Additional analytically exact solutions for three-anyons *

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Abstract

We present new family of exact analytic solutions for three anyons in a harmonic potential (or in free space) in terms of generalized harmonics on $S^3$, which supplement the known solutions. The new solutions satisfy the hard-core condition when $\alpha = \frac{1}{3}, 1$ ($\alpha$ being the statistical parameter) but otherwise, have finite non-vanishing two-particle colliding probability density, which is consistent with self-adjointness of the Hamiltonian. These solutions, however, do not have one-to-one mapping property between bosonic and fermionic spectra.

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Smooth interpolation between bosonic and (spinless) fermionic spectra has been a very attractive idea and appeared the concept of anyon [1] based on the homotopy group in two space dimensions. The anyons are described effectively by Schrödinger equation of bosonic or fermionic particles with statistical gauge field of Aharonov-Bohm type. One can also derive [2] this equation of motion from the second quantized abelian Chern-Simons gauge theory. It is believed that fractional quantum Hall effect is an example of physical phenomenon of anyons [3].

On the other hand, the detailed analysis of many anyons has not been successful. Even for simple system such as harmonic oscillator (HO), analytically known exact solutions [4–7] fall short of the numerical result in [8] which demonstrates smooth interpolation. Only perturbation approach [9] could reproduce the numerical result. The main difficulty of the problem lies in the non-trivial exchange property of anyons and hard-core condition on its wavefunction.

In this Letter, we present new family of analytically exact solutions of three particles in HO or in free space. These solutions supplement the presently known analytically exact solutions but unfortunately do not satisfy the hard-core condition except at $\alpha = 1/3, 1$ ($\alpha$ is the statistical parameter chosen as $0 \leq \alpha \leq 1$, $\alpha=0$ at for boson and 1 for spinless fermion). None the less, considering the situation of absence of analytic solutions for missing states, we think these solutions will provide valuable information about anyons.

Our analysis is done by adopting ordinary Hamiltonian with anyonic particle-exchange property imposed on wavefunction. We use coordinates system following [4]. Coordinates of three particles are represented as complex numbers, $z_a = x_a + iy_a$ where $a = 1, 2, 3$. We separate out center of mass (CM is irrelevant to the anyonic property) by using CM coordinate, $Z = \frac{(z_1+z_2+z_3)}{\sqrt{3}}$ and relative ones $u = \frac{(z_1+\eta z_2+z_3)}{\sqrt{3}}$ and $v = \frac{(z_1+\eta^2 z_2+\eta z_3)}{\sqrt{3}}$ with $\eta = e^{i\frac{2\pi}{3}}$.

Relative motion (RM) of free particles is described in terms of Hamiltonian (scaled as dimensionless), $H_{free} = -\frac{\partial}{\partial u} \frac{\partial}{\partial \bar{u}} - \frac{\partial}{\partial v} \frac{\partial}{\partial \bar{v}}$. We are considering systems with non-singular potential energy of the form, $V = V(r)$ where $r = \sqrt{uu + vv} \geq 0$. This includes the free
case, $V = 0$ and HO case, $V = r^2$. This Hamiltonian is invariant under exchange of any two particles. To see this in our notation, let us denote the second and third particle exchange operation, $(1, 2, 3) \rightarrow (1, 3, 2)$ as $E$, and cyclic operation, $(1, 2, 3) \rightarrow (2, 3, 1)$ as $P$. Any two-particle exchange is represented in combination of $E$’s and $P$’s. The definition of $u$ and $v$ shows that $P: (u, v) \rightarrow (\eta^2 u, \eta v)$ and $E: (u, v) \rightarrow (v, u)$. Therefore, RM is trivially invariant under $P$ and $E$.

For separation of variables, we introduce spherical coordinates, $u = r \sin(\xi/2) e^{i(\theta + \frac{\phi}{2})}$ and $v = r \cos(\xi/2) e^{i(\theta - \frac{\phi}{2})}$. $\xi$, $\theta$ and $\phi$ are angle variables whose fundamental domain is given as $0 \leq r$, $0 \leq \xi < \frac{\pi}{2}$, $-\frac{\pi}{3} \leq \phi < \frac{\pi}{3}$, $0 \leq \theta < 2\pi$ since under $P$ and $E$, we have $P: (r, \xi, \phi, \theta) \rightarrow (r, \xi, \phi + \frac{2\pi}{3}, \theta + \pi)$ and $E: (r, \xi, \phi, \theta) \rightarrow (r, \pi - \xi, -\phi, \theta)$ respectively. The angular domain is not $S^3$ but $\frac{S^3}{Z_2 \times Z_3}$. (Euler angles on $S^3$ or on $SU(2)$ are defined typically as $(\xi, \chi = -\phi, \psi \equiv 2\theta - 2\pi)$ whose ranges are given as $0 \leq \xi < \pi$, $0 \leq \chi < 2\pi$, $-2\pi \leq \psi < 2\pi$).

Anyon wavefunction is defined to have a phase $e^{i\alpha\pi}$ when any of two particles are interchanged. This requires the wavefunction have the phase under $P$ and $E$,

$$E: \Psi(r, \pi - \xi, -\phi, \theta) = e^{i\alpha\pi}\Psi(r, \xi, \phi, \theta)$$
$$P: \Psi(r, \xi, \phi + \frac{2\pi}{3}, \theta + \pi) = e^{i2\alpha\pi}\Psi(r, \xi, \phi, \theta)$$
$$R: \Psi(r, \xi, \phi, \theta + 2\pi) = e^{i6\alpha\pi}\Psi(r, \xi, \phi, \theta)$$

The operation $R$, $2\pi$ rotation of $\theta$ corresponds to a composite operation $EPEP$.

The Hamiltonian is written in $(r, \xi, \phi, \theta)$ coordinates as

$$H = -\frac{1}{4r^2} \frac{\partial}{\partial r} \left( r^3 \frac{\partial}{\partial r} \right) + \frac{1}{4r^2} M + V(r^2)$$

where $M$ is a Laplacian on $S^3$,

$$M = -\frac{4}{\sin \xi} \frac{\partial}{\partial \xi} \sin \xi \frac{\partial}{\partial \xi} + \frac{1}{\sin^2(\xi/2)} \left( \frac{1}{2i} \frac{\partial}{\partial \theta} + \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2 + \frac{1}{\cos^2(\xi/2)} \left( \frac{1}{2i} \frac{\partial}{\partial \theta} - \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2.$$ (3)

$M$ commutes with the Hamiltonian $H$. The relative angular momentum, $L$,

$$L = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} - \bar{u} \frac{\partial}{\partial \bar{u}} - \bar{v} \frac{\partial}{\partial \bar{v}} = \frac{1}{i} \frac{\partial}{\partial \theta},$$ (4)
is another useful mutually commuting operator. A simultaneous eigenstate of $H$, $M$ and $L$ is given in factorized form, $\Psi_{E,\mu,l}(r,\xi,\phi,\theta) = R_{E,\mu}(r) \Xi_{\mu,l}(\xi,\phi,\theta)$. $\Xi_{\mu,l}$ is a harmonic on $S^3$, whose two-particle analogue is $e^{\pm i\alpha \theta_{12}}$ where $0 \leq \theta_{12} \leq \pi$ is the angle of the relative coordinate. $E$, $\mu(\mu + 2)$ and $l$ are eigenvalues of $H$, $M$ and $L$ respectively.

Since one can easily solve the radial solution once the harmonics are found (for example, it is written as a Bessel function of the first kind for free particle and as a Laguerre for HO), we concentrate on finding the angular part. Boundary conditions for $\Xi$ are obtained from Eq. (1), consistently with self-adjointness of $L$ and $M$,

\[
\Xi(\xi,\phi,\theta = 2\pi) = e^{i\alpha \pi} \Xi(\xi,\phi,\theta = 0)
\]
\[
\Xi(\xi,\phi = \frac{\pi}{3},\theta = \pi) = e^{i2\alpha \pi} \Xi(\xi,\phi = -\frac{\pi}{3},\theta = 0)
\]
\[
\frac{\partial}{\partial \phi} \ln \Xi(\xi,\phi = \frac{\pi}{3},\theta = \pi) = \frac{\partial}{\partial \phi} \ln \Xi(\xi,\phi = \frac{\pi}{3},\theta = 0)
\]
\[
\Xi(\xi = \frac{\pi}{2},\phi,\theta) = e^{-i\alpha \pi} \Xi(\xi = \frac{\pi}{2},-\phi,\theta)
\]
\[
\frac{\partial}{\partial \xi} \ln \Xi(\xi = \frac{\pi}{2},\phi,\theta) = -\frac{\partial}{\partial \xi} \ln \Xi(\xi = \frac{\pi}{2},-\phi,\theta),
\]

where last two identities hold for $0 < \phi \leq \pi/3$. In addition, any current across the boundary needs to be finite, which requests

\[
\Xi(\xi = 0,\phi,\theta), \quad \lim_{\Delta \phi \to 0} \frac{\Delta \phi}{\Delta \xi} \Xi(\xi = \frac{\pi}{2}, \Delta \phi, \theta), \quad \frac{\partial}{\partial \phi} \Xi(\xi = \frac{\pi}{2}, \phi = \pm \frac{\pi}{3}, \theta)
\]

finite. We note that even though hard-core condition (vanishing at $\xi = \pi/2$ and $\phi = 0$) is consistent with this boundary conditions, it is not the unique choice for $\alpha \neq$ odd integer as we will see promptly.

The harmonics are not single- or double-valued but multi-valued since boundary conditions contain $e^{i\alpha \pi}$ instead of ordinary $\pm 1$. To find the harmonics, we employ ladder operators [11] satisfying the $su(2)$ Lie-algebra, $[K_3, K_{\pm}] = \pm K_{\pm}$ and $[K_+, K_-] = 2K_3$ where

\[
K_+ = e^{i2\theta} \{ i \frac{\partial}{\partial \xi} - \frac{1}{\sin \xi} \frac{\partial}{\partial \phi} - \frac{\cot \xi}{2} \frac{\partial}{\partial \theta} \}
\]
\[
K_- = e^{-i2\theta} \{ i \frac{\partial}{\partial \xi} + \frac{1}{\sin \xi} \frac{\partial}{\partial \phi} + \frac{\cot \xi}{2} \frac{\partial}{\partial \theta} \}
\]
\[
K_3 = \frac{1}{2i} \frac{\partial}{\partial \theta} = \frac{1}{2} L,
\]
where $K^+_l = K_+$. (Integration measure is $\sin 2 \xi d\xi d\phi d\theta$). $K_\pm$ and $K_3$ are invariant under $P$ and $R$ but get extra $-$ sign under $E$. Noting that $M = 4\{K_+K_- + K_-K_+\}/2 + K_3^2$, $\mu/2$ is Casimir number for harmonics and is semi-positive definite.

Well-known analytic solutions called type I (II) \cite{4,5} are reproduced in our approach if we apply $K^+_l$ ($K^+_3$) on states which are annihilated by $K^+_l$ ($K^+_3$). Of course, they satisfy the hard-core condition for $0 \leq \alpha \leq 1$ and $\mu = \text{integer} + 3\alpha \geq 0$ for type I ($\mu = \text{integer} - 3\alpha \geq 3$ for type II).

Newly found solutions cannot be constructed this way. They do not have states which are annihilated by $K^+_3$ or $K^+_2$. Suggestion of this new type has been made in \cite{12}. Similar solutions but approximate ones were considered in \cite{5}. These have $\mu = \text{integer} \pm \alpha \geq 2$ ($+ (-)$ for type III (IV)). (Note that the convention for the class of the solution in \cite{5} is different from ours: II, III and IV corresponding to our III, IV and II respectively).

A solution with $(\mu = 3 - \alpha, l = -3 + 3\alpha)$ (type IV), resulting in the interpolation between the fermionic ground state and a bosonic excited state of HO, is given as (using notations of $z \equiv v/r$ and $w \equiv u/r$)

$$\Xi_{3-\alpha,-3+3\alpha} = (z^3 - w^3)^\alpha P_{3-\alpha,-3+3\alpha}$$

with

$$P_{3-\alpha,-3+3\alpha} = \frac{(\bar{z} + \bar{w})^{3-2\alpha}}{(z^2 + zw + w^2)^\alpha} + 2 \text{ more cyclic permutations},$$

where 2 more added terms are obtained from the first term if $z$ ($w$) is replaced by $\eta z$ ($\eta^2 w$), and by $\eta^3 z$ ($\eta w$) respectively. Note that for $|w| < |z|$, $\eta^3 w = w$ but $\eta^3 z \neq z$ because of multi-valuedness of the factors in $\Xi_{3-\alpha,-3+3\alpha}$. The solution is equally put as

$$\Xi_{3-\alpha,-3+3\alpha} = (z - w)^\alpha(\bar{z} + \bar{w})^{3-2\alpha} + (z - \eta w)^\alpha(\bar{z} + \eta \bar{w})^{3-2\alpha} + (z - \eta^2 w)^\alpha(\bar{z} + \eta^2 \bar{w})^{3-2\alpha}$$

and more symbolically as $\Xi_{3-\alpha,-3+3\alpha} = S\{(z - w)^\alpha(\bar{z} + \bar{w})^{3-2\alpha}\}$. Its explicit form is given as

$$\Xi_{3,-3} = 3(\bar{z}^3 + \bar{w}^3)$$

in the bosonic limit, and $\Xi_{2,0} = 3(z\bar{z} - w\bar{w})$ in the fermionic limit.

One can check that $\Xi_{3-\alpha,-3+3\alpha}$ is an eigenfunction of $M$ using $K_+\bar{z} = -iw$, $K_+\bar{w} = iz$, $K_-z = -iw$ and $K_-w = iz$. $\Xi_{3-\alpha,-3+3\alpha}$ satisfies the boundary conditions in Eqs. (3,4), yet does not vanish but is finite when two of three particles coincide ($z = w = 1$), except for $3\alpha = \text{odd integer}$:

$$\Xi_{3-\alpha,-3+3\alpha}(z = w = 1) = -(1 - \eta)^\alpha e^{i2\pi \alpha}/3(1 + e^{i3\pi \alpha}).$$
We have used identities such as $\lim_{\delta \to 0}(1 + (1 - \delta)\eta) = e^{i\frac{\pi}{3}}$, $\lim_{\delta \to 0}(1 + (1 - \delta)\eta^2) = e^{i\frac{5\pi}{3}}$ and their complex conjugates. (Coordinates $(z_1, z_2, z_3)$ is not suited to proving Eq. (9) due to lack of faithful representation of $(-1)^\eta$). Two particles in the presence of the third can collide each other, as seen in two-particle case in the context of self-adjoint extension [13]. The chance for simultaneous collision of three particles is, however, strictly prohibited except the bosonic limit because radial part vanishes.

Will the enumeration of these solutions (FIG. 1, whose details will appear elsewhere) lead to smooth interpolation? A solution is missing, interpolating between $(\mu=2, l=-2)$ boson and $(\mu=3, l=1)$ fermion, but this only implies that $\mu \neq 2+\alpha$. A real potential problem against the smooth interpolation comes from the possibility of a two-to-one correspondence. For example, two bosonic states with $(\mu=5, l=-1)$ and $(\mu=3, l=-1)$ are mapped into a fermionic state with $(\mu=4, l=2)$:

$$
\Xi_{5-\alpha,-1+3\alpha} = S\{(z - w)^{2+\alpha}(\bar{z} + \bar{w})^{3-2\alpha}\},
$$

$$
\Xi_{3+\alpha,-1+3\alpha} = S\{(\bar{z} - \bar{w})^{2-\alpha}(z + w)^{1+2\alpha}\},
$$

$\Xi_{5-\alpha,-1+3\alpha}$ is an extra solution which has no counter-part to the numerical solution in Ref [8]. It satisfies the boundary conditions in Eqs. (5, 6) and is obtained by applying $K^2_+$ on $\Xi_{5-\alpha,-5+3\alpha} = S\{(z - w)^{\alpha}(\bar{z} + \bar{w})^{5-2\alpha}\}$. Since that hard-core condition is satisfied at $\alpha=1/3$, the relaxed boundary condition cannot be the cause of this trouble. It might be possible that the extra solution is an isolated one which cannot be interpolated smoothly to bosonic and fermionic limit satisfying the hard-core condition. Even so, it does not eliminate the fact that there are extra states beyond smooth interpolation.

For HO, energies of our solutions are linear in $\alpha$, since the energy is given as $\mu + 2 + 2n$ (n is a non-negative integer and $2n$ is due to the radial part). However, as we impose the hard-core condition, the wavefunctions will adjust themselves and energies will have a slight non-linear dependence on $\alpha$. This non-linear dependency should be manifest at the near side of the fermionic states of new solutions since all the solutions satisfy the hard-core condition.
at $\alpha = 1/3$. In fact, this property is seen in numerical [8], perturbative [9] and approximate estimation [3], even though $\mu$ at $\alpha = 1/3$ does not agree completely.

One can check that reflection symmetry in $\mu; \alpha$ by $2 - \alpha$ [6] for type I and II, and $\alpha$ by $1 - \alpha$ [14] for III and IV, although the latter is not perfect, being subject to hard-core boundary condition. We also expect that the tendency of linear dependence of $\mu$ on $\alpha$ for $N$ anyons, $(\frac{N(N-1)}{2})\alpha$, $(\frac{N(N-1)}{2} - 2)\alpha$, $\cdots$, $-(\frac{N(N-1)}{2} - 2)\alpha$, and $-(\frac{N(N-1)}{2})\alpha$ up to an integer, will persist if we relax the hard-core boundary condition. This trend is shown in the numerical analysis for four anyons [15].

In summary, we present four types of analytically exact normalizable wavefunctions for three anyons, two of which are newly found. The solutions are given in terms of generalized harmonics on $S^3$, with boundary conditions consistent with anyonic exchange property, self-adjointness of the Hamiltonian and no-infinite current across the boundary. The hard-core condition turns out to be an additional constraint on the harmonics, which is obeyed by new family of solutions at $\alpha=1/3$, 1. The analytic expressions for the whole solutions do not give a one-to-one correspondence mapping between the bosonic and fermionic spectra, whose way out is not seen at present.

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FIG. 1. Four types of interpolation between bosonic and fermionic states: Schematic diagram for Casimir number ($\mu$) v.s. statistical parameter ($\alpha$). Solid lines represent Type I and II, and dashed ones III and IV. Wiggly lines are extras which prevent smooth interpolation. Exponents on brackets denote degeneracy.