Correlation functions of the XXZ spin chain with the twisted boundary condition

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Abstract

The scalar products, form factors and correlation functions of the XXZ spin chain with twisted (or antiperiodic) boundary condition are obtained based on the inhomogeneous $T - Q$ relation and the Bethe states constructed via the off-diagonal Bethe Ansatz. It is shown that the scalar product of two off-shell Bethe states, the form factors and the two-point correlation functions can be expressed as the summation of certain determinants. The corresponding homogeneous limits are studied. The results are also checked by the numerical calculations.

Keywords: Quantum spin chain; Bethe Ansatz; Yang-Baxter equation

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1 Introduction

The quantum lattice models with integrable boundary conditions have drawn a wide attention during these years [1, 2, 3]. The typical quantization conditions of the energy spectrum include the periodic, twisted (or anti-periodic) [4, 5, 6, 7] and open boundary conditions [8, 9, 10, 11, 12]. For the integrable models with periodic or diagonal boundary reflection, the Bethe Ansatz methods have been applied successfully [13, 14]. However, for the models with twisted or off-diagonal boundary reflection, the \( U(1) \) symmetry of the system is broken and an obvious reference state is missing, which prevent us from applying the conventional Bethe Ansatz methods to solve the system exactly. Recently, the off-diagonal Bethe Ansatz (ODBA) is developed to study the exact solutions of quantum integrable models, especially for those with nontrivial integrable boundary reflections [15].

The Bethe Ansatz is a powerful method to calculate the partition functions [16] and the correlation functions [1] for one-dimensional integrable systems. In particular, for models related to the \( gl_2 \) symmetry, Slavnov [17] and the Gaudin-Korepin [18, 19] formulas lead to that the scalar product between an eigenstate and an arbitrary state, and that the norm of the eigenstates have a determinant representation. In [20, 21], it is shown the local operator can be expressed by the elements of monodromy matrix, which give a direct path to study the form factors and correlation functions [22]. The thermodynamical limit of the determinant is considered for XXZ model with a diagonal open boundary condition [23]. Recently, the determinant representations of the scalar products and correlation functions are extensively studied for models related to various boundary conditions especially those with non-diagonal boundary terms [7, 24, 25, 26, 27, 28, 29, 30]. Among them, the separation of variables (SOV) method [7, 32, 33] gives an attractive approach to obtain the determinant representations of the scalar product between an eigenstate and an arbitrary state with the open boundary conditions.

The XXZ spin chain with twisted boundary condition is a very interesting integrable model [1, 5, 6, 7]. By using the ODBA method, the spectrum and the inhomogeneous \( T-Q \) relation of the system are obtained [31]. The distribution of Bethe roots at the ground state and the low-energy excitation are studied in the ferromagnetic region [34]. Based on the inhomogeneous \( T-Q \) relations, the Bethe-type eigenstates, which have well defined homogeneous limit, are retrieved [35].
In this paper, we study the scalar products, the form factors and the two points correlation functions of the XXZ spin chain with twisted boundary condition. The corresponding determinant representations are obtained. We also provide the explicit forms of the homogeneous limit and check our results by the numerical calculations.

The paper is organized as follows. Section 2 serves as an introduction of the model and its exact solution. In section 3, we discuss the SoV basis. In section 4, the Bethe states are reviewed and the scalar product of two off-shell Bethe states are calculated. In section 5, we present a compact form for the local operators in terms of the elements of the monodromy matrix. In section 6, form factors are calculated and expressed by certain determinants. In section 7, we calculate the two points correlation functions. In section 8, the homogeneous limit of above results are studied. Numerical results are given in section 9 and the concluding remarks are given in section 10.

2 XXZ spin chain with twisted boundary conditions

The XXZ spin chain with twisted boundary condition is characterized by the Hamiltonian

\[ H = - \sum_{j=1}^{N} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh \eta \sigma_j^z \sigma_{j+1}^z], \quad (2.1) \]

where \( N \) is the number of sites, \( \eta \) is the crossing parameter and the boundary condition is the twisted one, namely,

\[ \sigma_{N+1}^\alpha = \sigma_1^\alpha \sigma_0^\alpha, \quad \text{for} \quad \alpha = x, y, z. \quad (2.2) \]

It is remarked that the twisted boundary condition breaks the \( U(1) \)-symmetry of the system.

The integrability of the model (2.1) is associated with the six-vertex \( R \)-matrix

\[ R_{0,j}(u) = \frac{1}{2} \left[ \frac{\sinh(u + \eta)}{\sinh \eta} (1 + \sigma_j^z \sigma_0^z) + \frac{\sinh u}{\sinh \eta} (1 - \sigma_j^z \sigma_0^z) \right] + \frac{1}{2} (\sigma_j^x \sigma_0^x + \sigma_j^y \sigma_0^y), \quad (2.3) \]

where \( u \) is the spectral parameter. The \( R \)-matrix satisfies the quantum Yang-Baxter equation (QYBE)

\[ R_{1,2}(u_1 - u_2)R_{1,3}(u_1 - u_3)R_{2,3}(u_2 - u_3) = R_{2,3}(u_2 - u_3)R_{1,3}(u_1 - u_3)R_{1,2}(u_1 - u_2). \quad (2.4) \]

From the \( R \)-matrix, we can define the monodromy matrix as

\[ T_0(u) = \sigma_0^z R_{0,N}(u - \theta_N) \cdots R_{0,1}(u - \theta_1) = \begin{pmatrix} C(u) & D(u) \\ A(u) & B(u) \end{pmatrix}, \quad (2.5) \]
where \( \{ \theta_j, j = 1, \cdots, N \} \) are the inhomogeneous parameters. Let us introduce further the left quasi-vacuum state \( \langle 0 | \) and the right quasi-vacuum state \( | 0 \rangle \)

\[
\langle 0 | = (1, 0)_{[1]} \otimes \cdots \otimes (1, 0)_{[N]}, \quad | 0 \rangle = \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right)_{[1]} \otimes \cdots \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_{[N]}.
\]

(2.6)

The elements of monodromy matrix \( (2.5) \) acting on the quasi-vacuum states give rise to

\[
\langle 0 | A(u) = a(u) \langle 0 |, \quad \langle 0 | B(u) = 0, \quad \langle 0 | C(u) \neq 0, \quad D(u) = d(u) \langle 0 |,
\]

\[
A(u) | 0 \rangle = a(u) | 0 \rangle, \quad B(u) | 0 \rangle \neq 0, \quad C(u) | 0 \rangle = 0, \quad D(u) | 0 \rangle = d(u),
\]

(2.7)

where

\[
a(u) = \prod_{k=1}^{N} \frac{\sinh(u - \theta_k + \eta)}{\sinh \eta}, \quad d(u) = \prod_{k=1}^{N} \frac{\sinh(u - \theta_k)}{\sinh \eta}.
\]

(2.8)

The operator \( B(u) \) [or \( C(u) \)] acts on the quasi-vacuum states \( | 0 \rangle \) [or \( \langle 0 | \)] as a creation operator. The \( R \)-matrix and the monodromy matrix \( T \) satisfy the RTT relation

\[
R_{0,0}(u - v)T_0(u)T_0(v) = T_0(v)T_0(u)R_{0,0}(u - v).
\]

(2.9)

From the RTT relation \( (2.9) \), we obtain the commutative relations among the elements of the monodromy matrix as

\[
[B(u), B(v)] = [C(u), C(v)] = 0,
\]

\[
A(u) B(v) = \frac{\sinh(u - v - \eta)}{\sinh(u - v)} B(v) A(u) + \frac{\sinh \eta}{\sinh(u - v)} B(u) A(v),
\]

\[
D(u) B(v) = \frac{\sinh(u - v + \eta)}{\sinh(u - v)} B(v) D(u) - \frac{\sinh \eta}{\sinh(u - v)} B(u) D(v),
\]

\[
C(u) A(v) = \frac{\sinh(u - v + \eta)}{\sinh(u - v)} A(v) C(u) - \frac{\sinh \eta}{\sinh(u - v)} A(u) C(v),
\]

\[
C(u) D(v) = \frac{\sinh(u - v - \eta)}{\sinh(u - v)} D(v) C(u) + \frac{\sinh \eta}{\sinh(u - v)} D(u) C(v),
\]

\[
[C(u), B(v)] = \frac{\sinh \eta}{\sinh(u - v)} [D(u) A(v) - D(v) A(u)].
\]

(2.10)

The transfer matrix of the system is defined as

\[
t(u) = tr_0 \{ T_0(u) \} = B(u) + C(u).
\]

(2.11)
Again, from the RTT relation (2.9), one can prove that the transfer matrices with different spectral parameters commute with each other,

$$ [t(u), t(v)] = 0. \quad (2.12) $$

Then the transfer matrix $t(u)$ serves as the generating functional of all the conserved quantities, which ensures the integrability of the system. The Hamiltonian (2.1) is chosen as the first order derivative of the logarithm of the transfer matrix, namely,

$$ H = -2 \sinh \eta \frac{\partial \ln t(u)}{\partial u} \bigg|_{u=0, \{\theta_j=0\}} + N \cosh \eta. \quad (2.13) $$

The transfer matrix (2.11) and the Hamiltonian (2.1) can be exactly solved by using the ODBA method [15, 31]. Let $|\Phi\rangle$ be the common eigenstates of the transfer matrix (2.11) and the Hamiltonian (2.1) with the eigenvalues $\Lambda(u)$ and $E$, respectively,

$$ t(u)|\Phi\rangle = \Lambda(u)|\Phi\rangle, \quad H|\Phi\rangle = E|\Phi\rangle. $$

The eigenvalue $\Lambda(u)$ can be parameterized into following inhomogeneous $T-Q$ relation

$$ \Lambda(u)Q(u) = a(u)e^uQ(u - \eta) - e^{-u-\eta}d(u)Q(u + \eta) - c(u)a(u)d(u), \quad (2.14) $$

where $Q(u)$ is a trigonometric polynomial with the form

$$ Q(u) = \prod_{j=1}^{N} \frac{\sinh(u - \lambda_j)}{\sinh \eta}, \quad (2.15) $$

the $\{\lambda_j\}$ are the Bethe roots and the coefficient $c(u)$ is given by

$$ c(u) = e^{u-N\eta+\sum_{l=1}^{N}(\theta_l-\lambda_l)} - e^{-u-N\eta+\sum_{l=1}^{N}(\theta_l-\lambda_l)}. \quad (2.16) $$

The $N$ Bethe roots $\{\lambda_j\}$ satisfy the Bethe Ansatz equations (BAEs)

$$ e^{\lambda_j}a(\lambda_j)Q(\lambda_j - \eta) - e^{-\lambda_j-\eta}d(\lambda_j)Q(\lambda_j + \eta) - c(\lambda_j)a(\lambda_j)d(\lambda_j) = 0, \quad j = 1, \cdots, N. \quad (2.17) $$

The eigen-energy of the Hamiltonian (2.1) is then expressed in terms of the Bethe roots as

$$ E = 2 \sinh \eta \sum_{j=1}^{N} \left[ \coth(\lambda_j + \eta) - \coth(\lambda_j) \right] - N \cosh \eta - 2 \sinh \eta. \quad (2.18) $$
### 3 SoV basis

A convenient SoV basis of the model (2.1) is

\[ |h_1, \ldots, h_N\rangle = \prod_{j=1}^{N} [B(\theta_j)]^{h_j} |0\rangle, \quad (3.1) \]

\[ \langle h_1, \ldots, h_N| = \langle 0| \prod_{j=1}^{N} [C(\theta_j)]^{h_j}, \quad (3.2) \]

where \( h_j = 0 \) or \( 1 \). Using the commutation relations (2.10), one can derive following orthogonal relations between the right states (3.1) and the left states (3.2)

\[ f(h) = \langle h_1, \ldots, h_N| h_1, \ldots, h_N\rangle = \prod_{i=1}^{N} [-a(\theta_i) d(\theta_i - \eta) e^{-\eta(N-1)}]^{h_i} \frac{|V(\theta_1, \ldots, \theta_N)|}{|V(\theta_1 - \eta h_1, \ldots, \theta_N - \eta h_N)|}, \quad (3.3) \]

where \(|V|\) means the determinant of matrix \( V \) which is given by

\[ V(x_1, \ldots, x_N) = \begin{pmatrix} 1 & e^{2x_1} & \cdots & e^{2x_1(N-1)} \\ 1 & e^{2x_2} & \cdots & e^{2x_2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{2x_N} & \cdots & e^{2x_N(N-1)} \end{pmatrix}. \quad (3.4) \]

The above determinant is a kind of Vandermonde determinant, which possesses the following properties

\[ |V(x_1, \ldots, x_N)| = \prod_{1 \leq j < i \leq N} (e^{2x_i} - e^{2x_j}), \]

\[ |V(x_1, \ldots, x_N, y)| = \prod_{1 \leq j < i \leq N} (e^{2x_i} - e^{2x_j}) \prod_{k=1}^{N} (e^{2y} - e^{2x_k}). \quad (3.5) \]

The right states (3.1) form an orthogonal right basis of the Hilbert space, while the left states (3.2) form an orthogonal left basis of the Hilbert space. Thus we have

\[ \text{id} = \sum_{\{h\}} |h_1, \ldots, h_N\rangle \langle h_1, \ldots, h_N| f^{-1}(h), \quad (3.6) \]

where id is the unitary matrix, \( \sum_{\{h\}} \) means sum over all possible values of \( \{h_1 \cdots h_N\} \), and the total number of the terms is \( 2^N \).
We note that the SoV basis (3.1) is the eigenstate of the operator $D(u)$,

$$D(u)|h_1,\cdots,h_N\rangle = \prod_{j=1}^{N} \frac{\sinh(u - \theta_j + \eta h_j)}{\sinh \eta} |h_1,\cdots,h_N\rangle. \quad (3.7)$$

Meanwhile, we also know the action of operator $C(u)$ acting on SoV basis (3.1)

$$C(u)|h_1,\cdots,h_N\rangle = \sum_{i=1}^{N} \prod_{j \neq i} \frac{\sinh(u - \theta_j + \eta h_j) \sinh(\theta_i - \theta_j - \eta h_j)}{\sinh(\theta_i - \theta_j)} a(\theta_i) \times |h_1,\cdots,h_i - 1,\cdots,h_N\rangle, \quad (3.8)$$

which will be used later.

### 4 Bethe states and scalar products

#### 4.1 Bethe states

The Bethe states of the system read

$$|\lambda_1,\cdots,\lambda_N\rangle = \prod_{j=1}^{N} D(\lambda_j)|\Omega; \{\theta_j\}\rangle, \quad (4.1)$$

$$\langle \lambda_1,\cdots,\lambda_N| = \langle \Omega; \{\theta_j\}|\prod_{j=1}^{N} D(\lambda_j), \quad (4.2)$$

where $|\Omega; \{\theta_j\}\rangle$ is the reference state which can be determined by the inner products

$$\langle h_1,\cdots,h_N|\Omega; \{\theta_j\}\rangle = \prod_{i=1}^{N} [a(\theta_i)d(\theta_i)]^{h_i}, \quad (4.3)$$

$$\langle \Omega; \{\theta_j\}|h_1,\cdots,h_N\rangle = \prod_{i=1}^{N} [a(\theta_i)d(\theta_i)]^{h_i}. \quad (4.4)$$

We express the Bethe states (4.1) by the SoV basis (3.1) as

$$|\lambda_1,\cdots,\lambda_N\rangle = \sum_{h} |h_1,\cdots,h_N\rangle \langle h_1,\cdots,h_N|f^{-1}(h)|\lambda_1,\cdots,\lambda_N\rangle$$

$$= \sum_{h} f^{-1}(h) \prod_{j=1}^{N} \bar{d}(\{\lambda_k\}, \theta_j, h_j) a^{h_j}(\theta_j) e^{h_j\theta_j} |h_1,\cdots,h_N\rangle, \quad (4.5)$$

where

$$\bar{d}(\{\lambda_k\}, u, h) = \prod_{k=1}^{N} \frac{\sinh(\lambda_k - u + \eta h)}{\sinh \eta}. \quad (4.6)$$
If the Bethe roots \( \{ \lambda_j, j = 1, \ldots, N \} \) satisfy the BAEs (2.17), then the Bethe states (4.1) are the common eigenstates of the Hamiltonian (2.1) and the transfer matrix (2.11). The eigenstates, denoted as \( |\Phi\{\lambda_k\}\rangle \), can also be expressed by the SoV basis (3.1) as

\[
|\Phi\{\lambda_k\}\rangle = \sum_h f^{-1}(h) \prod_{j=1}^{N} \Lambda^{h_j}(\{\lambda_k\}, \theta_j) |h_1, \ldots, h_N\rangle.
\] (4.7)

In the derivation, we have used following identities

\[
\langle h_1, \ldots, h_N|\Phi\{\lambda_k\}\rangle = N \prod_{j=1}^{N} \frac{d(u_j)}{d(\lambda_j)} \frac{\sinh(\theta_i - \theta_j + \eta h)}{\sinh(\theta_i - \theta_j)},
\]
\[
\langle \Phi\{u_k\}|h_1, \ldots, h_N\rangle = \prod_{j=1}^{N} \frac{d(u_j)}{d(\lambda_j)} \frac{\sinh(\theta_i - \theta_j + \eta)}{\sinh(\theta_i - \theta_j)},
\]
\[
\frac{\sinh(\theta_i - \theta_j - \eta h)}{\sinh(\theta_i - \theta_j)} \frac{\sinh(\theta_i - \theta_j - \eta + \eta h)}{\sinh(\theta_i - \theta_j)} = \sinh(\theta_i - \theta_j - \eta),
\] (4.8)

where \( h = 0 \) or \( 1 \).

### 4.2 Scalar product of two off-shell Bethe states

Now, we consider the scalar product of two off-shell Bethe states. After tedious calculation, we obtain

\[
S_N \equiv \langle u_1, \ldots, u_N|\lambda_1, \ldots, \lambda_N\rangle = \langle \Omega; \{\theta_j\}\rangle \prod_{k=1}^{N} D(u_k) D(\lambda_k)|\Omega; \{\theta_j\}\rangle
\]

\[
\equiv \sum_h f^{-1}(h) \prod_{j=1}^{N} \left\{ \frac{d(u_j)}{d(\lambda_j)} \frac{\sinh(\theta_i - \theta_j + \eta h)}{\sinh(\theta_i - \theta_j)} \prod_{k=1}^{N} \frac{\sinh(u_j - \theta_k + \eta h_k)}{\sinh(u_j - \theta_k)} \right\}. \tag{4.9}
\]

We also find that above result can be expressed by the ratio of two determinants

\[
S_N = \frac{|P|}{|V(\theta_1, \ldots, \theta_N)|},
\] (4.10)

where \( V(\theta_1, \ldots, \theta_N) \) is given by Eq.(3.4), \( P \) is the \( N \times N \) matrix with the elements

\[
P_{ij} = \sum_{h=0}^{1} \tau(\{u_l\}, \{\lambda_l\}, h, \theta_i) e^{2(\theta_i - \eta h)(j-1)},
\] (4.11)
and the function \( \tau(\{u_l\}, \{\lambda_l\}, h, u) \) is

\[
\tau(\{u_l\}, \{\lambda_l\}, h, u) = \bar{d}(\{u_k\}; u, h) \bar{d}(\{\lambda_k\}; u, h) \left[ \frac{-a(u)}{d(u - \eta)} \right]^h e^{2u_1 + \eta h(N-1)}.
\] (4.12)

4.3 Scalar product of one off-shell Bethe state and one on-shell Bethe state

Next, we consider the scalar product of an eigenstate and an off-shell Bethe state. Suppose the parameters \( \{u_k\} \) satisfy the BAEs (2.17), and we obtain

\[
\langle \Phi\{u_k\} | \lambda_1, \cdots, \lambda_N \rangle \equiv \sum_h f^{-1}(h) N \prod_{j=1}^{N} \left\{ d(u_j) d(\lambda_j) [a(\theta_j) e^{\theta_j} \Lambda(\{u_k\}, \theta_j)]^h \prod_{k=1}^{N} \frac{\sinh(\lambda_j - \theta_k + \eta h_k)}{\sinh(\lambda_j - \theta_k)} \right\}. \] (4.13)

We find that above equation can be expressed as

\[
\langle \Phi\{u_k\} | \lambda_1, \cdots, \lambda_N \rangle = \frac{|P_{NL}|}{|V(\theta_1, \cdots, \theta_N)|},
\] (4.14)

where \( V(\theta_1, \cdots, \theta_N) \) is given by Eq.(3.4) and \( P_{NL} \) is a \( N \times N \) matrix with the elements

\[
P_{ij}^{NL} = \sum_{h=0}^{1} e^{\eta h(N-1) + \theta_i h + 2(\theta_i - \eta)(j-1)} d(u_i) \left[ \frac{-\Lambda(\{u_k\}, \theta_i)}{d(\theta_i - \eta)} \right]^h \prod_{k=1}^{N} \frac{\sinh(\lambda_k - \theta_i + \eta h)}{\sinh \eta}. \] (4.15)

If the right Bethe state is an eigenstate and the left Bethe state is the off-shell one, the scalar product reads

\[
\langle u_1, \cdots, u_N | \Phi\{\lambda_k\} \rangle = \frac{|P_{NR}|}{|V(\theta_1, \cdots, \theta_N)|},
\] (4.16)

where \( P_{NR} \) is a \( N \times N \) matrix with the elements \( P_{ij}^{NR} = P_{ij}^{NL} |_{\lambda_k \leftrightarrow u_k} \).

5 Inverse problem

The inverse problem is to reconstruct the local operators by the elements of monodromy matrix. The general method to solve the inverse problem of the quantum integrable models was proposed in [20, 21]. The inverse problem of the closed XXZ spin chain was considered in [22] and the antiperiodic case was considered in [7]. Here, the main procedures to solve the quantum inverse problem are reviewed and a slightly different form of the results for the antiperiodic case are obtained (see (5.11)-(5.13) below).
At the point of \( u = 0 \), the \( R \)-matrix (2.3) degenerates into the permutation operator, \( R_{0,j}(0) = P_{0,j} \), which possesses the property

\[
R_{i,j}(u) = P_{0,i} R_{0,j}(u) P_{0,i}.
\] (5.1)

Considering the value of transfer matrix \( t(u) \) at the point of \( u = \theta_j \) and using the property (5.1), we obtain

\[
t(\theta_j) = tr_{0}\{\sigma_0^x R_{0,N}(\theta_j - \theta_N) \cdots R_{0,j+1}(\theta_j - \theta_{j+1}) P_{0,j} R_{0,j-1}(\theta_j - \theta_{j-1}) \cdots R_{0,1}(\theta_j - \theta_1)\}
\]

\[
= R_{j,j-1}(\theta_j - \theta_{j-1}) \cdots R_{j,1}(\theta_j - \theta_1) \sigma_j^x R_{j,N}(\theta_j - \theta_N) \cdots R_{j,j+1}(\theta_j - \theta_{j+1}),
\]

\[
\equiv (j, j-1) \cdots (j, 1) \sigma_j^x (j, N) \cdots (j, j+1),
\] (5.2)

where we have used the notation \((i, j) \equiv R_{i,j}(\theta_i - \theta_j)\). The production of \( t(\theta_j) \) gives

\[
\prod_{j=1}^{i} t(\theta_j) = \prod_{j=1}^{i-1} \prod_{k>j}^{i} \phi_{jk} \cdot \sigma_j^x \cdots \sigma_1^x [(1, N) \cdots (1, i+1)] [(2, N) \cdots (2, i+1)] \cdots \times [(i, N) \cdots (i, i+1)],
\] (5.3)

where

\[
\phi_{jk} = \sinh(\eta - \theta_j + \theta_k) \sinh(\eta + \theta_j - \theta_k) / \sinh^2 \eta.
\] (5.4)

Because the transfer matrices with different spectral parameters commute with each other, the inverse of quantity (5.3) is

\[
\prod_{j=1}^{i} t^{-1}(\theta_j) = \prod_{j=1}^{i-1} \prod_{k>j}^{i} \phi_{jk}^{-1} \cdot \prod_{m=1}^{N} \prod_{n=i+1}^{N} \phi_{mn}^{-1} \cdot [(i+1, i) \cdots (N, i)] \cdots [(i+1, 2) \cdots (N, 2)] \cdots \times [(i+1, 1) \cdots (N, 1)] \sigma_1^x \cdots \sigma_i^x.
\] (5.5)

Let \( i = N \) in (5.3), we obtain following identities

\[
\prod_{j=1}^{N} t(\theta_j) = \prod_{i=1}^{N-1} \prod_{j>i}^{N} \phi_{ij} \cdot \sigma_i^x \cdots \sigma_N^x,
\] (5.6)

\[
\prod_{j=1}^{N} t(\theta_j) \cdot \prod_{k=1}^{N} t(\theta_k) = \prod_{i=1}^{N-1} \prod_{j>i}^{N} \phi_{ij}^2 \times \text{id}.
\] (5.7)

By using Eqs.(5.3) and (5.5), we also have

\[
\prod_{j=1}^{i-1} t^{-1}(\theta_j) x_i \prod_{j=1}^{i} t(\theta_j) = [(i, i-1) \cdots (i, 1)] x_i \sigma_i^x [(i, N) \cdots (i, i+1)],
\] (5.8)
where $x_i$ is a local operator defined in $i$-th space. Meanwhile,

$$
tr_0\{x_0T_0(\theta_i)\} = tr_0\{x_0\sigma^x_0R_{0,N}(\theta_i - \theta_N) \cdots R_{0,i+1}(\theta_i - \theta_{i+1})P_{0,i}R_{0,i-1}(\theta_i - \theta_{i-1}) \cdots \\
\times R_{0,1}(\theta_1 - \theta_i)\} = R_{i,i-1}(\theta_i - \theta_{i-1}) \cdots R_{i,1}(\theta_i - \theta_1)x_i\sigma^x_iR_{1,N}(\theta_1 - \theta_N) \cdots R_{i,i+1}(\theta_i - \theta_{i+1}) = [(i, i - 1) \cdots (i, 1)]x_i\sigma^x_i[(i, N) \cdots (i, i + 1)].
$$

Comparing Eqs.(5.8) and (5.9), we arrive at the identity

$$
tr_0\{x_0T_0(\theta_i)\} = \prod_{j=1}^{i-1} t^{-1}(\theta_j)x_i \prod_{j=1}^{i} t(\theta_j).
$$

(5.10)

With the help of Eqs.(5.7) and (5.10), we express the generators of $su(2)$ algebra in terms of elements of the monodromy matrix as

$$
\sigma_i^- = \prod_{i=1}^{N-1} \prod_{j>i}^N \phi_{ij}^{-2} \cdot \prod_{j=1}^{i-1} t(\theta_j) \cdot D(\theta_i) \cdot \prod_{j=i+1}^{N} t(\theta_j),
$$

(5.11)

$$
\sigma_i^+ = \prod_{i=1}^{N-1} \prod_{j>i}^N \phi_{ij}^{-2} \cdot \prod_{j=1}^{i-1} t(\theta_j) \cdot A(\theta_i) \cdot \prod_{j=i+1}^{N} t(\theta_j),
$$

(5.12)

$$
\sigma_i^z = \prod_{i=1}^{N-1} \prod_{j>i}^N \phi_{ij}^{-2} \cdot \prod_{j=1}^{i-1} t(\theta_j) \cdot [2C(\theta_i) - t(\theta_i)] \cdot \prod_{j=i+1}^{N} t(\theta_j).
$$

(5.13)

which will be used to calculate the form factors and the correlation functions.

6 Form factors

Suppose $\langle \Phi\{u_k\}\rangle$ is the left eigenstate of the model (2.1), while $|\Phi\{\lambda_k\}\rangle$ is the right eigenstate. First, we calculate the form factor $\langle \Phi\{u_k\}|\sigma^-|\Phi\{\lambda_k\}\rangle$. Using the formula (5.11), we obtain

$$
\langle \Phi\{u_k\}|\sigma^-|\Phi\{\lambda_k\}\rangle = \prod_{i=1}^{N-1} \prod_{j>i}^N \phi_{ij}^{-2} \prod_{j=1}^{i-1} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \prod_{j=1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \\
\times \langle \Phi\{u_k\}|D(\theta_i)|\Phi\{\lambda_k\}\rangle.
$$

(6.1)

The factor $\langle \Phi\{u_k\}|D(\theta_i)|\Phi\{\lambda_k\}\rangle$ reads

$$
\langle \Phi\{u_k\}|D(\theta_i)|\Phi\{\lambda_k\}\rangle = \sum_{\{h\}} \prod_{j=1}^{N} d(u_j)d(\lambda_j) \left[ \frac{-a(\theta_j)e^{(-\eta(N-1)+2\theta_j)}h_j}{d(\theta_j - \eta)} \right]^h \frac{\sinh(\theta_i - \theta_j + \eta h_j)}{\sinh \eta}
$$

(3.6)
\[ \times \prod_{k=1}^{N} \frac{\sinh(u_j - \theta_k + \eta h_k) \sinh(\lambda_j - \theta_k + \eta h_k)}{\sinh(u_j - \theta_k) \sinh(\lambda_j - \theta_k)} \right\} \frac{|V(\theta_1 - \eta h_1, \ldots, \theta_N - \eta h_N)|}{|V(\theta_1, \ldots, \theta_N)|}, \]

which can be written as the determinant expression

\[ \langle \Phi\{u_k\}|D(\theta_i)|\Phi\{\lambda_k\}\rangle = \frac{|F^D|}{|V(\theta_1, \ldots, \theta_N)|}, \]

where \( F^D \) is the \( N \times N \) matrix with the elements

\[ F^D_{m,n} = \sum_{h=0}^{1} \tau(\{u_t\}, \{\lambda_i\}, h, \theta_m)e^{2(h\eta - \eta h)(n-1)} \frac{\sinh(\theta_i - \theta_m + \eta h)}{\sinh \eta}. \]

Therefore, we arrive at

\[ \langle \Phi\{u_k\}|\sigma_i^-|\Phi\{\lambda_k\}\rangle = \prod_{m=1}^{N-1} \prod_{n>m}^{N} \phi_{mn}^{-2} \prod_{j=1}^{i-1} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \prod_{j=1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \]

\[ \times \frac{|F^D|}{|V(\theta_1, \ldots, \theta_N)|} \]

Similarly, the form factor \( \langle \Phi\{u_k\}|\sigma_i^+|\Phi\{\lambda_k\}\rangle \) is related with the scalar product \( \langle \Phi\{u_k\}|A(\theta_i)|\Phi\{\lambda_k\}\rangle \), which can be calculated directly and the results are very complicated and omitted here. We note that the form factor \( \langle \Phi\{u_k\}|\sigma_i^+|\Phi\{\lambda_k\}\rangle \) can also be calculated by a simple way as follows. First one can construct another set of SoV basis from the eigenstates of the operator \( A(u) \) in (2.5). Then using this basis and following the same procedure as above, one can obtain the form factor \( \langle \Phi\{u_k\}|\sigma_i^+|\Phi\{\lambda_k\}\rangle \).

Next, we consider the \( \langle \Phi\{u_k\}|\sigma_i^+|\Phi\{\lambda_k\}\rangle \). By using the reconstruction of local operator \( \sigma_i^z \) (5.13), we have

\[ \langle \Phi\{u_k\}|\sigma_i^z|\Phi\{\lambda_k\}\rangle = \prod_{m=1}^{N-1} \prod_{n>m}^{N} \phi_{mn}^{-2} \prod_{j=1}^{i-1} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \prod_{j=1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \]

\[ \times \langle \Phi\{u_k\}|[2C(\theta_i) - \Lambda(\{u_k\}, \theta_i)]|\Phi\{\lambda_k\}\rangle. \]

With the help of Eq.(3.8), we obtain

\[ \langle \Phi\{u_k\}|C(\theta_i)|\Phi\{\lambda_k\}\rangle = \sum_{j=1}^{N} \sum_{\{h\}} (-1)^{j+N} \xi(\{u_l\}, \{\lambda_i\}, \theta_i, \theta_j) \prod_{k \neq j}^{N} \tau(\{u_l\}, \{\lambda_i\}, h_k, \theta_k) \]

\[ \times \frac{|V_j(\theta_1 - \eta h_1, \ldots, \theta_N - \eta h_N, \theta_1)|}{|V(\theta_1, \ldots, \theta_N)|}. \]

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Substituting Eqs. (6.10) into (6.6), we obtain

\[
V_j(x_1, \cdots, x_N, y) = \begin{pmatrix}
1 & e^{2x_1} & \cdots & e^{2x_1(N-1)} \\
1 & e^{2x_2} & \cdots & e^{2x_2(N-1)} \\
\vdots & \vdots & & \vdots \\
1 & e^{2x_{j-1}} & \cdots & e^{2x_{j-1}(N-1)} \\
1 & e^{2x_{j+1}} & \cdots & e^{2x_{j+1}(N-1)} \\
\vdots & \vdots & & \vdots \\
1 & e^{2x_N} & \cdots & e^{2x_N(N-1)} \\
1 & e^{2y} & \cdots & e^{2y(N-1)}
\end{pmatrix},
\]

(6.8)

and

\[
\xi(\{u_t\}, \{\lambda_t\}, \theta_i, \theta_j) = d(\{u_k\}, \theta_j, 1)d(\{\lambda_k\}, \theta_j, 1) \left[ -\frac{a(\theta_j)}{d(\theta_j - \eta)} \right] e^{\theta_i + \eta N} \times \prod_{k=1}^{N} \left[ e^{-\theta_i + \theta_j - \eta} \frac{\sinh(\theta_j - u_k)}{\sinh(\theta_j - u_k - \eta)} \frac{\sinh(\theta_j - \theta_k - \eta)}{\sinh \eta} \right].
\]

(6.9)

We find that the scalar product (6.7) also has a determinant expression

\[
\langle \Phi \{u_k\} | C(\theta_i) | \Phi \{\lambda_k\} \rangle = \frac{|F^C|}{|V(\theta_1, \cdots, \theta_N)|}.
\]

(6.10)

where \(F^C\) is the \((N+1) \times (N+1)\) matrix with the elements

\[
F^C_{m,n} = \sum_{h=0}^{1} \tau(\{u_t\}, \{\lambda_t\}, h, \theta_m)e^{2(\theta_m - \eta)(n-1)}, \quad 1 \leq m \leq N, 1 \leq n \leq N,
\]

\[
F^C_{m,N+1} = \xi(\{u_t\}, \{\lambda_t\}, \theta_i, \theta_m), \quad 1 \leq m \leq N,
\]

\[
F^C_{N+1,n} = e^{2\theta_i(n-1)}, \quad 1 \leq n \leq N,
\]

\[
F^C_{N+1,N+1} = 0.
\]

(6.11)

Substituting Eqs. (6.10) into (6.6), we obtain

\[
\langle \Phi \{u_k\} | \sigma^z_i | \Phi \{\lambda_k\} \rangle = \prod_{m=1}^{N-1} \prod_{n>m}^{N} \phi_{mn}^{-2} \prod_{j=1}^{i-1} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \prod_{j=1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \times \frac{|F^z|}{|V(\theta_1, \cdots, \theta_N)|}.
\]

(6.12)

where \(F^z\) is the \((N+1) \times (N+1)\) matrix with the elements

\[
F^z_{m,n} = \sum_{h=0}^{1} \tau(\{u_t\}, \{\lambda_t\}, h, \theta_m)e^{2(\theta_m - \eta)(n-1)}, \quad 1 \leq m \leq N, \quad 1 \leq n \leq N,
\]
\[
F_{m,N+1}^z = 2\xi(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_m), \quad 1 \leq m \leq N,
\]
\[
F_{N+1,n}^z = e^{2\theta_m(n-1)}, \quad 1 \leq n \leq N,
\]
\[
F_{N+1,N+1}^z = -\Lambda(\{\lambda_k\}, \theta_i).
\]

### 7 Correlation functions

Now, we consider the two points correlation function \(\langle \Phi\{u_k\}|\sigma_{i-1}^-\sigma_i^-|\Phi\{\lambda_k\}\rangle\). According to Eq. (5.11), we have

\[
\langle \Phi\{u_k\}|\sigma_{i-1}^-\sigma_i^-|\Phi\{\lambda_k\}\rangle = \prod_{m=1}^{N-1} \prod_{n>m}^{N} \phi_{mn}^{-2} \prod_{j=1}^{i-2} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^{N} \Lambda(\{\lambda_k\}, \theta_j)
\]
\[
\times \langle \Phi\{u_k\}|D(\theta_{i-1})D(\theta_i)|\Phi\{\lambda_k\}\rangle.
\]

Direct calculation shows

\[
\langle \Phi\{u_k\}|D(\theta_{i-1})D(\theta_i)|\Phi\{\lambda_k\}\rangle \quad \overset{(3,6)}{=} \quad \sum_{\{h\}} \prod_{j=1}^{N} (d(u_j)d(\lambda_j)) \left[ -a(\theta_j)e^{\eta(N-1)+2\theta_j} \right]^{h_j} \frac{\sinh(\theta_i - \theta_j + \eta h_j)}{\sinh \eta}
\]
\[
\times \frac{\sinh(\theta_{i-1} - \theta_j + \eta h_j)}{\sinh \eta} \prod_{k=1}^{N} \frac{\sinh(u_j - \theta_k + \eta h_k)}{\sinh (u_j - \theta_k)} \frac{\sinh(\lambda_j - \theta_k + \eta h_k)}{\sinh(\lambda_j - \theta_k)}
\]
\[
\times \frac{|V(\theta_1 - \eta h_1, \cdots, \theta_N - \eta h_N)|}{|V(\theta_1, \cdots, \theta_N)|}, \quad (7.2)
\]

which can be expressed as the determinant form

\[
\langle \Phi\{u_k\}|D(\theta_{i-1})D(\theta_i)|\Phi\{\lambda_k\}\rangle = \frac{|F_{DD}^{DD}|}{|V(\theta_1, \cdots, \theta_N)|}, \quad (7.3)
\]

where \(F_{DD}^{DD}\) is the \(N \times N\) matrix with the elements

\[
F_{m,n}^{DD} = \sum_{h=0}^{1} \tau(\{u_i\}, \{\lambda_i\}, h, \theta_m)e^{2(\theta_m - \eta h)(m-1)} \frac{\sinh(\theta_{i-1} - \theta_m + \eta h)}{\sinh \eta} \frac{\sinh(\theta_i - \theta_m + \eta h)}{\sinh \eta}. \quad (7.4)
\]

Then we arrive at

\[
\langle \Phi\{u_k\}|\sigma_{i-1}^-\sigma_i^-|\Phi\{\lambda_k\}\rangle = \prod_{m=1}^{N-1} \prod_{n>m}^{N} \phi_{mn}^{-2} \prod_{j=1}^{i-2} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^{N} \Lambda(\{\lambda_k\}, \theta_j)
\]
\[
\times \prod_{j=1}^{N} \Lambda(\{\lambda_k\}, \theta_j) \frac{|F_{DD}^{DD}|}{|V(\theta_1, \cdots, \theta_N)|}. \quad (7.5)
\]
Next, we consider the two points correlation function $\langle \Phi \{ u_k \} | \sigma_{i-1}^z \sigma_i^z | \Phi \{ \lambda_k \} \rangle$. From Eq. (5.13), we have

$$\langle \Phi \{ u_k \} | \sigma_{i-1}^z \sigma_i^z | \Phi \{ \lambda_k \} \rangle = \prod_{m=1}^{N-1} \prod_{n>m}^N \phi_{mn}^{-2} \prod_{j=1}^{i-2} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^N \Lambda(\{\lambda_k\}, \theta_j) \prod_{j=1}^N \Lambda(\{\lambda_k\}, \theta_j)$$

$$\times \langle \Phi \{ u_k \} | \{ 4C(\theta_{i-1})C(\theta_i) - 2C(\theta_{i-1})t(\theta_i) - t(\theta_{i-1})[2C(\theta_i) - t(\theta_i)] \} | \Phi \{ \lambda_k \} \rangle. \tag{7.6}$$

Direct calculation shows

$$\langle \Phi \{ u_k \} | C(\theta_{i-1})C(\theta_i) | \Phi \{ \lambda_k \} \rangle \tag{7.7}$$

$$\begin{align*}
&= \sum_{j'=1}^N \sum_{j\neq j'}^N \sum_{\{h\}} \left( -1 \right)^{j+j'+1+x} \frac{\sinh(\theta_j - \theta_{j'} - \eta)}{\sinh(\theta_j - \theta_{j'} - \eta)} \gamma_1(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_{j'}) \gamma_2(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_{j'}) \\
&\times \prod_{k\neq j, j'} \tau(\{u_i\}, \{\lambda_i\}, h_k, \theta_k) \frac{|V_{j',j}(\theta_1 - \eta h_1, \ldots, \theta_N - \eta h_N, \theta_{i-1}, \theta_i)|}{|V(\theta_1, \ldots, \theta_N)|},
\end{align*}$$

where $x = 0$ if $j' > j$; $x = 1$ if $j' < j$, $\sum_{\{h\}}$ means that the parameters $h_{j'}$ and $h_j$ are not included in the summation, $h_{j'} = 1$ and $h_j = 1$, $V_{j',j}(x_1, \ldots, x_N, y, z)$ is the $N \times N$ matrix

$$V_{j',j}(x_1, \ldots, x_N, y, z) = \begin{pmatrix}
1 & e^{2x_1} & \ldots & e^{2x_1(N-1)} \\
1 & e^{2x_2} & \ldots & e^{2x_2(N-1)} \\
& \vdots & \ddots & \vdots \\
1 & e^{2x_{j'-1}} & \ldots & e^{2x_{j'-1}(N-1)} \\
1 & e^{2x_{j'+1}} & \ldots & e^{2x_{j'+1}(N-1)} \\
& \vdots & \ddots & \vdots \\
1 & e^{2x_{j-1}} & \ldots & e^{2x_{j-1}(N-1)} \\
1 & e^{2x_{j+1}} & \ldots & e^{2x_{j+1}(N-1)} \\
& \vdots & \ddots & \vdots \\
1 & e^{2x_N} & \ldots & e^{2x_N(N-1)} \\
1 & e^{2y} & \ldots & e^{2y(N-1)} \\
1 & e^{2z} & \ldots & e^{2z(N-1)}
\end{pmatrix}, \tag{7.8}$$

and the functions $\gamma_1$ and $\gamma_2$ are given by

$$\gamma_1(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_{j'}) = -\frac{\sinh(\theta_{i-1} - \theta_{j'})}{\sinh(\theta_i - \theta_{i-1})} \xi(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_{j'}),$$

$$\gamma_2(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_j) = -\sinh(\theta_i - \theta_j + \eta) \xi(\{u_i\}, \{\lambda_i\}, \theta_{i-1}, \theta_j). \tag{7.9}$$

Again, Eq. (7.7) can be rewritten as the determinant expression

$$\langle \Phi \{ u_k \} | C(\theta_{i-1})C(\theta_i) | \Phi \{ \lambda_k \} \rangle = \sum_{j'=1}^N (-1)^{N+j'} \gamma_1(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_{j'}) \frac{|F_{j'}^{CC}|}{|V(\theta_1, \ldots, \theta_N)|}. \tag{7.10}$$
where $F_{j'}^{CC}$ is the reduced matrix of $F^{CC}$ with $j'$ row vanished and $F^{CC}$ is the $(N+2) \times (N+1)$ matrix with the elements

\[
F_{m,n}^{CC} = \sum_{h=0}^{1} \tau(\{u_i\}, \{\lambda_i\}, h, \theta_m) e^{2(\theta_m - \eta h)(n-1)}, \quad 1 \leq m \leq N, 1 \leq n \leq N,
\]

\[
F_{m,N+1}^{CC} = \gamma_2(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_m) / \sinh(\theta_m - \theta_{j'} - \eta), \quad 1 \leq m \leq N,
\]

\[
F_{N+1,n}^{CC} = e^{2g_i(n-1)}, \quad 1 \leq n \leq N,
\]

\[
F_{N+2,n}^{CC} = e^{2g_i(n-1)}, \quad 1 \leq n \leq N,
\]

\[
F_{n,N+1}^{CC} = 0, \quad n = N, N+1.
\]

(7.11)

With the help of Eqs. (4.10) and (6.10), we have

\[
\langle \Phi\{u_k\}|C(\theta_{i-1})t(\theta_i)|\Phi\{\lambda_k\}\rangle = \Lambda(\{\lambda_k\}, \theta_i) \frac{|F^C(\theta_{i-1})|}{|V(\theta_1, \cdots, \theta_N)|},
\]

(7.12)

and

\[
\langle \Phi\{u_k\}|t(\theta_{i-1})[2C(\theta_i) - t(\theta_i)]|\Phi\{\lambda_k\}\rangle = \Lambda(\{u_k\}, \theta_{i-1}) \frac{|F^z(\theta_i)|}{|V(\theta_1, \cdots, \theta_N)|}.
\]

(7.13)

Substituting Eqs. (7.10), (7.12) and (7.13) into (7.6), we obtain

\[
\langle \Phi\{u_k\}|\sigma_{i-1}^+\sigma_i^+|\Phi\{\lambda_k\}\rangle = \prod_{m=1}^{N-1} \prod_{n>m}^{N} \phi_{mn}^{-2} \prod_{j=1}^{i-2} \Lambda(\{u_k\}, \theta_j) \prod_{j=i+1}^{N} \Lambda(\{\lambda_k\}, \theta_j)
\]

\[
\times \left\{ 4 \sum_{j'=1}^{N} (-1)^{N+j'} \gamma_1(\{u_i\}, \{\lambda_i\}, \theta_i, \theta_{j'}) |F_{j'}^{CC}| - 2\Lambda(\{\lambda_k\}, \theta_i) |F_{i-1}^{CC}| 
\]

\[
- \Lambda(\{u_k\}, \theta_{i-1}) |F^z| |V(\theta_1, \cdots, \theta_N)|^{-1},
\]

(7.14)

where $F_{i-1}^{CC}$ is the $(N+1) \times (N+1)$ matrix with the elements

\[
(F_{i-1}^{CC})_{m,n} = \sum_{h=0}^{1} \tau(\{u_i\}, \{\lambda_i\}, h, \theta_m) e^{2(\theta_m - \eta h)(n-1)}, \quad 1 \leq m \leq N, 1 \leq n \leq N,
\]

\[
(F_{i-1}^{CC})_{m,N+1} = \xi(\{u_i\}, \{\lambda_i\}, \theta_{i-1}, \theta_m), \quad 1 \leq m \leq N,
\]

\[
(F_{i-1}^{CC})_{N+1,n} = e^{2g_i(n-1)}, \quad 1 \leq n \leq N,
\]

\[
(F_{i-1}^{CC})_{N+1,N+1} = 0.
\]

(7.15)
8 Homogeneous limit

In this section, we consider the homogeneous limit of above results by using the method suggested in [36]. From Eq. (3.5), the denominator of scalar product (4.10) is

\[ |V(\theta_1, \ldots, \theta_N)| = \prod_{1 \leq j < i \leq N} (e^{2\theta_i} - e^{2\theta_j}). \]  (8.1)

We define

\[ \varphi_j(u, \{\theta_k\}) = \sum_{h=0}^1 \tau(\{u_l\}, \{\lambda_i\}, h, u)e^{2(u-h)(j-1)}, \]  (8.2)

where the function \( \tau \) is given by Eq. (4.12). Then the scalar product (4.10) reads

\[
S_N = \begin{vmatrix}
\varphi_1(\theta_1, \{\theta_k\}) & \varphi_2(\theta_1, \{\theta_k\}) & \cdots & \varphi_N(\theta_1, \{\theta_k\}) \\
\varphi_1(\theta_2, \{\theta_k\}) & \varphi_2(\theta_2, \{\theta_k\}) & \cdots & \varphi_N(\theta_2, \{\theta_k\}) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1(\theta_N, \{\theta_k\}) & \varphi_2(\theta_N, \{\theta_k\}) & \cdots & \varphi_N(\theta_N, \{\theta_k\})
\end{vmatrix} \prod_{j<i} (e^{2\theta_i} - e^{2\theta_j})^{-1}. \]  (8.3)

Taking the Taylor expansion of function \( \varphi_j(u, \{\theta_k\}) \) at the point of \( \theta_1 \), we obtain

\[
\varphi_j(u, \{\theta_k\}) = \varphi_j(\theta_1, \{\theta_k\}) + (u - \theta_1)\varphi'_j(u, \{\theta_k\})|_{u=\theta_1} + \frac{1}{2}(u - \theta_1)^2\varphi''_j(u, \{\theta_k\})|_{u=\theta_1} + \cdots. \]  (8.4)

Put \( u = \theta_2 \) in Eq. (8.4) and we have

\[
\varphi_j(\theta_2, \{\theta_k\}) = \varphi_j(\theta_1, \{\theta_k\}) + (\theta_2 - \theta_1)\varphi'_j(u, \{\theta_k\})|_{u=\theta_1} + \cdots. \]  (8.5)

Substitute (8.5) into (8.3) and taking the limit \( \theta_2 \to \theta_1 \), we obtain

\[
S_N = \begin{vmatrix}
\varphi_1(\theta_1, \{\theta_k\}) & \varphi_2(\theta_1, \{\theta_k\}) & \cdots & \varphi_N(\theta_1, \{\theta_k\}) \\
\varphi'_1(u, \{\theta_k\})|_{u=\theta_1} & \varphi'_2(u, \{\theta_k\})|_{u=\theta_1} & \cdots & \varphi'_N(u, \{\theta_k\})|_{u=\theta_1} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1(\theta_N, \{\theta_k\}) & \varphi_2(\theta_N, \{\theta_k\}) & \cdots & \varphi_N(\theta_N, \{\theta_k\})
\end{vmatrix} \times \left[ 2e^{2\theta_1} \prod_{2 \leq j < i \leq N} (e^{2\theta_i} - e^{2\theta_j}) \right]^{-1}. \]  (8.6)

Let \( u = \theta_3 \) and Eq. (8.4) reads

\[
\varphi_j(\theta_3, \{\theta_k\}) = \varphi_j(\theta_1, \{\theta_k\}) + (\theta_3 - \theta_1)\varphi'_j(u, \{\theta_k\})|_{u=\theta_1} + \cdots. \]
+ \frac{1}{2}(\theta_3 - \theta_1)^2 \varphi_j^{(2)}(u, \{\theta_k\})|_{u=\theta_1} + \cdots. \quad (8.7)

Substitute (8.7) into (8.6) and taking the limit \(\theta_3 \to \theta_1\), we obtain

\[
S_N = \begin{pmatrix}
\varphi_1(\theta_1, \{\theta_k\}) & \varphi_2(\theta_1, \{\theta_k\}) & \cdots & \varphi_N(\theta_1, \{\theta_k\}) \\
\varphi'_1(u, \{\theta_k\})|_{u=\theta_1} & \varphi'_2(u, \{\theta_k\})|_{u=\theta_1} & \cdots & \varphi'_N(u, \{\theta_k\})|_{u=\theta_1} \\
\varphi_1^{(2)}(u, \{\theta_k\})|_{u=\theta_1} & \varphi_2^{(2)}(u, \{\theta_k\})|_{u=\theta_1} & \cdots & \varphi_N^{(2)}(u, \{\theta_k\})|_{u=\theta_1} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1^{(N-1)}(u, \{\theta_k\})|_{u=\theta_1} & \varphi_2^{(N-1)}(u, \{\theta_k\})|_{u=\theta_1} & \cdots & \varphi_N^{(N-1)}(u, \{\theta_k\})|_{u=\theta_1}
\end{pmatrix}
\times \left[2(2e^{2\theta_1})^3 \prod_{3 \leq j < i \leq N} (e^{2\theta_i} - e^{2\theta_j})\right]^{-1}. \quad (8.8)

Repeat above procedures and we arrive at

\[
S_N = \begin{pmatrix}
\varphi_1(\theta_1, \{\theta_k\}) & \varphi_2(\theta_1, \{\theta_k\}) & \cdots & \varphi_N(\theta_1, \{\theta_k\}) \\
\varphi_1^{(2)}(u, \{\theta_k\})|_{u=\theta_1} & \varphi_2^{(2)}(u, \{\theta_k\})|_{u=\theta_1} & \cdots & \varphi_N^{(2)}(u, \{\theta_k\})|_{u=\theta_1} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1^{(N-1)}(u, \{\theta_k\})|_{u=\theta_1} & \varphi_2^{(N-1)}(u, \{\theta_k\})|_{u=\theta_1} & \cdots & \varphi_N^{(N-1)}(u, \{\theta_k\})|_{u=\theta_1}
\end{pmatrix}
\times \left[(2e^{2\theta_1})^{N(N-1)/2} \prod_{k=1}^{N-1} k!\right]^{-1}. \quad (8.9)

Finally, let all the inhomogeneous parameters be zero and we obtain the scalar product (4.10) in the homogeneous limit

\[
S_N = |P^{\text{hom}}| \left[2^{N(N-1)/2} \prod_{k=1}^{N-1} k!\right]^{-1}. \quad (8.10)
\]

Here the elements of matrix \(P^{\text{hom}}\) are

\[
P^{\text{hom}}_{m,n} = \left. \frac{\partial^{m-1} \varphi_n(u)}{\partial u^{m-1}} \right|_{u=0},
\]

\[
\varphi_n(u) = \sum_{h=0}^{1} \tilde{\tau}(\{u_1\}, \{\lambda_l\}, h, u) e^{2(u-h)(n-1)}, \quad (8.11)
\]

where

\[
\tilde{\tau}(\{u_1\}, \{\lambda_l\}, h, u) = \tilde{d}(\{u_k\}, u, h) \tilde{d}(\{\lambda_k\}, u, h) \left[ \frac{-\sinh^N u + \eta}{\sinh^N u - \eta} \right]^h e^{2uh+\eta h(N-1)}, \quad (8.12)
\]

and \(\tilde{d}(\{\lambda_k\}, u, h)\) is given by Eq. (4.16). We note that the scalar product (8.10) is valid for both the on-shell and the off-shell Bethe states, which means that the values of parameters \(\{u_k\}\) and \(\{\lambda_k\}\) are arbitrary.
Using the same procedure, the homogeneous limit of the form factors and correlation functions can be obtained. After some algebras, we have

\[ \langle \Phi \{ u_k \} | \sigma_{\bar{i}}^{-1} | \Phi \{ \lambda_k \} \rangle = |F_{\text{hom},D}^{\Lambda^{-1}}(\{ u_k \}, 0)\Lambda^{2N-i}(\{ \lambda_k \}, 0) \left[ 2^{N(N-1)/2} \prod_{k=1}^{N-1} k! \right]^{-1}, \quad (8.13) \]

where the elements of matrix \( F_{\text{hom},D} \) are

\[ F_{m,n}^{\text{hom},D} = \left. \frac{\partial^{m-1} f_n(u)}{\partial u^{m-1}} \right|_{u=0}, \]

\[ f_n(u) = \sum_{h=0}^{1} \tilde{\tau}(\{ u_l \}, \{ \lambda_l \}, h, u) e^{2(u-\eta h)(n-1)} \frac{\sinh(-u + \eta h)}{\sinh \eta}. \quad (8.14) \]

The homogeneous limit of the form factor \( (6.12) \) is

\[ \langle \Phi \{ u_k \} | \sigma_{\bar{i}}^{-1} | \Phi \{ \lambda_k \} \rangle = |F_{\text{hom},z}^{\Lambda^{-1}}(\{ u_k \}, 0)\Lambda^{2N-i}(\{ \lambda_k \}, 0) \left[ 2^{N(N-1)/2} \prod_{k=1}^{N-1} k! \right]^{-1}, \quad (8.15) \]

where \( F_{\text{hom},z} \) is the \((N+1) \times (N+1)\) matrix with the elements

\[ F_{m,n}^{\text{hom},z} = \left. \frac{\partial^{m-1} \varphi_n(u)}{\partial u^{m-1}} \right|_{u=0}, \quad 1 \leq m \leq N, \quad 1 \leq n \leq N, \]

\[ F_{m,N+1}^{\text{hom},z} = 2\left. \frac{\partial^{m-1} \tilde{\xi}(u)}{\partial u^{m-1}} \right|_{u=0}, \quad 1 \leq m \leq N, \]

\[ F_{N+1,n}^{\text{hom},z} = 1, \quad 1 \leq n \leq N, \]

\[ F_{N+1,N+1}^{\text{hom},z} = -\Lambda(\{ \lambda_k \}, 0), \quad (8.16) \]

the \( \varphi_n(u) \) is given by Eq.\( (8.11) \) and the \( \tilde{\xi}(u) \) reads

\[ \tilde{\xi}(u) = -d(\{ u_k \}, u, 1)d(\{ \lambda_k \}, u, 1)e^{uN \sinh^N(u + \eta)} \prod_{k=1}^{N} \frac{\sinh(u - u_k)}{\sinh \eta}. \quad (8.17) \]

The homogeneous limit of the correlation function \( (7.5) \) is

\[ \langle \Phi \{ u_k \} | \sigma_{\bar{i}}^{-1} \sigma_{\bar{i}}^{-1} | \Phi \{ \lambda_k \} \rangle \]

\[ = |F_{\text{hom},DD}^{\Lambda^{-2}}(\{ u_k \}, 0)\Lambda^{2N-i}(\{ \lambda_k \}, 0) \left[ 2^{N(N-1)/2} \prod_{k=1}^{N-1} k! \right]^{-1}, \quad (8.18) \]

where elements of matrix \( F_{\text{hom},DD} \) are

\[ F_{m,n}^{\text{hom},DD} = \left. \frac{\partial^{m-1} f_n^{-}(u)}{\partial u^{m-1}} \right|_{u=0}, \]
\begin{equation}
    f_n^-(u) = \sum_{h=0}^{1} \tilde{\tau}(\{u_l\}, \{\lambda_l\}, h, u) e^{2(u-\eta h)(n-1)} \frac{\sinh^2(u-\eta h)}{\sinh^2 \eta}.
\end{equation}

The homogeneous limit of the correlation function (7.14) is quite complicated to calculate. Here we give it at \( N = 2 \) case

\begin{equation}
    \langle \Phi \{u_k\} | \sigma_1^z \sigma_2^z | \Phi \{\lambda_k\} \rangle = \Lambda^2(\{\lambda_k\}, 0) \times \left\{ 4\tilde{\xi}^2(0) - \Lambda(\{\lambda_k\}, 0) |F_{hom,C}^{|/2} - \Lambda(\{u_k\}, 0) |F_{hom,z}^{|/2} \right\},
\end{equation}

where \( F_{hom,C} \) is the \( 3 \times 3 \) matrix with the elements

\begin{align*}
    F_{m,n}^{hom,C} &= \left. \frac{\partial^{m-1} \varphi_n(u)}{\partial u^{m-1}} \right|_{u=0}, \quad 1 \leq m \leq 2, \quad 1 \leq n \leq 2, \\
    F_{m,3}^{hom,C} &= \left. \frac{\partial^{m-1} \tilde{\xi}(u)}{\partial u^{m-1}} \right|_{u=0}, \quad 1 \leq m \leq 2, \\
    F_{3,n}^{hom,C} &= 1, \quad 1 \leq n \leq 2, \\
    F_{3,3}^{hom,C} &= 0,
\end{align*}

where the \( \varphi_n(u) \) is given by Eq.(8.11) and the \( \tilde{\xi}(u) \) is given by (8.17).

\section{Numerical results}

In this section, we check above results by the numerical calculation. For simplicity, we consider the homogeneous case. All the scalar products, the form factors and the correlation functions can be obtained by two ways. One is from the definition and the other is from the analytical formula provided in previous sections. Therefore, we can compare these two results and check the valid of the analytical formula.

Put all the inhomogeneous parameters be zero in Eq.(4.1), and we obtain the Bethe states in the homogeneous limit. We note that during this process, the singularity does not arise. From Eq.(4.3), we obtain the homogeneous right reference state \(|\Omega\rangle\)

\begin{equation}
    |\Omega\rangle = \sum_{l=0}^{\infty} \frac{(B^-)^l}{[l]_q^!} |0\rangle = \sum_{l=0}^{N} \frac{(B^-)^l}{[l]_q^!} |0\rangle,
\end{equation}

where the \( q \)-integers \( \{[l]_q, l = 0, \cdots \} \) and the operator \( B^- \) are given by

\begin{equation}
    [l]_q = \frac{1 - q^{2l}}{1 - q^2}, \quad [0]_q = 1,
\end{equation}
\[ [l]_q! = [l]_q [l - 1]_q \cdots [1]_q, \quad q = e^\eta, \]
\[ B^- = \sum_{l=1}^{N} e^{\frac{(N-1)\eta}{2}} e^{\frac{2}{\sum_{k=l+1}^{N} \sigma^z_k}} e^{-\frac{2}{\sum_{k=l}^{l-1} \sigma^z_k}}. \] \tag{9.2}

Through the similar calculation, we obtain the homogeneous left reference state
\[ \langle \Omega | = \sum_{l=0}^{\infty} \frac{0}{[l]_q!} = \sum_{l=0}^{N} \frac{(C^+)^l}{[l]_q!}, \] \tag{9.3}
where the explicit expression of the operator \( C^+ \) is
\[ C^+ = \sum_{l=1}^{N} e^{\frac{(N-1)\eta}{2}} e^{\frac{2}{\sum_{k=l+1}^{N} \sigma^z_k}} e^{\frac{2}{\sum_{k=l}^{l-1} \sigma^z_k}}. \] \tag{9.4}

Put all the inhomogeneous parameters be zero in the BAEs (2.17) and solve them, we obtain the values of Bethe roots \( \{u_k\} \) and \( \{\lambda_k\} \), which are shown in Table 1.

| \( \{u_k\} \) | \( \{\lambda_k\} \) |
|----------------|------------------|
| -1.637416729786854 + 1.570796326794897i | -1.431625849182040 - 0.0000000000000000i |
| -0.500000000000000 + 1.570796326794896i | -0.500000000000000 - 0.0000000000000000i |
| 0.637416729786854 + 1.570796326794897i | 0.431625849182040 + 0.0000000000000000i |

Table 2: Numerical result of the scalar product \( \langle \Phi\{\lambda_k\} | \Phi\{\lambda_k\} \rangle \), where \( \eta = 1, N = 3. \)

| \( \text{definition} \) | \( \text{numerical value} \) |
|------------------|------------------|
| \( \varphi_1(0) \) | 0.003625763123158 |
| \( P_{\text{hom}} \) | 0.05801220970527 |
| \( \text{analytical formula} \) | 0.003625763123158 |

Table 3: Numerical result of the form factor \( \langle \Phi\{u_k\} | \sigma^z_1 | \Phi\{\lambda_k\} \rangle \), where \( \eta = 1, N = 3. \)

| \( \text{definition} \) | \( \text{numerical value} \) |
|------------------|------------------|
| \( \varphi_1(0) \) | 0.200953522733016i |
| \( \xi(0) \) | 4.041937264439135i |
| \( \Lambda(\{u_k\}, 0) \) | 1.000000000000000 |
| \( \Lambda(\{\lambda_k\}, 0) \) | -1.000000000000000 |
| \( F_{\text{hom},z} \) | -3.215256363728254i |
| \( \text{analytical formula} \) | 0.200953522733016i |
Table 4: Numerical result of the form factor $\langle \Phi^{\{u_k\}} | \sigma_{i-1}^{-} | \Phi^{\{\lambda_k\}} \rangle$, where $\eta = 1, N = 3$.

| definition               | $0.113108828168255i$ |
|--------------------------|----------------------|
| $f^{-}_i (0)$            | $4.674571757211132i$ |
| $\Lambda (\{u_k\}, 0)$  | $1.000000000000000$  |
| $\Lambda (\{\lambda_k\}, 0)$ | $-1.000000000000000$ |
| $|F_{\text{hom},D}\rangle$ | $-1.809741250692130i$ |
| analytical formula       | $0.113108828168258i$ |

Table 5: Numerical result of the correlation function $\langle \Phi^{\{\lambda_k\}} | \sigma^{-}_1 \sigma^{-}_2 | \Phi^{\{\lambda_k\}} \rangle$, where $\eta = 1, N = 3$.

| definition               | $0.001006270991793$ |
|--------------------------|---------------------|
| $f^{-}_i (0)$            | $0.587693133253497$ |
| $\Lambda (\{\lambda_k\}, 0)$ | $-1.000000000000000$ |
| $|F_{\text{hom},DD}\rangle$ | $0.016100335868683$ |
| analytical formula       | $0.001006270991793$ |

Now, we are ready to calculate all the scalar products. The scalar product of two Bethe states $\langle \Phi^{\{\lambda_k\}} | \Phi^{\{\lambda_k\}} \rangle$, the form factors $\langle \Phi^{\{u_k\}} | \sigma_{i}^{-} | \Phi^{\{\lambda_k\}} \rangle$ and $\langle \Phi^{\{u_k\}} | \sigma_{i-1}^{-} | \Phi^{\{\lambda_k\}} \rangle$, and the correlation function $\langle \Phi^{\{\lambda_k\}} | \sigma_{i}^{-} \sigma_{i+1}^{-} | \Phi^{\{\lambda_k\}} \rangle$ are shown in Tables 2-5, respectively. In the these Tables, the first row is the results obtained from the definition, while the last row is the results obtained from the analytical formula. We see that these two results agree with each other very well.

10 Conclusions

In this paper, based on the inhomogeneous $T - Q$ relation and Bethe states obtained in [31, 35], we investigate the scalar products, the form factors and the two-point correlation functions of the XXZ spin chain with twisted boundary condition. We find that the correlation function $\langle \Phi^{\{u_k\}} | \sigma_{i}^{-} \sigma_{i+1}^{-} | \Phi^{\{\lambda_k\}} \rangle$ can be expressed as a ratio of two determinants, while the correlation function $\langle \Phi^{\{u_k\}} | \sigma_{i-1}^{-} \sigma_{i}^{-} | \Phi^{\{\lambda_k\}} \rangle$ can be expressed as a linear combination of $N + 2$ determinants. We also study the homogeneous limits of these results and check them numerically. The analytical results and the numerical ones agree with each other very well. These results could be used to study the correlation length, critical behavior and dynamic properties of the system.
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