Bifurcation of crack pattern in two-dimensional periodic rectangular crack arrays under dynamic shear stress

Xiao-Ping Zhou

School of Civil Engineering, Chongqing University, Chongqing400045, P.R. China

Email: xiao_ping_zhou@126.com, zhouxiaopinga@sina.com

Abstract. Crack interaction leads to the bifurcation of crack pattern. The pseudo-traction method is used to analyze interaction among cracks. Bifurcation condition of crack growth pattern under dynamic shear stress depends upon the ratios of crack length to spacing ratio, and the crack length to the perpendicular distance between the two adjacent row ratios. Bifurcation of crack growth pattern occurs when the above ratios extend critical values. Bifurcation condition for periodic rectangular cracks subjected to dynamic shear stress is analyzed theoretically. The critical parameter is dependent on crack growth velocity and the velocity of a Rayleigh wave. It is shown from numerical result that the critical parameter increases with the velocity of a Rayleigh wave, and decreases with the increase of crack growth velocity.

1. Introduction
Shear stress wave loading appears during rock blasting and earthquake. The response of the crack-weakened rock masses to high dynamic loadings, which may last only for a few milliseconds, remains largely unknown. Rock masses often consist of multiple, near-parallel, planar fractures and on most occasions, such set (or sets) of parallel fractures control the mechanical behaviour of the rock masses. When a wave propagates through fractured rock masses, it is greatly attenuated (and slowed) due to the presence of these fractures. Bifurcation of crack pattern importantly affects the dynamic mechanical behaviors of the crack-weakened rock masses, such as the strength, the constitutive relationship, etc [1]. The previous works showed that a bifurcation of the crack pattern emerges as soon as the critical value is reached under static loading [2-3]. However, the bifurcation of crack growth pattern under dynamic loads is not researched in the previous works. The main objective of the present work is to derive the crack length to crack spacing critical ratio for periodic rectangular crack arrays subjected to dynamic shear stress.

2. Theoretical model
Consider rock masses that contains periodic rectangular crack arrays, as shown in Fig.1. For simplicity, it is assumed that all cracks are two-dimensional. The previous works showed that the bifurcation of crack pattern is due to the crack interactions. The dipole asymptotic method considers the interaction by superposing \( \tau^+ (t) \) with an additional stress term \( \tau_j (t) \) (called pseudo-traction by Horii and Nemat-Nasser[4]), obtained by adding all the far field dynamic stresses caused by other cracks on the surface of the jth crack. With the above assumptions, the shear pseudo-traction on the jth crack is

---

1 To whom any correspondence should be addressed.
\[ \tau_j(t) = \sum_{k=1}^{N} \Lambda_{jk} \left[ \tau^\infty(t) + \tau_k(t) \right] \]  

(1)

where \( k \neq j, \ j = 1, 2, 3, \ldots, N \), \( \Lambda_{jk} \) is the influence factor, \( \tau^\infty(t) \) is the far field dynamic shear stress.

The influence factor \( \Lambda_{jk} \) can be approximated by the far field dynamic stresses at a point \((r, \phi)\) caused by internal traction \( \tau^\infty(t) \) [2], as shown in Fig.2:

\[ \tau_{12}(t) = \frac{1}{2} \tau^\infty(t) \left( \frac{a}{r} \right)^2 \cos 4\phi \]  

(2)

Consider \( N \) cracks in a solid under uniform far field dynamic shear stresses, the size of the \( j \)th crack is \( 2a \), as shown in Fig.1. By combining (1) and (2), the \( N \) pseudo-tractions \( (\tau_j(t), j = 1, 2, \ldots, N) \) induced by collinear interactions and \( N \) crack sizes are governed by

\[ \tau_j(t) = \frac{1}{2} \sum_{k=1}^{N} \left( \frac{a}{H} \right)^2 \left[ \tau^\infty(t) + \tau_k(t) \right] \]  

(3)

where \( H \) is the crack spacing.

By combining Eq.(1) and Eq.(2), the \( N \) pseudo-tractions caused by stacked interactions can be obtained by

\[ \tau_j(t) = \frac{1}{2} \sum_{k=1}^{N} \left( \frac{a}{W} \right)^2 \left[ \tau^\infty(t) + \tau_k(t) \right] \]  

(4)

where \( W \) is the perpendicular distance between the two adjacent row

Using the corollary of the Riemann – Zeta function \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \) [5], Eq.(3) can be rewritten by

\[ \tau_k(t) = \frac{\pi^2 \lambda_2^2 \tau^\infty(t)}{6 - \pi^2 \lambda_2^2} \]  

(5)

Using the corollary of the Riemann – Zeta function \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} \) [5], Eq.(4) can be recasted by

\[ \tau_p(t) = \frac{\pi^2 \lambda_2^2 \tau^\infty(t)}{6 - \pi^2 \lambda_2^2} \]  

(6)

where \( \lambda_1 = a/H \), \( \lambda_2 = a/W \)

From(3) and (5), the \( N \) pseudo-tractions \( (\tau_j, j = 1, 2, \ldots, N) \) induced by collinear interactions is

\[ \tau_j(t) = \frac{1}{2} \sum_{k=1}^{N} \left( \frac{a}{H} \right)^2 \left[ \frac{1}{6 - \pi^2 \lambda_1^2} \tau^\infty(t) + \tau_k(t) \right] \]  

(7)

Under static shear loads, the mode II stress intensity factor \( K_{II} \) can be expressed as
Under dynamic loads, crack growth velocities have a great influence on the dynamic stress intensity factor. In most cases, the dynamic crack tip stress intensity factor $K_{HD}$ and the static crack tip stress intensity factor $K_{II}$ are related by:

$$K_{HD} = k_2(v)K_{II}$$  \hspace{1cm} (9)

When the crack is loaded by only a pair of concentrated dynamic forces at its center, an approximate formulation of $k_2(v)$ is given as[6-8]

$$k_2(v) = \frac{v_r - v}{v_r - 0.9v}$$  \hspace{1cm} (10)

where $v = \frac{\dot{a}}{t}$ is crack growth velocity, $v_r$ is the velocity of a Rayleigh wave in the rock material.

When the crack is loaded by only far field uniform dynamic stresses, an approximate formulation of $k_2(v)$ is obtained as

$$k_2(v) = \frac{v_r - v}{v_r - 0.65v}$$  \hspace{1cm} (11)

The length of crack growth is governed by the dynamic crack growth criterion
where \( K_{ii}' \) is the mode II critical dynamic stress intensity factor.

Taking the variation of Eq.(12), the following relation between \( \delta \tau_j(t) \) and \( \delta a_j \) can be obtained as:

\[
\delta \tau_j(t) = \left[ \frac{6 - 5\pi^2 \lambda_2^2 - \pi^4 \lambda_2^1 \lambda_1^2}{2(v_r - v)(6 - \pi^2 \lambda_2^2)} \right] - \frac{0.65v(6 - 5\pi^2 \lambda_2^2 - \pi^4 \lambda_2^1 \lambda_1^2)}{(v_r - 0.65v)(6 - \pi^2 \lambda_2^2)} \frac{2\pi^2 \lambda_1^2}{6 - \pi^2 \lambda_1^2} \]

Taking the variation of Eq.(7), the following expression can be obtained by

\[
\delta \tau_j(t) = \frac{3}{4} \sum_{k=1}^{N} \frac{\lambda_2}{(k-j)^2} \left[ \frac{1}{6 - \pi^2 \lambda_1^2} \tau^\infty(t) - \frac{\pi^2 \lambda_1^2 \tau^\infty(t)}{6 - \pi^2 \lambda_1^2} \right] \delta a_j
\]

This set of Equations can be solved by using the discrete Fourier transform defined here by[9]

\[
A(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \delta a_n e^{i\phi n}, \quad \delta a_n = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} A(\phi) e^{-i\phi} d\phi
\]

where \( i = (-1)^{1/2} \)

From Eq.(15), we can get

\[
\sum_{n=-\infty}^{\infty} \frac{e^{i\phi n}}{n^2} = \frac{(\pi - \phi)^2}{2} - \frac{\pi^2}{6}, \quad n \neq 0
\]

Combining Eqs(13),(14),(15) and (16), the following expression can be determined by

\[
3 \left[ \frac{6 - 5\pi^2 \lambda_2^2 - \pi^4 \lambda_2^1 \lambda_1^2}{(6 - \pi^2 \lambda_1^2)} \right] \lambda_2 \lambda_1 \left[ \frac{(\pi - \phi)^2}{2} - \frac{\pi^2}{6} \right]
\]

\[
= \frac{6 - 5\pi^2 \lambda_2^2 - \pi^4 \lambda_2^1 \lambda_1^2}{2(v_r - v)(6 - \pi^2 \lambda_2^2)} \left[ 2(v_r - v)(6 - \pi^2 \lambda_2^2) \right] - \frac{0.65v(6 - 5\pi^2 \lambda_2^2 - \pi^4 \lambda_2^1 \lambda_1^2)}{(v_r - 0.65v)(6 - \pi^2 \lambda_2^2)} \frac{2\pi^2 \lambda_1^2}{6 - \pi^2 \lambda_1^2}
\]

From(4) and (6), the N pseudo-tractions \( (\tau_j, j = 1,2,\ldots,N) \) induced by stacked interactions is

\[
\tau_j(t) = \frac{1}{2} a \left[ \frac{1}{(k-j)^2} \right] \left[ \frac{1}{6 - \pi^2 \lambda_2^2} \tau^\infty(t) + \tau_k(t) \right]
\]

The length of crack growth is governed by the crack growth criterion

\[
K_{ii} = \frac{\sqrt{v_r - v}}{\sqrt{v_r - 0.65v}} \left[ \frac{\tau^\infty(t)}{6 - \pi^2 \lambda_2^2} + \tau_k(t) \right] \sqrt{\pi a} = K_{ii}'
\]

Taking the variation of Eq.(18), the following relation between \( \delta \tau_j(t) \) and \( \delta a_j \) can be obtained as:
\[ \delta \tau_j(t) = \frac{3}{4} \sum_{i=1}^{\infty} \frac{\lambda_i}{(k-j)^2} \left( \frac{1}{6-\pi^2 \lambda_i^2} + \frac{\pi^2 \lambda_i^2}{6-\pi^2 \lambda_i^2} \right) \delta a_j \tau^r(t) \]  

(20)

Taking the variation of Eq.(19), we have

\[
\delta \tau_j(t) = \left[ \frac{6 - 5\pi^2 \lambda_i^2 - \pi^4 \lambda_i^2 \lambda_j^2}{2(v_r - v)(6 - \pi^2 \lambda_i^2)} \right] - \frac{0.65v(6 - 5\pi^2 \lambda_i^2 - \pi^4 \lambda_i^2 \lambda_j^2)}{(v_r - 0.65v)(6 - \pi^2 \lambda_i^2)} - \frac{2\pi^2 \lambda_i^2}{6 - \pi^2 \lambda_i^2} \right] \tau^r(t) \delta a_j 
\]

(21)

Combining Eqs(20),(21),(15) and (16), we can get

\[
\frac{3}{4} \left[ \frac{6 - 5\pi^2 \lambda_2^2 - \pi^4 \lambda_2^2 \lambda_1^2}{2(v_r - v)(6 - \pi^2 \lambda_2^2)} \right] \lambda_1 \lambda_2 \left( \frac{(\pi - \varphi)^2}{2} - \frac{\pi^2}{6} \right) 
\]

\[
\left( \frac{6 - 5\pi^2 \lambda_2^2 - \pi^4 \lambda_2^2 \lambda_1^2}{2(v_r - v)(6 - \pi^2 \lambda_2^2)} \right) - \frac{0.65v(6 - 5\pi^2 \lambda_1^2 - \pi^4 \lambda_1^2 \lambda_1^2)}{(v_r - 0.65v)(6 - \pi^2 \lambda_1^2)} - \frac{2\pi^2 \lambda_2^2}{6 - \pi^2 \lambda_2^2} 
\]

(22)

From Eqs(17) and (22), the solutions for both \( \lambda_i \) and \( \lambda_2 \) can determined. It is shown From (22) and (17)that the critical parameter \( \lambda_{cr} \) is dependent on crack growth velocity and the velocity of a Rayleigh wave.

3. Numerical results

To investigate the dependence of the critical parameter \( \lambda_{cr} \) on crack growth velocity and the velocity of a Rayleigh wave, it is assumed that the velocity of a Rayleigh wave is equal to \( v_r = 1500 \text{m/s} \), \( 2000 \text{m/s} \), \( 2500 \text{m/s} \). Numerical results are depicted in Fig.3.

It is shown from Fig.3 that the critical parameter \( \lambda_{cr} \) increases with the velocity of a Rayleigh wave, and the critical parameter \( \lambda_{cr} \) decreases with increasing crack growth velocity.

![Fig.3 The dependence of $\lambda$ on $v$ and $v_r$](image-url)
4. Conclusions

Under dynamic shear loads, bifurcation of crack growth pattern occurs if the crack length/spacing ratio, $\lambda_1 = a/H$ or $\lambda_1 = a/W$, is larger than a critical value $\lambda_{cr}$. That is, bifurcation of the crack growth pattern under dynamic shear loads is induced by crack interaction. The critical parameter $\lambda_{cr}$ is dependent on crack growth velocity and the velocity of a Rayleigh wave. It is found from numerical results that the critical parameter $\lambda_{cr}$ increases with the velocity of a Rayleigh wave, and the critical parameter $\lambda_{cr}$ decreases with an increase in crack growth velocity.

Acknowledgements - This work is supported by the National Natural Science Foundation of China (No. 50778184)

References

[1] Zhou XP 2004 *Int. J. Solid. Struct.* **41** 1725
[2] Zhou XP 2007 Int. J. Nonlinear. Sci. Numer. Sim. **8** 55
[3] Muhlhaus HB and Chau KTA 1996 *Int. J. Fract.* **77** 1
[4] Horii H and Nemat-Nasser S 1985 *Int. J. Solid. Struct.* **21** 731
[5] Abramowitz M and Stegun IA 1966 *Handbook of mathematical functions* Dover New York
[6] Freund LB 1973 *J. Mech. Phys. Solid.* **21** 47
[7] Freund LB 1990 *Dynamic fracture mechanics* Cambridge University Press
[8] Cherepanov GP 1979 Mechanics of brittle fracture, Translated by R de Wit and W C Ccoley, WcGraw-Hill, New York
[9] Rees CS, Shah SM, and Stanojevic CV 1981 Theory and applications of Fourier analysis, Marcel Dekker, New York