Interaction induced directed transport in ac-driven periodic potentials

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Abstract

We demonstrate that repulsive power law interactions can induce deterministic directed transport of particles in dissipative ac-driven periodic potentials, in regimes where the underlying noninteracting system exhibits localized oscillations. Contrasting the well-established single particle ratchet mechanism, this interaction induced transport is based on the collective behaviour of the interacting particles yielding a spatiotemporal nonequilibrium pattern comprising persistent travelling excitations.

1. Introduction

The ratchet effect describes the emergence of a directed particle transport due to breaking of parity and time reversal symmetry [1–5] and nonlinearity in a system where all applied forces and gradients vanish after averaging over time and space [1–3, 6–8]. Originally inspired by the desire of understanding how molecular motors convert chemical energy into directional motion [6], ratchets have now advanced to a widespread and topical paradigm with applications in various branches of nonequilibrium atomic [2, 3, 9], condensed matter [1, 2] and biophysics [1, 10].

Despite often considered as being of single particle character, ratchets can experience a profound impact by interactions [11–20]. Specifically, interactions can accelerate single particle transport [15–19, 21], enhance their controllability [15–19] and lead to one or multiple reversals [21–23] of the transport direction which could be observed experimentally e.g. for vortices in superconducting devices [22]. These setups in particular also allow for interacting ratchets without spatial asymmetry of the ratchet potential [24–26]. In Hamiltonian systems, interaction induced current reversals can even occur dynamically, as a consequence of a continuous conversion, from diffusive particle motion in one direction into ballistic motion in the opposite direction, taking place particle by particle [21].

While most of the above works explored the impact of either linear or short range interactions in a noisy environment, we focus here on a deterministic and dissipative setup [27–29] of power law interacting particles (see also [14]) in a laterally oscillating lattice. This is inspired by the exceptional progress of modern atomic cooling and trapping techniques [30, 31] which have allowed for particularly clean and versatile realizations of ratchets with cold atoms in ac-driven optical lattices [9, 32–36]. Here, Coulomb and dipole–dipole interactions, for example, occur naturally for particles like dipolar atoms or molecules [37, 38], charged colloids [39] or microscopic ions [40].

In the present work, we demonstrate that repulsive power law interactions can induce directed particle transport, in parameter regimes where friction suppresses any single particle transport. The many particle transport is based on a spatiotemporal organization of the dynamics of the individual particles in form of a nonequilibrium pattern comprising a systematic and unidirectional particle hopping process from lattice site to lattice site. Remarkably, this hopping process is made possible by the fact that repulsive interactions among the particles can reduce frictional energy losses compared to the underlying noninteracting system.
2. Setup

We consider $N$ repulsively power law interacting point particles in one dimension with coordinates $x_j$ and mass $m$ that are exposed to a frictional force $F_x = -\gamma \dot{x}$ and a spatially periodic and laterally oscillating lattice potential $V(x, t) = V_0 \cos^2[k(x - f(t))]$ (Figure 1). Here $L = \pi/k$ is the distance between two adjacent minima and $f(t) = a(\cos(\omega_0 t) + \sin(2\omega_0 t))$ is the biharmonic driving law with frequency $\omega_0$ and amplitude $a$, which is frequently used to break the space-time symmetries relevant for ratchets [2, 25, 41].

$$mx_j + \gamma \dot{x}_j + V_0 k \sin \left\{2k \left[ x_j - f(t) \right] \right\} = - \sum_{k=1, k \neq j}^{N} F_{jk} = 0,$$

This model might describe charged ($r = 1$) or dipolar particles ($r = 2$) in an ac-driven (optical) lattice potential including a dissipative frictional force [9] in the weak noise regime of low temperatures. Particularly for the case of dipolar particles in a quasi one-dimensional optical lattice, interactions can be tuned to be purely repulsive by applying a uniform external field which aligns the dipoles. Possible values for $a$ can then range from $\alpha \sim 1$ Debye for ground state diatomic heteronuclear molecules to many thousands of Debye in case of Rydberg atoms or molecules. Experimentally, the importance of the interaction can of course be effectively tuned by changing other parameters such as $\omega_0$, $V_0$ or $\gamma$ that are easier to control.

We focus here on the non-perturbative regime where driving, frictional and interaction forces are comparable. Specifically, we can distinguish (i) the force exerted by the instantaneous static lattice ($F_s := -V_0 k \sin(2kx) \sim V_0 k$). If dominant, this force drags the particles along the minima of the oscillating lattice. (ii) The time dependent pseudo force which acts on a particle in the comoving coordinate system of the lattice ($F_p := -m \ddot{f}(t) - \gamma f(t) \sim (\omega_0)^2 \sqrt{\gamma^2 + (m\omega_0)^2}$). (iii) The frictional force $F_f = -\gamma \dot{x} \sim \gamma \omega_0 L$ slowing down the particles in favour to keep them at fixed positions in the laboratory coordinate system. (iv) The repulsive interaction forces $F_{jk} \sim \alpha/L^{r+1}$. To be in a regime where the driving forces and the nonlinear components of $F_p$ as well as the power law shape of $F_{jk}$ play a significant role, $\omega_0$ is chosen close to the resonance of the underlying linearized single particle problem (a damped driven harmonic oscillator), i.e. $2\omega_0 \sim \sqrt{2Vk^2/m}$. This way, we are far from the comparatively simple regimes of adiabatically slow and fast driving resulting in effectively time independent problems.

In the following we investigate the impact of different interaction strengths $\alpha > 0$ on the transport properties of spatially uniform initial states with $x_j(t = 0) = L(j - N/2 + 1/2) + a (N \text{ even}; j = 0, 1 ... N - 1; a: \text{initial elongation of the lattice})$ and random velocities $v_j(t = 0)$, uniformly chosen in an interval $(-v_0, v_0)$. To avoid capture by specific low-energy asymptotic states (attractors) we choose $v_0$ to be large, such that frictional forces dominate at short time scales.

3. Results

We integrate equation (1) using the Runge–Kutta–Dormand–Prince integration scheme [42, 43] and periodic boundary conditions ($x + LN \rightarrow x$) [44] and calculate the time evolution of the ensemble averaged position

![Figure 1. Schematic illustration of the setup. Upper panel: in the absence of interactions all particles perform independent on site oscillations in the oscillating lattice. Lower panel: repulsive power law interactions can induce a directed particle transport based on collective oscillations of the underlying particles.](image-url)
\[ \langle d(t) \rangle = \langle 1/N \sum_j x_j/t \rangle \]

\( \langle d(t) \rangle \) refers to the time evolution of \( \langle \langle d(t) \rangle \rangle \), where \( \langle \langle . \rangle \rangle \) refers to the averaging over many \( N \)-particle ensembles. For vanishing interactions (\( \alpha = 0 \)) as well as for weak interactions (\( \alpha = 1.0 \)) \( \langle d(t) \rangle \) saturates after a short initial growth at \( t/T \sim 20 \) and we observe that all particles asymptotically perform pure on site oscillations. Increasing the strength of the repulsive interactions, one might expect that \( \langle d(t) \rangle \) saturates even earlier as repulsive interactions restrict the mobility of the particles dynamics even further thereby impeding the emergence of any single particle based transport in the system. However, for \( \alpha = 2; 5 \) (and all values of \( \alpha \) in between) \( \langle d(t) \rangle \) grows linearly, i.e. we observe a permanent directed current of the underlying ions that emerges spontaneously when the repulsive interaction exceeds a certain threshold.

Figure 2, inset (b) shows that apart from a few ensembles that stay close to their initial positions, after 1000 oscillations of the lattice \( d(t) \) is very similar for most ensembles which propagate with similar velocities through the lattice. For stronger interactions (\( \alpha = 20; 175 \)) \( \langle d(t) \rangle \) saturates again, i.e. the directed current dies out after a short initial drift and each of the underlying ions is trapped on its lattice site. Hence, we observe a directed current that is interaction induced but does not survive for strong interactions.

Asymptotically, for long times, the interacting particles move in a collective way. For interaction strength leading to directed transport, their trajectories form a spatiotemporal pattern shown in figure 3. In contrast, for all nontransporting cases (\( \alpha = 0; 1; 20; 175 \)) all particles perform pure on site-oscillations (not shown). The trajectories underlying the pattern perform on-site oscillations (exemplary encircled and marked with an O’) that are interrupted by propagation processes from one lattice site to its right neighbour (P’) and by propagation processes of two lattice sites (PP’). This is reminiscent of a situation where all particles oscillate approximately in phase while an additional travelling collective excitation represents the propagation process leading to directed transport.
A Fourier analysis of the transporting pattern (inset in figure 3) reveals that the pattern is of quasi-periodic character, meaning that the corresponding interaction induced current persists asymptotically. The low frequency part of the Fourier spectrum shows subharmonic frequencies of character, meaning that the corresponding interaction induced current persists asymptotically. The low motion of e.g. only system is a nonlinear, nonequilibrium system breaking both parity- and time-reversal symmetry it fulfills the general criteria for observing a directed current [3]. Indeed, in the single particle regime ($\alpha = 0$ or $N = 1$) a directed current can be observed, as long as $\gamma$ is sufficiently small, i.e. for weak dissipation (figure 4(a); blue curve). This current is represented by one or several coexisting transporting limit cycle attractors (figure 4(b); black) in the underlying single particle phase space ($x, v, t$). Physically such attractors exist as long as the energy transfer from the oscillating lattice to the particles can periodically balance the frictional energy loss, which often happens within a single period of the driving. The slowest possible directed motion which allows for such a balancing is typically a motion of a distance $L$ for a driving period $T$. In synchronization theory this is a 1:1 (or a $k$: $k, k \in \mathbb{N}$) phase locking resonance between the particle motion and the lattice oscillation [45, 46]. A slower motion of e.g. only $L/2$ within $T$ would correspond to a particle motion which alternatingly gains and looses energy to the lattice and does not allow for a balancing of the energy. Note that the existence of such a minimal transport velocity is in contrast to both the Hamiltonian regime ($\gamma = 0$) and the regime of strong noise (high temperature/low particle mass) where the directed transport emerges as an asymmetric (chaotic deterministic or Brownian) diffusion which can generally become arbitrarily slow [1]. In the considered dissipative system, the minimal asymptotic, time-averaged transport velocity of one particle in the lattice that is typically possible is $v_t = \omega_0/(2k) \approx 1.77$, corresponding to the long plateau in (figure 4(a); blue) between $\gamma \approx 2.1$ and $\gamma \approx 7.3$. When $\gamma$ becomes sufficiently large, the transporting attractors (figure 4(b); black) disappear and we have exclusively non transporting attractors (figure 4(b); blue) in the underlying phase space. Values for $\langle \gamma_j \rangle$ that are smaller than $\omega_0/(2k)$ in figure 4(a) are facilitated by non transporting attractors that coexist with transporting ones and attract a certain fraction of the initial ensemble. Larger transport velocities are possible due to higher resonances such as the peak at $\gamma \approx 1.1$ representing particles that move $4L$ within $2T$, i.e. a 4:2 phase locking.

In contrast to the single particle transport that is based on the existence of transporting attractors with a certain minimal velocity, the presence of interactions allows for the emergence of directed currents as a collective phenomenon. Specifically, while for $\alpha = 0$ the Fourier spectrum consists of peaks at $\omega_0$ and its higher
harmonics $2\omega_0, 3\omega_0, \ldots$ (not shown) for $\alpha = 5$ the Fourier spectrum (inset in figure 3) shows subharmonic frequencies representing the possibility of a directed current of the interacting ensemble without requiring that the underlying particles propagate by (at least) one lattice site per driving period. Indeed, in figure 2 the transporting ensembles propagate with $\langle \gamma \rangle = L/(8T) = \omega_0/(16k) < L/T = \omega_0/(2k)$. On the level of the $2N + 1$-dimensional $N$ particle phase space, the corresponding emerging collective current (figure 2) is represented by a transporting attractor (figure 4(b); green). For visualizing this high dimensional attractor we exploit the fact that all $N$ particle trajectories underlying the corresponding transporting pattern (figure 3) are identical except for certain phase shifts, i.e. the single trajectory shown in figure 4(b) is representative for the complete $N$ particle attractor.

Let us further illuminte the microscopic mechanisms underlying the interaction induced transport. The interaction among the particles can, in the course of the dynamics, reduce the loss of kinetic energy due to friction, which in turn of course facilitates the emergence of directed currents. Consider the time evolution of the kinetic energy $E_\alpha (t) := m\dot{x}^2 / 2$ corresponding to the relative motion $\Delta := x_2 - x_1$ of a free frictionic two-particle system $m\ddot{x}_{1,2} + \gamma \dot{x}_{1,2} + x_{1,2} = \alpha (x_{1,2} - x_1^2) = 0$. While the frictional energy loss concerning the centre of mass dynamics $E_c (t) := m\ddot{x}^2 / 2$ with $x = (x_1 + x_2) / 2$ is of course independent of the value of $\alpha$, $E_\alpha (t)$ initially decreases faster for $\alpha > 0$ as compared to $\alpha = 0$ (see figure 4(d)) but then grows again and becomes larger than for $\alpha = 0$ (where $E_\alpha (t) \propto \exp [-2\gamma t / m]$). The zero in $E_\alpha (t)$ (see blue curves in figure 4(d)) hereby indicates the close encounter i.e. collision of two particles. Accordingly, a particle travelling from one lattice site to an adjacent one, can reach the latter with a higher kinetic energy when encountering a collision on its path. Physically this collision induced reduction of frictional energy losses is based on the fact that upon a collision kinetic energy is converted into interaction energy which is not subject to frictional losses. In the case of the complete $N$ particle system in the oscillating lattice, this mechanism is still present and leads to a substantial increase of the asymptotic kinetic energies (figure 4(c)) for $\alpha = 2; 5$ (transport) as compared to the energies in the noninteracting $\alpha = 0$ and nontransporting $\alpha = 1; 20; 175$ cases. In these nontransporting cases the particles asymptotically exhibit almost equidistant on site oscillations.

We now discuss why the interaction induced directed transport is present only in some regime of the interaction strength (e.g. for $\alpha = 2; 5$) but vanishes for comparatively large values of $\alpha$ ($\alpha = 20; 175$). Clearly, in the limit $\alpha \to \infty$ of a stiff ion chain where all $N$ particles exhibit an identical dynamics, the $N$ particle transport breaks down at the same friction strength $\gamma$ as in the single particle case ($N = 1$ or $\alpha = 0$). Using linear stability analysis, we show that the $N$ particle dynamics reduces to that of a stiff ion chain already for finite values of $\alpha$, no matter how strong the driving is. Consider a two particle system (equation (1)) for $N = 2; r = 1$) with periodic boundary conditions and only next neighbour interactions. Transforming $x_i \to \tilde{x}_i := 2k_i (x_i - f (t))$ and then to centre of mass $x := (\tilde{x}_1 + \tilde{x}_2) / 2$ and relative $\Delta := \tilde{x}_2 - \tilde{x}_1$ coordinates yields after inserting $\Delta = 2\pi + \delta$ ($2\pi$ is the equilibrum distance) and truncating in linear order $\delta$:

$$0 = m\ddot{x} + \gamma \dot{x} - 2V_0 k^2 \sin (x) + F (t),$$

$$0 = m\ddot{\delta} + \gamma \dot{\delta} + 2V_0 k^2 \delta \cos (2x) + 8k^2 \alpha \delta / \pi^4,$$

$$F (t) = 2k \begin{bmatrix} m \dot{f} (t) + \gamma \dot{f} (t) \end{bmatrix}.$$  \(2\)

As $\cos (2x) \in (-1, 1)$ it follows from the linear equation (3) that for $\alpha > V_0 \pi / (4k)$ the asymptotic state for $\delta (t)$ is uniquely determined by a fixed point attractor at $\delta = 0$ with a global basin of attraction. Then, the state where the two considered ions exhibit an identical phase locked dynamics is stable (for arbitrary values of $\alpha$, $\omega_0$ and $\gamma$) and any initial state sufficiently close to it asymptotically approaches it. This result can be generalized to the $N$ particle case, yielding $\alpha > KV_0 \pi / (Qk)$ with $Q = 4; 3; 2; (5 - \sqrt{5}) / 2; 1$ for $N = 2, 3, \ldots$ and $Q > 0$ for $N < \infty$. In fact, for $\alpha > KV_0 \pi / (Qk)$ we numerically observe that the $N$ particle ensemble discussed in the context of figure 2 generically comes close to a translationally invariant state (after redistributing the nonuniform initial kinetic energies) and we observe a breakdown of the $N$ particle current at the same value of $\alpha$ as in the single particle case. The uniform (or phase locked) $N$ particle dynamics is then described by the equation of motion of its centre of mass coordinate $x := (1/N) \sum x_i$ (equation (2)), representing a biharmonic version of an ac-driven damped physical pendulum which is known to exhibit a period doubling route to chaos [47]. Note that the criterion $\alpha > KV_0 \pi / (Qk)$ represents only an upper limit for the stability of the spatially uniform dynamics; in practice stability can of course occur already for smaller values of $\alpha$.

5. Irregular transport

We finally demonstrate that interaction induced directed currents are even possible in a regime where the dynamics of the underlying ions is irregular (figure 5, inset (a)). Here, a current emerges already for comparatively weak interactions ($\alpha = 0.25$) and is accompanied, as before, by a significant increase of the mean kinetic energy per particle compared to the noninteracting ($\alpha = 0$) and the other nontransporting cases.
doorway towards interaction based transport phenomena in time dependent lattices. or alternatively with trapped ions additionally exposed to driven optical lattices [40]. Our results might open a
doorway towards interaction based transport phenomena in time dependent lattices.

6. Conclusions

We demonstrate here, that collective effects in repulsively power-law-interacting systems can induce directed particle currents even in parameter regimes where corresponding noninteracting systems do not exhibit transport. These interaction induced currents could be detected e.g. for cold thermal clouds of dipolar atoms and molecules [37, 38] and interacting colloids [39, 48] in optical lattices driven via standard techniques [38, 49] or alternatively with trapped ions additionally exposed to driven optical lattices [40]. Our results might open a
doorway towards interaction based transport phenomena in time dependent lattices.

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Figure 5. Time evolution $\langle d \rangle (t)$ for $N = 6$, averaged over $10^3$ ensembles for $\alpha = 0.25$; 2; 5 (three upper curves in green, blue and black, lying almost on top of each other); $\alpha = 20$ (cyan, second lowest curve); $\alpha = 175$ (purple, lowest curve) and $\alpha = 0$ (covered by the purple curve). Inset (a): like figure 3. Inset (b): like figure 4(c). Upper dots: $\alpha = 0.25$ (covered by the blue dot), $\alpha = 20$ (blue) and $\alpha = 175$ (black). Lower dots: $\alpha = 20$ (cyan) and $\alpha = 175$ (purple, covering the $\alpha = 0$ dot and partly also the cyan one) Parameters: $\omega_0 = 4.0; \alpha = m = 1.0, \gamma = 7.0; V = 25; \nu \in (−250, 250)$. (figure 5, inset (b)). The corresponding Fourier spectrum of the velocities of the ions (not shown) exhibits a continuous distribution. This provides the particle system with an enhanced flexibility to
dynamically transfer energy between the individual particles. The emerging current is larger than that of the
transporting regular excitation pattern discussed (\langle \langle d \rangle \rangle \sim 0.2\omega_0 / k) but still smaller than any single particle
transport corresponding to a 1:1 resonance \langle \langle d \rangle \rangle = \omega_0 / (2k).

Finally we note, that our results are robust with respect to weak (Gaussian) noise, and do survive when
truncating the interactions beyond next neighbour ones. Interaction induced directed transport does not
depend on details of the driving law such as the phase between the two harmonics.

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