Inversion of Tchebychev-Tchernov inequality

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Abstract

We derive in this article the lower bound for tail of distribution for the random variables (r.v.) through a lower estimate for its moment generating functions (MGF).

Key words and phrases: Random variable (r.v.), exponential and ordinary tail of distribution, upper and lower estimates, exponential inequalities, Cramer’s condition, tail function, moment generating functions (MGF), regular and slowly varying functions, natural function.

1 Definitions. Notations. Previous results. Statement of problem.

Let $(\Omega = \{ \omega \}, F, P)$ be certain non-trivial Probability Space. Let also a numerical valued r.v. $\eta = \eta(\omega)$ with correspondent tail function

$$T_\eta(x) \overset{\text{def}}{=} \max(P(\eta \geq x), P(\eta < -x)), \ x > 0,$$

and a numerical valued non-negative function $\nu = \nu(\lambda), \ \lambda \in [1, b), \ b = \text{const} \in (1, \infty]$, not necessary to be continuous or convex, be such that

$$E \exp(\lambda \eta) \leq \exp(\nu(\lambda)).$$

Then

$$T_\eta(x) \leq \exp(-\nu^*(x)),$$

be the classical Tchebychev-Tchernov (Markov, Cramer) inequality.

The function $\lambda \rightarrow E \exp(\lambda \eta)$ is named as a Moment Generating Function (MGF) for the r.v. $\eta$, write $\Theta_\eta(\lambda) = \Theta(\lambda) := E \exp(\lambda \eta)$, if of course it there exists in some non-trivial interval of the values $\lambda : |\lambda| < \lambda_0, \ \lambda_0 = \text{const} \in (0, \infty]$, (Cramer’s condition). It is closely related with the so-called Grand Lebesgue Spaces [GLS], see e.g. [1], [3]-[6], [9]-[12] etc. It is also widely used in the theory of Great Deviations.
Recall that the Cramér’s condition for the r.v. \( \eta \) is quite equivalent to the following tail estimate

\[
\exists \mu = \text{const} > 0 \Rightarrow T_\eta(x) \leq \exp(-\mu x), \ x \geq 0.
\]  

(1.2a)

For instance, the function

\[
\phi_\eta(\lambda) := \ln \mathbb{E} \exp(\lambda \eta)
\]

is named as the natural function for the r.v. \( \xi \), is generating function for the correspondent GLS.

Another name: deviation function for the r.v. \( \eta \), if of course this r.v. \( \eta \) satisfies the Cramér’s condition.

The complete description of such a functions \( \phi = \phi_\eta(\lambda) \) may be found in the book [??OstrMono], pp. 22-24.

Denote for simplicity

\[
T_\eta(x) := \exp(-G(x)) = \exp(-G_\eta(x)),
\]  

(1.3)

so that

\[
G_\xi(x) = |\ln T_\eta(x)|,
\]

then the estimate (1.2) may be rewritten as follows

\[
G(x) = G_\eta(x) \geq \nu^*(x).
\]  

(1.4)

This function \( G(x) = G_\eta(x) \) may be named as an exponential tail function for the correspondent r.v. \( \eta \).

Here and in the sequel the transform \( \nu \rightarrow \nu^* \) will be denote an ordinary Young - Fenchel, or Legendre operator

\[
\nu^*(x) \overset{\text{def}}{=} \sup_{\lambda \in [1,b]} (\lambda x - \nu(\lambda)),
\]  

(1.5)

To what extent can this inequality (1.2), or equally (1.4) be reversed?

To be more precisely, suppose the numerical valued r.v. \( \xi = \xi(\omega) \) with correspondent tail function \( T_\xi(x) = \exp(-G_\xi(x)) \) and a numerical valued non-negative function \( \phi = \phi(\lambda), \ \lambda \in [1,b], \ b = \text{const} \in [1,\infty] \), not necessary to be continuous or convex, be such that

\[
\mathbb{E} \exp(\lambda \xi) \geq \exp(\phi(\lambda)).
\]  

(1.6)

Can we conclude (under some natural conditions) that

\[
\exists c = c(\phi) \in (0, 1) \Rightarrow T_\xi(x) \geq \exp(-\phi^*(c x)), \ x \geq x_0 = \text{const} \in (0, \infty),
\]  

(1.7)
Our claim in this article is to ground under some additional conditions the inequality (1.7), i.e. to obtain the opposite result for the inequality (1.2).

2 Auxiliary estimates from the theory of saddle-point method.

We must use in advance one interest and needed further integrals. Namely, let $(X, M, \mu)$, $X \subset \mathbb{R}$ be non-trivial measurable space with non-trivial sigma finite measure $\mu$.

We assume at once $\mu(X) = \infty$, as long as the opposite case is trivial for us. We intend to estimate for sufficiently greatest values of real parameter $\lambda$, say $\lambda > e$, the following integral

$$I(\lambda) := \int_X e^{\lambda x - \zeta(x)} \mu(dx). \quad (2.1)$$

assuming of course its convergence for all the sufficiently greatest values of the parameter $\lambda$.

Here $\zeta = \zeta(x)$ is non-negative measurable function, not necessary to be convex.

We will use the main results obtained in the recent articles [10]-[12]. The offered below estimates may be considered in turn as a some generalizations of the saddle-point method.

We represent now two methods for upper estimate $I(\lambda)$ for sufficiently greatest values of the real parameter $\lambda$.

Note first of all that if in contradiction the measure $\mu$ is finite: $\mu(X) = M \in (0, \infty)$; then the integral $I(\lambda)$ allows a very simple estimate

$$I(\lambda) \leq M \cdot \sup_{x \in X} \exp (\lambda x - \zeta(x)) = M \cdot \exp (\zeta^*(\lambda)). \quad (2.2)$$

Let now $\mu(X) = \infty$ and let $\epsilon = \text{const} \in (0, 1)$; let us introduce the following auxiliary integral

$$K(\epsilon) = K[\zeta](\epsilon) := \int_X e^{-\epsilon \zeta(x)} \mu(dx). \quad (2.3)$$

It will be presumed its finiteness at last for some positive value $\epsilon_0 \in (0, 1)$; then

$$\forall \epsilon \geq \epsilon_0 \Rightarrow K(\epsilon) < \infty. \quad (2.4)$$

It is proved in particular in [12] that
\[ I(\lambda) \leq K(\epsilon) \cdot \exp \left\{ (1 - \epsilon) \zeta^* \left( \frac{\lambda}{1 - \epsilon} \right) \right\}. \] (2.5)

As a slight consequence:

\[ I(\lambda) \leq K(\epsilon) \cdot \exp \left\{ \zeta^* \left( \frac{\lambda}{1 - \epsilon} \right) \right\}, \] (2.6)

and of course

\[ I(\lambda) \leq \inf_{\epsilon \in (0, 1)} \left[ K(\epsilon) \cdot \exp \left\{ (1 - \epsilon) \zeta^* \left( \frac{\lambda}{1 - \epsilon} \right) \right\} \right]. \] (2.7)

An opposite method, which was introduced in particular case in [10], [11], sections 1.2. Indeed, let again \( \epsilon = \text{const} \in (0, 1) \). Introduce a new function

\[ R(\epsilon) := \int_X e^{\zeta^*(1-\epsilon)x - \zeta(x)} \mu(dx). \] (2.8)

Then

\[ I(\lambda) \leq R(\epsilon) \cdot e^{\zeta^*(\lambda/(1-\epsilon))}. \] (2.9)

Denote

\[ M[\zeta](\epsilon) := \min(K(\epsilon), R(\epsilon)), \epsilon \in (0, 1). \] (2.10)

We obtained actually the following compound estimate.

**Theorem 2.1.** Suppose

\[ \exists \epsilon \in (0, 1) \Rightarrow M[\zeta](\epsilon) < \infty. \] (2.11)

Then

\[ I(\lambda) \leq M[\zeta](\epsilon) \cdot e^{\zeta^*(\lambda/(1-\epsilon))}, \lambda > 0. \] (2.12)

As a slight consequence:

\[ I(\lambda) \leq \inf_{\epsilon \in (0, 1)} \left[ M[\zeta](\epsilon) \cdot e^{\zeta^*(\lambda/(1-\epsilon))} \right], \lambda > 0. \] (2.12a)

**Theorem 2.2.** Suppose the non-negative r.v. \( \xi \) satisfies the Cramer’s condition. Then

\[ \forall \epsilon > 0 \Rightarrow K(\epsilon) := K(G[\xi], \epsilon) < \infty. \] (2.13)

**Proof.** Denote for brevity \( G(x) = G[\xi](x) \), so that again

\[ T_\xi(x) = e^{-G(x)}, x \geq 0. \]
We have for the positive values $\lambda > 0$

$$\infty > \lambda^{-1} \Theta_\xi(\lambda) = \lambda^{-1} \mathbb{E} \exp(\lambda \xi) = \int_0^\infty \exp(\lambda x - G(x)) \, dx =$$

$$\sum_{n=0}^\infty \int_n^{n+1} \exp(\lambda x - G(x)) \, dx \geq \sum_{n=0}^\infty \exp(\lambda n - G(n + 1)),$$

therefore

$$G[\xi](n) \geq \lambda n - d(\lambda), \; \lambda > 0.$$

Further,

$$K[G](\epsilon) = \int_0^\infty \exp(-\epsilon G(x)) \, dx = \sum_{n=0}^\infty \int_n^{n+1} \exp(-\epsilon G(x)) \, dx \leq$$

$$\sum_{n=0}^\infty \exp(-\epsilon G(n)) \leq \sum_{n=0}^\infty \exp(-\epsilon(\lambda n - d(\lambda))) < \infty, \; \epsilon > 0,$$

Q.E.D.

3 Main result: exponential lower bound for tail of distribution. Unilateral approach.

So, let the inequality (1.6) be given. We need to introduce some another notations, definitions, and conditions.

We can and will suppose without loss of generality that the r.v. $\xi$ is non-negative, as long as we intend to investigate the right-hand tail behavior. The case of opposite tail may be studied analogously.

Further, define the auxiliary function

$$\phi_1(\lambda) = \phi_1[\phi](\lambda) := \ln \left[ \frac{\exp(\phi(\lambda)) - 1}{\lambda} \right], \; \lambda \geq 1.$$

(3.1)

Evidently $\phi(\lambda) > 1, \; \lambda \geq 1$.

**Definition 3.1.** We will say that $\phi$ – function belongs to the class $W : \phi(\cdot) \in W$, if the auxiliary function $\phi_1[\phi](\lambda)$ is equivalent to the source one in the following sense

$$\phi(\cdot) \in W \iff \exists c_1 = c_1[\phi] = \text{const} \in (0, 1], \; \phi_1(\lambda) \geq \phi(c_1 \lambda), \; \lambda \geq 1.$$  

(3.2)

(The inverse inequality is trivial).

This condition (3.2) is satisfied, if for example the function $\phi(\cdot)$ is regular varying:
\[ \phi(\lambda) = \lambda^r L(\lambda), \]
where \( r = \text{const} > 0 \) and \( L(\cdot) \) is positive continuous slowly varying at infinity function.

**Definition 3.2.** The exponential tail function \( G = G(x) \) is said to be **super convex**, write \( G = G(\cdot) \in SC \), iff it is convex and is also lower semi-continuous.

**Theorem 3.1. A.** Suppose for some \( \epsilon \in (0, 1) \) \( M[\phi_1](\epsilon) < \infty \). Then

\[ \exp \left\{ -G^*(\frac{\lambda}{1 - \epsilon}) \right\} \leq \exp \left( -\phi_{2,\epsilon}(\lambda) \right), \tag{3.3} \]
where

\[ \exp (-\phi_{2,\epsilon}(\lambda)) := \frac{1}{M[\phi_1](\epsilon)} \cdot \exp(-\phi_1(\lambda)). \tag{3.3a} \]

**B.** If in addition \( \phi(\cdot) \in W \), then

\[ \exp \left\{ -G^* \left( \frac{\lambda}{1 - \epsilon} \right) \right\} \leq \exp \left( -\phi(c_2(\epsilon) \lambda) \right), \tag{3.4} \]

**C.** If in addition the exponential tail function \( G(x) \) is super convex: \( G(\cdot) \in SC \), then

\[ G(x) \leq \phi^*(c_3(\epsilon) x), \quad x \geq 1, \tag{3.5} \]
or equally

\[ T_\xi(x) \geq \exp \{ -\phi^*(c_3(\epsilon) x) \}, \quad x \geq 1. \tag{3.5a} \]

**Proof.**

**Preview.** We represent for beginning the sketch of proof, not to be strict. Suppose in addition that the r.v. \( \xi \) has a logarithmical convex density \( f(x) \); this means by definition that

\[ f(x) = f_\xi(x) = e^{-v(x)}, \quad x \geq x_0 = \text{const} > 0, \]
where \( v = v(x) \) is convex lower semi-continuous function. We have

\[ e^{\phi(\lambda)} \leq \mathbb{E}e^{\lambda \xi} = \int_R e^{\lambda x - v(x)} \, dx =: J(\lambda). \]

The last integral may be estimated as \( \lambda \to \infty \) by means of the saddle - point method

\[ \ln J(\lambda) \asymp \sup_{x}(\lambda x - \phi(\lambda)) = \phi^*(\lambda), \]
and we conclude therefore that for all sufficiently great values \( \lambda, \lambda \geq \lambda_0 = \text{const} \geq 1 \)
\[
J(\lambda) \leq e^{v^*(C_1, \lambda)},
\]

following
\[
v^*(C_2 \lambda) \geq \phi(\lambda).
\]

Further,
\[
v^{**}(x) \leq \phi^*(C_2 x), \quad C_2 = C_2[\phi](\lambda_0) = \text{const} \in [1, \infty).
\]

By virtue of theorem Fenchel-Moreau \( v^{**}(x) = v(x) \), and we deduce finally
\[
f(x) \geq \exp(-\phi^*(c_1 x)), \quad T_\xi(x) \geq \exp(-\phi^*(c_2 x)), \quad x \geq x_0 = \text{const} \geq 1.
\]

where \( c_1, c_2 = \text{const} = c_1, c_2[\phi](x_0) \in (0, 1) \).

Recall that always
\[
T_\xi(x) \leq \exp(-\phi^*(x)), \quad x \geq x_0 = \text{const} \geq 0.
\]

Notice in addition that \( \lim_{x \to \infty} c_2(x_0) = 1 \). Thus, our estimations are essentially non-improvable as \( x \to \infty \).

The consequences of the last relation will be used further, in the seventh section.

Actually, let’s move on the complete proof. Denote by \( F(x) \) a function of distribution for the r.v. \( \xi : F(x) = P(\xi < x), \quad x \geq 0 \). We have by means of integration “by parts”
\[
E e^{\lambda \xi} = \int_0^\infty e^{\lambda x} dF(x) = -\int_0^\infty e^{\lambda x} dT_\xi(x) = P(\xi \geq 0) + \\
\lambda \int_0^\infty e^{\lambda x} T(x) \, dx = 1 + \lambda \int_0^\infty e^{\lambda x} T(x) \, dx \geq e^{\phi(\lambda)}, \quad \lambda \geq 1,
\]
therefore
\[
\int_0^\infty e^{\lambda x - G(x)} \, dx \geq e^{\phi_1(\lambda)}.
\] (3.6)

One can apply the inequality (2.12) of theorem 2.1:
\[
M[G](\epsilon) \cdot e^{G^*(\lambda/(1-\epsilon))} \geq e^{\phi_1(\lambda)}, \quad \lambda > 1,
\] (3.7)

which is equivalent to the propositions (3.3), (3.3a). The next assertion follows immediately from the direct definition of the set \( W \):
\[
G^* \left( \frac{\lambda}{1-\epsilon} \right) \geq \phi(c_2(\epsilon) \lambda),
\] (3.8)
following

$$G^{**}(\lambda) \leq \phi(c_4(\epsilon) \lambda).$$  \hspace{1cm} (3.9)

But as long as \(G \in SC\), then \(G^{**} = G\), (theorem of Fenchel-Moreau,) therefore

$$G(\lambda) \leq \phi(c_4(\epsilon) \lambda),$$  \hspace{1cm} (3.9)

and finally

$$T_{\xi}(x) \geq \exp \{ -\phi^*(c_3(\epsilon) x) \}, \ x \geq 1,$$

Q.E.D.

Remark 3.1. The estimate (3.5a) is trivially satisfied if the r.v. \(\xi\) does not satisfy the Cramer’s condition.

4 Main result. Exponential level. Bilateral approach.

Statement of the problem: given the bilateral inequality for some real valued random variable \(\xi\):

$$\exp (\phi_1(\lambda)) \leq E \exp (\lambda \xi) \leq \exp (\phi_2(\lambda)), \ |\lambda| < \lambda_0 = \text{const} \in (0, \infty],$$

$$\phi_1, \phi_2 \in \Phi; \ \phi_1(\lambda) \leq \phi_2(\lambda).$$

Our purpose in this section is obtaining the lower estimate for the tail function for the r.v. \(\xi\).

We significantly weaken condition of the last section imposed on the function \(\phi_2(\cdot)\) and intend to obtain more qualitative up to multiplicative constant fine estimations.

The investigated in this report problem but for the random variables satisfying the Cramer’s condition was considered, in particular, in the monograph [11], chapter 1, sections 1.3, 1.4. We intend to improve the obtained therein results.

We must introduce now some used notations.

$$S(\lambda, x) := \lambda x - \phi_2^*(x), \ \ \ \ S^*(\lambda) := \sup_x S(\lambda, x),$$

$$x_0 = x_0(\lambda) := \arg\max_{x \in \mathbb{R}} S(\lambda, x) = \left[ (\phi_2^*)' \right]^{-1}(\lambda),$$(4.2)

Further, let \(x_- = x_-(\lambda)\) be arbitrary variable from the set \((0, x_0(\lambda))\), and \(x_+ = x_+(\lambda)\) be arbitrary variable from the set \((x_0(\lambda), \infty)\). The set of all this
variables \( x_-, x_+ \) will be denoted by \( X_0 = X_0(\lambda) \).

For instance, one can take

\[
x_-^\Delta = x_-^{\Delta(\lambda)} := x_0(\lambda(1 - \Delta)),
\]

\[
x_+^\Delta = x_+^{\Delta(\lambda)} := x_0(\lambda(1 + \Delta)),
\]

\( \Delta = \text{const} \in (0, 1) \). More generally, one can choose also

\[
x_-^{\Delta(1)} = x_-^{\Delta(1)(\lambda)} := x_0(\lambda(1 - \Delta(1))),
\]

\[
x_+^{\Delta(2)} = x_+^{\Delta(2)(\lambda)} := x_0(\lambda(1 + \Delta(2))),
\]

\( \Delta(1), \Delta(2) = \text{const} \in (0, 1) \).

Introduce also the numerical valued function

\[
z \to G_-(z) = G[x_-, x_+](z), \quad z \geq e
\]
as follows:

\[
G_-(x_-(\lambda)) = G_-[x_-(\cdot), x_+()])(x_-(\lambda)) = G_-[x_-, x_+](x_-(\lambda)) \triangleq
\]

\[
e^{-\lambda \cdot x_+(\lambda)} \times \left\{ e^{\phi_1(\lambda)} - \lambda \frac{e^{S(\lambda, x_-(\lambda))}}{S'(\lambda, x_-(\lambda))} - \lambda \frac{e^{S(\lambda, x_+(\lambda))}}{S'(\lambda, x_+(\lambda))} \right\},
\]

as well as its “closure”

\[
\overline{G}_-(z) \triangleq \sup_{(x_-, x_+) \in X(\lambda)} G_-[x_-, x_+](z).
\]

**Theorem 4.1.** We conclude under formulated above notations and restrictions

\[
T_\xi(z) \geq \overline{G}_-(z), \quad z \geq 1.
\]

**Proof.** We have after integration by part assuming \( \lambda > 0 \)

\[
\lambda^{-1} e^{\phi_1(\lambda)} \leq \int_{\infty} e^{\lambda x} T_\xi(x) \, dx = I_1 + I_2 + I_3,
\]

where

\[
I_1 = I_1(\lambda) = \int_{\infty} e^{\lambda x} T_\xi(x) \, dx \leq \int_{\infty} e^{\lambda x - \phi_1(x)} \, dx =
\]

\[
\int_{-\infty} e^{S(\lambda, x)} \, dx \leq \int_{-\infty} \exp \left\{ S(\lambda, x_-) + S'(\lambda, x_-)(x - x_-) \right\} \, dx =
\]

\[
\exp \frac{S(\lambda, x_-)}{S'(\lambda, x_-)}.
\]
We find analogously
\[ I_3 = I_3(\lambda) = \int_{x_+}^{\infty} e^{\lambda x} T_\xi(x) \, dx \leq \int_{x_+}^{\infty} e^{\lambda x - \phi_2(x)} \, dx = \int_{x_+}^{\infty} e^{S(\lambda, x)} \, dx \leq \int_{x_+}^{\infty} \exp \left\{ S(\lambda, x_+) - |S'_x(\lambda, x_+)| (x - x_+) \right\} \, dx = \exp S(\lambda, x_+) \frac{\exp S'(\lambda, x_+)}{|S''_x(\lambda, x_+)|}.
\]
Finally,
\[ I_2 = I_2(\lambda) \leq \lambda \int_{x_-}^{x_+} T(x_-) \, e^{\lambda x} \, dx < T(x_-) \, e^{\lambda x_+}. \quad (4.6) \]
We deduce after substituting into (4.5) and solving obtained inequality relative the variable \( T(x_-) \)
\[ T_\xi(z) \geq G_-[x_-, x_+](z), \quad z \geq 1. \]
It remains to make the optimization subject to the natural limitation \((x_-, x_+) \in X(\lambda)\):
\[ T_\xi(z) \geq \sup_{(x_-, x_+) \in X(\lambda)} G_-[x_-, x_+](z) = \overline{G}_-(z), \quad z \geq 1, \]
Q.E.D.

Remark 4.1. Of course, one can take accept as the values \((x_-, x_+)\) the variable from (4.2b)
\[ x_- := x_0(\lambda(1 - \Delta(1))), \quad x_+ := x_0(\lambda(1 + \Delta(2))), \quad 0 < \Delta(1), \Delta(2) < 1. \quad (4.7) \]

5 Consequences. Particular cases.

It seems quite reasonable to choose the values \( x_\pm = x_\pm(\lambda) \) for sufficiently greatest values \( \lambda \), say by definiteness \( \lambda \geq e \), as before
\[ x_\pm(\lambda) := x_0(\lambda(1 \pm \Delta)), \quad \Delta = \text{const} \in [0, 1/2], \]
or for simplicity
\[ x_\pm(\lambda) := x_0(\lambda(1 + \Delta)), \quad \Delta = \text{const} \in [-1/2, 1/2]. \quad (5.0) \]
We impose also the following natural condition on the function \( \phi = \phi(\lambda) := \phi_2(\lambda) \)
\[
\inf_{0 \neq \Delta \in [-1/2, 1/2]} \inf_{\lambda \geq e} \left[ \frac{S(\lambda, x_0) - S(\lambda, x_0(\lambda(1 + \Delta)))}{S(\lambda, x_0) \Delta^2} \right] =: V[\phi] = V > 0. \quad (5.1)
\]

We consider in this section only the case when the function \( \phi_1(\lambda) \) satisfies the restrictions of the form

\[
\phi(\lambda) \geq \phi_1(\lambda) \geq (1 - \delta^2) \phi(\lambda), \quad \lambda \geq e, \quad \delta = \text{const} \in (0, 1/2), \quad (5.2)
\]

so that

\[
\exp \left( (1 - \delta^2) \phi(\lambda) \right) \leq \mathbb{E} \exp(\lambda \xi) \leq \exp(\phi(\lambda)), \quad |\lambda| < \lambda_0 = \text{const} \in (0, \infty]. \quad (5.2a)
\]

Further, note that

\[
\lambda x_0(\lambda) - \phi(\lambda) = \phi^*(x_0(\lambda)), \quad \lambda \geq 1.
\]

Therefore, it is naturally to suppose in addition

\[
\exists c_0 \in (0, \infty) \forall \lambda \geq e, \quad \delta \in (0, 1/2) \quad \Rightarrow \quad \lambda x_0(\lambda(1 + \delta)) - (1 - \delta^2)\phi(\lambda) \leq (1 + c_0\delta)\phi^*(x_0(\lambda(1 - \delta))). \quad (5.3)
\]

Both the conditions (3.1), (3.3) are satisfied if for example the function \( \lambda \to \phi(\lambda) \) is sufficiently smooth regular varying:

\[
\phi(\lambda) = \phi[p; L](\lambda) := p^{-1} |\lambda|^p L( |\lambda|), \quad |\lambda| \geq e, \quad (5.4)
\]

where \( p = \text{const} > 1 \), and \( L(\cdot) \) is some positive continuous differentiable slowly varying at infinity function. For instance, one can take

\[
\phi(\lambda) = \phi_2(\lambda) := 0.5 \lambda^2, \quad \lambda \in R, \quad -
\]

the so-called subgaussian case; as well as many popular examples

\[
\phi(\lambda) = \phi[p; r](\lambda) := p^{-1} |\lambda|^p \ln(e + |\lambda|)^r, \quad r = \text{const}, \quad |\lambda| \geq 1. \quad (5.4a)
\]

**Theorem 5.1.** Suppose that all the formulated before conditions, (2.1), (3.1), (3.2), (3.3) imposed on the r.v. \( \xi \) are satisfied. Then for some finite positive constant \( c = c(\phi) \in (0, 1/(2\delta)) \)

\[
T_\xi(z) \geq \exp \left\{ -(1 - c\delta)\phi^*(z/(1 - c\delta)) \right\}, \quad z \geq e. \quad (5.5)
\]

**Proof.** One need only to apply the assertion of theorem 2.1., choosing \( \Delta = C_2 \delta, \quad \Delta \in (0, 1/2) \). We omit some simple calculations.
Remark 5.1. Suppose that for certain function \( \phi(\cdot) \in \Phi \) and for some random variable \( \xi \)

\[
\exp \left( (1 - \delta^2) \phi(\lambda) \right) \leq \mathbb{E} \exp(\lambda \xi) \leq \exp (\phi(\lambda)),
\]

\( \forall \lambda : |\lambda| < \lambda_0 = \text{const} \in (0, \infty], \delta = \text{const} \in (0, 1/2]. \tag{5.6} \)

Since

\[
\left[ (1 - \delta^2) \phi \right]^* (x) = \sup_{\lambda} \left( \lambda x - (1 - \delta^2) \phi(\lambda) \right) =
\]

\[
(1 - \delta^2) \sup_{\lambda} \left( \frac{\lambda}{1 - \delta^2} x - \phi(\lambda) \right) = (1 - \delta^2) \phi^* \left( \frac{x}{1 - \delta^2} \right),
\]

it is natural to wait (our hypothesis!) that

\[
T_\xi (z) \geq \exp \left\{ - (1 - c_2 \delta^2) \phi^* (z/(1 - c_2 \delta^2)) \right\}, \ z \geq e;
\]

but we have grounded (under formulated above conditions) only the more weak estimate (3.5).

Example 3.1. Let \( \phi(\lambda) = 0.5 \lambda^2, \lambda \in \mathbb{R}, \delta = \text{const} \in (0, 1/2); \) then all the assumptions formulated above hold true. In detail: \( \delta = \text{const} \in (0, 1/2), \) and the (meal zero) r.v. \( \xi \) is such that

\[
\exp \left( (1 - \delta^2) \lambda^2 / 2 \right) \leq \mathbb{E} \exp(\lambda \xi) \leq \exp \left( \lambda^2 / 2 \right), \ \lambda \in \mathbb{R}, \tag{5.7} \]

a subgaussian case. We derive by virtue of theorems 2.1-3.1

\[
T_\xi (z) \geq \exp \left\{ -0.5 \ z^2 (1 + c \delta) \right\}, \ z \geq 0,
\]

but in accordance with our hypotheses

\[
T_\xi (z) \geq \exp \left\{ -0.5 \ z^2 (1 + c \delta^2) \right\}.
\]

This is an open problem.

6 Main result. Power level.

We intend in this section to deduce the lower bound for the tail of probability for the r.v. \( \xi : \ T_\xi (x), \ x \geq 1 \) through its moment estimates, or equally through its Lebesgue-Riesz \( L_p = L_p(\Omega) \) norm

\[
|\xi|_p = \left[ \mathbb{E} |\xi|^p \right]^{1/p}, \ p \in [1, b), \ b = \text{const} \in (1, \infty]. \tag{6.0} \]

This case may be easily reduced to the considered one. More detail, one can suppose
\[ |\xi|_p \geq \zeta(p), \ 1 \leq p < b, \ b = \text{const} \in (1, \infty], \]  
(6.1a)
an unilateral inequality or

\[ \zeta(p) \leq |\xi|_p \leq \psi(p), \ 1 \leq p < b, \ b = \text{const} \in (1, \infty], \]  
(6.1b)
a bilateral estimate.

The relation, e.g. (6.1b) may be rewritten as follows. Put \(|\xi| = \exp(\theta)|

\[ T_\xi(x) = T_\theta(\ln x), \ x \geq e, \]

and denote

\[ \phi_1(\lambda) = \lambda \ln \zeta(\lambda), \ \phi_2(\lambda) = \lambda \ln \psi(\lambda), \ \lambda \in [1, b); \]

then

\[ e^{\phi(\lambda)} \leq E e^{\lambda \theta} \leq e^{\phi_2(\lambda)}. \]  
(6.2)

It remains to apply the results of foregoing sections 2-5.

**Example 6.1.** Let the r.v. \( \eta \) be such that

\[ \forall p \in [1, b) \Rightarrow |\eta|_p \geq C (b - p)^{-\beta}, \]

\[ b = \text{const} \in (1, \infty), \ \beta = \text{const} \in (0, \infty). \]  
(6.3)

The examples of these variables may be found, e.g. in [1], [11] and so one. We conclude from (6.3) after some calculations

\[ P(|\xi| > x) \geq C(b, \beta) x^{-\gamma}, \ x \geq 1, \ \exists \gamma = \text{const} \in (1, b). \]  
(6.4)

**Example 6.2.** Suppose the r.v. \( X \) is such that

\[ |X|_p \asymp p^{1/m}, \ p \geq 1, \ m = \text{const} \in (0, \infty), \]  
(6.5)

the both extremal "boundary" cases \( m = 0 \) or \( m = \infty \) are trivial.

Many practical examples of such a r.v. may be found in the articles [2], [7]-[8], devoting to a mesoscopic physics. The authors obtained in particular the mild and great deviations for the sequence of these variables.

For instance, there exists the r.v. \( Y \) for which

\[ E|Y|^p = C D^p \frac{\prod_{j=1}^l \Gamma(a_j p + b_j)}{\prod_{k=1}^K \Gamma(a'_k p + b'_k)}, \]

where \( \Gamma(\cdot) \) is ordinary Gamma-function, see [7]-[8].

We conclude that under relation (6.5)
\[
\exp(-C_2x^m) \leq T_X(x) \leq \exp(-C_1x^m), \ x \geq 1,
\]

\[C_1, C_2 = \text{const} \in (0, \infty), C_1 \leq C_2. \tag{6.6}\]

Notice that the r.v. \(X\) from (6.5) satisfies the Cramer's condition only when \(m \geq 1\).

### 7 Tauberian theorem.

1. Ordinary approach.

**Theorem 7.1.** Assume the function \(\phi(\cdot)\) satisfies all the conditions of theorem 5.1. We assert that the following implication holds: the random variable \(\xi\) is such that

\[
\lim_{\lambda \to \infty} \phi^{-1}(\ln E \exp(\lambda \xi)) / \lambda = K = \text{const} \in (0, \infty) \tag{7.1}
\]

if and only if

\[
\lim_{x \to \infty} (\phi^*)^{-1}(|\ln P(\xi \geq x)|) / x = 1 / K. \tag{7.2}
\]

**Remark 7.1.** This result is the direct generalization one represented in [11], theorem 1.4.1. Wherein our proof is alike.

1. We can and will suppose without loss of generality \(K = 1\).

Suppose at first that the relation (7.2) be given. Let \(\delta = \text{const} \in (0, 1/e)\) be arbitrary "small" number. Then there exists a positive value \(x_0 = x_0(\delta)\) such that for all the values \(x \geq x_0(\delta)\) there holds

\[
\left| (\phi^*)^{-1}(\ln T_\xi(x))/x - 1 \right| \leq \delta.
\]

We have solving the last inequality relative the tail function \(T_\xi(x)\):

\[
\exp[-\phi^*(x(1 + \delta))] \leq T_\xi(x) \leq \exp[-\phi^*(x(1 - \delta))], \tag{7.3}
\]

and after integration by parts

\[
C_1 \lambda \int_R \exp[\lambda x - \phi^*(x(1 + \delta))] \ dx \leq E \exp(\lambda \xi) \leq C_2 \lambda \int_R \exp[\lambda x - \phi^*(x(1 - \delta))] \ dx, \ \lambda \geq 1. \tag{7.4}
\]

We conclude relaying the proposition of theorem 2.1 for sufficiently greatest values of parameter \(\lambda\)
\[
\exp \phi(\lambda(1-2\delta)) \leq E \exp(\lambda \xi) \leq \exp \phi(\lambda(1+2\delta)).
\] (7.5)

By logarithm and taking the inverse function, we arrive at (7.1).

2. Conversely, suppose (7.1) holds true. Then for greatest positive values \( \lambda \)
\[
\exp \phi(\lambda(1-\delta)) \leq E \exp(\lambda \xi) \leq \exp \phi(\lambda(1+\delta)).
\]

Theorem 5.1 common with Tchebychev-Tchernov inequality give us the following bilateral estimate
\[
\exp \left[ -\phi^*(x(1+C_3\delta)) \right] \leq T_\xi(x) \leq \exp \left[ -\phi^*(x(1-C_4\delta)) \right].
\] (7.6)
The relation (7.2) follows immediately from (7.6) after simple calculations.

2. Richter’s approach.

The proposition of theorem 5.1 is not true in the case when \( \delta = 0 \), i.e. when
\[
E e^{\lambda \xi} = e^{\phi(\lambda)}, \ \lambda \geq 1,
\] (7.7)
is the well-known Richter’s case. We assume in the sequel that the function \( \phi(\cdot) \) satisfies all the conditions of theorem 5.1.

But one can apply the proof of theorem 4.1, where as before
\[
\phi_1(\lambda) = \phi_2(\lambda) = \phi(\lambda), \ T_\xi(x) \leq \exp(-\phi^*(x)),
\]
\[
\Delta = c_1 = \text{const} > 0, \ \phi(\lambda) = \lambda x_0 - \phi^*(x_0).
\]
We obtain after some calculations from (7.7)
\[
T_\xi(x) \geq \exp(-\phi^*(x) - c_2[\phi] x), \ x \geq e.
\]
To summarize:

Theorem 7.2. We propose under formulated in this subsection conditions
\[
\exp(-\phi^*(x) - c_2[\phi] x) \leq T_\xi(x) \leq \exp(-\phi^*(x)), \ x \geq e.
\] (7.8)

8 Concluding remarks.

It is interest by our opinion to generalize obtained in this article results onto a multidimensional random vector.

References.
1. Buldygin V.V., Kozachenko Yu.V. *Metric Characterization of Random Variables and Random Processes.* 1998, Translations of Mathematics Monograph, AMS, v.188.

2. Peter Eichelsbacher and Lucas Knichel. *Fine asymptotics for models with Gamma type moments.* arXiv:1710.06484v1 [math.PR] 17 Oct 2017

3. A. Fiorenza. *Duality and reflexivity in grand Lebesgue spaces.* Collect. Math. **51**, (2000), 131 - 148.

4. A. Fiorenza and G.E. Karadzhov. *Grand and small Lebesgue spaces and their analogs.* Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picone, Sezione di Napoli, Rapporto tecnico 272/03, (2005).

5. Alberto Fiorenza, Maria Rosaria Formica, Amiran Gogatishvili, Tengiz Kopaliani, Michel Rakotoson. *Characterization of interpolation between Grand, small or classical Lebesgue spaces.* arXiv:1709.05892v1 [math.FA] 18 Sep 2017

6. T. Iwaniec and C. Sbordone. *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat.Mech. Anal., 119, (1992), 129-143.

7. Janson. *Further examples with moments of Gamma type.* arXiv:1204.56372v2, 2010.

8. Janson. *Moments of Gamma type and the Brownian supremum process area.* Probab. Surveys, **7**, (2010), 152. MR 2645216

9. Kozachenko Yu. V., Ostrovsky E.I. (1985). *The Banach Spaces of random Variables of subgaussian Type.* Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43-57.

10. Kozachenko Yu.V., Ostrovsky E., Sirota L. *Relations between exponential tails, moments and moment generating functions for random variables and vectors.* arXiv:1701.01901v1 [math.FA] 8 Jan 2017

11. Ostrovsky E.I. (1999). *Exponential estimations for Random Fields and its applications,* (in Russian). Moscow-Obninsk, OINPE.
12. Ostrovsky E. and Sirota L. *Vector rearrangement invariant Banach spaces of random variables with exponential decreasing tails of distributions.*
arXiv:1510.04182v1 [math.PR] 14 Oct 2015

13. Ostrovsky E. and Sirota L. *Non-asymptotical sharp exponential estimates for maximum distribution of discontinuous random fields.*
arXiv:1510.08945v1 [math.PR] 30 Oct 2015

14. Ostrovsky E.I. *About supports of probability measures in separable Banach spaces.* Soviet Math., Doklady, (1980), V. 255, N 6, p. 836-838, (in Russian).