Solution to the \( \beta \)-functions in Lorentz-violating theories as a decomposition into irreducible representations

Alejandro Ferrero

Departmento de Física
Universidad de los Andes
Bogotá, Colombia

Abstract

We analyze the \( \beta \)-functions of Yukawa and electromagnetic theories with Lorentz violation (LV) and propose an alternative method to find the scale dependence of the different fields that parametrize such violations. The method of solution consists of decomposing a family of parameters into their irreducible representations and thus generating a group of subfamilies that obey the same symmetries and transformation rules. This method allows us to decouple the differential equations describing the \( \beta \)-functions and find out if whether they are positive or not. For a set of parameters describing a Lorentz-violating theory, we expect their associated \( \beta \)-functions to be nonnegative or, otherwise, their scale dependence to be weak enough. These conditions rely on the fact that asymptotically-free parameters can leave high imprints of LV at low energies, which are ruled out by observations. Besides imposing some constrains on the coefficients that describe LV, this method can be used to extract irrelevant coefficients with no scale dependence.

\[^1\text{a.ferrero@uniandes.edu.co}\]
1 Introduction

Although LV has not been observed in our relatively low-energy experiments, there is no reason to assume that small violations of Lorentz and CPT symmetries are incompatible with quantum field theories \[1, 2\]. Nonetheless, any model that includes Lorentz or CPT violations, for instance, the Standard Model Extension (SME) \[1, 2, 3\], must be compatible with the phenomena observed in our universe.

The conservation of Lorentz and CPT symmetries should be related. In 2002, Greenberg proposed that CPT violations imply the violation of Lorentz invariance \[4\]. Nevertheless, Deutsch and Gracia-Bondía recently suggested that such claim is still on somewhat shaky ground \[5\]. While the relation between both symmetries might still be unclear, there are some fundamental causes that could induce their respective violations. Some possible scenarios include models with time-varying coupling constants \[6, 7, 8, 9, 10, 11\] and models with noncommutative geometry, space-time discretization, among others \[12, 13, 14, 15\].

The values of some of the parameters described by the SME can be constrained by means of experiments. Most of the analysis in the SME has been performed in the electromagnetic sector \[16, 17, 18, 19, 20\]. Additionally, some work has been performed in the gravitational \[21, 22\] and neutrino regimes \[23\]. A possible violation of Lorentz symmetry in the Yukawa sector has not been strongly emphasized in spite of its great importance \[24, 25\], specially with the great evidence supporting the existence of the Higgs particle coming from the LHC experiments \[27\]. The existence of the Higgs particle, which generates the masses of the particles in the standard model, opens the window to have a better understanding of the Yukawa interactions.

Besides the use of observations in order to test the validity of models that include Lorentz and CPT violations, the renormalization group and the scale dependence of the parameters that describe such violations are important to formulate a consistent theory. As it is well known, the scale-dependence of a physical quantity in a quantum field theory is described by its $\beta$-function. While a positive value of such function implies that the observed value of its associated physical quantity increases with the energy scale and a negative value describes the opposite behavior, the zeroes of the $\beta$-function provide information of the fixed points, in which the physical quantities have no scale dependence.

The renormalization of the electron charge in QED is a very famous case in which a positive $\beta$-function is obtained. In this case, therefore, we expect its observed value to decrease in the low energy regimes. Similarly, it is natural to expect the same behavior for a family of parameters describing LV in the SME. If some of their $\beta$-functions turn out to be negative, we should expect to observe (unless their scale dependence is weak enough) a relatively large imprint of LV at low energies, which, at the best of our knowledge, has not taken place. Therefore, we can impose constraints on some of the coefficients contained in Lorentz-violating theories by studying their scale dependence as well as test the validity of such models.
Indeed, if the coefficients describing LV turn out to have positive $\beta$-functions, we could say that Lorentz and CPT symmetries are low-energy emergent symmetries present in our universe \cite{28, 29}. While (if the described scenario is correct) the universe and its space-time structure might be described by very different physics and other symmetry principles at high enough energies, there could exist an energy scale from which such effects are practically washed out. This can explain why Lorentz symmetry seems to be exact.

The purpose of this paper is to provide a method to easily solve the $\beta$-functions in some theories with LV and impose constrains on such models by demanding the condition that their $\beta$-functions must be nonnegative. Particularly, we will apply this method to Yukawa \cite{30} and electromagnetic theories \cite{31, 32} at one-loop order. The $\beta$-functions for a Yukawa theory with Lorentz and CPT violations were already found in \cite{30}, however, they were not solved. On the other hand, the one-loop renormalization of Lorentz-violating electrodynamics in flat space-time was already studied in \cite{31}; nonetheless, the method used to solve for the running couplings does not decompose the tensor-like parameters into their irreducible representations, so different symmetry-like parameters are mixed leading to different solutions. In addition, by changing to diagonal bases, the subsequent method can be used to redefine (at one-loop order) a new set of coefficients where the effects of irrelevant parameters are extracted. It could also be interesting to apply this method to electroweak interactions \cite{33}, non-Abelian gauge theories \cite{34, 35}, or other scalar-fermion interactions \cite{30} with LV; however, this will not be done here.

This article is organized as follows: in section \ref{sec:Yukawa} we briefly describe the Lagrange density for a Yukawa theory with Lorentz and CPT violations as well as the different coefficients that describe such theory. Section \ref{sec:method} describes the general method of solution used to solve for the running couplings. Particularly, such method is applied to Yukawa theories in section \ref{sec:Yukawaapplications}, the application of such method in a known model such as electromagnetism is studied in section \ref{sec:electromagnetism} for comparison reasons. Finally, a brief summary and some conclusions are presented in section \ref{sec:conclusions}.

\section{Yukawa interactions with LV}

As stated in \cite{30}, a theory of Yukawa interactions with Lorentz and CPT violations can be described by the Lagrange density

\begin{equation}
    \mathcal{L}_Y = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} K^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 + \bar{\psi} \left( i \Gamma^\mu \partial_\mu - M \right) \psi - \phi \bar{\psi} G \psi,
\end{equation}

where $K^{\mu\nu}$ are the $\delta$-like terms of the theory, $\lambda$ is the Yukawa coupling, and $G$ is the field strength of the gauge field.
where
\[
\Gamma^\nu = \gamma^\nu + c^{\mu\nu}\gamma_\mu + d^{\mu\nu}\gamma_5\gamma_\mu + e^{\nu} + i\gamma_5 f^{\nu} + \frac{1}{2} g^{\lambda\mu\nu}\sigma_{\lambda\mu} , 
\]
\[
M = m + i\gamma_5 m' + a^{\mu}\gamma_\mu + b^{\mu}\gamma_5\gamma_\mu + \frac{1}{2} H^{\mu\nu}\sigma_{\mu\nu} , 
\]
\[
G = g + i\gamma_5 g' + I^{\mu}\gamma_\mu + J^{\mu}\gamma_5\gamma_\mu + \frac{1}{2} L^{\mu\nu}\sigma_{\mu\nu} . 
\]

When quantum corrections are introduced, the previous operators mix according to their symmetry properties. This fact allows us to introduce a set of groups whose elements obey the same symmetries; these groups are described in table 1.

Similarly, the described theory can be renormalized by defining the counter-terms [30]
\[
i\delta \Phi = i(p^2 + K_{\mu\nu}p^\mu p^\nu)\delta_\phi - i\delta_\rho + i p_\mu p_\nu \delta_{K_{\mu\nu}} , 
\]
\[
i\delta \Psi = i(\Gamma^\mu p_\mu - M_1)\delta_\psi - i\delta_m + \gamma_5 \delta_{m'} + i p_\mu \delta_\epsilon - p_\mu \gamma_5 \delta_f + i \gamma_\mu p_\nu \delta_{c^{\mu\nu}} 
+ i\gamma_5 \gamma_\mu p_\nu \delta_{d^{\mu\nu}} + \frac{i}{2} \sigma_{\lambda\mu} p_\nu \delta_{g^{\lambda\mu\nu}} - i\gamma_\mu \delta_a - i\gamma_5 \gamma_\mu \delta_b - \frac{i}{2} \sigma_{\mu\nu} \delta_{H^{\mu\nu}} , 
\]
\[-i\delta G = -i\delta_g + \gamma_5 \delta_g' - i\gamma_\mu \delta_I - i\gamma_5 \gamma_\mu \delta_J - \frac{i}{2} \sigma_{\mu\nu} \delta_{L^{\mu\nu}} , 
\]
\[-i\delta \Lambda = -i\delta_\lambda . 
\]
The set of coefficients \(\delta_{\mu_1 \cdots \mu_n}\) was already calculated in [30].
3 Renormalization

In our previous paper [30], we found expressions for the $\beta$-functions associated with all the parameters that describe a theory with Yukawa interactions with Lorentz-violating operators. Nonetheless, solving for the running couplings was not performed due to the complexity of the $\beta$-functions. Such difficulty arises when different parameters mix due to quantum corrections and so generating sets of coupled differential equations. Additionally, each group of parameters can in turn be decomposed into subgroups using the symmetry group; the action if this group is the permutation of their indices. For example, the coefficient $d_{\mu\nu}$ can be decomposed into three irreducible representations: a trace-like part written as $\eta_{\mu\nu}d_{\alpha\alpha}$, an antisymmetric part described by $d_{\mu\nu}^A = \frac{1}{2}(d_{\mu\nu} - d_{\nu\mu})$, and a traceless symmetric part given by $d_{\mu\nu}^S = \frac{1}{2}(d_{\mu\nu} + d_{\nu\mu}) - \frac{1}{4}\eta_{\mu\nu}d_{\alpha\alpha}$.

It is well known that, in this case, the trace-like representation has scalar (rank-zero tensor) properties; the antisymmetric and symmetric parts have vector (rank-one tensor) and rank-two tensor properties respectively. This situation also arises when we add the angular momentum of two spin-one particles.

If the mixing of the different coefficients did not exist in a Lorentz-violating theory, we would obtain $\beta$-functions of the form

$$(\beta_x)^{\mu_1\cdots\mu_n} = f_x(m, m', g, g')x^{\mu_1\cdots\mu_n}, \quad (4)$$

where $f_x(m, m', g, g')$ is a function that depends on the masses and the couplings of the Lorentz invariant theory (note that the sign of $(\beta_x)^{\mu_1\cdots\mu_n}$ depends on the sign of $f_x$). We use the same notation as in [30, 37] by defining the dimensionless quantity $\tilde{p} = p/M$, with $M$ a particular mass scale, to represent the scale associated with the renormalization group. The variable $x^{\mu_1\cdots\mu_n}$ or $x^{i_1\cdots i_n}_{i}^{\mu_1\cdots\mu_n}(\tilde{p})$ will indicate the value of such parameter at the scale $\tilde{p}$ and $\bar{x}^{\mu_1\cdots\mu_n}$ its value at $M$ ($\tilde{p} = 1$). For parameters satisfying Eq. (4), its scale dependence is given by [30]

$$x^{\mu_1\cdots\mu_n}(\tilde{p}) = \bar{x}^{\mu_1\cdots\mu_n}\exp\left[\int_1^{\tilde{p}} \frac{dp'}{p'} f_x(m, m', g, g')(\tilde{p}')\right]. \quad (5)$$

Nonetheless, when we attempt to obtain the scale dependence of the Lorentz-violating coefficients belonging to the same group, we will obtain coupled systems of differential equations of the form

$$(\beta_x)^{\mu_1\cdots\mu_n}_m = \sum_{k=1}^{N} M_{mk}x^{\mu_1\cdots\mu_n}_k, \quad (6)$$

where $M_{mk}$ are matrix elements that depend on the functions $f_x$ and $N$ is the number of elements belonging to the same group or subgroup. Each subgroup is in fact described by means of a decomposition into irreducible representations, where all elements belong to the same irreducible class. The symmetric group $S_n$ will be used to accomplish this goal.
If we introduce the diagonal matrix $M^{(d)}$ given by $M = AM^{(d)}A^{-1}$ we can decouple the linear system and obtain solutions of the form
\[ x_k^{(d)\mu_1...\mu_n}(\tilde{p}) \equiv \tilde{x}_k^{(d)\mu_1...\mu_n} S_{kk}, \quad \text{where} \]
\[ S_{kk} = \exp \left[ \int_1^{\tilde{p}} \frac{d\tilde{p}'}{\tilde{p}'} \lambda_k^x(m, m', g, g')(\tilde{p}') \right] \]
are the solutions to the $\beta$-functions in the diagonal basis. Here $\lambda_k^x$ are the eigenvalues of the matrix $M$; the initial conditions in the diagonal basis are related to those in the original basis by $\tilde{x}^{(d)\mu_1...\mu_n} = A^{-1}x^{\mu_1...\mu_n}$. Therefore, the solutions are given by
\[ x^{\mu_1...\mu_n}(\tilde{p}) = ASA^{-1}x^{\mu_1...\mu_n}. \] Notice that the set of eigenvalues $\lambda_k^x$ determine the sign of the $\beta$-functions in the diagonal basis. However, when we have a mixture of positive and negative eigenvalues the coefficients in the original basis can have the sum of contributions that grow and decrease with the energy scale. This still predicts a large imprint of LV for low-energy scales unless some very special conditions are satisfied or the scale dependence is weak enough. In the presence of fixed points we have null eigenvalues, which indicates that the $\beta$-function of at least one parameter can be written as the linear combination of the other ones. If this situation takes place, the number of relevant parameters that describe LV can be reduced (at least, at one-loop order).

The notation introduced in this section will we used in the following ones.

4 Solutions to the $\beta$-functions of a Yukawa theory

4.1 Lorentz invariant couplings

Since the $\beta$-functions of the Lorentz-violating parameters depend on the parameters associated with the Lorentz invariant theory, we should find the scale dependence of the latter first. As indicated in [30], neglecting $O(c_\alpha^2, K_\alpha)$ contributions, the couplings $g$ and $g'$ are given by
\[ g^2(\tilde{p}) = \tilde{g}^2 F(\tilde{p})^{-1}, \quad g'^2(\tilde{p}) = \tilde{g}'^2 F(\tilde{p})^{-1}, \] with
\[ F(\tilde{p}) \equiv 1 - 5\eta\tilde{g}^2 \ln \tilde{p}^2. \]
We will use the definitions $\eta^{-1} \equiv 16\pi^2$ and $\tilde{g}_\pm^2 \equiv \tilde{g}^2 \pm \tilde{g}'^2$. In this theory, both $g$ and $g'$ grow as the energy scale increases and present the usual Landau pole as $F(\tilde{p}) = 0$. Fortunately, the Landau pole takes place as $\tilde{p} = \exp \left[ \frac{8\pi^2}{5\eta \tilde{g}_+^2} \right]$, which is a very high momentum scale.
For $\bar{g} = \bar{g}' = \frac{1}{3}$, for instance, $\bar{p} \sim 10^{54}$, so the perturbative regime includes a very comprehensive energy spectrum.

Having found the scale dependence of $g$ and $g'$, we can now solve for $m$ and $m'$. In terms of the eigenvalues $\lambda_{\pm} = \frac{1}{2}(g_+^2 \pm \sqrt{(5g^2 + 3g'^2)^2 + 4g^2g'^2})$ we obtain

$$m_{\pm}^{(d)}(\bar{p}) = \bar{m}_{\pm}^{(d)} F(\bar{p}) \frac{8\pi^2 \lambda_{\pm}}{5\pi^2} \equiv \bar{m}_{\pm}^{(d)} F_{\pm}(\bar{p}). \quad (12)$$

Hence

$$m(\bar{p}) = \frac{\bar{m}(5g^2 - 3g'^2) + 8\bar{m}' \bar{g}g'}{4\eta^{-1}\lambda_{+} - 2\bar{g}_+^2} (F_+(\bar{p}) - F_-(\bar{p})) + \frac{1}{2} \bar{m}(F_+(\bar{p}) + F_-(\bar{p})), \quad (13a)$$

$$m'(\bar{p}) = -\frac{m'(5g^2 - 3g'^2) - 8\bar{m}' \bar{g}g'}{4\eta^{-1}\lambda_{+} - 2\bar{g}_+^2} (F_+(\bar{p}) - F_-(\bar{p})) + \frac{1}{2} \bar{m}'(F_+(\bar{p}) + F_-(\bar{p})). \quad (13b)$$

To find the momentum dependence of the boson mass and the scalar coupling $\lambda$, we need to solve, respectively, the differential equations

$$\bar{p} \frac{d\mu^2}{d\bar{p}} = 8\eta \left[ \mu^2 g_+^2 - 2(mg + m'g')^2 - (m^2 + m'^2)g_+^2 \right], \quad (14)$$

$$\bar{p} \frac{d\lambda}{d\bar{p}} = \eta \left[ 3\lambda^2 + 8\lambda g_+^2 - 48g_+^2 \right]. \quad (15)$$

where $O(c_{\mu}^\mu)$ and $O(K_{\mu}^\mu)$ have been neglected. The $\beta$-functions associated with the Lorentz-invariant coefficients do not have sign restrictions because of consistency conditions. Since we have not been able to obtain analytical solutions for the last set of equations, some numerical solutions are shown in figures [1] and [2].
4.2 Lorentz-violating parameters

Now we will solve for the running couplings associated with the Lorentz-violating theory. We will independently focus on each one of the four groups presented in table 1.

4.3 Solution to group 1

For this family we have the set of differential equations [30]

\[(\beta_K)^{\mu\nu} = 4\eta g_2^2 (K^{\mu\nu} + c^\alpha a^{\mu\nu} - c^{\mu\nu} - c^{\nu\mu}), \quad (16a)\]
\[(\beta_c)^{\mu\nu} = \frac{1}{3} \eta g_2^2 (c^{\mu\nu} + c^{\nu\mu} - K^{\mu\nu} + (c^\alpha + K^\alpha) \eta^{\mu\nu}), \quad (16b)\]

In order to solve previous equations we will use the rank-two irreducible decomposition

\[X^{\mu\nu}_A = \frac{1}{2} (X^{\mu\nu} - X^{\nu\mu}), \quad X^{\mu\nu}_S = \frac{1}{2} (X^{\mu\nu} + X^{\nu\mu}) - \frac{1}{2} \eta^{\mu\nu} X^\alpha. \quad (17)\]

Using this decomposition we find that \((\beta_K^A)^{\mu\nu} = 0\) and \((\beta_c^A)^{\mu\nu} = 0\); this is expected in the case of \(K^{\mu\nu}\) because it is contracted by the external momentum combination \(p^\mu p^\nu\). Writing our linear system as in Eq. (6) with \((x^{\mu\nu})^T = (K^{\mu\nu}_S, c^{\mu\nu}_S)\) (and something similar for the traces) we find the solutions

\[K^{(d)\alpha}_A (\tilde{p}) = \bar{K}^{(d)\alpha}_A F(\tilde{p})^{-3/5}, \quad c^{(d)\alpha}_A (\tilde{p}) = \bar{c}^{(d)\alpha}_A, \quad (18a)\]
\[K^{(d)\mu\nu}_S (\tilde{p}) = \bar{K}^{(d)\mu\nu}_S F(\tilde{p})^{-7/15}, \quad c^{(d)\mu\nu}_S (\tilde{p}) = \bar{c}^{(d)\mu\nu}_S. \quad (18b)\]

So, in the original basis

\[K^\alpha_A (\tilde{p}) = -\frac{4}{3} \bar{c}^\alpha_A (1 - F(\tilde{p})^{-3/5}) + \frac{1}{3} \bar{K}^\alpha_A (2 F(\tilde{p})^{-3/5} + 1), \quad (19a)\]
\[c^\alpha_A (\tilde{p}) = \frac{1}{3} \bar{c}^\alpha_A (F(\tilde{p})^{-3/5} + 2) - \frac{1}{6} \bar{K}^\alpha_A (1 - F(\tilde{p})^{-3/5}), \quad (19b)\]
\[K^{\mu\nu}_S (\tilde{p}) = \frac{12}{7} \bar{c}^{\mu\nu}_S (1 - F(\tilde{p})^{-7/15}) + \frac{1}{7} \bar{K}^{\mu\nu}_S (1 + 6 F(\tilde{p})^{-7/15}), \quad (19c)\]
\[c^{\mu\nu}_S (\tilde{p}) = \frac{1}{7} \bar{c}^{\mu\nu}_S (6 + F(\tilde{p})^{-7/15}) + \frac{1}{14} \bar{K}^{\mu\nu}_S (1 - F(\tilde{p})^{-7/15}). \quad (19d)\]

4.4 Group 2

Having solved for the coefficients \(c^{\mu\nu}\) and \(K^{\mu\nu}\), we proceed with the vector parameters, which are \(a^\mu, e^\mu, f^\mu\) and \(I^\mu\). Since all four tensors have vector representations, they do not need to be decomposed into irreducible representations. In this case, however, the \(\beta\)-functions of the linear system do not explicitly depend on \(a^\mu\), so its \(\beta\)-function is a linear combination of the other three couplings.
The $\beta$-functions satisfy the set of differential equations \[30\]

\[
\begin{pmatrix}
(\beta_1)^\mu \\
(\beta_2)^\mu \\
(\beta_3)^\mu
\end{pmatrix} = \eta \begin{pmatrix}
6g_1^2 & -g'_2g_2^2 & -gg_2^2 \\
-2g'_2 & 2g_2^2 & 2gg'_2 \\
-2g & 2gg' & 2g^2
\end{pmatrix} \begin{pmatrix}
I^\mu \\
f^\mu \\
ed^\mu
\end{pmatrix},
\]

which generate the solutions

\[ I^{(d)}\mu(\bar{p}) = \bar{I}^{(d)}\mu, \quad f^{(d)}\mu(\bar{p}) = \bar{f}^{(d)}\mu F_{+}(\bar{p}), \quad e^{(d)}\mu(\bar{p}) = \bar{e}^{(d)}\mu F_{-}(\bar{p}), \]

with \( F_{\pm}(\bar{p}) \equiv F(\bar{p})^{-(4 \pm \sqrt{6})/10}. \)

In the original basis we have

\[ I^\mu(\bar{p}) = \frac{1}{2\sqrt{\bar{\alpha}}} (\bar{g}e^{\mu} + \bar{g}'f^{\mu} - 2\bar{I}^\mu) (F_{+}(\bar{p}) - F_{-}(\bar{p})) + \frac{1}{4}\bar{I}^\mu (F_{+}(\bar{p}) + F_{-}(\bar{p})), \]

\[ f^\mu(\bar{p}) = \frac{\bar{g}}{2g_+^2} (\bar{g}f^{\mu} - \bar{g}'e^{\mu}) + \frac{\bar{g}'I^\mu}{\sqrt{6g_+^2}} (F_{+}(\bar{p}) - F_{-}(\bar{p})) + \frac{\bar{g}'(\bar{g}e^{\mu} + \bar{g}'f^{\mu})}{2\sqrt{6g_+^2}} ((2 + \sqrt{6})F_{+}(\bar{p}) - (2 - \sqrt{6})F_{-}(\bar{p})), \]

\[ e^\mu(\bar{p}) = \frac{\bar{g}'}{2g_+^2} (\bar{g}'e^{\mu} - \bar{g}f^{\mu}) + \frac{\bar{g}I^\mu}{\sqrt{6g_+^2}} (F_{+}(\bar{p}) - F_{-}(\bar{p})) + \frac{\bar{g}(\bar{g}e^{\mu} + \bar{g}'f^{\mu})}{2\sqrt{6g_+^2}} ((2 + \sqrt{6})F_{+}(\bar{p}) - (2 - \sqrt{6})F_{-}(\bar{p})). \]

On the other hand, \( a^\mu \) can be determined by solving

\[ a^\mu(\bar{p}) = \bar{a}^\mu + \eta \int_{1}^{\bar{p}} \frac{dp'}{p'} \left[ A(2m + g'm')I^\mu - g_+^2 (me^{\mu} + m'f^{\mu}) \right] \]

using Eqs. [10], [13a], [13b], and [22a]–[22c]. Although \( a^\mu \) can be solved analytically, its expression is quite long and so it will not be shown here. An interesting fact is to study its leading contribution at low momentum (\( \bar{p} \ll 1 \)); it is given by

\[ a^\mu(\bar{p}) \sim \bar{a}^\mu + A\bar{I}^\mu (1 - F(\bar{p})^\alpha), \]

where \( \alpha = \frac{1}{20} \left( 1 + 2\sqrt{6} + \frac{1}{9} \sqrt{(5g_+^2 + 3g'_2)^2 + 4g_+^2g_+^2} \right) \) and \( A \) is a constant with mass units. The maximum value of \( \alpha \) is achieved as \( g' = 0 \), thus \( \alpha_{\text{max}} = \frac{1}{10}(3 + \sqrt{6}) \). In spite that \( \alpha > 0 \) and so \( a^\mu \) increases as \( \bar{p} \to 0 \), we do not really expect to observe large values of \( a^\mu \) at low energy scales. For \( \bar{g} = 0.25, \bar{g}' = 0, \) and \( \bar{p} = 10^{-100} \), for example, \( F(\bar{p})^{\alpha_{\text{max}}} \simeq 1.42 \), so \( a^\mu = O(\bar{a}^\mu, A\bar{I}^\mu) \) at such low scales.
4.5 Group 3

In order to solve for $d^{\mu \nu}$, $H^{\mu \nu}$ and $L^{\mu \nu}$ (remember that $H^{\mu \nu}$ and $L^{\mu \nu}$ are antisymmetric by construction), we notice that they obey the set of differential equations \[30\]

\[(\beta_d)^{\mu \nu} = \eta \left\{ \frac{1}{3} g_+^2 (5d^{\mu \nu} - d_{\alpha}^{\mu \nu} \eta^{\alpha \mu} d_{\alpha}^{\nu}) - 2g'L^{\mu \nu}_A - 2gL^{\mu \nu}_A \right\}, \quad (25a)\]

\[(\beta_H)^{\mu \nu}_A = \eta \left\{ g_+^2 H^{\mu \nu}_A + 4(gm + g'm')L^{\mu \nu}_A + 4(mg' - m'g)L^{\nu \mu}_A \right. \]

\[\left. + 2(mg_2^2 + 2m'g')d_{\nu}^{\mu \nu} + 2(m'g_2^2 - 2m'g')d_{\nu}^{\mu \nu} \right\}, \quad (25b)\]

\[(\beta_L)^{\mu \nu}_A = \eta g_+^2 \left\{ 7L^{\mu \nu}_A + 2gd^{\mu \nu}_A - 2g'd^{\mu \nu}_A \right\}, \quad (25c)\]

where we used the Hodge dual representation $X^{\ast \mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} X_{\alpha \beta}$. (From now on we will use the convention $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.) After decomposing $d^{\mu \nu}$ into its irreducible parts we see that the trace and symmetric parts of $d^{\mu \nu}$ obey

\[d_{\alpha}^{\mu \nu}(\bar{p}) = \bar{d}_{\alpha}^{\mu \nu}, \quad d^{\mu \nu}_S(\bar{p}) = \bar{d}^{\mu \nu}_S F(\bar{p})^{-2/15}. \quad (26a)\]

The antisymmetric parts couple and satisfy

\[(\beta_d)^{\mu \nu}_A = \eta \left\{ 2g_+^2 d^{\mu \nu}_A - 2g'L^{\mu \nu}_A - 2gL^{\mu \nu}_A \right\}, \quad (27a)\]

\[(\beta_H)^{\mu \nu}_A = \eta \left\{ g_+^2 H^{\mu \nu}_A + 4(gm + g'm')L^{\mu \nu}_A + 4(mg' - m'g)L^{\nu \mu}_A \right. \]

\[\left. + 2(mg_2^2 + 2m'g')d_{\nu}^{\mu \nu} + 2(m'g_2^2 - 2m'g')d_{\nu}^{\mu \nu} \right\}, \quad (27b)\]

\[(\beta_L)^{\mu \nu}_A = \eta g_+^2 \left\{ 7L^{\mu \nu}_A + 2gd^{\mu \nu}_A - 2g'd^{\mu \nu}_A \right\}. \quad (27c)\]

In spite that last equations only involve antisymmetric representations, the Hodge dual components are also present. Last system cannot be solved directly because, although the Hodge dual representation of a tensor carries the same information and should have the same scale behavior than the original tensor itself, different components are mixed. For example, for $\mu = 0$ and $\nu = 1$, $L^{\mu \nu}_A = L^{01}_A$ and $L^{\ast \mu \nu}_A = L^{23}_A$. This would mix different components of $L^{\mu \nu}_A$, for instance, in Eq. \[(27a).\]

The solution to this issue is solved by taking the Hodge duals of Eqs. \[(27a) - (27c)\] to obtain a six-variables linear system. In this case we can define the six-component vector $(x^T)^{\mu \nu} = (d_A, d_A^*, H_A, H_A^*, L_A, L_A^*)^{\mu \nu}$, where, for example, we have the condition $(\beta_d)^{\mu \nu}_A = \eta \left\{ 2g_+^2 d^{\mu \nu}_A - 2g'L^{\mu \nu}_A + 2gL^{\mu \nu}_A \right\}$ from Eq. \[(27a).\] We thus find the solutions

\[d_A^{(d)\mu \nu}(\bar{p}) = \bar{d}_A^{(d)\mu \nu} F_1(\bar{p}), \quad d_A^{(d)\nu \mu}(\bar{p}) = \bar{d}_A^{(d)\nu \mu} F_1(\bar{p}), \quad F_1(\bar{p}) \equiv F(\bar{p})^{-1/10}, \quad (28a)\]

\[H_A^{(d)\mu \nu}(\bar{p}) = \bar{H}_A^{(d)\mu \nu} F_2(\bar{p}), \quad H_A^{(d)\nu \mu}(\bar{p}) = \bar{H}_A^{(d)\nu \mu} F_2(\bar{p}), \quad F_2(\bar{p}) \equiv F(\bar{p})^{-2(4-\sqrt{2})/5}, \quad (28b)\]

\[L_A^{(d)\mu \nu}(\bar{p}) = \bar{L}_A^{(d)\mu \nu} F_3(\bar{p}), \quad L_A^{(d)\nu \mu}(\bar{p}) = \bar{L}_A^{(d)\nu \mu} F_3(\bar{p}), \quad F_3(\bar{p}) \equiv F(\bar{p})^{-2(10-\sqrt{2})/9}. \quad (28c)\]
Using previous definitions (suppressing the dependence on $\tilde{p}$ in $F_i(\tilde{p})$), we find

\[ d_\lambda^{\mu\nu}(\tilde{p}) = \frac{1}{2} \tilde{d}_A^{\mu\nu} \left[ F_2 + F_3 + \frac{5}{\sqrt{41}} (F_2 - F_3) \right] + \frac{2}{\sqrt{41} g_+^2} (gL^{\mu\nu} + \bar{g}L_\lambda^{\mu\nu}) (F_2 - F_3), \]  
\[ H_\lambda^{\mu\nu}(\tilde{p}) = \bar{H}_A^{\mu\nu} F_1 - \frac{4}{\sqrt{41} g_+^2} \left[ (\bar{m}\tilde{g} + \bar{m}'\tilde{g}')L_A^{\mu\nu} + (\bar{m}\tilde{g}' - \bar{m}'\tilde{g})L_\lambda^{\mu\nu} \right] (F_2 - F_3) \]

\[ + \frac{1}{g_+^2} \left[ (2\bar{m}\tilde{g}' - \bar{m}'\tilde{g}^2)d_A^{\mu\nu} - (2\bar{m}'\tilde{g}' + \bar{m}\tilde{g}^2)d_\lambda^{\mu\nu} \right] \left[ 2F_1 - (1 + \frac{5}{\sqrt{41}})F_2 - (1 - \frac{5}{\sqrt{41}})F_3 \right], \]
\[ L_\lambda^{\mu\nu}(\tilde{p}) = \frac{1}{2} L_A^{\mu\nu} \left[ (1 - \frac{5}{\sqrt{41}})F_2 + (1 + \frac{5}{\sqrt{41}})F_3 \right] + \frac{2}{\sqrt{41}} \left( \bar{g}d_A^{\mu\nu} - \bar{g}d_\lambda^{\mu\nu} \right) (F_2 - F_3). \]

## 4.6 Group 4

This set of differential equations is the most difficult to handle; it couples the terms $g^{\lambda\mu\nu}$, $J^\mu$, and $b^\mu$, as well as their dual representations. The difficulty mainly arises from finding an appropriate irreducible decomposition for $g^{\lambda\mu\nu}$; a careful decomposition is performed in appendix A. An anti-symmetrization in the indices $\{\lambda, \mu\}$ of $g^{\lambda\mu\nu}$ must be performed so it can be written according to Eqs. (68a)–(68d). In terms of $g^{(x\lambda\mu)\nu} \equiv \frac{1}{2} \varepsilon^{\alpha\beta\lambda\mu} g_{\alpha\beta}^{\nu}$ we have [30]

\[ (\beta_g)^{\lambda\mu\nu} = 2\eta \left[ \frac{1}{3} (g^2 + g_2^2) g^{\lambda\mu\nu} - \frac{1}{3} g_2^2 g^{\lambda\mu\nu} + \frac{1}{6} g_2^2 \eta^{\mu\nu} g_2^\lambda - \frac{1}{6} g_2^2 \eta^{\lambda\mu} g_2^\nu + g^J,^{\lambda\mu\nu} \right. \]

\[ - g' \eta^{\mu\nu} J^\lambda + g' \eta^{\lambda\mu} J^\nu + \frac{1}{3} gg' g_{2}^{\nu\lambda\mu} + \frac{1}{3} gg' \left( g^{(x\lambda\mu)\nu} + 2g^{(x\lambda\nu)} \right) \right]. \]  

After decomposing into irreducible representations we have

\[ (\beta_g)_A^{\lambda\mu\nu} + \frac{4}{3} (\beta_g)_A^{\mu\nu} + \frac{4}{3} (\beta_g)_A^{\lambda\nu} = \frac{1}{3} \eta^{\lambda\mu} (\beta_g)_A^{\mu\nu} + \frac{1}{3} \eta^{\mu\nu} (\beta_g)_A^{\lambda\nu} = 2\eta \left[ \frac{2}{3} gg_A^{\lambda\mu\nu} + g^J^{\lambda\mu\nu} + \frac{2}{3} gg' g_{2}^{\nu\lambda\mu} \right. \]

\[ - \frac{2}{3} gg' g_{3}^{\nu\lambda\mu} + \frac{4}{3} (g^2 + g_2^2) g_1^{\nu\lambda\mu} - \frac{4}{3} g_2^2 g_{2}^{\nu\lambda\mu} + \eta^{\lambda\nu} \left( \frac{1}{18} (g^2 + 5g_2^2) g_2^\lambda + \frac{1}{2} g_2^2 g_{2}^{\nu\lambda\mu} + g', J^\lambda \right) \]

\[ - \eta^{\mu\nu} \left( \frac{1}{18} (g^2 + 5g_2^2) g_2^\lambda + \frac{1}{2} g_2^2 g_{2}^{\nu\lambda\mu} + g', J^\lambda \right) \right]. \]

Multiplying last equation by $\varepsilon_{\lambda\mu\nu}$ we obtain

\[ (\beta_g)_A^{\nu\mu} = \eta \left[ \frac{2}{3} gg_A^{\nu\mu} + 2gJ^\mu + \frac{4}{3} gg' g_{2}^{\nu\mu} + \frac{8}{3} gg' g_{2}^{\nu\mu} \right], \]

where we used the condition $g_1^\mu + g_2^\mu + g^\mu = 0$ (see appendix A for more details).

The $\beta$-function of $g_2^\mu$, as defined in Eq. (64), can be found by multiplying Eq. (31) by $\eta_{\lambda\nu}$. In order to find $(\beta_g)_A^{\mu}$ we must perform index permutations in Eq. (30) before
contracting the first two indices (this must be done because Eq. (31) does not provide enough information). Hence we find

\[(\beta_6)_\mu = \eta \left[ \frac{1}{3}(g^2 + 5g'^2)g_2^\mu + \frac{2}{3}g^2g_1^\mu + 6g'J^\mu + 6gg'g_A^\mu \right], \quad (33a)\]

\[(\beta_7)_\mu = \eta \left[ \frac{2}{7}(2g^2 + g'^2)g_1^\mu + \frac{7}{2}g^2g_2^\mu - 3g'J^\mu - 3gg'g_A^\mu \right]. \quad (33b)\]

The \(\beta\)-functions of \(b^\mu\) and \(J^\mu\) were already found in \([30]\), which are also used to obtain the linear system

\[
\begin{pmatrix}
(\beta_6)_\mu \\
(\beta_7)_\mu \\
(\beta_8)_\mu \\
(\beta_9)_\mu \\
(\beta_{10})_\mu
\end{pmatrix} = \eta
\begin{pmatrix}
g_2^2 & 2(mg + m'g') & 0 & \frac{1}{2}m'g_2^\mu & \frac{3}{2}mg_2^\mu \\
0 & 8g_2^2 & 0 & g_2'g_2^\mu & 3g_2^2 \mu \\
0 & -3g' & \frac{2}{3}(2g^2 + g'^2) & \frac{7}{2}g_2^2 & -3gg' \\
0 & 6g' & \frac{2}{3}g_2^2 & \frac{1}{2}(g^2 + 5g'^2) & 6gg' \\
0 & 2g & \frac{1}{3}gg' & \frac{8}{2}gg' & 2g^2
\end{pmatrix}
\begin{pmatrix}
b^\mu \\
J^\mu \\
g_1^\mu \\
g_2^\mu \\
g_A^\mu
\end{pmatrix} \quad . \quad (34)
\]

While one of the eigenvalues (the one associated with \(b^\mu\)) of the previous matrix is always \(\lambda_1 = \eta g_2^2\) and so is positive, the other four eigenvalues take a very complicated form and their sign depend on the values of \(g\) and \(g'\); this behavior is shown in figure 3. For some values of such couplings we have a negative eigenvalue and even complex results. The complex results associated with \(\lambda_3\) and \(\lambda_4\) in figure 3 are, however, just a mathematical artifact and the final results are real in the original basis (this will become more clear below); since their real parts are always positive, they do not generate an asymptotically-free behavior.

In spite that \(\lambda_5\) is negative for most values of \(g\) and \(g'\), the evolution of the parameters described by Eq. (34) also depends from contributions coming from the electromagnetic sector \([31]\). Hence, the inclusion of electromagnetic effects may change such tendency. Additionally, the magnitude of \(\lambda_5\) does not take considerably large values for typical values of \(\beta\) (this parameter is defined in figure 3). In figure 3 the smallest value of \(\lambda_5\) is \(\lambda_5 \approx -8.8 \times 10^{-3}\) and takes place at \(\beta = 3\). Using this value and \(\bar{g} = 0.25\), we see that \(F(\bar{p})^{-|\lambda_5|} \approx 0.96\) for \(\bar{p} = 10^{-100}\), which is very close to 1. Therefore, \(\lambda_5\) induces a negligible deviation at low energy scales.

If we neglect electromagnetic effects, there are still some conditions that could make all eigenvalues in Eq. (33) positive for arbitrary values of \(g\) and \(g'\). The solution is assuming that the values of some coefficients at the renormalization scale vanish, so there is no LV at any energy scale coming from such fields. Nevertheless, such conditions are very unlikely and restrictive and demand fine-tuning.

Using Eq. (33) into Eq. (31) and the respective \(\beta\)-functions for \(g^{\lambda\mu}\) and \(g^{\nu\lambda}\) (last ones can be found by permuting indices) we have that the other two irreducible repre-
Figure 3: Values of the eigenvalues $\lambda_2, \lambda_3, \lambda_4$ and $\lambda_5$ as a function of $\bar{g}$ and $\bar{g}'$. Here $\bar{g} = g_0$ and $\bar{g}' = \beta g_0$. The vertical dashed lines divide a set of intervals; the horizontal dashed line is the value 0. The first interval is $0 \leq \beta < \frac{1}{\sqrt{3}}$ and all eigenvalues except $\lambda_5$ are positive, like in the third interval given by $1 < \beta < 1.038$ and the fifth, where $\beta \geq 1.434$. In the second interval all eigenvalues are positive and $\frac{1}{\sqrt{3}} \leq \beta \leq 1$. In the fourth interval, $1.038 < \beta < 1.433$, $\lambda_3$ and $\lambda_4$ are complex. The upper small graph zooms the interval where $\lambda_5 \geq 0$, the lower small graph zooms the imaginary parts of $\lambda_3$ and $\lambda_4$. 
sentations satisfy
\[
(\beta_1)_{\lambda^\mu} = \frac{2}{3} \eta \left[ (2g^2 + g')g_1 - g_2^2g_{-2} + gg'g_{-1}^{(+\lambda\mu)\nu} + 2gg'g_{+2}^{(+\lambda\mu)\nu} \right], \tag{35a}
\]
\[
(\beta_{2+})_{\lambda^\mu} = \frac{2}{3} \eta \left[ (2g^2 + g')g_{2+} - g_2^2g_1 - gg'g_{1-}^{(+\lambda\mu)\nu} + 2gg'g_{2+}^{(+\lambda\mu)\nu} \right]. \tag{35b}
\]

Last system is, however, very difficult to solve. For convenience we will change to a different basis by defining \( g_\pm^{\lambda^\mu} = g_1^{\lambda^\mu} \pm g_{2+}^{\lambda^\mu} \). In this basis we have the linear system
\[
\begin{pmatrix}
(\beta_1)_{\lambda^\mu} \\
(\beta_2)_{\lambda^\mu} \\
(\beta_{2+})_{\lambda^\mu}
\end{pmatrix} = \frac{2}{3} \eta \begin{pmatrix}
3g^2 & 0 & 0 & 0 \\
0 & 3g^2 & 0 & 0 \\
3gg' & 0 & -gg' & g^2 + 2g'^2
\end{pmatrix} \begin{pmatrix}
g_1^{\lambda^\mu} \\
g_{-1}^{\lambda^\mu} \\
g_1^{\lambda^\mu} \\
g_{-1}^{\lambda^\mu}
\end{pmatrix}. \tag{36}
\]

Besides the eigenvalues \( \lambda_{1,2} = 2nq^2 \), last system has the complex eigenvalues \( \lambda_{3,4} = g^2 + 2g'^2 \mp igq' \). Fortunately, the evolution of the given parameters is real when we go back to the original basis (this also applies for the parameters described in figure [3]). In the diagonal basis we find
\[
g^{(d)\lambda^\mu}(p) = g^{(d)\lambda^\mu}(p_1)F_1(p)
\]
\[
g^{(d)\lambda^\mu}(p) = g^{(d)\lambda^\mu}(p_2)F_2(p), \quad g^{(d)\lambda^\mu}(p) = g^{(d)(\lambda\mu)\nu}F_2^*(p), \tag{37b}
\]
\[
F_1(p) = F(p) - \frac{g^2}{5g^2}F_1, \tag{37c}
\]
\[
F_2(p) = F(p) - \frac{(g^2 + 2g'^2)/(15g^2)}{\cos \left( \frac{\theta g'}{15g^2} \ln F(p) \right) + i \sin \left( \frac{\theta g'}{15g^2} \ln F(p) \right)} \tag{37d}
\]

In terms of \( F_2^R(p) \equiv \text{Re}[F_2(p)] \), and \( F_2^I(p) \equiv \text{Im}[F_2(p)] \) we have
\[
g_1^{\lambda^\mu}(p) = \frac{1}{2}g^{\lambda^\mu}F_1(p) + \frac{1}{2}g^{\lambda^\mu}F_2^R(p) - \frac{1}{2}g^{(\lambda^\mu)\nu}g^{\lambda^\nu}F_2^I(p) + \frac{3g^2g'^2g_1^{(+\lambda\mu)\nu} + 6g^2g'g_1^{(+\lambda\mu)\nu}}{8(g^2)^2 + 2g^2g'^2}F_2^I(p)
\]
\[
+ \frac{3g^2g'^2g_1^{(+\lambda\mu)\nu} - 6g^2g'g_1^{(+\lambda\mu)\nu}}{8(g^2)^2 + 2g^2g'^2}(F_1(p) - F_2^R(p)), \tag{38a}
\]
\[
g_2^{\lambda^\mu}(p) = g_1^{\lambda^\mu}(p) - g_1^{\lambda^\mu}F_1(p). \tag{38b}
\]

At first glance, one might be tempted to think that \( g_1^{\lambda^\mu}(p) \) and \( g_2^{\lambda^\mu}(p) \) have an oscillatory behaviour because of the definition of \( F_2(p) \) in Eq. \( \text{[37d]} \). Nevertheless, such oscillatory behavior does not take place. From Eq. \( \text{[37d]} \) we can easily see that \( F(p) \) does not take a very large value unless \( \tilde{p} \) is extremely small. For instance, for \( \tilde{g} = \tilde{g}' = 0.5 \) and \( \tilde{p} = 10^{-100} \), \( F(p) \approx 2.88 \). On the other hand, if \( \tilde{g}' = \beta \tilde{g} \), \( \frac{\theta g'}{15g^2} \) is maximum as \( \beta = 1 \), hence \( \frac{\theta g'}{15g^2} \leq \frac{1}{30} \).

Therefore, we can easily conclude that \( 0 \leq \frac{\theta g'}{15g^2} \ln F(p) < \frac{\pi}{2} \) for typical values of \( \tilde{g}, \tilde{g}' \), and \( \tilde{p} \).
Figure 4: $g_{1+}^{\lambda\mu\nu}$ as a function of the scale. Here the black lines represent the conditions $2g = g' = 0.2g_0$ and the gray lines the conditions $g = 2g' = 0.4g_0$. In the continuous lines $\bar{g}_1^{\lambda\mu\nu} = 2\bar{g}_2^{\lambda\mu\nu} = 2g_0^{\lambda\mu\nu}$; in the dashed lines $\bar{g}_1^{\lambda\mu\nu} = \bar{g}_2^{\lambda\mu\nu} = g_0^{\lambda\mu\nu}$.

Figure 5: $g_{2-}^{\lambda\mu\nu}$ as a function of the scale. We used the same initial conditions and labelings as in figure 4.

5 Application to Lorentz-violating QED

The method that we have used to decomposing into irreducible representations can also be applied to Lorentz-violating QED using the results from [31].

Let us remember that the evolution of the electron charge in usual QED is given by

$$e^2(\tilde{p}) = e^2Q^{-1}(\tilde{p}), \quad Q(\tilde{p}) \equiv 1 - (e^2/6\pi^2) \ln \tilde{p}. \quad (39)$$

The evolution of the mass operator, using $\beta_m = \frac{3e^2}{8\pi^2}m$, is [31]

$$m(\tilde{p}) = \bar{m}Q(\tilde{p})^{9/4}. \quad (40)$$

From [31] we can see that the parameters in Lorentz-violating QED can also be decomposed into four groups, which are

1. Set $\{\eta^{\lambda\mu\nu}c^{\mu\sigma}, (k_F)^{\lambda\mu\nu}\}$.
2. Set $\{a^\mu, e^\mu, f^\mu\}$.
3. Set $\{d^{\mu\nu}, H^{\mu\nu}\}$.
4. Set $\{b^\lambda\eta^{\mu\nu}, g^{\mu\nu}, (k_{AF})^{\lambda\eta^{\mu\nu}}\}$. 
5.1 Group 1

In Lorentz-violating QED we also have the parameter \((k_F)_{\lambda\mu\nu\sigma}\) associated with the electromagnetic field. Since its contribution to the Lorentz-violating QED lagrange density is \(\Delta L = -\frac{1}{2}(k_F)_{\lambda\mu\sigma}F_{\lambda\mu}F^{\nu\sigma}\), it has the symmetry properties \([31]\)

\[
\begin{align*}
(k_F)_{\lambda\mu\nu\sigma} &= (k_F)_{\nu\sigma\lambda\mu} = -(k_F)_{\mu\lambda\nu\sigma}, \\
(k_F)_{\lambda\mu\nu\sigma} + (k_F)_{\lambda\sigma\mu\nu} + (k_F)_{\lambda\sigma\mu\nu} &= 0, \\
(k_F)_{\mu\nu} &= 0.
\end{align*}
\]

From \([31]\) we have the \(\beta\)-functions

\[
\begin{align*}
(\beta_{k_F})^{\lambda\mu\nu\sigma} &= -\frac{\eta^{\lambda\nu}c_S^{\mu\sigma} + \eta^{\lambda\sigma}c_S^{\mu\nu} + \eta^{\mu\nu}c_S^{\lambda\sigma} - \eta^{\mu\sigma}c_S^{\lambda\nu}}{2
c^2\alpha \left( \eta^{\lambda\nu}\eta^{\mu\sigma} - \eta^{\lambda\sigma}\eta^{\mu\nu} \right)}, \\
(\beta_{c})^{\mu\alpha} &= \frac{c^2}{6\pi^2} \left[ 2c_S^{\mu\alpha} + \frac{1}{2} \eta^{\mu\alpha}c^a_a - (k_F)^{\alpha\mu}c^a_a \right].
\end{align*}
\]

Notice that previous relations satisfy the symmetry conditions stated in Eqs. \((41a)-(41c)\). From now on we will use the same definitions as in Eq. \((70)\) and deal with \((k_F)_{\lambda\mu\nu\sigma}\) as explained in appendix \([13]\). Notice that when we decompose \((k_F)_{\lambda\mu\nu\sigma}\) according to Eq. \((70)\) we have permutations in the indices \(\{\lambda, \mu, \nu, \sigma\}\) just like in Eq. \((42a)\). Hence, each permutation of indices satisfies the same equations and so a decomposition of the \(\beta\)-function into irreducible representations can be written as

\[
\begin{align*}
(\beta_{k_F}^t)^{\lambda\mu\nu\sigma} + \frac{1}{2} \eta^{\lambda\nu}(\beta_{k_F})^{\mu\sigma}_{2S} &= \frac{c^2}{6\pi^2} \left[ (k_F)^{\lambda\mu\nu\sigma} + \frac{1}{2} \eta^{\lambda\nu}(k_F)^{\mu\sigma}_{2S} - 2c_S^{\mu\sigma} \right] \\
&\quad - \frac{1}{2} c^\alpha_a \left( \eta^{\lambda\nu}\eta^{\mu\sigma} - \eta^{\lambda\sigma}\eta^{\mu\nu} \right), \\
(\beta_{c})^{\mu\alpha} &= \frac{c^2}{6\pi^2} \left[ 2c_S^{\mu\alpha} + \frac{1}{2} \eta^{\mu\alpha}c^a_a - (k_F)^{\alpha\mu}_{2S} \right].
\end{align*}
\]

Taking the double trace over Eq. \((43a)\) we see that \(c^\alpha_a = 0\). This is not inconsistent, however, because this parameter has no physical relevance because it can be absorbed by a redefinition in the fermion field \([38]\). Multiplying Eq. \((43a)\) by \(\eta_{\lambda\nu}\) we have the linear system

\[
\begin{pmatrix}
(\beta_{k_F})^{\mu\sigma}_{2S} \\
(\beta_{c})^{\mu\sigma}_{2S}
\end{pmatrix} = \frac{c^2}{6\pi^2} \begin{pmatrix}
1 & -2 \\
-1 & 2
\end{pmatrix} \begin{pmatrix}
(k_F)^{\mu\sigma}_{2S} \\
(c_S^{\mu\sigma})
\end{pmatrix}.
\]

The solutions are thus

\[
\begin{align*}
(k_F)^{\mu\sigma}_{2S}(\bar{p}) &= (k_F)^{\mu\sigma}_{2S} - \frac{1}{3}(k_F)^{\mu\sigma}_{2S}(1 - Q(\bar{p})^{-3}) + \frac{2}{3} c_S^{\mu\sigma}(1 - Q(\bar{p})^{-3}) \, , \\
c_S^{(d)\mu\sigma}(\bar{p}) &= c_S^{\mu\sigma} + \frac{1}{3}(k_F)^{\mu\sigma}_{2S}(1 - Q(\bar{p})^{-3}) - \frac{2}{3} c_S^{\mu\sigma}(1 - Q(\bar{p})^{-3}) \, .
\end{align*}
\]
Additionally, we have
\[(\beta g)_{\lambda \mu \nu} A = \frac{e^2}{6\pi^2} (k_F)_{\lambda \mu \nu}, \quad \text{so} \quad (k_F)_{\lambda \mu \nu} (\tilde{p}) = (\tilde{k}_F)_{\lambda \mu \nu} Q(\tilde{p})^{-1}. \quad (46a)\]

The solutions found, however, are different from those found in [31]. The reason, as mentioned, relies on the fact that here we are decomposing over irreducible representations, which is not done in [31]. It is necessary to decompose into irreducible representations to avoid the mixing of different components.

5.2 Group 2

The solution for this family is easily found. Using [31] we can easily verify that
\[e^\mu(\tilde{p}) = \bar{e}^\mu, \quad f^\mu(\tilde{p}) = \bar{f}^\mu, \quad (47)\]
\[a^\mu(\tilde{p}) = \bar{a}^\mu - \bar{m}e^\mu (1 - Q(\tilde{p})^{9/4}), \quad (48)\]
which is agreement with the results found in [31]. This agreement is expected because this group of parameters does not need to be decomposed into irreducible representations.

5.3 Group 3

The solution to this group can also be easily found, it is
\[d^\alpha(\tilde{p}) = \bar{d}^\alpha Q(\tilde{p})^{-2}, \quad d^\mu(\tilde{p}) = \bar{d}^\mu Q(\tilde{p})^{-2}, \quad (49)\]
and
\[d^\mu(\tilde{p}) = \bar{d}^\mu, \quad H^\mu(\tilde{p}) = H^\mu Q(\tilde{p})^{-3/4} + 4\bar{m}\bar{d}^\mu Q(\tilde{p})^{9/4}(1 - Q(\tilde{p})^{-3/4}). \quad (50)\]
We show, again, the difference in the results coming from the method that we have used.

5.4 Group 4

We will first focus on the solutions for the coefficients $g^{\lambda \mu \nu}$ [31]. By permuting indices and decomposing into irreducible representations we find the $\beta$-functions for $g^{\lambda \mu \nu}$ and $g^{\lambda \mu \nu}$, which are, respectively
\[(\beta g)^{\lambda \mu \nu} A = \frac{4}{3} (\beta g)^{\lambda \mu \nu} A - \frac{1}{3} \eta^\nu (\beta g)^{\lambda \mu} 1 - \frac{1}{3} \eta^\mu (\beta g)^{\lambda \nu} 2 - \frac{1}{3} \eta^\nu (\beta g)^{\lambda \nu} 2 = \frac{e^2}{8\pi^2} \left[ \frac{16}{3} g_1 - \frac{8}{3} g_2 + \frac{4}{3} \eta^\nu g_2 - \frac{4}{3} \eta^\nu g_2 + \frac{5}{3} \eta^\nu g_1 - \frac{5}{3} \eta^\mu g_1 \right], \quad (51)\]
\[\frac{16}{3} (\beta g)^{\lambda \mu \nu} 2 + \frac{4}{3} (\beta g)^{\lambda \mu} 1 + \frac{1}{3} \eta^\nu (\beta g)^{\lambda} = \frac{e^2}{8\pi^2} \left[ \frac{4}{3} g_2 + \frac{4}{3} g_1 - \frac{1}{3} \eta^\nu g_2 - \frac{1}{3} \eta^\nu g_2 + \frac{1}{6} \eta^\nu g_1 - \frac{1}{6} \eta^\nu g_1 \right], \quad (52)\]
where we used again the condition $g_1^μ + g_2^μ + g_3^μ = 0$. By multiplying by $ε_{λμσ}$ in either equation we see that $(β_g^A)^μ = 0$. Thus

$$g_A^λμ(\bar{p}) = \bar{g}_A^λμ, \quad \text{or} \quad g_A^μ(\bar{p}) = \bar{g}_A^μ. \quad (53)$$

Multiplying by either $η_μν$ or $η_λν$ in both equations we find the linear system

$$\begin{pmatrix} (β_1^g)^μ \\ (β_2^g)^μ \end{pmatrix} = \frac{e^2}{8π^2} \begin{pmatrix} -\frac{1}{2} & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} g_1^μ \\ g_2^μ \end{pmatrix} \quad (54)$$

leading to the solutions

$$
g_1^μ(\bar{p}) = \bar{g}_1^μ(5Q(\bar{p})^{-9/8} - 4Q(\bar{p})^{-3/2}) + 2g_2^μ(\bar{p})^{-9/8} - Q(\bar{p})^{-3/2}),$$
$$g_2^μ(\bar{p}) = \bar{g}_2^μ(5Q(\bar{p})^{-3/2} - 4Q(\bar{p})^{-9/8}) + 10\bar{g}_1^μ(\bar{p})^{-3/2} - Q(\bar{p})^{-9/8}). \quad (55)$$

The solutions for the other coefficients are easily found, they are

$$(k_{AF})^μ(\bar{p}) = (\bar{k}_{AF})^μQ(\bar{p})^{-1},$$
$$b^μ(\bar{p}) = \bar{b}^μ - \bar{m} g_A^μ (1-Q(\bar{p})^{9/4}) - \frac{5}{4}(\bar{k}_{AF})^μ (1-Q(\bar{p})^{-1}). \quad (56a)$$

The solutions are completed by the linear system

$$\begin{pmatrix} (β_1^g)^1_λμ \\ (β_2^g)^2_λμ \end{pmatrix} = \frac{e^2}{6π^2} \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} g_1^λμ \\ g_2^λμ \end{pmatrix}, \quad (57)$$

whose solutions are

$$g_1^λμ(\bar{p}) = \bar{g}_1^λμ(2Q(\bar{p})^{-3} - Q(\bar{p})^{-2}) - 2\bar{g}_2^λμQ(\bar{p})^{-3} - Q(\bar{p})^{-2}),$$
$$g_2^λμ(\bar{p}) = \bar{g}_2^λμ(2Q(\bar{p})^{-2} - Q(\bar{p})^{-3}) - 2\bar{g}_1^λμQ(\bar{p})^{-2} - Q(\bar{p})^{-3}). \quad (58a)$$

### 5.4.1 Inclusion of Yukawa terms in $g^{λμ}$

Let us remember that Eq. \[54\] described the set of differential equations for the parameters $b^μ, J^μ, g_1^μ, g_2^μ$, and $g_A^μ$ in the pure Yukawa regime. When we also include electromagnetic contributions we will find a linear system of the form $$(β_x)^μ = \sum_{m=1}^6 M_{mn} x_n^μ,$$ with $(x^T)^μ = (k_{AF}, b, J, g_1, g_2, g_A)^μ$ and

$$M = \eta \begin{pmatrix} \frac{2}{3} e^2 & 0 & 0 & 0 & 0 \\ 6e^2 g_3^μ & 2(mg + m'g') & 0 & \frac{1}{7} m'g_2^μ & \frac{3}{2} mg_1^μ - 6me^2 \\ 0 & 0 & 8g_2^μ & 0 & 0 \\ 0 & 0 & -3g' & \frac{2}{3}(2g^2 + g'^2) - e^2 & \frac{7}{2} g_2^μ - 2e^2 \\ 0 & 0 & 6g' & \frac{4}{3} g_2^μ + 10e^2 & \frac{1}{3}(g^2 + 5g'^2) + 8e^2 \\ 0 & 0 & 2g & \frac{8}{9} gg' & 2g^2 \end{pmatrix}. \quad (59)$$
Figure 6: $\tilde{\lambda}_6$ as a function of $\bar{g}$, $\bar{g}'$, and $\bar{e}$. Here we have $\bar{g} = g_0$, $\bar{g}' = \beta g_0$, and $\bar{e} = \sigma g_0$. For each one of the six plots we have $\sigma_1 = 0.25$, $\sigma_2 = 0.6$, $\sigma_3 = 1$, $\sigma_4 = 2.5$, $\sigma_5 = 6$, and $\sigma_6 = 20$. It can easily be seen that when electromagnetic effects are included in Eq. (34), the negative eigenvalue tends to become more positive. This effect is larger for larger $\sigma$.

Last matrix has 6 different eigenvalues. While two of them can be read automatically from last matrix and are $\lambda_1 = \frac{2g_0}{\eta} e^2$ and $\lambda_2 = \eta g_0^2$, the remaining four take a very complicated form. Nonetheless, three of them, name them $\lambda_3$, $\lambda_4$ and $\lambda_5$ are always positive for any combination of $\bar{g}$, $\bar{g}'$, and $\bar{e}$. $\lambda_6$ is not always positive for any combination of such parameters, which can be seen in figure 6.

6 Summary and Conclusions

We have used a method based on decomposing a tensor into its irreducible representations to solve the $\beta$-functions of two theories with Lorentz and CPT violations. As similarly performed in [31], we have changed to diagonal bases to easily solve the $\beta$-functions of a group of operators that satisfy the same symmetries.

Although we mainly focused on a Yukawa theory because the scale dependence of its coefficients had not been solved, we also applied this method to an electromagnetic theory to test this method. Nevertheless, we did not find the same results than those in [31] because in that work the different coefficients are not decomposed into their irreducible representations. Without such decomposition, operators described by different symmetries are mixed, leading to inaccurate solutions.

As stated, those operators whose scale dependence is described by negative $\beta$-functions could lead to an inconsistent theory if their scale dependence is strong enough, thus predicting a large imprint of Lorentz violation at low energies, which, at best of our knowledge, has not been observed. Fortunately, in their respective diagonal bases, most
coefficients are described by positive $\beta$-functions. In this way we could envision, within the scope of these models, Lorentz symmetry as an emergent low-energy property.

Some mass operators have negative $\beta$-functions; however, their scale dependence is weak enough and so their imprint of LV at low energies is negligible. This fact also took place in some of the representations of $g^{\lambda\mu\nu}$, but the same situation applies to them.

This method could also be applied to renormalize other Lorentz-violating theories with similar characteristics or any other theory that has similar properties.

Appendices

A Irreducible decomposition of three-rank tensors

The permutations of three indices form the group $S_3$. The irreducible representations of three-rank tensors can be found using the following set of Young diagrams

\[
\begin{array}{c}
\begin{array}{c}
\boxed{S} \\
\lambda \quad \mu \quad \nu
\end{array} \\
\begin{array}{c}
\boxed{A} \\
\lambda \\
\mu \\
\nu
\end{array} \\
\begin{array}{c}
1^+ \\
\lambda \quad \mu \\
\nu
\end{array} \\
\begin{array}{c}
1^- \\
\lambda \quad \nu \\
\mu
\end{array}
\end{array}
\]

Last set of diagrams generates the four irreducible representations

\[
\begin{align*}
X^\lambda_{S\mu\nu} &= \frac{1}{6} (X^{\lambda\mu\nu} + X^{\lambda\nu\mu} + X^{\mu\lambda\nu} + X^{\mu\nu\lambda} + X^{\nu\lambda\mu} + X^{\nu\mu\lambda}), \\
X^\lambda_{A\mu\nu} &= \frac{1}{6} (X^{\lambda\mu\nu} - X^{\lambda\nu\mu} + X^{\mu\lambda\nu} + X^{\mu\nu\lambda} - X^{\nu\lambda\mu} - X^{\nu\mu\lambda}), \\
X^\lambda_{1^+\mu\nu} &= \frac{1}{4} (X^{\lambda\mu\nu} + X^{\nu\mu\lambda} - X^{\mu\lambda\nu} - X^{\nu\lambda\mu}), \\
X^\lambda_{1^-\mu\nu} &= \frac{1}{4} (X^{\lambda\mu\nu} + X^{\nu\mu\lambda} - X^{\mu\lambda\nu} - X^{\nu\lambda\mu}).
\end{align*}
\]

However, last set only generates 4 linearly-independent representations. In order to complete six independent combinations, we will exchange the indices $\mu \to \nu$ in the tensors generated by last diagram. So we define

\[
X^\lambda_{2^+\mu\nu} = X^\lambda_{1^+\mu\nu}, \quad X^\lambda_{2^-\mu\nu} = X^\lambda_{1^-\mu\nu}.
\]
This set of six tensors can be inverted to find the relations

\[
\begin{pmatrix}
X^{\lambda\mu
} \\
X^{\lambda\nu} \\
X^{\mu\lambda} \\
X^{\mu\nu} \\
X^{\nu\lambda} \\
X^{\nu\mu}
\end{pmatrix}
= \frac{1}{3}
\begin{pmatrix}
3 & 3 & 4 & 0 & 4 & 0 \\
3 & -3 & 0 & 4 & 0 & 4 \\
3 & 3 & -4 & -4 & 0 & 4 \\
3 & -3 & 4 & 0 & -4 & -4 \\
3 & 3 & 0 & 4 & -4 & -4 \\
3 & -3 & -4 & -4 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
X^\lambda_{\mu
} \\
X^\lambda_{\nu} \\
X^\mu_{\lambda} \\
X^\mu_{\nu} \\
X^\nu_{\lambda} \\
X^\nu_{\mu}
\end{pmatrix},
\]

(62)

We can verify that \( X^*_{\lambda\mu
} = X^*_{A\lambda\mu
} \) as well as the following relations

\[
\begin{align*}
X^\lambda_{S\mu\nu} = X^\lambda_{S\nu\mu} = X^{\mu\lambda}_S = X^{\nu\lambda}_S = X^{\nu\mu}_S, \\
X^{\lambda\mu
}_A = -X^A_{\lambda\mu
} = -X^A_{\mu\lambda
} = X^{\nu\lambda}_A = -X^{\nu\mu}_A, \\
X^\lambda_{1+\mu\nu} = X^\lambda_{1+\nu\mu}, \\
X^\lambda_{1-\mu\nu} = -(X^\lambda_{1-\nu\mu} + X^\lambda_{2+\mu\nu}), \\
X^\lambda_{1-\mu\nu} = -X^\lambda_{1-\mu\nu}, \\
X^\lambda_{1-\mu\nu} = X^\lambda_{1-\mu\nu}.
\end{align*}
\]

(63a) (63b) (63c) (63d)

Now we want to isolate the “trace-like” components, which have different transformation rules. Let us define

\[
X^\sigma_1 \equiv X^{\sigma\alpha}_1, \quad X^\sigma_2 \equiv X^{\alpha\sigma}_2, \quad X^\sigma_3 \equiv X^{\alpha\sigma}_3.
\]

(64)

In order to isolate these contributions we must make the six irreducible tensors previously defined traceless in all of their components. So we make

\[
\begin{align*}
X^\lambda_{St\mu\nu} &= X^\lambda_{S\mu\nu} - \frac{1}{18} \eta^{\lambda\mu
}(X^\mu_{1+} + X^\mu_{2+} + X^\mu_{3+}) - \frac{1}{18} \eta^{\lambda\nu\mu}(X^\nu_{1+} + X^\nu_{2+} + X^\nu_{3+}) \\
&\quad - \frac{1}{18} \eta^{\mu\nu\lambda}(X^\lambda_{1+} + X^\lambda_{2+} + X^\lambda_{3+}), \\
X^{\lambda\mu
}_A &= X^{\lambda\mu
}_{A}, \\
X^{\lambda\mu
}_{1+} &= X^{\lambda\mu
}_{1+} + \frac{1}{12} \eta^{\lambda\mu
}(X^\mu_{1-} - X^\mu_{3-}) - \frac{1}{12} \eta^{\lambda\nu\mu}(X^\nu_{1-} - X^\nu_{3-}) - \frac{1}{12} \eta^{\mu\nu\lambda}(X^\lambda_{1-} - X^\lambda_{3-}), \\
X^{\lambda\mu
}_{1-} &= X^{\lambda\mu
}_{1-} - \frac{1}{12} \eta^{\lambda\mu
}(X^\mu_{1-} - X^\mu_{3-}) + \frac{1}{6} \eta^{\lambda\nu\mu}(X^\nu_{1-} - X^\nu_{2-}) - \frac{1}{12} \eta^{\mu\nu\lambda}(X^\lambda_{1-} - X^\lambda_{2-}),
\end{align*}
\]

(65a) (65b) (65c) (65d)

where the subindex “\( t \)” means that the tensors are traceless and, again, \( X^{\lambda\mu\nu}_{2t+} = X^{\lambda\mu\nu}_{1t+} \) and \( X^{\lambda\mu\nu}_{2t-} = X^{\lambda\mu\nu}_{1t-} \). The traceless representations of \( X^{\lambda\mu\nu}_t \) (and its permutations) can be found using Eq. (62).

From now on we will drop the traceless notation “\( t \)” and assume that the irreducible tensors in Eqs. (65a) – (65d) are traceless.

For our particular situation of interest, we will focus on the properties of \( X^{\lambda\mu\nu} \) when it is antisymmetric under the exchange \( \lambda \leftrightarrow \mu \) (this is the symmetry condition obeyed by \( g^{\lambda\mu\nu} \) as introduced in Eq. (2a)). This means that we can write \( X^{\lambda\mu\nu} = X^{[\lambda\mu\nu]} = \)

20
\( \frac{1}{2}(X^{\lambda\mu} - X^{\mu\lambda}) \). This condition reduces the number of linearly-independent irreducible representations (neglecting the “traces”) from six to three, which now are

\[
X_A^{[\lambda\mu]} = \frac{1}{6} \left( X^{\lambda\mu} - X^{\lambda\nu} + X^{\mu\nu} - X^{\lambda\rho} + X^{\rho\lambda} - X^{\mu\rho} \right),
\]

\[
X_{2+}^{[\lambda\mu]} = \frac{1}{4} \left( X^{\lambda\mu} - X^{\mu\nu} - \frac{1}{2} \left( X^{\rho\lambda} - X^{\rho\mu} \right) + \frac{1}{2} \left( X^{\lambda\rho} - X^{\mu\rho} \right) \right) + \frac{1}{2} \eta^{\lambda\nu} \left( X_1^{\mu} - X_3^{\mu} \right),
\]

\[
X_{1-}^{[\lambda\mu]} = \frac{1}{4} \left( X^{\lambda\mu} - X^{\mu\nu} - \frac{1}{2} \left( X^{\rho\lambda} - X^{\rho\mu} \right) + \frac{1}{2} \left( X^{\lambda\rho} - X^{\mu\rho} \right) \right) + \frac{1}{2} \eta^{\lambda\nu} \left( X_1^{\mu} - X_2^{\mu} \right).
\]

Notice that \( \eta_{\lambda\mu} X^{[\lambda\mu]} = 0, \eta_{\lambda\nu} X^{[\lambda\mu]} = \frac{1}{2} (X_2^\mu - X_1^\mu) \), and \( \eta_{\mu\nu} X^{[\lambda\mu]} = \frac{1}{2} (X_1^\lambda - X_3^\lambda) \). We will now use the definitions \( X_1^{[\alpha]} \equiv \frac{1}{2} (X_1^\alpha - X_3^\alpha) \), \( X_2^{[\alpha]} \equiv \frac{1}{2} (X_2^\alpha - X_1^\alpha) \), and \( X_3^{[\alpha]} \equiv \frac{1}{2} (X_3^\alpha - X_2^\alpha) \). By construction we have the property

\[
X_1^{[\alpha]} + X_2^{[\alpha]} + X_3^{[\alpha]} = 0.
\]

In order to simplify the notation we will assume from now on that \( X^{\lambda\mu} \) and its permutations are antisymmetric under the exchange \( \lambda \leftrightarrow \mu \) and so \( X^{[\lambda\mu]} \rightarrow X^{\lambda\mu} \). Relabeling \( X_i^{[\alpha]} \rightarrow X_i^{\alpha} \) we find

\[
X^{\lambda\mu} = X_A^{\lambda\mu} + \frac{4}{3} X_1^{\lambda\mu} + \frac{1}{3} \eta^{\lambda\nu} X_2^{\mu} - \frac{1}{3} \eta^{\mu\nu} X_2^{\lambda}, \quad \text{with} \quad \eta_{\mu\nu} X^{\lambda\mu} = -X_2^{\lambda},
\]

\[
X^{\lambda\nu} = -X_A^{\lambda\nu} + \frac{4}{3} X_2^{\lambda\nu} - \frac{1}{3} \eta^{\lambda\nu} X_1^{\mu} + \frac{1}{3} \eta^{\mu\nu} X_1^{\lambda}, \quad \text{with} \quad \eta_{\lambda\nu} X^{\lambda\mu} = -X_1^{\mu},
\]

\[
X^{\nu\lambda} = -X_A^{\nu\lambda} - \frac{4}{3} X_1^{\nu\lambda} + \frac{4}{3} X_2^{\nu\lambda} - \frac{1}{3} \eta^{\lambda\nu} X_3^{\mu} + \frac{1}{3} \eta^{\mu\nu} X_3^{\lambda}, \quad \text{with} \quad \eta_{\lambda\nu} X^{\lambda\mu} = -X_3^{\mu},
\]

\[
X^{\nu\mu} = -X_A^{\nu\mu}, \quad X^{\mu\lambda} = -X^{\lambda\mu}, \quad X^{\nu\lambda} = -X^{\lambda\nu},
\]

where \( X_i^{\alpha} \) satisfy Eq. (67).

## B Irreducible decomposition of four-rank tensors

A four-rank tensor \( X^{\lambda\mu\nu\sigma} \) must also be decomposed into irreducible representations under the group \( S_4 \); its associated Young diagrams are
and permutations of the indices \(\{\lambda, \mu, \nu, \sigma\}\) to complete 24 different representations. In terms of the set of Young diagrams we can write

\[
X^{\lambda\mu\nu\sigma} = x_1 X_1^{\lambda\mu\nu\sigma} + x_2 X_{1a}^{\lambda\mu\nu\sigma} + x_3 X_{1b}^{\lambda\mu\nu\sigma} + x_4 X_{1c}^{\lambda\mu\nu\sigma} + x_5 X_{2a}^{\lambda\mu\nu\sigma} + x_6 X_{2b}^{\lambda\mu\nu\sigma} + x_7 X_{3a}^{\lambda\mu\nu\sigma} + x_8 X_{3b}^{\lambda\mu\nu\sigma} + x_9 X_{3c}^{\lambda\mu\nu\sigma} + x_{10} X_4^{\lambda\mu\nu\sigma},
\]

where \(x_i\) are rational numbers. Fortunately, the \(\beta\)-functions of Lorentz violating QED that deal with four-rank tensors only include the combinations \((k_F)_{\lambda\mu\nu\sigma}\) and \((k_F)_{\lambda\alpha\beta}\). Hence, we do not need to find all irreducible representations, we just need to focus on the trace and double trace representations of \((k_F)_{\lambda\mu\nu\sigma}\). Let us define

\[
X_1^{\alpha\beta} \equiv X^{\rho\alpha\beta}_\rho, \quad X_2^{\alpha\beta} \equiv X^{\rho\alpha\beta}_\rho, \quad X_3^{\alpha\beta} \equiv X^{\rho\alpha\beta}_\rho, \\
X_4^{\alpha\beta} \equiv X^{\rho\alpha\beta}_\rho, \quad X_5^{\alpha\beta} \equiv X^{\rho\alpha\beta}_\rho, \quad X_6^{\alpha\beta} \equiv X^{\rho\alpha\beta}_\rho, \\
X_7 \equiv X^{\rho\rho \rho}, \quad X_8 \equiv X^{\rho\rho \rho}, \quad X_9 \equiv X^{\rho\rho \rho}.
\]
In terms of these definitions we can write the traceless and double traceless tensor $X_t^{\lambda\mu\nu\sigma}$ as

$$X_t^{\lambda\mu\nu\sigma} = X^{\lambda\mu\nu\sigma} + \frac{1}{96} \eta^{\lambda\mu} \left[ -31X_1 + X_1^T + 7X_2 - X_2^T - X_3 + 7X_3^T - X_4 + 7X_4^T - 3X_5 - 3X_5^T \right]^{\lambda\mu\nu\sigma} + \frac{1}{96} \eta^{\lambda\mu} \left[ 7X_1 - X_1^T - 31X_2 + X_2^T + 7X_3 - X_3^T - 3X_4 - 3X_4^T + 7X_5 - X_5^T - X_6 + 7X_6^T \right]^{\lambda\mu\nu\sigma} + \frac{1}{96} \eta^{\lambda\sigma} \left[ -X_1 + 7X_1^T + 7X_2 - X_2^T - 31X_3 + X_3^T + 7X_4 - X_4^T - 3X_5 - 3X_5^T - X_6 + 7X_6^T \right]^{\lambda\mu\nu\sigma} + \frac{1}{96} \eta^{\lambda\mu} \left[ 7X_1 - X_1^T + 7X_2 - X_2^T - 3X_3 - 3X_3^T + 7X_4 - X_4^T - 31X_5 + X_5^T + 7X_6 - X_6^T \right]^{\lambda\nu\sigma} + \frac{1}{96} \eta^{\lambda\sigma} \left[ -X_1 + 7X_1^T - 3X_2 - 3X_2^T + 7X_3 - X_3^T - 31X_4 + X_4^T + 7X_5 - X_5^T + 7X_6 - X_6^T \right]^{\lambda\sigma\nu\mu} + \frac{1}{96} \eta^{\lambda\mu} \left[ -3X_1 - 3X_1^T - X_2 + 7X_2^T - X_3 + 7X_3^T + 7X_4 - X_4^T + 7X_5 - X_5^T - 31X_6 + X_6^T \right]^{\lambda\mu\nu\sigma} + \frac{1}{144} \eta^{\lambda\mu\nu\sigma} \left[ 17X_7 - 7X_8 - 7X_9 \right] + \frac{1}{144} \eta^{\lambda\nu\sigma\mu} \left[ -7X_7 + 17X_8 - 7X_9 \right] + \frac{1}{144} \eta^{\lambda\sigma\mu\nu} \left[ -7X_7 - 7X_8 + 17X_9 \right], \quad (71)$$

where $(X^T)^{\alpha\beta} = X^{\beta\alpha}$.

We will now study the properties of four-rank tensors obeying Eqs. (41a)–(41c).

When we multiply by $\eta_{\lambda\nu}$, $\eta_{\mu\sigma}$ and $\eta_{\lambda\nu}\eta_{\mu\sigma}$ respectively in Eq. (41b) we obtain the conditions

$$(X_1)^{\mu\sigma}_S + (X_2)^{\mu\sigma}_S + (X_3)^{\mu\sigma}_S = 0, \quad (72a)$$

$$(X_4)^{\mu\nu}_S + (X_5)^{\lambda\nu}_S + (X_6)^{\lambda\nu}_S = 0, \quad (72b)$$

$$(X_7 + X_8 + X_9) = 0. \quad (72c)$$

Eq. (41c) implies $X_8 = 0$, so $X_7 = -X_9$. However, $X_7 = X^{\lambda\mu\sigma\eta\mu\eta\sigma}$ is zero because of the antisymmetry properties in Eq. (41a), so $X_9$ also vanishes. This implies that $X^{\lambda\mu\sigma\eta}$ is double traceless in all components.

Using the symmetry properties stated in Eq. (41a) we find that $X^{\lambda\mu\sigma\eta}$ can be simplified to

$$X^{\lambda\mu\sigma\eta} = X_t^{\lambda\mu\sigma\eta} + \frac{1}{8} \eta^{\lambda\mu} (X_a)^{\mu\sigma}_S - \frac{1}{8} \eta^{\mu\nu} (X_a)^{\lambda\sigma}_S - \frac{1}{8} \eta^{\lambda\nu} (X_a)^{\mu\sigma}_S - \frac{1}{8} \eta^{\mu\sigma} (X_a)^{\lambda\nu}_S, \quad (73)$$

where $X_a \equiv X_2 - X_3 + X_4 - X_5$.

Notice that the symmetry conditions obeyed by $X^{\lambda\mu\sigma\eta}$ also imply that

$$(X_2)^{\mu\sigma}_S = (X_4)^{\mu\sigma}_S = -(X_3)^{\mu\sigma}_S = -(X_5)^{\mu\sigma}_S = \frac{1}{4} (X_a)^{\mu\sigma}_S. \quad (74)$$
So we finally have

\[ X^\lambda_{\mu\nu\sigma} = X_t^\lambda_{\mu\nu\sigma} + \frac{1}{2}\eta^\lambda_{\mu}(X_2)^{\mu\sigma}_S - \frac{1}{2}\eta^\mu_{\nu}(X_2)^{\lambda\sigma}_S - \frac{1}{2}\eta^\lambda_{\sigma}(X_2)^{\mu\nu}_S + \frac{1}{2}\eta^\mu_{\sigma}(X_2)^{\lambda\nu}_S, \] (75)

which is enough to decompose \((k_F)^{\lambda\mu\nu\sigma}\).

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