Let $E$ be a commutative ring with identity and $P \in E[x]$ be a polynomial. In the present paper we consider digit representations in the residue class ring $E[x]/(P)$. In particular, we are interested in the question whether each $A \in E[x]/(P)$ can be represented modulo $P$ in the form $e_0 + e_1X + \cdots + e_hX^h$, where the $e_i \in E[x]/(P)$ are taken from a fixed finite set of digits. This general concept generalises both canonical number systems and digit systems over finite fields. Due to the fact that we do not assume that 0 is an element of the digit set and that $P$ need not be monic, several new phenomena occur in this context.

1. Introduction

In recent years, many different notions of number systems have been invented and thoroughly studied (see e.g. [6, 7] and the references therein). Several of them, like canonical number systems and digit systems over finite fields [13, 15] represent elements of a factor ring of the shape $E[x]/(P)$, where $E$ is a commutative ring with identity and $P \in E[x]$ is a polynomial. The representations obtained in these number systems have the shape $e_0 + e_1X + \cdots + e_hX^h$ where $X$ is the coset of $x$ modulo $P$, and the elements $e_i$ ($0 \leq i \leq h$) are taken from a finite set $\mathcal{N} \subset E[x]/(P)$ of digits.

The aim of the present paper is to establish a common general framework for all number systems of this kind. Indeed, we allow $E$ to be an arbitrary commutative ring with identity. The only requirement for the digit set $\mathcal{N}$ is that it has to be a system of coset representatives of the ring $E[x]/(P)$ modulo the basis $X$. (Note that $E[x]/(P, x)$ is isomorphic to $E/(P(0))$; thus this property is decided by looking at the constant coefficients of the digits alone.) In particular, we allow digit sets $\mathcal{N}$ that do not contain zero.

Although we are able to prove several theorems in this general context, new and interesting phenomena and difficulties occur throughout.

As a first point, treated in Section 2 if zero is not contained in the digit set, one has to be careful how to define finite expansions; also the well known fact that each element of $E[x]/(P)$ has a finite representation if and only if the dynamical system associated to the number system has only zero as a periodic point, is no longer true. In the general case,
there may well occur a cycle of length greater than 1 without the finite expansion property being violated.

Moreover, as we do not have the concept of an expanding polynomial in our general setting, one has to take care to adopt the right definition of periodicity. Indeed, let \( T \) be the dynamical system associated to the number system. We can easily show that an element with eventually periodic orbit admits a digit representation which is eventually periodic. However, the converse is only known when the defining polynomial \( P \) is expanding.

Secondly, in canonical number systems as well as in digit systems over finite fields, monic defining polynomials \( P \) have been considered almost exclusively (although [3], [10] and [18, Section 5.3.3] do treat nonintegral bases for number systems in \( \mathbb{Z} \) from various viewpoints). In Section 3 we explore the case of non-monic polynomials. Here the basis of the number system, taken as a root of the defining polynomial, need no longer be an algebraic integer. While \( \mathcal{E}[x]/(P) \) is a finitely generated free \( \mathcal{E} \) module if \( P \) is monic, neither property holds if \( P \) is not monic. Interestingly, we are able to exhibit a finitely generated (and in many cases even free) module \( \mathcal{R}_k \subset \mathcal{E}[x]/(P) \) with the following property. If each element of \( \mathcal{R}_k \) admits a finite expansion then the same is true for all elements of \( \mathcal{E}[x]/(P) \). This reduces the problem from a set with complicated algebraic structure to an easier framework.

In Section 4 given the exact sequence
\[
0 \to \mathcal{E}[x]/(P_2) \xrightarrow{P_1} \mathcal{E}[x]/(P_1P_2) \to \mathcal{E}[x]/(P_1) \to 0,
\]
of \( \mathcal{E} \)-modules, with number systems on the outer components, we construct a number system on the middle module and derive conditions when all elements in this number system have a finite expansion, or at least a periodic digit sequence.

In Section 5 we extend the definitions of canonical number systems and number systems over finite fields to possibly non-monic defining polynomials. Interestingly, for canonical number systems, this more general version is still covered by the theory of shift radix systems (in the sense of [1]).

In the final Section 6 we take the concept of a set of witnesses (due to Brunotte [9] in the context of canonical number systems) to our general setting. The idea is that the finite expansion property of a number system can be decided by looking only at the elements of a properly chosen subset of \( \mathcal{R} \). Whenever a number system possesses a finite set of witnesses, we can decide the finite expansion property by a finite algorithm. We prove the existence of finite witness sets for a large class of number systems.

Throughout the paper we illustrate our concepts with examples which show the difficulties and new phenomena occurring here.

2. Basic properties of digit systems

We start this section with a formal definition of a quite general notion of number system. Fix a commutative ring \( \mathcal{E} \) with identity.

**Definition 2.1.** Let \( d \geq 1 \) be an integer and
\[
P(x) = p_dx^d + \cdots + p_1x + p_0
\]
a polynomial with coefficients in \( \mathcal{E} \), such that \( p_d \) and \( p_0 \) are not zero divisors of \( \mathcal{E} \), and \( \mathcal{E}/(p_0) \) is finite. Furthermore, let \( \mathcal{R} = \mathcal{E}[x]/(P) \), and let \( \mathcal{N} \subset \mathcal{R} \) be a system of coset representatives of \( \mathcal{R}/(X) \). The digit system over \( \mathcal{E} \) defined by \( P \) and \( \mathcal{N} \) is the triple \((\mathcal{R}, X, \mathcal{N})\), where \( X \) is the image of \( x \) under the canonical epimorphism \( \mathcal{E}[x] \to \mathcal{E}[x]/(P) \).

Define the maps
\[
D_{\mathcal{N}} : \mathcal{R} \to \mathcal{N} : D_{\mathcal{N}}(A) = e, \text{ the unique } e \in \mathcal{N} \text{ with } A \equiv e \pmod{X},
\]
\[
T : \mathcal{R} \to \mathcal{R} : T(A) = \frac{A - D_{\mathcal{N}}(A)}{X}.
\]
We say that \((\mathcal{R}, X, \mathcal{N})\) has the periodic expansion property (PEP) if the sequence \((T^i(A))_{i \geq 0}\) is eventually periodic for each \( A \in \mathcal{R} \). We call the sequence \((D_{\mathcal{N}}(T^i(A)))_{i \geq 0}\) the digit sequence of \( A \) in the digit system \((\mathcal{R}, X, \mathcal{N})\).

**Lemma 2.2.** The PEP implies that each \( A \in \mathcal{R} \) has an eventually periodic digit sequence.

**Proof.** Trivial. \( \square \)

It is not clear to us whether the converse also holds; it holds trivially whenever the periodic set of the number system (cf. Definition 2.9) is finite.

One notes that \( \mathcal{R}/(X) \cong \mathcal{E}/(p_0) \), so that \( |\mathcal{N}| = |\mathcal{E}/(p_0)| \). The digit sequence of \( A \in \mathcal{R} \) clearly exists and is unique, because \( \mathcal{N} \) is a system of representatives of \( \mathcal{R} \) modulo \( X \). If \( \mathcal{N} \) were larger, we would have non-uniqueness and nondeterminism in the \( X \)-ary representation of \( A \). The repeated application of the map \( T \) gives the backward division algorithm, as defined in [10] and many later papers.

**Lemma 2.3.** There exists \( n \in \mathbb{N} \) such that
\[
(2.1) \quad A = \sum_{i=0}^{n-1} D_{\mathcal{N}}(T^i(A))X^i
\]
if and only if \( T^n(A) = 0 \) for some \( n \in \mathbb{N} \).

**Proof.** Assume that \( A \) has the representation \((2.1)\). We obviously have \( T(A) = \frac{A - D_{\mathcal{N}}(A)}{X} = \sum_{i=0}^{n-2} D_{\mathcal{N}}(T^{i+1}(A))X^i \). Continuing, by induction, we obtain \( T^n(A) = 0 \). Conversely, suppose there exists an \( n \in \mathbb{N} \) such that \( T^n(A) = 0 \). Then it is easy to see that \( A = \sum_{i=0}^{n-1} D_{\mathcal{N}}(T^i(A))X^i \).

\( \square \)

Lemma 2.3 motivates the following definition.

**Definition 2.4.** Let \((\mathcal{R}, X, \mathcal{N})\) be a digit system over \( \mathcal{E} \). \( (\mathcal{R}, X, \mathcal{N}) \) has the finite expansion property (FEP) if for each element \( A \in \mathcal{R} \) there exists an \( n \in \mathbb{N} \) with \( T^n(A) = 0 \). The representation \((2.1)\) is called the finite \( X \)-ary expansion of \( A \).

**Example 2.5.** Let \( \mathcal{E} = \mathbb{Z} \), \( P(x) = 3x^2 - 2x + 5 \), and \( \mathcal{N} = \{0, 1, 2, 3, 4\} \). We want to calculate the digit sequence of \(-X^k \) for \( k \geq 0 \) (where \( X^0 = 1 \)) in the digit system \((\mathbb{Z}[x]/(P), X, \mathcal{N})\). The difficulty in the non-monic case is that we cannot represent \(-X^k\)
in \( \mathbb{Z}[x]/(P) \) as a linear combination of smaller powers of \( X \). We have \(-1 \equiv 4 \pmod{X}\) and therefore

\[
T(-1) = \frac{-1 - D_X(-1)}{X} = \frac{-1 - 4}{X} = \frac{3X^2 - 2X}{X} = 3X - 2.
\]

Continuing in this way we obtain

\[
T^2(-1) = T(3X - 2) = \frac{3X - 2 - D_X(3X - 2)}{X} = \frac{3X - 2 - 3}{X} = \frac{3X^2 + X}{X} = 3X + 1
\]

\[
T^3(-1) = T(3X + 1) = 3
\]

\[
T^4(-1) = T(3) = 0.
\]

Thus we have \( (D_X(T^i(-1)))_{i \geq 0} = 4, 3, 1, 3, 0, 0, \ldots \) and obviously

\[
(D_X(T^i(-X^k)))_{i \geq 0} = 0, \ldots, 0, 4, 3, 1, 3, 0, 0, \ldots.
\]

We see that the element \(-X^k\) has the finite \( X \)-ary expansion

\[
-X^k = \sum_{j=0}^{k+3} D_N(T^j(-X^k))X^j = 4X^k + 3X^{k+1} + X^{k+2} + 3X^{k+3}.
\]

Alternatively, if we take \( \mathcal{M} = \{-2, -1, 0, 1, 2\} \) as digits, we easily obtain

\[
(D_M(T^i(-X^k)))_{i \geq 0} = 0, \ldots, 0, -1, 0, 0, \ldots.
\]

The finite \( X \)-ary expansion of \(-X^k\) is just \(-1X^k\).

We are going to deal with such generalisations of canonical number systems in Subsection 5.1. There we also will see whether \((\mathcal{R}, X, \mathcal{N})\) and \((\mathcal{R}, X, \mathcal{M})\), respectively, have the PEP or FEP.

**Lemma 2.6.** The finite expansion property implies the periodic expansion property.

**Proof.** Suppose that there exists an \( A \) having finite \( X \)-ary expansion and no periodic digit sequence. Then there exists an \( n \in \mathbb{N} \) with \( T^n(A) = 0 \). Since \( A \) is not periodic we have \( T^k(A) \neq 0 \) for all \( k > n \). Thus the element \( T^{n+1}(A) = T(0) \neq 0 \) cannot have a finite \( X \)-ary expansion.

\[\square\]

**Definition 2.7.** A zero cycle of a digit system \((\mathcal{R}, X, \mathcal{N})\) is a finite sequence \((d_0, \ldots, d_\ell)\), with \( d_i \in \mathcal{N} \) and \( \ell \geq 0 \), such that

\[
0 = \sum_{i=0}^{\ell} d_i X^i.
\]

The zero period of \((\mathcal{R}, X, \mathcal{N})\) is the length of a shortest zero cycle, if such a one exists, and undefined otherwise.

Note that the finite expansion \((22)\) of 0 given by a zero cycle is different from the trivial expansion of 0 by an empty sum.
Lemma 2.8. The finite expansion property implies the existence of a zero cycle.

Proof. If \(0 \in \mathcal{N}\), the assertion is trivial, because \((0)\) is a zero cycle. Otherwise, let \(a = T(0)\), so that \(aX + D_\mathcal{N}(0) = 0\). We have \(a \neq 0\); by assumption, there is a finite expansion \(a = \sum_{i=0}^{\ell} d_iX^i\ (d_i \in \mathcal{N})\). But then
\[
0 = \sum_{i=0}^{\ell} d_iX^{i+1} + D_\mathcal{N}(0).
\]
Thus \((D_\mathcal{N}(0), d_0, \ldots, d_\ell)\) is a zero cycle. \(\square\)

Note that zero cycles are uniquely determined by their length; under the periodic expansion property, if a zero cycle exists, it is a concatenation of copies of the shortest zero cycle.

Every finite expansion of an element \(A \in \mathcal{R}\) can be prolonged indefinitely by appending the sum (2.2) corresponding to the zero cycle; this generalises padding with zeros in case \(0 \notin \mathcal{N}\). This gives another (constructive) proof of Lemma 2.6.

Definition 2.9. The periodic set \(\mathcal{P}\) of a digit system \((\mathcal{R}, X, \mathcal{N})\) is the set of all elements \(A\) of \(\mathcal{R}\) with \(T^n(A) = A\) for some \(n \geq 1\).

\(\mathcal{P}\) is thus the set of all elements of \((\mathcal{R}, X, \mathcal{N})\) that are purely periodic under the action of \(T\). The map \(T\) permutes \(\mathcal{P}\), and we can consider the quotient \(\mathcal{P}/T\), which is the set of orbits in \(\mathcal{P}\) under the action of \(T\). Note that the orbits are finite by definition. Clearly, \(\mathcal{R}\) has the PEP if and only if for all \(A \in \mathcal{R}\) there exists some \(n \in \mathbb{N}\) with \(T^n(A) \in \mathcal{P}\).

In general, it is not clear if \(\mathcal{P}\) is, for example, nonempty, a singleton, or finite. However, in the special case where \(\mathcal{R}\) can be embedded in a finite-dimensional complex vector space, we can consider the expanding property on the defining polynomial \(P\), which requires that all zeros have modulus strictly greater than 1. This property at once implies the nonemptiness and finiteness of \(\mathcal{P}\), as well as the periodic expansion property (cf. Section 5.1). In this case, \(\mathcal{P}\) is also called the attractor of \((\mathcal{R}, X, \mathcal{N})\).

The next lemma gives another criterion for the finiteness of \(\mathcal{P}\). The essence of this result is well-known, cf. for example [16, Lemma 2.1]. An implication is that the FEP cannot hold whenever there exists some \(A \neq 0 \in \mathcal{R}\) with \(T(A) = A\).

Lemma 2.10. Assume that \((\mathcal{R}, X, \mathcal{N})\) has the PEP. Then \((\mathcal{R}, X, \mathcal{N})\) has the FEP if and only if \(0 \in \mathcal{P}\) and \(|\mathcal{P}/T| = 1\).

Proof. Since \((\mathcal{R}, X, \mathcal{N})\) has the PEP, it follows that the digit sequence of each element of \(\mathcal{R}\) ends up periodically, i.e. for every \(A \in \mathcal{R}\) there exists \(n \in \mathbb{N}\) with \(T^n(A) \in \mathcal{P}\). Now if \(|\mathcal{P}/T| = 1\) and \(0 \in \mathcal{P}\), it follows that for each \(A \in \mathcal{R}\) there exists an \(n \in \mathbb{N}\) with \(T^n(A) = 0\).

To show the converse, note that the orbits in \(\mathcal{P}/T\) are pairwise disjoint. The requirements \(0 \in \mathcal{P}\) and \(|\mathcal{P}/T| = 1\) are therefore equivalent to \(0 \in \mathcal{O}\) for all orbits \(\mathcal{O} \in \mathcal{P}/T\). Suppose that there exists an orbit \(\mathcal{O} \in \mathcal{P}/T\) with \(0 \notin \mathcal{O}\). Then for each element \(A \in \mathcal{O}\) we have \(T^n(A) \neq 0\) for all \(n \in \mathbb{N}\) and therefore \((\mathcal{R}, X, \mathcal{N})\) cannot have the FEP. \(\square\)
Example 2.11. Let $E = F_2[y]$, $P(x) = (y + 1)x^2 + yx + (y^2 + 1) \in E[x]$ and $R = E[x]/(P)$. Digit systems of this kind have been investigated in [13]. A system of representatives of $R$ is, for example, $N := \{1, y, y + 1, y^3 + y\}$. Note that this set does not include 0. It can be easily verified that $(y^3 + y, 1, 1, 1, y + 1)$ is the zero cycle of $(R, X, N)$; thus we have

$$0 = (y^3 + y) + X + X^2 + X^3 + (y + 1)X^4.$$ 

We immediately see that $P$ includes the orbit of 0, which consists of the elements

$$0 \to (y^2 + y)X + y^2 \to (y + 1)X + y^2 \to (y + 1)X + 1 \to y + 1 \to 0.$$ 

When $(R, X, N)$ has the PEP and $P$ consists only of these elements, then $(R, X, N)$ has also the FEP. We come back to this example in Section 5.2.

Several important properties of the digit system $(R, X, N)$ can be derived from the constant and leading coefficients of the defining polynomial $P$. First, we examine the pathological case when the constant coefficient of $P$ is a unit.

Lemma 2.12. Let the polynomial $P$ be such that the constant coefficient $p_0$ is a unit of $E$, and suppose that $(R, X, N)$ has the FEP. Then $R$ is finite.

Proof. For $p_0$ a unit, the digit set $N$ contains exactly one element $d$, and all elements of $(R, X, N)$ have the same digit sequence $(d, d, d, \ldots)$. Thus $P = R$. By the Lemma 2.10, $P$ is finite. \qed

The last lemma may seem curious. In order to illustrate it, we will consider the example of base 1 in $\mathbb{Z}$, that is, we take $E = \mathbb{Z}$ and $P = x - 1$, getting $R \cong E$. If the unique digit is chosen to be 0, then clearly only 0 has a finite expansion. If the digit is, say, 1, then it is true that all positive integers can be finitely expanded, in the way of (primitive) Roman numerals or tally marks, but negative integers have no finite expansion. If we change the base ring to $E = \mathbb{F}_p$ for some prime $p$, and take $P$ irreducible, the tally marks do cover all elements of $R$. In general, if $R$ is a finite ring and $X$ some non-zero element of $R$, it is not trivial to decide if every element of a finite ring $R$ can be written in the form $\sum_{i=0}^{t} dX^i$, for some fixed "digit" $d$. The problem is related to the theory of linear congruential sequences; see also [19, Lemma 4.13] and [11, Section 3.2.1].

Because of the strange properties of digit systems with $|N| = 1$, we will assume that $p_0$ is not a unit for the rest of the paper.

The other important coefficient of the defining polynomial $P$ is the leading coefficient $p_d$, which is usually taken to be 1, because of the better algebraic properties of the quotient ring $R$ in that case. Several monic cases, i.e., when $p_d$ is a unit, have already been investigated: $E = \mathbb{Z}$ and $N = \{0, \ldots, |p_0| - 1\}$ give the well analysed canonical number systems (see for instance [13]). If $E$ is the ring of polynomials over a finite field $\mathbb{F}$ we obtain the digit systems presented in [15]. Both concepts will be generalised in the next section.

3. Digit systems in the non-monic case

Most of the investigations on digit systems in the literature have been limited to the case where $P$ is monic. One of the reasons for this constraint is that the structure of
\( R = \mathcal{E}[x]/(P) \) regarded as an \( \mathcal{E} \)-module is much more complicated if \( P \) is not monic. In fact, if the leading coefficient \( p_d \) of \( P \) is a unit in \( \mathcal{E} \), then \( R \) is a free \( \mathcal{E} \)-module of finite rank \( d \); for example, the powers \( 1, X, \ldots, X^{d-1} \) of \( X \) form a basis. If \( p_d \) is not a unit, then \( R \) is no longer free, and it is not even finitely generated over \( \mathcal{E} \). We do have the following standard representation of its elements, relative to the choice of representatives of \( \mathcal{E} \) modulo \( p_d \).

**Lemma 3.1.** Let \( M \subset \mathcal{E} \) be a set of representatives of \( \mathcal{E}/(p_d) \). For each \( A \in R \) there exists a unique \( A' \in \mathcal{E}[x] \) with \( \deg A' < d \) and unique \( r_d, \ldots, r_k \in M \), with \( k \in \mathbb{N} \) minimal, such that

\[
A \equiv A' + \sum_{i=d}^{k} r_i x^i \mod P.
\]

**Proof.** Let \( A \in R \) be represented by

\[
f = \sum_{i=0}^{k} b_i x^i \in \mathcal{E}[x]
\]

with \( k \) minimal. If \( k \geq d \), there is a unique \( r_k \in M \) such that \( b_k = r_k + q_k p_d \), for some \( q_k \in \mathcal{E} \), and it follows that

\[
f = q_k P x^{k-d} + r_k x^k + f' \equiv r_k x^k + f' \pmod{P}
\]

where \( f' \in \mathcal{E}[x] \) has lower degree than \( f \). Continuing by induction, we find

\[
f = QP + \sum_{i=d}^{k} r_i x^i + \tilde{f},
\]

with \( Q \in \mathcal{E}[x], r_i \in M, \) and \( \tilde{f} \in \mathcal{E}[x] \) a polynomial of degree less than \( d \). In order to prove unicity, suppose that \( A \) is also represented by

\[
g = \sum_{i=d}^{k} r'_i x^i + \tilde{g},
\]

where \( \tilde{g} \in \mathcal{E}[x] \) has degree less than \( d \). Thus, the difference \( (f - QP) - g \) is a multiple of \( P \). We have \( r'_k = r_k \), because both are congruent modulo \( p_d \), and both are in \( M \). Continuing by induction, we find that \( \tilde{f} = \tilde{g} \), as desired. \( \square \)

The question whether a given digit system \((R, X, \mathcal{N})\) has periodic representations or even finite expansions can already be decided by looking at a properly chosen finitely generated \( \mathcal{E} \)-submodule of \( R \).

**Theorem 3.2.** Let \((R, X, \mathcal{N})\) be a digit system with \( R = \mathcal{E}[x]/(P) \) and let \( k \geq \deg P \) be minimal such that all digits in \( \mathcal{N} \) can be represented by polynomials in \( \mathcal{E}[x] \) of degree at most \( k \). Let \( R_k \) be the submodule of \( R \) generated by \( X^i \) for \( i = 0, \ldots, k - 1 \). Then \((R, X, \mathcal{N})\) has the FEP (PEP, resp.) if and only if every element of \( R_k \) has a finite \( X \)-ary expansion (periodic digit sequence, resp.).
Proof. Let \( A \in \mathcal{R} \) be represented by \( f \in \mathcal{E}[x] \). We will investigate the action of \( T \) on \( A \). Let \( c \in \mathcal{E}[x] \) be a representative of minimal degree of \( D_N(A) \). Thus, \( f - c \) is divisible by \( X \) modulo \( P \). Hence, for some \( Q \in \mathcal{E}[x] \), the polynomial \( f - c - QP \) has zero constant coefficient, and in fact we may take \( Q \) to be constant, so that the degree of the disturbing term \( c + QP \) is bounded by \( k \). Thus, if \( f \) has degree at most \( k - 1 \), then the same holds for the representative \( \frac{f - c - QP}{X} \) of \( T(A) \), so \( T(A) \in \mathcal{R}_k \); whereas if \( \deg f \geq k \), then \( \frac{f - c - QP}{X} \) has smaller degree than \( f \), so that \( T^i(A) \) will be in \( \mathcal{R}_k \) if \( i \) is large enough. Thus, it is enough to consider the finite expansion (the periodic expansion, resp.) of the elements of \( \mathcal{R}_k \). \( \square \)

Example 3.3. Consider \( P(x) = 2x + 3 \in \mathbb{Z}[x] \), which corresponds to numeration with basis \(-3/2\). If all digits have degree 0 or 1, then we can take \( k = 1 \) in the theorem, and it suffices to consider \( R_1 \), the set of polynomials having degree 0 representatives. Of course, \( R_1 \) is isomorphic to \( \mathbb{Z} \). If some digit can only be represented by polynomials of degree at least \( m > 1 \), then we have \( k \geq m \). Such classes exist: as soon as \( P \) is not monic, the monomials \( x^m \), for \( m \geq 0 \), cannot be further reduced modulo \( P \).

Remark 3.4. Note that, in general, when \( k \leq \deg P \), the module \( R_k \) is free, because we assume that the leading coefficient \( p_d \) of \( P \) is not a zero divisor. This case occurs for \( k = \deg P \) when all digits have low-degree representatives. If \( p_d \) is a unit in \( \mathcal{R} \), then for any \( k \) the module \( R_k \) is contained in the free \( \mathcal{E} \)-module generated by \( \frac{X^i}{p_d} \) for \( i = 0, \ldots, k \); this follows from the normal form given in Lemma 3.1.

Establishing the FEP for a given digit system is in general a difficult problem. Therefore, if we have a critical subset as in the theorem, we will be interested in making it as small as possible, so that we need to check as few elements as possible for representability. Below, we will show that a special independent set in \( \mathcal{R} \) called the Brunotte basis (cf. [9]) generates a rather small critical submodule of \( \mathcal{R} \), and furthermore brings the dynamic mapping \( T \) into an especially simple form. The Brunotte basis has already been used in the case where \( P \) is monic, or, in other words, in the CNS case (cf. [9 16]), where it is a basis of \( \mathcal{R} \) over \( \mathcal{E} \). We will now generalise this concept to the non-monic case.

Because the effect of choosing this basis on the backward division algorithm is only visible when the digit set \( N \) is chosen to be a subset of \( \mathcal{E} \), we will assume for the remainder of this section that each digit has a representative contained in \( \mathcal{E} \) and we will identify the digits with these representatives.

Definition 3.5. Let \( w_0 = p_d \) and \( w_k = Xw_{k-1} + p_{d-k} \) for \( k = 1, \ldots, d - 1 \); then \((w_0, \ldots, w_{d-1})\) is called the Brunotte basis of \( \mathcal{E}[x] \) modulo \( P \). The \( \mathcal{E} \)-submodule \( \Lambda_P \) of \( \mathcal{R} \) generated by the \( w_i \) will be called the Brunotte module of \( P \).

We now state some easy properties of the Brunotte basis. The proofs are straightforward and will be omitted.

Lemma 3.6. Let \((w_0, \ldots, w_{d-1})\) be the Brunotte basis of \( \mathcal{E}[x] \) modulo \( P \).

(i) For \( i = 0, \ldots, d - 1 \), the basis element \( w_i \) is exactly the integral (polynomial) part of \( P/X^{d-i} \). In particular, we have \( Xw_{d-1} + p_0 = P \).
(ii) The coordinate matrix of the $w_i$, with respect to the basis $1, X, \ldots, X^{d-1}$, is upper triangular, and all diagonal elements are equal to $p_d$.

We have the inclusions

$$\Lambda_P \subseteq \bigoplus_{i=0}^{d-1} \mathcal{E}X^i = \mathcal{R}_d \subseteq \mathcal{R}.$$  

The quotient $\mathcal{R}_d/\Lambda_P$ is isomorphic to $(\mathcal{E}/(p_d))^d$ and thus, $\mathcal{R}_d = \Lambda_P$ if and only if $p_d$ is a unit. Under the same condition, $\mathcal{R}_d = \mathcal{R}$.

Using the Brunotte basis, we modify the representation of elements in $\mathcal{R}$ given in Lemma 3.1.

**Lemma 3.7.** Let $M \subset \mathcal{E}$ be a set of representatives of $\mathcal{E}/(p_d)$. For each $A \in \mathcal{R}$ there exist unique $q_0, \ldots, q_{d-1} \in \mathcal{E}$ and unique $r_0, \ldots, r_k \in M$, with $k \in \mathbb{N}$ minimal, such that

$$A = \sum_{i=0}^{d-1} q_i w_i + \sum_{i=0}^{k} r_i X^i.$$  

**Proof.** Let $A \in \mathcal{R}$, and reduce it to the form $A' + \sum_{i=d}^{k} r_i X^i$, as in Lemma 3.1, with $\deg A' < d$. Now observe that with respect to the usual power basis $1, X, \ldots, X^{d-1}$, the elements $w_i$ of the Brunotte basis have leading coefficient $p_d$. We use the $w_i$ instead of $P$ to reduce the coefficients of $f$ further, from $w_{d-1}$ down to $w_0$, and we find

$$A' = \sum_{i=0}^{d-1} q_i w_i + r_{d-1} X^{d-1} + \cdots + r_1 X + r_0,$$

as desired. \( \square \)

**Definition 3.8.** Let $M \subset \mathcal{E}$ a set of representatives of $\mathcal{E}/(p_d)$. For an element $A \in \mathcal{R}$ the representation

$$A = \sum_{i=0}^{d-1} q_i w_i + r$$

from the above lemma is called the *standard representation* of $A$ with respect to $M$. We say that $r = \sum_{i=0}^{k} r_i X^i$ is the *residue polynomial* of $A$.

The main result of this section is the following theorem, which shows that we may take $\Lambda_P$ instead of $\mathcal{R}_d$ in Theorem 3.2.

**Theorem 3.9.** Let $(\mathcal{R}, X, \mathcal{N})$ be a digit system with $\mathcal{N} \subseteq \mathcal{E}$. Then $(\mathcal{R}, X, \mathcal{N})$ has the FEP (PEP, resp.) if and only if each $A \in \Lambda_P$ has a finite $X$-ary expansion (periodic digit sequence, resp.). On $\Lambda_P$, the dynamic mapping $T$ takes the form

$$T : \mathcal{E}^d \to \mathcal{E}^d, \quad (a_0, \ldots, a_{d-1}) \mapsto \left( a_1, \ldots, a_{d-1}, -\sum_{i=0}^{d-1} a_i p_{d-i} - e_0 \right),$$

where a general element $\sum_{i=0}^{d-1} a_i w_i \in \Lambda_P$ is represented by its coordinate sequence with respect to the basis $(w_i)$, and $e_0 \in \mathcal{N}$ is the unique digit that ensures divisibility by $p_0$. 
Proof. Let $A \in \mathcal{R}$, let $M \subset \mathcal{E}$ be a set of representatives of $\mathcal{E}/(p_d)$, and let

$$A = \sum_{i=0}^{d-1} a_i w_i + r$$

with $r = r_k X^k + \cdots + r_1 X + r_0$, $r_i \in M$, be the standard representation of $A$. We will investigate the action of $T$ on $A$. Let $e_0 \in \mathcal{E}$ represent the digit $D_{\mathcal{N}}(A)$. Taking

$$q = \frac{\sum_{i=0}^{d-1} a_i p_d - i + r_0 - e_0}{p_0},$$

we can write

$$A = e_0 + q p_0 + \sum_{i=0}^{d-1} a_i (w_i - p_{d-i}) + X \frac{r - r_0}{X} \in \mathcal{E}[X].$$

Observe that $w_0 - p_d = 0$, and $w_i - p_{d-i} = X w_{i-1}$ for $i = 1, \ldots, d - 1$; furthermore, we have $q p_0 \equiv -q X w_{d-1} \pmod{P}$. Set $a_d = -q$. Then the standard representation of $T(A)$ is

$$T(A) = \frac{A - e_0}{X} = \sum_{i=0}^{d-1} a_{i+1} w_i + \frac{r - r_0}{X}.$$

In other words: after one application of $T$ the degree of the residue polynomial $r$ decreases by one and the first $k$ coefficients do not change. Hence, after $k+1$ applications, we have $T^{k+1}(A) \in \Lambda_P$.

The first assertion of the theorem follows immediately. The second assertion follows from (3.2) by taking $r = r_0 = 0$ and taking $a_d = -q$ from (3.2). \hfill $\square$

We end the section with the special case where the base ring $\mathcal{E}$ is Euclidean. Here we obtain a necessary condition for the finite expansion property that is analogous to the usual expanding property of $P$.

**Theorem 3.10.** Let $\mathcal{E}$ be Euclidean with value function $g : \mathcal{E} \mapsto \mathbb{R}^+ \cup \{0, -\infty\}$ where $g(0) = -\infty$, and let $(\mathcal{R}, X, \mathcal{N})$ be a digit system satisfying $\mathcal{N} \subset \mathcal{E}$ and $g(e) < g(p_0)$ for all $e \in \mathcal{N}$. If $g(p_d) \geq g(p_0)$, then no element of $\Lambda_P$ but 0 has a finite $X$-ary expansion.

**Proof.** First note that the assumption on $\mathcal{N}$ implies $0 \in \mathcal{N}$. Let $\pi : \mathcal{E}[x] \to \mathcal{R}$ be the canonical epimorphism and $A \in \mathcal{E}[x]$. Because the leading coefficient of $w_k$ is $p_d$ for each $k \in \{0, \ldots, d - 1\}$, it is easy to see that $\pi(A) \in \Lambda_P$ implies that the leading coefficient of $A$ is a multiple of $p_d$. Now suppose that there is a $B \in \Lambda_P$, $B \neq 0$ with finite $X$-ary expansion

$$B = \sum_{i=0}^{h} e_i X^i, \quad e_i \in \mathcal{N}, e_h \neq 0.$$

By assumption we have that $g(e) < g(p_0)$ for $e \in \mathcal{N}$ and therefore $g(e_h) < g(p_0) \leq g(p_d)$. As observed above we also must have that $e_h = q p_d$ for some nonzero $q \in \mathcal{E}$. But $\mathcal{E}$ is Euclidean, so $q, p_d \neq 0$ implies $g(e_h) = g(q p_d) \geq g(p_d)$, which is a contradiction. \hfill $\square$
**Corollary 3.11.** With the above assumptions on $E$ and $N$, $g(p_0) < g(p_0)$ is necessary for $(R, X, N)$ to have the finite expansion property.

**Example 3.12.** We retrieve the following known result for a linear polynomial $P(x) = p_1 x + p_0$ with $p_1 \neq 0$: if $E = \mathbb{Z}$ and $|e| < |p_0|$ holds for all $e \in N$, then Corollary 3.11 tells us that $|p_0| > |p_1|$ is necessary for $(\mathbb{Z}[x]/(P), X, N)$ to have finite expansions. Of course, $|p_0| > |p_1|$ is equivalent to the polynomial $P$ being expanding.

4. Products of digit systems

Consider the exact sequence

$$0 \to E[x]/(P_2) \xrightarrow{P_1} E[x]/(P_1 P_2) \to E[x]/(P_1) \to 0,$$

of $E$-modules, and suppose we have defined number systems on the outer components. Below, we construct a naturally defined number system on the middle module and derive conditions when all elements in this number system have a finite expansion, or at least a periodic digit sequence.

**Theorem 4.1.** Let

$$P_1(x) = p_0 + p_1 x + \cdots + p_m x^m, \quad P_2(x) = p'_0 + p'_1 x + \cdots + p'_n x^n,$$

and for $i \in \{1, 2\}$, let $R_i = E[x]/(P_i)$ and $N_i$ be systems of representatives of $R_i/(X)$ with $N_i \subset E$. Let $M = \{d + eP_1 : d \in N_1, e \in N_2\}$. Then the following assertions hold:

- $(E[x]/(P_1 P_2), X, M)$ is a digit system.
- if $0 \in N_1$ and $(R_1, X, N_1)$ and $(R_2, X, N_2)$ have the FEP, then $(E[x]/(P_1 P_2), X, M)$ has the FEP.
- if $0 \in N_1$, $(R_1, X, N_1)$ has the FEP, and $(R_2, X, N_2)$ has the PEP, then $(E[x]/(P_1 P_2), X, M)$ has the PEP.

**Proof.** We only prove the second assertion, as the first is trivial, and the third follows analogously to the second.

Consider an element

$$A = a_0 + a_1 x + \cdots + a_N x^N \in E[x].$$

Let $d_0 \in N_1$, $e_0 \in N_2$, $k, \ell \in E$ such that

$$a_0 = d_0 + k p_0 \quad \text{and} \quad k = e_0 + \ell p'_0.$$

Let $A^{(0)} = A$ and $B^{(0)} = 0$. Since

$$p_0 \equiv -p_1 X - \cdots - p_m x^m + P_1 \pmod{P_1 P_2} \quad \text{and} \quad p'_0 \equiv (-p'_1 X - \cdots - p'_n x^n) P_1 \pmod{P_1 P_2},$$
we obtain that
\[ A = A^{(0)} + P_1 B^{(0)} \]
\[ \equiv d_0 + (a_1 - kp_1)X + \cdots + (a_m - kp_m)X^m + a_{m+1}X^{m+1} + \cdots + a_N X^N + kP_1 \]
\[ \equiv d_0 + (a_1 - kp_1)X + \cdots + (a_m - kp_m)X^m + a_{m+1}X^{m+1} + \cdots + a_N X^N + e_0P_1 + (-\ell p_1 X - \cdots - \ell p'_n X^n) P_1 \]
\[ \equiv d_0 + e_0 P_1 + X(A^{(1)} + P_1 B^{(1)}) \pmod{P_1 P_2} \]
with \( d_0 + e_0 P_1 \in \mathcal{M} \). Iterating this process, we get a recurrence for the coefficients of
\[ A^{(j)} = \sum_{i \geq 0} a_i^{(j)} X^i \quad \text{and} \quad B^{(j)} = \sum_{i \geq 0} b_i^{(j)} X^i. \]
Starting with
\[ a_i^{(0)} = \begin{cases} a_i & \text{for } i \leq N, \\ 0 & \text{for } i > N, \end{cases} \quad b_i^{(0)} = 0 \text{ for } i \geq 0 \]
we obtain for \( j \geq 0, \)
\[ d_j \equiv a_0^{(j)}, \quad k^{(j)} = (a_0^{(j)} - d_j)/p_0, \quad a_i^{(j+1)} = a_{i+1}^{(j)} - k^{(j)} p_{i+1}, \]
\[ e_j \equiv b_0^{(j)} + k^{(j)}, \quad \ell^{(j)} = (b_0^{(j)} + k^{(j)} - e_j)/p'_0, \quad b_i^{(j+1)} = b_{i+1}^{(j)} - \ell^{(j)} p'_{i+1}, \]
with \( d_j \in \mathcal{N}_1, e_j \in \mathcal{N}_2 \). Then
\[ A^{(j)} + P_1 B^{(j)} = d_j + e_j P_1 + X(A^{(j+1)} + P_1 B^{(j+1)}). \]
Note that the recurrence for \( k^{(j)}, a_i^{(j)} \) and \( d_j \) is just backward division algorithm for \( A \), considered as an element of \((\mathcal{R}_1, X, \mathcal{N}_1)\). Since \((\mathcal{R}_1, X, \mathcal{N}_1)\) has the FEP, there is an index \( j_0 \) such that \( k^{(j)} = 0 \) for all \( j \geq j_0 \). Then, for \( j \geq j_0 \), the recurrence for the \( b_i^{(j)} \) is no longer disturbed by the \( k^{(j)} \). Since \((\mathcal{R}_2, X, \mathcal{N}_2)\) has the FEP, there is an \( j_1 \geq j_0 \) such that \( \ell^{(j)} = 0 \) for all \( j \geq j_1 \). Thus, we obtain a finite expansion
\[ A \equiv \sum_{j \geq 0} (d_j + e_j P_1) X^j. \]
\[ \square \]

It is possible (but tedious) to extend this theorem to the case where the \( \mathcal{N}_i \) are no longer assumed to be subsets of \( \mathcal{E} \). Assuming this has been done, the following generalisation of Theorem 4.1 follows by induction on the number of factors.

**Corollary 4.2.** For \( i = 1, \ldots, k \), let \((\mathcal{E}[x]/(P_i), X, \mathcal{N}_i)\) be digit systems. Let
\[ \mathcal{M} = \{ d_1 + d_2 P_1 + d_3 P_1 P_2 + \cdots + d_k P_1 \cdots P_{k-1} \} \quad \text{with} \quad d_i \in \mathcal{N}_i. \]
Then \((\mathcal{E}[x]/(P_1 \cdots P_k), X, \mathcal{M})\) is a digit system. If \((\mathcal{E}[x]/(P_i), X, \mathcal{N}_i)\) has the FEP for all \( i \) and \( 0 \in \mathcal{N}_i \) for \( 1 \leq i \leq k - 1 \), then \((\mathcal{E}[x]/(P_1 \cdots P_k), X, \mathcal{M})\) also has the FEP.
Note that Theorem 4.1 and its corollary are asymmetric. Thus, by permutation of the $P$, one can obtain other digit systems.

5. Generalisation of known digit systems

Several known families of digit systems have the form $(E[x])/(P), X, N$ for a monic polynomial $P$, e.g. taking $E = \mathbb{Z}$ or $E = \mathbb{F}_q[y]$. These families will now be generalised to the case where $P$ is not necessarily monic.

5.1. Generalisation of canonical number systems. Assume that $E = \mathbb{Z}$. If $P \in \mathbb{Z}[x]$ is monic and we choose $N = \{0, \ldots, |p_0| - 1\}$, then $(R, X, N)$ has the finite expansion property if and only if the pair $(P, N)$ is a canonical number system (CNS) in the sense of [13]. What we have done so far, allows us to extend this definition as follows.

Definition 5.1. Let $P = p_dx^d + \cdots + p_0 \in \mathbb{Z}[x], R = \mathbb{Z}[x]/(P)$ and $N = \{0, \ldots, |p_0| - 1\}$. If $(R, X, N)$ has the finite expansion property, then we call $(P, N)$ a canonical number system (CNS).

It is an open problem to characterise all CNS, even in the monic case. Many partial results have been obtained (see [7, Section 3.1] and the literature given there), several of which immediately generalise to the non-monic case. One of the most promising directions here is the characterisation of CNS in terms of the discrete dynamical systems called shift radix systems (SRS).

Definition 5.2 (first given in [1]). Let $d$ be a positive integer. For a $r \in \mathbb{R}^d$ define the function

$$\tau_r : \mathbb{Z}^d \rightarrow \mathbb{Z}^d,$$

$$z = (z_0, \ldots, z_{d-1}) \mapsto (z_1, \ldots, z_{d-1}, -\lfloor rz \rfloor),$$

where $rz$ is the scalar product of $r$ and $z$. The mapping $\tau_r$ is called a shift radix system (SRS) if for any $z \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that the $k$-th iterate of $\tau_r$ satisfies $\tau_r^k(z) = 0$. Define the sets

$$D_d := \{ r \in \mathbb{R}^d \mid \text{each orbit of } \tau_r \text{ is ultimately periodic} \}$$

and

$$D_d^{(0)} := \{ r \in \mathbb{R}^d \mid \tau_r \text{ is an SRS} \}.$$

The main result relating CNS with SRS is proved for monic polynomials in [1] Theorem 3.1. It can be generalised also to non-monic polynomials as follows.

Theorem 5.3. Let $P(x) = p_dx^d + \cdots + p_0 \in \mathbb{Z}[x], R = \mathbb{Z}[x]/(P), N = \{0, \ldots, |p_0| - 1\}$ and

$$r = \left(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right).$$

Then the following assertions hold:

- $(R, X, N)$ has the FEP (i.e., $(P, N)$ is a CNS) if and only if $r \in D_d^{(0)}$.
- $(R, X, N)$ has the PEP if and only if $r \in D_d$. 
Proof. As in Definition 3.5 let \( \{w_0, \ldots, w_{d-1}\} \) be the Brunotte basis of \( \mathcal{E}[x] \) modulo \( P \). One easily computes that \( w_k = \sum_{i=0}^{d-1} p_i X^{d-i} \). By Theorem 3.9, \( (\mathcal{R}, X, \mathcal{N}) \) has the FEP if and only if each element \( A \) of the shape

\[
A = A^{(0)} = \sum_{i=0}^{d-1} a_i w_i, \quad a_i \in \mathbb{Z},
\]

has a finite \( X \)-ary expansion. In the proof of Theorem 3.9 we showed that an application of \( T \) yields \( A^{(0)} = e_0 + X A^{(1)} \) with

\[
A^{(1)} = T(A^{(0)}) = \sum_{i=0}^{d-1} a_{i+1} w_i
\]

and \( e_0 = D_{\mathcal{N}}(A^{(0)}) \in \mathcal{N} \). In order to find a value of \( a_d \), we apply (3.2); by the form of the digit set \( \mathcal{N} \) we find

\[
a_d = -q = - \lfloor r \cdot (a_0, \ldots, a_{d-1}) \rfloor.
\]

Thus \( A^{(1)} = (w_0, \ldots, w_{d-1}) \cdot \tau_r((a_0, \ldots, a_{d-1})) \). Recall that \( (\mathcal{R}, X, \mathcal{N}) \) has the FEP if and only if successive application of \( T \) to each \( A \in \Lambda_P \) ends up in 0, and we see that this is equivalent to \( \tau_r \) being an SRS. The second assertion can be shown analogously. \( \square \)

The connection just given allows for an easy proof of the following criterion, which was given for the monic case in [10, Proposition 7] and [12].

**Theorem 5.4.** Let \( P = p_0 x^d + \cdots + p_1 x + p_0 \in \mathbb{Z}[x] \) and \( \mathcal{N} = \{0, \ldots, |p_0| - 1\} \). Suppose \( p_0 \geq 2 \) and \( p_0 \geq p_1 \geq \ldots \geq p_d > 0 \).

Then \( (P, \mathcal{N}) \) is a CNS.

**Proof.** Theorem 3.5 of [2] tells us that if \( 0 \leq r_1 \leq \ldots \leq r_d < 1 \) then \( r \in D_d^{(0)} \). Therefore, the result follows directly from Theorem 5.3. \( \square \)

In [5] so-called symmetric canonical number systems (SCNS) have been introduced. The difference to usual CNS is the digit set: it is almost symmetric around 0, viz.

\[
\mathcal{N} = \left[ -\frac{|p_0|}{2}, \frac{|p_0|}{2} \right] \cap \mathbb{Z}.
\]

In [17] the definition has been generalised to any digit set consisting of \( |p_0| \) consecutive integers including 0. We will give these definitions in our formalism, generalised to not necessarily monic polynomials.

**Definition 5.5 (cf. [5, 17]).** Let \( \varepsilon \in [0, 1) \), \( P = p_d x^d + \cdots + p_0 \in \mathbb{Z}[x] \), \( \mathcal{R} = \mathbb{Z}[x]/(P) \) and \( \mathcal{N} := [-|p_0|, (1 - \varepsilon)|p_0|] \cap \mathbb{Z} \). If \( (\mathcal{R}, X, \mathcal{N}) \) has the finite expansion property, then we call \( (P, \mathcal{N}) \) an \( \varepsilon \)-canonical number system (\( \varepsilon \)-CNS). A \( \frac{1}{2} \)-CNS is also called symmetric canonical number system (SCNS).
Analogously to Theorem 5.3, $\varepsilon$-CNS are closely related to a slight modification of SRS. For an $\varepsilon \in [0, 1)$ and an $r \in \mathbb{R}^d$ let

$$
\tau_{r, \varepsilon} : \mathbb{Z}^d \to \mathbb{Z}^d,
$$

$$
z = (z_0, \ldots, z_{d-1}) \mapsto (z_1, \ldots, z_{d-1}, -[rz + \varepsilon]).
$$

Define

$$
\mathcal{D}_{d, \varepsilon} := \{ r \in \mathbb{R}^d \mid \text{each orbit of } \tau_{r, \varepsilon} \text{ is ultimately periodic} \}
$$

and

$$
\mathcal{D}_{d, \varepsilon}(0) := \{ r \in \mathbb{R}^d \mid \text{each orbit of } \tau_{r, \varepsilon} \text{ is ultimately zero} \}.
$$

Note that $\mathcal{D}_d = \mathcal{D}_{d, 0}$ and $\mathcal{D}_{d, \varepsilon}(0) = \mathcal{D}_{d, 0}(0)$. The following theorem can be proved in a similar way as Theorem 5.3.

**Theorem 5.6 (cf. [5, 17]).** Let $P(x) = p_dx^d + \cdots + p_0 \in \mathbb{Z}[x]$, $\mathcal{R} = \mathbb{Z}[x]/(P)$, $\varepsilon \in [0, 1)$, $\mathcal{N} = [-\varepsilon|p_0|, (1 - \varepsilon)|p_0|) \cap \mathbb{Z}$ and

$$
r = \left( \frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0} \right).
$$

Then the following assertions hold:

- $(\mathcal{R}, X, \mathcal{N})$ has the FEP (i.e., $(\mathcal{P}, \mathcal{N})$ is an $\varepsilon$-CNS) if and only if $r \in \mathcal{D}_{d, \varepsilon}(0)$.

- $(\mathcal{R}, X, \mathcal{N})$ has the PEP if and only if $r \in \mathcal{D}_{d, \varepsilon}$.

**Example 5.7.** We are now able to decide whether the digit systems in Example 2.5 have the PEP or the FEP by using Theorem 5.3 and Theorem 5.6. Set $P(x) = 3x^2 - 2x + 5$, $\mathcal{N} = \{0, 1, 2, 3, 4\}$ and $\mathcal{M} = \{-2, -1, 0, 1, 2\}$. Thus we are interested in the mappings $\tau_r$ and $\tau_{r, \frac{3}{5}}$ corresponding to the vector $r := \{\frac{3}{5}, -\frac{2}{5}\}$. By [2, Lemma 5.2] we have $r \in \mathcal{D}_{d, \frac{3}{5}}(0)$ and therefore $(\mathcal{R}, X, \mathcal{N})$ has the FEP. On the other hand, according to [5, Theorem 5.2], $r \not\in \mathcal{D}_{d, \frac{3}{5}}(0)$ but $r \in \mathcal{D}_{d, \frac{3}{5}}$ however. This shows that $(\mathcal{R}, X, \mathcal{M})$ has the PEP but not the FEP.

5.2. **Generalisation of digit systems over a finite field.** Let $\mathcal{E} = \mathbb{F}[y]$ be the ring of polynomials in $y$ over the finite field $\mathbb{F}$. It is well-known that $\mathbb{F}[y]$ is a Euclidean domain. The corresponding value function $g$ assigns to an element $q$ of $\mathbb{F}[y]$ its degree, which we denote by $\deg_y(q)$.

We choose a defining polynomial $P = p_dx^d + \cdots + p_0 \in \mathbb{F}[y][x]$ and put $\mathcal{R} = \mathbb{F}[y][x]/(P)$. For the digit set $\mathcal{N}$, the canonical choice is

$$
\mathcal{N} = \{ q \in \mathbb{F}[y] \mid \deg_y(q) < \deg_y(p_0) \}.
$$

This gives us a digit system $(\mathcal{R}, X, \mathcal{N})$. Scheicher and Thuswaldner [15] analysed the case where $P$ is monic in $y$, i.e., where the leading coefficient $p_d$ is a nonzero element of $\mathcal{F}$. In their terminology, the pair $(\mathcal{P}, \mathcal{N})$ is a *digit system* if $(\mathcal{R}, X, \mathcal{N})$ has the finite expansion property. Their main result is the observation that $(\mathcal{P}, \mathcal{N})$ is a digit system if and only if $\max_{i=1}^{d-1} \deg_y(p_i) < \deg_y(p_0)$. We will generalise this statement for any choice of $p_d$. 
Theorem 5.8 (cf. [15, Theorem 2.5]). Let $\mathbb{F}$ be a finite field, $P(x) = p_d x^d + \cdots + p_1 x + p_0 \in \mathbb{F}[y][x]$, $\mathcal{R} = \mathbb{F}[y][x]/(P)$ and $\mathcal{N} = \{ q \in \mathbb{F}[y] \mid \deg_y(q) < \deg_y(p_0) \}$. Then the following assertions hold:

- $(\mathcal{R}, X, \mathcal{N})$ has the FEP if and only if $\max_{i=1}^d \deg_y(p_i) < \deg_y(p_0)$.
- $(\mathcal{R}, X, \mathcal{N})$ has the PEP if and only if $\max_{i=1}^d \deg_y(p_i) \leq \deg_y(p_0)$.

Proof. Because of Theorem 3.9 it is enough to check finiteness and periodicity for the Brunotte-module $\Lambda_p$. In [15] Lemmas 2.2 and 2.4 the first assertion was shown for monic polynomials, i.e., where $\Lambda_p = \mathcal{R}$ (see also the much simpler proof in [8, Theorem 2.2]). The second assertion was shown in [15, Theorem 3.1]. These proofs can be immediately adapted to the non-monic case, since they do not use the monicity condition $\deg_y(p_d) = 0$. \qed

Remark 5.9. The first assertion of Theorem 5.8 remains valid even if $\mathbb{F}$ is not assumed to be finite (cf. [8]). However, this case does not fit in our framework as the digit set $\mathcal{N}$ is infinite in this case.

Example 5.10. We will prove that the number system $(\mathbb{F}_2[y][x]/(P), X, \mathcal{N})$ defined in Example 2.11 has the FEP. Recall that $P(x) = (y + 1)x^2 + yx + (y^2 + 1) \in \mathbb{F}_2[y][x]$ and $\mathcal{N} = \{1, y, y+1, y^3+y\}$. We cannot use Theorem 5.8 here, since the digit set $\mathcal{N}$ has the wrong shape. But we will show a strategy how to decide the problem with the aid of the theorem, which has also been used successfully in the case of number systems over $\mathbb{Z}$ (see [19, Section 3]). We will show that every $A \in \mathcal{R}$ has a finite $X$-ary expansion with digits in $\mathcal{N}$. For $A = 0$ we can take the empty expansion. For $A \neq 0$, however, we will make use of the zero cycle, which gives us the expansion $0 = (y^3+y) + X + X^2 + X^3 + (y+1)X^4$. Now let $A$ be an arbitrary nonzero element of $\mathcal{R}$. By Theorem 5.8 we know that $A$ has a finite $X$-ary expansion with digits in the standard digit set $\mathcal{N}' = \{0, 1, y, y+1\}$, say,

$$A = \sum_{j=0}^h b_j X^j$$

with $b_j \in \mathcal{N}'$ for all $j$ and with $h$ minimal. If all of the $b_j$ are contained in $\mathcal{N}$, i.e., $b_j \neq 0$ for all $j$, we are done. Otherwise obey the following instructions:

1. Let $i$ be the smallest index with $b_i = 0$.
2. Add successively $y^i + y$ to $b_i$, $1$ to $b_{i+1}, b_{i+2}$ and $b_{i+3}$, and $y + 1$ to $b_{i+4}$, if necessary changing the value of $h$ such that $b_h \neq 0$.
3. If $b_j \neq 0$ for $0 \leq j \leq h$, then $A = \sum_{j=0}^h b_j X^j$ is the finite $X$-ary expansion of $A$, otherwise return to (1).

We immediately see that $i$ increases. Furthermore, the addition of 1 or $Y + 1$, respectively, to an element of $\mathcal{N}'$ gives either 0 or an element of $\mathcal{N}$. Hence, at any moment we have $b_j \in \mathcal{N}$ for each $j < i$. The question is whether the procedure terminates. Note that from the moment that $h$ starts to increase, if it occurs, only the most significant 5 digits are of interest, and in fact, we can reformulate the above procedure as the iteration of a map.
Let \( S \), where \( S = \{(c_1, \ldots, c_4) \in \mathcal{N} \cup \mathcal{N}'\} \), defined as follows:

\[
\Phi(c_1, \ldots, c_4) = \begin{cases} 
(c_2, c_3, c_4, 0) & \text{if } c_1 \neq 0; \\
(c_2 + 1, c_3 + 1, c_4 + 1, y + 1) & \text{if } c_1 = 0.
\end{cases}
\]

Here, the new digit \( c_1 \), be it changed to \( y^3 + y \) or unchanged, is immediately discarded, and the 4-digit window on the expansion moves one step to the right. We are done when for every \((c_1, \ldots, c_4)\) with digits in \( \mathcal{N}' \), there exists an integer \( m \) such that \( \Phi^m(c_1, \ldots, c_4) = (0, 0, 0, 0) \). This is easily verified by computer. For example, if the original expansion ends (on the most significant side) in \( 1, 0, y, 1 \), we obtain

\[
(1, 0, y, 1) \rightarrow (0, y, 1, 0) \rightarrow (y + 1, 0, 1, y + 1) \rightarrow (0, 1, y + 1, 0) \rightarrow (0, y, 1, y + 1)
\]

\[
\rightarrow (y + 1, 0, y + 1) \rightarrow (0, y, y + 1, 0) \rightarrow (1, y + 1, y, y + 1)
\]

\[
\rightarrow (y + 1, y, y + 1, 0) \rightarrow (y, y + 1, 0, 0) \rightarrow (y + 1, 0, 0, 0) \rightarrow (0, 0, 0, 0).
\]

Thus, the procedure terminates in all cases and actually yields the finite \( X \)-ary expansion for \( A \), and the FEP for \((\mathcal{R}, X, \mathcal{N})\) has been established.

6. Sets of witnesses for the finite expansion property

In Section 3 above, we already gave some results to the effect that the PEP and FEP of a digit system \((\mathcal{R}, X, \mathcal{N})\) can be decided by looking at the expansions of elements in certain \( \mathcal{E} \)-submodules of \( \mathcal{R} \). Building on these results, we will show here that for a large class of digit systems, the PEP and FEP can be decided by checking a finite subset only.

**Definition 6.1.** Let \((\mathcal{R}, X, \mathcal{N})\) be a digit system, and \( S \) an arbitrary subset of \( \mathcal{R} \). A set \( V \subset S \) with the properties

1. every \( A \in S \) can be written as a finite sum of elements of \( V \);
2. for each \( e \in \mathcal{N} \), the set \( V \) is closed under \( A \mapsto T(A + e) \)

is called a set of witnesses of \( S \) with respect to \((\mathcal{R}, X, \mathcal{N})\).

**Remark 6.2.** As to \((\mathcal{R}, X, \mathcal{N})\), when \( S \) is an additive subgroup of \( \mathcal{R} \), the condition can be satisfied by including in \( V \) a set of generators of \( S \) as well as their (additive) inverses.

**Lemma 6.3.** Let \((\mathcal{R}, X, \mathcal{N})\) a digit system, \( S \) a subset of \( \mathcal{R} \) and \( V \) a set of witnesses of \( S \) with respect to \((\mathcal{R}, X, \mathcal{N})\). Then all elements of \( S \) admit finite \( X \)-ary expansions if and only if the same holds for all witnesses \( v \in V \).

**Proof.** The implication “\( \Rightarrow \)” is clear. Thus suppose that all elements of \( V \) have finite \( X \)-ary expansions. This means that for each element \( v \) of \( V \) there exists an \( n \in \mathbb{N} \) with \( T^n(v) = 0 \).

Let \( A \in S \) and \( v \in V \). Then

\[
T(A + v) = \frac{A + v - D_N(A + v)}{X} = \frac{A - D_N(A) + v + D_N(A) - D_N(A + v)}{X}
\]

\[
= \frac{A - D_N(A)}{X} + v + \frac{D_N(A) - D_N(A + v)}{X} = T(A) + v',
\]

where \( v' \in V \) by property (2) of Definition 6.1.
Now suppose there exist elements in $S$ having no finite $X$-ary expansion; let $A = v_1 + \cdots + v_k$ be such an element for which $k$ is minimal. Using (6.1) iteratively for $k - 1$ times, we have

$$T(A) = T(v_1) + v'_2 + \cdots + v'_k$$

for $v'_2, \ldots, v'_k \in \mathcal{V}$. As $v_1 \in \mathcal{V}$ there exists an $n$ such that $T^n(v_1) = 0$. Thus, repeating the above procedure $n$ times, we get

$$T^n(A) = T^n(v_1) + v''_2 + \cdots + v''_k = v''_2 + \cdots + v''_k.$$

The element $v''_2 + \cdots + v''_k$ also cannot have a finite $X$-ary expansion, because this would imply $T^m(A) = 0$ for some $m$. As $v''_2, \ldots, v''_k \in \mathcal{V}$ this contradicts the minimality of $k$. □

As above, we write $R_k$ for the submodule of $R$ generated by $1, X, \ldots, X^{k-1}$.

**Theorem 6.4.** Let $(R, X, N)$ be a digit system, let $k \geq \deg P$ be minimal such that all digits in $N$ can be represented by polynomials in $E[x]$ of degree at most $k$, and let $\mathcal{V}$ be a set of witnesses of $R_k$. Then $(R, X, N)$ has the FEP if and only if every element in $\mathcal{V}$ has a finite $X$-ary expansion.

**Proof.** This follows directly by combining the above lemma with Theorem 3.2. □

The concept of witness sets is useful mainly in those situations where we can construct a finite witness set. This is the case for a large class of number systems, as we will now show.

First observe that one can construct a set of witnesses by the following iterative approach. Let $\mathcal{V}_0 \subseteq S$ be an arbitrary subset satisfying condition 1. Then, for $i \geq 0$, define

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cup \{T(v + e) \mid v \in \mathcal{V}_i, e \in N\}, \text{ and } \mathcal{V} = \bigcup_{i \geq 0} \mathcal{V}_i.$$

We apply this construction to the module $R_k$ as in Theorem 6.4. As this module is finitely generated, we can start off with a finite set $\mathcal{V}_0$, in view of Remark 6.2. Now the question is whether the sequence of the $\mathcal{V}_i$ is eventually stable, and whether the resulting set $\mathcal{V}$ is finite.

An important special case where this can be proved is the case where $R$ can be embedded as a discrete set in a finite-dimensional complex vector space. Here we examine whether the operator $A \mapsto XA$ is expanding, i.e., whether all its eigenvalues have modulus greater than 1. It is well known that the expanding property implies the PEP in this case (special instances occur in [1, 14]; the proof in the general case is the same).

**Proposition 6.5.** Suppose $R$ is embedded in a finite-dimensional complex vector space, let $\lambda$ be the operator norm of $A \mapsto A/X$ on $R$, and assume that $\lambda < 1$. Then in the construction (6.2), we have

$$\max_{v \in \mathcal{V}} \|v\| \leq \max_{v \in \mathcal{V}_0} \|v\|, \frac{2\lambda}{1 - \lambda} \max_{e \in N} \|e\|.$$

In particular, if $R$ is a discrete set and $\mathcal{V}_0$ is finite, then $\mathcal{V}$ is finite.
Proof. We have
\[ \|T(A + e)\| = \left\| \frac{A + e - D_N(A + e)}{X} \right\| \leq \lambda (\|A\| + \|e\| + \|D_N(A + e)\|); \]
thus
\[ \|T(A + e)\| < \|A\| \text{ unless } \|A\| < \frac{2\lambda}{1 - \lambda} \max_{e \in \mathcal{N}} \|e\|. \]

Example 6.6. We will illustrate the usage of a set of witnesses by the following example. Let \( E = \mathbb{Z}[i] \) be the ring of Gaussian integers, \( P(x) = (1 + i)x + (1 + 2i) \in \mathbb{Z}[i][x], \mathcal{R} = \mathbb{Z}[i][x]/(P) \) and \( \mathcal{N} = \{0, 1, 2, 3, 4\}; \mathcal{N} \) is a complete set of residues of \( \mathcal{R}/(X) \), because they are a constant polynomials and form a complete set of residues of \( \mathbb{Z}[i]/(1 + 2i) \). Since \( |1 + 2i| = \sqrt{5} \) we see that \( P \) is expanding and therefore \( (\mathcal{R}, X, \mathcal{N}) \) has the PEP. The Brunotte basis of \( \mathcal{E}[x] \) modulo \( P \) is composed of the single element \( w_0 = 1 + i \). As \( \mathcal{N} \subset \mathbb{Z}[i] \) we know by Theorem 3.9 that \( (\mathcal{R}, X, \mathcal{N}) \) has the FEP if and only if each element of \( \Lambda_P = (1 + i)\mathbb{Z}[i] \) has a finite \( X \)-ary expansion. Moreover, by the expanding property of \( P \), Proposition 6.5 implies the existence of a finite set of witnesses \( V \) for \( \Lambda_P \); we claim that \( V = \{0, \pm 1 \pm i, \pm 2, \pm (3 - i), \pm (4 - 2i), \pm (2 - 2i)\} \) is such a set. Of course, the first condition of Definition 6.1 is satisfied, since \( 1 + i \) and \( -1 + i = i(1 + i) \) are additive generators of the Brunotte module \( (1 + i)\mathbb{Z}[i] \), and they and their negatives are in \( V \). To show the second condition we need to check that
\[ \{T(v + e) \mid v \in V, e \in \mathcal{N}\} \subseteq V. \]
This can be checked by direct calculation. For example, for \( v = 3 - i \) we get
\[ \{T(3 - i + e) \mid e \in \mathcal{N}\} = \{-1 + i, -4 + 2i\} \subseteq V. \]
The other elements can be treated likewise. It remains to check that the orbit \( (T^n(v))_{n \geq 0} \) contains zero for each \( v \in V \). In Figure 1 the action of \( T \) on the elements of \( V \) is indicated by arrows.
This shows that all orbits of $V$ end in 0. This shows that the periodic set of the digit system contains just the loop at 0. By Lemma 2.10, the digit system $(\mathcal{R}, X, \mathcal{V})$ has the FEP.

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