The Euler-Maruyama Scheme for SDEs with Irregular Drift: Convergence Rates via Reduction to a Quadrature Problem.

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Abstract

We study the strong convergence order of the Euler-Maruyama scheme for scalar stochastic differential equations with additive noise and irregular drift. We provide a novel framework for the error analysis by reducing it to a weighted quadrature problem for irregular functions of Brownian motion. Assuming Sobolev-Slobodeckij-type regularity of order \( \kappa \in (0, 1) \) for the drift, our analysis of the quadrature problem yields the convergence order \( \min\{3/4, (1 + \kappa)/2\} - \epsilon \) for the equidistant Euler-Maruyama scheme (for arbitrarily small \( \epsilon > 0 \)). The cut-off of the convergence order at \( \kappa = 3/4 \) can be overcome by using a suitable non-equidistant discretization, which yields the strong convergence order of \( (1 + \kappa)/2 - \epsilon \) for the corresponding Euler-Maruyama scheme.

Keywords: stochastic differential equations, Euler-Maruyama scheme, strong convergence, quadrature problem, non-equidistant discretization, Sobolev-Slobodeckij regularity

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1 Introduction and Main Results

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}) \) be a filtered probability space, where the filtration satisfies the usual conditions and let \( W = (W_t)_{t \in [0,T]} \) be a standard Brownian motion adapted to \( (\mathcal{F}_t)_{t \in [0,T]} \). We consider Itô-stochastic differential equations (SDEs) of the form

\[
X_t = \xi + \int_0^t \mu(X_s) ds + W_t, \quad t \in [0,T],
\]

where \( T \in (0, \infty) \), the drift coefficient \( \mu: \mathbb{R} \to \mathbb{R} \) is measurable and bounded, and the initial condition \( \xi \) is independent of \( W \). Existence and uniqueness of a strong solution \( X = (X_t)_{t \in [0,T]} \) to \( (1) \) is provided, e.g., in [25].

For \( n \in \mathbb{N} \) let \( x^{(\pi_n)} = (x_t^{(\pi_n)})_{t \in [0,T]} \) be the continuous-time Euler-Maruyama (EM) scheme based on the discretization

\[
\pi_n = \{t_0, t_1, \ldots, t_n\} \quad \text{with} \quad 0 = t_0 < t_1 < \ldots < t_n = T,
\]

i.e.

\[
x_t^{(\pi_n)} = \xi + \int_0^t \mu(x_s^{(\pi_n)}) ds + W_t, \quad t \in [0,T],
\]

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where \( t = \max \{ t_k : t_k \leq t \} \). Our goal is to analyse the pointwise \( L^2 \)-approximation error at time \( T \), that is
\[
\left( \mathbb{E} \left[ |X_T - x_T^{(\pi_n)}|^2 \right] \right)^{1/2},
\]
and in particular its dependence on \( n \), i.e. the scheme’s convergence order.

The error analysis of EM-type schemes for SDEs with discontinuous drift coefficient has become – after two pioneering articles by Gyöngy [4] and Halidias and Kloeden [5] – a topic of growing interest in the recent years, see e.g. [9, 17, 10, 18, 19]. The best known results for Euler-Maruyama schemes for SDEs with discontinuous drift coefficients and non-additive noise are

(i) \( L^2 \)-order \( 1/4 - \varepsilon \) for arbitrarily small \( \varepsilon > 0 \) of the EM scheme for multi-dimensional SDEs with piecewise Lipschitz and bounded drift and bounded, possibly degenerate diffusion coefficient, see Leobacher and Szölgyenyi [11];

(ii) \( L^2 \)-order \( 1/2 - \varepsilon \) for arbitrarily small \( \varepsilon > 0 \) of an EM scheme with adaptive time-stepping for multi-dimensional SDEs with piecewise Lipschitz drift and possibly degenerate diffusion coefficient, see Neuenkirch et al. [15];

(iii) \( L^p \)-order \( 1/2 \) of the EM scheme for scalar SDEs with piecewise Lipschitz drift and possibly degenerate diffusion coefficient, see Müller-Gronbach and Yaroslavtseva [13].

Recently, also a transformation-based Milstein-type scheme has been analyzed for scalar SDEs by Müller-Gronbach and Yaroslavtseva [14]. They obtain \( L^p \)-order \( 3/4 \) for drift coefficients, which are piecewise Lipschitz with piecewise Lipschitz derivative, and possibly degenerate diffusion coefficient.

Lower error bounds for the strong approximation of scalar SDEs with possibly irregular coefficients have been studied in Hefter et al. [6]. Assuming smoothness of the coefficients only locally in a small neighbourhood of the initial value, the authors obtain for arbitrary methods that use a finite number of evaluations of the driving Brownian motion a lower error bound of order one for the pointwise \( L^1 \)-error. Lower bounds will be also addressed in a forthcoming work by Müller-Gronbach and Yaroslavtseva.

We will provide a general framework for the analysis of the scheme (2) for the SDE (1) under the following assumptions:

**Assumption 1.1.** Assume that \( \mu : \mathbb{R} \to \mathbb{R} \) with \( \mu \neq 0 \) can be decomposed into a regular and an irregular part \( a, b : \mathbb{R} \to \mathbb{R} \), that is \( \mu = a + b \), such that:

(i) (boundedness) \( a, b : \mathbb{R} \to \mathbb{R} \) are bounded,

(ii) (regular part) \( a \in C^2_b(\mathbb{R}) \), i.e. \( a \) is twice continuously differentiable with bounded derivatives,

(iii) (irregular part) \( b \in L^1(\mathbb{R}) \).

Moreover, we assume that

(iv) (initial value) \( \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}) \).

**Assumption 1.2.** There exists \( \kappa \in (0, 1) \) such that
\[
|b|_{\kappa} := \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|b(x) - b(y)|^2}{|x - y|^{2\kappa + 1}} \, dx \, dy \right)^{1/2} < \infty.
\]

\(^1\)According to a remark in [15] the boundedness of the coefficients is not necessary.
We call \( | \cdot |_\kappa \) Sobolev-Slobodeckij semi-norm. Note that the decomposition of \( \mu \) is only required for the error analysis and not for the actual implementation of the scheme.

Assumption 1.1 is required for our perturbation analysis, where we use a suitable transformation of the state space and a Girsanov transform to show that for all \( \varepsilon \in (0, 1) \) there exists a constant \( C^{(R)}_{\varepsilon,a,b,T} > 0 \) such that

\[
\mathbb{E} \left[ \left| X_T - x_T^{(\pi_n)} \right|^2 \right] \leq C^{(R)}_{\varepsilon,a,b,T} \cdot \left( \| \pi_n \|^2 + | W^{(\pi_n)} |^{1 - \varepsilon} \right),
\]

where

\[
\| \pi_n \| := \max_{k=0,\ldots,n-1} | t_{k+1} - t_k |
\]

and

\[
W^{(\pi_n)} = \mathbb{E} \left[ \left| \int_0^T \exp \left( -2 \int_0^W b(z + \xi) dz \right) \left[ b(W_s + \xi) - b(W_s) \right] ds \right|^2 \right],
\]

see Theorem 2.4. The term \( W^{(\pi_n)} \) corresponds to a quadrature problem, see Remark 2.5.

We would like to point out that

- this provides a novel framework for the error analysis of numerical schemes for SDEs,
- which can be used to analyse the convergence behaviour of the Euler-Maruyama scheme under very general assumptions on the drift coefficient by various means for various discretizations.

We assume Sobolev-Slobodeckij regularity of order \( \kappa \in (0, 1) \) for \( b \), i.e. Assumption 1.2 and estimate \( W^{(\pi_n)} \) for two different discretizations. For an equidistant discretization \( \pi_{n}^{\text{equi}} \) given by

\[
t_k^{\text{equi}} = T \frac{k}{n}, \quad k = 0, \ldots, n,
\]

we obtain that \( W^{(\pi_n)} \) is of order \( \min \{ 3/2, 1 + \kappa \} \) and consequently we have

\[
\left( \mathbb{E} \left[ \left| X_T - x_T^{(\pi_n^{\text{equi}})} \right|^2 \right] \right)^{1/2} \leq C^{(EM),\text{equi}}_{\varepsilon,\mu,T,\kappa} \cdot \left[ \frac{1}{n^{(1+\kappa)/2 - \varepsilon}} + \frac{1}{n^{3/4 - \varepsilon}} \right] \quad (3)
\]

for \( \varepsilon > 0 \) arbitrarily small and a constant \( C^{(EM),\text{equi}}_{\varepsilon,\mu,T,\kappa} > 0 \), independent of \( n \), see Theorem 3.7 and Corollary 3.9. To overcome the cut-off of the convergence order for \( \kappa = 3/4 \), we use a non-equidistant discretization \( \pi_n^{*} \) given by

\[
t_k^{*} = T \left( \frac{k}{n} \right)^2, \quad k = 0, \ldots, n.
\]

Similar non-equidistant nets have been used, e.g., in [12] to deal with weak error estimates for non-smooth functionals and in [3] to deal with hedging errors in the presence of non-smooth pay-offs. We obtain that \( W^{(\pi_n^{*})} \) is up to a log-term of order \( 1 + \kappa \) and therefore we have

\[
\left( \mathbb{E} \left[ \left| X_T - x_T^{(\pi_n^{*})} \right|^2 \right] \right)^{1/2} \leq C^{(EM),*}_{\varepsilon,\mu,T,\kappa} \cdot \frac{1}{n^{(1+\kappa)/2 - \varepsilon}} \quad (4)
\]

for \( \varepsilon > 0 \) arbitrarily small and a constant \( C^{(EM),*}_{\varepsilon,\mu,T,\kappa} > 0 \), independent of \( n \), see Theorem 3.7 and Corollary 3.9.
Remark 1.1. (i) Our set-up covers a wide range of irregular perturbations. It is well known that indicator functions of bounded intervals have Sobolev-Slobodeckij regularity of all order $\kappa < 1/2$. Also functions, which are $\kappa$-Hölder continuous and have compact support, have Sobolev-Slobodeckij regularity of order $\kappa$.

(ii) Our assumptions also cover step functions as drift, i.e.

$$\mu(x) = \sum_{\ell=1}^{L} \gamma_{\ell} \cdot \text{sign}(x - x_{\ell}), \quad x \in \mathbb{R},$$

with $L \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_L \in \mathbb{R}$, and $-\infty < x_1 < x_2 < \ldots < x_L < \infty$. This can be seen from the following: let $\mu(x) = \text{sign}(x)$ and $\alpha \in (0, \infty)$. Then the decomposition $a_{\alpha}, b_{\alpha} : \mathbb{R} \to \mathbb{R}$, $\mu(x) = a_{\alpha}(x) + b_{\alpha}(x)$, which satisfies Assumption 1.1 and Assumption 1.2 for all $\kappa < 1/2$ and all $\alpha \in (0, \infty)$, can be chosen as

$$a_{\alpha}(x) = \begin{cases} 1, & x \in (\alpha, \infty), \\ \frac{2}{\alpha} \int_{\alpha}^{x} (2\alpha - y)^2 (y - \alpha)^2 dy - 1, & x \in (-\alpha, \alpha), \\ -1, & x \in (-\infty, -\alpha), \end{cases}$$

and $b_{\alpha}(x) = 1_{(0,\alpha)}(x) \cdot (1 - a_{\alpha}(x)) + 1_{(-\alpha,0)}(x) \cdot (1 - a_{\alpha}(x))$. Figure 1 illustrates this decomposition. Recall that such a decomposition of $\mu$ is only required for the error analysis and not for the actual implementation of the scheme.

(iii) In particular for bounded $C_{b}^{2} (\mathbb{R})$-drift coefficients, which are perturbed by a step function (5), we obtain convergence order $3/4 - \varepsilon$ for all $\varepsilon > 0$, similar to the transformation-based Milstein-type method in Müller-Gronbach and Yaroslavtseva [14]. Moreover, for Lipschitz-continuous drift coefficients with bounded support we obtain convergence order $1 - \varepsilon$ for all $\varepsilon > 0$, similar to the drift-randomized Milstein-type scheme analyzed in Kruse and Wu [8] under structurally different assumptions on the coefficient.

Remark 1.2. Lamperti’s transformation, i.e.

$$\lambda : \mathbb{R} \to \mathbb{R}, \quad \lambda(x) = \int_{x_0}^{x} \frac{1}{\sigma(z)} \, dz,$$

with $x_0 \in \mathbb{R}$, reduces general scalar SDEs

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad t \in [0, T], \quad X_0 = x_0,$$
with sufficiently smooth elliptic diffusion coefficient $\sigma: \mathbb{R} \to \mathbb{R}$ to SDEs of the form
\[ dY_t = g(Y_t)dt + dW_t, \quad t \in [0, T], \quad Y_0 = \lambda(x_0), \]
with additive noise, where
\[ g(x) = \frac{\mu(\lambda^{-1}(x))}{\sigma(\lambda^{-1}(x))} - \frac{1}{2}(\sigma\sigma')(\lambda^{-1}(x)), \quad x \in \mathbb{R}, \]
and $X(t) = \lambda^{-1}(Y(t)), t \in [0, T]$. If $\mu$ satisfies Assumptions 1.1 and 1.2 and if $\sigma$ is twice continuously differentiable with bounded derivatives and
\[ 0 < \inf_{x \in \mathbb{R}} \sigma(x) \leq \sup_{x \in \mathbb{R}} \sigma(x) < \infty, \]
then $g$ satisfies Assumptions 1.1 and 1.2. So, if $\lambda$, $\lambda^{-1}$ and $g$ are explicitly known, then $X_T$ can be approximated by $\lambda^{-1}(Y_T^{(\pi_n)})$ and the error bounds (3) and (4) carry over.

2 Reduction to a quadrature problem for irregular functions of Brownian motion

In this section we will relate the analysis of the pointwise $L^2$-error of the Euler-Maruyama scheme to a quadrature problem which will be simpler to analyse.

In the whole paper we will denote the expectation w.r.t. $\mathbb{P}$ by $\mathbb{E}$, the expectation w.r.t. any other measure $\mathbb{Q}$ by $\mathbb{E}_\mathbb{Q}$, and the Lipschitz constant of a Lipschitz continuous function $f$ by $L_f$. For notational simplicity we will drop the superscript $(\pi_n)$, wherever possible.

2.1 Notation and preliminaries

First, we introduce a transformation $\varphi$ of the state space, which allows us to deal with the irregular part $b$ of the drift coefficient of SDE (1).

**Lemma 2.1.** Let Assumption 1.1 hold. Let $\varphi: \mathbb{R} \to \mathbb{R}$ be defined by
\[ \varphi(x) = \int_0^x \exp\left(-2 \int_0^y b(z)dz\right)dy, \quad x \in \mathbb{R}. \]
Then

(i) the map $\varphi$ is differentiable with bounded derivative $\varphi'$, which is absolutely continuous with bounded Lebesgue density $\varphi''': \mathbb{R} \to \mathbb{R}$;

(ii) the map $\varphi$ is invertible with $\varphi^{-1} \in C^1_b(\mathbb{R})$;

(iii) the maps $\varphi' \circ \varphi^{-1}: \mathbb{R} \to \mathbb{R}$ and $(\varphi' a) \circ \varphi^{-1}: \mathbb{R} \to \mathbb{R}$ are globally Lipschitz.

**Proof.** First note that $b$ is bounded. So, by construction and the fundamental theorem of Lebesgue-integral calculus we have
\[ \varphi'(x) = \exp\left(-2 \int_0^x b(z)dz\right), \quad \varphi''(x) = -2b(x)\varphi'(x), \quad x \in \mathbb{R}. \]

Since by assumption $b \in L^1(\mathbb{R})$, we have that
\[ \exp(-2\|b\|_{L^1}) \leq \varphi'(x) \leq \exp(2\|b\|_{L^1}), \quad x \in \mathbb{R}, \tag{6} \]
which shows item [ii]. The last equation also implies that \( \varphi \) is invertible. Moreover, we have
\[
(\varphi^{-1})'(y) = \frac{1}{\varphi'(\varphi^{-1})(y)},
\]
so [\(6\)] implies that \( \varphi^{-1} \in C_1^1(\mathbb{R}) \). This proves item [iii]. The Lipschitz property of \( \varphi' \circ \varphi^{-1} \) and \( (\varphi')' \circ \varphi^{-1} \) follows from the boundedness of \( a, \varphi' \) and the Lipschitz property of \( \varphi', \varphi^{-1} \), and \( a \). This proves item [iii].

The previous lemma implies in particular that \( \varphi \) is twice differentiable almost everywhere and solves
\[
b(x)\varphi'(x) + \frac{1}{2} \varphi''(x) = 0 \quad \text{for almost all } x \in \mathbb{R}. \tag{7}
\]

A similar transformation was introduced by Zvonkin in [26] and the use of such techniques for the numerical analysis of SDEs goes back until [25].

Now, define the transformed process \( Y = (Y_t)_{t \in [0,T]} \) as \( Y_t = \varphi(X_t) \). By Itô’s formula, \( \mu = a + b \), and [7] we have
\[
Y_t = \varphi(\xi) + \int_0^t \varphi'(X_s) a(X_s) \, ds + \int_0^t \varphi'(X_s) \, dW_s, \quad t \in [0,T].
\]
Moreover, define the transformed EM scheme \( y = (y_t)_{t \in [0,T]} \) as \( y_t = \varphi(x_t) \). Itô’s formula, [2], \( \mu = a + b \), and [7] give
\[
y_t = \varphi(\xi) + \int_0^t \left( \varphi'(x_s)(a + b)(x_s) + \frac{1}{2} \varphi''(x_s) \right) \, ds + \int_0^t \varphi'(x_s) \, dW_s
\]
\[
= \varphi(\xi) + \int_0^t \varphi'(x_s) ((a + b)(x_s) - (a + b)(\xi)) \, ds
\]
\[
+ \int_0^t \varphi'(x_s) a(x_s) \, ds + \int_0^t \varphi'(x_s) \, dW_s, \quad t \in [0,T].
\]

Next, let \( L^{(\nu)}_T = \frac{dQ}{d\mathcal{Q}} \) be the Radon-Nikodym derivative for which \( x^{(\nu)} - \xi = (x_t^{(\nu)} - \xi)_{t \in [0,T]} \) is a Brownian motion under \( \mathcal{Q} \), that is
\[
L^{(\nu)}_T = \exp \left( -\int_0^T \mu(x_2^{(\nu)}) \, dW_s - \frac{1}{2} \int_0^T \mu^2(x_2^{(\nu)}) \, ds \right).
\]

We will require the following moment bound:

**Lemma 2.2.** Let Assumption [1] hold. For all \( \varepsilon > 0 \) there exists a constant \( c^{(L)}_{\mu,T,\varepsilon} > 0 \) such that
\[
\mathbb{E}_Q \left[ \left| L^{(\nu)}_T \right|^{-\frac{1}{\varepsilon}} \right] \leq c^{(L)}_{\mu,T,\varepsilon}.
\]

**Proof.** First, note that
\[
\mathbb{E}_Q \left[ \left| L^{(\nu)}_T \right|^{-\frac{1}{\varepsilon}} \right] = \mathbb{E} \left[ \left| L^{(\nu)}_T \right|^{-\frac{1}{\varepsilon}} \right]
\]
\[
= \mathbb{E} \left[ \exp \left( \frac{\varepsilon - 1}{\varepsilon} \left[ -\int_0^T \mu(x_2) \, dW_s - \frac{1}{2} \int_0^T \mu^2(x_2) \, ds \right] \right) \right]
\]
\[
= \mathbb{E} \left[ \exp \left( \frac{1 - \varepsilon}{\varepsilon} \left[ \int_0^T \mu(x_2) \, dW_s + \frac{1}{2} \int_0^T \mu^2(x_2) \, ds \right] \right) \right]
\]
\[
\leq \exp \left( \frac{1 - \varepsilon}{2\varepsilon} T \| \mu \|_\infty^2 \right) \mathbb{E} \left[ \exp \left( \frac{1 - \varepsilon}{\varepsilon} \int_0^T \mu(x_2) \, dW_s \right) \right].
\]
Itô-integrals with bounded integrands have Gaussian tails, i.e.,
\[
\mathbb{P} \left( \left| \int_0^T \mu(x_s) dW_s \right| \geq \delta \right) \leq 2 \exp \left( -\frac{\delta^2}{2||\mu||_\infty^2} \right), \quad \delta > 0,
\]
see, e.g., \cite{20} (A.5) in Appendix A.2. Since positive random variables \(Z\) satisfy
\[
\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z \geq z) \, dz,
\]
it follows that
\[
\mathbb{E} \left[ \exp \left( \frac{1 - \varepsilon}{\varepsilon} \int_0^T \mu(x_s) dW_s \right) \right]
\leq \mathbb{E} \left[ \exp \left( \frac{1 - \varepsilon}{\varepsilon} \left| \int_0^T \mu(x_s) dW_s \right| \right) \right]
= \int_0^\infty \mathbb{P} \left( \left| \int_0^T \mu(x_s) dW_s \right| \geq z \right) \, dz
\leq 1 + \int_1^\infty \mathbb{P} \left( \int_0^T \mu(x_s) dW_s \geq \log(z) \left| \frac{\varepsilon}{1 - \varepsilon} \right| \right) \, dz
\leq 1 + 2 \int_0^\infty \exp \left( -\frac{(\log(z))^2 \varepsilon^2}{2(1 - \varepsilon)^2 ||\mu||_\infty^2} \right) \, dz
= 1 + 2 \int_{-\infty}^\infty \exp \left( \delta - \frac{\delta^2 \varepsilon^2}{2(1 - \varepsilon)^2 ||\mu||_\infty^2} \right) \, d\delta
\leq 1 + \frac{\sqrt{8\pi(1 - \varepsilon)||\mu||_\infty^2}}{\varepsilon} \exp \left( \frac{(1 - \varepsilon)^2 ||\mu||_\infty^2}{2\varepsilon^2} \right) < \infty,
\]
where the last step follows, e.g., from the moment generating function for a centred Gaussian variable with variance \(\frac{(1 - \varepsilon)^2 ||\mu||_\infty^2}{\varepsilon^2}\).

Finally, we establish a technical, but straightforward estimate of weighted sums of iterated (Itô)-integrals.

**Lemma 2.3.** Let \(\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}\) be bounded and measurable functions. Then
\[
\mathbb{E} \left[ \left| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi_1(x_{t_k}) \int_{t_k}^s \psi_2(x_u) \, dW_u \, ds \right|^2 \right] \leq \frac{T}{2} \|\psi_1\|_\infty^2 \|\psi_2\|_\infty^2 \cdot ||\pi_n||^2.
\]

**Proof.** Since \(\psi_1(x_s)\) is \(\mathcal{F}_s\)-measurable for \(\tau \in [0, T]\) we have
\[
\mathbb{E} \left[ \psi_1(x_s) \int_s^T \psi_2(x_u) \, dW_u \, \psi_1(x_t) \int_t^s \psi_2(x_v) \, dW_v \right]
= \mathbb{E} \left[ \int_s^T \psi_1(x_s) \psi_2(x_u) \, dW_u \int_t^s \psi_1(x_t) \psi_2(x_v) \, dW_v \right], \quad s, t \in [0, T].
\]
Assume now that \(t \geq s\). Conditioning on \(\mathcal{F}_s\) yields that
\[
\mathbb{E} \left[ \psi_1(x_s) \int_s^T \psi_2(x_u) \, dW_u \, \psi_1(x_t) \int_t^s \psi_2(x_v) \, dW_v \right]
= \mathbb{E} \left[ \int_s^T \psi_1(x_s) \psi_2(x_u) \, dW_u \, \mathbb{E} \left[ \int_t^s \psi_1(x_t) \psi_2(x_v) \, dW_v \bigg| \mathcal{F}_s \right] \right] = 0.
\]
since \( \int_x^s \psi_1(x_u) \psi_2(x_u) dW_u \) is \( F_x \)-measurable and

\[
E \left[ \int_t^s \psi_1(x_u) \psi_2(x_v) dW_u \mid F_t \right] = 0.
\]

Therefore, we have

\[
E \left[ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi_1(x_k) \int_{t_k}^s \psi_2(x_u) dW_u ds \right]^2
= \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} E \left[ \left( \int_{t_k}^{t_{k+1}} \psi_1(x_k) \int_{t_k}^s \psi_2(x_u) dW_u ds \right) \left( \int_{t_\ell}^{t_{\ell+1}} \psi_1(x_\ell) \int_{t_\ell}^t \psi_2(x_v) dW_v dt \right) \right]
= \sum_{k=0}^{n-1} E \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^s \psi_1(x_k) \psi_2(x_u) dW_u ds \right]^2.
\]

Applying the Cauchy-Schwartz inequality and using the Itô-isometry and the boundedness of \( \psi_1, \psi_2 \) yields

\[
\sum_{k=0}^{n-1} E \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^s \psi_1(x_k) \psi_2(x_u) dW_u ds \right]^2
\leq \| \pi_n \| \| \psi_1 \|_\infty \| \psi_2 \|_\infty \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (s - t_k) ds
\leq \frac{1}{2} \| \pi_n \| \| \psi_1 \|_\infty^2 \| \psi_2 \|_\infty^2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2
\leq \frac{T}{2} \| \pi_n \| \| \psi_1 \|_\infty^2 \| \psi_2 \|_\infty^2.
\]

\(\square\)

### 2.2 Reduction to a quadrature problem

Now we relate the error of the Euler-Maruyama scheme \( x^{(\pi_n)}(t) = (x^{(\pi_n)}(t))_{t \in [0,T]} \) to the error of a weighted quadrature problem.

**Theorem 2.4.** Let Assumption 1.1 hold. Then, for all \( \varepsilon \in (0,1) \) there exists a constant \( C_{\varepsilon,a,b,T}^\infty > 0 \) such that

\[
E \left[ \| X_T - x_T^{(\pi_n)} \|^2 \right] \leq C_{\varepsilon,a,b,T}^\infty \left( \| \pi_n \|^2 + \left( W^{(\pi_n)} \right)^{1-\varepsilon} \right),
\]

where

\[
W^{(\pi_n)} = E \left[ \left| \int_0^T \varphi'(W_s + \xi) (b(W_s + \xi) - b(W_s + \xi)) ds \right|^2 \right].
\]

**Proof.** *Step 1.* First note that by Lemma 2.1 we have

\[
E \left[ |X_t - x_t|^2 \right] = E \left[ \varphi^{-1}(Y_t) - \varphi^{-1}(y_t) \right]^2 \leq L_{\varphi^{-1}}^2 E \left[ |Y_t - y_t|^2 \right], \quad t \in [0,T].
\]
Furthermore, we have for all $t \in [0, T]$ that
\[
Y_t - y_t = E_t + \int_0^t ((\varphi'(a)(\varphi^{-1}(Y_s)) - (\varphi'(a)(\varphi^{-1}(y_s))) \, ds
\]
\[
+ \int_0^t (\varphi'(\varphi^{-1}(Y_s)) - \varphi'(\varphi^{-1}(y_s))) \, dW_s,
\]
where
\[
E_t = \int_0^t \varphi'(x_s) ((a + b)(x_s) - (a + b)(x_2)) \, ds.
\]

Applying the representation \(\text{(9)}\), the Cauchy-Schwartz inequality, the Itô-isometry, and Lemma \(2.1\) we obtain for all $t \in [0, T]$ that
\[
E \left[ |Y_t - y_t|^2 \right] \leq 3E\left[ |E_t|^2 \right] + 3E \left[ \int_0^t \left((\varphi'(a)(\varphi^{-1}(Y_s)) - (\varphi'(a)(\varphi^{-1}(y_s))) \, ds \right]^2 \right]
\]
\[
+ 3E \left[ \int_0^t \left((\varphi'(\varphi^{-1}(Y_s)) - \varphi'(\varphi^{-1}(y_s))) \, dW_s \right]^2 \right]
\]
\[
\leq 3E \left[ |E_t|^2 \right] + 3 \left(T L^2 \varphi^{-1} \right) \int_0^t E \left[ |Y_s - y_s|^2 \right] \, ds.
\]
This estimate, Gronwall’s lemma, and \(\text{(8)}\) establish that there exists a constant $c_{a,b,T}^{(1)} > 0$ such that
\[
E \left[ |X_t - x_t|^2 \right] \leq c_{a,b,T}^{(1)} E \left[ |E_t|^2 \right], \quad t \in [0, T].
\]

Clearly, we have that
\[
E \left[ |E_T|^2 \right] = E \left[ \int_0^T \varphi'(x_s) ((a + b)(x_s) - (a + b)(x_2)) \, ds \right]^2 \leq 3(E_1 + E_2 + E_3),
\]
where
\[
E_1 = E \left[ \int_0^T \varphi'(x_s) (a(x_s) - a(x_2)) \, ds \right]^2,
\]
\[
E_2 = E \left[ \int_0^T (\varphi'(x_s) - \varphi'(x_2)) (a(x_s) - a(x_2)) \, ds \right]^2,
\]
\[
E_3 = E \left[ \int_0^T \varphi'(x_s) ((b(x_s) - b(x_2)) \, ds \right]^2.
\]

We will first deal with $E_1$ and $E_2$ using standard tools, then we will rewrite $E_3$ using a Girsanov transform.

**Step 2.** For estimating $E_1$ and $E_2$ note that for all $s \in [0, T]$ we have
\[
E[|x_s - x_2|^4] = E\left[ \int_2^s (a + b)(x_t) \, dt + (W_s - W_2)^4 \right]
\]
\[
\leq 8E\left[ \int_2^s |(a + b)(x_t)|^4 \right] + 8E[|W_s - W_2|^4]
\]
\[
\leq 8(s - s)^4E \sup_{t \in [0, T]} |(a + b)(x_t)|^4 + 24(s - s)^2
\]
\[
\leq 8\|a + b\|_\infty^4 \|\pi_n\|^4 + 24\|\pi_n\|^2.
\]
To estimate $\mathcal{E}_2$ we apply the Cauchy-Schwarz inequality and $2xy \leq x^2 + y^2$ to obtain that
\[\mathcal{E}_2 \leq T \int_0^T \mathbb{E} \left[ |\varphi'(x_s) - \varphi'(x_2)|^2 |a(x_s) - a(x_2)|^2 \right] ds \]
\[\leq \frac{T}{2} \int_0^T \mathbb{E} \left[ |\varphi'(x_s) - \varphi'(x_2)|^4 + |a(x_s) - a(x_2)|^4 \right] ds.\]

Since $\varphi'$ and $a$ are globally Lipschitz, (12) yields
\[\mathcal{E}_2 \leq \frac{T}{2} \left( L_0^4 + L_0^4 \right) \int_0^T \mathbb{E} \left[ |x_s - x_2|^4 \right] ds \]
\[\leq \frac{T^2}{2} \left( L_0^4 + L_0^4 \right) (8\|a + b\|_{\infty}^4 \|\pi_{\pi}\|^4 + 24\|\pi_{\pi}\|^2). \tag{13}\]

Recall that $a \in C^2_0(\mathbb{R})$. So, Itô's formula yields
\[\mathcal{E}_1 = \int_0^T \varphi'(x_2) \left( a(x_s) - a(x_2) \right) ds \]
\[= \int_0^T \varphi'(x_2) \left( \int_{\frac{s}{2}}^s \left( a'(x_u)(a + b)(x_2) + \frac{1}{2}a''(x_u) \right) du + \int_{\frac{s}{2}}^s a'(x_u)dW_u \right) ds.\]

Hence, we have
\[\mathcal{E}_1 \leq 2 \mathbb{E} \left[ \int_0^T \varphi'(x_2) \int_{\frac{s}{2}}^s \left( a'(x_u)(a + b)(x_2) + \frac{1}{2}a''(x_u) \right) du ds \right]^2 \]
\[+ 2 \mathbb{E} \left[ \int_0^T \varphi'(x_2) \int_{\frac{s}{2}}^s a'(x_u)dW_u ds \right]^2 \]
\[\leq 2 \mathbb{E} \left[ \int_0^T \int_{\frac{s}{2}}^s |\varphi'(x_2)| \left| a'(x_u)(a + b)(x_2) + \frac{1}{2}a''(x_u) \right| du ds \right]^2 \]
\[+ 2 \mathbb{E} \left[ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \varphi'(x_{t_k}) \int_{t_k}^s a'(x_u)dW_u ds \right]^2. \tag{14}\]

Using that $a, b, a', a'', \varphi'$ are bounded, gives
\[\sup_{u,s \in [0,T]} |\varphi'(x_2)| \left| a'(x_u)(a + b)(x_2) + \frac{1}{2}a''(x_u) \right| \leq \|\varphi'\|_{\infty} \left( \|a'\|_{\infty} \|a + b\|_{\infty} + \frac{1}{2}\|a''\|_{\infty} \right). \]

So we obtain
\[\mathbb{E} \left[ \int_0^T \int_{\frac{s}{2}}^s \varphi'(x_2) \left( a'(x_u)(a + b)(x_2) + \frac{1}{2}a''(x_u) \right) du ds \right]^2 \]
\[\leq T^2 \|\varphi'\|_{\infty}^2 \left( \|a'\|_{\infty} \|a + b\|_{\infty} + \frac{1}{2}\|a''\|_{\infty} \right)^2 \|\pi_{\pi}\|^2. \tag{15}\]

Combining (14) with (15) and applying Lemma 2.3 to the second summand of (14) yield
\[\mathcal{E}_1 \leq 2T^2 \|\varphi'\|_{\infty}^2 \left( \|a'\|_{\infty} \|a + b\|_{\infty} + \frac{1}{2}\|a''\|_{\infty} \right)^2 \|\pi_{\pi}\|^2 + T\|a'\|_{\infty}^2 \|\varphi'\|_{\infty}^2 \|\pi_{\pi}\|^2. \tag{16}\]

Thus, (16) and (13) imply that there exists a constant $c^{(2)}_{a,b,T} > 0$ such that
\[\mathcal{E}_1 + \mathcal{E}_2 \leq c^{(2)}_{a,b,T} \|\pi_{\pi}\|^2. \tag{17}\]
So, combining (10), (11), and (17), we obtain that there exists a constant \( c_{a,b,T}^{(3)} > 0 \) such that
\[
\mathbb{E}[|X_T - x_T|^2] \leq c_{a,b,T}^{(3)} (\| \pi_n \|^2 + \mathcal{E}_3).
\] (18)

**Step 3:** Now we use the Girsanov-transform with density \( L_T \). For \( \varepsilon \in (0,1) \), Hölder’s inequality and Lemma 2.2 yield
\[
\mathcal{E}_3 = \mathbb{E}_Q \left[ L_T^{-1} \left| \int_0^T \varphi'(x_s) (b(x_s) - b(x_s)) \, ds \right|^2 \right]
\leq \left( \mathbb{E}_Q \left[ |L_T|^{-\frac{1}{2}} \right] \right)^{2\varepsilon} \left( \mathbb{E}_Q \left[ \left| \int_0^T \varphi'(x_s) (b(x_s) - b(x_s)) \, ds \right|^2 \right] \right)^{1-2\varepsilon}
\leq c_{\mu,T,\varepsilon}^{(L)} \left( \mathbb{E} \left[ \left| \int_0^T \varphi'(W_s + \xi) (b(W_s + \xi) - b(W_s + \xi)) \, ds \right|^2 \right] \right)^{1-\varepsilon}.
\]

Since
\[
\left| \int_0^T \varphi'(W_s + \xi) (b(W_s + \xi) - b(W_s + \xi)) \, ds \right|^2 = \left| \int_0^T \varphi'(W_s + \xi) (b(W_s + \xi) - b(W_s + \xi)) \, ds \right|^{2\varepsilon} \left| \int_0^T \varphi'(W_s + \xi) (b(W_s + \xi) - b(W_s + \xi)) \, ds \right|^{2(1-\varepsilon)}
\leq (2T\|\varphi'\|_\infty)^{2\varepsilon} \int_0^T \varphi'(W_s + \xi) (b(W_s + \xi) - b(W_s + \xi)) \, ds \right|^2,
\]
we obtain that
\[
\mathcal{E}_3 \leq c_{\mu,T,\varepsilon}^{(L)} (2T\|\varphi'\|_\infty)^{2\varepsilon} (\mathcal{W}(\pi_n))^{1-\varepsilon}. \tag{19}
\]
Combining (18) and (19) proves the theorem.

**Remark 2.5.** The term \( \mathcal{W}(\pi_n) \) corresponds to the mean-square error of a weighted quadrature problem, namely the prediction of
\[
I = \int_0^T \mathcal{Y}_s Z_s \, ds
\]
by the quadrature rule
\[
I(\pi_n) = \sum_{k=0}^{n-1} Z_{t_k} \int_{t_k}^{t_{k+1}} \mathcal{Y}_s \, ds,
\]
where
\[
\mathcal{Y}_t = \varphi'(W_t + \xi) = \exp \left( -2 \int_0^W b(z + \xi) \, dz \right), \quad t \in [0, T],
\]
is a random weight function, and the process \( Z \) given by
\[
Z_t = b(W_t + \xi), \quad t \in [0, T],
\]
is evaluated at \( t_0, \ldots, t_{n-1} \). Related unweighted integration problems, i.e. with \( \mathcal{Y} = 1 \) and \( Z \) given by irregular functions of stochastic processes such as (fractional) Brownian motion, SDE solutions, or general Markov processes, have recently been studied in [16, 7, 1]. The study
3 Analysis of the quadrature problem

For the analysis of
\[ W(\pi_n) = \mathbb{E} \left[ \left| \int_0^T \varphi'(W_s + \xi) \left( b(W_s + \xi) - b(W_s) \right) ds \right|^2 \right] \]
we assume additionally Assumption [1.2] i.e. that the irregular part of the drift has Sobolev-Slobodeckij regularity of order \( \kappa \in (0, 1) \).

3.1 Analytic preliminaries

As a preparation we need:

**Lemma 3.1.** Let Assumptions [1.1] and [1.2] hold. Then we have \( |\varphi'|_{\kappa} < \infty \).

**Proof.** We can write
\[ (\varphi'(x) - (\varphi'(y) = \varphi'(x)(b(x) - b(y)) + b(y)(\varphi'(x) - \varphi'(y)) \]
Since \( \varphi' \) is bounded, we have that
\[ \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\varphi'(x)(b(x) - b(y))|^2}{|x-y|^{1+2\kappa}} \; dx \; dy \leq \|\varphi'\|_{\infty}^2 \|b\|_{\kappa}^2. \]
Moreover, the boundedness of \( \varphi'' \) implies
\[ |b(y)(\varphi'(x) - \varphi'(y))|^2 \leq |b(y)|^2 \|\varphi''\|_{\infty}^2 |x-y|^2. \]
Since \( b \) is bounded and \( b \in L^1(\mathbb{R}) \), it follows that \( b \in L^2(\mathbb{R}) \). Hence, it follows that for all \( \kappa \in (0, 1) \) we have
\[ \int_{\mathbb{R}} \int_{y-1}^{y+1} \frac{|b(y)(\varphi'(x) - \varphi'(y))|^2}{|x-y|^{1+2\kappa}} \; dx \; dy \leq 2\|\varphi''\|_{\infty}^2 \int_{\mathbb{R}} |b(y)|^2 \int_{y-1}^{y+1} |x-y|^{-1-2\kappa} \; dx \; dy \]
\[ = \frac{1}{1-\kappa} \|\varphi''\|_{\infty}^2 \|b\|_{L^2}^2 < \infty. \]
Furthermore, the boundedness of \( \varphi' \) yields
\[ \int_{\mathbb{R}} \int_{y+1}^{\infty} \frac{|b(y)(\varphi'(x) - \varphi'(y))|^2}{|x-y|^{1+2\kappa}} \; dx \; dy \leq 4\|\varphi'\|_{\infty}^2 \int_{\mathbb{R}} |b(y)|^2 \int_{y+1}^{\infty} |x-y|^{-1-2\kappa} \; dx \; dy \]
\[ = \frac{2}{\kappa} \|\varphi'\|_{\infty}^2 \|b\|_{L^2}^2 < \infty, \]
and analogously
\[ \int_{\mathbb{R}} \int_{-\infty}^{y-1} \frac{|b(y)(\varphi'(x) - \varphi'(y))|^2}{|x-y|^{1+2\kappa}} \; dx \; dy \leq \frac{2}{\kappa} \|\varphi'\|_{\infty}^2 \|b\|_{L^2}^2 < \infty. \]
Thus, the assertion follows. \( \square \)
Since the Sobolev-Slobodeckij semi-norm is shift invariant, Lemma 3.1 also yields:

**Corollary 3.2.** Let Assumptions 1.1 and 1.2 hold. Then we have \( P \)-a.s. that

\[ |\varphi' b(\cdot + \xi)|_\kappa = |\varphi' b|_\kappa < \infty. \]

In the following, we will frequently use that for all \( p \geq 0 \) there exists a constant \( c_p > 0 \) such that for all \( w \in \mathbb{R} \) we have

\[ |w|^p \exp(-w^2/2) \leq c_p \exp(-w^2/4). \tag{20} \]

A crucial tool will be the following bound on the Gaussian density:

**Lemma 3.3.** Let \( t > s > 0 \) and

\[ p_t,s(x, y) = \frac{1}{2\pi} \left( \frac{1}{\sqrt{s(t-s)}} \right) \exp \left( -\frac{(x-y)^2}{2(t-s)} \right), \quad x, y \in \mathbb{R}. \tag{21} \]

Then we have

\[ \frac{\partial^2}{\partial t \partial s} p_t,s(x, y) = \frac{1}{4} p_t,s(x, y) \left( \frac{y^2}{s^2} - \frac{1}{s} \right) \left( \frac{(y-x)^2}{(t-s)^2} - \frac{1}{t-s} \right) \]

\[ - \frac{1}{4} p_t,s(x, y) \left( \frac{(y-x)^2}{(t-s)^2} - \frac{1}{t-s} \right)^2 \]

\[ + \frac{1}{2} p_t,s(x, y) \left( \frac{2(y-x)^2}{(t-s)^3} - \frac{1}{(t-s)^2} \right) \tag{22} \]

and there exists a constant \( c_n^{(p)} > 0 \) such that

\[ -|x-y|^{1+2\kappa} \frac{\partial^2}{\partial t \partial s} p_t,s(x, y) \leq c_n^{(p)} \left( |t-s|^{\kappa-2}s^{-1/2} + |t-s|^{\kappa-1}s^{-3/2} \right). \]

**Proof.** Straightforward calculations yield the first assertion (22).

Moreover, we have

\[ -|x-y|^{1+2\kappa} \frac{\partial^2}{\partial t \partial s} p_t,s(x, y) \]

\[ \leq \left[ \frac{3}{4} \frac{1}{(t-s)^2} + \frac{1}{4} \frac{(x-y)^4}{(t-s)^4} + \frac{1}{4} \frac{y^2}{s^2} \frac{1}{t-s} + \frac{1}{4} \frac{1}{s(t-s)^2} \right] |x-y|^{1+2\kappa} p_t,s(x, y) \]

\[ = \frac{1}{8\pi} \left( \frac{1}{\sqrt{s(t-s)}} \right) \left( \frac{1}{|t-s|^{5/2-\kappa}} \right) \exp \left( -\frac{(x-y)^2}{2(t-s)} \right) \exp \left( -\frac{y^2}{2s} \right) \]

\[ \times \left[ \frac{3}{4} \frac{|x-y|^{1+2\kappa}}{|t-s|^{1/2+\kappa}} + \frac{|x-y|^{5+2\kappa}}{|t-s|^{5/2+\kappa}} \right] \]

\[ + \frac{1}{8\pi} \left( \frac{1}{\sqrt{s(t-s)}} \right) \left( \frac{1}{|t-s|^{1/2-\kappa}} \right) \exp \left( -\frac{(x-y)^2}{2(t-s)} \right) \exp \left( -\frac{y^2}{2s} \right) \]

\[ \times \left[ \frac{y^2}{s} \frac{|x-y|^{1+2\kappa}}{|t-s|^{1/2+\kappa}} + \frac{|x-y|^{3+2\kappa}}{|t-s|^{3/2+\kappa}} \right]. \tag{23} \]

Setting \( w^2 = x^2/(t-s) \) respectively \( w^2 = y^2/s \) in (20), we obtain that for every \( p \geq 0 \) there exists a constant \( c_{2p} > 0 \) such that for all \( t > s > 0 \) and \( x, y \in \mathbb{R} \) it holds

\[ \frac{|x-y|^{2p}}{|t-s|^p} \exp \left( -\frac{|x-y|^2}{2(t-s)} \right) \leq c_{2p} \exp \left( -\frac{|x-y|^2}{4(t-s)} \right), \]

\[ \frac{y^{2p}}{s^p} \exp \left( -\frac{y^2}{2s} \right) \leq c_{2p} \exp \left( -\frac{y^2}{4s} \right). \]
This and \[ (23) \] establish that there exist constants \( c_{1+2\kappa}, c_{5+2\kappa}, c_2, c_{3+2\kappa} > 0 \) such that

\[
-8\pi \sqrt{s(t-s)} |x-y|^{1+2\kappa} \frac{\partial^2}{\partial t \partial s} p_{t,s}(x,y) \\
\leq \left[ \frac{3c_{1+2\kappa}}{t-s} \right]^{3/2-\kappa} \exp \left( -\frac{(x-y)^2}{4(t-s)} - \frac{y^2}{2s} \right) + \frac{c_{5+2\kappa}}{s(t-s)^{3/2-\kappa}} \exp \left( -\frac{(x-y)^2}{4(t-s)} - \frac{y^2}{2s} \right) \\
+ \frac{c_{2}c_{1+2\kappa}}{s(t-s)^{1/2-\kappa}} \exp \left( -\frac{(x-y)^2}{4(t-s)} - \frac{y^2}{4s} \right) + \frac{c_{3+2\kappa}}{s(t-s)^{1/2-\kappa}} \exp \left( -\frac{(x-y)^2}{4(t-s)} - \frac{y^2}{2s} \right) .
\]

Hence, there exists a constant \( c_{\kappa}^{(p)} > 0 \) such that

\[
-|x-y|^{1+2\kappa} \frac{\partial^2}{\partial t \partial s} p_{t,s}(x,y) \leq c_{\kappa}^{(p)} |t-s|^{\kappa-3/2}, \frac{1}{\sqrt{s(t-s)}} \exp \left( -\frac{(x-y)^2}{4(t-s)} - \frac{y^2}{2s} \right) \\
+ c_{\kappa}^{(p)} |t-s|^{\kappa-1} \frac{1}{s} \exp \left( -\frac{(x-y)^2}{4(t-s)} - \frac{y^2}{4s} \right) .
\]

Using that the exponential terms above are bounded by one, we have

\[
-|x-y|^{1+2\kappa} \frac{\partial^2}{\partial t \partial s} p_{t,s}(x,y) \leq c_{\kappa}^{(p)} |t-s|^{\kappa-2} s^{-1/2} + c_{\kappa}^{(p)} |t-s|^{\kappa-1} s^{-3/2} .
\]

\[ \square \]

Finally, we will need:

**Lemma 3.4.** Let \( p \in (0, 1) \). It holds that

\[
\sum_{k=1}^{n-1} t_k^{-p} (t_{k+1} - t_k) = \int_{t_1}^{T} \left( \frac{1}{s} \right)^p ds \leq \frac{T^{1-p}}{1-p} .
\]

**Proof.** We have that

\[
\sum_{k=1}^{n-1} t_k^{-p} (t_{k+1} - t_k) = \int_{t_1}^{T} \left( \frac{1}{s} \right)^p ds \leq \int_{0}^{T-t_1} \left( \frac{1}{s} \right)^p ds \leq \int_{0}^{T} \left( \frac{1}{s} \right)^p ds = \frac{T^{1-p}}{1-p} .
\]

\[ \square \]

### 3.2 Stochastic preliminaries

We denote by \( \phi_\vartheta \) the function \( \phi_\vartheta(x) = \frac{1}{\sqrt{2\pi \vartheta}} \exp \left( -\frac{x^2}{2\vartheta} \right) \), \( x \in \mathbb{R} \), \( \vartheta > 0 \). We require the following auxiliary result.

**Lemma 3.5.** Let \( \kappa \in (0, 1) \), and let \( f : \mathbb{R} \to \mathbb{R} \) be measurable such that \( |f|_\kappa < \infty \). Then there exists a constant \( c_\kappa > 0 \) such that for all \( 0 < s \leq t \leq T \) we have

\[
\mathbb{E} \left[ |f(W_t + \xi) - f(W_s + \xi)|^2 \right] \leq c_\kappa |f|^2 \cdot (t-s)^{\kappa s^{-1/2}} .
\]

**Proof.** Clearly, we have

\[
\mathbb{E} \left[ |f(W_t + \xi) - f(W_s + \xi)|^2 \right] = \mathbb{E} \left[ \mathbb{E} \left[ |f(W_t + \xi) - f(W_s + \xi)|^2 | \mathcal{F}_0 \right] \right] .
\]

Since \( W \) is independent of \( \mathcal{F}_0 \), we obtain

\[
\mathbb{E} \left[ |f(W_t + \xi) - f(W_s + \xi)|^2 | \mathcal{F}_0 \right] = \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x + y + \xi) - f(y + \xi))^2 \phi_{t-s}(x) \phi_s(y) dy dx .
\]
Now write
\[
\int \int _{\mathbb{R}^{2}} (f(x + y + \xi) - f(y + \xi))^2 \phi_{t-s}(x) \phi_{s}(y) dy dx
\]
\[
= (t-s)^{1/2+\kappa} \int \int _{\mathbb{R}^{2}} \frac{(f(x + y + \xi) - f(y + \xi))^2}{|x|^{1+2\kappa}} \phi_{t-s}(x) \phi_{s}(y) dy dx.
\]
Next we use (20) with \( w^2 = x^2/(t-s) \). This yields for all \( x \in \mathbb{R} \),
\[
\frac{|x|^{1+2\kappa}}{(t-s)^{1/2+\kappa}} \phi_{t-s}(x) = \frac{|x|^{1+2\kappa}}{(t-s)^{1/2+\kappa}} \exp \left( -\frac{x^2}{2(t-s)} \right)
\]
\[
\leq c_{1+2\kappa} \frac{1}{2\pi(t-s)^{1/2+\kappa}} \exp \left( -\frac{x^2}{4(t-s)} \right) \leq c_{1+2\kappa} \frac{1}{2\pi(t-s)^{1/2+\kappa}}.
\]
Since moreover \( \phi_{s}(y) \leq \frac{1}{\sqrt{2\pi}s} \), Corollary 3.2 yields
\[
\mathbb{E}[|f(W_{t} + \xi) - f(W_{s} + \xi)|^2] \leq \frac{c_{1+2\kappa}}{2\pi} (t-s)^{\kappa} s^{-1/2} \int \int _{\mathbb{R}^{2}} \mathbb{E} \left[ \frac{(f(z + \xi) - f(y + \xi))^2}{|z - y|^{1+2\kappa}} \right] dy dz
\]
\[
= \frac{c_{1+2\kappa}}{2\pi} (t-s)^{\kappa} s^{-1/2} |f|_2^2,
\]
which is the desired statement.

The following Lemma deals with an integration problem seemingly similar to \( \mathcal{W}(\pi_n) \). However, the transformation of \( W + \xi \) has significantly more smoothness here.

**Lemma 3.6.** Let \( \kappa > 0 \) and \( \psi_3, \psi_4 : \mathbb{R} \to \mathbb{R} \) be bounded and measurable functions. Moreover, let \( \psi_3 \) be absolutely continuous with bounded Lebesgue-density \( \psi_3' : \mathbb{R} \to \mathbb{R} \) that satisfies \( |\psi_3'|_\kappa < \infty \). Then, there exists a constant \( c_{\psi_3,\psi_4,\kappa,T}^{(qs)} > 0 \) such that
\[
\mathbb{E} \left[ \left( \int _{0}^{T} (\psi_3(W_{s} + \xi) - \psi_3(W_{s} + \xi)) \psi_4(W_{s} + \xi) ds \right)^2 \right] \leq c_{\psi_3,\psi_4,\kappa,T}^{(qs)}\|\pi_n\|^{1+\kappa}.
\]

**Proof.** The fundamental theorem of Lebesgue-integral calculus implies
\[
\mathbb{E} \left[ \left( \int _{0}^{T} (\psi_3(W_{s} + \xi) - \psi_3(W_{s} + \xi)) \psi_4(W_{s} + \xi) ds \right)^2 \right]
\]
\[
= \mathbb{E} \left[ \left( \int _{0}^{T} \int _{0}^{1} (W_{s} - W_{s}^2) \psi_3' (\xi + W_{s} + \gamma(W_{s} - W_{s}^2)) \psi_4(W_{s} + \xi) d\gamma ds \right)^2 \right] \tag{24}
\]
\[
\leq 2 (\mathcal{E}_1 + \mathcal{E}_2),
\]
where
\[
\mathcal{E}_1 = \mathbb{E} \left[ \left( \int _{0}^{T} \int _{0}^{1} \left[ \psi_3'(W_{s} + \xi + \gamma(W_{s} - W_{s}^2)) - \psi_3'(W_{s} + \xi) \right] (W_{s} - W_{s}^2) \psi_4(W_{s} + \xi) d\gamma ds \right)^2 \right],
\]
\[
\mathcal{E}_2 = \mathbb{E} \left[ \left( \int _{0}^{T} (\psi_3' \psi_4)(W_{s} + \xi) (W_{s} - W_{s}^2) ds \right)^2 \right].
\]
For the second term, we apply Lemma 2.3 with \( \psi_1 = \psi_3 \psi_4 \), \( \psi_2 = 1 \) and obtain
\[
\mathcal{E}_2 \leq \frac{T}{2} \|\psi_3 \psi_4\|_\infty^2 \|\pi_n\|^2 \tag{25}.
\]
For $\mathcal{E}_1$ the Cauchy Schwartz inequality gives
\[
\mathcal{E}_1 \leq T \int_0^T \int_0^1 \mathbb{E} \left[ \left( \psi_3'(W_{z, s} + \xi + \gamma(W_s - W_z)) - \psi_3'(W_{z, s}) \right) (W_s - W_z) \psi_4(W_{z, s} + \xi) \right]^2 \, d\gamma ds.
\]
Splitting the time integral yields
\[
\mathcal{E}_1 \leq 4T \|\psi_3'\|_\infty^2 \|\psi_4\|_\infty^2 T^2 + T \tilde{\mathcal{E}}_1 \leq 4T \|\psi_3'\|_\infty^2 \|\psi_4\|_\infty^2 \|\pi_n\|^2 + T \tilde{\mathcal{E}}_1,
\]
where
\[
\tilde{\mathcal{E}}_1 = \int_t^T \int_0^1 \mathbb{E} \left[ \left( \psi_3'(W_{z, s} + \xi + \gamma(W_s - W_z)) - \psi_3'(W_{z, s}) \right) (W_s - W_z) \psi_4(W_{z, s} + \xi) \right]^2 \, d\gamma ds.
\]
Now write
\[
\tilde{\mathcal{E}}_1 = \int_t^T \int_0^1 \mathbb{E} \left[ \left( \psi_3'(y + \xi + x) - \psi_3'(y + \xi) \right)^2 \left( \frac{x}{\gamma} \right)^2 (\psi_4(y + \xi))^2 \right. \times \phi_{\gamma^2(s-z)}(x) \phi_2(y) \, d\gamma ds dy dx.
\]
With $\phi_2(y) \leq \frac{1}{\sqrt{2\pi\gamma}}$ for all $y \in \mathbb{R}$, we obtain
\[
\tilde{\mathcal{E}}_1 \leq \frac{\|\psi_4\|_\infty^2}{\sqrt{2\pi}} \mathbb{E} \left[ \int_R \int_R \int_t^T \int_0^1 (s^{-1/2} |\psi_3'(y + \xi + x) - \psi_3'(y + \xi)|^2) \left( \frac{x}{\gamma} \right)^2 \phi_{\gamma^2(s-z)}(x) \, d\gamma ds dy dx \right]
\]
\[
= \frac{\|\psi_4\|_\infty^2}{\sqrt{2\pi}} \mathbb{E} \left[ \int_R \int_R \int_t^T \int_0^1 \gamma^{1+2\kappa} s^{-1/2} (s-z)^{3/2+\kappa} \left( |\psi_3'(y + \xi + x) - \psi_3'(y + \xi)|^2 \right)^2 \frac{x^{1+2\kappa}}{\gamma^{1+2\kappa}} \phi_{\gamma^2(s-z)}(x) \, d\gamma ds dy dx \right].
\]
Setting $u^2 = x^2/\gamma^2(s-z)$ in (20) we get that for all $x \in \mathbb{R}, s \in (0, T], \gamma \in (0, 1]$ there exists a constant $c_{3+2\kappa} > 0$ such that
\[
\frac{|x|^{3+2\kappa}}{(\gamma^2(s-z))^{3/2+\kappa}} \phi_{\gamma^2(s-z)}(x) = \frac{|x|^{3+2\kappa}}{(\gamma^2(s-z))^{3/2+\kappa}} \exp \left( -\frac{x^2}{2\gamma^2(s-z)} \right) \frac{1}{\sqrt{2\pi\gamma^2(s-z)}} \leq c_{3+2\kappa} \exp \left( -\frac{x^2}{4\gamma^2(s-z)} \right) \frac{1}{\sqrt{2\pi\gamma^2(s-z)}} \leq c_{3+2\kappa} \frac{1}{\sqrt{2\pi\gamma^2(s-z)}}.
\]
Therefore,
\[
\tilde{\mathcal{E}}_1 \leq \frac{c_{3+2\kappa}}{2\pi} \mathbb{E} \left[ \int_R \int_R \int_t^T \int_0^1 \gamma^{2\kappa} s^{-1/2} (s-z)^{1+\kappa} \phi_{\gamma^2(s-z)}(x) \, d\gamma ds dy dx \right]
\]
\[
= \frac{c_{3+2\kappa}}{2\pi} \int_0^T \int_0^1 \gamma^{2\kappa} s^{-1/2} (s-z)^{1+\kappa} \mathbb{E} \left[ \left| \psi_3'(y + \xi + x) - \psi_3'(y + \xi) \right|^2 \right] \, d\gamma ds dy dx.
\]
Corollary 3.2 gives
\[
\mathbb{E} \left[ \left| \psi_3'(y + \xi + x) - \psi_3'(y + \xi) \right|^2 \right] = |\psi_3'|_\kappa^2,
\]
and hence we obtain
\[
\tilde{\mathcal{E}}_1 \leq \frac{c_{3+2\kappa}}{2\pi} \| \psi_4 \|^2 \| \psi_4' \|^2 \int_{t_{11}}^T \int_0^1 \gamma^{2\kappa} s^{-1/2} (s - \xi)^{1+\kappa} d\gamma ds.
\]
Lemma 3.4 now yields that
\[
\tilde{\mathcal{E}}_1 \leq \frac{c_{3+2\kappa}}{2\pi} \| \psi_4 \|^2 \| \psi_4' \|^2 \sqrt{T} \| \pi_n \|^{1+\kappa}.
\] (27)
Combining (24), (25), (26), and (27) concludes the proof. \(\square\)

### 3.3 Error analysis of the quadrature problem

Now we will consider two specific discretizations: an equidistant discretization \(\pi_{\text{equi}}^n\) given by
\[
t_{k}^{\text{equi}} = T \frac{k}{n}, \quad k = 0, \ldots, n,
\] (28)
and the non-equidistant discretization \(\pi_n^*\) given by
\[
t_{k}^{*} = T \left( \frac{k}{n} \right)^2, \quad k = 0, \ldots, n.
\] (29)
Clearly, we have
\[
t_{k+1}^{*} - t_{k}^{*} = \frac{2k + 1}{n} \cdot \frac{T}{n}, \quad k = 0, \ldots, n - 1,
\]
and
\[
\| \pi_n^* \| = \max_{k=0,\ldots,n-1} |t_{k+1}^{*} - t_{k}^{*}| = \left( 2 - \frac{1}{n} \right) \cdot \frac{T}{n} \leq \frac{2T}{n}.
\] (30)

Our main result is:

**Theorem 3.7.** Let Assumptions 1.1 and 1.2 hold. Then there exist constants \(C_{(Q),\text{equi}}^{(Q)} > 0\) and \(C_{(Q),*}^{(Q)} > 0\) such that
\[
\mathcal{W}(\pi_{\text{equi}}^n) \leq C_{b,T,\kappa}^{(Q),\text{equi}} \cdot \left( \frac{1}{n^{1+\kappa}} + \frac{1}{n^{3/2}} \right)
\]
and
\[
\mathcal{W}(\pi_n^*) \leq C_{b,T,\kappa}^{(Q),*} \cdot \frac{\log(n)}{n^{1+\kappa}}.
\]

**Proof.** We will start with an arbitrary discretization and specialize only in the final steps to \(\pi_{\text{equi}}^n\) or \(\pi_n^*\). For estimating \(\mathcal{W}(\pi_n)\) we use that
\[
\mathcal{W}(\pi_n) \leq 2 (\mathcal{W}_1 + \mathcal{W}_2),
\] (31)
where
\[
\mathcal{W}_1 = \mathbb{E} \left[ \left| \int_0^T \left[ (\varphi' b)(W_x + \xi) - (\varphi' b)(W_x + \xi) \right] ds \right|^2 \right],
\]
\[
\mathcal{W}_2 = \mathbb{E} \left[ \left| \int_0^T \left[ \varphi'(W_x + \xi) - \varphi'(W_x + \xi) \right] b(W_x + \xi) ds \right|^2 \right].
\]
Step 1. Setting $\psi_3 = \varphi'$ and $\psi_4 = b$, noting that $\varphi'' = -2b\varphi'$, and using Lemma 3.4 we obtain that Lemma 3.6 can be applied to estimate $W_2$. Thus, there exists a constant $c_{\varphi',b,K,T}^{(qs)} > 0$ such that

$$W_2 \leq c_{\varphi',b,K,T}^{(qs)} ||\pi_n||^{1+\kappa}. \quad (32)$$

Step 2. For the remaining term, note that

$$|\varphi' b(\cdot + \xi)|_\kappa = |\varphi' b|_\kappa < \infty$$

by Corollary 3.2 and that

$$W_1 \leq 8||\varphi' b||_2^2 t_1^2 + 2\mathbb{E} \left[ \int_{t_1}^T \left[ (\varphi' b)(W_s + \xi) - (\varphi' b)(W_2 + \xi) \right] ds \right]^2 \quad (33)$$

$$\leq 8||\varphi' b||_2^2 ||\pi_n||^2 + 2\mathbb{E} \left[ \int_{t_1}^T \left[ (\varphi' b)(W_s + \xi) - (\varphi' b)(W_2 + \xi) \right] ds \right]^2.$$

In the following, denote

$$I^{k,\ell} = \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} \left( (\varphi' b)(W_s + \xi) - (\varphi' b)(W_{t_k} + \xi) \right) \left( (\varphi' b)(W_t + \xi) - (\varphi' b)(W_{t_\ell} + \xi) \right) dt ds.$$

We have that

$$\mathbb{E} \left[ \left( \int_{t_1}^T \left[ (\varphi' b)(W_s + \xi) - (\varphi' b)(W_2 + \xi) \right] ds \right)^2 \right] = 2 \sum_{k=2}^{n-1} \sum_{\ell=1}^{n-1} \mathbb{E}[I^{k,\ell}] + \sum_{k=1}^{n-1} \mathbb{E}[I^{k,k}]. \quad (34)$$

Step 3. Using $2xy \leq x^2 + y^2$ for $x, y \in \mathbb{R}$ we obtain that

$$\sum_{k=1}^{n-1} \mathbb{E}[I^{k,k}] = \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \left( (\varphi' b)(W_s + \xi) - (\varphi' b)(W_{t_k} + \xi) \right) \times \left( (\varphi' b)(W_t + \xi) - (\varphi' b)(W_{t_k} + \xi) \right) \right] dt ds$$

$$\leq \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \left| (\varphi' b)(W_s + \xi) - (\varphi' b)(W_{t_k} + \xi) \right|^2 \right] dt ds.$$

For $s \geq t_k \geq t_\ell$, Lemma 3.5 shows that there exist a constant $c_\kappa > 0$ such that

$$\sum_{k=1}^{n-1} \mathbb{E}[I^{k,k}] \leq \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} c_\kappa |\varphi' b|^2 (s - t_k)^{\kappa} t_k^{-1/2} dt ds$$

$$\leq c_\kappa |\varphi' b|^2 ||\pi_n||^{1+\kappa} \sum_{k=1}^{n-1} t_k^{-1/2} (t_{k+1} - t_k).$$

Now, Lemma 3.4 gives

$$\sum_{k=1}^{n-1} \mathbb{E}[I^{k,k}] \leq 2c_\kappa T^{1/2} |\varphi' b|^2 ||\pi_n||^{1+\kappa}. \quad (35)$$

It remains to take care of the off-diagonal terms with $k - \ell \geq 1$.
Step 4. Consider the case $\ell = k - 1 \neq 0$. Again using $2xy \leq x^2 + y^2$ for $x, y \in \mathbb{R}$, Lemma 3.4 and Lemma 3.5 we get that

\[
2 \sum_{k=2}^{n-1} \mathbb{E}[I^{k,k-1}] = 2 \sum_{k=2}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} \mathbb{E}\left( ((\varphi'b)(W_s + \xi) - (\varphi'b)(W_{t_k} + \xi)) \times \left( (\varphi'b)(W_t + \xi) - (\varphi'b)(W_{t_k} + \xi) \right) \right) dt ds
\]

\[
\leq \sum_{k=2}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[ |(\varphi'b)(W_s + \xi) - (\varphi'b)(W_{t_k} + \xi)|^2 \right] dt ds
\]

\[
+ \sum_{k=2}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[ |(\varphi'b)(W_t + \xi) - (\varphi'b)(W_{t_k} + \xi)|^2 \right] dt ds
\]

\[
\leq \sum_{k=2}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} c_n |\varphi'b|^2_{\infty}\kappa^s_1 t_k^{-1/2} dt ds
\]

\[
+ \sum_{k=2}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} c_n |\varphi'b|^2_{\infty}(t - t_{k-1})^{-1/2} dt ds
\]

\[
\leq 2c_n |\varphi'b|^2_{\infty} \|\pi_n\|^{1+\kappa} \sum_{k=1}^{n-1} t_k^{-1/2} (t_{k+1} - t_k) \leq 4c_n T^{1/2} |\varphi'b|^2_{\infty} \|\pi_n\|^{1+\kappa}.
\]

Step 5. Now assume $k \geq \ell + 2$ and use (21). We get

\[
\mathbb{E}[I^{k,\ell}|\mathcal{F}_0] = \int_{\mathbb{R}} \int_{\mathbb{R}} ((\varphi'b)(x + \xi)(\varphi'b)(y + \xi)) \int_{t_k}^{t_{k+1}} \int_{t_{\ell}}^{t_{\ell+1}} (p_{s,t}(x,y) - p_{t_k,t}(x,y) - p_{s,t}(x,y) + p_{t_k,t}(x,y)) dtds dx dy.
\]

First note that

\[
p_{s,t}(x,y) - p_{t_k,t}(x,y) - p_{s,t}(x,y) + p_{t_k,t}(x,y) = \int_{t_k}^{s} \int_{t_k}^{t} \frac{\partial^2}{\partial u \partial v} p_{u,v}(x,y) dudv.
\]

Now observe that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} ((\varphi'b)(x + \xi)^2 (p_{s,t}(x,y) - p_{t_k,t}(x,y) - p_{s,t}(x,y) + p_{t_k,t}(x,y)) dxy = (\mathbb{E}[|(\varphi'b)(W_s + \xi)|^2|\mathcal{F}_0] - \mathbb{E}[|(\varphi'b)(W_t + \xi)|^2|\mathcal{F}_0])
\]

\[
- (\mathbb{E}[|(\varphi'b)(W_s + \xi)|^2|\mathcal{F}_0] - \mathbb{E}[|(\varphi'b)(W_t + \xi)|^2|\mathcal{F}_0]) = 0
\]

and analogously

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} ((\varphi'b)(y + \xi)^2 (p_{s,t}(x,y) - p_{t_k,t}(x,y) - p_{s,t}(x,y) + p_{t_k,t}(x,y)) dxy = 0.
\]

Combining this with (37) and (38) we obtain

\[
\mathbb{E}[I^{k,\ell}|\mathcal{F}_0] = -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |(\varphi'b)(x + \xi) - (\varphi'b)(y + \xi)|^2
\]

\[
\times \int_{t_k}^{t_{k+1}} \int_{t_{\ell}}^{t_{\ell+1}} \int_{t_k}^{s} \int_{t_k}^{t} \frac{\partial^2}{\partial u \partial v} p_{u,v}(x,y) dudv dtds dx dy
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|(\varphi'b)(x + \xi) - (\varphi'b)(y + \xi)|^2}{|x - y|^{1+2\kappa}}
\]

\[
\times \int_{t_k}^{t_{k+1}} \int_{t_{\ell}}^{t_{\ell+1}} \int_{t_k}^{s} \int_{t_k}^{t} |x - y|^{1+2\kappa} \frac{\partial^2}{\partial u \partial v} p_{u,v}(x,y) dudv dtds dx dy.
\]
Corollary 3.2 and Lemma 3.3 ensure that there exists a constant $c^{(p)}_c > 0$ such that

$$\mathbb{E}[I_{k,\ell}^{c}|\mathcal{F}_0] \leq \frac{c^{(p)}_c}{2} |\varphi'|^2_{b,\kappa}\int_{t_k}^{t_{\ell+1}} \int_{t_\ell}^{t_{\ell+1}} \int_{t_k}^{t_{\ell+1}} \left(|u - v|^{\kappa-2} v^{-1/2} + |u - v|^{\kappa-1} v^{-3/2}\right) dv du \, ds$$

$$\leq \frac{c^{(p)}_c}{2} |\varphi'|^2_{b,\kappa}(t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} \left(|u - v|^{\kappa-2} v^{-1/2} + |u - v|^{\kappa-1} v^{-3/2}\right) dv du.$$

Hence,

$$2 \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} \mathbb{E}[I_{k,\ell}^{c}] = 2 \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} \mathbb{E}[\mathbb{E}[I_{k,\ell}^{c}|\mathcal{F}_0]]$$

$$\leq c^{(p)}_c |\varphi'|^2_{b,\kappa} \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} \left(|u - v|^{\kappa-2} v^{-1/2} + |u - v|^{\kappa-1} v^{-3/2}\right) dv du.$$

Summarizing the above estimates (31), (32), (33), (34), (35), (36), and (39), establishes that

$$\mathcal{W}^{(p,\kappa)}(\pi_n) \leq \left(2c^{(p,\kappa)}_{\varphi, b, \kappa, 2} + 16 |\varphi'|^2_{b,\kappa} \mathbb{E}[\mathbb{E}[I_{k,\ell}^{c}|\mathcal{F}_0]] + 24 c_{\kappa} T^{1/2} |\varphi'|^2_{b,\kappa} \right) \mathbb{E}[|\pi_n|^1]^{1+\kappa}$$

$$+ 4 c^{(p)}_c |\varphi'|^2_{b,\kappa} \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} \left(|u - v|^{\kappa-2} v^{-1/2} + |u - v|^{\kappa-1} v^{-3/2}\right) dv du.$$  

**Step 6, Case 1.** First consider the non-equidistant discretization [29]. Observe that

$$2(1 - \kappa) \int_{t_\ell}^{t_{\ell+1}} |u - v|^{\kappa-2} v^{-1/2} dv = \int_{t_\ell}^{t_{\ell+1}} |u - v|^{\kappa-1} v^{-3/2} dv + 2 |u - v|^{\kappa-1} v^{-3/2} \big|_{v = t_{\ell+1}}.$$

Thus we have

$$\sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} \left(|u - v|^{\kappa-2} v^{-1/2} + |u - v|^{\kappa-1} v^{-3/2}\right) dv du$$

$$= \left(1 + \frac{1}{2(1 - \kappa)}\right) T^{(1)} + \frac{1}{1 - \kappa} T^{(2)}$$

with

$$T^{(1)} = \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} |u - v|^{\kappa-1} v^{-3/2} dv du,$$

$$T^{(2)} = \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \left(|u - t_{\ell+1}|^{\kappa-1} t_{\ell+1}^{-1/2} - |u - t_\ell|^{\kappa-1} t_\ell^{-1/2}\right) du.$$
Since \( k \geq \ell + 2, \kappa \in (0, 1) \) and \( x^\kappa - y^\kappa \leq |x-y|^\kappa \) for \( x > y \geq 0 \) we have

\[
\int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} |u - v|^\kappa - 1 v^{-3/2} \, dv \, du \\
\leq \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} |u - t_{\ell+1}|^\kappa - 1 v^{-3/2} \, dv \, du \\
= 2 \left( t_{\ell}^{-1/2} - t_{\ell+1}^{-1/2} \right) \int_{t_k}^{t_{k+1}} |u - t_{\ell+1}|^\kappa - 1 du \\
= \frac{2}{\kappa} \left( t_{\ell}^{-1/2} - t_{\ell+1}^{-1/2} \right) (|t_{k+1} - t_{\ell+1}|^\kappa - |t_k - t_{\ell+1}|^\kappa) \\
\leq \frac{2}{\kappa} \left( t_{\ell}^{-1/2} - t_{\ell+1}^{-1/2} \right) |t_{k+1} - t_k|^\kappa.
\]

Thus it follows

\[
\sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} |u - v|^\kappa - 1 v^{-3/2} \, dv \, du \\
\leq \frac{2}{\kappa} \| \pi_n^\kappa \|^\kappa \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \left( t_{\ell}^{-1/2} - t_{\ell+1}^{-1/2} \right) \\
\leq \frac{2T}{\kappa} \| \pi_n^\kappa \|^\kappa \sum_{\ell=1}^{n-3} \left( t_{\ell+1} - t_\ell \right) \left( t_{\ell}^{-1/2} - t_{\ell+1}^{-1/2} \right) \\
= \frac{2T^{3/2}}{\kappa} \| \pi_n^\kappa \|^\kappa \sum_{\ell=1}^{n-3} \frac{2\ell + 1}{n^2} \left( n - \frac{n}{\ell + 1} \right) \\
= \frac{2T^{3/2}}{\kappa} \| \pi_n^\kappa \|^\kappa \frac{1}{n} \sum_{\ell=1}^{n-3} \frac{2\ell + 1}{\ell(\ell + 1)} \\
\leq \frac{2^{2+\kappa}T^{3/2+\kappa}}{\kappa} \sum_{\ell=1}^{n-3} \frac{1}{\ell},
\]

where we have used (30) and that \( t_{\ell+1} - t_\ell = T(2\ell + 1)n^{-2} \). Since

\[
\sum_{\ell=1}^{n-1} \frac{1}{\ell} \leq \log(n),
\]

we have

\[
I_n^{(1)} \leq \frac{2^{2+\kappa}T^{3/2+\kappa}}{\kappa} \frac{\log(n)}{n^{1+\kappa}}.
\]

So, the remaining term to estimate is

\[
I_n^{(2)} = \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k)(t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \left( |u - t_{\ell+1}|^\kappa - 1 t_{\ell+1}^{-1/2} - |u - t_\ell|^\kappa - 1 t_\ell^{-1/2} \right) \, du.
\]
We get
\[
I_n^{(2)} = \sum_{\ell=1}^{n-3} \sum_{k=\ell+2}^{n-1} (t_{k+1} - t_k) (t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \left( |u - t_{\ell+1}|^{\kappa-1} t_{\ell+1}^{-1/2} - |u - t_\ell|^{\kappa-1} t_\ell^{-1/2} \right) du
\]
\[
\leq \left\| \pi_n^\ast \right\| \sum_{\ell=1}^{n-3} (t_{\ell+1} - t_\ell) \int_{t_\ell}^{T} \left( |u - t_{\ell+1}|^{\kappa-1} t_{\ell+1}^{-1/2} - |u - t_\ell|^{\kappa-1} t_\ell^{-1/2} \right) du
\]
\[
= \frac{\left\| \pi_n^\ast \right\|}{\kappa} \sum_{\ell=1}^{n-3} (t_{\ell+1} - t_\ell) \left[ |T - t_{\ell+1}|^{\kappa} - |t_{\ell+2} - t_{\ell+1}|^{\kappa} \right] t_{\ell+1}^{-1/2} - \left( |T - t_\ell|^{\kappa} - |t_{\ell+2} - t_\ell|^{\kappa} \right) t_\ell^{-1/2}.
\]
Using (29), (30), and estimating negative terms from above by zero we obtain that
\[
I_n^{(2)} \leq \frac{\left\| \pi_n^\ast \right\|}{\kappa} \sum_{\ell=1}^{n-3} (t_{\ell+1} - t_\ell) \left[ |T - t_{\ell+1}|^{\kappa} - |T - t_\ell|^{\kappa} \right] t_{\ell+1}^{-1/2} + |t_{\ell+2} - t_\ell|^{\kappa} t_\ell^{-1/2}
\]
\[
\leq \frac{2\kappa \left\| \pi_n^\ast \right\|}{\kappa} \sum_{\ell=1}^{n-3} (t_{\ell+1} - t_\ell) t_\ell^{-1/2}
\]
\[
\leq \frac{2^{1+2\kappa} T^{1+\kappa}}{\kappa} \sum_{\ell=1}^{n-3} (t_{\ell+1} - t_\ell) t_\ell^{-1/2}.
\]
Finally, Lemma 3.4 establishes
\[
I_n^{(2)} \leq \frac{2^{2+2\kappa} T^{3/2+\kappa}}{\kappa} \sum_{\ell=1}^{n-3} (t_{\ell+1} - t_\ell) t_\ell^{-1/2}.
\]
Combining (40) with (42), (43), and (44) finishes the analysis of \( W(\pi_n^\ast) \).

**Step 6, Case 2.** Now consider the equidistant discretization [28]. We make use of (41) in a different way than above. It holds that
\[
\sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} (t_{k+1} - t_k) (t_{\ell+1} - t_\ell) \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} \left( |u - v|^{\kappa-2} v^{-1/2} + |u - v|^{\kappa-1} v^{-3/2} \right) dudu
\]
\[
= (3 - 2\kappa) \sum_{\ell=1}^{n-1} \sum_{k=3}^{\ell-2} T_n^{\text{equiv},(1)} - 2 T_n^{\text{equiv},(2)}
\]
with
\[
T_n^{\text{equiv},(1)} = \frac{T^2}{n^2} \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} |u - v|^{\kappa-2} v^{-1/2} dudu,
\]
\[
T_n^{\text{equiv},(2)} = \frac{T^2}{n^2} \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} \int_{t_k}^{t_{k+1}} \left( |u - t_{\ell+1}|^{\kappa-1} t_{\ell+1}^{-1/2} - |u - t_\ell|^{\kappa-1} t_\ell^{-1/2} \right) du.
\]
Exploiting the telescoping sum in the second term, we get
\[ T_{n}^{\text{equi},(2)} = \frac{T^2}{n^2} \sum_{k=3}^{n-1} \int_{t_k}^{t_{k+1}} \left( |u - t_{k-1}|^{\kappa-1} t_{k-1}^{-1/2} - |u - t_1|^{\kappa-1} t_1^{-1/2} \right) \, du \]
\[ \geq -\frac{T^2}{n^2} \int_{t_1}^{T} |u - t_1|^{\kappa-1} t_1^{-1/2} \, du \]
\[ \geq -\frac{T^{3/2 + \kappa}}{\kappa} \frac{1}{n^{3/2}}, \]
since \( t_1 = T/n \). It follows that
\[ -2T_{n}^{\text{equi},(2)} \leq \frac{2T^{3/2 + \kappa}}{\kappa} \frac{1}{n^{3/2}}. \]

Moreover, we have
\[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} |u - v|^{\kappa-2} v^{-1/2} \, dv \, du \leq \frac{T}{n} \int_{t_k}^{t_{k+1}} |t_k - v|^{\kappa-2} v^{-1/2} \, dv \]
\[ \leq \frac{T}{n} |t_k - t_{\ell+1}|^{\kappa-2} \int_{t_k}^{t_{\ell+1}} v^{-1/2} \, dv \]
\[ = \frac{T^{\kappa-1}}{n^{\kappa-1}} |k - \ell - 1|^{\kappa-2} \int_{t_k}^{t_{\ell+1}} v^{-1/2} \, dv. \]

Thus, we end up with
\[ T_{n}^{\text{equi},(1)} \leq \frac{T^{1+\kappa}}{n^{1+\kappa}} \sum_{k=3}^{n-1} \sum_{\ell=1}^{k-2} |k - \ell - 1|^{\kappa-2} \int_{t_k}^{t_{\ell+1}} v^{-1/2} \, dv \]
\[ = \frac{T^{1+\kappa}}{n^{1+\kappa}} \sum_{\ell=1}^{n-3} \int_{t_1}^{t_{\ell+1}} v^{-1/2} \, dv \sum_{k=\ell+2}^{n-1} |k - \ell - 1|^{\kappa-2} \]
\[ \leq \frac{T^{1+\kappa}}{n^{1+\kappa}} \int_{0}^{T} v^{-1/2} \, dv \sum_{j=1}^{n-3} j^{\kappa-2} \]
\[ \leq \frac{2T^{3/2 + \kappa}}{n^{1+\kappa}} \sum_{j=1}^{n} j^{\kappa-2}, \]

where \( \kappa \in (0, 1) \) implies \( \sum_{j=1}^{\infty} j^{\kappa-2} < \infty \).

Combining (40) with (45), (46), and (47) finishes the analysis of \( \mathcal{W}(\pi_n^{\text{equi}}) \) and the proof of this theorem. \( \square \)

**Remark 3.8.** If the initial condition \( \xi \) has additionally a bounded Lebesgue-density, the analysis of the term \( \mathcal{W}_1 \) in the proof of Theorem 3.7 can be replaced by Theorem 3.6 in Altmeyer [1] to show that
\[ \mathcal{W}(\pi_n^{\text{equi}}) \leq c(Q, \text{equi}, \ell) \frac{\log(n)}{n^{1+\kappa}}, \]
i.e. there is no cut-off of the convergence order for \( \kappa \in [3/4, 1) \) for equidistant discretizations. Due to the independence of \( \xi \) and \( \mathcal{W} \), the assumption of a bounded Lebesgue-density \( \varsigma \) for \( \mathbb{P}^\xi \) leads to a smoothing effect in the integration problem; roughly spoken, \( (\varphi' b)(\cdot + \xi) \) can be replaced by the convolution \( \int_{\mathbb{R}} (\varphi b)(\cdot + z) \varsigma(z) \, dz \).
We finally obtain the following statement for the convergence rate of the Euler-Maruyama schemes $x^{(π^e)}_n$ and $x^{(π^∗)}_n$.

**Corollary 3.9.** Let Assumptions 1.1 and 1.2 hold. Then, for all $\epsilon \in (0, 1)$ there exist constants $C_{\epsilon,\mu,T,\kappa}^{(EM),e^{\text{equi}}}>0$ and $C_{\epsilon,\mu,T,\kappa}^{(EM),e^{\text{∗}}} > 0$ such that

$$E \left[ \left| X_T - x_T^{(π^e)} \right|^2 \right] \leq C_{\epsilon,\mu,T,\kappa}^{(EM),e^{\text{equi}}} \left( \frac{1}{n^{1+\kappa-\epsilon}} + \frac{1}{n^{3/2-\epsilon}} \right)$$

and

$$E \left[ \left| X_T - x_T^{(π^∗)} \right|^2 \right] \leq C_{\epsilon,\mu,T,\kappa}^{(EM),e^{\text{∗}}} \cdot \frac{1}{n^{1+\kappa-\epsilon}}.$$

**Proof.** Theorems 2.4 and 3.7 yield that there exist constants $C_{\epsilon,a,b,T}^{(R)}, C_{b,T,\kappa}^{(Q),e^{\text{∗}}} > 0$ such that

$$E \left[ \left| X_T - x_T^{(π^∗)} \right|^2 \right] \leq C_{\epsilon,a,b,T}^{(R)} \left( \left\| π^\ast_n \right\|^2 + \left( C_{b,T,\kappa}^{(Q),e^{\text{∗}}} \log(n) \right) \frac{1-\epsilon}{n^{1+\kappa}} \right)$$

$$\leq C_{\epsilon,a,b,T}^{(R)} \left[ \frac{4T^2}{n^2} + \left( C_{b,T,\kappa}^{(Q),e^{\text{∗}}} \log(n) \right) \frac{1-\epsilon}{n^{1+\kappa}} \right],$$

where we used (30). The estimate for the equidistant discretization is obtained analogously. □

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