Simulating Galaxy Clusters

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1. – Introduction

Galaxy clusters are the largest gravitationally bound objects in the universe. They are also among the rarest objects in the universe. While these two facts about galaxy clusters may seem disparate, they are in fact intimately related. Our current theory of the origin and evolution of galaxy clusters places them within the broader context of cosmological structure formation in which galaxies, galaxy groups, galaxy clusters, and galaxy superclusters all arise from gravitational instability amplifying perturbations in the cold dark matter density field in an expanding universe. At early times the perturbations are linear in amplitude, and are extremely well described by a Gaussian random field with a known power spectrum (the $\Lambda CDM$ power spectrum; cf. Fig. 2). At later times, density perturbations become nonlinear and collapse into gravitationally bound systems. The shape of the $\Lambda CDM$ power spectrum is such that structure forms from the “bottom up”, with galaxies forming first and clusters forming later. It just so happens that we live in a universe in which cluster-scale perturbations collapsed rather recently (since $z \sim 1$), which accounts for their rarity as well as their sometimes complex substructure.

As cluster-scale perturbations collapse, they bring in all matter within a sphere of comoving radius of about 15 Mpc, which includes galaxies, intergalactic gas, and anything else in that patch of the universe. Because the escape velocity of galaxy clusters is of order 1000 km/s, everything but relativistic particles become trapped in the cluster potential well. For this reason it is often said that clusters represent a fair sample of the universe. This is true from the standpoint of their matter content. However, from the standpoint of cosmological structure this could not be further from the truth. Galaxy clusters form and evolve in the rarest peaks ($\sim 3\sigma$) of the density field (Fig. 1). Galaxy formation begins sooner in such regions, and the galaxies evolve due to internal and external processes which are somewhat different from the general field (e.g., ram pressure stripping).

Galaxy clusters are interesting objects in their own right, and for decades have been extensively studied in the optical, X-ray, and radio wavebands [65]. More recently this has been extended to the microwave, infrared, and extreme UV [66, 67], motivated in part by the fact that galaxy clusters are excellent cosmological probes. Because of their large size and high X-ray luminosities, they can be seen to great distances. As discussed in this volume and in [67], galaxy clusters can also be seen in absorption/emission against the cosmic microwave background (CMB) via the Sunyaev-Zeldovich effect (SZE). As discussed by Rephaeli elsewhere in these proceedings, the SZE is redshift independent, meaning that deep microwave surveys should detect all galaxy clusters in a particular
Fig. 1. – AMR hydrodynamic cosmological simulation of cosmic structure in a 700 Mpc volume of the universe. Up to seven levels of adaptive mesh refinement (AMR) resolve the distribution of baryons within and between galaxy clusters, for an effective resolution of $65,536^3$. Volume rendering of baryon density. From [70].

One of the most challenging measurements in modern observational cosmology is the dark energy equation of state which describes the time rate of change of the vacuum energy density $\dot{\Lambda}$ responsible for the accelerating expansion of the universe [5]. Galaxy clusters were identified by the US Dark Energy Task Force (DETF) as one of four complementary methods for doing this. However in order to measure $\dot{\Lambda}$ we must measure the cluster mass distribution function versus redshift to very high accuracy. Accurately measuring the mass of a galaxy cluster is actually quite difficult despite the number of ingenious techniques that have been developed. This motivates attempts to measure the “mass-observable” relationships in observed samples of clusters, and find the ones with the least scatter and least bias.

Numerical simulations are helpful in this regard, as one can in principle calibrate the mass-observable relationships by comparing simulated observations with in situ measurements. Where this has been done, a discouraging result is found: most of the observables of a given simulated cluster depend sensitively on numerical resolution and assumed baryonic physics, including radiative cooling, star formation, galactic and AGN feedback processes. However it has been shown that the integrated SZE signal is rather insensitive to assumed baryonic physics [48, 71], perhaps relaxing requirements on modelers and giving encouragement to SZE cluster observers that the dark energy program may
be feasible after all. The point is that the astrophysics of galaxy clusters and their utility as cosmological probes are inextricably linked, and both are worthwhile of study.

I was asked by the organizers to lecture on three topics relevant to the theme of the summer school. First I was asked to review the standard cosmological framework and basic results from the theory of cosmological structure formation within which galaxy clusters can be understood. Second, I was asked to review how galaxy clusters are simulated on a computer, and summarize the basic findings. I do this in the next two sections, which are slightly updated and abbreviated versions of the lecture notes I published in the 2004 Varenna Summer School volume [67]. Finally I was asked to give a lecture of my choosing, which was on recent progress in galaxy cluster modeling focusing on the incorporation of additional baryonic physics and simulating SZE surveys. These topics are presented in Sections 5 and 6 respectively.

In line with the character of the summer school, I have attempted to be pedagogical, emphasizing the key concepts and results that a student needs to know if s/he wants to understand the current literature or do research in this area. Literature citations are kept to a minimum, except for textbooks, reviews, and research papers that I found to be particularly helpful in preparing this article. I am indebted to Dr. Rocky Kolb whose slides much of Section 2 are based upon.

2. Cosmological framework and perturbation growth in the linear regime

Our modern theory of the structure and evolution of the universe, along with the observational data which support it, is admirably presented in the textbook by Dodelson [4]. Remarkable observational progress has been made in the past two decades which has strengthened our confidence in the correctness of the hot, relativistic, expanding universe model (Big Bang), has measured the universe’s present mass-energy contents and kinematics, and lent strong support to the notion of a very early, inflationary phase. Moreover, observations of high redshift supernovae unexpectedly have revealed that the cosmic expansion is accelerating at the present time, implying the existence of a pervasive, dark energy field with negative pressure [5]. This surprising discovery has enlivened observational efforts to accurately measure the cosmological parameters over as large a fraction of the age of the universe as possible, especially over the redshift interval \( 0 < z < 1.5 \) which, according to current estimates, spans the deceleration-acceleration transition. These efforts include large surveys of galaxy large scale structure, galaxy clusters, weak lensing, the Lyman alpha forest, and high redshift supernovae, all of which span the relevant redshift range. Except for the supernovae, all other techniques rely on measurements of cosmological structure in order to deduce cosmological parameters.

2.1. Cosmological standard model. – The dynamics of the expanding universe is described by the two Friedmann equations derived from Einstein’s theory of general relativity under the assumption of homogeneity and isotropy. The expansion rate at time \( t \) is given by

\[
H^2(t) \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{k}{a^2} + \frac{\Lambda}{3}
\]

where \( H(t) \) is the Hubble parameter and \( a(t) \) is the FRW scale factor at time \( t \). The first term on the RHS is proportional to the sum over all energy densities in the universe \( \rho_i \), including baryons, photons, neutrinos, dark matter and dark energy. We have explicitly
pulled the dark energy term out of the sum and placed it in the third term assuming it is a constant (the cosmological constant). The second term is the curvature term, where $k = 0, \pm 1$ for zero, positive, negative curvature, respectively. Equation (1) can be cast in a form useful for numerical integration if we introduce $\Omega$ parameters:

$$\Omega_i = \frac{8\pi G}{3H^2\rho_i}, \quad \Omega_\Lambda = \frac{8\pi G}{3H^2\rho_\Lambda} = \frac{\Lambda}{3H^2}, \quad \Omega_k = \frac{-k}{(aH)^2}$$

Dividing equation (1) by $H^2$ we get the sum rule $1 = \Omega_m + \Omega_k + \Omega_\Lambda$, which is true at all times, where $\Omega_m$ is the sum over all $\Omega_i$ excluding dark energy. At the present time $H(t) = H_0, a = 1$, and cosmological density parameters become

$$\Omega_i(0) = \frac{8\pi G}{3H_0^2\rho_i(0)}, \quad \Omega_\Lambda(0) = \frac{\Lambda}{3H_0^2}, \quad \Omega_k(0) = \frac{-k}{H_0^2}$$

Equation (1) can then be manipulated into the form

$$\frac{\dot{a}}{a} = H_0[\Omega_m(0)(a^{-1} - 1) + \Omega_\gamma(0)(a^{-2} - 1) + \Omega_\Lambda(0)(a^2 - 1) + 1]^{1/2}$$

Here we have explicitly introduced a density parameter for the background radiation field $\Omega_\gamma$ and used the fact that matter and radiation densities scale as $a^{-3}$ and $a^{-4}$, respectively, and we have used the sum rule to eliminate $\Omega_k$. Equation (4) is equation (1) expressed in terms of the current values of the density and Hubble parameters, and makes explicit the scale factor dependence of the various contributions to the expansion rate. In particular, it is clear that the expansion rate is dominated first by radiation, then by matter, and finally by the cosmological constant.

Current measurements of the cosmological parameters by different techniques [73] yield the following numbers [(0) notation suppressed]:

$$h = \frac{H_0}{(100\text{km/s/Mpc})} \approx 0.72$$
$$\Omega_{\text{total}} \approx 1, \quad \Omega_\Lambda \approx 0.73, \quad \Omega_m = \Omega_{\text{cdm}} + \Omega_b \approx 0.27, \quad \Omega_k \approx 0$$
$$\Omega_b \approx 0.04, \quad \Omega_\nu \approx 0.005, \quad \Omega_\gamma \approx 0.00005$$

This set of parameters is referred to as the concordance model [7], and describes a spatially flat, low matter density, high dark energy density universe in which baryons, neutrinos, and photons make a negligible contribution to the large scale dynamics. Most of the matter in the universe is cold dark matter (CDM) whose dynamics is discussed below. As we will also see below, baryons and photons make an important contribution to shaping the matter power spectrum despite their small contribution to the present-day energy budget. Understanding the evolution of baryons in nonlinear structure formation is essential to interpret X-ray and SZE observations of galaxy clusters.

The second Friedmann equation relates the second time derivative of the scale factor to the cosmic pressure $p$ and energy density $\rho$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \quad \rho = \sum_i \rho_i = \rho_m + \rho_\gamma + \rho_\Lambda$$
\( p \) and \( \rho \) are related by an equation of state \( p_i = w_i \rho_i \), with \( w_m = 0 \), \( w_\gamma = 1/3 \), and \( w_\Lambda = -1 \). We thus have

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho_m + 2\rho_\gamma - 2\rho_\Lambda).
\]

Expressed in terms of the current values for the cosmological parameters we have

\[
\frac{\ddot{a}}{a} = -\frac{1}{2} H_0^2 [\Omega_m(0) a^{-3} + 2\Omega_\gamma(0) a^{-4} - 2\Omega_\Lambda(0)].
\]

Evaluating equation 7 using the concordance parameters, we see the universe is currently accelerating \( \ddot{a} \approx 0.6 H_0^2 \). Assuming the dark energy density is a constant, the acceleration began when

\[
a = \frac{1}{1 + z} = \left( \frac{\Omega_m(0)}{2\Omega_\Lambda(0)} \right)^{1/3} \approx 0.57
\]

or \( z \sim 0.75 \).

2.2. The Linear power spectrum. – Cosmic structure results from the amplification of primordial density fluctuations by gravitational instability. The power spectrum of matter density fluctuations has now been measured with considerable accuracy across roughly four decades in scale. Figure 2 shows the latest results, taken from reference [8]. Combined in this figure are measurements using cosmic microwave background (CMB) anisotropies, galaxy large scale structure, weak lensing of galaxy shapes, and the Lyman alpha forest, in order of decreasing comoving wavelength. In addition, there is a single data point for galaxy clusters, whose current space density measures the amplitude of the power spectrum on \( 8 \, h^{-1} \) Mpc scales [9]. Superimposed on the data is the predicted \( \Lambda \)CDM linear power spectrum at \( z=0 \) for the concordance model parameters. As one can see, the fit is quite good. In actuality, the concordance model parameters are determined by fitting the data. A rather complex statistical machinery underlies the determination of cosmological parameters, and is discussed in Dodelson (2003, Ch. 11). The fact that modern CMB and LSS data agree over a substantial region of overlap gives us confidence in the correctness of the concordance model. In this section, we define the power spectrum mathematically, and review the basic physics which determines its shape. Readers wishing a more in depth treatment are referred to references [4, 10].

At any epoch \( t \) (or \( a \) or \( z \)) express the matter density in the universe in terms of a mean density and a local fluctuation:

\[
\rho(\vec{x}) = \bar{\rho}(1 + \delta(\vec{x})�)
\]

where \( \delta(\vec{x}) \) is the density contrast. Expand \( \delta(\vec{x}) \) in Fourier modes:

\[
\delta(\vec{x}) \equiv \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}} = \int \delta(\vec{k}) \exp(-i\vec{k} \cdot \vec{x}) d^3k.
\]
The autocorrelation function of $\delta(\vec{x})$ defines the power spectrum through the relations

$$
\langle \delta(\vec{x})\delta(\vec{x}) \rangle = \int_0^\infty \frac{dk}{k} \frac{k^3}{2\pi^2} \left| \Delta^2(k) \right| = \int_0^\infty \frac{dk}{k} \frac{k^3 P(k)}{2\pi^2} = \int_0^\infty \frac{dk}{k} \Delta^2(k)
$$

where we have the definitions

$$
P(k) = \left| \delta^2(\vec{k}) \right|, \quad \Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}.
$$

The quantity $\Delta^2(k)$ is called the dimensionless power spectrum and is an important function in the theory of structure formation. $\Delta^2(k)$ measures the contribution of perturbations per unit logarithmic interval at wavenumber $k$ to the variance in the matter...
density fluctuations. The ΛCDM power spectrum asymptotes to \( P(k) \sim k^1 \) for small \( k \), and \( P(k) \sim k^{-3} \) for large \( k \), with a peak \( a k^* \sim 2 \times 10^{-2} \) h Mpc\(^{-1} \) corresponding to \( \lambda^* \sim 350 \) h\(^{-1} \) Mpc. \( \Delta^2(k) \) is thus asymptotically flat at high \( k \), but drops off as \( k^4 \) at small \( k \). We therefore see that most of the variance in the cosmic density field in the universe at the present epoch is on scales \( \lambda < \lambda^* \).

Fig. 3. – The tale of two fluctuations. A fluctuation which is superhorizon scale at matter-radiation equality grows always, while a fluctuation which enters the horizon during the radiation dominated era stops growing in amplitude until the matter dominated era begins.

What is the origin of the power spectrum shape? Here we review the basic ideas. Within the inflationary paradigm, it is believed that quantum mechanical (QM) fluctuations in the very early universe were stretched to macroscopic scales by the large expansion factor the universe underwent during inflation. Since QM fluctuations are random, the primordial density perturbations should be well described as a Gaussian random field. Measurements of the Gaussianity of the CMB anisotropies [11] have confirmed this. The primordial power spectrum is parameterized as a power law \( P_p(k) \propto k^n \), with \( n = 1 \) corresponding to scale-invariant spectrum proposed by Harris and Zeldovich on the grounds that any other value would imply a preferred mass scale for fluctuations entering the Hubble horizon. Large angular scale CMB anisotropies measure the primordial power spectrum directly since they are superhorizon scale. Observations with the WMAP satellite yield a value very close to \( n = 1 \) [73].

To understand the origin of the spectrum, we need to understand how the amplitude of a fluctuation of fixed comoving wavelength \( \lambda \) grows with time. Regardless of its wavelength, the fluctuation will pass through the Hubble horizon as illustrated in Fig. 3. This is because the Hubble radius grows linearly with time, while the proper wavelength \( a \lambda \) grows more slowly with time. It is easy to show from Eq. 1 that in the radiation-dominated era, \( a \sim t^{1/2} \), and in the matter-dominated era (prior to the onset of cosmic acceleration) \( a \sim t^{2/3} \). Thus, inevitably, a fluctuation will transition from superhorizon to subhorizon scale. We are interested in how the amplitude of the fluctuation evolves
during these two phases. Here we merely state the results of perturbation theory (e.g., Dodelson 2003, Ch. 7).

Fig. 4. – a) Evolution of the primordial power spectrum on superhorizon scales during the radiation dominated era. b) Scale-free spectrum produces a constant contribution to the density variance per logarithmic wavenumber interval entering the Hubble horizon (no preferred scale) c) resulting matter power spectrum, super- and sub-horizon. Figures courtesy Rocky Kolb.

2.3. Growth of fluctuations in the linear regime. – To calculate the growth of superhorizon scale fluctuations requires general relativistic perturbation theory, while sub-horizon scale perturbations can be analyzed using a Newtonian Jeans analysis. We are interested in scalar density perturbations, because these couple to the stress tensor of the matter-radiation field. Vector perturbations (e.g., fluid turbulence) are not sourced by the stress-tensor, and decay rapidly due to cosmic expansion. Tensor perturbations are gravity waves, and also do not couple to the stress-tensor. A detailed analysis for the scalar perturbations yields the following results. In the radiation dominated era,

\[ \delta_+(t) = \delta_+(t_i)(t/t_i) \text{ superhorizon scales} \]
\[ \delta_+(t) = \text{constant} \quad \text{subhorizon scales} \]
while in the matter dominated era, 

$$\delta_+(t) = \delta_+(t_i) (t/t_i)^{2/3} \text{ superhorizon scales}$$

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This is summarized in Fig. 3, where we consider two fluctuations of different comoving wavelengths, which we will call large and small. The large wavelength perturbation remains superhorizon through matter-radiation equality (MRE), and enters the horizon in the matter dominated era. Its amplitude will grow as $t$ in the radiation dominated era, and as $t^{2/3}$ in the matter dominated era. It will continue to grow as $t^{2/3}$ after it becomes subhorizon scale. The small wavelength perturbation becomes subhorizon before MRE. Its amplitude will grow as $t$ while it is superhorizon scale, remain constant while it is subhorizon during the radiation dominated era, and then grow as $t^{2/3}$ during the matter-dominated era.

Armed with these results, we can understand what is meant by a scale-free primordial power spectrum (the Harrison-Zeldovich power spectrum.) We are concerned with perturbation growth in the very early universe during the radiation dominated era. Superhorizon scale perturbation amplitudes grow as $t$, and then cease to grow after they have passed through the Hubble horizon. We can define a Hubble wave number $k_H \equiv 2\pi/R_H \propto t^{-1}$. Fig. 4a shows the primordial power spectrum at three instants in time for $k<k_H$. We see that the fluctuation amplitude at $k=k_H(t)$ depends on primordial power spectrum slope $n$. The scale-free spectrum is the value of $n$ such that $\Delta^2(k_H(t))=\text{constant}$ for $k>k_H$. A simple analysis shows that this implies $n=1$. Since $\Delta^2(k) \propto k^3 P(k)$, we then have

$$P(k) \propto k^1, k \leq k_H$$

$$P(k) \propto k^{-3}, k > k_H$$

In actuality, the power spectrum has a smooth maximum, rather than a peak as shown in Fig. 4c. This smoothing is caused by the different rates of growth before and after matter-radiation equality. The transition from radiation to matter-dominated is not instantaneous. Rather, the expansion rate of the universe changes smoothly through equality, as given by Eq. 1, and consequently so do the temporal growth rates. The position of the peak of the power spectrum is sensitive to the time when the universe reached matter-radiation equality, and hence is a probe of $\Omega_\gamma/\Omega_m$.

Once a fluctuation becomes sub-horizon, dissipative processes modify the shape of the power spectrum in a scale-dependent way. Collisionless matter will freely stream out of overdense regions and smooth out the inhomogeneities. The larger the particle, the larger its free streaming length. Particles which are relativistic at MRE, such as light neutrinos, are called hot dark matter (HDM). They have a large free-streaming length, and consequently damp the power spectrum over a large range of $k$. Weakly Interacting Massive Particles (WIMPs) which are nonrelativistic at MRE, are called cold dark matter (CDM), and modify the power spectrum very little (Fig. 5). Baryons are tightly coupled to the radiation field by electron scattering prior to recombination. During recombination, the photon mean-free path becomes large. As photons stream out of dense regions, they drag baryons along, erasing density fluctuations on small scales. This process is called Silk damping, and results in damped oscillations of the baryon-photon fluid once they become subhorizon scale. The magnitude of this effect is sensitive to the ratio of baryons to collisionless matter, as shown in Fig. 5.
3. Analytic models for nonlinear growth, virial scaling relations, and halo statistics

Here we introduce a few concepts and analytic results from the theory of structure formation which underly the use of galaxy clusters as cosmological probes. These provide us with the vocabulary which pervades the literature on analytic and numerical models of galaxy cluster evolution. Material in this section has been derived from three primary sources: Padmanabhan (1993) [12] for the spherical top-hat model for nonlinear collapse, Dodelson (2003) [4] for Press-Schechter theory, and Bryan & Norman (1998) [13] for virial scaling relations.

3.1. Nonlinearity defined. In the linear regime, both super- and sub-horizon scale perturbations grow as $t^{2/3}$ in the matter-dominated era. This means that after recombination, the linear power spectrum retains its shape while its amplitude grows as $t^{4/3}$ before the onset of cosmic acceleration (Fig. 6a). When $\Delta^2(k)$ for a given $k$ approaches unity linear theory no longer applies, and some other method must be used to determine the fluctuation’s growth. In general, numerical simulations are required to model the nonlinear phase of growth because in the nonlinear regime, the modes do not grow independently. Mode-mode coupling modifies both the shape and amplitude of the power spectrum over the range of wavenumbers that have gone nonlinear.

At any given time, there is a critical wavenumber which we shall call the nonlinear wavenumber $k_{nl}$ which determines which portion of the spectrum has evolved into the nonlinear regime. Modes with $k<k_{nl}$ are said to be linear, while those for which $k>k_{nl}$ are nonlinear (Fig. 6b). Conventionally, one defines the nonlinear wavenumber such that $\Delta(k_{nl}, z) = 1$. From this one can derive a nonlinear mass scale $M_{nl}(z) = \frac{4\pi}{3} \bar{\rho}(z) \left( \frac{2\pi}{k_{nl}} \right)^3$.

A more useful and rigorous definition of the nonlinear mass scale comes from evaluating the amplitude of mass fluctuations within spheres or radius $R$ at epoch $z$. The enclosed mass is $M = \frac{4\pi}{3} \bar{\rho}(z) R^3$. The mean square mass fluctuations (variance) is

$$
\langle (\delta M/M)^2 \rangle \equiv \sigma^2(M) = \int d^3k W_f^2(kR) P(k, z),
$$

(13)
Fig. 6. – Two ways of looking at the growth of post-recombination matter fluctuations in the linear regime. Left: 3D matter power spectrum increases uniformly proportional to the linear growth factor $D(z)$. Measurements are from SDSS galaxy large scale structure data. Right: the evolution of the dimensionless power spectrum $\Delta^2(k)$. Nonlinearity is defined where $\Delta^2(k_{nl}) = 1$. All scales with $k \geq k_{nl}$ have collapsed into bound objects, and do so in a “bottom-up” fashion.

where $W$ is the Fourier transform of the top-hat window function

$$W(x) = \begin{cases} 3/4\pi R^3, & |x| < R \\ 0, & |x| \geq R \end{cases} \rightarrow W_T(kR) = 3 \left[ \frac{\sin(kR)/kR - \cos(kR)}{(kR)^2} \right].$$

If we approximate $P(k)$ locally with a power-law $P(k, z) = D^2(z)k^m$, where $D$ is the linear growth factor, then $\sigma^2(M) \propto D^2 R^{-(3+m)} \propto D^2 M^{-(3+m)/3}$. From this we see that the RMS fluctuations are a decreasing function of $M$. At very small mass scales, $m \to -3$, and the fluctuations asymptote to a constant value. We now define the nonlinear mass scale by setting $\sigma(M_{nl}) = 1$. We get that (15)

$$M_{nl}(z) \propto D(z)^{6/(3+m)} (\propto (1 + z)^{-6/(3+m)} \text{ for EdS}).$$

For $m > -3$, the smallest mass scales become nonlinear first. This is the origin of hierarchical (“bottom-up”) structure formation (Fig. 6b).

3.2. Spherical Top-Hat Model. – We now ask what happens when a spherical volume of mass $M$ and radius $R$ exceeds the nonlinear mass scale. The simplest analytic model of the nonlinear evolution of a discrete perturbation is called the spherical top-hat model. In it, one imagines as spherical perturbation of radius $R$ and some constant overdensity $\bar{\delta} = 3M/4\pi\bar{\rho}R^3$ in an Einstein-de Sitter (EdS) universe. By Birkhoff’s theorem the equation of motion for $R$ is

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} = -\frac{4\pi G}{3} \bar{\rho}(1+\bar{\delta})R$$

whereas the background universe expands according to Eq. 6

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3} \bar{\rho}a.$$
Comparing these two equations, we see that the perturbation evolves like a universe of a different mean density, but with the same initial expansion rate. Integrating Eq. 16 once with respect to time gives us the first integral of motion:

\[
\frac{1}{2} \left( \frac{dR}{dt} \right)^2 - \frac{GM}{R} = E,
\]

where \( E \) is the total energy of the perturbation. If \( E < 0 \), the perturbation is bound, and obeys

\[
\frac{R}{R_m} = \frac{1 - \cos \theta}{2}, \quad \frac{t}{t_m} = \frac{\theta - \sin \theta}{\pi}
\]

where \( R_m \) and \( t_m \) are the radius and time of “turnaround”. At turnaround (as \( \theta \to \pi \)), the fluctuation reaches its maximum proper radius (see Fig. 7). As \( t \to 2t_m, R \to 0 \), and we say the fluctuation has collapsed.

A detailed analysis of the evolution of the top-hat perturbation is given in Padmanabhan (1993, Ch. 8) for general \( \Omega_m \). Here we merely quote results for an EdS universe. The mean linear overdensity at turnaround; i.e., the value one would predict from the linear growth formula \( \delta \sim t^{2/3} \), is 1.063. The actual overdensity at turnaround using the nonlinear model is 4.6. This illustrates that nonlinear effects set in well before the amplitude of a linear fluctuation reaches unity. As \( R \to 0 \), the nonlinear overdensity becomes infinite. However, the linear overdensity at \( t = 2t_m \) is only 1.686. As the fluctuation collapses, other physical processes (pressure, shocks, violent relation) become important which establish a gravitationally bound object in virial equilibrium before infinite density is reached. Within the framework of the spherical top-hat model, we say virialization has occurred when the kinetic and gravitational energies satisfy virial equilibrium: \( |U| = 2K \).

It is easy to show from conservation of energy that this occurs when \( R = R_m/2 \); in other words, when the fluctuation has collapsed to half its turnaround radius. The nonlinear overdensity at virialization \( \Delta_c \) is not infinite since the radius is finite. For an EdS uni-
verse, $\Delta_c = 18\pi^2 \approx 180$. Fitting formulae for non-EdS models are provided in the next section.

3.3. Virial Scaling Relations. – The spherical top-hat model can be scaled to perturbations of arbitrary mass. Using virial equilibrium arguments, we can predict various physical properties of the virialized object. The ones that interest us most are those that relate to the observable properties of gas in galaxy clusters, such as temperature, X-ray luminosity, and SZ intensity change. Kaiser [14] first derived virial scaling relations for clusters in an EdS universe. Here we generalize the derivation to non-EdS models of interest. In order to compute these scaling laws, we must assume some model for the distribution of matter as a function of radius within the virialized object. A top-hat distribution with a density $\rho = \Delta_c \bar{\rho}(z)$ is not useful because it is not in mechanical equilibrium. More appropriate is the isothermal, self-gravitating, equilibrium sphere for the collisionless matter, whose density profile is related to the one-dimensional velocity dispersion [15]

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}.$$  

(20)

If we define the virial radius $r_{\text{vir}}$ to be the radius of a spherical volume within which the mean density is $\Delta_c$ times the critical density at that redshift ($M = 4\pi r_{\text{vir}}^3 \rho_{\text{crit}} \Delta_c / 3$), then there is a relation between the virial mass $M$ and $\sigma$:

$$\sigma = M^{1/3} [H^2(z) \Delta_c G^2 / 16]^{1/6} \approx 476 f_\sigma \left( \frac{M}{10^{15} M_\odot} \right)^{1/3} (h^2 \Delta_c E^2)^{1/6} \text{ km s}^{-1}. $$  

(21)

Here we have introduced a normalization factor $f_\sigma$ which will be used to match the normalization from simulations. The redshift dependent Hubble parameter can be written as $H(z) = 100 h E(z)$ km s$^{-1}$ with the function $E^2(z) = \Omega_m (1 + z)^3 + \Omega_k (1 + z)^2 + \Omega_\Lambda$, where the $\Omega$’s have been previously defined.

The value of $\Delta_c$ is taken from the spherical top-hat model, and is $18\pi^2$ for the critical EdS model, but has a dependence on cosmology through the parameter $\Omega(z) = \Omega_m (1 + z)^3 / E^2(z)$. Bryan and Norman (1998) provided fitting formulae for $\Delta_c$ for the critical for both open universe models and flat, lambda-dominated models

$$\Delta_c = 18\pi^2 + 82x - 39x^2 \text{ for } \Omega_k = 0, \Delta_c = 18\pi^2 + 60x - 32x^2 \text{ for } \Omega_\Lambda = 0$$  

(22)

where $x = \Omega(z)-1$.

If the distribution of the baryonic gas is also isothermal, we can define a ratio of the “temperature” of the collisionless material ($T_\sigma = \mu m_p \sigma^2 / k$) to the gas temperature:

$$\beta = \frac{\mu m_p \sigma^2}{kT}.$$  

(23)

Given equations (22) and (23), the relation between temperature and mass is then

$$kT = \frac{GM^{2/3} \mu m_p}{2\beta} \left[ \frac{H^2(z) \Delta_c}{2G} \right]^{1/3} \approx 1.39 f_T \left( \frac{M}{10^{15} M_\odot} \right)^{2/3} (h^2 \Delta_c E^2)^{1/3} \text{ keV},$$  

(24)

where in the last expression we have added the normalization factor $f_T$ and set $\beta=1$.  


The scaling behavior for the object’s X-ray luminosity is easily computed by assuming bolometric bremsstrahlung emission and ignoring the temperature dependence of the Gaunt factor: 

\[ L_{\text{bol}} \propto \int \rho^2 T^{1/2} dV \propto M_b T^{1/2} \]

where \( M_b \) is the baryonic mass of the cluster. This is infinite for an isothermal density distribution, since \( \rho \) is singular. Observationally and computationally, it is found that the baryon distribution rolls over to a constant density core at small radius. A procedure is described in Bryan and Norman (1998) which yields a finite luminosity:

\[ L_{\text{bol}} = 1.3 \times 10^{45} \left( \frac{M}{10^{15} M_\odot} \right)^{4/3} (h^2 \Delta_c E^2)^{7/6} \left( \frac{\Omega_b}{\Omega_m} \right)^2 \text{ erg s}^{-1}. \]

Eliminating \( M \) in favor of \( T \) in Eq. 25 we get

\[ L_{\text{bol}} = 6.8 \times 10^{44} \left( \frac{kT}{f_T 1.0 \text{ keV}} \right)^2 (h^2 \Delta_c E^2)^{1/2} \left( \frac{\Omega_b}{\Omega_m} \right)^2 \text{ erg s}^{-1}. \]

The scaling of the SZ “luminosity” is likewise easily computed. If we define \( L_{SZ} \) as the integrated SZ intensity change: 

\[ L_{SZ} = \int dA \int n_e \sigma_T \left( \frac{kT}{m_e c^2} \right) dl \propto M_b T, \]

then

\[ L_{SZ} = \frac{GM^5}{3 \beta m_e c^2} \left[ H^2(z) \Delta_c \right]^{1/3} \left( \frac{\Omega_b}{\Omega_m} \right). \]

We note that cosmology enters these relations only with the combination of parameters \( h^2 \Delta_c E^2 \), which comes from the relation between the cluster’s mass and the mean density of the universe at redshift \( z \). The redshift variation comes mostly from \( E(z) \), which is equal to \((1+z)^{3/2}\) for an EdS universe.

3.4. Statistics of hierarchical clustering: Press-Schechter theory. – Now that we have a simple model for the nonlinear evolution of a spherical density fluctuation and its observable properties as a function of its virial mass, we would like to estimate the number of virialized objects of mass \( M \) as a function of redshift given the matter power spectrum. This is the key to using surveys of galaxy clusters as cosmological probes. While large scale numerical simulations can and have been used for this purpose (see below), we review a powerful analytic approach by Press and Schechter [16] which turns out to be remarkably close to the numerical results. The basic idea is to imagine smoothing the cosmological density field at any epoch \( z \) on a scale \( R \) such that the mass scale of virialized objects of interest satisfies 

\[ M = \frac{4 \pi}{3} \rho(z) R^3. \]

Because the density field (both smoothed and unsmoothed) is a Gaussian random field, the probability that the mean overdensity in spheres of radius \( R \) exceeds a critical overdensity \( \delta_c \) is

\[ p(R, z) = \frac{2}{\sqrt{2\pi} \sigma(R, z)} \int_0^\infty d\delta \exp \left( -\frac{\delta^2}{2\sigma^2(R, z)} \right) \]

where \( \sigma(R, z) \) is the RMS density variation in spheres of radius \( R \) as discussed above. Press and Schechter suggested that this probability be identified with the fraction of particles which are part of a nonlinear lump with mass exceeding \( M \) if we take \( \delta_c = 1.686 \), the linear overdensity at virialization. This assumption has been tested against numerical
simulations and found to be quite good [9] (however, see below). The fraction of the volume collapsed into objects with mass between $M$ and $M + dM$ is given by $(dp/dM) dM$. Multiply this by the average number density of such objects $\rho_m / M$ to get the number density of collapsed objects between $M$ and $M + dM$:

\[
\frac{dn}{dM}(M, z) = -\frac{\bar{\rho}}{M} \frac{dp(M(R), z)}{dM} dM.
\]

The minus sign appears here because $p$ is a decreasing function of $M$. Carrying out the derivative using the fact that $dM/dR = 3M/R$, we get

\[
\frac{dn}{dM}(M, z) = \sqrt{\frac{2}{\pi}} \frac{\bar{\rho} \delta_c}{3M^2 \sigma} e^{-\delta_c^2 / 2\sigma^2} \left[ -\frac{d \ln \sigma}{d \ln R} \right].
\]

The term is square brackets is related to the logarithmic slope of the power spectrum, which on the mass scale of galaxy clusters is close to unity. Eq. 30 is called the halo mass function, and it has the form of a power law multiplied by an exponential. To make this more explicit, approximate the power spectrum on scales of interest as a power law as we have done above. Substituting the scaling relations for $\sigma$ in Eq. 30 one gets the result [17]

\[
\frac{dn}{dM} = \left( \frac{2}{\pi} \right)^{1/2} \frac{\bar{\rho}}{M^2} \left( 1 + \frac{m}{3} \right) \left[ \frac{M}{M_{nl}(z)} \right]^{2m-3} \exp \left[ -\left( \frac{M}{M_{nl}(z)} \right)^{2m} / 2 \right].
\]

Here, $M_{nl}(z)$ is the nonlinear mass scale. To be more consistent with the spherical top-hat model, it satisfies the relation $\sigma(M_{nl}, z) = \delta_c$; i.e., those fluctuations in the smoothed density field that have reached the linear overdensity for which the spherical top-hat model predicts virialization.

### 3.5. Validating the Halo Mass Function using N-body Simulations.

Below we discuss how one goes about numerically simulating the nonlinear evolution of the density fluctuations described by the $\Lambda$CDM power spectrum. Here we simply mention two works which made detailed comparisons of the PS formula with halo populations found in dark matter N-body simulations. The first is by Jenkins et al. (2001)[74] who analyzed the results of the “Hubble Volume” simulation—a simulation of dark matter clustering carried out in a cubic volume 3 Gpc/h on a side with $1024^3$ dark matter particles (Fig. 8). This yields a dark matter particle mass of $2.2 \times 10^{12} M_\odot$, implying that a galaxy cluster halo would typically contain $10^2 - 10^3$ particles. The relatively poor mass resolution is offset by the very large volume, which permits exploring the cluster mass function across a broad range of masses including the very high mass end. Fig. 8 shows a slice through the simulation volume on which the dark matter density field is plotted. Jenkins et al. (2001) identified dark matter halos using the friends-of-friends algorithm [52] and found that while the PS formula gives a good approximation to the numerical data, it underpredicts the number of rare, massive objects, and overpredicts the number of “typical” objects as shown in the right panel of Fig. 8.

Warren et al. (2006)[75] were interested in testing the validity of the PS formula over a wider range of mass scales than can be obtained from a single simulation. They simulated 16 boxes of different physical size but the same number of DM particles ($1024^3$) nested in such a way that together they define a composite halo mass function covering
5 orders of magnitude in mass scale. They derived a fitting formula for the composite halo mass function for the WMAP3 concordance cosmological parameters by assuming a parameterized form for the \( f(\sigma) \) of:

\[
f(\sigma) = A(\sigma^{-a} + b) \exp^{-c/\sigma^2}
\]

where the multiplicity function \( f(\sigma) \) is related to the mass function \( n(M) \) via

\[
f(\sigma) = \frac{M}{\rho} \frac{dn}{d\ln \sigma^{-1}}
\]

where \( A, a, b \) and \( c \) come from the fitting procedure, and are documented in [75].

To illustrate one application of this formula, Fig. 9 shows the cluster halo mass function computed using the Warren et al. (2006) fit for three different redshifts for the cosmological parameters adopted in the simulation shown in Fig. 1. Overplotted on the semi-analytic predictions are the halo mass functions obtained from the simulation itself.
The departure of the simulation from the predictions at the low mass end are due to finite resolution effects discussed in Sec. 4 below.

Fig. 9. – Cumulative number of galaxy clusters exceeding mass $m$ for three redshifts. Solid lines: $z = 0.1$, short-dashed lines $z = 1$, long-dashed lines $z = 2$. The thick lines are from the simulation shown in Fig. 1, while the thin lines are predictions using the Warren [75] fitting function. Note the rapid redshift evolution of the number of massive clusters. The departure of the simulation from the predictions at the low mass end are due to finite resolution effects. From [77].

$3.6$. Application to galaxy clusters. – The aerial density of galaxy clusters can be calculated by multiplying the redshift-dependent halo mass function using the Warren fitting formula with the redshift-dependent differential volume element for one square degree on the sky. Fig. 10a shows the result varying the amplitude of the matter power spectrum $\sigma_8$ holding all other cosmological parameters fixed, while Fig. 10b shows the result varying the matter density $\Omega_m$, holding all other cosmological parameters fixed. In general we see a rapid rise in the number of clusters with increasing redshift due to the increasing volume element. However, the space density of clusters declines rapidly with redshift (see Fig. 9), and thus the aerial density peaks at $z \approx 1$ and then declines rapidly toward higher redshift.

From these curves and the virial scaling relations given above it is easy to predict the expected number of clusters of a given X-ray temperature, X-ray luminosity, or SZ
luminosity as a function of redshift [18, 13], bearing in mind that real clusters may not perfectly obey the virial scaling relations. In fact they don’t as discussed in my 2004 Varenna lectures [69] and by Borgani in these proceedings.

4. – Numerical simulations of gas in galaxy clusters

The central task is for a given cosmological model, calculate the formation and evolution of a population of clusters from which synthetic X-ray and SZ catalogs can be derived. These can be used to calibrate simpler analytic models, as well as to build synthetic surveys (mock catalogs) which can be used to assess instrumental effects and survey biases. One would like to directly simulate $n(M, z), n(L_x, z), n(T, z), n(Y, z)$ from the governing equations for collisionless and collisional matter in an expanding universe. Clearly, the quality of these statistical predictions relies on the ability to adequately resolve the internal structure and thermodynamical evolution of the ICM.

Since X-ray emission and the SZE are both consequences of hot plasma bound in the cluster’s gravitational potential well, the requirements to faithfully simulate X-ray clusters and SZ clusters are essentially the same. Numerical progress can be characterized as a quest for higher resolution and essential baryonic physics. In this section I describe the technical challenges involved and the numerical methods that have been developed to overcome them. I then discuss the effects of assumed baryonic physics on ICM structure. Our point of reference is the non-radiative (so-called adiabatic) case, which has been the subject of an extensive code comparison [20].

In Norman (2003) [19] I provided a historical review of the progress that has been made in simulating the evolution of gas in galaxy clusters motivated by X-ray observations. In Norman (2004) [69] I discuss the statistical properties of simulated galaxy cluster samples and how they depend on assumed baryonic physics. The key result of this work is that while $L_x$ is highly sensitive to input physics and numerical resolution, $L_{SZ}$ is not,
and therefore potentially a useful proxy for the cluster mass and thereby a cosmological probe. I discuss recent progress on increasing the physics fidelity of individual cluster simulations in Sec. 5, and the use of cluster SZ surveys as cosmological probes in Sec. 6.

4.1. Dynamic range considerations. – Fig. 11 illustrates the dynamic range difficulties encountered with simulating a statistical ensemble of galaxy clusters, while at the same time resolving their internal structure. Massive clusters are rare at any redshift, yet these are the ones most that are most sensitive to cosmology. From the cluster mass function (Fig. 9), in order to get adequate statistics, one deduces that one must simulate a survey volume many hundreds of megaparsecs on a side (Fig. 11, left panel). A massive cluster has a virial radius of ~2 Mpc. It forms via the collapse of material within a comoving Lagrangian volume of ~15 Mpc. However, tidal effects from a larger region (50-100 Mpc) are important on the dynamics of cluster formation. The internal structure of cluster's ICM is shown in Fig. 11, center panel. While clusters are not spherical, two important radii are generally used to characterize them: the virial radius, which is the approximate location of the virialization shock wave that thermalizes infalling gas to 10-100 million K, and the core radius, within which the baryon densities plateau and the highest X-ray emissions and SZ intensity changes are measured. A typical radius is ~200 kpc. Within the core, radiative cooling and possibly other physical processes are important. Outside the core, cooling times are longer than the Hubble time, and the ICM gas is effectively adiabatic. If we wanted to achieve a spatial resolution of 1/10 of a core radius everywhere within the survey volume, we would need a spatial dynamic range of D=500 Mpc/20 kpc = 25,000. The mass dynamic range is more severe. If we want 1 million dark matter particles within the virial radius of a $10^{15} M_{\odot}$ cluster, then we would need $N_{\text{particle}} = M_{\text{box}}/M_{\text{particle}} = \Omega_m \rho_{\text{crit}} L^3/10^9 \approx 10^{11}$ if they were uniformly distributed in the survey volume.

Two solutions to spatial dynamic range problem have been developed: tree codes for gridless N-body methods [21, 22] and adaptive mesh refinement (AMR) for Eulerian particle-mesh/hydrodynamic methods [23, 24, 25, 26]. Both methods increase the spatial resolution automatically in collapsing regions as described below. The solution to the mass dynamic range problem is the use of multi-mass initial conditions in which a hierarchy of particle masses is used, with many low mass particles concentrated in the region of interest. This approach has most recently used by Springel et al. (2000) [27],
Fig. 12. – Cosmological simulations are generally carried out in a frame of reference that is comoving with the expanding universe. Initial conditions are generated from the input power spectrum at a starting redshift and then advanced in time using equations 34–36.

who simulated the formation of a galaxy cluster dark matter halo with $N = 6.9 \times 10^6$ dark matter particles, resolving the dark matter halos down to the mass scale of the Fornax dwarf spheroidal galaxy. The spatial dynamic range achieved in this simulation was $R = 2 \times 10^5$. Such dynamic ranges have not yet been achieved in galaxy cluster simulations with gas.

4.2. Simulating cluster formation. – Simulations of cosmological structure formation are done in a cubic domain which is comoving with the expanding universe (cf. Fig. 12). Matter density and velocity fluctuations are initialized at the starting redshift chosen such that all modes in the volume are still in the linear regime. Once initialized, these fluctuations are then evolved to $z=0$ by solving the equations for collisionless N-body dynamics for cold dark matter, and the equations of ideal gas dynamics for the baryons in an expanding universe. Making the transformation from proper to comoving coordinates $\vec{r} = a(t)\vec{x}$, Newton’s laws for the collisionless dark matter particles become

\[
\frac{d\vec{x}_{dm}}{dt} = \vec{v}_{dm}, \quad \frac{d\vec{v}_{dm}}{dt} = -\frac{2}{a} \frac{\dot{a}}{a^2} \vec{v}_{dm} - \frac{1}{a^2} \nabla_x \phi
\]

where $x$ and $v$ are the particle’s comoving position and peculiar velocity, respectively, and $\phi$ is the comoving gravitational potential that includes baryonic and dark matter contributions. The hydrodynamical equations for mass, momentum, and energy conservation
in an expanding universe in comoving coordinates are ([28])

\[
\frac{\partial \rho_b}{\partial t} + \nabla \cdot (\rho_b \vec{v}_b) + \frac{3}{a^2} \rho_b = 0,
\]

\[
\frac{\partial (\rho_b \vec{v}_b,)}{\partial t} + \nabla \cdot [(\rho_b \vec{v}_b, \vec{v}_b) \vec{v}_b + 5 \frac{3}{a^2} \rho_b \vec{v}_b,] = -\frac{1}{a^2} \frac{\partial \rho}{\partial x_i} - \rho_b \frac{\partial \phi}{\partial x_i},
\]

\[
\frac{\partial e}{\partial t} + \nabla \cdot (e \vec{v}_b) + p \nabla \cdot \vec{v}_b + 3 \frac{3}{a^2} e = \Gamma - \Lambda,
\]

where \( \rho_b, p \) and \( e \), are the baryonic density, pressure and internal energy density defined in the proper reference frame, \( \vec{v}_b \) is the comoving peculiar baryonic velocity, \( a = 1/(1+z) \) is the cosmological scale factor, and \( \Gamma \) and \( \Lambda \) are the microphysical heating and cooling rates. The baryonic and dark matter components are coupled through Poisson’s equation for the gravitational potential

\[
\nabla^2 \phi = 4\pi Ga^2 (\rho_b + \rho_{dm} - \bar{\rho}(z))
\]

where \( \bar{\rho}(z) = 3H_0\Omega_m(0)/8\pi Ga^3 \) is the proper background density of the universe.

The cosmological scale factor \( a(t) \) is obtained by integrating the Friedmann equation (Eq. 4). To complete the specification of the problem we need the ideal gas equation of state \( p = (\gamma - 1)e \), and the gas heating and cooling rates. When simulating the ICM, the simplest approximation is to assume \( \Gamma \) and \( \Lambda = 0 \); i.e., no heating or cooling of the gas other than by adiabatic processes and shock heating. Such simulations are referred to as adiabatic (despite entropy-creating shock waves), and are a reasonable first approximation to real clusters because except in the cores of clusters, the radiative cooling time is longer than a Hubble time, and gravitational heating is much larger than sources of astrophysical heating. However, as discussed in the paper by Cavaliere in this volume, there is strong evidence that the gas in cores of clusters has evolved non-adiabatically. This is revealed by the entropy profiles observed in clusters [29] which deviate substantially from adiabatic predictions. In the simulations presented below, we consider radiative cooling due to thermal bremsstrahlung, and mechanical heating due to galaxy feedback, details of which are described below.

4.3. Numerical methods overview. – A great deal of literature exists on the gravitational clustering of CDM using N-body simulations. A variety of methods have been employed including the fast grid-based methods particle-mesh (PM), and particle-particle+particle-mesh (P3M) [30], spatially adaptive methods such as adaptive P3M [31], adaptive mesh refinement [24], tree codes [32, 33], and hybrid methods such as TreePM [34]. Because of the large dynamic range required, spatially adaptive methods are favored, with Tree and TreePM methods the most widely used today. Fig. 13 shows a high resolution N-body simulation of the substructure within a dark matter halo performed by Springel using the GADGET code [78].

When gas dynamics is included, only certain combinations of hydrodynamics algorithms and collisionless N-body algorithms are “natural”. Dynamic range considerations have led to two principal approaches: P3MSPH and TreeSPH, which marries a P3M or tree code for the dark matter with the Lagrangian smoothed-particle-hydrodynamics (SPH) method [35, 21, 22], and adaptive mesh refinement (AMR), which marries PM with Eulerian finite-volume gas dynamics schemes on a spatially adaptive mesh [23, 26, 25, 36]. Pioneering hydrodynamic simulations using non-adaptive Eulerian grids [37, 38, 13] yielded some important insights about cluster formation and statistics, but generally have inadequate resolution to resolve their internal structure in large survey volumes. In the following we concentrate on our latest results using the AMR code Enzo [26].
The reader is also referred to the paper by Borgani et al. [39] which presents recent, high-resolution results from a large TreeSPH simulation.

*Enzo* is a grid-based hybrid code (hydro + N-body) which uses the block-structured AMR algorithm of Berger & Collela [40] to improve spatial resolution in regions of large gradients, such as in gravitationally collapsing objects. The method is attractive for cosmological applications because it: (1) is spatially- and time-adaptive, (2) uses accurate and well-tested grid-based methods for solving the hydrodynamics equations, and (3) can be well optimized and parallelized. The central idea behind AMR is to solve the evolution equations on a grid, adding finer meshes in regions that require enhanced resolution. Mesh refinement can be continued to an arbitrary level, based on criteria involving any combination of overdensity (dark matter and/or baryon), Jeans length, cooling time, etc., enabling us to tailor the adaptivity to the problem of interest. The code solves the following physics models: collisionless dark matter and star particles, using the particle-mesh N-body technique [41]; gravity, using FFTs on the root grid and multigrid relaxation on the subgrids; cosmic expansion; gas dynamics, using the piecewise parabolic method (PPM)[42]; multispecies nonequilibrium ionization and H$_2$ chemistry, using backward Euler time differencing [28]; radiative heating and cooling, using subcycled forward Euler time differencing [43]; and a parameterized star formation/feedback recipe [44]. At the present time, magnetic fields and radiation transport are being installed. *Enzo* is publicly available at [http://lca.ucsd.edu/projects/enzo](http://lca.ucsd.edu/projects/enzo).

4.4. *Structure of nonradiative clusters: the Santa Barbara test cluster.* – In Frenk et al. [20] 12 groups compared the results of a variety of hydrodynamic cosmological algorithms on a standard test problem. The test problem, called the Santa Barbara cluster, was to simulate the formation of a Coma-like cluster in a standard CDM cosmology ($\Omega_m = 1$) assuming the gas is nonradiative. Groups were provided with uniform initial conditions and were asked to carry out a “best effort” computation, and analyze their results at $z=0.5$ and $z=0$ for a set of specified outputs. These outputs included global
integrated quantities, radial profiles, and column-integrated images. The simulations varied substantially in their spatial and mass resolution owing to algorithmic and hardware limitations. Nonetheless, the comparisons brought out which predicted quantities were robust, and which were not yet converged. In Fig. 14 we show a few figures from Frenk et al. (1999) which highlight areas of agreement (top row) and disagreement (bottom row).

Fig. 14. – The Santa Barbara test cluster. Top row, left to right: profiles of dark matter density, gas density, and pressure. Bottom row, left to right: profiles of gas temperature, gas entropy, and X-ray emissivity. Different symbols correspond to different code results. From [20].

The top row shows profile of dark matter density, baryon density, and pressure for the different codes. All are in quite good agreement for the mechanical structure of the cluster. The dark matter profile is well described by an NFW profile which has a central cusp [45]. The baryon density profiles show more dispersion, but all codes agree that the profile flattens at small radius, as observed. All codes agree extremely well on the gas pressure profile, which is not surprising, since mechanical equilibrium is easy to achieve for all methods even with limited resolution. This bodes well for the interpretation of SZE observations of clusters, since the Compton y parameter is proportional to the projected pressure distribution. In section 5 we show results from a statistical ensemble of clusters which bear this out.

The bottom row shows the thermodynamic structure of the cluster, as well as the profile of X-ray emissivity. The temperature profiles show a lot of scatter within about one-third the virial radius (≈2.7 Mpc). Systematically, the SPH codes produce nearly isothermal cores, while the grid codes produce temperature profiles which continue to rise as r→0. The origin of this discrepancy has not been resolved, but improved SPH formulations come closer to reproducing the AMR results [51]. This discrepancy is reflected in the entropy profiles. Again, agreement is good in the outer two-thirds of the
cluster, but the profiles show a lot of dispersion in the inner one third. Discounting the codes with inadequate resolution, one finds the SPH codes produce an entropy profile which continues to fall as $r \to 0$, while the grid codes show an entropy core, which is more consistent with observations [29]. The dispersion in the density and temperature profiles are amplified in the X-ray emissivity profile, since $\varepsilon_x \propto \rho_b^2 T^{1/2}$. The different codes agree on the integrated X-ray luminosity of the cluster only to within a factor of 2. This is primarily because the density profile is quite sensitive to resolution in the core; any underestimate in the core density due to inadequate resolution is amplified by the density squared dependence of the emissivity. This suggests that quite high resolution is needed, as well as a good grasp on non-adiabatic processes operating in cluster cores, before simulations will be able to accurately predict X-ray luminosities.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig15.png}
\caption{Effect of physical processes on simulated galaxy cluster ICM observables. Left: Columns show X-ray surface brightness, projected temperature, and Compton y-parameter for a $M = 2 \times 10^{15} M_\odot$ cluster assuming different baryonic physics. Field of view is 5 h$^{-1}$ Mpc. Right: Corresponding spherically averaged radial temperature profiles. From [46].}
\end{figure}

4.5. Effect of additional physics. – Within $r = 0.15 r_{vir}$, Vikhlinin et al. [50] found large variation in temperature profiles, but in all cases the gas is cooler than the cluster mean. This suggests that radiative cooling is important in cluster cores, and possibly other effects as well. It has been long known that $\sim 60$ percent of nearby, luminous X-ray clusters have central X-ray excesses, which has been interpreted as evidence for the presence of a cluster-wide cooling flows [64]. More recently, Ponman et al. [29] have used X-ray observations to deduce the entropy profiles in galaxy groups and clusters. They find an entropy floor in the cores of clusters indicative of extra, non-gravitational heating, which they suggest is feedback from galaxy formation. It is easy to imagine
cooling and heating both may be important to the thermodynamic evolution of ICM gas. To explore the effects of additional physics on the ICM, we recomputed the entire sample of clusters changing the assumed baryonic physics, keeping initial conditions the same. Three additional samples of about 100 clusters each were simulated: The “radiative cooling” sample assumes no additional heating, but gas is allowed to cool due to X-ray line and bremsstrahlung emission in a 0.3 solar metallicity plasma. The “star formation” sample uses the same cooling, but additionally cold gas is turned into collisionless star particles at a rate \( \dot{\rho}_{sf} = \varepsilon_{sf} \rho_{b} \max(\tau_{cool}^{-1}, \tau_{dyn}^{-1}) \), where \( \varepsilon_{sf} \) is the star formation efficiency factor \( \sim 0.1 \), and \( \tau_{cool} \) and \( \tau_{dyn} \) are the local cooling time and freefall time, respectively. This locks up cold baryons in a non-X-ray emitting component, which has been shown to have an important effect of the entropy profile of the remaining hot gas \([56, 57]\). Finally, we have the “star formation feedback” sample, which is similar to the previous sample, except that newly formed stars return a fraction of their rest mass energy as thermal and mechanical energy. The source of this energy is high velocity winds and supernova energy from massive stars. In Enzo, we implement this as thermal heating in every cell forming stars: \( \Gamma_{sf} = \varepsilon_{SN} \dot{\rho}_{SF} c^2 \). The feedback parameter depends on the assumed stellar IMF the explosion energy of individual supernovae. It is estimated to be in the range \( 10^{-6} \leq \varepsilon_{SN} \leq 10^{-5} \) \([44]\). We treat it as a free parameter.

Fig. 15 shows synthetic maps of X-ray surface brightness, temperature, and Compton y-parameter for a \( M = 2 \times 10^{15} M_\odot \) cluster at \( z=0 \) for the three cases indicated. The “star formation” case is omitted because the images are very similar to the “star formation feedback” case (see reference \([46]\).) The adiabatic cluster shows that the X-ray emission is highly concentrated to the cluster core. The projected temperature distribution shows a lot of substructure, which is true for the adiabatic sample as a whole \([47]\). A complex virialization shock is toward the edge of the frame. The y-parameter is smooth, relatively symmetric, and centrally concentrated. The inclusion of radiative cooling has a strong effect on the temperature and X-ray maps, but relatively little effect on the SZE map. The significance of this is discussed in Section 5. In simulations with radiative cooling only, dense gas in merging subclusters cools to \( 10^4 \) K and is brought into the cluster core intact \([48]\). These cold lumps are visible as dark spots in the temperature map. They appear as X-ray bright features. The inclusion of star formation and energy feedback erases these cold lumps, producing maps in all three quantities that resemble slightly smoothed versions of the adiabatic maps. However, an analysis of the radial temperature profiles (Fig. 15) reveal important differences in the cluster core. The temperature continues to rise toward smaller radii in the adiabatic case, while it plummets to \( \sim 10^4 \) K for the radiative cooling case. While the temperature profile looks qualitatively similar to observations of so-called cooling flow clusters, our central temperature is too low and the X-ray brightness too high. The star formation feedback case converts the cool gas into stars, and yields a temperature profile which follows the UTP at \( r \geq 0.15 r_{vir} \), but flattens out at smaller radii. This is consistent with the high resolution Chandra observations of Vikhlinin et al. \([50]\).

5. Recent Progress in Galaxy Cluster Modeling

5.1 Improved Treatment of Galaxy Feedback: GALCONS. – The problem with existing brute force approaches to modeling star formation and feedback within the context of galaxy cluster simulations is that they produce too few galaxies per cluster (unless extraordinarily high resolution is used), the galaxies form stars at too high a rate at late time (i.e., they are blue when they should be red), and the fraction of a cluster’s baryonic
Fig. 16. – Galaxy cluster simulation using Galcons. Galcons, short for galaxy constructs, are programmable subgrid models for galaxies that have finite mass and size, are evolved kinematically as N-bodies, and feed back mass, energy, metals, etc. to the ICM through a resolved spherical boundary at their extremities. Small circles denote the location of 89 galcons, superimposed on a contour map of the baryon density. Large circle denotes the virial radius of the cluster. From [79].

mass in stars is too high [72]. In this section we highlight a new approach to incorporating galaxies and their feedbacks into hydrodynamic cosmological simulations of galaxy clusters which improves the agreement between simulated and observed clusters [79, 80]. The key idea is rather than attempt to simulate the internal processes of galaxies, these are modeled analytically or taken from observations. This is done through the introduction of a galaxy construct (Galcon) – one per galaxy dark matter halo – whose motion through the cluster is simulated as an N-body of finite size and mass, but whose internal processes are programmable. The star formation history of each Galcon is an input function rather than an output. This ensures agreement with observations. Feedbacks of energy and metals is taken to be proportional the the instantaneous star formation rate, again calibrated by observations. Feedback is done in a well-resolved spherical shell at the Galcon’s outer radius rather than from an unresolved point source, which ensures that the energy and metals get out into the IGM. Finally, and perhaps most importantly, a cluster with hundreds if not thousands of galaxies can be simulated economically because the mass and spatial resolution requirement is not so high. One requires only enough resolution to be able to find the galaxy dark matter halos at high redshift which fall into the cluster.

Fig. 16 shows the results of a Galcon simulation described in more detail in [79, 80].
The procedure is as follows. We set up a standard Enzo simulation of a single galaxy cluster including dark matter, gas but no radiative cooling and \textit{in situ} star formation and feedback. In order to achieve the required mass and spatial resolution in the cluster forming region we employ the nested grid+AMR strategy described in Sec. 4. We run this simulation to the “replacement redshift” of \(z=3\), where the output is analyzed for galaxy dark matter halos. For each halo we introduce a Galcon with an analytic mass model for stellar and gas distribution derived from a fit to the simulated baryon distribution. We then restart the simulation and run it to \(z=0\). We assume each Galcon’s star formation history follows the observationally determined cosmic star formation history with an
appropriate weighting reflecting its mass. Mass loss through galactic winds and ram pressure stripping are modeled analytically.

The upper left panel in Fig. 17 compares the SFR density in the Galcon simulation with another Enzo simulation with identical initial conditions but using the star formation and feedback recipe of Cen & Ostriker (1992). Observational data points are also superposed. We see a sharply declining SFR in the Galcon simulation after z=3, while the standard simulation rises and remains quite high to z=0 in conflict with observations. Since feedback by galactic winds is assumed to be proportional to the SFR, this implies that the ICM in the two simulations have very different heating histories. In the standard simulation, the heating is confined to a small number of massive, centrally located galaxies late in time. In the Galcon simulation heating occurs early in an extended region of space by nearly 100 galaxies before the cluster collapses. This difference can be seen by comparing the redshift evolution of the volume of gas being heated by galaxy feedback (Fig. 17, upper right panel).

The different heating histories and spatial distribution has an effect on the spherically averaged gas temperature profiles at z=0, as shown in the lower left panel of Fig. 17. The Galcon simulation produces an isothermal core out to r=0.1r_{vir}, while the standard simulation produces a temperature profile which continues to rise to smaller radii. The former is in better agreement with observations (shaded region), although still a bit high in the core region, while the latter appears to be inconsistent with observations.

The different feedback histories in the two simulations are also reflected in the different metallicity profiles at z=0 shown in lower right panel of Fig. 17. The Galcon simulation produces a flat metallicity profile at $Z = 0.4Z_\odot$ out to a radius of 600 kpc, in good agreement with observations, while the standard simulation shows a sharply declining metallicity gradient. This difference can be understood by looking at the contribution of metals due to galactic winds and ram pressure stripping in the Galcon simulation, shown as two separate curves in Fig. 17. A metallicity floor of about $Z = 0.15Z_\odot$ is contributed by galactic winds driven by early star formation when the galaxies were spatially extended. Ram pressure stripping removes metal enriched ISM gas from the Galcons and deposits it preferentially in the central regions of the cluster at late times where the ICM gas is denser.

This model can be significantly improved upon in several ways. First, Galcons could be introduced dynamically rather than at one time, and starting earlier. This would require running a halo finder inline with the calculation, which we can now do with Enzo [81]. Second, we could import star formation histories more appropriate to cluster galaxies, rather than assume the globally average rate. If one believes that Lyman break galaxies are the precursors of cluster galaxies [109], then their high rates of star formation and strong outflows would provide more heating and hence higher entropies in the cluster core than we have simulated. This could help establish the observed entropy floor that current simulations fail to produce. Finally, we could allow our Galcons to merge using simple kinematic rules, which we presently do not do. We could introduce a burst of star formation in a way which is consistent with observations, and possibly even fuel AGN activity which could provide extra heating.

5.2. AGN Feedback: X-ray Cavity Formation by a Magnetized Jet. – High-resolution X-ray images of galaxy clusters by Chandra have revealed giant cavities and weak shock fronts in the hot gas [82, 83, 84] which are commonly associated with energetic radio lobes [85, 86] and suggest that magnetic fields play an important role. Numerical simulations of hot, underdense bubbles in galaxy clusters have been performed by a number of authors
Fig. 18. – Snapshots of the jet-lobe evolution in a realistic cluster ICM driven by the magnetic energy output of an AGN. Each image is 672 kpc on a side. Columns from left to right are slices through the cluster center of gas density, gas temperature, magnetic energy density, and X-ray emissivity. Rows correspond to evolutionary times after the turn-on of the AGN, which remains in the on state for 36 Myr. As can be seen, well defined X-ray cavities with sharp, smooth boundaries are formed which remain intact for hundreds of Myr as they buoyantly rise through the ICM. From [105].

It is generally possible to inject a large amount of energy into the ICM via AGNs, but it is not exactly clear how the AGN energy can be efficiently utilized [91]. One of the most interesting characteristics of the radio bubbles is that they are intact, whereas most hydrodynamic simulations [95, 89, 96] have shown that purely hydrodynamic bubbles will disintegrate on timescales of much less than $10^8$ yr, markedly different from observations.

Using the first cosmological AMR MHD simulations, Xu et al. (2008)[105] have shown that intact X-ray cavities can be produced with properties similar to those observed [82, 83]. The simulations model the formation of a galaxy cluster within its proper cosmological context with magnetic energy feedback from an active galactic nucleus (AGNs). The X-ray cavities are produced by the magnetically dominated jet-lobe system that is supported by a central axial current. The cavities are magnetically dominated, and their morphology is determined jointly by the magnetic fields and the background cluster pressure profile. The expansion and motion of the cavities are driven initially by the Lorentz
force of the magnetic fields, and the cavities only become buoyant at late stages (1500 Myr). Interestingly, Xu et al. find that up to 80–90% of the injected magnetic energy goes into doing work against the hot cluster medium, heating it, and lifting it in the cluster potential.

The simulation used the newly developed cosmological AMR-MHD module for the Enzo code described in [97]. The procedure is evolve a galaxy cluster from cosmological initial conditions to $z=0.05$. Magnetic feedback of a SMBH in the cluster center is modeled by injecting both poloidal and toroidal magnetic flux in a divergence-free way into the central few kpc for $3 \times 10^7$ yr. The magnetic configuration is not force-free, but rather develops bipolar magnetically dominated “towers” that extend along an axis. The total amount of magnetic energy injected is $6 \times 10^{60}$ erg. After the source switches off, the cluster is evolved for another 650 Myr.

Fig. 18 shows snapshots of the jet-lobe evolution driven by the magnetic energy output of an AGN. Each image is 672 kpc on the side. Columns from left to right are slices of density, temperature, the averaged magnetic energy density, and the integrated X-ray luminosity, respectively. The top row shows the cluster with the jet lobe at the end of magnetic energy injection. The middle and bottom rows show the well-developed bubbles moving out of the cluster center. The bubbles are driven by magnetic forces at all stages and might become buoyant only after $t > 500$ Myr. As can be seen, the X-ray cavities remain intact with sharply defined boundaries until the end of the simulation.

Fig. 19. – Shocks and turbulence from an Enzo AMR simulation of a galaxy cluster using a new AMR refinement criterion based on the velocity field. Left: magnitude of the gas velocity on a slice 7.5 Mpc on a side with a resolution of 18 kpc throughout the cluster. Right: map of Mach number (colors) and turbulent velocity vectors on the slice. The virialization shock and internal shocks are kept very sharp using the new refinement criterion. From [108].

5.3. ICM Turbulence. – It is now appreciated that a non-negligible fraction of a cluster’s binding energy is in the form of fluid turbulence. Major mergers are the most likely source of energy, stirring the ICM on scales approaching the virial radius. Direct
evidence for cluster turbulence would be Doppler broadening of X-ray lines, or the kinetic SZ effect [68]. As yet, these velocity signatures have not been detected due to instrumental limitations. Indirect evidence is the strong magnetic fields observed in cluster cores via Faraday rotation measurements (see discussion below). Here, strong fields would be the result of turbulent amplification of some small seed field.

Cluster turbulence was first studied by Norman & Bryan (1999) [98] using the newly developed ENZO AMR code. They simulated the formation of a rich cluster in a SCDM cosmological model assuming non-radiative gas dynamics. They found turbulent velocities and bulk flows in the core of around 0.25 $\sigma_{\text{vir}}$, increasing to as large as 0.6 $\sigma_{\text{vir}}$ near the virial radius. This translates into a turbulent pressure of about 10% of the total in the core region, increasing to about 30% near the virial radius, although distinguishing between what is an ordered versus a disordered flow becomes problematic near the virial radius. These results were broadly confirmed through the Santa Barbara Cluster code comparison project, in which a number of different numerical methods were applied to evolve a common set of initial conditions leading to the formation of a Coma-like cluster [20].

![Fig. 20. Radial profiles of turbulent kinetic, bulk kinetic, and thermal energies for the same cluster simulated with different resolutions and choices for the AMR refinement criterion. Left: as fractions of the total energy (turbulent+bulk+thermal). Right. Unnormalized profiles. From [108].](image)

Recently Vazza et al. (2009) [108] have implemented a new adaptive mesh refinement criterion into Enzo based on velocity jumps that provides higher resolution within the cluster virial radius and thus a better characterization of the turbulent state of the ICM gas. They found an outer scale for the turbulence of about 300 kpc and a velocity power spectrum consistent with Kolmogorov for incompressible turbulence $E_k \sim k^{-5/3}$. They found that compared to the standard density-based refinement criterion, their clusters have lower central gas densities, flatter entropy profiles, and a higher level of turbulence at all radii. The ratio of turbulent kinetic energy to thermal energy is found to be $\sim 5$ percent within $0.1R_{\text{vir}}$, increasing to $\sim 10 - 20$ percent within $R_{\text{vir}}$. The trend of increasing levels of turbulence with radius is consistent with Norman & Bryan (1999), however the absolute levels are somewhat lower because Vazza et al. did a more careful job separating the velocity field into bulk and turbulent components. They find bulk flows (velocity fields which are ordered on scales of 300 kpc or more) also contribute substantially to the nonthermal energy budget, consistent with the findings of Norman.
Fig. 21. – Enzo AMR MHD cosmological simulation of the magnetization of the ICM of a galaxy cluster due to a single AGN injection event at $z = 3$. Snapshots of projected baryon density (upper rows) and magnetic energy density (bottom rows) for different epochs of the evolving cluster. A major merger at $z = 2$ induces cluster turbulence which spreads and amplifies the magnetic field throughout the cluster by $z = 0$. This model predicts an RMS field strength which is nearly constant with radius out to a Mpc at late times. From [106].

& Bryan (1999).

Fig. 19 shows the velocity field on a slice through the cluster center, and Fig. 20 shows radial profiles of turbulent, bulk kinetic, and thermal energies for the same cluster simulated with different choices for the AMR refinement criterion.

5.4. Origin of Cluster-Wide Magnetic Fields. – There is growing evidence that the intra-cluster medium (ICM) is permeated with magnetic fields, as indicated by the detec-
The simulation described above suggests a way to magnetize the ICM of a galaxy cluster. Magnetic flux deposited into the IGM at high redshift by one or more AGN could provide a seed field for subsequent turbulent amplification and mixing driven by cluster mergers. Norman & Bryan (1999) showed that cluster formation produces cluster-wide turbulence with Mach numbers ranging from 0.1 in the core to 0.3 near the virial radius. This could drive a fast dynamo, amplifying the seed field to observed levels.

To test this hypothesis Xu et al. (2009)[106] carried out a cosmological AMR MHD simulation using the ENZO+MHD code in which an AGN is switched on in a subcluster at a redshift of z=3. Just as in the calculation described above, a magnetically-dominated jet-lobe system is formed by injecting magnetic energy for 36 Myr. Thereafter the AGN is switched off, and the magnetic fields evolve passively subject to the fluid dynamics of cluster assembly.
Fig. 21 shows images of the projected baryon density (upper row) and magnetic energy density (bottom row) for various times in the evolution of the cluster. One can see that magnetic energy, initially deposited in a small volume, is amplified and distributed throughout the cluster by the end of the simulation at z=0.

Fig. 22 shows the evolution of the magnetic field in both time and space. A total of about $2 \times 10^{60}$ erg of magnetic energy is injected over 36 Myr. It then decreases due to expansion to about $1/4$ that value by $z=1.5$. Then it is amplified to about $10^{61}$ erg by turbulence during cluster formation. The lower panel shows how turbulent diffusion spreads the magnetic field throughout the cluster by z=0, with an average value of about a microgauss out to a radius of 1 Mpc.

To see that turbulence is indeed the agent for field amplification and spreading, the right panel of Fig. 22 shows power spectra for gas kinetic energy and magnetic energy. By forcing the AMR code to refine to the maximum level everywhere within the virial radius, we have an effective uniform grid resolution of $600^3$ in the cluster region. We see a kinetic energy spectrum which is consistent with the Kolmogorov self-similar scaling result over the wavenumber range $0.01 \leq k \leq 0.1 \text{ kpc}^{-1}$. Turbulent energy is damped at smaller scales due to numerical dissipation, and back-reaction of the magnetic field which is amplified the most at these scales.

Fig. 23. – Growth of cosmic structure “on the lightcone”, derived from the Hubble Volume Simulation [76].

6. – Statistical Ensembles and Lightcones

With a volume as large as the Hubble Volume simulation, and an adequate number of intermediate snapshots in time, it is possible to portray the growth of cosmic structure “on the lightcone”; that is, just as we observe the real universe. A graphical depiction of that is shown in Fig. 23; here only a thin slice through the narrow 3D lightcone is shown. Here one sees the time evolution of the galaxy cluster population over the redshift interval $0 \leq z \leq 3$. Despite the low spatial and mass resolution of the simulation, it is plain that cluster size objects appear rather late in the evolution of the universe.

At present, it is not possible to simulate volumes this large and at the same time resolve the internal structure of galaxy clusters including the baryonic component. However, by simulating somewhat smaller (but still large) volumes, and exploiting the periodicity of the boundary conditions, it is still possible to generate lightcones for the purposes of synthetic deep redshift cluster surveys. An example of this is shown in Fig. 24 and described in more detail in Hallman et al. (2007)[77]. It shows a 100 square degree projected lightcone image of the Compton y parameter due to the thermal SZE from galaxy clusters over the redshift interval $0.1 \leq z \leq 3$. In order to produce this image, 27 redshift outputs from the AMR hydrodynamic simulation shown in Fig. 1. are stacked at $\Delta z$ intervals of 0.1. The simulation has a dark matter particle mass of $7.3 \times 10^{10} M_{\odot}$ and a maximum spatial resolution of $7.8 h^{-1} \text{comoving kpc.}$ This allows a
fairly complete sample of clusters above $M_{\text{halo}} \approx 4 \times 10^{13} M_\odot$ (Fig. 9). In order to avoid repeating structures, each redshift “shell” of the lightcone is generated by projecting through the cube along a different random axis, and then exploiting periodicity to shift and fill the required solid angle. Using different random seeds for orientation and shift, multiple lightcone realizations can be made from a single simulation in order quantify sampling errors due to cosmic variance.

**Fig. 24.** Left: 100 deg$^2$ projection lightcone image of the Compton y-parameter from a 512$^3$ the Enzo AMR simulation shown in Fig. 1. Right: same as left panel, except removing the contribution of all virialized gas inside clusters of mass $M \geq 5 \times 10^{13} M_\odot$. From [77]

6.1. **Mock SZE Surveys.** With such lightcone datasets, it is possible to mock up the blind SZE surveys that are being carried out by Planck, ACT, and SPT/APEX-SZ [77]. First, an observing frequency is chosen. Then an image similar to that shown in Fig. 24 is generated using the appropriate frequency-dependent SZE signal (see Rephaeli, these proceedings). Second, the image is convolved with the PSF of the relevant instrument. For the instruments listed above, the beam size assumed is 7.1, 1.7, and 1 arcmin, respectively at 144 GHz. Finally, noise is added to the image at the level of the design sensitivities for the instruments, which are assumed to be 6, 2 and 10 $\mu$K per beam, respectively. These images are shown in Fig. 25. The graph at the upper right hand corner of Fig. 25 shows the angular power spectrum computed for the three instruments, as well as the raw image before degrading resolution and sensitivity. The solid lines show the average power spectrum over 200 lightcone realizations, while the dotted lines show the range in which 90% of the power is found. One can see the effects of finite angular resolution by the sharp turnover of the angular power at high multipoles. All instruments recover the “theoretical” power at low multipoles, however. Cosmic variance can change the measured power by factors of 5-8, implying that converged measurements of the real power will require very large areas of the sky to be mapped.

With simulated lightcones, one can ask whether projection effects damage the potential of SZE surveys to constrain cosmological parameters. As discussed in Sec. 3.6 above, counting clusters down to a limiting mass/flux versus redshift is a strong probe of cosmol-
ogy, especially the parameters $\Omega_m$ and $\sigma_8$. This requires calibrating the Y-M relation, where Y is the integrated SZE and M is the cluster mass. Motl et al. (2005)[46] and Nagai (2006)[71] showed using hydrodynamic cosmological simulations that there is a tight correlation between these two quantities that is rather insensitive to assumed baryonic processes (cooling, star formation, feedback, etc.) However, because the SZE signal is redshift independent, when galaxy clusters are observed projected against the sky, foreground objects such as other clusters, hot gas outside the virial radius, and Warm-Hot Intergalactic Medium (WHIM) may contribute to the total signal and spoil the tight
Fig. 26. – Left: Scatter plot of Y-M relation (corrected for cosmological evolution) taken directly from simulation data. Right: same as left panel except taken from lightcone projections which include SZE signal from hot gas along the LOS to the cluster. Tight correlation is severely degraded by projection effects. From [77]

correlation. Hallman et al. (2007) showed that indeed this is the case, as indicated in Fig. 26. On the left is the Y-M relation scatter plot deduced from the simulation data itself, while on the right we see the clusters in projection. The tight correlation intrinsic to individual clusters is destroyed by other contributions to the SZ signal along the LOS, especially for the more numerous low mass clusters. As visual proof of this we show in the right panel of Fig. 24 the projected SZ signal removing the contribution of all cluster gas inside the virial radii of every cluster with $M \geq 5 \times 10^{13}M_\odot$. Roughly 1/3 of the flux of the original image remains after this subtraction is done. A majority of it arises in the hot gas immediately outside the virial radius of clusters which was heated by prior epochs of structure formation, while some comes from the WHIM [107]. This signal is entirely absent in analytic or semi-analytic models of the SZE in which the baryons are painted on a halo population taken from a Press-Schechter analysis or pure dark matter N-body simulation. This highlights the importance of performing self-consistent hydrodynamic simulations and including the effects of foregrounds through detailed lightcone analyses.

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