A preconditioned fast collocation method for a linear bond-based peridynamic model

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Abstract

We develop a fast collocation method for a static bond-based peridynamic model. Based on the analysis of the structure of the stiffness matrix, a fast matrix-vector multiplication technique was found, which can be used in the Krylov subspace iteration method. In this paper, we also present an effective preconditioner to accelerate the convergence of the Krylov subspace iteration method. Using the block-Toeplitz–Toeplitz-block (BTTB)-type structure of the stiffness matrix, we give a block-circulant-circulant-block (BCCB)-type preconditioner. The numerical experiments show the utility of the preconditioned fast collocation method.

Keywords: Nonlocal models; Peridynamic model; Preconditioner; Fast collocation method

1 Introduction

In the last decades, the peridynamic model has been applied in many research fields, such as failure and damage in composite laminates, crack propagation and branching, crack nucleation, phase transformations in solids, impact damage, damage in concrete, and so on [1–7]. In contrast to the classical theory of solid or fluid mechanics, which are usually modeled by partial differential equations [8–10], the peridynamic model does not explicitly involve any spatial derivatives of the displacement. Thus it provides a more natural description for problems with spontaneous formation of discontinuities or other singularities.

To date, several numerical methods have been developed and analyzed for the nonlocal diffusion model and peridynamic model as well as other related nonlocal models, such as finite element discretizations, finite difference method, finite volume method, collocation method, and the meshfree method [1, 11–15]. However, mathematically speaking, because of the nonlocality of these models, the stiffness matrices resulting from finite element method or collocation method are usually full or dense diagonal. The widely used Krylov subspace iteration method requires $O(N^2)$ of memory to store the stiffness matrix and $O(N^2)$ computational work in each iteration to solve the associated linear system where $N$ is the number of spatial nodes. Therefore, the simulation of peridynamic model is usually time-consuming especially for large-scale problems (e.g., multi-dimensional peridynamic models).
Several research works have been devoted to reducing the memory requirement and computations of the corresponding numerical schemes for peridynamic model [15–18]. These works are based on the analysis of Toeplitz or Toeplitz-type structure of the stiffness matrices and on the fast matrix-vector multiplication technique which can be used in the Krylov subspace iteration method. In [18], a fast collocation method for a two-dimensional bond-based linear peridynamic model was developed by carefully exploring the BTTB-type structure of the stiffness matrix $A_{2N}$. However, from the numerical experiments we can find that the convergence is slow, especially when the singularity of the kernel function is stronger.

In this paper, we follow the work started in [18] and develop a preconditioned fast collocation method for a linear peridynamic model. We provide an effective preconditioner to accelerate the convergence of the Krylov subspace iteration method. The rest of the paper is organized as follows. In Sect. 2, we recall the bilinear collocation method developed in [18]. However, the stiffness matrix is rewritten by reordering the unknowns differently from the unknowns’ arrangement in [18]. In Sect. 3, we analyze that each block matrix of the stiffness matrix is a BTTB matrix. Based on this analysis, the operations in each Krylov subspace iteration and the storage of the stiffness matrix can be reduced. In Sect. 4, to accelerate the convergence of the Krylov subspace iteration method, a BCCB-type preconditioner is provided. In Sect. 5, we carry out several numerical experiments to investigate the performance of this preconditioner. In Sect. 6, we draw concluding remarks.

2 The bilinear collocation method

We consider the following two-dimensional bond-based linear peridynamic model:

$$
\int_{B_\delta(x)} C(x', x)(u(x') - u(x)) \, dx' = f(x), \quad x \in \Omega,
$$

(1)

$$
u(x) = g(x), \quad x \in \Omega_c.
$$

Here $C$ is the micromodulus function

$$
C(x', x) = \sigma(|x' - x|) \left[ \frac{(x' - x)}{|x' - x|} \otimes \frac{(x' - x)}{|x' - x|} \right],
$$

(2)

$\Omega := (0, x_R) \times (0, y_R)$ represents a rectangular domain; $x := (x, y)^T$ and $x' := (x', y')^T$ are positions of the particles in the reference configuration; $u(x) := [v(x), w(x)]^T$ and $u(x') := [v(x'), w(x')]^T$ represent the displacements of particles $x$ and $x'$ with respect to the reference configuration, respectively. $\sigma(\cdot)$ is the kernel function; $\delta > 0$ is the horizon of the material; $B_\delta(x)$ is assumed to be an open disk that is centered at $x$ with $\delta$ being the radius; $\Omega_c$ denotes a boundary zone surrounding $\Omega$ with width $\delta$; $f(x) := [f^v(x), f^w(x)]^T$ represents the external force density; $g(x) := [g^v(x), g^w(x)]^T$ is the prescribed nonlocal boundary data imposed on the boundary zone $\Omega_c$. For clarity, we show the domains $\Omega$, $\Omega_c$, and $B_\delta(x)$ in Fig. 1.

Given two $n$-dimensional vectors

$$
V = \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix}, \quad W = \begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{pmatrix},
$$
the tensor product ‘⊗’ in (2) is defined as

$$V \otimes W = \begin{pmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n w_1 & v_n w_2 & \cdots & v_n w_n \end{pmatrix}.$$

For the convenience of the following chapter, we also give the definition of tensor product in the matrix case. Given two matrices $A = (a_{ij}) \in \mathbb{R}^{p \times q}$ and $B \in \mathbb{R}^{r \times s}$, the tensor product of $A$ and $B$ is defined as follows:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{pq}B & a_{p2}B & \cdots & a_{p,q}B \end{pmatrix}.$$

Let $N_x$ and $N_y$ be positive integers. We define a spatial partition on $\bar{\Omega}$ by $x_i := ih_x$ for $i = 0, 1, \ldots, N_x$ and $y_j := jh_y$ for $j = 0, 1, \ldots, N_y$, where $h_x := x_R/N_x$ and $h_y := y_R/N_y$ are the mesh sizes in the $x$ and $y$ directions, respectively. To handle the discretization on the boundary zone $\Omega_c$, we extend the partition to $(x_i, y_j)$ for $i = -K + 1, \ldots, -1, 0, 1, \ldots, N_x, N_x + 1, \ldots, N_x + K - 1$ and $j = -L + 1, \ldots, -1, 0, 1, \ldots, N_y, N_y + 1, \ldots, N_y + L - 1$. Here,

$$K := \lceil \delta/h_x \rceil, \quad L := \lceil \delta/h_y \rceil$$

are the ceilings of $\delta/h_x$ and $\delta/h_y$, respectively.

Let $\psi(\xi) = 1 - |\xi|$ for $\xi \in [-1, 1]$ and zero otherwise. The two-dimensional pyramid functions $\phi_{ij}(x, y)$ centered at $(x_i, y_j)$ can be expressed as

$$\phi_{ij}(x, y) = \psi\left( \frac{x - x_i}{h_x} \right) \psi\left( \frac{y - y_j}{h_y} \right), \quad 0 \leq i \leq N_x, 0 \leq j \leq N_y.$$
Then the trial functions $v$ and $w$ in the displacement vector $\mathbf{u}$ can be written as

$$
v(x, y) = \sum_{i'=-K+1}^{N_x+K-1} \sum_{j'=-L+1}^{N_y+L-1} v_{i',j'} \phi_{i',j'}(x, y),
$$

$$
w(x, y) = \sum_{i'=-K+1}^{N_x+K-1} \sum_{j'=-L+1}^{N_y+L-1} w_{i',j'} \phi_{i',j'}(x, y).
$$

(5)

We choose $(x_i, y_j)$ for $i = 1, \ldots, N_x - 1$ and $j = 1, \ldots, N_y - 1$ as our collocation points. By substituting (5) into (1), we obtain the following collocation scheme:

$$
\int_{B_4(x_i,y_j)} \frac{\sigma((x' - x_i, y' - y_j))}{(x' - x_i)^2 + (y' - y_j)^2} \left( x' - x_i \right)^2 \left[ v_{i,j} - \sum_{i'=-K+1}^{N_x+K-1} \sum_{j'=-L+1}^{N_y+L-1} v_{i',j'} \phi_{i',j'}(x', y') \right]
+ \left( x' - x_i \right) (y' - y_j) \left[ w_{i,j} - \sum_{i'=-K+1}^{N_x+K-1} \sum_{j'=-L+1}^{N_y+L-1} w_{i',j'} \phi_{i',j'}(x', y') \right] \, dx' \, dy' = f^v(x_i, y_j),
$$

$$
\int_{B_4(x_i,y_j)} \frac{\sigma((x' - x_i, y' - y_j))}{(x' - x_i)^2 + (y' - y_j)^2} \left( y' - y_j \right)^2 \left[ w_{i,j} - \sum_{i'=-K+1}^{N_x+K-1} \sum_{j'=-L+1}^{N_y+L-1} w_{i',j'} \phi_{i',j'}(x', y') \right]
+ \left( x' - x_i \right) (y' - y_j) \left[ v_{i,j} - \sum_{i'=-K+1}^{N_x+K-1} \sum_{j'=-L+1}^{N_y+L-1} v_{i',j'} \phi_{i',j'}(x', y') \right] \, dx' \, dy' = f^w(x_i, y_j),
$$

(6)

$$
1 \leq i \leq N_x - 1, 1 \leq j \leq N_y - 1.
$$

Let $N = (N_x - 1)(N_y - 1)$ be the number of unknowns. If we define the vector of unknowns $\mathbf{u}_{2N}$ and the vector of right-hand side as follows:

$$
\mathbf{u}_{2N} := [v_{1,1}, v_{2,1}, \ldots, v_{N_x-1,1}, v_{1,2}, \ldots, v_{N_x-1,2},
\ldots, v_{1,N_y-1}, v_{2,N_y-1}, \ldots, v_{N_x-1,N_y-1},
\ldots, v_{1,N_y-1}, \ldots, v_{1,N_y-1}^T],
$$

$$
\mathbf{f}_{2N} := [f^v_{1,1}, f^v_{2,1}, \ldots, f^v_{N_x-1,1}, f^v_{1,2}, \ldots, f^v_{N_x-1,2},
\ldots, f^v_{1,N_y-1}, \ldots, f^v_{1,N_y-1}^T],
$$

$$
\ldots, f^w_{1,1}, f^w_{2,1}, \ldots, f^w_{N_x-1,1}, f^w_{1,2}, \ldots, f^w_{N_x-1,2},
\ldots, f^w_{1,N_y-1}, \ldots, f^w_{1,N_y-1}^T],
$$

(7)

then the collocation scheme can be written as the following matrix form:

$$
A_{2N} \mathbf{u}_{2N} = \mathbf{f}_{2N},
$$

(8)
where the $2N$-by-$2N$ stiffness matrix $A_{2N}$ can be expressed as the following form in which each matrix block is of $N$ order:

$$A_{2N} = \begin{pmatrix} A_{N}^{vp} & A_{N}^{vw} \\ A_{N}^{wv} & A_{N}^{ww} \end{pmatrix}.$$  \hspace{1cm} (9)

In (9), the entries in the submatrix $A_{N}^{vp}$ are defined as

$$A_{m,n}^{vp} := \int_{B_{2}(x_{i},y_{j})} \frac{\sigma ((|x' - x_{i},y' - y_{j}|))}{(x' - x_{i})^{2} + (y' - y_{j})^{2}} (x' - x_{i})^2 \left( \delta_{m,n} - \phi_{i,j} (x',y') \right) \, dx' \, dy'. \hspace{1cm} (10)$$

Similarly, the entries in the submatrices $A_{N}^{vw}$ and $A_{N}^{ww}$ are given by

$$A_{m,n}^{vw} := \int_{B_{2}(x_{i},y_{j})} \frac{\sigma ((|x' - x_{i},y' - y_{j}|))}{(x' - x_{i})^{2} + (y' - y_{j})^{2}} (y' - y_{j})^2 \left( k_{m,n} - \phi_{i,j} (x',y') \right) \, dx' \, dy', \hspace{1cm} (11)$$

and

$$A_{m,n}^{ww} := \int_{B_{2}(x_{i},y_{j})} \frac{\sigma ((|x' - x_{i},y' - y_{j}|))}{(x' - x_{i})^{2} + (y' - y_{j})^{2}} (y' - y_{j})^2 \left( k_{m,n} - \phi_{i,j} (x',y') \right) \, dx' \, dy', \hspace{1cm} (12)$$

respectively. $A_{N}^{wp} = A_{N}^{vp}$. The function $\phi_{i,j}(x,y)$ denotes the two-dimensional bilinear basis function at the grid point $(x_{i},y_{j})$. The global indices $m$ and $n$ are related to the indices $(i,j)$ and $(i',j')$ by

$$m = (j - 1) \times (N_{x} - 1) + i, \hspace{1cm} 1 \leq i \leq N_{x} - 1, 1 \leq j \leq N_{y} - 1,$$

$$n = (j' - 1) \times (N_{x} - 1) + i', \hspace{1cm} 1 \leq i \leq N_{x} - 1, 1 \leq j' \leq N_{y} - 1.$$  \hspace{1cm} (13)

In (7) $f_{ij}^{v}$ and $f_{ij}^{w}$ are defined as follows:

$$f_{ij}^{v} = f^{v}(x_{i},y_{j}) + \sum_{-K_{1} \leq i' \leq K_{1}} \sum_{-K_{2} \leq j' \leq K_{2}} \int_{B_{2}(x_{i},y_{j})} \frac{\sigma ((|x' - x_{i},y' - y_{j}|))}{(x' - x_{i})^{2} + (y' - y_{j})^{2}} \times (x' - x_{i})^2 g(x_{i},y_{j}) \phi_{i',j'} (x',y') \, dx' \, dy'$$

$$+ \sum_{-K_{1} \leq i' \leq K_{1}} \sum_{-K_{2} \leq j' \leq K_{2}} \int_{B_{2}(x_{i},y_{j})} \frac{\sigma ((|x' - x_{i},y' - y_{j}|))}{(x' - x_{i})^{2} + (y' - y_{j})^{2}} \times (y' - y_{j})^2 g_{y}(x_{i},y_{j}) \phi_{i,j'} (x',y') \, dx' \, dy', \hspace{1cm} (14)$$

$$f_{ij}^{w} = f^{w}(x_{i},y_{j}) + \sum_{-K_{1} \leq i' \leq K_{1}} \sum_{-K_{2} \leq j' \leq K_{2}} \int_{B_{2}(x_{i},y_{j})} \frac{\sigma ((|x' - x_{i},y' - y_{j}|))}{(x' - x_{i})^{2} + (y' - y_{j})^{2}} \times (x' - x_{i}) (y' - y_{j})^2 g(x_{i},y_{j}) \phi_{i,j'} (x',y') \, dx' \, dy'$$

$$+ \sum_{-K_{1} \leq i' \leq K_{1}} \sum_{-K_{2} \leq j' \leq K_{2}} \int_{B_{2}(x_{i},y_{j})} \frac{\sigma ((|x' - x_{i},y' - y_{j}|))}{(x' - x_{i})^{2} + (y' - y_{j})^{2}} \times (y' - y_{j}) (y' - y_{j})^2 g_{y}(x_{i},y_{j}) \phi_{i,j} (x',y') \, dx' \, dy'. \hspace{1cm} (14)$$
3 The structure of stiffness matrix

Theorem 1 Each of the submatrices $A_v^v$, $A^w_v$, and $A^w_w$ has a BTTB structure. More precisely, each matrix can be expressed as a $(N_v-1)$-by-$(N_v-1)$ block-banded Toeplitz matrix with a block bandwidth $2L + 1$,

$$A_N^v = \begin{pmatrix}
    T_0^v & \cdots & T_L^v & 0 & \cdots & 0 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    T_{-L}^v & \cdots & T_{-1}^v & 0 & \cdots & 0 & 0 & \cdots & 0 \\
    0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 & \cdots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \ddots & 0 & \ddots & T_0^v & \cdots & 0 & \cdots & \ddots \\
    \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & 0 & \cdots & T_{-L}^v & \cdots & T_0^v \\
\end{pmatrix}, \quad (15)$$

where the superscript $I = (v,v), (v,w)$, or $(w,w)$. Furthermore, each matrix block $T_j^I$, with $-L \leq j \leq L$, is an $(N_x-1)$-by-$(N_x-1)$ banded Toeplitz matrix with a bandwidth $2K + 1$

$$T_j^I = \begin{pmatrix}
    t_{0,j} & \cdots & t_{K,j} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    t_{-K,j} & \cdots & t_{0,j} & 0 & \cdots & 0 & 0 & \cdots & \ddots \\
    0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 & \cdots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & 0 & \ddots & \ddots & t_{0,j} & \cdots & 0 & \cdots & \ddots \\
    \vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & 0 & \cdots & t_{-K,j} & \cdots & t_{0,j} \\
\end{pmatrix}. \quad (16)$$

Proof We only investigate the structure of $A_N^v$. The analysis of $A_N^w$ and $A_N^w$ is similar. By expanding the matrix $A_N^v$, we can easily find that $A_N^v$ can be written as the following form:

$$A_N^v = \begin{pmatrix}
    B_{1,1} & B_{1,1} & \cdots & B_{1,N_v-1} \\
    B_{2,1} & B_{2,2} & \cdots & B_{2,N_v-1} \\
    \vdots & \ddots & \ddots & \ddots \\
    B_{N_v-1,1} & B_{N_v-1,2} & \cdots & B_{N_v-1,N_v-1} \\
\end{pmatrix}. \quad (17)$$

Here, each block matrix $B_{j,j'}$ of order $N_v - 1$ denotes the interaction of row $j$ and row $j'$ in the discrete system for $1 \leq j \leq N_v - 1$ and $1 \leq j' \leq N_v - 1$. From (11) and (13), we can find
that the entry $A_{m,n}^{v,v} \neq 0$ if and only if

$$\text{supp}(\phi_{i,j'}) \cap B_3(x_i,y_j) \neq \emptyset.$$  \hspace{1cm} (18)

Therefore all the matrix blocks $B_{j,j'}$ with $|j - j'| > L$ in (17) vanish. Then $A_N^{v,v}$ is a $2L + 1$ block-banded matrix and can be expressed as the following form:

$$A_N^{v,v} = \begin{pmatrix}
B_{1,1} & B_{1,2} & \cdots & B_{1,L+1} & 0 & \cdots & 0 \\
B_{2,1} & B_{2,2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
B_{L+1,1} & B_{L+1,2} & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & B_{N_L-L+1,N_L-1} \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & B_{N_L-1,N_L-L-1} & \cdots & B_{N_L-1,N_L-1}
\end{pmatrix}. \hspace{1cm} (19)$$

Furthermore, in each matrix block $B_{i,j'}$, we have $A_{m,n}^{v,v} = 0$ for $|i - i'| > K$. That is, $B_{i,j'}$ is a $2K + 1$ banded matrix expressed as

$$B_{i,j'} = \begin{pmatrix}
\delta_{j,1}^{i,1} & \delta_{j,2}^{i,1} & \cdots & \delta_{j,K}^{i,1} & 0 & \cdots & 0 \\
\delta_{j,1}^{i,2} & \delta_{j,2}^{i,2} & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\delta_{j,K+1}^{i,1} & \delta_{j,K+1}^{i,2} & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & \delta_{j,N_L-K-1}^{i,N_L-K-1} & \cdots & \delta_{j,N_L-K-1}^{i,N_L-1}
\end{pmatrix}. \hspace{1cm} (20)$$

By introducing the following translations:

$$\xi_1 = x_i - x_i', \quad \xi_2 = y_j - y_j', \hspace{1cm} (21)$$

the entries $A_{m,n}^{v,v}$ given in (11) can be reduced to

$$A_{m,n}^{v,v} = \int_{B_3(0,0)} \xi_1^2 \xi_2^2 \left( \xi_1 + \xi_2 \right) \left( \psi \left( \frac{\xi_1 - x_{i' - 1}}{h_x} \right) \psi \left( \frac{\xi_2 - y_{j' - 1}}{h_y} \right) \right) d\xi_1 d\xi_2. \hspace{1cm} (22)$$
Let \( j_1' - j_1 = j_2' - j_2 = l, -L \leq l \leq L \), and let
\[
\begin{align*}
  m_1 &= (j_1 - 1)(N_x - 1) + i, & 1 \leq i \leq N_x - 1, 1 \leq j_1 \leq N_y - 1, \\
  n_1 &= (j_1' - 1)(N_x - 1) + i', & 1 \leq i' \leq N_x - 1, 1 \leq j_1' \leq N_y - 1, \\
  m_2 &= (j_2 - 1)(N_x - 1) + i, & 1 \leq i \leq N_x - 1, 1 \leq j_2 \leq N_y - 1, \\
  n_2 &= (j_2' - 1)(N_x - 1) + i', & 1 \leq i' \leq N_x - 1, 1 \leq j_2' \leq N_y - 1.
\end{align*}
\]

Then we observe that, for \( 1 \leq i, i' \leq N_x - 1 \),
\[
\begin{align*}
  b_{j_1',j_1}^{i,i'} &= A_{m_1,n_1}^{i,i}
  &= \int_{B_4(0,0)} \frac{\xi_1^2 \sigma(\xi_1, \xi_2)}{\xi_1^2 + \xi_2^2} \left( \delta_{m_1,n_1} - \psi \left( \frac{\xi_1 - x_{i'-i}}{h_x} \right) \psi \left( \frac{\xi_2 - y_{j'-j_1}}{h_y} \right) \right) \, d\xi_1 \, d\xi_2 \\
  &= \int_{B_4(0,0)} \frac{\xi_1^2 \sigma(\xi_1, \xi_2)}{\xi_1^2 + \xi_2^2} \left( \delta_{m_2,n_2} - \psi \left( \frac{\xi_1 - x_{i'-i}}{h_x} \right) \psi \left( \frac{\xi_2 - y_{j'-j_2}}{h_y} \right) \right) \, d\xi_1 \, d\xi_2 \\
  &= A_{m_2,n_2}^{i,i} = b_{j_2',j_2}^{i,i'}.
\end{align*}
\]

According to (24), we have proved \( B_{j_1,j_1'} = B_{j_2,j_2'} \) if the block matrices \( B_{j_1,j_1'} \) and \( B_{j_2,j_2'} \) are on the same diagonal.

Let \( i_3' - i_3 = i_4' - i_4 = k, -K \leq k \leq K \), and let
\[
\begin{align*}
  m_3 &= (j - 1)(N_x - 1) + i_3, & 1 \leq i_3 \leq N_x - 1, 1 \leq j \leq N_y - 1, \\
  n_3 &= (j' - 1)(N_x - 1) + i_3', & 1 \leq i_3' \leq N_x - 1, 1 \leq j' \leq N_y - 1, \\
  m_4 &= (j - 1)(N_x - 1) + i_4, & 1 \leq i_4 \leq N_x - 1, 1 \leq j \leq N_y - 1, \\
  n_4 &= (j' - 1)(N_x - 1) + i_4', & 1 \leq i_4' \leq N_x - 1, 1 \leq j' \leq N_y - 1.
\end{align*}
\]

Then we observe that
\[
\begin{align*}
  b_{j_1',j_1}^{i_3,i_3'} &= A_{m_3,n_3}^{i_3,i_3'}
  &= \int_{B_4(0,0)} \frac{\xi_1^2 \sigma(\xi_1, \xi_2)}{\xi_1^2 + \xi_2^2} \left( \delta_{m_3,n_3} - \psi \left( \frac{\xi_1 - x_{i_3'-i_3}}{h_x} \right) \psi \left( \frac{\xi_2 - y_{j_3'-j_3}}{h_y} \right) \right) \, d\xi_1 \, d\xi_2 \\
  &= \int_{B_4(0,0)} \frac{\xi_1^2 \sigma(\xi_1, \xi_2)}{\xi_1^2 + \xi_2^2} \left( \delta_{m_4,n_4} - \psi \left( \frac{\xi_1 - x_{i_4'-i_4}}{h_x} \right) \psi \left( \frac{\xi_2 - y_{j_4'-j_4}}{h_y} \right) \right) \, d\xi_1 \, d\xi_2 \\
  &= A_{m_4,n_4}^{i_4,i_4'} = b_{j_2',j_2}^{i_4,i_4'}.
\end{align*}
\]

According to (26), we conclude that each block matrix \( B_{j,j'} \) is a banded Toeplitz matrix. Combining (24) and (26), we complete the proof.

\[\square\]

**Corollary 1** The stiffness matrix \( A_{2N} \) can be stored in \( O(N) \) memories.

**Proof** From the structure of \( A_{2N} \), we only prove that the block matrix \( A_N^{i,j} \) can be stored in \( O(N) \) memories. From Theorem 1 we can find that the matrix \( A_N^{i,j} \) can be stored only
by storing the following \((2K + 1)\)-by-\((2L + 1)\) entries:

\[
G = \begin{pmatrix}
    t_{v,0} \quad \cdots \quad t_{v,K}
    \\
    \vdots & \ddots & \vdots
    \\
    t_{v,L} \quad \cdots \quad t_{v,0}
\end{pmatrix}
\]  

(27)

Hence, \(A_{2N}^2\) can be stored in \(O(4 \times (2K + 1) \times (2L + 1)) = O(N)\) memories.

\[\square\]

**Corollary 2** For any vector \(u \in \mathbb{R}^{2N}\), the matrix-vector multiplication \(A_{2N}^2 u\) can be carried out in \(O(N \log N)\) operations.

**Proof** We divide the vector \(u\) in half. That is,

\[
u = \begin{pmatrix}
u_1 \\
u_2\end{pmatrix},
\]  

(28)

in which \(u_1, u_1 \in \mathbb{R}^N\). Therefore

\[
A_{2N}^2 u = \begin{pmatrix}
    A_{v,v}^{v,v} & A_{v,v}^{v,w} \\
    A_{w,v}^{v,v} & A_{w,v}^{w,w}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2\end{pmatrix}
= \begin{pmatrix}
    A_{v,v}^{v,v} u_1 + A_{v,w}^{v,w} u_2 \\
    A_{w,v}^{v,v} u_1 + A_{w,w}^{w,w} u_2
\end{pmatrix}.
\]  

(29)

Because of the BTTB structure of the matrices \(A_{v,v}^{v,v}\), \(A_{v,v}^{v,w}\), and \(A_{w,v}^{w,w}\), the matrix-vector multiplications \(A_{v,v}^{v,v} u_1\), \(A_{v,v}^{v,w} u_2\), \(A_{w,v}^{v,v} u_1\), and \(A_{w,w}^{w,w} u_2\) can be computed in \(O(N \log N)\) operations [19]. Accordingly, the total matrix-vector multiplication \(A_{2N}^2 u\) can be evaluated in \(O(4N \log N) = O(N \log N)\) operations.

\[\square\]

**4 A preconditioned fast Krylov subspace method**

In the Krylov subspace iteration method, each iteration consists of matrix-vector multiplications and the related vector operations. Thus a fast Krylov subspace iteration method can be developed due to the fast matrix-vector multiplication proved in the previous section. It can reduce the computational work from \(O(N^2)\) to \(O(N \log N)\) per Krylov subspace iteration. Furthermore, from Corollary 2, the memory requirement can also be reduced from \(O(N^2)\) to \(O(N)\). However, the number of iterations may still be large due to the singularity of kernel function \(\sigma\). Accordingly, we need to introduce an effective preconditioner to reduce the number of iterations and the overall computational cost.

For a BTTB-type matrix \(A_{2N}^2\), a straightforward preconditioner can be chosen as

\[
C_{F,1} = \begin{pmatrix}
    C_{F,1}^{v,v} & C_{F,1}^{v,w} \\
    C_{F,1}^{w,v} & C_{F,1}^{w,w}
\end{pmatrix} \in \mathbb{R}^{2N},
\]  

(30)
where $C_{F,v}^{w,v} = C_{F,v}^{w,w}$ and each matrix $C_{F,v}^{I} \in \mathbb{R}^{N \times N}$, with $I = (v, v), (v, w)$ or $(w, w)$, is written as [19–21]

$$C_{F,v}^{I} = \begin{pmatrix}
C_{0}^{I} & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
C_{L}^{I} & 0 & \cdots & 0 & \cdots & 0 \\
0 & C_{0}^{I} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & C_{0}^{I} & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & 0 & C_{L}^{I} \\
0 & \cdots & 0 & \cdots & 0 & C_{0}^{I} \\
\end{pmatrix}, \tag{31}$$

in which the matrix block $C_{l}^{I}$, with $-L \leq l \leq L$, is the Chan’s circulant matrix of the Toeplitz matrix $T_{l}^{I}$:

$$C_{l}^{I} = \begin{pmatrix}
\tilde{c}_{0,l}^{I} & \tilde{c}_{1,l}^{I} & \cdots & \tilde{c}_{Nx-3,l}^{I} & \tilde{c}_{Nx-2,l}^{I} \\
\tilde{c}_{-1,l}^{I} & \tilde{c}_{0,l}^{I} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\tilde{c}_{-Ny,l}^{I} & \cdots & \tilde{c}_{0,l}^{I} & \cdots & \tilde{c}_{l}^{I} \\
\tilde{c}_{-Ny,l}^{I} & \cdots & \tilde{c}_{-1,l}^{I} & \cdots & \tilde{c}_{0,l}^{I} \\
\end{pmatrix}, \tag{32}$$

In (32), the diagonals of $C_{l}^{I}$ are given by

$$c_{k,l}^{I} = \begin{cases} 
\frac{(N_{x}+1-k)\tilde{c}_{k+1,l}^{I}+k\tilde{c}_{N_{x}-1-k,l}^{I}}{N_{x}-1}, & 0 \leq k \leq N_{x} - 2, \\
\frac{\tilde{c}_{1-N_{x}-k,l}^{I}}{N_{x}-1}, & 2 - N_{x} \leq k < 0.
\end{cases} \tag{33}$$

However, this preconditioner is not easy to invert. So we consider the following preconditioner called BCCB-type preconditioner in this paper:

$$C_{F,2}^{I} = \begin{pmatrix}
C_{F,v}^{w,v} & C_{F,v}^{w,w} \\
C_{F,w}^{v,v} & C_{F,w}^{w,w}
\end{pmatrix}, \tag{34}$$

where for $I = (v, v), (v, w)$ or $(w, w)$, $C_{F,2}^{I}$ represents the BCCB matrix of $A_{I}^{v}$, and it can be generated by $C_{F,1}^{I}$ defined in (31). More precisely, $C_{F,2}^{I}$ can be expressed as

$$C_{F,2}^{I} = \begin{pmatrix}
\tilde{C}_{0}^{I} & \tilde{C}_{1}^{I} & \cdots & \tilde{C}_{N_{x}-3}^{I} & \tilde{C}_{N_{x}-2}^{I} \\
\tilde{C}_{-1}^{I} & \tilde{C}_{0}^{I} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\tilde{C}_{-N_{x}}^{I} & \cdots & \tilde{C}_{0}^{I} & \cdots & \tilde{C}_{l}^{I} \\
\tilde{C}_{-N_{x}}^{I} & \cdots & \tilde{C}_{-1}^{I} & \cdots & \tilde{C}_{0}^{I} \\
\end{pmatrix}, \tag{35}$$
where each circulant matrix block is defined by
\[
\tilde{C}_{I-k} = \begin{cases} 
\frac{(N_y-1-k)C_{I-k} \cdot kC_{N_y-1-k}}{N_y-1}, & 0 \leq k \leq N_y - 2, \\
C_{I_1-N_y-k}^v, & 2-N_y \leq k < 0.
\end{cases}
\] (36)

Let \( F_n \) be the \( n \)-order discrete Fourier transform matrix and \( I_2 \) be the 2-order identity matrix
\[
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \( F_2 = F_{N_y-1} \otimes F_{N_x-1} \) be the two-dimensional discrete Fourier transform matrix and \( F_2^* = (F_{N_y-1} \otimes F_{N_x-1})^* \) be the two-dimensional inverse Fourier transform matrix. Then the matrix \( C_{F,2} \) can be decomposed into the following form:
\[
C_{F,2} = (I_2 \otimes F_2) \Lambda (I_2 \otimes F_2^*)
= (I_2 \otimes F_2) \begin{pmatrix} A^v & A^w \end{pmatrix} (I_2 \otimes F_2^*) \Lambda^v \Lambda^w.
\] (37)

Here \( A^I \) is a diagonal matrix in which the entries on the main diagonal are the eigenvalues of the matrix \( C_{F,2}^I \) for \( I = (v, v), (v, w), \) or \( (w, w) \). That is, \( A^I \) can be written as
\[
A^I = \begin{pmatrix} \lambda^I_{1,1} & 0 & \cdots & 0 \\
0 & \lambda^I_{2,2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda^I_{N,N} \end{pmatrix}.
\] (38)

To invert the matrix \( C_{F,2} \) fast, we consider the computational cost of solving the linear equation
\[
C_{F,2} y = d
\] (39)
for some \( d \in \mathbb{R}^{2N} \).

**Theorem 2** For any vector \( d \in \mathbb{R}^{2N} \), linear system (39) can be solved in \( O(N \log N) \) operations.

**Proof** Recall that the eigenvalues of \( C_{F,2}^I \) can be computed in \( O(N \log N) \) operations by two-dimensional fast Fourier transform (2DFFT). Thus, the computation of \( A \) requires \( O(N \log N) \) operations. From (37), to inverse \( C_{F,2} \) efficiently, we need to compute the inverse of \( A \) efficiently.

We introduce a permutation matrix \( P \) such that
\[
P^* A P = \begin{pmatrix} \hat{A}_{1,1} & 0 & \cdots & 0 \\
0 & \hat{A}_{2,2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{A}_{N,N} \end{pmatrix},
\] (40)
where for \(1 \leq k \leq N\),
\[
\tilde{A}_{k,k} = \begin{pmatrix}
\tilde{\lambda}_{k,k} & \tilde{\lambda}_{k,k}^\ast \\
\tilde{\lambda}_{k,k} & \tilde{\lambda}_{k,k}^\ast
\end{pmatrix}.
\] (41)

Therefore, each matrix block \(\tilde{A}_{k,k}\) with \(1 \leq k \leq N\) is a 2-by-2 matrix. It requires \(O(1)\) operations to inverse \(\tilde{A}_{k,k}\). Accordingly, the overall computational work to inverse the matrix \(\Lambda\) is \(O(N)\) operations. To inverse the matrix \(C_{F,2}\), we also have to compute \((I_2 \otimes F_2^* \mathbf{u})\) for some \(\mathbf{u} \in \mathbb{R}^{2N}\). It requires \(O(N \log N)\) operations by using 2DFFT and two-dimensional inverse fast Fourier transform (2DIFFM).

More precisely, we give the following procedures to solve the linear system (39):

- Evaluate the matrix \(\Lambda\) by 2DFFT. Note that it needs to be computed only once before the startup of the Krylov subspace iterative method. It requires \(O(N \log N)\) operations.
- Divide the vector \(\mathbf{d}\) in half. That is, \(\mathbf{d} = [\mathbf{d}_1, \mathbf{d}_2]^T\), where \(\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^N\). Carry out the 2DFTs \(\mathbf{z}_1 = F_2 \mathbf{d}_1\) and \(\mathbf{z}_2 = F_2 \mathbf{d}_2\). It requires \(O(N \log N)\) operations.
- For \(1 \leq i \leq N\), we solve the matrix equation
\[
\tilde{A}_{k,k} \begin{pmatrix}
\mathbf{w}_1(i) \\
\mathbf{w}_2(i)
\end{pmatrix} = \begin{pmatrix}
\mathbf{z}_1(i) \\
\mathbf{z}_2(i)
\end{pmatrix},
\] (42)

where \(\mathbf{w}_1(i)\) and \(\mathbf{w}_2(i)\) are the \(i\)th elements of the vectors \(\mathbf{w}_1\) and \(\mathbf{w}_2\), respectively. It requires \(O(N)\) operations.
- Carry out the two-dimensional inverse fast Fourier transform (2DIFFM) \(\mathbf{y}_1 = F_2^* \mathbf{w}_1\) and \(\mathbf{y}_2 = F_2^* \mathbf{w}_2\). Then we obtain the solution
\[
\mathbf{y} := [\mathbf{y}_1, \mathbf{y}_2]^T.
\] (43)

It requires \(O(N \log N)\) operations.

5 Numerical experiments

In this section, we carry out some numerical experiments to investigate the performance of the preconditioned fast collocation method with the BCCB-type preconditioner discussed in Sect. 3. In the numerical experiments, we consider the peridynamic model (1) with the kernel function [15]
\[
\sigma \left( \left| (x, y) \right| \right) = \frac{1}{(x^2 + y^2)^{1+s}}.
\] (44)

Let \(\Omega = (0, 1) \times (0, 1)\). The radius of the horizon \(\delta = 1/8\). The exact solution \(u(x, y) = (v(x, y), w(x, y))\) is chosen to be
\[
v(x, y) = w(x, y) = x(1-x)y(1-y), \quad (x, y) \in \Omega.
\]

We also use \(v\) and \(w\) to define \(g^v\) and \(g^w\) on the boundary zone \(\Omega_c\). The right-hand side external forcing term \(f(x, y)\) is computed by polar coordinates transformation as fol-
For simplicity, we choose $h_x = h_y = h$.

We solve linear equation (8) by the conjugate gradient squared method (CGS), the fast conjugate gradient squared method (FCGS) without any preconditioners, and the fast conjugate gradient squared method with BCCB-type preconditioner (BCCB-FCGS). The following numerical experiments are performed on a computer with 8-GB memory. The accuracy requirement for iteration termination is $10^{-8}$.

Example 1 In this numerical example, we choose $s = 3/8$ in (44). We present the $L^2$ errors of the numerical solutions generated by the CGS solver, the FCGS solver, and the BCCB-FCGS solver for the mesh size from $1/2^4$ to $1/2^9$ in Table 1. We also present the CPU time consumed by these solvers and the number of iterations in the iterative methods in Table 1. In addition to the advantages of fast methods (FCGS and BCCB-FCGS) over traditional method (CGS) in memory requirement and CPU time, we observe from Table 1 that the BCCB-type preconditioner can significantly reduce the number of iterations and CPU time in contrast to FCGS without any preconditioners. For example, while $h$ is equal to $2^9$, the FCGS solver takes 455 iterations and nearly 3 hours of CPU time, but the BCCB solver needs only 58 iterations and nearly 8 minutes of CPU time without any loss of accuracy. However, while we carried out the numerical test with $h = 1/2^{10}$, we found a small increase in $L^2$ error. This small increase may be caused by singular integral when we calculate the main-diagonal entries of $A^{vv}$, $A^{vw}$, and $A^{ww}$.

In this numerical experiment, we also use a linear regression to fit the convergence rate $\alpha$ and the associated constant $C_\alpha$ in the error estimate

$$\|u - u_h\|_{L^2(\Omega)} \leq C_\alpha h^\alpha.$$  

We also present these results in Table 1. We see that the convergence rate $\alpha$ is sublinear.

Example 2 We choose $s = 0$ in (44). We present the corresponding numerical results in Table 2 as in Example 1. We have the similar observations as in Example 1. However, in comparison with the results showed in Table 1, an improved accuracy of the numerical solutions and a reduced number of iterations in these three iteration methods are observed in Table 2.

6 Conclusions

Firstly, a fast bilinear collocation method for a static bond-based peridynamic model was developed by reordering the unknowns, which is different from [18], and by carefully analyzing the structure of the stiffness matrix. This work significantly reduces the computational work to solve the resulting linear system from $O(N^2)$ to $O(N \log N)$ per Krylov
Table 1 Performance of CGS, FCGS, and BCCB-FCGS in Example 1

| h   | \(|\mathbf{u}_h - \mathbf{u}\|_{L^2}\)    | # of iter. | CPUs       |
|-----|---------------------------------|-----------|------------|
| CGS | 1/2^4 3.2905 \times 10^{-2}    | 46        | 0.1 s      |
|     | 1/2^5 2.3835 \times 10^{-2}    | 85        | 1.28 s     |
|     | 1/2^6 1.4337 \times 10^{-2}    | 140       | 33.57 s    |
|     | 1/2^7 7.4180 \times 10^{-3}    | 215       | >10 h      |
|     | 1/2^8 out of memory             |           |            |
|     | 1/2^9 out of memory             |           |            |
|     | \(C_\alpha = 0.26, \alpha = 0.60\) |
| FCGS| 1/2^4 3.2905 \times 10^{-2}    | 47        | 0.95 s     |
|     | 1/2^5 2.3835 \times 10^{-2}    | 85        | 4.5 s      |
|     | 1/2^6 1.4337 \times 10^{-2}    | 140       | 22.2 s     |
|     | 1/2^7 7.4180 \times 10^{-3}    | 215       | 2 m 38 s   |
|     | 1/2^8 5.8471 \times 10^{-3}    | 302       | 13 m 50 s  |
|     | 1/2^9 2.4611 \times 10^{-3}    | 455       | 2 h 58 m 13 s |
|     | \(C_\alpha = 0.28, \alpha = 0.73\) |
| BCCB-FCGS| 1/2^4 3.2899 \times 10^{-2}   | 20        | 1.08 s     |
|      | 1/2^5 2.3816 \times 10^{-2}   | 28        | 2.2 s      |
|      | 1/2^6 1.4506 \times 10^{-2}   | 36        | 9.8 s      |
|      | 1/2^7 7.3757 \times 10^{-3}   | 44        | 32.8 s     |
|      | 1/2^8 5.8005 \times 10^{-3}   | 51        | 2 m 3 s    |
|      | 1/2^9 2.4611 \times 10^{-3}   | 58        | 7 m 48 s   |
|      | 1/2^{10} 2.7433 \times 10^{-3} | 76      | 1 h 13 m 16 s |
|      | \(C_\alpha = 0.21, \alpha = 0.66\) |

Table 2 Performance of CGS, FCGS, and BCCB-FCGS in Example 2

| h   | \(\|\mathbf{u}_h - \mathbf{u}\|_{L^2}\)    | # of iter. | CPUs       |
|-----|---------------------------------|-----------|------------|
| CGS | 1/2^4 1.6387 \times 10^{-2}    | 35        | 0.04 s     |
|     | 1/2^5 6.8624 \times 10^{-3}    | 48        | 0.93 s     |
|     | 1/2^6 2.3722 \times 10^{-3}    | 59        | 18.05 s    |
|     | 1/2^7 9.4000 \times 10^{-4}    | 78        | >10 h      |
|     | 1/2^8 out of memory             |           |            |
|     | 1/2^9 out of memory             |           |            |
|     | \(C_\alpha = 0.81, \alpha = 1.39\) |
| FCGS| 1/2^4 1.6387 \times 10^{-2}    | 35        | 0.8 s      |
|     | 1/2^5 6.8624 \times 10^{-3}    | 49        | 3.4 s      |
|     | 1/2^6 2.3722 \times 10^{-3}    | 59        | 15.9 s     |
|     | 1/2^7 9.4000 \times 10^{-4}    | 78        | 1 m 10 s   |
|     | 1/2^8 7.2921 \times 10^{-4}    | 99        | 4 m 11 s   |
|     | 1/2^9 2.2423 \times 10^{-4}    | 113       | 38 m 58 s  |
|     | \(C_\alpha = 0.41, \alpha = 1.20\) |
| BCCB-FCGS| 1/2^4 1.6374 \times 10^{-2}   | 19        | 1 s        |
|      | 1/2^5 6.8154 \times 10^{-3}   | 23        | 2.4 s      |
|      | 1/2^6 2.3057 \times 10^{-3}   | 26        | 8.8 s      |
|      | 1/2^7 8.6494 \times 10^{-4}   | 29        | 22.9 s     |
|      | 1/2^8 6.9507 \times 10^{-4}   | 34        | 1 m 18 s   |
|      | 1/2^9 1.8282 \times 10^{-4}   | 31        | 8 m 24 s   |
|      | 1/2^{10} 8.6238 \times 10^{-4} | 36       | 1 h 2 m 57 s |
|      | \(C_\alpha = 0.13, \alpha = 0.89\) 

subspace iteration. It also reduces storage of the stiffness matrix from \(O(N^2)\) to \(O(N)\). However, the number of iterations in the Krylov subspace iteration method seems to be large especially for the peridynamic model in which the kernel function has stronger singularity. In this paper, using the structure of the stiffness matrix, we construct an efficient BCCB-type preconditioner to improve the performance of the convergence behavior of
the fast collocation method. The numerical experiments show the efficiency of this preconditioner.

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Availability of data and materials
All of the authors declare that all the data can be accessed in our manuscript in the numerical simulation section.

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