Constants of motion for the magnetic force: the angular momentum and the Laplace-Runge-Lenz vector

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It is well-known that an electric charge under a uniform magnetic field has a bidimensional motion if its initial position and velocity are perpendicular to this magnetic field. Although some constants of motion, as the energy and angular momentum, have been identified for this system, its features hide others. In this work, we build generalizations of the angular momentum and the Laplace-Runge-Lenz vector and show that these vectors are constants of motion. Moreover, from them, we find four dynamically independent conserved qualities.

PACS numbers: 52.20.Dq, 45.20.D-, 45.20.df

I. INTRODUCTION

An important approach to understanding and studying a classical mechanics system is to identify its constants of the motion (CsM). There are several methods to find them for a given system. The methods that rely on a systematic and direct calculation are often efficient in finding the CsM but leave the open question as whether there are other CsM.

The dynamics of a charged particle in uniform electric and magnetic field have been widely studied by diverse methods. Many CsM as the energy, angular momentum and pseudomomentum have been identified for this system. However, the LRLV has been overlooked as a constant of motion (CM) of the system.

The concept of the Laplace-Runge-Lenz vector (LRLV) was found for an isotropic harmonic oscillator with central potential \( \kappa r^2/2 \). The LRLV vector was also obtained for the classical motion of two charged particles, confined to two dimensions and embedded in a constant magnetic field, a component of the LRLV was obtained even though in this case \( \mathbf{L}_{\text{mech}} \) is not conserved and the magnetic force is not a central one. However, the LRLV has been overlooked as a constant of motion (CM) of the system comprised of a charged particle in uniform and perpendicular electric and magnetic fields.

The aim of this work is to study some of the CsM of a charged particle in uniform electric and magnetic perpendicular fields. Particularly, we introduce LRLV as a CM for such a system.

This work is organized as follows. In Sec. I we present the classical analysis of a charged particle in uniform electric and magnetic fields in order to introduce the main concepts and physical quantities. Here we also show that the angular momentum is a CM and, from it, we obtain the charged particle’s trajectory. In Sec. II we calculate the LRLV and show that it is a CM. By using the obtained LRLV we integrate charged particle’s orbit. By means of the Liouville theorem, we demonstrate that the angular momentum, the LRLV and the pseudomomentum are in fact CsM. Moreover, we prove that they are four dynamically independent conserved qualities. In Sec. IV we summarize.

II. THE LORENTZ FORCE AND THE NEWTON’S SECOND LAW. THE ORBIT EQUATION

In order to study the charged particle motion we introduce Lorentz force into Newton’s second law of motion

\[
\frac{d}{dt} \left( m \dot{\mathbf{R}} \right) = q \left( \mathbf{E} + \mathbf{R} \times \mathbf{B} \right)
\]

where \( m \) is the mass, \( q \) is the charge, \( \mathbf{R} \) is the particle’s position, and \( \mathbf{E} \) and \( \mathbf{B} \) are the uniform external electric and magnetic fields respectively. As a first step we move into the frame of the guiding center coordinates by defining \( \mathbf{R} = r + (E/B)tu \) where \( E/B \) is the drift velocity and \( u \) is a unitary vector perpendicular to \( \mathbf{E} \) and \( \mathbf{B} \) such that \( u = \mathbf{B} \times \mathbf{E}/(BE) \). In this frame Newton’s second law takes the form

\[
\frac{d}{dt} (m \dot{\mathbf{r}}) = q \dot{\mathbf{r}} \times \mathbf{B}.
\]

From the previous results it is clear that the electron orbits around the center guiding coordinates \( (E/B)tu \), and in turn moves at constant speed \( E/B \) in a direction \( \mathbf{u} \) perpendicular to the electric and magnetic fields.

Let us choose the polar coordinates \((r, \theta)\) to obtain the orbit equation. The equations of motion are:

\[
\ddot{r} - r \dot{\theta}^2 = r \dot{\theta} \omega, \quad 2 \dot{r} \dot{\theta} + r \ddot{\theta} = -\omega \dot{r},
\]

where \( \omega = qB/m \) is the cyclotron frequency. The last expression can be expressed as a total time derivative in the following form

\[
\frac{d}{dt} \left( m r^2 \dot{\theta} + \frac{1}{2} q^2 (B^2) \right) = 0,
\]
therefore, as is shown in the next section, the component parallel to $B$ of the angular momentum with respect to the guiding coordinates

$$L = mr^2\dot{\theta} + \frac{1}{2}qr^2B$$

is a CM\([3, 4]\).

By introducing the solution for $\dot{\theta}$ from Eq. (4) into (3), we obtain

$$\dot{r} - \frac{L^2}{m^2r^3} + \frac{1}{4}\omega^2r = 0,$$  \hspace{1cm} (6)

that corresponds to the motion equation of an isotropic harmonic oscillator. Thus, the particle’s orbit with respect to the guiding coordinate frame is (see Appendix A)

$$r^2 - 2rr_0\cos(\theta - \theta_0) + r_0^2 = \frac{2L}{m\omega} = a^2;$$  \hspace{1cm} (7)

which is the polar equation for a circle with radius $a$ centered in $(r_0 \cos \theta_0, r_0 \sin \theta_0)$.

III. THE ANGULAR MOMENTUM, THE LAPLACE-RUNGE-LENZ VECTOR AND THE PSEUDOMOMENTUM

Let us rewrite Newton’s second law of motion (2) for a charged particle in the guiding coordinate frame

$$\frac{d}{dt}(m\dot{r} + qA) = -\nabla(q\phi - q\dot{r} \cdot A) - qE$$  \hspace{1cm} (8)

where $\phi$ is the scalar potential and $A$ is the vector potential\([2, 4]\) that follow the usual relations $E = -\nabla\phi - \partial A/\partial t$ and $B = \nabla \times A$. Notice that the left side of the previous equation corresponds to the time variation of the minimal momentum

$$P = m\dot{r} + qA.$$  \hspace{1cm} (9)

Now we choose a gauge such that $\phi = -r \cdot E$ with uniform electric and magnetic fields. By taking a cross product of (8) with $r$ for the left side we have

$$\frac{d}{dt}[r \times (m\dot{r} + qA)] = q[r \times \nabla(\dot{r} \cdot A) + \dot{r} \times A]$$  \hspace{1cm} (10)

For an uniform and a constant magnetic field $B$ the vector potential in the Landau gauge is $A = (1/2)B \times r$.

By replacing the explicit form of the vector potential, we obtain

$$\frac{d}{dt}\left[m\dot{r} + \frac{1}{2}r^2qB\right] = \frac{q}{2}[r \times \nabla(\dot{r} \cdot B \times r) + \dot{r} \times (B \times r)].$$  \hspace{1cm} (11)

It is straightforward to show that the bottom side of this equation vanishes and therefore the angular momentum, $L = r \times P$, is a CM. The modulus of $L$ is given by\([5]\) in polar coordinates and it has the same direction as the magnetic field, while Cartesian coordinates it is given by

$$L = m(x\dot{y} - y\dot{x}) + \frac{1}{2}m\omega(x^2 + y^2).$$  \hspace{1cm} (12)

On the other hand, by taking the cross product of (2) with $L$ for the right side, we have

$$\frac{d}{dt}(m\dot{r}) \times L = q(\dot{r} \times B) \times L.$$  \hspace{1cm} (13)

By using the facts that $L$ is a CM, $(d/dt)(m\dot{r}) \times L = (d/dt)(m\dot{r} \times L)$ and that $L$ is perpendicular to $\dot{r}$ we obtain

$$(\dot{r} \times B) \times L = -(d/dt)(BLr).$$  \hspace{1cm} (14)

We thus obtain a vectorial CM given by

$$T = m\dot{r} \times L + qBLr.$$  \hspace{1cm} (15)

Given that we followed a similar method to the one used to calculate the LRLV in the Kepler problem, we name it the LRLV. It is perpendicular to the angular momentum and the magnetic field thus $L \cdot T = 0$ and $B \cdot T = 0$.

Since the LRLV is a CM, it can used as a basis to integrate the trajectory of the electron. As a first step we express $T$ in Cartesian coordinates as

$$T = T_x \hat{i} + T_y \hat{j},$$  \hspace{1cm} (16)

where

$$T_x = Lm(\dot{y} + \omega x)$$  \hspace{1cm} (17)

and

$$T_y = -Lm(\dot{x} - \omega y)$$  \hspace{1cm} (18)

are the Cartesian components of the LRLV. Second, we obtain the modulus of the LRLV as

$$T = L\sqrt{2m(E + L\omega)}.$$  \hspace{1cm} (19)

Finally, the equation for the particle’s trajectory is obtained by solving $\dot{x}$ and $\dot{y}$ from (17) and (18) respectively and substituting the result in (12), giving

$$\left(x - \frac{T_x}{Lm\omega}\right)^2 + \left(y - \frac{T_y}{Lm\omega}\right)^2 = \frac{2E}{m\omega^2}.$$  \hspace{1cm} (20)

This expression is the circle equation with center at $(T_x/Lm\omega, T_y/Lm\omega)$ and radius $a = \sqrt{2E/m\omega^2}$, where the mechanical energy is associated with the initial momentum $(E = (1/2m)p_x^2 + p_y^2)$. In polar coordinates we obtain an orbit equation which is identical to Eq. (7), if we set $\tan \theta_0 = T_y/T_x$ and $r_0 = T/m\omega L$. Notice that the LRLV is parallel to the center of the trajectory, in fact

$$r_0 = \frac{T}{m\omega L} = \left(\frac{\dot{y}}{\omega} + x\right)\hat{i} - \left(\frac{\dot{x}}{\omega} - y\right)\hat{j}.$$  \hspace{1cm} (21)
For a first guess as to the direction of the vector \( \mathbf{r}_0 \) it is
helpful to compute \( \mathbf{r}_0 \cdot \mathbf{L} \). Because of the orthogonality
of \( \mathbf{L} \) to both terms in the definition of \( \mathbf{r}_0 \) this dot product
vanishes. From this result it follows that \( \mathbf{r}_0 \) must lie in the particle’s orbit plane of motion. As we
calculated above, this interpretation of the LRLV implies that
\( \mathbf{r}_0 \) should be conserved because the position and geometry
of a bound orbit does not change over time and therefore it should depend on the initial conditions e.
g. the particle’s initial position and velocity. Let us set
\( T_x/Lm = \omega x_0 \) and \( T_y/Lm = \omega y_0 \) where \( x_0 = r_0 \cos \theta_0 \)
and \( y_0 = r_0 \sin \theta_0 \) and Eqs. (17) and (18) describe the
movement of the charge particle.

According to Ref. [3], the last conserved vector can be obtained by rewriting Eq. (2) as
\[
\frac{d}{dt} (m \dot{r} - q \mathbf{r} \times \mathbf{B}) = 0.
\]
In this way, we have a third conserved vector
\[
\mathbf{P}_s = m \dot{r} - q \mathbf{r} \times \mathbf{B},
\] (22)
this vector is not the minimal momentum. From now on, we name as the pseudomomentum. In Cartesian coordinates, this vector is given by
\[
\mathbf{P}_s = m (\dot{x} - \omega y) \mathbf{i} + m (\dot{y} + \omega x) \mathbf{j}.
\] (23)

Notice that \( \mathbf{P}_s \cdot \mathbf{B} = 0 \), \( \mathbf{P}_s \cdot \mathbf{L} = 0 \) and \( \mathbf{P}_s \cdot \mathbf{T} = 0 \), by
taking the cross product with \( \mathbf{r}_0 \), we get
\[
\mathbf{r}_0 \times \mathbf{P}_s = \frac{2}{\omega} (E + L \omega) \mathbf{k}
\] (24)
and its modulus is
\[
\mathbf{P}_s = \sqrt{2m(E + L \omega)}.
\] (25)

The LRLV is also proportional to \( E + L \omega \) as can be seen in Eq. [19], moreover \( \mathbf{T} = \mathbf{P}_s \mathbf{L} \). It can be shown that this quantity is a CM [3].

The Liouville Theorem is a well-known approach to test whether or not a quality is conserved. By defining the Poisson bracket in such way that
\[
\{r, p_r\} = 1
\] (26)
\[
\{\theta, p_\theta\} = 1
\] (27)
where \( p_r = m \dot{r} \) and \( p_\theta = L = m r^2 \left( \dot{\theta} + \omega/2 \right) \), we can find that the vectors \( \mathbf{L}, \mathbf{T} \) and \( \mathbf{P}_s \) are CsM. With those three vectors, the system has ten CsM: the energy, the three components of the angular momentum, the three components of the LRLV and the three components of the pseudomomentum. Because this system has six initial conditions, the three components of the position vector and the three components of the initial momentum, there must exist four relations that turn these ten dependent CsM into six independent ones, namely, there must be four relations connecting these qualities. Such relations are the orthogonality of \( \mathbf{B}, \mathbf{L}, \mathbf{T} \) and \( \mathbf{P}_s \), i.e. \( \mathbf{L} \cdot \mathbf{T} = 0 \), \( \mathbf{L} \cdot \mathbf{P}_s = 0 \), \( \mathbf{P}_s \cdot \mathbf{T} = 0 \), \( \mathbf{B} \cdot \mathbf{T} = 0 \) and \( \mathbf{B} \cdot \mathbf{P}_s = 0 \). The
first and second relations imply that \( \mathbf{L} \), Eq. (19), is a CM; \( \mathbf{P}_s \cdot \mathbf{T} = 0 \) indicates that \( E + L \omega \) is another CM; and, \( \mathbf{B} \cdot \mathbf{T} = 0 \) and \( \mathbf{B} \cdot \mathbf{P}_s = 0 \) show that \( \omega \) is also a CM. Finally, the four relations
\[
\omega = \frac{qB}{m}
\] (28)
\[
L = mr^2 \left( \dot{\theta} + \frac{\omega}{2} \right)
\] (29)
\[
E = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{L \omega}{2} + \frac{1}{8} m \omega^2 r^2
\] (30)
\[
\frac{T^2}{2mL^2} = E + L \omega.
\] (31)
are dynamically independent, because they are in involution, i.e. \( \{C_i, C_j\} = 0 \) where \( C_i = \omega, L, E \) or \( T^2/2mL^2 \).
Those relations, after some rearranging, give the nine dependent components of \( \mathbf{L}, \mathbf{T} \) and \( \mathbf{P}_s \) in terms of the cyclotron frequency, the energy, the angular momentum and \( T^2/2mL^2 \).

IV. CONCLUSIONS

We have studied the CsM of a charged particle in uniform electric and magnetic fields. Aside from the well known CsM as the energy [1–3], we found that a vector, obtained by similar means as the LRLV, is also a CM connected to the center of the particle’s orbit. The particle’s trajectory was integrated from it. Additionally we have proved that the cyclotron frequency \( \omega \), the angular momentum \( L \), the energy \( E \) and the LRLV \( T \) are four dynamically conserved qualities.

Appendix A: The particle’s trajectory

Here we obtain the particle’s trajectory by using the angular momentum and later on the LRLV. We start by expressing \( r \) as a function of \( \theta \) and performing the variable change \( r = u^{-2} \) in the differential equation (6)
\[
\frac{dr}{dt} = \frac{dr}{du} \frac{d\theta}{dt}
\]
\[
\frac{dr}{dt} = - \left( \frac{L}{m} \frac{d}{d\theta} u^2 + \frac{1}{2} \omega \frac{d}{d\theta} u^{-2} \right).
\]
Doing a new variable change
\[
g = \frac{L}{m} u^2 + \frac{1}{2} \omega u^{-2}
\]
we can obtain \( \dot{r} \)
\[
\dot{r} = \left( \frac{L}{m} u^4 - \frac{1}{2} \omega \right) \frac{dg}{dt^2}
\]
FIG. 1: Eq. $r^2 - 2rr_0 \cos(\theta - \theta_0) + 2L/m\omega = 0$ is plotted in the plane $xy$ and it describes a circle of radius $a = \sqrt{r_0^2 - 2L/m\omega}$ and centered at $(x_0, y_0)$ where $x_0 = r_0 \cos \theta_0$ and $y_0 = r_0 \sin \theta_0$.

On the right hand side of the previous equation, the two last terms can be factorized as

$$-\frac{L^2}{m^2}u^6 + \frac{1}{4}\omega u^{-2} = -\left(\frac{L}{m} u^4 - \frac{1}{2}\omega\right) g.$$ 

With the two previous expressions, we can write down Eq. (6) in the following form

$$\frac{d^2 g}{d\theta^2} + g = 0. \quad (A1)$$

The solution of the previous differential equation yields

$$r^2 - 2rr_0 \cos(\theta - \theta_0) + \frac{2L}{m\omega} = 0, \quad (A2)$$

In Cartesian coordinates, this expression can be rewritten as

$$(x - x_0)^2 + (y - y_0)^2 = a^2. \quad (A3)$$

It describes a circular trajectory with radius $a = \sqrt{r_0^2 - 2L/m\omega}$ and center at $(x_0, y_0)$ where $x_0 = r_0 \cos \theta_0$ and $y_0 = r_0 \sin \theta_0$ as shown in Fig. 1.

Similarly, by solving $\dot{r}$ from (30), using (31) and applying the previous variable change we obtain

$$\left(\frac{dg}{d\theta}\right)^2 = 2\frac{1}{m} (E + L\omega) - g^2 = \frac{T^2}{m^2L^2} - g^2,$$

Finally we integrate and obtain the orbit expressed in Eq. (A2).

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