ON A SMOOTHNESS CHARACTERIZATION FOR GOOD MODULI SPACES

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Abstract. Let \( \mathcal{X} \) be a smooth Artin stack with properly stable good moduli space \( \mathcal{X} \xrightarrow{\pi} X \). The purpose of this paper is to prove that a simple geometric criterion can often characterize when the moduli space \( X \) is smooth and the morphism \( \pi \) is flat.

Part I. Main Results

1. Introduction

Let \( K \) be an algebraically closed field of characteristic 0 and \( \mathcal{X} \) be a \( K \)-smooth Deligne–Mumford stack with coarse space \( p: \mathcal{X} \to X \). Applying the Purity of the Branch Locus Theorem to the proper quasi-finite morphism \( p \) yields a necessary condition for \( X \) to be smooth: the branch locus of \( p \) must be pure of codimension-one; here the branch locus is the complement of the largest open set \( U \subset X \) over which \( p \) is étale. On the other hand, the beautiful theorem of Chevalley–Shephard–Todd gives a simple group-theoretic criterion which is sufficient for determining when \( X \) is smooth. Specifically, if \( x \in \mathcal{X}(K) \) with stabilizer group \( G_x \), then \( X \) is smooth at \( p(x) \) if and only if the \( G_x \)-action on the tangent space \( T_{X,x} \) is generated by pseudo-reflections, i.e. \( G \) is generated by elements \( g \in G_x \) whose fixed locus is a hyperplane. Whenever \( X \) is smooth, \( p \) is automatically flat.

For smooth Artin stacks we consider the following situation analogous to the Deligne–Mumford setting. Let \( \mathcal{X} \) be a smooth Artin stack with properly stable good moduli space \( p: \mathcal{X} \to X \); this means that there is a dense set of points \( x \) in \( \mathcal{X} \) which have 0-dimensional stabilizer and are also \( p \)-saturated, i.e., \( p^{-1}p(x) = x \) [ER, Definition 2.5]. In this case the good moduli space morphism is not separated but shares some properties of a proper quasi-finite morphism: it is universally closed and if \( x \in X \) is a closed point, then there is a unique closed point in the fiber of \( p \) over \( x \). Such \( \mathcal{X} \) arise naturally in the context of GIT, e.g. if \( G \) is a reductive group with properly stable action on a variety \( U \) and if \( U^{ss} \) denotes the semistable locus, then \( p: [U^{ss}/G] \to U/G \) is a properly stable good moduli space.

Despite the fact that smooth Artin stacks with properly stable good moduli spaces are analogous to Deligne–Mumford stacks, there are no general necessary and sufficient criteria to determine when \( X \) is smooth and \( p \) is flat. Indeed, one cannot invoke the Purity of the Branch Locus Theorem as \( p \) is not proper quasi-finite, and there is no known analogue of the Chevalley–Shephard–Todd Theorem since smooth Artin stacks do not have tangent bundles.

The starting point for this paper is to instead take a GIT point of view. By [AHR, Theorem 4.12] and [Lun], at a closed point \( x \) of \( \mathcal{X} \), the map \( p \) is étale locally isomorphic to \([V/G_x] \to V/G\) for some representation \( V \) of the stabilizer group \( G_x \). Thus, the problem of determining when \( X \) is smooth and \( p \) is flat reduces to the case where \( \mathcal{X} = [V/G] \) and \( V \) is a representation of a linearly reductive group \( G \). A natural analogue of the branch locus is then the image in \( X \) of points in \( V \) which have a positive dimensional stabilizer group. This is exactly the image of the GIT strictly
semi-stable points of $X$. Inspired by the Purity of the Branch Locus Theorem, a naïve guess is that the following condition is necessary for $X$ to be smooth:

(*) The image of the strictly semi-stable points must be of pure codimension-one.

The main results of this paper imply that condition (*) goes a long way toward determining when $V/G$ is smooth and $p$ is flat. Specifically, we prove for irreducible representations of simple groups, condition (*) is both necessary and sufficient for $V/G$ to be smooth and $[V/G] \rightarrow V/G$ to be flat. In addition we show that when $G$ is a torus, a slight strengthening of condition (*) is necessary and sufficient to characterize when $V/G$ is smooth and $[V/G] \rightarrow V/G$ is flat.

To state our results precisely we introduce Definition 1.2 after recalling some basic notions.

**Definition 1.1.** Let $V$ be a representation of a reductive group $G$. A vector $v \in V$ is $G$-stable if $Gv$ is closed and $v$ is not contained in the closure of any other orbit. A vector $v \in V$ is $G$-properly stable if $v$ is stable and $\dim Gv = \dim G$.

A representation $V$ is stable (resp. properly stable) if it contains a stable (resp. properly stable vector). In this case, the set $V^s = V^s(G)$ of $G$-stable (resp. properly stable) vectors is Zariski open. We denote by $V^{sss} = V^{sss}(G)$ the closed subset $V \setminus V^s$; vectors $v \in V \setminus V^s$ are said to be $G$-strictly semi-stable.

**Definition 1.2.** Let $V$ be a stable representation of a connected reductive group $G$ and let $\pi : V \rightarrow V/G$ be the quotient map. Then $V$ is pure if $\pi(V^{sss})$ is pure of codimension-one in $V/G$.

Despite the fact that $p$ is not proper quasi-finite and that $\pi(V^{sss})$ is not a perfect analogue of the branch locus of $p$, our crude GIT analogy is already remarkably powerful, as witnessed by Theorem 1.3 below. Recall that a $G$-representation $V$ is called coregular if $V/G$ is smooth, and is called cofree if it is coregular and $\pi : V \rightarrow V/G$ is flat.

**Theorem 1.3.** Let $V$ be an irreducible stable representation of a simple Lie group $G$. Then $V$ is cofree if and only if $V$ is pure.

For groups $G$ with non-trivial characters, a more refined notion of purity is needed. By Remark 1.5, this more refined notion is equivalent to purity for simple Lie groups.

**Definition 1.4.** Let $V$ be a stable representation of a connected reductive group $G$ and let $\pi : V \rightarrow V/G$ be the quotient map. We say $V$ is coprincipal if it is pure and every irreducible component of $V^{sss}$ (with its reduced subscheme structure) maps to a principal divisor under $\pi$.

**Remark 1.5.** If $G$ has no non-trivial characters then it is easy to show (Lemma 1.1.3) that the notions of pure and coprincipal coincide, and that a sufficient condition for $V$ to be pure is that $\text{codim}(V^{sss}) = 1$. In contrast, when $G$ is a torus, we exhibit pure representations which are not coprincipal; we also give representations where $V^{sss}$ is a divisor but $V$ is not pure, i.e. $\pi(V^{sss})$ is not pure of codimension-one. See Example 1.2.3.

Further illustrating the utility of our GIT analogy, we prove:

**Theorem 1.6.** A stable torus representation is cofree if and only if it is coprincipal.

**Remark 1.7.** Furthermore, we prove in Proposition 1.1.2.4 that if $V$ is a pure representation of a torus which is not coprincipal then $V/G$ has worse than finite quotient singularities.

**Remark 1.8.** The restriction to stable representations in Theorem 1.6 is relatively insignificant because Wehlau [Veh, Lemma 2] proved that any torus representation $V$ has a (canonical) stable submodule $V'$ such that $V'/T = V/T$ with the properties that $V' = V$ if and only if $V$ is stable, and $V'$ is cofree if and only if $V$ is cofree. Thus Theorem 1.6 can be restated as saying that if $V$ is

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1Note that any locally principal divisor on $V/G$ is principal because $\text{Pic}(V/G) = 0$ [KKV]
an arbitrary representation of a torus with non-trivial invariant ring, then \( V \) is cofree if and only if the stable submodule \( V' \) is coprincipal.

Our main results relating coregularity, cofreeness, purity, and coprincipality for stable representations are summarized in the diagram below. Note that when \( V \) is irreducible and \( G \) is simple, all four notions coincide.

![Diagram](image)

### 1.1. Questions and examples

The fact that coprincipality characterize cofreeness for irreducible representations of simple Lie groups as well as reducible representations of tori suggests that it may be a useful class of representations in greater generality. We pose the following questions.

**Question 1.9.** Let \( V \) be a stable representation of a connected reductive group \( G \).

1. If \( V \) is cofree, then is it coprincipal?
2. Let \( G \) be semisimple and \( V \) be irreducible. If \( V \) is pure (equivalently coprincipal), then is it cofree?

**Remark 1.10 (Relationship to a result of Brion).** Michel Brion pointed us to a result of his \([\text{Bri}, 4.3 \text{ Corollaire 1}]\) which gives some evidence for an affirmative answer to Question 1.9(1). Precisely, Brion proves that if \( V \) is a properly stable representation of a reductive group (not necessarily connected), and if \( \text{codim}(V \setminus V^\text{pr}) \geq 2 \), then \( K[V] \) cannot be a free \( K[V]^G \) module. Here \( V^\text{pr} \) is the locus of orbits of principal type. Since \( V^\text{pr} \subset V^s \), it follows that such representations are not pure.

**Remark 1.11 (Reducible representations).** We note that the irreducibility assumption in Question 1.9(2) cannot be dropped, even when \( G \) is simple. The smallest example we know to illustrate this, which we learned from Gerald Schwarz, is the \( \text{SL}_3 \)-representation \( V = \text{Sym}^2(C^3) \oplus (C^3)^{\oplus 2} \). The fact that the irreducibility assumption cannot be dropped is completely analogous to the picture for finite groups. Indeed, for a faithful representation \( V \) of a finite group \( G \), if \( V^f \) denotes the open set on which \( G \) acts freely, then the condition that \( V \setminus V^f \) is a divisor is necessary but not sufficient for cofreeness of \( V \). For a simple example, consider the \( \mu_4 \)-action on \( C^2 \) with weights \((1,2)\).

Although reducible, pure, non-cofree representations do exist, the conditions of purity and cofreeness are both quite rare for reducible representations. Indeed, for semisimple groups, any representation with no trivial summands and at least two properly stable summands cannot be cofree; similarly, for a reductive group, any representation with at least two properly stable summands cannot be pure. To see the former statement, note that any properly stable \( G \)-representation has dimension at least \( \dim G + 1 \), and by [PV, Theorem 8.9] if \( G \) is semisimple then any coregular (and thus cofree) representation with no trivial summands has dimension at most \( 2 \dim G \). To see the latter statement, note that if \( V \) and \( W \) are properly stable representations then the Hilbert–Mumford criterion implies that \((V^s \oplus W) \cup (V \oplus W^s) \subset (V \oplus W)^s\). Since \( \text{codim}_V(V \setminus V^s) \geq 1 \) and \( \text{codim}_W(W \setminus W^s) \geq 1 \), we see that \( \text{codim}_{V \oplus W}((V \oplus W) \setminus (V \oplus W)^s) \geq 2 \).
2. OUTLINE OF THE PROOFS OF THE MAIN THEOREMS

2.1. Theorem 1.3: “only if” direction. This is the easier direction of Theorem 1.3. Recall that a representation of $V$ is polar if there is a subspace $c \subset V$ and a finite group $W$ such that $K[V]^G = K[c]^W$. The basic example of a polar representation is the adjoint representation $g$; here $c$ is a Cartan subalgebra and $W$ is the Weyl group. Using results of Dadok and Kac [DK] we prove that any stable polar representation (not necessarily irreducible) is pure, see Proposition II.2.1. On the other hand, Dadok and Kac proved that any irreducible cofree representation of a simple group is polar. Thus, we conclude that any stable irreducible cofree representation of a simple group is pure.

2.2. Theorem 1.3: “if” direction. This is the most involved result of the paper. In Section II.3 we show that if $G$ is reductive and $V$ is a pure $G$-representation, then there is a hyperplane $H$ in the character lattice of $V$ tensored with $\mathbb{R}$ satisfying the following special condition: $H$ contains at least $\dim V - \dim G + 1$ weights when counted with multiplicity. In particular, this implies that when $G$ is semisimple, every irreducible pure representation $V$ has dimension bounded by a cubic in $\text{rk}(G)$. In Section II.4 we further show that if $V$ is pure, then its highest weight lies on a ray (or possibly a 2-dimensional face, if $G = \text{SL}_n$) of the Weyl chamber. Comparing with the known list of cofree representations of simple groups, we are reduced to checking that 11 infinite families and 94 more sporadic cases, are not pure. These calculations, performed in Section II.5 are mostly done by computer, but a number must be done by hand, and will show the nature of the computer analysis done.

2.3. Theorem 1.6. We prove Theorem 1.6 for tori $T$ by inducting on $\dim V$. The key to the proof is showing in Proposition III.1.6 that if $V$ is a coprincipal representation of a torus, then $V$ splits as a sum of $T$-representations $V = V_1 \oplus V_2$ such that $V/T = V_1/T \times V_2/T$ and $V_1/T$ is 1-dimensional. This argument makes essential use of the fact that the images of the irreducible components of $V^{\text{ssss}}$ are Cartier divisors.

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Part II. Representations of simple groups: proof of Theorem I.1.3

This part is organized as follows. In §1 we prove some basic facts about pure representations that we will use throughout. In §2 we prove that every stable cofree irreducible representation of a connected simple group $G$ is pure, that is, we prove the “only if” direction of Theorem I.1.3. In §3 we prove the key result that if $V$ is any pure representation, then there is a hyperplane in the weight lattice containing most of the weights. This implies that up to isomorphism there are a finite number of pure representations not containing a trivial summand.

In §4 we apply the criteria in §3 to create a finite list on which all irreducible pure representations of a simple group may be found. Then, in §5 we demonstrate that representations on this list which are not cofree are also not pure, thus proving Theorem I.1.3.
1. Basic facts about pure representations

Lemma 1.1. If $V$ is a representation of a reductive group $H$, and $f : G \to H$ is a surjection, then $V$ is pure when considered as an $H$-representation if and only if $V$ is pure when considered as a $G$-representation.

Proof. Since $f$ is surjective, $Hv = f(G)v$, which is $Gv$ by definition of the $G$-action. So all the orbits of the actions are the same, and hence their purity is the same. □

In the event that $G$ is a simple Lie group which is not simply connected, it has a universal covering $G' \to G$ with finite kernel $K$. There is a one-to-one correspondence between $G$-representations and $G'$-representations which are trivial when viewed as a $K$-representations. By Lemma 1.1 we have

Corollary 1.2. To prove Theorem 1.3 for all simple Lie groups, it suffices to prove it for the exceptional groups and the groups $SL_n, Sp_{2n}, Spin_n$ for all $n$.

Lemma 1.3. Let be $G$ a reductive group with no non-trivial characters. If $V$ is stable representation for which $V^{ss}$ is pure of codimension-one, then $V$ is coprincipal. In particular, every pure representation of $G$ is coprincipal.

Proof. Since $G$ is connected, every component of $V^{ss}$ is $G$-invariant. Thus, the equation $f \in K[V]$ of the component must be an eigenfunction for the action of $G$ on $K[V]$; that is, $g \cdot f = \lambda(g)f$ for all $g \in G$. Since $G$ has no non-trivial characters, $f$ must in fact be invariant. Hence the image of $V(f)$ is the Cartier divisor defined by $f$ in $Spec K[V]^G$. □

Lemma 1.4. If $\rho, \rho'$ are representations of $G$ on a vector space $V$, and there exists an automorphism $f$ of $G$ exchanging $\rho$ and $\rho'$, then $\rho$ is pure (resp. cofree) if and only if $\rho'$ is pure (resp. cofree).

Proof. Purity and cofreeness are both determined at the level of the image of the $\rho'(G) = \rho(f(G)) = \rho(G) \subseteq GL(V)$. □

Remark 1.5. Lemma 1.4 will be used for the spin groups in the following fashion: for each spin group $Spin_{2n}$ of even order, there exists an outer automorphism exchanging the half-spinor representation $\Gamma_{\omega_{2n}}$ and the half-spinor representation $\Gamma_{\omega_{2n-1}}$, and it follows that the positive half-spinor representation is cofree and pure if and only if the negative half-spinor representation is cofree and pure.

For $Spin_8$ there is a triality which gives automorphisms exchanging the half-spinor representations and the standard representation. Since the standard representation is cofree and pure, both half-spinor representations of $Spin_8$ are cofree and pure. Moreover, these outer automorphisms send the symmetric square $Sym^2 C^8$ with highest weight $2\omega_1$ to the representations $\Gamma_{2\omega_7}, \Gamma_{2\omega_8}$, which are the irreducible components of the wedge product $\wedge^4 C^8$. Since $Sym^2 C^8$ is cofree and pure as a $Spin_8$ representation, so too are both irreducible components of $\wedge^4 C^8$.

For low dimensional special orthogonal Lie algebras, there exist exceptional isomorphisms $sp(4) = so(5)$ and $sl(4) = so(6)$. These obviously preserve purity and cofreeness of representations, and so it suffices for the classical groups to prove Theorem 1.3 for $SL_n$ when $n \geq 2$, $Spin_{2n+1}$ when $n \geq 2$, $Sp_{2n}$ when $n \geq 3$, and $Spin_{2n}$ when $n \geq 4$.

2. Proof of the “only if” direction of Theorem 1.3

The proof of the “only if” direction of Theorem 1.3 is relatively straightforward thanks to the work of Dadok and Kac on polar representations [DK]. Recall that a representation $V$ of a reductive group $G$ is polar if there exists a subspace $\mathfrak{c}$, called a Cartan subspace, such that the map $\mathfrak{c} \to Spec K[V^G]$ is finite and surjective.
In [DK] Theorem 2.9, Dadok and Kac proved that if \( V \) is polar with Cartan subspace \( c \), then the group \( W = N_G(c)/Z_G(c) \) is finite and \( K[V]^G = K[c]^W \). By [DK] Theorem 2.10, every polar representation is cofree. Furthermore, using the classification of irreducible cofree representations of simple groups, they showed that every irreducible cofree representation of a simple group is polar.

As a result, to prove the “only if” direction of Theorem 1.3, it is enough to show that polar representations are pure. We are grateful to Ronan Terpereau for suggesting this proof.

**Proposition 2.1.** If \( V \) is a stable polar representation (not necessarily irreducible), then it is pure.

**Proof.** Let \( c \) be a Cartan subspace, and following [DK] p. 506, let \( c^{reg} \) be the set of regular points. By definition, \( v \in c^{reg} \) if and only if \( Gv \) is closed and of maximal dimension among closed orbits. If \( V \) is a stable representation then this condition is equivalent to the stability of \( v \). Since the Cartan subspace contains a point of each closed \( G \) orbit, it follows that \( Gc^{reg} = V^s \). By [DK] Lemma 2.11, \( c_{sing} = c \setminus c^{reg} \) is a finite union of hyperplanes, and \( V^{sss} = Gc_{sing} \) by definition. If \( W \) is as above, then the image of \( c_{sing} \) under the quotient map \( p: c \to c/W \) is a divisor. The composition \( c \hookrightarrow V \xrightarrow{\pi} V/G \) is finite by [DK] Proposition 2.2]. Under the identification \( V/G = c/W \) this finite map is just the quotient map \( c \to c/W \). Thus every irreducible component of \( V^{sss} = \pi^{-1}(p(c_{sing})) \) is a divisor, because \( V \to V/G \) is flat, as \( V \) is cofree by [DK] Theorem 2.10]. By By Lemma 1.3 we see \( V \) is pure. \( \square \)

**Remark 2.2.** As noted by Victor Kac, there are polar representations with non-trivial rings of invariants where our proposition does not apply. However, an analogous statement holds with \( V^s \) replaced by the \( G \)-saturation of the locus of closed orbits, which are of maximal dimension among closed orbits.

3. **Bounding the dimension of a pure representation**

We begin by obtaining results that show pure representations are relatively rare; specifically, any simple group has a finite number of pure representations that do not contain a trivial summand.

The following result holds for any reductive group, not just simple or semi-simple ones.

**Proposition 3.1.** Let \( V \) be a stable representation of a reductive group \( G \). Suppose \( V^{sss} \) contains a divisorial component that maps to a divisor in \( V/G \), e.g. \( V \) is pure. Then there exists one-parameter subgroup \( \lambda \) such that

\[
\dim V_\lambda^0 \geq \dim V - \dim G - 1,
\]

where \( V_\lambda^0 \subset V \) denotes the 0-weight space of \( \lambda \); said differently, there are at most \( \dim G + 1 \) weights that do not lie on the hyperplane of the weight space defined by \( \lambda \).

**Proof.** First assume that \( V \) is properly stable. In this case, every stable vector has finite dimensional stabilizer. Hence, a vector \( v \) is not stable if and only if it contains a point with positive dimensional stabilizer in its orbit closure. Any closed orbit in a representation is affine so its stabilizer is reductive by [MM], so if it is positive dimensional then it contains a one-parameter subgroup. Thus \( v \in V^{sss} \) if and only if there is a 1-parameter subgroup \( \lambda \) such that \( v \in V_\lambda^{\geq 0} \), where \( V_\lambda^{\geq 0} \) is the subspace of \( V \) whose vectors have non-negative weight with respect to \( \lambda \).

Since all one-parameter subgroups are \( G \)-conjugate, it follows that \( V^{sss} = \bigcup_{\lambda \in N(T)} GV_\lambda^{\geq 0} \) where \( N(T) \) is the group of one-parameter subgroups of a fixed maximal torus \( T \). Since \( V \) is finite dimensional, it contains a finite number of weights, so there are only finitely many distinct subspaces \( V_\lambda^{\geq 0} \) as \( \lambda \) runs through the elements of \( N(T) \). Hence, there exists a one-parameter subgroup \( \lambda \) such that \( GV_\lambda^{\geq 0} \) is the divisorial component of \( V^{sss} \). Since \( GV_\lambda^{\geq 0} \) is the \( G \)-saturation of the fixed locus \( V_\lambda^0 \), it follows that \( \pi(GV_\lambda^{\geq 0}) = \pi(V_\lambda^0) \), where \( \pi: V \to V/G \) is the quotient map. Hence \( \pi(V_\lambda^0) \) is a divisor in \( V/G \), and it therefore has dimension \( \dim V - \dim G - 1 \). Thus \( \dim V_\lambda^0 \geq \dim V - \dim G - 1 \).
When $V$ is stable but not properly stable, it is still the case that any strictly semi-stable point contains a point with positive dimensional stabilizer in its orbit closure. The same argument used above implies that $V^{ss} \subset \bigcup_{\lambda \in N(T)} GV^\geq_0$. It follows that any divisorial component of $V^{ss}$ is contained in $GV^\geq_0$ for some 1-parameter subgroup $\lambda$. If this divisorial component maps to a divisor, then image of $V^0_\lambda$ contains a divisor, so we conclude that $\dim V^0_\lambda \geq \dim V/G - 1 \geq \dim V - \dim G - 1$. 

Remark 3.2. It is clear that if a representation can be shown to not be pure by Proposition 3.1 then its dual representation will also not be pure, since it will have the same number of weights in a hyperplane.

Example 3.3. In this example, we illustrate that if $V$ is not properly stable, then $V^{ss}$ may be a proper subset of $\bigcup_{\lambda \in N(T)} GV^\geq_0$. Let $V$ be the adjoint representation of $SL_2$. Then the strictly semi-stable locus $V^{ss}$ is the divisor defined by the vanishing of the determinant. However, since the torus of $SL_2$ is rank one, the fixed locus $V^0_\lambda$ is the same for all $\lambda$ and is one dimensional. In this case, $V = \bigcup_{\lambda \in N(T)} GV^\geq_0$ and $\pi(V^0_\lambda) = \pi(V) = A^1$. 

We are grateful to the referee for pointing out the following consequence of Proposition 3.1 which has been used to significantly simplify the computations in the paper:

Lemma 3.4. Suppose $\{t_1, \ldots, t_k\}$ are a complete set of Weyl group orbit representatives of order $\ell$ toral elements in the maximal torus of a simple Lie group $G$. If $\pi(V^{ti}) \subset V/G$ has codimension at least 2 for all $i$, then $\codim V^{ss} \geq 2$. Moreover,

$$\dim \pi(V^{ti}) \leq \dim V^{ti}/N_{t_i} \leq \dim V^{ti},$$

where $N_{t_i}$ denotes the normalizer of $t_i$.

Proof. Any 1-parameter subgroup $\lambda$ contains some toral element $t$ of order $\ell$. Thus an upper bound for $\dim \pi(V^\lambda)$ is $\max\{\dim \pi(V^t) \mid t \text{ of order } \ell\}$. In particular, if each $\pi(V^t)$ has codimension at least 2, then $\pi(V^\lambda)$ has codimension at least 2, and by Proposition 3.1, $\dim V^{ss} \geq 2$. Moreover, the morphism $V^t \to \pi(V^t)$ factors through $V^t/N_{t_i}$ of $t$, whence $\dim \pi(V^t) \leq \dim V^{ti}/N_{t_i} \leq \dim V^{ti}$. 

In practice, Lemma 3.4 is much easier to use than Proposition 3.1. Software packages like LiE [LCL] are capable of computing $\dim V^t$ given a representation of a semisimple Lie group and a toral element $t$ of finite order, and so it is possible to show many representations are not pure by checking that $\dim V^t \leq \dim V - \dim G - 2$ for all $t$ up to the action of the Weyl group. For representations where this bound is not a priori clear and for representations occurring in infinite families, a more careful analysis can be done by hand. In some exceptional cases, it is still easier to count the number of weights occurring in a hyperplane.

An important use of Proposition 3.1 is to show that a simple Lie group has finitely many pure representations. This can be shown by establishing a bound on the dimension of a pure representation which is polynomial in the rank of the group. To show this bound, we separately bound with multiplicity the number of zero and non-zero weights that can occur in the representation.

Proposition 3.5. Let $V$ be a (stable) representation of a simple Lie group $G$, and suppose that $V^{ss}$ contains a divisorial component. Then there are at most $\text{rk}(G)(\dim(G) + 1)$ non-zero weights counted with multiplicity.

Proof. Since $G$ is simple, the image of a divisorial component of $V^{ss}$ in $V/G$ is also a divisor by Lemma 3.1. Hence by Proposition 3.1 there is a 1-parameter subgroup $\lambda$ such that $V^0_\lambda$ contains at

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1That is, elements contained in some fixed choice of maximal torus of $G$ that act on $V$ with eigenvalues of the form $\exp(2\pi i k/\ell)$ for some $k$. 

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least $\dim V - \dim(G) - 1$ weights. Let $H$ be the hyperplane in the character lattice determined by this one-parameter subgroup, so Proposition 3.1 says that $H$ contains at least $\dim V - (\dim(G) + 1)$ such weights. The Weyl group conjugates of $H$ also contain at least $\dim V - (\dim(G) + 1)$ weights. We claim that $H$ has at least $\rk(G)$ linearly independent conjugates under the Weyl group. Suppose $v$ is a normal vector to $H$. If $W$ is the Weyl group, then it acts linearly on $\mathfrak{h}^*$, and the representation so obtained is irreducible by [FH, Lemma 14.31]. Since the linear span of the orbit $Wv$ is a non-zero $W$-invariant subspace of $\mathfrak{h}^*$, it must be all of $\mathfrak{h}^*$, and so the number of linearly independent conjugates of $v$ under the Weyl group is $\dim \mathfrak{h}^* = \dim \mathfrak{h} = \rk(G)$.

Let $H = H_1, \ldots, H_{\rk(G)}$ be $\rk(G)$ such conjugate hyperplanes whose normal vectors are linearly independent. By inclusion-exclusion, $H_1 \cap H_2$ contains at least $\dim V - 2(\dim(G) + 1)$ weights counted with multiplicity, since $H_1 \cup H_2$ contains at most $\dim V$ weights. Assume by induction that $H_1 \cap \cdots \cap H_k$ contains at least $\dim V - k(\dim(G) + 1)$ weights counted with multiplicity. Since $(H_1 \cap \cdots \cap H_k) \cup H_{k+1}$ still contains at most $\dim V$ weights, the inclusion-exclusion principle implies that $(H_1 \cap \cdots \cap H_k) \cap H_{k+1}$ contains at least $\dim V - (k + 1)(\dim(G) + 1)$ weights counted with multiplicity. Hence, $\{0\} = H_1 \cap \cdots \cap H_{\rk(G)}$ contains at least $\dim V - \rk(G)(\dim(G) + 1)$ weights counted with multiplicity. In other words, the multiplicity of the 0 weight in $V$ is at least $\dim V - \rk(G)(\dim(G) + 1)$.

Lemma 3.6. Let $V$ be an irreducible representation of a semi-simple Lie group $G$, and let $\alpha_i$ be the positive simple roots. If $a$ is any non-highest weight for $V$, then

$$
\dim(V_a) \leq \sum_i \dim(V_{a + \alpha_i}).
$$

In particular, if $V$ is not the trivial representation, then $\dim(V_0) \leq \sum_i \dim(V_{\alpha_i}).$

Proof. Consider the linear map $V_a \to \bigoplus_i V_{a + \alpha_i}$ given by $v \mapsto (e_i(v))$ where $e_i$ is the root vector in the Lie algebra for $\alpha_i$. Since $a$ is not a highest weight, $V_a$ does not contain a highest weight vector, that is, no vector $v \in V_a$ is killed by all positive simple roots. Hence, the above map is injective.

Lemma 3.7. Let $V$ be a representation of a simple Lie group $G$ which contains no trivial summands. Then

$$
\frac{\dim(V_0)}{\dim(V)} \leq \frac{\rk(G)}{\dim(G) - \rk(G)}.
$$

Proof. It clearly suffices to prove the lemma for every non-trivial irreducible subrepresentation of $V$, and so we may assume $V$ is irreducible. Let $d = \dim V$, let $d_0 = \dim V_0$ be the dimension of the 0-weight space, and let $d_\alpha = \dim V_{\alpha}$ be the dimension of the weight space of any root $\alpha$. There are $\dim(G) - \rk(G)$ total roots, so we obtain the inequality $(\dim(G) - \rk(G))d_\alpha + d_0 \leq d$. Of the $\dim(G) - \rk(G)$ roots, $\rk(G)$ of them are simple, so by Lemma 3.6, $d_0 \leq \rk(G)d_\alpha$. Thus,

$$
\frac{d_0}{d} \leq \frac{\rk(G)d_\alpha}{(\dim(G) - \rk(G))d_\alpha + d_0} \leq \frac{\rk(G)}{\dim(G) - \rk(G)}.
$$

Proposition 3.8. Let $V$ be a (stable) representation of a simple Lie group $G$ which contains no trivial summands and such that $V^{\text{ss}}$ contains a divisor. If 0 is not a weight of $V$, then $\dim V \leq \rk(G)(\dim(G) + 1)$. If 0 is a weight of $V$, then $\dim V \leq \frac{\rk(G)(\dim(G) + 1)(\dim(G) - \rk(G))}{\dim(G) - 2\rk(G)}$.

Proof. Proposition 3.5 proves the claim when 0 is not a weight. Assume 0 is a weight. Since $V^{\text{ss}}$ is a divisor, it follows from Proposition 3.5 that $d \leq \rk(G)(\dim(G) + 1) + d_0$ where $d = \dim V, d_0 = \dim V_0$. If $0$ is not a weight of $V$.
\[
\dim V_0. \text{ Then }
\]
\[
d \leq \text{rk}(G)(\dim(G) + 1) + d_0 \leq \text{rk}(G)(\dim(G) + 1) + \frac{\text{rk}(G)}{\dim(G) - \text{rk}(G)}d
\]
by Lemma 3.7, which implies that
\[
d \leq \frac{\text{rk}(G)(\dim(G) + 1)(\dim(G) - \text{rk}(G))}{\dim(G) - 2 \text{rk}(G)}.
\]

For the classical groups, we make use of slightly weaker bounds than those in Proposition 3.8.

**Definition 3.9.** Let \( G \) be a classical group and \( V \) an irreducible \( G \)-representation. Then \( V \) is **small enough** if \( \dim V \leq \kappa(G) \), where
\[
\kappa(G) = \begin{cases} 
  n^3, & G = \text{SL}_n \\
  2(n^3 + n^2 + n + 1), & G = \text{Sp}_{2n} \\
  2(n^3 + n^2 + n + 1) + 1, & G = \text{SO}_{2n+1} \\
  139, & G = \text{SO}_8 \\
  2(n^3 + n + 3/2) + 1, & G = \text{SO}_{2n}, n > 4.
\end{cases}
\]

In this case, we often say the highest weight of \( V \) is small enough.

**Remark 3.10.** By Proposition 3.8, a pure irreducible representation of a simple Lie group is small enough. The problem of enumerating the pure representations may thus be reduced to first enumerating the small enough representations, and then determining which of them are pure.

**Remark 3.11.** The bound of Proposition 3.8 holds for representations of semisimple Lie groups also. Unlike the case of simple Lie groups, for semisimple Lie groups difficulties arise in trying to create lists of representations meeting these bounds primarily because if \( G_1 \) is a simple Lie group with large rank and \( G_2 \) is a simple Lie group with much smaller rank, then \( V_1 \otimes V_2 \) frequently gives a representation which is small enough but is not pure or cofree. The sizes of the lists in this case are infeasible to handle with the methods presented here.

## 4. Small enough representations

Let \( \mathfrak{g} \) be a simple Lie algebra with Cartan algebra \( \mathfrak{h} \) and fundamental weights \( \omega_1, \ldots, \omega_n \in \mathfrak{h}^* \). Denote by \( \Gamma_{a_1\omega_1 + \cdots + a_n\omega_n} \) the representation of \( \mathfrak{g} \) with highest weight vector \( a_1\omega_1 + \cdots + a_n\omega_n \). Define the **width** of the vector \( a_1\omega_1 + \cdots + a_n\omega_n \) to be \( a_1 + \cdots + a_n \), and define its **support** to be the number of \( i \) such that \( a_i \neq 0 \). If \( \omega = a_1\omega_1 + \cdots + a_n\omega_n \) is a weight, and \( i, j \) are such that \( a_i, a_j \neq 0 \), define the **shift** \( \omega_{i \rightarrow j} \) to be \( \omega + a_i(\omega_j - \omega_i) \). Note that \( \omega \) and \( \omega_{i \rightarrow j} \) both have the same width, and the support of \( \omega_{i \rightarrow j} \) is one less than the support of \( \omega \). We use the following lemma, which is proven along the way to [GGS] Lemma 2.1.

**Lemma 4.1.** Suppose \( (a_1, \ldots, a_n) \in \mathbb{N}^n \), and let \( \omega = a_1\omega_1 + \cdots + a_n\omega_n \). If \( i, j \) are distinct indices such that \( a_i, a_j \neq 0 \), then
\[
\dim \Gamma_\omega \geq \min \left( \dim \Gamma_{\omega_{i \rightarrow j}}, \dim \Gamma_{\omega_{j \rightarrow i}} \right).
\]

In particular, if \( \omega \) is small enough, then one of \( \omega_{i \rightarrow j}, \omega_{j \rightarrow i} \) must be small enough. This suggests the following algorithm to classify the small enough representations:

1. Find a closed form for the dimension of the irreducible representation with highest weight \( \omega_s \) for each \( s \), and classify the 1-supported small enough weights using these closed forms.
2. If \( \omega \) is a small enough weight with support \( \{i, j\} \), use Lemma 4.1 to restrict the possibilities for \( \omega \) and classify the 2-supported small enough weights.
If the list of 2-supported small enough weights is not empty, repeat the above procedure for 3-supported small enough weights, and so forth, until the list of $K$-supported small enough weights is empty for some $K$.

If $\omega$ is a weight with support greater than $K$, by repeatedly applying Lemma 4.1 its dimension can be bounded below by that of a $K$-supported weight. Since no $K$-supported weight is small enough, it follows that $\omega$ is not small enough for any support greater than $K$, and the list is complete.

**Reading the tables.** Each of the families of Lie groups in this section are accompanied by a list of all their small enough representations, see Tables 1–4. The representations are described by their highest weights, in the notation of [FH].

The final column in each table describes whether the indicated representation is cofree and/or pure for the indicated values of $n$. By the “only if” direction of Theorem I.1.3, cofree implies pure for a stable irreducible representation of a simple group, so “cofree” is written to mean “cofree, pure, and stable”. Since no $K$-supported weight is small enough, it follows that $\omega$ is not small enough for any support greater than $K$, and the list is complete.

For $\text{SL}_2$, every irreducible representation is of the form $\text{Sym}^k V$ for some $k$, where $V$ is the defining representation. The maximum number of weights contained in a hyperplane is then 0 if $k$ is odd, or 1 if $k$ is even. So if $k > 4$, then $\text{Sym}^k V$ is not pure by Proposition 3.1; one can check that when $k \leq 4$, the representation is both cofree and pure. In the sequel we consider representations of $\text{SL}_n$ for $n > 2$.

**Lemma 4.2.** The small enough irreducible representations of $\text{SL}_n$ with 1-supported highest weights are the entries (1)–(20) of Table 1.

Proof. Recall that $\Gamma_{\omega_s} = \bigwedge^s \text{SL}_n$ and so has dimension $\binom{n}{s}$. When $4 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$, it is clear that this dimension is asymptotically larger than $n^3$, and so there can be at most finitely many small enough such $\omega_s$. We verify that the following are the only possible representations:

$$\omega_4(8 \leq n \leq 29), \omega_5(10 \leq n \leq 15), \omega_6(12 \leq n \leq 13).$$

We may also verify that no non-trivial scalar multiples of these representations are small enough.
Since $\Gamma_{\omega_1} = \text{Sym}^t SL_n$, it has dimension $\binom{n+t-1}{t}$, and we see again that for $t \geq 4$ there can be at most finitely many solutions. We can verify that these solutions are exactly

$$4\omega_1 (3 \leq n \leq 17), 5\omega_1 (3 \leq n \leq 4), 6\omega_1 (n = 2), 7\omega_1 (n = 2).$$

By [GGS Lemma 2.2], $\dim \Gamma_{t\omega_s} \geq \dim \Gamma_{t\omega_1}$ for all $t, s$; using this we check that $t\omega_2$ and $t\omega_3$ are never small enough for $t \geq 4$.

For $t = 2, 3$, we have $\dim \Gamma_{t\omega_1} \leq n^3$ for all $n$, and so these representations are both small enough. It remains only to determine the $n$ for which $2\omega_2$, $2\omega_3$, $3\omega_2$ and $3\omega_3$ are small enough. The dimension of $2\omega_2$ is $n^2(n^2-1)/12$, which is at most $n^3$ when $4 \leq n \leq 12$, and we check for $n$ in this range that $3\omega_2$ is only small enough when $n = 4$. The weight $2\omega_3$ is only small enough when $n = 6$, and $3\omega_3$ is not small enough for this $n$. This completes the list of 1-supported small enough weights.

□

**Lemma 4.3.** The small enough 2-supported highest weights of $SL_n$ are the entries (21)–(29) of Table 1.

|    | Highest weight | Restrictions on $n$ | Purity/Cofreeness |
|----|----------------|---------------------|-------------------|
| 2  | $k\omega_1$    | $n = 2, 1 \leq k \leq 4$ | Cofree            |
| 3  | $k\omega_1$    | $n = 2, 5 \leq k \leq 8$ | Impure            |
| 4  | $\omega_1$     | $n \geq 3$          | Unstable          |
| 5  | $2\omega_1$    | $n \geq 3$          | Cofree            |
| 6  | $3\omega_1$    | $n \leq 3$          | Cofree            |
| 7  | $3\omega_1$    | $n \geq 4$          | Impure, Lemma 5.4 |
| 8  | $4\omega_1$    | $3 \leq n \leq 17$  | Impure            |
| 9  | $5\omega_1$    | $3 \leq n \leq 4$   | Impure            |
| 10 | $\omega_2$     | $n \geq 4$          | Cofree; Unstable for odd $n$ |
| 11 | $2\omega_2$    | $n = 4$             | Cofree            |
| 12 | $2\omega_2$    | $5 \leq n \leq 12$  | Impure            |
| 13 | $3\omega_2$    | $n = 6$             | Impure            |
| 14 | $\omega_3$     | $6 \leq n \leq 9$   | Cofree            |
| 15 | $\omega_3$     | $n \geq 10$         | Impure, Lemma 5.1 |
| 16 | $2\omega_3$    | $n = 6$             | Impure            |
| 17 | $\omega_4$     | $n = 8$             | Cofree            |
| 18 | $\omega_4$     | $9 \leq n \leq 29$  | Impure            |
| 19 | $\omega_5$     | $10 \leq n \leq 15$ | Impure            |
| 20 | $\omega_6$     | $12 \leq n \leq 13$ | Impure            |
| 21 | $\omega_1 + \omega_{n-1}$ | $n \geq 3$ | Cofree (Adjoint) |
| 22 | $2\omega_1 + \omega_{n-1}$ | $n \geq 3$ | Impure, Lemma 5.4 |
| 23 | $\omega_1 + \omega_2$ | $n \geq 3$ | Impure, Lemma 5.4 |
| 24 | $\omega_1 + \omega_{n-2}$ | $n \geq 5$ | Impure, Lemma 5.4 |
| 25 | $\omega_1 + \omega_3$ | $n = 8$ | Impure |
| 26 | $\omega_1 + \omega_3$ | $6 \leq n \leq 10$ | Impure |
| 27 | $\omega_1 + \omega_{n-3}$ | $7 \leq n \leq 9$ | Impure |
| 28 | $\omega_2 + \omega_3$ | $5 \leq n \leq 6$ | Impure |
| 29 | $\omega_2 + \omega_4$ | $n = 6$ | Impure |
| 30 | $\omega_1 + \omega_2 + \omega_3$ | $n = 4$ | Impure |

Table 1. Small enough representations of $SL_n$.
Proof. By Lemma 4.1 if \( \omega = \omega_1 + \omega_j \) is small enough, then one of \( \omega_{i-j} = 2 \omega_j \) and \( \omega_{j-i} = 2 \omega_i \) must be small enough. Therefore, assuming without loss of generality that \( i \leq \lfloor n/2 \rfloor \), we have that \( i \) is one of 1, 2, 3, with the latter two cases only possible if \( n \leq 12 \) and \( n = 6 \) respectively. If \( i < j \), then we check that \( i = 3 \) is never small enough, and for \( i = 2 \) we have the possible cases \( \omega_2 + \omega_3 \) for \( n = 5 \), and \( \omega_2 + \omega_3, \omega_2 + \omega_1 \) for \( n = 6 \).

Otherwise, \( i = 1 \). From [FH] Proposition 15.25(i), we find that \( \Gamma_{\omega_1 + \omega_j} \otimes \Lambda^{j+1} SL_n \simeq \Lambda^j SL_n \otimes SL_n \). Therefore,

\[
\dim \Gamma_{\omega_1 + \omega_j} = n \left( \begin{array}{c} n \hfill \\
 2 \end{array} \right) - \left( \begin{array}{c} n \hfill \\
 j + 1 \end{array} \right).
\]

If \( \omega_1 + \omega_j \) is small enough, it must also be the case that \( \omega_j \) is small enough. Therefore, \( j \in \{2, 3, n - 3, n - 2, n - 1\} \) unless \( n \) is in some of the ranges \([8, 29], [10, 15], [12, 13]\), where \( j \) could also be in \( \{4, n - 4\}, \{5, n - 5\} \), and \( \{6, n - 6\} \) respectively. In the latter case, \( \omega_1 + \omega_4 \) is small enough for \( n = 8 \), no \( \omega_1 + \omega_5 \) are small enough for \( n \in [10, 15] \), and no \( \omega_1 + \omega_6 \) are small enough for \( n = 12, 13 \). When \( j \in \{2, n - 2, n - 1\} \), this formula shows that \( \omega_1 + \omega_j \) is small enough. For \( j = 3 \) we use the dimension formula to find that \( \omega_1 + \omega_3 \) is small enough for \( n \leq 10 \), finally, we use the formula again to find that \( \omega_1 + \omega_{n-3} \) is small enough for \( n \leq 9 \).

Other than the weights \( \omega_1 + \omega_2, \omega_1 + \omega_{n-2} \), and \( \omega_1 + \omega_{n-1} \), the two-supported weights of weight two form a finite list, and we may check that increasing the width will not give a small enough weight; for example, \( \omega_2 + \omega_3 \) for \( n = 5 \) is small enough, but neither \( 2 \omega_2 + \omega_3 \) nor \( \omega_2 + 2 \omega_3 \) is. In the infinite families \( \omega_1 + \omega_2, \omega_1 + \omega_{n-2}, \omega_1 + \omega_{n-1} \), we may again apply [FH] Proposition 15.25(i) to see that all of \( 2 \omega_1 + \omega_2, \omega_1 + 2 \omega_2, 2 \omega_1 + \omega_{n-2}, \omega_1 + 2 \omega_{n-2} \) are not small enough. The only remaining case is \( 2 \omega_1 + \omega_{n-1} \), which is small enough for all \( n \). Finally, we verify that neither of \( 3 \omega_1 + \omega_{n-1}, 2(\omega_1 + \omega_{n-1}) \) are small enough, and so this list of two-supported small enough highest weights is complete.

Lemma 4.4. The only small enough 3-supported highest weight of \( SL_n \) is \( \omega_1 + \omega_2 + \omega_3 \) for \( n = 4 \).

Proof. A small enough 3-supported highest weight \( \omega \) must have width exactly three, since there is no small enough 2-supported weight with width larger than 3. Thus, \( \omega = \omega_i + \omega_j + \omega_k \) for some \( i, j, k \), and since one of \( \omega_{j-i}, \omega_{j-k} \) must be small enough, we have either \( \omega_{j-i} = 2 \omega_1 + \omega_{n-1} \) or \( \omega_{j-k} = 2 \omega_1 + \omega_{n-1} \). Suppose it is the former without loss of generality. Then \( i = 1 \) and \( k = n - 1 \), so \( \omega = \omega_1 + \omega_j + \omega_{n-1} \). Note that \( \Gamma_\omega \) is contained in \( \Gamma_{\omega_1 + \omega_j} \otimes V^* \). The exact decomposition is given by [FH] Proposition 15.25(ii): we conclude that for \( j = 2 \),

\[
\dim \Gamma_\omega = n \left( \begin{array}{c} n \hfill \\
 2 \end{array} \right) - \left( \begin{array}{c} n \hfill \\
 3 \end{array} \right) - \left( \begin{array}{c} n + 1 \hfill \\
 2 \end{array} \right) - \left( \begin{array}{c} n \hfill \\
 2 \end{array} \right),
\]

and so \( \Gamma_\omega \) is small enough in this case only when \( n = 4 \) and \( \omega = \omega_1 + \omega_2 + \omega_3 \). When \( 2 < j < n - 1 \), there are only finitely many \( (j, n) \) for which \( \omega_1 + \omega_j \) is small enough, and we may check that none of them yield small enough \( \omega_1 + \omega_j + \omega_{n-1} \).

Proposition 4.5. No \( k \)-supported highest weight of \( SL_n \) is small enough for \( k > 3 \), and hence the lists of the previous three lemmas are complete.

Proof. If \( \omega \) is \( k \)-supported for \( k > 3 \), then \( \omega \) has width at least \( k \). By applying \( k - 3 \) shift operations in every possible way, we get a list of \( 3 \)-supported weights \( \alpha_1, \ldots, \alpha_m \), all of width at least \( k \), such that

\[
\dim \Gamma_\omega \geq \min_i \dim \Gamma_{\alpha_i}
\]

by Lemma 4.1. But the only small enough weights of support 3 have width 3, so none of the \( \alpha_i \) are small enough, and hence \( \omega \) is not small enough.
The cases of $\text{Sp}_{2n}$, $\text{Spin}_{2n+1}$, and $\text{Spin}_{2n}$ proceed in the same fashion as $\text{SL}_n$ and are, in fact, easier to handle so their proofs are omitted.

| Highest weight | Restrictions on $n$ | Purity/Cofreeness |
|----------------|---------------------|-------------------|
| $\omega_1$     | $n \geq 3$          | Unstable          |
| $2\omega_1$    | $n \geq 3$          | Cofree            |
| $3\omega_1$    | $n \geq 3$          | Impure, Lemma 5.3 |
| $\omega_2$     | $n \geq 3$          | Cofree            |
| $\omega_3$     | $n \geq 4$          | Impure, Lemma 5.1 |
| $\omega_4$     | $n = 4$             | Cofree            |
| $\omega_5$     | $5 \leq n \leq 6$   | Impure            |
| $\omega_6$     | $n = 5$             | Impure            |
| $\omega_7$     | $n = 6$             | Impure            |
| $\omega_8$     | $n = 7$             | Impure, Lemma 5.1 |
| $\omega_9$     | $n = 8$             | Impure, Lemma 5.9 |
| $\omega_1 + \omega_3$ | $n = 3$ | Impure |
| $\omega_1 + \omega_2$ | $3 \leq n \leq 4$ | Impure |

**Table 2.** Small enough representations of $\text{Sp}_{2n}$

4.2. **The case of $\text{Sp}_{2n}$.** Listed in Table 2 are the small enough representations of $\text{Sp}_{2n}$ for $n \geq 3$. A representation $V$ of $\text{Sp}_{2n}$ is small enough if $\dim V \leq \kappa(\text{Sp}_{2n}) = 2(n^3 + n^2 + n + 1) + 1$.

**Proposition 4.6.** Table 2 is a complete list of the highest weights of the small enough irreducible rational representations of the symplectic groups $\text{Sp}_{2n}$ for $n \geq 3$.

| Highest weight | Restrictions on $n$ | Purity/Cofreeness |
|----------------|---------------------|-------------------|
| $\omega_1$     | $n \geq 2$          | Cofree            |
| $2\omega_1$    | $n \geq 2$          | Cofree            |
| $3\omega_1$    | $n \geq 3$          | Impure, 5.6       |
| $\omega_2$     | $n \geq 3$          | Cofree (Adjoint)  |
| $2\omega_2$    | $n = 2$             | Cofree (Adjoint)  |
| $2\omega_3$    | $n = 3$             | Impure, 5.6       |
| $\omega_3$     | $n \geq 4$          | Impure, 5.6       |
| $\omega_4$     | $2 \leq n \leq 6$   | Cofree            |
| $\omega_7$     | $n = 7$             | Impure, Lemma 5.1 |
| $\omega_8$     | $n = 8$             | Impure, Lemma 5.9 |
| $\omega_9$     | $9 \leq n \leq 11$  | Impure            |
| $2\omega_4$    | $n = 4$             | Impure            |
| $3\omega_2$    | $n = 2$             | Impure            |
| $\omega_1 + \omega_4$ | $2 \leq n \leq 4$ | Impure |

**Table 3.** Small enough representations of $\text{Spin}_{2n+1}$

4.3. **The case of $\text{Spin}_{2n+1}$.** Listed in Table 3 are the small enough representations of $\text{Spin}_{2n+1}$ for $n \geq 2$. Let $L_1, \ldots, L_n$ be the standard dual basis in $\mathfrak{h}^*$, so the fundamental weights are $\omega_i = L_1 + \cdots + L_i$ for $i < n$ and $\omega_n = \frac{1}{2}(L_1 + \cdots + L_n)$. A representation $V$ of $\text{Spin}_{2n+1}$ is small enough if $\dim V \leq \kappa(\text{Spin}_{2n+1}) = 2(n^3 + n^2 + n + 1) + 1$.

**Proposition 4.7.** Table 3 is a complete list of the highest weights of the small enough irreducible rational representations of the odd spin groups $\text{Spin}_{2n+1}$ for $n \geq 2$. 
| Highest weight | Restrictions on $n$ | Purity/Cofreeness |
|---------------|-------------------|-----------------|
| $\omega_{n-1}, \omega_n$ | $n = 5, 6, 7, 8$ | Cofree |
| $\omega_8, \omega_9$ | $n = 9$ | Impure, Lemma 5.11 |
| $\omega_1$ | $n \geq 4$ | Cofree |
| $\omega_2$ | $n \geq 4$ | Cofree (Adjoint) |
| $\omega_3$ | $n \geq 5$ | Impure, Lemma 5.10 |
| $2\omega_1$ | $n \geq 4$ | Cofree |
| $3\omega_1$ | $n \geq 4$ | Impure, Lemma 5.10 |
| $\omega_3, \omega_4$ | $n = 4$ | Cofree, Remark 1.5 |
| $2\omega_3, 2\omega_4$ | $n = 4$ | Cofree, Remark 1.5 |
| $3\omega_3, 3\omega_4$ | $n = 4$ | Impure |
| $2\omega_4, 2\omega_5$ | $n = 5$ | Unstable |
| $\omega_{n-1} + \omega_n$ | $n = 4, 5$ | Impure |
| $\omega_1 + \omega_{n-1}, \omega_1 + \omega_n$ | $n = 4, 5, 6$ | Impure |

Table 4. Small enough representations of $\text{Spin}_{2n}$

4.4. The case of $\text{Spin}_{2n}$. Listed in Table 4 are the small enough representations of $\text{Spin}_{2n}$ for $n \geq 4$. Let $L_1, \ldots, L_n$ be the standard basis in $h^*$, and then we have the fundamental weights $\omega_i = L_1 + \cdots + L_i$ for $i \leq n - 2$, and $\omega_{n-1} = \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$, $\omega_n = \frac{1}{2}(L_1 + \cdots + L_n)$. A representation of $\text{Spin}_{2n}$ is small enough if $\dim V \leq \kappa(\text{Spin}_{2n}) = 2(n^3 + n + 3/2) + 1$ for $n > 4$, or $\dim V \leq 139$ when $n = 4$.

Proposition 4.8. Table 4 is a complete list of the highest weights of the small enough irreducible rational representations of the odd spin groups $\text{Spin}_{2n}$ for $n \geq 4$.

4.5. The exceptional groups.

Proposition 4.9. Theorem 1.3 holds for all exceptional groups.

Proof. A computer check reveals that any highest weight for an exceptional group either has dimension greater than the bound of Proposition 3.8, is the adjoint representation, the smallest non-trivial representation, or is the representation $2\omega_1$ of $G_2$. The adjoints and smallest non-trivial representations are cofree for all the exceptional groups. If $V$ is the $G_2$-representation $2\omega_1$, one checks that every order 3 toral element $t$ has $\dim V^t = 9 \leq \dim V - \dim G_2 - 2 = 27 - 14 - 2 = 11$, and so $2\omega_1$ is also not pure by Lemma 3.4. Thus pure implies cofree for irreducible representations of exceptional groups. \qed

5. Remaining cases

In this section we complete the proof of Theorem 1.3 by demonstrating that the representations in Tables 1–4 that were not cofree are not pure. By Lemma 3.4 it suffices to check that $\dim V^t/N_t$ is at most $\dim V - \dim G - 2$ for all toral elements of any fixed order, up to the action of the Weyl group, which we will do for every representation but the spinor representation of $\text{Spin}_{15}$.

5.1. The case of $\text{SL}_n$.

Lemma 5.1. The representation $\Gamma_{\omega_3} = \bigwedge^3 \text{SL}_n$ is not pure for $n \geq 10$.

Proof. Let $t$ be a toral element of order 2 and let $k = \dim \text{SL}_n^t$; note that $n - k$ is always even.

As a representation of $G'' := \text{SL}_k \times \text{SL}_{n-k}$, we have,

$$\left(\bigwedge^3 \text{SL}_n\right)^t = \bigwedge^3 \text{SL}_k \oplus \left(\text{SL}_k \otimes \bigwedge^2 \text{SL}_{n-k}\right).$$
Let $H$ be the stabilizer of a generic point for the $G'$-action on $(\Lambda^3 \text{SL}_n)^t$. Since $G'$ normalizes $t$, from Lemma 3.4 it follows that if

$$\dim(\Lambda^3 \text{SL}_n)^t / G' \leq \dim(\Lambda^3 \text{SL}_n) - \dim \text{SL}_n - 2 = \binom{n}{3} - (n^2 - 1) - 2, \tag{5.2}$$

then $\Lambda^3 \text{SL}_n$ is not a pure representation of $\text{SL}_n$.

Note that $H \subseteq H' \times \text{SL}_{n-k}$, where $H'$ is the stabilizer of a generic point for the action of $\text{SL}_k$ on $\Lambda^3 \text{SL}_k$. For $k \geq 10$, this representation is properly stable, so $\dim H' = 0$; for $k \leq 9$, $H'$ is listed in the generic stabilizer table of [PV, p. 261]. So

$$\dim \left( \left( \Lambda^3 \text{SL}_n \right)^t / G' \right) \leq \dim \left( \Lambda^3 \text{SL}_n \right)^t - (k^2 - 1) + \dim H' \leq \binom{k}{3} + k \left( \binom{n-k}{2} - (k^2 - 1) \right).$$

Comparing with the previous equation, it is enough to show

$$\dim H' \leq \binom{n}{3} - (n^2 - 1) - 2 - \binom{k}{3} - k \left( \binom{n-k}{2} - (k^2 - 1) \right). \tag{5.3}$$

The table of [PV, p. 261] reveals that $\dim H' \leq 16$ always, and as a result we can compute that if $n \geq 15$, then (5.3) holds for any $k$. When $10 \leq n \leq 14$, one checks that (5.3) holds for all $(n, k)$ except for

$$(10, 2), (10, 4), (10, 6), (11, 1), (11, 3), (11, 5), (12, 2), (12, 4), (12, 6), (13, 3), (13, 5), (14, 4), (14, 6);$$

for example, if $k = 7$, then [PV, p. 261] shows that $\dim H' = 14$ and (5.3) holds for $n \in \{11, 13\}$.

It remains to handle the $(n, k)$ pairs listed above. We begin with the case where $k = 1$. Then $(\Lambda^3 \text{SL}_n)^t / G' = \Lambda^2 \text{SL}_{n-1} / \text{SL}_{n-1}$ which is 1-dimensional, hence (5.2) holds.

When $k = 2$, our $G'$-representation $(\Lambda^3 \text{SL}_n)^t$ is $\text{SL}_2 \otimes \Lambda^2 \text{SL}_{n-2}$. From [E2, Table 6], we see the dimension of the generic stabilizer is $\frac{3}{2}(n-2)$, so

$$\dim \left( \left( \Lambda^3 \text{SL}_n \right)^t / G' \right) \leq 3 \left( \frac{n-2}{2} \right) - \dim G' + \frac{3}{2} (n-2) \leq \binom{n}{3} - (n^2 - 1) - 2,$$

again showing (5.2) holds.

With the exception of $(n, k) = (10, 6)$, for all remaining $(n, k)$, the $G'$-representation $\text{SL}_k \otimes \Lambda^2 \text{SL}_{n-k}$ falls within Case 6 of [E2], but does not appear on Table 6 of (loc. cit.), so $\dim H = 0$. It follows that

$$\dim \left( \left( \Lambda^3 \text{SL}_n \right)^t / G' \right) \leq \binom{k}{3} + k \left( \binom{n-k}{2} - \dim G' \leq \binom{n}{3} - (n^2 - 1) - 2,$$

and so (5.2) holds.

We now turn to the last case: $(n, k) = (10, 6)$. We must show that the quotient of $V := (\text{SL}_6 \otimes \Lambda^2 \text{SL}_4) \oplus \Lambda^3 \text{SL}_6$ by the normalizer $N \subseteq \text{SL}_{10}$ of $t = \text{diag}(1, 1, 1, 1, 1, 1, -1, -1, -1, -1)$ has dimension at most $\dim (\Lambda^3 \text{SL}_{10} - \dim \text{SL}_{10} - 2 = 19$.

First, note that $N$ contains the image of the map $\text{GL}_6 \times \text{SL}_4 \to \text{SL}_{10}$ given by $(g, h) \mapsto \text{diag}(g, \det g^{-1}h)$. The kernel of this map is finite, so $\dim V/N \leq \dim V/((\text{GL}_6 \times \text{SL}_4)$. We will show that the latter quotient has dimension at most 19.

To simplify notation, let $G = \text{GL}_6$ and $H = \text{SL}_4$. Note that $(V \oplus W)/(G \times H) = ((V \oplus W)/G)/H$. As $G = \text{GL}_6$-module, $V = \text{SL}_6^g$, and $G$ acts generically freely on $V$; in particular, the quotient is 0-dimensional. Thus $\dim(V + W)/G = \dim V/G + \dim W = 20$. To prove the assertion, we need to show that $H$ acts non-trivially on this quotient and that the generic orbit is closed. The action
of \( h \in H \) on a \( G \)-orbit \( (v, w) \) is given by \( h(v, w) = (hv, hw) = (hv, w) \), where the last equality is because \( H = \text{SL}_4 \) acts trivially on \( W = \mathbb{F}^3 \text{SL}_6 \).

If \( v \in V \) is a generic vector in \( V \) then \( \text{Stab}_G v = 1 \). In addition \( H \) acts trivially on \( V/G \) (as it is 0-dimensional), so if \( h \in H \), there is a unique \( g \in V \) such that \( hv = gv \). This defines a morphism \( \varphi_v : H \to G \), and we can rewrite the action of \( H \) on \( (V + W)/G \) as \( h(v, w) = (hv, w) = (v, \varphi(v)^{-1} w) \).

The map \( \varphi_v \) must necessarily have finite kernel, because \( H = \text{SL}_4 \) is a simple group and the image is non-trivial—\( H \) acts non-trivially on \( V \), so we can pick \( v \) such that \( \text{Stab}_G H \neq H \). Thus \( h \) stabilizes \( (v, w) \) if and only \( \varphi_v(h) \in \text{Stab}_G w \). Since \( w \) is independent of \( v \), we claim that there must exist \( w \in W \) such that the image of \( \varphi_v(H) \) is not contained in \( \text{Stab}_G w \). This would mean that it is a strictly smaller dimensional subgroup of \( \varphi_v(H) \), since a proper subgroup of a connected group always has strictly smaller dimension.

To prove the claim, suppose to the contrary that \( \varphi_v(H) \subseteq \text{Stab}_G w \) for all \( w \) in a dense open set. Then \( \varphi_v(H) \) is a subgroup of the kernel of the \( G \) action on \( \bigwedge^3 \text{SL}_6 \). But the kernel of this action is trivial. \( \square \)

\textbf{Lemma 5.4.} For \( n \geq 4 \), the representations \( \Gamma_{\omega_1 + \omega_2} \), \( \Gamma_{2\omega_1 + \omega_{n-1}} \), \( \Gamma_{\omega_1 + \omega_{n-2}} \), and \( \Gamma_{3\omega_1} \) are not pure.

\textbf{Proof.} We first consider the case of \( V := \Gamma_{\omega_1 + \omega_2} \). From the decomposition

\[ V \oplus \bigwedge^3 \text{SL}_n \cong \text{SL}_n \otimes \bigwedge^2 \text{SL}_n \]

we see that if \( t = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \) is an order 2 toral element with \( k \) ones, then

\[ \dim V^t = \dim \left( \text{SL}_n \otimes \bigwedge^2 \text{SL}_n \right)^t - \dim \left( \bigwedge^3 \text{SL}_n \right)^t = k(n - k)^2 + k \binom{k}{2} - \binom{k}{3}. \]

Since \( t \neq 1 \), we have \( 0 \leq k \leq n - 2 \). With these constraints, one checks that the above cubic is maximized at \( k = n - 2 \), so for \( n \geq 7 \), we have

\[ \dim V^t \leq n \binom{n}{2} - \binom{n}{3} - (n^2 - 1) - 2 = \dim V - \dim \text{SL}_n - 2 \]

showing that \( V \) is not pure by Lemma 3.4.

The cases \( 4 \leq n \leq 6 \) are handled as follows. When \( n = 6 \), one may use LiE to directly verify that \( \dim V^t \leq \dim V - \dim \text{SL}_6 - 2 \) for all \( t \) in a set of toral representatives of order 3, and so \( V \) is not pure by Lemma 3.4. When \( n = 5 \), one check by computer that the maximum number of weights lying on a hyperplane in the weight space for \( V \) is \( 14 < 15 = \dim V - \dim \text{SL}_5 - 1 \), so by Proposition 3.1 it is not pure.

For \( n = 4 \), we check using toral elements of order four that we always have \( \dim V^t/N_t \leq 3 \), branching the representation as an \( \text{SL}_2 \times \text{SL}_2 \) module as necessary. We are grateful for the referee for suggesting a simpler proof of this case. There are five toral elements of order four up to Weyl group conjugacy, of which two have no fixed points, and the remaining three are the elements \( \text{diag}(i, -i, -1, 1) \), \( \text{diag}(i, -i, 1, 1) \), \( \text{diag}(i, -i, -1, -1) \) which is normalized by \( \{ 1 \} \times \text{SL}_2 \). As a representation of \( \text{SL}_2 \times \text{SL}_2 \), we have \( V = C^2 \otimes \text{SL}_2 \otimes \text{SL}_2 \otimes C^2 \otimes \text{SL}_2 \otimes \text{Sym}^2 \text{SL}_2 \otimes \text{Sym}^2 \text{SL}_2 \otimes \text{SL}_2 \), and so \( V^t = C^2 \otimes \text{SL}_2 \otimes (\text{Sym}^2 \text{SL}_2)^0 \otimes \text{SL}_2 \), where \( (\text{Sym}^2 \text{SL}_2)^0 \) denotes the 0 weight space of \( \text{Sym}^2 \text{SL}_2 \). From here we can see that \( \dim V^t/N_t \leq \dim V^t/\{ 1 \} \times \text{SL}_2 \leq 3 \). A similar analysis can be performed for the other two toral elements, showing that \( V \) is not pure for \( n = 4 \).

The cases \( \Gamma_{2\omega_1 + \omega_{n-1}} \) and \( \Gamma_{\omega_1 + \omega_{n-2}} \) follow similarly using the decompositions \( \Gamma_{2\omega_1 + \omega_{n-1}} \otimes \text{SL}_n \cong \text{SL}_n \otimes \text{Sym}^2 \text{SL}_n \) and \( \text{SL}_n \otimes \bigwedge^{n-2} \text{SL}_n \cong \text{SL}_n^* \otimes \Gamma_{\omega_1 + \omega_{n-2}} \). In both cases, \( \dim V - \dim \text{SL}_n - 2 - \dim V^t \) are again cubic in \( \dim \text{SL}_n \). In the former case, this cubic is always non-negative for \( n > 6 \), meaning that the representation is not pure for \( n \geq 6 \) by Lemma 3.4. In the latter case, this cubic is always non-negative for \( n > 6 \) and \( 1 \leq \dim \text{SL}_n^t \leq 6 \), and in the last case where \( \dim \text{SL}_n^t = 0 \), then \( t = -I \).
The remaining for both representations where $n = 4, 5, 6$ are handled with LiE using toral elements of order 3 and using Lemma 3.4.

Lastly, for $V := \Gamma_{3\omega_1} \simeq \text{Sym}^3 \text{SL}_n$, we have $V^t = \text{Sym}^3 \text{SL}_k \oplus (\text{SL}_k \otimes \text{Sym}^2 \text{SL}_{n-k})$. When $n \geq 7$, then one can check $\dim V^t \leq \dim V - \dim \text{SL}_n - 2$ for all $k$ except $k = n$ when $n$ is even; but then $t = -I$. For $4 \leq n \leq 6$ we adopt a different approach. It is proved in [MPK] that every smooth cubic hypersurface of degree 3 is GIT stable. This implies that if $V = \text{Sym}^3 \text{SL}_n$ then $V^{\text{ss}}$ is contained in the discriminant divisor. For $n = 4, 5, 6$ (cubic surfaces, threefolds and fourfolds) work in GIT [ACT, All, Laz] implies that the generic singular hypersurface is GIT stable. Since the discriminant divisor is irreducible this implies that $V^{\text{ss}}$ cannot have codimension one since it is a proper algebraic subset of the discriminant. □

5.2. The case of $\text{Sp}_{2n}$.

**Lemma 5.5.** The $\text{Sp}_{2n}$-representations $\Gamma_{\omega_3}$ and $\Gamma_{3\omega_1}$ for $n \geq 5$ are not pure.

**Proof.** The representation $\Gamma_{\omega_3}$ is the kernel of the contraction $\wedge^3 \text{Sp}_{2n} \to \text{Sp}_{2n}$ using the bilinear form, of dimension $\binom{2n}{3} - 2n$. Let $t$ be a toral element of order 2, and let $k = \dim \text{Sp}_{2n}^t$. Note that since with respect to the standard maximal torus, $t = \text{diag}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$ for some $x_i = \pm 1$, it follows that $k$ is even. By decomposing $\text{Sp}_{2n}^t$ into its positive and negative eigenspaces under $t$ and expanding the exterior product as a sum of tensors, we find by the exactness of taking invariants that

$$\dim \Gamma_{\omega_3}^t = \binom{k}{3} + k\binom{2n - k}{2} - k.$$

For $0 \leq k \leq 2n - 2$, this cubic is maximized at $k = 2n - 2$ for $n \geq 6$, and if $k > 2n - 2$, then since $k$ must be even, in fact $k = 2n$ and $t = 1$ is not of order 2. Thus we have an upper bound $\dim \Gamma_{\omega_3}^t \leq \binom{2n - 2}{3}$ for all $n \geq 6$. We then check that $\binom{2n - 2}{3} \leq \dim \Gamma_{\omega_3} - \dim \text{Sp}_{2n}^t - 2$ for $n \geq 6$, showing that $\Gamma_{\omega_3}$ is not pure for such $n$ by Lemma 3.4.

For $n = 5$, we find that $\dim V^t \leq \dim V - \dim \text{Sp}_{2n}^t - 2$ for toral elements of order 3 except $t = \text{diag}(\zeta, 1, 1, 1, 1, 1, \zeta^2, 1, 1, 1, 1)$, up to Weyl group conjugacy. For this $t$, we have

$$\Gamma_{\omega_3}^t = \left(\ker \wedge^3 \text{Sp}_{10} \to \text{Sp}_{10}\right)^t \simeq \ker \left(\left(\wedge^3 \text{Sp}_{10}\right)^t \to \text{Sp}_{10}^t\right) \simeq \text{Sp}_8 \oplus \wedge^3 \text{Sp}_8$$

by the exactness of taking invariants. Since $\text{Sp}_8 \times \text{Sp}_2 \subseteq N_t$, we find

$$\dim \Gamma_{\omega_3}^t / N_t \leq \dim \left(\wedge^3 \text{Sp}_8\right)/(\text{Sp}_8 \times \text{Sp}_2) = \dim \left(\wedge^3 \text{Sp}_8\right)/\text{Sp}_8.$$

However, since $\text{Sp}_8 \oplus \wedge^3 \text{Sp}_8$ is a reducible representation of a simple Lie group that does not appear on the table of [E1], it follows that $\text{Sp}_8$ acts with trivial generic stabilizer on this representation, and so

$$\dim \Gamma_{\omega_3}^t / N_t \leq \dim \wedge^3 \text{Sp}_8 - \dim \text{Sp}_8 = 64 - 36 = 28 \leq 53 = \dim V - \dim \text{Sp}_{10}^t - 2,$$

and the representation is not pure by Lemma 3.4.

The proof for $\Gamma_{3\omega_1} = \text{Sym}^3 \text{Sp}_{2n}$ is similar: let $t$ be a toral element of order 2, and let $k = \dim \text{Sp}_{2n}^t$. Then we compute

$$\dim \Gamma_{3\omega_1}^t = \binom{k + 2}{3} + k\binom{2n - k + 1}{2},$$

and find again that on $0 \leq k \leq 2n - 2$, it is maximized at $k = 2n - 2$. So $\dim \Gamma_{3\omega_1}^t \leq \binom{2n}{3}$, and we check that $\binom{2n}{3} \leq \dim \Gamma_{3\omega_1} - \dim \text{Sp}_{2n} - 2$ for all $n \geq 5$, so these representations are not pure by Lemma 3.4. □
5.3. The case of Spin\(_{2n+1}\).

**Lemma 5.6.** The representations \(\Gamma_{\omega_3}\) for \(n \geq 4\) and \(\Gamma_{3\omega_1}\) for \(n \geq 2\) of Spin\(_{2n+1}\) are not pure.

**Proof.** The representation \(\Gamma_{3\omega_1}\) has dimension \(\left(\frac{2n+1+2}{3}\right) - (2n+1)\), and is the kernel of the contraction \(\text{Sym}^3\text{Spin}_{2n+1} \rightarrow \text{Spin}_{2n+1}\) by the symmetric bilinear form. For any toral element \(t\), exactness of taking invariants gives

\[
\dim (\ker \text{Sym}^3 \text{Spin}_{2n+1} \rightarrow \text{Spin}_{2n+1})^t = \dim (\text{Sym}^3 \text{Spin}_{2n+1})^t - \dim \text{Spin}_{2n+1}^t,
\]

so if \(t\) is of order 2 and \(k = \dim \text{Spin}_{2n+1}^t\), then

\[
\dim \Gamma_{3\omega_1}^t = \left(\frac{k + 2}{3}\right) + k \left(\frac{2n - k + 2}{2}\right) - k.
\]

If \(k = 2n + 1\) then \(t = 1\) is not of order 2. Otherwise, note that an order two toral element acting on Spin\(_{2n+1}\) always has odd-dimensional positive eigenspace; as a matrix, it can be taken to be of the form \(\text{diag}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, 1)\) in the standard maximal torus, where each \(x_i = \pm 1\), and so the number of entries with value 1 is always odd.

One may check that as a function of \(k\) on the range \([0, 2n]\), \(\dim \Gamma_{3\omega_1}^t\) is maximized at \(k = 2n - 1\). This gives us an upper bound \(\dim \Gamma_{3\omega_1}^t \leq \left(\frac{2n+1}{3}\right)\), and we may check that \(\left(\frac{2n+1}{3}\right) \leq \dim \Gamma_{3\omega_1} - \dim \text{Spin}_{2n+1}^t - 2\) always, so by Lemma 3.4, \(\Gamma_{3\omega_1}\) is not a pure representation of Spin\(_{2n+1}\).

For \(\Gamma_{\omega_3}\), we note that it is equal to the third exterior power \(\bigwedge^3\text{Spin}_{2n+1}\) for \(n \geq 4\). As a result, if \(t\) is a toral element of order 2, and \(k = \dim \text{Spin}_{2n+1}^t\), then

\[
\dim \Gamma_{\omega_3}^t = \left(\frac{k}{3}\right) + k \left(\frac{2n + 1 - k}{2}\right),
\]

and it can be checked that this is less than \(\dim \Gamma_{\omega_3} - \dim \text{Spin}_{2n+1}^t - 2\) for all \((n, k)\) when \(n \geq 4\) and \(0 \leq k \leq 2n + 1\) except when \(k \geq 2n\). As we remarked above, since \(k\) must be odd, \(k \geq 2n\) implies \(k = 2n + 1\) and then \(t = 1\) is not of order 2, and thus by Lemma 3.4, \(\Gamma_{\omega_3}\) is not pure for any \(n \geq 4\). \(\square\)

**Lemma 5.7.** The spinor representation of Spin\(_{15}\) is not pure.

**Proof.** Let \(V = \text{Spin}_{16}^\dagger\) be the positive half-spinor representation, which is a cofree representation of Spin\(_{16}\), hence pure by Proposition 2.1. Then the spinor representation Spin\(_{15}\) is the restriction of \(V\) under the natural inclusion Spin\(_{15}\) \(\subset\) Spin\(_{16}\) of Lie groups. To show Spin\(_{15}\) is not pure, we employ the following strategy. Since Spin\(_{15}\) and Spin\(_{16}\) are properly stable representations, by the Hilbert–Mumford Criterion, we have an inclusion

\[
V^{\text{sss}}(\text{Spin}_{15}) \subseteq V^{\text{sss}}(\text{Spin}_{16}).
\]

We know that \(V^{\text{sss}}(\text{Spin}_{16})\) is pure of codimension 1, so to prove Spin\(_{15}\) is not pure, it suffices to show the above inclusion is strict.

To do so, consider the set

\[
S = \{\{1, 2, 3, 4, 5, 6, 7\}, \{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.
\]

For each \(S \in S\), let \(a_{S, i} = 1\) if \(i \in S\) and \(a_{S, i} = -1\) if \(i \notin S\). We then obtain a set of 8 weights

\[
\mathcal{W} = \left\{ \sum_{i=1}^{7} \frac{a_{S, i}}{2} L_i + \frac{1}{2} L_8 \mid S \in S \right\}
\]

for Spin\(_{16}^\dagger\). For each \(\mu \in \mathcal{W}\), choose a non-zero weight vector \(v_\mu\) and let \(v = \sum_\mu v_\mu\). Since every \(\mu\) has a positive \(L_8\)-coefficient, \(v \in V^{\text{sss}}(\text{Spin}_{16})\).
We claim that \( v \notin V^{ss}(\text{Spin}_15) \). To see this, let \( \mu' \) be the weight of \( \text{Spin}_15 \) induced by \( \mu \). Then when \( V \) is viewed as a \( \text{Spin}_15 \)-representation, our set of weights \( \mathcal{W} \) map to the set
\[
\mathcal{W}' = \left\{ \frac{1}{2} \sum_{i=1}^{7} a_{S,i} L_i \mid S \in S \right\},
\]
and moreover, \( \sum_{\mu' \in \mathcal{W}'} \mu' = 0 \). Therefore, the origin is the interior of the convex hull of \( \mathcal{W}' \). We further see that the weights of \( \mathcal{W}' \) satisfy a strong orthogonality relation: any two \( \mu'_1, \mu'_2 \in \mathcal{W}' \) differ by 4 sign flips, so \( \mu'_1 - \mu'_2 \) is not a root. It follows from \([\text{DK}, \text{Proposition 1.2}]\) that \( v \in V^s(\text{Spin}_15) \), the complement of \( V^{ss}(\text{Spin}_15) \).

**Remark 5.8.** Lemma \(5.7\) did not rely on Proposition \(3.1\) to show that the spinor representation of \( \text{Spin}_15 \) was not pure, and so the content of Remark \(3.2\) does not apply. However, since this representation is self-dual, the dual representation does not need to be handled differently.

**Lemma 5.9.** The spinor representation of \( \text{Spin}_17 \) is not pure.

**Proof.** Let \( V = \text{Spin}_17 \) be the spinor representation of the group \( \text{Spin}_17 \) and let \( t = \text{diag}(\zeta^{m_1}, \ldots, \zeta^{m_s}) \) be a toral element with \( \zeta \) a primitive 3rd root of unity. Using the fact that \( t \) acts trivially on the weight space of \( \frac{1}{2} \sum_{i=1}^{8} a_i L_i \) if and only if \( \sum_{i=1}^{8} a_i m_i = 0 \mod 3 \), one checks that \( \dim V^t \leq 118 = 2^8 - (\binom{17}{2}) - 2 \) unless \( t = \text{diag}(1, \ldots, 1, \zeta, \zeta^m) \) with \( m = \pm 1 \).

To handle this remaining case, we begin by constructing a copy of \( \text{Spin}_{13} \) in the centralizer of \( t \). In the notation of \([\text{FH}, \text{p. 370 (23.8)}]\), we have
\[
t = w(1, \ldots, 1, \zeta, \zeta^m) = \frac{1}{4}(\zeta e_7 e_{15} + \zeta^{-1} e_{15} e_7)(\zeta^m e_8 e_{16} + \zeta^{-m} e_{16} e_8).
\]
Let \( S = \{7, 8, 15, 16\} \). Since \( e_i \) is orthogonal to \( e_j \) for \( i \notin S \) and \( j \in S \), the even Clifford algebra generated by the \( e_i \) for \( i \notin S \) yields a copy of \( \text{Spin}_{13} \subset \text{Spin}_m \) that commutes with \( t \).

Next, since \( V^t \) is the direct sum of the weight spaces with weights \( \frac{1}{2} \sum_{i=1}^{8} a_i L_i \) and \( a_7 + ma_8 = 0 \mod 3 \), as a \( \text{Spin}_{13} \)-representation, we have
\[
V^t = (\text{Spin}_{13})^{\oplus 2},
\]
i.e., it is two copies of the spinor representation. From \([\text{PV}, \text{p. 262}]\), we see the generic stabilizer of \( \text{Spin}_{13} \) is 16-dimensional, so
\[
\dim(V^t/N_t) \leq \dim(V^t/\text{Spin}_{13}) \leq 2^7 - \binom{13}{2} + 16 = 66 \leq 118.
\]
Thus, \( V \) is not pure. \( \square \)

### 5.4. The case of \( \text{Spin}_{2n} \).

**Lemma 5.10.** The representations \( \Gamma_{\omega_3} \) for \( n \geq 5 \) and \( \Gamma_{\omega_3} \) for \( n \geq 3 \) of \( \text{Spin}_{2n} \) are not pure.

**Proof.** This proof is very similar to the proof of Lemma \(5.6\) so we carry it out with somewhat less detail. Note that \( \Gamma_{\omega_3} = \Lambda^3 \text{Spin}_{2n} \) for \( n \geq 5 \), and so if \( t \) is an order 2 toral element and \( k = \dim \text{Spin}_{2n}^t \), then
\[
\dim \Gamma_{\omega_3}^t = \binom{k}{3} + k\binom{2n-k}{2}.
\]
As we remarked previously for the odd spin groups, the only possibilities for \( k \) will be even, and if \( k = 2n \) then \( t = 1 \) is not of order 2; so only \( 0 \leq k \leq 2n - 2 \) must be considered. Except for \( n = 5 \), the maximum is always reached at the endpoint \( 2n - 2 \), where we can check that \( \dim \Gamma_{\omega_3}^t \leq \dim \Gamma_{\omega_3} - \dim \text{Spin}_{2n} - 2 \), and so none of these representations are pure by Lemma \(3.4\). When \( n = 5 \) we can check again that no choice of \( 0 \leq k \leq 8 \) gives \( \dim \Gamma_{\omega_3}^t \leq \dim \Gamma_{\omega_3} - \dim \text{Spin}_{10} - 2 \), so once more by Lemma \(3.4\) \( \Gamma_{\omega_3} \) is not pure for \( n = 5 \).
The proof for $\Gamma_{3\omega_1}$ is similar to the case of the third symmetric power for odd spin groups: we have $\Gamma_{3\omega_1} = \ker(\text{Sym}^3\text{Spin}_{2n} \to \text{Spin}_{2n})$ and so for a toral element $t$ of order two,
\[
\dim \Gamma^t_{3\omega_1} = \dim(\ker \text{Sym}^3\text{Spin}_{2n} \to \text{Spin}_{2n})^t = \dim(\text{Sym}^3\text{Spin}_{2n})^t - \dim\text{Spin}_{2n}^t.
\]
If we write $k = \dim\text{Spin}_{2n}^t$, then
\[
\dim \Gamma^t_{3\omega_1} = \binom{k+2}{3} + k\binom{2n-k+1}{2} - k
\]
and we can again check that for $0 \leq k \leq 2n-2$, this is less than $\dim \Gamma_{3\omega_1} - \dim\text{Spin}_{2n} - 2$, so by Lemma 5.3, these representations are not pure.

Lemma 5.11. The half-spinor representations of Spin$_{18}$ are not pure.

Proof. Since there is an outer automorphism of Spin$_{18}$ interchanging the two half-spinor representations, it suffices by Remark 1.5 to consider the representation $V = \text{Spin}_{18}^+$. Then the weights of $V$ are $\frac{1}{2}\sum_{i=1}^{9} a_i L_i$ with $a_i = \pm 1$, and an even number of $a_i = -1$. We follow the same strategy of proof as in Lemma 5.9. One checks that if $t$ is an order 3 toral element, then $\dim V^t \leq 101 = 2^8 - \binom{18}{2} - 2$ unless $t = \text{diag}(1, \ldots, 1, \zeta^m)$ with $m = \pm 1$. It remains to handle this latter case.

As in the proof of Lemma 5.9, there is again a copy of Spin$_{14}$ in the centralizer of $t$. Note that $V^t$ is the direct sum of the weight spaces where weights $\frac{1}{2}\sum_{i=1}^{9} a_i L_i$ with $a_i = \pm 1$, $a_8 + ma_9 = 0$ mod 3, and $\prod_i a_i = 1$; said another way, it is a direct sum of weight spaces with weights of the form $\frac{1}{2}(\sum_{i=7}^{9} a_i L_i + a_8(L_8 - m L_9))$, where $-m = \prod_{i=1}^{7} a_i$. Thus, viewing $V^t$ as a Spin$_{14}$-representation, we have
\[
V^t \simeq \begin{cases} (\text{Spin}_1{14}^+) \otimes^2, & m = 1 \\ (\text{Spin}_1{14}^+) \otimes^2, & m = -1. \end{cases}
\]
From [PV, p. 262], we see the generic stabilizer of Spin$_{14}$ is 28-dimensional, so
\[
\dim(V^t/N_t) \leq \dim(V^t/\text{Spin}_{14}) \leq 2^7 - \binom{14}{2} + 28 = 65 \leq 101.
\]
Thus, $V$ is not pure. \hfill \Box

Part III. Actions of tori

We now turn our attention to torus representations. We prove Theorem III.6 in §1. In §2 we give examples that distinguish the classes of representations pure, coprincipal, and coregular. The most subtle of these is Example 2.3 which shows that coprincipal is not equivalent to pure; this is in contrast to Lemma III.3 which shows that pure and coprincipal are equivalent for connected $G$ with no non-trivial characters.

1. Proof of Theorem III.6

Our initial goal is to prove the following proposition. This is done after several preliminary lemmas. Throughout this section, if $V$ is a $G$-representation, then we denote by $V^{\text{ss}}(G)$ the strictly semi-stable locus for the action of $G$.

Proposition 1.1. Let $V_1$ and $V_2$ be stable representations of a torus $T$. Let $V = V_1 \oplus V_2$ be a decomposition as $T$-representations and assume that $V/T = V_1/T \times V_2/T$. Then $V$ is cofree (resp. coprincipal) if and only $V_1$ and $V_2$ are cofree (coprincipal) representations.

Remark 1.2. Note the condition that $V/T = V_1/T \times V_2/T$ is a very strong since it implies that $K[V]^T = K[V_1]^T \otimes_K K[V_2]^T$. 
Lemma 1.3. If $V$ is a stable representation of a reductive group $G$ such that $\dim V/G = 1$, then $V$ is cofree and coprincipal.

Proof. Since $\dim V/G = 1$, $K[V]^G$ is a polynomial ring in one variable and hence $K[V]$ is free over $K[V]^G$ as it is torsion free. If $f \in K[V]^G$ generates $K[V]^G$ as a $K$-algebra, then $V(f) = V^{sss}$, so $V^{sss}$ is a Cartier divisor whose image is the Cartier divisor $0 \in \text{Spec } K[V]^G$. □

Lemma 1.4. Let $V_1$ and $V_2$ be representations of a reductive algebraic group $G$ and let $V = V_1 \oplus V_2$ with the product $G \times G$ action. Then

1. $V^{sss}(G \times G) = (V_1)^{sss}(G) \times (V_2)^{sss}(G)$.
2. If $G$ is a torus then $V^{sss}(T) \subset V^{sss}(T \times T)$ where the $T$-action is the diagonal action.

Proof. We first show that $V^{sss}(G \times G) \subset (V_1)^{sss}(G) \times (V_2)^{sss}(G)$. A vector $v = (v_1, v_2) \in V$ is $(G \times G)$-strictly semi-stable if and only if the orbit $(G \times G)v$ is not saturated with respect to the quotient map, i.e. if there exists $v' = (v'_1, v'_2)$ such $v'$ has the same image in $(V_1 \oplus V_2)/G \times G$ and $v'$ is in the same $G \times G$ orbits as $(v_1, v_2)$. Since $V_1 \oplus V_2$ has the product action we must have either $v'_1 \notin Gv_1$ or $v'_2 \notin Gv_2$. Assume without loss of generality that $v'_1 \notin Gv_1$.

Since $v'$ has the same image as $v$ under the quotient map, $h(v) = h(v')$ for all $h \in K[V]^{G \times G} = K[V_1]^G \times K[V_2]^G$. In particular for all $f_1 \in K[V_1]^G, (f_1 \otimes 1)(v) = (f_1 \otimes 1)(v')$. But $(f_1 \otimes 1)(v) = f_1(v_1)$ and $(f_1 \otimes 1)(v') = f_1(v'_1)$. Thus, $v_1$ and $v'_1$ have the same value on all $G$-invariant functions on $V_1$ but do not lie in the same orbit, so $v_1 \in V_1^{sss}(G)$.

To prove $(V_1)^{sss}(G) \times (V_2)^{sss}(G) \subset V^{sss}(G \times G)$, by symmetry, it enough to show $V_1^{sss}(G) \times V_2 \subset V^{sss}(G \times G)$. If $v_1 \in V_1^{sss}(G)$ then we know there is a vector $v'_1 \notin Gv_1$ such that $f_1(v'_1) = f(v_1)$ for all $f_1 \in K[V_1]^G$. Hence, if $v_2 \in V_2$ is any vector then $v = (v_1, v_2)$ and $v' = (v'_1, v_2)$ are in the same $G \times G$ orbit, but $(f'_1 \otimes f_2)(v) = (f_1 \otimes f_2)(v')$ for all $f_1 \in K[V_1]^G$ and $f_2 \in K[V_2]^G$. Since $K[V]^{G \times G} = K[V_1]^G \otimes K[V_2]^G$ it follows that any $(G \times G)$-invariant function has the same value on $v$ and $v'$, but these two vectors are not in the same orbit. Hence $v \in V^{sss}(G \times G)$. This proves part (1).

We now prove (2). First note that if $V$ is any representation of a reductive group $G$ then a vector $v \in V$ is strictly semi-stable if and only if there is a vector $v' \in \overline{Gv}$ such that $\dim G_{v'} > d$ where $d$ is the generic stabilizer dimension of $V$. When $G = T$, then for all vectors $v, G_v = G_0$ where $G_0$ is the kernel of the action and the generic stabilizer equals $G_0$. In particular if $\dim G_{v'} > d$ then $G_{v'}$ contains a 1-parameter subgroup not contained in $G_0$. It follows that $v \in V^{sss}(T)$ if and only if the following condition holds: there is a 1-parameter subgroup $\lambda$ not contained in $G_0$ such that $v$ has only non-negative weights for the action of $\lambda$.

Given $V = V_1 \oplus V_2$, let $K_1$ and $K_2$ be the kernels of the actions of $T$ on $V_1$ and $V_2$, respectively. Then the kernel of the diagonal action of $T$ on $V$ is $K_1 \cap K_2 \subset T$ and the kernel of the action of $T \times T$ is $K_1 \times K_2$. Suppose that $(v_1, v_2) \in V^{sss}(T)$. Then there is a 1-parameter subgroup $\lambda$ of $T$ not contained in $K_1 \cap K_2$ such that $(v_1, v_2)$ has only non-negative weights with respect to the action of $\lambda$. The image of $\lambda$ in $T \times T$ under the diagonal embedding is not contained in $K_1 \times K_2$. Therefore $(v_1, v_2)$ is also in $V^{sss}(T \times T)$.

Lemma 1.5. Let $G$ be a reductive group. Suppose that $V = V_1 \oplus V_2$ and $V/G = V_1/G \times V_2/G$. Then $V^{sss}(G) \supset V^{sss}(G \times G)$ where the action of $G$ is the diagonal action.

Proof. Suppose that $v = (v_1, v_2)$ is a $(G \times G)$-strictly semistable point. By Lemma 1.4, we may assume without loss of generality that $v_1 \in V_1^{sss}(G)$; so $v_1$ is not saturated with respect to the quotient map $V_1 \rightarrow V_1/G$. In other words there is a point $v'_1 \notin Gv_1$ such that $f(v'_1) = f(v_1)$ for all $f \in K[V_1]^G$.

We claim that $(v'_1, v_2)$ is in the $G$-saturation of $(v_1, v_2)$, i.e. $h(v'_1, v_2) = h(v_1, v_2)$ for all $h \in K[V]^G$. To see this note that our assumption implies that $K[V]^G = K[V_1]^G \otimes K[V_2]^G$ so $h \in K[V]^G$ can be
expressed as $h = \sum a_i b_i$ where $a_i \in K[V_1]_G$ and $b_j \in K[V_2]_G$. Then $h(v'_1, v_2) = \sum a_i(v'_i) b_i(v_2) = \sum a_i(v_1) b_i(v_2) = h(v_1, v_2)$ as claimed.

Given the claim it follows that $(v_1, v_2)$ is not strictly semi-stable since $(v'_1, v_2)$ is not in the $G$-orbit of $(v_1, v_2)$. \hfill \Box

Proof of Proposition 1.6. Note that a representation $V$ of a group $G$ is cofree if and only if $V/G$ is smooth and the quotient map $\pi: V \to V/G$ is flat.

If $V/G = V_1/G \times V_2/G$ then $V/G$ is smooth if and only if $V_1/G$ and $V_2/G$ are smooth. By hypothesis the quotient map $\pi$ factors as $\pi = \pi_1 \times \pi_2$ where $\pi_1: V \to V_1/G$ and $\pi_2: V \to V_2/G$ are corresponding quotient maps. Hence $\pi$ is flat if and only $\pi_1$ and $\pi_2$ are flat. It follows that $V$ is cofree if and only if $V_1$ and $V_2$ are cofree.

If $G = T$ is a torus then by Lemmas [1.4] and [1.5] we know that $V^{ss}(T) = (V_1^{ss} \times V_2) \cup (V_1 \times V_2^{ss})$ so we see that $V^{ss}$ is a union of divisors if and only if $V_1^{ss}$ and $V_2^{ss}$ are. Hence $V^{ss}$ is pure of codimension-one if and only if $V_1^{ss}$ and $V_2^{ss}$ are pure of codimension-one.

Now if $D = D_1 \times V_2$ is a divisor in $V^{ss}$ then $D_1$ is a divisor in $V_1^{ss}$ and $\pi(D) = \pi_1(D_1) \times V_2/G$. Hence $\pi(D)$ is a Cartier divisor if and only if $\pi_1(D_1)$ is Cartier. A similar statement holds for divisors in $V^{ss}$ of the form $V_1 \times D_2$. Therefore $V$ is coprincipal if and only if $V_1$ and $V_2$ are. \hfill \Box

We now come to the key proposition required to prove the Theorem 11.6.

Proposition 1.6. Let $V$ be a coprincipal representation of a torus $T$. Then there are $T$-representations $V_i$ such that $V = V_1 \oplus V_2$ as $T$-representations, $V/T = V_1/T \times V_2/T$, and $V_1/T$ is one-dimensional.

Proof. Let $x_1, \ldots, x_n$ be coordinates on $V$ diagonalizing the $T$-action. Any invariant $f \in K[x_1, \ldots, x_n]^T$ is necessarily a sum of invariant monomials, i.e. $K[V]^T$ is generated by monomials. Let $f_1, \ldots, f_r$ be a minimal set of monomials that generate $K[V]^T$. If $r = 1$ then the statement is trivial so we assume that $r \geq 2$.

Since $T$ acts diagonally, $V^{ss}$ is the union of linear subspaces. By purity, there is a divisorial component of $V^{ss}$, which after reordering coordinates, we can assume is $V(x_1)$. Since $V(x_1) \subset V^{ss}$ there is a non-trivial invariant function vanishing on $V(x_1)$. Since such a function is a polynomial in the monomials $f_1, \ldots, f_r$, we must have that $x_1 | f_i$ for some $i$. After reordering we may assume that $x_1 | f_1$.

By assumption the image of $V(x_1)$ is Cartier. Since $V/T$ is an affine toric variety, Pic($V/T$) = 0 so the ideal $I = (x_1) \cap K[V]^T$ defining $\pi(V(x_1))$ is principal. We claim that minimality of $f_1, \ldots, f_r$ implies that $I = (f_1)$ and $x_1 | f_i$ for $i \neq 1$.

To prove the claim we argue as follows. Let $p = f_1^{a_1} \cdots f_r^{a_r}$ be a monomial generator of $I$. Since $f_1 \in I$ we can write $f_1 = q f_1^{a_1} \cdots f_r^{a_r}$. Since this equation also holds in the polynomial ring $K[V]$ we conclude that either $q = 1$ and $p = f_1$ or that $f_1$ can be expressed as a monomial in $f_2, \ldots, f_r$ which contradicts the minimality of $f_1, \ldots, f_r$.

We now claim that if $x_i | f_1$ then $x_i | f_k$ for $i \neq 1$. To see this suppose that $x_2 | f_1$ and $x_2 | f_2$. Then the image of $V(x_2)$ is contained in the subvariety of $V/T = \text{Spec} K[f_1, f_2, \ldots, f_r]$ defined by the ideal $(f_1, f_2)$. Note that $f_1, f_2$ are necessarily algebraically independent in $\text{Spec} K[f_1, f_2, \ldots, f_r]$ because $f_1$ is the only generator divisible by $x_1$. Hence it follows that the image of $V(x_2)$ is not a divisor.

On the other hand, we will show $V(x_2) \subset V^{ss}$ so by assumption on the representation $V$, we know that the image of $V(x_2)$ is a divisor. This will lead to a contradiction. If $V(x_2)$ is not in $V^{ss}$ then $V(x_2)$ has dense intersection with the open set of stable points $V^s$. The quotient map $\pi_s: V^s \to V^s/G$ has constant dimensional fibers which are orbits. In particular, any $T$-invariant subvariety of $V^s$ is saturated, so the image of the $T$-invariant divisor $V(x_2) \cap V^s$ in $V^s/T$ would have codimension one.
Given the claim we can, after reordering the coordinates, assume that \(x_1, \ldots, x_s | f_1 \) and \(x_j \nmid f_1 \) if \( j > s \) and \( x_i \nmid f_k \) if \( i \leq s \) and \( k \neq 1 \). (Note that we must have \( s < n \) since \( K[V]^T \) is generated by at least two invariants.) Hence the invariant ring \( K[V]^T \) is generated by \( f_1 = x_1^{a_1} \cdots x_s^{a_s} \) with \( a_i > 0 \) and monomials \( f_2, \ldots, f_r \) in the variables \( x_{s+1}, \ldots, x_n \). So we can split \( V = V_1 \oplus V_2 \) where \( V_1 \) is the subspace spanned by the coordinates \( x_1, \ldots, x_s \) and \( V_2 \) is the subspace spanned by the coordinates \( x_{s+1}, \ldots, x_n \). The invariant ring \( K[V_1]^T \) consists of those elements of \( K[V]^T \) that only involve \( x_1, \ldots, x_s \). Since these variables do not divide \( f_2, \ldots, f_r \), we know \( K[V_1]^T \) is generated by \( f_1 \). Likewise, any \( T \)-invariant monomial in \( x_{s+1}, \ldots, x_n \) is a product of \( f_2, \ldots, f_r \), so \( K[V_2]^T \) is generated by \( f_2, \ldots, f_r \). Since \( f_1 \) is algebraically independent from \( f_2, \ldots, f_r \), we have \( K[V]^T = K[f_1][f_2, \ldots, f_r] \) if \( T \) is regular, so \( V \) is coregular. However, \( V^{ss} \) is the union of two codimension-two subspaces \( V(x, z) \) and \( V(y, z) \).

Proof of Theorem 1.6 The theorem follows by induction on the dimension of \( V/G \) and Propositions 1.4 and 1.6.

2. Further results and examples for torus actions

In this section, we give examples to illustrate how coregular, pure, and coprincipal differ.

2.1. Example to show coregular does not imply pure. Not surprisingly, there are stable coregular representations of tori which are not pure. Here is a simple example.

Example 2.1. Let \( T = G_m \) act on a 3-dimensional vector space \( V \) with weights \((1, -1, 0)\). If we identify \( K[V] = K[x, y, z] \) then \( K[V]^G = K[xy, z] \) is regular, so \( V \) is coregular. However, \( V^{ss} \) is the union of two codimension-two subspaces \( V(x, z) \) and \( V(y, z) \).

2.2. Example to show that \( V^{ss} \) being pure of codimension-one does not imply pure.

Example 2.2. Consider the \( G_m^2 \)-action on \( A^5 \) with weights
\[
\begin{align*}
x &= (2, 0), & y &= (0, 1), & z &= (-2, -1), & u_1 &= (-1, 0), & u_2 &= (-1, 0)
\end{align*}
\]
This representation is stable and \( V^{ss} = V(x) \cup V(y) \cup V(z) \). One checks that
\[
K[x, y, z, u_1, u_2]^{G_m^2} = K[xy, xu_1^2, xu_2^2, xu_1 u_2].
\]
So, the quotient is \( A^1 \) times an \( A_1 \)-singularity, hence it is not smooth but has finite quotient singularities. We see that \( V \) is not pure as \( V(x) \) maps to a point under the quotient map.

2.3. Example to show that pure does not imply coprincipal.

Example 2.3. Consider the action of \( G_m^3 \) on \( A^6 \) with weights
\[
\begin{align*}
u_1 &= (0, 1, 0), & u_2 &= (1, -1, 0), & u_3 &= (1, 0, 0), & u_4 &= (-1, 0, 0), & y_1 &= (0, 0, 1), & y_2 &= (-1, 0, -1)
\end{align*}
\]
We calculate the invariants. Let \( H = (0, 0, 1) \) be a hyperplane in the character lattice tensored with \( \mathbb{R} \). Note that the \( u_i \in H \), and that \( y_1 \) and \( y_2 \) are on opposite sides of \( H \). As a result, every monomial invariant \( \prod u_i^{b_i} \) must have \( a_1 = a_2 \). Hence,
\[
K[u_1, u_2, u_3, u_4, y_1, y_2]^{G_m^3} = K[u_1, u_2, u_3, u_4, y_1y_2]^{G_m},
\]
where \( G_m^2 \) is the subtorus \( G_m^2 \times 1 \subset G_m^3 \). Said another way, \( A^6/G_m^3 \cong A^5/G_m^2 \), where \( G_m^2 \) acts on \( A^5 \) with weights
\[
\begin{align*}
u_1' &= (0, 1), & u_2' &= (1, -1), & u_3' &= (1, 0), & u_4' &= w = (-1, 0)
\end{align*}
\]
Now notice that the weights \( u_3' \) and \( u_4' = w \) are contained on the line \( L = (0, 1)^\perp \), and that \( u_1' \) and \( u_2' \) live on opposite sides of \( L \). So by the same reasoning as above,
\[
K[u_1, u_2, u_3, u_4, y_1y_2]^{G_m} = K[u_1u_2, u_3, u_4, y_1y_2]^{G_m},
\]
or said another another way, \( \mathbb{A}^5/G_m^2 \simeq \mathbb{A}^4/G_m \) where \( G_m \) acts on \( \mathbb{A}^4 \) with weights 1, 1, −1, −1. This quotient is the non-simplicial toric variety given by the cone over the quadratic surface. We have therefore shown
\[
K[u_1, u_2, u_3, u_4, y_1, y_2]^{G_m} = K[u_1u_2y_1y_2, u_1u_2u_4, u_3y_1y_2, u_3u_4].
\]
One checks that
\[
V^{\text{sss}} = V(u_1) \cup V(u_2) \cup V(y_1) \cup V(y_2)
\]
and that each of these components maps to a divisor, so \( V \) is pure. However, \( V \) is not coprincipal since all of these components map to Weil divisors which are not Cartier, e.g. \( x \) the coordinates we know that \( V(G) \)
\[
dim
\]
\[
\text{Polar representations}
\]
and that each of these components maps to a divisor, so \( V \) is pure. However, \( V \) is not coprincipal since all of these components map to Weil divisors which are not Cartier, e.g. \( V(u_1) \subset \mathbb{A}^6 \) maps to \( V(u_1u_2y_1y_2, u_1u_2u_4) \subset \mathbb{A}^6/G_m^2 \) which is the divisor \( a = b = 0 \) in the quotient \( \text{Spec} K[a, b, c, d]/(ad−bc) \).

Note that by contrast if \( V \) is a pure representation of a connected reductive group \( G \) such that \( \dim V/G = 2 \), then it follows from Kempf [Kem] (cf. [PV, Theorem 8.6]) that \( V \) is cofree and hence coprincipal if Question 1.9 has an affirmative answer.

### 2.4. Co-orbifold and pure implies coprincipal.

The pure representation of Example 2.3 is not coprincipal but has worse than finite quotient singularities. The following proposition shows that this is not an isolated phenomena.

**Proposition 2.4.** If \( V \) is an pure representation of a torus \( T \) for which \( V/G \) is singular, then \( V/G \) has worse than finite quotient singularities.

**Proof.** We will show that if \( V \) is pure and the image has finite quotient singularities then it is in fact coprincipal and hence cofree by Theorem 1.6.

If \( V/G \) has finite quotient singularities then any divisor on \( V/G \) is \( \mathbb{Q} \)-Cartier and the proposition follows from the following lemma.

**Lemma 2.5.** Let \( V \) be a stable representation of a torus \( T \) and let \( \pi: V \to V/T \) be the quotient map. Let \( Z \) be a divisorial component of \( V^{\text{sss}} \). Then the effective Weil divisor \( [\pi(Z)] \) is \( \mathbb{Q} \)-Cartier if and only it is Cartier.

**Proof.** We use an argument similar to that used in the proof of Proposition 1.6. As above choose coordinates \( x_1, \ldots, x_n \) diagonalizing the \( T \) action and let \( f_1, \ldots, f_r \) be a minimal set of monomials that generate \( K[V]^T \). After reordering the coordinates we may assume that \( Z = V(x_1) \). The image \( \pi(Z) \) is the subvariety of \( V/T \) defined by the contracted ideal \( I = (x_1) \cap K[V]^T \). Since the map \( V \to V/T \) is toric the ideal \( I \) is generated by monomials \( (p_1, \ldots, p_s) \). Since \( \pi(Z) \) is \( \mathbb{Q} \)-Cartier there is a monomial \( p \) such that \( \sqrt{(p)} = I \). Write \( p = f_1^{a_1} \cdots f_r^{a_r} \). Since \( p \in (x_1) \), after possibly reordering the coordinates we know that \( x_1|f_1^{a_1} \) and hence \( x_1|f_1 \). Thus \( f_1 \in I = \sqrt{(p)} \) so \( f_1 = qf_1^{a_1} \cdots f_r^{a_r} \). for some monomial \( q \in K[f_1, \ldots, f_r] \). Since this equation also holds in the UFD \( K[V] \) we conclude that \( q = 1 \) and \( p = f_1 \). As in the proof of proposition 1.6 this implies that no other \( x_i \nmid f_i \) for \( i \neq 1 \). Hence \( I \) must be generated by powers of \( f_1 \) so \( I = (f_1) \) since \( f_1 \in I \). Therefore \( \pi(Z) \) is defined by a single equation as claimed.

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