Multitemporal generalization of the Tangherlini solution

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The n-time generalization of the Tangherlini solution [1] is considered. The equations of geodesics for the metric are integrated. For $n = 2$ it is shown that the naked singularity is absent only for two sets of parameters, corresponding to the trivial extensions of the Tangherlini solution. The motion of a relativistic particle in the multitemporal background is considered. This motion is governed by the gravitational mass tensor. Some generalizations of the solution, including the multitemporal analogue of the Myers-Perry charged black hole solution, are obtained.

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1 Introduction

In ref. [3] the Tangherlini solution [1,2] \((O(d + 1)\)-symmetric analogue of the Schwarzschild solution) was generalized on the case of \(\bar{n}\) internal Ricci-flat spaces. The metric of this solution is defined on the manifold

\[
M = M^{(2+d)} \times M_1 \times \ldots \times M_{\bar{n}},
\]

and has the following form

\[
g = -f^a dt \otimes dt + f^{b-1} dR \otimes dR + f^b R^2 d\Omega_2^2 + \sum_{i=1}^{\bar{n}} f^{a_i} g^{(i)},
\]

where \(M^{(2+d)}\) is \((2 + d)\)-dimensional space-time \((d \geq 2)\), \((M_i, g^{(i)})\) is Ricci-flat manifold \((g^{(i)}\) is metric on \(M_i)\), \(dim M_i = N_i, i = 1, \ldots, \bar{n}\); \(d\Omega_2^2\) is canonical metric on \(d\)-dimensional sphere \(S^d\),

\[
f = f(R) = 1 - BR^{1-d},
\]

\[
b = (1 - a - \sum_{i=1}^{\bar{n}} a_i N_i)/(d - 1),
\]

\(B = const\), and the parameters \(a, a_1, \ldots, a_{\bar{n}}\) satisfy the relation

\[
(a + \sum_{i=1}^{\bar{n}} a_i N_i)^2 + (d - 1)(a^2 + \sum_{i=1}^{\bar{n}} a_i^2 N_i) = d.
\]

(Here the notations slightly differ from those of ref. [3]). The metric (1.2) with the relations (1.3)-(1.5) imposed satisfies the Einstein equations or, equivalently,

\[
R_{MN}[g] = 0.
\]

Some special cases of the solution (1.2)-(1.5) were considered earlier in the following publications: [4-6] \((d = 2; \bar{n} = 1; N_1 = 1)\), [7,8] \((d = 2; \bar{n} = 2, 3; N_1 = \ldots = N_{\bar{n}} = 1)\), [9] \((d = 2; \bar{n} = 1)\), [10] \((\bar{n} = 1; a = (1 - (d + N_1)^{-1})/(1 - d^{-1})])^{1/2}; a_1 = -a/(d + N_1 - 1))\); [11] \((\bar{n} = 2, N_2 = 1)\), [12] \((d = 2; \bar{n} is arbitrary)\).

It was shown in [3] that in the \((2 + d)\)-dimensional section of the metric (1.2) a horizon exists only in the trivial case: \(a - 1 = a_1 = \ldots = a_{\bar{n}}\) (this proposition was also suggested in [12]). We also note that the cosmological analogue of the solution [3] was presented in [14], where the tree-generalizations of the solution were considered. (Such tree-generalizations may be also constructed for spherically-symmetric case [3]).

In this paper we consider an interesting special case of the solution [3]. This is the \(n\)-time generalization of the Tangherlini solution. (The multitemporal analogue of the Schwarzschild solution \((d = 2)\) was considered earlier in [15].)

We note, that the space-time manifolds with extra time dimensions were considered in gravitational context by many authors (see, for example, [17-26]). Some revival of the interest in this direction was inspired recently by supergravity and string models [22-27].
note that the idea of the existence of multidimensional domains with several times in the (multidimensional) Universe was suggested by Sakharov in [20].

The paper is organized as following. In Sec. 2 the metric of the multitemporal solution is considered. The explicit expression for the Riemann tensor squared, corresponding to this solution is presented. The proposition concerning the singularity of the solution at \( R = L > 0 \) (\( L \) is the parameter) for any set of dimensionless parameters \((a_i)\) except \( n \) ”tangherlinian” sets is suggested. This proposition is proved for \( n = 2 \). In Sec. 3 the equations of the geodesics, corresponding to the solution are integrated. The notion of the multitemporal horizon is introduced. It is proved that for all non-exceptional sets of \( a_i\)-parameters the multitemporal horizon is absent. In Sec. 4 the motion of the relativistic particle is considered. The multitemporal \( O(d + 1) \)- symmetric analogue of the Newton’s formula for this case is obtained. In multitemporal case the inertial and gravitational masses are defined as matrices (or it may be defined also as tensors). In Sec. 5 the vacuum solution [3] is generalized on the electro-scalar- vacuum case for the model with exponential scalar-electro-magnetic coupling. Some infinite-dimensional generalizations of the solution (including infinite-temporal and Grassmann-Banach analogues) are presented.

2 The metric

We consider the special case of the solution (1.1)-(1.5) with \( \bar{n} = n - 1 \) one-dimensional internal spaces: \( M_i = R \), \( g^{(i)} = -dt^i \otimes dt^i, i = 1, \ldots, n - 1 \). Denoting \( t = t^n \) and \( a = a_n \) we get from (1.2)

\[
g = - \sum_{i=1}^{n} (1 - BR^{1-d}a_i)dt^i \otimes dt^i + (1 - BR^{1-d})^{-1}dR \otimes dR + (1 - BR^{1-d})^{-1}dR^2d\Omega^2_d,
\]

where

\[
b = (1 - \sum_{i=1}^{n} a_i) / (d - 1)
\]

and the parameters \( a_1, \ldots, a_n \) satisfy the relations

\[
(\sum_{i=1}^{n} a_i)^2 + (d - 1) \sum_{i=1}^{n} a_i^2 = d.
\]

The metric (2.1) with the parameters satisfying (2.2) and (2.3) is the solution of the \( (n+d+1) \)-dimensional Einstein equations (or, equivalently, of the Ricci-flatness eqs. (1.6)).

The solution (2.1)-(2.3) is multitemporal \((n\text{-time})\) generalization of the Tangherlini solution [1].

Let \( E(d, n) \) be the set of points \( a = (a_1, \ldots, a_n) \in R^n \), satisfying the relation (2.3). Clearly, that \( E(d, n) \) is ellipsoid. We denote the solution (2.1), corresponding to \( a \in E(d, n) \), \( B \in R \) by \( g = g(a, B) \). An interesting fact is that the metrics \( g(a, B) \) and \( g(-a, -B) \) are equivalent (for any \( a \in E(d, n) \), \( B \in R \)), i.e.

\[
g(-a, -B) = \varphi^* g(a, B)
\]
for some diffeomorphism $\varphi$. This diffeomorphism $\varphi = \varphi_B$ is defined by the relations
\[
\varphi : (t^i, R^*, \theta^\alpha) \mapsto (t^i, R, \theta^\alpha), \quad R^{d-1} = R^{d-1}_s + B.
\]  \hfill (2.5)

**Remark 1.** An analogous equivalence takes place for the metric (1.2).

Due to relation (2.4) it is quite sufficient to restrict our consideration by the case $B > 0$ (the case $B = 0$ is trivial).

We introduce the following notations
\[
T_1 \equiv (1, 0, \ldots, 0), \ldots, T_n \equiv (0, \ldots, 0, 1),
\]  \hfill (2.6)
\[
T \equiv \{T_1, \ldots, T_n\} \subset E(d, n).
\]  \hfill (2.7)

We call the points (2.6) as Tangherlini points and the set (2.7) as Tangherlini set. The metric (2.1) for $a = T_k$ has a rather simple form
\[
g(T_k, B) = g^{(k)}_T - \sum_{i \neq k} dt^i \otimes dt^i;
\]  \hfill (2.8)

where $g^{(k)}_T$ is the Tangherlini solution with the time variable $t = t^k$, $k = 1, \ldots, n$. The metric (2.8) is a trivial (cylindrical) extension of the Tangherlini solution with the time $t^k$.

**Singularity.** The Riemann tensor squared for the metric (2.1) has the following form
\[
I[g] \equiv R_{MNPQ}R^{MNPQ} = \bar{I}[g]/8f^{2(b-1)},
\]  \hfill (2.9)
\[
\bar{I}[g] = 16d(d-1)R^{-4}f^{-2} - 8d(d-1)R^{-2}f^{-1}(b\frac{f'}{f} + \frac{2}{R})^2
\]
\[
- d(b\frac{f''}{f} + \frac{2}{R})^2 + 2d[2b\frac{f''}{f} - b(\frac{f'}{f})^2 + \frac{2(b+1)}{R}f']^2 - \sum_{i=1}^n \left\{ -a_i^4(\frac{f'}{f})^4 + 2a_i^2[2\frac{f''}{f} + (a_i - 1 - b)(\frac{f'}{f})^2] - \frac{2}{R} \right\} - \sum_{i=1}^n a_i^2(\frac{f'}{f})^2,
\]  \hfill (2.10)

where $f' \equiv df/dR$ and $f$ is defined by (1.3). We denote $L = L_B \equiv B^{1/(d-1)}$ for $B > 0$. The relation (2.10) may be obtained from the formula presented in the Appendix.

**Proposition 1.** Let $B > 0$ and $a = (a_i) \in E(d, n) \setminus T$, i.e. the set of parameters $a$ is non-tangherlinian. Then the quadratic invariant (2.9) for the metric (2.1) $g = g(a, B)$ is divergent: $I[g] \to \infty$, as $R \to L$.

**Proof.** Here we prove the proposition for the case $n = 2$. (The case $n > 2$ will be considered in a separate publication [28]).

From eqs. (2.9)-(2.10) we get the following asymptotical formula (here $n$ is arbitrary)
\[
I[g] = A\left[ f'(L) \right]^4[f(R)]^{-2b-2}[1 + O(L - R)],
\]  \hfill (2.11)
as \( R \to L \), where

\[
A = A(a) = -db^4 + 2db^2 + (db^2 + \sum_{i=1}^{n} a_i^2)^2 + \sum_{i=1}^{n} [-a_i^4 + 2a_i^2(a_i - 1 - b)^2].
\] (2.12)

The formula (2.11) is valid, when \( A \neq 0 \). We note that

\[
(1 - r)/(d - 1) \leq b \leq (1 + r)/(d - 1),
\] (2.13)

where \( r \equiv \sqrt{dn/(d+n-1)} \) (see also Remark 2 below). It follows from (2.13) that

\[
1 + b > 0
\] (2.14)

and consequently (see (2.11)) \( I[g] \to \infty \) as \( R \to L \), when

\[
A \neq 0
\] (2.15)

Now, we prove the inequality (2.15) for \( n = 2 \) and \( a \in E(d, n) \setminus T \). For \( n = 2 \) we have

\[
A = \frac{1}{2}d(d+1)b^2\bar{A},
\] (2.16)

where

\[
\bar{A} = -(d-1)(d+2)b^2 + 8b + 8.
\] (2.17)

Using inequalities (2.13) it is not difficult to verify that \( \bar{A} = \bar{A}(b) > 0 \) (see also Remark 3 below). On the other hand (in the case \( n = 2 \)) \( b = b(a) = 0 \) only for the tangherlinian points \( a = (0, 1), (1, 0) \). So, the inequality (2.15) takes place for all \( a \in E(d, 2) \setminus T \) (\( n = 2 \)). The proposition 1 is proved for \( n = 2 \).

Remark 2. In the coordinates

\[
\bar{a}_1 = (a_1 + \ldots + a_n)/\sqrt{n},
\]
\[
\bar{a}_2 = (a_1 - a_2)/\sqrt{2},
\]
\[
\bar{a}_3 = (a_1 + a_2 - 2a_3)/\sqrt{6}, \quad (n > 2) \ldots
\]
\[
\bar{a}_n = (a_1 + \ldots + a_{n-1} - (n-1)a_n)/\sqrt{n(n-1)},
\]

the ellipsoid equation (2.3) reads

\[
(n + d - 1)\bar{a}_1^2 + (d - 1)\sum_{i=2}^{n} \bar{a}_i^2 = d. \] (2.18)

The inequalities (2.13) can be easily obtained from (2.18) and the relation

\[
b = (1 - \sqrt{n\bar{a}_1})/(d - 1).
\] (2.19)
Remark 3. The inequality $\bar{A} > 0$ may be proved, using (2.13) and the following inequalities

\[
1 + \sqrt{2d/(d+1)} < \frac{5}{2} < b_+(d-1),
\]
\[
1 - \sqrt{2d/(d+1)} > -\frac{1}{2} > b_-(d-1),
\]

where $b_\pm = [4 \pm \sqrt{8d(d+1)/(d-1)(d+2)}]$ are zeros of the quadratic polynomial $\bar{A}(b)$.

Thus for $n = 2$, $a \in E(d, 2) \setminus T$, $B > 0$ the metric (2.1) $g = g(a, B)$ is singular at $R = L$.

In the case $a \in T$, $B > 0$ ($n$ is arbitrary) the metric (2.1) $g = g(a, B)$ is regular for $R > 0$ and

\[
I[g] = B^2 R^{-2} - d^2(d^2 - 1).
\]

Remark 4. In this case the metric has form (2.8). We remind that the regularity of the Tangherlini metric for $R > 0$ may be easily seen using the coordinates

\[
\bar{t} = t + \int dx \varphi(x)(f(x))^{-1}, \quad \bar{R} = R + \int dx (\varphi(x))^{-1}(f(x))^{-1},
\]

where $\varphi(x) = (L/x)^{(d-1)/2}$.

3 The geodesic equations

We consider the geodesic equations for the metric (2.1)

\[
\ddot{x}^M + \Gamma^M_{NP}[g] \dot{x}^N \dot{x}^P = 0.
\]

Here and below $x^M = x^M(\tau)$ and $\dot{x}^M = dx^M/d\tau$.

These equations are equivalent to the Lagrange equations for the Lagrangian

\[
L = \frac{1}{2} g_{MN}(x) \dot{x}^M \dot{x}^N
\]

\[
= \frac{1}{2} [ f^{b-1}(\dot{R})^2 + f^b R^2 \kappa_{ij}(\theta) \dot{\theta}^i \dot{\theta}^j - \sum_{i=1}^n f^{a_i}(\dot{\epsilon}^i)^2 ].
\]

where the function $f = f(R)$ is defined in eq. (1.3) and

\[
\kappa = d\theta^1 \otimes d\theta^1 + \sin^2 \theta^1 d\theta^2 \otimes d\theta^2 + \ldots + \sin^2 \theta^1 \ldots \sin^2 \theta^{d-1} d\theta^d \otimes d\theta^d
\]

is standard metric on $S^d$. Here $0 < \theta^1, \ldots, \theta^{d-1} < \pi$, $0 < \theta^d = \varphi < 2\pi$.

The complete set of integrals of motion for the Lagrange system (3.2) is following

\[
f^{a_i} \dot{\epsilon}^i = \epsilon^i,
\]

\[
f^b R^2 \dot{\varphi} = \jmath,
\]

\[
f^{b-1}(\dot{R})^2 + j^2 f^{-b} R^{-2} - \sum_{i=1}^n (\epsilon^i)^2 f^{-a_i} = 2E_L
\]
\( i = 1, \ldots, n. \) We put here \( \theta^1 = \ldots = \theta^{d-1} = \frac{\pi}{2} \). (This may be done for any trajectory by a suitable choice of coordinate system.) The radial equation

\[
(f^{b-1} \dot{R}) + \frac{j^2}{2} (f^{-b} R^{-2})' - \frac{1}{2} (\dot{R})^2 (f^{b-1})' - \frac{1}{2} \sum_{i=1}^{n} (\epsilon_i)^2 (f^{-a_i})' = 0 \tag{3.7}
\]

(here \((.)' = d(.)/dR\)) is generated by the Lagrangian

\[
L_R = \frac{1}{2} [f^{b-1} (\dot{R})^2 - j^2 f^{-b} R^{-2} + \sum_{i=1}^{n} (\epsilon_i)^2 f^{-a_i}] \tag{3.8}
\]

We note, that the case \( 2E_L = 2L > 0 \) in (3.6) correspond to a tachion.

**Multitemporal horizon.** Here we consider the null geodesics. Putting \( E_L = 0 \) in (3.6) we get for a light "moving" to the center

\[
\dot{R} = - \sqrt{\frac{\sum_{i=1}^{n} (\epsilon_i)^2 f^{1-b-a_i} - j^2 f^{1-2b} R^{-2}}{}} \tag{3.9}
\]

and consequently

\[
t^i - t_i^0 = - \int_{R_0}^{R} \frac{\epsilon_i [f(x)]^{-a_i}}{\sqrt{\sum_{i=1}^{n} (\epsilon_i)^2 [f(x)]^{1-b-a_i} - j^2 [f(x)]^{1-2b} x^{-2}}} \, dx \tag{3.10}
\]

\( i = 1, \ldots, n. \)

**Definition.** Let \( B > 0, \epsilon = (\epsilon^i) \neq 0 \) and \( a \in E = E(d, n). \) We say that the \( \epsilon \)-horizon takes place for the metric \( g(a, B) \) at \( R = L \equiv B^{1/(d-1)} \) if and only if

\[
||t - t_0|| \equiv \sum_{i=1}^{n} |t^i - t_i^0| \to +\infty, \tag{3.11}
\]

as \( R \to L \) for all \( t_0 \) and \( j \).

**Proposition 2.** Let \( B > 0, \) and \( a \in E = E(d, n) \setminus T. \) Then the \( \epsilon \)-horizon for the metric \( g(a, B) \) at \( R = L \) is absent for any \( \epsilon \neq 0. \)

**Proof.** We put \( j = 0. \) It is sufficient to prove that all integrals in (3.10) are convergent, when \( R \to L \). The integrals in (3.10) are convergent only if

\[
s_i = -a_i - \frac{1}{2} \min_{\epsilon} (1 - b - a_i) > -1 \tag{3.12}
\]

for all \( i \in K_\epsilon \equiv \{j|\epsilon^j \neq 0\}. \) Here

\[
\min_{\epsilon} (u_i) \equiv \min\{u_i|i \in K_\epsilon\}. \tag{3.13}
\]

Indeed, the integrand in the \( i \)-th integral in (3.10) behaves like \( \epsilon^i (L - x)^{s_i} \) as \( x \to L. \) The set of inequilities (3.12) may be rewritten as following

\[
2a_i < \max_{\epsilon} (a_i) + 1 + b, \tag{3.14}
\]

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for all $i \in K_\varepsilon$, where $\max_\varepsilon$ is defined analogously to $\min_\varepsilon$. It can be easily verified that the set of inequalities (3.14) is equivalent to the following inequality

$$\max_\varepsilon (a_i) < 1 + b.$$  

(3.15)

This inequality follows from

$$\max (a_i) < 1 + b.$$  

(3.16)

Now we prove (3.16) for all $a \in E \setminus T$. Let us consider the tangent hypersurface to the ellipsoid $E$ in the point $T_1 = (1, 0, \ldots, 0)$. The equation for this hypersurface has following form

$$d(a_1 - 1) + a_2 + \ldots + a_n = 0.$$  

(3.17)

It is clear, that for all $a \in E \setminus T_1$

$$d(a_1 - 1) + a_2 + \ldots + a_n < 0,$$  

(3.18)

or, equivalently,

$$a_1 < 1 + b.$$  

(3.19)

In analogous manner it may be proved that

$$a_i < 1 + b.$$  

(3.20)

for all $a \in E \setminus T_i, i = 1, \ldots, n$. The inequalities (3.20) imply (3.16). The proposition is proved.

Now we consider the case $a \in T$. Without loss of generality we put $a = T_1 = (1, 0, \ldots, 0)$. It is not difficult to verify that in this case the $\varepsilon$-horizon takes place only if $\varepsilon^1 \neq 0$.

4 Relativistic particle

Here we consider the motion of the relativistic particle in the gravitational field, corresponding to the metric (2.1). The Lagrangian of the particle is well-known

$$L_1 = -m\sqrt{-g_{MN}(x)\dot{x}^M\dot{x}^N},$$  

(4.1)

where $m$ is the mass of the particle ($\dot{x}^M = dx^M/d\tau$).

The Lagrange equations for (4.1) in the proper time gauge

$$g_{MN}(x)\dot{x}^M\dot{x}^N = -1$$  

(4.2)

coincide with the geodesic equations (3.1). In this case $(E^i) = (m\varepsilon^i)$ is the energy vector and $J = mj$ is the angular momentum (see (3.4) and (3.5)). For fixed values of $\varepsilon^i$ the (d+1)-dimensional part of the equations of motion is generated by the Lagrangian

$$L_* = \frac{m}{2} [f^b \tilde{g}_{T,\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta + \sum_{i=1}^{n} (\varepsilon^i)^2 f^{-a_i}],$$  

(4.3)

where $\tilde{g}_T$ is the space section of the Tangherlini metric.
Now, we restrict our consideration by the non-relativistic motion at large distances: $R \gg L_B$. In this approximation: $t^i = \varepsilon^i \tau$, $\sum_{i=1}^{n} (\varepsilon^i)^2 = 1$. It follows from (4.3) that in this approximation we get a non-relativistic particle of mass $m$, moving in the potential

$$V = -\frac{m}{2} \sum_{i=1}^{n} (\varepsilon^i)^2 \frac{a_i B}{R^{d-1}} = -G \frac{m(\varepsilon^i M_{ij} \varepsilon^j)}{R^{d-1}}, \quad (4.4)$$

where $G$ is the gravitational constant and

$$M_{ij} = a_i \delta_{ij} B/2G, \quad (4.5)$$

are the components of the gravitational mass matrix.

It is interesting to note that the relation (4.4) may be rewritten as following

$$V = -G \frac{\text{tr}(MM_T)}{R^{d-1}}, \quad (4.6)$$

where $M_T = (m \varepsilon^i \varepsilon^j)$ is the inertial mass matrix of the particle.

The solution (2.1) may be also rewritten in the matrix form

$$g = -[(1 - BR^{1-d} A)]_{ij} d\bar{t}^i \otimes d\bar{t}^j + (1 - BR^{1-d} b^{-1}) dR \otimes dR + (1 - BR^{1-d} b^2) d\Omega_d^2, \quad (4.7)$$

where $A$ is a real symmetric $n \times n$-matrix satisfying the relation

$$(\text{tr}A)^2 + (d - 1)\text{tr}(A^2) = d. \quad (4.8)$$

and

$$b = (1 - \text{tr}A)/(d - 1). \quad (4.9)$$

Here $x^A \equiv \exp(A \ln x)$ for $x > 0$. The metric (4.7) can be reduced to the metric (2.1) by the diagonalization of the $A$-matrix: $A = S^T (a_i \delta_{ij}) S$, $S^T S = 1_n$ and the reparametrization of the time variables: $\bar{t} = S_{ij} t^j$. In this case the gravitational mass matrix is

$$(M_{ij}) = (A_{ij} B/2G). \quad (4.10)$$

We may also define the gravitational mass tensor as

$$\mathcal{M} = M_{ij} d\bar{t}^i \otimes d\bar{t}^j. \quad (4.11)$$

We call the extended object, corresponding to the solution (4.7)-(4.9) as multitemporal hedgehog. At large distances $R^{d-1} \gg B$ this object is described by the matrix analogue of the Newton’s potential

$$\Phi_{ij} = -\frac{1}{2} BR^{1-d} A_{ij} = -GR^{1-d} M_{ij}. \quad (4.12)$$

Clearly that this potential for the diagonal case (2.1) $A = a_i \delta_{ij}$ is a superposition of the potentials, corresponding to "pure" black hole states (2.8).
Remark 5. It is interesting to note that the formula
\[ A = Q_i R^{1-d} dt^i \] (4.13)
describe the multitemporal \( O(d+1) \)-analogue of the well-known electrostatic solution of the Maxwell equations. In this case the charge \( Q = (Q_i) \) is a vector (or we may also define the charge as the 1-form \( Q_i dt^i \)).

Remark 6. Let us consider the solution (2.1) for \( n = 2 \) with \( a_1 > 0 \) and \( a_2 < 0 \). In this case under a suitable choice of the \( \varepsilon^i \)-parameters a point \( R > L \), may be a libration point, i.e. the point of equilibrium. In this case
\[ a_1(\varepsilon^1)^2 + a_2(\varepsilon^2)^2[f(R)]^{a_2-a_1} = 0 \] (4.14)
and \( \varepsilon^2 \neq 0 \). An analogous situation takes place for arbitrary \( n \), when there exist positive and negative \( a_i \)-th parameters.

5 Some generalizations

Here we present some generalizations of the considered above solutions. First, we consider the model described by the following action
\[ S = \int d^Dx \sqrt{|g|} \{ \frac{1}{2\kappa^2} R[g] - \frac{1}{2\kappa^2} \partial_M \varphi \partial_N \varphi g^{MN} - \frac{1}{4} \exp(2\lambda \varphi) F_{MN} F^{MN} \}, \] (5.1)
where \( g = g_{MN} dx^M \otimes dx^N \) is the metric , \( F = \frac{1}{2} F_{MN} dx^M \wedge dx^N = dA \) is the strength of the electromagnetic field and \( \varphi \) is the scalar field. Here \( \lambda \) is constant. The action (5.1) describes for certain values of parameters \( \lambda \) and \( D \) a lot of interesting physical models including standard Kaluza-Klein theory, dimensionally reduced Einstein-Maxwell theory, supergravity theories (see, for example [34]). We present the spherically-\( O(d+1) \)-symmetric solutions of the field equations corresponding to the action (5.1) with the topology (1.1) [28]. The solution is following
\[ g = -f_1^{(D-3)/A(\lambda)} f_\varphi^2 dt \otimes dt \]
\[ + f_1^{-1/A(\lambda)} (f_2^{-1} f_\varphi^2 f^2)^{1/(1-d)} [f_2 du \otimes du + d\Omega_d^2] \]
\[ + \sum_{i=1}^n f_1^{-1/A(\lambda)} \exp(2A_i u + 2D_i) g^{(i)}, \] (5.2)
\[ F = Q f_1 du \wedge dt, \] (5.3)
\[ \exp \varphi = f_1^{(2-D)\lambda/2A(\lambda)} f_\varphi. \] (5.4)

In (5.2)-(5.4)
\[ f_1 = f_1(u) = C_1 (D-2)/\kappa^2 Q^2 A(\lambda) \sinh^2(\sqrt{C_1(u-u_1)}), \] (5.5)
\[ f_2 = f_2(u) = C_2/(d-1)^2 \sinh^2(\sqrt{C_2(u-u_2)}), \] (5.6)
\[ f_\varphi = f_\varphi(u) = \exp(Bu + D_\varphi), \quad (5.7) \]
\[ f = f(u) = \exp[\sum_{i=1}^{\bar{n}} N_i(A_i u + D_i)], \quad (5.8) \]
\[ A(\lambda) = D - 3 + \lambda^2(D - 2), \quad (5.9) \]

and \( Q \neq 0, D_i, D_\varphi, u_1, u_2 \) are constants and the parameters \( C_1, C_2, B, A_i \) satisfy the relation

\[
\frac{C_2d}{d-1} = \frac{C_1(D-2)}{D-3+\lambda^2(D-2)} + B^2(1+\lambda^2) + \frac{1}{d-1}(\lambda B + \sum_{i=1}^{\bar{n}} A_i N_i)^2 + \sum_{i=1}^{\bar{n}} A_i^2 N_i. \quad (5.10)
\]

The solution (5.2)-(5.10) generalizes the well-known Myers-Perry charged black hole solution [2] for the model (5.1) on the case of \( \bar{n} \) internal Ricci-flat spaces. (We remind that \((M_i, g^{(i)})\) is Ricci-flat space of dimension \(N_i, i = 1, \ldots, \bar{n}.\)) For \( \varphi = 0 \) the presented here solution was obtained in [29] (the case \( d = 2 \) was considered previously in [30-32]). Some special cases of this solution were considered also in [33,34]. For \( \bar{n} = n - 1 \), \( M_i = R, g^{(i)} = -dt^i \otimes dt^i, i = 1, \ldots, n - 1, t = t^n \) we get from (5.2) the multitemporal generalization of the solution [2] for the action (5.1).

In the zero charge case \( F = 0 \) we have

\[
g = -\exp(2A_{-1} u + 2D_{-1})dt \otimes dt + \exp[2(1-d)^{-1}\sum_{\nu} N_\nu(A_\nu u + D_\nu)]f_2^{1/(d-1)}[f_2 du \otimes du + d\Omega_d^2]
+ \sum_{i=1}^{\bar{n}} \exp(2A_i u + 2D_i)g^{(i)}, \quad (5.11)\]
\[ \varphi = Bu + D_\varphi. \quad (5.12) \]

The integration constants satisfy the relation

\[
\frac{C_2d}{d-1} = \frac{1}{d-1}(\sum_{\nu} A_{\nu} N_\nu)^2 + \sum_{\nu} A_{\nu}^2 N_\nu + B^2. \quad (5.13)\]

Hear \( \nu = -1, 1, \ldots, n; N_{-1} = 1. \)

The solution (2.1)-(2.3) may be also generalized on the infinite-time case: \( n = \infty. \) In this case the following restriction on the parameters \( a_i \) should be imposed (see also [14])

\[
\sum_{i=1}^{\infty} |a_i| < +\infty. \quad (5.14)\]

This relation implies

\[
\sum_{i=1}^{\infty} |a_i|^2 < +\infty. \quad (5.15)\]
In this case the metric (2.1) is correctly defined on a proper infinite-dimensional (Banach) manifold and satisfies the Einstein equations. We note that an infinite-dimensional version of the Einstein gravity was considered earlier by Kalitzin [16].

Remark 7. Another infinite-dimensional extension of the considered here solution may be obtained if the field of real numbers \( \mathbb{R} \) is replaced by the even part \( G_0 \) of the infinite-dimensional Grassmann-Banach algebra \( G = G_0 + G_1 \) [35,36]. In this case all coordinates and the parameters of the solution (2.1) are elements of \( G_0 \). (The \( d \)-dimensional sphere with the metric on it should be replaced by its trivial \( G_0 \)-extensions.)

6 Conclusion

In this paper the multitemporal analogue of the Tangherlini solution was considered. It was shown that in the case of two time directions the solution describes a naked singularity for any non-trivial (non-tangherlinian) set of parameters. We have integrated the geodesic equations for the considered solution. It was obtained the multitemporal analogue for the Newton’s formula (eq. (4.6)), describing the interaction between the massive particle and the multitemporal extended object (”multitemporal hedgehog”), corresponding to the solution. It was shown that in the multitemporal case the inertial and gravitational masses are matrices. (It may be defined also as tensors). We have also obtained the generalization of the Myers-Perry charged black hole solution on the case of a chain of Ricci-flat internal spaces (this solution contains the multitemporal analogue as a special case).

Appendix

Here we present the expression for the tensor Riemann squared (2.9) corresponding to the cosmological metric

\[
g = -B(t)dt \otimes dt + \sum_{i=1}^{n} A_i(t)g^{(i)},
\]
defined on the manifold \( M = R \times M_1 \times \ldots \times M_n \), where \( g^{(i)} \) is a metric on the manifold \( M_i, \dim M_i = N_i, i = 1, \ldots, n \). By a straightforward calculation the following relation was obtained

\[
I[g] = \sum_{i=1}^{n} \{ A_i^{-2}I[g^{(i)}] + A_i^{-3}B^{-1}\ddot{A}_i^2R[g^{(i)}] - \frac{1}{8}N_iB^{-2}A_i^{-4}\dddot{A}_i^4 + \frac{1}{4}N_iB^{-2}(2A_i^{-1}\dddot{A}_i - B^{-1}\dot{B}A_i^{-1}\dddot{A}_i - A_i^{-2}\ddot{A}_i^2)^2\} + \frac{1}{8}B^{-2}\sum_{i=1}^{n} N_i(A_i^{-1}\dddot{A}_i)^2.\]

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