Consecutive Sequential Probability Ratio Tests of Multiple Statistical Hypotheses *

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Abstract

In this paper, we develop a simple approach for testing multiple statistical hypotheses based on the observations of a number of probability ratios enumerated consecutively with respect to the index of hypotheses. Explicit and tight bounds for the probability of making wrong decisions are obtained for choosing appropriate parameters for the proposed tests. In the special case of testing two hypotheses, our tests reduce to Wald’s sequential probability ratio tests.

1 Introduction

Consider a continuous-time stochastic process \((X_t)_{t \in [0, \infty)}\) defined in a probability space \((\Omega, \mathcal{F}, P)\). Suppose that the stochastic process \((X_t)_{t \in [0, \infty)}\) is parameterized by \(\theta \in \Theta\). In many applications of engineering and sciences, it is desirable to infer the true value of \(\theta\) based on the observation of such stochastic processes. This topic can be formulated as a general problem of testing \(m\) mutually exclusive and exhaustive composite hypotheses:

\[ H_0 : \theta \in \Theta_0, \quad H_1 : \theta \in \Theta_1, \quad \ldots, \quad H_{m-1} : \theta \in \Theta_{m-1}, \]

where \(\Theta_i = \{\theta \in \Theta : \theta_i < \theta \leq \theta_{i+1}\}, \quad i = 0, 1, \cdots, m - 1\) with \(-\infty = \theta_0 < \theta_1 < \cdots < \theta_{m-1} < \theta_m = \infty\). To control the probabilities of making wrong decisions, for pre-specified numbers \(\delta_i \in (0, 1), \quad i = 0, 1, \cdots, m-1\), it is typically required that

\[ \Pr\{\text{Reject } H_i \mid \theta\} \leq \delta_i, \quad \forall \theta \in \Theta_i, \quad i = 0, 1, \cdots, m-1 \]

where \(\Theta_i = \{\theta \in \Theta : \theta_i' < \theta \leq \theta_{i+1}'\}, \quad i = 0, 1, \cdots, m - 1\) with \(\theta_i', \theta_{i}' \in \Theta, \quad i = 1, \cdots, m - 1\) satisfying \(-\infty = \theta_0' < \theta_1' < \theta_i < \theta_{i}' \leq \theta_{i+1}' < \theta_{i+1} < \theta_{i}' = \infty\) for \(i = 1, \cdots, m - 2\). The set \(\bigcup_{i=1}^{m-1} (\theta_i', \theta_i')\) is referred to as the indifference zone, since no specification on risk is imposed for the set. Here we consider continuous-time processes for the sake of generality, since discrete-time stochastic processes can be treated as right-continuous processes in continuous time.

The hypothesis testing problem defined by \((1)\) and \((2)\) has been studied extensively for more than a half century (see, [8, 9] and the references therein). In particular, for the special problem of testing two hypotheses, Wald [12] invented the famous Sequential Probability Ratio Tests (SPRTs). Armitage

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extended Wald’s SPRTs to the general problem of testing multiple hypotheses. Lorden \[10\] proposed sequential likelihood ratio tests for the same problem. Baum \[2\] established multiple sequential probability ratio tests in a Bayesian framework. At present the general theory of tests on multiple statistical hypotheses is much less developed than for the two-decision situation. Existing methods suffer from one or more of the following drawbacks: (i) There is no rigorous method for controlling the risk of making wrong decisions; (ii) The method of bounding the risk of making wrong decisions is too conservative; (iii) The application is limited to simple hypotheses; (iv) The application is limited by the number of hypotheses. Motivated by this situation, we develop a new class of tests, referred to as Consecutive Sequential Probability Ratio Tests (CSPRTs) based on the principle of probabilistic comparison proposed in \[3\] \[4\] \[7\].

The remainder of this paper is organized as follows. In Section \[2\] we introduce the connection between multi-hypotheses testing and sequential random intervals. In Section \[3\] we describe the principle of probabilistic comparison. In Section \[4\] we apply the principle of probabilistic comparison to develop consecutive sequential probability ratio tests. In Section \[5\] we establish consecutive sequential probability ratio tests on parameters of continuous-time processes. Section \[6\] is the conclusion. All proofs are given in Appendices. The main results of this paper have been appeared in our conference paper \[7\].

Throughout this paper, we shall use the following notations. The empty set is denoted by \(\emptyset\). The set of positive integers is denoted by \(\mathbb{N}\). The notation \(\Pr\{E \mid \theta\}\) denotes the probability of the event \(E\) associated with parameter \(\theta\). The expectation of a random variable is denoted by \(\mathbb{E}[\cdot]\). The support of a random variable \(Z\) is denoted by \(I_Z\). In the discrete-time case, the stochastic process \((X_t)_{t \in [0, \infty)}\) is actually a sequence of random variables \(X_1, X_2, \ldots\). For simplicity of notations, let \(X_n = (X_1, \ldots, X_n)\) for \(n \in \mathbb{N}\). Let \(x_n = (x_1, \ldots, x_n)\) denote the realization of \(X_n\). Let \(f_n(x_n; \theta)\) denote the probability density function (PDF) or probability mass function (PMF) of \((X_1, \ldots, X_n)\) parameterized by \(\theta \in \Theta\). Accordingly, replacing \(x_n\) in \(f_n(x_n; \theta)\) by \(X_n\) gives the likelihood function \(f_n(X_n; \theta)\). For \(\theta', \theta'' \in \Theta\) and \(\kappa > 0\), we use \(\Upsilon_n(X_n; \theta', \theta'') \sim \kappa\) to represent \(f_n(X_n; \theta'') \sim \kappa f_n(X_n; \theta')\), where \(\sim\) is a relation such as \(<, =, >, \leq, \geq\), corresponding to “less than, equal, greater than, less or equal, greater or equal”, respectively. The notation \(\Upsilon_n(X_n; \theta', \theta'')\) can be interpreted as the likelihood ratio \(\frac{f_n(X_n; \theta')}{f_n(X_n; \theta'')}\) whenever \(f_n(X_n; \theta')\) is not equal to \(0\). We shall frequently use the concept of unimodal function. A function is said to be unimodal with respect to \(\theta \in \Theta\) if there exists a number \(\theta^*\) such that the function is non-decreasing with respect to \(\theta \in \Theta\) no greater than \(\theta^*\) and is non-increasing with respect to \(\theta \in \Theta\) no less than \(\theta^*\). The other notations and concepts will be made clear as we proceed.

## 2 Multi-hypotheses Testing and Sequential Random Intervals

As demonstrated in \[3\], the general hypothesis testing problem defined by \[1\] and \[2\] can be cast into the framework of constructing a sequential random interval with pre-specified coverage probabilities. This can be illustrated in the sequel.

To reach a fast decision, it is desirable to solve the hypothesis testing problem by a multistage approach such that the sampling procedure is divided into \(s\) stages with observational times \(t_\ell, \ell = 1, \ldots, s\), where \(t_\ell\) is the observational time at the \(\ell\)-th stage. Starting from \(\ell = 1\), at the \(\ell\)-th stage, based on the observation of \((X_t)_{0 \leq t \leq t_\ell}\), pre-determined stopping and decision rules are applied to check whether the accumulated observational data is sufficient to accept a hypothesis and terminate the sampling procedure. If the observational data is considered to be insufficient for making a decision, then proceed to the next stage of observation. The observation is continued stage by stage until a hypothesis is accepted at some stage. Although the number of stages \(s\) may be infinity, for practical considerations, the stopping and decision rules are required to guarantee that the sampling procedure will surely eventually terminate with a finite number of stages. Central to a multistage procedure are the stopping and decision rules, which
can be related to a sequential random interval described as follows. Let \( \theta_0' = -\infty \) and \( \theta_m'' = \infty \). For \( i = 0, 1, \ldots, m - 1 \), let \( \mathcal{J}_i \) denote the open interval \( (\theta_i', \theta_i'' \cdot 1) \). Let \( l \) be the index of stage at the termination of the sampling procedure. Let \( \mathcal{L} \) and \( \mathcal{U} \) be random variables defined in terms of samples of the stochastic process up to the \( l \)-th stage such that the sequential random interval \( \mathcal{L}, \mathcal{U} \) has \( m \) possible outcomes \( \mathcal{J}_i \), \( i = 0, 1, \ldots, m - 1 \) and that \( \Pr\{ \mathcal{L} < \theta < \mathcal{U} \mid \theta \} > 1 - \delta_i \) for any \( \theta \in \Theta_i \) and \( i = 0, 1, \ldots, m - 1 \). Given that the sequential random interval \( \mathcal{L}, \mathcal{U} \) satisfying such requirements is constructed, the risk requirement \((2)\) can be satisfied by using \( \mathcal{L}, \mathcal{U} \) to define a decision rule such that, for \( i = 0, 1, \ldots, m - 1 \), hypothesis \( \mathcal{H}_i \) is accepted when the sequential random interval \( \mathcal{L}, \mathcal{U} \) takes \( \mathcal{J}_i \) as its outcome at the termination of the sampling process. It follows that \( \{ \text{Accept } \mathcal{H}_i \} = \{ \mathcal{L} < \theta < \mathcal{U} \} \) for any \( \theta \in \Theta_i \) and \( i = 0, 1, \ldots, m - 1 \). Therefore, to solve the multi-valued decision problem defined by \((1)\) and \((2)\), the objective is to ensure that \( \theta \) is included in the sequential random interval with pre-specified probabilities. In the sequel, we shall propose a general approach for defining stopping and decision rules for the construction of such sequential random interval.

3 Principle of Probabilistic Comparison

In \cite{3,5,7}, a general methodology has been proposed for constructing sequential random intervals with prescribed specifications of coverage probabilities. The main idea is to use one-sided confidence sequences to control the coverage probability of the sequential random interval. Assume that the number of stages, \( s \), and the observational times, \( t_\ell, \ell = 1, \ldots, s \), are given. Assume that for \( \ell = 1, \ldots, s \) and \( i = 1, \ldots, m - 1 \), random variables \( L_{\ell,i} \) and \( U_{\ell,i} \) can be defined in terms of positive numbers \( \zeta, \alpha, \beta \) and the set of random variables \( \{ X_t \}_{0 \leq t \leq t_\ell} \) such that \( \Pr\{ L_{\ell,i} \geq \theta \mid \theta \} \) and \( \Pr\{ U_{\ell,i} \leq \theta \mid \theta \} \) can be made arbitrarily small by decreasing \( \zeta \alpha_i \) and \( \zeta \beta_i \) respectively. Due to such assumption, we call \( (-\infty, L_{\ell,i}] \) and \( [U_{\ell,i}, \infty) \) one-sided confidence intervals for \( \theta \). Accordingly, \( (-\infty, L_{\ell,i}], \ell = 1, \ldots, s \) and \( [U_{\ell,i}, \infty), \ell = 1, \ldots, s \) are said to be one-sided confidence sequences for \( \theta \). In view of the controllability of the coverage probabilities of the one-sided confidence intervals, the number \( \zeta \) is referred to as the coverage tuning parameter, and \( \alpha_i, \beta_i, i = 1, \ldots, m - 1 \) are called weighting coefficients. Given that \( \zeta \) is sufficiently small, \( \theta > \theta_i' \) will be credible if \( L_{\ell,i} > \theta_i' \) is observed. Similarly, \( \theta < \theta_i'' \) will be credible if \( U_{\ell,i} < \theta_i'' \) is observed. To figure out the general structure of stopping and decision rules, imagine that the sampling procedure is stopped at the \( \ell \)-th stage and \( \mathcal{J}_i \) is to be designated as the outcome of the sequential random interval. Since \( \mathcal{J}_i \) contains \( [\theta_i', \theta_{i+1}' \cdot 1] \), it follows that for \( \theta \in [\theta_i', \theta_{i+1}' \cdot 1] \), it is true that \( \theta < \theta_j'' \) for \( j > i \) and \( \theta > \theta_j' \) for \( j \leq i \). This implies that, if the coverage tuning parameter \( \zeta \) is sufficiently small, then it is very likely to observe that \( U_{\ell,j} < \theta_j'' \) for \( j > i \) and \( L_{\ell,j} > \theta_j' \) for \( j \leq i \). Therefore, turning this thinking around leads to the following stopping and decision rules:

Continue observing the stochastic processes until for some \( i \in \{0,1,\ldots,m-1\} \), the event \( \{ U_{\ell,j} < \theta_j' \} \) for \( j > i \) and \( L_{\ell,j} > \theta_j' \) for \( j \leq i \) occurs at some stage with index \( \ell \in \{1,\ldots,s\} \).

At the termination of the sampling process, make the following decision: If such index \( i \) is unique, then designate \( \mathcal{J}_i \) as the outcome of the sequential random interval. If there are multiple indexes satisfying the condition, then pick one of them and assign the corresponding interval \( \mathcal{J}_i \) as the outcome of the sequential random interval based on a predetermined policy.

The idea in the derivation of the above stopping and decision rules is to infer the location of \( \theta \) relative to the sequential random interval by comparing the confidence limits with the endpoints of the sequential random interval. Due to the probabilistic nature of the comparison, such method of constructing stopping and decision rules is referred to as the Principle of Probabilistic Comparison. It should be noted that similar principles have been proposed in \cite{4,6} for multistage estimation of parameters. The properties of the above stopping and decision rules are indicated by the following probabilistic result.
Theorem 1 Let \( a_0 = b_0 = -\infty \), \( a_m = b_m = \infty \) and \( a_i < b_i \leq a_{i+1} < b_{i+1} \) for \( i = 1, \ldots, m-2 \). Let \( \Theta_0 = (-\infty, a_1) \), \( \Theta_{m-1} = [b_{m-1}, \infty) \) and \( \Theta_i = [b_i, a_{i+1}) \) for \( i = 1, \ldots, m-2 \). Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \Pr) \) be a filtered space. Let \( \tau \) be a proper stopping time with a support \( I_\tau \). For all \( \ell \in I_\tau \), let \( L_{\ell,m} = -\infty \), \( U_{\ell,0} = \infty \) and let \( L_{\ell,i}, U_{\ell,i}, i = 1, \ldots, m-1 \) be random variables measurable in \( \mathcal{F}_\ell \). Let \( \mathcal{L} \) and \( \mathcal{U} \) be random variables such that \( \bigcup_{i=0}^{m-1} \{ \mathcal{L} = a_i, \mathcal{U} = b_{i+1}\} = \Omega \) and that \( \{\tau = \ell, \mathcal{L} = a_j, \mathcal{U} = b_{j+1}\} \subseteq \{L_{\ell,i} \leq a_i, 0 < i \leq j \text{ and } U_{\ell,i} \leq b_j, j < i < m\} \) for all \( \ell \in I_\tau \) and \( j = 0, 1, \ldots, m-1 \). Then, \( \Pr[\mathcal{L} \geq \theta] = \Pr[\mathcal{L} \geq a_{i+1}] \leq \Pr[L_{\ell,i+1} \geq a_{i+1}] \) for some \( \ell \in I_\tau \) and \( \Pr[\mathcal{U} \leq \theta] = \Pr[\mathcal{U} \leq b_i] \leq \Pr[U_{\ell,i} \leq b_i \text{ for some } \ell \in I_\tau \text{ for } i = 0, 1, \ldots, m-1 \text{ and } \theta \in \Theta_i \).

See Appendix A for a proof.

4 Consecutive Sequential Probability Ratio Tests

In this section, we shall apply the principle of probabilistic comparison and Theorem 1 to develop a new class of tests for solving the multi-valued decision problem defined by \( (1) \) and \( (2) \) regarding the parameter \( \theta \in \Theta \) associated with a discrete process \( (X_n)_{n \in \mathcal{N}} \), where \( \mathcal{N} \) is a subset of positive integers. For generality, we do not restrict \( \mathcal{N} \) as an unbounded set such as \( \mathbb{N} \). Our purpose is to accommodate the situation that the sequence of \( X_n \) can be of finite length. A familiar example can be found in the context of sampling without replacement from a finite population of \( N \) units, among which \( Np \) units having a certain attribute. If we define a Bernoulli random variable \( X_n \) such that \( X_n \) assumes values 1 or 0 in accordance with whether the \( n \)-th drawn unit has the attribute, then we have a sequence of dependent Bernoulli random variables \( (X_n)_{n \in \mathcal{N}} \) with \( \mathcal{N} = \{1, 2, \ldots, N\} \). Throughout the remainder of this paper, we use symbol \( N^* \) to denote \( \infty \) if \( \mathcal{N} \) is unbounded and otherwise the maximum of \( \mathcal{N} \).

4.1 Confidence Sequences

For the purpose of deriving sequential tests based on the principle of probabilistic comparison, we need a method for constructing confidence sequences as described by the following theorem.

Theorem 2 For \( n \in \mathcal{N} \), let \( \mathcal{X}_n \) be random variables parameterized by \( \theta \in \Theta \) and let the likelihood function be denoted by \( f_n(\mathcal{X}_n; \theta) \). Let \( \delta \in (0,1) \) and let \( \theta_0, \theta_1 \in \Theta \) with \( \theta_0 < \theta_1 \). Define random variables \( L_n(\mathcal{X}_n) = \inf\{\theta \in \Theta : T_n(\mathcal{X}_n; \theta_1, \theta) > \frac{\delta}{2}\} \) and \( U_n(\mathcal{X}_n) = \sup\{\theta \in \Theta : T_n(\mathcal{X}_n; \theta_0, \theta) > \frac{\delta}{2}\} \). The following statements hold true.

(I) For all \( \theta \in \Theta \),
\[
\Pr[L_n(\mathcal{X}_n) \leq \theta \mid \mathcal{N} \} \geq 1 - \frac{\delta}{2}, \\
\Pr[U_n(\mathcal{X}_n) \leq \theta \mid \mathcal{N} \} \geq 1 - \frac{\delta}{2}, \\
\Pr[L_n(\mathcal{X}_n) \geq \theta \mid \mathcal{N} \} \geq 1 - \frac{\delta}{2}.
\]

(II) For all \( n \in \mathcal{N} \),
\[
\{L_n(\mathcal{X}_n) > \theta_0\} \subseteq \left\{ T_n(\mathcal{X}_n; \theta_1, \theta_0) \leq \frac{\delta}{2} \right\}, \\
\{U_n(\mathcal{X}_n) < \theta_1\} \subseteq \left\{ T_n(\mathcal{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2} \right\}.
\]

(III) If \( f_n(\mathcal{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \), then
\[
\{L_n(\mathcal{X}_n) \geq \theta_0\} \supseteq \left\{ T_n(\mathcal{X}_n; \theta_1, \theta_0) \leq \frac{\delta}{2} \right\}, \\
\{U_n(\mathcal{X}_n) \leq \theta_1\} \supseteq \left\{ T_n(\mathcal{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2} \right\}
\]
for all \( n \in \mathcal{N} \).
See Appendix \[\text{I}\] for a proof.
Assuming that \(f_n(\mathbf{X}_n; \theta)\) is unimodal with respect to \(\theta \in \Theta\) and that \(\Theta\) is a discrete set or \(f_n(\mathbf{X}_n; \theta)\) is continuous with respect to \(\theta \in \Theta\), we have

\[
\{L_n(\mathbf{X}_n) \geq \theta_0\} = \left\{Y_n(\mathbf{X}_n; \theta_1, \theta_0) \leq \frac{\delta}{2}\right\}, \quad \{U_n(\mathbf{X}_n) \leq \theta_1\} = \left\{Y_n(\mathbf{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2}\right\}
\]

for all \(n \in \mathbb{N}\).

### 4.2 CSPRTs on Multiple Composite Hypotheses

In order to construct a sequential test, choose \(\alpha_i, \beta_i \in (0, 1)\) for \(i = 1, \ldots, m - 1\) and \(\alpha_m = \beta_0 = 0\). Define lower confidence limit

\[
L_{n,i} = \inf\{\theta \in \Theta : Y_n(\mathbf{X}_n; \theta_0', \theta) > \alpha_i\}
\]

and upper confidence limit

\[
U_{n,i} = \sup\{\theta \in \Theta : Y_n(\mathbf{X}_n; \theta_i', \theta) > \beta_i\}
\]

for \(i = 1, \ldots, m - 1\). Making use of the principle of probabilistic comparison, we propose stopping and decision rules as follows:

Continue the sampling process until there exists an index \(j \in \{0, 1, \ldots, m - 1\}\) such that

\[
L_{n,i} \geq \theta_i', \quad 0 < i \leq j \quad \text{and} \quad U_{n,i} \leq \theta_i'', \quad j < i < m.
\]

At the termination of the sampling process, accept \(\mathcal{H}_j\) with the index \(j\) satisfying the stopping condition.

As a consequence of Theorems \[\text{I}\] and \[\text{II}\] we have that if the sampling process will eventually terminate with probability 1, then \(\Pr\{\text{Reject } \mathcal{H}_i | \theta\} \leq \alpha_{i+1} + \beta_i\) for \(0 \leq i < m\) and \(\theta \in \Theta_i\).

Under the assumption that \(f_n(\mathbf{X}_n; \theta)\) is unimodal with respect to \(\theta \in \Theta\) and that \(\Theta\) is a discrete set or \(f_n(\mathbf{X}_n; \theta)\) is continuous with respect to \(\theta \in \Theta\), it follows from Theorem \[\text{II}\] that \(\{L_{n,i} \geq \theta_i'\} = \{Y_n(\mathbf{X}_n; \theta_i', \theta_i') \leq \alpha_i\}\) and \(\{U_{n,i} \leq \theta_i''\} = \{Y_n(\mathbf{X}_n; \theta_i', \theta_i'') \leq \beta_i\}\) for \(0 < i < m\). Hence, the stopping and decision rules can be simplified as follows:

Continue the sampling process until there exists an index \(j\) in the set \(\{0, 1, \ldots, m - 1\}\) such that

\[
Y_n(\mathbf{X}_n; \theta_i', \theta_i'') \geq \frac{1}{\alpha_i} \quad \text{for} \ 0 < i \leq j \quad \text{and} \quad Y_n(\mathbf{X}_n; \theta_j', \theta_j'') \leq \beta_i \quad \text{for} \ j < i < m. \quad \text{At the termination of the sampling process, accept } \mathcal{H}_j \text{ with the index } j \text{ satisfying the stopping condition.}
\]

A salient feature of our test is that \(m-1\) consecutive probability ratios are used for defining the stopping and decision rules. The name Consecutive Sequential Probability Ratio Test is derived from such nature of the test. We have established that the consecutive sequential probability ratio test has the following properties.

**Theorem 3** Assume that the likelihood function \(f_n(\mathbf{X}_n; \theta)\) is unimodal with respect to \(\theta \in \Theta\) for any \(n \in \mathbb{N}\). If the sampling process will eventually terminate according to the stopping rule with probability 1, then the following statements (I)–(III) hold true:

I) \(\Pr\{\text{Reject } \mathcal{H}_i | \theta\} \leq \alpha_{i+1} + \beta_i\) for \(0 \leq i \leq m - 1\) and \(\theta \in \Theta_i\).

II) For \(j = 1, \ldots, m - 1\), \(\Pr\{\text{Accept } \mathcal{H}_i \text{ with some index } i \text{ no less than } j | \theta\}\) is no greater than \(\alpha_j\) and is non-decreasing with respect to \(\theta \in \Theta\) no greater than \(\theta_j'\).

III) For \(j = 1, \ldots, m - 1\), \(\Pr\{\text{Accept } \mathcal{H}_i \text{ with some index } i \text{ less than } j | \theta\}\) is no greater than \(\beta_j\) and is non-increasing with respect to \(\theta \in \Theta\) no less than \(\theta_j''\).

Moreover, the sampling process will eventually terminate according to the stopping rule with probability 1, provided that the following additional assumption is satisfied: For arbitrary \(\alpha, \beta \in (0, 1)\) and \(\theta \in \Theta\),

\[
\Pr \left\{ \beta \leq Y_n(\mathbf{X}_n; \theta_i', \theta_i'') \leq \frac{1}{\alpha} \theta \right\} \to 0, \quad i = 1, \ldots, m - 1
\]

(3)
as the sample number \( n \) tends to \( N^* \).

See Appendix [3] for a proof. It should be emphasized that throughout this paper, the notion of “the sampling process will eventually terminate according to the stopping rule” is that the stopping rule is satisfied for some \( n \in \mathbb{N} \).

Statement (I) of Theorem [3] provides a simple method for controlling the risk of making wrong decisions. To satisfy the risk requirement [2], it suffices to choose \( \alpha_i \) and \( \beta_i \) such that \( \alpha_{i+1} + \beta_i \leq \delta_i \) for \( 0 \leq i \leq m-1 \). Specially, one can simply use \( \alpha_1 = \delta_0 \), \( \beta_m = \delta_m \), and \( \alpha_{i+1} = \beta_i = \frac{\delta_i}{2} \) for \( 1 \leq i \leq m-2 \) in the stopping and decision rules for purpose of ensuring [2].

### 4.3 CSPRTs on Multiple Simple Hypotheses

In some situations, it may be interesting to test multiple simple hypotheses

\[
\mathcal{H}_0 : \theta = \theta_0, \quad \mathcal{H}_1 : \theta = \theta_1, \quad \ldots, \quad \mathcal{H}_{m-1} : \theta = \theta_{m-1}. \tag{4}
\]

For risk control purpose, it is typically required that, for prescribed numbers \( \delta_i \in (0,1) \),

\[
\Pr \{ \text{Reject } \mathcal{H}_i \mid \theta_i \} \leq \delta_i, \quad i = 0, 1, \ldots, m-1. \tag{5}
\]

As before, let \( \alpha_i, \beta_i \in (0,1) \) for \( i = 1, \ldots, m-1 \) and \( \alpha_m = \beta_0 = 0 \). Define lower confidence limit

\[
L_{n,i} = \inf \{ \theta \in \Theta : \Upsilon_n(\mathcal{X}_n; \theta_{i+1}, \theta) > \alpha_i \}
\]

and upper confidence limit

\[
U_{n,i} = \sup \{ \theta \in \Theta : \Upsilon_n(\mathcal{X}_n; \theta_i, \theta) > \beta_i \}
\]

for \( i = 1, \ldots, m-1 \). By the principle of probabilistic comparison, we propose the following stopping and decision rules:

Continue the sampling process until there exists an index \( j \in \{0, 1, \ldots, m-1\} \) such that

\[
L_{n,i} \geq \theta_i, \quad 0 \leq i < j \quad \text{and} \quad U_{n,i} \leq \theta_{i+1}, \quad j \leq i < m-2.
\]

At the termination of the sampling process, accept \( \mathcal{H}_j \) with the index \( j \) satisfying the stopping condition.

Under the assumption that \( f_n(\mathcal{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \), it follows from Theorem [2] that

\[
\{ L_{n,i} \geq \theta_i \} = \{ \Upsilon_n(\mathcal{X}_n; \theta_{i+1}, \theta_i) \leq \alpha_i \} \quad \text{and} \quad \{ U_{n,i} \leq \theta_{i+1} \} = \{ \Upsilon_n(\mathcal{X}_n; \theta_i, \theta_{i+1}) \leq \beta_i \} 
\]

for \( 0 \leq i \leq m-2 \). Hence, the stopping and decision rules can be simplified as follows:

| Continue the sampling process until there exists an index \( j \) in the set \( \{0, 1, \ldots, m-1\} \) such that \( \Upsilon_n(\mathcal{X}_n; \theta_{i+1}, \theta_i) \geq \frac{1}{\alpha_i} \) for \( 0 < i \leq j \) and \( \Upsilon_n(\mathcal{X}_n; \theta_{i-1}, \theta_i) \leq \beta_i \) for \( j < i < m \). At the termination of the sampling process, accept \( \mathcal{H}_j \) with the index \( j \) satisfying the stopping condition. |

We have shown that the above consecutive sequential probability ratio test has the following properties.

**Theorem 4** If the sampling process will eventually terminate according to the stopping rule with probability 1, then \( \Pr \{ \text{Reject } \mathcal{H}_i \mid \theta_i \} \leq \alpha_{i+1} + \beta_i \) for \( 0 \leq i \leq m-1 \). Moreover, the sampling process will eventually terminate according to the stopping rule with probability 1, provided that the likelihood function \( f_n(\mathcal{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \) for any positive integer \( n \), and that for arbitrary \( \alpha, \beta \in (0,1) \) and \( \theta \in \Theta \),

\[
\Pr \left\{ \beta < \Upsilon_n(\mathcal{X}_n; \theta_{i-1}, \theta_i) < \frac{1}{\alpha} \mid \theta \right\} \rightarrow 1, \quad i = 1, \ldots, m-1
\]

as the sample number \( n \) tends to \( N^* \).

See Appendix [3] for a proof.

According to Theorem [4], it suffices to choose \( \alpha_i \) and \( \beta_i \) such that \( \alpha_{i+1} + \beta_i \leq \delta_i \) for \( 0 \leq i \leq m-1 \). Particularly, one can use \( \alpha_1 = \delta_0 \), \( \beta_m = \delta_m \), and \( \alpha_{i+1} = \beta_i = \frac{\delta_i}{2} \) for \( 1 \leq i \leq m-2 \) in the stopping and decision rules to ensure that \( \Pr \{ \text{Reject } \mathcal{H}_i \mid \theta_i \} \leq \delta_i, \quad i = 0, 1, \ldots, m-1. \)
4.4 General Termination Properties

In Theorems 3 and 4, one of the assumptions that we use to establish the termination properties is that the likelihood functions are unimodal on \( \Theta \). Actually, with regard to the CSPRT’s on composite and simple hypotheses proposed in Sections 4.2 and 4.3, the termination properties are valid under fairly general assumptions, as asserted by the following results.

**Theorem 5** The sampling process will eventually terminate according to the stopping rule with probability 1, provided that the following assumptions are satisfied:

(I) For arbitrary \( \alpha, \beta \in (0, 1) \) and \( \theta, \theta', \theta'' \in \Theta \) with \( \theta' < \theta'' \),

\[
\Pr \left\{ \beta < \Upsilon_n(X_n; \theta', \theta'') < \frac{1}{\alpha} \mid \theta \right\} \to 0
\]  

as the sample number \( n \) tends to \( N^* \).

(II) For arbitrary \( \alpha \in (0, 1) \) and \( \theta, \theta', \theta'' \in \Theta \) with \( \theta' < \theta'' \leq \theta \),

\[
\Pr \left\{ \Upsilon_n(X_n; \theta', \theta'') \geq \frac{1}{\alpha} \mid \theta \right\} \to 1
\]  

as the sample number \( n \) tends to \( N^* \).

(III) For arbitrary \( \beta \in (0, 1) \) and \( \theta, \theta', \theta'' \in \Theta \) with \( \theta \leq \theta' < \theta'' \),

\[
\Pr \left\{ \Upsilon_n(X_n; \theta', \theta'') \leq \beta \mid \theta \right\} \to 1
\]  

as the sample number \( n \) tends to \( N^* \).

Theorem 5 can be established by mimicking the argument of the termination property of Theorem 3 as in Appendix C.

It should be noted that (I) implies

\[
\Pr \left\{ \beta < \Upsilon_n(X_n; \theta', \theta'') < \frac{1}{\alpha} \text{ for all } n \in \mathcal{N} \mid \theta \right\} = 0,
\]  

which has been established in [12] Appendix A.1], under a very general assumption, for the termination property of Wald’s sequential probability ratio tests on two hypotheses. However, (II) does not imply (I).

Actually, in the case that \( X_1, X_2, \cdots \) are i.i.d samples of \( X \) parameterized by \( \theta \in \Theta \), the assumption (I) of Theorem 5 holds under fairly general conditions, as can be seen by the following result.

**Theorem 6** Let \( \theta, \theta', \theta'' \in \Theta \). Assume that \( \Pr \{ f(X; \theta') f(X; \theta'') = 0 \mid \theta \} = 0 \) and that the variance of \( \ln \frac{f(X; \theta')}{f(X; \theta'')} \) is positive and finite. Then, for arbitrary \( \alpha, \beta \in (0, 1) \),

\[
\lim_{n \to \infty} \Pr \left\{ \beta < \Upsilon_n(X_n; \theta', \theta'') < \frac{1}{\alpha} \mid \theta \right\} = 0.
\]  

See Appendix C for a proof. It should be noted that if \( \Pr \{ f(X; \theta') f(X; \theta'') = 0 \mid \theta \} > 0 \), then \( \lim_{n \to \infty} \Pr \left\{ \beta < \Upsilon_n(X_n; \theta', \theta'') < \frac{1}{\alpha} \mid \theta \right\} = 0 \).

In the case that \( X_1, X_2, \cdots \) are i.i.d samples of \( X \) parameterized by \( \theta \in \Theta \), the assumptions (II) and (III) of Theorem 5 are valid under fairly general conditions, as can be seen by the following result.

**Theorem 7** Let \( \theta, \theta', \theta'' \in \Theta \) and \( \alpha, \beta \in (0, 1) \). Assume that \( \Pr \{ f(X; \theta') f(X; \theta'') = 0 \mid \theta \} = 0 \) and that the variances of \( \ln f(X; \theta') \) and \( \ln f(X; \theta'') \) associated with \( \theta \) are positive and finite. Then,

(I) \( \lim_{n \to \infty} \Pr \left\{ \Upsilon_n(X_n; \theta', \theta'') \geq \frac{1}{\alpha} \mid \theta \right\} = 1 \) holds under the additional assumption that \( \mathbb{E}[\ln f(X; \theta') \mid \theta] < \mathbb{E}[\ln f(X; \theta'') \mid \theta] \).

(II) \( \lim_{n \to \infty} \Pr \left\{ \Upsilon_n(X_n; \theta', \theta'') \leq \beta \mid \theta \right\} = 1 \) holds under the additional assumption that \( \mathbb{E}[\ln f(X; \theta') \mid \theta] > \mathbb{E}[\ln f(X; \theta'') \mid \theta] \).
Similarly, if \( \Pr \{ f(X; \theta') = 0 \mid \theta \} > 0 \), then \( \lim_{n \to \infty} \Pr \{ \Upsilon_n(\mathbf{X}_n; \theta', \theta'') \geq \frac{1}{\alpha} \mid \theta \} = 1 \).

Similarly, if \( \Pr \{ f(X; \theta'') = 0 \mid \theta \} > 0 \), then \( \lim_{n \to \infty} \Pr \{ \Upsilon_n(\mathbf{X}_n; \theta', \theta'') \leq \beta \mid \theta \} = 1 \).

### 4.5 One-sided Hypotheses

It should be noted that in the special context of testing two hypotheses, our CSPRTs reduce to Wald’s SPRTs.

For the problem of testing simple hypotheses \( \mathcal{H}_0 : \theta = \theta_0 \) versus \( \mathcal{H}_1 : \theta = \theta_1 \), the likelihood function \( f_n(\mathbf{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \), since there are only two values in the parameter space \( \Theta \). Therefore, the required assumption of our CSPRT is the same as that of Wald’s SPRT.

For the problem of testing composite hypotheses \( \mathcal{H}_0 : \theta \leq \theta_0 \) versus \( \mathcal{H}_1 : \theta \geq \theta_1 \), our CSPRT requires the assumption that the test will surely eventually terminate and that the likelihood function \( f_n(\mathbf{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \). It has been previously known that the SPRT is applicable to the composite hypotheses under the assumption that the SPRT will surely eventually terminate and that the relevant likelihood ratio is monotone.

We would like to point out that there are some situations where the relevant likelihood ratio does not possess the monotonicity property, but the likelihood function \( f_n(\mathbf{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \). To illustrate, consider hypotheses regarding the distribution of random variable \( X \) uniformly distributed on \([\theta - a, \theta + a]\) with known \( a > 0 \) and unknown parameter \( \theta \). Suppose one wish to test hypotheses on \( \theta \) based on i.i.d. samples \( X_1, X_2, \cdots \) of \( X \). Since for any sample number \( n \), the likelihood ratio needs to be expressed in terms of \( \min\{X_1, \cdots, X_n\} \) and \( \max\{X_1, \cdots, X_n\} \), we can conclude that the likelihood ratio does not possess the monotonicity property. However, it can be readily shown that the likelihood function \( f_n(\mathbf{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \). From this discussion, it can be seen that our result in Theorem 3 has extended the applications of Wald’s SPRTs to a wider variety of composite hypotheses.

### 4.6 Two-sided Hypotheses

Consider a classical problem of testing two-sided hypotheses \( H_0 : \theta = \vartheta_0 \) versus \( H_1 : \theta \neq \vartheta_0 \), with \( \vartheta_0 \in \Theta \). As pointed out by Wald [12, Section 4.4.4, page 77], it is a common contention that the acceptance of \( H_0 \) will not be considered a serious error if \( \theta \neq \vartheta_0 \) but is near \( \vartheta_0 \). However, there will be, in general, two parameter values \( a \) and \( b \) with \( a < \vartheta_0 < b \) such that the acceptance of \( H_0 \) is considered an error of practical importance if (and only if) \( \theta \notin (a, b) \). Thus, the region of preference for rejection may be defined as the set of all values \( \theta \) for which \( \theta \notin (a, b) \). The region of preference for acceptance will consist of the single value \( \vartheta_0 \), and the region of indifference will be the set of all values \( \theta \) for which \( (a, \vartheta_0) \cup (\vartheta_0, b) \). To control the risk of making wrong decision, it is typically required that

\[
\Pr\{\text{Reject } H_0 \mid \theta \} \leq \alpha \quad \text{for } \theta = \vartheta_0
\]

and

\[
\Pr\{\text{Accept } H_0 \mid \theta \} \leq \beta \quad \text{for } \theta \in \Theta \text{ such that } \theta \notin (a, b).
\]

To solve this problem, Wald proposed the principle of weight function. However, an appropriate weight function is difficult to find, especially for discrete distributions. We propose to solve the problem by constructing CSPRT for the following three new hypotheses

\[
\mathcal{H}_0 : \theta \leq \frac{a + \vartheta_0}{2}, \quad \mathcal{H}_1 : \frac{a + \vartheta_0}{2} < \theta \leq \frac{b + \vartheta_0}{2}, \quad \mathcal{H}_2 : \theta > \frac{b + \vartheta_0}{2}
\]

See Appendix 3 for a proof.
so that
\[
\Pr\{\text{Reject } \mathcal{H}_0 \mid \theta\} \leq \frac{\beta}{2} \quad \text{for } \theta \leq a; \quad \Pr\{\text{Reject } \mathcal{H}_1 \mid \theta\} \leq \alpha \quad \text{for } \theta = \theta_0; \quad \Pr\{\text{Reject } \mathcal{H}_2 \mid \theta\} \leq \frac{\beta}{2} \quad \text{for } \theta \geq b.
\]

This can be accomplished by applying the CSPRT with \(m = 3\), \(\delta_0 = \frac{\beta}{2}\), \(\delta_1 = \alpha\), \(\delta_2 = \frac{\beta}{2}\) and
\[
\begin{align*}
\theta_1 &= \frac{a + \theta_0}{2}, \quad \theta_2 = \frac{b + \theta_0}{2}, \quad \theta'_1 = a, \quad \theta''_1 = \theta_0, \quad \theta'_2 = \theta_0, \quad \theta''_2 = b, \\
\alpha_1 &= \frac{\beta}{2}, \quad \alpha_2 = \frac{\alpha}{2}, \quad \alpha_3 = 0, \quad \beta_0 = 0, \quad \beta_1 = \frac{\alpha}{2}, \quad \beta_2 = \frac{\beta}{2}.
\end{align*}
\]

At the termination of the CSPRT, the decision on the original hypotheses \(H_0\) versus \(H_1\) is made based on the decision on the new hypotheses \(\mathcal{H}_0\), \(\mathcal{H}_1\) and \(\mathcal{H}_2\) by the following rule:

Accept \(H_0\) if \(\mathcal{H}_1\) is accepted; Reject \(H_0\) if either \(\mathcal{H}_0\) or \(\mathcal{H}_2\) is accepted.

Based on this proposal, it can be readily shown that the risk requirements \([\ref{eq:rho1}]\) and \([\ref{eq:rho2}]\) are satisfied.

### 4.7 CSPRTs on Parameters of Exponential Family

In this section, we shall show that the CSPRTs can be applied to the parameters of the exponential family under mild assumptions. Let \(X\) be a random variable with PDF or PMF of the form
\[
f_X(x; \theta) = h(x) \exp[u(\theta)T(x) - v(\theta)],
\]
where \(T(x)\) and \(h(x)\) are functions of \(x\), and \(u(\theta)\), \(v(\theta)\) are functions of \(\theta \in \Theta\). We have obtained the following results.

**Theorem 8** Assume that \(\frac{du(\theta)}{d\theta} = \theta \frac{du(\theta)}{d\theta}\) and that \(\frac{du(\theta)}{d\theta} > 0\) for \(\theta \in \Theta\). Let \(X_1, X_2, \cdots\) be i.i.d. samples of \(X\). Then, for any \(n \in \mathbb{N}\), the likelihood function \(f_n(X_n; \theta)\) is unimodal with respect \(\theta \in \Theta\). Moreover, for arbitrary \(\alpha, \beta \in (0, 1)\) and \(\theta, \theta', \theta'' \in \Theta\) with \(\theta' < \theta''\),
\[
\lim_{n \to \infty} \Pr\left\{ \beta < \mathcal{Y}_n(X_n; \theta', \theta'') < \frac{1}{\alpha} \mid \theta \right\} = 0.
\]

See Appendix \([\ref{app:proof}]\) for a proof. It can be readily verified that the assumption of Theorem \([\ref{thm:exponential}]\) is satisfied for the binomial, Poisson, normal, exponential, gamma, geometric and negative binomial distributions.

### 4.8 CSPRTs on Proportion of Finite Population

Consider a finite population of \(N\) units among which there are \(Np\) units having a certain attribute, where \(p \in \Theta \overset{\text{def}}{=} \{\frac{i}{N} : i = 0, 1, \cdots, N\}\). Many practical problems can be formulated as the multiple hypotheses testing problem defined by \([\ref{eq:image_1}]\) and \([\ref{eq:image_2}]\), with the parameter \(\theta\) identified as \(p\). For such a problem, consider sampling without replacement. As before, define a Bernoulli random variable \(X_n\) such that \(X_n\) assumes values 1 or 0 in accordance with whether the \(n\)-th drawn unit has the attribute. This leads to a sequence of dependent Bernoulli random variables \(X_1, \cdots, X_N\) parameterized by \(p \in \Theta\). The following analysis shows that our CSPRTs can be applied to the general multiple hypotheses testing problem.

Clearly, the likelihood function is
\[
f_n(X_n; p) = \frac{\binom{Np}{K_n} \binom{N-Np}{n-K_n}}{\binom{n}{K_n}}
\]
where \(K_n = \sum_{i=1}^{n} X_i\). Let \(\alpha, \beta \in (0, 1)\) and \(p, p', p'' \in \Theta\) with \(p' < p''\).
In the case of \( p \leq p' \), we have \( f_N(\mathbf{X}_N;p') = 0 \). Thus, \( f_N(\mathbf{X}_N;p'') > \beta f_N(\mathbf{X}_N;p') \) is violated.

In the case of \( p \geq p'' \), we have \( f_N(\mathbf{X}_N;p') = 0 \). Thus, \( f_N(\mathbf{X}_N;p') > \alpha f_N(\mathbf{X}_N;p'') \) is violated.

In the case of \( p' < p < p'' \), it must be true that \( f_N(\mathbf{X}_N;p') = f_N(\mathbf{X}_N;p'') = 0 \), which implies that both \( f_N(\mathbf{X}_N;p'') > \beta f_N(\mathbf{X}_N;p') \) and \( f_N(\mathbf{X}_N;p') > \alpha f_N(\mathbf{X}_N;p'') \) are violated.

This proves that

\[
\Pr \left\{ \beta < Y_n(\mathbf{X}_n;p',p'') < \frac{1}{\alpha} \mid p \right\} \to 0
\]
as \( n \to N \). It can be shown by direct computation that \( f_n(\mathbf{X}_n;p) \) is unimodal with respect to \( p \).

4.9 Unimodal Property of Various Distributions

In addition to the exponential family and the distribution associated with a sampling without replacement from a finite population, the likelihood functions of a wide variety of distributions have the desired unimodal properties which permit the applications of CSPRTs. A few of such distributions are outlined in the sequel.

4.9.1 Positive Power Law Distribution

A random variable \( X \) is said to have a positive power law distribution if the density function of \( X \) is given by

\[
f_X(x; \gamma, \kappa) = \begin{cases} \frac{\kappa+1}{\gamma} x^{\kappa} & \text{for } x \in [0, \gamma], \\ 0 & \text{for } x \notin [0, \gamma], \end{cases}
\]

where \( \kappa \geq 0 \) and \( \gamma > 0 \). Clearly, taking \( \kappa = 0 \) gives the uniform distribution. It can be checked that for a given \( \gamma > 0 \), the likelihood function \( f_n(\mathbf{X}_n; \gamma, \kappa) \) is unimodal with respect to \( \kappa \). On the other hand, when \( \kappa \geq 0 \) is fixed, \( f_n(\mathbf{X}_n; \gamma, \kappa) \) is unimodal with respect to \( \gamma > 0 \).

4.9.2 Pareto Distribution

The Pareto distribution is given in density-function form by

\[
f_X(x; \gamma, \kappa) = \begin{cases} \frac{\gamma}{x} \left( \frac{\gamma}{x} \right)^{\kappa+1} & \text{for } x \in [\gamma, \infty), \\ 0 & \text{for } x \notin [\gamma, \infty), \end{cases}
\]

where \( \kappa > 0 \) and \( \gamma > 0 \). It can be shown that for any given \( \gamma > 0 \), \( f_n(\mathbf{X}_n; \gamma, \kappa) \) is unimodal with respect to \( \kappa \). When \( \kappa \) is fixed, \( f_n(\mathbf{X}_n; \gamma, \kappa) \) is unimodal with respect to \( \gamma > 0 \).

4.9.3 Normal Distribution with Known Mean

The normal distribution is given in density-function form by

\[
f_X(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right),
\]

where \(-\infty < \mu < \infty\) and \( \sigma > 0 \). It can be shown that for any given \( \mu \), the likelihood function \( f_n(\mathbf{X}_n; \mu, \sigma) \) is unimodal with respect to \( \sigma \).
4.9.4 Laplace Distribution
A random variable $X$ is said to have a Laplace distribution if the density function of $X$ is given by

$$f_X(x; \mu, \nu) = \frac{1}{2\nu} \exp\left(-\frac{|x-\mu|}{\nu}\right),$$

where $-\infty < \mu < \infty$ and $\nu > 0$. It can be shown that for any given $\mu$, the likelihood function $f_n(\mathbf{X}_n; \mu, \nu)$ is unimodal with respect to $\nu$.

4.9.5 Negative Exponential Distribution
The negative exponential distribution is given in density-function form by

$$f_X(x; \mu, \nu) = \begin{cases} \frac{1}{\nu} \exp\left(-\frac{x-\mu}{\nu}\right) & \text{for } x \in [\mu, \infty), \\ 0 & \text{for } x \notin [\mu, \infty), \end{cases}$$

where $-\infty < \mu < \infty$ and $\nu > 0$. Clearly, for any given $\mu$, the likelihood function $f_n(\mathbf{X}_n; \mu, \nu)$ is unimodal with respect to $\nu > 0$. On the other hand, when $\nu > 0$ is fixed, $f_n(\mathbf{X}_n; \mu, \nu)$ is unimodal with respect to $\mu$.

4.9.6 Weibull Distribution
The Weibull distribution is given in density-function form by

$$f_X(x; \lambda, \kappa) = \frac{\kappa}{\lambda} \left(\frac{x}{\lambda}\right)^{\kappa-1} \exp\left(-\left(\frac{x}{\lambda}\right)^\kappa\right), \quad x > 0, \quad \kappa > 0, \quad \lambda > 0$$

It can be shown that for any given $\kappa > 0$, the likelihood function $f_n(\mathbf{X}_n; \lambda, \kappa)$ is unimodal with respect to $\lambda > 0$.

5 Continuous-Time Stochastic Processes
By a similar approach as that of the CSPRTs for the discrete-time process $(X_n)_{n \in \mathbb{N}}$, we can develop CSPRTs for a continuous-time processes $(X_t)_{t \in [0, \infty)}$ parameterized by $\theta \in \Theta$. Throughout Sections 5.1 and 5.2 let $(X_t)_{t \in [0, \infty)}$ be a right-continuous stochastic process parameterized by $\theta \in \Theta$ and let the probability mass or density function of $X_t$ be denoted by $f_t(x; \theta)$ for $t \in [0, \infty)$. Assume that $f_t(x; \theta)$ is right-continuous with respect to $t \in [0, \infty)$ for any $\theta \in \Theta$ and $x \in \mathbb{R}$.

5.1 Maximal Inequality and Confidence Sequences
For parameter values $\theta', \theta'' \in \Theta$, define likelihood ratio $\Upsilon_t(X_i; \theta', \theta'') = \frac{f_t(X_i; \theta'')}{f_t(X_i; \theta')}$ for $t \in [0, \infty)$. We have established the following results on maximal inequalities and confidence sequences.

**Theorem 9** Assume that for arbitrary integer $n$ and real numbers $t_i$, $i = 0, \cdots, n$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$, the conditional probability mass or density function of $X_{t_i}$, $i = 0, 1, \cdots, n-1$ given the value of $X_t$ does not depend on $\theta$. Let $\theta_0, \theta_1 \in \Theta$ and $\delta \in (0, 1)$. Then,

$$\text{Pr}\left\{\Upsilon_t(X_i; \theta_0, \theta_1) > \frac{1}{\delta} \text{ for some } t \in [0, \infty) \mid \theta_0\right\} \leq \delta. \quad (11)$$

Moreover, $\text{Pr}\{L_t(X_t) \leq \theta \text{ for all } t \mid \theta\} \geq 1 - \frac{\delta}{2}$, $\text{Pr}\{U_t(X_t) \geq \theta \text{ for all } t \mid \theta\} \geq 1 - \frac{\delta}{2}$, and $\text{Pr}\{L_t(X_t) \leq \theta \leq U_t(X_t) \text{ for all } t \mid \theta\} \geq 1 - \delta$ for all $\theta \in \Theta$, where $L_t(X_t) = \inf\{\vartheta \in \Theta : \Upsilon_t(X_t; \vartheta, \theta) \geq \frac{1}{\delta}\}$ and $U_t(X_t) = \sup\{\vartheta \in \Theta : \Upsilon_t(X_t; \theta_0, \vartheta) \geq \frac{1}{\delta}\}$.
See Appendix [1] for a proof. If the likelihood function \( f_t(X_t; \theta) \) is unimodal with respect to \( \theta \in \Theta \), then there exists an estimator \( \hat{\theta}_t \) for \( \theta \) such that \( f_t(X_t; \theta) \) is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( \hat{\theta}_t \) and is non-increasing with respect to \( \theta \in \Theta \) no less than \( \hat{\theta}_t \). Hence, it must be true that \( \{ \hat{\theta}_t \leq \theta_0 \} \subseteq \{ \hat{\theta}_t \leq \theta_0 \} \subseteq \{ \hat{\theta}_t \leq \theta_0 \} = 0 \). It follows that

\[
\begin{align*}
\left\{ \hat{Y}_t(X_t; \theta_1, \theta_0) < \frac{\delta}{2} \right\} &= \left\{ \hat{Y}_t(X_t; \theta_1, \theta_0) < \frac{\delta}{2}, \hat{\theta}_t < \theta_0 \right\} \\
&\subseteq \left\{ \hat{Y}_t(X_t; \theta_1, \theta) < \frac{\delta}{2} \text{ for all } \theta \leq \theta_0 \right\} \\
&\subseteq \left\{ \hat{L}_t(X_t) \geq \theta_0 \right\},
\end{align*}
\]

where (12) is also a consequence of the assumption that the likelihood function \( f_t(X_t; \theta) \) is unimodal with respect to \( \theta \in \Theta \).

5.2 CSPRTs on Multiple Hypotheses

For the multi-hypotheses testing problem defined by (1) and (2), we propose a CSPRT with stopping and decision rules as follows:

\[
\text{Continue observing } (X_t)_{t \in [0, \infty)} \text{ until there exists an index } j \text{ in the set } \{0, 1, \ldots, m-1\} \text{ such that } \\
\hat{Y}_t(X_t; \theta_j, \theta_j') > \frac{1}{\alpha_i} \text{ for } 0 < i \leq j \text{ and } \hat{Y}_t(X_t; \theta_j, \theta_j') < \beta_j \text{ for } j < i < m. \text{ At the termination of the observational procedure, accept } \mathcal{H}_j 	ext{ with the index } j \text{ satisfying the stopping condition.}
\]

We have established that the above CSPRT has the following properties.

**Theorem 10** Assume that for arbitrary integer \( n \) and real numbers \( t_i, i = 0, \ldots, n \) with \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \), the conditional probability mass or density function of \( X_{t_i} \), \( i = 0, 1, \ldots, n-1 \) given the value of \( X_t \) does not depend on \( \theta \). Assume that the likelihood function \( f_t(X_t; \theta) \) is unimodal with respect to \( \theta \in \Theta \) for any positive number \( t \). If the observational process will eventually terminate according to the stopping rule with probability 1, then the following statements (I)–(III) hold true:

(I) \( \Pr\{\text{Reject } \mathcal{H}_i | \theta \} \leq \alpha_{i+1} + \beta_i \) for \( 0 \leq i \leq m-1 \) and \( \theta \in \Theta_i \).

(II) For \( j = 1, \ldots, m-1 \), \( \Pr\{\text{Accept } \mathcal{H}_i \text{ with some index } i \text{ no less than } j | \theta \} \) is no greater than \( \alpha_j \) and is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( \theta_j' \).

(III) For \( j = 1, \ldots, m-1 \), \( \Pr\{\text{Accept } \mathcal{H}_i \text{ with some index } i \text{ less than } j | \theta \} \) is no greater than \( \beta_j \) and is non-increasing with respect to \( \theta \in \Theta \) no less than \( \theta_j' \).

Moreover, the sampling process will eventually terminate according to the stopping rule with probability 1, provided that the following additional assumption is satisfied: For arbitrary \( \alpha, \beta \in (0, 1) \) and \( \theta \in \Theta \),

\[
\lim_{t \to \infty} \Pr \left\{ \beta \leq \hat{Y}_t(X_t; \theta_i, \theta_i') \leq \frac{\alpha}{i} | \theta \right\} = 0, \quad i = 1, \ldots, m-1
\]

The proof of Theorem 10 is similar to that of Theorem 8.

For testing simple hypotheses defined by (1) and (5), we propose a CSPRT with stopping and decision rules as follows:

\[
\text{Continue observing } (X_t)_{t \in [0, \infty)} \text{ until there exists an index } j \text{ in the set } \{0, 1, \ldots, m-1\} \text{ such that } \\
\hat{Y}_t(X_t; \theta_{i-1}, \theta_i) > \frac{1}{\alpha_i} \text{ for } 0 < i \leq j \text{ and } \hat{Y}_t(X_t; \theta_{i-1}, \theta_i) < \beta_i \text{ for } j < i < m. \text{ At the termination of the observational procedure, accept } \mathcal{H}_j \text{ with the index } j \text{ satisfying the stopping condition.}
\]

We have established that such CSPRT possesses the following properties.
Theorem 11 Assume that for arbitrary integer \( n \) and real numbers \( t_i, i = 0, \cdots, n \) with \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \), the conditional probability mass or density function of \( X_{t_i}, i = 0, 1, \cdots, n-1 \) given the value of \( X_t \) does not depend on \( \theta \). If the observational process will eventually terminate according to the stopping rule with probability 1, then \( \Pr\{\text{Reject } \mathcal{H}_i \mid \theta_i \} \leq \alpha_i + 1 + \beta_i \) for \( 0 \leq i \leq m - 1 \). Moreover, the observational process will eventually terminate according to the stopping rule with probability 1, provided that the likelihood function \( f_t(X_t; \theta_i) \) is unimodal with respect to \( \theta \in \Theta \), and that for arbitrary \( \alpha, \beta \in (0, 1) \) and \( \theta \in \Theta \),

\[
\lim_{t \to 0} \Pr\left\{ \beta \leq T_t(X_t; \theta_i-1, \theta_i) \leq \frac{1}{\alpha} \mid \theta \right\} = 0, \quad i = 1, \cdots, m - 1.
\]

The proof of Theorem 11 is similar to that of Theorem 4.

5.3 CSPRTS on Arrival Rates of Poisson Processes

Consider a Poisson process \((X_t)_{t \in [0, \infty)}\) with an arrival rate \( \lambda > 0 \). Note that for \( \gamma > 0 \),

\[
\left\{ \begin{array}{l}
\frac{f_t(X_t; \lambda_1)}{f_t(X_t; \lambda_0)} > \gamma \\
\frac{\lambda_1 - \lambda_0}{\ln \frac{\lambda_1}{\lambda_0}}
\end{array} \right\} = \left\{ X_t > \frac{(\lambda_1' - \lambda_0')(t + \ln \gamma)}{\ln \frac{\lambda_1'}{\lambda_0'}} \right\}.
\]

For testing multiple composite hypotheses defined by (1) and (2), with \( \theta, \theta_i', \theta_i, \theta_i'' \) identified as \( \lambda, \lambda_1', \lambda_i, \lambda_i'' \) respectively, we propose a CSPRT with stopping and decision rules as follows:

Continue observing \((X_t)_{t \in [0, \infty)}\) until there exists an index \( j \) in the set \( \{0, 1, \cdots, m-1\} \) such that

\[
X_t > \frac{(\lambda_i' - \lambda_i)(t + \ln \gamma)}{\ln \frac{\lambda_i'}{\lambda_i}} \quad \text{for } 0 < i < j \quad \text{and} \quad X_t < \frac{(\lambda_i'' - \lambda_i')(t + \ln \gamma)}{\ln \frac{\lambda_i''}{\lambda_i'}} \quad \text{for } j < i < m.
\]

At the termination of the observational procedure, accept \( \mathcal{H}_j \) with the index \( j \) satisfying the stopping condition.

Regarding the above CSPRT, we have shown the following result.

Theorem 12 The observational process will eventually terminate according to the stopping rule with probability 1. Moreover, the following statements (I)–(III) hold true:

(I) \( \Pr\{\text{Reject } \mathcal{H}_i \mid \lambda \} \leq \alpha_i + 1 + \beta_i \) for \( 0 \leq i \leq m - 1 \) and \( \lambda \in \Theta_i \).

(II) For \( j = 1, \cdots, m - 1 \), \( \Pr\{\text{Accept } \mathcal{H}_i \mid \lambda \} \) is no greater than \( \alpha_j \) and is non-decreasing with respect to \( \lambda \in \Theta \) no greater than \( \lambda_j' \).

(III) For \( j = 1, \cdots, m - 1 \), \( \Pr\{\text{Accept } \mathcal{H}_i \mid \lambda \} \) is no greater than \( \beta_j \) and is non-increasing with respect to \( \lambda \in \Theta \) no less than \( \lambda_j'' \).

See Appendix I for a proof.

For testing multiple simple hypotheses defined by (4) and (5), with \( \theta, \theta_i \) identified as \( \lambda, \lambda_i \) respectively, we propose a CSPRT with stopping and decision rules as follows:

Continue observing \((X_t)_{t \in [0, \infty)}\) until there exists an index \( j \) in the set \( \{0, 1, \cdots, m-1\} \) such that

\[
X_t > \frac{(\lambda_i - \lambda_i-1)(t + \ln \gamma)}{\ln \frac{\lambda_i}{\lambda_i-1}} \quad \text{for } 0 < i < j \quad \text{and} \quad X_t < \frac{(\lambda_i'' - \lambda_i')(t + \ln \gamma)}{\ln \frac{\lambda_i''}{\lambda_i'}} \quad \text{for } j < i < m.
\]

At the termination of the observational procedure, accept \( \mathcal{H}_j \) with the index \( j \) satisfying the stopping condition.

Regarding the above CSPRT, we have shown the following result.

Theorem 13 The observational process will eventually terminate according to the stopping rule with probability 1. Moreover, \( \Pr\{\text{Reject } \mathcal{H}_i \mid \lambda_i \} \leq \alpha_i + 1 + \beta_i \) for \( 0 \leq i \leq m - 1 \).

Theorem 13 is a direct consequence of Theorem 11.
### 5.4 CSPRTS on Parameters of Brownian Motions

Consider a Brownian motion \((X_t)_{t \in [0, \infty)}\) with unknown drift \(\mu\) and known variance \(\sigma^2\) per unit time. Note that for \(\gamma > 0\),

\[
\left\{ \frac{f_i(X_t; \mu_1, \sigma)}{f_i(X_t; \mu_0, \sigma)} > \gamma \right\} = \left\{ X_t > \frac{(\mu_0 + \mu_1)t}{2} + \frac{\sigma^2}{\mu_1 - \mu_0} \ln \gamma \right\}.
\]

For testing multiple composite hypotheses defined by (1) and (2) with \(\theta, \theta_i', \theta_i, \theta_i''\) identified as \(\mu, \mu_i', \mu_i, \mu_i''\) respectively, we propose a CSPRT with stopping and decision rules as follows:

| Continue observing \((X_t)_{t \in [0, \infty)}\) until there exists an index \(j\) in the set \([0, 1, \cdots, m-1]\) such that \(X_t > \frac{\mu_i' + \mu_i''}{2} + \frac{\sigma^2}{\mu_i' - \mu_i''} \ln \frac{1}{\sigma_i}\) for \(0 < i < j\) and \(X_t < \frac{\mu_i' + \mu_i''}{2} + \frac{\sigma^2}{\mu_i' - \mu_i''} \ln \beta_i\) for \(j < i < m\). |
|---|
| At the termination of the observational procedure, accept \(H_j\) with the index \(j\) satisfying the stopping condition. |

With regard to above CSPRT, we have shown the following results.

**Theorem 14** The observational process will eventually terminate according to the stopping rule with probability 1. Moreover, the following statements (I)-(III) hold true:

(I) \(\text{Pr}\{\text{Reject } H_i \mid \mu\} \leq \alpha_{i+1} + \beta_i\) for \(0 \leq i \leq m-1\) and \(\mu \in \Theta_i\).

(II) For \(j = 1, \cdots, m-1\), \(\text{Pr}\{\text{Accept } H_i \text{ with some index } i \text{ no less than } j \mid \mu\} \) is no greater than \(\alpha_j\) and is non-decreasing with respect to \(\mu \in \Theta\) no greater than \(\mu_j'\).

(III) For \(j = 1, \cdots, m-1\), \(\text{Pr}\{\text{Accept } H_i \text{ with some index } i \text{ less than } j \mid \mu\} \) is no greater than \(\beta_j\) and is non-increasing with respect to \(\mu \in \Theta\) no less than \(\mu_j''\).

See Appendix [I] for a proof.

For testing multiple simple hypotheses defined by (4) and (5) with \(\theta, \theta_i\) identified as \(\mu, \mu_i\) respectively, we propose a CSPRT with stopping and decision rules as follows:

| Continue observing \((X_t)_{t \in [0, \infty)}\) until there exists an index \(j\) in the set \([0, 1, \cdots, m-1]\) such that \(X_t > \frac{\mu_i' + \mu_i''}{2} + \frac{\sigma^2}{\mu_i' - \mu_i''} \ln \frac{1}{\sigma_i}\) for \(0 < i < j\) and \(X_t < \frac{\mu_i' + \mu_i''}{2} + \frac{\sigma^2}{\mu_i' - \mu_i''} \ln \beta_i\) for \(j < i < m\). |
|---|
| At the termination of the observational procedure, accept \(H_j\) with the index \(j\) satisfying the stopping condition. |

Same results as in Theorem [I] hold for above CSPRT.

### 6 Conclusion

In this paper, we have established consecutive sequential probability ratio tests for testing multiple statistical hypotheses. Our tests are derived based on the principle of probabilistic comparison. Simple analytic formulae are derived for controlling the risk of making wrong decisions. We have demonstrated that the new tests can be applied to a wide variety of statistical hypotheses.

### A Proof of Theorem [I]

By the assumption that \(\cup_{i=0}^{m-1}\{\mathcal{L} = a_i, \mathcal{U} = b_{i+1}\} = \Omega\), we have \(\cup_{i=0}^{m-1}\{\mathcal{L} = a_i\} = \Omega\) and \(\cup_{i=0}^{m-1}\{\mathcal{U} = b_{i+1}\} = \Omega\). Therefore, for \(\theta \in \Theta\), we have

\[
\{\mathcal{L} \geq \theta\} = \bigcup_{\ell \in I_\tau} \{\tau = \ell, \mathcal{L} \geq \theta\} = \bigcup_{\ell \in I_\tau} \{\tau = \ell, \mathcal{L} \geq a_{i+1}\} = \{\mathcal{L} \geq a_{i+1}\},
\]

\[
\{\mathcal{U} \leq \theta\} = \bigcup_{\ell \in I_\tau} \{\tau = \ell, \mathcal{U} \leq \theta\} = \bigcup_{\ell \in I_\tau} \{\tau = \ell, \mathcal{U} \leq b_i\} = \{\mathcal{U} \leq b_i\}.
\]
For $i = m - 1$, we have $\Theta_i = \Theta_{m-1} = \{b_{m-1}, \infty\}$, $L_{\ell,i+1} = L_{\ell,m} = -\infty$, $a_{i+1} = a_m = \infty$ and hence, $\Pr\{L \geq \theta\} = \Pr\{L \geq a_{i+1}\} = 0 = \Pr\{L_{\ell,1} \geq a_{i+1}\}$ for some $\ell \in I_{\tau}$. As a consequence of the assumption that $\bigcup_{\ell=0}^{m-1} \{L = a_i, \ U = b_{i+1}\} = \Omega$, we have

$$\{\tau = \ell, \ L = a_j\} \cap \{\tau = \ell\} = \{L = a_j\} \cap \bigcup_{i=0}^{m-1} \{\tau = \ell, \ L = a_i, \ U = b_{i+1}\}$$

for $j = 0, 1, \ldots, m - 1$. Hence, for $i = 0, 1, \ldots, m - 2$ and $\theta \in \Theta_i$, we have

$$\{L \geq \theta\} = \{L \geq a_{i+1}\} = \bigcup_{\ell \in I_{\tau}} \{\tau = \ell, \ L \geq a_{i+1}\} = \bigcup_{\ell \in I_{\tau}} \bigcup_{j \geq i} \{\tau = \ell, \ L = a_j, \ U = b_{j+1}\}$$

$$\subseteq \bigcup_{\ell \in I_{\tau}} \bigcup_{j \geq i} \{L_{\ell,k} \geq a_k, \ 0 < k \leq j \text{ and } U_{\ell,k} \leq b_k, \ j < k < m\}$$

$$\subseteq \bigcup_{\ell \in I_{\tau}} \{L_{\ell,i+1} \geq a_{i+1}\} = \{L_{\ell,i+1} \geq a_{i+1} \text{ for some } \ell \in I_{\tau}\}.$$

For $i = 0$, we have $\Theta_i = \Theta_0 = (-\infty, a_1]$, $U_{\ell,i} = U_{\ell,0} = \infty$, $b_0 = b_0 = -\infty$ and hence, $\Pr\{\mathcal{U} \leq \theta\} = \Pr\{\mathcal{U} \leq b_i\} = 0 = \Pr\{U_{\ell,i} \leq b_i \text{ for some } \ell \in I_{\tau}\} = 0$. As a consequence of the assumption that $\bigcup_{\ell=0}^{m-1} \{L = a_i, \ U = b_{i+1}\} = \Omega$, we have

$$\{\tau = \ell, \ \mathcal{U} = b_{j+1}\} \subseteq \{\tau = \ell, \ \mathcal{U} = b_{j+1}\} \cap \{\tau = \ell\}$$

$$= \{\mathcal{U} = b_{j+1}\} \cap \bigcup_{\ell \in I_{\tau}} \{\tau = \ell, \ L = a_i, \ U = b_{i+1}\}$$

$$= \{\mathcal{U} = b_{j+1}\} \cap \bigcup_{\ell \in I_{\tau}} \{\tau = \ell, \ \mathcal{U} = b_{j+1}\}$$

for $j = 0, 1, \ldots, m - 1$. Hence, for $i = 1, \ldots, m - 1$ and $\theta \in \Theta_i$, we have

$$\{\mathcal{U} \leq \theta\} = \{\mathcal{U} \leq b_i\} = \bigcup_{\ell \in I_{\tau}} \{\tau = \ell, \ \mathcal{U} \leq b_i\} = \bigcup_{\ell \in I_{\tau}} \bigcup_{j \geq i} \{\tau = \ell, \ \mathcal{U} = b_{j+1}\}$$

$$= \bigcup_{\ell \in I_{\tau}} \bigcup_{j < i} \{\tau = \ell, \ \mathcal{U} = b_{j+1}\}$$

$$\subseteq \bigcup_{\ell \in I_{\tau}} \bigcup_{j < i} \{L_{\ell,k} \geq a_k, \ 0 < k \leq j \text{ and } U_{\ell,k} \leq b_k, \ j < k < m\}$$

$$\subseteq \bigcup_{\ell \in I_{\tau}} \{U_{\ell,i} \leq b_i\} = \{U_{\ell,i} \leq b_i \text{ for some } \ell \in I_{\tau}\}.$$

This completes the proof of the theorem.

**B Proof of Theorem 2**

We need a preliminary result stated as follows.

**Lemma 1** Let $\alpha \in (0, 1)$ and let $\theta', \ \theta''$ be two parameter values in $\Theta$. Then,

$$\Pr\{Y_n(X_n; \theta'; \theta'') \geq \frac{1}{\alpha} \text{ for some } n \in \mathcal{N} \mid \theta'\} \leq 2\alpha.$$

Actually, the result of Lemma 1 is due to Ville [11], which was rediscovered by Wald [12, page 146].

We are now in a position to prove the theorem. By the definition of the lower confidence limit, we have $\{L_n(X_n) \leq \theta_0\} \supseteq \{Y_n(X_n; \theta_1, \theta_0) > \frac{1}{\alpha}\}$. This implies that $\{L_n(X_n) > \theta_0\} \subseteq \{Y_n(X_n; \theta_1, \theta_0) \leq \frac{1}{\alpha}\}$.
and consequently, \( \Pr\{L_n(\mathbf{X}_n) > \theta \} \geq \frac{\delta}{2} \) for some \( n \in \mathcal{N} \mid \theta \}

for \( \theta \in \Theta \). It follows from Lemma \( \ref{lem:prmnt} \) that \( \Pr\{L_n(\mathbf{X}_n) > \theta \} \geq \frac{\delta}{2} \) for \( \theta \in \Theta \).

Similarly, it follows from the definition of the upper confidence limit that \( \{Y_n(\mathbf{X}_n; \theta_0, \theta_1) > \frac{\delta}{2}\} \subseteq \{U_n(\mathbf{X}_n) \geq \theta_1\} \). This implies that \( \{U_n(\mathbf{X}_n) < \theta_1\} \subseteq \{Y_n(\mathbf{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2}\} \) and consequently, \( \Pr\{U_n(\mathbf{X}_n) < \theta \} \leq \frac{\delta}{2} \) for some \( n \in \mathcal{N} \mid \theta \} \) for \( \theta \in \Theta \). It follows from Lemma \( \ref{lem:prmnt} \) that \( \Pr\{U_n(\mathbf{X}_n) < \theta \} \leq \frac{\delta}{2} \) for \( \theta \in \Theta \). So, by virtue of Bonferroni’s inequality, we have \( \Pr\{L_n(\mathbf{X}_n) \leq \theta \leq U_n(\mathbf{X}_n)\} \) for all \( n \in \mathcal{N} \mid \theta \} \geq 1 - \delta \). This completes the proof of statements (I) and (II).

By the assumption that \( f_n(\mathbf{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \), there exists an estimator \( \hat{\theta}_n \) of \( \theta \) such that \( f_n(\mathbf{X}_n; \theta) \) is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( \hat{\theta}_n \) and is non-increasing with respect to \( \theta \in \Theta \) no less than \( \hat{\theta}_n \). Such estimator is referred to as a unimodal-likelihood estimator (ULE) of \( \theta \). To show statement (III), note that as a consequence of the existence of a ULE \( \hat{\theta}_n \) for \( \theta \), it must be true that \( \{\hat{\theta}_n \leq \theta_0\} \subseteq \{Y_n(\mathbf{X}_n; \theta_1, \theta_0) \geq 1\} \) and consequently, \( \{Y_n(\mathbf{X}_n; \theta_1, \theta_0) \leq \frac{\delta}{2}, \hat{\theta}_n \leq \theta_0\} \subseteq \{Y_n(\mathbf{X}_n; \theta_1, \theta_0) < 1, \hat{\theta}_n \leq \theta_0\} = \emptyset \). It follows that

\[
\{Y_n(\mathbf{X}_n; \theta_1, \theta_0) \leq \frac{\delta}{2}\} = \{Y_n(\mathbf{X}_n; \theta_1, \theta_0) \leq \frac{\delta}{2}, \hat{\theta}_n > \theta_0\} \subseteq \{Y_n(\mathbf{X}_n; \theta_1, \theta_0) \leq \frac{\delta}{2} \text{ for all } \theta \leq \theta_0\} \subseteq \{L_n(\mathbf{X}_n) \geq \theta_0\}.
\]

Similarly, note that as a consequence of the assumption that there exists a ULE \( \tilde{\theta}_n \) for \( \theta \), it must be true that \( \{\tilde{\theta}_n \geq \theta_1\} \subseteq \{Y_n(\mathbf{X}_n; \theta_0, \theta_1) \geq 1\} \) and consequently, \( \{Y_n(\mathbf{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2}, \tilde{\theta}_n \geq \theta_1\} \subseteq \{Y_n(\mathbf{X}_n; \theta_0, \theta_1) < 1, \tilde{\theta}_n \geq \theta_1\} = \emptyset \). It follows that

\[
\{Y_n(\mathbf{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2}\} = \{Y_n(\mathbf{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2}, \tilde{\theta}_n < \theta_1\} \subseteq \{Y_n(\mathbf{X}_n; \theta_0, \theta_1) \leq \frac{\delta}{2} \text{ for all } \theta \geq \theta_1\} \subseteq \{U_n(\mathbf{X}_n) \leq \theta_1\}.
\]

This completes the proof of the theorem.

### Proof of Theorem \( \ref{thm:prmnt} \)

We need to develop some preliminary results based on the assumptions of the theorem.

**Lemma 2** Let \( \alpha \in (0, 1) \) and let \( \theta' < \theta'' \) be two parameter values in \( \Theta \). Then,

\[
\{Y_n(\mathbf{X}_n; \theta', \theta'') \geq \frac{1}{\alpha}\} \subseteq \{Y_n(\mathbf{X}_n; \theta, \theta'') \geq \frac{1}{\alpha}\} \quad \text{for } \theta \in (-\infty, \theta'] \cap \Theta. \quad (13)
\]

Similarly,

\[
\{Y_n(\mathbf{X}_n; \theta', \theta'') \leq \alpha\} \subseteq \{Y_n(\mathbf{X}_n; \theta', \theta'') \leq \alpha\} \quad \text{for } \theta \in [\theta'', \infty) \cap \Theta. \quad (14)
\]

**Proof.** As pointed out in the proof of Theorem \( \ref{thm:prmnt} \) in Appendix \( \ref{app:thm:prmnt} \), by the assumption that \( f_n(\mathbf{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \), there exists a ULE \( \hat{\theta}_n \) for \( \theta \).

To show \( \ref{eq:lem2_1} \), note that \( \{Y_n(\mathbf{X}_n; \theta', \theta'') \geq \frac{1}{\alpha}, \hat{\theta}_n \leq \theta'\} = \emptyset \) and that \( \{Y_n(\mathbf{X}_n; \theta', \theta'') \geq \frac{1}{\alpha}, \hat{\theta}_n > \theta'\} \subseteq \{Y_n(\mathbf{X}_n; \theta, \theta'') \geq \frac{1}{\alpha}, \hat{\theta}_n > \theta'\} \subseteq \{Y_n(\mathbf{X}_n; \theta, \theta'') \geq \frac{1}{\alpha}\} \) for \( \theta \in (-\infty, \theta'] \cap \Theta \). It follows that \( \{Y_n(\mathbf{X}_n; \theta', \theta'') \geq \frac{1}{\alpha}\} = \{Y_n(\mathbf{X}_n; \theta, \theta'') \geq \frac{1}{\alpha}\} \) for \( \theta \in (-\infty, \theta'] \cap \Theta \).

To show \( \ref{eq:lem2_2} \), note that \( \{Y_n(\mathbf{X}_n; \theta', \theta'') \leq \alpha, \hat{\theta}_n \geq \theta''\} = \emptyset \) and that \( \{Y_n(\mathbf{X}_n; \theta', \theta'') \leq \alpha, \hat{\theta}_n < \theta''\} \subseteq \{Y_n(\mathbf{X}_n; \theta', \theta'') \leq \alpha\} \) for \( \theta \in [\theta'', \infty) \cap \Theta \). It follows that \( \{Y_n(\mathbf{X}_n; \theta', \theta'') \leq \alpha\} = \{Y_n(\mathbf{X}_n; \theta', \theta'') \leq \alpha\} \) for \( \theta \in [\theta'', \infty) \cap \Theta \).

\[\square\]

**Lemma 3** \( \{\mathcal{H}_\ell \text{ with some } \ell > j \text{ is accepted}\} \subseteq \{Y_n(\mathbf{X}_n; \theta, \theta''_j) \geq \frac{1}{\alpha_{j+1}} \text{ for some } n \in \mathcal{N}\} \) for \( 0 \leq j \leq m - 2 \) and \( \theta \in (-\infty, \theta''_{j+1}] \cap \Theta \).

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Proof. By (13) of Lemma 3 and the definition of the stopping and decision rules,

\[ \{ \mathcal{H}_\ell \text{ with some } \ell > j \text{ is accepted} \} \subseteq \bigcup_{\ell > j} \left\{ \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) \geq \frac{1}{\alpha_i}, 1 \leq i \leq \ell \text{ for some } n \in \mathcal{N} \right\} \]

\[ \subseteq \left\{ \gamma_n(\mathcal{X}_n; \theta'_{j+1}, \theta''_{j+1}) \geq \frac{1}{\alpha_{j+1}} \text{ for some } n \in \mathcal{N} \right\} \]

\[ \subseteq \left\{ \gamma_n(\mathcal{X}_n; \theta, \theta''_{j+1}) \geq \frac{1}{\alpha_{j+1}} \text{ for some } n \in \mathcal{N} \right\} \]

for \( 0 \leq j \leq m - 2 \) and \( \theta \in (\infty, \theta'_{j+1}] \cap \Theta \).

\[ \square \]

Lemma 4 \( \{ \mathcal{H}_\ell \text{ with some } \ell < j \text{ is accepted} \} \subseteq \{ \gamma_n(\mathcal{X}_n; \theta'_j, \theta) \leq \beta_j \text{ for some } n \in \mathcal{N} \} \) for \( 1 \leq j \leq m - 1 \) and \( \theta \in [\theta''_j, \infty) \cap \Theta \).

Proof. By (14) of Lemma 3 and the definition of the stopping and decision rules,

\[ \{ \mathcal{H}_\ell \text{ with some } \ell < j \text{ is accepted} \} \subseteq \bigcup_{\ell < j} \left\{ \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) \leq \beta_i, \ell < i < m \text{ for some } n \in \mathcal{N} \right\} \]

\[ \subseteq \{ \gamma_n(\mathcal{X}_n; \theta'_j, \theta''_{j}) \leq \beta_j \text{ for some } n \in \mathcal{N} \} \]

\[ \subseteq \{ \gamma_n(\mathcal{X}_n; \theta'_j, \theta) \leq \beta_j \text{ for some } n \in \mathcal{N} \} \]

for \( 1 \leq j \leq m - 1 \) and \( \theta \in [\theta''_j, \infty) \cap \Theta \).

\[ \square \]

Lemma 5 Let \( 0 < j < m \) and \( \theta \in (\theta''_j, \infty) \cap \Theta \). Then, \( \Pr \left\{ \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) \geq \frac{1}{\alpha_i}, 0 < i \leq j \mid \theta \right\} \to 1 \) as the sample number \( n \) tends to \( N^* \).

Proof. Let \( \beta \in (0, 1) \). By (13) of Lemma 3 for \( 0 < j < m \) and \( \theta \in (\theta''_j, \infty) \cap \Theta \),

\[ \Pr \left\{ \text{There exists some } i \text{ such that } 0 < i \leq j \text{ and } \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) < \frac{1}{\alpha_i} \mid \theta \right\} \]

\[ \leq \sum_{i=1}^{j} \Pr \left\{ \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) < \frac{1}{\alpha_i} \mid \theta \right\} \]

\[ \leq \sum_{i=1}^{j} \left[ \Pr \left\{ \beta \leq \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) < \frac{1}{\alpha_i} \mid \theta \right\} + \Pr \left\{ \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) \leq \beta \mid \theta \right\} \right] \]

\[ \leq \sum_{i=1}^{j} \left[ \Pr \left\{ \beta \leq \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) < \frac{1}{\alpha_i} \mid \theta \right\} + \Pr \left\{ \gamma_n(\mathcal{X}_n; \theta'_i, \theta) \leq \beta \mid \theta \right\} \right] \]

\[ \leq \sum_{i=1}^{j} \left[ \Pr \left\{ \beta \leq \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) < \frac{1}{\alpha_i} \mid \theta \right\} + \beta \right] \to j \beta \]

as the sample number \( n \) tends to \( N^* \). But this holds for arbitrarily small \( \beta \in (0, 1) \).

\[ \square \]

Lemma 6 Let \( 0 < j < m \) and \( \theta \in (\infty, \theta'_j) \cap \Theta \). Then, \( \Pr \{ \gamma_n(\mathcal{X}_n; \theta'_i, \theta''_{i}) \leq \beta_i \text{ for } j \leq i < m \mid \theta \} \to 1 \) as the sample number \( n \) tends to \( N^* \).
Proof. Let $\alpha \in (0,1)$. By (13) of Lemma 2 for $0 < j < m$ and $\theta \in (-\infty, \theta_j') \cap \Theta$,

$$
\Pr \{ \text{There exists some } i \text{ such that } j \leq i < m \text{ and that } \mathcal{Y}_n(\mathcal{X}_n; \theta_i', \theta_j') > \beta_i | \theta \}
\leq \sum_{i=j}^{m-1} \Pr \{ \mathcal{Y}_n(\mathcal{X}_n; \theta_i', \theta_j') > \beta_i | \theta \}
\leq \sum_{i=j}^{m-1} \left[ \Pr \left\{ \beta_i < \mathcal{Y}_n(\mathcal{X}_n; \theta_i', \theta_j') < \frac{1}{\alpha} | \theta \right\} + \Pr \left\{ \mathcal{Y}_n(\mathcal{X}_n; \theta_i', \theta_j') \geq \frac{1}{\alpha} | \theta \right\} \right]
\leq \sum_{i=j}^{m-1} \left[ \Pr \left\{ \beta_i < \mathcal{Y}_n(\mathcal{X}_n; \theta_i', \theta_j') < \frac{1}{\alpha} | \theta \right\} + \Pr \left\{ \mathcal{Y}_n(\mathcal{X}_n; \theta_i', \theta_j') \geq \frac{1}{\alpha} | \theta \right\} \right]
\leq \sum_{i=j}^{m-1} \left[ \Pr \left\{ \beta_i < \mathcal{Y}_n(\mathcal{X}_n; \theta_i', \theta_j') < \frac{1}{\alpha} | \theta \right\} + \alpha \right] \to (m-j)\alpha
$$
as the sample number $n$ tends to $N^*$. But this holds for arbitrarily small $\alpha \in (0,1)$. $\square$

We are now in a position to prove the theorem.

C.1 Proof of Statements (I)–(III)

We shall show statements (I)–(III) based on the assumption that the likelihood function $f_n(\mathcal{X}_n; \theta)$ is unimodal with respect to $\theta \in \Theta$ and that the sampling process will eventually terminate according to the stopping rule.

Statement (I) can be shown as follows. Invoking Lemmas 1 and 3, we have

$$
\Pr \{ \mathcal{H}_i \text{ with some } i \geq j \text{ is accepted } | \theta \} \leq \Pr \left\{ \mathcal{Y}_n(\mathcal{X}_n; \theta, \theta'_j) \geq \frac{1}{\alpha_j} \text{ for some } n \in \mathcal{N} | \theta \right\} \leq \alpha_j \tag{15}
$$

for $j = 1, \cdots, m-1$ and $\theta \in (-\infty, \theta'_j) \cap \Theta$. Making use of Lemmas 1 and 2, we have

$$
\Pr \{ \mathcal{H}_i \text{ with some } i < j \text{ is accepted } | \theta \} \leq \Pr \{ \mathcal{Y}_n(\mathcal{X}_n; \theta'_j, \theta) \leq \beta_{m-1} \text{ for some } n \in \mathcal{N} | \theta \} \leq \beta_{m-1} \tag{16}
$$

for $j = 1, \cdots, m-1$ and $\theta \in (\theta'_j, \infty) \cap \Theta$. Therefore,

$$
\Pr \{ \text{Reject } \mathcal{H}_0 | \theta \} = \Pr \{ \mathcal{H}_i \text{ with some } i \geq 1 \text{ is accepted } | \theta \} \leq \alpha_1 \quad \text{for } \theta \in (-\infty, \theta'_j) \cap \Theta,

\Pr \{ \text{Reject } \mathcal{H}_{m-1} | \theta \} = \Pr \{ \mathcal{H}_i \text{ with some } i < m-1 \text{ is accepted } | \theta \} \leq \beta_{m-1} \quad \text{for } \theta \in (\theta''_{m-1}, \infty) \cap \Theta
$$

and

$$
\Pr \{ \text{Reject } \mathcal{H}_j | \theta \} = \Pr \{ \mathcal{H}_i \text{ with some } i > j \text{ is accepted } | \theta \} + \Pr \{ \mathcal{H}_i \text{ with some } i < j \text{ is accepted } | \theta \}
\leq \alpha_{j+1} + \beta_j
$$

for $0 < j \leq m-2$ and $\theta \in (\theta''_j, \theta'_{j+1}) \cap \Theta$. This proves statement (I).

To show statement (II), let $n$ denote the sample number at the termination of the sampling process. Since the likelihood function $f_n(\mathcal{X}_n; \theta)$ is unimodal with respect to $\theta \in \Theta$, we have that for every value $n$ in the support of $n$, there exists an estimator $\hat{\theta}_n$, defined in terms of $\mathcal{X}_n$, such that $f_n(\mathcal{X}_n; \theta)$ is nondecreasing with respect to $\theta \in \Theta$ no greater than $\hat{\theta}_n$ and is non-increasing with respect to $\theta \in \Theta$ no less than $\hat{\theta}_n$. Define a sequential estimator $\hat{\theta}$ by replacing $n$ with $n$, that is $\hat{\theta} = \hat{\theta}_n$. Then, $\hat{\theta}$ is a ULE of $\theta$. By the definition of the stopping and decision rules, we have, for $j = 1, \cdots, m-1$ and every $n$ in the support of $n$,

$$
\{ \text{Accept } \mathcal{H}_i \text{ with some index } i \text{ no less than } j \} \cap \{ n = n \} \subseteq \left\{ \mathcal{Y}_n(\mathcal{X}_n; \theta_j', \theta_j') \geq \frac{1}{\alpha_j}, n = n \right\} \subseteq \{ \hat{\theta} \geq \theta_j', n = n \}.
$$
It follows from Lemma 5 that \( \Pr \{ \text{Accept } H_i \text{ with some index } i \text{ no less than } j \} \subseteq \{ \hat{\theta} \geq \theta_j^i \} \) for \( j = 1, \cdots, m - 1 \).

According to the second statement of Lemma 3 of [4, version 32, Appendix A3, page 127], we have that \( \Pr \{ \text{Accept } H_i \text{ with some index } i \text{ no less than } j \mid \theta \} \) is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( \theta_j^i \) for \( j = 1, \cdots, m - 1 \). This result together with the proven inequality (13) complete the proof of Statement (II). Similarly, to show statement (III), note that for \( j = 1, \cdots, m - 1 \) and every \( n \) in the support of \( n \),

\[
\{ \text{Accept } H_i \text{ with some index } i \text{ no less than } j \} \cap \{ n = n \} \subseteq \{ \Upsilon_n(\mathbf{X}_n; \theta_j^i, \theta_j^{i''}) \leq \beta_j, n = n \} \subseteq \{ \theta \leq \theta_j^{i''}, n = n \}.
\]

According to the first statement of Lemma 3 of [4, version 32, Appendix A3, page 127], we have that \( \Pr \{ \text{Accept } H_i \text{ with some index } i \text{ no less than } j \mid \theta \} \) is non-increasing with respect to \( \theta \in \Theta \) no less than \( \theta_j^{i''} \) for \( j = 1, \cdots, m - 1 \). This result together with the proven inequality (14) complete the proof of Statement (III).

### C.2 Proof of the Termination Property

We shall show that the sampling process will eventually terminate according to the stopping rule under the assumption that the likelihood function \( f_{\beta}(\mathbf{X}_n; \theta) \) is unimodal with respect to \( \theta \in \Theta \) and that (3) is satisfied. Note that for all \( n \) and \( \theta \in (-\infty, \theta_1^i) \cap \Theta \),

\[
\Pr \{ \text{The sampling process will eventually terminate according to the stopping rule } \mid \theta \} \geq \Pr \{ \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \leq \beta_i \text{ for } 0 < i < m \mid \theta \}
\]

It follows from Lemma 6 that \( \Pr \{ \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \leq \beta_i \text{ for } 0 < i < m \mid \theta \} \to 1 \) as the sample number \( n \) tends to \( N^* \). It must be true that

\[
\Pr \{ \text{The sampling process will eventually terminate according to the stopping rule } \mid \theta \} = 1 \quad \text{for } \theta \in (-\infty, \theta_1^i) \cap \Theta.
\]

Note that for all \( n \) and \( \theta \in (\theta_{m-1}^i, \infty) \cap \Theta \),

\[
\Pr \{ \text{The sampling process will eventually terminate according to the stopping rule } \mid \theta \} \geq \Pr \{ \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \geq \frac{1}{\alpha_i} \text{ for } 0 < i < m \mid \theta \}
\]

It follows from Lemma 5 that \( \Pr \{ \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \geq \frac{1}{\alpha_i} \text{ for } 0 < i < m \mid \theta \} \to 1 \) as the sample number \( n \) tends to \( N^* \). It must be true that

\[
\Pr \{ \text{The sampling process will eventually terminate according to the stopping rule } \mid \theta \} = 1 \quad \text{for } \theta \in (\theta_{m-1}^i, \infty) \cap \Theta.
\]

By Lemmas 5 and Bonferroni’s inequality, we have

\[
\Pr \{ \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \geq \frac{1}{\alpha_i}, 0 < i \leq j \text{ and } \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \leq \beta_i, j < i < m \mid \theta \} \to 1
\]

for \( j = 1, \cdots, m - 2 \) and \( \theta \in (\theta_j^{i''}, \theta_{j+1}^{i''}) \cap \Theta \), as the sample number \( n \) tends to \( N^* \). Note that for all \( n \), \( j = 1, \cdots, m - 2 \) and \( \theta \in (\theta_j^{i''}, \theta_{j+1}^{i''}) \cap \Theta \),

\[
\Pr \{ \text{The sampling process will eventually terminate according to the stopping rule } \mid \theta \} \geq \Pr \{ \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \geq \frac{1}{\alpha_i}, 0 < i \leq j \text{ and } \Upsilon_n(\mathbf{X}_n; \theta_i^i, \theta_i^{i''}) \leq \beta_i, j < i < m \mid \theta \},
\]
Thus, it must be true that
\[ \Pr(\text{The sampling process will eventually terminate according to the stopping rule } | \theta) = 1 \]
for \( j = 1, \ldots, m - 2 \) and \( \theta \in (\theta''_j, \theta''_{j+1}) \cap \Theta \).

It remains to show that the sampling process will eventually terminate according to the stopping rule for \( \theta \in [\theta'_j, \theta''_j] \cap \Theta \) with \( j = 1, \ldots, m - 1 \). By Lemma 4 for \( 1 < j < m \) and all \( \theta \in [\theta'_j, \theta''_j] \cap \Theta \subseteq (\theta''_{j-1}, \infty) \cap \Theta \),
\[
\Pr \left\{ \bigcup_{1 < i < j} \left( \Theta_n(U_n, \theta'_{j-1}) \cap \Theta \right) \right\} \to 1
\]
as the sample number \( n \) tends to \( N^* \). By Lemma 3 for \( 0 \leq j < m - 1 \) and \( \theta \in [\theta'_j, \theta''_j] \cap \Theta \subseteq (\infty, \theta'_{j+1}) \cap \Theta \),
\[
\Pr \left\{ \bigcup_{1 < i < j} \left( \Theta_n(U_n, \theta'_{j-1}) \cap \Theta \right) \right\} \to 1
\]
as the sample number \( n \) tends to \( N^* \). By the assumption associated with (3), for \( j = 1, \ldots, m - 1 \) and \( \theta \in [\theta'_j, \theta''_j] \cap \Theta \),
\[
\Pr \left\{ \bigcup_{1 < i < j} \left( \Theta_n(U_n, \theta'_{j-1}) \cap \Theta \right) \right\} \to 1
\]
as the sample number \( n \) tends to \( N^* \). Note that
\[
\{\text{The sampling process will eventually terminate according to the stopping rule}\}
\geq \left\{ \bigcup_{1 < i < j} \left( \Theta_n(U_n, \theta'_{j-1}) \cap \Theta \right) \right\} \bigcap \{\text{The stopping rule for } \theta \}
\]
for \( \theta \in [\theta'_{n-1}, \theta''_{n-1}] \cap \Theta \), as the sample number \( n \) tends to \( N^* \). Therefore, we have shown that, with probability 1, the sampling process will eventually terminate according to the stopping rule for \( \theta \in [\theta'_j, \theta''_j] \) for \( j = 1, \cdots, m-1 \). This completes the proof of the theorem.

D Proof of Theorem 4

We need some preliminary results.

Lemma 7 For \( 0 \leq j < m - 1 \),

\[
\{ \mathcal{H}_\ell \text{ with some } \ell > j \text{ is accepted} \} \subseteq \left\{ Y_n(\mathbf{X}_n; \theta_j, \theta_{j+1}) \geq \frac{1}{\alpha_j} \text{ for some } n \in \mathcal{N} \right\} \tag{20}
\]

Similarly,

\[
\{ \mathcal{H}_\ell \text{ with some } \ell < j \text{ is accepted} \} \subseteq \left\{ Y_n(\mathbf{X}_n; \theta_{j-1}, \theta_j) \leq \beta_j \text{ for some } n \in \mathcal{N} \right\} \tag{21}
\]

for \( 1 \leq j < m \).

Proof. By the definition of the stopping and decision rules,

\[
\{ \mathcal{H}_\ell \text{ with some } \ell > j \text{ is accepted} \} \subseteq \bigcup_{\ell > j} \left\{ Y_n(\mathbf{X}_n; \theta_{i-1}, \theta_i) \geq \frac{1}{\alpha_i}, 1 \leq i \leq \ell \text{ for some } n \in \mathcal{N} \right\}
\]

for \( 0 \leq j < m - 1 \). Similarly, by the definition of the stopping and decision rules,

\[
\{ \mathcal{H}_\ell \text{ with some } \ell < j \text{ is accepted} \} \subseteq \bigcup_{\ell < j} \left\{ Y_n(\mathbf{X}_n; \theta_{i-1}, \theta_i) \leq \beta_i, \ell < i \leq m - 1 \text{ for some } n \in \mathcal{N} \right\}
\]

for \( 1 \leq j < m \). 

We are now in a position to prove the theorem. It follows from (20) of Lemma 7 that

\[
\Pr\{ \text{Reject } \mathcal{H}_0 \mid \theta_0 \} = \Pr\{ \mathcal{H}_\ell \text{ with some } \ell > 0 \text{ is accepted } \mid \theta_0 \} 
\leq \Pr\left\{ Y_n(\mathbf{X}_n; \theta_0, \theta_1) \geq \frac{1}{\alpha_1} \text{ for some } n \in \mathcal{N} \mid \theta_0 \right\} \leq \alpha_1.
\]

It follows from (21) of Lemma 7 that

\[
\Pr\{ \text{Reject } \mathcal{H}_{m-1} \mid \theta_{m-1} \} = \Pr\{ \mathcal{H}_\ell \text{ with some } \ell < m - 1 \text{ is accepted } \mid \theta_{m-1} \}
\leq \Pr\left\{ Y_n(\mathbf{X}_n; \theta_{m-2}, \theta_{m-1}) \leq \beta_{m-1} \text{ for some } n \in \mathcal{N} \mid \theta_{m-1} \right\} \leq \beta_{m-1}.
\]

It follows from (20) and (21) of Lemma 7 that

\[
\Pr\{ \text{Reject } \mathcal{H}_j \mid \theta_j \}
= \Pr\{ \mathcal{H}_\ell \text{ with some } \ell > j \text{ is accepted } \mid \theta_j \} + \Pr\{ \mathcal{H}_\ell \text{ with some } \ell < j \text{ is accepted } \mid \theta_j \}
\leq \Pr\left\{ Y_n(\mathbf{X}_n; \theta_j, \theta_{j+1}) \geq \frac{1}{\alpha_{j+1}} \text{ for some } n \in \mathcal{N} \mid \theta_j \right\} + \Pr\left\{ Y_n(\mathbf{X}_n; \theta_{j-1}, \theta_j) \leq \beta_j \text{ for some } n \in \mathcal{N} \mid \theta_j \right\}
\leq \alpha_{j+1} + \beta_j
\]

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for $1 \leq j \leq m - 2$.

To show that the sampling process will eventually terminate according to the stopping rule with probability $1$, it suffices to apply the argument of the proof of the termination property of Theorem 3 in Appendix C to the following hypotheses

$$H_0 : \theta \leq \vartheta_1, \quad H_1 : \vartheta_1 < \theta \leq \vartheta_2, \quad \ldots, \quad H_{m-2} : \vartheta_{m-2} < \theta \leq \vartheta_{m-1}, \quad H_{m-1} : \theta > \vartheta_{m-1}$$

with $\vartheta_i = \frac{\theta_i + \theta_{i+1}}{2}$, $i = 1, \ldots, m - 1$ and indifference zone $\bigcup_{i=1}^{m-1} (\vartheta_{i-1}, \vartheta_i)$. This concludes the proof of the theorem.

## E Proof of Theorem 6

For simplicity of notations, define $Y = \ln \frac{f(X; \theta^n)}{f(X; \theta^0)}$. Let $\mu$ and $\nu$ denote, respectively, the mean and variance of $Y$ associated with $\theta \in \Theta$. Let $Y_1, Y_2, \cdots$ be i.i.d. samples of $Y$. Define $Z_n = \sum_{i=1}^{n} (Y_i - \mu)$ for $n = 1, 2, \cdots$.

By the central limit theorem, $Z_n$ converges in distribution to a Gaussian random variable, $Z$, with zero mean and unit variance. Note that

$$\Pr\left\{ \ln \beta < \sum_{i=1}^{n} Y_i < \ln \frac{1}{\alpha} \mid \theta \right\} = \Pr\left\{ \frac{\ln \beta - n\mu}{\sqrt{n\nu}} < Z < \frac{\ln \frac{1}{\alpha} - n\mu}{\sqrt{n\nu}} \mid \theta \right\}, \quad n = 1, 2, \cdots$$

for $\theta \in \Theta$.

In the case of $\mu > 0$, we have

$$\Pr\left\{ \ln \beta < \sum_{i=1}^{n} Y_i < \ln \frac{1}{\alpha} \mid \theta \right\} \leq \Pr\left\{ Z_n < \frac{\ln \frac{1}{\alpha} - n\mu}{\sqrt{n\nu}} \mid \theta \right\} \to 0$$

for $\theta \in \Theta$ as $n \to \infty$. To show this, let $\varepsilon > 0$. Let $z$ be a number such that $\Pr(Z < z) < \frac{\varepsilon}{2}$. Let $n$ be chosen such that $z > \frac{\ln \frac{1}{\alpha} - n\mu}{\sqrt{n\nu}}$ and that $|\Pr(Z_n < z) - \Pr(Z < z)| < \frac{\varepsilon}{2}$. By the triangle inequality,

$$\Pr\left\{ Z_n < \frac{\ln \frac{1}{\alpha} - n\mu}{\sqrt{n\nu}} \mid \theta \right\} \leq \Pr\left\{ Z_n < z \mid \theta \right\} \leq \Pr\{Z < z\} + |\Pr\{Z_n < z\} - \Pr\{Z < z\}| < \varepsilon.$$

In the case of $\mu = 0$, we have

$$\Pr\left\{ \ln \beta < \sum_{i=1}^{n} Y_i < \ln \frac{1}{\alpha} \mid \theta \right\} = \Pr\left\{ \frac{\ln \beta}{\sqrt{n\nu}} < Z_n < \frac{\ln \frac{1}{\alpha}}{\sqrt{n\nu}} \mid \theta \right\} \to 0$$

for $\theta \in \Theta$ as $n \to \infty$.

In the case of $\mu < 0$, we have

$$\Pr\left\{ \ln \beta < \sum_{i=1}^{n} Y_i < \ln \frac{1}{\alpha} \mid \theta \right\} \leq \Pr\left\{ Y_n > \frac{\ln \frac{1}{\alpha} - n\mu}{\sqrt{n\nu}} \mid \theta \right\} \to 0$$

for $\theta \in \Theta$ as $n \to \infty$.

This completes the proof of the theorem.

## F Proof of Theorem 7

As in the proof of Theorem 6 in Appendix E for simplicity of notations, define $Y = \ln \frac{f(X; \theta^n)}{f(X; \theta^0)}$. Let $\mu$ and $\nu$ denote, respectively, the mean and variance of $Y$ associated with $\theta \in \Theta$. By the assumption of the
theorem, we have $\mu > 0$ and $0 < \nu < \infty$. Let $Y_1, Y_2, \cdots$ be i.i.d. sample of $Y$. Note that
\[
\Pr \left\{ Y_n(X_n; \theta', \theta'') < \frac{1}{\alpha} | \theta \right\} = \Pr \left\{ \sum_{i=1}^{n} Y_i \geq \ln \frac{1}{\alpha} | \theta \right\} \\
= \Pr \left\{ \frac{\sum_{i=1}^{n} (Y_i - \mu)}{n} < \frac{\ln \frac{1}{\alpha} - \mu}{\theta} | \theta \right\} \\
\leq \Pr \left\{ \frac{\sum_{i=1}^{n} (Y_i - \mu)}{n} < \frac{\mu}{2} | \theta \right\}
\]
for $n > \frac{2 \ln \frac{1}{\mu}}{\nu}$. By Chebyshev’s inequality,
\[
\Pr \left\{ Y_n(X_n; \theta', \theta'') < \frac{1}{\alpha} | \theta \right\} \leq \Pr \left\{ \left| \frac{\sum_{i=1}^{n} (Y_i - \mu)}{n} \right| > \frac{\mu}{2} | \theta \right\} \leq \frac{4 \nu}{n \mu^2} \rightarrow 0
\]
as $n \rightarrow \infty$. This establishes statement (I). In a similar manner, we can show statement (II). This completes the proof of the theorem.

**G Proof of Theorem 8**

For simplicity of notations, define $Y = T(X)$. We need some preliminary results.

**Lemma 8** The derivative of $\exp (u(\theta)z - v(\theta))$ with respect to $\theta$ is equal to $(z - \theta) \exp (u(\theta)z - v(\theta)) \frac{du(\theta)}{d\theta}$.

**Proof.** Since $\frac{du(\theta)}{d\theta} = \theta \frac{du(\theta)}{d\theta}$ for $\theta \in \Theta$, by the chain rule of differentiation, we have that the derivative of $\exp (u(\theta)z - v(\theta))$ with respect to $\theta$ is equal to $(z - \theta) \exp (u(\theta)z - v(\theta)) \frac{du(\theta)}{d\theta}$.

\[\Box\]

**Lemma 9** The expectation of $Y$ is equal to $\theta$.

**Proof.** Let $\psi(.)$ be the inverse function of $u(.)$ such that $u(\psi(\zeta)) = \zeta$ for $\zeta \in \{u(\theta) : \theta \in \Theta\}$. Define compound function $w(.)$ such that $w(\zeta) = v(\psi(\zeta))$ for $\zeta \in \{u(\theta) : \theta \in \Theta\}$. For simplicity of notations, we abbreviate $\psi(\zeta)$ as $\psi$ which can be done without causing confusion. Putting $\zeta = u(\theta)$, we have
\[
\mathbb{E} [\exp (tY)] = \mathbb{E} [\exp (tT(X))] = \int h(x) \exp ((\zeta + t)T(x) - w(\zeta)) \, dx \\
= \exp (w(\zeta + t) - w(\zeta)) \int h(x) \exp ((\zeta + t)T(x) - w(\zeta + t)) \, dx = \exp (w(\zeta + t) - w(\zeta)).
\]
By the defining relationship $u(\psi(\zeta)) = \zeta$, the assumption that $\frac{du(\theta)}{d\theta} = \theta \frac{du(\theta)}{d\theta}$, and the chain rule of differentiation, we have
\[
\frac{dw(\zeta)}{d\zeta} = \frac{dv(\psi)}{dv} \frac{d\psi}{d\zeta} = \psi \frac{du(\psi)}{d\psi} \frac{d\psi}{d\zeta} = \psi \frac{du(\psi)}{d\zeta} = \psi(\zeta).
\]
By virtue of (22), the derivative of $w(\zeta + t) - w(\zeta)$ with respect to $t$ is given by $\frac{dw(\zeta + t)}{dt} = \psi(\zeta + t)$, which is equal to $\psi(\zeta) = \theta$ for $t = 0$. Thus, $\mathbb{E}[Y] = \theta$, which implies that the sample mean of $Y$ is also an unbiased estimator of $\theta$.

\[\Box\]

**Lemma 10** The variance of $Y$ is equal to $\frac{1}{n \mu^2}$.
Proof. Now we are in a position to compute the variance of Y. Recall that
\[
\frac{dE[\exp(tY)]}{dt} = \frac{dw(\zeta + t)}{d(\zeta + t)} \exp(w(\zeta + t) - w(\zeta)) = \psi(\zeta + t) \exp(w(\zeta + t) - w(\zeta)).
\]
Hence,
\[
\frac{d^2E[\exp(tY)]}{dt^2} = \psi^2(\zeta + t) \exp(w(\zeta + t) - w(\zeta)) + \frac{d\psi(\zeta + t)}{dt} \exp(w(\zeta + t) - w(\zeta))
\]
\[
= \psi^2(\zeta + t) \exp(w(\zeta + t) - w(\zeta)) + \frac{d\psi(\zeta + t)}{d(\zeta + t)} \exp(w(\zeta + t) - w(\zeta)).
\]
Therefore, \(E[Y^2] = \psi^2(\zeta) + \frac{d\psi(\zeta)}{d\zeta} \). To compute \(\frac{d\psi(\zeta)}{d\zeta}\), we differentiate both sides of the defining relationship with respect to \(\zeta\) to obtain \(\frac{du}{d\zeta} \frac{dw}{d\zeta} = 1\), which implies that \(\frac{dw}{d\zeta} = \frac{1}{\frac{du}{d\zeta}} = \frac{1}{\frac{du}{d\theta}}\), where we have used \(\theta = \psi(\zeta)\) to obtain the last equality. Therefore, \(E[Y^2] = \psi^2(\zeta) + \frac{1}{\frac{du}{d\theta}} = \theta^2 + \frac{1}{\frac{du}{d\theta}}\), which implies that
\[
\text{Var}[Y] = E[Y^2] - E^2[Y] = \theta^2 + \frac{1}{\frac{du}{d\theta}(\theta)} - \theta^2 = \frac{1}{\frac{du}{d\theta}(\theta)}.
\]

We are now in a position to prove the theorem. Since \(\frac{du(\theta)}{d\theta} > 0\) for \(\theta \in \Theta\), from Lemma 8 we have that the derivative of \(\exp(u(\theta)z - v(\theta))\) with respect to \(\theta\) is positive for \(\theta < z\) and negative for \(\theta > z\). This implies that \(\exp(u(\theta)z - v(\theta))\) is monotonically increasing with respect to \(\theta\) less than \(z\) and monotonically decreasing with respect to \(\theta\) greater than \(z\). Since \(f_n(X_n; \theta) = \left[\exp\left(u(\theta)\sum^n_{i=1}T(X_i) - v(\theta)\right)\right]^n\prod_{i=1}^n h(X_i)\), it follows that \(f_n(X_n; \theta)\) is unimodal with respect to \(\theta \in \Theta\).

Let \(X_1, X_2, \ldots\) be i.i.d. samples of \(X\). For parameter values \(\theta'\), \(\theta''\) in \(\Theta\) with \(\theta' < \theta''\), the likelihood ratio is
\[
\gamma_n(X_n; \theta', \theta'') = \frac{\exp[u(\theta') \sum^n_{i=1}T(X_i) - n\nu(\theta')]}{\exp[u(\theta'') \sum^n_{i=1}T(X_i) - n\nu(\theta'')]}.
\]
Note that for \(n = 1, 2, \ldots\),
\[
\Pr\left\{\beta < \gamma_n(X_n; \theta', \theta'') < \frac{1}{\alpha}\right\} = \Pr\left\{\frac{n\nu(\theta'') - \nu(\theta') + \ln \beta}{u(\theta'') - u(\theta')} < \sum^n_{i=1}T(X_i) < \frac{n\nu(\theta') - \nu(\theta') + \ln \frac{1}{\alpha}}{u(\theta') - u(\theta')}\right\}
\]
\[
= \Pr\left\{n\rho - a < \sum^n_{i=1}T(X_i) < n\rho + b\right\}
\]
\[
= \Pr\left\{\frac{n(\rho - \theta) - a}{\sqrt{n\sigma}} < Z_n < \frac{n(\rho - \theta) + b}{\sqrt{n\sigma}}\right\}
\]
where
\[
\rho = \frac{\nu(\theta'') - \nu(\theta')}{u(\theta'') - u(\theta')}, \quad a = -\frac{\ln \beta}{u(\theta'') - u(\theta')}, \quad b = \frac{\ln \frac{1}{\alpha}}{u(\theta') - u(\theta')}
\]
and
\[
Z_n = \frac{\sum^n_{i=1}T(X_i) - n\theta}{\sqrt{n\sigma}}, \quad n = 1, 2, \ldots
\]
with \(\sigma^2 = \frac{1}{n\rho} u'\) being the variance of \(T(X)\). From Lemmas 9 and 10 we know that \(T(X)\) is a random variable with mean \(\theta\) and variance \(\sigma^2\). By the central limit theorem, \(Z_n\) converges to a Gaussian random variable with zero mean and unit variance as \(n\) tends to infinity. Consequently,
\[
\Pr\left\{\frac{n(\rho - \theta) - a}{\sqrt{n\sigma}} < Z_n < \frac{n(\rho - \theta) + b}{\sqrt{n\sigma}}\right\} \to 0
\]
as \(n \to \infty\), which can be readily shown by considering the cases of \(\theta > \rho\), \(\theta = \rho\) and \(\theta < \rho\) as in the proof of Theorem 9 in Appendix E. This completes the proof of the theorem.
H Proof of Theorem 9

We need a preliminary result.

**Lemma 11** Suppose that $(X_n)_{n \in \mathbb{N}}$ is a discrete-time process parameterized by $\theta \in \Theta$ such that for any $n$, the conditional probability density or mass function of $X_1, \cdots, X_{n-1}$ given the value of $X_n$ does not depend on $\theta$. Let $\{\mathcal{F}_n\}$ be a natural filtration such that for $n \in \mathbb{N}$, where $\mathcal{F}_n$ is $\sigma$-algebra generated by $X_1, \cdots, X_n$. Then, for any parameter values $\theta_0$ and $\theta_1$, $\{f_n(X_n, \theta_0)\}_{n \in \mathbb{N}}$ is a martingale process with respect to the filtration $\{\mathcal{F}_n\}$ and the probability measure associated with $\theta_0$.

**Proof.** For simplicity of notations, let $x_n = (x_1, \cdots, x_n)$ for $n = 1, 2, \cdots$. First, consider the case that the PDF exists. By the assumption of the lemma, we have

$$
\frac{f_{X_n}(x_n; \theta_1)}{f_{X_n}(x_n; \theta_0)} = \frac{f_{X_n}(x_n; \theta_1)}{f_{X_n}(x_n; \theta_0)},
$$

for any $n \in \mathbb{N}$. For simplicity of notations, let $f_{X_n}(x_n; \theta_0) = \delta_{x_n}$ for any $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ and any $x_n \in \mathbb{R}^n$, let $f_{X_n+1}(x_{n+1}; \theta_0) = \delta_{x_{n+1}}$. Therefore,

$$
\int_{X_n \in S} \frac{f_{X_n+1}(x_{n+1}; \theta_1)}{f_{X_n}(x_n; \theta_0)} \, dx_n = \int_{x_n \in S} \frac{f_{X_n+1}(x_{n+1}; \theta_1)}{f_{X_n+1}(x_{n+1}; \theta_0)} \, dx_{n+1} = \int_{x_n \in S} f_{X_n+1}(x_{n+1}; \theta_0) \, dx_{n+1},
$$

which implies that $\{f_n(X_n, \theta_1), \mathcal{F}_n\}_{n \in \mathbb{N}}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$ and the probability measure associated with $\theta_0$. In the case that the PMF exists, the integration in the above is replaced by summation.

We are now in a position to prove the theorem. Define $Y_t = \Upsilon_t(X_t; \theta_0, \theta_1)$ for $t \in [0, \infty)$. Define

$$
Q_k = \{0\} \cup \left\{ \frac{q}{p} : \gcd(p, q) = 1; \ p, q \in \mathbb{N}; \ p \leq k \right\}
$$

for $k = 1, 2, \cdots$, where $\gcd(p, q)$ denotes the greatest common divider of $p$ and $q$. Let $\tilde{Q}$ denote the set of non-negative rational numbers. Define $E_j = \{\omega \in \Omega : \sup_{t \in Q_j} Y_t(\omega) > \frac{1}{j}\}$. Then,

$$
Q_j \subseteq Q_{j+1} \Rightarrow \sup_{t \in Q_j} Y_t(\omega) \leq \sup_{t \in Q_{j+1}} Y_t(\omega) \Rightarrow E_j \subseteq E_{j+1}.
$$

Define $E_\infty = \{\omega \in \Omega : \sup_{t \in \tilde{Q}} Y_t(\omega) > \frac{1}{j}\}$. It is easy to show that $E_\infty = \cup_{j=0}^\infty E_j$. As a consequence of the continuity of the probability measure, $\Pr\{E_\infty\} = \lim_{n \to \infty} \Pr\{E_n\}$. By Lemma 11, $\{Y_t, t \in Q_j\}$ is
a martingale process. It follows from Doob’s super-martingale inequality that \( \Pr \{ E_j \} \leq \delta \ E[Y_0] \). This implies that \( \Pr \{ E_{\infty} \} = \lim_{J \to \infty} \Pr \{ E_j \} \leq \delta \ E[Y_0] \). We claim that \( \sup_{t \in Q} Y_t(\omega) = \sup_{t \in [0, \infty)} Y_t(\omega) \). To show this claim, note that for any \( t \in [0, \infty) \), there exists a sequence \( \{ \eta_j \}_{j=1}^{\infty} \) no less than \( t \) such that \( Y_t(\omega) = \lim_{J \to \infty} Y_{\eta_j}(\omega) \). That is, the sample path of \( Y_t \) is right-continuous. Observing that \( Y_{\eta_j}(\omega) \leq \sup_{t \in Q} Y_t(\omega) \), \( \forall t \in [0, \infty) \), we have \( Y_t(\omega) \leq \sup_{t \in Q} Y_t(\omega) \), which implies that \( \sup_{t \in [0, \infty)} Y_t(\omega) \) and thus the claim is established. This proves (11), that is, \( \Pr \{ Y_t > \frac{\delta}{\alpha} \} \) for some \( t \in [0, \infty) \) \( \{ \Theta \} \leq \delta \).

By the definition of the lower confidence limit, we have \( \{ L_t(X_t) \leq \theta_0 \} \subseteq \{ Y_t(X_t; \theta_1, \theta_0) \geq \frac{\delta}{\alpha} \} \). This implies that \( \{ L_t(X_t) > \theta_0 \} \subseteq \{ Y_t(X_t; \theta_1, \theta_0) < \frac{\delta}{\alpha} \} \) and consequently, \( \Pr \{ L_t(X_t) > \theta \) for some \( t \mid \theta \} \leq \Pr \{ Y_t(X_t; \theta_1, \theta_0) < \frac{\delta}{\alpha} \) for some \( t \mid \theta \} \) for \( \theta \in \Theta \). It follows from the proven inequality (11) that \( \Pr \{ L_t(X_t) > \theta \) for some \( t \mid \theta \} \leq \frac{\delta}{\alpha} \).

Similarly, from the definition of the upper confidence limit, we have \( \{ U_t(X_t; \theta_0, \theta_1) \geq \frac{\delta}{\alpha} \} \subseteq \{ U_t(X_t) \geq \theta_1 \} \). This implies that \( \{ U_t(X_t) < \theta_1 \} \subseteq \{ Y_t(X_t; \theta_0, \theta_1) < \frac{\delta}{\alpha} \} \) and consequently, \( \Pr \{ U_t(X_t) < \theta \) for some \( t \mid \theta \} \leq \Pr \{ Y_t(X_t; \theta_0, \theta_1) < \frac{\delta}{\alpha} \) for some \( t \mid \theta \} \) for \( \theta \in \Theta \). It follows from (11) that \( \Pr \{ U_t(X_t) < \theta \) for some \( t \mid \theta \} \leq \frac{\delta}{\alpha} \). So, by virtue of Bonferroni’s inequality, we have \( \Pr \{ L_t(X_t) \leq \theta \leq U_t(X_t) \) for all \( t \mid \theta \} \geq 1 - \delta \). This completes the proof of the theorem.

# Proof of Theorem 12

We need some preliminary results.

**Lemma 12** For arbitrary \( \alpha, \beta \in (0, 1) \) and \( \lambda, \lambda', \lambda'' \in (0, \infty) \) with \( \lambda' < \lambda'' \),

$$
\lim_{t \to \infty} \Pr \left\{ \beta \leq Y_n(X_n; \lambda', \lambda'') \leq \frac{1}{\alpha} \mid \lambda \right\} = 0.
$$

**Proof.** Note that

$$
\Pr \left\{ \beta \leq Y_n(X_n; \lambda', \lambda'') \leq \frac{1}{\alpha} \mid \lambda \right\} = \Pr \left\{ \frac{(\lambda'' - \lambda')t + \ln \beta}{\ln \lambda'} \leq X_t \leq \frac{(\lambda'' - \lambda')t + \ln \frac{1}{\alpha}}{\ln \lambda} \mid \lambda \right\}.
$$

Therefore, \( \Pr \{ \beta \leq Y_n(X_n; \lambda', \lambda'') \leq \frac{1}{\alpha} \mid \lambda \} \) can be written as \( \Pr \{ pt - a \leq X_t \leq pt + b \mid \lambda \} \), where \( \rho = \frac{(\lambda'' - \lambda')}{\ln \lambda} \) and \( a, b \) are some positive numbers. Define \( Y_t = \frac{X_t - \lambda t}{\sqrt{\lambda t}} \). Then,

$$
\Pr \{ pt - a \leq X_t \leq pt + b \mid \lambda \} = \Pr \left\{ \frac{(\rho - \lambda)t - a}{\sqrt{\lambda t}} \leq Y_t \leq \frac{(\rho - \lambda)t + b}{\sqrt{\lambda t}} \mid \lambda \right\}.
$$

Noting that

$$
\mathbb{E} \left[ \exp \left( \frac{X_t - \lambda t}{\sqrt{\lambda t}} \right) \right] = \exp(-s\sqrt{\lambda t}) \mathbb{E} \left[ \exp \left( \frac{X_t}{\sqrt{\lambda t}} \right) \right] = \exp \left\{ -s\sqrt{\lambda t} + \lambda t \left[ \exp \left( \frac{s}{\sqrt{\lambda t}} \right) - 1 \right] \right\}
$$

and that

$$
\lim_{t \to \infty} \left\{ -s\sqrt{\lambda t} + \lambda t \left[ \exp \left( \frac{s}{\sqrt{\lambda t}} \right) - 1 \right] \right\} = \lim_{t \to \infty} \left\{ -st + t^2 \left[ \exp \left( \frac{s}{t} \right) - 1 \right] \right\} = \lim_{t \to \infty} \left\{ -st + t^2 \left[ 1 + \frac{s}{t} + \frac{s^2}{2t^2} + O \left( \frac{1}{t^3} \right) - 1 \right] \right\} = \frac{s^2}{2},
$$

we have that \( Y_t = \frac{X_t - \lambda t}{\sqrt{\lambda t}} \) converges to a Gaussian random variable with zero mean and unit variance as \( t \to \infty \). Consequently,

$$
\Pr \{ pt - a \leq X_t \leq pt + b \mid \lambda \} = \Pr \left\{ \frac{(\rho - \lambda)t - a}{\sqrt{\lambda t}} \leq Y_t \leq \frac{(\rho - \lambda)t + b}{\sqrt{\lambda t}} \mid \lambda \right\} \to 0
$$

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Lemma 13 For arbitrary integer \( n \) and real numbers \( t_i, i = 0, \cdots, n \) with \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \), the conditional probability mass function of \( X_{t_i}, i = 0, 1, \cdots, n - 1 \) given the value of \( X_{t_n} \) does not depend on \( \lambda \).

**Proof.** Note that for a Poisson process \( (X_t)_{t \in [0, \infty)} \) with an arrival rate \( \lambda > 0 \), we have

\[
\frac{\Pr\{X_{t_i} = x_i, i = 1, \cdots, n\}}{\Pr\{X_{t_n} = x_n\}} = \frac{\prod_{i=1}^{n} \frac{[(t_i - t_{i-1})\lambda]^x_i e^{-\lambda(t_i - t_{i-1})}}{(x_i!)}}{(t_n\lambda)^x_n e^{-\lambda t_n}} = \frac{x_n!}{(t_n)^x_n} \prod_{i=1}^{n} \frac{(t_i - t_{i-1})^{x_i} e^{-\lambda(t_i - t_{i-1})}}{(x_i - x_{i-1})!},
\]

where \( x_0 = 0 \). This implies that the conditional PMF of \( X_{t_i}, i = 1, \cdots, n - 1 \) given the value of \( X_{t_n} \) does not involve \( \lambda \).

\[\square\]

We are now in a position to prove the theorem. It can be readily checked that \( f_t(X_t; \lambda) \) is unimodal with respect to \( \lambda > 0 \). Applying this fact and Lemma 12 leads to the conclusion that the observational process will eventually terminate with probability 1. As a consequence of the proven termination property and Lemma 13 statements (I), (II) and (III) of Theorem 12 follow from Theorem 10. This completes the proof of the theorem.

**J Proof of Theorem 14**

We need some preliminary results.

Lemma 14 For arbitrary \( \alpha, \beta \in (0, 1) \) and \( \mu, \mu', \mu'' \in (-\infty, \infty) \) with \( \mu < \mu'' \),

\[
\lim_{t \to \infty} \Pr \left\{ \beta \leq Y_t(X_t; \mu', \mu'') \leq \frac{1}{\alpha} \mid \mu \right\} = 0.
\]

**Proof.** Note that

\[
\Pr \left\{ \beta \leq Y_t(X_t; \mu', \mu'') \leq \frac{1}{\alpha} \mid \mu \right\} = \Pr \left\{ \frac{(\mu' + \mu'')t}{\frac{\sigma^2}{\mu'' - \mu'}} + \frac{\sigma^2}{\mu'' - \mu'} \ln \beta \leq X_t \leq \frac{(\mu' + \mu'')t}{\frac{\sigma^2}{\mu'' - \mu'}} + \frac{\sigma^2}{\mu'' - \mu'} \ln \frac{1}{\alpha} \mid \mu \right\}.
\]

Thus, \( \Pr \{ \beta \leq Y_t(X_t; \mu', \mu'') \leq \frac{1}{\alpha} \mid \mu \} \) can be written as \( \Pr \{ \rho t - a \leq X_t \leq \rho t + b \mid \mu \} \), where \( \rho = \frac{\mu' + \mu''}{\frac{\sigma^2}{\mu'' - \mu'}} \) and \( a, b \) are some positive numbers. Define \( Y_t = \frac{X_t - \mu}{\sigma \sqrt{t}} \). Then, \( Y_t \) is a Gaussian random variable with zero mean and unit variance. It follows that

\[
\Pr \{ \rho t - a \leq X_t \leq \rho t + b \mid \mu \} = \Pr \left\{ \frac{(\rho - \mu)t - a}{\sigma \sqrt{t}} \leq Y_t \leq \frac{(\rho - \mu)t + b}{\sigma \sqrt{t}} \mid \mu \right\} \to 0
\]
as \( t \to \infty \), which can be readily shown by considering the cases of \( \mu < \rho, \mu = \rho \) and \( \mu > \rho \).

\[\square\]

Lemma 15 For arbitrary integer \( n \) and real numbers \( t_i, i = 0, \cdots, n \) with \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \), the conditional probability density function of \( X_{t_i}, i = 0, 1, \cdots, n - 1 \) given the value of \( X_{t_n} \) does not depend on \( \mu \).
Proof. Define $Z_i = X_{t_i} - X_{t_{i-1}}$ for $i = 1, \ldots, n$, where $X_{t_0} = X_0 = 0$. Then, $Z_i$ are independent Gaussian variables with PDFs

$$f_i(z_i) = \frac{1}{\sqrt{2\pi(t_i - t_{i-1})\sigma}} \exp \left( -\frac{(z_i - (t_i - t_{i-1})\mu)^2}{2(t_i - t_{i-1})\sigma^2} \right), \quad i = 1, \ldots, n.$$  

Note that $\Pr\{X_{t_i} \leq x_i, \ i = 1, \ldots, n\} = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f_1(z_1) \cdots f_n(z_n) \, dz_1 \cdots dz_n$, where $S = \{(z_1, \ldots, z_n) : \sum_{i=1}^n z_i \leq x_j \text{ for } j = 1, \ldots, n\}$. Define $y_j = \sum_{i=1}^j z_i$ for $j = 1, \ldots, n$. Then, $z_1 = y_1$ and $z_j = y_j - y_{j-1}$ for $j = 2, \ldots, n$. Note that the determinant of the Jacobian of the transformation is equal to 1 and thus

$$\Pr\{X_{t_i} \leq x_i, \ i = 1, \ldots, n\} = \int_{-\infty}^{x_1} f_1(y_1) \cdots \int_{-\infty}^{x_j} f_j(y_j - y_{j-1}) \cdots \int_{-\infty}^{x_n} f_n(y_n - y_{n-1}) \, dy_n \cdots dy_j \cdots dy_1.$$

Sequentially taking partial derivatives of the multiple integral with respect to $x_n, x_{n-1}, \ldots, x_1$ gives

$$\frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} \Pr\{X_{t_i} \leq x_i, \ i = 1, \ldots, n\} = \prod_{i=1}^n f_i(x_i - x_{i-1}), \quad x_0 \overset{\text{def}}{=} 0.$$

It can be checked that

$$\frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} \Pr\{X_{t_n} \leq x_n\} = \prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})\sigma}} \exp \left( -\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})\sigma^2} \right),$$

which is independent of $\mu$.

We are now in a position to prove the theorem. It can be readily checked that $f_i(X_t; \mu, \sigma)$ is unimodal with respect to $\mu \in (-\infty, \infty)$. This fact together with Lemma 14 lead to the conclusion that the observational process will eventually terminate with probability 1. As a consequence of the proven termination property and Lemma 15 statements (I), (II) and (III) of Theorem 14 follow from Theorem 10. This completes the proof of the theorem.

References

[1] P. Armitage, “Sequential analysis with more than two alternative hypotheses, and its relation to discriminant function analysis,” J. Roy. Statist. Soc. B, vol. 12, pp. 137–144, 1950.

[2] C. W. Baum and V. V. Veeravalli, “A sequential procedure for multihypothesis testing,” IEEE Trans. Inform. Theory, vol. 40, pp. 1994–2007, Nov. 1994.

[3] X. Chen, “Sequential tests of statistical hypotheses with confidence limits,” arXiv.1007.4278 [math.ST], July 2010 – present, multiple versions.

[4] X. Chen, “A new framework of multistage estimation,” arXiv.0809.1241 [math.ST], September 2008 – present, multiple versions.

[5] X. Chen, “A new framework of multistage hypothesis tests,” arXiv.0809.3170 [math.ST], September 2008 – present, multiple versions.

[6] X. Chen, “Confidence interval for the mean of a bounded random variable and its applications in point estimation,” arXiv:0802.3458v2 [math.ST], 2009.
[7] X. Chen, “A computational approach for statistical learning and inference,” *Proceeding of SPIE Conference*, Baltimore, Maryland, April 24–27, 2012.

[8] B. Eisenberg, *Multihypothesis Problems*, in *Handbook of Sequential Analysis*, pp. 229–244, edited by B. K. Ghosh and P.K. Sen, Marcel Dekker, 1991.

[9] B. K. Ghosh, *Sequential Tests of Statistical Hypotheses*, Addison Wesley, 1970.

[10] G. Lorden, “Likelihood ratio tests for sequential $k$-decision problems,” *Ann. Math. Stat.*, vol. 43, pp. 1412–1427, 1972.

[11] J. Ville, *Étude critique de la notion de collectif*, Paris 1939.

[12] A. Wald, *Sequential Analysis*, Wiley, 1947.