Lyapunov-Based Reinforcement Learning State Estimator

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Abstract—In this paper, we consider the state estimation problem for nonlinear stochastic discrete-time systems. We combine Lyapunov’s method in control theory and deep reinforcement learning to design the state estimator. We theoretically prove the convergence of the bounded estimate error solely using the data simulated from the model. An actor-critic reinforcement learning algorithm is proposed to learn the state estimator approximated by a deep neural network. The convergence of the algorithm is analysed. The proposed Lyapunov-based reinforcement learning state estimator is compared with a number of existing nonlinear filtering methods through Monte Carlo simulations, showing its advantage in terms of estimate convergence even under some system uncertainties such as covariance shift in system noise and randomly missing measurements. To the best of our knowledge, this is the first reinforcement learning based nonlinear state estimator with bounded estimate error performance guarantee.

Index Terms—Nonlinear filtering, deep reinforcement learning, Lyapunov stability.

I. INTRODUCTION

State estimation, inferring unknown states using noisy measurements and underlying model of the dynamic system, has found its important application in different areas ranging from control engineering, robotics, tracking and navigation to machine learning [1]–[4]. For linear stochastic systems with Gaussian noise, the renowned Kalman filter [5] is proved to give the optimal estimate with elegant properties of convergence in estimate error covariance. For nonlinear stochastic systems, since the probability distribution of the state does not preserve the property of Gaussian, a closed-form estimator like Kalman filter generally does not exist. As a result, many nonlinear state estimation methods based on different approximation techniques have been proposed, among which the extended Kalman filter (EKF), unscented Kalman filter (UKF) [6] and particle filter (PF) [7] have been most widely used. For nonlinear systems with Gaussian noise, the EKF and UKF approximate the system state by a Gaussian distribution using the linearisation technique and the deterministic sampling method, respectively. For nonlinear non-Gaussian systems, the PF uses sequential Monte Carlo methods to approximate distribution of the state by a finite number of particles.

A fundamental problem of state estimation is how to design a state estimator such that the estimate error can be guaranteed to be bounded. This problem is challenging and scarcely addressed only in certain scenarios under some assumptions [8]–[10]. In [8], it is proved that if the linearisation error is negligible, the local asymptotic stability of the EKF can be ensured. In [9], the estimate error of the discrete-time EKF remains bounded only if the initial estimate error is sufficiently small and system noise intensity is small. In [10], similar results can be also found for the UKF. Nonetheless, it is hard to reduce the initial estimate errors and linearisation errors in the deployment of the EKF and UKF, making these convergence conditions less applicable. For the PF, though it is theoretically proved that the estimate performance converges asymptotically to the optimal estimates as the number of particles goes to infinity in [11], only a limited particles can be used in practice due to the expensive online computation as the number of particle increases. Unfortunately, there are not any guarantees on estimation performance for the PF with limited particles. By summarising the observations of the EKF, UKF and PF, it is expected that an ideal state estimator can estimate the state with bounded estimate error performance guarantee for nonlinear non-Gaussian systems. Moreover, the performance guarantee should not rely on conditions of the initial estimate error, the system noise level, or the degree of non-linearity of the underlying dynamic system. In this paper, we are interested in seeking an reinforcement learning (RL) based method to design such an state estimator.

RL was first applied to the state estimation problem in [12], where impressive estimation performance was shown. However, the theoretical guarantee on the convergence of bounded estimate error were unavailable thus make it less applicable for practical applications [13]. More recently, an RL based Kalman filter design method was proposed in [14], in which the bounded estimate error can be guaranteed by using finite samples for linear stable systems. Inspired by these works, we are going to combine deep learning and reinforcement learning, a.k.a., deep reinforcement learning (DRL), to design a state estimator in which the filter gain function will be learned using a deep neural network (DNN) from the sequence of estimate errors. Once the estimator is trained well offline, it can be deployed online and supposed to be efficient given the advance in DNN microprocessor for online applications. The key questions are how to prove the bounded estimate error guarantee in a RL setup and design an efficient learning algorithm with such a guarantee.

Motivated by [8]–[10], we plan to prove the stability of the estimate error dynamics, thereafter the bounded estimate error can be guaranteed. In particular, we make no assumptions typically used in these results such as initial estimate error, linearisation errors or assumptions on model nonlinearities, etc. Similar to [12], the mathematical model can be treated as a simulator and the training of estimator is solely from data simulated using the mathematical model. Unfortunately, the data-based stability analysis of closed-loop systems in a model-
free RL manner is still an open problem \cite{13, 15}. Typically, Lyapunov’s method in control theory is widely used to analyse the stability of dynamical systems. In \cite{16}, the stability of a deterministic nonlinear system was analysed using Lyapunov’s method, assuming that the discount of the infinite-horizon cost is sufficiently close to 1. However, such an assumption makes it difficult to guarantee the optimality of the learned policy without introducing extra certain assumptions \cite{17–19}. As a basic tool in control theory, the construction/learning of the Lyapunov function is not trivial \cite{20, 21}. In \cite{22}, the RL agent controls the switch between designed controllers using Lyapunov domain knowledge, so that any policy is safe and reliable. \cite{23} proposes a straightforward approach to construct the Lyapunov functions for nonlinear systems using DNNs. In \cite{24, 25}, a learning-based approach for constructing Lyapunov neural networks to ensure stability was proposed based on the assumption that the learned model is a Gaussian process, Lipschitz continuous and on discretised points in the subset of low-dimensional state space. Only until recently, the asymptotic stability in model-free RL is given for robotic control tasks \cite{26}. In \cite{27, 28}, the stability of a system with a combination of a classic baseline controller and a RL controller is proved for autonomous surface vehicles with collisions.

In summary, we will combine Lyapunov’s method in control theory and deep reinforcement learning to design state estimators for nonlinear stochastic discrete-time systems with bounded estimate error guarantee. The theoretical result is obtained by solely using the data simulated from the model. In our method, the estimator is trained/learned offline and deployed directly using a DNN. Moreover, our state estimator is shown to be robust to system uncertainties, i.e., unknown measurement noise, missing measurement and non-Gaussian noise, which makes our method more applicable. The main contribution of the paper has twofold:

1) For the first time, a deep reinforcement learning method has been employed for nonlinear state estimator design;
2) The bounded estimate error can be theoretically guaranteed by solely using data regardless of the degree of model nonlinearities and noise distribution.

To the best of our knowledge, this is the first reinforcement learning based nonlinear state estimator design method with bounded estimate error guarantee.

The rest of the paper is organised as follows. In Section II, the state estimation problem of nonlinear stochastic discrete-time systems is formulated. In Section III, the theoretical result on bounded estimate error guarantee is proved. In Section IV, the learning algorithm is derived and the algorithm convergence is analysed. In Section V, our method is compared with EKF, UKF and PF in simulations. Conclusion is given in Section VI.

Notation: The notation used here is fairly standard except where otherwise stated. \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times m}\) denote, respectively, the \(n\) dimensional Euclidean space and the set of all \(n \times m\) real matrices. \(A^\top\) represents the transpose of \(A\), and \(E\{x\}\) stands for the expectation of the stochastic variable \(x\). \(x \sim \mathcal{N}(m, N)\) with \(m \in \mathbb{R}^n\) and \(N \in \mathbb{R}^{n \times n}\) denotes the probability function of the random variable \(x\) follows a Gaussian distribution with \(m\) and \(N\) as the expectation and covariance, respectively. \(|x|\) denotes 2-norm of the vector \(x\), i.e., \(|x| = x^\top x\).

II. PROBLEM FORMULATION

In this paper, we consider the state estimation problem for the following discrete-time nonlinear stochastic systems:

\[
\begin{align*}
    x_{k+1} &= f(x_k) + w_k, \\
    y_k &= g(x_k) + v_k,
\end{align*}
\]

where \(x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^m\) are the state and measurement, respectively, and \(w_k\) and \(v_k\) are stationary stochastic noise. The nonlinear functions \(f(\cdot)\) and \(g(\cdot)\) and the probability distributions of process noise \(w_k\) and measurement noise \(v_k\) are assumed to be all known, and the noise may be non-Gaussian.

A. State estimation

To estimate the state of system (1), the state estimator is typically designed in the following form:

\[
\begin{align*}
    \hat{x}_{k+1} &= f(\hat{x}_k) + a(\hat{x}_k)e_{k+1}, \\
    e_{k+1} &= y_{k+1} - g(f(\hat{x}_k))
\end{align*}
\]

where \(\hat{x}_k\) is the estimate of state \(x_k\), \(a(\cdot)\) is a linear/nonlinear function that can be calculated using various approximation methods and the measurement prediction error is given as follows:

\[
e_{k+1} = y_{k+1} - g(f(\hat{x}_k))
\]

The form of the state estimator in (2) is standard and widely used in some existing estimation algorithms, such as the Kalman filter (KF) and extended Kalman filter (EKF). In the EKF, the state estimator of (1) is given as follows:

\[
\hat{x}_{k+1} = f(\hat{x}_k) + K_k e_{k+1}
\]

where \(K_k\) is the estimator gain at time instant \(k\) that is calculated using partial derivatives of the \(f\) and \(g\) at \(\hat{x}_k\). As a result, the estimator gain \(K_k\) is actually a function of \(\hat{x}_k\). Similarly, in the unscented Kalman filter (UKF), \(a(\hat{x}_k)\) in (2) is instantiated using deterministic sampling methods. As for the particle filter (PF), even though no estimator gain is used explicitly, the importance weights of particles could be viewed as the function of \(a(\hat{x}_k)\) in the sampling form. In this paper, our high-level plan is to approximate \(a(\cdot)\) as a generic nonlinear function, i.e., a deep neural network, which can be learned from estimate errors data \(x_k - \hat{x}_k\) over time.

Remark 1. In this paper, we will focus on the state estimation problem while two other closely related problems need to be remarked, i.e., state prediction and smoothing. In state prediction, measurement up to the current time instant is used to predict the state in the future; in state smoothing, measurement up to the current time instant is used to interpolate the state in the past. Our method proposed can be easily adapted to state prediction and smoothing problems as well.

Definition. \cite{9} The estimate error \(\tilde{x}_k \triangleq x_k - \hat{x}_k\) is said to be exponentially bounded in mean square if \(\exists \eta > 0\) and \(0 < \varphi < 1\), such that

\[
E[|\tilde{x}_k|^2] \leq \eta|\tilde{x}_0|^2 \varphi^k + p,
\]
holds at all the time instants $k \geq 0$, where $p$ is a positive constant number.

In this paper, our aim is to learn the state estimator policy function $a(\cdot)$ in the estimator $[2]$ using deep reinforcement learning such that the estimate error of the estimator $[2]$ is guaranteed to converge exponentially to a positive bound in mean square, as defined in Definition 1.

III. DATA-BASED LYAPUNOV STABILITY ANALYSIS

A. Markov Decision Process

In the RL state estimator design, the estimate error dynamics can be described by a Markov decision process (MDP), which is defined as a tuple $< S, A, P, C, \gamma >$. Here, $S$ is the state space, $A$ is the action space, $P$ is the transition probability distribution, $C$ is the cost and $\gamma \in [0, 1)$ is the discount factor. Then we have

\[ \tilde{x}_{k+1} \sim P(\tilde{x}_{k+1} | \tilde{x}_k, a_k), \forall k \in \mathbb{Z}_+, \]

where $\tilde{x}_k$ and $C_k$ are the state and cost at time instant $k$ and $\tilde{x}_k \in S$. An action $a(\cdot)$ at the state $s(\cdot)$ is sampled from the policy $\pi(a(\cdot) | s(\cdot))$. The standard state estimator $[2]$ is naturally a special case of $[6]$.

B. Data-based Stability Analysis

The convergence of bounded estimate error $\tilde{x}_k$ is essentially equivalent to show the stability of $[6]$. In this paper, we are interested to establish the stability theorem by only using samples $\{\tilde{x}_{k+1}, \tilde{x}_k, a_k\}, \forall k$. The most useful and general approach for studying the stability of a dynamical system is Lyapunov’s method $[29]–[31]$. In Lyapunov’s method, a suitable “energy-like” Lyapunov function $\mathcal{L}(x)$ is selected and its derivative along the system trajectories is ensured to be negative semi-definite, i.e., $\mathcal{L}(\tilde{x}_{k+1}) - \mathcal{L}(\tilde{x}_k) \leq 0$ for all time instants and states, so that the state goes in the direction of decreasing the value of Lyapunov function and eventually converges to the origin or a sub-level set of the Lyapunov function. In this subsection, such a function is introduced to analyse the stability of the estimate error dynamical system $[9]$. Different from the model-based stability analysis literature, we will learn a Lyapunov function instead of an explicit expression regardless of the degree of nonlinearity of the system and the time-consuming human expert design. The Lyapunov candidate $\mathcal{L}_k$ can be chosen as the Q-function $Q_\pi(\tilde{x}_k, a_k)$ $[22]–[25]$. It is well-known that the Q-function $Q_\pi(\tilde{x}_k, a_k)$ is related to the cost $C_k$. Thus, there always exist a positive function

\[ \hat{\mathcal{L}}_k = C_k + \delta_k = Q_\pi(\tilde{x}_k, a_k) \]

where $Q_\pi(\tilde{x}_k, a_k)$ can be parameterised by a DNN. Here, $\delta_k > 0$ should be satisfied due to the property of the Q-function, and it should be also non-increasing. Once the trace of the estimate error covariance converges, $\delta_k$ will converge to a constant number. We first prove the stability theorem by exploiting the function $\hat{\mathcal{L}}_k$ and the characteristic of $\delta_k$. The details of the learning algorithm will be discussed in Section IV.

In this paper, we assume that a Markov chain induced by a policy $\pi$ is ergodic with a unique distribution probability $q_\pi(\tilde{x}_k)$ with $q_\pi(\tilde{x}_k) = \lim_{k \to \infty} P(\tilde{x}_k | \rho, \pi, k)$. This is known as a common assumption in RL literature $[32]–[34]$. Then we prove our main theorem in the following.

Theorem 1. If there exists a constant $\beta \geq 0$ such that the following inequality holds

\[ E_{\tilde{x}_k} [E_{\tilde{x}_{k+1}} - \mathcal{L}_{k+1}] = -\beta E_{\tilde{x}_k} [||\tilde{x}_k||^2] + \delta_k. \]  

Then the estimate error $\tilde{x}_k$ is guaranteed to be exponentially bounded in mean square, i.e.,

\[ E_{\tilde{x}_k} [||\tilde{x}_k||^2] \leq \sigma_k^2 E_{\tilde{x}_k} [||\tilde{x}_0||^2] + \delta_k - \sigma_k \delta_0, \]

where

\[ \mu_\pi(\tilde{x}_k) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} P(\tilde{x}_k | \rho, \pi, k) \]

is the estimate error distribution, and $\sigma \in (0, 1)$.

Proof. The existence of the sampling distribution $\mu_\pi(\tilde{x}_k)$ is guaranteed by the existence of $\pi(\tilde{x}_k)$. Since the sequence $\{P(\tilde{x}_k | \rho, \pi, k), k \in \mathbb{Z}_+\}$ converges to $q_\pi(\tilde{x}_k)$ as $k \to \infty$, then by the Abelian theorem that if a sequence or function behaves regularly, then some average of it behaves regularly. We have the sequence $\frac{1}{N} \sum_{k=0}^{N} P(\tilde{x}_k | \rho, \pi, k), N \in \mathbb{Z}_+$ also converges and $\mu_\pi(\tilde{x}_k) = q_\pi(\tilde{x}_k)$. Then, $[8]$ can be rewritten as

\[ \int_{S} \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} P(\tilde{x}_k | \rho, \pi, k) (E_{P_\pi(\tilde{x}_{k+1} | \tilde{x}_k)} [L_{k+1} - L_k]) d\tilde{x}_k \]

\[ \leq -\beta E_{\tilde{x}_k} [||\tilde{x}_k||^2] + \delta_k \]

If $L_k$ is bounded, the probability density function (PDF) $P(\tilde{x}_k | \rho, \pi, k)$ over $\tilde{x}_k, \forall k$, $P(\tilde{x}_k | \rho, \pi, k)L_k$ is bounded. Besides, the sequence $\left\{\frac{1}{N} \sum_{k=0}^{N} P(\tilde{x}_k | \rho, \pi, k)L_k\right\}$ converges point-wise to the function $q_\pi(\tilde{x}_k)L_k$. According to the Lebesgue’s Dominated convergence theorem $[35]$, if a sequence $f_n(\tilde{x}_k)$ converges point-wise to a function $f$ and is dominated by some integrable function $g$ in the sense that,

\[ |f_n(\tilde{x}_k)| \leq g(\tilde{x}_k), \forall \tilde{x}_k \in S, \forall n. \]

Then, we can obtain

\[ \lim_{n \to \infty} \int_{S} f_n(\tilde{x}_k) d\tilde{x}_k = \int_{S} \lim_{n \to \infty} f_n(\tilde{x}_k) d\tilde{x}_k \]

We will use cost instead of reward in this paper which is often used in control literature. Maximisation in RL setup will be minimisation instead.
Accordingly, the following inequality holds

\[
\int \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} P(\tilde{x}_k | \rho, \pi, k) \times \left( \int_{S} P_{\pi}(\tilde{x}_{k+1} | \tilde{x}_k) \mathcal{L}_{k+1} d\tilde{x}_{k+1} - \mathcal{L}_k \right) d\tilde{x}_k
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \left( \sum_{k=1}^{N+1} E_{P(\tilde{x}_k | \rho, \pi, k)} \mathcal{L}_{k+1} - \sum_{t=0}^{N} E_{P(\tilde{x}_k | \rho, \pi, k)} \mathcal{L}_k \right)
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} (E_{P(\tilde{x}_k | \rho, \pi, k+1)} \mathcal{L}_{k+1} - E_{P(\tilde{x}_k | \rho, \pi, k)} \mathcal{L}_k).
\]

There always exists a scalar \( \sigma \) such that the following equation holds

\[
\left( \frac{1}{\sigma} - 1 \right) - \frac{\beta}{\sigma} = 0.
\]

To conclude, from [7], there always exist a function \( \mathcal{L}_k = \mathcal{C}_k + \delta_k \) that equals to \( Q_\pi(\tilde{x}_k, a_k) \) (i.e., \( \mathcal{L}_k \)), we can derive

\[
\frac{1}{\sigma^{t+1}} E_{P(\tilde{x}_{k+1} | \rho, \pi, k+1)} \mathcal{L}_{k+1} - \frac{1}{\sigma^t} E_{P(\tilde{x}_{k} | \rho, \pi, k)} \mathcal{L}_k
\]

\[
+ \frac{1}{\sigma^t} \left( \frac{1}{\sigma} - 1 \right) E_{P(\tilde{x}_{k} | \rho, \pi, k)} \mathcal{L}_k
\]

\[
\leq \frac{1}{\sigma^t} \left( \frac{1}{\sigma} - 1 \right) \| \tilde{x}_t \|^2
\]

\[
+ \frac{1}{\sigma^t} \left( \frac{1}{\sigma} - 1 \right) \delta_t,
\]

which implies

\[
\frac{1}{\sigma^{t+1}} E_{P(\tilde{x}_{k+1} | \rho, \pi, k+1)} \mathcal{L}_{k+1} - \frac{1}{\sigma^t} E_{P(\tilde{x}_{k} | \rho, \pi, k)} \mathcal{L}_k
\]

\[
\leq \frac{1}{\sigma^t} \left( \frac{1}{\sigma} - 1 \right) \delta_t.
\]

To sum the above inequality from \( t = 0, 1, \ldots, k-1 \) yields

\[
\frac{1}{\sigma^k} E_{P(\tilde{x}_{k} | \rho, \pi, k)} \mathcal{L}_k - E_{P(\tilde{x}_{0} | \rho, \pi, 0)} \mathcal{L}_0
\]

\[
\leq \delta_k / \sigma^k - \delta_0,
\]

which implies

\[
E_{\tilde{x}_k \sim \mu_{\pi}} \left[ \left\| \tilde{x}_k \right\|^2 \right] \leq \sigma^k E_{\tilde{x}_0 \sim \mu_{\pi}} \left[ \left\| \tilde{x}_0 \right\|^2 \right] + \delta_k - \sigma^k \delta_0.
\]

Thus, the estimate error is exponential bounded in mean square therefore complete the proof.

IV. REINFORCEMENT LEARNING FILTER DESIGN

In this section, the reinforcement learning algorithm design and implementation will be discussed. We plan to design the algorithm based on the actor-critic RL algorithms which are widely used in continuous control tasks [35] which is a similar setup for filter design. In actor-critic RL, DNN is used to approximate the “critic” and the “actor”. In the following, we will introduce how such algorithms can be designed by incorporating the theoretical results. The convergence of the algorithm will be analysed.

A. Reinforcement learning

In this paper, the objective is to find an optimal policy to minimise the expected accumulated cost as a value function:

\[
V_\pi(\tilde{x}_k) = \sum_{k=1}^{\infty} \pi(a_k | x_k) \sum_{k=1}^{\mathcal{P}_{k+1}|k} (C_k + \gamma V_\pi(\tilde{x}_{k+1})),
\]

where \( \mathcal{P}_{k+1}|k = P(\tilde{x}_{k+1} | \tilde{x}_k, a_k) \) is the transition probability of the dynamical system, \( C_k = C(\tilde{x}_k, a_k) \) is the cost function with \( C_k = \tilde{x}_k^T \tilde{x}_k, \gamma \in [0, 1) \) is a constant discount factor, and \( \pi(a_k | x_k) \) is a policy to be learned. In RL, the policy (nonlinear filter gain function in this paper) \( \pi(a_k | x_k) \) is typically a Gaussian distribution:

\[
\pi(a|x) = \mathcal{N}(a|x),
\]

from which an action \( a_k \in \mathcal{U} \) at the state \( \tilde{x}_k \in \mathcal{S} \) is sampled [36]. During the inference, the mean value \( a(\tilde{x}) \) is applied.

During the training process, a Q-function \( Q_\pi(\tilde{x}_k, a_k) \) (i.e., the action-value function) is practically minimised. \( Q_\pi(\tilde{x}_k, a_k) \) is given as

\[
Q_\pi(\tilde{x}_k, a_k) = \mathcal{C}_k + \gamma E_{\tilde{x}_{k+1} \sim P} [V_\pi(\tilde{x}_{k+1})]
\]

\[
- \alpha H(\pi(a_k | \tilde{x}_{k+1})), \tag{14}
\]

where \( E_{\tilde{x}_{k+1} \sim P} = \sum_{\tilde{x}_{k+1}} \mathcal{P}_{k+1}|k \) is an expectation operator over the distribution of \( \tilde{x}_{k+1} \).

Soft actor-critic (SAC) algorithm is one of the state-of-the-art off-policy actor-critic RL algorithms [37]. In SAC, an entropy item is added to the Q-function, with which the exploration performance becomes adjustable. Based on (14), the Q-function with the entropy item is described as

\[
Q_\pi(\tilde{x}_k, a_k) = \mathcal{C}_k + \gamma E_{\tilde{x}_{k+1} \sim P} [V_\pi(\tilde{x}_{k+1})]
\]

\[
- \alpha H(\pi(a_k | \tilde{x}_{k+1})), \tag{15}
\]

where \( H(\pi(a_k | \tilde{x}_{k+1})) = - \sum a_k \pi(a_k | \tilde{x}_k) \ln (\pi(a_k | \tilde{x}_k)) = -E_{\tilde{x}} \ln (\pi(a_k | \tilde{x}_k)) \) is the entropy of the policy, and \( \alpha \) is a temperature parameter. It is defined to determine the relative importance of the entropy term [37].

Thus, the algorithm is to solve the following optimisation problem:

\[
\pi^* = \arg \min_{\pi \in \Pi} \left( \mathcal{C}_k + \gamma E_{\tilde{x}_{k+1} \sim P} [V_\pi(\tilde{x}_{k+1})]
\]

\[
- \alpha H(\pi(a_k | \tilde{x}_{k+1})), \tag{16}
\]

where \( \Pi \) is the policy set.

An optimal policy \( \pi^*(a | \tilde{x}) = \mathcal{N}(a^* | \tilde{x}) \) can be obtained by solving (16). Here, \( \sigma^* \) is close to 0, which further implies the optimal policy \( a^* \) converges to a deterministic mean value. Once such a policy is obtained, it can be deployed to the target system. For \( a^* \), it can be parameterised as a DNN and learned using stochastic gradient descent algorithms.

Training/learning process will repeatedly execute policy evaluation and policy improvement. In the policy evaluation, the Q-value in (15) is computed by applying a Bellman operation

\[
\mathcal{T}^\pi Q_\pi(x_k, a_{l,t}) = \mathcal{T}^\pi Q_\pi(x_k, a_{l,t})
\]

\[
= R_k + \gamma E_{\tilde{x}_{k+1} \sim P} [E_{\pi} [Q_\pi(x_{k+1}, a_{k+1})]], \tag{17}
\]

where \( Q_\pi(x_k, a_k) = Q_\pi(x_k, a_k) + \alpha \ln (\pi(a_k | x_k)) \).
In the policy improvement, the policy is updated by
\[
\pi_{\text{new}} = \arg \min_{\pi \in \Pi} \mathcal{D}_{KL} \left( \pi' (\cdot | x_k) \| \frac{e^{\frac{1}{|B|} Q_{\pi_{\text{old}}} (x_k, \cdot)}}{Z_{\pi_{\text{old}}}} \right),
\]
where \( \pi_{\text{old}} \) denotes the policy from the last update, \( Q_{\pi_{\text{old}}} \) is the Q-value of \( \pi_{\text{old}} \), \( \mathcal{D}_{KL} \) denotes the Kullback-Leibler (KL) divergence, and \( Z_{\pi_{\text{old}}} \) is a normalisation factor. The objective can be transformed into
\[
\pi^* = \arg \min_{\pi \in \Pi} \mathbb{E}_\pi \left[ \alpha \ln (\pi (a_k | x_k)) + Q (x_k, a_k) \right].
\]
More details can be found in [37].

B. Deep neural networks approximation

The DNNs are constructed by fully connected multiple layer perceptrons (MLP), in which the rectified linear unit (ReLU) nonlinearities are chosen as the activation functions [38]. The ReLU nonlinearities are defined as \( \rho (z) = \max \{ z, 0 \} \). For a vector \( z = [z_1, \ldots, z_n]^T \in \mathbb{R}^n \), there exists \( \rho (z) = [\rho (z_1), \ldots, \rho (z_n)]^T \). An example of a MLP with two hidden layers is given as
\[
\text{MLP}_w^2 (z) = w_2 \left[ \rho \left( w_1 \left[ \rho \left( w_0 \left[ z \right] \right) \right] \right) \right]^T ,
\]
where \( [z^T, 1]^T \) is a vector composed of \( z \) and a bias 1, the superscript “2” denotes the total number of hidden layers, the subscript “w” denotes the parameter set to be trained in a MLP with \( w = \{w_0, w_1, w_2\} \), and \( w_0, w_1 \), and \( w_2 \) are weight matrices with appropriate dimensions.

If there is a set of inputs \( z = \{z_1, \ldots, z_L\} \) for the MLP in (20) with \( z_1, \ldots, z_L \) denoting vector signals, we have
\[
\text{MLP}_w^2 (z) = \text{MLP}_w^\ast (z_1^T, \ldots, z_L^T).
\]
Besides, \( \text{MLP}_w^\ast (z_1, z_2) = \text{MLP}_w^\ast (z_1^T, z_2^T) \) for two vector inputs \( z_1 \) and \( z_2 \). If \( z_1 = \{z_1, \ldots, z_L\} \) is a set of vectors, \( \text{MLP}_w^\ast (z_1, z_2) = \text{MLP}_w^\ast (z_1^T, \ldots, z_L^T, z_2^T) \).

In this paper, the constructed MLP is used to approximate the “critic” \( Q (\tilde{x}_k, a_k) \) and the “actor” \( \pi (a_k | \tilde{x}_k) \). We respectively use \( \theta \) and \( \phi \) to parameterise \( Q (\tilde{x}_k, a_k) \) and \( \pi (a_k | \tilde{x}_k) \), i.e., \( Q_{\theta} (\tilde{x}_k, a_k) \) and \( \pi_{\phi} (a_k | \tilde{x}_k) \). As discussed in Section III-B the Q-function \( Q (\tilde{x}_k, a_k) \) is regarded as a Lyapunov candidate \( L (k) \) in our paper. Namely, the “critic” is the Lyapunov function. In the following context, we replace \( Q (\tilde{x}_k, a_k) \) with \( L (k) \), and replace \( Q_{\theta} (\tilde{x}_k, a_k) \) with \( L_{\theta} (k) \). It is noted that the direct output of the constructed MLP may not satisfy the requirements of a Lyapunov function \( L_k \), for example, \( L_0 = 0 \) and \( L_k > 0 \) for all \( k > 0 \). To this end, it is necessary to modifiy the output of such a MLP. Following (20) and (21), the Lyapunov function approximation \( L_{\theta} (k) \) is
\[
L_{\theta} (k) = \left( \text{MLP}_{K_1}^w (\tilde{x}_k) \right)^2 ,
\]
where \( \theta = \{ \theta_0, \ldots, \theta_{K_1} \} \) with \( \theta_i \) for \( 0 \leq i \leq K_1 \) denoting the weight matrices with proper dimensions. The DNN for \( L_{\theta} \) is illustrated in Fig. 1.

The filter gain \( a_k \) is also approximated using a MLP. The approximated filter gain of \( a_k \) with a parameter set \( \phi \) is
\[
a_{\phi} = \text{MLP}_{K_2}^w (\tilde{x}_k) ,
\]
where Π is the policy set, π_{old} is the last updated policy, Q_{π_{old}} is the action value function of π_{old}, D_{KL} means the Kullback-Leibler divergence, and Z_{π_{old}} is a partition function which is introduced to normalise the distribution.

**Remark 2.** Different from the SAC algorithm, the Lyapunov constraint, that is, \( L_\phi(k+1) - L(k) \leq -\beta \text{Tr}(\hat{x}_k \hat{x}_k^\top) + \delta_k \) is considered when the policy improvement step is executed. In this way, we can guarantee that the steady estimate error always converges to a positive constant in mean square, which will be proved in the later content.

Then we can rewrite (26) as

\[
\pi_{\text{new}} = \underset{\pi \in \Pi}{\arg \min} \mathbb{E} \left[ \alpha \ln(\pi(a_k|\hat{x}_k)) + L(k) \right] \\
\text{s.t. } L_\phi(k+1) - L(k) \leq -\beta \text{Tr}(\hat{x}_k \hat{x}_k^\top) + \delta_k.
\]

(27)

For the optimisation of (27), the Lagrangian multiplier can be introduced to deal with the constraint. Thus, (27) can be further described as

\[
\pi_{\text{new}} = \underset{\pi \in \Pi}{\arg \min} \mathbb{E} \left[ \alpha \ln(\pi(a_k|\hat{x}_k)) + L(k) \right] + \lambda \left( L_\phi(k+1) - L(k) + \beta \text{Tr}(\hat{x}_k \hat{x}_k^\top) - \delta_k \right),
\]

(28)

where \( \lambda \) is a Lagrangian multiplier.

Based on the setup of RL, the policy improvement in (28) is converted into finding \( \pi^* \) by minimising the following function

\[
\mathcal{J}_\pi(\phi) = \mathbb{E}(\hat{x}_k, a_k \sim \mathcal{M}) \left\{ \alpha \ln(\pi_\phi) + L_\theta(k) \right\},
\]

whose gradient in terms of \( \phi \) is derived as

\[
\nabla_\phi \mathcal{J}_\pi(\phi) = \sum (\alpha \nabla_{a_k} \ln \pi_\phi + \nabla_a L_\theta(k)) \nabla_a \pi_\phi + \alpha \nabla_\phi \ln \pi_\phi |_{\mathcal{B}}.
\]

For the temperature \( \alpha \), it is updated by minimising the following function

\[
\mathcal{J}_\alpha = \mathbb{E}_\pi \left\{ -\alpha \ln \pi(a_k|\hat{x}_k) - \alpha \mathcal{H} \right\},
\]

where \( \mathcal{H} \) is a target entropy.

For the Lagrangian multiplier \( \lambda \), it is learned by maximising

\[
\mathcal{J}(\lambda) = \mathbb{E} \left[ L_\phi(k+1) - L(k) + \beta \text{Tr}(\hat{x}_k \hat{x}_k^\top) - \delta_k \right].
\]

Our algorithm is implemented based SAC algorithm [37], in which \( \tau_\mathcal{L}, \tau_\pi, \tau_\alpha, \) and \( \tau_\lambda \) are the positive learning rates, and \( \tau > 0 \) is a constant scalar. The optimal parameters for the DNN in (22) and (23) will be learned and the filter gain policy will be approximated by \( \pi_{\phi^*} \) from which an action will be sampled.

During inference, the mean value of \( \pi_{\phi^*} \) will be deployed since policy is often assumed to be Gaussian distributed in SAC [37].

The inference procedure is illustrated in Fig. 3. The learned policy is deployed to tune the error \( y_{k+1} - g(\hat{x}_k) \) in the estimator. We sample the measurement output signal \( y_{k+1} \) from the real system. Then, the estimator starts to estimate states for the real system.

**Algorithm 1** Lyapunov-Based Reinforcement Learning Filter Algorithm (LRLF)

1. Set the initial parameters \( \theta \) for the Lyapunov function \( L_\theta \), \( \phi \) for the filtering policy \( \pi_\phi \), \( \lambda \) for the Lagrangian multiplier, \( \alpha \) for the temperature parameter, and the replay memory \( \mathcal{M} \)
2. Set the target parameter \( \bar{\theta} \) as \( \bar{\theta} \leftarrow \theta \)
3. **while** Training **do**
   4. **for** each data collection step **do**
   5. Choose \( l_k \) using \( \pi_\phi(\hat{x}_k|l_k) \)
   6. Run the system \( 1 \) and the filter system \( 2 \) and collect data \( \hat{x}_k \)
   7. \( \mathcal{M} \leftarrow \mathcal{M} \cup \hat{x}_k \)
4. **end for**
9. **for** each gradient step **do**
   10. \( \theta \leftarrow \theta - \tau_\mathcal{L} \nabla_\theta L_\mathcal{L} (\theta) \)
   11. \( \phi \leftarrow \phi - \tau_\pi \nabla_\phi \mathcal{J}_\pi (\phi) \)
   12. \( \alpha \leftarrow \alpha - \tau_\alpha \nabla_\alpha \mathcal{J}_\alpha (\alpha) \)
   13. \( \lambda \leftarrow \lambda - \tau_\lambda \nabla_\lambda \mathcal{J}_\lambda (\lambda) \)
   14. \( \phi^*_\bar{\theta} \leftarrow \tau \theta + (1 - \tau) \phi^*_\theta \)
   15. **end for**
16. **end while**
17. Output optimal parameters \( \theta^*, \phi^*, \lambda^* \), and \( \alpha^* \)

**Remark 3.** Bayesian nonlinear filtering methods such as the EKF, UKF, PF adjust the estimator gains or sampling weights via online computation. The proposed LRLF is trained offline and the filter gain is approximated by a DNN which will be deployed directly for online applications. In this paper, like

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**Fig. 2:** Offline training process of LRLF

**Fig. 3:** Estimator structure
other methods, the training needs the full knowledge of the mathematical model, i.e., \([1]\). It should also be noted that the filter \((2)\) is trained by only using the samples simulated from \([1]\) instead of any other assumptions on the model. In other words, the mathematical model is actually a simulator and our training is performed in a model-free manner. As shown in Figs. 2 and 3, the statistical information of the system’s noise such as the covariance are not directly used by the LRLF, which is different from the EKF, UKF and PF where such statistical information is used explicitly.

D. Algorithm convergence analysis

Next, a lemma is given to show that the policy evaluation can guarantee the action value function to converge. Since the proof is standard, it is omitted and can be referred to \([37]\) for more details.

Lemma 1. (Policy evaluation) Consider the backup operator \(\mathcal{T}^\pi\) in \((25)\) and define \(\mathcal{L}^{\pi,t+1}(k) \triangleq \mathcal{T}^\pi \mathcal{L}^\pi(k)\). The sequence \(\mathcal{L}^{\pi,t+1}(k)\) can converge to a soft value \(\mathcal{L}^\pi\) of the policy \(\pi\) as \(t \to \infty\).

For the policy improvement, a lemma is given to show that the updated policy is better than the last one.

Lemma 2. (Policy improvement) Considering the last updated policy \(\pi_{\text{old}}\) and the new policy \(\pi_{\text{new}}\) to be obtained from \((27)\), \(\mathcal{L}^{\pi_{\text{new}}}(k) \leq \mathcal{L}^{\pi_{\text{old}}}(k)\) holds for \(\forall \tilde{x}_k \in \mathcal{S}\) and \(\forall a_k \in \mathcal{A}\).

Proof. According to \((27)\), we can obtain
\[
\mathbb{E}_{\pi_{\text{new}}} [\alpha \ln(\pi_{\text{new}}(a_k|\tilde{x}_k)) + \mathcal{L}^{\pi_{\text{old}}}(k)] \leq \mathbb{E}_{\pi_{\text{old}}} [\alpha \ln(\pi_{\text{old}}(a_k|\tilde{x}_k)) + \mathcal{L}^{\pi_{\text{old}}}(k)],
\]
which implies
\[
\mathbb{E} [\mathcal{L}^{\pi_{\text{old}}} + \alpha \ln(\pi_{\text{new}}(a_k|\tilde{x}_k))] \leq \mathcal{V}_{\pi_{\text{old}}} (\tilde{x}_k).
\]
Then, the following inequality holds
\[
\mathcal{L}^{\pi_{\text{old}}}(k) = C_k + \gamma \mathcal{E}_{\tilde{x}_{k+1}} [\mathcal{V}_{\pi_{\text{old}}} (\tilde{x}_k)] \geq C_k + \gamma \mathcal{E}_{\tilde{x}_{k+1}} [\mathbb{E}_{\pi_{\text{new}}} \{ \mathcal{L}^{\pi_{\text{new}}}(k + 1) - \alpha \ln(\pi_{\text{new}}(a_{k+1}|\tilde{x}_{k+1})) \}]
\]
which completes the proof.

Next, a theorem is derived to show that \(\mathbb{E}[\|\tilde{x}_k\|^2]\) converges to \(p\) as \(k \to \infty\) after repeatedly executing the policy evaluation and policy improvement.

Theorem 2. Starting from any policy \(\pi_0\) belonging to the set \(\Pi\), define \(\pi_i\) \((i = 1, 2, \ldots, \infty)\) as the policy obtained at the \(i\)-th policy improvement step. \(\pi_i\) can converge to an optimal policy \(\pi^\star\), which ensures that \(\mathbb{E}[\|\tilde{x}_k\|^2]\) converges to a constant \(p\) as \(k \to \infty\) \(\mathbb{E}[\|\tilde{x}_k\|^2]\) converges to a constant based on the conclusion in Lemma 1.

V. SIMULATION

The experiment setup for the LRLF is a three-stage procedure. First, a number of \(N\) estimation policies are trained for different initial conditions and noise is sampled from a known distribution. During training, the simulator will generate sample trajectories \(\{x_i^k, y_i^k, i = 1, \ldots, I, k = 1, \ldots, K\}\), where \(I, K\) are the number of training trajectories and that of the trajectory length respectively. Second, the estimation performance of each DRL-based state estimator \((2)\) will be evaluated by running \(M\) Monte Carlo simulations, again for different initial conditions and noise sampled from a known distribution. Finally, the state estimator with the lowest trace of estimate error covariance during inference (unless diverged) will be deployed online and used for comparison with other nonlinear filtering algorithms.

We first consider a free-pendulum tracking example widely used as a benchmark for nonlinear state estimation \([4], [12]\). \([39]\). The pendulum has unity mass of 1 kg and length of 1 m. The discrete-time dynamics of the pendulum is given as follows:
\[
x_{1,k+1} = x_{2,k} + \frac{\delta t}{2} \left( x_{1,k} \right), \quad x_{2,k+1} = \frac{\delta t}{2} \left( x_{2,k} - g \sin(x_{1,k}) \right) + w_k
\]
where \(x_{1,k} = \theta_k, x_{2,k} = \dot{\theta}_k\) are the angle and angle velocity of the pendulum at time instant \(k\), respectively. \(\delta t\) is the sampling time and set as 0.1 second in the simulation. The process noise \(w_k\) is Gaussian distributed as
\[
w_k \sim \mathcal{N}(\mathbf{0}, \delta t^2 \mathbf{Q}_1)
\]
The measurement equation is given as:
\[
y_k = \sin(x_{1,k}) + v_k
\]
where the measurement noise is also Gaussian distributed with \(v_k \sim \mathcal{N}(0, 0.01)\). Since the scalar measurement solely depends on the angle \((x_1)\), the estimate of the latent state \(x_2\) has to be reconstructed using the cross-correlation information between the angle and the angular velocity in the dynamics \((29)\).

To test if the estimate error of LRLF converges regardless of the initial state and estimate, for each trajectory in the training, the initial state were sampled from uniform distribution:
\[
\theta(0) \sim U[-0.5\pi, 0.5\pi], \quad \omega(0) \sim U[-0.5\pi, 0.5\pi],
\]
and the initial estimate was sampled from uniform distribution:
\[
\hat{\theta}(0) \sim U[-0.25\pi + \theta(0), 0.25\pi + \theta(0)], \quad \hat{\omega}(0) \sim U[-0.25\pi + \omega(0), 0.25\pi + \omega(0)].
\]
We compared LRLF with three other classic nonlinear Bayesian estimation algorithms, the EKF, UKF and PF (10^3 and 10^4 particles respectively) \([4]\). The same initial state and state estimate in \([31]\) and \([32]\) were used for all estimation algorithms.

In the training, each trajectory has \(K = 100\) data points (10 seconds simulation of \((29)\)). We trained \(N = 10\) policy.
networks and evaluated each network for \( M = 500 \) Monte Carlo simulations. The training details are given as follows:

### TABLE I: Hyperparameters of LRLF

| Hyperparameters       | Pendulum | Vehicle |
|-----------------------|----------|---------|
| Time horizon \( K \)  | 100      | 100     |
| Minibatch size        | 256      | 256     |
| Actor learning rate   | 1e-4     | 1e-4    |
| Critic learning rate  | 3e-4     | 3e-4    |
| Lyapunov learning rate| 3e-4     | 3e-4    |
| Target entropy        | NaN      | NaN     |
| Soft replacement(\( \tau \)) | 0.005 | 0.005    |
| Discount(\( \gamma \)) | 0.995   | 0.995   |
| \( \alpha_0 \)        | 0.1      | 0.1     |
| Structure of \( \alpha \) | (32,16) | (32,16) |
| Structure of \( L_\theta \) | (64,32) | (64,32) |

For the LRLF, there are two networks: the policy network and the Lyapunov critic network. For the policy network, we use a fully-connected MLP with one hidden layer, outputting the mean and SD of a Gaussian distribution. As mentioned in section IV, it should be noted that the output of the Lyapunov critic network is a square term, which is always non-negative. More specifically, we use a fully-connected MLP with one hidden layer and one output layer with different units as in Table 1, outputting the feature vector \( Q_\theta \) (see Fig. 1). The Lyapunov critic function is obtained by \( L_c(s, a) = Q_\theta(s, a)Q_\theta(s, a) \). All the hidden layers use ReLu activation function and we adopt the same invertible squashing function as [40] to the output layer of the policy network.

The proposed LRLF was evaluated for the following aspects:

1. Algorithm convergence: does the proposed training algorithm converge with random initial states and estimate initialisation?
2. Estimate error convergence: does the estimate error variance converge compared with other state estimation algorithms?
3. Performance comparison: how does the proposed LRLF perform compared with other state estimation algorithms under various initial state conditions?
4. Robustness to uncertainty: how do the trained estimator perform during inference when faced with uncertainties unseen during training, such as noise with covariance shift and randomly-occurring missing measurement?

#### A. Algorithm convergence

The convergence guarantee can be validated by checking the values of Lagrange multipliers. When the Lyapunov constraint in (28) is satisfied, the parameter \( \lambda \) should continuously decrease to zero. In Fig. 4, the value of \( \lambda \) during training is demonstrated. In all training trials of the 10 policy networks, \( \lambda \) converges to zero eventually, which implies the convergence guarantee of the state estimate.

#### B. Estimation performance

The proposed LRLF is compared with the EKF, UKF and PF. For each method, a number of 500 Monte Carlo tests were run with random initial state given in (31) and (32). The average value and SD of estimate error \( \tilde{x}_k \) are illustrated in the left column of Fig. 5 (see Figs. 5a, d, g, j, m) for EKF, UKF, PF and LRLF. These comparisons show that only the PF with a larger number of particles (10^4) has comparable performance with the LRLF.

To verify estimation performance of the LRLF for nonlinear systems with non-Gaussian noise, we tested the LRLF over three typical non-Gaussian distributions as shown in Fig. 6. Specifically, the measurement noise \( \nu_k \) of the pendulum model in (29) -(30) is assumed to take three kinds of non-Gaussian probability distributions: 1) a Gaussian distribution \( N(0,0.01) \) is truncated, 2) a uniform distribution at the interval \([-0.3,0.3] \), or 3) an exponential distribution with the PDF \( p(w) = \left\{ \begin{array}{ll} 0.04 \exp(-0.04w) & \text{if } w \geq 0 \\ 0 & \text{if } w < 0 \end{array} \right. \). A number of 500 Monte Carlo simulations with random initial state in (31) and (32) were tested for each kind of measurement noise. The average value and SD of estimate error \( \tilde{x}_k \) are illustrated in Fig. 7. It is found that the LRLF still generates state estimate with bounded estimate errors under all the three kinds of non-Gaussian noises.

#### C. Robustness of the estimator

We considered the robustness of state estimators for two scenarios. (1) Against larger measurement noise: in the inference, the variance of the measurement noise increases from 0.01 to 0.1. (2) Against missing measurement: \( y_k \) is sampled/contaminated randomly with Bernoulli distribution, in which \( p(\text{missing}) = p(\text{not missing}) = 0.5 \). This means half of the measurements are not received/set to zeros during inference.

1. Larger measurement noise: Similarly, we run 500 Monte Carlo tests with random initial states and initial state estimates given in (31) and (32) for each state estimator. The average value and variance of estimate error \( \tilde{x}_k \) are illustrated in Figs. 5(b), (e), (h), (k), (n). From Figs. 5(b), (e), (h), it can be found that under a large measurement noise with variance 0.1, the estimates of the EKF, UKF and PF (10^4 particles) all diverge, while the estimate error of our proposed LRLF and PF (10^4 particles) increase compared with those under noise with variance 0.01.
Fig. 5: State estimate error of different estimation methods under different conditions. The left column corresponds to those with simulation setup with the measurement noise variance $R = 0.01$; the middle column corresponds to those with the increased measurement noise variance $R = 0.1$; the right column corresponds to those with measurement missing happening with the probability 0.5. Solid line indicates the average estimate error $\hat{x}_{1,k}$ (red) and $\hat{x}_{2,k}$ (blue) and shadowed region for the 1-SD confidence interval. All results are obtained from 500 Monte Carlo runs.
where $x$, $p$, $w_k$ and the measurement model is given as follows:

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$$

and the measurement model is given as follows:

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$$

where $x_{1,k}$, $x_{2,k}$ are the distance and speed of the vehicle at time instant $k$, respectively. The Gaussian noise $w_k \sim \mathcal{N}(0,0.01)$ and $v_k \sim \mathcal{N}(0,0.02)$, and the initial state and the initial state estimate are $x_0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$, $\hat{x}_0 \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}\right)$, respectively.

We use the same experiment setup as in the previous example of tracking pendulum. The estimate error of the LRLF is obtained from 500 Monte Carlo simulations, as shown in Fig. 8. The SD of state estimate error of the Kalman filter is plotted in Fig. 8 for comparison. It can be found that our proposed LRLF estimate both the distance and speed of the vehicle quite accurately, with slightly bigger estimate error than that of the optimal Kalman filter, which is expected since no state estimators can perform better than the optimal Kalman filter for linear stochastic systems with Gaussian noise.

VI. CONCLUSION

In this paper, we have combined Lyapunov’s method in control theory and deep reinforcement learning to design state estimator for discrete-time nonlinear stochastic systems. We theoretically prove the convergence of bounded estimate error solely using the data in a model-free manner. In our approach, the filter gain is approximated by a deep neural network and trained offline. During inference, the learned filter can be deployed efficiently without extensive online computations. Simulation results show the superiority of our state estimator design method over existing nonlinear filters, in terms of estimate convergence even under some system uncertainties such as covariance shift in system noise and randomly missing measurements.

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Fig. 7: State estimate error of the LRLF with different kinds of measurement noise shown in Fig. 6. Solid line indicates the average estimate error $\hat{x}_{1,k}$ (red) and $\hat{x}_{2,k}$ (blue) and shadowed region for the 1-SD confidence interval. The results are obtained from 500 Monte Carlo runs.

Fig. 8: State estimate error of the LRLF vs the SD of state estimate of the Kalman filter. Solid line indicates the average estimate error $\hat{x}_{1,k}$ (red) and $\hat{x}_{2,k}$ (blue) and shadowed region for the 1-SD confidence interval of the LRLF. The results of LRLF is obtained with 500 Monte Carlo runs.

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