl. The Grothendieck group of algebraic stacks. *arXiv preprint arXiv:0903.3143*, 2009.

[7] H. W. Lenstra, Jr. Rational functions invariant under a finite abelian group. *Invent. Math.*, 25:299–325, 1974.

[8] Ivan Martino. The Ekedahl invariants for finite groups. *Journal of Pure and Applied Algebra*, 220(4):1294–1309, 2016.

[9] Alexander Merkurjev. Invariants of algebraic groups and retract rationality of classifying spaces. *Algebraic Groups: Structure and Actions*, 94:277, 2017.

[10] Emmy Noether. Gleichungen mit vorgeschriebener Gruppe. *Mathematische Annalen*, 78(1):221–229, 1917.

[11] Emmanuel Peyre. Unramified cohomology of degree 3 and Noether’s problem. *Invent. Math.*, 171(1):191–225, 2008.

[12] David J. Saltman. Noether’s problem over an algebraically closed field. *Invent. Math.*, 77(1):71–84, 1984.

[13] Richard G. Swan. Invariant rational functions and a problem of Steenrod. *Invent. Math.*, 7:148–158, 1969.

[14] Burt Totaro. The motive of a classifying space. *Geom. Topol.*, 20(4):2079–2133, 2016.
[15] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.

[16] Claire Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.

[17] V. E. Voskresenski˘ı. Birational properties of linear algebraic groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:3–19, 1970.