The Schwinger Nonet Mass and Sakurai Mass-Mixing Angle Formulae Reexamined

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Abstract

We study the origins of the inaccuracies of Schwinger’s none t mass, and the Sakurai mass-mixing angle, formulae for the pseudoscalar meson nonet, and suggest new versions of them, modified by the inclusion of the pseudoscalar decay constants. We use these new formulae to determine the pseudoscalar decay constants and mixing angle. The results obtained, $f_8/f_\pi = 1.185 \pm 0.040$, $f_9/f_\pi = 1.095 \pm 0.020$, $f_9/f_\pi = 1.085 \pm 0.025$, $f_\eta'/f_\pi = 1.195 \pm 0.035$, $\theta = (-21.4 \pm 1.0)\degree$, are in excellent agreement with experiment.

Key words: Schwinger’s formula, Gell-Mann–Okubo, chiral Lagrangian, pseudoscalar mesons

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1 Introduction

Schwinger’s original nonet mass formula [1] (here the symbol for the meson stands either for its mass or mass squared),

\[(4K - 3\eta - \pi)(3\eta' + \pi - 4K) = 8(K - \pi)^2, \tag{1}\]

and the Sakurai mass-mixing angle formula [2],

\[\tan^2 \theta = \frac{4K - 3\eta - \pi}{3\eta' + \pi - 4K}, \tag{2}\]

both relate the masses of the isovector ($\pi$), isodoublet ($K$) and isoscalar mostly octet ($\eta$) and mostly singlet ($\eta'$) states of a meson nonet, and the nonet mixing angle ($\theta$). We alert the reader that, although we use notation suggestive of masses below, each formula is to be reprised in terms of mass or mass squared values in this introductory section.

The relations (1) and (2) are usually derived in the following way: For a meson nonet, the isoscalar octet-singlet mass matrix,

\[M = \begin{pmatrix} M_{88} & M_{89} \\ M_{98} & M_{99} \end{pmatrix}, \tag{3}\]

is diagonalized by the masses of the physical $\eta$ and $\eta'$ states:

\[M = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta' \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \tag{4}\]

where $\theta$ is the nonet mixing angle, which is determined by comparing the corresponding quadrants of the matrices (3) and (4), by any of the three following relations:

\[\tan^2 \theta = \frac{M_{88} - \eta}{\eta' - M_{88}}, \tag{5}\]

\[\tan^2 \theta = \frac{\eta' - M_{99}}{M_{99} - \eta}, \tag{6}\]

\[\sin 2\theta = \frac{2M_{89}}{\eta' - \eta}. \tag{7}\]

It is easily seen that Eqs. (5) and (6) are identical, since, due to the trace invariance of $M$,

\[\eta + \eta' = M_{88} + M_{99}, \tag{8}\]

and therefore, $M_{88} - \eta = \eta' - M_{99}$, and $\eta' - M_{88} = M_{99} - \eta$. Eliminating $\theta$ from (5),(7), or (6),(7), with the help of $\sin 2\theta = 2\tan \theta/(1 + \tan^2 \theta)$, leads, respectively, to

\[(\eta - M_{88})(M_{88} - \eta) = M_{89}^2. \tag{9}\]
\[(M_{99} - \eta)(\eta' - M_{99}) = M_{89}^2, \] (10)

which again are identical, through (8).

We note that the “ideal” structure of a meson nonet,

\[(\eta = 2K - \pi, \ \eta' = \pi), \ \text{or} \ \ (\eta = \pi, \ \eta' = 2K - \pi), \] (11)

and the corresponding pure quark physical state valence flavor wavefunctions,

\[
\omega_\eta = -s \bar{s}, \ \ \omega_{\eta'} = \frac{u \bar{u} + d \bar{d}}{\sqrt{2}} \equiv n \bar{n}, \ \text{or} \ \ \omega_\eta = n \bar{n}, \ \omega_{\eta'} = s \bar{s},
\] (12)

given by the mixing

\[
\omega_\eta = \omega_8 \cos \theta - \omega_9 \sin \theta, \ \ \omega_{\eta'} = \omega_8 \sin \theta + \omega_9 \cos \theta
\] (13)

with the “ideal” nonet mixing angle,

\[
\theta = \arctan \frac{1}{\sqrt{2}} \approx 35.3^\circ, \ \ \text{or} \ \ \theta = -\arctan \sqrt{2} \approx -54.7^\circ, \] (14)

and the definitions

\[
\omega_8 = \frac{u \bar{u} + d \bar{d} - 2s \bar{s}}{\sqrt{6}}, \ \ \omega_9 = \frac{u \bar{u} + d \bar{d} + s \bar{s}}{\sqrt{3}},
\] (15)

are the unique solution to Eqs. (9),(10) under the quark model inspired conditions

\[
M_{88} = \frac{4K - \pi}{3}, \ \ M_{99} = \frac{2K + \pi}{3}, \ \ M_{89} = -\frac{2\sqrt{2}}{3}(K - \pi), \] (16)

where the first of the three relations in (16) is the standard Gell-Mann–Okubo mass formula [3].

For all well established meson nonets, except the pseudoscalar (and, we expect, scalar) one(s), both linear and quadratic versions of Eqs. (5)-(10) are in good agreement with experiment. For example, for vector mesons, if one assumes the validity of the Gell-Mann–Okubo formula \(\omega_8 = (4K^* - \rho)/3\), then one obtains from Eq. (9) with the measured meson masses [4], \(M_{89} = -0.209 \pm 0.001\) GeV\(^2\) in the quadratic case, and \(-0.113 \pm 0.001\) GeV in the linear case. Note that this is entirely consistent with \(-0.196 \pm 0.005\) GeV\(^2\) and \(-0.118 \pm 0.003\) GeV, respectively, which follow from the third element of (16). For tensor mesons, a similar comparison gives \(-0.305 \pm 0.020\) GeV\(^2\) vs. \(-0.287 \pm 0.015\) GeV\(^2\), and \(-0.105 \pm 0.008\) GeV vs. \(-0.104 \pm 0.005\) GeV, respectively.

However, for the pseudoscalar nonet, one obtains from Eq. (5) with meson masses squared, \(\theta \approx -11^\circ\), in sharp disagreement with experiment, which favors the \(\eta-\eta'\) mixing angle in the vicinity of \(-20^\circ\) [4, 5, 6]. Although using linear meson masses in Eq. (5) does give \(\theta \approx -24^\circ\), in better agreement with data than its mass-squared counterpart, the value of \(M_{89}\), as given by (9), is now \(-0.165 \pm 0.004\) GeV, vs.
-2\sqrt{2}/3 (K - \pi) = -0.338 \pm 0.004 \text{ GeV}. This emphasizes that neither Schwinger’s nonet mass formula nor the mass-mixing angle relations (including Sakurai’s) (5)-(7) hold for the pseudoscalar nonet.

It is well known, however, that the pseudoscalar (and, probably, scalar) mass spectrum does not follow the “ideal” structure, Eq. (16), since the mass of the pseudoscalar isoscalar singlet state is shifted up from its “ideal” value of (2K + \pi)/3, presumably by the instanton-induced ’t Hooft interaction \[7\] which breaks axial U(1) symmetry \[8, 9, 10\]. However, the use of \(M_{99} = (2K + \pi)/3 + A, A \neq 0\), in Eqs. (5)-(8) will again lead to Schwinger’s formula (9), which does not hold for the pseudoscalar mesons, as just demonstrated. [In fact, the structure of this formula does not depend at all on \(M_{99}\), as seen in (9).] Therefore, instanton, as well as any other effects which may shift the mass of the pseudoscalar isoscalar singlet state, cannot constitute the explanation of the failure of Schwinger’s quartic mass and the Sakurai mass-mixing angle formulae for the pseudoscalar nonet. We believe, however, that the following analysis can resolve this problem.

2 Pseudoscalar meson mass squared matrix

It is known that the observed mass splitting among the pseudoscalar nonet may be induced (in terms of the 1/\(N_c\) expansion) by the following symmetry breaking terms \[8\].

\[
L_m^{(0)} = \frac{\tilde{f}^2}{4} \left( B \text{ Tr} M \left( U + U^\dagger \right) + \frac{\varepsilon}{6N_c} \left[ \text{ Tr} \left( \ln U - \ln U^\dagger \right) \right]^2 \right),
\]

with \(M\) being the quark mass matrix,

\[
M = \text{ diag } (m_u, m_d, m_s) = m_s \text{ diag } (x, y, 1), \quad x \equiv \frac{m_u}{m_s}, \quad y \equiv \frac{m_d}{m_s},
\]

\(N_c\) the number of colors, \(\tilde{f}\) the pseudoscalar decay constant in the limit of exact nonet symmetry, and \(B, \varepsilon = \text{ const}\). This symmetry breaking Lagrangian term is to be added to the U(3)\(_L\) × U(3)\(_R\) invariant non-linear Lagrangian

\[
L^{(0)} = \frac{\tilde{f}^2}{4} \text{ Tr } \left( \partial_\mu U \partial^\mu U^\dagger \right),
\]

with

\[
U = \exp(i\pi/\tilde{f}), \quad \pi \equiv \lambda_a \pi^a, \quad a = 0, 1, \ldots, 8,
\]

which incorporates the constraints of current algebra for the light pseudoscalars \(\pi^a\) \[11\].

As pointed out in ref. \[12\], chiral corrections can be important, the kaon mass being half the typical 1 GeV chiral symmetry breaking scale. Such large corrections are clearly required from the study of the octet-singlet mass squared matrix \(M^2\). In the isospin limit \(x = y\) one has \[13\] (with \(m_n \equiv (m_u + m_d)/2\)

\[
M^2 = B \left( \frac{2}{3} \left( 2m_s + m_n \right) \quad \frac{2\sqrt{3}}{3} \left( m_n - m_s \right) \quad \frac{2}{3} \left( m_s + 2m_n \right) + \frac{\varepsilon}{B N_c} \right),
\]
which, on the most naive level, through the schematic Gell-Mann–Oakes-Renner relations (to first order in chiral symmetry breaking) \[14\],

\[
\begin{align*}
\pi^2 &= 2B m_n, \\
K^2 &= B (m_s + m_n), \\
\eta^2 &= \frac{2}{3} B (2m_s + m_n),
\end{align*}
\]

(21)

which we discuss in more detail below, reduces to

\[
M^2 = \frac{1}{3} \begin{pmatrix}
4K^2 - \pi^2 & -2\sqrt{2} (K^2 - \pi^2) & 2K^2 + \pi^2 + 3\tilde{A} \\
-2\sqrt{2} (K^2 - \pi^2) & 2K^2 + \pi^2 + 3\tilde{A} \\
2\sqrt{2} (K^2 - \pi^2) & 2K^2 + \pi^2 + 3\tilde{A}
\end{pmatrix}, \quad \tilde{A} \equiv \frac{\varepsilon}{BN_c}.
\]

(22)

Also, the Nambu–Jona-Lasinio model with the instanton-induced 't Hooft interaction, initiated by Hatsuda and Kunihiro \[15\] and Bernard et al. \[16\], and then extensively studied by Dmitrasinovic \[9, 10\], provides shifts of both the pseudoscalar and scalar isoscalar singlet masses by the same amount, but in opposite directions (viz., the pseudoscalar isoscalar singlet mass is increased, while the scalar one is decreased). Thus, for the pseudoscalar mesons, rather than using the model-dependent \(\tilde{A}\), as defined in (22), we introduce the quantity \(A\) below, which may be considered as the sum of all possible contributions to the shift of the isoscalar singlet mass (from instanton effects, \(1/N_c\)-expansion diagrams, gluon annihilation diagrams, etc.).

Here, however, we suggest that the actual form of the mass squared matrix for the pseudoscalar mesons (and the corresponding symmetry breaking terms in (17)) is as follows:

\[
\bar{f}^2 M^2 = \frac{1}{3} \begin{pmatrix}
4f_K^2 K^2 - f_\pi^2 \pi^2 & -2\sqrt{2} (f_K^2 K^2 - f_\pi^2 \pi^2) & 2f_K^2 K^2 + f_\pi^2 \pi^2 + 3f_\pi^2 A \\
-2\sqrt{2} (f_K^2 K^2 - f_\pi^2 \pi^2) & 2f_K^2 K^2 + f_\pi^2 \pi^2 + 3f_\pi^2 A \\
2\sqrt{2} (f_K^2 K^2 - f_\pi^2 \pi^2) & 2f_K^2 K^2 + f_\pi^2 \pi^2 + 3f_\pi^2 A
\end{pmatrix},
\]

(23)

where \(f\)'s are the pseudoscalar decay constants defined below.

Indeed, the form of such a mass squared matrix is determined by the form of Gell-Mann–Okubo type relations among the masses of the isovector, isodoublet, and isoscalar octet and singlet states (which in our case are Eqs. (26)-(28) below), since this matrix must be equivalent to that of the form (3), which in the case we are considering is

\[
\bar{f}^2 M^2 = \begin{pmatrix}
f_{\pi}^2 \eta_{88}^2 & f_{\pi} f_{\pi} \eta_{89}^2 & f_{\pi} f_{\pi} \eta_{99}^2 \\
f_{\pi} f_{\pi} \eta_{89}^2 & f_{\pi}^2 \eta_{89}^2 & f_{\pi}^2 \eta_{99}^2 \\
f_{\pi} f_{\pi} \eta_{99}^2 & f_{\pi} f_{\pi} \eta_{99}^2 & f_{\pi}^2 \eta_{99}^2
\end{pmatrix},
\]

(24)

and is diagonalized by the physical \(\eta\) and \(\eta'\) meson masses and decay constants:

\[
\bar{f}^2 M^2 = \begin{pmatrix}
f_\pi^2 \eta_{88}^2 & 0 & 0 \\
0 & f_\pi^2 \eta_{99}^2 & 0 \\
0 & 0 & f_\pi^2 \eta_{99}^2
\end{pmatrix}.
\]

(25)

The equivalence of the matrices (23) and (24) is guaranteed by the validity of the following relations:

\[
f_{\pi}^2 \eta_{88}^2 = -\frac{1}{3} \left[ m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle + 4m_s \langle \bar{s}s \rangle \right] = \frac{4f_K^2 K^2 - f_\pi^2 \pi^2}{3},
\]

(26)
\[
f_{8}f_{9}\eta_{99}^{2} = -\frac{\sqrt{2}}{3}\left[m_{u}\langle\bar{u}u\rangle + m_{d}\langle\bar{d}d\rangle - 2m_{s}\langle\bar{s}s\rangle\right] = \frac{2\sqrt{2}}{3}\left(f_{\pi}^{2}\pi^{2} - f_{K}^{2}K^{2}\right),
\]
with
\[
f_{9}\eta_{99}^{2} = f_{9}^{2}A - \frac{2}{3}\left[m_{u}\langle\bar{u}u\rangle + m_{d}\langle\bar{d}d\rangle + m_{s}\langle\bar{s}s\rangle\right] = f_{9}^{2}A + \frac{2f_{K}^{2}K^{2} + f_{\pi}^{2}\pi^{2}}{3},
\]
(28)

as suggested by Dmitrasinovic\[10\], on the basis of the (precise) Gell-Mann–Oakes-Renner formulae which relate the pseudoscalar masses and decay constants to the quark masses and condensates\[14\]:

\[
f_{\pi}^{2}\pi^{2} = -\left[m_{u}\langle\bar{u}u\rangle + m_{d}\langle\bar{d}d\rangle\right],
\]
(30)

\[
f_{K}^{2}\left(K^{\pm}\right)^{2} = -\left[m_{u}\langle\bar{u}u\rangle + m_{s}\langle\bar{s}s\rangle\right],
\]
(31)

\[
f_{K}^{2}\left(K^{0}\right)^{2} = -\left[m_{d}\langle\bar{d}d\rangle + m_{s}\langle\bar{s}s\rangle\right].
\]
(32)

(We ignore Dashen’s theorem violating effects\[17\], and only approximately take into account isospin violating effects via (29), as we are not concerned here with accuracies better than 1%).

Note that in the limit of exact nonet symmetry,
\[
f_{\pi} = f_{K} = f_{88} = f_{99} \equiv \bar{f}, \quad \langle\bar{u}u\rangle = \langle\bar{d}d\rangle = \langle\bar{s}s\rangle \equiv \langle\bar{q}q\rangle,
\]
(33)

one has the mass squared matrix (20) with \(B = -\langle\bar{q}q\rangle/f^{2}\), which further reduces to (22). The real world, and the mass squared matrix (23) associated with it, however (as we shall see), corresponds to the situation when the first set of the relations (33) is broken, but the second one remains (approximately) satisfied, i.e., the amount of SU(3) flavor symmetry breaking in terms of the quark condensates is much smaller than that in terms of the pseudoscalar decay constants.

### 3 Modified Gell-Mann–Okubo mass formula

Here, we shall only explicitly demonstrate the validity of the modified Gell-Mann–Okubo formula (26) (the remaining relations (27) and (28) may be checked in a similar way).

First, this relation, as well as the mass squared matrix (23),(24), may be obtained through relating the vacuum expectation values of the equal time axial divergence-axial current commutators (which are “sigma” commutators in Eq. (34),(35) below) to integrals over the pseudoscalar meson spectrum, as done by Gensini\[18\] following Gatto et al.\[19\]. For the symmetry realized through a set of massless Goldstone pseudoscalar mesons, only the pole terms survive in the first order of the symmetry breaking and are expected to dominate over the continuum, which contributes only at the second-order level. We therefore have the identities\[19\]

\[
\langle\sigma_{ab}(0)\rangle = f_{a}f_{b}M_{ab}^{2} + \int d\mu \rho_{ab}(\mu),
\]
(34)
which define the current quark mass contributions to the pseudoscalar masses. These may be decomposed as

\[ M_{ab}^2 = \frac{\langle \sigma_{ab}(0) \rangle}{f_{a} f_{b}} + (M_{e.m.})_{ab} + (M_{g}^2)_{ab} + (M_{h.o.t.}^2)_{ab}, \]  

(35)

where \( M_{e.m.} \) is the long-range electromagnetic contribution, \( M_{g}^2 \) is the gluon term which stands for both perturbative two-gluon annihilation and nonperturbative instanton effects. (Both are responsible for the shift of the isoscalar singlet mass prior to its mixing with the isoscalar octet, leading to the physical \( \eta \) and \( \eta' \) masses.) The latter must therefore act only on the singlet-singlet state matrix element:

\[ (M_{g}^2)_{ab} = A \delta_{a9} \delta_{b9}. \]  

(36)

Finally, \( M_{h.o.t.}^2 \) stands for the contribution of the higher order terms.

Mass terms, which are the only explicit symmetry-breaking terms, have the general form

\[ L_m(x) = \varepsilon_0 S_0(x) + \varepsilon_3 S_3(x) + \ldots + \varepsilon_{N^2-1} S_{N^2-1}(x), \]  

(37)

where \( S_i = \bar{\psi}(x) \lambda_i/2 \psi(x), \lambda_0 = (2/N)^{1/2} \mathbf{1} \), and \( \lambda_{M^2-1} \) are the Gell-Mann-type matrices which constitute the SU(M) basis, \( M \leq N \). Restricting the indices to a subgroup SU\((N')\), \( N' < N \), does not introduce higher mass quark fields anywhere but in the singlet-singlet part, where they can be absorbed into the gluon term within the definition of higher-order terms given above. This restriction is meaningful only if the mixing of low mass \( \bar{\psi}\psi \) pairs with higher mass ones is small enough. This is precisely what preserves the approximate SU(3) symmetry of hadronic interactions, independent of the total number of flavors.

In terms of the current quark masses we also have

\[ \varepsilon_0 = \left( \frac{N}{2} \right)^{-1/2} \sum_{i=1}^{N} m_i, \]  

(38)

\[ \varepsilon_{M^2-1} = [M(M-1)]^{-1/2} \left( \sum_{i=1}^{M-1} m_i - (M-1) m_M \right). \]  

(39)

Assuming the SU(3)-invariant vacuum, fixing \( M = 3 \), and neglecting long-range electromagnetic effects, one obtains in the lowest order of the symmetry breaking

\[ (\pi^\pm)^2 = \frac{f_\pi^2}{f_{K^0}^2} B (m_u + m_d), \]  

\[ (K^\pm)^2 = \frac{f_\pi^2}{f_{K^0}^2} B (m_u + m_s), \]  

\[ (K^0)^2 = \frac{f_\pi^2}{f_{K^0}^2} B (m_d + m_s). \]  

(40)

\[ \text{We use the notations of ref. [18].} \]
in the charged sector, and

\[
\begin{align*}
\eta^2_{33} & = \frac{f^2}{f^2_\pi} B (m_u + m_d), \\
\eta^2_{38} & = \frac{f^2}{f^2_\pi f_8} B \frac{m_u - m_d}{\sqrt{3}}, \\
\eta^2_{39} & = \frac{f^2}{f^2_\pi f_9} B \sqrt{\frac{2}{3}} (m_u - m_d), \\
\eta^2_{88} & = \frac{f^2}{f^2_8} B \frac{4m_s + m_u + m_d}{3}, \\
\eta^2_{89} & = -\frac{f^2}{f^2_8} B \frac{\sqrt{3}}{3} (2m_u - m_u - m_d), \\
\eta^2_{99} & = \frac{f^2}{f^2_9} B \frac{2}{3} (m_s + m_u + m_d) + A
\end{align*}
\] (41)

in the neutral \( Y = I_3 = 0 \) sector. Eliminating the quark masses from the above two sets of relations, one obtains

\[
\begin{align*}
f^2_\pi \eta^2_{33} & = f^2_\pi (\pi^\pm)^2, \\
f^2_8 \eta^2_{88} & = \frac{1}{3} \left[ 2f^2_K \left( (K^\pm)^2 + 2(K^0)^2 \right) - f^2_\pi (\pi^\pm)^2 \right], \\
f_8 f_9 \eta^2_{89} & = -\frac{\sqrt{2}}{3} \left[ f^2_K \left( (K^\pm)^2 + (K^0)^2 \right) - 2f^2_\pi (\pi^\pm)^2 \right], \\
f^2_9 \eta^2_{99} & = \frac{1}{3} \left[ f^2_K \left( (K^\pm)^2 + (K^0)^2 \right) + f^2_\pi (\pi^\pm)^2 \right] + f^2_9 A, \quad (42)
\end{align*}
\]

which corresponds to Eqs. (26)-(28). Using the definition (20), one can also obtain the mass squared matrix (23),(24) from Eqs. (40)-(42). (We have, consistently, ignored small \( \pi^0 - \eta \) and \( \eta' [\eta_{38}, \eta_{39}] \) mixing effects of electromagnetic and isospin breaking origin).

Independently, the relation (26) may be obtained from the following expressions for the pseudoscalar meson masses calculated by Li [20] in the chiral effective field theory of mesons \( (\mu, \Lambda = \text{const}) \):

\[
\begin{align*}
\pi^2 & = \frac{16N_c \mu^3}{(4\pi)^2 f^2_\pi} \left( \ln \frac{\Lambda^2}{\mu^2} - \gamma + 1 \right) (m_u + m_d), \\
(K^\pm)^2 & = \frac{16N_c \mu^3}{(4\pi)^2 f^2_K} \left( \ln \frac{\Lambda^2}{\mu^2} - \gamma + 1 \right) (m_u + m_s), \\
(K^0)^2 & = \frac{16N_c \mu^3}{(4\pi)^2 f^2_K} \left( \ln \frac{\Lambda^2}{\mu^2} - \gamma + 1 \right) (m_d + m_s), \\
\eta^2_{88} & = \frac{16N_c \mu^3}{(4\pi)^2 f^2_{88}} \left( \ln \frac{\Lambda^2}{\mu^2} - \gamma + 1 \right) \frac{m_u + m_d + 4m_s}{3}. \quad (43)
\end{align*}
\]
Additionally, even at lowest order, \( O(p^2) \), in generalized chiral perturbation theory, the inclusion of decay constants is required to form a relation among the octet pseudoscalar mesons \[21\]. Finally, this formula may be also obtained in standard chiral perturbation theory \[22\], as follows.

Standard chiral perturbation theory leads to the following expressions for the pseudoscalar meson masses and decay constants \[22\]:

\[
\begin{align*}
\pi^2 &= 2m_n B \left( 1 + \mu_\pi - \frac{1}{3} \mu_8 + 2m_n K_3 + K_4 \right), \\
K^2 &= (m_n + m_s) B \left( 1 + \frac{2}{3} \mu_8 + (m_n + m_s) K_3 + K_4 \right), \\
\eta_{88}^2 &= \frac{2}{3} (m_n + 2m_s) B \left( 1 + 2\mu_K - \frac{4}{3} \mu_8 + \frac{2}{3} (m_n + 2m_s) K_3 + K_4 \right) \\
&\quad + 2m_n B \left( -\mu_\pi + \frac{2}{3} \mu_K + \frac{1}{3} \mu_8 \right) + K_5, \\
f_\pi &= \bar{f} \left( 1 - 2\mu_\pi - \mu_K + 2m_n K_6 + K_7 \right), \\
f_K &= \bar{f} \left( 1 - \frac{3}{4} \mu_\pi - \frac{3}{2} \mu_K - \frac{3}{4} \mu_8 + (m_n + m_s) K_6 + K_7 \right), \\
f_8 &= \bar{f} \left( 1 - 3\mu_K + \frac{2}{3} (m_n + 2m_s) K_6 + K_7 \right),
\end{align*}
\]

where \( \mu \)'s are chiral logarithms, and the constants \( K_i \) are proper combinations of the low energy coupling constants \( L_i \). It then follows from these relations that the standard Gell-Mann–Okubo formula is broken in first nonleading order \[22\],

\[
\Delta_{\text{GMO}} \equiv 4K^2 - 3\eta_{88}^2 - \pi^2 = -2 \left( 4K^2 \mu_K - 3\eta_{88}^2 \mu_8 - \pi^2 \mu_\pi \right) + \ldots \\
= 4B \left[ m_n (\mu_\pi + \mu_8 - 2\mu_K) + m_s (2\mu_8 - 2\mu_K) \right] + \ldots ,
\]

where \( \ldots \) stands for the higher order terms.

However, the modified Gell-Mann–Okubo formula remains valid in this order, and is violated only by second order SU(3)-flavor breaking effects:

\[
\Delta'_{\text{GMO}} \equiv \frac{1}{f^2} \left( 4f_K^2 K^2 - 3f_8^2 \eta_{88}^2 - f_\pi^2 \pi^2 \right) = 4B (m_s - m_n) \left( \mu_K + \frac{1}{2} \mu_8 - \frac{3}{2} \mu_\pi \right) + \ldots ,
\]

since the second factor on the r.h.s. of (46) must vanish in the SU(3)-flavor limit. Thus, the above analyses show, in an almost model independent way, that the modified Gell-Mann–Okubo formula (26) is the only valid relation among the octet pseudoscalar mesons, and therefore, the form of the mass squared matrix (23) is completely justified. \[3\]

\[2\] A search for a relation of a more general form, \( 4f_K^2 K^2 = 3f_8^2 \eta_{88}^2 + f_\pi^2 \pi^2 \), which would hold in the first nonleading order of standard chiral perturbation theory, results in \( a = 2 \).

\[3\] It has been suggested in the literature that the pseudoscalar decay constants should enter relations like (26)-(28) in the first rather than second power \[23\]. As discussed above, such relations are expected to be less accurate than ours, according to chiral perturbation theory.
4 The Schwinger and Sakurai formulae reexamined

Starting with the mass squared matrix (23), the considerations which lead to Eqs. (5)-(10) above, will now lead, through (26)-(28), to the following two relations,

\[ \sin^2 \theta = \frac{4f_K^2 K^2 - 3f_\eta^2 \eta^2 - f_\pi^2 \pi^2}{3f_\eta^2 \eta^2 + f_\pi^2 \pi^2 - 4f_K^2 K^2}, \quad (47) \]

\[ \left(4f_K^2 K^2 - 3f_\eta^2 \eta^2 - f_\pi^2 \pi^2\right) \left(3f_\eta^2 \eta^2 + f_\pi^2 \pi^2 - 4f_K^2 K^2\right) = 8 \left(f_K^2 K^2 - f_\pi^2 \pi^2\right)^2, \quad (48) \]

which we refer to as “the Schwinger nonet mass, and Sakurai mass-mixing angle (respectively), formulae reexamined”.

In contrast to \(f_\pi\) and \(f_K\), the values of which are well established experimentally [4],

\[ \sqrt{2} f_K = 159.8 \pm 1.6 \text{ MeV}, \quad \sqrt{2} f_\pi = 130.7 \pm 0.3 \text{ MeV}, \quad \frac{f_K}{f_\pi} = 1.22 \pm 0.01 \quad (49) \]

the values of \(f_\eta\), \(f_\eta'\) and \(\theta\) are known rather poorly. We now wish to calculate the values of \(f_\eta\), \(f_\eta'\) and \(\theta\), using the relations (47),(48), and compare the results with available experimental data. It is obvious that the two relations are not enough for determining the three unknowns. Note that the additional relation, independent of (47),(48) (the trace condition for (23),(25)),

\[ f_\eta^2 \eta^2 + f_\eta'^2 \eta'^2 = 2f_K^2 K^2 + f_\theta^2 A, \quad (50) \]

introduces an additional unknown, \(A\). We therefore develop another independent relation among \(f_\eta\), \(f_\eta'\) and \(\theta\), as follows.

The light pseudoscalar decay constants are defined by the matrix elements

\[ \langle 0 | \bar{\psi}(0) \gamma^\mu \gamma^5 \frac{\lambda^j}{2} \psi(0) | P(p) \rangle = i\delta^{jp} f_P p^\mu, \quad (51) \]

where \(\psi = (u,d,s)\) is the fundamental representation of SU(3)\(_f\), and \(P = (\pi^0, \eta_{88}, \eta_{99})\) means the corresponding SU(3)\(_f\) indices 3,8,9, thus picking out the diagonal \((j = 3,8)\) SU(3)\(_f\) Gell-Mann matrices \(\lambda^j\), and \(\lambda^9 \equiv \sqrt{2/3} \mathbf{I}\). The neutral pseudoscalar wave functions, \(P\), may be expressed in terms of the quark basis states \(qq\):

\[ |P\rangle = \sum_q \frac{\chi^P_{qq}}{\sqrt{2}} |qq\rangle \equiv \sum_q c_q^P |qq\rangle, \quad q = u,d,s, \quad (52) \]

where for \(P = \pi^0\), \(c_u^8 = 1/\sqrt{2} = \lambda^3_{11}/\sqrt{2} = -\lambda^3_{22}/\sqrt{2} = -c_d^3\), \(c_s^3 = 0\), for \(P = \eta_{88}\), \(c_u^8 = c_d^8 = 1/\sqrt{6} = \lambda^8_{11}/\sqrt{2} = \lambda^8_{22}/\sqrt{2}\), \(c_s^8 = -2/\sqrt{6} = \lambda^8_{33}/\sqrt{2}\), and for \(P = \eta_{99}\), \(c_u^9 = c_d^9 = c_s^9 = 1/\sqrt{3} = (\lambda^9/\sqrt{2})_{qq}\).
The pseudoscalar decay constants defined in (51) can now be expressed as

\[ f_P = \sum_q \frac{(\lambda^P_{qq})^2}{2} f_{qq}, \tag{53} \]

where we have introduced the auxiliary decay constants \( f_{qq} \) defined as the decay constants of the \( qq \) pseudoscalar bound states having the masses \( M(qq) \). In the isospin limit, \( f_{uu} = f_{dd} = f_{ud} = f_{u^0} = f_{\pi^+} \). Using this approximation, and evaluating the appropriate matrix elements leads to the following relations:

\[ f_\eta = \left( \frac{\cos \theta - \sqrt{2} \sin \theta}{\sqrt{3}} \right)^2 f_\pi + \left( \frac{\sin \theta + \sqrt{2} \cos \theta}{\sqrt{3}} \right)^2 f_{s\bar{s}}, \tag{54} \]

\[ f_{\eta'} = \left( \frac{\sin \theta + \sqrt{2} \cos \theta}{\sqrt{3}} \right)^2 f_\pi + \left( \frac{\cos \theta - \sqrt{2} \sin \theta}{\sqrt{3}} \right)^2 f_{s\bar{s}}. \tag{55} \]

Now we have four equations, (47),(48),(54),(55), which allow us to determine three unknowns, \( f_\eta, f_{\eta'}, \theta \), as well as the additional quantity introduced, namely, \( f_{s\bar{s}} \). The solution to these four equations is

\[ \frac{f_\eta}{f_\pi} = 1.085 \pm 0.025, \tag{56} \]

\[ \frac{f_{\eta'}}{f_\pi} = 1.195 \pm 0.035, \tag{57} \]

\[ \frac{f_{s\bar{s}}}{f_\pi} = 1.280 \pm 0.060, \tag{58} \]

\[ \theta = (-21.4 \pm 1.0)^\circ. \tag{59} \]

[The \( \pi \) and \( K \) electromagnetic mass differences, and the uncertainties in the values of \( f_\pi \) and \( f_K \), (see (49)), are taken as a measure of the uncertainties of the results.]

Before comparing the solution obtained with experiment, let us also calculate the values of \( f_8 \) and \( f_9 \) which are obtained from (54),(55) in the no-mixing case (\( \theta = 0 \)):

\[ f_8 = \frac{1}{3} f_\pi + \frac{2}{3} f_{s\bar{s}}, \tag{60} \]

\[ f_9 = \frac{2}{3} f_\pi + \frac{1}{3} f_{s\bar{s}}. \tag{61} \]

Therefore, as follows from (58),(59),

\[ \frac{f_8}{f_\pi} = 1.185 \pm 0.040, \tag{62} \]

\[ \frac{f_9}{f_\pi} = 1.095 \pm 0.020. \tag{63} \]

The \( \eta-\eta' \) mixing angle, as given in (59), is in agreement with most of experimental data which concentrate around \(-20^\circ \) [1, 3, 4]. Also, the values for \( f_8/f_\pi, f_9/f_\pi \) and \( \theta \) are consistent with those suggested in the literature, as we show in Table I.
Note that (62),(63) are almost identical to (57),(56), respectively. This suggests that the quark content of the states must be the same. However, the relative phases, of course, cannot be. As we have suggested elsewhere [30], this conundrum may be resolved by identifying the wavefunctions in (15) with \( \eta \) and \( \eta' \) respectively, but with the signs of the \( s\bar{s} \) terms of each reversed, viz.,

\[
\eta \approx \frac{u\bar{u} + d\bar{d} - s\bar{s}}{\sqrt{3}}, \quad \eta' \approx \frac{u\bar{u} + d\bar{d} + 2s\bar{s}}{\sqrt{6}}.
\]

### 5 Comparison with data

We now wish to compare the values obtained above for the ratios \( f_8/f_\pi, f_9/f_\pi \), and for the \( \eta-\eta' \) mixing angle with available experimental data. We shall first consider in more detail the well-known \( \pi^0, \eta, \eta' \rightarrow \gamma\gamma \) decays, for which experimental data are more complete than those for other processes involving light neutral pseudoscalar mesons, and then briefly mention the \( \eta, \eta' \rightarrow \pi^+\pi^-\gamma \), and \( J/\psi \rightarrow \eta\gamma, \eta'\gamma \) decays.

#### 5.1 \( P^0 \rightarrow \gamma\gamma \) decays

In the case of \( \pi^0 \rightarrow \gamma\gamma \), the anomalous Wess-Zumino-Witten chiral lagrangian predicts

\[
A_{\pi^0\rightarrow\gamma\gamma} = \frac{\alpha N_c}{3\pi f} \epsilon^{\mu \nu \alpha \beta} \epsilon_{1\mu} \epsilon_{2\nu} k_{1\alpha} k_{2\beta}, \tag{64}
\]

where \( \alpha = 1/137.036 \) is the fine structure constant, and \( \epsilon_i, k_i \) are the polarization and momenta of the outgoing photons. A leading-log calculation of the chiral corrections reveals that the dominant effect is simply to replace \( \bar{f} \) by the physical value \( f_\pi \). The resulting amplitude is guaranteed by general theorems to remain unchanged in higher chiral orders. One then finds that the predicted amplitude, as extracted from Eq. (71) below, with the experimentally measured width \[ \Gamma(\pi^0 \rightarrow \gamma\gamma) = (7.7 \pm 0.6) \text{ eV}, \]

\[
F_{\pi\gamma\gamma}(0) = \frac{\alpha N_c}{3\pi f_\pi} = 0.025 \text{ GeV}^{-1}, \tag{65}
\]
is in excellent agreement with experiment [3]:

\[ F_{\pi\gamma\gamma}(0) = (0.025 \pm 0.001) \, \text{GeV}^{-1}, \]  

(66)

thus providing the confidence that one may analyze the corresponding \( \eta, \eta' \) decays with a similar precision.

In the case of the \( \eta, \eta' \rightarrow \gamma\gamma \) decays, one should include both the \( \eta-\eta' \) mixing and the renormalization of the octet-singlet couplings, which leads to the predicted amplitudes

\[
F_{\eta\gamma\gamma}(0) = \frac{\alpha N_c}{3\sqrt{3}} \left( \frac{f_\pi}{f_8} \cos \theta - 2\sqrt{2} \frac{f_\pi}{f_9} \sin \theta \right),
\]

(67)

\[
F_{\eta'\gamma\gamma}(0) = \frac{\alpha N_c}{3\sqrt{3}} \left( \frac{f_\pi}{f_8} \sin \theta + 2\sqrt{2} \frac{f_\pi}{f_9} \cos \theta \right).
\]

(68)

The values of these amplitudes, as extracted from data, are [28]

\[
F_{\eta\gamma\gamma}(0) \approx 0.024 \pm 0.001 \, \text{GeV}^{-1},
F_{\eta'\gamma\gamma}(0) \approx 0.031 \pm 0.001 \, \text{GeV}^{-1}.
\]

(69)

Calculation with the help of Eqs. (40),(49),(54),(55),(58),(59) yields

\[
F_{\eta\gamma\gamma}(0) = 0.025 \pm 0.001 \, \text{GeV}^{-1},
F_{\eta'\gamma\gamma}(0) = 0.030 \pm 0.001 \, \text{GeV}^{-1},
\]

(70)

in excellent agreement with (69). Note that one can similarly compare the \( \eta, \eta' \rightarrow \gamma\gamma \) widths, as given by the relation (see, e.g., ref. [13])

\[
\Gamma(P^0 \rightarrow \gamma\gamma) = \frac{F_{P^0 \gamma\gamma}(0) M^2(P^0)}{64\pi},
\]

(71)

with \( F_{P^0 \gamma\gamma}(0) \) defined in (65),(67),(68), with those measured. Such a comparison gives (in keV): 0.51 \pm 0.05 vs. [4] 0.46 \pm 0.04 for \( \eta \rightarrow \gamma\gamma \), and 4.04 \pm 0.27 vs. [4] 4.26 \pm 0.19 for \( \eta' \rightarrow \gamma\gamma \).

Also, one can compare the values for the \( \eta-\eta' \) mixing-independent \( R \)-ratio, given by (using (65),(67),(68),(71))

\[
R \equiv \left[ \frac{\Gamma(\eta \rightarrow \gamma\gamma)}{\eta^3} + \frac{\Gamma(\eta' \rightarrow \gamma\gamma)}{\eta'^3} \right] \frac{\pi^3}{\Gamma(\pi \rightarrow \gamma\gamma)} = \frac{1}{3} \left( \frac{f_8^2}{f_\pi^2} + 8 \frac{f_9^2}{f_\pi^2} \right).
\]

With the values for \( f_8/f_\pi \) and \( f_9/f_\pi \) obtained above, our result is \( R = 2.25 \pm 0.15 \), vs. 2.45 \pm 0.35, as follows from using the experimentally measured masses and widths in the above expression.
5.2 \( \eta, \eta' \rightarrow \pi^+\pi^-\gamma \) decays

These, as well as the \( P^0 \rightarrow \gamma\gamma \), processes were extensively studied by Venugopal and Holstein [28] in chiral perturbation theory. The analysis of experimental data for both of these processes done in ref. [28] yields

\[
\begin{align*}
\frac{f_8}{f_\pi} &= 1.38 \pm 0.22, \\
\frac{f_9}{f_\pi} &= 1.06 \pm 0.03, \\
\theta &= (-22.0 \pm 3.3)^\circ,
\end{align*}
\]

in good agreement with our Eqs. (59), (62), (63).

5.3 \( J/\psi \rightarrow \eta\gamma, \eta'\gamma \) decays

These processes were studied by Kisselev and Petrov [27]. The values of \( f_8/f_\pi \), \( f_9/f_\pi \) and \( \theta \) extracted in ref. [27] from the experimentally measured \( P^0 \rightarrow \gamma\gamma \) and \( J/\psi \rightarrow \eta\gamma, \eta'\gamma \) widths, as given in the last column of Table I of ref. [27], which corresponds to conventional mass-mixing angle relations, are

\[
\begin{align*}
\frac{f_8}{f_\pi} &= 1.12 \pm 0.14, \\
\frac{f_9}{f_\pi} &= 1.04 \pm 0.08, \\
\theta &= (-18.9 \pm 2.0)^\circ,
\end{align*}
\]

again in good agreement with our Eqs. (59), (62), (63).

Thus, the three values of \( f_8/f_\pi \), \( f_9/f_\pi \) and \( \theta \) agree with experiment (at least, as far as the processes considered above are concerned). As to the remaining \( f_\eta/f_\pi \), \( f_\eta'/f_\pi \) ratios also calculated in the paper, the experimental values of them, as extracted from data by the CELLO [34] and TPC/2\( \gamma \) [35] collaborations, are, respectively,

\[
\begin{align*}
\frac{f_\eta}{f_\pi} &= 1.12 \pm 0.12, \quad (72) \\
\frac{f_\eta'}{f_\pi} &= 1.06 \pm 0.10, \quad (73)
\end{align*}
\]

and

\[
\begin{align*}
\frac{f_\eta}{f_\pi} &= 1.09 \pm 0.10, \quad (74) \\
\frac{f_\eta'}{f_\pi} &= 0.93 \pm 0.09. \quad (75)
\end{align*}
\]

While the value calculated for \( f_\eta/f_\pi \), Eq. (56), clearly agrees with both experimental values (72) and (74), the value calculated for \( f_\eta'/f_\pi \), Eq. (57), only marginally agrees with (73), and disagrees with (75) by almost 3 standard deviations.

To clarify this point, let us note that the values of \( f_\eta/f_\pi \) and \( f_\eta'/f_\pi \) were extracted by both CELLO and TPC/2\( \gamma \) from experimental data on the transition form-factors \( T_{\eta(\eta')}(0,-Q^2) \), assuming that the pole mass \( \Lambda_{\eta(\eta')} \), which parametrizes their fits to the data, can be identified with \( 2\pi\sqrt{2} f_{\eta(\eta')} \). Then, these pole fits to the data are presumed to join smoothly, as \( Q^2 \rightarrow \infty \), to the perturbative QCD predictions for
\( T_{\eta(\eta')} (0, -Q^2) \) [36], i.e., these fits would then agree with both the QCD asymptotic form \( \sim 1/Q^2 \) and its coefficient. However, the values of both \( f_{\eta} \) and \( f_{\eta'} \) quoted by the two groups, (in MeV) (94.0 ± 7.1, 89.1 ± 4.9) [34], and (91.2 ± 5.7, 77.8 ± 4.9) [35], respectively, are all close to \( M(\rho)/(2\pi\sqrt{2}) \approx 86.5 \) MeV, thus indicating on possible connection with the vector meson dominance interpretation of \( \Lambda_{\eta(\eta')} \approx M(\rho) \) in the range of \( Q^2 \) investigated.

On the other hand, as remarked in ref. [37], the model independent calculation by Gasser and Leutwyler [22], testing the Goldberger-Treiman relations by Scadron [38], and the calculation by Burden et al. [39], all agree that both \( f_{\eta} \) and \( f_{\eta'} \) should be noticeably larger than \( f_\pi \). Our results \( f_{\eta} \sim 1.1f_\pi \), \( f_{\eta'} \sim 1.2f_\pi \) are in agreement with this.

It therefore seems that the extraction of the values of \( f_{\eta} \) and \( f_{\eta'} \) from the transition form-factors \( T_{\eta(\eta')} (0, -Q^2) \) cannot be done accurately enough, at least in the range of \( Q^2 \) investigated so far. That this may indeed be the case is indicated by the experimental value \( f_{\rho^0} = 84.1 \pm 2.8 \) MeV [22], extracted by the same method, which is again close to \( M(\rho)/(2\pi\sqrt{2}) \). The central value of this \( f_{\rho^0} \), 84.1 MeV, is \( \sim 10\% \) below the well established value given in Eq. (49), 92.4 ± 0.2 MeV. Such a large discrepancy cannot be explained by, e.g., small isospin violation, indicating therefore that the extracted values for both \( f_{\eta} \) and \( f_{\eta'} \) may well have been underestimated too.

6 Concluding remarks

As a lagniappe, we note that Eq. (50) may be combined with our extracted values for \( f_9, f_\eta, f_{\eta'} \) to obtain the value of \( A \) :

\[
A = 0.78 \pm 0.12 \text{ GeV}^2.
\]

This is consistent with the usual value 0.73 GeV\(^2\) determined by the trace condition for (23),(25) without \( f' \)'s [30].

Let us briefly summarize the findings of this work:

i) We have found that many theoretical approaches suggest that the more natural object to study is the pseudoscalar mass squared matrix modified by the inclusion of the squared factors of the pseudoscalar decay constants.

ii) We have shown that this modified mass squared matrix leads to new Schwinger’s quartic mass and the Sakurai mass-mixing angle relations for the pseudoscalar meson nonet.

iii) We have used these new relations for calculation of the pseudoscalar decay constants and mixing angle. We have demonstrated that, except where questions may be raised regarding the reliability of the extraction of the relevant quantities from direct experimental data, the results obtained are in excellent agreement with available data.
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